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ON THE COST OF OBSERVABILITY IN SMALL TIMES FOR THE ONE-DIMENSIONAL HEAT EQUATION

JÉRÉMI DARDÉ AND SYLVAIN ERVEDOZA

We aim at presenting a new estimate on the cost of observability in small times of the one-dimensional heat equation, which also provides a new proof of observability for the one-dimensional heat equation. Our proof combines several tools. First, it uses a Carleman-type estimate borrowed from our previous work (*SIAM J. Control Optim.* **56**:3 (2018), 1692–1715), in which the weight function is derived from the heat kernel and which is therefore particularly easy. We also use explicit computations in the Fourier domain to compute the high-frequency part of the solution in terms of the observations. Finally, we use the Phragmén–Lindelöf principle to estimate the low-frequency part of the solution. This last step is done carefully with precise estimations coming from conformal mappings.

1. Introduction

Setting. The goal of this work is to analyze the cost of observability in small times of the one-dimensional heat equation. To fix the ideas, let $L, T > 0$ and consider the following heat equation, set in the bounded interval $(-L, L)$ and among some time interval $(0, T)$:

$$\begin{cases} \partial_t u - \partial_x^2 u = 0 & \text{in } (0,T)\times(-L,L), \\ u(t,-L) = u(t,L) = 0 & \text{in } (0,T), \\ u(0,x) = u_0(x) & \text{in } (-L,L). \end{cases} \tag{1-1}$$

In (1-1), the state $u = u(t,x)$ satisfies a heat equation, with initial datum $u_0 \in H_0^1(-L, L)$.

Our main goal is to study the cost of observability in small times T of the problem (1-1) observed from both sides $x = -L$ and $x = +L$. To be more precise, let us recall that it is by now well known that there exists $C_0(T, L)$ such that all solutions u of (1-1) with initial datum $u_0 \in H_0^1(-L, L)$ satisfy

$$\|u(T)\|_{L^2(-L,L)} \leqslant C_0(T,L)\big(\|\partial_x u(t,-L)\|_{L^2(0,T)} + \|\partial_x u(t,L)\|_{L^2(0,T)}\big). \tag{1-2}$$

In fact, the existence of the constant $C_0(T, L)$ is a consequence of the null controllability results in small times obtained by [Egorov 1963; Fattorini and Russell 1971] in the one-dimensional case. From now on, we denote by $C_0(T, L)$ the best constant in the observability inequality (1-2).

A precise description of the constant $C_0(T, L)$ as $T \to 0$ is still missing, despite several contributions in this direction, which we would like to briefly recall here. First, [Seidman 1984] showed that

$$\limsup_{T\to 0} T \log C_0(T,L) < \infty, \tag{1-3}$$

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while [Güichal 1985] proved that

$$\liminf_{T\to 0} T \log C_0(T, L) > 0. \tag{1-4}$$

Besides, due to the scaling of the equation, $C_0(T, L)$ depends only on the ratio L^2/T. Therefore, the quantity $T \log C_0(T, L)$ should be compared to L^2. We list below several contributions:

$$\liminf_{T\to 0} T \log C_0(T, L) \geqslant \tfrac{1}{4}L^2 \qquad \text{[Miller 2004]},$$

$$\liminf_{T\to 0} T \log C_0(T, L) \geqslant \tfrac{1}{2}L^2 \qquad \text{[Lissy 2015]},$$

$$\limsup_{T\to 0} T \log C_0(T, L) \leqslant 2\left(\tfrac{36}{37}\right)^2 L^2 \qquad \text{[Miller 2006]},$$

$$\limsup_{T\to 0} T \log C_0(T, L) \leqslant \tfrac{3}{4}L^2 \qquad \text{[Tenenbaum and Tucsnak 2007]}.$$

Main result. Our contribution comes in this context. Namely we prove the following result:

Theorem 1.1. *Setting*

$$K_0 = \frac{1}{4} + \frac{\Gamma\left(\frac{1}{4}\right)^2}{8\sqrt{2}\pi^2} \sum_{n\in\mathbb{N}} \frac{(-1)^n}{(2n+1)} \frac{\Gamma\left(n+\frac{1}{4}\right)}{\Gamma\left(n+\frac{7}{4}\right)} \quad (K_0 \simeq 0.6966), \tag{1-5}$$

where Γ *denotes the gamma function, we have*

$$\limsup_{T\to 0} T \log C_0(T, L) \leqslant K_0 L^2. \tag{1-6}$$

In fact, for all $K > K_0$, *there exists a constant* $C > 0$ *such that for all* $T \in (0, 1]$, *for all solutions* u *of* (1-1) *with initial datum* $u_0 \in H_0^1(-L, L)$,

$$\left\| u(T) \exp\left(\frac{x^2}{4T}\right) \right\|_{L^2(-L,L)} \leqslant C \exp\left(\frac{KL^2}{T}\right) \left(\|\partial_x u(t, -L)\|_{L^2(0,T)} + \|\partial_x u(t, L)\|_{L^2(0,T)} \right). \tag{1-7}$$

Remark 1.2. The constant K_0 in (1-5) can alternatively be written as

$$K_0 = \frac{1}{4} + \frac{2}{\pi} \frac{\int_0^{\frac{\pi}{2}} \ln\left(\cot\left(\frac{t}{2}\right)\right) \sqrt{\cos(t)}\, dt}{\int_0^{\frac{\pi}{2}} \sqrt{\cos(t)}\, dt}; \tag{1-8}$$

see Proposition 2.3 in Section 2.

Theorem 1.1 slightly improves the cost of observability in small times when compared to [Tenenbaum and Tucsnak 2007]. However, we do not claim that this bound is sharp, and this remains, to our knowledge, an open problem. In particular, we shall comment in Section 4F a possible path to improve the estimates given in Theorem 1.1.

In fact, we believe that Theorem 1.1 is interesting mostly by its proof, presented in Section 2, which combines several arguments. In particular, it uses a Carleman-type estimate, which was already used in [Dardé and Ervedoza 2018] to derive a good description of the reachable set for the one-dimensional heat equation in terms of domains of holomorphic extension of the states. This Carleman-type estimate is used to reduce the problem of observability to an estimate of the low-frequency part of the solution of (1-1). Then, we shall use Fourier analysis on the conjugated heat equation to get an exact formula for the high-frequency part of the solution of (1-1) in terms of the observations. The last part of the argument is a complex analysis argument based on the Phragmén–Lindelöf principle. We refer to Sections 2 and 3 for the detailed proof of Theorem 1.1.

Let us also mention that Theorem 1.1 is strongly connected to control theory. Indeed, let us consider the following null-controllability problem: given $T > 0$ and $y_0 \in L^2(-L, L)$, find control functions $v_-, v_+ \in L^2(0, T)$ such that the solution y of

$$\begin{cases} \partial_t y - \partial_x^2 y = 0 & \text{in } (0,T)\times(-L,L), \\ y(t,-L) = v_-(t) & \text{in } (0,T), \\ y(t,+L) = v_+(t) & \text{in } (0,T), \\ y(0,x) = y_0(x) & \text{in } (-L,L), \end{cases} \tag{1-9}$$

satisfies

$$y(T,x) = 0 \quad \text{in } (-L,L). \tag{1-10}$$

It is well known, see, e.g., [Egorov 1963; Fattorini and Russell 1971], that for any $T > 0$, one can find controls v_-, v_+ of minimal $(L^2(0,T))^2$ norm, depending linearly on $y_0 \in L^2(-L,L)$, such that the controlled trajectory, i.e., the solution of (1-9), satisfies (1-10). Besides, the $\mathscr{L}(L^2(-L,L); (L^2(0,T))^2)$-norm of the linear map $y_0 \mapsto (v_-, v_+)$ is precisely $C_0(T, L)$. In other words, $C_0(T, L)$ also characterizes the cost of controllability for the one-dimensional heat equation.

We emphasize that Theorem 1.1 also allows us to tackle some multidimensional settings. Namely, as a consequence of Theorem 1.1 and the control transmutation method, see [Miller 2006], one gets the following corollary:

Corollary 1.3. *Let Ω be a smooth bounded domain of $\mathbb{R}^d$, and let Γ_0 be an open subset of $\partial\Omega$. Let $a = a(x) \in L^\infty(\Omega; M_d(\mathbb{R}))$ and $\rho \in L^\infty(\Omega; \mathbb{R})$ be such that there exist strictly positive numbers ρ_-, ρ_+, a_- and a_+ such that for all $x \in \Omega$ and $\xi \in \mathbb{R}^d$,*

$$a_-|\xi|^2 \leqslant a(x)\xi\cdot\xi \leqslant a_+|\xi|^2, \quad \rho_- \leqslant \rho(x) \leqslant \rho_+.$$

Further assume that there exist a time $S_0 > 0$ and a constant $C > 0$ such that, for any $(w_0, w_1) \in H_0^1(\Omega)\times L^2(\Omega)$, the solution w of

$$\begin{cases} \rho(x)\,\partial_{ss} w - \operatorname{div}(a(x)\nabla w) = 0 & \text{in } (0,S)\times\Omega, \\ w(s,x) = 0 & \text{on } (0,S)\times\partial\Omega, \\ (w(0,x), \partial_s w(0,x)) = (w_0(x), w_1(x)) & \text{in } \Omega \end{cases} \tag{1-11}$$

satisfies $a(x)\nabla w\cdot n \in L^2((0,S_0)\times\Gamma_0)$ and

$$\|(w_0,w_1)\|_{H_0^1(\Omega)\times L^2(\Omega)} \leqslant C\|a(x)\nabla w\cdot n\|_{L^2((0,S_0)\times\Gamma_0)}. \tag{1-12}$$

We define $C_0(T,\Omega,\Gamma_0)$ as the best constant in the following observability inequality: for all $u_0 \in H_0^1(M)$, the solution u of

$$\begin{cases} \rho(x)\,\partial_t u - \operatorname{div}(a(x)\nabla u) = 0 & \text{in } (0,T)\times\Omega, \\ u(t,x) = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega \end{cases} \tag{1-13}$$

satisfies

$$\|u(T)\|_{L^2(M)} \leqslant C_0(T,\Omega,\Gamma_0)\|a(x)\nabla u\cdot n\|_{L^2((0,T)\times\Gamma_0)}. \tag{1-14}$$

Then we have

$$\limsup_{T\to 0} T\log C_0(T,\Omega,\Gamma_0) \leqslant K_0 S_0^2. \tag{1-15}$$

Corollary 1.3 uses the transmutation method and therefore the observability of the corresponding wave equation (1-11), which has been well-studied in the literature. In particular, if the coefficients ρ and a are $C^2(\overline{\Omega})$, according to [Bardos et al. 1988; 1992; Burq and Gérard 1997], the wave equation (1-11) satisfies the observability inequality (1-12) if and only if all the rays of geometric optics meet Γ_0 in a nondiffractive point in time less than S_0. In the case of coefficients ρ and a which are less regular, let us quote [Fanelli and Zuazua 2015] in the one-dimensional case with ρ and a in the Zygmund class, and [Dehman and Ervedoza 2017] in the multidimensional case for coefficients $\rho \in C^0(\overline{\Omega})$ and $a = 1$, with ρ satisfying a multiplier-type condition similar to the one in [Ho 1986; Lions 1988] in the sense of distributions (and ρ locally C^1 close to the boundary, see [Dehman and Ervedoza 2017, Section 4.2]).

Let us emphasize that Corollary 1.3 can be applied in the one-dimensional case as well for coefficients in the Zygmund class [Fanelli and Zuazua 2015], thus allowing a more general class of coefficients than in the analysis of [Miller 2004; Tenenbaum and Tucsnak 2007], which is done for $\rho = 1$ and $a \in C^2$ (and, possibly, a continuous potential). But even in the case $\Omega = (-L,L)$, $\Gamma_0 = \{-L,L\}$, $\rho(x) = 1$, $a(x) = 1$, we get $S_0 = 2L$ and thus we obtain an estimate on the cost of observability of the form

$$\limsup_{T\to 0} T\log C_0(T,(-L,L),\{-L,L\}) \leqslant 4K_0L^2,$$

instead of (1-6). In other words, we have a loss of a factor 4, so that the results in [Miller 2004; Tenenbaum and Tucsnak 2007] are better than ours in the one-dimensional case for a coefficient a in (1-13) which belongs to C^2. Therefore, we shall also explain how Theorem 1.1 can be extended to a multidimensional case directly when the observation is performed on the whole boundary; see Theorems 4.1–4.2.

Let us mention that the proofs of the observability inequality of the heat equation for general smooth bounded domains Ω and observation in an open subset Γ_0 of the boundary in [Fursikov and Imanuvilov 1996; Lebeau and Robbiano 1995] yield that

$$\limsup_{T\to 0} T\log C_0(T,\Omega,\Gamma_0) < \infty,$$

while on the other hand, [Miller 2004] proves

$$\liminf_{T\to 0} T\log C_0(T,\Omega,\Gamma_0) \geqslant \tfrac{1}{4}\sup_{\Omega} d(x,\Gamma_0)^2.$$

To our knowledge, getting more intrinsic geometric upper estimates on the cost of observability in small times in such general settings is still out of reach. In fact [Laurent and Léautaud 2018] shows that upper estimates on the cost of observability in small times cannot be linked only to the maximal distance to the control set and are deeply related to the geometry of the domain and of the observation set; see Remark 4.3. However, in geometrical cases which can be obtained by tensorization, some estimates can be obtained; see [Miller 2005] and Section 4B for more details.

We shall also mention that estimating the observability constant in small times for the heat equation in the one-dimensional case is related to the uniform controllability of viscous approximations of the transport equation; see [Coron and Guerrero 2005; Glass 2010; Lissy 2012; 2015]. We refer in particular to Section 4G for a more precise discussion on this problem. In particular, the proof in [Lissy 2012], when combined with Theorem 1.1, easily yields an improvement of the results known on this problem; see Section 4G and Theorem 4.10 for more details.

As we have seen in the above discussion, there are still some open questions on the observability of the one-dimensional constant-coefficient parabolic equations, despite the efficiency and robustness of the approach based on Carleman estimates [Fursikov and Imanuvilov 1996; Lebeau and Robbiano 1995]. This has justified the development of new manners to derive controllability of parabolic equations, and we shall in particular quote the flatness method developed in [Martin et al. 2014; 2016], a heat packet decomposition [Gimperlein and Waters 2017] and the backstepping approach [Coron and Nguyen 2017]. Our method comes in this context and provides what seems to be another approach to obtain observability results for the heat equation.

Outline. Section 2 presents the main strategy of the proof of Theorem 1.1 using several technical results that will be proved afterwards, in Section 3 for the ones using new arguments and in the Appendix for a Carleman-type estimate (Theorem 2.1) which can be found also in [Dardé and Ervedoza 2018] in a slightly different form. Section 4 provides several comments on Theorem 2.1 and its generalization, including a discussion on what can be done in the multidimensional setting (Section 4A), when the geometry has a tensorized form (Section 4B), or when the observation is on one side of the domain (Section 4C) or on some distributed open subset (Section 4D). We also present in Section 4E an alternative proof of a weaker version of Theorem 1.1 based on the uncertainty principles of [Landau and Pollak 1961] and the result in [Fuchs 1964], recovering the result of [Tenenbaum and Tucsnak 2007]. This will lead us to discuss the possibility of improving the estimate of the cost of observability in small times in Theorem 1.1 by using a better bound than the one provided by the use of Phragmén–Lindelöf principle for entire functions; see Section 4F for more details. We end up in Section 4G by giving a consequence of our result on the problem of uniform controllability of viscous approximations of transport equations. The Appendix gives the detailed proof of a rather easy Carleman estimate which is one of the building blocks of our analysis.

2. Proof of Theorem 1.1: main steps

As said in the Introduction, the proof of Theorem 1.1 relies on several steps.

The first step is the following Carleman-type estimate.

Theorem 2.1. *For any smooth solution u of (1-1), setting*

$$z(t,x) = u(t,x)\exp\left(\frac{x^2-L^2}{4t}\right), \quad (t,x) \in (0,T)\times(-L,L), \tag{2-1}$$

we have the inequality

$$\int_{-L}^{L} |\partial_x z(T,x)|^2\,dx - \frac{L^2}{4T^2}\int_{-L}^{L} |z(T,x)|^2\,dx \leqslant \frac{L}{T^2}\int_0^T t(|\partial_x u(t,-L)|^2 + |\partial_x u(t,L)|^2)\,dt. \tag{2-2}$$

Theorem 2.1 is based on the study of the equation satisfied by z in (2-1). As u satisfies the heat equation (1-1), the function z in (2-1) satisfies

$$\begin{cases} \partial_t z + \dfrac{x}{t}\partial_x z + \dfrac{1}{2t}z - \partial_x^2 z - \dfrac{L^2}{4t^2}z = 0, & (t,x)\in(0,\infty)\times(-L,L), \\ z(t,-L) = z(t,L) = 0, & t\in(0,\infty), \\ z(0,x) = 0, & x\in(-L,L). \end{cases} \tag{2-3}$$

One can therefore perform energy estimates on (2-3), which will eventually lead to (2-2). In the Appendix, we prove a slightly more general result, encompassing also some multidimensional settings, see Proposition A.1, from which one immediately derives Theorem 2.1 by setting $\Omega = (-L, L)$ and $g \equiv 0$.

Note that Theorem 2.1 was used in [Dardé and Ervedoza 2018] in time $T > L^2/\pi$ in order to describe the reachable set of the one-dimensional heat equation. Estimate (2-2) is somehow a Carleman estimate even if here no parameter appears in the proof. In fact, it rather corresponds to a *limiting Carleman estimate* as the conjugated operator (2-3) does not satisfy the usual strict pseudoconvexity conditions of [Hörmander 1985]. We refer in particular to [Dos Santos Ferreira et al. 2009] for other instances of limiting Carleman weights in another context, namely elliptic operators.

The second step of our analysis amounts to realizing that the solutions z of (2-3) could be explicitly solved using Fourier analysis if one extends the solution z of (2-3) by zero outside the space interval $(-L, L)$. We therefore introduce, for $t \in (0, T]$,

$$w(t,x) = \begin{cases} z(t,x) \quad \left(= u(t,x)\exp\left(\dfrac{x^2-L^2}{4t}\right)\right) & \text{for } x\in(-L,L), \\ 0 & \text{for } x\notin(-L,L). \end{cases} \tag{2-4}$$

In view of the above definition, it is then natural to set $w(0,\cdot) = 0$, since it is consistent with the above definition when taking the limit $t \to 0$. This function w satisfies

$$\begin{cases} \partial_t w + \dfrac{x}{t}\partial_x w + \dfrac{1}{2t}w - \partial_x^2 w - \dfrac{L^2}{4t^2}w = \partial_x u(t,L)\delta_L - \partial_x u(t,-L)\delta_{-L}, & (t,x)\in(0,\infty)\times\mathbb{R}, \\ w(0,x) = 0, & x\in\mathbb{R}. \end{cases} \tag{2-5}$$

Using Fourier transform, one can then compute explicitly

$$\hat{w}(T,\xi) = \int_{\mathbb{R}} w(T,x)e^{-\imath\xi x}\,dx,$$

at least for some frequency $\xi \in \mathbb{C}$:

Proposition 2.2. *For $\alpha \geqslant 0$, define the sets (see Figure 1)*

$$\mathscr{C}_\alpha = \{\xi = a + \imath b \in \mathbb{C} : (a, b) \in \mathbb{R}^2 \text{ with } |a| \geqslant |b| + \alpha\}. \tag{2-6}$$

Let w be given by (2-4) corresponding to some smooth solution u of (1-1).

Then, for any $\xi \in \mathscr{C}_{L/(2T)}$,

$$\hat{w}(T, \xi) = \int_0^T \sqrt{\frac{T}{t}}\big(-\partial_x u(t, -L) e^{\imath \frac{\xi L T}{t}} + \partial_x u(t, L) e^{-\imath \frac{\xi L T}{t}}\big) e^{-(\xi^2 T^2 - \frac{L^2}{4})(\frac{1}{t} - \frac{1}{T})}\, dt. \tag{2-7}$$

In particular, for any $\alpha > L/(2T)$, setting

$$C_\alpha(T) = \frac{1}{\sqrt{L(\alpha - L/(2T))}}, \tag{2-8}$$

for all $\xi \in \mathscr{C}_\alpha$, we have

$$|\hat{w}(T, \xi)| \leqslant C_\alpha(T)\sqrt{T} e^{|\Im(\xi)| L}\big(\|\partial_x u(\cdot, L)\|_{L^2(0,T)} + \|\partial_x u(\cdot, -L)\|_{L^2(0,T)}\big). \tag{2-9}$$

The proof of Proposition 2.2 is done in Section 3A and relies on explicit computations. In particular, it gives a precise L^∞ bound on the high-frequency component of $w(T)$ given by (2-4) corresponding to a smooth solution u of (1-1).

The third step of our analysis consists in the recovery of the low-frequency part of w given by (2-4). In order to do that, we recall that $\hat{w}(T, \cdot)$ is the Fourier transform of a function supported in $[-L, L]$. Therefore, its growth as $|\Im(\xi)| \to \infty$ is known, while $\hat{w}(T, \cdot)$ is holomorphic in the whole complex plane $\mathbb{C}$. Combined with the fact that we have nice estimates on $\hat{w}(T, \cdot)$ in $\mathscr{C}_\alpha$ for $\alpha > L^2/(2T)$, we are in the position to use Phragmén–Lindelöf principles to estimate $\hat{w}(T, \cdot)$ everywhere in the complex plane, but more importantly on the real axis $\mathbb{R}$.

Proposition 2.3. *Let $L > 0$, $\alpha > 0$ and f be a holomorphic function on $\mathscr{O}_\alpha = \mathbb{C} \setminus \mathscr{C}_\alpha$ (see Figure 1) such that:*

- *There exists a constant C_0 such that*

$$\text{for all } \xi \in \partial\mathscr{O}_\alpha, \quad |f(\xi)| \leqslant C_0 \exp(|\Im(\xi)| L). \tag{2-10}$$

- *There exists a constant C_1 such that*

$$\text{for all } \xi \in \mathscr{O}_\alpha, \quad |f(\xi)| \leqslant C_1 \exp(|\Im(\xi)| L). \tag{2-11}$$

Defining

$$\widetilde{\mathscr{O}}_1 = \{(a, b) \in \mathbb{R}^2 : |a| < |b| + 1\},$$

there exists a unique function $\tilde{\varphi}$ satisfying

$$\begin{cases} \Delta\tilde{\varphi} = -2\delta_{(-1,1)\times\{0\}} & \text{in } \widetilde{\mathscr{O}}_1, \\ \tilde{\varphi} = 0 & \text{on } \partial\widetilde{\mathscr{O}}_1, \\ \lim_{|b|\to\infty} \sup_{a \in (-|b|-1, |b|+1)} |\tilde{\varphi}(a, b)| = 0, \end{cases} \tag{2-12}$$

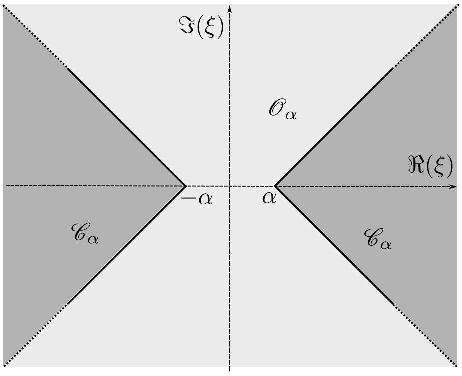

Figure 1. The complex plane, with domains $\mathscr{C}_\alpha$ and $\mathscr{O}_\alpha$.

and we define the function φ on $\mathscr{O}_1$ by

$$\varphi(\xi) = \tilde{\varphi}(\Re(\xi), \Im(\xi)), \quad \xi \in \mathscr{O}_1. \tag{2-13}$$

Then we have the bound

$$\text{for all } \xi \in \mathscr{O}_\alpha, \quad |f(\xi)| \leqslant C_0 \exp(|\Im(\xi)|L) \exp\left(L\alpha\varphi\left(\frac{\xi}{\alpha}\right)\right). \tag{2-14}$$

Besides, the maximum of φ on $\mathscr{O}_1$ is attained in 0:

$$\sup_{\mathscr{O}_1} \varphi = \varphi(0) = \frac{\Gamma\left(\frac{1}{4}\right)^2}{4\sqrt{2}\pi^2} \sum_{n\in\mathbb{N}} \frac{(-1)^n}{(2n+1)} \frac{\Gamma\left(n+\frac{1}{4}\right)}{\Gamma\left(n+\frac{7}{4}\right)} \quad (\simeq 0.893204), \tag{2-15}$$

which can be alternatively written as

$$\varphi(0) = \frac{2}{\pi} \frac{\int_0^{\frac{\pi}{2}} \ln\left(\cot\left(\frac{t}{2}\right)\right)\sqrt{\cos(t)}\,dt}{\int_0^{\frac{\pi}{2}} \sqrt{\cos(t)}\,dt}. \tag{2-16}$$

Proposition 2.3 mainly reduces to the application of Phragmén–Lindelöf principle for holomorphic functions. In fact, the main point in Proposition 2.3 is that the maximum of the harmonic function $\tilde{\varphi}$ can be explicitly computed. This is done using conformal maps to link the solution of the Laplace equation in the domain $\widetilde{\mathscr{O}}_1$ with solutions of the Laplace operator in the half-strip, in which explicit solutions can be computed using Fourier decomposition techniques. We refer to Section 3B for the proof of Proposition 2.3.

Of course, we shall apply Proposition 2.3 to the function $f = \hat{w}(T, \cdot)$, which, according to (2-9), satisfies (2-10) for any $\alpha > L/(2T)$ with

$$C_0 = C_\alpha(T)\sqrt{T}\left(\|\partial_x u(\cdot, L)\|_{L^2(0,T)} + \|\partial_x u(\cdot, -L)\|_{L^2(0,T)}\right),$$

while (2-11) holds with

$$C_1 = \|w(T)\|_{L^1(-L,L)} \leqslant \sqrt{2L}\|u(T)\|_{L^2(-L,L)} \leqslant \sqrt{2L}\|u_0\|_{L^2(-L,L)}.$$

We then immediately deduce the following corollary.

Corollary 2.4. *Let w be given by (2-4) corresponding to some smooth solution u of (1-1). Then, for any $\alpha > L/(2T)$,*

$$\text{for all } \xi \in \mathscr{O}_\alpha \cap \mathbb{R}, \quad |\hat{w}(T,\xi)| \leqslant C_\alpha(T)\sqrt{T} e^{L\alpha\varphi(0)} \big(\|\partial_x u(\cdot, L)\|_{L^2(0,T)} + \|\partial_x u(\cdot,-L)\|_{L^2(0,T)} \big), \tag{2-17}$$

where $C_\alpha(T)$ denotes the constant in (2-8).

End of the proof of Theorem 1.1. Let $\varepsilon > 0$, and choose $\alpha = (1+\varepsilon)L/(2T)$. Combining (2-17) and (2-9), we see that

$$\text{for all } \xi \in \mathbb{R}, \quad |\hat{w}(T,\xi)| \leqslant \sqrt{\frac{2}{\varepsilon}\frac{T}{L}} \exp\Big((1+\varepsilon)\frac{L^2}{2T}\varphi(0) \Big) \big(\|\partial_x u(\cdot, L)\|_{L^2(0,T)} + \|\partial_x u(\cdot,-L)\|_{L^2(0,T)} \big). \tag{2-18}$$

Then, using Theorem 2.1 and the identity

$$\int_{-L}^{L} |\partial_x z(T,x)|^2 \, dx - \frac{L^2}{4T^2} \int_{-L}^{L} |z(T,x)|^2 \, dx = \int_{\mathbb{R}} \Big(|\xi|^2 - \frac{L^2}{4T^2} \Big) |\hat{w}(T,\xi)|^2 \, d\xi$$

we have

$$\frac{3L^2}{4T^2} \int_{|\xi|>L/T} |\hat{w}(T,\xi)|^2 \, d\xi \leqslant \frac{L}{T} \big(\|\partial_x u(\cdot, L)\|^2_{L^2(0,T)} + \|\partial_x u(\cdot,-L)\|^2_{L^2(0,T)} \big) + \frac{L^2}{4T^2} \int_{|\xi|<L/(2T)} |\hat{w}(T,\xi)|^2 \, d\xi.$$

Combined with (2-18), we obtain

$$\int_{|\xi|>L/T} |\hat{w}(T,\xi)|^2 \, d\xi \leqslant \Big(\frac{4T}{3L} + \frac{4T}{3L\varepsilon} \exp\Big((1+\varepsilon)\frac{L^2}{T}\varphi(0) \Big) \Big) \big(\|\partial_x u(\cdot, L)\|^2_{L^2(0,T)} + \|\partial_x u(\cdot,-L)\|^2_{L^2(0,T)} \big) \tag{2-19}$$

and

$$\int_{|\xi|<L/T} |\hat{w}(T,\xi)|^2 \, d\xi \leqslant \frac{8T}{\varepsilon L} \exp\Big((1+\varepsilon)\frac{L^2}{T}\varphi(0) \Big) \big(\|\partial_x u(\cdot, L)\|^2_{L^2(0,T)} + \|\partial_x u(\cdot,-L)\|^2_{L^2(0,T)} \big). \tag{2-20}$$

Using Parseval's identity and the explicit form of w in (2-4), we easily get, for some constant $C_\varepsilon(T)$ that goes to zero as $T \to 0$, that

$$\Big\| u(T,x) \exp\Big(\frac{x^2 - L^2}{4T} \Big) \Big\|_{L^2(-L,L)} \leqslant C_\varepsilon(T) \exp\Big(\frac{L^2}{2T}(1+\varepsilon)\varphi(0) \Big) \big(\|\partial_x u(\cdot, L)\|_{L^2(0,T)} + \|\partial_x u(\cdot,-L)\|_{L^2(0,T)} \big),$$

which we rewrite as

$$\left\| u(T,x)\exp\left(\frac{x^2}{4T}\right)\right\|_{L^2(-L,L)} \leqslant C_\varepsilon(T)\exp\left(\frac{L^2}{T}\left(\tfrac{1}{4}+\tfrac{1}{2}(1+\varepsilon)\varphi(0)\right)\right)\left(\|\partial_x u(\cdot,L)\|_{L^2(0,T)}+\|\partial_x u(\cdot,-L)\|_{L^2(0,T)}\right). \quad (2\text{-}21)$$

This concludes the proof of Theorem 1.1, as $C_\varepsilon(T) \leqslant C_\varepsilon(1) = C_\varepsilon$ for T small enough, for some C_ε independent of T. □

Remark 2.5. Note that the constant C_ε in the above proof blows up as ε goes to zero. If it were not the case, one could pass to the limit $\varepsilon \to 0$ in (2-21), so that one could choose $K = K_0$ in the observability inequality (1-7). So far, we do not know if this choice is allowed in the observability inequality (1-7) or not.

We have thus reduced the proof of Theorem 1.1 to the proofs of Theorem 2.1 and Propositions 2.2 and 2.3. The proof of Theorem 2.1 is postponed to the Appendix in which a slightly more general result is proved (Proposition A.1), while the proofs of Propositions 2.2 and 2.3 are detailed in Section 3.

Remark 2.6. The above approach allows us in fact to recover an explicit formula to compute $\hat{w}(T)$ in terms of the observations. Namely, for $\xi \in \mathbb{R}$ with $|\xi| \geqslant L/(2T)$, formula (2-7) yields

$$\hat{w}(T,\xi) = \int_0^T \sqrt{\frac{T}{t}}\left(-\partial_x u(t,-L)e^{\imath\frac{\xi LT}{t}} + \partial_x u(t,L)e^{-\imath\frac{\xi LT}{t}}\right)e^{-(\xi^2T^2-\frac{L^2}{4})(\frac{1}{t}-\frac{1}{T})}\,dt. \quad (2\text{-}22)$$

On the other hand, combining the formula (2-7) and Remark 3.2 allowing us to get an explicit expression under the assumptions of Proposition 2.3, we get: for all $\alpha_* > \alpha > L/(2T)$, for all $\xi \in \mathbb{R}$ with $|\xi| < L/(2T)$,

$$\begin{aligned}\hat{w}(T,\xi) = &-\int_0^T \sqrt{\frac{T}{t}}\partial_x u(t,-L)\frac{1}{2\imath\pi}\int_{\gamma_\alpha}\frac{e^{L\alpha_*(\phi(\xi/\alpha)-\phi(\zeta/\alpha))}}{\zeta-\xi}e^{\imath\frac{\zeta LT}{t}}e^{-(\zeta^2T^2-\frac{L^2}{4})(\frac{1}{t}-\frac{1}{T})}\,d\zeta\,dt\\ &+\int_0^T \sqrt{\frac{T}{t}}\partial_x u(t,L)\frac{1}{2\imath\pi}\int_{\gamma_\alpha}\frac{e^{L\alpha_*(\phi(\xi/\alpha)-\phi(\zeta/\alpha))}}{\zeta-\xi}e^{-\imath\frac{\zeta LT}{t}}e^{-(\zeta^2T^2-\frac{L^2}{4})(\frac{1}{t}-\frac{1}{T})}\,d\zeta\,dt, \quad (2\text{-}23)\end{aligned}$$

where ϕ is a holomorphic function on $\mathscr{O}_1$ such that $\Re(\phi(\xi)) = \varphi(\xi) + |\Im(\xi)|$ for all $\xi \in \mathscr{O}_1$ (see Section 3B2 for the existence of such function ϕ), and γ_α is the union of the two connected components of $\partial\mathscr{O}_\alpha$ oriented counterclockwise. But this formula does not seem easy to deal with as the kernels

$$K_\mp(t,\xi) = \frac{1}{2\imath\pi}\int_{\gamma_\alpha}\frac{e^{L\alpha_*(\phi(\xi/\alpha)-\phi(\zeta/\alpha))}}{\zeta-\xi}e^{\pm\imath\frac{\zeta LT}{t}}e^{-(\zeta^2T^2-\frac{L^2}{4})(\frac{1}{t}-\frac{1}{T})}\,d\zeta, \quad (t,\xi) \in (0,T)\times\left(-\frac{L}{2T},\frac{L}{2T}\right),$$

are difficult to estimate directly.

3. Proof of Theorem 1.1: intermediate results

3A. ***Proof of Proposition 2.2.*** Let w be as in Proposition 2.2. Then w satisfies (2-5). When taking its Fourier transform in the space variable, we easily check that

$$\hat{w}(t,\xi) = \int_{\mathbb{R}} w(t,x)e^{-\imath\xi x}\,dx, \quad (t,\xi) \in [0,T]\times\mathbb{R},$$

solves the equation

$$\begin{cases} \partial_t \hat{w} - \dfrac{\xi}{t}\partial_\xi \hat{w} - \dfrac{1}{2t} w + \xi^2 \hat{w} - \dfrac{L^2}{4t^2}\hat{w} = \partial_x u(t,L)e^{-\imath\xi L} - \partial_x u(t,-L)e^{\imath \xi L}, & (t,\xi)\in(0,\infty)\times\mathbb{R}, \\ \hat{w}(0,\xi) = 0, & \xi \in \mathbb{R}. \end{cases} \tag{3-1}$$

We are thus back to the study of a transport equation. For each $\xi_0 \in \mathbb{R}$, we therefore introduce the characteristics $\xi(t,\xi_0)$ reaching ξ_0 at time T,

$$\frac{d\xi}{dt}(t,\xi_0) = -\frac{\xi(t,\xi_0)}{t}, \quad t\in(0,T], \qquad \xi(T,\xi_0) = \xi_0, \tag{3-2}$$

which is explicitly given by

$$\xi(t,\xi_0) = \frac{\xi_0 T}{t}, \quad t \in (0,T].$$

We can thus write, for all $t\in(0,T]$,

$$\frac{d}{dt}\left(\hat{w}\left(t,\frac{\xi_0 T}{t}\right)\right) + \left(\frac{1}{t^2}\left(\xi_0^2 T^2 - \frac{L^2}{4}\right) - \frac{1}{2t}\right)\hat{w}\left(t,\frac{\xi_0 T}{t}\right) = \partial_x u(t,L)e^{-\imath\frac{\xi_0 LT}{t}} - \partial_x u(t,-L)e^{\imath\frac{\xi_0 LT}{t}}.$$

This yields the formula

$$\frac{d}{dt}\left(\hat{w}\left(t,\frac{\xi_0 T}{t}\right)t^{-\frac12}e^{-(\xi_0^2T^2-\frac{L^2}{4})/t}\right) = \left(\partial_x u(t,L)e^{-\imath\frac{\xi_0 LT}{t}} - \partial_x u(t,-L)e^{\imath\frac{\xi_0 LT}{t}}\right)t^{-\frac12}e^{-(\xi_0^2T^2-\frac{L^2}{4})/t}.$$

For any $\eta > 0$, we can integrate this formula between η and T to get

$$\begin{aligned} &\hat{w}(T,\xi_0)T^{\frac12}e^{-(\xi_0^2T^2-\frac{L^2}{4})/T} - \hat{w}(\eta,\xi_0)\eta^{\frac12}e^{-(\xi_0^2T^2-\frac{L^2}{4})/\eta} \\ &\qquad = \int_\eta^T t^{-\frac12}\left(\partial_x u(t,L)e^{-\imath\frac{\xi_0 LT}{t}} - \partial_x u(t,-L)e^{\imath\frac{\xi_0 LT}{t}}\right)e^{-(\xi_0^2T^2-\frac{L^2}{4})/t}\,dt. \end{aligned}$$

It is not difficult to check that for $\xi_0\in\mathbb{R}$ with $|\xi_0| > L/(2T)$, the integral on the right-hand side converges when η goes to zero, and

$$\lim_{\eta\to 0}\hat{w}(\eta,\xi_0)\eta^{-\frac12}e^{-(\xi_0^2T^2-\frac{L^2}{4})/\eta} = 0.$$

Therefore, provided $\xi_0\in\mathbb{R}$ satisfies $|\xi_0| > L/(2T)$, one gets the formula

$$\hat{w}(T,\xi_0) = \int_0^T \sqrt{\frac{T}{t}}\left(\partial_x u(t,L)e^{-\imath\frac{L\xi_0 T}{t}} - \partial_x u(t,-L)e^{\imath\frac{L\xi_0 T}{t}}\right)e^{-(\xi_0^2T^2-\frac{L^2}{4})(\frac1t-\frac1T)}\,dt. \tag{3-3}$$

This formula coincides with the one in (2-7) for $\xi_0 \in \mathscr{C}_{L^+/2T}\cap\mathbb{R}$ (here, we use the notation L^+ to denote any constant strictly larger than L). As $\hat{w}(T,\cdot)$ is holomorphic on $\mathbb{C}$, we only have to check that the right-hand side of formula (3-3) can be extended holomorphically to $\mathscr{C}_{L^+/2T}$. In fact, writing $\xi = a + \imath b$ with $(a,b)\in\mathbb{R}^2$, the right-hand side of (3-3) can be extended holomorphically in the domain in which

$$\begin{cases} \Re\left(+\imath\xi LT - \left(\xi^2T^2 - \dfrac{L^2}{4}\right)\right) = -bLT - \left((a^2-b^2)T^2 - \dfrac{L^2}{4}\right) < 0, \\ \Re\left(-\imath\xi LT - \left(\xi^2T^2 - \dfrac{L^2}{4}\right)\right) = +bLT - \left((a^2-b^2)T^2 - \dfrac{L^2}{4}\right) < 0, \end{cases}$$

which is equivalent to

$$|a| > |b| + \frac{L}{2T},$$

i.e., $\xi \in \mathscr{C}_{L+/(2T)}$. We have thus proved that for all $\xi \in \mathscr{C}_{L+/(2T)}$, $\hat{w}(T,\xi)$ is given by the formula (2-7). In fact, by continuity, this formula also holds for $\xi \in \mathscr{C}_{L/2T}$.

In order to deduce (2-9), we start from the formula (2-7) and we use a Cauchy–Schwarz estimate: for $\xi \in \mathscr{C}_\alpha$ with $\alpha > L/(2T)$,

$$\begin{aligned}|\hat{w}(T,\xi)| \leqslant \sqrt{T}\|\partial_x u(t,L)\|_{L^2(0,T)} \left\| t^{-\frac{1}{2}} \exp\left(-\frac{i\xi LT}{t} - \left(\xi^2T^2 - \frac{L^2}{4}\right)\left(\frac{1}{t}-\frac{1}{T}\right)\right)\right\|_{L^2(0,T)} \\ + \sqrt{T}\|\partial_x u(t,-L)\|_{L^2(0,T)} \left\| t^{-\frac{1}{2}} \exp\left(+\frac{i\xi LT}{t} - \left(\xi^2T^2 - \frac{L^2}{4}\right)\left(\frac{1}{t}-\frac{1}{T}\right)\right)\right\|_{L^2(0,T)}. \quad (3\text{-}4)\end{aligned}$$

Writing $\xi \in \mathscr{C}_\alpha$ for $\alpha > L/(2T)$ as $\xi = a + \imath b$ with $(a,b) \in \mathbb{R}^2$ and using the fact that

$$\begin{aligned}\Re\left(\mp \imath \xi LT - \left(\xi^2T^2 - \frac{L^2}{4}\right)\right) &\leqslant |b|LT - \left((a^2-b^2)T^2 - \frac{L^2}{4}\right) \\ &\leqslant -T^2\left(a^2 - \left(|b| + \frac{L}{2T}\right)^2\right) \\ &\leqslant -T^2\left(|a| - \left(|b| + \frac{L}{2T}\right)\right)\left(|a| + |b| + \frac{L}{2T}\right) \leqslant -\frac{LT}{2}\left(\alpha - \frac{L}{2T}\right),\end{aligned}$$

we have the estimates, for $s \in \{-1, 1\}$,

$$\begin{aligned}&\left\| t^{-\frac{1}{2}} \exp\left(s\frac{\imath\xi LT}{t} - \left(\xi^2T^2 - \frac{L^2}{4}\right)\left(\frac{1}{t} - \frac{1}{T}\right)\right)\right\|_{L^2(0,T)} \\ &\qquad \leqslant \left\| t^{-\frac{1}{2}} \exp\left(|b|L + \left(|b|LT - \left((a^2-b^2)T^2 - \frac{L^2}{4}\right)\left(\frac{1}{t} - \frac{1}{T}\right)\right)\right)\right\|_{L^2(0,T)} \\ &\qquad \leqslant e^{|b|L}\left\| t^{-\frac{1}{2}} \exp\left(-\frac{LT}{2}\left(\alpha - \frac{L}{2T}\right)\left(\frac{1}{t} - \frac{1}{T}\right)\right)\right\|_{L^2(0,T)}.\end{aligned}$$

Now, doing the change of variable $\mu = LT\left(\alpha - \frac{L}{2T}\right)\left(\frac{1}{t} - \frac{1}{T}\right)$, we easily get, for all $\xi \in \mathscr{C}_\alpha$,

$$\begin{aligned}\left\| t^{-\frac{1}{2}} \exp\left(-\frac{LT}{2}\left(\alpha - \frac{L}{2T}\right)\left(\frac{1}{t} - \frac{1}{T}\right)\right)\right\|_{L^2(0,T)}^2 &= \int_0^\infty e^{-\mu}\frac{d\mu}{\mu + L(\alpha - L/(2T))} \\ &\leqslant \frac{1}{L(\alpha - L/(2T))}.\end{aligned}$$

Combining (3-4) and this last estimate, we easily conclude estimate (2-9).

3B. ***Proof of Proposition 2.3.*** We shall start the proof of Proposition 2.3 by proving the existence of a function $\tilde{\varphi}$ satisfying (2-12), and we will then explain how it can be used to derive the bound in (2-14).

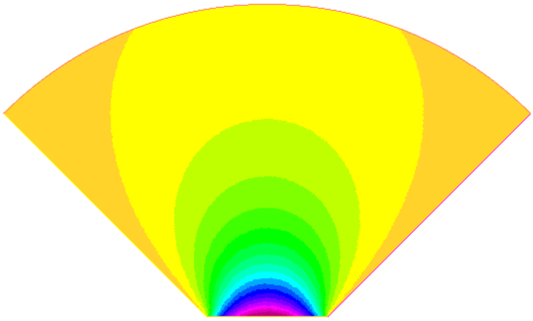

Figure 2. Approximation of $\tilde{\varphi}$ solving (3-5), obtained by a finite-element approach (using FreeFem++ [Hecht 2012]).

Notation. In the following arguments, to avoid ambiguities, we will write differently complex sets and their identification as a part of $\mathbb{R}^2$; for instance we write $\mathscr{O}_1 = \{\xi \in \mathbb{C} : |\Re(\xi)| < |\Im(\xi)| + 1\}$ and $\widetilde{\mathscr{O}}_1 = \{(a,b) \in \mathbb{R}^2 : |a| < |b|+1\}$ as in Proposition 2.3. To be consistent with this notation, we will also distinguish functions of the complex variable ξ from the corresponding ones considered as functions of the real variables (a,b) using a tilde notation for the function viewed as depending on real variables, as in (2-13).

3B1. *Existence and uniqueness of a function $\tilde{\varphi}$ satisfying* (2-12). The first remark is that the uniqueness of a function $\tilde{\varphi}$ satisfying (2-12) is rather easy to prove. Indeed, if two functions $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ satisfy (2-12), then their difference $\tilde{\varphi}_2 - \tilde{\varphi}_1$ is harmonic in $\mathscr{O}_1$ and vanishes on $\partial\widetilde{\mathscr{O}}_1$ as well as at infinity. Therefore, the minimum and maximum of $\tilde{\varphi}_2 - \tilde{\varphi}_1$ is zero, and $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ coincide.

Thus, we will focus on the existence of a function $\tilde{\varphi}$ as in (2-12). In fact, by uniqueness, we see that necessarily $\tilde{\varphi}(a,b) = \tilde{\varphi}(a,|b|)$ for all $(a,b) \in \mathscr{O}_1$. We will thus only look for a solution $\tilde{\varphi}$ in $\widetilde{\mathscr{O}}_1^+ = \widetilde{\mathscr{O}}_1 \cap (\mathbb{R} \times \mathbb{R}_+^*)$ of the problem

$$\begin{cases} \Delta\tilde{\varphi} = 0 & \text{in } \widetilde{\mathscr{O}}_1^+, \\ \tilde{\varphi} = 0 & \text{on } \partial\widetilde{\mathscr{O}}_1^+ \setminus (-1,1), \\ \partial_b\tilde{\varphi}(a,0) = -1 & \text{for } a \in (-1,1), \end{cases} \tag{3-5}$$

with the condition at infinity

$$\lim_{b\to\infty} \sup_{a\in(-|b|-1,|b|+1)} |\tilde{\varphi}(a,b)| = 0. \tag{3-6}$$

Let us introduce

$$\begin{aligned} \Gamma_\ell &= \{\xi \in \mathbb{C} : \Im(\xi) > 0 \text{ and } -\Re(\xi) = 1 + \Im(\xi)\}, \\ \Gamma_r &:= \{\xi \in \mathbb{C} : \Im(\xi) > 0 \text{ and } \Re(\xi) = 1 + \Im(\xi)\}, \\ \Gamma_b &:= \{\xi \in \mathbb{C} : (\Re(\xi), \Im(\xi)) \in [-1,1] \times \{0\}\}, \end{aligned}$$

the three components of the boundary of $\mathscr{O}_1^+ = \mathscr{O}_1 \cap \{\Im(\xi) > 0\}$.

Our goal is to show the existence of a function $\tilde{\varphi}$ satisfying (3-5). In order to do so, we will rely on two Schwarz–Christoffel conformal mappings [Henrici 1974, Chapter 5.12].

The first one, $F_{\frac{3}{4}}$, is defined for all $\zeta \in \mathbb{C}^+ = \{\zeta \in \mathbb{C} : \Im(\zeta) \geqslant 0\}$ by

$$F_{\frac{3}{4}}(\zeta) = \frac{2}{K_{\frac{3}{4}}} \int_{-1}^{\zeta} (1-z^2)^{-\frac{1}{4}}\, dz - 1, \quad \text{with } K_{\frac{3}{4}} = \int_{-1}^{1} (1-x^2)^{-\frac{1}{4}}\, dx = \sqrt{\pi}\,\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)},$$

where the path integration is arbitrary in $\mathbb{C}^+$.

The map $F_{\frac{3}{4}}$ conformally maps $\mathbb{C}^+$ into $\overline{\mathscr{O}}_1^+$, and satisfies the properties

$$F_{\frac{3}{4}}(-1) = -1, \quad F_{\frac{3}{4}}(0) = 0, \quad F_{\frac{3}{4}}(1) = 1,$$

and

$$F_{\frac{3}{4}}((-\infty,-1)) = \Gamma_\ell, \quad F_{\frac{3}{4}}((-1,1)) = \Gamma_b, \quad F_{\frac{3}{4}}((1,\infty)) = \Gamma_r, \quad F_{\frac{3}{4}}(\imath\mathbb{R}^+) = \imath\mathbb{R}^+.$$

The second conformal mapping we will use is defined, for any $\zeta \in \mathbb{C}^+$, by

$$F_{\frac{1}{2}}(\zeta) = \frac{2}{\pi}\arcsin(\zeta) = \frac{2}{\pi}\int_{-1}^{\zeta} (1-z^2)^{-\frac{1}{2}}\, dz - 1,$$

which conformally maps $\mathbb{C}^+$ into the closure of the half strip $\mathscr{S}_1^+ = \{\Xi = A + \imath B : A \in (-1,1),\ B > 0\}$ with the properties

$$F_{\frac{1}{2}}(-1) = -1, \quad F_{\frac{1}{2}}(0) = 0, \quad F_{\frac{1}{2}}(1) = 1,$$

and

$$F_{\frac{1}{2}}((-\infty,-1]) = -1 + \imath\mathbb{R}^+, \quad F_{\frac{1}{2}}((-1,1)) = (-1,1), \quad F_{\frac{1}{2}}([1,\infty)) = 1 + \imath\mathbb{R}^+, \quad F_{\frac{1}{2}}(\imath\mathbb{R}^+) = \imath\mathbb{R}^+.$$

Finally, we define the conformal mapping

$$F = F_{\frac{1}{2}} \circ F_{\frac{3}{4}}^{-1},$$

which maps $\mathscr{O}_1^+$ into $\mathscr{S}_1^+$.

For any $\xi = a + \imath b \in \mathscr{O}_1^+$, we define $\Xi = A + \imath B = F(\xi)$. Using a standard computation from conformal transplantation [Henrici 1974, Chapter 5.6], we see that $\tilde{\varphi}$ solves (3-5) in $\widetilde{\mathscr{O}}_1^+$ if and only if $\widetilde{\Phi}$ given by $\widetilde{\Phi}(A,B) = \tilde{\varphi}(a,b)$ for $A + \imath B = F(a + \imath b)$ solves the following problem posed in the half-strip $\widetilde{\mathscr{S}}_1^+$:

$$\begin{cases} \Delta_{A,B}\widetilde{\Phi} = 0 & \text{for } A \in (-1,1),\ B > 0, \\ \widetilde{\Phi}(-1,B) = \widetilde{\Phi}(1,B) = 0 & \text{for } B > 0, \\ \partial_B\widetilde{\Phi}(A,0) = -\dfrac{\pi}{K_{\frac{3}{4}}}\sqrt{\cos\left(\frac{\pi}{2}A\right)} & \text{for } A \in (-1,1). \end{cases} \tag{3-7}$$

If the first two equations are standard, the last one deserves additional details. In fact, it comes from the identity [Henrici 1974, Theorem 5.6a]

$$\operatorname{grd}_\xi \varphi(\xi) = \operatorname{grd}_\Xi \Phi(F(\xi))\,\overline{F'(\xi)}, \tag{3-8}$$

applied to $\xi = a \in (-1,1)$, (implying $F(\xi) = A \in (-1,1)$), where grd is the complex gradient: for $\xi = a + \imath b$, $\operatorname{grd}_\xi \varphi(\xi) = \partial_a\tilde{\varphi}(a,b) + \imath\,\partial_b\tilde{\varphi}(a,b)$ and for $\Xi = A + \imath B$, $\operatorname{grd}_\Xi \Phi(\Xi) = \partial_A\widetilde{\Phi}(A,B) + \imath\,\partial_B\widetilde{\Phi}(A,B)$.

We therefore have to compute $F'(\xi) = (F_{\frac{1}{2}} \circ F_{\frac{3}{4}}^{-1})'(\xi) = F_{\frac{1}{2}}'(F_{\frac{3}{4}}^{-1}(\xi))(F_{\frac{3}{4}}^{-1})'(\xi)$. To do so, let us define $\zeta = F_{\frac{3}{4}}^{-1}(\xi) \in \mathbb{C}^+$. By definition,

$$F_{\frac{1}{2}}'(F_{\frac{3}{4}}^{-1}(\xi)) = F_{\frac{1}{2}}'(\zeta) = \frac{2}{\pi}\frac{1}{\sqrt{1-\zeta^2}},$$

whereas

$$(F_{\frac{3}{4}}^{-1})'(\xi) = (F_{\frac{3}{4}}^{-1})'(F_{\frac{3}{4}}(\zeta)) = \frac{1}{F_{\frac{3}{4}}'(\zeta)} = \frac{K_{\frac{3}{4}}}{2}\sqrt[4]{1-\zeta^2}.$$

Therefore,

$$F'(\xi) = \frac{K_{\frac{3}{4}}}{\pi}\frac{1}{\sqrt[4]{1-\zeta^2}},$$

with $\zeta = F_{\frac{3}{4}}^{-1}(\xi)$. In particular, for $\xi = a \in (-1, 1)$, we have $\zeta \in (-1, 1)$ and therefore $F'(\xi) \in \mathbb{R}$ and

$$\partial_B \widetilde{\Phi}(A, 0) = \partial_b \tilde{\varphi}(a, 0)\frac{1}{F'(a)} = -\frac{\pi}{K_{\frac{3}{4}}}\sqrt[4]{1-\zeta^2}, \quad \text{with } \zeta = F_{\frac{3}{4}}^{-1}(a).$$

To conclude, we just note that $\zeta = F_{\frac{1}{2}}^{-1}(A)$ if and only if $\zeta = \sin(A\pi/2)$, and the third identity in (3-7) follows.

Problem (3-7) has the advantage of being explicitly solvable. Indeed, as $\widetilde{\Phi}$ is harmonic in $(-1, 1) \times (0, \infty)$, and satisfies $\widetilde{\Phi}(-1, B) = \widetilde{\Phi}(1, B) = 0$ for all $B > 0$, it necessarily has the decomposition

$$\widetilde{\Phi}(A, B) = \sum_{k \geqslant 1} (\alpha_k e^{-k\frac{\pi}{2}B} + a_k e^{k\frac{\pi}{2}B}) \sin\bigl(k\tfrac{\pi}{2}(A+1)\bigr), \quad (A, B) \in \widetilde{\mathscr{S}}_1^+.$$

Recalling condition (3-6) on $\tilde{\varphi}$, we wish to have $\widetilde{\Phi}$ going to zero as $B \to \infty$. We thus choose $a_k = 0$ for all $k \geqslant 1$, so that $\widetilde{\Phi}$ can be written as

$$\widetilde{\Phi}(A, B) = \sum_{k \geqslant 1} \alpha_k e^{-k\frac{\pi}{2}B} \sin\bigl(k\tfrac{\pi}{2}(A+1)\bigr), \quad (A, B) \in \widetilde{\mathscr{S}}_1^+.$$

But the boundary condition on $B = 0$ is equivalent to

$$\frac{\pi}{2}\sum_{k \geqslant 1} k\alpha_k \sin\bigl(k\tfrac{\pi}{2}(A+1)\bigr) = \frac{\pi}{K_{\frac{3}{4}}}\sqrt{\cos\bigl(\tfrac{\pi}{2}A\bigr)},$$

which explicitly yields the coefficients α_k:

$$\text{for all } k \in \mathbb{N}, \quad \alpha_k = \frac{2}{k}\frac{1}{K_{\frac{3}{4}}}\int_{-1}^{1} \sin\bigl(k\tfrac{\pi}{2}(A+1)\bigr)\sqrt{\cos\bigl(\tfrac{\pi}{2}A\bigr)}\, dA.$$

As $\sqrt{\cos(A\pi/2)}$ is an even function and $\sin(k\pi(A+1)/2)$ is an odd function for all even k, we have $\alpha_k = 0$ for all even k. On the other hand, we have for any $n \in \mathbb{N}$, see [Gradshteyn and Ryzhik 2007, equation 3.631.9],

$$\begin{aligned}\int_{-1}^{1} \sin\bigl((2n+1)\tfrac{\pi}{2}(A+1)\bigr)\sqrt{\cos\bigl(\tfrac{\pi}{2}A\bigr)}\, dA &= (-1)^n \int_{-1}^{1} \cos\bigl((2n+1)\tfrac{\pi}{2}A\bigr)\sqrt{\cos\bigl(\tfrac{\pi}{2}A\bigr)}\, dA \\ &= (-1)^n \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \cos((2n+1)t)\sqrt{\cos(t)}\, dt = \frac{1}{2\sqrt{\pi}}\frac{\Gamma\bigl(n+\frac{1}{4}\bigr)}{\Gamma\bigl(n+\frac{7}{4}\bigr)},\end{aligned}$$

where $\Gamma(\cdot)$ stands for the Gamma function, so in the end we obtain

$$\alpha_{2n+1} = \frac{1}{\pi} \frac{1}{2n+1} \frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \frac{\Gamma\left(n+\frac{1}{4}\right)}{\Gamma\left(n+\frac{7}{4}\right)},$$

which can be slightly simplified using that $\Gamma\left(\frac{5}{4}\right) = \Gamma\left(\frac{1}{4}\right)/4$ and $\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\pi/\Gamma\left(\frac{1}{4}\right)$, giving

$$\alpha_{2n+1} = \frac{\Gamma\left(\frac{1}{4}\right)^2}{4\sqrt{2}\pi^2} \frac{1}{(2n+1)} \frac{\Gamma\left(n+\frac{1}{4}\right)}{\Gamma\left(n+\frac{7}{4}\right)}.$$

So finally, we have

$$\widetilde{\Phi}(A, B) = \frac{\Gamma\left(\frac{1}{4}\right)^2}{4\sqrt{2}\pi^2} \sum_{n\in\mathbb{N}} \frac{1}{(2n+1)} \frac{\Gamma\left(n+\frac{1}{4}\right)}{\Gamma\left(n+\frac{7}{4}\right)} e^{-(2n+1)\frac{\pi}{2}B} \sin\left((2n+1)\tfrac{\pi}{2}(A+1)\right), \quad (A, B) \in \mathscr{S}_1^+, \tag{3-9}$$

and

$$\widetilde{\Phi}(0,0) = \frac{\Gamma\left(\frac{1}{4}\right)^2}{4\sqrt{2}\pi^2} \sum_{n\in\mathbb{N}} \frac{(-1)^n}{(2n+1)} \frac{\Gamma\left(n+\frac{1}{4}\right)}{\Gamma\left(n+\frac{7}{4}\right)}. \tag{3-10}$$

Note that, according to [Lebedev 1972, (1.4.25)],

$$\frac{1}{2n+1} \frac{\Gamma\left(n+\frac{1}{4}\right)}{\Gamma\left(n+\frac{7}{4}\right)} \underset{n\to\infty}{\simeq} \frac{1}{2n^{\frac{5}{2}}};$$

hence the above series are well-defined. In particular, the identity (3-9) can be understood pointwise and $\widetilde{\Phi}(\cdot, B)$ goes to zero as $B \to \infty$:

$$\sup_{A\in(-1,1)} \{|\widetilde{\Phi}(A, B)| + |\partial_A \widetilde{\Phi}(A, B)|\} \leqslant C \exp\left(-\frac{\pi B}{2}\right), \quad B \geqslant 0. \tag{3-11}$$

Let us also note that, because $\widetilde{\Phi}(0,0)$ is defined through a converging alternating series, we have

$$\widetilde{\Phi}(0,0) < \frac{\Gamma\left(\frac{1}{4}\right)^2}{4\sqrt{2}\pi^2} \sum_{n=0}^{2} \frac{(-1)^n}{(2n+1)} \frac{\Gamma\left(n+\frac{1}{4}\right)}{\Gamma\left(n+\frac{7}{4}\right)} < \frac{9}{10}.$$

Computing the 100th partial sum of the series using Octave [Eaton et al. 2014], we obtain

$$\widetilde{\Phi}(0,0) \sim 0.893204.$$

A different expression for $\widetilde{\Phi}(0,0)$ is

$$\widetilde{\Phi}(0,0) = \frac{2}{\pi} \frac{\int_0^{\frac{\pi}{2}} \ln\left(\cot\left(\frac{t}{2}\right)\right) \sqrt{\cos(t)}\, dt}{\int_0^{\frac{\pi}{2}} \sqrt{\cos(t)}\, dt}, \tag{3-12}$$

which easily comes from the equality $\widetilde{\Phi}(0,0) = \sum_{n\in\mathbb{N}} (-1)^n \alpha_{2n+1}$, the fact that

$$\alpha_{2n+1} = (-1)^n \frac{8}{(2n+1)\pi} \frac{1}{K_{\frac{3}{4}}} \int_0^{\frac{\pi}{2}} \cos((2n+1)t) \sqrt{\cos(t)}\, dt,$$

the definition of $K_{\frac{3}{4}}$ and the identity, see [Gradshteyn and Ryzhik 2007, identity 1.442.2 p. 46],

$$\sum_{n\in\mathbb{N}} \frac{\cos((2n+1)t)}{2n+1} = \tfrac{1}{2}\ln\bigl(\cot\bigl(\tfrac{t}{2}\bigr)\bigr).$$

Note in particular that under the form (3-12), one immediately checks that

$$\widetilde{\Phi}(0,0) > 0. \tag{3-13}$$

In agreement with Figure 2, we then show that the maximum of $\widetilde{\Phi}$ is attained at $(A, B) = (0, 0)$. We first note that the function $\widetilde{\Phi}$ given by (3-9) is positive in the strip $\widetilde{\mathscr{S}}_1^+$. Indeed, since $\widetilde{\Phi}$ is harmonic in the half strip $\widetilde{\mathscr{S}}_1^+$ and is not constant, its minimum is attained at the boundary $\widetilde{\mathscr{S}}_1^+$ or at infinity [Gilbarg and Trudinger 1998, Lemma 3.4, Theorem 3.5]. The boundary conditions on $\partial\widetilde{\mathscr{S}}_1^+$ and the behavior of $\widetilde{\Phi}$ as $B \to \infty$ in (3-11) implies that the minimum value of $\widetilde{\Phi}$ is 0 and is attained on the lateral boundaries $\{-1, 1\} \times \mathbb{R}_+$ of the half strip. Consequently, the function $\widetilde{\Phi}$ is positive in $\widetilde{\mathscr{S}}_1^+$, and its minimal value is 0.

Besides, as $\widetilde{\Phi}$ vanishes on the lateral boundaries $\{-1, 1\} \times \mathbb{R}^+$ of the half strip, $\partial_A\widetilde{\Phi}(1, \cdot)$ is strictly negative by the Hopf maximum principle [Protter and Weinberger 1984, Chapter 2, Theorem 7]. We then consider the function $\widetilde{\Phi}_A = \partial_A\widetilde{\Phi}$. Formula (3-9) easily yields that $\widetilde{\Phi}_A(0, B) = 0$ for $B > 0$, so that $\widetilde{\Phi}_A$ satisfies

$$\begin{cases} \Delta\widetilde{\Phi}_A = 0 & \text{in } \widetilde{\mathscr{S}}_1^+ \cap \{A > 0\}, \\ \widetilde{\Phi}_A(0, B) = 0 & \text{for } B > 0, \\ \widetilde{\Phi}_A(1, B) < 0 & \text{for } B > 0, \\ \partial_B\widetilde{\Phi}_A(A, 0) \geqslant 0 & \text{for } A \in (0, 1), \\ \lim_{|B|\to\infty} \sup_{A\in(0,1)} |\widetilde{\Phi}_A(A, B)| = 0. \end{cases}$$

It easily follows that the maximum of $\widetilde{\Phi}_A$ is necessarily nonpositive in $\widetilde{\mathscr{S}}_1^+ \cap \{A > 0\}$ by the application of the maximum principle.

Finally, as $\widetilde{\Phi}$ is harmonic in the half-strip $\widetilde{\mathscr{S}}_1^+$ and is strictly positive in $(0, 0)$, see (3-13), the maximum of $\widetilde{\Phi}$ on the half strip $\widetilde{\mathscr{S}}_1^+$ is necessarily attained on the boundary of the half-strip or at infinity, and therefore on $(-1, 1) \times \{0\}$ according to the boundary conditions satisfied by $\widetilde{\Phi}$ in (3-7) and the conditions (3-11) as $B \to \infty$. Now, $\partial_A\widetilde{\Phi}$ is nonpositive in $\widetilde{\mathscr{S}}_1^+ \cap \{A > 0\}$ and $\widetilde{\Phi}(A, B) = \widetilde{\Phi}(|A|, B)$ in the half-strip $\widetilde{\mathscr{S}}_1^+$ according to (3-9), so the maximum of $\widetilde{\Phi}$ is necessarily attained in $(A, B) = (0, 0)$.

We then come back to the problem (3-5)–(3-6) and check that the function $\tilde{\varphi}$ given by

$$\tilde{\varphi}(a, b) = \widetilde{\Phi}(A, B) \quad \text{for } A + \imath B = F(a + \imath b), \ \ (a, b) \in \widetilde{\mathscr{O}}_1^+, \tag{3-14}$$

with $\widetilde{\Phi}$ as in (3-9), satisfies (3-5)–(3-6).

By construction, $\tilde{\varphi}$ automatically satisfies (3-5) and its maximum is attained in $(a, b) = (0, 0)$ and takes value $\tilde{\varphi}(0, 0) = \widetilde{\Phi}(0, 0)$. We thus only have to check the condition (3-6). In order to do that, let us introduce the real functions $\widetilde{A} = \widetilde{A}(a, b)$ and $\widetilde{B} = \widetilde{B}(a, b)$ given for $(a, b) \in \widetilde{\mathscr{O}}_1^+$ by

$$F(a + \imath b) = \widetilde{A}(a, b) + \imath\widetilde{B}(a, b), \tag{3-15}$$

and let us check that

$$\lim_{b\to\infty}\inf_{|a|<b+1}\widetilde{B}(a,b)=+\infty. \tag{3-16}$$

Indeed, if it were not the case, we could find real sequences $(a_n,b_n)_{n\in\mathbb{N}}$ with

$$\lim_{n\to\infty} b_n=+\infty,\qquad \text{for all } n\in\mathbb{N},\quad |a_n|\leqslant b_n+1,\qquad \text{and}\qquad \sup_n \widetilde{B}(a_n,b_n)<\infty. \tag{3-17}$$

Then, if we set $\zeta_n=F_{\frac34}^{-1}(a_n+\imath b_n)$, by construction,

$$F_{\frac12}(\zeta_n)=\widetilde{A}(a_n,b_n)+\imath\widetilde{B}(a_n,b_n).$$

Therefore, according to the definition of $F_{\frac12}$,

$$\zeta_n=\sin\bigl(\tfrac{\pi}{2}(\widetilde{A}(a_n,b_n)+\imath\widetilde{B}(a_n,b_n))\bigr),$$

so that the sequence (ζ_n) is uniformly bounded in $\mathbb{C}$ as $n\to\infty$. Then the sequence (a_n,b_n) is given by $a_n+\imath b_n=F_{\frac34}(\zeta_n)$. But $F_{\frac34}$ maps bounded sets of $\mathbb{C}$ into bounded sets of $\mathbb{C}$, so this is in contradiction with (3-17), and the property (3-16) holds.

We can thus use (3-11) to get that for all $b\geqslant 0$,

$$\sup_{|a|<b+1}\{|\tilde{\varphi}(a,b)|\}\leqslant C\exp\bigl(-\tfrac{\pi}{2}\inf_{|a|<b+1}\widetilde{B}(a,b)\bigr),$$

which, according to (3-16), implies (3-6).

Remark 3.1. Another approach to obtain information on $\tilde{\varphi}$, the solution of (3-5), is through integral equations. More precisely, for $((a,b),(a_0,b_0))\in(\widetilde{\mathcal{O}}_1^+)^2$, we define $\mathcal{G}$ as

$$\widetilde{\mathcal{G}}(a,b,a_0,b_0)=\frac{1}{4\pi}\ln\left(\frac{\bigl((a-a_0)^2+(b-b_0)^2\bigr)\bigl((a+a_0)^2+(b+b_0+2)^2\bigr)}{\bigl((a+b_0+1)^2+(b+a_0+1)^2\bigr)\bigl((a-b_0-1)^2+(a_0-b-1)^2\bigr)}\right).$$

It is readily verified that for any $(a_0,b_0)\in\widetilde{\mathcal{O}}_1^+$, $\widetilde{\mathcal{G}}(\cdot,\cdot,a_0,b_0)$ satisfies

$$\begin{cases}\Delta_{a,b}\widetilde{\mathcal{G}}(\cdot,\cdot,a_0,b_0)=\delta_{(a_0,b_0)} & \text{in } \widetilde{\mathcal{O}}_1^+,\\ \widetilde{\mathcal{G}}(a,b,a_0,b_0)=0 & \text{for } (a,b) \text{ such that } |a|=|b|+1.\end{cases}$$

Indeed, this comes from the fact that $\widetilde{\mathcal{G}}$ is the suitable combination of the fundamental solution of the Laplace operator in the sectors $\{(a,b)\in\mathbb{R}^2: b=|a|-1\}$ and $\{(a,b)\in\mathbb{R}^2: b=1-|a|\}$.

Then, standard computations show that $\tilde{\varphi}$ is a solution of (3-5) if and only if it satisfies the integral equation

$$\tilde{\varphi}(a_0,b_0)=-\int_{-1}^{1}\partial_b\widetilde{\mathcal{G}}(a,0,a_0,b_0)\,\tilde{\varphi}(a,0)\,da+\int_{-1}^{1}\widetilde{\mathcal{G}}(a,0,a_0,b_0)\,da\quad \text{for all } (a_0,b_0)\in\widetilde{\mathcal{O}}_1^+. \tag{3-18}$$

We then introduce $\widetilde{\mathscr{G}}$ defined by

$$\widetilde{\mathscr{G}}(a,a_0,b_0)=-\partial_b\widetilde{\mathcal{G}}(a,0,a_0,b_0)-\frac{1}{2\pi}\frac{b_0}{b_0^2+(a-a_0)^2}.$$

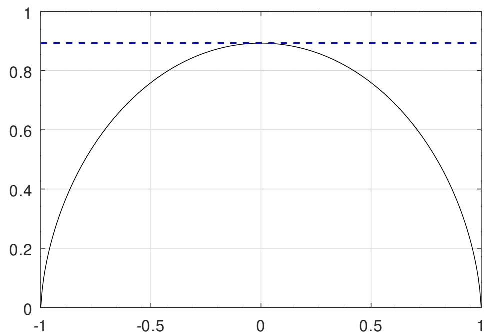

Figure 3. The solid line shows $\tilde{\varphi}(a_0, 0)$ for $a_0 \in (-1, 1)$, obtained by discretization of (3-19). The dashed line shows $\widetilde{\Phi}(0, 0) = \tilde{\varphi}(0, 0)$.

It is easily seen that for any $a_0 \in (-1, 1)$,

$$\lim_{b_0 \to 0} \int_{-1}^{1} \widetilde{\mathscr{G}}(a, a_0, b_0)\, \tilde{\varphi}(a, 0)\, da = \int_{-1}^{1} \widetilde{\mathscr{G}}(a, a_0, 0)\, \tilde{\varphi}(a, 0)\, da,$$
$$\lim_{b_0 \to 0} \int_{-1}^{1} \widetilde{\mathcal{G}}(a, 0, a_0, b_0)\, da = \int_{-1}^{1} \widetilde{\mathcal{G}}(a, 0, a_0, 0)\, da,$$

whereas

$$\lim_{b_0 \to 0} \frac{1}{2\pi} \int_{-1}^{1} \frac{b_0}{b_0^2 + (a - a_0)^2}\, \tilde{\varphi}(a, 0)\, da = \tfrac{1}{2} \tilde{\varphi}(a_0, 0).$$

Therefore, choosing $a_0 \in (-1, 1)$ and taking the limit $b_0 \to 0$ in (3-18) leads to the integral equation

$$\tfrac{1}{2} \tilde{\varphi}(a_0, 0) = \int_{-1}^{1} \widetilde{\mathscr{G}}(a, a_0, 0)\, \tilde{\varphi}(a, 0)\, da + \int_{-1}^{1} \widetilde{\mathcal{G}}(a, 0, a_0, 0)\, da. \tag{3-19}$$

Discretizing (3-19), we can obtain a good approximation of $\tilde{\varphi}(a_0, 0)$ for $a_0 \in (-1, 1)$ (see Figure 3).

3B2. *Phragmén–Lindelöf principle.* With $\tilde{\varphi}$ as in (2-12), the function $(a, b) \mapsto \tilde{\varphi}(a, b) + |b|$ is harmonic in $\widetilde{\mathscr{O}}_1$, and it is therefore the real part of some holomorphic function ϕ in $\mathscr{O}_1$:

$$\text{for all } (a, b) \in \widetilde{\mathscr{O}}_1, \quad \Re(\phi(a + \imath b)) = \tilde{\varphi}(a, b) + |b|,$$

or, equivalently, for all $\xi \in \mathscr{O}_1$, $\Re(\phi(\xi)) = \varphi(\xi) + |\Im(\xi)|$.

For each $\alpha_* > \alpha$, we consider the function g_{α_*} defined for $\xi \in \mathscr{O}_\alpha$ by

$$g_{\alpha_*}(\xi) = f(\xi) \exp\left(-L\alpha_* \phi\left(\frac{\xi}{\alpha}\right)\right). \tag{3-20}$$

By construction, g_{α_*} is holomorphic in $\mathscr{O}_\alpha$ and satisfies

$$\text{for all } \xi \in \partial\mathscr{O}_\alpha, \quad |g_{\alpha_*}(\xi)| \leqslant C_0, \quad \text{and} \quad \lim_{|\Im(\xi)| \to \infty} \Bigl(\sup_{|\Re(\xi)| < |\Im(\xi)| + \alpha} |g_{\alpha_*}(\xi)| \Bigr) = 0.$$

Therefore, g_{α_*} attains its maximum on $\partial\mathscr{O}_\alpha$, so that

$$\text{for all } \xi \in \mathscr{O}_\alpha, \quad |f(\xi)| \leqslant C_0 \exp\left(\frac{\alpha_*}{\alpha}|\Im(\xi)|L\right) \exp\left(L\alpha_*\varphi\left(\frac{\xi}{\alpha}\right)\right).$$

Taking the limit $\alpha_* \to \alpha$, we immediately have

$$\text{for all } \xi \in \mathscr{O}_\alpha, \quad |f(\xi)| \leqslant C_0 \exp(|\Im(\xi)|L) \exp\left(L\alpha\varphi\left(\frac{\xi}{\alpha}\right)\right), \tag{3-21}$$

that is, (2-14).

Remark 3.2. Let us remark that we can obtain from the above proof an explicit formula for f. Indeed, for $\alpha_* > \alpha > L/(2T)$, we can use the Cauchy formula for the function g_{α_*} in (3-20) on the contour given by

$$\gamma_{\alpha,R} = \partial(\mathscr{O}_\alpha \cap \{\Im(\xi) < R\}) \quad (\text{with } R > 0)$$

oriented in a counterclockwise manner, which yields, for all $\xi \in \mathbb{R}$ with $|\xi| < L/(2T)$,

$$g_{\alpha_*}(\xi) = \frac{1}{2\imath\pi}\int_{\gamma_{\alpha,R}} \frac{g_{\alpha_*}(\zeta)}{\zeta - \xi}\, d\zeta.$$

Now, due to the decay of g_{α_*} at infinity, one can pass to the limit in the above formula as $R \to \infty$: for all $\xi \in \mathbb{R}$ with $|\xi| < L/(2T)$,

$$g_{\alpha_*}(\xi) = \frac{1}{2\imath\pi}\int_{\gamma_\alpha} \frac{g_{\alpha_*}(\zeta)}{\zeta - \xi}\, d\zeta,$$

where γ_α is the union of the two connected components of $\partial\mathscr{O}_\alpha$ oriented counterclockwise. Recalling the definition of g_{α_*}, we end up with the following formula: for all $\xi \in \mathbb{R}$ with $|\xi| < L/(2T)$,

$$f(\xi) = \frac{1}{2\imath\pi}\int_{\gamma_\alpha} e^{L\alpha_*(\phi(\xi/\alpha)-\phi(\zeta/\alpha))}\frac{f(\zeta)}{\zeta - \xi}\, d\zeta. \tag{3-22}$$

4. Further comments

4A. *Higher-dimensional settings.* The method developed above applies also to the cost of observability of the heat equation in multidimensional balls. More precisely, we consider the following heat equation, set in the ball of radius $L > 0$ of $\mathbb{R}^d$ $(d \geqslant 1)$, denoted by $\mathcal{B}_L$ in the following, and in the time interval $(0, T)$:

$$\begin{cases} \partial_t u - \Delta_x u = 0 & \text{in } (0,T)\times\mathcal{B}_L, \\ u(t,x) = 0 & \text{in } (0,T)\times\partial\mathcal{B}_L, \\ u(0,x) = u_0(x) & \text{in } \mathcal{B}_L, \end{cases} \tag{4-1}$$

where the initial datum u_0 belongs to $H_0^1(\mathcal{B}_L)$. In that setting, we have the following result:

Theorem 4.1. *Setting K_0 as in Theorem 1.1, for any $K > K_0$, there exists a constant $C > 0$ such that for all $T \in (0,1]$, for all solutions u of (4-1) with initial datum $u_0 \in H_0^1(\mathcal{B}_L)$,*

$$\left\| u(T)\exp\left(\frac{|x|^2}{4T}\right)\right\|_{L^2(\mathcal{B}_L)} \leqslant C\exp\left(\frac{KL^2}{T}\right)\|\partial_\nu u\|_{L^2((0,T)\times\partial\mathcal{B}_L)}. \tag{4-2}$$

Here and in the following, $|\cdot|$ denotes the euclidean norm in $\mathbb{R}^d$. The proof of Theorem 4.1 closely follows the one of Theorem 1.1; therefore we only sketch its proof, explaining the main differences with the proof of Theorem 1.1.

Sketch of the proof of Theorem 4.1. We start by considering a smooth solution u of (4-1), and define

$$z(t,x) = u(t,x)\exp\left(\frac{|x|^2 - L^2}{4t}\right), \qquad (t,x) \in (0,T)\times \mathcal{B}_L,$$

which satisfies

$$\begin{cases} \partial_t z + \dfrac{x}{t}\cdot\nabla_x z + \dfrac{d}{2t} z - \Delta_x z - \dfrac{L^2}{4t^2} z = 0 & \text{in } (0,\infty)\times\mathcal{B}_L, \\ z(t,x) = 0 & \text{in } (0,T)\times\partial\mathcal{B}_L, \\ z(0,x) = 0 & \text{in } \mathcal{B}_L, \end{cases}$$

Proposition A.1 with $\Omega = \mathcal{B}_L$ and $g \equiv 0$ implies directly the following estimate for z:

$$\int_{\mathcal{B}_L} |\nabla_x z(T,x)|^2\,dx - \frac{L^2}{4T^2}\int_{\mathcal{B}_L} |z(T,x)|^2\,dx \leqslant \frac{L}{T^2}\int_0^T\int_{\partial\mathcal{B}_L} t|\nabla_x z(t,x)\cdot\nu|^2\,ds(x)\,ds. \tag{4-3}$$

We define w as the extension of z by 0 outside $\mathcal{B}_L$: w satisfies the equations

$$\begin{cases} \partial_t w + \dfrac{x}{t}\cdot\nabla_x w + \dfrac{d}{2t} w - \Delta_x w - \dfrac{L^2}{4t} w = \nabla_x u(t,x)\cdot\nu\,\delta_{\partial\mathcal{B}_L} & \text{in } (0,\infty)\times\mathbb{R}^d, \\ w(0,x) = 0, & x \in \mathbb{R}^d. \end{cases}$$

Thus, its Fourier transform, defined for $(t,\xi)\in(0,T)\times\mathbb{C}^d$ by

$$\hat{w}(t,\xi) = \int_{\mathbb{R}^d} w(t,x)e^{-\imath\xi\cdot x}\,dx$$

satisfies

$$\begin{cases} \partial_t \hat{w} - \dfrac{\xi}{t}\cdot\nabla_\xi\hat{w} - \dfrac{d}{2t}\hat{w} + \xi^2\hat{w} - \dfrac{L^2}{4t^2}\hat{w} = \displaystyle\int_{\partial\mathcal{B}_L} \nabla_x u(t,x)\cdot\nu e^{-\imath\xi\cdot x}\,ds(x), & (t,\xi)\in(0,\infty)\times\mathbb{R}^d, \\ \hat{w}(0,\xi) = 0, & \xi\in\mathbb{R}^d. \end{cases} \tag{4-4}$$

As in the one-dimensional case, (4-3) gives a high-frequency ($|\xi| > L/(2T)$) L^2-estimate of $w(T,\cdot)$ depending on the observation and the low-frequency ($|\xi| \leqslant L/(2T)$) L^2-norm of $w(T,\cdot)$, on which we focus from now. To do so, much as in Section 3A, we solve the transport equation (4-4), and obtain, for $\xi_0\in\mathbb{R}^d$ such that $|\xi_0| > L/(2T)$,

$$\hat{w}(T,\xi_0) = \int_0^T \left(\frac{T}{t}\right)^{\frac{d}{2}} \int_{\partial\mathcal{B}_L} \nabla_x u(t,x)\cdot\nu e^{-\imath\frac{x\cdot\xi_0 T}{t} - (\xi_0^2T^2 - \frac{L^2}{4})(\frac{1}{t}-\frac{1}{T})}\,ds(x)\,dt, \tag{4-5}$$

with $\xi_0^2 = \xi_0\cdot\xi_0$.

Once here, we consider $\xi_0 = (\xi_1,\tilde{\xi})$, with $\tilde{\xi}\in\mathbb{R}^{d-1}$ fixed, and $\xi_1 = a + \imath b$, $a, b\in\mathbb{R}$, and define $f(\xi_1) = \hat{w}(T,\xi_1,\tilde{\xi})$, which is an entire function satisfying (2-11). Besides, with computations similar to those in Section 3A, it is easy to obtain that for all $\alpha > L^2/(2T)$, there exists $C_\alpha(T) > 0$, which may blow up polynomially in T as $T\to 0$ (contrarily to what happens in the one-dimensional setting, the constant $C_\alpha(T)$ may now blow up as $T \to 0$, but only polynomially in T, so that it will not significantly affect the cost of observability in small times in (4-2), which blows up as an exponential of $1/T$ as $T \to 0$), such that for all $\xi_1 \in \mathscr{C}_\alpha$ as in (2-6), we have

$$|f(\xi_1)| \leqslant C_\alpha e^{|\Im(\xi_1)|L} \|\partial_\nu u\|_{L^2((0,T)\times\partial\mathcal{B}_L)}.$$

From that, we end the proof of Theorem 4.1 exactly as in the one-dimensional case, with the use of Proposition 2.3. □

Actually, the method developed above works not only for balls, but also for any bounded domain $\Omega \subset \mathbb{R}^d$. More precisely:

Theorem 4.2. *Let Ω be a smooth bounded domain of $\mathbb{R}^d$. If we set*

$$L_\Omega = \inf_{x\in\Omega} \sup_{y\in\partial\Omega} |x-y|,$$

and we choose $\bar{x} \in \overline{\Omega}$ such that

$$\sup_{y\in\partial\Omega} |\bar{x}-y| = L_\Omega,$$

then for any $K > K_0$, there exists $C > 0$ such that any smooth solution u of

$$\begin{cases} \partial_t u - \Delta_x u = 0 & \text{in } (0,T)\times\Omega, \\ u(t,x) = 0 & \text{in } (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{4-6}$$

satisfying

$$\left\| u(T)\exp\left(\frac{|x-\bar{x}|^2}{4T}\right)\right\|_{L^2(\Omega)} \leqslant C \exp\left(\frac{KL_\Omega^2}{T}\right)\|\partial_\nu u\|_{L^2((0,T)\times\partial\Omega)}.$$

Note that this is a geometrical setting in which Corollary 1.3 applies but yields a different estimate on the cost of observability. Indeed, when the observation is done on the whole boundary, one easily checks that the choice $S_0 = S_\Omega^+$, where

$$S_\Omega = \sup\{\text{length of segments included in } \Omega\},$$

is suitable for the application of Corollary 1.3. In particular, when Ω is convex, $L_\Omega \leqslant S_\Omega \leqslant 2L_\Omega$ and Theorem 4.2 always yields at least the estimate given by Corollary 1.3 when the observation is done on the whole boundary of Ω, and a better one in general (as in the case of a ball discussed in Theorem 4.1).

Remark 4.3. The above discussion, and Theorem 4.2 in particular, might suggest that the cost of observability in small times is linked only to the maximal distance to the control set. This is not the case, as it is strongly underlined by [Laurent and Léautaud 2018]. There, among other results, an analysis of the observability constant $C_0(T, \mathbb{B}(0,1), \mathbb{B}(0,r))$ for the heat equation is done when the domain of interest is $\Omega = \mathbb{B}(0,1) \subset \mathbb{R}^2$, the unit ball of the plane, and the observation set is $\mathbb{B}(0,r)$. To be more precise, $C_0(T, \mathbb{B}(0,1), \mathbb{B}(0,r))$ is the best constant in the following estimate: for any solution u of (4-6) with $\Omega = \mathbb{B}(0,1)$ with initial datum $u_0 \in H_0^1(\Omega)$,

$$\|u(T)\|_{L^2(\Omega)} \leqslant C_0(T, \mathbb{B}(0,1), \mathbb{B}(0,r))\|u\|_{L^2((0,T)\times\mathbb{B}(0,r)))}.$$

The work [Laurent and Léautaud 2018] then shows the following result: there exist $C > 0$ and $r_0 < 1$ such that for all $r \in (0, r_0)$

$$\liminf_{T\to 0} T \log(C_0(T, \mathbb{B}(0,1), \mathbb{B}(0,r))) \geqslant C \log(r)^2. \tag{4-7}$$

This shows that the behavior of the cost of observability in small times is in fact strongly linked to the geometry under consideration. Indeed, Theorem 4.2 in fact corresponds to a geometrical setting in which the wave equation is observable in small times, while the result (4-7) proved in [Laurent and Léautaud 2018] focuses on a case in which the geometric control condition for the observability of the wave equation fails due to whispering gallery phenomena.

4B. ***Tensorized equations.*** Another application of our method concerns the cost of observability of the heat equation on a tensorized domain. More precisely, we consider the heat equation set in a tensorized spatial domain $\Omega = \Omega_x \times \Omega_y$, and want to know the cost of observability in small time when the solution is observed on $\partial\Omega_x \times \Omega_y$. Note that the answer is already known: the cost is the same as the one for the heat equation set on Ω_x only, when the observation is done on the whole boundary $\partial\Omega_x$ [Miller 2005, Theorem 1.5]. Our purpose is therefore just to underline that our approach also applies in that context and allows us to retrieve easily this result.

To fix ideas, we focus on the case $\Omega_x = (-L, L)$ (when Ω_x is a multidimensional domain, similar arguments can be developed, under appropriate geometric conditions, by using Theorem 4.2 instead of Theorem 1.1). Hence we are interested in the following heat equation, set in the domain $\Omega = (-L, L)\times\Omega_y$, with $L > 0$ and Ω_y a smooth bounded domain of $\mathbb{R}^{d_y}$, in some time interval $(0, T)$, $T > 0$:

$$\begin{cases} \partial_t u - \partial_x^2 u - \Delta_y u = 0 & \text{for } (t,x,y) \in (0,T)\times(-L,L)\times\Omega_y, \\ u(t,L,y) = u(t,-L,y) = 0 & \text{for } (t,y) \in (0,T)\times\Omega_y, \\ u(t,x,y) = 0 & \text{for } (t,x,y) \in (0,T)\times(-L,L)\times\partial\Omega_y, \\ u(0,x,y) = u_0(x,y) & \text{in } (-L,L)\times\Omega_y. \end{cases} \tag{4-8}$$

As usual, the initial datum u_0 belongs to $H_0^1((-L, L)\times\Omega_y)$. We have the following:

Theorem 4.4. *Setting K_0 as in Theorem 1.1, for any $K > K_0$, there exists a constant $C > 0$ such that for all $T \in (0, 1]$, for all solutions u of (4-8),*

$$\begin{aligned} &\left\| u(T,x,y) \exp\left(\frac{x^2}{4T}\right)\right\|_{L^2((-L,L)\times\Omega_y)} \\ &\qquad \leqslant C \exp\left(\frac{KL^2}{T}\right)\big(\|\partial_x u(t,-L,y)\|_{L^2((0,T)\times\Omega_y)} + \|\partial_x u(t,L,y)\|_{L^2((0,T)\times\Omega_y)}\big). \end{aligned} \tag{4-9}$$

Sketch of the proof of Theorem 4.4. Let us denote by (v_n, λ_n^2) the family of normalized eigenfunctions and eigenvalues of the Dirichlet–Laplace operator set in Ω_y, that is,

$$\begin{cases} -\Delta_y v_n = \lambda_n^2 v_n & \text{in } \Omega_y, \\ v_n = 0 & \text{on } \partial\Omega_y, \\ \|v_n\|_{L^2(\Omega_y)} = 1. \end{cases}$$

Expanding u, a solution of (4-8), on the $L^2(\Omega_y)$ Hilbert basis (v_n), that is,

$$u(t,x,y)=\sum_{n\in\mathbb{N}} u_n(t,x)v_n(y),$$

we see that each u_n solves a one-dimensional heat equation with potential λ_n^2 set in $(0,T)\times(-L,L)$:

$$\begin{cases}\partial_t u_n-\partial_x^2 u_n+\lambda_n^2 u_n=0 & \text{in } (0,T)\times(-L,L),\\ u_n(t,-L)=u_n(t,L)=0 & \text{in } (0,T),\\ u_n(0,x)=u_{n,0}(x) & \text{in } (-L,L),\end{cases} \tag{4-10}$$

with

$$u_{n,0}(x)=\int_\Omega u_0(x,y)\,v_n(y)\,dy.$$

To prove Theorem 4.4, it is sufficient to prove that each u_n satisfies the observability inequality

$$\left\|u_n(T,x)\exp\left(\frac{x^2}{4T}\right)\right\|_{L^2(-L,L)}\leqslant C\exp\left(\frac{KL^2}{T}\right)\left(\|\partial_x u_n(t,-L)\|_{L^2(0,T)}+\|\partial_x u_n(t,L)\|_{L^2(0,T)}\right), \tag{4-11}$$

with a constant C independent of n. To do so, we consider $\tilde{u}_n=u_n e^{\lambda_n^2 t}$, which satisfies

$$\begin{cases}\partial_t \tilde{u}_n-\partial_x^2 \tilde{u}_n=0 & \text{in } (0,T)\times(-L,L),\\ \tilde{u}_n(t,-L)=\tilde{u}_n(t,L)=0 & \text{in } (0,T),\\ \tilde{u}_n(0,x)=u_{n,0}(x) & \text{in } (-L,L).\end{cases}$$

Applying Theorem 1.1, we get

$$\left\|\tilde{u}_n(T,x)\exp\left(\frac{x^2}{4T}\right)\right\|_{L^2(-L,L)}\leqslant C\exp\left(\frac{KL^2}{T}\right)\left(\|\partial_x \tilde{u}_n(t,-L)\|_{L^2(0,T)}+\|\partial_x \tilde{u}_n(t,L)\|_{L^2(0,T)}\right),$$

which directly gives (4-11) as $e^{\lambda_n^2(t-T)}\leqslant 1$ for all $t\in(0,T)$, and therefore ends the proof. □

4C. *Observation from one side of the domain: symmetrization argument.* In this section, we are interested in the cost of observability for the one-dimensional heat equation when observed on one side of the domain. In other words, for $L,T>0$ and $u_0\in H_0^1(0,L)$, we consider the system

$$\begin{cases}\partial_t u-\partial_x^2 u=0 & \text{in } (0,T)\times(0,L),\\ u(t,0)=u(t,L)=0 & \text{in } (0,T),\\ u(0,x)=u_0(x) & \text{in } (0,L).\end{cases} \tag{4-12}$$

We have the following:

Theorem 4.5. *Setting K_0 as in Theorem 1.1, for any $K>K_0$, there exists a constant $C>0$ such that for all $T\in(0,1]$, for all solutions u of (4-12) with $u_0\in H_0^1(0,L)$,*

$$\left\|u(T)\exp\left(\frac{x^2}{4T}\right)\right\|_{L^2(0,L)}\leqslant C\exp\left(\frac{KL^2}{T}\right)\|\partial_x u(t,L)\|_{L^2(0,T)}. \tag{4-13}$$

Proof. The proof is based on a classical symmetrization argument: for u a solution of (4-12), we define

$$u_s(t,x) = \begin{cases} u(t,x) & \text{for } (t,x) \in (0,T) \times (0,L), \\ -u(t,-x) & \text{for } (t,x) \in (0,T) \times (-L,0). \end{cases}$$

It is readily seen that u_s satisfies system (1-1). Therefore, Theorem 1.1 gives

$$\left\| u_s(T) \exp\left(\frac{x^2}{4T}\right) \right\|_{L^2(-L,L)} \leqslant C \exp\left(\frac{KL^2}{T}\right) \bigl(\|\partial_x u_s(t,-L)\|_{L^2(0,T)} + \|\partial_x u_s(t,L)\|_{L^2(0,T)} \bigr).$$

The result follows easily, as $\partial_x u_s(t,-L) = \partial_x u_s(t,L) = \partial_x u(t,L)$ for all $t \in (0,T)$. □

4D. *Distributed observations.* One is sometimes interested in distributed observations, in which case the corresponding observability inequality reads

$$\|u(T)\|_{L^2(0,L)} \leqslant C(T,L,a,b) \|u\|_{L^2((0,T)\times(a,b))} \tag{4-14}$$

for smooth solutions u of (4-12), where $a, b \in \mathbb{R}$ are such that $(a,b) \subset (0,L)$ and $a < b$.

We claim the following:

Theorem 4.6. *Let $0 \leqslant a < b \leqslant L$. Setting K_0 as in Theorem 1.1, for any $K > K_0$, there exists a constant $C > 0$ such that for all $T \in (0,1]$, for all solutions u of (4-12),*

$$\|u(T)\|_{L^2(0,L)} \leqslant C \exp\left(\frac{K \min\{a^2, (L-b)^2\}}{T}\right) \|u\|_{L^2(0,T;H^1(a,b))}. \tag{4-15}$$

Proof. As in the proof of Theorem 4.5, we start by symmetrizing the function u, and we call u_s its symmetric extension. We then take $\varepsilon > 0$ small enough to have $a + 2\varepsilon < b$ and we choose an even cut-off function ρ taking value 1 on $(-a-\varepsilon, a+\varepsilon)$ and vanishing for $|x| > a + 2\varepsilon$. Then the function

$$z(t,x) = \begin{cases} \rho(x)\, u_s(t,x) \exp\left(\dfrac{x^2 - (a+2\varepsilon)^2}{4t}\right) & \text{for } |x| < a + 2\varepsilon, \\ 0 & \text{for } |x| > a + 2\varepsilon \end{cases}$$

satisfies, much as in (2-3),

$$\begin{cases} \partial_t z + \dfrac{x}{t} \partial_x z + \dfrac{1}{2t} z - \partial_x^2 z - \dfrac{(a+2\varepsilon)^2}{4t^2} z = g, & (t,x) \in (0,\infty) \times (-a-2\varepsilon, a+2\varepsilon), \\ z(t, -a-2\varepsilon) = z(t, a+2\varepsilon) = 0, & t \in (0,\infty), \\ z(0,x) = 0, & x \in (-a-2\varepsilon, a+2\varepsilon), \end{cases} \tag{4-16}$$

where

$$g(t,x) = \exp\left(\frac{x^2 - (a+2\varepsilon)^2}{4t}\right) (2\partial_x \rho\, \partial_x u(t,x) + \partial_{xx} \rho u(t,x)).$$

One can then follow the approach developed in Section 2 (using Proposition A.1 instead of Theorem 2.1 and the fact that $\partial_x z(t, -a-2\varepsilon) = \partial_x z(t, a+2\varepsilon) = 0$) to show that for all $K_1 > K_0$, there exists C such that for all $T \in (0, 1]$,

$$\left\| u_s(T) \exp\left(\frac{x^2}{4T}\right) \right\|_{L^2(-a-2\varepsilon, a+2\varepsilon)} \leqslant C \exp\left(\frac{K_1(a+2\varepsilon)^2}{T}\right) \|g\|_{L^2((0,T)\times(-a-2\varepsilon,a+2\varepsilon))}.$$

Using the definition of g, one easily gets

$$\|u(T)\|_{L^2(0,a+\varepsilon)} \leqslant C \exp\left(\frac{K_1(a+2\varepsilon)^2}{T}\right) \|u\|_{L^2(0,T;H^1(a,a+2\varepsilon))}.$$

Similarly, one can obtain

$$\|u(T)\|_{L^2(b-\varepsilon,L)} \leqslant C \exp\left(\frac{K_1(L-b+2\varepsilon)^2}{T}\right) \|u\|_{L^2(0,T;H^1(b-2\varepsilon,b))}.$$

It is straightforward to show that

$$\|u(T)\|_{L^2(a+\varepsilon,b-\varepsilon)} \leqslant C \|u\|_{L^2(0,T;H^1(a,b))},$$

for instance by looking at $v(t,x) = \eta(t)\, u(t,x)\, \rho_0(x)$, where $\eta = \eta(t)$ is a smooth function of time taking value 0 at $t = 0$ and 1 at $t = T$, and $\rho_0 = \rho_0(x)$ taking value 1 on $(a+\varepsilon, b-\varepsilon)$ and vanishing for $x \notin (a,b)$, and doing energy estimates.

Combining the three above estimates, we easily conclude (4-15) by taking $K_1 \in (K_0, K)$ and $\varepsilon > 0$ small enough. □

Note that the above argument is only based on suitable cut-off arguments. It can therefore be applied as well in multidimensional settings, provided some geometric assumptions compatible with Theorem 4.2 are satisfied, namely if the distributed observation set is a neighborhood of the whole boundary.

4E. *Related uncertainty principles.* One key point to obtain Theorem 1.1 is the complex analysis argument developed in Section 3B, based principally on the Schwarz–Christoffel conformal mapping and the Phragmén–Lindelöf principle. It is nevertheless possible to develop a purely *real analysis* argument, but it only allows us to retrieve the cost of observability for the one-dimensional heat equation known since [Tenenbaum and Tucsnak 2007]:

Theorem 4.7. *For all $K > \frac{3}{4}$, there exists a constant $C > 0$ such that for all $T \in (0, 1]$, all solutions u of (1-1) with initial datum $u_0 \in H_0^1(-L, L)$ satisfy (1-2).*

The proof of Theorem 4.7 is based on the following *uncertainty principle result.*

Proposition 4.8 [Landau and Pollak 1961; Fuchs 1964]. *Let $A, B > 0$. Let $f \in L^2(\mathbb{R})$ be supported in $[-A, A]$ and $\hat{f}$ its Fourier transform. Then*

$$\int_{-B}^{B} |\hat{f}(\xi)|^2 \, d\xi \leqslant \lambda_0 \int_{\mathbb{R}} |\hat{f}(\xi)|^2 \, d\xi, \tag{4-17}$$

where $\lambda_0 = \lambda_0(AB)$ satisfies $0 < \lambda_0 < 1$ and

$$\lambda_0 = 1 - 4\sqrt{\pi}\sqrt{AB}e^{-2AB}(1 + \varepsilon_{AB}), \tag{4-18}$$

where $\varepsilon_{AB} \to 0$ as $AB \to \infty$.

Relation (4-17) is a particular case of [Landau and Pollak 1961, Theorem p. 68], whereas the proof of the asymptotic behavior of λ_0 can be found in [Fuchs 1964, Theorem 1, p. 319].

Proof of Theorem 4.7. We start from formula (2-7), which we recall: for any $\xi_0 \in \mathbb{R}$ such that $|\xi_0| > L/(2T)$, we have

$$\hat{w}(T,\xi_0) = -\int_0^T \sqrt{\frac{T}{t}}\partial_x u(t,-L) e^{\imath \frac{L\xi_0 T}{t} - (\xi_0^2 T^2 - \frac{L^2}{4})(\frac{1}{t}-\frac{1}{T})}\,dt + \int_0^T \sqrt{\frac{T}{t}}\partial_x u(t,L) e^{-\imath \frac{L\xi_0 T}{t} - (\xi_0^2 T^2 - \frac{L^2}{4})(\frac{1}{t}-\frac{1}{T})}\,dt.$$

Therefore, we directly obtain, for $\xi_0 \in \mathbb{R}$ with $|\xi_0| > L/(2T)$,

$$|\hat{w}(T,\xi_0)|^2 \leqslant T\big(\|\partial_x u(t,-L)\|^2_{L^2(0,T)} + \|\partial_x u(t,L)\|^2_{L^2(0,T)}\big)\int_0^T e^{-2T^2(\xi_0^2 - \frac{L^2}{4T^2})(\frac{1}{t}-\frac{1}{T})}\,\frac{dt}{t}.$$

For $\eta > 1$, we choose $\xi_0 \in \mathbb{R}$ with $|\xi_0| \geqslant \eta L/(2T)$, which implies

$$\xi_0^2 - \frac{L^2}{4T^2} \geqslant \frac{\eta^2-1}{\eta^2}\xi_0^2$$

and

$$\int_0^T e^{-2T^2(\xi_0^2 - \frac{L^2}{4T^2})(\frac{1}{t}-\frac{1}{T})}\,\frac{dt}{t} \leqslant \int_0^T e^{-2T^2\frac{\eta^2-1}{\eta^2}\xi_0^2(\frac{1}{t}-\frac{1}{T})}\,\frac{dt}{t} \leqslant \frac{\eta^2}{2T(\eta^2-1)\xi_0^2}.$$

Hence we obtain, for $\xi_0 \in \mathbb{R}$ with $|\xi_0| > \eta L/(2T)$,

$$|\hat{w}(T,\xi_0)|^2 \leqslant \frac{\eta^2}{2(\eta^2-1)\xi_0^2}\big(\|\partial_x u(t,-L)\|^2_{L^2(0,T)} + \|\partial_x u(t,L)\|^2_{L^2(0,T)}\big)$$

and

$$\int_{|\xi_0|>\eta\frac{L}{2T}} |\hat{w}(T,\xi_0)|^2\,d\xi_0 \leqslant \frac{2T\eta}{(\eta^2-1)L}\big(\|\partial_x u(t,-L)\|^2_{L^2(0,T)} + \|\partial_x u(t,L)\|^2_{L^2(0,T)}\big).$$

Now, from (4-17) applied to $f = \hat{w}(T)$ with $A = L,\ B = \eta L/(2T)$ and $\lambda_0 = \lambda_0(\eta L^2/(2T))$, we have

$$\int_{\mathbb{R}} |\hat{w}(T,\xi_0)|^2\,d\xi_0 = \int_{|\xi_0|<\eta\frac{L}{2T}} |\hat{w}(T,\xi_0)|^2\,d\xi_0 + \int_{|\xi_0|>\eta\frac{L}{2T}} |\hat{w}(T,\xi_0)|^2\,d\xi_0 \leqslant \lambda_0 \int_{\mathbb{R}} |\hat{w}(T,\xi_0)|^2\,d\xi_0 + \int_{|\xi_0|>\eta\frac{L}{2T}} |\hat{w}(T,\xi_0)|^2\,d\xi_0,$$

and thus

$$\int_{\mathbb{R}} |\hat{w}(T,\xi_0)|^2\,d\xi_0 \leqslant \frac{1}{1-\lambda_0}\int_{|\xi_0|>\eta\frac{L}{2T}} |\hat{w}(T,\xi_0)|^2\,d\xi_0.$$

We have therefore obtained

$$\int_{-L}^{L} |w(T,x)|^2\,dx = \int_{\mathbb{R}} |\hat{w}(T,\xi_0)|^2\,d\xi_0 \leqslant \frac{1}{1-\lambda_0}\frac{2T\eta}{(\eta^2-1)L}\big(\|\partial_x u(t,-L)\|^2_{L^2(0,T)} + \|\partial_x u(t,L)\|^2_{L^2(0,T)}\big),$$

which implies from Proposition 4.8 and (4-18) the existence of a constant C such that for T small enough

$$\|w(T)\|_{L^2(-L,L)} \leqslant C e^{\eta\frac{L^2}{2T}}\big(\|\partial_x u(t,-L)\|_{L^2(0,T)} + \|\partial_x u(t,L)\|_{L^2(0,T)}\big).$$

The result of Theorem 4.7 follows from the definition of w. □

4F. ***On a possible improvement of Theorem 1.1.*** As we said in the Introduction, we do not know if the estimate on the cost of observability in small times given by Theorem 1.1 is sharp or not. In fact, when looking at the main steps of the proof of Theorem 1.1 given in Section 2, it seems that one step in which our estimates are not sharp may be the one using Phragmén–Lindelöf principles, i.e., Proposition 2.3.

Indeed, introducing the class

$$\mathscr{E}_\alpha = \{ f \in \mathrm{Hol}(\mathcal{O}_\alpha) : f(\xi) e^{-|\Im(\xi)|} \in L^\infty(\mathscr{O}_\alpha) \text{ and for all } \xi \in \partial \mathcal{O}_\alpha,\ |f(\xi)| \leqslant e^{|\Im(\xi)|} \},$$

Proposition 2.3 shows that for all $\alpha \in \mathbb{R}_+^*$,

$$\sup_{f \in \mathscr{E}_\alpha} \Big(\sup_{x \in [-\alpha,\alpha]} \{|f(x)|\} \Big) \leqslant \exp(\alpha \varphi(0)), \tag{4-19}$$

where $\varphi(0)$ is given by (2-15). Besides, this estimate is sharp as we can construct a holomorphic function ϕ in $\mathscr{O}_1$ whose real part coincides with $\varphi(\xi) + |\Im(\xi)|$ given by (2-12)–(2-13) and check that $f_\phi(\xi) = \exp(\alpha \phi(\xi/\alpha))$ belongs to $\mathscr{E}_\alpha$ and saturates the estimate (4-19), so that for all $\alpha \in \mathbb{R}_+^*$,

$$\max_{f \in \mathscr{E}_\alpha} \Big(\max_{x \in [-\alpha,\alpha]} \{|f(x)|\} \Big) = \exp(\alpha \varphi(0)). \tag{4-20}$$

Now, in our approach (in the case $L = 1$, which can always be assumed by a scaling argument), we apply estimate (4-19) to the function $f = \hat{w}(T,\cdot)/\|\hat{w}(T,\xi) e^{-|\Im(\xi)|}\|_{L^\infty(\mathscr{C}_\alpha)}$, which in fact belongs to a smaller class

$$\mathscr{E}_\alpha^* = \{ f \in \mathrm{Hol}(\mathbb{C}) : f(\xi) e^{-|\Im(\xi)|} \in L^\infty(\mathbb{C}) \text{ and for all } \xi \in \mathscr{C}_\alpha, |f(\xi)| \leqslant e^{|\Im(\xi)|} \}.$$

Therefore, our proof requires an estimate on the constant

$$C^*(\alpha) = \sup_{f \in \mathscr{E}_\alpha^*} \Big(\sup_{x \in [-\alpha,\alpha]} \{|f(x)|\} \Big) \tag{4-21}$$

in the asymptotics $\alpha \to \infty$. It is clear that

$$C^*(\alpha) \leqslant \exp(\alpha \varphi(0)), \tag{4-22}$$

which is precisely the estimate we use, but there is no evidence to support the idea that this estimate gives the good asymptotics as $\alpha \to \infty$.

Let us in particular point out that:

- The function f_ϕ given above to show that estimate (4-19) is sharp does not belong to the class $\mathscr{E}_\alpha^*$.
- The constant $C^*(\alpha)$ in (4-21) blows up at least like $\exp(\alpha/2)$ as $\alpha \to \infty$, as otherwise the proof given in Section 2 would yield a cost of observability smaller than $\exp(L^2/2T)$ in small times, which is known to be false due to [Lissy 2015].
- Looking at the 2-parameter family of functions of the form

$$f_{A,\gamma}(\xi) = \cos(A\sqrt{\xi^2 - \gamma^2})$$

for parameters $A \in [0, 1]$ and $\gamma \in [0, \alpha]$, we find that

$$\sup_{f \in \{f_{A,\gamma}\} \cap \mathscr{E}_\alpha^*} \Big(\sup_{x \in [-\alpha,\alpha]} \{|f(x)|\}\Big) = \cosh\left(\frac{\alpha}{2}\right),$$

and is achieved when taking $A = 1/\sqrt{2}$ and $\gamma = \alpha/\sqrt{2}$, i.e.,

$$f(\xi) = \cos\left(\frac{1}{\sqrt{2}}\sqrt{\xi^2 - \frac{\alpha^2}{2}}\right).$$

This function yields evidence of the fact that

$$\liminf_{\alpha\to\infty} \frac{1}{\alpha} \log(C^*(\alpha)) \geqslant \frac{1}{2}.$$

Let us finally emphasize that if we were able to show that

$$\limsup_{\alpha\to\infty} \frac{1}{\alpha} \log(C^*(\alpha)) \leqslant \frac{1}{2},$$

the proof given in Section 2 would yield a cost of observability in small times $C_0(T, L)$ satisfying

$$\limsup_{T\to 0} T \log(C_0(T, L)) \leqslant \frac{L^2}{2}.$$

Combined with [Lissy 2015], this would give that

$$\lim_{T\to 0} T \log(C_0(T, L)) = \frac{L^2}{2}.$$

4G. ***Uniform controllability of viscous approximations of the transport equation.*** The problem we considered in this article is intimately related to the question of uniform controllability of viscous approximations of the transport equation raised in [Coron and Guerrero 2005]. Namely, for all $\varepsilon > 0$, one considers the following viscous approximation of the transport equation at velocity $M \in \mathbb{R}$:

$$\begin{cases} \partial_t y_\varepsilon - \varepsilon\, \partial_x^2 y_\varepsilon + M\, \partial_x y_\varepsilon = 0, & (t, x) \in (0, T) \times (0, L), \\ y_\varepsilon(t, 0) = v_\varepsilon(t), & t \in (0, T), \\ y_\varepsilon(t, L) = 0, & t \in (0, T), \\ y_\varepsilon(0, \cdot) = y_0(x), & x \in (0, L). \end{cases} \tag{4-23}$$

For each $\varepsilon > 0$, the equation (4-23) is null-controllable in any time $T > 0$, and the map $\mathcal{V}_{\varepsilon,T} : y_0 \to v_\varepsilon$ which to any $y_0 \in L^2(0, L)$ associates the control v_ε of minimal $L^2(0, T)$-norm is linear. The problem raised in [Coron and Guerrero 2005] is the following one: give conditions on the time T guaranteeing that

$$\limsup_{\varepsilon\to 0} \|\mathcal{V}_{\varepsilon,T}\|_{\mathscr{L}(L^2(0,L);L^2(0,T))} < \infty. \tag{4-24}$$

It is clear that if $|M|T < L$, (4-24) cannot happen, as otherwise the convergence of (4-23) as $\varepsilon \to 0$ would imply the null-controllability of the transport equation in a time which is not enough to make the characteristics go out of the domain.

Several conditions on the time T ensuring (4-24) were then proposed in the literature, namely in [Coron and Guerrero 2005; Glass 2010; Lissy 2012]. In fact, to our knowledge, the best results are the ones obtained in [Lissy 2012], which we recall now:

Theorem 4.9 [Lissy 2012]. *If $M \neq 0$ and*

$$|M|T > L(2\sqrt{3}+1-\operatorname{sign}(M)) \quad (2\sqrt{3} \approx 3.4641),$$

where $\operatorname{sign}(M) = 1$ *if* $M > 0$ *and* $= -1$ *if* $M < 0$, *we have*

$$\limsup_{\varepsilon \to 0} \|\mathcal{V}_{\varepsilon,T}\|_{\mathscr{L}(L^2(0,L);L^2(0,T))} = 0.$$

These results are based on the knowledge of the cost of observability of the one-dimensional heat equation in small time obtained in [Tenenbaum and Tucsnak 2007]. Therefore, as Theorem 4.5 improves the one in that paper, following the proof of [Lissy 2012] immediately improves the known result on the uniform controllability of the viscous approximations (4-23) of the transport equation:

Theorem 4.10. *Let K_0 as in* (1-5). *Then, if $M \neq 0$ and*

$$|M|T > L(4\sqrt{K_0}+1-\operatorname{sign}(M)) \quad (4\sqrt{K_0} \approx 3.3385),$$

we have

$$\limsup_{\varepsilon \to 0} \|\mathcal{V}_{\varepsilon,T}\|_{\mathscr{L}(L^2(0,L);L^2(0,T))} = 0. \tag{4-25}$$

As the proof of Theorem 4.10 follows line to line the one of [Lissy 2012], as it is explained in Section 3, item (i) of that paper, it is left to the reader.

We are currently investigating if one can do better than the combination of the cost of observability of the one-dimensional heat equation in small times and of the arguments in [Lissy 2012] to obtain better sufficient conditions on the ratio $|M|T/L$ to guarantee (4-25). We believe that a direct approach following the strategy in Section 2 could help in improving Theorem 4.10.

Appendix: Carleman-type estimate

We consider the equation

$$\begin{cases} \partial_t z - \Delta_x z + \dfrac{1}{2t}(2x \cdot \nabla_x z + dz) - \dfrac{L^2}{4t^2} z = g & \text{in } (0,T) \times \Omega, \\ z(t,x) = 0 & \text{on } (0,T) \times \partial\Omega, \\ \lim_{t\to 0} \|z(t)\|_{L^2(\Omega)} = 0, \\ \lim_{t \to 0} t\|\nabla z(t)\|_{L^2(\Omega)} = 0, \end{cases} \tag{A-1}$$

with $T > 0$, Ω a bounded domain of $\mathbb{R}^d$, $d \geqslant 1$,

$$L = \sup_{x \in \Omega} |x| \tag{A-2}$$

and

$$g \in L^2((0,T) \times \Omega).$$

We then have the following result:

Proposition A.1. *Any smooth solution z of* (A-1) *with $g \in L^2((0,T)\times\Omega)$ satisfies the estimate*

$$\int_\Omega \Bigl(|\nabla_x z(T)|^2 - \frac{L^2}{4T^2}|z(T)|^2 \Bigr)\,dx \leqslant \frac{L}{T^2}\int_0^T \Bigl(t \int_{\Gamma_+} |\nabla_x z(t,x)\cdot\nu|^2\,ds(x) \Bigr)\,dt + \frac{1}{T^2}\int_0^T\!\!\int_\Omega t^2|g|^2\,dx\,dt, \tag{A-3}$$

with $\Gamma_+ = \{x \in \partial\Omega : x\cdot\nu > 0\}$, and L is given by (A-2).

Proof. We define the spatial operators

$$Sz = -\Delta_x z - \frac{L^2}{4t^2}z, \qquad Az = \frac{1}{2t}(2x\cdot\nabla_x z + d\,z),$$

so that z is a solution of (A-1) satisfying

$$\partial_t z + Sz + Az = g \quad \text{in } (0,T)\times\Omega.$$

Note that S and A respectively correspond to the symmetric and skew-symmetric parts of the operator in (A-1).

We then consider

$$D(t) := \int_\Omega \Bigl(|\nabla_x z(t,x)|^2 - \frac{L^2}{4t^2}|z(t,x)|^2 \Bigr)\,dx = \int_\Omega (Sz)(t,x)z(t,x)\,dx.$$

A direct calculation shows that

$$\begin{aligned} D'(t) &= \frac{L^2}{2t^3}\int_\Omega |z|^2\,dx + 2\int_\Omega Sz\,\partial_t z\,dx \\ &= \frac{L^2}{2t^3}\int_\Omega |z|^2\,dx - 2\int_\Omega |Sz|^2\,dx - 2\int_\Omega Sz Az\,dx + 2\int_\Omega Sz\,g\,dx. \end{aligned}$$

Furthermore, as A is a skew-symmetric operator, we have

$$-2\int_\Omega Sz\,Az\,dx = 2\int_\Omega \Delta_x z\,Az\,dx = \frac{1}{t}\int_\Omega \Delta_x z(2x\cdot\nabla_x z + d\,z)\,dx.$$

On one hand, we obviously have

$$\int_\Omega \Delta_x z\,d\,z\,dx = -d\int_\Omega |\nabla_x z|^2\,dx.$$

On the other hand, we note that

$$\begin{aligned} \int_\Omega \Delta_x z\,2x\cdot\nabla_x z\,dx &= 2\int_{\partial\Omega} (\nabla_x z\cdot\nu)(x\cdot\nabla_x z)\,ds(x) - 2\int_\Omega \nabla_x z\cdot\nabla_x(x\cdot\nabla_x z)\,dx \\ &= 2\int_{\partial\Omega} (x\cdot\nu)|\nabla_x z\cdot\nu|^2\,ds(x) - 2\int_\Omega \nabla_x z\cdot\nabla_x(x\cdot\nabla_x z)\,dx. \end{aligned}$$

Here, we have used that as $z = 0$ on $\partial\Omega$, $\nabla_x z = (\nabla_x z\cdot\nu)\nu$ on $\partial\Omega$. As

$$\nabla_x z\cdot\nabla_x(x\cdot\nabla_x z) = |\nabla_x z|^2 + \frac{x}{2}\cdot\nabla_x(|\nabla_x z|^2),$$

we have

$$\begin{aligned}\int_\Omega \nabla_x z\cdot\nabla_x(x\cdot\nabla_x z)\,dx &= \int_\Omega |\nabla_x z|^2\,dx + \int_\Omega \frac{x}{2}\cdot\nabla_x(|\nabla_x z|^2)\,dx\\ &= \int_\Omega |\nabla_x z|^2\,dx + \frac{1}{2}\int_{\partial\Omega}(x\cdot\nu)|\nabla_x z|^2\,ds(x) - \frac{d}{2}\int_\Omega |\nabla_x z|^2\,dx\\ &= \int_\Omega |\nabla_x z|^2\,dx + \frac{1}{2}\int_{\partial\Omega}(x\cdot\nu)|\nabla_x z\cdot\nu|^2\,ds(x) - \frac{d}{2}\int_\Omega |\nabla_x z|^2\,dx.\end{aligned}$$

Gathering the above computations, we get

$$\begin{aligned}D'(t)+2\int_\Omega |Sz|^2\,dx &= \frac{L^2}{2t^3}\int_\Omega |z|^2\,dx - \frac{2}{t}\int_\Omega |\nabla_x z|^2\,dx + \frac{1}{t}\int_{\partial\Omega}(x\cdot\nu)|\nabla_x z\cdot\nu|^2\,ds(x) + 2\int_\Omega Sz\,g\,dx\\ &\leqslant -\frac{2}{t}D(t) + \frac{1}{t}\int_{\partial\Omega}(x\cdot\nu)|\nabla_x z\cdot\nu|^2\,ds(x) + \int_\Omega |Sz|^2\,dx + \int_\Omega |g|^2\,dx,\end{aligned}$$

which implies in particular

$$(t^2D(t))' \leqslant t\int_{\Gamma_+}(x\cdot\nu)|\nabla_x z\cdot\nu|^2\,ds(x) + t^2\int_\Omega |g|^2\,dx. \tag{A-4}$$

Using the assumption on z in the third and fourth lines of (A-1), one easily checks $\lim_{t\to 0} t^2D(t) = 0$; hence we can integrate (A-4) between 0 and T, which gives (A-3), as $|(x\cdot\nu)| \leqslant L$ for all $x \in \overline{\Omega}$. □

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JÉRÉMI DARDÉ: jeremi.darde@math.univ-toulouse.fr
Institut de Mathématiques de Toulouse, UMR 5219, Université de Toulouse, CNRS, Toulouse, France

SYLVAIN ERVEDOZA: sylvain.ervedoza@math.univ-toulouse.fr
Institut de Mathématiques de Toulouse, UMR 5219, Université de Toulouse, CNRS, Toulouse, France

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