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SPACELIKE RADIAL GRAPHS OF PRESCRIBED MEAN CURVATURE IN THE LORENTZ–MINKOWSKI SPACE

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We investigate the existence and uniqueness of spacelike radial graphs of prescribed mean curvature in the Lorentz–Minkowski space $\mathbb{L}^{n+1}$, for $n \geq 2$, spanning a given boundary datum lying on the hyperbolic space $\mathbb{H}^n$.

1. Introduction

A radial graph is a hypersurface Σ such that each ray emanating from the origin intersects Σ once at most. In the euclidean context the problem of finding radial graphs of prescribed mean curvature has been extensively studied over the years. In the first paper on the subject, Radó [1932] proved that for any given Jordan curve $\Gamma \subset \mathbb{R}^3$, with one-to-one radial projection onto a convex subset of the unit sphere $\mathbb{S}^2$, there exists a minimal graph spanning Γ. Later, Tausch [1981] proved that area-minimizing disk-type hypersurfaces spanning a boundary datum Γ which can be expressed as a radial graph over $\partial\Omega$, where $\Omega \subset \mathbb{S}^n$ is a convex subset, have a local representation as a radial graph. The case of variable mean curvature was investigated by Serrin [1969], and a recent result of radial representation for H-surfaces in cones was given in [Caldiroli and Iacopetti 2016]. Treibergs and Wei [1983] studied the case of closed hypersurfaces, i.e., compact hypersurfaces without boundary. Lopez [2003] and de Lira [2002] studied the case of radial graphs of constant mean curvature.

The Lorentz–Minkowski space, denoted by $\mathbb{L}^{n+1}$, is defined as the vector space $\mathbb{R}^{n+1}$ equipped with the symmetric bilinear form

$$\langle x, y\rangle := x_1 y_1 + \cdots + x_n y_n - x_{n+1} y_{n+1},$$

where $x = (x_1, \dots, x_{n+1})$, $y = (y_1, \dots, y_{n+1}) \in \mathbb{R}^{n+1}$. The bilinear form $\langle \cdot, \cdot \rangle$ is a nondegenerate bilinear form of index 1, see [Spivak 1975, Section A], where the index of a bilinear form on a real vector space is defined as the largest dimension of a negative definite subspace. The modulus of $v \in \mathbb{L}^{n+1}$ is defined as $|v| := \sqrt{|\langle v, v\rangle|}$.

The interest in finding spacelike hypersurfaces of prescribed mean curvature in the Lorentz–Minkowski space comes from the theory of relativity, in which maximal and constant-mean-curvature spacelike

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hypersurfaces play an important role, see [Bartnik and Simon 1982], where spacelike means that the restriction of the Lorentz metric to the tangent plane, at every point, is positive definite. In the literature, several result are available for spacelike vertical graphs, i.e., hypersurfaces which are expressed as a cartesian graph. Entire maximal spacelike hypersurfaces were studied by Cheng and Yau [1976] and later Treibergs [1982] tackled the general case of entire spacelike hypersurfaces of constant mean curvature. The Dirichlet problem for spacelike vertical graphs in $\mathbb{L}^{n+1}$ was solved by Bartnik and Simon [1982], and Gerhardt [1983] extended those results to the case of vertical graphs contained in Lorentzian manifolds which can be expressed as a product of a Riemannian manifold times an interval. Bayard [2003] studied the more general problem of prescribed scalar curvature. On the contrary, for radial graphs, to our knowledge, the only available result concerns entire spacelike hypersurfaces with prescribed scalar curvature which are asymptotic to the light-cone; see [Bayard and Delanoë 2009].

The geometry of Lorentz–Minkowski spaces plays an in important role in the setting of the problem. A first relevant fact is that there cannot exist spacelike closed hypersurfaces (see Proposition 2.5, or [López 2014] for the case of surfaces in $\mathbb{L}^3$). Therefore $\mathbb{S}^n$-type surfaces are ruled out, and the model hypersurface in $\mathbb{L}^{n+1}$ for describing spacelike radial graphs is the hyperbolic space $\mathbb{H}^n$ (see Definition 2.6). Another important feature of Lorentz–Minkowski spaces is that, given a domain, there exist spacelike hypersurfaces of arbitrarily large (in modulus) mean curvature, see [López 2013], while in the euclidean context this is not true in general. This fact will be crucial in our paper to construct barriers.

We state now the problem. Let Ω be a smooth bounded domain of $\mathbb{H}^n$. For $u:\overline{\Omega}\to\mathbb{R}$, we define the associated radial graph over Ω as the set

$$\Sigma(u) := \{p = e^{u(q)}q \in \mathbb{L}^{n+1} : q \in \overline{\Omega}\}.$$

Let $\mathcal{C}_{\overline{\Omega}}$ be the cone spanned by $\overline{\Omega}$ (minus the origin), i.e., $\mathcal{C}_{\overline{\Omega}} := \{p = \rho q \in \mathbb{L}^{n+1} : q \in \overline{\Omega},\ \rho > 0\}$, and let $H:\mathcal{C}_{\overline{\Omega}}\to\mathbb{R}$.

Definition 1.1. A H-bump (over Ω) is a radial graph Σ whose boundary coincides with $\partial\Omega$ and such that the mean curvature of Σ at every (interior) point equals H.

The Dirichlet problem for spacelike H-bumps is given by

$$\begin{cases} \sum_{i,j=1}^{n}((1-|\nabla u|^2)\delta_{ij}+u_iu_j)u_{ij} = n(1-|\nabla u|^2)-n(1-|\nabla u|^2)^{3/2}e^uH(e^uq) & \text{in } \Omega,\\ |\nabla u|<1 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1-1}$$

where u_i, u_{ij} are the covariant derivatives of u, ∇u is the gradient with respect to the Levi-Civita connection of $(\mathbb{H}^n, g)$ (see Section 3), and $g = dx_1\otimes dx_1+\cdots+dx_n\otimes dx_n - dx_{n+1}\otimes dx_{n+1}$ is the induced Riemannian metric on $\mathbb{H}^n$ (see Section 2).

Definition 1.2. Let $0<r_1\le 1\le r_2$, with $r_1\neq r_2$. The hyperbolic conical cap of radii r_1, r_2 spanned by $\overline{\Omega}$ is the set

$$\mathcal{C}_{\overline{\Omega}}(r_1,r_2) := \{p=\rho q\in\mathbb{L}^{n+1} : q\in\overline{\Omega},\ r_1\le\rho\le r_2\}.$$

The main result of our paper is the following existence theorem.

Theorem 1.3. *Let $\alpha \in (0,1)$, $0 < r_1 \le 1 \le r_2$, with $r_1 \ne r_2$. Assume Ω is a bounded domain of $\mathbb{H}^n$ of class $C^{3,\alpha}$ that satisfies a uniform exterior geodesic ball condition. If $H \in C^{1,\alpha}(\mathcal{C}_{\overline{\Omega}}(r_1, r_2))$ is positive and satisfies*

(i) *$H(r_1 q) > r_1^{-1}$ and $H(r_2 q) < r_2^{-1}$ for any $q \in \overline{\Omega}$,*

(ii) *$(\partial/\partial\lambda)(\lambda H(\lambda q)) \le 0$ for all $q \in \overline{\Omega}$, $\lambda \in [r_1, r_2]$,*

then there exists a unique solution of problem (1-1) *whose associated radial graph is contained in $\mathcal{C}_{\overline{\Omega}}(r_1, r_2)$.*

Let Ω, r_1, r_2 be in the statement of Theorem 1.3. Let $m \ge 1$, let $\omega : \overline{\Omega} \to \mathbb{R}^+$ be a smooth positive function such that $r_1^{m-1} < \omega < r_2^{m-1}$ and let $H_{m,\omega} : \mathcal{C}_{\overline{\Omega}}(r_1, r_2) \to \mathbb{R}^+$, defined by

$$H_{m,\omega}(x) := \frac{\omega(x/|x|)}{|x|^m}. \tag{1-2}$$

One easily verifies that $H_{m,\omega}$ satisfies the hypotheses (i) and (ii) of Theorem 1.3. In particular, this shows the existence of spacelike radial graphs of prescribed mean curvature even for nonhomogeneous functions H, a case which is not contemplated for instance in [Bayard and Delanoë 2009], where the k-th scalar curvature is prescribed just on $\mathbb{H}^n$.

We remark that (1-1) can be put in divergence form, namely

$$\begin{cases} -\operatorname{div}_{\mathbb{H}^n}(\nabla u/\sqrt{1-|\nabla u|^2}) + n/\sqrt{1-|\nabla u|^2} = n e^u H(e^u q) & \text{in } \Omega, \\ |\nabla u| < 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1-3}$$

where $\operatorname{div}_{\mathbb{H}^n}$ denotes the divergence operator for $(\mathbb{H}^n, g)$. The principal part of this operator appears in the Born–Infeld theory of electromagnetism [1934], which is a particular example of what is usually known as a nonlinear electrodynamics. We therefore stress that Theorem 1.3 provides existence and uniqueness of solutions for some specific Born–Infeld equations in which appear nontrivial nonlinearities involving both the gradient and the function; see also [Bonheure et al. 2016; Bonheure and Iacopetti 2019].

The proof of Theorem 1.3 relies on the combination of several tools. For the existence, we apply a variant of the classical Leray–Schauder fixed point theorem due to Potter [1972]. To this aim, we make use of suitable comparison theorems and we prove fine a priori estimates for the solutions and their gradient. Regarding uniqueness, we take advantage of the Hopf maximum principle as in the version stated by Pucci and Serrin [2004].

We point out that the uniform exterior geodesic ball condition allows us to construct barriers for the gradient of the solutions at the boundary. Such construction strongly depends on the shape of the mean curvature operator for spacelike hypersurfaces in the Lorentz–Minkowski space, and we remark that Theorem 1.3 grants existence of spacelike radial graphs over arbitrarily large and even nonconvex domains of $\mathbb{H}^n$. We note that it is not possible to mimic this construction in the euclidean framework, and in fact the problem of finding radial graphs over proper (possibly nonconvex) domains of $\mathbb{S}^n$ which are not contained in a hemisphere is still open.

Concerning global a priori estimates for the gradient, which is the key step in the proof, we derive a quite complex technical result, see Proposition 8.1, which is inspired from [Gerhardt 1983] and is based on the introduction of an ad hoc differential operator, Stampacchia's truncation method and fine estimates of the L^p-norm of the quantity $\nu(u) = 1/\sqrt{1-|\nabla u|^2}$.

In this paper we also introduce a new definition of admissible couple (Ω, H) and triple (Ω, H, θ), see Definition 4.4, where $\theta \in (0,1)$. This notion of admissibility is very general and works even for nonsmooth domains and just for continuous functions H. However, given a couple (Ω, H), it is not easy in general to verify whether it is admissible or not. In Section 4 we provide trivial examples of admissible couples and in Proposition 4.7 we exhibit a class of functions H such that (Ω, H) is admissible whenever Ω satisfies a uniform exterior geodesic condition. Using the notion of admissible couple, we can extend Theorem 1.3 to a wider class of domains and mean curvature functions.

Theorem 1.4. *Let $\alpha \in (0,1)$, $0 < r_1 \le 1 \le r_2$, with $r_1 \neq r_2$. Assume that Ω is a bounded domain of $\mathbb{H}^n$ of class $C^{3,\alpha}$ and $H \in C^{1,\alpha}(\mathcal{C}_{\overline{\Omega}}(r_1, r_2))$ satisfies conditions* (i) *and* (ii) *of Theorem 1.3. Assume that (Ω, H) is admissible. Then there exists a unique solution of problem* (1-1) *whose associated radial graph is contained in $\mathcal{C}_{\overline{\Omega}}(r_1, r_2)$.*

A further existence result for problem (1-1), under more restrictive assumptions, is as follows.

Theorem 1.5. *Let $\alpha \in (0,1)$ and Ω be a bounded domain of $\mathbb{H}^n$ of class $C^{3,\alpha}$. Assume $\theta \in (0,1)$, $0 < r_1 \le 1 \le r_2$, with $r_1 \neq r_2$, and $H \in C^{1,\alpha}(\mathcal{C}_{\overline{\Omega}}(r_1, r_2))$ satisfies*

(a) *$H(r_1 q) > r_1^{-1}$ and $H(r_2 q) < r_2^{-1}$ for any $q \in \overline{\Omega}$,*

(b) *$(\partial/\partial\lambda)(\lambda H(\lambda q)) < -1/(r_1(\theta - \theta^2/4)^{1/2})$ for all $q \in \overline{\Omega}$, $\lambda \in [r_1, r_2]$,*

(c) *$\|\nabla_0^T H(x)\|_{n+1} < (1-\theta)/(n^{3/2} r_2^2)$ for all $x \in \mathcal{C}_{\overline{\Omega}}(r_1, r_2)$, where $\nabla_0^T H$ is the euclidean tangential component of $\nabla_0 H(x)$ on $T_{x/|x|}\mathbb{H}^n$ (see Definition 6.2), $\nabla_0 H$ is the gradient of H with respect to the euclidean flat metric, and $\|\cdot\|_{n+1}$ is the euclidean norm in $\mathbb{R}^{n+1}$.*

Assume at last that (Ω, H, θ) is admissible according to Definitions 4.4 and 4.10. Then there exists a unique spacelike H-bump contained in $\mathcal{C}_{\overline{\Omega}}(r_1, r_2)$.

We mention this result because the proof quite differs from that of Theorem 1.4 and better shows the differences and difficulties with respect to the euclidean case. The proof is this time based on the classical Leray–Schauder theorem; see for instance [Gilbarg and Trudinger 1977, Theorem 11.3]. The first step is to solve a suitable regularized equation associated to (1-1); see (4-2) and Theorem 5.1. The idea of solving such a regularized equation is taken from [Treibergs 1982], where the author constructs barriers for the gradient at the boundary. The way back to the original Dirichlet problem then uses a gradient maximum principle [Treibergs and Wei 1983, Proposition 6]. In contrast with [Treibergs 1982], we deal here with equations which do not satisfy, in general, a gradient maximum principle [Gilbarg and Trudinger 1977, Theorem 15.1]. In fact, in our case, when passing to local coordinates, we see that the regularized operator associated to (1-1) does not satisfy, in general [loc. cit., condition (15.11)], and the principal part depends both on the gradient and on the domain variables. We refer to Lemma 4.1 below for more details. In order to overcome this difficulty, and eventually deduce a global a priori C^1 estimate, we perform the regularization in a proper way. We then use the admissibility condition to control the gradient at the boundary, whereas we use two different strategies, see Lemma 4.12, for the interior estimate. The first one which is based on the properties of harmonic functions, works only in dimension 2. The other proof works in any dimension and is based on the global gradient bound given by [loc. cit., Theorem 15.2].

Finally, in the spirit of [Treibergs and Wei 1983], we prove a new kind of interior gradient estimate, see Proposition 6.4, so that, under the hypotheses of Theorem 1.5, the solution of the regularized problem is a solution of (1-1). It is important to note that, in contrast to [loc. cit.], since $\mathbb{H}^n$ has negative Ricci curvature and since we deal with hypersurfaces with boundary, the mere gradient estimate of Proposition 6.4 is not sufficient for getting a global a priori C^1-estimate. We refer to Remark 6.5 for more details.

When Ω satisfies a uniform exterior geodesic condition, thanks to Proposition 4.7, Remark 4.8 and Remark 4.9, it is possible to show that the functions given by (1-2) satisfy the hypotheses of Theorem 1.5 for suitable choices of r_1, r_2, m, for ω close to 1 (in the C^1-topology), and for some $\theta_* \in (0, 1)$.

As a future goal, it would be natural to investigate if it is possible to remove the monotonicity assumption on H and to extend Theorem 1.3 also to sign-changing mean curvature functions.

The outline of the paper is the following. In Section 2, we fix the notation and we collect some known facts which are useful in the remainder of the paper. In Section 3, we derive the equation for spacelike H-bumps and in Section 4 we prove Proposition 4.7 and some a priori estimates. Section 5 is dedicated to the proof of existence and uniqueness of solutions for the regularized Dirichlet problem associated to problem (1-1). In Section 6, we work out an interior gradient estimate, namely Proposition 6.4, and in Section 7 we prove Theorem 1.5. In Section 8, we prove a global a priori estimate for the gradient. We finally prove Theorems 1.3 and 1.4 in Section 9.

2. Notation and preliminary results

Let $n \geq 2$; we denote by $\mathbb{L}^{n+1}$ the $(n+1)$-dimensional Lorentz–Minkowski space, which is $\mathbb{R}^{n+1}$ equipped with the symmetric bilinear form

$$\langle x, y\rangle := x_1 y_1 + \cdots + x_n y_n - x_{n+1} y_{n+1}.$$

We classify the vectors of $\mathbb{L}^{n+1}$ in three types.

Definition 2.1. A vector $v \in \mathbb{L}^{n+1}$ is said to be

- spacelike if $\langle v, v\rangle > 0$ or $v = 0$;
- timelike if $\langle v, v\rangle < 0$;
- lightlike if $\langle v, v\rangle = 0$ and $v \neq 0$.

The modulus of $v \in \mathbb{L}^{n+1}$ is defined as $|v| := \sqrt{|\langle v, v\rangle|}$. We also denote by $(x, y)_{n+1} = x_1 y_1 + \cdots + x_{n+1} y_{n+1}$ the euclidean scalar product, and by $\|x\|_{n+1} = \sqrt{x_1^2 + \cdots + x_{n+1}^2}$ the euclidean norm in $\mathbb{R}^{n+1}$. Given a vector subspace V of $\mathbb{L}^{n+1}$, we consider the induced metric $\langle \cdot, \cdot\rangle_V$ defined in the natural way

$$\langle v, w\rangle_V := \langle v, w\rangle,\ \ v, w \in V.$$

According to Definition 2.1 we classify the subspaces of $\mathbb{L}^{n+1}$ as follows.

Definition 2.2. A vector subspace V of $\mathbb{L}^{n+1}$ is said to be

- spacelike if the induced metric is positive definite;
- timelike if the induced metric has index 1;
- lightlike if the induced metric is degenerate.

In this paper, we deal only with hypersurfaces in $\mathbb{L}^{n+1}$, and thus we identify the tangent space of $M \subset \mathbb{L}^{n+1}$ at $p \in M$, denoted by T_pM, with a vector subspace of dimension n in $\mathbb{L}^{n+1}$. In particular, by abuse of notation, if $\phi : U \to M$, where U is an open subset of $\mathbb{R}^n$, is a local parametrization, we still use the symbol ∂_i to denote the vector $\partial\phi/\partial x_i$.

Definition 2.3. Let $M \subset \mathbb{L}^{n+1}$ be a hypersurface. We say that M is spacelike (resp. timelike, lightlike) if, for any $p \in M$, the vector subspace T_pM is spacelike (resp. timelike, lightlike). We say that M is a nondegenerate hypersurface if M is spacelike or timelike.

Definition 2.4. A timelike vector $v \in \mathbb{L}^{n+1}$ is said to be future-oriented if $\langle v, E_{n+1}\rangle < 0$ and past-oriented if $\langle v, E_{n+1}\rangle > 0$, where $E_{n+1} := (0, \dots, 0, 1)$.

We observe that for a spacelike (resp. timelike) surface M and $p \in M$, we have the decomposition $\mathbb{L}^{n+1} = T_pM \oplus (T_pM)^\perp$, where $(T_pM)^\perp$ is a timelike (resp. spacelike) subspace of dimension 1; see [López 2014]. A Gauss map is a differentiable map $N : M \to \mathbb{L}^{n+1}$ such that $|N(p)| = 1$ and $N(p) \in (T_pM)^\perp$ for all $p \in M$. If M is spacelike, the Gauss map pointing to the future is a map $N : M \to \mathbb{H}^n$.

We recall now a result which is simple but crucial because it marks a relevant difference between the euclidean geometry and the geometry of Lorentz–Minkowski spaces.

Proposition 2.5. *Let $M \subset \mathbb{L}^{n+1}$ be a compact spacelike, timelike or lightlike hypersurface. Then $\partial M \neq \varnothing$.*

Proof. Assume that $\partial M = \varnothing$ and that M is spacelike (resp. timelike or lightlike). Let $a \in \mathbb{L}^{n+1}$ be a spacelike (resp. timelike) vector. Since M is compact, there exists a minimum (or a maximum) $p_0 \in M$ for the function $f(p) = \langle p, a\rangle$. Since $\partial M = \varnothing$, we know p_0 is a critical point of the function f and thus $\langle v, a\rangle = 0$ for all $v \in T_pM$. Hence $a \in (T_pM)^\perp$, but this gives a contradiction because $(T_pM)^\perp$ is timelike (resp. spacelike or lightlike). □

In other words, the previous result tells us that a closed hypersurface (i.e., compact without boundary) must be degenerate (see Definition 2.3). Therefore closed surfaces are not relevant in the Lorentz–Minkowski space, and this is deeply in contrast to euclidean geometry. For the sake of completeness, we also point out that Proposition 2.5, as well the previous definitions, can be extended to general hypersurfaces; see, e.g., [López 2014, Section 3].

Definition 2.6. The hyperbolic space of center $p_0 \in \mathbb{L}^{n+1}$ and radius $r > 0$ is the hypersurface defined by

$$\mathbb{H}^n(p_0, r) := \{p \in \mathbb{L}^{n+1} : \langle p - p_0, p - p_0\rangle = -r^2,\ \langle p - p_0, E_{n+1}\rangle < 0\},$$

where $E_{n+1} = (0, \dots, 0, 1)$.

From the euclidean point of view, this hypersurface is the "upper sheet" of a hyperboloid of two sheets.

Remark 2.7. The hyperbolic space is a spacelike hypersurface; see [López 2014; Spivak 1975]. In fact, let $v \in T_p\mathbb{H}^n(p_0, r)$ and let $\sigma = \sigma(s)$ be a curve in $\mathbb{H}^n(p_0, r)$ such that $\sigma'(0) = v$. Then, differentiating with respect to s the relation $\langle \sigma(s) - p_0, \sigma(s) - p_0 \rangle = -r^2$ at $s = 0$, we obtain $\langle v, p - p_0 \rangle = 0$. This implies $T_p\mathbb{H}^n(p_0, r) = \operatorname{Span}\{p - p_0\}^\perp$. Since $p - p_0$ is a timelike vector, it follows that $\mathbb{H}^n(p_0, r)$ is a spacelike hypersurface. Moreover $N(p) = (p - p_0)/r$ is a Gauss map.

When p_0 is the origin of $\mathbb{L}^{n+1}$, and $r = 1$, the hyperbolic space is denoted by $\mathbb{H}^n$; that is,

$$\mathbb{H}^n := \{(x_1, \dots, x_{n+1}) \in \mathbb{L}^{n+1} : x_1^2 + \dots + x_n^2 - x_{n+1}^2 = -1,\ x_{n+1} > 0\}.$$

In view of the previous remark, for any $p \in \mathbb{H}^n$, the induced metric on $T_p\mathbb{H}^n$ is positive definite, and hence the tensor $g = dx_1 \otimes dx_1 + \dots + dx_n \otimes dx_n - dx_{n+1} \otimes dx_{n+1}$ is a Riemannian metric for $\mathbb{H}^n$. Another model for $\mathbb{H}^n$ is the Poincaré model in the unit disk $\mathbb{B}^n := \{y \in \mathbb{R}^n : \|y\|_n < 1\}$, where $\|\cdot\|_n$ is the euclidean norm in $\mathbb{R}^n$. The hyperbolic metric in $\mathbb{B}^n$ is defined by

$$\tilde{g} = \frac{4}{(1 - \|y\|_n^2)^2} \sum_{i=1}^{n} dy_i \otimes dy_i,$$

which is conformally equivalent to the flat metric in $\mathbb{B}^n$. The isometry between $(\mathbb{H}^n, g)$ and $(\mathbb{B}^n, \tilde{g})$ is given by the map $F : \mathbb{H}^n \to \mathbb{B}^n$ defined by

$$F(x) := x_0 - \frac{2(x - x_0)}{\langle x - x_0, x - x_0 \rangle} = \left(\frac{x_1}{1 + x_{n+1}}, \dots, \frac{x_n}{1 + x_{n+1}} \right), \tag{2-1}$$

where $x_0 = (0, \dots, 0, -1) \in \mathbb{R}^{n+1}$; see [Lee 1997, Proposition 3.5]. The map F is also known as hyperbolic stereographic projection, and from a geometrical point of view, F sends a point $x \in \mathbb{H}^n$ to the intersection between the line joining x and x_0 with the hyperplane $\{y \in \mathbb{R}^{n+1} : y_{n+1} = 0\}$.

We conclude this section by recalling a variant of the Leray–Schauder fixed point theorem which will be used in the proof of Theorem 1.3.

Theorem 2.8 (A. J. B. Potter [1972]). *Let X be a locally convex linear Hausdorff topological space and U a closed convex subset of X such that the zero element of X is contained in the interior of U. Let $T : [0, 1] \times U \to X$ be a continuous map such that $T([0, 1] \times U)$ is relatively compact in X. Assume that*

(a) *$T(t, x) \neq x$ for all $x \in \partial U$ and $t \in [0, 1]$;*

(b) *$T(0 \times \partial U) \subset U$.*

Then, there is an element $\bar{x}$ of U such that $\bar{x} = T(1, \bar{x})$.

3. Derivation of the equation

Let Ω be a proper smooth bounded domain of the hyperbolic space $\mathbb{H}^n$. Let us denote by $\mathcal{T}(\Omega)$ the space of tangent vector fields to Ω and denote by ∇^0 the Levi-Civita connection of $\mathbb{L}^{n+1}$. We recall that ∇^0 coincides with the flat connection of $\mathbb{R}^{n+1}$, and we denote by ∇ the induced Levi-Civita connection on Ω. Let u be a smooth function defined on Ω. We denote by du the differential of u and by ∇u the gradient of u, which is the only vector field on Ω such that

$$du(X) = \langle X, \nabla u\rangle \quad \text{for any } X \in \mathcal{T}(\Omega).$$

The second covariant derivative of u is defined as

$$\nabla_{X,Y} u := \nabla_X \nabla_Y u - \nabla_X Y(u) = \nabla_X \nabla_Y u - \nabla_{\nabla_X Y} u \quad \text{for any } X, Y \in \mathcal{T}(\Omega),$$

and the Hessian of u, denoted by $\nabla^2 u$, is the symmetric 2-tensor given by

$$\nabla^2 u\ (X, Y) := \nabla_{X,Y} u \quad \text{for any } X, Y \in \mathcal{T}(\Omega).$$

The Laplacian of u, denoted by Δu, is the trace of the Hessian.

Let $\{e_1, \dots, e_n\}$ be a local orthonormal frame field for Ω and let $\{\omega^1, \dots, \omega^n\}$ be the dual coframe field; i.e., $\omega^i(e_j) = \delta_{ij}$ for any $i, j = 1, \dots, n$. The connection forms ω_{ij}'s defined by

$$\omega_{ij}(X) := \langle \nabla_X e_j, e_i\rangle, \quad X \in \mathcal{T}(\Omega), \tag{3-1}$$

and thus we have

$$\nabla_{e_i} e_j = \sum_{k=1}^{n} \omega_{kj}(e_i) e_k. \tag{3-2}$$

We also recall that the connection forms are skew symmetric, i.e., $\omega_{ij} + \omega_{ji} = 0$, for any $i, j \in \{1, \dots, n\}$. In terms of the dual coframe field the exterior derivative of u (i.e., the differential) can be written as

$$du = \sum_{i=1}^{n} u_i \omega^i,$$

where u_i denotes the covariant derivative $\nabla_{e_i} u$. We will also use the notation ∇_i to denote ∇_{e_i}.

For the second covariant derivatives, taking $X = e_i$, $Y = e_j$ and using (3-1) we have

$$\nabla_{e_i, e_j} u = \nabla_{e_i} u_j - \sum_{k=1}^{n} \omega_{kj}(e_i) u_k. \tag{3-3}$$

From now on we will use the notation u_{ij} to denote $\nabla_{e_i, e_j} u$. In particular the Hessian of u can be written as $u_{ij}\omega_j \otimes \omega_i$ and the Laplacian of u as $\Delta u = \sum_{i=1}^{n} u_{ii}$.

Definition 3.1. Let $A \subset \mathbb{L}^{n+1}$; we define the cone spanned by A as the set

$$\mathcal{C}_A := \{\rho q \in \mathbb{L}^{n+1} : q \in A,\ \rho > 0\}.$$

Remark 3.2. Observe that setting $e_{n+1}(x) := x/|x|$ for $x \in \mathcal{C}_\Omega$, and extending the e_i's as constant along radii, i.e., $e_i(x) = e_i(x/|x|)$, $x \in \mathcal{C}_\Omega$, for $i = 1, \dots, n$, we get that $\{e_1, \dots, e_{n+1}\}$ is a local orthonormal frame field for $\mathcal{C}_\Omega$, where e_{n+1} is the future-oriented unit radial direction, i.e., $\langle e_{n+1}, e_{n+1}\rangle = -1$, $\langle e_{n+1}, E_{n+1}\rangle < 0$. We also observe that by direct computation we have $\nabla^0_i e_{n+1} = e_i$ for any $i = 1, \dots, n$. We remark that by definition $e_{n+1}(q) = q$ for any $q \in \Omega$, and by abuse of notation when writing $\nabla^0_w q$, where $w \in \mathcal{T}(\mathbb{R}^{n+1})$, it will be always understood that we are computing $\nabla^0_w e_{n+1}$ at $x = q$, and $\nabla^0_q w$ will stand for $\nabla^0_{e_{n+1}} w$.

In order to derive the equation of spacelike H-bumps, one can argue as in [Treibergs and Wei 1983, Section 1] with minor adjustments. Indeed, we only need to take into account the changes due to the bilinear form $\langle\cdot,\cdot\rangle$, and the definition of mean curvature for spacelike hypersurfaces [López 2014, Section 3.2]. For the sake of completeness we derive the equation following the scheme of [loc. cit., Section 2].

Let $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$, let Σ be the associated radial graph and let $\mathcal{Y} : \Omega \to \mathbb{R}^{n+1}$ be the map defined as $\mathcal{Y}(q) := e^{u(q)}q$. From Remark 3.2 it holds that $\nabla^0_i q = e_i$ and thus

$$\nabla^0_i \mathcal{Y} = \nabla^0_i (e^u q) = e^u u_i q + e^u e_i. \tag{3-4}$$

Therefore a local basis for $T_{\mathcal{Y}(q)}\Sigma$ is given by

$$E_i(q) = e^u(e_i + u_i q), \quad i = 1, \dots, n,$$

and the components of the metric are

$$g_{ij} = \langle E_i, E_j\rangle = e^{2u}(\langle e_i, e_j\rangle + u_i u_j \langle q, q\rangle) = e^{2u}(\delta_{ij} - u_i u_j).$$

Since we look for a spacelike hypersurface we must have $|\nabla u|^2 < 1$, and by elementary computations we see that the inverse matrix (g^{ij}) is given by

$$g^{ij} = e^{-2u}\left(\delta_{ij} + \frac{u_i u_j}{1 - |\nabla u|^2}\right). \tag{3-5}$$

For the Gauss map we have

$$N(\mathcal{Y}(q)) = \frac{q + \sum_{i=1}^n u_k e_k}{(1 - |\nabla u|^2)^{1/2}}.$$

Indeed it is elementary to verify that $\langle N(\mathcal{Y}(q)), E_i\rangle = 0$ for any $i = 1, \dots, n$ and

$$\langle N(\mathcal{Y}(q)), N(\mathcal{Y}(q))\rangle = \frac{-1 + |\nabla u|^2}{1 - |\nabla u|^2} = -1.$$

Moreover, as $u = 0$ on $\partial\Omega$, there exists $q_1 \in \Omega$ such that $\nabla u(q_1) = 0$ and by definition $N(\mathcal{Y}(q_1)) = q_1$ and thus $\langle N(\mathcal{Y}(q_1)), E_{n+1}\rangle < 0$. Therefore, since $N \circ \mathcal{Y} \in C^0(\overline{\Omega}, \mathbb{R}^{n+1})$ and Ω is connected, it follows that $N(\mathcal{Y}(\Omega)) \subset \mathbb{H}^n$, so that N is future-oriented. The coefficients of the second fundamental form are given by

$$\sigma_{ij} = \langle N, \nabla^0_i \nabla^0_j \mathcal{Y}\rangle = \frac{e^u(-\delta_{ij} + u_i u_j - u_{ij})}{(1 - |\nabla u|^2)^{1/2}}. \tag{3-6}$$

Indeed, recalling Remark 3.2 and (3-4), by direct computation we have

$$\nabla^0_i(\nabla^0_j \mathcal{Y}) = e^u(u_i u_j q + \nabla^0_i \nabla^0_j u\, q + u_j e_i + u_i e_j + \nabla^0_i e_j).$$

Hence, by using the relations $\langle e_i, e_j\rangle = \delta_{ij}$, $\langle e_i, q\rangle = 0$, and regrouping the terms, we deduce that

$$\langle N, \nabla^0_i \nabla^0_j \mathcal{Y}\rangle = \frac{e^u}{(1 - |\nabla u|^2)^{1/2}}\left(u_i u_j - \nabla^0_i \nabla^0_j u + \langle \nabla^0_i e_j, q\rangle + \sum_{k=1}^n u_k \langle \nabla^0_i e_j, e_k\rangle\right). \tag{3-7}$$

Since $\langle \nabla_i^0 e_j, q\rangle = -\langle e_j, \nabla_i^0 q\rangle = -\langle e_j, e_i\rangle = -\delta_{ij}$ and

$$\nabla_i^0\nabla_j^0 u - \sum_{k=1}^n u_k\langle \nabla_i^0 e_j, e_k\rangle = \nabla_i\nabla_j u - \sum_{k=1}^n u_k\langle \nabla_i e_j, e_k\rangle = u_{ij},$$

from (3-7) we finally get (3-6).

At the end, from [López 2014, Definition 3.3], the mean curvature of a spacelike hypersurface at $p = \mathcal{Y}(q) \in \Sigma$ is given by

$$nH(\mathcal{Y}(q)) = -\sum_{i,j=1}^n g^{ij}\sigma_{ij}.$$

Therefore, from (3-5) and (3-6), we deduce that u must satisfy the equation

$$\sum_{i,j=1}^n ((1-|\nabla u|^2)\delta_{ij} + u_iu_j)u_{ij} = n(1-|\nabla u|^2) - ne^u(1-|\nabla u|^2)^{3/2}H(\mathcal{Y}(q)).$$

4. A priori estimates

Let $\epsilon > 0$ and let $\eta_\epsilon \in C_0^\infty([0,+\infty))$ be such that $r\eta_\epsilon \in C_0^\infty([0,+\infty))$, $r \mapsto \eta_\epsilon(r)r$, is increasing in $(0, 2/\epsilon)$ and decreasing in $(2/\epsilon, +\infty)$. Assume moreover that

$$\eta_\epsilon(r)r = \begin{cases} r & \text{for } r < 1-\epsilon, \\ 1-\epsilon/2 & \text{for } 1-\epsilon/2 < r < 2/\epsilon, \\ 0 & \text{for } r > 3/\epsilon. \end{cases}$$

We define the regularized equation as

$$\begin{aligned}\sum_{i,j=1}^n \big((1-\eta_\epsilon^2(|\nabla u|)|\nabla u|^2)\delta_{ij} + \eta_\epsilon^2(|\nabla u|)u_iu_j\big)u_{ij} \\ = n(1-\eta_\epsilon(|\nabla u|)^2|\nabla u|^2)(1-\sqrt{1-\eta_\epsilon(|\nabla u|)^2|\nabla u|^2}e^uH(e^uq)).\end{aligned} \tag{4-1}$$

To simplify the notation we will write $\eta_\epsilon^2|\nabla u|^2$ instead of $\eta_\epsilon^2(|\nabla u|)|\nabla u|^2$. The regularized Dirichlet problem for spacelike H-bumps is

$$\begin{cases} \sum_{i,j=1}^n ((1-\eta_\epsilon^2|\nabla u|^2)\delta_{ij} + \eta_\epsilon^2 u_iu_j)u_{ij} = n(1-\eta_\epsilon^2|\nabla u|^2)(1-\sqrt{1-\eta_\epsilon^2|\nabla u|^2}e^uH(e^uq)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4-2}$$

We denote by Q_ϵ the operator

$$Q_\epsilon(u) := \sum_{i,j=1}^n ((1-\eta_\epsilon^2|\nabla u|^2)\delta_{ij} + \eta_\epsilon^2 u_iu_j)u_{ij} - n(1-\eta_\epsilon^2|\nabla u|^2) + n(1-\eta_\epsilon^2|\nabla u|^2)^{3/2}e^uH(e^uq).$$

We claim that, in hyperbolic stereographic coordinates, the operator Q_ϵ is uniformly elliptic. This is the content of the next lemma.

Lemma 4.1. *For any $\epsilon \in (0,1)$, the operator Q_ϵ, in hyperbolic stereographic coordinates, is uniformly elliptic with ellipticity constants depending only on ϵ and Ω.*

Proof. Let $F:\mathbb{H}^n\to\mathbb{B}^n$ be the hyperbolic stereographic projection. By definition we have that Ω is mapped into a smooth proper domain $\Lambda=F(\Omega)\Subset\mathbb{B}^n$, $\phi=F^{-1}:\Lambda\to\Omega$ is a global parametrization, and there exist $c_1,c_2>0$ depending only on Ω such that

$$c_1\le \tfrac{1}{4}(1-\|y\|_n^2)^2\le c_2\quad\text{for all } y\in\Lambda. \tag{4-3}$$

Let us set

$$\lambda(y):=\frac{2}{1-\|y\|_n^2},\qquad y\in\mathbb{B}^n. \tag{4-4}$$

We recall that F is an isometry and the hyperbolic metric in $\mathbb{B}^n$ is $\tilde g=\lambda^2\sum_{i=1}^n dy_i\otimes dy_i$ (see Section 2). In particular, $\langle\partial_i,\partial_j\rangle=\delta_{ij}\lambda^2$, where ∂_i denotes the vector $\partial\phi/\partial y_i$, and the Christoffel symbols of the hyperbolic Levi-Civita connection are given by

$$\Gamma_{ij}^k=\frac{\lambda_i}{\lambda}\delta_{jk}+\frac{\lambda_j}{\lambda}\delta_{ik}-\sum_{l=1}^n\delta_{kl}\frac{\lambda_l}{\lambda}\delta_{ij}, \tag{4-5}$$

where $\lambda_i=\partial\lambda/\partial y_i$. In local coordinates the gradient is given by

$$\nabla u=\sum_{i,j=1}^n\tilde g^{ij}\frac{\partial\tilde u}{\partial y_i}\partial_j=\lambda^{-2}\sum_{i=1}^n\frac{\partial\tilde u}{\partial y_i}\partial_i, \tag{4-6}$$

and thus

$$|\nabla u|^2=\lambda^{-2}\|\nabla_0\tilde u\|_n^2, \tag{4-7}$$

where $\tilde u=u\circ F^{-1}$ and $\nabla_0\tilde u$ is the gradient of $\tilde u$ with respect to the euclidean flat metric. Using the well-known expression for the Hessian and the Laplacian in local coordinates we have

$$\nabla^2u(\partial_i,\partial_j)=\frac{\partial^2\tilde u}{\partial y_j\,\partial y_i}-\sum_{k=1}^n\Gamma_{ji}^k\frac{\partial\tilde u}{\partial y_k}, \tag{4-8}$$

$$\Delta u=\sum_{i,j=1}^n\tilde g^{ij}\biggl(\frac{\partial^2\tilde u}{\partial y_i\,\partial y_j}-\sum_{k=1}^n\Gamma_{ij}^k\frac{\partial\tilde u}{\partial y_k}\biggr)=\lambda^{-2}\sum_{i=1}^n\biggl(\frac{\partial^2\tilde u}{\partial y_i^2}-\sum_{k=1}^n\Gamma_{ii}^k\frac{\partial\tilde u}{\partial y_k}\biggr). \tag{4-9}$$

By using the previous identities and (4-5) we infer that

$$\sum_{i,j=1}^n u_iu_ju_{ij}=\lambda^{-4}\sum_{h,k=1}^n\frac{\partial\tilde u}{\partial y_h}\frac{\partial\tilde u}{\partial y_k}\frac{\partial^2\tilde u}{\partial y_h\,\partial y_k}+\Phi, \tag{4-10}$$

where Φ is a term which does not involve second-order partial derivatives. From (4-4), (4-7), (4-9) and (4-10) we deduce that the principal part of the operator Q_ϵ, in hyperbolic stereographic coordinates, is

$$\lambda^{-2}\biggl[\sum_{i,j=1}^n(1-\eta_\epsilon^2\lambda^{-2}\|\nabla_0\tilde u\|_n^2)\delta_{ij}\frac{\partial^2\tilde u}{\partial y_i\,\partial y_j}+\eta_\epsilon^2\lambda^{-2}\frac{\partial\tilde u}{\partial y_i}\frac{\partial\tilde u}{\partial y_j}\frac{\partial^2\tilde u}{\partial y_i\,\partial y_j}\biggr],$$

where $\eta_\epsilon=\eta_\epsilon(\lambda^{-1}\|\nabla_0\tilde u\|_n)$. For any $i,j=1,\dots,n$, we define, for $y\in\Lambda$, $p=(p_1,\dots,p_n)\in\mathbb{R}^n$,

$$\tilde a_\epsilon^{ij}(y,p):=\lambda^{-2}[(1-\eta_\epsilon^2(\lambda^{-1}\|p\|_n)\lambda^{-2}\|p\|_n^2)\delta_{ij}+\eta_\epsilon^2(\lambda^{-1}\|p\|_n)\lambda^{-2}p_ip_j]. \tag{4-11}$$

Now, for any $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n,\ y \in \Lambda,\ p \in \mathbb{R}^n$ we claim that

$$c_2\|\xi\|_n^2 \geq \sum_{i,j=1}^{n} \tilde{a}_\epsilon^{ij}(y,p)\xi_i\xi_j \geq \tfrac{1}{2}\epsilon c_1\|\xi\|_n^2, \tag{4-12}$$

where the constants c_1, c_2 are given by (4-3). Indeed by the definition of η_ϵ for any $y \in \Lambda,\ p \in \mathbb{R}^n$ it holds that

$$0 \leq \eta_\epsilon^2(\lambda^{-1}\|p\|_n)\lambda^{-2}\|p\|_n^2 \leq \left(1 - \tfrac{1}{2}\epsilon\right)^2$$

and thus

$$\begin{aligned}\sum_{i,j=1}^{n} \tilde{a}_\epsilon^{ij}(y,p)\xi_i\xi_j &= \lambda^{-2}\left[(1-\eta_\epsilon^2\lambda^{-2}\|p\|_n^2)\|\xi\|_n^2 + \eta_\epsilon^2\lambda^{-2}\left(\sum_{i=1}^{n} p_i\xi_i\right)\left(\sum_{j=1}^{n} p_j\xi_j\right)\right] \\ &= \lambda^{-2}[(1-\eta_\epsilon^2\lambda^{-2}\|p\|_n^2)\|\xi\|_n^2 + \eta_\epsilon^2\lambda^{-2}(p,\xi)_n^2] \\ &\geq \lambda^{-2}(1-\eta_\epsilon^2\lambda^{-2}\|p\|_n^2)\|\xi\|_n^2 \geq \lambda^{-2}\left(1-\left(1-\tfrac{1}{2}\epsilon\right)^2\right)\|\xi\|_n^2 \geq \tfrac{1}{2}c_1\epsilon\|\xi\|_n^2,\end{aligned}$$

where $(\cdot\,,\cdot)_n$ denotes the euclidean scalar product in $\mathbb{R}^n$. The proof of the other inequality in (4-12) is similar and we omit the details. □

For $t \in [0, 1]$, we define the operator

$$Q_\epsilon^t(u) := \sum_{i,j=1}^{n} ((1-\eta_\epsilon^2|\nabla u|^2)\delta_{ij} + \eta_\epsilon^2 u_i u_j)u_{ij} - nt(1-\eta_\epsilon^2|\nabla u|^2) + nt(1-\eta_\epsilon^2|\nabla u|^2)^{3/2}e^u H(e^u q).$$

For u such that $|\nabla u|_{\infty,\Omega} < 1$, we also define the operator $\mathcal{Q}^t(u)$ as

$$\mathcal{Q}^t(u) := \sum_{i,j=1}^{n} ((1-|\nabla u|^2)\delta_{ij} + u_i u_j)u_{ij} - nt(1-|\nabla u|^2) + nt(1-|\nabla u|^2)^{3/2}e^u H(e^u q). \tag{4-13}$$

Remark 4.2. By definition, for any fixed $\epsilon \in (0, 1)$, if u is such that $|\nabla u|_{\infty,\Omega} \leq 1-\epsilon$, we have $Q_\epsilon^t(u) = \mathcal{Q}^t(u)$ for any $t \in [0, 1]$. Moreover, in view of Lemma 4.1 and since the principal parts of Q_ϵ^t, $\mathcal{Q}^t$ are independent of t, they are uniformly elliptic even with respect to t, when passing to hyperbolic stereographic coordinates.

Remark 4.3. As seen in the proof of Lemma 4.1 we can write an explicit expression of the operator Q_ϵ^t in hyperbolic stereographic coordinates defined in the whole $\Lambda = F(\Omega)$. For our purposes we just observe that the transformed operator is of the form

$$\tilde{Q}_\epsilon^t(\tilde{u}) = \sum_{i,j=1}^{n} \tilde{a}_\epsilon^{ij}(y, \nabla_0\tilde{u})\tilde{u}_{ij} + \tilde{b}_{\epsilon,t}(y, u, \nabla_0\tilde{u}),$$

where $\tilde{a}_\epsilon^{ij} = \tilde{a}_\epsilon(y,p) : \Lambda \times \mathbb{R}^n \to \mathbb{R}$ is given by (4-11), and $\tilde{b}_{\epsilon,t} : \Lambda \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is given by

$$\begin{aligned}\tilde{b}_{\epsilon,t}(y,z,p) := -(1-\eta_\epsilon^2\lambda^{-2}\|p\|_n^2)\sum_{k=1}^{n} G_k(y)p_k - \eta_\epsilon^2\lambda^{-2}\sum_{h,k,r=1}^{n} G_{hkr}(y)p_h p_k p_r \\ - nt(1-\eta_\epsilon^2\lambda^{-2}\|p\|_n^2) + nt(1-\eta_\epsilon^2\lambda^{-2}\|p\|_n^2)^{3/2}e^z H(e^z F(y)),\end{aligned}$$

where λ is defined in (4-4), G_k, G_{hkr} are smooth functions defined in Λ, where $h, k, r \in \{1, \dots, n\}$. We point out that $\tilde{a}_\epsilon^{ij}$ does not depend on z and $\tilde{a}_\epsilon^{ij} = O(1)$, $\tilde{b}_{\epsilon,t} = O(\|p\|_n)$, as $\|p\|_n \to +\infty$, uniformly for $y \in \Lambda$, and z in compact subsets of $\mathbb{R}$. In particular, according to the notation of [Gilbarg and Trudinger 1977], setting $\mathcal{E} := \sum_{ij=1}^n \tilde{a}_\epsilon^{ij} p_i p_j$, we have that $\mathcal{E}$ does not depend on z and $\mathcal{E} = O(\|p\|_n^2)$ as $\|p\|_n \to +\infty$, uniformly for $y \in \Lambda$.

These properties will be useful in the sequel. In addition, since $(\tilde{a}_\epsilon^{ij})$ is symmetric and positive definite, when applying the results of [loc. cit., Section 15], it will be understood that we take $(\tilde{a}_\epsilon^{ij})^* = \tilde{a}_\epsilon^{ij}$ and $c_i = 0$; see [loc. cit., (15.23)].

We define now the class of admissible domains.

Definition 4.4. Let Ω be a bounded domain of $\mathbb{H}^n$ and let $H \in C^0(\mathcal{C}_{\overline{\Omega}})$. We say that (Ω, H) is admissible if there exists a constant $\theta \in (0, 1)$ such that for any $q_0 \in \partial\Omega$ and for any $t \in [0, 1]$, there exist two functions $\varphi_1, \varphi_2 \in C^2(\overline{\Omega})$ satisfying

(i) $\sup_\Omega |\nabla\varphi_i| \le 1 - \theta$ for $i = 1, 2$,

(ii) $\varphi_1(q_0) = 0$ and $\varphi_1(q_0) \le 0$ on $\partial\Omega$,

(iii) $\varphi_2(q_0) = 0$ and $\varphi_2(q_0) \ge 0$ on $\partial\Omega$,

(iv) $\mathcal{Q}^t(\varphi_1) \ge 0$, $\mathcal{Q}^t(\varphi_2) \le 0$ in Ω.

We denote by $\mathcal{A}$ the set of admissible couples (Ω, H). Given $\theta \in (0, 1)$, and given Ω and H as above, we say that (Ω, H, θ) is admissible if (Ω, H) is admissible with constant θ.

Remark 4.5. We observe that $\mathcal{A} \ne \varnothing$. In fact for any given domain $\Omega \subset \mathbb{H}^n$ for any fixed $m > 0$, the function $H(x) = 1/|x|^m$, $x \in \mathcal{C}_{\overline{\Omega}}$, is such that $(\Omega, H) \in \mathcal{A}$. In fact it is easy to see that $Q_t(0) = 0$ for any $t \in [0, 1]$, so that the functions $\varphi_1 = 0$, $\varphi_2 = 0$ satisfy (i)–(iv) for any $\theta \in (0, 1)$. More generally, for any domain Ω and for any function $H \in C^0(\mathcal{C}_{\overline{\Omega}})$ such that $H|_\Omega = 1$, we have $(\Omega, H) \in \mathcal{A}$, and (Ω, H, θ) is admissible for any $\theta \in (0, 1)$.

This condition of admissibility is very general. If we impose some regularity on $\partial\Omega$, and if we assume that H is positive, smooth and not increasing along radii, then every couple (Ω, H) is admissible. This is the content of the next result. We introduce first the following definition.

Definition 4.6. Let Ω be a bounded domain of $\mathbb{H}^n$. We say that Ω satisfies a uniform exterior geodesic ball condition if there exist $\sigma > 0$ and a map $\Xi : \partial\Omega \to \mathbb{H}^n$ of class C^2 such that for any $q_0 \in \partial\Omega$ there exists a geodesic ball in $\mathbb{H}^n$ of radius σ centered at $\xi = \Xi(q_0) \in \mathbb{H}^n \setminus \overline{\Omega}$, and denoted by $B_\sigma(\xi)$, such that $q_0 \in \partial B_\sigma(\xi)$ and $B_\sigma(\xi) \subset \mathbb{H}^n \setminus \overline{\Omega}$.

Proposition 4.7. *Let Ω be a bounded domain of $\mathbb{H}^n$ satisfying a uniform exterior geodesic ball condition. Let $H \in C^1(\mathcal{C}_{\overline{\Omega}})$ be such that $H > 0$ and $(\partial/\partial\lambda)(\lambda H(\lambda q)) \le 0$ for all $q \in \overline{\Omega}$, $\lambda > 0$. Then (Ω, H) is admissible.*

Proof. Let $\mathrm{dist}_{\mathbb{H}^n}(\cdot, \cdot)$ be the geodesic distance in $\mathbb{H}^n$. Let $\sigma > 0$ be the number given by Definition 4.6 for Ω. In particular, by definition, it follows that for any $q_0 \in \partial\Omega$ there exists $\xi = \xi(q_0) \notin \overline{\Omega}$ such that $\mathrm{dist}_{\mathbb{H}^n}(\xi, \partial\Omega) = \mathrm{dist}_{\mathbb{H}^n}(\xi, q_0) = \sigma$.

Let $q_0 \in \partial\Omega$, $t \in [0,1]$ and let $\xi = \xi(q_0)$ satisfying the above properties. Since every geodesic ball of $\mathbb{H}^n$ is geodesically convex, see [Papadopoulos 2005, Section 2.5], we can take $R > 0$ sufficiently large so that $\overline{\Omega}$ is contained in the geodesically convex ball $B_R(\xi)$. We observe that since Ω is bounded and $\mathrm{dist}_{\mathbb{H}^n}(\xi, \partial\Omega) = \sigma$, up to a new choice of a larger R, we can assume that R is uniform with respect to the choice of $q_0 \in \partial\Omega$.

Arguing as in proof of [Gerhardt 1983, Theorem 2.1], we set $|||q||| := \mathrm{dist}_{\mathbb{H}^n}(q, \xi)$ to denote the geodesic distance from ξ and we define

$$\delta^+(q) := \int_{|||q_0|||}^{|||q|||} (1+\gamma(s))^{-1/2}\, ds, \quad \gamma(s) := \alpha e^{\beta s},$$

where α, β are positive constants to be determined later. By construction it holds that $\delta^+ \in C^2(\overline{B_R(\xi)} \setminus \{\xi\})$, $\delta^+ \in C^2(\overline{\Omega})$, $\delta^+(q_0) = 0$ and $\delta^+ \geq 0$ in $\overline{\Omega}$ because of the exterior ball condition.

Let us consider the operator

$$\mathcal{Q}^t_{\mathrm{div}}(u) := -\mathrm{div}_{\mathbb{H}^n}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) + \frac{nt}{\sqrt{1-|\nabla u|^2}} - nte^u H(e^u q)$$

(which is the divergence form of $-\mathcal{Q}^t$). We set

$$A(u) := -\mathrm{div}_{\mathbb{H}^n}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right), \quad \nu(u) := \frac{1}{\sqrt{1-|\nabla u|^2}}.$$

We observe that $|\nabla |||q||||| = 1$ for any $q \in B_R(\xi) \setminus \{\xi\}$. This property is known for general manifolds when R is sufficiently small so that $B_R(\xi)$ is contained in a normal neighborhood of ξ; see [Lee 1997, Corollaries 6.9 and 6.11]. In our case, as a consequence of the Cartan–Hadamard theorem, since $\mathbb{H}^n$ has negative sectional curvature it admits global normal coordinates and we are done.

Therefore, since the covariant derivatives of δ^+ are given by $(\delta^+)_i = (1+\gamma)^{-1/2} |||q|||_i$, we obtain that for any $q \in \overline{\Omega}$

$$|\nabla \delta^+| = (1+\gamma)^{-1/2} < 1,$$

and

$$\nu(\delta^+) = \gamma^{-1/2}(1+\gamma)^{1/2}.$$

In addition, by direct computation, see [Gerhardt 1983, (2.14)–(2.16)], it holds that

$$A(\delta^+) = (1+\gamma)^{-1/2}\big(\tfrac{1}{2}\beta - \Delta |||q|||\big)\nu(\delta^+).$$

We observe that $\Delta |||q|||$ is smooth and bounded in compact subsets of $\overline{B_R(\xi)} \setminus \{\xi\}$ and it is singular as $q \to \xi$. Indeed, see [Gerhardt 1983, (2.17)–(2.18)], we have

$$-\Delta |||q||| = -\frac{n-1}{|||q|||} + \Psi, \tag{4-14}$$

where Ψ is a bounded term which is given, in normal coordinates centered at ξ, by

$$\Psi = -\sum_{i,j,k=1}^{n} g^{ij}\,\Gamma^k_{ij}\, |||q|||_k.$$

In particular, in view of the uniform exterior geodesic ball condition, since $\mathrm{dist}_{\mathbb{H}^n}(q,\xi(q_0)) \geq \sigma$ for any $q \in \overline{\Omega}$, for any $q_0 \in \partial\Omega$, from (4-14) we infer that $\Delta|||q|||$ is bounded in $\overline{\Omega}$ by a constant depending only on n, σ, Ω, q_0. In addition, by definition the map $q_0 \mapsto \xi$ is of class $C^2(\partial\Omega, \mathbb{H}^n)$ and thus, by compactness of $\partial\Omega$, it follows that $\Delta|||q|||$ is bounded by a constant depending only n, σ, Ω.

Now, by the previous relations we have

$$\begin{aligned} A(\delta^+) + tn\nu(\delta^+) &= \big[(1+\gamma)^{-1/2}\big(\tfrac{1}{2}\beta - \Delta|||q|||\big) + tn\big](\gamma^{-1/2}(1+\gamma)^{1/2}) \\ &\geq \big(\tfrac{1}{2}\beta - \Delta|||q|||\big)\gamma^{-1/2} = \big(\tfrac{1}{2}\beta - \Delta|||q|||\big)\alpha^{-1/2}e^{-\beta s/2}. \end{aligned}$$

Setting $\overline{H} := \max_{q\in\overline{\Omega}} H(q) > 0$, we can choose β sufficiently large so that $\frac{1}{2}\beta - \Delta|||q||| > 0$ for any $q \in \overline{\Omega}$. With this choice of β we choose α sufficiently small so that $\big(\frac{1}{2}\beta - \Delta|||q|||\big)\alpha^{-1/2}e^{\beta s/2} \geq n\overline{H}$ for any $x \in \overline{\Omega},\ s \in [|||q_0|||, \sup_{q\in\overline{\Omega}} |||q|||]$. Therefore, since $\delta^+ \geq 0$ in $\overline{\Omega}$, and in view of the monotonicity assumption on H, it follows that

$$\begin{aligned} A(\delta^+) + nt\nu(\delta^+) - nte^{\delta^+}H(e^{\delta^+}q) &\geq A(\delta^+) + nt\nu(\delta^+) - nte^0H(e^0q) \\ &\geq A(\delta^+) + nt\nu(\delta^+) - nt\overline{H} \\ &\geq 0. \end{aligned}$$

Hence, $\mathcal{Q}^t_{\mathrm{div}}(\delta^+) \geq 0$ in $\overline{\Omega}$, which is equivalent to $\mathcal{Q}^t(\delta^+) \leq 0$ in $\overline{\Omega}$, and in addition by construction we have $\delta^+ \geq 0$ on $\partial\Omega$, $\delta^+(x_0) = 0$, and $|\nabla\delta^+| = (1+\gamma)^{-1/2} \leq 1 - \theta_+$ for some number $\theta_+ = \theta_+(\alpha, \beta) \in (0,1)$.

As pointed out before, in view of the uniform exterior ball condition, $-\Delta|||q|||$ is uniformly bounded by a constant depending only on n, σ, Ω, and by construction $\sup_{q\in\overline{\Omega}} |||q||| \leq R$. Therefore, the numbers α, β can be chosen in a uniform way with respect to the base point $q_0 \in \partial\Omega$ (and also with respect to $t \in [0,1]$). Hence, there exists $\theta_+ \in (0,1)$ such that for any $q_0 \in \partial\Omega,\ t \in [0,1]$, the function $\varphi_2 := \delta^+$ (which depends on the choice of q_0 but not on t) satisfies (i)–(iv) of Definition 4.4 with $\theta = \theta^+$. For the other barrier is suffices to take $\varphi_1 := \delta^-$, where

$$\delta^- := -\int_{|||q_0|||}^{|||q|||} (1+\gamma(s))^{-1/2}\,ds,$$

and to argue as in the previous case. We observe that in this case the choice of α, β has to be made in a different way but it is still uniform with respect to q_0, and t.

In fact

$$\begin{aligned} A(\delta^-) + tn\nu(\delta^-) &= -\gamma^{-1/2}\big(\tfrac{1}{2}\beta - \Delta|||q||| - tn(1+\gamma)^{1/2}\big) \\ &= -\alpha^{-1/2}e^{-\beta s/2}\big(\tfrac{1}{2}\beta - \Delta|||q||| - tn(1+\alpha e^{\beta s})^{1/2}\big). \end{aligned}$$

Taking $\alpha = e^{-\beta \sup_{q\in\overline{\Omega}} |||q|||}$, it follows that $n(1+\alpha e^{\beta s})^{1/2} \leq \sqrt{2}n$ for any $\beta > 0,\ s \in [|||q_0|||, \sup_{q\in\overline{\Omega}} |||q|||]$. With this choice of α, we choose β such that

$$\tfrac{1}{2}\beta - \Delta|||q||| - 2n \geq 0$$

for $x \in \overline{\Omega}$. At the end, we have $A(\delta^-) + tn\nu(\delta^-) \leq 0$, and thus since $H > 0$ it holds that

$$A(\delta^-) + tn\nu(\delta^-) - tne^{\delta^-}H(e^{\delta^-}q) \leq A(\delta^-) + tn\nu(\delta^-) \leq 0 \quad \text{in } \Omega.$$

As before we find a uniform $\theta_- \in (0,1)$ such that for any $q_0 \in \partial\Omega$, $t \in [0,1]$, the function $\varphi_1 := \delta^-$ satisfies $|\nabla\varphi_1| \le 1-\theta_-$ and (ii)–(iv) of Definition 4.4. At the end, choosing $\theta := \min\{\theta_-, \theta_+\}$ we have that for any $q_0 \in \partial\Omega$, $t \in [0,1]$, the functions φ_1, φ_2 satisfy (i)–(iv) of Definition 4.4, and hence (Ω, H) is admissible. □

Remark 4.8. It is important to note that in the previous proof the choice of θ depends only on n, σ, Ω and depends on H just by the number $\overline{H} := \max_{q\in\overline{\Omega}} H(q) > 0$ because of the monotonicity assumption. In particular θ does not depend on the derivatives of H.

If $H \in C^1(C_{\overline{\Omega}}(r_1, r_2))$ we define a canonical extension of H to a mapping on the cone $\mathcal{C}_{\overline{\Omega}}$ in the following way: set

$$h_1(q) := \left[\frac{\partial}{\partial\rho}\rho H(\rho q)\right]_{\rho=r_1}, \quad h_2(q) := \left[\frac{\partial}{\partial\rho}\rho H(\rho q)\right]_{\rho=r_2}$$

and

$$\widehat{H}(\rho q) := \begin{cases} (r_1/\rho)H(r_1 q) + (1-r_1/\rho)h_1(q) & \text{for } \rho \in (0, r_1), \\ H(\rho q) & \text{for } \rho \in [r_1, r_2], \\ (r_2/\rho)H(r_2 q) + (1-r_2/\rho)h_2(q) & \text{for } \rho \in (r_2, +\infty). \end{cases} \tag{4-15}$$

Remark 4.9. It is elementary to check that $\widehat{H} \in C^1(\mathcal{C}_{\overline{\Omega}})$, and if H satisfies $(\partial/\partial\lambda)(\lambda H(\lambda q)) \le 0$ for all $q \in \overline{\Omega}$, $\lambda \in [r_1, r_2]$, it follows that

$$\frac{\partial}{\partial\lambda}(\lambda \widehat{H}(\lambda q)) \le 0 \quad \text{for all } q \in \overline{\Omega},\ \lambda > 0.$$

Therefore, since $\widehat{H}(x) = H(x)$ for $x \in \overline{\Omega}$, by Remark 4.8 if Ω satisfies the hypotheses of Proposition 4.7 and H is positive, it follows that $(\widehat{H}, \Omega)$ is admissible with constant which does not depend on the choice of r_1, r_2, and the derivatives of $\widehat{H}$.

In view of the previous remark, the following definition makes sense:

Definition 4.10. Let Ω be a bounded domain of $\mathbb{H}^n$, let $0 < r_1 \le 1 \le r_2$ and let $H \in C^1(C_{\overline{\Omega}}(r_1, r_2))$. We say that (Ω, H) is admissible if $(\Omega, \widehat{H})$ is admissible, where $\widehat{H}$ is the extension of H defined in (4-15), and for $\theta \in (0,1)$ we say that (Ω, H, θ) is admissible if $(\Omega, \widehat{H}, \theta)$ is admissible.

Now we have all the tools to prove the a priori estimates. Let us fix some notation: let $k \in \mathbb{N}$, $\alpha \in (0,1)$, and we consider the subspaces $C_0^{k,\alpha}(\overline{\Omega}) := \{u \in C^{k,\alpha}(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$, $C_0^k(\overline{\Omega}) := \{u \in C^k(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$, endowed, respectively, with the usual norms $|\cdot|_{k,\alpha}$, $|\cdot|_k$. We point out that $C_0^{k,\alpha}(\overline{\Omega})$, $C_0^k(\overline{\Omega})$ are closed subspaces of Banach spaces and thus they are Banach too. When needed we will specify also the domain in the norms; otherwise it will be understood that the domain is Ω. Moreover, for the $C^0(\overline{\Omega})$-norm we will use the notation $|\cdot|_\infty$, $|\cdot|_{\infty,\Omega}$, and $\|\cdot\|_\infty$, $\|\cdot\|_{\infty,\Omega}$ when working in the euclidean setting.

We define $\widehat{Q}_\epsilon^t$ as the operator obtained from Q_ϵ^t by replacing H with its extension $\widehat{H}$, and in the class of functions satisfying $|\nabla u|_{\infty,\Omega} < 1$ we define $\widehat{\mathcal{Q}}^t$ as

$$\widehat{\mathcal{Q}}^t(u) := \sum_{i,j=1}^n ((1-|\nabla u|^2)\delta_{ij} + u_i u_j)u_{ij} - nt(1-|\nabla u|^2) + nt(1-|\nabla u|^2)^{3/2} e^u \widehat{H}(e^u q). \tag{4-16}$$

In order to simplify the notation we set $L_{\epsilon,u}u := \sum_{i,j=1}^{n}((1-\eta_\epsilon^2|\nabla u|^2)\delta_{ij} + \eta_\epsilon^2 u_i u_j)u_{ij}$. The first result we prove is about a priori C^0 estimates for solutions of $\widehat{Q}_\epsilon^t(u) = 0$.

Lemma 4.11 (a priori C^0 estimates). *Let Ω be a bounded domain and let $r_1 \neq r_2$ such that $0 < r_1 \leq 1 \leq r_2$. Assume that $H \in C^1(\mathcal{C}_{\overline{\Omega}}(r_1, r_2))$ satisfies*

$$H(r_1 q) > r_1^{-1} \quad \textit{and} \quad H(r_2 q) < r_2^{-1} \quad \textit{for any } q \in \overline{\Omega}, \tag{4-17}$$

and

$$\frac{\partial}{\partial\lambda}(\lambda H(\lambda q)) \leq 0 \quad \textit{for all } q \in \overline{\Omega},\ \lambda \in [r_1, r_2]. \tag{4-18}$$

For $\epsilon \in (0,1)$, for every $t \in [0,1]$, if $u \in C_0^2(\overline{\Omega})$ is a solution of $\widehat{Q}_\epsilon^t(u) = 0$ then

$$\log r_1 \leq u(q) \leq \log r_2 \quad \textit{for every } q \in \overline{\Omega}.$$

Proof. Let us observe that since we are assuming (4-17), (4-18), it holds that

$$\widehat{H}(x) > |x|^{-1} \quad \text{if } |x| \leqslant r_1,\ x \in \mathcal{C}_{\overline{\Omega}} \qquad \text{and} \qquad \widehat{H}(x) < |x|^{-1} \quad \text{if } |x| \geqslant r_2,\ x \in \mathcal{C}_{\overline{\Omega}}. \tag{4-19}$$

Let $\epsilon \in (0,1)$, let $t \in [0,1]$ and let $u \in C_0^2(\overline{\Omega})$ such that $\widehat{Q}_\epsilon^t(u) = 0$. By definition u is a classical solution of the Dirichlet problem

$$\begin{cases} L_{\epsilon,u}u = nt(1-\eta_\epsilon^2|\nabla u|^2)(1-\sqrt{1-\eta_\epsilon^2|\nabla u|^2}\, e^u \widehat{H}(e^u q)) & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega. \end{cases} \tag{4-20}$$

Let $q_0 \in \overline{\Omega}$ such that $u(q_0) = \max_{\overline{\Omega}} u$. Assume by contradiction that $u(q_0) > \log r_2$. Then $q_0 \in \Omega$ because $r_2 \geq 1$ and $u = 0$ on $\partial\Omega$. Hence $\nabla u(q_0) = 0$, and since q_0 is a maximum point, it holds that $\Delta u(q_0) \leq 0$, and by the definition of $L_{u,\epsilon}$ this reads as

$$L_{u,\epsilon}u \leq 0.$$

On the other hand it must be that $t > 0$ because otherwise if $t = 0$ then $u \equiv 0$. Moreover

$$L_{u,\epsilon}u(q_0) = nte^{u(q_0)}\left(\frac{1}{e^{u(q_0)}} - \widehat{H}(e^{u(q_0)}q_0)\right) > 0,$$

because $\widehat{H}(x) < |x|^{-1}$ as $|x| > r_2$. Thus we reach a contradiction. The same argument holds to show that $\min_{\overline{\Omega}} u \geq \log r_1$. □

Lemma 4.12 (a priori $C^{1,\alpha}$ estimates). *Let $\epsilon \in (0,1)$ and let Ω be a bounded domain of class C^2. Assume that H satisfies* (4-17), (4-18). *Then, there exist two positive constants M, C and $\alpha_0 \in (0,1)$ such that for all $t \in [0,1]$ if $u \in C_0^2(\overline{\Omega})$ is such that $|\nabla u|_{\infty,\partial\Omega} \leq 1-\epsilon$ and is a solution of the equation $\widehat{Q}_\epsilon^t(u) = 0$, then*

$$|\nabla u|_{\infty,\Omega} \leq M, \quad |u|_{1,\alpha_0} \leq C.$$

Proof. Let us fix $\epsilon \in (0,1)$, let $t \in [0,1]$ and let $u = u_t$ be a solution of $\widehat{Q}_\epsilon^t(u) = 0$. From Lemma 4.11 we have $\log r_1 \leq u \leq \log r_2$ and thus by definition u also satisfies $Q_\epsilon^t(u) = 0$. Therefore, from now on we can work just with the operator Q_ϵ^t.

Let us set $b_{\epsilon,t}(q,u,\nabla u) := nt(1-\eta_\epsilon^2|\nabla u|^2)(1-\sqrt{1-\eta_\epsilon^2|\nabla u|^2}e^u H(e^u q))$. In view of Remark 4.2, passing to hyperbolic stereographic coordinates, the operator Q_ϵ^t is uniformly elliptic with constants independent of t; moreover, by definition and thanks to Lemma 4.11, the term $b_{\epsilon,t}(q,u,\nabla u)$ is uniformly bounded with respect to t.

Now there are only two possibilities: there exists a constant M independent of t such that $|\nabla u|_{\infty,\Omega} \le M$ for all $t \in [0,1]$ or there exists a subsequence $(t_k) \subset [0,1]$ such that $|\nabla u_{t_k}|_{\infty,\Omega} \to +\infty$ as $k \to +\infty$. We claim that the second case cannot happen. To this end we will give two proofs of this fact; one works only in dimension 2, the other one works in any dimension.

Case of dimension 2: Assume that $|\nabla u_{t_k}|_{\infty,\Omega} \to +\infty$. Let us set $\Omega_k' := \{x \in \overline{\Omega} : |\nabla u_{t_k}| \ge 3/\epsilon\}$, and $q_k \in \Omega_k'$ such that $|\nabla u_{t_k}|_{\infty,\Omega} = |\nabla u_{t_k}(q_k)|$. We observe that Ω_k' is closed and hence is a compact subset of $\overline{\Omega}$, and since $|\nabla u_t|_{\infty,\partial\Omega} \le 1-\epsilon$, we have $\Omega_k' \cap \partial\Omega = \varnothing$ for all k. Let Ω_k'' be the connected component of Ω_k' containing q_k. Consider now the auxiliary problem

$$\begin{cases} \Delta v_{t_k} = nt_k(1-\eta_\epsilon^2|\nabla u_{t_k}|^2)(1-\sqrt{1-\eta_\epsilon^2|\nabla u_{t_k}|^2}e^{u_{t_k}} H(e^{u_{t_k}} q)) & \text{in } \Omega, \\ v_{t_k} = 0 & \text{in } \partial\Omega. \end{cases} \tag{4-21}$$

We observe that since u_{t_k} is uniformly bounded, by construction and standard regularity theory we get that v_{t_k} and its gradient are uniformly bounded with respect to k. By definition $w_{t_k} := u_{t_k} - v_{t_k}$ is harmonic in Ω_k''. Therefore, considering the isometry $F : \mathbb{H}^2 \to \mathbb{B}^2$, and since harmonicity is preserved through composition with isometries, see [Hélein and Wood 2008, Section 2.2], we know $\tilde{w}_{t_k} := F \circ w_{t_k}$ is harmonic in $\widetilde{\Omega}_k'' := F(\Omega_k'') \Subset \mathbb{B}^2$. Now, since the hyperbolic metric $\tilde{g}$ is conformal to the euclidean metric g_0 in $\mathbb{B}^2$ (see Section 2), we have that $\tilde{w}_{t_k}$ is harmonic also in $(\widetilde{\Omega}_k'', g_0)$. We point out that, in general, this fact is false in other dimensions. Hence, since $\tilde{w}_{t_k}$ is harmonic, it follows that also $\nabla_0 \tilde{w}_{t_k}$ is harmonic in $\widetilde{\Omega}_k''$, so $\|\nabla_0 \tilde{w}_{t_k}\|_n$ achieves its maximum on the boundary, and thus $\|\nabla_0 \tilde{w}_{t_k}\|_\infty = \|\nabla_0 \tilde{w}_{t_k}\|_{\infty,\partial\widetilde{\Omega}_k''} \to +\infty$ as $k \to +\infty$. On the other hand, by construction and (4-7) we have that

$$\begin{aligned} \|\nabla_0 \tilde{w}_{t_k}\|_{\infty,\partial\widetilde{\Omega}_k''} = \sup_{y \in \partial\widetilde{\Omega}_k''} \|\nabla_0 \tilde{w}_{t_k}(y)\|_n &= \sup_{q \in \partial\Omega_k''} \frac{4}{(1-\|F(q)\|_n^2)^2} |\nabla w_{t_k}(q)| \\ &\le \sup_{q \in \partial\Omega_k''} \frac{4}{(1-\|F(q)\|_n^2)^2} \Big(\frac{3}{\epsilon} + |\nabla v_k(q)|\Big) \end{aligned}$$

is uniformly bounded and thus we get a contradiction.

Case of any dimension $n \ge 2$: Consider $\tilde{u} := u \circ F^{-1}$, where $F : \mathbb{H}^n \to \mathbb{B}^n$ is the hyperbolic stereographic projection. Then $\tilde{u}$ is a solution of a uniformly elliptic equation which satisfies the hypotheses of [Gilbarg and Trudinger 1977, Theorem 15.2]; see [loc. cit., (i), p. 367]. In fact, thanks to Remark 4.3, writing Q_ϵ^t in local coordinates we see by elementary computations that the natural conditions of [loc. cit., (i), p. 367], are satisfied (uniformly in t). In particular, introducing the operator

$$\delta = \frac{\partial}{\partial z} + \sum_{k=1}^n \|p\|_n^{-2} p_k \frac{\partial}{\partial y_k},$$

we see that $\delta\tilde{a}^{ij}_\epsilon$, $\delta\tilde{b}_{\epsilon,t}$ satisfy, as $\|p\|_n \to +\infty$ (uniformly for $(y,z)\in\Lambda\times[\log r_1, \log r_2]$, and in $t\in[0,1]$), the growth conditions of [loc. cit., (15.36)], and thus the hypotheses of [loc. cit., Theorem 15.2] are satisfied with $c\le 0$.

Thanks to Lemma 4.11 the oscillation of u is uniformly bounded; moreover, since $|\nabla u|_{\infty,\partial\Omega}\le 1-\epsilon$ and the structural conditions are satisfied uniformly in t, we have that the constant given by [loc. cit., Theorem 15.2] is uniformly bounded with respect to t. Hence there exists C independent of t such that $\|\nabla_0\tilde{u}\|_{\infty,F(\Omega)}\le C$, and hence, in view of (4-7), the same holds for $|\nabla u|_{\infty,\Omega}$. Therefore, it cannot happen that there exists a sequence (t_k) such that $|\nabla u_{t_k}|_{\infty,\Omega}\to+\infty$, and we are done.

Conclusion: From the previous discussion the only possibility is that there exists a constant M such that $|\nabla u|_{\infty,\Omega}\le M$ for all $t\in[0,1]$. From this fact, up to passing to local coordinates, since Q^t_ϵ is uniformly elliptic (with ellipticity constant independent of t) and $b_{\epsilon,t}(q,u,\nabla u)$ is uniformly bounded in t, thanks to [loc. cit., Theorem 13.7], there exists $\alpha_0\in(0,1)$ and a positive constant C, both depending only on n, Ω, $|\nabla u|_{\infty,\Omega}$, Ω, and the ratio between the uniform bound on $b_{\epsilon,t}$ and the lower ellipticity constant, such that

$$[\nabla u]_{0,\alpha_0}\le C,$$

where $[\,\cdot\,]_{0,\alpha_0}$ denotes the C^{0,α_0} seminorm. At the end, from this fact and Lemma 4.11, we conclude that

$$|u|_{1,\alpha_0}\le C_1$$

for some constant C_1 not depending on t, and the proof is complete. □

5. Existence and uniqueness of solutions for the regularized problem

The aim of this section is to prove the following:

Theorem 5.1. *Let $\alpha\in(0,1)$, $0<r_1\le 1\le r_2$, with $r_1\ne r_2$, let Ω be a bounded domain of $\mathbb{H}^n$, with boundary of class $C^{2,\alpha}$. Let $H\in C^1(\mathcal{C}_{\overline{\Omega}}(r_1,r_2))$ satisfy hypotheses* (i), (ii) *of Theorem 1.3. Assume that (Ω,H) is admissible. Then, there exists $\bar\epsilon\in(0,1)$ such that for any $\epsilon\in(0,\bar\epsilon)$ equation* (4-2) *has a solution. Moreover such a solution is the unique solution of problem* (4-2) *whose associated radial graph is contained in $\mathcal{C}_{\overline{\Omega}}(r_1,r_2)$.*

Proof. We divide the proof into several steps.

Step 1: Choice of $\bar\epsilon\in(0,1)$.

Let $\widehat{H}$ be the extension of H defined in (4-15). Since $(\Omega,\widehat{H})$ is admissible we choose $\bar\epsilon=\theta$, where θ is given by Definition 4.4.

Let $\epsilon\in(0,1)$ such that $\epsilon<\theta$, let $\alpha_0\in(0,1)$ be the number given by Lemma 4.12 and set $\beta:=\min\{\alpha,\alpha_0\}$. For any fixed $w\in C^{1,\beta}(\overline{\Omega})$ we define the operator $L_{w,\epsilon}: C^{2,\beta}_0(\overline{\Omega})\to C^{0,\beta}(\overline{\Omega})$ as

$$L_{w,\epsilon}u:=\sum_{i,j=1}^n((1-\eta^2_\epsilon|\nabla w|^2)\delta_{ij}+\eta^2_\epsilon w_iw_j)u_{ij}.$$

Step 2: For every $w\in C^{1,\beta}(\overline{\Omega})$ the operator $L_{w,\epsilon}$ is a bijection of $C^{2,\beta}_0(\overline{\Omega})$ onto $C^{0,\beta}(\overline{\Omega})$.

A mapping $u \in C_0^{2,\beta}(\overline{\Omega})$ belongs to the kernel of $L_{w,\epsilon}$ if and only if u solves the Dirichlet problem

$$\begin{cases} \sum_{i,j=1}^{n} ((1-\eta_\epsilon^2|\nabla w|^2)\delta_{ij} + \eta_\epsilon^2 w_i w_j)u_{ij} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5-1}$$

Since $L_{w,\epsilon}$ is uniformly elliptic (see Lemma 4.1), by the maximum principle, because $u = 0$ on $\partial\Omega$ we obtain that $u = 0$ in Ω, and this means that $L_{w,\epsilon}$ is injective. In order to prove that $L_{w,\epsilon}$ is onto we use the continuity method. Let $t \in [0, 1]$; we introduce the family of operators

$$\mathcal{L}_{t,w,\epsilon} : C_0^{2,\beta}(\overline{\Omega}) \to C^{0,\beta}(\overline{\Omega})$$

defined by

$$\mathcal{L}_{t,w,\epsilon} = (1-t)\Delta + tL_{w,\epsilon}.$$

We observe that $\mathcal{L}_{0,w,\epsilon} = \Delta$ and for every $f \in C^{0,\beta}(\overline{\Omega})$ the Dirichlet problem

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega \end{cases}$$

admits a solution $C^{2,\beta}(\overline{\Omega})$. That is, $\mathcal{L}_{0,w,\epsilon}$ sends $C^{0,\beta}(\overline{\Omega})$ onto $C_0^{2,\beta}(\overline{\Omega})$. Now we claim that there exists a constant $C > 0$ such that

$$|u|_{2,\beta} \le C|\mathcal{L}_{t,w,\epsilon}|_{0,\beta} \tag{5-2}$$

for every $t \in [0, 1]$, for every $u \in C_0^{2,\beta}(\overline{\Omega})$. In view of the method of continuity, this is enough to infer that $\mathcal{L}_{1,w,\epsilon} = L_{w,\epsilon}$ is onto. If (5-2) is false then there exist sequences $(t_k) \subset [0, 1]$ and $(u_k) \subset C_0^{2,\beta}(\overline{\Omega})$ such that

$$|\mathcal{L}_{t,w,\epsilon}|_{0,\beta} \to 0 \quad \text{and} \quad |u_k|_{2,\beta} = 1. \tag{5-3}$$

By compactness, in particular using also the Ascoli–Arzelà theorem, there exist $t \in [0, 1]$ and $u \in C_0^{2,\beta}(\overline{\Omega})$ such that, up to subsequences,

$$t_k \to t \quad \text{and} \quad u_k \to u \quad \text{in } C^2(\overline{\Omega}).$$

By continuity we have $\mathcal{L}_{t,w,\epsilon}u = 0$. Since, up to passing to hyperbolic stereographic coordinates, $\mathcal{L}_{t,w,\epsilon}$ is a convex combination of elliptic operators, it follows that $u = 0$. In particular

$$u_k \to 0 \quad \text{in } C^0(\overline{\Omega}). \tag{5-4}$$

We observe that

$$\mathcal{L}_{t,w,\epsilon}u = \sum_{i,j=1}^{n} a_{t,\epsilon}^{ij} u_{ij},$$

where $a_{t,\epsilon}^{ij} = ((1-t\eta_\epsilon^2|\nabla w|^2)\delta_{ij} + t\eta_\epsilon^2 w_i w_j)$, and $\mathcal{L}_{t,w,\epsilon}$ is uniformly elliptic; moreover, arguing as in the proof of Lemma 4.1 we see that the ellipticity constants are independent of t. Since the boundary is smooth we can apply global Schauder estimates and we get

$$|u_k|_{2,\beta} \le C(|u_k|_\infty + |\mathcal{L}_{t,w,\epsilon}|_{0,\beta}),$$

with C independent of k. This yields a contradiction with (5-3), (5-4). Hence (5-2) is true and the proof of Step 2 is complete.

Step 3: For every $C > 0$ there exists $K > 0$ such that if $|w|_{1,\beta} \le C$ then $|u|_{2,\beta} \le K|L_{w,\epsilon}u|_{0,\beta}$ for every $u \in C_0^{2,\beta}(\overline{\Omega})$.

We argue by contradiction as in the last part of the proof of Step 2. If the result is false then there exist a bounded sequence (w_k) in $C^{1,\beta}(\overline{\Omega})$ and a sequence (u_k) in $C_0^{2,\beta}(\overline{\Omega})$ such that

$$|u_k|_{2,\beta} = 1 \quad \text{and} \quad |L_{w_k,\epsilon}u_k|_{0,\beta} \to 0. \tag{5-5}$$

By compactness, there exist $w \in C^1(\overline{\Omega})$ and $u \in C_0^2(\overline{\Omega})$ such that, up to subsequences,

$$w_k \to w \quad \text{in } C^1(\overline{\Omega}) \qquad \text{and} \qquad u_k \to u \quad \text{in } C^2(\overline{\Omega}).$$

By continuity we get $L_{w,\epsilon}u = 0$. Then $u = 0$, by Step 2. Taking into account Lemma 4.1 we observe that the operators $L_{w_k,\epsilon}$ are uniformly elliptic with ellipticity constants independent of k. Using standard Schauder estimates we obtain that

$$|u_k|_{2,\beta} \le C_1(|u_k|_\infty + |L_{w_k,\epsilon}u_k|_{0,\beta}),$$

where C_1 is a constant independent of k. Since $u_k \to 0$ in $C^0(\overline{\Omega})$ and by (5-5) we reach a contradiction. The proof of Step 3 is complete.

Step 4: Let (w_k) be a bounded sequence in $C^{1,\beta}(\overline{\Omega})$ and let (f_k) be a bounded sequence in $C^{0,\beta}(\overline{\Omega})$. Then the sequence (u_k) of solutions of

$$\begin{cases} L_{w_k,\epsilon}u_k = f_k & \text{in } \Omega, \\ \qquad u_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{5-6}$$

is bounded in $C^{2,\beta}(\overline{\Omega})$.

The existence of a solution u_k of (5-6) is given by Step 2, and the thesis follows from Step 3.

Step 5: Let us consider the map $T_\epsilon : C_0^{1,\beta}(\overline{\Omega}) \to C_0^{1,\beta}(\overline{\Omega})$, defined as follows: for every $w \in C_0^{1,\beta}(\overline{\Omega})$ we set $T_\epsilon(w) := u$, where $u = u(w,\epsilon)$ is the unique solution of the problem

$$\begin{cases} \displaystyle\sum_{i,j=1}^{n} ((1-\eta_\epsilon^2|\nabla w|^2)\delta_{ij} + \eta_\epsilon^2 w_i w_j)u_{ij} = n(1-\eta_\epsilon^2|\nabla w|^2)(1-\sqrt{1-\eta_\epsilon^2|\nabla w|^2}e^w \hat{H}(e^w q)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5-7}$$

We claim that T_ϵ is a compact operator.

We first observe T_ϵ is well-defined; in fact, as proved in Step 2, for a given $w \in C_0^{1,\beta}(\overline{\Omega})$, the operator $L_{w,\epsilon}$ is a bijection between $C_0^{2,\beta}(\overline{\Omega})$ and $C^{0,\beta}(\overline{\Omega})$. In addition is T_ϵ is a linear map. It remains to prove that T_ϵ maps bounded families of $C_0^{1,\beta}(\overline{\Omega})$ into relatively compact subsets of $C_0^{1,\beta}(\overline{\Omega})$.

Let (w_λ) be a bounded family of $C_0^{1,\beta}(\overline{\Omega})$; then (u_λ), where $u_\lambda = Tw_\lambda$, is a family of solutions of (5-7). Hence $u_\lambda \in C_0^{2,\beta}(\overline{\Omega})$ and since we assumed that there exists $C > 0$ such that $|w_\lambda|_{1,\beta} \le C$, by Step 3 we have

$$|u_\lambda|_{2,\beta} \le K\big|n(1-\eta_\epsilon^2|\nabla w_\lambda|^2)(1-\sqrt{1-\eta_\epsilon^2|\nabla w_\lambda|^2}e^{w_\lambda}\hat{H}(e^{w_\lambda}q))\big|_{0,\beta} \le K_1,$$

where K_1 is a positive constant not depending on the family. Hence (u_λ) is uniformly bounded in $C_0^{2,\beta}(\overline{\Omega})$, and in particular by Step 4 and the Ascoli–Arzelà theorem, it is relatively compact in $C_0^{1,\beta}(\overline{\Omega})$. This proves that (u_λ) is relatively compact in $C_0^{1,\beta}(\overline{\Omega})$ and we are done.

Step 6: There exists a constant $C > 0$ such that $|u|_{1,\beta} \le C$ for any $u \in C_0^{1,\beta}(\overline{\Omega})$ satisfying $u = tT_\epsilon u$, where $t \in [0, 1]$.

We first observe that by definition and standard elliptic regularity theory any $u \in C_0^{1,\beta}(\overline{\Omega})$ satisfying $u = tT_\epsilon u$ is of class $C_0^{2,\beta}(\overline{\Omega})$ and satisfies $\widehat{Q}_\epsilon^t(u) = 0$. Thanks to Lemma 4.12, and since $\beta \le \alpha_0$, there exists $C > 0$ such that $|u|_{1,\beta} \le C$, provided that $|\nabla u|_{\infty,\partial\Omega} \le 1-\epsilon$. Therefore, in order to conclude, it is sufficient to check this boundary estimate for the gradient.

Let $q_0 \in \partial\Omega$ such that $|\nabla u(q_0)| = |\nabla u|_{\infty,\partial\Omega}$. If $\nabla u(q_0) = 0$, it follows that $\nabla u = 0$ on $\partial\Omega$ and hence there is nothing to prove. Therefore, let us assume that $\nabla u(q_0) \neq 0$.

Since $(\Omega, \widehat{H})$ is admissible, for any $t \in [0, 1]$ there exist $\varphi_1, \varphi_2 \in C^2(\overline{\Omega})$ satisfying (i)–(iv) of Definition 4.4 at q_0. Hence, taking into account of the choice of ϵ and Remark 4.2, we have

$$\widehat{Q}_\epsilon^t(\varphi_1) \ge \widehat{Q}_\epsilon^t(u) \ge \widehat{Q}_\epsilon^t(\varphi_2) \quad \text{in } \Omega,$$

and $\varphi_1 \le u \le \varphi_2$ on $\partial\Omega$. Let us write

$$\widehat{Q}_\epsilon^t(u) = \sum_{i,j=1}^{n} a_\epsilon^{ij} u_{ij} + \hat{b}_{\epsilon,t}(q, u, \nabla u),$$

where

$$\hat{b}_{\epsilon,t}(q, u, \nabla u) := -nt(1-\eta_\epsilon^2|\nabla u|^2) + nt(1-\eta_\epsilon^2|\nabla u|^2)^{3/2} e^u \widehat{H}(e^u q).$$

Notice that thanks to assumption (ii) and Remark 4.9, it follows that for any fixed $q \in \Omega$ the map $z \mapsto e^z \widehat{H}(e^z q)$ is not increasing.

Thanks to Lemma 4.1 and Remark 4.3, under the hyperbolic stereographic projection $F : \mathbb{H}^n \to \mathbb{B}^n$, the operator $\widehat{Q}_\epsilon^t$ is transformed into a uniformly elliptic operator of the form

$$\widetilde{Q}_\epsilon^t \tilde{u} = \sum_{i,j=1}^{n} \tilde{a}_\epsilon^{ij}(y, \nabla_0 \tilde{u}) \tilde{u}_{ij} + \tilde{b}_{\epsilon,t}(y, \tilde{u}, \nabla_0 \tilde{u}),$$

where $y \in F(\Omega)$, $\nabla_0 \tilde{u}$ is the euclidean gradient, and $\tilde{u}_{ij}$ are the second partial derivatives of $\tilde{u} = u \circ F^{-1}$. In view of Remark 4.3 and assumption (ii) the principal part $\tilde{a}_\epsilon^{ij}(y, p)$ does not depend on z, and for each $(y, p) \in F(\Omega) \times \mathbb{R}^n$ the map $z \mapsto \tilde{b}_{\epsilon,t}(y, z, p)$ is nonincreasing. Hence the comparison principle applies, see [Gilbarg and Trudinger 1977, Theorem 10.1], and thus setting $\tilde{\varphi}_i := \varphi_i \circ F^{-1}$, for $i = 1, 2$, from $\widetilde{Q}_\epsilon^t(\tilde{\varphi}_1) \ge \widetilde{Q}_\epsilon^t(\tilde{u}) \ge \widetilde{Q}_\epsilon^t(\tilde{\varphi}_2)$ in $F(\Omega)$, and $\tilde{\varphi}_1 \le \tilde{u} \le \tilde{\varphi}_2$ on $\partial F(\Omega)$, it follows that $\tilde{\varphi}_1 \le \tilde{u} \le \tilde{\varphi}_2$ in $F(\Omega)$. Therefore we obtain

$$\varphi_1 \le u \le \varphi_2 \quad \text{in } \Omega. \tag{5-8}$$

We observe that since $u = 0$ on $\partial\Omega$, we know $\nabla u(q_0)$ is orthogonal to $T_{q_0}\partial\Omega$, where $T_{q_0}\partial\Omega$ is the tangent space at q_0 for $\partial\Omega$, and we have the orthogonal decomposition $\mathrm{Span}\{\nabla u(q_0)\} \oplus T_{q_0}\partial\Omega = T_{q_0}\mathbb{H}^n$.

Let us set $\hat{w} := \nabla u(q_0)/|\nabla u(q_0)|$ and consider a curve $\psi : (-\delta, \delta) \to \mathbb{H}^n$ such that $\psi(0) = q_0$, $\psi(s) \in \Omega$ for $s \in (0, \delta)$ and $\psi'(0) = \hat{w}$ if $\hat{w}$ points towards the interior of Ω (otherwise we take $\psi'(0) = -\hat{w}$). Since Ω has a smooth boundary we can always find a curve satisfying these properties. From (5-8), and since $u(q_0) = \varphi_1(q_0) = \varphi_2(q_0) = 0$, we deduce that for all sufficiently small $h > 0$

$$\frac{\varphi_1(\psi(h)) - \varphi_1(\psi(0))}{h} \leq \frac{u(\psi(h)) - u(\psi(0))}{h} \leq \frac{\varphi_2(\psi(h)) - \varphi_2(\psi(0))}{h}. \tag{5-9}$$

Passing to the limit as $h \to 0^+$ we get

$$d\varphi_1(q_0)[\hat{w}] \leq du(q_0)[\hat{w}] \leq d\varphi_2(q_0)[\hat{w}]$$

(if $\hat{w}$ points in the opposite direction, (5-9) holds but with the reversed inequalities). Thus, it follows that

$$|du(q_0)[\hat{w}]| \leq \max\{|d\varphi_1(q_0)[\hat{w}]|, |d\varphi_2(q_0)[\hat{w}]|\}.$$

Since $\mathbb{H}^n$ a spacelike hypersurface, for any $q \in \mathbb{H}^n$ the Cauchy–Schwarz inequality holds in $T_q\mathbb{H}^n$ for $\langle \cdot, \cdot \rangle_{T_q\mathbb{H}^n}$ (we point out that, in general, the Cauchy–Schwarz inequality does not hold in $\mathbb{L}^{n+1}$; see [López 2013]). In particular $|d\varphi_i(q_0)[\hat{w}]| = |\langle \nabla\varphi_i(q_0), \hat{w}\rangle| \leq |\varphi_i(q_0)|\,|\hat{w}| = |\varphi_i(q_0)|$.

Hence, by the previous discussion and by Definition 4.4 we have

$$|\nabla u(q_0)| = |\langle \nabla u(q_0), \hat{w}\rangle| = |du(q_0)[\hat{w}]| \leq \max\{|d\varphi_1[\hat{w}], d\varphi_2[\hat{w}]\} \leq 1 - \theta.$$

Finally, since $\epsilon < \theta$ we get that $|\nabla u|$ satisfies the desired boundary estimate, and thus from the initial discussion, the proof of Step 6 is complete.

Step 7: Existence of a solution of problem (4-2). Thanks to Steps 5 and 6 it follows that the operator $T_\epsilon : C_0^{1,\beta}(\overline{\Omega}) \to C_0^{1,\beta}(\overline{\Omega})$ satisfies the hypotheses of the Leray–Schauder theorem, see [Gilbarg and Trudinger 1977, Theorem 11.3], and thus there exists $u \in C_0^{1,\beta}(\overline{\Omega})$ which solves $u = T_\epsilon u$. Hence, $u \in C_0^{2,\beta}(\overline{\Omega})$, and by the definition of $\widehat{H}$ and Lemma 4.11, u is a solution of problem (4-2). The proof of Step 7 is complete.

Step 8: Uniqueness.

For the uniqueness of the solution it is sufficient to argue as in [Caldiroli and Gullino 2013, Section 2.3]. For the sake of completeness we give a sketch of the proof.

Let us fix $\epsilon \in (0, 1)$ and let $u_1, u_2 \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be two solutions of problem (4-2) such that the corresponding radial graphs are contained in $\mathcal{C}_{\overline{\Omega}}(r_1, r_2)$. If $u_1 \neq u_2$ then there exists $\bar{q} \in \Omega$ such that $u_1(\bar{q}) \neq u_2(\bar{q})$. Without loss of generality we can assume that $u_1(\bar{q}) < u_2(\bar{q})$. Then there exists $\mu > 0$ such that $u_1(q) + \mu \geqslant u_2(q)$ for every $q \in \Omega$ and $u_1(q_0) + \mu = u_2(q_0)$ at some $q_0 \in \Omega$. Set $\bar{u}_1 := u_1 + \mu$ and observe that $\bar{u}_1$ satisfies

$$\sum_{i,j=1}^{n} ((1 - \eta_\epsilon^2|\nabla\bar{u}_1|^2)\delta_{ij} + \eta_\epsilon^2\bar{u}_{1i}\bar{u}_{1j})\bar{u}_{1ij} \leqslant n(1 - |\eta_\epsilon^2\nabla\bar{u}_1|^2)(1 - \sqrt{1 - \eta_\epsilon^2|\nabla\bar{u}_1|^2}e^{\bar{u}_1}\widehat{H}(e^{\bar{u}_1}q)) \quad \text{in } \Omega$$

because of (ii) and $\mu > 0$. Notice that the radial graph defined by $\bar{u}_1$ stays over (in the radial direction) that one corresponding to u_2 and they intersect at the point $X_0 = q_0 e^{u_2(q_0)}$. Now, in order to conclude, it sufficient to compare $\bar{u}_1$ and u_2 by means of the Hopf maximum principle. To this end we use the version stated in [Pucci and Serrin 2004, Theorem 2.3] for the operator

$$Q_\epsilon(u) = \sum_{i,j=1}^{n} ((1-\eta_\epsilon^2|\nabla u|^2)\delta_{ij} + u_i u_j)u_{ij} - n(1-\eta_\epsilon^2|\nabla u|^2)(1-\sqrt{1-\eta_\epsilon^2|\nabla u|^2}e^u \widehat{H}(e^u q)).$$

It is easy to see that, up to passing to hyperbolic stereographic coordinates, the assumptions of [Pucci and Serrin 2004, Theorem 2.3] are fulfilled, and applying the theorem as in [Caldiroli and Gullino 2013], we deduce that $\bar{u}_1 = u_2$ in Ω. But this gives a contradiction since $\bar{u}_1|_{\partial\Omega} = \mu > 0 = u_2|_{\partial\Omega}$. Hence it must be that $u_1 = u_2$ and we are done. □

6. An interior estimate for the gradient

In this section we prove an estimate for the gradient when the maximum point of its modulus lies in the interior of the domain Ω. We begin with a preliminary elementary result of linear algebra.

Lemma 6.1. *Let $A = (a_{ij})$, $B = (b_{ij}) \in \mathcal{M}_n(\mathbb{R})$ be two symmetric matrices. Assume that A is positive semidefinite and B is negative semidefinite. Then*

$$\sum_{i,j=1}^{n} a_{ij} b_{ij} \le 0.$$

Proof. Since A, B are symmetric we have $\sum_{i,j=1}^{n} a_{ij} b_{ij} = \operatorname{trace}(AB)$, and there exist two invertible matrices P, Q such that $P^{-1}AP = D_A$ and $Q^{-1}BQ = D_B$ are diagonal. Thanks to the assumptions we have that D_A has nonnegative elements on the diagonal, while D_B has nonpositive elements on the diagonal. Therefore, since the trace is invariant under similitude, and diagonal matrices commute in the product, we have

$$\operatorname{trace}(AB) = \operatorname{trace}(P^{-1}APP^{-1}BP) = \operatorname{trace}(D_A P^{-1}BP) = \operatorname{trace}(P^{-1}D_A BP) = \operatorname{trace}(D_A B).$$

Now, by the same argument we get

$$\operatorname{trace}(D_A B) = \operatorname{trace}(Q^{-1}D_A Q Q^{-1}BQ) = \operatorname{trace}(D_A Q^{-1}QD_B) = \operatorname{trace}(D_A D_B).$$

Therefore, $\sum_{i,j=1}^{n} a_{ij} b_{ij} = \operatorname{trace}(AB) = \operatorname{trace}(D_A D_B) \le 0$, and the proof is complete. □

Definition 6.2. Let $H \in C^1(\mathcal{C}_{\overline{\Omega}})$, and let $\nabla_0 H$ be gradient of H in $\mathbb{R}^{n+1}$ with respect to the flat metric. We define the (euclidean) tangential component of $\nabla_0 H$ on $T_{x/|x|}\mathbb{H}^n$ as the vector

$$\nabla_0^T H(x) := \nabla_0 H(x) - (\nabla_0 H(x), \hat{r}(x))_{n+1}\hat{r}(x), \quad x \in \mathcal{C}_{\overline{\Omega}},$$

where

$$\hat{r}(x) := \frac{(x_1, \dots, x_n, -x_{n+1})}{\|x\|_{n+1}}.$$

Remark 6.3. We point out that by definition $\nabla_0^T H(x) = \nabla_0 H(x) - \langle \nabla_0 H(x), x/\|x\|_{n+1}\rangle \hat{r}(x)$, and if $v \in \mathbb{R}^{n+1}$ is such that $(\hat{r}(x), v)_{n+1} = 0$ then $\langle x/|x|, v\rangle = 0$, and vice versa. In particular $(\hat{r}(x), v)_{n+1} = 0$ for any $v \in T_{x/|x|}\mathbb{H}^n$, $x \in \mathcal{C}_{\overline{\Omega}}$.

In the sequel we will make use also of the following formulas for the second and third covariant derivatives of a smooth function u defined over Ω, see [Yau 1975, Section 2]:

$$\sum_{j=1}^{n} u_{ij}\omega^j = du_i - \sum_{j=1}^{n} u_j\omega_{ji}, \tag{6-1}$$

$$\sum_{k=1}^{n} u_{ijk}\omega^k = du_{ij} - \sum_{k=1}^{n} u_{kj}\omega_{ki} - \sum_{k=1}^{n} u_{ik}\omega_{kj}. \tag{6-2}$$

Proposition 6.4. *Let Ω be a bounded domain of $\mathbb{H}^n$, let $H \in C^1(\mathcal{C}_{\overline{\Omega}})$, $\epsilon \in (0,1)$, and let $u \in C^3(\overline{\Omega})$ be a solution of*

$$\sum_{i,j=1}^{n} ((1-\eta_\epsilon^2|\nabla u|^2)\delta_{ij} + \eta_\epsilon^2 u_i u_j)u_{ij} = n(1-\eta_\epsilon^2|\nabla u|^2)(1-\sqrt{1-\eta_\epsilon^2|\nabla u|^2}e^u H(e^u q)). \tag{6-3}$$

Then, if the maximum point q_0 of $|\nabla u|$ lies in the interior of Ω, we have $|\nabla u(q_0)| = 0$ or

$$\Big[-(n-1) - n(1-\eta_\epsilon^2|\nabla u(q_0)|^2)^{1/2} e^{u(q_0)} \frac{\partial}{\partial\lambda}(\lambda H(\lambda q))\Big|_{\lambda=e^{u(q_0)}}\Big]|\nabla u(q_0)| \\ -n^{3/2}(1-\eta_\epsilon^2|\nabla u(q_0)|^2)^{1/2} e^{2u(q_0)} \|\nabla_0^T H(e^{u(q_0)}q_0)\|_{n+1} \le 0, \tag{6-4}$$

where $\nabla_0^T H$ is the (euclidean) tangential component of $(\nabla_0 H)(e^{u(q_0)}q_0)$ on $T_{q_0}\mathbb{H}^n$.

Proof. We will prove a more general version of (6-4). Let us fix a smooth positive function $f : \mathbb{R} \to \mathbb{R}^+$ and consider the auxiliary function $\varphi := f(2Cu)|\nabla u|^2$, where $C \in \mathbb{R}$ is a fixed constant. In order to simplify the notation we set $v := |\nabla u|^2$; hence $\varphi = f(2Cu)v$. Assume that φ has a maximum point at some q_0 lying in the interior of Ω. Hence $\nabla\varphi(q_0) = 0$ and the Hessian $(\varphi_{ij}(q_0))$ is negative semidefinite.

By direct computation we have $v_i = 2\sum_{j=1}^n u_j u_{ji}$ and from $\nabla\varphi(q_0) = 0$ we get

$$\sum_{h=1}^{n} f(2Cu)u_h u_{hi} + f'(2Cu)Cvu_i = 0 \quad \text{for all } i = 1, \dots, n, \tag{6-5}$$

which implies

$$\sum_{i,h=1}^{n} u_i u_{ih} u_h = -C\frac{f'}{f}v^2, \tag{6-6}$$

where, f, f' stand, respectively, for $f(2Cu)$, $f'(2Cu)$. By a simple computation, from (6-6), we get

$$\sum_{i,h,k=1}^{n} u_i u_{ih} u_{hk} u_k = \left(\frac{f'}{f}\right)^2 C^2 v^3. \tag{6-7}$$

Let us set

$$a_\epsilon^{ij} := ((1-\eta_\epsilon^2|\nabla u|^2)\delta_{ij} + \eta_\epsilon^2 u_i u_j),$$
$$b_\epsilon := n(1-\eta_\epsilon^2|\nabla u|^2)(1-\sqrt{1-\eta_\epsilon^2|\nabla u|^2}e^u H(e^u q)).$$

Since (a_ϵ^{ij}) is a positive definite symmetric matrix and $(\varphi_{ij}(q_0))$ is symmetric negative semidefinite, from Lemma 6.1 it follows that

$$\sum_{i,j=1}^{n} a_\epsilon^{ij}\varphi_{ij}(q_0) \le 0. \tag{6-8}$$

In order to get an estimate for $v = |\nabla u|^2$, the idea is to use (6-8). To this end we compute explicitly $\varphi_{ij}(q_0)$. Recalling that

$$\varphi_i = 2\biggl(\sum_{h=1}^{n} f(2Cu)u_h u_{hi} + f'(2Cu)Cvu_i\biggr),$$

and using (6-5), (6-1) we have

$$\sum_{j=1}^{n}\varphi_{ij}(q_0)\omega^j = 2\sum_{j=1}^{n}\biggl[\sum_{h=1}^{n}(2Cf'u_ju_hu_{hi}+fu_{hj}u_{hi} + fu_{hij}u_h)+2C^2f''u_ju_iv+Cf'u_{ij}v+\sum_{h=1}^{n}2Cf'u_iu_hu_{hj}\biggr]\omega^j,$$

and thus from (6-8) we infer that

$$\begin{aligned}2\biggl[\sum_{i,h=1}^{n}&\bigl(4Cf'(1-\eta_\epsilon^2v)u_iu_hu_{hi}+f(1-\eta_\epsilon^2v)u_{hi}^2+f(1-\eta_\epsilon^2v)u_{hii}u_h\bigr)\\ &+2C^2f''(1-\eta_\epsilon^2v)v^2+\sum_{i=1}^{n}Cf'(1-\eta_\epsilon^2v)u_{ii}v\\ &+\sum_{i,h=1}^{n}2Cf'\eta_\epsilon^2u_iu_hu_{hi}+\sum_{i,j,h=1}^{n}(f\eta_\epsilon^2u_iu_ju_{hj}u_{hi}+f\eta_\epsilon^2u_iu_ju_hu_{hij})\\ &+2C^2f''\eta v^3+\sum_{i,j=1}^{n}Cf'\eta_\epsilon^2u_iu_ju_{ij}v+\sum_{j,h=1}^{n}2Cf'\eta_\epsilon^2u_ju_hu_{hj}v\biggr]\le 0.\end{aligned} \tag{6-9}$$

Now we estimate and rewrite the terms involving the second and third covariant derivatives. We may choose a coordinate frame at q_0 satisfying $\delta_{1i}v^{1/2} = u_i$. If $v(q_0) = 0$, then $\max_{\overline{\Omega}}\varphi = \varphi(q_0) = 0$ and the thesis follows immediately. Otherwise in these coordinates, from (6-5), it follows that

$$u_{11} = -\frac{f'}{f}Cv, \tag{6-10}$$

which implies

$$\sum_{i,j=1}^{n} u_{ij}u_{ij} \ge \left(\frac{f'}{f}\right)^2 C^2v^2. \tag{6-11}$$

Since u is a solution of (6-3), computing at q_0 in these coordinates we have

$$\sum_{i=1}^{n}(1-\eta_\epsilon^2v)u_{ii} + \eta_\epsilon^2vu_{11} = b_\epsilon,$$

and from (6-10) we obtain

$$\sum_{i=1}^{n} u_{ii} = \left(b_\epsilon + C\frac{f'}{f}\eta_\epsilon^2 v^2\right)(1-\eta_\epsilon^2 v)^{-1}. \tag{6-12}$$

Recalling that $\eta_\epsilon(|\nabla u|) = \eta_\epsilon(v^{1/2})$, by direct computation we infer

$$\nabla_k(\eta_\epsilon^2 v) = \sum_{h=1}^{n}\left(2\eta_\epsilon\eta'_\epsilon v^{1/2} u_h u_{hk} + 2\eta_\epsilon^2 u_h u_{hk}\right), \tag{6-13}$$

where it is understood that η'_ϵ stands for $\eta'_\epsilon(v^{1/2})$. By differentiating (6-3), taking into account (6-1),(6-2) and (6-13), after some standard computations we deduce that

$$\sum_{k=1}^{n}\Bigg[-\sum_{i,h=1}^{n}(2\eta_\epsilon\eta'_\epsilon v^{1/2}u_h u_{hk}+2\eta_\epsilon^2 u_h u_{hk})u_{ii} + \sum_{i,j,h=1}^{n} 2\eta_\epsilon\eta'_\epsilon v^{-1/2}u_i u_j u_h u_{hk} u_{ij} + \sum_{i,j=1}^{n}\left(\eta_\epsilon^2 u_j u_{ik} u_{ij} + \eta_\epsilon^2 u_i u_{jk} u_{ij} + ((1-\eta_\epsilon^2 v)\delta_{ij} + \eta_\epsilon^2 u_i u_j)u_{ijk}\right)\Bigg]\omega^k = \sum_{k=1}^{n}(b_\epsilon)_k\omega^k.$$

Now, contracting the equation with u_k, we get

$$\begin{aligned}\sum_{i,h,k=1}^{n}&(-2\eta_\epsilon\eta'_\epsilon v^{1/2}u_h u_k u_{hk} u_{ii} - 2\eta_\epsilon^2 u_h u_k u_{hk} u_{ii}) \\ &+ \sum_{i,j,h,k=1}^{n} 2\eta_\epsilon\eta'_\epsilon v^{-1/2} u_i u_j u_k u_h u_{hk} u_{ij} \\ &\quad+ \sum_{i,j,k=1}^{n}(\eta_\epsilon^2 u_j u_k u_{ik} u_{ij} + \eta_\epsilon^2 u_i u_k u_{jk} u_{ij}) \\ &\qquad+ \sum_{i,k=1}^{n}(1-\eta_\epsilon^2 v)u_k u_{iik} + \sum_{i,j,k=1}^{n}\eta_\epsilon^2 u_i u_j u_k u_{ijk} = \sum_{k=1}^{n}(b_\epsilon)_k u_k.\end{aligned} \tag{6-14}$$

Since the Ricci curvature of the hyperbolic space is $R_{ij} = -(n-1)\delta_{ij}$, the Ricci formula, see formula (2.11) in [Yau 1975], gives

$$\sum_{k=1}^{n} u_k u_{kii} = \sum_{k=1}^{n} u_k u_{iik} - (n-1)v. \tag{6-15}$$

Hence, using (6-15), and taking into account (6-6), (6-7),(6-12), we rewrite (6-14) as

$$\begin{aligned}\sum_{i,k=1}^{n}(1-\eta_\epsilon^2 v)u_k u_{kii} + \sum_{i,j,k=1}^{n}\eta_\epsilon^2 u_i u_j u_k u_{ijk} &= -(n-1)v(1-\eta_\epsilon^2 v) - 2\eta_\epsilon\eta'_\epsilon C\frac{f'}{f}v^{5/2}(1-\eta_\epsilon^2 v)^{-1}\left(b_\epsilon + C\frac{f'}{f}\eta_\epsilon^2 v^2\right) \\ &\quad -2\eta_\epsilon^2 C v^2(1-\eta_\epsilon^2 v)^{-1}(b_\epsilon + C\eta_\epsilon^2 v^2) \\ &\quad -2\eta_\epsilon\eta'_\epsilon C^2\left(\frac{f'}{f}\right)^2 v^{7/2} - 2\eta_\epsilon^2 C^2\left(\frac{f'}{f}\right)^2 v^3 + \sum_{k=1}^{n}(b_\epsilon)_k u_k.\end{aligned} \tag{6-16}$$

Now, from (6-9) and by using (6-6), (6-7), (6-11), (6-12) (6-16), we deduce that

$$
\begin{aligned}
-4C^2(1-\eta_\epsilon^2 v)\frac{(f')^2}{f}v^2+C^2(1-\eta_\epsilon^2 v)\frac{(f')^2}{f}v^2 \\
+f\bigg[-(n-1)v(1-\eta_\epsilon^2 v)-2\eta_\epsilon\eta_\epsilon' C\frac{f'}{f}v^{5/2}(1-\eta_\epsilon^2 v)^{-1}\Big(b_\epsilon+C\frac{f'}{f}\eta_\epsilon^2 v^2\Big) \\
-2C\eta_\epsilon^2 v^2(1-\eta_\epsilon^2 v)^{-1}\Big(b_\epsilon+C\frac{f'}{f}\eta_\epsilon^2 v^2\Big) \\
-2C^2\eta_\epsilon\eta_\epsilon'\Big(\frac{f'}{f}\Big)^2 v^{7/2}-2C^2\Big(\frac{f'}{f}\Big)^2 v^3+\sum_{k=1}^n (b_\epsilon)_k u_k\bigg] \\
+2C^2 f''(1-\eta_\epsilon^2 v)v^2+Cf'v\Big(b_\epsilon+C\frac{f'}{f}\eta_\epsilon^2 v^2\Big) \\
-2C^2\frac{(f')^2}{f}\eta_\epsilon^2 v^2+C^2\frac{(f')^2}{f}\eta_\epsilon^2 v^3+2C^2 f''\eta_\epsilon^2 v^3 \\
-C^2\frac{(f')^2}{f}\eta_\epsilon^2 v^3-2C^2\frac{(f')^2}{f}\eta_\epsilon^2 v^3 \le 0. \quad (6\text{-}17)
\end{aligned}
$$

Now we compute the term $\sum_{k=1}^n (b_\epsilon)_k u_k$. To this end let us observe that

$$
\begin{aligned}
&(e^u H(e^u q))_k(q_0) \\
&\quad = e^{u(q_0)}u_k H(e^{u(q_0)}q_0)+e^{2u(q_0)}u_k(q_0)(\nabla_0 H)(e^{u(q_0)}q_0)\cdot q_0+e^{2u(q_0)}(\nabla_0 H)(e^{u(q_0)}q_0)\cdot e_k(q_0),
\end{aligned}
$$

where $\cdot$ denotes the standard euclidean product of $\mathbb{R}^{n+1}$ and $\nabla_0 H$ is the gradient of H with respect to the flat metric in $\mathbb{R}^{n+1}$. Then, after some computations and taking into account (6-13), we get

$$
\begin{aligned}
\sum_{k=1}^n (b_\epsilon)_k u_k(q_0) &= 2nC\frac{f'}{f}\eta_\epsilon\eta_\epsilon' v^{5/2}(1-\sqrt{1-\eta_\epsilon^2 v}\; e^{u(q_0)}H(e^{u(q_0)}q_0)) \\
&\quad + 2nC\frac{f'}{f}(\eta_\epsilon^2 v)v(1-\sqrt{1-\eta_\epsilon^2 v}\; e^{u(q_0)}H(e^{u(q_0)}q_0)) \\
&\quad - nC\frac{f'}{f}\eta_\epsilon\eta_\epsilon'\sqrt{1-\eta_\epsilon^2 v}(e^{u(q_0)}H(e^{u(q_0)}q_0))v^2 \\
&\quad - nC\frac{f'}{f}(\eta_\epsilon^2 v)(1-\eta_\epsilon^2 v)(e^{u(q_0)}H(e^{u(q_0)}q_0))v \\
&\quad - n(1-\eta_\epsilon^2 v)^{3/2}e^{u(q_0)}\big(e^{u(q_0)}H(e^{u(q_0)}q_0)+e^{2u(q_0)}\nabla_0 H(e^{u(q_0)}q_0)\cdot q_0\big)v \\
&\quad - n(1-\eta_\epsilon^2 v)^{3/2}e^{2u(q_0)}\nabla_0 H(e^{u(q_0)}q_0)\cdot\nabla u, \qquad (6\text{-}18)
\end{aligned}
$$

Computing $(\partial/\partial\lambda)(\lambda H(\lambda q_0))|_{\lambda=e^{u(q_0)}}$ and taking into account Remark 6.3, we rewrite the last two terms of (6-18) as

$$
-n(1-\eta_\epsilon^2 v)^{3/2}e^{u(q_0)}\frac{\partial}{\partial\lambda}(\lambda H(\lambda q_0))\Big|_{\lambda=e^{u(q_0)}}v-n(1-\eta_\epsilon^2 v)^{3/2}e^{2u(q_0)}\nabla_0^T H(e^{u(q_0)}q_0)\cdot\nabla u, \quad (6\text{-}19)
$$

where $\nabla_0^T H$ is the euclidean tangential component of $(\nabla_0 H)$ on $T_{q_0}\mathbb{H}^n$. Hence, from (6-17), (6-18), (6-19), regrouping and simplifying terms, we get

$$\begin{aligned}
&-2C^2(f')^2[1-\eta_\epsilon^2+\eta_\epsilon'(\eta_\epsilon v^{1/2})]v^3\\
&\quad -C^2[(f')^2(3-4\eta_\epsilon^2 v+(\eta_\epsilon^2 v))+2f'f(\eta_\epsilon^2 v)^2-2f''f(1-\eta_\epsilon^2 v)]v^2\\
&\quad\quad -nC(1-\eta_\epsilon^2 v)^{3/2}f'f\eta_\epsilon'(\eta_\epsilon v^{1/2})e^{u(q_0)}H(e^{u(q_0)}q_0)v^{3/2}\\
&\quad\quad\quad +\Big[-(n-1)f^2(1-\eta_\epsilon^2 v)^2-C^2(f')^2 f(\eta_\epsilon^2 v)(1-\eta_\epsilon^2 v)\\
&\quad\quad\quad\quad -nCff'(1-\eta_\epsilon^2 v)^2(\eta_\epsilon^2 v)e^{u(q_0)}H(e^{u(q_0)}q_0)\\
&\quad\quad\quad\quad\quad -nf^2(1-\eta_\epsilon^2 v)^{5/2}e^{u(q_0)}\frac{\partial}{\partial\lambda}(\lambda H(\lambda q))\Big|_{\lambda=e^{u(q_0)}}\Big]v\\
&\quad\quad\quad -nf^2(1-\eta_\epsilon^2 v)^{5/2}e^{2u(q_0)}\nabla_0^T H(e^{u(q_0)}q_0)\cdot\nabla u\\
&\quad\quad\quad\quad +nCf'f(1-\eta_\epsilon^2 v)^2(1-\sqrt{1-\eta_\epsilon^2 v}\; e^{u(q_0)}H(e^{u(q_0)}q_0))\le 0, \qquad (6\text{-}20)
\end{aligned}$$

and the proof of the general inequality is complete. Now we prove (6-4). Taking $f\equiv 1$, and dividing (6-20) by $(1-\eta_\epsilon^2 v)^2$, we get

$$\begin{aligned}
&\Big[-(n-1)-n(1-\eta_\epsilon^2 v)^{1/2}e^{u(q_0)}\frac{\partial}{\partial\lambda}(\lambda H(\lambda q))\Big|_{\lambda=e^{u(q_0)}}\Big]v\\
&\qquad\qquad -n(1-\eta_\epsilon^2 v)^{1/2}e^{2u(q_0)}\nabla_0^T H(e^{u(q_0)}q_0)\cdot\nabla u\ \le 0. \qquad (6\text{-}21)
\end{aligned}$$

Assume that $v(q_0)\neq 0$ (otherwise $v\equiv 0$ and there is nothing to prove). To conclude the proof it remains to estimate the term $\nabla_0^T H(e^{u(q_0)}q_0)\cdot\nabla u$. To this end, recalling the notation used in the proof of Lemma 4.1, we define $\tilde h\in\mathbb{R}^n$ as the vector whose i-th component is

$$\tilde h_i := \nabla_0^T H(e^{u(q_0)}q_0)\cdot\frac{\partial_i}{\|\partial_i\|_{n+1}},$$

$i=1,\dots,n$, where $\partial_i=(\partial\phi/\partial y_i)(F(q_0))$. Then, by construction and the Cauchy–Schwarz inequality we have

$$\|\tilde h\|_n^2=\sum_{i=1}^n\Big(\nabla_0^T H(e^{u(q_0)}q_0)\cdot\frac{\partial_i}{\|\partial_i\|_{n+1}}\Big)^2\le n\|\nabla_0^T H(e^{u(q_0)}q_0)\|_{n+1}^2. \qquad (6\text{-}22)$$

Now, exploiting (4-6) we have

$$\nabla_0^T H(e^{u(q_0)}q_0)\cdot\nabla u=\lambda^{-2}\sum_{i=1}^n\frac{\partial\tilde u}{\partial y_i}\nabla_0^T H(e^{u(q_0)}q_0)\cdot\partial_i=\lambda^{-1}\sum_{i=1}^n\frac{\partial\tilde u}{\partial y_i}\tilde h_i=\lambda^{-1}(\nabla_0\tilde u,\tilde h)_n,$$

and thus, from (4-7), (6-22) we deduce that

$$|\nabla_0^T H(e^{u(q_0)}q_0)\cdot\nabla u|=\lambda^{-1}|(\nabla_0\tilde u,\tilde h)_n|\le\lambda^{-1}\|\nabla_0\tilde u\|_n\|\tilde h\|_n\le\sqrt{n}|\nabla u|\|\nabla_0^T H(e^{u(q_0)}q_0)\|_{n+1}. \quad (6\text{-}23)$$

Finally, combining (6-21), (6-23) and dividing by $v^{1/2}$, we obtain (6-4). □

Remark 6.5. Applying the gradient estimate (6-4) to the solutions of $Q^t_\epsilon(u)=0$, we obtain

$$\Big[-(n-1)-nt(1-\eta_\epsilon^2|\nabla u(q_0)|^2)^{1/2}e^{u(q_0)}\frac{\partial}{\partial\lambda}(\lambda H(\lambda q))\Big|_{\lambda=e^{u(q_0)}}\Big]|\nabla u(q_0)| \\ -n^{3/2}t(1-\eta_\epsilon^2|\nabla u(q_0)|^2)^{1/2}e^{2u(q_0)}\|\nabla_0^T H(e^{u(q_0)}q_0)\|_{n+1}\le 0. \quad (6\text{-}24)$$

Hence, it is not possible, by using only this strategy, to get a uniform bound with respect to t for $|\nabla u|_\infty$ as in [Treibergs and Wei 1983]. In fact here we deal with functions defined on a manifold with negative Ricci curvature, and thus in (6-24) we have a term $-(n-1)$, while in [loc. cit.], for the sphere, this term has the opposite sign. We also point out that this trouble does not depend on the choice of the auxiliary function in the proof of Proposition 6.4, as shown by (6-20), where the leading term v^3 has a negative coefficient.

7. Proof of Theorem 1.5

Proof of Theorem 1.5. We first observe that by definition (Ω, H) is admissible with constant θ, and thus in the proof of Theorem 5.1 we can take $\bar\epsilon=\theta$. Therefore, for any $\epsilon\in(0,\theta)$, there exists a solution u_ϵ of the regularized problem (4-2). Let us choose $\epsilon\in(0,\theta)$ sufficiently close to θ so that

$$\frac{\partial}{\partial\lambda}(\lambda H(\lambda q)) < -\frac{1}{r_1(\epsilon-\epsilon^2/4)^{1/2}} \quad \text{for all } q\in\overline\Omega,\ \lambda\in[r_1,r_2], \quad (7\text{-}1)$$

$$\|\nabla_0^T H(X)\|_{n+1} < \frac{1-\epsilon}{n^{3/2}r_2^2}, \qquad X\in\mathcal{C}_{\overline\Omega}(r_1,r_2), \quad (7\text{-}2)$$

and let u be the solution of the regularized problem (4-2).

Let $q_0\in\overline\Omega$ be the maximum point of $|\nabla u|$, and set $v=|\nabla u(q_0)|^2$. There are only two possibilities: $v<(1-\epsilon)^2$ or $v\ge(1-\epsilon)^2$. In the first case there is nothing to prove; in fact, by definition of η_ϵ we have that u is a solution of problem (1-1) and we are done. Therefore let us assume that $v\ge(1-\epsilon)^2$. We point out that in this case q_0 cannot belong to $\partial\Omega$ because by Step 6 of the proof of Theorem 5.1 and since $\epsilon<\theta$, we have

$$\sup_{\partial\Omega}|\nabla u|^2\le(1-\theta)^2<(1-\epsilon)^2.$$

Hence $q_0\in\Omega$. We also observe that $u\in C^{3,\beta}(\overline\Omega)$, for some $\beta\in(0,\alpha]$. In fact, by Theorem 5.1 we know that $u\in C^{2,\beta}_0(\overline\Omega)$. Thanks to Lemma 4.11 we know that $\Sigma(u)$ is contained in $\mathcal{C}_{\overline\Omega}(r_1,r_2)$ and since $H\in C^{1,\alpha}(\mathcal{C}_{\overline\Omega}(r_1,r_2))$, $\partial\Omega\in C^{3,\alpha}$, by standard regularity results, see [Gilbarg and Trudinger 1977], we get $u\in C^{3,\beta}(\overline\Omega)$. Therefore, we can apply Proposition 6.4 and recalling that by definition

$$1-\eta_\epsilon^2 v = 1-\eta_\epsilon^2(|\nabla u(q_0)|)|\nabla u(q_0)|^2,$$

we have

$$\Big[-(n-1)-n(1-\eta_\epsilon^2 v)^{1/2}e^{u(q_0)}\frac{\partial}{\partial\lambda}(\lambda H(\lambda q))\Big|_{\lambda=e^{u(q_0)}}\Big]v^{1/2} \\ -n^{3/2}(1-\eta_\epsilon^2 v)^{1/2}e^{2u(q_0)}\|\nabla_0^T H(e^{u(q_0)}q_0)\|_{n+1}\le 0, \quad (7\text{-}3)$$

but on the other hand, since we are assuming that $v \geq (1-\epsilon)^2$, by the definition of η_ϵ we have $1 \geq 1-\eta_\epsilon^2 v \geq 1-\left(1-\frac{1}{2}\epsilon\right)^2$, and in view of (7-1), (7-2) we have

$$\begin{aligned}
\Big[-(n-1) - n(1-\eta_\epsilon^2 v)^{1/2} e^{u(q_0)} \frac{\partial}{\partial\lambda}(\lambda H(\lambda q))\Big|_{\lambda=e^{u(q_0)}}\Big] v^{1/2}& \\
-n^{3/2}(1-\eta_\epsilon^2 v)^{1/2} e^{2u(q_0)} \|\nabla^T H(e^{u(q_0)} q_0)\|_{n+1}& \\
> \left[-(n-1) + n\frac{e^{u(q_0)}}{r_1}\frac{(1-\eta_\epsilon^2 v)^{1/2}}{\left(\epsilon-\frac{1}{4}\epsilon^2\right)^{1/2}}\right] v^{1/2} - \frac{e^{2u(q_0)}}{r_2^2}(1-\epsilon)& \\
\geq \left[-(n-1) + n\frac{\left(\epsilon-\frac{1}{4}\epsilon^2\right)^{1/2}}{\left(\epsilon-\frac{1}{4}\epsilon^2\right)^{1/2}}\right] v^{1/2} - (1-\epsilon)& \\
\geq (1-\epsilon)-(1-\epsilon) = 0&
\end{aligned}$$

and thus we contradict (7-3). Therefore the only possibility is $v < (1-\epsilon)^2$, and by the definition of η_ϵ this means that u is a solution of problem (1-1). Moreover, as proved in Theorem 5.1, such a solution is the unique solution whose associated radial graph is contained in $\mathcal{C}_{\overline{\Omega}}(r_1, r_2)$, and this completes the proof. □

8. A finer gradient estimate

In this section we prove an a priori estimate for the gradient of the solutions of

$$\begin{cases} -\operatorname{div}_{\mathbb{H}^n}(\nabla u/\sqrt{1-|\nabla u|^2}) + nt/\sqrt{1-|\nabla u|^2} = nte^u H(e^u q) & \text{in } \Omega, \\ |\nabla u| < 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{8-1}$$

where $t \in [0,1]$. As in Section 4 we introduce the function $\nu = 1/\sqrt{1-|\nabla u|^2}$.

Proposition 8.1. *Let Ω be a bounded domain of $\mathbb{H}^n$, let $H \in C^1(\mathcal{C}_{\overline{\Omega}})$, let $r_1, r_2 \in \mathbb{R}$ be such that $r_1 \neq r_2$, $0 < r_1 \leq 1 \leq r_2$, and let $\nu_0 > 0$ be a positive number. Then, there exists a constant $C = C(r_1, r_2, \nu_0, \Omega, H) > 0$ such that for any $t \in [0,1]$, for any solution $u \in C^3(\overline{\Omega})$ of* (8-1) *satisfying $\log r_1 \leq u \leq \log r_2$ and $\sup_{\partial\Omega} \nu \leq \nu_0$, we have*

$$\sup_\Omega \nu \leq C.$$

Proof. Let $u \in C^3(\overline{\Omega})$ be a solution of (8-1) satisfying $\log r_1 \leq u \leq \log r_2$ and $\sup_{\partial\Omega} \nu \leq \nu_0$. Clearly $\nu \in C^0(\overline{\Omega})$ and we can introduce the differential operator $P_u : C^1(\overline{\Omega}) \to C^0(\overline{\Omega})$ defined by

$$P_u w := \nu \sum_{k=1}^n u_k w_k,$$

where u_k, w_k are the covariant derivatives with respect to a orthonormal frame field. Applying P_u to both sides of the equation in (8-1) and arguing as in Proposition 6.4 we deduce that ν satisfies the equation

$$\sum_{i,j=1}^n \nabla_i(\nu^{-2} f_{ij} \nu_j) + \nu^{-2}|\nabla\nu|^2 + |\langle\nabla u, \nabla\nu\rangle|^2 + \nu \sum_{i,j,k=1}^n f_{ij} u_{jk} u_{ik} + \sum_{i,j=1}^n \nu^2 R_{ij} u_i u_j + nt\nu\langle\nabla u, \nabla\nu\rangle = \nu nt \sum_{k=1}^n u_k \nabla_k(e^u H(e^u q)), \tag{8-2}$$

where $f_{ij} := \nu\delta_{ij} + \nu^3 u_i u_j$ and $R_{ij} = -(n-1)\delta_{ij}$ is the Ricci curvature tensor of $\mathbb{H}^n$, $i, j = 1, \dots, n$. This relation resembles that appearing in [Gerhardt 1983, (4.8)], and it can be proved by direct computation taking into account of the identities $\nu^{-2} = 1 - |\nabla u|^2$, $\nu_i = \nu^3 \sum_{l=1}^n u_l u_{li}$, $\langle \nabla u, \nabla \nu \rangle = \nu^3 \sum_{l,m=1}^n u_l u_m u_{lm}$, $|\nabla \nu|^2 = \nu^6 \sum_{i=1}^n \bigl(\sum_{l=1}^n u_l u_{li}\bigr)^2$, and [Yau 1975, (2.6)]. In order to estimate the terms appearing in (8-2) we first observe that

$$\sum_{i,j,k=1}^n f_{ij} u_{jk} u_{ik} = \sum_{i,j,k=1}^n (\nu\delta_{ij} + \nu^3 u_i u_j) u_{jk} u_{ik} = \nu \sum_{i,k=1}^n u_{ik}^2 + \nu^3 \sum_{k=1}^n \biggl(\sum_{i=1}^n u_i u_{ik} \biggr)^2 \geq \nu |D^2 u|^2, \tag{8-3}$$

where $|D^2 u|^2 := \sum_{i,k=1}^n u_{ik}^2$ is the square of the matrix norm of the Hessian. For the term $nt\langle \nabla u, \nabla \nu \rangle$, we write the equation in (8-1) in nondivergence form as

$$-\nu \Delta u - \langle \nabla u, \nabla \nu \rangle + nt\nu = nte^u H(e^u q). \tag{8-4}$$

Then, multiplying each side by $nt\nu$, recalling that $\nu \geq 1$, $e^u H(e^u q)$ is uniformly bounded with respect to t, and using the inequality $|\Delta u| \leq \sqrt{n} |D^2 u|$ we deduce that

$$|nt\nu \langle \nabla u, \nabla \nu \rangle| \leq c_1 \nu^2 (1 + |\Delta u|) \leq c_2 \nu^2 (1 + |D^2 u|)$$

for some constants $c_1, c_2 > 0$ depending on n, r_1, r_2 and $\|H\|_{\infty, \mathcal{C}_{\overline{\Omega}}}$, but not on t. From now on c_3, c_4, etc. will denote positive constants which do not depend on t. Now, if $|D^2 u| < c_2(1 + \sqrt{1 + 1/c_2})$, we get $|nt\nu \langle \nabla u, \nabla \nu \rangle| \leq c_3 \nu^2$, where c_3 depends just on c_2, and thus $nt\nu \langle \nabla u, \nabla \nu \rangle \geq -c_3 \nu^2$. On the other hand, if $|D^2 u| \geq c_2(1 + \sqrt{1 + 1/c_2})$, by an elementary computation we infer that

$$-c_2 \nu^2 (1 + |D^2 u|) + \tfrac{1}{2} \nu^2 |D^2 u|^2 \geq 0.$$

Hence, in view of (8-3) and the previous inequalities we obtain

$$\sum_{i,j,k=1}^n f_{ij} u_{jk} u_{ik} + nt\nu \langle \nabla u, \nabla \nu \rangle \geq -c_4 \nu^2 + \tfrac{1}{2} \nu^2 |D^2 u|^2. \tag{8-5}$$

Therefore, from (8-2), (8-5) we have

$$-\sum_{i,j=1}^n \nabla_i (\nu^{-2} f_{ij} \nu_j) + |\langle \nabla u, \nabla \nu \rangle|^2 + \tfrac{1}{2} \nu^2 |D^2 u|^2 \leq c_5 \nu^2 + \nu nt \sum_{k=1}^n u_k \nabla_k (e^u H(e^u q)). \tag{8-6}$$

Now, writing (8-4) as $-\nu \Delta u - \langle \nabla u, \nabla \nu \rangle = nte^u H(e^u q) - nt\nu$ and squaring, by using elementary inequalities we get

$$\nu^2 |\Delta u|^2 - 2\nu |\Delta u| |\langle \nabla u, \nabla \nu \rangle| + |\langle \nabla u, \nabla \nu \rangle|^2 \leq 2n^2 e^{2u} H^2(e^u q) + 2n^2 \nu^2. \tag{8-7}$$

Multiplying (8-4) by ν, and using $|\Delta u| \leq \sqrt{n} |D^2 u|$, we deduce that

$$\nu |\langle \nabla u, \nabla \nu \rangle| \leq c_5 \nu^2 (1 + |D^2 u|).$$

Hence, from this, using again $|\Delta u| \le \sqrt{n}|D^2u|$, and (8-7) we obtain

$$-n\nu^2|D^2u|^2 - 2\sqrt{n}c_5\nu^2(1+|D^2u|) + |\langle\nabla u, \nabla\nu\rangle|^2 \le 2n^2e^{2u}H^2(e^uq) + 2n^2\nu^2,$$

and thus by elementary computations we deduce that

$$-c_6\nu^2|D^2u|^2 + |\langle\nabla u, \nabla\nu\rangle|^2 \le 2n^2e^{2u}H^2(e^uq) + c_7\nu^2. \tag{8-8}$$

Therefore, dividing (8-8) by $C := 2c_6 + 1$ and summing with (8-6) we deduce

$$\begin{aligned}-\sum_{i,j=1}^n \nabla_i(\nu^{-2}f_{ij}\nu_j) + (1+2c_*)|\langle\nabla u, \nabla\nu\rangle|^2 + c_*\nu^2|D^2u|^2 \\ \le c_8\nu^2 + c_8e^{2u}H^2(e^uq) + \nu nt\sum_{k=1}^n u_k\nabla_k(e^uH(e^uq)),\end{aligned} \tag{8-9}$$

where $c_* = \frac{1}{2} - c_6/(2c_6+1) > 0$ does not depend on t. From (8-9), by arguing as in [Gerhardt 1983, Theorem 4.1], we can conclude the proof. In fact, using Stampacchia's truncation method (for the details see the Appendix in [loc. cit.]), multiplying (8-9) with

$$\psi_l := \nu\max\{\nu - l, 0\}, \quad l \ge \nu_0,$$

and integrating by parts we deduce

$$\sup_\Omega \nu \le \nu_0 + c_9(1+|\nu|^3_{2n,\Omega}), \tag{8-10}$$

where $c_9 > 0$ is a constant depending on n, Ω, r_1, r_2 and $\|H\|_{\infty,\mathcal{C}_{\overline{\Omega}}}$ but not on t, and $|\cdot|_{p,\Omega}$ denotes the standard L^p-norm. Therefore, in order to conclude the proof it suffices to prove a uniform estimate for the L^{2n}-norm of ν with respect to the parameter t. To this end, recalling that $\nu \ge 1$, and that $e^uH(e^u)$ is uniformly bounded by a constant depending only on $r_1, r_2, \|H\|_{\infty,\mathcal{C}_{\overline{\Omega}}}$, we can rewrite the right-hand side of (8-9) in a simpler way:

$$\begin{aligned}-\sum_{i,j=1}^n \nabla_i(\nu^{-2}f_{ij}\nu_j) + (1+2c_*)|\langle\nabla u, \nabla\nu\rangle|^2 + c_*\nu^2|D^2u|^2 \\ \le c_{10}\nu^2 + \nu nt\sum_{k=1}^n u_k\nabla_k(e^uH(e^uq)).\end{aligned} \tag{8-11}$$

Now, let $p \ge 2$ be any fixed real number, let $\lambda > 0$ be a real number to be chosen later and multiply (8-11) by

$$\rho_l := \nu_l^p e^{\lambda u},$$

where $\nu_l := \max\{\nu - l, 0\}$ and l is any fixed number such that $l \ge \nu_0$. Since $\nu_l^p e^{\lambda u} \in H_0^{1,q}(\Omega)$, for any $q \in [1, +\infty)$, we can integrate by parts and thus we obtain

$$\begin{aligned}p\sum_{i,j=1}^n \int_\Omega \nu^{-2}f_{ij}\nu_j\nu_i\nu_l^{p-1}e^{\lambda u} + \lambda\sum_{i,j=1}^n\int_\Omega \nu^{-2}f_{ij}\nu_ju_i\nu_l^pe^{\lambda u} \\ + (1+2c_*)\int_\Omega |\langle\nabla u, \nabla\nu\rangle|^2\nu_l^pe^{\lambda u} + c_*\int_\Omega \nu^2|D^2u|^2\nu_l^pe^{\lambda u} \\ \le c_{11}\int_\Omega \nu^2\nu_l^pe^{\lambda u} + c_{11}(p+1)\int_\Omega \nu\nu_l^{p-1}|\langle\nabla u, \nabla\nu\rangle|e^{\lambda u} + c_{11}\lambda\int_\Omega \nu\nu_l^p|D^2u|e^{\lambda u}.\end{aligned} \tag{8-12}$$

Now let us observe that

$$\sum_{i,j=1}^{n} f_{ij}\nu_i\nu_j = \sum_{i,j=1}^{n} (\delta_{ij}\nu + \nu^3 u_i u_j)\nu_i\nu_j = \nu|\nabla\nu|^2 + \nu^3|\langle\nabla u, \nabla\nu\rangle|^2 \ge \nu^3|\langle\nabla u, \nabla\nu\rangle|^2. \tag{8-13}$$

In addition, by direct computation we have

$$\begin{aligned}\lambda \sum_{i,j=1}^{n} \int_\Omega \nu^{-2} f_{ij}\nu_j u_i \nu_l^p e^{\lambda u} &= \lambda \sum_{i,j=1}^{n} \int_\Omega (1-|\nabla u|^2)\nu(\delta_{ij}\nu + \nu^3 u_i u_j)\nu_j u_i \nu_l^p e^{\lambda u} \\ &= \lambda \int_\Omega \langle\nabla u, \nabla\nu\rangle \nu\nu_l^p e^{\lambda u}.\end{aligned} \tag{8-14}$$

Furthermore, fixing a large constant C_1 and splitting the domains of the integrals into two parts $|\langle\nabla u, \nabla\nu\rangle| \le C_1$ and $|\langle\nabla u, \nabla\nu\rangle| > C_1$, by elementary computations it follows that for a suitable large constant $c_{12} > 0$ it holds that

$$\begin{aligned}c_* \int_\Omega \nu_l^p |\langle\nabla u, \nabla\nu\rangle|^2 e^{\lambda u} - c_{11}(p+1)\int_\Omega \nu\nu_l^{p-1}|\langle\nabla u, \nabla\nu\rangle| e^{\lambda u} \\ \ge -c_{12}\int_\Omega \nu_l^p e^{\lambda u} - c_{12}\int_\Omega \nu\nu_l^{p-1} e^{\lambda u}.\end{aligned} \tag{8-15}$$

Again by elementary considerations we obtain the further estimate

$$c_* \int_\Omega \nu^2|D^2u|^2\nu_l^p e^{\lambda u} - c_{11}\lambda \int_\Omega \nu|D^2u|\nu_l^p e^{\lambda u} \ge -c_{13}\int_\Omega \nu_l^p e^{\lambda u}. \tag{8-16}$$

Indeed, since it is always possible to find a constant $c_{11} > 0$ such that $c_* x^2 - c_{11}\lambda x + c_{13} > 0$ for all $x \ge 0$, then, taking $x = \nu|D^2u|$ we obtain the desired inequality. Therefore, from (8-12), and using the estimates (8-13)–(8-16), we deduce that

$$\begin{aligned}(1+p+c_*)&\int_\Omega |\langle\nabla u, \nabla\nu\rangle|^2 \nu_l^p e^{\lambda u} \\ &\le \lambda \int_\Omega |\langle\nabla u, \nabla\nu\rangle| \nu\nu_l^p e^{\lambda u} + c_{14}\int_\Omega \nu^2\nu_l^p e^{\lambda u} + \underbrace{c_{14}\int_\Omega \nu_l^p e^{\lambda u} + c_{14}\int_\Omega \nu\nu_l^{p-1} e^{\lambda u}}_{(I)}.\end{aligned} \tag{8-17}$$

Observe that (I) contains only powers of the form $\nu^a\nu_l^b$, with $a, b \ge 0$ such that $a + b \le p + 1$. From now on we will denote by I_1, I_2, etc. terms which are finite sums of integrals of the form $c\int_\Omega \nu^a\nu_l^b e^{\lambda u}$, where $a + b \le p + 1$, $a, b \ge 0$ and c is a constant which does not depend on t. The strategy to conclude the proof is to obtain an estimate of the kind

$$\int_\Omega \nu^2\nu_l^p e^{\lambda u} \le I. \tag{8-18}$$

To this aim, from (8-17), dividing each side by $(p + 1 + c_*)$ and using the elementary inequality $xy \le \frac{1}{2}x^2 + \frac{1}{2}y^2$, we obtain that

$$\int_\Omega |\langle\nabla u, \nabla\nu\rangle|^2 \nu_l^p e^{\lambda u} \le \frac{\lambda^2}{(1+p+c_*)^2}\int_\Omega \nu^2\nu_l^p e^{\lambda u} + \frac{2c_{12}}{1+p+c_*}\int_\Omega \nu^2\nu_l^p e^{\lambda u} + I_1. \tag{8-19}$$

Now, multiplying (8-4) by $\varphi = \nu\nu_l^p e^{\lambda u}$, integrating by parts, taking into account that $\nabla\varphi = \nabla\nu\nu_l^p e^{\lambda u} + p\nu\nabla\nu\nu_l^{p-1} e^{\lambda u} + \lambda\nu\nu_l^p \nabla u$, and $p \geq 2$, we get

$$\lambda \int_\Omega \nu^2 \nu_l^p |\nabla u|^2 e^{\lambda u} \leq c_{15} \int_\Omega \nu^2 \nu_l^p e^{\lambda u} + (p+1) \int_\Omega \nu^2 \nu_l^{p-1} |\langle \nabla u, \nabla \nu \rangle| e^{\lambda u} + I_2. \tag{8-20}$$

Now, choosing $\lambda > c_{15}$ and recalling that $\nu^{-2} = 1 - |\nabla u|^2$, from (8-20) we obtain

$$\lambda \int_\Omega \nu^2 \nu_l^p e^{\lambda u} \leq \frac{(p+1)^2}{(\lambda - c_{15})^2} \int_\Omega \nu_l^p |\langle \nabla u, \nabla \nu \rangle|^2 e^{\lambda u} + I_3. \tag{8-21}$$

From the combination of (8-19) and (8-21), for a large λ such that

$$\frac{\lambda^2 (p+1)^2}{(p+1+c_*)^2 (\lambda - c_{15})^2} + \frac{2c_{12}(p+1)^2}{(p+1+c_*)(\lambda - c_{15})^2} < 1$$

it follows that

$$\int_\Omega \nu_l^p |\langle \nabla u, \nabla \nu \rangle|^2 e^{\lambda u} \leq I_4,$$

and then, from this and (8-21), we conclude that

$$\int_\Omega \nu^2 \nu_l^p e^{\lambda u} \leq I_5,$$

which gives the desired inequality (8-18). Therefore, from (8-18) and the arbitrariness of p we deduce that $|\nu|_{2n,\Omega}$ is uniformly bounded in t and thus from (8-10) we deduce the thesis. □

9. Proofs of Theorems 1.3 and 1.4

The proofs of Theorem 1.3 and Theorem 1.4 are identical except for a small part and thus we give a unified proof in which at some point we distinguish between the two cases.

Proof. Let α, r_1, r_2, Ω and H be as in the statement of the theorem. Recalling the definition of the operators $\mathcal{Q}^t$, $\widehat{\mathcal{Q}}^t$ (see (4-13), (4-16)), by the same proof as that of Lemma 4.11 it follows that, for any $t \in [0,1]$, if $u \in C_0^2(\overline{\Omega})$ is a solution of $\widehat{\mathcal{Q}}^t(u) = 0$ and satisfies $|\nabla u|_{\infty,\Omega} < 1$ then

$$\log r_1 \leq u(q) \leq \log r_2 \quad \text{for any } q \in \overline{\Omega}. \tag{9-1}$$

Hence, by the definition of $\mathcal{Q}^t$, we have also a uniform bound with respect to t on the L^∞ norm of the solutions of $\mathcal{Q}^t(u) = 0$. In order to get a uniform bound on the gradient we use Proposition 8.1. To this end, in the case of Theorem 1.3 since Ω satisfies a uniform exterior geodesic condition and $H > 0$, thanks to Proposition 4.7 we have that (Ω, H) is admissible, and by arguing as in Step 6 of the proof of Theorem 5.1 we obtain that there exists $\theta \in (0,1)$ such that for any $t \in [0,1]$, if $u \in C_0^2(\overline{\Omega})$ is a solution of $\mathcal{Q}^t(u) = 0$ and satisfies $|\nabla u|_{\infty,\Omega} < 1$, then

$$|\nabla u(q)| \leq 1 - \theta \quad \text{for any } q \in \partial\Omega.$$

Indeed, if $|\nabla u|_{\infty,\Omega} < 1$ and $u \in C^1(\overline{\Omega})$, then, by the same proof as that of Lemma 4.1 we get that $\mathcal{Q}^t$ is uniformly elliptic in Ω (when passing to hyperbolic stereographic coordinates) and thus, thanks to the hypotheses on H, we can apply [Gilbarg and Trudinger 1977, Theorem 10.1] and argue as in Step 6 of the proof of Theorem 5.1. In the case of Theorem 1.4, the proof of this fact is identical and we use directly the hypothesis that (Ω, H) is admissible without invoking Proposition 4.7.

Since Ω is of class $C^{3,\alpha}$, $H \in C^{1,\alpha}(\mathcal{C}_{\overline{\Omega}}(r_1, r_2))$ and thanks to (9-1), by standard elliptic regularity theory, see [Gilbarg and Trudinger 1977], any solution $u \in C_0^{2,\alpha}(\overline{\Omega})$ of $\mathcal{Q}^t(u) = 0$ such that $|\nabla u| < 1$ in $\overline{\Omega}$ turns out to be of class $C^{3,\alpha}(\overline{\Omega})$. Hence, setting $\nu_0 := 1/\sqrt{1-\theta^2}$, by Proposition 8.1, it follows that there exists $\theta_* \in (0,1)$, depending only on $n, r_1, r_2, \nu_0, \Omega, H$ but not on t, such that for any solution $u \in C^3(\overline{\Omega})$ of $\mathcal{Q}^t(u) = 0$ satisfying $|\nabla u| < 1$ in Ω it holds that

$$|\nabla u(q)| \le 1 - \theta_* \quad \text{for any } q \in \overline{\Omega}. \tag{9-2}$$

Let us fix $\delta > 0$ sufficiently small so that $1 - \theta_* + \delta < 1$ and consider the set

$$U := \{w \in C_0^{1,\alpha}(\overline{\Omega}) : |\nabla w|_{\infty,\Omega} \le 1 - \theta_* + \delta\}.$$

Clearly U is a convex and closed subset of $C_0^{1,\alpha}(\overline{\Omega})$. We define the map $T : [0,1] \times U \to C_0^{1,\alpha}(\overline{\Omega})$, $T(t, w) := u$, where u is the unique solution of

$$\begin{cases} \sum_{i,j=1}^{n} ((1-|\nabla w|^2)\delta_{ij} + w_i w_j)u_{ij} = nt(1-|\nabla w|^2)(1-\sqrt{1-|\nabla w|^2}e^w \widehat{H}(e^w q)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We observe that T is well-defined. Indeed, for a fixed $w \in U$, considering the linear operator $L_w u := \sum_{i,j=1}^n ((1-|\nabla w|^2)\delta_{ij} + w_i w_j)u_{ij}$, and arguing as in Step 2 of the proof of Theorem 5.1, we see that $L_{w,\epsilon} : C_0^{2,\alpha}(\overline{\Omega}) \to C^{0,\alpha}(\overline{\Omega})$ is a bijection. Hence

$$T(w) = t L_w^{-1}\big(n(1-|\nabla w|^2)(1-\sqrt{1-|\nabla w|^2}e^w \widehat{H}(e^w q))\big)$$

is defined and we are done.

It is easy to verify that T is continuous and, arguing as in the proof of Step 5 of Theorem 5.1, we have that $T([0,1]\times U)$ is a relatively compact subset of $C_0^{1,\alpha}(\overline{\Omega})$. Moreover 0 lies in the interior of U and $T(0 \times \partial U) \subset U$. To conclude the proof it suffices to prove that if $(t,u) \in [0,1]\times U$ satisfies $T(t,u) = u$ then $u \notin \partial U$. Indeed, if $T(t,u) = u$ then $u \in C_0^{2,\alpha}(\overline{\Omega})$ is a solution of $\widehat{\mathcal{Q}}^t(u) = 0$ and thus from (9-1) we have $\mathcal{Q}^t(u) = 0$. Then, since $u \in U$ we have $|\nabla u|_{\infty,\Omega} \le 1 - \theta^* + \delta < 1$ and thus $\mathcal{Q}^t$ is uniformly elliptic. Therefore by elliptic regularity theory $u \in C^{3,\alpha}(\overline{\Omega})$ and thanks to (9-2) it follows that $|\nabla u|_{\infty,\Omega} \le 1 - \theta_* < 1 - \theta_* + \delta$, thus u cannot belong to ∂U and we are done.

Finally, from Theorem 2.8 we conclude that there exists $\bar{u} \in U$ which solves $T(1,\bar{u}) = \bar{u}$; i.e., $\bar{u}$ is a solution of (1-1). For the uniqueness it suffices to argue as in Step 8 of the proof of Theorem 5.1. □

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