

ANALYSIS & PDE

Volume 12

No. 7

2019

M. CRISTINA CAPUTO, THIERRY GOUDON AND ALEXIS F. VASSEUR

**SOLUTIONS OF THE 4-SPECIES QUADRATIC
REACTION-DIFFUSION SYSTEM ARE BOUNDED AND
 C^∞ -SMOOTH, IN ANY SPACE DIMENSION**

SOLUTIONS OF THE 4-SPECIES QUADRATIC REACTION-DIFFUSION SYSTEM ARE BOUNDED AND C^∞ -SMOOTH, IN ANY SPACE DIMENSION

M. CRISTINA CAPUTO, THIERRY GOUDON AND ALEXIS F. VASSEUR

We establish the boundedness of solutions of reaction-diffusion systems with quadratic (in fact slightly superquadratic) reaction terms that satisfy a natural entropy dissipation property, in any space dimension $N > 2$. This bound implies the C^∞ -regularity of the solutions. This result extends the theory which was restricted to the two-dimensional case. The proof heavily uses De Giorgi's iteration scheme, which allows us to obtain local estimates. The arguments rely on duality reasoning in order to obtain new estimates on the total mass of the system, both in the $L^{(N+1)/N}$ norm and in a suitable weak norm. The latter uses C^α regularization properties for parabolic equations.

1. Introduction

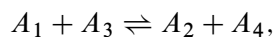
This paper is mainly concerned with the system of reaction-diffusion equations

$$\begin{aligned} \partial_t a_i - \nabla \cdot (D_i \nabla a_i) &= Q_i(a), \quad i \in \{1, 2, 3, 4\}, \quad t \geq 0, \quad x \in \mathbb{R}^N, \\ Q_i(a) &= (-1)^{i+1} (a_2 a_4 - a_1 a_3), \end{aligned} \quad (1)$$

with initial condition

$$a|_{t=0} = a^0 = (a_1^0, a_2^0, a_3^0, a_4^0). \quad (2)$$

This system arises in chemistry where four species interact according to the reactions



the unknowns $(t, x) \mapsto a_i(t, x)$ in (1) being the local mass concentrations of the species labeled by $i \in \{1, 2, 3, 4\}$: $\int_{\mathbb{R}^N} a_i(t, x) dx$ is interpreted as the mass of the constituent i at time t . It is thus physically relevant to consider initial data a_i^0 which are nonnegative integrable functions. The reactants are subjected to space diffusion and the diffusion coefficients depend on the considered species. In full generality, D_i can be a function of the space variable with values in the space of $N \times N$ matrices. Throughout this paper, we restrict to the case of scalar and constant matrices

$$D_i(x) = d_i \mathbb{I}, \quad d_i > 0 \text{ constant},$$

with coefficients that satisfy

$$0 < \delta_\star \leq d_i \leq \delta^\star. \quad (3)$$

Assuming that the initial data are smooth, say $a_i^0 \in C^\infty(\mathbb{R}^N)$, existence-uniqueness of smooth and nonnegative solutions for (1)–(2) can be justified at least on a small time interval, by using a standard

MSC2010: 35K45, 35B65, 35K57.

Keywords: reaction-diffusion systems, global regularity, blow-up methods.

fixed-point reasoning; see for instance [Goudon and Vasseur 2010, Proposition A.2] or [Pierre 2010, Lemma 1.1]. Global existence of weak solutions is established in [Desvillettes et al. 2007]. We address the question of the boundedness of the solutions, which will imply that solutions are globally defined and remain infinitely smooth [Goudon and Vasseur 2010, Proposition A.1].

The difficulty comes from the fact we are dealing with different diffusion coefficients. As already noticed in [Goudon and Vasseur 2010], the question becomes trivial when all the D_i 's vanish: in this case, we are concerned with a mere system of ODEs which clearly satisfies a maximum principle. The answer is also immediate when all the diffusion coefficients are equal to the same constant $d_i = \delta_\star$. Indeed, in this situation, the total mass

$$M(t, x) = \sum_{i=1}^4 a_i(t, x)$$

satisfies the heat equation $\partial_t M = \delta_\star \Delta M$, which, again, easily leads to a maximum principle. In the general situation, one may wonder whether or not the system has the explosive behavior of nonlinear heat equations [Weissler 1985]. Counterexamples of systems with polynomial nonlinearities presented in [Pierre and Schmitt 1997] show that this question is relevant and nontrivial; see also [Pierre 2010, Theorem 4.1]. We refer the reader to the survey [Pierre 2010] for a general presentation of the problem, further references, and many deep comments on the mathematical difficulties raised by such systems.

Two properties are crucial for the analysis of the problem. First of all, system (1) conserves mass:

$$\frac{d}{dt} \sum_{i=1}^4 \int_{\mathbb{R}^N} a_i \, dx = 0. \quad (4)$$

Second of all, it dissipates entropy:

$$\sum_{i=1}^4 Q_i(a) \ln(a_i) = -(a_2 a_4 - a_1 a_3) \ln\left(\frac{a_2 a_4}{a_1 a_3}\right) \leq 0. \quad (5)$$

These properties suggest to consider more general systems, involving more reactants and possibly more intricate nonlinearities. To be more specific, we extend the discussion to systems that read

$$\begin{aligned} \partial_t a_i - \nabla \cdot (D_i \nabla a_i) &= Q_i(a), \quad i \in \{1, \dots, p\}, \quad t \geq 0, \quad x \in \mathbb{R}^N, \\ Q_i : a \in \mathbb{R}^p &\longmapsto \mathbb{R}^p, \end{aligned} \quad (6)$$

endowed with the initial condition

$$a|_{t=0} = a^0 = (a_1^0, \dots, a_p^0), \quad (7)$$

where the reaction term fulfills the following conditions:

- (h1) There exists $\mathcal{Q} > 0$ and $q > 0$ such that for any $a \in \mathbb{R}^p$ and $i \in \{1, \dots, p\}$, we have $|\nabla_a Q_i(a)| \leq \mathcal{Q}|a|^{q-1}$.
- (h2) For any $i \in \{1, \dots, p\}$, if $a_i \leq 0$ then $Q_i(a) \leq 0$.
- (h3) $\sum_{i=1}^p Q_i(a) = 0$.
- (h4) $\sum_{i=1}^p Q_i(a) \ln(a_i) \leq 0$.

Assumption (h1) governs the growth of the nonlinearity. In what follows, we will be concerned with quadratic and superquadratic growth: $q \geq 2$ (but q is not necessarily assumed to be an integer). Assumption (h2) relies on the preservation of nonnegativity of the solutions, and it is thus physically relevant. Assumptions (h3) and (h4) imply mass conservation and entropy dissipation, respectively. Note that the entropy dissipation actually provides an estimate on (nonlinear) derivatives of the unknown since it leads to

$$\frac{d}{dt} \sum_{i=1}^p \int_{\mathbb{R}^N} a_i \ln(a_i) dx + 4\delta_\star \sum_{i=1}^p \int_{\mathbb{R}^N} |\nabla \sqrt{a_i}|^2 dx \leq 0. \quad (8)$$

In view of (h3) and (h4), it is thus natural to consider initial data such that

$$\begin{aligned} a_i^0 : x \in \mathbb{R}^N &\mapsto a_i^0(x) \geq 0, \\ \sup_{i \in \{1, \dots, p\}} \int_{\mathbb{R}^N} a_i^0 (1 + \ln(a_i^0) + |x|) dx &= \mathcal{M}^0 < \infty. \end{aligned} \quad (9)$$

We refer the reader to Proposition 2.1 below for a more precise statement in terms of a priori estimates. It means that the initial concentrations have finite mass and entropy. The moment condition controls the spreading of the mass. However, while (8) has a clear physical meaning, it does not provide enough estimates for the analysis of the problem: note that with $u, u \ln(u) \in L^1$ and $\nabla \sqrt{u} \in L^2$, it is still not clear how the nonlinear term $Q(u)$ can make sense in \mathcal{D}' ! For this reason, a notion of *renormalized* solutions is introduced in [Fischer 2015], and existence of solutions in this framework can be established. Entropy dissipation plays also a central role in the analysis of the asymptotic trend towards equilibrium [Desvillettes and Fellner 2006; Fellner et al. 2016; Pierre et al. 2017a; 2017b].

In the specific quadratic and two-dimensional case ($q = 2$, $N = 2$) the question is fully answered in [Goudon and Vasseur 2010]: starting from $L^\infty \cap C^\infty$ initial data, the solution remains bounded and smooth and the problem is globally well-posed. In fact [loc. cit.] proves a *regularizing* effect: with data satisfying (9) only, the solution becomes *instantaneously* bounded and smooth, which implies global well-posedness. The proof in [loc. cit.] relies on De Giorgi's approach [1957]; it uses entropy dissipation, see (8), to get a nonlinear control on level sets of the solution, which eventually leads to the L^∞ bound. The result is extended for higher space dimensions in [Cañizo et al. 2014], which handles, with different techniques, the quadratic case when the diffusion coefficients are close enough to the same constant (how small the distance between the d_j 's should be depends on the space dimension, in an explicit way; see also [Fellner et al. 2016; Pierre et al. 2017a]), and in [Caputo and Vasseur 2009], which handles subquadratic nonlinearities ($q < 2$ in (h1), not necessarily integer). Two ingredients are crucial in the approach of [loc. cit.]:

- First, [loc. cit.] uses systematically rescaled quantities

$$a_i^{(\epsilon)}(s, y) = \epsilon^{2/(q-1)} a_i(t + \epsilon^2 s, x + \epsilon y), \quad (10)$$

with $\epsilon > 0$: $a^{(\epsilon)}$ satisfies the same evolution equation as a . Note that in the quadratic case ($q = 2$), for $N = 2$, the rescaling leaves invariant the natural norms of the problem $\|a\|_{L^\infty(0, \infty; L^1(\mathbb{R}^2))}$ and $\|\nabla \sqrt{a}\|_{L^2((0, \infty) \times \mathbb{R}^2)}$.

• Second, the parabolic regularity is obtained by adapting De Giorgi's techniques, and by working with a certain norm of the rescaled unknown which becomes small as $\epsilon \rightarrow 0$. It turns out that the necessary estimate holds in a weak sense. Namely, one has to consider the set of distributions

$$T \in \mathcal{D}'((0, T) \times \mathbb{R}^N) \text{ such that } T = \Delta \Phi, \text{ with } \Phi \in L^\infty((0, \infty) \times \mathbb{R}^N).$$

The corresponding rescaled norm behaves like $\mathcal{O}(\epsilon^{(4-2q)/(q-1)})$, which indeed tends to 0 as $\epsilon \rightarrow 0$ for subquadratic nonlinearities $q < 2$. The idea of using such a weak norm also appeared in the regularity analysis for the Navier–Stokes equation [Vasseur 2010]. We also refer the reader to [Caffarelli and Vasseur 2010; Vasseur 2007] for further applications of De Giorgi's techniques to the analysis of fluid mechanics systems and to [Alonso et al. 2016; Goudon and Urrutia 2016] for the study of models for populations dynamics governed by “chemotactic-like” mechanisms. This approach is also useful for the analysis of the preservation of bounds by numerical schemes when solving nonlinear convection-diffusion systems [Chainais-Hillairet et al. 2017]. In the reasoning adopted in [Caputo and Vasseur 2009], a special role is played by the total mass $M = \sum_{i=1}^p a_i$, which satisfies the diffusion equation

$$\partial_t M - \Delta(dM) = 0, \quad d(t, x) = \frac{\sum_{i=1}^p d_i a_i(t, x)}{\sum_{i=1}^p a_i(t, x)}, \quad (11)$$

where, by virtue of (3), the diffusion coefficient d satisfies

$$0 < \delta_\star \leq d(t, x) \leq \delta^\star.$$

This relation can be used to establish, through an elegant duality argument, an estimate in $L^2((0, T) \times \mathbb{R}^N)$; see [Pierre and Schmitt 1997; Desvillettes et al. 2007]. This estimate is a key for proving the global existence of weak solutions for the quadratic problem (1)–(2) in [Desvillettes et al. 2007]: at least, it is worth pointing out that with this L^2 estimate the right-hand side $Q_i(a)$ in (1) makes sense, while the estimates based on the mass conservation and entropy dissipation were not enough. However, the L^2 estimate does not shrink the rescaled solutions $a^{(\epsilon)}$ as $\epsilon \rightarrow 0$ and it is thus not enough to provide global boundedness and regularity. This is where we can take advantage of using a weak norm.

In the present work, we wish to fill the gap in the boundedness theory and to provide a complete answer for the quadratic case in any dimension. In fact, our analysis also covers higher nonlinearities, but with an implicit condition on the growth condition. Our main results can be stated as follows.

Theorem 1.1. *Let $N \in \mathbb{N}$, with $N \geq 3$. For any initial data $a^0 = (a_1^0, a_2^0, a_3^0, a_4^0)$ in $(C^\infty(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))^4$ such that $a_i(x) \geq 0$ for any $x \in \mathbb{R}^N$ and $i \in \{1, \dots, 4\}$, there exists a unique, globally defined, solution $a = (a_1, a_2, a_3, a_4)$ to (1)–(2) which is nonnegative, bounded on $[0, T] \times \mathbb{R}^N$ for any $0 < T < \infty$, and C^∞ -smooth.*

Theorem 1.2. *Let $N \in \mathbb{N}$, with $N \geq 3$. Consider a system (6) satisfying (h1)–(h4). There exists $v_0 > 0$ depending on N, δ_\star and δ^\star such that if (h1) holds with $2 \leq q \leq 2 + v_0 \leq 2(N + 1)/N$, then for any nonnegative $a^0 \in C^\infty(\mathbb{R}^N; \mathbb{R}^p) \cap L^\infty(\mathbb{R}^N; \mathbb{R}^p)$, there exists a unique, globally defined, solution a to (6)–(7) which is nonnegative, bounded on $[0, T] \times \mathbb{R}^N$ for any $0 < T < \infty$, and C^∞ -smooth.*

Theorem 1.1 thus appears as a consequence of **Theorem 1.2**. The extra power ν_0 allowed on the nonlinearities depends on N , δ_\star and δ^\star in an implicit way and our method does not provide any precise estimate. It seems unlikely that it can correspond to a physically relevant threshold. The problem of regularity remains open for higher nonlinearities. The proof still follows the De Giorgi strategy, and relies on a refinement of the weak norm estimate obtained in [Caputo and Vasseur 2009] (which, though, remains a crucial ingredient of the proof). To be more specific, we are going to upgrade the L^∞ estimate to a C^α estimate, working with the set of distributions

$$T \in \mathcal{D}'((0, T) \times \mathbb{R}^N) \text{ such that } T = \Delta \Phi, \text{ with } \Phi \in L^\infty(0, \infty; C^\alpha(\mathbb{R}^N)),$$

for a certain regularity coefficient $0 < \alpha \leq 1$. This is combined with a $L^{(N+1)/N}$ estimate on the total mass, obtained through a duality argument. This argument is directly inspired by the derivation of elliptic estimates in [Fabes and Stroock 1984] and it appears as a dual version of the Alexandrof–Bakelman–Pucci–Krylov–Tso (ABPKT) estimate [Alexandrof 1966; Bakelman 1961; Pucci 1966; Krylov 1976; Tso 1985]. We point out that, contrarily to the approach in [Caputo and Vasseur 2009], we do not use here the bounds derived from the entropy dissipation (8).

The paper is organized as follows. In **Section 2**, we give an overview of the main steps of the proof. **Section 3** is concerned with the weak estimate on the total mass. It relies on a Hölderian regularity analysis for parabolic equations. This is combined with a duality argument which uses crucially the nonnegativity of the solution. **Section 4** is devoted to a complementary estimate in a suitable Lebesgue space, which, again, relies on a duality approach. **Section 5** explains how the arguments combine to end the proof of the main results. The paper is completed by a quite long appendix which details how the De Giorgi machinery arises in the justification of the intermediate steps of the proof.

Remark 1.3. The key estimates of the proof involve the bound from below $0 < \delta_\star \leq d(t, x)$; hence the result cannot be extended to cases with degenerate diffusion coefficients. That the total mass M satisfies the diffusion equation (11) uses crucially the fact that the diffusion coefficients are given by scalar matrices. It also uses the assumption that the coefficients d_i are constant; otherwise an additional convection term $\nabla \cdot (uM)$ arises with the velocity field $(t, x) \mapsto u(t, x)$ having components

$$u_j(t, x) = \frac{\sum_{i=1}^p a_i(t, x) \partial_{x_j} d_i(x)}{\sum_{i=1}^p a_i(t, x)}.$$

It is likely that, up to suitable technical requirements on the d_i 's, the analysis could cover such a situation as well. Our analysis can be adapted to handle problems in bounded domains with Neumann boundary conditions; the situation of Dirichlet conditions is more subtle since there are difficulties to obtain useful estimates up to the boundaries [Pierre et al. 2017a].

2. Main steps of the proof

A priori estimates: boundedness, global existence and regularity of the solutions. In what follows, we are going to establish several a priori estimates satisfied by the solutions of (6). To this end, we will perform various manipulations, such as integrations by parts, permutations of integrals and derivation,

etc. These manipulations apply to the smooth solutions of the problem that can be shown to exist on a small enough time interval; see [Goudon and Vasseur 2010, Proposition A.2]. They equally apply to solutions of suitable approximations of the problem (6). The construction of such an approximation (by regularizing data, coefficients, cutting-off the nonlinearities...) can be a delicate issue in order to preserve the structural features of the original equation, and to admit a globally defined smooth solution. We refer the reader on this issue to [Desvillettes et al. 2007]. As it will be clear in the forthcoming discussion, the estimates we are going to derive do not depend on the regularization parameter, but only on N , δ_\star , δ^\star , and \mathcal{Q} , p , q (see (h1)), which, eventually, allows us to conclude by getting rid of the regularization parameter. The very first estimate is a direct consequence of the mass conservation and entropy dissipation properties of the system. The following claim, see [Goudon and Vasseur 2010, Proposition 2.1], applies without any restriction on the number of species p , the degree of nonlinearity q nor on the space dimension N .

Proposition 2.1 [Goudon and Vasseur 2010]. *Assume (h1)–(h4). Let $a_0 = (a_1^0, \dots, a_p^0)$, with nonnegative components, satisfy (9). Then, for any $0 < T < \infty$, there exists $0 < C(T) < \infty$ such that*

$$\sup_{0 \leq t \leq T} \left\{ \sum_{i=1}^p \int_{\mathbb{R}^N} a_i (1 + |x| + |\ln(a_i)|)(t, x) \, dx \right\} + \sum_{i=1}^p \int_0^T \int_{\mathbb{R}^N} |\nabla \sqrt{a_i}|^2(s, x) \, dx \, ds + \sum_{i=1}^p \int_0^T \int_{\mathbb{R}^N} Q_i(a) \ln(a_i) \, dx \, ds \leq C(T).$$

The entropy dissipation (8) tells us that $\sum_{i=1}^p \int_{\mathbb{R}^N} a_i \ln(a_i)(t, x) \, dx$ is a nonincreasing function of the time variable. However, this quantity has no sign. To make this information a useful estimate, involving the nonnegative quantities $a_i |\ln(a_i)|$ we need a control on the first-order space moments $\int_{\mathbb{R}^N} |x| a_i(t, x) \, dx$. We refer the reader to [Goudon and Vasseur 2010] for details. This estimate will not be used in our reasoning; nevertheless the entropy dissipation still has a crucial role in the proof of Theorems 1.1 and 1.2. By the way note that the counterexamples of systems that produce blow up in [Pierre and Schmitt 1997] very likely do not satisfy the entropy dissipation property.

As said above, for data in $C^\infty \cap L^\infty(\mathbb{R}^N)$, we can construct a C^∞ and bounded solution defined on a small enough interval. Let T_{\max} be the lifespan of such a solution. Standard bootstrapping arguments tell us that if $T_{\max} < \infty$ then we have

$$\limsup_{t \rightarrow T_{\max}} \|a(t, \cdot)\|_{L^\infty(\mathbb{R}^N)} = +\infty.$$

In what follows, we are going to obtain a uniform bound satisfied by $\|a(t, \cdot)\|_{L^\infty(\mathbb{R}^N)}$ on the time interval $[0, T_{\max})$, depending only on T_{\max} and the assumptions on the data, which thus contradicts the occurrence of a blow-up of the solution in finite time. Therefore, the L^∞ estimate implies that the lifespan of the solutions of (6)–(7) is infinite. Moreover, boundedness also implies the regularity of the solution, by a bootstrap argument; see [Goudon and Vasseur 2010, Proposition A.1].

The key intermediate statements. The main ingredient consists in showing that the local boundedness can be obtained from a local estimate in L^r , with $r > 1$; see [Caputo and Vasseur 2009, Proposition 4].

We thus work on balls

$$B_\rho = \{x \in \mathbb{R}^N : |x| \leq \rho\}.$$

Lemma 2.2 (De Giorgi-type lemma, [Caputo and Vasseur 2009]). *We suppose that $2 \leq q < 2(N+1)/N$. We also suppose that (h1)–(h4) holds. Let a be a nonnegative solution to (6) on $(-1, 0) \times B_1$. Then, for any $r > 1$, there exists a universal constant $\delta_r > 0$ such that, if $a = (a_1, \dots, a_p)$ satisfies*

$$\sum_{i=1}^p \|a_i\|_{L^r((-1,0) \times B_1)} \leq \delta_r,$$

then $0 \leq a_i(0, 0) \leq 1$ for $i \in \{1, \dots, p\}$.

The proof relies on De Giorgi's techniques [1957]; see also [Alikakos 1979] for an alternative approach. For the sake of completeness we describe the main steps in Appendix B; it is also important to detail this proof since this is where the entropy dissipation plays a central role. At first sight this information does not look very useful since the natural estimates for (6)–(7) in Proposition 2.1 do not involve L^r norms for an exponent r larger than 1. However, we will be able to identify further estimates, which shrink for the rescaled solutions (10) as $\epsilon \rightarrow 0$. Namely, we will find that $r = (N+1)/N$ plays a specific role since the rescaled total mass satisfies $\lim_{\epsilon \rightarrow 0} \|M^{(\epsilon)}\|_{L^{(N+1)/N}((-1,0) \times B_1)} = 0$. Thus, for ϵ small enough the rescaled solution fulfills the criterion in Lemma 2.2.

Lemma 2.3. *There exists $\epsilon_0 > 0$ and $\nu_0 > 0$ depending on N , δ_\star and δ^\star such that if (h1) holds with $2 \leq q \leq 2 + \nu_0 \leq 2(N+1)/N$, then for all $0 < \epsilon \leq \epsilon_0$ we have*

$$\sum_{i=1}^p \|a_i^{(\epsilon)}\|_{L^{(N+1)/N}((-1,0) \times B_1)} \leq \delta,$$

with $\delta = \delta_{(N+1)/N}$ as defined in Lemma 2.2.

Coming back to the original variables, we obtain the L^∞ estimate.

Corollary 2.4. *Let ϵ_0 be defined in Lemma 2.3. Then, for all $T_{\max}/2 < t < T_{\max}$, we have*

$$\sum_{i=1}^p \|a_i(t, \cdot)\|_{L^\infty(\mathbb{R}^N)} \leq \epsilon_0^{-2/(q-1)}.$$

Proof. Pick x_0 in \mathbb{R}^N and $t_0 \in (T_{\max}/2, T_{\max})$. Applying Lemma 2.2 to $a^{(\epsilon_0)}$ yields

$$0 \leq \sum_{i=1}^p a_i(t_0, x_0) = \epsilon_0^{-2/(q-1)} \sum_{i=1}^p a_i^{(\epsilon_0)}(0, 0) \leq \epsilon_0^{-2/(q-1)}. \quad \square$$

Having this statement at hand allows us to conclude the proof of Theorem 1.2. Let $2 \leq q \leq 2 + \nu_0 \leq 2(N+1)/N$. Let $a = (a_1, \dots, a_p)$ be a solution to (6)–(7), and let T_{\max} be the lifespan of a . Assume that T_{\max} is finite. Then, for each $i \in \{1, \dots, p\}$, Corollary 2.4 tells us that $a_i(t, \cdot)$ is uniformly bounded for all $T_{\max}/2 < t < T_{\max}$ and thus the sup norm does not blow up as $t \rightarrow T_{\max}$. This contradicts the fact that T_{\max} is the maximal time of existence of a smooth solution of (6)–(7). \square

Therefore the cornerstone of the proof consists in proving [Lemma 2.3](#) and identifying the specific role played by the norm $L^{(N+1)/N}$. The argument is two-fold and it uses the diffusion equation (11) satisfied by the total mass $M(t, x) = \sum_{i=1}^p a_i(t, x)$. On the one hand, we shall show that the norm $L^{(N+1)/N}$ of M can be controlled by means of the norm $L^\infty(0, \infty; L^1(\mathbb{R}^N))$. On the other hand, we shall obtain a new estimate on a *weak norm* of M , which will allow us to conclude that

$$\lim_{\epsilon \rightarrow 0} \|M^{(\epsilon)}\|_{L^\infty(0, \infty; L^1(\mathbb{R}^N))} = 0, \quad \text{with } M^{(\epsilon)}(s, y) = \epsilon^{2/(q-1)} M(t + \epsilon^2 s, x + \epsilon y).$$

This analysis is based on duality arguments and regularization properties of parabolic equations. Accordingly, we can conclude to the shrinking as $\epsilon \rightarrow 0$ of the $L^{(N+1)/N}$ norm of the rescaled solutions.

3. Weak norm estimates on the total mass and shrinking of the rescaled total mass

Our approach relies on the following statement.

Proposition 3.1. *Let $\Phi : (0, T) \times \mathbb{R}^N \rightarrow \mathbb{R}$ such that*

- (a) Φ lies in $L^\infty((0, T) \times \mathbb{R}^N)$;
- (b) $\Delta \Phi = M \geq 0$;
- (c) Φ satisfies $\partial_t \Phi - d \Delta \Phi = 0$ on $(0, T) \times \mathbb{R}^N$, with a coefficient $d : (0, T) \times \mathbb{R}^N \rightarrow \mathbb{R}$ satisfying $0 < \delta_\star \leq d(t, x) \leq \delta^\star < \infty$ for a.e. $(t, x) \in (0, T) \times \mathbb{R}^N$.

Then, there exists $\alpha \in (0, 1]$ such that $\Phi \in C^{[\alpha/2, \alpha]}([t_0, T] \times \mathbb{R}^N)$ for any $t_0 > 0$, which means that we can find $C > 0$ such that, for any $(t, x) \in [t_0, T] \times \mathbb{R}^N$ and $(\tau, h) \in \mathbb{R} \times \mathbb{R}^N$ with $t + \tau \geq t_0$, we have

$$\frac{|\Phi(t + \tau, x + h) - \Phi(t, x)|}{|\tau|^{\alpha/2} + |h|^\alpha} \leq C \|\Phi\|_{L^\infty}.$$

This Hölder regularity estimate for nonconservative parabolic equations dates back to [\[Krylov and Safonov 1979; 1980\]](#). In fact, the result of those papers does not need the sign property (b). However, as it will be explained below, this sign property naturally appears for the system under consideration, and it plays a further crucial role throughout the analysis. Let us explain the interest of this statement for our purpose. As said above, the total mass M satisfies the diffusion equation (11). Of course, by definition, M is a nonnegative function which lies in $L^\infty(0, \infty; L^1(\mathbb{R}^N))$. Let Φ satisfy $\Delta \Phi = M \geq 0$. Since $d(t, x)$ is bounded above by δ^\star , Φ also satisfies the evolution equation

$$\partial_t \Phi - \delta^\star \Delta \Phi = (d - \delta^\star) \Delta \Phi = (d - \delta^\star) M \leq 0.$$

This observation is the cornerstone of the analysis performed in [\[Caputo and Vasseur 2009\]](#). In particular, we will make use of the following crucial property established in Proposition 11 and Corollary 12 of that paper.

Proposition 3.2. *Let $N \in \mathbb{N}$, with $N \geq 3$. Let $\Phi = \Delta^{-1} M$ with M the total mass associated to a solution of (6). Then, we have*

$$\|\Phi\|_{L^\infty((0, T) \times \mathbb{R}^N)} \leq \|\Phi(0, \cdot)\|_{L^\infty(\mathbb{R}^N)} \leq K_N \|M(0, \cdot)\|_{L^\infty(\mathbb{R}^N)}^{1-2/N} \|M(0, \cdot)\|_{L^1(\mathbb{R}^N)}^{2/N},$$

where $K_N > 0$ is a certain universal constant, which only depends on the space dimension.

Proposition 3.1 thus strengthens the results of [Caputo and Vasseur 2009] in the sense that it provides, beyond the L^∞ estimate on Φ , a Hölder-regularity estimate. Since the estimate in **Proposition 3.2** is not evident at first sight, we give the main steps of the proof in **Appendix C** for the sake of completeness. We shall use the following consequence of **Proposition 3.1**, which is precisely the estimate that allows us to go beyond the subquadratic nonlinearities dealt with in [loc. cit.].

Lemma 3.3. *Let M be a nonnegative solution of (11), and let $\Phi = \Delta^{-1}M$. Let $t \geq t_0 > 0$ and $x \in \mathbb{R}^N$. For $\epsilon > 0$, we set $M^{(\epsilon)}(s, y) = \epsilon^{2/(q-1)}M(t + \epsilon^2s, x + \epsilon y)$. We suppose that $M^{(\epsilon)}$ lies in $L^\infty(-4, 0; L^1(\mathbb{R}^N))$. Then, there exists $c > 0$ and $0 < \alpha \leq 1$ (provided by **Proposition 3.1**), depending only on N, δ_\star and δ^\star , such that for any $0 < \epsilon \leq \sqrt{t_0}/2$,*

$$\sup_{-4 \leq s \leq 0} \int_{B_2} M^{(\epsilon)}(s, y) \, dy \leq c \|\Phi\|_{L^\infty} \epsilon^{\alpha-2+2/(q-1)}.$$

Proof. Let $\zeta \in C_c^\infty(\mathbb{R}^N)$ be such that $\text{supp}(\zeta) \subset B_2$ and $\zeta(x) = 1$ for any $x \in B_1$. Since $M^{(\epsilon)} \geq 0$, we get

$$\begin{aligned} \int_{B_1} M^{(\epsilon)}(s, y) \, dy &\leq \int_{B_2} \zeta M^{(\epsilon)}(s, y) \, dy = \int_{B_2} \zeta \Delta \Phi^{(\epsilon)}(s, y) \, dy \\ &\leq \int_{B_2} \Delta \zeta(y) (\Phi^{(\epsilon)}(s, y) - \Phi^{(\epsilon)}(0, 0)) \, dy. \end{aligned}$$

By virtue of **Proposition 3.1**, we can write

$$\begin{aligned} \int_{B_1} M^{(\epsilon)}(s, y) \, dy &\leq \epsilon^{-2+2/(q-1)} \int_{B_2} \Delta \zeta(y) (\Phi(t + \epsilon^2s, x + \epsilon y) - \Phi(t, x)) \, dy \\ &\leq C \|\zeta\|_{W^{2,\infty}(\mathbb{R}^N)} \|\Phi\|_{L^\infty} \epsilon^{\alpha-2+2/(q-1)} \end{aligned}$$

for any $s \in (-4, 0)$ and $0 < \epsilon^2 < t_0/4$. In the first inequality, the exponent $2/(q-1)$ comes from the rescaling that defines $M^{(\epsilon)}$, and the exponent -2 comes from the relation

$$M^{(\epsilon)}(s, y) = \frac{1}{\epsilon^2} \Delta(\Phi^{(\epsilon)}(t + \epsilon^2s, x + \epsilon y)). \quad \square$$

The information in **Lemma 3.3** is relevant when the exponent $\alpha - 2 + 2/(q-1)$ is positive. This implies a restriction on $q \leq 1 + 2/(2 - \alpha)$, where we remind the reader that $\alpha \in (0, 1]$ depends on $N, \delta_\star, \delta^\star$ and we note that the bound from above increases to 3 as $\alpha \rightarrow 1$. This combines with the constraint $q \leq 2(N+1)/N$, which is of different nature; see **Lemma 2.2**, **Lemma 2.3** and **Lemma B.2**.

As indicated above, the Hölder estimate in **Proposition 3.1** is a standard result due to [Krylov and Safonov 1979; 1980]. For the sake of completeness, we provide in **Appendix D** an alternative proof, fully based on De Giorgi's arguments. Note however that this analysis uses the additional assumption (b), which appears naturally in the problem under consideration.

4. $L^{(N+1)/N}$ estimate on the total mass

This section is devoted to the proof of the following statement.

Proposition 4.1. *There exists a constant $K > 0$ (depending on N, δ_*, δ^*) such that, $M \geq 0$ being a solution of (11) in $Q_2 = (-4, 0) \times B_2$ and defining $Q_1 = (-1, 0) \times B_1$, we have*

$$\|M\|_{L^{(N+1)/N}(Q_1)} \leq K \sup_{-4 \leq t \leq 0} \int_{B_2} M(t, x) \, dx.$$

Proof. Let f be in $C_c^\infty(Q_1)$ such that

$$\|f\|_{L^{N+1}(Q_1)} \leq 1.$$

We consider the solution of the end-value problem

$$\begin{aligned} \partial_t u + d \Delta u &= f \quad \text{in } (0, T) \times \mathbb{R}^N, \\ u(T, x) &= 0, \quad u|_{\partial B_2} = 0. \end{aligned} \tag{12}$$

We start by reminding the reader of the Alexandroff–Bakelman–Pucci–Krylov–Tso (ABPKT) inequality [Alexandroff 1966; Bakelman 1961; Pucci 1966; Krylov 1976; Tso 1985]: there exists a constant $\mathcal{C} > 0$ such that

$$\sup_{(t,x) \in Q_2} |u(t, x)| \leq \mathcal{C} \|f\|_{L^{N+1}(Q_2)}. \tag{13}$$

In order to obtain an estimate on the $L^{(N+1)/N}(Q_1)$ norm of M , a solution of (11), we proceed by duality, bearing in mind the definition

$$\|M\|_{L^{(N+1)/N}(Q_1)} = \sup \left\{ \left| \iint_{Q_1} M f \, dx \, dt \right| : f \in C_c^\infty(Q_1), \|f\|_{L^{N+1}(Q_1)} \leq 1 \right\}.$$

Let ζ be a cut-off function: $\zeta \in C_c^\infty(B_{3/2})$, $\zeta(x) = 1$ for any $x \in B_1$, and $0 \leq \zeta(x) \leq 1$ for any $x \in \mathbb{R}^N$. Note that

$$\iint_{Q_2} \zeta M f \, dx \, dt = \iint_{Q_1} M f \, dx \, dt,$$

since $\text{supp}(f) \subset Q_1$. We compute this integral by using (12):

$$\begin{aligned} \iint_{Q_2} \zeta M f \, dx \, dt &= \iint_{Q_2} \zeta M (\partial_t u + d \Delta u) \, dx \, dt \\ &= \int_{-2}^0 \frac{d}{dt} \left(\int_{B_2} \zeta M u \, dx \right) dt - \iint_{Q_2} \zeta u \Delta(dM) \, dx \, dt + \iint_{Q_2} \zeta M d \Delta u \, dx \, dt \\ &= \int_{B_2} \zeta M u(0, x) \, dx - 2 \iint_{Q_2} dM \nabla \zeta \cdot \nabla u \, dx \, dt - \iint_{Q_2} u dM \Delta \zeta \, dx \, dt. \end{aligned}$$

We have used several integrations by parts where the boundary terms vanish owing to the fact that $\text{supp}(\zeta) \subset B_{3/2} \subset B_2$. The integrand of the penultimate term in the right-hand side can be rewritten as $\sqrt{dM} \nabla u \cdot \sqrt{dM} \nabla \zeta$, and then we use the Cauchy–Schwarz inequality and the Young inequality

$$ab = \sqrt{\kappa} a \frac{b}{\sqrt{\kappa}} \leq \frac{1}{2} \left(\kappa a^2 + \frac{b^2}{\kappa} \right).$$

We thus arrive at the following estimate:

$$\left| \iint_{Q_2} \zeta f M \, dx \, dt \right| \leq \left| \int_{B_2} \zeta M u(0, x) \, dx \right| + \kappa \iint_{Q_2} dM |\nabla u|^2 \, dx \, dt + \frac{1}{\kappa} \iint_{Q_2} dM |\nabla \zeta|^2 \, dx \, dt + \left| \iint_{Q_2} u dM \Delta \zeta \, dx \, dt \right|, \quad (14)$$

where $\kappa \in (0, 1)$ is a parameter that will be determined later on. Inspired from [Fabes and Stroock 1984, proof of Theorem 2.1], in order to estimate the second integral in the right-hand side, we use the elementary relation

$$|\nabla u|^2 = \frac{1}{2} \Delta(u^2) - u \Delta u.$$

Going back to (12), we are thus led to

$$d|\nabla u|^2 = \frac{1}{2} d\Delta(u^2) + \frac{1}{2} \partial_t(u^2) - uf.$$

The advantage of this formulation relies on the fact that, denoting by ν the outward unit normal on ∂B_2 ,

$$u|_{\partial B_2} = u^2|_{\partial B_2} = 0, \quad \nabla u^2 \cdot \nu|_{\partial B_2} = 2u \nabla u \cdot \nu|_{\partial B_2} = 0,$$

which allows us to perform further integration by parts. We get

$$\begin{aligned} & \iint_{Q_2} dM |\nabla u|^2 \, dx \, dt \\ &= \frac{1}{2} \iint_{Q_2} dM \Delta(u^2) \, dx \, dt + \frac{1}{2} \iint_{Q_2} M \partial_t(u^2) \, dx \, dt - \iint_{Q_2} M u f \, dx \, dt \\ &= -\frac{1}{2} \iint_{Q_2} \nabla(dM) \cdot \nabla(u^2) \, dx \, dt + \frac{1}{2} \int_{B_2} M u^2(0, x) \, dx - \frac{1}{2} \iint_{Q_2} \Delta(dM) u^2 \, dx \, dt - \iint_{Q_2} M u f \, dx \, dt \\ &= \frac{1}{2} \int_{B_2} M u^2(0, x) \, dx - \iint_{Q_2} M u f \, dx \, dt. \end{aligned}$$

For the last term, since $\text{supp}(f) \subset Q_1$, the integral actually reduces over Q_1 only. The Hölder inequality then yields

$$\begin{aligned} \left| \iint_{Q_2} M u f \, dx \, dt \right| &= \left| \iint_{Q_1} M u f \, dx \, dt \right| \\ &\leq \|u\|_{L^\infty(Q_1)} \|M\|_{L^{(N+1)/N}(Q_1)} \|f\|_{L^{N+1}(Q_1)} \\ &\leq \mathcal{C} \|f\|_{L^{N+1}(Q_1)}^2 \|M\|_{L^{(N+1)/N}(Q_1)}, \end{aligned}$$

by using (13). Additionally, still by using (13) and $\text{supp}(f) \subset Q_1$, we get

$$\begin{aligned} \frac{1}{2} \iint_{Q_2} M u^2(0, x) \, dx &\leq \frac{1}{2} \|u\|_{L^\infty(Q_2)}^2 \|M\|_{L^\infty(-4, 0; L^1(Q_2))} \\ &\leq \mathcal{C}^2 \|f\|_{L^{N+1}(Q_1)}^2 \|M\|_{L^\infty(-4, 0; L^1(Q_2))}. \end{aligned}$$

The last two terms in the right-hand side of (14) are estimated as follows: we get

$$\iint_{Q_2} dM |\nabla \zeta|^2 dx dt \leq 4\delta^* \|\zeta\|_{W^{1,\infty}(B_2)}^2 \|M\|_{L^\infty((-4,0);L^1(B_2))}$$

and

$$\begin{aligned} \left| \iint_{Q_2} u dM \Delta \zeta dx dt \right| &\leq 4\delta^* \|\zeta\|_{W^{2,\infty}(B_2)} \|u\|_{L^\infty(Q_2)} \|M\|_{L^\infty((-4,0);L^1(B_2))} \\ &\leq 4\delta^* \|\zeta\|_{W^{2,\infty}(B_2)} \mathcal{C} \|f\|_{L^{N+1}(Q_1)} \|M\|_{L^\infty((-4,0);L^1(B_2))}. \end{aligned}$$

The first integral in the right-hand side of (14) is dominated by

$$\|u\|_{L^\infty(Q_2)} \|M\|_{L^\infty(-4,0;L^1(B_2))} \leq \mathcal{C} \|f\|_{L^{N+1}(Q_2)} \|M\|_{L^\infty(-4,0;L^1(B_2))}.$$

Finally, we have found a constant $C > 0$ such that for any $f \in C_c^\infty(Q_1)$, with $\|f\|_{L^{N+1}(Q_1)} \leq 1$, we have

$$\left| \iint_{Q_1} f M dx dt \right| \leq C \left(\left(1 + \kappa + \frac{1}{\kappa}\right) \|M\|_{L^\infty(-4,0;L^1(B_2))} + \kappa \|M\|_{L^{(N+1)/N}(Q_1)} \right).$$

Taking the supremum over such f 's makes the dual norm $L^{(N+1)/N}(Q_1)$ appear. We choose κ small enough, so that $1 - \kappa C > 1$, and we conclude that

$$\|M\|_{L^{(N+1)/N}(Q_1)} \leq \frac{C(1 + \kappa + 1/\kappa)}{1 - \kappa C} \|M\|_{L^\infty(-4,0;L^1(B_2))}$$

holds. □

5. End of proof of Theorem 1.2: proof of Lemma 2.3

Let $0 < \epsilon_0 < \sqrt{T_{\max}/2}$. For each component $a_i^{(\epsilon)}$, Proposition 4.1 gives

$$\|a_i^{(\epsilon)}\|_{L^{(N+1)/N}(Q_1)} \leq \|M^{(\epsilon)}\|_{L^{(N+1)/N}(Q_1)} \leq K \|M^{(\epsilon)}\|_{L^\infty(-4,0;L^1(B_2))}. \quad (15)$$

Next, Lemma 3.3, yields

$$\|M^{(\epsilon)}\|_{L^\infty(-4,0;L^1(B_2))} \leq c \|\Phi\|_{L^\infty} \epsilon^{\alpha-2+2/(q-1)}. \quad (16)$$

Combining (15) and (16) with Proposition 3.2 leads to

$$\sum_{i=1}^p \|a_i^{(\epsilon)}\|_{L^{(N+1)/N}(Q_1)} \leq \mathcal{K} \|a^0\|_{L^\infty(\mathbb{R}^N)}^{1-2/N} \|a^0\|_{L^1(\mathbb{R}^N)}^{2/N} \epsilon^{\alpha-2+2/(q-1)} \quad (17)$$

for a constant \mathcal{K} which depends on p and N . This information is useful as far as the degree of nonlinearities is such that the exponent remains positive, which means $q \leq 2 + \alpha/(2 - \alpha)$. It ends the proof of Lemma 2.3.

As explained in Section 2, having at hand this property of the rescaled solution we go back to the original unknown, and we deduce the L^∞ bound of the solution, see Corollary 2.4. Theorem 1.2, and therefore Theorem 1.1 too, is fully justified. □

Remark 5.1. The estimates discussed above differ from [Caputo and Vasseur 2009] (see specifically Corollary 14 and Lemma 15), and in particular the smallness condition on ϵ_0 does not involve the initial entropy (9).

Appendix A: A basic iteration lemma

The De Giorgi approach leads us to construct sequences, based on energy-entropy estimates, where the parameter of the sequence controls level sets of the solution and space-time localization. Roughly speaking, we obtain a nonlinear control of the n -th level by the $(n-1)$ -th level. Namely, if u_n characterizes a level set associated to a value $\eta_n > 0$, over a domain Q_n , we obtain inequalities like $u_n \leq \Lambda u_{n-1}^\gamma$. We can finally conclude to a local property of the solution, letting n go to ∞ by using the following simple result.

Lemma A.1. *Let $(u_n)_{n \in \mathbb{N}}$ be a sequence of nonnegative real numbers. We suppose that it satisfies, for any $n \in \mathbb{N} \setminus \{0\}$,*

$$u_n \leq \Lambda^n u_{n-1}^\gamma,$$

where $\Lambda, \gamma > 1$. Then, there exists $\kappa > 0$ (depending on Λ, γ) such that, if $0 \leq u_0 \leq \kappa$, then $\lim_{n \rightarrow \infty} u_n = 0$.

Proof. We set $v_n = \ln(u_n)$ which satisfies

$$v_n \leq n \ln(\Lambda) + \gamma v_{n-1},$$

and thus

$$v_n \leq \ln(\Lambda) \sum_{j=0}^n j \gamma^{n-j} + v_0 \gamma^n \leq \gamma^n \ln(\Lambda^{F(\gamma)} u_0),$$

with

$$F(\gamma) = \frac{1}{\gamma} \sum_{j=0}^{\infty} j \left(\frac{1}{\gamma}\right)^{j-1} = \frac{1}{\gamma} \frac{d}{dx} \left(\frac{1}{1-x} \right) \Big|_{x=1/\gamma} = \frac{1}{\gamma} \left(\frac{1}{1-1/\gamma} \right)^2.$$

Therefore v_n tends to $-\infty$, and u_n tends to 0, as $n \rightarrow \infty$ provided u_0 is small enough. \square

Appendix B: Proof of Lemma 2.2

The proof is based on the De Giorgi techniques [1957] and it is reminiscent of the method introduced by Alikakos [1979]. We exploit the dissipative properties of the system by considering the following nonnegative, nondecreasing, convex, and C^1 function

$$H(z) = \begin{cases} (1+z) \ln(1+z) - z & \text{if } z \geq 0, \\ 0 & \text{if } z \leq 0. \end{cases}$$

Let us introduce the sequences, for $j \in \mathbb{N}$,

$$k_j = 1 - 2^{-j}, \quad t_j = \frac{1}{4} + 2^{-j-2}.$$

Henceforth, we set $\mathcal{B}_j = B_{t_j}$ and $\mathcal{Q}_j = (-t_j, 0) \times \mathcal{B}_j$. Note that

$$\begin{aligned} B(0, \tfrac{1}{4}) &\subset \mathcal{B}_j \subset \mathcal{B}_{j-1} \subset B(0, \tfrac{1}{2}), \\ (-\tfrac{1}{4}, 0) \times B(0, \tfrac{1}{4}) &\subset \mathcal{Q}_j \subset \mathcal{Q}_{j-1} \subset (-\tfrac{1}{2}, 0) \times B(0, \tfrac{1}{2}). \end{aligned}$$

We also introduce a family of cut-off functions that satisfies the properties

$$\begin{aligned} \zeta_j : \mathbb{R}^N &\rightarrow [0, \infty), \quad \zeta_j \in C_c^\infty(\mathbb{R}^N), \\ 0 &\leq \zeta_j(x) \leq 1, \\ \zeta_j(x) &= 1 \text{ for } x \in \mathcal{B}_j, \quad \zeta_j(x) = 0 \text{ for } x \in \mathbb{R}^N \setminus \mathcal{B}_{j-1}, \end{aligned}$$

and

$$\sup_{l, m \in \{1, \dots, N\}, x \in \mathbb{R}^N} |\partial_{l, m}^2 \zeta_j(x)| \leq C 2^{2j} \quad \text{for a certain constant } C > 0.$$

Lemma B.1. *There exists a constant $\hat{C} > 0$, which depends only on δ_\star , δ^\star , and on (h1)–(h4), such that for any solution $a = (a_1, \dots, a_p)$ of (6) and any $\eta \in [0, 1]$, we have*

$$\begin{aligned} \sup_{-t_j \leq t \leq 0} \sum_{i=1}^p \int_{\mathcal{B}_j} H(a_i - \eta)(t, x) \, dx + 4\delta_\star \sum_{i=1}^p \iint_{\mathcal{Q}_j} |\nabla_x \sqrt{1 + [a_i - \eta]_+}|^2(\tau, x) \, dx \, d\tau \\ \leq \hat{C} \left(2^{2j} \sum_{i=1}^p \int_{-t_{j-1}}^0 \int_{\mathcal{B}_{j-1}} H(a_i - \eta)(s, x) \, dx \, ds \right. \\ \left. + \sum_{i=1}^p \int_{-t_{j-1}}^0 \int_{\mathcal{B}_{j-1}} (1 + [a_i - \eta]_+)^{q-1} \ln(1 + [a_i - \eta]_+)(\tau, x) \, dx \, d\tau \right). \end{aligned}$$

Proof. Multiply (6) by $\zeta_j H'(a_i - \eta)$, integrate over \mathcal{B}_{j-1} and sum. We get

$$\frac{d}{dt} \sum_{i=1}^p \int_{\mathcal{B}_{j-1}} \zeta_j H(a_i - \eta) \, dx = \sum_{i=1}^p \int_{\mathcal{B}_{j-1}} d_i \Delta a_i H'(a_i - \eta) \zeta_j \, dx + \sum_{i=1}^p \int_{\mathcal{B}_{j-1}} Q_i(a) H'(a_i - \eta) \zeta_j \, dx. \quad (18)$$

The first term in the right-hand side of (18) can be written as

$$- \sum_{i=1}^p \int_{\mathcal{B}_{j-1}} d_i |\nabla a_i|^2 H''(a_i - \eta) \zeta_j \, dx + \sum_{i=1}^p \int_{\mathcal{B}_{j-1}} d_i H(a_i - \eta) \Delta \zeta_j \, dx,$$

where, on the one hand,

$$\sum_{i=1}^p \int_{\mathcal{B}_{j-1}} d_i |\nabla_x a_i|^2 H''(a_i - \eta) \zeta_j \, dx \geq 4\delta_\star \sum_{i=1}^p \int_{\mathcal{B}_j} |\nabla_x \sqrt{1 + [a_i - \eta]_+}|^2 \, dx,$$

and, on the other hand,

$$\sum_{i=1}^p \int_{\mathcal{B}_{j-1}} d_i H(a_i - \eta) \Delta \zeta_j \, dx \leq C \delta^\star 2^{2j} \int_{\mathcal{B}_{j-1}} H(a_i - \eta) \, dx.$$

For the second term in the right-hand side of (18), we get

$$\begin{aligned}
 \sum_{i=1}^p \int_{\mathcal{B}_{j-1}} Q_i(a) H'(a_i - \eta) \zeta_j \, dx &= \sum_{i=1}^p \int_{\mathcal{B}_{j-1}} (Q_i(a) - Q_i(1 + [a - \eta]_+)) \ln(1 + [a_i - \eta]_+) \zeta_j \, dx \\
 &\quad + \underbrace{\sum_{i=1}^p \int_{\mathcal{B}_{j-1}} Q_i(1 + [a - \eta]_+) \ln(1 + [a_i - \eta]_+) \zeta_j \, dx}_{\leq 0 \text{ by (h4)}} \\
 &\leq 2p\mathcal{Q} \sum_{i=1}^p \int_{\mathcal{B}_{j-1}} (1 + [a_i - \eta]_+)^{q-1} \ln(1 + [a_i - \eta]_+) \, dx.
 \end{aligned}$$

The last estimate is a consequence of (h1) and of the elementary inequality

$$|1 + [a - \eta]_+ - a| \leq 1 + |[a - \eta]_+ - a| \leq 1 + \eta \leq 2;$$

see [Goudon and Vasseur 2010, proof of Lemma 3.1] or [Caputo and Vasseur 2009, Lemma 3]. We arrive at

$$\begin{aligned}
 \frac{d}{dt} \sum_{i=1}^p \int_{\mathcal{B}_{j-1}} \zeta_j H(a_i - \eta) \, dx &+ 4\delta_\star \sum_{i=1}^p \int_{\mathcal{B}_j} |\nabla_x \sqrt{1 + [a_i - \eta]_+}|^2 \, dx \\
 &\leq C\delta^\star 2^{2j} \sum_{i=1}^p \int_{\mathcal{B}_{j-1}} H(a_i - \eta) \, dx + 2p\mathcal{Q} \sum_{i=1}^p \int_{\mathcal{B}_{j-1}} (1 + [a_i - \eta]_+)^{q-1} \ln(1 + [a_i - \eta]_+) \, dx.
 \end{aligned}$$

We integrate this relation over (s, t) , with $-t_j \leq t \leq 0$ and $-t_{j-1} \leq s \leq t_j$, and next we average with respect to $s \in (-t_{j-1}, -t_j)$, taking into account that $t_{j-1} - t_j = 2^{-j-2}$. We obtain

$$\begin{aligned}
 \sum_{i=1}^p \int_{\mathcal{B}_j} H(a_i - \eta)(t, x) \, dx &+ 4\delta_\star \sum_{i=1}^p \int_{-t_j}^t \int_{\mathcal{B}_j} |\nabla_x \sqrt{1 + [a_i - \eta]_+}|^2(\tau, x) \, dx \, d\tau \\
 &\leq \sum_{i=1}^p \int_{\mathcal{B}_{j-1}} \zeta_j H(a_i - \eta)(t, x) \, dx \\
 &\quad + 4\delta_\star \sum_{i=1}^p \frac{1}{2^{-j-2}} \int_{-t_{j-1}}^{-t_j} \int_s^t \int_{\mathcal{B}_j} |\nabla_x \sqrt{1 + [a_i - \eta]_+}|^2(\tau, x) \, dx \, d\tau \, ds \\
 &\leq \sum_{i=1}^p \frac{1}{2^{-j-2}} \int_{-t_{j-1}}^{-t_j} \int_{\mathcal{B}_{j-1}} \zeta_j H(a_i - \eta)(s, x) \, dx \, ds \\
 &\quad + C\delta^\star 2^{2j} \sum_{i=1}^p \frac{1}{2^{-j-2}} \int_{-t_{j-1}}^{-t_j} \int_s^t \int_{\mathcal{B}_{j-1}} H(a_i - \eta)(\tau, x) \, dx \, d\tau \, ds \\
 &\quad + 2p\mathcal{Q} \sum_{i=1}^p \frac{1}{2^{-j-2}} \int_{-t_{j-1}}^{-t_j} \int_s^t \int_{\mathcal{B}_{j-1}} (1 + [a_i - \eta]_+)^{q-1} \ln(1 + [a_i - \eta]_+)(\tau, x) \, dx \, d\tau \, ds
 \end{aligned}$$

$$\begin{aligned} &\leq 2^{j+2} \sum_{i=1}^p \int_{-t_{j-1}}^{-t_j} \int_{B_{j-1}} H(a_i - \eta)(s, x) \, dx \, ds + C \delta^* 2^{2j} \sum_{i=1}^p \int_{-t_{j-1}}^t \int_{B_{j-1}} H(a_i - \eta)(\tau, x) \, dx \, d\tau \\ &\quad + 2p \mathcal{Q} \sum_{i=1}^p \int_{-t_{j-1}}^t \int_{B_{j-1}} (1 + [a_i - \eta]_+)^{q-1} \ln(1 + [a_i - \eta]_+)(\tau, x) \, dx \, d\tau. \end{aligned}$$

We conclude by taking the supremum over $t \in (-t_j, 0)$. \square

Next, we specify the level set considered in these estimates: we use [Lemma B.1](#) with $\eta = k_j$ and we set

$$\mathcal{U}_j = \left(\sup_{-t_j \leq t \leq 0} \sum_{i=1}^p \int_{B_j} H(a_i - k_j) \, dx + \sum_{i=1}^p \iint_{\mathcal{Q}_j} |\nabla_x \sqrt{1 + [a_i - k_j]_+}|^2 \, dx \, ds \right).$$

Lemma B.2. *Let $2 \leq q < 2(N+1)/N$. Then:*

(i) *For any $r > 1$ there exists a universal constant $c_r > 0$ such that*

$$\mathcal{U}_0 \leq c_r \sum_{i=1}^p (\|a_i\|_{L^r((-1,0) \times B_1)}^r + \|a_i\|_{L^r((-1,0) \times B_1)}^{1/2} + \|a_i\|_{L^r((-1,0) \times B_1)}).$$

(ii) *There exists a constant $\Lambda > 1$ such that*

$$\mathcal{U}_j \leq \Lambda^j \mathcal{U}_{j-1}^{1+N/2}$$

for any $j \geq j_0$. Consequently, there exists $\delta > 0$ such that $\mathcal{U}_0 \leq \delta$ implies $\lim_{j \rightarrow \infty} \mathcal{U}_j = 0$.

Proof. Throughout the proof, we simply denote by c a constant that depends only on the parameters of the model, and on the Lebesgue exponent, without paying attention to the possible changes of the value of the constant from a line to another.

For proving (i), we go back to the definition

$$\mathcal{U}_0 = \left(\sup_{-1/2 \leq t \leq 0} \sum_{i=1}^p \int_{B_0} H(a_i) \, dx + \sum_{i=1}^p \iint_{\mathcal{Q}_0} |\nabla_x \sqrt{a_i + 1}|^2 \, dx \, d\tau \right),$$

where we remind the reader that $B_0 = B_{1/2}$ and $\mathcal{Q}_0 = (-\frac{1}{2}, 0) \times B_{1/2}$. We make use of the elementary inequalities

$$H(z) \leq c(z(1 + |\ln(z)|)), \quad (19)$$

$$|\nabla \sqrt{1+a}| \leq |\nabla \sqrt{a}|, \quad (20)$$

which hold for any $z \geq 0$ and any (smooth enough) function $a : \mathbb{R}^N \rightarrow [0, \infty)$, respectively. We consider $\zeta_0 \in C_c^\infty(\mathbb{R}^N)$, supported in B_1 , such that $0 \leq \zeta_0(x) \leq 1$ on \mathbb{R}^N and $\zeta_0(x) = 1$ on B_0 . We get

$$\frac{d}{dt} \sum_{i=1}^p \int_{B_1} \zeta_0(x) a_i(t, x) \, dx = \sum_{i=1}^p \int_{B_1} d_i \Delta \zeta_0(x) a_i(t, x) \, dx \leq \delta^* \|\Delta \zeta_0\|_{L^\infty} \sum_{i=1}^p \int_{B_1} a_i(t, x) \, dx.$$

Let $t \in (-\frac{1}{2}, 0)$ and $\tau \in (-1, t)$. We integrate over the time interval (τ, t) , and then we average over $\tau \in (-1, -\frac{1}{2})$. We are led to

$$\sup_{-1/2 \leq t \leq 0} \sum_{i=1}^p \int_{B_0} a_i(t, x) dx \leq c \sum_{i=1}^p \int_{-1}^0 \int_{B_1} a_i(\tau, x) dx d\tau.$$

Similarly, the localized version of the entropy dissipation becomes

$$\begin{aligned} \frac{d}{dt} \sum_{i=1}^p \int_{B_1} \zeta_0(x) a_i \ln(a_i) dx + \int_{B_1} \zeta_0(x) \frac{d_i |\nabla_x a_i|^2}{a_i} dx &= \sum_{i=1}^p \int_{B_1} \Delta \zeta_0 d_i (a_i \ln(a_i) - a_i) dx. \\ &\leq \delta^* \|\Delta \zeta_0\|_{L^\infty} \sum_{i=1}^p \int_{B_1} (a_i |\ln(a_i)| + a_i) dx. \end{aligned}$$

Again we integrate with respect to the time variable. We shall also use the trick

$$u |\ln(u)| = u \ln(u) \mathbf{1}_{u \geq 1} - u \ln(u) \mathbf{1}_{0 \leq u < 1} \leq u \ln(u) \mathbf{1}_{u \geq 1} + \frac{2}{e} \sqrt{u} \mathbf{1}_{0 \leq u < 1},$$

which allows us to dominate

$$u |\ln(u)| \leq c(u^r + \sqrt{u}).$$

It follows that

$$\begin{aligned} \sup_{-1/2 \leq t \leq 0} \sum_{i=1}^p \int_{B_0} a_i |\ln(a_i)| dx + 4\delta_* \sum_{i=1}^p \int_{-1/2}^0 \int_{B_0} |\nabla_x \sqrt{a_i}|^2 dx d\tau \\ \leq c \sum_{i=1}^p \left(\int_{-1}^0 \int_{B_1} (a_i^r + \sqrt{a_i} + a_i) dx d\tau \right) \\ \leq c \sum_{i=1}^p \left(\int_{-1}^0 \int_{B_1} |a_i|^r dx d\tau + \left(\int_{-1}^0 \int_{B_1} |a_i|^r dx d\tau \right)^{1/(2r)} \text{meas}(B_1)^{1-1/(2r)} \right. \\ \left. + \left(\int_{-1}^0 \int_{B_1} |a_i|^r dx d\tau \right)^{1/r} \text{meas}(B_1)^{1-1/r} \right), \end{aligned}$$

by using the Hölder inequality.

We turn to the proof of (ii). The estimate in [Lemma B.1](#) can be recast as

$$\begin{aligned} \mathcal{W}_j \leq C \left(2^{2j} \sum_{i=1}^p \iint_{\mathcal{Q}_{j-1}} H(a_i - k_j)(s, x) dx ds \right. \\ \left. + \sum_{i=1}^p \iint_{\mathcal{Q}_{j-1}} (1 + [a_i - k_j]_+)^{q-1} \ln(1 + [a_i - k_j]_+)(s, x) dx ds \right). \quad (21) \end{aligned}$$

Let us set

$$\Psi(z) = \sqrt{1+z} - 1.$$

For any $\gamma \geq 1$, $\beta > 0$, we can find a constant $c_{\gamma, \beta}$ such that

$$(1+z)^\gamma \ln(1+z) \leq c_\beta \Psi(z)^{2(\gamma+\beta)}.$$

Moreover, for $z \geq k_j \geq k_{j-1}$ we have

$$1 \leq \frac{z - k_{j-1}}{k_j - k_{j-1}} = 2^j (z - k_{j-1}).$$

Hence, we can estimate both integrals in the right-hand side of (21) by an expression like

$$\begin{aligned} \sum_{i=1}^p \iint_{Q_{j-1}} 2^{\gamma j} (1 + [a_i - k_{j-1}]_+)^{\gamma} \ln(1 + [a_i - k_{j-1}]_+) \, dx \, ds \\ \leq c_{\gamma, \beta} 2^{\gamma j} \sum_{i=1}^p \iint_{Q_{j-1}} \Psi([a_i - k_{j-1}]_+)^{\gamma + \beta} \, dx \, ds. \end{aligned}$$

We can play with the exponents γ and β for both terms so that we obtain a common bound from above, and we arrive at

$$\mathcal{U}_j \leq c 2^{4j} \sum_{i=1}^p \iint_{Q_{j-1}} \Psi([a_i - k_{j-1}]_+)^{2(N+2)/N} \, dx \, ds.$$

This is possible as far as $2(q-1) \leq 2(N+2)/N$, that is to say $q \leq 2(N+1)/N$. We shall conclude by using an interpolation argument. Indeed, on the one hand, we obviously have

$$\sup_{-t_{j-1} \leq s \leq 0} \int_{B_{j-1}} |\Psi([a_i - k_{j-1}]_+)|^2(s, x) \, dx \leq \mathcal{U}_{j-1},$$

while the Gagliardo–Nirenberg–Sobolev inequality, see [Nirenberg 1959, Theorem, p. 125], yields

$$\begin{aligned} \int_{-t_{j-1}}^0 \left(\int_{B_{j-1}} |\Psi([a_i - k_{j-1}]_+)|^{2N/(N-2)}(s, x) \, dx \right)^{(N-2)/N} ds \\ \leq c \iint_{Q_{j-1}} |\nabla \Psi([a_i - k_{j-1}]_+)|^2(s, x) \, dx \, ds \leq c \mathcal{U}_{j-1}. \end{aligned}$$

By using the interpolation

$$\frac{N+2}{N} = \theta \frac{2N}{N-2} + 2(1-\theta), \quad \theta = \frac{N-2}{N} \in (0, 1),$$

we combine these into

$$\begin{aligned} \iint_{Q_{j-1}} |\Psi([a_i - k_{j-1}]_+)|^{2(N+2)/N}(s, x) \, dx \, ds \\ \leq \int_{-t_{j-1}}^0 \left(\int_{B_{j-1}} |\Psi([a_i - k_{j-1}]_+)|^{2N/(N-2)}(s, x) \, dx \right)^{\theta} \left(\int_{B_{j-1}} |\Psi([a_i - k_{j-1}]_+)|^2(s, x) \, dx \right)^{1-\theta} ds \\ \leq \mathcal{U}_{j-1}^{1-\theta} \int_{-t_{j-1}}^0 \left(\int_{B_{j-1}} |\Psi([a_i - k_{j-1}]_+)|^{2N/(N-2)}(s, x) \, dx \right)^{(N-2)/N} ds \leq c \mathcal{U}_{j-1}^{1+2/N}. \end{aligned}$$

We conclude by applying Lemma A.1. □

Once we know that $\lim_{j \rightarrow \infty} \mathcal{U}_j = 0$ we deduce that

$$\lim_{j \rightarrow \infty} \frac{1}{t_j} \sum_{i=1}^p \iint_{-\mathcal{Q}_j} H(a_i - k_j) \, dx \, dt = 0 \geq 4 \sum_{i=1}^p \int_{-1/4}^0 \int_{B(0, 1/2)} H(a_i - 1) \, dx \, dt.$$

It implies that $0 \leq a_i(t, x) \leq 1$ holds for a.e. $(t, x) \in (-\frac{1}{4}, 0) \times B(0, \frac{1}{4})$.

Appendix C: Proof of Proposition 3.2

It is worth giving some hints for the proof of Proposition 3.2, which is fully detailed in [Caputo and Vasseur 2009, Proposition 11, Corollary 12]. Again, the proof heavily relies on duality arguments. The main step consists in showing that

$$\|\Phi(t, \cdot)\|_{L^\infty(\mathbb{R}^N)} \leq \|\Phi(0, \cdot)\|_{L^\infty(\mathbb{R}^N)}. \quad (22)$$

Indeed, we remind the reader that $\Phi(t, x)$ is determined by the convolution formula (for $N > 2$)

$$\Phi(t, x) = -C_N \int_{\mathbb{R}^N} \frac{M(t, y)}{|x - y|^{N-2}} \, dy,$$

where

$$C_N = \frac{1}{(N-2)\sigma_N},$$

with $\sigma_N = 2\pi^{N/2}/\Gamma(N/2)$ the measure of the unit sphere of \mathbb{R}^N . Thus, given $R > 0$, we simply split

$$\Phi(0, x) = -C_N \int_{|x-y| \leq R} \frac{M(0, y)}{|x - y|^{N-2}} \, dy - C_N \int_{|x-y| > R} \frac{M(0, y)}{|x - y|^{N-2}} \, dy,$$

which yields

$$|\Phi(0, x)| \leq C_N \|M(0, \cdot)\|_{L^\infty(\mathbb{R}^N)} \frac{\sigma_N R^2}{2} + \frac{C_N}{R^{N-2}} \|M(0, \cdot)\|_{L^1(\mathbb{R}^N)}.$$

Optimizing with respect to R , we get

$$|\Phi(0, x)| \leq K_N \|M(0, \cdot)\|_{L^\infty(\mathbb{R}^N)}^{1-2/N} \|M(0, \cdot)\|_{L^1(\mathbb{R}^N)}^{2/N},$$

where $K_N > 0$ depends only on the space dimension $N \geq 3$.

In order to justify (22), we need to introduce a mollified diffusion coefficient. Indeed, as the a_i 's are smooth on $[0, T_{\max}) \times \mathbb{R}^N$, M is smooth too; thus $(t, x) \mapsto d(t, x)$ is a smooth function, except possibly at the points where $M(t, x)$ vanishes. Given $\mu > 0$, we denote by $d_\mu(t, x)$ a smooth function satisfying

$$d_\mu(t, x) = d(t, x) \quad \text{when } M(t, x) \geq \mu, \quad 0 < \delta_\star \leq d_\mu(t, x) \leq \delta^\star.$$

The proof of (22) splits into two steps.

Let $0 < T < \infty$. Let $\zeta \in C_c^\infty(\mathbb{R}^N)$ and consider the solution of the *end-value* equation

$$\partial_t \varphi + d_\mu \Delta \varphi = 0, \quad \varphi(T, x) = \zeta(x), \quad (23)$$

together with the initial value problem

$$\partial_t \rho - \Delta(d_\mu \rho) = 0, \quad \rho(0, x) = \rho^0(x).$$

We assume that

$$\|\zeta\|_{L^\infty(\mathbb{R}^N)} \leq 1.$$

The maximum principle, see for instance [Evans 1998, Theorem 8, Chapter 7], implies

$$\sup_{0 \leq t \leq T} \|\varphi(t, \cdot)\|_{L^\infty(\mathbb{R}^N)} \leq \|\zeta\|_{L^\infty(\mathbb{R}^N)} \leq 1.$$

We have, by integrating by parts,

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^N} \rho(t, x) \varphi(t, x) \, dx &= \int_{\mathbb{R}^N} (\partial_t \rho(t, x) \varphi(t, x) + \rho(t, x) \partial_t \varphi(t, x)) \, dx \\ &= \int_{\mathbb{R}^N} (\Delta(d_\mu \rho(t, x)) \varphi(t, x) - \rho(t, x) d_\mu(t, x) \Delta \varphi(t, x)) \, dx \\ &= - \int_{\mathbb{R}^N} \nabla(d_\mu \rho(t, x)) \cdot \nabla \varphi(t, x) \, dx + \int_{\mathbb{R}^N} \nabla(\rho(t, x) d_\mu(t, x)) \cdot \nabla \varphi(t, x) \, dx \\ &= 0. \end{aligned}$$

Now, we integrate over $[0, T]$ by using the conditions at $t = 0$ for ρ and $t = T$ for φ . It follows that

$$\left| \int_{\mathbb{R}^N} \rho(T, x) \zeta(x) \, dx \right| = \left| \int_{\mathbb{R}^N} \rho^0(x) \varphi(0, x) \, dx \right| \leq \|\rho^0\|_{L^1(\mathbb{R}^N)}.$$

By virtue of the Hahn–Banach theorem, we conclude that

$$\begin{aligned} \|\rho(T, \cdot)\|_{L^1(\mathbb{R}^N)} &= \sup \left\{ \left| \int_{\mathbb{R}^N} \rho(T, x) \zeta(x) \, dx \right| : \zeta \in C_c^\infty(\mathbb{R}^N), \|\zeta\|_{L^\infty(\mathbb{R}^N)} \leq 1 \right\} \\ &\leq \|\rho^0\|_{L^1(\mathbb{R}^N)}. \end{aligned}$$

Next, we shall apply similar reasoning in order to make the norm $\|\Delta \zeta\|_{L^1(\mathbb{R}^N)}$ appear. For $0 < T < \infty$ and φ a solution of (23), let us set

$$\rho(t, x) = \Delta \varphi(T - t, x),$$

which satisfies

$$\partial_t \rho - \Delta(d_\mu \rho) = 0, \quad \rho(0, x) = \Delta \zeta(x) \in L^1(\mathbb{R}^N).$$

The previous step thus tells us that

$$\|\rho(T, \cdot)\|_{L^1(\mathbb{R}^N)} = \|\Delta \varphi(0, \cdot)\|_{L^1(\mathbb{R}^N)} \leq \|\rho(0, \cdot)\|_{L^1(\mathbb{R}^N)} = \|\Delta \zeta\|_{L^1(\mathbb{R}^N)}.$$

Going back to the equation for the total mass, we get

$$\frac{d}{dt} \int_{\mathbb{R}^N} M \varphi(t, x) \, dx = \int_{\mathbb{R}^N} M(d - d_\mu) \Delta \varphi(t, x) \, dx.$$

Let $0 < T < T_{\max}$. Integrating over $(0, T)$ yields

$$\begin{aligned}
 \left| \int_{\mathbb{R}^N} M\varphi(T, x) \, dx \right| &= \left| \int_{\mathbb{R}^N} \Delta \Phi \varphi(T, x) \, dx \right| = \left| \int_{\mathbb{R}^N} \Phi(T, x) \Delta \zeta(x) \, dx \right| \\
 &= \left| \int_{\mathbb{R}^N} M\varphi(0, x) \, dx + \int_0^T \int_{\mathbb{R}^N} M(d - d_\mu) \Delta \varphi(t, x) \, dx \, dt \right| \\
 &= \left| \int_{\mathbb{R}^N} \Delta \Phi \varphi(0, x) \, dx + \int_0^T \int_{\mathbb{R}^N} M(d - d_\mu) \Delta \varphi(t, x) \, dx \, dt \right| \\
 &\leq \left| \int_{\mathbb{R}^N} \Phi \Delta \varphi(0, x) \, dx \right| + \left| \int_0^T \int_{\mathbb{R}^N} M(d - d_\mu) \Delta \varphi(t, x) \, dx \, dt \right| \\
 &\leq \|\Phi(0, \cdot)\|_{L^\infty(\mathbb{R}^N)} \|\Delta \varphi(0, \cdot)\|_{L^1(\mathbb{R}^N)} + 2T \delta^* \mu \|\Delta \varphi\|_{L^\infty(0, T; L^1(\mathbb{R}^N))} \\
 &\leq (\|\Phi(0, \cdot)\|_{L^\infty(\mathbb{R}^N)} + 2T \delta^* \mu) \|\Delta \zeta\|_{L^1(\mathbb{R}^N)},
 \end{aligned}$$

where the penultimate inequality holds since

$$|d - d_\mu| M = |d - d_\mu| M \mathbf{1}_{M \leq \mu} \leq 2\delta^* \mu.$$

This relation holds for any $\mu > 0$ and $\zeta \in C_c^\infty(\mathbb{R}^N)$. Therefore, we can conclude that (22) holds, which ends the proof. \square

Appendix D: Proof of Proposition 3.1 by De Giorgi's approach

For the sake of completeness, we provide here an alternative proof of Proposition 3.1, which, however, uses the additional assumption (b). The interest of this proof is that it entirely relies on energy estimates and De Giorgi's methods, which gives a unified viewpoint on the whole argumentation of the paper. Since the result stated in Proposition 3.1 is standard, the remainder of this section can be safely skipped by the reader not interested in such an alternative proof (the original proof relies on a probabilistic interpretation of the equation and uses arguments from the theory of diffusion processes).

Here and below, given $\rho > 0$, with B_ρ the ball $\{x \in \mathbb{R}^N : |x| \leq \rho\}$, we define

$$Q_\rho = (-\rho^2, 0) \times B_\rho.$$

In fact, we shall work within Q_2 , considered as a reference domain. From an equation satisfied on Q_2 we wish to establish qualitative properties on a smaller domain, say Q_1 or $Q_{1/2}$. It is also convenient to introduce the domain

$$\tilde{Q} = \left(-\frac{9}{4}, -1\right) \times B_1.$$

We refer the reader to Figure 1; having the picture of the subdomains of Q_2 might be helpful in following the arguments.

The argument for proving Proposition 3.1 relies on a technical lemma that controls oscillations. From now on, for a function φ defined on $\Omega \subset \mathbb{R}^d$, we set

$$\text{osc}(\varphi, \Omega) = \sup_{x \in \Omega} \varphi(x) - \inf_{x \in \Omega} \varphi(x).$$

Lemma D.1 (decay of oscillations). *Let Φ satisfy the assumptions of [Proposition 3.1](#). There exists $\lambda \in (0, 1)$, which depends only on N and δ_\star , such that*

$$\text{osc}(\Phi, Q_{1/2}) \leq \lambda \text{osc}(\Phi, Q_2).$$

Let us assume temporarily that [Lemma D.1](#) holds true. We pick $(t, x) \in (t_0, T) \times \mathbb{R}^N$, where $0 < t_0 < T < \infty$, and we set

$$\Phi_k(s, y) = \Phi(t + 2^{-2k}s, x + 2^{-k}y),$$

where $k \in \mathbb{N}$ is large enough so that the time variable remains larger than t_0 when $-4 \leq s \leq 0$; namely, we have

$$k \geq k_0 = \ln\left(\frac{t - t_0}{4}\right) \frac{1}{2 \ln(\frac{1}{2})}.$$

The function Φ_k is defined on Q_2 and it satisfies

$$\partial_s \Phi_k = d_k \Delta_y \Phi_k,$$

where

$$d_k(s, y) = d(t + 2^{-2k}s, x + 2^{-k}y).$$

Moreover, we still have

$$-\|\Phi\|_{L^\infty} \leq \Phi_k(s, y) \leq \|\Phi\|_{L^\infty}.$$

Applying [Lemma D.1](#) yields

$$\text{osc}(\Phi_k, Q_{1/2}) \leq \lambda \text{osc}(\Phi_k, Q_2),$$

which can be rewritten as

$$\text{osc}(\Phi(t + \cdot, x + \cdot), Q_{2^{-k-1}}) \leq \lambda \text{osc}(\Phi(t + \cdot, x + \cdot), Q_{2^{-k+1}}).$$

We deduce that

$$\text{osc}(\Phi(t + \cdot, x + \cdot), Q_{2^{-k}}) \leq \sqrt{\lambda}^k \times C_0, \quad C_0 = \frac{2}{\sqrt{\lambda}^{k_0}} \|\Phi\|_{L^\infty}.$$

(We should bear in mind the fact that C_0 depends on t_0 through the definition of k_0 and it is proportional to $\|\Phi\|_{L^\infty}$.) Let $x' \in \mathbb{R}^N$ and $t' > t_0$; there exists a unique $k \in \mathbb{N}$ such that $x' - x \in B_{2^{-k+1}} \setminus B_{2^{-k}}$, $2^{-2k} \leq |t' - t| \leq 2^{-2(k-1)}$. It follows that

$$\frac{|\Phi(t', x') - \Phi(t, x)|}{|t' - t|^{\alpha/2} + |x' - x|^\alpha} \leq \frac{C_0}{\sqrt{\lambda}} (\sqrt{\lambda} 2^\alpha)^k.$$

If $0 < \sqrt{\lambda} \leq \frac{1}{2}$, the right-hand side remains obviously bounded, uniformly with respect to k , for any $0 < \alpha \leq 1$; otherwise we choose

$$0 < \alpha = \frac{\ln(1/\sqrt{\lambda})}{\ln(2)} < 1.$$

Hence [Proposition 3.1](#) follows from [Lemma D.1](#). □

We are thus left with the task of proving [Lemma D.1](#). To this end, we shall apply the following statement.

Proposition D.2. *Let $(t, x) \mapsto v(t, x)$ satisfy*

- *the differential inequality $\partial_t v - \delta^* \Delta v \leq 0$ on Q_2 ;*
- *$-1 \leq v(t, x) \leq +1$ on Q_2 ;*
- *$\text{meas}(\{(t, x) \in \tilde{Q} : v(t, x) \leq 0\}) \geq \mu \text{meas}(\tilde{Q})$ for some $\mu > 0$.*

Then, there exists $0 < \eta < 1$ such that

$$v(t, x) \leq \eta \quad \text{on } Q_{1/2}.$$

The function

$$\tilde{\Phi}(t, x) = \frac{2}{\text{osc}(\Phi, Q_2)} \left(\Phi(t, x) - \frac{\sup_{Q_2} \Phi + \inf_{Q_2} \Phi}{2} \right)$$

satisfies the first two assumptions of [Proposition D.2](#). Suppose that

$$\text{meas}(\{(t, x) \in Q_2 : \tilde{\Phi}(t, x) \leq 0\}) \geq \frac{\text{meas}(Q_2)}{2}.$$

(Otherwise, we shall apply the same reasoning to $-\tilde{\Phi}$.) [Proposition D.2](#) tells us that $\tilde{\Phi}(t, x) \leq \eta$ on $Q_{1/2}$, which yields $\text{osc}(\tilde{\Phi}, Q_{1/2}) \leq 1 + \eta$ (since $\inf_{Q_{1/2}} \tilde{\Phi} \geq -1$), and thus

$$\text{osc}(\Phi, Q_{1/2}) \leq \frac{1 + \eta}{2} \text{osc}(\Phi, Q_2).$$

It justifies [Lemma D.1](#), with $\lambda = (1 + \eta)/2 \in (0, 1)$. □

The proof of [Proposition D.2](#) relies on a series of intermediate statements.

Lemma D.3. *Let $-\infty < a, b < \infty$ and let Ω be a smooth bounded domain in \mathbb{R}^N . We define $Q = (a, b) \times \Omega$.*

(a) *Let $u \in L^\infty(a, b; L^2(\Omega)) \cap L^2(a, b; H^1(\Omega))$ such that*

$$\partial_t u - \delta^* \Delta u + \mu = 0$$

holds in $\mathcal{D}'(Q)$, with μ a nonnegative measure on Q . Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be a nondecreasing convex function. We assume that $F(0) = 0$ and $F \in W_{\text{loc}}^{1,\infty}(\mathbb{R})$. Then, there exists a nonnegative measure ν such that $v = F(u)$ satisfies $\partial_t v - \delta^ \Delta v + \nu = 0$ in $\mathcal{D}'(Q)$.*

(b) *Let $v \in L^\infty((a, b) \times \Omega) \cap L^2(a, b; H^1(\Omega))$ be a nonnegative solution of $\partial_t v - \delta^* \Delta v + \nu = 0$, with ν a nonnegative measure on Q . Then, for any trial function $\varphi \in C_c^\infty(\Omega)$ there exists $C > 0$, which depends only on δ_\star , $\|v\|_{L^\infty}$ and φ , such that, for a.e. $a < s < t < b$, the following energy inequality holds:*

$$\frac{1}{2} \int_\Omega v^2(t, x) \varphi^2(x) \, dx + \delta_\star \int_s^t \int_\Omega |\nabla(\phi v)|^2(\tau, x) \, dx \, d\tau \leq \frac{1}{2} \int_\Omega v^2(s, x) \varphi^2(x) \, dx + C(t - s).$$

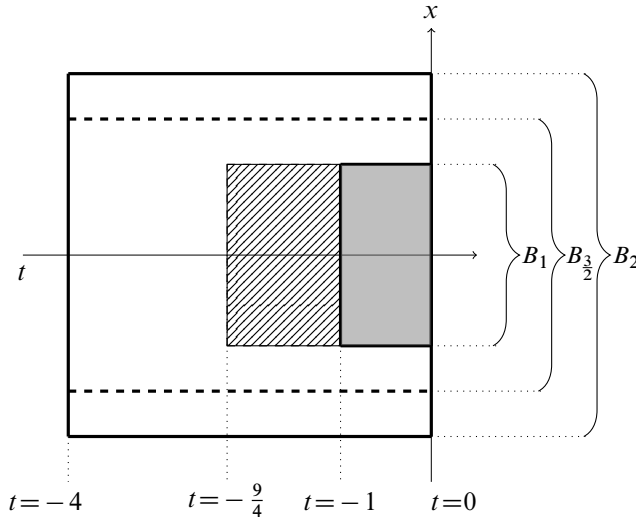


Figure 1. The domains Q_2 (the largest box), \tilde{Q} (the dashed box) and Q_1 (the gray box).

Proof. Note that $v = F(u)$ also lies in $L^\infty(a, b; L^2(\Omega)) \cap L^2(a, b; H^1(\Omega))$; see, e.g., [Brezis 1983, Proposition IX.5]. Item (a) follows from the computation

$$\partial_t F(u) = -F'(u)\mu + F'(u)\delta_\star \Delta u = \underbrace{-F'(u)\mu - \delta^\star F''(u)|\nabla u|^2}_{\leq 0} + \delta_\star \Delta F(u).$$

The argument can be made rigorous by working on the weak variational formulation of the equation, with suitable approximation of the solution u .

For proving item (b), we compute

$$\begin{aligned} \frac{1}{2} \partial_t (v^2 \varphi^2) &= \delta^\star \varphi^2 v \nabla \cdot \nabla v - v \varphi^2 v \\ &= \delta^\star \nabla \cdot (\varphi^2 v \nabla v) - v \varphi^2 v - \delta^\star \nabla v \cdot \nabla (\varphi^2 v) \\ &= \delta^\star \nabla \cdot (\varphi^2 v \nabla v) - v \varphi^2 v - \delta^\star |\nabla(\varphi v)|^2 + \delta^\star v^2 |\nabla \varphi|^2. \end{aligned}$$

The second and third terms of the right-hand side are nonpositive; the integral of the last term is dominated by $\delta_\star \|v\|_{L^\infty(Q)}^2 \|\varphi\|_{H^1(\Omega)}$. Again a full justification proceeds through an approximation argument. \square

For proving Proposition D.2, we shall work with several subdomains of Q_2 , as indicated by Figure 1 which might help to follow the arguments.

Lemma D.4. *Let u satisfy $\partial_t u - \delta^\star \Delta u \leq 0$ and $-1 \leq u(t, x) \leq +1$ in Q_2 . Let us set*

$$\begin{aligned} \mathcal{A} &= \{(t, x) \in Q_1 : u(t, x) \geq \tfrac{1}{2}\}, \\ \mathcal{B} &= \{(t, x) \in \tilde{Q} : u(t, x) \leq 0\}, \\ \mathcal{C} &= \{(t, x) \in Q_1 \cup \tilde{Q} : 0 < u(t, x) < \tfrac{1}{2}\}. \end{aligned}$$

There exists $\alpha > 0$ such that if $\text{meas}(\mathcal{A}) \geq \eta$ and $\text{meas}(\mathcal{B}) \geq \frac{1}{2} \text{meas}(\tilde{Q})$, then $\text{meas}(\mathcal{C}) \geq \alpha$.

Proof. We argue by contradiction, assuming that a sequence $(u_k)_{k \in \mathbb{N}}$ of solutions of $\partial_t u_k - \delta^* \Delta u_k \leq 0$ in Q_2 satisfies $-1 \leq u_k(t, x) \leq +1$ and

$$\begin{aligned} \text{meas}(\mathcal{A}_k) &\geq \eta, & \text{with } \mathcal{A}_k &= \{(t, x) \in Q_1 : u_k(t, x) \geq \tfrac{1}{2}\}, \\ \text{meas}(\mathcal{B}_k) &\geq \tfrac{1}{2} \text{meas}(\tilde{Q}), & \text{with } \mathcal{B}_k &= \{(t, x) \in \tilde{Q} : u_k(t, x) \leq 0\}, \\ \text{meas}(\mathcal{C}_k) &\leq \tfrac{1}{k}, & \text{with } \mathcal{C}_k &= \{(t, x) \in Q_1 \cup \tilde{Q} : 0 < u_k(t, x) < \tfrac{1}{2}\}. \end{aligned} \quad (24)$$

We focus our interest on the positive part $v_k = [u_k]_+$, with $[z]_+ = \max(z, 0)$, which is still uniformly bounded: $0 \leq v_k(t, x) \leq 1$. By virtue of [Lemma D.3\(a\)](#), it satisfies

$$\partial_t v_k - \delta^* \Delta v_k + \mu_k = 0, \quad (25)$$

with μ_k a nonnegative measure. The strategy can be recapped as follows. We shall establish the compactness of v_k in the reduced domain $(-4, 0) \times B_{3/2}$. It allows us to assume that v_k converges to a certain function v . Roughly speaking, we are going to show that $v(s, x)$ vanishes on B_1 for certain times $-\frac{3}{2} < s < -1$, which will imply that v vanishes over Q_1 . It will eventually lead to a contradiction by considering the behavior of the sets $\mathcal{A}_k, \mathcal{B}_k, \mathcal{C}_k$ as $k \rightarrow \infty$.

Let us pick a trial function $\zeta \in C_c^\infty(B_2)$ such that $\zeta(x) = 1$ for any $x \in B_{3/2}$ and $0 \leq \zeta(x) \leq 1$ for any $x \in \mathbb{R}^N$. By using [Lemma D.3\(b\)](#), we get for $-4 < t_1 < t_2 < 0$

$$\int \zeta^2 |v_k|^2(t_2, x) dx + \delta^* \int_{t_1}^{t_2} \int |\nabla(\zeta v_k)|^2(s, x) dx ds \leq \int \zeta^2 v_k^2(t_1, x) dx + C(t_2 - t_1) \quad (26)$$

for a certain constant $C > 0$. In particular, we have $(\zeta v_k)_{k \in \mathbb{N}}$ is bounded in $L^\infty(-4, 0; L^2(B_2)) \cap L^2(-4, 0; H^1(B_2))$. Going back to (25), since $\mu_k \geq 0, v_k \geq 0$, we observe that

$$\begin{aligned} 0 &\leq \int_{t_1}^{t_2} \int_{B_{3/2}} \mu_k dx ds \leq \int_{t_1}^{t_2} \int_{B_2} \zeta \mu_k dx ds \\ &\leq \int_{B_2} \zeta v_k(t_1, x) dx - \delta^* \int_{t_1}^{t_2} \int_{B_2} \nabla v_k \cdot \nabla \zeta dx ds \\ &\leq \|\zeta\|_{L^1} + 2\delta^* \|\nabla v_k\|_{L^2(Q_2)} \|\nabla \zeta\|_{L^2(B_2)} \end{aligned}$$

is bounded uniformly with respect to k . Coming back to (25), we deduce that $(\partial_t v_k)_{k \in \mathbb{N}}$ is bounded in $\mathcal{M}^1((-4, 0) \times B_{3/2}) + L^2(-4, 0; H^{-1}(B_{3/2}))$. By virtue of the Aubin–Lions–Simon lemma [[Simon 1987](#)] (in fact we use the extended version [[Moussa 2016](#), Theorem 1], which allows us to deal with measure-valued time derivatives), we conclude that $(v_k)_{k \in \mathbb{N}}$ is compact in $L^2((-4, 0) \times B_{3/2})$. We can thus assume that v_k (possibly relabeling sequence) converges to some v in $L^2((-4, 0) \times B_{3/2})$. The Bienaymé–Tchebyshev inequality yields

$$\text{meas}(\{(t, x) \in ((-4, 0) \times B_1) : |v_k(t, x) - v(t, x)| \geq \epsilon\}) \leq \frac{\|v_k - v\|_{L^2((-4, 0) \times B_1)}^2}{\epsilon^2} \xrightarrow{k \rightarrow \infty} 0$$

for any $\epsilon > 0$.

Let $(t, x) \in (-4, 0) \times B_1$ be such that $\epsilon \leq v(t, x) \leq \frac{1}{2} - \epsilon$. Then we distinguish the following two cases: either $|v - v_k|(t, x) \geq \epsilon$ or $0 \leq v_k(t, x) = (v_k - v)(t, x) + v(t, x) \leq |v - v_k|(t, x) + v(t, x) \leq \frac{1}{2}$. It follows that

$$\begin{aligned} \text{meas}(\{(t, x) \in Q_1 \cup \tilde{Q} : \epsilon \leq v(t, x) \leq \tfrac{1}{2} - \epsilon\}) &\leq \text{meas}(\{(t, x) \in Q_1 \cup \tilde{Q} : |v - v_k|(t, x) \geq \epsilon\}) \\ &\quad + \underbrace{\text{meas}(\{(t, x) \in Q_1 \cup \tilde{Q} : 0 \leq v_k(t, x) \leq \tfrac{1}{2}\})}_{\text{meas}(\mathcal{C}_k)} \\ &\leq \text{meas}(\{(t, x) \in Q_1 \cup \tilde{Q} : |v - v_k|(t, x) \geq \epsilon\}) + \frac{1}{k}, \end{aligned}$$

by using (24). Letting k go to ∞ yields

$$\text{meas}(\{(t, x) \in Q_1 \cup \tilde{Q} : \epsilon \leq v(t, x) \leq \tfrac{1}{2} - \epsilon\}) = 0.$$

Since this property holds for any ϵ , the monotone convergence property leads to

$$\text{meas}(\{(t, x) \in Q_1 \cup \tilde{Q} : 0 < v(t, x) < \tfrac{1}{2}\}) = 0.$$

Therefore, we have

$$\text{for a.e. } t \in (-\tfrac{9}{4}, 0), \text{ either } v(t, x) = 0 \text{ or } v(t, x) \geq \tfrac{1}{2} \text{ in } B_1. \quad (27)$$

Similarly, let $(t, x) \in (-4, 0) \times B_1$ be such that $v_k(t, x) = 0$. We distinguish the following two cases: either $|v - v_k|(t, x) \geq \epsilon$ or $0 \leq v(t, x) = (v - v_k)(t, x) \leq |v - v_k|(t, x) \leq \epsilon$. Coming back to (24), we get

$$\begin{aligned} \tfrac{1}{2} \text{meas}(\tilde{Q}) &\leq \text{meas}(\mathcal{B}_k) \\ &\leq \text{meas}(\{(t, x) \in \tilde{Q} : |v - v_k|(t, x) \geq \epsilon\}) + \text{meas}(\{(t, x) \in \tilde{Q} : v(t, x) \leq \epsilon\}). \end{aligned}$$

Letting k go to ∞ we obtain

$$\tfrac{1}{2} \text{meas}(\tilde{Q}) \leq \text{meas}(\{(t, x) \in \tilde{Q} : v(t, x) \leq \epsilon\}).$$

By monotone convergence, as $\epsilon \rightarrow 0$, we arrive at

$$\tfrac{1}{2} \text{meas}(\tilde{Q}) \leq \text{meas}(\{(t, x) \in \tilde{Q} : v(t, x) = 0\}).$$

Consequently, we can find a nonnegligible set of times $s \in (-\frac{3}{2}, -1)$ such that $v(s, x) = 0$ holds for a.e. $x \in B_1$. Letting k go to ∞ in (25), we obtain $\partial_t v - \delta^* \Delta v + v = 0$ on $(-4, 0) \times B_{3/2}$, with v a nonnegative measure. Let $\zeta \in C_c^\infty(B_{3/2})$ be a nonnegative trial function such that $\zeta(x) = 1$ for any $x \in B_1$. We apply Lemma D.3(b), and we obtain for a.e. $t \in (s, 0)$,

$$\int_{B_1} v^2(t, x) \, dx \leq \int_{B_{3/2}} v^2(t, x) \zeta^2(x) \, dx \leq \int_{B_{3/2}} v^2(s, x) \zeta(x) \, dx + C(t - s) = C(t - s),$$

where, owing to (27), we also know that the left-hand side is either null or larger than $\text{meas}(B_1)/4$. We deduce that, actually, v vanishes on Q_1 . We are going to show that it contradicts (24).

Indeed, let us consider $(t, x) \in Q_1$ such that $v_k(t, x) \geq \frac{1}{2}$. Then, for any $\epsilon > 0$, either $|v - v_k|(t, x) \geq \epsilon$ or $v(t, x) = v_k(t, x) + (v - v_k)(t, x) \geq v_k(t, x) - |v - v_k|(t, x) \geq \frac{1}{2} - \epsilon$. With the first property in (24),

it follows that

$$\eta \leq \text{meas}(\mathcal{A}_k) \leq \text{meas}(\{(t, x) \in Q_1 : |v - v_k|(t, x) \geq \epsilon\}) + \text{meas}(\{(t, x) \in Q_1 : v(t, x) \geq \tfrac{1}{2} - \epsilon\}).$$

Letting k go to ∞ yields

$$\eta \leq \text{meas}(\{(t, x) \in Q_1 : v(t, x) \geq \tfrac{1}{2} - \epsilon\}).$$

Since this inequality holds for any $\epsilon > 0$, we conclude, by monotone convergence, that

$$\eta \leq \text{meas}(\{(t, x) \in Q_1 : v(t, x) \geq \tfrac{1}{2}\})$$

holds, a contradiction. □

Proof of Proposition D.2. We consider $(t, x) \mapsto v(t, x)$ such that $-1 \leq v(t, x) \leq +1$,

$$\text{meas}(\{(t, x) \in \tilde{Q} : v(t, x) \leq 0\}) \geq \mu \text{meas}(\tilde{Q}),$$

and v satisfies $\partial_t v - \delta^* \Delta v \leq 0$ in Q_2 . The proof splits into two steps.

Step 1: For $k \in \mathbb{N}$, set

$$v_k(t, x) = 2^k \left(v(t, x) - \left(1 - \frac{1}{2^k} \right) \right).$$

We shall show that the integral

$$\iint_{Q_1} [v_k]_+^2 \, dx \, dt$$

can be made as small as we wish, by choosing k large enough. Observe that

$$v_k = 2^k(v - 1) + 1 = 2v_{k-1} - 1,$$

which implies that $v_k \leq 1$ and

$$\{(t, x) \in \tilde{Q} : v(t, x) \leq 0\} \subset \left\{ (t, x) \in \tilde{Q} : v(t, x) \leq 1 - \frac{1}{2^k} \right\} = \{(t, x) \in \tilde{Q} : v_k(t, x) \leq 0\}.$$

Thus, by assumption on v , we have

$$\text{meas}(\{(t, x) \in \tilde{Q} : v_k(t, x) \leq 0\}) \geq \text{meas}(\{(t, x) \in \tilde{Q} : v(t, x) \leq 0\}) \geq \mu \text{meas}(\tilde{Q}).$$

Let us suppose that, for any $k \in \mathbb{N}$,

$$\iint_{Q_1} [v_k]_+^2 \, dx \, dt \geq \delta$$

holds for a certain $\delta > 0$. Since this integral is dominated by

$$\text{meas}(\{(t, x) \in Q_1 : v_k(t, x) \geq 0\}) = \text{meas}(\{(t, x) \in Q_1 : v_{k-1}(t, x) \geq \tfrac{1}{2}\}),$$

we infer

$$\text{meas}(\{(t, x) \in Q_1 : v_{k-1}(t, x) \geq \tfrac{1}{2}\}) \geq \delta$$

independently of k . Applying Lemma D.4 yields

$$\text{meas}(\{(t, x) \in Q_1 \cup \tilde{Q} : 0 < v_{k-1}(t, x) < \tfrac{1}{2}\}) \geq \alpha,$$

still independently of k . It follows that

$$\begin{aligned} & \text{meas}(\{(t, x) \in Q_1 \cup \tilde{Q} : v_k(t, x) \leq 0\}) \\ &= \text{meas}(\{(t, x) \in Q_1 \cup \tilde{Q} : 2v_{k-1}(t, x) - 1 \leq 0\}) \\ &= \text{meas}(\{(t, x) \in Q_1 \cup \tilde{Q} : v_{k-1}(t, x) \leq 0\}) + \text{meas}(\{(t, x) \in Q_1 \cup \tilde{Q} : 0 < v_{k-1}(t, x) \leq \tfrac{1}{2}\}) \\ &\geq \text{meas}(\{(t, x) \in Q_1 \cup \tilde{Q} : v_{k-1}(t, x) \leq 0\}) + \alpha. \end{aligned}$$

Since $\text{meas}(\{(t, x) \in Q_1 \cup \tilde{Q} : v_0(t, x) \leq 0\}) \geq \text{meas}(\{(t, x) \in \tilde{Q} : v_0(t, x) \leq 0\}) \geq \mu \text{meas}(\tilde{Q})$, this recursion formula leads to

$$\text{meas}(\{(t, x) \in Q_1 \cup \tilde{Q} : v_k(t, x) \leq 0\}) \geq \mu \text{meas}(\tilde{Q}) + k\alpha.$$

However, this cannot occur for any k since the left-hand side is bounded by $\text{meas}(Q_2)$. We conclude that, given $\delta > 0$, there exists $k_\star \in \mathbb{N}$ such that

$$\iint_{Q_1} [v_{k_\star}]_+^2 \, dx \, dt \leq \delta.$$

Step 2: The second step relies on De Giorgi's analysis. Let us set $w(t, x) = v_{k_\star}(t, x)$. We shall show that, provided δ is small enough (which means k_\star large enough), $w(t, x) \leq \frac{1}{2}$ on $Q_{1/2}$. To this end, let us set, for $\ell \in \mathbb{N}$,

$$m_\ell = \frac{1}{2} \left(1 - \frac{1}{2^\ell}\right), \quad w_\ell(t, x) = [w(t, x) - m_\ell]_+, \quad r_\ell = \frac{1}{2} \left(1 + \frac{1}{2^\ell}\right), \quad t_\ell = -r_\ell^2 = -\frac{1}{4} \left(1 + \frac{1}{2^\ell}\right)^2.$$

We are going to work in the domains $Q_{1/2} \subset Q_{r_\ell} \subset Q_1$, which shrink to $Q_{1/2}$ as $\ell \rightarrow \infty$; see [Figure 2](#).

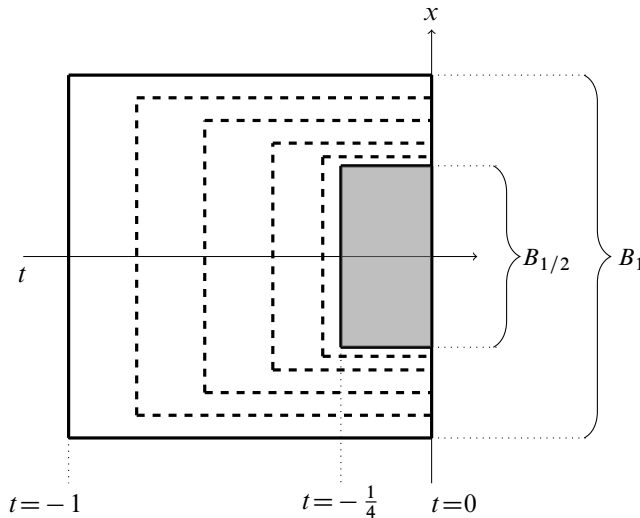


Figure 2. The domains Q_1 , Q_{r_ℓ} and $Q_{1/2}$ (the gray box).

We consider a sequence of functions $\zeta_\ell \in C_c^\infty(B_{r_{\ell-1}})$ such that $0 \leq \zeta_\ell(x) \leq 1$ on $B_{r_{\ell-1}}$ and $\zeta_\ell(x) = 1$ on B_{r_ℓ} . We shall use the basic estimate

$$|\nabla \zeta_\ell(x)| \leq C 2^\ell, \quad \frac{1}{t_\ell - t_{\ell-1}} \leq C 2^{2\ell}.$$

We already know that $0 \leq w_\ell(t, x) \leq 1$, by definition. We can apply the energy estimate in [Lemma D.3](#), which reads

$$\begin{aligned} \frac{1}{2} \int_{B_1} w_\ell^2(t, x) \zeta_\ell^2(x) dx + \delta^\star \int_s^t \int_{B_1} |\nabla(\zeta_\ell w_\ell)|^2(\tau, x) dx d\tau \\ \leq \frac{1}{2} \int_{B_1} w_\ell^2(s, x) \zeta_\ell^2(x) dx + \delta^\star \int_s^t \int_{B_1} w_\ell^2 |\nabla \zeta_\ell|^2(\tau, x) dx d\tau \end{aligned} \quad (28)$$

for $-1 < s < t_\ell < t < 0$ (note that here we keep explicit the integral in the right-hand side that is roughly estimated by a constant in [Lemma D.3](#)). Averaging over $s \in (t_{\ell-1}, t_\ell)$ (and using the fact that the integral of a positive quantity over (s, t) is thus bounded below by the integral over (t_ℓ, t) and above by the integral over $(t_{\ell-1}, t)$) yields

$$\begin{aligned} \frac{1}{2} \int_{B_1} w_\ell^2(t, x) \zeta_\ell^2(x) dx + \delta^\star \int_{t_\ell}^t \int_{B_1} |\nabla(\zeta_\ell w_\ell)|^2(\tau, x) dx d\tau \\ \leq \left(\frac{1}{2} + \delta^\star\right) C 2^{2\ell} \int_{t_{\ell-1}}^0 \int_{\text{supp}(\zeta_\ell)} |w_\ell|^2(\tau, x) dx d\tau. \end{aligned}$$

Let us set

$$\begin{aligned} \mathcal{U}_\ell &= \int_{t_\ell}^0 \int_{B_{r_\ell}} |w_\ell|^2(t, x) dx dt, \\ \mathcal{E}_\ell &= \sup_{t_\ell \leq t \leq 0} \int_{B_1} w_\ell^2(t, x) \zeta_\ell^2(x) dx + \int_{t_\ell}^0 \int_{B_1} |\nabla(\zeta_\ell w_\ell)|^2(\tau, x) dx d\tau. \end{aligned}$$

We wish to establish a nonlinear recursion for \mathcal{U}_ℓ , which will allow us to justify that it tends to 0 as $\ell \rightarrow \infty$. On the one hand, since

$$w_\ell \leq w_{\ell-1} \quad \text{and} \quad \text{supp}(\zeta_\ell) \subset B_{\ell-1},$$

we note that (28) yields

$$\mathcal{E}_\ell \leq \left(2 + \frac{1}{\delta^\star}\right) \left(\frac{1}{2} + \delta^\star\right) C 2^{2\ell} \mathcal{U}_{\ell-1}.$$

On the other hand, we observe that

$$\begin{aligned} \mathcal{U}_\ell &\leq \int_{t_\ell}^0 \int_{B_{r_\ell}} |\zeta_\ell w_\ell|^2(t, x) dx dt \\ &\leq \left(\int_{t_\ell}^0 \int_{B_{r_\ell}} |\zeta_\ell w_\ell|^{2(N+2)/N}(t, x) dx dt \right)^{N/(N+2)} \left(\text{meas}(\{(t, x) \in (t_\ell, 0) \times B_{r_\ell} : \zeta_\ell w_\ell > 0\}) \right)^{2/(N+2)}, \end{aligned}$$

by using Hölder's inequality. Note that

$$w - m_{\ell-1} = w - m_\ell + \frac{1}{2^{\ell+1}},$$

which leads to

$$\begin{aligned} \text{meas}(\{(t, x) \in (t_\ell, 0) \times B_{r_\ell} : \zeta_\ell w_\ell > 0\}) &\leq \text{meas}(\{(t, x) \in (t_{\ell-1}, 0) \times B_{r_{\ell-1}} : w_{\ell-1} > 2^{-\ell-1}\}) \\ &\leq 2^{2\ell+2} \mathcal{U}_{\ell-1}, \end{aligned}$$

by virtue of the Bienaymé–Tchebyshev inequality. Next, we use the Gagliardo–Nirenberg–Sobolev inequality, see [Nirenberg 1959, Theorem p. 125],

$$\left(\int_{B_{r_\ell}} |\zeta_\ell w_\ell|^{2N/(N-2)}(t, x) \, dx \right)^{(N-2)/N} \leq C_S \int_{B_{r_\ell}} |\nabla(\zeta_\ell w_\ell)|^2(t, x) \, dx.$$

Mind that we have integrated with respect to the space variable only. We can write

$$\frac{N+2}{N} = \theta \frac{2N}{N-2} + 2(1-\theta), \quad \theta = \frac{N-2}{N} \in (0, 1),$$

so that

$$\begin{aligned} \int_{t_\ell}^0 \int_{B_{r_\ell}} |\zeta_\ell w_\ell|^{2(N+2)/N}(t, x) \, dx \, dt &\leq \int_{t_\ell}^0 \left(\int_{B_{r_\ell}} |\zeta_\ell w_\ell|^{2N/(N-2)}(t, x) \, dx \right)^\theta \underbrace{\left(\int_{B_{r_\ell}} |\zeta_\ell w_\ell|^2(t, x) \, dx \right)^{1-\theta}}_{\leq \mathcal{E}_\ell^{1-\theta}} \, dt \\ &\leq C_S^\theta \mathcal{E}_\ell^{2-\theta}. \end{aligned}$$

Therefore, gathering all this together, we obtain

$$\mathcal{U}_\ell \leq \Lambda^\ell \mathcal{U}_{\ell-1}^{1+2/(N+2)}$$

for a certain constant $\Lambda > 1$. Owing to Lemma A.1, we deduce that $\lim_{\ell \rightarrow \infty} \mathcal{U}_\ell = 0$ provided \mathcal{U}_0 is small enough. The smallness condition on \mathcal{U}_0 is precisely ensured by the definition $w = v_{k_\star}$ coming from Step 1. Since

$$\frac{1}{|t_\ell|} \int_{t_\ell}^0 \int_{B_{r_\ell}} |w_\ell|^2(t, x) \, dx \, dt \leq \mathcal{U}_\ell,$$

we conclude, by applying Fatou's lemma, that

$$2 \iint_{Q_{1/2}} \left[w - \frac{1}{2} \right]_+^2(t, x) \, dx \, dt \leq \liminf_{\ell \rightarrow \infty} \frac{1}{t_\ell} \int_{t_\ell}^0 \int_{B_{r_\ell}} |w_\ell|^2(t, x) \, dx \, dt = 0$$

so that, finally, $w(t, x) \leq \frac{1}{2}$ holds a.e. on $Q_{1/2}$.

Coming back to the change of unknown,

$$w(t, x) = v_{k_\star}(t, x) = 2^{k_\star} \left(v(t, x) - \left(1 - \frac{1}{2^{k_\star}} \right) \right) \leq \frac{1}{2}$$

becomes

$$v(t, x) \leq 1 + \frac{1}{2^{k_\star+1}} - \frac{1}{2^{k_\star}} = 1 - \frac{1}{2^{k_\star+1}} < 1.$$

□

Note

After the completion of this work, we learned about results in a similar direction by J. I. Kanel [1990]. This approach shares similar ideas and assumptions, but with different techniques; it has been recently revisited by P. Souplet [2018] to deal with problems endowed with Neumann boundary conditions and nonlinearities with a quadratic growth.

Acknowledgements

Goudon acknowledges the warm welcome of the Math Department of UT-Austin, where a couple of visits have made possible progress on this question.

References

- [Alexandrov 1966] A. D. Aleksandrov, “Majorants of solutions of linear equations of order two”, *Vestnik Leningrad. Univ.* **21**:1 (1966), 5–25. In Russian; translated in *Amer. Math. Soc. Trans.* **68**:2 (1968), 120–143. [MR](#) [Zbl](#)
- [Alikakos 1979] N. D. Alikakos, “ L^p bounds of solutions of reaction-diffusion equations”, *Comm. Partial Differential Equations* **4**:8 (1979), 827–868. [MR](#) [Zbl](#)
- [Alonso et al. 2016] R. Alonso, P. Amorim, and T. Goudon, “Analysis of a chemotaxis system modeling ant foraging”, *Math. Models Methods Appl. Sci.* **26**:9 (2016), 1785–1824. [MR](#) [Zbl](#)
- [Bakelman 1961] I. J. Bakelman, “On the theory of quasilinear elliptic equations”, *Sibirsk. Mat. Ž.* **2** (1961), 179–186. In Russian. [MR](#) [Zbl](#)
- [Brezis 1983] H. Brezis, *Analyse fonctionnelle: théorie et applications*, Masson, Paris, 1983. [MR](#) [Zbl](#)
- [Cañizo et al. 2014] J. A. Cañizo, L. Desvillettes, and K. Fellner, “Improved duality estimates and applications to reaction-diffusion equations”, *Comm. Partial Differential Equations* **39**:6 (2014), 1185–1204. [MR](#) [Zbl](#)
- [Caffarelli and Vasseur 2010] L. A. Caffarelli and A. Vasseur, “Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation”, *Ann. of Math.* (2) **171**:3 (2010), 1903–1930. [MR](#) [Zbl](#)
- [Caputo and Vasseur 2009] M. C. Caputo and A. Vasseur, “Global regularity of solutions to systems of reaction-diffusion with sub-quadratic growth in any dimension”, *Comm. Partial Differential Equations* **34**:10 (2009), 1228–1250. [MR](#) [Zbl](#)
- [Chainais-Hillairet et al. 2017] C. Chainais-Hillairet, B. Merlet, and A. F. Vasseur, “Positive lower bound for the numerical solution of a convection-diffusion equation”, pp. 331–339 in *Finite volumes for complex applications, VIII: Methods and theoretical aspects* (Lille, 2017), edited by C. Cancès and P. Omnes, Springer Proc. Math. Stat. **199**, Springer, 2017. [MR](#) [Zbl](#)
- [De Giorgi 1957] E. De Giorgi, “Sulla differenziabilità e l’analiticità delle estremali degli integrali multipli regolari”, *Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat.* (3) **3** (1957), 25–43. [MR](#) [Zbl](#)
- [Desvillettes and Fellner 2006] L. Desvillettes and K. Fellner, “Exponential decay toward equilibrium via entropy methods for reaction-diffusion equations”, *J. Math. Anal. Appl.* **319**:1 (2006), 157–176. [MR](#) [Zbl](#)
- [Desvillettes et al. 2007] L. Desvillettes, K. Fellner, M. Pierre, and J. Vovelle, “Global existence for quadratic systems of reaction-diffusion”, *Adv. Nonlinear Stud.* **7**:3 (2007), 491–511. [MR](#) [Zbl](#)
- [Evans 1998] L. C. Evans, *Partial differential equations*, Graduate Studies in Math. **19**, Amer. Math. Soc., Providence, RI, 1998. [MR](#) [Zbl](#)
- [Fabes and Stroock 1984] E. B. Fabes and D. W. Stroock, “The L^p -integrability of Green’s functions and fundamental solutions for elliptic and parabolic equations”, *Duke Math. J.* **51**:4 (1984), 997–1016. [MR](#) [Zbl](#)
- [Fellner et al. 2016] K. Fellner, E. Latos, and T. Suzuki, “Global classical solutions for mass-conserving, (super-)quadratic reaction-diffusion systems in three and higher space dimensions”, *Discrete Contin. Dyn. Syst. Ser. B* **21**:10 (2016), 3441–3462. [MR](#) [Zbl](#)
- [Fischer 2015] J. Fischer, “Global existence of renormalized solutions to entropy-dissipating reaction-diffusion systems”, *Arch. Ration. Mech. Anal.* **218**:1 (2015), 553–587. [MR](#) [Zbl](#)

- [Goudon and Urrutia 2016] T. Goudon and L. Urrutia, “Analysis of kinetic and macroscopic models of pursuit-evasion dynamics”, *Commun. Math. Sci.* **14**:8 (2016), 2253–2286. [MR](#) [Zbl](#)
- [Goudon and Vasseur 2010] T. Goudon and A. Vasseur, “Regularity analysis for systems of reaction-diffusion equations”, *Ann. Sci. Éc. Norm. Supér. (4)* **43**:1 (2010), 117–142. [MR](#) [Zbl](#)
- [Kanel 1990] Y. I. Kanel, “Solvability in the large of a system of reaction-diffusion equations with the balance condition”, *Diff. Uravneniya* **26**:3 (1990), 448–458. In Russian; translated in *Diff. Equations* **26**:3 (1990), 331–339. [MR](#) [Zbl](#)
- [Krylov 1976] N. V. Krylov, “Sequences of convex functions, and estimates of the maximum of the solution of a parabolic equation”, *Sibirsk. Mat. Ž.* **17**:2 (1976), 290–303. In Russian; translated in *Siberian Math. J.* **17**:2 (1976), 226–236. [MR](#) [Zbl](#)
- [Krylov and Safonov 1979] N. V. Krylov and M. V. Safonov, “An estimate for the probability of a diffusion process hitting a set of positive measure”, *Dokl. Akad. Nauk SSSR* **245**:1 (1979), 18–20. In Russian. [MR](#)
- [Krylov and Safonov 1980] N. V. Krylov and M. V. Safonov, “A property of the solutions of parabolic equations with measurable coefficients”, *Izv. Akad. Nauk SSSR Ser. Mat.* **44**:1 (1980), 161–175. In Russian; translated in *Math. USSR-Izv.* **16**:1 (1981), 151–164. [MR](#) [Zbl](#)
- [Moussa 2016] A. Moussa, “Some variants of the classical Aubin–Lions lemma”, *J. Evol. Equ.* **16**:1 (2016), 65–93. [MR](#) [Zbl](#)
- [Nirenberg 1959] L. Nirenberg, “On elliptic partial differential equations”, *Ann. Scuola Norm. Sup. Pisa (3)* **13**:2 (1959), 115–162. [MR](#) [Zbl](#)
- [Pierre 2010] M. Pierre, “Global existence in reaction-diffusion systems with control of mass: a survey”, *Milan J. Math.* **78**:2 (2010), 417–455. [MR](#) [Zbl](#)
- [Pierre and Schmitt 1997] M. Pierre and D. Schmitt, “Blowup in reaction-diffusion systems with dissipation of mass”, *SIAM J. Math. Anal.* **28**:2 (1997), 259–269. [MR](#) [Zbl](#)
- [Pierre et al. 2017a] M. Pierre, T. Suzuki, and H. Umakoshi, “Global-in-time behavior of weak solutions to reaction-diffusion systems with inhomogeneous Dirichlet boundary condition”, *Nonlinear Anal.* **159** (2017), 393–407. [MR](#) [Zbl](#)
- [Pierre et al. 2017b] M. Pierre, T. Suzuki, and R. Zou, “Asymptotic behavior of solutions to chemical reaction-diffusion systems”, *J. Math. Anal. Appl.* **450**:1 (2017), 152–168. [MR](#) [Zbl](#)
- [Pucci 1966] C. Pucci, “Limitazioni per soluzioni di equazioni ellittiche”, *Ann. Mat. Pura Appl. (4)* **74**:1 (1966), 15–30. [MR](#) [Zbl](#)
- [Simon 1987] J. Simon, “Compact sets in the space $L^p(0, T; B)$ ”, *Ann. Mat. Pura Appl. (4)* **146** (1987), 65–96. [MR](#) [Zbl](#)
- [Souplet 2018] P. Souplet, “Global existence for reaction-diffusion systems with dissipation of mass and quadratic growth”, *J. Evol. Equ.* **18**:4 (2018), 1713–1720. [MR](#) [Zbl](#)
- [Tso 1985] K. Tso, “On an Aleksandrov–Bakel’man type maximum principle for second-order parabolic equations”, *Comm. Partial Differential Equations* **10**:5 (1985), 543–553. [MR](#) [Zbl](#)
- [Vasseur 2007] A. F. Vasseur, “A new proof of partial regularity of solutions to Navier–Stokes equations”, *Nonlinear Diff. Equations Appl.* **14**:5-6 (2007), 753–785. [MR](#) [Zbl](#)
- [Vasseur 2010] A. Vasseur, “Higher derivatives estimate for the 3D Navier–Stokes equation”, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **27**:5 (2010), 1189–1204. [MR](#) [Zbl](#)
- [Weissler 1985] F. B. Weissler, “An L^∞ blow-up estimate for a nonlinear heat equation”, *Comm. Pure Appl. Math.* **38**:3 (1985), 291–295. [MR](#) [Zbl](#)

Received 28 Nov 2017. Revised 22 Aug 2018. Accepted 25 Oct 2018.

M. CRISTINA CAPUTO: caputo@math.utexas.edu

Department of Mathematics, University of Texas at Austin, Austin, TX, United States

THIERRY GOUDON: thierry.goudon@inria.fr

Université Côte d’Azur, Inria, CNRS, LJAD, Nice, France

ALEXIS F. VASSEUR: vasseur@math.utexas.edu

Department of Mathematics, University of Texas at Austin, Austin, TX, United States

Analysis & PDE

msp.org/apde

EDITORS

EDITOR-IN-CHIEF

Patrick Gérard

patrick.gerard@math.u-psud.fr

Université Paris Sud XI

Orsay, France

BOARD OF EDITORS

Massimiliano Berti	Scuola Intern. Sup. di Studi Avanzati, Italy berti@sissa.it	Gilles Pisier	Texas A&M University, and Paris 6 pisier@math.tamu.edu
Michael Christ	University of California, Berkeley, USA mchrist@math.berkeley.edu	Tristan Rivière	ETH, Switzerland riviere@math.ethz.ch
Charles Fefferman	Princeton University, USA cf@math.princeton.edu	Igor Rodnianski	Princeton University, USA irod@math.princeton.edu
Ursula Hamenstaedt	Universität Bonn, Germany ursula@math.uni-bonn.de	Sylvia Serfaty	New York University, USA serfaty@cims.nyu.edu
Vadim Kaloshin	University of Maryland, USA vadim.kaloshin@gmail.com	Yum-Tong Siu	Harvard University, USA siu@math.harvard.edu
Herbert Koch	Universität Bonn, Germany koch@math.uni-bonn.de	Terence Tao	University of California, Los Angeles, USA tao@math.ucla.edu
Izabella Laba	University of British Columbia, Canada ilaba@math.ubc.ca	Michael E. Taylor	Univ. of North Carolina, Chapel Hill, USA met@math.unc.edu
Richard B. Melrose	Massachusetts Inst. of Tech., USA rhm@math.mit.edu	Gunther Uhlmann	University of Washington, USA gunther@math.washington.edu
Frank Merle	Université de Cergy-Pontoise, France Frank.Merle@u-cergy.fr	András Vasy	Stanford University, USA andras@math.stanford.edu
William Minicozzi II	Johns Hopkins University, USA minicozz@math.jhu.edu	Dan Virgil Voiculescu	University of California, Berkeley, USA dvv@math.berkeley.edu
Clément Mouhot	Cambridge University, UK c.mouhot@dpmms.cam.ac.uk	Steven Zelditch	Northwestern University, USA zelditch@math.northwestern.edu
Werner Müller	Universität Bonn, Germany mueller@math.uni-bonn.de	Maciej Zworski	University of California, Berkeley, USA zworski@math.berkeley.edu

PRODUCTION

production@msp.org

Silvio Levy, Scientific Editor

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2019 is US \$310/year for the electronic version, and \$520/year (+\$60, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow[®] from MSP.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2019 Mathematical Sciences Publishers

ANALYSIS & PDE

Volume 12 No. 7 2019

Generalized q -gaussian von Neumann algebras with coefficients, I: Relative strong solidity	1643
MARIUS JUNGE and BOGDAN UDREA	
Complex interpolation and Calderón–Mityagin couples of Morrey spaces	1711
MIECZYŚLAW MASTYŁO and YOSHIHIRO SAWANO	
Monotonicity and local uniqueness for the Helmholtz equation	1741
BASTIAN HARRACH, VALTER POHJOLA and MIKKO SALO	
Solutions of the 4-species quadratic reaction-diffusion system are bounded and C^∞ -smooth, in any space dimension	1773
M. CRISTINA CAPUTO, THIERRY GOUDON and ALEXIS F. VASSEUR	
Spacelike radial graphs of prescribed mean curvature in the Lorentz–Minkowski space	1805
DENIS BONHEURE and ALESSANDRO IACOPETTI	
Square function estimates, the BMO Dirichlet problem, and absolute continuity of harmonic measure on lower-dimensional sets	1843
SVITLANA MAYBORODA and ZIHUI ZHAO	