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We show that *among sets of finite perimeter* balls are the only volume-constrained critical points of the perimeter functional.

1. Introduction

1.1. ***Sets of finite perimeter and the isoperimetric problem.*** The Euclidean isoperimetric theorem is probably the most basic result in the calculus of variations. There are many different proofs of the isoperimetry of balls in different classes of competitors, thus motivating the question: which is the natural competition class in which the isoperimetric theorem can be formulated? From the perspective of the modern calculus of variations, the answer is found by looking at the relaxation of the perimeter functional. Following the seminal work of De Giorgi [1954; 1955] we consider as particularly natural his formulation of the Euclidean isoperimetric problem in the class of sets of finite perimeter. The characterization of Euclidean balls as the only isoperimetric sets among sets of finite perimeter was achieved in [De Giorgi 1958]. By using the compactness properties of sets of finite perimeter, De Giorgi shows the existence of global minimizers (isoperimetric sets). Next, he shows that distributional perimeter is decreased under Steiner symmetrization, thus deducing that Steiner symmetrization applied to an isoperimetric set leads to an equality case in the Steiner perimeter inequality. He finally derives some necessary conditions for being an equality case in the Steiner perimeter inequality, in order to deduce the sphericity of isoperimetric sets.

Despite the intimate connection between sets of finite perimeter and the isoperimetric problem, a characterization of balls as the only *critical points* in the isoperimetric problem *among sets of finite perimeter* is currently missing. The main result of this paper is showing the validity of this characterization.

The problem is already subtle in the case of local minimizers. By a local minimizer we mean a set of finite perimeter which minimizes perimeter among variations compactly supported in a fixed neighborhood of its own boundary. In particular, local minimality does not allow for perimeter comparison with sets obtained by symmetrization, thus ruling out the use of De Giorgi's original argument. In Euclidean spaces of dimension less than or equal to 7 the problem can be settled by the means of the regularity theory for local perimeter minimizers. In fact, in these dimensions any local minimizer is a bounded smooth set with constant mean curvature. One can then combine the strong maximum principle with the geometric construction known as the moving planes method (Alexandrov's theorem [1962]) to deduce the sphericity of the boundary. But this strategy fails in dimension 8 or larger, as boundaries of local

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perimeter minimizers could have, in principle, singular points, where local graphicality fails [Simons 1968]. Actually, it has been recently shown that local volume-constrained perimeter minimizers in nonconvex perturbations of the unit ball may indeed have singularities [Sternberg and Zumbrun 2018].

The problem is open in every dimension for critical points, that is, sets of finite perimeter and finite volume such that the first variation of perimeter under volume-fixing flows vanishes. These sets have constant mean curvature in a very natural (distributional) sense. However, at variance with the case of local minimizers, there seems to be no obvious way, even in low dimensions, to exploit regularity theorems and the moving planes method to conclude their sphericity.

Here we approach this problem by combining regularity theorems and maximum principles with various geometric constructions inspired by the proof of Alexandrov's theorem in [Montiel and Ros 1991]. We thus extend De Giorgi's isoperimetric theorem from the case of global minimizers to that of critical points in the isoperimetric problem.

Theorem 1. *Among sets of finite perimeter and finite volume, finite unions of balls with equal radii are the unique critical points of the Euclidean isoperimetric problem.*

Remark. Theorem 1 is stated in terms of finite unions of balls. By assuming indecomposability (the measure-theoretic analogue of connectedness) of our critical points, we can change "finite unions of balls" to "a single ball". However, it seems natural to consider finite unions of mutually tangent balls as genuinely distinct critical points of the perimeter functional. Indeed, as proved in [Ciraolo and Maggi 2017; Delgadino et al. 2018] (and as it has been known for a much longer time in the case of parametrized surfaces [Brezis and Coron 1984; Struwe 1984]), finite unions of mutually tangent balls are the unique limits of sequences of bounded connected smooth sets with bounded perimeters and scalar mean curvatures which converge to a constant. In short, finite unions of mutually tangent balls are the limit points of Palais–Smale sequences for the isoperimetric problem among connected open sets with smooth boundary.

Remark. Wente's torus [1986] provides an example of an integer rectifiable varifold with multiplicity 1 in $\mathbb{R}^3$ which has constant distributional mean curvature and is not a sphere. Clearly, Wente's torus is not the boundary of a set of finite perimeter. From this point of view, Theorem 1 seems to identify the most general family of surfaces such that constant distributional mean curvature implies sphericity.

While uniqueness and symmetry results for global minimizers can be obtained by a wealth of methods (symmetrization, mass transportation, etc.), the methods employed in the case of critical points/solutions to geometric PDEs, that we are aware of, require a sufficient degree of smoothness (e.g., the classical Alexandrov theorem [1962]). Addressing this kind of issue without assuming smoothness seems a novel aspect of Theorem 1. This point could be particularly useful in proving convergence of geometric flows to unions of balls. Indeed, without strong assumptions like convexity or star-shapedness, global-in-time existence results for geometric flows hold only in a weak (either distributional or viscous) sense. Corollary 2 below should be useful in this context. To better illustrate this point, and to state the corollary itself, we introduce some terminology. In Theorem 1 we consider Borel sets Ω in $\mathbb{R}^{n+1}$ with the following properties:

(i) *Finite perimeter*: There exists a Borel set $\partial^*\Omega$ which is covered, up to an $\mathcal{H}^n$-negligible set, by countably many graphs of C^1 functions from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$, and a Borel vector field $\nu_\Omega : \partial^*\Omega \to \mathbb{S}^n$ such that a generalized version of the divergence theorem holds:

$$\int_\Omega \operatorname{div} X = \int_{\partial^*\Omega} X \cdot \nu_\Omega \, d\mathcal{H}^n \quad \text{for all } X \in C^1_c(\mathbb{R}^{n+1}; \mathbb{R}^{n+1}). \tag{1-1}$$

Here $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure on $\mathbb{R}^{n+1}$.

(ii) *Constant distributional mean curvature*: There exists $\lambda \in \mathbb{R}$ such that

$$\int_{\partial^*\Omega} \operatorname{div}^{\partial^*\Omega} X \, d\mathcal{H}^n = \lambda \int_{\partial^*\Omega} X \cdot \nu_\Omega \, d\mathcal{H}^n \quad \text{for all } X \in C^1_c(\mathbb{R}^{n+1}; \mathbb{R}^{n+1}). \tag{1-2}$$

Here $\operatorname{div}^{\partial^*\Omega} X = \operatorname{div} X - \nu_\Omega \cdot (\nabla X)[\nu_\Omega]$ is the tangential divergence of X along $\partial^*\Omega$. Condition (1-2) is equivalent to asking that Ω be a *critical point in the Euclidean isoperimetric problem*, that is,

$$\frac{d}{dt}\Big|_{t=0} P(f_t(\Omega)) = 0 \tag{1-3}$$

whenever $\{f_t\}_{|t|<1}$ is a volume-preserving variation of Ω. Namely, each f_t is a diffeomorphism with $f_t = \mathrm{Id}$ outside of a compact set, $f_0 \equiv \mathrm{Id}$, and $|f_t(\Omega)| = |\Omega|$ for every $|t| < 1$, where $|\Omega|$ denotes the Lebesgue measure, or volume, of Ω. When Ω is an open bounded set with C^2-boundary, as in Alexandrov's theorem, one simply has $\partial^*\Omega = \partial\Omega$ and (1-3) is equivalent to asking that $\partial\Omega$ have constant mean curvature.

With this terminology in place, we can state the following corollary of Theorem 1.

Corollary 2. *If $\{\Omega_j\}_{j\in\mathbb{N}}$ and Ω are sets of finite perimeter in $\mathbb{R}^{n+1}$ such that*

$$\lim_{j\to\infty} |\Omega_j \Delta \Omega| = 0, \qquad \lim_{j\to\infty} P(\Omega_j) = P(\Omega), \tag{1-4}$$

and if the distributional mean curvatures of the Ω_j converge to a constant $\lambda \in \mathbb{R}$, i.e.,

$$\lim_{j\to\infty} \int_{\partial^*\Omega_j} (\operatorname{div}^{\partial^*\Omega_j} X - \lambda X \cdot \nu_{\Omega_j}) \, d\mathcal{H}^n = 0 \quad \textit{for all } X \in C^1_c(\mathbb{R}^{n+1}; \mathbb{R}^{n+1}), \tag{1-5}$$

then $\lambda = nP(\Omega)/(n+1)|\Omega|$ and Ω is a finite union of balls of radius n/λ.

Remark. Notice that (1-5) holds whenever each Ω_j has distributional mean curvature $H_{\Omega_j} \in L^p(\mathcal{H}^n \llcorner \partial^*\Omega_j)$ for some $p \geq 1$ (see (2-7) and (2-16) below) and

$$\lim_{j\to\infty} \int_{\partial^*\Omega_j} |H_{\Omega_j} - \lambda|^p \, d\mathcal{H}^n = 0. \tag{1-6}$$

Remark. Global-in-time weak solutions of the volume-preserving mean curvature flow have been constructed in [Mugnai et al. 2016] following the method proposed by Almgren, Taylor and Wang [Almgren et al. 1993] and Luckhaus and Sturzenhecker [1995]. Considering [Mugnai et al. 2016, Theorem 2.3.2] and (1-5), it seems reasonable to conjecture that, for a large class of initial data and along time subsequences $t_j \to \infty$, the evolution $\{\Omega(t) : t \geq 0\}$ should converge to finite union of balls. This is indeed the case, with a single ball as the limit for $t \to \infty$, when the initial data is uniformly smooth and convex, as proved in a classical theorem of [Huisken 1987]. As geometric evolutions unavoidably produce singularities, Theorem 1 should turn out to be a fundamental ingredient in attacking such questions.

1.2. ***The Montiel–Ros argument.*** Our starting point is the beautiful proof of Alexandrov's theorem in [Montiel and Ros 1991], which we now recall. Assume that Ω is a bounded open set with smooth boundary and positive mean curvature H_Ω with respect to its outer unit normal ν_Ω. Denote by $\{\kappa_i\}_{i=1}^n$ the principal curvatures of $\partial\Omega$, indexed in increasing order so that $\kappa_n \ge H_\Omega/n > 0$, set $u(y) = \mathrm{dist}(y, \partial\Omega)$ for each $y \in \Omega$, and define

$$Z = \left\{(x,t) \in \partial\Omega \times \mathbb{R} : 0 < t \le \frac{1}{\kappa_n(x)}\right\}, \quad \zeta(x,t) = x - t\nu_\Omega(x), \quad (x,t) \in Z. \tag{1-7}$$

Let us denote by $B_\rho(x)$ the Euclidean ball in $\mathbb{R}^{n+1}$ with center at x and radius ρ. If $y \in \Omega$, then $B_{u(y)}(y)$ touches Ω from inside at a point $x \in \partial\Omega$, where $\kappa_n(x) \le 1/u(y)$, i.e., $u(y) \le 1/\kappa_n(x)$. In particular,

$$\Omega \subset \zeta(Z) \tag{1-8}$$

and by the area formula, with $J^Z\zeta$ denoting the tangential Jacobian of ζ along Z,

$$|\Omega| \le |\zeta(Z)| \le \int_{\zeta(Z)} \mathcal{H}^0(\zeta^{-1}(y))\,dy = \int_Z J^Z\zeta\,d\mathcal{H}^{n+1} = \int_{\partial\Omega} d\mathcal{H}^n_x \int_0^{1/\kappa_n(x)} \prod_{i=1}^n (1 - t\kappa_i(x))\,dt.$$

By the arithmetic-geometric mean inequality and by $\kappa_n \ge H_\Omega/n$,

$$\begin{aligned} |\Omega| &\le \int_{\partial\Omega} d\mathcal{H}^n_x \int_0^{1/\kappa_n(x)} \left(\frac{1}{n}\sum_{i=1}^n (1 - t\kappa_i(x))\right)^n dt \\ &\le \int_{\partial\Omega} d\mathcal{H}^n_x \int_0^{n/H_\Omega(x)} \left(1 - t\frac{H_\Omega(x)}{n}\right)^n dt = \frac{n}{n+1}\int_{\partial\Omega} \frac{d\mathcal{H}^n}{H_\Omega}, \end{aligned} \tag{1-9}$$

so that we have proved the *Heintze–Karcher inequality*

$$|\Omega| \le \frac{n}{n+1}\int_{\partial\Omega} \frac{d\mathcal{H}^n}{H_\Omega}. \tag{1-10}$$

If H_Ω is constantly equal to some $\lambda \in \mathbb{R}$, then, by combining the divergence theorems (1-1) and (1-2) (see (2-24) below), we find $\lambda = n\mathcal{H}^n(\partial\Omega)/(n+1)|\Omega|$. Hence equality holds throughout the argument, $\partial\Omega$ is umbilical, and thus is a sphere. In this way the Montiel–Ros argument provides a very effective proof of Alexandrov's theorem.

1.3. ***The Montiel–Ros argument revisited.*** As the Montiel–Ros argument heavily relies on the smoothness of $\partial\Omega$, it does not seem obvious how to adapt it to the case when Ω is a set with finite volume, finite perimeter and constant distributional mean curvature.

From the point of view of regularity of $\partial\Omega$, the starting point is given by the regularity theory of [Allard 1972]; see [Simon 1983; De Lellis 2008]. Up to modifying Ω on a set of volume zero, we can assume that Ω is open and that its topological boundary $\partial\Omega$ can be split into a closed subset Σ with $\mathcal{H}^n(\Sigma) = 0$, and a relatively open subset $\partial^*\Omega = \partial\Omega \setminus \Sigma$ which is locally an analytic constant mean curvature hypersurface, characterized by the property that for every $x \in \partial\Omega$

$$x \in \partial^*\Omega \quad \text{if and only if} \quad \lim_{\rho\to 0^+} \frac{\mathcal{H}^n(B_\rho(x) \cap \partial\Omega)}{\rho^n} = \omega_n,$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. It is thus natural to redefine Z by replacing $\partial\Omega$ with $\partial^*\Omega$ in (1-7); i.e.,

$$Z = \left\{(x,t) \in \partial^*\Omega \times \mathbb{R} : 0 < t \le \frac{1}{\kappa_n(x)}\right\}, \tag{1-11}$$

where it is still true that the largest principal curvature κ_n is positive along $\partial^*\Omega$.

Given this choice of Z, in order to obtain (1-8) we would need to show that, for every $y \in \Omega$, $B_{u(y)}(y)$ touches $\partial\Omega$ at a point $x \in \partial^*\Omega$. This is not obvious as we just know that $\Sigma = \partial\Omega \setminus \partial^*\Omega$ is $\mathcal{H}^n$-negligible. Actually, this is false for an arbitrary point $y \in \Omega$: this is the case when Ω is a union of two mutually tangent balls, x is a tangency point between two balls, and y is any point between x and the center of one of the balls. A cheap argument (see Lemma 3) shows that at each touching point x, $\partial\Omega$ blows up a hyperplane *with integer multiplicity possibly larger than* 1. So, near a touching point x, $\partial\Omega$ consists of finitely many sheets that are mutually tangent at x. The union of these sheets has constant mean curvature in the distributional sense defined by (1-2), although it is not immediate to extract information on the mean curvature of each separate sheet. A deep result of [Schätzle 2004] implies that the lower and upper sheets (with respect to any given direction) satisfy a measure-theoretic version of the strong maximum principle. This is crucial information, which is delicate to exploit, but fundamental to our argument.

We now describe our argument by referring to the main steps of the proof of Theorem 1, which is contained in detail in Section 3. We start by identifying a large subset $\Omega^\star$ of good points of Ω, meaning that

$$|\Omega^\star \setminus \zeta(Z)| = 0, \qquad |\Omega \setminus \Omega^\star| = 0. \tag{1-12}$$

In other words, the projection of almost every point in $\Omega^\star$ onto $\partial\Omega$ is contained in $\partial^*\Omega$, and $\Omega^\star$ is equivalent to Ω. The definition of $\Omega^\star$ is as follows. First, for every $s > 0$, we set

$$\Omega_s = \{y \in \Omega : u(y) > s\}, \qquad \partial\Omega_s = \{y \in \Omega : u(y) = s\}. \tag{1-13}$$

Clearly Ω_s satisfies an exterior ball condition of radius s at each point of $\partial\Omega_s$, but otherwise Ω_s is just a set of finite perimeter (for a.e. $s > 0$). We can also obtain an interior ball condition, restricting ourselves to the following subset. Setting $t > s > 0$, we define

$$\Gamma_s^t = \left\{y \in \partial\Omega_s : y = \left(1 - \frac{s}{t}\right)x + \frac{s}{t}z \quad \text{for some } z \in \partial\Omega_t,\ x \in \partial\Omega\right\}. \tag{1-14}$$

Notice that Γ_s^t is just a compact subset of $\partial\Omega_s$, which could be very porous inside $\partial\Omega_s$. Some technical effort (see Step 1) is put into showing that Γ_s^t can be covered by countably many $C^{1,1}$-images of $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$, and that ∇u is tangentially differentiable along Γ_s^t (with bounds on the tangential derivatives corresponding to the exterior/interior ball conditions). Once these technical aspects are settled, we are allowed to use $\mathrm{Id} - r\nabla u$ to change variables between Γ_s^t and Γ_{s-r}^t and we can prove that $|\Omega \setminus \Omega^\star| = 0$, where $\Omega^\star$ is defined by

$$\Gamma_s^+ = \bigcup_{t>s} \Gamma_s^t, \quad \Omega^\star = \bigcup_{s>0} \Gamma_s^+. \tag{1-15}$$

This is done in Step 2 of the proof.

Showing that $|\Omega^\star \setminus \zeta(Z)| = 0$, see Steps 3 and 4, is considerably more delicate. We have to exclude that the points in a given Γ_s^t that are projected into the singular set $\Sigma = \partial\Omega \setminus \partial^*\Omega$ have positive $\mathcal{H}^n$-measure; in other words, we want

$$\mathcal{H}^n((\mathrm{Id} - s\nabla u)^{-1}(\Sigma) \cap \Gamma_s^t) = 0.$$

This may seem obvious, as $\mathrm{Id} - s\nabla u$ is almost injective on Γ_s^t (see (3-43)) and it is Lipschitz on each piece of a countable decomposition of Γ_s^t (see (3-16)), while at the same time $\mathcal{H}^n(\Sigma) = 0$. However we cannot derive a straightforward contradiction from the area formula, as the tangential Jacobian of $\mathrm{Id} - s\nabla u$ along Γ_s^t may be zero $\mathcal{H}^n$-a.e. In fact, this is the information that we obtain from the area formula; namely, the least principal curvature of Γ_s^t is equal to $-1/s$ along points in $(\mathrm{Id} - s\nabla u)^{-1}(\Sigma) \cap \Gamma_s^t$. Heuristically, this curvature for Γ_s^t can only be obtained when $\partial\Omega$ has a inward corner, which is ruled out by absolute continuity of the mean curvature. Following this guiding example, we change variable to show that the least principal curvature of Γ_{s-r}^t at corresponding points is thus as negative as we wish. This indicates that $\partial\Omega_{s-r}$ has negative mean curvature on a set of positive $\mathcal{H}^n$-measure for any r close enough to s. By the almost-everywhere second-order differentiability of u, swiping r over an interval we can find a paraboloid with negative mean curvature, locally contained inside $\partial\Omega_{s-r}$. By translating this object until it touches $\partial\Omega$ (at Σ) we can apply Schätzle's maximum principle and derive a contradiction.

As pointed out to us by a referee, our argument up to this point shares some similarities with the strategy adopted by Almgren [1986] in proving the isoperimetric inequality in higher codimension. Almgren's goal in that paper is showing that an upper bound on the length of the mean curvature vector implies a lower bound on the area, which is saturated by spheres. His arguments are also based on a viscosity approach, where sliding constructions and the maximum principle are combined to infer regularity properties. The referee's insight is that Almgren's argument could be adapted to our setting by updating some technical aspects along the lines of the recent work [Santilli 2017], or, better said, of a possible generalization of that paper to the bounded mean curvature case. This approach could provide a proof of (1-12) independent of Schätzle's maximum principle.

Having proved (1-12), we are ready to argue as Montiel and Ros. We thus find, from the equality case in their argument, that

$$|\zeta(Z) \setminus \Omega| = 0, \tag{1-16}$$

$$\mathcal{H}^0(\zeta^{-1}(y)) = 1, \quad \text{for a.e. } y \in \Omega, \tag{1-17}$$

$$\kappa_i(x) = \frac{H_\Omega}{n} \quad \text{for every } x \in \partial^*\Omega,\ i = 1, \dots, n. \tag{1-18}$$

Condition (1-18) implies that $\partial^*\Omega$ is umbilical, in addition to having constant mean curvature. In particular, $\partial^*\Omega$ consists of at most countably many open pieces of spheres with same curvature. Should these pieces be finitely many, one could conclude from the distributional constant mean curvature condition, in a rather direct way, that each piece is equal to a complete sphere. But as the number of the pieces could indeed be infinite, the pieces may have smaller and smaller areas and combine themselves in particular ways to achieve constant distributional mean curvature, creating at the same time a large singular set $\partial\Omega \setminus \partial^*\Omega$. To rule out this possibility, we exploit the information contained in (1-16) and (1-17) through a geometric argument. In this last step, we make once again use of Schätzle's strong maximum principle; see in particular (3-56).

We conclude with two remarks. First, as a by-product of this analysis, we obtain a *Heintze–Karcher inequality for sets of finite perimeter* which are mean convex in a viscous sense; see Theorem 8 below. This result is actually not needed to prove Theorem 1, but it is included as it may be considered of independent interest. Second, as recently shown by Brendle [2013], the Montiel–Ros approach to Alexandrov's theorem is quite flexible, as it allows one to show that constant mean curvature implies umbilicality in many warped product manifolds of physical and geometric interest. The methods of this paper should be naturally adaptable to these more general contexts. In this direction, in a preliminary version of this manuscript [Delgadino and Maggi 2017, Section 5], we prove that Wulff shapes are the only volume-constrained *local minimizers* of smooth uniformly elliptic surface tension energies. Of course the assumption of local minimality is considerably stronger than criticality.

1.4. ***Organization of the paper.*** The paper is organized as follows. In Section 2 we gather some background material from geometric measure theory. In Section 3 we prove Theorem 1 and Corollary 2. The generalized Heintze–Karcher inequality for sets of finite perimeter is stated and proved in Section 4.

2. Background material from geometric measure theory

In this section we review some preliminaries from the theory of rectifiable sets (Section 2.1), rectifiable varifolds (Section 2.2) and sets of finite perimeter (Section 2.3). We refer to [Simon 1983; Ambrosio et al. 2000; Maggi 2012; Evans and Gariepy 1992] for detailed accounts. Finally, in Section 2.4, we discuss some basic properties of volume-constrained critical points of the perimeter functional.

2.1. ***Rectifiable sets.*** Denote by $\mathcal{H}^n$ the Hausdorff measure on $\mathbb{R}^{n+1}$. A Borel set $M \subset \mathbb{R}^{n+1}$ is a *locally $\mathcal{H}^n$-rectifiable set* if M can be covered, up to a $\mathcal{H}^n$-negligible set, by countably many Lipschitz images of $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$, and if $\mathcal{H}^n \llcorner M$ is locally finite on $\mathbb{R}^{n+1}$. We say that M is *$\mathcal{H}^n$-rectifiable* if in addition $\mathcal{H}^n(M) < \infty$, and that M is *normalized* if $M = \operatorname{spt} \mathcal{H}^n \llcorner M$, i.e.,

$$x \in M \qquad \text{if and only if} \qquad \mathcal{H}^n(B_\rho(x) \cap M) > 0 \quad \text{for all } \rho > 0.$$

Basic properties of rectifiable sets needed in the sequel are:

(i) For $\mathcal{H}^n$-a.e. $x \in M$ there exists $T_x M \in G(n, n+1)$ (the space of n-dimensional planes in $\mathbb{R}^{n+1}$) such that

$$\lim_{\rho \to 0^+} \int_{(M-x)/\rho} \varphi \, d\mathcal{H}^n = \int_{T_x M} \varphi \, d\mathcal{H}^n \quad \text{for all } \varphi \in C_c^0(\mathbb{R}^{n+1}); \tag{2-1}$$

see [Maggi 2012, Theorem 10.2]. The plane $T_x M$ is called the *approximate tangent plane to M at x*.

(ii) If M_1 and M_2 are locally $\mathcal{H}^n$-rectifiable sets, then

$$T_x M_1 = T_x M_2 \quad \mathcal{H}^n\text{-a.e. on } M_1 \cap M_2; \tag{2-2}$$

see [Maggi 2012, Proposition 10.5].

(iii) Lipschitz functions are differentiable along approximate tangent planes; that is, if $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a Lipschitz function, then, for $\mathcal{H}^n$-a.e. $x \in M$ such that $T_x M$ exists, the restriction of f to $x + T_x M$ is differentiable at x, and the limit

$$(\nabla^M f)_x[\tau] = \lim_{h \to 0^+} \frac{f(x + h\tau) - f(x)}{h} \quad \text{for all } \tau \in T_x M$$

defines the *tangential gradient* $\nabla^M f(x) = (\nabla^M f)_x$ *of* f *along* M *at* x; see [Maggi 2012, Theorem 11.4].

(iv) The tangential gradient just depends on the restriction of f to M. In other words, if $f : M \to \mathbb{R}^{n+1}$ is a Lipschitz function, and $F, G : \mathbb{R}^{n+1} \to \mathbb{R}$ are Lipschitz functions such that $F = G = f$ on M, then

$$\nabla^M F = \nabla^M G \quad \mathcal{H}^n\text{-a.e. on } M. \tag{2-3}$$

(v) Finally, given a Lipschitz function $f : M \to \mathbb{R}^{n+1}$, the *tangential Jacobian of* f *along* M is defined at $\mathcal{H}^n$-a.e. $x \in M$ by

$$J^M f(x) = \sqrt{\det(\nabla^M f(x)^* \nabla^M f(x))} = \left| \bigwedge_{i=1}^{n} (\nabla^M f)_x[\tau_i(x)] \right|$$

provided $\{\tau_i(x)\}_{i=1}^n$ is an orthonormal basis of $T_x M$, and the area formula

$$\int_{f(M)} \mathcal{H}^0(f^{-1}(y))\, d\mathcal{H}^n_y = \int_M J^M f(x)\, d\mathcal{H}^n_x \tag{2-4}$$

holds [Maggi 2012, Theorem 11.6].

For the lack of precise reference we justify property (iv). If $\psi : \mathbb{R}^n \to \mathbb{R}^{n+1}$ is a Lipschitz map and $E \subset \mathbb{R}^n$ is a Borel set, then by [Maggi 2012, Lemmas 10.4 and 11.5] we have $T_x M = (\nabla \psi)_{\psi^{-1}(x)}[\mathbb{R}^n]$ for $\mathcal{H}^n$-a.e. $x \in M \cap \psi(E)$, with

$$(\nabla^M F)_x[\tau] = \nabla(F \circ \psi)_{\psi^{-1}(x)}[(\nabla \psi)_x^{-1}[\tau]] \quad \text{for all } \tau \in T_x M. \tag{2-5}$$

Since $F = G$ on M implies $\nabla(F \circ \psi) = \nabla(G \circ \psi)$ $\mathcal{H}^n$-a.e. on $E \cap \psi^{-1}(M)$ [Maggi 2012, Lemma 7.6] we deduce (2-3) from (2-5).

2.2. *Integer rectifiable varifolds.* If M is a C^2-hypersurface without boundary in $\mathbb{R}^{n+1}$, then the mean curvature vector $\boldsymbol{H}_M \in C^0(M; \mathbb{R}^{n+1})$ of M is such that

$$\int_M \operatorname{div}^M X\, d\mathcal{H}^n = \int_M \boldsymbol{H}_M \cdot X\, d\mathcal{H}^n \quad \text{for all } X \in C^1_c(\mathbb{R}^{n+1}; \mathbb{R}^{n+1}), \tag{2-6}$$

with $\boldsymbol{H}_M(x) \cdot \tau = 0$ for every $\tau \in T_x M$. This basic fact motivates the following definitions.

Let M be a locally $\mathcal{H}^n$-rectifiable set, and consider a Borel measurable function $\theta \in L^1_{\mathrm{loc}}(\mathcal{H}^n \llcorner M; \mathbb{N})$. The *integer rectifiable varifold* $\mathrm{var}(M, \theta)$ defined by M and θ is the Radon measure on $\mathbb{R}^{n+1} \times G(n, n+1)$ defined as

$$\int_{\mathbb{R}^{n+1}\times G(n,n+1)} \Phi \, d \,\mathrm{var}(M, \theta) = \int_M \Phi(x, T_x M)\theta(x)\, d\mathcal{H}^n_x$$

for every bounded, compactly supported Borel function Φ on $\mathbb{R}^{n+1} \times G(n, n+1)$. To each $X \in C^1_c(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$ we associate the test function

$$\Phi_X(x, T) = (\mathrm{div}^T X)(x), \quad (x, T) \in \mathbb{R}^{n+1} \times G(n, n+1),$$

where $\mathrm{div}^T X$ is the divergence of X with respect to T. Motivated by (2-6), we say that $\mathrm{var}(M, \theta)$ has *distributional mean curvature vector* $\boldsymbol{H}_M \in L^1_{\mathrm{loc}}(\theta\mathcal{H}^n \llcorner M; \mathbb{R}^{n+1})$ if

$$\int_M \mathrm{div}^M X\theta \, d\mathcal{H}^n = \int_M \boldsymbol{H}_M \cdot X\theta \, d\mathcal{H}^n \quad \text{for all } X \in C^1_c(\mathbb{R}^{n+1}; \mathbb{R}^{n+1}). \tag{2-7}$$

(The dependency of $\boldsymbol{H}_M$ from θ is omitted.) When $|\boldsymbol{H}_M|$ is constant ($\mathcal{H}^n$-a.e. on M) we say that $\mathrm{var}(M, \theta)$ has *constant distributional mean curvature on* $\mathbb{R}^{n+1}$; when $\boldsymbol{H}_M = 0$ we say that $\mathrm{var}(M, \theta)$ is *stationary on* $\mathbb{R}^{n+1}$. For example, if M is a union of finitely many *possibly intersecting* spheres with same radius, then M has constant distributional mean curvature in $\mathbb{R}^{n+1}$. Similarly, a finite union of hyperplanes is stationary in $\mathbb{R}^{n+1}$.

In the proof of Theorem 1 we will exploit two forms of the maximum principle for integer rectifiable varifolds. The first one is a simple fact, well-known to experts, whose proof is included for the sake of clarity.

Lemma 3. *Let M be a normalized locally $\mathcal{H}^n$-rectifiable set such that* $\mathrm{var}(M, \theta)$ *is stationary on* $\mathbb{R}^{n+1}$. *If M is a cone* (*that is, $M = tM$ for every $t > 0$*), *and M is contained in a closed half-space H with $0 \in \partial H$, then $M = \partial H$ and θ is constant. In particular, M cannot be contained in the convex intersection of two distinct, nonopposite half-spaces containing the origin.*

Proof. Let $H = \{z \in \mathbb{R}^{n+1} : z \cdot \nu < 0\}$, where $\nu \in \mathbb{S}^n$. Given $\varphi \in C^\infty_c([0, \infty))$ with $0 \le \varphi \le 1$, $\varphi(r) = 1$ on $[0, \varepsilon)$ for some $\varepsilon > 0$, and $\varphi'(r) < 0$ on $\{0 < \varphi < 1\}$, let us set $X(x) = \varphi(|x|)\nu$ for $x \in \mathbb{R}^{n+1}$. Then $X \in C^\infty_c(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$ and $\nabla X = \varphi'(|x|)\nu \otimes \hat{x}$, where $\hat{x} = x/|x|$ if $x \neq 0$. Let $\nu_M : M \to \mathbb{S}^n$ be a Borel vector field such that $T_x M = \nu_M(x)^\perp$ for $\mathcal{H}^n$-a.e. $x \in M$. Since M is a cone, we have $\hat{x} \cdot \nu_M(x) = 0$ for $\mathcal{H}^n$-a.e. $x \in M$, and hence

$$\mathrm{div}^M X = \mathrm{div}\, X - \nu_M \cdot \nabla X[\nu_M] = \varphi'(|x|)(\nu \cdot \hat{x} - (\nu_M \cdot \nu)(\nu_M \cdot \hat{x})) = \varphi'(|x|)(\nu \cdot \hat{x}),$$

and thus, by the stationarity of M,

$$0 = \int_M \mathrm{div}^M X\theta \, d\mathcal{H}^n = \int_M \varphi'(|x|)(\nu \cdot \hat{x})\theta(x)\, d\mathcal{H}^n(x).$$

Since $M \subset H$ implies $\hat{x} \cdot \nu \le 0$ for every $x \in M$, $x \neq 0$, thanks to the arbitrariness of φ we find $\nu \cdot \hat{x} = 0$ for $\mathcal{H}^n$-a.e. $x \in M$. The lemma is proved. □

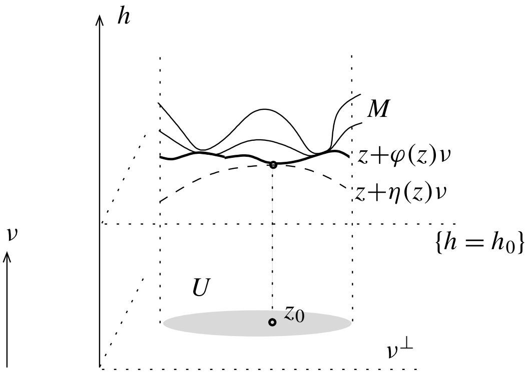

Figure 1. The strong maximum principle for integer varifolds. The rectifiable set M may consist of multiple sheets which, combined with the multiplicity function θ, have distributional mean curvature $\boldsymbol{H}_M$ in some L^p. The sheets may overlap in complicated ways along sets of positive area, so there is a nontrivial relation between the mean curvature vector $\boldsymbol{H}_M$ of the whole configuration and that of a single sheet. The function φ describes the lower sheet of M *above height* h_0 *with respect to the direction* ν and projecting over an open set $U \subset \nu^\perp$. This lower sheet is shown to satisfy a strong maximum principle. Notice that the role of h_0 is that of localizing the part of the varifold we are looking at. For example, in this picture, M could have many more points of the form $z + h\nu$ with $h < h_0$ and $z \in U$, but these points will not contribute to the definition of φ.

The second tool we shall use is a much deeper result, namely, Schätzle's strong maximum principle [2004] for integer rectifiable varifolds with sufficiently summable distributional mean curvature. The statement we adopt here is a slightly simplified version, still sufficient for our purposes, of [Schätzle 2004, Theorem 6.2].

Theorem 4. *Let M be a normalized locally $\mathcal{H}^n$-rectifiable set with distributional mean curvature vector $\boldsymbol{H}_M \in L^p(\theta\mathcal{H}^n \llcorner M; \mathbb{R}^{n+1})$ for some $p > \max\{2, n\}$.*

Pick $\nu \in \mathbb{S}^n$, $h_0 \in \mathbb{R}$, and consider a connected open set $U \subset \nu^\perp$ such that

$$\varphi(z) = \inf\{h > h_0 : z + h\nu \in M\}, \quad z \in U, \tag{2-8}$$

satisfies $\varphi(z) \in (h_0, \infty)$ for every $z \in U$.

If $\eta \in W^{2,p}(U; (h_0, \infty))$ is such that $\eta \le \varphi$ on U and $\eta(z_0) = \varphi(z_0)$ for some $z_0 \in U$, then it cannot be that

$$-\operatorname{div}\left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right)(z) \le \boldsymbol{H}_M(z+\varphi(z)\nu) \cdot \frac{-\nabla\varphi(z)+\nu}{\sqrt{1+|\nabla\varphi(z)|^2}} \tag{2-9}$$

for $\mathcal{H}^n$-a.e. $z \in U$, unless $\eta = \varphi$ on U.

The signs in (2-9) and the geometric intuition behind Theorem 4 are illustrated in Figure 1. The left-hand side is the mean curvature of the subgraph of η with respect to its outer unit normal $(-\nabla\eta+\nu)/\sqrt{1+|\nabla\eta|^2}$, and, similarly, the right-hand side is the mean curvature of the subgraph of φ with respect to its outer unit normal. So, if η touches φ from below at z_0, it cannot be that the subgraph of η is in average bent upwards at least as much as the subgraph of η, unless $\eta = \varphi$. The considerable difficulty of the theorem lies in the fact that $\boldsymbol{H}_M$ does not come into play as the mean curvature of the graph of φ, but rather as the mean curvature of a more complex structure (the integer rectifiable varifold $\mathrm{var}(M, \theta)$), of which φ only represents a sort of lower envelope localized in the cylinder $\{z + t\nu : z \in U, t > h_0\}$.

2.3. ***Sets of finite perimeter.*** A Borel set $\Omega \subset \mathbb{R}^{n+1}$ has *locally finite perimeter* if there exists an $\mathbb{R}^{n+1}$-valued Radon measure μ_Ω on $\mathbb{R}^{n+1}$ such that

$$\int_\Omega \operatorname{div} X = \int_{\mathbb{R}^{n+1}} X \cdot d\mu_\Omega \quad \text{for all } X \in C^1_c(\mathbb{R}^{n+1}; \mathbb{R}^{n+1}). \tag{2-10}$$

The *perimeter of* Ω *relative to an open set* A is defined as $P(\Omega; A) = |\mu_\Omega|(A)$, where $|\mu_\Omega|$ is the total variation of μ_Ω, and Ω has *finite perimeter* if $P(\Omega) = P(\Omega; \mathbb{R}^{n+1}) < \infty$. In this case, either Ω or its complement has finite volume. By exploiting (2-10), the support of μ_Ω is seen to satisfy

$$\operatorname{spt} \mu_\Omega = \{x \in \mathbb{R}^{n+1} : 0 < |B_\rho(x) \cap \Omega| < \omega_n \rho^n \text{ for all } \rho > 0\} \subset \partial\Omega; \tag{2-11}$$

see [Maggi 2012, Proposition 12.19]. Notice that $\operatorname{spt}\mu_\Omega$ is invariant by zero-volume modifications of Ω, while of course $\partial\Omega$ is not. The *reduced boundary* of a set of locally finite perimeter Ω is defined as the set of points such that

$$\nu_\Omega(x) = \lim_{\rho \to 0^+} \frac{\mu_\Omega(B_\rho(x))}{|\mu_\Omega|(B_\rho(x))} \quad \text{exists and belongs to } \mathbb{S}^n. \tag{2-12}$$

The Borel vector field $\nu_\Omega : \partial^*\Omega \to \mathbb{S}^n$ is called the *measure-theoretic outer unit normal to* Ω, and we always have

$$\overline{\partial^*\Omega} = \operatorname{spt} \mu_\Omega. \tag{2-13}$$

Moreover by [Maggi 2012, Theorem 15.9], the reduced boundary is locally $\mathcal{H}^n$-rectifiable, with

$$\mu_\Omega = \nu_\Omega \mathcal{H}^n \llcorner \partial^*\Omega, \qquad P(\Omega; A) = \mathcal{H}^n(A \cap \partial^*\Omega)$$

for every open set $A \subset \mathbb{R}^{n+1}$, and thus (2-10) takes the form

$$\int_\Omega \operatorname{div} X = \int_{\partial^*\Omega} X \cdot \nu_\Omega \, d\mathcal{H}^n \quad \text{for all } X \in C^1_c(\mathbb{R}^{n+1}; \mathbb{R}^{n+1}). \tag{2-14}$$

In addition, for every $x \in \partial^*\Omega$, $\nu_\Omega(x)^\perp = T_x(\partial^*\Omega)$ is the approximate tangent plane to $\partial^*\Omega$ at x and in particular we have

$$\lim_{\rho \to 0^+} \frac{\mathcal{H}^n(B_\rho(x) \cap \partial^*\Omega)}{\rho^n} = \omega_n \quad \text{for all } x \in \partial^*\Omega. \tag{2-15}$$

To every set Ω of locally finite perimeter we can always associate in a natural way an integer rectifiable varifold $\mathrm{var}(\partial^*\Omega, 1)$. If $\mathrm{var}(\partial^*\Omega, 1)$ admits a distributional mean curvature vector $\boldsymbol{H}_{\partial^*\Omega}$, then the *distributional mean curvature of* Ω is defined by setting

$$H_\Omega = \boldsymbol{H}_{\partial^*\Omega} \cdot \nu_\Omega. \tag{2-16}$$

The subscript Ω on H_Ω is a reminder that we have used the outer orientation of Ω to specify the scalar curvature. With this notation, $H_{B_r} = n/r$ for every $r > 0$.

2.4. *Basic properties of critical points.* Here we prove some properties of critical points in the isoperimetric problem which descend from generally known facts about integer varifolds and sets of finite perimeter. A set of finite perimeter and finite volume Ω is a *critical point for the isoperimetric problem* if

$$\frac{d}{dt}\bigg|_{t=0} P(f_t(\Omega)) = 0 \tag{2-17}$$

whenever $\{f_t\}_{|t|<1}$ is a one-parameter family of diffeomorphisms with $f_0 = \mathrm{Id}$, $|f_t(\Omega)| = |\Omega|$ and $\mathrm{spt}(f_t - \mathrm{Id}) \Subset \mathbb{R}^{n+1}$ for every $|t| < 1$. By [Maggi 2012, Theorem 17.20], (2-17) is equivalent to the existence of a constant $\lambda \in \mathbb{R}$ such that

$$\int_{\partial^*\Omega} \mathrm{div}^{\partial^*\Omega} X \, d\mathcal{H}^n = \lambda \int_{\partial^*\Omega} X \cdot \nu_\Omega \, d\mathcal{H}^n \quad \text{for all } X \in C^1_c(\mathbb{R}^{n+1}; \mathbb{R}^{n+1}). \tag{2-18}$$

Lemma 5. *If $\Omega \subset \mathbb{R}^{n+1}$ is a critical point for the isoperimetric problem, then Ω is (equivalent modulo sets of volume zero to) a bounded open set such that $\partial\Omega = \mathrm{spt}\,\mu_\Omega$ and $\mathcal{H}^n(\partial\Omega \setminus \partial^*\Omega) = 0$. Moreover, the constant λ in (2-18) is equal to*

$$H^0_\Omega = \frac{n P(\Omega)}{(n+1)|\Omega|}; \tag{2-19}$$

that is, $H_\Omega \equiv H^0_\Omega$. Finally,

$$\partial^*\Omega = \left\{ x \in \partial\Omega : \lim_{\rho \to 0^+} \frac{\mathcal{H}^n(B_\rho(x) \cap \partial\Omega)}{\rho^n} = \omega_n \right\}$$

is locally an analytic hypersurface with constant mean curvature, relatively open in $\partial\Omega$.

Proof. By [Simon 1983, Theorem 17.6], condition (2-18) implies that for every $x \in \mathbb{R}^{n+1}$,

$$e^{|\lambda|\rho} \frac{\mathcal{H}^n(B_\rho(x) \cap \partial^*\Omega)}{\rho^n} \quad \text{is increasing on } \rho > 0, \tag{2-20}$$

which combined with (2-15) and (2-13) gives

$$\mathcal{H}^n(B_\rho(x) \cap \partial^*\Omega) \ge \omega_n e^{-|\lambda|} \rho^n \quad \text{for all } \rho \in (0, 1),\ x \in \mathrm{spt}\,\mu_\Omega. \tag{2-21}$$

A first consequence of the lower bound (2-21) is that

$$\mathcal{H}^n(\mathrm{spt}\,\mu_\Omega \setminus \partial^*\Omega) = 0; \tag{2-22}$$

see, e.g., [Maggi 2012, Exercise 17.19]. Moreover, by combining (2-21) with $P(\Omega) < \infty$ and a covering argument, we see that $\mathrm{spt}\,\mu_\Omega$ is bounded.

Let us now consider the open set Ω_1 of those $x \in \mathbb{R}^{n+1}$ such that $|\Omega \cap B_\rho(x)| = |B_\rho(x)|$ for every ρ small enough, and the open set Ω_0 of those $x \in \mathbb{R}^{n+1}$ such that $|\Omega \cap B_\rho(x)| = 0$ for every ρ small enough, so that

$$\mathrm{spt}\,\mu_\Omega = \mathbb{R}^{n+1} \setminus (\Omega_0 \cup \Omega_1), \tag{2-23}$$

thanks to (2-11). If $\Omega^{(1)}$ denotes the set of points of density 1 of Ω, then $\Omega_1 \subset \Omega^{(1)}$, while

$$|\Omega^{(1)} \setminus \Omega_1| = |\Omega^{(1)} \cap \Omega_0| + |\Omega^{(1)} \cap \operatorname{spt} \mu_\Omega| = |\Omega^{(1)} \cap \operatorname{spt} \mu_\Omega| = 0$$

as $\mathcal{H}^n(\operatorname{spt} \mu_\Omega) < \infty$ thanks to (2-22). Thus $|\Omega^{(1)} \Delta \Omega_1| = 0$, and then $|\Omega \Delta \Omega_1| = 0$ by the Lebesgue density theorem. Since Ω_0 and Ω_1 are disjoint open sets, (2-23) implies $\partial \Omega_1 \subset \operatorname{spt} \mu_\Omega$. At the same time, $|\Omega \Delta \Omega_1| = 0$ and the inclusion in (2-11) imply $\operatorname{spt} \mu_\Omega \subset \partial \Omega_1$. Hence $\operatorname{spt} \mu_\Omega = \partial \Omega_1$, and since $\operatorname{spt} \mu_\Omega = \partial \Omega_1$ is bounded and $|\Omega_1| < \infty$, we have that Ω_1 is bounded. The first part of the statement is proved.

We show that λ in (2-18) satisfies $\lambda = H^0_\Omega$ with H^0_Ω defined in (2-19). Since Ω is bounded we can test both (2-14) and (2-18) with $X \in C^1_c(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$, where $X(x) = x$ for x in a neighborhood of Ω. Hence,

$$\begin{aligned}(n+1)|\Omega| &= \int_\Omega \operatorname{div}(x)\, dx = \int_\Omega \operatorname{div} X = \int_{\partial^*\Omega} X \cdot \nu_\Omega \, d\mathcal{H}^n = \frac{1}{\lambda} \int_{\partial^*\Omega} \operatorname{div}^{\partial^*\Omega} X \, d\mathcal{H}^n \\ &= \frac{1}{\lambda} \int_{\partial^*\Omega} \operatorname{div}^{\partial^*\Omega}(x)\, d\mathcal{H}^n_x = \frac{n P(\Omega)}{\lambda}, \end{aligned} \tag{2-24}$$

and thus $\lambda = H^0_\Omega$.

Finally, by applying Allard's regularity theorem (see [Simon 1983, Theorem 24.2] or [De Lellis 2008]) to $\operatorname{var}(\partial\Omega, 1)$, we see that $\partial\Omega$ is an analytic constant mean curvature hypersurface in a neighborhood of every $x \in \partial\Omega$ such that

$$\lim_{\rho \to 0^+} \frac{\mathcal{H}^n(B_\rho(x) \cap \partial\Omega)}{\rho^n} = \omega_n. \tag{2-25}$$

In particular, if $x \in \partial\Omega$ satisfies (2-25) then there exists $\rho > 0$ such that $B_\rho(x) \cap \Omega$ is the epigraph of an analytic function, and thus $x \in \partial^*\Omega$. Vice versa, (2-25) holds everywhere on $\partial^*\Omega$ thanks to (2-15). □

We also notice a simple consequence of Lemma 3.

Lemma 6. *If $\Omega \subset \mathbb{R}^{n+1}$ is a critical point for the isoperimetric problem, $x \in \partial\Omega$, and $y_1, y_2 \in \Omega$ are such that $|y_i - x| = \operatorname{dist}(y_i, \partial\Omega)$ and $|x - y_1| = |x - y_2|$, then $x - y_1 = y_2 - x$.*

Proof. Since $\operatorname{var}(\partial\Omega, 1)$ is an integer varifold of constant distributional mean curvature, it admits at least one blow-up limit in the weak convergence of varifolds at x, and each such limit varifold is stationary and supported on a cone M; see [Simon 1983, Chapter 46]. By construction, M is contained in the half-spaces $\{z \cdot \nu_i \le 0\}$ defined by $\nu_i = (x - y_i)/|x - y_i|$, $i = 1, 2$. If $y_1 \ne y_2$, then $\nu_1 \ne \nu_2$, and Lemma 3 implies that $\nu_1 = -\nu_2$. □

3. Critical points of the isoperimetric problem

Referring to the Introduction for the general strategy, we now present the proof of Theorem 1. At the end of the section we also prove Corollary 2.

Proof of Theorem 1. Let Ω be a set with finite perimeter and finite volume which is a critical point for the isoperimetric problem. The conclusion of Lemma 5 is the starting point of our analysis, aimed at showing that Ω is a finite union of disjoint balls of radius n/H^0_Ω. We rescale Ω so that $H^0_\Omega = n$.

Properties of the distance function: We set $u(y) = \operatorname{dist}(y, \partial\Omega)$ for $y \in \mathbb{R}^{n+1}$ so that

$$N(y) = \nabla u(y) \in \mathbb{S}^n \quad \text{exists for a.e. } y \in \Omega, \tag{3-1}$$

thanks to Rademacher's theorem. For $s > 0$ we set

$$\Omega_s = \{y \in \Omega : u(y) > s\}, \quad \partial\Omega_s = \{y \in \Omega : u(y) = s\},$$

and recall that, by the coarea formula [Maggi 2012, Theorems 13.1 and 18.1], Ω_s is a set of finite perimeter for a.e. $s > 0$, and for every Borel set $E \subset \mathbb{R}^{n+1}$,

$$|E| = \int_0^\infty \mathcal{H}^n(E \cap \partial^*\Omega_s)\, ds = \int_0^\infty \mathcal{H}^n(E \cap \partial\Omega_s)\, ds. \tag{3-2}$$

In particular,

$$\mathcal{H}^n(\partial\Omega_s \setminus \partial^*\Omega_s) = 0 \quad \text{for a.e. } s > 0. \tag{3-3}$$

We recall that for a.e. $y \in \Omega$, u admits a second-order Taylor expansion at y. Indeed, given $A \subset \Omega$ and $y \in \Omega$, denote by $\overline{\Theta}(u, A)(y)$ the infimum of the constants $c > 0$ such that for $a \in \mathbb{R}$ and $b \in \mathbb{R}^{n+1}$ we have

$$a + b \cdot z + c\frac{|z|^2}{2} \geq u(z) \quad \text{for all } z \in A,$$

with equality at y. For any $y \in \Omega$ we can pick $x \in \partial\Omega$ such that $|x - y| = u(y)$,

$$u(z) = \operatorname{dist}(z, \partial\Omega) \leq \operatorname{dist}(z, \{x\}) = |z - x| \quad \text{for all } z \in \Omega, \tag{3-4}$$

that is, $z \mapsto |z - x|$ touches u from above at y over Ω. At the same time we can construct a second-order polynomial that touches $z \mapsto |z - x|$ from above at y over $\mathbb{R}^{n+1}$. Indeed, it holds

$$|z - x| \leq |y - x| + \frac{y - x}{|y - x|} \cdot (z - y) + \frac{|z - y|^2}{2|y - x|} \quad \text{for all } z \in \mathbb{R}^{n+1}. \tag{3-5}$$

To check this set $y = x + tv$ for $t > 0$ and $|v| = 1$, and set $w = z - y$, so that (3-5) becomes

$$|tv + w| \leq t + v \cdot w + \frac{|w|^2}{2t} \quad \text{for all } w \in \mathbb{R}^{n+1}.$$

Taking squares this is equivalent to

$$\begin{aligned} t^2 + 2tv \cdot w + |w|^2 &\leq t^2 + 2tv \cdot w + |w|^2 + (v \cdot w)^2 + \frac{(v \cdot w)|w|^2}{t} + \frac{|w|^4}{4t^2} \\ &= t^2 + 2tv \cdot w + |w|^2 + \left(v \cdot w + \frac{|w|^2}{2t}\right)^2, \end{aligned}$$

which clearly holds for every $w \in \mathbb{R}^{n+1}$. Thanks to (3-5) there exists $a, b \in \mathbb{R}$ such that

$$|z - x| \leq a + b \cdot z + \frac{|z|^2}{2|y - x|} \quad \text{for all } z \in \mathbb{R}^{n+1},$$

with equality if $z = y$, so that, by the definition of $\overline{\Theta}$ and by (3-4)

$$\overline{\Theta}(u,\Omega)(y) \le \frac{1}{u(y)} \quad \text{for all } y \in \Omega. \tag{3-6}$$

Arguing as in [Caffarelli and Cabré 1995, Proposition 1.6], we see that u is twice differentiable a.e. in Ω.

Preliminary properties of the sets Γ_s^t: For every $t > s > 0$, we consider the compact set

$$\Gamma_s^t = \left\{ y \in \partial\Omega_s : y = \left(1 - \frac{s}{t}\right)x + \frac{s}{t}z \text{ for some } z \in \partial\Omega_t,\ x \in \partial\Omega \right\}. \tag{3-7}$$

By definition, if $y \in \Gamma_s^t$, then there exist $x \in \partial\Omega$ and $z \in \partial\Omega_t$ such that

$$B_{t-s}(z) \subset \Omega_s \subset \mathbb{R}^{n+1} \setminus B_s(x), \qquad \{y\} = \partial B_{t-s}(z) \cap \partial B_s(x). \tag{3-8}$$

In particular x and z are uniquely determined by the uniqueness of limits in L^1_{loc}. Indeed, when $\rho \to 0^+$,

$$\frac{\Omega_s - y}{\rho} \to [x - z]^- \quad \text{as characteristic functions in } L^1_{\text{loc}}(\mathbb{R}^{n+1}), \tag{3-9}$$

where $[v]^-$ denotes the negative half-space defined by, $v \neq 0$,

$$[v]^- = \{w \in \mathbb{R}^{n+1} : w \cdot v < 0\}.$$

Notice also that $\text{Lip}(u; \mathbb{R}^{n+1}) \le 1$ and the inclusion $B_{s+\varepsilon}(y - \varepsilon(x-z)/|x-z|) \subset \Omega$ (which holds for $\varepsilon > 0$ small since $t > s$) imply that y has a unique projection onto $\partial\Omega$. This shows that u is differentiable at $y \in \Gamma_s^t$ with

$$N(y) = -\frac{x-z}{|x-z|} \quad \text{for all } y = \left(1 - \frac{s}{t}\right)x + \frac{s}{t}z \in \Gamma_s^t. \tag{3-10}$$

In turn, (3-10) gives

$$y + rN(y) \in \partial\Omega_{s-r} \quad \text{for all } r \in [-s, t-s],\ y \in \Gamma_s^t. \tag{3-11}$$

By (3-11), if $y, y' \in \Gamma_s^t$ then

$$s^2 \le |y - sN(y) - y'|^2 = s^2 - 2sN(y)\cdot(y-y') + |y-y'|^2,$$
$$(t-s)^2 \le |y + (t-s)N(y) - y'|^2 = (t-s)^2 + 2(t-s)N(y)\cdot(y-y') + |y-y'|^2;$$

that is

$$|N(y)\cdot(y-y')| \le \max\left\{\frac{1}{s}, \frac{1}{t-s}\right\} \frac{|y-y'|^2}{2} \quad \text{for all } y, y' \in \Gamma_s^t. \tag{3-12}$$

Using (3-10) we easily see that N is continuous on Γ_s^t so that $(u, N) \in C^0(\Gamma_s^t; \mathbb{R} \times \mathbb{R}^{n+1})$ and satisfies (3-12). By Whitney's extension theorem, there exists $\phi \in C^1(\mathbb{R}^{n+1})$ such that $(\phi, \nabla\phi) = (u, N)$ on Γ_s^t. In particular, this implies the $\mathcal{H}^n$-rectifiability of Γ_s^t.

Decomposition of Ω *and covering by* $\zeta(Z)$: We define

$$\Gamma_s^+ = \bigcup_{t>s} \Gamma_s^t, \quad \Omega^\star = \bigcup_{s>0} \Gamma_s^+ \subset \Omega, \quad Z = \left\{ (x,t) \in \partial^*\Omega \times \mathbb{R} : 0 < t \le \frac{1}{\kappa_n(x)} \right\}, \tag{3-13}$$

and set $\zeta(x,t) = x - t\nu_\Omega(x)$. We claim that

$$|\Omega \setminus \Omega^\star| = 0, \qquad |\Omega^\star \setminus \zeta(Z)| = 0. \tag{3-14}$$

We divide the proof of (3-14) into four steps.

Step 1: We prove that N is tangentially differentiable along Γ_s^t at $\mathcal{H}^n$-a.e. $y \in \Gamma_s^t$, with

$$\begin{cases} \nabla^{\Gamma_s^t} N(y) = -\sum_{i=1}^n (\kappa_s^t)_i(y)\tau_i(y) \otimes \tau_i(y), \\ -\dfrac{1}{s} \le (\kappa_s^t)_i(y) \le (\kappa_s^t)_{i+1}(y) \le \dfrac{1}{t-s}, \end{cases} \tag{3-15}$$

where $\{\tau_i(y)\}_{i=1}^n$ is an orthonormal basis of $T_y\Gamma_s^t$. To this end, we first prove that Γ_s^t can be covered by compact sets $\{\mathcal{U}_j\}_{j\in\mathbb{N}}$ in such a way that the restriction of N to $\mathcal{U}_j$ is a Lipschitz map, that is,

$$|N(y_1) - N(y_2)| \le C_j|y_1 - y_2| \quad \text{for all } y_1, y_2 \in \mathcal{U}_j. \tag{3-16}$$

(In passing we notice that (3-16) implies the $C^{1,1}$-rectifiability of Γ_s^+, that is to say, the possibility of covering Γ_s^+ by graphs of $C^{1,1}$ functions from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$.)

We start by defining the sets $\mathcal{U}_j$. Let us denote by

$$\boldsymbol{C}(N,\rho) = \{z + hN : z \in N^\perp,\ |z| < \rho,\ |h| < \rho\}$$

the open cylinder centered at the origin with axis along $N \in \mathbb{S}^n$, radius $\rho > 0$, and height 2ρ. Notice that, by the interior/exterior ball condition, Γ_s^t admits an approximate tangent plane at $\mathcal{H}^n$-a.e. of its points, and this plane is then necessarily equal to $N(y)^\perp$; that is,

$$T_y\Gamma_s^t = N(y)^\perp \quad \text{for } \mathcal{H}^n\text{-a.e. } y \in \Gamma_s^t.$$

In particular (2-1) implies

$$\lim_{\rho\to 0^+} \frac{\mathcal{H}^n\big(\Gamma_s^t \cap (y + \boldsymbol{C}(N(y),\rho))\big)}{\rho^n} = \omega_n \quad \text{for } \mathcal{H}^n\text{-a.e. } y \in \Gamma_s^t.$$

By Egoroff's theorem, we can find compact sets $\mathcal{U}_j$ covering Γ_s^t such that

$$\mu_j^*(\rho) = \sup_{y\in\mathcal{U}_j}\left|1 - \frac{\mathcal{H}^n\big(\Gamma_s^t \cap (y + \boldsymbol{C}(N(y),\rho))\big)}{\omega_n\rho^n}\right| \to 0 \quad \text{as } \rho \to 0^+. \tag{3-17}$$

Consider the function ϕ constructed in proving the $\mathcal{H}^n$-rectifiability of Γ_s^t. Since $\nabla\phi(y) = N(y) \ne 0$ at each $y \in \Gamma_s^t$, we can apply the implicit function theorem at y and find that Γ_s^t is a C^1-graph over a disk of radius ρ_y in a neighborhood of y. We can thus pick any sequence $\rho_j \to 0^+$, and up to further subdivision of $\mathcal{U}_j$ and relabeling the resulting pieces, we can assume that each $\mathcal{U}_j$ has the following property: for each $y \in \mathcal{U}_j$ there exists

$$\psi_j \in C^1(N(y)^\perp), \quad \psi_j(0) = 0, \quad \nabla\psi_j(0) = 0, \quad \|\nabla\psi_j\|_{C^0(N(y)^\perp)} \le 1 \tag{3-18}$$

such that, if

$$\mathcal{U}_j' = \text{projection of } \mathcal{U}_j \text{ on } N(y)^\perp \cap \{|z| < \rho_j\}, \tag{3-19}$$

then

$$\mathcal{U}_j \cap (y + \boldsymbol{C}(N(y), \rho_j)) = \Gamma_s^t \cap (y + \boldsymbol{C}(N(y), \rho_j)) = y + \{z + \psi_j(z)N(y) : z \in \mathcal{U}_j'\}. \tag{3-20}$$

(Notice that both ψ_j and $\mathcal{U}_j'$ depend on the point $y \in \mathcal{U}_j$ at which we are considering the "graphicality" property of $\mathcal{U}_j$, but that this dependency is not stressed to simplify the notation.) If we set

$$\mu_j(\rho) = \max\Big\{\mu_j^*(\rho), \max_{|z|\le\rho} |\nabla\psi_j(z)|\Big\}, \qquad \rho \in (0, \rho_j], \tag{3-21}$$

then $\mu_j(\rho) \to 0$ as $\rho \to 0^+$ by (3-17) and continuity of $\nabla\psi_j$. This completes the definition of the sets $\mathcal{U}_j$.

We now prove (3-16). Fix $y_1, y_2 \in \mathcal{U}_j$. Let ρ_j and ψ_j be the functions associated to $\mathcal{U}_j$ and $y_2 \in \mathcal{U}_j$ as we have just described. For $r_j < \rho_j/3$ to be chosen, we can directly assume that

$$y_1 \in y_2 + \boldsymbol{C}(N(y_2), r_j) \tag{3-22}$$

for otherwise $|y_1 - y_2| \ge c(n)r_j$ and, trivially, $|N(y_1) - N(y_2)| \le 2 \le C_j|y_1 - y_2|$. Next we assume, as we can do without loss of generality up to a rigid motion, that

$$y_2 = (0, 0) \in \mathbb{R}^n \times \mathbb{R}, \qquad N(y_2) = (0, 1) \in \mathbb{R}^n \times \mathbb{R}, \qquad N(y_2)^\perp = \mathbb{R}^n.$$

In this way (3-20) takes the form

$$\{(z, h) \in \Gamma_s^t : |z| < \rho_j, |h| < \rho_j\} = \{(z, \psi_j(z)) : z \in \mathcal{U}_j'\}, \tag{3-23}$$

with

$$\psi_j \in C^1(\mathbb{R}^n), \qquad \psi_j(0) = 0, \qquad \nabla\psi_j(0) = 0, \qquad \|\nabla\psi_j\|_{C^0(\mathbb{R}^n)} \le 1. \tag{3-24}$$

By (3-22), $y_1 = (z_1, \psi_j(z_1))$ for some $z_1 \in \mathcal{U}_j'$ with $|z_1| < r_j$. By continuity of N along Γ_s^t and since $N(0) = (0, 1)$, we find

$$N(y_1) = \frac{(-\nabla\psi_j(z_1), 1)}{\sqrt{1 + |\nabla\psi_j(z_1)|^2}}.$$

In particular,

$$\frac{|N(y_1) - N(y_2)|^2}{2} = 1 - \frac{1}{\sqrt{1 + |\nabla\psi_j(z_1)|^2}} \le \frac{|\nabla\psi_j(z_1)|^2}{2},$$

while at the same time $|y_1 - y_2|^2 = |z_1|^2 + \psi_j(z_1)^2 \ge |z_1|^2$. We are thus left to show

$$|\nabla\psi_j(z_1)| \le C_j|z_1|. \tag{3-25}$$

To this end we would like to exploit (3-12) with $y = y_1$ and $y' = y_0$ where $y_0 = (z_0, h_0)$ is defined, in terms of a suitable $e_0 \in \mathbb{S}^n$ (see (3-30) below), as

$$z_0 = z_1 - |z_1|e_0, \qquad h_0 = \psi_j(z_0). \tag{3-26}$$

Since Γ_s^t may be very "porous", that is, its projection over $\{|z| < \rho_j\}$ could have lots of holes, it is not generally true that $y_0 \in \Gamma_s^t$ and thus that $y' = y_0$ is an admissible choice in (3-12). But when this is the case, by (3-12)

$$C|y_1-y_0|^2 \geq N(y_1)\cdot(y_1-y_0) = |z_1|\frac{\nabla\psi_j(z_1)\cdot(-e_0)}{\sqrt{1+|\nabla\psi_j(z_1)|^2}} + \frac{\psi_j(z_1)-\psi_j(z_0)}{\sqrt{1+|\nabla\psi_j(z_1)|^2}}. \tag{3-27}$$

Now, in order to exploit (3-27), we notice that

$$|\psi_j(z)| \leq C|z|^2 \quad \text{for all } |z|<\rho_j \text{ such that } (z,\psi_j(z))\in\Gamma_s^t, \tag{3-28}$$

which is an immediate consequence of the fact that, around $(0,0)=(0,\psi_j(0))$, Γ_s^t is trapped between two tangent balls (notice that we do not know this about the graph of ψ_j, and so we can apply (3-28) only to the points of this graph that lie in Γ_s^t). Since $|z_0|\leq 2|z_1|<2r_j<\rho_j$, still *assuming* that $y_0=(z_0,h_0)\in\Gamma_s^t$, by (3-28) we find that

$$|y_1-y_0|^2 = |z_1|^2+(\psi_j(z_1)-\psi_j(z_0))^2 \leq C|z_1|^2,$$
$$\left|\frac{\psi_j(z_1)-\psi_j(z_0)}{\sqrt{1+|\nabla\psi_j(z_1)|^2}}\right| \leq |\psi_j(z_1)|+|\psi_j(z_0)| \leq C|z_1|^2,$$

and thus (3-27) takes the form

$$C|z_1|^2 \geq |z_1|\frac{\nabla\psi_j(z_1)\cdot(-e_0)}{\sqrt{1+|\nabla\psi_j(z_1)|^2}}. \tag{3-29}$$

Our choice of e_0 is thus clear; we want

$$e_0 = -\frac{\nabla\psi_j(z_1)}{|\nabla\psi_j(z_1)|} \tag{3-30}$$

to have a chance of proving (3-25).

We are now ready to prove (3-25). Set $y_0=(z_0,h_0)$ for e_0 as in (3-30) and z_0 and h_0 as in (3-26). If $z_0\in\mathcal{U}_j'$, and thus $y_0\in\Gamma_s^t$, then, as explained, we are done. Otherwise, let ε_0 be the largest $\varepsilon>0$ such that

$$\{|z-z_0|<\varepsilon\}\cap\mathcal{U}_j' = \varnothing.$$

Since $z_1\in\mathcal{U}_j'$ and $|z_0-z_1|=|z_1|$, we have $\varepsilon_0\leq|z_1|$. In particular, since $|z_0|\leq 2|z_1|$, the ball $\{|z-z_0|<\varepsilon_0\}$ is contained in $\{|z|<3|z_1|\}\subset\{|z|<\rho_j\}$ thanks to $3r_j<\rho_j$. By the definition of ε_0, there exists $z_*\in\mathcal{U}_j'$ with $|z_*-z_0|=\varepsilon_0$ and

$$\begin{aligned}\omega_n|z_0-z_*|^n &= \mathcal{H}^n(\{|z-z_0|<\varepsilon_0\}) \\ &\leq \mathcal{H}^n(\{|z|<3|z_1|\}\setminus\mathcal{U}_j') = \omega_n(3|z_1|)^n - \mathcal{H}^n(\mathcal{U}_j'\cap\{|z|<3|z_1|\}).\end{aligned} \tag{3-31}$$

On the one hand, since $\mathcal{U}_j$ is the graph of the Lipschitz function ψ_j over $\mathcal{U}_j'$,

$$\begin{aligned}\mathcal{H}^n(\mathcal{U}_j'\cap\{|z|<3|z_1|\}) &\leq \int_{\mathcal{U}_j'\cap\{|z|<3|z_1|\}}\sqrt{1+|\nabla\psi_j|^2} = \mathcal{H}^n(\mathcal{U}_j\cap\boldsymbol{C}(N(y_2),3|z_1|)) \\ &= \mathcal{H}^n(\Gamma_s^t\cap\boldsymbol{C}(N(y_2),3|z_1|)) \\ &\leq \omega_n(3|z_1|)^n(1+\mu_j(3|z_1|))\end{aligned}$$

thanks to (3-21); on the other hand, again by the definition (3-21) of μ_j,

$$\begin{aligned}\mathcal{H}^n(\mathcal{U}_j'\cap\{|z|<3|z_1|\}) &= \int_{\mathcal{U}_j'\cap\{|z|<3|z_1|\}}\frac{\sqrt{1+|\nabla\psi_j|^2}}{\sqrt{1+|\nabla\psi_j|^2}}\\ &\geq \frac{\mathcal{H}^n(\Gamma_s^t\cap \boldsymbol{C}(N(y_2),3|z_1|))}{\sqrt{1+\mu_h(3|z_1|)^2}} \geq \frac{1-\mu_j(3|z_1|)}{\sqrt{1+\mu_h(3|z_1|)^2}}\omega_n(3|z_1|)^n.\end{aligned}$$

Combining the last two estimates into (3-31) we find

$$\omega_n|z_0-z_*|^n \leq C\mu_j(3|z_1|)\omega_n(3|z_1|)^n;$$

that is,

$$|z_0-z_*|\leq C\mu_j(3|z_1|)^{1/n}|z_1|. \tag{3-32}$$

In other words, after scaling out $|z_1|$, the best point we can use, z_*, is as close as we want to the point we would like to use, z_0. We conclude the argument setting $y_*=(z_*,\psi_j(z_*))$. Since $z_*\in\mathcal{U}_j'$, we have $y_*\in\Gamma_s^t$. We can apply (3-12) with $y=y_1=(z_1,\psi_j(z_1))$ and $y'=y_*$ to find

$$\begin{aligned}C|y_1-y_*|^2 &\geq N(y_1)\cdot(y_1-y_*)\\ &\geq \frac{(-\nabla\psi_j(z_1))\cdot(z_1-z_*)}{\sqrt{1+|\nabla\psi_j(z_1)|^2}}+\frac{\psi_j(z_1)-\psi_j(z_*)}{\sqrt{1+|\nabla\psi_j(z_1)|^2}}\\ &\geq \frac{(-\nabla\psi_j(z_1))\cdot(z_1-z_*)}{\sqrt{1+|\nabla\psi_j(z_1)|^2}}-C(|z_1|^2+|z_*|^2)\\ &\geq |\nabla\psi_j(z_1)|(1-C\mu_j(3|z_1|)^{1/n})\frac{|z_1|}{C}-C(|z_1|^2+|z_*|^2),\end{aligned} \tag{3-33}$$

where we have first applied (3-28) to z_1 and z_*, and then have decomposed z_1-z_* as the sum of $z_1-z_0=e_0|z_1|$ and of z_0-z_*, have recalled the definition of e_0, and have used (3-32). Similarly,

$$\begin{aligned}|y_1-y_*| &\leq |z_1-z_*|+|\psi_j(z_1)-\psi_j(z_*)|\\ &\leq |z_1-z_0|+|z_0-z_*|+C(|z_1|^2+|z_*|^2)\leq C|z_1|,\end{aligned}$$

and thus (3-33) implies (3-25). This concludes the proof of (3-16). We now prove (3-15).

As noticed in Section 2.2, since N is a Lipschitz function on each $\mathcal{U}_j$, and since the $\mathcal{U}_j$ are covering Γ_s^t, we deduce that N is tangentially differentiable along Γ_s^t, and that its tangential gradient along Γ_s^t can be computed by looking at any Lipschitz extension of N to $\mathbb{R}^{n+1}$. Moreover, by (2-2), it is enough to work with $\mathcal{U}_j$ in place of Γ_s^t.

To construct a convenient extension of N we go back to the proof of the $\mathcal{H}^n$-rectifiability of Γ_s^t, and this time we construct $\phi\in C^{1,1}(\mathbb{R}^{n+1})$ such that $(u,N)=(\phi,\nabla\phi)$ on $\mathcal{U}_j$ by taking (3-12) and (3-16) into account. Then we can go back to the construction of the sets $\mathcal{U}_j$, and apply the $C^{1,1}$-implicit function theorem to deduce that for each $y\in\mathcal{U}_j$ there exists

$$\psi_j\in C^{1,1}(N(y)^\perp),$$

satisfying (3-18) and (3-20). In particular, we can consider the Lipschitz extension N_* of N from $\mathcal{U}_j \cap (y + \boldsymbol{C}(N(y), \rho_j))$ to $y + \boldsymbol{C}(N(y), \rho_j)$ given by

$$N_*(y+z+hN(y)) = \frac{-\nabla\psi_j(z) + N(y)}{\sqrt{1+|\nabla\psi_j(z)|^2}} \quad \text{for all } z \in N(y)^\perp,\ |z| < \rho_j,\ |h| < \rho_j.$$

Setting $\Psi_j(z) = y + z + \psi_j(z)N(y)$ for $|z| < \rho_j$, by (2-5) we have that for $\mathcal{H}^n$-a.e. $y' \in \mathcal{U}_j$,

$$(\nabla^{\mathcal{U}_j} N)_{y'}[\tau] = \nabla(N_* \circ \Psi_j)_{\Psi_j^{-1}(y')}[e],$$

where $\tau \in T_{y'}\mathcal{U}_j$ and $e = (\nabla\Psi_j)_{\Psi_j^{-1}(y')}[\tau] \in \mathbb{R}^n$. When $\psi_j \in C^2(N(y)^\perp)$, a classical computation shows that

$$\nabla(N_* \circ \Psi_j)_z[e] = A_j(\Psi_j(z))[\tau],$$

where A_j denotes the second fundamental form to the graph of ψ_j, which is symmetric thanks to the commutativity property of the second derivatives of ψ_j, and where the eigenvalues of A_j are bounded from below by $-1/s$ and from above by $1/(t-s)$ thanks to $\mathcal{U}_j \subset \Gamma_s^t$. In our case the same computations hold for a.e. $|z| < \rho_j$ by the chain rule for Lipschitz functions, where the symmetry of A_j is guaranteed by the fact that $\nabla^2\psi_j$ is both a distributional gradient and an a.e. classical differential of $\nabla\psi_j$. Finally, the a.e.-pointwise estimates on the eigenvalues are deduced a.e. on $\mathcal{U}_j'$ thanks to the fact that $\nabla^2\psi_j$ is an a.e. classical differential. This proves (3-15).

Step 2: We claim that for every $t > s > 0$ we have

$$\mathcal{H}^n(\partial\Omega_t) \le (t/s)^n \mathcal{H}^n(\Gamma_s^t), \tag{3-34}$$

and then use (3-34) to prove

$$|\Omega\Delta\Omega^\star| = 0. \tag{3-35}$$

Indeed, for $r \in [-s, t-s]$ let us consider the map

$$f_r : \Gamma_s^t \to \partial\Omega_{s+r}, \qquad f_r(y) = y + rN(y), \qquad y \in \Gamma_s^t. \tag{3-36}$$

The fact that $f_r(y) \in \partial\Omega_{s+r}$ is immediate as every $y \in \Gamma_s^t$ has the form $y = (1-(s/t))x + (s/t)z$ for $x \in \partial\Omega$, $z \in \partial\Omega_t$. Notice that, again by the definition of Γ_s^t, the map f_{t-s} is surjective; that is, $\partial\Omega_t = f_{t-s}(\Gamma_s^t)$. Thus

$$\mathcal{H}^n(\partial\Omega_t) = \mathcal{H}^n(f_{t-s}(\Gamma_s^t)) \le \int_{f_{t-s}(\Gamma_s^t)} \mathcal{H}^0(f_{t-s}^{-1}(z))\, d\mathcal{H}^n_z = \int_{\Gamma_s^t} J^{\Gamma_s^t} f_{t-s}\, d\mathcal{H}^n,$$

where by (3-15), and in particular by the lower bound on $(\kappa_s^t)_i$,

$$J^{\Gamma_s^t} f_{t-s} = \prod_{i=1}^n (1-(t-s)(\kappa_s^t)_i) \le \Big(1 + \frac{t-s}{s}\Big)^n \quad \mathcal{H}^n\text{-a.e. on } \Gamma_s^t.$$

This proves (3-34). To prove (3-35), we first apply the coarea formula (3-2) to find

$$|\Omega\Delta\Omega^\star| = \int_0^\infty \mathcal{H}^n((\Omega\Delta\Omega^\star) \cap \partial\Omega_s)\, ds = \int_0^\infty \mathcal{H}^n(\partial\Omega_s \setminus \Gamma_s^+)\, ds, \tag{3-37}$$

where $\Gamma_s^+ \subset \partial\Omega_s$. Again by the coarea formula, for a.e. $s > 0$,

$$\mathcal{H}^n(\partial\Omega_s) = \lim_{\varepsilon\to 0} \frac{|\Omega_s| - |\Omega_{s+\varepsilon}|}{\varepsilon} = \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\int_0^\varepsilon \mathcal{H}^n(\partial\Omega_{s+r})\,dr.$$

where by (3-34)

$$\frac{1}{\varepsilon}\int_0^\varepsilon \mathcal{H}^n(\partial\Omega_{s+r})\,dr \le \frac{1}{\varepsilon}\int_0^\varepsilon \Big(1+\frac{r}{s}\Big)^n \mathcal{H}^n(\Gamma_s^{s+r})\,dr \le \Big(1+\frac{\varepsilon}{s}\Big)^n \mathcal{H}^n(\Gamma_s^+).$$

Since $\Gamma_s^+ \subset \partial\Omega_s$, this proves

$$\mathcal{H}^n(\Gamma_s^+) = \mathcal{H}^n(\partial\Omega_s) \quad \text{for a.e. } s > 0, \tag{3-38}$$

which, combined with (3-37) gives in turn (3-35).

Step 3: For $r \in (0, s)$, let us consider the map

$$g_r : \Gamma_s^+ \to \Gamma_{s-r}^+, \quad g_r(y) = y - rN(y), \quad y \in \Gamma_s^+,$$

which is (clearly) a bijection between Γ_s^t and Γ_{s-r}^t for each $t > 0$. We claim that if y is a point of tangential differentiability of N along Γ_s^t, then $g_r(y)$ is a point of tangential differentiability of N along Γ_{s-r}^t, and

$$(\kappa_{s-r}^t)_i(g_r(y)) = \frac{(\kappa_s^t)_i(y)}{1 + r(\kappa_s^t)_i(y)} \quad \text{for all } i = 1, \dots, n. \tag{3-39}$$

Indeed, it is easily seen that

$$N(y) = N(g_r(y)) = N(y - rN(y)) \quad \text{for all } y \in \Gamma_s^t, \tag{3-40}$$

so that if y is a point of tangential differentiability of N along Γ_s^t and $\tau \in T_y\Gamma_s^t$, then $\tau \in T_{g_r(y)}\Gamma_s^t$ and

$$(\nabla^{\Gamma_s^t} N)_y[\tau] = (\nabla^{\Gamma_{s-r}^t} N)_{g_r(y)}[\tau - r(\nabla^{\Gamma_s^t} N)_y[\tau]].$$

Plugging in $\tau = \tau_i(y)$ as in (3-15) we find

$$-(\kappa_s^t)_i(y)\tau_i(y) = (1 + r(\kappa_s^t)_i(y))(\nabla^{\Gamma_{s-r}^t} N)_{g_r(y)}[\tau_i(y)];$$

that is,

$$-\tau_i(y)\cdot(\nabla^{\Gamma_{s-r}^t} N)_{g_r(y)}[\tau_i(y)] = \frac{(\kappa_s^t)_i(y)}{1 + r(\kappa_s^t)_i(y)}.$$

Thus $\{\tau_i(y)\}_{i=1}^n$ is an orthonormal basis for $T_{g_r(y)}\Gamma_{s-r}^t = T_y\Gamma_s^t$ made up of eigenvalues of $\nabla^{\Gamma_{s-r}^t} N(g_r(y))$, and the last formula is just (3-39).

Step 4: We prove that

$$|\Omega^\star \setminus \zeta(Z)| = 0. \tag{3-41}$$

By the coarea formula (3-2) and by (3-38)

$$\begin{aligned}|\Omega^\star \setminus \zeta(Z)| &= \int_0^\infty \mathcal{H}^n((\Omega^\star \setminus \zeta(Z)) \cap \partial\Omega_s)\,ds = \int_0^\infty \mathcal{H}^n((\Omega^\star \setminus \zeta(Z)) \cap \Gamma_s^+)\,ds \\ &= \int_0^\infty \mathcal{H}^n(\Gamma_s^+ \setminus \zeta(Z))\,ds.\end{aligned}$$

Since $x \in \partial^*\Omega$ and $y \in \Gamma_s^+$ are such that $y = x - s\nu_\Omega(x)$ if and only if $x = y - sN(y) = g_s(y)$, with g_s as in Step 3, we have

$$\zeta(Z) \cap \Gamma_s^+ = g_s^{-1}(\partial^*\Omega) \quad \text{for all } s > 0.$$

Taking into account that $\partial\Omega \setminus \partial^*\Omega = \Sigma$ (recall Lemma 5) and that $g_s^{-1}(\partial\Omega) \subset \Gamma_s^+$, in order to prove (3-41) we are left to show that for a.e. $s > 0$

$$\mathcal{H}^n(g_s^{-1}(\Sigma)) = 0. \tag{3-42}$$

In other words, the points in Γ_s^+ that, projected over $\partial\Omega$, end up on the singular set, have negligible $\mathcal{H}^n$-measure. We are actually going to show that (3-42) holds for every $s > 0$ such that $\mathcal{H}^n(\Gamma_s^+) = \mathcal{H}^n(\partial\Omega_s)$. We shall argue by contradiction, assuming that $\mathcal{H}^n(\Gamma_s^+) = \mathcal{H}^n(\partial\Omega_s)$ and

$$\mathcal{H}^n(g_s^{-1}(\Sigma)) > 0.$$

In particular, there exists $t > s$, such that $\mathcal{H}^n(\Gamma_s^t \cap g_s^{-1}(\Sigma)) > 0$.

As a preliminary step to derive a contradiction we first notice that

$$\mathcal{H}^0(g_s^{-1}(x)) \le 2 \quad \text{for all } x \in \partial\Omega. \tag{3-43}$$

Otherwise, $g_s^{-1}(x)$ would contain at least two points y_1, y_2 such that $(x-y_1)/|x-y_1|$ and $(x-y_2)/|x-y_2|$ are not antipodal. Any blow-up of $\operatorname{var}(\partial\Omega, x)$ would then be a stationary varifold contained in the intersection of two nonopposite half-spaces, a contradiction to Lemma 3. By (3-43) and by $\mathcal{H}^n(\Sigma) = 0$ (recall (3-3)) we find that

$$0 = 2\mathcal{H}^n(\Sigma) \ge \int_\Sigma \mathcal{H}^0(g_s^{-1}(x))\, d\mathcal{H}^n = \int_{g_s^{-1}(\Sigma)} J^{\Gamma_s^t} g_s \, d\mathcal{H}^n,$$

where

$$J^{\Gamma_s^t} g_s = \prod_{i=1}^n (1 + s(\kappa_s^t)_i) \ge 0 \quad \text{on } \Gamma_s^t$$

thanks to $-1/s \le (\kappa_s^t)_i$; see (3-15). Having assumed $\mathcal{H}^n(g_s^{-1}(\Sigma)) > 0$, and since $\{(\kappa_s^t)_i\}_i$ are ordered increasingly on i, we deduce in particular that

$$\mathcal{H}^n\Bigl(\Bigl\{y \in \Gamma_s^t : (\kappa_s^t)_1(y) = -\frac{1}{s}\Bigr\}\Bigr) \ge \mathcal{H}^n(\Gamma_s^t \cap g_s^{-1}(\Sigma)) > 0. \tag{3-44}$$

By (3-39) we see that

$$\Bigl\{\tilde{y} \in \Gamma_{s-r}^t : (\kappa_{s-r}^t)_1(\tilde{y}) = -\frac{1}{s-r}\Bigr\} = g_r\Bigl(\Bigl\{y \in \Gamma_s^t : (\kappa_s^t)_1(y) = -\frac{1}{s}\Bigr\}\Bigr).$$

Since $g_r : \Gamma_s^t \to \Gamma_{s-r}^t$ is injective, by the area formula

$$\mathcal{H}^n\Bigl(\Bigl\{\tilde{y} \in \Gamma_{s-r}^t : (\kappa_{s-r}^t)_1(\tilde{y}) = -\frac{1}{s-r}\Bigr\}\Bigr) = \int_{\{y \in \Gamma_s^t : (\kappa_s^t)_1(y) = -1/s\}} J^{\Gamma_s^t} g_r \, d\mathcal{H}^n.$$

Using again that $(\kappa_s^t)_i \geq -1/s$ on Γ_s^t, we have

$$J^{\Gamma_s^t} g_r = \prod_{i=1}^n (1 + r(\kappa_s^t)_i) \geq \left(1 - \frac{r}{s}\right)^n > 0 \quad \text{for all } r \in (0, s),$$

so that (3-44) implies that for every $r \in (0, s)$

$$\mathcal{H}^n(\Lambda_{s-r}^t) > 0 \quad \text{for } \Lambda_{s-r}^t = \left\{\tilde{y} \in \Gamma_{s-r}^t : (\kappa_{s-r}^t)_1(\tilde{y}) = -\frac{1}{s-r}\right\}. \tag{3-45}$$

By using (3-39) and the fact that $a \mapsto a/(1+ra)$ is increasing on $a \geq 0$, we see that for every $\tilde{y} \in \Lambda_{s-r}^t$, $\tilde{y} = g_r(y)$, we have

$$\sum_{i=1}^n (\kappa_{s-r}^t)_i(\tilde{y}) = -\frac{1}{s-r} + \sum_{i=2}^n \frac{(\kappa_s^t)_i(y)}{1 + r(\kappa_s^t)_i(y)} \leq -\frac{1}{s-r} + (n-1)\frac{1/(t-s)}{1 + (r/(t-s))} \leq 0, \tag{3-46}$$

provided $r \in (r_0, s)$ for $r_0 = r_0(s, t)$ suitably close to s, depending on s and t. Here the choice of 0 on the right-hand side of (3-46) is arbitrary. Any constant strictly less than n would suffice for the rest of the argument.

Now consider the set

$$\Lambda = \bigcup_{r_0 < r < s} \Lambda_{s-r}^t$$

so that by the coarea formula and (3-45)

$$|\Lambda| = \int_{r_0}^s \mathcal{H}^n(\Lambda \cap \partial\Omega_{s-r})\, dr = \int_{r_0}^s \mathcal{H}^n(\Lambda_{s-r}^t)\, dr > 0.$$

By the a.e. second-order differentiability of u, there exists $y_0 \in \Lambda$ such that u admits a second-order Taylor expansion at y_0. Moreover there exists $r \in (r_0, s)$ such that $y_0 \in \Lambda_{s-r}^t \subset \Gamma_{s-r}^t$, so that $\nabla^2 u(y_0)[N(y_0)] = 0$ by (3-40), and thus

$$\nabla^2 u(y_0) = \nabla^{\Gamma_{s-r}^t} N(y_0) = -\sum_{i=1}^n (\kappa_{s-r}^t)_i(y_0)\tau_i(y_0) \otimes \tau_i(y_0), \tag{3-47}$$

thanks to (3-15). Moreover, by (3-46), we definitely have

$$\sum_{i=1}^n (\kappa_{s-r}^t)_i(y_0) \leq 0. \tag{3-48}$$

Let us now set $\nu = -N(y_0)$ and

$$\boldsymbol{D}_\rho = \{z \in \nu^\perp : |z| < \rho\}, \quad \boldsymbol{C}_\rho = \{z + h\nu : z \in \boldsymbol{D}_\rho, |h| < \rho\}, \quad \rho > 0.$$

For every $\varepsilon > 0$, the second-order differentiability of u at y_0, (3-48) and (3-47) imply the existence of $\rho > 0$ and of a second-order polynomial $\eta : \nu^\perp \equiv \mathbb{R}^n \to \mathbb{R}$ such that $\eta(0) = 0$, $\nabla\eta(0) = 0$,

$$-\operatorname{div}\left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right)(z) \leq -\operatorname{div}\left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right)(0) + \varepsilon \leq \sum_{i=1}^n (\kappa_{s-r}^t)_i(y_0) + 2\varepsilon \leq 2\varepsilon \tag{3-49}$$

for every $z \in \boldsymbol{D}_\rho$ and

$$y_0 + \{z + h\nu : z \in \boldsymbol{D}_\rho, -\rho < h < \eta(z)\} \subset (y_0 + \boldsymbol{C}_\rho) \cap \Omega_{s-r}. \tag{3-50}$$

If we translate Ω by $(s-r)N(y_0)$, then

$$\Omega_{s-r} \subset (\Omega + (s-r)N(y_0)) \quad \text{with } y_0 \in \partial\Omega_{s-r} \cap \partial(\Omega + (s-r)N(y_0)).$$

We are now in the position to apply Theorem 4 with

$$M = \partial(\Omega + (s-r)N(y_0) - y_0),$$

$\nu = -N(y_0)$, $U = \boldsymbol{D}_\rho$, $z_0 = 0$, $h_0 = \nu \cdot y_0 - \rho$ and η as in (3-49). Indeed by (3-50) we have that if we set

$$\varphi(z) = \inf\{h \in (h_0, \infty) : z + h\nu \in M\}, \quad z \in \boldsymbol{D}_\rho,$$

then $\infty > \varphi \geq \eta > h_0$ on $\boldsymbol{D}_\rho$, as well as $\varphi(0) = \eta(0) = 0$. However, by (3-49),

$$2\varepsilon \geq -\operatorname{div}\left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right)(z) \quad \text{for all } z \in \boldsymbol{D}_\rho,$$

while by the constant mean curvature condition $n = H^0_\Omega = \boldsymbol{H}_{\partial\Omega} \cdot \nu_\Omega$ on $\partial^*\Omega$ we have

$$n = \boldsymbol{H}_M(z + \varphi(z)\nu) \cdot \frac{-\nabla\varphi(z) + \nu}{\sqrt{1+|\nabla\varphi(z)|^2}} \quad \text{for a.e. } z \in \boldsymbol{D}_\rho.$$

This is a contradiction to Theorem 4; hence we obtain (3-41).

Conclusion of the proof: Having proved (3-41), we can now apply the Montiel–Ros argument. By (3-35) and (3-41),

$$|\Omega| = |\Omega^\star| \leq |\zeta(Z)| \leq \int_Z \mathcal{H}^0(\zeta^{-1}(y))\,dy = \int_{\partial^*\Omega} d\mathcal{H}^n_x \int_0^{1/\kappa_n(x)} \prod_{i=1}^n (1 - t\kappa_i(x))\,dt,$$

where $Z = \{(x,t) \in \partial^*\Omega \times \mathbb{R} : 0 < t \leq 1/\kappa_n(x)\}$ and $\zeta(x,t) = x - t\nu_\Omega(x)$. Here we have used the fact that Z is a locally $\mathcal{H}^{n-1}$-rectifiable set in $\mathbb{R}^{n+1} \times \mathbb{R}$ with

$$\mathcal{H}^{n+1} \llcorner ((\partial^*\Omega) \times \mathbb{R}) = (\mathcal{H}^n \llcorner \partial^*\Omega) \times \mathcal{H}^1, \tag{3-51}$$

see [Maggi 2012, Exercise 18.10], and that $J^Z\zeta = \prod_{i=1}^n (1 - t\kappa_i)$. By the arithmetic-geometric mean inequality and by $\kappa_n \geq H^0_\Omega/n$, arguing as in (1-9) we thus find

$$\begin{aligned}
\int_{\partial^*\Omega} d\mathcal{H}^n_x \int_0^{1/\kappa_n(x)} \prod_{i=1}^n (1 - t\kappa_i(x))\,dt &\leq \int_{\partial\Omega} d\mathcal{H}^n_x \int_0^{1/\kappa_n(x)} \left(\frac{1}{n}\sum_{i=1}^n (1 - t\kappa_i(x))\right)^n dt \\
&\leq \int_{\partial\Omega} d\mathcal{H}^n_x \int_0^{n/H^0_\Omega} (1 - tH^0_\Omega)^n\,dt \\
&= \frac{n}{n+1} \int_{\partial\Omega} \frac{d\mathcal{H}^n}{H^0_\Omega} = |\Omega|,
\end{aligned}$$

so that equalities hold everywhere and

$$|\zeta(Z)\setminus\Omega| = 0, \tag{3-52}$$

$$\mathcal{H}^0(\zeta^{-1}(y)) = 1 \quad \text{for a.e. } y\in\Omega, \tag{3-53}$$

$$\kappa_i(x) = \frac{H^0_\Omega}{n} \quad \text{for every } x\in\partial^*\Omega,\ i=1,\dots,n. \tag{3-54}$$

Recall that we have rescaled Ω so that $H^0_\Omega = n$. By (3-54), since $\partial^*\Omega$ is relatively open in $\partial\Omega$, we can find a family $\{S_i\}_{i\in I}$, $I\subset\mathbb{N}$, of mutually disjoint subsets of $\partial^*\Omega$ with $S_i\subset\partial B_1(x_i)$ for points $x_i\in\mathbb{R}^{n+1}$ such that

$$\partial^*\Omega = \bigcup_{i\in I} S_i, \qquad S_i \text{ is relatively open in } \partial\Omega, \qquad S_i \text{ is connected.} \tag{3-55}$$

Because $S_i\subset\partial\Omega$, we know that $u(x_i)\le 1$.

We claim that $u(x_i)=1$ for every $i\in I$. Indeed if $\delta>0$ and $i\in I$ are such that $u(x_i)=1-4\delta$, then $B_\delta(x_i)\cap A_i\subset\Omega$, where $A_i=\zeta(S_i\times(0,1))$ is an open subset of Ω. For any $y\in B_\delta(x_i)\cap A_i$, the triangle inequality implies $u(y)<1-3\delta$, while clearly $d(y,S_i)\ge d(y,\partial B_1(x_i))\ge 1-\delta$. In particular, if $x\in\partial\Omega$ is such that $|x-y|=u(y)$, then $x\notin S_i$. Since (3-35) and (3-41) imply that for a.e. $y\in\Omega$ there exists $x\in\partial^*\Omega$ such that $|x-y|=u(y)$, we conclude from (3-55) that for a.e. $y\in B_\delta(x_i)\cap A_i$ there exist $j\ne i$ and $x\in S_j$ such that $|x-y|=u(y)$; in particular, $B_\delta(x_i)\cap A_i\cap A_j$ is nonempty, and since it is an open set, we have

$$0<|B_\delta(x_i)\cap A_i\cap A_j|, \quad \text{where, if } i\ne j,\quad A_i\cap A_j\subset\{y\in\Omega:\mathcal{H}^0(\zeta^{-1}(y))\ge 2\}.$$

This is a contradiction to (3-53). Thus $u(x_i)=1$ for every $i\in I$.

Now let T_i denote the closure of S_i in $\partial B_1(x_i)$. Since $u(x_i)=1$ for every $i\in I$, we can apply Theorem 4 to $M=\partial\Omega$ at each $x\in T_i$ to find $\rho_x>0$ such that

$$\partial\Omega\cap B_{\rho_x}(x) = \partial B_1(x_i)\cap B_{\rho_x}(x). \tag{3-56}$$

This in turn proves that $T_i=\partial B_1(x_i)$, and thus that $\partial B_1(x_i)\subset\partial\Omega$ for every $i\in I$.

Since $\mathcal{H}^n(\partial B_1(x)\cap\partial B_1(y))=0$ unless $x=y$, $P(\Omega)<\infty$ implies that I is finite. Since $\partial^*\Omega$ is covered by the S_i, Ω is the finite union of the balls $B_1(x_i)$, and owing to $\partial B_1(x_i)\subset\partial\Omega$, these balls must be disjoint (their closures can of course intersect). This completes the proof of Theorem 1. □

Proof of Corollary 2. Condition (1-4) implies that the vector-valued Radon measures

$$\mu_{\Omega_j} = \nu_{\Omega_j}\mathcal{H}^n \llcorner \partial^*\Omega_j$$

converge in weak-star sense to μ_Ω with $|\mu_{\Omega_j}| \overset{*}{\rightharpoonup} |\mu_\Omega|$ on $\mathbb{R}^{n+1}$. By Reshetnyak's continuity theorem [Ambrosio et al. 2000, Theorem 2.39]

$$\lim_{j\to\infty}\int_{\mathbb{R}^{n+1}}\Phi\Big(x,\frac{d\mu_{\Omega_j}}{d|\mu_{\Omega_j}|}(x)\Big)\,d|\mu_{\Omega_j}| = \int_{\mathbb{R}^{n+1}}\Phi\Big(x,\frac{d\mu_\Omega}{d|\mu_\Omega|}(x)\Big)\,d|\mu_\Omega|$$

whenever $\Phi \in C^0_c(\mathbb{R}^{n+1} \times \mathbb{S}^n)$. Given $X \in C^1_c(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$,

$$\Phi(x, \nu) = \operatorname{div} X(x) - \nu \cdot \nabla X(x)[\nu], \quad (x, \nu) \in \mathbb{R}^{n+1} \times \mathbb{S}^n,$$

belongs to $C^0_c(\mathbb{R}^{n+1} \times \mathbb{S}^n)$ and thus we find

$$\lim_{j\to\infty} \int_{\partial^*\Omega_j} \operatorname{div}^{\partial^*\Omega_j} X \, d\mathcal{H}^n = \int_{\partial^*\Omega} \operatorname{div}^{\partial^*\Omega} X \, d\mathcal{H}^n.$$

By (1-5) and by $\mu_{\Omega_j} \overset{*}{\rightharpoonup} \mu_\Omega$

$$\lim_{j\to\infty} \int_{\partial^*\Omega_j} \operatorname{div}^{\partial^*\Omega_j} X \, d\mathcal{H}^n = \lambda \lim_{j\to\infty} \int_{\partial^*\Omega_j} X \cdot \nu_{\Omega_j} \, d\mathcal{H}^n = \lambda \int_{\partial^*\Omega} X \cdot \nu_\Omega \, d\mathcal{H}^n.$$

We have thus proved that Ω is a set of finite perimeter, finite volume and constant distributional mean curvature. We conclude by Theorem 1. □

4. The Heintze–Karcher inequality for sets of finite perimeter

The proof of Theorem 1 also shows that the Heintze–Karcher inequality can be generalized to sets of finite perimeter. In this section we explain how this is done. As usual, set $u(y) = \operatorname{dist}(y, \partial\Omega)$ for $y \in \Omega$.

Lemma 7. *If Ω is an open set with finite perimeter and finite volume in $\mathbb{R}^{n+1}$, then $\Omega_s = \{y \in \Omega : u(y) > s\}$ is an open set of finite perimeter with $\mathcal{H}^n(\partial\Omega_s \setminus \Gamma_s^+) = 0$ for a.e. $s > 0$, where $\Gamma_s^+ = \bigcup_{t>0} \Gamma_s^t$ and Γ_s^t is defined as in (1-14). Moreover:*

(i) *For every $s > 0$, Γ_s^+ can be covered by countably many graphs of $C^{1,1}$-functions from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$.*

(ii) *For every $s > 0$, the principal curvatures $(\kappa_s)_i$ of Γ_s^+ are defined $\mathcal{H}^n$-a.e. on Γ_s^+ by setting*

$$(\kappa_s)_i = (\kappa_s^t)_i \quad \text{on } \Gamma_s^t \text{ for each } t > s,$$

for $(\kappa_s^t)_i$ as in (3-15). Correspondingly, $\mathcal{H}^n$-a.e. on Γ_s^+ we can define

$$H_{\Omega_s} = \sum_{i=1}^n (\kappa_s)_i, \quad |A_{\Omega_s}|^2 = \sum_{i=1}^n (\kappa_s)_i^2$$

as natural generalizations of the mean curvature and of the length of the second fundamental form of $\partial\Omega_s$ with respect to ν_{Ω_s} at points in $\Gamma_s^+ \subset \partial\Omega_s$.

(iii) *For every $r < s < t$, the map $g_r : \Gamma_s^t \to \Gamma_{s-r}^t$, defined by $g(y) = y - r\nabla u(y)$ for $y \in \Gamma_s^t$, is a Lipschitz bijection from Γ_s^t to Γ_{s-r}^t, with*

$$J^{\Gamma_s^t} g_r(y) = \prod_{i=1}^n (1 + r(\kappa_s)_i(y)), \quad (\kappa_{s-r})_i(g_r(y)) = \frac{(\kappa_s)_i(y)}{1 + r(\kappa_s)_i(y)} \tag{4-1}$$

for $\mathcal{H}^n$-a.e. $y \in \Gamma_s^t$.

Proof. All these conclusions are contained in Steps 1, 2 and 3 of the proof of Theorem 1, where at no stage the constant distributional mean curvature condition, or the regularity of $\partial^*\Omega$ implied by it, have been used. □

As a consequence of Lemma 7, we see that for every $x \in g_s(\Gamma_s^+) \subset \partial\Omega$, the limit

$$\kappa_i(x) = \lim_{r \to s^-} (\kappa_{s-r})_i(x) \in [-\infty, \infty) \tag{4-2}$$

exists by monotonicity; see (4-1). We thus give the following definitions: given an open set of finite perimeter and finite volume $\Omega \subset \mathbb{R}^{n+1}$ we define the *viscosity boundary of* Ω as

$$\partial^v \Omega = \bigcup_{s>0} g_s(\Gamma_s^+)$$

and the *viscosity mean curvature of* Ω by

$$H_\Omega^v(x) = \sum_{i=1}^n \kappa_i(x) \quad \text{for all } x \in \partial^v\Omega. \tag{4-3}$$

Notice that $\partial^v\Omega$ is covered by countably many $\mathcal{H}^n$-rectifiable sets, although it may contain points of $\operatorname{spt}\mu_\Omega$ that are outside the reduced boundary, or that have density 1 for Ω. It is not obvious if, at this level of generality, $\partial^v\Omega$ is $\mathcal{H}^n$-finite. In any case, our only reason for introducing these concepts is to formulate the following definition: a set of finite perimeter and finite volume Ω is *mean convex in the viscosity sense* if H_Ω^v defined in (4-3) is positive along $\partial^v\Omega$. It is easy to see that if $\partial\Omega$ is C^2, then $\partial^v\Omega = \partial\Omega$ and $H_\Omega^v(x) = H_\Omega(x)$ for any $x \in \partial\Omega$. Hence, the viscosity notion generalizes the mean convexity in the classical sense.

This said, following Brendle's point of view [2013] on the Montiel–Ros argument, we have the following generalized form of the Heintze–Karcher inequality; see (4-4) below.

Theorem 8 (Heintze–Karcher inequality for sets of finite perimeter). *If $\Omega \subset \mathbb{R}^{n+1}$ is an open set of finite perimeter and finite volume which is mean convex in the viscosity sense, then for every $s > 0$*

$$|\Omega_s| \le \frac{n}{n+1} \int_{\Gamma_s^+} \frac{d\mathcal{H}^n}{H_{\Omega_s}}. \tag{4-4}$$

Moreover, the limit of the right-hand side of (4-4) as $s \to 0^+$ always exists in $(0, \infty]$.

Proof. The mean convexity assumption on Ω and the monotonicity property behind the definition (4-2) of κ_i imply that $\sum_{i=1}^n (\kappa_s)_i > 0$ on Γ_s^+. We define for every $s > 0$

$$Q(s) = \int_{\Gamma_s^+} \frac{d\mathcal{H}^n}{H_{\Omega_s}} > 0.$$

Moreover, for every $t > 0$ we define $Q^t : (0, t) \to (0, \infty)$ by setting

$$Q^t(s) = \int_{\Gamma_s^t} \frac{d\mathcal{H}^n}{H_{\Omega_s}}, \quad s \in (0, t).$$

Notice that

$$Q(s) \ge Q^t(s) \ge Q^{t+\varepsilon}(s) \quad \text{for all } t > s,\ \varepsilon > 0, \tag{4-5}$$

and recall that $\mathcal{H}^n(\Gamma_s^t)$ converges monotonically to $\mathcal{H}^n(\Gamma_s^+)$ as $t \to s^+$, so that

$$Q(s) = \lim_{t \to s^+} Q^t(s) = \sup_{t>s} Q^t(s) \quad \text{for every } s > 0. \tag{4-6}$$

For $r \in (0, s)$ by Lemma 7(iii) we have

$$Q^t(s-r) - Q^t(s) = \int_{\Gamma_s^t} \left(\frac{\prod_{i=1}^n (1 + r(\kappa_s)_i)}{\sum_{i=1}^n (\kappa_s)_i / (1 + r(\kappa_s)_i)} - \frac{1}{H_{\Omega_s}} \right) d\mathcal{H}^n$$
$$= \int_{\Gamma_s^t} \left(\frac{1 + r H_{\Omega_s} + O_t(r^2)}{H_{\Omega_s} - r|A_{\Omega_s}|^2 + O_t(r^2)} - \frac{1}{H_{\Omega_s}} \right) d\mathcal{H}^n,$$

where $O_t(r^2)/r \to 0$ uniformly on Γ_s^t as $r \to 0$. We thus find that Q^t is differentiable on $(0, t)$ with

$$(Q^t)'(s) = -\int_{\Gamma_s^t} 1 + \frac{|A_{\Omega_s}|^2}{H_{\Omega_s}^2} \, d\mathcal{H}^n \quad \text{for all } s \in (0, t).$$

By the Cauchy–Schwarz inequality, $H_{\Omega_s}^2 \le n|A_{\Omega_s}|^2$. Hence,

$$(Q^t)'(s) \le -\frac{n+1}{n} \mathcal{H}^n(\Gamma_s^t) \quad \text{for all } s \in (0, t). \tag{4-7}$$

If $0 < s_1 < s_2$, then by (4-6), (4-5) and (4-7) respectively, we have

$$Q(s_1) - Q(s_2) = \lim_{\varepsilon \to 0^+} Q^{s_1+\varepsilon}(s_1) - Q^{s_2+\varepsilon}(s_2)$$
$$\ge \lim_{\varepsilon \to 0^+} Q^{s_2+\varepsilon}(s_1) - Q^{s_2+\varepsilon}(s_2) = Q^{s_2}(s_1) - Q^{s_2}(s_2)$$
$$\ge \frac{n+1}{n} \int_{s_1}^{s_2} \mathcal{H}^n(\Gamma_s^{s_2}) \, ds, \tag{4-8}$$

and, in particular, Q is decreasing on $(0, \infty)$. Again by Lemma 7(iii)

$$\mathcal{H}^n(\Gamma_{s-r}^t) = \int_{\Gamma_s^t} \prod_{i=1}^n (1 + r(\kappa_i)_s) \, d\mathcal{H}^n,$$

where $1 + r(\kappa_i)_s \to 1$ uniformly on Γ_s^t as $r \to 0$ thanks to $1/(t-s) \ge (\kappa_s)_i \ge -1/s$ for every $i = 1, \dots, n$. Thus $\mathcal{H}^n(\Gamma_s^t)$ is continuous on $s \in (0, t)$, and

$$\int_{s_1}^{s_2} \mathcal{H}^n(\Gamma_s^{s_2}) \, ds = (s_2 - s_1) \mathcal{H}^n(\Gamma_{s^*}^{s_2})$$

for a suitable $s_* \in (s_1, s_2)$. But (3-34) implies

$$\liminf_{s \to (s_2)^-} \mathcal{H}^n(\Gamma_s^{s_2}) \ge \mathcal{H}^n(\partial\Omega_{s_2})$$

so that, in conclusion,

$$\liminf_{s_1 \to (s_2)^-} \frac{1}{s_2 - s_1} \int_{s_1}^{s_2} \mathcal{H}^n(\Gamma_s^{s_2}) \, ds \ge \mathcal{H}^n(\partial\Omega_{s_2}) \quad \text{for all } s_2 > 0.$$

Coming back to (4-8), and noticing that $Q'(s)$ exists for a.e. $s > 0$ by monotonicity, we conclude that

$$-Q'(s) \ge \frac{n+1}{n} \mathcal{H}^n(\partial\Omega_s) \quad \text{for a.e. } s > 0.$$

We integrate this inequality over (s, ∞) to complete the proof of (4-4). □

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