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DIMENSIONAL CROSSOVER WITH A CONTINUUM OF CRITICAL EXPONENTS FOR NLS ON DOUBLY PERIODIC METRIC GRAPHS

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We investigate the existence of ground states for the focusing nonlinear Schrödinger equation on a prototypical doubly periodic metric graph. When the nonlinearity power is below 4, ground states exist for every value of the mass, while, for every nonlinearity power between 4 (included) and 6 (excluded), a mark of L^2-criticality arises, as ground states exist if and only if the mass exceeds a threshold value that depends on the power. This phenomenon can be interpreted as a continuous transition from a two-dimensional regime, for which the only critical power is 4, to a one-dimensional behavior, in which criticality corresponds to the power 6. We show that such a dimensional crossover is rooted in the coexistence of one-dimensional and two-dimensional Sobolev inequalities, leading to a new family of Gagliardo–Nirenberg inequalities that account for this continuum of critical exponents.

1. Introduction

Since the first appearance of branched structures in the modeling of organic molecules [Ruedenberg and Scherr 1953], through the development of the mathematical theory of quantum graphs [Berkolaiko and Kuchment 2013; Post 2012], networks (or metric graphs) have provided a general and flexible tool to describe dynamics in complex structures like systems of quantum wires, Josephson junctions, propagation of signals through waveguides, and some related technologies. Pioneering studies about nonlinear systems on metric graphs appeared in [Ali Mehmeti 1994; Ali Mehmeti et al. 2001], but more recently the research on such topics has grown rapidly, and several results have been achieved on propagation of solitary waves [Adami et al. 2011; Caudrelier 2015; Sobirov et al. 2010] and on stationary states [Sabirov et al. 2013; Cacciapuoti et al. 2015; Noja 2014; Noja et al. 2015; Pelinovsky and Schneider 2017; Gnutzmann and Waltner 2016].

In a series of recent works [Adami et al. 2015a; 2015b; 2016] we investigated the problem of existence of ground states for the energy functional associated to the focusing, L^2-subcritical and critical nonlinear Schrödinger (NLS) equation

$$i\,\partial_t u(t) = -u''(t) - |u(t)|^{p-2}u(t) \tag{1}$$

on finite noncompact metric graphs, i.e., branched structures with a finite number of vertices and edges, and at least one infinite edge (i.e., a half-line).

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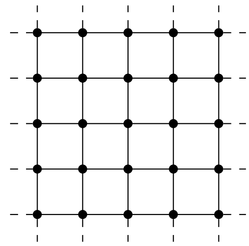

Figure 1. The grid $\mathcal{G}$.

Specifically, by *ground state* on a metric graph $\mathcal{G}$ we mean every global minimizer of the energy functional

$$E_p(u) = \frac{1}{2}\int_{\mathcal{G}} |u'|^2\,dx - \frac{1}{p}\int_{\mathcal{G}} |u|^p\,dx \tag{2}$$

in the class of $H^1(\mathcal{G})$ functions with fixed L^2-norm (or mass) $\mu > 0$. The constraint is dynamically meaningful as the mass, as well as the energy, is conserved by the NLS flow, and the problem of the existence of ground states is particularly relevant in the physics of Bose–Einstein condensates; see, e.g., [Adami et al. 2015a; 2015b; 2016; 2017b, Section 1].

In this paper we extend the analysis of the existence of ground states to a prototypical *doubly periodic* metric graph $\mathcal{G}$, particularly relevant in the applications, for which the techniques developed in previous works (where noncompactness was due to one or more unbounded edges) do not apply: a two-dimensional infinite grid isometrically embedded in $\mathbb{R}^2$, with vertices on the lattice $\mathbb{Z}^2$ and edges of unit length (see Figure 1).

Schrödinger equations on periodic metric graphs have received considerable attention in the last few years. Linear problems have been extensively studied, and a fairly complete spectral analysis is now available for different types of coupling conditions. We refer for instance to the early papers [Exner 1996; Exner and Gawlista 1996] treating rectangular lattices, as well as to Chapter 4 in [Berkolaiko and Kuchment 2013] for a more up-to-date overview of several results in a general periodic setting. Concerning the square grid we focus on, we specifically quote [Exner and Turek 2010] for some results strictly rooted in the two-dimensional nature of the domain.

More recently, nonlinear problems have been addressed too. For instance, [Pelinovsky and Schneider 2017] considers a specific example of a structure periodic along a single direction, the so-called *necklace* graph, via bifurcation techniques. From a variational point of view, the first investigation for very general periodic graphs can be found in [Pankov 2018], where the approach is based on the Nehari method. We notice that, for this reason, in that paper the problem of the existence of ground states with prescribed mass cannot be dealt with.

Let us now discuss our results. We first note, roughly speaking, that macroscopically the grid $\mathcal{G}$ has dimension 2, while microscopically it is of dimension 1. This peculiarity is absent in graphs with a finite number of half-lines, where the two-dimensional scale is lacking, as well as in other two-dimensional structures like $\mathbb{Z}^2$, where edges are missing and there is of course no microscopic one-dimensional structure

[Weinstein 1999]. The presence of two scales in $\mathcal{G}$ results in a transition from a one-dimensional to a two-dimensional behavior, which emerges in functional inequalities and influences the existence of ground states. We shall refer to this phenomenon as *dimensional crossover*.

Before commenting further on this point, it is convenient to state our main results in a precise form. We define, for $\mu > 0$, the mass-constrained set

$$H^1_\mu(\mathcal{G}) = \left\{u \in H^1(\mathcal{G}) : \textstyle\int_{\mathcal{G}} |u|^2\,dx = \mu\right\} \tag{3}$$

and the corresponding "ground-state energy level"

$$\mathcal{E}_p(\mu) = \inf_{u \in H^1_\mu(\mathcal{G})} E_p(u), \tag{4}$$

considered as a function $\mathcal{E}_p : (0, +\infty) \to \mathbb{R} \cup \{-\infty\}$ of the mass μ. By a "ground states of mass μ" we mean a function $u \in H^1_\mu(\mathcal{G})$ such that

$$E_p(u) = \mathcal{E}_p(\mu).$$

When $p \in (2, 4)$, ground states exist for every prescribed mass.

Theorem 1.1 (subcritical case). *Assume $2 < p < 4$. Then for every $\mu > 0$ there exists a ground state of mass μ, and $\mathcal{E}_p(\mu) < 0$.*

The picture changes as the exponent of the nonlinearity increases.

Theorem 1.2 (dimensional crossover). *For every $p \in [4, 6]$ there exists a critical mass $\mu_p > 0$ such that*:

(i) *If $p \in (4, 6)$ then ground states of mass μ exist if and only if $\mu \geq \mu_p$, and*

$$\begin{aligned} &\mathcal{E}_p(\mu) = 0 \quad \text{if } \mu \leq \mu_p, \\ &\mathcal{E}_p(\mu) < 0 \quad \text{if } \mu > \mu_p. \end{aligned} \tag{5}$$

(ii) *If $p = 4$ then ground states of mass μ exist if $\mu > \mu_4$, whereas they do not exist if $\mu < \mu_4$. Moreover* (5) *is valid also when $p = 4$.*

(iii) *If $p = 6$ then there are no ground states, regardless of the value of μ, and*

$$\begin{aligned} &\mathcal{E}_6(\mu) = 0 \qquad \text{if } \mu \leq \mu_6, \\ &\mathcal{E}_6(\mu) = -\infty \quad \text{if } \mu > \mu_6. \end{aligned} \tag{6}$$

We point out that, when $p = 4$, the existence of ground states of mass $\mu = \mu_4$ is still an open problem. For the sake of completeness, we also mention that when $p > 6$ one has $\mathcal{E}_p(\mu) \equiv -\infty$ for every μ, as one can easily see by a scaling argument.

In order to interpret Theorems 1.1 and 1.2, let us recall that in $\mathbb{R}^d$, for the minimization of the NLS energy under a mass constraint, there exists a *critical exponent* p^*_d such that

(1) if $p < p^*_d$, for every mass $\mu > 0$ the ground-state energy level is finite and negative, and is attained by a ground state;

(2) if $p > p^*_d$, for every mass $\mu > 0$ the ground-state energy level equals $-\infty$.

It is well known [Cazenave 2003] that $p_d^* = 4/d + 2$ for the NLS in $\mathbb{R}^d$, yielding $p_1^* = 6$ for $\mathbb{R}$ and $p_2^* = 4$ for $\mathbb{R}^2$. Furthermore, it has been proved in [Adami et al. 2015b; 2016] that for finite noncompact graphs (i.e., graphs with finitely many edges, at least one of them being unbounded) the critical exponent is 6, exactly as for $\mathbb{R}$. Thus the exponents considered in Theorem 1.1 are subcritical both in dimension 1 and 2, which reflects into the typical subcritical flavor of the result.

In fact, the main novelty of the paper emerges in Theorem 1.2 and lies in the "splitting" of the critical exponent p_d^* induced by the twofold nature (one-/two-dimensional) of the grid. Indeed, on the grid $\mathcal{G}$:

(1) $p = 4$ is the supremum of those exponents p such that $\mathcal{E}_p(\mu)$ is finite and negative (and attained by a ground state) for every $\mu > 0$.

(2) $p = 6$ is the infimum of those exponents p such that $\mathcal{E}_p(\mu) = -\infty$ for every $\mu > 0$.

Besides, let us stress another remarkable aspect of the dimensional crossover. In $\mathbb{R}^d$, as well as on noncompact finite graphs, the critical exponent is characterized by the existence of a *critical mass* in the following sense: for smaller masses every function has positive energy, while for larger masses there are functions with negative energy (as already mentioned, on a noncompact finite graph such a critical mass arises only when $p = 6$).

On the contrary, on the grid $\mathcal{G}$ a similar notion of critical mass (the number μ_p in Theorem 1.2) arises for every $p \in [4, 6]$, so that, in this respect, *every exponent within this range is, in fact, critical* (see Remark 2.5). Beyond this critical mass, however, the energy is still bounded from below and a ground state exists, as if the problem had kept track of the subcriticality of the exponent $p < 6$ at the microscopic scale.

From the point of view of functional analysis, the dimensional crossover is due to the simultaneous validity, for every function $u \in W^{1,1}(\mathcal{G})$, of the two inequalities

$$\|u\|_{L^\infty(\mathcal{G})} \le \|u'\|_{L^1(\mathcal{G})}, \quad \|u\|_{L^2(\mathcal{G})} \le \|u'\|_{L^1(\mathcal{G})}. \tag{7}$$

Of these, the former is typical of dimension 1, modeled on the well-known inequality

$$\|v\|_{L^\infty(\mathbb{R})} \le \tfrac{1}{2}\|v'\|_{L^1(\mathbb{R})} \quad \text{for all } v \in W^{1,1}(\mathbb{R}), \tag{8}$$

while the latter is the formal analogue of the Sobolev inequality in $\mathbb{R}^2$

$$\|v\|_{L^2(\mathbb{R}^2)} \le C\|\nabla v\|_{L^1(\mathbb{R}^2)} \quad \text{for all } v \in W^{1,1}(\mathbb{R}^2),$$

and is typical of dimension 2. As discussed in Section 2, either inequality in (7) yields a particular version of the Gagliardo–Nirenberg inequality in $H^1(\mathcal{G})$ ((12) and (18) respectively). By interpolation, one obtains the *critical* Gagliardo–Nirenberg inequalities

$$\int_{\mathcal{G}} |u|^p\,dx \le K_p \left(\int_{\mathcal{G}} |u|^2\,dx \right)^{(p-2)/2} \int_{\mathcal{G}} |u'|^2\,dx \quad \text{for all } u \in H^1(\mathcal{G}), \tag{9}$$

which, being valid for *every* exponent $p \in [4, 6]$, give rise to a continuum of critical exponents (see also Remark 2.5). Indeed, using (9), the NLS energy in (2) can be estimated from below as

$$E_p(u) \geq \frac{1}{2}\left(1 - \frac{2K_p}{p}\mu^{(p-2)/2}\right)\int_{\mathcal{G}} |u'|^2\,dx,$$

which shows that $E_p(u) \geq 0$ for every $u \in H^1_\mu(\mathcal{G})$, as soon as

$$\mu \leq \left(\frac{p}{2K_p}\right)^{2/(p-2)} =: \mu_p.$$

The number in the right-hand side of this inequality is thus the critical mass μ_p of Theorem 1.2.

Finally we would like to point out that we have chosen the grid $\mathcal{G}$ to illustrate our results because it is the simplest doubly periodic metric graph, on which computations and proofs are particularly transparent. It should be clear however that many other doubly periodic graphs can be treated with the methods developed in the present work. Among these, we explicitly mention the hexagonal grid, a model for *graphene*.

At the core of the results stands the double periodicity of the graph, which is responsible for the occurrence of phenomena such as the dimensional crossover. To exploit the double periodicity on a concrete given graph one must of course alter some parts of the proofs presented in this paper (e.g., the proof of Theorem 2.2) to adapt them to the particular features of the graph under study. We plan to illustrate this with the detailed study of some other particular graphs, significantly relevant for the applications, in forthcoming papers.

2. Inequalities

In this section we establish some fundamental inequalities for functions on the grid.

For notational purposes, it is convenient to describe the grid $\mathcal{G}$ as isometrically embedded in $\mathbb{R}^2$, with the lattice $\mathbb{Z}^2$ as set of vertices, and an edge of length 1 joining every pair of adjacent vertices. In this way, it is natural to interpret $\mathcal{G}$ as the union of horizontal lines $\{H_j\}$ and vertical lines $\{V_k\}$, which cross at every vertex $(k, j) \in \mathbb{Z}^2$.

As on any metric graph, to deal with the energy functional (2), the natural functional framework is given by the standard spaces $L^p(\mathcal{G})$ and $H^1(\mathcal{G})$. With the notation for $\mathcal{G}$ introduced above, for the L^p norms we have

$$\|u\|^p_{L^p(\mathcal{G})} = \sum_{j\in\mathbb{Z}} \|u\|^p_{L^p(H_j)} + \sum_{k\in\mathbb{Z}} \|u\|^p_{L^p(V_k)} = \sum_{j\in\mathbb{Z}} \int_{H_j} |u(x)|^p\,dx + \sum_{k\in\mathbb{Z}} \int_{V_k} |u(x)|^p\,dx < \infty \tag{10}$$

and

$$\|u\|_{L^\infty(\mathcal{G})} = \sup_{j,k}\{\|u\|_{L^\infty(H_j)}, \|u\|_{L^\infty(V_k)}\}, \tag{11}$$

while

$$\|u\|^2_{H^1(\mathcal{G})} = \|u\|^2_{L^2(\mathcal{G})} + \|u'\|^2_{L^2(\mathcal{G})}.$$

Here, as usual, $H^1(\mathcal{G})$ denotes the space of functions on $\mathcal{G}$ whose restriction to every horizontal and vertical line belongs to $H^1(\mathbb{R})$, and that, in addition, are continuous at every vertex of $\mathcal{G}$. In Theorem 2.2 we shall also need the space $W^{1,1}(\mathcal{G})$, similarly defined as the space of functions on $\mathcal{G}$ whose restriction to every horizontal and vertical line belongs to $W^{1,1}(\mathbb{R})$ and that, in addition, are continuous at every vertex.

Remark. In the following, symbols like $\|u\|_p$ stand for $\|u\|_{L^p(\mathcal{G})}$. When the domain of integration is different from $\mathcal{G}$, it will always be indicated in the norm.

First we recall the standard Gagliardo–Nirenberg inequality, which (up to a multiplicative constant $C > 1$ on the right-hand side) is valid on any noncompact metric graph; a proof in the general framework can be found in [Adami et al. 2016]. Here, for the sake of completeness, we shall give a short proof tailored to the grid $\mathcal{G}$ which, by the way, yields a slightly sharper estimate.

Theorem 2.1 (one-dimensional Gagliardo–Nirenberg inequality). *For every $p \in [2, \infty)$ one has*

$$\|u\|_p \le \|u\|_2^{1/2+1/p} \|u'\|_2^{1/2-1/p} \quad \textit{for all } u \in H^1(\mathcal{G}) \tag{12}$$

and, moreover,

$$\|u\|_\infty \le \|u\|_2^{1/2} \|u'\|_2^{1/2} \quad \textit{for all } u \in H^1(\mathcal{G}). \tag{13}$$

Proof. Since $\|u\|_p \le \|u\|_\infty^{1-2/p} \|u\|_2^{2/p}$, it suffices to prove (13). On the other hand, given $u \in H^1(\mathcal{G})$, we have $u^2 \in W^{1,1}(H_j)$ for every horizontal line H_j of $\mathcal{G}$. Then, applying (8) with $v = u^2$ on H_j yields

$$\|u\|^2_{L^\infty(H_j)} \le \int_{H_j} |u(x)u'(x)|\,dx \le \|u\|_{L^2(H_j)} \|u'\|_{L^2(H_j)} \le \|u\|_{L^2(\mathcal{G})} \|u'\|_{L^2(\mathcal{G})}.$$

Since clearly this inequality remains true if we replace H_j with any vertical line V_k, (13) follows immediately from (11). □

As already mentioned, inequalities like (12) and (13) hold for every noncompact graph. On the contrary, the next inequality, and its consequences below, rely on the two-dimensional web structure of the grid $\mathcal{G}$.

Theorem 2.2 (two-dimensional Sobolev inequality). *For every $u \in W^{1,1}(\mathcal{G})$,*

$$\|u\|_2 \le \tfrac{1}{2} \|u'\|_1. \tag{14}$$

Proof. Given $u \in W^{1,1}(\mathcal{G})$, we have

$$\|u\|_2^2 = \sum_{j \in \mathbb{Z}} \int_{H_j} |u(x)|^2\,dx + \sum_{k \in \mathbb{Z}} \int_{V_k} |u(y)|^2\,dy. \tag{15}$$

First observe that, for each k, using (8) we obtain

$$\int_{V_k} |u(y)|^2\,dy \le \|u\|_{L^\infty(V_k)} \int_{V_k} |u(y)|\,dy \le \tfrac{1}{2} \|u'\|_{L^1(V_k)} \int_{V_k} |u(y)|\,dy. \tag{16}$$

Then, for each $j \in \mathbb{Z}$, consider the horizontal lines H_j and H_{j+1}, and denote by P_j the path in $\mathcal{G}$ obtained by joining together the half-line of H_j to the left of V_k, the vertical segment of V_k between H_j and H_{j+1} (which we denote by I_j), and the half-line of H_{j+1} to the right of V_k (see Figure 2).

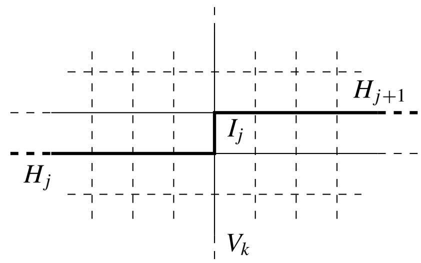

Figure 2. The path P_j (thick in the picture).

Since in particular $u \in W^{1,1}(P_j)$, and the metric graph P_j is isometric to $\mathbb{R}$, we find from (8)

$$|u(y)| \le \frac{1}{2} \int_{P_j} |u'(x)|\, dx \quad \text{for all } y \in I_j$$

and, since I_j has length 1, integrating this inequality over I_j yields

$$\int_{I_j} |u(y)|\, dy \le \frac{1}{2} \int_{P_j} |u'(x)|\, dx \quad \text{for all } j \in \mathbb{Z}. \tag{17}$$

Now observe that

$$V_k = \bigcup_{j\in\mathbb{Z}} I_j, \qquad \bigcup_{j\in\mathbb{Z}} P_j = V_k \cup \bigcup_{j\in\mathbb{Z}} H_j,$$

and moreover, up to a negligible set, the paths $\{P_j\}$ $(j \in \mathbb{Z})$ are mutually disjoint: therefore, summing (17) over $j \in \mathbb{Z}$ yields

$$\int_{V_k} |u(y)|\, dy \le \frac{1}{2}\left(\int_{V_k} |u'(y)|\, dy + \sum_j \int_{H_j} |u'(x)|\, dx \right) = \frac{1}{2}\left(v_k + \sum_j h_j \right)$$

having set, for brevity, $v_k = \int_{V_k} |u'(y)|\, dy$ and $h_j = \int_{H_j} |u'(x)|\, dx$. Combining with (16), and summing over k, one obtains

$$\sum_k \int_{V_k} |u(y)|^2\, dy \le \frac{1}{4} \sum_k v_k \left(v_k + \sum_j h_j \right).$$

Of course, by the symmetry of $\mathcal{G}$, we also have

$$\sum_j \int_{H_j} |u(x)|^2\, dx \le \frac{1}{4} \sum_j h_j \left(h_j + \sum_k v_k \right),$$

and summing the last two inequalities we find

$$\|u\|^2_{L^2(\mathcal{G})} \le \frac{1}{4}\left(\sum_k (h_k^2 + v_k^2) + 2 \sum_{j,k} h_j v_k \right) \le \frac{1}{4}\left(\sum_k h_k + v_k \right)^2 = \frac{1}{4} \|u'\|^2_{L^1(\mathcal{G})}. \qquad \square$$

Theorem 2.3 (two-dimensional Gagliardo–Nirenberg inequality). *For every $p \in [2, \infty)$ one has*

$$\|u\|_p \le C \|u\|_2^{2/p} \|u'\|_2^{1-2/p} \quad \textit{for all } u \in H^1(\mathcal{G}), \tag{18}$$

where C is an absolute constant.

Proof. Given $p \in [2, \infty)$, we have

$$\|u\|_p \le \|u\|_2^{1-\theta} \|u\|_{p+2}^{\theta}, \tag{19}$$

where

$$\frac{1-\theta}{2} + \frac{\theta}{p+2} = \frac{1}{p}, \quad \text{i.e.,} \quad \theta = 1 - \frac{4}{p^2}. \tag{20}$$

Now observe that $u \in L^\infty(\mathcal{G})$ by (13), and hence $u^{1+p/2}$ belongs to $W^{1,1}(\mathcal{G})$ since $p \ge 2$. Therefore, we can replace u with $u^{1+p/2}$ in (14), thus obtaining

$$\|u\|_{p+2}^{1+p/2} \le \frac{p+2}{4} \int_{\mathcal{G}} |u(x)|^{p/2} |u'(x)|\, dx \le \frac{p+2}{4} \|u\|_p^{p/2} \|u'\|_2.$$

Raising to the power $2/(p+2)$ we find

$$\|u\|_{p+2} \le C \|u\|_p^{p/(p+2)} \|u'\|_2^{2/(p+2)}, \quad C = \sup_{p \ge 2} \left(\frac{p+2}{4} \right)^{2/(p+2)}; \tag{21}$$

one may take, e.g., $C = \frac{3}{2}$. Plugging this inequality into (19) gives

$$\|u\|_p \le \|u\|_2^{1-\theta} C^\theta \|u\|_p^{\theta p/(p+2)} \|u'\|_2^{2\theta/(p+2)}$$

and (18) follows using (20), after elementary computations. □

Corollary 2.4 (interdimensional Gagliardo–Nirenberg inequality). *There exists a universal constant $C > 0$ such that, for every $p \in [2, \infty)$,*

$$\|u\|_p \le C \|u\|_2^{1-\alpha} \|u'\|_2^{\alpha} \quad \text{for all } \alpha \in \left[\frac{p-2}{2p}, \frac{p-2}{p} \right], \text{ for all } u \in H^1(\mathcal{G}). \tag{22}$$

In particular, for every $p \in [4, 6]$ there exists a constant K_p, depending only on p, such that

$$\|u\|_p^p \le K_p \|u\|_2^{p-2} \|u'\|_2^2 \quad \text{for all } u \in H^1(\mathcal{G}). \tag{23}$$

Proof. Observe that (22) reduces to (12) (where $C = 1$) when $\alpha = (p-2)/(2p)$, while it reduces to (18), where $C \le \frac{3}{2}$ by (21), when $\alpha = (p-2)/p$. Then (22) is established also for every intermediate value of α, since the right-hand side is a convex function of α, with a constant C independent of p and α.

Finally, when $p \in [4, 6]$, (23) is obtained letting $\alpha = 2/p$ in (22) (the condition $p \in [4, 6]$ guarantees that this choice of α is admissible). The constant K_p in (23) is the best possible (i.e., the smallest); of course $K_p \le C^p$ for every $p \in [4, 6]$, where C is the constant appearing in (22). □

Remark 2.5. In $\mathbb{R}^d$, when dealing with the NLS energy

$$\frac{1}{2} \|\nabla u\|_{L^2(\mathbb{R}^d)}^2 - \frac{1}{p} \|u\|_{L^p(\mathbb{R}^d)}^p$$

in the presence of an L^2 mass constraint, the relevant version of the Gagliardo–Nirenberg (G-N) inequality is

$$\|u\|_{L^p(\mathbb{R}^d)} \le C \|u\|_{L^2(\mathbb{R}^d)}^{1-\alpha} \|\nabla u\|_{L^2(\mathbb{R}^d)}^{\alpha}, \quad \alpha = \frac{d(p-2)}{2p}, \tag{24}$$

valid as soon as $\alpha \in [0, 1)$; see [Leoni 2009]. When $p = 2 + 4/d$, this inequality becomes *critical* for the NLS energy because $\alpha = 2/p$ (i.e., the exponents in the inequality become as in (23)), and a critical mass μ_p comes into play. Now, while in (24) this *critical exponent* $p = 2 + 4/d$ is uniquely determined by the ambient space $\mathbb{R}^d$, on the grid $\mathcal{G}$ *every* $p \in [4, 6]$ is critical for the NLS energy, since one can let $\alpha = 2/p$ in (22) (and obtain (23)) not just for one particular p, but for every $p \in [4, 6]$.

Formally, solving for d in (24), for fixed α we can interpret (22) as a G-N inequality in dimension $d = 2\alpha p/(p-2)$: we call (22) *interdimensional* since d ranges over $[1, 2]$ as α varies (this is in contrast with (24), where the exponent α is uniquely determined by p and the space dimension d). With this interpretation, (23) (which is just (22) with $\alpha = 2/p$) can be seen as a critical G-N inequality in dimension $d = 4/(p-2)$ so that, formally, every $p \in [4, 6]$ can be seen as the critical exponent $p = 2+4/d$, in a fractal scaling dimension $d \in [1, 2]$.

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1.

Remark 3.1. Note that, for every $\mu > 0$ and $p < 6$, the one-dimensional Gagliardo–Nirenberg inequality (12) ensures that $\mathcal{E}_p(\mu)$ is finite and E_p is coercive on $H^1_\mu(\mathcal{G})$ [Adami et al. 2016].

Recalling (3) and (4), we first prove a dichotomy lemma for minimizing sequences, which is useful in proving the existence of ground states.

Lemma 3.2 (dichotomy). *Given $\mu > 0$ and $p \in (2, 6)$, let $\{u_n\} \subset H^1_\mu(\mathcal{G})$ be a minimizing sequence for E_p, i.e.,*

$$\lim_{n\to\infty} E_p(u_n) = \mathcal{E}_p(\mu),$$

and assume that $u_n \rightharpoonup u$ weakly in $H^1(\mathcal{G})$ and pointwise a.e. on $\mathcal{G}$. If

$$m := \mu - \|u\|_2^2 \in [0, \mu] \tag{25}$$

denotes the loss of mass in the limit, then either $m = 0$ or $m = \mu$.

Proof. We assume that $0 < m < \mu$ and seek a contradiction. According to the Brezis–Lieb lemma [1983], we can write

$$E_p(u_n) = E_p(u_n - u) + E_p(u) + o(1) \quad \text{as } n \to \infty, \tag{26}$$

and, since $u_n \rightharpoonup u$ in $L^2(\mathcal{G})$,

$$\|u_n - u\|_2^2 = \|u_n\|_2^2 + \|u\|_2^2 - 2\langle u_n, u\rangle_2 \to \mu - \|u\|_2^2 = m \tag{27}$$

as $n \to \infty$. Now, for n large enough,

$$\begin{aligned}\mathcal{E}_p(\mu) &\le E_p\left(\frac{\sqrt{\mu}}{\|u_n - u\|_2}(u_n - u)\right)\\ &= \frac{1}{2}\frac{\mu}{\|u_n - u\|_2^2}\|u_n' - u'\|_2^2 - \frac{1}{p}\frac{\mu^{p/2}}{\|u_n - u\|_2^p}\|u_n - u\|_p^p < \frac{\mu}{\|u_n - u\|_2^2}E_p(u_n - u),\end{aligned}$$

since $\|u_n - u\|_p \neq 0$ and $\|u_n - u\|_2^2 < \mu$. Thus,

$$E_p(u_n - u) > \frac{\|u_n - u\|_2^2}{\mu}\,\mathcal{E}_p(\mu),$$

and by (27)

$$\liminf_n E_p(u_n - u) \geq \frac{m}{\mu}\,\mathcal{E}_p(\mu).$$

Thus, taking the liminf in (26) we find

$$\mathcal{E}_p(\mu) \geq \frac{m}{\mu}\,\mathcal{E}_p(\mu) + E_p(u). \tag{28}$$

Similarly, since $u \not\equiv 0$ we also have

$$\mathcal{E}_p(\mu) \leq E_p\left(\frac{\sqrt{\mu}}{\sqrt{\mu - m}}\,u\right) \leq \frac{1}{2}\frac{\mu}{\mu - m}\|u'\|_2^2 - \frac{1}{p}\left(\frac{\mu}{\mu - m}\right)^{p/2}\|u\|_p^p < \frac{\mu}{\mu - m}E_p(u) \tag{29}$$

and, as $\mathcal{E}_p(\mu) > -\infty$ by Remark 3.1, from (28) we finally obtain

$$\mathcal{E}_p(\mu) > \frac{m}{\mu}\,\mathcal{E}_p(\mu) + \frac{\mu - m}{\mu}\,\mathcal{E}_p(\mu) = \mathcal{E}_p(\mu),$$

a contradiction. □

Proposition 3.3. *Assume $p < 6$ and $\mathcal{E}_p(\mu)$ is strictly negative. Then there exists $u \in H^1_\mu(\mathcal{G})$ such that*

$$E_p(u) = \mathcal{E}_p(\mu).$$

Proof. Let $\{u_n\} \subset H^1_\mu(\mathcal{G})$ be a minimizing sequence for E_p. Since $p < 6$, Remark 3.1 yields that $\mathcal{E}_p(\mu) > -\infty$ and u_n is bounded in $H^1(\mathcal{G})$, and by translating each u_n (exploiting the periodicity of $\mathcal{G}$) we can also assume that u_n attains its L^∞-norm on a compact set $\mathcal{K} \subset \mathcal{G}$ independent of n. Therefore, up to subsequences, u_n converges weakly in $H^1(\mathcal{G})$, and strongly in $L^\infty_{\text{loc}}(\mathcal{G})$, to some function $u \in H^1(\mathcal{G})$. Setting $m := \mu - \|u\|_2^2$, from Lemma 3.2 one sees that either $m = 0$ or $m = \mu$. If $m = \mu$ then $u \equiv 0$, but in this case $u_n \to 0$ in $L^\infty(\mathcal{G})$, since in particular, $u_n \to u \equiv 0$ uniformly on $\mathcal{K}$. Therefore we would have

$$E_p(u_n) \geq -\frac{1}{p}\|u_n\|_\infty^{p-2}\int_{\mathcal{G}} |u_n|^2\,dx = -\frac{\mu}{p}\|u_n\|_\infty^{p-2} \to 0,$$

contradicting the fact that $\mathcal{E}_p(\mu) < 0$.

Thus it must be that $m = 0$, so that $u_n \to u$ strongly in $L^2(\mathcal{G})$ and therefore $u \in H^1_\mu(\mathcal{G})$. Moreover, since u_n is bounded in $L^\infty(\mathcal{G})$, $u_n \to u$ strongly also in $L^p(\mathcal{G})$. Then

$$\mathcal{E}_p(\mu) \leq E_p(u) \leq \liminf_n E_p(u_n) = \mathcal{E}_p(\mu)$$

by weak lower semicontinuity, and the proof is complete. □

Remark 3.4. It is interesting to compare Proposition 3.3 with Theorem 3.3 in [Adami et al. 2016]. According to that result, in a finite noncompact graph the energy threshold under which the existence of a ground state of a given mass is guaranteed equals the energy of the soliton on $\mathbb{R}$ with the same mass. On the contrary, on the grid $\mathcal{G}$ the absence of half-lines and the periodicity pushes the energy threshold up to zero. This makes some proofs easier, since finding a function with negative energy is far easier than finding a function whose energy lies below a particular negative number. In fact, this task is immediately accomplished when $p < 4$, as we now show.

Proof of Theorem 1.1. In view of Proposition 3.3, it suffices to construct a function in $H^1_\mu(\mathcal{G})$ with negative energy. Given $\mu > 0$, for $\varepsilon > 0$ let

$$\kappa_\varepsilon = \left(\frac{\varepsilon\mu}{2} \frac{1-e^{-2\varepsilon}}{1+e^{-2\varepsilon}} \right)^{1/2} \tag{30}$$

and consider the function of two variables

$$\varphi_\varepsilon(x, y) = \kappa_\varepsilon e^{-\varepsilon(|x|+|y|)}, \quad (x, y) \in \mathbb{R}^2.$$

Now, as described in Section 2, we can consider $\mathcal{G}$ isometrically embedded in $\mathbb{R}^2$, with its vertices on the lattice $\mathbb{Z}^2$, and we can define $u_\varepsilon : \mathcal{G} \to \mathbb{R}$ as the restriction of φ_ε to the grid $\mathcal{G}$. Observe that, on every horizontal line H_j of $\mathcal{G}$, u takes the form $\kappa_\varepsilon e^{-\varepsilon(|x|+|j|)}$, and a similar expression holds on vertical lines. Since for every $\lambda > 0$

$$\int_{\mathbb{R}} e^{-\lambda\varepsilon|x|}\,dx = \frac{2}{\lambda\varepsilon} \quad \text{and} \quad \sum_{j\in\mathbb{Z}} e^{-\lambda\varepsilon|j|} = \frac{1+e^{-\lambda\varepsilon}}{1-e^{-\lambda\varepsilon}},$$

recalling (30) we obtain

$$\int_{\mathcal{G}} |u_\varepsilon|^2\,dx = 2\sum_{j\in\mathbb{Z}} \int_{H_j} |u_\varepsilon|^2\,dx = 2\kappa_\varepsilon^2 \sum_{j\in\mathbb{Z}} e^{-2\varepsilon|j|} \int_{\mathbb{R}} e^{-2\varepsilon|x|}\,dx = \mu$$

and, since $|u_\varepsilon'(x)| = \varepsilon|u_\varepsilon(x)|$,

$$\int_{\mathcal{G}} |u_\varepsilon'|^2\,dx = \varepsilon^2\mu.$$

This shows in particular that $u_\varepsilon \in H^1_\mu(\mathcal{G})$. Similarly, observing that $\kappa_\varepsilon \sim \varepsilon\sqrt{\mu/2}$ as $\varepsilon \to 0$, we obtain the expansion

$$\int_{\mathcal{G}} |u_\varepsilon|^p\,dx = 2\sum_{j\in\mathbb{Z}} \int_{H_j} |u_\varepsilon|^p\,dx = 2\kappa_\varepsilon^p \frac{2}{\varepsilon p} \frac{1+e^{-\varepsilon p}}{1-e^{-\varepsilon p}} \sim C\mu^{p/2}\varepsilon^{p-2} \quad \text{as } \varepsilon \to 0,$$

where C depends only on p. Therefore, as $\varepsilon \to 0$,

$$E_p(u_\varepsilon) \sim \frac{1}{2}\varepsilon^2\mu - \frac{1}{p}C\mu^{p/2}\varepsilon^{p-2}, \tag{31}$$

so that $E_p(u_\varepsilon) < 0$ (for ε small enough) when $p < 4$. This proves that, when $p < 4$, $\mathcal{E}_p(\mu) < 0$ for every $\mu > 0$. Moreover, since in particular $p < 6$, Remark 3.1 guarantees that $\mathcal{E}_p(\mu)$ is finite. The result then follows from Proposition 3.3. □

4. Proof of Theorem 1.2

In the following we assume that the constants K_p in the Gagliardo–Nirenberg inequality (23) are the smallest possible. In other words, for $p \in [4, 6]$ we let

$$K_p = \sup_{\substack{u\in H^1(\mathcal{G})\\ u\not\equiv 0}} Q_p(u), \quad \text{where } Q_p(u) = \frac{\|u\|_p^p}{\|u\|_2^{p-2}\|u'\|_2^2}. \tag{32}$$

The critical masses μ_p mentioned in Theorem 1.2 are defined in terms of the constants K_p as follows.

Definition 4.1. For every $p \in [4, 6]$ we define the *critical mass* μ_p as the positive number

$$\mu_p = \left(\frac{p}{2K_p}\right)^{2/(p-2)}. \tag{33}$$

This definition is natural due to the identity

$$E_p(u) = \frac{1}{2}\|u'\|_2^2\Big(1 - \frac{2}{p}Q_p(u)\mu^{(p-2)/2}\Big) \quad \text{for all } u \in H^1_\mu(\mathcal{G}), \tag{34}$$

which, using $Q_p(u) \le K_p$ and (33), leads to the lower bound

$$E_p(u) \ge \frac{1}{2}\|u'\|_2^2\left(1 - \left(\frac{\mu}{\mu_p}\right)^{(p-2)/2}\right) \quad \text{for all } u \in H^1_\mu(\mathcal{G}), \tag{35}$$

which will be widely used in the sequel.

Remark 4.2. On the real line $\mathbb{R}$, when $p = 6$ the ground-state level

$$\mathcal{E}_6^{\mathbb{R}}(\mu) = \inf\big\{\tfrac{1}{2}\|w'\|^2_{L^2(\mathbb{R})} - \tfrac{1}{6}\|w\|^6_{L^6(\mathbb{R})} \;\big|\; w \in H^1_\mu(\mathbb{R})\big\}, \quad \mu > 0, \tag{36}$$

is attained by a ground state if and only if $\mu = \mu_6^{\mathbb{R}}$, where the number

$$\mu_6^{\mathbb{R}} = \frac{\pi\sqrt{3}}{2} \tag{37}$$

is the critical mass of the real line; see [Adami et al. 2017a]. Up to sign and translations, the ground states (of mass $\mu_6^{\mathbb{R}}$) are the soliton $\varphi(x) = \operatorname{sech}(2x/\sqrt{3})^{1/2}$ together with all its mass-preserving rescalings $\varphi_\lambda(x) = \sqrt{\lambda}\varphi(\lambda x)$ $(\lambda > 0)$. There holds

$$\begin{aligned} \mathcal{E}_6^{\mathbb{R}}(\mu) &= 0 && \text{if } \mu \le \mu_6^{\mathbb{R}}, \\ \mathcal{E}_6^{\mathbb{R}}(\mu) &= -\infty && \text{if } \mu < \mu_6^{\mathbb{R}} \end{aligned} \tag{38}$$

so that in particular ground states have zero energy. Another related quantity is the optimal constant in the Gagliardo–Nirenberg inequality on $\mathbb{R}$, i.e., the number

$$K_6^{\mathbb{R}} = \sup_{\substack{w \in H^1(\mathbb{R}) \\ w \not\equiv 0}} \frac{\|w\|^6_{L^6(\mathbb{R})}}{\|w\|^4_{L^2(\mathbb{R})}\|w'\|^2_{L^2(\mathbb{R})}} = \frac{4}{\pi^2} \tag{39}$$

(note that $\mu_6^{\mathbb{R}} = (3/K_6^{\mathbb{R}})^{1/2}$, which is formally consistent with (33) when $p = 6$).

The following proposition gives a complete picture of the problem on the grid $\mathcal{G}$ when $p = 6$ and, moreover, provides the exact values of μ_6 and K_6.

Proposition 4.3. *There hold* $\mu_6 = \mu_6^{\mathbb{R}} = \pi\sqrt{3}/2$ *and* $K_6 = K_6^{\mathbb{R}} = 4/\pi^2$*. Moreover there holds* $\mathcal{E}_6(\mu) = \mathcal{E}_6^{\mathbb{R}}(\mu)$ *for every* $\mu > 0$*, but the infimum*

$$\mathcal{E}_6(\mu) = \inf\big\{\tfrac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} - \tfrac{1}{6}\|u\|^6_{L^6(\mathcal{G})} \;\big|\; u \in H^1_\mu(\mathcal{G})\big\}, \quad \mu > 0, \tag{40}$$

is never attained.

Proof. By a density argument, the infimum in (36) can be restricted to functions $w \in H^1_\mu(\mathbb{R})$ having compact support. In fact, by a mass-preserving transformation $w(x) \mapsto w(x/\varepsilon^2)/\varepsilon$, one can restrict to functions supported in the interval $I = \left[-\frac{1}{2}, \frac{1}{2}\right]$. Then, by interpreting this interval as one of the edges of the grid $\mathcal{G}$, any function $w \in H^1_\mu(\mathbb{R})$ supported in I can be embedded in $H^1_\mu(\mathcal{G})$ by setting $w \equiv 0$ on $\mathcal{G} \setminus I$, thus providing an admissible function in (40). This proves that $\mathcal{E}_6(\mu) \le \mathcal{E}_6^{\mathbb{R}}(\mu)$ for every $\mu > 0$. Similarly, starting from the supremum in (39), by the same argument one proves that $K_6 \ge K_6^{\mathbb{R}}$.

To prove the opposite inequalities we argue as follows. Given a nonnegative function $u \in H^1(\mathcal{G})$ ($u \not\equiv 0$), let $x_0 \in \mathcal{G}$ be a point where u achieves its absolute maximum $\|u\|_\infty$, and let P be any path in $\mathcal{G}$ such that $x_0 \in P$ and P is isometric to the real line $\mathbb{R}$ (a natural choice for P is the horizontal/vertical line of $\mathcal{G}$ that contains x_0). Since $u(x_0) = \|u\|_\infty$ and $u(x) \to 0$ as $x \to \pm\infty$ along P (in both directions away from x_0), the continuity of u guarantees that $N(t) \ge 2$ for every $t \in (0, \|u\|_\infty)$, where

$$N(t) = \#\{x \in \mathcal{G} \mid u(x) = t\} \tag{41}$$

counts the number of preimages in $\mathcal{G}$. Then, if $\hat{u} \in H^1(\mathbb{R})$ denotes the symmetric rearrangement of u on $\mathbb{R}$, applying Proposition 3.1 of [Adami et al. 2015b] we obtain

$$\|(\hat{u})'\|_{L^2(\mathbb{R})} \le \|u'\|_{L^2(\mathcal{G})}, \qquad \|\hat{u}\|_{L^r(\mathbb{R})} = \|u\|_{L^r(\mathcal{G})} \quad \text{for all } r \tag{42}$$

so that, by the definition of $K_6^{\mathbb{R}}$ in (39), we can estimate

$$\|u\|^6_{L^6(\mathcal{G})} = \|\hat{u}\|^6_{L^6(\mathbb{R})} \le K_6^{\mathbb{R}} \|\hat{u}\|^4_{L^2(\mathbb{R})} \|(\hat{u})'\|^2_{L^2(\mathbb{R})} \le K_6^{\mathbb{R}} \|u\|^4_{L^2(\mathcal{G})} \|u'\|^2_{L^2(\mathcal{G})}.$$

Therefore, $K_6 \le K_6^{\mathbb{R}}$ by (32). Similarly, for the NLS energy we have

$$\tfrac{1}{2}\|(\hat{u})'\|^2_{L^2(\mathbb{R})} - \tfrac{1}{6}\|\hat{u}\|^6_{L^6(\mathbb{R})} \le \tfrac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} - \tfrac{1}{6}\|u\|^6_{L^6(\mathcal{G})} \tag{43}$$

and, since $\hat{u} \in H^1_\mu(\mathbb{R})$ whenever $u \in H^1_\mu(\mathcal{G})$, this proves that $\mathcal{E}_6^{\mathbb{R}}(\mu) \le \mathcal{E}_6(\mu)$ for every $\mu > 0$.

Now assume that, for some μ, a function $u \in H^1_\mu(\mathcal{G})$ achieves the infimum $\mathcal{E}_6(\mu)$ in (40). Then, since $\mathcal{E}_6^{\mathbb{R}}(\mu) = \mathcal{E}_6(\mu)$, (43) shows that, necessarily (i) $\hat{u}$ achieves the infimum $\mathcal{E}_6^{\mathbb{R}}(\mu)$ in (36); (ii) equality must occur in (43), i.e., in (42). Now, condition (i) gives that $\hat{u}$ is a soliton on $\mathbb{R}$ (necessarily of mass $\mu_6^{\mathbb{R}}$), while (ii) implies, see Proposition 3.1 of [Adami et al. 2015b], that $N(t) = 2$ in (41), i.e., that $u^{-1}(t)$ has exactly two elements for almost every $t \in (0, \|u\|_\infty)$; then, since every vertex of $\mathcal{G}$ has degree 4, u must vanish at every vertex and is necessarily supported in a single edge of $\mathcal{G}$. So $\hat{u}$ has compact support too, which is incompatible with $\hat{u}$ being a soliton. This contradiction shows the infimum in (40) is not achieved.

Finally, (33) with $p = 6$ yields $\mu_6 = \sqrt{3/K_6} = \pi\sqrt{3}/2$; hence $\mu_6 = \mu_6^{\mathbb{R}}$ by (37). □

Proof of Theorem 1.2. The case where $p = 6$ has already been proved through Proposition 4.3. The rest of the proof is divided into three parts.

Computation of $\mathcal{E}_p(\mu)$ when $p \in [4, 6)$. First observe that, in the proof of Theorem 1.1, no restriction on p was used to construct u_ε and obtain (31), which is therefore valid also when $p \ge 4$. As a consequence, in this case, letting $\varepsilon \to 0$ in (31) we obtain

$$\mathcal{E}_p(\mu) \le \liminf_{\varepsilon\to 0} E_p(u_\varepsilon) \le 0 \quad \text{for all } p \ge 4, \text{ for all } \mu > 0. \tag{44}$$

Moreover, (35) shows that $\mathcal{E}_p(\mu) \ge 0$ when $\mu \le \mu_p$. This, combined with (44), proves the first part of (5), also when $p = 4$.

Now fix a mass $\mu > \mu_p$ and a number $\varepsilon > 0$. Since the quotient $Q_p(u)$ in (32) is unaltered if u is replaced with λu, there exists $u \in H^1_\mu(\mathcal{G})$ such that

$$Q_p(u) = \frac{\|u\|_p^p}{\mu^{(p-2)/2}\|u'\|_2^2} \ge K_p - \varepsilon. \tag{45}$$

Plugging this into (34), and then using (33), we can estimate

$$E_p(u) \le \frac{1}{2}\|u'\|_2^2\left(1 - \frac{2}{p}(K_p - \varepsilon)\mu^{(p-2)/2}\right) = \frac{1}{2}\|u'\|_2^2\left(1 - \left(\frac{\mu}{\mu_p}\right)^{(p-2)/2} + \frac{2\varepsilon}{p}\mu^{(p-2)/2}\right).$$

Since $\mu > \mu_p$, this quantity is strictly negative if ε is small enough. Thus, for $\mu > \mu_p$, $\mathcal{E}_p(\mu) < 0$. Moreover, when $p < 6$, $\mathcal{E}_p(\mu) > -\infty$ by Remark 3.1. This proves the second part of (5), also when $p = 4$.

Ground states when $p \in [4, 6)$ and $\mu \ne \mu_p$. When $\mu > \mu_p$, (5) (valid also when $p = 4$) shows that $\mathcal{E}_p(\mu)$ is finite and negative; hence a ground state exists by Proposition 3.3. When $\mu < \mu_p$, $\mathcal{E}_p(\mu) = 0$ by (5), but (35) reveals that $E_p(u) > 0$ for every $u \in H^1_\mu(\mathcal{G})$. Therefore, no ground state exists in this case.

Ground states when $p \in (4, 6)$ and $\mu = \mu_p$. Since by (5) $\mathcal{E}_p(\mu_p) = 0$, we can no longer rely on Proposition 3.3, and another argument is needed to show that $\mathcal{E}_p(\mu_p)$ is in fact achieved.

Arguing as for (45), let $u_n \in H^1_{\mu_p}(\mathcal{G})$ be a sequence of functions such that

$$\lim_n Q_p(u_n) = \lim_n \frac{\|u_n\|_p^p}{\mu_p^{(p-2)/2}\|u_n'\|_2^2} = K_p. \tag{46}$$

We shall bound $Q_p(u_n)$ in two different ways. First, from the Gagliardo–Nirenberg inequality (12) we obtain

$$Q_p(u_n) \le \frac{\|u_n\|_2^{p/2+1}\|u_n'\|_2^{p/2-1}}{\mu_p^{(p-2)/2}\|u_n'\|_2^2} = \frac{\mu_p^{(6-p)/4}}{\|u_n'\|_2^{(6-p)/2}}.$$

Secondly, interpolating and then using (23) with $p = 4$, we obtain

$$Q_p(u_n) \le \frac{\|u_n\|_\infty^{p-4}\|u_n\|_4^4}{\mu_p^{(p-2)/2}\|u_n'\|_2^2} \le \|u_n\|_\infty^{p-4}\frac{K_4\|u_n\|_2^2\|u_n'\|_2^2}{\mu_p^{(p-2)/2}\|u_n'\|_2^2} = \|u_n\|_\infty^{p-4}\frac{K_4}{\mu_p^{(p-4)/2}}.$$

Recalling (46), from these two bounds we infer that $\|u_n'\|_2 \le C$ (compactness) and $\|u_n\|_\infty \ge C^{-1}$ (nondegeneracy) for some constant $C > 0$ independent of n. Thus $\{u_n\}$ is bounded in $H^1(\mathcal{G})$ and, up to translations, we can also assume that each u_n achieves its L^∞ norm on some compact set $\mathcal{K} \subset \mathcal{G}$ independent of n. Then, up to subsequences, $u_n \rightharpoonup u$ in $H^1(\mathcal{G})$ for some $u \in H^1(\mathcal{G})$, and $u_n \to u$ in $L^\infty_{\text{loc}}(\mathcal{G})$; in particular, $u_n \to u$ uniformly on $\mathcal{K}$ and, since $\|u_n\|_{L^\infty(\mathcal{K})} > C^{-1}$, u is not identically zero.

Finally, writing (34) with $u = u_n$ and $\mu = \mu_p$, since $\|u_n'\|_2 \le C$ we find

$$|E_p(u_n)| \le \frac{C^2}{2}\left|1 - \frac{2}{p} Q_p(u_n)\mu_p^{(p-2)/2}\right| = \frac{C^2}{2}\left|1 - \frac{Q_p(u_n)}{K_p}\right|,$$

having used (33). Therefore, $E_p(u_n) \to 0$ by (46) and, since $\mathcal{E}_p(\mu_p) = 0$, u_n is a minimizing sequence for E_p, so that Lemma 3.2 applies: since we already know that u is not identically zero, we obtain that $\|u\|_2^2 = \mu_p$, i.e., $u \in H^1_{\mu_p}(\mathcal{G})$. But then u is the required minimizer: indeed, $u_n \to u$ strongly in $L^2(\mathcal{G})$ hence also in $L^p(\mathcal{G})$, and by weak lower semicontinuity we obtain

$$\mathcal{E}_p(\mu_p) \le E_p(u) \le \liminf_n E_p(u_n) = \mathcal{E}_p(\mu_p). \qquad \square$$

References

[Adami et al. 2011] R. Adami, C. Cacciapuoti, D. Finco, and D. Noja, "Fast solitons on star graphs", *Rev. Math. Phys.* **23**:4 (2011), 409–451. MR Zbl

[Adami et al. 2015a] R. Adami, E. Serra, and P. Tilli, "Lack of ground state for NLSE on bridge-type graphs", pp. 1–11 in *Mathematical technology of networks*, edited by D. Mugnolo, Springer Proc. Math. Stat. **128**, Springer, 2015. MR Zbl

[Adami et al. 2015b] R. Adami, E. Serra, and P. Tilli, "NLS ground states on graphs", *Calc. Var. Partial Differential Equations* **54**:1 (2015), 743–761. MR Zbl

[Adami et al. 2016] R. Adami, E. Serra, and P. Tilli, "Threshold phenomena and existence results for NLS ground states on metric graphs", *J. Funct. Anal.* **271**:1 (2016), 201–223. MR Zbl

[Adami et al. 2017a] R. Adami, E. Serra, and P. Tilli, "Negative energy ground states for the L^2-critical NLSE on metric graphs", *Comm. Math. Phys.* **352**:1 (2017), 387–406. MR Zbl

[Adami et al. 2017b] R. Adami, E. Serra, and P. Tilli, "Nonlinear dynamics on branched structures and networks", *Riv. Math. Univ. Parma (N.S.)* **8**:1 (2017), 109–159. MR Zbl

[Ali Mehmeti 1994] F. Ali Mehmeti, *Nonlinear waves in networks*, Mathematical Research **80**, Akademie, Berlin, 1994. MR Zbl

[Ali Mehmeti et al. 2001] F. Ali Mehmeti, J. von Below, and S. Nicaise (editors), *Partial differential equations on multistructures* (Luminy, 1999), Lecture Notes in Pure and Applied Mathematics **219**, Marcel Dekker, New York, 2001. MR Zbl

[Berkolaiko and Kuchment 2013] G. Berkolaiko and P. Kuchment, *Introduction to quantum graphs*, Mathematical Surveys and Monographs **186**, American Mathematical Society, Providence, RI, 2013. MR Zbl

[Brezis and Lieb 1983] H. Brezis and E. Lieb, "A relation between pointwise convergence of functions and convergence of functionals", *Proc. Amer. Math. Soc.* **88**:3 (1983), 486–490. MR Zbl

[Cacciapuoti et al. 2015] C. Cacciapuoti, D. Finco, and D. Noja, "Topology-induced bifurcations for the nonlinear Schrödinger equation on the tadpole graph", *Phys. Rev. E* (3) **91**:1 (2015), art. id. 013206. MR

[Caudrelier 2015] V. Caudrelier, "On the inverse scattering method for integrable PDEs on a star graph", *Comm. Math. Phys.* **338**:2 (2015), 893–917. MR Zbl

[Cazenave 2003] T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics **10**, Courant Institute of Mathematical Sciences, New York, 2003. MR Zbl

[Exner 1996] P. Exner, "Contact interactions on graph superlattices", *J. Phys. A* **29**:1 (1996), 87–102. MR Zbl

[Exner and Gawlista 1996] P. Exner and R. Gawlista, "Band spectra of rectangular graph superlattices", *Phys. Rev. B* **53**:11 (1996), 7275–7286.

[Exner and Turek 2010] P. Exner and O. Turek, "High-energy asymptotics of the spectrum of a periodic square lattice quantum graph", *J. Phys. A* **43**:47 (2010), art. id. 474024. MR Zbl

[Gnutzmann and Waltner 2016] S. Gnutzmann and D. Waltner, "Stationary waves on nonlinear quantum graphs: general framework and canonical perturbation theory", *Phys. Rev. E* **93**:3 (2016), art. id. 032204. MR

[Leoni 2009] G. Leoni, *A first course in Sobolev spaces*, Graduate Studies in Mathematics **105**, American Mathematical Society, Providence, RI, 2009. MR Zbl

[Noja 2014] D. Noja, "Nonlinear Schrödinger equation on graphs: recent results and open problems", *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **372**:2007 (2014), art. id. 20130002. MR Zbl

[Noja et al. 2015] D. Noja, D. Pelinovsky, and G. Shaikhova, "Bifurcations and stability of standing waves in the nonlinear Schrödinger equation on the tadpole graph", *Nonlinearity* **28**:7 (2015), 2343–2378. MR Zbl

[Pankov 2018] A. Pankov, "Nonlinear Schrödinger equations on periodic metric graphs", *Discrete Contin. Dyn. Syst.* **38**:2 (2018), 697–714. MR Zbl

[Pelinovsky and Schneider 2017] D. Pelinovsky and G. Schneider, "Bifurcations of standing localized waves on periodic graphs", *Ann. Henri Poincaré* **18**:4 (2017), 1185–1211. MR Zbl

[Post 2012] O. Post, *Spectral analysis on graph-like spaces*, Lecture Notes in Mathematics **2039**, Springer, 2012. MR Zbl

[Ruedenberg and Scherr 1953] K. Ruedenberg and C. W. Scherr, "Free-electron network model for conjugated systems, I: Theory", *J. Chem. Phys.* **21**:9 (1953), 1565–1581.

[Sabirov et al. 2013] K. K. Sabirov, Z. A. Sobirov, D. Babajanov, and D. U. Matrasulov, "Stationary nonlinear Schrödinger equation on simplest graphs", *Phys. Lett. A* **377**:12 (2013), 860–865. MR Zbl

[Sobirov et al. 2010] Z. Sobirov, D. Matrasulov, K. Sabirov, S. Sawada, and K. Nakamura, "Integrable nonlinear Schrödinger equation on simple networks: connection formula at vertices", *Phys. Rev. E* (3) **81**:6 (2010), art. id. 066602. MR

[Weinstein 1999] M. I. Weinstein, "Excitation thresholds for nonlinear localized modes on lattices", *Nonlinearity* **12**:3 (1999), 673–691. MR Zbl

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