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DISCRETELY SELF-SIMILAR SOLUTIONS TO THE NAVIER–STOKES EQUATIONS WITH DATA IN L^2_{loc} SATISFYING THE LOCAL ENERGY INEQUALITY

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Chae and Wolf recently constructed discretely self-similar solutions to the Navier–Stokes equations for any discretely self-similar data in L^2_{loc}. Their solutions are in the class of local Leray solutions with projected pressure and satisfy the "local energy inequality with projected pressure". In this note, for the same class of initial data, we construct discretely self-similar suitable weak solutions to the Navier–Stokes equations that satisfy the classical local energy inequality of Scheffer and Caffarelli–Kohn–Nirenberg. We also obtain an explicit formula for the pressure in terms of the velocity. Our argument involves a new purely local energy estimate for discretely self-similar solutions with data in L^2_{loc} and an approximation of divergence-free, discretely self-similar vector fields in L^2_{loc} by divergence-free, discretely self-similar elements of L^3_w.

1. Introduction

The Navier–Stokes equations describe the evolution of a viscous incompressible fluid's velocity field v and associated scalar pressure π. In particular, v and π are required to satisfy

$$\partial_t v - \Delta v + v \cdot \nabla v + \nabla \pi = 0, \tag{1-1}$$

$$\nabla \cdot v = 0, \tag{1-2}$$

in the sense of distributions. For our purposes, (1-1) is applied on $\mathbb{R}^3 \times (0, \infty)$ and v evolves from a prescribed, divergence-free initial data $v_0 : \mathbb{R}^3 \to \mathbb{R}^3$. Solutions to (1-1) exhibit a natural scaling: if v satisfies (1-1), then for any $\lambda > 0$

$$v^\lambda(x, t) = \lambda v(\lambda x, \lambda^2 t) \tag{1-3}$$

is also a solution with pressure

$$\pi^\lambda(x, t) = \lambda^2 \pi(\lambda x, \lambda^2 t) \tag{1-4}$$

and initial data

$$v_0^\lambda(x) = \lambda v_0(\lambda x). \tag{1-5}$$

A solution is called self-similar (SS) if $v^\lambda(x, t) = v(x, t)$ for all $\lambda > 0$ and is discretely self-similar with factor λ (i.e., v is λ-DSS) if this scaling invariance holds for a given $\lambda > 1$. Similarly, v_0 is self-similar (a.k.a. (-1)-homogeneous) if $v_0(x) = \lambda v_0(\lambda x)$ for all $\lambda > 0$ or λ-DSS if this holds for a given $\lambda > 1$.

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These solutions can be either forward or backward if they are defined on $\mathbb{R}^3 \times (0, \infty)$ or $\mathbb{R}^3 \times (-\infty, 0)$ respectively. In this note we work exclusively with forward solutions and omit the qualifier "forward".

Self-similar solutions satisfy an ansatz for v in terms of a time-independent profile u, namely,

$$v(x,t) = \frac{1}{\sqrt{t}} u\left(\frac{x}{\sqrt{t}}\right), \tag{1-6}$$

where u solves the *Leray equations*

$$\begin{aligned} -\Delta u - \tfrac{1}{2}u - \tfrac{1}{2}y \cdot \nabla u + u \cdot \nabla u + \nabla p = 0, \\ \nabla \cdot u = 0 \end{aligned} \quad \text{in } \mathbb{R}^3, \tag{1-7}$$

in the variable $y = x/\sqrt{t}$. Discretely self-similar solutions are determined by their behavior on the time interval $1 \le t \le \lambda^2$ and satisfy the ansatz

$$v(x,t) = \frac{1}{\sqrt{t}} u(y,s), \tag{1-8}$$

where

$$y = \frac{x}{\sqrt{t}}, \quad s = \log t. \tag{1-9}$$

The vector field u is T-periodic with period $T = 2\log\lambda$ and solves the *time-dependent Leray equations*

$$\begin{aligned} \partial_s u - \Delta u - \tfrac{1}{2}u - \tfrac{1}{2}y \cdot \nabla u + u \cdot \nabla u + \nabla p = 0, \\ \nabla \cdot u = 0 \end{aligned} \quad \text{in } \mathbb{R}^3 \times \mathbb{R}. \tag{1-10}$$

Note that the *similarity transform* (1-8)–(1-9) gives a one-to-one correspondence between solutions to (1-1) and (1-10). Moreover, when v_0 is SS or DSS, the initial condition $v|_{t=0} = v_0$ corresponds to a boundary condition for u at spatial infinity; see [Korobkov and Tsai 2016; Bradshaw and Tsai 2017a; 2017b].

Self-similar solutions are interesting in a variety of contexts as candidates for ill-posedness or finite time blow-up of solutions to the 3-dimensional Navier–Stokes equations; see [Guillod and Šverák 2017; Jia and Šverák 2014; 2015; Leray 1934; Nečas et al. 1996; Tsai 1998] and the discussion in [Bradshaw and Tsai 2017a]. Forward self-similar solutions are compelling candidates for nonuniqueness [Jia and Šverák 2015; Guillod and Šverák 2017]. Until recently, the existence of forward self-similar solutions was only known for small data [Barraza 1996; Cannone and Planchon 1996; Giga and Miyakawa 1989; Koch and Tataru 2001; Kato 1992]. Such solutions are necessarily unique. Jia and Šverák [2014] constructed forward self-similar solutions for large data where the data is assumed to be Hölder continuous away from the origin. This result has been generalized in a number of directions by a variety of authors [Bradshaw and Tsai 2017a; 2017b; 2018; Chae and Wolf 2018; Korobkov and Tsai 2016; Lemarié-Rieusset 2016; Tsai 2014]. This paper can be understood in the context of [Bradshaw and Tsai 2017a; Chae and Wolf 2018; Lemarié-Rieusset 2016] and we briefly recall the main results of these papers.

In [Bradshaw and Tsai 2017a], we generalize [Jia and Šverák 2014] in two ways. First, all smoothness assumptions on the initial data are removed; we only require $v_0 \in L^3_w$ (and v_0 divergence-free and SS or DSS). Second, we allow the data to be DSS for any $\lambda > 1$, in which case we obtain DSS solutions as opposed to SS solutions — in contrast, the method of [Jia and Šverák 2014] can be adapted to give DSS solutions but only when λ is close to 1 [Tsai 2014]. The method of proof in [Bradshaw and Tsai 2017a] has since been extended to the half-space in [Bradshaw and Tsai 2017b] and to initial data in the Besov spaces $\dot{B}^{3/p-1}_{p,\infty}$ when $3 < p < 6$ [Bradshaw and Tsai 2018]. Solutions which satisfy a rotationally corrected scaling invariance are also constructed in [Bradshaw and Tsai 2017b].

The solutions of [Bradshaw and Tsai 2017a] belong to the class of *local Leray solutions*. This class was introduced in [Lemarié-Rieusset 2002] to provide a local analogue of Leray's weak solutions [1934]. We recall the definition of local Leray solutions in full. For $q \in [1, \infty)$, we say $f \in L^q_{\text{uloc}}$ if

$$\|f\|_{L^q_{\text{uloc}}} = \sup_{x\in\mathbb{R}^3} \|f\|_{L^q(B(x,1))} < \infty.$$

Definition 1.1 (local Leray solutions). A vector field $v \in L^2_{\text{loc}}(\mathbb{R}^3 \times [0, \infty))$ is a local Leray solution to (1-1) with divergence-free initial data $v_0 \in L^2_{\text{uloc}}$ if:

(1) For some $\pi \in L^{3/2}_{\text{loc}}(\mathbb{R}^3 \times [0, \infty))$, the pair (v, π) is a distributional solution to (1-1).

(2) For any $R > 0$, the vector field v satisfies

$$\operatorname*{ess\,sup}_{0\le t<R^2} \sup_{x_0\in\mathbb{R}^3} \int_{B_R(x_0)} \tfrac{1}{2}|v(x,t)|^2\,dx + \sup_{x_0\in\mathbb{R}^3} \int_0^{R^2}\int_{B_R(x_0)} |\nabla v(x,t)|^2\,dx\,dt < \infty.$$

(3) For all compact subsets K of $\mathbb{R}^3$ we have $v(t) \to v_0$ in $L^2(K)$ as $t \to 0^+$.

(4) v is suitable in the sense of Caffarelli–Kohn–Nirenberg; i.e., for all cylinders Q compactly supported in $\mathbb{R}^3 \times (0, \infty)$ and all nonnegative $\phi \in C^\infty_0(Q)$, we have

$$\int |v(t)|^2\phi\,dx+2\iint |\nabla v|^2\phi\,dx\,dt \le \iint |v|^2(\partial_t\phi+\Delta\phi)\,dx\,dt +\iint (|v|^2+2\pi)(v\cdot\nabla\phi)\,dx\,dt. \quad (1\text{-}11)$$

(5) For every $x_0 \in \mathbb{R}^3$, there exists $c_{x_0} \in L^{3/2}(0, T)$ such that

$$p(x,t) - c_{x_0}(t) = -\frac{1}{3}|v(x,t)|^2 + \frac{1}{4\pi}\int_{B_2(x_0)} K(x-y) : v(y,t)\otimes v(y,t)\,dy + \frac{1}{4\pi}\int_{\mathbb{R}^3\setminus B_2(x_0)} (K(x-y) - K(x_0-y)) : v(y,t)\otimes v(y,t)\,dy$$

in $L^{3/2}(0, T; L^{3/2}(B_1(x_0)))$, where $K(x) = \nabla^2(1/|x|)$.

Lemarié-Rieusset [2002] constructed global-in-time local Leray solutions if v_0 belongs to E^2, the closure of C^∞_0 in the $L^2_{\text{uloc}}(\mathbb{R}^3)$ norm. See [Kikuchi and Seregin 2007] for another construction which treats the pressure carefully. Note that [Lemarié-Rieusset 2002; Kikuchi and Seregin 2007; Jia and Šverák 2014; 2015] contain alternative definitions of local Leray solutions. On one hand, [Kikuchi and Seregin 2007] requires the pressure satisfy a certain formula (we will establish a similar pressure formula for our solutions; see Theorem 1.2). In [Jia and Šverák 2014; 2015], the explicit pressure formula is replaced by a decay condition imposed on the solution at spatial infinity, namely, for all $R > 0$

$$\lim_{|x_0|\to\infty} \int_0^{R^2}\int_{B(x_0,R)} |v|^2\,dx\,dt = 0.$$

Jia and Šverák [2014; 2015] claim that, if v exhibits this decay, then the pressure formula from [Kikuchi and Seregin 2007] is valid. Since the decay property is easier to directly establish for a given solution, this justifies using it in place of the explicit pressure formula in the definition of local Leray solutions. It turns out that these properties are equivalent when $v_0 \in E^2$. This can be proved using ideas contained in a recent preprint of Maekawa, Miura, and Prange [Maekawa et al. 2019] on the construction of local energy solutions in the half-space.

Local Leray solutions are known to satisfy a useful a priori bound. Let $\mathcal{N}(v_0)$ denote the class of local Leray solutions with initial data v_0. The following estimate is well known for local Leray solutions (see [Jia and Šverák 2014]): for all $\tilde{v} \in \mathcal{N}(v_0)$ and $r > 0$ we have

$$\operatorname*{ess\,sup}_{0\le t\le \sigma r^2} \sup_{x_0\in\mathbb{R}^3} \int_{B_r(x_0)} \tfrac{1}{2}|\tilde{v}(x,t)|^2\,dx + \sup_{x_0\in\mathbb{R}^3} \int_0^{\sigma r^2}\int_{B_r(x_0)} |\nabla\tilde{v}|^2\,dx\,dt < CA, \tag{1-12}$$

where

$$A = \sup_{x_0\in\mathbb{R}^3} \int_{B_r(x_0)} \tfrac{1}{2}|v_0|^2\,dx, \quad \sigma(r) = c_0 \min\{r^2A^{-2}, 1\}, \tag{1-13}$$

for a small universal positive constant c_0.

Concurrently to the publication of [Bradshaw and Tsai 2017a], the book [Lemarié-Rieusset 2016] was published, which includes a chapter on the self-similar solutions of [Jia and Šverák 2014]. Here, Lemarié-Rieusset generalizes the space of initial data to include any L^2_{loc}, divergence-free, self-similar vector field. The main elements of his argument are as follows. He first uses the Leray–Schauder approach of [Jia and Šverák 2014] to construct self-similar solutions for initial data v_0 satisfying $|v_0(x)| \lesssim |x|^{-1}$. This construction is more general than that in [Jia and Šverák 2014] but less general than that in [Bradshaw and Tsai 2017a]. But, provided v_0 is self-similar, $v_0 \in L^2_{\mathrm{loc}}$ if and only if $v_0 \in L^2_{\mathrm{uloc}}$. And, furthermore, if v_0 is self-similar and belongs to L^2_{uloc}, then it can be approximated by a sequence $v_0^{(k)}$ where each $|v_0^{(k)}(x)| \lesssim |x|^{-1}$. Then, the first construction gives local Leray solutions for each $v_0^{(k)}$ and, because local Leray solutions satisfy the a priori bound (1-12) depending only on the L^2_{uloc} norm of their initial data, these will converge to an SS local Leray solution with L^2_{loc} data. This argument breaks down for DSS solutions since $L^2_{\mathrm{loc}} \cap \mathrm{DSS} \neq L^2_{\mathrm{uloc}} \cap \mathrm{DSS}$ (see (1-15) for an example) and, therefore, we cannot get the uniform bound (1-12) on a sequence of approximating solutions for free.

Chae and Wolf [2018], on the other hand, introduced an entirely new method to construct λ-DSS solutions for any $\lambda > 1$ and initial data $v_0 \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$. These solutions live in the class of "local Leray solutions with projected pressure", which means they satisfy a modified local energy inequality instead of the classical local energy inequality (1-11) of [Caffarelli et al. 1982]. To construct these solutions, Chae and Wolf use a fixed-point argument to solve the mollified Navier–Stokes equations (this is the same system studied in [Bradshaw and Tsai 2017a], but written in physical variables as opposed to the similarity variables, see (3-4) and (3-5)). To apply the fixed-point argument, Chae and Wolf first prove existence for the (mollified) linearized equations where the given drift velocity is DSS. They then apply a fixed-point theorem (the space for the fixed-point argument is a bounded set of the DSS subspace of $L^{18/5}(0,T;L^3(B_1))$ — B_r denotes the ball of radius r centered at the origin — defined below [Chae and Wolf 2018, (3.1)]) to prove that there exists a drift velocity which matches the solution. This gives existence of a DSS solution to the mollified Navier–Stokes equations. Note that the approximations satisfy the a priori (energy) bound [Chae and Wolf 2018, (2.35)] and the norm of the mollification term can be absorbed for T sufficiently small.

In this paper we give a simple, alternative proof of the result in [Chae and Wolf 2018]. The following theorem is our main result.

Theorem 1.2. *Assume $v_0 \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ is a divergence-free λ-DSS vector field for some $\lambda > 1$. Then there exists a λ-DSS distributional solution v to* (1-1) *and associated pressure π so that v is suitable in the sense of [Caffarelli et al. 1982] and satisfies*

$$\lim_{t\to 0^+} \|v(t) - v_0\|_{L^2(K)} = 0$$

for every compact subset K of $\mathbb{R}^3$. Moreover, for any $T > 0$ and compact subset K of $\mathbb{R}^3$, we have $v \in L^\infty(0, T; L^2(K)) \cap L^2(0, T; H^1(K))$ and $\pi \in L^{3/2}(0, T; L^{3/2}(K))$. Furthermore, for any $(x, t) \in \mathbb{R}^3 \times (0, \infty)$, the pressure satisfies the formula

$$\pi(x, t) = -\tfrac{1}{3}|v|^2(x, t) + \lim_{\delta\to 0} \int_{|y|>\delta} K_{ij}(x - y) v_i(y, t) v_j(y, t)\, dy \tag{1-14}$$

in $L^{3/2}_{\mathrm{loc}}(\mathbb{R}^3 \times (0, \infty))$.

Comments on Theorem 1.2. (1) In [Chae and Wolf 2018], the data also belongs to L^2_{loc}, but the solution is not shown to satisfy the local energy inequality of [Caffarelli et al. 1982]. Instead, it satisfies a "local energy inequality with projected pressure". Since the solution constructed in Theorem 1.2 satisfies the traditional local energy inequality, this theorem is a slight refinement of the main result of [Chae and Wolf 2018]. Furthermore, we are careful to give a precise formulation (1-14) of the pressure and its connection to the velocity. The relationship between v and π is less clear in [loc. cit.].

(2) The integral in (1-14) is not a Calderón–Zygmund singular integral because we do not have a global bound for v. It is defined in $L^{3/2}_{\mathrm{loc}}$ using the DSS property.

(3) Our method of proof is by approximation and is similar to the argument from [Lemarié-Rieusset 2016]. The main difference is that we need to construct a sequence of approximating solutions and establish a new a priori bound for these solutions for DSS data — in [loc. cit.] the bound (1-12) is sufficient (and free). Note that an approximation argument using (1-12) was also used by the authors in [Bradshaw and Tsai 2017a] to construct SS solutions as a limit of DSS solutions where the scaling factors are converging to 1.

(4) Generally, the solution v is not necessarily a local Leray solution because v_0 may not be in L^2_{uloc}, and we do not assert the uniform bounds in Definition 1.1(2). Consider the DSS function in L^2_{loc} for $0 < a < \frac{3}{2}$

$$f_a(x) = \sum_{k\in\mathbb{Z}} \lambda^k f_{a,0}(\lambda^k x), \qquad f_{a,0}(x) = |x - x_0|^{-a} \chi(x - x_0), \tag{1-15}$$

where $1+r<|x_0|<\lambda-r$ for some $r>0$, and χ is the characteristic function of the ball $B_r(0)$. It is not in L^2_{uloc} when $1<a<\frac{3}{2}$ for its behavior at infinity. It is in L^2_{uloc} when $0<a\le 1$. The function $f_1(x)$ for $a=1$ is given in Comment 4 after [Bradshaw and Tsai 2017a, Theorem 1.2] as an inapplicable example since it is not in $L^{3,\infty}(\mathbb{R}^3)$.

(5) If $v_0\in L^2_{\mathrm{uloc}}$, then it is not difficult to obtain uniform bounds on v in the sense of Definition 1.1(2). Furthermore, Definition 1.1(5) can be established whenever $v_0\in E^2$; see [Maekawa et al. 2019]. Thus, our construction yields DSS local Leray solutions whenever the data is DSS, divergence-free, and in E^2.

Our strategy for proving Theorem 1.2 is to approximate a solution with data in L^2_{loc} using solutions constructed in [Bradshaw and Tsai 2017a]. There are several steps. First we need to prove that DSS data in L^2_{loc} can be approximated in $L^2(B_1)$ by DSS data in L^3_w. This is the subject of Section 4A. Then, [loc. cit.] gives us a sequence of DSS solutions in the local Leray class. To prove that these solutions converge to a solution with L^2_{loc} data satisfying the desired pressure formula, we need to establish new a priori bounds for the solutions from [loc. cit.] which are independent of the L^3_w norm of the initial data (this is done in Section 3) and also prove that they satisfy the pressure formula (see Section 2). In Sections 4B and 4C, we put these ingredients together to prove Theorem 1.2.

As a last remark, in [Chae and Wolf 2018] it is unclear if the solution is suitable in the classical sense. The referee for this paper suggested a compelling argument to address this. In particular, the discretely self-similar ansatz and the boundedness of the solution in $L^2_{\mathrm{loc}}(\mathbb{R}^3\times[0,\infty))$ should make it possible to define $\mathbb{P}\nabla\cdot(u\otimes u)$. Then, starting with a solution of [loc. cit.], a pressure p could be constructed in $\mathcal{D}'$. It then could be shown that $\nabla p+\mathbb{P}(u\cdot\nabla u)=0$. This should follow from the slow growth of u at spatial infinity and using the fact that $\nabla p+\mathbb{P}(u\cdot\nabla u)$ is spatially harmonic.

2. A limiting pressure formula for DSS solutions

In this section we will prove that, under certain conditions, the limiting pressure distribution of an approximation scheme for (1-1) inherits the structure of the approximate pressure distributions. This result will be applied in Sections 3 and 4C.

Lemma 2.1. *Fix $\lambda>1$ and $T>0$. Let $v_0\in L^2_{\mathrm{loc}}$ be a given divergence-free, λ-DSS vector field and assume $\{v_0^{(k)}\}\subset L^2_{\mathrm{loc}}$ is a sequence of divergence-free, λ-DSS vector fields so that $v_0^{(k)}\to v_0$ in $L^2(B_1)$. Assume v_k and $\tilde v_k$ are divergence-free, λ-DSS vector fields and that there exists a distribution π_k so that the following conditions are satisfied:*

- *v_k, $\tilde v_k$, and π_k solve the system*

$$\partial_t v_k-\Delta v_k+\tilde v_k\cdot\nabla v_k+\nabla\pi_k=0,\quad (x,t)\in\mathbb{R}^3\times[0,T],$$

for the initial data $v_0^{(k)}$ and both v_k and $\tilde v_k$ converge to $v_0^{(k)}$ in L^2_{loc}.

- *v_k and $\tilde v_k$ are uniformly bounded in $L^\infty(0,T;L^2(B_1))\cap L^2(0,T;H^1(B_1))$ over all $k\in\mathbb{N}$.*

• *For all* $0 < t \le T$, *the distribution* π_k *satisfies the formula*

$$\pi_k(x,t) = -\tfrac{1}{3}[\tilde{v}_k\cdot v_k](x,t) + \lim_{\delta\to 0}\int_{|y|>\delta} K_{ij}(x-y)(\tilde{v}_k)_i(y,t)(v_k)_j(y,t)\,dy. \tag{2-1}$$

• *There exists a* λ*-DSS solution* v *in* $L^\infty(0,T;L^2(B_1))\cap L^2(0,T;H^1(B_1))$ *with pressure* π *in* $L^{3/2}(0,T;L^{3/2})$ *so that*

$$\begin{aligned} v_k \text{ and } \tilde{v}_k \to v \text{ weakly} &\quad \text{in } L^2(0,T;H^1(B_1)),\\ v_k \text{ and } \tilde{v}_k \to v &\quad \text{in } L^2(0,T;L^2(B_1)),\\ \pi_k \to \pi \text{ weakly} &\quad \text{in } L^{3/2}(0,T;L^{3/2}(B_1)). \end{aligned}$$

Then, for a.e. $0 < t \le T$ *and* $x \in B_\lambda$, *the pressure* π *satisfies the formula*

$$\pi(x,t) = -\tfrac{1}{3}|v|^2(x,t) + \lim_{\delta\to 0}\int_{|y|>\delta} K_{ij}(x-y)(v)_i(y,t)(v)_j(y,t)\,dy \tag{2-2}$$

in $L^{3/2}((0,T)\times B_\lambda)$.

Remark 2.2. The purpose of this lemma is to establish the pressure formula (2-2), which, ultimately, will allow us to prove (1-14). It is, however, not needed to establish the other conclusions of Theorem 1.2.

Proof. Note that since v_k, $\tilde{v}_k$, and v are all uniformly bounded in $L^\infty(0,T;L^2(B_1))\cap L^2(0,T;H^1(B_1))$, convergence in $L^2(0,T;L^2(B_1))$, Hölder's inequality, Sobolev embedding, using the equation to get uniform bound of $\partial_t v_k$, and rescaling the solution imply

$$v_k \text{ and } \tilde{v}_k \to v \quad \text{in } L^3(0,T;L^3(B_1)).$$

It also shows that v_k, $\tilde{v}_k$, and v are all uniformly bounded in $L^3(0,T;L^3(B_1))$ (at least for k sufficiently large).

Let

$$\begin{aligned} \pi_k^1(x,t) &= -\tfrac{1}{3}[\tilde{v}_k\cdot v_k](x,t),\\ \pi_k^2(x,t) &= \lim_{\delta\to 0}\int_{\lambda^2>|y|>\delta} K_{ij}(x-y)(\tilde{v}_k)_i(y,t)(v_k)_j(y,t)\,dy,\\ \pi_k^3(x,t) &= \int_{y\ge\lambda^2} K_{ij}(x-y)(\tilde{v}_k)_i(y,t)(v_k)_j(y,t)\,dy. \end{aligned}$$

Also let

$$\begin{aligned} \pi^1(x,t) &= -\tfrac{1}{3}|v|^2(x,t),\\ \pi^2(x,t) &= \lim_{\delta\to 0}\int_{\lambda^2>|y|>\delta} K_{ij}(x-y)v_i(y,t)v_j(y,t)\,dy,\\ \pi^3(x,t) &= \int_{y\ge\lambda^2} K_{ij}(x-y)v_i(y,t)v_j(y,t)\,dy. \end{aligned}$$

Since v_k and $\tilde{v}_k \to v$ in $L^3(0,T;L^3(B_\lambda))$, we have

$$\pi_k^1 \to \pi^1 \quad \text{in } L^{3/2}(0,T;L^{3/2}(B_\lambda)).$$

Let

$$\begin{aligned} h_{i,j}(y,t) &= (\tilde v_k)_i (v_k)_j - v_i v_j \\ &= \{(\tilde v_k)_i[(v_k)_j - v_j] + [(\tilde v_k)_i - v_i]v_j\}(y,t). \end{aligned}$$

Using the Calderón–Zygmund theory we clearly have

$$\begin{aligned} \int_0^T\int_{B_\lambda} |\pi_k^2(x,t)-\pi^2(x,t)|^{3/2}\,dx\,dt &\le C\int_0^T\int_{B_{\lambda^2}} |h_{i,j}(x,t)|^{3/2}\,dx\,dt \\ &\le C\Big(\int_0^T\int_{B_{\lambda^2}} \tilde v_k^3\,dx\,dt\Big)^{1/2}\Big(\int_0^T\int_{B_{\lambda^2}} (v_k - v)^3\,dx\,dt\Big)^{1/2} \\ &\quad + C\Big(\int_0^T\int_{B_{\lambda^2}} v^3\,dx\,dt\Big)^{1/2}\Big(\int_0^T\int_{B_{\lambda^2}} (\tilde v_k - v)^3\,dx\,dt\Big)^{1/2}. \end{aligned} \tag{2-3}$$

Rescaling gives

$$\int_0^T\int_{B_{\lambda^2}} (\tilde v_k - v)^3(x,t)\,dx\,dt = \lambda^4\int_0^{T\lambda^{-4}}\int_{B_1} (\tilde v_k - v)^3(z,\tau)\,dz\,d\tau$$

for the obvious choice of z and τ. Since the right-hand side of the equation above vanishes as $k\to\infty$, as does the identical term but with $\tilde v_k$ replaced by v_k, we conclude that π_k^2 converges to π^2 in $L^{3/2}(0,T;L^{3/2}(B_1))$.

Establishing the convergence of π_k^3 to π^3 is more difficult. Let

$$p_k(x,t) = \pi_k^3(x,t) - \pi^3(x,t) = \int_{|y|\ge\lambda^2} K_{ij}(x-y)h_{i,j}(y,t)\,dy.$$

Fix $x\in B_\lambda$. Then

$$\begin{aligned} |p_k(x,t)|^{3/2} &\le C\Big|\int_{|y|\ge\lambda^2} \frac{1}{|y|^3}|h_{i,j}(y,t)|\,dy\Big|^{3/2} \\ &\le C\Big(\int_{|y|\ge\lambda^2}\frac{1}{|y|^4}\,dy\Big)^{1/2}\int_{|y|\ge\lambda^2}\frac{1}{|y|^{5/2}}|h_{i,j}(y,t)|^{3/2}\,dy \\ &= C\int_{|y|\ge\lambda^2}\frac{1}{|y|^{5/2}}|h_{i,j}(y,t)|^{3/2}\,dy. \end{aligned}$$

Let $A_k = \{x : \lambda^{k-1}\le |x| < \lambda^k\}$ for $k\in\mathbb{Z}$. Then, using the scaling properties of h,

$$\begin{aligned} \int_{|y|\ge\lambda^2}\frac{1}{|y|^{5/2}}|h_{i,j}(y,t)|^{3/2}\,dy &= \sum_{k=3}^\infty\int_{A_k}\frac{1}{|y|^{5/2}}|h_{i,j}(y,t)|^{3/2}\,dy \\ &\le C(\lambda)\sum_{k=3}^\infty\frac{1}{\lambda^{5k/2}}\int_{A_k}|h_{i,j}(y,t)|^{3/2}\,dy \\ &\le C(\lambda)\sum_{k=3}^\infty\frac{1}{\lambda^{5k/2}}\int_{B_1}|h_{i,j}(z,t\lambda^{-2k})|^{3/2}\,dz. \end{aligned}$$

Thus,

$$\begin{aligned}\int_0^T\int_{B_\lambda} |p_k(x,t)|^{3/2}\,dt &\le \lambda^3 C(\lambda)\int_0^T \sum_{k=3}^\infty \frac{1}{\lambda^{5k/2}}\int_{B_1} |h_{i,j}(z,t\lambda^{-2k})|^{3/2}\,dz\,dt\\ &\le C(\lambda)\sum_{k=3}^\infty \frac{1}{\lambda^{k/2}}\int_0^{T\lambda^{-2k}}\int_{B_1}|h_{i,j}(z,\tau)|^{3/2}\,dz\,d\tau\\ &\le C(\lambda)\int_0^T\int_{B_1}|h_{i,j}(z,\tau)|^{3/2}\,dz\,d\tau.\end{aligned}$$

Therefore this term is bounded as (2-3).

We have now shown that $\pi_k(x,t)$ converges weakly to both $\pi^1(x,t)+\pi^2(x,t)+\pi^3(x,t)$ and $\pi(x,t)$ in $L^{3/2}(0,T;L^{3/2}(B_\lambda))$, implying that $\pi(x,t)=\pi^1(x,t)+\pi^2(x,t)+\pi^3(x,t)$ as distributions. In other words, $\pi(x,t)$ satisfies (2-2) in $L^{3/2}((0,T)\times B_\lambda)$. □

3. Properties of DSS solutions with data in L^3_w

The goal of this section is to obtain a bound on the local evolution of DSS solutions v constructed in [Bradshaw and Tsai 2017a] that is independent of both the L^3_w and L^2_{uloc} norms of v and to establish an explicit representation formula for the pressure.

Assume $v_0 \in L^3_w(\mathbb{R}^3)$ and v is a DSS solution evolving from v_0 as constructed in [loc. cit.]. For a generic solution to (1-1), we cannot close energy estimates for ϕv solely in terms of $v_0|_{B_\lambda}$ — there is always some spillover. Proposition 3.1 states that this is possible for DSS solutions as a result of their scaling properties. In our argument, we must work with a quantity that is continuous in time. This is not known for $\int_{B_1}|v(t)|^2\,dx$ when v is a local Leray solution. Hence, we need to work at the level of a *mollified approximation scheme* [loc. cit., (2.24)] (see (3-4) below). Note that in [loc. cit.], the mollified scheme is used to approximate a solution to the time-periodic Leray equations and the mollification is time-independent. Undoing the similarity transformation results in a time-dependent mollification of the drift component of the nonlinear term of the solution in the physical variables (see (3-5) below); this matches the mollification used in [Chae and Wolf 2018].

Proposition 3.1. *Fix $\lambda > 1$. Assume $v_0 \in L^3_w(\mathbb{R}^3)$ is λ-DSS and divergence-free, and v is a λ-DSS local Leray solution evolving from v_0 constructed in [Bradshaw and Tsai 2017a]* (*in particular, it is the limit of the mollified approximation scheme* (2.24) *in that paper*) *and π is its associated pressure. Let $\alpha_0 = \|v_0\|^2_{L^2(B_\lambda)}$. Then, there exist positive $T = T(\alpha_0,\lambda)$ and $C(\alpha_0,\lambda)$ independent of $\|v_0\|_{L^2_{\mathrm{uloc}}}$ and $\|v_0\|_{L^3_w}$ so that*

$$\operatorname*{ess\,sup}_{0\le t\le T}\int_{B_1}|v(x,t)|^2\,dx + \int_0^T\int_{B_1}|\nabla v|^2\,dx\,dt < C(\alpha_0,\lambda), \tag{3-1}$$

and

$$\int_0^T\int_{B_1}|\pi(x,t)|^{3/2}\,dx\,dt < C(\alpha_0,\lambda). \tag{3-2}$$

Moreover, for $x \in B_1$ and $t \in (0, T)$, the pressure satisfies the formula

$$\pi(x,t) = -\tfrac{1}{3}|v|^2(x,t) + \lim_{\delta\to 0}\int_{|y|>\delta} K_{ij}(x-y)v_i(y,t)v_j(y,t)\,dy \tag{3-3}$$

in $L^{3/2}(B_1 \times (0, T))$.

Typically, the best pressure decompositions we have for local Leray solutions depend on a particular ball containing the spatial point at which the pressure is being computed. The resulting formula consists of a local Calderón–Zygmund part and a far-field part with a singular kernel that is decaying faster than the kernel of K. The formula (3-3) does not involve such a decomposition, and, as is evident in the proof, the integral in (3-3) is defined using the DSS property.

The proof of [Bradshaw and Tsai 2017a] shows that the left sides of (3-1) and (3-2) are bounded by constants depending on v_0, in particular its $L^3_w(\mathbb{R}^3)$-norm. For this application, we need a bound depending only on $\|v_0\|_{L^2(B_\lambda)}$ and λ.

Proof. Since v is a solution from [Bradshaw and Tsai 2017a], its image under the similarity transform (1-9) solves the time-periodic Leray equations and is the limit of a mollified approximation scheme [loc. cit., (2.24)]. In particular, for each $\epsilon > 0$, there exists a time-periodic solution u_ϵ to the problem

$$\big(\partial_s u_\epsilon - \Delta u_\epsilon - \tfrac{1}{2}u_\epsilon - \tfrac{1}{2}y\cdot\nabla u_\epsilon + (\eta_\epsilon * u_\epsilon)\cdot\nabla u_\epsilon + \nabla p_\epsilon\big)(y,s) = 0, \tag{3-4}$$

where $\eta_\epsilon(y) = (1/\epsilon^3)\eta(y/\epsilon)$ and η is in $C_0^\infty(\mathbb{R}^3)$, is nonnegative, and satisfies $\int \eta(y)\,dy = 1$. Applying (1-8)–(1-9) we obtain a λ-DSS vector field v_ϵ satisfying

$$\partial_t v_\epsilon(x,t) - \Delta v_\epsilon(x,t) + (\eta_{\epsilon\sqrt{t}} * v_\epsilon)\cdot\nabla v_\epsilon(x,t) + \nabla\pi_\epsilon(x,t) = 0. \tag{3-5}$$

Note the time dependence of the convolution kernel $\eta_{\epsilon\sqrt{t}}$ in (3-5).

By the convergence properties of $u_\epsilon(y,s)$ to $u(y,s) = \sqrt{t}\,v(x,t)$ [loc. cit., p. 1108] and discretely self-similar scaling (to extend the estimates down to $t = 0$), it follows that for all $T > 0$ and all compact sets $K \subset \mathbb{R}^3$,

$$\begin{aligned} v_\epsilon \to v \text{ weakly} \quad & \text{in } L^2(0,T;H^1(K)),\\ v_\epsilon \to v \text{ strongly} \quad & \text{in } L^2(0,T;L^2(K)),\\ v_\epsilon(s) \to v(s) \text{ weakly} \quad & \text{in } L^2(K) \text{ for all } s \in [0,T]. \end{aligned}$$

Note also that $v_\epsilon(t) \to v_0$ in L^2_{loc}; i.e., the mollification does not affect the initial data. Furthermore, because each v_ϵ is smooth on $\mathbb{R}^3 \times (0,\infty)$ and right continuous in L^2_{loc} at $t = 0$, it follows that

$$\alpha_\epsilon(t) = \int_{B_1} |v_\epsilon(x,t)|^2\,dx$$

and

$$\tilde{\alpha}_\epsilon(t) = \sup_{0\le\tau\le t} \alpha_\epsilon(\tau)$$

are continuous as functions of t. This is not clearly true for $\int_{B_1} |v(x,t)|^2\,dx$.

Note that, for any $k \in \mathbb{Z}$ and $q \in [1, \infty)$, since $v_\epsilon(x,t) = \lambda^{-k} v_\epsilon(\lambda^{-k}x, \lambda^{-2k}t)$,

$$\int_{B_{\lambda^k}} |v_\epsilon(x,t)|^q \, dx = \lambda^{(3-q)k} \int_{B_1} |v_\epsilon(\tilde{x}, \lambda^{-2k}t)|^q \, d\tilde{x}. \tag{3-6}$$

Our goal is to establish local-in-time a priori bounds for $\alpha_\epsilon(t)$ that are independent of ϵ. Note that v_ϵ satisfies the local energy equality; i.e.,

$$\begin{aligned}\int |v_\epsilon|^2 \phi(t)\, dx + 2\int_0^t \int |\nabla v_\epsilon|^2 \phi \, dx\, ds \\ = \int |v_0|^2 \phi\, dx + \int_0^t \int |v_\epsilon|^2 (\partial_s \phi + \Delta\phi)\, dx\, ds \\ + \int_0^t \int (|v_\epsilon|^2 ((\eta_{\epsilon\sqrt{s}} * v_\epsilon) \cdot \nabla \phi)\, dx\, ds + \int_0^t \int 2\pi_\epsilon (v_\epsilon \cdot \nabla\phi)\, dx\, ds\end{aligned} \tag{3-7}$$

for any nonnegative $\phi \in C_0^\infty(\mathbb{R}^3 \times [0,\infty))$. Fix $\chi \in C^\infty(\mathbb{R})$ with $\chi(t) = 1$ if $t \le 1$ and $\chi(t) = 0$ if $t \ge \lambda$. We now fix ϕ in (3-7) as

$$\phi(x,t) = \chi^2(|x|) \cdot \chi(t).$$

We will estimate the terms on the right-hand side of (3-7) for $0 < t \le 1$, and we can treat ϕ as t-independent from now on. The first term is bounded by α_0. For the second, using the scaling properties (3-6) of v_ϵ, we have

$$\int_0^t \int |v_\epsilon|^2 (\partial_s\phi + \Delta\phi)\, dx\, ds \le C \int_0^t \int_{B_\lambda} |v_\epsilon|^2\, dx\, ds \le C\lambda^3 \int_0^{t/\lambda^2} \int_{B_1} |v_\epsilon|^2 \, dx\, ds \le C(\lambda) \int_0^t \tilde{\alpha}_\epsilon(s)\, ds.$$

For the cubic term, we begin by using Young's inequality to obtain

$$\int_0^t \int |v_\epsilon|^2 ((\eta_{\epsilon\sqrt{s}} * v_\epsilon) \cdot \nabla\phi)\, dx\, ds \le C \int_0^t \int_{B_\lambda} |v_\epsilon|^3 \, dx\, ds + C \int_0^t \int_{B_\lambda} |(\eta_{\epsilon\sqrt{s}} * v_\epsilon)|^3 \, dx\, ds.$$

Rescaling the unmollified term and making the obvious change of variables results in the estimate

$$\int_0^t \int_{B_\lambda} |v_\epsilon|^3 \, dx\, ds \le C(\lambda) \int_0^{t/\lambda^2} \int_{B_1} |v_\epsilon|^3 \, dy\, d\tau \le C(\lambda) \int_0^t \int |v_\epsilon|^3 \phi^{3/2} \, dx\, ds.$$

For the term involving the mollifier, note that $\eta \in C_0^\infty$ and $\operatorname{supp} \eta \subset B_\rho$ for some $\rho > 0$. By taking ϵ sufficiently small we can ensure that $\operatorname{supp} \eta_{\epsilon\sqrt{s}} \subset B_{\lambda-1}$ whenever $s < 1$. Note $\lambda^k + (\lambda - 1) \le \lambda^{k+1}$ for all $k \ge 0$. Thus, for $x \in B_\lambda$,

$$\begin{aligned}|(\eta_{\epsilon\sqrt{s}} * v_\epsilon)(x,s)| &\le \int \eta_{\epsilon\sqrt{s}}(y) |v_\epsilon(x-y,s)|\, dy \\ &= \int \eta_{\epsilon\sqrt{s}}(y) |v_\epsilon(x-y,s)| \chi_{B_{\lambda^2}}(x-y)\, dy \\ &= (\eta_{\epsilon\sqrt{s}} * (\chi_{B_{\lambda^2}} |v_\epsilon|))(x,s)\end{aligned}$$

whenever ϵ is sufficiently small and $s<1$. Therefore, under the same assumptions and after rescaling we see that, for any $1<q<\infty$,

$$\|(\eta_{\epsilon\sqrt{s}}*v_\epsilon)(s)\|_{L^q(B_\lambda)} \le C(q,\eta)\|v_\epsilon(s)\|_{L^q(B_{\lambda^2})} \le C(q,\eta,\lambda)\|v_\epsilon(\lambda^{-4}s)\|_{L^q(B_1)}, \tag{3-8}$$

where C is independent of s and ϵ. Note that this estimate is also valid if B_λ is replaced by B_{λ^2} but with a different choice of constants, smallness condition on ϵ, and right-hand side determined at time $\lambda^{-6}s$.

Using standard inequalities and (3-8) with $q=3$ thus leads to the estimate

$$\int_0^t\int |v_\epsilon|^2((\eta_{\epsilon\sqrt{s}}*v_\epsilon)\cdot\nabla\phi)\,dx\,ds \le C(\eta,\lambda)\int_0^t\int |v_\epsilon|^3\phi^{3/2}\,dx\,ds. \tag{3-9}$$

By the Gagliardo–Nirenberg inequality and rescaling (3-6), we have, for any $s>0$, that

$$\begin{aligned}\|\phi^{1/2}v_\epsilon(s)\|_{L^3} &\le C\|\nabla\otimes(\phi^{1/2}v_\epsilon)\|_{L^2}^{1/2}\|\phi^{1/2}v_\epsilon\|_{L^2}^{1/2}(s)\\ &\le C(\lambda)(\tilde\alpha_\epsilon(s)^{1/2}+\|\phi^{1/2}\nabla v_\epsilon(s)\|_{L^2})^{1/2}(\tilde\alpha_\epsilon(s))^{1/4}.\end{aligned}$$

Hence, for any $\gamma>0$,

$$\|\phi^{1/2}v_\epsilon(s)\|_{L^3}^3 \le C(\lambda)(\gamma^{-3}\tilde\alpha_\epsilon(s)^3+\gamma\tilde\alpha_\epsilon(s)^1+\gamma\|\phi^{1/2}\nabla v_\epsilon(s)\|_2^2).$$

Thus,

$$\begin{aligned}\int_0^t\int |v_\epsilon|^2((\eta_{\epsilon\sqrt{s}}*v_\epsilon)\cdot\nabla\phi)\,dx\,ds\\ \le C(\lambda,\gamma,\eta)\int_0^t(\tilde\alpha_\epsilon(s)^3+\tilde\alpha_\epsilon(s)^1)\,ds + C(\lambda)\gamma\int_0^t\int|\nabla v_\epsilon|^2\phi\,dx\,ds.\end{aligned} \tag{3-10}$$

Provided γ is small enough, the gradient term can be absorbed into the left-hand side of (3-7).

We next estimate the pressure term. For this we need a formula for the pressure, which we presently justify. Let $w_\epsilon = v_\epsilon - V_0$, where $V_0(x,t)=e^{t\Delta}v_0$. We have

$$\partial_t w_\epsilon - \Delta w_\epsilon + \nabla\pi_\epsilon = g,\quad \operatorname{div} w_\epsilon = 0,$$

where $g_i = -\partial_j G_{ji}$ with

$$\begin{aligned}G &= (\eta_{\epsilon\sqrt{t}}*v_\epsilon)\otimes v_\epsilon\\ &= (\eta_{\epsilon\sqrt{t}}*w_\epsilon+\eta_{\epsilon\sqrt{t}}*V_0)\otimes(w_\epsilon+V_0).\end{aligned}$$

For $0<t_1<t_2<\infty$, we have

$$\begin{aligned}&V_0\in C([t_1,t_2];\,L^4(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)),\\ &w_\epsilon\in L^\infty(t_1,t_2;\,L^2(\mathbb{R}^3))\cap L^2(t_1,t_2;\,L^6(\mathbb{R}^3))\subset L^4(t_1,t_2;\,L^3(\mathbb{R}^3)).\end{aligned}$$

By Young's convolution inequality,

$$\|G\|_{L^2(t_1,t_2;L^2)} \lesssim \|\eta_{\epsilon\sqrt{t}}\|_{L^\infty(t_1,t_2;L^{6/5}\cap L^1)}(\|w_\epsilon\|_{L^4(t_1,t_2;L^3(\mathbb{R}^3))}+\|V_0\|_{L^4(\mathbb{R}^3\times[t_1,t_2])})^2.$$

Since $g\in L^2([t_1,t_2];H^{-1})$, [Caffarelli et al. 1982, Lemma A.2] implies $w_\epsilon\in C([t_1,t_2];L^2)$ (after modification on a set of time of measure zero; since the modified vector field still satisfies the above system distributionally, this does not effect our argument).

Consider the following nonstationary Stokes system with forcing g:

$$\partial_t V - \Delta V + \nabla P = g, \quad \operatorname{div} V = 0,$$

with initial data $V_0 = w_\epsilon(t_1) \in L^2(\mathbb{R}^3)$. It is well known that if $g \in L^\infty(t_1, t_2; H^{-1})$ and $V_0 \in L^2$, then there exists a unique $V \in C_w([t_1, t_2]; L^2(\mathbb{R}^3)) \cap L^2([t_1, t_2]; H^1(\mathbb{R}^3))$ and unique ∇P solving the nonstationary stokes system given above; see [Bradshaw and Tsai 2017a, p. 1107–1108]. Letting $V = w_\epsilon$ and $P = \pi_\epsilon$, this implies that w_ϵ and $\nabla \pi_\epsilon$ are unique. Up to a function $\pi_*(t)$ independent of x,

$$\pi_\epsilon(x,t) - \pi_*(t) = -\tfrac{1}{3}[(\eta_{\epsilon\sqrt{t}} * v_\epsilon)\cdot v_\epsilon](x,t) + \lim_{\delta\to 0}\int_{|y|>\delta} K_{ij}(x-y)(\eta_{\epsilon\sqrt{t}} * v_\epsilon)_i(y,t)(v_\epsilon)_j(y,t)\,dy, \quad (3\text{-}11)$$

where

$$K_{ij}(x) = \partial_i \partial_j \frac{1}{4\pi |x|}.$$

The right-hand side of (3-11) is defined in $L^2([t_1, t_2]; L^2(\mathbb{R}^3))$. Since the only appearance of π_ϵ in (3-5) is $\nabla \pi_\epsilon$, we can redefine π_ϵ to equal $\pi_\epsilon - \pi_*(t)$ and, therefore, can drop $\pi_*(t)$ from (3-11).

The pressure π_ϵ given by (3-11) is already bounded in $L^2([t_1, t_2]; L^2(\mathbb{R}^3))$ for any $0 < t_1 < t_2 < \infty$ but the bound depends on t_1, t_2 and ϵ. We now bound it in $L^{3/2}(0, T; L^{3/2}(B_\lambda))$. Bounding the first term from (3-11) is simple given Hölder's inequality, (3-8), and (3-9). In particular, we have for any $\gamma > 0$

$$\int_0^t \big\| \tfrac{1}{3} |(\eta_{\epsilon\sqrt{s}} * v_\epsilon)(\cdot, s)|\, |v_\epsilon(\cdot, s)| \big\|_{L^{3/2}(B_\lambda)}^{3/2} ds \le C(\lambda, \gamma, \eta) \int_0^t (\tilde\alpha_\epsilon(s)^3 + \tilde\alpha_\epsilon(s)^1)\, ds + \gamma \int_0^t \int |\nabla v_\epsilon|^2 \phi \, dx\, ds.$$

To bound the principal value integral in (3-11), we need to split the integral into local and nonlocal parts as follows:

$$\begin{aligned}
&\lim_{\delta\to 0}\int_{|y|>\delta} K(x-y)(\eta_{\epsilon\sqrt{t}} * v_\epsilon)(y,t) v_\epsilon(y,t)\, dy \\
&\quad = \lim_{\delta\to 0}\int_{B_{\lambda^2}\setminus B_\delta} K(x-y)(\eta_{\epsilon\sqrt{t}} * v_\epsilon)(y,t) v_\epsilon(y,t)\chi_{B_{\lambda^2}}(y)\, dy + \int_{|y|>\lambda^2} K(x-y)(\eta_{\epsilon\sqrt{t}} * v_\epsilon)(y,t) v_\epsilon(y,t)\, dy \\
&\quad =: \pi_{\mathrm{near}}(x,t) + \pi_{\mathrm{far}}(x,t).
\end{aligned}$$

To bound π_{near} note that, by the Calderón–Zygmund theory,

$$\|\pi_{\mathrm{near}}(\cdot, t)\|_{L^{3/2}(B_\lambda)} \le \|(\eta_{\epsilon\sqrt{t}} * v_\epsilon)(\cdot, t) v_\epsilon(\cdot, t)\|_{L^{3/2}(B_{\lambda^2})},$$

and, arguing as above using (3-8) but with B_{λ^2} in place of B_λ (see the note following (3-8)), it follows that

$$\int_0^t \|\pi_{\mathrm{near}}(\cdot, s)\|_{L^{3/2}(B_\lambda)}^{3/2} ds \le C(\lambda, \gamma, \eta) \int_0^t (\tilde\alpha_\epsilon(s)^3 + \tilde\alpha_\epsilon(s)^1)\, ds + \gamma \int_0^t \int |\nabla v_\epsilon|^2 \phi\, dx\, ds.$$

Bounding the term π_{far} is more complicated. Let

$$A_k = \{x : \lambda^{k-1} \le |x| < \lambda^k\}.$$

We start with the following pointwise estimate which is valid whenever $x \in B_\lambda$:

$$\begin{aligned}|\pi_{\text{far}}(x,t)| &\le C\sum_{k=3}^{\infty}\int_{A_k}\frac{1}{|x-y|^3}|(\eta_{\epsilon\sqrt{t}}*v_\epsilon)(y,t)||v_\epsilon(y,t)|\,dy\\ &\le C(\lambda)\sum_{k=3}^{\infty}\frac{1}{\lambda^{3k}}\int_{A_k}|(\eta_{\epsilon\sqrt{t}}*v_\epsilon)(y,t)||v_\epsilon(y,t)|\,dy\\ &= C(\lambda)\sum_{k=3}^{\infty}\frac{1}{\lambda^{2k}}\int_{A_0}|(\eta_{\epsilon\sqrt{t\lambda^{-2k}}}*v_\epsilon)(z,t\lambda^{-2k})||v_\epsilon(z,t\lambda^{-2k})|\,dz\\ &\le C(\lambda)\sum_{k=3}^{\infty}\frac{1}{\lambda^{2k}}\|(\eta_{\epsilon\sqrt{t\lambda^{-2k}}}*v_\epsilon)(t\lambda^{-2k})\|_{L^2(B_1)}\|v_\epsilon(t\lambda^{-2k})\|_{L^2(B_1)}\\ &\le C(\lambda)\sum_{k=3}^{\infty}\frac{1}{\lambda^{2k}}\|v_\epsilon(t\lambda^{-2k})\|^2_{L^2(B_{\lambda^2})}\le C(\lambda)\tilde{\alpha}_\epsilon(t),\end{aligned}$$

where we have used (3-6), (3-8) and rescaled the solution. Therefore,

$$\int_0^t\|\pi_{\text{far}}(\,\cdot\,,s)\|^{3/2}_{L^{3/2}(B_\lambda)}\,ds\le C(\lambda)\int_0^t\tilde{\alpha}_\epsilon(s)^{3/2}\,ds.$$

After using Hölder's inequality, (3-9), the bounds above, and $\alpha^{3/2}\le\alpha+\alpha^3$ for $\alpha>0$, it is clear that

$$\int_0^t\|\pi_\epsilon(\,\cdot\,,s)\|^{3/2}_{L^{3/2}(B_\lambda)}\,ds+\int_0^t\int 2\pi_\epsilon(v_\epsilon\cdot\nabla\phi)\,dx\,ds \\ \le C(\lambda,\gamma,\eta)\int_0^t(\tilde{\alpha}_\epsilon(s)^3+\tilde{\alpha}_\epsilon(s)^1)\,ds+\gamma\int_0^t\int|\nabla v_\epsilon|^2\phi\,dx\,ds.$$

Combining the estimates above (and taking γ sufficiently small to absorb the gradient terms on the right-hand side), we obtain

$$\alpha_\epsilon(t)+\int_0^t\int_{B_1}|\nabla v_\epsilon|^2\,dx\,ds\le\alpha_0+C(\lambda,\eta,\gamma)\int_0^t(\tilde{\alpha}_\epsilon(s)^3+\tilde{\alpha}_\epsilon(s)^1)\,ds. \tag{3-12}$$

Therefore,

$$\tilde{\alpha}_\epsilon(t)\le\alpha_0+C\int_0^t(\tilde{\alpha}_\epsilon(s)^3+\tilde{\alpha}_\epsilon(s)^1)\,ds. \tag{3-13}$$

By continuity of $\alpha_\epsilon(t)$, we have

$$\tilde{\alpha}_\epsilon(t)\le 2\alpha_0\quad\text{for all } t<T, \tag{3-14}$$

for some $T>0$. By a continuity argument, we may take $T=(C(2+8\alpha_0^2))^{-1}$.

Letting $\epsilon\to 0$ yields

$$(v,\chi_{B_1}v)(t)\le\liminf_{\epsilon\to 0}(v_\epsilon,\chi_{B_1}v_\epsilon)_{L^2}(t)\le 2\alpha_0$$

for all $t\le T$. Note that (3-12) gives uniform (in ϵ) control of

$$\int_0^T\int_{B_1}|\nabla v_\epsilon|^2\,dx\,dt\le C(\alpha_0,\lambda)$$

for some constant $C(\alpha_0,\lambda)$. From [Bradshaw and Tsai 2017a] we have that v_ϵ converges weakly to v in $L^2(1/k,T;H^1(B_1))$ for every $k\in\mathbb{N}$. Hence,

$$\int_{1/k}^T\int_{B_1}|\nabla v|^2\,dx\,dt\le\sup_{\epsilon>0}\int_0^T\int_{B_1}|\nabla v_\epsilon|^2\,dx\,dt,$$

and, letting $k\to\infty$, it follows that

$$\int_0^T\int_{B_1}|\nabla v|^2\,dx\,dt\le C(\alpha_0,\lambda).$$

Similarly, since $\pi_\epsilon\in L^{3/2}(0,T;L^{3/2}(B_1))$ with uniformly bounded norms, it follows that

$$\pi\in L^{3/2}(0,T;L^{3/2}(B_1)).$$

Applying Lemma 2.1 yields the desired pressure representation in $L^{3/2}(0,T;L^{3/2}(B_1))$ and concludes the proof. □

4. DSS solutions with data in $L^2_{\mathrm{loc}}(\mathbb{R}^3)$

In this section we prove Theorem 1.2. To do this, we need to approximate DSS data in L^2_{loc} by divergence-free DSS vector fields in L^3_w and also characterize discrete self-similarity on $\mathbb{R}^3\times(0,\infty)$ in terms of a neighborhood of the origin.

4A. *Approximation of DSS data in L^2_{loc}.*

Lemma 4.1. *Let $f\in L^2_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R}^3)$ be a given divergence-free λ-DSS vector field for some $\lambda>0$. There exists a sequence of divergence-free λ-DSS vector fields $\phi^{(k)}$ so that $\phi^{(k)}\in L^3_w(\mathbb{R}^3)$ and $\|\phi^{(k)}-f\|_{L^2(B_1)}\to 0$ as $k\to\infty$ (B_1 is the ball of radius 1 centered at the origin).*

The main difficulty in proving this lemma is that each $f^{(k)}$ must be divergence-free. We thus need to use the Bogovski map [1980], which we presently recall.

Lemma 4.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $2\le n<\infty$. There is a linear map Ψ that maps a scalar $f\in L^q(\Omega)$ with $\int_\Omega f=0$, $1<q<\infty$, to a vector field $v=\Psi f\in W^{1,q}_0(\Omega;\mathbb{R}^n)$ and*

$$\operatorname{div} v=f,\qquad \|v\|_{W^{1,q}_0(\Omega)}\le c(\Omega,q)\|f\|_{L^q(\Omega)}.$$

The map Ψ is independent of q for $f\in C^\infty_c(\Omega)$.

Proof of Lemma 4.1. Let $Z_0(x)\in C^\infty(\mathbb{R}^3)$ satisfy

$$Z_0(x)=\begin{cases}1, & |x|>1,\\ \text{radial, increasing}, & \lambda^{-1}\le|x|\le 1,\\ 0, & |x|<\lambda^{-1}.\end{cases}$$

Note that $\nabla\cdot(Z_0f)=f\cdot\nabla Z_0$; i.e., Z_0f is not divergence-free. We can correct this using Lemma 4.2 with $q=2$ for the scalar $-f\cdot\nabla Z_0$ noting that f is locally square integrable and

$$\int -f\cdot\nabla Z_0\,dx=0,$$

because f is divergence-free. Denote by Φ_0 the image of $-f\cdot\nabla Z_0$ under a Bogovski mapping with domain $\{x:\lambda^{-1}\le|x|\le 1\}$. Then, $\Phi_0\in W_0^{1,2}(B_1\setminus B_{\lambda^{-1}})$ and

$$\nabla\cdot(Z_0 f+\Phi_0)=0.$$

Let $Z_i(x)=Z_0(x/\lambda^i)$ and $\Phi_i(x)=\lambda^{-i}\Phi_0(\lambda^{-i}x)$ for all $i\in\mathbb{Z}$. It follows that

$$\nabla\cdot(Z_i f+\Phi_i)=0$$

for all $i\in\mathbb{Z}$. Note that $\operatorname{supp}(Z_j-Z_{j+2})=\{x:\lambda^{j-1}\le|x|\le\lambda^{j+2}\}$. Let

$$f_i=\tfrac{1}{2}(Z_i-Z_{i+2})f+\tfrac{1}{2}(\Phi_i-\Phi_{i+2}).$$

Then each f_i is divergence-free and supported on $B_{\lambda^{i+2}}\setminus B_{\lambda^{i-1}}$. Furthermore,

$$f=\sum_{i\in\mathbb{Z}}f_i,$$

where convergence is understood in the pointwise sense for all $x\neq 0$. To confirm this note that if x satisfies $\lambda^i\le|x|<\lambda^{i+1}$ then $x\in\operatorname{supp}(Z_j-Z_{j+2})$ if and only if $j\in\{i-1,i,i+1\}$. It follows that

$$\sum_{j\in\mathbb{Z}}(Z_j-Z_{j+2})(x)=2.$$

On the other hand, $\operatorname{supp}\Phi_j=\{x:\lambda^{j-1}\le|x|\le\lambda^j\}$ and, therefore,

$$\sum_{j\in\mathbb{Z}}(\Phi_j(x)-\Phi_{j+2}(x))=\Phi_{i+1}(x)-\Phi_{i+1}(x)=0.$$

It follows that $f=\sum_{i\in\mathbb{Z}}f_i$.

Assume $\phi_0^{(k)}$ is a sequence of divergence-free vector fields in $C_0^\infty(B_{\lambda^2}\setminus B_{\lambda^{-1}})$ so that $\phi_0^{(k)}\to f_0$ in $L^2(B_{\lambda^2}\setminus B_{\lambda^{-1}})$. Let $\phi_i^{(k)}=\lambda^{-i}\phi_0^{(k)}(\lambda^{-i}x)$. Then the vector field

$$\phi^{(k)}=\sum_{i\in\mathbb{Z}}\phi_i^{(k)}$$

is a divergence-free, λ-DSS vector field, and satisfies

$$|\phi^{(k)}(x)|\le c_k|x|^{-1}$$

(where the proportionality constants c_k are *not* uniformly bounded with respect to k). Hence, $\phi^{(k)}\in L^3_w$. We finish by arguing that $\phi^{(k)}\to f$ in $L^2(B_1)$. We know that $\int_{B_{\lambda^2}\setminus B_{\lambda^{-1}}}(\phi_0^{(k)}-f)^2\,dx\to 0$ as $k\to\infty$. Using the definition of $\phi^{(k)}$ and the fact that f is discretely self-similar we have, letting $A_i=B_{\lambda^i}\setminus B_{\lambda^{i-1}}$, that

$$\begin{aligned}\int_{B_1}(\phi^{(k)}-f)^2\,dx&=\sum_{i\le 0}\int_{A_i}(\phi^{(k)}-f)^2\,dx\\&=\sum_{i\le 0}\lambda^i\int_{A_0}(\phi^{(k)}-f)^2\,dx=\frac{\lambda}{\lambda-1}\int_{A_0}(\phi^{(k)}-f)^2\,dx.\end{aligned}$$

In A_0, we have $\phi^{(k)} - f = \sum_{i=-2}^{0}(\phi_i^{(k)} - f_i)$. Thus

$$\|\phi^{(k)} - f\|_{L^2(A_0)} \le \sum_{i=-2}^{0} \|\phi_i^{(k)} - f_i\|_{L^2(A_0)} = \sum_{k=0}^{2} \lambda^{-k/2}\|\phi_0^{(k)} - f_0\|_{L^2(A_k)} \le 3\|\phi_0^{(k)} - f_0\|_{L^2(B_{\lambda^2}\setminus B_{\lambda^{-1}})},$$

which completes the proof. □

4B. ***DSS solutions in a neighborhood of the origin.*** In the Introduction we saw that any time-periodic solution u to (1-10) corresponds to a DSS solution v after the change of variables (1-9). Distributionally, u is a time-periodic solution to (1-10) if and only if

$$\int_{s'}^{s'+T} \big((u, \partial_s f) - (\nabla u, \nabla f) + \big(\tfrac{1}{2}u + \tfrac{1}{2}y\cdot\nabla u - u\cdot\nabla u, f\big)\big)\, ds = 0 \tag{4-1}$$

holds for all $s' \in \mathbb{R}$ and $f \in \mathcal{D}_T$, where $\mathcal{D}_T$ denotes the collection of all smooth divergence-free vector fields in $\mathbb{R}^3 \times \mathbb{R}$ which are time-periodic with period T and whose supports are compact in space. In [Bradshaw and Tsai 2017a], this definition was used with $s' = 0$ since the goal was to extend a solution on $[0, T]$ to $\mathbb{R}$ using periodicity. The same modification can be made here based on the observations that if u satisfies (4-1) then u can be extended to a time-periodic solution on $\mathbb{R}$ and if u is a time-periodic solution on $\mathbb{R}$ then u satisfies (4-1).

Since there is a one-to-one correspondence between time-periodic solutions to (1-10) and DSS solutions, an equivalent characterization of DSS solutions is obtained by reformulating (4-1) in the physical variables. For $f \in \mathcal{D}_T$ let $\zeta_f(x,t) = t^{-1} f(y,s)$. Note $\zeta_f(x,t) = \lambda^2\zeta_f(\lambda x, \lambda^2 t)$. Then, v is λ-DSS if and only if

$$\int_t^{\lambda^2 t} ((v, \partial_t\zeta_f) - (\nabla v, \nabla\zeta_f) - (v\cdot\nabla v, \zeta_f))\, d\tau = 0 \tag{4-2}$$

for all $t > 0$ and $f \in \mathcal{D}_T$, since (4-1) is just (4-2) in similarity variables. Note that $(v, \zeta_f)|_{\tau=\lambda^2 t} = (v, \zeta_f)|_{\tau=t}$. It follows that, if v is a solution to (1-1) that satisfies (4-2) for $t = 1$, then $v|_{\tau\in[1,\lambda^2]}$ can be extended to a λ-DSS solution for all positive times.

Fix $k \in \mathbb{Z}$ and let $Q_k = B_{\lambda^k}(0) \times (0, \lambda^{2k})$. Our goal is to give a third characterization of discrete self-similarity on Q_k. Let $f \in \mathcal{D}_T$ be given and ζ_f be as above. Let R be large enough so that, for all $t \in [1, \lambda^2]$, the support of $\zeta_f(t)$ is a subset of $B_R(0)$ and choose $m = m(f) \in \mathbb{Z}$ so that $R/\lambda^m < \lambda^k$ and $\lambda^{2-2m} < \lambda^{2k}$. It follows that

$$B_{R/\lambda^m}(0) \times [\lambda^{-2m}, \lambda^{2-2m}] \subset Q_k.$$

Extend ζ_f to all $t > 0$ using the following scaling: for $(x,t) \in \mathbb{R}^3 \times (0,\infty)$, let

$$\zeta_f(x,t) = \lambda^{2i}\zeta_f(\lambda^i x, \lambda^{2i} t),$$

where i is chosen so that $\lambda^{2i} t \in [1, \lambda^2]$. Since $\zeta_f|_{\mathbb{R}^3\times[1,\lambda^2]}$ is compactly supported in space, its spatial support shrinks as $t \to 0^+$. In particular, for $t \in [\lambda^{-2m}, \lambda^{2-2m}]$, we have $\operatorname{supp}\zeta_f \subset Q_k$. For $m \in \mathbb{Z}$, let

$$\begin{aligned}\mathcal{D}^m_{Q_k} = \{\phi \in C^\infty(\mathbb{R}^3\times(0,\infty)) : \operatorname{supp}\phi|_{t\in[\lambda^{-2m},\lambda^{2-2m}]} \subset Q_k \\ \text{and } \forall (x,t) \in \mathbb{R}^3\times(0,\infty), \exists f \in \mathcal{D}_T \text{ such that } \phi(x,t) = \zeta_f(x,t)\}.\end{aligned} \tag{4-3}$$

It is easy to see that

$$\bigcup_{m\in\mathbb{Z}} \mathcal{D}^m_{Q_k} = \mathcal{D}_T.$$

Rescaling (4-2) gives

$$\int_{\lambda^{-2m}}^{\lambda^{2-2m}} ((v, \partial_t \zeta_f) - (\nabla v, \nabla \zeta_f) - (v\cdot\nabla v, \zeta_f))\,dt' = 0, \tag{4-4}$$

where $t' = t/\lambda^{2m}$ and the inner products are taken with respect to the rescaled spatial variable $x' = x/\lambda^m$. In particular, the integral is computed over a subset of Q_k and is identical to the same integral with ζ_f replaced by ϕ for some $\phi \in \mathcal{D}^m_{Q_m}$. Thus, if v is a solution to (1-1), and $\phi \in \mathcal{D}^m_{Q_k}$ for some $m \in \mathbb{Z}$, then (4-2) is satisfied if and only if (4-4) is satisfied for the $f \in \mathcal{D}_T$ for which $\zeta_f = \phi$. This leads to the following extendability property: if v is a solution to (1-1) on Q_k and satisfies (4-4) for every $m \in \mathbb{Z}$ and $\phi \in \mathcal{D}^m_{Q_k}$, then v can be extended to a discretely self-similar solution on $\mathbb{R}^3 \times (0,\infty)$; in other words, if a solution is DSS in a neighborhood of the origin, then it can be extended to a DSS solution on $\mathbb{R}^3 \times (0,\infty)$.

4C. *Construction of DSS solutions.*

Proof of Theorem 1.2. Fix $\lambda > 1$ and assume $v_0 \in L^2_{\mathrm{loc}}$ is a divergence-free λ-DSS vector field. Let $\{v_0^{(k)}\}$ be the sequence of vector fields $\{\phi^{(k)}\}$ from Lemma 4.1 applied to v_0. Then, the values $\|v_0^{(k)}\|_{L^2(B_1)}$ are uniformly bounded and $\|v_0^{(k)} - v_0\|_{L^2(B_1)} \to 0$ as $k \to \infty$. Since $v_0^{(k)} \in L^3_w$ and is λ-DSS, by [Bradshaw and Tsai 2017a] there exists a λ-DSS local Leray solution v_k to (1-1) and an associated pressure π_k having initial data $v_0^{(k)}$ for every $k \in \mathbb{N}$. By Proposition 3.1, v_k are uniformly bounded in $L^\infty(0,T; L^2(B_1)) \cap L^2(0,T; H^1(B_1))$ (hence also in $L^{10/3}(0,T; L^{10/3}(B_1))$) for some T which depends only on λ and $\|v_0^{(k)}\|_{L^2(B_1)}$. As usual, see [Bradshaw and Tsai 2017a; Kikuchi and Seregin 2007; Lemarié-Rieusset 2016], there exists a distribution v and a subsequence of $\{v_k\}$ (still indexed by k for simplicity) so that v_k converges to v in the weak star topology on $L^\infty(0,T; L^2(B_1))$, in the weak topology on $L^2(0,T; H^1(B_1))$, and in $L^2(0,T; L^2(B_1))$. Since they are uniformly bounded in $L^{10/3}(0,T; L^{10/3}(B_1))$, they also converge in $L^q(0,T; L^q(B_1))$ for any $q < \frac{10}{3}$. By the pressure estimate (3-2) in Proposition 3.1, π_k are uniformly bounded in $L^{3/2}(0,T; L^{3/2}(B_1))$ by $C(\lambda, \|v_0\|_{L^2(B_\lambda)})$ and, therefore, we may extract a subsequence which converges weakly to a distribution $\pi \in L^{3/2}(0,T; L^{3/2}(B_1))$.

Fix $\kappa \in \mathbb{Z}$ so that $\lambda^\kappa < 1$ and $\lambda^{2\kappa} < T$. Then, $Q_\kappa = B_{\lambda^\kappa} \times (0, \lambda^{2\kappa}) \subset B_1 \times (0,T)$. Therefore v_k satisfies (1-1) on Q_κ and satisfies (4-4) for every $m \in \mathbb{Z}$ and $\phi \in \mathcal{D}^m_{Q_\kappa}$. Thus, v can be extended to a DSS solution on $\mathbb{R}^3 \times (0,\infty)$ (which we still denote by v).

For compact subsets K of B_1, we automatically have $\lim_{t\to 0^+} \|v - v_0\|_{L^2(K)} = 0$. For a general compact subset K of $\mathbb{R}^3$, we have $K' = \lambda^m K \subset B_1$ for some $m \in \mathbb{Z}$, and

$$\int_K |v(x,t) - v_0(x)|^2\,dx = \lambda^{-m} \int_{K'} |v(x', \lambda^{2m} t) - v_0(x')|^2\,dx'.$$

It follows that $\lim_{t\to 0^+} \|v(t) - v_0\|_{L^2(K)} = 0$ for every compact set $K \subset \mathbb{R}^3$. A similar rescaling argument also implies that $v \in L^\infty(0,T'; L^2(K)) \cap L^2(0,T'; H^1(K))$ and $\pi \in L^{3/2}(0,T'; L^{3/2}(K))$ for any $T' > 0$ and compact subset K of $\mathbb{R}^3$.

To confirm that v satisfies the local energy inequality, first note that each v_k satisfies the local energy inequality

$$\int |v_k(t)|^2\phi\,dx + 2\iint |\nabla v_k|^2\phi\,dx\,dt \le \int |v_0^{(k)}|^2\phi\,dx + \iint |v_k|^2(\partial_t\phi + \Delta\phi)\,dx\,dt + \iint (|v_k|^2 + 2\pi_k)(v_k\cdot\nabla\phi)\,dx\,dt$$

for all nonnegative $\phi \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R}^3_+)$. Furthermore, the right-hand sides of the energy inequality for $v^{(k)}$ converge to the right-hand side of the energy inequality for v as $k \to \infty$, while the left-hand sides are lower semicontinuous; see [Caffarelli et al. 1982, (A.51)]. The local energy inequality for v plainly follows.

Finally, note that π_k satisfies the formula (3-3). Applying Lemma 2.1 to the sequence and limit above implies that π satisfies the desired pressure formula in $L^{3/2}(0, T; L^{3/2}(B_1))$. Rescaling establishes the formula in $L^{3/2}_{\mathrm{loc}}(\mathbb{R}^3 \times (0, \infty))$. □

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