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COMPLEX INTERPOLATION AND CALDERÓN–MITYAGIN COUPLES OF MORREY SPACES

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We study interpolation spaces between global Morrey spaces and between local Morrey spaces. We prove that for a wide class of couples of these spaces the upper complex Calderón spaces are not described by the K-method of interpolation. A by-product of our results is that couples of Morrey spaces belonging to this class are not Calderón–Mityagin couples. A Banach couple (X_0, X_1) is said to have the universal K-property if all relative interpolation spaces from any Banach couple to (X_0, X_1) are relatively K-monotone. A couple of local Morrey spaces is proved to have the universal K-property once it is a Calderón–Mityagin couple.

1. Introduction

The theory of Calderón–Mityagin couples is a central topic in abstract interpolation theory, since the interpolation spaces relative to such couples are isomorphic to generalized real interpolation spaces. We are interested in unifying this collection of results on Calderón–Mityagin couples. This attempt at unification forms an important component in the general program of describing all interpolation spaces with respect to a given compatible couple of Banach spaces. There is a simple characterization of Calderón–Mityagin couples in terms of the so-called submajorization of the K-functional and orbits. Besides the fundamental example of the couple (L^1, L^∞), which was independently discovered by Calderón [1966] and Mityagin [1965], many other examples were found out later by many mathematicians in interpolation theory, like couples of weighted L^p or of certain types of rearrangement invariant spaces. Unfortunately, it is still difficult to prove or disprove that a given couple of Banach spaces is a Calderón–Mityagin couple. Nevertheless, many Calderón–Mityagin couples have been discovered; we refer, e.g., to [Cwikel and Nilsson 2003; Kalton 1993; Mastyło and Sinnamon 2017] for more about this topic. In this paper we handle couples of Morrey spaces and local Morrey spaces as examples and counterexamples of Calderón–Mityagin couples. Based on the results, we consider the interpolation of Morrey spaces.

Cwikel [1981] conjectured that all interpolation spaces with respect to a given Banach couple are described by K-method whenever all complex interpolation spaces have this property. However, in [Mastyło and Ovchinnikov 1997] the authors disproved this conjecture. This motivates us to study classes of Banach couples for which Cwikel's conjecture is true.

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The main purpose of the present paper is to study Cwikel's conjecture for the class of Morrey spaces, which play an important role in nonlinear potential analysis and harmonic analysis; see [Adams and Xiao 2004; 2012]. These Banach spaces were used for the first time by Morrey [1938] to prove that certain systems of partial differential equations have Hölder continuous solutions. Morrey spaces are widely used to investigate the local behavior of solutions of partial differential equations, including the Navier–Stokes equations; see, e.g., [Lemarié-Rieusset 2012; Mazzucato 2003; Taylor 1997].

Before we state the main results of the present paper, we introduce some fundamental definitions. For $1 \le q \le p \le \infty$ the (global) Morrey space $\mathcal{M}_q^p$ over $\mathbb{R}^n$ is defined to be the space of all q-locally integrable functions f on $\mathbb{R}^n$ ($f \in L^q_{\mathrm{loc}}$ for short) such that

$$\|f\|_{\mathcal{M}_q^p} := \sup_{(x,r)\in\mathbb{R}^n\times\mathbb{R}_+} |B(x,r)|^{1/p-1/q} \left(\int_{B(x,r)} |f(y)|^q\,dy\right)^{1/q} < \infty.$$

Here and below we write $\mathbb{R}_+ = (0,\infty)$. The symbol $|A|$ stands for the Lebesgue measure of any Lebesgue measurable set A in $\mathbb{R}^n$, and $B(x,r)$ is the open ball in $\mathbb{R}^n$ centered at x of radius $r > 0$. In particular, by the Lebesgue differentiation theorem $\mathcal{M}_q^\infty = L^\infty$ with identical norms. For simplicity of notation, we abbreviate $B(0,r)$ to $B(r)$.

Note that for these spaces sometimes other symbols, such as $\mathcal{L}^{q,\lambda}$ [Peetre 1969] and $L^{q,\lambda}$ [Nakai 2008], are used. Apart from the choice of a different letter $\mathcal{L}$, the second parameter λ is also introduced into the norm in a way different from the above; namely for a measurable function f we define

$$\|f\|_{\mathcal{L}^{q,\lambda}} = \|f\|_{L^{q,\lambda}} = \sup_{x\in\mathbb{R}^n\times\mathbb{R}_+} \left(\frac{1}{r^\lambda}\int_{B(x,r)} |f(y)|^q\,dy\right)^{1/q},$$

where $1 \le q < \infty$, $0 \le \lambda < n$. Among various function spaces above, we note the following relation:

$$\mathcal{M}_q^p = L^{q,\lambda} = \mathcal{L}^{q,\lambda}, \quad \lambda = n - \frac{nq}{p}.$$

We point out that for technical reasons it is convenient to use a norm equivalent to the original norm of the Morrey space $\mathcal{M}_q^p$ given by

$$\|f\|^*_{\mathcal{M}_q^p} = \sup_Q |Q|^{1/p-1/q}\left(\int_Q |f(y)|^q\,dy\right)^{1/q}, \quad f \in \mathcal{M}_q^p,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$ with sides parallel to coordinate axes.

It seems worth investigating its local counterpart, which is related to the Beurling algebra B^q and Wiener spaces. The local version of Morrey spaces, where only balls centered at the origin are taken into account, has a connection with studies of N. Wiener [1930; 1932], who considered the spaces of all measurable functions f such that for given $q \in \{1,2\}$

$$\frac{1}{T}\int_0^T |f(y)|^q\,dy$$

is bounded in T or tends to 0 as $T \to \infty$.

In the multidimensional case, a variant of these spaces defined by the norm

$$\|f\|_{B^q} = \sup_{r>0}\left(\frac{1}{|B(r)|}\int_{B(r)}|f(y)|^q\,dy\right)^{1/q}$$

appeared in [Beurling 1964] as the dual of the so-called Beurling algebra.

A local variant of Morrey spaces appeared in [García-Cuerva and Herrero 1994]. The local Morrey space $L\mathcal{M}^p_q$ is defined to be the set of all $f \in L^q_{\mathrm{loc}}$ such that

$$\|f\|_{L\mathcal{M}^p_q} := \sup_{r>0}|B(r)|^{1/p-1/q}\left(\int_{B(r)}|f(y)|^q\,dy\right)^{1/q} < \infty.$$

We note that

$$\|f\|^*_{L\mathcal{M}^p_q} := \sup_{r>0}|Q(r)|^{1/p-1/q}\left(\int_{Q(r)}|f(y)|^q\,dy\right)^{1/q}$$

is an equivalent norm in $L\mathcal{M}^p_q$, where $Q(r) := [-r, r]^n$.

Interpolation properties of classical Morrey spaces were obtained in [Campanato and Murthy 1965; Peetre 1969; Stampacchia 1964]. More and more attention is now being paid to the interpolation of Morrey spaces due to certain properties of Morrey spaces that have become clear recently. For example, as the function $|x|^{-n/p}$ shows, $\mathcal{M}^p_q$ does not contain L^∞_{c} densely. Complex interpolation of Morrey spaces has been studied in [Lemarié-Rieusset 2012; 2013; Yuan et al. 2015]. We mention that Lemarié–Rieusset [2013, case (a), p. 750] proved that if

$$1 \le q_j < p_j < \infty, \quad j \in \{0,1\}, \qquad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \tag{1-1}$$

then for every $\theta \in (0,1)$,

$$[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta \ne \mathcal{M}^p_q$$

whenever $q_0/p_0 \ne q_1/p_1$. For the case when $q_0/p_0 = q_1/p_1$, Lemarié–Rieusset [2013] proved that

$$[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]^\theta = \mathcal{M}^p_q.$$

Here and below $[\,\cdot\,]_\theta$ and $[\,\cdot\,]^\theta$ denote the lower and the upper (Calderón) complex methods of interpolation defined in [Calderón 1964], respectively. Lemarié–Rieusset also studied real interpolation of Morrey spaces. In particular under conditions (1-1) we have

$$(L^{q_0}, L^{q_1})_{\theta,q} = L^q,$$

and hence

$$(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})_{\theta,q} \hookrightarrow \mathcal{M}^p_q$$

in the sense of continuous embedding. Meanwhile,

$$\mathcal{M}^p_q \hookrightarrow (\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})_{\theta,\infty}$$

if and only if $q_0/p_0 = q_1/p_1$. We refer to [Lemarié-Rieusset 2013; Yuan et al. 2015] for more details.

In addition to Morrey spaces and local Morrey spaces, we will also consider in our paper the "weak" Morrey space $\mathrm{w}\mathcal{M}_q^p$ and the "weak" local Morrey space $\mathrm{w}L\mathcal{M}_q^p$. For given $1 \le q \le p < \infty$ the weak Morrey space $\mathrm{w}\mathcal{M}_q^p$ is defined to be the quasi-Banach space of all Lebesgue measurable functions f endowed with the quasinorm

$$\|f\|_{\mathrm{w}\mathcal{M}_q^p} = \sup_{\lambda>0} \lambda \|\chi_{\{|f|>\lambda\}}\|_{\mathcal{M}_q^p};$$

Nakai [2008] used the norm

$$\|f\|_{\mathrm{w}L^{q,\lambda}} = \sup_{\lambda>0} \lambda \|\chi_{\{|f|>\lambda\}}\|_{L^{q,\lambda}}$$

to define weak Morrey spaces, while the weak local Morrey space $\mathrm{w}L\mathcal{M}_q^p$ is defined to be the quasi-Banach space of all Lebesgue measurable functions f for which

$$\|f\|_{\mathrm{w}L\mathcal{M}_q^p} = \sup_{\lambda>0} \lambda \|\chi_{\{|f|>\lambda\}}\|_{L\mathcal{M}_q^p} < \infty.$$

When $\Phi(r) = r^q$ and $\phi(r) = r^{-1+\lambda/n}$ and $\lambda/n = 1-q/p$, this space $\mathrm{w}\mathcal{M}_q^p$ is the same as Nakai's space $L_{\mathrm{weak}}^{(\Phi,\varphi)}$ [Nakai 2008, Definition 6.1, p. 207]. In particular, for every $s \in [1,\infty)$, $L_{\mathrm{weak}}^{(1,n-n/s)}$ coincides with $\mathrm{w}\mathcal{M}_1^s$.

We add some comments on difficulties related to interpolation of Morrey spaces. First we notice that until now there is no complete description of all complex or real interpolation spaces of all couples $(\mathcal{M}_{q_0}^{p_0}, \mathcal{M}_{q_1}^{p_1})$ of Morrey spaces with $1 \le q_0 \le p_0 < \infty$ and $1 \le q_1 \le p_1 < \infty$. Regarding real interpolation, it should be pointed out that a general formula remains unknown, even within equivalence for the K-functional of these couples. It is apparently very difficult to find such a formula, and this indeed is one of the nontrivial difficulties of dealing with Morrey couples, especially in the description of interpolation spaces with respect these couples, and in particular real interpolation spaces which just involve the K-functional. Interestingly the situation is completely different in the setting of local Morrey spaces (see Section 5).

In this paper we provide a solution to Cwikel's conjecture in the Morrey space setting. Our new results show that Cwikel's conjecture is still valid for a wide class of global and local Morrey spaces. In particular, because of the fact that a wide class of these couples are not Calderón–Mityagin couples, we have to declare that the problem related to the description of all interpolation spaces for all couples of Morrey spaces is extremely difficult in general.

Let us now describe more precisely the contents of the present paper. In Section 2 we introduce some fundamental definitions and notation used in the paper. In Section 3 we study the upper complex method of interpolation $[\,\cdot\,]^\theta$ for any $\theta \in (0,1)$. We prove that for any couple (X_0, X_1) of complex Banach lattices on an arbitrary measure space (Ω, Σ, μ) the Gagliardo completion of $[X_0, X_1]^\theta$ with respect to $X_0 + X_1$ coincides isometrically with the Gagliardo completion of the Calderón product $X_0^{1-\theta} X_1^\theta$ with respect to $X_0 + X_1$. In particular we show that if (Ω, Σ, μ) is a σ-finite measure space, then $[X_0, X_1]^\theta = X_0^{1-\theta} X_1^\theta$ with equality of norms whenever each of X_0 and X_1 has the Fatou property. Applying this result to Morrey spaces, we recover the results above due to [Lemarié-Rieusset 2013].

In Sections 4 and 5 we provide a general sufficient condition on a Banach couple (A_0, A_1), which guarantees that (A_0, A_1) is not a Calderón–Mityagin couple. As a by-product, we prove in Section 4 that both couples $(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})$ and $(\mathcal{M}^{p_0}_{q_0}, L^\infty)$ are not Calderón–Mityagin couples provided that $q_0 \neq q_1$, $p_0 \neq q_0$ and $p_1 \neq q_1$.

Finally, in Section 5, we describe real interpolation of local Morrey spaces by the upper complex method $[\,\cdot\,]^\theta$ and the classical real method $(\,\cdot\,)_{\theta,\infty}$ for all $\theta \in (0, 1)$. These results are applied to prove that $(L\mathcal{M}^{p_0}_{q_0}, L\mathcal{M}^{q_1}_{p_1})$ is a Calderón–Mityagin couple if and only if $q_0 = q_1$, and in this case this couple has the universal K-property, i.e., (A_0, A_1) and $(L\mathcal{M}^{p_0}_{q_0}, L\mathcal{M}^{q_1}_{p_1})$ are relative Calderón–Mityagin couples for any Banach couple (A_0, A_1). We stress that the key point here is that fact the inclusion $L\mathcal{M}^p_q \hookrightarrow \mathrm{w}L\mathcal{M}^p_q$ is proper for every $1 \le q \le p < \infty$; see Lemma 4.7(ii) for the proof, where we use the maximal Hardy–Littlewood operator.

We will use standard notation; in particular given two nonnegative functions f and g defined on the same set A, we write $f \prec g$ or $g \succ f$ if there is a constant $c > 0$ such that $f(x) \le cg(x)$ for all $x \in A$, while $f \asymp g$ means that both $f \prec g$ and $g \prec f$ hold. If X and Y are topological linear spaces, then $X \hookrightarrow Y$ means that $X \subset Y$ and the inclusion map is continuous. In the case when X and Y are Banach spaces, we write $X \cong Y$ whenever $X = Y$, with *equality* of norms. Throughout the entire paper, C will denote a constant which may have a different value in different appearances.

2. Preliminaries

We will use the standard notation in the theory of Banach spaces and the theory of integration. If X is a Banach space, we denote its (closed) unit ball by B_X. For any measure space (Ω, Σ, μ), the space of all μ-equivalence classes of real-valued Σ-measurable μ-almost everywhere finite functions will be denoted by $L^0(\mu) := L^0(\Omega, \Sigma, \mu)$. This space is a vector lattice under the natural order: $f \le g$ if and only if $f(s) \le g(s)$ for μ-almost everywhere $s \in \Omega$.

A linear subspace E of $L^0(\Omega, \Sigma, \mu)$ is called an *ideal* if it is solid, i.e., $|f| \le |g|$ for some $g \in E$ implies $f \in E$. We will consider Banach lattices on an arbitrary measure space (in general we do not need to assume that the measure is σ-finite, which is usually found in the literature). We recall that a Banach space $X \subset L^0(\mu)$, which is an ideal with a monotone norm (meaning $\|f\|_X \le \|g\|_X$ for all $f, g \in X$ satisfying $|f| \le |g|$) is said to be a *Banach lattice* on (Ω, Σ, μ). It is well known that in the theory of Banach lattices on measure spaces we may assume without loss of generality that the measure spaces are complete. A Banach lattice X is called *σ-order continuous* if $x_n \downarrow 0$ implies $\|x_n\|_X \to 0$.

We note that for all choices of two Banach lattices X_0 and X_1 on an arbitrary measure space, $\vec{X} := (X_0, X_1)$ forms a Banach couple in the sense of interpolation theory; see, e.g., [Krein et al. 1982, Corollary 1, p. 42] in the case of σ-finite measures, and for an arbitrary measure space, [Cwikel and Nilsson 2003, Remark 1.41, pp. 34–35].

Let (Ω, Σ, μ) be a measure space, and let $1 \le p < \infty$. We recall that the *weak Lebesgue* or the *Marcinkiewicz space* $L^{p,\infty}(\mu)$ is made up of all functions $f \in L^0(\mu)$ such that

$$\|f\|_{p,\infty} := \sup_{\lambda>0} \lambda\, \mu(\{x \in \Omega : |f(x)| > \lambda\})^{1/p} < \infty.$$

If $p > 1$, then the quasinorm $\|\cdot\|_{p,\infty}$ is equivalent to the norm

$$\|f\|^*_{p,\infty} := \sup_{A\subset\Omega} \mu(A)^{1/p-1} \int_A |f|\,d\mu,$$

where the supremum is taken over all measurable subsets A of Ω with $\mu(A) > 0$.

If $X \subset L^0(\mu)$ is a Banach lattice and $p \in (1, \infty)$, then its *p-convexification* $X^{(p)}$ is the Banach lattice of all elements $f \in L^0(\mu)$ such that $|f|^p \in X$ with a norm $\|f\|_{X^{(p)}} = \||f|^p\|_X^{1/p}$.

Most of our notation and terminology from interpolation theory is standard and can be found in [Bergh and Löfström 1976; Brudnyi and Krugljak 1991]. For the reader's convenience, we recall some of them.

Let $\vec{X} = (X_0, X_1)$ and $\vec{Y} = (Y_0, Y_1)$ be Banach couples and let $\mathcal{L}(\vec{X}, \vec{Y})$ be the Banach space of all linear operators $T : \vec{X} \to \vec{Y}$ (meaning, as usual, that $T : X_0 + X_1 \to Y_0 + Y_1$ is linear and $T : X_j \to Y_j$ boundedly for $j = 0, 1$), where the norm is given by $\|T\|_{\vec{X}\to\vec{Y}} = \max_{j=0,1} \|T\|_{X_j\to Y_j}$.

Let X be an intermediate space with respect to a Banach couple $\vec{X} = (X_0, X_1)$. The *Gagliardo completion* or *relative completion* of X with respect to $\vec{X}$ is the Banach space X^{c} of all limits in $X_0 + X_1$ of sequences that are bounded in X and endowed with the norm $\|x\|_{X^{\mathrm{c}}} = \inf\{\sup_{k\geq 1} \|x_k\|_X\}$, where the infimum is taken over all bounded sequences $\{x_k\}_{k=1}^\infty$ in X whose limit in $X_0 + X_1$ equals x.

For every Banach couple $\vec{X} = (X_0, X_1)$ and $\theta \in (0, 1)$ the Peetre K-functional of $x \in X_0 + X_1$ is defined by

$$K(t, x; \vec{X}) = K(t, x; X_0, X_1) := \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1\}, \quad t > 0,$$

and the real interpolation space $\vec{X}_{\theta,\infty}$ is defined to be the collection of all $x \in X_0 + X_1$ such that

$$\|x\|_{\theta,\infty} := \sup_{t>0} t^{-\theta} K(t, x; \vec{X}) < \infty.$$

Let X and Y be intermediate spaces with respect to $\vec{X}$ and $\vec{Y}$ respectively. We say that they are *relative interpolation spaces* with respect to $\vec{X}$ and $\vec{Y}$ if every $T \in \mathcal{L}(\vec{X}, \vec{Y})$ maps X into Y. The relative interpolation spaces X and Y are said to be *relative K-monotone spaces* with respect to $\vec{X}$ and $\vec{Y}$ if, whenever $x \in X$ and $y \in Y_0 + Y_1$ satisfy $K(t, y; \vec{Y}) \leq K(t, x; \vec{X})$ for all $t > 0$, it follows that $y \in Y$. If $\vec{X} = \vec{Y}$ and $X = Y$, then X is said to be a K-monotone space with respect to $\vec{X}$. Note that $K(t, Tx; \vec{Y}) \leq \|T\|_{\vec{X}\to\vec{Y}} K(t, x; \vec{X})$ for $x \in \vec{X}$. So, if X and Y are relative K-monotone, then they are relative interpolation spaces. If all relative interpolation spaces for $\vec{X}$ and $\vec{Y}$ are relative K-monotone, then we say that $\vec{X}$ and $\vec{Y}$ are relative Calderón–Mityagin couples. In particular, if also $\vec{X} = \vec{Y}$, then $\vec{X}$ is said to be a Calderón–Mityagin couple. We remark here that in a number of papers, various alternative terminologies, such as C-couple or K-adequate couple, are used for the notion of Calderón–Mityagin couples. It is well known and easy to prove, see, e.g., [Cwikel and Nilsson 2003, Remark 1.31], that $\vec{X}$ and $\vec{Y}$ are relative Calderón–Mityagin couples if and only if, for every $x \in X_0 + X_1$ and $y \in Y_0 + Y_1$, the inequality

$$K(t, y; \vec{Y}) \leq K(t, x; \vec{X}), \quad t > 0,$$

implies that there exists an operator $T : \vec{X} \to \vec{Y}$ such that $Tx = y$.

Let $\lambda \geq 1$ be a fixed constant. If we can arrange that $\|T\|_{\vec{X}\to\vec{Y}} \leq \lambda$ for all x and y above, then we say that $\vec{X}$ and $\vec{Y}$ are λ-relatively uniform Calderón–Mityagin couples. An interpolation couple $\vec{X}$ is said to be a uniform Calderón–Mityagin couple if $\vec{X}$ and $\vec{X}$ are λ-relatively uniform Calderón–Mityagin couples for some λ.

If Φ is a Banach lattice of Lebesgue measurable functions on $\mathbb{R}_+$ that contain the function $\min\{1, t\}$, then we can define the Banach space $(X_0, X_1)_\Phi$ of all $x \in X_0 + X_1$ such that $K(\cdot, x; \vec{X}) \in \Phi$ with the norm

$$\|x\| = \|K(\cdot, x; \vec{X})\|_\Phi.$$

The space $(X_0, X_1)_\Phi$ is called the K-space generated by Φ. It is a fundamental result of Brudnyi and Krugljak [1991, Theorem 3.3.20, p. 355] that if (X_0, X_1) is a uniform Calderón–Mityagin couple, then every interpolation space X with respect to (X_0, X_1) is (up to equivalence of norm) a K-space for some Φ. The key ingredient of this result is the K-divisibility theorem first proved by Brudnyi and Krugljak [1981; 1991] and later refined in [Cwikel 1984].

Let (Ω, Σ, μ) be a σ-finite measure space. It is well known that $(L^1(\mu), L^\infty(\mu))$ is a 1-uniform C-couple; see [Calderón 1966]. Several years later, Sedaev and Semenov [1971] proved that every weighted couple $(L^1(w_0), L^1(w_1))$ is a uniform C-couple. For more examples of uniform Calderón–Mityagin couples of Banach lattices we refer to [Cwikel 1981; Cwikel and Nilsson 2003; Kalton 1993; Mastyło and Sinnamon 2017].

3. Upper complex interpolation between Banach lattices

We will use complex methods of interpolation introduced in the fundamental paper [Calderón 1964] to prove isometric relationships between Banach lattices generated by standard operations applied to the Calderón product in the setting of couples of complex Banach lattices on an arbitrary measure space. We will apply these results to couples of Banach lattices enjoying the Fatou property and so in particular to Morrey spaces later.

Let $S := \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$ be an open strip on the plane. For a given $\theta \in (0, 1)$ and any couple $\vec{X} = (X_0, X_1)$ we denote by $\mathcal{F}(\vec{X})$ the Banach space of all bounded continuous functions $f : \overline{S} \to X_0 + X_1$ on the closure $\overline{S}$ that are analytic on S, and $\mathbb{R} \ni t \mapsto f(j + it) \in X_j$ is a bounded continuous function, for each $j = 0, 1$, and endowed with the norm

$$\|f\|_{\mathcal{F}(\vec{X})} = \max_{j=0,1} \sup_{t\in\mathbb{R}} \|f(j + it)\|_{X_j}.$$

The lower/first complex interpolation space is defined by

$$[X_0, X_1]_\theta := \{f(\theta) : f \in \mathcal{F}(\vec{X})\}$$

and is endowed with the quotient norm. This definition is slightly different from those in [Bergh and Löfström 1976; Calderón 1964], however it gives the same interpolation spaces; see, e.g., [Krein et al. 1982]. We recall that in the original definition it is required in addition that $f \in \mathcal{F}(\vec{X})$ satisfies

$$\lim_{|t|\to\infty} \|f(j + it)\|_{X_j} = 0, \quad j \in \{0, 1\}.$$

Calderón defined a different interpolation method as follows. Let $\mathcal{G}(\vec{X})$ be the Banach space of all continuous functions $g : \overline{S} \to X_0 + X_1$ that are analytic on S, for which there exists $C = C(g) > 0$ such that $\|g(z)\|_{X_0+X_1} \le C(1+|z|)$ for all $z \in S$, and that are endowed with the norm

$$\|g\|_{\mathcal{G}(\vec{X})} := \max_{j=0,1} \left\{ \sup_{s \neq t} \frac{\|g(j+is) - g(j+it)\|_{X_j}}{|s-t|} \right\}.$$

The upper/second complex interpolation space is defined by

$$[X_0, X_1]^\theta := \{g'(\theta) : g \in \mathcal{G}(\vec{X})\}$$

and is endowed with the quotient norm.

Throughout the paper, when the complex methods are applied to a couple (X_0, X_1) of Banach lattices, we mean that $X_j := X_j(\mathbb{C})$ is a complexification of X_j for each $j = 0, 1$.

We need the following lemma:

Lemma 3.1. *Let $\vec{X} = (X_0, X_1)$ be a complex Banach couple, and let $\theta \in (0, 1)$:*

(i) $([X_0, X_1]^\theta)^c \cong ([X_0, X_1]_\theta)^c$.

(ii) $[X_0, X_1]^\theta \cong ([X_0, X_1]_\theta)^c$ *if and only if the unit ball of* $[X_0, X_1]^\theta$ *is closed in* $X_0 + X_1$.

Proof. We claim that $[X_0, X_1]^\theta \hookrightarrow ([X_0, X_1]_\theta)^c$ with norm of continuous inclusion less than or equal to 1. Fix $x \in [X_0, X_1]^\theta$. For $\varepsilon > 0$ there exists $g \in \mathcal{G}(\vec{X})$ such that $x = g'(\theta)$ and

$$\|g\|_{\mathcal{G}(\vec{X})} \le \|x\|_{[X_0,X_1]^\theta} + \varepsilon. \tag{3-1}$$

Consider the sequence $\{g_m\}_{m=1}^\infty$ given by

$$g_m(z) = \frac{g(z + i/m) - g(z)}{i/m}, \quad z \in \overline{S}.$$

Observe that for each $m \in \mathbb{N}$ we have

$$\max_{j=0,1} \sup_{t \in \mathbb{R}} \|g_m(j+it)\|_{X_0+X_1} \le \max_{j=0,1} \sup_{t \in \mathbb{R}} \|g_m(j+it)\|_{X_j} \le \|g\|_{\mathcal{G}(\vec{X})}.$$

Thus it follows by the Phragmén–Lindelöf principle for Banach spaces that

$$\|g_m(z)\|_{X_0+X_1} \le \|g\|_{\mathcal{G}(\vec{X})}, \quad z \in S.$$

We clearly have that each function $g_m : \overline{S} \to X_0 + X_1$ is continuous and analytic on the strip S. Thus we conclude that $g_m \in \mathcal{F}(\vec{X})$ with $\|g_m\|_{\mathcal{F}(\vec{X})} \le \|g\|_{\mathcal{G}(\vec{X})}$. Hence $g_m(\theta) \in [X_0, X_1]_\theta$ and

$$\|g_m(\theta)\|_{[X_0,X_1]_\theta} \le \|g\|_{\mathcal{G}(\vec{X})}, \quad m \in \mathbb{N}.$$

Since

$$\lim_{m\to\infty} g_m(\theta) = g'(\theta) = x \quad \text{(convergence in } X_0 + X_1\text{)},$$

$x \in [X_0, X_1]_\theta^c$ and so we deduce by (3-1) that

$$\|x\|_{[X_0,X_1]_\theta^c} \le \|x\|_{[X_0,X_1]^\theta} + \varepsilon.$$

Letting ε tend to 0, this estimate completes the proof of claim. Applying the well-known continuous inclusion map with norm less than or equal to 1, see [Bergh and Löfström 1976, Theorem 4.3.1], we learn

$$[X_0, X_1]_\theta \hookrightarrow [X_0, X_1]^\theta$$

which completes the proof of (i).

The proof of (ii) is obvious by (i) and the fact that the unit ball of $([X_0, X_1]_\theta)^{\mathrm{c}}$ is closed in X_0+X_1. □

We will also need results on relationships between the upper complex space $[X_0, X_1]^\theta$ and the *Calderón product* $X_0^{1-\theta} X_1^\theta$ defined for any couple (X_0, X_1) of Banach lattices over a measure space (Ω, Σ, μ), which consists of all $f \in L^0(\mu)$ such that $|f| \le \lambda\, |f_0|^{1-\theta} |f_1|^\theta$ μ-a.e. for some $\lambda > 0$ and $f_j \in B_{X_j}$, $j \in \{0, 1\}$. It is well known, see [Calderón 1964, 13.5, p. 123], that $X_0^{1-\theta} X_1^\theta$ is a Banach lattice endowed with the norm

$$\|f\|_{X_0^{1-\theta} X_1^\theta} = \inf\{\lambda > 0 : |f| \le \lambda\, |f_0|^{1-\theta} |f_1|^\theta,\ f_0 \in B_{X_0},\ f_1 \in B_{X_1}\}$$

and that $X_0^{1-\theta} X_1^\theta$ is continuously embedded into $X_0 + X_1$, which is also a Banach lattice.

We come now to the first theorem of this section.

Theorem 3.2. *For any couple of Banach lattices $\vec{X} = (X_0, X_1)$ over a measure space (Ω, Σ, μ), the following continuous inclusion relation holds with norm less than or equal to* 1:

$$X_0^{1-\theta} X_1^\theta \hookrightarrow [X_0, X_1]^\theta.$$

Before we turn to the proof of this theorem, some comments seem called for. First, in the case of σ-finite measure, this result follows from the vector-valued inclusion proved in [Calderón 1964, p. 125] by taking $B = B_0 = B_1 = \mathbb{C}$. For the reader's convenience, we include a different and transparent proof without assuming the σ-finiteness of the underlying measure space.

Proof of Theorem 3.2. Let $f \in X_0^{1-\theta} X_1^\theta$, so that the estimate $|f| \le |f_0|^{1-\theta} |f_1|^\theta$ holds for some $f_0 \in X_0$ and $f_1 \in X_1$. Note that the support of f is contained in the intersection of the supports of f_0 and f_1. Hence without loss of generality we may suppose that f_0 and f_1 are not equal to zero on Ω.

We define functions

$$F(z) := |f_0|^{1-z} |f_1|^z, \quad G(z) := \int_\theta^z F(w)\, dw, \quad z \in \overline{S}.$$

We claim that $G \in \mathcal{G}(\vec{X})$. To show this we fix $s, t \in \mathbb{R}$ and $j \in \{0, 1\}$. We observe that

$$G(j + is) - G(j + it) = \int_{j+it}^{j+is} |f_0|^{1-w} |f_1|^w\, dw$$

yields $|G(j + is) - G(j + it)| \le |s - t|\, |f_j|$. This implies

$$\max_{j=0,1} \sup_{-\infty<s<t<\infty} \frac{\|G(j + is) - G(j + it)\|_{X_j}}{|s - t|} \le \max_{j=0,1} \|f_j\|_{X_j}. \tag{3-2}$$

We will show that $G : S \to X_0 + X_1$ is analytic. To see this, consider the functions F_0 and F_1 defined by

$$F_0(z) = \chi_{\{|f_0|\ge|f_1|\}}|f_0|^{1-z}|f_1|^z, \quad F_1(z) = F(z) - F_0(z), \quad z \in \overline{S}.$$

We estimate

$$|F_0(z)| = \chi_{\{|f_0|\ge|f_1|\}}|f_0|^{1-\operatorname{Re} z}|f_1|^{\operatorname{Re} z} \le |f_0|.$$

Likewise we have

$$|F_1(z)| \le |f_1|.$$

We define functions

$$G_0(z) = \int_\theta^z F_0(w)\,dw, \quad G_1(z) = \int_\theta^z F_1(w)\,dw \quad z \in \overline{S}.$$

Since $|G_0(z+h) - G_0(z)| \le |h||f_0|$ for all $z, h \in \mathbb{C}$ such that $z+h, z \in \overline{S}$, it follows that $G_0 : \overline{S} \to X_0$ is a continuous function. Similarly, we can establish that $G_1 : \overline{S} \to X_1$ is also continuous.

We now show that the mapping $G_0 : S \to X_0$ is analytic. To this end we fix $0 < \varepsilon < \frac{1}{2}$ and consider the open strip $S_\varepsilon = \{z \in S : \varepsilon < \operatorname{Re} z < 1 - \varepsilon\}$. We note that $F_j(z) \in X_j$ for $z \in S_\varepsilon$.

Since

$$\chi_{\{|f_0|\ge|f_1|\}}\left|\left(\frac{|f_1|}{|f_0|}\right)^\varepsilon \exp\left(h\log\left(\frac{|f_1|}{|f_0|}\right)\right) - \left(\frac{|f_1|}{|f_0|}\right)^\varepsilon\left(1 + h\log\left(\frac{|f_1|}{|f_0|}\right)\right)\right| \prec |h|^2,$$

we conclude that

$$G_0(z+h) - G_0(z) - hF_0(z) = \mathrm{O}(|h|^2) \quad \text{as } h \to 0,$$

in X_0 uniformly over $z \in S_\varepsilon$. Similarly, we can show that

$$G_1(z+h) - G_1(z) - hF_1(z) = \mathrm{O}(|h|^2) \quad \text{as } h \to 0,$$

in X_1 uniformly over $z \in S_\varepsilon$. Combining these calculations, we see that $G|_S = G_0|_S + G_1|_S : S \to X_0 + X_1$ is analytic.

To finish the proof of the claim, we need only to observe that

$$\|G_j(z)\|_{X_j} \le \|G_j(z) - G_j(\operatorname{Re}(z))\|_{X_j} + \|G_j(\operatorname{Re}(z)) - G_j(\theta)\|_{X_j} = \mathrm{O}(|z|+1)$$

for $j \in \{0, 1\}$ keeping in mind that $G_j(\theta) = 0$.

Now observe that $G'(\theta) = |f_0|^{1-\theta}|f_1|^\theta$ and by (3-2) $\|G\|_{\mathcal{G}(\vec{X})} \le \max_{j=0,1}\|f_j\|_{X_j}$. Thus we deduce that $|f| \in [X_0, X_1]^\theta$. Since $[X_0, X_1]^\theta$ is a Banach lattice and f_0 and f_1 are arbitrary, we conclude that

$$X_0^{1-\theta}X_1^\theta \hookrightarrow [X_0, X_1]^\theta,$$

with norm of the continuous inclusion map less than or equal to 1. □

Remark 3.3. The inclusion in the above theorem is proper in general. To see this we recall that Lozanovskii [1972] constructed a closed Banach sublattice Y_0 of a weighted Banach lattice $L^\infty(w)$ on $((0,1), m)$ with Lebesgue measure m, where

$$L^\infty(w) = \{f \in L^\infty(0,1) : wf \in L^\infty(0,1)\}$$

with $w(t) = t$ for all $t \in (0,1)$ and endowed with the norm $\|f\|_{L^\infty(w)} = \|wf\|_{L^\infty}$, such that $(L^\infty(w))^{1-\theta}(L^\infty)^\theta$ and $Y_0^{1-\theta}(L^\infty)^\theta$ are not relative interpolation spaces with respect to $(L^\infty(w), L^\infty)$ and (Y_0, L^∞) for any $\theta \in (0,1)$. We complexify these spaces. Since $[\,\cdot\,]^\theta$ is an interpolation functor in the class of complex Banach spaces, it follows that for the couple $(X_0, X_1) = (Y_0(\mathbb{C}), L^\infty(\mathbb{C}))$ the inclusion

$$X_0^{1-\theta} X_1^\theta \subset [X_0, X_1]^\theta$$

is proper for an arbitrary $\theta \in (0,1)$.

The following result shows isometric relationships between Banach lattices generated by standard operations and Calderón's constructions in the setting of couples of complex Banach lattices on an arbitrary measure space:

Theorem 3.4. *Let $\vec{X} = (X_0, X_1)$ be a couple of complex Banach lattices on an arbitrary measure space (Ω, Σ, μ). Then the following statements are true for all $\theta \in (0,1)$:*

(i) $[X_0, X_1]_\theta \cong (X_0^{1-\theta} X_1^\theta)^\circ$.

(ii) $([X_0, X_1]^\theta)^{\mathrm{c}} \cong (X_0^{1-\theta} X_1^\theta)^{\mathrm{c}}$.

(iii) $[X_0, X_1]^\theta \cong X_0^{1-\theta} X_1^\theta$ *whenever the unit ball of $X_0^{1-\theta} X_1^\theta$ is closed in $X_0 + X_1$.*

Proof. We begin with (i). Since $[\vec{X}]_\theta \cong ([X_0, X_1]^\theta)^\circ$ is a closed subspace of $[\vec{X}]^\theta$ and the norm in $[\vec{X}]_\theta$ is the restriction of the norm in $[\vec{X}]^\theta$, see [Bergh 1979], if follows from Theorem 3.2 that

$$(X_0^{1-\theta} X_1^\theta)^\circ \hookrightarrow ([X_0, X_1]^\theta)^\circ \cong [X_0, X_1]_\theta,$$

with norm of the continuous inclusion map less or equal to 1.

To obtain the reverse inclusion, we recall that Calderón proved

$$[X_0, X_1]_\theta \hookrightarrow (X_0^{1-\theta} X_1^\theta)^\circ$$

for any σ-finite measure space. The proof of the following continuous inclusion map with norm less than or equal to 1 given in [Calderón 1964, (i), p. 125], see also [Krein et al. 1982, pp. 240–241], works for any measure space: combining two the above continuous inclusions, we obtain the statement (i).

To finish the proofs of (ii) and (iii) at the same time, we observe that the above inclusion, combined with Lemma 3.1 and Theorem 3.2, yields continuous inclusion maps with norm less than or equal to 1,

$$X_0^{1-\theta} X_1^\theta \hookrightarrow [X_0, X_1]^\theta \hookrightarrow ([X_0, X_1]_\theta)^{\mathrm{c}} \hookrightarrow ((X_0^{1-\theta} X_1^\theta)^\circ)^{\mathrm{c}} \hookrightarrow (X_0^{1-\theta} X_1^\theta)^{\mathrm{c}}.$$

Clearly these inclusions complete the proofs of statements (ii) and (iii). □

Remark 3.5. We note that in the case of σ-finite measure spaces the above statement (i) was proved by Shestakov [1974], who extended Calderón's result [1964] proved under the assumption that $X_0^{1-\theta} X_1^\theta$ is σ-order continuous.

We will apply Theorem 3.4 to some class of Banach lattices. In what follows we assume that a measure space (Ω, Σ, μ) is such that $L^0(\Omega, \Sigma, \mu)$ is a Dedekind complete vector lattice (i.e., every subset of $L^0(\mu)$ order bounded from above has a supremum). We refer to [Fremlin 1974, Theorem 64 B, p. 170] for a description and general examples of such measure spaces. Let us just notice here that such measure spaces are semifinite (i.e., for any $A \in \Sigma$ with $\mu(A) > 0$, there is $B \in \Sigma$ such that $B \subset A$ and $0 < \mu(B) < \infty$). Since $L^0(\Omega, \Sigma, \mu)$ is a Dedekind complete vector lattice, it follows that for any subset $E \subset L^0(\Omega, \Sigma, \mu)$, the set $\{\chi_{\mathrm{supp}\, x} : x \in E\}$ is order bounded in $L^0(\mu)$. If we put $x_0 := \sup\{\chi_{\mathrm{supp}\, x} : x \in E\}$, then $\mathrm{supp}(E)$ exists and is given by $\mathrm{supp}(E) = \mathrm{supp}\, x_0$.

We shall need to use some results on Köthe duality. We recall that the *Köthe dual* space X' of any Banach lattice X on (Ω, Σ, μ) is defined to be the space of all $x' \in L^0(\mu)$ with $\mathrm{supp}\, x \subset \mathrm{supp}(X)$ such that $xx' \in L^1(\mu)$ for all $x \in X$. It is well known that X' is a Banach lattice on (Ω, Σ, μ) equipped with the norm

$$\|x'\|_{X'} := \sup\left\{ \int_\Omega |xx'|\, d\mu : \|x\|_X \leq 1 \right\}, \quad x' \in X'.$$

As usual, the Köthe dual space of X' is denoted by X''. If a Banach lattice X is such that $X \cong X''$, then X is said to be *maximal*.

We note that if X is a Banach lattice on a σ-finite measure space (Ω, Σ, μ), then $X \cong X''$ if and only if X has the Fatou property; see, [Kantorovich and Akilov 1982, Theorem 7, p. 191]. We recall that X is said to have the *Fatou property* if for any sequence $\{f_m\}_{m=1}^\infty$ of nonnegative elements from X such that $f_m \uparrow f$ for $f \in L^0(\Omega)$ and $\sup_{m \geq 1} \|f_m\|_X < \infty$, one has $f \in X$ and $\|f_m\|_X \uparrow \|f\|_X$.

We will need the following lemma.

Lemma 3.6. *Let X_0, X_1 and X be Banach lattices on a measure space (Ω, Σ, μ). If X is an intermediate space with respect to (X_0, X_1), then $X^{\mathrm{c}} \hookrightarrow X''$ with the norm inclusion less than or equal to* 1.

Proof. Fix $x \in X^{\mathrm{c}}$. Then in a similar fashion to the proof of [Cwikel and Nilsson 2003, Lemma 1.16], we claim that for a given $\varepsilon > 0$, there exists a sequence $\{y_m\}$ in X such that $0 \leq y_m \uparrow |x|$ μ-a.e., $\|y_m\|_X \leq (1+\varepsilon)\|x\|_{X^{\mathrm{c}}}$ for each $m \in \mathbb{N}$ and $y_m \to y$ in $X_0 + X_1$. In fact, it follows by the definition of X^{c} that we can find a sequence $\{z_m\}$ in X such that $\|z_m\|_X \leq (1+\varepsilon)\|x\|_{X^{\mathrm{c}}}$ for each $m \in \mathbb{N}$ and $z_m \to z$ in $X_0 + X_1$. If we set $y_m = \min\{\max\{0, z_m\}, |x|\}$ for each $m \in \mathbb{N}$, then we obtain the desired sequence. We conclude by Lebesgue's monotone convergence theorem that for any $x' \in X'$,

$$\int_\Omega |xx'|\, d\mu = \lim_{m \to \infty} \int_\Omega y_m |x'|\, d\mu \leq (1+\varepsilon)\|x\|_{X^{\mathrm{c}}} \|x'\|_{X'}.$$

Since $\varepsilon > 0$ is arbitrary, the desired continuous inclusion follows. □

We are now ready to state the following result.

Theorem 3.7. *Assume that a measure space (Ω, Σ, μ) is such that $L^0(\Omega, \Sigma, \mu)$ is a Dedekind complete vector lattice and it satisfies the following condition: if $A \subset \Omega$ is such that $A \cap B \in \Sigma$ for every set $B \in \Sigma$, $\mu(B) < \infty$, then $A \in \Sigma$. Let $\vec{X} = (X_0, X_1)$ be an arbitrary couple of complex Banach lattices on (Ω, Σ, μ). If X_0 and X_1 are both maximal and $\mathrm{supp}(X_0) = \mathrm{supp}(X_1) = \Omega$, then*

$$[X_0, X_1]^\theta \cong X_0^{1-\theta} X_1^\theta.$$

Proof. Since both X_0 and X_1 are maximal, it follows by the original Lozanovskii duality formula [1978]

$$(X_0^{1-\theta}X_1^{\theta})' \cong (X_0')^{1-\theta}(X_1')^{\theta}$$

that $X_0^{1-\theta}X_1^{\theta}$ is also a maximal Banach lattice. Thus, we deduce from Lemma 3.6 that

$$X_0^{1-\theta}X_1^{\theta} \hookrightarrow (X_0^{1-\theta}X_1^{\theta})^c \hookrightarrow (X_0^{1-\theta}X_1^{\theta})'' \cong X_0^{1-\theta}X_1^{\theta}$$

with the norm of the inclusion maps less than or equal to 1. □

Corollary 3.8. *Let $\vec{X} = (X_0, X_1)$ be an arbitrary couple of complex Banach lattices with the Fatou property on a σ-finite measure space* (Ω, Σ, μ). *If* $\operatorname{supp}(X_0) = \operatorname{supp}(X_1) = \Omega$, *then*

$$[X_0, X_1]^{\theta} \cong X_0^{1-\theta}X_1^{\theta}.$$

Proof. Clearly any σ-finite measure space (Ω, Σ, μ) satisfies the desired condition from Theorem 3.7. Moreover, it is well known that $L^0(\Omega, \Sigma, \mu)$ is a Dedekind complete vector lattice, see [Kantorovich and Akilov 1982], and so the desired statement follows from Theorem 3.7. □

Remark 3.9. Lozanovskii proved that for all $\theta \in (0, 1)$ we have

$$(X_0^{1-\theta}X_1^{\theta})' = (X_0')^{1-\theta}(X_1')^{\theta}$$

with equality of norms; see [Lozanovskii 1969, Theorem 2]. Using this result for $\theta = \frac{1}{2}$, Lozanovskii [1969] showed $X^{1/2}(X')^{1/2} \simeq L^2$ for any Banach lattice on a given σ-finite measure space. Thus taking X which does not enjoy the Fatou property, we conclude that the Fatou property of $X_0^{1-\theta}X_1^{\theta}$ does not always imply that the Fatou property holds for X_0 and X_1. For further examples refer to [Reisner 1993], where among others it is shown (see Example 2) that there exist σ-order continuous Banach sequence lattices X and Y that do not enjoy the Fatou property such that $X^{1-\theta}Y^{\theta}$ is σ-order continuous and has the Fatou property.

Simple calculation shows that for any Banach lattice X and every $1 < r < \infty$ we have $X^{1/r}(L^{\infty})^{1-1/r} \cong X^{(r)}$; thus by Theorem 3.7 we obtain the following useful formula:

Corollary 3.10. *Let X be a Banach lattice with the Fatou property on a σ-finite measure space. Then, for any* $\theta \in (0, 1)$,

$$[X, L^{\infty}]^{\theta} \cong X^{1-\theta}(L^{\infty})^{\theta} \cong X^{(r)},$$

where $r = (1-\theta)^{-1}$.

An immediate application of our results is the following variant of the Riesz–Thorin interpolation theorem:

Theorem 3.11. *Let (X_0, X_1) and (Y_0, Y_1) be couples of complex Banach lattices on measure spaces. Then for every linear operator $T : (X_0, X_1) \to (Y_0, Y_1)$ and all $\theta \in (0, 1)$, T is bounded from $X_0^{1-\theta}X_1^{\theta}$ into $(Y_0^{1-\theta}Y_1^{\theta})^{\mathrm{c}}$ and satisfies*

$$\|T\|_{X_0^{1-\theta}X_1^{\theta} \to (Y_0^{1-\theta}Y_1^{\theta})^{\mathrm{c}}} \le \|T\|_{X_0 \to Y_0}^{1-\theta}\|T\|_{X_1 \to Y_1}^{\theta}.$$

In particular if Y_0 *and* Y_1 *are Banach lattices on a* σ*-finite measure space* (Ω, Σ, μ) *with* $\operatorname{supp}(Y_0) = \operatorname{supp}(Y_1) = \Omega$ *and both enjoy the Fatou property, then* T *is bounded from* $X_0^{1-\theta} X_1^\theta$ *into* $Y_0^{1-\theta} Y_1^\theta$ *with the norm estimate*

$$\|T\|_{X_0^{1-\theta} X_1^\theta \to Y_0^{1-\theta} Y_1^\theta} \le \|T\|_{X_0 \to Y_0}^{1-\theta} \|T\|_{X_1 \to Y_1}^\theta.$$

Proof. We have $\|T\|_{[X_0, X_1]^\theta \to [X_0, X_1]^\theta} \le \|T\|_{X_0 \to Y_0}^{1-\theta} \|T\|_{X_1 \to Y_1}^\theta$ according to [Bergh and Löfström 1976, Theorem 4.1.4]. Since $X_0^{1-\theta} X_1^\theta \hookrightarrow [X_0, X_1]^\theta$ by Theorem 3.2 with the inclusion constant less than or equal to 1 and $[Y_0, Y_1]^\theta \hookrightarrow (Y_0^{1-\theta} Y_1^\theta)^{\mathrm{c}}$ by Lemma 3.1 again with the inclusion constant less than or equal to 1, the first required estimate follows. This estimate combined with the continuous inclusions shown in the proof of Theorem 3.7 yields the second estimate. □

The following lemma is surely well known to specialists, but we include a proof for the sake of completeness.

Lemma 3.12. *If* $0 < \theta < 1$ *then for any couple* $\vec{X} = (X_0, X_1)$ *of complex Banach spaces, we have* $[X_0, X_1]^\theta \hookrightarrow (X_0, X_1)_{\theta,\infty}$ *with the norm of the inclusion map less or equal to* 1.

Proof. It is well known that for any Banach couples $\vec{X} = (X_0, X_1)$ and $\vec{Y} = (Y_0, Y_1)$ and any operator $T : \vec{X} \to \vec{Y}$, we have the following estimate for restrictions of T (see [Bergh and Löfström 1976, Theorem 4.1.4]):

$$\|T\|_{[\vec{X}]^\theta \to [\vec{Y}]^\theta} \le \|T\|_{X_0 \to Y_0}^{1-\theta} \|T\|_{X_1 \to Y_1}^\theta. \tag{3-3}$$

We fix $t > 0$ and $x \in [X_0, X_1]^\theta$. By the Hahn–Banach theorem, there exists a continuous linear functional $x^* \in (X_0 + X_1)^*$ such that $x^*(x) = K(t, x; \vec{X})$ and

$$|x^*(y)| \le K(t, y; \vec{X}), \quad y \in X_0 + X_1.$$

This implies that the linear operator $T : X_0 + X_1 \to \mathbb{C}$ defined by $T(y) = x^*(y)$ for all $y \in X_0 + X_1$ satisfies $T : (X_0, X_1) \to (\mathbb{C}, \mathbb{C})$ with $\|T\|_{X_0 \to \mathbb{C}} \le 1$ and $\|T\|_{X_1 \to \mathbb{C}} \le t$.

Now we apply the estimate (3-3) to $(Y_0, Y_1) = (\mathbb{C}, \mathbb{C})$ and the obvious equality $[\mathbb{C}, \mathbb{C}]^\theta \cong \mathbb{C}$ to get that

$$K(t, x; \vec{X}) = x^*(x) = T(x) \le t^\theta \|x\|_{[\vec{X}]^\theta}.$$

Since $t > 0$ and $x \in [X_0, X_1]^\theta$ are arbitrary, the proof is complete. □

We conclude with the remark that if (X_0, X_1) is a couple of complex Banach lattices which enjoy the Fatou property, then the formula $[X_0, X_1]^\theta = X_0^{1-\theta} X_1^\theta$ (up to equivalence of norms) is a consequence of abstract interpolation results, combined with relationships between the Köthe duality results and the orbital descriptions of the upper complex method and of other interpolation constructions; see [Ovchinnikov 1984, pp. 474–492] for more details.

4. On Calderón–Mityagin couples of Morrey spaces

As is mentioned in the Introduction, one of the fundamental problems in the theory of interpolation spaces is the description of all interpolation spaces X with respect to a given compatible couple of Banach spaces $\vec{X} = (X_0, X_1)$. Cwikel [1981] posed the question of whether in fact *all* Calderón–Mityagin couples can be identified by checking whether their complex interpolation spaces are K-spaces. In [Mastyło and Ovchinnikov 1997] the authors provided counterexamples, which give a negative answer to this question.

Also as is mentioned in the Introduction there is no complete description of the complex interpolation spaces between Morrey spaces in the general case. We show in this section that the complex spaces with respect to any couple of Morrey spaces which are not L^p-spaces cannot be described by the K-method of interpolation. In particular this implies that these couples are not Calderón–Mityagin couples.

Before we state and prove the main results of this section we need some technical observations. Suppose that we have a nonnegative measurable function on $\mathbb{R}^n$ which is rotationally symmetric, so that it can be expressed as $f(|x|)$, where $f : [0, \infty) \to \mathbb{R}_+$ is a measurable function. Recall that $|B(x, r)| = v_n r^n$ for all $x \in \mathbb{R}^n$, where $v_n := \pi^{n/2}/\Gamma(1 + n/2)$ is the measure of the unit ball $B(1)$.

Then a standard calculation via applying spherical coordinates gives

$$\int_{\mathbb{R}^n} f(|x|)\,dx = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_{\mathbb{R}_+} f(t)t^{n-1}\,dt.$$

Since the ball $B(x, r)$ has the same measure as $B(0, r)$ for every $r > 0$, by choosing $f = \chi_{(0,r)}$, we obtain

$$|B(x, r)| = |B(0, r)| = \int_{\mathbb{R}^n} \chi_{(0,r)}(|y|)\,dy = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^r t^{n-1}\,dt = v_n r^n.$$

Combining these formulas we conclude that if $1 < s < \infty$ and $f(t) = t^{-n/s}\chi_{(0,r)}$ for all $t \geq 0$, then

$$\int_{B(0,r)} |x|^{-n/s}\,dx = v_n \frac{s}{s-1} r^{n-n/s}.$$

This shows that the function $x \mapsto |x|^{-n/s}$ belongs to $L\mathcal{M}_1^s$ and its norm is equal to ${v_n}^{1/s} s/(s-1)$. In what follows we will use that this function belongs to the Morrey space $\mathcal{M}_1^s$. For the sake of completeness we include a proof of this fact.

Proposition 4.1. *If $1 < s < n$, then the function $x \mapsto |x|^{-n/s}$ belongs to $\mathcal{M}_1^s$ and its norm is equal to ${v_n}^{1/s} s/(s-1)$.*

Proof. Let $g(x) = |x|^{-n/s}$ for all $x \in \mathbb{R}^n$ (we put $g(0) := 0$). We observe that

$$|\{y \in \mathbb{R}^n : |g(y)| > \tau\}| = |B(0, \tau^{-s/n})| = v_n \tau^{-s}, \quad \tau > 0.$$

Hence, for any $x \in \mathbb{R}^n$ and any $r > 0$, we have

$$\begin{aligned}\int_{B(x,r)} |g(y)|\,dy &= \int_0^\infty |\{y \in \mathbb{R}^n : \chi_{B(x,r)}(y)g(y) > \tau\}|\,d\tau \\ &\leq \int_0^\infty \min\bigl\{|B(x, r)|,\ |\{y \in \mathbb{R}^n : g(y) > \tau\}|\bigr\}\,d\tau = v_n \int_0^\infty \min\{r^n, \tau^{-s}\}\,d\tau \\ &= v_n r^{n(1-1/s)} + v_n \int_{r^{-n/s}}^\infty \tau^{-s}\,d\tau = v_n \frac{s}{s-1} r^{n(1-1/s)} \\ &= {v_n}^{1/s} \frac{s}{s-1} |B(x, r)|^{1-1/s}.\end{aligned}$$

The above estimate combined with previous calculation completes the proof. □

According to Lemma 3.12, for a Banach couple (A_0, A_1), $[A_0, A_1]^\theta$ is a subspace of $(A_0, A_1)_{\theta,\infty}$ for any $\theta \in (0,1)$. More precisely, $[A_0, A_1]^\theta \hookrightarrow (A_0, A_1)_{\theta,\infty}$ with the norm of the inclusion map less than or equal to 1. Concerning this inclusion, we will also need the following useful Proposition 4.2:

Here and below in this paper we let $1 \le q_0 \le p_0 \le \infty$ and $1 \le q_1 \le p_1 \le \infty$ satisfy $\min(p_0, p_1) < \infty$.

We will apply Proposition 4.2 below to Morrey spaces. Again according to Lemma 3.12,

$$[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]^\theta \hookrightarrow (\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})_{\theta,\infty}.$$

We have the following result.

Proposition 4.2. *Let $\vec{A} = (A_0, A_1)$ be a Banach couple, and let A be any interpolation space with respect to $\vec{A}$ that is a proper subspace of $\vec{A}_{\theta,\infty}$ for some $\theta \in (0,1)$. If there exists $a_\infty \in A$ such that one of the following conditions is satisfied, then $\vec{A}$ is not a Calderón–Mityagin couple:*

(i) *$K(t, a_\infty; \vec{A}) \succ t^\theta$ for all $t \in (0,\infty)$.*

(ii) *$A_1 \hookrightarrow A_0$ and there exists $t_0 > 0$ such that $K(t, a_\infty; \vec{A}) \succ t^\theta$ for all $t \in (0, t_0]$.*

Proof. (i) For every $a \in \vec{A}_{\theta,\infty}$ we set $x_a = \|a\|_{\vec{A}_{\theta,\infty}} a_\infty \in A$. Our hypothesis gives that

$$K(t, x_a; \vec{A}) \succ t^\theta \|a\|_{\vec{A}_{\theta,\infty}} \ge K(t, a; \vec{A}), \quad t > 0.$$

Suppose that A is a proper subset of $\vec{A}_{\theta,\infty}$. If $\vec{A}$ were a Calderón–Mityagin couple, then we would get $a = T(x_a)$ for some operator $T : \vec{A} \to \vec{A}$, and therefore, by interpolation, $a \in A$. Since $a \in \vec{A}_{\theta,\infty}$ is arbitrary, this would imply that $\vec{A}_{\theta,\infty} \subset A$ and

$$A = \vec{A}_{\theta,\infty},$$

which is a contradiction with our hypothesis.

(ii) There is no loss of generality in assuming that t_0 is equal to $\|\mathrm{id}\|_{A_1 \to A_0}$, the operator norm of the embedding of A_1 into A_0. It is clear that $K(t, x; \vec{A}) = \|a\|_{A_0}$ for every $a \in A_0$ and $t \ge t_0$. In this case our hypothesis implies that for each $a \in \vec{A}_{\theta,\infty}$ there exists an element $x_a \in A$ which is a suitable scalar multiple of a_∞ which satisfies

$$K(t, a; \vec{A}) \prec K(t, x_a; \vec{A}), \quad t \in (0, t_0],$$

and also $\|a\|_{A_0} \le \|x_a\|_{A_0}$. Therefore, $K(t, a; \vec{A}) \prec K(t, x_a; \vec{A})$ for all $t > 0$ and this enables us to complete the proof via the same reasoning as in part (i). $\square$

Theorem 4.3. *Let $1 \le q_0 < p_0 < \infty$ and $1 \le q_1 \le p_1 \le \infty$ with $q_0 \ne q_1$. Then for any $\theta \in (0,1)$ the inclusion $[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]^\theta \subset (\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})_{\theta,\infty}$ is proper.*

Proof. Define p and q by

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

We distinguish two cases: (1) $p_1 < \infty$ and (2) $p_1 = \infty$.

Case (1): $p_1 < \infty$. Since $p_0 < \infty$ and $q_0 \neq q_1$, the proof of [Lemarié-Rieusset 2013, case (b), p. 751] shows that $(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})_{\theta,\infty}$ is not embedded into $\mathcal{M}^p_q$. Meanwhile, if we go through an argument similar to the one to prove $(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})_{\theta,q} \hookrightarrow \mathcal{M}^p_q$ using the result $(L^{q_0}, L^{q_1})_{\theta,q} = L^q$ by Calderón, we have $[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]^\theta \hookrightarrow \mathcal{M}^p_q$. Combining these observations, we conclude that $[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]^\theta$ is a proper subspace of $(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})_{\theta,\infty}$.

Case (2): $p_1 = \infty$. We have $\mathcal{M}^{p_1}_{q_1} = L^\infty$ isometrically for all choices of q_1 in the permitted range $[1, \infty]$.

We apply [Cwikel and Gulisashvili 2000, Theorem 4] again to see that $(X, L^\infty)_{\theta,\infty}$ is the space of all $f \in L^0$ in a quasi-Banach lattice X_θ endowed with the quasinorm

$$\|f\|_{X_\theta} := \sup_{\lambda>0} \lambda \, \|\chi_{\{|f|>\lambda\}}\|_X^{1-\theta} < \infty,$$

whenever X is a Banach function lattice on a σ-finite complete nonatomic measure space (Ω, Σ, μ). By Corollary 3.10, we have $[\mathcal{M}^{p_0}_{q_0}, L^\infty]^\theta = (\mathcal{M}^{p_0}_{q_0})^{(r)} = \mathcal{M}^{rp_0}_{rq_0}$. Furthermore, the inclusion $\mathcal{M}^p_q \subset \mathrm{w}\mathcal{M}^p_q$ is proper for every $1 \le q < p < \infty$ (see Lemma 4.7(ii)), we conclude that the inclusion

$$[\mathcal{M}^{p_0}_{q_0}, L^\infty]^\theta = \mathcal{M}^{rp_0}_{rq_0} \hookrightarrow (\mathcal{M}^{p_0}_{q_0}, L^\infty)_{\theta,\infty} = \mathrm{w}\mathcal{M}^{rp_0}_{rq_0}$$

is also proper, and so the proof is complete. □

We also will need the following results: the first one is motivated by [Brudnyi and Krugljak 1991, Theorem 4.5.5].

Theorem 4.4. *Let $\vec{X} = (X_0, X_1)$ be a Banach couple, and let $r \in (0, 1)$, $\theta \in (0, 1)$ and $\gamma \in (1, \infty)$ be fixed. Assume that for each $j \in \mathbb{J} = \mathbb{Z}$ (resp. $j \in \mathbb{J} = \mathbb{Z}_+$) there exists $v_j \in X_0 + X_1$ such that*

$$\min\{1, r^{-j} t\} \le K(t, v_j; \vec{X}) \le \gamma \min\{1, r^{-j} t\}, \quad t > 0 \ \ (\text{resp. } t \in (0, 1]).$$

Then, for a certain positive integer N which depends on r, θ and γ, the element $x_\theta \in X_0 + X_1$ defined by

$$x_\theta = \sum_{j \in \mathbb{J} \cap N\mathbb{Z}} r^{j\theta} v_j$$

satisfies

$$K(t, x_\theta, \vec{X}) \asymp t^\theta \quad \text{for all } t > 0 \ \ (\text{resp. for all } t \in (0, 1]).$$

Proof. Fix $\theta \in (0, 1)$ and $r \in (0, 1)$. It is easy to check that there is a constant $C = C(r, \theta) > 1$ such that for each positive integer N we have

$$\sum_{j \in \mathbb{J} \cap N\mathbb{Z}} r^{j\theta} \min\{1, r^{-j} t\} \le C t^\theta, \quad t > 0. \tag{4-1}$$

Thus, we conclude by our hypothesis that for $\mathbb{J} = \mathbb{Z}$ (resp. $\mathbb{J} = \mathbb{Z}_+$) the series

$$x_\theta := \sum_{j \in \mathbb{J} \cap N\mathbb{Z}} r^{j\theta} v_j$$

converges in $X_0 + X_1$.

Combining (4-1) with the right-hand inequality of our hypothesis yields that for x_θ we have, for all $t \in (0,\infty)$ (resp. $t \in (0,1]$),

$$K(t, x_\theta; \vec{X}) \le \sum_{j \in \mathbb{J}} r^{j\theta} K(t, v_j; \vec{X}) \le \gamma \sum_{j \in \mathbb{J}} r^{j\theta} \min\{1, r^{-j} t\} \le \gamma C t^\theta.$$

The sums $\sum_{m=1}^{\infty} r^{N\theta m}$ and $\sum_{m=-\infty}^{-1} r^{-N(1-\theta)m}$ can be made arbitrarily small by choosing $N \in \mathbb{N}$ large enough. So it is clear that we can choose a positive integer N which depends only on r, θ and γ and which is large enough to satisfy

$$\gamma \sum_{m \in \mathbb{Z} \setminus \{0\}} \min\{r^{N\theta m}, r^{-N(1-\theta)m}\} \le \frac{1}{2}.$$

For this N and for each $j \in \mathbb{J} \cap N\mathbb{Z}$, setting $j_0 = j/N$, we consequently have

$$\begin{aligned} \gamma \sum_{k \in \mathbb{J} \cap N\mathbb{Z} \setminus \{j\}} r^{k\theta} \min\{1, r^{j-k}\} &\le r^{j\theta} \gamma \sum_{k \in N\mathbb{Z} \setminus \{j\}} \min\{r^{(k-j)\theta}, r^{(j-k)(1-\theta)}\} \\ &\le r^{j\theta} \gamma \sum_{m \in \mathbb{Z} \setminus \{j_0\}} \min\{r^{N(m-j_0)\theta}, r^{N(j_0-m)(1-\theta)}\} \\ &= r^{j\theta} \gamma \sum_{m \in \mathbb{Z} \setminus \{0\}} \min\{r^{Nm\theta}, r^{-Nm(1-\theta)}\} \le \frac{1}{2} r^{j\theta}. \end{aligned}$$

So, we get that for each $j \in \mathbb{J} \cap N\mathbb{Z}$,

$$\begin{aligned} K(r^j, x_\theta; \vec{X}) &\ge K(r^j, r^{j\theta} v_j; \vec{X}) - K\Bigl(r^j, \sum_{k \in \mathbb{J} \cap N\mathbb{Z} \setminus \{j\}} r^{k\theta} v_k; \vec{X}\Bigr) \\ &\ge r^{j\theta} - \sum_{k \in \mathbb{J} \cap N\mathbb{Z} \setminus \{j\}} r^{k\theta} K(r^j, v_k; \vec{X}) \\ &\ge r^{j\theta} - \gamma \sum_{k \in \mathbb{J} \cap N\mathbb{Z} \setminus \{j\}} r^{k\theta} \min\{1, r^{j-k}\} \ge \frac{1}{2} r^{j\theta}. \end{aligned}$$

To conclude the proof, observe that for a given $t > 0$ (resp. $t \in (0,1]$) there is an integer $j \in N\mathbb{Z}$ (resp. $j \in \mathbb{Z}_+ \cap N\mathbb{Z}$) such that $r^j \le t \le r^{j-N}$. Then the above estimate yields

$$K(t, x_\theta; \vec{X}) \ge K(r^j, x_\theta; \vec{X}) \ge \tfrac{1}{2} r^{j\theta} \ge \tfrac{1}{2} r^{N\theta} t^\theta.$$

Thus $K(t, x_\theta; \vec{X}) \asymp t^\theta$ for all $t > 0$ (resp. $t \in (0,1]$), as required. □

As an application we obtain the following result:

Lemma 4.5. *Let the set $\mathbb{J}$ be either $\mathbb{Z}$ or $\mathbb{Z}_+$. Let $\mathcal{T}$ be the interval $(0,\infty)$ if $\mathbb{J} = \mathbb{Z}$ or $(0,1]$ if $\mathbb{J} = \mathbb{Z}_+$. Let (X_0, X_1) be a couple of Banach lattices on a complete σ-finite measure space (Ω, Σ, μ), and let $r \in (0,1)$, $\theta \in (0,1)$ be fixed. Assume that there exists a sequence $\{F_j\}_{j \in \mathbb{J}}$ in Σ with positive measures such that $\|\chi_{F_j}\|_{X_1} \asymp r^{-j} \|\chi_{F_j}\|_{X_0}$ and that $K(t, \chi_{F_j}; \vec{X}) \asymp \min\{\|\chi_{F_j}\|_{X_0}, t\|\chi_{F_j}\|_{X_1}\}$ for each $j \in \mathbb{J}$*

and all $t \in \mathcal{T}$. Then there exists a positive integer N for which the function f_θ defined by

$$f_\theta = \sum_{j \in \mathbb{J} \cap N\mathbb{Z}} r^{j\theta} \frac{\chi_{F_j}}{\|\chi_{F_j}\|_{X_0}}$$

has the following two properties:

(i) *$f_\theta \in X_0 + X_1$ and $K(t, f_\theta; \vec{X}) \asymp t^\theta$ for all $t \in \mathcal{T}$.*

(ii) *If furthermore $\{F_j\}_{j \in \mathbb{J}}$ is a nondecreasing sequence and if $\chi_{\bigcup_{j \in \mathbb{J} \cap N\mathbb{Z}} F_j} \in X_0$ and if $f_\theta^{1/\theta} \in X_1$, then $f_\theta \in {X_0}^{1-\theta} {X_1}^\theta$.*

Proof. (i) If we let $v_j := \chi_{F_j}/\|\chi_{F_j}\|_{X_0}$ we have, for each $j \in \mathbb{J}$,

$$K(t, v_j; \vec{X}) \asymp \min\{1, r^{-j} t\}, \quad t \in \mathcal{T}.$$

Thus the statement (i) follows from Theorem 4.4.

(ii) This is a direct consequence of the decomposition:

$$f_\theta = (\chi_{\bigcup_{j \in \mathbb{J} \cap N\mathbb{Z}} F_j})^{1-\theta} (f_\theta^{1/\theta})^\theta \in X_0^{1-\theta} X_1^\theta. \qquad \square$$

For $s > 1$, we write $s' = s/(s-1)$. We will need the following useful lemma:

Lemma 4.6. *For a given $s > 1$ we put $\alpha = 2^{-s'}$. For each $\varepsilon \in \{0,1\}^n$, we define an affine map $f_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ by*

$$f_\varepsilon(x) = \alpha x + (1-\alpha)\varepsilon, \quad x \in \mathbb{R}^n.$$

We also define two sequences $\{F_j\}_{j=0}^\infty$ and $\{E_j\}_{j=0}^\infty$ of subsets of $\mathbb{R}^n$ by $F_j = \alpha^{-j} E_j$ for each $j \geq 0$, where $E_0 := [0,1]^n$ and E_j are given by

$$E_j := \bigcup_{\varepsilon \in \{0,1\}^n} f_\varepsilon(E_{j-1}), \quad j \in \mathbb{N}.$$

Then the following statements are true:

(i) *$F_j \subset F_{j+1}$ for each $j \geq 0$.*

(ii) *F_j is made up of 2^{jn} pairwise disjoint cubes of volume 1 for each $j \in \mathbb{N}$.*

(iii) *For all $1 < u < \infty$ and each $j \geq 0$ we have*

$$\|\chi_{F_j}\|_{\mathcal{M}_1^u} \asymp \max\{1, \alpha^{jn/u - jn} 2^{-jn}\} = \max\{1, \alpha^{jn/u - jn/s}\},$$

where the constants of equivalence do not depend on j.

(iv) *Let $1 \leq q_0 \leq p < \infty$ satisfy $s = p/q_0$. Then $\chi_{\bigcup_{j \in \mathbb{Z}_+} F_j} \in \mathcal{M}_{q_0}^p$.*

(v) *For every $x \in \mathbb{R}^n$ and each $j \in \mathbb{N}$, we have*

$$M\chi_{F_j}(x) \succ \sum_{k=1}^{j} \frac{\chi_{F_k}(\alpha^{j-k} x)}{\|\chi_{F_k}(\alpha^{j-k} \cdot)\|_{L\mathcal{M}_1^s}^*},$$

where $\|\cdot\|^*_{L\mathcal{M}^s_1}$ *is the local Morrey norm generated by cubes*:

$$\|f\|^*_{L\mathcal{M}^s_1} := \sup_{r>0} r^{n/p-n} \int_{[-r,r]^n} |f(y)|\,dy < \infty.$$

Proof. The statements (i) and (ii) are obvious. So we concentrate on other parts.

(iii) The statement follows from the equivalence below obtained via standard calculations using an equivalent norm in Morrey spaces generated by cubes, and the fact that α satisfies $2\alpha = \alpha^{1/s}$:

$$\|\chi_{E_j}\|^*_{\mathcal{M}^u_1} \asymp \max\{\alpha^{jn/u}, 2^{jn}\alpha^{jn}\} = 2^{jn}\alpha^{jn}\max\{1,\ \alpha^{jn/u-jn}2^{-jn}\}, \quad j \geq 0.$$

We include the proof of the equivalence $\|\chi_{E_j}\|^*_{\mathcal{M}^u_1} \asymp \max\{\alpha^{jn/u}, 2^{jn}\alpha^{jn}\}$ for the reader's convenience. From the definition of $\|\cdot\|^*_{\mathcal{M}^u_1}$, we have

$$\|\chi_{E_j}\|^*_{\mathcal{M}^u_1} = \sup_Q |Q|^{1/u-1}\left(\int_Q \chi_{E_j}(y)\,dy\right) = \sup_Q |Q|^{1/u-1}|Q \cap E_j|,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$ with sides parallel to coordinate axes. Since $E_j \subset [0,1]^n$, we get

$$\|\chi_{E_j}\|^*_{\mathcal{M}^u_1} = \sup_Q |Q|^{1/u-1}|Q \cap [0,1]^n \cap E_j|.$$

Thus it follows that we may suppose that Q intersects $[0,1]^n$. Assume that Q is such a cube. Translate Q to have a cube R of the same volume as Q so that $Q \cap [0,1]^n \subset R$ and $R \cap [0,1]^n$ is a cube. Then

$$\begin{aligned}|Q|^{1/u-1}|Q \cap [0,1]^n \cap E_j| &\leq |R|^{1/u-1}|R \cap [0,1]^n \cap E_j| \\ &\leq |R \cap [0,1]^n|^{1/u-1}|R \cap [0,1]^n \cap E_j|.\end{aligned}$$

So, we arrive at

$$\|\chi_{E_j}\|^*_{\mathcal{M}^u_1} = \sup_Q |Q|^{1/u-1}|Q \cap E_j|,$$

where the supremum is taken over all cubes Q in $[0,1]^n$ with sides parallel to coordinate axes.

Using the cubes $[0,1]^n$ and $[0,\alpha^j]^n$, we have

$$\|\chi_{E_j}\|^*_{\mathcal{M}^u_1} \geq \max\{\alpha^{jn/u}, 2^{jn}\alpha^{jn}\}.$$

To show the opposite estimate, we notice that if $|Q| \leq \alpha^{jn}$, then we have

$$|Q|^{1/u-1}|Q \cap E_j| \leq |Q|^{1/u} \leq \alpha^{jn/u}.$$

Assume that $\alpha^{kn} \leq |Q| \leq \alpha^{(k-1)n}$ for some $k \in \{1,\ldots,j\}$. We first observe that it follows by $E_{k-1} \supset E_j$ that

$$|Q \cap E_j| = |Q \cap E_{k-1} \cap E_j|.$$

Now note that E_{k-1} is made up of $2^{n(k-1)}$ disjoint compact cubes of volume $\alpha^{(k-1)n}$. In view of the size of Q, we know Q can intersect at most 2^n of them, say, $Q^1, \dots, Q^L$ with $L \le 2^n$. Then we have

$$|Q \cap E_j| = |Q \cap E_{k-1} \cap E_j| \le \sum_{l=1}^{L} |Q^l \cap E_j| = L2^{n(j-k+1)}\alpha^{jn} \le 2^{n+n(j-k+1)}\alpha^{jn},$$

since $Q^l \cap E_j$ is made up of $2^{n(j-l)}$ disjoint cubes of volume α^{jn}. As a consequence,

$$|Q|^{1/u-1}|Q \cap E_j| \le \alpha^{kn/u-kn}2^{n+n(j-k+1)}\alpha^{jn} \le 4^n \max\{\alpha^{jn/u}, 2^{jn}\alpha^{jn}\},$$

as required.

(iv) Since $\mathcal{M}^u_w = (\mathcal{M}^{u/w}_1)^{(w)}$ for every $1 \le u < w < \infty$, (iii) yields that for all integers $j \ge 0$, we have the equivalence

$$\|\chi_{F_j}\|_{\mathcal{M}^p_{q_0}} \asymp \max\{1, \alpha^{-jn/s+jn/s}\} = 1.$$

(v) Fix $j \in \mathbb{N}$. Let

$$G_k = \{x \in \mathbb{R}^n : \alpha^{j-k}x \in F_k\}, \quad k \in \{0, \dots, j\}.$$

Then $G_0 = [0, \alpha^{-j}]^n \supset G_1 \supset \cdots \supset G_j = F_j$. If $x \in G_j$, then the conclusion is clear since $F_j = G_j$. If $x \in \mathbb{R}^n \setminus G_0$, then the conclusion is again clear since the right-hand side is zero. Assume otherwise; $x \in G_k \setminus G_{k+1}$ for some $k \in \{0, \dots, j-1\}$. Let $H_k(x)$ be the connected component of G_k containing x. By translation we may assume that $x \in [0, \alpha^{-j+k}]^n = H_k(x)$. Then

$$\begin{aligned} M\chi_{F_j}(x) &\ge \frac{|F_j \cap [0, \alpha^{-j+k}]^n|}{|H_k(x)|} = 2^{(j-k)n}\alpha^{(j-k)n} \\ &\succ \frac{\chi_{F_k}(\alpha^{j-k}x)}{\|\chi_{F_k}(\alpha^{j-k}\cdot)\|^*_{L\mathcal{M}^s_1}} \succ \sum_{m=1}^{j} \frac{\chi_{F_m}(\alpha^{j-m}x)}{\|\chi_{F_m}(\alpha^{j-m}\cdot)\|^*_{L\mathcal{M}^s_1}}. \end{aligned}$$ □

Lemma 4.7 below is somewhat known. However we include a proof for completeness since we will use it later. In the proof we will use the Hardy–Littlewood maximal operator $M : L^1_{\mathrm{loc}} \to L^0$, which is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy, \quad x \in \mathbb{R}^n.$$

Lemma 4.7. *The following statements are true*:

(i) *The Hardy–Littlewood maximal operator M is unbounded both on $\mathcal{M}^s_1$ and on $L\mathcal{M}^s_1$ for every $s \in [1, \infty)$.*

(ii) *The inclusions $L\mathcal{M}^p_q \hookrightarrow \mathrm{w}L\mathcal{M}^p_q$ and $\mathcal{M}^p_q \hookrightarrow \mathrm{w}\mathcal{M}^p_q$ are proper for all $1 \le q \le p < \infty$.*

Proof. (i) Let $s = 1$. It is a classical fact that if Mf is in L^1 for $f \in L^1_{\mathrm{loc}}$, then $f = 0$ a.e. and M cannot be bounded on L^1. Let $1 < s < \infty$. The statement that M is not bounded in $\mathcal{M}^s_1$ is an immediate consequence of Nakai's result [2008, Corollary 2.5, p. 205] on necessary and sufficient conditions for the boundedness of M on generalized Orlicz–Morrey spaces $L^{(\Phi,\phi)}$. In fact the Morrey space $\mathcal{M}^s_1$ is the

Orlicz–Morrey space generated by the Young function $\Phi(t) = t$ and the function $\phi(t) = t^{-1/s}$ for all $t \geq 0$. Recall that $\mathcal{M}_1^s = L^{(1,\lambda)}$ and $\mathrm{w}\mathcal{M}_1^s = L^{(1,\lambda)}_{\mathrm{weak}}$ if $\lambda = n - n/s$.

The necessary condition $\Phi \in \nabla_2$ (i.e., that there exists $k \geq 1$ such that $\Phi(t) \leq \frac{1}{2k}\Phi(kt)$ for all $t > 0$) is not satisfied. A careful analysis of the proof of the mentioned result of Nakai based on a key observation in [Nakai 2008, Lemma 4.10] also gives that the operator M is not bounded in $L\mathcal{M}_1^s$. We point out here that by using the sequence $\{F_j\}_{j=1}^{\infty}$ defined in the proof of Theorem 4.8 below, we can also disprove that M is bounded on $L\mathcal{M}_1^s$ for all $s > 1$. We include a short and transparent proof of our own, for the reader's convenience. Let $\{F_j\}_{j=0}^{\infty}$ be the sequence constructed in the proof of Lemma 4.6. Let $j \in \mathbb{N}$ be arbitrary. Then from Lemma 4.6(v), we get

$$|[-\alpha^{-j}, \alpha^{-j}]^n\|^{1/s-1} \int_{[-\alpha^{-j},\alpha^{-j}]^n} M\chi_{F_j}(x)\,dx \succ \sum_{k=1}^{j} \frac{\|\chi_{F_k}(\alpha^{j-k}\cdot)\|^*_{\mathcal{M}_1^s}}{\|\chi_{F_k}(\alpha^{j-k}\cdot)\|^*_{\mathcal{M}_1^s}} = j.$$

This proves that M is not bounded on $L\mathcal{M}_1^s$.

(ii) We apply the fact that M is bounded from $\mathcal{M}_1^s$ to $\mathrm{w}\mathcal{M}_1^s$, see, e.g.,. [Nakai 2008, Corollary 6.3, p. 207], and also that M is bounded from $L\mathcal{M}_1^s$ to $\mathrm{w}L\mathcal{M}_1^s$. The second fact easily follows from [Burenkov and Guliyev 2004, Lemma 10], which yields that there exists a positive constant C such that for all $f \in L^1_{\mathrm{loc}}$ and $r > 0$,

$$\sup_{\lambda>0} \lambda|\{x \in B(r) : Mf(x) > \lambda\}| \leq Cr^n \int_r^{\infty} \left(\frac{1}{t^{n+1}} \int_{B(t)} |f(x)|\,dx\right) dt.$$

If $f \in L\mathcal{M}_1^s$, then simple calculus yields that

$$\sup_{\lambda>0} \lambda|\{x \in B(r) : Mf(x) > \lambda\}| \leq Cr^n \int_r^{\infty} t^{-n/s-1}\|f\|_{L\mathcal{M}_1^s}\,dt = Cr^{n-n/s}\|f\|_{L\mathcal{M}_1^s}.$$

As a result we get

$$\|Mf\|_{\mathrm{w}\mathcal{M}_1^s} = \sup_{\lambda,r>0} |B(r)|^{1/s-1}|\{x \in B(r) : Mf(x) > \lambda\}| \leq C\|f\|_{L\mathcal{M}_1^s},$$

as required. Let $f_0 = \lim_{j\to\infty} \chi_{F_j} \in L\mathcal{M}_1^s$, where each F_j is as in Lemma 4.6. From the proof of (i), Lemma 4.6(iv) and (v), $Mf_0 \in \mathrm{w}L\mathcal{M}_1^s \setminus L\mathcal{M}_1^s$. If $1 < q < p < \infty$, then we let $s := p/q$ and define $g_0 = (Mf_0)^{1/q}$. Combining these observations, we get $g_0 \in (\mathrm{w}L\mathcal{M}_1^s)^{(q)} = \mathrm{w}L\mathcal{M}_q^p$ and $g_0 \notin (L\mathcal{M}_1^s)^{(q)} = L\mathcal{M}_q^p$ and so $L\mathcal{M}_q^p \neq \mathrm{w}L\mathcal{M}_q^p$. Similarly we can show that $\mathcal{M}_q^p \neq \mathrm{w}\mathcal{M}_q^p$.

Since the case $p = q$ reduces to the well-known fact that $L^p \neq L^{p,\infty}$, the proof is complete. □

With all these preliminary results, we are now ready to state our main result of this section, which shows that Cwikel's conjecture is valid in a wide class of Morrey spaces.

Theorem 4.8. *Let $1 \leq q_0 < p_0 < \infty$ and $1 \leq q_1 < p_1 \leq \infty$ with $q_0 \neq q_1$. Then for any $\theta \in (0,1)$ the upper complex interpolation space $[\mathcal{M}_{q_0}^{p_0}, \mathcal{M}_{q_1}^{p_1}]^{\theta}$ is not a K-monotone couple with respect to $(\mathcal{M}_{q_0}^{p_0}, \mathcal{M}_{q_1}^{p_1})$, and so $(\mathcal{M}_{q_0}^{p_0}, \mathcal{M}_{q_1}^{p_1})$ is not a Calderón–Mityagin couple.*

Proof. It follows from Proposition 4.2 and Theorem 4.3 that it is suffices to prove that for every $\theta \in (0,1)$ there exists $f \in [\mathcal{M}_{q_0}^{p_0}, \mathcal{M}_{q_1}^{p_1}]^{\theta}$ such that $t^{\theta} \asymp K(t, f; \mathcal{M}_{q_0}^{p_0}, \mathcal{M}_{q_1}^{p_1})$.

We will distinguish three cases:

Case 1: $p_0, p_1 < \infty$ with $p_0 \neq p_1$. Let $1/p = (1-\theta)/p_0 + \theta/p_1$ and $1/q = (1-\theta)/q_0 + \theta/q_1$. Consider the functions f, g_0 and g_1 given by $f(x) = |x|^{-n/p}$ and $g_j(x) = |x|^{-n/p_j}$ for all $x \in \mathbb{R}^n \setminus \{0\}$, for $j \in \{0, 1\}$. By the preceding discussion in introduction of this section, it follows that $g_0 \in \mathcal{M}^{p_0}_{q_0}$ and $g_1 \in \mathcal{M}^{p_1}_{q_1}$.

Since Morrey spaces enjoy the Fatou property, it follows from Theorem 3.7 that

$$(\mathcal{M}^{p_0}_{q_0})^{1-\theta}(\mathcal{M}^{p_1}_{q_1})^{\theta} \cong [\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]^{\theta}.$$

Combining the above facts with $f = g_0^{1-\theta} g_1^{\theta}$, we get $f \in [\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]^{\theta}$.

We claim that

$$t^{\theta} \prec K(t, f; \mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}), \quad t > 0.$$

First we notice that $\mathcal{M}^{p_j}_{q_j} \hookrightarrow \mathcal{M}^{p_j}_{1}$ with the norm of the inclusion map 1 for $j \in \{0, 1\}$:

$$\begin{aligned} K(t, f; \mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}) &\geq K(t, f; \mathcal{M}^{p_0}_{1}, \mathcal{M}^{p_1}_{1}) = \inf\{\|f_0\|_{\mathcal{M}^{p_0}_1} + t\,\|f_1\|_{\mathcal{M}^{p_1}_1} : f_0 + f_1 = f\} \\ &\geq \inf_{f_0+f_1=f} \sup_{x \in \mathbb{R}^n, r>0} \int_{B(x,r)} \big(|B(x,r)|^{1/p_0-1}|f_0(y)| + t\,|B(x,r)|^{1/p_1-1}|f_1(y)|\big)\,dy. \end{aligned}$$

Thus applying the formula which was explained at the beginning of this section,

$$|B(0,r)|^{1/s-1} \int_{B(0,r)} |x|^{-n/s}\,dx = C(s),$$

we obtain the following estimates for all $t > 0$:

$$\begin{aligned} &K(t, f; \mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}) \\ &\quad \geq \inf_{f_0+f_1=f} \sup_{x \in \mathbb{R}^n, r>0} \min\{|B(x,r)|^{1/p_0-1}, t\,|B(x,r)|^{1/p_1-1}\} \int_{B(x,r)} (|f_0(y)| + |f_1(y)|)\,dy \\ &\quad \geq \sup_{x \in \mathbb{R}^n, r>0} \min\{|B(x,r)|^{1/p_0-1}, t\,|B(x,r)|^{1/p_1-1}\} \int_{B(x,r)} |f(y)|\,dy \\ &\quad \geq \sup_{r>0} \min\{|B(r)|^{1/p_0-1}, t\,|B(r)|^{1/p_1-1}\} \int_{B(r)} |f(y)|\,dy \\ &\quad \asymp \sup_{r>0} \min\{|B(r)|^{1/p_0-1/p}, t\,|B(r)|^{1/p_1-1/p}\}. \end{aligned}$$

If we calculate the last expression, we obtain

$$K(t, f; \mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}) \asymp \sup_{r>0} \min\left\{1, \frac{t}{|B(r)|^{1/p_0-1/p_1}}\right\} |B(r)|^{\theta/p_0-\theta/p_1} = \sup_{s>0} \min\left\{1, \frac{t}{s}\right\} s^{\theta} = t^{\theta}.$$

Case 2: $p_1 = \infty$. We will use [Cwikel and Gulisashvili 2000, Lemma 6] from which it follows that if X is a Banach function lattice on a σ-finite complete measure space (Ω, Σ, μ) and f is a nonnegative function in $(X, L^{\infty})_{\theta,\infty}$ for some $\theta \in (0, 1)$, then for each $y > 0$ we have $\chi_{E(y,f)} \in X$ and

$$\|f\chi_{E(y,f)}\|_X \leq K(\|\chi_{E(y,f)}\|_X, f; X, L^{\infty}), \tag{4-2}$$

where $\chi_{E(y,f)} = \{x \in \Omega : f(x) > y\}$.

At this stage we observe that it follows from Corollary 3.10 and Proposition 4.1 that for the function f given by $f(x) = |x|^{-n(1-\theta)/p_0}$ for all $x \in \mathbb{R}^n \setminus \{0\}$ with $f(0) := 0$ we have

$$f \in (\mathcal{M}^{p_0}_{q_0})^{1/(1-\theta)} = [\mathcal{M}^{p_0}_{q_0}, L^\infty]^\theta \hookrightarrow (\mathcal{M}^{p_0}_{q_0}, L^\infty)_{\theta,\infty}.$$

Now we apply the above estimate (4-2) of the K-functional for the couple $(\mathcal{M}^{p_0}_{q_0}, L^\infty)$. First notice that it is easy to check that

$$\|\chi_{B(r)}\|_{\mathcal{M}^{p_0}_{q_0}} = |B(r)|^{1/p_0} = (v_n r)^{n/p_0}, \quad r > 0,$$

and that, for function f shown above, we have

$$E(y, f) = \{x \in \mathbb{R}^n : f(x) > y\} = B(1/y^{p_0/n(1-\theta)}).$$

For $t > 0$ let us take $y := y(t) = (v_n^{n/p_0} t^{-1})^{1-\theta}$. Then we get $\|\chi_{E(y,f)}\|_{\mathcal{M}^{p_0}_{q_0}} = t$. Hence, we obtain

$$K(t, f; \mathcal{M}^{p_0}_{q_0}, L^\infty) \geq \|f\chi_{E(y,f)}\|_{\mathcal{M}^{p_0}_{q_0}} \geq y\|\chi_{E(y,f)}\|_{\mathcal{M}^{p_0}_{q_0}} = (v_n^{n/p_0} t^{-1})^{1-\theta} t = v_n^{n(1-\theta)/p_0} t^\theta.$$

<u>Case 3</u>: $1 \leq q_0 < q_1 < p := p_0 = p_1 < \infty$. First observe that if $\vec{X} = (X_0, X_1)$ is a Banach couple such that the norm of the inclusion map $X_1 \hookrightarrow X_0$ is less than or equal to 1, then $K(t, x; \vec{X}) = \|x\|_{X_0}$ for every $t \geq 1$. We learn from Hölder's inequality that the couple $(X_0, X_1) := (\mathcal{M}^p_{q_0}, \mathcal{M}^p_{q_1})$ enjoys this property.

To finish we will apply Lemmas 4.5(ii) and 4.6. To do this we will use a sequence $\{F_j\}_{j\geq 0}$ of Lebesgue measurable subsets in $\mathbb{R}^n$, constructed in the proof of Lemma 4.6, which satisfies the conditions of the Lemma 4.5(ii). As a result,

$$g_\theta := \sum_{j=0}^{\infty} r^{j\theta} \chi_{F_j} \in (\mathcal{M}^p_{q_0})^{1-\theta} (\mathcal{M}^p_{q_1})^\theta$$

and

$$K(t, g_\theta; \mathcal{M}^p_{q_0}, \mathcal{M}^p_{q_1}) \asymp t^\theta, \quad t \in (0, 1]. \qquad \square$$

We conclude this section with the following result:

Proposition 4.9. *Assume there exists $\theta \in (0, 1)$ such that the inclusion $[\mathcal{M}^{s_0}_1, \mathcal{M}^{s_1}_1]^\theta \hookrightarrow (\mathcal{M}^{s_0}_1, \mathcal{M}^{s_1}_1)_{\theta,\infty}$ is proper for every $s_0, s_1 \in (1, \infty)$ with $s_0 \neq s_1$. Then $(\mathcal{M}^{p_0}_q, \mathcal{M}^{p_1}_q)$ is not a Calderón–Mityagin couple for all $p_0, p_1 \in (1, \infty)$ with $p_0 \neq p_1$ and all $1 \leq q < \min\{p_0, p_1\}$.*

Proof. It is easy to verify that for any couple (X_0, X_1) of Banach lattices and every $1 < q < \infty$, we have

$$(X_0^{(q)})^{1-\theta} (X_1^{(q)})^\theta \cong (X_0^{1-\theta} X_1^\theta)^{(q)}.$$

Thus thanks to the well-known equivalence

$$K(t, f; X_0^{(q)}, X_1^{(q)}) \asymp K(t^q, |f|^q; X_0, X_1)^{1/q}, \quad f \in X_0^{(q)} + X_1^{(q)},$$

up to equivalence of norms, we get

$$(X_0^{(q)}, X_1^{(q)})_{\theta,\infty} = (X_0, X_1)^{(q)}_{\theta,\infty}.$$

These formulas combined with the Fatou property of $\mathcal{M}_q^{p_j} = (\mathcal{M}^{p_j/q})^{(q)}$ imply for $(X_0, X_1) := (\mathcal{M}_1^{p_0/q}, \mathcal{M}_1^{p_1/q})$ that the inclusion

$$[\mathcal{M}_q^{p_0}, \mathcal{M}_q^{p_1}]^\theta \hookrightarrow (\mathcal{M}_q^{p_0}, \mathcal{M}_q^{p_1})_{\theta,\infty}$$

is proper. Since $p_0 \neq p_1$, it follows from the proof of Theorem 4.8 that there exists $f \in [\mathcal{M}_q^{p_0}, \mathcal{M}_q^{p_1}]^\theta$ such that

$$K(t, f; \mathcal{M}_q^{p_0}, \mathcal{M}_q^{p_1}) \asymp t^\theta.$$

The required statement now follows from Proposition 4.2. □

To conclude this section, we note that it is natural to ask whether $[\mathcal{M}_1^{p_0}, \mathcal{M}_1^{p_1}]^\theta \neq (\mathcal{M}_1^{p_0}, \mathcal{M}_1^{p_1})_{\theta,\infty}$ in the set-theoretical sense for all $\theta \in (0, 1)$ and $p_0, p_1 \in (1, \infty)$ with $p_0 \neq p_1$.

5. On Calderón–Mityagin couples of local Morrey spaces

We now study Calderón–Mityagin couples of local Morrey spaces. The following interpolation results will play a key role. We proceed in a couple of simple steps, which seem independently interesting themselves.

Lemma 5.1. *If $1 \le q_0 < p_0 < \infty$ and $1/p = (1-\theta)/p_0$ and $1/q = (1-\theta)/q_0$ for $\theta \in (0, 1)$, then the following formulas are true*:

(i) $[L\mathcal{M}_{q_0}^{p_0}, L^\infty]^\theta \cong (L\mathcal{M}_{q_0}^{p_0})^{1-\theta}(L^\infty)^\theta \cong L\mathcal{M}_q^p$.

(ii) $(L\mathcal{M}_{q_0}^{p_0}, L^\infty)_{\theta,\infty} = \mathrm{w}L\mathcal{M}_q^p$.

If we reexamine the proof of Theorem 5.2, we obtain the proof of Lemma 5.1 as a special case of Theorem 5.2. However, we give a proof using what we have shown.

Proof. Since $L\mathcal{M}_{q_0}^{p_0}$ has the Fatou property, statement (i) follows from Corollary 3.10. In a similar fashion to the proof of Theorem 4.3, we explain that (ii) follows by [Cwikel and Gulisashvili 2000, Theorem 4]. □

We now handle the case $q_1 < p_1 < \infty$ by a different method.

Theorem 5.2. *Let $1 \le q_j < p_j < \infty$ for $j \in \{0, 1\}$ and $\theta \in (0, 1)$. If $1/p = (1-\theta)/p_0 + \theta/p_1$ and $1/q = (1-\theta)/q_0 + \theta/q_1$, then we have the following properties*:

(i) $[L\mathcal{M}_{q_0}^{p_0}, L\mathcal{M}_{q_1}^{p_1}]^\theta = L\mathcal{M}_q^p$.

(ii) *If $q = q_0 = q_1$, then $(L\mathcal{M}_{q_0}^{p_0}, L\mathcal{M}_{q_1}^{p_1})_{\theta,\infty} = L\mathcal{M}_q^p$.*

(iii) *If $q_0 \neq q_1$, then $(L\mathcal{M}_{q_0}^{p_0}, L\mathcal{M}_{q_1}^{p_1})_{\theta,\infty} = \mathrm{w}L\mathcal{M}_q^p$.*

Proof. We will use an equivalent norm on local Morrey spaces. It is obvious that for any $1 \le q \le p < \infty$ the functional $\|\cdot\|'$ defined on $L\mathcal{M}_q^p$ by

$$\|f\|' = \sup_{k\in\mathbb{Z}} |B(2^k)|^{1/p-1/q}\left(\int_{B(2^k)} |f(y)|^q\,dy\right)^{1/q}, \quad f \in L\mathcal{M}_q^p,$$

is a norm equivalent to the original norm on $L\mathcal{M}_q^p$.

We claim that if in addition $q \neq p$, then the formula

$$\|f\|^* := \sup_{k\in\mathbb{Z}} |B(2^k)|^{1/p-1/q}\left(\int_{B(2^k)\setminus B(2^{k-1})} |f(y)|^q\,dy\right)^{1/q}$$

also gives an equivalent norm on $L\mathcal{M}^p_q$.

From the definition it is clear that $\|\cdot\|^* \geq \|\cdot\|'$. To prove the converse inequality, we observe that

$$\left(\int_{B(2^j)\setminus B(2^{j-1})} |f(y)|^q\,dy\right)^{1/q} \leq |B(2^j)|^{-1/p+1/q}\|f\|^*, \quad j\in\mathbb{Z}.$$

If we use the triangle inequality, we get that for each $k\in\mathbb{Z}$

$$\begin{aligned}\left(\int_{B(2^k)} |f(y)|^q\,dy\right)^{1/q} &= \left\|\sum_{j=-\infty}^{k} f\chi_{B(2^j)\setminus B(2^{j-1})}\right\|_q\\ &\leq \left(\sum_{j=0}^{\infty} 2^{jn/p-jn/q}\right)|B(2^k)|^{-1/p+1/q}\|f\|^*\\ &= c_{p,q}|B(2^k)|^{-1/p+1/q}\|f\|^*.\end{aligned}$$

This implies that $\|f\|' \leq c_{p,q}\|f\|^*$ for all $f\in L\mathcal{M}^p_q$.

In what follows we will need some general interpolation formulas. In order to state them we introduce some additional notation. For a given sequence $\{\vec{A}^k\}_{k\in\mathbb{Z}}$ of Banach couples, where $\vec{A}^k = (A_0^k, A_1^k)$, we define a Banach couple

$$\ell_\infty(\{\vec{A}_k\}) := (\ell_\infty(\{A_0^k\}_{k\in\mathbb{Z}}), \ell_\infty(\{A_1^k\}_{k\in\mathbb{Z}})),$$

where $\ell_\infty(\{A_j^k\}_{k\in\mathbb{Z}})$ for each $j\in\{0,1\}$ is a Banach space of all bounded sequences $\{a_j^k\}_{k\in\mathbb{Z}} \in \prod_{k\in\mathbb{Z}} A_j^k$ endowed with the uniform norm.

We omit the standard proofs of the following formulas:

$$[\ell_\infty(\{A_0^k\}_{k\in\mathbb{Z}}), \ell_\infty(\{A_1^k\}_{k\in\mathbb{Z}})]^\theta = \ell_\infty(\{[A_0^k, A_1^k]^\theta\}_{k\in\mathbb{Z}}),$$
$$(\ell_\infty(\{A_0^k\}_{k\in\mathbb{Z}}), \ell_\infty(\{A_1^k\}_{k\in\mathbb{Z}}))_{\theta,\infty} = \ell_\infty(\{(A_0^k, A_1^k)_{\theta,\infty}\}_{k\in\mathbb{Z}}).$$

We consider a sequence $\{(A_0^k, A_1^k)\}_{k\in\mathbb{Z}}$ given by $A_j^k := w_j^k L^{q_j}$, with $w_j^k = |B(2^k)|^{1/p_j-1/q_j}$ for each $k\in\mathbb{Z}$ and $j\in\{0,1\}$, and endowed with norms $\|f\|_{A_j^k} = w_j^k\|f\|_{L^{q_j}}$.

Since $[L^{q_0}, L^{q_1}]^\theta = L^q$ if $q_0\neq q_1$, $(L^{q_0}, L^{q_1})_{\theta,\infty} = L^{q,\infty}$ and $(L^q, L^q)_{\theta,\infty} = L^q$, then the above vector-valued formulas easily yield

$$[\ell_\infty(\{A_0^k\}_{k\in\mathbb{Z}}), \ell_\infty(\{A_1^k\}_{k\in\mathbb{Z}})]^\theta = \ell_\infty(\{[w_0^k L^{q_0}, w_1^k L^{q_1}]^\theta\}_{k\in\mathbb{Z}}) = \ell_\infty(\{w^k L^q\}_{k\in\mathbb{Z}}),$$
$$(\ell_\infty(\{A_0^k\}_{k\in\mathbb{Z}}), \ell_\infty(\{A_1^k\}_{k\in\mathbb{Z}}))_{\theta,\infty} = \ell_\infty(\{(w_0^k L^q, w_1^k L^q)_{\theta,\infty}\}_{k\in\mathbb{Z}}) = \ell_\infty(\{w^k L^{q,\infty}\}_{k\in\mathbb{Z}}),$$
$$(\ell_\infty(\{A_0^k\}_{k\in\mathbb{Z}}), \ell_\infty(\{A_1^k\}_{k\in\mathbb{Z}}))_{\theta,\infty} = \ell_\infty(\{(w_0^k L^{q_0}, w_1^k L^{q_1})_{\theta,\infty}\}_{k\in\mathbb{Z}}) = \ell_\infty(\{w^k L^{q,\infty}\}_{k\in\mathbb{Z}}),$$

where $\{w^k\}_{k\in\mathbb{Z}} = \{|B(2^k)|^{1/p-1/q}\}$ for each $k \in \mathbb{Z}$ and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

It is easy to check that $\mathrm{w}L\mathcal{M}_q^p = \ell_\infty(\{w^k L^{q,\infty}\})$ with equality of quasinorms and so the last formula can take the form

$$(\ell_\infty(\{A_0^k\}_{k\in\mathbb{Z}}), \ell_\infty(\{A_1^k\}_{k\in\mathbb{Z}}))_{\theta,\infty} = \mathrm{w}L\mathcal{M}_q^p, \quad q_0 \neq q_1.$$

Now by the discussion before about equivalent norms on Morrey spaces, it follows that

$$\|f\|_{L\mathcal{M}_q^p} \asymp \sup_{k\in\mathbb{Z}} |B(2^k)|^{1/p-1/q} \left(\int_{B(2^k)\setminus B(2^{k-1})} |f(y)|^q \, dy \right)^{1/q}, \quad f \in L\mathcal{M}_q^p.$$

This equivalence implies that operators U and V given by

$$\begin{aligned} Uf &= \{f\chi_{B(2^k)\setminus B(2^{k-1})}\}_{k\in\mathbb{Z}}, \quad f \in L\mathcal{M}_{q_0}^{p_0} + L\mathcal{M}_{q_1}^{p_1}, \\ V(\{f_k\}_{k\in\mathbb{Z}}) &= \sum_{k\in\mathbb{Z}} f_k \chi_{B(2^k)\setminus B(2^{k-1})}, \quad \{f_k\}_{k\in\mathbb{Z}} \in \ell_\infty(\{A_0^k\}_{k\in\mathbb{Z}}) + \ell_\infty(\{A_1^k\}_{k\in\mathbb{Z}}) \end{aligned}$$

are bounded between the Banach couples

$$\begin{aligned} U &: (L\mathcal{M}_{q_0}^{p_0}, L\mathcal{M}_{q_1}^{p_1}) \to (\ell_\infty(\{A_0^k\}_{k\in\mathbb{Z}}), \ell_\infty(\{A_1^k\}_{k\in\mathbb{Z}})), \\ V &: (\ell_\infty(\{A_0^k\}_{k\in\mathbb{Z}}), \ell_\infty(\{A_1^k\}_{k\in\mathbb{Z}})) \to (L\mathcal{M}_{q_0}^{p_0}, L\mathcal{M}_{q_1}^{p_1}). \end{aligned}$$

We conclude by the vector-valued interpolation formulas shown above that

$$U : [L\mathcal{M}_{q_0}^{p_0}, L\mathcal{M}_{q_1}^{p_1}]^\theta \to \ell_\infty(\{|B(2^k)|^{1/p-1/q} L^q\})$$

is bounded. In particular this yields the continuous inclusion

$$[L\mathcal{M}_{q_0}^{p_0}, L\mathcal{M}_{q_1}^{p_1}]^\theta \hookrightarrow L\mathcal{M}_q^p.$$

The boundedness of an operator V from $\ell_\infty(\{|B(2^k)|^{1/p-1/q} L^q\}_{k\in\mathbb{Z}})$ into $[L\mathcal{M}_{q_0}^{p_0}, L\mathcal{M}_{q_1}^{p_1}]^\theta$ yields the reverse continuous inclusion

$$L\mathcal{M}_q^p \hookrightarrow [L\mathcal{M}_{q_0}^{p_0}, L\mathcal{M}_{q_1}^{p_1}]^\theta$$

and so $[L\mathcal{M}_{q_0}^{p_0}, L\mathcal{M}_{q_1}^{p_1}]^\theta = L\mathcal{M}_q^p$, as required.

Similarly we obtain the remaining formulas and this completes the proof. □

Remark 5.3. We notice that using maps U and V defined in the proof of Theorem 5.2 we easily conclude the following equivalence for the K-functional of local Morrey couples: if $1 \le q_j < p_j < \infty$ for $j \in \{0, 1\}$, then for all $f \in L\mathcal{M}_{q_0}^{p_0} + L\mathcal{M}_{q_1}^{p_1}$ and $t > 0$,

$$\begin{aligned} K(t, f; L\mathcal{M}_{q_0}^{p_0}, L\mathcal{M}_{q_1}^{p_1}) &\asymp K\big(t, \{f\chi_{B(2^k)\setminus B(2^{k-1})}\}_{k\in\mathbb{Z}}; \ell_\infty(\{w_0^k L^{q_0}\}_{k\in\mathbb{Z}}), \ell_\infty(\{w_1^k L^{q_1}\}_{k\in\mathbb{Z}})\big) \\ &\asymp \sup_{k\in\mathbb{Z}} K(t, f\chi_{B(2^k)\setminus B(2^{k-1})}; w_0^k L^{q_0}, w_1^k L^{q_1}), \end{aligned}$$

where $w_j^k = |B(2^k)|^{1/p_j-1/q_j}$ for each $k \in \mathbb{Z}$ and $j \in \{0, 1\}$.

We apply the above results to study Calderón–Mityagin couples of local Morrey spaces. Our main result of this section shows that Cwikel's conjecture is valid in the class of local Morrey spaces.

Theorem 5.4. *Let* $1 \le q_0 < p_0 < \infty$ *and* $1 \le q_1 < p_1 < \infty$. *The following are equivalent*:

(i) $q_0 = q_1$.

(ii) $(L\mathcal{M}^{p_0}_{q_0}, L\mathcal{M}^{p_1}_{q_1})$ *has the universal* K*-property.*

(iii) $(L\mathcal{M}^{p_0}_{q_0}, L\mathcal{M}^{p_1}_{q_1})$ *is a Calderón–Mityagin couple.*

Proof. (i) $\Rightarrow$ (ii). We invoke the following result from [Cwikel and Peetre 1981]: for any Banach couple (A_0, A_1) and any numbers $\theta_0, \theta_1 \in (0,1)$ the couple of interpolation spaces $((A_0, A_1)_{\theta_0,\infty}, (A_0, A_1)_{\theta_1,\infty})$ has the universal K-property. To establish that the couple $(L\mathcal{M}^{p_0}_q, L\mathcal{M}^{p_1}_q)$ falls under this scope, we fix $0 < \varepsilon < \min\{p_0 - q, p_1 - q\}$ and set $u_0 = p_0 - \varepsilon$, $u_1 = p_0 + \varepsilon$, $v_0 = p_1 - \varepsilon$ and $v_1 = p_1 + \varepsilon$. Then for $\theta_0, \theta_1 \in (0,1)$ given by $\theta_0 = \frac{1}{2}(1 + \varepsilon/p_0)$ and $\theta_1 = \frac{1}{2}(1 + \varepsilon/p_1)$ we have

$$\frac{1}{p_0} = \frac{1-\theta_0}{u_0} + \frac{\theta_0}{u_1}, \qquad \frac{1}{p_1} = \frac{1-\theta_1}{v_0} + \frac{\theta_1}{v_1}.$$

If $q = q_0 = q_1$, then it follows from Theorem 5.2 that

$$(L\mathcal{M}^{u_0}_q, L\mathcal{M}^{u_1}_q)_{\theta_0,\infty} = L\mathcal{M}^{p_0}_q, \quad (L\mathcal{M}^{v_0}_q, L\mathcal{M}^{v_1}_q)_{\theta_1,\infty} = L\mathcal{M}^{p_1}_q.$$

(ii) $\Rightarrow$ (iii). This is obvious.

(iii) $\Rightarrow$ (i). We consider the contrapositive. Suppose that $q_0 \ne q_1$. We have two cases:

Case 1: $p_0 \ne p_1$. Then as before the function f defined by $f(x) = |x|^{-n/p}$ for almost all $x \in \mathbb{R}^n$ is in $L\mathcal{M}^p_q$, and

$$K(t, f; L\mathcal{M}^{p_0}_{q_0}, L\mathcal{M}^{p_1}_{q_1}) \asymp t^\theta, \quad t > 0.$$

Meanwhile, we conclude from Theorem 5.2 that

$$f \in [L\mathcal{M}^{p_0}_{q_0}, L\mathcal{M}^{p_1}_{q_1}]^\theta = L\mathcal{M}^p_q$$

and so it follows from Lemma 4.7 and Theorem 5.2 that the inclusion

$$[L\mathcal{M}^{p_0}_{q_0}, L\mathcal{M}^{p_1}_{q_1}]^\theta \hookrightarrow (L\mathcal{M}^{p_0}_{q_0}, L\mathcal{M}^{p_1}_{q_1})_{\theta,\infty}$$

is proper. Applying Proposition 4.2(i), we deduce that $(L\mathcal{M}^{p_0}_{q_0}, L\mathcal{M}^{p_1}_{q_1})$ is not a Calderón–Mityagin couple and so we get a contradiction.

Case 2: $p := p_0 = p_1$. Since the cube $[0,1]^n$ appears in the definition of the sequence $\{E_j\}_{j=0}^\infty$ constructed in the proof of Lemma 4.6, the same conclusion as in the case of Morrey spaces yields that for $\{F_j\}_{j\ge 0} := \{\alpha^{-j} E_j\}_{j\ge 0}$ with the same $\alpha \in \left(0, \frac{1}{2}\right)$, we get

$$\|\chi_{F_j}\|_{L\mathcal{M}^u_1} \asymp \max\{1, \alpha^{-jn/u + jn} 2^{-jn}\} = \max\{1, \alpha^{-jn/u + jn/s}\}, \quad j \ge 0.$$

Then as we did in the proof of Theorem 4.8, we apply Lemma 4.5 for the couple $(X_0, X_1) := (L\mathcal{M}^p_{q_0}, L\mathcal{M}^p_{q_1})$ to find $f \in [L\mathcal{M}^{p_0}_{q_0}, L\mathcal{M}^{p_1}_{q_1}]^\theta = L\mathcal{M}^p_q$ such that

$$K(t, f; L\mathcal{M}^{p_0}_{q_0}, L\mathcal{M}^{p_1}_{q_1}) \asymp t^\theta, \quad t \in (0,1].$$

Then similar to the above proof of Case 1, applying Proposition 4.2(ii), we deduce that $(L\mathcal{M}^{p_0}_{q_0}, L\mathcal{M}^{p_1}_{q_1})$ is not a Calderón–Mityagin couple. But this is a contradiction. ☐

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References

[Adams and Xiao 2004] D. R. Adams and J. Xiao, "Nonlinear potential analysis on Morrey spaces and their capacities", *Indiana Univ. Math. J.* **53**:6 (2004), 1629–1663. MR Zbl

[Adams and Xiao 2012] D. R. Adams and J. Xiao, "Morrey spaces in harmonic analysis", *Ark. Mat.* **50**:2 (2012), 201–230. MR Zbl

[Bergh 1979] J. Bergh, "On the relation between the two complex methods of interpolation", *Indiana Univ. Math. J.* **28**:5 (1979), 775–778. MR Zbl

[Bergh and Löfström 1976] J. Bergh and J. Löfström, *Interpolation spaces: an introduction*, Grundlehren der Math. Wissenschaften **223**, Springer, 1976. MR Zbl

[Beurling 1964] A. Beurling, "Construction and analysis of some convolution algebras", *Ann. Inst. Fourier* (*Grenoble*) **14**:2 (1964), 1–32. MR Zbl

[Brudnyi and Krugljak 1991] Y. A. Brudnyi and N. Y. Krugljak, *Interpolation functors and interpolation spaces, I*, North-Holland Math. Library **47**, North-Holland, Amsterdam, 1991. MR Zbl

[Brudnyi, Y. A. and Kruglyak, N. Y. 1981] Y. A. Brudnyĭ and N. Y. Kruglyak, "Functors of real interpolation", *Dokl. Akad. Nauk SSSR* **256**:1 (1981), 14–17. In Russian; translated in *Soviet Math. Dokl.* **23**:1 (1981), 5–8. MR

[Burenkov and Guliyev 2004] V. I. Burenkov and H. V. Guliyev, "Necessary and sufficient conditions for boundedness of the maximal operator in local Morrey-type spaces", *Studia Math.* **163**:2 (2004), 157–176. MR Zbl

[Calderón 1964] A. P. Calderón, "Intermediate spaces and interpolation: the complex method", *Studia Math.* **24** (1964), 113–190. MR Zbl

[Calderón 1966] A. P. Calderón, "Spaces between L^1 and L^∞ and the theorem of Marcinkiewicz", *Studia Math.* **26** (1966), 273–299. MR Zbl

[Campanato and Murthy 1965] S. Campanato and M. K. V. Murthy, "Una generalizzazione del teorema di Riesz–Thorin", *Ann. Scuola Norm. Sup. Pisa* (3) **19**:1 (1965), 87–100. MR Zbl

[Cwikel 1981] M. Cwikel, "Monotonicity properties of interpolation spaces, II", *Ark. Mat.* **19**:1 (1981), 123–136. MR Zbl

[Cwikel 1984] M. Cwikel, "K-divisibility of the K-functional and Calderón couples", *Ark. Mat.* **22**:1 (1984), 39–62. MR Zbl

[Cwikel and Gulisashvili 2000] M. Cwikel and A. Gulisashvili, "Interpolation on families of characteristic functions", *Studia Math.* **138**:3 (2000), 209–224. MR Zbl

[Cwikel and Nilsson 2003] M. Cwikel and P. G. Nilsson, "Interpolation of weighted Banach lattices", pp. 1–105 in Mem. Amer. Math. Soc. **787**, Amer. Math. Soc., Providence, RI, 2003. MR Zbl

[Cwikel and Peetre 1981] M. Cwikel and J. Peetre, "Abstract K and J spaces", *J. Math. Pures Appl.* (9) **60**:1 (1981), 1–49. MR Zbl

[Fremlin 1974] D. H. Fremlin, *Topological Riesz spaces and measure theory*, Cambridge Univ. Press, 1974. MR Zbl

[García-Cuerva and Herrero 1994] J. García-Cuerva and M.-J. L. Herrero, "A theory of Hardy spaces associated to the Herz spaces", *Proc. London Math. Soc.* (3) **69**:3 (1994), 605–628. MR Zbl

[Kalton 1993] N. J. Kalton, "Calderón couples of rearrangement invariant spaces", *Studia Math.* **106**:3 (1993), 233–277. MR Zbl

[Kantorovich and Akilov 1982] L. V. Kantorovich and G. P. Akilov, *Functional analysis*, 2nd ed., Pergamon Press, Oxford, 1982. MR Zbl

[Krein et al. 1982] S. G. Kreĭn, Y. I. Petunin, and E. M. Semenov, *Interpolation of linear operators*, Trans. Math. Monographs **54**, Amer. Math. Soc., Providence, R.I., 1982. MR Zbl

[Lemarié-Rieusset 2012] P. G. Lemarié-Rieusset, "The role of Morrey spaces in the study of Navier–Stokes and Euler equations", *Eurasian Math. J.* **3**:3 (2012), 62–93. MR Zbl

[Lemarié-Rieusset 2013] P. G. Lemarié-Rieusset, "Multipliers and Morrey spaces", *Potential Anal.* **38**:3 (2013), 741–752. Correction in **41**:4 (2014), 1359–1362. MR Zbl

[Lozanovskii 1969] G. Y. Lozanovskii, "On some Banach lattices", *Sibirsk. Mat. Ž.* **10**:3 (1969), 584–599. In Russian; translated in *Siberian Math. J.* **10**:3 (1969), 419–431. MR Zbl

[Lozanovskii 1972] G. Y. Lozanovskii, "A remark on a certain interpolation theorem of Calderón", *Funkcional. Anal. i Priložen.* **6**:4 (1972), 89–90. In Russian; translated in *Funct. Anal. Appl.* **6**:4 (1972), 333–334. MR

[Lozanovskii 1978] G. Y. Lozanovskii, "Transformations of ideal Banach spaces by means of concave functions", pp. 122–148 in *Qualitative and approximate methods for the investigation of operator equations, III*, edited by P. P. Zabreiko, Yaroslavl Gos. Univ., 1978. In Russian. MR Zbl

[Mastyło and Ovchinnikov 1997] M. Mastyło and V. I. Ovchinnikov, "On the relation between complex and real methods of interpolation", *Studia Math.* **125**:3 (1997), 201–218. MR Zbl

[Mastyło and Sinnamon 2017] M. Mastyło and G. Sinnamon, "Calderón–Mityagin couples of Banach spaces related to decreasing functions", *J. Funct. Anal.* **272**:11 (2017), 4460–4482. MR Zbl

[Mazzucato 2003] A. L. Mazzucato, "Besov–Morrey spaces: function space theory and applications to non-linear PDE", *Trans. Amer. Math. Soc.* **355**:4 (2003), 1297–1364. MR Zbl

[Mityagin 1965] B. Mityagin, "An interpolation theorem for modular spaces", *Mat. Sb.* **66(108)**:4 (1965), 473–482. In Russian; translated in *Interpolation spaces and allied topics in analysis*, edited by M. Cwikel and J. Peetre, Lecture Notes in Math. **1070** (1984), Springer, 10–23. Zbl

[Morrey 1938] C. B. Morrey, Jr., "On the solutions of quasi-linear elliptic partial differential equations", *Trans. Amer. Math. Soc.* **43**:1 (1938), 126–166. MR Zbl

[Nakai 2008] E. Nakai, "Orlicz–Morrey spaces and the Hardy–Littlewood maximal function", *Studia Math.* **188**:3 (2008), 193–221. MR Zbl

[Ovchinnikov 1984] V. I. Ovchinnikov, "The method of orbits in interpolation theory", *Math. Rep.* **1**:2 (1984), 349–515. MR Zbl

[Peetre 1969] J. Peetre, "On the theory of $\mathscr{L}_{p,\lambda}$ spaces", *J. Funct. Anal.* **4** (1969), 71–87. MR Zbl

[Reisner 1993] S. Reisner, "Some remarks on Lozanovskyi's intermediate normed lattices", *Bull. Polish Acad. Sci. Math.* **41**:3 (1993), 189–196. MR Zbl

[Sedaev and Semenov 1971] A. A. Sedaev and E. M. Semenov, "The possibility of describing interpolation spaces in terms of Peetre's K-method", *Optimizacija* **4(21)** (1971), 98–114. In Russian. MR Zbl

[Shestakov 1974] V. A. Shestakov, "Complex interpolation in Banach spaces of measurable functions", *Vestnik Leningrad Univ.* **19** (1974), 64–68. In Russian. MR Zbl

[Stampacchia 1964] G. Stampacchia, "$\mathscr{L}^{(p,\lambda)}$-spaces and interpolation", *Comm. Pure Appl. Math.* **17** (1964), 293–306. MR Zbl

[Taylor 1997] M. E. Taylor, "Microlocal analysis on Morrey spaces", pp. 97–135 in *Singularities and oscillations* (Minneapolis, 1994/1995), edited by J. Rauch and M. Taylor, IMA Vol. Math. Appl. **91**, Springer, 1997. MR Zbl

[Wiener 1930] N. Wiener, "Generalized harmonic analysis", *Acta Math.* **55**:1 (1930), 117–258. MR Zbl

[Wiener 1932] N. Wiener, "Tauberian theorems", *Ann. of Math.* (2) **33**:1 (1932), 1–100. MR Zbl

[Yuan et al. 2015] W. Yuan, W. Sickel, and D. Yang, "Interpolation of Morrey–Campanato and related smoothness spaces", *Sci. China Math.* **58**:9 (2015), 1835–1908. MR Zbl

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