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RENJIE FENG AND DONG YAO

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ZEROS OF REPEATED DERIVATIVES OF RANDOM POLYNOMIALS

RENJIE FENG AND DONG YAO

It has been shown that zeros of Kac polynomials $K_n(z)$ of degree n cluster asymptotically near the unit circle as $n \to \infty$ under some assumptions. This property remains unchanged for the l-th derivative of the Kac polynomials $K_n^{(l)}(z)$ for any fixed order l. So it's natural to study the situation when the number of the derivatives we take depends on n, i.e., $l = N_n$. We will show that the limiting behavior of zeros of $K_n^{(N_n)}(z)$ depends on the limit of the ratio N_n/n. In particular, we prove that when the limit of the ratio is strictly positive, the property of the uniform clustering around the unit circle fails; when the ratio is close to 1, the zeros have some rescaling phenomenon. Then we study such problem for random polynomials with more general coefficients. But things, especially the rescaling phenomenon, become very complicated for the general case when $N_n/n \to 1$, where we compute the case of the random elliptic polynomials to illustrate this.

1. Introduction

There are many well-known results regarding the nontrivial relations between zeros and critical points of polynomials. The classical Gauss–Lucas theorem states that all the critical points of a polynomial are in the convex hull of its zeros; in particular, if all the zeros are real, then so are the zeros of the derivative. Differentiating a polynomial which has only real zeros will even out zero spacings [Farmer and Rhoades 2005]; in the case of random trigonometric polynomials, it's proved in [Farmer and Yerrington 2006] that the repeated differentiation causes the roots of the function to approach equal spacing, which can be viewed as a toy model of crystallization in one dimension. For random polynomials under some mild assumptions, the distribution of critical points and the distribution of its zeros are asymptotically the same as the degree tends to infinity. This is because, roughly speaking, the coefficients of the derivative of a random polynomial are not changed dramatically. Actually, such result holds for any fixed number of derivatives [Feng $\geq$ 2019; Kabluchko and Zaporozhets 2014]. In this article, we are primarily interested in the case when the number of the derivatives we take for the random polynomials is not fixed but grows to infinity with the degree.

Our starting point is the classical Kac polynomials. Let $\xi_0, \xi_1, \ldots$ be nondegenerate, independent and identically distributed (i.i.d.) complex random variables. The Kac polynomials are defined as

$$K_n(z) = \sum_{k=0}^{n} \xi_k z^k. \tag{1}$$

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The Kac polynomials have degree n almost surely by assuming

$$\mathbb{P}(\xi_0 = 0) = 0. \tag{2}$$

The distribution of zeros of Kac polynomials has been studied for decades; we refer to [Bloom and Shiffman 2007; Hough et al. 2009; Ibragimov and Zeitouni 1997; Ibragimov and Zaporozhets 2013; Kac 1943; Kabluchko and Zaporozhets 2013; 2014; Sodin and Tsirelson 2004; Shepp and Vanderbei 1995]. It's proved that if

$$\mathbb{E}\log(1 + |\xi_0|) < \infty, \tag{3}$$

then with probability 1, the empirical measure of zeros of Kac polynomials converges weakly to the uniform probability measure on the unit circle as n tends to infinity [Ibragimov and Zeitouni 1997; Ibragimov and Zaporozhets 2013; Kabluchko and Zaporozhets 2013; 2014; Shepp and Vanderbei 1995]. If the assumption (3) is removed, then zeros of $K_n(z)$ may not concentrate around the unit circle; see [Ibragimov and Zaporozhets 2013; Kabluchko and Zaporozhets 2013] for the case when $|\xi_0|$ has some logarithmic tails.

The property of clustering around the unit circle remains unchanged for the l-th derivative of the Kac polynomials $K_n^{(l)}(z)$ for any fixed l as n tends to infinity [Feng $\geq$ 2019; Kabluchko and Zaporozhets 2014]. But things become interesting if the number of the derivatives we take depends on n, e.g., $l = N_n$. For the extreme case when $N_n = n$, there is no zero for $K_n^{(n)}$ almost surely. Hence, some natural questions are: What is the critical growth order of N_n so that the property of clustering around the unit circle for the Kac polynomials $K_n^{(N_n)}$ fails? When it fails, what is the distribution of zeros of $K_n^{(N_n)}$? And how does the distribution depend on the growth order of N_n? In this article, we will answer these questions for the Kac polynomials completely. The estimates we derive for the Kac case can be applied to the general random polynomials. But there are some issues for the general random polynomials, where we will compute the case of the random elliptic polynomials to illustrate this.

1.1. ***Notation.*** Before we state our main results, we need to introduce some notation. We denote by

$$p_n(z) = \sum_{k=0}^{n} \xi_k p_{k,n} z^k \tag{4}$$

the random polynomials of degree n with general coefficients, where $p_{k,n}$ are deterministic coefficients and ξ_k are nondegenerate i.i.d. complex random variables. Throughout the article, we assume the random variable ξ_0 satisfies the conditions (2) and (3).

We denote by $p_n^{(N_n)}(z)$ the N_n-th derivative of $p_n(z)$ with the degree

$$D_n = n - N_n. \tag{5}$$

Without loss of generality, we may assume the convergence of

$$\frac{N_n}{n} \to a \in [0, 1]. \tag{6}$$

The random measure of zeros of $p_n(z)$ is denoted by

$$\mu_n = \sum_{z : p_n(z) = 0} \delta_z, \tag{7}$$

and we use the notation

$$\mu_{D_n} = \sum_{z : p_n^{(N_n)}(z)=0} \delta_z \tag{8}$$

for the random measure of zeros of $p_n^{(N_n)}(z)$ of degree D_n.

Similarly, we denote by μ_n^K and $\mu_{D_n}^K$ the random measures of zeros of $K_n(z)$ and $K_n^{(N_n)}(z)$ for the Kac polynomials, respectively, and we denote by μ_n^E and $\mu_{D_n}^E$ the random elliptic polynomials. We denote by $\mathbb{D}_r$ the open disk of radius r centered at the origin in the complex plane. The convergence of the random measures ν_n to ν in probability (or in distribution) means the convergence in probability (or in distribution) in the weak sense, i.e., $\int_X \phi \nu_n(dx) \to \int_X \phi \nu(dx)$ in probability (or in distribution) for any smooth test function ϕ with compact support. Given a measure ν on the complex plane, we define the scaling operator $(\mathcal{S}_h \nu)(B) = \nu(B/h)$ for $h > 0$ where B is any Borel set in $\mathbb{C}$. In the end, we set $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$ and set $\log 0 = -\infty$.

1.2. ***Kabluchko–Zaporozhets theorem.*** There are many well-known results regarding the global distribution of zeros of some special Gaussian random analytic functions where the ensembles are usually invariant under some group action, such as the Gaussian elliptic polynomials and Gaussian hyperbolic analytic functions [Hough et al. 2009; Sodin and Tsirelson 2004]. Recently, a remarkable result proved in [Kabluchko and Zaporozhets 2014] deals with more general random analytic functions. Kabluchko and Zaporozhets [2014] proved that under certain assumptions on the coefficients of the random analytic functions, the distribution of zeros will converge to a deterministic rotationally invariant measure on a domain of the complex plane. Such measure can be explicitly characterized in terms of the coefficients. To be more precise, let's consider the random analytic function in the form of

$$F_n(z) = \sum_{k=0}^{\infty} \xi_k p_{k,n} z^k, \tag{9}$$

where ξ_k are nondegenerate i.i.d. complex random variables satisfying condition (3) and the coefficients $p_{k,n}$ satisfy the following assumptions.

Assumptions 1. Assume there are a function $p : [0, \infty) \to [0, \infty)$ and a number $T_0 \in (0, \infty]$ such that

(1) $p(t) > 0$ for $t < T_0$ and $p(t) = 0$ for $t > T_0$,

(2) p is continuous on $[0, T_0)$, and in the case $T_0 < \infty$, left continuous at T_0,

(3) $\lim_{n\to\infty} \sup_{k\in[0,An]} ||p_{k,n}|^{1/n} - p(k/n)| = 0$ for every $A > 0$, and

(4) $R_0 = \liminf_{t\to\infty} p(t)^{-1/t} \in (0, \infty]$, $\liminf_{k\to\infty} |p_{k,n}|^{-1/k} \geq R_0$ for every fixed $n \in \mathbb{N}$ and additionally, $\liminf_{n,k/n\to\infty} |p_{k,n}|^{-1/k} \geq R_0$.

Roughly speaking, the major assumption is that the coefficients $p_{k,n}$ are approximately $e^{n \log p(k/n)}$ for some p, which is positive on some interval $[0, T_0)$, continuous in $[0, T_0]$, and equal to 0 in (T_0, ∞).

Theorem 1 [Kabluchko and Zaporozhets 2014]. *Under Assumptions 1 and* (3), *let $I(s)$ be the Legendre–Fenchel transform of $-\log p$, i.e., $I(s) = \sup_{t\geq 0}(st + \log p(t))$; then the random measure $(1/n)\mu_{F_n}$*

of zeros of $F_n(z)$ converges in probability to a deterministic measure μ in $\mathbb{D}_{R_0}$, which is rotationally invariant and satisfies

$$\mu(\mathbb{D}_r) = I'(\log r), \quad r \in (0, R_0).$$

As a convention, I' is the left derivative of I. A typical example to which to apply the Kabluchko–Zaporozhets theorem is the Kac polynomials where we have

$$p_{k,n} = 1_{k \le n}, \qquad p(t) = 1_{t \le 1}, \qquad T_0 = 1. \tag{10}$$

By some computations, we have $I(s) = s \vee 0$ and thus the limiting distribution satisfies

$$\mu(\mathbb{D}_r) = \begin{cases} 0, & 0 \le r \le 1, \\ 1, & r > 1, \end{cases} \tag{11}$$

i.e., the uniform probability measure on the unit circle.

But we cannot apply the Kabluchko–Zaporozhets theorem directly in our case to derive the distribution of zeros of $K_n^{(N_n)}$ or that of the general random polynomials $p_n^{(N_n)}$. For example, if $N_n = n - \lfloor \log n \rfloor$, then the degree of $p_n^{(N_n)}$ is $D_n = \lfloor \log n \rfloor$; therefore, one cannot find some A so that Assumption 1(3) is satisfied. We need to modify their theorem to deal with our situation more conveniently. We consider the random polynomials in the form of

$$F_n(z) = \sum_{k=0}^{(T_0-\delta_n)L_n} \xi_k p_{k,n} z^k, \tag{12}$$

where $(T_0 - \delta_n)L_n$ is an integer and we assume that $F_n(z)$ satisfies the following assumptions:

Assumptions 2. There exist a function $p : [0, \infty) \to [0, \infty)$, a positive number $T_0 \in (0, \infty)$, a sequence of positive integers L_n going to ∞ as $n \to \infty$, and a sequence of numbers $\delta_n \in (-T_0, T_0)$ (not necessarily positive) that goes to 0 as $n \to \infty$ such that

(1) $p(t) > 0$ for $t \in [0, T_0)$ and $p(t) = 0$ for $t > T_0$,

(2) p is continuous in $[0, T_0]$, and

(3) $\lim_{n\to\infty} \sup_{0\le k \le (T_0-\delta_n)L_n} ||p_{k,n}|^{1/L_n} - p((k/L_n) \wedge T_0)| = 0$.

Then we have the following theorem whose proof is sketched in the Appendix.

Theorem 2. *For random polynomials $F_n(z)$ in the form of* (12) *which satisfy Assumptions 2, let $I(s)$ be the Legendre–Fenchel transform of $-\log p$; then the random measure $(1/L_n)\mu_{F_n}$ of zeros will converge in probability to a deterministic rotationally invariant measure μ where*

$$\mu(\mathbb{D}_r) = I'(\log r), \quad r > 0. \tag{13}$$

Throughout the article, we often make use of the estimate

$$\lim_{n\to\infty} \sup_{0 \le k \le (T_0-\delta_n)L_n} \left| \frac{1}{L_n} \log|p_{k,n}| - \log p\left(\frac{k}{L_n} \wedge T_0\right) \right| = 0. \tag{14}$$

This estimate implies the main Assumption 2(3), which is the direct consequence of the inequality

$$|x-y| \le (x \wedge y)e^{|\log x - \log y|}|\log x - \log y|$$

for any $x, y > 0$. The main advantage of (14) is the convenience in computations.

1.3. ***Main results.*** We first state our main results for the Kac polynomials, which will answer the questions we raised at the beginning of the article.

Kac polynomials. The main result is that the limiting behavior of the distribution of zeros of $K_n^{(N_n)}$ will depend on the limit of the ratio N_n/n. We will divide our discussions into two categories: D_n goes to infinity and D_n remains a fixed number, where $D_n = n - N_n$ is the degree of the random polynomials $K_n^{(N_n)}$. Without loss of generality, we consider the four different cases ① $N_n/n \to 0$, ② $N_n/n \to a \in (0, 1)$, ③ $N_n/n \to 1$ and $D_n \to \infty$, e.g., $N_n = n - \lfloor \log n \rfloor$ and $D_n = \lfloor \log n \rfloor$, and ④ $N_n/n \to 1$ but $D_n = m < \infty$, i.e., $K_n^{(N_n)}$ has a fixed degree m.

In the cases of ①–③ where $D_n \to \infty$, we will show that the coefficients of $K_n^{(N_n)}$ or its rescaling will satisfy Assumptions 2 with different choices of L_n, δ_n, T_0, and p; then we apply Theorem 2 to prove:

Theorem 3. *Assume $D_n \to \infty$ as $n \to \infty$; we have the following results regarding the empirical measure of zeros of derivatives of Kac polynomials $K_n^{(N_n)}$:*

(1) *If* $\lim_{n\to\infty} N_n/n = 0$, *then* $(1/D_n)\mu^K_{D_n}$ *converges in probability to the uniform probability measure on the unit circle*, *i.e.*, *the measure defined in* (11).

(2) *If* $\lim_{n\to\infty} N_n/n = a \in (0, 1)$, *then* $(1/D_n)\mu^K_{D_n}$ *converges in probability to a rotationally invariant measure* μ^K_a *on* $\mathbb{C}$ *defined by*

$$\mu^K_a(\mathbb{D}_r) = \begin{cases} ar/((1-a)(1-r)), & 0 < r < 1-a, \\ 1, & r \ge 1-a. \end{cases} \tag{15}$$

(3) *If* $\lim_{n\to\infty} N_n/n = 1$, *then globally we have the convergence in probability*

$$\frac{1}{D_n}\mu^K_{D_n} \to \delta_0. \tag{16}$$

If we set $R_n = n/D_n$ *as the quotient of the degrees of* K_n *and* $K_n^{(N_n)}$ *and consider the rescaling Kac polynomials* $\widetilde{K}_n(z) := K_n^{(N_n)}(z/R_n)$, *then the empirical measure* $(1/D_n)\mu^{\widetilde{K}}_{D_n}$ *which is the same as* $(1/D_n)\mathcal{S}_{R_n}(\mu^K_{D_n})$ *converges in probability to a rotationally invariant measure* $\tilde{\mu}^K$ *where*

$$\tilde{\mu}^K(\mathbb{D}_r) = \begin{cases} r, & r < 1, \\ 1, & r \ge 1. \end{cases} \tag{17}$$

In particular, *the density for the measure* $\tilde{\mu}^K$ *is*

$$\tilde{d}^K(z) = \frac{1}{2\pi|z|}1_{|z|\le 1}. \tag{18}$$

In the case ④ when D_n remains a fixed number, we will show that the measure of zeros of the rescaling polynomials $K_n^{(N_n)}(z/n)$ will converge to some random measure. The main tool to prove this result is Rouché's theorem in complex analysis. Our result is as follows.

Theorem 4. *Suppose* $\lim_{n\to\infty} N_n/n = 1$ *and* $D_n = m$ *for all* n; *then globally*

$$\frac{1}{m}\mu^K_{D_n} \to \delta_0, \tag{19}$$

where the convergence is in probability. Furthermore, *we have the rescaling limit*

$$\mathcal{S}_n(\mu^K_{D_n}) \to \mu_{f^K_m}, \tag{20}$$

where the convergence is in distribution and $\mu_{f^K_m}$ *is the random measure of zeros of the random polynomial*

$$f^K_m(z) = \sum_{k=0}^{m} \frac{\xi_k}{k!} z^k. \tag{21}$$

Remark. The relationship between the results in Theorems 3(3) and 4 has an intuitive explanation. Consider the case in Theorem 3(3). We can zoom in zeros of $K_n^{(N_n)}(z)$ in two steps. First we zoom in the zeros of $K_n^{(N_n)}(z)$ by a factor of n; then by Theorem 4 (treating D_n as fixed for this moment) the scaled zeros will be close to the zeros of $f^K_{D_n}(z)$. Here $f^K_{D_n}(z)$ is just the function in (21) with m replaced by D_n. If we then zoom out zeros of $f^K_{D_n}$ by a factor of D_n (which is the degree of the polynomial $f^K_{D_n}$), then as a whole we get something close to zooming in the zeros of $K_n^{(N_n)}(z)$ by a factor of n/D_n. Taking n to infinity we should get the limit in Theorem 3(3). This is in accordance with the fact that (17) is also the limit of the empirical measure of zeros of $f^K_{D_n}(D_n z)$ as $m \to \infty$, as shown in Theorem 2.3 of [Kabluchko and Zaporozhets 2014]. Note that in the zooming out process, we can also replace $\sum_{k=0}^{D_n}(\xi_k/k!)(D_n z)^k$ by $\sum_{k=0}^{\infty}(\xi_k/k!)(D_n z)^k$ since Theorem 2.1 of [Kabluchko and Zaporozhets 2014] shows the empirical measure of $\sum_{k=0}^{\infty}(\xi_k/k!)(D_n z)^k$ restricted to unit disk also converges to the measure in (17).

As a summary, we show that the clustering property of zeros around the unit circle for the derivatives of Kac polynomials holds if and only if $N_n/n \to 0$; the conclusion (3) in Theorem 3 together with Theorem 4 imply that, if $N_n/n \to 1$, zeros will converge to the origin with the average decay rate D_n/n which is the quotient of the degrees of $K_n^{(N_n)}$ and K_n. Thus we will completely answer the questions we proposed at the beginning of the article.

General random polynomials. We can extend the above results for the Kac polynomials to the general random polynomials where the coefficients satisfy Assumptions 1 in the Kabluchko–Zaporozhets theorem.

Theorem 5. *Suppose the random polynomial* $p_n(z)$ *of* (4) *satisfies Assumptions 1 with some function* $p(t)$; *then regarding the zeros of* $p_n^{(N_n)}$, *we have*:

(1) *If* $\lim_{n\to\infty} N_n/n = 0$, *let* $I(s)$ *be the Legendre–Fenchel transform of* $-\log p$; *then* $(1/D_n)\mu_{D_n}$ *converges in probability to a rotationally invariant measure* μ *given by*

$$\mu(\mathbb{D}_r) = I'(\log r), \quad r > 0.$$

That is, $(1/D_n)\mu_{D_n}$ *has the same limit as* $(1/n)\mu_n$.

(2) *If* $\lim_{n\to\infty} N_n/n = a \in (0, 1)$, *let* $\log u_a = \log p(t+a) + (t+a)\log(t+a) - t\log t + (1-a)\log(1-a)$ *if* $0 \le t \le 1-a$ *and* $-\infty$ *if* $t > 1-a$. *Let* $I_a(s)$ *be the Legendre–Fenchel transform of* $-\log u_a$; *then* $(1/D_n)\mu_{D_n}$ *converges in probability to a rotationally invariant measure* μ_a *given by*

$$\mu_a(\mathbb{D}_r) = \frac{1}{1-a} I_a'(\log r), \quad r > 0.$$

Compared with Theorems 3 and 4 for the Kac case, things become complicated for the general random polynomials when the ratio N_n/n tends to 1. First, one cannot conclude that $(1/D_n)\mu_{D_n}$ converges in probability to δ_0. To see this, let's consider the following example where the coefficients of the random polynomials p_n are

$$p_{k,n} = \begin{cases} 1, & 0 \le k < N_n, \\ n!\,(k-N_n)!/(k!\,D_n!), & N_n \le k \le n, \end{cases}$$

where

$$D_n = \lfloor \log n \rfloor \quad \text{and} \quad N_n = n - D_n.$$

We let

$$p(t) = 1_{0\le t\le 1}.$$

We claim that $p_{k,n}$ and p satisfy Assumptions 1. Indeed, when $0 \le k < N_n$, we have

$$p_{k,n}^{1/n} = p\Big(\frac{k}{n}\Big).$$

Therefore, it remains to prove

$$\lim_{n\to\infty} \sup_{N_n\le k\le n} |p_{k,n}^{1/n} - 1| = 0.$$

By (14), it's enough to show

$$\lim_{n\to\infty} \sup_{N_n\le k\le n} \left|\frac{1}{n}\log p_{k,n}\right| = 0. \tag{22}$$

For $N_n \le k \le n$, we have $1 \le n!\,(k-N_n)!/(k!\,D_n!) \le n!/k!$; then

$$\sup_{N_n\le k\le n} \left|\frac{1}{n}\log p_{k,n}\right| \le \sup_{N_n\le k\le n} \frac{1}{n}\log\frac{n!}{k!} \le \frac{1}{n}\log n^{D_n} \le \frac{\log^2 n}{n},$$

where (22) follows as $n\to\infty$, which completes the proof of the claim. But the N_n-th derivative of p_n is

$$p_n^{(N_n)} = \frac{n!}{D_n!}\sum_{k=0}^{D_n} \xi_{k+N_n} z^k,$$

which is in the form of Kac polynomials; thus, the empirical measure of zeros will converge to the uniform probability measure on the circle instead of the delta function at the origin.

Secondly, even if zeros converge to δ_0, one cannot easily find the rescaling limit of the empirical measure of zeros if there exists one. The rescaling property should highly depend on the properties of coefficients, such as the convergent rate of $p_{n,k}$ to $p(t)$ and the monotonicity of $p_{k,n}$ for each fixed n. The following results regarding the elliptic polynomials provide such an example.

Random elliptic polynomials. The random elliptic polynomials are in the form of

$$E_n(z) = \sum_{k=0}^{n} \xi_k \sqrt{\binom{n}{k}} z^k. \tag{23}$$

If ξ_k are i.i.d. complex Gaussian random variables, then the random elliptic polynomials are also called Gaussian SU(2) polynomials. The Gaussian SU(2) polynomials can be viewed as meromorphic functions defined on the complex projective space $\mathbb{CP}^1 \cong S^2$, and a basic fact is that the distribution of its zeros is invariant under the SU(2) action. The Gaussian SU(2) polynomial is the standard model when one tries to generalize the random polynomials to random holomorphic sections on the complex manifolds [Bleher et al. 2000; Hough et al. 2009].

One can show that the coefficients of the random elliptic polynomials satisfy all of Assumptions 1 with the associated function (see also [Kabluchko and Zaporozhets 2014])

$$\log p^E(t) = -\tfrac{1}{2} t \log t - \tfrac{1}{2}(1-t)\log(1-t) \quad \text{for } 0 \le t \le 1. \tag{24}$$

Theorem 6. *For the random elliptic polynomials $E_n(z)$ defined in* (23), *we have*:

(1) *The conclusions in Theorem 5 hold for $(1/D_n)\mu^E_{D_n}$ with p replaced by p^E defined in* (24).

(2) *If $\lim_{n\to\infty} N_n/n = 1$, then we have the global convergence in probability*

$$\frac{1}{D_n}\mu^E_{D_n} \to \delta_0.$$

Furthermore, if $D_n \to \infty$, then in probability, we have

$$\frac{1}{D_n}\mathcal{S}_{\sqrt{R_n}}(\mu^E_{D_n}) \to \mu,$$

where $R_n = n/D_n$ as before and μ is the rotationally invariant probability measure defined as

$$\mu(\mathbb{D}_r) = \frac{r(\sqrt{4+r^2}-r)}{2}, \quad r \in (0,\infty). \tag{25}$$

If $D_n = m < \infty$, then the following rescaling limit holds in distribution:

$$\mathcal{S}_{\sqrt{n}}(\mu^E_{D_n}) \to \mu_{f^E_m},$$

where $\mu_{f^E_m}$ is the random measure of zeros of $f^E_m = \sum_{k=0}^{m}(\xi_k/(k!\sqrt{(m-k)!}))z^k$.

1.4. ***Further remarks.*** Let's compare Theorem 6 with Theorems 3(3) and 4 for the case when $N_n/n \to 1$. Both the empirical measures of zeros of derivatives tend to the point mass at the origin, but the interesting result is that they converge with different decay rates. Zeros converge to the origin with the average decay rate D_n/n for the Kac case and $\sqrt{D_n/n}$ for the elliptic case, which indicates that Assumptions 1 is not enough to extract the complete information about the convergence of zeros of the N_n-th derivative of general random polynomials; i.e., the main assumption $\lim_{n\to\infty}\sup_{k\in[0,An]}||p_{k,n}|^{1/n} - p(k/n)| = 0$ for every $A > 0$ is not enough. It seems that we need to impose additional assumptions on the rate of the convergence of $p_{k,n}$ to p for $N_n \le k \le n$ and the growth order of $p_{k,n}$. As in (14), we may alternatively consider the quantities

$$\eta_n := \sup_{N_n \le k \le n} \left| \frac{1}{n} \log|p_{k,n}| - \log p\left(\frac{k}{n}\right) \right| \tag{26}$$

and

$$b_n := \sup_{N_n \le k \le n} |p_{k,n}|. \tag{27}$$

The asymptotic properties of η_n and b_n may play important roles in the case when $N_n/n \to 1$. Note that η_n is identical to 0 for the Kac polynomials and asymptotic to $(\log D_n)/(4n) + O(1/n)$ for the random elliptic polynomials. Two questions are raised: What are the asymptotic properties of η_n and b_n so that zeros of $p_n^{(N_n)}$ tend to the origin? And if zeros tend to the origin, how does the decay rate depend on η_n and b_n? We postpone these two problems for further investigation.

Along with Kac polynomials, there is another important type of random polynomial defined via the orthogonal polynomials. Given a bounded simply connected domain Ω in the complex plane with analytic boundary C of length L and a positive weight function $w(z)$, we define the inner product

$$\langle f, g \rangle = \frac{1}{L} \int_C f(z)\overline{g(z)}w(z)|dz|. \tag{28}$$

Then we can find an orthonormal basis $\{p_n^w(z)\}$ with respect to this inner product, where $p_n^w(z)$ is a polynomial of degree n in which the coefficient of z^n is real and positive. Shiffman and Zelditch [2003] prove that the empirical measure of zeros of

$$P_n(z) = \sum_{k=0}^{n} \xi_k p_k^w(z), \tag{29}$$

where ξ_k are i.i.d. standard complex Gaussian random variables, tends to the equilibrium measure of Ω as n tends to infinity. Such result is then generalized by Bloom and Shiffman [2007] to higher dimensions where they get rid of the analytic assumption and replace it by the Bernstein–Markov condition. In [Feng $\ge$ 2019], the author further studied zeros of the l-th derivative of $P_n^{(l)}$ for any fixed l as $n \to \infty$, and proved that zeros of derivatives of any fixed order also tend to the equilibrium measure. The method used in [Feng $\ge$ 2019; Shiffman and Zelditch 2003] is quite different from that of [Kabluchko and Zaporozhets 2014]. One needs to apply the classical Szegő theorem [1975] on orthogonal polynomials together with the conformal transformation between the bounded domain and the unit disk. Then it's a natural problem to study the behavior of zeros of derivatives of $P_n^{(N_n)}$. As indicated by Theorem 3 for the Kac polynomials, it seems that zeros will still converge to the equilibrium measure if $N_n/n \to 0$, but the results for the case when $N_n/n \to a \in (0, 1]$ are quite hard to predict. One may prove the results with the aid of the conformal transformation, but the strategy is unclear to the authors.

The paper is organized as follows. We will prove Theorems 3 and 4 for the Kac polynomials in great details in Section 2. The estimates for the Kac case can be applied to prove Theorem 5 for the general random polynomials in Section 3. In the end, we will prove Theorem 6 for the random elliptic polynomials. In the Appendix, we will sketch the proof of Theorem 2.

2. Kac polynomials

In this section, we will prove Theorems 3 and 4 for the Kac polynomials.

Let $K_n^{(N_n)}$ be the N_n-th derivative of the Kac polynomials. Since we want to prove the empirical measure of zeros converges to a deterministic limit, it suffices to prove the convergence in distribution. By the fact that ξ_k are i.i.d., it's equivalent to consider

$$K_n^{(N_n)}(z) = \sum_{k=0}^{D_n} \xi_k (k+1)\cdots(k+N_n) z^k. \tag{30}$$

Observing that the random measure of zeros is invariant by the dilation, i.e., $\mu_{cf} = \mu_f$ for any nonzero c, we can alternatively consider the following normalized random polynomial so that the leading-order term is $\xi_{D_n} z^{D_n}$:

$$K_n^{(N_n)}(z) = \sum_{k=0}^{D_n} \xi_k f_{k,n} z^k, \tag{31}$$

where throughout the article, we set

$$f_{k,n} := \frac{(k+N_n)!\, D_n!}{k!\, n!}. \tag{32}$$

Stirling's formula reads

$$k! = c_k \sqrt{2\pi k}\left(\frac{k}{e}\right)^k, \tag{33}$$

where c_k is a sequence of positive numbers tending to 1 as k tends to ∞ and hence uniformly bounded. Then we have

$$\begin{aligned}
\frac{1}{L_n}\log f_{k,n} &= \frac{1}{L_n}\Bigg[(k+N_n)\log(k+N_n)-(k+N_n)+\frac{\log(k+N_n)}{2}+D_n\log D_n-D_n+\frac{\log D_n}{2} \\
&\qquad -\left(k\log k-k+\frac{\log k}{2}+n\log n-n+\frac{\log n}{2}\right)\Bigg] \\
&\qquad +\frac{1}{L_n}(\log c_{k+N_n}+\log c_{D_n}-\log c_k-\log c_n) \\
&= \frac{1}{L_n}[(k+N_n)\log(k+N_n)+D_n\log D_n-n\log n-k\log k] \\
&\qquad +\frac{1}{2L_n}(\log(k+N_n)+\log D_n-\log n-\log k)+\frac{1}{L_n}(\log c_{k+N_n}+\log c_{D_n}-\log c_k-\log c_n) \\
&:= I_1(k,n)+I_2(k,n)+I_3(k,n).
\end{aligned} \tag{34}$$

When $k=0$, we set $c_k=1$ and set $I_1(0,n)=(1/L_n)(N_n\log N_n+D_n\log D_n-n\log n)$, $I_2(0,n)=(1/(2L_n))(\log N_n+\log D_n-\log n)$, and $I_3(0,n)=(1/L_n)(\log c_{N_n}+\log c_{D_n}-\log c_n)$ to be consistent with the definitions. The expressions of I_j are different according to the choices of L_n (only differ by the front factor L_n), but we use the same notation I_j for different cases throughout the article to reduce the notation we use.

In the following computations, we will let $L_n \to \infty$ (although we choose different L_n for different cases); hence, $I_3(k,n)$ will tend to 0 uniformly by the uniform bound of c_k, which means the third term $I_3(k,n)$ is always negligible.

2.1. *Case* ①. Let's first consider the case ① when

$$\lim_{n\to\infty} \frac{N_n}{n} = 0. \tag{35}$$

For this case, we need to choose $L_n = n$ in (34). We first simply have

$$\lim_{n\to\infty} \sup_{0\le k\le D_n} |I_2(k,n)| \le \lim_{n\to\infty} \frac{2}{n} \log n = 0. \tag{36}$$

For $I_1(k,n)$, we observe that for each fixed n, $I_1(k,n)$ is increasing with respect to k by considering the function $I(x) = (x+N_n)\log(x+N_n) - x\log x$ where $I'(x) = \log(x+N_n) - \log x \ge 0$. We combine this with the fact that $I_1(D_n, n) = (1/n)((D_n+N_n)\log(D_n+N_n) + D_n \log D_n - n\log n - D_n \log D_n) = 0$; we first have

$$\sup_{0\le k\le D_n} |I_1(k,n)| \le |I_1(0,n)| \vee |I_1(D_n,n)| = |I_1(0,n)|,$$

which further reads

$$\begin{aligned}\sup_{0\le k\le D_n} |I_1(k,n)| &\le \frac{1}{n}|n\log n - N_n \log N_n - D_n \log D_n| \\ &= \frac{1}{n}|N_n \log n + D_n \log n - N_n \log N_n - D_n \log D_n| \\ &= \left|-\frac{N_n}{n}\log\left(\frac{N_n}{n}\right) - \frac{D_n}{n}\log\left(\frac{D_n}{n}\right)\right|.\end{aligned}$$

Thus, we have

$$\lim_{n\to\infty} \sup_{0\le k\le D_n} |I_1(k,n)| = 0, \tag{37}$$

since $N_n/n \to 0$ and $D_n/n = 1 - N_n/n \to 1$ as $n \to \infty$.

Combining (36)–(37) and the fact that I_3 always tends to 0, we get

$$\lim_{n\to\infty} \sup_{0\le k\le D_n} \left|\frac{1}{n}\log f_{k,n}\right| = 0. \tag{38}$$

Hence, the coefficients $f_{k,n}$ satisfy Assumptions 2 with $L_n = n$, $T_0 = 1$, and $\delta_n = N_n/n$ so that $(1-\delta_n)L_n = D_n$ and $\log f(t) = 0$ for $0 \le t \le 1$ and $\log f = -\infty$ for $t > 1$. Therefore, zeros of $K_n^{(N_n)}$ will have the same distribution as the Kac polynomials by computations in (10) and (11) as $n \to \infty$.

2.2. *Case* ②. Let's consider the case when

$$\lim_{n\to\infty} \frac{N_n}{n} = a \in (0,1). \tag{39}$$

Let's choose $L_n = n$ in (34) again. By the same arguments as in Case ①, I_2 and I_3 converge to 0 uniformly for $0 \le k \le D_n$ as $n \to \infty$. Therefore, it remains to estimate I_1. Let's put $N_n/n = a + \delta_n$ where $\delta_n \to 0$. Assume n is large enough so that

$$|\delta_n| \le \frac{1-a}{2} \wedge \frac{a}{2}. \tag{40}$$

For $k \ge 1$, we rewrite

$$\begin{aligned}
I_1 &= \frac{1}{n}[(k+N_n)\log(k+N_n) + D_n \log D_n - n\log n - k\log k] \\
&= \frac{1}{n}[(n-D_n+k)\log(k+N_n) - n\log n - k\log k + D_n\log k - D_n\log k + D_n\log D_n] \\
&= \frac{1}{n}\left[n\log\left(\frac{k}{n}+\frac{N_n}{n}\right) + (k-D_n)\log(k+N_n) - (k-D_n)\log k + D_n\log\left(\frac{D_n}{k}\right)\right] \\
&= \log\left(\frac{k}{n}+\frac{N_n}{n}\right) + \left[\left(\frac{k}{n}-\frac{D_n}{n}\right)\log\left(1+\frac{N_n}{k}\right) + \frac{D_n}{n}\log\left(\frac{D_n}{k}\right)\right] \\
&:= I_4 + I_5.
\end{aligned}$$

To estimate I_4 and I_5, we will make use of the following inequality which is the direct consequence of the intermediate value theorem:

$$0 \le \log y - \log x \le \frac{1}{c}(y-x) \quad \text{for } 0 < c \le x \le y. \tag{41}$$

We can rewrite I_4 as $\log(k/n + a + \delta_n)$; by (40)–(41), we have

$$\left|I_4 - \log\left(\frac{k}{n}+a\right)\right| \le \left|\frac{2\delta_n}{a}\right|$$

for all $1 \le k \le D_n$. So we have

$$\lim_{n\to\infty} \sup_{1\le k\le D_n} \left|I_4 - \log\left(\frac{k}{n}+a\right)\right| = 0. \tag{42}$$

For I_5, since $N_n/n = a + \delta_n$ and $D_n/n = 1 - a - \delta_n$, we can rewrite it as

$$\begin{aligned}
I_5 &= \left(\frac{k}{n} - (1-a) + \delta_n\right)\log\left(1 + \frac{(a+\delta_n)n}{k}\right) + (1-a-\delta_n)\log\frac{(1-a-\delta_n)n}{k} \\
&= \frac{k}{n}\log\left(1 + a\frac{n}{k} + \delta_n\frac{n}{k}\right) + (1-a-\delta_n)\log\left(-1 + \frac{n+k}{k+(a+\delta_n)n}\right).
\end{aligned}$$

Then we have

$$\begin{aligned}
&\left|I_5 - \left[\frac{k}{n}\log\left(1+a\frac{n}{k}\right) + (1-a)\log\left(-1+\frac{n+k}{k+an}\right)\right]\right| \\
&\quad \le \frac{k}{n}\left|\log\left(1+a\frac{n}{k}+\delta_n\frac{n}{k}\right) - \log\left(1+a\frac{n}{k}\right)\right| \\
&\qquad + (1-a)\left|\log\left(-1+\frac{n+k}{k+(a+\delta_n)n}\right) - \log\left(-1+\frac{n+k}{k+an}\right)\right| + |\delta_n|\left|\log\left(-1+\frac{n+k}{k+(a+\delta_n)n}\right)\right| \\
&\quad := I_6 + I_7 + I_8.
\end{aligned}$$

By (40)–(41) again, we have

$$I_6 \le \frac{k}{n}\frac{1}{1+(a/2)(n/k)}|\delta_n|\frac{n}{k} \le |\delta_n| \to 0.$$

For I_7, since $|\delta_n| \le (1-a)/2$, we know $k+(a+\delta_n)n \le k+((1+a)/2)n$. Therefore,

$$-1+\frac{n+k}{k+(a+\delta_n)n} \ge -1+\frac{n+k}{k+((1+a)/2)n} = \frac{(1-a)n/2}{((1+a)/2)n+k} \ge \frac{(1-a)n/2}{((1+a)/2)n+n} = \frac{1-a}{3+a}. \quad (43)$$

We also have

$$-1+\frac{k+n}{k+an} \ge \frac{1-a}{3+a}.$$

Thus, by (41), we have

$$\begin{aligned} I_7 &= (1-a)\left|\log\left(-1+\frac{n+k}{k+(a+\delta_n)n}\right)-\log\left(-1+\frac{n+k}{k+an}\right)\right| \\ &\le (1-a)\frac{3+a}{1-a}\left|\frac{k+n}{k+(a+\delta_n)n}-\frac{k+n}{k+an}\right| \\ &\le (3+a)\frac{(k+n)|\delta_n|n}{(k+an/2)^2} \le (3+a)\frac{(n+n)n|\delta_n|}{(an/2)^2} \le \frac{8(3+a)|\delta_n|}{a^2} \to 0. \end{aligned}$$

For I_8, taking into account (40) and (43), we have

$$\frac{1-a}{3+a} \le -1+\frac{k+n}{k+(a+\delta_n)n} = \frac{(1-a-\delta_n)n}{k+(a+\delta_n)n} \le \frac{[(1-a)+(1-a)/2]n}{(a-a/2)n} \le \frac{3(1-a)}{a};$$

it follows that

$$I_8 \le \left(\left|\log\left(\frac{3(1-a)}{a}\right)\right| \vee \left|\log\left(\frac{1-a}{3+a}\right)\right|\right)|\delta_n| \to 0.$$

If we combine the estimates of I_6, I_7, and I_8, we conclude that

$$\lim_{n\to\infty}\sup_{1\le k\le D_n}\left|I_5-\left[\frac{k}{n}\log\left(1+a\frac{n}{k}\right)+(1-a)\log\left(-1+\frac{n+k}{k+an}\right)\right]\right| = 0. \quad (44)$$

If we set

$$\begin{aligned} \log f_1(t) &= \log(t+a)+t\log\left(1+\frac{a}{t}\right)+(1-a)\log\left(\frac{1-a}{t+a}\right) \\ &= (t+a)\log(t+a)-t\log t+(1-a)\log(1-a), \quad t>0, \end{aligned} \quad (45)$$

then the estimates (42) and (44) for I_4 and I_5 imply

$$\lim_{n\to\infty}\sup_{1\le k\le D_n}\left|I_1-\log f_1\left(\frac{k}{n}\right)\right| = 0. \quad (46)$$

The estimate of I_1 in the case $k=0$ can be achieved by the same way, and actually (46) holds with the supremum taken over $0 \le k \le D_n$.

Let's set $f = f_1$ for $0 \le t \le 1-a$ and 0 for $t > 1-a$. Let's set

$$\Delta(b) = \sup_{1-a\le t\le s\le 1,\, s-t\le b}|\log f_1(t)-\log f_1(s)|;$$

then we have

$$\sup_{0\le k\le D_n}\left|I_1-\log f\left(\frac{k}{n}\wedge(1-a)\right)\right|\le \sup_{0\le k\le D_n}\left|I_1-\log f_1\left(\frac{k}{n}\right)\right|+\Delta(|\delta_n|).$$

Observing that $\log f_1$ is uniformly continuous on $[1-a, 1]$, combining (46) and the fact that $\delta_n \to 0$, then we have

$$\lim_{n\to\infty}\sup_{0\le k\le D_n}\left|I_1-\log f\left(\frac{k}{n}\wedge(1-a)\right)\right|=0.$$

Therefore, if we combine the estimates of I_1, I_2, and I_3 we derived above for Case ②, we have

$$\lim_{n\to\infty}\sup_{0\le k\le D_n}\left|\frac{1}{n}f_{k,n}-\log f\left(\frac{k}{n}\wedge(1-a)\right)\right|=0. \tag{47}$$

As a summary, in the case when $N_n/n \to a \in (0, 1)$, by defining $f(t)$ above, the coefficients $f_{k,n}$ will satisfy Assumptions 2 with $T_0 = 1 - a$, $L_n = n$, and $\delta_n = N_n/n - a$ (note that $D_n = (T_0 - \delta_n)L_n$ again). The Legendre–Fenchel transform of $-\log f$ is

$$I(s)=\begin{cases}a\log(a/(e^{-s}-1)+a)+(1-a)\log(1-a), & s<\log(1-a),\\ s(1-a), & s\ge\log(1-a).\end{cases}$$

Therefore, by Theorem 2, the limiting measure for the sequence of the random measure $(1/L_n)\mu^K_{D_n}$ (which is $(1/n)\mu^K_{D_n}$) satisfies

$$\hat{\mu}(\mathbb{D}_r)=\begin{cases}ar/(1-r), & 0<r<1-a,\\ 1-a, & r\ge 1-a.\end{cases}$$

Since $D_n/n \to 1-a$, the limit of the empirical measure $(1/D_n)\mu^K_{D_n}$ will thus be $(1/(1-a))\hat{\mu}(\mathbb{D}_r)$, which is (15).

2.3. ***Case*** **③.** In the case when

$$\lim_{n\to\infty}\frac{N_n}{n}=1 \quad\text{and}\quad D_n\to\infty, \tag{48}$$

we only prove (17), which implies (16). To prove (17), we need to consider

$$\widetilde{K}_n(z):=R_n^{D_n}K_n^{(N_n)}\left(\frac{z}{R_n}\right)=\sum_{k=0}^{D_n}\xi_k\tilde{f}_{k,n}z^k, \tag{49}$$

where

$$\tilde{f}_{k,n}=f_{k,n}R_n^{D_n-k} \quad\text{and}\quad R_n=\frac{n}{D_n}. \tag{50}$$

It's enough to study $\widetilde{K}_n(z)$ since it has the same zeros as $K_n^{(N_n)}(z/R_n)$.

In this case, we need to choose $L_n = D_n$ in (34) with the decomposition

$$\frac{1}{D_n}\log f_{k,n}:=I_1(k,n)+I_2(k,n)+I_3(k,n).$$

Thus, we have the decomposition

$$\begin{aligned} \frac{1}{D_n}\log \tilde{f}_{k,n} &= \left(1-\frac{k}{D_n}\right)\log R_n + \frac{1}{D_n}\log f_{k,n} \\ &= \left[\left(1-\frac{k}{D_n}\right)\log R_n + I_1\right] + I_2 + I_3. \end{aligned} \tag{51}$$

As before, I_3 goes to 0 uniformly again since $D_n \to \infty$ as $n \to \infty$.

We note that

$$I_2(k,n) = \frac{1}{2D_n}(\log(k+N_n) + \log D_n - \log n - \log k)$$

is decreasing with respect to $k \geq 1$ for fixed N_n, D_n, and n; thus, we simply have $\sup_{0\leq k\leq D_n}|I_2(k,n)| = |I_2(1,n)| \vee |I_2(D_n,n)|$. Since $I_2(D_n,n) = 0$, we further have

$$\sup_{0\leq k\leq D_n}|I_2(k,n)| = |I_2(1,n)| = \frac{1}{2D_n}|\log(N_n+1) + \log D_n - \log n|.$$

By assumption (48), we can choose n large enough so that $N_n \geq \frac{1}{2}n$; thus, we have

$$\sup_{0\leq k\leq D_n}|I_2(k,n)| \leq \frac{1}{2D_n}\left(\log\left(\frac{n}{N_n}\right) + \log D_n\right) \leq \frac{\log 2}{2D_n} + \frac{\log D_n}{2D_n} \to 0,$$

since $D_n \to \infty$ as $n \to \infty$.

For I_1, we rewrite it as

$$\begin{aligned} I_1 &= \frac{1}{D_n}((k+n-D_n)\log(k+N_n) - n\log n + D_n\log D_n - k\log k) \\ &= \frac{1}{D_n}\left(n\log\frac{k+N_n}{n} + (k-D_n)\log(k+N_n) + D_n\log D_n - k\log k\right) \\ &= \frac{1}{D_n}\left(n\log\left(\frac{k+N_n}{n}\right) + (k-D_n)\log\left(\frac{k+N_n}{n}\right) + (k-D_n)\log n + D_n\log D_n - k\log D_n - k\log\left(\frac{k}{D_n}\right)\right) \\ &= \frac{n}{D_n}\log\left(\frac{n+k-D_n}{n}\right) - \frac{k}{D_n}\log\left(\frac{k}{D_n}\right) + \left(\frac{k}{D_n}-1\right)(\log n - \log D_n) + \left(\frac{k}{D_n}-1\right)\log\left(\frac{k+N_n}{n}\right). \end{aligned}$$

Thus, we can rewrite

$$\begin{aligned} \tilde{I}_1 &:= \left(1-\frac{k}{D_n}\right)\log R_n + I_1 \\ &= \frac{n}{D_n}\log\left(\frac{n+k-D_n}{n}\right) - \frac{k}{D_n}\log\left(\frac{k}{D_n}\right) + \left(\frac{k}{D_n}-1\right)\log\left(\frac{k+N_n}{n}\right). \end{aligned}$$

Now we put

$$\log\tilde{f} = t - 1 - t\log t \quad \text{for } 0 \leq t \leq 1 \qquad \text{and} \qquad \log\tilde{f} = -\infty \quad \text{for } t > 1. \tag{52}$$

Then we can write $\tilde{I}_1$ as

$$\tilde{I}_1 = \log\tilde{f}\left(\frac{k}{D_n}\right) + I_9, \tag{53}$$

where

$$I_9 = \frac{n}{D_n}\left[\log\left(1+\frac{k-D_n}{n}\right)-\frac{k-D_n}{n}\right]+\left(\frac{k}{D_n}-1\right)\log\left(\frac{k+N_n}{n}\right).$$

Since $|\log(1+x)| \le |x|$ and $|\log(1+x)-x| \le x^2$ when $|x|$ is small, then we have the uniform estimate

$$\left|\log\left(1+\frac{k-D_n}{n}\right)-\frac{k-D_n}{n}\right| \le \left(\frac{k-D_n}{n}\right)^2 \le \left(\frac{D_n}{n}\right)^2$$

as n becomes large enough, which implies the first term in I_9 tends to 0.

Note that $1 \ge (k+N_n)/n \ge N_n/n$; thus, $|\log((k+N_n)/n)| \le |\log(N_n/n)| = |\log(1-D_n/n)| \le D_n/n$. If we combine this with the fact $|k/D_n - 1| \le 1$, we prove that the second term in I_9 also tends to 0. Hence, $I_9 \to 0$ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} \sup_{0\le k\le D_n} \left|\tilde{I}_1 - \log \tilde{f}\left(\frac{k}{D_n}\right)\right| = 0.$$

If we combine the estimates of $\tilde{I}_1$, I_2, and I_3 above, we have proved

$$\lim_{n\to\infty} \sup_{0\le k\le D_n} \left|\frac{1}{D_n}\log \tilde{f}_{k,n} - \log \tilde{f}\left(\frac{k}{D_n}\right)\right| = 0. \tag{54}$$

As a summary, the coefficients $\tilde{f}_{k,n}$ satisfy Assumptions 2 with $L_n = D_n$, $T_0 = 1$, $\delta_n = 0$, and $\tilde{f}$. The Legendre–Fenchel transform $I(s) = \sup_{0\le t\le 1}(st + \log \tilde{f}(t))$ is

$$I(s) = \begin{cases} e^s - 1, & s < 0, \\ s, & s \ge 0. \end{cases}$$

Thus, the explicit expression (17) of the limiting measure $\tilde{\mu}^K$ follows by Theorem 2.

2.4. *Case* ④. Now we prove Theorem 4 for the case where D_n remains a fixed positive integer m. The proof makes use of Rouché's theorem. We start with the following proposition regarding the convergence of zeros of a sequence of deterministic polynomials.

Proposition 7. *Let $G = \sum_{k=0}^m g_k z^k$, where $\{g_k\}$ are deterministic constants and $g_m \ne 0$. Let $G_n = \sum_{k=0}^m g_{k,n} z^k$, where $\{g_{k,n}\}$ are also deterministic. Assume $g_{k,n}$ converges to g_k for each fixed k. Then the measure of zeros μ_{G_n} will converge to μ_G in the sense of distribution.*

Proof. Let's choose ϕ as the smooth test function with compact support and pick $\epsilon > 0$ small enough. We first claim that for each zero z_0 of G with multiplicity α_0, for n large enough, G_n has exactly α_0 zeros in $\mathbb{D}(z_0, \epsilon)$, the open disc centered at z_0 with radius ϵ. Once this is done, since G has m zeros (m is a finite number), we can pick a common N_0 such that when $n > N_0$, G_n will have exactly α_i zeros in $\mathbb{D}(z_i, \epsilon)$ for any z_i in the zero set of G with multiplicity α_i. This means that we can make an appropriate ordering of the zero set of G (denoted by z_i, $1 \le i \le m$) and the zero set of G_n (denoted by $z_{i,n}$, $1 \le i \le m$) such that $|z_i - z_{i,n}| \le \epsilon$ for all i. Then we have

$$|\mu_{G_n}(\phi) - \mu_G(\phi)| \le \sum_{1\le i\le m} |\phi(z_i) - \phi(z_{i,n})| \le mK\epsilon, \tag{55}$$

where K is the sup norm of the derivative of ϕ. Since ϵ is arbitrarily small, this implies the weak convergence of μ_{G_n}. All the rest is to prove the claim.

Let's choose $\epsilon < 1$ small enough such that z_0 is the only zero of G with multiplicity $\alpha \geq 1$ in the closure of $\mathbb{D}(z_0, \epsilon)$. Assume $|z_0| + 1 \leq R$ for some R. For any $z \in \overline{\mathbb{D}(z_0, \epsilon)}$, we have

$$|G_n - G| \leq \sum_{k=0}^{m} |g_{n,k} - g_k| R^k. \tag{56}$$

Let's set

$$\eta(\epsilon) = \min_{z \in \partial\mathbb{D}(z_0,\epsilon)} |G(z)|;$$

then as n becomes large enough, we have

$$\sum_{k=0}^{m} |g_{n,k} - g_k| R^k < \eta(\epsilon),$$

which implies that

$$|G_n(z) - G(z)| < |G(z)| \quad \text{for any } z \in \partial\mathbb{D}(z_0, \epsilon).$$

Hence, G_n and G have the same number of zeros in $\mathbb{D}(z_0, \epsilon)$ by Rouché's theorem. This completes the proof of the claim and hence Proposition 7. □

Let's apply Proposition 7 to prove Theorem 4. In the case of $D_n = m$ and $N_n = n - m$, (31) reads

$$K_n^{(n-m)}(z) = \sum_{k=0}^{m} \xi_k f_{k,n} z^k.$$

To study the limiting behavior of zeros of $K_n^{(n-m)}(z/n)$, we may alternatively consider the random polynomials $G_n(z) = n^m K_n^{(n-m)}(z/n)$. The coefficients of G_n are

$$g_{k,n} = n^{m-k} f_{k,n} = \frac{m!}{k!} \frac{n^{m-k}}{n(n-1)\cdots(n-(m-k)+1)}.$$

Since k and m are both fixed when $n \to \infty$, we have

$$\lim_{n\to\infty} g_{k,n} = \frac{m!}{k!}.$$

By Proposition 7, the measure of zeros μ_{G_n} will converge to $\mu_{f_m^K}$ almost surely, where $\mu_{f_m^K}$ is the random measure of zeros of $f_m^K(z) = \sum_{k=0}^{m} (\xi_k/k!) z^k$. The limit (20) follows from this since $K_n^{(n-m)}(z/n)$ have the same zeros as G_n. In particular, the empirical measure of zeros of $K_n^{(n-m)}$ will converge to δ_0.

3. General random polynomials

In this section, we will apply the estimates we derived for the Kac polynomials in Section 2 to prove Theorem 5 for the general random polynomials.

Let p_n be the general random polynomials of degree n defined in (4). Let's assume that the coefficients $p_{k,n}$ satisfy Assumptions 1 with the associated continuous function p that is positive on [0, 1) and

$$\lim_{n\to\infty} \sup_{k\in[0,n]} \left| |p_{k,n}|^{1/n} - p\left(\frac{k}{n}\right) \right| = 0. \tag{57}$$

The N_n-th derivative of p_n is

$$p_n^{(N_n)} = \sum_{k=0}^{D_n} \xi_{k+N_n} p_{k+N_n,n} f_{k,n} z^k, \tag{58}$$

where $f_{k,n}$ is defined in (32). Since ξ_k are i.i.d., it's equivalent to consider the random polynomials

$$p_n^{(N_n)} = \sum_{k=0}^{D_n} \xi_k p_{k+N_n,n} f_{k,n} z^k, \tag{59}$$

where (58) and (59) have the same distribution of zeros. We set

$$u_{k,n} = p_{k+N_n,n} f_{k,n};$$

then we rewrite

$$p_n^{(N_n)} = \sum_{k=0}^{D_n} \xi_k u_{k,n} z^k.$$

We now verify that $u_{k,n}$ satisfy Assumptions 2 with some associated function u.

3.1. ***Case 1.*** ($N_n/n \to 0$). As in Case ① of Kac polynomials, we take $L_n = n$, $\delta_n = N_n/n$, and $T_0 = 1$. For fixed n, $f_{k,n}$ is increasing with k since

$$\frac{f_{k+1,n}}{f_{k,n}} = \frac{k+1+N_n}{k+1} > 1.$$

Since $f_{D_n,n} = 1$, it follows that $f_{k,n} \le 1$ for all n and $0 \le k \le D_n$. By Assumptions 1, p is continuous on [0, 1] and therefore is bounded by C. Hence,

$$\begin{aligned}
&\sup_{0\le k\le D_n} \left| |u_{k,n}|^{1/n} - p\left(\frac{k}{n}\right) \right| \\
&\quad \le \sup_{0\le k\le D_n} \left| |p_{k+N_n,n}|^{1/n} - p\left(\frac{k}{n}\right) \right| |f_{k,n}|^{1/n} + \sup_{0\le k\le D_n} \left| |f_{k,n}|^{1/n} - 1 \right| p\left(\frac{k}{n}\right) \\
&\quad \le \sup_{0\le k\le D_n} \left| |p_{k+N_n,n}|^{1/n} - p\left(\frac{k+N_n}{n}\right) \right| + \sup_{0\le k\le D_n} \left| p\left(\frac{k+N_n}{n}\right) - p\left(\frac{k}{n}\right) \right| + C \sup_{0\le k\le D_n} \left| |f_{k,n}|^{1/n} - 1 \right| \\
&\quad := J_1 + J_2 + J_3.
\end{aligned}$$

Our assumption (57) implies that J_1 converges to 0. J_2 converges to 0 since p is uniformly continuous on [0, 1] and N_n/n converges to 0 under the definition of the case. J_3 also converges to 0 by the estimate (38) which we have already proved for the Kac polynomials. Hence, the coefficients $u_{k,n}$ satisfy Assumptions 2 with $L_n = n$, $\delta_n = N_n/n$, $T_0 = 1$, and the associated function p. The conclusion (1) of Theorem 5 then follows.

3.2. ***Case 2.*** $(N_n/n \to a \in (0,1))$. As in Case ② of Section 2.2, we set $L_n = n$, $\delta_n = N_n/n - a$, and $T_0 = 1-a$; then $(T_0-\delta_n)L_n = D_n$. Let's choose f_1 as in (45) and set that f coincides with f_1 in $[0, 1-a]$ and equals 0 in $[1-a, \infty)$ as in the Kac case. Proceeding like Case 1 above, we have

$$\begin{aligned}
&\sup_{0\le k\le D_n}\Big||u_{k,n}|^{1/n} - p\Big(\Big(\frac{k}{n}+a\Big)\wedge 1\Big) f\Big(\frac{k}{n}\wedge T_0\Big)\Big| \\
&\quad\le \sup_{0\le k\le D_n} |f_{k,n}|^{1/n}\Big||p_{k+N_n,n}|^{1/n} - p\Big(\Big(\frac{k}{n}+a\Big)\wedge 1\Big)\Big| + \sup_{0\le k\le D_n} p\Big(\Big(\frac{k}{n}+a\Big)\wedge 1\Big)\Big||f_{k,n}|^{1/n} - f\Big(\frac{k}{n}\wedge T_0\Big)\Big| \\
&\quad\le \sup_{0\le k\le D_n} |f_{k,n}|^{1/n}\Big||p_{k+N_n,n}|^{1/n} - p\Big(\frac{n+N_n}{n}\Big)\Big| + \sup_{0\le k\le D_n} |f_{k,n}|^{1/n}\Big|p\Big(\frac{k+N_n}{n}\Big) - p\Big(\Big(\frac{k}{n}+a\Big)\wedge 1\Big)\Big| \\
&\qquad\qquad + \sup_{0\le k\le D_n} p\Big(\Big(\frac{k}{n}+a\Big)\wedge 1\Big)\Big||f_{k,n}|^{1/n} - f\Big(\frac{k}{n}\wedge T_0\Big)\Big| \\
&\quad:= J_1 + J_2 + J_3.
\end{aligned}$$

As in Case 1, our assumptions of p imply that J_1 converges to 0; J_3 converges to 0, which is equivalent to (47) as in the Kac case. Again using the boundedness of $f_{k,n}$ and the uniform continuity of p together with the fact that

$$\sup_{0\le k\le D_n}\Big|\Big(\Big(\frac{k}{n}+a\Big)\wedge 1\Big) - \frac{k+N_n}{n}\Big| \le |\delta_n|,$$

we have $J_2 \to 0$ since $\delta_n \to 0$. Hence, the coefficients $u_{k,n}$ satisfy Assumptions 2 with $u^a(t) = f(t)p(t+a)$; this will complete the proof of Theorem 5(2).

4. Random elliptic polynomials

In this section, we will prove Theorem 6 for the random elliptic polynomials E_n defined in (23). Let's denote by

$$p^E_{k,n} = \sqrt{\binom{n}{k}}$$

the coefficients. By Stirling's formula, one can prove that the coefficients $p^E_{k,n}$ satisfy Assumptions 1 with the associated function p^E given in (24). Thus, Theorem 6(1) is the direct consequence of Theorem 5. Now let's prove Theorem 6(2), which is the interesting part, and the nontrivial ingredient is to find the rescaling factor.

As in (59), the N_n-th derivative of E_n is equivalent to

$$E_n^{(N_n)} = \sum_{k=0}^{D_n} \xi_k p^E_{k+N_n,n} f_{k,n} z^k := \sum_{k=0}^{D_n} \xi_k u^E_{k,n} z^k. \tag{60}$$

Let's first consider the case when $N_n/n \to 1$ and $D_n \to \infty$. By discarding a negligible lower-order term and by Stirling's formula, we have

$$\begin{aligned}\frac{1}{D_n}\log p^E_{k+N_n,n} &\sim \frac{1}{2D_n}(n\log n-(k+N_n)\log(k+N_n)-(D_n-k)\log(D_n-k))\\ &=\frac{1}{2}\left(\frac{k+N_n}{D_n}\log\left(\frac{n}{k+N_n}\right)+\frac{D_n-k}{D_n}\log\left(\frac{n}{D_n-k}\right)\right)\\ &=\frac{1}{2}\left(-\frac{n+k-D_n}{D_n}\log\left(\frac{n-D_n+k}{n}\right)-\frac{D_n-k}{D_n}\log\left(\frac{D_n-k}{D_n}\right)+\frac{D_n-k}{D_n}\log\left(\frac{n}{D_n}\right)\right)\\ &=I_{1,1}+I_{1,2}+I_{1,3}. \end{aligned} \tag{61}$$

By $|\log(1+x)-x|\le x^2$ when $|x|$ is small, we can get the uniform estimate

$$\left|I_{1,1}-\frac{1}{2}\left(-\frac{n+k-D_n}{D_n}\frac{-D_n+k}{n}\right)\right|\le\frac{n}{2D_n}\left(\frac{-D_n+k}{n}\right)^2\le\frac{n}{2D_n}\left(\frac{D_n}{n}\right)^2\to 0.$$

We also have the uniform estimate

$$\left|\frac{1}{2}\left(-\frac{n+k-D_n}{D_n}\frac{-D_n+k}{n}\right)-\frac{D_n-k}{2D_n}\right|=\frac{(D_n-k)^2}{2nD_n}\le\frac{D_n}{2n}\to 0;$$

it follows that if we define

$$h_1=\tfrac{1}{2}(1-t),$$

then

$$\lim_{n\to\infty}\sup_{0\le k\le D_n}\left|I_{1,1}-h_1\left(\frac{k}{D_n}\right)\right|=0. \tag{62}$$

Let's put

$$h_2=-\tfrac{1}{2}(1-t)\log(1-t);$$

then we can rewrite

$$I_{1,2}=h_2\left(\frac{k}{D_n}\right). \tag{63}$$

The trick now is to eliminate $I_{1,3}$ by a rescaling factor. To be more explicit, let's put $R_n=n/D_n$ again and put

$$\tilde{p}^E_{k+N_n,n}=p^E_{k+N_n,n}R_n^{-(D_n-k)/2}. \tag{64}$$

By defining in this way, we note that

$$\frac{1}{D_n}\log R_n^{-(D_n-k)/2}=-I_{1,3}; \tag{65}$$

hence, if we combine (61)–(65) and define the function

$$\log\tilde{p}^E(x)=h_1+h_2=\tfrac{1}{2}(1-t)-\tfrac{1}{2}(1-t)\log(1-t), \tag{66}$$

then we have proved

$$\lim_{n\to\infty}\sup_{0\le k\le D_n}\left|\frac{1}{D_n}\log\tilde{p}^E_{k+N_n,n}-\log\tilde{p}^E\left(\frac{k}{D_n}\right)\right|=0. \tag{67}$$

Let's further recall (50) in the proof of Case ③ for the Kac case where

$$\tilde{f}_{k,n}=f_{k,n}R_n^{D_n-k}; \tag{68}$$

then we can rewrite (60) as

$$E_n^{(N_n)}(z) = \sum_{k=0}^{D_n} \xi_k \tilde{p}^E_{k+N_n,n} \tilde{f}_{k,n} z^k R_n^{-(D_n-k)/2}.$$

Therefore, the rescaling random polynomials read

$$E_n^{(N_n)}\left(\frac{z}{\sqrt{R_n}}\right) = R_n^{-D_n/2} \sum_{k=0}^{D_n} \xi_k \tilde{p}^E_{k+N_n,n} \tilde{f}_{k,n} z^k. \tag{69}$$

Let's define

$$\widetilde{E}_n^{(N_n)}(z) := \sum_{k=0}^{D_n} \xi_k \tilde{p}^E_{k+N_n,n} \tilde{f}_{k,n} z^k.$$

Let's derive the limit of the empirical measure of zeros of $E_n^{(N_n)}(z/\sqrt{R_n})$, which is the same as $\widetilde{E}_n^{(N_n)}(z)$. To do this, let's define the coefficients $\tilde{u}^E_{k,n} := \tilde{p}^E_{k+N_n,n} \tilde{f}_{k,n}$; then the estimates (54) and (67) imply that $\tilde{u}^E_{k,n}$ satisfy Assumptions 2 with $L_n = D_n$, $\delta_n = 0$, and $T_0 = 1$ and the associated function $\tilde{u}^E$ is given by $\log \tilde{u}^E = \log \tilde{p}^E + \log \tilde{f}$. By (52) and (66), we have

$$\log \tilde{u}^E(t) = \begin{cases} \frac{1}{2}(t-1) - \frac{1}{2}(1-t)\log(1-t) - t\log t, & 0 \le t \le 1, \\ -\infty, & t > 1. \end{cases}$$

Therefore, $(1/D_n)\mu_{D_n}^{\widetilde{E}}$, or equivalently $(1/D_n)\mathcal{S}_{\sqrt{R_n}}(\mu^E_{D_n})$, converges in probability to a deterministic measure. To find out the limit, we compute the Legendre–Fenchel transform of $-\log \tilde{u}^E$ as

$$I(s) = \sup_{0\le t\le 1} (st + \log \tilde{u}(t)) = \tfrac{1}{2}(t_s - 1) - \tfrac{1}{2}\log(1-t_s),$$

where $t_s = (-1 + \sqrt{1+4e^{-2s}})/(2e^{-2s})$. Therefore, (25) follows by Theorem 2.

The analysis for the case when D_n remains a fixed number m follows exactly the same approach as in Section 2.4 for the Kac case. Recall the definition of $u^E_{k,n}$ in (60); if we replace $D_n = m$ and $N_n = n-m$, then we can rewrite

$$u^E_{k,n} = \left(\frac{n!}{(k+n-m)!\,(m-k)!}\right)^{1/2} \frac{(k+n-m)!\,m!}{k!\,n!} = \frac{m!}{k!}\left(\frac{(n-m+k)!}{n!\,(m-k)!}\right)^{1/2}.$$

Now we consider the rescaling random polynomials

$$\widetilde{E}_n^m(z) := n^{m/2} E_n^{(n-m)}\left(\frac{z}{\sqrt{n}}\right) = \sum_{k=0}^{m} \tilde{u}^E_{k,n} \xi_k z^k,$$

where $\tilde{u}^E_{k,n} = u^E_{k,n} n^{(m-k)/2}$. Since m and k are both fixed when $n \to \infty$, we get

$$\lim_{n\to\infty} \tilde{u}^E_{k,n} = \frac{m!}{k!\,((m-k)!)^{1/2}}.$$

Therefore, since $\widetilde{E}_n^m(z)$ have the same zeros as $E_n^{(n-m)}(z/\sqrt{n})$, then by Proposition 7, the limiting measure $\mathcal{S}_{\sqrt{n}}(\mu^E_{D_n})$ when $D_n = m$ will tend to the random zeros of

$$f_m^E = \sum_{k=0}^{m} \frac{1}{k!\,((m-k)!)^{1/2}} \xi_k z^k$$

in distribution, which completes the proof of Theorem 6.

Appendix: Proof of Theorem 2

Now we sketch the proof of Theorem 2 by modifying the one in [Kabluchko and Zaporozhets 2014].

Let's first recall the proof of Theorem 1 in [Kabluchko and Zaporozhets 2014]. For random analytic functions $F(z)$ defined in (9) where the coefficients satisfy Assumptions 1, if one establishes the convergence in probability

$$\frac{1}{n} \log|F_n(z)| \to I(\log|z|) \tag{70}$$

as $n \to \infty$, then Theorem 1 follows by the classical Poincaré–Lelong formula. Kabluchko and Zaporozhets proved (70) by establishing some appropriate upper and lower bounds for $|F_n(z)|$; see estimates (22) and (27) in [Kabluchko and Zaporozhets 2014].

Under Assumptions 2, the convergence radius is automatically infinity because we are now dealing with a finite sum for any fixed n. Given random polynomials F_n in the form of (12) satisfying Assumptions 2, to prove Theorem 2, it's enough to derive the analogue convergence

$$\frac{1}{L_n} \log|F_n(z)| \to I(\log|z|) \tag{71}$$

as $n \to \infty$, where the convergence is also in probability. To prove this, we need the same upper and lower bounds as in [Kabluchko and Zaporozhets 2014].

For the upper bound, for any $\epsilon > 0$, we have

$$|F_n(z)| \le M e^{L_n(I(\log|z|)+3\epsilon+\delta_n^-(\log|z|)^+)} \quad \text{for } n \text{ large enough,} \tag{72}$$

where M is an almost surely finite random variable depending on ϵ. Here we use the convention that for any real number w, w^+ and w^- are the positive and negative parts of w, i.e., $w^+ = w \vee 0$ and $w^- = (-w) \vee 0$.

We also need to show the lower bound estimate

$$\mathbb{P}(|F_n(z)| < e^{L_n(I(\log|z|)-4\epsilon)}) = O\left(\frac{1}{\sqrt{L_n}}\right) \quad \text{as } n \to \infty. \tag{73}$$

Recall Lemma 4.4 in [Kabluchko and Zaporozhets 2014]; we know that for any $A > 0$, there exists an almost surely finite random variable M' such that $|\xi_k| \le M' e^{Ak}$ for all k with probability 1. If we set $A = \epsilon/(2T_0)$, then for all $0 \le k \le (T_0 - \delta_n)L_n$, we have

$$|\xi_k| \le M' e^{\epsilon k/(2T_0)} \le M' e^{\epsilon L_n}. \tag{74}$$

To prove (72), if we apply the bound (74) together with Assumptions 2, for n large enough and δ small enough, we have

$$\begin{aligned}|F_n(z)| &= \Bigg|\sum_{0\le k\le (T_0-\delta_n)L_n} \xi_k p_{k,n} z^k\Bigg| \le \sum_{0\le k\le (T_0-\delta_n)L_n} |\xi_k||p_{k,n}||z|^k \\ &\le M' e^{\epsilon L_n}\Bigg(\sum_{0\le k\le (T_0-\delta_n^+)L_n} |p_{k,n}||z|^k + \sum_{T_0 L_n < k \le (T_0+\delta_n^-)L_n} |p_{k,n}||z|^k\Bigg) \\ &\le M' e^{\epsilon L_n} \sum_{0\le k\le (T_0-\delta_n^+)L_n} (e^{(k/L_n)\log|z| + \log p(k/L_n)} + \delta |z|^{k/L_n})^{L_n} \\ &\qquad + M' e^{\epsilon L_n} \sum_{T_0 L_n < k \le (T_0+\delta_n^-)L_n} (e^{(k/L_n - T_0)\log|z| + (T_0 \log|z| + \log p(T_0))} + \delta |z|^{k/L_n})^{L_n}.\end{aligned}$$

By the definition of the Legendre–Fenchel transform, we further have

$$\begin{aligned}|F_n(z)| &\le M' e^{2\epsilon L_n} (e^{I(\log|z|)} + \delta(1\vee |z|^{T_0}))^{L_n} + M' e^{2\epsilon L_n} e^{\delta_n^-(\log|z|)^+ L_n} (e^{I(\log|z|)} + \delta(1\vee |z|^{2T_0}))^{L_n} \\ &\le M'' e^{L_n(I(\log|z|) + 3\epsilon + \delta_n^-(\log|z|)^+)},\end{aligned}$$

where M'' is another almost surely finite random variable, which completes the proof of the upper bound.

For the lower bound (73), if we choose the set J as the one in the proof of (27) in [Kabluchko and Zaporozhets 2014], then the assumptions $L_n \to \infty$ and $\delta_n \to 0$ imply that the set $\{k : 0 \le k \le (T_0 - \delta_n)L_n,\ k/L_n \in J\}$ has cardinality bounded below by $(|J|/2)L_n$. The rest proof follows the one in [Kabluchko and Zaporozhets 2014] by replacing n by L_n and hence the lower bound follows.

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RENJIE FENG: renjie@math.pku.edu.cn
Beijing International Center for Mathematical Research, Peking University, Beijing, China

DONG YAO: dong.yao@duke.edu
Duke University, Durham, NC, United States

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