ANALYSIS & PDE

Volume 12 No. 8 2019

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Analysis & PDE

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

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ANALYSIS AND PDE
Vol. 12, No. 8, 2019
dx.doi.org/10.2140/apde.2019.12.1891

TANGENT MEASURES OF ELLIPTIC MEASURE AND APPLICATIONS

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Tangent measure and blow-up methods are powerful tools for understanding the relationship between the infinitesimal structure of the boundary of a domain and the behavior of its harmonic measure. We introduce a method for studying tangent measures of elliptic measures in arbitrary domains associated with (possibly nonsymmetric) elliptic operators in divergence form whose coefficients have vanishing mean oscillation at the boundary. In this setting, we show the following for domains $\omega \subset \mathbb{R}^{n+1}$, $n \geq 2$:

(1) We extend the results of Kenig, Preiss, and Toro (*J. Amer. Math. Soc.* **22**:3 (2009), 771–796) by showing mutual absolute continuity of interior and exterior elliptic measures for *any* domains implies the tangent measures are a.e. flat and the elliptic measures have dimension n.

(2) We generalize the work of Kenig and Toro (*J. Reine Agnew. Math.* **596** (2006), 1–44) and show that VMO equivalence of doubling interior and exterior elliptic measures for general domains implies the tangent measures are always supported on the zero sets of elliptic polynomials.

(3) In a uniform domain that satisfies the capacity density condition and whose boundary is locally finite and has a.e. positive lower n-Hausdorff density, we show that if the elliptic measure is absolutely continuous with respect to n-Hausdorff measure then the boundary is rectifiable. This generalizes the work of Akman, Badger, Hofmann, and Martell (*Trans. Amer. Math. Soc.* **369:8** (2017), 5711–5745).

1. Introduction

1A. ***Background.*** In this paper, we study how the relationships between the elliptic measures of two complementary domains in $\mathbb{R}^{n+1}$, for $n \geq 2$, dictate the geometry of their common boundaries. We shall denote those domains by Ω^+ and Ω^- and the respective elliptic measures by ω^+ and ω^-. Bishop,

MSC2010: 31A15, 28A75, 28A78, 28A33.
Keywords: harmonic measure, elliptic measure, capacity density condition, Δ-regular domains, tangent measures, absolute continuity, rectifiability.

Carleson, Garnett and Jones [Bishop et al. 1989] showed that, for disjoint simply connected planar domains with mutually absolutely continuous harmonic measures, the boundary has tangents on a set of positive measure. Kenig, Preiss, and Toro [Kenig et al. 2009] showed that if $\Omega^{\pm}$ are both nontangentially accessible (or NTA) domains in $\mathbb{R}^{n+1}$, with $n \geq 2$, and the interior and exterior harmonic measures are mutually absolutely continuous, then at every point of the common boundary except for a set of harmonic measure zero, $\partial\Omega^+$ looks flatter and flatter as we zoom in. We will not define NTA but refer the reader to its inception in [Jerison and Kenig 1982]. Recently, the authors of the current paper, along with Tolsa [Azzam et al. 2017b], as well as with Tolsa and Volberg [Azzam et al. 2016c], showed that additionally the boundary is n-rectifiable in the sense that, off a set of harmonic measure zero, the boundary is a union of Lipschitz images of $\mathbb{R}^{n+1}$, and in fact Ω^+ and Ω^- need not be NTA but just connected.

These are, however, almost everywhere phenomena, so it is interesting to ask what assumptions we need on $\omega^{\pm}$ to guarantee some nice limiting behavior of our blow-ups at *every* point. Kenig and Toro [2006] showed that if Ω^+ is 2-sided NTA and $\log(d\omega^-/d\omega^+) \in \mathrm{VMO}(d\omega^+)$, then as we zoom in on any point of the boundary for a particular sequence of scales, $\partial\Omega^+$ begins to look more and more like the zero set of a harmonic polynomial (see Section 6 for the definition of VMO). In [Badger 2011], it is further shown that these harmonic polynomials are always homogeneous, and [Badger 2013] investigates the topological properties of sets where the boundary is approximated by zero sets of harmonic polynomials in this way.

To explain these results in more detail, we need to discuss what we mean by "blow-ups" and what it means for these to look like not necessarily one object but any one of a class of objects as we zoom in on harmonic measure. There are two ways we can consider this. Firstly, we can look at the Hausdorff convergence of rescaled copies of the support of a measure as we zoom in. To do this, we follow the framework of [Badger and Lewis 2015].

Definition 1.1. Let $A \subset \mathbb{R}^{n+1}$ be a set. For $x \in A$, $r > 0$, and $\mathscr{S}$ a collection of sets, define

$$\Theta_A^{\mathscr{S}}(x,r) = \inf_{S \in \mathscr{S}} \max\left\{ \sum_{a \in A \cap B(x,r)} \frac{\mathrm{dist}(a, x+S)}{r}, \sum_{z \in (x+S)\cap B(x,r)} \frac{\mathrm{dist}(z, A)}{r} \right\}.$$

We say $x \in A$ is a $\mathscr{S}$ *point* of A if $\lim_{r\to 0} \Theta_A^{\mathscr{S}}(x,r) = 0$. We say A is *locally bilaterally well approximated by* $\mathscr{S}$ (or simply LBWA($\mathscr{S}$)) if, for all $\varepsilon > 0$ and all compact sets $K \subset A$, there is $r_{\varepsilon,K} > 0$ such that $\Theta_A^{\mathscr{S}}(x,r) < \varepsilon$ for all $x \in K$ and $0 < r < r_{\varepsilon,K}$.

Thus, for $x \in A$ to be an $\mathscr{S}$-point means that, as we zoom in on A at the point x, the set A resembles more and more an element of $\mathscr{S}$ (though that element may change as we zoom in).

Secondly, we can look at the weak convergence of rescaled copies of the measure itself. To do this, we follow the framework of [Preiss 1987]. For $a \in \mathbb{R}^{n+1}$ and $r > 0$, set

$$T_{a,r}(x) = \frac{x-a}{r}.$$

Note that $T_{a,r}(B(a,r)) = B(0,1)$. Given a Radon measure μ, the notation $T_{a,r}[\mu]$ is the image measure of μ by $T_{a,r}$; that is,

$$T_{a,r}[\mu](A) = \mu(rA + a), \quad A \subset \mathbb{R}^{n+1}.$$

Here and later, for a function f and a measure μ, we write $f[\mu]$ to denote the push-forward measure $f[\mu](A) = \mu(f^{-1}(A))$.

Definition 1.2. We say that ν is a *tangent measure* of μ at a point $a \in \mathbb{R}^{n+1}$ if ν is a nonzero Radon measure on $\mathbb{R}^{n+1}$ and there are sequences $c_i > 0$ and $r_i \downarrow 0$ so that $c_i T_{a,r_i}[\mu]$ converges weakly to ν as $i \to \infty$ and write $\nu \in \operatorname{Tan}(\mu, a)$.

That is, ν is a tangent measure of μ at a point ξ if, as we zoom in on μ at ξ for a sequence of scales, the rescaled μ converges weakly to ν.

The collections of measures and sets that we will consider are associated to zero sets of harmonic functions. Let H denote the set of harmonic functions vanishing at the origin, $P(k)$ denote the set of harmonic polynomials h of degree k such that $h(0) = 0$ and $F(k)$ denote the set of homogeneous polynomials of degree k. For $h \in H$, we define

$$\Sigma_h = \{h=0\}, \quad \Omega_h = \{h>0\},$$

and

$$\mathscr{H} = \{\omega_h : h \in H\}, \quad \mathscr{P}(k) = \{\omega_h : h \in P(k)\}, \quad \mathscr{F}(k) = \{\omega_h : h \in F(k)\},$$

where

$$\omega_h = -\nu_{\Omega_h} \cdot \nabla h \, d\sigma_{\Sigma_h}.$$

Also set

$$\mathscr{P}_\Sigma(k) = \{\Sigma_h : h \in P(1) \cup \cdots \cup P(k)\}, \quad \mathscr{F}_\Sigma(k) = \{\Sigma_h : h \in F(k)\}$$

and

$$\mathcal{H}_\Sigma = \{\Sigma_h : h \in H\}.$$

Here $\nu_{\Omega_h}(x)$ stands for the measure-theoretic unit outward normal of Ω_h at $x \in \partial^*\Omega_h$, the reduced boundary of Ω_h. Now h is a harmonic function and thus, real analytic, which implies that Σ_h is an n-dimensional real analytic variety; hence, Ω_h is a set of locally finite perimeter and one can prove that $\mathcal{H}^n(\partial\Omega_h \setminus \partial^*\Omega_h) = 0$, where $\mathcal{H}^n$ stands for the n-Hausdorff measure. Notice now that $\nu_{\Omega_h}(x)$ is defined at $\mathcal{H}^n$-almost every point of Σ_h and σ_{Σ_h} is the usual surface measure. For a detailed proof of this see [Azzam et al. 2017b, p. 21].

In the rest of the paper we will be dealing with unbounded domains, i.e., open and connected sets in $\mathbb{R}^{n+1}$, with $n \geq 2$.

We summarize the best results to date. We first mention a result by the authors, Tolsa, and Volberg.

Theorem 1.3 [Azzam et al. 2016c; 2017b]. *Let $\Omega^\pm \subset \mathbb{R}^{n+1}$ be two disjoint domains and $\omega^\pm = \omega_{\Omega^\pm}^{x_\pm}$ for some $x_\pm \in \Omega^\pm$. If $\omega^\pm$ are mutually absolutely continuous on E, then for $\omega^\pm$-a.e. $\xi \in E$ we have* $\operatorname{Tan}(\omega^\pm, \xi) \subset \mathscr{F}(1)$ *and $\omega^+|_E$ can be covered up to a set of ω^+-measure zero by n-dimensional Lipschitz graphs. Furthermore, if $\partial\Omega^\pm$ are CDC, then* $\lim_{r\to 0} \Theta_{\partial\Omega^+}^{\mathscr{F}_\Sigma(1)}(\xi, r) = 0$ *for ω^+-a.e. $\xi \in E$.*

This was originally shown by Bishop, Carleson, Garnett, and Jones [Bishop et al. 1989] for simply connected planar domains. Later, Kenig, Preiss and Toro showed that, under the same assumptions, provided that the domain is also 2-sided locally NTA, it holds that $\dim \omega^+ = n$ (but not that ω^+ is rectifiable).

Below we summarize the results so far in the situation when Ω is 2-sided NTA and the interior and exterior harmonic measures are VMO equivalent, which brings together results and techniques from Badger [2011; 2013] and Kenig and Toro [2006].

Theorem 1.4. *Let $\Omega^+ \subset \mathbb{R}^{n+1}$ and $\Omega^- = \operatorname{ext}(\Omega^+)$ be NTA domains, and let $\omega^\pm$ be the harmonic measure in $\Omega^\pm$ with pole $x^\pm \in \Omega^\pm$. Assume that ω^+ and ω^- are mutually absolutely continuous and $f := d\omega^-/d\omega^+$ satisfies $\log f \in \mathrm{VMO}(d\omega^+)$. Then, there exists $d \in \mathbb{N}$ (depending on n and the NTA constants) such that the boundary $\partial\Omega^+$ is $\mathrm{LBWA}(\mathscr{P}_\Sigma(d))$ and may be decomposed into sets $\Gamma_1, \dots, \Gamma_d$ satisfying the following*:

(1) *For* $1 \le k \le d$, $\Gamma_k = \{\xi \in \partial\Omega^+ : \operatorname{Tan}(\omega^+, \xi) \subset \mathscr{F}(k)\}$.

(2) $\Gamma_1 \cup \cdots \cup \Gamma_d = \partial\Omega^+$.

(3) $\lim_{r\to 0} \Theta^{\mathscr{F}_\Sigma(k)}_{\partial\Omega^+}(\xi, r) = 0$ *for* $\xi \in \Gamma_k$.

The work of [Badger et al. 2017] studies the geometric structure of the set as well as the tangent measure structure using the conclusions of the results above. We refer to their work for more details.

1B. ***Blowups of elliptic measures.*** In this paper, our objective is to recreate some parts of these results for a class of elliptic measures. Admittedly, there are more results that could be generalized to this setting, like Tsirelson's theorem (using the method of [Tolsa and Volberg 2018]), but we content ourselves with the present results to convey the flexibility of the method.

Let $\Omega \subset \mathbb{R}^{n+1}$ be open and $A = A(\cdot) = (a_{ij}(\cdot))_{1\le i,j\le n+1}$ be a matrix with real measurable coefficients in Ω. We say that A is a *uniformly elliptic matrix* in Ω with constant $\Lambda \ge 1$ and write $A \in \mathscr{A}$ if it satisfies the following conditions:

$$\Lambda^{-1}|\xi|^2 \le \langle A(x)\xi, \xi\rangle \quad \text{for a.e. } x \in \Omega \text{ and for all } \xi \in \mathbb{R}^{n+1}, \tag{1-1}$$

$$\langle A(x)\xi, \eta\rangle \le \Lambda|\xi|\,|\eta| \quad \text{for a.e. } x \in \Omega \text{ and for all } \xi, \eta \in \mathbb{R}^{n+1}. \tag{1-2}$$

Notice that the matrix is *possibly nonsymmetric* and has variable coefficients. If $A \in \mathscr{A}$, we define a *uniformly elliptic operator* associated with A by

$$L_A = -\operatorname{div}(A(\cdot)\nabla).$$

We will let $\omega^{A,x}_\Omega$ denote the L_A*-harmonic measure* in Ω with pole at x (see Section 11 in [Heinonen et al. 1993] for the definition), which we also call *elliptic measure*. It is clear that the transpose matrix of A, which we denote by A^T, is also uniformly elliptic in Ω. Finally, a function $u : \Omega \to \mathbb{R}$ that satisfies the equation $L_A u = 0$ in the weak sense is called L_A*-harmonic*. We will denote by $\mathscr{C}$ the subclass of $\mathscr{A}$ consisting of matrices with constant entries.

To make sense of tangent measures of an elliptic measure at a point ξ in its support, we need to assume that the coefficients A do not oscillate too much there on small scales.

Definition 1.5. Let $\Omega \subset \mathbb{R}^{n+1}$ and let L_A be an elliptic operator on Ω. For a compact set $K \subset \partial\Omega$, we will say that the coefficients of L_A have *vanishing mean oscillation on K* with respect to Ω (or just

$L_A \in \mathrm{VMO}(\Omega, K)$) if

$$\lim_{r\to 0} \sup_{\xi\in K} \frac{1}{r^{n+1}} \inf_{C\in\mathscr{C}} \int_{B(\xi,r)\cap\Omega} |A(x) - C|\, dx = 0. \tag{1-3}$$

We also say the coefficients of L_A have VMO *at* $\xi \in \partial\Omega$ if

$$\lim_{r\to 0} \frac{1}{r^{n+1}} \inf_{C\in\mathscr{C}} \int_{B(\xi,r)\cap\Omega} |A(x) - C|\, dx = 0. \tag{1-4}$$

Much like the harmonic case, the tangent measures we will obtain are supported on zero sets of elliptic polynomials associated with an elliptic operator with constant coefficients. For a constant-coefficient matrix A with real entries, we will denote by H_A the set of L_A-harmonic functions u vanishing at zero, i.e., those functions u for which

$$\int A\nabla u \nabla\varphi\, dx = 0 \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^{n+1}) \qquad \text{and} \qquad u(0) = 0.$$

We also let $P_A(k)$ denote the set of L_A-harmonic polynomials of degree k vanishing at the origin, and $F_A(k) \subset P_A(k)$ the subset of homogeneous L_A-harmonic polynomials of degree k. When $A = I$, we will simply write $F(k)$, $P(k)$ and H in place of $F_A(k)$, $P_A(k)$ and H_A.

For $h \in H_A$, we will write

$$d\omega_h^A = -\nu_{\Omega_h} \cdot A\nabla h\, d\sigma_{\Sigma_h},$$

where σ_S stands for the surface measure on a surface S and ν is the outward normal vector at $x \in \partial^*\Omega_h$, the reduced boundary of Ω_h. Once more, we used that h is real analytic since A has constant coefficients and $L_A h = 0$; see, e.g., Proposition 11.3 in [Mitrea 2013]. Again, when A is the identity, we will drop the superscripts and, for example, write ω_h in place of ω_h^A. For $\mathscr{S} \subset \mathscr{C}$, we write

$$\mathscr{H}_{\mathscr{S}} = \{\omega_h^A : h \in H_A,\ A \in \mathscr{S}\}, \quad \mathscr{P}_{\mathscr{S}}(k) = \{\omega_h^A : h \in P_A(k),\ A \in \mathscr{S}\}, \quad \mathscr{F}_{\mathscr{S}}(k) = \{\omega_h^A : h \in F_A(k),\ A \in \mathscr{S}\},$$
$$\mathscr{H}_A = \mathscr{H}_{\{A\}}, \quad \mathscr{P}_A = \mathscr{P}_{\{A\}}, \quad \mathscr{F}_A = \mathscr{F}_{\{A\}},$$

and define $\mathscr{H}_{\mathscr{S},\Sigma}$, $\mathscr{P}_{\mathscr{S},\Sigma}$, and $\mathscr{F}_{\mathscr{S},\Sigma}$ as we did before. Observe that $\mathscr{F}_{\mathscr{C}}(1) = \mathscr{F}_A(1) = \mathscr{F}(1)$ for any $A \in \mathscr{C}$.

Our results also recover some LBWA properties implied in previous results if we consider domains satisfying the capacity density condition (CDC), whose complements also satisfy the CDC (see Definition 3.3 below) and whose associated elliptic measures are doubling. Examples of domains satisfying these conditions are NTA domains and, by [Martio 1979, Theorem 3.1], any uniform domain Ω for which there is $s > n - 1$ such that $\mathcal{H}_\infty^s(B(\xi, r) \cap \partial\Omega)/r^s \geq c > 0$ for all $\xi \in \partial\Omega$ and $r > 0$ is a CDC domain.

Our first result extends the work of [Kenig et al. 2009] to the elliptic case, and for domains beyond NTA. First, recall the dimension of a measure μ.

For a Borel measure μ in $\mathbb{R}^{n+1}$, we define the *Hausdorff dimension of* μ by

$$\dim(\mu) = \inf\{\dim(Z) : \mu(\mathbb{R}^{n+1} \setminus Z) = 0\}.$$

In practice, it is easier to compute this dimension as follows. Define *lower* and *upper pointwise dimension* at a point $x \in \operatorname{supp}\mu$ to be

$$\underline{d}_\mu(x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \quad \text{and} \quad \bar{d}_\mu(x) = \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

We call the common value $\underline{d}_\mu(x) = \bar{d}_\mu(x) = d_\mu(x)$, if it exists, the *pointwise dimension* of μ at $x \in \operatorname{supp}\mu$. It is shown in [Barreira and Wolf 2006, Proposition 3] that

$$\dim(\mu) = \operatorname{ess\,sup}\{\underline{d}_\mu(x) : x \in \Lambda\}.$$

Theorem I. *Let $\Omega^\pm \subset \mathbb{R}^{n+1}$ be two disjoint domains and let L_A be a uniformly elliptic operator on $\Omega^+ \cup \Omega^-$. Let also $\omega^\pm = \omega_{\Omega^\pm}^{L_A, x_\pm}$ for some $x_\pm \in \Omega^\pm$ be the L_A-harmonic measures in the respective domains and L_A be in $\mathrm{VMO}(\Omega^+ \cup \Omega^-, \xi)$ at ω^+-almost every $\xi \in E \subset \partial\Omega^+ \cap \partial\Omega^-$ with respect to either $\Omega^\pm$. If $\omega^\pm$ are mutually absolutely continuous on E, then for $\omega^\pm$-a.e. $\xi \in E$ we have $\operatorname{Tan}(\omega^\pm, \xi) \subset \mathscr{F}(1)$ and $\dim \omega^\pm|_E = n$. Furthermore, if $\partial\Omega^\pm$ are CDC, then $\lim_{r\to 0} \Theta_{\partial\Omega^+}^{\mathscr{F}_\Sigma(1)}(\xi, r) = 0$ for ω^+-a.e. $\xi \in E$.*

Kenig, Preiss, and Toro originally showed this if $\Omega^\pm$ were both NTA domains, and the dimension was computed by estimating the Hausdorff dimension directly from above and then using the monotonicity formula of Alt, Caffarelli, and Friedman [Alt et al. 1984] to estimate it from below. The latter is not available for L-harmonic functions when L satisfies the VMO condition above. For this reason, we use instead the fact that the tangent measures are all flat, which forces $\omega^\pm$ to decay like a planar n-dimensional Hausdorff measure on small scales.

Assuming a VMO condition on the interior and exterior elliptic measures, we can also obtain the results of [Kenig and Toro 2006] and [Badger 2011] for elliptic measures on domains that do not have to be NTA. We first state a pointwise version of these.

Theorem II. *Let Ω^+ be a domain in $\mathbb{R}^{n+1}$, let $\Omega^- := \operatorname{ext}(\Omega^+)$ be its exterior, and let L_A be a uniformly elliptic operator in $\Omega^+ \cup \Omega^-$. Denote by $\omega^\pm$ the L_A-harmonic measures of $\Omega^\pm$ with poles at some points $x^\pm \in \Omega^\pm$, and assume that $\omega^\pm$ are mutually absolutely continuous with $f = d\omega^-/d\omega^+$. If for a fixed $\xi \in \partial\Omega^+ \cap \partial\Omega^+$ it holds that $L_A \in \mathrm{VMO}(\Omega^+ \cup \Omega^-, \xi)$,*

$$\lim_{r\to 0} \left(\fint_{B(\xi,r)} f \, d\omega^+ \right) \exp\left(-\fint_{B(\xi,r)} \log f \, d\omega^+ \right) = 1, \tag{1-5}$$

and $\operatorname{Tan}(\omega^+, \xi) \neq \varnothing$, then $\operatorname{Tan}(\omega^+, \xi) \subset \mathscr{F}_{\mathscr{C}}(k)$ for some k and

$$\limsup_{r\to 0} \frac{\omega^+(B(\xi, 2r))}{\omega^+(B(\xi, r))} < \infty. \tag{1-6}$$

If $\Omega^\pm$ have the CDC, then additionally

$$\lim_{r\to 0} \Theta_{\partial\Omega^+}^{\mathscr{F}_{\mathscr{C},\Sigma}(k)}(\xi, r) = 0.$$

It is well known that $\operatorname{Tan}(\omega^+, \xi) \neq \varnothing$ whenever ω^+ satisfies the pointwise doubling condition (1-6). In our situation, however, we do not assume that, but we get it for free since $\mathscr{F}_{\mathscr{C}}(k)$ is compact (see [Badger 2011, Lemma 4.10] for the harmonic case and Theorem 2.4 below).

One might have guessed that a pointwise version of Theorem 1.4 would have assumed instead that

$$\lim_{r\to 0} \fint_{B(\xi,r)} \left| f - \fint_{B(\xi,r)} \log f \, d\omega^+ \right| d\omega^+ = 0,$$

but we were not able to show that this implies Theorem II. However, under certain conditions they are equivalent. We will discuss this matter in depth in Section 6 below.

Next, we state a global version.

Theorem III. *Let $\Omega^\pm \subset \mathbb{R}^{n+1}$ be two disjoint domains in $\mathbb{R}^{n+1}$ with common boundary, and let L_A be a uniformly elliptic operator in $\Omega^+ \cup \Omega^-$ such that $L_A \in \mathrm{VMO}(\Omega^+ \cup \Omega^-, \xi)$ at every $\xi \in \partial\Omega^+ \cap \partial\Omega^-$. Denote by $\omega^\pm$ the L_A-harmonic measures of $\Omega^\pm$ with poles at some points $x^\pm \in \Omega^\pm$. If ω^+ is C-doubling, $\omega^\pm$ are mutually absolutely continuous, and $\log f = \log(d\omega^-/d\omega^+) \in \mathrm{VMO}(d\omega^+)$, then there is d depending on n and the doubling constant so that, for every compact subset $K \subseteq \partial\Omega^+$,*

$$\lim_{r\to 0} \sup_{\xi\in K} d_1(T_{\xi,r}[\omega^+], \mathscr{P}_{\mathscr{C}}(d)) = 0. \tag{1-7}$$

If additionally $\Omega^\pm$ are CDC domains, then for any compact set $K \subseteq \partial\Omega$

$$\lim_{r\to 0} \sup_{\xi\in K} \Theta_{\partial\Omega^+}^{\mathscr{P}_{\mathscr{C},\Sigma}(d)}(\xi, r) = 0.$$

That is, $\partial\Omega^+ \in \mathrm{LBWA}(\mathscr{P}_{\mathscr{C},\Sigma}(d))$.

See Section 2 for the definition of $d_1(\,\cdot\,, \mathscr{P}_{\mathscr{C}}(d))$, which is essentially a distance between measures and the set $\mathscr{P}_{\mathscr{C}}(d)$.

The proof of Theorem II involves some useful lemmas about tangent measures that may be of independent interest. Specifically, we refer the reader to Lemma 2.10.

Over the course of working on this manuscript, we also resolved a question left open in [Badger 2011] (see the discussion on page 861 of that work).

Proposition I. *The d-cone $\mathscr{P}(k)$ has compact basis for each $k \in (0, n]$.*

See Section 2 for the definition of compact bases. A consequence of this result is that we can improve on the following theorem of Badger.

Theorem 1.6 [Badger 2011, Theorem 1.1]. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an NTA domain with harmonic measure ω and let $\xi \in \partial\Omega$. If $\mathrm{Tan}(\omega, \xi) \subset \mathscr{P}(d)$, then $\mathrm{Tan}(\omega, \xi) \subset \mathscr{F}(k)$ for some $k \le d$.*

In the proof of this result, Badger relied on the NTA assumption to conclude that $\mathrm{Tan}(\omega, \xi)$ was compact. By using Proposition I (whose proof is rather short), the compactness of $\mathscr{F}(k)$ (to which much of the proof of Theorem 1.6 is dedicated), and a connectivity theorem of Preiss, we can improve this by showing that, to get the same conclusion, no a priori information about the geometry of ω is needed; it need not have been a harmonic measure, let alone one for an NTA domain:

Proposition II. *Let ω be a Radon measure in $\mathbb{R}^{n+1}$ and $\xi \in \mathbb{R}^{n+1}$ such that $\mathrm{Tan}(\omega, \xi) \subset \mathscr{P}(k)$ for some integer k. If $\mathrm{Tan}(\omega, \xi) \cap \mathscr{F}(k) \neq \varnothing$ for some integer k, then $\mathrm{Tan}(\omega, \xi) \subset \mathscr{F}(k)$.*

1C. ***Rectifiability and elliptic measure for uniform domains.*** The blow-up arguments we use also have an application to studying the relationship between rectifiability and harmonic measure, a subject in which there have been a flurry of results in the last few years. For simply connected planar domains, the problem of when harmonic measure is absolutely continuous with respect to $\mathcal{H}^1$ is classical. Bishop and Jones [1990] showed that, if Ω is simply connected, $\omega_\Omega^x \ll \mathcal{H}^1$ on the subset of any Lipschitz curve intersecting $\partial\Omega$. Conversely, Pommerenke [1986] showed that if $\omega_\Omega \ll \mathcal{H}^1$ on a subset $E \subset \partial\Omega$, then that set can be covered by Lipschitz graphs up to a set of harmonic measure zero. In fact, a much earlier result of the Riesz brothers says that any Jordan domain has harmonic measure and is $\mathcal{H}^1$ mutually absolutely continuous if and only if the boundary is rectifiable; see [Riesz and Riesz 1920] or [Garnett and Marshall 2005, Chapter VI.1].

In higher dimensions, the problem is more delicate. There are some examples of simply connected domains $\Omega \subset \mathbb{R}^{n+1}$ with n-rectifiable boundaries of finite $\mathcal{H}^n$-measure so that either $\omega_\Omega \not\ll \mathcal{H}^n$ or $\mathcal{H}^n \not\ll \omega_\Omega$; see [Wu 1986; Ziemer 1974]. David and Jerison [1990] showed that mutual absolute continuity occurs for NTA domains with Ahlfors–David regular boundaries. Building on that, Badger [2012] showed that $\mathcal{H}^n \ll \omega_\Omega$ if Ω is an NTA domain whose boundary simply has locally finite $\mathcal{H}^n$-measure, although we showed with Tolsa that the converse relation $\omega_\Omega \ll \mathcal{H}^n$ could be false for such domains [Azzam et al. 2017c].

However, in [Azzam et al. 2016b], along with Hofmann, Martell, Mayboroda, Tolsa, and Volberg, we showed that for *any* domain $\Omega \subset \mathbb{R}^{n+1}$ and $E \subset \partial\Omega$ with $\omega_\Omega(E) > 0$ and $\mathcal{H}^n(E) < \infty$, if $\omega_\Omega \ll \mathcal{H}^n$ on E, then E may be covered up to ω_Ω-measure zero by Lipschitz graphs. By a theorem of Wolff, harmonic measure in the plane lies on a set of σ-finite $\mathcal{H}^1$-measure, and so the assumption that $\mathcal{H}^1(E) < \infty$ is unnecessary in this case (although very necessary in higher dimensions due to the existence of Wolff snowflakes). With Akman, we developed a converse for domains $\Omega \subset \mathbb{R}^{n+1}$ with *big complements*, meaning

$$\mathcal{H}^n_\infty(B(\xi, r)\backslash\Omega) \geq cr^n \quad \text{for all } \xi \in \partial\Omega \text{ and } 0 < r < \operatorname{diam} \partial\Omega. \tag{1-8}$$

We showed that, for such domains, $\omega_\Omega \ll \mathcal{H}^n$ on the subset of any n-dimensional Lipschitz graph [Akman et al. 2019], and hence, for these domains, we know that absolute continuity is equivalent to rectifiability of harmonic measure (versus rectifiability of the boundary).

There are fewer positive results concerning absolute continuity and rectifiability of *elliptic* measures. Even in the case of the half-plane, without some extra assumptions on the behavior of the elliptic coefficients, elliptic measure can be singular [Caffarelli et al. 1981; Sweezy 1992; Wu 1994], and some sort of Dini condition on the coefficients near the boundary is needed [Fabes et al. 1984; Fefferman et al. 1991]. For example, Kenig and Pipher [2001], considered the following condition.

Definition 1.7. Let $\delta(x) = \operatorname{dist}(x, \partial\Omega)$. We will say that an elliptic operator $L = -\operatorname{div} A\nabla$ satisfies the *Kenig–Pipher condition (or KP-condition)* if $A = (a_{ij}(x))$ is a uniformly elliptic real matrix that has distributional derivatives such that

$$\varepsilon^L_\Omega(z) := \sup\bigl\{\delta(x)|\nabla a_{ij}(x)|^2 : x \in \tfrac{1}{2}B(z, \delta(z)),\ 1 \leq i, j \leq n+1\bigr\} \tag{1-9}$$

is a Carleson measure in Ω, by which we mean that for all $x \in \partial\Omega$ and $r \in (0, \operatorname{diam} \partial\Omega)$,

$$\int_{B(x,r)\cap\Omega} \varepsilon_\Omega^L(z)\, dz \le C r^n.$$

In [Kenig and Pipher 2001], they showed that for Lipschitz domains in $\mathbb{R}^{n+1}$, elliptic operators satisfying the KP-condition give rise to elliptic measures which are A_∞-equivalent to surface measure. In fact, it was proved in [Hofmann et al. 2017] that the same result can be obtained under the following more general assumptions on the coefficients:

$$(\widetilde{\mathrm{KP}}) = \begin{cases} \nabla a_{ij} \in \mathrm{Lip}_{\mathrm{loc}}(\Omega), \\ \|\delta_\Omega |\nabla a_{ij}|\|_{L^\infty(\Omega)} < \infty, \\ \delta(x)|\nabla a_{ij}(x)|^2 \text{ is a Carleson measure} \end{cases} \tag{1-10}$$

for $1 \le i, j \le n+1$. Akman, Badger, Hofmann, and Martell observed in [Akman et al. 2017, Section 3.2] that, using the same arguments in [David and Jerison 1990], this result can be extended to NTA domains with Ahlfors–David regular boundaries. They used this fact to show that, on a *uniform domain* Ω (see Definition 8.1 below) with Ahlfors–David regular boundary, if L_A is a *symmetric* elliptic operator satisfying a local L^1 version of (1-9), i.e., $A \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ and $\sup\{|\nabla a_{ij}(x)| : x \in \frac{1}{2}B(z, \delta(z)),\ 1 \le i, j \le n+1\}$ is a Carleson measure with Carleson constant depending on the ball, then $\mathcal{H}^n \ll \omega_\Omega^L$ implies n-rectifiability of the boundary.

Using our blow-up arguments, we can obtain the following improvement.

Theorem IV. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a uniform CDC domain so that $\mathcal{H}^n|_{\partial\Omega}$ is locally finite. Let $\omega_\Omega^{L_A}$ be the L_A-harmonic measure associated to a (possibly nonsymmetric) elliptic operator satisfying* (1-1) *and* (1-2). *Let $E \subseteq \partial\Omega$ be a set with $\mathcal{H}^n(E) > 0$ such that $\mathcal{H}^n \ll \omega_\Omega^{L_A}$ on E and for $\mathcal{H}^n$-a.e. $\xi \in E$*

$$\theta^n_{\partial\Omega,*}(\xi, r) := \liminf_{r\to 0} \frac{\mathcal{H}^n(B(\xi, r)\cap\partial\Omega)}{(2r)^n} > 0$$

and A has vanishing mean oscillation at ξ. Then E is n-rectifiable.

Surprisingly, to get this improvement requires a very different set of techniques than originally considered in [Akman et al. 2017]. Let us point out that the argument therein uses the symmetry hypothesis on the coefficients in a significant way and does not seem easy to extend to the nonsymmetric case unless one additionally assumes that $\mathcal{H}^n \ll \omega_\Omega^{L_{A^T}}$.

Having VMO coefficients $\mathcal{H}^n$-a.e. on $\partial\Omega$ is natural as it is implied by the Carleson condition considered in [Akman et al. 2017; Kenig and Pipher 2001] by the following proposition:

Proposition III. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a uniform domain and suppose that A is an elliptic matrix satisfying* (1-1) *and* (1-2) *such that $A \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ and, for some ball B_0 centered on $\partial\Omega$,*

$$\int_{B_0} \delta(x)|\nabla a_{ij}(x)|^2\, dx < \infty. \tag{1-11}$$

Then $L_A \in \mathrm{VMO}(\Omega, \xi)$ for $\mathcal{H}^n$-a.e. $\xi \in B_0 \cap \partial\Omega$.

Discussion of related results. Near the completion of this work, we learned that Toro and Zhao [2017] simultaneously proved that $\mathcal{H}^n \ll \omega_\Omega$ implies rectifiability of the boundary if $\Omega \subseteq \mathbb{R}^{n+}$ is a uniform domain with Ahlfors–David n-regular boundary and the elliptic coefficients are in $W^{1,1}(\Omega)$. They also exploit the vanishing oscillation of the coefficients at almost every boundary point (which they show is implied by the $W^{1,1}$ condition) in the context of uniform domains, though, their proof is distinct by their use of pseudotangents and stopping-time arguments.

1D. ***Notation.*** We will write $a \lesssim b$ if there is $C > 0$ so that $a \leq Cb$ and $a \lesssim_t b$ if the constant C depends on the parameter t. We write $a \approx b$ to mean $a \lesssim b \lesssim a$ and define $a \approx_t b$ similarly.

2. Tangent measures

2A. ***Cones and compactness.*** Given two Radon measures μ and σ, we set

$$F_B(\mu, \sigma) = \sup_f \int f\, d(\mu - \sigma),$$

where the supremum is taken over all the nonnegative 1-Lipschitz functions supported on B. For $r > 0$, we write

$$F_r(\mu, \nu) = F_{B(0,r)}, \qquad F_r(\mu) = F_r(\mu, 0) = \int (r - |z|)_+\, d\mu.$$

A set of Radon measures $\mathscr{M}$ is a *d-cone* if $cT_{0,r}[\mu] \in \mathscr{M}$ for all $\mu \in \mathscr{M},\ c > 0$ and $r > 0$. We say a d-cone has *closed* (*resp. compact*) *basis* if its basis $\{\mu \in \mathscr{M} : F_1(\mu) = 1\}$ is closed (resp. compact) with respect to the weak topology.

For a d-cone $\mathscr{M},\ r > 0$, and μ a Radon measure with $0 < F_r(\mu) < \infty$, we define the *distance* between μ and $\mathscr{M}$ as

$$d_r(\mu, \mathscr{M}) = \inf\left\{ F_r\left(\frac{\mu}{F_r(\mu)}, \nu\right) : \nu \in \mathscr{M},\ F_r(\nu) = 1 \right\}.$$

Lemma 2.1 [Kenig et al. 2009, Section 2]. *Let μ be a Radon measure in $\mathbb{R}^{n+1}$ and $\mathscr{M}$ a d-cone. For $\xi \in \mathbb{R}^{n+1}$ and $r > 0$:*

(1) $T_{\xi,r}[\mu](B(0, s)) = \mu(B(\xi, sr))$.

(2) $\int f\, dT_{\xi,r}[\mu] = \int f \circ T_{\xi,r}\, d\mu$.

(3) $F_{B(\xi,r)}(\mu) = r F_1(T_{\xi,r}[\mu])$.

(4) $F_{B(\xi,r)}(\mu, \nu) = r F_1(T_{\xi,r}[\mu], T_{\xi,r}[\nu])$.

(5) *$\mu_i \to \mu$ weakly if and only if $F_r(\mu_i, \mu) \to 0$ for all $r > 0$.*

(6) $d_r(\mu, \mathscr{M}) \leq 1$.

(7) $d_r(\mu, \mathscr{M}) = d_1(T_{0,r}[\mu], \mathscr{M})$.

(8) *If $\mu_i \to \mu$ weakly and $F_r(\mu) > 0$, then $d_r(\mu_i, \mathscr{M}) \to d_r(\mu, \mathscr{M})$.*

Lemma 2.2 [Kenig et al. 2009, Remark 2.13]. *A d-cone $\mathscr{M}$ of Radon measures in $\mathbb{R}^{n+1}$ has a closed basis if and only if it is a relatively closed subset of the nonzero Radon measures in $\mathbb{R}^{n+1}$.*

Proof. One direction is obvious, so suppose $\mathscr{M}$ has closed basis and $\mu_i \in \mathscr{M}$ converges weakly to some nonzero Radon measure μ. Then $F_r(\mu) > 0$ for some $r > 0$. The set $\{\nu \in \mathscr{M} : F_1(\nu) = 1\}$ is closed by assumption, and since $\mathscr{M}$ is a d-cone, the set $\{\nu \in \mathscr{F} : F_r(\nu) = 1\}$ is also closed. Hence, since $\mu_i/F_r(\mu_i) \to \mu/F_r(\mu)$, we know $\mu/F_r(\mu) \in \mathscr{M}$, and thus $\mu \in \mathscr{M}$. □

Lemma 2.3. *If μ is a nonzero Radon measure and $\mathscr{M}$ is a d-cone with closed basis, then $\mu \in \mathscr{M}$ if and only if $d_r(\mu, \mathscr{M}) = 0$ for all $r > 0$ for which $F_r(\mu) > 0$.*

Proof. Suppose $d_r(\mu, \mathscr{M}) = 0$ for all $r > 0$ for which $F_r(\mu) > 0$. For $j \in \mathbb{N}$ large enough, we can find a sequence $\mu_{j,k} \in \mathscr{M}$ such that

$$F_j(\mu_{j,k}) = 1 \quad \text{and} \quad \lim_{k\to\infty} F_j\left(\frac{\mu}{F_j(\mu)}, \mu_{j,k}\right) = 0. \tag{2-1}$$

In particular, we can pass to a subsequence so that $\mu_{j,k}$ converges weakly in $B(0, j)$ to a measure μ_j supported in $B(0, j)$ with $F_j(\mu_j) = 1$. In view of (2-1), the latter implies $\mu = F_j(\mu)\mu_j$ in $B(0, j)$, and thus

$$F_j(\mu)\mu_j \rightharpoonup \mu.$$

Since $\mu_{j,k} \rightharpoonup \mu_j$ and $F_j(\mu) \neq 0$ for j large, we can pick k_j so that

$$F_j(\mu_{j,k_j}, \mu_j) < \frac{1}{jF_j(\mu)}.$$

In particular, for any $r > 0$ and $j > r$,

$$\begin{aligned} F_r(\mu_{j,k_j}F_j(\mu), \mu) &\le F_r(\mu_{j,k_j}F_j(\mu), \mu_j F_j(\mu)) + F_r(\mu_j F_j(\mu), \mu) \\ &\le F_j(\mu_{j,k_j}F_j(\mu), \mu_j F_j(\mu)) + F_r(\mu_j F_j(\mu), \mu) \\ &< \frac{1}{j} + F_r(\mu_j F_j(\mu), \mu) \to 0. \end{aligned}$$

Thus, $\mu_{j,k_j}F_j(\mu) \rightharpoonup \mu$. By Lemma 2.2, $\mathscr{M}$ is closed, and since we have $\mu_{j,k_j}F_j(\mu) \in \mathscr{M}$ for all j, this implies $\mu \in \mathscr{M}$. The other implication is trivial. □

Theorem 2.4 [Preiss 1987, Corollary 2.7]. *Let μ be a Radon measure on $\mathbb{R}^{n+1}$ and $\xi \in \operatorname{supp}\mu$. Then* $\operatorname{Tan}(\mu, \xi)$ *has compact basis if and only if*

$$\limsup_{r\to 0} \frac{\mu(B(\xi, 2r))}{\mu(B(\xi, r))} < \infty. \tag{2-2}$$

In this case, for any $\nu \in \operatorname{Tan}(\mu, \xi)$, it holds that $0 \in \operatorname{supp}\nu$ and

$$\frac{\nu(B(0, 2r))}{\nu(B(0, r))} \le \limsup_{\rho\to 0} \frac{\mu(B(\xi, 2\rho))}{\mu(B(\xi, \rho))} \quad \text{for all } r > 0.$$

Lemma 2.5 [Mattila 1995, Theorem 14.3]. *Let μ be a Radon measure on $\mathbb{R}^{n+1}$. If $\xi \in \mathbb{R}^{n+1}$ and* (2-2) *holds, then every sequence $r_i \downarrow 0$ contains a subsequence such that*

$$\frac{T_{\xi,r_j}[\mu]}{\mu(B(\xi, r_j))} \rightharpoonup \nu \tag{2-3}$$

for some measure $\nu \in \operatorname{Tan}(\mu, \xi)$.

Having tangent measures that arise as limits of the form (2-3) is very convenient, but this limit does not always converge weakly to something. This may happen if μ is not pointwise doubling at the point a. However, all tangent measures are at least dilations of tangent measures arising in this way.

Lemma 2.6 [Mattila 1995, Remark 14.4(1)]. *Let μ be a nonzero Radon measure, $\xi \in \operatorname{supp}\mu$, and $\nu \in \operatorname{Tan}(\mu, \xi)$. Then there are $\rho_j \downarrow 0$ and $\rho, c > 0$ so that*

$$\frac{T_{\xi,\rho_j}[\mu]}{\mu(B(\xi, \rho_j))} \rightharpoonup cT_{0,\rho}[\nu] \quad \textit{and} \quad cT_{0,\rho}[\nu](\mathbb{B}) > 0.$$

Proposition 2.7 [Preiss 1987, Proposition 2.2]. *Let $\mathscr{M}$ be a d-cone. Then $\mathscr{M}$ has compact basis if and only if for every $\lambda > 1$ there is $\tau > 1$ such that*

$$F_{\tau r}(\Psi) \leq \lambda F_r(\Psi) \quad \textit{for every } \Psi \in \mathscr{M} \textit{ and } r > 0. \tag{2-4}$$

In this case, $0 \in \operatorname{supp}\Psi$ for all $\Psi \in \mathscr{M}$.

Theorem 2.8 [Mattila 1995, Theorem 14.16]. *Let μ be a Radon measure on $\mathbb{R}^{n+1}$. For μ-almost every $x \in \mathbb{R}^{n+1}$, if $\nu \in \operatorname{Tan}(\mu, x)$, the following hold:*

(1) $T_{y,r}[\nu] \in \operatorname{Tan}(\mu, x)$ *for all* $y \in \operatorname{supp}\nu$ *and* $r > 0$.

(2) $\operatorname{Tan}(\nu, y) \subset \operatorname{Tan}(\mu, x)$ *for all* $y \in \operatorname{supp}\nu$.

Lemma 2.9 [Badger 2011, Lemma 2.6]. *Let μ be a nonzero Radon measure on $\mathbb{R}^{n+1}$ and $x \in \operatorname{supp}(\mu)$. If $\nu \in \operatorname{Tan}(\mu, x)$, then $\operatorname{Tan}(\nu, 0) \subset \operatorname{Tan}(\mu, x)$.*

2B. ***Connectivity of cones.*** The main tool from [Kenig et al. 2009; Badger 2011] is the following "connectivity" lemma, which was originally shown in [Kenig et al. 2009, Corollary 2.16] under the assumption that $\mathscr{M}$ had compact basis. For our purposes, we need to remove this assumption.

Lemma 2.10. *Let $\mathscr{F}$ and $\mathscr{M}$ be d-cones and assume $\mathscr{F}$ has compact basis. Furthermore, suppose that there is $\varepsilon_0 > 0$ such that for $\mu \in \mathscr{M}$, if there is $r_0 > 0$ so that $d_r(\mu, \mathscr{F}) \leq \varepsilon$ for all $r \geq r_0$, then $\mu \in \mathscr{F}$. For a Radon measure η and $x \in \operatorname{supp}\eta$, if $\operatorname{Tan}(\eta, x) \subset \mathscr{M}$ and $\operatorname{Tan}(\eta, x) \cap \mathscr{F} \neq \varnothing$, then $\operatorname{Tan}(\eta, x) \subset \mathscr{F}$.*

We will first require some lemmas.

Lemma 2.11. *Let $\mathscr{F}$ be a d-cone with compact basis. There is $\beta > 0$ depending only on $\mathscr{F}$ so that the following holds. Suppose ω is a Radon measure in $\mathbb{R}^{n+1}$, $\xi \in \operatorname{supp}\omega$, $\operatorname{Tan}(\omega, \xi) \cap \mathscr{F} \neq \varnothing$ and*

$$\limsup_{r\to 0} d_{r_0}(T_{\xi,r}[\omega], \mathscr{F}) \geq \varepsilon_0 > 0 \quad \textit{for some } r_0 > 0.$$

Then for $\varepsilon < \varepsilon_0$ small enough, we may find $\mu \in \operatorname{Tan}(\omega, \xi) \backslash \mathscr{F}$ so that

(1) $d_{r_0}(\mu, \mathscr{F}) = \varepsilon$,

(2) $d_r(\mu, \mathscr{F}) \leq \varepsilon$ *for all* $r > r_0$, *and*

(3) $\mu(B(0, r)) \leq r^\beta \mu(B(0, 4r_0))$ *for all* $r \geq r_0$.

This is an adaptation of the proof of [Kenig et al. 2009, Corollary 2.16], but with some extra care.

Proof. Without loss of generality, we will assume $r_0 = 1$. Let $c_j > 0$ and $r_j \downarrow 0$ be such that $c_j T_{\xi,r_j}[\omega] \to \nu \in \mathscr{F}$. Since $\mathscr{F}$ is compact, by Proposition 2.7, $0 \in \operatorname{supp}\nu$ and so $\nu(\mathbb{B}) > 0$. Thus, by Lemma 2.1(5), $c_j T_{\xi,r_j}[\omega](\mathbb{B}) > 0$ for j large. By Lemma 2.1(8), we have that, given $\varepsilon > 0$, for j large enough,

$$d_1(T_{\xi,r_j}[\omega], \mathscr{F}) = d_1(c_j T_{\xi,r_j}[\omega], \mathscr{F}) < \varepsilon. \tag{2-5}$$

Note that $0 \in \operatorname{supp} T_{\xi,r_j}[\omega]$ since $\xi \in \operatorname{supp}\omega$, and so there is no accidental dividing by zero in the definition of d_1. By assumption, there is also $s_j \downarrow 0$ so that

$$d_1(T_{\xi,s_j}[\omega], \mathscr{F}) > \varepsilon. \tag{2-6}$$

We can assume $s_j < r_j$ by passing to a subsequence. Then by (2-5) and (2-6), let $\rho_j \in (s_j, r_j)$ be the maximal number such that

$$d_1(T_{\xi,\rho_j}[\omega], \mathscr{F}) = \varepsilon. \tag{2-7}$$

Then, by the maximality of ρ_j,

$$\sup_{t\in[\rho_j,r_j]} d_1(T_{\xi,t}[\omega], \mathscr{F}) \le \varepsilon. \tag{2-8}$$

We claim $\rho_j/r_j \to 0$. If not, then since $\rho_j/r_j \le 1$, we may pass to a subsequence so that $\rho_j/r_j \to t \in (0, 1)$, and so

$$c_j T_{\xi,\rho_j}[\omega] = T_{0,\rho_j/r_j}[c_j T_{\xi,r_j}[\omega]] \to T_{0,t}[\nu] \in \mathscr{F},$$

which contradicts (2-7). Thus, $\rho_j/r_j \to 0$, and so (2-8) implies that for $\alpha \ge 1$, if j is large enough, we have $1 \le \alpha < r_j/\rho_j$. If $\omega_j = T_{\xi,\rho_j}[\omega]$, then by Lemma 2.1(7), it holds that

$$d_\alpha(\omega_j, \mathscr{F}) = d_\alpha(T_{\xi,\rho_j}[\omega], \mathscr{F}) = d_1(T_{\xi,\alpha\rho_j}[\omega], \mathscr{F}) \overset{(2\text{-}8)}{\le} \varepsilon, \tag{2-9}$$

which by (2-7) implies

$$d_1(\omega_j, \mathscr{F}) = \varepsilon > 0 \qquad \text{and} \qquad \limsup_{j\to\infty} d_r(\omega_j, \mathscr{F}) \le \varepsilon \quad \text{for } r > 1. \tag{2-10}$$

For $r \ge 1$, let $\mu_{j,r} \in \mathscr{F}$ be such that $F_{\tau r}(\mu_{j,r}) = 1$ and

$$F_{\tau r}\left(\frac{\omega_j}{F_{\tau r}(\omega_j)}, \mu_{j,r}\right) < \tfrac{3}{2} d_{\tau r}(\omega_j, \mathscr{F}).$$

By (2-10), for j large enough,

$$F_r\left(\frac{\omega_j}{F_{\tau r}(\omega_j)}, \mu_{j,r}\right) \le F_{\tau r}\left(\frac{\omega_j}{F_{\tau r}(\omega_j)}, \mu_{j,r}\right) < \tfrac{3}{2} d_{\tau r}(\omega_j, \mathscr{F}) < 2\varepsilon. \tag{2-11}$$

Since $\mathscr{F}$ has compact basis, by Proposition 2.7 with $\lambda = 2$, there is $\tau > 1$ depending only on $\mathscr{F}$ so that (2-4) holds for $\mathscr{M} = \mathscr{F}$. Thus, if $\varepsilon < \frac{1}{8}$, by the triangle inequality for F_r and (2-11),

$$\frac{F_r(\omega_j)}{F_{\tau r}(\omega_j)} \ge F_r(\mu_{j,r}) - 2\varepsilon \ge \tfrac{1}{2} F_{\tau r}(\mu_{j,r}) - 2\varepsilon = \tfrac{1}{2} - 2\varepsilon > \tfrac{1}{4}. \tag{2-12}$$

Hence, for any $r \ge 1$,

$$F_{\tau r}(\omega_j) \le 4 F_r(\omega_j).$$

Set $\mu_j = \omega_j / F_1(\omega_j)$. Then iterating the above inequality and letting $j \to \infty$, we get that for all $\ell \in \mathbb{N}$,

$$\limsup_{j\to\infty} F_{\tau^\ell}(\mu_j) \le 4^\ell.$$

This implies that we can pass to a subsequence so that μ_j converges weakly to a measure $\mu \in \operatorname{Tan}(\omega, \xi)$. In particular, for $r \ge 1$, since $F_1(\mu_j) = 1$, we may compute

$$d_1(\mu, \mathscr{F}) = \lim_{j\to\infty} d_1(\mu_j, \mathscr{F}) = \lim_{j\to\infty} d_1(\omega_j, \mathscr{F}) \overset{(2\text{-}10)}{=} \varepsilon,$$

$$d_r(\mu, \mathscr{F}) = \lim_{j\to\infty} d_r(\mu_j, \mathscr{F}) = \lim_{j\to\infty} d_r(\omega_j, \mathscr{F}) \overset{(2\text{-}10)}{\le} \varepsilon,$$

and

$$\tau^\ell \mu(B(0, \tau^\ell)) \le F_{2\tau^\ell}(\mu) \le 4^\ell F_2(\mu) \quad \text{for all } \ell \in \mathbb{N}. \tag{2-13}$$

Since $\tau > 1$, for any $r \ge 1$, there exists $\ell > 0$ such that $\tau^{\ell-1} < r \le \tau^\ell$. If $\tau \in (1, 4)$, then (2-13) implies

$$\tau^\ell \mu(B(0, \tau^\ell)) \le \tau^\alpha r^\alpha \mu(\overline{B(0, 2)}),$$

where $\alpha = 1/\log_4 \tau \in (1, \infty)$ and we used that $4^\ell = \tau^{\ell\alpha}$. Therefore,

$$\mu(B(0, r)) \le \tau^{\alpha-\ell} r^\alpha \mu(\overline{B(0, 2)}),$$

and notice that $\tau^{\alpha-\ell} \le 1$ whenever $\tau^\ell \ge 4$; i.e., the constant is independent of τ. In the case that $1 \le r \le \tau^\ell < 4$, we simply use that $B(0, r) \subset B(0, 4)$ to conclude that

$$\mu(B(0, r)) \le \mu(B(0, 4)).$$

If $\tau \ge 4$, then (2-13) trivially gives

$$\tau^\ell \mu(B(0, \tau^\ell)) \le 4^\ell \mu(\overline{B(0, 2)}) \le \tau^\ell \mu(\overline{B(0, 2)}),$$

which can only be true if $r \le \tau^\ell \le 2$. Thus, $B(0, r) \subset B(0, 2)$ and (3) readily follows. □

Corollary 2.12. *Let $\mathscr{F}$ be a d-cone with compact basis. There is $\beta > 0$ so that the following holds. Suppose μ is a Radon measure in $\mathbb{R}^{n+1}$ so that*

(1) $\operatorname{Tan}(\mu, \xi) \cap \mathscr{F} \neq \varnothing$ *and*

(2) $\operatorname{Tan}(\mu, \xi) \backslash \mathscr{F} \neq \varnothing$.

Then there is $r_0 > 0$ so that for any $\varepsilon > 0$ sufficiently small, the conclusion of Lemma 2.11 holds.

Proof. Let $\nu \in \operatorname{Tan}(\mu, \xi) \backslash \mathscr{F}$. By Lemma 2.3, there exists $r_0 > 0$ so that $F_{r_0}(\nu) > 0$ and $d_{r_0}(\nu, \mathscr{F}) > 0$. Let $c_j > 0$ and $r_j \downarrow 0$ be so that $c_j T_{\xi, r_j}[\mu] \to \nu$. Then, for j large enough, $d_{r_0}(T_{\xi, r_j}[\mu], \mathscr{F}) > \frac{1}{2} d_{r_0}(\nu, \mathscr{F}) > 0$. The corollary now follows from Lemma 2.11 with $\varepsilon_0 = \frac{1}{2} d_{r_0}(\nu, \mathscr{F})$. □

Proof of Lemma 2.10. If $\operatorname{Tan}(\eta, x) \backslash \mathscr{F} \neq \varnothing$, then, by Corollary 2.12, we may find $\mu \in \operatorname{Tan}(\eta, x) \setminus \mathscr{F}$ and $\varepsilon, r_0 > 0$ so that $d_{r_0}(\mu, \mathscr{F}) = \varepsilon$ and $d_r(\mu, \mathscr{F}) \le \varepsilon$ for all $r > r_0$. By assumption, this implies $\mu \in \mathscr{F}$, which is a contradiction. Thus, $\operatorname{Tan}(\eta, x) \subset \mathscr{F}$. □

3. Elliptic measures

3A. ***Uniformly elliptic operators in divergence form.*** Let A be a real matrix with measurable coefficients that satisfies (1-1) and (1-2). We consider the second-order elliptic operator $L = -\operatorname{div} A\nabla$ and we say that a function $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a *weak solution* of the equation $Lu = 0$ in Ω (or just *L-harmonic*) if

$$\int A\nabla u \cdot \nabla\varphi = 0 \quad \text{for all } \varphi \in C_0^\infty(\Omega). \tag{3-1}$$

We also say that $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a *supersolution* (resp. *subsolution*) for L in Ω or just *L-superharmonic* (resp. *L-subharmonic*) if $\int A\nabla u\nabla\varphi \geq 0$ (resp. $\int A\nabla u\nabla\varphi \leq 0$) for all nonnegative $\varphi \in C_0^\infty(\Omega)$.

In this section, we assume $n \geq 2$.

3B. ***Regularity of the domain and Dirichlet problem.*** We say that a point $x_0 \in \partial\Omega$ is *Sobolev L-regular* if, for each function $\varphi \in W^{1,2}(\Omega) \cap C(\overline{\Omega})$, the L-harmonic function h in Ω with $h - \varphi \in W_0^{1,2}(\Omega)$ satisfies

$$\lim_{x\to x_0} h(x) = \varphi(x_0).$$

Theorem 3.1 [Heinonen et al. 1993, Theorem 6.27]. *If for $x_0 \in \partial\Omega$ it holds that*

$$\int_0^1 \frac{\operatorname{cap}(B(x_0,r)\cap\Omega^c, B(x_0,2r))}{\operatorname{cap}(B(x_0,r), B(x_0,2r))}\,\frac{dr}{r} = +\infty,$$

then x_0 is Sobolev L-regular. Here $\operatorname{cap}(\,\cdot\,,\,\cdot\,)$ *stands for the variational* 2*-capacity of the condenser* $(\,\cdot\,,\,\cdot\,)$ (see, e.g., [Heinonen et al. 1993, p. 27]).

We say that a point $x_0 \in \partial\Omega$ is *Wiener regular* if, for each function $f \in C(\partial\Omega;\mathbb{R})$, the L-harmonic function H_f constructed by the Perron's method satisfies

$$\lim_{x\to x_0} H_f(x) = f(x_0).$$

See [Heinonen et al. 1993, Chapter 9].

Lemma 3.2 [Heinonen et al. 1993, Theorem 9.20]. *Suppose that $x_0 \in \partial\Omega$. If x_0 is Sobolev L-regular then it is also Wiener regular.*

The aforementioned result from [Heinonen et al. 1993] is only stated for Ω bounded but in fact it holds for unbounded domains, since the only part of the proof that requires the domain to be bounded is the existence of a unique solution of the Dirichlet problem with Sobolev Dirichlet data in bounded domains. This is true though in the unbounded case as well. See, e.g., on p. 11 in [Azzam et al. 2016a] where this is shown. Moreover, ∞ is also a Wiener regular point for each unbounded $\Omega \subset \mathbb{R}^{n+1}$, if $n \geq 2$; see, e.g., Theorem 9.22 in [Heinonen et al. 1993].

We say that Ω is Sobolev L-regular (resp. Wiener regular) if all the points in $\partial\Omega$ are Sobolev L-regular (resp. Wiener regular).

Definition 3.3. A domain $\Omega \subset \mathbb{R}^{n+1}$ is called *regular* if every point of $\partial\Omega$ is regular (i.e., if the classical Dirichlet problem is solvable in Ω for the elliptic operator $\mathcal{L}$), where $\partial\Omega$ denotes the boundary of Ω. For $K \subset \partial\Omega$, we say that Ω has the *capacity density condition* (*CDC*) if, for all $x \in \partial\Omega$ and $0 < r < \operatorname{diam} \partial\Omega$,

$$\operatorname{cap}(B(x,r) \cap \Omega^c, B(x,2r)) \gtrsim r^{n-1}.$$

Note that if $n \geq 2$, by Wiener's criterion, domains satisfying the CDC are both Wiener regular and L-Sobolev regular.

Let $\Omega \subset \mathbb{R}^{n+1}$ be Wiener regular and $x \in \Omega$. If $f \in C(\partial\Omega)$, then the map $f \mapsto \overline{H}_f(x)$ is a bounded linear functional on $C(\partial\Omega)$. Therefore, by the Riesz representation theorem and the maximum principle, there exists a probability measure ω^x on $\partial\Omega$ (associated to L and the point $x \in \Omega$) defined on Borel subsets of $\partial\Omega$ so that

$$\overline{H}_f(x) = \int_{\partial\Omega} f \, d\omega^x \quad \text{for all } x \in \Omega.$$

We call ω^x the *elliptic measure* or *L-harmonic measure* associated to L and x.

3C. *Green's function and PDE estimates.*

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an open, connected set so that $\partial\Omega$ is Sobolev L-regular. There exists a Green's function $G : \Omega \times \Omega \setminus \{(x,y) : x = y\} \to \mathbb{R}$ associated with L which satisfies the following. For $0 < a < 1$, there are positive constants C and c depending on a, n and Λ such that for all $x, y \in \Omega$ with $x \neq y$, it holds that*

$$0 \leq G(x,y) \leq C|x-y|^{1-n},$$
$$G(x,y) \geq c|x-y|^{1-n} \quad \text{if } |x-y| \leq a\delta_\Omega(x),$$
$$G(x,\cdot) \in C(\overline{\Omega} \setminus \{x\}) \cap W^{1,2}_{\text{loc}}(\Omega \setminus \{x\}) \quad \text{and} \quad G(x,\cdot)|_{\partial\Omega} \equiv 0,$$
$$G(x,y) = G^T(y,x),$$

where G^T is the Green's function associated with the operator L_{A^T}, and for every $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$

$$\int_{\partial\Omega} \varphi \, d\omega^x - \varphi(x) = -\int_\Omega A^T(y) \nabla_y G(x,y) \cdot \nabla\varphi(y) \, dy \quad \text{for a.e. } x \in \Omega. \tag{3-2}$$

In the statement of (3-2), one should understand that the integral on right-hand side is absolutely convergent for a.e. $x \in \Omega$ and a proof of it can be found in Lemma 2.6 in [Azzam et al. 2016a]. The rest were proved in [Grüter and Widman 1982; Hofmann and Kim 2007].

The lemma below is frequently called Bourgain's lemma, as he proved a similar estimate for harmonic measure in [Bourgain 1987].

Lemma 3.5 [Heinonen et al. 1993, Lemma 11.21]. *Let $\Omega \subset \mathbb{R}^{n+1}$ be any domain satisfying the CDC condition, $x_0 \in \partial\Omega$, and $r > 0$ so that $\Omega \setminus B(x_0, 2r) \neq \varnothing$. Then*

$$\omega_\Omega^{L,x}(B(x_0,2r)) \geq c > 0 \quad \text{for all } x \in \Omega \cap B(x_0,r), \tag{3-3}$$

where c depends on d and the constant in the CDC.

Lemma 3.6. *For $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, and the assumptions of Lemma 3.4, if B is centered on $\partial\Omega$, then*

$$G(x, y) r_B^{n-1} \inf_{z \in 2B} \omega^{L,z}(4B) \lesssim \omega^{L,y}(4B) \quad \text{for } x \in B \cap \Omega \text{ and } y \in \Omega \backslash 2B. \tag{3-4}$$

In particular, for a CDC domain, we have

$$G(x, y) r_B^{n-1} \lesssim \omega^{L,y}(4B) \quad \text{for } x \in B \cap \Omega \text{ and } y \in \Omega \backslash 2B.$$

Proof. This was originally shown for harmonic measure in [Azzam et al. 2016b], but we cover the details here.

By Bourgain's estimate, $\omega^{L,y}(4B) \gtrsim 1$ for $y \in 2B \cap \Omega$, and so for $y \in \Omega \backslash 2B$ and $x \in B \cap \Omega$

$$\inf_{z \in 2B} \omega^{L,z}(4B) G(x, y) r_B^{n-1} \lesssim \frac{\inf_{z \in 2B} \omega^{L,z}(4B)}{|x-y|^{n-1}} r_B^{n-1} \lesssim \inf_{z \in 2B} \omega^{L,z}(4B)$$

and since $G(x, \cdot)$ vanishes on $\partial\Omega$, we thus have that, for some constant $C > 0$,

$$\limsup_{y \to \xi} C\omega^{L,y}(4B) - \inf_{z \in 2B} \omega^{L,z}(4B) G(x, y) r_B^{n-1} \geq 0 \quad \text{for all } \xi \in \partial(\Omega \backslash 2B)$$

and so (3-4) follows from the maximum principle [Heinonen et al. 1993, Theorem 11.9]. □

By an iteration argument using Lemma 3.5, one can obtain the following lemma.

Lemma 3.7. *Let $\Omega \subsetneq \mathbb{R}^{n+1}$ be open with the CDC. Let $x \in \partial\Omega$ and $0 < r < \operatorname{diam}\Omega$. Let u be a nonnegative L-harmonic function in $B(x, 4r) \cap \Omega$ and continuous in $B(x, 4r) \cap \overline{\Omega}$ so that $u \equiv 0$ in $\partial\Omega \cap B(x, 4r)$. Then extending u by 0 in $B(x, 4r) \setminus \overline{\Omega}$, there exists a constant $\alpha > 0$ such that*

$$u(y) \leq C \left(\frac{\delta_\Omega(y)}{r} \right)^\alpha \sup_{B(x,2r)} u \quad \text{for all } y \in B(x, r), \tag{3-5}$$

where C and α depend on n, Λ and the CDC constant, and $\delta_\Omega(y) = \operatorname{dist}(y, \Omega^c)$. In particular, u is α-Hölder continuous in $B(x, r)$.

The following lemma is standard but we provide a proof for the sake of completeness.

Lemma 3.8. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, and assume that A is an elliptic matrix and $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a bi-Lipschitz map. Set*

$$\tilde{A} := |\det D_\Phi| D_{\Phi^{-1}} (A \circ \Phi) D_{\Phi^{-1}}^T.$$

Then u is a weak solution of $L_A u = 0$ in $\Phi(\Omega)$ if and only if $\tilde{u} = u \circ \Phi$ is a weak solution of $L_{\tilde{A}} \tilde{u} = 0$ in Ω.

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$ and $\varphi = \psi \circ \Phi$. Then by change of variables and the chain rule

$$\begin{aligned}
\int_{\Phi(\Omega)} A \nabla u \cdot \nabla \psi &= \int_\Omega (A \circ \Phi) \nabla u \circ \Phi \cdot \nabla \psi \circ \Phi |\det D_\Phi| \\
&= \int_\Omega (A \circ \Phi) D_{\Phi^{-1}}^T \nabla(u \circ \Phi) \cdot D_{\Phi^{-1}}^T \nabla(\psi \circ \Phi) |\det D_\Phi| \\
&= \int_\Omega |\det D_\Phi| D_{\Phi^{-1}} (A \circ \Phi) D_{\Phi^{-1}}^T \nabla(u \circ \Phi) \cdot \nabla(\psi \circ \Phi) = \int_\Omega \tilde{A} \nabla \tilde{u} \cdot \nabla \varphi.
\end{aligned}$$

The lemma readily follows. □

We will usually apply the above lemma when $\Phi(x) = Sx$ for some matrix S, in which case

$$\tilde{A} = (\det S) S^{-1}(A \circ S)(S^{-1})^T. \tag{3-6}$$

Lemma 3.9. *With the same assumptions as Lemma 3.8, and assuming Ω is a Wiener regular domain, we have that for any set $E \subset \Phi(\partial\Omega) = \partial\Phi(\Omega)$ and $x \in \Omega$*

$$\omega_{\Phi(\Omega)}^{L_A, \Phi(x)}(E) = \omega_{\Omega}^{L_{\tilde{A}}, x}(\Phi^{-1}(E)). \tag{3-7}$$

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$. Since the function

$$v(x) = \int \varphi \, d\omega_{\Phi(\Omega)}^{L,x}$$

is L_A-harmonic for $x \in \Phi(\Omega)$, by the previous lemma we know that the function

$$\tilde{v}(x) = \int \varphi \, d\omega_{\Phi(\Omega)}^{L,\Phi(x)}$$

is $L_{\tilde{A}}$-harmonic for $x \in \Omega$. If $\xi \in \partial\Omega$, then as $x \to \xi$ in Ω, $\Phi(x) \to \Phi(\xi)$ in $\Phi(\Omega)$, and so

$$\tilde{v}(x) = \int \varphi \, d\omega_{\Phi(\Omega)}^{L,\Phi(x)} \to \varphi(\Phi(\xi)).$$

Thus, $\tilde{v}$ is the $L_{\tilde{A}}$-harmonic extension of $(\varphi \circ \Phi)|_{\partial\Omega}$ to Ω, and so

$$\int_{\partial\Phi(\Omega)} \varphi \, d\omega_{\Phi(\Omega)}^{L_A,\Phi(x)} = \int_{\partial\Omega} \varphi \circ \Phi \, d\omega_{\Omega}^{L_{\tilde{A}},x} \quad \text{for all } x \in \Omega.$$

Since this holds for all such φ, we get that for any set $E \subset \partial\Phi(\Omega) = \Phi(\partial\Omega)$,

$$\omega_{\Phi(\Omega)}^{L_A,\Phi(x)}(E) = \omega_{\Omega}^{L_{\tilde{A}},x}(\Phi^{-1}(E)),$$

which gives the lemma. □

The following lemma will help us relate measures generated by elliptic polynomials to just measures generated by harmonic polynomials. In particular, if A is an elliptic matrix with constant and real coefficients, by the change of variables described below (which is just a linear transformation), if h is a harmonic polynomial solution in an open set Ω and $S = \sqrt{A_s}$ (where A_s is the symmetric part of A), then $\tilde{h} = h \circ S^{-1}$ is a polynomial solution of $-\operatorname{div} A\nabla u = 0$ in $S(\Omega)$. So, there is a bijection between the set of harmonic polynomials and the set of polynomial solutions of $-\operatorname{div} A\nabla u = 0$ in $S(\Omega)$ (for a fixed constant elliptic matrix A). Recall also that p is a harmonic polynomial in an open set if and only if it is a harmonic polynomial in $\mathbb{R}^{n+1}$. So, if A is as above, there is an abundance of nontrivial polynomial solutions of $-\operatorname{div} A\nabla u = 0$ in any open subset of $\mathbb{R}^{n+1}$ (including $\mathbb{R}^{n+1}$ itself). In fact, Theorem 2 in [Abramov and Petkovšek 2012] states that for such L_A, for any $k \in \mathbb{N}$, there exists a polynomial solution of $L_A h = 0$ of degree k.

Lemma 3.10. *Let A be an elliptic constant matrix, $A_s = \frac{1}{2}(A + A^T)$, and $S = \sqrt{A_s}$. Let $h \in H_A$ and $\tilde{h} = h \circ S$. Then $\tilde{A} = (\det S)I$, $\tilde{h} \in H$ and*

$$\omega_{\tilde{h}} = (\det S)^{-1} S^{-1}[\omega_h^A]. \tag{3-8}$$

Proof. Note that since L_A has constant coefficients, $L_{A_s} = L_A$ by the fact that for $u \in C^2$

$$L_A u = \sum_{i,j} a_{ij}\, \partial_i \partial_j u = \frac{1}{2}\sum_{i,j} a_{ij}\, \partial_i \partial_j u + \frac{1}{2}\sum_{i,j} a_{ij}\, \partial_j \partial_i u$$
$$= \sum_{i,j} \frac{(a_{ij}+a_{ji})}{2}\, \partial_i \partial_j u = L_{A_s} u.$$

Thus, if h is an L_A-harmonic function, it is also an L_{A_s}-harmonic function. Moreover, for any $\psi \in C_c^\infty(\mathbb{R}^{n+1})$

$$\int \psi\, d\omega_h^{A_s} = \int_{\Omega_h} h L_{A_s}(\psi) = \int_{\Omega_h} h L_A(\psi) = \int \psi\, d\omega_h^A.$$

In fact, without loss of generality, we may assume that $A = A_s$.

Recall now that since A_s is a symmetric, positive definite and invertible matrix with constant real entries, then it has a unique real symmetric positive definite square root $S = \sqrt{A_s}$ which is also invertible. Hence, by Lemma 3.8 and (3-6) with $A = A_s$, we have that $\tilde{A} = (\det S) I$ and $\tilde{h}$ is $L_{(\det S)I}$-harmonic, and thus just harmonic.

Let now $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$ and $\psi \circ S = \varphi$. By Green's formula and the fact that S is also symmetric, we have

$$(\det S) \int \varphi\, d\omega_{\tilde{h}} = (\det S) \int_{\Omega_{\tilde{h}}} \tilde{h}\Delta\varphi = -(\det S)\int_{\Omega_{\tilde{h}}} \nabla\tilde{h}\cdot\nabla\varphi$$
$$= -(\det S)\int_{\Omega_{\tilde{h}}} S^T\nabla h \circ S \cdot S^T \nabla\psi \circ S$$
$$= -\int_{S^{-1}(\Omega_h)} S S^T \nabla h \circ S \cdot \nabla\psi \circ S$$
$$= -\int_{\Omega_h} A_s \nabla h \cdot \nabla \psi = \int_{\Omega_h} h L_{A_s}(\psi)$$
$$= \int_{\Omega_h} h L_A(\psi) = \int \psi\, d\omega_h^A = \int \varphi\, dS^{-1}[\omega_h^A]. \qquad \square$$

Let us recall some simple facts from linear algebra which help us understand how the geometry of Ω is affected by the linear transformation above. Note that S is orthogonally diagonalizable since it is symmetric, which means that it represents a linear transformation with scaling in mutually perpendicular directions. Hence S^{-1} is a special bi-Lipschitz change of variables that takes balls to ellipsoids, where eigenvectors determine directions of semiaxes, eigenvalues determine lengths of semiaxes and its maximum eccentricity is given by $\sqrt{(\lambda_{\max}/\lambda_{\min})}$ (where $\lambda_{\max}$ are $\lambda_{\min}$ are the maximal and minimal eigenvalues of S^{-1}), which is in turn bounded below by $\sqrt{\Lambda}^{-1}$ and above by $\sqrt{\Lambda}$. In particular, $S^{-1}(\partial\Omega) = \partial(S^{-1}(\Omega))$, $\Lambda^{-1/2} \le \|S^{-1}\| \le \Lambda^{1/2}$; i.e., S^{-1} distorts distances by at most a constant depending on ellipticity.

3D. ***The main blow-up lemma.*** We now introduce the main tool of this paper, which is a variant of previous blow-up arguments, first introduced by Kenig and Toro [2006] for NTA domains, then extended to CDC domains in [Azzam et al. 2017b]. Both these cases apply to harmonic measure but can be extended to elliptic measures with a VMO condition on the coefficients.

Lemma 3.11. *Let $\Omega^+ \subset \mathbb{R}^{n+1}$ be a CDC domain, $K \subset \partial\Omega^+$ a compact set, $\xi_j \in K$ a sequence of points, and $L = -\operatorname{div} A\nabla$ a uniformly elliptic operator in Ω^+ such that*

$$\lim_{r\to 0} \sup_{\xi\in K} \frac{1}{r^{n+1}} \inf_{C\in\mathscr{C}} \int_{B(\xi,r)\cap\Omega^+} |A(x)-C|\,dx = 0. \tag{3-9}$$

Let ω^+ be the elliptic measure for Ω^+ and $c_j \geq 0$, and $r_j \to 0$ such that $\omega_j^+ = c_j T_{\xi_j,r_j}[\omega^+] \rightharpoonup \omega_\infty^+$ for some nonzero measure ω_∞^+. Let $\Omega_j^+ = T_{\xi_j,r_j}(\Omega^+)$. Then there is a subsequence and a closed set $\Sigma \subset \mathbb{R}^{n+1}$ such that:

(a) *For all $R > 0$ sufficiently large, $B(0,R) \cap \partial\Omega_j^+ \neq \varnothing$ and $\partial\Omega_j^+ \cap \overline{B(0,R)} \to \Sigma \cap \overline{B(0,R)}$ in the Hausdorff metric.*

(b) *$\Sigma^c = \Omega_\infty^+ \cup \Omega_\infty^-$, where Ω_∞^+ is a nonempty open set and Ω_∞^- is also open but possibly empty. Further, they satisfy that for any ball B with $\overline{B} \subset \Omega_\infty^\pm$, a neighborhood of $\overline{B}$ is contained in $\Omega_j^\pm$ for all j large enough.*

(c) $\operatorname{supp}\omega_\infty^+ \subset \Sigma$.

(d) *Let $u^+(x) = G_{\Omega^+}(x, x^+)$ on Ω^+ and $u^+(x) = 0$ on $(\Omega^+)^c$. Set*

$$u_j^+(x) = c_j u^+(xr_j + \xi_j) r_j^{n-1}.$$

Then u_j^+ converges locally uniformly in $\mathbb{R}^{n+1}$ and in $W^{1,2}_{\text{loc}}(\mathbb{R}^{n+1})$ to a nonzero function u_∞^+ which is continuous in $\mathbb{R}^{n+1}$, vanishes in $(\Omega_\infty^+)^c$, and satisfies

$$u_\infty^+(y) \lesssim \omega_\infty^+(\overline{B(x,4r)})r^{1-n} \tag{3-10}$$

for $x \in \Sigma$, $r > 0$, and $y \in B(x,r) \cap \Omega_\infty^+$. Moreover, there is A_0^+ a constant elliptic matrix so that if $L_0^+ = -\operatorname{div} A_0^+\nabla$, then

$$\int \varphi\, d\omega_\infty^+ = \int_{\mathbb{R}^{n+1}} u_\infty^+ L_0^+ \varphi \quad \textit{for any } \varphi \in C_c^\infty(\mathbb{R}^{n+1}). \tag{3-11}$$

Suppose now that $\Omega^- = \mathbb{R}^{n+1} \setminus \overline{\Omega}^+$, so that $\partial\Omega^+ = \partial\Omega^-$ and Ω^- is also connected and has the CDC. Define analogously ω_j^-, u^-, u_j^-, and u_∞^-. Assume that A is uniformly elliptic in $\Omega^+ \cup \Omega^-$, (3-9) *holds for $\Omega^+ \cup \Omega^-$ in place of Ω^+ and ω_j^- converges weakly to $\omega_\infty^- = c\omega_\infty^+$ for some number $c \in (0,\infty)$. Then $\Omega_\infty^- \neq \varnothing$ and for a suitable subsequence,* (d) *holds for u_j^-, u_∞^-, and Ω_∞^-. Furthermore, if we set $u_\infty = u_\infty^+ - c^{-1}u_\infty^-$, then*:

(e) *u_∞ extends to a continuous function on $\mathbb{R}^{n+1}$ which satisfies $L_0 u_\infty = 0$ in $\mathbb{R}^{n+1}$.*

(f) *$\Sigma = \{u_\infty = 0\}$, with $u_\infty > 0$ on Ω_∞^+ and $u_\infty < 0$ on Ω_∞^-. Further, Σ is a real analytic variety of dimension n.*

(g) *$d\omega_\infty^+ = -(\partial u_\infty/\partial\nu_{A_0})\, d\sigma_{\partial\Omega_\infty^+}$, where σ_S stands for the surface measure on a surface S and $\partial/\partial\nu_{A_0} = \nu \cdot A_0\nabla$ is the outward conormal derivative.*

Proof. The proof of this lemma can be found in [Azzam et al. 2017b] for harmonic measure for the case that $K = \{\xi\}$ (i.e., so that (1-4) holds). The proof for general K is essentially the same in this setting with minor changes. Here we shall only record the required modifications (some of which are quite substantial) for the $K=\{\xi\}$ case in order for the same proof to work for any elliptic measure as well. In this case, $\xi_j=\xi$ for all j. We set

$$A_j(x):=A(r_jx+\xi),\quad u_j^{\pm}(x):=c_jr_j^{n-1}u^{\pm}(r_jx+\xi),\quad \varphi_j(x):=\varphi\Big(\frac{x-\xi}{r_j}\Big).$$

Without loss of generality we can only work with u^+ since the results for u^- can be proved analogously.

Notice now that for j large enough, the pole x^+ is not in $\operatorname{supp}(\varphi_j)$. In fact, for any ball B centered at the boundary of Ω_j, we can find $j_0\in\mathbb{N}$ such that for all $j\geq j_0$, we have $x^+\notin T_{\xi,r_j}(B)$. Moreover, for $x\in B\cap\Omega_j$ and j large enough,

$$u_j^+(x)=c_jr_j^{n-1}u^+(r_jx+\xi)\overset{(3\text{-}4)}{\lesssim}c_jr_j^{n-1}(r_jr_B)^{1-n}\omega^+(4r_jB+\xi)=r_B^{1-n}\omega_j^+(4B). \tag{3-12}$$

Proof of (b): We only need to prove the existence of $B\subset\Omega_j^+$ for large $j\in\mathbb{N}$. Suppose there is no such ball. Let φ be any continuous compactly supported nonnegative function for which $\int\varphi\,d\omega_\infty^+\neq 0$, and let $M>0$ be so that $\operatorname{supp}\varphi\subset B(0,M)$. Thus, there must be $x_0\in B(0,M)\cap\operatorname{supp}\omega_\infty^+$. We set

$$\delta_j:=\sup\{\operatorname{dist}(x,(\Omega_j^+)^c):x\in B(0,2M)\},$$

which goes to zero by assumption. For $x\in B(0,2M)$ and $j\in\mathbb{N}$, let $\zeta_j(x)\in(\Omega_j^+)^c$ be closest to x so that $|x-\zeta_j(x)|\leq\delta_j\leq 2M$ (the second inequality holds because $0\in\partial\Omega_j^+$). It also holds that for all $x\in B(0,2M)$, we have $|x-x_0|\leq|x|+|x_0|<3M$.

Notice now that for any j big enough, u_j^+ is a solution in $B(0,2M)\cap\Omega_j^+$ and a subsolution in $B(0,2M)$. Moreover, if $x\in\Omega_j^+$, then $\zeta_j(x)\in\partial\Omega_j^+$. Thus, for j large, by Cauchy–Schwarz, Caccioppoli's inequality in $B(0,M)$ (which also holds for subsolutions), and the fact that u_j^+ and φ are supported in Ω_j^+ and $B(0,M)$ respectively,

$$\begin{aligned}
0 &< \int\varphi\,d\omega_j^+=\int_{\Omega_j^+}A_j\nabla u_j^+\cdot\nabla\varphi\lesssim_{\lambda,\Lambda,n,M}\|\nabla\varphi\|_\infty\Big(\int_{B(0,2M)}|u_j^+|^2\Big)^{1/2}\\
&\overset{(3\text{-}5)}{\lesssim}\Big(\int_{\Omega_j^+\cap B(0,2M)}\big(\sup_{B(\zeta_j(x),2M)}u_j^+\big)^2\Big(\frac{x-\zeta_j(x)}{2M}\Big)^{2\alpha}dx\Big)^{1/2}\\
&\overset{(3\text{-}12)}{\lesssim}\Big(\int_{\Omega_j^+\cap B(0,2M)}[\omega_j^+(B(\zeta_j(x),8M))(2M)^{1-n}]^2\,dx\Big)^{1/2}\Big(\frac{\delta_j}{2M}\Big)^{\alpha}\\
&\lesssim (2M)^{(n+1)/2}\omega_j^+(B(x_0,13M))(2M)^{1-n}\Big(\frac{\delta_j}{2M}\Big)^{\alpha},
\end{aligned}$$

and thus

$$\begin{aligned}
0<\int\varphi\,d\omega_\infty^+ &\lesssim_{\lambda,\Lambda,n,M,\varphi}\big(\limsup_{j\to\infty}\omega_j^+(B(x_0,13M))\big)\lim_j\delta_j^\alpha\\
&\leq\omega_\infty^+(\overline{B(x_0,13M)})\cdot 0=0,
\end{aligned}$$

which is a contradiction. Thus, there is $B\subset\Omega_j$ for all large j (after passing to a subsequence).

Proof of (d): Arguing as in [Azzam et al. 2017b], there exists u_∞^+ which is continuous in $\mathbb{R}^{n+1}$ and vanishes on $(\Omega_\infty^+)^c$ such that (after passing to a subsequence) $u_j^+ \to u_\infty^+$ uniformly on compact sets of $\mathbb{R}^{n+1}$. Moreover, it is not hard to see that $u_j^+ \in W^{1,2}(B)$ for large j. Indeed, by (3-12), it is clear that

$$\|u_j^+\|_{L^2(B)}^2 \lesssim r_B^{3-n}[\omega_j^+(4B)]^2, \tag{3-13}$$

while by Caccioppoli's inequality and (3-12),

$$\int_B |\nabla u_j^+|^2 \lesssim r_B^{-2}\int_B |u_j^+|^2 \lesssim r_B^{-2}[r_B^{1-n}\omega_j^+(4B)]^2 r_B^{n+1} = r_B^{1-n}[\omega_j^+(4B)]^2. \tag{3-14}$$

In view of (3-13) and (3-14) we have

$$\begin{aligned}\limsup_{j\to\infty}\|u_j^+\|_{W^{1,2}(B)} &\lesssim r_B^{(1-n)/2}(1+r_B)\limsup_{j\to\infty}\omega_j^+(4B)\\ &\leq r_B^{(1-n)/2}(1+r_B)\omega_\infty^+(\overline{4B}) < \infty.\end{aligned}$$

Therefore, by [Heinonen et al. 1993, Theorem 1.32], $u_\infty^+ \in W^{1,2}_{\rm loc}(\mathbb{R}^{n+1})$ and there exists a further subsequence of u_j^+ that converges weakly to u_∞^+ in $W^{1,2}_{\rm loc}(\mathbb{R}^{n+1})$.

Notice that

$$-\int_{\Omega_j^+} A_j\nabla u_j^+\cdot\nabla\varphi = \int \varphi\, d\omega_j^+.$$

Indeed, by a change of variables, and letting $\varphi_j = \varphi\circ T_{\xi,r_j}$ and $\varphi_j = \varphi\circ T_{\xi,r_j}$,

$$\begin{aligned}\int\varphi\, d\omega_j^+ = c_j\int\varphi_j\, d\omega^+ &= \int_{\Omega^+} A\nabla u^+\cdot\nabla\varphi_j\\ &= c_j r_j^n\int_{\Omega_j^+} A(r_jx+\xi)\nabla u^+(r_jx+\xi)\cdot\nabla\varphi(x)\, dx\\ &= \int_{\Omega_j^+} A_j\nabla u_j^+\cdot\nabla\varphi.\end{aligned}$$

Let $C_{j,k}$ be a constant elliptic matrix so that

$$\lim_j (kr_j)^{-1-n}\int_{B(\xi,kr_j)\cap\Omega^+}|A-C_{j,k}| = 0.$$

By a diagonalization argument and compactness, we may pass to a subsequence so that for each k, $C_{j,k}$ converges to a uniformly elliptic matrix C_k with constant coefficients. It is not hard to check that we must in fact have that $C_k = A_0^+$ for some fixed matrix A_0^+ (using the fact that $\inf\delta_j > 0$). Thus, we have

$$\lim_j (Mr_j)^{-1-n}\int_{B(\xi,Mr_j)\cap\Omega^+}|A-A_0^+| = 0 \quad \text{for all } M\geq 1. \tag{3-15}$$

To see the ellipticity of A_0^+ is pretty easy but we show the details for completeness. Note that since A is uniformly elliptic for a.e. $x\in\Omega^+$, for $\xi\in\mathbb{R}^{n+1}$

$$\Lambda^{-1}|\xi|^2 \leq A(x)\xi\cdot\xi = (A(x)-A_0^+)\xi\cdot\xi + A_0^+\xi\cdot\xi.$$

Then, if we take averages over $B(\xi, Mr_j)\cap\Omega$, use the existence of corkscrew balls in Ω_j for large j proved in (b), and then take limits as $j\to\infty$, by (3-15) we have

$$\Lambda^{-1}|\xi|^2 \le A_0^+\xi\cdot\xi.$$

The upper bound follows by a similar argument and the proof is omitted.

We will now estimate the difference

$$\int_{\Omega_j^+} A_j\nabla u_j^+\cdot\nabla\varphi - \int_{\Omega_\infty^+} A_0^+\nabla u_\infty^+\cdot\nabla\varphi \tag{3-16}$$

for sufficiently large j.

To this end, let $\operatorname{supp}(\varphi)\subset B(0,M)$. Note that

$$|(3\text{-}16)| \le \left|\int_{\Omega_j^+}(A(r_jx+\xi)-A_0^+)\nabla u_j^+\cdot\nabla\varphi\right| + \left|\int_{B(0,M)}(\nabla u_j^+\mathbb{1}_{\Omega_j}-\nabla u_\infty^+\mathbb{1}_{\Omega_\infty})\cdot A_0^{+,T}\nabla\varphi\right| \le I_1+I_2.$$

Note that $u_j^+, u_\infty^+\in W^{1,2}(\mathbb{R}^{n+1})$, $u_j^+>0$ only in Ω_j^+, and $u_\infty^+>0$ only in Ω_∞^+. Since the extension of the gradient of a function $f\in W_0^{1,2}(\Omega)$ by zero to $\mathbb{R}^{n+1}$ (where Ω is any domain) is the same as the gradient of the extension of f by zero,[1] we have that in $W^{1,2}(B(0,M))$

$$\nabla u_j^+\mathbb{1}_{\Omega_j^+} = \nabla(u_j^+\mathbb{1}_{\Omega_j^+}) = \nabla u_j^+ \rightharpoonup \nabla u_\infty^+ = \nabla(u_\infty^+\mathbb{1}_{\Omega_\infty^+}) = \nabla u_\infty^+\mathbb{1}_{\Omega_\infty^+},$$

so we have $I_2\to 0$. On the other hand, since A and $A_0^+\in L^\infty(\Omega)$,

$$\begin{aligned} I_1 &\le \|\nabla u_j^+\|_{L^2(B(0,M))}\|\nabla\varphi\|_\infty\left(\int_{B(0,M)\cap\Omega_j^+}|A(r_jx+\xi)-A_0^+|^2\,dx\right)^{1/2} \\ &\overset{(3\text{-}14)}{\lesssim_\Lambda} M^{(1-n)/2}\omega_\infty^+(\overline{B(0,4M)})\left(\frac{1}{r_j^{1+n}}\int_{B(0,Mr_j)\cap\Omega^+}|A(x)-A_0^+|\,dx\right)^{1/2} \\ &\overset{(3\text{-}15)}{\to} 0. \end{aligned}$$

Thus, combining the above estimates and taking $j\to\infty$, we infer that

$$-\int_{\Omega_\infty^+} A_0^+\nabla u_\infty^+\cdot\nabla\varphi = \int\varphi\,d\omega_\infty^+.$$

In particular, u_∞^+ is a continuous weak solution of

$$L_0^+w = -\operatorname{div} A_0^+\nabla w = 0 \quad \text{in } \Omega_\infty^+.$$

Since L_0^+ is a second-order elliptic operator with constant coefficients, u_∞^+ is real analytic in Ω_∞^+. Thus, by the definition of u_∞^+ and since the gradient of its extension by zero is the extension by zero of the gradient, we have

$$\int_{\Omega_\infty^+} A_0^+\nabla u_\infty^+\cdot\nabla\varphi = \int_{\mathbb{R}^{n+1}} A_0^+\nabla u_\infty^+\cdot\nabla\varphi.$$

[1] See Proposition 9.18 in [Brezis 2011]. It is stated for C^1-domains, but the direction we need holds for general Ω.

We now use the divergence theorem along with the fact that $\operatorname{supp}(\nabla\varphi) \subset B(0.M)$ and obtain (writing $L_0^{+,T} = L_{A_0^{+,T}}$)

$$\int \varphi \, d\omega_\infty^+ = -\int_{\mathbb{R}^{n+1}} \operatorname{div}[u_\infty^+ A_0^{+,T} \nabla\varphi] + \int_{\mathbb{R}^{n+1}} u_\infty^+ L_0^{+,T} \varphi = -0 + \int_{\mathbb{R}^{n+1}} u_\infty^+ L_0^{+,T} \varphi,$$

which finishes the proof of (d). The rest of the proof is almost identical since one only uses that u_∞ is real analytic in $\mathbb{R}^{n+1}$ and Liouville's theorem for positive solutions of uniformly elliptic equations; see, e.g., [Heinonen et al. 1993, Corollary 6.11].

One may argue similarly in the case of u_j^-. Notice that in this case, we will obtain a constant-coefficient uniformly elliptic matrix A_0 such that

$$\lim_j (Mr_j)^{-1-n} \int_{B(\xi, Mr_j)\cap(\Omega^+\cup\Omega^-)} |A - A_0| = 0 \quad \text{for all } M \geq 1. \qquad \square$$

Now we prove a slightly weaker version of this result in the next two lemmas. Again, this is based on the details in the proof of [Azzam et al. 2016c, Lemma 5.3], but with some adjustments for elliptic measure.

Lemma 3.12. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain. Let $\xi_j \in \partial\Omega$ and $L = -\operatorname{div} A\nabla$ be a uniformly elliptic operator in Ω such that* (1-3) *holds with $K = \{\xi_j\}$ and, if $\omega = \omega_\Omega^{L_A, x_0}$ is its L_A-harmonic measure with pole at $x_0 \in \Omega$, there is $r_j \to 0$ and $c_j > 0$ so that*

$$\omega_j := c_j T_{\xi_j, r_j}[\omega] \to \omega_\infty, \tag{3-17}$$

$$\liminf_j \frac{|\Omega \cap B(\xi_j, r_j)|}{r_j^{n+1}} > 0, \tag{3-18}$$

$$\omega^z(B(\xi_j, 2r_j)) \gtrsim 1 \quad \text{for all } j \text{ and } z \in B(\xi_j, r_j) \cap \Omega. \tag{3-19}$$

Then there is a subsequence such that the following hold: If $u(x) = G_\Omega(x, x_0)$ on Ω and $u(x) = 0$ on Ω^c, and

$$u_j(x) = c_j u(x r_j + \xi_j) r_j^{n-1},$$

then u_j converges in $L^2_{\mathrm{loc}}\left(\frac{1}{2}\mathbb{B}\right)$ to a nonzero function u_∞ which is L_{A_0}-harmonic in $\{x : u_\infty > 0\} \cap \left(\frac{1}{2}\mathbb{B}\right)$ for constant uniformly elliptic matrix A_0 and such that

$$\|u_\infty\|_{L^2(\mathbb{B}/2)} \lesssim \omega_\infty(\overline{B(0,2)}), \tag{3-20}$$

and for any $\varphi \in C_c^\infty\left(\frac{1}{2}\mathbb{B}\right)$

$$\int \varphi \, d\omega_\infty = \int_{\mathbb{R}^{n+1}} u_\infty L_{A_0} \varphi. \tag{3-21}$$

If $\xi = \xi_j$ and A is continuous at ξ, then A_0 is just the value of A at ξ.

Proof. Recall that we let $\mathbb{B} = B(0, 1)$. Again, to simplify notation, we'll just prove the case when $\xi_j = \xi \in \partial\Omega$.

By (3-19), without loss of generality, we can scale the c_j so that

$$\omega_\infty\left(\tfrac{1}{4}\mathbb{B}\right) = 1. \tag{3-22}$$

Let $\Omega_j = T_{\xi,r_j}(\Omega)$. By (3-19) and (3-4),

$$\omega(B(\xi, 2r_j)) \gtrsim r_j^{n-1} u(x) \quad \text{for all } x \in B(\xi, r_j) \cap \Omega_1, \tag{3-23}$$

and so,

$$\omega_j(2\mathbb{B}) \gtrsim u_j(x) \quad \text{for all } x \in \mathbb{B} \cap \Omega_1^j, \tag{3-24}$$

By Caccioppoli's inequality for L-subharmonic functions and the uniform boundedness of u in $\mathbb{B}$, we deduce that, for $i = 1, 2$,

$$\limsup_{j\to\infty} \|\nabla u_j\|_{L^2(\mathbb{B}/2)} \lesssim \limsup_{j\to\infty} \|u_j\|_{L^2(\mathbb{B})} \lesssim \limsup_{j\to\infty} \omega_j(2\mathbb{B}) \le \omega_\infty(\overline{2\mathbb{B}}).$$

By the Rellich–Kondrachov theorem, the unit ball of the Sobolev space $W^{1,2}\big(\frac{1}{2}\mathbb{B}\big)$ is relatively compact in $L^2\big(\frac{1}{2}\mathbb{B}\big)$, and thus there exists a subsequence of the functions u_j which converges *strongly* in $L^2\big(\frac{1}{2}\mathbb{B}\big)$ to another function $u_\infty \in L^2\big(\frac{1}{2}\mathbb{B}\big)$. This and the above inequality imply (3-20).

By the same diagonalization argument as in the proof of the previous lemma (although using (3-18) instead of $\inf \delta_j > 0$ that we used in the previous lemma), we can pass to a subsequence so that, for some uniformly elliptic matrix A_0 with constant coefficients,

$$\lim_j (Mr_j)^{-1-n} \int_{B(\xi, Mr_j)\cap\Omega} |A(x) - A_0| = 0 \quad \text{for all } M \ge 1. \tag{3-25}$$

It easy to check that

$$\int \varphi \, d\omega_j = \int A_j \nabla u_j \cdot \nabla\varphi \, dx$$

for any C^∞ function φ compactly supported in $\frac{1}{2}\mathbb{B}$. Then passing to a limit, it follows that

$$\int \varphi \, d\omega_\infty = \int A_0 \nabla u_\infty \cdot \nabla\varphi \, dx, \quad \text{for any } \varphi \in C_c^\infty\big(\tfrac{1}{2}\mathbb{B}\big). \qquad \square$$

Theorem 3.13. *Let $\Omega^\pm \subset \mathbb{R}^{n+1}$ be disjoint domains. Let $\xi_j \in \partial\Omega^+ \cap \partial\Omega^-$ and $L = -\operatorname{div} A\nabla$ be a uniformly elliptic operator in $\Omega^+ \cup \Omega^-$ such that* (1-3) *holds with $K = \{\xi_j\}$ with respect to $\Omega^+ \cup \Omega^-$. If $\omega^\pm = \omega_{\Omega^\pm}^{L_A, x^\pm}$ is the L_A-harmonic measure with pole at $x^\pm \in \Omega^\pm$, and if there is $r_j \to 0$ and $c_j > 0$ so that*

$$\omega_j^+ := c_j T_{\xi_j, r_j}[\omega^+] \to \omega_\infty,$$
$$\omega_j^- := c_j T_{\xi_j, r_j}[\omega^-] \to c\omega_\infty$$

for some constant $c > 0$, then there is a subsequence such that the following hold. If $u^\pm(x) = G_{\Omega^\pm}(x, x^\pm)$ on $\Omega^\pm$, $u(x) = 0$ on $(\Omega^\pm)^c$ and

$$u_j^\pm(x) = c_j u^\pm(x r_j + \xi_j) r_j^{n-1}, \tag{3-26}$$

then $u_j := u_j^+ - c^{-1} u_j^-$ converges in $L^2\big(\frac{1}{2}\mathbb{B}\big)$ to a nonzero function u_∞, which is L_{A_0}-harmonic in $\frac{1}{2}\mathbb{B}$ for some constant uniformly elliptic matrix A_0, and moreover,

$$\tfrac{1}{2}\mathbb{B} \cap \operatorname{supp}\omega_\infty = \{u_\infty = 0\} \cap \tfrac{1}{2}\mathbb{B} \tag{3-27}$$

and (3-20) *and* (3-21) *hold. If $\xi_j = \xi$ and A is continuous at ξ, then A_0 is just the value of A at ξ.*

By applying this result to the sequences $c_j T_{\xi_j, ar_j}[\omega^{\pm}]$ for all $a > 0$, we see that u_∞ extends to an L_{A_0}-harmonic function on $\mathbb{R}^{n+1}$ so that for $r > 0$

$$\|u_\infty\|_{L^2(B(0,r))} \lesssim r^{1-n}\omega_\infty(\overline{B(0,4r)}), \tag{3-28}$$

and for any $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$

$$\int \varphi\, d\omega_\infty = \int_{\mathbb{R}^{n+1}} u_\infty L_{A_0}\varphi. \tag{3-29}$$

Proof. The proof is mostly the same as the proof of [Azzam et al. 2016c, Lemma 5.3], but we provided some of the details here to show the differences. Again, we assume $\xi_j = \xi$. Note that since Ω^+ and Ω^- are disjoint, we may assume without loss of generality that

$$\left|B\left(\xi, \tfrac{1}{8}r_j\right)\backslash\Omega^+\right| \geq \tfrac{1}{2}\left|B\left(\xi, \tfrac{1}{8}r_j\right)\right|$$

and so Bourgain's estimate implies

$$\omega^{+,z}(B(\xi, 2r_j)) \gtrsim 1 \quad \text{for all } z \in B(\xi, r_j).$$

Hence, the conclusions of Lemma 3.12 apply to $\omega = \omega^+$, $\Omega = \Omega^+$, and $u = u^+$. In particular, (3-24) in our scenario is

$$\omega_j^+(2\mathbb{B}) \gtrsim u_j^+(x) \quad \text{for all } x \in \mathbb{B} \cap \Omega_1^j. \tag{3-30}$$

Again, by rescaling, we can assume that $\omega_\infty\left(\frac{1}{4}\mathbb{B}\right) = 1$.

Observe now that for any nonnegative $\varphi \in C_c^\infty\left(\frac{1}{2}\mathbb{B}\right)$ with $\varphi = 1$ in $\frac{1}{4}\mathbb{B}$, by Cauchy–Schwarz and Caccioppoli's inequality (since $u_j^{\pm}$ is positive and L_{A_j}-harmonic in $\mathbb{B} \cap \Omega_j^{\pm}$ and zero in $\mathbb{B} \setminus \Omega_j^{\pm}$) we have

$$\begin{aligned}
1 &= \omega_\infty\left(\tfrac{1}{4}\mathbb{B}\right) \leq \int \varphi\, d\omega_\infty = \int A_0 \nabla u_\infty^+ \cdot \nabla\varphi\, dx \\
&= \lim_j \int_{\Omega_j^+} A_j \nabla u_j^+ \cdot \nabla\varphi\, dx \\
&\leq \|A\|_{L^\infty}\|\nabla\varphi\|_{L^\infty(\mathbb{B})} \lim_j \int_{\Omega_j^+\cap\mathbb{B}/2} |\nabla u_j^+| \\
&\lesssim \|A\|_{L^\infty}\|\nabla\varphi\|_{L^\infty(\mathbb{B})} \lim_j \left(\int_{\Omega_j^+\cap\mathbb{B}} |u_j^+|^2\right)^{1/2} \\
&\lesssim \lim_j \left(\int_{\mathbb{B}\cap\Omega_j^+\cap\{u_j^+>t\}} |u_j^+|^2\, dx + \int_{\mathbb{B}\cap\Omega_j^+\cap\{u_j^+\leq t\}} |u_j^+|^2\, dx\right)^{1/2} \\
&\lesssim \liminf_j (|\{x \in \mathbb{B}\cap\Omega_j^+ : u_j^+ > t\}|^{1/2} \cdot \|u_j^+\|_{L^\infty(\mathbb{B}\cap\Omega_j^+)}) + t \\
&\overset{(3\text{-}30)}{\lesssim} \liminf_j (|\{x \in \mathbb{B}\cap\Omega_j^+ : u_j^+ > t\}|^{1/2}\omega_\infty(\overline{2\mathbb{B}}) + t),
\end{aligned}$$

and so, for t small enough,

$$|\mathbb{B}\cap\Omega_j^+| \geq |\{x \in \mathbb{B}\cap\Omega_j^+ : u_j^+(x) > t\}| \gtrsim \omega_\infty(\overline{2\mathbb{B}})^{-2}.$$

In particular,

$$|B(\xi, r_j)\backslash\Omega^-| \geq |B(\xi, r_j)\cap\Omega^+| \gtrsim r_j^{n+1}\omega_\infty(\overline{2\mathbb{B}})^{-2}. \tag{3-31}$$

Thus, by the same arguments as earlier in proving (3-24), we have that for j large

$$\omega_j^-(B(\xi, 2r_j)) \gtrsim u_j^-(x)\omega_\infty(\overline{2\mathbb{B}})^{-2} \quad \text{for all } x \in B(\xi, r_j)\cap\Omega^-. \tag{3-32}$$

Thus, we can apply Lemma 3.12 and can pass to a subsequence so that u_j^- converges in $L^2\big(\frac{1}{2}\mathbb{B}\big)$ to a function u_∞^-. Hence, $u_j^+ - c^{-1}u_j^- \to u_\infty^+ - c^{-1}u_\infty^- =: u_\infty$ and

$$c\int \varphi\, d\omega_\infty = \int L_{A_0^*}\varphi u_\infty^-\, dx \quad \text{for any } \varphi \in C_c^\infty\big(\tfrac{1}{2}\mathbb{B}\big). \tag{3-33}$$

In particular, we can show that u_∞ is L_{A_0}-harmonic in $\frac{1}{2}\mathbb{B}$, and the rest of the proof is exactly as in [Azzam et al. 2016c] starting from equation (5.15). □

4. Harmonic polynomial measures

4A. ***Preliminaries.*** We now review and collect some lemmas that will help us work with the quantities ω_h^A.

Lemma 4.1. *Let $h \in H_A$ and $r > 0$. Then*

$$T_{0,r}[\omega_h^A] = r^{n-1}\omega^A_{h\circ T_{0,r}^{-1}}, \tag{4-1}$$

$$F_r(\omega_h^A) = r^n F_1(\omega^A_{h\circ T_{0,r}^{-1}}). \tag{4-2}$$

Proof. By Lemma 3.10, it suffices to prove this in the case that $h \in H$. Note that if h is a harmonic function and $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$, then

$$\begin{aligned}\int \varphi\, dT_{0,r}[\omega_h] &= \int \varphi\circ T_{0,r}\, d\omega_h \\ &= \int h\Delta(\varphi\circ T_{0,r})\, dx = r^{-2}\int h\Delta\varphi\circ T_{0,r}\, dx \\ &= r^{n-1}\int h\circ T_{0,r}^{-1}\Delta\varphi\, dx = r^{n-1}\int \varphi\, d\omega_{h\circ T_{0,r}^{-1}},\end{aligned} \tag{4-3}$$

and so (4-1) follows. Moreover, by Lemma 2.1(3),

$$F_r(\omega_h) = rF_1(T_{0,r}[\omega_h]) \overset{(4\text{-}1)}{=} r^n F_1(\omega_{h\circ T_{0,r}^{-1}}). \qquad \square$$

Lemma 4.2. *Let $h \in F_A(k)$ and $r > 0$. Then*

$$F_r(\omega_h^A) = r^{n+k}F_1(\omega_h^A). \tag{4-4}$$

Proof. Note that since h is homogeneous of degree k,

$$h\circ T_{0,r}^{-1}(x) = h(rx) = r^k h(x),$$

and thus, by (4-2),

$$F_r(\omega_h^A) = r^n F_1(\omega^A_{h\circ T_{0,r}^{-1}}) = r^n F_1(\omega^A_{r^k h}) = r^{n+k}F_1(\omega_h^A). \qquad \square$$

The following is an immediate consequence of Lemma 4.1.

Lemma 4.3 [Badger 2011, Lemma 4.1]. *Since $\mathscr{F}_A(k)$, $\mathscr{P}_A(k)$, and $\mathscr{H}_A$ are d-cones, so are $\mathscr{F}_{\mathscr{S}}(k)$, $\mathscr{P}_{\mathscr{S}}(k)$, and $\mathscr{H}_{\mathscr{S}}$ for any $\mathscr{S}\subset\mathscr{C}$.*

Lemma 4.4. *Let $A_j\in\mathscr{C}$ converge to a matrix $A\in\mathscr{C}$ and let $h_j\in H_{A_j}$ converge uniformly on compact subsets to some $h\in H_A$. Then $\omega_{h_j}^{A_j}\to\omega_h^A$ weakly.*

Proof. First we will deal with the case that $A_j=A=I$ for all j.

We first claim that, since h and h_j are harmonic, $\mathbb{1}_{\Omega_{h_j}}\to\mathbb{1}_{\Omega_h}$ a.e. Indeed, if $\mathbb{1}_{\Omega_h}(x)=1$, then $h(x)>0$, and by uniform convergence, $h_j(x)>0$ for all large j, and so $\mathbb{1}_{\Omega_{h_j}}(x)=1$ for all large j; similarly, if $\mathbb{1}_{\Omega}(x)=0$, then either $x\in\partial\Omega_h$ (which has measure zero) or $h_j(x)<0$ for all large j, in which case $\mathbb{1}_{\Omega_{h_j}}(x)=0$ for all large j. Thus, $\mathbb{1}_{\Omega_{h_j}}\to\mathbb{1}_{\Omega_h}$ pointwise everywhere in $(\partial\Omega_h)^c$ and thus a.e. in $\mathbb{R}^{n+1}$. In particular, $h_j\mathbb{1}_{\Omega_j}\to h\mathbb{1}_{\Omega}$ a.e. Hence, for $\varphi\in C_c^\infty(\mathbb{R}^{n+1})$, by the dominated convergence theorem,

$$\lim_{j\to\infty}\int\varphi\,d\omega_{h_j}=\lim_{j\to\infty}\int_{\Omega_{h_j}}h_j\Delta\varphi=\int_{\Omega_h}h\Delta\varphi=\int\varphi\,d\omega_h,$$

which implies $\omega_{h_j}\rightharpoonup\omega_h$ as $j\to\infty$.

Now we handle the general case. Let $A_{j,s}=\frac{1}{2}(A_j+A_j^T)$, and $S_j=\sqrt{A_{j,s}}$, and define A_s and S similarly. Let $\tilde{A}_j$ and $\tilde{A}$ be defined as in (3-6), and let $\tilde{h}=h\circ S$ and $\tilde{h}_j=h_j\circ S_j$. Since $\sqrt{\cdot}$ is continuous on the set of real symmetric matrices, $\tilde{h}_j\to\tilde{h}$ uniformly on compact subsets and both are harmonic. Thus, $\omega_{\tilde{h}_j}\rightharpoonup\omega_{\tilde{h}}$, and so

$$\lim_{j\to\infty}\omega_{h_j}^A\overset{\text{(3-8)}}{=}\lim_{j\to\infty}(\det S_j)S_j[\omega_{\tilde{h}_j}]=(\det S)S[\omega_{\tilde{h}}]\overset{\text{(3-8)}}{=}\omega_h^A.\qquad\square$$

Lemma 4.5. *If $A\in\mathscr{C}$ and $h\in P_A(k)$ for some $k\in\mathbb{N}$, then*

$$\|h\|_{L^\infty(\mathbb{B})}\lesssim_{k,\Lambda}F_1(\omega_h^A).\tag{4-5}$$

Proof. Suppose instead that there exist $A_j\in\mathscr{C}$ and $h_j\in P_{A_j}(k)$ for which $\|h_j\|_{L^\infty(\mathbb{B})}>jF_1(\omega_{h_j}^{A_j})$. Without loss of generality, we may assume $\|h_j\|_{L^\infty(\mathbb{B})}=1$, and thus $F_1(\omega_{h_j}^{A_j})\to 0$. Using Cauchy estimates (see, e.g., Proposition 11.3 [Mitrea 2013]), $\{h_j\}_{j=1}^\infty$ forms a normal family in $\mathbb{B}$, and thus we can pass to a subsequence so that h_j converges uniformly on compact subsets of $\mathbb{B}$ and so that A_j converges to some $A\in\mathscr{C}$. Since all h_j are polynomials of order k, we know that the coefficients of h_j converge, which, in turn, implies that h_j converges to some function $h\in\mathscr{P}_{\mathscr{C}}(k)$ uniformly on compact subsets of $\mathbb{R}^{n+1}$. By Lemma 4.4, $\omega_{h_j}^{A_j}\to\omega_h^A$. In particular,

$$F_1(\omega_h^A)=\lim_{j\to\infty}F_1(\omega_{h_j}^{A_j})=0.$$

Thus, $\omega(B(0,r))=0$ for all $r<1$, and so $0\notin\operatorname{supp}\omega_h$. We will now show that in fact $0\in\operatorname{supp}\omega_h^A$ in order to get a contradiction.

First, by Lemma 3.10, we can assume without loss of generality that $A=I$ and $\omega_h^A=\omega_h$. Secondly, notice that as $h_j\in\mathscr{P}_{\mathscr{C}}(k)$, we have $h\in\mathscr{P}(k)$ and so $h(0)=0$. By Lojasiewicz's structure theorem for real analytic varieties (see, e.g., [Krantz and Parks 1992, Theorem 6.3.3, p. 168]), if U is a small enough neighborhood of a point $0 \in \Sigma_h$, we have

$$U \cap \Sigma_h = V^n \cup V^{n-1} \cup \cdots \cup V^0,$$

where V^0 is either the empty set or the singleton $\{0\}$ and for each $i \in \{1, \ldots, n\}$, we may write V^i as a finite, disjoint union $V^i = \bigcup_{j=1}^{N_k} \Gamma_j^i$ of i-dimensional real analytic submanifolds. Further, for each $1 \le i \le n-1$,

$$U \cap \overline{V}^i \supset V^{i-1} \cup \cdots \cup V^0.$$

Moreover, for $1 \le k \le n$ and $1 \le j \le N_k$, $U \cap \partial \Gamma_j^i$ is a union of sets of the form Γ_m^ℓ for $1 \le \ell < i$ and $1 \le m \le N_\ell$ and possibly V^0.

By the main result in [Cheeger et al. 2015], $\dim\{\nabla h = 0\} \le n-1$, and thus $V^n \cap \{\nabla h = 0\}$ is a closed set of relatively empty interior in V^n, so in particular

$$\overline{V^n \backslash \{\nabla h = 0\}} \cap U = \overline{V}^n \cap U = \Sigma_h \cap U \ni 0.$$

For $\zeta \in U \cap V^n \backslash \{\nabla h = 0\}$, the derivative of h at ζ tangent to V^n is always zero, as h is zero on V^n, which forces ∇h to be perpendicular to V^n. Since the normal derivative is nonzero,

$$U \cap V^n \backslash \{\nabla h = 0\} \subset \left\{ \zeta \in U \cap V^n : \frac{\partial h}{\partial \nu} \neq 0 \right\} \subset U \cap V^n \cap \operatorname{supp} \omega_h.$$

Thus, $0 \in U \cap \overline{V^n \backslash \{\nabla h = 0\}} \subset \operatorname{supp} \omega_h$, which gives us the contradiction and concludes the proof. □

4B. *Proof of Proposition I.* Proposition I is a consequence of the following more general result.

Lemma 4.6. *Let $\mathscr{S} \subset \mathscr{C}$ be closed (hence compact). Then $P_{\mathscr{S}}(k)$ and $\mathscr{F}_{\mathscr{S}}(k)$ have compact bases*

Proof. Let $h_j \in P_{A_j}(k)$ with $A_j \in \mathscr{S}$ and assume $\mathscr{F}(\omega_{h_j}^{A_j}) = 1$. Then by (4-5) and Cauchy estimates, we can bound each coefficient of the polynomials h_j uniformly, and then pass to a subsequence so that $A_j \to A \in \mathscr{S}$ and h_j converges on compact subsets of $\mathbb{R}^{n+1}$ to a function $h \in P_A(k) \subset P_{\mathscr{S}}(k)$. By Lemma 4.4, we have $\omega_{h_j} \to \omega_h$, which implies that $\mathscr{P}_{\mathscr{S}}(k)$ has compact basis. The proof for $\mathscr{F}_{\mathscr{S}}(k)$ is similar. □

As a corollary, we show the following stronger version of (4-5).

Corollary 4.7. *For $h \in P_{\mathscr{C}}(k)$ and $r > 0$,*

$$\|h\|_{L^\infty(r\mathbb{B})} \approx_k r^{-n} F_r(\omega_h). \tag{4-6}$$

Proof. Let $h \in P_{\mathscr{C}}(k)$ and $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$ be such that $\mathbb{1}_{\mathbb{B}/2} \le \varphi \le \mathbb{1}_{\mathbb{B}}$. Since $\mathscr{P}_{\mathscr{C}}(k)$ has compact basis by Lemma 4.6, we can estimate

$$F_1(\omega_h) \overset{(2\text{-}4)}{\lesssim} F_{1/2}(\omega_h) \le \int \varphi \, d\omega_h = \int_{\Omega_h} h \Delta\varphi \le \|\Delta\varphi\|_\infty \int_{\mathbb{B}} |h| \lesssim \|h\|_{L^\infty(\mathbb{B})} \overset{(4\text{-}5)}{\lesssim} F_1(\omega_h).$$

For $r \neq 1$, by the previous inequalities we have

$$F_r(\omega_h) \overset{(4\text{-}2)}{=} r^n F_1(\omega_{h \circ T_{0,r}^{-1}}) \approx r^n \|h \circ T_{0,r}^{-1}\|_{L^\infty(\mathbb{B})} \approx r^n \|h\|_{L^\infty(r\mathbb{B})}.$$ □

4C. *Proof of Proposition II.*

Lemma 4.8. *Let $h \in H_A$, $A \in \mathscr{C}$, and*

$$h(x) = \sum_{j=m}^{\infty} \sum_{|\alpha|=j} \frac{D^\alpha h(0)}{\alpha!} x^\alpha = \sum_{j=m}^{\infty} h_j(x)$$

be its Taylor series (where $m > 0$ and $h_m \neq 0$), which converges uniformly to h on compact subsets of $\mathbb{R}^{n+1}$. Then $\operatorname{Tan}(\omega_h^A, 0) = \{c\omega_{h_m}^A : c > 0\}$.

Proof. For notational convenience, we will just consider the case $A = I$; the general case is identical. Note that as $r \to 0$, we have $r^{-m} h \circ T_{0,r}^{-1} \to h_m$ uniformly on compact subsets of $\mathbb{R}^{n+1}$. Indeed, fix $R > 0$. Then the series

$$r^{-m} \sum_{j=m}^{\infty} \sum_{|\alpha|=j} \frac{D^\alpha h(0)}{\alpha!} (rx)^\alpha = \sum_{j=m}^{\infty} \sum_{|\alpha|=j} \frac{D^\alpha h(0)}{\alpha!} x^\alpha r^{|\alpha|-m}$$

converges uniformly to $r^{-m} h \circ T_{0,r}^{-1}$ on compact subsets of $B(0, R)$, provided r is small enough. In fact, by Cauchy estimates,

$$|D^\alpha h(0)| \lesssim_n |\alpha|^{|\alpha|},$$

and since there exists a constant $C > 1$ such that $k^k/k! \lesssim C^k$, for $x \in B(0, R)$ and $r \in (0, 1/(CR))$ we have

$$\begin{aligned} |r^{-m} h \circ T_{0,r}^{-1}(x) - h_m(x)| &\le \sum_{j=m+1}^{\infty} \sum_{|\alpha|=j} \left| \frac{D^\alpha h(0)}{\alpha!} \right| R^{|\alpha|} r^{|\alpha|-m} \\ &\lesssim_{n,m} \sum_{j=m+1}^{\infty} C^j R^j r^{j-m} \lesssim r^{-m} (CRr)^{m+1} = (CR)^{m+1} r \overset{r\downarrow 0}{\longrightarrow} 0. \end{aligned}$$

Let now

$$\nu_r := r^{-m-n+1} T_{0,r}[\omega_h] \overset{(4\text{-}1)}{=} r^{-m} \omega_{h \circ T_{0,r}^{-1}} = \omega_{r^{-m} h \circ T_{0,r}^{-1}}.$$

By Lemma 4.4, $\nu_r \rightharpoonup \omega_{h_m} \in \mathscr{F}(m)$. In particular, every tangent measure of ω_h at zero must be a multiple of this one. □

We now state an interesting consequence of these results: if a portion of tangent measures of an arbitrary Radon measure are in $\mathscr{P}(k)$, then in fact they are all in $\mathscr{F}(k)$ (that is, we did not have to assume the original measure was special like harmonic measure).

Lemma 4.9. *Let ω be a Radon measure, $\xi \in \operatorname{supp} \omega$, and k be the minimal integer such that $\operatorname{Tan}(\omega, \xi) \cap \mathscr{P}(k) \neq \varnothing$; then $\operatorname{Tan}(\omega, \xi) \cap \mathscr{P}(k) \subset \mathscr{F}(k)$.*

We follow the proof in [Badger 2011, Lemma 5.9], which originally supposed that ω was a harmonic measure for an NTA domain.

Proof. If $k = 1$, then $\mathscr{P}(1) = \mathscr{F}(1)$. Now suppose $k > 1$ and there is $h \in P(k)$ nonhomogeneous such that $\omega_h \in \operatorname{Tan}(\omega, \xi) \cap \mathscr{P}(k)$. Since $h \in \mathscr{P}(k)$, we may write

$$h(x) = \sum_{j=m}^{k} \sum_{|\alpha|=j} \frac{D^\alpha h(0)}{\alpha!} x^\alpha = \sum_{j=m}^{k} h_m(x),$$

where $m < k$ since $h \in \mathscr{P}(k)$ is not homogeneous. By Lemma 4.8, $\operatorname{Tan}(\omega_h, 0) = \{c\omega_{h_m} : c > 0\} \subset \mathscr{F}(m)$, and since $\operatorname{Tan}(\omega_h, 0) \subset \operatorname{Tan}(\omega, \xi)$ by Lemma 2.9, $\operatorname{Tan}(\omega, \xi) \cap \mathscr{F}(m) \neq \varnothing$, contradicting the minimality of k. Thus, $\operatorname{Tan}(\omega, \xi) \cap \mathscr{P}(k) \subset \mathscr{F}(k)$. □

We will also need the following result.

Lemma 4.10 [Badger 2011, Lemma 4.7]. *Suppose $h \in P(m)$ for some m. There exist $\varepsilon = \varepsilon(n, m, k) > 0$ and $r_0 > 0$ so that if $d_r(\omega_h, \mathscr{F}(k)) < \varepsilon$ for all $r \geq r_0$, then $m = k$.*

Proof of Proposition II. Suppose $\operatorname{Tan}(\omega, \xi) \subset \mathscr{P}(k)$. Let m be the minimal integer for which $\operatorname{Tan}(\omega, \xi) \cap \mathscr{P}(m) \neq \varnothing$, so $m \leq k$. Then, by Lemma 4.9, $\operatorname{Tan}(\omega, \xi) \cap \mathscr{P}(m) \subset \mathscr{F}(m)$. In particular, $\operatorname{Tan}(\omega, \xi) \cap \mathscr{F}(m) \neq \varnothing$. Since, by Proposition I, $\mathscr{P}(k)$ has compact basis, we can use Lemmas 4.10 and 2.10 to conclude $\operatorname{Tan}(\omega, \xi) \subset \mathscr{F}(m)$. □

5. Proof of Theorem I

Lemma 5.1. *Let $\mathscr{S} \subset \mathscr{C}$ be closed and $\omega = \omega_\Omega^{A,x}$ be an L_A-harmonic measure where $A \in \mathscr{A}$ and $L_A \in \operatorname{VMO}(\Omega, \xi)$ at $\xi \in \operatorname{supp} \omega$. Also assume we have $\operatorname{Tan}(\omega, \xi) \subset \mathscr{H}_{\mathscr{S}}$. Let k be the smallest integer for which $\operatorname{Tan}(\omega, \xi) \cap \mathscr{F}_{\mathscr{S}}(k) \neq \varnothing$. Then $\operatorname{Tan}(\omega, \xi) \subset \mathscr{F}_{\mathscr{S}}(k)$. In particular,*

$$\lim_{r \to 0} \frac{\log \omega(B(\xi, r))}{\log r} = n + k - 1; \tag{5-1}$$

i.e., the pointwise dimension of harmonic measure at the point ξ is $n + k - 1$.

Proof. If $\operatorname{Tan}(\omega, \xi) \not\subset \mathscr{F}_{\mathscr{S}}(k)$, then by Corollary 2.12, there is $r_0 > 0$ so that for any $\varepsilon > 0$ small we may find $\nu \in \operatorname{Tan}(\omega, \xi) \backslash \mathscr{F}_{\mathscr{S}}(k)$ so that $d_{r_0}(\nu, \mathscr{F}_{\mathscr{S}}(k)) = \varepsilon$ and $d_r(\nu, \mathscr{F}_{\mathscr{S}}(k)) \leq \varepsilon$ for all $r \geq r_0$. Without loss of generality, we can assume $r_0 = 1$. For each $r > 1$, choose $\mu_r \in \mathscr{F}_{\mathscr{S}}(k)$ such that $F_r(\mu_r) = 1$ and

$$F_r\left(\frac{\nu}{F_r(\nu)}, \mu_r\right) < 2\varepsilon.$$

Then for $r \geq 1$,

$$\begin{aligned} \frac{F_r(\nu)}{F_{2r}(\nu)} &= \int (r - |x|)_+ \, d\frac{\nu}{F_{2r}(\nu)} < 2\varepsilon + \int (r - |x|)_+ \, d\mu_{2r} = 2\varepsilon + F_r(\mu_{2r}) \\ &\overset{(4\text{-}4)}{=} 2\varepsilon + 2^{-n-k} F_{2r}(\mu_{2r}) = 2\varepsilon + 2^{-n-k} = 2^{-n-k+\beta} \end{aligned}$$

for some $\beta > 0$ that goes to zero as $\varepsilon \to 0$. Similarly,

$$\frac{F_r(\nu)}{F_{2r}(\nu)} \geq 2^{-n-k-\beta}.$$

Hence, for $\ell \in \mathbb{N}$,

$$2^{\ell(n+k-\beta)} \leq \frac{F_{2^\ell r}(\nu)}{F_r(\nu)} \leq 2^{\ell(n+k+\beta)}. \tag{5-2}$$

Note that $\nu = \omega_h^A$ for some $h \in \mathscr{H}_A$ by Theorem 3.13 and $A \in \mathscr{S}$, and so

$$\|h\|_{L^\infty(2^\ell \mathbb{B})} \overset{(3\text{-}28)}{\lesssim} 2^{\ell(1-n)} \omega_h(B(0, 2^{\ell+1})) \leq 2^{-\ell n - 1} F_{2^{\ell+2}}(\omega_h) \overset{(5\text{-}2)}{\leq} 2^{\ell(k+\beta)-1} F_{2^2}(\omega_h). \tag{5-3}$$

Let α be a multi-index of length $|\alpha| > k$. Then we can pick $\varepsilon > 0$ small enough so that β is so small that $|\alpha| - k - \beta > 0$ holds. Thus, by Cauchy estimates,

$$|\partial^\alpha h(0)| \lesssim_\alpha 2^{-\ell|\alpha|} \|h\|_{L^\infty(2^\ell \mathbb{B})} \overset{(5\text{-}3)}{\lesssim} 2^{-\ell(|\alpha|-k-\beta)} F_{2^2}(\omega_h) \to 0$$

as $\ell \to \infty$, and so $h \in \mathscr{P}_A(k)$.

Suppose $h = \sum_{j=1}^k h_j$. If $\omega_h \notin \mathscr{F}_A(k)$, then there exists $j < k$ such that $h_j \neq 0$, and by Lemma 4.8, we infer that $\operatorname{Tan}(\omega_h^A, 0)$ contains an element of $\mathscr{F}_A(j)$. Since $\omega_h^A \in \operatorname{Tan}(\omega, \xi)$, we know that $\operatorname{Tan}(\omega_h^A, 0) \subset \operatorname{Tan}(\omega, \xi)$ by Lemma 2.9 and thus, $\operatorname{Tan}(\omega, \xi) \cap \mathscr{F}_A(j) \neq \varnothing$. Hence $\operatorname{Tan}(\omega, \xi) \cap \mathscr{F}_{\mathscr{S}}(j) \neq \varnothing$, contradicting the minimality of k. This proves $\operatorname{Tan}(\omega, \xi) \subset \mathscr{F}_{\mathscr{S}}(k)$.

For the final equality, note that $\operatorname{Tan}(\omega, \xi) \subset \mathscr{F}_{\mathscr{S}}(k)$ and so $\operatorname{Tan}(\omega, \xi)$ has compact basis. In particular, by Lemma 2.11,

$$\lim_{r \to 0} d_1(T_{\xi,r}[\omega], \mathscr{F}_{\mathscr{S}}(k)) = 0.$$

Thus, for $\varepsilon > 0$, there is $r_0 > 0$ such that for each $r \leq r_0$ there exists $\mu_r \in \mathscr{F}_{\mathscr{S}}(k)$ so that $F_1(\mu_r) = 1$ and

$$F_1\left(\frac{T_{\xi,r}[\omega]}{F_1(T_{\xi,r}[\omega])}, \mu_r\right) < \varepsilon.$$

Setting $\nu_r = r^{-1} T_{\xi,r}^{-1}[\mu_r]$, this gives $F_r(\nu_r) = 1$ and

$$F_r\left(\frac{\omega}{F_r(\omega)}, \nu_r\right) < \varepsilon.$$

By the same arguments as earlier, we can show that there exists $\gamma > 0$, which goes to zero as $\varepsilon \to 0$, so that for all $\ell \geq 0$ and $r < 2^{-\ell-1} r_0$

$$2^{\ell(n+k-\gamma)} \leq \frac{F_{2^\ell r}(\omega)}{F_r(\omega)} \leq 2^{\ell(n+k+\gamma)}. \tag{5-4}$$

Hence, if we set $d = n + k - 1$, we get

$$\begin{aligned} \omega(B(\xi, 2^\ell r)) &= T_{\xi,r}[\omega](B(0, 2^\ell)) \leq 2^{-\ell} F_{2^{\ell+1}}(T_{\xi,r}[\omega]) \\ &\leq 2^{(\ell+1)(n+k+\gamma)-\ell} F_1(T_{\xi,r}[\omega]) \\ &\leq 2^{\ell(d+\gamma)+n+k+\gamma} T_{\xi,r}[\omega](B(0,1)) \\ &= 2^{\ell(d+\gamma)+n+k+\gamma} \omega(B(\xi, r)). \end{aligned}$$

Similarly,

$$\begin{aligned} \omega(B(\xi, r)) &= T_{\xi,r}[\omega](B(0,1)) \leq F_2(T_{\xi,r}[\omega]) \\ &\leq 2^{-(\ell-1)(n+k-\gamma)} F_{2^\ell}(T_{\xi,r}[\omega]) \\ &\leq 2^{-(\ell-1)(n+k-\gamma)+\ell} \omega(B(\xi, 2^\ell r)) \\ &= 2^{-\ell(d-\gamma)+n+k-\gamma} \omega(B(\xi, 2^\ell r)). \end{aligned}$$

For $r < \frac{1}{2} r_0$, let $\ell \in \mathbb{N}$ be so that $2^{-\ell-1} r_0 \leq r \leq 2^{-\ell} r_0$. Then

$$\begin{aligned} \omega(B(\xi, r)) \leq \omega(B(\xi, 2^{-\ell} r_0)) &\leq 2^{-\ell(d-\gamma)+n+k-\gamma} \omega(B(\xi, r_0)) \\ &\leq 2^{1+(n+k-\gamma)} r^{d-\gamma} \omega(B(\xi, r_0)). \end{aligned}$$

Hence, recalling that these logs are negative, we conclude

$$\liminf_{r\to 0}\frac{\log\omega(B(\xi,r))}{\log r}\geq \liminf_{r\to 0}\frac{\log(2^{1+(n+k-\gamma)}\omega(B(\xi,r_0)))}{\log r}+d-\gamma=d-\gamma.$$

A similar estimate gives

$$\limsup_{r\to 0}\frac{\log\omega(B(\xi,r))}{\log r}\leq d+\gamma.$$

If we let $\gamma\to 0$, then (5-1) follows. □

Proof of Theorem I. We set

$$E^*=\left\{\xi\in E:\lim_{r\to 0}\frac{\omega^+(E\cap B(\xi,r))}{\omega^+(B(\xi,r))}=\lim_{r\to 0}\frac{\omega^-(E\cap B(\xi,r))}{\omega^-(B(\xi,r))}=1\right\},$$
$$E^{**}=\{\xi\in E^*:\text{(1-4) holds}\}.$$

Notice that by [Mattila 1995, Corollary 2.14(1)] and because ω_1 and ω_2 are mutually absolutely continuous on E,

$$\omega^+(E\backslash E^{**})=\omega^-(E\backslash E^{**})=0.$$

Also, set

$$\Lambda_1=\left\{\xi\in E^{**}\colon 0<h(\xi):=\frac{d\omega^-}{d\omega^+}(\xi)=\lim_{r\to 0}\frac{\omega^-(B(\xi,r))}{\omega^+(B(\xi,r))}=\lim_{r\to 0}\frac{\omega^-(E\cap B(\xi,r))}{\omega^+(E\cap B(\xi,r))}<\infty\right\},$$
$$\Gamma=\{\xi\in\Lambda_1:\xi\text{ is a Lebesgue point for }h\text{ with respect to }\omega^+\}.$$

Again, by Lebesgue differentiation for measures (see [Mattila 1995, Corollary 2.14(2) and Remark 2.15(3)]), Γ has full measure in E^{**} and hence in E.

Next, we record a lemma which was proven in [Azzam et al. 2017b, Lemma 5.8] (which in turn is based on the work of [Kenig et al. 2009]) in the case of the harmonic functions in domains that satisfy the CDC condition, but its proof goes through unchanged for L-harmonic functions in general domains.

Lemma 5.2. *Let $\xi\in\Gamma$, $c_j\geq 0$, and $r_j\to 0$ be so that $\omega_j^+=c_jT_{\xi,r_j}[\omega^+]\to\omega_\infty$. Then $\omega_j^-=c_jT_{\xi,r_j}[\omega^-]\to h(\xi)\omega_\infty$.*

We define

$$\mathscr{F}:=\{c\mathcal{H}^n|_V:c>0,\ V\text{ a }d\text{-dimensional plane containing the origin}\}.$$

It is not hard to show that $\mathscr{F}$ has compact basis.

Lemma 5.3. *For ω^+-a.e. $\xi\in\Gamma$,*

$$\operatorname{Tan}(\omega^+,\xi)\cap\mathscr{F}\neq\varnothing.$$

Proof. We can pick $\xi\in\Gamma$ so that $\operatorname{Tan}(\omega^+,\xi)\neq\varnothing$, let $\omega_\infty\in\operatorname{Tan}(\omega^+,\xi)$, so there is $c_j>0$ and $r_j\downarrow 0$ so that $c_jT_{\xi,r_j}[\omega^+]\to\omega_\infty$. By Lemma 5.2, we also have $c_jT_{\xi,r_j}[\omega^-]\to h(\xi)\omega_\infty$. By Theorem 3.13, (3-27) holds.

In particular, $\frac{1}{2}\mathbb{B}\cap\operatorname{supp}\omega_\infty$ is a smooth real analytic variety, and arguing as in [Azzam et al. 2016c], for example, one deduces that

$$d\omega_\infty|_{\mathbb{B}/2}=-c_n(\nu_{\Omega_\infty^+}\cdot A_0\nabla u_\infty)\,d\mathcal{H}^n|_{\partial^*\Omega_\infty^+\cap\mathbb{B}/2},$$

where A_0 is the matrix from Theorem 3.13, $\partial^*\Omega_\infty^+$ is the reduced boundary of $\Omega_\infty^+ = \{u_\infty > 0\}$ and $\nu_{\Omega_\infty^+}$ is the measure-theoretic outer unit normal. Hence, ω_∞ is absolutely continuous with respect to surface measure of $\partial\Omega_\infty^+$ in $\frac{1}{2}\mathbb{B}$. Thus, since the tangent measure at $\mathcal{H}^n$-almost every point of $\partial\Omega_\infty^+$ is contained in $\mathscr{F}$, we can take another tangent measure of ω_∞ that is in $\mathscr{F}$ and apply Theorem 2.8 to conclude the proof. □

By Lemmas 5.1 and 5.3, we also have that $\dim \omega^+|_E = n$. It remains to show that if $\Omega^\pm$ both have the CDC, then $\lim_{r\to 0} \Theta^{\mathscr{F}}_{\partial\Omega^+}(\xi, r) = 0$ for ω^+-a.e. $\xi \in E$. But this follows almost immediately because, for almost every $\xi \in \Gamma$ and any $r_j \downarrow 0$, we may pass to a subsequence so that, by Lemma 3.11(a) and (f), $\lim_{j\to\infty} \Theta^{\mathscr{F}}_{\partial\Omega^+}(\xi, r_j) = 0$. This concludes the proof of Theorem I. □

6. BMO, VMO and vanishing A_∞

In this section, we will prove some estimates relating the logarithm of a Radon–Nikodym derivative to the mutual absolute continuity properties of two measures. We will apply them to the specific case of elliptic measure, but we will prove them for general measures.

Definition 6.1. Let μ be a Radon measure on a metric space X. We say that a function $f \in L^1_{\mathrm{loc}}(\mu)$ is of *bounded mean oscillation* and write $f \in \mathrm{BMO}(\mu)$ if there exists a constant $C > 0$ such that

$$\sup_{r\in(0,\infty)} \sup_{x\in\operatorname{supp}\mu} \fint_{B(x,r)} |f - f_{B(x,r)}|\, d\mu \le C, \tag{6-1}$$

where $f_A := \fint_A f\, d\mu := \mu(A)^{-1} \int_A f\, d\mu$ for any $A \subset X$ with $\mu(A) > 0$. We define the space of *vanishing mean oscillation* $\mathrm{VMO}(\mu)$ to be the closure in the $\mathrm{BMO}(\mu)$ norm of the set of bounded uniformly continuous functions defined on X. Equivalently, we say $f \in \mathrm{VMO}(\mu)$ if $f \in L^1_{\mathrm{loc}}(\mu)$ and

$$\lim_{r\to 0} \sup_{x\in\operatorname{supp}\mu} \fint_{B(x,r)} |f - f_{B(x,r)}|\, d\mu = 0. \tag{6-2}$$

Definition 6.2. For two measures μ and ν on a metric space X, we will say $\nu \in A_\infty(\mu)$ if $\mu \ll \nu$ and there is $K = K(\mu, \nu)$ so that for any ball B centered on the support of μ

$$\fint_B \frac{d\nu}{d\mu}\, d\mu \, \exp\biggl(-\fint_B \log \frac{d\nu}{d\mu}\, d\mu\biggr) \le K(\mu, \nu). \tag{6-3}$$

We will say $\nu \in A'_\infty(\mu)$ if there are $\varepsilon, \delta \in (0, 1)$ so that for all $B \subseteq X$ and $E \subseteq B$

$$\frac{\mu(E)}{\mu(B)} < \delta \implies \frac{\nu(E)}{\nu(B)} < \varepsilon. \tag{6-4}$$

We will say $\nu \in VA_\infty(\mu)$ (or *vanishing A_∞ with respect to μ*) if

$$\lim_{r\to 0} \sup_{\xi\in\operatorname{supp}\mu} \fint_B \frac{d\nu}{d\mu}\, d\mu \, \exp\biggl(-\fint_B \log \frac{d\nu}{d\mu}\, d\mu\biggr) = 1 \tag{6-5}$$

and $\nu \in VA'_\infty(\mu)$ if for all $r > 0$ there is $\varepsilon_r \in (0, 1)$ so that $\lim_{r\to 0} \varepsilon_r = 0$ and $\delta_r > 0$ so that for all balls $B \subset X$ with $r_B < r$ and $E \subset B$

$$\frac{\mu(E)}{\mu(B)} < \delta_r \implies \frac{\nu(E)}{\nu(B)} < \varepsilon_r. \tag{6-6}$$

In the case that $X=\mathbb{R}^{n+1}$ and μ is equal to the $(n+1)$-dimensional Lebesgue measure, A_∞-equivalence is the same as A'_∞-equivalence, and this is from [Reimann and Rychener 1975], although it was also shown later in [Khrushchev 1984; García-Cuerva and Rubio de Francia 1985].

We recall a notion introduced in [Korey 1998].

Definition 6.3. A probability space (X,μ) is *halving* if every subset $E\subset X$ of positive measure has a subset $F\subset E$ so that $\mu(F)=\frac{1}{2}\mu(E)$.

We will first focus on proving the following after a series of other lemmas.

Lemma 6.4. *Let (X,μ) be a metric measure space, $\nu\ll\mu$, and $f=d\nu/d\mu$:*

(1) *If $\nu\in A'_\infty(\mu)$ and $\log f\in \mathrm{BMO}(\mu)$, then $\nu\in A_\infty(\mu)$. If X is also halving, then $\nu\in A_\infty(\mu)$ implies $\nu\in A'_\infty(\mu)$ and $\log f\in\mathrm{BMO}(\mu)$.*

(2) *If $\nu\in VA'_\infty(\mu)$ and $\log f\in \mathrm{VMO}(\mu)$, then $\nu\in VA_\infty(\mu)$. If X is also halving, then $\nu\in VA_\infty(\mu)$ implies $\nu\in VA'_\infty(\mu)$ and $\log f\in\mathrm{VMO}(\mu)$.*

The first implication of the second half of (1) of the lemma is a consequence of the following theorem.

Theorem 6.5 [Khrushchev 1984, Theorem 1]. *Suppose $\nu\ll\mu$, B is a ball centered on $\operatorname{supp}\mu$, and*

$$\fint_B \frac{d\nu}{d\mu}\,d\mu\,\exp\left(-\fint_B \log\frac{d\nu}{d\mu}\,d\mu\right)\le C.$$

Then there are $\varepsilon,\delta>0$ so that, for any $F\subset B\cap\operatorname{supp}\mu$,

$$\frac{\mu(F)}{\mu(B)}<\delta \implies \frac{\nu(F)}{\nu(B)}<\varepsilon. \tag{6-7}$$

Moreover, there is $\delta>0$ so that

$$\frac{\mu(F)}{\mu(B)}<\delta \implies \frac{\nu(F)}{\nu(B)}<2(C-1). \tag{6-8}$$

In particular, if $\nu\in A_\infty(\mu)$, then $\nu\in A'_\infty(\nu)$, and if $\nu\in VA_\infty(\mu)$, then $\nu\in VA'_\infty(\mu)$.

Proof. We follow the proof from [Khrushchev 1984, Theorem 1], since he proves (6-7) but not (6-8). Let $\delta\in(0,1)$ to be chosen later, let $F\subseteq B$ and suppose $\mu(F)=\delta\mu(B)$; we will pick δ later. Let $f=d\nu/d\mu$, $E=B\backslash F$, and set

$$t=\frac{\nu(E)}{\nu(F)}.$$

Let $g_B=\fint_B f\,d\mu$. Then

$$\log C\ge(\log f^{-1})_B+\log f_B=\frac{\mu(E)}{\mu(B)}(\log f^{-1})_E+\frac{\mu(F)}{\mu(B)}(\log f^{-1})_F+\log f_B. \tag{6-9}$$

By Jensen's inequality, for any set S

$$(\log f^{-1})_S=-(\log f)_S\ge-\log f_S,$$

and applying this to $S = E, F$, we have

$$\begin{aligned}\log C &\geq -\frac{\mu(E)}{\mu(B)}\log f_E - \frac{\mu(F)}{\mu(B)}\log f_F + \log f_B\\ &\geq -\frac{\mu(E)}{\mu(B)}\log f_E - \frac{\mu(F)}{\mu(B)}\log f_E + \frac{\mu(F)}{\mu(B)}\log\frac{\mu(F)}{\mu(E)} + \frac{\mu(F)}{\mu(B)}\log t + \log f_B\\ &= -\log f_E + \frac{\mu(F)}{\mu(B)}\log\frac{\mu(F)}{\mu(E)} + \frac{\mu(F)}{\mu(B)}\log t + \log f_B.\end{aligned}$$

Now observe that

$$-\log f_E = \log\left(\frac{\mu(E)}{\mu(B)}\frac{\mu(B)}{\nu(B)}\frac{\nu(B)}{\nu(E)}\right) = \log\frac{\mu(E)}{\mu(B)} - \log f_B + \log\left(1+\frac{1}{t}\right)$$

and so we have

$$\begin{aligned}\log C &\geq \log\frac{\mu(E)}{\mu(B)} + \log\left(1+\frac{1}{t}\right) + \frac{\mu(F)}{\mu(B)}\log\frac{\mu(F)}{\mu(E)} + \frac{\mu(F)}{\mu(B)}\log t\\ &= \frac{\mu(F)}{\mu(B)}\log\frac{\mu(F)}{\mu(B)} + \frac{\mu(E)}{\mu(B)}\log\frac{\mu(E)}{\mu(B)} + \log(1+t) + \frac{\mu(E)}{\mu(B)}\log\frac{1}{t}\\ &= \underbrace{\delta\log\delta + (1-\delta)\log(1-\delta)}_{=:\varphi(\delta)} + \log(1+t) + \frac{\mu(E)}{\mu(B)}\log\frac{1}{t}.\end{aligned}$$

Note that $\lim_{\delta\to 0}\varphi(\delta) = 0$. Let $\alpha > 0$ and pick $\delta > 0$ so that $|\varphi(\delta)| < \alpha\log C$. Then

$$(1+\alpha)\log C \geq \log(1+t) + \frac{\mu(E)}{\mu(B)}\log\frac{1}{t}. \tag{6-10}$$

We restrict δ further so that $\delta < \alpha$. If $t > 1$, then

$$\frac{\mu(E)}{\mu(B)}\log\frac{1}{t} \geq \log\frac{1}{t};$$

otherwise,

$$\frac{\mu(E)}{\mu(B)}\log\frac{1}{t} \geq (1-\alpha)\log\frac{1}{t}$$

since $\mu(E)/\mu(B) = 1-\delta > 1-\alpha$. Thus, in any case, we have

$$\frac{1+\alpha}{1-\alpha}\log C > \log\frac{1}{t}. \tag{6-11}$$

This implies $t \geq c = C^{-(1+\alpha)/(1-\alpha)}$, and so

$$\nu(F) = \frac{\nu(F)}{1+t} + \frac{t\nu(F)}{1+t} = \frac{\nu(F)+\nu(E)}{1+t} = \frac{\nu(B)}{1+t} \leq \frac{\nu(B)}{1+c}.$$

This proves (6-7) with $\varepsilon = (1+c)^{-1}$. To prove (6-8), we go back to (6-10) with the same bound on δ. Then, since $t \geq c$,

$$\begin{aligned}(1+\alpha)\log C &\geq \log(1+t) + \frac{\mu(E)}{\mu(B)}\log\frac{1}{t} = \log\left(1+\frac{1}{t}\right) + \frac{\mu(F)}{\mu(B)}\log t\\ &\geq \log\left(1+\frac{1}{t}\right) - \delta\frac{1+\alpha}{1-\alpha}\log C.\end{aligned}$$

Since $\delta < \alpha$, this implies

$$\log\left(1+\frac{1}{t}\right) < \left(1+\alpha+\delta\frac{1+\alpha}{1-\alpha}\right)\log C = (1+\alpha)\left(1+\frac{\delta}{1-\alpha}\right)\log C < \frac{1+\alpha}{1-\alpha}\log C,$$

and so

$$C^{(1+\alpha)/(1-\alpha)} - 1 > \frac{1}{t}.$$

We now pick α so that $C^{(1+\alpha)/(1-\alpha)} - 1 = 2(C-1)$, and we are done. □

Korey showed that VA_∞ implies the logarithm of the density is VMO.

Theorem 6.6 [Korey 1998, Theorem 4 and Section 3.5]. *There is a universal constant $c > 0$ so that the following holds. Let (X, μ) be a halving probability space, and suppose that*

$$\frac{\left(\fint_X \exp g\, d\mu\right)}{\exp\left(\fint_X g\, d\mu\right)} \le K. \tag{6-12}$$

Then

$$\fint_X \left| g - \fint_X g\, d\mu \right| d\mu \le \log 2K \tag{6-13}$$

and as $K \to 1$,

$$\fint_X \left| g - \fint_X g\, d\mu \right| d\mu \le c\sqrt{K-1}. \tag{6-14}$$

Lemma 6.7. *Let (X, μ) be a metric probability space and suppose $\nu \ll \mu$. Let $\varepsilon, \delta \in (0, 1)$ be so that for any $E \subset X$*

$$\mu(E) < \delta\mu(X) \implies \nu(E) < \varepsilon\nu(X). \tag{6-15}$$

Set $f = d\nu/d\mu$ and assume

$$\fint_X \left| \log f - \fint_X \log f\, d\mu \right| d\mu < \eta. \tag{6-16}$$

Then

$$1 \le \fint_X f\, d\mu \exp\left(-\fint_X \log f\, d\mu\right) \le \frac{e^{\eta/\delta}}{1-\varepsilon}. \tag{6-17}$$

Proof. Without loss of generality, we may assume $\mu(X) = \nu(X) = 1$. Let $\varepsilon > 0$ and pick δ so that (6-15) holds.

Let $c = \fint_X \log f\, d\mu$ and

$$G = \{|\log f - c| < \rho := \eta\delta^{-1}\}, \quad F = G^c. \tag{6-18}$$

Then, by Chebyshev's inequality and (6-16), we infer that $\mu(F) < \delta$, which, in turn, by (6-15), implies

$$\nu(F) < \varepsilon. \tag{6-19}$$

Moreover, on the set G,

$$\frac{\eta}{\delta} > |\log f - c|$$

and so

$$f \le e^{c+\eta/\delta} \quad \text{on } G. \tag{6-20}$$

Then,

$$1=\frac{\nu(X)}{\mu(X)}=\int_X f\,d\mu \overset{\text{(6-20)}}{\le} \left(\int_G e^{c+\eta/\delta}\,d\mu+\int_F f\,d\mu\right)$$
$$\le\; e^{c+\eta/\delta}+\nu(F)\overset{\text{(6-19)}}{<} e^{c+\eta/\delta}+\varepsilon.$$

Thus,

$$(1-\varepsilon)\int_X f\,d\mu = 1-\varepsilon < e^{c+\eta/\delta}$$

and so

$$\int_X f\,d\mu < \frac{e^{c+\eta/\delta}}{1-\varepsilon}. \tag{6-21}$$

This and Jensen's inequality imply

$$1\le e^{-c}\int_X f\,d\mu < e^{-c}\frac{1}{1-\varepsilon}e^{c+\eta/\delta}=\frac{1}{1-\varepsilon}e^{\eta/\delta}. \qquad \square$$

Corollary 6.8. *Let (X,μ) be a metric measure space. Set $f=d\nu/d\mu$ and assume that for some sequence of balls B_j in X*

$$\lim_j ⨍_{B_j}\left|\log f-⨍_{B_j}\log f\,d\mu\right|d\mu=0 \tag{6-22}$$

and for all $\varepsilon>0$ there is $\delta>0$ so that for j sufficiently large

$$\frac{\mu(E)}{\mu(B_j)}<\delta \implies \frac{\nu(E)}{\nu(B_j)}<\varepsilon. \tag{6-23}$$

Then

$$\lim_{j\to\infty}⨍_{B_j} f\,d\mu\,\exp\left(-⨍_{B_j}\log f\,d\mu\right)=1. \tag{6-24}$$

In particular, if $\log f\in \mathrm{VMO}(d\mu)$ *and* $\nu\in VA'_\infty(\mu)$, *then* $\nu\in VA_\infty(\mu)$.

Proof. Let $\varepsilon,\eta>0$ and let $\delta>0$ be so that (6-23) holds for j large enough. Then (6-16) holds (with B_j in place of X and $\mu|_{B_j}$ in place of μ). Then (6-17) must hold. In particular,

$$\limsup_{j\to\infty}⨍_{B_j} f\,d\mu\,\exp\left(-⨍_{B_j}\log f\,d\mu\right)\le\frac{e^{\eta/\delta}}{1-\varepsilon}.$$

As ε and δ did not depend on η, we can send $\eta\to 0$, and then $\varepsilon\to 0$ since δ now vanishes from the inequality, and then we obtain (6-24). □

Proof of Lemma 6.4. The second halves of (1) and (2) follow from Theorems 6.5 and 6.6. The first half of (1) follows from Lemma 6.7, and the first half of (2) is from Corollary 6.8. □

Lemma 6.9. *Let $\Omega\subset\mathbb{R}^{n+1}$ be any connected domain and $\omega=\omega_\Omega^{L_A,x}$ where $A\in\mathscr{A}(\Omega)$. Then ω is halving.*

Proof. Suppose there is $E\subset\partial\Omega$ with $\omega(E)>0$ that is not halving. For $t\in\mathbb{R}$ and $v\in\mathbb{S}^{n-1}$, let $H_{t,v}=\{x\in\mathbb{R}^{n+1}:x\cdot v\ge t\}$. Then $t\mapsto\omega(H_{t,v}\cap E)$ is not continuous for any $v\in\mathbb{S}^n$, and so there is t_v so that $\omega(\partial H_{t_v,v}\cap E)>0$. Let $V_v=\partial H_{t_v,v}$, which is an n-dimensional plane. Since $\mathbb{S}^n$ is uncountable, there is $\varepsilon>0$ so that $\omega(V_v\cap E)>\varepsilon>0$ for all v in some uncountable set $A\subset\mathbb{S}^n$. Let $A'\subset A$ be countable. Note that for any $u, v \in A'$ distinct, $V_u \cap V_v$ is an $(n-1)$-dimensional subspace. This implies $V_u \cap V_v$ has 2-capacity zero [Heinonen et al. 1993, Theorem 2.27]; hence it is a polar set for ω [loc. cit., Theorem 10.1] and polar sets have L_A-harmonic measure zero [loc. cit., Theorem 11.15]. Thus, if we set

$$W_u := V_u \setminus \bigcup_{\substack{v \in A' \\ v \neq u}} V_v,$$

we have $\omega(W_u \cap E) = \omega(V_u \cap E) \geq \varepsilon$ and W_u are mutually disjoint. But since A' is infinite, this implies $\omega(E) = \infty$, which is a contradiction. □

Lemma 6.10. *Let $\Omega^+ \subset \mathbb{R}^{n+1}$ be a connected domain with connected complement $\Omega^- = \operatorname{ext}(\Omega^+)$ and let L_A be a uniformly elliptic operator with real coefficients. If $\omega^\pm$ denote the L_A-harmonic measures of $\Omega^\pm$ with fixed poles $x^\pm \in \Omega^\pm$, then $\omega^- \in A_\infty(\omega^+)$ if and only if $\omega^- \in A'_\infty(\omega^+)$ and $\log(d\omega^-/d\omega^+) \in \mathrm{BMO}(d\omega^+)$. Moreover, $\omega^- \in VA_\infty(\omega^+)$ if and only if $\omega^- \in VA'_\infty(\omega^+)$ and $\log(d\omega^-/d\omega^+) \in \mathrm{VMO}(d\omega^+)$.*

Proof. This follows from Lemmas 6.4 and 6.9. □

7. Proofs of Theorems II and III

Lemma 7.1. *Let $\omega^\pm$ be two halving Radon measures with equal supports and set $f = \log(d\omega^-/d\omega^+)$. Suppose there are $r_j \downarrow 0$ and $\xi_j \in \partial\Omega^+$ so that $\omega_j^+ = T_{\xi_j, r_j}[\omega^+]/\omega(B(\xi_j, r_j))$ converges weakly to some measure ω with $\omega(\mathbb{B}) > 0$. Further assume that for all $M > 0$*

$$\lim_j \fint_{B(\xi_j, Mr_j)} f\, d\omega^+ \exp\left(-\fint_{B(\xi_j, Mr_j)} \log f\, d\omega^+\right) = 1. \tag{7-1}$$

Then $\omega_j^- \rightharpoonup \omega$ as well.

The proof is similar to that of [Kenig and Toro 2006, Theorem 4.4], though using the techniques of the previous section, we no longer require the doubling assumption.

Proof. Let $B_j = B(\xi_j, r_j)$ and for a ball B set $c_B = \fint_B \log f$. By assumption, for each $M > 0$,

$$e^{-c_{MB_j}} \frac{\omega^-(MB_j)}{\omega^+(MB_j)} \to 1 \quad \text{as } j \to \infty. \tag{7-2}$$

Let $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$ with support in $B(0, M)$ for some $M > 0$ and let $\varphi_j = \varphi \circ T_{\xi_j, r_j}$. Then $\operatorname{supp} \varphi_j \subset MB_j$. Let $\varepsilon > 0$. By (7-2), for j large enough, we have

$$0 \leq e^{-c_{B_j}} \frac{\omega^-(B_j)}{\omega^+(B_j)} - 1 < \varepsilon \quad \text{and} \quad 0 \leq e^{-c_{MB_j}} \frac{\omega^-(MB_j)}{\omega^+(MB_j)} - 1 < \varepsilon. \tag{7-3}$$

Let now $\eta = c\sqrt{1-\varepsilon}$, where c is the constant in (6-14). For j large enough, Theorem 6.6 and (7-2) imply

$$\fint_{B_j} |\log f - c_{B_j}|\, d\omega^+ < \eta \quad \text{and} \quad \fint_{MB_j} |\log f - c_{MB_j}|\, d\omega^+ < \eta. \tag{7-4}$$

Note that ε is independent of η. For fixed $\delta > 0$ and for a ball B, we set

$$G_B = \{\xi \in B \cap \partial\Omega^+ : |\log f(\xi) - c_B| \leq \eta/\delta\}, \quad F_B = B \setminus G_B.$$

Then, Chebyshev's inequality and (7-4) imply

$$\omega^+(F_{B_j}) < \delta\omega^+(B_j) \quad \text{and} \quad \omega^+(F_{MB_j}) < \delta\omega^+(MB_j), \tag{7-5}$$

and for $\delta > 0$ small enough and j large enough Theorem 6.5 and (7-2) imply

$$\omega^-(F_{B_j}) < \varepsilon, \omega^-(B_j) \quad \text{and} \quad \omega^-(F_{MB_j}) < \varepsilon\omega^-(MB_j). \tag{7-6}$$

Let $C = 2\omega(\overline{M\mathbb{B}})/\omega(\mathbb{B})$. Since $\omega(\mathbb{B}) > 0$, we know

$$\limsup_{j\to\infty} \frac{\omega^+(MB_j)}{\omega^+(B_j)} = \limsup_{j\to\infty} \frac{\omega_j^+(M\mathbb{B})}{\omega_j^+(\mathbb{B})} \le \frac{\omega(\overline{M\mathbb{B}})}{\omega(\mathbb{B})} = \tfrac{1}{2}C,$$

and so for j large enough,

$$\omega^+(MB_j) \le C\omega^+(B_j). \tag{7-7}$$

Also, note that for j large enough,

$$\begin{aligned}
|c_{B_j} - c_{MB_j}| &= \left| \fint_{B_j} (c_{B_j} - c_{MB_j}) \right| d\omega^+ \\
&\le \fint_{B_j} |c_{B_j} - \log f|\, d\omega^+ + \fint_{B_j} |\log f - c_{MB_j}|\, d\omega^+ \\
&\overset{(7\text{-}4)}{<} \eta + \frac{\omega^+(MB_j)}{\omega^+(B_j)} \fint_{MB_j} |\log f - c_{MB_j}|\, d\omega^+ \overset{\substack{(7\text{-}4)\\(7\text{-}7)}}{<} (1+C)\eta.
\end{aligned} \tag{7-8}$$

Hence,

$$\begin{aligned}
\omega^-(MB_j) &\overset{(7\text{-}3)}{\le} \omega^+(MB_j)(1+\varepsilon)e^{c_{MB_j}} \overset{\substack{(7\text{-}7)\\(7\text{-}8)}}{<} C\omega^+(B_j)(1+\varepsilon)e^{c_{B_j}+(1+C)\eta} \\
&\overset{(7\text{-}3)}{\le} C\omega^-(B_j)(1+\varepsilon)e^{(1+C)\eta} \le 2Ce^{(1+C)}\omega^-(B_j) \lesssim_C \omega^-(B_j).
\end{aligned} \tag{7-9}$$

Then

$$\begin{aligned}
\int \varphi\, d\omega_j^- - \int \varphi\, d\omega_j^+ &= \frac{1}{\omega^-(B_j)} \int_{MB_j} \varphi_j\, d\omega^- - \frac{1}{\omega^+(B_j)} \int_{MB_j} \varphi_j\, d\omega^+ \\
&= \underbrace{\frac{1}{\omega^-(B_j)} \int_{MB_j \cap F_{MB_j}} \varphi_j f\, d\omega^+}_{=:I_1} + \underbrace{\frac{1}{\omega^-(B_j)} \int_{MB_j \cap G_{MB_j}} (f - e^{c_{MB_j}})\varphi_j\, d\omega^+}_{=:I_2} \\
&\quad - \underbrace{\frac{e^{c_{MB_j}}}{\omega^-(B_j)} \int_{MB_j \cap F_{MB_j}} \varphi_j\, d\omega^+}_{=:I_3} + \underbrace{\frac{e^{c_{MB_j}}}{\omega^-(B_j)} \int_{MB_j} \varphi_j\, d\omega^+ - \frac{1}{\omega^+(B_j)} \int_{MB_j} \varphi_j\, d\omega^+}_{=:I_4} \\
&= I_1 + I_2 - I_3 + I_4.
\end{aligned}$$

We will estimate each of these terms separately, with the understanding that j is large enough (depending on M and η):

$$|I_1| \le \frac{\|\varphi\|_\infty}{\omega^-(B_j)} \int_{MB_j} \mathbb{1}_{F_{MB_j}} f\, d\omega^+ = \frac{\|\varphi\|_\infty \omega^-(F_{MB_j})}{\omega^-(B_j)} = \frac{\omega^-(MB_j)}{\omega^-(B_j)} \frac{\|\varphi\|_\infty \omega^-(F_{MB_j})}{\omega^-(MB_j)} \overset{\substack{(7\text{-}6)\\(7\text{-}9)}}{\lesssim}_{C,M,\|\varphi\|_\infty} \varepsilon.$$

Next, for points in G_{MB_j},

$$e^{-\eta/\delta}e^{c_{MB_j}} \le f \le e^{\eta/\delta}e^{c_{MB_j}}$$

and so

$$e^{c_{MB_j}}(e^{-\eta/\delta}-1) \le f - e^{c_{MB_j}} \le e^{c_{MB_j}}(e^{\eta/\delta}-1).$$

Thus, for $\eta > 0$ small enough (i.e., for j large enough), we can make

$$|f - e^{c_{MB_j}}| < \delta e^{c_{MB_j}} \quad \text{on } G_{MB_j}.$$

Therefore,

$$\begin{aligned} |I_2| &\le \frac{\delta e^{c_{MB_j}}\|\varphi\|_\infty}{\omega^-(B_j)}\omega^+(G_{MB_j}) \le \frac{\delta e^{c_{MB_j}}\|\varphi\|_\infty}{\omega^-(B_j)}\omega^+(MB_j) \\ &= e^{c_{MB_j}}\frac{\omega^+(MB_j)}{\omega^-(MB_j)}\frac{\delta\|\varphi\|_\infty\omega^-(MB_j)}{\omega^-(B_j)} \overset{\substack{(7\text{-}9)\\(7\text{-}3)}}{\lesssim}_{\|\varphi\|_\infty,C,M} \delta, \end{aligned}$$

$$\begin{aligned} |I_3| &\le \frac{e^{c_{MB_j}}\|\varphi\|_\infty}{\omega^-(B_j)}\omega^+(F_{MB_j}) \overset{(7\text{-}5)}{<} \delta\frac{e^{c_{MB_j}}\|\varphi\|_\infty}{\omega^-(B_j)}\omega^+(MB_j) \\ &= \delta\frac{e^{c_{MB_j}}\|\varphi\|_\infty\omega^-(MB_j)}{\omega^-(B_j)}\frac{\omega^+(MB_j)}{\omega^-(MB_j)} \overset{\substack{(7\text{-}3)\\(7\text{-}9)}}{\lesssim}_{C,M,\|\varphi\|_\infty} \delta. \end{aligned}$$

Finally,

$$|I_4| \le \left(e^{c_{MB_j}}\frac{\omega^+(B_j)}{\omega^-(B_j)}-1\right)\frac{\omega^+(MB_j)}{\omega^+(B_j)}\fint_{MB_j}\varphi_j\,d\omega^+ \overset{\substack{(7\text{-}3)\\(7\text{-}7)}}{\lesssim}_{C,\|\varphi\|_\infty,M} \varepsilon.$$

Since these estimates hold for all j large enough, we can conclude

$$\limsup_{j\to\infty}\left|\int\varphi\,d\omega_j^- - \int\varphi_j\,d\omega_j^+\right| \lesssim_{C,M,\|\varphi\|_\infty} \varepsilon+\delta.$$

Now send δ to zero since it only had to be small enough depending on ε. Finally, ε was arbitrarily chosen, which implies that the above limit is zero. Since this holds for all φ, we get that $\omega_j^\pm$ have the same weak limit. □

Proof of Theorem II. Let $\omega \in \mathrm{Tan}(\omega^+,\xi)$. We claim that $\omega \in \mathscr{H}_{\mathscr{C}}$. By Lemma 2.6, $\omega = cT_{0,r}(\mu)$ for some constants $c, r > 0$ and some measure μ of the form $\mu = \lim_{j\to 0} T_{\xi,r_j}[\omega^+]/\omega^+(B(\xi,r_j))$ for some $r_j \downarrow 0$, where $\mu(\mathbb{B}) > 0$. By Lemma 7.1, $\mu = \lim_{j\to 0} T_{\xi,r_j}[\omega^-]/\omega^-(B(\xi,r_j))$ as well. By Theorem 3.13 (or Lemma 3.11(g) if $\Omega^\pm$ have the CDC), $\mu \in \mathscr{H}_{\mathscr{C}}$, and since $\mathscr{H}_{\mathscr{C}}$ is a d-cone by Lemma 4.3, we also have that $\omega \in \mathscr{H}_{\mathscr{C}}$, which proves the claim.

Hence, $\omega = \omega_u$ for some $u \in H_A$ and some $A \in \mathscr{C}$. By Lemma 4.8, for some $k > 0$,

$$\mathrm{Tan}(\omega_u,0) = \{c\omega_{u_k} : c > 0\} \subset \mathscr{F}_A(k) \subset \mathscr{F}_{\mathscr{C}}(k),$$

and since $\mathrm{Tan}(\omega_u,0) \subset \mathrm{Tan}(\omega^+,\xi)$ by Lemma 2.9, we now know that $\mathrm{Tan}(\omega^+,\xi)\cap\mathscr{F}_{\mathscr{C}}(k) \ne \varnothing$ as well. By Lemma 5.1, $\mathrm{Tan}(\omega^+,\xi) \subset \mathscr{F}_{\mathscr{C}}(k)$. The proof that $\Theta_{\partial\Omega^+}^{\mathscr{F}_{\Sigma,\mathscr{C}}(k)}(\xi,r) \to 0$ if $\Omega^\pm$ have the CDC is similar to the proof of Theorem I. □

Proof of Theorem III. Let K be any compact subset of $\partial\Omega^+$. Suppose there is a sequence of radii $r_j \downarrow 0$ and $\xi_j \in K$ so that

$$d_1(T_{\xi_j,r_j}[\omega^+], \mathscr{P}_{\mathscr{C}}(d)) \geq \varepsilon > 0, \tag{7-10}$$

where d will be chosen later, but it will depend only on n and the doubling constant of ω^+.

Since ω^+ is doubling, we may pass to a subsequence so that $\omega_j^+ := T_{\xi_j,r_j}[\omega^+]/\omega^+(B(\xi_j, r_j))$ converges weakly to some measure ω.

If $f = d\omega^-/d\omega^+$ satisfies $\log f \in \mathrm{VMO}(\omega^-)$, then doubling also implies that $\omega^- \in VA'_\infty(\omega^+)$. Indeed, if ω^+ is doubling, then the John–Nirenberg theorem holds, and the VMO condition tells us that on small enough balls, f is a traditional A_p-weight (see [Garnett 2007, Chapter 6.2]). This easily implies $f\,d\omega^+ = d\omega^- \in VA'_\infty(\omega^+)$. Thus, by Corollary 6.8, we know $\omega^- \in VA_\infty(\omega^+)$ and that (7-1) holds for every $M > 0$. By Lemma 7.1, $\omega_j^- \rightharpoonup \omega$ as well. Thus, we can pass to a subsequence so that the conclusions of Theorem 3.13 hold. In particular, $\omega = \omega_h$ for some L_0-harmonic function h, where L_0 is a uniformly elliptic operator with constant coefficients, and also, for any $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$, (3-21) holds.

Now we apply the same standard trick from [Kenig and Toro 2006]. Notice that since ω^+ is doubling, so is ω_h, which combined with Cauchy estimates implies that there exists $\beta > 0$ such that for any $\ell \in \mathbb{N}$ and any multi-index α

$$\begin{aligned} |\partial_\alpha h(0)| &\lesssim 2^{-|\alpha|\ell} \|h\|_{L^\infty(2^\ell \mathbb{B})} \overset{(3\text{-}28)}{\lesssim} 2^{\ell(-|\alpha|+1-n)} \omega_h(B(0, 2^{\ell+1})) \\ &\lesssim 2^{\ell(-|\alpha|+1-n+\beta)} \omega_h(B(0, 2)). \end{aligned} \tag{7-11}$$

Hence, if $|\alpha| > 1 - n + \beta$, letting $\ell \to \infty$ gives $|\partial_\alpha h(0)| = 0$, which implies h is a polynomial of degree at most $1 - n + \beta$. Setting $d = \lceil 1 - n + \beta \rceil$ gives a contradiction to (7-10). The proof of (1-7) is similar to the proof of Theorem I, where we use instead Lemma 3.11 instead of Theorem 3.13. □

8. Proof of Theorem IV

All elliptic operators in this section will be assumed to satisfy (1-1) and (1-2). We will require a few lemmas about elliptic measures in uniform domains as well as some new notation.

Definition 8.1. Let $\Omega \subseteq \mathbb{R}^{n+1}$:

- We say Ω satisfies the *corkscrew condition* if, for some uniform constant $c > 0$ and every ball B centered on $\partial\Omega$ with $0 < r_B < \operatorname{diam}(\partial\Omega)$, there is a ball $B(x_B, cr_B) \subseteq \Omega \cap B$. The point x_B is called a *corkscrew point relative to* B.

- We say Ω satisfies the *Harnack chain condition* if there is a uniform constant C such that for every $\rho > 0$, $\Lambda \geq 1$, and every pair of points $x, y \in \Omega$ with $\delta(x)$, $\delta(y) \geq \rho$ and $|x - y| < \Lambda\rho$ there is a chain of open balls $B_1, \dots, B_N \subset \Omega$, $N \leq C(\Lambda)$, with $x \in B_1$, $y \in B_N$, $B_k \cap B_{k+1} \neq \varnothing$ and $C^{-1} \operatorname{diam}(B_k) \leq \operatorname{dist}(B_k, \partial\Omega) \leq C \operatorname{diam}(B_k)$. The chain of balls is called a *Harnack chain*.

Definition 8.2. If Ω satisfies both the corkscrew and the Harnack chain conditions, then we say that Ω is a *uniform domain*.

Theorem 8.3. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a uniform domain with the CDC and u a nonnegative L_A-elliptic function vanishing on $2B \cap \partial\Omega$, where B is a ball with $r_B < \operatorname{diam} \partial\Omega$ and $A \in \mathscr{A}(\Omega)$. Then*

$$\sup_{x \in B \cap \Omega} u(x) \lesssim u(x_B). \tag{8-1}$$

This was originally shown in Section 4 of [Jerison and Kenig 1982] for NTA domains, but the proof only uses the Hölder continuity of u at the boundary and the fact that NTA domains are uniform, and so the proof of the above result is exactly the same.

Theorem 8.4. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a uniform domain with the CDC and L_A an elliptic operator satisfying* (1-1) *and* (1-2). *Then, for all B centered on $\partial\Omega$,*

$$\omega^{L_A,x}(B) \approx r_B^{n-1} G_\Omega(x, x_B) \quad \text{for all } x \in \Omega \backslash 2B. \tag{8-2}$$

This follows from [Aikawa and Hirata 2008]. Their proof is originally for harmonic measures, but an inspection of the proof shows that it carries through for elliptic measure as well.

Theorem 8.5. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a uniform domain with the CDC. If L_A is an elliptic operator satisfying* (1-1) *and* (1-2), *B is a ball centered on $\partial\Omega$, and $E \subset B \cap \partial\Omega$ is Borel, then*

$$\omega_\Omega^{L_A,x_B}(E) \approx \frac{\omega_\Omega^{L_A,x}(E)}{\omega_\Omega^{L_A,x}(B)}. \tag{8-3}$$

Again, this is [Jerison and Kenig 1982, Lemma 4.11], and since the previous two lemmas are available, the proof is exactly the same for elliptic measures modulo the proof of [loc. cit., Lemma 4.10]. The latter can also be proved by building a subuniform domain as in [loc. cit.], and then showing as in [Akman et al. 2019, Lemma 2.26] that the resulting domain is also CDC (all of this instead of a geometric localization theorem due to Jones, which only works for NTA domains).

Lemma 8.6. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a uniform domain with the CDC and L_A an elliptic operator satisfying* (1-1) *and* (1-2), *and also* (1-4) *at ξ. If $\xi \in \partial\Omega$ and $\omega_j = \omega^{L_A,x_0}(B(\xi, r_j))^{-1} T_{\xi,r_j}(\omega^{L_A,x_0})$ converges weakly to a tangent measure $\omega_\infty \in \operatorname{Tan}(\omega^{L_A,x_0}, \xi)$, then there is a uniform domain Ω_∞ and a constant matrix $A_0 \in \mathscr{C}$ such that, for each $x \in \Omega_\infty$, $\omega_{\Omega_j}^x \rightharpoonup \omega_{\Omega_\infty}^x$ and for all balls $B' \subset B$ centered on $\partial\Omega_\infty$, if x_B is a corkscrew point in $\Omega_\infty \cap B$,*

$$\omega_{\Omega_\infty}^{L_{A_0},x_B}(B') \approx \frac{\omega_\infty(B')}{\omega_\infty(B)}. \tag{8-4}$$

This was originally shown in [Azzam and Mourgoglou 2018] for harmonic measure. In our situation, the proof is much shorter, so we provide it here.

Proof. By Lemma 3.11, there is $A_0 \in \mathscr{C}$ so that we can pass to a subsequence so that $u_j(x) = c_j u(x r_j + \xi) r_j^{n-1}$ converges uniformly in $\mathbb{R}^{n+1}$ to a nonzero L_{A_0}-elliptic function u_∞ and also so that, if $\Omega_j = T_{\xi,r_j}(\Omega)$, then $\partial\Omega_j$ converges in the Hausdorff metric on compact subsets. Let $\Omega_\infty = \{u_\infty > 0\}$.

Claim. Ω_∞ is uniform. If $x, y \in \Omega_\infty$ with $\operatorname{dist}(\{x, y\}, \partial\Omega) \geq \varepsilon|x - y|$, then they are contained in Ω_j and $\operatorname{dist}(\{x, y\}, \partial\Omega_j) \geq \frac{1}{2}\varepsilon|x - y|$ for sufficiently large j. Since the Ω_j are uniform, for each j we can find a Harnack chain of length $N = N(\varepsilon)$ contained in Ω_j. By passing to a subsequence, we can assume the length of this chain is constant and their centers and radii are converging, and hence the chain converges to a Harnack chain in Ω_∞ of length no more than N. A similar proof shows that Ω_∞ is a corkscrew domain. Hence, Ω_∞ is uniform.

Suppose $B' \subset \mathbb{B}$ are centered on $\partial\Omega_\infty$. Let

$$\omega_{\Omega_j}^{T_{\xi,r_j}(x)} = T_{\xi,r_j}[\omega^{L_A,x}].$$

If $x_j = T_{\xi,r_j}(x_0)$, then

$$\omega_{\Omega_j}^{x_B}(B') \approx \frac{\omega_{\Omega_j}^{x_j}(B')}{\omega_{\Omega_j}^{x_j}(B)} = \frac{\omega_{\Omega_j}^{x_j}(\mathbb{B})}{\omega_{\Omega_j}^{x_j}(B)} \frac{\omega_{\Omega_j}^{x_j}(B')}{\omega_{\Omega_j}^{x_j}(\mathbb{B})} = \frac{\omega_j(B')}{\omega_j(B)}.$$

Since ω_j and ω_{Ω_j} are doubling measures, we have

$$\omega_{\Omega_\infty}^{x_B}(B') \le \liminf_{j\to\infty} \omega_{\Omega_j}^{x_B}(B') \lesssim \limsup_{j\to\infty} \frac{\omega_j(B')}{\omega_j(B)} \le \frac{\omega_\infty(\bar{B}')}{\omega_\infty(B)} \lesssim \frac{\omega_\infty(B')}{\omega_\infty(B)}.$$

A similar estimate gives the reverse inequality, and hence proves (8-4). □

We will use the following criterion for uniform rectifiability due to Hofmann, Martell, and Uriarte-Tuero. See Theorem 1.23, equation 1.22, and Remark 1.25 in [Hofmann et al. 2014]; for a local version see Corollary 11.2 in [Mourgoglou and Tolsa 2017].

Theorem 8.7. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a uniform domain with n-regular boundary and let ω_Ω be the harmonic measure defined in Ω. Suppose there is $q > 1$ so that, for any balls $B' \subset B$ centered on $\partial\Omega$, if $k_B = d\omega_\Omega^{x_B}/(d\mathcal{H}^n|_{\partial\Omega})$, then*

$$\left(\fint_{B'\cap\partial\Omega} k_B^q \, d\mathcal{H}^n\right)^{1/q} \lesssim \fint_{2B'\cap\partial\Omega} k_B \, d\mathcal{H}^n.$$

Then $\partial\Omega$ is uniformly rectifiable.

Recall that, by the main result of [Aikawa and Hirata 2008], harmonic measure is doubling in uniform domains satisfying the CDC, and thus, by (8-3), the right side of this inequality is comparable to $\fint_{B'\cap\partial\Omega} k_B \, d\mathcal{H}^n$ (that is, with B' instead of $2B'$), which we will use below.

Remark 8.8. This result still holds for constant coefficients. Indeed, it is easy to see that the A_∞-property is preserved under linear transformations that map balls to ellipsoids, as is the one in Lemma 3.10 (see the paragraph after the proof of this lemma), using that such weights are doubling.Thus, by Lemma 3.10 and the fact that being a uniformly rectifiable set, by its very definition, is invariant under bi-Lipschitz maps, $\partial\Omega_\infty$ is uniformly rectifiable.

Recall that an Ahlfors n-regular set E is *uniformly rectifiable* if there are $c, L > 0$ so that, for every ball B centered on E with $r_B < \operatorname{diam} E$, there is an L-Lipschitz map $f : B(0, r_B) \cap \mathbb{R}^n \to \mathbb{R}^{n+1}$ so that

$$\mathcal{H}^n(f(B(0, r_B)) \cap E) \ge c r_B^n.$$

Now we prove Theorem IV. Let $\Omega \subset \mathbb{R}^{n+1}$ be a uniform CDC domain so that $\mathcal{H}^n|_{\partial\Omega}$ is locally finite. Let $\omega = \omega_\Omega^{L_A}$ be the L_A-harmonic measure associated to a (possibly nonsymmetric) elliptic operator satisfying (1-1) and (1-2). Let $E \subseteq \partial\Omega$ be a set with $\mathcal{H}^n(E) > 0$ such that $\mathcal{H}^n \ll \omega_\Omega^{L_A}$ on E and for $\mathcal{H}^n$-a.e. $\xi \in E$,

$$\theta^n_{\partial\Omega,*}(\xi, r) := \liminf_{r\to 0} \frac{\mathcal{H}^n(B(\xi, r)\cap\partial\Omega)}{(2r)^n} > 0$$

and A has vanishing mean oscillation at ξ.

Assume $\mathcal{H}^n(E) > 0$ (otherwise the theorem is trivial). Then we may find a subset E' of full $\mathcal{H}^n$-measure, where ω and $\mathcal{H}^n$ are mutually absolutely continuous (in particular, $\mathcal{H}^n = g\omega$ for some function g, so we pick $E' = \{x : g(x) > 0\}$). For $\mathcal{H}^n|_{\partial\Omega}$-a.e. $\xi \in E'$, we also have

$$0 < \theta^n_*(\mathcal{H}^n|_{\partial\Omega}, \xi) \le \theta^{n,*}(\mathcal{H}^n|_{\partial\Omega}, \xi) < \infty. \tag{8-5}$$

The lower bound is by assumption, and the upper bound is from [Mattila 1995, Theorem 6.2]. By [loc. cit., Theorem 14.7], for $\mathcal{H}^n|_{\partial\Omega}$-a.e. $\xi \in E'$, $\mathrm{Tan}(\mathcal{H}^n|_{\partial\Omega}, \xi)$ consists of Ahlfors–David n-regular measures. By [loc. cit., Lemmas 14.5 and 14.6], for $\mathcal{H}^n|_{\partial\Omega}$-a.e. $\xi \in E'$,

$$\mathrm{Tan}(\mathcal{H}^n|_{\partial\Omega}, \xi) = \mathrm{Tan}(\mathcal{H}^n|_{E'}, \xi) = \mathrm{Tan}(\omega, \xi)$$

and $\mathrm{Tan}(\omega, \xi)$ consists only of Ahlfors–David n-regular measures. Let $E'' \subset E'$ be the set of points where this holds.

By the Besicovitch decomposition theorem, we can split E'' into two sets F_1 and F_2, where F_1 is n-rectifiable and F_2 is purely n-unrectifiable. Suppose $\mathcal{H}^n(F_2) > 0$. Let $\xi \in F_2$ be a point of density of F_2 with respect to $\mathcal{H}^n$.

Let $r_j \downarrow 0$ be so that $\omega_j := \omega^{L_A,x_0}(B(\xi, r_j))^{-1} T_{\xi,r_j}(\omega^{L_A,x_0})$ converges weakly to some Ahlfors–David n-regular measure $\omega_\infty \in \mathrm{Tan}(\omega, \xi)$. By Lemma 8.6, we may find a uniform domain Ω_∞ so that $\mathrm{supp}\,\omega_\infty = \partial\Omega_\infty$ and, for any balls $B' \subset B$ centered on $\partial\Omega$,

$$\omega^{L_{A_0},x_B}_{\Omega_\infty}(B') \approx \frac{\omega_\infty(B')}{\omega_\infty(B)} \approx \frac{r^n_{B'}}{r^n_B}$$

for some $A_0 \in \mathscr{C}$. If $\sigma = \mathcal{H}^n|_{\partial\Omega_\infty}$, then σ is Ahlfors–David n-regular and so if we set

$$k_B := \frac{d\omega^{L_{A_0},x_B}_{\Omega_\infty}}{d\sigma},$$

then we have that for σ-a.e. $x \in B \cap \partial\Omega$

$$k_B(x) = \lim_{r\to 0} \frac{\omega^{L_{A_0},x_B}_{\Omega_\infty}(B(x, r))}{\sigma(B(x, r))} \approx \frac{r^n/r^n_B}{r^n} = r_B^{-n}.$$

Hence, if $B' \subset B$ is centered on $\partial\Omega$,

$$\left(\fint_{B'} k_B^2\, d\sigma\right)^{1/2} \approx r_B^{-n} \approx \fint_{B'} k_B\, d\sigma.$$

Thus, in light of Remark 8.8, $\partial\Omega_\infty$ is uniformly rectifiable. By the main result of [Azzam et al. 2017a], Ω_∞ is an NTA domain. In particular, we can find corkscrew balls $B_1 \subset \mathbb{B} \cap \Omega_\infty$ and $B_2 \subseteq \mathbb{B}\backslash\Omega_\infty$. We claim that, for all j sufficiently large, $\frac{1}{2}B_1 \subset \Omega_j \cap \mathbb{B}$ and $\frac{1}{2}B_2 \subset \mathbb{B}\backslash\Omega_j$. Indeed, if $\frac{1}{2}B_i \cap \partial\Omega_j \neq \varnothing$ for infinitely many j, then since ω_j is doubling, $\omega_j\left(\frac{2}{3}B_i\right) \sim \omega_j(\mathbb{B}) = 1$ for all j, and so $\omega_\infty(B_i) > 0$, and in particular $\partial\Omega_\infty \cap B_i \neq \varnothing$, which is a contradiction. Thus, B_1 and B_2 do not intersect $\partial\Omega_j$ for sufficiently large j. They cannot both be in Ω_j for all large j, since otherwise, if they were both in Ω_j for infinitely many j then in each such Ω_j, they would be connected by a Harnack chain in Ω_j of bounded length; passing to a subsequence, this implies there is a Harnack chain connecting B_1 to B_2, and since $B_1 \subseteq \Omega_\infty$, the whole chain, including B_2, must be in Ω_∞, which is a contradiction. Thus, at least one of these balls is in Ω_j^c for all j large. By the proof of Lemma 8.6, $\Omega_\infty = \{u_\infty > 0\}$, and since $u_j \to u_\infty$ uniformly on compact subsets of Ω_∞ and $u_\infty > 0$ on B_1, we have $B_1 \subset \Omega_j$ for j large, and so $B_2 \subset \Omega_j^c$ for j large. This proves the claim.

Now there is a small angle of directions around the vector parallel to the line between the centers of B_1 and B_2 where the orthogonal projection of $\partial\Omega_j \cap \mathbb{B}$ has Lebesgue measure comparable to 1. By the Besicovitch–Federer projection theorem, the purely unrectifiable part of $\partial\Omega_j$ has zero Lebesgue measure projection in almost all of these directions, and so $\partial\Omega_j \cap \mathbb{B}$ contains an n-rectifiable set of $\mathcal{H}^n$-measure $\gtrsim 1$ (with constant depending on the sizes of B_1 and B_2). Thus,

$$\liminf_{j\to\infty} \frac{\mathcal{H}^n(B(\xi, r_j) \cap \partial\Omega \backslash F_2)}{\mathcal{H}^n(B(\xi, r_j) \cap \partial\Omega)} \gtrsim \liminf_{j\to\infty} \frac{r_j^n}{\mathcal{H}^n(B(\xi, r_j) \cap \partial\Omega)} \overset{(8\text{-}5)}{>} 0.$$

But this contradicts that ξ is a point of density for F_2. Therefore, $\mathcal{H}^n(F_2) = 0$, and we have now shown that $\mathcal{H}^n$-almost all of E' is rectifiable, and thus ω^{x_0}-almost all of E is contained in a countable union of Lipschitz graphs. This finishes the proof of Theorem IV.

9. Proof of Proposition III

Assume the conditions of the proposition. We recall the following result.

Theorem 9.1 [Hurri-Syrjänen 1994, Theorem 1.3]. *Suppose that $\Omega \subset \mathbb{R}^{n+1}$ is a bounded C-uniform*[2] *domain. If*

$$p \le q \le \frac{(n+1)p}{n+1-p(1-\delta)} \quad \textit{and} \quad p(1-\delta) < n+1,$$

then for all $u \in L^1_{\mathrm{loc}}(\Omega)$ such that $\nabla u(x) d(x, \partial\Omega)^\delta \in L^p(\Omega)$,

$$\inf_{a\in\mathbb{R}} \|u(x) - a\|_{L^q(\Omega)} \lesssim_{n,p,q,\delta,C} |\Omega|^{(1-\delta)/(n+1)+1/q-1/p} \|\nabla u \,\mathrm{dist}(\,\cdot\,, \Omega^c)^\delta\|_{L^p(\Omega)}. \tag{9-1}$$

(The explicit constant in (9-1) is written at the end of the proof on page 218 of [Hurri-Syrjänen 1994].) We will use this in the case that $\delta = \frac{1}{2}$ and $p = q = 2$, so (9-1) becomes

$$\inf_{a\in\mathbb{R}} \|u(x) - a\|_{L^2(\Omega)} \lesssim_{n,p,q,\delta,C} |\Omega|^{1/(2(n+1))} \|\nabla u \,\mathrm{dist}(\,\cdot\,, \Omega^c)^{1/2}\|_{L^2(\Omega)}. \tag{9-2}$$

Lemma 9.2. *Suppose $E \subset \mathbb{R}^{n+1}$ is a closed set and $\varepsilon : E^c \to [0, \infty]$ is a function such that for some ball B_0 centered on E*

$$\int_{E^c \cap B_0} \varepsilon(z)\, dz < \infty.$$

[2] In fact it holds for John domains.

Then for $\mathcal{H}^n$-a.e. $x \in E \cap B_0$,

$$\lim_{r\to 0} r^{-n} \int_{E^c \cap B(x,r)} \varepsilon(z)\, dz = 0.$$

Proof. Without loss of generality, we can assume $E \subset B_0$. Let $d\mu(z) = \varepsilon(z)\, dz|_{E^c}$. For $x \in E$ and $r > 0$, set

$$a(x,r) = \frac{\mu(B(x,r))}{r^n} = r^{-n} \int_{E^c \cap B(x,r)} \varepsilon(z)\, dz.$$

Suppose there is $F \subset E$ with $\mathcal{H}^n(F) > 0$ such that

$$\limsup_{r\to 0} a(x,r) > 0.$$

Then there is $t > 0$ and a compact set $G \subset F$ with $\mathcal{H}^n_\infty(G) > 0$ and

$$\limsup_{r\to 0} a(x,r) > t > 0 \quad \text{for all } x \in G.$$

For each $x \in G$, pick $r_{x,1} > 0$ so that $B(x, r_{x,1}) \subset B_0$ and $a(x, r_{x,1}) > t$. Let B_j^1 be a Besicovitch subcovering from $\mathscr{G}_1 := \{B(x, r_x^1) : x \in G\}$, that is, a countable collection of balls in $\mathscr{G}_1$ so that

$$\mathbb{1}_G \le \sum_j \mathbb{1}_{B_j^1} \lesssim_n 1.$$

Since the B_j^1 come from $\mathscr{G}$, we have that for all j

$$\frac{\mu(B_j^1)}{r_{B_j^1}^n} = a(x_{B_j^1}, r_{B_j^1}) > t.$$

Let

$$L_1 = \bigcup B_j^1 \backslash E.$$

Then since the B_j^1 have bounded overlap and come from $\mathscr{G}_1$,

$$\mu(L_1) = \int_{L_1} d\mu \gtrsim \int_{L_1} \sum_j \mathbb{1}_{B_j^1}\, d\mu = \sum_j \mu(B_j^1) > t \sum_j r_{B_j^1}^n \ge t\mathcal{H}^n_\infty(G).$$

Since $\mu(G) = 0$, there is $\delta_1 > 0$ so that if $G_{\delta_1} = \{x \in \mathbb{R}^n : \operatorname{dist}(x, G) < \delta_1\}$ and $L^1 = L_1 \backslash G_{\delta_1}$, then

$$\mu(L^1) > \tfrac{1}{2}\mu(L_1) \ge \tfrac{1}{2} t \mathcal{H}^n_\infty(G).$$

Now inductively, suppose we have constructed disjoint sets $L^1, \dots, L^k \subset B_0$, where

$$\mu(L^j) \gtrsim t\mathcal{H}^n_\infty(G) \quad \text{for all } j = 1, 2, \dots, k,$$

and there is $\delta_k > 0$ so that $L^1 \cup \dots \cup L^k \cap G_{\delta_k} = \varnothing$.

For each $x \in G$, we may find $r_{x,k+1} \in (0, \delta_k)$ so that $B(x, r_{x,k+1}) \subset B_0$ and $a(x, r_{x,k+1}) > t$. Let $\{B_j^{k+1}\}$ be a Besicovitch subcovering of the collection $\mathscr{G}_{k+1} = \{B(x, r_{x,k+1}) : x \in G\}$, so

$$\mathbb{1}_G \le \sum_j \mathbb{1}_{B_j^{k+1}} \lesssim_n \mathbb{1}_{L_{k+1}},$$

where $L_{k+1}=\bigcup_j B_j^{k+1}$. Since G has $\mu(G)=0$, there is $\delta_{k+1}\in(0,\delta_k)$ so that $L^{k+1}=L_{k+1}\backslash G_{\delta_{k+1}}$ has

$$\mu(L^{k+1})\geq\frac{\mu(L_{k+1})}{2}=\frac{1}{2}\int\mathbb{1}_{L_{k+1}}\,d\mu\gtrsim\int\sum_j\mu(B_j^{k+1})\geq t\sum_j r_{B_j^{k+1}}^n\gtrsim t\mathcal{H}_\infty^n(G).$$

Also note that by our induction hypothesis

$$L^{k+1}\subset L_{k+1}\subset G_{\delta_k}\subset(L^1\cup\cdots\cup L^k)^c.$$

Thus, by induction, we can come up with a sequence of disjoint sets $L^k\subset B_0$ so that $\mu(L^k)\gtrsim t\mathcal{H}_\infty^n(G)$ for all k, which contradicts the finiteness of μ since ε is locally integrable. □

Now we finish the proof of Proposition III. By the previous lemma, for $\varepsilon(z)=|\nabla A(z)|^2\operatorname{dist}(z,\Omega^c)$ and $E=\partial\Omega$, we have that for $\mathcal{H}^n$-a.e. $\xi\in B_0\cap\partial\Omega$

$$\lim_{r\to 0}r^{-n}\int_{B(\xi,r)\cap\Omega}|\nabla A|^2\operatorname{dist}(z,\Omega^c)\,dz=0. \tag{9-3}$$

Let $\xi\in B_0\cap\partial\Omega$ be such a point. There is a universal constant M depending on the uniformity constants so that, for all $r>0$, there is an MC-uniform domain Ω_r such that

$$\Omega\cap B(\xi,r)\subset\Omega_r\subset\Omega\cap B(\xi,Mr).$$

This follows from the proof of [Hofmann and Martell 2014, Lemma 3.61]. See also [Azzam 2016, Lemma 4.1; Jerison and Kenig 1982, Lemma 6.3].

Hence, by Cauchy–Schwarz inequality,

$$\begin{aligned}\inf_C r^{-(n+1)}\int_{B(\xi,r)\cap\Omega}|A-C| &\lesssim \inf_C\left(r^{-(n+1)}\int_{B(\xi,r)\cap\Omega}|A-C|^2\right)^{1/2}\\ &\leq \inf_C\left(r^{-(n+1)}\int_{\Omega_r}|A-C|^2\right)^{1/2}\\ &\overset{(9\text{-}2)}{\lesssim} |\Omega_r|^{1/(2(n+1))}\left(\frac{1}{r^{n+1}}\int_{\Omega_r}|\nabla A|^2\operatorname{dist}(z,\Omega_r^c)\,dz\right)^{1/2}\\ &\lesssim \left(r^{-n}\int_{\Omega\cap B(\xi,Mr)}|\nabla A|^2\operatorname{dist}(z,\Omega^c)\,dz\right)^{1/2}\to 0\quad\text{as } r\to 0.\end{aligned}$$

Acknowledgements

Both authors were supported by the ERC grant 320501 of the European Research Council (FP7/2007-2013). Mourgoglou was supported by Ikerbasque and partially supported by the grant MTM-2017-82160-C2-2-P of the Ministerio de Economía y Competitividad (Spain) and by IT-641-13 (Basque Government).

This project was started while we were postdocs under Tolsa's ERC grant 320501 of the European Research Council (FP7/2007-2013) between 2015 and 2016, and also during mutual visits between our home institutions of Universitat Autònoma de Barcelona (UAB), the Basque Center for Applied Mathematics, Universidad del País Vasco and the University of Edinburgh. We are grateful to these institutions for their hospitality, to Xavier Tolsa for his helpful discussions and encouragement on the project, José Conde for his clear presentation of [Badger 2011] in our working seminar at the UAB, and to Tatiana Toro for her comments on our paper. We would like to thank the anonymous referee for reading our paper carefully and providing us with many useful comments.

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Received 15 Aug 2017. Revised 19 Oct 2018. Accepted 30 Nov 2018.

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ANALYSIS AND PDE
Vol. 12, No. 8, 2019
dx.doi.org/10.2140/apde.2019.12.1943

DISCRETELY SELF-SIMILAR SOLUTIONS TO THE NAVIER–STOKES EQUATIONS WITH DATA IN L^2_{loc} SATISFYING THE LOCAL ENERGY INEQUALITY

ZACHARY BRADSHAW AND TAI-PENG TSAI

Chae and Wolf recently constructed discretely self-similar solutions to the Navier–Stokes equations for any discretely self-similar data in L^2_{loc}. Their solutions are in the class of local Leray solutions with projected pressure and satisfy the "local energy inequality with projected pressure". In this note, for the same class of initial data, we construct discretely self-similar suitable weak solutions to the Navier–Stokes equations that satisfy the classical local energy inequality of Scheffer and Caffarelli–Kohn–Nirenberg. We also obtain an explicit formula for the pressure in terms of the velocity. Our argument involves a new purely local energy estimate for discretely self-similar solutions with data in L^2_{loc} and an approximation of divergence-free, discretely self-similar vector fields in L^2_{loc} by divergence-free, discretely self-similar elements of L^3_w.

1. Introduction

The Navier–Stokes equations describe the evolution of a viscous incompressible fluid's velocity field v and associated scalar pressure π. In particular, v and π are required to satisfy

$$\partial_t v - \Delta v + v \cdot \nabla v + \nabla \pi = 0, \tag{1-1}$$

$$\nabla \cdot v = 0, \tag{1-2}$$

in the sense of distributions. For our purposes, (1-1) is applied on $\mathbb{R}^3 \times (0, \infty)$ and v evolves from a prescribed, divergence-free initial data $v_0 : \mathbb{R}^3 \to \mathbb{R}^3$. Solutions to (1-1) exhibit a natural scaling: if v satisfies (1-1), then for any $\lambda > 0$

$$v^\lambda(x, t) = \lambda v(\lambda x, \lambda^2 t) \tag{1-3}$$

is also a solution with pressure

$$\pi^\lambda(x, t) = \lambda^2 \pi(\lambda x, \lambda^2 t) \tag{1-4}$$

and initial data

$$v_0^\lambda(x) = \lambda v_0(\lambda x). \tag{1-5}$$

A solution is called self-similar (SS) if $v^\lambda(x, t) = v(x, t)$ for all $\lambda > 0$ and is discretely self-similar with factor λ (i.e., v is λ-DSS) if this scaling invariance holds for a given $\lambda > 1$. Similarly, v_0 is self-similar (a.k.a. (-1)-homogeneous) if $v_0(x) = \lambda v_0(\lambda x)$ for all $\lambda > 0$ or λ-DSS if this holds for a given $\lambda > 1$.

MSC2010: 35Q30, 76D05.
Keywords: Navier–Stokes equations, self-similar solution, weak solution.

These solutions can be either forward or backward if they are defined on $\mathbb{R}^3 \times (0, \infty)$ or $\mathbb{R}^3 \times (-\infty, 0)$ respectively. In this note we work exclusively with forward solutions and omit the qualifier "forward".

Self-similar solutions satisfy an ansatz for v in terms of a time-independent profile u, namely,

$$v(x,t) = \frac{1}{\sqrt{t}} u\left(\frac{x}{\sqrt{t}}\right), \tag{1-6}$$

where u solves the *Leray equations*

$$\begin{aligned} -\Delta u - \tfrac{1}{2}u - \tfrac{1}{2}y \cdot \nabla u + u \cdot \nabla u + \nabla p = 0, \\ \nabla \cdot u = 0 \end{aligned} \quad \text{in } \mathbb{R}^3, \tag{1-7}$$

in the variable $y = x/\sqrt{t}$. Discretely self-similar solutions are determined by their behavior on the time interval $1 \le t \le \lambda^2$ and satisfy the ansatz

$$v(x,t) = \frac{1}{\sqrt{t}} u(y,s), \tag{1-8}$$

where

$$y = \frac{x}{\sqrt{t}}, \quad s = \log t. \tag{1-9}$$

The vector field u is T-periodic with period $T = 2\log\lambda$ and solves the *time-dependent Leray equations*

$$\begin{aligned} \partial_s u - \Delta u - \tfrac{1}{2}u - \tfrac{1}{2}y \cdot \nabla u + u \cdot \nabla u + \nabla p = 0, \\ \nabla \cdot u = 0 \end{aligned} \quad \text{in } \mathbb{R}^3 \times \mathbb{R}. \tag{1-10}$$

Note that the *similarity transform* (1-8)–(1-9) gives a one-to-one correspondence between solutions to (1-1) and (1-10). Moreover, when v_0 is SS or DSS, the initial condition $v|_{t=0} = v_0$ corresponds to a boundary condition for u at spatial infinity; see [Korobkov and Tsai 2016; Bradshaw and Tsai 2017a; 2017b].

Self-similar solutions are interesting in a variety of contexts as candidates for ill-posedness or finite time blow-up of solutions to the 3-dimensional Navier–Stokes equations; see [Guillod and Šverák 2017; Jia and Šverák 2014; 2015; Leray 1934; Nečas et al. 1996; Tsai 1998] and the discussion in [Bradshaw and Tsai 2017a]. Forward self-similar solutions are compelling candidates for nonuniqueness [Jia and Šverák 2015; Guillod and Šverák 2017]. Until recently, the existence of forward self-similar solutions was only known for small data [Barraza 1996; Cannone and Planchon 1996; Giga and Miyakawa 1989; Koch and Tataru 2001; Kato 1992]. Such solutions are necessarily unique. Jia and Šverák [2014] constructed forward self-similar solutions for large data where the data is assumed to be Hölder continuous away from the origin. This result has been generalized in a number of directions by a variety of authors [Bradshaw and Tsai 2017a; 2017b; 2018; Chae and Wolf 2018; Korobkov and Tsai 2016; Lemarié-Rieusset 2016; Tsai 2014]. This paper can be understood in the context of [Bradshaw and Tsai 2017a; Chae and Wolf 2018; Lemarié-Rieusset 2016] and we briefly recall the main results of these papers.

In [Bradshaw and Tsai 2017a], we generalize [Jia and Šverák 2014] in two ways. First, all smoothness assumptions on the initial data are removed; we only require $v_0 \in L^3_w$ (and v_0 divergence-free and SS or DSS). Second, we allow the data to be DSS for any $\lambda > 1$, in which case we obtain DSS solutions as opposed to SS solutions — in contrast, the method of [Jia and Šverák 2014] can be adapted to give DSS solutions but only when λ is close to 1 [Tsai 2014]. The method of proof in [Bradshaw and Tsai 2017a] has since been extended to the half-space in [Bradshaw and Tsai 2017b] and to initial data in the Besov spaces $\dot B^{3/p-1}_{p,\infty}$ when $3 < p < 6$ [Bradshaw and Tsai 2018]. Solutions which satisfy a rotationally corrected scaling invariance are also constructed in [Bradshaw and Tsai 2017b].

The solutions of [Bradshaw and Tsai 2017a] belong to the class of *local Leray solutions*. This class was introduced in [Lemarié-Rieusset 2002] to provide a local analogue of Leray's weak solutions [1934]. We recall the definition of local Leray solutions in full. For $q \in [1, \infty)$, we say $f \in L^q_{\mathrm{uloc}}$ if

$$\|f\|_{L^q_{\mathrm{uloc}}} = \sup_{x\in\mathbb{R}^3} \|f\|_{L^q(B(x,1))} < \infty.$$

Definition 1.1 (local Leray solutions). A vector field $v \in L^2_{\mathrm{loc}}(\mathbb{R}^3 \times [0, \infty))$ is a local Leray solution to (1-1) with divergence-free initial data $v_0 \in L^2_{\mathrm{uloc}}$ if:

(1) For some $\pi \in L^{3/2}_{\mathrm{loc}}(\mathbb{R}^3 \times [0, \infty))$, the pair (v, π) is a distributional solution to (1-1).

(2) For any $R > 0$, the vector field v satisfies

$$\operatorname*{ess\,sup}_{0\le t<R^2} \sup_{x_0\in\mathbb{R}^3} \int_{B_R(x_0)} \tfrac12 |v(x,t)|^2\,dx + \sup_{x_0\in\mathbb{R}^3} \int_0^{R^2}\int_{B_R(x_0)} |\nabla v(x,t)|^2\,dx\,dt < \infty.$$

(3) For all compact subsets K of $\mathbb{R}^3$ we have $v(t) \to v_0$ in $L^2(K)$ as $t \to 0^+$.

(4) v is suitable in the sense of Caffarelli–Kohn–Nirenberg; i.e., for all cylinders Q compactly supported in $\mathbb{R}^3 \times (0, \infty)$ and all nonnegative $\phi \in C_0^\infty(Q)$, we have

$$\int |v(t)|^2\phi\,dx + 2\iint |\nabla v|^2\phi\,dx\,dt \le \iint |v|^2(\partial_t\phi + \Delta\phi)\,dx\,dt + \iint (|v|^2 + 2\pi)(v\cdot\nabla\phi)\,dx\,dt. \tag{1-11}$$

(5) For every $x_0 \in \mathbb{R}^3$, there exists $c_{x_0} \in L^{3/2}(0, T)$ such that

$$\begin{aligned} p(x,t) - c_{x_0}(t) = -\frac13 |v(x,t)|^2 &+ \frac{1}{4\pi}\int_{B_2(x_0)} K(x-y) : v(y,t)\otimes v(y,t)\,dy \\ &+ \frac{1}{4\pi}\int_{\mathbb{R}^3\setminus B_2(x_0)} (K(x-y) - K(x_0-y)) : v(y,t)\otimes v(y,t)\,dy \end{aligned}$$

in $L^{3/2}(0, T; L^{3/2}(B_1(x_0)))$, where $K(x) = \nabla^2(1/|x|)$.

Lemarié-Rieusset [2002] constructed global-in-time local Leray solutions if v_0 belongs to E^2, the closure of C_0^∞ in the $L^2_{\mathrm{uloc}}(\mathbb{R}^3)$ norm. See [Kikuchi and Seregin 2007] for another construction which treats the pressure carefully. Note that [Lemarié-Rieusset 2002; Kikuchi and Seregin 2007; Jia and Šverák 2014; 2015] contain alternative definitions of local Leray solutions. On one hand, [Kikuchi and Seregin 2007] requires the pressure satisfy a certain formula (we will establish a similar pressure formula for our solutions; see Theorem 1.2). In [Jia and Šverák 2014; 2015], the explicit pressure formula is replaced by a decay condition imposed on the solution at spatial infinity, namely, for all $R > 0$

$$\lim_{|x_0|\to\infty} \int_0^{R^2}\int_{B(x_0,R)} |v|^2\,dx\,dt = 0.$$

Jia and Šverák [2014; 2015] claim that, if v exhibits this decay, then the pressure formula from [Kikuchi and Seregin 2007] is valid. Since the decay property is easier to directly establish for a given solution, this justifies using it in place of the explicit pressure formula in the definition of local Leray solutions. It turns out that these properties are equivalent when $v_0 \in E^2$. This can be proved using ideas contained in a recent preprint of Maekawa, Miura, and Prange [Maekawa et al. 2019] on the construction of local energy solutions in the half-space.

Local Leray solutions are known to satisfy a useful a priori bound. Let $\mathcal{N}(v_0)$ denote the class of local Leray solutions with initial data v_0. The following estimate is well known for local Leray solutions (see [Jia and Šverák 2014]): for all $\tilde{v} \in \mathcal{N}(v_0)$ and $r > 0$ we have

$$\operatorname*{ess\,sup}_{0\le t\le\sigma r^2} \sup_{x_0\in\mathbb{R}^3} \int_{B_r(x_0)} \tfrac{1}{2}|\tilde{v}(x,t)|^2\,dx + \sup_{x_0\in\mathbb{R}^3} \int_0^{\sigma r^2}\int_{B_r(x_0)} |\nabla\tilde{v}|^2\,dx\,dt < CA, \tag{1-12}$$

where

$$A = \sup_{x_0\in\mathbb{R}^3} \int_{B_r(x_0)} \tfrac{1}{2}|v_0|^2\,dx, \quad \sigma(r) = c_0 \min\{r^2A^{-2}, 1\}, \tag{1-13}$$

for a small universal positive constant c_0.

Concurrently to the publication of [Bradshaw and Tsai 2017a], the book [Lemarié-Rieusset 2016] was published, which includes a chapter on the self-similar solutions of [Jia and Šverák 2014]. Here, Lemarié-Rieusset generalizes the space of initial data to include any L^2_{loc}, divergence-free, self-similar vector field. The main elements of his argument are as follows. He first uses the Leray–Schauder approach of [Jia and Šverák 2014] to construct self-similar solutions for initial data v_0 satisfying $|v_0(x)| \lesssim |x|^{-1}$. This construction is more general than that in [Jia and Šverák 2014] but less general than that in [Bradshaw and Tsai 2017a]. But, provided v_0 is self-similar, $v_0 \in L^2_{\mathrm{loc}}$ if and only if $v_0 \in L^2_{\mathrm{uloc}}$. And, furthermore, if v_0 is self-similar and belongs to L^2_{uloc}, then it can be approximated by a sequence $v_0^{(k)}$ where each $|v_0^{(k)}(x)| \lesssim |x|^{-1}$. Then, the first construction gives local Leray solutions for each $v_0^{(k)}$ and, because local Leray solutions satisfy the a priori bound (1-12) depending only on the L^2_{uloc} norm of their initial data, these will converge to an SS local Leray solution with L^2_{loc} data. This argument breaks down for DSS solutions since $L^2_{\mathrm{loc}} \cap \mathrm{DSS} \neq L^2_{\mathrm{uloc}} \cap \mathrm{DSS}$ (see (1-15) for an example) and, therefore, we cannot get the uniform bound (1-12) on a sequence of approximating solutions for free.

Chae and Wolf [2018], on the other hand, introduced an entirely new method to construct λ-DSS solutions for any $\lambda > 1$ and initial data $v_0 \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$. These solutions live in the class of "local Leray solutions with projected pressure", which means they satisfy a modified local energy inequality instead of the classical local energy inequality (1-11) of [Caffarelli et al. 1982]. To construct these solutions, Chae and Wolf use a fixed-point argument to solve the mollified Navier–Stokes equations (this is the same system studied in [Bradshaw and Tsai 2017a], but written in physical variables as opposed to the similarity variables, see (3-4) and (3-5)). To apply the fixed-point argument, Chae and Wolf first prove existence for the (mollified) linearized equations where the given drift velocity is DSS. They then apply a fixed-point theorem (the space for the fixed-point argument is a bounded set of the DSS subspace of $L^{18/5}(0,T;L^3(B_1))$ — B_r denotes the ball of radius r centered at the origin — defined below [Chae and Wolf 2018, (3.1)]) to prove that there exists a drift velocity which matches the solution. This gives existence of a DSS solution to the mollified Navier–Stokes equations. Note that the approximations satisfy the a priori (energy) bound [Chae and Wolf 2018, (2.35)] and the norm of the mollification term can be absorbed for T sufficiently small.

In this paper we give a simple, alternative proof of the result in [Chae and Wolf 2018]. The following theorem is our main result.

Theorem 1.2. *Assume $v_0 \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ is a divergence-free λ-DSS vector field for some $\lambda > 1$. Then there exists a λ-DSS distributional solution v to* (1-1) *and associated pressure π so that v is suitable in the sense of [Caffarelli et al. 1982] and satisfies*

$$\lim_{t\to 0^+} \|v(t) - v_0\|_{L^2(K)} = 0$$

for every compact subset K of $\mathbb{R}^3$. Moreover, for any $T > 0$ and compact subset K of $\mathbb{R}^3$, we have $v \in L^\infty(0, T; L^2(K)) \cap L^2(0, T; H^1(K))$ and $\pi \in L^{3/2}(0, T; L^{3/2}(K))$. Furthermore, for any $(x, t) \in \mathbb{R}^3 \times (0, \infty)$, the pressure satisfies the formula

$$\pi(x, t) = -\tfrac{1}{3}|v|^2(x, t) + \lim_{\delta\to 0} \int_{|y|>\delta} K_{ij}(x - y) v_i(y, t) v_j(y, t)\, dy \tag{1-14}$$

in $L^{3/2}_{\mathrm{loc}}(\mathbb{R}^3 \times (0, \infty))$.

Comments on Theorem 1.2. (1) In [Chae and Wolf 2018], the data also belongs to L^2_{loc}, but the solution is not shown to satisfy the local energy inequality of [Caffarelli et al. 1982]. Instead, it satisfies a "local energy inequality with projected pressure". Since the solution constructed in Theorem 1.2 satisfies the traditional local energy inequality, this theorem is a slight refinement of the main result of [Chae and Wolf 2018]. Furthermore, we are careful to give a precise formulation (1-14) of the pressure and its connection to the velocity. The relationship between v and π is less clear in [loc. cit.].

(2) The integral in (1-14) is not a Calderón–Zygmund singular integral because we do not have a global bound for v. It is defined in $L^{3/2}_{\mathrm{loc}}$ using the DSS property.

(3) Our method of proof is by approximation and is similar to the argument from [Lemarié-Rieusset 2016]. The main difference is that we need to construct a sequence of approximating solutions and establish a new a priori bound for these solutions for DSS data — in [loc. cit.] the bound (1-12) is sufficient (and free). Note that an approximation argument using (1-12) was also used by the authors in [Bradshaw and Tsai 2017a] to construct SS solutions as a limit of DSS solutions where the scaling factors are converging to 1.

(4) Generally, the solution v is not necessarily a local Leray solution because v_0 may not be in L^2_{uloc}, and we do not assert the uniform bounds in Definition 1.1(2). Consider the DSS function in L^2_{loc} for $0 < a < \frac{3}{2}$

$$f_a(x) = \sum_{k\in\mathbb{Z}} \lambda^k f_{a,0}(\lambda^k x), \qquad f_{a,0}(x) = |x - x_0|^{-a} \chi(x - x_0), \tag{1-15}$$

where $1+r < |x_0| < \lambda - r$ for some $r > 0$, and χ is the characteristic function of the ball $B_r(0)$. It is not in L^2_{uloc} when $1 < a < \frac{3}{2}$ for its behavior at infinity. It is in L^2_{uloc} when $0 < a \le 1$. The function $f_1(x)$ for $a = 1$ is given in Comment 4 after [Bradshaw and Tsai 2017a, Theorem 1.2] as an inapplicable example since it is not in $L^{3,\infty}(\mathbb{R}^3)$.

(5) If $v_0 \in L^2_{\mathrm{uloc}}$, then it is not difficult to obtain uniform bounds on v in the sense of Definition 1.1(2). Furthermore, Definition 1.1(5) can be established whenever $v_0 \in E^2$; see [Maekawa et al. 2019]. Thus, our construction yields DSS local Leray solutions whenever the data is DSS, divergence-free, and in E^2.

Our strategy for proving Theorem 1.2 is to approximate a solution with data in L^2_{loc} using solutions constructed in [Bradshaw and Tsai 2017a]. There are several steps. First we need to prove that DSS data in L^2_{loc} can be approximated in $L^2(B_1)$ by DSS data in L^3_w. This is the subject of Section 4A. Then, [loc. cit.] gives us a sequence of DSS solutions in the local Leray class. To prove that these solutions converge to a solution with L^2_{loc} data satisfying the desired pressure formula, we need to establish new a priori bounds for the solutions from [loc. cit.] which are independent of the L^3_w norm of the initial data (this is done in Section 3) and also prove that they satisfy the pressure formula (see Section 2). In Sections 4B and 4C, we put these ingredients together to prove Theorem 1.2.

As a last remark, in [Chae and Wolf 2018] it is unclear if the solution is suitable in the classical sense. The referee for this paper suggested a compelling argument to address this. In particular, the discretely self-similar ansatz and the boundedness of the solution in $L^2_{\mathrm{loc}}(\mathbb{R}^3 \times [0,\infty))$ should make it possible to define $\mathbb{P}\nabla \cdot (u \otimes u)$. Then, starting with a solution of [loc. cit.], a pressure p could be constructed in $\mathcal{D}'$. It then could be shown that $\nabla p + \mathbb{P}(u \cdot \nabla u) = 0$. This should follow from the slow growth of u at spatial infinity and using the fact that $\nabla p + \mathbb{P}(u \cdot \nabla u)$ is spatially harmonic.

2. A limiting pressure formula for DSS solutions

In this section we will prove that, under certain conditions, the limiting pressure distribution of an approximation scheme for (1-1) inherits the structure of the approximate pressure distributions. This result will be applied in Sections 3 and 4C.

Lemma 2.1. *Fix $\lambda > 1$ and $T > 0$. Let $v_0 \in L^2_{\mathrm{loc}}$ be a given divergence-free, λ-DSS vector field and assume $\{v_0^{(k)}\} \subset L^2_{\mathrm{loc}}$ is a sequence of divergence-free, λ-DSS vector fields so that $v_0^{(k)} \to v_0$ in $L^2(B_1)$. Assume v_k and $\tilde v_k$ are divergence-free, λ-DSS vector fields and that there exists a distribution π_k so that the following conditions are satisfied*:

- *v_k, $\tilde v_k$, and π_k solve the system*

$$\partial_t v_k - \Delta v_k + \tilde v_k \cdot \nabla v_k + \nabla \pi_k = 0, \quad (x,t) \in \mathbb{R}^3 \times [0,T],$$

for the initial data $v_0^{(k)}$ and both v_k and $\tilde v_k$ converge to $v_0^{(k)}$ in L^2_{loc}.

- *v_k and $\tilde v_k$ are uniformly bounded in $L^\infty(0,T;L^2(B_1)) \cap L^2(0,T;H^1(B_1))$ over all $k \in \mathbb{N}$.*

- *For all* $0 < t \le T$, *the distribution* π_k *satisfies the formula*

$$\pi_k(x,t) = -\tfrac{1}{3}[\tilde v_k \cdot v_k](x,t) + \lim_{\delta\to 0}\int_{|y|>\delta} K_{ij}(x-y)(\tilde v_k)_i(y,t)(v_k)_j(y,t)\,dy. \tag{2-1}$$

- *There exists a* λ*-DSS solution* v *in* $L^\infty(0,T;L^2(B_1))\cap L^2(0,T;H^1(B_1))$ *with pressure* π *in* $L^{3/2}(0,T;L^{3/2})$ *so that*

$$\begin{aligned} v_k \text{ and } \tilde v_k \to v \text{ weakly} &\quad \text{in } L^2(0,T;H^1(B_1)),\\ v_k \text{ and } \tilde v_k \to v &\quad \text{in } L^2(0,T;L^2(B_1)),\\ \pi_k \to \pi \text{ weakly} &\quad \text{in } L^{3/2}(0,T;L^{3/2}(B_1)). \end{aligned}$$

Then, for a.e. $0 < t \le T$ *and* $x \in B_\lambda$, *the pressure* π *satisfies the formula*

$$\pi(x,t) = -\tfrac{1}{3}|v|^2(x,t) + \lim_{\delta\to 0}\int_{|y|>\delta} K_{ij}(x-y)(v)_i(y,t)(v)_j(y,t)\,dy \tag{2-2}$$

in $L^{3/2}((0,T)\times B_\lambda)$.

Remark 2.2. The purpose of this lemma is to establish the pressure formula (2-2), which, ultimately, will allow us to prove (1-14). It is, however, not needed to establish the other conclusions of Theorem 1.2.

Proof. Note that since v_k, $\tilde v_k$, and v are all uniformly bounded in $L^\infty(0,T;L^2(B_1))\cap L^2(0,T;H^1(B_1))$, convergence in $L^2(0,T;L^2(B_1))$, Hölder's inequality, Sobolev embedding, using the equation to get uniform bound of $\partial_t v_k$, and rescaling the solution imply

$$v_k \text{ and } \tilde v_k \to v \quad \text{in } L^3(0,T;L^3(B_1)).$$

It also shows that v_k, $\tilde v_k$, and v are all uniformly bounded in $L^3(0,T;L^3(B_1))$ (at least for k sufficiently large).

Let

$$\begin{aligned} \pi_k^1(x,t) &= -\tfrac{1}{3}[\tilde v_k\cdot v_k](x,t),\\ \pi_k^2(x,t) &= \lim_{\delta\to 0}\int_{\lambda^2>|y|>\delta} K_{ij}(x-y)(\tilde v_k)_i(y,t)(v_k)_j(y,t)\,dy,\\ \pi_k^3(x,t) &= \int_{y\ge\lambda^2} K_{ij}(x-y)(\tilde v_k)_i(y,t)(v_k)_j(y,t)\,dy. \end{aligned}$$

Also let

$$\begin{aligned} \pi^1(x,t) &= -\tfrac{1}{3}|v|^2(x,t),\\ \pi^2(x,t) &= \lim_{\delta\to 0}\int_{\lambda^2>|y|>\delta} K_{ij}(x-y)v_i(y,t)v_j(y,t)\,dy,\\ \pi^3(x,t) &= \int_{y\ge\lambda^2} K_{ij}(x-y)v_i(y,t)v_j(y,t)\,dy. \end{aligned}$$

Since v_k and $\tilde v_k \to v$ in $L^3(0,T;L^3(B_\lambda))$, we have

$$\pi_k^1 \to \pi^1 \quad \text{in } L^{3/2}(0,T;L^{3/2}(B_\lambda)).$$

Let

$$h_{i,j}(y,t) = (\tilde{v}_k)_i (v_k)_j - v_i v_j$$
$$= \{(\tilde{v}_k)_i[(v_k)_j - v_j] + [(\tilde{v}_k)_i - v_i]v_j\}(y,t).$$

Using the Calderón–Zygmund theory we clearly have

$$\begin{aligned}\int_0^T \int_{B_\lambda} |\pi_k^2(x,t) - \pi^2(x,t)|^{3/2}\,dx\,dt &\le C \int_0^T \int_{B_{\lambda^2}} |h_{i,j}(x,t)|^{3/2}\,dx\,dt \\ &\le C\left(\int_0^T \int_{B_{\lambda^2}} \tilde{v}_k^3\,dx\,dt\right)^{1/2} \left(\int_0^T \int_{B_{\lambda^2}} (v_k - v)^3\,dx\,dt\right)^{1/2} \\ &\quad + C\left(\int_0^T \int_{B_{\lambda^2}} v^3\,dx\,dt\right)^{1/2} \left(\int_0^T \int_{B_{\lambda^2}} (\tilde{v}_k - v)^3\,dx\,dt\right)^{1/2}. \qquad (2\text{-}3)\end{aligned}$$

Rescaling gives

$$\int_0^T \int_{B_{\lambda^2}} (\tilde{v}_k - v)^3(x,t)\,dx\,dt = \lambda^4 \int_0^{T\lambda^{-4}} \int_{B_1} (\tilde{v}_k - v)^3(z,\tau)\,dz\,d\tau$$

for the obvious choice of z and τ. Since the right-hand side of the equation above vanishes as $k \to \infty$, as does the identical term but with $\tilde{v}_k$ replaced by v_k, we conclude that π_k^2 converges to π^2 in $L^{3/2}(0,T;L^{3/2}(B_1))$.

Establishing the convergence of π_k^3 to π^3 is more difficult. Let

$$p_k(x,t) = \pi_k^3(x,t) - \pi^3(x,t) = \int_{|y|\ge\lambda^2} K_{ij}(x-y)h_{i,j}(y,t)\,dy.$$

Fix $x \in B_\lambda$. Then

$$\begin{aligned}|p_k(x,t)|^{3/2} &\le C\left|\int_{|y|\ge\lambda^2} \frac{1}{|y|^3}|h_{i,j}(y,t)|\,dy\right|^{3/2} \\ &\le C\left(\int_{|y|\ge\lambda^2} \frac{1}{|y|^4}\,dy\right)^{1/2} \int_{|y|\ge\lambda^2} \frac{1}{|y|^{5/2}}|h_{i,j}(y,t)|^{3/2}\,dy \\ &= C\int_{|y|\ge\lambda^2} \frac{1}{|y|^{5/2}}|h_{i,j}(y,t)|^{3/2}\,dy.\end{aligned}$$

Let $A_k = \{x : \lambda^{k-1} \le |x| < \lambda^k\}$ for $k \in \mathbb{Z}$. Then, using the scaling properties of h,

$$\begin{aligned}\int_{|y|\ge\lambda^2} \frac{1}{|y|^{5/2}}|h_{i,j}(y,t)|^{3/2}\,dy &= \sum_{k=3}^{\infty} \int_{A_k} \frac{1}{|y|^{5/2}}|h_{i,j}(y,t)|^{3/2}\,dy \\ &\le C(\lambda)\sum_{k=3}^{\infty} \frac{1}{\lambda^{5k/2}} \int_{A_k} |h_{i,j}(y,t)|^{3/2}\,dy \\ &\le C(\lambda)\sum_{k=3}^{\infty} \frac{1}{\lambda^{5k/2}} \int_{B_1} |h_{i,j}(z,t\lambda^{-2k})|^{3/2}\,dz.\end{aligned}$$

Thus,

$$\begin{aligned}\int_0^T\int_{B_\lambda}|p_k(x,t)|^{3/2}\,dt &\le \lambda^3 C(\lambda)\int_0^T\sum_{k=3}^{\infty}\frac{1}{\lambda^{5k/2}}\int_{B_1}|h_{i,j}(z,t\lambda^{-2k})|^{3/2}\,dz\,dt\\ &\le C(\lambda)\sum_{k=3}^{\infty}\frac{1}{\lambda^{k/2}}\int_0^{T\lambda^{-2k}}\int_{B_1}|h_{i,j}(z,\tau)|^{3/2}\,dz\,d\tau\\ &\le C(\lambda)\int_0^T\int_{B_1}|h_{i,j}(z,\tau)|^{3/2}\,dz\,d\tau.\end{aligned}$$

Therefore this term is bounded as (2-3).

We have now shown that $\pi_k(x,t)$ converges weakly to both $\pi^1(x,t)+\pi^2(x,t)+\pi^3(x,t)$ and $\pi(x,t)$ in $L^{3/2}(0,T;L^{3/2}(B_\lambda))$, implying that $\pi(x,t)=\pi^1(x,t)+\pi^2(x,t)+\pi^3(x,t)$ as distributions. In other words, $\pi(x,t)$ satisfies (2-2) in $L^{3/2}((0,T)\times B_\lambda)$. □

3. Properties of DSS solutions with data in L^3_w

The goal of this section is to obtain a bound on the local evolution of DSS solutions v constructed in [Bradshaw and Tsai 2017a] that is independent of both the L^3_w and L^2_{uloc} norms of v and to establish an explicit representation formula for the pressure.

Assume $v_0\in L^3_w(\mathbb{R}^3)$ and v is a DSS solution evolving from v_0 as constructed in [loc. cit.]. For a generic solution to (1-1), we cannot close energy estimates for ϕv solely in terms of $v_0|_{B_\lambda}$ — there is always some spillover. Proposition 3.1 states that this is possible for DSS solutions as a result of their scaling properties. In our argument, we must work with a quantity that is continuous in time. This is not known for $\int_{B_1}|v(t)|^2\,dx$ when v is a local Leray solution. Hence, we need to work at the level of a *mollified approximation scheme* [loc. cit., (2.24)] (see (3-4) below). Note that in [loc. cit.], the mollified scheme is used to approximate a solution to the time-periodic Leray equations and the mollification is time-independent. Undoing the similarity transformation results in a time-dependent mollification of the drift component of the nonlinear term of the solution in the physical variables (see (3-5) below); this matches the mollification used in [Chae and Wolf 2018].

Proposition 3.1. *Fix $\lambda>1$. Assume $v_0\in L^3_w(\mathbb{R}^3)$ is λ-DSS and divergence-free, and v is a λ-DSS local Leray solution evolving from v_0 constructed in [Bradshaw and Tsai 2017a] (in particular, it is the limit of the mollified approximation scheme* (2.24) *in that paper) and π is its associated pressure. Let $\alpha_0=\|v_0\|^2_{L^2(B_\lambda)}$. Then, there exist positive $T=T(\alpha_0,\lambda)$ and $C(\alpha_0,\lambda)$ independent of $\|v_0\|_{L^2_{\mathrm{uloc}}}$ and $\|v_0\|_{L^3_w}$ so that*

$$\operatorname*{ess\,sup}_{0\le t\le T}\int_{B_1}|v(x,t)|^2\,dx+\int_0^T\int_{B_1}|\nabla v|^2\,dx\,dt<C(\alpha_0,\lambda), \tag{3-1}$$

and

$$\int_0^T\int_{B_1}|\pi(x,t)|^{3/2}\,dx\,dt<C(\alpha_0,\lambda). \tag{3-2}$$

Moreover, for $x \in B_1$ and $t \in (0,T)$, the pressure satisfies the formula

$$\pi(x,t) = -\tfrac{1}{3}|v|^2(x,t) + \lim_{\delta\to 0}\int_{|y|>\delta} K_{ij}(x-y)v_i(y,t)v_j(y,t)\,dy \tag{3-3}$$

in $L^{3/2}(B_1\times(0,T))$.

Typically, the best pressure decompositions we have for local Leray solutions depend on a particular ball containing the spatial point at which the pressure is being computed. The resulting formula consists of a local Calderón–Zygmund part and a far-field part with a singular kernel that is decaying faster than the kernel of K. The formula (3-3) does not involve such a decomposition, and, as is evident in the proof, the integral in (3-3) is defined using the DSS property.

The proof of [Bradshaw and Tsai 2017a] shows that the left sides of (3-1) and (3-2) are bounded by constants depending on v_0, in particular its $L^3_w(\mathbb{R}^3)$-norm. For this application, we need a bound depending only on $\|v_0\|_{L^2(B_\lambda)}$ and λ.

Proof. Since v is a solution from [Bradshaw and Tsai 2017a], its image under the similarity transform (1-9) solves the time-periodic Leray equations and is the limit of a mollified approximation scheme [loc. cit., (2.24)]. In particular, for each $\epsilon > 0$, there exists a time-periodic solution u_ϵ to the problem

$$\big(\partial_s u_\epsilon - \Delta u_\epsilon - \tfrac{1}{2}u_\epsilon - \tfrac{1}{2}y\cdot\nabla u_\epsilon + (\eta_\epsilon * u_\epsilon)\cdot\nabla u_\epsilon + \nabla p_\epsilon\big)(y,s) = 0, \tag{3-4}$$

where $\eta_\epsilon(y) = (1/\epsilon^3)\eta(y/\epsilon)$ and η is in $C_0^\infty(\mathbb{R}^3)$, is nonnegative, and satisfies $\int \eta(y)\,dy = 1$. Applying (1-8)–(1-9) we obtain a λ-DSS vector field v_ϵ satisfying

$$\partial_t v_\epsilon(x,t) - \Delta v_\epsilon(x,t) + (\eta_{\epsilon\sqrt{t}} * v_\epsilon)\cdot\nabla v_\epsilon(x,t) + \nabla\pi_\epsilon(x,t) = 0. \tag{3-5}$$

Note the time dependence of the convolution kernel $\eta_{\epsilon\sqrt{t}}$ in (3-5).

By the convergence properties of $u_\epsilon(y,s)$ to $u(y,s) = \sqrt{t}v(x,t)$ [loc. cit., p. 1108] and discretely self-similar scaling (to extend the estimates down to $t=0$), it follows that for all $T>0$ and all compact sets $K\subset\mathbb{R}^3$,

$$\begin{aligned} v_\epsilon \to v \text{ weakly} &\quad \text{in } L^2(0,T;H^1(K)),\\ v_\epsilon \to v \text{ strongly} &\quad \text{in } L^2(0,T;L^2(K)),\\ v_\epsilon(s) \to v(s) \text{ weakly} &\quad \text{in } L^2(K) \text{ for all } s\in[0,T]. \end{aligned}$$

Note also that $v_\epsilon(t)\to v_0$ in L^2_{loc}; i.e., the mollification does not affect the initial data. Furthermore, because each v_ϵ is smooth on $\mathbb{R}^3\times(0,\infty)$ and right continuous in L^2_{loc} at $t=0$, it follows that

$$\alpha_\epsilon(t) = \int_{B_1} |v_\epsilon(x,t)|^2\,dx$$

and

$$\tilde{\alpha}_\epsilon(t) = \sup_{0\le\tau\le t}\alpha_\epsilon(\tau)$$

are continuous as functions of t. This is not clearly true for $\int_{B_1}|v(x,t)|^2\,dx$.

Note that, for any $k \in \mathbb{Z}$ and $q \in [1, \infty)$, since $v_\epsilon(x,t) = \lambda^{-k} v_\epsilon(\lambda^{-k}x, \lambda^{-2k}t)$,

$$\int_{B_{\lambda^k}} |v_\epsilon(x,t)|^q \, dx = \lambda^{(3-q)k} \int_{B_1} |v_\epsilon(\tilde{x}, \lambda^{-2k}t)|^q \, d\tilde{x}. \tag{3-6}$$

Our goal is to establish local-in-time a priori bounds for $\alpha_\epsilon(t)$ that are independent of ϵ. Note that v_ϵ satisfies the local energy equality; i.e.,

$$\begin{aligned} \int |v_\epsilon|^2 \phi(t)\, dx + 2\int_0^t \int |\nabla v_\epsilon|^2 \phi \, dx\, ds \\ = \int |v_0|^2 \phi \, dx + \int_0^t \int |v_\epsilon|^2 (\partial_s \phi + \Delta\phi)\, dx\, ds \\ + \int_0^t \int (|v_\epsilon|^2 ((\eta_{\epsilon\sqrt{s}} * v_\epsilon) \cdot \nabla\phi)\, dx\, ds + \int_0^t \int 2\pi_\epsilon (v_\epsilon \cdot \nabla \phi)\, dx\, ds \end{aligned} \tag{3-7}$$

for any nonnegative $\phi \in C_0^\infty(\mathbb{R}^3 \times [0,\infty))$. Fix $\chi \in C^\infty(\mathbb{R})$ with $\chi(t) = 1$ if $t \le 1$ and $\chi(t) = 0$ if $t \ge \lambda$. We now fix ϕ in (3-7) as

$$\phi(x,t) = \chi^2(|x|) \cdot \chi(t).$$

We will estimate the terms on the right-hand side of (3-7) for $0 < t \le 1$, and we can treat ϕ as t-independent from now on. The first term is bounded by α_0. For the second, using the scaling properties (3-6) of v_ϵ, we have

$$\int_0^t \int |v_\epsilon|^2 (\partial_s\phi + \Delta\phi)\, dx\, ds \le C \int_0^t \int_{B_\lambda} |v_\epsilon|^2 \, dx\, ds \le C\lambda^3 \int_0^{t/\lambda^2} \int_{B_1} |v_\epsilon|^2 \, dx\, ds \le C(\lambda) \int_0^t \tilde{\alpha}_\epsilon(s)\, ds.$$

For the cubic term, we begin by using Young's inequality to obtain

$$\int_0^t \int |v_\epsilon|^2 ((\eta_{\epsilon\sqrt{s}} * v_\epsilon) \cdot \nabla\phi)\, dx\, ds \le C \int_0^t \int_{B_\lambda} |v_\epsilon|^3 \, dx\, ds + C \int_0^t \int_{B_\lambda} |(\eta_{\epsilon\sqrt{s}} * v_\epsilon)|^3 \, dx\, ds.$$

Rescaling the unmollified term and making the obvious change of variables results in the estimate

$$\int_0^t \int_{B_\lambda} |v_\epsilon|^3 \, dx\, ds \le C(\lambda) \int_0^{t/\lambda^2} \int_{B_1} |v_\epsilon|^3 \, dy\, d\tau \le C(\lambda) \int_0^t \int |v_\epsilon|^3 \phi^{3/2} \, dx\, ds.$$

For the term involving the mollifier, note that $\eta \in C_0^\infty$ and $\operatorname{supp} \eta \subset B_\rho$ for some $\rho > 0$. By taking ϵ sufficiently small we can ensure that $\operatorname{supp} \eta_{\epsilon\sqrt{s}} \subset B_{\lambda - 1}$ whenever $s < 1$. Note $\lambda^k + (\lambda - 1) \le \lambda^{k+1}$ for all $k \ge 0$. Thus, for $x \in B_\lambda$,

$$\begin{aligned} |(\eta_{\epsilon\sqrt{s}} * v_\epsilon)(x,s)| &\le \int \eta_{\epsilon\sqrt{s}}(y) |v_\epsilon(x-y,s)|\, dy \\ &= \int \eta_{\epsilon\sqrt{s}}(y) |v_\epsilon(x-y,s)| \chi_{B_{\lambda^2}}(x-y)\, dy \\ &= (\eta_{\epsilon\sqrt{s}} * (\chi_{B_{\lambda^2}} |v_\epsilon|))(x,s) \end{aligned}$$

whenever ϵ is sufficiently small and $s<1$. Therefore, under the same assumptions and after rescaling we see that, for any $1<q<\infty$,

$$\|(\eta_{\epsilon\sqrt{s}}*v_\epsilon)(s)\|_{L^q(B_\lambda)} \le C(q,\eta)\|v_\epsilon(s)\|_{L^q(B_{\lambda^2})} \le C(q,\eta,\lambda)\|v_\epsilon(\lambda^{-4}s)\|_{L^q(B_1)}, \tag{3-8}$$

where C is independent of s and ϵ. Note that this estimate is also valid if B_λ is replaced by B_{λ^2} but with a different choice of constants, smallness condition on ϵ, and right-hand side determined at time $\lambda^{-6}s$.

Using standard inequalities and (3-8) with $q=3$ thus leads to the estimate

$$\int_0^t\int |v_\epsilon|^2((\eta_{\epsilon\sqrt{s}}*v_\epsilon)\cdot\nabla\phi)\,dx\,ds \le C(\eta,\lambda)\int_0^t\int |v_\epsilon|^3\phi^{3/2}\,dx\,ds. \tag{3-9}$$

By the Gagliardo–Nirenberg inequality and rescaling (3-6), we have, for any $s>0$, that

$$\begin{aligned}\|\phi^{1/2}v_\epsilon(s)\|_{L^3} &\le C\|\nabla\otimes(\phi^{1/2}v_\epsilon)\|_{L^2}^{1/2}\|\phi^{1/2}v_\epsilon\|_{L^2}^{1/2}(s)\\ &\le C(\lambda)(\tilde\alpha_\epsilon(s)^{1/2}+\|\phi^{1/2}\nabla v_\epsilon(s)\|_{L^2})^{1/2}(\tilde\alpha_\epsilon(s))^{1/4}.\end{aligned}$$

Hence, for any $\gamma>0$,

$$\|\phi^{1/2}v_\epsilon(s)\|_{L^3}^3 \le C(\lambda)(\gamma^{-3}\tilde\alpha_\epsilon(s)^3+\gamma\tilde\alpha_\epsilon(s)^1+\gamma\|\phi^{1/2}\nabla v_\epsilon(s)\|_2^2).$$

Thus,

$$\begin{aligned}&\int_0^t\int |v_\epsilon|^2((\eta_{\epsilon\sqrt{s}}*v_\epsilon)\cdot\nabla\phi)\,dx\,ds\\ &\qquad\le C(\lambda,\gamma,\eta)\int_0^t(\tilde\alpha_\epsilon(s)^3+\tilde\alpha_\epsilon(s)^1)\,ds + C(\lambda)\gamma\int_0^t\int|\nabla v_\epsilon|^2\phi\,dx\,ds.\end{aligned} \tag{3-10}$$

Provided γ is small enough, the gradient term can be absorbed into the left-hand side of (3-7).

We next estimate the pressure term. For this we need a formula for the pressure, which we presently justify. Let $w_\epsilon=v_\epsilon-V_0$, where $V_0(x,t)=e^{t\Delta}v_0$. We have

$$\partial_t w_\epsilon-\Delta w_\epsilon+\nabla\pi_\epsilon=g,\quad \operatorname{div} w_\epsilon=0,$$

where $g_i=-\partial_j G_{ji}$ with

$$\begin{aligned}G&=(\eta_{\epsilon\sqrt{t}}*v_\epsilon)\otimes v_\epsilon\\ &=(\eta_{\epsilon\sqrt{t}}*w_\epsilon+\eta_{\epsilon\sqrt{t}}*V_0)\otimes(w_\epsilon+V_0).\end{aligned}$$

For $0<t_1<t_2<\infty$, we have

$$\begin{aligned}&V_0\in C([t_1,t_2];\,L^4(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)),\\ &w_\epsilon\in L^\infty(t_1,t_2;\,L^2(\mathbb{R}^3))\cap L^2(t_1,t_2;\,L^6(\mathbb{R}^3))\subset L^4(t_1,t_2;\,L^3(\mathbb{R}^3)).\end{aligned}$$

By Young's convolution inequality,

$$\|G\|_{L^2(t_1,t_2;L^2)} \lesssim \|\eta_{\epsilon\sqrt{t}}\|_{L^\infty(t_1,t_2;L^{6/5}\cap L^1)}(\|w_\epsilon\|_{L^4(t_1,t_2;L^3(\mathbb{R}^3))}+\|V_0\|_{L^4(\mathbb{R}^3\times[t_1,t_2])})^2.$$

Since $g\in L^2([t_1,t_2];H^{-1})$, [Caffarelli et al. 1982, Lemma A.2] implies $w_\epsilon\in C([t_1,t_2];L^2)$ (after modification on a set of time of measure zero; since the modified vector field still satisfies the above system distributionally, this does not effect our argument).

Consider the following nonstationary Stokes system with forcing g:

$$\partial_t V - \Delta V + \nabla P = g, \quad \operatorname{div} V = 0,$$

with initial data $V_0 = w_\epsilon(t_1) \in L^2(\mathbb{R}^3)$. It is well known that if $g \in L^\infty(t_1, t_2; H^{-1})$ and $V_0 \in L^2$, then there exists a unique $V \in C_w([t_1, t_2]; L^2(\mathbb{R}^3)) \cap L^2([t_1, t_2]; H^1(\mathbb{R}^3))$ and unique ∇P solving the nonstationary stokes system given above; see [Bradshaw and Tsai 2017a, p. 1107–1108]. Letting $V = w_\epsilon$ and $P = \pi_\epsilon$, this implies that w_ϵ and $\nabla \pi_\epsilon$ are unique. Up to a function $\pi_*(t)$ independent of x,

$$\pi_\epsilon(x,t) - \pi_*(t) = -\tfrac{1}{3}[(\eta_{\epsilon\sqrt{t}} * v_\epsilon)\cdot v_\epsilon](x,t) + \lim_{\delta\to 0}\int_{|y|>\delta} K_{ij}(x-y)(\eta_{\epsilon\sqrt{t}} * v_\epsilon)_i(y,t)(v_\epsilon)_j(y,t)\,dy, \quad (3\text{-}11)$$

where

$$K_{ij}(x) = \partial_i \partial_j \frac{1}{4\pi |x|}.$$

The right-hand side of (3-11) is defined in $L^2([t_1, t_2]; L^2(\mathbb{R}^3))$. Since the only appearance of π_ϵ in (3-5) is $\nabla \pi_\epsilon$, we can redefine π_ϵ to equal $\pi_\epsilon - \pi_*(t)$ and, therefore, can drop $\pi_*(t)$ from (3-11).

The pressure π_ϵ given by (3-11) is already bounded in $L^2([t_1, t_2]; L^2(\mathbb{R}^3))$ for any $0 < t_1 < t_2 < \infty$ but the bound depends on t_1, t_2 and ϵ. We now bound it in $L^{3/2}(0, T; L^{3/2}(B_\lambda))$. Bounding the first term from (3-11) is simple given Hölder's inequality, (3-8), and (3-9). In particular, we have for any $\gamma > 0$

$$\int_0^t \big\| \tfrac{1}{3} |(\eta_{\epsilon\sqrt{s}} * v_\epsilon)(\cdot, s)|\, |v_\epsilon(\cdot, s)| \big\|^{3/2}_{L^{3/2}(B_\lambda)} ds \le C(\lambda, \gamma, \eta) \int_0^t (\tilde\alpha_\epsilon(s)^3 + \tilde\alpha_\epsilon(s)^1)\,ds + \gamma \int_0^t \int |\nabla v_\epsilon|^2 \phi\,dx\,ds.$$

To bound the principal value integral in (3-11), we need to split the integral into local and nonlocal parts as follows:

$$\begin{aligned}
&\lim_{\delta\to 0}\int_{|y|>\delta} K(x-y)(\eta_{\epsilon\sqrt{t}} * v_\epsilon)(y,t) v_\epsilon(y,t)\,dy \\
&\quad = \lim_{\delta\to 0}\int_{B_{\lambda^2}\setminus B_\delta} K(x-y)(\eta_{\epsilon\sqrt{t}} * v_\epsilon)(y,t) v_\epsilon(y,t)\chi_{B_{\lambda^2}}(y)\,dy + \int_{|y|>\lambda^2} K(x-y)(\eta_{\epsilon\sqrt{t}} * v_\epsilon)(y,t) v_\epsilon(y,t)\,dy \\
&\quad =: \pi_{\mathrm{near}}(x,t) + \pi_{\mathrm{far}}(x,t).
\end{aligned}$$

To bound π_{near} note that, by the Calderón–Zygmund theory,

$$\|\pi_{\mathrm{near}}(\cdot, t)\|_{L^{3/2}(B_\lambda)} \le \|(\eta_{\epsilon\sqrt{t}} * v_\epsilon)(\cdot, t) v_\epsilon(\cdot, t)\|_{L^{3/2}(B_{\lambda^2})},$$

and, arguing as above using (3-8) but with B_{λ^2} in place of B_λ (see the note following (3-8)), it follows that

$$\int_0^t \|\pi_{\mathrm{near}}(\cdot, s)\|^{3/2}_{L^{3/2}(B_\lambda)} ds \le C(\lambda, \gamma, \eta) \int_0^t (\tilde\alpha_\epsilon(s)^3 + \tilde\alpha_\epsilon(s)^1)\,ds + \gamma \int_0^t \int |\nabla v_\epsilon|^2 \phi\,dx\,ds.$$

Bounding the term π_{far} is more complicated. Let

$$A_k = \{x : \lambda^{k-1} \le |x| < \lambda^k\}.$$

We start with the following pointwise estimate which is valid whenever $x \in B_\lambda$:

$$\begin{aligned}|\pi_{\text{far}}(x,t)| &\le C\sum_{k=3}^{\infty}\int_{A_k}\frac{1}{|x-y|^3}|(\eta_{\epsilon\sqrt{t}}*v_\epsilon)(y,t)|\,|v_\epsilon(y,t)|\,dy\\ &\le C(\lambda)\sum_{k=3}^{\infty}\frac{1}{\lambda^{3k}}\int_{A_k}|(\eta_{\epsilon\sqrt{t}}*v_\epsilon)(y,t)|\,|v_\epsilon(y,t)|\,dy\\ &= C(\lambda)\sum_{k=3}^{\infty}\frac{1}{\lambda^{2k}}\int_{A_0}|(\eta_{\epsilon\sqrt{t\lambda^{-2k}}}*v_\epsilon)(z,t\lambda^{-2k})|\,|v_\epsilon(z,t\lambda^{-2k})|\,dz\\ &\le C(\lambda)\sum_{k=3}^{\infty}\frac{1}{\lambda^{2k}}\|(\eta_{\epsilon\sqrt{t\lambda^{-2k}}}*v_\epsilon)(t\lambda^{-2k})\|_{L^2(B_1)}\|v_\epsilon(t\lambda^{-2k})\|_{L^2(B_1)}\\ &\le C(\lambda)\sum_{k=3}^{\infty}\frac{1}{\lambda^{2k}}\|v_\epsilon(t\lambda^{-2k})\|^2_{L^2(B_{\lambda^2})}\le C(\lambda)\tilde{\alpha}_\epsilon(t),\end{aligned}$$

where we have used (3-6), (3-8) and rescaled the solution. Therefore,

$$\int_0^t\|\pi_{\text{far}}(\,\cdot\,,s)\|^{3/2}_{L^{3/2}(B_\lambda)}\,ds\le C(\lambda)\int_0^t\tilde{\alpha}_\epsilon(s)^{3/2}\,ds.$$

After using Hölder's inequality, (3-9), the bounds above, and $\alpha^{3/2}\le\alpha+\alpha^3$ for $\alpha>0$, it is clear that

$$\int_0^t\|\pi_\epsilon(\,\cdot\,,s)\|^{3/2}_{L^{3/2}(B_\lambda)}\,ds+\int_0^t\!\!\int 2\pi_\epsilon(v_\epsilon\cdot\nabla\phi)\,dx\,ds \le C(\lambda,\gamma,\eta)\int_0^t(\tilde{\alpha}_\epsilon(s)^3+\tilde{\alpha}_\epsilon(s)^1)\,ds+\gamma\int_0^t\!\!\int|\nabla v_\epsilon|^2\phi\,dx\,ds.$$

Combining the estimates above (and taking γ sufficiently small to absorb the gradient terms on the right-hand side), we obtain

$$\alpha_\epsilon(t)+\int_0^t\!\!\int_{B_1}|\nabla v_\epsilon|^2\,dx\,ds\le\alpha_0+C(\lambda,\eta,\gamma)\int_0^t(\tilde{\alpha}_\epsilon(s)^3+\tilde{\alpha}_\epsilon(s)^1)\,ds. \tag{3-12}$$

Therefore,

$$\tilde{\alpha}_\epsilon(t)\le\alpha_0+C\int_0^t(\tilde{\alpha}_\epsilon(s)^3+\tilde{\alpha}_\epsilon(s)^1)\,ds. \tag{3-13}$$

By continuity of $\alpha_\epsilon(t)$, we have

$$\tilde{\alpha}_\epsilon(t)\le 2\alpha_0\quad\text{for all } t<T, \tag{3-14}$$

for some $T>0$. By a continuity argument, we may take $T=(C(2+8\alpha_0^2))^{-1}$.

Letting $\epsilon\to 0$ yields

$$(v,\chi_{B_1}v)(t)\le\liminf_{\epsilon\to 0}(v_\epsilon,\chi_{B_1}v_\epsilon)_{L^2}(t)\le 2\alpha_0$$

for all $t\le T$. Note that (3-12) gives uniform (in ϵ) control of

$$\int_0^T\!\!\int_{B_1}|\nabla v_\epsilon|^2\,dx\,dt\le C(\alpha_0,\lambda)$$

for some constant $C(\alpha_0,\lambda)$. From [Bradshaw and Tsai 2017a] we have that v_ϵ converges weakly to v in $L^2(1/k,T;H^1(B_1))$ for every $k\in\mathbb{N}$. Hence,

$$\int_{1/k}^T\int_{B_1}|\nabla v|^2\,dx\,dt\le\sup_{\epsilon>0}\int_0^T\int_{B_1}|\nabla v_\epsilon|^2\,dx\,dt,$$

and, letting $k\to\infty$, it follows that

$$\int_0^T\int_{B_1}|\nabla v|^2\,dx\,dt\le C(\alpha_0,\lambda).$$

Similarly, since $\pi_\epsilon\in L^{3/2}(0,T;L^{3/2}(B_1))$ with uniformly bounded norms, it follows that

$$\pi\in L^{3/2}(0,T;L^{3/2}(B_1)).$$

Applying Lemma 2.1 yields the desired pressure representation in $L^{3/2}(0,T;L^{3/2}(B_1))$ and concludes the proof. □

4. DSS solutions with data in $L^2_{\mathrm{loc}}(\mathbb{R}^3)$

In this section we prove Theorem 1.2. To do this, we need to approximate DSS data in L^2_{loc} by divergence-free DSS vector fields in L^3_w and also characterize discrete self-similarity on $\mathbb{R}^3\times(0,\infty)$ in terms of a neighborhood of the origin.

4A. *Approximation of DSS data in L^2_{loc}.*

Lemma 4.1. *Let $f\in L^2_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R}^3)$ be a given divergence-free λ-DSS vector field for some $\lambda>0$. There exists a sequence of divergence-free λ-DSS vector fields $\phi^{(k)}$ so that $\phi^{(k)}\in L^3_w(\mathbb{R}^3)$ and $\|\phi^{(k)}-f\|_{L^2(B_1)}\to 0$ as $k\to\infty$ (B_1 is the ball of radius 1 centered at the origin).*

The main difficulty in proving this lemma is that each $f^{(k)}$ must be divergence-free. We thus need to use the Bogovski map [1980], which we presently recall.

Lemma 4.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $2\le n<\infty$. There is a linear map Ψ that maps a scalar $f\in L^q(\Omega)$ with $\int_\Omega f=0$, $1<q<\infty$, to a vector field $v=\Psi f\in W^{1,q}_0(\Omega;\mathbb{R}^n)$ and*

$$\operatorname{div}v=f,\qquad \|v\|_{W^{1,q}_0(\Omega)}\le c(\Omega,q)\|f\|_{L^q(\Omega)}.$$

The map Ψ is independent of q for $f\in C^\infty_c(\Omega)$.

Proof of Lemma 4.1. Let $Z_0(x)\in C^\infty(\mathbb{R}^3)$ satisfy

$$Z_0(x)=\begin{cases}1, & |x|>1,\\ \text{radial, increasing}, & \lambda^{-1}\le|x|\le 1,\\ 0, & |x|<\lambda^{-1}.\end{cases}$$

Note that $\nabla\cdot(Z_0f)=f\cdot\nabla Z_0$; i.e., Z_0f is not divergence-free. We can correct this using Lemma 4.2 with $q=2$ for the scalar $-f\cdot\nabla Z_0$ noting that f is locally square integrable and

$$\int -f\cdot\nabla Z_0\,dx=0,$$

because f is divergence-free. Denote by Φ_0 the image of $-f\cdot\nabla Z_0$ under a Bogovski mapping with domain $\{x:\lambda^{-1}\le|x|\le 1\}$. Then, $\Phi_0\in W_0^{1,2}(B_1\setminus B_{\lambda^{-1}})$ and

$$\nabla\cdot(Z_0 f+\Phi_0)=0.$$

Let $Z_i(x)=Z_0(x/\lambda^i)$ and $\Phi_i(x)=\lambda^{-i}\Phi_0(\lambda^{-i}x)$ for all $i\in\mathbb{Z}$. It follows that

$$\nabla\cdot(Z_i f+\Phi_i)=0$$

for all $i\in\mathbb{Z}$. Note that $\operatorname{supp}(Z_j-Z_{j+2})=\{x:\lambda^{j-1}\le|x|\le\lambda^{j+2}\}$. Let

$$f_i=\tfrac{1}{2}(Z_i-Z_{i+2})f+\tfrac{1}{2}(\Phi_i-\Phi_{i+2}).$$

Then each f_i is divergence-free and supported on $B_{\lambda^{i+2}}\setminus B_{\lambda^{i-1}}$. Furthermore,

$$f=\sum_{i\in\mathbb{Z}} f_i,$$

where convergence is understood in the pointwise sense for all $x\neq 0$. To confirm this note that if x satisfies $\lambda^i\le|x|<\lambda^{i+1}$ then $x\in\operatorname{supp}(Z_j-Z_{j+2})$ if and only if $j\in\{i-1,i,i+1\}$. It follows that

$$\sum_{j\in\mathbb{Z}}(Z_j-Z_{j+2})(x)=2.$$

On the other hand, $\operatorname{supp}\Phi_j=\{x:\lambda^{j-1}\le|x|\le\lambda^j\}$ and, therefore,

$$\sum_{j\in\mathbb{Z}}(\Phi_j(x)-\Phi_{j+2}(x))=\Phi_{i+1}(x)-\Phi_{i+1}(x)=0.$$

It follows that $f=\sum_{i\in\mathbb{Z}} f_i$.

Assume $\phi_0^{(k)}$ is a sequence of divergence-free vector fields in $C_0^\infty(B_{\lambda^2}\setminus B_{\lambda^{-1}})$ so that $\phi_0^{(k)}\to f_0$ in $L^2(B_{\lambda^2}\setminus B_{\lambda^{-1}})$. Let $\phi_i^{(k)}=\lambda^{-i}\phi_0^{(k)}(\lambda^{-i}x)$. Then the vector field

$$\phi^{(k)}=\sum_{i\in\mathbb{Z}}\phi_i^{(k)}$$

is a divergence-free, λ-DSS vector field, and satisfies

$$|\phi^{(k)}(x)|\le c_k|x|^{-1}$$

(where the proportionality constants c_k are *not* uniformly bounded with respect to k). Hence, $\phi^{(k)}\in L^3_w$. We finish by arguing that $\phi^{(k)}\to f$ in $L^2(B_1)$. We know that $\int_{B_{\lambda^2}\setminus B_{\lambda^{-1}}}(\phi_0^{(k)}-f)^2\,dx\to 0$ as $k\to\infty$. Using the definition of $\phi^{(k)}$ and the fact that f is discretely self-similar we have, letting $A_i=B_{\lambda^i}\setminus B_{\lambda^{i-1}}$, that

$$\begin{aligned}\int_{B_1}(\phi^{(k)}-f)^2\,dx&=\sum_{i\le 0}\int_{A_i}(\phi^{(k)}-f)^2\,dx\\&=\sum_{i\le 0}\lambda^i\int_{A_0}(\phi^{(k)}-f)^2\,dx=\frac{\lambda}{\lambda-1}\int_{A_0}(\phi^{(k)}-f)^2\,dx.\end{aligned}$$

In A_0, we have $\phi^{(k)} - f = \sum_{i=-2}^{0}(\phi_i^{(k)} - f_i)$. Thus

$$\|\phi^{(k)} - f\|_{L^2(A_0)} \le \sum_{i=-2}^{0} \|\phi_i^{(k)} - f_i\|_{L^2(A_0)} = \sum_{k=0}^{2} \lambda^{-k/2}\|\phi_0^{(k)} - f_0\|_{L^2(A_k)} \le 3\|\phi_0^{(k)} - f_0\|_{L^2(B_{\lambda^2}\setminus B_{\lambda^{-1}})},$$

which completes the proof. □

4B. ***DSS solutions in a neighborhood of the origin.*** In the Introduction we saw that any time-periodic solution u to (1-10) corresponds to a DSS solution v after the change of variables (1-9). Distributionally, u is a time-periodic solution to (1-10) if and only if

$$\int_{s'}^{s'+T} \Big((u, \partial_s f) - (\nabla u, \nabla f) + \big(\tfrac{1}{2}u + \tfrac{1}{2}y\cdot\nabla u - u\cdot\nabla u, f\big)\Big)\, ds = 0 \tag{4-1}$$

holds for all $s' \in \mathbb{R}$ and $f \in \mathcal{D}_T$, where $\mathcal{D}_T$ denotes the collection of all smooth divergence-free vector fields in $\mathbb{R}^3 \times \mathbb{R}$ which are time-periodic with period T and whose supports are compact in space. In [Bradshaw and Tsai 2017a], this definition was used with $s' = 0$ since the goal was to extend a solution on $[0, T]$ to $\mathbb{R}$ using periodicity. The same modification can be made here based on the observations that if u satisfies (4-1) then u can be extended to a time-periodic solution on $\mathbb{R}$ and if u is a time-periodic solution on $\mathbb{R}$ then u satisfies (4-1).

Since there is a one-to-one correspondence between time-periodic solutions to (1-10) and DSS solutions, an equivalent characterization of DSS solutions is obtained by reformulating (4-1) in the physical variables. For $f \in \mathcal{D}_T$ let $\zeta_f(x,t) = t^{-1} f(y,s)$. Note $\zeta_f(x,t) = \lambda^2 \zeta_f(\lambda x, \lambda^2 t)$. Then, v is λ-DSS if and only if

$$\int_t^{\lambda^2 t} \big((v, \partial_t \zeta_f) - (\nabla v, \nabla \zeta_f) - (v\cdot\nabla v, \zeta_f)\big)\, d\tau = 0 \tag{4-2}$$

for all $t > 0$ and $f \in \mathcal{D}_T$, since (4-1) is just (4-2) in similarity variables. Note that $(v, \zeta_f)|_{\tau=\lambda^2 t} = (v, \zeta_f)|_{\tau=t}$. It follows that, if v is a solution to (1-1) that satisfies (4-2) for $t = 1$, then $v|_{\tau\in[1,\lambda^2]}$ can be extended to a λ-DSS solution for all positive times.

Fix $k \in \mathbb{Z}$ and let $Q_k = B_{\lambda^k}(0) \times (0, \lambda^{2k})$. Our goal is to give a third characterization of discrete self-similarity on Q_k. Let $f \in \mathcal{D}_T$ be given and ζ_f be as above. Let R be large enough so that, for all $t \in [1, \lambda^2]$, the support of $\zeta_f(t)$ is a subset of $B_R(0)$ and choose $m = m(f) \in \mathbb{Z}$ so that $R/\lambda^m < \lambda^k$ and $\lambda^{2-2m} < \lambda^{2k}$. It follows that

$$B_{R/\lambda^m}(0) \times [\lambda^{-2m}, \lambda^{2-2m}] \subset Q_k.$$

Extend ζ_f to all $t > 0$ using the following scaling: for $(x,t) \in \mathbb{R}^3 \times (0,\infty)$, let

$$\zeta_f(x,t) = \lambda^{2i}\zeta_f(\lambda^i x, \lambda^{2i} t),$$

where i is chosen so that $\lambda^{2i} t \in [1, \lambda^2]$. Since $\zeta_f|_{\mathbb{R}^3\times[1,\lambda^2]}$ is compactly supported in space, its spatial support shrinks as $t \to 0^+$. In particular, for $t \in [\lambda^{-2m}, \lambda^{2-2m}]$, we have $\operatorname{supp}\zeta_f \subset Q_k$. For $m \in \mathbb{Z}$, let

$$\mathcal{D}^m_{Q_k} = \{\phi \in C^\infty(\mathbb{R}^3\times(0,\infty)) : \operatorname{supp}\phi|_{t\in[\lambda^{-2m},\lambda^{2-2m}]} \subset Q_k \text{ and } \forall (x,t) \in \mathbb{R}^3\times(0,\infty), \exists f \in \mathcal{D}_T \text{ such that } \phi(x,t) = \zeta_f(x,t)\}. \tag{4-3}$$

It is easy to see that

$$\bigcup_{m\in\mathbb{Z}} \mathcal{D}^m_{Q_k} = \mathcal{D}_T.$$

Rescaling (4-2) gives

$$\int_{\lambda^{-2m}}^{\lambda^{2-2m}} ((v, \partial_t \zeta_f) - (\nabla v, \nabla \zeta_f) - (v\cdot\nabla v, \zeta_f))\, dt' = 0, \tag{4-4}$$

where $t' = t/\lambda^{2m}$ and the inner products are taken with respect to the rescaled spatial variable $x' = x/\lambda^m$. In particular, the integral is computed over a subset of Q_k and is identical to the same integral with ζ_f replaced by ϕ for some $\phi \in \mathcal{D}^m_{Q_m}$. Thus, if v is a solution to (1-1), and $\phi \in \mathcal{D}^m_{Q_k}$ for some $m\in\mathbb{Z}$, then (4-2) is satisfied if and only if (4-4) is satisfied for the $f\in\mathcal{D}_T$ for which $\zeta_f = \phi$. This leads to the following extendability property: if v is a solution to (1-1) on Q_k and satisfies (4-4) for every $m\in\mathbb{Z}$ and $\phi\in\mathcal{D}^m_{Q_k}$, then v can be extended to a discretely self-similar solution on $\mathbb{R}^3\times(0,\infty)$; in other words, if a solution is DSS in a neighborhood of the origin, then it can be extended to a DSS solution on $\mathbb{R}^3\times(0,\infty)$.

4C. *Construction of DSS solutions.*

Proof of Theorem 1.2. Fix $\lambda > 1$ and assume $v_0 \in L^2_{\mathrm{loc}}$ is a divergence-free λ-DSS vector field. Let $\{v_0^{(k)}\}$ be the sequence of vector fields $\{\phi^{(k)}\}$ from Lemma 4.1 applied to v_0. Then, the values $\|v_0^{(k)}\|_{L^2(B_1)}$ are uniformly bounded and $\|v_0^{(k)} - v_0\|_{L^2(B_1)} \to 0$ as $k\to\infty$. Since $v_0^{(k)} \in L^3_w$ and is λ-DSS, by [Bradshaw and Tsai 2017a] there exists a λ-DSS local Leray solution v_k to (1-1) and an associated pressure π_k having initial data $v_0^{(k)}$ for every $k\in\mathbb{N}$. By Proposition 3.1, v_k are uniformly bounded in $L^\infty(0,T;L^2(B_1))\cap L^2(0,T;H^1(B_1))$ (hence also in $L^{10/3}(0,T;L^{10/3}(B_1))$) for some T which depends only on λ and $\|v_0^{(k)}\|_{L^2(B_1)}$. As usual, see [Bradshaw and Tsai 2017a; Kikuchi and Seregin 2007; Lemarié-Rieusset 2016], there exists a distribution v and a subsequence of $\{v_k\}$ (still indexed by k for simplicity) so that v_k converges to v in the weak star topology on $L^\infty(0,T;L^2(B_1))$, in the weak topology on $L^2(0,T;H^1(B_1))$, and in $L^2(0,T;L^2(B_1))$. Since they are uniformly bounded in $L^{10/3}(0,T;L^{10/3}(B_1))$, they also converge in $L^q(0,T;L^q(B_1))$ for any $q<\frac{10}{3}$. By the pressure estimate (3-2) in Proposition 3.1, π_k are uniformly bounded in $L^{3/2}(0,T;L^{3/2}(B_1))$ by $C(\lambda, \|v_0\|_{L^2(B_\lambda)})$ and, therefore, we may extract a subsequence which converges weakly to a distribution $\pi\in L^{3/2}(0,T;L^{3/2}(B_1))$.

Fix $\kappa\in\mathbb{Z}$ so that $\lambda^\kappa<1$ and $\lambda^{2\kappa}<T$. Then, $Q_\kappa = B_{\lambda^\kappa}\times(0,\lambda^{2\kappa})\subset B_1\times(0,T)$. Therefore v_k satisfies (1-1) on Q_κ and satisfies (4-4) for every $m\in\mathbb{Z}$ and $\phi\in\mathcal{D}^m_{Q_\kappa}$. Thus, v can be extended to a DSS solution on $\mathbb{R}^3\times(0,\infty)$ (which we still denote by v).

For compact subsets K of B_1, we automatically have $\lim_{t\to0^+}\|v-v_0\|_{L^2(K)}=0$. For a general compact subset K of $\mathbb{R}^3$, we have $K' = \lambda^m K\subset B_1$ for some $m\in\mathbb{Z}$, and

$$\int_K |v(x,t)-v_0(x)|^2\,dx = \lambda^{-m}\int_{K'} |v(x',\lambda^{2m}t)-v_0(x')|^2\,dx'.$$

It follows that $\lim_{t\to0^+}\|v(t)-v_0\|_{L^2(K)}=0$ for every compact set $K\subset\mathbb{R}^3$. A similar rescaling argument also implies that $v\in L^\infty(0,T';L^2(K))\cap L^2(0,T';H^1(K))$ and $\pi\in L^{3/2}(0,T';L^{3/2}(K))$ for any $T'>0$ and compact subset K of $\mathbb{R}^3$.

To confirm that v satisfies the local energy inequality, first note that each v_k satisfies the local energy inequality

$$\int |v_k(t)|^2\phi\,dx + 2\iint |\nabla v_k|^2\phi\,dx\,dt \le \int |v_0^{(k)}|^2\phi\,dx + \iint |v_k|^2(\partial_t\phi+\Delta\phi)\,dx\,dt + \iint (|v_k|^2+2\pi_k)(v_k\cdot\nabla\phi)\,dx\,dt$$

for all nonnegative $\phi \in C_0^\infty(\mathbb{R}^3\times\mathbb{R}^3_+)$. Furthermore, the right-hand sides of the energy inequality for $v^{(k)}$ converge to the right-hand side of the energy inequality for v as $k\to\infty$, while the left-hand sides are lower semicontinuous; see [Caffarelli et al. 1982, (A.51)]. The local energy inequality for v plainly follows.

Finally, note that π_k satisfies the formula (3-3). Applying Lemma 2.1 to the sequence and limit above implies that π satisfies the desired pressure formula in $L^{3/2}(0,T;L^{3/2}(B_1))$. Rescaling establishes the formula in $L^{3/2}_{\text{loc}}(\mathbb{R}^3\times(0,\infty))$. □

Acknowledgments

The research of Tsai was partially supported by the NSERC grant 261356-13 (Canada).

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Received 24 Jan 2018. Revised 16 Oct 2018. Accepted 30 Nov 2018.

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ANALYSIS AND PDE
Vol. 12, No. 8, 2019
dx.doi.org/10.2140/apde.2019.12.1963

CONTINUITY PROPERTIES FOR DIVERGENCE FORM BOUNDARY DATA HOMOGENIZATION PROBLEMS

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We study the asymptotic behavior at rational directions of the effective boundary condition in periodic homogenization of oscillating Dirichlet data. We establish a characterization for the directional limits at a rational direction in terms of a relatively simple two-dimensional boundary layer problem for the homogenized operator. Using this characterization we show continuity of the effective boundary condition for divergence form linear systems, and for divergence form nonlinear equations we give an example of discontinuity.

1. Introduction

In this work we will study the following type of boundary layer problem in dimension $d \geq 2$:

$$\begin{cases} -\nabla \cdot a(y, \nabla v_n^s) = 0 & \text{in } P_n^s = \{y \cdot n > s\}, \\ v_n^s(y) = \varphi(y) & \text{on } \partial P_n^s. \end{cases} \tag{1-1}$$

Here $n \in S^{d-1}$ is a unit vector, $s \in \mathbb{R}$, φ is continuous and $\mathbb{Z}^d$ periodic, the operator a is also $\mathbb{Z}^d$ periodic in y and will satisfy a uniform ellipticity assumption. This work will consider both nonlinear scalar equations and linear systems, so, for now, we do not specify the assumptions on a any further.

The *boundary layer limit* of the system (1-1) is defined by

$$\varphi_*(n, s) := \lim_{R \to \infty} v_n(Rn + y) \quad \text{if the limit exists and is independent of } y \in \partial P_n^s.$$

If, additionally, the boundary layer limit is independent of s then we say that the cell equation (1-1) homogenizes. Typically φ_* is independent of s for irrational directions n and we write $\varphi_*(n)$, while for rational directions $n \in \mathbb{R}\mathbb{Z}^d$ the limits above exist but depend on s.

The focus of this article is on the limiting behavior of φ_* at rational directions. As a consequence of this study we will be able to establish continuity or discontinuity of φ_* on S^{d-1}. We will see that continuity of φ_* is intrinsically linked with linearity of the operator $a(x, p)$. In the case of a linear system we show continuity of φ_*, while in the case of nonlinear scalar equations we give an example where φ_* is discontinuous; this indicates generic discontinuity for nonlinear equations.

The main result established in this paper is that the directional limits of φ_* at a rational direction are determined by a "second cell problem", which is a boundary layer problem for the homogenized operator a^0. From this asymptotic formula it becomes relatively straightforward to address questions of

MSC2010: 35B27, 35J57, 35J60.
Keywords: homogenization, boundary layers, oscillating boundary data, nonlinear elliptic equations, elliptic systems.

continuity or discontinuity of φ_* at rational directions. Let us take $\xi \in \mathbb{Z}^d \setminus \{0\}$ to be an irreducible lattice vector and $\hat{\xi}$ to be the corresponding rational unit vector in the same direction. Then the cell equation (1-1) solution v_ξ^s exists for each $s \in \mathbb{R}$ and has a boundary layer limit,

$$\varphi_*(\xi, s) := \lim_{R\to\infty} v_\xi^s(R\xi),$$

but that limit typically is not independent of the translation s applied to the half-space domain P_ξ. We will see that $\varphi_*(\xi, s)$ is a $1/|\xi|$-periodic function on $\mathbb{R}$. Now suppose that we have a sequence of directions $n_k \to \hat{\xi}$ such that

$$\frac{\hat{\xi} - n_k}{|\hat{\xi} - n_k|} \to \eta, \qquad \text{where } \eta \text{ is a unit vector with } \eta \perp \xi.$$

Call η the approach direction of the sequence n_k to ξ. We will show that the limit of $\varphi_*(n_k)$ is determined by the following boundary layer problem. Call $P_\xi = P_\xi^0 = \{x \cdot \xi > 0\}$ and define

$$\begin{cases} -\nabla \cdot a^0(\nabla w_{\xi,\eta}) = 0 & \text{in } P_\xi, \\ w_{\xi,\eta} = \varphi_*(\xi, x\cdot\eta) & \text{on } \partial P_\xi \end{cases} \quad \text{and} \quad L(\xi,\eta) = \lim_{R\to\infty} w_{\xi,\eta}(R\xi). \tag{1-2}$$

Then it holds

$$\lim_{k\to\infty} \varphi_*(n_k) = L(\xi,\eta). \tag{1-3}$$

We will see below that $L(\xi,\eta)$ is continuous in $\eta \in S^{d-1}$. Thus the directional limits of φ_* at ξ are determined by the boundary layer limit of a half-space problem for the homogenized operator. This limit structure was first observed in [Choi and Kim 2014] and developed further by the first author and Kim [Feldman and Kim 2017]; both papers studied nondivergence form and possibly nonlinear equations. We will explain in this paper how the second cell problem follows purely from *qualitative* features which are shared by a wide class of elliptic equations, including divergence form linear systems, and both divergence and nondivergence form nonlinear equations. We are somewhat vague about the hypotheses, which will be explained in detail in Sections 3 and 4.

Once we have established (1-3), the question of qualitative continuity/discontinuity of φ_* is reduced to a much simpler problem. For linear equations the homogenized operator a^0 is linear and translation-invariant and so a straightforward argument, for example by the Riesz representation theorem, shows that

$$L(\xi,\eta) = \lim_{R\to\infty} w_{\xi,\eta}(R\xi) = |\xi| \int_0^{1/|\xi|} \varphi_*(\xi, s)\, ds;$$

i.e., it is the average over a period of $\varphi_*(\xi,\cdot)$. Evidently this does not depend on the approach direction η. Thus qualitative continuity of φ_* for linear problems follows easily once we establish (1-3).

In the case of nonlinear equations the formula (1-3) allows us to construct examples where discontinuities do occur; see Theorem 1.3 below. Our conjecture is that discontinuities are generic for the class of quasilinear equations we consider. Note that when φ_* is not continuous at ξ, the asymptotic formula (1-3) still contains interesting information; it explains the structure of the discontinuity. In particular, the blow up of φ_* at a discontinuity is 0-homogeneous and continuous away from the origin.

Before we state our main theorems we give a brief explanation about where (1-1) arises and why one should be interested in the continuity/discontinuity of φ_*. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and consider the homogenization problem with oscillating Dirichlet boundary data,

$$\begin{cases} -\nabla \cdot \big(a\big(\tfrac{x}{\varepsilon}, \nabla u^\varepsilon\big)\big) = 0 & \text{in } \Omega, \\ u^\varepsilon(x) = g\big(x, \tfrac{x}{\varepsilon}\big) & \text{on } \partial\Omega, \end{cases} \tag{1-4}$$

where $\varepsilon > 0$ is a small parameter, $g(x, y)$ is continuous in x, y and $\mathbb{Z}^d$ periodic in y. This system is natural to consider in its own right, but also it arises naturally in the study of homogenization with nonoscillatory Dirichlet data when one studies the higher-order terms in the asymptotic expansion; see [Gérard-Varet and Masmoudi 2012] where this is explained.

The interest in studying (1-4) is the asymptotic behavior of the u^ε solutions as $\varepsilon \to 0$. This problem has been studied recently by a number of authors starting with [Gérard-Varet and Masmoudi 2011; 2012] and followed by [Aleksanyan, Shahgholian, and Sjölin 2015; Aleksanyan 2017; Choi and Kim 2014; Feldman and Kim 2017; Feldman 2014; Prange 2013; Zhang 2017; Armstrong, Kuusi, Mourrat, and Prange 2017; Guillen and Schwab 2016]. It has been established that solutions u^ε converge, at least in $L^2(\Omega)$, to some u^0 which is a unique solution to

$$\begin{cases} -\nabla \cdot (a^0(\nabla u^0)) = 0 & \text{in } \Omega, \\ u^0(x) = \varphi^0(x) & \text{on } \partial\Omega, \end{cases} \tag{1-5}$$

where a^0 and $\varphi^0(x)$ are called respectively the homogenized operator and homogenized boundary data. The identification of the homogenized operator a^0 is a classical topic. The homogenized boundary φ^0 is determined by the boundary layer equation (1-1),

$$\varphi^0(x) = \varphi_*(n_x), \quad \text{when } n_x \text{ is the inward unit normal to } \Omega \text{ and } \varphi(y) = g(x, y).$$

That is, (1-1) can be viewed as a kind of cell problem associated with the homogenization of (1-4). At least for linear equations this definition makes sense as long as the set of boundary points of $\partial\Omega$ where (1-1) does not homogenize, i.e., those with rational normal, has zero harmonic measure. The convergence of u^ε to u^0 has been established rigorously for linear systems by Gérard-Varet and Masmoudi [2012], and further investigations have yielded optimal rates of convergence; see [Armstrong, Kuusi, Mourrat, and Prange 2017; Shen and Zhuge 2018]. For nonlinear divergence form equations, to our knowledge, the problem has not been studied yet. This is the source of our interest in the fine properties of φ_*: quantitative continuity estimates for φ_* lead to quantitative continuity estimates for u^0 and u^ε, and are used to establish rates of convergence $u^\varepsilon \to u^0$. Meanwhile, characterizing the type of discontinuities of φ_*, when they are present, leads to understanding the qualitative features of u^ε and u^0.

Now we return to state our main results. The first is the validity of the "second cell problem" formula (1-3) for the directional limits of φ_*.

Theorem 1.1. *The limit characterization* (1-3) *holds for divergence form linear systems and nonlinear equations satisfying a uniform ellipticity condition. The* 0*-homogeneous profile* $L(\xi, \eta)$ *at direction* $\xi \in \mathbb{Z}^d \setminus \{0\}$ *is continuous in* $\eta \in S^{d-1}$.

Our arguments to derive (1-3) can be quantified to obtain a modulus of continuity, which we make explicit below, however so far we cannot push the method to obtain the optimal modulus of continuity. In a very nice recent work Shen and Zhuge [2017] obtain an almost Lipschitz modulus of continuity by a different method; we will compare their approach with ours below.

Theorem 1.2. *For elliptic linear systems, $d \geq 2$, for any $0 < \alpha < \frac{1}{d}$ there is a constant $C \geq 1$ depending on α as well as universal parameters associated with the system (see Section 3) such that, for any n_1, n_2 irrational,*

$$|\varphi_*(n_1) - \varphi_*(n_2)| \leq C \|\varphi\|_{C^5} |n_1 - n_2|^\alpha.$$

We note that in the course of proving Theorem 1.2 we actually show Hölder regularity for every $0 < \alpha < 1$ at each lattice direction $\xi \in \mathbb{Z}^d \setminus \{0\}$; the modulus of continuity however depends on the rational direction and degenerates as $|\xi| \to \infty$. This is why we only end up with (almost) Hölder-$\frac{1}{d}$ continuity in the end.

For nonlinear problems our conjecture is that φ_* is discontinuous at rational directions, at least for generic boundary data and operators. A result of this kind was established for nondivergence form equations in [Feldman and Kim 2017]. In the divergence form nonlinear case we have constructed an explicit example showing that discontinuity is possible.

Theorem 1.3. *For $d \geq 3$ there exist smooth boundary data φ and uniformly elliptic, positively 1-homogeneous, nonlinear operators $a(x, p)$ such that φ_* is discontinuous at some rational direction.*

We compare with [Shen and Zhuge 2017], which studies continuity properties of φ_* for linear divergence form systems. They show, in the linear systems case, that φ_* is in $W^{1,p}$ for every $p < \infty$. They establish Lipschitz estimates on the Diophantine directions which only grow subpolynomially in the Diophantine parameter, and thereby obtain the $W^{1,p}$ estimates and extend continuously to the rational directions. As can be seen, for example, by Theorem 1.3, this type of result would not be possible for quasilinear elliptic equations. Our approach is to compute the directional limits at each rational direction via the second cell problem formula (1-3). Although this method does not yet yield an optimal quantitative estimate, it applies to both linear and nonlinear equations including both divergence form, as established here, and nondivergence form, as in [Feldman and Kim 2017]. We establish (suboptimal) quantitative continuity for linear systems, and also we can classify the types of discontinuities which are present in the nonlinear setting.

Finally we compare with the work of the first author and Kim [Feldman and Kim 2017] in the nondivergence form case. As we try to emphasize in Section 2, the broad outline of the arguments for Theorems 1.1 and 1.2 are the same in divergence and nondivergence form. However, at the level of the proofs there are many technical differences; we will try to highlight the most interesting throughout the paper. The idea, from [Feldman and Kim 2017], for the construction of nonlinear operators with discontinuous φ_* does not work at all in the divergence form setting. We needed a completely different construction for Theorem 1.3.

Generally speaking, for linear systems we need to replace arguments with maximum principle by large-scale estimates on the Poisson kernel in half-spaces and cone-type domains. These estimates come from [Avellaneda and Lin 1991] or are adapted from the arguments there. For nonlinear equations we do have a maximum principle, but many new arguments need to be developed since, as far as we are aware, this is the first paper on the boundary layer problem for quasilinear divergence form equations.

1A. ***Notation.*** We go over some of the notation and terminology used in the paper. We will refer to constants which depend only on the dimension or fundamental parameters associated with the operator $a(x, p)$ (to be made specific below), e.g., ellipticity ratio or smoothness norm, as universal constants. We will write C or c for universal constants which may change from line to line. Given some quantities A, B we write $A \lesssim B$ if $A \leq CB$ for a universal constant C. If the constants depend on an additional nonuniversal parameter α we may write $A \lesssim_\alpha B$.

We will use various standard L^p and Hölder $C^{k,\alpha}$ norms. For Hölder seminorms, which omit the zeroth-order sup norm term, we write $[f]_{C^{k,\alpha}}$. Given a measurable set $E \subset \mathbb{R}^d$ we will also use the $L^p_{\text{avg}}(E)$ norm, which is defined by

$$\|f\|_{L^p_{\text{avg}}(E)} = \left(\frac{1}{|E|}\int_E |f|^p\right)^{1/p}.$$

The oscillation is a convenient quantity for us since the solution property for the equations we consider is preserved under addition of constant functions. This is usually defined for a scalar-valued function $u : E \to \mathbb{R}$ on a set $E \subset \mathbb{R}^d$ as $\operatorname{osc}_E u = \sup_E u - \inf_E u$. We use a slightly different definition which also makes sense for vector-valued $u : E \to \mathbb{R}^N$,

$$\operatorname*{osc}_E u := \inf\left\{r > 0 : \text{there exists } u_0 \in \mathbb{R}^N \text{ such that } \|u - u_0\|_{L^\infty(E)} \leq \tfrac{1}{2}r\right\}.$$

2. Explanation of the limit structure at rational directions

We give a high-level description of the asymptotics of the boundary layer limit at rational directions. What we would like to emphasize throughout this description is that the argument is basically geometric, and has to do with the way that ∂P_n intersects the unit periodicity cell in the asymptotic limit as n approaches a rational direction. This calculation relies only on certain qualitative features of Dirichlet problems for elliptic equations which are true both for divergence and nondivergence form both linear (including systems) and nonlinear. To emphasize the level of abstraction we will write the boundary layer problem in the form

$$\begin{cases} F[v_n, x] = 0 & \text{in } P_n := \{x \cdot n > 0\}, \\ v_n = \varphi & \text{on } \partial P_n. \end{cases} \tag{2-1}$$

Always F and φ will share $\mathbb{Z}^d$ periodicity in the x-variable. In order to carry out the heuristic argument we will need the following properties of the class of equations/systems. We emphasize that the following properties are not stated very precisely, they are merely meant to be illustrative:

(i) (homogenization) There is an elliptic operator F^0 in the same class such that if u^ε is a sequence of solutions of $F\left[u^\varepsilon, \frac{x}{\varepsilon}\right] = 0$ in a domain Ω converging to some u^0 then $F[u^0] = 0$ in Ω.

(ii) (continuity with respect to boundary data in L^∞) There exists $C > 0$ so that if $n \in S^{d-1}$ and u_1, u_2 are bounded solutions of (2-1) with respective boundary data φ_1 and φ_2 then

$$\sup_{P_n} |u_1 - u_2| \leq C \sup_{\partial P_n} |\varphi_1 - \varphi_2|.$$

(iii) (large-scale interior and boundary regularity estimates) There is $\alpha \in (0,1)$ such that for any $r > 0$ if $F[u,x] = 0$ in $B_r \cap P_n$, where B_r is some ball of radius r,

$$[u]_{C^\alpha(B_{r/2}\cap P_n)} \lesssim r^{-\alpha} \operatorname*{osc}_{B_r\cap P_n} u + [g]_{C^\alpha(B_r\cap\partial P_n)}.$$

The heuristic outline below applies to a wide class of elliptic equations; already the arguments were carried out rigorously for nondivergence nonlinear equations by Choi and Kim [2014] and the first author and Kim [Feldman and Kim 2017] and similar ideas were used for parabolic equations in moving domains by the second author in [Zhang 2017]. Here we will be studying divergence form equations, linear systems and nonlinear scalar equations.

To begin we need to understand the boundary layer limit at a rational direction. Let $\xi \in \mathbb{Z}^d \setminus \{0\}$ and consider the solution $v_\xi^s(x)$ of,

$$\begin{cases} F[v_\xi^s, x] = 0 & \text{in } P_\xi^s = \{x \cdot n > s\}, \\ v_\xi^s = \varphi & \text{on } \partial P_\xi^s. \end{cases} \tag{2-2}$$

Translating the half-space, by changing s, changes the part of the data φ seen by the boundary condition. Thus the boundary layer limit of v_ξ^s can depend on the parameter s; we define

$$\varphi_*(\xi, s) = \lim_{R\to\infty} v_\xi^s(R\xi).$$

As will become clear, this particular parametrization of the boundary layer limits is naturally associated with the asymptotic structure of the boundary layer limits for directions n near ξ.

The next step is to understand the geometry near ξ. Let $n \in S^{d-1}$ be a direction near ξ and v_n be the corresponding half-space solution. We can write,

$$n = (\cos\varepsilon)\hat{\xi} - (\sin\varepsilon)\eta \quad \text{for some small angle } \varepsilon \text{ and a unit vector } \eta \perp \xi.$$

We obtain an asymptotic for v_n at an intermediate length scale.

Let $x \in \partial P_n$, then the hyperplanes ∂P_n and $\partial P_\xi^{x\cdot\hat{\xi}}$ are close in a large neighborhood, any scale $o\big(\frac{1}{\varepsilon}\big)$, of x. By using the local up-to-the-boundary regularity we see that v_n and v_ξ^s, with $s = x \cdot \hat{\xi}$, are close on the boundary of their common domain, at least in this $o\big(\frac{1}{\varepsilon}\big)$ neighborhood of x. Now v_ξ^s has a boundary layer limit $\varphi_*(\xi, s)$, and the length scale $|\xi|$ associated with the boundary layer depends on ξ, but not on ε. Thus for ε small and $|\xi| \ll R \ll \frac{1}{\varepsilon}$

$$v_n(x + Rn) = \varphi_*(\xi, x \cdot \hat{\xi}) + o_\varepsilon(1) = \varphi_*(\xi, \tan\varepsilon(x \cdot \eta)) + o_\varepsilon(1).$$

Here $o_\varepsilon(1)$ depends only on $|\xi|$, ε, and universal parameters of the problem. This is one of the main places where we use the large-scale boundary regularity estimates, property (iii) above. Thus, moving into the domain by Rn and rescaling to the scale $1/\tan\varepsilon$, i.e., letting $w^\varepsilon(x) \sim v_n((x + Rn)/\tan\varepsilon)$, we find that the boundary layer limit is well approximated by the boundary layer limit of

$$\begin{cases} F[w^\varepsilon, x/\tan\varepsilon] = 0 & \text{in } P_\xi, \\ w^\varepsilon = \varphi_*(\xi, x\cdot\eta) & \text{on } \partial P_\xi \end{cases} \tag{2-3}$$

in the limit as $\varepsilon \to 0$. Now taking the limit as $\varepsilon \to 0$ of in (2-3) we find the "second cell problem"

$$\begin{cases} F^0[w_{\xi,\eta}] = 0 & \text{in } P_\xi, \\ w_{\xi,\eta} = \varphi_*(\xi, x\cdot\eta) & \text{on } \partial P_\xi. \end{cases} \tag{2-4}$$

Thus we characterize the directional limits at the rational direction ξ as the boundary layer limits of the associated second cell problem

$$\lim_{k\to\infty} \varphi_*(n_k) = \lim_{R\to\infty} w_{\xi,\eta}(R\xi) \quad \text{if } \frac{\hat{\xi}-n_k}{|\hat{\xi}-n_k|} \to \eta.$$

With this characterization the *qualitative* continuity and discontinuity of φ_* can be investigated solely by studying (2-4).

In the following, Sections 3 and 4, we will explain background regularity results for linear systems and nonlinear divergence form equations and the well-posedness of Dirichlet problems in half-spaces. In particular we will prove that properties we used in the heuristic arguments above do hold for the type of equations/systems we consider. In Section 5 we will go into more detail about the boundary layer equation (1-1) in rational and irrational half-spaces. In Section 6 we will make rigorous the above outline obtaining intermediate-scale asymptotics which lead to the second cell equation (2-4). In Section 7 we show how to derive continuity of φ_* from the second cell problem for linear problems, and in Section 8 we show how nonlinearity can cause discontinuity of φ_*.

3. Linear systems background results

In this section we will recall some results about divergence form linear systems. Let Ω be a domain of $\mathbb{R}^d$ and $N \ge 1$; we consider solutions of the elliptic linear system

$$-\nabla\cdot(A(x)\nabla u) = 0 \quad \text{in } \Omega,$$

where $u \in H^1(\Omega;\mathbb{R}^N)$ is at least a weak solution. Here we use the notation $A = (A^{\alpha\beta}_{ij}(x))$ for $1 \le \alpha,\beta \le d$ and $1 \le i, j \le N$ defined for $x \in \mathbb{R}^d$, where we mean, using the summation convention,

$$(\nabla\cdot(A(x)\nabla u^\varepsilon))_i = \partial_{x_\alpha}(A^{\alpha\beta}_{ij}(x)\partial_{x_\beta}u^\varepsilon_j).$$

We assume that A satisfies the following hypotheses:

(i) Periodicity:

$$A(x+z) = A(x) \quad \text{for all } x \in \mathbb{R}^d,\ z \in \mathbb{Z}^d. \tag{3-1}$$

(ii) Ellipticity: for some $\lambda > 0$ and all $\xi \in \mathbb{R}^{d\times N}$,

$$\lambda \xi^i_\alpha \xi^i_\alpha \le A^{\alpha\beta}_{ij}\xi^i_\alpha\xi^j_\beta \le \xi^i_\alpha\xi^i_\alpha. \tag{3-2}$$

(iii) Regularity: for some $M > 0$,

$$\|A\|_{C^5(\mathbb{R}^d)} \le M. \tag{3-3}$$

We remark that the regularity on A is far more than is necessary for most of the results below. When we say that C is a universal constant below we mean that it depends only on the parameters, d, N, λ, M.

3A. ***Integral representation.*** Consider the following boundary layer problem, which will be the main object of our study:

$$\begin{cases} -\nabla\cdot(A(x)\nabla u) = \nabla\cdot f + g & \text{in } P_n,\\ u(x) = \varphi(x) & \text{on } \partial P_n \end{cases} \tag{3-4}$$

for f, g smooth vector-valued functions with compact support and φ continuous and bounded. A solution is given by the Green's function formula

$$u(x) = \int_{P_n} \nabla G(x,y)\cdot f(y)\,dy + \int_{P_n} G(x,y)g(y)dy + \int_{\partial P_n} P(x,y)\varphi(y)\,dy.$$

Here G, P are the Green matrix and Poisson kernel corresponding to our operator. For $y \in P_n$, G solves

$$\begin{cases} -\nabla_x\cdot(A(x)\nabla_x G(x,y)) = \delta(x-y)I_N & \text{in } P_n,\\ G(x,y) = 0 & \text{on } \partial P_n, \end{cases} \tag{3-5}$$

and the Poisson kernel is given, for $x \in P_n$ and $y \in \partial P_n$, by

$$P(x,y) = -n\cdot(A^t(y)\nabla_y G(x,y)),$$

that is,

$$P_{ij}(x,y) = -n_\alpha A^{\beta\alpha}_{ki}(y)\partial_{y_\beta} G_{kj}(x,y).$$

Following from [Avellaneda and Lin 1991], and exactly stated in [Gérard-Varet and Masmoudi 2012, Proposition 5], G and P satisfy the same bounds as for a constant coefficient operator:

Theorem 3.1. *Call $\delta(y) := \operatorname{dist}(y, \partial P_n)$. For all $x \neq y$ in P_n, one has*

$$\begin{aligned}
|G(x,y)| &\le \frac{C}{|x-y|^{d-2}} && \text{for } d \ge 3,\\
|G(x,y)| &\le C(|\log|x-y|| + 1) && \text{for } d = 2,\\
|G(x,y)| &\le \frac{C\delta(x)\delta(y)}{|x-y|^{d}} && \text{for all } d,\\
|\nabla_x G(x,y)| &\le \frac{C}{|x-y|^{d-1}} && \text{for all } d,\\
|\nabla_x G(x,y)| &\le C\left(\frac{\delta(y)}{|x-y|^{d}} + \frac{\delta(x)\delta(y)}{|x-y|^{d+1}}\right) && \text{for all } d.
\end{aligned}$$

For all $x \in P_n$ and $y \in \partial P_n$, one has

$$|P(x,y)| \le \frac{C\delta(x)}{|x-y|^{d}},$$
$$|\nabla P(x,y)| \le C\left(\frac{1}{|x-y|^{d}} + \frac{\delta(x)}{|x-y|^{d+1}}\right).$$

Although it is not precisely stated there, the methods of [Avellaneda and Lin 1991] also can achieve the same bounds for the Green's function and Poisson kernel associated with the operator $-\nabla\cdot(A(x)\nabla)$

in the strip-type domains

$$\Pi_n(0, R) := \{0 < x \cdot n < R\},$$

with constants independent of R. This will be useful later.

From the Poisson kernel bounds we can derive the L^∞ estimate which replaces the maximum principle for linear systems.

Lemma 3.2. *Suppose that u_1, u_2 are bounded solutions of* (3-4) *with respective boundary data φ_1, φ_2 and zero right-hand side. Then,*

$$\sup_{P_n} |u_1 - u_2| \le C \|\varphi_1 - \varphi_2\|_{L^\infty(\partial P_n)},$$

where C is a universal constant. The same holds for solutions in $\Pi_n(0, R)$.

For the solutions given by the Poisson kernel representation formula, the result of Lemma 3.2 follows from a standard calculation using Theorem 3.1. There is some subtlety in showing uniqueness; see [Gérard-Varet and Masmoudi 2012, Section 2.2] for a proof.

3B. ***Large-scale boundary regularity.*** In this section we consider the large-scale boundary regularity used in the heuristic argument of Section 2 for linear elliptic systems. We will need a boundary regularity result [Avellaneda and Lin 1987, Theorem 1]. For the following we assume Ω is some domain with $0 \in \partial\Omega$ and that u^ε solves

$$-\nabla \cdot \Big(A\Big(\frac{x}{\varepsilon}\Big)\nabla u^\varepsilon\Big) = 0 \quad \text{in } \Omega \cap B_1 \qquad \text{and} \qquad u^\varepsilon = g \quad \text{on } \partial\Omega \cap B_1.$$

Lemma 3.3. *For every $0 < \alpha < 1$ there is a constant C depending on α and universal quantities such that, if $\Omega = \{x_d > 0\} \cap B_1 =: B_1^+$,*

$$[u^\varepsilon]_{C^\alpha(B^+_{1/2})} \le C(\|\nabla g\|_{L^\infty(\{x_d=0\}\cap B_1)} + \|u^\varepsilon - g(0)\|_{L^2(B_1^+)}),$$

and for every $\nu > 0$

$$\|\nabla u^\varepsilon\|_{L^\infty(B^+_{1/2})} \le C(\|\nabla g\|_{C^{0,\nu}(\{x_d=0\}\cap B_1)} + \|u^\varepsilon - g(0)\|_{L^2(B_1^+)}).$$

We need the Hölder regularity result in cone-type domains which are the intersection of two half-spaces with normal directions n_1, n_2 very close to each other. We will consider the more general class of domains Ω which are a Lipschitz graph over $\mathbb{R}^{d-1}$ with small Lipschitz constant. In particular we assume that there is an $f : \mathbb{R}^{d-1} \to \mathbb{R}$ Lipschitz with $f(0) = 0$ such that

$$\Omega \cap B_1 = \{(x', x_d) : x_N > f(x')\} \cap B_1.$$

Lemma 3.4. *For every $0 < \alpha < 1$ there is a $\delta(\alpha) > 0$ universal such that, if Ω as above with $\|\nabla f\|_\infty \le \delta$, then*

$$[u^\varepsilon]_{C^\alpha(\Omega\cap B_{1/2})} \le C(\|\nabla g\|_{L^\infty(\partial\Omega\cap B_1)} + \|u^\varepsilon - g(0)\|_{L^2(\Omega\cap B_1)}).$$

The proof is by compactness; we postpone it to Appendix A.

3C. ***Poisson kernel in half-space intersection.*** From the regularity estimates of the previous subsection we can derive estimates on the Poisson kernel in the intersection of nearby half-space domains. Consider two unit vectors n_1, n_2 with $|n_1 - n_2| \sim \varepsilon$ small. For simplicity we suppose that

$$n_j = (\cos\varepsilon)e_d + (-1)^j(\sin\varepsilon)e_1.$$

Set

$$K = P_{n_1} \cap P_{n_2}.$$

Define $G_K(x, y)$ to be the Green's matrix. Although the domain is Lipschitz, G_K still satisfies the bound (via [Avellaneda and Lin 1987]), in $d \geq 3$,

$$|G_K(x, y)| \lesssim \frac{1}{|x-y|^{d-2}}.$$

We set $P_K(x, y)$, for $x \in K$ and $y \in \partial K$, to be the Poisson kernel for K, which is well-defined as long as $y_1 \neq 0$. Call $\delta(x) = \mathrm{dist}(x, \partial K)$.

Lemma 3.5. *For any $\alpha \in (0,1)$ and ε sufficiently small depending on α and universal quantities,*

$$|P_K(x, y)| \lesssim_\alpha \begin{cases} \dfrac{\delta(x)^\alpha}{|x-y|^{d-1+\alpha}} & \text{for } |y_1| \geq \frac{1}{2}|x-y|, \\ \dfrac{1}{|y_1|}\dfrac{\delta(x)^\alpha}{|x-y|^{d-2+\alpha}} & \text{for } |y_1| \leq \frac{1}{2}|x-y|. \end{cases}$$

The proof is postponed to Appendix A; we show how the estimates are used. Suppose $\psi : \partial K \to \mathbb{R}^N$ satisfies

$$|\psi(x)| \leq \min\{|x_1|, 1\}.$$

We consider the Poisson kernel solution of the Dirichlet problem,

$$u(x) = \int_{\partial K} P_K(x, y)\psi(y)\, dy.$$

In particular we are interested in the continuity at 0; we only consider really $x = te_d$ for some $t > 0$ (or $x = tn_1$ or tn_2 but this is basically the same) so we restrict to that case. Now for $y \in \partial K$, $|x-y| \sim t + |y|$ and so $|x - y| \gtrsim |y_1|$ and the first bound in Lemma 3.5 implies the second. Thus we can compute

$$\begin{aligned} |u(te_d)| &\lesssim \int_{\partial K} \frac{1}{|y_1|}\frac{t^\alpha}{(t+|y|)^{d-2+\alpha}} \min\{|y_1|, 1\}\, dy \\ &\lesssim \int_{\partial K} \frac{t^\alpha}{(t+|y|)^{d-2+\alpha}} \min\left\{1, \frac{1}{|y_1|}\right\} dy \\ &\lesssim \int_{\mathbb{R}}\int_{\mathbb{R}^{d-2}} \min\left\{1, \frac{1}{|y_1|}\right\} \frac{t^\alpha}{(t+|y_1|+|z|)^{d-2+\alpha}}\, dz\, dy_1. \end{aligned}$$

Computing the inner integrals, we have

$$\int_{\mathbb{R}^{d-2}} \frac{1}{(t+|y_1|+|z|)^{d-2+\alpha}}\, dz = \frac{1}{(t+|y_1|)^\alpha}\int_{\mathbb{R}^{d-2}} \frac{1}{(1+|w|)^{d-2+\alpha}}\, dw \lesssim \frac{1}{(t+|y_1|)^\alpha}.$$

Then

$$|u(te_d)| \lesssim \int_{\mathbb{R}} \min\left\{1, \frac{1}{|y_1|}\right\} \frac{t^\alpha}{(t+|y_1|)^\alpha}\, dy_1 \lesssim t^\alpha \quad \text{for } t \le 1.$$

We state the result of a slight generalization of this calculation as a lemma.

Lemma 3.6. *Suppose that $K = P_{n_1} \cap P_{n_2}$, $\alpha \in (0,1)$ and $\varepsilon = |n_1 - n_2|$ is sufficiently small so that the estimates of Lemma 3.5 hold, $\psi : \partial K \to \mathbb{R}$ smooth and satisfies the bound*

$$|\psi(x)| \le \min\{\delta^\beta |x \cdot (n_1 - n_2)|^\beta, 1\}$$

for some $\delta > 0$ and $1 \ge \beta > \alpha$, Then for any bounded solution u of

$$-\nabla \cdot (A(x)\nabla u) = 0 \quad \textit{in } K \textit{ with } u = \psi \textit{ on } \partial K,$$

it holds

$$|u(te_d)| \lesssim \delta^\alpha t^\alpha \quad \textit{for } t \le \frac{1}{\delta}.$$

There is an additional subtlety which is the uniqueness of the bounded solution of the Dirichlet problem in K; the argument is the same as in the half-space case; see [Gérard-Varet and Masmoudi 2012]. To derive Lemma 3.6 from the previous calculation just do a rescaling to $u\left(\frac{\cdot}{\delta}\right)$; the domain K is scaling-invariant and the Poisson kernel associated with $A\left(\frac{\cdot}{\delta}\right)$ satisfies the same bounds as for A.

4. Nonlinear equations background results

In this section we consider the boundary layer problem for nonlinear operators. To explain the assumptions we write out the problem in a general domain

$$\begin{cases} -\nabla \cdot a\left(\frac{x}{\varepsilon}, \nabla u^\varepsilon\right) = 0 & \text{in } \Omega, \\ u^\varepsilon(x) = g\left(x, \frac{x}{\varepsilon}\right) & \text{on } \partial\Omega. \end{cases} \tag{4-1}$$

This type of equation would arise as the Euler–Lagrange equation of a variational problem,

$$\text{minimize} \ \ E(u) = \int_\Omega F\left(\frac{x}{\varepsilon}, \nabla u\right) dx \ \ \text{over} \ \ u \in H_0^1(\Omega) + g\left(\cdot, \frac{\cdot}{\varepsilon}\right).$$

A natural uniform ellipticity assumption on the functional F is

$$F \text{ is convex with } 1 \ge D^2 F \ge \lambda > 0.$$

Then $a = DF$ is 1-Lipschitz continuous in p and has the monotonicity property

$$(a(x,p) - a(x,q)) \cdot (p - q) \ge \lambda |p - q|^2 \quad \text{for all } p, q \in \mathbb{R}^d.$$

Now we consider how to determine the effective boundary conditions for the homogenization equation (4-1). We zoom in at a boundary point $x_0 \in \partial\Omega$ defining

$$v^\varepsilon(y) = u^\varepsilon(x_0 + \varepsilon y), \quad \text{which solves } \begin{cases} -\nabla \cdot a\left(y + \frac{x_0}{\varepsilon}, \frac{1}{\varepsilon}\nabla v^\varepsilon\right) = 0 & \text{in } \frac{1}{\varepsilon}(\Omega - x_0), \\ v^\varepsilon(y) = g\left(x_0 + \varepsilon y, y + \frac{x_0}{\varepsilon}\right) & \text{on } \frac{1}{\varepsilon}\partial(\Omega - x_0). \end{cases}$$

Now in order to have a unique equation in the limit $\varepsilon \to 0$ the following limit needs to exist:

$$a_*(y, p) = \lim_{t\to 0} ta(y, t^{-1}p).$$

Note that, if said limit exists, it is always 1-homogeneous in p,

$$a_*(y, \lambda p) = \lim_{t\to 0} ta(y, (\lambda^{-1}t)^{-1}p) = \lambda a_*(y, p).$$

In other words we need a to be 1-homogeneous in p at ∞; then the operator a_* is this limiting homogeneous profile of a at x_0.

The above discussion motivates our assumption on the operators we study in the half-space problem:

(i) Periodicity:

$$a(x+z, p) = a(x, p) \quad \text{for all } x \in \mathbb{R}^d, z \in \mathbb{Z}^d, p \in \mathbb{R}^d. \tag{4-2}$$

(ii) Ellipticity: for some $\lambda > 0$ and all $p, q \in \mathbb{R}^d$

$$(a(x, p) - a(x, q)) \cdot (p-q) \geq \lambda |p-q|^2 \quad \text{and} \quad |a(x, p - a(x, q)| \leq |p-q|. \tag{4-3}$$

(iii) Positive homogeneity: for all x, p and $t > 0$,

$$a(x, tp) = ta(x, p). \tag{4-4}$$

For convenience will also assume $a(x, p)$ is C^1 in x so that, by the De Giorgi regularity theorem, solutions are locally $C^{1,\alpha}$ for some universal $\alpha > 0$.

4A. ***Regularity estimates for nonlinear equations.*** In this section we explain the regularity estimates which we use to obtain (1) existence of boundary layer limits and (2) the characterization of limits at rational directions. For both results we need the De Giorgi estimates respectively for the interior and boundary. As is the usual approach for regularity of nonlinear equations, we can reduce to considering actually the regularity of linear equations but with only bounded measurable coefficients.

For what follows we will take $A : \mathbb{R}^d \to M_{d\times d}$ to be measurable and elliptic,

$$\lambda \leq A(x) \leq 1.$$

Recall that results for bounded measurable coefficients imply results for solutions of nonlinear uniformly elliptic equations and for the difference of two solutions. If $u_1, u_2 \in H^1_{\text{loc}}(\Omega)$ solve

$$-\nabla \cdot a(x, \nabla u_j) = 0 \quad \text{in } \Omega$$

then $w = u_1 - u_2$ solves

$$-\nabla \cdot (A(x)\nabla w) = 0 \quad \text{in } \Omega \quad \text{with } A(x) = \int_0^1 D_p a(x, s\nabla u_1 + (1-s)\nabla u_2)\, ds, \tag{4-5}$$

and one can easily check that $\lambda \leq A(x) \leq 1$.

We remind that, despite the overlap of notation, the results in this section apply to solutions of scalar equations not systems.

Theorem 4.1 (De Giorgi–Nash–Moser). *There is an $\alpha \in (0,1)$ and $C > 0$ depending on d, λ so that if u solves*

$$-\nabla \cdot (A(x)\nabla u) = 0 \quad \text{in } B_1$$

then,

$$[u]_{C^\alpha(B_{1/2})} \leq C \inf_{c \in \mathbb{R}} \|u - c\|_{L^2(B_1)}.$$

A similar result holds up to the boundary for regular domains. We say that Ω is a regular domain of $\mathbb{R}^d$ if there are $r_0, \mu > 0$ so that for every $x \in \partial\Omega$ and every $0 < r < r_0$,

$$|\Omega^C \cap B_r(x)| \geq \mu |B_r|.$$

Lemma 4.2. *Suppose that Ω is a regular domain, $r_0 \geq 1$ and $0 \in \partial\Omega$, and $\varphi \in C^\beta$. There is an $\alpha_0(d, \lambda, \mu) \in (0,1)$ such that for $0 < \alpha < \min\{\alpha_0, \beta\}$ there is $C(d, \lambda, \mu, \alpha) > 0$ so that if u solves*

$$-\nabla \cdot (A(x)\nabla u) = 0 \quad \text{in } B_1 \cap \Omega, \quad \text{with } u = \varphi \text{ on } \partial\Omega,$$

then for every $r \leq 1$,

$$\operatorname*{osc}_{B_r} u \leq C\big([\varphi]_{C^\beta(B_1)} + \inf_{c \in \mathbb{R}} \|u - c\|_{L^2(B_1)}\big) r^\alpha.$$

The proof is postponed to Appendix A. We make a remark on the optimality of this estimate. Using these results one can show local $C^{1,\alpha}$ estimates for solutions of nonlinear uniformly elliptic equations. Large-scale $C^{1,\alpha}$ estimates are not possible due to the x-dependence, but in the spirit of [Avellaneda and Lin 1991] one can prove large-scale Lipschitz estimates; this was done in [Moser and Struwe 1992]. See also [Armstrong and Smart 2016] for the stochastic case. These estimates however are for *solutions*, we seem to require the result of Lemma 4.2 for *differences* of solutions (i.e., basically it is a C^α estimate of a derivative). It is not clear, therefore, whether we can do better than Lemma 4.2.

4B. *Half-space problem.* We consider the basic well-posedness results for nonlinear problems set in half-spaces. Consider

$$\begin{cases} -\nabla \cdot a(x, \nabla u) = 0 & \text{in } P_n, \\ u = \varphi(x) & \text{on } \partial P_n. \end{cases} \tag{4-6}$$

Then the maximum principle holds.

Lemma 4.3. *Suppose u_1 and u_2 are respectively bounded subsolutions and supersolutions of* (4-6) *with boundary data $\varphi_1 \leq \varphi_2$ on ∂P_n; then,*

$$u_1 \leq u_2 \quad \text{in } P_n.$$

The result follows from Lemma 4.2 or, more precisely, its proof. The proof is postponed to Appendix A.

4C. *Homogenization of nonoscillatory Dirichlet problem.* In this section we recall quantitative homogenization results for nonlinear divergence form problems in bounded domains with regular Dirichlet boundary condition. We will refer mainly to [Armstrong and Smart 2016]; they considered the stochastic case but their arguments also apply to the periodic case. The problem has also been studied in [Cardone, Pastukhova, and Zhikov 2005; Pastukhova 2008].

More precisely we study the limit

$$\begin{cases} -\nabla\cdot a\big(\frac{x}{\varepsilon},\nabla u^\varepsilon\big)=0 & \text{in } \Omega,\\ u^\varepsilon(x)=g(x) & \text{on } \partial\Omega \end{cases} \quad\text{to}\quad \begin{cases} -\nabla\cdot a^0(\nabla u^0)=0 & \text{in } \Omega,\\ u^0(x)=g(x) & \text{on } \partial\Omega, \end{cases} \tag{4-7}$$

where the boundary data g is a trace of $g\in W^{1,p}(\Omega)$ for some $p>2$. The following result is a combination of Proposition 4.1 and Corollary 4.2 in [Armstrong and Smart 2016] adapted to the periodic setting.

Theorem 4.4 [Armstrong and Smart 2016]. *Let $\Omega\subseteq\mathbb{R}^d$ be a bounded Lipschitz domain and $p>2$. Fix $\varepsilon\in(0,1]$ and let $u^\varepsilon, u^0\in g+H_0^1(\Omega)$ satisfying* (4-7). *There exist constants $C(d,\lambda,p,\Omega)\geq 1$ and $\beta(d,\lambda,p)\in(0,1]$ such that*

$$\|u^\varepsilon-u^0\|_{L^2_{\mathrm{avg}}(\Omega)}\leq C\varepsilon^\beta\|\nabla g\|_{L^p_{\mathrm{avg}}(\Omega)}.$$

By interpolating the L^2 estimate with the interior and boundary regularity, Theorem 4.1 and Lemma 4.2, there exist constants $C'(d,\lambda,\Omega)\geq 1$ and $\beta'(d,\lambda)\in(0,1]$ such that

$$\sup_\Omega|u^\varepsilon-u^0|\leq C'\varepsilon^{\beta'}\|\nabla g\|_{L^\infty(\partial\Omega)}.$$

Actually Corollary 4.2 of [Armstrong and Smart 2016] only does the interpolation argument for interior points; adding in the boundary regularity Lemma 4.2 to get the uniform estimate up to $\partial\Omega$ is an elementary argument. There are additional error terms in [Armstrong and Smart 2016] but these can be made zero in the periodic setting using the existence of periodic correctors.

5. Boundary layers limits

In this section we will discuss the boundary layer problem for divergence form elliptic problems in rational and irrational half-spaces. The results that we need for this paper are valid for both nonlinear scalar equations and linear systems and the proofs have only minor differences. For that reason, in this section and the next, we will discuss both types of equations in a unified way. We use the nonlinear notation for the PDE. We consider the cell problem

$$\begin{cases} -\nabla\cdot a(y,\nabla v_n^s)=0 & \text{in } P_n^s,\\ v_n^s=\varphi(y) & \text{on } \partial P_n^s. \end{cases} \tag{5-1}$$

We will first consider the case when $n\in S^{d-1}\setminus\mathbb{R}\mathbb{Z}^d$ is irrational.

5A. ***Irrational half-spaces.*** For linear systems, (5-1) in irrational half-spaces has been much studied [Gérard-Varet and Masmoudi 2011; 2012; Aleksanyan, Shahgholian, and Sjölin 2015; Aleksanyan 2017; Armstrong, Kuusi, Mourrat, and Prange 2017; Prange 2013; Shen and Zhuge 2017]. Typically the focus has been on the Diophantine irrational directions. We do not give the definition, since it is not needed for our work, but basically the Diophantine condition is a quantification of the irrationality. Under this assumption strong quantitative results can be derived for the convergence to the boundary layer limit.

For the purposes of this paper we are only interested in the qualitative result, the existence of a boundary layer limit for (5-1) in a generic irrational half-space (no Diophantine assumption). The existence of a boundary layer tail in general irrational half-spaces was originally proven by Prange [2013] for divergence form linear systems, and for nonlinear nondivergence form equations by the first author in [Feldman and Kim 2017] (following [Choi and Kim 2014] on the Neumann problem). To our knowledge the case of nonlinear divergence form equations has not been studied yet.

What we would like to explain here is that the proof of [Feldman and Kim 2017] applies also to the problems we consider in this paper, careful inspection shows that the proof of [Feldman and Kim 2017] only required the interior regularity, continuity up to the boundary (small-scale), and the L^∞ estimate (or maximum principle) with respect to the boundary data.

Theorem 5.1. *Suppose that $n \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$. Then there exists $\varphi_*(n)$ such that*

$$\sup_s \sup_{y \in \partial P_n} |v_n^s(y + Rn) - \varphi_*(n)| \to 0 \quad \text{as } R \to \infty.$$

One consequence of this theorem is that, for irrational directions, we can just study $v_n = v_n^0$. We give a sketch of the proof following [Feldman 2014].

Proof (*sketch*). The boundary data, and hence the solution v_n^s as well by uniqueness (L^∞ estimate Lemma 3.2 or maximum principle Lemma 4.3), satisfies an almost periodicity property in the directions parallel to ∂P_n. More precisely, given $N \geq 1$ there is a modulus $\omega_n(N) \to 0$ as $N \to \infty$ (uses n irrational) so that for any $y \in \partial P_n$ there is a lattice vector $z \in \mathbb{Z}^d$ with $|z - y| \leq N$ and $|z \cdot n - s| \leq \omega(N)$; see [Feldman and Kim 2017, Lemma 2.3]. Define z' to be the projection of z onto ∂P_n^s; then

$$|\varphi(x + z') - \varphi(x)| \leq |\varphi(x + z) - \varphi(x + z')| \leq \|\nabla\varphi\|_\infty \omega(N).$$

The same estimate, up to a universal constant, holds for $|v_n^s(x + z') - v_n^s(x)|$ by Lemma 3.2 or Lemma 4.3.

Since $v_n^0(\,\cdot + z)$ solves the same equation in $P_n^{z \cdot n}$, we can use the up-to-the-boundary Hölder continuity and the L^∞ estimate (or maximum principle) to see that

$$\|v_n^s(\cdot) - v_n^0(\,\cdot + z)\|_{L^\infty(P_n^s \cap P_n^{z\cdot n})} \lesssim \|\nabla\varphi\|_\infty \omega_n(N)^\alpha.$$

Sending N to ∞ we see that if v_n^0 has a boundary layer limit then so does v_n^s and they have the same value.

Then we just need to argue for v_n^0. Given $y \in \partial P_n$ the same argument as above shows there is $\bar{z} \in \partial P_n$ with $|\bar{z} - y| \leq N$ and

$$|v_n^0(\cdot) - v_n^0(\,\cdot + \bar{z})| \lesssim \|\nabla\varphi\|_\infty \omega_n(N)^\alpha.$$

Then using the L^∞ estimate Lemma 3.2 (or the maximum principle) and the large-scale interior regularity estimates, Theorem 4.1 above for the nonlinear case or Lemma 9 in [Avellaneda and Lin 1987] for the linear systems case,

$$\begin{aligned} \operatorname*{osc}_{y\cdot n \geq R} v_n^s(y) \lesssim \operatorname*{osc}_{y \cdot n = R} v_n^s(y) &\leq \operatorname*{osc}_{y \in B_N(0) \cap \partial P_n} v_n^s(y + Rn) + C\|\nabla\varphi\|_\infty \omega_n(N)^\alpha \\ &\lesssim \|\nabla\varphi\|_\infty \Big(\Big(\frac{N}{R}\Big)^\alpha + \omega_n(N)^\alpha\Big). \end{aligned}$$

Choosing N large first to make $\omega_n(N)$ small and then $R \gg N$ gets the existence of a boundary layer limit. ☐

5B. ***Rational half-spaces.*** Next we consider the case of a rational half-space. Let $\xi \in \mathbb{Z}^d \setminus \{0\}$ be an irreducible lattice direction, and v_ξ^s be the corresponding half-space problem solution. In this case φ is periodic with respect to a $(d-1)$-dimensional lattice parallel to ∂P_ξ. There exist $\ell_1, \dots, \ell_{d-1}$ with $\ell_j \perp \xi$ and $|\ell_j| \le |\xi|$ which are periods of φ. Then by uniqueness ℓ_j are also periods of v_ξ^s. In this special situation it is possible to show that there is a boundary layer limit with an exponential rate of convergence.

We give a general set-up. We consider the half-space problem

$$\begin{cases} -\nabla \cdot a(x, \nabla v) = \nabla \cdot f & \text{in } \mathbb{R}^d_+, \\ v = \psi(x') & \text{on } \partial \mathbb{R}^d_+, \end{cases} \tag{5-2}$$

where $\psi : \partial \mathbb{R}^d_+ \to \mathbb{R}$ and f are smooth, and ψ, f, and $a(\,\cdot\,, p)$ all share $d-1$ linearly independent periods $\ell_1, \dots, \ell_{d-1} \in \partial \mathbb{R}^d_+$ such that

$$\max_{1 \le j \le d-1} |\ell_j| \le M.$$

The operators a, as always, will also satisfy the assumptions of either Section 3 or Section 4. For now we will take $f = 0$; this covers most of the situations we will run into in this paper. Then v has a boundary layer limit with exponential rate of convergence.

Lemma 5.2. *There exists a value $c_*(\psi)$ such that*

$$\sup_{y \in \partial \mathbb{R}^d_+} |v(y + Re_d) - c_*| \le C(\operatorname{osc} \psi) e^{-cR/M},$$

with $C, c > 0$ depending only on λ, d.

The proof of this result is the same as the proof of the analogous result, [Feldman and Kim 2017, Lemma 3.1], so we only include a sketch. The only tools necessary are the maximum principle (or L^∞ estimate Lemma 3.2) and the large-scale interior Hölder estimates via De Giorgi–Nash–Moser for nonlinear equations (Theorem 4.1) or [Avellaneda and Lin 1987, Lemma 9] for linear systems.

Proof (*sketch*). Let $L \ge 1$ to be chosen, call Q to be the unit periodicity cell of ψ which has diameter at most $\sim M$. Apply the De Giorgi interior Hölder estimates or the Avellaneda–Lin large-scale Hölder estimates to find

$$\operatorname*{osc}_{\partial P_n + LMn} v = \operatorname*{osc}_{y \in Q} v(y + LMn) \le CL^{-\alpha} \operatorname*{osc}_{P_n} u \le CL^{-\alpha} \operatorname{osc} \psi \le \tfrac{1}{2} \operatorname{osc} \psi.$$

The second inequality is by the maximum principle or the L^∞ estimate Lemma 3.2; for the third inequality we have chosen $L \ge 1$ universal to make $CL^{-\alpha} \le \frac{1}{2}$. Then iterate the argument with the new boundary data on $\partial P_n + LMn$ with oscillation decayed by a factor of $\frac{1}{2}$. □

We will also need a slight variant of the above result when the operator a does not share the same periodicity as the boundary data, but instead has oscillations at a much smaller scale. We assume that ψ has periods $\ell_1, \dots, \ell_{d-1}$ as before, and now we also assume that there are $e_1, \dots, e_d$ which are periods of a and

$$\max_{1 \le j \le d} |e_j| \le \varepsilon.$$

For example this is the case with $a\big(\frac{x}{\varepsilon}, p\big)$ when $a(\cdot, p)$ is $\mathbb{Z}^d$-periodic. In this situation we do not quite have a boundary layer limit with exponential rate, but at least there is an exponential decay of the oscillation down to a scale $\sim \varepsilon^{\alpha}$.

Lemma 5.3. *There exists a value $c_*(\psi)$ such that for any $\beta \in (0, 1]$*

$$\sup_{y \in \partial \mathbb{R}^d_+} |v(y + Re_d) - c_*| \leq C(\operatorname{osc} \psi) e^{-cR/M} + C\|\psi\|_{C^\beta} \varepsilon^{\alpha}$$

for some universal $\alpha(\beta) \in (0, 1)$ (nonlinear case) or for every $\alpha \in (0, \beta)$ (linear case), with $c, C > 0$ universal and C depending on α as well.

Again the proof of this result mirrors the proof of Lemma 3.2 in [Feldman and Kim 2017] and we do not include it. Briefly, the idea is the same as Lemma 5.2 except that the lattice vectors generated by $\ell_1, \dots, \ell_{d-1}$ are no longer periods of v; instead for each lattice vector there is a nearby vector (distance at most ε) which is a period of the operator. This vector will almost be a period of v, with error of ε^{α} which comes from the boundary continuity estimate Lemma 4.2 (nonlinear) or Lemma 3.4 (linear system).

Finally we discuss the boundary layer equation (5-1) with nonzero right-hand side f. We will restrict to the case of linear systems. We need to put a decay assumption on f to guarantee even the existence of a solution. We will assume that there are $K, b > 0$ so that

$$\sup_{y_d \geq R} |f(y)| \leq \frac{K}{R} e^{-bR/M}. \tag{5-3}$$

Such assumption arises naturally; it is exactly the decay obtained for ∇v when v solves (5-1) with $f = 0$. The $\frac{1}{R}$ polynomial decay is important since we will care about the dependence on $M \gg 1$; the exponential does not take effect until $R \gg M$, while the $\frac{1}{R}$ decay begins at the unit scale.

Lemma 5.4. *Suppose that f satisfies the bound* (5-3) *and v is the solution of the half-space equation* (5-1) *for a linear system satisfying the standard assumptions of Section 3. Then there exists $c_*(\psi, f)$ such that*

$$\sup_{y \in \partial \mathbb{R}^d_+} |v(y + Re_d) - c_*| \leq C((\operatorname{osc} \psi) + K \log M) e^{-b_0 R/M},$$

where the constants C and b_0 depend on universal parameters as well as b from (5-3).

See the Appendix and [Feldman and Kim 2017, Lemma A.4] for more details.

5C. ***Interior homogenization of a boundary layer problem.*** In this section we will consider the *interior* homogenization of half-space problems with periodic boundary data; as explained in Section 2 such a problem arises in the course of computing the directional limits of φ_* at a rational direction:

$$\begin{cases} -\nabla \cdot a\big(\frac{x}{\varepsilon}, \nabla u^{\varepsilon}\big) = 0 & \text{in } P_n, \\ u^{\varepsilon} = \psi(x) & \text{on } \partial P_n \end{cases} \tag{5-4}$$

homogenizing to

$$\begin{cases} -\nabla \cdot a^0(\nabla u^0) = 0 & \text{in } P_n, \\ u^0 = \psi(x) & \text{on } \partial P_n. \end{cases} \tag{5-5}$$

Here $\psi : \partial P_n \to \mathbb{R}^N$, as in the previous section, will be smooth and periodic with respect to $d-1$ linearly independent translations parallel to ∂P_n, which we call $\ell_1, \dots, \ell_{d-1} \in \partial P_n$. As before we call $M = \max_j |\ell_j|$ and assume that $M \gg \varepsilon$. For convenience we can assume that $M = 1$; general results can be derived by scaling.

This problem is quite similar to the standard homogenization problem for Dirichlet boundary data, the unboundedness of the domain is compensated by the periodicity of the boundary data and by the existence of a boundary layer limit which is a kind of (free) boundary condition at infinity. The main result of this section is the *uniform* convergence of u^ε to u^0, and hence also (importantly for us) the convergence of the boundary layer limits.

Proposition 5.5. *Homogenization holds for* (5-4) *with estimates*:

(i) (nonlinear equations) *For every* $\beta \in (0,1)$, *there exists* $0 < \alpha(\beta, \lambda, d) \le \beta$ *such that, for all* $\varepsilon \le \frac{1}{2}$,

$$\sup_{P_n} |u^\varepsilon - u^0| \lesssim_\beta [\psi]_{C^\beta} \varepsilon^\alpha.$$

(ii) (linear systems) *For every* $\varepsilon \le \frac{1}{2}$,

$$\sup_{P_n} |u^\varepsilon - u^0| \lesssim [\psi]_{C^4} \varepsilon \Big(\log \frac{1}{\varepsilon}\Big)^3.$$

We will follow the idea of [Feldman and Kim 2017, Lemma 4.5]; there is a slight additional difficulty since for divergence form nonlinear problems it is not possible to add a linear function $n \cdot x$ and preserve the solution property, even for the homogenized problem. The C^4 norm we require for ψ in the linear systems case is more than necessary.

For convenience we will make some additional assumptions so that u^ε shares the periods of the boundary data ψ. Assume that $n = \xi/|\xi|$ for an irreducible lattice direction $\xi \in \mathbb{Z}^d \setminus \{0\}$. In that case $a(\cdot, p)$ is periodic with respect to the lattice $\xi^\perp \cap \mathbb{Z}^d = \{k \in \mathbb{Z}^d : k \cdot \xi = 0\}$. Then we assume that the periods of ψ are also periods of $a\big(\frac{\cdot}{\varepsilon}, p\big)$,

$$\ell_1, \dots, \ell_{d-1} \in \varepsilon \xi^\perp \cap \mathbb{Z}^d. \tag{5-6}$$

Then by the uniqueness of bounded solutions to (5-4) the solution u^ε also has $\ell_1, \dots, \ell_{d-1}$ as periods. The result of Proposition 5.5 should hold without this assumption, as was proven in the nondivergence form case in [Feldman and Kim 2017, Lemma 4.5].

The proof will use known results about homogenization of Dirichlet boundary value problems in bounded domains; specifically we consider the problem in a strip-type domain,

$$\begin{cases} -\nabla \cdot a\big(\frac{x}{\varepsilon}, \nabla u_R^\varepsilon\big) = 0 & \text{in } \Pi_n(0,R) = \{0 < x \cdot n < R\}, \\ u_R^\varepsilon = \psi(x) & \text{on } \partial \Pi_n(0,R) = \{x \cdot n \in \{0, R\}\}, \end{cases} \tag{5-7}$$

where we make some choice to extend ψ to $x \cdot n = R$, preserving the regularity and periodic structure. The solution of the homogenized problem u_R^0 is defined analogously. Because of (5-6), u_R^ε and u_R^0 have periods $\ell_1, \dots, \ell_{d-1}$, so although the domain $\Pi_n(0,R)$ is unbounded, actually we can consider (5-7) as a homogenization problem on the bounded domain $\mathbb{T}^{d-1} \times [0, R]$, or rather a rotation/rescaling of this domain.

For linear systems we have, for $R \geq 1$, the rate for convergence

$$\sup_{\Pi_n(0,R)} |u^\varepsilon_R - u^0_R| \leq CR^4 \|\psi\|_{C^4}(R^{-1}\varepsilon), \tag{5-8}$$

which can be derived from the rate of convergence proved in [Avellaneda and Lin 1991] by scaling. The C^4 regularity on ψ is sufficient; we did not state the precise regularity requirement on ψ which can be found in [Avellaneda and Lin 1991]. With less regularity on ψ one can also obtain an algebraic rate of convergence $O(\varepsilon^\alpha)$.

For nonlinear equations there is an algebraic rate of convergence, for any $\beta \in (0, 1)$,

$$\sup_{\Pi_n(0,R)} |u^\varepsilon_R - u^0_R| \leq CR^\beta \|\psi\|_{C^{0,\beta}}(R^{-1}\varepsilon)^\alpha, \tag{5-9}$$

with some $\alpha = \alpha(\beta) \in (0, 1)$ universal. This result was recounted above in Section 4C, and can be found in [Armstrong and Smart 2016; Pastukhova 2008].

Proof of Proposition 5.5. We define the boundary layer limits of, respectively, the ε-problem and the homogenized problem in (5-4). We have not proven that the ε-problem has a boundary layer limit; however Lemma 5.3 gives that the limit values are concentrated in a set of diameter $o_\varepsilon(1)$. So we define,

$$\mu^\varepsilon \in \lim_{R\to\infty} u^\varepsilon(Rn) \quad \text{and} \quad \mu^0 = \lim_{R\to\infty} u^0(Rn),$$

where μ^ε can be any subsequential limit and satisfies, again via Lemma 5.3,

$$|\mu^\varepsilon - u^\varepsilon(Rn)| \leq C\|\nabla\psi\|_\infty(\varepsilon^\alpha + e^{-cR}) \quad \text{(nonlinear case)}, \tag{5-10}$$

$$|\mu^\varepsilon - u^\varepsilon(Rn)| \leq C\|\nabla\psi\|_{C^{0,\nu}}(\varepsilon + e^{-cR}) \quad \text{(linear system case)}. \tag{5-11}$$

Instead of arguing directly with u^ε and u^0 we consider

$$\begin{cases} -\nabla\cdot a\big(\frac{x}{\varepsilon}, \nabla u^\varepsilon_R\big) = 0 & \text{in } \Pi_n(0, R), \\ u^\varepsilon_R = \psi(x) & \text{on } x\cdot n = 0, \\ u^\varepsilon_R = \mu^\varepsilon & \text{on } x\cdot n = R \end{cases} \tag{5-12}$$

and, for $j \in \{0, \varepsilon\}$

$$\begin{cases} -\nabla\cdot a^0(\nabla u^0_{R,j}) = 0 & \text{in } \Pi_n(0, R), \\ u^0_{R,j} = \psi(x) & \text{on } x\cdot n = 0, \\ u^0_{R,j} = \mu^j & \text{on } x\cdot n = R. \end{cases} \tag{5-13}$$

We will choose $R = R(\varepsilon)$ below to balance the various errors. The error in replacing u^ε by u^ε_R is given by

$$|u^\varepsilon(x) - u^\varepsilon_R(x)| \leq C\|\nabla\psi\|_\infty(\varepsilon^\alpha + e^{-cR}) \quad \text{for } x \in \Pi_n(0, R),$$

and replacing u^0 by $u^0_{R,0}$ by

$$|u^0(x) - u^0_{R,0}(x)| \leq C(\operatorname{osc}\psi)e^{-cR} \quad \text{for } x \in \Pi_n(0, R);$$

the estimates hold on $\partial\Pi_n(0, R)$ by (5-10) (or for linear we use (5-11) instead), and therefore by the maximum principle (or by Lemma 3.2 for linear systems) they hold on the interior as well. To estimate the error in replacing $u^0_{R,0}$ by $u^0_{R,\varepsilon}$ we need to estimate the difference $\mu^\varepsilon - \mu^0$, which is basically the goal of the proof; this will be achieved below.

By Lemma 4.2 (or Lemma 3.3 in the linear systems case) there exists a universal $\delta_0(\lambda, d) > 0$ so that if B is uniformly elliptic and q solves

$$\begin{cases} -\nabla\cdot(B(x)\nabla q) = 0 & \text{in } \Pi_n(0,1), \\ q = 0 & \text{on } x\cdot n = 0, \\ |q| = 1 & \text{on } x\cdot n = 1; \end{cases} \tag{5-14}$$

then $|q(x)| \le \frac{1}{2}$ for $x\cdot n \le \delta_0$. Now set

$$q^\varepsilon = u^0_{R,0} - u^0_{R,\varepsilon}, \quad \text{which solves } \begin{cases} -\nabla\cdot(B(x)\nabla q^\varepsilon) = 0 & \text{in } 0 < x\cdot n < R, \\ q^\varepsilon = 0 & \text{on } x\cdot n = 0, \\ q^\varepsilon = \mu^0 - \mu^\varepsilon & \text{on } x\cdot n = R, \end{cases}$$

with $B(x) = A^0$ in the linear case, or

$$B(x) = \int_0^1 Da^0(t\nabla u^0_{R,0}(x) + (1-t)\nabla u^0_{R,\varepsilon}(x))\,dt \quad \text{uniformly elliptic,}$$

in the nonlinear case. Now $(1/|\mu^0 - \mu^\varepsilon|)q(Rx)$ solves an equation of the type (5-14) and so,

$$|q(\delta_0 Rn)| \le \tfrac{1}{2}|\mu^0 - \mu^\varepsilon|.$$

Now we apply the homogenization error estimates (5-9) and (5-8) for the domain $\Pi_n(0, R)$ to (5-12)

$$|u^0_{R,\varepsilon} - u^\varepsilon_R| \le CR\|\nabla\psi\|_\infty(R^{-1}\varepsilon)^\gamma$$

or respectively in the linear system case

$$|u^0_{R,\varepsilon} - u^\varepsilon_R| \le CR^4\|\psi\|_{C^4}(R^{-1}\varepsilon).$$

Now we estimate the error in $\mu^\varepsilon - \mu^0$ for the nonlinear case

$$\begin{aligned} |\mu^\varepsilon - \mu^0| &\le |u^\varepsilon(\delta_0 Rn) - u^0(\delta_0 Rn)| + C\|\nabla\psi\|_\infty(\varepsilon^\alpha + e^{-cR}) \\ &\le |u^\varepsilon_R(\delta_0 Rn) - u^0_{R,\varepsilon}(\delta_0 Rn)| + |q^\varepsilon(\delta_0 Rn)| + C\|\nabla\psi\|_\infty(\varepsilon^\alpha + e^{-cR}) \\ &\le CR\|\nabla\psi\|_\infty(R^{-1}\varepsilon)^\gamma + \tfrac{1}{2}|\mu^\varepsilon - \mu^0| + C\|\nabla\psi\|_\infty(\varepsilon^\alpha + e^{-cR}). \end{aligned}$$

Moving the middle term above to the left-hand side we find,

$$|\mu^\varepsilon - \mu^0| \le C\|\nabla\psi\|_\infty(R(R^{-1}\varepsilon)^\gamma + \varepsilon^\alpha + e^{-cR}) \le C\|\nabla\psi\|_\infty\varepsilon^{\alpha'},$$

where finally we have chosen $R = C\log\frac{1}{\varepsilon}$ and $\alpha' < \min\{\alpha, \gamma\}$. The same argument in the linear case yields,

$$|\mu^\varepsilon - \mu^0| \le C[\psi]_{C^4}(R^4(R^{-1}\varepsilon) + \varepsilon + e^{-cR}) \le C[\psi]_{C^4}\varepsilon\Big(\log\frac{1}{\varepsilon}\Big)^3. \qquad \square$$

6. Asymptotics near a rational direction

We study asymptotic behavior of the cell problems as $n \in S^{d-1}$ approaches a rational direction $\xi \in \mathbb{Z}^d \setminus \{0\}$. We call v_ξ^s the solution of the cell problem

$$\begin{cases} -\nabla \cdot a(x+s\xi, \nabla v_\xi^s) = 0 & \text{in } P_\xi, \\ v_\xi^s(x) = \varphi(x+s\xi) & \text{on } \partial P_\xi. \end{cases} \tag{6-1}$$

The boundary layer limit of the above cell problem depends on the parameter s and we define

$$\varphi_*(\xi, s) := \lim_{R\to\infty} v_\xi^s(x+R\xi), \tag{6-2}$$

which is well-defined and the limit is independent of x; see Lemma 5.2. It follows from Bézout's identity that φ_* is a $1/|\xi|$-periodic function on $\mathbb{R}$; see [Feldman and Kim 2017, Lemma 2.9]. As long as we can we will combine the arguments for linear systems and nonlinear equations.

6A. ***Regularity of*** $\varphi_*(\xi, \cdot)$. To begin we need to establish some regularity of $\varphi_*(\xi, \cdot)$. For quantitative purposes it is important to control the dependence of the regularity on $|\xi|$. We just state the results, postponing the proofs until the end of the section. A modulus of continuity for $\varphi_*(\xi, \cdot)$ which is uniform in $|\xi|$ is not difficult to establish. This follows from the continuity up to the boundary Lemma 4.2 (or Lemma 3.3) and the maximum principle Lemma 4.3 (or the L^∞ estimate Lemma 3.2).

Lemma 6.1. *The boundary layer limits* $\varphi_*(\xi, s)$ *are continuous in* s:

(i) (*nonlinear equations*)

$$[\varphi_*(\xi, \cdot)]_{C^\alpha} \le C \|\nabla\varphi\|_\infty,$$

which holds for some universal $C \ge 1$ *and* $\alpha \in (0, 1)$.

(ii) (*linear systems*) *Hölder estimates as above hold for all* $\alpha \in (0, 1)$ *and moreover*,

$$\left\| \frac{d}{ds}\varphi_*(\xi, \cdot) \right\|_\infty \le C \|\nabla\varphi\|_{C^{0,\nu}} \quad \textit{for any } 0 < \nu \le 1.$$

To optimize our estimates, in the linear case we will also need higher regularity of φ_* which is (almost) uniform in $|\xi|$; this is somewhat harder to establish.

Lemma 6.2 (linear systems). *For any* $\xi \in \mathbb{Z}^d \setminus \{0\}$, *suppose* $\varphi_*(\xi, s)$ *is defined as above. Then for all* $j \in \mathbb{N}^d$ *and any* $\nu > 0$ *there exists some constant* C_j *universal such that*

$$\sup_s \left| \frac{d^j}{ds^j}\varphi_*(\xi, s) \right| \le C_j \|\varphi\|_{C^{j,\nu}} \log^j(1+|\xi|).$$

Note that Lemma 6.2 is a bit weaker than Lemma 6.1 in the case $j = 1$; this is because we take a different approach which is suboptimal in the $j = 1$ case; it is not clear if the logarithmic terms are necessary when $j > 1$. The proof is similar to [Feldman and Kim 2017, Lemma 7.2], taking the derivative of v_ξ^s with respect to s and estimating based on the PDE. Probably more precise Sobolev estimates are possible but we did not pursue this.

6B. ***Intermediate-scale asymptotics.*** Consider an irrational direction n close to a lattice direction $\xi \in \mathbb{Z}^d \setminus \{0\}$. Let $\varepsilon > 0$ small and we write

$$n = (\cos\varepsilon)\hat{\xi} - (\sin\varepsilon)\eta \quad \text{for some } \xi \in \mathbb{Z}^d \setminus \{0\} \text{ and a unit vector } \eta \perp \xi.$$

We will assume below that $|\varepsilon| \le \frac{\pi}{6}$. We consider the cell problem in P_n

$$\begin{cases} -\nabla \cdot a(y, \nabla v_n) = 0 & \text{in } P_n, \\ v_n = \varphi(y) & \text{on } \partial P_n. \end{cases} \tag{6-3}$$

The first step of the argument is to show, with error estimate, that the boundary layer limit of v_n is close to the boundary layer limit of the problem

$$\begin{cases} -\nabla \cdot a(y/\tan\varepsilon, \nabla v_n^{\text{int}}) = 0 & \text{in } P_n, \\ v_n^{\text{int}} = \varphi_*(\xi, y \cdot \eta) & \text{on } \partial P_n. \end{cases} \tag{6-4}$$

The solution v_n^{int} approximates v_n, asymptotically as $\varepsilon \to 0$, starting at an intermediate scale $1 \ll R \ll \frac{1}{\varepsilon}$ away from ∂P_n. The argument is by direct comparison of v_n with v_ξ^s in their common domain.

Since (6-4) has a boundary layer of size uniform in ε we can replace, again with small error, by a problem in a fixed domain

$$\begin{cases} -\nabla \cdot a(y/\tan\varepsilon, \nabla w_{\xi,\eta}^{\varepsilon}) = 0 & \text{in } P_\xi, \\ w_{\xi,\eta}^{\varepsilon} = \varphi_*(\xi, y \cdot \eta) & \text{on } \partial P_\xi. \end{cases} \tag{6-5}$$

We note that there may be some confusion due to similarities in the notation between v_ξ^s and $w_{\xi,\eta}^{\varepsilon}$. The boundary value problem for $w_{\xi,\eta}^{\varepsilon}$, or its homogenized version introduced later, will always be set in P_ξ, so there will be no need for the translation parameter s.

We remark that for both (6-4) and (6-5) we have not proven the existence of a boundary layer limit; rather we use Lemma 5.3. For convenience we will state estimates on $\lim_{R\to\infty} v_n^{\text{int}}(Rn)$ or on $\lim_{R\to\infty} w_{\xi,\eta}^{\varepsilon}(R\hat{\xi})$, but technically we will mean that the estimate holds for every subsequential limit.

Proposition 6.3. *Let $\xi \in \mathbb{Z}^d \setminus \{0\}$ and $n = (\cos\varepsilon)\hat{\xi} - (\sin\varepsilon)\eta$ with $\varepsilon > 0$ small and a unit vector $\eta \perp \xi$:*

(i) (*nonlinear equations*) *There is universal $\alpha \in (0, 1)$ such that*

$$\Big|\varphi_*(n) - \lim_{R\to\infty} w_{\xi,\eta}^{\varepsilon}(R\hat{\xi})\Big| \lesssim \|\nabla\varphi\|_\infty |\xi|^\alpha \varepsilon^\alpha,$$

where we mean that the estimate holds for any subsequential limit of $w_{\xi,\eta}^{\varepsilon}(R\hat{\xi})$ as $R \to \infty$.

(ii) (*linear systems*) *For every $\alpha \in (0, 1)$ and any $\nu > 0$*

$$\Big|\varphi_*(n) - \lim_{R\to\infty} w_{\xi,\eta}^{\varepsilon}(R\hat{\xi})\Big| \lesssim_{\alpha,\nu} [\varphi]_{C^{1,\nu}} |\xi|^\alpha \varepsilon^\alpha,$$

where again we mean that the estimate holds for any subsequential limit of $w_{\xi,\eta}^{\varepsilon}(R\hat{\xi})$ as $R \to \infty$.

The first step is to compare the boundary layer limits of (6-3) and (6-4).

Lemma 6.4. *Fix any* $x \in \partial P_n$, $1 \le R \le \frac{1}{\varepsilon}$ *and let* $s = x \cdot \eta \tan \varepsilon$:

(i) (*nonlinear equations*) *There is a universal* $\alpha \in (0,1)$ *such that*

$$|v_n - v_\xi^s|(x + Rn) \lesssim \|\nabla \varphi\|_\infty (R\varepsilon)^\alpha.$$

(ii) (*linear systems*) *For every* $\alpha \in (0,1)$

$$|v_n - v_\xi^s|(x + Rn) \lesssim_\alpha \|\nabla \varphi\|_\infty (R\varepsilon)^\alpha.$$

Proof. Let us define the cone domains

$$K(x) := (P_\xi + x) \cap P_n \quad \text{and} \quad K_R(x) = K(x) \cap B_R(x);$$

we may simply write K, K_R if $x = 0$. Let $x_0 \in \partial P_n$; we compute using $n \cdot x_0 = 0$ and $n = (\cos\varepsilon)\hat{\xi} - (\sin\varepsilon)\eta$ that

$$x_0 \cdot \hat{\xi} = (x_0 \cdot \eta) \tan \varepsilon.$$

Let $x \in \partial K(x_0)$; then $x \in \partial P_n$ (or $x \in \partial P_\xi + x_0$) and there exists $y \in \partial P_\xi + x_0$ (or respectively ∂P_n) with

$$|x - y| \le |x - x_0| \sin \varepsilon \le \varepsilon |x - x_0|.$$

Nonlinear equations: Applying the De Giorgi boundary continuity estimates Lemma 4.2 for small enough $\alpha \in (0,1)$ universal, for all $x \in \partial K(x_0)$,

$$|v_\xi^s(x) - v_n(x)| \le |v_\xi^s(x) - \varphi(y)| + |\varphi(y) - v_n(x)| \lesssim \|\nabla \varphi\|_\infty \varepsilon^\alpha |x - x_0|^\alpha.$$

Now since $v_\xi^s(x) - v_n(x)$ is a difference of solutions we can apply the boundary continuity estimate from Lemma 4.2 again,

$$|v_\xi^s(x) - v_n(x)| \lesssim \|\nabla \varphi\|_\infty \varepsilon^\alpha |x - x_0|^\alpha \quad \text{for } x \in K(x_0),$$

with perhaps a slightly smaller $\alpha(d, \lambda)$.

Linear systems: We have, by almost the same argument as above now using instead Lemma 3.3, for any $\alpha \in (0,1)$

$$|v_\xi^s(x) - v_n(x)| \lesssim \|\nabla \varphi\|_\infty \varepsilon^\alpha |x - x_0|^\alpha \quad \text{on } \partial K(x_0).$$

Now by the Poisson kernel bounds in $K(x_0)$, Lemmas 3.5 and 3.6, for a slightly smaller α and ε sufficiently small depending on α

$$|v_\xi^s(x) - v_n(x)| \lesssim \|\nabla \varphi\|_\infty \varepsilon^\alpha |x - x_0|^\alpha \quad \text{for } x \in K(x_0).$$

The remainder of the proof is the same as the case of scalar equations. ☐

Now we derive some consequences of Lemma 6.4. Let's assume that $\|\nabla \varphi\|_\infty \le 1$ to simplify the exposition; the general inequalities can of course be derived by rescaling. Combining Lemma 5.2 with Lemma 6.4 we find that for any $R \ge 1$

$$|v_n(x + Rn) - \varphi_*(\xi, x \cdot \eta \tan \varepsilon)| \lesssim [(R\varepsilon)^\alpha + e^{-cR/|\xi|}] \quad \text{for } x \in \partial P_n.$$

Choosing $R = |\xi| \log \frac{1}{\varepsilon}$ we obtain,

$$|v_n(x + Rn) - \varphi_*(\xi, x \cdot \eta \tan \varepsilon)| \lesssim |\xi|^\alpha \varepsilon^\alpha \quad \text{for } x \in \partial P_n, \tag{6-6}$$

either for a slightly smaller universal α in the nonlinear case, or again for every $\alpha \in (0, 1)$ in the case of linear systems.

Now consider the rescaling

$$\tilde{v}_n^{\text{int}}(y) = v_n\left([Rn] + \frac{y}{\tan \varepsilon}\right) \quad \text{defined for } y \in P_n, \tag{6-7}$$

where $[Rn] \in \mathbb{Z}^d$ is the lattice point such that $Rn - [Rn] \in [0, 1)^d$.

Lemma 6.5. *Let $R = |\xi| \log \frac{1}{\varepsilon}$ and $\tilde{v}_n^{\text{int}}$ be defined as above in* (6-7). *Then*:

(i) (*nonlinear equations*) *There is universal $\alpha \in (0, 1)$ such that*

$$\sup_{P_n} |\tilde{v}_n^{\text{int}} - v_n^{\text{int}}| \lesssim \|\nabla \varphi\|_\infty |\xi|^\alpha \varepsilon^\alpha.$$

(ii) (*linear systems*) *For every $\alpha \in (0, 1)$*

$$\sup_{P_n} |\tilde{v}_n^{\text{int}} - v_n^{\text{int}}| \lesssim_\alpha \|\nabla \varphi\|_\infty |\xi|^\alpha \varepsilon^\alpha.$$

Proof. Again assume that $\|\nabla \varphi\|_\infty \le 1$ to simplify the exposition. Note that

$$\tilde{v}_n^{\text{int}}(y) = v_n\Big(Rn + \frac{1}{\tan \varepsilon}(y + ([Rn] - Rn) \tan \varepsilon)\Big)$$

so by (6-6)

$$|\tilde{v}_n^{\text{int}}(y) - \varphi_*(\xi, (y + ([Rn] - Rn) \tan \varepsilon) \cdot \eta)| \lesssim_\alpha |\xi|^\alpha \varepsilon^\alpha.$$

Then applying the regularity of φ_* from Lemma 6.1

$$|\tilde{v}_n^{\text{int}}(y) - \varphi_*(\xi, y \cdot \eta)| \lesssim_\alpha |\xi|^\alpha \varepsilon^\alpha.$$

Thus $\tilde{v}_n^{\text{int}}$ solves

$$\begin{cases} -\nabla \cdot a(y/\tan \varepsilon, \nabla \tilde{v}_n^{\text{int}}) = 0 & \text{in } P_n, \\ |\tilde{v}_n^{\text{int}}(y) - \varphi_*(\xi, y \cdot \eta)| \le C|\xi|^\alpha \varepsilon^\alpha & \text{on } \partial P_n. \end{cases} \tag{6-8}$$

This is almost the same as (6-4) solved by v_n^{int}. The L^∞-estimate Lemma 3.2 (or the maximum principle) implies

$$\sup_{P_n} |v_n^{\text{int}} - \tilde{v}_n^{\text{int}}| \lesssim_\alpha |\xi|^\alpha \varepsilon^\alpha \tag{6-9}$$

either for every $\alpha \in (0, 1)$ in the linear systems case, or for some universal α in the nonlinear case. □

To complete the proof of Proposition 6.3 we just need to compare the solutions v_n^{int} of (6-4) and $w_{\xi,\eta}^\varepsilon$ of (6-5). The width of the boundary layer is now of uniform size in ε so this is not a problem; we will just need to use the boundary continuity estimate (Lemmas 3.4 and 4.2) and the continuity estimate of $\varphi_*(\xi, \cdot)$ Lemma 6.1.

Lemma 6.6. *The following estimates hold for the boundary layers of* v_n^{int} *and* $w^\varepsilon_{\xi,\eta}$:

(i) (*nonlinear equations*) *There is* $\alpha \in (0,1)$ *universal such that*

$$\Big|\lim_{R\to\infty} v_n^{\mathrm{int}}(Rn) - \lim_{R\to\infty} w^\varepsilon_{\xi,\eta}(R\hat\xi)\Big| \lesssim \|\nabla\varphi\|_\infty |\xi|^\alpha \varepsilon^\alpha,$$

where technically we mean that the estimate holds for any pair of subsequential limits.

(ii) (*linear systems*) *For every* $\alpha \in (0,1)$ *and any* $\nu > 0$

$$\Big|\lim_{R\to\infty} v_n^{\mathrm{int}}(Rn) - \lim_{R\to\infty} w^\varepsilon_{\xi,\eta}(R\hat\xi)\Big| \lesssim_{\alpha,\nu} [\varphi]_{C^{1,\nu}} |\xi|^\alpha \varepsilon^\alpha,$$

where technically we mean that the estimate holds for any pair of subsequential limits.

Proof. We compare the two solutions in their common domain. As before let $K = P_n \cap P_\xi$ and

$$u = v_n^{\mathrm{int}} - w^\varepsilon_{\xi,\eta}.$$

Nonlinear equations: We have

$$-\nabla\cdot(A(x)\nabla u) = 0 \quad \text{in } K \text{ with some } \lambda \le A(x) \le 1 \text{ as in (4-5).}$$

We compute the error on ∂K in the same way that we did in Lemma 6.4. Using Lemma 4.2 we find for $x \in \partial K$,

$$|u(x)| = |v_n^{\mathrm{int}}(x) - w^\varepsilon_{\xi,\eta}(x)| \lesssim \|\varphi_*(\xi,\cdot)\|_{C^{\alpha'}} \varepsilon^\alpha |x|^\alpha \lesssim \|\nabla\varphi\|_\infty \varepsilon^\alpha |x|^\alpha,$$

where α' is the universal, continuity modulus from Lemma 6.1 and $\alpha < \alpha'$. Next we use the De Giorgi boundary continuity estimate, Lemma 4.2 to obtain, again with a slightly smaller α,

$$|u(x)| \lesssim \|\nabla\varphi\|_\infty \varepsilon^\alpha |x|^\alpha \quad \text{for } x \in K. \tag{6-10}$$

Next we use that the size of the boundary layers for v_n^{int} and $w^\varepsilon_{\xi,\eta}$ are uniformly bounded in ε, via Lemma 5.3, to find for all $R_0 \ge 1$,

$$\sup_{y\in\partial P_n} \Big|v_n^{\mathrm{int}}(y + R_0 n) - \lim_{R\to\infty} v_n^{\mathrm{int}}(Rn)\Big| \lesssim \|\varphi_*(\xi,\cdot)\|_{C^{\alpha'}} \varepsilon^\alpha + (\operatorname{osc}\varphi_*) e^{-R_0/|\xi|},$$

where again we mean that the estimate holds for any subsequential limit of $v_n^{\mathrm{int}}(Rn)$. An analogous estimate holds for $w^\varepsilon_{\xi,\eta}$ replacing Rn with $R\hat\xi$. Using our assumption that $\varepsilon \le \frac{\pi}{4}$ we have $n\cdot\hat\xi \ge \frac{1}{\sqrt2}$ and so we have

$$\max\Big\{\Big|v_n^{\mathrm{int}}(R_0\hat\xi) - \lim_{R\to\infty} v_n^{\mathrm{int}}(Rn)\Big|, \Big|w^\varepsilon_{\xi,\eta}(R_0\hat\xi) - \lim_{R\to\infty} w^\varepsilon_{\xi,\eta}(R\hat\xi)\Big|\Big\} \\ \lesssim \|\varphi_*(\xi,\cdot)\|_{C^{\alpha'}} \varepsilon^\alpha + (\operatorname{osc}\varphi_*) e^{-R_0/|\xi|}. \tag{6-11}$$

Finally we combine (6-10) with (6-11), choosing $R_0 = |\xi| \log 1/(|\xi|\varepsilon)$, to find

$$\begin{aligned}\Big|\lim_{R\to\infty} v_n^{\mathrm{int}}(Rn) - \lim_{R\to\infty} w^\varepsilon_{\xi,\eta}(R\hat\xi)\Big| &\le |v_n^{\mathrm{int}}(R_0\hat\xi) - w^\varepsilon_{\xi,\eta}(R_0\hat\xi)| + C\|\nabla\varphi\|_\infty |\xi|^\alpha \varepsilon^\alpha \\ &\lesssim \|\nabla\varphi\|_\infty \varepsilon^\alpha R_0^\alpha \\ &\lesssim \|\nabla\varphi\|_\infty |\xi|^\alpha \varepsilon^\alpha \Big(\log\frac{1}{|\xi|\varepsilon}\Big)^\alpha.\end{aligned}$$

Making α slightly smaller we can remove the logarithmic term.

Linear systems: We have

$$-\nabla \cdot (A(x)\nabla u) = 0 \quad \text{in } K.$$

Using Lemma 3.3 we find, for $x \in \partial K$ and any $\nu > 0$,

$$|u(x)| = |v_n^{\text{int}}(x) - w^{\varepsilon}_{\xi,\eta}(x)| \lesssim_\alpha \|\nabla \varphi_*(\xi,\cdot)\|_\infty \varepsilon^\alpha |x|^\alpha \lesssim_\nu \|\nabla \varphi\|_{C^{0,\nu}} \varepsilon^\alpha |x|^\alpha.$$

By the Poisson kernel bounds in K, Lemmas 3.5 and 3.6, we have for a slightly smaller $\alpha \in (0, 1)$ and ε sufficiently small depending on α

$$|u(x)| \lesssim_\alpha [\varphi]_{C^{1,\nu}} \varepsilon^\alpha |x|^\alpha \quad \text{for } x \in K.$$

The remainder of the proof is the same as the case of scalar equations. ☐

Proposition 6.3 follows combining Lemmas 6.5 and 6.6.

6C. ***Interior homogenization of the intermediate-scale problem.*** We take $\varepsilon \to 0$ in (6-5) and derive the second cell problem

$$\begin{cases} -\nabla \cdot a(x/\tan\varepsilon, \nabla w^{\varepsilon}_{\xi,\eta}) = 0 & \text{in } P_\xi, \\ w^{\varepsilon}_{\xi,\eta}(x) = \varphi_*(\xi, x\cdot\eta) & \text{on } \partial P_\xi, \end{cases} \tag{6-12}$$

which homogenizes to

$$\begin{cases} -\nabla \cdot a^0(\nabla w_{\xi,\eta}) = 0 & \text{in } P_\xi, \\ w_{\xi,\eta}(x) = \varphi_*(\xi, x\cdot\eta) & \text{on } \partial P_\xi, \end{cases} \tag{6-13}$$

where a^0 is the homogenized operator associated with $a\big(\frac{x}{\varepsilon}, \cdot\big)$.

We make the definition

$$L(\xi,\eta) = \lim_{R\to\infty} w_{\xi,\eta}(x + R\xi).$$

As we will show below $L(\xi,\cdot)$ is the limiting 0-homogeneous profile of φ_* at the direction ξ,

$$\lim_{j\to\infty} \varphi_*(n_j) = L(\xi,\eta)$$

for any sequence of n_j irrational with $n_j \to \xi$ and $(\hat{\xi} - n_j)/|\hat{\xi} - n_j| \to \eta$. This characterization is the first main result of the paper Theorem 1.1.

We make a further remark about the second cell problem in (1-2). It is straightforward to see that $w_{\xi,\eta}$ is actually a function only of two variables $x\cdot\xi$ and $x\cdot\eta$. The boundary data $\varphi_*(\xi, x\cdot\eta)$ is invariant with respect to translations which are perpendicular to both ξ and η, and so by uniqueness the solution $w_{\xi,\eta}$ is invariant in those directions as well. Note that we are using the spatial homogeneity of the operator here; the same is not true of $w^{\varepsilon}_{\xi,\eta}$. This property was useful in [Feldman and Kim 2017] since solutions of nonlinear nondivergence form elliptic problems in dimension $d = 2$ have better regularity properties. Although we do not use this in a significant way here, we point it out anyway since it could be potentially useful in the future.

Now we state the quantitative version of Theorem 1.1:

Theorem 6.7. *Let $\xi \in \mathbb{Z}^d \setminus \{0\}$ be irreducible and $n = (\cos \varepsilon)\hat{\xi} - (\sin \varepsilon)\eta$ be an irrational direction. Then*:

(i) (*nonlinear equations*) *There is a universal $\alpha \in (0, 1)$ such that*

$$|\varphi_*(n) - L(\xi, \eta)| \lesssim \|\nabla\varphi\|_\infty |\xi|^\alpha \varepsilon^\alpha.$$

(ii) (*linear systems*) *For every $\alpha \in (0, 1)$*

$$|\varphi_*(n) - L(\xi, \eta)| \lesssim_\alpha [\varphi]_{C^5} |\xi|^\alpha \varepsilon^\alpha.$$

We will need one more lemma in the proof of Theorem 6.7, which is independently interesting since it gives the continuity of $L(\xi, \eta)$ in η.

Lemma 6.8. *Let $\xi \in \mathbb{Z}^d \setminus \{0\}$ be irreducible and $\eta, \eta' \perp \xi$. Then*

$$\Big|\lim_{R\to\infty} w^\varepsilon_{\xi,\eta}(R\xi) - \lim_{R\to\infty} w^\varepsilon_{\xi,\eta'}(R\xi)\Big| \lesssim_\alpha \|\varphi\|_{C^k} (|\xi|^{-\alpha}|\eta - \eta'|^\alpha + \varepsilon^\alpha)$$

and

$$|L(\xi, \eta) - L(\xi, \eta')| \lesssim_\alpha \|\varphi\|_{C^k} |\xi|^{-\alpha} |\eta - \eta'|^\alpha$$

either for a universal $\alpha \in (0, 1)$ and $k = 1$ (nonlinear case), or for every $\alpha \in (0, 1)$ and $k = 3$ (linear systems case). For the first estimate we mean that the inequality holds for any pair of subsequential limits of $w^\varepsilon_{\xi,\eta}(R\xi)$, $w^\varepsilon_{\xi,\eta'}(R\xi)$ as $R \to \infty$.

Proof of Theorem 6.7. The ingredients have all been established elsewhere, we just need to combine them.

There is some set up to use Proposition 5.5 since the (5-6) does not necessarily hold for (6-12). Recall that $\xi^\perp \cap \mathbb{Z}^d$ is spanned by $d-1$ linearly independent vectors $\ell_1, \ldots, \ell_{d-1}$ with norms $|\ell_j| \le |\xi|$. Then for each $\varepsilon > 0$ we can choose a vector $\eta_\varepsilon \in \varepsilon\xi^\perp \cap \mathbb{Z}^d$, i.e., a period of $a(\cdot/\tan\varepsilon, p)$ with

$$|\eta_\varepsilon - \eta| \le C\varepsilon|\xi|. \tag{6-14}$$

Now Proposition 5.5 will apply to get a quantitative estimate of the difference $w^\varepsilon_{\xi,\eta_\varepsilon} - w_{\xi,\eta_\varepsilon}$; we will use this below.

Nonlinear equations: By Proposition 6.3, there is universal $\alpha \in (0, 1)$ such that

$$\Big|\varphi_*(n) - \lim_{R\to\infty} w^\varepsilon_{\xi,\eta}(R\hat{\xi})\Big| \lesssim \|\nabla\varphi\|_\infty |\xi|^\alpha \varepsilon^\alpha,$$

where we mean that the estimate holds for any subsequential limit of $w^\varepsilon_{\xi,\eta}(R\hat{\xi})$ as $R \to \infty$. Proposition 5.5, homogenization of problems in half-space-type domains, applies to $w^\varepsilon_{\xi,\eta_\varepsilon}$

$$\sup_{P_\xi} |w^\varepsilon_{\xi,\eta_\varepsilon} - w_{\xi,\eta_\varepsilon}| \lesssim [\varphi_*(\xi, \cdot)]_{C^\beta} |\xi|^{\alpha-\beta} \varepsilon^\alpha \lesssim \|\nabla\varphi\|_\infty \varepsilon^\alpha$$

for some universal $\beta > \alpha \in (0, 1)$. We have used Lemma 6.1 to estimate the Hölder norm of $\varphi_*(\xi, \cdot)$. Then Lemma 6.8 and (6-14) implies

$$\Big|\lim_{R\to\infty} w^\varepsilon_{\xi,\eta}(R\hat{\xi}) - \lim_{R\to\infty} w^\varepsilon_{\xi,\eta_\varepsilon}(R\hat{\xi})\Big| + |L(\xi, \eta_\varepsilon) - L(\xi, \eta)| \lesssim \|\nabla\varphi\|_\infty (|\xi|^{-\alpha}|\eta_\varepsilon - \eta|^\alpha + \varepsilon^\alpha) \lesssim \|\nabla\varphi\|_\infty \varepsilon^\alpha.$$

Combining these

$$|\varphi_*(n) - L(\xi,\eta)| \lesssim \|\nabla\varphi\|_\infty |\xi|^\alpha \varepsilon^\alpha.$$

Linear systems: By Proposition 6.3, for every $\alpha \in (0,1)$ and any $\nu > 0$

$$\big|\varphi_*(n) - \lim_{R\to\infty} w^\varepsilon_{\xi,\eta}(R\hat{\xi})\big| \lesssim_{\alpha,\nu} [\varphi]_{C^{1,\nu}} |\xi|^\alpha \varepsilon^\alpha,$$

where again we mean that the estimate holds for any subsequential limit of $w^\varepsilon_{\xi,\eta}(R\hat{\xi})$ as $R \to \infty$. Now Proposition 5.5 (properly rescaled) applies to $w^\varepsilon_{\xi,\eta_\varepsilon}$,

$$\begin{aligned}\sup_{P_\xi} |w_{\xi,\eta_\varepsilon} - w^\varepsilon_{\xi,\eta_\varepsilon}| &\lesssim_\alpha [\varphi_*(\xi,\cdot)]_{C^4} |\xi|^{\alpha-1} \varepsilon^\alpha \\ &\lesssim \|\varphi\|_{C^5} |\xi|^{\alpha-1} \log^4(1+|\xi|)\varepsilon^\alpha\end{aligned}$$

for every $\alpha \in (0,1)$. We have used Lemma 6.2 to obtain the C^4 regularity of $\varphi_*(\xi,\cdot)$. We also have $|\xi|^{\alpha-1}\log^4(1+|\xi|) \lesssim_\alpha 1$. Then Lemma 6.8 and (6-14) imply

$$\big|\lim_{R\to\infty} w^\varepsilon_{\xi,\eta}(R\hat{\xi}) - \lim_{R\to\infty} w^\varepsilon_{\xi,\eta_\varepsilon}(R\hat{\xi})\big| + |L(\xi,\eta_\varepsilon) - L(\xi,\eta)| \lesssim_\alpha [\varphi]_{C^3}(|\xi|^{-\alpha}|\eta_\varepsilon - \eta|^\alpha + \varepsilon^\alpha) \lesssim [\varphi]_{C^3}\varepsilon^\alpha.$$

Combining these, for any $\alpha \in (0,1)$,

$$|\varphi_*(n) - L(\xi,\eta)| \lesssim_\alpha [\varphi]_{C^5} |\xi|^\alpha \varepsilon^\alpha. \qquad \square$$

Proof of Lemma 6.8. We just argue for $W = w^\varepsilon_{\xi,\eta} - w^\varepsilon_{\xi,\eta'}$; the argument for $w_{\xi,\eta} - w_{\xi,\eta'}$ is almost the same and slightly simpler.

Nonlinear equations: Note that $W(0) = 0$ and the boundary data for W on ∂P_ξ has

$$|\varphi_*(x\cdot\eta) - \varphi_*(x\cdot\eta')| \le \|\varphi_*\|_{C^\alpha} |\eta-\eta'|^\alpha |x|^\alpha \lesssim \|\nabla\varphi\|_\infty |\eta-\eta'|^\alpha |x|^\alpha$$

for a universal $\alpha \in (0,1)$ by Lemma 6.1. By the boundary regularity Lemma 4.2 and the maximum principle,

$$|W(x)| \lesssim \|\nabla\varphi\|_\infty |\eta-\eta'|^\alpha |x|^\alpha \quad \text{for } x \in P_\xi \cap B_R,$$

for a, possibly smaller, universal α. Now by Lemma 5.3 applied to $w^\varepsilon_{\xi,\eta}, w^\varepsilon_{\xi,\eta'}$ separately, there is $c_* \in \mathbb{R}$ such that for all $R \ge 1$

$$\sup_{x\cdot\hat{\xi}\ge R} |W(x) - c_*| \lesssim [\varphi_*]_{C^\alpha}\left(\frac{1}{|\xi|^\alpha}\exp(-c|\xi|R) + \varepsilon^{\alpha'}\right),$$

where α universal is from Lemma 6.1, and $\alpha' < \alpha$ universal. Combining the two estimates with $R = c^{-1}|\xi|^{-1}|\log|\eta-\eta'||$ we get

$$|c_*| \lesssim_\alpha \|\nabla\varphi_*\|_\infty (|\xi|^{-\alpha}|\eta-\eta'|^\alpha + \varepsilon^\alpha),$$

again with a possibly different universal $\alpha \in (0,1)$.

Linear systems: Note that $W(0) = 0$ and the boundary data for W on ∂P_ξ has

$$|\nabla(\varphi_*(x\cdot\eta) - \varphi_*(x\cdot\eta'))| \le \left\|\frac{d}{ds}\varphi_*\right\|_\infty |\eta-\eta'| + \left\|\frac{d^2}{ds^2}\varphi_*\right\|_\infty |\eta-\eta'|\,|x|.$$

By the boundary regularity Lemma 3.3, for any $\alpha \in (0,1)$,

$$|W(x)| \lesssim_\alpha \left\|\frac{d}{ds}\varphi_*\right\|_{C^2}(1+R)|\eta-\eta'|\,|x|^\alpha \quad \text{for } x \in P_\xi \cap B_R.$$

Now by Lemma 5.3 applied to $w^\varepsilon_{\xi,\eta}, w^\varepsilon_{\xi,\eta'}$ separately, there is $c_* \in \mathbb{R}$ such that for all $R \ge 1$

$$\sup_{x\cdot\hat{\xi}\ge R} |W(x) - c_*| \lesssim \left\|\frac{d}{ds}\varphi_*\right\|_\infty \Big(\frac{1}{|\xi|}\exp(-c|\xi|R) + \varepsilon^\alpha\Big).$$

Combining the two estimates with $R = c^{-1}|\xi|^{-1}|\log|\eta-\eta'||$, we get

$$|c_*| \lesssim_\alpha \|\varphi_*\|_{C^2}(|\xi|^{-\alpha}|\eta-\eta'|^\alpha + \varepsilon^\alpha).$$

We are ignoring some negative powers of $|\xi|$ since they are ≤ 1. □

6D. *Proofs of regularity estimates of* φ_*. We return to prove the regularity estimates of φ_* Lemma 6.1 and Lemma 6.2. The Hölder regularity Lemma 6.1 is relatively straightforward, while the higher regularity Lemma 6.2 requires some more careful estimates.

Proof of Lemma 6.1. We will show an upper bound for $|\varphi_*(\xi,h) - \varphi_*(\xi,0)|$ with $h < 0$; the proof works also for nonzero s and $h \in \mathbb{R}$. Consider v^0_ξ a solution in P_ξ and v^h_ξ a solution in $P_\xi + h\hat{\xi} \supset P_\xi$. By the boundary continuity estimates for v^h_ξ, for every $y \in \partial P_\xi$,

$$|v^h_\xi(y) - v^0_\xi(y)| = |v^h_\xi(y) - \varphi(y)| \le |v^h_\xi(y) - \varphi(y-h\hat{\xi})| + \|\nabla\varphi\|_\infty h \le C\|\nabla\varphi\|_\infty h^\alpha$$

for some $\alpha \in (0,1)$ by Lemma 4.2. For the case of linear systems we have similarly,

$$|v^h_\xi(y) - v^0_\xi(y)| = |v^h_\xi(y) - \varphi(y)| \le C[\varphi]_{C^{1,\nu}} h$$

for any $\nu > 0$ by the boundary gradient estimates for smooth coefficient linear systems. Then the maximum principle, or respectively the L^∞ estimate for systems Lemma 3.2, implies the same bound holds in all of P_ξ and therefore also for the boundary layer limits. □

Proof of Lemma 6.2. In order to get estimates on higher derivatives of v^s_ξ in s, the method for Lemma 6.1 doesn't work; we need to differentiate in the equation. Since we only consider one normal direction $\xi \in \mathbb{Z}^d \setminus \{0\}$ we drop the dependence $v^s = v^s_\xi$ on ξ. We denote derivatives with respect to s by ∂ and then

$$\begin{cases} -\nabla\cdot(A(x+s\hat{\xi})\nabla\partial^k v^s) = \nabla\cdot f & \text{in } P_\xi, \\ \partial^k v^s = (\hat{\xi}\cdot\nabla)^k\varphi(x+s\hat{\xi}) & \text{on } \partial P_\xi, \end{cases} \tag{6-15}$$

where f involves derivatives $\partial^j v$ for $j < k$,

$$f = \sum_{j=0}^{k-1}\binom{k}{j}(\hat{\xi}\cdot\nabla)^{k-j}A(x+s\hat{\xi})\nabla\partial^j v^s.$$

Let $p > d$ arbitrary but fixed. We will suppose, inductively, that we can prove for any $R \geq 0$ and every $j < k$,

$$\sup_{y \in \partial P_\xi, R' \geq R} \|\nabla \partial^j v^s\|_{L^p_{\mathrm{avg}}(B_{R'/2}(y+R'\hat{\xi}))} \leq C_j [\varphi]_{C^{j+1,\nu}} \frac{1}{R} \log^j(1+|\xi|) e^{-c_j R/|\xi|},$$

where the constants depend on j, $[A]_{C^j}$ and universal parameters. The case $R \leq 1$ corresponds basically to an L^∞ bound on P_ξ.

Then by Lemma B.1

$$\|\partial^k v^s\|_{L^\infty(P_\xi)} \leq C \|(\hat{\xi} \cdot \nabla)^k \varphi\|_\infty + C \log^k(1+|\xi|)[\varphi]_{C^{k,\nu}}. \tag{6-16}$$

Furthermore, by Lemma B.2, $\partial^k v^s$ has a boundary layer limit

$$\mu_k = \frac{d^k}{ds^k} \varphi_*(\xi, s),$$

with

$$|\partial^k v^s - \mu_k| \leq C \log^k(1+|\xi|)[\varphi]_{C^{k,\nu}} e^{-cR/|\xi|}.$$

Now we aim to establish the inductive hypothesis. The following argument will also establish the base case when $j = 0$. First we consider the case $R \leq 1$. This follows from (6-16) and the up-to-the-boundary gradient estimates (Lemma 3.3),

$$\|\nabla \partial^k v^s\|_{L^\infty(P_\xi)} \leq C \|(\hat{\xi} \cdot \nabla)^k \varphi\|_{C^{1,\nu}} + C \log^k(1+|\xi|)[\varphi]_{C^{k,\nu}} \leq C \log^k(1+|\xi|)[\varphi]_{C^{k+1,\nu}}.$$

In the case $R \geq 1$, by the Avellaneda–Lin large-scale interior $W^{1,p}$ estimates and the inductive hypothesis,

$$\begin{aligned} \|\nabla \partial^k v^s\|_{L^p_{\mathrm{avg}}(B_{R/2}(y+R\hat{\xi}))} &\leq C \frac{1}{R} \operatorname*{osc}_{B_{3R/4}(y+R\hat{\xi})} \partial^k v^s + \|f\|_{L^p_{\mathrm{avg}}(B_{3R/4}(y+R\hat{\xi}))} \\ &\leq C \frac{1}{R} \log^k(1+|\xi|)[\varphi]_{C^{k,\nu}} e^{-cR/|\xi|}. \end{aligned}$$

Combining the cases $R \leq 1$ and $R \geq 1$ establishes the inductive hypothesis for $j = k$. The bound on $\|\partial^k v^s\|_{L^\infty}$ and hence on the boundary layer limit μ_k, which is also a consequence of the induction, is the desired result. □

7. Continuity estimate for homogenized boundary data associated with linear systems

In this section we use the limiting structure at rational directions established above to prove that the homogenized boundary condition associated with a linear system is continuous. We recall the second cell problem; let $\xi \in \mathbb{Z}^d \setminus \{0\}$ a rational direction and suppose that we have a sequence of directions $n_k \to \hat{\xi}$ such that

$$\frac{\hat{\xi} - n_k}{|\hat{\xi} - n_k|} \to \eta, \quad \text{where } \eta \text{ is a unit vector with } \eta \perp \xi.$$

Then the limit of $\varphi_*(n_k)$ is determined by the second cell problem

$$\begin{cases} -\nabla \cdot (A^0 \nabla w_{\xi,\eta}) = 0 & \text{in } P_\xi, \\ w_{\xi,\eta} = \varphi_*(\xi, x \cdot \eta) & \text{on } \partial P_\xi, \end{cases} \tag{7-1}$$

and thus $\lim_{k\to\infty}\varphi_*(n_k)=\lim_{R\to\infty} w_{\xi,\eta}(R\xi)$, where A^0, constant, is the homogenized matrix associated with $A\left(\frac{\cdot}{\varepsilon}\right)$ and $\varphi_*(\xi,\cdot)$ defined in (6-2) is a $1/|\xi|$ periodic function on $\mathbb{R}$ (see [Feldman and Kim 2017, Lemma 2.9] where the period of φ_* is explained).

First we state the qualitative result, identifying the limit and showing continuity at rational directions. Continuity of φ_* at the irrational directions has been established, for example in [Prange 2013]. Combining those results shows that φ_* extends to a continuous function on S^{d-1}.

Lemma 7.1. *Let $\xi\in\mathbb{Z}^d\setminus\{0\}$; then for any sequence $n_k\to\hat{\xi}$,*

$$\lim_{k\to\infty}\varphi_*(n_k)=|\xi|\int_0^{1/|\xi|}\varphi_*(\xi,t)\,dt.$$

From this we know that $L(\xi,\eta)$, defined in Section 6C, is independent of η in the linear case. And we will simply write $L(\xi)=L(\xi,\eta)$.

Proof. By rotation and rescaling we can reduce to proving that the boundary layer limit associated with the half-space problem

$$\begin{cases}-\nabla\cdot(A^0\nabla v)=0 & \text{in } \mathbb{R}^d_+,\\ v=g(x_1,\dots,x_{d-1}) & \text{on } \partial\mathbb{R}^d_+,\end{cases}\tag{7-2}$$

where A^0 is a constant and uniformly elliptic and g is a $\mathbb{Z}^{d-1}$-periodic continuous function $\mathbb{R}^{d-1}\to\mathbb{R}^N$, is

$$\lim_{R\to\infty} v(Re_d)=\int_{[0,1]^{d-1}} g(x')\,dx'.$$

We will actually give two proofs of this result, especially since it plays a key role in our main results.

Riesz representation: Consider the (linear) map $T:C(\mathbb{R}^{d-1}/\mathbb{Z}^{d-1})\to\mathbb{R}^N$ mapping $g\mapsto\lim_{R\to\infty} v(Re_d)$. The L^∞ estimates Lemma 3.2 imply that T is continuous. Since A^0 is constant, translating g parallel to $\partial\mathbb{R}^d_+$ just translates the solution v and so we also get translation invariance, for any $y'\in\mathbb{T}^{d-1}$,

$$Tg(\cdot-y')=Tg.$$

The Riesz representation theorem implies that $Tg=\int_{\mathbb{R}^{d-1}/\mathbb{Z}^{d-1}} g(x')\,d\mu(x')$ for some (vector-valued) measure μ. The translation invariance of T implies translation invariance of μ which means it is a constant multiple of the Haar measure, Lebesgue measure in this case. Then $T1=1$ implies that $d\mu=dx'$.

Direct method: Consider

$$\mu(t)=\int_{[0,1]^{d-1}} v(x',t)\,dx'.$$

If we can show that μ is constant we are done. Compute, using a summation convention,

$$\begin{aligned}A^{0,ij}_{dd}\,\mu_j''(t)&=\int_{[0,1]^{d-1}} A^{0,ij}_{dd}\,\partial_d^2 v_j(x',t)\,dx'\\ &=-\int_{[0,1]^{d-1}}\sum_{\alpha\beta\neq dd} A^{0,ij}_{\alpha\beta}\,\partial^2_{\alpha\beta}v_j(x',t)\,dx'=0.\end{aligned}$$

Now note that for each derivative $\partial^2_{\alpha\beta}$ appearing in the sum either α or β is $\neq d$ and so we are integrating the derivative of a periodic function over its unit cell. Thus

$$A^{0,ij}_{dd}\,\mu_j''(t) = 0 \quad \text{for all } 1 \le i \le N.$$

Let $\bar{\xi} \in \mathbb{R}^N$; applying (3-2) with the vector $\xi^i_\alpha = \bar{\xi}^i \delta_{\alpha d}$ gives

$$\lambda|\bar{\xi}|^2 = \lambda \xi^i_\alpha \xi^i_\alpha \le A^{0,ij}_{\alpha\beta}\,\xi^i_\alpha \xi^j_\beta = A^{0,ij}_{dd}\,\bar{\xi}^i \bar{\xi}^j.$$

In particular the $N \times N$ matrix with coefficients $A^{0,ij}_{dd}$ is invertible and therefore

$$\mu''(t) = 0 \quad \text{for all } t \ge 0.$$

Thus μ is linear, since μ is bounded it must be constant. □

The next result is quantitative; the argument, which is the same as in [Feldman and Kim 2017], uses the Dirichlet approximation theorem. We recall that number-theoretic result here.

Theorem 7.2 (Dirichlet approximation). *For given real numbers $\alpha_1, \dots, \alpha_n$ and $N \in \mathbb{N}$, there are integers $p_1, \dots, p_n, q \in \mathbb{Z}$ with $1 \le q \le N$ such that*

$$|q\alpha_i - p_i| \le \frac{1}{N^{1/n}}.$$

This is proved by the pigeonhole principle.

Theorem 7.3. *Let $\varphi_*(\cdot)$ be defined the boundary layer limit associated with* (1-1) *defined for $n \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$. Then for every $\alpha < \frac{1}{d}$ and all $n_1, n_2 \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$,*

$$|\varphi_*(n_1) - \varphi_*(n_2)| \lesssim_\alpha \|\varphi\|_{C^5} |n_1 - n_2|^\alpha.$$

Proof. Let n_1, n_2 be a pair of irrational unit vectors and set $\delta = |n_1 - n_2|$. Assume $\delta \le 2^{-d} d^{-d/2}$. Let $M = \delta^{-s/(s+1)}$ with $s = d-1$. By Dirichlet's approximation theorem, there exists $\xi \in \mathbb{Z}^d \setminus \{0\}$ and $k \in \mathbb{Z}$ with $1 \le k \le M$ such that

$$\left| \frac{n_1}{|n_1|_\infty} - k^{-1}\xi \right| \le k^{-1} M^{-1/s}.$$

Now $k^{-1}|n_1|_\infty \xi$ is not a unit vector, but by the above inequality

$$k^{-1}|n_1|_\infty |\xi| \ge 1 - \sqrt{d}\,\delta^{1/d} \ge \tfrac{1}{2}.$$

Then, since the map $x \mapsto x/|x|$ is Lipschitz on $\mathbb{R}^d \setminus B_{1/2}$,

$$\left| n_1 - \frac{\xi}{|\xi|} \right| \lesssim_d k^{-1} M^{-1/s}, \qquad \left| n_2 - \frac{\xi}{|\xi|} \right| \le \delta + C k^{-1} M^{-1/s}.$$

Note also that

$$|\xi| \le \frac{k}{|n_1|_\infty} + M^{-1/s} \le \sqrt{d}\,k + 1 \lesssim k.$$

Thus, for $j = 1, 2$,

$$|\xi|\left|n_j - \frac{\xi}{|\xi|}\right| \le |\xi||\delta + Ck^{-1}M^{-1/s}| \lesssim M\delta + M^{-1/s} \sim \delta^{1/(s+1)} = \delta^{1/d},$$

where we chose M at the beginning so that the two terms are of the same size.

For appropriate choices of $\eta_j \perp \xi$,

$$n_j = (\cos\varepsilon_j)\hat{\xi} - (\sin\varepsilon_j)\eta_j,$$

with $\varepsilon_j \sim |n_j - \xi/|\xi||$.

Now apply Theorem 6.7, noting that $L(\xi, \eta_1) = L(\xi, \eta_2) = L(\xi)$ by Lemma 7.1. For any $0 < \alpha < 1$ we have

$$\begin{aligned}|\varphi_*(n_1) - \varphi_*(n_2)| &\le |\varphi_*(n_1) - L(\xi)| + |L(\xi) - \varphi_*(n_2)| \\ &\lesssim_\alpha \|\varphi\|_{C^5}\left(|\xi|^\alpha\left|n_1 - \frac{\xi}{|\xi|}\right|^\alpha + |\xi|^\alpha\left|n_2 - \frac{\xi}{|\xi|}\right|^\alpha\right) \\ &\lesssim \|\varphi\|_{C^5}\delta^{\alpha/d}.\end{aligned}$$

This completes the proof for $|n_1 - n_2|$ small; for general $n_1, n_2 \in S^{d-1}$ just use the boundedness of φ_*. □

8. A nonlinear equation with discontinuous homogenized boundary data

In this final section we study the second cell equation (1-2) for nonlinear equations. We give an example of a nonlinear divergence form equation, with smooth boundary condition, for which the boundary layer limit of (1-2) depends on the approach direction η.

We consider the nonlinear operator

$$a(p_1, p_2, p_3) = (p_1, p_2, p_3 + f(p_1, p_3))^t,$$

where

$$f(p_1, p_3) := \tfrac{1}{8}(\sqrt{8p_1^2 + 9p_3^2} + p_3).$$

Here f is a solution of

$$8f^2 - 2p_3 f - (p_1^2 + p_3^2) = 0.$$

It is easy to check that f is positively 1-homogeneous and uniformly elliptic.

We will take $\xi = e_3$ and $\eta = e_1$ or e_2 and we will set $(x_1, x_2, x_3) = (x, y, z)$. For the boundary condition we choose

$$\varphi(y) = \tfrac{1}{3} + \cos(y \cdot \xi) \quad \text{so that} \quad \varphi_*(\xi, s) = \tfrac{1}{3} + \cos(s).$$

It is worthwhile to note that arbitrary $\varphi_*(\xi, s)$ can be achieved by choosing $\varphi(y) = \varphi_*(\xi, y \cdot \xi)$. We aim to compute $L(\xi, \eta)$.

If $\eta = e_1$, (1-2) becomes

$$\begin{cases} -\nabla \cdot (u_x, u_y, u_z + f(u_x, u_z)) = 0 & \text{in } \mathbb{R}^3_+, \\ u(x, y, 0) = \frac{1}{3} + \cos x & \text{in } \mathbb{R}^3_+. \end{cases} \tag{8-1}$$

The operator and boundary data were chosen to make the solution

$$u(x,y,z)=\big(\tfrac{1}{3}+\cos x\big)e^{-z}.$$

Note that

$$f(u_x,u_z)=\tfrac{1}{3}e^{-z}$$

and so

$$(u_x,u_y,u_z+f(u_x,u_z))=(-\sin x\, e^{-z},\ 0,\ -\cos x\, e^{-z}),$$

from which it is easy to verify that u solves (8-1). The boundary layer limit in this case is 0 and so, by its definition, $L(\xi,e_1)=0$.

If $\eta=e_2$ then the equation becomes

$$\begin{cases}-\nabla\cdot(u_x,u_y,u_z+f(u_x,u_z))=0 & \text{in } \mathbb{R}^3_+,\\ u(x,y,0)=\frac{1}{3}+\cos y & \text{in } \mathbb{R}^3_+.\end{cases} \tag{8-2}$$

This reduces to the following two-dimensional problem for $v(y,z)=u(x,y,z)$:

$$\begin{cases}-\nabla\cdot\big(v_y,\frac{9}{8}v_z+\frac{3}{8}|v_z|\big)=0 & \text{in } \mathbb{R}^2_+,\\ v(y,0)=\frac{1}{3}+\cos y & \text{on } \partial\mathbb{R}^2_+.\end{cases} \tag{8-3}$$

Let v be the solution of (8-3). Consider $w(y,z):=\big(\frac{1}{3}+\cos y\big)e^{-z}$, the solution from before,

$$-\nabla\cdot\big(w_y,\tfrac{9}{8}w_z+\tfrac{3}{8}|w_z|\big)=\big[\big(-\tfrac{4}{9}-\tfrac{1}{3}\cos y\big)\mathbf{1}_{\{\cos y<0\}}+\tfrac{1}{4}(\cos y-1)\mathbf{1}_{\{\cos y>0\}}\big]e^{-z}\le 0. \tag{8-4}$$

Thus w is a subsolution of (8-3); from Lemma 4.3 we have $w\le v$.

The operator $\big(v_y,\frac{9}{8}v_z+\frac{3}{8}|v_z|\big)$ is uniformly elliptic and Lipschitz continuous. We use a strong maximum principle [Serrin 1970, Theorem 1′]; in any bounded domain, we either have $w\equiv v$ or $w<v$. Since the inequality in (8-4) is strict, except when $y=0 \bmod 2\pi$, the case must be $w<v$. Since both w,v are 1-periodic in the y-direction, restricting to the set $z=1$, $w(y,1)\le v(y,1)-\delta$ for some $\delta>0$. Then by comparing w and $v-\delta$ on $z\ge 1$, again using Lemma 4.3, we deduce that $w\le v-\delta$; in particular

$$\lim_{z\to\infty} v\ge \lim_{z\to\infty} w+\delta=\delta.$$

Thus $L(\xi,e_2)<0=L(\xi,e_1)$ and therefore $\varphi_*(n)$ is discontinuous at the direction e_3.

Appendix A

Hölder estimate in cone domain. We complete the proof of Lemma 3.4, the Hölder estimate in the flat cone domain which we used above.

Proof of Lemma 3.4. Suppose that

$$\|\nabla g\|_{L^\infty(\partial\Omega\cap B_1)}\le 1 \quad\text{and}\quad \fint_{B_1\cap\Omega}|u^\varepsilon-g(0)|^2\le 1.$$

Let some $\alpha < \alpha' < 1$; by Lemma 3.3 there is a $1 > \theta > 0$ so that if $K_\Sigma = P_n$ for some $n \in S^{d-1}$ then

$$\sup_{B_\theta \cap P_n} |u^\varepsilon - g(0)| \le \theta^{\alpha'}.$$

We prove by compactness that there exists $\delta > 0$ sufficiently small such that for any solution u^ε as above

$$\left(\fint_{B_\theta \cap \Omega} |u^\varepsilon - g(0)|^2 \right)^{1/2} \le \theta^\alpha. \tag{A-1}$$

To achieve the Hölder estimate from (A-1) is the standard iteration argument.

Suppose that the previous statement fails; that is, there exists f_k and corresponding Ω_k with $\delta_k = \|\nabla f_k\|_\infty \to 0$, A_k satisfying the standard assumptions, $\varepsilon_k > 0$, g_k with Lipschitz norm at most 1 and corresponding u_k solving the equation with boundary data g_k on $\partial\Omega_k \cap B_1$ and

$$\left(\fint_{B_\theta \cap \Omega_k} |u_k - g_k(0)|^2 \right)^{1/2} > \theta^\alpha.$$

By taking subsequences we can assume that $A_k \to A$ uniformly, $g_k \to g$ uniformly and the u_k converge to some u weakly in H^1 and strongly in L^2. Then, assuming that $\varepsilon_k \to \varepsilon > 0$, we claim u solves

$$-\nabla \cdot A\left(\frac{x}{\varepsilon}\right) \nabla u = 0 \quad \text{in } \Omega \cap B_1 \text{ with } u = g \text{ on } \{x_d = 0\} \cap B_1. \tag{A-2}$$

If $\varepsilon_k \to 0$ or $\varepsilon_k \to \infty$ then we replace $A(x/\varepsilon)$ by A^0 or $A(0)$ respectively.

The only part which is not the same as in [Avellaneda and Lin 1987] is to check the boundary condition. Consider the transformations

$$\Phi_k(x) = (x', x_d + f_k(x')) \text{ mapping } \Phi_k : \{x_d > 0\} \to \{x_d > f_k(x')\}.$$

Define $v_k = u_k \circ \Phi_k$. Note that $|\Phi_k - x| \le \delta_k$, $\nabla v_k = \nabla \Phi_k \nabla u_k$ and $\|\nabla \Phi_k - I\|_{L^\infty} \le \delta_k$. Therefore, up to taking a subsequence, the v_k converge weakly in $H^1(B_1^+)$ and strongly in $H^{1/2}(B_1^+)$ to the same limit u. Since the trace operator is continuous $T : H^{1/2}(B_1^+) \to L^2(\{x_d = 0\} \cap B_1)$, we have that the trace of v is the limit of the traces g_k of the v_k.

Then, once we have established the limit (A-2), from the regularity estimate in the flat domain

$$\theta^\alpha \le \left(\fint_{B_\theta \cap P_n} |u - g(0)|^2 \right)^{1/2} \le \theta^{\alpha'},$$

which is a contradiction since $\alpha < \alpha'$ and $\theta < 1$. □

Poisson kernel bounds in half-space intersection. We return to prove the Poisson kernel bounds in the intersection of nearby half-spaces, Lemma 3.5.

Proof of Lemma 3.5. The proof basically follows the proof of the Poisson kernel bounds in a smooth domain in [Avellaneda and Lin 1987, Lemma 21] except we need to be careful to deal with the singularity of the boundary. We do the case $d \ge 3$; the $d = 2$ case is a similar modification of the arguments in

[Avellaneda and Lin 1987, Lemma 21]. Let $x, y \in K$ and call $r = |y - x|$. We have the Green's function bound holding for $x, y \in K$ (see Theorem 13 in [Avellaneda and Lin 1987] and the remark below)

$$|G_K(x, y)| \lesssim \frac{1}{r^{d-2}}.$$

We will first improve the Green's function bound; the bound on the Poisson kernel will follow.

If $\delta(x) > \frac{1}{3}r$ then $|G_K(x, y)| \lesssim \delta(x)/r^{d-1}$. Consider the case $\delta(x) < \frac{1}{3}r$. Let $\bar{x} \in \partial K$ with $|x - \bar{x}| = \delta(x)$. Then $G_K(\cdot\,, y)$ is a solution of the system in $B\big(\bar{x}, \frac{1}{2}r\big) \cap K$. For ε sufficiently small depending on α the boundary Hölder estimates Lemma 3.4 apply and

$$G(z, y) \lesssim \frac{\delta(z)^\alpha}{r^{d-2+\alpha}} \quad \text{for all } z \in B\big(\bar{x}, \tfrac{1}{3}r\big) \cap K; \tag{A-3}$$

in particular the bound holds at $z = x$.

Now we make a similar argument in the y-variable starting from (A-3); however Hölder regularity is not sufficient anymore so we need to deal more directly with the singularity. Since we will send $y \to \partial K \setminus \{y_1 = 0\}$ we can just consider the case $\delta(y) \le \min\big\{\frac{1}{3}r, \frac{1}{2}|y_1|\big\}$. Let $\bar{y} \in \partial K$ with $|y - \bar{y}| = \delta(y)$. Then $G_K(x, \cdot)$ is a solution of the adjoint equation in $B\big(\bar{y}, \frac{1}{2}r\big) \cap K$. If $|y_1| \ge \frac{1}{2}r$ then $|\bar{y}_1| \ge |y_1| \ge \frac{1}{2}r$ and $B\big(\bar{y}, \frac{1}{2}r\big) \cap K$ is the intersection of a half-space with the ball $B\big(\bar{y}, \frac{1}{2}r\big)$. The boundary Lipschitz estimate of [Avellaneda and Lin 1987] applies and

$$|G(x, z)| \lesssim \frac{\delta(z)\delta(x)^\alpha}{r^{d-1+\alpha}} \quad \text{for all } z \in B\big(\bar{y}, \tfrac{1}{3}r\big) \cap K;$$

since $\delta(y) \le \frac{1}{3}r$ we get the bound at $z = y$. If $|y_1| \le \frac{1}{2}r$ then we instead apply the boundary Lipschitz estimate in $B(\bar{y}, |y_1|)$ to find

$$|G(x, z)| \lesssim \frac{\delta(z)\delta(x)^\alpha}{|y_1| r^{d-2+\alpha}} \quad \text{for all } z \in B(\bar{y}, |y_1|/2) \cap K;$$

since $\delta(y) \le \frac{1}{2}|y_1|$ we get the bound at $z = y$. The bounds for the Poisson kernel follow by taking appropriate difference quotients. □

Large-scale boundary regularity nonlinear equations. We return to prove the De Giorgi boundary Hölder estimates, Lemma 4.2, for scalar equations with bounded uniformly elliptic coefficients.

Proof of Lemma 4.2. Without loss we can assume that $\operatorname{osc}_{\Omega \cap B_1} u = 1$ and $0 \le u \le 1$ in $\Omega \cap B_1$. Call $M = \max_{\partial\Omega \cap B_1} \varphi$ and consider

$$v = (u - M)_+, \quad \text{which is a subsolution of } -\nabla \cdot (A(x)\nabla v) \le 0 \text{ in } B_1.$$

Now since

$$|\{v \le 0\} \cap B_1| \ge \mu,$$

we apply the De Giorgi weak Harnack inequality to find

$$v \le (1 - \delta)\Big(\max_{\Omega \cap B_1} u - M\Big) \quad \text{in } B_{1/2}$$

for some $\delta > 0$ depending on μ, d, λ. Making the same argument for $-u$ we find

$$\operatorname*{osc}_{\Omega\cap B_{1/2}} u \le (1-\delta) \operatorname*{osc}_{\Omega\cap B_1} u + \delta \operatorname*{osc}_{\partial\Omega\cap B_1} \varphi.$$

Iterating this argument we obtain,

$$\operatorname*{osc}_{\Omega\cap B_{1/2^k}} u \le (1-\delta)^k \operatorname*{osc}_{\Omega\cap B_1} u + \sum_{j=0}^{k-1} \delta(1-\delta)^{k-j-1} \operatorname*{osc}_{\partial\Omega\cap B_{1/2^j}} \varphi.$$

Using the Hölder continuity of φ,

$$\operatorname*{osc}_{\Omega\cap B_{1/2^k}} u \le (1-\delta)^k \biggl(\operatorname*{osc}_{\Omega\cap B_1} u + [\varphi]_{C^\beta} \sum_{j=0}^{k-1} \delta(1-\delta)^{-j-1} 2^{-j\beta} \biggr).$$

Choosing δ smaller if necessary so that

$$2^{-\beta} < (1-\delta),$$

the summation is bounded independent of k and

$$\operatorname*{osc}_{\Omega\cap B_{1/2^k}} u \le C(\alpha)\bigl(\operatorname*{osc}_{\Omega\cap B_1} u + [\varphi]_{C^\beta}\bigr) 2^{-\alpha k}, \quad \text{with } \alpha = -\frac{\log(1-\delta)}{\log 2} < \beta. \qquad \square$$

Proof of Lemma 4.3. Set $w = u_1 - u_2$; then by (4-5) w solves a uniformly elliptic equation in P_n:

$$-\nabla\cdot(A(x)\nabla w) = 0 \quad \text{in } \Omega \quad \text{with } A(x) = \int_0^1 D_p a(x, s\nabla u_1 + (1-s)\nabla u_2)\, ds,$$

with $w \le 0$ on ∂P_n, and $w \le M$ for some $M > 0$. Define

$$v = w_+ = \max\{w, 0\}, \quad \text{which is a subsolution of } -\nabla\cdot(A(x)\nabla v) \le 0 \text{ in } \mathbb{R}^d.$$

Now since,

$$|\{v \le 0\} \cap B_r| \ge \tfrac{1}{2}|B_r|$$

for any $r \ge 1$ we apply the De Giorgi weak Harnack inequality to find

$$\max_{B_{r/2}\cap P_n} w_+ \le \max_{B_{r/2}} v \le (1-\delta) \max_{B_r\cap P_n} w \quad \text{in } B_{r/2}.$$

Iterating this argument we find

$$\max_{B_r\cap P_n} w_+ \le (1-\delta)^k \max_{B_{2^k r}\cap P_n} w_+ \le (1-\delta)^k M.$$

Sending $k \to \infty$ and then $r \to \infty$ we find $w_+ \equiv 0$. $\square$

Appendix B

In this section we complete the proof of Lemma 5.4. Recall that we are considering the boundary layer problem with

$$\begin{cases} -\nabla\cdot(A(x)\nabla v) = \nabla\cdot f & \text{in } \mathbb{R}^d_+, \\ v = \psi(x') & \text{on } \partial\mathbb{R}^d_+, \end{cases} \tag{B-1}$$

where $\psi : \partial\mathbb{R}^d_+ \to \mathbb{R}$ and f are smooth, A satisfies the usual assumptions from Section 3 and, furthermore, ψ, f, and A all share $d-1$ linearly independent periods $\ell_1, \dots, \ell_{d-1} \in \partial\mathbb{R}^d_+$ such that, for some $M > 2$,

$$\max_{1\le j\le d-1} |\ell_j| \le M.$$

The following maximal function-type norms turn out to be useful:

$$M_p(f, R) := \sup_{y\cdot e_d=0, R'\ge R} \|f\|_{L^p_{\mathrm{avg}}(B_{R'/2}(y+R'e_d))}, \tag{B-2}$$

$$I_p(f) := M_p(f, 0) + \sum_{N\in 2^{\mathbb{N}}} N M_p(f, N). \tag{B-3}$$

Note that $M_p(f, 0) = \|f\|_{L^\infty(\mathbb{R}^d_+)}$.

We write v by the Green's function formula,

$$v(x) = \int_{\partial\mathbb{R}^d_+} P(x, y)\psi(y)\, dy + \int_{\mathbb{R}^d_+} \nabla_x G(x, y) f(y)\, dy.$$

The first result is an L^∞ estimate:

Lemma B.1. *For any $p > d$,*

$$\operatorname*{osc}_{\mathbb{R}^d_+} v \lesssim_p \operatorname*{osc}_{\partial\mathbb{R}^d_+} \psi + I_p(f).$$

Proof. The bound for the Poisson integral is already done in Lemma 3.2. For the Green's function term we use the Avellaneda–Lin bounds in Theorem 3.1 along with a Whitney-type decomposition.

Let $x\in\mathbb{R}^d_+$; without loss of generality $x=(0,x_d)$. If $x_d\ge 1$, let $N_x\in 2^{\mathbb{N}}$ be the unique dyadic such that $N_x\le x_d<2N_x$. Then define $\alpha\in[1,2)$ such that $\alpha N_x=x_d$. If $x_d\le 1$ define $\alpha=2$. Now we make a cube decomposition $Q_{N,j}:=\alpha N(j,1)+\alpha\left[-\frac14 N,\frac12 N\right)^d$ for $2\le N\in 2^{\mathbb{N}}$ and $j\in\mathbb{Z}^{d-1}$, with side length comparable to the distance to $x_d=0$. For $N=1$ we define $Q_{1,j}=\alpha(j,1)+\alpha\left[-1,\frac12\right)^d$. In this set up $(0,x_d)\in Q_{N_x,0}$.

Now we bound the Green's function integral by

$$\begin{aligned}\int_{\mathbb{R}^d_+} |\nabla_x G(x, y)||f(y)|\, dy &= \sum_Q \int_Q |\nabla_x G(x, y)||f(y)|\, dy\\ &\le \sum_Q |Q|\,\|\nabla_x G(x,\cdot)\|_{L^p_{\mathrm{avg}}(Q)}\|f\|_{L^p_{\mathrm{avg}}(Q)}\\ &\le \sum_N\sum_j N^d \|\nabla_x G(x,\cdot)\|_{L^{p'}_{\mathrm{avg}}(Q)} M_p(f, N).\end{aligned}$$

We claim that, for any $p > d$ and any $N \in 2^{\mathbb{N}}$, $j \in \mathbb{Z}^{d-1}$,

$$\|\nabla_x G(x,\cdot)\|_{L^{p'}_{\mathrm{avg}}(Q)} \lesssim_p N^{1-d}(1+|j|)^{-d}. \tag{B-4}$$

Taking the bound for granted we can complete the computation,

$$\int_{\mathbb{R}^d_+} |\nabla_x G(x, y)||f(y)|\, dy \lesssim \sum_N\sum_j N(1+|j|)^{-d} M_p(f, N) \lesssim I_p(f),$$

where for the last inequality we used that $(1+|j|)^{-d}$ is summable on $\mathbb{Z}^{d-1}$.

Now we finish by proving (B-4) using the Avellaneda–Lin bounds, Theorem 3.1. When $j = 0$ and $N = N_x$ we bound

$$\begin{aligned}\left(\frac{1}{|Q|}\int_Q |\nabla_x G(x,y)|^{p'}\,dy\right)^{1/p'} &\lesssim \left(\frac{1}{|Q|}\int_Q |x-y|^{(1-d)p'}\,dy\right)^{1/p'}\\ &\lesssim N^{-d/p'}\left(\int_0^{CN} r^{(1-d)p'} r^{d-1}\,dr\right)^{1/p'}\\ &\lesssim N^{-d/p'} N^{((1-p')(d-1)+1)/p'} = N^{1-d},\end{aligned}$$

where we have used $p > d$ so that $(p'-1)(d-1) < 1$ and the integral in the second line converges.

When $j \neq 0$ and/or $N \neq N_x$ we have that $|x-y| \gtrsim \max\{N(1+|j|), N_x\}$ for $y \in Q_{N,j}$. In this case,

$$\begin{aligned}|\nabla_x G(x,y)| &\lesssim \frac{y_d}{|x-y|^d} + \frac{x_d y_d}{|x-y|^{d+1}}\\ &\lesssim N^{1-d}(1+|j|)^{-d} + N N_x \max\{N(1+|j|), N_x\}^{-(d+1)}\\ &\lesssim N^{1-d}(1+|j|)^{-d},\end{aligned}$$

which was the desired estimate. □

Next we prove the existence of a boundary layer limit with convergence rate. We assume the following exponential-type bounds on f, which are well suited to the boundary layer problem: there are $K, b > 0$ so that, for all $R > 0$,

$$M_p(f,R) \le \frac{K}{1+R} e^{-bR/M}. \tag{B-5}$$

From (B-5) one can compute,

$$I_p(f) \lesssim_b K \log M,$$

and also

$$I_p(f,R) := \sum_{2^{\mathbb{N}} \ni N \ge R} N M_p(f,N) \lesssim_b K \log M e^{-bR/M}.$$

Lemma B.2. *Let v, f, ψ and A as above in* (B-1) *with f satisfying the exponential bound* (B-5). *There exists $c_* \in \mathbb{R}^m$ such that*

$$\sup_{y\cdot e_d \ge R} |v(y) - c_*| \lesssim_b ((\operatorname{osc}\psi) + K\log M) e^{-c_0 R/M},$$

where the rate c_0 depends on b and universal constants.

The proof is almost the same as [Feldman and Kim 2017, Lemma A.4] so we omit it.

Acknowledgements

Feldman was partially supported by the National Science Foundation RTG grant DMS-1246999. Zhang was partially supported by National Science Foundation grant DMS-1566578.

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Received 7 Feb 2018. Revised 28 Sep 2018. Accepted 30 Nov 2018.

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ANALYSIS AND PDE
Vol. 12, No. 8, 2019
dx.doi.org/10.2140/apde.2019.12.2003

DYNAMICS OF ONE-FOLD SYMMETRIC PATCHES FOR THE AGGREGATION EQUATION AND COLLAPSE TO SINGULAR MEASURE

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We are concerned with the dynamics of one-fold symmetric patches for the two-dimensional aggregation equation associated to the Newtonian potential. We reformulate a suitable graph model and prove a local well-posedness result in subcritical and critical spaces. The global existence is obtained only for small initial data using a weak damping property hidden in the velocity terms. This allows us to analyze the concentration phenomenon of the aggregation patches near the blow-up time. In particular, we prove that the patch collapses to a collection of disjoint segments and we provide a description of the singular measure through a careful study of the asymptotic behavior of the graph.

1. Introduction

This paper is devoted to the study of the two-dimensional aggregation equation with the Newtonian potential:

$$\begin{cases} \partial_t \rho + \operatorname{div}(v\rho) = 0, \quad t \geq 0,\ x \in \mathbb{R}^2, \\ v(t,x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} (x-y)/|x-y|^2 \rho(t,y)\, dy, \\ \rho(0,x) = \rho_0(x). \end{cases} \tag{1-1}$$

This model with more general potential interactions, with or without dissipation, is used to explain some behavior in physics and population dynamics. As a matter of fact, it appears in vortex densities in superconductors [Ambrosio and Serfaty 2008; Du and Zhang 2003; Keller and Segel 1970], material sciences [Holm and Putkaradze 2006; Nieto et al. 2001], cooperative controls and biological swarming

MSC2010: 35B44, 35A07, 35Q92.
Keywords: aggregation equations, concentration, vortex patches.

[Bernoff and Topaz 2011; Breder 1954; Boi et al. 2000; Gazi and Passino 2003; Mogilner and Edelstein-Keshet 1999; Morale et al. 2005; Topaz and Bertozzi 2004], etc. During the last few decades, a lot of intensive research activity has been devoted to exploring several mathematical and numerical aspects of this equation. It is known according to [Bertozzi et al. 2012; Nieto et al. 2001] that classical solutions can be constructed for short times. They develop a finite-time singularity if and only if the initial data is strictly positive at some points and the blow-up time is explicitly given by $T_\star = 1/\max \rho_0$. This follows from the equivalent form

$$\partial_t \rho + v \cdot \nabla \rho = \rho^2,$$

which, written with Lagrangian coordinates, gives exactly a Riccati equation. Note that similarly to Yudovich's result [1963] for Euler equations, weak unique solutions in $L^1 \cap L^\infty$ can be constructed following the same strategy; for more details see [Bertozzi et al. 2009; 2011; 2012; Bertozzi and Laurent 2007; Bertozzi and Brandman 2010; Fetecau et al. 2011; Fetecau and Huang 2013; Dong 2011; Laurent 2007; Li and Rodrigo 2009]. Since the L^1 norm is conserved at least at the formal level, a lot of effort was made to extend the classical solutions beyond the first blow-up time. Poupaud [2002] established the existence of global generalized solutions with defect measure when the initial data is a nonnegative bounded Radon measure. He also showed that when the second moment of the initial data is bounded, for such solutions the atomic part appears in finite time. This result is to some extent in contrast with what is established for Euler equations. Indeed, according to Delort's result [1991] global weak solutions without defect measure can be established when the initial vorticity is a nonnegative bounded Radon measure and the associated velocity has finite local energy. During the time, those solutions do not develop atomic part, contrary to the aggregation equation. This illustrates somehow the gap between both equations, not only at the level of classical solutions but also for the weak solutions. The literature dealing with measure-valued solutions for the aggregation equation with different potentials is very abundant and we refer the reader to [Bodnar and Velazquez 2006; Carrillo et al. 2006; 2011; Carrillo and Rosado 2010; Masmoudi and Zhang 2005].

Now we shall discuss another subject concerning the aggregation patches. Assume that the initial data takes the patch form

$$\rho_0 = \mathbf{1}_{D_0},$$

with D_0 a bounded domain; then solutions can be uniquely constructed up to the time $T^\star = 1$ and one can check that

$$\rho(t) = \frac{1}{1-t}\mathbf{1}_{D_t}, \quad \text{with } (\partial_t + v \cdot \nabla)\mathbf{1}_{D_t} = 0.$$

Note that v is computed from ρ through the Biot–Savart law. To filter the time factor in the velocity field and find an analogous equation to the Euler equations, it is more convenient to rescale the time as was done in [Bertozzi et al. 2012]. Indeed, set

$$\tau = -\ln(1-t), \quad u(\tau, x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2}\mathbf{1}_{\widetilde{D}_\tau}(y)\,dy, \quad \widetilde{D}_\tau = D_t;$$

then we get

$$(\partial_\tau + u \cdot \nabla)\mathbf{1}_{\widetilde{D}_\tau} = 0, \quad \widetilde{D}_0 = D_0.$$

We observe that with this formulation, the blow-up occurs at infinite time and so the solutions do exist globally in time. To simplify the notation we shall write this latter equation with the initial variables. Hence the vortex patch problem is reduced to understanding the evolution equation

$$\begin{cases} \partial_t \rho + v \cdot \nabla \rho = 0, \quad t \geq 0, \\ v(t,x) = -\frac{1}{2\pi}\int_{D_t} (x-y)/|x-y|^2\, dy, \\ \rho(0) = \mathbf{1}_{D_0}. \end{cases} \tag{1-2}$$

Let us point out that the area of the domain D_t shrinks to zero exponentially; that is,

$$\text{for all } t \geq 0, \quad \|\rho(t)\|_{L^1} = e^{-t}|D_0|. \tag{1-3}$$

The solution to this problem is global in time and takes the form $\rho(t) = \mathbf{1}_{D_t},\ D_t = \psi(t, D_0)$, where ψ denotes the flow associated to the velocity v. Similarly to the Euler equations [Bertozzi and Constantin 1993; Chemin 1993], Bertozzi, Garnett, Laurent and Verdera [Bertozzi et al. 2016] proved the global-in-time persistence of the boundary regularity in Hölder spaces $C^{1+s},\ s \in (0,1)$. However the asymptotic behavior of the patches for large time is still not well understood despite some interesting numerical simulations giving some indications on the concentration dynamics. Notice first that the area of the patch shrinks to zero, which gives that the associated domains will converge in Hausdorff distance to negligible sets. The geometric structure of such sets is not well explored and hereafter we will give two pedagogic and interesting simple examples illustrating the concentration, and one can find more details in [Bertozzi et al. 2012]. The first example is the disc which shrinks to its center, leading after a normalization procedure to the convergence to Dirac mass. The second one is the ellipse patch which collapses to a segment along the big axis and the normalized patch converges weakly to Wigner's semicircle law of density

$$x_1 \mapsto \frac{2\sqrt{\boldsymbol{x_0}^2 - x_1^2}}{\pi \boldsymbol{x_0}^2}\mathbf{1}_{[-x_0,x_0]}, \quad \boldsymbol{x_0} = a - b.$$

It seems that the mechanisms governing the concentration are very complex and related in part for some special class to the initial distribution of the local mass. Indeed, the numerical experiments implemented in [Bertozzi et al. 2012] for some regular shapes indicate that generically the concentration is organized along a skeleton structure. The aim of this paper is to investigate this phenomenon and try to give a complete answer for a special class of initial data where the concentration occurs along disjoint segments lying in the same line. More precisely, we will deal with a one-fold symmetric patch, and by rotation invariance we can suppose that its axis of symmetry coincides with the real axis. We assume in addition that the boundary of the upper part is the graph of a slightly smooth function with small amplitude. Then we will show that we can track the dynamics of the graph globally in time and prove that the normalized solution converges weakly towards a probability measure supported in the union of disjoint segments lying in the real axis. The results will be formulated rigorously in Section 2. The paper is organized as follows. In next section we formulate the graph equation and state our main results. In Sections 3, 4 and 5 we shall discuss basic tools that we use frequently throughout the paper. In Section 6 we prove the local well-posedness for the graph equation. The global existence with small initial data is proved in Section 7, and Section 8 deals with the asymptotic behavior of the normalized density and its convergence towards a singular measure.

2. Graph reformulation and main results

The main purpose of this section is to describe the boundary motion of the patch associated to (1-2) under suitable symmetry structure. One of the basic properties of the aggregation equation that we shall use in a crucial way concerns its group of symmetry, which is much richer than for Euler equations. Actually, in addition to rotation and translation invariance, the aggregation equation is in fact invariant by reflection. To check this property and without loss of generality we can look for the invariance with respect to the real axis. Set

$$X = (x, y) \in \mathbb{R}^2 \quad \text{and} \quad \overline{X} = (x, -y)$$

and introduce

$$\hat{\rho}(t, X) = \rho(t, \overline{X}), \quad \hat{v}(t, X) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{X - Y}{|X - Y|^2} \, \hat{\rho}(t, Y) \, dY.$$

Using straightforward change of variables, it is quite easy to get

$$v(t, X) = \overline{\hat{v}(t, \overline{X})}, \quad \operatorname{div}(v\,\rho)(t, X) = \operatorname{div}(\hat{v}\hat{\rho})(t, \overline{X}).$$

Therefore we find that $\hat{\rho}$ satisfies also the aggregation equation

$$\partial_t \hat{\rho} + \operatorname{div}(\hat{v}\,\hat{\rho}) = 0.$$

Combining this property with the uniqueness of Yudovich's solutions, it follows that if the initial data belong to $L^1 \cap L^\infty$ and admit an axis of symmetry then the solution remains invariant with respect to the same axis. In the framework of the vortex patches this result means that if the initial data are given by $\rho_0 = \mathbf{1}_{D_0}$ and the domain D_0 is symmetric with respect to the real axis, the domain D_t defining the solution $\rho(t) = \mathbf{1}_{D_t}$ remains symmetric with respect to the same axis for any positive time. Recall that in the form (1-2) Yudovich-type solutions are global in time. To be precise about the terminology, here and contrary to the standard definition in topology, where "domain" means a connected open set, we mean by "domain" any measurable set of strictly positive measure. In addition, a patch whose domain is symmetric with respect to the real axis (or any axis) is called one-fold symmetric.

In the current study, we shall focus on the domains D_0 such that the boundary part lying in the upper half-plane is described by the graph of a C^1 positive function $f_0 : \mathbb{R} \to \mathbb{R}_+$ with compact support. This is equivalent to

$$D_0 = \{(x, y) \in \mathbb{R}^2 : x \in \operatorname{supp} f_0, \, -f_0(x) \le y \le f_0(x)\}.$$

We point out that concretely we shall consider the evolution not of D_0 but of its extended set defined by

$$\widehat{D}_0 = \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{R}, \, -f_0(x) \le y \le f_0(x)\}.$$

This does not matter since the domain D_t remains symmetric with respect to the real axis and then we can simply track its evolution by knowing the dynamics of its extended domain: we just remove the extra lines located on the real axis.

One of the main objectives of this paper is to follow the dynamics of the graph and investigate local and global well-posedness issues in different function spaces. In the next lines, we shall derive the evolution equation governing the motion of the initial graph f_0. Assume that in a short time interval $[0, T]$ the part of the boundary in the upper half-plane is described by the graph of a C^1 function $f_t:\mathbb{R}\to\mathbb{R}_+$. This forces the points of the boundary ∂D_t located on the real axis to be cusp singularities. As a material point located at the boundary remains on the boundary, any parametrization $s\mapsto\gamma_t(s)$ of the boundary should satisfy

$$(\partial_t\gamma_t(s)-v(t,\gamma_t(s)))\cdot\vec{n}(\gamma_t(s))=0,$$

with $\vec{n}(\gamma_t)$ being a normal unit vector to the boundary at the point $\gamma_t(s)$. Now take the parametrization in the graph form $\gamma_t:x\mapsto(x,f(t,x))$; then the preceding equation reduces to the nonlinear transport equation

$$\begin{cases}\partial_t f(t,x)+u_1(t,x)\,\partial_x f(t,x)=u_2(t,x), & t\geq 0,\ x\in\mathbb{R},\\ f(0,x)=f_0(x),\end{cases}\tag{2-1}$$

where $(u_1,u_2)(t,x)$ is the velocity $(v_1,v_2)(t,X)$ computed at the point $X=(x,f(t,x))$. Throughout this paper we use the notation

$$f_t(x)=f(t,x)\quad\text{and}\quad f'(t,x)=\partial_x f(t,x).$$

To reformulate (2-1) in a closed form we shall recover the velocity components with respect to the graph parametrization. We start with the computation of $v_1(X)$. Here and for the sake of simplicity we drop the time parameter from the graph and the domain of the patch. One writes according to Fubini's theorem

$$-2\pi v_1(X)=\int_D\frac{x-y_1}{|X-Y|^2}\,dY=\int_{\mathbb{R}}(x-y_1)\int_{-f(y_1)}^{f(y_1)}\frac{dy_2}{(x-y_1)^2+(f(x)-y_2)^2}\,dy_1,$$

where $Y=(y_1,y_2)$. Using the change of variables $y_2-f(x)=(x-y_1)Z$ we find

$$\begin{aligned}2\pi v_1(X)&=\int_{\mathbb{R}}\left[\arctan\left(\frac{f(y)-f(x)}{y-x}\right)+\arctan\left(\frac{f(y)+f(x)}{y-x}\right)\right]dy\\&=\int_{\mathbb{R}}\left[\arctan\left(\frac{f(x+y)-f(x)}{y}\right)+\arctan\left(\frac{f(x+y)+f(x)}{y}\right)\right]dy.\end{aligned}$$

To compute v_2 in terms of f we proceed as before and we find

$$-2\pi v_2(X)=\int_D\frac{f(x)-y_2}{|X-Y|^2}\,dA(Y)=\int_{\mathbb{R}}\int_{-f(y_1)}^{f(y_1)}\frac{f(x)-y_2}{(x-y_1)^2+(f(x)-y_2)^2}\,dy_2\,dy_1.$$

Therefore we obtain the expression

$$4\pi v_2(x,f(x))=\int_{\mathbb{R}}\log\left(\frac{y^2+(f(x+y)-f(x))^2}{y^2+(f(x+y)+f(x))^2}\right)dy.$$

With the notation adopted before for (u_1,u_2) we finally get the formulas

$$\begin{aligned}u_1(t,x)&=\frac{1}{2\pi}\int_{\mathbb{R}}\left[\arctan\left(\frac{f_t(x+y)-f_t(x)}{y}\right)+\arctan\left(\frac{f_t(x+y)+f_t(x)}{y}\right)\right]dy,\\u_2(t,x)&=\frac{1}{4\pi}\int_{\mathbb{R}}\log\left(\frac{y^2+(f_t(x+y)-f_t(x))^2}{y^2+(f_t(x+y)+f_t(x))^2}\right)dy.\end{aligned}\tag{2-2}$$

We emphasize that for the coherence of the model the graph equation (2-1) is supplemented with the initial condition $f_0(x)\geq 0$. According to Proposition 6.2, the positivity is preserved for enough smooth solutions. Furthermore, and once again according to this proposition we have a maximum principle estimate:

$$\text{for all } t \geq 0, \text{ for all } x \in \mathbb{R}, \quad 0 \leq f(t,x) \leq \|f_0\|_{L^\infty}.$$

Notice that the model remains meaningful even when the function f_t changes sign. In this case the geometric domain of the patch is simply obtained by looking to the region delimited by the curve of f_t and it is symmetric with respect to the real axis. This is also equivalent to dealing with a positive function f_t but its graph will be less regular and belongs only to the Lipschitz class. Another essential element that will be analyzed later in Proposition 6.2 concerns the support of the solutions, which remains confined through the time interval. More precisely, if supp $f_0 \subset [a,b]$ with $a < b$ then provided that the graph exists for $t \in [0,T]$ one has

$$\operatorname{supp} f(t) \subset [a,b].$$

This follows from the fact that the flow associated to the horizontal velocity u_1 is contractive on the boundary. It is not clear whether global weak solutions satisfying the maximum principle can be constructed. However, to deal with classical solutions one should control higher regularity of the graph and it seems from the transport structure of the equation that the optimal scaling for local well-posedness theory is Lipschitz class. Thus, in what follows we say that a function space is critical if it scales as a Lipschitz class and subcritical if it scales above like Hölder spaces C^{1+s}, $s > 0$. Denote by $g(t,x) = \partial_x f(t,x)$ the slope of the graph; then it is quite obvious from (2-1) that

$$\partial_t g + u_1 \partial_x g = -\partial_x u_1 g + \partial_x u_2. \tag{2-3}$$

For the computation of the source term we proceed in a classical way using the differentiation under the integral sign and we get successively

$$2\pi \partial_x u_1(x) = \text{p.v.} \int_{\mathbb{R}} \frac{f'(x+y) - f'(x)}{y^2 + (f(x+y) - f(x))^2} y \, dy + \text{p.v.} \int_{\mathbb{R}} \frac{f'(x+y) + f'(x)}{y^2 + (f(x+y) + f(x))^2} y \, dy \tag{2-4}$$

and

$$2\pi \partial_x u_2(x) = \text{p.v.} \int_{\mathbb{R}} \frac{(f(x+y) - f(x))(f'(x+y) - f'(x))}{y^2 + (f(x+y) - f(x))^2} \, dy \\ - \text{p.v.} \int_{\mathbb{R}} \frac{(f(x+y) + f(x))(f'(x+y) + f'(x))}{y^2 + (f(x+y) + f(x))^2} \, dy,$$

where the notation "p.v." is the Cauchy principal value. It is worth pointing out that the first two integrals appearing in the right-hand side of the expressions of $\partial_x u_1$ and $\partial_x u_2$ are in fact connected to the Cauchy operator associated to the curve f defined in (5-1). This operator is well-studied in the literature and some details will be given later in Section 5. Next, we shall check that the integrals appearing in the right-hand side of the preceding formulas can actually be restricted over a compact set related to the support of f. Let $[-M, M]$ be a symmetric segment containing the set $K_0 - K_0$, with K_0 being the convex hull of the support of f_0, which is denoted by supp f_0. It is clear that the support of $\partial_x u_1 f'$ is contained in K_0 and thus for $x \in K_0$ one has

$$\text{p.v.} \int_{\mathbb{R}} \frac{f'(x+y)-f'(x)}{y^2+(f(x+y)-f(x))^2}\, y\, dy = \text{p.v.} \int_{-M}^{M} \frac{f'(x+y)-f'(x)}{y^2+(f(x+y)-f(x))^2}\, y\, dy.$$

Consequently, we obtain for $x \in \mathbb{R}$

$$2\pi f'(x)\partial_x u_1(x) = f'(x)\, \text{p.v.} \int_{-M}^{M} \frac{f'(x+y)-f'(x)}{y^2+(f(x+y)-f(x))^2}\, y\, dy \\ + f'(x)\, \text{p.v.} \int_{-M}^{M} \frac{f'(x+y)+f'(x)}{y^2+(f(x+y)+f(x))^2}\, y\, dy.$$

Coming back to the integral representation defining $\partial_x u_2$ one can see, using a cancellation between both integrals, that the support of $\partial_x u_2$ is contained in K_0. Furthermore, for $x \in K_0$ one may write

$$2\pi \partial_x u_2(x) = \text{p.v.} \int_{-M}^{M} \frac{(f(x+y)-f(x))(f'(x+y)-f'(x))}{y^2+(f(x+y)-f(x))^2}\, dy \\ - \text{p.v.} \int_{-M}^{M} \frac{(f(x+y)+f(x))(f'(x+y)+f'(x))}{y^2+(f(x+y)+f(x))^2}\, dy.$$

Gathering the preceding identities we deduce that

$$2\pi(-\partial_x u_1 f'(x) + \partial_x u_2) = F(x) - G(x), \tag{2-5}$$

with

$$F(x) \triangleq \text{p.v.} \int_{-M}^{M} \frac{[f(x+y)-f(x)-yf'(x)](f'(x+y)-f'(x))}{y^2+(f(x+y)-f(x))^2}\, dy,$$
$$G(x) \triangleq \text{p.v.} \int_{-M}^{M} \frac{[f(x+y)+f(x)+yf'(x)](f'(x+y)+f'(x))}{y^2+(f(x+y)+f(x))^2}\, dy.$$

One should keep in mind that the integrals above can also be extended to the full real axis. Sometimes in order to reduce the size of the integral representation we use the notation

$$\Delta_y^{\pm} f(x) = f(x+y) \pm f(x). \tag{2-6}$$

Thus F and G take the form

$$F(x) = \text{p.v.} \int_{-M}^{M} \frac{[\Delta_y^- f(x) - yf'(x)]\Delta_y^- f'(x)}{y^2+(\Delta_y^- f(x))^2}\, dy, \tag{2-7}$$

$$G(x) = \text{p.v.} \int_{-M}^{M} \frac{[\Delta_y^+ f(x) + yf'(x)]\Delta_y^+ f'(x)}{y^2+(\Delta_y^+ f(x))^2}\, dy. \tag{2-8}$$

The first main result of this paper is devoted to the local well-posedness issue. We shall discuss two results related to subcritical and critical regularities. Denote by X one of the following spaces: Hölder spaces $C^s(\mathbb{R})$ with $s \in (0,1)$ or the Dini space $C^\star(\mathbb{R})$. For more details about classical properties of these spaces we refer the reader to Section 4.

Theorem 2.1. *Let f_0 be a positive compactly supported function such that $f_0' \in X$. Then, the following results hold true*:

(1) *Equation* (2-1) *admits a unique local solution such that $f' \in L^\infty([0,T], X)$, where the time existence T is related to the norm $\|f_0'\|_X$ and the size of the support of f_0. In addition, the solution satisfies the maximum principle*

$$\text{for all } t \in [0,T], \quad \|f(t)\|_{L^\infty} \le \|f_0\|_{L^\infty}.$$

(2) *There exists a constant $\varepsilon > 0$ depending only on s and the size of the support of f_0 such that if*

$$\|f_0'\|_{C^s} < \varepsilon \tag{2-9}$$

then (2-1) *admits a unique global solution $f' \in L^\infty(\mathbb{R}_+; C^s(\mathbb{R}))$. Moreover*,

$$\text{for all } t \ge 0, \quad \|\partial_x f(t)\|_{L^\infty} \le C_0 e^{-t},$$

with C_0 a constant depending only on $\|f_0'\|_{C^s}$.

Before outlining the strategy of the proof, some comments are in order.

Remarks. (1) The global existence result is only proved for the subcritical case (C^s). The critical case (Dini case) is more delicate to handle due to the lack of strong damping, which is only proved in the subcritical case (see Proposition 7.1). Roughly speaking, the damping comes from the linearization of the nonlinear term. Indeed, one finds that the equation

$$\partial_t f' + u_1\,\partial_x f' = \frac{1}{2\pi}(F(x) - G(x)) = -f' + L_1(x) + \text{nonlinear},$$

where (see Proposition 7.1) the term "nonlinear" has superlinear C^s-type estimates. If the term $L_1(x)$ were identically zero, then one can use the damping term $-f'$ to obtain exponentially decaying global solutions with small initial data. On the other hand, as it turns out, the almost-linear-type term $L_1(x)$ admits estimates of the form

$$\|L_1\|_s \le (\|f'\|_s + 2\|f'\|_{L^\infty}) + C\|f'\|_{L^\infty}^s \|f'\|_s,$$
$$\|L_1\|_{L^\infty} \le C\min(\|f\|_{L^\infty}^s \|f'\|_s, \|f'\|_{L^\infty}).$$

The key improvement here is the first estimate in the L^∞ estimate of L_1, which is in some sense superlinear. By using Proposition 6.2 one can obtain an exponential decay estimate of $\|f\|_1$ through an area argument. This important estimate together with some interpolation estimates (and an exponential decay estimate of $\|\partial_x u_1\|_\infty$) and the strong damping term $-f'$ then yields global well-posedness for small data.

(2) Coming back to the patch domain, we see that it admits cusp-like singularities located on the axis of symmetry. This is not covered by the preceding result [Bertozzi et al. 2016] where the boundary is assumed to be more regular than C^1. From the proof of Theorem 2.1 we deduce that the graph solutions generate a Lipschitz velocity. This allows us to easily propagate a weak notion for the order of a cusp. More precisely, let $\alpha > 0$ be the order of a cusp x_0; that is, for small r, we have $|D \cap B(x_0, r)| = O(r^{2+\alpha})$, and then for the solutions constructed in Theorem 2.1 we get $|D_t \cap B(x_t, r)| = O(r^{2+\alpha})$, with x_t the image of x_0 by the flow. Notice that this problem was studied for Euler equations in [Danchin 2000].

(3) From Sobolev embeddings we deduce according to the assumption on f_0 listed in Theorem 2.1 that f_0 belongs to the space $C_c^1(\mathbb{R})$ of compactly supported C^1 functions.

(4) The maximum principle holds true globally in time; however, it is not clear whether some suitable weak global solutions could be constructed in this setting.

Now we shall give some details about the proofs. First we establish local-in-time a priori estimates based on the transport structure of the equation combined with some refined studies on modified curved Cauchy operators implemented in Section 5 and essentially based on standard arguments from singular integrals. The construction of the solutions done in Section 6C is slightly more intricate than the usual schemes used for transport equations. This is due to the fact that the establishment of the a priori estimates is not purely energetic. First, at some levels we use some nonlinear rigidity of the equation like in Theorem 2.1(3), where the factor f' behind the operator should be the derivative of the function f that appears inside the operator. Second, we use at some point the fact that the support is confined in time. Last we use at different steps the positivity of the solution. Hence it seems quite difficult to find a linear scheme taking into account all of those constraints. The idea is to implement a nonlinear scheme with two regularizing parameters ε and n. The first one is used to smooth out the singularity of the kernel and the second to smooth the solution through a nonlinear scheme. We first establish that one has uniform a priori estimates on n but on some small interval depending on ε. We are also able to pass to the limit on n and get a solution for a modified nonlinear problem. Second we check that the a priori estimates still be valid uniformly on ε. This ensures that the time existence can be in fact pushed up to the time given by the a priori estimates obtained for the initial equation (2-1). As a consequence we get a uniform time existence with respect to ε and finally we establish the convergence towards a solution of the initial value problem using standard compactness arguments.

The global existence for small initial data requires much more careful analysis because there is no apparent dissipation or damping mechanisms in the equation. Notice that the estimate of the source term G contains some linear parts as it is stated in Proposition 6.1. The basic ingredient to get rid of those linear parts is to implement a kind of linearization allowing us to capture a weak damping effect in G that can just absorb the growth of the linear part. We do not know if the damping proved for lower regularity still happens in the resolution space. As to the nonlinear terms, they are always associated with some subcritical norms and thus using an interpolation argument with the exponential decay of the L^1 norm we get a global-in-time control that leads to the global existence.

The second result that we shall discuss deals with the asymptotic behavior of the solutions to (1-2) and (2-1). We shall study the collapse of the support to a collection of disjoint segments located at the axis of symmetry. Another interesting issue that will be covered by this discussion concerns the characterization of the limit behavior of the probability measure

$$dP_t \triangleq e^t \frac{\mathbf{1}_{D_t}}{|D_0|} dA, \tag{2-10}$$

with dA being Lebesgue measure and $|D_0|$ denoting the Lebesgue measure of D_0. Our result reads as follows.

Theorem 2.2. *Let f_0 be a positive compactly supported function such that $f_0' \in C^s(\mathbb{R})$, with $s \in (0,1)$. Assume that* supp f_0 *is the union of n disjoint segments and satisfies the smallness condition* (2-9). *Then there exists a compact set $D_\infty \subset \mathbb{R}$ composed of exactly of n disjoint segments and a constant $C > 0$ such that*

$$\text{for all } t \geq 0, \quad d_H(D_t, D_\infty) \leq Ce^{-t}, \quad |D_\infty| \geq \tfrac{1}{2}|D_0|,$$

with d_H being the Hausdorff distance and $|D_\infty|$ the one-dimensional Lebesgue measure of D_∞. In addition, the probability measures $\{dP_t\}_{t\geq 0}$ defined in (2-10) *converge weakly as t goes to ∞ to the probability measure*

$$dP_\infty := \Phi\, \delta_{D_\infty \otimes \{0\}},$$

with Φ being a compactly supported function in D_∞ belonging to $C^\alpha(\mathbb{R})$ for any $\alpha \in (0,1)$ and can be expressed in the form

$$\Phi(x) = \frac{f_0(\psi_\infty^{-1}(x))}{\|f_0\|_{L^1}} e^{g(x)}, \tag{2-11}$$

with g a function that can be implicitly recovered from the full dynamics of solution $\{f_t : t \geq 0\}$ and

$$\psi_\infty = \lim_{t\to\infty} \psi(t).$$

Note that $\psi(t)$ is the one-dimensional flow associated to u_1 defined in (6-26) *and*

$$D_t = \{(x, y) : x \in \operatorname{supp} f_t,\ -f_t(x) \leq y \leq f_t(x)\}.$$

Remark 2.3. The regularity of the profile Φ might be improved and we expect that Φ keeps the same regularity as the graph.

The proof of the collapse of the support to a disjoint union of segments can be easily derived from the formula (2-11) which ensures that the support of the limit measure is exactly the image of the support of f_0 by the limit flow ψ_∞, which is a homeomorphism of the real axis. To get the convergence with the Hausdorff distance we just use the exponential damping of the amplitude of the curve. As to the characterization of the limit measure it is based on the exponential decay of the amplitude of graph combined with the scattering as t goes to infinity of the normalized solution $e^t f(t)$. In fact, we prove that the density is nothing but the formal quantity

$$\Phi(x) = 2 \lim_{t\to\infty} e^t f(t, x)$$

whose existence is obtained using the transport structure of the equation through the method of characteristics combined with the damping effects of the nonlinear source terms.

3. Generalities on the limit shapes

In this short section we shall discuss a simple result dealing with the role of symmetry in the structure of the limit shape D_∞. Roughly speaking, we shall prove that thin initial domains along their axis of symmetry generate concentration to segments. Notice that

$$D_\infty \triangleq \big\{\lim_{t\to\infty} \psi(t,x) : x \in D_0\big\},$$

where ψ is the flow associated to the velocity v and defined through the ODE

$$\begin{cases} \partial_t \psi(t,x) = v(t,\psi(t,x)), & t \geq 0,\ x \in \mathbb{R}^2, \\ \psi(0,x) = x. \end{cases} \tag{3-1}$$

The existence of the set D_∞ will be proved below. We intend to prove the following.

Proposition 3.1. *The following assertions hold*:

(1) *If D_0 is a bounded domain of $\mathbb{R}^2$, then for any $x \in \mathbb{R}^2$ the quantity $\lim_{t\to\infty} \psi(t,x)$ exists.*

(2) *Let D_0 be a simply connected bounded domain symmetric with respect to an axis Δ. Denote by $d_0 = \operatorname{Length}(D_0 \cap \Delta)$. There exists an absolute constant C such that if*

$$d_0 > C|D_0|^{\frac{1}{2}}$$

then the shape D_∞ contains an interval of the size $d_0 - C|D_0|^{\frac{1}{2}}$.

Proof. (1) Integrating in time the flow equation (3-1) yields

$$\psi(t,x) = x + \int_0^t v(\tau,\psi(\tau,x))\,d\tau.$$

Now observe that pointwisely

$$|v(t,x)| \leq \frac{1}{2\pi}\left(\frac{1}{|\cdot|^2} \star |\rho(t)|\right)(x).$$

Thus interpolation inequalities combined with (1-3) lead to

$$\|v(t)\|_{L^\infty} \leq C\|\rho(t)\|_{L^1}^{\frac{1}{2}}\|\rho(t)\|_{L^\infty}^{\frac{1}{2}} \leq Ce^{-\frac{t}{2}}|D_0|^{\frac{1}{2}}, \tag{3-2}$$

with C an absolute constant. This implies that the integral $\int_0^\infty v(\tau,\psi(\tau,x))\,d\tau$ converges absolutely and therefore $\lim_{t\to\infty}\psi(t,x)$ exists in $\mathbb{R}^2$. This allows us to define the limit shape D_∞ as

$$D_\infty = \big\{\lim_{t\to\infty} \psi(t,x) : x \in D_0\big\}.$$

(2) Without loss of generality we will suppose that the straight line Δ coincides with the real axis. Since D is a simply connected bounded domain, there exist two different points $X_0^-, X_0^+ \in \mathbb{R}$ such that

$$\overline{D}_0 \cap \Delta = [X_0^-, X_0^+].$$

Then it is clear that $\operatorname{Length}(\overline{D}_0 \cap \Delta) = X_0^+ - X_0^- := d_0$. By assumption D_0 is symmetric with respect to Δ; then the domain D_t remains also symmetric with respect to the same axis and the points $X_0^\pm$ move along this axis. Set

$$X^\pm(t) = \psi(t, X_0^\pm);$$

then as the flow is a homeomorphism

$$\overline{D}_t \cap \Delta = [X^-(t), X^+(t)].$$

Now we wish to follow the evolution of the distance $d(t) := X^+(t) - X^-(t)$ and find a sufficient condition such that this distance remains away from zero up to infinity. Notice from the first point that $\lim_{t\to\infty} d(t)$ exists and is equal to some positive number d_∞. From the triangle inequality, one easily gets

$$d(t) \ge d_0 - 2\int_0^t \|v(\tau)\|_{L^\infty}\, d\tau.$$

Inequality (3-2) ensures that

$$d(t) \ge d_0 - C|D_0|^{\frac{1}{2}}$$

and therefore $d_\infty \ge d_0 - C|D_0|^{\frac{1}{2}}$. Consequently, if $d_0 > C|D_0|^{\frac{1}{2}}$ then the points $\{X^\pm(t)\}$ do not collide up to infinity and thus the set D_∞ contains a nontrivial interval as claimed. □

4. Basic properties of Dini and Hölder spaces

We now set up some function spaces that we shall use and review some of their important properties. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function; we define its modulus of continuity $\omega_f : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\omega_f(r) = \sup_{|x-y|\le r} |f(x) - f(y)|.$$

This is a nondecreasing function satisfying $\omega_f(0) = 0$ and it is subadditive; that is, for $r_1, r_2 \ge 0$ we have

$$\omega_f(r_1 + r_2) \le \omega_f(r_1) + \omega_f(r_2). \tag{4-1}$$

Now we intend to recall Dini and Hölder spaces. The Dini space denoted by $C^\star(\mathbb{R})$ is the set of continuous bounded functions f such that

$$\|f\|_{L^\infty} + \|f\|_D < \infty, \quad \text{with } \|f\|_D = \int_0^1 \frac{\omega_f(r)}{r}\, dr.$$

Another space that we frequently use throughout this paper is the Hölder space. Let $s \in (0, 1)$; we denote by $C^s(\mathbb{R})$ the set of functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\|f\|_{L^\infty} + \|f\|_s < \infty, \quad \text{with } \|f\|_s = \sup_{0<r<1} \frac{\omega_f(r)}{r^s}.$$

Let K be a compact set of $\mathbb{R}$; we define $C^\star_K$ as the subspace of $C^\star(\mathbb{R})$ whose elements are supported in K. Note that $C^\star_K \hookrightarrow L^\infty(\mathbb{R})$, which means that a constant C depending only on the diameter of the compact K exists such that

$$\text{for all } f \in C^\star_K, \quad \|f\|_{L^\infty} \le C\|f\|_D. \tag{4-2}$$

This follows easily from the observation

$$\text{for all } r \in \left(0, \tfrac{1}{2}\right], \quad \omega(r)\ln 2 \le \|f\|_D.$$

From (4-2) we deduce that for any $A \ge 1$

$$\int_0^A \frac{\omega_f(r)}{r}\, dr \le \|f\|_D + 2\|f\|_{L^\infty} \ln A \le C\|f\|_D(1 + \ln A). \tag{4-3}$$

Coming back to the definition of Dini seminorm one deduces the product laws: for $f, g \in C_K^\star$

$$\|fg\|_D \le \|f\|_{L^\infty}\|g\|_D + \|g\|_{L^\infty}\|f\|_D \quad \text{and} \quad \|fg\|_D \le C\|f\|_D\|g\|_D. \tag{4-4}$$

Another useful space is C_K^s, which is the subspace of $C^s(\mathbb{R})$ whose functions are supported on compact K. It is quite obvious that

$$C_K^s \hookrightarrow C_K^\star \hookrightarrow L^\infty. \tag{4-5}$$

We point out that all these spaces are complete. Another property which will be very useful is the following composition law. If $f \in C^s(\mathbb{R})$ with $0 < s < 1$ and $\psi : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function then $f \circ \psi \in C^s(\mathbb{R})$ and

$$\|f \circ \psi\|_s \le (\|f\|_s + 2\|f\|_{L^\infty})\|\nabla\psi\|_{L^\infty}^s. \tag{4-6}$$

It is worth pointing out that in the case of the Dini space $C^\star(\mathbb{R})$ we get a more precise estimate of logarithmic type,

$$\|f \circ \psi\|_D \le C(\|f\|_D + \|f\|_{L^\infty})(1 + \ln_+(\|\nabla\psi\|_{L^\infty})), \tag{4-7}$$

with the notation

$$\ln_+ x \triangleq \begin{cases} \ln x & \text{if } x \ge 1, \\ 0 & \text{otherwise.} \end{cases}$$

Another estimate of great interest is the following product law:

$$\|fg\|_s \le \|f\|_{L^\infty}\|g\|_s + \|g\|_{L^\infty}\|f\|_s. \tag{4-8}$$

In the next task we will be concerned with a pointwise estimate connecting a positive smooth function to its derivative and explore how this property is affected by the regularity. This kind of property will be required in Section 5 in studying Cauchy operators with special forms.

Lemma 4.1. *Let K be a compact set of $\mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}_+$ be a continuous positive function supported in K such that $f' \in C^\star(\mathbb{R})$. Then we have*

$$\text{for all } x \in \mathbb{R}, \quad |f'(x)| \le C\frac{\|f'\|_D + \|f'\|_{L^\infty}}{1 + \ln_+(\|f'\|_D / f(x))}.$$

A weak version of this inequality is

$$\text{for all } x \in \mathbb{R}, \quad |f'(x)| \le C\frac{(\|f'\|_D + \|f'\|_{L^\infty})(1 + \ln_+(1/\|f'\|_D))}{1 + \ln_+(1/f(x))},$$

with C an absolute constant. If in addition $f' \in C^s(\mathbb{R})$ with $s \in (0,1)$, then

$$\text{for all } x \in \mathbb{R}, \quad |f'(x)| \le C\|f'\|_s^{\frac{1}{1+s}}[f(x)]^{\frac{s}{1+s}}$$

and the constant C depends only on s.

Proof. Let x be a given point; without any loss of generality one can assume that $f'(x) \geq 0$. Now let $h \in [0,1]$; then using the mean value theorem, there exists $c_h \in [x-h, x)$ such that

$$\begin{aligned} f(x-h) &= f(x) - hf'(c_h) \\ &= f(x) - hf'(x) - h[f'(c_h) - f'(x)] \\ &\leq f(x) - hf'(x) + h\,\omega_{f'}(h). \end{aligned}$$

From the positivity of the function f we deduce that for any $h \in [0,1]$ one gets

$$f(x) - hf'(x) + h\,\omega_{f'}(h) \geq 0.$$

Then dividing by h^2 and integrating in h between ε and 1, with $\varepsilon \in (0,1]$, we get

$$f(x)\frac{1}{\varepsilon} + f'(x)\ln\varepsilon + \|f'\|_D \geq 0.$$

Multiplying by ε we obtain

$$\text{for all } \varepsilon \in (0,1), \quad f(x) + f'(x)\,\varepsilon\ln\varepsilon + \|f'\|_D\,\varepsilon \geq 0. \tag{4-9}$$

By studying the variation with respect to ε we find that the suitable value of ε is given by

$$\ln\varepsilon = -1 - \frac{\|f'\|_D}{f(x)}.$$

Inserting this choice into (4-9) we find that

$$\varepsilon f'(x) \leq f(x);$$

that is,

$$e^{-1-\|f'\|_D/f'(x)} f'(x) \leq f(x).$$

From the inequality $te^{-t} \leq e^{-1}$ we deduce that

$$e^{-1} \geq \frac{\|f'\|_D}{f'(x)} e^{-\|f'\|_D/f'(x)},$$

which implies in turn that

$$e^{-1-\|f'\|_D/f'(x)} f'(x) \geq e^{-2\|f'\|_D/f'(x)} \|f'\|_D.$$

Consequently we get

$$e^{-2\|f'\|_D/f'(x)} \|f'\|_D \leq f(x).$$

Thus when $f(x)/\|f'\|_D > 1$ this estimate does not give any useful information and then we simply write

$$f'(x) \leq \|f'\|_{L^\infty}.$$

However for $f(x)/\|f'\|_D < 1$ we get

$$f'(x) \leq C\frac{\|f'\|_D}{1 + \ln_+(\|f'\|_D/f(x))},$$

from which we deduce that

$$f'(x) \le C \frac{\|f'\|_D(1+\ln_+(1/\|f'\|_D)}{1+\ln_+(1/f(x))}.$$

Indeed, one may use the estimate

$$\text{for all } x>0, \quad \frac{1+\ln_+(1/x)}{1+\ln_+(a/x)} \le 1+\ln_+(1/a),$$

which can be verified easily by studying the variation of the fractional function.

Now let us move to the proof when f' is assumed to belong to the Hölder space C^s, with $s \in (0,1)$. Following the same proof as before one deduces that under the assumption $f'(x) \ge 0$ one obtains for any $h \in \mathbb{R}_+$

$$f(x) - hf'(x) + h^{1+s}\|f'\|_s \ge 0.$$

By studying the variation of this function with respect to h we find that the best choice of h is given by

$$h^s = \frac{f'(x)}{(1+s)\|f'\|_s},$$

which implies the desired result, that is,

$$f'(x) \le C\|f'\|_s^{\frac{1}{1+s}}[f(x)]^{\frac{s}{1+s}}. \qquad \square$$

5. Modified curved Cauchy operators

This section is devoted to the study of some variants of Cauchy operators which are closely connected to the operators arising in (2-4) and (2-5). Let us first recall the classical Cauchy operator associated to the graph of a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$,

$$\mathcal{C}_f g(x) = \int_{\mathbb{R}} \frac{g(x+y)-g(x)}{y+i(f(x+y)-f(x))}\,dy, \tag{5-1}$$

which is well-defined at least for a smooth function g. According to a famous theorem of Coifman, McIntosh and Meyer [Coifman et al. 1982], this operator can be extended as a bounded operator from L^p to L^p for $1<p<\infty$. By adapting the proof of [Wittmann 1987], this operator can also be extended continuously from C^s_K to $C^s(\mathbb{R})$ for $0<s<1$, provided that f belongs to $C^{1+s}(\mathbb{R})$. However this operator fails to be extended continuously from the Dini space $C^\star_K$ to itself, as can be checked from Hilbert transform. The structure of the operators that we have to deal with, as one may observe from the expression of F following (2-5), is slightly different from the Cauchy operators. It can be associated to the truncated bilinear Cauchy operator defined as follows: for given $M>0$, $\theta \in [0,1]$,

$$\mathcal{C}_f^\theta(g,h)(x) = \int_{-M}^{M} \frac{(g(x+\theta y)-g(x))(h(x+y)-h(x))}{y+i(f(x+y)-f(x))}\,dy.$$

The real and imaginary parts of this operator are given respectively by

$$\mathcal{C}_f^{\theta,\Re}(g,h)(x)=\int_{-M}^{M}\frac{y(g(x+\theta y)-g(x))(h(x+y)-h(x))}{y^2+[f(x+y)-f(x)]^2}\,dy \tag{5-2}$$

and

$$\mathcal{C}_f^{\theta,\Im}(g,h)(x)=-\int_{-M}^{M}\frac{(f(x+y)-f(x))(g(x+\theta y)-g(x))(h(x+y)-h(x))}{y^2+[f(x+y)-f(x)]^2}\,dy.$$

In what follows we denote by X one of the spaces C_K^s, with $0<s<1$, or $C_K^\star$. The result that we shall discuss deals with the continuity of the preceding bilinear operators on the spaces X. This may have been discussed in the literature, but as we need to control the continuity constant we shall give a detailed proof.

Proposition 5.1. *Let K be a compact set of $\mathbb{R}$ and f be a compactly supported function such that $f'\in X$. Then the following assertions hold true: The bilinear operator $\mathcal{C}_f^\theta: X\times X\to X$ is well-defined and continuous. More precisely, there exists a constant C independent of θ such that for any $g,h\in X$*

$$\|\mathcal{C}_f^{\theta,\Re}(g,h)\|_X\le C(1+\|f'\|_{L^\infty}\|f'\|_X)(\|g\|_D\|h\|_X+\|h\|_D\|g\|_X),$$
$$\|\mathcal{C}_f^{\theta,\Im}(g,h)\|_X\le C\|f'\|_X(1+\|f'\|_{L^\infty}^2)(\|g\|_D\|h\|_X+\|g\|_X\|h\|_D).$$

Proof. We shall first establish the result for the real-part operator given by (5-2). First we note that one may rewrite the expression using the notation (2-6) as follows:

$$\mathcal{C}_f^{\theta,\Re}(g,h)(x)=\int_{-M}^{M}\frac{y\Delta_{\theta y}g(x)\Delta_y h(x)}{y^2+(\Delta_y f(x))^2}\,dy,$$

where we simply replace the notation Δ_y^- by Δ_y. Using the product laws (4-4) and (4-8) one obtains

$$\|\mathcal{C}_f^{\theta,\Re}(g,h)\|_X\le\int_{-M}^{M}\|\Delta_{\theta y}g\Delta_y h\|_X\frac{dy}{|y|}+\int_{-M}^{M}|y|\,\|\Delta_{\theta y}g\Delta_y h\|_{L^\infty}\left\|\frac{1}{y^2+(\Delta_y f)^2}\right\|_X dy.$$

Using once again the product law, it becomes

$$\begin{aligned}\|\Delta_{\theta y}g\Delta_y h\|_X&\le\|\Delta_{\theta y}g\|_{L^\infty}\|\Delta_y h\|_X+\|\Delta_{\theta y}g\|_X\|\Delta_y h\|_{L^\infty}\\&\le\omega_g(|y|)\|h\|_X+2\|g\|_X\,\omega_h(|y|),\end{aligned}$$

where we have used that for $\theta\in[0,1]$, $y\in\mathbb{R}$

$$\|\Delta_y h\|_X\le 2\|h\|_X,\quad \|\Delta_{\theta y}h\|_{L^\infty}\le\omega_h(|y|). \tag{5-3}$$

Consequently

$$\int_{-M}^{M}\|\Delta_{\theta y}g\Delta_y h\|_X\frac{dy}{|y|}\le C(\|g\|_D\|h\|_X+\|h\|_D\|g\|_X). \tag{5-4}$$

By the definition it is quite easy to check that for any function $\varphi\in X\cap L^\infty(\mathbb{R})$

$$\left\|\frac{1}{y^2+\varphi^2}\right\|_X\le\frac{2\|\varphi\|_{L^\infty}}{y^4}\|\varphi\|_X.$$

Hence we get

$$\begin{aligned}\left\|\frac{1}{y^2+(\Delta_y f)^2}\right\|_X &\le 2\frac{\|\Delta_y f\|_{L^\infty}}{y^4}\|\Delta_y f\|_X\\ &\le Cy^{-2}\|f'\|_{L^\infty}\|f'\|_X,\end{aligned} \tag{5-5}$$

where we have used the inequalities

$$\|\Delta_y f\|_{L^\infty}\le |y|\,\|f'\|_{L^\infty}\quad\text{and}\quad \omega_{\Delta_y f}(r)\le |y|\omega_{f'}(r).$$

Therefore we get in view of (5-3),

$$\begin{aligned}\int_{-M}^{M}|y|\,\|\Delta_{\theta y}g\Delta_y h\|_{L^\infty}\left\|\frac{1}{y^2+(\Delta_y f)^2}\right\|_X dy &\le C\|f'\|_{L^\infty}\|f'\|_X\,\|h\|_{L^\infty}\int_{-M}^{M}\frac{\omega_g(|y|)}{|y|}\,dy\\ &\le C\|f'\|_{L^\infty}\|f'\|_X\,\|h\|_{L^\infty}\|g\|_D.\end{aligned}$$

Combining this last estimate with (5-4) we find that

$$\|\mathcal{C}_f^{\theta,\Re}(g,h)\|_X\le C(\|g\|_D\,\|h\|_X+\|h\|_D\,\|g\|_X+\|f'\|_{L^\infty}\|f'\|_X\,\|h\|_{L^\infty}\|g\|_D).$$

To deduce the result it is enough to use (4-5).

We are left with the task of estimating the imaginary part, which takes the form

$$\mathcal{C}_f^{\theta,\Im}(g,h)(x)=\int_{-M}^{M}\frac{\Delta_y f(x)\Delta_{\theta y}g(x)\Delta_y h(x)}{y^2+(\Delta_y f(x))^2}\,dy.$$

Note that we have dropped the minus sign before the integral, which of course has no consequence on the computations. Using Taylor's formula we get

$$\Delta_y f(x)=y\int_0^1 f'(x+\tau y)\,d\tau$$

and thus

$$\mathcal{C}_f^{\theta,\Im}(g,h)(x)=\int_{-M}^{M}\int_0^1\frac{yf'(x+\tau y)\Delta_{\theta y}g(x)\Delta_y h(x)}{y^2+(\Delta_y f(x))^2}\,dy\,d\tau.$$

It suffices to reproduce the preceding computations using in particular the estimates

$$\|f'(\cdot+\tau y)\Delta_{\theta y}g\Delta_y h\|_{L^\infty}\le\|f'\|_{L^\infty}\|h\|_{L^\infty}\omega_g(|y|)$$

and

$$\begin{aligned}\|f'(\cdot+\tau y)\Delta_{\theta y}g\Delta_y h\|_X &\le\|f'\|_{L^\infty}\|\Delta_{\theta y}g\Delta_y h\|_X+\|f'\|_X\,\|\Delta_{\theta y}g\Delta_y h\|_{L^\infty}\\ &\le\|f'\|_{L^\infty}\big(\omega_g(|y|)\|h\|_X+\|g\|_X\,\omega_h(|y|)\big)+2\|f'\|_X\,\|g\|_{L^\infty}\omega_h(|y|).\end{aligned}$$

This implies, according to the Sobolev embeddings (4-5),

$$\int_{-M}^{M}\int_0^1\|f'(\cdot+\tau y)\Delta_{\theta y}g\Delta_y h\|_X\,\frac{dy}{|y|}\,d\tau\le C\|f'\|_X(\|g\|_D\,\|h\|_X+\|g\|_X\,\|h\|_D).$$

Using (5-5) one may easily get

$$\int_{-M}^{M} |y| \, \|f'(\cdot + \tau y)\Delta_{\theta y} g \Delta_y h\|_{L^\infty} \left\| \frac{1}{y^2 + (\Delta_y f)^2} \right\|_X dy \le C \|f'\|_{L^\infty}^2 \|f'\|_X \|h\|_{L^\infty} \|g\|_D,$$

which gives the desired result using the Sobolev embeddings (4-5). □

The second kind of Cauchy integrals that we have to deal with, and that are related to the integral terms in (2-4) and (2-5), are given by the linear operators

$$T_f^{\alpha,\beta} g(x) = \text{p.v.} \int_{\mathbb{R}} \frac{y g(\alpha x + \beta y)}{y^2 + [f(x) + f(x+y)]^2} \, dy,$$

with α and β being two given parameters. The continuity of these operators in classical Banach spaces is not in general easy to establish and could fail for some special cases. We point out that it is not our purpose in this exposition to implement a complete study of these operators. A more complete theory may be achieved but this topic exceeds the scope of this paper and we shall restrict ourselves to some special configurations that fit with the application to the aggregation equation. Our result in this direction reads as follows.

Theorem 5.2. *Let $\alpha, \beta \in [0, 1]$, K be a compact set of $\mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}_+$ be a compactly supported continuous positive function such that $f' \in C_K^\star$. Then the following assertions hold true*:

(1) *The operator $T_f^{\alpha,\beta} : C_K^\star \to L^\infty(\mathbb{R})$ is well-defined and continuous and*

$$\|T_f^{\alpha,\beta} g\|_{L^\infty} \le C(1 + \|f'\|_{L^\infty}^2 + \|f'\|_{L^\infty} \|f'\|_D) \|g\|_D,$$

with C a constant depending only on K and not on α and β.

(2) *The modified operator $f' T_f^{\alpha,\beta} : C_K^\star \to C_K^\star$ is continuous. More precisely*,

$$\|f' T_f^{\alpha,\beta} g\|_D \le C \|f'\|_D (C_\beta \ln_+(1/\|f'\|_D) + \|f'\|_D^{14}) \|g\|_D,$$

with C a constant depending only on K and

$$C_\beta \triangleq \begin{cases} (1 - \ln \beta), & \beta \in (0, 1], \\ 1, & \beta = 0. \end{cases}$$

(3) *Let $s \in (0, 1)$ and assume that $f' \in C_K^s$; then $f' T_f^{\alpha,\beta} : C_K^s \to C_K^s(\mathbb{R})$ is well-defined and continuous. More precisely, there exists a constant C depending only on the compact K and s such that*

$$\|f' T_f^{\alpha,\beta} g\|_s \le C(C_\beta \|f'\|_{L^\infty}^{\frac{1}{1+s}} + \|f'\|_s^{14}) \|g\|_s. \tag{5-6}$$

In addition, one has the refined estimate

$$\|f' T_f^{\alpha,\beta} g\|_s \le C \|f'\|_{L^\infty}^{\frac{1}{2+s}} (\|f'\|_s^{\frac{1}{2+s}} C_\beta + \|f'\|_s^{14}) \|g\|_s + C \|g\|_{L^\infty}^{\frac{1}{2+s}} \|g\|_s^{\frac{1+s}{2+s}} \|f'\|_s,$$

with

$$C_\beta \triangleq \begin{cases} \beta^{-\frac{1}{2}}, & \beta \in (0, 1], \\ 1, & \beta = 0. \end{cases}$$

Proof. To simplify the notation we shall throughout this proof write $T_f g$ instead of $T_f^{\alpha,\beta} g$.

(1) By symmetrizing we get

$$\begin{aligned} T_f g(x) &= \int_0^\infty \frac{y\,[g(\alpha x+\beta y)-g(\alpha x-\beta y)]}{y^2+[f(x)+f(x+y)]^2}\,dy \\ &\qquad + \lim_{\varepsilon\to 0}\int_\varepsilon^\infty \frac{y\,g(\alpha x-\beta y)[f(x-y)-f(x+y)]\,[\Delta_y^+ f(x)+\Delta_{-y}^+ f(x)]}{(y^2+[\Delta_y^+ f(x)]^2)(y^2+[\Delta_{-y}^+ f(x)]^2)}\,dy \\ &\triangleq T_f^1 g(x)+T_f^2 g(x). \end{aligned} \tag{5-7}$$

Without loss of generality we can assume that $K=[-1,1]$ and $\operatorname{supp} g\subset[-1,1]$ and deal only with $x\ge 0$. We shall distinguish two cases $0\le\alpha x\le 2$ and $\alpha x\ge 2$. In the first case, reasoning on the support of g we simply get

$$T_f^1 g(x)=\int_{\{0\le\beta y\le 3\}} \frac{y[g(\alpha x+\beta y)-g(\alpha x-\beta y)]}{y^2+[f(x)+f(x+y)]^2}\,dy.$$

Hence we obtain according to the definition of the modulus of continuity, a change of variables and (4-3)

$$|T_f^1 g(x)|\le \int_{\{0\le\beta y\le 3\}} \frac{\omega_g(2\beta y)}{y}\,dy\le C\|g\|_D. \tag{5-8}$$

Coming back to the case $\alpha x\ge 2$ one may write

$$\begin{aligned} |T_f^1 g(x)| &\le \int_{\{\alpha x-1\le\beta y\le 1+\alpha x\}} \frac{\omega_g(2\beta y)}{y}\,dy \\ &\le 2\|g\|_{L^\infty}\int_{\alpha x-1}^{1+\alpha x}\frac{1}{y}\,dy \\ &\le \|g\|_{L^\infty}\ln\Big(\frac{1+\gamma}{-1+\gamma}\Big), \qquad \gamma=\alpha x\ge 2 \\ &\le C\|g\|_{L^\infty}. \end{aligned}$$

Combining this last inequality with (5-8) we deduce that

$$\|T_f^1 g\|_{L^\infty}\le C\|g\|_D. \tag{5-9}$$

For the second term $T_f^2 g$ we split it into two parts as follows:

$$\begin{aligned} T_f^2 g(x) &= \lim_{\varepsilon\to 0} 4f(x)\int_\varepsilon^\infty \frac{y\,g(\alpha x-\beta y)[f(x-y)-f(x+y)]}{(y^2+[f(x)+f(x+y)]^2)(y^2+[f(x)+f(x-y)]^2)}\,dy \\ &\qquad + \int_0^\infty \frac{y\,g(\alpha x-\beta y)[f(x-y)-f(x+y)]\psi(x,y)}{(y^2+[f(x)+f(x+y)]^2)(y^2+[f(x)+f(x-y)]^2)}\,dy \\ &\triangleq T_f^{2,1} g(x)+T_f^{2,2} g(x), \end{aligned} \tag{5-10}$$

with

$$\psi(x,y)=f(x+y)+f(x-y)-2f(x)=y\int_0^1 [f'(x+\theta y)-f'(x-\theta y)]\,d\theta.$$

The first term $T_f^{2,1}g$ is easily estimated. Indeed, one can assume that $f(x)>0$; otherwise the integral vanishes. Thus using the mean value theorem and a change of variables we obtain

$$\begin{aligned}|T_f^{2,1}g(x)| &\le 8\|g\|_{L^\infty}\|f'\|_{L^\infty} f(x)\int_0^\infty \frac{y^2}{(y^2+[f(x)]^2)^2}\,dy\\ &\le 8\|g\|_{L^\infty}\|f'\|_{L^\infty}\int_0^\infty \frac{y^2}{(y^2+1)^2}\,dy\\ &\le C\|g\|_{L^\infty}\|f'\|_{L^\infty}.\end{aligned} \tag{5-11}$$

As for the term $T_f^{2,2}$, straightforward arguments yield

$$\begin{aligned}|T_f^{2,2}g(x)| &\le 8\|g\|_{L^\infty}\|f\|_{L^\infty}^2\int_{y\ge\frac12}\frac{1}{y^3}\,dy+2\|g\|_{L^\infty}\|f'\|_{L^\infty}\int_0^{\frac12}\frac{|\psi(x,y)|}{y^2}\,dy\\ &\le C\|g\|_{L^\infty}\biggl(\|f\|_{L^\infty}^2+C\|f'\|_{L^\infty}\int_0^{\frac12}\frac{\omega_{f'}(2y)}{y}\,dy\biggr)\\ &\le C\|g\|_{L^\infty}(\|f'\|_{L^\infty}^2+C\|f'\|_{L^\infty}\|f'\|_D),\end{aligned}$$

where we have used the fact

$$|\psi(x,y)|\le 2y\,\omega_{f'}(2y).$$

Consequently we obtain

$$\|T_f^2 g\|_{L^\infty}\le C\|g\|_{L^\infty}(\|f'\|_{L^\infty}^2+\|f'\|_{L^\infty}\|f'\|_D+\|f'\|_{L^\infty}). \tag{5-12}$$

Putting together this estimate with (5-11) and (4-2) we obtain the desired estimate.

(2) First, recall from part (1) of this proof the decomposition

$$T_f g(x)=T_f^1 g(x)+T_f^{2,1}g(x)+T_f^{2,2}g(x). \tag{5-13}$$

The second term is easier to deal with and one has

$$\|T_f^{2,1}g\|_D\le C\|g\|_D\,\|f'\|_D(1+\|f'\|_{L^\infty}^{13}). \tag{5-14}$$

This implies in view of the product laws (4-4) and (5-11) that

$$\|T_f^{2,1}g\|_D\le C\|g\|_D\,\|f'\|_D(\|f'\|_{L^\infty}+\|f'\|_{L^\infty}^{14}). \tag{5-15}$$

To establish (5-14) we first note that if $f(x)=0$ then $T_f^{2,1}g(x)=0$. However for $f(x)>0$, using the mean value theorem and the change of variables $y\to f(x)y$ we get

$$T_f^{2,1}g(x)=-4\int_0^\infty \frac{y^2\,g(\alpha x-\beta f(x)y)\int_0^1[f'(x+\theta f(x)y)+f'(x-\theta f(x)y)]\,d\theta}{\varphi(x,y)\varphi(x,-y)}\,dy, \tag{5-16}$$

with

$$\varphi(x,y)=y^2+\biggl[2+y\int_0^1 f'(x+\theta f(x)y)\,d\theta\biggr]^2.$$

Observe that the identity (5-16) is meaningful even for $f(x)=0$ and we can check easily that it vanishes. This follows from the fact that owing to the positivity of f when $f(x)=0$ we have $f'(x)=0$. To simplify the expressions we introduce the functions

$$\begin{aligned}\mathcal{N}_1(x,y)&=g(\alpha x-\beta f(x)y)\int_0^1[f'(x+\theta f(x)y)+f'(x-\theta f(x)y)]\,d\theta,\\ \mathcal{D}_1(x,y)&=\varphi(x,y)\varphi(x,-y).\end{aligned}$$

Then by (4-4) we obtain for fixed y

$$\|\mathcal{N}_1(\cdot,y)\|_D\le 2\|g\circ(\alpha\mathrm{Id}-\beta yf)\|_D\,\|f'\|_{L^\infty}+\|g\|_{L^\infty}\int_0^1[\|f'\circ(\mathrm{Id}+\theta yf)\|_D+\|f'\circ(\mathrm{Id}-\theta yf)\|_D]\,d\theta.$$

Using the composition law (4-7) we get successively

$$\begin{aligned}\|g\circ(\alpha\mathrm{Id}-\beta yf)\|_D&\le C\|g\|_D(1+\ln_+(\alpha+\beta\|f'\|_{L^\infty}y)),\\ \|f'\circ(\mathrm{Id}+\theta yf)\|_D&\le C\|f'\|_D(1+\ln(1+\theta\|f'\|_{L^\infty}y)).\end{aligned}$$

This implies

$$\begin{aligned}&\|\mathcal{N}_1(\cdot,y)\|_D\\ &\quad\le C\|g\|_D(1+\ln_+(\alpha+\beta\|f'\|_{L^\infty}y))\,\|f'\|_{L^\infty}+C\|g\|_{L^\infty}\,\|f'\|_D\int_0^1(1+\ln(1+\theta\|f'\|_{L^\infty}y))\,d\theta.\end{aligned}$$

Since

$$\ln\Bigl(1+\prod_{i=1}^n x_i\Bigr)\le\sum_{i=1}^n\ln(1+x_i)\quad\text{for all } x_i\ge 0,$$

we have

$$\|\mathcal{N}_1(\cdot,y)\|_D\le C\|g\|_D\,\|f'\|_D\,(1+\ln_+\|f'\|_{L^\infty}+\ln_+ y). \tag{5-17}$$

On the other hand it is clear that

$$\|\mathcal{N}_1(\cdot,y)\|_{L^\infty}\le C\|g\|_{L^\infty}\|f'\|_{L^\infty}. \tag{5-18}$$

To estimate $1/\mathcal{D}_1(\cdot,y)$ in the Dini space $C_K^\star$ we come back to the definition, which implies

$$\|1/\mathcal{D}_1(\cdot,y)\|_D\le\|\mathcal{D}_1(\cdot,y)\|_D\,\|1/\mathcal{D}_1(\cdot,y)\|_{L^\infty}^2. \tag{5-19}$$

Now using the product law (4-4) we deduce that

$$\|\mathcal{D}_1(\cdot,y)\|_D\le\|\varphi(\cdot,y)\|_{L^\infty}\,\|\varphi(\cdot,-y)\|_D+\|\varphi(\cdot,y)\|_D\,\|\varphi(\cdot,-y)\|_{L^\infty}.$$

From simple calculations we get

$$\|\varphi(\cdot,\pm y)\|_{L^\infty}\le y^2+(2+y\|f'\|_{L^\infty})^2\le C(1+\|f'\|_{L^\infty}^2)(1+y^2).$$

Applying (4-4) and (4-7) to the expression of φ it is quite easy to check that

$$\begin{aligned}\|\varphi(\cdot,\pm y)\|_D&\le C(1+y\|f'\|_{L^\infty})y\int_0^1\|f'\circ(\mathrm{Id}\pm\theta yf)\|_D\,d\theta\\ &\le C(y+y^2\|f'\|_{L^\infty})\,\|f'\|_D\,(1+\ln_+\|f'\|_{L^\infty}+\ln_+ y).\end{aligned}$$

Thus combining the preceding estimates we find

$$\begin{aligned}\|\mathcal{D}_1(\cdot\,,y))\|_D &\le C(y+y^2\|f'\|_{L^\infty})\,\|f'\|_D\,(1+\ln_+\|f'\|_{L^\infty}+\ln_+ y)(1+\|f'\|_{L^\infty}^2)(1+y^2)\\ &\le C(1+y^4\ln_+ y)\,\|f'\|_D\,(1+\ln_+\|f'\|_{L^\infty})(1+\|f'\|_{L^\infty}^3).\end{aligned}\tag{5-20}$$

Now we shall use the following inequalities, which can be proved in a straightforward way: for any $y\in\mathbb{R}_+$ and for any $a,b\in\mathbb{R}$ with $|a|\le b$, one has

$$y^2+(2+ya)^2\ge y^2+(2-ya)^2\ge\frac{1+y^2}{1+a^2}\ge\frac{1+y^2}{1+b^2}.\tag{5-21}$$

It follows that

$$\|1/\varphi(\cdot\,,\pm y)\|_{L^\infty}\le\frac{1+\|f'\|_{L^\infty}^2}{1+y^2}.\tag{5-22}$$

Putting this estimate together with (5-20) and (5-19) yields

$$\begin{aligned}\|1/\mathcal{D}_1(\cdot\,,y)\|_D &\le C\,\frac{1+y^4\ln_+ y}{1+y^8}\,\|f'\|_D(1+\ln_+\|f'\|_{L^\infty})(1+\|f'\|_{L^\infty}^{11})\\ &\le C\,\frac{1+\ln_+ y}{1+y^4}\,\|f'\|_D(1+\|f'\|_{L^\infty}^{12}).\end{aligned}$$

Therefore we obtain using (5-17), (5-18) and (5-22)

$$\begin{aligned}\|(\mathcal{N}_1/\mathcal{D}_1)(\cdot\,,y)\|_D &\le \|(\mathcal{N}_1(\cdot\,,y)\|_{L^\infty}\,\|1/\mathcal{D}_1)(\cdot\,,y)\|_D+\|(\mathcal{N}_1(\cdot\,,y)\|_D\,\|1/\mathcal{D}_1)(\cdot\,,y)\|_{L^\infty}\\ &\le C\,\|g\|_{L^\infty}\,\|f'\|_{L^\infty}\frac{1+\ln_+ y}{1+y^4}\,\|f'\|_D(1+\|f'\|_{L^\infty}^{12})\\ &\qquad\qquad + C\,\|g\|_D\,\|f'\|_D\,\frac{1+\ln_+\|f'\|_{L^\infty}+\ln_+ y}{1+y^4}(1+\|f'\|_{L^\infty}^4)\\ &\le C\,\|g\|_D\,\|f'\|_D\,\frac{1+\ln_+ y}{1+y^4}(1+\|f'\|_{L^\infty}^{13}).\end{aligned}$$

Plugging this estimate into (5-16) we find

$$\|T_f^{2,1}g\|_D\le 4\int_0^\infty y^2\|(\mathcal{N}_1/\mathcal{D}_1)(\cdot\,,y)\|_D\,dy\le C\,\|g\|_D\,\|f'\|_D(1+\|f'\|_{L^\infty}^{13}).\tag{5-23}$$

This concludes the proof of (5-14).

Now we intend to estimate $\|T_f^1 g\|_D$, which is trickier. Let $r\in(0,1)$ and $x_1,x_2\in\mathbb{R}$ such that $|x_1-x_2|\le r$. We shall decompose $T_f^1 g$ as follows:

$$T_f^1 g=T_{f,\mathrm{int}}^{r,1}g+T_{f,\mathrm{ext}}^{r,1}g,\tag{5-24}$$

with

$$\begin{aligned}T_{f,\mathrm{int}}^{r,1}g(x)&=\int_0^r\frac{y[g(\alpha x+\beta y)-g(\alpha x-\beta y)]}{y^2+[f(x)+f(x+y)]^2}\,dy,\\ T_{f,\mathrm{ext}}^{r,1}g(x)&=\int_r^\infty\frac{y[g(\alpha x+\beta y)-g(\alpha x-\beta y)]}{y^2+[f(x)+f(x+y)]^2}dy.\end{aligned}$$

From the subadditivity of the modulus of continuity we get

$$|f'(x)T^{r,1}_{f,\mathrm{int}}\,g(x)| \le C|f'(x)|\int_0^r \frac{y\omega_g(y)}{y^2+[f(x)]^2}\,dy \le C|f'(x)|\int_0^r \frac{\omega_g(y)}{y+f(x)}\,dy.$$

Using Lemma 4.1 we find

$$|f'(x)T^{r,1}_{f,\mathrm{int}}\,g(x)| \le C\frac{\gamma(f)}{1+\ln_+(1/f(x))}\int_0^r \frac{\omega_g(y)}{y+f(x)}\,dy, \tag{5-25}$$

where

$$\gamma(f) \triangleq \|f'\|_D(1+\ln_+(1/\|f'\|_D)). \tag{5-26}$$

Now we claim that for $y\in(0,1)$

$$\sup_{\varepsilon>0}\frac{1}{1+\ln_+(1/\varepsilon)}\frac{1}{y+\varepsilon} \le \frac{C}{y(1+|\ln y|)}+\frac{1}{1+y} \tag{5-27}$$

for some universal constant $C>0$. To prove this result it is enough to get

$$\sup_{\varepsilon\in(0,1)}\frac{1}{1+\ln(1/\varepsilon)}\frac{1}{y+\varepsilon} \le \frac{C}{y(1+|\ln y|)}.$$

Indeed, we shall consider the two cases $\varepsilon\ge\sqrt{y}$ and $\varepsilon\le\sqrt{y}$. In the first case we observe

$$\frac{1}{y+\varepsilon}\le\frac{1}{\sqrt{y}} \quad\text{and}\quad \frac{1}{1+\ln(1/\varepsilon)}\le 1,$$

which implies

$$\frac{1}{1+\ln(1/\varepsilon)}\frac{1}{y+\varepsilon}\le\frac{1}{\sqrt{y}}\le\frac{C}{y(1+|\ln y|)}.$$

However in the second case $\varepsilon\le\sqrt{y}$ we write simply that

$$\frac{1}{y+\varepsilon}\le\frac{1}{y} \quad\text{and}\quad \frac{1}{1+\ln(1/\varepsilon)}\le\frac{1}{1+\frac{1}{2}\ln(1/y)},$$

which gives the desired result. Coming back to (5-25) and using (5-27) we deduce that

$$\begin{aligned}\sup_x|f'(x)T^{r,1}_{f,\mathrm{int}}\,g(x)| &\le C\gamma(f)\int_0^r \sup_x\frac{\omega_g(y)}{(1+\ln_+(1/f(x)))(y+f(x))}\,dy\\ &\le C\gamma(f)\bigg(\int_0^r\frac{\omega_g(y)}{y(1+|\ln y|)}\,dy+\int_0^r\frac{\omega_g(y)}{1+y}\,dy\bigg).\end{aligned} \tag{5-28}$$

Consequently

$$\sup_{|x_1-x_2|\le r}\big|f'(x_1)T^{r,1}_{f,\mathrm{int}}\,g(x_1)-f'(x_2)T^{r,1}_{f,\mathrm{int}}\,g(x_2)\big| \le C\gamma(f)\bigg(\int_0^r\frac{\omega_g(y)}{y(1+|\ln y|)}\,dy+\int_0^r\omega_g(y)\,dy\bigg).$$

Therefore we get by using Fubini's theorem

$$\int_0^1 \sup_{|x_1-x_2|\le r} |f'(x_1)T^{r,1}_{f,\mathrm{int}}\, g(x_1)-f'(x_2)T^{r,1}_{f,\mathrm{int}}\, g(x_2)|\,\frac{dr}{r}$$
$$\le C\gamma(f)\int_0^1 \frac{\omega_g(y)}{y}\frac{|\ln y|}{(1+|\ln y|)}\,dy + C\gamma(f)\int_0^1 |\ln y|\omega_g(y)\,dy$$
$$\le C\gamma(f)\|g\|_D.$$

As for $T^{r,1}_{f,\mathrm{ext}}g$, we write

$$\begin{aligned} f'(x_1)T^{r,1}_{f,\mathrm{ext}}g(x_1) - f'(x_2)T^{r,1}_{f,\mathrm{ext}}g(x_2) &= (f'(x_1)-f'(x_2))T^{r,1}_{f,\mathrm{ext}}g(x_2) + f'(x_1)(T^{r,1}_{f,\mathrm{ext}}g(x_1) - T^{r,1}_{f,\mathrm{ext}}g(x_2)) \\ &\triangleq \mu_1(x_1,x_2)+\mu_2(x_1,x_2). \end{aligned} \tag{5-29}$$

Our current goal is to prove that for $j\in\{1,2\}$

$$\int_0^1 \sup_{|x_1-x_2|\le r}\frac{\mu_j(x_1,x_2)}{r}\,dr$$

is well-estimated. For the first term we use (5-9) leading to

$$\int_0^1 \sup_{|x_1-x_2|\le r}\frac{\mu_1(x_1,x_2)}{r}\,dr \le \|T^{r,1}_{f,\mathrm{ext}}g\|_{L^\infty}\int_0^1 \frac{\omega_{f'}(r)}{r}\,dr \le C\|g\|_D\|f'\|_D.$$

The second term is subtler. First note that if $|x_1-x_2|\le 1$ then the quantity

$$f'(x_1)T^{r,1}_{f,\mathrm{ext}}g(x_1) - f'(x_2)T^{r,1}_{f,\mathrm{ext}}g(x_2)$$

vanishes for x_1,x_2 outside a compact set related only to the support of f. Therefore the integrals defining $\mu_2(x_1,x_2)$ may be restricted to the set $\{\beta r\le\beta y\le B\}$, with B being some constant related to the size of the supports of f and g, and without loss of generality we can take $B=1$. It follows that

$$\begin{aligned} \mu_2(x_1,x_2) &= f'(x_1)\int_{\{\beta r\le\beta y\le 1\}} \frac{y\,[\hat g(x_1,y)-\hat g(x_2,y)]}{y^2+[f(x_1)+f(x_1+y)]^2}\,dy \\ &\quad + f'(x_1)\int_{\{\beta r\le\beta y\le 1\}} \frac{y\,\hat g(x_2,y)[\Delta_y^+ f(x_2)-\Delta_y^+ f(x_1)]\,[\Delta_y^+ f(x_2)+\Delta_y^+ f(x_1]}{(y^2+[\Delta_y^+ f(x_1)]^2)(y^2+[\Delta_y^+ f(x_2)]^2)}\,dy \\ &\triangleq \mu_{2,1}(x_1,x_2)+\mu_{2,2}(x_1,x_2), \end{aligned} \tag{5-30}$$

with

$$\hat g(x,y)\triangleq g(\alpha x+\beta y)-g(\alpha x-\beta y) \quad\text{and}\quad \Delta_y^+ f(x) = f(x+y)+f(x).$$

To estimate $\mu_{2,1}$ we shall use the following inequality, which is a consequence of Lemma 4.1:

$$\int_0^L \frac{|f'(x)|}{y+f(x)}\,dy = |f'(x)|\ln\Bigl(1+\frac{L}{f(x)}\Bigr) \le C\gamma(f)(1+\ln_+ L),$$

with C an absolute constant. This implies

$$\mu_{2,1}(x_1,x_2) \le C\omega_g(\alpha|x_1-x_2|)|f'(x_1)|\int_0^{\frac{1}{\beta}} \frac{1}{y+f(x_1)}\,dy$$
$$\le C\omega_g(|x_1-x_2|)\gamma(f)(1+|\ln\beta|).$$

Consequently, we find that

$$\sup_{|x_1-x_2|\le r} |\mu_{2,1}(x_1,x_2)| \le C\omega_g(r)\gamma(f)(1+|\ln\beta|)$$

and therefore

$$\int_0^1 \sup_{|x_1-x_2|\le r} |\mu_{2,1}(x_1,x_2)|\,\frac{dr}{r} \le C\gamma(f)(1+|\ln\beta|)\|g\|_D.$$

We emphasize that for $\beta=0$ one can still get an estimate since $\mu_{2,1}(x_1,x_2)=0$ and therefore we get the desired estimate.

Now we shall move to the estimate of $\mu_{2,2}(x_1,x_2)$. We start with using the estimate

$$\sup_{a>0} \frac{a}{y^2+a^2} \le \frac{1}{2|y|},$$

which implies

$$\frac{y|\hat{g}(x_2,y)||\Delta_y^+ f(x_2)-\Delta_y^+ f(x_1)||\Delta_y^+ f(x_2)+\Delta_y^+ f(x_1|}{(y^2+[\Delta_y^+ f(x_1)]^2)(y^2+[\Delta_y^+ f(x_2)]^2)} \le C|x_2-x_1|\|f'\|_{L^\infty}\frac{\omega_g(2\beta y)}{y^2}.$$

Thus

$$\sup_{|x_1-x_2|\le r} \mu_{2,2}(x_1,x_2) \le Cr\|f'\|_{L^\infty}^2 \int_r^{\frac{1}{\beta}} \frac{\omega_g(2\beta y)}{y^2}\,dy,$$

which yields in view of Fubini's theorem

$$\int_0^1 \sup_{|x_1-x_2|\le r} \mu_{2,2}(x_1,x_2)\,\frac{dr}{r} \le C\|f'\|_{L^\infty}^2 \int_0^1\int_{\{\beta r\le\beta y\le 1\}} \frac{\omega_g(2\beta y)}{y^2}\,dy\,dr$$
$$\le C\|f'\|_{L^\infty}^2 \int_{\{0\le\beta y\le 1\}} \frac{\omega_g(2\beta y)}{y}\,dy$$
$$\le C\|f'\|_{L^\infty}^2 \int_0^2 \frac{\omega_g(y)}{y}\,dy$$
$$\le C\|f'\|_{L^\infty}^2\|g\|_D.$$

Note that the last constant does not depend on β. Putting together the preceding estimates we find that

$$\|f'T_f^1 g\|_D \le C\|g\|_D((1+|\ln\beta|)\gamma(f)+\|f'\|_{L^\infty}^2), \tag{5-31}$$

where $\gamma(f)$ was defined in (5-26). As noted before, the case $\beta=0$ has a special structure and one gets

$$\|f'T_f^1 g\|_D \le C\|g\|_D(\gamma(f)+\|f'\|_{L^\infty}^2).$$

Now let us move to the estimate of $f'(x)T_f^{2,2}g$ given by

$$T_f^{2,2}g(x)=\int_0^\infty \frac{y\,g(\alpha x-\beta y)[f(x-y)-f(x+y)]\psi(x,y)}{(y^2+[f(x)+f(x+y)]^2)(y^2+[f(x)+f(x-y)]^2)}\,dy$$
$$=T_{f,\mathrm{int}}^{r,2,2}g(x)+T_{f,\mathrm{ext}}^{r,2,2}g(x), \tag{5-32}$$

where

$$\psi(x,y)=y\int_0^1[f'(x+\theta y)-f'(x-\theta y)]\,d\theta$$

and the cut-off operators are given by

$$T_{f,\mathrm{int}}^{r,2,2}g(x)\triangleq\int_0^r \frac{y\,g(\alpha x-\beta y)[f(x-y)-f(x+y)]\psi(x,y)}{(y^2+[\Delta_y^+f(x)]^2)(y^2+[\Delta_{-y}^+f(x)]^2)}\,dy$$

and

$$T_{f,\mathrm{ext}}^{r,2,2}g(x)=\int_r^1 \frac{y\,g(\alpha x-\beta y)[f(x-y)-f(x+y)]\psi(x,y)}{(y^2+[\Delta_y^+f(x)]^2)(y^2+[\Delta_{-y}^+f(x)]^2)}\,dy\triangleq\int_r^1\frac{\mathcal{N}(x,y)}{\mathcal{D}(x,y)}\,dy. \tag{5-33}$$

We shall proceed in a similar way to $T_f^1 g$. Let us start with $f'(x)T_{f,\mathrm{int}}^{r,2,2}g$. Since

$$|\psi(x,y)|\le 2y\omega_{f'}(y), \tag{5-34}$$

one has

$$|f'(x)T_{f,\mathrm{int}}^{r,2,2}g(x)|\le C\|g\|_{L^\infty}\|f'\|_{L^\infty}|f'(x)|\int_0^r\frac{y^3\omega_{f'}(y)}{(y^2+[f(x)]^2)^2}\,dy$$
$$\le C\|g\|_{L^\infty}\|f'\|_{L^\infty}|f'(x)|\int_0^r\frac{\omega_{f'}(y)}{y+f(x)}\,dy.$$

Thus following the same steps as for (5-28) we obtain

$$\sup_{|x_1-x_2|\le r}|f'(x_1)T_{f,\mathrm{int}}^{r,2,2}g(x_1)-f'(x_2)T_{f,\mathrm{int}}^{r,2,2}g(x_2)|$$
$$\le C\|g\|_{L^\infty}\|f'\|_{L^\infty}\gamma(f)\int_0^r\frac{\omega_{f'}(y)}{y(1+|\ln y|)}\,dy+C\|g\|_{L^\infty}\|f'\|_{L^\infty}\gamma(f)\int_0^r|\omega_{f'}(y)\,dy.$$

Therefore Fubini's theorem and (4-2) imply

$$\int_0^1\sup_{|x_1-x_2|\le r}|f'(x_1)T_{f,\mathrm{int}}^{r,2,2}g(x_1)-f'(x_2)T_{f,\mathrm{int}}^{r,2,2}g(x_2)|\,\frac{dr}{r}\le C\|g\|_{L^\infty}\|f'\|_D^2\gamma(f).$$

What is left is to estimate the quantity $f'(x)T_{f,\mathrm{ext}}^{r,2,2}g$. First, it is obvious that

$$f'(x_1)T_{f,\mathrm{ext}}^{r,2,2}g(x_1)-f'(x_2)T_{f,\mathrm{ext}}^{r,2,2}g(x_2)$$
$$=(f'(x_1)-f'(x_2))T_{f,\mathrm{ext}}^{r,2,2}(x_2)+f'(x_1)(T_{f,\mathrm{ext}}^{r,2,2}(x_1)-T_{f,\mathrm{ext}}^{r,2,2}(x_2)). \tag{5-35}$$

The first term of the right-hand side is easy to estimate. Indeed,

$$|(f'(x_1)-f'(x_2))T_{f,\mathrm{ext}}^{r,2,2}(x_2)|\le\omega_{f'}(|x_1-x_2|)\|T_{f,\mathrm{ext}}^{r,2,2}\|_{L^\infty}.$$

It is clear that

$$|T^{r,2,2}_{f,\text{ext}} g(x)| \le C\|g\|_{L^\infty}\|f'\|_{L^\infty}\int_r^1 \frac{\omega_{f'}(y)}{y}\,dy \le C\|g\|_{L^\infty}\|f'\|_D^2.$$

Hence

$$\int_0^1 \sup_{|x_1-x_2|\le r} |(f'(x_1)-f'(x_2))T^{r,2,2}_{f,\text{ext}}(x_2)|\,\frac{dr}{r} \le C\|g\|_{L^\infty}\|f'\|_D^3.$$

To deal with the second term we proceed as for the term $\mu_2(x_1,x_2)$ in (5-30). From (5-33) one has

$$\begin{aligned} f'(x_1)(T^{r,2,2}_{f,\text{ext}}(x_1) - T^{r,2,2}_{f,\text{ext}}(x_2)) \\ = f'(x_1)\int_r^1 \frac{\mathcal{N}(x_1,y)-\mathcal{N}(x_2,y)}{\mathcal{D}(x_1,y)}\,dy + f'(x_1)\int_r^1 \frac{\mathcal{N}(x_2,y)(\mathcal{D}(x_2,y)-\mathcal{D}(x_1,y))}{\mathcal{D}(x_1,y)\mathcal{D}(x_2,y)}\,dy. \end{aligned} \tag{5-36}$$

It is quite obvious from some straightforward computations using in particular (5-34) that for $|x_1-x_2|\le r$

$$|\mathcal{N}(x_1,y)-\mathcal{N}(x_2,y)| \le C\|f'\|_{L^\infty} y^2(\omega_g(\alpha r)\omega_{f'}(y)y + \|g\|_{L^\infty}\omega_{f'}(r)y + \|g\|_{L^\infty} r\,\omega_{f'}(y)).$$

Since

$$\frac{1}{\mathcal{D}(x,y)} \le \frac{C}{[y+f(x)]^4} \le \frac{C}{y^4},$$

we get

$$\frac{|\mathcal{N}(x_1,y)-\mathcal{N}(x_2,y)|}{\mathcal{D}(x_1,y)} \le C\|f'\|_{L^\infty}\biggl[\omega_g(\alpha r)\frac{\omega_{f'}(y)}{y} + \|g\|_{L^\infty}\frac{\omega_{f'}(r)}{y+f(x_1)} + \|g\|_{L^\infty} r\,\frac{\omega_{f'}(y)}{y^2}\biggr].$$

This gives, in view of (4-2),

$$\begin{aligned} |f'(x_1)|\int_r^1 \frac{|\mathcal{N}(x_1,y)-\mathcal{N}(x_2,y)|}{\mathcal{D}(x_1,y)}\,dy \\ \le C\|f'\|_D\biggl[\|f'\|_D^2\omega_g(\alpha r) + \|g\|_D\omega_{f'}(r)\int_0^1 \frac{|f'(x_1)|}{y+f(x_1)}\,dy\biggr] + \|g\|_D\|f'\|_D^2\, r\int_r^1 \frac{\omega_{f'}(y)}{y^2}\,dy, \end{aligned} \tag{5-37}$$

which implies according to (5-31)

$$\int_0^1 \sup_{|x_1-x_2|\le r} |f'(x_1)|\int_r^1 \frac{|\mathcal{N}(x_1,y)-\mathcal{N}(x_2,y)|}{\mathcal{D}(x_1,y)}\,dy\,\frac{dr}{r} \le C(\|f'\|_D^3 + \|f'\|_D^2\gamma(f))\|g\|_D.$$

Now straightforward computations show that

$$\frac{|\mathcal{N}(x_2,y)(\mathcal{D}(x_2,y)-\mathcal{D}(x_1,y))|}{\mathcal{D}(x_1,y)\mathcal{D}(x_2,y)} \le C\|g\|_{L^\infty}\|f'\|_{L^\infty}^2|x_1-x_2|\frac{\omega_{f'}(y)}{y^2}. \tag{5-38}$$

Therefore using Fubini's theorem we get

$$\int_0^1 \sup_{|x_1-x_2|\le r} |f'(x_1)|\int_r^1 \frac{|\mathcal{N}(x_2,y)(\mathcal{D}(x_2,y)-\mathcal{D}(x_1,y))|}{\mathcal{D}(x_1,y)\mathcal{D}(x_2,y)}\,dy\,\frac{dr}{r} \le C\|f'\|_D^4\|g\|_D.$$

Putting together the preceding estimates we find that

$$\begin{aligned}\|f' T_f^{2,2} g\|_D &\le C\|g\|_D(\|f'\|_D^2 + \|f'\|_D^2\gamma(f) + \|f'\|_D^3)\\ &\le C\|g\|_D(\|f'\|_D^2 + \|f'\|_D^4),\end{aligned} \tag{5-39}$$

with C a constant depending only on the diameter of the compact K. To get the desired estimate it suffices to put together (5-15), (5-31) and (5-39).

(3) We shall proceed as in the proof of part (2) of Theorem 5.2. We use exactly the same splitting with similar estimates and to avoid redundancy we shall only give the basic estimates with some details for the terms that require new treatment. We use the decomposition described in (5-13). To estimate $T_f^{2,1} g$ in C^s we use the expression (5-16). Then following the same lines using in particular the product law (4-8) and the composition law (4-6), one has

$$\|\mathcal{N}_1(\cdot,y)\|_s \le C\|g\|_s(\alpha^s + \beta^s\|f'\|_{L^\infty}^s y^s)\|f'\|_{L^\infty} + C\|g\|_{L^\infty}\|f'\|_s \int_0^1 (1+\theta^s\|f'\|_{L^\infty}^s y^s)\,d\theta.$$

Since $\alpha,\beta \in [0,1]$ we deduce

$$\|\mathcal{N}_1(\cdot,y)\|_s \le C(\|g\|_s\|f'\|_{L^\infty} + \|g\|_{L^\infty}\|f'\|_s)(1+\|f'\|_{L^\infty}^s y^s).$$

Similarly we get

$$\begin{aligned}\|\varphi(\cdot,\pm y)\|_s &\le C(1+y\|f'\|_{L^\infty})y\int_0^1 \|f'\circ(\mathrm{Id}\pm\theta y f)\|_s\,d\theta\\ &\le C(y+y^2\|f'\|_{L^\infty})\|f'\|_s(1+\|f'\|_{L^\infty}^s y^s).\end{aligned}$$

This implies

$$\|\mathcal{D}_1(\cdot,y)\|_s \le C(1+y^{4+s})(1+\|f'\|_{L^\infty}^{3+s})\|f'\|_s$$

and

$$\|1/\mathcal{D}_1(\cdot,y)\|_s \le \frac{C}{1+y^{4-s}}(1+\|f'\|_{L^\infty}^{11+s})\|f'\|_s.$$

Consequently for $s \in (0,1)$

$$\begin{aligned}\|(\mathcal{N}_1/\mathcal{D}_1)(\cdot,y)\|_s &\le \|(\mathcal{N}_1(\cdot,y)\|_{L^\infty}\|1/\mathcal{D}_1)(\cdot,y)\|_s + \|\mathcal{N}_1(\cdot,y)\|_s\|1/\mathcal{D}_1(\cdot,y)\|_{L^\infty}\\ &\le \frac{C}{1+y^{4-s}}(1+\|f'\|_{L^\infty}^{11+s})\|f'\|_s\|g\|_s.\end{aligned}$$

Therefore we get similarly to (5-23)

$$\begin{aligned}\|T_f^{2,1} g\|_s &\le C(1+\|f'\|_{L^\infty}^{11+s})\|f'\|_s\|g\|_s\int_0^\infty \frac{y^2}{1+y^{4-s}}\,ds\\ &\le C(1+\|f'\|_{L^\infty}^{11+s})\|f'\|_s\|g\|_s.\end{aligned}$$

Combining product laws with Sobolev embeddings and (5-11) we get

$$\begin{aligned}\|f' T_f^{2,1} g\|_s &\le \|f'\|_{L^\infty}\|T_f^{2,1} g\|_s + \|f'\|_s\|T_f^{2,1} g\|_{L^\infty}\\ &\le C(1+\|f'\|_{L^\infty}^{11+s})\|f'\|_s\|f'\|_{L^\infty}\|g\|_s + \|g\|_{L^\infty}\|f'\|_D\|f'\|_s\\ &\le C(1+\|f'\|_{L^\infty}^{11+s})\|f'\|_s\|f'\|_D\|g\|_s.\end{aligned}$$

Using once again Sobolev embeddings we get

$$\|f'T_f^{2,1}g\|_s \le C(\|f'\|_s + \|f'\|_s^{13})\|f'\|_D\|g\|_s. \tag{5-40}$$

Now to estimate $T_f^1 g$ we come back to the decomposition (5-24) and we easily get

$$\|T_{f,\mathrm{int}}^{r,1}g\|_{L^\infty} \le C\|g\|_s \int_0^r y^{-1+s}\,dy \le \|g\|_s r^s.$$

Hence we obtain, since $r = |x_1 - x_2|$,

$$|T_{f,\mathrm{int}}^{r,1}g(x_1) - T_{f,\mathrm{int}}^{r,1}g(x_2)| \le C\|g\|_s|x_1-x_2|^s.$$

and we also get

$$|f'(x_1)T_{f,\mathrm{int}}^{r,1}g(x_1) - f'(x_2)T_{f,\mathrm{int}}^{r,1}g(x_2)| \le C\|f'\|_{L^\infty}\|g\|_s|x_1-x_2|^s.$$

To estimate the term $f'T_{f,\mathrm{ext}}^{r,1}g$ we come back to (5-29) and (5-30) and following the same estimates one gets

$$\begin{aligned}|\mu_1(x_1,x_2)| &\le |x_1-x_2|^s\|f'\|_s\|T_{f,\mathrm{ext}}^{r,1}g\|_{L^\infty}\\ &\le C|x_1-x_2|^s\|f'\|_s\|g\|_D.\end{aligned}$$

Moreover

$$|\mu_2(x_1,x_2)| \le |\mu_{2,1}(x_1,x_2)| + |\mu_{2,2}(x_1,x_2)|$$

and

$$\begin{aligned}|\mu_{2,2}(x_1,x_2)| &\le C|x_2-x_1|\,\|f'\|_{L^\infty}^2\|g\|_s\int_{\{\beta r\le\beta y\le 1\}}(\beta y)^s y^{-2}\,dy\\ &\le C\|f'\|_{L^\infty}^2\|g\|_s|x_1-x_2|^s.\end{aligned}$$

To deal with the term $\mu_{2,1}(x_1,x_2)$ in (5-30) one obtains in view of (5-31)

$$\begin{aligned}|\mu_{2,1}(x_1,x_2)| &\le |x_1-x_2|^s\|g\|_s|f'(x_1)|\int_{\{\beta r\le\beta y\le 1\}}\frac{y}{y^2+f^2(x_1)}\,dy\\ &\le |x_1-x_2|^s\|g\|_s|f'(x_1)|\int_0^{\frac{1}{\beta}}\frac{1}{y+f(x_1)}\,dy.\end{aligned}$$

Using the second part of Lemma 4.1 one finds for $s' \in (0,s]$

$$|f'(x_1)|\int_0^{\frac{1}{\beta}}\frac{1}{y+f(x_1)}\,dy \le C\|f'\|_{s'}^{\frac{1}{1+s'}}|f(x_1)|^{\frac{s'}{1+s'}}\int_0^{\frac{1}{\beta}}\frac{1}{y+f(x_1)}\,dy.$$

Combining this inequality with

$$\sup_{a>0}\frac{a^{\frac{s'}{1+s'}}}{y+a} \le Cy^{-\frac{1}{1+s'}}$$

we get

$$\sup_{x_1\in\mathbb{R}}|f'(x_1)|\int_0^{\frac{1}{\beta}}\frac{1}{y+f(x_1)}\,dy \le C\|f'\|_{s'}^{\frac{1}{1+s'}}\beta^{-\frac{s'}{1+s'}} \tag{5-41}$$

and therefore

$$|\mu_{2,1}(x_1,x_2)| \le |x_1-x_2|^s \, \|g\|_s \, \|f'\|_{s'}^{\frac{1}{1+s'}} \beta^{-\frac{s'}{1+s'}}.$$

Hence

$$\begin{aligned}|f'(x_1)T^{r,1}_{f,\mathrm{ext}}g(x_1) - f'(x_2)T^{r,1}_{f,\mathrm{ext}}g(x_2)|\\ \le C\|g\|_D \, \|f'\|_s |x_1-x_2|^s + C\|f'\|^2_{L^\infty} \|g\|_s |x_1-x_2|^s + C|x_1-x_2|^s \, \|g\|_s \, \|f'\|_{s'}^{\frac{1}{1+s'}} \beta^{-\frac{s'}{1+s'}}.\end{aligned}$$

It follows that

$$\|f'T^1_f g\|_s \le C\|g\|_s(\|f'\|_{s'}^{\frac{1}{1+s'}} \beta^{-\frac{s'}{1+s'}} + \|f'\|^2_{L^\infty}) + C\|g\|_D \|f'\|_s. \tag{5-42}$$

It remains to estimate $f'T^{2,2}_f g$ described in (5-32) and (5-33). First one may write

$$\begin{aligned}|T^{r,2,2}_{f,\mathrm{int}} g(x)| &\le C\|g\|_{L^\infty} \|f'\|_{L^\infty} \|f'\|_s \int_0^r y^{s-1}\, dy\\ &\le C\|g\|_{L^\infty} \|f'\|_{L^\infty} \|f'\|_s |x_1-x_2|^s.\end{aligned}$$

Therefore

$$|f'(x_1)T^{r,2,2}_{f,\mathrm{int}} g(x_1) - f'(x_2)T^{r,2,2}_{f,\mathrm{int}} g(x_2)| \le C\|g\|_{L^\infty} \|f'\|^2_{L^\infty} \|f'\|_s |x_1-x_2|^s.$$

By Sobolev embeddings we get

$$|f'(x_1)T^{r,2,2}_{f,\mathrm{int}} g(x_1) - f'(x_2)T^{r,2,2}_{f,\mathrm{int}} g(x_2)| \le C\|g\|_s \|f'\|_{L^\infty} \|f'\|^2_s |x_1-x_2|^s. \tag{5-43}$$

From (5-35) and the analysis following that identity one has

$$\begin{aligned}|(f'(x_1)-f'(x_2))T^{r,2,2}_{f,\mathrm{ext}}(x_2)| &\le \|f'\|_s \|T^{r,2,2}_{f,\mathrm{ext}} g\|_{L^\infty} |x_1-x_2|^s\\ &\le C\|g\|_{L^\infty} \|f'\|^2_s \|f'\|_{L^\infty} |x_1-x_2|^s\end{aligned}$$

Using (5-36), (5-37) and (5-41) (with $s'=s$) combined with Sobolev embeddings, one deduces

$$|f'(x_1)| \int_r^1 \frac{|\mathcal{N}(x_1,y)-\mathcal{N}(x_2,y)|}{\mathcal{D}(x_1,y)}\, dy \le C\|f'\|_{L^\infty} \|g\|_s (\|f'\|^2_s + \|f'\|_s).$$

From (5-38) we get

$$\frac{|\mathcal{N}(x_2,y)(\mathcal{D}(x_2,y)-\mathcal{D}(x_1,y))|}{\mathcal{D}(x_1,y)\mathcal{D}(x_2,y)} \le C\|g\|_{L^\infty} \|f'\|_{L^\infty} \|f'\|_s |x_1-x_2|\, y^{s-2}.$$

Therefore we get

$$|f'(x_1)| \int_r^1 \frac{|\mathcal{N}(x_2,y)(\mathcal{D}(x_2,y)-\mathcal{D}(x_1,y))|}{\mathcal{D}(x_1,y)\mathcal{D}(x_2,y)}\, dy \le C\|g\|_{L^\infty} \|f'\|^2_{L^\infty} \|f'\|_s |x_1-x_2|^s.$$

Hence plugging the preceding estimates into (5-35) and (5-36), we find

$$\begin{aligned}|f'(x_1)T^{r,2,2}_{f,\mathrm{ext}}(x_1)-f'(x_2)T^{r,2,2}_{f,\mathrm{ext}}(x_2)| &\le C\|g\|_{L^\infty}\|f'\|_s^2\|f'\|_{L^\infty}|x_1-x_2|^s\\ &\quad + C\|f'\|_{L^\infty}\|g\|_s(\|f'\|_s^2+\|f'\|_s)|x_1-x_2|^s\\ &\quad + C\|g\|_{L^\infty}\|f'\|^2_{L^\infty}\|f'\|_s|x_1-x_2|^s.\end{aligned}$$

Using standard embeddings we get

$$|f'(x_1)T^{r,2,2}_{f,\mathrm{ext}}(x_1)-f'(x_2)T^{r,2,2}_{f,\mathrm{ext}}(x_2)| \le C\|g\|_s\|f'\|_{L^\infty}|x_1-x_2|^s(\|f'\|_s+\|f'\|_s^2). \tag{5-44}$$

Putting together (5-43), (5-44) and (5-32) we obtain

$$\|f'T_f^{2,2}g\|_s \le C\|g\|_s\|f'\|_{L^\infty}(\|f'\|_s+\|f'\|_s^2). \tag{5-45}$$

Combining (5-40), (5-42) and (5-45) we get for any $s'\in(0,s]$

$$\|f'T_fg\|_s \le C\|g\|_s\|f'\|_D(\|f'\|_s+\|f'\|_s^{13}) + C\|g\|_s\|f'\|_{s'}^{\frac{1}{1+s'}}\beta^{-\frac{s'}{1+s'}} + C\|g\|_D\|f'\|_s.$$

Now using the embedding $C^s\hookrightarrow C^{s'}\hookrightarrow D$ we get

$$\|f'T_fg\|_s \le C\|g\|_s(\beta^{-\frac{s}{1+s}}\|f'\|_s^{\frac{1}{1+s}}+\|f'\|_s^{14}) \le C\|g\|_s(\beta^{-\frac12}\|f'\|_s^{\frac{1}{1+s}}+\|f'\|_s^{14}).$$

Another useful estimate that one can get from taking $s'=\frac{s}{2}$ and using the interpolation inequalities

$$\|f'\|_D \le C\|f'\|_s^{\frac{1+s}{2+s}} \le C\|f'\|_{L^\infty}^{\frac{1}{2+s}}\|f'\|_s^{\frac{1+s}{2+s}},\quad \|f'\|_{\frac{s}{2}} \le C\|f'\|_{L^\infty}^{\frac12}\|f'\|_s^{\frac12},\quad \beta^{-\frac{s}{2+s}}\le\beta^{-\frac12},$$

is the following:

$$\|f'T_fg\|_s \le C\|g\|_s\|f'\|_{L^\infty}^{\frac{1}{2+s}}(\|f'\|_s^{\frac{1}{2+s}}\beta^{-\frac12}+\|f'\|_s^{14}) + C\|g\|_{L^\infty}^{\frac{1}{2+s}}\|g\|_s^{\frac{1+s}{2+s}}\|f'\|_s.$$

This completes the proof of Theorem 5.2. □

6. Local well-posedness proof

The main objective of this section is to prove the local well-posedness result stated in the first part of Theorem 2.1. The approach that we shall follow is classical and will be done in several steps. We start with a priori estimates of smooth solutions in suitable Banach spaces and this will be the main concern of Sections 6A and 6B. The rigorous construction of classical solutions will be conducted in Section 6C.

6A. *Estimates of the source terms.* The main goal of this section is to establish the following a priori estimates for the source terms F and G described in (2-7) and (2-8).

Proposition 6.1. *Let K be a compact set of $\mathbb{R}$ and $s\in(0,1)$. We denote by X one of the spaces $C_K^\star$ or C_K^s. There exists a constant $C>0$ depending only on K such that the following estimates hold true*:

(1) *For any $f\in X$ we have*

$$\|F\|_{L^\infty} \le C\|f'\|_{L^\infty}\|f'\|_D,\quad \|F\|_X \le C\|f'\|_D(\|f'\|_X+\|f'\|_X^3).$$

(2) *For any* $f \in X$ *we have*

$$\|G\|_{L^\infty} \le C\|f'\|_{L^\infty}(1+\|f'\|_D^3), \quad \|G\|_X \le C(1+\|f'\|_D^{\frac{1}{3}})(\|f'\|_X+\|f'\|_X^{16}).$$

Proof. For simplicity throughout this proof we denote the operator Δ_y^- by Δ_y.

(1) The estimate of F in L^∞ is quite easy. Indeed, it is obvious according to (4-3) that

$$\begin{aligned}\|F\|_{L^\infty} &\le C\|f'\|_{L^\infty}\int_{-M}^{M}\sup_{x\in\mathbb{R}}\frac{|f'(x+y)-f'(x)|}{|y|}\,dy\\ &\le C\|f'\|_{L^\infty}\int_{-M}^{M}\frac{\omega_{f'}(|y|)}{|y|}\,dy\\ &\le C\|f'\|_{L^\infty}\|f'\|_D.\end{aligned}$$

Now let us move to the estimate of F in the function space X, which is the Dini space $C_K^\star$ or the Hölder space C_K^s. For this purpose we shall transform slightly F in order to apply Proposition 5.1. In fact from Taylor's formula one can write

$$F(x) = \int_{-M}^{M}\int_0^1 \frac{y\,\Delta_{\theta y}f'(x)\Delta_y f'(x)}{y^2+(\Delta_y f(x))^2}\,dy\,d\theta.$$

From the notation (5-2) one has

$$F(x) = \int_0^1 \mathcal{C}_f^{\theta,\Re}(f',f')(x)\,d\theta.$$

At this stage it suffices to apply Proposition 5.1, which implies

$$\|F\|_X \le C(\|f'\|_D\|f'\|_X + \|f'\|_{L^\infty}\|f'\|_D\|f'\|_X^2)$$

and gives in turn the desired result according to the embedding $X \hookrightarrow L^\infty$.

(2) The expression of G is given in (2-8) and for simplicity we shall assume throughout this part that $M=1$. We shall first split G as follows:

$$\begin{aligned}G(x) &= \text{p.v.}\int_{-1}^{1}\frac{[2f(x)+\Delta_y^- f(x)+yf'(x)](f'(x+y)+f'(x))}{y^2+(f(x+y)+f(x))^2}\,dy\\ &= 2f(x)\ \text{p.v.}\int_{-1}^{1}\frac{f'(x+y)+f'(x)}{y^2+(f(x+y)+f(x))^2}\,dy + \text{p.v.}\int_{-1}^{1}\frac{[\Delta_y^- f(x)+yf'(x)](2f'(x)+\Delta_y^- f'(x))}{y^2+(f(x+y)+f(x))^2}\,dy\\ &\triangleq G_1(x)+G_2(x).\end{aligned} \tag{6-1}$$

The estimate G_1 in L^∞ is quite easy. To see this we can first assume that $f(x)>0$; otherwise the integral is vanishing. Thus by change of variables we get

$$|G_1(x)| \le 4\|f'\|_{L^\infty}\int_{-1}^{1}\frac{|f(x)|}{y^2+f^2(x)}\,dy \le C\|f'\|_{L^\infty}.$$

Note that for $x \in \operatorname{supp} f$ we have $f(x+y)=0$ for all $y \notin [-1,1]$. Thus

$$\begin{aligned}G_1(x) &= 2f(x)\int_{\mathbb{R}} \frac{f'(x+y)+f'(x)}{y^2+(f(x+y)+f(x))^2}\,dy - 4f(x)f'(x)\int_1^{\infty} \frac{1}{y^2+(f(x))^2}\,dy \\ &= 2f(x)\int_{\mathbb{R}} \frac{f'(x+y)+f'(x)}{y^2+(f(x+y)+f(x))^2}\,dy - 4f'(x)\arctan(f(x)) \\ &\triangleq G_{11}+G_{12}. \end{aligned} \tag{6-2}$$

The estimate of G_{12} in L^∞ is elementary:

$$\|G_{12}\|_{L^\infty} \le 4\|f'\|_{L^\infty}\|f\|_{L^\infty}. \tag{6-3}$$

However, to estimate G_{12} in X we use the product law (4-3) leading to

$$\|f' \arctan f\|_X \le \|\arctan f\|_{L^\infty}\|f'\|_X + \|f'\|_{L^\infty}\|\arctan f\|_X.$$

It is easy to check from the mean value theorem that

$$\|\arctan f\|_{L^\infty} \le \|f\|_{L^\infty} \quad \text{and} \quad \omega_{\arctan f}(r) \le \omega_f(r),$$

which implies in view of the embedding $\mathrm{Lip} \hookrightarrow X$ that

$$\|\arctan f\|_X \le \|f\|_X \le C\|f'\|_{L^\infty}.$$

Therefore we obtain from the classical embeddings

$$\|G_{12}\|_X \le C(\|f\|_{L^\infty}\|f'\|_X + C\|f'\|_{L^\infty}^2) \le C\|f'\|_{L^\infty}\|f'\|_X. \tag{6-4}$$

We shall now estimate the term G_{11} in the space X. First we use Taylor's formula

$$f(x+y)+f(x) = 2f(x) + y\int_0^1 f'(x+\theta y)\,d\theta,$$

which implies after the change of variables $y = f(x)z$ (assuming that $f(x) > 0$)

$$\begin{aligned}G_{11}(x) &= 2f(x)\int_{\mathbb{R}} \frac{f'(x)+f'(x+y)}{y^2+[2f(x)+y\int_0^1 f'(x+\theta y)\,d\theta]^2}\,dy \\ &= 2\int_{\mathbb{R}} \frac{f'(x)+f'(x+f(x)z)}{\varphi(x,z)}\,dz, \end{aligned} \tag{6-5}$$

with

$$\varphi(x,z) = z^2 + \left(2 + z\int_0^1 f'(x+\theta f(x)z)\,d\theta\right)^2.$$

Note that for $f(x)=0$ we have from the definition $G_{11}(x)=0$, which agrees with the expression (6-5) because $f'(x)=0$. The estimate in L^∞ is easy to get in view of (5-22):

$$\|G_{11}\|_{L^\infty} \le 4\|f'\|_{L^\infty}\int_{\mathbb{R}} \|1/\varphi(\cdot,z)\|_{L^\infty}\,dz \le C(\|f'\|_{L^\infty} + \|f'\|_{L^\infty}^3).$$

From the product laws (4-4) and (4-8) we deduce that

$$\begin{aligned}\|G_{11}\|_X &= 2\int_{\mathbb{R}} \|f'+f'\circ(\mathrm{Id}+zf)\|_X \, \|1/\varphi(\cdot,z)\|_{L^\infty}\, dz + 2\int_{\mathbb{R}} \|f'+f'\circ(\mathrm{Id}+zf)\|_{L^\infty}\, \|1/\varphi(\cdot,z)\|_X \, dz\\ &\triangleq \ell_1+\ell_2.\end{aligned}$$

According to the product laws (4-6) and (4-7), one may write

$$\|f'+f'\circ(\mathrm{Id}+zf)\|_X \le \|f'\|_X(1+\mu(1+|z|\,\|f'\|_{L^\infty})),$$

with

$$\mu(r) \triangleq \begin{cases} \ln r & \text{if } X = C^\star_K,\\ r^s & \text{if } X = C^s.\end{cases}$$

Observe that we can unify both cases through the estimate

$$\begin{aligned}\|f'+f'\circ(\mathrm{Id}+zf)\|_X &\le C\|f'\|_X(1+(1+|z|\,\|f'\|_{L^\infty})^s)\\ &\le C\|f'\|_X(1+|z|^s\,\|f'\|^s_{L^\infty}).\end{aligned} \tag{6-6}$$

Putting together (6-6) and (5-22) we find for any $s\in(0,1)$

$$\begin{aligned}\ell_1 &\le C\|f'\|_X(1+\|f'\|^2_{L^\infty})\int_{\mathbb{R}} \frac{1+|z|^s\,\|f'\|^s_{L^\infty}}{1+z^2}\,dz\\ &\le C\|f'\|_X(1+\|f'\|^3_{L^\infty}).\end{aligned} \tag{6-7}$$

To estimate ℓ_2 we use the elementary estimate

$$\|f'+f'\circ(\mathrm{Id}+zf)\|_{L^\infty} \le 2\|f'\|_{L^\infty}.$$

Notice from the definition of the spaces X and (5-22) that one can deduce

$$\|1/\varphi(\cdot,z)\|_X \le \|1/\varphi(\cdot,z)\|^2_{L^\infty}\,\|\varphi(\cdot,z)\|_X \le C\,\frac{1+\|f'\|^4_{L^\infty}}{1+z^4}\,\|\varphi(\cdot,z)\|_X. \tag{6-8}$$

Moreover by the product laws we find

$$\|\varphi(\cdot,z)\|_X \le 2|z|(2+|z|\,\|f'\|_{L^\infty})\int_0^1 \|f'\circ(\mathrm{Id}+\theta zf)\|_X\,d\theta,$$

and this implies according to (6-6)

$$\begin{aligned}\|\varphi(\cdot,z)\|_X &\le C|z|(2+|z|\,\|f'\|_{L^\infty})\|f'\|_X(1+|z|^s\,\|f'\|^s_{L^\infty})\\ &\le C(1+|z|^{2+s})(1+\|f'\|^{1+s}_{L^\infty})\|f'\|_X.\end{aligned}$$

Putting together this estimate with (6-8) we find

$$\|1/\varphi(\cdot,z)\|_X \le C\,\frac{(1+\|f'\|^{5+s}_{L^\infty})\|f'\|_X}{1+|z|^{2-s}}. \tag{6-9}$$

Therefore we deduce that

$$\ell_2 \le C\|f'\|_{L^\infty}(1+\|f'\|^{5+s}_{L^\infty})\|f'\|_X \le C(1+\|f'\|^7_{L^\infty})\|f'\|_X.$$

Combining this estimate with (6-7) we obtain

$$\|G_{11}\|_X \le C(1+\|f'\|_{L^\infty}^7)\|f'\|_X.$$

It follows from this latter estimate, (6-4) and (6-2) that

$$\|G_1\|_X \le C(1+\|f'\|_{L^\infty}^7)\|f'\|_X. \tag{6-10}$$

What is left is to estimate G_2. For this purpose we write according to Taylor's formula

$$\begin{aligned} G_2(x) &= \text{p.v.}\int_{\mathbb{R}} \frac{yf'(x)\big[f'(x)\chi(y)+2\int_0^1 f'(x+\theta y)\,d\theta+f'(x+y)\big]}{y^2+(f(x)+f(x+y))^2}\,dy \\ &\quad + \text{p.v.}\int_{\mathbb{R}} \frac{\Delta_y f(x)\Delta_y f'(x)}{y^2+(f(x)+f(x+y))^2}\,dy + 2f(x)f'(x)\int_1^\infty \frac{dy}{y^2+f^2(x)} \\ &\triangleq G_{2,1}(x)+G_{2,2}(x)+2f'(x)\arctan(f(x)), \end{aligned}$$

where $\chi:\mathbb{R}\to\mathbb{R}$ is an even continuous compactly supported function belonging to X and taking the value 1 on the neighborhood of $[-1,1]$. Note that we have used in the first line the identity, for any $x\in K$,

$$\text{p.v.}\int_{-1}^1 \frac{y}{y^2+[f(x+y)+f(x)]^2}\,dy = \text{p.v.}\int_{\mathbb{R}} \frac{y\chi(y)}{y^2+[f(x+y)+f(x)]^2}\,dy,$$

which follows from the fact that $f(x+y)=0$ for all $y\notin[-1,1]$. Therefore we may write

$$G_{2,1}(x) = (f'(x))^2(T_f^{0,1}\chi)(x) + 2\int_0^1 f'(x)(T_f^{1,\theta}f')(x)\,d\theta + f'(x)(T_f^{1,1}f')(x),$$

where we use the notation $T_f^{\alpha,\beta}$ from Theorem 5.2. The estimate of $G_{2,1}$ in L^∞ is quite easy and follows from Theorem 5.2:

$$\|G_{2,1}\|_{L^\infty} \le C\|f'\|_{L^\infty}\|f'\|_D(1+\|f'\|_D^2).$$

However to estimate $G_{2,2}$ in L^∞ it is more convenient to write it in the form

$$G_{2,2}(x) = \text{p.v.}\int_{-1}^1 \frac{\Delta_y f(x)\Delta_y f'(x)}{y^2+(f(x)+f(x+y))^2}\,dy + 2f'(x)\arctan(f(x)).$$

Thus using the mean value theorem we find

$$\|G_{2,2}\|_{L^\infty} \le C\|f'\|_{L^\infty}\|f'\|_D.$$

Combining these estimates with (4-2) we obtain

$$\|G_2\|_{L^\infty} \le C\|f'\|_{L^\infty}(\|f'\|_D+\|f'\|_D^3). \tag{6-11}$$

We shall now implement the estimates in X and start with the term $G_{2,1}$. According to Theorem 5.2 one can unify the estimates in $C_K^\star$ and C^s and get the weak estimate

$$\|f'T_f^{\alpha,\beta}g\|_X \le C\|g\|_X(\|f'\|_X^{\frac12}\beta^{-\frac12}+\|f'\|_X^{15}). \tag{6-12}$$

From the product laws (4-4) and (4-8) one has

$$\|(f')^2(T_f^{0,1}\chi)\|_X \le \|f'\|_{L^\infty}\|f'T_f^{0,1}\chi\|_X + \|f'\|_X\|f'\|_{L^\infty}\|T_f^{0,1}\chi\|_{L^\infty}.$$

Hence we find

$$\begin{aligned}\|(f')^2(T_f^{0,1}\chi)\|_X &\le C\|f'\|_{L^\infty}(\|f'\|_X^{\frac12}+\|f'\|_X^{15}) + \|f'\|_X\|f'\|_{L^\infty}(1+\|f'\|_X^2)\\ &\le C\|f'\|_{L^\infty}(\|f'\|_X^{\frac12}+\|f'\|_X^{15}).\end{aligned}$$

Using (6-12) we get successively

$$\|f'T_f^{0,\theta}f'\|_X \le C\|f'\|_X(\|f'\|_X^{\frac12}\theta^{-\frac12}+\|f'\|_X^{15}) \tag{6-13}$$

and

$$\|f'T_f^{1,1}f'\|_X \le C\|f'\|_X(\|f'\|_X^{\frac12}+\|f'\|_X^{15}).$$

Thus using the inequalities above we deduce that

$$\begin{aligned}\|G_{2,1}\|_X &\le C\|f'\|_{L^\infty}(\|f'\|_X^{\frac12}+\|f'\|_X^{15}) + C\|f'\|_X(\|f'\|_X^{\frac12}+\|f'\|_X^{15})\\ &\le C(\|f'\|_X^{\frac32}+\|f'\|_X^{16}).\end{aligned} \tag{6-14}$$

When $X = C^s$ we can give a refined estimate for (6-13) using (5-7),

$$\|f'T_f^{0,\theta}f'\|_s \le C\|f'\|_{L^\infty}^{\frac{1}{2+s}}(\|f'\|_s^{\frac{3+2s}{2+s}}\theta^{-\frac12}+\|f'\|_s^{15}),$$

which implies

$$\begin{aligned}\|G_{2,1}\|_s &\le C\|f'\|_{L^\infty}(\|f'\|_s^{\frac12}+\|f'\|_s^{15}) + C\|f'\|_{L^\infty}^{\frac{1}{2+s}}(\|f'\|_s^{\frac{3+2s}{2+s}}+\|f'\|_s^{15})\\ &\le C\|f'\|_{L^\infty}^{\frac13}(\|f'\|_s+\|f'\|_s^{16}).\end{aligned} \tag{6-15}$$

Hence one can combine (6-14) and (6-15):

$$\|G_{2,1}\|_X \le C\|f'\|_D^{\frac13}(\|f'\|_X+\|f'\|_X^{16}). \tag{6-16}$$

As for the term $G_{2,2}$, we may write

$$\begin{aligned}G_{2,2}(x) &= 2f'(x)\arctan(f(x))\\ &\quad + \int_{-M}^{M}\frac{[\Delta_y f(x)-yf'(x)]\Delta_y f'(x)}{y^2+(f(x+y)+f(x))^2}\,dy + \text{p.v.}\int_{\mathbb{R}}\frac{yf'(x)\Delta_y f'(x)}{y^2+(f(x+y)+f(x))^2}\,dy\\ &\triangleq 2f'(x)\arctan(f(x)) + G_{2,2}^1(x) + G_{2,2}^2(x).\end{aligned}$$

The last term was treated in the preceding estimates and we obtain as in (6-16)

$$\|G_{2,2}^2\|_X \le C\|f'\|_D^{\frac13}(\|f'\|_X+\|f'\|_X^{16}). \tag{6-17}$$

It remains to estimate $G_{2,2}^1$, which can be split into two terms

$$G_{2,2}^1(x) = \widehat{G}_{\text{int},r}(x) + \widehat{G}_{\text{ext},r}(x),$$

with

$$\widehat{G}_{\mathrm{int},r}(x)=\int_{|y|\le r}\frac{[\Delta_y f(x)-yf'(x)]\Delta_y f'(x)}{y^2+(f(x+y)+f(x))^2}\,dy,$$
$$\widehat{G}_{\mathrm{ext},r}(x)=\int_{M\ge|y|\ge r}\frac{[\Delta_y f(x)-yf'(x)]\Delta_y f'(x)}{y^2+(f(x+y)+f(x))^2}\,dy.$$

Now we shall proceed as in the proof of Theorem 5.2. Let $r\in(0,1)$ and $x_1,x_2\in\mathbb{R}$ such that $|x_1-x_2|\le r$. First it is clear that

$$|\Delta_y f'(x)|\le\omega_{f'}(|y|). \tag{6-18}$$

In addition, using Taylor formula we get

$$|\Delta_y f(x)-yf'(x)|\le|y|\omega_{f'}(|y|). \tag{6-19}$$

Therefore

$$|\widehat{G}_{\mathrm{int},r}(x)|\le\int_{|y|\le r}\frac{[\omega_{f'}(|y|)]^2}{|y|}\,dy.$$

It follows that

$$\sup_{|x_1-x_2|\le r}|\widehat{G}_{\mathrm{int},r}(x_2)-\widehat{G}_{\mathrm{int},r}(x_1)|\le 4\int_0^r\frac{[\omega_{f'}(y)]^2}{y}\,dy. \tag{6-20}$$

Hence by Fubini's theorem

$$\int_0^1\sup_{|x_1-x_2|\le r}|\widehat{G}_{\mathrm{int},r}(x_1)-\widehat{G}_{\mathrm{int},r}(x_1)|\,\frac{dr}{r}\le 4\int_0^1\frac{[\omega_{f'}(y)]^2}{y}|\ln y|\,dy.$$

From the definition and the monotonicity of the modulus of continuity one deduces that for any $r\in(0,1)$

$$|\ln r|\omega_{f'}(r)\le\int_r^1\frac{\omega_{f'}(y)}{y}\,dy\le\|f'\|_D,$$

which implies

$$\int_0^1\sup_{|x_1-x_2|\le r}|\widehat{G}_{\mathrm{int},r}(x_1)-\widehat{G}_{\mathrm{int},r}(x_1)|\,\frac{dr}{r}\le 4\|f'\|_D^2. \tag{6-21}$$

To get the suitable estimate in C^s we come back to (6-20), which gives

$$\sup_{|x_1-x_2|\le r}|\widehat{G}_{\mathrm{int},r}(x_2)-\widehat{G}_{\mathrm{int},r}(x_1)|\le 4\|f'\|_s^2\int_0^r y^{2s-1}\,dy\le C\|f'\|_s^2 r^{2s},$$

and thus

$$\sup_{|x_1-x_2|\le 1}\frac{|\widehat{G}_{\mathrm{int},r}(x_2)-\widehat{G}_{\mathrm{int},r}(x_1)|}{|x_1-x_2|^s}\le C\|f'\|_s^2. \tag{6-22}$$

As for $\widehat{G}_{\mathrm{ext},r}$, one writes

$$\widehat{G}_{\mathrm{ext},r}(x_1)-\widehat{G}_{\mathrm{ext},r}(x_2)=\int_{M\ge|y|\ge r}\frac{\mathcal{N}(x_1,y)-\mathcal{N}(x_2,y)}{\mathcal{K}(x_1)}\,dy+\int_{M\ge|y|\ge r}\frac{\mathcal{N}(x_2,y)[\mathcal{K}(x_2,y)-\mathcal{K}(x_1,y)]}{\mathcal{K}(x_1,y)\mathcal{K}(x_2,y)}\,dy,$$

with

$$\mathcal{N}(x,y)=[\Delta_y f(x)-yf'(x)]\Delta_y f'(x)\quad\text{and}\quad \mathcal{K}(x,y)=y^2+(f(x)+f(x+y))^2.$$

Notice that from (6-18) and (6-19) one gets

$$|\mathcal{N}(x_1,y)-\mathcal{N}(x_2,y)|\le C|y|\omega_{f'}(r)\omega_{f'}(|y|)\quad\text{and}\quad |\mathcal{N}(x,y)|\le 2|y|\omega_{f'}(|y|)\|f'\|_{L^\infty}. \tag{6-23}$$

In addition, using straightforward calculus we obtain

$$|\mathcal{K}(x_1,y)-\mathcal{K}(x_2,y)|\le Cr\|f'\|_{L^\infty}(\sqrt{\mathcal{K}(x_1,y)}+\sqrt{\mathcal{K}(x_2,y)}).$$

Thus

$$\sup_{|x_1-x_2|\le r}\frac{|\mathcal{N}(x_2,y)||\mathcal{K}(x_2,y)-\mathcal{K}(x_1,y)|}{\mathcal{K}(x_1,y)\mathcal{K}(x_2,y)}\le Cr\|f'\|_{L^\infty}^2\frac{\omega_{f'}(|y|)}{|y|^2}.$$

Hence we get by Fubini's theorem and (4-3)

$$\int_0^1\sup_{|x_1-x_2|\le r}\int_{\{M\ge|y|\ge r\}}\frac{|\mathcal{N}(x_1,y)-\mathcal{N}(x_2,y)|}{\mathcal{K}(x_1)}\,dy\,\frac{dr}{r}\le\int_0^1\int_{\{M\ge|y|\ge r\}}\omega_{f'}(r)\omega_{f'}(|y|)\,\frac{dy}{|y|}\,\frac{dr}{r}\le C\|f'\|_D^2$$

and

$$\begin{aligned}\int_0^1\sup_{|x_1-x_2|\le r}\int_{\{M\ge|y|\ge r\}}\frac{|\mathcal{N}(x_2,y)||\mathcal{K}(x_2,y)-\mathcal{K}(x_1,y)|}{\mathcal{K}(x_1,y)\mathcal{K}(x_2,y)}\,dy\,\frac{dr}{r}&\le C\|f'\|_{L^\infty}^2\int_0^1\int_{\{M\ge|y|\ge r\}}\frac{\omega_{f'}(|y|)}{|y|^2}\,dy\,dr\\&\le C\|f'\|_{L^\infty}^2\|f'\|_D.\end{aligned}$$

Finally we obtain

$$\int_0^1\sup_{|x_1-x_2|\le r}|\widehat{G}_{\text{ext},r}(x_1)-\widehat{G}_{\text{ext},r}(x_2)|\,\frac{dr}{r}\le C\|f'\|_D^2+C\|f'\|_{L^\infty}^2\|f'\|_D.$$

As to the estimate in C^s we use (6-23) which implies

$$\begin{aligned}\int_{\{r\le|y|\le M\}}\frac{|\mathcal{N}(x_1,y)-\mathcal{N}(x_2,y)|}{\mathcal{K}(x_1)}\,dy&\le C\|f'\|_s r^s\int_{\{r\le|y|\le M\}}\frac{\omega_{f'}(|y|)}{|y|}\,dy\\&\le C\|f'\|_s\|f'\|_D r^s\end{aligned}$$

and

$$\begin{aligned}\int_{\{M\ge|y|\ge r\}}\frac{|\mathcal{N}(x_2,y)||\mathcal{K}(x_2,y)-\mathcal{K}(x_1,y)|}{\mathcal{K}(x_1,y)\mathcal{K}(x_2,y)}\,dy&\le C\|f'\|_{L^\infty}^2\|f'\|_s r\int_{\{M\ge|y|\ge r\}}\frac{dy}{|y|^{2-s}}\\&\le C\|f'\|_{L^\infty}^2\|f'\|_s r^s.\end{aligned}$$

It follows from Sobolev embedding $C^s\hookrightarrow L^\infty$ that

$$\sup_{|x_1-x_2|\le r}\frac{|\widehat{G}_{\text{ext},r}(x_1)-\widehat{G}_{\text{ext},r}(x_2)|}{|x_1-x_2|^s}\le C\|f'\|_D\|f'\|_s+C\|f'\|_D\|f'\|_s^2.$$

Combining the estimates above with (6-21) and (6-22) we deduce

$$\|G_{2,2}^1\|_X\le C\|f'\|_D(\|f'\|_X+\|f'\|_X^2).$$

Putting together this estimate with (6-16) and (6-17) we get

$$\|G_2\|_X \le C\|f'\|_D^{\frac{1}{3}}(\|f'\|_X + \|f'\|_X^{16}). \tag{6-24}$$

Now using (6-10) and (6-24) we find

$$\begin{aligned}\|G\|_X &\le C\|f'\|_X(1+\|f'\|_D^7) + C\|f'\|_D^{\frac{1}{3}}(\|f'\|_X + \|f'\|_X^{16})\\ &\le C(1+\|f'\|_D^{\frac{1}{3}})(\|f'\|_X + \|f'\|_X^{16}),\end{aligned}$$

which ends the proof of Proposition 6.1. □

6B. ***A priori estimates.*** The aim of this section is to establish weak and strong a priori estimates for solutions to (2-1). This part is the cornerstone of the local well-posedness theory. The main result of this section reads as follows.

Proposition 6.2. *Let* $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ *be a smooth solution for the graph equation* (2-1). *Assume that the initial data is positive and with compact support* K_0. *Then the following assertions hold true*:

(1) *For any* $t \in [0, T]$, *the function* f_t *is positive and*

$$\textit{for all } t \in [0, T], \quad \|f(t)\|_{L^\infty} \le \|f_0\|_{L^\infty}.$$

(2) *For any* $t \in [0, T]$, *we have*

$$\|f(t)\|_{L^1} = \|f_0\|_{L^1} e^{-t}.$$

(3) *The support* supp f_t *is contained in the convex hull of* K_0; *that is*,

$$\textit{for all } t \in [0, T], \quad \operatorname{supp} f(t) \subset \operatorname{Conv} K_0.$$

(4) *Set* $X = C_K^\star$ *or* $X = C_K^s$, *with* $s \in (0, 1)$. *If* $f_0' \in X$ *then there exists* T *depending only on* $\|f_0'\|_X$ *such that* $f' \in L^\infty([0, T]; X)$.

Proof. (1) To get the first part about the persistence of the positivity of we shall prove that

$$\text{for all } x \in \mathbb{R}, \quad u_2(t, x) = f(t, x)U(t, x), \tag{6-25}$$

with

$$\|U(t)\|_{L^\infty} \le C(1 + \|f'(t)\|_D^6)$$

and C being a constant depending only on the size of the support of f_t. Note from part (3) of the current proposition that the support of f_t is contained in a fixed compact set and therefore the constant C can be taken independent of the time variable. Assume for a while (6-25) and let us see how to propagate the positivity. Denote by ψ the flow associated to the velocity u_1, that is, the solution of the ODE

$$\partial_t \psi(t, x) = u_1(t, \psi(t, x)), \quad \psi(0, x) = x. \tag{6-26}$$

Recall that

$$u_1(t, x) = \frac{1}{2\pi}\int_{\mathbb{R}}\left[\arctan\left(\frac{f(t, x+y) - f(t, x)}{y}\right) - \arctan\left(\frac{f(t, x+y) + f(t, x)}{y}\right)\right] dy.$$

Set

$$\eta(t,x)=f(t,\psi(t,x));$$

then

$$\partial_t\eta(t,x)=u_2(t,\psi(t,x))=\eta(t,x)U(t,\psi(t,x)). \tag{6-27}$$

Consequently

$$\eta(t,x)=f_0(x)e^{\int_0^t U(\tau,\psi(\tau,x))\,d\tau}.$$

Since the flow $\psi(t):\mathbb{R}\to\mathbb{R}$ is a diffeomorphism we get the representation

$$f(t,x)=f_0(\psi^{-1}(t,x))e^{\int_0^t U[\tau,\psi(\tau,\psi^{-1}(t,x))]\,d\tau}. \tag{6-28}$$

As an immediate consequence we get the persistence through the time of the positivity of the solution. Let us now come back to the proof of the identity (6-25). To simplify the notation we remove the variable t from the functions. Applying Taylor's formula to the function

$$\tau\in[0,f(x)]\mapsto g(\tau)\triangleq\log\left[\frac{y^2+(\tau-f(x+y))^2}{y^2+(\tau+f(x+y))^2}\right]$$

yields

$$\begin{aligned}
&-2\pi u_2(x)\\
&=f(x)\int_0^1\int_{-M}^M\frac{f(x+y)-\tau f(x)}{y^2+[f(x+y)-\tau f(x)]^2}\,d\tau\,dy+f(x)\int_0^1\int_{-M}^M\frac{f(x+y)+\tau f(x)}{y^2+[f(x+y)+\tau f(x)]^2}\,d\tau\,dy\\
&\triangleq f(x)V_1(x)+f(x)V_2(x).
\end{aligned}$$

Using once again Taylor's formula we get the expressions

$$\begin{aligned}
&V_1(x)\\
&=\int_0^1\int_{-M}^M\frac{(1-\tau)f(x)}{y^2+\big[(1-\tau)f(x)+y\int_0^1 f'(x+\theta y)\,d\theta\big]^2}\,d\tau\,dy+\text{p.v.}\int_0^1\int_{-M}^M\frac{y\int_0^1 f'(x+\theta y)\,d\theta}{y^2+[f(x+y)-\tau f(x)]^2}\,d\tau\,dy\\
&\triangleq V_{1,1}(x)+V_{1,2}(x)
\end{aligned}$$

and

$$\begin{aligned}
&V_2(x)\\
&=\int_0^1\int_{-M}^M\frac{(1+\tau)f(x)}{y^2+\big[(1+\tau)f(x)+y\int_0^1 f'(x+\theta y)\,d\theta\big]^2}\,d\tau\,dy+\text{p.v.}\int_0^1\int_{-M}^M\frac{y\int_0^1 f'(x+\theta y)\,d\theta}{y^2+[f(x+y)+\tau f(x)]^2}\,d\tau\,dy\\
&\triangleq V_{2,1}(x)+V_{2,2}(x).
\end{aligned}$$

To estimate $V_{1,1}$ and $V_{2,1}$ we can assume that $f(x)>0$. Then making the change of variables $z\mapsto y=(1-\tau)f(x)z$ leads to

$$V_{1,1}(x)=\int_0^1\int_{-\frac{M}{(1-\tau)f(x)}}^{\frac{M}{(1-\tau)f(x)}}\frac{d\tau\,dz}{z^2+\big[1+z\int_0^1 f'(x+\theta(1-\tau)f(x)z)\,d\theta\big]^2}. \tag{6-29}$$

Using (5-22) we deduce that

$$\|V_{1,1}\|_{L^\infty} \le C(1+\|f'\|_{L^\infty}^2). \tag{6-30}$$

Similarly we get

$$\|V_{2,1}\|_{L^\infty} \le C(1+\|f'\|_{L^\infty}^2). \tag{6-31}$$

Let us now bound $V_{j,2}$, $j=1,2$. First by symmetry we write

$$V_{1,2}(x) = \int_0^1\int_0^M \frac{y\int_0^1 f'(x+\theta y)\,d\theta[f(x-y)-f(x+y)]\psi_\tau(x,y)}{(y^2+[f(x+y)-\tau f(x)]^2)(y^2+[f(x-y)-\tau f(x)]^2)}\,dy\,d\tau + \int_0^1\int_0^M \frac{y\int_0^1[f'(x+\theta y)-f'(x-\theta y)]\,d\theta}{y^2+[f(x-y)-\tau f(x)]^2}\,dy\,d\tau,$$

where

$$\psi_\tau(x,y) = f(x+y)+f(x-y)-2\tau f(x) = 2(1-\tau)f(x)+y\int_0^1[f'(x+\theta y)-f'(x-\theta y)]\,d\theta.$$

Thus

$$\|V_{1,2}\|_{L^\infty} \le C\int_0^1\int_0^M \frac{\|f'\|_{L^\infty}^2 y^2[(1-\tau)f(x)+y\omega_{f'}(y)]}{(y^2+[f(x+y)-\tau f(x)]^2)(y^2+[f(x-y)-\tau f(x)]^2)}\,dy\,d\tau + C\int_0^1\int_0^M \frac{\omega_{f'}(y)}{y}\,dy\,d\tau.$$

Similarly to $V_{1,1}$ one gets

$$\int_0^1\int_0^M \frac{y^2(1-\tau)f(x)\,dy\,d\tau}{(y^2+[f(x+y)-\tau f(x)]^2)(y^2+[f(x-y)-\tau f(x)]^2)} \le C(1+\|f'\|_{L^\infty}^4).$$

It follows that

$$\begin{aligned}\|V_{1,2}\|_{L^\infty} &\le C\|f'\|_{L^\infty}^2\Big(1+\|f'\|_{L^\infty}^4+\int_0^M \frac{\omega_{f'}(y)}{y}\,dy\Big)+C\|f'\|_D \\ &\le C\|f'\|_{L^\infty}^2(1+\|f'\|_{L^\infty}^4+\|f'\|_D)+C\|f'\|_D. \end{aligned}\tag{6-32}$$

The estimate of $V_{2,2}$ can be done in a similar way and one obtains

$$\|V_{2,2}\|_{L^\infty} = C\|f'\|_{L^\infty}^2(1+\|f'\|_{L^\infty}^4+\|f'\|_D)+C\|f'\|_D. \tag{6-33}$$

Combining both last estimates with (6-30) and (6-31) we finally get according to the embedding (4-2)

$$\|U\|_{L^\infty} \le C(1+\|f'\|_D^6),$$

where the constant C depends only on the size of the support of f.

Now let us establish the maximum principle. From (2-2) combined with the positivity of f_t one gets

$$\text{for all } t\in[0,T],\ \text{for all } x\in\mathbb{R},\quad u_2(t,x)\le 0.$$

Coming back to (6-27) we deduce that

$$\partial_t\eta(t,x)\le 0,$$

which implies in turn that

$$\text{for all } t \in [0,T], \text{ for all } x \in \mathbb{R}, \quad f(t,x) \le f_0(\psi^{-1}(t,x)).$$

Combined with the positivity of $f(t)$ we deduce immediately the maximum principle

$$\text{for all } t \in [0,T], \quad \|f(t)\|_{L^\infty} \le \|f_0\|_{L^\infty}.$$

Now we intend to provide a more refined identity that we shall use later in studying the asymptotic behavior of the solution. Actually we have

$$u_2(t,x) = -f(t,x)(1+R(t,x)), \tag{6-34}$$

with

$$\|R(t)\|_{L^\infty} \le C\|f'(t)\|_D(1+\|f'(t)\|_{L^\infty}^5).$$

First note that $R = \sum_{i,j=1}^2 V_{i,j}$. The estimates of $V_{1,2}$ and $V_{2,2}$ are done in (6-32) and (6-33). However to deal with $V_{1,1}$ and similarly $V_{2,1}$ we return to the expression (6-29). Set

$$\tau \mapsto K(\tau) = \frac{1}{z^2+[1+z\tau]^2}.$$

Easy computations using (5-22) show the existence of a positive constant C such that

$$\text{for all } \tau, z \in \mathbb{R}, \quad |K'(\tau)| = \frac{2|z||1+z\tau|}{(z^2+[1+zt]^2)^2} \le \frac{1}{z^2+[1+z\tau]^2} \le C\frac{1+\tau^2}{1+z^2}.$$

Applying the mean value theorem yields

$$\left|K(\tau) - \frac{1}{1+z^2}\right| \le C|\tau|\frac{1+\tau^2}{1+z^2}.$$

Therefore we get

$$\left|V_{1,1}(x) - \int_0^1\int_{-\frac{M}{(1-\tau)f(x)}}^{\frac{M}{(1-\tau)f(x)}} \frac{dz\,d\tau}{1+z^2}\right| \le C\|f'\|_{L^\infty}(1+\|f'\|_{L^\infty}^2),$$

which implies that

$$|V_{1,1}(x)-\pi| \le C\|f'\|_{L^\infty}(1+\|f'\|_{L^\infty}^2) + C\|f\|_{L^\infty}. \tag{6-35}$$

Similarly we obtain

$$|V_{2,1}(x)-\pi| \le C\|f'\|_{L^\infty}(1+\|f'\|_{L^\infty}^2) + C\|f\|_{L^\infty}. \tag{6-36}$$

Putting together (6-32), (6-33), (6-35), (6-36) we get (6-34).

(2) Integrating (1-2) in the space variable we get after integration by parts

$$\frac{d}{dt}\int_{\mathbb{R}} \rho(t,x)\,dx = \int_{\mathbb{R}} \operatorname{div}\, v(t,x)\rho(t,x)\,dx = -\int_{\mathbb{R}} \rho^2(t,x)\,dx = -\int_{\mathbb{R}} \rho(t,x)\,dx,$$

where in the last line we have used that for the characteristic function one has $\rho^2 = \rho$. The time decay follows then easily.

(3) According to the representation of the solution given by (6-28) we have easily that the support of $f(t)$ is the image by the flow $\psi(t)$ of the initial support, that is,

$$K_t = \psi(t, K_0). \tag{6-37}$$

We have to check that if $K_0 \subset [a,b]$, with $a < b$, then $K_t \subset [a,b]$. To do so it is enough to prove that

$$\psi(t,[a,b]) \subset [a,b].$$

This means that the flow is contractive on the boundary of the support. As $\psi(t)$ is a homeomorphism, we have $\psi(t,[a,b]) = [\psi(t,a), \psi(t,b)]$. Hence to get the desired inclusion it suffices to establish that

$$a_t \triangleq \psi(t,a) \geq a \quad \text{and} \quad b_t \triangleq \psi(t,b) \leq b.$$

This reduces to studying the derivative in time of a_t and b_t. First one has

$$\dot{a}_t = u_1(t,a_t) \quad \text{and} \quad \dot{b}_t = u_1(t,b_t).$$

Since $f(t,y) = 0$, for all $y \notin (a_t, b_t)$ and f_t positive everywhere,

$$u_1(t,a_t) = \frac{1}{\pi}\int_0^{b_t-a_t} \arctan\left(\frac{f_t(a_t+y)}{y}\right) dy \geq 0.$$

Hence $\dot{a}_t \geq 0$ and therefore $a_t \geq a$, for any $t \in [0,T]$.

Similarly we get

$$u_1(t,b_t) = -\frac{1}{\pi}\int_0^{b_t-a_t} \arctan\left(\frac{f_t(b_t-y)}{y}\right) dy \leq 0,$$

which implies that $b_t \leq b$ for any $t \in [0,T]$. This ends the proof of part (3).

(4) Recall from (2-3) and (2-5) that $g \triangleq f'$ satisfies the equation

$$\partial_t g + u_1 \partial_1 g = \frac{1}{2\pi}(F - G).$$

Set $h(t,x) = g(t,\psi(t,x))$, where ψ is the flow defined in (6-26). Then

$$\partial_t h(t,x) = \frac{1}{2\pi}(F(t,\psi(t,x)) - G(t,\psi(t,x))).$$

Thus

$$g(t,x) = g_0(\psi^{-1}(t,x)) + \frac{1}{2\pi}\int_0^t (F-G)\big(\tau, \psi(\tau,\psi^{-1}(t,x))\big)\, d\tau.$$

Recall the classical estimate

$$\|\partial_x[\psi(\tau,\psi^{-1}(t,\cdot))]\|_{L^\infty} \leq e^{\int_\tau^t \|\partial_x u_1(t',\cdot)\|_{L^\infty}\, dt'}, \tag{6-38}$$

which we may combine with the composition laws (4-6) and (4-7) to get

$$\|g(t)\|_X \leq C e^{V(t)}\left[\|g_0\|_X + \int_0^t \|(F-G)(\tau)\|_X\, d\tau\right], \quad V(t) \triangleq \int_0^t \|\partial_x u_1(\tau)\|_{L^\infty}\, d\tau. \tag{6-39}$$

To estimate $\|\partial_x u_1(t)\|_{L^\infty}$ we come back to (2-4). The first integral term can be restricted to a compact set $[-M, M]$ and thus

$$\left|\text{p.v.}\int_{-M}^{M}\frac{f'(x+y)-f'(x)}{y^2+(f(x+y)-f(x))^2}y\,dy\right|\le 2\int_0^M\frac{\omega_{f'}(y)}{y}\,dy\le C\|f'\|_D.$$

As for the second term, the integral can be restricted to $[-M, M]$ and we simply write

$$\begin{aligned}\text{p.v.}\int_{\mathbb{R}}&\frac{f'(x+y)+f'(x)}{y^2+(f(x+y)+f(x))^2}y\,dy\\&=\text{p.v.}\int_{-M}^{M}\frac{f'(x+y)-f'(x)}{y^2+(f(x+y)+f(x))^2}y\,dy+\text{p.v.}\int_{\mathbb{R}}\frac{2f'(x)}{y^2+(f(x+y)+f(x))^2}y\,dy.\end{aligned}$$

The first term of the right-hand side is controlled as before:

$$\left|\text{p.v.}\int_{-M}^{M}\frac{f'(x+y)-f'(x)}{y^2+(f(x+y)+f(x))^2}y\,dy\right|\le C\|f'\|_D.$$

However, the last term can be estimated as in the proof of Theorem 5.2(1). One gets in view of (5-7), (5-10) and (5-11)

$$\left|\text{p.v.}\int_{\mathbb{R}}\frac{y}{y^2+(f(x+y)+f(x))^2}\,dy\right|\le C(\|f'\|_{L^\infty}^2+\|f'\|_{L^\infty}\|f'\|_D+\|f'\|_{L^\infty}).$$

Hence using the embedding $X\hookrightarrow C_K^\star\hookrightarrow L^\infty$ we find

$$\begin{aligned}\|\partial_x u_1(t)\|_{L^\infty}&\le C(\|f'\|_D+\|f'\|_{L^\infty}\|f'\|_D)\\&\le C(\|f'(t)\|_X+\|f'(t)\|_X^2),\end{aligned}\tag{6-40}$$

which implies

$$V(t)\le Ct(\|f'\|_{L_t^\infty X}+\|f'\|_{L_t^\infty X}^2).\tag{6-41}$$

Using Proposition 6.1 we obtain

$$\|(F-G)(t)\|_X\le C(\|f'(t)\|_X+\|f'(t)\|_X^{17}).\tag{6-42}$$

Plugging (6-41) and (6-42) into (6-39) we obtain

$$\|f'\|_{L_T^\infty X}\le e^{CT(\|f'\|_{L_T^\infty X}+\|f'\|_{L_T^\infty X}^2)}(\|f_0'\|_X+T(\|f'\|_{L_T^\infty X}+\|f'\|_{L_T^\infty X}^{17})).$$

This shows the existence of small T depending only on $\|f_0'\|_X$ and such that

$$\|f'\|_{L_T^\infty X}\le 2\|f_0'\|_X,$$

which ends the proof of the proposition. □

6C. ***Scheme construction of the solutions.*** This section is devoted to the construction of the solutions to (2-3) in short time. Before giving a precise description about the method used here and based on a double regularization, let us explain the main ideas of the strategy. The a priori estimates developed in the previous sections require some rigid properties like the confinement of the support, the positivity of the solution and some nonlinear effects in order to control some singular terms, as was mentioned in Theorem 5.2. So it appears hard to find a linear scheme that respects all of those constraints. The idea is to proceed with a nonlinear double regularization scheme. First, we fix a small parameter $\varepsilon > 0$ used to regularize the singularity of the kernels around the origin, and second we elaborate an iterative nonlinear scheme giving rise to a family of solutions $(f_n^\varepsilon)_n$ that may violate some of the mentioned constraints. With this scheme we are able to derive a priori estimates uniformly with respect to n during a short time $T_\varepsilon > 0$, but this time may shrink to zero as ε goes to zero. By compactness arguments we prove that these approximate solutions $(f_n^\varepsilon)_n$ converge as n goes to infinity to a solution f^ε living in our function space during the time interval $[0, T_\varepsilon]$. Now the function f^ε satisfies a modified nonlinear problem but the important fact is that all the a priori estimates developed in the preceding sections hold uniformly on ε. This allows us by a classical procedure to implement the bootstrap argument and prove that the family $(f^\varepsilon)_\varepsilon$ is actually defined on some time interval $[0, T]$ independently on ε. To conclude it remains to pass to the limit when ε goes to zero and this allows us to construct a solution for our initial problem.

Let us now give more details about this double scheme regularization. Consider the iterative scheme

$$\begin{cases} \partial_t f_{n+1}^\varepsilon + u_1^\varepsilon(f_n^\varepsilon)\partial_x f_{n+1}^\varepsilon = u_2^\varepsilon(f_{n+1}^\varepsilon), \quad n \in \mathbb{N}, \\ f_0^\varepsilon(t,x) = f_0(x), \\ f_{n+1}^\varepsilon(0,x) = f_0(x), \end{cases} \tag{6-43}$$

with

$$\begin{aligned} u_1^\varepsilon(g)(t,x) &\triangleq \frac{1}{2\pi}\chi(x)\int_{|y|\geq\varepsilon}\chi(y)\left[\arctan\left(\frac{g(t,x+y)-g(t,x)}{y}\right)+\arctan\left(\frac{g(t,x+y)+g(t,x)}{y}\right)\right]dy, \\ u_2^\varepsilon(g)(t,x) &\triangleq \frac{1}{4\pi}\int_{|y|\geq\varepsilon}\chi(y)\log\left(\frac{y^2+(g(t,x+y)-g(t,x))^2}{y^2+(g(t,x+y)+g(t,x))^2}\right)dy. \end{aligned} \tag{6-44}$$

The function χ is a positive smooth cut-off function taking the value 1 on some interval $[-M, M]$ such that

$$K_0,\, K_0 - K_0 \subset [-M, M],$$

with K_0 being the convex hull of supp f_0. The function χ is introduced in order to guarantee the convergence of the integrals. We shall see later by using the support structure of the solutions that one can in fact remove this cut-off function. Define

$$\mathcal{E}_T = \{f : f \in L^\infty([0,T]\times\mathbb{R}),\, f' \in L^\infty([0,T], X)\}$$

equipped with the norm

$$\|f\|_{\mathcal{E}_T} = \|f\|_{L^\infty([0,T]\times\mathbb{R})} + \|\partial_x f\|_{L^\infty([0,T],X)},$$

where X denotes the Dini space $C^\star$ or Hölder space $C^s(\mathbb{R})$, $0 < s < 1$, and for simplicity we shall during this part work only with the Hölder space. We intend to explain the approach without giving all the details, because some of them are classical. Using the method of characteristics, one can transform

(6-43) into a fixed-point problem

$$f_{n+1}=\mathcal{N}_n^{\varepsilon}(f_{n+1}),\quad \text{with } \mathcal{N}(f)(t,x)=f_0(\psi_{n,\varepsilon}^{-1}(t,x))+\int_0^t u_2^{\varepsilon}(f)(\tau,\psi_{n,\varepsilon}(\tau,\psi_{n,\varepsilon}^{-1}(t,x)))\,d\tau,$$

with $\psi_{n,\varepsilon}$ being the one-dimensional flow associated to $u_1^n(f_n^{\varepsilon})$, that is, the solution of the ODE

$$\psi_{n,\varepsilon}(t,x)=x+\int_0^t u_1^n(f_n^{\varepsilon})(\tau,\psi_{n,\varepsilon}(\tau,x))\,d\tau. \tag{6-45}$$

It is clear that

$$\|\mathcal{N}(f)(t)\|_{L^\infty}\le \|f_0\|_{L^\infty}+\int_0^t \|u_2^{\varepsilon}(f)(\tau)\|_{L^\infty}\,d\tau.$$

Applying the elementary inequality, for $a>0,\ b,c\in\mathbb{R}_+$,

$$\left|\log\left(\frac{a+b}{a+c}\right)\right|\le\frac{b+c}{a},$$

we get from (6-44) that

$$|u_2^{\varepsilon}(f)(t,x)|\le\frac{1}{4\pi}\int_{|y|\ge\varepsilon}\chi(y)\frac{f^2(t,x+y)+f^2(t,x)}{y^2}\,dy\le C\varepsilon^{-2}\|f(t)\|_{L^\infty}^2.$$

It follows that

$$\|\mathcal{N}(f)\|_{L_T^\infty L^\infty}\le\|f_0\|_{L^\infty}+C\varepsilon^{-2}T\|f\|_{L_T^\infty L^\infty}^2. \tag{6-46}$$

We shall move to the estimate of $\|\partial_x\mathcal{N}(f)\|_{L_T^\infty X}$. Let us first start with the estimate of $\|\partial_x\{f_0(\psi_{n,\varepsilon}^{-1})\}\|_{L_T^\infty X}$. By straightforward computations using product laws (4-8), composition laws (4-6) in Hölder spaces and the classical estimate on the flow

$$\|\partial_x\psi_{n,\varepsilon}^{\pm1}\|_{L_T^\infty X}\le Ce^{C\|\partial_x(u_1^{\varepsilon}(f_n^{\varepsilon}))\|_{L_T^1L^\infty}}(1+\|\partial_x(u_1^{\varepsilon}(f_n^{\varepsilon}))\|_{L_T^1X}),$$

one gets

$$\begin{aligned}\|\partial_x\{f_0(\psi_{n,\varepsilon}^{-1})\}\|_{L_T^\infty X}&\le\|\{\partial_x f_0\}(\psi_{n,\varepsilon}^{-1})\|_{L_T^\infty X}\,\|\partial_x\psi_{n,\varepsilon}^{-1}\|_{L_T^\infty X}\\&\le C\|\partial_x f_0\|_X e^{C\|\partial_x(u_1^{\varepsilon}(f_n^{\varepsilon}))\|_{L_T^1L^\infty}}(1+\|\partial_x(u_1^{\varepsilon}(f_n^{\varepsilon}))\|_{L_T^1X}).\end{aligned}$$

Differentiating the expression of $u_1^{\varepsilon}(f_n^{\varepsilon})$ in (6-44) and making standard estimates we get easily

$$\begin{aligned}\|\partial_x\{u_1^{\varepsilon}(f_n^{\varepsilon})(t)\}\|_X&\le C+C\varepsilon^{-1}\|\partial_x f_n^{\varepsilon}(t)\|_X+C\varepsilon^{-3}\|\partial_x f_n^{\varepsilon}(t)\|_{L^\infty}\|f_n^{\varepsilon}(t)\|_X^2\\&\le C+C\varepsilon^{-1}\|f_n^{\varepsilon}\|_{\mathcal{E}_T}+C\varepsilon^{-3}\|f_n^{\varepsilon}\|_{\mathcal{E}_T}^3,\end{aligned}$$

where we have used

$$\left\|\frac{1}{y^2+f^2}\right\|_X\le C\|f\|_X^2y^{-4}.$$

Therefore

$$\|\partial_x\{f_0(\psi_{n,\varepsilon}^{-1})\}\|_{L_T^\infty X}\le C\|\partial_x f_0\|_X e^{CT+C\varepsilon^{-1}T\|f_n^{\varepsilon}\|_{\mathcal{E}_T}+C\varepsilon^{-3}T\|f_n^{\varepsilon}\|_{\mathcal{E}_T}^3}, \tag{6-47}$$

$$\|\partial_x\psi_{n,\varepsilon}^{\pm1}\|_{L_T^\infty X}\le Ce^{CT+CT\varepsilon^{-1}\|f_n^{\varepsilon}\|_{\mathcal{E}_T}+CT\varepsilon^{-3}\|f_n^{\varepsilon}\|_{\mathcal{E}_T}^3}. \tag{6-48}$$

Similarly we get

$$\begin{aligned}\|\partial_x\{u_2^\varepsilon(f)\}\|_{L_T^\infty X} &\le C\varepsilon^{-2}\|\partial_x f\|_{L_T^\infty X}\|f\|_{L_T^\infty X} + C\varepsilon^{-4}\|\partial_x f\|_{L_T^\infty L^\infty}\|f\|_{L_T^\infty L^\infty}\|f\|_{L_T^\infty X}^2\\ &\le C\varepsilon^{-2}\|f\|_{\mathcal{E}_T}^2 + C\varepsilon^{-4}\|f\|_{\mathcal{E}_T}^4.\end{aligned}$$

Combining this estimate with product laws and (6-48) we deduce that

$$\|\partial_x\{u_2^\varepsilon(f)(\tau,\psi_{n,\varepsilon}(\tau,\psi_{n,\varepsilon}^{-1}))\}\|_X \le C(\varepsilon^{-2}\|f\|_{\mathcal{E}_T}^2 + \varepsilon^{-4}\|f\|_{\mathcal{E}_T}^4)e^{CT+CT\varepsilon^{-1}\|f_n^\varepsilon\|_{\mathcal{E}_T}+CT\varepsilon^{-3}\|f_n^\varepsilon\|_{\mathcal{E}_T}^3}.$$

Putting together this estimate with (6-47) we find that

$$\|\partial_x\mathcal{N}(f)\|_{L_T^\infty X} \le C(\|\partial_x f_0\|_X + T\varepsilon^{-2}\|f\|_{\mathcal{E}_T}^2 + T\varepsilon^{-4}\|f\|_{\mathcal{E}_T}^4)e^{CT+CT\varepsilon^{-1}\|f_n^\varepsilon\|_{\mathcal{E}_T}+CT\varepsilon^{-3}\|f_n^\varepsilon\|_{\mathcal{E}_T}^3},$$

which yields in view of (6-46)

$$\|\mathcal{N}(f)\|_{\mathcal{E}_T} \le C(\|f_0\|_{L^\infty} + \|\partial_x f_0\|_X + T\varepsilon^{-2}\|f\|_{\mathcal{E}_T}^2 + T\varepsilon^{-4}\|f\|_{\mathcal{E}_T}^4)e^{CT+CT\varepsilon^{-1}\|f_n^\varepsilon\|_{\mathcal{E}_T}+CT\varepsilon^{-3}\|f_n^\varepsilon\|_{\mathcal{E}_T}^3}.$$

We can assume that $0 < T \le 1$ and then

$$\|\mathcal{N}(f)\|_{\mathcal{E}_T} \le C(\|f_0\|_{L^\infty} + \|\partial_x f_0\|_X + T\varepsilon^{-4}\|f\|_{\mathcal{E}_T}^4)e^{CT\varepsilon^{-3}\|f_n^\varepsilon\|_{\mathcal{E}_T}^3}.$$

Consider now the closed ball

$$B = \{f \in \mathcal{E}_T : \|f\|_{\mathcal{E}_T} \le 2C(\|f_0\|_{L^\infty} + \|\partial_x f_0\|_X)e^{CT\varepsilon^{-3}\|f_n^\varepsilon\|_{\mathcal{E}_T}^3}\};$$

if we choose T such that

$$16C^3\varepsilon^{-4}T(\|f_0\|_{L^\infty} + \|\partial_x f_0\|_X)^3 e^{5CT\varepsilon^{-3}\|f_n^\varepsilon\|_{\mathcal{E}_T}^3} \le 1 \tag{6-49}$$

then $\mathcal{N} : B \to B$ is well-defined and proceeding as before we can show under this condition that it is also a contraction. This implies the existence in this ball of a unique solution to the fixed-point problem and so one can construct a solution $f_{n+1}^\varepsilon \in \mathcal{E}_T$ to (6-43) and we have the estimates

$$\text{for all } n \in \mathbb{N}, \quad \|f_{n+1}^\varepsilon\|_{\mathcal{E}_T} \le 2C(\|f_0\|_{L^\infty} + \|\partial_x f_0\|_X)e^{CT\varepsilon^{-3}\|f_n^\varepsilon\|_{\mathcal{E}_T}^3}.$$

Now we select T such that it satisfies also

$$64C^4(\|f_0\|_{L^\infty} + \|\partial_x f_0\|_X)^3 T\varepsilon^{-3} \le \ln 2; \tag{6-50}$$

then we get the uniform estimates

$$\text{for all } n \in \mathbb{N}, \quad \|f_n\|_{\mathcal{E}_T} \le 4C(\|f_0\|_{L^\infty} + \|\partial_x f_0\|_X).$$

In order to satisfy mutually the conditions (6-49) and (6-50) it suffices to take

$$T_\varepsilon := C_0\varepsilon^2, \tag{6-51}$$

with C_0 depending only on $\|f_0\|_{L^\infty} + \|\partial_x f_0\|_X$ such that

$$\text{for all } n \in \mathbb{N}, \quad \|f_n\|_{\mathcal{E}_T} \le 4C(\|f_0\|_{L^\infty} + \|\partial_x f_0\|_X). \tag{6-52}$$

Now we shall check that we can remove the localization in space through the cut-off function χ. To do so, it suffices to get suitable information on the support of (f_n^ε). We shall prove that

$$\text{for all } n\in\mathbb{N},\quad \operatorname{supp} f_n^\varepsilon(t)\subset K_0, \tag{6-53}$$

where K_0 is the convex hull of the support of f_0. Before giving the proof let us assume for a while this property and see how to get rid of the localizations in the velocity fields. From the expression of $u_2^\varepsilon(f_{n+1}^\varepsilon)$ one has

$$u_2^\varepsilon(f_{n+1}^\varepsilon)(t,x)=\frac{1}{4\pi}\int_{|y|\geq\varepsilon}\log\left(\frac{y^2+(f_{n+1}^\varepsilon(t,x+y)-f_{n+1}^\varepsilon(t,x))^2}{y^2+(f_{n+1}^\varepsilon(t,x+y)+f_{n+1}^\varepsilon(t,x))^2}\right)dy \\ -\frac{1}{4\pi}\int_{|y|\geq\varepsilon}[1-\chi(y)]\log\left(\frac{y^2+(f_{n+1}^\varepsilon(t,x+y)-f_{n+1}^\varepsilon(t,x))^2}{y^2+(f_{n+1}^\varepsilon(t,x+y)+f_{n+1}^\varepsilon(t,x))^2}\right)dy.$$

Since for all $x\notin K_0$ we have $f_{n+1}(t,x)=0$, it follows that $u_2^\varepsilon(f_{n+1}^\varepsilon)(t,x)=0$; hence $\operatorname{supp} u_2^\varepsilon(f_{n+1}^\varepsilon)(t)\subset K_0$. Thus for all $x\in K_0$

$$\int_{|y|\geq\varepsilon}[1-\chi(y)]\log\left(\frac{y^2+(f_{n+1}^\varepsilon(t,x+y)-f_{n+1}^\varepsilon(t,x))^2}{y^2+(f_{n+1}^\varepsilon(t,x+y)+f_{n+1}^\varepsilon(t,x))^2}\right)dy \\ =\int_{\{|y|\geq|\varepsilon\}\cap K_0-K_0}[1-\chi(y)]\log\left(\frac{y^2+(f_{n+1}^\varepsilon(t,x+y)-f_{n+1}^\varepsilon(t,x))^2}{y^2+(f_{n+1}^\varepsilon(t,x+y)+f_{n+1}^\varepsilon(t,x))^2}\right)dy=0$$

because $\chi=1$ on K_0-K_0. Now we claim that in the advection term $u_1^\varepsilon(f_{n+1}^\varepsilon)(t,x)\partial_x f_{n+1}^\varepsilon$ of (6-43) one can remove the cut-off function. Since $\partial_x f_{n+1}^\varepsilon=0$ outside K_0, one gets immediately $\chi(x)\partial_x f_{n+1}^\varepsilon=\partial_x f_{n+1}^\varepsilon$. Similarly one has

$$u_1^\varepsilon(g)(t,x)\triangleq\frac{1}{2\pi}\int_{|y|\geq\varepsilon}\left[\arctan\left(\frac{g(t,x+y)-g(t,x)}{y}\right)+\arctan\left(\frac{g(t,x+y)+g(t,x)}{y}\right)\right]dy \\ -\frac{1}{2\pi}\int_{|y|\geq\varepsilon}(1-\chi(y))\left[\arctan\left(\frac{g(t,x+y)-g(t,x)}{y}\right)+\arctan\left(\frac{g(t,x+y)+g(t,x)}{y}\right)\right]dy,$$

and for $x\in K_0$ it is clear that

$$\int_{|y|\geq\varepsilon}(1-\chi(y))\left[\arctan\left(\frac{g(t,x+y)-g(t,x)}{y}\right)+\arctan\left(\frac{g(t,x+y)+g(t,x)}{y}\right)\right]dy \\ =\int_{\{y|\geq\varepsilon\}\cap K_0-K_0}(1-\chi(y))\left[\arctan\left(\frac{g(t,x+y)-g(t,x)}{y}\right)+\arctan\left(\frac{g(t,x+y)+g(t,x)}{y}\right)\right]dy=0.$$

Now let us come back to the proof of (6-53) and provide further qualitative properties. Similarly to the identity (6-25) one obtains

$$u_2^{n+1,\varepsilon}(t,x)=f_{n+1}^\varepsilon(t,x)(1+U_{n+1,\varepsilon}(t,x)),\quad \|U_{n+1,\varepsilon}(t)\|_{L^\infty}\leq C(1+\|f_{n+1}(t)\|_D^6).$$

So following the same lines as in the proof of Proposition 6.2 we get a similar formula to (6-28) which implies the positivity result

$$f_{n+1}(t,x)\geq 0,$$

where we have used in particular that the initial data satisfies $f^{\varepsilon}_{n+1}(0,x)=f_0(x)\geq 0$. Thus we obtain

$$\text{for all } n\in\mathbb{N},\quad f_n(t,x)\geq 0.$$

As $u^2_{n,\varepsilon}(t,x)\leq 0$, following the same lines as the proof of Proposition 6.2 we get the maximum principle

$$\text{for all } n\in\mathbb{N},\quad \|f^{\varepsilon}_n(t)\|_{L^\infty}\leq\|f_0\|_{L^\infty}.$$

The proof of the confinement of the support (6-53) follows exactly the same lines as the proof of Proposition 6.2(3). Now we shall study the strong convergence of the sequence $(f^{\varepsilon}_n)_n$. Set

$$\theta^{\varepsilon}_n(t,x):=f_{n+1}(t,x)-f_n(t,x).$$

Then

$$\partial_t\theta^{\varepsilon}_{n+1}+u^{\varepsilon}_1(f^{\varepsilon}_{n+1})\partial_x\theta^{\varepsilon}_{n+1}=-(u^{\varepsilon}_1(f^{\varepsilon}_{n+1})-u^{\varepsilon}_1(f^{\varepsilon}_n))\partial_x f^{\varepsilon}_{n+1}+u^{\varepsilon}_2(f^{\varepsilon}_{n+2})-u^{\varepsilon}_2(f^{\varepsilon}_{n+1}).$$

According to the mean value theorem one has for $a>0,\ x,y\in\mathbb{R}$,

$$|\arctan(x)-\arctan(y)|\leq|x-y|\quad\text{and}\quad|\log(a+|x|)-\log(a+|y|)|\leq|x-y|a^{-1},$$

which imply

$$\begin{aligned}\|u^{\varepsilon}_1(f^{\varepsilon}_{n+1})(t)-u^{\varepsilon}_1(f^{\varepsilon}_n)(t)\|_{L^\infty}&\leq C\|f^{\varepsilon}_{n+1}(t)-f^{\varepsilon}_n(t)\|_{L^\infty}\int_{|y|\geq\varepsilon}\frac{\chi(y)}{|y|}\,dy\\&\leq C\varepsilon^{-1}\|f^{\varepsilon}_{n+1}(t)-f^{\varepsilon}_n(t)\|_{L^\infty}.\end{aligned}$$

Similarly, we obtain

$$\|u^{\varepsilon}_2(f^{\varepsilon}_{n+1})(t)-u^{\varepsilon}_2(f^{\varepsilon}_n)(t)\|_{L^\infty}\leq C\varepsilon^{-2}(\|f^{\varepsilon}_{n+1}(t)\|_{L^\infty}+\|f^{\varepsilon}_n(t)\|_{L^\infty})\|f^{\varepsilon}_{n+1}(t)-f^{\varepsilon}_n(t)\|_{L^\infty}.$$

Using the uniform estimates (6-52) we get for any $t\in[0,T_\varepsilon]$

$$\|u^{\varepsilon}_2(f^{\varepsilon}_{n+1})(t)-u^{\varepsilon}_2(f^{\varepsilon}_n)(t)\|_{L^\infty}\leq C\|f_0'\|_X\varepsilon^{-2}\|f^{\varepsilon}_{n+1}(t)-f^{\varepsilon}_n(t)\|_{L^\infty},\quad\|\partial_x f^{\varepsilon}_{n+1}(t)\|_{L^\infty}\leq C\|f_0'\|_X.$$

Using the maximum principle for the transport equation allows us to get for any $t\in[0,T_\varepsilon]$

$$\|\theta_{n+1}(t)\|_{L^\infty}\leq C\varepsilon^{-2}\|f_0'\|_X\|\int_0^t[\|\theta_{n+1}(\tau)\|_{L^\infty}+\|\theta_n(\tau)\|_{L^\infty}]\,d\tau.$$

By virtue of Gronwall's lemma one finds that for any $t\in[0,T_\varepsilon]$

$$\|\theta_{n+1}(t)\|_{L^\infty}\leq e^{C\varepsilon^{-2}\|f_0'\|_X T_\varepsilon}\int_0^t\|\theta_n(\tau)\|_{L^\infty}\,d\tau.$$

Hence we obtain in view of (6-51)

$$\|\theta_{n+1}(t)\|_{L^\infty}\leq C_0\int_0^t\|\theta_n(\tau)\|_{L^\infty}\,d\tau.$$

By induction we find

$$\text{for all } n\in\mathbb{N},\ \text{for all } t\in[0,T_\varepsilon],\quad\|\theta_n\|_{L^\infty_t L^\infty}\leq C_0^n\frac{t^n}{n!}\|\theta_0\|_{L^\infty_t L^\infty}.$$

This implies the convergence of the series

$$\sum_{n\in\mathbb{N}} \|\theta_{n+1}\|_{L^\infty_{T_\varepsilon}L^\infty} < \infty.$$

Therefore $(f_n^\varepsilon)_n$ converges strongly in $L^\infty_{T_\varepsilon}L^\infty$ to an element $f^\varepsilon \in L^\infty_{T_\varepsilon}L^\infty$. From the uniform estimates (6-52) we deduce that $f^\varepsilon \in \mathcal{E}_{T_\varepsilon}$. This allows us to pass to the limit in (6-43) and obtain that f^ε is a solution to

$$\begin{cases} \partial_t f^\varepsilon + u_1^\varepsilon(f^\varepsilon)\partial_x f^\varepsilon = u_2^\varepsilon(f^\varepsilon), \\ f_0^\varepsilon(t,x) = f_0(x), \end{cases} \tag{6-54}$$

with

$$\begin{aligned} u_1^\varepsilon(f^\varepsilon)(t,x) &\triangleq \frac{1}{2\pi}\int_{|y|\geq\varepsilon}\left[\arctan\left(\frac{f^\varepsilon(t,x+y)-f^\varepsilon(t,x)}{y}\right)+\arctan\left(\frac{f^\varepsilon(t,x+y)+f^\varepsilon(t,x)}{y}\right)\right]dy, \\ u_2^\varepsilon(f^\varepsilon)(t,x) &\triangleq \frac{1}{4\pi}\int_{|y|\geq\varepsilon}\log\left(\frac{y^2+(f^\varepsilon(t,x+y)-f^\varepsilon(t,x))^2}{y^2+(f^\varepsilon(t,x+y)+f^\varepsilon(t,x))^2}\right)dy. \end{aligned} \tag{6-55}$$

Now, the proofs used to get the a priori estimates can be adapted to (6-54) supplemented with (6-55). For instance the a priori estimates obtained in Proposition 6.2 hold for the modified equation (6-54) independently on vanishing ε. In particular one can bound uniformly in ε the solution f^ε in the space X_{T_ε} and therefore T_ε is not maximal and by a standard bootstrap argument we can continue the solution up to the local time T constructed in Proposition 6.2. It follows that f^ε belongs to $\mathcal{E}_T$ uniformly with respect to small ε. This yields according once again to Proposition 6.2 and the inequalities (6-25) and (6-40)

$$\sup_{\varepsilon\in[0,1]} \|\partial_t f^\varepsilon\|_{L^\infty_T L^\infty} \leq \|u_1^\varepsilon(f^\varepsilon)\|_{L^\infty_T L^\infty}\|\partial_x f^\varepsilon\|_{L^\infty_T L^\infty} + \|u_2^\varepsilon(f^\varepsilon)\|_{L^\infty_T L^\infty} \leq C_0,$$

and C_0 is a constant depending on the size of the initial data. Now from the compact embedding $C_K^s \to C_b$ and Ascoli's lemma we deduce that up to a subsequence (f^ε) converges strongly in $L^\infty_T L^\infty$ to some element f which belongs in turn to $\mathcal{E}_T$. This allows us to pass to the limit in (6-54) and (6-55) and find a solution to the initial value problem (6-43). We point out that by working more one may obtain the strong convergence of the full sequence (f^ε) to f. Note finally that the uniqueness follows easily from the arguments used to prove that (θ_n) is a Cauchy sequence.

7. Global well-posedness

We are concerned here with the global existence of strong solutions already constructed in Theorem 2.1. This will be established under a smallness condition on the initial data and it is probable that for arbitrary large initial data the graph structure might be destroyed in finite time. The basic ingredient which allows us to balance the energy amplification during the time evolution is a damping effect generated by the source terms. Note that this damping effect is plausible from the graph equation (2-1) according to the identity (6-34). However, as we shall see in the next section, it is quite complicated to extend this behavior for higher regularity at the level of the resolution space due to the existence of a linear part in the source term governing the motion of the slope (2-3). This part could in general bring an amplification in time of the energy. To circumvent this difficulty we establish a weakly damping property of the linearized operator associated to the source term that we combine with the time decay of the solution for weak regularity using an interpolation argument.

7A. ***Weak and strong damping behavior of the source term.*** Note from Proposition 6.1 that F does not contribute at the linear level, which is not the case of the functional G. We shall prove that actually there is no linear contribution for G. This will be done by establishing a damping property that occurs at least at the linear level. This is described by the following proposition.

Proposition 7.1. *Let K be a compact set of $\mathbb{R}$ and $s\in(0,1)$; then for any $f\in C^s_K$ we have the decomposition*

$$G(x)=2\pi f'(x)+L(x)+N(x),$$

with

$$\|L\|_s\le 2\pi(\|f'\|_s+2\|f'\|_{L^\infty})+C\|f'\|^s_{L^\infty}\|f'\|_s \quad \text{and} \quad \|N\|_s\le C\|f'\|^{\frac{1}{3}}_D(\|f'\|_s+\|f'\|^{16}_s),$$

where $C>0$ is a constant depending only on K and s. Moreover,

$$\|L\|_{L^\infty}\le C\min(\|f\|^s_{L^\infty}\|f'\|_s,\|f'\|_{L^\infty}) \quad \text{and} \quad \|N\|_{L^\infty}\le C\|f'\|_{L^\infty}(\|f'\|_D+\|f'\|^3_D).$$

Proof. In view of (6-1), (6-2), (6-4), (6-16), (6-17) and (6-24) one gets

$$G(x)=G_{11}(x)+H(x),\quad H=G_{12}+G_2,$$

with

$$\|H\|_s\le C\|f'\|^{\frac{1}{3}}_D(\|f'\|_s+\|f'\|^{16}_s). \tag{7-1}$$

Note also that from (6-3) and (6-11) we get

$$\|H\|_{L^\infty}\le C\|f'\|_{L^\infty}(\|f'\|_s+\|f'\|^3_s). \tag{7-2}$$

Now from (6-5) we get

$$G_{11}(x)=2\int_{\mathbb{R}}\frac{f'(x)+f'(x+f(x)z)}{\varphi(x,z)}\,dz,$$

with

$$\varphi(x,z)=z^2+\left(2+z\int_0^1 f'(x+\theta f(x)z)\,d\theta\right)^2.$$

We shall split again G_{11} as follows:

$$\begin{aligned}G_{11}(x)&=2\int_{\mathbb{R}}\frac{f'(x)+f'(x+f(x)z)}{z^2+4}\,dz-2\int_{\mathbb{R}}\frac{[f'(x)+f'(x+f(x)z)]\psi(x,z)}{\varphi(x,z)(z^2+4)}\,dz\\&\triangleq\mathcal{L}(x)+\mathcal{N}(x),\end{aligned}$$

with

$$\psi(x,z)\triangleq 4z\int_0^1 f'(x+\theta f(x)z)\,d\theta+z^2\left(\int_0^1 f'(x+\theta f(x)z)\,d\theta\right)^2.$$

From (5-22) one gets

$$\|\mathcal{N}\|_{L^\infty}\le C\|f'\|^2_{L^\infty}(1+\|f'\|^3_{L^\infty}). \tag{7-3}$$

Using the product law (4-8) we get

$$\left\|\frac{[f'+f'\circ(\mathrm{Id}+zf)]\psi(\cdot,z)}{\varphi(\cdot,z)}\right\|_s \le 2\|f'\|_{L^\infty}\|\psi(\cdot,z)\|_{L^\infty}\|1/\varphi(\cdot,z)\|_s + 2\|f'\|_{L^\infty}\|\psi(\cdot,z)\|_s\|1/\varphi(\cdot,z)\|_{L^\infty} + (\|f'\|_s+\|f'\circ(\mathrm{Id}+zf)\|_s)\|\psi(\cdot,z)\|_{L^\infty}\|1/\varphi(\cdot,z)\|_{L^\infty}.$$

In addition, it is clear that

$$\|\psi(\cdot,z)\|_{L^\infty} \le 4|z|\,\|f'\|_{L^\infty} + |z|^2\,\|f'\|_{L^\infty}^2.$$

Performing the composition law (4-6) we deduce that

$$\|\psi(\cdot,z)\|_s \le C|z|\,\|f'\|_s(1+|z|^s\,\|f'\|_{L^\infty}^s) + C|z|^2\,\|f'\|_{L^\infty}\,\|f'\|_s(1+|z|^s\,\|f'\|_{L^\infty}^s).$$

Combining this latter estimate with (6-9) and (5-22) yields

$$\left\|\frac{[f'+f'\circ(\mathrm{Id}+zf)]\psi(\cdot,z)}{\varphi(\cdot,z)}\right\|_s \le C\,\|f'\|_{L^\infty}\,\|f'\|_s(1+\|f'\|_{L^\infty}^{7+s})(1+|z|^s).$$

Hence we get according to the embedding $C_K^s \hookrightarrow L^\infty$

$$\begin{aligned}\|\mathcal{N}\|_s &\le C\,\|f'\|_{L^\infty}\,\|f'\|_s(1+\|f'\|_{L^\infty}^{7+s})\\ &\le C\,\|f'\|_{L^\infty}^{\frac{1}{3}}(\|f'\|_s^{\frac{5}{3}}+\|f'\|_s^{\frac{26}{3}+s})\\ &\le C\,\|f'\|_{L^\infty}^{\frac{1}{3}}(\|f'\|_s+\|f'\|_s^{10}).\end{aligned}$$

Setting $N=\mathcal{N}+H$ and combining the latter estimate with (7-1) we find the desired estimate for N stated in the proposition. Putting together (7-2) and (7-3) combined with Sobolev embedding we find

$$\|N\|_{L^\infty} \le C\,\|f'\|_{L^\infty}(\|f'\|_s+\|f'\|_s^4).$$

Coming back to $\mathcal{L}$ one may write

$$\mathcal{L}(x) = 4f'(x)\int_{\mathbb{R}}\frac{1}{z^2+4}\,dz + 2\int_{\mathbb{R}}\frac{f'(x+f(x)z)-f'(x)}{z^2+4}\,dz \triangleq 2\pi f'(x) + L(x). \tag{7-4}$$

To estimate L in C^s we simply write

$$\|L\|_s \le 2\int_{\mathbb{R}}\frac{\|f'\circ(\mathrm{Id}+zf)\|_s+\|f'\|_s}{z^2+4}\,dz.$$

Combined with (4-6) we find

$$\begin{aligned}\|f'\circ(\mathrm{Id}+zf)\|_s &\le (\|f'\|_s+2\|f'\|_{L^\infty})(1+|z|\|f'\|_{L^\infty})^s\\ &\le (\|f'\|_s+2\|f'\|_{L^\infty})(1+|z|^s\|f'\|_{L^\infty}^s),\end{aligned}$$

where in the last line we have used the inequality, for all $s\in(0,1)$, for all $x, y\ge 0$ one has

$$(x+y)^s \le x^s + y^s.$$

Using (4-2), it follows that

$$\|L\|_s \le 2\pi(\|f'\|_s + 2\|f'\|_{L^\infty}) + C\|f'\|_s \|f'\|_{L^\infty}^s.$$

The estimate of L in L^∞ is easier and one gets according to (7-4),

$$|L(x)| \le 2|f(x)|^s \|f'\|_s \int_{\mathbb{R}} \frac{|z|^s}{z^2+4}\, dz \le C|f(x)|^s \|f'\|_s.$$

Therefore we obtain

$$\|L\|_{L^\infty} \le C\|f\|_{L^\infty}^s \|f'\|_s.$$

We point out that we have obviously

$$\|L\|_{L^\infty} \le 2\pi\|f'\|_{L^\infty}.$$

It follows that

$$\|L\|_{L^\infty} \le C \min(\|f\|_{L^\infty}^s \|f'\|_s, \|f'\|_{L^\infty}). \tag{7-5}$$

This completes the proof of Proposition 7.1. □

7B. ***Global a priori estimates.*** The main goal of this section is to show how we may use the weakly damping effect of the source terms stated in Proposition 7.1 in order to get global a priori estimates when the initial data is small enough. The basic result reads as follows.

Proposition 7.2. *Let K be a compact set of $\mathbb{R}$ and $s \in (0,1)$. There exists a constant $\varepsilon > 0$ such that if $\|f_0'\|_s \le \varepsilon$ then* (2-1) *admits a unique global solution*

$$f' \in L^\infty(\mathbb{R}_+; C_K^s).$$

Moreover, there exists a constant C_0 depending on the initial data such that

$$\text{for all } t \ge 0, \quad \|f'(t)\|_{L^\infty} \le C_0 e^{-t}.$$

Proof. According to the decomposition of Proposition 7.1 combined with (2-3) and (2-5) we get that $g = \partial_x f$ satisfies the equation

$$\partial_t g(t,x) + u_1(t,x)\,\partial_1 g(t,x) + g(t,x) = \mathcal{R}(t,x), \quad \mathcal{R} \triangleq \tfrac{1}{2\pi}(F - L - N). \tag{7-6}$$

Using Propositions 6.1 and 7.1 combined with the (4-2) we find

$$\|\mathcal{R}\|_s \le \|f'\|_s + 2\|f'\|_{L^\infty} + C\|f'\|_D(\|f'\|_s + \|f'\|_s^3) + C\|f'\|_{L^\infty}^s \|f'\|_s + C\|f'\|_{L^\infty}^{\frac13}(\|f'\|_s + \|f'\|_s^{16}).$$

The embedding $C_K^{\frac{s}{2}} \subset C_K^\star$ combined with interpolation inequalities in Hölder spaces yields

$$\|f'\|_D \le C\|f'\|_{L^\infty}^{\frac12} \|f'\|_s^{\frac12}. \tag{7-7}$$

Set $s_0 = \min\big(s, \frac13\big)$; then it is easy to get

$$\|\mathcal{R}\|_s \le \|f'\|_s + 2\|f'\|_{L^\infty} + C\|f'\|_{L^\infty}^{s_0}(\|f'\|_s + \|f'\|_s^{16}). \tag{7-8}$$

Let $h(t,x) \triangleq g(t,\psi(t,x))$, where ψ is the flow introduced in (6-26). Then it is obvious that

$$\partial_t h(t,x) + h(t,x) = \mathcal{R}(t,\psi(t,x)).$$

This allows us to deduce the Duhamel integral representation

$$e^t g(t,x) = g_0(\psi^{-1}(t,x)) + \int_0^t e^\tau \mathcal{R}(\tau,\psi(\tau,\psi^{-1}(t,x)))\,d\tau.$$

Thus

$$e^t \|g(t)\|_s \le \|g_0(\psi^{-1}(t))\|_s + \int_0^t e^\tau \|\mathcal{R}(\tau,\psi(\tau,\psi^{-1}(t)))\|_s\,d\tau.$$

According to (6-38) and (4-6) we obtain

$$\|g_0(\psi^{-1}(t))\|_s \le C\|g_0\|_s e^{V(t)}, \quad V(t) = \int_0^t \|\partial_x u_1(\tau)\|_{L^\infty}\,d\tau$$

and

$$\|\mathcal{R}(\tau,\psi(\tau,\psi^{-1}(t)))\|_s \le (\|\mathcal{R}(\tau)\|_s + 2\|\mathcal{R}(\tau)\|_{L^\infty})e^{V(t)-V(\tau)}.$$

Note that the estimate of $\mathcal{R}$ in C^s has been already stated in (7-8). However to get a suitable estimate in L^∞ we use Propositions 6.1 and 7.1 combined with Sobolev embedding,

$$\begin{aligned}\|\mathcal{R}(t)\|_{L^\infty} &\le C\|f'(t)\|_{L^\infty}(\|f'(t)\|_D + \|f'(t)\|_D^3) + C\min(\|f(t)\|_{L^\infty}^s\|f'(t)\|_s, \|f'(t)\|_{L^\infty})\\ &\le C(\|f'(t)\|_{L^\infty} + \|f'(t)\|_{L^\infty}^s)(\|f'(t)\|_s + \|f'(t)\|_s^3)\\ &\le C\|f'(t)\|_{L^\infty}^{s_0}(\|f'(t)\|_s + \|f'(t)\|_s^4).\end{aligned} \tag{7-9}$$

It follows that

$$\begin{aligned}&\|\mathcal{R}(\tau,\psi(\tau,\psi^{-1}(t)))\|_s\\ &\qquad \le (\|f'(\tau)\|_s + 2\|f'(\tau)\|_{L^\infty})e^{V(t)-V(\tau)} + C\|f'(\tau)\|_{L^\infty}^{s_0}(\|f'(\tau)\|_s + \|f'(\tau)\|_s^{16})e^{V(t)-V(\tau)}.\end{aligned}$$

Set $K(t) = e^{-V(t)}e^t\|f'(t)\|_s$ and

$$S(t) = Ce^t e^{-V(t)}(\|f'(t)\|_{L^\infty} + \|f'(t)\|_{L^\infty}^{s_0}(\|f'(t)\|_s + \|f'(t)\|_s^{16})).$$

Then

$$K(t) \le CK(0) + \int_0^t K(\tau)\,d\tau + \int_0^t S(\tau)\,d\tau.$$

By virtue of Gronwall's lemma we deduce that

$$K(t) \le Ce^t K(0) + \int_0^t e^{t-\tau} S(\tau)\,d\tau.$$

This implies

$$\begin{aligned}&\|f'(t)\|_s \le Ce^{V(t)}\|f_0'\|_s\\ &\qquad + Ce^{V(t)}\int_0^t \|f'(\tau)\|_{L^\infty}\,d\tau + e^{V(t)}\int_0^t \|f'(\tau)\|_{L^\infty}^{s_0}(\|f'(\tau)\|_s + \|f'(\tau)\|_s^{16})\,d\tau.\end{aligned} \tag{7-10}$$

Combining the interpolation inequality

$$\|f_t'\|_{L^\infty} \le C\|f_t\|_{L^1}^{\frac{s}{2+s}}\|f_t'\|_s^{\frac{2}{2+s}},$$

with Proposition 6.2(2) we obtain

$$\|f'(t)\|_{L^\infty} \le Ce^{-\frac{s}{2+s}t}\|f_0\|_{L^1}^{\frac{s}{2+s}}\|f'(t)\|_s^{\frac{2}{2+s}}. \tag{7-11}$$

Plugging this estimate into (6-40) we find

$$\|\partial_x u_1(t)\|_{L^\infty} \le Ce^{-\frac{s}{2+s}t}\|f_0\|_{L^1}^{\frac{s}{2+s}}\big(\|f'(t)\|_s^{\frac{2}{2+s}}+\|f'(t)\|_s^{\frac{4+s}{2+s}}\big). \tag{7-12}$$

It is quite obvious from (4-2) and the compactness of the support that

$$\|f_0\|_{L^1} \le C\|f_0'\|_s,$$

with C a constant depending on the size of the support of f_0. Set

$$\rho(T) = \sup_{t\in[0,T]}\|f'(t)\|_s.$$

Then combining (7-10) with (7-11) and (7-12) yields

$$\rho(T) \le Ce^{C\|f_0'\|_s^{\frac{s}{2+s}}([\rho(T)]^{\frac{2}{2+s}}+[\rho(T)]^{\frac{4+s}{2+s}})}\mu(T),$$

with

$$\mu(T) = \|f_0'\|_s + \|f_0'\|_s^{\frac{s}{2+s}}[\rho(T)]^{\frac{2}{2+s}} + \|f_0'\|_s^{\frac{ss_0}{2+s}}[\rho(T)]^{\frac{2s_0}{2+s}}(\rho(T)+[\rho(T)]^{16}).$$

This implies the existence of small number $\varepsilon > 0$ depending only on C, and thus on the size of the support of f_0, such that

$$\|f_0'\|_s \le \varepsilon \quad \Longrightarrow \quad \text{for all } T>0, \quad \rho(T) \le \delta(\|f_0'\|_s), \tag{7-13}$$

with $\lim_{x\to 0}\delta(x) = 0$. This gives the global a priori estimates.

What is left is to establish the precise time decay of $\|f'(t)\|_{L^\infty}$ stated in Proposition 7.2. From (7-6) it is easy to establish the following estimate using the method of characteristics:

$$\|g(t)\|_{L^\infty} \le e^{-t}\|g_0\|_{L^\infty} + \int_0^t e^{-(t-\tau)}\|\mathcal{R}(\tau)\|_{L^\infty}\,d\tau. \tag{7-14}$$

According to (7-9) we obtain

$$e^t\|f'(t)\|_{L^\infty} \le \|f_0'\|_{L^\infty} + C\int_0^t e^\tau\|f'(\tau)\|_{L^\infty}(\|f'(\tau)\|_D + \|f'\|_D^3)\,d\tau.$$

Using Gronwall's lemma we obtain

$$e^t\|f'(t)\|_{L^\infty} \le \|f_0'\|_{L^\infty}e^{W(t)}, \quad W(t) = C\int_0^t(\|f'(\tau)\|_D + \|f'\|_D^3)\,d\tau.$$

Putting together (7-7) with (7-11) we obtain

$$\|f'(t)\|_D \le Ce^{-\frac{s}{4+2s}t}\|f_0\|_{L^1}^{\frac{s}{4+2s}}\|f'(t)\|_s^{\frac{4+s}{4+2s}}.$$

Hence we deduce from (7-13) that

$$\text{for all } t \geq 0, \quad W(t) \leq C_0,$$

and therefore

$$\text{for all } t \geq 0, \quad \|f'(t)\|_{L^\infty} \leq C_0 e^{-t}, \quad \|f'(t)\|_D \leq C_0 e^{-\frac{s}{4+2s}t} \tag{7-15}$$

for a suitable constant C_0 depending on the initial data. Inserting these estimates into (7-9) we obtain

$$\text{for all } t \geq 0, \quad \|\mathcal{R}(t)\|_{L^\infty} \leq C_0 e^{-t}. \tag{7-16}$$

Since f_t is compactly supported in a fixed compact set

$$\text{for all } t \geq 0, \quad \|f(t)\|_{L^\infty} \leq C_1 e^{-t}. \tag{7-17}$$

Finally, we point out that all the constants involved in the preceding estimates are time independent. Indeed, they are related to the support of f_t which is confined in the convex hull of the support of the initial data, as has been stated in Proposition 6.2(3). □

8. Scattering and collapse to singular measure

The aim of the last section is to analyze and identify the long time behavior of the global solutions stated in Theorem 2.2. It attempts to investigate the time evolution of the probability measure

$$dP_t(x) \triangleq \frac{\rho(t,x)}{\|\rho_t\|_{L^1}}\, dA(x) = e^t \mathbf{1}_{D_t}(x)\, dA(x),$$

where dA denotes the usual Lebesgue measure. Note that without loss of generality we have assumed in the last line that $\|\rho_0\|_{L^1} = 1$. As we shall see, this measure converges weakly as t goes to infinity to a probability measure concentrated on the real line and absolutely continuous with respect to Lebesgue measure on the real line. The description of the density and the support of this limiting measure will be the subject of the next two sections.

8A. ***Structure of the singular measure.*** In this section we shall prove the part of Theorem 2.2 dealing with the weak convergence of the measure dP_t when t goes to ∞. First, it is obvious that the probability measure dP_t is absolutely continuous with respect to the Lebesgue measure. The convergence of the family $\{dP_t : t \geq 0\}$ will be done in a weak sense as follows. Let $\varphi \in \mathcal{D}(\mathbb{R}^2)$ be a test function; one can write using Fubini's theorem

$$I_t \triangleq \int_{\mathbb{R}^2} \varphi(x,y)\, dP_t = e^t \int_{\mathbb{R}} \int_{-f_t(x)}^{f_t(x)} \varphi(x,y)\, dy.$$

According to Taylor expansion in the second variable one gets

$$\text{for all } (x,y) \in \mathbb{R}^2, \quad \varphi(x,y) = \varphi(x,0) + y\psi(x,y) \quad \text{and} \quad \|\psi\|_{L^\infty} \leq C.$$

This implies

$$I_t = 2e^t \int_{\mathbb{R}} f_t(x)\varphi(x,0)\, dx + I_t^1, \quad I_t^1 \triangleq e^t \int_{\mathbb{R}} \int_{-f_t(x)}^{f_t(x)} y\psi(x,y)\, dy. \tag{8-1}$$

We shall check that the term I_t^1 does not contribute in the limiting behavior. Actually it vanishes for t going to infinity. Indeed,

$$|I_t^1| \le e^t \|\psi\|_{L^\infty} \int_{\mathbb{R}} [f_t(x)]^2 \, dx.$$

Using (7-17) and the localization of the support of f_t in the convex hull of the initial support, we deduce that

$$|I_t^1| \le Ce^{-t},$$

and thus

$$\lim_{t\to\infty} I_t^1 = 0.$$

Combining (2-1), (6-34), (7-15), (7-17) and (7-13) we deduce that

$$\partial_t f(t,x) + u_1 \partial_x f(t,x) + f(t,x) = -f(t,x)R(t,x), \tag{8-2}$$

with

$$\|R(t)\|_{L^\infty} \le \|f'(t)\|_D (1 + \|f'(t)\|_\infty^5) \le Ce^{-\frac{s}{4+2s}t}. \tag{8-3}$$

From the method of characteristics developed in studying (7-6) we get the representation

$$e^t f(t,\psi(t,x)) = f_0(x) e^{\int_0^t R(\tau,\psi(\tau,x))\,d\tau}. \tag{8-4}$$

From the integrability property (8-3) we deduce that $\{e^t f(t,\psi(t))\}$ converges uniformly as t goes to ∞ to the positive function

$$x \mapsto f_0(x) e^{\int_0^\infty R(\tau,\psi(\tau,x))\,d\tau} \triangleq R_2(x). \tag{8-5}$$

More precisely, using straightforward computations we easily get

$$\|e^t f_t \circ \psi(t) - R_2\|_{L^\infty} \le \|R_2\|_{L^\infty} \int_t^\infty \|R(\tau)\|_{L^\infty} \, d\tau \le Ce^{-\frac{s}{4+2s}t}. \tag{8-6}$$

The next goal is prove that the flow $\psi(t)$ converges uniformly as t goes to infinity to some homeomorphism $\psi_\infty : \mathbb{R} \to \mathbb{R}$ which belongs to the bi-Lipschitz class. First, recall from the definition (6-26) that

$$\psi(t,x) = x + \int_0^t u_1(\tau, \psi(\tau,x)) \, d\tau.$$

Recall from Section 2 that $u_1(x) = v_1(x, f(x))$ and the velocity is computed from the density ρ according to the second equation of (1-2). Hence we get

$$\|u_1(t)\|_{L^\infty} \le \|\Delta^{-1}\nabla\rho\|_{L^\infty}.$$

Now using the classical interpolation inequality

$$\|\Delta^{-1}\nabla\rho\|_{L^\infty} \le C\|\rho\|_{L^1}^{\frac{1}{2}} \|\rho\|_{L^\infty}^{\frac{1}{2}}$$

combined with the decay rate stated in Proposition 6.2(2) we deduce that

$$\|u_1(t)\|_{L^\infty} \le Ce^{-\frac{t}{2}}. \tag{8-7}$$

Consequently, it follows that $\psi(t)$ converges uniformly to the function

$$\psi_\infty(x) \triangleq x + \int_0^\infty u_1(\tau, \psi(\tau, x))\, d\tau.$$

More precisely, we have

$$\|\psi(t) - \psi_\infty\|_{L^\infty} \le \int_t^\infty \|u_1(\tau)\|_{L^\infty}\, d\tau \le Ce^{-\frac{t}{2}}. \tag{8-8}$$

It remains to check that ψ_∞ is bi-Lipschitz. First we know that

$$\|\partial_x \psi(t)\|_{L^\infty} \le e^{V(t)}, \quad V(t) = \int_0^t \|\partial_x u_1(\tau)\|_{L^\infty}\, d\tau.$$

Using (7-12) and (7-13) we deduce that

$$\text{for all } t \ge 0,\ \|\partial_x \psi(\tau)\|_{L^\infty} \le C, \quad \|\partial_x u_1(t)\|_{L^\infty} \le C\varepsilon^{\frac{s}{2+s}} e^{-\frac{s}{2+s}t}. \tag{8-9}$$

Differentiating ψ_∞ and using the triangle inequality we get

$$1 - \int_0^\infty \|\partial_x u_1(\tau)\|_{L^\infty} \|\partial_x \psi(\tau)\|_{L^\infty}\, d\tau \le \psi_\infty'(x) \le 1 + \int_0^\infty \|\partial_x u_1(\tau)\|_{L^\infty} \|\partial_x \psi(\tau)\|_{L^\infty}\, d\tau.$$

Therefore we obtain

$$\text{for all } x \in \mathbb{R}, \quad 1 - C\varepsilon^{\frac{s}{2+s}} \le \psi_\infty'(x) \le 1 + C\varepsilon^{\frac{s}{2+s}}.$$

Taking ε small enough, meaning that the initial data is very small, we get

$$\text{for all } x \in \mathbb{R}, \quad \tfrac{1}{2} \le \psi_\infty'(x) \le \tfrac{3}{2}. \tag{8-10}$$

This shows that ψ_∞ is a bi-Lipschitz function from $\mathbb{R}$ to $\mathbb{R}$. Furthermore, it is obvious that

$$\psi_\infty(x) = \psi(t, x) + \int_t^\infty u_1(\tau, \psi(\tau, x))\, d\tau,$$

and hence

$$\psi_\infty(\psi^{-1}(t, x)) = x + \int_t^\infty u_1(\tau, \psi(\tau, \psi^{-1}(t, x)))\, d\tau.$$

Combining this identity with $\psi_\infty \circ \psi_\infty^{-1} = \mathrm{Id}$ and (8-7) yields

$$|\psi_\infty(\psi^{-1}(t, x)) - \psi_\infty(\psi_\infty^{-1} x)| \le \int_t^\infty \|u_1(\tau)\|_{L^\infty}\, d\tau \le Ce^{-\frac{t}{2}}.$$

Applying (8-10) we deduce that

$$\|\psi^{-1}(t) - \psi_\infty^{-1}\|_{L^\infty} \le Ce^{-\frac{t}{2}}.$$

This shows that $\psi^{-1}(t)$ converges uniformly to ψ_∞^{-1} with an exponential rate. Set

$$\Phi = R_2 \circ \psi_\infty^{-1} \tag{8-11}$$

and assume for a while that R_2 belongs to C^α for any $\alpha \in (0,1)$; then we deduce from the preceding estimates, especially (8-6) and (8-4), that

$$\begin{aligned}\|e^t f(t)-\Phi\|_{L^\infty} &\le \|e^t f(t)-R_2\circ\psi^{-1}(t)\|_{L^\infty}+\|R_2\circ\psi^{-1}(t)-R_2\circ\psi_\infty^{-1}\|_{L^\infty}\\ &\le Ce^{-\frac{s}{4+2s}t}+\|R_2\|_\alpha\|\psi^{-1}(t)-\psi_\infty^{-1}\|_{L^\infty}^\alpha\\ &\le Ce^{-\frac{s}{4+2s}t}+Ce^{-\alpha\frac{t}{2}}.\end{aligned}$$

Taking $\alpha=\frac{2s}{4+2s}$ we get

$$\|e^t f(t)-\Phi\|_{L^\infty}\le Ce^{-\frac{s}{4+2s}t}. \tag{8-12}$$

Let us now check that R_2 belongs to C^α for any $\alpha\in(0,1)$. For this goal we shall express differently the function R_2. Set $R_1(t,x)=-f(t,x)R(t,x)$; then from the method of characteristics the solution to (8-2) may be recovered as follows:

$$e^t f(t,\psi(t,x))=f_0(x)+\int_0^t e^\tau R_1(\tau,\psi(\tau,x))\,d\tau.$$

Putting together (8-3) and (7-17) we deduce that

$$\|R_1(\tau,\psi(\tau))\|_{L^\infty}\le Ce^{-\frac{4+3s}{4+2s}t}. \tag{8-13}$$

Therefore we find the identity

$$R_2(x)=f_0(x)+\int_0^\infty e^\tau R_1(\tau,\psi(\tau,x))\,d\tau. \tag{8-14}$$

We shall now study the regularity of R_2 through the use of this representation.

Differentiating (8-2) in x and comparing it to (7-6) we get the identity

$$\partial_x R_1(t,x)=\mathcal{R}(t,x)+\partial_x u_1(t,x)\,\partial_x f(t,x).$$

Using (7-15), (7-16) and (8-9) we find

$$\text{for all } t\ge 0,\quad \|\partial_x R_1(t)\|_{L^\infty}\le Ce^{-t}.$$

Combining this latter estimate with the Leibniz formula and (8-9) implies

$$\text{for all } t\ge 0,\quad \|\partial_x(R_1(t,\psi(t,\cdot)))\|_{L^\infty}\le Ce^{-t}. \tag{8-15}$$

It suffices now to apply the following classical interpolation inequality: for any $\alpha\in(0,1)$ there exists $C>0$ such that

$$\|h\|_\alpha\le C\|h\|_{L^\infty}^{1-\alpha}\|h'\|_{L^\infty}^\alpha,$$

which implies that according to (8-13) and (8-15)

$$\text{for all } t\ge 0,\quad \|R_1(t,\psi(t,\cdot))\|_\alpha\le Ce^{-t}e^{-t(1-\alpha)\frac{s}{4+2s}}. \tag{8-16}$$

Returning to the identity (8-14), one obtains in view of (8-16)

$$\|R_2\|_\alpha\le\|f_0\|_\alpha+\int_0^\infty e^\tau\|R_1(\tau,\psi(\tau,\cdot))\|_\alpha\,d\tau\le C$$

for any $\alpha \in (0,1)$. As an immediate consequence of (8-11), (8-10) and (4-6) we find that Φ belongs to C^α for any $\alpha \in (0,1)$. We guess the profile Φ to keep the same regularity as f_0, that is, in C^{1+s}, but this could require much more refined analysis.

Now coming back to (8-1) we find in view of (8-12) and the Lebesgue theorem

$$\lim_{t\to\infty} I(t) = 2\int_{\mathbb{R}} \Phi(x)\varphi(x,0)\,dx.$$

This is equivalent to writing in the weak sense

$$\lim_{t\to\infty} dP_t = 2\Phi\,\delta_{\mathbb{R}\otimes\{0\}} \triangleq dP_\infty. \tag{8-17}$$

Now we shall discuss some properties of Φ. From (8-5) and (8-11) we have

$$\operatorname{supp}\Phi = \psi_\infty(K_0), \quad K_0 = \operatorname{supp} f_0. \tag{8-18}$$

According to (8-10), the measure of $\operatorname{supp}\Phi$ is strictly positive with

$$|\operatorname{supp}\Phi| \ge \tfrac{1}{2}|K_0|. \tag{8-19}$$

It remains to check that dP_∞ is a probability measure on the real axis, which reduces to verifying that

$$2\int_{\mathbb{R}} \Phi(x)\,dx = 1.$$

First note that using Proposition 6.2(2) one obtains for any $t \ge 0$

$$1 = 2\int_{\mathbb{R}} e^t f(t,x)\,dx.$$

To exchange the limit and integral it suffices to apply the Lebesgue theorem thanks to the condition (8-12) and the fact that $\operatorname{supp} f_t \in \operatorname{Conv} K_0$ (recall that for simplicity we have assumed that $\|\rho_0\|_{L^1} = 1$):

$$\lim_{t\to\infty} \int_{\mathbb{R}} e^t f(t,x)\,dx = \int_{\mathbb{R}} \Phi(x)\,dx.$$

This provides the desired result. We point out that with the normalization $\|\rho_0\|_{L^1} = 1$ one gets instead of (8-17)

$$dP_\infty = \frac{\Phi}{\|f_0\|_{L^1}}\delta_{\mathbb{R}\otimes\{0\}},$$

which gives the structure of the limiting measure stated in Theorem 2.2 thanks to (8-5) and (8-11).

8B. ***Concentration of the support.*** In this section we shall complete the study of the limiting measure dP_∞ and identify its support, denoted by K_∞. What is left to conclude the proof of Theorem 2.2 is just to check that the support D_t of the solution ρ_t converges in the Hausdorff sense to K_∞. Recall that K_0 is the support of f_0 and is assumed to be a finite collection of increasing segments $[a_i; b_i]$, $i = 1, \dots, n$, such that $a_i < b_i < a_{i+1}$. According to (8-18) one has

$$\operatorname{supp}\Phi = \psi_\infty(K_0) \triangleq K_\infty.$$

Since Ψ_∞ is strictly increasing due to (8-10) one deduces easily that

$$\operatorname{supp}\Phi = \cup_{i=1}^n [a_i^\infty, b_i^\infty], \quad a_i^\infty \triangleq \psi_\infty(a_i), \quad b_i^\infty \triangleq \psi_\infty(b_i).$$

Using once again (8-10) one may easily obtain that

$$\text{for all } i, \quad |a_i^\infty - b_i^\infty| \geq \tfrac{1}{2}|a_i - b_i|.$$

Now to establish the convergence in the Hausdorff sense of D_t towards K_∞ it suffices to prove the result for each connected component, that is,

$$\text{for all } i = 1, \dots, n, \quad d_H(\Gamma_t^i, [a_i^\infty, b_i^\infty]) \leq Ce^{-t},$$

with

$$\Gamma_t^i \triangleq \{(x, f_t(x)) : x \in [a_i^t, b_i^t]\}.$$

By straightforward analysis using (7-17) one obtains

$$d_H(\Gamma_t^i, [a_i^\infty, b_i^\infty]) \leq Ce^{-t} + C\max(|a_i^t - a_i^\infty|, |b_i^t - b_i^\infty|).$$

From (8-8) one gets

$$\max(|a_i^t - a_i^\infty|, |b_i^t - b_i^\infty|) \leq Ce^{-t}$$

and therefore

$$\text{for all } t \geq 0, \quad d_H(D_t, K_\infty) \leq Ce^{-t}.$$

The proof of Theorem 2.1 is now complete.

Acknowledgements

Hmidi gratefully acknowledges the hospitality of Dong Li and the Hong Kong University of Science and Technology where portions of this work were completed. Li is partially supported by Hong Kong RGC grants GRF 16307317 and 16309518.

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Received 3 Apr 2018. Revised 26 Oct 2018. Accepted 30 Nov 2018.

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ANALYSIS AND PDE
Vol. 12, No. 8, 2019
dx.doi.org/10.2140/apde.2019.12.2067

COUPLED KÄHLER–RICCI SOLITONS ON TORIC FANO MANIFOLDS

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We prove a necessary and sufficient condition in terms of the barycenters of a collection of polytopes for existence of coupled Kähler–Einstein metrics on toric Fano manifolds. This confirms the toric case of a coupled version of the Yau–Tian–Donaldson conjecture and as a corollary we obtain an example of a coupled Kähler–Einstein metric on a manifold which does not admit Kähler–Einstein metrics. We also obtain a necessary and sufficient condition for existence of torus-invariant solutions to a system of soliton-type equations on toric Fano manifolds.

1. Introduction

Given a compact Kähler manifold (X, ω), an important question in complex geometry is the problem of finding a metric of constant scalar curvature in the Kähler class $[\omega]$. It has been known for a long time that there are deep obstructions to existence of these metrics. In the case when $[\omega] = \pm c_1(X)$, constant scalar curvature metrics coincide with Kähler–Einstein metrics, i.e., metrics that are proportional to their Ricci tensor. It was recently shown [Chen et al. 2015a; 2015b; 2015c] that existence of such metrics is equivalent to a certain algebraic stability condition: K-polystability (see also [Tian 2015]). A similar stability condition for general Kähler classes is conjectured to be equivalent to existence of constant scalar curvature metrics. However, except for some special classes of manifolds (see [Donaldson 2009]) this is open. It should also be pointed out that even in light of [Chen et al. 2015a; 2015b; 2015c], determining if a given manifold admits a Kähler–Einstein metric is not a straightforward task. The condition of K-polystability is not readily checkable. On the other hand, a large class of manifolds where existence of Kähler–Einstein metrics reduces to a simple criterion is given by toric Fano manifolds. Here, as was originally proved in [Wang and Zhu 2004], existence of Kähler–Einstein metrics is equivalent to the condition that the barycenter of the polytope associated to the anticanonical polarization is the origin. In addition, Wang and Zhu [2004] proved that any toric Fano manifold admits a Kähler–Ricci soliton, in other words a metric ω such that

$$\operatorname{Ric}\omega = L_V(\omega) + \omega \tag{1}$$

for a holomorphic vector field V. Here L_V denotes Lie derivative along V. These appear as natural long-time solutions to the Kähler–Ricci flow and have attracted great interest over the years; see for example [Hamilton 1993; 1995; Cao 1997; Tian 1997].

MSC2010: 32Q15, 32Q20, 32Q26, 53C25.
Keywords: coupled Kähler–Einstein metrics, Kähler–Einstein metrics, Monge–Ampère equations, Kähler manifolds.

In a recent paper Witt Nyström together with the present author introduced the concept of coupled Kähler–Einstein metrics [Hultgren and Nyström 2018]. These are k-tuples of Kähler metrics $(\omega_1, \dots, \omega_k)$ on a compact Kähler manifold X satisfying

$$\operatorname{Ric}\omega_1 = \cdots = \operatorname{Ric}\omega_k = \pm \sum_i \omega_i. \tag{2}$$

These generalize Kähler–Einstein metrics in the sense that for $k = 1$ this equation reduces to the classical equation

$$\operatorname{Ric}\omega_1 = \pm\omega_1$$

defining Kähler–Einstein metrics. Moreover, (2) implies a cohomological condition on $\omega_1, \dots, \omega_k$, namely

$$\sum_i [\omega_i] = \pm c_1(X). \tag{3}$$

We see that, much as for Kähler–Einstein metrics, the theory splits into two cases: $c_1(X) < 0$ and $c_1(X) > 0$. Now, as in [Hultgren and Nyström 2018] we will say that a k-tuple of Kähler classes $(\alpha_1, \dots, \alpha_k)$ such that $\sum_i \alpha_i = \pm c_1(X)$ is a *decomposition of* $\pm c_1(X)$ and given a decomposition of $c_1(X)$ we will say that it admits a coupled Kähler–Einstein metric if there is a coupled Kähler–Einstein metric $(\omega_1, \dots, \omega_k)$ such that $[\omega_i] = \alpha_i$ for all i. In [Hultgren and Nyström 2018] it was shown that fixing a decomposition of $c_1(X)$ imposes the right boundary conditions on (2) in the sense that:

- If $c_1(X) < 0$, then any decomposition of $-c_1(X)$ admits a unique coupled Kähler–Einstein metric.
- If $c_1(X) > 0$, then any coupled Kähler–Einstein metric admitted by a given decomposition of $c_1(X)$ is unique up to the flow of holomorphic vector fields.

Moreover, it was shown that if $c_1(X) > 0$ and $(\omega_1, \dots, \omega_k)$ is a coupled Kähler–Einstein metric, then the associated k-tuple of Kähler classes $([\omega_1], \dots, [\omega_k])$ satisfies a certain algebraic stability condition which, by analogy, was called K-polystability. It was also conjectured that the converse of this holds, providing a "coupled" Yau–Tian–Donaldson conjecture:

Conjecture 1 [Hultgren and Nyström 2018]. *Assume $c_1(X) > 0$. Then a decomposition of $c_1(X)$ admits a coupled Kähler–Einstein metric if and only if it is K-polystable.*

Our main theorem confirms this conjecture in the toric case and provides a simple condition for K-polystability in terms of the barycenters of a collection of polytopes associated to $(\alpha_1, \dots, \alpha_k)$. More precisely, consider the anticanonical line bundle $-K_X$ over a toric Fano manifold X. Fixing the action of $(\mathbb{C}^*)^n$ on X, this defines a polytope P_{-K_X} in the vector space $M \otimes \mathbb{R}$, where M is the character lattice of $(\mathbb{C}^*)^n$. For a general Kähler class that arises as the curvature of a toric line bundle, this correspondence is well-defined up to translation of the polytope (or equivalently, up to choice of action on the toric line bundle). Moreover, the correspondence trivially extends to all Kähler classes that can be written as linear combinations with positive real coefficients of Kähler classes of this type. By general facts (see Lemma 23 and the discussion following it) this holds for any Kähler class on a toric Fano manifold. This means that a decomposition of $c_1(X)$ determines (up to translations) a set of polytopes $P_1, \dots, P_k$ in $\mathbb{R}^n$. Moreover, the condition $\sum_i \alpha_i = c_1(X)$ means the polytopes can be chosen so that the Minkowski sum satisfies

$$\sum_i P_i = P_{-K_X}. \tag{4}$$

Enforcing this, we note that the polytopes associated to a decomposition of $c_1(X)$ are well-defined up to translations

$$(P_1, \dots, P_k) \mapsto (P_1 + c_1, \dots, P_k + c_k),$$

where $c_1, \dots, c_k \in \mathbb{R}^n$ satisfies $\sum_i c_i = 0$.

Now, given a polytope P in $\mathbb{R}^n$ we will let $b(P)$ be the (normalized) barycenter of P,

$$b(P) = \frac{1}{\mathrm{Vol}(P)} \int_P p\, dp,$$

where dp is the uniform measure on P and $\mathrm{Vol}(P) = \int_P dp$. Note that $b(P+c) = b(P) + c$; hence, assuming (4), the quantity $\sum_i b(P_i)$ is independent of the choices of translation of $P_1, \dots, P_k$. Our main theorem is:

Theorem 2. *Let X be a toric Fano manifold. Assume (α_i) is a decomposition of $c_1(X)$ and $P_1, \dots, P_k$ are the associated polytopes. Then the following are equivalent*:

(i) (α_i) *admits a coupled Kähler–Einstein tuple.*

(ii) (α_i) *is K-polystable in the sense of [Hultgren and Nyström 2018].*

(iii) $\sum_i b(P_i) = 0$.

Remark 3. One important point is $\sum_i b(P_i)$ is not in general equal to

$$b\Big(\sum_i P_i\Big) = b(P_{-K_X});$$

hence the condition on $P_1, \dots, P_k$ in Theorem 2 is not (a priori) equivalent to existence of a classical Kähler–Einstein metric. In fact, none of these conditions imply the others. By Corollary 4 below, there is an example of a manifold that doesn't admit Kähler–Einstein metrics but does admit coupled Kähler–Einstein metrics. Moreover, by Remark 6 there is an example of a Kähler–Einstein manifold with decompositions of $c_1(X)$ that doesn't admit coupled Kähler–Einstein metrics.

Corollary 4. *Let E be the rank-2 vector bundle*

$$E = \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$$

over $\mathbb{P}^2 \times \mathbb{P}^1$ and consider the toric four-manifold $X = \mathbb{P}(E)$. Then X does not admit a Kähler–Einstein metric. On the other hand, let $\pi : X \to \mathbb{P}^1$ be the natural projection onto $\mathbb{P}^1$ and $\beta_1, \beta_2 \in H^{(1,1)}(X)$ be the classes corresponding to the divisors given by $\pi^{-1}(0)$ and $\pi^{-1}(\infty)$, respectively. Then

$$\alpha_1 = \tfrac{1}{2} c_1(X) - \tfrac{1}{4}\sqrt{\tfrac{5}{7}}(\beta_1 + \beta_2), \quad \alpha_2 = \tfrac{1}{2} c_1(X) + \tfrac{1}{4}\sqrt{\tfrac{5}{7}}(\beta_1 + \beta_2) \tag{5}$$

are Kähler and the decomposition of $c_1(X)$ given by (α_1, α_2) admits a coupled Kähler–Einstein metric.

Remark 5. It would be interesting to see if there are simpler examples than the one given in Corollary 4 of manifolds which admit coupled Kähler–Einstein metrics but no Kähler–Einstein metrics. However, by Corollary 1.6 in [Hultgren and Nyström 2018], the automorphism group of any manifold that admits a coupled Kähler–Einstein metric is reductive. Among other things, this rules out $\mathbb{P}^2$ blown up in one or two points.

Remark 6. The following is an example of a decomposition of $c_1(X)$ on an Einstein manifold that does not admit a coupled Kähler–Einstein metric. Let X be the toric Fano manifold acquired by blowing up $\mathbb{P}^2$ in three points and D be the $(S^1)^n$-invariant divisor in X that corresponds to the ray generated by (1, 1) in the fan of X. Let $D_t = -\frac{1}{2}K_X + tD$. We have $D_t + D_{-t} = -K_X$. Computer calculations show that

$$b(P_{D_t}) + b(P_{D_{-t}}) \neq 0$$

for small t; in other words the decomposition of $c_1(X)$ given by $(c_1(D_t), c_1(D_{-t}))$ does not admit a coupled Kähler–Einstein metric for small t.

Remark 7. As discussed in [Hultgren and Nyström 2018], fixing a Kähler class α on X we get a family of decompositions of $c_1(X)$

$$\{(t\alpha, c_1(X) - t\alpha) : t \in (0, t_\alpha)\},$$

where $t_\alpha = \sup\{t : c_1(X) - t\alpha > 0\}$. Assuming they admit coupled Kähler–Einstein metrics (η_1^t, η_2^t) we get a canonical family of metrics $\{\omega_t := \eta_1^t / t\}$ in α. Now, let X be a toric Fano surface. By Theorem 2, $(t\alpha, \alpha - c_1(X))$ admits a coupled Kähler–Einstein metric if and only if

$$t b(P_{L_\alpha}) + b(P_{-K_X - tL_\alpha}) = 0, \tag{6}$$

where L_α is a toric ($\mathbb{R}$-)line bundle such that $c_1(L_\alpha) = \alpha$. On the other hand, it was proven in [Donaldson 2009] that α admits a constant scalar curvature metric if and only if

$$\frac{\int_{\partial P_{L_\alpha}} f \, d\sigma}{\int_{\partial P_{L_\alpha}} d\sigma} - \frac{\int_{P_{L_\alpha}} f \, dp}{\int_{P_{L_\alpha}} dp} \geq 0 \tag{7}$$

for every convex function f on the closure of P_{L_α}, with equality if and only if f is affine linear. Here $d\sigma$ is the measure on ∂P_{L_α} defined by the identity

$$\frac{d}{dt}\left(\int_{P_{L_\alpha} + tP_{-K_X}} h \, dp\right)\bigg|_{t=0} = \int_{\partial P_{L_\alpha}} h \, d\sigma$$

for all functions h continuous in a neighborhood of P. In particular, for affine linear functions f, (7) reduces to the barycenter condition

$$b(P_{L_\alpha}) = b(d\sigma) = \frac{\int_{\partial P_{L_\alpha}} \sigma \, d\sigma}{\int_{\partial P_{L_\alpha}} d\sigma}. \tag{8}$$

It would be interesting to understand the relationship of (6) with the conditions (7) and (8).

Our second result considers a more general (soliton-type) version of (2), namely, given holomorphic vector fields $V_1, \dots, V_k$

$$\operatorname{Ric}\omega_1 - L_{V_1}(\omega_1) = \cdots = \operatorname{Ric}\omega_k - L_{V_k}(\omega_k) = \sum_i \omega_i. \tag{9}$$

We will say that a k-tuple of Kähler metrics satisfying (2) is a *coupled Kähler–Ricci soliton*. When $k = 1$, (9) reduces to (1) and defines classical Kähler–Ricci solitons. As mentioned above these appear as natural solutions to the Kähler–Ricci flow. In fact, a similar interpretation in terms of natural solutions to a geometric flow can be given for (9). Given k Kähler metrics $\omega_1^0, \dots, \omega_k^0$ we may consider the flow defined by

$$\frac{d}{dt}\omega_1^t = \operatorname{Ric}\omega_1^t - \sum_i \omega_i^t, \quad \dots, \quad \frac{d}{dt}\omega_k^t = \operatorname{Ric}\omega_k^t - \sum_i \omega_i^t \tag{10}$$

for $t \in [0, \infty)$. Stationary solutions to (10) are given by coupled Kähler–Einstein metrics, i.e., solutions to (2). On the other hand, putting $V_1 = \cdots = V_k = V$ and letting (ω_i^t) be the flow along V of a k-tuple (ω_i^0) satisfying (9) means (ω_i^t) will satisfy (9) for each t. Plugging this into the right-hand side of (10) gives

$$\operatorname{Ric}\omega_j^t - \sum_i \omega_i^t = L_V(\omega_j^t)$$

for all j. By definition $(d/dt)\omega_j^t = L_V(\omega_j^t)$ for all j; hence (ω_i^t) satisfies (10).

To state our second result we need some terminology. Note that a point in the vector space that is dual to $M \otimes \mathbb{R}$, namely $N \otimes \mathbb{R}$ where N is the lattice consisting of one-parameter subgroups in $(\mathbb{C}^*)^n$, determines a holomorphic vector field on X. We will call any holomorphic vector field on X that arises in this manner a *toric vector field*. These can be given a concrete description in the following way: By definition, the action of $(\mathbb{C}^*)^n$ on X admits an open, dense and free orbit. Identifying $(\mathbb{C}^*)^n$ with this orbit and letting $\sigma_1, \dots, \sigma_n$ be the standard logarithmic coordinates on $(\mathbb{C}^*)^n$ the toric vector fields are simply the vector fields that arise as linear combinations of the coordinate vector fields $\partial/\partial\sigma_1, \dots, \partial/\partial\sigma_k$. We will often identify a toric vector field with its associated point in $N \otimes \mathbb{R}$.

In this context there is a natural vector-valued invariant $\mathcal{A}_V(P)$ determined by a polytope P in $\mathbb{R}^n = M \otimes \mathbb{R}$ and a point V in the dual vector space $N \otimes \mathbb{R}$. To define it we first introduce the V-weighted volume of P

$$\operatorname{Vol}_V(P) = \int_P e^{\langle V, p\rangle}\, dp.$$

Then $\mathcal{A}_V(P)$ is given by

$$\mathcal{A}_P(V) = \frac{1}{\operatorname{Vol}_V(P)} \int_P p e^{\langle V, p\rangle}\, dp. \tag{11}$$

With respect to this we have:

Theorem 8. *Let $V_1, \dots, V_k$ be toric vector fields on a toric Fano manifold X. Assume $(\alpha_1, \dots, \alpha_k)$ is a decomposition of $c_1(X)$ and $P_1, \dots, P_k$ are the associated polytopes. Then there is an $(S^1)^n$-invariant solution $(\omega_1, \dots, \omega_k)$ to* (9) *such that $\omega_i \in \alpha_i$ for each i if and only if*

$$\sum_i \mathcal{A}_{P_i}(V_i) = 0. \tag{12}$$

Remark 9. Much as in Theorem 2, the polytopes $P_1, \dots, P_k$ associated to $(\alpha_1, \dots, \alpha_k)$ are only well-defined up to translations $P_i \to P_i + c_i$ for $c_i \in \mathbb{R}^n$ satisfying $\sum_i c_i = 0$. On the other hand, much as the barycenter, $\mathcal{A}_V(P)$ satisfies

$$\mathcal{A}_{P+c}(V) = \mathcal{A}_P(V) + c,$$

and hence the left-hand side of (12) is invariant under such translations.

Remark 10. Theorem 8 is a generalization of Wang and Zhu's theorem [2004] on the existence of Kähler–Ricci solitons on toric manifolds. See also [Berman and Berndtsson 2013; Delcroix 2017] for generalizations in other directions.

A straightforward corollary of Theorem 8, using that (11) is the gradient of a strictly convex and proper function on $\mathbb{R}^n$, is:

Corollary 11. *Let (α_i) be a decomposition of $c_1(X)$ on a toric Fano manifold. Then there is a unique toric vector field V such that (α_i) admits an $(S^1)^n$-invariant coupled Kähler–Ricci soliton where $V_1 = \cdots = V_k = V$.*

Remark 12. Naturally, we expect solutions of the flow (10) to converge to the Kähler–Ricci solitons in Corollary 11. This parallels the theory in the case $k = 1$ (see [Tian and Zhu 2007]). On the other hand, it is interesting to note that by Theorem 8 there exists a large class of solitons that do not appear as natural solutions to (10) in the sense discussed above (this happens whenever $V_i \neq V_j$ for some i and j). This suggests that there is a more general flow, which includes (10) as a special case, and where the solitons of Theorem 8 appear as natural solutions.

A second corollary of Theorem 8 is related to the corresponding real Monge–Ampère equation. Let $f_1, \dots, f_k$ be twice differentiable convex functions on $\mathbb{R}^n$. Let ∇f_i denote the gradient of f_i. Then, given a decomposition $(\alpha_1, \dots, \alpha_k)$ and associated polytopes $P_1, \dots, P_k$, existence of coupled Kähler–Ricci solitons is equivalent to the solvability of the equation

$$\frac{e^{\langle V_1, \nabla f_1\rangle}}{\mathrm{Vol}_{V_1}(P_1)} \det\left(\frac{d^2 f_1}{dx_l\, dx_m}\right) = \cdots = \frac{e^{\langle V_k, \nabla f_k\rangle}}{\mathrm{Vol}_{V_k}(P_k)} \det\left(\frac{d^2 f_k}{dx_l\, dx_m}\right) = e^{-\sum_i f_i} \tag{13}$$

under the boundary conditions

$$\overline{\nabla f_i(\mathbb{R}^n)} = P_i, \tag{14}$$

where the left-hand side of (14) denotes the closure of the image of ∇f_i in $\mathbb{R}^n$. We will say that a k-tuple of polytopes in $\mathbb{R}^n$ is *toric Fano* if it is defined by a decomposition of $c_1(X)$ on a toric Fano manifold.

Corollary 13. *Assume $P_1, \dots, P_k$ is a toric Fano k-tuple of polytopes and $V_1, \dots, V_k \in \mathbb{R}^n$. Then* (13) *admits a solution satisfying* (14) *if and only if*

$$\sum_i \mathcal{A}_{P_i}(V_i) = 0.$$

In particular, if $V_1 = \cdots = V_k = 0$ then (13) *admits a solution satisfying* (14) *if and only if*

$$\sum_i b(P_i) = 0.$$

Theorem 2 essentially follows from considering the case $V_1 = \cdots = V_k = 0$ in Theorem 8. Doing this gives that (iii) in Theorem 2 implies (i). As mentioned above, by a previous result [Hultgren and Nyström 2018, Theorem 1.15] (i) implies (ii). Finally, an explicit formula for the (coupled) Donaldson–Futaki invariant of test configurations induced by toric vector fields shows that (ii) implies (iii). To be more precise, if V is a toric vector field and (α_i) is a decomposition of $c_1(X)$ with associated polytopes $P_1, \ldots, P_k$, then the test configuration for (α_i) induced by V has Donaldson–Futaki invariant

$$\left\langle V, \sum_i b(P_i) \right\rangle.$$

It follows that if $\sum_i b(P_i) \neq 0$, then there is a test configuration for (α_i) with negative Donaldson–Futaki invariant. By definition, this means (α_i) is not K-polystable (see Section 3.2 for a detailed argument).

The main point in the proof of Theorem 8 is to establish a priori C^0-estimates along an associated continuity path. More precisely, let $\theta_1, \ldots, \theta_k$ be Kähler metrics such that $[\theta_i] = \alpha_i$. Assume, using the Calabi–Yau theorem, that ω_0 is a Kähler form such that $\operatorname{Ric}\omega_0 = \sum_i \theta_i$ and $\int_X \omega_0^n = 1$. For each i, let $g_i = g_{\theta_i, V_i}$ be a θ_i-plurisubharmonic function on X such that

$$dd^c g_i = L_{V_i}(\theta_i)$$

and $\int_X e^{g_i}\theta_i^n = 1$ (see Lemma 17). For $t \in [0, 1]$ we will consider the equation

$$e^{g_1 + V_1(\phi_1)}(\theta_1 + dd^c\phi_1)^n = \cdots = e^{g_k + V_k(\phi_k)}(\theta_k + dd^c\phi_k)^n = e^{-t\sum_i \phi_i}\omega_0^n. \tag{15}$$

Moreover, fixing a point $x_0 \in X$ we will assume solutions to (15) are normalized according to

$$\phi_1(x_0) = \cdots = \phi_k(x_0). \tag{16}$$

The significance of these equations is that for $t = 1$, a k-tuple of functions $\phi_1, \ldots, \phi_k$ such that each ϕ_i is θ_i-plurisubharmonic solves (15) if and only if the k-tuple of Kähler metrics $(\theta_i + dd^c\phi_i)$ is a coupled Kähler–Ricci soliton. We prove:

Theorem 14. *Let V_i, α_i and P_i be as in Theorem 8 and assume* (12) *holds. Let x_0 be the point in X that, under the identification of $(\mathbb{C}^*)^n$ with its open, dense and free orbit, corresponds to the identity element in $(\mathbb{C}^*)^n$. Then, for any $t_0 > 0$ there is a constant C such that any solution $(\phi_1, \ldots, \phi_k)$ of* (15) *for $t \geq t_0$, normalized according to* (16), *satisfies*

$$\sup_X |\phi_i| < C$$

for all i.

Pingali [2018] reduced existence of coupled Kähler–Einstein metrics to a priori C^0-estimates. This means that Theorem 8 in the special case when $V_1 = \cdots = V_k = 0$, and thus Theorem 2, follows from Theorem 14 above and Pingali's work. For the general case we adapt the argument of Pingali to the soliton setting, essentially following the computations in [Tian and Zhu 2000]. Letting $\operatorname{Aut}(X)$ be the automorphism group of X we prove:

Theorem 15. *Let X be a Fano manifold and $V_1, \dots, V_k$ be holomorphic vector fields in the reductive part of the Lie algebra of* $\operatorname{Aut}(X)$ *such that* $\operatorname{Im} V_i$ *generate a compact one-parameter subgroup in* $\operatorname{Aut}(X)$ *for each i. Let (α_i) be a decomposition of $c_1(X)$ with representatives $\theta_1, \dots, \theta_k$ such that* $\operatorname{Im} L_{V_i}\theta_i = 0$ *for all i. Assume also C^0-estimates hold for* (15); *in other words, for each $t_0 > 0$, there is a constant C such that any solution (ϕ_i) to* (15) *at $t > t_0$ satisfies*

$$\sup_X |\phi_i| < C$$

for all i. Then (α_i) admits a solution to (9).

We get that the positive part of Theorem 8 follows directly from Theorems 14 and 15. The negative part of Theorem 8 follows directly from a change of variables in (13) (see Lemma 28).

Remark 16. Berman and Berndtsson [2013] used a variational approach to prove existence of Kähler–Ricci solitons on toric log Fano varieties. They give a direct argument for coercivity of the associated Ding functional on $(S^1)^n$-invariant metrics. It would be interesting if this coercivity estimate could be extended to the coupled setting. This would provide a stronger result than this paper in two respects: First of all, it would cover the singular setting of log Fano varieties. Secondly, since this bypasses the higher-order a priori estimates from complex geometry it would provide a version of Corollary 13 that is valid for all k-tuples of polytopes, not only the ones that are defined by decompositions of $c_1(X)$ on toric Fano manifolds.

This paper is organized in the following way: Sections 2.1 and 2.2 are devoted to the proof of Theorem 15. In Section 2.1 we prove openness along the continuity path and solvability at $t = 0$. In Section 2.2 we prove $C^{2,\alpha}$-estimates assuming C^0-estimates, thus finishing the proof of Theorem 15. In Section 3 we set up the real convex geometric framework and in Section 3.1 we use this to prove the C^0-estimate of Theorem 14. Finally, at the end of Section 3.1 we prove Theorem 8, Corollary 11 and Corollary 13 and in Section 3.2 we prove Theorem 2.

2. Openness and higher-order estimates

This section is devoted to proving Theorem 15.

The following lemma is well known. However, as a courtesy to the reader we include a proof of it.

Lemma 17. *Assume X is a Fano manifold, V a holomorphic vector field on X and θ a Kähler form on X such that the imaginary part of $L_V(\theta)$ vanishes. Then there is a smooth real-valued function g on X such that*

$$dd^c g = L_V(\theta).$$

Proof. Since V is a holomorphic vector field, the contraction operator i_V anticommutes with $\bar\partial$; hence $i_V\theta$ is a $\bar\partial$-closed $(0, 1)$-form. By the Kodaira vanishing theorem, since X is Fano, the sheaf cohomology group satisfies

$$H^1(X, \mathcal{O}) = H^1(X, -K_X + K_X) = 0.$$

This means the Dolbeault cohomology group satisfies

$$H^{(0,1)}(X) \cong H^1(X, \mathcal{O}) = 0;$$

hence $i_V\theta_i$ is also $\bar\partial$-exact. Let g be a smooth function such that $\sqrt{-1}\bar\partial g = i_V\theta$. As $L_V(\theta)$ is real, so is g. Moreover,

$$dd^c g = i\partial\bar\partial g = \partial i_V\theta = L_V(\theta). \qquad \square$$

For each i, let $\mathrm{PSH}(X, \theta_i)$ be the space of θ_i-plurisubharmonic functions on X, in other words the space of upper semicontinuous and locally integrable functions ϕ_i satisfying $dd^c\phi_i + \theta_i \geq 0$. Note that if ϕ_i is a smooth function in $\mathrm{PSH}(X, \theta_i)$, then

$$\begin{aligned} L_{V_i}(dd^c\phi) &= \partial i_{V_i}\sqrt{-1}\partial\bar\partial\phi_i \\ &= \sqrt{-1}\partial\bar\partial i_{V_i}\partial\phi_i = dd^c V_i(\phi_i); \end{aligned}$$

hence $dd^c(g_i + V_i(\phi_i)) = L_V(\theta_i + dd^c\phi_i)$. This means that, much as in [Hultgren and Nyström 2018], we get:

Lemma 18. *Let X be a Fano manifold, $V_1, \dots, V_k$ holomorphic vector fields on X and (α_i) a k-tuple of Kähler classes on X such that $\sum\alpha_i = c_1(X)$. Assume each class α_i has a representative θ_i such that* $\mathrm{Im}\, L_V(\theta_i) = 0$ *and, for each i, let ϕ_i be a smooth function in* $\mathrm{PSH}(X, \theta_i)$. *Then $(\phi_1, \dots, \phi_k)$ is a solution to* (15) *at $t = 1$ if and only if the k-tuple of Kähler metrics $(\theta_i + dd^c\phi_i)$ is a coupled Kähler–Ricci soliton.*

2.1. ***Openness.*** Here we will prove the first part of Theorem 15, namely that the set of t such that (15) is solvable is open.

We will use the Banach spaces

$$\begin{aligned} A &= \{(\phi_1, \dots, \phi_k) : \phi_i \in C^{4,\alpha}(X)\}, \\ B &= \{(v_1, \dots, v_k) : v_i \in C^{2,\alpha}(X)\}. \end{aligned}$$

Moreover, let A_{PSH} be the open subset of A given by

$$A_{\mathrm{PSH}} = \{(\phi_1, \dots, \phi_k) : \phi_i \in C^{4,\alpha}(X) \cap \mathrm{PSH}(X, \theta_i)\}.$$

Let $F : \mathbb{R} \times A_{\mathrm{PSH}} \to B$ be defined by

$$F(t, (\phi_i)) = \begin{pmatrix} \log((\theta_1 + dd^c\phi_1)^n/\omega_0^n) + g_1 + V_1(\phi_1) + t\sum\phi_i \\ \vdots \\ \log((\theta_k + dd^c\phi_k)^n/\omega_0^n) + g_k + V_k(\phi_k) + t\sum\phi_i \end{pmatrix}.$$

Note that $F(t, (\phi_i)) = 0$ if and only if (ϕ_i) defines a solution to (15) at t. Moreover, in this case the measure

$$\mu := (\theta_i + dd^c\phi_i)^n e^{g_i + V_i(\phi_i)}$$

is independent of i.

Lemma 19. *The Fréchet derivative of F at $(t, (\phi_i))$ with respect to the second argument is given by $H : A \to B$ defined by*

$$H(v_1, \ldots, v_k) = \begin{pmatrix} -\Delta_{\omega_1} v_1 + V_1(v_1) + t \sum v_i \\ \vdots \\ -\Delta_{\omega_k} v_k + V_k(v_k) + t \sum v_i \end{pmatrix}, \tag{17}$$

where $\omega_i = \theta_i + dd^c \phi_i$ and Δ_{ω_i} is the associated Laplace–Beltrami operator. Moreover, H is elliptic. Finally, assume $F(t, \phi) = 0$ and let $\langle \cdot , \cdot \rangle$ be the inner product on B given by

$$\langle (u_i), (v_i) \rangle = \sum_i \int_X u_i v_i \mu.$$

Then

$$\langle H(u_1, \ldots, u_k), (v_i) \rangle = \langle (u_i), H(v_1, \ldots, v_k) \rangle$$

for any $(u_i), (v_i) \in B$.

Proof. Equation (17) follows from straightforward differentiation and the well-known identity

$$\frac{d}{ds} \log \frac{(\theta_i + dd^c(\phi_i + s v_i))^n}{\theta_i^n} \bigg|_{s=0} = n \frac{dd^c v (\theta_i + dd^c \phi_i)^{n-1}}{(\theta_i + dd^c \phi_i)^n} = \Delta_{\omega_i} v_i.$$

Now, H takes the following form in local coordinates:

$$(u_i) \mapsto (v_j) = \left(\sum_{i,l,m} a_{ij}^{lm}(x) \frac{\partial^2 u_i}{\partial x_l \partial x_m} + \text{lower-order terms} \right),$$

where $a_{ij}^{lm} = 0$ if $i \neq j$ and $\{a_{ii}^{lm}\}_{l,m}$ are the coefficients for the Laplace operator Δ_{ω_i}. Recall that H is elliptic if the matrix

$$\left(\sum_{l,m} a_{ij}^{lm}(x) \xi_l \xi_m \right) \tag{18}$$

is invertible for all $p \in X$ and all nonzero $\xi = \sum \xi_l (\partial / \partial x_l) \in T_p X$, but this follows immediately. To see this note that

$$\sum_{l,m} a_{ij}^{lm}(x) \xi_l \xi_m$$

is 0 if $i \neq j$ and, by ellipticity of Δ_{ω_i}, positive if $i = j$. This means (18) is a diagonal matrix with positive entries on the diagonal; hence it is invertible.

We will now prove the last statement in the lemma. It is a consequence of the following identity for functions $u, v \in C^{2,\alpha}(X)$ (see Lemma 2.2 in [Tian and Zhu 2000]):

$$\int_X (\Delta_{\omega_i} v + V_i(v)) u \mu = - \int_X \langle dv, du \rangle_{\omega_i} \mu. \tag{19}$$

We get

$$\sum_i \int_X \Big(\Delta_{\omega_i} v_i + V_i(v_i) + \sum_j v_j\Big) u_i \mu = -\sum_i \int_X \langle dv_i, du_i\rangle \mu + \sum_{i,j} \int_X v_j u_i \mu$$
$$= \sum_i \int_X v_i \Big(\Delta_{\omega_i} u_i + V_i(u_i) + \sum_j u_j\Big)\mu,$$

and the last statement in the lemma follows. □

Lemma 20. *Assume $t \in [0, 1)$ and $(v_i) \in A$ are not all constant and satisfy*

$$\Delta_{\omega_1} v_1 + V_1(v_1) = \cdots = \Delta_{\omega_k} v_k + V_k(v_k) = \lambda \sum_i v_i \tag{20}$$

for a k-tuple $\omega_1, \ldots, \omega_k$ satisfying

$$\operatorname{Ric}\omega_1 - L_{V_1}(\omega_1) = \cdots = \operatorname{Ric}\omega_k - L_{V_k}(\omega_k) = t\sum_i \omega_i + (1-t)\sum_i \theta_i. \tag{21}$$

Then $\lambda > t$.

Proof. Let $\partial_{\omega_i} v$ denote the gradient of v with respect to the metric ω_i. Moreover, we will use the notation $\operatorname{Ric}_{\omega_i} = \operatorname{Ric}(\omega_i)$. The proof is based on the following Weitzenböck identity (see [Tian and Zhu 2000], equation 2.7, page 277):

$$-\int_X \langle d(\Delta_{\omega_i} v + V_i(v)), dv\rangle_{\omega_i} \mu \geq \int_X (\operatorname{Ric}_{\omega_i} - L_V(\omega_i))(\partial_{\omega_i} v, \overline{\partial_{\omega_i} v})\mu.$$

Combining this with (21) and (20) gives

$$\lambda^2 \int_X \Big(\sum_j v_j\Big)^2 \mu = \int_X (\Delta_{\omega_i} v_i + V_i(v_i))^2 \mu = -\int_X \langle d(\Delta_{\omega_i} v_i + V_i(v_i)), dv\rangle_{\omega_i} \mu$$
$$\geq \int_X (\operatorname{Ric}_{\omega_i} + L_V(\omega_i))(\partial_{\omega_i} v_i, \overline{\partial_{\omega_i} v_i})\mu \geq t \int_X \sum_j |\partial_{\omega_i} v_i|^2_{\omega_j} \mu. \tag{22}$$

Moreover, we claim that (20) implies

$$\int_X |\partial_{\omega_i} v_i|^2_{\omega_j} \mu \geq \int_X |dv_j|^2_{\omega_j} \mu \tag{23}$$

for any i and j. Assuming that this is true we see that (22) implies

$$\lambda^2 \int_X \Big(\sum_j v_j\Big)^2 \mu \geq t \int_X \sum_j |\partial_{\omega_j} v_j|^2_{\omega_j} \mu = t \int_X \sum_j |dv_j|^2_{\omega_j} \mu$$
$$= t \int_X \sum_j (\Delta_{\omega_j} v_j + V_i(v_j)) v_j \mu$$
$$= t\lambda \int_X \sum_j \Big(\sum_i v_i\Big) v_j \mu = t\lambda \int_X \Big(\sum_j v_j\Big)^2 \mu.$$

We conclude that $\lambda \geq t$. Moreover, if $\lambda = t$ then equality holds in all inequalities above. In particular, equality holds in the last inequality of (22); hence, by (21),

$$\begin{aligned} 0 &= \int_X (\mathrm{Ric}_{\omega_i} - L_V(\omega_i))(\partial_{\omega_i} v_i, \overline{\partial_{\omega_i} v_i})\mu - t\int_X \sum_j |\partial_{\omega_i} v_i|^2_{\omega_j}\mu \\ &= (1-t)\int_X \sum_j |\partial_{\omega_i} v_i|^2_{\theta_j}\mu \end{aligned}$$

from which it follows that v_i is constant for every i. This means that to finish the proof of the lemma it suffices to prove (23). To do this, note that for any i and j, by (20)

$$\begin{aligned} \int_X |dv_j|^2_{\omega_j}\mu &= \int_X (\Delta_{\omega_j} v_j + V_i(v_j))v_j\mu \\ &= \int_X (\Delta_{\omega_i} v_i + V_i(v_i))v_j\mu = \int_X \langle dv_i, dv_j\rangle_{\omega_i}\mu. \end{aligned}$$

Moreover, choosing coordinates $(z_1, \ldots, z_n)$ that are normal with respect to ω_j and such that ω_i is diagonal with eigenvalues $\beta_1, \ldots, \beta_n$ at a point p we get

$$|\langle dv_i, dv_j\rangle_{\omega_i}| = \left|\sum_l \frac{1}{\beta_l}\frac{\partial v_i}{\partial z_l}\overline{\frac{\partial v_j}{\partial z_l}}\right| \leq \sqrt{\sum_l \left|\frac{1}{\beta_l}\frac{\partial v_i}{\partial z_l}\right|^2}\sqrt{\sum_l \left|\frac{\partial v_j}{\partial z_l}\right|^2} = |\partial_{\omega_i} v_i|_{\omega_j}|dv_j|_{\omega_j}.$$

Combining this with the Cauchy–Schwarz inequality we get

$$\int_X |dv_j|^2_{\omega_j}\mu = \int_X \langle dv_i, dv_j\rangle_{\omega_i}\mu \leq \int_X |\partial_{\omega_i} v_i|_{\omega_j}|dv_j|_{\omega_j}\mu \leq \sqrt{\int_X |\partial_{\omega_i} v_i|^2_{\omega_j}\mu}\sqrt{\int_X |dv_j|^2_{\omega_j}\mu},$$

and (23) follows. □

We can now prove the first part of Theorem 15.

Proof of Theorem 15. First part: *openness and the case* $t = 0$. The theorem is proved using the continuity method along the path defined by (15). Here we will prove that the set of t such that (15) is solvable is nonempty and open in [0, 1]. At the end of Section 2.2 we will prove that it is also closed in [0, 1], hence that (15) is solvable for all $t \in [0, 1]$.

First of all, to see that the set of t such that (15) is solvable is nonempty, note that for $t = 0$, (15) reduces to the collection of equations

$$(\theta_j + dd^c\phi_j)^n e^{g_j + V_j(\phi_j)} = \omega_0^n. \tag{24}$$

This means that for each i we can apply the Main Theorem in [Zhu 2000] to get ϕ_i such that

$$(\theta_j + dd^c\phi_j)^n e^{g_j + V_j(\phi_j) + c_j} = \omega_0^n \tag{25}$$

for some $c_j \in \mathbb{R}$. Integrating both sides of this and using the fact that

$$\int_X e^{g_i + V_i(\phi_i)}(\theta_i + dd^c\phi_i)^n = \int_X e^{g_i}\theta_i^n = 1 = \int_X \omega_0^n \tag{26}$$

for all smooth $\phi_i \in \mathrm{PSH}(X, \theta_i)$ we see that $c_j = 0$ for all j; in other words $(\phi_1, \dots, \phi_k)$ provides a solution to (15) at $t = 0$.

Now, (26) is well known but for completeness we provide an argument for it here. Consider the variation of the left-hand side of (26) with respect to ϕ_i

$$\int_X (\Delta_{\omega_i}\dot{\phi}_i + V(\dot{\phi}_i))\mu_i, \tag{27}$$

where we use the notation $\mu_i = e^{g_i + V_i(\phi_i)}(\theta_i + dd^c\phi_i)^n$. By (19),

$$\int_X (\Delta_{\omega_i}\dot{\phi}_i + V(\dot{\phi}_i))(\dot{\phi}_i + 1)\mu_i = \int_X |d\dot{\phi}|^2_{\omega_i}\mu_i = \int_X (\Delta_{\omega_i}\dot{\phi}_i + V(\dot{\phi}_i))\dot{\phi}_i\mu_i, \tag{28}$$

and hence (27) vanishes. This proves (26).

The fact that the set of t such that (15) is solvable is open follows from Lemmas 19 and 20 and a standard application of the implicit function theorem. More precisely, H is elliptic by Lemma 19. This means the image of $H : (W^{2,2}(X))^k \to (L^2(X))^k$ is closed (see for example Theorem 10.4.7 in [Nicolaescu 1996]). Taking (v_i) in the orthogonal complement of the image of H gives

$$\langle (v_i), H(u_i)\rangle = 0$$

for all $(u_i) \in (W^{2,2}(X))^k$. In particular, it holds for all $(u_i) \in (C^\infty(X))^k$. By the last point in Lemma 19 this means $H(v_i) = 0$ as a distribution. By elliptic regularity (see for example Corollary 10.3.10 in [Nicolaescu 1996]) $(v_i) \in (C^\infty(X))^k$ and hence, by Lemma 20, $(v_i) = (C_i)$ for constants $C_1, \dots, C_k$. As $H(C_i) = 0$ we get $\sum C_i = 0$. Using this and elliptic regularity again (see for example Theorem 10.3.11(b) in [Nicolaescu 1996]), we may conclude that the kernel of H is $\{C_1, \dots, C_k : \sum C_i = 0\}$ and the image of H is

$$\widehat{B} = \left\{(v_i) \in B : \int_X v_1\mu = \cdots = \int_X v_k\mu\right\}. \tag{29}$$

It follows that H is invertible as a map from

$$\hat{A} = \{(v_i) \in A : v_1(x_0) = \cdots = v_k(x_0)\}$$

to $\widehat{B}$. Moreover, the derivative of F with respect to t, $(t, (\phi_i)) \mapsto \left(\sum \phi_i, \dots, \sum \phi_i\right)$, trivially maps to $\widehat{B}$. Thus, applying the implicit function theorem to F restricted to $\hat{A} \cap A_{\mathrm{PSH}}$ completes the proof of the theorem. □

2.2. ***Higher-order estimates.*** We begin with:

Lemma 21. *Assume* (ϕ_i) *satisfies* (15) *for some* $t \in [0, 1]$. *Then*

$$\sup_X |\Delta_{\theta_j}\phi_j| \le C,$$

where C *depends only on* $\sup_i \|\phi_i\|_{C^0(X)}$.

We will use the following lemma from [Zhu 2000] (page 768, Corollary 5.3):

Lemma 22. *Let X be a compact Kähler manifold, ω a Kähler form on X and V a holomorphic vector field on X. Assume $\phi \in \mathrm{PSH}(X,\omega)$ is smooth and $X(\phi)$ is a real-valued function. Then*

$$\sup_X |V(\phi)| < C$$

for a constant C that is independent of ϕ.

Proof of Lemma 21. We start with the following inequality originating in [Yau 1978] (see for example equation 2.3 on page 1587 in [Chen and He 2012]): assume ω is a Kähler form and v is a smooth function satisfying

$$(\omega + dd^c v)^n = e^F \omega^n.$$

Then there are constants C_1, C_2 and C_3, independent of v, such that

$$\Delta_{\omega+dd^c v}(e^{-C_1 v}(n+\Delta_\omega v)) \geq e^{-C_1 v}\Delta_\omega F + C_2(n+\Delta_\omega v)^{n/(n-1)} - C_3. \tag{30}$$

For each j, we have that ϕ_j satisfies the equation

$$(\theta_j + dd^c\phi_j) = e^{-g_j - V_i(\phi_j) - t\sum_i \phi_i + \log(\omega_0^n/\theta_j^n)}\theta_j^n. \tag{31}$$

Applying (30) to this and letting

$$u_j = e^{-C_1\phi_j}(n + \Delta_{\theta_j}\phi_j),$$

for all j we get

$$\Delta_{\omega_j} u_j \geq e^{-C_1\phi_j}\Delta_{\theta_j}\Bigl(-g_j - V_j(\phi_j) - t\sum_i \phi_i + \log(\omega_0^n/\theta_j^n)\Bigr) + C_2(n+\Delta_{\theta_j}\phi_j)^{n/(n-1)} - C_3. \tag{32}$$

Note that $dd^c\phi_i > -\theta_i$; hence

$$\Delta_{\theta_j}\phi_j = n\frac{(dd^c\phi_j)\wedge\theta_j^{n-1}}{\theta_j^n} > -n.$$

This means $u_j > 0$ for all j. Moreover, $u_j - e^{-C_1\phi_j}\Delta_{\theta_j}\phi_j = ne^{-C_1\phi_j}$. Hence, adjusting C_2 and C_3 in a way which only depends on $\sup_i \|\phi_i\|_{C^0(X)}$, we get

$$\Delta_{\omega_j} u_j \geq -e^{-C_1\phi_j}\Delta_{\theta_j}(g_j + V_j(\phi_j)) - t\sum_i u_i + C_2 u_j^{n/(n-1)} - C_3. \tag{33}$$

Now, let

$$V_j = \sum V_m^j \frac{\partial}{\partial z_m} \quad \text{and} \quad \theta_j = \sum \theta_{m\bar{l}}^j \, dz_m \, d\bar{z}_l.$$

As in [Tian and Zhu 2000], we compute

$$\begin{aligned}\Delta_{\theta_j}(g_j + V_j(\phi_j)) &= \sum_{m,l}\frac{\partial}{\partial z_l}\Bigl(V_m^j\Bigl(\theta_{m\bar{l}}^j + \frac{\partial\phi_j}{\partial z_m\partial\bar{z}_l}\Bigr)\Bigr)\\ &= \sum_{m,l}\frac{\partial V_m^j}{\partial z_l}\Bigl(\theta_{m\bar{l}}^j + \frac{\partial^2\phi_j}{\partial z_m\partial\bar{z}_l}\Bigr) + V_m^j\Bigl(\frac{\partial\theta_{m\bar{l}}^j}{\partial z_l} + \frac{\partial^3\phi_j}{\partial z_m\partial z_l\partial\bar{z}_l}\Bigr).\end{aligned} \tag{34}$$

We will be interested in this at a point, p, where u_j attains its maximum. Choosing coordinates around p that are normal with respect to θ_j and such that $\omega_j = \theta_j + dd^c\phi_j$ is diagonal, (34) reduces to

$$\sum_m \frac{\partial V_m^j}{\partial z_m}\left(1 + \frac{\partial^2 \phi_j}{\partial z_m \partial \bar{z}_m}\right) + V_j(\Delta\phi_j).$$

The first term of this can be bounded by

$$\sup_m \left|\frac{\partial V_m^j}{\partial z_m}\right|(1 + \Delta_{\theta_j}\phi_j).$$

Moreover, as u_j is stationary at p we get that

$$V_j(u_j) = C_1 V_j(\phi_j)u_j - e^{-C_1\phi_j} V_j(\Delta_{\theta_j}\phi_j)$$

vanishes at p; hence

$$(e^{-C_1\phi_j} V_j(\Delta_{\theta_j}\phi_j))|_p = (C_1 V_j(\phi_j)u_j)|_p.$$

We conclude that

$$e^{-C_1\phi_j}\Delta_{\theta_j}(g_j + V_j(\phi_j)) \le \left(\sup_m \left|\frac{\partial V_m^j}{\partial z_m}\right| + C_1 V_j(\phi_j)\right)u_j.$$

By Lemma 22 this is bounded by Cu_j for a uniform constant C.

We will now plug this into (33). By the maximum principle $\Delta_{\omega_j} u_j \le 0$ at p. Letting $M_i = \max_X u_i \ge 0$ we get

$$0 \ge -Cu_j - t\sum_i M_i + C_2 u_j^{n/(n-1)} - C_3$$

at p. Summing over j and using Young's inequality $a \le \epsilon a^{n/(n-1)} + C(n,\epsilon)$ we get, after adjusting C_3,

$$\begin{aligned} 0 &\ge -C\sum M_i - kt\sum M_i + \frac{C_2}{\epsilon}\sum M_i - C_3 \\ &= \left(-C - kt + \frac{C_2}{\epsilon}\right)\sum M_i - C_3. \end{aligned}$$

Choosing ϵ small enough that the expression in the parentheses is positive gives an upper bound on $\sum M_j$. Since $M_i \ge 0$ for all i, this implies a bound on $\sup M_i = \sup|u_i|$. This proves the lemma. ☐

Proof of Theorem 15. Second part: *$C^{2,\alpha}$-estimates.* Here we will prove that the set of t such that (15) is solvable is closed.

By Lemma 21, $|\Delta_{\theta_i}\phi_i|$ is bounded by a constant that depends only on $\|\phi_i\|_{C^0(X)}$ for all i. We wish to apply Theorem 1 in [Wang 2012]. To do this we need uniform bounds on the Hölder norms of ϕ_i and $V_i(\phi_i)$. These are implied by the uniform bounds on $\Delta_{\theta_i}\phi_i$. To see this, choose coordinates that are normal with respect to θ_i and such that $\theta_i + dd^c\phi_i$ is diagonal at a point p. Since

$$\theta_i + dd^c\phi_i = \sum\left(1 + \frac{\partial^2\phi_i}{\partial z_m \partial \bar{z}_m}\right) dz_m\, d\bar{z}_m > 0$$

we get that $\partial^2\phi_i/(\partial z_m\partial\bar{z}_m) > -1$ for all m. Together with the bound

$$\Delta_{\theta_i}\phi_i = \sum_m \frac{\partial^2\phi_i}{\partial z_m\partial\bar{z}_m} \le C$$

this gives uniform bounds on $|\partial^2\phi_i/(\partial z_m\partial\bar{z}_l)|$ for all m and l and the bounds on the Hölder norms follow.

Combining this with the argument at the end of Section 2.1, we conclude that the set of t such that (15) is solvable is nonempty, open and closed in [0, 1]. It follows that (15) has a solution (ϕ_i) at $t = 1$. Consequently, by Lemma 18 $(\theta_i + dd^c\phi_i)$ solves (9). □

3. C^0-estimates

In this section X will always be a toric Fano manifold. In other words $c_1(X) > 0$ and, letting $n = \dim X$, there is an n-dimensional complex torus $(\mathbb{C}^*)^n$ acting on X by biholomorphisms such that the action admits an open, dense and free orbit. The purpose of the section is to prove Theorem 14. We will begin by recalling the well-known correspondence between metrics on line bundles over toric varieties and convex functions in $\mathbb{R}^n$. As in the Introduction we fix an action of $(\mathbb{C}^*)^n$ on X and identify $(\mathbb{C}^*)^n$ with its open, dense and free orbit. Let θ be an $(S^1)^n$-invariant Kähler form on X that arises as the curvature of a metric $\|\cdot\|$ on a toric line bundle over X. Let P be the polytope associated to this toric line bundle. Assume s_0 is the $(\mathbb{C}^*)^n$-invariant section corresponding to the point $0 \in P$. By the invariance s_0 is nonvanishing on $(\mathbb{C}^*)^n$ and the metric can be represented by a plurisubharmonic function ψ on $(\mathbb{C}^*)^n$ by

$$\psi = -\log\|s_0\|^2.$$

Then ψ satisfies $dd^c\psi = \theta$. Using toric coordinates

$$(x_1, \dots, x_n) = (\log|z_1|, \dots, \log|z_n|) \in \mathbb{R}^n$$

ψ defines a convex function on $\mathbb{R}^n$

$$f(x_1, \dots, x_n) := \psi(e^{x_1}, \dots, e^{x_n})$$

which will have the property $\overline{\nabla f(\mathbb{R}^n)} = P$. Moreover, in logarithmic coordinates $\sigma_i = \log z_i$ we have

$$\sum_{ij} \frac{\partial^2 f}{\partial x_i\partial x_j}\, d\sigma_i\, d\bar{\sigma}_j = dd^c\psi = \theta. \tag{35}$$

Now, for a convex polytope P, let $E(P)$ be the space of smooth, strictly convex functions f such that

$$\overline{\nabla f(\mathbb{R}^n)} = P.$$

Then it is well known (see for example Proposition 3.3, page 687 in [Berman and Berndtsson 2013]) that (35) gives a one-to-one correspondence between the $(S^1)^n$-invariant elements in $[\theta]$ and $E(P)$.

As noted in the Introduction, the correspondence above extends trivially to any θ such that $[\theta]$ can be written as a linear combination with positive real coefficients of Kähler classes that arise as the curvature of toric line bundles. On the other hand, we have the following general principle which we record for the convenience of the reader:

Lemma 23. *Let α be a Kähler class on a Fano manifold X. Then there are some ample line bundles $L_1, \dots, L_m$ over X and positive real coefficients $\lambda_1, \dots, \lambda_m$ such that*

$$\alpha = \sum_i \lambda_i c_1(L_i). \tag{36}$$

Proof. First of all, any Kähler class α can be written as (36) where the line bundles L_i are not necessarily ample and the constants λ_i are not necessarily positive. To see this, recall that the map

$$c_1 : H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z})$$

is part of the exact sequence

$$H^1(X, \mathcal{O}^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}).$$

By the Kodaira vanishing theorem, since X is Fano,

$$H^2(X, \mathcal{O}) = H^2(X, K_X - K_X) = 0.$$

It follows that c_1 is surjective, and hence any element in $H^2_{DR}(X) \cong H^2(X, \mathbb{R})$ can be written as a linear combination over $\mathbb{R}$ of elements in the image of c_1. Note that this means the set of rational classes, in other words the set of classes of the form $qc_1(L)$ for some rational number q and some line bundle L, is dense in $H^{(1,1)}(X)$.

Now, the cone of Kähler classes K is open in $H^{(1,1)}(X)$. This means we can take a set of rational classes $\eta_1, \dots, \eta_j$ in K that span $H^{(1,1)}(X)$ over $\mathbb{R}$. Moreover, these classes define an open subcone of K,

$$C = \left\{ \sum_i \lambda_i \eta_i : \lambda_i \in \mathbb{R}_+ \right\}.$$

For any $\alpha \in K$ we may take a rational class η_0 in the open set $(\alpha - C) \cap K$ which is nonempty since α is in the interior of K. This means $\alpha = \eta_0 + \kappa$, where $\kappa \in C$ and (36) follows. □

Noting that any divisor on a toric manifold is linearly equivalent to an $(S^1)^n$-invariant divisor, Lemma 23 and the discussion preceding it gives:

Lemma 24. *Let α be a Kähler class on X and P be the polytope corresponding to α. Then* (35) *gives a one-to-one correspondence between the $(S^1)^n$-invariant elements in α and $E(P)$. Moreover, if $\alpha = c_1(L)$, where L is a toric line bundle over X, then this correspondence is given by $\theta \mapsto f$, where*

$$f(\log |z_1|, \dots, \log |z_n|) := -\log \|s_0\|^2,$$

where s_0 is the $(S^1)^n$-invariant (meromorphic) section corresponding to the point $0 \in \mathcal{M} \otimes \mathbb{R}$ and $\| \cdot \|$ is the metric on L with curvature θ.

For each i, let $h_i : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$h_i(x) = \log \frac{1}{N_P} \sum_y e^{\langle y, x\rangle},$$

where the sum is taken over all vertices of the polytope P_i and N_P is the number of vertices of the polytope P_i. These functions are smooth, strictly convex and satisfy $\overline{\nabla h_i(\mathbb{R}^n)} = P_i$; hence $h_i \in E(P_i)$. For each i, let θ_i be the element in α_i corresponding to h_i. Then there is a one-to-one correspondence between $E(P_i)$ and the smooth $(S^1)^n$-invariant elements of $\mathrm{PSH}(X, \theta_i)$ given by

$$f_i(x) - h_i(x) = \phi_i(e^x). \tag{37}$$

Moreover, $h_i(0) = 0$ for each i. This means the normalization (16) is equivalent to

$$f_1(0) = \cdots = f_k(0). \tag{38}$$

Using the correspondence in (37), it is possible to rewrite (15) as a real Monge–Ampère equation.

Lemma 25. *Assume (ϕ_i) and (f_i) are related as in* (37). *Then, for $t \in [0, 1]$, (ϕ_i) satisfies* (15) *if and only if (f_i) satisfies*

$$\frac{e^{\langle V_1, \nabla f_1\rangle}}{\mathrm{Vol}_{V_1}(P_1)} \det\left(\frac{\partial^2 f_1}{\partial x_m \partial x_l}\right) = \cdots = \frac{e^{\langle V_k, \nabla f_k\rangle}}{\mathrm{Vol}_{V_k}(P_k)} \det\left(\frac{\partial^2 f_k}{\partial x_m \partial x_l}\right) = e^{-t\sum_i f_i - (1-t)\sum_i h_i}. \tag{39}$$

Proof. First of all, using (35) we see that

$$(\theta_i + dd^c\phi_i)^n = \left(\sum_{m,l} \frac{\partial^2 f_i}{\partial x_m \partial x_l}\, d\sigma_j\, d\bar{\sigma}_l\right)^n = \det\left(\frac{\partial^2 f_i}{\partial x_m \partial x_l}\right) d\sigma\, d\bar{\sigma}, \tag{40}$$

where $d\sigma\, d\bar{\sigma} = d\sigma_1 \cdots d\sigma_n\, d\bar{\sigma}_1 \cdots d\bar{\sigma}_n$.

Abusing notation, we may think of f_i and h_i as $(S^1)^n$-invariant plurisubharmonic functions on $(\mathbb{C}^*)^n \subset X$. We will show that

$$e^{-t\sum_i \phi_i} \omega_0^n = e^{-t\sum_i (f_i - h_i)} \omega_0^n = e^{-t\sum_i f_i - (1-t)\sum_i h_i}\, d\sigma\, d\bar{\sigma}. \tag{41}$$

This will follow if we show that

$$e^{\sum h_i} \omega_0^n = d\sigma\, d\bar{\sigma}. \tag{42}$$

To do this, we note that by convexity

$$\overline{\nabla\left(\sum_i h_i\right)(\mathbb{R}^n)} = \overline{\sum_i \nabla h_i(\mathbb{R}^n)} = \sum P_i = P_{-K_X}.$$

By Lemma 24, $\sum h_i$ defines a metric on $-K_X$ of curvature $\sum \theta_i$ by the relation

$$\|s_0\|^2_{\sum h_i} = e^{-\sum h_i},$$

where s_0 is the unique $(\mathbb{C}^*)^n$-invariant section of $-K_X$, in other words

$$s_0 = \frac{\partial}{\partial \sigma_1} \wedge \cdots \wedge \frac{\partial}{\partial \sigma_k} = d\sigma^{-1}.$$

Moreover, the volume form ω_0^n defines a metric on $-K_X$ by the relation

$$\|d\sigma^{-1}\|^2_{\omega_0^n} = \frac{\omega_0^n}{d\sigma\, d\bar{\sigma}}.$$

The curvature of $\|\cdot\|_{\omega_0^n}$ is $\operatorname{Ric}\omega_0 = \sum \theta_i$. By uniqueness in the Calabi–Yau theorem $\|\cdot\|_{\sum h_k} = \|\cdot\|_{\omega_0^n}$ and (42) follows.

It remains to show that

$$\frac{e^{\langle V_i, \nabla f_i\rangle}}{\operatorname{Vol}_{V_i}(P_i)} = e^{g_i + V_i(\phi_i)}. \tag{43}$$

We will first show that

$$\langle V_i, \nabla f_i\rangle + C_i = g_i + V_i(\phi_i) \tag{44}$$

for some $C_i \in R$. Abusing notation again, and thinking of f_i as an $(S^1)^n$-invariant plurisubharmonic function on $(\mathbb{C}^*)^n \subset X$, we compute

$$\begin{aligned} dd^c\langle V_i, \nabla f_i\rangle = dd^c\bigg(\sum_m \frac{\partial f_i}{\partial x_m} a_m\bigg) &= \sum_{m,j,l} \frac{\partial^3 f_i}{\partial x_j \partial x_l \partial x_m} a_m\, d\sigma_j\, d\bar{\sigma}_l \\ &= \partial i_V \bigg(\sum_{m,l} \frac{\partial^2 f_i}{\partial x_m \partial x_l}\, d\sigma_m\, d\bar{\sigma}_l\bigg) \\ &= \partial i_V(\theta_i + dd^c\phi_i) = L_V(\theta_i) = dd^c(g_i + V_i(\phi_i)) \end{aligned}$$

and (44) follows by the maximum principle. To get (43), note that the push forward of $d\sigma\, d\bar{\sigma}$ under the map $(z_1, \ldots, z_n) \mapsto (\log|z_1|, \ldots, \log|z_n|)$ is the Euclidean measure dx on $\mathbb{R}^n$. This means, by (40) and (44),

$$\int_X e^{g_i + V_i(\phi_i)}(\theta_i + dd^c\phi_i)^n = \int_{\mathbb{R}^n} \det\bigg(\frac{\partial^2 f_i}{\partial x_m \partial x_l}\bigg) e^{\langle V_i, \nabla f_i\rangle + C_i}\, dx. \tag{45}$$

Performing the change of variables $\nabla f_i = p$ we get

$$(45) = e^{C_i} \int_{P_i} e^{\langle V_i, p\rangle}\, dp.$$

By (26)

$$\int_X e^{g_i + V_i(\phi_i)}(\theta_i + dd^c\phi_i)^n = \int_X e^{g_i}\theta_i^n = 1.$$

This means $C = \log \operatorname{Vol}_{V_i}(P_i)$ and (43) follows.

Using (40), (41) and (43) we conclude that (f_i) satisfies (13) if and only if (ϕ_i) satisfies (15) on $(\mathbb{C}^*)^n$. As (ϕ_i) is assumed to be smooth, the lemma follows. □

3.1. ***Estimates.*** To prove Theorem 14 we need to prove that for all $t_0 > 0$ there is a constant C such that any solution (f_i) to (39) at $t > t_0$, normalized according to (38), satisfies

$$\sup_X |f_i - h_i| \le C \tag{46}$$

for all i.

For each i, let u_i be the Legendre transform of f_i. Recall that f_i is a smooth, strictly convex function on $\mathbb{R}^n$ such that $\overline{\nabla f_i(\mathbb{R}^n)} = P_i$. This means each u_i is a smooth, strictly convex function on P_i. Moreover, a standard property of the Legendre transform is that

$$\sup_{\mathbb{R}^n} |f_i - h_i| = \sup_{P_i} |u_i - h_i^*|,$$

where h_i^* is the Legendre transform of h_i. Since h_i^* is bounded on P_i (this is easy to verify) we have that (46) is equivalent to a uniform bound on $\sup_{P_i} |u_i|$.

We will use a variant of the method of [Wang and Zhu 2004] (see also [Donaldson 2008]). The first step is to establish bounds on the function

$$w = w_t = \sum_i (t f_i + (1-t) h_i).$$

Since w is strictly convex and 0 is in the interior of $P_{-K_X} = \overline{\nabla w(\mathbb{R}^n)}$ we have that w is bounded from below and attains its minimal value at a unique point. Let $m = \inf w$ and let x_w be the minimal point of w.

Lemma 26. *Assume $t_0 > 0$ and* (12) *holds. Then there are constants C and ϵ such that if (f_i) is a solution to* (39) *at $t > t_0$, then*

$$w \geq \epsilon |x - x_w| - C \tag{47}$$

and

$$|x_w| \leq C. \tag{48}$$

The proof of Lemma 26 follows one of the arguments in [Donaldson 2008], which is based on [Wang and Zhu 2004]. The main point is the following convex geometric fact (see Proposition 2 in [Donaldson 2008]).

Lemma 27. *Assume f is a convex function on $\mathbb{R}^n$ attaining minimal value* 0, *and suppose*

$$\det\left(\frac{\partial^2 f}{\partial x_m \partial x_l}\right) \geq \lambda$$

on $K = \{f \leq 1\}$. Then

$$\operatorname{Vol}(K) \leq C \lambda^{-1/2}$$

for some constant C depending only on the dimension n.

Using Lemma 27 we can prove Lemma 26.

Proof of Lemma 26. The proof proceeds in four steps:

Step 1: m is bounded from below. Let ρ_{-K_X} be the support function of P_{-K_X} defined by

$$\rho_{-K_X}(x) = \sup_{p \in P_{-K_X}} \langle x, p \rangle.$$

Since $\nabla w(\mathbb{R}^n) = P_{-K_X}$ we have $w \leq m + \rho_{-K_X}$. Moreover, by the change of variables $p = \nabla f_i$

$$1 = \frac{\int_{P_i} e^{\langle V_i, p \rangle}\, dp}{\operatorname{Vol}_{V_i}(P_i)} = \int_{\mathbb{R}^n} \frac{e^{\langle V_i, \nabla f_i \rangle}}{\operatorname{Vol}_{V_i}(P_i)} \det\left(\frac{\partial^2 f_i}{\partial x_m \partial x_l}\right) dx = \int_{\mathbb{R}^n} e^{-w}\, dx \geq C e^{-m} \int_{\mathbb{R}^n} e^{-\rho_{-K_X}}\, dx \geq C e^{-m},$$

possibly changing C in the last inequality. This means m is bounded from below by a uniform constant.

Step 2: m is bounded from above. By monotonicity of the determinant function and convexity we have

$$\det\left(\frac{\partial^2 w}{\partial x_m \partial x_l}\right) = \det\left[t \sum_i \left(\frac{\partial^2 f_i}{\partial x_m \partial x_l}\right) + (1-t) \sum_i \left(\frac{\partial^2 h_i}{\partial x_m \partial x_l}\right)\right]$$
$$\geq t_0^n \det\left(\frac{\partial^2 f_j}{\partial x_m \partial x_l}\right) = t_0^n \operatorname{Vol}_{V_j}(P_j) e^{-\langle V_j, \nabla f_j\rangle - w} \geq C e^{-w} dx,$$

where the last inequality follows from the fact that $\overline{\nabla f_j(\mathbb{R}^n)} = P_j$ is bounded. This means

$$\det\left(\frac{\partial^2 w}{\partial x_m \partial x_l}\right) \geq C e^{-m-1}$$

on $K = \{w \leq m+1\}$. By Lemma 27, possibly redefining C,

$$\operatorname{Vol}(K) \leq C e^{m/2}. \tag{49}$$

Convexity of w and the coarea formula give

$$1 = \int_{\mathbb{R}^n} e^{-w}\, dx \leq C e^{-m/2}.$$

This means m is bounded from above.

Step 3: $w \geq \epsilon|\cdot - x_w| - m + 1$ for uniform constants ϵ and C. Since $\overline{\nabla w(\mathbb{R}^n)} = P_{-K_X}$ and P_{-K_X} is bounded we have that there is a uniform constant $r > 0$ such that K contains a small ball centered at x_w of radius r. If there was a point in K far from x_w then the volume of K would be very big, contradicting (49). This means K is contained in a ball centered at x_w of radius R for some uniform constant R. Convexity of w gives

$$w(x) \geq \begin{cases} R^{-1}|x - x_w| + m & \text{if } x \notin K, \\ m & \text{if } x \in K. \end{cases}$$

Moreover, $R^{-1}|x - x_w| \leq 1$ on K. This means putting $\epsilon = 1/R$ finishes Step 3.

Step 4: $|x_w|$ is bounded. In this step we will use the assumption (12). By the divergence theorem, since $e^{-w} \to 0$ exponentially as $|x| \to \infty$,

$$\int_{\mathbb{R}^n} \nabla w e^{-w}\, dx = \int_{\mathbb{R}^n} \operatorname{div} \nabla(e^{-w})\, dx = 0.$$

Moreover,

$$\int_{\mathbb{R}^n} \nabla\left(\sum_i f_i\right) e^{-w}\, dx = \sum_i \int_{\mathbb{R}^n} \nabla f_i e^{-w}\, dx$$
$$= \sum_i \frac{1}{\operatorname{Vol}_{V_i}(P_i)} \int_{\mathbb{R}^n} \nabla f_i e^{\langle V_i, \nabla f_i\rangle} \det\left(\frac{\partial^2 f_i}{\partial x_m \partial x_l}\right) dx$$
$$= \sum_i \frac{1}{\operatorname{Vol}_{V_i}(P_i)} \int_{P_i} p e^{\langle V_i, p\rangle}\, dp = 0,$$

where the last two equalities are given by performing the change of variables $p = \nabla f_i(x)$ in each summand and (12). This means

$$\int_{\mathbb{R}^n} \nabla\Big(\sum_i h_i\Big) e^{-w}\, dx = 0. \tag{50}$$

Recall that $\sum h_i$ is convex and hence $\nabla\big(\sum_i h_i\big)$ is monotone. Hence, if $|x_w|$ is large then, putting $v = x_w/|x_w|$, we get that $\langle x, v\rangle$ is positive and bounded away from 0 on some large ball centered at x_w. By (47) the mass of $e^{-w}\, dx$ is concentrated around x_w. This contradicts (50). □

We can now prove Theorem 14.

Proof of Theorem 14. First of all, by the change of variables $x = \nabla u_i(p)$ and (15) we have

$$\begin{aligned}\int_{P_i} |\nabla u_i|^q\, dp &= \int_{\mathbb{R}^n} |x|^q \det\Big(\frac{\partial^2 f_i}{\partial x_m \partial x_l}\Big)\, dx \\ &\le \operatorname{Vol}_{V_i}(P_i) \int_{\mathbb{R}^n} |x|^q e^{-\langle V_i, \nabla f_i\rangle - w}\, dx \\ &\le C \int_{\mathbb{R}^n} |x|^q e^{-w}\, dx \le C_q,\end{aligned} \tag{51}$$

where the second inequality follows from the fact that $\overline{\nabla f_i(\mathbb{R}^n)} = P_i$ is bounded and the last inequality follows from Lemma 26. Put $q = n+1$ and

$$\hat{u}_i = \frac{1}{\operatorname{Vol}(P_i)} \int_{P_i} u_i\, dp.$$

By Morrey's inequality (see [Haškovec and Schmeiser 2009]) we have

$$\begin{aligned}\|u_i - \hat{u}_i\|_{C^{0,\gamma}(P_i)} &\le C\|u_i - \hat{u}_i\|_{W^{1,q}(P_i)} \\ &= C\|u_i - \hat{u}_i\|_{L^q(P_i)} + C\|\nabla u_i\|_{L^q(P_i)},\end{aligned} \tag{52}$$

where $\gamma = 1 - n/q$. By the Poincaré–Wirtinger inequality this can be bounded by

$$C\|\nabla u_i\|_{L^q(P_i)}$$

for some C. This is bounded by (51). Since P_i is bounded we may conclude from this that

$$\sup_{p_1, p_2 \in P_i} |u_i(p_1) - u_i(p_2)| \le C\|u_i - \hat{u}_i\|_{C^{0,\gamma}(P_i)} \le C. \tag{53}$$

This means it suffices to bound each u_i in some point.

To bound each u_i in some point, note that by general properties of the Legendre transform $f_i(0) = -u_i(\nabla f_i(0))$. This means

$$|u_i(\nabla f_i(0))| = |f_i(0)| = \frac{1}{k}\Big|\sum_j f_j(0)\Big| = \frac{1}{k}|w(0)|,$$

where the last two equalities follow from (38) and the fact that $h_i(0) = 0$ for all i. Since $|x_w|$ is bounded and $\nabla w \in P_{-K_X}$ is bounded we have that $|w(0) - w(x_w)|$ is bounded. By Lemma 26, $|w(x_w)| = |m|$ is bounded. This means $|u_i(\nabla f_i(0))|$ and hence, by (53), $\sup_{P_i}|u_i|$ is bounded for each i. By the discussion following (46) this proves the theorem. □

Proof of Theorem 8. Assuming (12) holds, existence of coupled Kähler–Ricci solitons follow directly from Theorem 14 and Theorem 15. Indeed, any toric holomorphic vector field V_i is in the reductive part of the Lie algebra of $\mathrm{Aut}(X)$. Moreover, $\mathrm{Im}\, V_i$ generates a compact one-parameter subgroup of $\mathrm{Aut}(X)$ and, since θ_i is $(S^1)^n$-invariant, $\mathrm{Im}\, L_V(\theta_i)=0$.

Assume that (α_i) admits a coupled Kähler–Ricci soliton. By Lemmas 18 and 25, (13) admits a solution. Then (12) follows from Lemma 28 below. □

Lemma 28. *Assume* (13) *admits a solution. Then*

$$\sum_i \mathcal{A}_{P_i}(V_i)=0.$$

Proof. Let (f_i) be a solution to (13). As in the proof of Lemma 26, by the divergence theorem, since $e^{-\sum f_i}\to 0$ exponentially as $|x|\to\infty$,

$$\int_{\mathbb{R}^n}\Big(\sum_i \nabla f_i\Big)e^{-\sum_i f_i}\,dx=\int_{\mathbb{R}^n}\operatorname{div}\nabla(e^{-\sum_i f_i})\,dx=0. \tag{54}$$

On the other hand, by (13)

$$(54)=\sum_i\int_{\mathbb{R}^n}\nabla f_i e^{-\sum_i f_i}\,dx=\sum_i\int_{\mathbb{R}^n}\nabla f_i\frac{e^{\langle V_i,\nabla f_i\rangle}}{\mathrm{Vol}_{V_i}(P_i)}\det\Big(\frac{\partial^2 f_i}{\partial x_m\partial x_l}\Big)\,dx.$$

Performing the change of variables $\nabla f_i=p$ in each summand gives that the right-hand side of this equals

$$\sum_i\frac{1}{\mathrm{Vol}_{V_i}(P_i)}\int_{P_i}pe^{\langle V_i,p\rangle}\,dp=\sum_i\mathcal{A}_{P_i}(V_i).$$ □

Proof of Corollary 11. Note that

$$\sum_i\mathcal{A}_{P_i}(V) \tag{55}$$

is the gradient of the function on $\mathbb{R}^n$ defined by

$$V\mapsto\sum_i\log\int_{P_i}e^{\langle V,p\rangle}\,dp.$$

This is strictly convex and proper (in fact, its gradient image is $\sum_i P_i=P_{-K_X}$, which contains zero as an interior point); hence it admits a unique minimum. Letting V be this minimum means (12) is fulfilled. The corollary then follows from Theorem 8. □

Proof of Corollary 13. The corollary follows from Theorem 8 and Lemma 25. □

3.2. ***Toric test configurations and proof of Theorem 2.*** Theorem 2 will follow from Theorem 8 combined with Theorem 1.15 in [Hultgren and Nyström 2018] and an explicit calculation of the Donaldson–Futaki invariant of test configurations induced by toric vector fields.

In [Hultgren and Nyström 2018] a type of test configuration for decompositions of $c_1(X)$ was defined. The data defining them is essentially given by k test configurations $(\mathcal{X}_1, \mathcal{L}_1), \dots, (\mathcal{X}_k, \mathcal{L}_k)$, where $\mathcal{X}_1 = \cdots = \mathcal{X}_k =: \mathcal{X}$, such that $(\mathcal{X}, \sum_i \mathcal{L}_i)$ defines a test configuration for $(X, -K_X)$. The Donaldson–Futaki invariant associated to this data is defined as the intersection number

$$\mathrm{DF}(\mathcal{X}, (\mathcal{L}_i)) = -\sum_i \frac{\mathcal{L}^{n+1}}{|\alpha_i|} - (n+1)\frac{\big(-K_{\mathcal{X}/\mathbb{P}^1} - \sum_i \mathcal{L}_i\big)\cdot\big(\sum_i \mathcal{L}_i\big)^n}{(-K_X)^n}, \tag{56}$$

where $|\alpha_i| = \int_X \theta^n$ for any θ such that $[\theta] = \alpha$. We point out that the notation here differs from [Hultgren and Nyström 2018] in that here $(\mathcal{X}, \mathcal{L}_i)$ are the ($\mathbb{C}^*$-invariantly) compactified test configurations over $\mathbb{P}^1$.

Now, recall that if L is a toric line bundle over a toric manifold X, then a toric vector field V induces a test configuration $(\mathcal{X}^V, \mathcal{L}^V)$ for (X, L). This can be described in the following way: Let $d_1, \dots, d_k \in N \otimes \mathbb{R}$ and $c_1, \dots, c_k \in \mathbb{R}$ be the data defining the polytope P_L, i.e.,

$$P_L = \{\langle d_i, \cdot\rangle \geq -c_i\}.$$

Then, the polytope of $\mathcal{L}^V$ can be arranged to be

$$P_{\mathcal{L}^V} = \{\langle d_i, \cdot\rangle \geq -c_i\} \cap \{\langle d_0 + V, \cdot\rangle \geq 0\} \cap \{\langle -d_0, \cdot\rangle \geq -C_{\mathcal{L}^V}\},$$

where d_0 corresponds to the divisor given by the central fiber of $\mathcal{X}$ and $C_{\mathcal{L}^V}$ is a number that can be modified without changing the Donaldson–Futaki invariant by adding a multiple $\mathcal{O}_{\mathbb{P}^1}(1)$ to $\mathcal{L}^V$. In particular, as long as $C_{\mathcal{L}^V}$ is big enough for $\mathcal{L}^V$ to be ample,

$$(\mathcal{L}^V)^{n+1} = \mathrm{Vol}(P_{\mathcal{L}^V}) = \mathrm{Vol}(P_L)(C_{\mathcal{L}^V} + \langle V, b(P_L)\rangle).$$

This also gives

$$\begin{aligned}(n+1)\mathcal{O}_{\mathbb{P}^1}(1)\cdot(\mathcal{L}^V)^n &= \frac{d}{dt}(\mathcal{L}^V + t\,O_{\mathbb{P}^1}(1))^{n+1} \\ &= \frac{d}{dt}\,\mathrm{Vol}(P_{\mathcal{L}^V + t\,O_{\mathbb{P}^1}(1)}) = \mathrm{Vol}(P_L).\end{aligned} \tag{57}$$

Finally, we note that if $L = -K_X$ then $\mathcal{L}^V$ is the relative canonical bundle of $\mathcal{X}^V$ up to a twist determined by $C_{\mathcal{L}^V}$.

$$\mathcal{L}^V = -K_{\mathcal{X}^V/\mathbb{P}^1} + C_{\mathcal{L}^V}\mathcal{O}_{\mathbb{P}^1}(1). \tag{58}$$

Proof of Theorem 2. Putting $V_1 = \cdots = V_k = 0$ gives

$$\sum_i \mathcal{A}_{P_i}(V_i) = \sum_i b(P_i);$$

hence it follows from Theorem 8 that part (iii) of the theorem implies part (i). Moreover, (i) implies (ii) by Theorem 1.15 in [Hultgren and Nyström 2018]. Thus, to finish the proof of Theorem 2, it suffices to prove that (ii) implies (iii).

We will prove the contrapositive. Assume $\sum_i b(P_i) \neq 0$; in other words $\sum_i \langle V, b(P_i)\rangle < 0$ for some toric vector field V. Let $(\mathcal{X}^V, (\mathcal{L}_i^V))$ be the associated test configuration. As $(\mathcal{X}^V, \sum_i \mathcal{L}_i^V)$ is a test configuration for $-K_X$ we get, using (58) and $|\alpha_i| = \mathrm{Vol}(P_i)$,

$$\begin{aligned} \mathrm{DF}(\mathcal{X}^V, (\mathcal{L}_i^V)) &= \sum_i \frac{(\mathcal{L}_i^V)^{n+1}}{\mathrm{Vol}(P_i)} - (n+1)\frac{\big(\sum_i C_{\mathcal{L}_i^V}\big)\mathcal{O}_{\mathbb{P}^1}(1)\cdot\big(\sum_i \mathcal{L}_i^V\big)^n}{\mathrm{Vol}(P_{-K_X})} \\ &= \sum_i (C_{\mathcal{L}_i^V} + \langle V, b(P_L)\rangle) - \sum C_{\mathcal{L}_i^V} \\ &= \sum_i \langle V, b(P_i)\rangle < 0, \end{aligned} \tag{59}$$

and hence (α_i) is not K-polystable. □

3.3. *Proof of Corollary 4.*

Proof of Corollary 4. First of all, by [Futaki et al. 1990] (see also [Futaki 1983; Wang 1991]) the Futaki invariant of X is nonzero; hence X does not admit a Kähler–Einstein metric. To prove the rest of the corollary, we fix a $(\mathbb{C}^*)^4$-action on X in the following way: Consider the standard embeddings of $\mathcal{O}_{\mathbb{P}^2}(-1)$ and $\mathcal{O}_{\mathbb{P}^1}(-1)$ in to $\mathbb{C}^3 \times \mathbb{P}^2$ and $\mathbb{C}^2 \times \mathbb{P}^1$ respectively:

$$\mathcal{O}_{\mathbb{P}^2}(-1) = \{((z_0, z_1, z_2), (a_0 : a_1 : a_2)) : z_0a_1{=}z_1a_0,\ z_1a_2{=}z_2a_1\},$$
$$\mathcal{O}_{\mathbb{P}^1}(-1) = \{((w_0, w_1), (b_0 : b_1)) : w_0b_1{=}w_1b_0\}.$$

We get an embedding of $X = \mathbb{P}(E)$ into $\mathbb{P}^4 \times \mathbb{P}^2 \times \mathbb{P}^1$ as

$$X = \{((z_0 : z_1 : z_2 : w_0 : w_1), (a_0 : a_1 : a_2), (b_0 : b_1)) : z_0a_1 = z_1a_0,\ z_1a_2 = z_2a_1,\ w_0b_1 = w_1b_0\}.$$

We define a $(\mathbb{C}^*)^4$-action by letting an element $(t_1, t_2, t_3, t_4) \in (\mathbb{C}^*)^4$ act on X by

$$((z_0 : z_1 : z_2 : w_0 : w_1), (a_0 : a_1 : a_2), (b_0 : b_1)) \mapsto ((z_0 : t_1z_1 : t_2z_2 : t_4w_0 : t_4t_3w_1), (a_0 : t_1a_1 : t_2a_2), (b_0 : t_3b_1)).$$

The invariant divisors are

$$D_1 = \{z_0{=}a_0{=}0\},\quad D_2 = \{z_1{=}a_1{=}0\},\quad D_3 = \{z_2{=}a_2{=}0\},\quad D_4 = \{w_0{=}b_0{=}0\},$$
$$D_5 = \{w_1{=}b_1{=}0\},\quad D_6 = \{z_0{=}z_1{=}z_2{=}0\},\quad D_7 = \{w_0{=}w_1{=}0\}$$

corresponding to the following elements in the lattice $N \cong \mathbb{Z}^4$ of one-parameter subgroups of $(\mathbb{C}^*)^4$:

$$d_1 = (-1, -1, 0, -1),\quad d_2 = (1, 0, 0, 0),\quad d_3 = (0, 1, 0, 0),\quad d_4 = (0, 0, -1, 1),$$
$$d_5 = (0, 0, 1, 0),\quad d_6 = (0, 0, 0, -1),\quad d_7 = (0, 0, 0, 1).$$

The divisor corresponding to $-K_X$ is $\sum_{i=1}^7 D_i$. For $c \in \left(\frac{1}{4}, \frac{3}{4}\right)$, we will be interested in divisors of the form

$$D(c) = c(D_4 + D_5) + \sum_{i \neq 4,5} \tfrac{1}{2} D_i$$

corresponding to polytopes

$$P(c)=\left\{y\in\mathbb{R}^4:\langle y,d_i\rangle\le\tfrac{1}{2},\ i\ne 4,5,\ \langle y,d_i\rangle\le c,\ i=4,5\right\}. \tag{60}$$

Note that the two classes in (5) are given by $D(c)$ and $D(1-c)$ for

$$c=\tfrac{1}{2}+\tfrac{1}{4}\sqrt{\tfrac{5}{7}}\in\left(\tfrac{1}{4},\tfrac{3}{4}\right). \tag{61}$$

To prove the corollary we will verify the following two facts:

- As long as $c\in\left(\frac{1}{4},\frac{3}{4}\right)$, none of the conditions in (60) are redundant. (By standard theory for toric varieties this implies $D(c)$ and $D(-c)$ are ample and hence β_1 and β_2 are Kähler.)
- We have
$$\frac{\int_{P(c)}y\,dy}{\int_{P(c)}dy}+\frac{\int_{P(1-c)}y\,dy}{\int_{P(1-c)}dy}=0$$
when c is given by (61).

Note that both these conditions are invariant under linear transformations of R^n. Applying to the generators $d_1,\dots,d_7$ the linear transformation

$$A=\begin{bmatrix}1&0&0&-2\\0&1&0&-2\\0&0&1&3\\0&0&0&6\end{bmatrix}$$

gives new generators

$$\begin{aligned}&d_1'=(-1,-1,0,-2), && d_2'=(1,0,0,-2), && d_3'=(0,1,0,-2), && d_4'=(0,0,-1,3),\\ &d_5'=(0,0,1,3), && d_6'=(0,0,0,6), && d_7'=(0,0,0,-6),\end{aligned}$$

and a new polytope

$$P'(c)=\left\{y\in\mathbb{R}^4:\langle y,d_i'\rangle\le\tfrac{1}{2},\ i\ne 4,5,\ \langle y,d_i'\rangle\le c,\ i=4,5\right\}. \tag{62}$$

It is straightforward to check that as long as $c\in\left(\frac{1}{4},\frac{3}{4}\right)$, none of the conditions in (62) are redundant; hence $D(c)$ is ample for any $c\in\left(\frac{1}{4},\frac{3}{4}\right)$. Moreover, the sets $\{d_1',d_2',d_3',d_6',d_7'\}$ and $\{d_4',d_5'\}$ are both invariant under the linear transformation

$$B=\begin{bmatrix}0&-1&0&0\\1&-1&0&0\\0&0&-1&0\\0&0&0&1\end{bmatrix}.$$

It follows that $P'(c)$ and hence the barycenter of $P'(c)$ is invariant under B. As any fixed point of B is parallel to $(0,0,0,1)$ we conclude that

$$\int_{P'(c)}y_1\,dy=\int_{P'(c)}y_2\,dy=\int_{P'(c)}y_3\,dy=0.$$

Moreover, we denote by S_2 the two-dimensional simplex corresponding to the anticanonical bundle of $\mathbb{P}^2$

$$S_2 = \{y \in \mathbb{R}^2 : y_1 \le 1,\ y_2 \le 1,\ -y_1 - y_2 \le 1\}$$

and note that $(y_1, \ldots, y_4) \in P'(c)$ if and only if $y_4 \in \left(-\frac{1}{12}, \frac{1}{12}\right)$, $|y_3| \le c - 3y_4$ and $(y_1, y_2) \in \left(\frac{1}{2} + 2y_4\right)S_2$. We get

$$\begin{aligned}\int_{P'(c)} y_4\, dy &= \int_{\frac{1}{12}[-1,1]} y_4 \left(\int_{(\frac{1}{2}+2y_4)S_2} dy_1\, dy_2\right)\left(\int_{(c-3y_4)[-1,1]} dy_3\right) dy_4 \\ &= 2\,\mathrm{Vol}(S_2) \int_{\frac{1}{12}[-1,1]} y_4 \left(\tfrac{1}{2} + 2y_4\right)^2 (c - 3y_4)\, dy_4 = \frac{5c-2}{720}\end{aligned}$$

and similarly

$$\int_{P'(c)} dy = 2\,\mathrm{Vol}(S_2) \int_{\frac{1}{12}[-1,1]} \left(\tfrac{1}{2} + 2y_4\right)^2 (c - 3y_4)\, dy_4 = \frac{56c-3}{144}.$$

It follows that

$$\frac{\int_{P'(c)} y_4\, dy}{\int_{P'(c)} dy} + \frac{\int_{P'(1-c)} y_4\, dy}{\int_{P'(1-c)} dy} = \frac{1}{5}\left(\frac{5c-2}{56c-3} + \frac{5(1-c)-2}{56(1-c)-3}\right) = \frac{(112c^2 - 112c + 23)}{(56c-53)(56c-3)}, \tag{63}$$

which vanishes as

$$c = \tfrac{1}{2} \pm \tfrac{1}{4}\sqrt{\tfrac{5}{7}} \in \left(\tfrac{1}{4}, \tfrac{3}{4}\right). \qquad \square$$

Acknowledgements

The author would like to thank David Witt Nyström for many fruitful discussions relating to this paper and Thibaut Delcroix for his suggestion to allow the k vector fields in (9) to be different from one another. Moreover, the author would like to thank Yanir Rubinstein for directing him to [Futaki 1983] when looking for an example of a manifold with nonzero Futaki invariant but reductive automorphism group. The latter suggestion led to Corollary 4. Finally, the author would like to thank Erlend F. Wold for discussions related to the combinatorial aspects of Corollary 4.

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Received 9 Apr 2018. Accepted 18 Oct 2018.

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ANALYSIS AND PDE
Vol. 12, No. 8, 2019
dx.doi.org/10.2140/apde.2019.12.2095

CARLESON MEASURE ESTIMATES AND THE DIRICHLET PROBLEM FOR DEGENERATE ELLIPTIC EQUATIONS

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We prove that the Dirichlet problem for degenerate elliptic equations $\operatorname{div}(A\nabla u)=0$ in the upper half-space $(x,t)\in\mathbb{R}^{n+1}_+$ is solvable when $n\geq 2$ and the boundary data is in $L^p_\mu(\mathbb{R}^n)$ for some $p<\infty$. The coefficient matrix A is only assumed to be measurable, real-valued and t-independent with a degenerate bound and ellipticity controlled by an A_2-weight μ. It is not required to be symmetric. The result is achieved by proving a Carleson measure estimate for all bounded solutions in order to deduce that the degenerate elliptic measure is in A_∞ with respect to the μ-weighted Lebesgue measure on $\mathbb{R}^n$. The Carleson measure estimate allows us to avoid applying the method of ϵ-approximability, which simplifies the proof obtained recently in the case of uniformly elliptic coefficients. The results have natural extensions to Lipschitz domains.

1. Introduction

We consider the Dirichlet boundary value problem for the degenerate elliptic equation $\operatorname{div}(A\nabla u)=0$ in the upper half-space $\mathbb{R}^{n+1}_+$ when $n\geq 2$ and which we make precise below. The boundary $\mathbb{R}^n\times\{0\}$ is identified with $\mathbb{R}^n$ and we adopt the notation $X=(x,t)$ for points $X\in\mathbb{R}^{n+1}_+$ with coordinates $x\in\mathbb{R}^n$ and $t\in(0,\infty)$. The gradient $\nabla:=(\nabla_x,\partial_t)$ and divergence $\operatorname{div}:=\operatorname{div}_x+\partial_t$ are with respect to all $(n+1)$-coordinates. The coefficient A denotes an $(n+1)\times(n+1)$ matrix of measurable, real-valued and t-independent functions on $\mathbb{R}^{n+1}_+$. The matrix $A(x):=A(x,t)$ is not required to be symmetric. We suppose that there exist constants $0<\lambda\leq\Lambda<\infty$ and an A_2-weight μ on $\mathbb{R}^n$ such that the degenerate bound and ellipticity

$$|\langle A(x)\xi,\zeta\rangle|\leq\Lambda\mu(x)|\xi||\zeta| \quad\text{and}\quad \langle A(x)\xi,\xi\rangle\geq\lambda\mu(x)|\xi|^2 \tag{1.1}$$

hold for all $\xi,\zeta\in\mathbb{R}^{n+1}$ and almost every $x\in\mathbb{R}^n$. We use $\langle\cdot,\cdot\rangle$ and $|\cdot|$ to denote the Euclidean inner product and norm. An A_2-weight μ on $\mathbb{R}^n$ refers to a nonnegative locally integrable function

MSC2010: 35J25, 35J70, 42B20, 42B25.
Keywords: square functions, nontangential maximal functions, harmonic measure, Radon–Nikodym derivative, Carleson measure, divergence form elliptic equations, Dirichlet problem, A_2 Muckenhoupt weights, reverse Hölder inequality.

$\mu : \mathbb{R}^n \to [0, \infty]$ such that

$$[\mu]_{A_2(\mathbb{R}^n)} := \sup_Q \left(\frac{1}{|Q|} \int_Q \mu(x)\, dx \right) \left(\frac{1}{|Q|} \int_Q \frac{1}{\mu(x)}\, dx \right) < \infty,$$

where $\sup_Q$ denotes the supremum over all cubes Q in $\mathbb{R}^n$ with volume $|Q|$. We also use μ to denote the measure $\mu(Q) := \int_Q \mu(x)\, dx$ and consider the Lebesgue space $L^p_\mu(\mathbb{R}^n)$ with the norm $\|f\|_{L^p_\mu(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |f|^p \, d\mu\right)^{1/p}$ for all $p \in [1, \infty)$. There is also the notation $⨍_Q f \, d\mu := \mu(Q)^{-1} \int_Q f \, d\mu$, whilst $⨍_Q f := |Q|^{-1} \int_Q f(x)\, dx$.

If μ is identically 1, then A is called uniformly elliptic. The solvability of the Dirichlet problem for general nonsymmetric coefficients in that case was obtained only recently by Hofmann, Kenig, Mayboroda and Pipher [Hofmann et al. 2015a]. The result in dimension $n = 1$ had been obtained previously by Kenig, Koch, Pipher and Toro [Kenig et al. 2000]. These results assert that for each uniformly elliptic coefficient matrix A there exists some $p < \infty$ for which the Dirichlet problem is solvable for L^p-boundary data. Conversely, counterexamples in [Kenig et al. 2000] show that for each $p < \infty$, there exists a uniformly elliptic coefficient matrix A for which the Dirichlet problem is not solvable for L^p-boundary data. In contrast, solvability of the Dirichlet problem for symmetric coefficients in the uniformly elliptic case is well understood, and we mention only that it was obtained by Jerison and Kenig [1981] for L^p-boundary data when $2 \le p < \infty$.

The solvability of the Dirichlet problem in the uniformly elliptic case has also been established for a variety of complex coefficient structures; see, for instance, [Auscher and Stahlhut 2014; Hofmann et al. 2015a; 2015b]. A significant portion of that theory was recently extended to the degenerate elliptic case by Auscher, Rosén and Rule [Auscher et al. 2015] for L^2-boundary data. That extension did not include, however, the results for general nonsymmetric coefficients in [Hofmann et al. 2015a]. This paper complements the progress made in [Auscher et al. 2015] by extending the solvability obtained for the Dirichlet problem in [Hofmann et al. 2015a] to the degenerate elliptic case.

For solvability on the upper half-space $\mathbb{R}^{n+1}_+$, the A_2-weight μ on $\mathbb{R}^n$ is extended to the t-independent A_2-weight $\mu(x,t) := \mu(x)$ on $\mathbb{R}^{n+1}$ (and $[\mu]_{A_2(\mathbb{R}^{n+1})} = [\mu]_{A_2(\mathbb{R}^n)}$). We then say that u is a *solution* of the equation $\operatorname{div}(A\nabla u) = 0$ in an open set $\Omega \subseteq \mathbb{R}^{n+1}$ when $u \in W^{1,2}_{\mu,\mathrm{loc}}(\Omega)$ and $\int_{\mathbb{R}^{n+1}_+} \langle A\nabla u, \nabla \Phi \rangle = 0$ for all smooth compactly supported functions $\Phi \in C^\infty_c(\Omega)$. The solution space is the local μ-weighted Sobolev space $W^{1,2}_{\mu,\mathrm{loc}}$ defined in Section 2. The convergence of solutions to boundary data is afforded by estimates for the nontangential maximal function $N_* u$ of solutions u, defined by

$$(N_* u)(x) := \sup_{(y,t) \in \Gamma(x)} |u(y,t)| \quad \text{for all } x \in \mathbb{R}^n,$$

where $\Gamma(x)$ is the cone $\{(y,t) \in \mathbb{R}^{n+1}_+ : |y - x| < t\}$. If $p \in (1, \infty)$, then the Dirichlet problem for $L^p_\mu(\mathbb{R}^n)$-boundary data, or simply $(D)_{p,\mu}$, is said to be *solvable* when for each $f \in L^p_\mu(\mathbb{R}^n)$ there exists a solution u such that

$$\begin{cases} \operatorname{div}(A\nabla u) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ N_* u \in L^p_\mu(\mathbb{R}^n), \\ \lim_{t \to 0} u(\cdot, t) = f, \end{cases} \tag{$(D)_{p,\mu}$}$$

where the limit is required to converge in $L^p_\mu(\mathbb{R}^n)$-norm and in the nontangential sense whereby $\lim_{\Gamma(x)\ni(y,t)\to(x,0)} u(y,t) = f(x)$ for almost every $x \in \mathbb{R}^n$. Note that this definition of solvability is distinct from *well-posedness*, which requires that such solutions are unique. We are able to obtain a uniqueness result for solutions that converge uniformly to 0 at infinity, but the question of well-posedness more generally remains open (see Theorem 5.34 and the preceding discussion).

A nonnegative Borel measure ω on a cube Q_0 in $\mathbb{R}^n$ is said to be in the A_∞-class with respect to μ, written $\omega \in A_\infty(\mu)$, when there exist constants $C, \theta > 0$, which we call the $A_\infty(Q_0)$-constants, such that

$$\omega(E) \le C\left(\frac{\mu(E)}{\mu(Q)}\right)^\theta \omega(Q)$$

for all cubes $Q \subseteq Q_0$ and all Borel sets $E \subseteq Q$. This is a scale-invariant version of the absolute continuity of ω with respect to μ. It is well known, at least in the uniformly elliptic case, that solvability of the Dirichlet problem for L^p-boundary data for some $p < \infty$ is equivalent to the property that an adapted harmonic measure (elliptic measure) belongs to A_∞ with respect to the Lebesgue measure on $\mathbb{R}^n$; see Theorem 1.7.3 in [Kenig 1994]. In the degenerate case, an adapted harmonic measure ω^X, which we call degenerate elliptic measure, can also be defined at each $X \in \mathbb{R}^{n+1}_+$ (see Section 5). We prove that this degenerate elliptic measure is in A_∞ with respect to μ and then deduce the solvability of $(D)_{p,\mu}$ stated in the theorem below. This requires the notation associated with cubes Q in $\mathbb{R}^n$, where x_Q and $\ell(Q)$ denote the centre and side length of Q, respectively, and $X_Q := (x_Q, \ell(Q))$ denotes the corkscrew point in $\mathbb{R}^{n+1}_+$ relative to Q.

Theorem 1.2. *If $n \ge 2$ and the t-independent coefficient matrix A satisfies the degenerate bound and ellipticity in* (1.1) *for some constants $0 < \lambda \le \Lambda < \infty$ and an A_2-weight μ on $\mathbb{R}^n$, then there exists $p \in (1, \infty)$ such that $(D)_{p,\mu}$ is solvable. Moreover, on each cube Q in $\mathbb{R}^n$, the degenerate elliptic measure $\omega := \omega^{X_Q} \lfloor Q$ satisfies $\omega \in A_\infty(\mu)$ with $A_\infty(Q)$-constants that depend only on n, λ, Λ and $[\mu]_{A_2}$.*

In contrast to the proof of solvability in the uniformly elliptic case in [Hofmann et al. 2015a], we avoid the need to apply the method of ϵ-approximability by first establishing the Carleson measure estimate in the theorem below. This crucial estimate facilitates the main results of the paper. The connection between the Carleson measure estimate and solvability was first established in the uniformly elliptic case by Kenig, Kirchheim, Pipher and Toro [Kenig et al. 2016], and we follow their approach here, adapting it to the degenerate elliptic setting (see Lemma 5.24 below). In particular, the A_∞-property of degenerate elliptic measure is obtained by combining the Carleson measure estimate (1.4) with the notion of good ϵ-coverings introduced in [Kenig et al. 2000].

Theorem 1.3. *If $n \ge 2$ and the t-independent coefficient matrix A satisfies the degenerate bound and ellipticity in* (1.1) *for some constants $0 < \lambda \le \Lambda < \infty$ and an A_2-weight μ on $\mathbb{R}^n$, then any solution $u \in L^\infty(\mathbb{R}^{n+1}_+)$ of $\operatorname{div}(A\nabla u) = 0$ in $\mathbb{R}^{n+1}_+$ satisfies the Carleson measure estimate*

$$\sup_Q \frac{1}{\mu(Q)} \int_0^{\ell(Q)} \int_Q |t\nabla u(x,t)|^2 \, d\mu(x) \frac{dt}{t} \le C \|u\|_\infty^2, \tag{1.4}$$

where C depends only on n, λ, Λ and $[\mu]_{A_2}$.

Using the Carleson measure estimate in this way allows us to bypass the need to establish norm-equivalences between the nontangential maximal function $N_* u$ and the square function Su of solutions u, defined by

$$(Su)(x) := \left(\iint_{\Gamma(x)} |t\nabla u(y,t)|^2 \frac{d\mu(y)}{\mu(\Delta(x,t))} \frac{dt}{t} \right)^{1/2} \quad \text{for all } x \in \mathbb{R}^n,$$

where $\Delta(x,t)$ is the surface ball $\{y \in \mathbb{R}^n : |y-x| < t\}$. It was shown by Dahlberg, Jerison and Kenig [Dahlberg et al. 1984], however, that such estimates are a consequence of the A_∞-property of degenerate elliptic measure, which provides the following result.

Theorem 1.5. *If $n \geq 2$ and the t-independent coefficient matrix A satisfies the degenerate bound and ellipticity in* (1.1) *for some constants $0 < \lambda \leq \Lambda < \infty$ and an A_2-weight μ on $\mathbb{R}^n$, then any solution of* $\operatorname{div}(A\nabla u) = 0$ *in $\mathbb{R}^{n+1}_+$ satisfies*

$$\|Su\|_{L^p_\mu(\mathbb{R}^n)} \leq C \|N_* u\|_{L^p_\mu(\mathbb{R}^n)} \quad \textit{for all } p \in (0,\infty),$$

and if, in addition, $u(X_0) = 0$ for some $X_0 \in \mathbb{R}^{n+1}_+$, then

$$\|N_* u\|_{L^p_\mu(\mathbb{R}^n)} \leq C \|Su\|_{L^p_\mu(\mathbb{R}^n)} \quad \textit{for all } p \in (0,\infty),$$

where C depends only on X_0, p, n, λ, Λ and $[\mu]_{A_2}$.

The paper is structured as follows. Technical preliminaries concerning weights and degenerate elliptic operators are in Section 2, whilst estimates for weighted maximal operators are in Section 3. The Carleson measure estimate in Theorem 1.3 is obtained in Section 4. The degenerate elliptic measure is constructed in Section 5 and then the A_∞-estimates in Theorem 1.2 are deduced as part of Theorem 5.30. The square function and nontangential maximal function estimates in Theorem 1.5 are included in the more general result in Theorem 5.31, whilst the solvability of the Dirichlet problem in Theorem 1.2 is finally deduced in Theorem 5.34, where a uniqueness result is also obtained.

We state and prove our results in the upper half-space, but we note that they extend immediately to the case that the domain is the region above a Lipschitz graph, by a well-known pull-back technique which preserves the t-independence of the coefficients. In turn, our results concerning the A_∞-property of degenerate elliptic measure may then be extended to the case of a bounded star-like Lipschitz domain, with radially independent coefficients, by a standard localization argument using the maximum principle.

The convention is adopted whereby C denotes a finite positive constant that may change from one line to the next. For $a, b \in \mathbb{R}$, the notation $a \lesssim b$ means that $a \leq Cb$, whilst $a \eqsim b$ means that $a \lesssim b \lesssim a$. We write $a \lesssim_p b$ when $a \leq Cb$ and we wish to emphasise that C depends on a specified parameter p.

2. Preliminaries

We dispense with some technical preliminaries concerning general A_p-weights μ for $p \in (1,\infty)$ and degenerate elliptic operators on $\mathbb{R}^n$ for $n \in \mathbb{N}$. All cubes Q and balls B in $\mathbb{R}^n$ are assumed to be open (except in Section 5D where the standard dyadic cubes S in $\mathbb{D}(\mathbb{R}^n)$ are assumed to be closed to provide genuine coverings of $\mathbb{R}^n$). For $\alpha > 0$, let αQ and αB denote the concentric dilates of Q

and B respectively. For $x \in \mathbb{R}^n$ and $r > 0$, define the ball $B(x,r) := \{y \in \mathbb{R}^n : |y-x| < r\}$. An A_p-weight refers to a nonnegative locally integrable function μ on $\mathbb{R}^n$ with the property that $[\mu]_{A_p(\mathbb{R}^n)} := \sup_Q \big(\fint_Q \mu\big)\big(\fint_Q \mu^{-1/(p-1)}\big)^{p-1} < \infty$. The measure associated with such a weight satisfies the doubling property

$$\mu(\alpha B) \leq [\mu]_{A_p} \alpha^{np} \mu(B) \tag{2.1}$$

for all $\alpha \geq 1$; see, for instance, Section 1.5 in Chapter V of [Stein 1993].

For an open set $\Omega \subseteq \mathbb{R}^n$, the Sobolev space $W^{1,p}_\mu(\Omega)$ is defined as the completion, in the ambient space $L^p_\mu(\Omega)$, of the normed space of all $f \in C^\infty(\Omega)$ with finite norm

$$\|f\|^p_{W^{1,p}_\mu(\Omega)} := \int_\Omega |f|^p \, d\mu + \int_\Omega |\nabla f|^p \, d\mu < \infty. \tag{2.2}$$

The embedding of the completion $W^{1,p}_\mu(\Omega)$ in $L^p_\mu(\Omega)$ relies on the A_p-property of the weight (to the extent that it implies both μ and $\mu^{-1/(p-1)}$ are in $L^1_{\text{loc}}(\Omega)$), which ensures that if $(f_j)_j$ is a $W^{1,p}_\mu(\Omega)$-Cauchy sequence in $C^\infty(\Omega)$ converging to 0 in $L^p_\mu(\Omega)$, then $(f_j)_j$ converges to 0 in $W^{1,p}_\mu(\Omega)$-norm; see Section 2.1 in [Fabes et al. 1982b]. Therefore, since $C^\infty(\Omega)$ is dense in $W^{1,p}_\mu(\Omega)$, the gradient extends to a bounded operator $\nabla : W^{1,p}_\mu(\Omega) \to L^p_\mu(\Omega, \mathbb{R}^n)$, thereby extending (2.2) to all $f \in W^{1,p}_\mu(\Omega)$. The Sobolev space $W^{1,p}_{0,\mu}(\Omega)$ is defined as the closure of $C^\infty_c(\Omega)$ in $W^{1,p}_\mu(\Omega)$. It can be shown that $W^{1,p}_{0,\mu}(\mathbb{R}^n) = W^{1,p}_\mu(\mathbb{R}^n)$ by following the proof in the unweighted case from Proposition 1 of Chapter V in [Stein 1970] but instead using Lemma 2.2 in [Auscher et al. 2015] to deduce the convergence of the regularization in $L^p_\mu(\mathbb{R}^n)$. The local space $W^{1,p}_{\mu,\text{loc}}(\Omega)$ is then defined as the set of all $f \in L^p_{\mu,\text{loc}}(\Omega)$ such that $f \in W^{1,p}_\mu(\Omega')$ for all open sets Ω' with compact closure $\overline{\Omega}' \subset \Omega$ (henceforth denoted by $\Omega' \Subset \Omega$). Finally, the weighted Sobolev and Poincaré inequalities obtained for continuous functions in Theorems 1.2 and 1.5 in [Fabes et al. 1982b] have the following immediate extensions.

Theorem 2.3. *Let $n \geq 2$ and suppose that $B \subset \mathbb{R}^n$ denotes a ball with radius $r(B)$. If $p \in (1,\infty)$ and μ is an A_p-weight on $\mathbb{R}^n$, then there exists $\delta > 0$ such that*

$$\left(\fint_B |f|^{p(\frac{n}{n-1}+\delta)} \, d\mu\right)^{1/(p(\frac{n}{n-1}+\delta))} \lesssim r(B) \left(\fint_B |\nabla f|^p \, d\mu\right)^{1/p} \tag{2.4}$$

for all $f \in W^{1,p}_{0,\mu}(B)$, and

$$\left(\fint_B |f(x) - c_B|^p \, d\mu\right)^{1/p} \lesssim r(B) \left(\fint_B |\nabla f|^p \, d\mu\right)^{1/p} \tag{2.5}$$

for all $f \in W^{1,p}_\mu(B)$ and $c_B \in \{\fint_B f \, d\mu, \fint_B f\}$, where the implicit constants depend only on n, p and $[\mu]_{A_p}$. The estimates also hold when the ball B and the radius $r(B)$ are replaced by a cube Q and the side length $\ell(Q)$.

For $n \in \mathbb{N}$, constants $0 < \lambda \leq \Lambda < \infty$ and an A_2-weight μ on $\mathbb{R}^n$, let $\mathcal{E}(n, \lambda, \Lambda, \mu)$ denote the set of all $n \times n$ matrices $\mathcal{A}$ of measurable real-valued functions on $\mathbb{R}^n$ satisfying the degenerate bound and ellipticity

$$|\langle \mathcal{A}(x)\xi, \zeta\rangle| \le \Lambda\mu(x)|\xi||\zeta| \quad \text{and} \quad \langle \mathcal{A}(x)\xi, \xi\rangle \ge \lambda\mu(x)|\xi|^2 \tag{2.6}$$

for all $\xi, \zeta \in \mathbb{R}^n$ and almost every $x \in \mathbb{R}^n$. These properties allow us to define

$$\mathcal{L}_{\mu,\Omega} : \mathrm{Dom}(\mathcal{L}_{\mu,\Omega}) \subseteq L^2_\mu(\Omega) \to L^2_\mu(\Omega)$$

as the maximal accretive operator in $L^2_\mu(\Omega)$ associated with the bilinear form defined by

$$\mathfrak{a}_\Omega(f, g) := \int_\Omega \langle \mathcal{A}\nabla f, \nabla g\rangle = \int_\Omega \Big\langle \frac{1}{\mu}\mathcal{A}\nabla f, \nabla g\Big\rangle \, d\mu \tag{2.7}$$

for all $f, g \in W^{1,2}_{0,\mu}(\Omega)$. The domain of $\mathcal{L}_{\mu,\Omega}$ is dense in $L^2_\mu(\Omega)$, and in particular

$$\mathrm{Dom}(\mathcal{L}_{\mu,\Omega}) = \bigg\{ f \in W^{1,2}_{0,\mu}(\Omega) : \sup_{g \in C^\infty_c(\Omega)} \frac{|\mathfrak{a}_\Omega(f, g)|}{\|g\|_{L^2_\mu(\Omega)}} < \infty \bigg\},$$

with

$$\int_\Omega (\mathcal{L}_{\mu,\Omega} f) g \, d\mu = \mathfrak{a}_\Omega(f, g) \tag{2.8}$$

for all $f \in \mathrm{Dom}(\mathcal{L}_{\mu,\Omega})$ and $g \in W^{1,2}_{0,\mu}(\Omega)$. It is equivalent to define $\mathcal{L}_{\mu,\Omega}$ as the composition $-\mathrm{div}_{\mu,\Omega}((1/\mu)\mathcal{A}\nabla)$ of unbounded operators, where $-\mathrm{div}_{\mu,\Omega}$ is the adjoint ∇^* of the closed densely defined operator $\nabla : W^{1,2}_{0,\mu}(\Omega) \subseteq L^2_\mu(\Omega) \to L^2_\mu(\Omega, \mathbb{R}^n)$, that is,

$$\int_\Omega (-\mathrm{div}_{\mu,\Omega}\, \boldsymbol{f}) g \, d\mu = \int_\Omega \langle \boldsymbol{f}, \nabla g\rangle \, d\mu \tag{2.9}$$

for all $\boldsymbol{f} \in \mathrm{Dom}(\mathrm{div}_{\mu,\Omega}) := \mathrm{Dom}(\nabla^*)$ and $g \in W^{1,2}_{0,\mu}(\Omega)$. In view of (2.7) and (2.8), we have the formal identities $\mathrm{div}_{\mu,\Omega} = (1/\mu)\,\mathrm{div}_\Omega\, \mu$ and $\mathcal{L}_{\mu,\Omega} = -(1/\mu)\,\mathrm{div}_\Omega(\mathcal{A}\nabla)$.

Now let $\Omega = Q$ for some cube $Q \subset \mathbb{R}^n$ and denote the space of bounded linear functionals on $W^{1,2}_{0,\mu}(Q)$ by $W^{-1,2}_{0,\mu}(Q)$. The inclusions $W^{1,2}_{0,\mu}(Q) \subseteq L^2_\mu(Q) \subseteq W^{-1,2}_{0,\mu}(Q)$ are interpreted in the standard way by identifying $f \in L^2_\mu(Q)$ with the functional ℓ_f defined by $\ell_f(g) := \int_Q fg \, d\mu$ for all $g \in W^{1,2}_{0,\mu}(Q)$. Thus, setting

$$\mathcal{L}_{\mu,Q} f(g) := \mathfrak{a}_Q(f, g) \quad \text{and} \quad -\mathrm{div}_{\mu,Q}\, \boldsymbol{f}(g) := \int_Q \langle \boldsymbol{f}, \nabla g\rangle \, d\mu$$

for all $f, g \in W^{1,2}_{0,\mu}(Q)$ and $\boldsymbol{f} \in L^2(Q, \mathbb{R}^n)$, we obtain an extension of $\mathcal{L}_{\mu,Q}$ from (2.8) to a bounded invertible operator from $W^{1,2}_{0,\mu}(Q)$ onto $W^{-1,2}_{0,\mu}(Q)$, and an extension of $\mathrm{div}_{\mu,Q}$ from (2.9) to a bounded operator from $L^2_\mu(Q)$ into $W^{-1,2}_{0,\mu}(Q)$. The surjectivity of $\mathcal{L}_{\mu,Q}$ relies on (2.4) and the Lax–Milgram theorem. These definitions imply

$$\|\nabla \mathcal{L}^{-1}_{\mu,Q}\, \mathrm{div}_{\mu,Q}\, \boldsymbol{f}\|_{L^2_\mu(Q,\mathbb{R}^n)} \lesssim \|\boldsymbol{f}\|_{L^2_\mu(Q,\mathbb{R}^n)}$$

for all $\boldsymbol{f} \in L^2_\mu(Q, \mathbb{R}^n)$. The topological direct sum or $W^{1,2}_{0,\mu}(Q)$-Hodge decomposition

$$L^2_\mu(Q, \mathbb{R}^n) = \Big\{ \frac{1}{\mu}\mathcal{A}\nabla g : g \in W^{1,2}_{0,\mu}(Q) \Big\} \oplus \{ \boldsymbol{h} \in L^2_\mu(Q, \mathbb{R}^n) : \mathrm{div}_{\mu,Q}\, \boldsymbol{h} = 0 \} \tag{2.10}$$

follows by writing

$$\boldsymbol{f} = -\frac{1}{\mu}\mathcal{A}\nabla\mathcal{L}_{\mu,Q}^{-1}\operatorname{div}_{\mu,Q}\boldsymbol{f} + \Big(\boldsymbol{f} + \frac{1}{\mu}\mathcal{A}\nabla\mathcal{L}_{\mu,Q}^{-1}\operatorname{div}_{\mu,Q}\boldsymbol{f}\Big) =: \frac{1}{\mu}\mathcal{A}\nabla g + \boldsymbol{h},$$

since then $\operatorname{div}_{\mu,Q}\boldsymbol{h} = \operatorname{div}_{\mu,Q}\boldsymbol{f} - \mathcal{L}_{\mu,Q}\mathcal{L}_{\mu,Q}^{-1}\operatorname{div}_{\mu,Q}\boldsymbol{f} = 0$. This decomposition also extends to $L^p_\mu(Q,\mathbb{R}^n)$ for all $p \in [2, 2+\epsilon)$ and some $\epsilon > 0$ by recent work of Le [2015], although we do not need it here.

Now let $\Omega = \mathbb{R}^n$ and consider $\operatorname{div}_\mu := \operatorname{div}_{\mu,\mathbb{R}^n}$ as in (2.9) so $\mathcal{L}_\mu := -\operatorname{div}_\mu((1/\mu)\mathcal{A}\nabla)$ is maximal accretive, thus having a maximal accretive square root $\mathcal{L}_\mu^{1/2}$, in $L^2_\mu(\mathbb{R}^n)$. The solution of the Kato square root problem in [Auscher et al. 2002] was recently extended to degenerate elliptic equations by Cruz-Uribe and Rios [2015]. This shows that $\|\mathcal{L}_\mu^{1/2} f\|_{L^2_\mu(\mathbb{R}^n)} \eqsim \|\nabla f\|_{L^2_\mu(\mathbb{R}^n,\mathbb{R}^n)}$ for all $f \in W^{1,2}_\mu(\mathbb{R}^n)$; hence $\operatorname{Dom}(\mathcal{L}_\mu^{1/2}) = W^{1,2}_\mu(\mathbb{R}^n)$.

The operator $\mathcal{L}_\mu$ is also injective and type-$S_{\omega+}$ in $L^2_\mu(\mathbb{R}^n)$ for some $\omega \in \big(0, \frac{\pi}{2}\big)$, so it has a bounded $H^\infty(S^o_{\theta+})$-functional calculus in $L^2_\mu(\mathbb{R}^n)$ for each $\theta \in (\omega, \pi)$, where $S^o_{\theta+} := \{z \in \mathbb{C}\setminus\{0\} : |\arg z| < \theta\}$. See Section 2.2 of [Auscher 2007] for the uniformly elliptic case and Theorems F and G in [Albrecht et al. 1996] for the general theory. An equivalent property is the validity of the quadratic estimate

$$\int_0^\infty \|\psi(t\mathcal{L}_{\preceq})f\|^2_{L^2_\mu(\mathbb{R}^n)}\frac{dt}{t} \eqsim \|f\|^2_{L^2_\mu(\mathbb{R}^n)} \quad \text{for all } f \in L^2_\mu(\mathbb{R}^n), \tag{2.11}$$

for each holomorphic ψ on $S^o_{\theta+}$ satisfying $|\psi(z)| \lesssim \min\{|z|^\alpha, |z|^{-\beta}\}$ for some $\alpha, \beta > 0$, where the bounded operator $\psi(t\mathcal{L}_{\preceq})$ on $L^2_\mu(\mathbb{R}^n)$ is defined by a Cauchy integral. More generally, the relationship between bounded holomorphic functional calculi and quadratic estimates is developed in the seminal articles [McIntosh 1986; Cowling et al. 1996].

The functional calculus then defines a bounded operator $\varphi(\mathcal{L}_\mu)$ on $L^2_\mu(\mathbb{R}^n)$ for each bounded holomorphic function φ on $S^o_{\theta+}$ and $\|\varphi(\mathcal{L}_\mu)\|_{L^2_\mu(\mathbb{R}^n)\to L^2_\mu(\mathbb{R}^n)} \lesssim_\theta \|\varphi\|_\infty$. Another consequence is that $-\mathcal{L}_\mu$ generates a holomorphic contraction semigroup $(e^{-\zeta\mathcal{L}_\mu})_{\zeta\in S^o_{\pi/2-\omega}\cup\{0\}}$ on $L^2_\mu(\mathbb{R}^n)$; thus $e^{-t\mathcal{L}_{\preceq}} f \in \operatorname{Dom}(\mathcal{L}_\mu)$ and $\partial_t(e^{-t\mathcal{L}_{\preceq}} f) = \mathcal{L}_\mu e^{-t\mathcal{L}_{\preceq}} f$ for all $f \in L^2_\mu(\mathbb{R}^n)$ and $t > 0$. The functional calculus also extends to define an unbounded operator $\phi(\mathcal{L}_\mu)$ on $L^2_\mu(\mathbb{R}^n)$ for each holomorphic function ϕ on $S^o_{\theta+}$ satisfying $|\phi(z)| \lesssim \max\{|z|^\alpha, |z|^{-\beta}\}$ for some $\alpha, \beta > 0$, but the algebra homomorphism property of the functional calculus ($\phi_1(\mathcal{L}_\mu)\phi_2(\mathcal{L}_\mu) = (\phi_1\phi_2)(\mathcal{L}_\mu)$) must then be interpreted in the sense of unbounded linear operators. This allows us to interpret both the semigroup and the square root of $\mathcal{L}_\mu$ in terms of the functional calculus in order to justify some otherwise formal manipulations, beginning with (2.15) in the proof of the following corollary of the solution of the Kato problem in [Cruz-Uribe and Rios 2015].

Theorem 2.12. *Let $n \geq 1$ and suppose that $\mathcal{A} \in \mathcal{E}(n, \lambda, \Lambda, \mu)$ for some constants $0 < \lambda \leq \Lambda < \infty$ and an A_2-weight μ on $\mathbb{R}^n$. The operator $\mathcal{L}_\mu := -\operatorname{div}_\mu((1/\mu)\mathcal{A}\nabla)$ satisfies*

$$\int_0^\infty \|t\mathcal{L}_\mu e^{-t^2\mathcal{L}_\mu} f\|^2_{L^2_\mu(\mathbb{R}^n)}\frac{dt}{t} \eqsim \|\nabla f\|^2_{L^2_\mu(\mathbb{R}^n,\mathbb{R}^n)}, \tag{2.13}$$

$$\int_0^\infty \|t^2\nabla_{x,t}\mathcal{L}_\mu e^{-t^2\mathcal{L}_\mu} f\|^2_{L^2_\mu(\mathbb{R}^n,\mathbb{R}^{n+1})}\frac{dt}{t} \lesssim \|\nabla f\|^2_{L^2_\mu(\mathbb{R}^n,\mathbb{R}^n)} \tag{2.14}$$

for all $f \in W^{1,2}_\mu(\mathbb{R}^n)$, where the implicit constants depend only on n, λ, Λ and $[\mu]_{A_2}$.

Proof. The functional calculus of $\mathcal{L}_\mu$ justifies the identity

$$\mathcal{L}_\mu e^{-t^2\mathcal{L}_\mu} f = \mathcal{L}_\mu^{1/2} e^{-t^2\mathcal{L}_\mu} \mathcal{L}_\mu^{1/2} f = e^{-(t^2/2)\mathcal{L}_\mu} \mathcal{L}_\mu e^{-(t^2/2)\mathcal{L}_\mu} f \tag{2.15}$$

for all $f \in \mathrm{Dom}(\mathcal{L}_\mu^{1/2})$ and $t > 0$. The first equality in (2.15), the quadratic estimate in (2.11) and the solution of the Kato problem in [Cruz-Uribe and Rios 2015] imply

$$\begin{aligned}\int_0^\infty \|t\mathcal{L}_\mu e^{-t^2\mathcal{L}_\mu} f\|^2_{L^2_\mu(\mathbb{R}^n)} \frac{dt}{t} &= \int_0^\infty \|(\tau\mathcal{L}_\mu)^{1/2} e^{-\tau\mathcal{L}_\mu} \mathcal{L}_\mu^{1/2} f\|^2_{L^2_\mu(\mathbb{R}^n)} \frac{d\tau}{\tau} \\ &\eqsim \|\mathcal{L}_\mu^{1/2} f\|^2_{L^2_\mu(\mathbb{R}^n)} \\ &\eqsim \|\nabla f\|^2_{L^2_\mu(\mathbb{R}^n,\mathbb{R}^n)}\end{aligned}$$

for all $f \in \mathrm{Dom}(\mathcal{L}_\mu^{1/2}) = W^{1,2}_\mu(\mathbb{R}^n)$, which proves (2.13).

The bounded $H^\infty(S^o_{\theta+})$-functional calculus of $\mathcal{L}_\mu$ implies the uniform estimate

$$\begin{aligned}\|t\nabla_{x,t} e^{-t^2\mathcal{L}_\mu} g\|^2_{L^2_\mu(\mathbb{R}^n,\mathbb{R}^{n+1})} &= \|t\partial_t e^{-t^2\mathcal{L}_\mu} g\|^2_{L^2_\mu(\mathbb{R}^n)} + \|t\nabla_x e^{-t^2\mathcal{L}_\mu}\|^2_{L^2_\mu(\mathbb{R}^n,\mathbb{R}^n)} \\ &\lesssim \|t^2\mathcal{L}_\mu e^{-t^2\mathcal{L}_\mu} g\|^2_{L^2_\mu(\mathbb{R}^n)} + \int_{\mathbb{R}^n} t^2 \langle \mathcal{A}\nabla_x e^{-t^2\mathcal{L}_\mu} g, \nabla_x e^{-t^2\mathcal{L}_\mu} g\rangle \\ &\lesssim \|g\|^2_{L^2_\mu(\mathbb{R}^n)} + \|t^2\mathcal{L}_\mu e^{-t^2\mathcal{L}_\mu} g\|_{L^2_\mu(\mathbb{R}^n)} \|e^{-t^2\mathcal{L}_\mu} g\|_{L^2_\mu(\mathbb{R}^n)} \\ &\lesssim \|g\|^2_{L^2_\mu(\mathbb{R}^n)}\end{aligned}$$

for all $g \in L^2_\mu(\mathbb{R}^n)$ and $t > 0$. Thus, the second equality in (2.15) and the vertical square function estimate in (2.13), which we have already proved, imply

$$\int_0^\infty \|t^2\nabla_{x,t}\mathcal{L}_\mu e^{-t^2\mathcal{L}_\mu} f\|^2_{L^2_\mu(\mathbb{R}^n,\mathbb{R}^{n+1})} \frac{dt}{t} \lesssim \int_0^\infty \|t\mathcal{L}_\mu e^{-(t^2/2)\mathcal{L}_\mu} f\|^2_{L^2_\mu(\mathbb{R}^n)} \frac{dt}{t} \lesssim \|\nabla f\|^2_{L^2_\mu(\mathbb{R}^n,\mathbb{R}^n)}$$

for all $f \in W^{1,2}_\mu(\mathbb{R}^n)$, which proves (2.14). □

Now let us return to the case when $\Omega \subseteq \mathbb{R}^n$ is an arbitrary open set and suppose that $\boldsymbol{f} : \Omega \to \mathbb{R}^n$ is a measurable function for which $(1/\mu)\boldsymbol{f} \in L^\infty(\Omega)$. A *solution* of the inhomogeneous equation $\mathrm{div}(\mathcal{A}\nabla u) = \mathrm{div}\,\boldsymbol{f}$ in $\Omega \subseteq \mathbb{R}^n$ refers to any function $u \in W^{1,2}_{\mu,\mathrm{loc}}(\Omega)$ such that $\int_{\mathbb{R}^n} \langle \mathcal{A}\nabla u - \boldsymbol{f}, \nabla\Phi\rangle = 0$ for all $\Phi \in C^\infty_c(\Omega)$. All solutions u of the homogeneous equation $\mathrm{div}(\mathcal{A}\nabla u) = 0$ in Ω are locally bounded and Hölder continuous in the sense that

$$\|u\|_{L^\infty(B)} \lesssim \left(\fint_{2B} |u|^2\, d\mu\right)^{1/2} \tag{2.16}$$

and there exists $\alpha > 0$ such that

$$|u(x) - u(y)| \lesssim \left(\frac{|x-y|}{r(B)}\right)^\alpha \left(\fint_{2B} |u|^2\, d\mu\right)^{1/2} \quad \text{for all } x, y \in B, \tag{2.17}$$

and if, in addition, $u \geq 0$ almost everywhere on Ω, there is the Harnack inequality

$$\sup_B u \lesssim \inf_B u \tag{2.18}$$

for all balls B of radius $r(B)$ such that $2B \subseteq \Omega$, where α and the implicit constants depend only on n, λ, Λ and $[\mu]_{A_2}$. These properties follow from Corollary 2.3.4, Lemma 2.3.5 and Theorem 2.3.12 in [Fabes et al. 1982b] by observing that the proofs do not use the assumption therein that $\mathcal{A}$ is symmetric. The estimates also hold when the balls B are replaced by (open) cubes Q, and also when the dilate $2B$ is replaced by $C_0 B$ for any $C_0 > 1$, provided the implicit constants are understood to depend on C_0.

The following local boundedness estimate for solutions of the inhomogeneous equation is needed in Lemma 4.3, although only for $p = 2$. This is a simpler version of Theorem 8.17 in [Gilbarg and Trudinger 1977], which we have adapted to degenerate elliptic equations. In fact, the result for $p \geq 2$ is already proven in [Fabes et al. 1982b] by combining Corollary 2.3.4 with estimates (2.3.7) and (2.3.13) therein. The proof is included here for the reader's convenience and since it implies (2.16) as a special case, which in turn is the well-known starting point for establishing (2.17).

Theorem 2.19. *Let $n \geq 2$ and suppose that $\mathcal{A} \in \mathcal{E}(n, \lambda, \Lambda, \mu)$ for some constants $0 < \lambda \leq \Lambda < \infty$ and an A_2-weight μ on $\mathbb{R}^n$. Let $\Omega \subseteq \mathbb{R}^n$ denote an open set and suppose that $\boldsymbol{f} : \Omega \to \mathbb{R}^n$ is a measurable function such that $(1/\mu)\boldsymbol{f} \in L^\infty(\Omega)$. If $p \in (1, \infty)$ and $\operatorname{div}(\mathcal{A}\nabla u) = \operatorname{div} \boldsymbol{f}$ in Ω, then*

$$\|u\|_{L^\infty(B)} \lesssim \left(\fint_{2B} |u|^p \, d\mu \right)^{1/p} + r(B) \left\| \frac{1}{\mu} \boldsymbol{f} \right\|_{L^\infty(\Omega)} \tag{2.20}$$

for all balls B of radius $r(B) > 0$ such that $2B \subseteq \Omega$, where the implicit constant depends only on p, n, λ, Λ and $[\mu]_{A_2}$.

Proof. Suppose that $\operatorname{div}(\mathcal{A}\nabla u) = \operatorname{div} \boldsymbol{f}$ in Ω and consider a ball B such that $2B \subseteq \Omega$. First, assume that u is nonnegative and in $L^\infty(2B)$. Let $\epsilon > 0$, set $k = r(B)\|(1/\mu)\boldsymbol{f}\|_{L^\infty(\Omega)}$ and $\bar{u}_\epsilon := u + k + \epsilon$. Let B_r denote the ball concentric to B with radius $r > 0$ and recall the index $\delta > 0$ from the Sobolev inequality in Theorem 2.3. We claim that if $\gamma \in [p, \infty)$ and $r(B) \leq r_1 < r_2 \leq 2r(B)$, then

$$\left(\fint_{B_{r_1}} \bar{u}_\epsilon^{\gamma(\frac{n}{n-1}+\delta)} \, d\mu \right)^{1/(\gamma(\frac{n}{n-1}+\delta))} \lesssim \left(\gamma \frac{r_1}{r_2 - r_1} \right)^{2/\gamma} \left(\fint_{B_{r_2}} \bar{u}_\epsilon^\gamma \, d\mu \right)^{1/\gamma}, \tag{2.21}$$

where the implicit constant depends only on p, n, λ, Λ and $[\mu]_{A_2}$. To prove (2.21), fix $\eta \in C_c^\infty(\Omega)$ such that $\eta : \Omega \to [0, 1]$, $\eta \equiv 1$ on B_{r_1}, $\eta \equiv 0$ on $\Omega \setminus B_{r_2}$ and $\|\nabla\eta\|_\infty \leq 2/(r_2 - r_1)$. Set $\beta := \gamma - 1$ and $v := \eta^2 \bar{u}_\epsilon^\beta$. Note that $v \in W^{1,2}_{0,\mu}(\Omega)$ with

$$\nabla v = 2\eta\nabla\eta\bar{u}_\epsilon^\beta + \beta\eta^2\bar{u}_\epsilon^{\beta-1}\nabla u,$$

since $0 < \epsilon \leq \bar{u}_\epsilon(x) \leq \|u\|_{L^\infty(2B)} + k + \epsilon < \infty$ for almost every $x \in 2B$; thus

$$\int_{\mathbb{R}^n} \langle \mathcal{A}\nabla u - \boldsymbol{f}, 2\eta\nabla\eta\bar{u}_\epsilon^\beta \rangle = -\int_{\mathbb{R}^n} \langle \mathcal{A}\nabla u - \boldsymbol{f}, \beta\eta^2\bar{u}_\epsilon^{\beta-1}\nabla u \rangle.$$

We then use this identity and Cauchy's inequality with $\sigma > 0$ to obtain

$$\begin{aligned}\int_{\mathbb{R}^n} \eta^2 \bar{u}_\epsilon^{\beta-1} |\nabla u|^2 \, d\mu &\lesssim_\lambda \int_{\mathbb{R}^n} \eta^2 \bar{u}_\epsilon^{\beta-1} \langle \mathcal{A}\nabla u, \nabla u\rangle \\ &= -2\beta^{-1} \int_{\mathbb{R}^n} \eta \bar{u}_\epsilon^{\beta} \langle \mathcal{A}\nabla u - \boldsymbol{f}, \nabla \eta\rangle + \int_{\mathbb{R}^n} \eta^2 \bar{u}_\epsilon^{\beta-1} \langle \boldsymbol{f}, \nabla u\rangle \\ &\lesssim_\Lambda (p-1)^{-1} \int_{\mathbb{R}^n} \eta \bar{u}_\epsilon^{\beta} \Big(|\nabla u| + \Big|\frac{1}{\mu}\boldsymbol{f}\Big|\Big) |\nabla \eta| \, d\mu + \int_{\mathbb{R}^n} \eta^2 \bar{u}_\epsilon^{\beta-1} \Big|\frac{1}{\mu}\boldsymbol{f}\Big| |\nabla u| \, d\mu \\ &\lesssim_p \sigma \int_{\mathbb{R}^n} \eta^2 \bar{u}_\epsilon^{\beta-1} |\nabla u|^2 \, d\mu + \sigma^{-1} \int_{\mathbb{R}^n} \bar{u}_\epsilon^{\beta+1} |\nabla \eta|^2 \, d\mu \\ &\qquad + \int_{\mathbb{R}^n} \bar{u}_\epsilon^{\beta+1} |\nabla \eta|^2 \, d\mu + \int_{\mathbb{R}^n} \Big(\frac{\eta}{r(B)}\Big)^2 \bar{u}_\epsilon^{\beta+1} \, d\mu \\ &\qquad + \sigma \int_{\mathbb{R}^n} \eta^2 \bar{u}_\epsilon^{\beta-1} |\nabla u|^2 \, d\mu + \sigma^{-1} \int_{\mathbb{R}^n} \Big(\frac{\eta}{r(B)}\Big)^2 \bar{u}_\epsilon^{\beta+1} \, d\mu,\end{aligned}$$

where in the second inequality we used the assumption that $\beta := \gamma - 1 \geq p - 1$ and in the final inequality we used the fact that $|(1/\mu)\boldsymbol{f}| \leq k/r(B) \leq \bar{u}_\epsilon / r(B)$ on Ω. Next, choose $\sigma > 0$ small enough, depending only on p, λ and Λ, to deduce that

$$\int_{B_{r_1}} \bar{u}_\epsilon^{\beta-1} |\nabla u|^2 \, d\mu \lesssim_{p,\lambda,\Lambda} \int_{\mathbb{R}^n} \bar{u}_\epsilon^{\beta+1} \Big(|\nabla \eta|^2 + \Big(\frac{\eta}{r(B)}\Big)^2 \Big) \, d\mu \lesssim \frac{1}{(r_2 - r_1)^2} \int_{B_{r_2}} \bar{u}_\epsilon^{\beta+1} \, d\mu,$$

where in the final inequality we used the fact that $r(B) \geq r_2 - r_1$. Now combine this estimate with the Sobolev inequality (2.4) and recall that $\beta := \gamma - 1$ to obtain

$$\begin{aligned}\Big(\fint_{B_{r_1}} \bar{u}_\epsilon^{\gamma(\frac{n}{n-1}+\delta)} \, d\mu \Big)^{1/(\frac{n}{n-1}+\delta)} &\lesssim r_1^2 \fint_{B_{r_1}} |\nabla(\bar{u}_\epsilon^{(\beta+1)/2})|^2 \, d\mu \\ &\lesssim ((\beta+1) r_1)^2 \fint_{B_{r_1}} \bar{u}_\epsilon^{\beta-1} |\nabla u|^2 \, d\mu \\ &\lesssim \Big(\gamma \frac{r_1}{r_2 - r_1} \Big)^2 \fint_{B_{r_2}} \bar{u}_\epsilon^{\gamma} \, d\mu,\end{aligned}$$

where the implicit constants depend only on p, n, λ, Λ and $[\mu]_{A_2}$, proving (2.21).

We now apply the Moser iteration technique to prove (2.20). Set $\chi := n/(n-1) + \delta$ and define $\Phi(q, r) := \big(\fint_{B_r} \bar{u}_\epsilon^q \, d\mu \big)^{1/q}$ for $q, r > 0$. Estimate (2.21) implies

$$\Phi(\gamma\chi, r_1) \leq \Big(C\gamma \frac{r_1}{r_2 - r_1} \Big)^{2/\gamma} \Phi(\gamma, r_2),$$

where C depends only on p, n, λ, Λ and $[\mu]_{A_2}$, and it follows by induction that

$$\Phi(p\chi^m, (1 + 2^{-m}) r(B)) \leq (4Cp)^{(2/p)\sum_{k=0}^{m-1} \chi^{-k}} (2\chi)^{(2/p)\sum_{k=0}^{m-1} k\chi^{-k}} \Phi(p, 2r(B)) \lesssim \Phi(p, 2r(B))$$

for all $m \in \mathbb{N}$. This shows that

$$\|\bar{u}_\epsilon\|_{L^\infty(B)} = \lim_{m\to\infty} \Phi(p\chi^m, r(B)) \lesssim \Phi(p, 2r(B)) = \left(\fint_{2B} \bar{u}_\epsilon^p \, d\mu\right)^{1/p}$$

and therefore

$$\|u\|_{L^\infty(B)} \le \|\bar{u}_\epsilon\|_{L^\infty(B)} \lesssim \left(\fint_{2B} \bar{u}_\epsilon^p \, d\mu\right)^{1/p} \lesssim \left(\fint_{2B} u^p \, d\mu\right)^{1/p} + r(B)\left\|\frac{1}{\mu} f\right\|_{L^\infty(\Omega)} + \epsilon$$

for all $\epsilon > 0$, which implies (2.20).

Finally, it remains to remove the assumption that u is nonnegative and bounded. This is achieved by setting $\bar{u}_\epsilon := \max\{u, 0\} + k + \epsilon$ and $\bar{u}_\epsilon := -\min\{u, 0\} + k + \epsilon$ respectively and in each case adjusting the proof above to incorporate the truncated test function $v := \eta^2 h_N(\bar{u}_\epsilon)\bar{u}_\epsilon$, where

$$h_N(x) := \begin{cases} x^{\beta-1}, & x \le N + k + \epsilon, \\ (N + k + \epsilon)^{\beta-1}, & x > N + k + \epsilon. \end{cases}$$

We leave the standard details to the reader. □

The following self-improvement property for Carleson measures will be used in conjunction with the local Hölder continuity estimate for solutions in (2.17). The result is proved in the unweighted case in Lemma 2.14 in [Auscher et al. 2001]. In that proof, the Lebesgue measure on $\mathbb{R}^n$ can in fact be replaced by any doubling measure, since the Whitney decomposition of open sets can be adapted to any such measure; see, for instance, Lemma 2 in Chapter I of [Stein 1993]. The result below then follows.

Lemma 2.22. *Let $n \ge 1$ and suppose that μ is an A_2-weight on $\mathbb{R}^n$. Let α, $\beta_0 > 0$ and suppose that $(v_t)_{t>0}$ is a collection of Hölder continuous functions on a cube $Q \subset \mathbb{R}^n$ satisfying*

$$0 \le v_t(x) \le \beta_0 \quad \text{and} \quad |v_t(x) - v_t(y)| \le \beta_0\left(\frac{|x-y|}{t}\right)^\alpha$$

for all $x, y \in Q$. If there exists $\eta \in (0, 1]$, $\beta > 0$ and, for each cube $Q' \subseteq Q$, a measurable set $F' \subset Q'$ such that

$$\mu(F') \ge \eta\mu(Q') \quad \text{and} \quad \frac{1}{\mu(Q')}\int_0^{l(Q')}\int_{F'} v_t(x)\, d\mu(x)\frac{dt}{t} \le \beta,$$

then

$$\frac{1}{\mu(Q)}\int_0^{\ell(Q)}\int_Q v_t(x)\, d\mu(x)\frac{dt}{t} \lesssim_{\alpha,\eta} \beta + \beta_0,$$

where the implicit constant depends only on α, η, n and $[\mu]_{A_2}$.

3. Estimates for maximal operators

We obtain estimates for a variety of maximal operators (M_μ, $D_{*,\mu}$, N_*^η and $\widetilde{N}_{*,\mu}^\eta$) adapted to an A_2-weight μ and degenerate elliptic operators $\mathcal{L}_\mu := -\operatorname{div}_\mu((1/\mu)\mathcal{A}\nabla)$ on $\mathbb{R}^n$ for $n \ge 2$. These will be used to prove the Carleson measure estimate from Theorem 1.3 in Section 4. We first define the maximal operators M_μ and $D_{*,\mu}$ by

$$M_\mu f(x) := \sup_{r>0} \fint_{B(x,r)} |f(y)|\, d\mu(y),$$

$$D_{*,\mu} g(x) := \sup_{r>0} \left(\fint_{B(x,r)} \left(\frac{|g(x)-g(y)|}{|x-y|} \right)^2 d\mu(y) \right)^{1/2}$$

for all $f \in L^1_{\mu,\mathrm{loc}}(\mathbb{R}^n)$, $g \in W^{1,2}_{\mu,\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. The usual unweighted and centred Hardy–Littlewood maximal operator is abbreviated by M. The maximal operator M_μ is bounded on $L^p_\mu(\mathbb{R}^n)$ for all $p \in (1,\infty)$ and satisfies the weak-type estimate

$$\mu(\{x \in \mathbb{R}^n : |M_\mu f(x)| > \kappa\}) \lesssim \kappa^{-1} \|f\|_{L^1_\mu(\mathbb{R}^n)} \quad \text{for all } \kappa > 0, \tag{3.1}$$

for all $f \in L^1_\mu(\mathbb{R}^n)$; see, for instance, Theorem 1 in Chapter I of [Stein 1993]. There is also the following weak-type estimate for the maximal operator $D_{*,\mu}$.

Lemma 3.2. *Let $n \geq 2$. If μ is an A_2-weight on $\mathbb{R}^n$, then*

$$\mu(\{x \in \mathbb{R}^n : |D_{*,\mu} f(x)| > \kappa\}) \lesssim \kappa^{-2} \|\nabla f\|^2_{L^2_\mu(\mathbb{R}^n,\mathbb{R}^n)} \quad \textit{for all } \kappa > 0, \tag{3.3}$$

for all $f \in W^{1,2}_\mu(\mathbb{R}^n)$, where the implicit constant depends only on n and $[\mu]_{A_2}$.

Proof. If $f \in C^\infty_c(\mathbb{R}^n)$, then a version of Morrey's inequality [1966, Theorem 3.5.2] shows that

$$\frac{|f(x)-f(y)|}{|x-y|} \lesssim M(\nabla f)(x) + M(\nabla f)(y)$$

for almost every $x, y \in \mathbb{R}^n$; hence

$$D_{*,\mu} f(x) \lesssim M(\nabla f)(x) + \big(M_\mu[M(\nabla f)]^2(x)\big)^{1/2}.$$

Estimate (3.3) then follows from the weak-type bound for M_μ in (3.1), the fact that M is bounded on $L^2_\mu(\mathbb{R}^n)$ (see, for instance, Theorem 1 in Chapter V of [Stein 1993]) and the density of $C^\infty_c(\mathbb{R}^n)$ in $W^{1,2}_\mu(\mathbb{R}^n)$. □

We now define the nontangential maximal operators N^η_* and $\widetilde{N}^\eta_{*,\mu}$, for $\eta > 0$, by

$$N^\eta_* u(x) := \sup_{(y,t)\in\Gamma_\eta(x)} |u(y,t)|, \quad \widetilde{N}^\eta_{*,\mu} v(x) := \sup_{(y,t)\in\Gamma_\eta(x)} \left(\fint_{B(y,tat)} |v(z,t)|^2\, d\mu(z) \right)^{1/2}$$

for all measurable functions u, v on $\mathbb{R}^{n+1}_+$ (such that $v(\cdot,t) \in L^2_{\mu,\mathrm{loc}}(\mathbb{R}^n)$ for a.e. $t > 0$) and $x \in \mathbb{R}^n$, where $\Gamma_\eta(x) := \{(y,t) \in \mathbb{R}^{n+1}_+ : |y-x| < \eta t\}$ is the conical nontangential approach region in $\mathbb{R}^{n+1}_+$ with vertex at x and aperture η.

Now suppose that $\mathcal{A} \in \mathcal{E}(n,\lambda,\Lambda,\mu)$, as defined by (2.6). In particular, since $\mathcal{A}$ has real-valued coefficients, there exists an integral kernel $W_t(x,y)$ such that

$$e^{-t\mathcal{L}_\mu} f(x) = \int_{\mathbb{R}^n} W_t(x,y) f(y)\, d\mu(y) \tag{3.4}$$

for all $f \in L^2_\mu(\mathbb{R}^n)$, and there exists constants $C_1, C_2 > 0$ such that

$$|W_t(x,y)| \leq \frac{C_1}{\mu(B(x,\sqrt{t}))} \exp\biggl(-C_2 \frac{|x-y|^2}{t}\biggr) \tag{3.5}$$

for all $t > 0$ and $x, y \in \mathbb{R}^n$. This was proved by Cruz-Uribe and Rios for $f \in C_c^\infty(\mathbb{R}^n)$ under the assumption that $\mathcal{A}$ is symmetric; see Theorem 1 and Remark 3 in [Cruz-Uribe and Rios 2014]. The symmetry assumption can be removed, however, by following their proof and applying the Harnack inequality for degenerate parabolic equations obtained by Ishige [1999, Theorem A], which does not require symmetric coefficients, instead of the version recorded in Proposition 3.8 of [Cruz-Uribe and Rios 2008]. The results also extend to $f \in L^2_\mu(\mathbb{R}^n)$ by density, Schur's lemma and the doubling property of μ.

We now consider the semigroup generated by $\mathcal{L}_\mu := -\operatorname{div}_\mu((1/\mu)\mathcal{A}\nabla)$ with elliptic homogeneity (t replaced by t^2) and denoted by $\mathcal{P}_t := e^{-t^2\mathcal{L}_\mu}$ in the estimates below.

Lemma 3.6. *Let $n \geq 2$ and suppose that $\mathcal{A} \in \mathcal{E}(n,\lambda,\Lambda,\mu)$ for some constants $0 < \lambda \leq \Lambda < \infty$ and an A_2-weight μ on $\mathbb{R}^n$. Let $p \in (1,\infty)$ and suppose that μ is also an A_p-weight on $\mathbb{R}^n$. If $x \in \mathbb{R}^n$, $\eta > 0$ and $\alpha \geq 1$, then*

$$\sup_{(y,t)\in\Gamma_\eta(x)} |(\eta t)^{-1}[\mathcal{P}_{\eta t}(f - c_{B(x,\alpha\eta t)})](y)|^2 \lesssim_\alpha [M_\mu(|\nabla f|^p)(x)]^{2/p} \tag{3.7}$$

for all $f \in W^{1,p}_\mu(\mathbb{R}^n)$ and $c_{B(x,\alpha\eta t)} \in \bigl\{⨍_{B(x,\alpha\eta t)} f\, d\mu, ⨍_{B(x,\alpha\eta t)} f\bigr\}$, and

$$|N^\eta_*(\partial_t \mathcal{P}_t f)(x)|^2 \lesssim_\eta [M_\mu(|\nabla f|^p)(x)]^{2/p}, \tag{3.8}$$

$$|\eta^{-1} N^\eta_*(\partial_t \mathcal{P}_{\eta t} f)(x)|^2 \lesssim [M_\mu(|\nabla f|^p)(x)]^{2/p}, \tag{3.9}$$

$$|\widetilde{N}^\eta_{*,\mu}(\nabla_x \mathcal{P}_{\eta t} f)(x)|^2 \lesssim M_\mu\bigl([M_\mu(|\nabla f|^p)]^{2/p}\bigr)(x) + M_\mu(|\nabla f|^2)(x) \tag{3.10}$$

for all $f \in W^{1,2}_\mu(\mathbb{R}^n) \cap W^{1,p}_{\mu,\mathrm{loc}}(\mathbb{R}^n)$, where the implicit constants depend only on n, λ, Λ, p, $[\mu]_{A_2}$ and $[\mu]_{A_p}$, as well as on α in (3.7) and on η in (3.8).

Proof. Let $x \in \mathbb{R}^n$, $(y,t) \in \Gamma^\eta(x)$, $f \in W^{1,2}_\mu(\mathbb{R}^n) \cap W^{1,p}_{\mu,\mathrm{loc}}(\mathbb{R}^n)$, $f_{B(x,t)} := ⨍_{B(x,t)} f$ and $\tilde{f}_{B(x,t)} := ⨍_{B(x,t)} f\, d\mu$. To prove (3.7), it suffices to assume that $\eta = 1$ and $\alpha \geq 1$. We set $C_0(t) := B(x,\alpha t)$ and define the dyadic annulus $C_j(t) := B(x, 2^j\alpha t) \setminus B(x, 2^{j-1}\alpha t)$ for all $j \in \mathbb{N}$. The Gaussian kernel estimates in (3.4) and (3.5) imply

$$\begin{aligned}
|t^{-1}[\mathcal{P}_t(f - f_{B(x,\alpha t)})](y)| &= t^{-1}\biggl|\int_{\mathbb{R}^n} W_{t^2}(y,z)[f(z) - f_{B(x,\alpha t)}]\, d\mu(z)\biggr| \\
&\leq \sum_{j=0}^\infty t^{-1} \frac{C_1}{\mu(B(y,t))} \int_{C_j(t)} \exp\biggl(-C_2\frac{|y-z|^2}{t^2}\biggr) |f(z) - f_{B(x,\alpha t)}|\, d\mu(z) \\
&=: \sum_{j=0}^\infty I_j.
\end{aligned}$$

To estimate I_0, note that $B(x,\alpha t) \subseteq B(y,(1+\alpha)t)$ and apply the doubling property of μ, followed by the L^p_μ-Poincaré inequality in (2.5) with $c_B = \fint_{B(x,\alpha t)} f$, to obtain

$$I_0 \lesssim_\alpha t^{-1} \fint_{B(x,\alpha t)} |f(z) - f_{B(x,\alpha t)}| \, d\mu(z) \lesssim \left(\fint_{B(x,\alpha t)} |\nabla f|^p \, d\mu \right)^{1/p} \lesssim [M_\mu(|\nabla f|^p)(x)]^{1/p}.$$

To estimate I_j, for each $j \in \mathbb{N}$, expand $f(z) - f_{B(x,\alpha t)}$ as a telescoping sum to write

$$\begin{aligned} I_j &\le C_1 e^{-C_2(2^{j-1}\alpha - 1)^2} \frac{\mu(B(x, 2^j \alpha t))}{\mu(B(y,t))} t^{-1} \\ &\qquad \times \left(\fint_{B(x,2^j\alpha t)} |f - \tilde{f}_{B(x,2^j\alpha t)}| \, d\mu + \sum_{i=1}^{j} |\tilde{f}_{B(x,2^i\alpha t)} - \tilde{f}_{B(x,2^{i-1}\alpha t)}| + |\tilde{f}_{B(x,\alpha t)} - f_{B(x,\alpha t)}| \right) \\ &\lesssim e^{-C_2(2^{j-1}\alpha - 1)^2} \frac{\mu(B(y,(1+2^j\alpha)t))}{\mu(B(y,t))} \sum_{i=0}^{j} t^{-1} \fint_{B(x,2^i\alpha t)} |f - \tilde{f}_{B(x,2^i\alpha t)}| \, d\mu \\ &\lesssim e^{-C_2(2^{j-1}\alpha - 1)^2} (1+2^j\alpha)^{2n} \sum_{i=0}^{j} 2^i \alpha \left(\fint_{B(x,2^i\alpha t)} |\nabla f|^p \, d\mu \right)^{1/p} \\ &\lesssim_\alpha e^{-C4^j} 4^{nj} [M_\mu(|\nabla f|^p)(x)]^{1/p}, \end{aligned}$$

where the second inequality relies on the inclusion $B(x, 2^j\alpha t) \subseteq B(y,(1+2^j\alpha)t)$, whilst the third inequality uses the doubling property of μ in (2.1) with $p = 2$, and the L^p_μ-Poincaré inequality in (2.5) with $c_B = \fint_{B(x,2^i\alpha t)} f \, d\mu$. Altogether, we have

$$|t^{-1}[\mathcal{P}_t(f - f_{B(x,\alpha t)})](y)| \lesssim_\alpha \left(\sum_{j=0}^{\infty} e^{-C4^j} 4^{nj} \right) [M_\mu(|\nabla f|^p)(x)]^{1/p} \lesssim [M_\mu(|\nabla f|^p)(x)]^{1/p},$$

which proves (3.7) when $c_{B(x,\alpha t)} = \fint_{B(x,\alpha t)} f$. The proof when $c_{B(x,\alpha t)} = \fint_{B(x,\alpha t)} f \, d\mu$ follows as above by replacing $f_{B(x,\alpha t)}$ with $\tilde{f}_{B(x,\alpha t)}$, since (2.5) can still be applied.

To prove (3.8) and (3.9), suppose that $\eta > 0$. The Gaussian kernel estimate for $e^{-t\mathcal{L}_\mu}$ in (3.5) implies that $t\partial_t \mathcal{P}_t f(y)$ has an integral kernel $\widetilde{W}_{t^2}(y,z)$ satisfying

$$|\widetilde{W}_{t^2}(y,z)| \le \frac{C_1}{\mu(B(y,t))} \exp\left(-C_2 \frac{|y-z|^2}{t^2} \right)$$

and the conservation property $\int_{\mathbb{R}^n} \widetilde{W}_{t^2}(y,z) \, d\mu(y) = 0$ for all $z \in \mathbb{R}^n$ and $t > 0$. This follows from Theorem 5 in [Cruz-Uribe and Rios 2014], where the assumption that $\mathcal{A}$ is symmetric can be removed as per the remarks preceding this lemma. Therefore, we may write

$$|\partial_t \mathcal{P}_t f(y)| = t^{-1} \left| \int_{\mathbb{R}^n} \widetilde{W}_{t^2}(y,z)[f(z) - f_{B(x,\eta t)}] \, d\mu(z) \right|$$

and a change of variables implies

$$\sup_{(y,t)\in\Gamma^\eta(x)} |\partial_t \mathcal{P}_t f(y)| = \sup_{(y,t)\in\Gamma(x)} t^{-1} \left| \int_{\mathbb{R}^n} \eta \widetilde{W}_{(t/\eta)^2}(y,z)[f(z) - f_{B(x,t)}] \, d\mu(z) \right|.$$

We can then obtain (3.8) by following the proof of (3.7) with $\alpha = 1$ in order to show that this is bounded by $[M_\mu(|\nabla f|^p)(x)]^{1/p}$, since the doubling property of μ ensures that

$$|\eta \widetilde{W}_{(t/\eta)^2}(y,z)| \leq \frac{C_{1,\eta}}{\mu(B(y,t))} \exp\left(-C_{2,\eta}\frac{|y-z|^2}{t^2}\right)$$

for some positive constants $C_{1,\eta}$ and $C_{2,\eta}$ that depend on η. We obtain (3.9) as an immediate consequence of (3.8) and the fact that $\eta^{-1}\partial_t \mathcal{P}_{\eta t} = (\partial_s \mathcal{P}_s)|_{s=\eta t}$.

To prove (3.10), let $\eta > 0$, set $u_{\eta t} := \mathcal{P}_{\eta t} f$ and choose a nonnegative function $\Phi \in C_c^\infty(B(y, 2\eta t))$ such that $\Phi \equiv 1$ on $B(y, \eta t)$ and $|\nabla_x \Phi| \lesssim (\eta t)^{-1}$. Let $c > 0$ denote a constant that will be chosen later. The definition of $\mathcal{L}_\mu$ implies

$$\begin{aligned}
&⨍_{B(y,\eta t)} |\nabla_x \mathcal{P}_{\eta t} f|^2 \, d\mu \\
&\qquad \leq \frac{1}{\mu(B(y,\eta t))} \int_{\mathbb{R}^n} |\nabla_x u_{\eta t}|^2 \Phi^2 \, d\mu \\
&\qquad \lesssim \frac{1}{\mu(B(y,\eta t))} \int_{\mathbb{R}^n} \langle \mathcal{A}\nabla_x u_{\eta t}, \nabla_x(u_{\eta t} - c)\rangle \Phi^2 \\
&\qquad = \frac{1}{\mu(B(y,\eta t))} \int_{\mathbb{R}^n} \{\langle \mathcal{A}\nabla_x u_{\eta t}, \nabla_x[(u_{\eta t} - c)\Phi^2]\rangle - 2\langle \mathcal{A}\nabla_x u_{\eta t}, \nabla_x \Phi (u_{\eta t} - c)\rangle \Phi\} \\
&\qquad \lesssim \frac{1}{\mu(B(y,\eta t))} \int_{\mathbb{R}^n} \{(\mathcal{L}_\mu u_{\eta t})(u_{\eta t} - c)\Phi^2 + |\nabla_x u_{\eta t}||\nabla_x \Phi||(u_{\eta t} - c)\Phi|\} \, d\mu \\
&\qquad \leq \frac{1}{\mu(B(y,\eta t))} \int_{B(y,2\eta t)} \left(\frac{1}{2\eta^2 t}|\partial_t u_{\eta t}||u_{\eta t} - c|\Phi^2 + |\nabla_x u_{\eta t}||\nabla_x \Phi||u_{\eta t} - c|\Phi\right) d\mu \\
&\qquad =: I + II.
\end{aligned}$$

Now fix $c := \tilde{f}_{B(x,3\eta t)}$. To estimate I, we use Cauchy's inequality and the doubling property of μ, combined with the fact that $B(x,\eta t) \subseteq B(y, 2\eta t) \subseteq B(x, 3\eta t)$, to obtain

$$I \lesssim ⨍_{B(x,3\eta t)} \big(|\eta^{-1}\partial_t u_{\eta t}|^2 + (\eta t)^{-2}|u_{\eta t} - f|^2 + (\eta t)^{-2}|f - \tilde{f}_{B(x,3\eta t)}|^2\big) \, d\mu =: I_1 + I_2 + I_3.$$

It is immediate that $I_1 \leq M_\mu(|\eta^{-1} N_*^\eta(\partial_t \mathcal{P}_{\eta t} f)|^2)(x)$, whilst the semigroup property

$$|u_{\eta t}(z) - f(z)| = \left|\int_0^{\eta t} \partial_s u_s(z) \, ds\right| \leq \eta t N_*(\partial_s u_s)(z)$$

implies that $I_2 \lesssim M_\mu(|N_*(\partial_s u_s)|^2)(x)$, and the L^2_μ-Poincaré inequality in (2.5) shows that $I_3 \lesssim M_\mu(|\nabla f|^2)(x)$; hence

$$I \leq M_\mu(|\eta^{-1} N_*^\eta(\partial_t \mathcal{P}_{\eta t} f)|^2)(x) + M_\mu(|N_*(\partial_s u_s)|^2)(x) + M_\mu(|\nabla f|^2)(x).$$

To estimate II, we use Cauchy's inequality with $\epsilon > 0$ to obtain

$$II \lesssim \frac{\epsilon}{\mu(B(y,\eta t))} \int_{\mathbb{R}^n} |\nabla_x u_{\eta t}|^2 \Phi^2 \, d\mu + \epsilon^{-1}(I_2 + I_3).$$

A sufficiently small choice of $\epsilon > 0$ allows the ϵ-term to be subtracted, yielding

$$\fint_{B(y,\eta t)} |\nabla_x \mathcal{P}_{\eta t} f|^2 \, d\mu \lesssim I + II \lesssim M_\mu(|\eta^{-1} N_*^\eta(\partial_t \mathcal{P}_{\eta t} f)|^2 + |N_*(\partial_t \mathcal{P}_t f)|^2 + |\nabla f|^2)(x),$$

which, combined with (3.8) and (3.9), implies (3.10). □

The pointwise estimates in Lemma 3.6 have the following corollary.

Corollary 3.11. *Let $n \geq 2$ and suppose that $\mathcal{A} \in \mathcal{E}(n, \lambda, \Lambda, \mu)$ for some constants $0 < \lambda \leq \Lambda < \infty$ and an A_2-weight μ on $\mathbb{R}^n$. If $\eta > 0$, then*

$$\mu(\{x \in \mathbb{R}^n : |N_*^\eta(\partial_t \mathcal{P}_t f)(x)| > \kappa\}) \lesssim_\eta \kappa^{-2} \|\nabla f\|^2_{L^2_\mu(\mathbb{R}^n,\mathbb{R}^n)}, \tag{3.12}$$

$$\mu(\{x \in \mathbb{R}^n : |\eta^{-1} N_*^\eta(\partial_t \mathcal{P}_{\eta t} f)(x)| > \kappa\}) \lesssim \kappa^{-2} \|\nabla f\|^2_{L^2_\mu(\mathbb{R}^n,\mathbb{R}^n)}, \tag{3.13}$$

$$\mu(\{x \in \mathbb{R}^n : |\widetilde{N}_{*,\mu}^\eta(\nabla_x \mathcal{P}_{\eta t} f)(x)| > \kappa\}) \lesssim \kappa^{-2} \|\nabla f\|^2_{L^2_\mu(\mathbb{R}^n,\mathbb{R}^n)} \tag{3.14}$$

for all $\kappa > 0$ and $f \in W^{1,2}_\mu(\mathbb{R}^n)$, where the implicit constants depend only on n, λ, Λ and $[\mu]_{A_2}$, as well as on η in (3.12).

Proof. Estimates (3.12) and (3.13) follow respectively from (3.8) and (3.9), in the case $p = 2$, since M_μ satisfies the weak-type estimate in (3.1). To prove (3.14), note that there exists $1 < q < 2$ such that μ is an A_q-weight on $\mathbb{R}^n$; see, for instance, Section 3 in Chapter V of [Stein 1993]. Therefore, combining (3.10) in the case $p = q$ with (3.1) and noting that $2/q > 1$, we obtain

$$\begin{aligned}\mu(\{x \in \mathbb{R}^n : |\widetilde{N}_{*,\mu}^\eta(\nabla_x \mathcal{P}_{\eta t} f)(x)| > \kappa\}) &\lesssim \kappa^{-2}(\|M_\mu(|\nabla f|^q)\|^{2/q}_{L^{2/q}_\mu(\mathbb{R}^n)} + \|\nabla f\|^2_{L^2_\mu(\mathbb{R}^n,\mathbb{R}^n)}) \\ &\lesssim \kappa^{-2} \|\nabla f\|^2_{L^2_\mu(\mathbb{R}^n,\mathbb{R}^n)}\end{aligned}$$

for all $\kappa > 0$ and $f \in W^{1,2}_\mu(\mathbb{R}^n)$ (since $W^{1,2}_\mu(\mathbb{R}^n) \subseteq W^{1,q}_{\mu,\text{loc}}(\mathbb{R}^n)$), as required. □

4. The Carleson measure estimate

The purpose of this section is to prove the Carleson measure estimate (1.4) in Theorem 1.3. We adopt the strategy outlined at the end of Section 3.1 in [Hofmann et al. 2015a], although the crucial technical estimate, stated here as Theorem 4.10, is not at all an obvious extension of the uniformly elliptic case. Moreover, establishing the Carleson measure estimate directly allows us to avoid "good-λ" inequalities and thus apply a change of variables based on the $W^{1,2}_{0,\mu}$-Hodge decomposition in (2.10), instead of the $W^{1,2+\epsilon}_0$-version (for a sufficiently small $\epsilon > 0$) required in [Hofmann et al. 2015a].

The technical result in Theorem 4.10 establishes (1.4) on certain "big pieces" of all cubes. The passage to the general estimate ultimately follows from the self-improvement property for Carleson measures in Lemma 2.22. This requires, however, that the Carleson measure estimate on the full gradient ∇u of a solution u can be controlled by the same estimate on its transversal derivative $\partial_t u$, which is the content of Lemma 4.2. We briefly postpone the statement and proof of Lemma 4.2 and Theorem 4.10, however, in order to deduce Theorem 1.3 from those results below.

In contrast to the previous two sections, the results here concern solutions of the equation $\mathrm{div}(A\nabla u) = 0$ in open sets $\Omega \subseteq \mathbb{R}^{n+1}_+$ when $n \geq 2$ and A is a t-independent coefficient matrix that satisfies (1.1) for some $0 < \lambda \leq \Lambda < \infty$ and an A_2-weight μ on $\mathbb{R}^n$. In particular, in Section 2, weighted Sobolev spaces were defined on open sets in $\mathbb{R}^d$ and matrix coefficients $\mathcal{A} \in \mathcal{E}(d, \lambda, \Lambda, \mu)$ were considered for all $d \in \mathbb{N}$. Those results also hold here on open sets in the upper half-space with the weight $\mu(x,t) := \mu(x)$ and the coefficients $A(x,t) := A(x)$ for all $(x,t) \in \mathbb{R}^{n+1}$, since then $[\mu]_{A_2(\mathbb{R}^{n+1})} = [\mu]_{A_2(\mathbb{R}^n)}$ and $A \in \mathcal{E}(n+1, \lambda, \Lambda, \mu)$. In particular, the solution space $W^{1,2}_{\mu,\mathrm{loc}}(\Omega)$ is defined and the regularity estimates in (2.16), (2.17) and (2.18) hold when $\Omega \subseteq \mathbb{R}^{n+1}_+$.

We will also use, without reference, the well-known fact that if u is a solution of $\mathrm{div}(A\nabla u) = 0$ in $\Omega \subseteq \mathbb{R}^{n+1}_+$, then $\partial_t u$ is also a solution in Ω. In particular, to see that $\partial_t u$ is in $W^{1,2}_{\mu,\mathrm{loc}}(\Omega)$, a Whitney decomposition of Ω reduces matters to showing that $\partial_t u$ is in $W^{1,2}_\mu(R)$ for all cubes $R \subset \Omega$ satisfying $\ell(R) < \frac{1}{2}\,\mathrm{dist}(R, \partial\Omega)$. To this end, define the difference quotients $D^h_i u(X) := (1/h)[u(X + he_i) - u(X)]$ for all $X \in R$ and $h < \mathrm{dist}(R, \partial\Omega)$, where e_i is the unit vector in the i-th coordinate direction in $\mathbb{R}^{n+1}$. The t-independence of the coefficients implies that $D^h_{n+1}u$ is a solution in R, so we use the identity $D^h_{n+1}(\partial_i u) = \partial_i(D^h_{n+1}u)$ and Caccioppoli's inequality to obtain

$$\begin{aligned}\iint_R |D^h_{n+1}(\partial_i u)|^2\, d\mu \leq \iint_R |\nabla(D^h_{n+1}u)|^2\, d\mu &\lesssim \ell(R)^2 \iint_{2R} |D^h_{n+1}u|^2\, d\mu \\ &\leq \ell(R)^2 \iint_{2R} |\partial_t u|^2\, d\mu =: K \quad \text{for all } h < \mathrm{dist}(R, \partial\Omega),\end{aligned}$$

where the implicit constant depends only on n, λ, Λ and $[\mu]_{A_2}$, and the final bound holds uniformly in h because u is in $W^{1,2}_\mu(R)$; see Lemma 7.23 in [Gilbarg and Trudinger 1977]. We can then use Lemma 7.24 in the same reference to deduce that $\partial_t u$ is in $W^{1,2}_\mu(R)$ with the estimate

$$\|\partial_i \partial_t u\|^2_{L^2_\mu(R)} = \|\partial_t \partial_i u\|^2_{L^2_\mu(R)} \leq K$$

for all $i \in \{1, \ldots, n+1\}$, as required. Note that the proofs of Lemmas 7.23 and 7.24 in [Gilbarg and Trudinger 1977] extend immediately to the weighted context considered here because $C^\infty(R)$ is still dense in $W^{1,2}_\mu(R)$.

Proof of Theorem 1.3 from Lemma 4.2 and Theorem 4.10. Let $Q \subset \mathbb{R}^n$ denote a cube and suppose that $u \in L^\infty(\mathbb{R}^{n+1}_+)$ solves $\mathrm{div}(A\nabla u) = 0$ in $\mathbb{R}^{n+1}_+$. It follows a fortiori from Theorem 4.10 that there exist constants $C, c_0 > 0$ and, for each cube $Q' \subseteq Q$, a measurable set $F' \subset Q'$ such that $\mu(F') \geq c_0 \mu(Q')$ and

$$\frac{1}{\mu(Q')} \int_0^{l(Q')} \int_{F'} |t\partial_t u(x,t)|^2\, d\mu(x)\, \frac{dt}{t} \leq C\|u\|^2_\infty,$$

where C and c_0 depend only on n, λ, Λ and $[\mu]_{A_2}$.

The coefficient matrix A is t-independent, so $\partial_t u$ is also a solution and thus the degenerate version of Moser's estimate in (2.16), followed by Caccioppoli's inequality, shows that $\|t\partial_t u\|_\infty \lesssim \|u\|_\infty$. Moreover, the degenerate version of the de Giorgi–Nash Hölder regularity for solutions in (2.17) shows that

$$|t\partial_t u(x,t) - t\partial_t u(y,t)| \lesssim \left(\frac{|x-y|}{t}\right)^{\alpha} \|t\partial_t u\|_\infty \lesssim \|u\|_\infty \left(\frac{|x-y|}{t}\right)^{\alpha}$$

for all $x, y \in Q$ and $t > 0$, where all of the implicit constants and the exponent $\alpha > 0$ depend only on n, λ, Λ and $[\mu]_{A_2}$. Therefore, we may apply Lemma 2.22 with

$$\{v_t, \alpha, \beta_0, \eta, \beta\} := \{(t\partial_t u)^2, \alpha, C\|u\|_\infty^2, c_0, C\|u\|_\infty^2\}$$

to obtain

$$\frac{1}{\mu(Q)} \int_0^{\ell(Q)} \int_Q |t\partial_t u(x,t)|^2 \, d\mu(x) \frac{dt}{t} \lesssim \|u\|_\infty^2, \tag{4.1}$$

where the implicit constant depends only on n, λ, Λ and $[\mu]_{A_2}$. This estimate holds for all cubes Q, so by Lemma 4.2, we conclude that (1.4) holds. □

We now dispense with the following lemma, which was used in the proof of Theorem 1.3 above to reduce to a Carleson measure estimate on the transversal derivative of solutions. The proof is adapted from Section 3.1 of [Hofmann et al. 2015a].

Lemma 4.2. *Let $n \geq 2$ and consider a cube $Q \subset \mathbb{R}^n$. If A is a t-independent coefficient matrix that satisfies the degenerate bound and ellipticity in* (1.1) *for some constants $0 < \lambda \leq \Lambda < \infty$ and an A_2-weight μ on $\mathbb{R}^n$, then any solution $u \in L^\infty(4Q \times (0, 4\ell(Q)))$ of* $\operatorname{div}(A\nabla u) = 0$ *in* $4Q \times (0, 4\ell(Q))$ *satisfies*

$$\int_0^{\ell(Q)} \int_Q |t\nabla u(x,t)|^2 \, d\mu(x) \frac{dt}{t} \lesssim \int_0^{4\ell(Q)} \int_{4Q} |t\partial_t u(x,t)|^2 \, d\mu(x) \frac{dt}{t} + \mu(Q)\|u\|_\infty^2,$$

where the implicit constant depends only on n, λ, Λ and $[\mu]_{A_2}$.

Proof. Let $0 < \delta < \frac{1}{2}$ and set $\Phi_Q(t) := \Phi(t/\ell(Q))$, where $\Phi : \mathbb{R} \to [0,1]$ denotes a C^∞-function such that $\Phi(t) = 1$ for all $2\delta \leq t \leq 1$, whilst $\Phi(t) = 0$ for all $t \leq \delta$ and $t \geq 2$. Integrating by parts with respect to the t-variable and noting that $\|\partial_t \Phi\|_{L^\infty([1,2])} \lesssim 1$, whilst $\|\partial_t \Phi\|_{L^\infty([\delta,2\delta])} \lesssim 1/\delta$, we obtain

$$\begin{aligned}
\boldsymbol{I} &:= \int_Q \int_0^{2\ell(Q)} |\nabla u(x,t)|^2 \Phi_Q(t) t \, dt \, d\mu(x) \\
&\eqsim \int_Q \int_0^{2\ell(Q)} \partial_t(|\nabla u(x,t)|^2 \Phi_Q(t)) t^2 \, dt \, d\mu(x) \\
&\lesssim \int_Q \int_0^{2\ell(Q)} \langle \nabla\partial_t u(x,t), \nabla u(x,t)\rangle \Phi_Q(t) t^2 \, dt \, d\mu(x) \\
&\qquad + \int_Q \fint_{\ell(Q)}^{2\ell(Q)} |\nabla u(x,t)|^2 t^2 \, dt \, d\mu(x) + \int_Q \fint_{\delta\ell(Q)}^{2\delta\ell(Q)} |\nabla u(x,t)|^2 t^2 \, dt \, d\mu(x) \\
&=: \boldsymbol{I}' + \boldsymbol{I}'' + \boldsymbol{I}'''.
\end{aligned}$$

For the term $\boldsymbol{I}'$, we apply Cauchy's inequality with an arbitrary $\epsilon > 0$ to obtain

$$\boldsymbol{I}' \leq \epsilon \boldsymbol{I} + \frac{1}{\epsilon} \int_Q \int_0^{2\ell(Q)} |\nabla\partial_t u(x,t)|^2 t^3 \, dt \, d\mu(x).$$

For the term $\boldsymbol{I}''$, we apply Caccioppoli's inequality, the doubling property of μ and the fact that $t \approx \ell(Q)$ in the domain of the integration to obtain

$$\begin{aligned}\boldsymbol{I}'' &\eqsim \ell(Q)\int_Q \int_{\ell(Q)}^{2\ell(Q)} |\nabla u(x,t)|^2\,dt\,d\mu(x)\\ &\lesssim \frac{1}{\ell(Q)}\int_{2Q}\int_{\ell(Q)/2}^{5\ell(Q)/2} |u(x,t)|^2\,dt\,d\mu(x) \lesssim \mu(Q)\|u\|_\infty^2.\end{aligned}$$

For the term $\boldsymbol{I}'''$, the same reasoning shows that $\boldsymbol{I}''' \lesssim \mu(Q)\|u\|_\infty^2$. We now fix $\epsilon > 0$, depending only on allowable constants, such that altogether

$$\boldsymbol{I} \lesssim \int_Q \int_0^{2\ell(Q)} |\nabla \partial_t u(x,t)|^2 t^3\,dt\,d\mu(x) + \mu(Q)\|u\|_\infty^2,$$

which is justified since $\boldsymbol{I} < \infty$ by Caccioppoli's inequality and the support of Φ_Q.

To complete the estimate, we let $\{W_j : j \in J\}$ denote a collection of Whitney boxes (from a Whitney decomposition of $\mathbb{R}^{n+1}_+$) such that $W_j \cap (Q \times (0, 2\ell(Q))) \neq \varnothing$ and $\sum_{j\in J} \mathbb{1}_{2W_j}(x,t) \lesssim 1$. The coefficient matrix A is t-independent, so $\partial_t u$ is also a solution of $\operatorname{div}(A\nabla u) = 0$ in each set W_j; hence we may apply Caccioppoli's inequality in combination with the fact that $t \eqsim l(W_j)$ in W_j to obtain

$$\begin{aligned}\int_{2\delta\ell(Q)}^{\ell(Q)}\int_Q |t\nabla u(x,t)|^2\,d\mu(x)\,\frac{dt}{t} &\lesssim \sum_{j\in J}\iint_{W_j} |\nabla\partial_t u(x,t)|^2 t^3\,dt\,d\mu(x) + \mu(Q)\|u\|_\infty^2\\ &\lesssim \sum_{j\in J} l(W_j)\iint_{2W_j} |\partial_t u(x,t)|^2\,dt\,d\mu(x) + \mu(Q)\|u\|_\infty^2\\ &\lesssim \int_0^{4\ell(Q)}\int_{4Q} |t\partial_t u(x,t)|^2\,d\mu(x)\,\frac{dt}{t} + \mu(Q)\|u\|_\infty^2,\end{aligned}$$

where the implicit constants do not depend on δ. The final result is then obtained by applying Fatou's lemma to estimate the limit as δ approaches 0. □

The remainder of this section is dedicated to the proof of the crucial technical estimate, Theorem 4.10, that was used to prove Theorem 1.3. The proof adapts the change of variables from Section 3.2 of [Hofmann et al. 2015a] to the degenerate elliptic case. This is used to pull back solutions to certain sawtooth domains where the Carleson measure estimate can be verified by reducing matters to the vertical square function estimates in Theorem 2.12, which we recall were obtained from the solution of the Kato problem in [Cruz-Uribe and Rios 2015]. The following technical lemma, which reprises the notation $\mathcal{P}_t := e^{-t^2\mathcal{L}_\mu}$ for $\mathcal{L}_\mu := -\operatorname{div}_\mu((1/\mu)\mathcal{A}\nabla)$ and $\mathcal{A} \in \mathcal{E}(n,\lambda,\Lambda,\mu)$ as in (2.6) and Lemma 3.6, will be used to justify these changes of variables.

Lemma 4.3. *Let $n \geq 2$ and suppose that $\mathcal{A} \in \mathcal{E}(n,\lambda,\Lambda,\mu)$ for some constants $0 < \lambda \leq \Lambda < \infty$ and an A_2-weight μ on $\mathbb{R}^n$. Let $Q \subset \mathbb{R}^n$ denote a cube and suppose that $\boldsymbol{f} : 5Q \to \mathbb{R}^n$ is a measurable function such that $(1/\mu)\boldsymbol{f} \in L^\infty(5Q)$. Let $\phi \in W^{1,2}_{0,\mu}(5Q)$ and suppose that* $\operatorname{div}(\mathcal{A}\nabla\phi) = \operatorname{div}\boldsymbol{f}$ *in $5Q$. If $\kappa_0 > 0$,*

$0<\eta<\frac{1}{2}$ and $x_0\in Q$ satisfy $\Lambda(\eta,\phi,\mathcal{A})(x_0)\le\kappa_0$, where

$$\Lambda(\eta,\phi,\mathcal{A}):=\eta^{-1}N_*^\eta(\partial_t\mathcal{P}_{\eta t}\phi)+N_*(\partial_t\mathcal{P}_t\phi)+[M_\mu(|\nabla_x\phi|^2)]^{1/2}+D_{*,\mu}\phi, \tag{4.4}$$

then

$$|\partial_t\mathcal{P}_{\eta t}\phi(x)|\le\eta\kappa_0 \quad \text{for all } (x,t)\in\Gamma_\eta(x_0) \tag{4.5}$$

and

$$|(I-\mathcal{P}_{\eta t})\phi(x)|\lesssim\eta\Big(\kappa_0+\Big\|\frac{1}{\mu}\boldsymbol{f}\Big\|_\infty\Big)t \quad \text{for all } (x,t)\in\Gamma_\eta(x_0)\cap(2Q\times(0,4\ell(Q))), \tag{4.6}$$

where the implicit constant depends only on n, λ, Λ and $[\mu]_{A_2}$.

Proof. Suppose that $\kappa_0>0$, $0<\eta<\frac{1}{2}$ and $x_0\in Q$ satisfy $\Lambda(\eta,\phi,\mathcal{A})(x_0)\le\kappa_0$. It follows a fortiori that $\eta^{-1}N_*^\eta(\partial_t\mathcal{P}_{\eta t}\phi)(x_0)\le\kappa_0$, so (4.5) holds for all $(x,t)\in\Gamma_\eta(x_0)$.

To prove (4.6), first note that the properties of the semigroup imply

$$|(I-\mathcal{P}_{\eta t})\phi(x_0)|=\Big|\int_0^{\eta t}\partial_s\mathcal{P}_s\phi(x_0)\,ds\Big|\le\eta t\kappa_0 \tag{4.7}$$

for all $t>0$, since $N_*(\partial_s\mathcal{P}_s\phi)(x_0)\le\kappa_0$. Now let $(x,t)\in\Gamma_\eta(x_0)\cap(2Q\times(0,4\ell(Q)))$. We set $\phi_{x_0,\eta t}:=\fint_{B(x_0,2\eta t)}\phi(y)\,dy$ and apply estimate (3.7) with $\alpha=2$ to obtain

$$|\mathcal{P}_{\eta t}(\phi-\phi_{x_0,\eta t})(x)|\lesssim\eta t[M_\mu(|\nabla_x\phi|^2)(x_0)]^{1/2}\le\eta t\kappa_0. \tag{4.8}$$

Next, since $\operatorname{div}(\mathcal{A}\nabla(\phi-\phi(x_0)))=\operatorname{div}(\mathcal{A}\nabla\phi)=\operatorname{div}\boldsymbol{f}$ in $5Q$, and since $0<\eta<\frac{1}{2}$ ensures that $B(x_0,2\eta t)\subseteq 5Q$, we may apply the degenerate version of Moser's estimate for inhomogeneous equations in (2.20) to obtain

$$\begin{aligned}|\phi(x)-\phi(x_0)|&\lesssim\Big(\fint_{B(x_0,2\eta t)}|\phi(y)-\phi(x_0)|^2\,d\mu(y)\Big)^{1/2}+\eta t\Big\|\frac{1}{\mu}\boldsymbol{f}\Big\|_\infty\\&\lesssim\eta t\Big(D_{*,\mu}\phi(x_0)+\Big\|\frac{1}{\mu}\boldsymbol{f}\Big\|_\infty\Big)\\&\lesssim\eta t\Big(\kappa_0+\Big\|\frac{1}{\mu}\boldsymbol{f}\Big\|_\infty\Big).\end{aligned} \tag{4.9}$$

Combining estimates (4.7), (4.8) and (4.9), we obtain

$$\begin{aligned}|(I-\mathcal{P}_{\eta t})\phi(x)|&\le|\phi(x)-\phi(x_0)|+|(I-\mathcal{P}_{\eta t})\phi(x_0)|+|\mathcal{P}_{\eta t}(\phi-\phi_{x_0,\eta t})(x_0)|+|\mathcal{P}_{\eta t}(\phi-\phi_{x_0,\eta t})(x)|\\&\lesssim\eta\Big(\kappa_0+\Big\|\frac{1}{\mu}\boldsymbol{f}\Big\|_\infty\Big)t,\end{aligned}$$

which proves (4.6), as the implicit constant depends only on n, λ, Λ and $[\mu]_{A_2}$. □

We now present the main technical result of this section. The proof is adapted from Section 3.2 of [Hofmann et al. 2015a], although some arguments have been simplified as detailed at the beginning of this section, and the additional justification required in the degenerate elliptic case has been emphasised.

The strategy of the original proof in [Hofmann et al. 2015a] was motivated in part by the fact that integration by parts is sufficient to establish the required estimate in the case when A has a certain block upper-triangular structure. A key idea in that paper was to account for the presence of lower-triangular coefficients $\boldsymbol{c}$ (and upper-triangular coefficients) by decomposing them according to a $W_0^{1,2+\epsilon}$-Hodge decomposition. This was done locally on a given cube Q and the idea has been adapted here. First, the $W_{0,\mu}^{1,2}$-Hodge decomposition $\boldsymbol{c}\mathbb{1}_{5Q} = \mu\boldsymbol{h} - A_\parallel^*\nabla\varphi$ is introduced in (4.13), where $A_\parallel$ is the $n \times n$ submatrix of A shown in (4.12). After integrating by parts, the divergence-free component $\mu\boldsymbol{h}$ provides valuable cancellation, whilst the adapted gradient vector field $A_\parallel^*\nabla\varphi$ facilitates a reduction to the square function estimates in Theorem 2.12, which are implied by the solution to the Kato problem in [Cruz-Uribe and Rios 2015], for the boundary operator $L_{\parallel,\mu}^* := -\operatorname{div}_\mu((1/\mu)A_\parallel^*\nabla_x)$.

The latter estimates, however, require that $L_{\parallel,\mu}^*$ acts on the range of $P_t^* := e^{-t^2 L_{\parallel,\mu}^*}$ and this is arranged by initially making the Dahlberg–Kenig–Stein-type pull-back $t \mapsto t - (I - P_{\eta t}^*)\varphi(x)$ so that the lower-triangular coefficients become $\mu\boldsymbol{h} - A_\parallel^*\nabla_x P_{\eta t}^*\varphi$. This change of variables is justified by choosing $\eta > 0$ small enough so that the pull-back is bi-Lipschitz in t. Once this is in place, a set F is introduced that contains a "big piece" of Q and on which the various maximal functions in Lemma 4.3 are bounded. The integration on $F \times (0, \ell(Q))$ is then performed by introducing a smooth test function Ψ_δ that equals 1 on $F \times (2\delta\ell(Q), 2\ell(Q))$ and is supported on a certain truncated sawtooth domain $\Omega_{\eta/8,Q,\delta}$ over F, where $\delta > 0$ is an arbitrary (small) parameter that provides for a smooth truncation in the t-direction near the boundary of $\mathbb{R}_+^{n+1}$. The main integration by parts is then performed in (4.32). The two principal terms $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ arise from the tangential and transversal integration by parts, respectively, where the former is taken with respect to the measure μ and thus requires additional justification from the uniformly elliptic case. These and numerous error terms are then shown to be appropriately under control.

Theorem 4.10. *Let $n \geq 2$ and consider a cube $Q \subset \mathbb{R}^n$. If A is a t-independent coefficient matrix that satisfies the degenerate bound and ellipticity in* (1.1) *for some constants $0 < \lambda \leq \Lambda < \infty$ and an A_2-weight μ on $\mathbb{R}^n$, then for any solution $u \in L^\infty(4Q \times (0, 4\ell(Q)))$ that solves $\operatorname{div}(A\nabla u) = 0$ in $4Q \times (0, 4\ell(Q))$, there exist constants $C, c_0 > 0$ and a measurable set $F \subset Q$ such that $\mu(F) \geq c_0\mu(Q)$ and*

$$\frac{1}{\mu(Q)} \int_0^{\ell(Q)} \int_F |t\nabla u(x,t)|^2 \, d\mu(x) \, \frac{dt}{t} \leq C\|u\|_\infty^2, \tag{4.11}$$

where C and c_0 depend only on n, λ, Λ and $[\mu]_{A_2}$.

Proof. We begin by expressing the matrix A and its adjoint A^* (which is just the transpose A^{t}, since the matrix coefficients are real-valued) in the form

$$A = \left[\begin{array}{c|c} A_\parallel & \boldsymbol{b} \\ \hline \boldsymbol{c}^{\mathrm{t}} & d \end{array}\right], \quad A^* = \left[\begin{array}{c|c} A_\parallel^* & \boldsymbol{c} \\ \hline \boldsymbol{b}^{\mathrm{t}} & d \end{array}\right], \tag{4.12}$$

where $A_\parallel$ denotes the $n \times n$ submatrix of A with entries $(A_\parallel)_{i,j} := A_{i,j}$, $1 \leq i, j \leq n$, whilst $\boldsymbol{b} := (A_{i,n+1})_{1\leq i\leq n}$ is a column vector, $\boldsymbol{c}^{\mathrm{t}} := (A_{n+1,j})_{1\leq j\leq n}$ is a row vector and $d := A_{n+1,n+1}$ is a scalar.

Now consider a cube $Q \subset \mathbb{R}^n$. The aim is to construct a set $F \subset Q$ with the required properties. To this end, we apply the Hodge decomposition from (2.10) to the space $L_\mu^2(5Q, \mathbb{R}^n)$ in order to write

$$\frac{1}{\mu}\boldsymbol{c}\mathbb{1}_{5Q} = -\frac{1}{\mu}A_\parallel^*\nabla\varphi + \boldsymbol{h}, \quad \frac{1}{\mu}\boldsymbol{b}\mathbb{1}_{5Q} = -\frac{1}{\mu}A_\parallel\nabla\tilde{\varphi} + \tilde{\boldsymbol{h}}, \tag{4.13}$$

where $\varphi, \tilde{\varphi} \in W^{1,2}_{\mu,0}(5Q)$ and $\boldsymbol{h}, \tilde{\boldsymbol{h}} \in L^2_\mu(5Q, \mathbb{R}^n)$ are such that $\operatorname{div}_\mu \boldsymbol{h} = \operatorname{div}_\mu \tilde{\boldsymbol{h}} = 0$ and

$$\fint_{5Q} (|\nabla\varphi(x)|^2 + |\boldsymbol{h}(x)|^2)\, d\mu(x) \lesssim \fint_{5Q} \left|\frac{\boldsymbol{c}(x)}{\mu}\right|^2 d\mu(x) \lesssim 1, \tag{4.14}$$

$$\fint_{5Q} (|\nabla\tilde{\varphi}(x)|^2 + |\tilde{\boldsymbol{h}}(x)|^2)\, d\mu(x) \lesssim \fint_{5Q} \left|\frac{\boldsymbol{b}(x)}{\mu}\right|^2 d\mu(x) \lesssim 1. \tag{4.15}$$

We extend each of $\varphi, \tilde{\varphi}, \boldsymbol{h}, \tilde{\boldsymbol{h}}$ to functions on $\mathbb{R}^n$ by setting them equal to 0 on $\mathbb{R}^n \setminus 5Q$.

In Sections 2 and 3, we investigated the operators $\mathcal{L}_\mu := -\operatorname{div}_\mu((1/\mu)\mathcal{A}\nabla)$ and $\mathcal{P}_t := e^{-t^2\mathcal{L}_\mu}$ for arbitrary coefficient matrices $\mathcal{A}$ in $\mathcal{E}(n, \lambda, \Lambda, \mu)$. We now set

$$\begin{aligned} L_{\parallel,\mu} &:= -\operatorname{div}_\mu\Big(\frac{1}{\mu} A_\parallel \nabla_x\Big), \quad P_t := e^{-t^2 L_{\parallel,\mu}}, \\ L^*_{\parallel,\mu} &:= -\operatorname{div}_\mu\Big(\frac{1}{\mu} A^*_\parallel \nabla_x\Big), \quad P^*_t := e^{-t^2 L^*_{\parallel,\mu}} \end{aligned} \tag{4.16}$$

in order to apply those results in the cases $\mathcal{A} = A_\parallel$ and $\mathcal{A} = A^*_\parallel$.

We now introduce two constants $\kappa_0, \eta > 0$, which will be fixed shortly, and recall the function $\Lambda(\eta, \phi, \mathcal{A})$ from (4.4) to define the set $F \subset Q$ by

$$F := \{x \in Q : \Lambda(\eta, \varphi, A^*_\parallel)(x) + \Lambda(\eta, \tilde{\varphi}, A_\parallel)(x) + \widetilde{N}^\eta_{*,\mu}(\nabla_x P^*_{\eta t}\varphi)(x) + \widetilde{N}^\eta_{*,\mu}(\nabla_x P_{\eta t}\tilde{\varphi})(x) \le \kappa_0\}. \tag{4.17}$$

Applying the weak-type bounds in (3.1), (3.3), (3.13) and (3.14) followed by the estimates from the Hodge decomposition in (4.14) and (4.15), we obtain

$$\mu(Q \setminus F) \lesssim \kappa_0^{-2}(\|\nabla\varphi\|^2_{L^2_\mu(\mathbb{R}^n,\mathbb{R}^n)} + \|\nabla\tilde{\varphi}\|^2_{L^2_\mu(\mathbb{R}^n,\mathbb{R}^n)}) \lesssim \kappa_0^{-2}\mu(Q),$$

where the implicit constants depend only on n, λ, Λ and $[\mu]_{A_2}$. This allows us to now fix $\kappa_0 > 1$ and some constant $c_0 > 0$ such that $\mu(F) \ge c_0\mu(Q)$, where both κ_0 and c_0 depend only on the allowed constants, and thus are independent of η.

We now fix the value of η as follows. First, for $0 \le \alpha \le 4$ and $\beta > 0$, let

$$\Omega_\beta := \bigcup\nolimits_{x\in F} \Gamma_\beta(x), \quad \Omega_{\beta,Q,\alpha} := \Omega_\beta \cap (2Q \times (\alpha\ell(Q), 4\ell(Q))) \quad \text{and} \quad \Omega_{\beta,Q} := \Omega_{\beta,Q,0}$$

denote the sawtooth domains in $\mathbb{R}^{n+1}_+$ spanned by cones centred on F of aperture β. Next, note that the properties of the Hodge decomposition in (4.13) imply $-\operatorname{div}(A^*_\parallel\nabla\varphi) = \operatorname{div}(\boldsymbol{c}\mathbb{1}_{5Q})$ and $-\operatorname{div}(A_\parallel\nabla\tilde{\varphi}) = \operatorname{div}(\boldsymbol{b}\mathbb{1}_{5Q})$ in $5Q$. Therefore, we now fix $0 < \eta < \frac{1}{2}$ in accordance with (4.5) and (4.6) such that

$$\max\{|\partial_t P^*_{\eta t}\varphi(x)|, |\partial_t P_{\eta t}\tilde{\varphi}(x)|\} \le \eta\kappa_0 < \tfrac{1}{8} \quad \text{for all } (x,t) \in \Omega_\eta \tag{4.18}$$

and

$$\begin{aligned} &\max\{|(I - P^*_{\eta t})\varphi(x)|, |(I - P_{\eta t})\tilde{\varphi}(x)|\} \\ &\qquad \lesssim \eta\Big(\kappa_0 + \max\Big\{\Big\|\frac{1}{\mu}\boldsymbol{c}\Big\|_\infty, \Big\|\frac{1}{\mu}\boldsymbol{b}\Big\|_\infty\Big\}\Big)t \lesssim \eta\kappa_0 t < \tfrac{1}{8}t \quad \text{for all } (x,t) \in \Omega_{\eta,Q}, \end{aligned} \tag{4.19}$$

where η and the implicit constants depend only on n, λ, Λ and $[\mu]_{A_2}$.

It remains to prove (4.11). We will achieve this by changing variables in the transversal direction using the mapping $t \mapsto \tau(x,t)$, with $x \in \mathbb{R}^n$ fixed, defined by

$$\tau(x,t) := t - (I - P^*_{\eta t})\varphi(x)$$

and having Jacobian denoted by

$$J(x,t) := \partial_t \tau(x,t) = 1 + \partial_t P^*_{\eta t}\varphi(x). \tag{4.20}$$

In order to justify such changes of variables, we note from (4.18) and (4.19) that

$$\tfrac{7}{8}t < \tau(x,t) < \tfrac{9}{8}t \quad \text{and} \quad \tfrac{7}{8} < J(x,t) < \tfrac{9}{8} \quad \text{for all } (x,t) \in \Omega_{\eta,Q}. \tag{4.21}$$

In particular, for each $x \in F$ and $0 \le \alpha \le \frac{1}{8}$, this implies that the mapping $t \mapsto \tau(x,t)$ is bi-Lipschitz in t on $(2\alpha\ell(Q), 2\ell(Q))$ with range

$$(4\alpha\ell(Q), \ell(Q)) \subseteq \tau(x,\cdot)\big((2\alpha\ell(Q), 2\ell(Q))\big) \subseteq (\alpha\ell(Q), 4\ell(Q)). \tag{4.22}$$

Moreover, for each $0 < \beta \le \eta$, the mapping $(x,t) \mapsto \rho(x,t)$ defined by

$$\rho(x,t) := (x, \tau(x,t)) = (x, t + P^*_{\eta t}\varphi(x) - \varphi(x))$$

is bi-Lipschitz in t on $\Omega_{\beta,Q}$ with range

$$\Omega_{8\beta/9,Q} \subseteq \rho(\Omega_{\beta,Q}) \subseteq \Omega_{8\beta/7,Q}. \tag{4.23}$$

Now consider a bounded solution u satisfying $\operatorname{div}(A\nabla u) = 0$ in $4Q \times (0, 4\ell(Q))$. The pull-back $u_1 := u \circ \rho$ is in $L^\infty(\Omega_{\eta,Q})$ and $\operatorname{div}(A_1 \nabla u_1) = 0$ in $\Omega_{\eta,Q}$, where

$$A_1 := \left[\begin{array}{c|c} JA_{\parallel} & \boldsymbol{b} + A_{\parallel}\nabla_x\varphi - A_{\parallel}\nabla_x P^*_{\eta t}\varphi \\ \hline (\mu\boldsymbol{h} - A^*_{\parallel}\nabla_x P^*_{\eta t}\varphi)^{\mathrm{t}} & \langle A\boldsymbol{p}, \boldsymbol{p}\rangle / J \end{array}\right]$$

and

$$\boldsymbol{p}(x,t) := \begin{bmatrix} \nabla_x \tau(x,t) \\ -1 \end{bmatrix} = \begin{bmatrix} \nabla_x P^*_{\eta t}\varphi(x) - \nabla_x\varphi(x) \\ -1 \end{bmatrix}. \tag{4.24}$$

Our statement that $\operatorname{div}(A_1\nabla u_1) = 0$ in $\Omega_{\eta,Q}$ does not mean that A_1 satisfies (1.1), only that $u_1 \in W^{1,2}_{\mu,\mathrm{loc}}(\Omega_{\eta,Q})$ and that $\int_{\mathbb{R}^{n+1}_+} \langle A_1\nabla u_1, \nabla\Phi\rangle = 0$ for all $\Phi \in C^\infty_c(\Omega_{\eta,Q})$. To prove this, we combine the pointwise identity

$$\langle A((\nabla u)\circ\rho), (\nabla v)\circ\rho\rangle J = \langle A_1 \nabla(u\circ\rho), \nabla(v\circ\rho)\rangle \quad \text{for all } v \in W^{1,2}_{0,\mu}(\rho(\Omega_{\eta,Q})) \tag{4.25}$$

with the change of variables $(x,t) \mapsto \rho(x,t)$ on $\Omega_{\eta,Q}$, which is justified because ρ is bi-Lipschitz in t on $\Omega_{\eta,Q}$ with range $\rho(\Omega_{\eta,Q}) \subset 4Q \times (0, 4\ell(Q))$ by (4.23). Also, we note for later use that $\|\mathbb{1}_{\Omega_{\eta,Q}} u_1\|_\infty \le \|u\|_\infty$ and, using (4.21), that

$$|\nabla u_1| \lesssim \left| \begin{bmatrix} \nabla_x u_1 - (\nabla_x\tau)(\partial_t u_1)/J \\ (\partial_t u_1)/J \end{bmatrix} \right| + |\nabla_x\tau||\partial_t u_1| = |(\nabla u)\circ\rho| + |\nabla_x\tau||\partial_t u_1| \tag{4.26}$$

on $\Omega_{\eta,Q}$.

Next, in order to work with the pull-back solution u_1, we consider an arbitrary constant $0 < \delta \le \frac{1}{8}$ and define a smooth cut-off function Ψ_δ adapted to $\Omega_{\eta,Q}$ as follows. Let $\delta_F(x) := \operatorname{dist}(x, F)$, fix a C^∞-function $\Phi : \mathbb{R} \to [0,1]$ satisfying $\Phi(t) = 1$ when $t < \frac{1}{16}$ and $\Phi(t) = 0$ when $t \ge \frac{1}{8}$, and then define

$$\Psi_\delta(x,t) := \Phi\biggl(\frac{\delta_F(x)}{\eta t}\biggr)\Phi\biggl(\frac{t}{32\ell(Q)}\biggr)\biggl(1-\Phi\biggl(\frac{t}{16\delta\ell(Q)}\biggr)\biggr) \quad \text{for all } (x,t) \in \mathbb{R}^{n+1}_+.$$

This function is designed so that $\Psi_\delta \equiv 1$ on $F \times (2\delta\ell(Q), 2\ell(Q))$, and since $\eta < \frac{1}{2}$, we have $\operatorname{supp}\Psi_\delta \subseteq \Omega_{\eta/8,Q,\delta}$ and

$$|\nabla_{x,t}\Psi_\delta(x,t)| \lesssim \frac{\mathbb{1}_{E_1}(x,t)}{t} + \frac{\mathbb{1}_{E_2}(x,t)}{\ell(Q)} + \frac{\mathbb{1}_{E_3}(x,t)}{\delta\ell(Q)} \quad \text{for all } (x,t) \in \Omega_{\eta/8,Q,\delta}, \tag{4.27}$$

where

$$\begin{aligned} E_1 &:= \bigl\{(x,t) \in 2Q \times (0, 4\ell(Q)) : \tfrac{1}{16}\eta t \le \delta_F(x) \le \tfrac{1}{8}\eta t\bigr\}, \\ E_2 &:= 2Q \times (2\ell(Q), 4\ell(Q)), \\ E_3 &:= 2Q \times (\delta\ell(Q), 2\delta\ell(Q)). \end{aligned}$$

In contrast to Section 3.2 in [Hofmann et al. 2015a], the cut-off function Ψ_δ introduced here incorporates an additional truncation in the t-direction at the boundary. This is done to simplify subsequent integration-by-parts arguments, since it ensures that Ψ_δ vanishes on the boundary of $\mathbb{R}^{n+1}_+$. For later purposes, it is also convenient to isolate the following general fact here.

Remark 4.28. For each $k \in \mathbb{Z}$, let $\mathbb{D}^\eta_k$ denote the grid of dyadic cubes $Q' \subset \mathbb{R}^n$ such that

$$\tfrac{1}{64}\eta 2^{-k} \le \operatorname{diam} Q' < \tfrac{1}{32}\eta 2^{-k}.$$

If $C_0 > 0$ and $(v_t)_{t>0}$ is a collection of nonnegative measurable functions such that

$$\sup_{t\in[2^{-k},2^{-k+1}]} ⨍_{Q'} v_t(x)\,d\mu(x) \le C_0 \quad \text{for all } k \in \mathbb{Z},\ \text{for all } Q' \in \mathbb{D}^\eta_k,$$

then

$$\iint_{\mathbb{R}^{n+1}_+} \biggl(\frac{\mathbb{1}_{E_1}(x,t)}{t} + \frac{\mathbb{1}_{E_2}(x,t)}{\ell(Q)} + \frac{\mathbb{1}_{E_3}(x,t)}{\delta\ell(Q)}\biggr) v_t(x)\,d\mu(x)\,dt \lesssim C_0\mu(Q), \tag{4.29}$$

where the implicit constant depends only on n, λ, Λ and $[\mu]_{A_2}$. To see this, first observe that since δ_F is a Lipschitz mapping with constant 1, we have

$$\begin{aligned} Q^{(1)} \times [2^{-k}, 2^{-k+1}] &\subseteq \widetilde{E}_1 := \biggl\{(x,t) \in 4Q \times (0, 4\ell(Q)) : \frac{\eta t}{C} \le \delta_F(x) \le C\eta t\biggr\}, \\ Q^{(2)} \times [2^{-k}, 2^{-k+1}] &\subseteq 4Q \times (\ell(Q), 8\ell(Q)), \\ Q^{(3)} \times [2^{-k}, 2^{-k+1}] &\subseteq 4Q \times \bigl(\tfrac{1}{2}\delta\ell(Q), 4\delta\ell(Q)\bigr) \end{aligned}$$

whenever $E_i \cap (Q^{(i)} \times [2^{-k}, 2^{-k+1}]) \neq \varnothing$ and $i \in \{1,2,3\}$. The estimate in (4.27) and the doubling property of μ then imply that the left side of (4.29) is bounded by

$$C_0\Big(\sum_{k\in\mathbb{Z}}\sum_{Q'\in\mathbb{D}_k^\eta}\int_{2^{-k}}^{2^{-k+1}}\int_{Q'}\mathbb{1}_{\widetilde{E}_1}\,d\mu\,\frac{dt}{t}+C\fint_{\ell(Q)}^{8\ell(Q)}\mu(Q)\,dt+C\fint_{\frac{1}{2}\delta\ell(Q)}^{4\delta\ell(Q)}\mu(Q)\,dt\Big)$$
$$\lesssim C_0\Big(\int_{4Q}\int_{(1/(C\eta))\delta_F(x)}^{(C/\eta)\delta_F(x)}\frac{dt}{t}\,d\mu(x)+\mu(Q)\Big)\lesssim C_0\mu(Q),$$

as required.

We now proceed to prove (4.11). First, note that it suffices to show that

$$\sup_{0<\delta\le 1/8}\int_{4\delta\ell(Q)}^{\ell(Q)}\int_F|t\nabla u(x,t)|^2\,d\mu(x)\,\frac{dt}{t}\lesssim\|u\|_\infty^2\mu(Q), \tag{4.30}$$

since we may then obtain (4.11) by using Fatou's lemma to pass to the limit as δ approaches 0. To this end, we use (4.22), followed by the bi-Lipschitz-in-t change of variables $t\mapsto\tau(x,t)$ on $(\delta\ell(Q),2\ell(Q))$ for each $x\in F$, estimate (4.21) and identity (4.25) to obtain

$$\begin{aligned}\int_{4\delta\ell(Q)}^{\ell(Q)}\int_F|t\nabla u(x,t)|^2\,d\mu(x)\,\frac{dt}{t}&\lesssim\int_F\int_{4\delta\ell(Q)}^{\ell(Q)}\langle A\nabla u,\nabla u\rangle\,t\,dt\,dx\\&\lesssim\int_F\int_{2\delta\ell(Q)}^{2\ell(Q)}\langle A_1\nabla u_1,\nabla u_1\rangle\,t\,dt\,dx\\&\le\iint_{\mathbb{R}^{n+1}_+}\langle A_1\nabla u_1,\nabla u_1\rangle\Psi_\delta^2\,t\,dx\,dt.\end{aligned}$$

Thus, in order to prove (4.30) and ultimately (4.11), it suffices to show that

$$\iint_{\mathbb{R}^{n+1}_+}\langle A_1\nabla u_1,\nabla u_1\rangle\Psi_\delta^2\,t\,dx\,dt\lesssim\|u\|_\infty^2\mu(Q)\quad\text{for all }0<\delta\le\tfrac{1}{8}, \tag{4.31}$$

where the implicit constant depends only on n, λ, Λ and $[\mu]_{A_2}$.

Next, we recall that $\operatorname{div}(A_1\nabla u_1)=0$ in $\Omega_{\eta,Q}$, noting that $u_1\Psi_\delta^2 t\in W_{0,\mu}^{1,2}(\Omega_{\eta,Q})$, and then integrate by parts to obtain

$$\begin{aligned}&\iint_{\mathbb{R}^{n+1}_+}\langle A_1\nabla u_1,\nabla u_1\rangle\Psi_\delta^2\,t\,dx\,dt\\&=-\frac{1}{2}\iint_{\mathbb{R}^{n+1}_+}\langle A_1\nabla(u_1^2),\nabla(\Psi_\delta^2 t)\rangle\,dx\,dt\\&=-\frac{1}{2}\iint_{\mathbb{R}^{n+1}_+}\Big\langle\nabla(u_1^2),\frac{1}{\mu}A_1^*e_{n+1}\Big\rangle\Psi_\delta^2\,d\mu\,dt-\frac{1}{2}\iint_{\mathbb{R}^{n+1}_+}\langle A_1\nabla(u_1^2),\nabla(\Psi_\delta^2)\rangle\,t\,dx\,dt\\&=\frac{1}{2}\iint_{\mathbb{R}^{n+1}_+}u_1^2(L_{\parallel,\mu}^*P_{\eta t}^*\varphi)\Psi_\delta^2\,d\mu\,dt+\frac{1}{2}\iint_{\mathbb{R}^{n+1}_+}u_1^2\partial_t\Big(\frac{\langle A\boldsymbol{p},\boldsymbol{p}\rangle}{J}\Big)\Psi_\delta^2\,dx\,dt\\&\qquad-\frac{1}{2}\iint_{\mathbb{R}^{n+1}_+}\langle A_1\nabla(u_1^2),\nabla(\Psi_\delta^2)\rangle\,t\,dx\,dt+\frac{1}{2}\iint_{\mathbb{R}^{n+1}_+}u_1^2\langle e_{n+1},A_1\nabla(\Psi_\delta^2)\rangle\,dx\,dt\\&=:\boldsymbol{S}_1+\boldsymbol{S}_2+\boldsymbol{E}_1+\boldsymbol{E}_2,\end{aligned} \tag{4.32}$$

where $e_{n+1} := (0, \dots, 0, 1)$ denotes the unit vector in the t-direction. In particular, note that the tangential integration by parts

$$\int_{\mathbb{R}^n} \Big\langle \nabla_x(u_1^2), \boldsymbol{h} - \frac{1}{\mu} A_\parallel^* \nabla_x P_{\eta t}^* \varphi \Big\rangle \Psi_\delta^2 \, d\mu = \int_{\mathbb{R}^n} u_1^2 \operatorname{div}_\mu \Big[\Big(\boldsymbol{h} - \frac{1}{\mu} A_\parallel^* \nabla_x P_{\eta t}^* \varphi \Big) \Psi_\delta^2 \Big] \, d\mu,$$

with respect to the measure μ, is justified by the definition of the operator div_μ, since $P_{\eta t}^* \varphi \in \operatorname{Dom}(L_{\parallel,\mu}^*)$ and $\operatorname{div}_\mu \boldsymbol{h} = 0$ imply $(\boldsymbol{h} - (1/\mu) A_\parallel^* \nabla_x P_{\eta t}^* \varphi) \Psi_\delta^2 \in \operatorname{Dom}(\operatorname{div}_\mu)$ (recall (2.8), (2.9) and (4.16)). Meanwhile, the transversal integration by parts

$$\int_0^\infty \partial_t(u_1^2) \Big(\frac{\langle A\boldsymbol{p}, \boldsymbol{p} \rangle}{J} \Big) \Psi_\delta^2 \, dt = - \int_0^\infty u_1^2 \partial_t \Big[\Big(\frac{\langle A\boldsymbol{p}, \boldsymbol{p} \rangle}{J} \Big) \Psi_\delta^2 \Big] \, dt$$

is justified because Ψ_δ vanishes on the boundary of $\mathbb{R}_+^{n+1}$.

We proceed to prove that, for all $\sigma \in (0, 1)$, each term in (4.32) is controlled by

$$\boldsymbol{S}_1 + \boldsymbol{S}_2 + \boldsymbol{E}_1 + \boldsymbol{E}_2 \lesssim \sigma \iint_{\mathbb{R}_+^{n+1}} \langle A_1 \nabla u_1, \nabla u_1 \rangle \Psi_\delta^2 \, t \, dx \, dt + \sigma^{-1} \|u\|_\infty^2 \mu(Q), \tag{4.33}$$

where the implicit constant depends only on n, λ, Λ and $[\mu]_{A_2}$. Estimate (4.31) will then follow by fixing a sufficiently small $\sigma \in (0, 1)$, depending only on allowed constants, to move the integral in (4.33) to the left side of (4.32). This is justified because the integral in (4.33) is finite by Caccioppoli's inequality and the fact that Ψ_δ vanishes in a neighbourhood of the boundary of $\mathbb{R}_+^{n+1}$ ($\operatorname{supp} \Psi_\delta \subseteq \Omega_{\eta/8,Q,\delta}$).

We now prove (4.33) in three steps to complete the proof.

Step 1: estimates for the error terms $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ in (4.32).

We first apply Cauchy's inequality with σ to write

$$\begin{aligned} \boldsymbol{E}_1 &\le \Big| \frac{1}{2} \iint_{\mathbb{R}_+^{n+1}} \langle A_1 \nabla(u_1^2), \nabla(\Psi_\delta^2) \rangle \, t \, dx \, dt \Big| \\ &= 2 \Big| \iint_{\mathbb{R}_+^{n+1}} \langle A_1 \nabla u_1, \nabla \Psi_\delta \rangle u_1 \Psi_\delta \, t \, dx \, dt \Big| \\ &\lesssim \sigma \iint_{\mathbb{R}_+^{n+1}} \langle A_1 \nabla u_1, \nabla u_1 \rangle \Psi_\delta^2 \, t \, dx \, dt + \sigma^{-1} \iint_{\mathbb{R}_+^{n+1}} u_1^2 \langle A_1 \nabla \Psi_\delta, \nabla \Psi_\delta \rangle \, t \, dx \, dt \\ &=: \sigma \iint_{\mathbb{R}_+^{n+1}} \langle A_1 \nabla u_1, \nabla u_1 \rangle \Psi_\delta^2 \, t \, dx \, dt + \sigma^{-1} \boldsymbol{E}_1'. \end{aligned}$$

We then use $\mu \boldsymbol{h} = \boldsymbol{c} \mathbb{1}_{5Q} + A_\parallel^* \nabla \varphi$ from (4.13), the degenerate bound in (1.1) for A, the bound $\|\mathbb{1}_{\Omega_{\eta,Q}} u_1\|_\infty \lesssim \|u\|_\infty$ and the estimate for $\nabla \Psi_\delta$ from (4.27) to obtain

$$\boldsymbol{E}_1' + \boldsymbol{E}_2 \lesssim \|u\|_\infty^2 \iint_{\Omega_{\eta/8,Q}} \Big(\frac{\mathbb{1}_{E_1}}{t} + \frac{\mathbb{1}_{E_2}}{\ell(Q)} + \frac{\mathbb{1}_{E_3}}{\delta \ell(Q)} \Big) (1 + |\nabla_x (I - P_{\eta t}^*) \varphi|^2) \, d\mu \, dt,$$

where (4.27) ensures that $|\nabla(\Psi_\delta^2)|$ and $|\nabla\Psi_\delta|^2 t$ can be controlled in the same manner. In order to apply Remark 4.28 with $v_t = \mathbb{1}_{\Omega_{\eta/8,Q}}(1+|\nabla_x(I-P_{\eta t}^*)\varphi|^2)$, we observe that if $k\in\mathbb{Z}$, $Q'\in\mathbb{D}_k^\eta$ and $\Omega_{\eta/8,Q,\delta}\cap(Q'\times[2^{-k},2^{-k+1}])\neq\varnothing$, then there exists $x_0\in F$ such that $Q'\subseteq\Delta(x_0,\eta 2^{-k})\subseteq CQ'$, where Δ is used to denote balls in $\mathbb{R}^n$; hence

$$Q'\times[2^{-k},2^{-k+1}]\subseteq\Omega_{\eta,2Q,\delta/4} \tag{4.34}$$

and the doubling property of μ implies

$$\begin{aligned}\fint_{Q'}|\nabla_x(I-P_{\eta t}^*)\varphi|^2\,d\mu &\lesssim \fint_{\Delta(x_0,\eta t)}|\nabla_x P_{\eta t}^*\varphi|^2\,d\mu+\fint_{\Delta(x_0,\eta 2^{-k})}|\nabla_x\varphi|^2\,d\mu\\ &\lesssim [\widetilde{N}_{*,\mu}^\eta(\nabla_x P_{\eta t}^*\varphi)(x_0)]^2+M_\mu(|\nabla_x\varphi|^2)(x_0)\\ &\lesssim \kappa_0^2\lesssim 1\quad\text{for all } t\in[2^{-k},2^{-k+1}],\end{aligned} \tag{4.35}$$

where in the last line we used the definition of the set F in (4.17) and the weighted maximal operators $\widetilde{N}_{*,\mu}$ and M_μ from Section 3. It thus follows from (4.29) that $\boldsymbol{E}_1'+\boldsymbol{E}_2\lesssim\|u\|_\infty^2\mu(Q)$, so altogether we have

$$\boldsymbol{E}_1+\boldsymbol{E}_2\lesssim\sigma\iint_{\mathbb{R}_+^{n+1}}\langle A_1\nabla u_1,\nabla u_1\rangle\Psi_\delta^2\,t\,dx\,dt+\sigma^{-1}\|u\|_\infty^2\mu(Q)\quad\text{for all }\sigma\in(0,1). \tag{4.36}$$

Step 2: estimates for the term $\boldsymbol{S}_1$ in (4.32).

We note that $\partial_t P_{\eta t}^* = -2\eta^2 t L_{\|,\mu}^* P_{\eta t}^*$ on $L_\mu^2(\mathbb{R}^n)$ and integrate by parts in t to write

$$\begin{aligned}\boldsymbol{S}_1 &= \frac{1}{2}\iint_{\mathbb{R}_+^{n+1}}u_1^2(L_{\|,\mu}^*P_{\eta t}^*\varphi)\Psi_\delta^2\,d\mu\,dt\\ &= -\frac{1}{2}\iint_{\mathbb{R}_+^{n+1}}u_1^2\partial_t(L_{\|,\mu}^*P_{\eta t}^*\varphi)\Psi_\delta^2\,t\,d\mu\,dt\\ &\qquad+\frac{1}{2\eta^2}\iint_{\mathbb{R}_+^{n+1}}(u_1\partial_t u_1)(\partial_t P_{\eta t}^*\varphi)\Psi_\delta^2\,d\mu\,dt+\frac{1}{2\eta^2}\iint_{\mathbb{R}_+^{n+1}}u_1^2(\partial_t P_{\eta t}^*\varphi)\Psi_\delta\partial_t\Psi_\delta\,d\mu\,dt\\ &=: \boldsymbol{S}_1'+\boldsymbol{S}_1''+\boldsymbol{S}_1''',\end{aligned}$$

where there is no boundary term because Ψ_δ vanishes on the boundary of $\mathbb{R}_+^{n+1}$.

To estimate $\boldsymbol{S}_1'''$, we use the definition of the set F in (4.17), the estimate for $|\nabla\Psi_\delta|$ from (4.27), and Remark 4.28 in the case $v_t\equiv 1$, to obtain

$$\begin{aligned}\boldsymbol{S}_1''' &\lesssim \|u\|_\infty^2\iint_{\Omega_{\eta/8,Q}}N_*^\eta(\partial_t P_{\eta t}^*\varphi)\,|\partial_t\Psi_\delta|\,d\mu\,dt\\ &\lesssim \eta\kappa_0\|u\|_\infty^2\mu(Q)\lesssim\|u\|_\infty^2\mu(Q).\end{aligned}$$

To estimate $\boldsymbol{S}_1'$, we observe that $\partial_t(L_{\|,\mu}^*P_{\eta t}^*\varphi)=L_{\|,\mu}^*(\partial_t P_{\eta t}^*\varphi)$, since $\varphi\in W_{\mu,0}^{1,2}(\mathbb{R}^n)$ and $\partial_t P_{\eta t}^* = -2\eta^2 t P_{\eta t}^* L_{\|,\mu}^*$ on the dense subset $\mathrm{Dom}(L_{\|,\mu}^*)$ of $W_{0,\mu}^{1,2}(\mathbb{R}^n)$ (note also that $t\nabla_x P_{\eta t}^*$ and hence its adjoint are bounded operators on L_μ^2, as can be seen from the proof of Theorem 2.12). We then apply Cauchy's inequality with σ to write

$$\begin{aligned}
\boldsymbol{S}_1' &\le \left| \iint_{\mathbb{R}^{n+1}_+} L^*_{\|,\mu}(\partial_t P^*_{\eta t}\varphi) u_1^2 \Psi_\delta^2 \, t \, d\mu \, dt \right| \\
&\lesssim \left| \iint_{\mathbb{R}^{n+1}_+} \Big\langle \frac{1}{\mu} A^*_{\|} \nabla_x (\partial_t P^*_{\eta t}\varphi), \nabla_x u_1 \Big\rangle u_1 \Psi_\delta^2 \, t \, d\mu \, dt \right| \\
&\qquad + \left| \iint_{\mathbb{R}^{n+1}_+} \Big\langle \frac{1}{\mu} A^*_{\|} \nabla_x (\partial_t P^*_{\eta t}\varphi), \nabla_x \Psi_\delta \Big\rangle u_1^2 \Psi_\delta \, t \, d\mu \, dt \right| =: \boldsymbol{J} + \boldsymbol{K} \\
&\lesssim \sigma \iint_{\mathbb{R}^{n+1}_+} |\nabla_x u_1|^2 \Psi_\delta^2 \, t \, d\mu \, dt + (\sigma^{-1}+1) \iint_{\mathbb{R}^{n+1}_+} u_1^2 |\nabla_x \partial_t P^*_{\eta t}\varphi|^2 \Psi_\delta^2 \, t \, d\mu \, dt \\
&\qquad + \iint_{\mathbb{R}^{n+1}_+} u_1^2 |\nabla_x \Psi_\delta|^2 \, t \, d\mu \, dt =: \sigma \boldsymbol{S}_{11}' + (\sigma^{-1}+1)\boldsymbol{S}_{12}' + \boldsymbol{S}_{13}', \quad (4.37)
\end{aligned}$$

where the integration by parts in x, with respect to the measure μ, is justified by the definition of the operator $L^*_{\|,\mu}$ (recall (2.8), (2.9) and (4.16)). The terms $\boldsymbol{J}$ and $\boldsymbol{K}$ are highlighted above for reference in Step 3.

To estimate $\boldsymbol{S}_{13}'$, we use the estimate for $|\nabla \Psi_\delta|$ from (4.27) and Remark 4.28 in the case $v_t \equiv 1$ to obtain $\boldsymbol{S}_{13}' \lesssim \|u\|_\infty^2 \mu(Q)$.

To estimate $\boldsymbol{S}_{12}'$, we observe that $\nabla_x \partial_t P^*_{\eta t} = -2\eta^2 t \nabla_x L^*_{\|,\mu} P^*_{\eta t}$ on $L^2_\mu(\mathbb{R}^n)$ and then apply the vertical square function estimate from (2.14) followed by the $W^{1,2}_{0,\mu}(5Q)$-Hodge estimate for φ from (4.14) to obtain

$$\begin{aligned}
\boldsymbol{S}_{12}' &\lesssim \iint_{\mathbb{R}^{n+1}_+} u_1^2 |\nabla_x \partial_t P^*_{\eta t}\varphi|^2 \Psi_\delta^2 \, t \, d\mu \, dt \lesssim \|u\|_\infty^2 \iint_{\mathbb{R}^{n+1}_+} |t^2 \nabla_x L^*_{\|,\mu} P^*_{\eta t}\varphi|^2 \, d\mu \, \frac{dt}{t} \\
&\lesssim \|u\|_\infty^2 \|\nabla \varphi\|^2_{L^2_\mu(\mathbb{R}^n,\mathbb{R}^n)} \lesssim \|u\|_\infty^2 \mu(Q).
\end{aligned}$$

The terms $\boldsymbol{S}_{11}'$ and $\boldsymbol{S}_1''$ will now be estimated together. We again apply Cauchy's inequality with σ, followed by the vertical square function estimate from (2.13) with $\mathcal{L}_\mu = L^*_{\|,\mu}$ and the $W^{1,2}_{0,\mu}(5Q)$-Hodge estimate for φ from (4.14) to obtain

$$\begin{aligned}
\sigma \boldsymbol{S}_{11}' + \boldsymbol{S}_1'' &\lesssim \sigma \iint_{\mathbb{R}^{n+1}_+} |\nabla_x u_1|^2 \Psi_\delta^2 \, t \, d\mu \, dt + \left| \iint_{\mathbb{R}^{n+1}_+} (u_1 \partial_t u_1)(\partial_t P^*_{\eta t}\varphi) \Psi_\delta^2 \, d\mu \, dt \right| \\
&\lesssim \sigma \iint_{\mathbb{R}^{n+1}_+} |\nabla u_1|^2 \Psi_\delta^2 \, t \, d\mu \, dt + \sigma^{-1} \|u\|_\infty^2 \iint_{\mathbb{R}^{n+1}_+} |\partial_t P^*_{\eta t}\varphi|^2 \, d\mu \, \frac{dt}{t} \\
&\lesssim \sigma \iint_{\mathbb{R}^{n+1}_+} \langle A_1 \nabla u_1, \nabla u_1 \rangle \Psi_\delta^2 \, t \, dx \, dt + \sigma \iint_{\mathbb{R}^{n+1}_+} |\nabla_x \tau|^2 |\partial_t u_1|^2 \Psi_\delta^2 \, t \, d\mu \, dt + \sigma^{-1} \|u\|_\infty^2 \mu(Q),
\end{aligned}$$

where we combined the pointwise estimates for ∇u_1 and J from (4.26) and (4.21) with identity (4.25) and the ellipticity of A to deduce the final inequality.

We use the dyadic decomposition from Remark 4.28 to write

$$\iint_{\mathbb{R}^{n+1}_+} |\nabla_x \tau|^2 |\partial_t u_1|^2 \Psi_\delta^2 \, t \, d\mu \, dt \le \sum_{k \in \mathbb{Z}} \sum_{Q' \in \mathbb{D}^\eta_k} \int_{2^{-k}}^{2^{-k+1}} \int_{Q'} \mathbb{1}_{\Omega_{\eta,Q,\delta}} |\nabla_x \tau|^2 |\partial_t u_1|^2 \, t \, d\mu \, dt. \quad (4.38)$$

Observe that if $k \in \mathbb{Z}$, $Q' \in \mathbb{D}_k^\eta$ and $\Omega_{\eta/8,Q,\delta} \cap (Q' \times [2^{-k}, 2^{-k+1}]) \neq \varnothing$, then, as in (4.34) and (4.35), it holds that $Q' \times [2^{-k}, 2^{-k+1}] \subseteq \Omega_{\eta,2Q,\delta/4}$ and

$$\fint_{Q'} |\nabla_x \tau(x,t)|^2 \, d\mu(x) \lesssim \kappa_0^2 \quad \text{for all } t \in [2^{-k}, 2^{-k+1}].$$

Also, we have $\frac{7}{8}t < \tau(x,t) < \frac{9}{8}t$ and $J(x,t) \eqsim 1$ on $Q' \times [2^{-k}, 2^{-k+1}]$ by (4.21), so the degenerate version of Moser's estimate in (2.16) and t-independence show that

$$\sup_{x \in Q'} |\partial_t u_1(x,t)|^2 = \sup_{x \in Q'} |J(x,t)\partial_\tau u(x,\tau(x,t))|^2 \lesssim \fint_{2Q'} \fint_{t/2}^{2t} |\partial_s u(y,s)|^2 \, ds \, d\mu(y)$$

for all $t \in [2^{-k}, 2^{-k+1}]$. In particular, note that

$$2Q' \times [2^{-k-1}, 2^{-k+2}] \subseteq \Omega^* := \big\{(y,s) \in \mathbb{R}^{n+1}_+ : \delta_F(y) < \tfrac{5}{8}\eta s,\ \tfrac{1}{2}\delta\ell(Q) < s < 8\ell(Q)\big\},$$

since there exists $(x_0, t_0) \in Q' \times [2^{-k}, 2^{-k+1}]$ satisfying $\delta_F(x_0) < \frac{1}{8}\eta t_0$, whence

$$\delta_F(y) < \operatorname{diam}(2Q') + \tfrac{1}{8}\eta t_0 \le \tfrac{5}{16}\eta 2^{-k} \le \tfrac{5}{8}\eta s \quad \text{for all } y \in 2Q' \text{ and } s \ge 2^{-k-1},$$

whilst $\delta\ell(Q) < t_0 < 4\ell(Q)$ implies $[2^{-k}, 2^{-k+1}] \subseteq \big(\frac{1}{2}\delta\ell(Q), 8\ell(Q)\big)$.

The observations in the preceding paragraph show that (4.38) is bounded by

$$\begin{aligned}
&\sum_{k \in \mathbb{Z}} \sum_{Q' \in \mathbb{D}_k^\eta} \int_{2^{-k}}^{2^{-k+1}} \bigg(\fint_{Q'} |\nabla_x \tau|^2 \, d\mu\bigg)\bigg(\int_{2Q'} \fint_{t/2}^{2t} |\partial_s u(y,s)|^2 \mathbb{1}_{\Omega^*}(y,s) \, ds \, d\mu(y)\bigg) dt \\
&\qquad \lesssim \sum_{k \in \mathbb{Z}} \sum_{Q' \in \mathbb{D}_k^\eta} \int_{2^{-k-1}}^{2^{-k+2}} \int_{2Q'} |\partial_s u(y,s)|^2 \mathbb{1}_{\Omega^*}(y,s) \, s \, d\mu(y) \, ds \\
&\qquad \lesssim \bigg(\iint_{\Omega^{**}} |\partial_s u(y,s)|^2 \, s \, d\mu(y) \, ds + \iint_{\Omega^* \setminus \Omega^{**}} |\partial_s u(y,s)|^2 \, s \, d\mu(y) \, ds\bigg) := \boldsymbol{M} + \boldsymbol{E},
\end{aligned}$$

where we used the fact that $\sum_{k \in \mathbb{Z}} \sum_{Q' \in \mathbb{D}_k^\eta} \mathbb{1}_{2Q' \times [2^{-k-1}, 2^{-k+2}]} \lesssim \mathbb{1}_{\mathbb{R}^{n+1}_+}$ and introduced

$$\Omega^{**} := \big\{(y,s) \in \mathbb{R}^{n+1}_+ : \delta_F(y) < \tfrac{1}{18}\eta s,\ 4\delta\ell(Q) < s < \ell(Q)\big\}.$$

To estimate the main term $\boldsymbol{M}$, we use (4.21)–(4.23) to observe that

$$\rho^{-1}(\Omega^{**}) \subseteq \Omega_{\eta/16} \cap (2Q \times (2\delta\ell(Q), 2\ell(Q))).$$

Thus, since $\Psi_\delta \equiv 1$ on these sets, the change of variables $(y,s) \mapsto \rho(y,s)$ gives

$$\boldsymbol{M} \lesssim \iint_{\mathbb{R}^{n+1}_+} |(\partial_t u) \circ \rho|^2 J \, \Psi_\delta^2 \, t \, d\mu \, dt \lesssim \iint_{\mathbb{R}^{n+1}_+} \langle A_1 \nabla u_1, \nabla u_1 \rangle \Psi_\delta^2 \, t \, dx \, dt,$$

where we used identity (4.25) and the ellipticity of A to deduce the final inequality.

To estimate the error term $\boldsymbol{E}$, recall that the degenerate version of Moser's estimate in (2.16), followed by Caccioppoli's inequality, ensures that $\|s\partial_s u\|_\infty \lesssim \|u\|_\infty$. Thus, by the definition of $\Omega^* \setminus \Omega^{**}$ and the doubling property of μ, we obtain

$$\boldsymbol{E} \lesssim \|u\|_\infty^2 \int_{2Q} \biggl(\int_{(8/5\eta)\delta_F(y)}^{(18/\eta)\delta_F(y)} \frac{ds}{s} + \int_{\ell(Q)}^{8\ell(Q)} \frac{ds}{s} + \int_{(\delta/2)\ell(Q)}^{4\delta\ell(Q)} \frac{ds}{s} \biggr) \, d\mu(y) \lesssim \|u\|_\infty^2 \mu(Q).$$

This shows that

$$\sigma \boldsymbol{S}_{11}' + \boldsymbol{S}_1'' \lesssim \sigma \iint_{\mathbb{R}^{n+1}_+} \langle A_1 \nabla u_1, \nabla u_1 \rangle \Psi_\delta^2 \, t \, dx \, dt + \sigma^{-1} \|u\|_\infty^2 \mu(Q);$$

hence

$$\boldsymbol{S}_1 \lesssim \sigma \iint_{\mathbb{R}^{n+1}_+} \langle A_1 \nabla u_1, \nabla u_1 \rangle \Psi_\delta^2 \, t \, dx \, dt + \sigma^{-1} \|u\|_\infty^2 \mu(Q) \quad \text{for all } \sigma \in (0,1). \tag{4.39}$$

<u>Step 3</u>: estimates for the term $\boldsymbol{S}_2$ in (4.32).

We observe that since A is t-independent it is possible to write

$$\begin{aligned} 2\boldsymbol{S}_2 &= \iint_{\mathbb{R}^{n+1}_+} u_1^2 \partial_t \biggl(\frac{\langle A\boldsymbol{p}, \boldsymbol{p} \rangle}{J} \biggr) \Psi_\delta^2 \, dx \, dt \\ &= \iint_{\mathbb{R}^{n+1}_+} u_1^2 \partial_t \biggl(\frac{1}{J} \biggr) \langle A\boldsymbol{p}, \boldsymbol{p} \rangle \Psi_\delta^2 \, dx \, dt \\ &\qquad + \iint_{\mathbb{R}^{n+1}_+} \biggl(\frac{u_1^2}{J} \biggr) \langle \partial_t \boldsymbol{p}, A^* \boldsymbol{p} \rangle \Psi_\delta^2 \, dx \, dt + \iint_{\mathbb{R}^{n+1}_+} \biggl(\frac{u_1^2}{J} \biggr) \langle A\boldsymbol{p}, \partial_t \boldsymbol{p} \rangle \Psi_\delta^2 \, dx \, dt \\ &=: \boldsymbol{I} + \boldsymbol{II} + \boldsymbol{III}. \end{aligned}$$

To estimate $\boldsymbol{I}$, we recall the Jacobian $J(x,t) = 1 + \partial_t P_{\eta t}^* \varphi(x)$ from (4.20) and then integrate by parts in t to write

$$\begin{aligned} \boldsymbol{I} &= -\iint_{\mathbb{R}^{n+1}_+} u_1^2 \frac{\partial_t^2 P_{\eta t}^* \varphi}{J^2} \langle A\boldsymbol{p}, \boldsymbol{p} \rangle \Psi_\delta^2 \, dx \, dt \\ &= \iint_{\mathbb{R}^{n+1}_+} \partial_t(u_1^2) \frac{\partial_t P_{\eta t}^* \varphi}{J^2} \langle A\boldsymbol{p}, \boldsymbol{p} \rangle \Psi_\delta^2 \, dx \, dt + \iint_{\mathbb{R}^{n+1}_+} u_1^2 \frac{\partial_t P_{\eta t}^* \varphi}{J^2} \partial_t(\langle A\boldsymbol{p}, \boldsymbol{p} \rangle) \Psi_\delta^2 \, dx \, dt \\ &\qquad + \iint_{\mathbb{R}^{n+1}_+} u_1^2 \partial_t P_{\eta t}^* \varphi \, \partial_t(J^{-2}) \langle A\boldsymbol{p}, \boldsymbol{p} \rangle \Psi_\delta^2 \, dx \, dt + \iint_{\mathbb{R}^{n+1}_+} u_1^2 \frac{\partial_t P_{\eta t}^* \varphi}{J^2} \langle A\boldsymbol{p}, \boldsymbol{p} \rangle \partial_t(\Psi_\delta^2) \, dx \, dt \\ &=: \boldsymbol{I}_1 + \boldsymbol{I}_2 + \boldsymbol{I}_3 + \boldsymbol{I}_4, \end{aligned}$$

where there is no boundary term because Ψ_δ vanishes on the boundary of $\mathbb{R}^{n+1}_+$.

To estimate $\boldsymbol{I}_1$, we recall that $J \eqsim 1$ on $\operatorname{supp} \Psi_\delta \subseteq \Omega_{\eta/8,Q,\delta}$ by (4.21) and then apply Cauchy's inequality with σ to obtain

$$\begin{aligned} |\boldsymbol{I}_1| &\lesssim \sigma \iint_{\mathbb{R}^{n+1}_+} |\partial_t u_1|^2 |\boldsymbol{p}|^2 \Psi_\delta^2 \, t \, d\mu \, dt + \sigma^{-1} \iint_{\mathbb{R}^{n+1}_+} u_1^2 |\partial_t P_{\eta t} \varphi|^2 |\boldsymbol{p}|^2 \Psi_\delta^2 \, d\mu \, \frac{dt}{t} \\ &=: \sigma \boldsymbol{I}_1' + \sigma^{-1} \boldsymbol{I}_1''. \end{aligned} \tag{4.40}$$

To estimate $\boldsymbol{I}_1'$, recall that $|\boldsymbol{p}|^2 = 1 + |\nabla_x \tau|^2$ by the definition of $\boldsymbol{p}$ in (4.24), so we follow the treatment of (4.38) above to obtain

$$\boldsymbol{I}_1' \lesssim \iint_{\mathbb{R}^{n+1}_+} \langle A_1 \nabla u_1, \nabla u_1 \rangle \Psi_\delta^2 \, t \, dx \, dt + \|u\|_\infty^2 \mu(Q).$$

To estimate $\boldsymbol{I}_1''$, recall that $\|\mathbb{1}_{\Omega_{\eta,Q}} u_1\|_\infty \lesssim \|u\|_\infty$ and use the dyadic decomposition from Remark 4.28 to obtain

$$\begin{aligned}
\boldsymbol{I}_1'' &\lesssim \|u\|_\infty^2 \sum_{k\in\mathbb{Z}} \sum_{Q'\in\mathbb{D}_k^\eta} \|\partial_t P_{\eta t}^* \varphi\|^2_{L^\infty(Q'\times[2^{-k},2^{-k+1}])} \int_{2^{-k}}^{2^{-k+1}} \int_{Q'} |\boldsymbol{p}|^2 \Psi_\delta^2 \, d\mu \, \frac{dt}{t} \\
&\lesssim \|u\|_\infty^2 \sum_{k\in\mathbb{Z}} \sum_{Q'\in\mathbb{D}_k^\eta} \mu(Q') \|\partial_t P_{\eta t}^* \varphi\|^2_{L^\infty(Q'\times[2^{-k},2^{-k+1}])} \\
&\lesssim \|u\|_\infty^2 \sum_{k\in\mathbb{Z}} \sum_{Q'\in\mathbb{D}_k^\eta} \mu(Q') \fint_{2^{-k-1}}^{2^{-k+2}} \fint_{2Q'} |\partial_t P_{\eta t}^* \varphi|^2 \, d\mu \, dt \\
&\lesssim \|u\|_\infty^2 \sum_{k\in\mathbb{Z}} \sum_{Q'\in\mathbb{D}_k^\eta} \int_{2^{-k-1}}^{2^{-k+2}} \int_{2Q'} |\partial_t P_{\eta t}^* \varphi|^2 \, d\mu \, \frac{dt}{t} \\
&\lesssim \|u\|_\infty^2 \iint_{\mathbb{R}^{n+1}_+} |t L_{\parallel,\mu}^* e^{-t^2 L_{\parallel,\mu}^*} \varphi|^2 \, d\mu \, \frac{dt}{t} \\
&\lesssim \|u\|_\infty^2 \|\nabla \varphi\|^2_{L^2_\mu(\mathbb{R}^n,\mathbb{R}^n)} \lesssim \|u\|_\infty^2 \mu(Q),
\end{aligned} \tag{4.41}$$

where the second line uses the pointwise bound $|\boldsymbol{p}|^2 \Psi_\delta^2 \le \mathbb{1}_{\Omega_{\eta/8,Q,\delta}} (1 + |\nabla_x (I - P_{\eta t}^*) \varphi|^2)$ and estimate (4.35), the third line uses the parabolic version of the degenerate Moser-type estimate in (2.16) (see Theorem B in [Fernandes 1991]), noting that $v := \partial_t (e^{-t L_{\parallel,\mu}^*} \varphi)$ solves $\partial_t v = -L_{\parallel,\mu}^* v$, whilst $|\partial_t P_{\eta t}^* \varphi(x)| \lesssim |t \, v(x, \eta^2 t^2)|$, and the final line uses the vertical square function estimate from (2.13) with $\mathcal{L}_\mu = L_{\parallel,\mu}^*$ and the $W^{1,2}_{0,\mu}(5Q)$-Hodge estimate for φ from (4.14).

To estimate $\boldsymbol{I}_2$, we again use the bound $J \eqsim 1$ on $\operatorname{supp} \Psi_\delta \subseteq \Omega_{\eta/8,Q,\delta}$ from (4.21), and then recall the definition $\boldsymbol{p} := (\nabla_x (P_{\eta t}^* - I) \varphi, -1)$ from (4.24) to obtain

$$|\boldsymbol{I}_2| \lesssim \iint_{\mathbb{R}^{n+1}_+} u_1^2 |\nabla_x \partial_t P_{\eta t}^* \varphi|^2 \Psi_\delta^2 \, t \, d\mu \, dt + \iint_{\mathbb{R}^{n+1}_+} u_1^2 |\partial_t P_{\eta t}^* \varphi|^2 |\boldsymbol{p}|^2 \Psi_\delta^2 \, d\mu \, \frac{dt}{t}. \tag{4.42}$$

The first integral in (4.42) is the same as $\boldsymbol{S}_{12}'$ from (4.37), whilst the second integral is the same as $\boldsymbol{I}_1''$ from (4.40); hence $|\boldsymbol{I}_2| \lesssim \|u\|_\infty^2 \mu(Q)$.

To estimate $\boldsymbol{I}_3$, we use the bound $|\partial_t P_{\eta t}^* \varphi| < \frac{1}{8}$ guaranteed by (4.18) to deduce that

$$|\partial_t (J^{-2})| = |\partial_t (1 + \partial_t P_{\eta t}^* \varphi)^{-2}| \lesssim |\partial_t^2 P_{\eta t}^* \varphi|$$

on $\operatorname{supp} \Psi_\delta \subseteq \Omega_{\eta/8,Q,\delta}$ and write

$$|\boldsymbol{I}_3| \lesssim \iint_{\mathbb{R}^{n+1}_+} u_1^2 |\partial_t P_{\eta t}^* \varphi|^2 |\boldsymbol{p}|^2 \Psi_\delta^2 \, d\mu \, \frac{dt}{t} + \iint_{\mathbb{R}^{n+1}_+} u_1^2 |\partial_t^2 P_{\eta t}^* \varphi|^2 |\boldsymbol{p}|^2 \Psi_\delta^2 \, t \, d\mu \, dt =: \boldsymbol{I}_3' + \boldsymbol{I}_3''.$$

To estimate $\boldsymbol{I}_3'$, we note that it is the same as $\boldsymbol{I}_1''$ from (4.40); thus $\boldsymbol{I}_3' \lesssim \|u\|_\infty^2 \mu(Q)$.

To estimate $\boldsymbol{I}_3''$, we follow the estimates and justification provided for (4.41), noting in addition that $\partial_t v = \partial_t^2 (e^{-tL_{\|,\mu}^*}\varphi)$ solves $\partial_t(\partial_t v) = -L_{\|,\mu}^*(\partial_t v)$, to obtain

$$\begin{aligned}
\boldsymbol{I}_3'' &\lesssim \|u\|_\infty^2 \sum_{k\in\mathbb{Z}} \sum_{Q'\in\mathbb{D}_k^\eta} \|t\partial_t^2 P_{\eta t}^*\varphi\|_{L^\infty(Q'\times[2^{-k},2^{-k+1}])}^2 \int_{2^{-k}}^{2^{-k+1}} \int_{Q'} |\boldsymbol{p}|^2 \Psi_\delta^2 \, d\mu \, \frac{dt}{t} \\
&\lesssim \|u\|_\infty^2 \sum_{k\in\mathbb{Z}} \sum_{Q'\in\mathbb{D}_k^\eta} \mu(Q') \| |tL_{\|,\mu}^* P_{\eta t}^*\varphi| + |t^2\partial_t(L_{\|,\mu}^* P_{\eta t}^*\varphi)| \|_{L^\infty(Q'\times[2^{-k},2^{-k+1}])}^2 \\
&\lesssim \|u\|_\infty^2 \sum_{k\in\mathbb{Z}} \sum_{Q'\in\mathbb{D}_k^\eta} \mu(Q') \fint_{2^{-k-1}}^{2^{-k+2}} \fint_{2Q'} (|tL_{\|,\mu}^* P_{\eta t}^*\varphi|^2 + |t^2\partial_t(L_{\|,\mu}^* P_{\eta t}^*\varphi)|^2) \, d\mu \, dt \\
&\lesssim \|u\|_\infty^2 \iint_{\mathbb{R}_+^{n+1}} |tL_{\|,\mu}^* P_{\eta t}^*\varphi|^2 \, d\mu \, \frac{dt}{t} + \|u\|_\infty^2 \iint_{\mathbb{R}_+^{n+1}} |t^2\nabla_{x,t}(L_{\|,\mu}^* P_{\eta t}^*\varphi)|^2 \, d\mu \, \frac{dt}{t} \\
&\lesssim \|u\|_\infty^2 \|\nabla\varphi\|_{L_\mu^2(\mathbb{R}^n,\mathbb{R}^n)}^2 \lesssim \|u\|_\infty^2 \mu(Q),
\end{aligned}$$

where the second line uses $|\partial_t^2 P_{\eta t}^*\varphi| \lesssim |\partial_t(tL_{\|,\mu}^* P_{\eta t}^*\varphi)| \lesssim |L_{\|,\mu}^* P_{\eta t}^*\varphi| + |t\partial_t(L_{\|,\mu}^* P_{\eta t}^*\varphi)|$, the third line uses $|L_{\|,\mu}^* P_{\eta t}^*\varphi(x)| = |v(x,\eta^2 t^2)|$ and $|\partial_t(L_{\|,\mu}^* P_{\eta t}^*\varphi)(x)| \lesssim |t(\partial_t v)(x,\eta^2 t^2)|$, and the final line uses the vertical square function estimates from (2.13) and (2.14) with $\mathcal{L}_\mu = L_{\|,\mu}^*$; hence $|\boldsymbol{I}_3| \lesssim \|u\|_\infty^2 \mu(Q)$.

To estimate $\boldsymbol{I}_4$, we use $|\partial_t P_{\eta t}^*\varphi| \lesssim 1$, $J \eqsim 1$ and $|\boldsymbol{p}|^2 \leq (1 + |\nabla_x(I - P_{\eta t}^*)\varphi|^2)$, which hold on $\operatorname{supp}\Psi_\delta \subseteq \Omega_{\eta/8,Q,\delta}$ by (4.18), (4.21) and (4.24), to reduce to the estimate obtained for $\boldsymbol{E}_1' + \boldsymbol{E}_2$; hence $|\boldsymbol{I}_4| \lesssim \|u\|_\infty^2 \mu(Q)$.

To estimate $\boldsymbol{II}$, we use the definition $\boldsymbol{p} := (\nabla_x(P_{\eta t}^* - I)\varphi, -1)$ from (4.24) to note that $\partial_t \boldsymbol{p} = (\nabla_x \partial_t P_{\eta t}^*\varphi, 0)$ and use the Hodge decomposition from (4.13) to write

$$\langle \partial_t \boldsymbol{p}, A^* \boldsymbol{p}\rangle = \langle \nabla_x \partial_t P_{\eta t}^*\varphi, A_\|^* \nabla_x (P_{\eta t}^* - I)\varphi - \boldsymbol{c}\rangle = \langle \nabla_x \partial_t P_{\eta t}^*\varphi, A_\|^* \nabla_x P_{\eta t}^*\varphi - \mu \boldsymbol{h}\rangle \tag{4.43}$$

for all $x \in 5Q$ and $t > 0$. Using this and recalling that $\operatorname{div}_\mu \boldsymbol{h} = 0$, it follows that

$$\begin{aligned}
\boldsymbol{II} &= \iint_{\mathbb{R}_+^{n+1}} \left(\frac{u_1^2}{J}\right) \langle \nabla_x \partial_t P_{\eta t}^*\varphi, A_\|^* \nabla_x P_{\eta t}^*\varphi - \mu\boldsymbol{h}\rangle \Psi_\delta^2 \, dx \, dt \\
&= \iint_{\mathbb{R}_+^{n+1}} \left(\frac{u_1^2}{J}\right) (\partial_t P_{\eta t}^*\varphi)(L_{\|,\mu}^* P_{\eta t}^*\varphi) \Psi_\delta^2 \, d\mu \, dt \\
&\quad - \iint_{\mathbb{R}_+^{n+1}} \partial_t P_{\eta t}^*\varphi \langle \nabla_x \left(\frac{u_1^2}{J}\right), A_\|^* \nabla_x P_{\eta t}^*\varphi - \mu\boldsymbol{h}\rangle \Psi_\delta^2 \, dx \, dt \\
&\quad - \iint_{\mathbb{R}_+^{n+1}} \left(\frac{u_1^2}{J}\right) \partial_t P_{\eta t}^*\varphi \langle \nabla_x (\Psi_\delta^2), A_\|^* \nabla_x P_{\eta t}^*\varphi - \mu\boldsymbol{h}\rangle \, dx \, dt \\
&=: \boldsymbol{II}_1 + \boldsymbol{II}_2 + \boldsymbol{II}_3,
\end{aligned} \tag{4.44}$$

where the integration by parts in x, with respect to the measure μ, is justified by the definition of the operator $L_{\|,\mu}^*$ (recall (2.8), (2.9) and (4.16)).

To estimate $\boldsymbol{II}_1$, we use $J \eqsim 1$ and $L^*_{\parallel,\mu} P^*_{\eta t}\varphi = -(2\eta^2 t)^{-1}\partial_t P^*_{\eta t}\varphi$ to show that it can be treated the same way as $\boldsymbol{I}''_1$ in (4.40), without $|\boldsymbol{p}|^2$; hence $|\boldsymbol{II}_1| \lesssim \|u\|^2_\infty \mu(Q)$.

To estimate $\boldsymbol{II}_2$, we use $J \eqsim 1$,

$$|\nabla_x(J^{-1})| = |\nabla_x(1+\partial_t P^*_{\eta t}\varphi)^{-1}| \lesssim |\nabla_x \partial_t P^*_{\eta t}\varphi|$$

and apply Cauchy's inequality with σ to obtain

$$|\boldsymbol{II}_2| \lesssim \sigma \iint_{\mathbb{R}^{n+1}_+} |\nabla_x u_1|^2 \Psi^2_\delta \, t \, d\mu \, dt + \iint_{\mathbb{R}^{n+1}_+} u_1^2 |\nabla_x \partial_t P^*_{\eta t}\varphi|^2 \Psi^2_\delta \, t \, d\mu \, dt \\ + (\sigma^{-1}+1) \iint_{\mathbb{R}^{n+1}_+} u_1^2 |\partial_t P^*_{\eta t}\varphi|^2 (|\nabla_x P^*_{\eta t}\varphi|^2 + |\boldsymbol{h}|^2) \Psi^2_\delta \, d\mu \, \frac{dt}{t}. \quad (4.45)$$

The first integral is the same as $\boldsymbol{S}'_{11}$ from (4.37), whilst the remaining two integrals are the same as those that bound $\boldsymbol{I}_2$ in (4.42), except $(|\nabla_x P^*_{\eta t}\varphi|^2 + |\boldsymbol{h}|^2)$ replaces $|\boldsymbol{p}|^2$. This factor is controlled in the same way, however, since the Hodge decomposition in (4.13) implies

$$|\boldsymbol{h}|^2 = \left|\frac{1}{\mu}\boldsymbol{c}\mathbb{1}_{5Q} + \frac{1}{\mu}A^*_\parallel \nabla_x \varphi\right|^2 \lesssim 1 + |\nabla_x \varphi|^2;$$

hence by (4.35) we obtain

$$|\boldsymbol{II}_2| \lesssim \sigma \iint_{\mathbb{R}^{n+1}_+} \langle A_1 \nabla u_1, \nabla u_1 \rangle \Psi^2_\delta \, t \, dx \, dt + \sigma^{-1} \|u\|^2_\infty \mu(Q).$$

To estimate $\boldsymbol{II}_3$, we use $J \eqsim 1$ and Cauchy's inequality to write

$$|\boldsymbol{II}_3| \lesssim \iint_{\mathbb{R}^{n+1}_+} u_1^2 |\nabla_x \Psi_\delta|^2 \, t \, d\mu \, dt + \iint_{\mathbb{R}^{n+1}_+} u_1^2 |\partial_t P^*_{\eta t}\varphi|^2 (|\nabla_x P^*_{\eta t}\varphi|^2 + |\boldsymbol{h}|^2) \Psi^2_\delta \, d\mu \, \frac{dt}{t}.$$

The first term above is the same as $\boldsymbol{S}'_{13}$ in (4.37), whilst the remaining term is the same as the last integral in (4.45); hence $|\boldsymbol{II}_3| \lesssim \|u\|^2_\infty \mu(Q)$.

To estimate $\boldsymbol{III}$, we observe by analogy with (4.43) that

$$\begin{aligned} \langle A\boldsymbol{p}, \partial_t \boldsymbol{p}\rangle &= \langle A_\parallel \nabla_x (P^*_{\eta t} - I)\varphi - \boldsymbol{b}, \nabla_x \partial_t P^*_{\eta t}\varphi\rangle \\ &= \langle A_\parallel \nabla_x (P^*_{\eta t}\varphi - \varphi) + A_\parallel \nabla_x \tilde{\varphi} - \mu \tilde{\boldsymbol{h}}, \nabla_x \partial_t P^*_{\eta t}\varphi\rangle \\ &= \langle A_\parallel \nabla_x [(P^*_{\eta t}\varphi - \varphi) - (P_{\eta t}\tilde{\varphi} - \tilde{\varphi})] + A_\parallel \nabla_x P_{\eta t}\tilde{\varphi} - \mu \tilde{\boldsymbol{h}}, \nabla_x \partial_t P^*_{\eta t}\varphi\rangle \end{aligned}$$

for all $x \in 5Q$ and $t > 0$ and then write

$$\begin{aligned} \boldsymbol{III} = &\iint_{\mathbb{R}^{n+1}_+} \left(\frac{u_1^2}{J}\right) \langle \nabla_x [(P^*_{\eta t}\varphi - \varphi) - (P_{\eta t}\tilde{\varphi} - \tilde{\varphi})], A^*_\parallel \nabla_x \partial_t P^*_{\eta t}\varphi\rangle \Psi^2_\delta \, dx \, dt \\ &+ \iint_{\mathbb{R}^{n+1}_+} \left(\frac{u_1^2}{J}\right) \langle A_\parallel \nabla_x P_{\eta t}\tilde{\varphi} - \mu \tilde{\boldsymbol{h}}, \nabla_x \partial_t P^*_{\eta t}\varphi\rangle \Psi^2_\delta \, dx \, dt \\ =: &\ \boldsymbol{III}_1 + \boldsymbol{III}_2. \end{aligned}$$

To estimate $\boldsymbol{III}_1$, we integrate by parts in x with respect to the measure μ to write

$$\begin{aligned}\boldsymbol{III}_1 = \iint_{\mathbb{R}^{n+1}_+} \left(\frac{u_1^2}{J}\right)[(P^*_{\eta t}\varphi-\varphi)-(P_{\eta t}\tilde{\varphi}-\tilde{\varphi})](L^*_{\|,\mu}\partial_t P^*_{\eta t}\varphi)\Psi_\delta^2\,d\mu\,dt \\ - \iint_{\mathbb{R}^{n+1}_+}[(P^*_{\eta t}\varphi-\varphi)-(P_{\eta t}\tilde{\varphi}-\tilde{\varphi})]\left\langle\nabla_x\left(\frac{u_1^2\Psi_\delta^2}{J}\right), A^*_\|\nabla_x\partial_t P^*_{\eta t}\varphi\right\rangle\,dx\,dt \\ =: \boldsymbol{III}'_1+\boldsymbol{III}''_1,\end{aligned}$$

which is justified by the definition of $L^*_{\|,\mu}$ (recall (2.8), (2.9) and (4.16)).

To estimate $\boldsymbol{III}'_1$, we use Hardy's inequality (see, for instance, page 272 in [Stein 1970]) to observe, for the semigroups $\mathcal{P}_t \in \{e^{-t^2 L^*_{\|,\mu}}, e^{-t^2 L_{\|,\mu}}\}$, the estimate

$$\int_0^\infty |\mathcal{P}_{\eta t} f - f|^2\,\frac{dt}{t^3} \le \int_0^\infty\left(\int_0^{\eta t}|\partial_s\mathcal{P}_s f|\,ds\right)^2\frac{dt}{t^3} \lesssim \int_0^\infty |\partial_t\mathcal{P}_t f|^2\,\frac{dt}{t} \quad \text{for all } f\in L^2_\mu(\mathbb{R}^n).$$

We then recall that $\|\mathbb{1}_{\Omega_{\eta,Q}}u_1\|_\infty \lesssim \|u\|_\infty$ and $J \eqsim 1$ on $\operatorname{supp}\Psi_\delta \subseteq \Omega_{\eta/8,Q,\delta}$ to obtain

$$\begin{aligned}|\boldsymbol{III}'_1| &\lesssim \|u\|_\infty^2\int_{\mathbb{R}^n}\int_0^\infty(|P^*_{\eta t}\varphi-\varphi|+|P_{\eta t}\tilde{\varphi}-\tilde{\varphi}|)\,|L^*_{\|,\mu}\partial_t P^*_{\eta t}\varphi|\,dt\,d\mu \\ &\lesssim \|u\|_\infty^2\int_{\mathbb{R}^n}\left(\int_0^\infty|P^*_{\eta t}\varphi-\varphi|^2+|P_{\eta t}\tilde{\varphi}-\tilde{\varphi}|^2\,\frac{dt}{t^3}\right)^{1/2}\left(\int_0^\infty|t^2L^*_{\|,\mu}\partial_t P^*_{\eta t}\varphi|^2\,\frac{dt}{t}\right)^{1/2}d\mu \\ &\lesssim \|u\|_\infty^2\left(\iint_{\mathbb{R}^{n+1}_+}|\partial_t P^*_t\varphi|^2+|\partial_t P_t\tilde{\varphi}|^2\,d\mu\,\frac{dt}{t}\right)^{1/2}\left(\iint_{\mathbb{R}^{n+1}_+}|t^2\partial_t L^*_{\|,\mu}P^*_{\eta t}\varphi|^2\,d\mu\,\frac{dt}{t}\right)^{1/2} \\ &\lesssim \|u\|_\infty^2(\|\nabla\varphi\|^2_{L^2_\mu(\mathbb{R}^n,\mathbb{R}^n)}+\|\nabla\tilde{\varphi}\|^2_{L^2_\mu(\mathbb{R}^n,\mathbb{R}^n)})^{1/2}\|\nabla\varphi\|_{L^2_\mu(\mathbb{R}^n,\mathbb{R}^n)} \lesssim \|u\|_\infty^2\mu(Q),\end{aligned}$$

where the final line uses the vertical square function estimates from (2.13)–(2.14) for $\mathcal{L}_\mu \in \{L^*_{\|,\mu}, L_{\|,\mu}\}$ and the $W^{1,2}_{0,\mu}(5Q)$-Hodge estimates for φ, $\tilde{\varphi}$ from (4.14)–(4.15).

To estimate $\boldsymbol{III}''_1$, recall that $|P^*_{\eta t}\varphi-\varphi| \lesssim t$ and $|P_{\eta t}\tilde{\varphi}-\tilde{\varphi}| \lesssim t$ on $\operatorname{supp}\Psi_\delta \subseteq \Omega_{\eta/8,Q,\delta}$ by (4.19), whilst $J \eqsim 1$ and $|\nabla_x(J^{-1})| \lesssim |\nabla_x\partial_t P^*_{\eta t}\varphi|$, so distributing ∇_x over u_1^2, Ψ_δ^2 and $1/J$ yields terms that can be controlled in the same way as $\boldsymbol{J}$, $\boldsymbol{K}$ and $\boldsymbol{S}'_{12}$ in (4.37).

To estimate $\boldsymbol{III}_2$, note that the estimates used to control φ and $P_{\eta t}\varphi$ also hold for $\tilde{\varphi}$ and $P_{\eta t}\tilde{\varphi}$ by (4.14)–(4.15) and (4.18)–(4.19), whilst $\operatorname{div}_\mu \boldsymbol{h} = \operatorname{div}_\mu \tilde{\boldsymbol{h}} = 0$ by (4.13); hence $\boldsymbol{III}_2$ can be estimated in the same way as $\boldsymbol{II}$ in (4.44).

This gives

$$|\boldsymbol{III}''_1|+|\boldsymbol{III}_2| \lesssim \sigma\iint_{\mathbb{R}^{n+1}_+}\langle A_1\nabla u_1,\nabla u_1\rangle\Psi_\delta^2\,t\,dx\,dt+\sigma^{-1}\|u\|_\infty^2\mu(Q);$$

hence

$$\boldsymbol{S}_2 \lesssim \sigma\iint_{\mathbb{R}^{n+1}_+}\langle A_1\nabla u_1,\nabla u_1\rangle\Psi_\delta^2\,t\,dx\,dt+\sigma^{-1}\|u\|_\infty^2\mu(Q) \quad \text{for all } \sigma\in(0,1). \tag{4.46}$$

We combine (4.36), (4.39) and (4.46) to obtain (4.33), as required. □

5. Solvability of the Dirichlet problem

This section is dedicated to the proof of Theorem 1.2. We first consider the construction and properties of a degenerate elliptic measure ω^X for degenerate elliptic equations $\operatorname{div}(A\nabla u) = 0$ in the upper half-space, where $X = (x, t) \in \mathbb{R}^{n+1}_+$ and $n \geq 2$. The t-independent coefficient matrix A is assumed throughout to satisfy the degenerate bound and ellipticity in (1.1) for some constants $0 < \lambda \leq \Lambda < \infty$ and an A_2-weight μ on $\mathbb{R}^n$. This is necessary as the literature only seems to treat bounded domains, whilst the passage to unbounded domains in the uniformly elliptic case (see Section 10 in [Littman et al. 1963] and [Hofmann and Kim 2007]) relies on a global version of the Sobolev embedding in (2.4), which is not known for A_2-weights in general. The degenerate elliptic measure is then shown to be in the A_∞-class with respect to μ on the boundary $\mathbb{R}^n$ in Theorem 5.30 and the solvability of the Dirichlet problem follows in Theorem 5.34. These results together prove Theorem 1.2.

5A. ***Boundary estimates for solutions.*** We require some estimates for solutions near the boundary $\partial\Sigma$ of a bounded Lipschitz domain $\Sigma \subset \mathbb{R}^n$ (see Section 2 of [Caffarelli et al. 1981] for the standard definition). These estimates require some regularity on the domain boundary but no attempt is made here to obtain the minimal such regularity, as the focus is to define and analyse a degenerate elliptic measure on $\mathbb{R}^n$.

The Lipschitz regularity of the boundary $\partial\Sigma$ ensures that the smooth class $C^\infty(\bar{\Sigma})$ and the Lipschitz class $C^{0,1}(\bar{\Sigma})$ are both dense in $W^{1,2}_\mu(\Sigma)$; see Theorem 3.4.1 in [Morrey 1966] and page 29 in [Kinderlehrer and Stampacchia 1980]. This allows the usual definition, for $E \subseteq \partial\Sigma$ and $u \in W^{1,2}_\mu(\Sigma)$, whereby $u \geq 0$ on E in the $W^{1,2}_\mu(\Sigma)$-sense means there exists a sequence u_j in $C^{0,1}(\bar{\Sigma})$ that converges to u in $W^{1,2}_\mu(\Sigma)$ with $u_j(x) \geq 0$ for all $x \in E$. This induces definitions for inequalities $\leq$, $\geq$ and $=$, between functions and/or constants, on E in the $W^{1,2}_\mu(\Sigma)$-sense; see, for instance, Definition 5.1 in [Kinderlehrer and Stampacchia 1980]. Moreover, with $\sup_{\partial\Sigma} u := \inf\{k \in \mathbb{R} : u \leq k \text{ on } \partial\Sigma \text{ in the } W^{1,2}_\mu(\Sigma)\text{-sense}\}$ and $\inf_{\partial\Sigma} := -\sup_{\partial\Sigma}(-u)$, the weak maximum principle holds [Fabes et al. 1982b, Theorem 2.2.2], and the strong version follows by the Harnack inequality in (2.18) [Fabes et al. 1982b, Corollary 2.3.10].

We can now state a Hölder continuity estimate and a Harnack inequality for certain solutions near the boundary. For a cube $Q \subset \mathbb{R}^n$, recall the corkscrew point $X_Q := (x_Q, \ell(Q))$ and denote the Carleson box in $\mathbb{R}^{n+1}_+$ by $T_Q := Q \times (0, \ell(Q))$. Also, recall that $\mu(x, t) := \mu(x)$, so $d\mu(x, t) = \mu(x)\, dx\, dt$, for $(x, t) \in \mathbb{R}^{n+1}$. If $u \in W^{1,2}_\mu(T_{2Q})$ is a solution of $\operatorname{div}(A\nabla u) = 0$ in T_{2Q}, and $u = 0$ on $2Q$ in the $W^{1,2}_\mu(T_{2Q})$-sense, then

$$|u(x,t)| \lesssim \left(\frac{t}{\ell(Q)}\right)^\alpha \left(\fint_{T_{2Q}} |u|^2 \, d\mu\right)^{1/2} \quad \text{for all } (x,t) \in T_Q, \tag{5.1}$$

and if, in addition, $u \geq 0$ almost everywhere on T_{2Q}, then

$$u(X) \lesssim u(X_Q) \quad \text{for all } X \in T_Q, \tag{5.2}$$

where α is from (2.17) and the implicit constants depend only on n, λ, Λ and $[\mu]_{A_2}$. Estimate (5.1) follows from standard reflection arguments and the interior Hölder continuity estimate in (2.17), as observed on page 102 in [Fabes et al. 1982b]. Estimate (5.2) can then be deduced from (5.1) and the interior Harnack inequality in (2.18), as in the uniformly elliptic case; see the proof of Theorem 1.1 in [Caffarelli et al. 1981], which does not use the assumption therein that A is symmetric.

5B. ***Definition and properties of degenerate elliptic measure.*** For $X \in \mathbb{R}^{n+1}$, $x \in \mathbb{R}^n$ and $r > 0$, we use $B(X,r) := \{Y \in \mathbb{R}^{n+1} : |Y - X| < r\}$ to denote balls in $\mathbb{R}^{n+1}$ and $\Delta(x,r) := \{y \in \mathbb{R}^n : |x| < r\}$ to denote balls in $\mathbb{R}^n$, where $\Delta(x,r)$ is identified with the surface ball $B((x,0),r) \cap \partial\mathbb{R}^{n+1}_+$ in $\mathbb{R}^{n+1}$. For each $R > 0$, consider the bounded Lipschitz domain $\Sigma_R := B(0,R) \cap \mathbb{R}^{n+1}_+$ with Lipschitz constant at most 1. For each $X \in \Sigma_R$, the degenerate elliptic measure ω^X_R is the measure on $\partial\Sigma_R$, as defined on page 583 in [Fabes et al. 1983], such that $u(X) = \int_{\partial\Sigma_R} h\, d\omega^X_R$ solves the Dirichlet problem for continuous boundary data $h \in C(\partial\Sigma_R)$ in the sense that $\operatorname{div}(A\nabla u) = 0$ in Σ_R and $u \in C(\overline{\Sigma}_R)$ with $u|_{\partial\Sigma_R} = h$.

We now define the degenerate elliptic measure on $\mathbb{R}^n$. If $f \in C_c(\mathbb{R}^n)$, fix $R_0 > 0$ such that $\operatorname{supp} f \subseteq \Delta(0, R_0)$ and set f equal to zero on $\mathbb{R}^{n+1}_+$, so then $f^\pm \in C(\partial\Sigma_R)$ for all $R \geq R_0$, where $f^\pm(X) := \max\{\pm f(X), 0\}$; thus

$$u^\pm_R(X) := \int_{\partial\Sigma_R} f^\pm\, d\omega^X_R \quad \text{for all } X \in \Sigma_R$$

solve the Dirichlet problem as above in Σ_R for all $R \geq R_0$. The maximum principle then implies that $u^\pm_{R_1}(X) \leq u^\pm_{R_2}(X)$, whenever $R_0 \leq R_1 \leq R_2$ and $X \in \Sigma_{R_1}$, and that $\sup_{R>0} \|u^\pm_R\|_\infty \leq \|f\|_\infty$. This allows us to define

$$u(X) := \lim_{R\to\infty} [u^+_R(X) - u^-_R(X)] \quad \text{for all } X \in \mathbb{R}^{n+1}_+, \tag{5.3}$$

and since the mapping $f \mapsto u(X)$ is a positive linear functional on $C_c(\mathbb{R}^n)$, the Riesz representation theorem implies that there exists a regular Borel probability measure (the degenerate elliptic measure) ω^X on $\mathbb{R}^n$ such that $u(X) = \int_{\mathbb{R}^n} f\, d\omega^X$.

The function u from (5.3) solves $\operatorname{div}(A\nabla u) = 0$ in $\mathbb{R}^{n+1}_+$. To prove this, note that $\|u\|_\infty \leq \|f\|_\infty$, so for each compact set $K \subset \mathbb{R}^{n+1}_+$, the Hölder continuity of solutions in (2.17) ensures the equicontinuity required to apply the Arzelà–Ascoli theorem and extract a subsequence u_{R_j} that converges to u uniformly on K. This combined with Caccioppoli's inequality shows that u_{R_j} converges to u in $W^{1,2}_\mu(K)$; hence $u \in W^{1,2}_{\mu,\mathrm{loc}}(\mathbb{R}^{n+1}_+)$. Moreover, if $\varphi \in C^\infty_c(\mathbb{R}^{n+1}_+)$ and $K = \operatorname{supp}\varphi \subset \Sigma_R$, then

$$\left| \int_K \langle A\nabla(u - u_R), \nabla\varphi\rangle \right| \leq \Lambda \|\nabla\varphi\|_\infty \mu(K)^{1/2} \|u - u_R\|_{W^{1,2}_\mu(K)}, \tag{5.4}$$

from which it follows that $\int_{\mathbb{R}^{n+1}_+} \langle A\nabla u, \nabla\varphi\rangle = 0$, as required.

We note by (5.3) that, when restricted to any bounded Borel subset of $\mathbb{R}^n$, the measures ω^X_R converge weakly to ω^X, so Theorem 1 on page 54 of [Evans and Gariepy 1992] shows that

$$\omega^X(U) \leq \liminf_{R\to\infty} \omega^X_R(U), \quad \omega^X(K) \geq \limsup_{R\to\infty} \omega^X_R(K), \quad \omega^X(\mathcal{B}) = \lim_{R\to\infty} \omega^X_R(\mathcal{B}) \tag{5.5}$$

for all bounded open sets $U \subset \mathbb{R}^n$, all compact sets $K \subset \mathbb{R}^n$, and all bounded Borel sets $\mathcal{B} \subset \mathbb{R}^n$ such that $\omega^X(\partial\mathcal{B}) = 0$. This construction of the degenerate elliptic measure also provides for the following expected properties.

Lemma 5.6. *If $X_0, X_1 \in \mathbb{R}^{n+1}_+$ and $E \subseteq \mathbb{R}^n$ is a Borel set, then $\omega^{X_0}(E) = 0$ if and only if $\omega^{X_1}(E) = 0$. Moreover, the nonnegative function $u(X) := \omega^X(E)$ is a solution of* $\operatorname{div}(A\nabla u) = 0$ *in $\mathbb{R}^{n+1}_+$ and the boundary Hölder continuity estimate*

$$|u(x,t)| \lesssim \left(\frac{t}{\ell(Q)}\right)^{\alpha} u(X_Q) \quad \textit{for all } (x,t) \in T_Q \tag{5.7}$$

holds on all cubes Q such that $2Q \subseteq \mathbb{R}^n \setminus E$, where α is from (2.17) *and the implicit constants depend only on n, λ, Λ and $[\mu]_{A_2}$,*

Proof. The proof follows that of Lemma 1.2.7 in [Kenig 1994], except we must account for the fact that the solution to the Dirichlet problem in $\mathbb{R}^{n+1}_+$ defined by (5.3) requires boundary data to have compact support, which is easily done as we now show. Suppose that $\omega^{X_0}(E) = 0$ and that $K \subseteq E$ is a compact set. The regularity of the measure implies that $\omega^{X_0}(K) = 0$ and, for each $\epsilon > 0$, there exists a bounded open set $U \supset K$ such that $\omega^{X_0}(U) < \epsilon$. In particular, we may assume that U is *bounded* because K is compact, so by Urysohn's lemma there exists $g \in C_c(\mathbb{R}^n)$ such that $g(x) = 1$ on K, $0 \le g(x) \le 1$ on U, and $\operatorname{supp} g \subset U$. It follows that $u(X) = \int_{\mathbb{R}^n} g \, d\omega^X$ is the solution to the Dirichlet problem in $\mathbb{R}^{n+1}_+$ defined by (5.3) with boundary data g. Applying the Harnack inequality from (2.18) and connecting X_0 with X_1 via a Harnack chain then shows that there exists $C > 0$, depending on X_0 and X_1, such that

$$\omega^{X_1}(K) \le u(X_1) \le C u(X_0) \le C\omega^{X_0}(U) \le C\epsilon \quad \text{for all } \epsilon > 0;$$

hence $\omega^{X_1}(K) = 0$ for all compact sets $K \subseteq E$, and so $\omega^{X_1}(E) = 0$ by regularity.

The proof that $u(X) := \omega^X(E)$ is a solution of $\operatorname{div}(A\nabla u) = 0$ in $\mathbb{R}^{n+1}_+$ also follows that of Lemma 1.2.7 in [Kenig 1994]. It remains to prove that the boundary Hölder continuity estimate holds on all cubes Q such that $2Q \subseteq \mathbb{R}^n \setminus E$. We first consider when E is bounded. In that case, let U_δ denote the open δ-neighbourhood of E and set $\chi_{\epsilon,\delta} := \varphi_\epsilon * \mathbb{1}_{U_\delta}$ for all $\delta > \epsilon > 0$, where $\varphi_\epsilon(x) := \epsilon^{-n}\varphi(x/\epsilon)$ and $\varphi \in C^\infty_c(\Delta(0,1))$ is a fixed nonnegative function with $\int_{\mathbb{R}^n} \varphi = 1$. In particular, since U_δ is open, we have $\mathbb{1}_E \le \mathbb{1}_{U_\delta} \le \liminf_{\epsilon\to 0} \chi_{\epsilon,\delta}$. Consequently, if $X = (x,t) \in \mathbb{R}^{n+1}_+$, then

$$u(X) = \omega^X(E) \le \omega^X(U_\delta) \le \int_{\mathbb{R}^n} \liminf_{\epsilon\to 0} \chi_{\epsilon,\delta} \, d\omega^X \le \liminf_{\epsilon\to 0} \int_{\mathbb{R}^n} \chi_{\epsilon,\delta} \, d\omega^X. \tag{5.8}$$

The function $\chi_{\epsilon,\delta}$ belongs to $C^\infty_c(\mathbb{R}^n)$ and thus extends to a function in $C^\infty_c(\mathbb{R}^{n+1})$. The construction of the degenerate elliptic measure (see pages 580–583 in [Fabes et al. 1983], which was the starting point for our extension to the upper half-space above) thus implies $v_\epsilon(X) := \int_{\mathbb{R}^n} \chi_{\epsilon,\delta} \, d\omega^X$ is in $W^{1,2}(T_{(3/2)Q})$ and vanishes on $\frac{3}{2}Q$ whenever $0 < \epsilon < \delta < \frac{1}{4}\ell(Q)$, so estimate (5.8) combined with the boundary Hölder continuity estimate in (5.1) and the boundary Harnack inequality in (5.2) shows that

$$u(x,t) \le \liminf_{\epsilon\to 0} v_\epsilon(x,t) \lesssim \left(\frac{t}{\ell(Q)}\right)^{\alpha} \liminf_{\epsilon\to 0} v_\epsilon(X_Q) \quad \text{for all } (x,t) \in T_Q. \tag{5.9}$$

We now let $U_{\delta,\epsilon}$ denote the open ϵ-neighbourhood of U_δ, in which case $\chi_{\epsilon,\delta} \le \mathbb{1}_{U_{\delta,\epsilon}}$ and $v_\epsilon(X) \le \omega^X(U_{\delta,\epsilon})$, so by (5.9) and the regularity of the degenerate elliptic measure we have

$$u(x,t) \lesssim \left(\frac{t}{\ell(Q)}\right)^{\alpha} \liminf_{\epsilon\to 0} \omega^{X_Q}(U_{\delta,\epsilon}) \lesssim \left(\frac{t}{\ell(Q)}\right)^{\alpha} \omega^{X_Q}(U_\delta) \quad \text{for all } (x,t) \in T_Q.$$

This proves (5.7) if E is bounded, since the regularity of the measure also implies that $\omega^{X_Q}(U_\delta)$ approaches $\omega^{X_Q}(E) = u(X_Q)$ as δ approaches 0. If E is not bounded, then applying (5.7) on the bounded sets $E_k := \mathbb{1}_{2^{k+1}Q\setminus 2^k Q} E$, for $k \in \mathbb{N}$, shows that

$$u(x,t) = \sum_{k=1}^{\infty} \omega^X(E_k) \lesssim \sum_{k=1}^{\infty} \left(\frac{t}{\ell(Q)}\right)^{\alpha} \omega^{X_Q}(E_k) = \left(\frac{t}{\ell(Q)}\right)^{\alpha} \omega^{X_Q}(E) \quad \text{for all } (x,t) \in T_Q,$$

as required. □

5C. *Preliminary estimates for degenerate elliptic measure.* In the uniformly elliptic case, there is a rich theory for the Green's function on bounded domains, and specifically, estimates and connections with elliptic measure; see, for instance, Theorem 1.2.8 and Corollary 1.3.6 in [Kenig 1994]. This theory also extends to unbounded domains; see Section 10 in [Littman et al. 1963] and [Hofmann and Kim 2007]. In the degenerate elliptic case, the theory was developed on bounded domains in [Fabes et al. 1982a; 1982b; 1983], but it is not clear if there is always such a Green's function on unbounded domains. In particular, the construction in [Hofmann and Kim 2007] for the uniformly elliptic case relies on the (unweighted) global version of the Sobolev embedding in (2.4), which is not known for a general A_2-weight. In what follows, we combine the properties of the Green's function on the bounded domain $\Sigma_R := B(0,R) \cap \mathbb{R}^{n+1}_+$ with the limit properties in (5.5) to deduce estimates for degenerate elliptic measure on $\mathbb{R}^n$. These will be used to prove Lemma 5.24 and ultimately Theorem 5.30.

For each $R > 0$, the Green's function $g_R : \overline{\Sigma}_R \times \overline{\Sigma}_R \mapsto [0,\infty]$ is constructed by following Proposition 2.4 in [Fabes et al. 1982a]. In particular, for each $Y \in \Sigma_R$, the mapping $X \mapsto g_R(X,Y)$ is the Hölder continuous function in $\overline{\Sigma}_R \setminus \{Y\}$ that vanishes on $\partial\Sigma_R$ and satisfies $\int_{\Sigma_R} \langle A\nabla g_R(\cdot\,, Y), \nabla\Phi\rangle = \Phi(Y)$ for all $\Phi \in C_c^\infty(\Sigma_R)$. As explained on page 583 in [Fabes et al. 1983], these properties are valid on any NTA domain, hence a fortiori on Σ_R. The proofs do not rely on the assumption therein that A is symmetric, although the symmetry property $g_R(X,Y) = g_R(Y,X)$ is no longer guaranteed, as $g_R^*(X,Y) := g_R(Y,X)$ is the Green's function for the adjoint operator $-\operatorname{div}(A^*\nabla)$. We will rely on the following two lemmas, which are immediate from Theorem 4 and Lemma 3 in [Fabes et al. 1983], respectively, to estimate the Green's function g_R and the degenerate elliptic measure ω_R on Σ_R.

Lemma 5.10. *If $X, Y \in \Sigma_R$ and $|X-Y| < \frac{1}{2}\operatorname{dist}(Y, \partial\Sigma_R)$, then*

$$g_R(X,Y) \eqsim \int_{|X-Y|}^{\operatorname{dist}(Y,\partial\Sigma_R)} \frac{s^2}{\mu(B(Y,s))} \frac{ds}{s},$$

where the implicit constants depend only on n, λ, Λ and $[\mu]_{A_2}$.

Lemma 5.11. *If $R > 0$ and Q is a cube in $\mathbb{R}^n$ such that $T_{2Q} \subset \Sigma_R$, then*

$$\frac{g_R(X_Q, Y)}{\ell(Q)} \eqsim \omega_R^Y(Q) \frac{\ell(Q)}{\mu(T_Q)} = \frac{\omega_R^Y(Q)}{\mu(Q)} \quad \textit{for all } Y \in \Sigma_R \setminus T_{2Q},$$

where the implicit constants depend only on n, λ, Λ and $[\mu]_{A_2}$.

The degenerate elliptic measure ω_R^X satisfies the doubling property $\omega_R^X(2Q) \le C_0 \omega_R^X(Q)$ for all cubes Q in $\mathbb{R}^n$ such that $T_{2Q} \subset \Sigma_R$ and all $X \in \Sigma_R \setminus T_{2Q}$, where the doubling constant $C_0 > 0$ depends only on n, λ, Λ and $[\mu]_{A_2}$. This is proved in Lemma 1 on page 584 of [Fabes et al. 1983] by using the estimates in Lemma 5.11, the Harnack inequality in (2.18), and the doubling property of μ. The doubling constant C_0 does not depend on R, which allows us to use the inequalities in (5.5) to show that the degenerate elliptic measure ω^X is locally doubling on $\mathbb{R}^n$, in the sense that

$$\omega^X(2Q) \le \liminf_{R\to\infty} \omega_R^X(2Q) \lesssim \liminf_{R\to\infty} \omega_R^X\big(\tfrac{1}{2}Q\big) \le \limsup_{R\to\infty} \omega_R^X\big(\tfrac{1}{2}\overline{Q}\big) \le \omega^X(Q) \tag{5.12}$$

for all cubes $Q \subset \mathbb{R}^n$ and all $X \in \mathbb{R}^{n+1}_+ \setminus T_{2Q}$, where the implicit constant is C_0^2. In particular, the doubling property implies $\omega^X(\partial Q) = 0$ for all cubes $Q \subset \mathbb{R}^n$ (see page 403 in [García-Cuerva and Rubio de Francia 1985] or Proposition 6.3 in [Hofmann and Martell 2014]), so (5.12) actually improves to $\omega^X(2Q) \le C_0 \omega^X(Q)$, since by the equality in (5.5) we now have

$$\omega^X(Q) = \lim_{R\to\infty} \omega_R^X(Q) \tag{5.13}$$

for all cubes $Q \subset \mathbb{R}^n$ and all $X \in \mathbb{R}^{n+1}_+ \setminus T_{2Q}$. This provides the following estimate for degenerate elliptic measure.

Lemma 5.14. *If Q is a cube in $\mathbb{R}^n$, then $\omega^{X_Q}(Q) \gtrsim 1$, where the implicit constant depends only on n, λ, Λ and $[\mu]_{A_2}$.*

Proof. Let Q denote a cube in $\mathbb{R}^n$ and fix $R_0 > 0$ such that $T_{2Q} \subset \Sigma_{R_0}$. The Hölder continuity at the boundary in (5.1) and the Harnack inequality in (2.18) imply (see the proof of Lemma 3 on page 585 in [Fabes et al. 1983]) that

$$\omega_R^{X_Q}(Q) \gtrsim 1 \quad \text{for all } R \ge R_0,$$

where the implicit constant depends only on n, λ, Λ and $[\mu]_{A_2}$, and so does not depend on R. The result follows by using Harnack's inequality to shift the pole (from X_{2Q} to X_Q) in (5.12)–(5.13) to obtain $\omega^{X_Q}(Q) = \lim_{R\to\infty} \omega_R^{X_Q}(Q) \gtrsim 1$. □

The estimates in Lemma 5.11 also imply the following comparison principle. The result is stated on page 585 in [Fabes et al. 1983] and the proof is the same as in the uniformly elliptic case; see Theorem 1.4 in [Caffarelli et al. 1981] or Lemma 1.3.7 in [Kenig 1994], neither of which use the assumption therein that A is symmetric.

Lemma 5.15 (comparison principle). *Let Q denote a cube in $\mathbb{R}^n$ and suppose that $u, v \in W^{1,2}_\mu(T_{2Q}) \cap C(\overline{T}_{2Q})$ with $u, v \ge 0$ on T_{2Q}. If $\operatorname{div}(A\nabla u) = \operatorname{div}(A\nabla v) = 0$ in T_{2Q} and $u = v = 0$ on $2Q$, then*

$$\frac{u(X)}{v(X)} \approx \frac{u(X_Q)}{v(X_Q)} \quad \textit{for all } X \in T_Q,$$

where the implicit constants depend only on n, λ, Λ and $[\mu]_{A_2}$.

The following corollary of these preliminaries will be used in Proposition 5.18 to estimate Radon–Nikodym derivatives of the degenerate elliptic measure.

Lemma 5.16. *If Q_0 and Q are cubes in $\mathbb{R}^n$ such that $Q \subseteq Q_0$, then*

$$\omega^{X_{Q_0}}(Q) \eqsim \frac{\omega^X(Q)}{\omega^X(Q_0)} \quad \text{for all } X \in \mathbb{R}^{n+1}_+ \setminus T_{2Q_0},$$

where the implicit constants depend only on n, λ, Λ and $[\mu]_{A_2}$.

Proof. Let $Q \subset Q_0$ be cubes in $\mathbb{R}^n$, suppose that $X \in \mathbb{R}^{n+1}_+ \setminus T_{2Q_0}$ and consider $R > 0$ large enough so that $X \in \Sigma_R$ and $T_{4Q_0} \subset \Sigma_R$. Lemma 5.11 shows that

$$\omega_R^X(Q_0)\,\ell(Q_0) \eqsim \mu(Q_0)\, g_R(X_{Q_0}, X),$$
$$\omega_R^X(Q)\,\ell(Q) \eqsim \mu(Q)\, g_R(X_Q, X)$$
$$\omega_R^{X_{3Q_0}}(Q)\,\ell(Q) \eqsim \mu(Q)\, g_R(X_Q, X_{3Q_0}).$$

If $u(Y) = g_R(Y, X)$ and $v(Y) = g_R(Y, X_{3Q_0})$, then $\operatorname{div}(A\nabla u) = \operatorname{div}(A\nabla u)v = 0$ in T_{2Q_0} and $u = v = 0$ on $2Q_0$, so the comparison principle in Lemma 5.15 shows that

$$\frac{g_R(X_Q, X)}{g_R(X_Q, X_{3Q_0})} = \frac{u(X_Q)}{v(X_Q)} \eqsim \frac{u(X_{Q_0})}{v(X_{Q_0})} = \frac{g_R(X_{Q_0}, X)}{g_R(X_{Q_0}, X_{3Q_0})}.$$

Also, Lemma 5.10 shows that $g_R(X_{Q_0}, X_{3Q_0}) \eqsim \ell(Q_0)/\mu(Q_0)$, so together we obtain

$$\frac{\omega_R^X(Q)}{\omega_R^X(Q_0)} \eqsim \frac{g_R(X_Q, X)}{g_R(X_{Q_0}, X)} \frac{\mu(Q)}{\ell(Q)} \frac{\ell(Q_0)}{\mu(Q_0)} \eqsim \frac{g_R(X_Q, X_{3Q_0})}{g_R(X_{Q_0}, X_{3Q_0})} \frac{\mu(Q)}{\ell(Q)} \frac{\ell(Q_0)}{\mu(Q_0)} \eqsim \omega_R^{X_{3Q_0}}(Q).$$

The Harnack inequality from (2.18) then shows that $\omega_R^X(Q) \eqsim \omega_R^X(Q_0)\omega_R^{X_{Q_0}}(Q)$ and the result follows by using (5.13) to estimate the limit as R approaches infinity. □

If $X, X_0 \in \mathbb{R}^{n+1}_+$, then Lemma 5.6 shows that ω^X and ω^{X_0} are mutually absolutely continuous, so the Lebesgue differentiation theorem for the locally doubling measure ω^{X_0} implies that the Radon–Nikodym derivative of ω^X satisfies

$$K(X_0, X, y) := \frac{d\omega^X}{d\omega^{X_0}}(y) = \lim_{s \to 0} \frac{\omega^X(Q(y,s))}{\omega^{X_0}(Q(y,s))}, \quad \omega^{X_0}\text{-a.e. } y \in \mathbb{R}^n, \tag{5.17}$$

where $Q(y, s)$ denotes the cube in $\mathbb{R}^n$ with centre y and side length s. The following decay estimate for the kernel function K extends Lemma 2 on page 584 in [Fabes et al. 1983]. It is the final property of degenerate elliptic measure needed to prove Lemma 5.24.

Proposition 5.18. *If Q_0 and Q are cubes in $\mathbb{R}^n$ such that $Q \subseteq Q_0$, then*

$$K(X_{Q_0}, X_Q, y) \lesssim \frac{1}{\omega^{X_{Q_0}}(Q)} \max\left\{ \frac{|y - x_Q|}{\ell(Q)}, 1 \right\}^{-\alpha}, \quad \omega^{X_{Q_0}}\text{-a.e. } y \in Q_0,$$

where $\alpha > 0$ from (2.17) *and the implicit constant depend only on n, λ, Λ and $[\mu]_{A_2}$.*

Proof. Let $Q \subseteq Q_0$ denote cubes in $\mathbb{R}^n$ and fix $J \in \mathbb{N}$ such that $2^{J-1}Q \subseteq Q_0 \subseteq 2^J Q$. If $y \in Q$, then Lemma 5.16 and the Harnack inequality in (2.18) show that

$$\omega^{X_Q}(Q(y,s)) \approx \frac{\omega^{X_{2Q_0}}(Q(y,s))}{\omega^{X_{2Q_0}}(Q)} \approx \frac{\omega^{X_{Q_0}}(Q(y,s))}{\omega^{X_{Q_0}}(Q)}$$

whenever $0 < s < \operatorname{dist}(y, \mathbb{R}^n \setminus Q)$. If $y \in 2^j Q \setminus 2^{j-1}Q$ for some $j \in \{1, \dots, J\}$, then the boundary Hölder continuity estimate in (5.7) combined with Lemma 5.16 and the Harnack inequality in (2.18) show that

$$\omega^{X_Q}(Q(y,s)) \lesssim \left(\frac{\ell(Q)}{2^{j-2}\ell(Q)}\right)^{\alpha} \omega^{X_{2^{j-2}Q}}(Q(y,s)) \approx \left(\frac{\ell(Q)}{|y-x_Q|}\right)^{\alpha} \frac{\omega^{X_{Q_0}}(Q(y,s))}{\omega^{X_{Q_0}}(2^j Q)}$$

whenever $0 < s < \operatorname{dist}(y, \mathbb{R}^n \setminus (2^j Q \setminus 2^{j-2}Q))$, where $\alpha > 0$ from (2.17) and the implicit constants depend only on n, λ, Λ and $[\mu]_{A_2}$. The result follows by using these two estimates to bound the limit as s approaches zero in (5.17). ☐

5D. ***The A_∞-estimate for degenerate elliptic measure.*** We now combine the properties of degenerate elliptic measure with good ϵ_0-coverings for sets, as introduced in [Kenig et al. 2000] and defined below (see also [Kenig et al. 2016]), to construct bounded solutions that satisfy the truncated square function estimate in Lemma 5.24. This result, combined with the Carleson measure estimate from Theorem 1.3, allows us to prove the A_∞-estimate for the degenerate elliptic measure in Theorem 5.30. This avoids the need to apply the method of ϵ-approximability, as was done in [Hofmann et al. 2015a], and so simplifies the proof in the uniformly elliptic case.

Let $\mathbb{D}(\mathbb{R}^n)$ denote the standard collection $\{2^k(j+[0,1]^n) : k \in \mathbb{Z},\ j \in \mathbb{Z}^n\}$ of all *closed* dyadic cubes S in $\mathbb{R}^n$. For each $S \in \mathbb{D}(\mathbb{R}^n)$ and $\eta = 2^{-K}$, where $K \in \mathbb{N}$, define $\mathbb{D}(S) := \{S' \in \mathbb{D}(\mathbb{R}^n) : S' \subseteq S\}$ and

$$\mathbb{D}^\eta(S) := \{S' \in \mathbb{D}(S) : \ell(S') = 2^{-K}\ell(S)\}, \tag{5.19}$$

so $\mathbb{D}^\eta(S)$ is precisely the set of all *dyadic descendants* of S at scale $2^{-K}\ell(S)$.

Definition 5.20. Suppose that Q_0 is a cube in $\mathbb{R}^n$. If $\epsilon_0 > 0$, $k \in \mathbb{N}$, $Q \subseteq Q_0$ is a cube and $E \subseteq Q$, then a *good ϵ_0-cover of E of length k in Q* is a collection $\{O_l\}_{l=1}^k$ of nested open sets that satisfy $E \subseteq O_k \subseteq O_{k-1} \subseteq \cdots \subseteq O_1 \subseteq Q$ and each of which has a decomposition $O_l = \bigcup_{i=1}^\infty S_i^l$ given by a collection $\{S_i^l\}_{i\in\mathbb{N}} \subseteq \mathbb{D}(\mathbb{R}^n)$ of dyadic cubes with pairwise disjoint interiors such that

$$\omega^{X_{2Q_0}}(O_l \cap S_i^{l-1}) \le \epsilon_0\, \omega^{X_{2Q_0}}(S_i^{l-1}) \quad \text{for all } i \in \mathbb{N},\ \text{for all } l \in \{2, \dots, k\}. \tag{5.21}$$

Let us record a few important consequences of this definition that will be needed. It is proved on page 243 in [Kenig et al. 2000] that for each $i \in \mathbb{N}$ and $l \in \{2, \dots, k\}$, there exists a unique $j \in \mathbb{N}$ such that S_i^l is a *proper* subset of S_j^{l-1}; thus $\ell(S_i^l) \le \frac{1}{2}\ell(S_j^{l-1})$. Also, for $m \in \{2, \dots, k\}$, iterating (5.21) as in Lemma 2.5 of [Kenig et al. 2000] shows that

$$\omega^{X_{2Q_0}}(O_l \cap S_i^m) \le \epsilon_0^{l-m} \omega^{X_{2Q_0}}(S_i^m) \quad \text{for all } i \in \mathbb{N},\ \text{for all } l \in \{m, \dots, k\}. \tag{5.22}$$

In the uniformly elliptic case, the following result is Lemma 2.3 from [Kenig et al. 2016]. The proof extends to the degenerate elliptic case, since it only relies on the fact that the degenerate elliptic measure $\omega^{X_{2Q_0}}$ is doubling when restricted to the cube Q_0.

Lemma 5.23. *Suppose that Q_0 is a cube in $\mathbb{R}^n$. If $\epsilon_0 > 0$, then there exists $\delta_0 > 0$, depending only on ϵ_0, n, λ, Λ and $[\mu]_{A_2}$, such that the following property holds*:

If $Q \subseteq Q_0$ is a cube and $E \subseteq Q_0$ such that $\omega^{X_{2Q_0}}(E) \leq \delta_0$, then there exists a good ϵ_0-cover of E of length k in Q for some natural number $k \eqsim \log(\omega^{X_{2Q_0}}(E))/\log \epsilon_0$, where the implicit constants depend only on n, λ, Λ and $[\mu]_{A_2}$.

We can now prove the following lemma by adapting the proof in [Kenig et al. 2016] to the degenerate elliptic case. The original argument has also been somewhat modified.

Lemma 5.24. *Suppose that Q_0 is a cube in $\mathbb{R}^n$. If $M \geq 1$, then there exists $\delta_M > 0$, depending only on M, n, λ, Λ and $[\mu]_{A_2}$, such that the following property holds*:

If $Q \subseteq Q_0$ is a cube and $E \subseteq Q$ and $\omega^{X_{2Q_0}}(E) \leq \delta_M$, then there is a Borel subset $\mathcal{B}$ of $\mathbb{R}^n$ such that the solution $u(X) := \omega^X(\mathcal{B})$ of $\operatorname{div}(A\nabla u) = 0$ in $\mathbb{R}^{n+1}_+$ satisfies

$$M \leq \int_0^{\gamma\ell(Q)} \int_{\Delta(x,\gamma t)} |t\nabla u(y,t)|^2 \,\frac{d\mu(y)}{\mu(\Delta(x,t))}\,\frac{dt}{t} \quad \textit{for all } x \in E,$$

where $\gamma > 0$ is a constant that depends only on n, λ, Λ and $[\mu]_{A_2}$.

Proof. We introduce three constants $\epsilon_0, \delta, \eta \in (0,1)$ that will be chosen with $\delta \leq \delta_0$, where δ_0 is determined by ϵ_0 as in Lemma 5.23, and $\eta = 2^{-K}$ for some $K \in \mathbb{N}$. Therefore, if $E \subseteq Q \subseteq Q_0$ and $\omega^{X_{2Q_0}}(E) \leq \delta$, then there exists a good ϵ_0-cover of E of length k in Q such that $k \eqsim \log(\omega^{X_{2Q_0}}(E))/\log \epsilon_0$. This cover is denoted by $\{O_l\}_{l=1}^k$ with $O_l = \bigcup_{i=1}^\infty S_i^l$ as in Definition 5.20, and for each such cube S_i^l, a dyadic descendant $\widetilde{S}_i^l$ in $\mathbb{D}^\eta(S_i^l)$ that contains the centre of S_i^l is now fixed and

$$\widetilde{O}_l := \bigcup_{i=1}^\infty \widetilde{S}_i^l, \tag{5.25}$$

where we note that $\ell(\widetilde{S}_i^l) = \eta\ell(S_i^l)$ in accordance with (5.19).

We claim that there exists a Borel subset $\mathcal{B}$ of $\mathbb{R}^n$ such that $\mathbb{1}_{\mathcal{B}} = \sum_{l=2}^k \mathbb{1}_{\widetilde{O}_{l-1}\setminus O_l}$. To see this, suppose that $\sum_{l=2}^k \mathbb{1}_{\widetilde{O}_{l-1}\setminus O_l}(x) \neq 0$ and let l_0 denote the smallest integer $l \in [2,k]$ such that $\mathbb{1}_{\widetilde{O}_{l-1}}\setminus O_l(x) = 1$. It must hold that $x \in \widetilde{O}_{l_0-1} \setminus O_{l_0}$, so then $x \notin O_{l_0}$, which implies $x \notin O_l$ and $x \notin \widetilde{O}_l$ for all $l \geq l_0$; hence $\mathbb{1}_{\widetilde{O}_{l-1}\setminus O_l}(x) = 0$ for all $l > l_0$ and the claim follows.

We now aim to choose $\epsilon_0, \eta \in (0,1)$ such that $u(X) := \omega^X(\mathcal{B})$ on $\mathbb{R}^{n+1}_+$ satisfies

$$|u(X_{\eta S_i^l}) - u(X_{\eta \widehat{S}_i^l})| \gtrsim 1 \quad \text{for all } \widehat{S}_i^l \in \mathbb{D}^\eta(S_i^l),\ \text{for all } i \in \mathbb{N},\ \text{for all } l \in \{1,\dots,k\}, \tag{5.26}$$

where the implicit constant depends only on the allowed constants n, λ, Λ and $[\mu]_{A_2}$, and if x_i^l and $\hat{x}_i^l$ denote the centres of S_i^l and $\widehat{S}_i^l$, then the relevant corkscrew points are precisely $X_{\eta S_i^l} = (x_i^l, \eta\ell(S_i^l))$ and $X_{\eta \widehat{S}_i^l} = (\hat{x}_i^l, \eta^2\ell(S_i^l))$. To this end, we proceed to obtain estimates for $u(X_{\eta S_i^l})$ and $u(X_{\eta \widehat{S}_i^l})$.

To estimate $u(X_{\eta S_i^l})$, write

$$u(X_{\eta S_i^l}) = \int_{\mathbb{R}^n \setminus S_i^l} \mathbb{1}_{\mathcal{B}} \, d\omega^{X_{\eta S_i^l}} + \int_{S_i^l} \mathbb{1}_{\mathcal{B}} \, d\omega^{X_{\eta S_i^l}} =: I + II.$$

The boundary Hölder continuity in (5.7) shows that $I \le \omega^{X_{\eta S_i^l}}(\mathbb{R}^n \setminus S_i^l) \le C_0 \eta^\alpha$, where $C_0, \alpha > 0$ depend only on the allowed constants. To estimate II, write

$$\begin{aligned} II &= \sum_{j=2}^{l} \int_{S_i^l} \mathbb{1}_{\widetilde{O}_{j-1} \setminus O_j} \, d\omega^{X_{\eta S_i^l}} + \sum_{j=l+2}^{k} \int_{S_i^l} \mathbb{1}_{\widetilde{O}_{j-1} \setminus O_j} \, d\omega^{X_{\eta S_i^l}} + \int_{S_i^l} \mathbb{1}_{\widetilde{O}_l \setminus O_{l+1}} \, d\omega^{X_{\eta S_i^l}} \\ &=: II_1 + II_2 + II_3. \end{aligned}$$

First, observe that $II_1 = 0$, since if $m \in \{2, \dots, l\}$, then $S_i^l \subseteq O_l \subseteq O_j$ and so $(\widetilde{O}_{j-1} \setminus O_j) \cap S_i^l = \varnothing$. To estimate II_2, the kernel function representation in (5.17) and estimates in Proposition 5.18, the local doubling property of the degenerate elliptic measure in (5.12) and property (5.22) of the good ϵ_0-covering, show that

$$\begin{aligned} II_2 &= \sum_{j=l+2}^{k} \int_{(\widetilde{O}_{j-1} \setminus O_j) \cap S_i^l} K(X_{2Q_0}, X_{\eta S_i^l}, y) \, d\omega^{X_{2Q_0}}(y) \\ &\le \frac{C_\eta}{\omega^{X_{2Q_0}}(S_i^l)} \sum_{j=l+2}^{k} \omega^{X_{2Q_0}}((\widetilde{O}_{j-1} \setminus O_j) \cap S_i^l) \\ &\le \frac{C_\eta}{\omega^{X_{2Q_0}}(S_i^l)} \sum_{j=l+2}^{k} \omega^{X_{2Q_0}}(O_{j-1} \cap S_i^l) \\ &\le \frac{C_\eta}{\omega^{X_{2Q_0}}(S_i^l)} \sum_{j=l+2}^{k} \epsilon_0^{j-1-l} \omega^{X_{2Q_0}}(S_i^l) \le \frac{C_\eta \epsilon_0}{1 - \epsilon_0}, \end{aligned}$$

where the constant $C_\eta > 0$ depends only on η and the allowed constants.

To estimate II_3, observe that $S_i^l \cap \widetilde{O}_l = \widetilde{S}_i^l$ by the definition of $\widetilde{O}_l$ in (5.25); hence

$$II_3 = \int_{\widetilde{S}_i^l} d\omega^{X_{\eta S_i^l}} - \int_{\widetilde{S}_i^l \cap O_{l+1}} d\omega^{X_{\eta S_i^l}} =: II_3' - II_3''.$$

The term II_3'' is estimated in the same way as II_2 above to show that

$$II_3'' \le \frac{C_\eta}{\omega^{X_{2Q_0}}(S_i^l)} \omega^{X_{2Q_0}}(O_{l+1} \cap \widetilde{S}_i^l) \le \frac{C_\eta}{\omega^{X_{2Q_0}}(S_i^l)} \omega^{X_{2Q_0}}(O_{l+1} \cap S_i^l) \le C_\eta \epsilon_0.$$

We estimate II_3' from above and below. First, note that $X_{\eta S_i^l} = (x_i^l, \eta \ell(S_i^l))$, $x_i^l \in \widetilde{S}_i^l$ and $\ell(\widetilde{S}_i^l) = \eta \ell(S_i^l)$, so $\omega^{X_{\eta S_i^l}}(\widetilde{S}_i^l) \eqsim \omega^{X_{\widetilde{S}_i^l}}(\widetilde{S}_i^l)$ by the Harnack inequality in (2.18), whilst $\omega^{X_{\widetilde{S}_i^l}}(\widetilde{S}_i^l) \gtrsim 1$ by Lemma 5.14. Thus, there exists $c_0 \in (0, 1)$ depending only on the allowed constants such that $II_3' = \omega^{X_{\eta S_i^l}}(\widetilde{S}_i^l) \ge c_0$. Next, choose a different dyadic descendant $\underset{\sim}{S}_i^l \ne \widetilde{S}_i^l$ in $\mathbb{D}^\eta(S_i^l)$ that contains the centre of S_i^l. The preceding argument shows that $\omega^{X_{\eta S_i^l}}(\underset{\sim}{S}_i^l) \geq c_0$, whilst $\omega^{X_{\eta S_i^l}}(\underset{\sim}{S}_i^l \cap \widetilde{S}_i^l) \leq \omega^{X_{\eta S_i^l}}(\partial \widetilde{S}_i^l) = 0$; hence

$$c_0 \leq II_3' = \omega^{X_{\eta S_i^l}}(\widetilde{S}_i^l) = 1 - \omega^{X_{\eta S_i^l}}(\mathbb{R}^n \setminus \widetilde{S}_i^l) \leq 1 - \omega^{X_{\eta S_i^l}}(\underset{\sim}{S}_i^l) \leq 1 - c_0.$$

The above estimates together show that if $\epsilon_0 \in \left(0, \frac{1}{2}\right)$, then

$$c_0 \leq u(X_{\eta S_i^l}) \leq C_0 \eta^\alpha + 3C_\eta \epsilon_0 + 1 - c_0. \tag{5.27}$$

To estimate $u(X_{\eta \widehat{S}_i^l})$, write

$$u(X_{\eta \widehat{S}_i^l}) = \int_{\mathbb{R}^n \setminus \widehat{S}_i^l} \mathbb{1}_{\mathcal{B}}\, d\omega^{X_{\eta \widehat{S}_i^l}} + \int_{\widehat{S}_i^l} \mathbb{1}_{\mathcal{B}}\, d\omega^{X_{\eta \widehat{S}_i^l}} =: \hat{I} + \widehat{II},$$

as well as

$$\begin{aligned}\widehat{II} &= \sum_{j=2}^{l} \int_{\widehat{S}_i^l} \mathbb{1}_{\widetilde{O}_{j-1} \setminus O_j}\, d\omega^{X_{\eta \widehat{S}_i^l}} + \sum_{j=l+2}^{k} \int_{\widehat{S}_i^l} \mathbb{1}_{\widetilde{O}_{j-1} \setminus O_j}\, d\omega^{X_{\eta \widehat{S}_i^l}} + \int_{\widehat{S}_i^l} \mathbb{1}_{\widetilde{O}_l \setminus O_{l+1}}\, d\omega^{X_{\eta \widehat{S}_i^l}} \\ &=: \widehat{II}_1 + \widehat{II}_2 + \widehat{II}_3.\end{aligned}$$

The arguments used to estimate I, II_1 and II_2 show that $\hat{I} \leq \omega^{X_{\eta \widehat{S}_i^l}}(\mathbb{R}^n \setminus \widehat{S}_i^l) \leq C_0 \eta^\alpha$, $\widehat{II}_1 = 0$ and $\widehat{II}_2 \leq C_\eta \epsilon_0 / (1 - \epsilon_0)$. To estimate $\widehat{II}_3$, observe that

$$\widehat{S}_i^l \cap (\widetilde{O}_l \setminus O_{l+1}) = (\widehat{S}_i^l \cap \widetilde{S}_i^l) \setminus O_{l+1},$$

where either $\omega^{X_{\eta \widehat{S}_i^l}}(\widehat{S}_i^l \cap \widetilde{S}_i^l) = 0$ and $\widehat{II}_3 = 0$, or $\widehat{S}_i^l = \widetilde{S}_i^l$ and

$$\widehat{II}_3 = \int_{\widehat{S}_i^l} d\omega^{X_{\eta \widehat{S}_i^l}} - \int_{\widehat{S}_i^l \cap O_{l+1}} d\omega^{X_{\eta \widehat{S}_i^l}} =: \widehat{II}_3' - \widehat{II}_3''.$$

The boundary Hölder continuity estimate in (5.7) shows that

$$\widehat{II}_3' = \omega^{X_{\eta \widehat{S}_i^l}}(\widehat{S}_i^l) = 1 - \omega^{X_{\eta \widehat{S}_i^l}}(\mathbb{R}^n \setminus \widehat{S}_i^l) \geq 1 - C_0 \eta^\alpha,$$

whilst repeating the arguments used to estimate II_3'' shows that

$$\widehat{II}_3'' \leq \frac{C_\eta}{\omega^{X_{2Q_0}}(\widehat{S}_i^l)} \omega^{X_{2Q_0}}(O_{l+1} \cap \widehat{S}_i^l) \leq \frac{C_\eta}{\omega^{X_{2Q_0}}(S_i^l)} \omega^{X_{2Q_0}}(O_{l+1} \cap S_i^l) \leq C_\eta \epsilon_0.$$

These estimates together show that if $\epsilon_0 \in \left(0, \frac{1}{2}\right)$, then either

$$0 \leq u(X_{\eta \widehat{S}_i^l}) \leq C_0 \eta^\alpha + 3C_\eta \epsilon_0 \quad \text{or} \quad u(X_{\eta \widehat{S}_i^l}) \geq 1 - (C_0 \eta^\alpha + C_\eta \epsilon_0). \tag{5.28}$$

The estimates (5.27) and (5.28) together imply

$$|u(X_{\eta S_i^l}) - u(X_{\eta \widehat{S}_i^l})| \geq c_0 - 2C_0 \eta^\alpha - 4C_\eta \epsilon_0.$$

We thus obtain (5.26) by first choosing $\eta \in (0, 1)$ so that $2C_0 \eta^\alpha \leq \frac{1}{4} c_0$ and then choosing $\epsilon_0 \in \left(0, \frac{1}{2}\right)$ (depending on η) so that $4C_\eta \epsilon_0 \leq \frac{1}{4} c_0$. These choices of η and ϵ_0, which depend only on the allowed constants, are now fixed.

To complete the proof, suppose that $M \geq 1$ and $x \in E$, and recall that $\delta \in (0, \delta_0)$ remains to be chosen, where δ_0 is now fixed by our choice of ϵ_0 as in Lemma 5.23. First, fix a cube S^k in $\{S_i^k\}_{i\in\mathbb{N}}$ such that $x \in S^k$. The remarks after Definition 5.20 then imply that for each $l \in \{1, \dots, k-1\}$, there exists a unique cube S^l in $\{S_i^l\}_{i\in\mathbb{N}}$ such that $x \in S^l$ and $S^{l+1} \subset S^l$; thus $\ell(S^{l+1}) \leq \frac{1}{2}\ell(S^l)$. Next, for each $l \in \{1, \dots, k\}$, fix a dyadic descendant $\widehat{S}^l$ in $\mathbb{D}^\eta(S^l)$ such that $x \in \widehat{S}^l$.

Observe that, for some $\tau \in (0, 1)$ sufficiently close to 1 and depending only on η, the corkscrew points $X_{\eta S^l}$ and $X_{\eta \widehat{S}^l}$ both belong to the dilate τQ_η^l of the cube

$$Q_\eta^l := \big\{(y,t) \in \mathbb{R}^{n+1}_+ : |y-x|_\infty < \big(\tfrac{1}{2} + \tfrac{1}{4}\eta^2\big)\ell(S^l),\ \tfrac{1}{2}\eta^2\ell(S^l) < t < (1+\eta^2)\ell(S^l)\big\},$$

with $\ell(Q_\eta^l) = (1+\frac{1}{2}\eta^2)\ell(S^l)$. Therefore, if $c^l := \fint_{Q_\eta^l} u$, then the Moser-type estimate in (2.16), the Poincaré inequality in (2.5) and the doubling property of μ show that

$$\begin{aligned} |u(X_{\eta S^l}) - u(X_{\eta \widehat{S}^l})|^2 &\lesssim |u(X_{\eta S^l}) - c^l|^2 + |u(X_{\eta \widehat{S}^l}) - c^l|^2 \lesssim \|u - c^l\|^2_{L^\infty(\tau Q_\eta^l)} \\ &\lesssim_\eta \fint_{Q_\eta^l} |u - c^l|^2\, d\mu \lesssim \ell(Q_\eta^l)^2 \fint_{Q_\eta^l} |\nabla u|^2\, d\mu \\ &\lesssim \frac{\ell(S^l)}{\mu\big(\Delta\big(x, \big(1+\frac{1}{2}\eta^2\big)\ell(S^l)\big)\big)} \int_{Q_\eta^l} |\nabla u|^2\, d\mu \\ &\lesssim \iint_{Q_\eta^l} |t\nabla u(y,t)|^2 \frac{d\mu(y)}{\mu(\Delta(x,t))}\frac{dt}{t}. \end{aligned} \tag{5.29}$$

Iterating the bound $\ell(S^{l+1}) \leq \frac{1}{2}\ell(S^l)$ shows that $\ell(S^{l'}) \leq 2^{l-l'}\ell(S^l)$ when $l' \geq l$. This implies that the collection $\{Q_\eta^1, \dots, Q_\eta^k\}$ has the bounded intersection property whereby, for each $l \in \{1, \dots, k\}$, there are at most $3 + 2\log_2(1/\eta^2 + 1))$ such cubes $Q_\eta^{l'}$ satisfying $Q_\eta^{l'} \cap Q_\eta^l \neq \varnothing$. This allows us to sum estimate (5.29) over $l \in \{1, \dots, k\}$ and then apply (5.26) to obtain

$$k \lesssim_\eta \iint_{\bigcup_{l=1}^k Q_\eta^l} |t\nabla u(y,t)|^2 \frac{d\mu(y)}{\mu(\Delta(x,t))}\frac{dt}{t} \lesssim \int_0^{\gamma\ell(Q)}\!\!\int_{\Delta(x,\gamma t)} |t\nabla u(y,t)|^2 \frac{d\mu(y)}{\mu(\Delta(x,t))}\frac{dt}{t}$$

for some $\gamma > 0$ that depends only on $\eta > 0$ and thus only on the allowed constants.

To conclude, recall that $k \gtrsim \log(\omega^{X_{2Q_0}}(E)^{-1})/\log(1/\epsilon_0) \geq \log(1/\delta)/\log(1/\epsilon_0)$, since $\omega^{X_{2Q_0}}(E) \leq \delta < 1$. Therefore, the result follows by choosing $\delta \in (0, \delta_0]$ such that $M \leq \log(1/\delta)$, since $\delta_M := \delta$ depends only on M and the allowed constants. □

We now combine the above technical lemma with the Carleson measure estimate from Theorem 1.3 to prove the main A_∞-estimate for degenerate elliptic measure.

Theorem 5.30. *Suppose that Q_0 is a cube in $\mathbb{R}^n$. If $X \in \mathbb{R}^{n+1}_+ \setminus T_{Q_0}$ and $\omega := \omega^X \lfloor Q_0$ denotes the degenerate elliptic measure restricted to Q_0, then $\omega \in A_\infty(\mu)$ and the following equivalent properties hold*:

(1) *For each $\epsilon \in (0,1)$, there exists $\delta \in (0,1)$, depending only on ϵ, n, λ, Λ and $[\mu]_{A_2}$, such that the following property holds*: *if $Q \subseteq Q_0$ is a cube and $E \subseteq Q$ such that $\omega(E) \leq \delta\omega(Q)$, then $\mu(E) \leq \epsilon\mu(Q)$.*

(2) *The measure ω is absolutely continuous with respect to μ and there exists $q \in (1, \infty)$ such that the Radon–Nikodym derivative $k := d\omega/d\mu$ satisfies, on all surface balls $\Delta \subseteq Q_0$, the reverse Hölder estimate*

$$\left(\fint_\Delta k^q \, d\mu \right)^{1/q} \lesssim \fint_\Delta k \, d\mu,$$

where q and the implicit constant depend only on n, λ, Λ and $[\mu]_{A_2}$.

(3) *There exist $C, \theta > 0$, depending only on n, λ, Λ and $[\mu]_{A_2}$, such that*

$$\omega(E) \le C \left(\frac{\mu(E)}{\mu(Q)} \right)^\theta \omega(Q)$$

for all cubes $Q \subseteq Q_0$ and all Borel sets $E \subseteq Q$.

Proof. It is well known that (1)–(3) are equivalent; see Theorem 1.4.13 in [Kenig 1994]. Moreover, by Lemma 5.16, it suffices to prove (1) when $X = X_{2Q_0}$. In that case, by Lemma 5.24, the Carleson measure estimate in Theorem 1.3, Fubini's theorem and the doubling property of μ, it follows that for each $M \ge 1$, there exists $\delta_M > 0$, depending only on M and the allowed constants, such that the following property holds: if $Q \subseteq Q_0$ is a cube and $E \subseteq Q$ such that $\omega(E) \le \delta_M \omega(Q)$, then there exists a solution u of the equation $\operatorname{div}(A\nabla u) = 0$ in $\mathbb{R}^{n+1}_+$ with $\|u\|_\infty \le 1$ such that

$$\begin{aligned} M\mu(E) &\le \int_E \int_0^{\gamma\ell(Q)} \int_{\Delta(x,\gamma t)} |t\nabla u(y,t)|^2 \frac{d\mu(y)}{\mu(\Delta(x,t))} \frac{dt}{t} \, d\mu(x) \\ &\lesssim \int_0^{\tilde{\gamma}\ell(Q)} \int_{\tilde{\gamma}Q} |t\nabla u(y,t)|^2 \, d\mu(y) \frac{dt}{t} \lesssim \mu(Q), \end{aligned}$$

where the implicit constants and $\tilde{\gamma} > \gamma > 0$ depend only on the allowed constants. Therefore, if $\epsilon \in (0, 1)$, we choose $M(\epsilon) \ge 1$ and thus $\delta_{M(\epsilon)} \in (0, 1)$, depending only on ϵ and the allowed constants, such that $\mu(E) \le \epsilon\mu(Q)$, as required. □

5E. *The square function and nontangential maximal function estimates.* The $L^p_\mu(\mathbb{R}^n)$-norm equivalence between the square function Su and the nontangential maximal function N_*u of solutions u in Theorem 1.5 is now a corollary of the main A_∞-estimate for the degenerate elliptic measure in Theorem 5.30. This was proved by Dahlberg, Jerison and Kenig in Theorem 1 of [Dahlberg et al. 1984], which actually provides the more general result in Theorem 5.31 below. In particular, the degenerate elliptic case is treated on page 106 of the same paper, noting that the normalisation $u(X_0) = 0$ assumed therein is actually only required for the so-called $N \lesssim S$-estimate.

Theorem 5.31. *Suppose that $\Phi : [0,\infty) \to [0,\infty)$ is an unbounded, nondecreasing, continuous function with $\Phi(0) = 0$ and $\Phi(2t) \le C\Phi(t)$ for all $t > 0$ and some $C > 0$. If $\operatorname{div}(A\nabla u) = 0$ in $\mathbb{R}^{n+1}_+$, then*

$$\int_{\mathbb{R}^n} \Phi(Su) \, d\mu \lesssim \int_{\mathbb{R}^n} \Phi(N_*u) \, d\mu,$$

and if, in addition, $u(X_0) = 0$ *for some* $X_0 \in \mathbb{R}^{n+1}_+$, *then*

$$\int_{\mathbb{R}^n} \Phi(N_* u)\, d\mu \lesssim \int_{\mathbb{R}^n} \Phi(Su)\, d\mu,$$

where the implicit constants depend only on X_0, Φ, n, λ, Λ *and* $[\mu]_{A_2}$.

The next result is also a consequence of the main A_∞-estimate in Theorem 5.30. It will allow us to construct solutions to the Dirichlet problem $(D)_{p,\mu}$ as integrals of $L^p_\mu(\mathbb{R}^n)$-boundary data with respect to degenerate elliptic measure.

Lemma 5.32. *Suppose that* $1/p + 1/q = 1$, *where* $q \in (1,\infty)$ *is the reverse Hölder exponent from Theorem 5.30. If* $X = (x,t) \in \mathbb{R}^{n+1}_+$, *then the Radon–Nikodym derivative* $k(X,\cdot) := d\omega^X/d\mu$ *is in* $L^q_\mu(\mathbb{R}^n)$ *and*

$$\int_{\mathbb{R}^n} k((x,t),y)^q\, d\mu(y) \lesssim \mu(\Delta(x,t))^{1-q}.$$

Moreover, if $f \in L^p_\mu(\mathbb{R}^n)$ *and* $u(X) := \int_{\mathbb{R}^n} f(y)\, d\omega^X$, *then* $\|N_* u\|_{L^p_\mu(\mathbb{R}^n)} \lesssim \|f\|_{L^p_\mu(\mathbb{R}^n)}$. *The implicit constant in each estimate depends only on* n, λ, Λ *and* $[\mu]_{A_2}$.

Proof. Suppose that $X = (x,t) \in \mathbb{R}^{n+1}_+$. The proof of Proposition 5.18 shows that

$$k((x,t),y) \lesssim 2^{-j\alpha} \frac{k((x,2^j t),y)}{\omega^{(x,2^j t)}(\Delta(x,2^j t))} \quad \text{for all } y \in \Delta(x,2^j t) \setminus \Delta(x,2^{j-1}t), \text{ for all } j \in \mathbb{N}.$$

Applying the reverse Hölder estimate from Theorem 5.30 then shows that

$$\begin{aligned}\int_{\mathbb{R}^n} k((x,t),y)^q\, d\mu(y) &= \int_{\Delta(x,t)} k((x,t),y)^q\, d\mu(y) + \sum_{j=1}^{\infty} \int_{\Delta(x,2^j t)\setminus\Delta(x,2^{j-1}t)} k((x,t),y)^q\, d\mu(y)\\ &\lesssim \mu(\Delta(x,t))^{1-q} + \sum_{j=1}^{\infty} 2^{-j\alpha q} \mu(\Delta(x,2^j t))^{1-q} \lesssim \mu(\Delta(x,t))^{1-q}.\end{aligned}$$

To obtain the nontangential maximal function estimate, it suffices to consider the case when $f \geq 0$, since in general we may then decompose $f = f^+ - f^-$ into its positive and negative parts $f^+, f^- \geq 0$. To this end, suppose that $x_0 \in \mathbb{R}^n$ and that $X = (x,t) \in \mathbb{R}^{n+1}_+$ in order to write

$$f = f\mathbb{1}_{\Delta(x_0,2t)} + \sum_{j=1}^{\infty} f\mathbb{1}_{\Delta(x_0,2^{j+1}t)\setminus\Delta(x_0,2^j t)} =: \sum_{j=0}^{\infty} f_j$$

and define

$$u_j(X) := \int_{\mathbb{R}^n} f_j(y)\, d\omega^X(y) = \int_{\mathbb{R}^n} f_j(y)\, k(X,y)\, d\mu(y).$$

The self-improvement property of the reverse Hölder estimate from Theorem 5.30 (see Theorem 1.4.13 in [Kenig 1994]) implies that there exists an exponent $r > q$ such that

$$\left(\fint_{\Delta} k((x,t),y)^r\, d\mu(y)\right)^{1/r} \lesssim \fint_{\Delta} k((x,t),y)\, d\mu(y) \leq \frac{1}{\mu(\Delta)} \tag{5.33}$$

for all surface balls $\Delta \subseteq \Delta\big(x, \tfrac{1}{2}t\big)$.

Now suppose that $X = (x, t) \in \Gamma(x_0)$. To estimate u_0, we apply the interior Harnack inequality in (2.18) followed by Hölder's inequality and (5.33) to obtain

$$\begin{aligned} u_0(x,t) \eqsim u_0(x,6t) &\le \int_{\Delta(x_0,2t)} f(y)\,k((x,6t),y)\,d\mu(y) \\ &\le \Big(\int_{\Delta(x_0,2t)} |k((x,6t),y)|^r\,d\mu(y)\Big)^{1/r}\Big(\int_{\Delta(x_0,2t)} f(y)^{r'}\,d\mu(y)\Big)^{1/r'} \\ &\lesssim \mu(\Delta(x_0,2t))^{-1/r'}\Big(\int_{\Delta(x_0,2t)} f(y)^{r'}\,d\mu(y)\Big)^{1/r'} \\ &\le [M_\mu(f^{r'})(x_0)]^{1/r'}. \end{aligned}$$

To estimate u_j when $j \in \mathbb{N}$, we apply the boundary Hölder continuity estimate from (5.7) and then proceed as in the estimate above to obtain

$$\begin{aligned} u_j(x,t) &\lesssim \Big(\frac{t}{2^j t}\Big)^{\alpha} u_j(x_0,2^j t) \eqsim 2^{-j\alpha} u_j(x_0,2^{j+2}t) \\ &\le 2^{-j\alpha}\int_{\Delta(x_0,2^{j+1}t)} f(y)\,k((x_0,2^{j+2}t),y)\,d\mu(y) \\ &\le 2^{-j\alpha}\Big(\int_{\Delta(x_0,2^{j+1}t)} k((x_0,2^{j+2}t),y)^r\,d\mu(y)\Big)^{1/r}\Big(\int_{\Delta(x_0,2^{j+1}t)} f(y)^{r'}\,d\mu(y)\Big)^{1/r'} \\ &\lesssim 2^{-j\alpha}\Big(\fint_{\Delta(x_0,2^{j+1}t)} f(y)^{r'}\,d\mu(y)\Big)^{1/r'} \\ &\le 2^{-j\alpha}[M_\mu(f^{r'})(x_0)]^{1/r'}. \end{aligned}$$

The above estimates together show that

$$N_* u(x_0) \lesssim [M_\mu(f^{r'})(x_0)]^{1/r'}$$

for all $x_0 \in \mathbb{R}^n$, and since $r' < q' = p$, it follows that $\|N_* u\|_{L^p_\mu} \lesssim \|f\|_{L^p_\mu}$, as required. □

We conclude the paper by using the preceding lemma to obtain solvability of the Dirichlet problem $(D)_{p,\mu}$. A uniqueness result is also obtained but only for solutions that converge uniformly to 0 at infinity. This restriction does not appear in the uniformly elliptic case; see Theorem 1.7.7 in [Kenig 1994]. It arises here because of the absence of a Green's function for degenerate elliptic equations on unbounded domains (see Section 5C) and it is not clear to us whether this can be improved.

Theorem 5.34. *Suppose that $1/p + 1/q = 1$, where $q \in (1,\infty)$ is the reverse Hölder exponent from Theorem 5.30. The Dirichlet problem for $L^p_\mu(\mathbb{R}^n)$-boundary data is solvable in the sense that for each $f \in L^p_\mu(\mathbb{R}^n)$, there exists a solution u such that*

$$\begin{cases} \operatorname{div}(A\nabla u) = 0 \quad \text{in } \mathbb{R}^{n+1}_+, \\ N_* u \in L^p_\mu(\mathbb{R}^n), \\ \lim_{t\to 0} u(\cdot,t) = f, \end{cases} \tag{$(D)_{p,\mu}$}$$

where the limit converges in $L^p_\mu(\mathbb{R}^n)$-norm and in the nontangential sense whereby

$$\lim_{\Gamma(x)\ni(y,t)\to(x,0)} u(y,t) = f(x)$$

for almost every $x \in \mathbb{R}^n$. Moreover, if f has compact support, then there is a unique solution u of $(D)_{p,\mu}$ that converges uniformly to 0 at infinity in the sense that $\lim_{R\to\infty} \|u\|_{L^\infty(\mathbb{R}^{n+1}_+ \setminus B(0,R))} = 0$.

Proof. Suppose that $f \in L^p_\mu(\mathbb{R}^n)$ and define $u(X) := \int_{\mathbb{R}^n} f \, d\omega^X$ for all $X \in \mathbb{R}^{n+1}_+$. We first prove that $\operatorname{div}(A\nabla u) = 0$ in $\mathbb{R}^{n+1}_+$. Let $(f_j)_j$ denote a sequence in $C_c(\mathbb{R}^n)$ that converges to f in $L^p_\mu(\mathbb{R}^n)$ and consider the solutions $u_j(X) := \int_{\mathbb{R}^n} f_j \, d\omega^X$. The $L^q(\mathbb{R}^n)$-estimate for the Radon–Nikodym derivative $d\omega^X/d\mu$ from Lemma 5.32 and the doubling property of μ show that

$$\|u_j - u\|_{L^\infty(K)} \lesssim_{\mu,K} \|f_j - f\|_{L^p_\mu(\mathbb{R}^n)}$$

for all $j \in \mathbb{N}$ and any compact set $K \subset \mathbb{R}^{n+1}_+$, so u_j converges to u in $L^2_{\mu,\mathrm{loc}}(\mathbb{R}^n)$. Moreover, Cacioppoli's inequality and the arguments preceding (5.4) show that u_j converges to a solution v in $W^{1,2}_{\mu,\mathrm{loc}}(\mathbb{R}^n)$, so then $u = v$ is a solution in $\mathbb{R}^{n+1}_+$ as required.

The nontangential maximal function estimate $\|N_* u\|_{L^p_\mu(\mathbb{R}^n)} \lesssim \|f\|_{L^p_\mu(\mathbb{R}^n)}$ is given by Lemma 5.32. To prove the nontangential convergence to the boundary datum, first recall that $u_j \in C(\bar{\mathbb{R}}^{n+1}_+)$ with $u_j|_{\mathbb{R}^n} := f_j$, so $\lim_{\Gamma(x)\ni(y,t)\to(x,0)} u_j(y,t) = f_j(x)$ (see Section 5B). We combine this fact with the bound

$$|u(y,t) - f(x)| \le |u(y,t) - u_j(y,t)| + |u_j(y,t) - f_j(x)| + |(f_j - f)(x)|$$

to obtain

$$\limsup_{\Gamma(x)\ni(y,t)\to(x,0)} |u(y,t) - f(x)| \le |N_*(u - u_j)(x)| + |(f - f_j)(x)|$$

for all $x \in \mathbb{R}^n$. For any $\eta > 0$, we then apply Chebyshev's inequality and the nontangential maximal function estimate from Lemma 5.32 to show that

$$\begin{aligned}
\mu\Big(\Big\{x \in \mathbb{R}^n : \limsup_{\Gamma(x)\ni(y,t)\to(x,0)} |u(y,t) - f(x)| > \eta\Big\}\Big) \\
&\le \mu\big(\big\{x \in \mathbb{R}^n : N_*(u - u_j)(x) > \tfrac{1}{2}\eta\big\}\big) + \mu\big(\big\{x \in \mathbb{R}^n : |(f - f_j)(x)| > \tfrac{1}{2}\eta\big\}\big) \\
&\lesssim \eta^{-p}\big(\|N_*(u - u_j)\|^p_{L^p_\mu(\mathbb{R}^n)} + \|f - f_j\|^p_{L^p_\mu(\mathbb{R}^n)}\big) \\
&\lesssim \eta^{-p}\|f - f_j\|^p_{L^p_\mu(\mathbb{R}^n)}.
\end{aligned}$$

It follows, since f_j converges to f in $L^p_\mu(\mathbb{R}^n)$, that

$$\lim_{\Gamma(x)\ni(y,t)\to(x,0)} u(y,t) = f(x)$$

for almost every $x \in \mathbb{R}^n$, as required. The norm convergence $\lim_{t\to 0} \|u(\cdot\,,t) - f\|_{L^p_\mu(\mathbb{R}^n)}$ then follows by Lebesgue's dominated convergence theorem.

It remains to prove that u is the unique solution satisfying $\lim_{|X|\to\infty} \|u(X)\|_\infty = 0$ when f has compact support. In that case, fix $R_0 > 0$ such that f is supported in the surface ball $\Delta(0, R_0)$. If

$X \in \mathbb{R}^{n+1}_+$ and $|X| > 2R_0$, then the reverse Hölder estimate in Theorem 5.30 shows that

$$\begin{aligned} |u(X)| &\le \int_{\Delta(0,R_0)} |f(y)|\, k(X,y)\, d\mu(y) \\ &\le \|f\|_{L^p_\mu(\mathbb{R}^n)} \biggl(\int_{\Delta(0,|X|/2)} k(X,y)^q\, d\mu(y) \biggr)^{1/q} \\ &\lesssim \|f\|_{L^p_\mu(\mathbb{R}^n)} \mu\bigl(\Delta(0, \tfrac{1}{2}|X|)\bigr)^{1/q} ⨍_{\Delta(0,|X|/2)} k(X,y)\, d\mu(y) \\ &\le \|f\|_{L^p_\mu(\mathbb{R}^n)} \mu\bigl(\Delta(0, \tfrac{1}{2}|X|)\bigr)^{-1/p}, \end{aligned}$$

whilst $\lim_{R\to\infty} \mu(\Delta(0,R)) = \infty$, since μ is in the A_∞-class with respect to Lebesgue measure on $\mathbb{R}^n$; thus $\lim_{R\to\infty} \|u\|_{L^\infty(\mathbb{R}^{n+1}_+ \setminus B(0,R))} = 0$. The maximum principle allows us to conclude that any solution of $(D)_{p,\mu}$ with this decay must be unique. □

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Received 2 May 2018. Revised 17 Oct 2018. Accepted 30 Nov 2018.

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Analysis & PDE

Volume 12 No. 8 2019

2157-5045(2019)12:8;1-I