ANALYSIS & PDE

Volume 12 No. 8 2019

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ANALYSIS AND PDE
Vol. 12, No. 8, 2019
dx.doi.org/10.2140/apde.2019.12.1963

CONTINUITY PROPERTIES FOR DIVERGENCE FORM BOUNDARY DATA HOMOGENIZATION PROBLEMS

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We study the asymptotic behavior at rational directions of the effective boundary condition in periodic homogenization of oscillating Dirichlet data. We establish a characterization for the directional limits at a rational direction in terms of a relatively simple two-dimensional boundary layer problem for the homogenized operator. Using this characterization we show continuity of the effective boundary condition for divergence form linear systems, and for divergence form nonlinear equations we give an example of discontinuity.

1. Introduction

In this work we will study the following type of boundary layer problem in dimension $d \geq 2$:

$$\begin{cases} -\nabla \cdot a(y, \nabla v_n^s) = 0 & \text{in } P_n^s = \{y \cdot n > s\}, \\ v_n^s(y) = \varphi(y) & \text{on } \partial P_n^s. \end{cases} \tag{1-1}$$

Here $n \in S^{d-1}$ is a unit vector, $s \in \mathbb{R}$, φ is continuous and $\mathbb{Z}^d$ periodic, the operator a is also $\mathbb{Z}^d$ periodic in y and will satisfy a uniform ellipticity assumption. This work will consider both nonlinear scalar equations and linear systems, so, for now, we do not specify the assumptions on a any further.

The *boundary layer limit* of the system (1-1) is defined by

$$\varphi_*(n,s) := \lim_{R\to\infty} v_n(Rn + y) \quad \text{if the limit exists and is independent of } y \in \partial P_n^s.$$

If, additionally, the boundary layer limit is independent of s then we say that the cell equation (1-1) homogenizes. Typically φ_* is independent of s for irrational directions n and we write $\varphi_*(n)$, while for rational directions $n \in \mathbb{R}\mathbb{Z}^d$ the limits above exist but depend on s.

The focus of this article is on the limiting behavior of φ_* at rational directions. As a consequence of this study we will be able to establish continuity or discontinuity of φ_* on S^{d-1}. We will see that continuity of φ_* is intrinsically linked with linearity of the operator $a(x, p)$. In the case of a linear system we show continuity of φ_*, while in the case of nonlinear scalar equations we give an example where φ_* is discontinuous; this indicates generic discontinuity for nonlinear equations.

The main result established in this paper is that the directional limits of φ_* at a rational direction are determined by a "second cell problem", which is a boundary layer problem for the homogenized operator a^0. From this asymptotic formula it becomes relatively straightforward to address questions of

MSC2010: 35B27, 35J57, 35J60.
Keywords: homogenization, boundary layers, oscillating boundary data, nonlinear elliptic equations, elliptic systems.

continuity or discontinuity of φ_* at rational directions. Let us take $\xi \in \mathbb{Z}^d \setminus \{0\}$ to be an irreducible lattice vector and $\hat{\xi}$ to be the corresponding rational unit vector in the same direction. Then the cell equation (1-1) solution v_ξ^s exists for each $s \in \mathbb{R}$ and has a boundary layer limit,

$$\varphi_*(\xi, s) := \lim_{R\to\infty} v_\xi^s(R\xi),$$

but that limit typically is not independent of the translation s applied to the half-space domain P_ξ. We will see that $\varphi_*(\xi, s)$ is a $1/|\xi|$-periodic function on $\mathbb{R}$. Now suppose that we have a sequence of directions $n_k \to \hat{\xi}$ such that

$$\frac{\hat{\xi} - n_k}{|\hat{\xi} - n_k|} \to \eta, \qquad \text{where } \eta \text{ is a unit vector with } \eta \perp \xi.$$

Call η the approach direction of the sequence n_k to ξ. We will show that the limit of $\varphi_*(n_k)$ is determined by the following boundary layer problem. Call $P_\xi = P_\xi^0 = \{x \cdot \xi > 0\}$ and define

$$\begin{cases} -\nabla \cdot a^0(\nabla w_{\xi,\eta}) = 0 & \text{in } P_\xi, \\ w_{\xi,\eta} = \varphi_*(\xi, x\cdot\eta) & \text{on } \partial P_\xi \end{cases} \quad \text{and} \quad L(\xi,\eta) = \lim_{R\to\infty} w_{\xi,\eta}(R\xi). \tag{1-2}$$

Then it holds

$$\lim_{k\to\infty} \varphi_*(n_k) = L(\xi, \eta). \tag{1-3}$$

We will see below that $L(\xi,\eta)$ is continuous in $\eta \in S^{d-1}$. Thus the directional limits of φ_* at ξ are determined by the boundary layer limit of a half-space problem for the homogenized operator. This limit structure was first observed in [Choi and Kim 2014] and developed further by the first author and Kim [Feldman and Kim 2017]; both papers studied nondivergence form and possibly nonlinear equations. We will explain in this paper how the second cell problem follows purely from *qualitative* features which are shared by a wide class of elliptic equations, including divergence form linear systems, and both divergence and nondivergence form nonlinear equations. We are somewhat vague about the hypotheses, which will be explained in detail in Sections 3 and 4.

Once we have established (1-3), the question of qualitative continuity/discontinuity of φ_* is reduced to a much simpler problem. For linear equations the homogenized operator a^0 is linear and translation-invariant and so a straightforward argument, for example by the Riesz representation theorem, shows that

$$L(\xi,\eta) = \lim_{R\to\infty} w_{\xi,\eta}(R\xi) = |\xi| \int_0^{1/|\xi|} \varphi_*(\xi, s)\, ds;$$

i.e., it is the average over a period of $\varphi_*(\xi, \cdot)$. Evidently this does not depend on the approach direction η. Thus qualitative continuity of φ_* for linear problems follows easily once we establish (1-3).

In the case of nonlinear equations the formula (1-3) allows us to construct examples where discontinuities do occur; see Theorem 1.3 below. Our conjecture is that discontinuities are generic for the class of quasilinear equations we consider. Note that when φ_* is not continuous at ξ, the asymptotic formula (1-3) still contains interesting information; it explains the structure of the discontinuity. In particular, the blow up of φ_* at a discontinuity is 0-homogeneous and continuous away from the origin.

Before we state our main theorems we give a brief explanation about where (1-1) arises and why one should be interested in the continuity/discontinuity of φ_*. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and consider the homogenization problem with oscillating Dirichlet boundary data,

$$\begin{cases} -\nabla \cdot \left(a\left(\frac{x}{\varepsilon}, \nabla u^\varepsilon\right)\right) = 0 & \text{in } \Omega, \\ u^\varepsilon(x) = g\left(x, \frac{x}{\varepsilon}\right) & \text{on } \partial\Omega, \end{cases} \tag{1-4}$$

where $\varepsilon > 0$ is a small parameter, $g(x, y)$ is continuous in x, y and $\mathbb{Z}^d$ periodic in y. This system is natural to consider in its own right, but also it arises naturally in the study of homogenization with nonoscillatory Dirichlet data when one studies the higher-order terms in the asymptotic expansion; see [Gérard-Varet and Masmoudi 2012] where this is explained.

The interest in studying (1-4) is the asymptotic behavior of the u^ε solutions as $\varepsilon \to 0$. This problem has been studied recently by a number of authors starting with [Gérard-Varet and Masmoudi 2011; 2012] and followed by [Aleksanyan, Shahgholian, and Sjölin 2015; Aleksanyan 2017; Choi and Kim 2014; Feldman and Kim 2017; Feldman 2014; Prange 2013; Zhang 2017; Armstrong, Kuusi, Mourrat, and Prange 2017; Guillen and Schwab 2016]. It has been established that solutions u^ε converge, at least in $L^2(\Omega)$, to some u^0 which is a unique solution to

$$\begin{cases} -\nabla \cdot (a^0(\nabla u^0)) = 0 & \text{in } \Omega, \\ u^0(x) = \varphi^0(x) & \text{on } \partial\Omega, \end{cases} \tag{1-5}$$

where a^0 and $\varphi^0(x)$ are called respectively the homogenized operator and homogenized boundary data. The identification of the homogenized operator a^0 is a classical topic. The homogenized boundary φ^0 is determined by the boundary layer equation (1-1),

$$\varphi^0(x) = \varphi_*(n_x), \quad \text{when } n_x \text{ is the inward unit normal to } \Omega \text{ and } \varphi(y) = g(x, y).$$

That is, (1-1) can be viewed as a kind of cell problem associated with the homogenization of (1-4). At least for linear equations this definition makes sense as long as the set of boundary points of $\partial\Omega$ where (1-1) does not homogenize, i.e., those with rational normal, has zero harmonic measure. The convergence of u^ε to u^0 has been established rigorously for linear systems by Gérard-Varet and Masmoudi [2012], and further investigations have yielded optimal rates of convergence; see [Armstrong, Kuusi, Mourrat, and Prange 2017; Shen and Zhuge 2018]. For nonlinear divergence form equations, to our knowledge, the problem has not been studied yet. This is the source of our interest in the fine properties of φ_*: quantitative continuity estimates for φ_* lead to quantitative continuity estimates for u^0 and u^ε, and are used to establish rates of convergence $u^\varepsilon \to u^0$. Meanwhile, characterizing the type of discontinuities of φ_*, when they are present, leads to understanding the qualitative features of u^ε and u^0.

Now we return to state our main results. The first is the validity of the "second cell problem" formula (1-3) for the directional limits of φ_*.

Theorem 1.1. *The limit characterization* (1-3) *holds for divergence form linear systems and nonlinear equations satisfying a uniform ellipticity condition. The* 0*-homogeneous profile* $L(\xi, \eta)$ *at direction* $\xi \in \mathbb{Z}^d \setminus \{0\}$ *is continuous in* $\eta \in S^{d-1}$.

Our arguments to derive (1-3) can be quantified to obtain a modulus of continuity, which we make explicit below, however so far we cannot push the method to obtain the optimal modulus of continuity. In a very nice recent work Shen and Zhuge [2017] obtain an almost Lipschitz modulus of continuity by a different method; we will compare their approach with ours below.

Theorem 1.2. *For elliptic linear systems, $d \geq 2$, for any $0 < \alpha < \frac{1}{d}$ there is a constant $C \geq 1$ depending on α as well as universal parameters associated with the system (see Section 3) such that, for any n_1, n_2 irrational,*

$$|\varphi_*(n_1) - \varphi_*(n_2)| \leq C \|\varphi\|_{C^5} |n_1 - n_2|^\alpha.$$

We note that in the course of proving Theorem 1.2 we actually show Hölder regularity for every $0 < \alpha < 1$ at each lattice direction $\xi \in \mathbb{Z}^d \setminus \{0\}$; the modulus of continuity however depends on the rational direction and degenerates as $|\xi| \to \infty$. This is why we only end up with (almost) Hölder-$\frac{1}{d}$ continuity in the end.

For nonlinear problems our conjecture is that φ_* is discontinuous at rational directions, at least for generic boundary data and operators. A result of this kind was established for nondivergence form equations in [Feldman and Kim 2017]. In the divergence form nonlinear case we have constructed an explicit example showing that discontinuity is possible.

Theorem 1.3. *For $d \geq 3$ there exist smooth boundary data φ and uniformly elliptic, positively 1-homogeneous, nonlinear operators $a(x, p)$ such that φ_* is discontinuous at some rational direction.*

We compare with [Shen and Zhuge 2017], which studies continuity properties of φ_* for linear divergence form systems. They show, in the linear systems case, that φ_* is in $W^{1,p}$ for every $p < \infty$. They establish Lipschitz estimates on the Diophantine directions which only grow subpolynomially in the Diophantine parameter, and thereby obtain the $W^{1,p}$ estimates and extend continuously to the rational directions. As can be seen, for example, by Theorem 1.3, this type of result would not be possible for quasilinear elliptic equations. Our approach is to compute the directional limits at each rational direction via the second cell problem formula (1-3). Although this method does not yet yield an optimal quantitative estimate, it applies to both linear and nonlinear equations including both divergence form, as established here, and nondivergence form, as in [Feldman and Kim 2017]. We establish (suboptimal) quantitative continuity for linear systems, and also we can classify the types of discontinuities which are present in the nonlinear setting.

Finally we compare with the work of the first author and Kim [Feldman and Kim 2017] in the nondivergence form case. As we try to emphasize in Section 2, the broad outline of the arguments for Theorems 1.1 and 1.2 are the same in divergence and nondivergence form. However, at the level of the proofs there are many technical differences; we will try to highlight the most interesting throughout the paper. The idea, from [Feldman and Kim 2017], for the construction of nonlinear operators with discontinuous φ_* does not work at all in the divergence form setting. We needed a completely different construction for Theorem 1.3.

Generally speaking, for linear systems we need to replace arguments with maximum principle by large-scale estimates on the Poisson kernel in half-spaces and cone-type domains. These estimates come from [Avellaneda and Lin 1991] or are adapted from the arguments there. For nonlinear equations we do have a maximum principle, but many new arguments need to be developed since, as far as we are aware, this is the first paper on the boundary layer problem for quasilinear divergence form equations.

1A. ***Notation.*** We go over some of the notation and terminology used in the paper. We will refer to constants which depend only on the dimension or fundamental parameters associated with the operator $a(x, p)$ (to be made specific below), e.g., ellipticity ratio or smoothness norm, as universal constants. We will write C or c for universal constants which may change from line to line. Given some quantities A, B we write $A \lesssim B$ if $A \leq CB$ for a universal constant C. If the constants depend on an additional nonuniversal parameter α we may write $A \lesssim_\alpha B$.

We will use various standard L^p and Hölder $C^{k,\alpha}$ norms. For Hölder seminorms, which omit the zeroth-order sup norm term, we write $[f]_{C^{k,\alpha}}$. Given a measurable set $E \subset \mathbb{R}^d$ we will also use the $L^p_{\text{avg}}(E)$ norm, which is defined by

$$\|f\|_{L^p_{\text{avg}}(E)} = \left(\frac{1}{|E|}\int_E |f|^p\right)^{1/p}.$$

The oscillation is a convenient quantity for us since the solution property for the equations we consider is preserved under addition of constant functions. This is usually defined for a scalar-valued function $u : E \to \mathbb{R}$ on a set $E \subset \mathbb{R}^d$ as $\operatorname{osc}_E u = \sup_E u - \inf_E u$. We use a slightly different definition which also makes sense for vector-valued $u : E \to \mathbb{R}^N$,

$$\operatorname*{osc}_E u := \inf\left\{r > 0 : \text{there exists } u_0 \in \mathbb{R}^N \text{ such that } \|u - u_0\|_{L^\infty(E)} \leq \tfrac{1}{2}r\right\}.$$

2. Explanation of the limit structure at rational directions

We give a high-level description of the asymptotics of the boundary layer limit at rational directions. What we would like to emphasize throughout this description is that the argument is basically geometric, and has to do with the way that ∂P_n intersects the unit periodicity cell in the asymptotic limit as n approaches a rational direction. This calculation relies only on certain qualitative features of Dirichlet problems for elliptic equations which are true both for divergence and nondivergence form both linear (including systems) and nonlinear. To emphasize the level of abstraction we will write the boundary layer problem in the form

$$\begin{cases} F[v_n, x] = 0 & \text{in } P_n := \{x \cdot n > 0\}, \\ v_n = \varphi & \text{on } \partial P_n. \end{cases} \tag{2-1}$$

Always F and φ will share $\mathbb{Z}^d$ periodicity in the x-variable. In order to carry out the heuristic argument we will need the following properties of the class of equations/systems. We emphasize that the following properties are not stated very precisely, they are merely meant to be illustrative:

(i) (homogenization) There is an elliptic operator F^0 in the same class such that if u^ε is a sequence of solutions of $F\left[u^\varepsilon, \frac{x}{\varepsilon}\right] = 0$ in a domain Ω converging to some u^0 then $F[u^0] = 0$ in Ω.

(ii) (continuity with respect to boundary data in L^∞) There exists $C > 0$ so that if $n \in S^{d-1}$ and u_1, u_2 are bounded solutions of (2-1) with respective boundary data φ_1 and φ_2 then

$$\sup_{P_n} |u_1 - u_2| \leq C \sup_{\partial P_n} |\varphi_1 - \varphi_2|.$$

(iii) (large-scale interior and boundary regularity estimates) There is $\alpha \in (0,1)$ such that for any $r > 0$ if $F[u,x] = 0$ in $B_r \cap P_n$, where B_r is some ball of radius r,

$$[u]_{C^\alpha(B_{r/2}\cap P_n)} \lesssim r^{-\alpha} \operatorname*{osc}_{B_r\cap P_n} u + [g]_{C^\alpha(B_r\cap \partial P_n)}.$$

The heuristic outline below applies to a wide class of elliptic equations; already the arguments were carried out rigorously for nondivergence nonlinear equations by Choi and Kim [2014] and the first author and Kim [Feldman and Kim 2017] and similar ideas were used for parabolic equations in moving domains by the second author in [Zhang 2017]. Here we will be studying divergence form equations, linear systems and nonlinear scalar equations.

To begin we need to understand the boundary layer limit at a rational direction. Let $\xi \in \mathbb{Z}^d \setminus \{0\}$ and consider the solution $v^s_\xi(x)$ of,

$$\begin{cases} F[v^s_\xi, x] = 0 & \text{in } P^s_\xi = \{x\cdot n > s\}, \\ v^s_\xi = \varphi & \text{on } \partial P^s_\xi. \end{cases} \tag{2-2}$$

Translating the half-space, by changing s, changes the part of the data φ seen by the boundary condition. Thus the boundary layer limit of v^s_ξ can depend on the parameter s; we define

$$\varphi_*(\xi, s) = \lim_{R\to\infty} v^s_\xi(R\xi).$$

As will become clear, this particular parametrization of the boundary layer limits is naturally associated with the asymptotic structure of the boundary layer limits for directions n near ξ.

The next step is to understand the geometry near ξ. Let $n \in S^{d-1}$ be a direction near ξ and v_n be the corresponding half-space solution. We can write,

$$n = (\cos\varepsilon)\hat{\xi} - (\sin\varepsilon)\eta \quad \text{for some small angle } \varepsilon \text{ and a unit vector } \eta \perp \xi.$$

We obtain an asymptotic for v_n at an intermediate length scale.

Let $x \in \partial P_n$, then the hyperplanes ∂P_n and $\partial P^{x\cdot\hat{\xi}}_\xi$ are close in a large neighborhood, any scale $o\big(\frac{1}{\varepsilon}\big)$, of x. By using the local up-to-the-boundary regularity we see that v_n and v^s_ξ, with $s = x\cdot\hat{\xi}$, are close on the boundary of their common domain, at least in this $o\big(\frac{1}{\varepsilon}\big)$ neighborhood of x. Now v^s_ξ has a boundary layer limit $\varphi_*(\xi, s)$, and the length scale $|\xi|$ associated with the boundary layer depends on ξ, but not on ε. Thus for ε small and $|\xi| \ll R \ll \frac{1}{\varepsilon}$

$$v_n(x + Rn) = \varphi_*(\xi, x\cdot\hat{\xi}) + o_\varepsilon(1) = \varphi_*(\xi, \tan\varepsilon(x\cdot\eta)) + o_\varepsilon(1).$$

Here $o_\varepsilon(1)$ depends only on $|\xi|$, ε, and universal parameters of the problem. This is one of the main places where we use the large-scale boundary regularity estimates, property (iii) above. Thus, moving into the domain by Rn and rescaling to the scale $1/\tan\varepsilon$, i.e., letting $w^\varepsilon(x) \sim v_n((x + Rn)/\tan\varepsilon)$, we find that the boundary layer limit is well approximated by the boundary layer limit of

$$\begin{cases} F[w^\varepsilon, x/\tan\varepsilon] = 0 & \text{in } P_\xi, \\ w^\varepsilon = \varphi_*(\xi, x\cdot\eta) & \text{on } \partial P_\xi \end{cases} \tag{2-3}$$

in the limit as $\varepsilon \to 0$. Now taking the limit as $\varepsilon \to 0$ of in (2-3) we find the "second cell problem"

$$\begin{cases} F^0[w_{\xi,\eta}] = 0 & \text{in } P_\xi, \\ w_{\xi,\eta} = \varphi_*(\xi, x\cdot\eta) & \text{on } \partial P_\xi. \end{cases} \tag{2-4}$$

Thus we characterize the directional limits at the rational direction ξ as the boundary layer limits of the associated second cell problem

$$\lim_{k\to\infty} \varphi_*(n_k) = \lim_{R\to\infty} w_{\xi,\eta}(R\xi) \quad \text{if } \frac{\hat{\xi}-n_k}{|\hat{\xi}-n_k|} \to \eta.$$

With this characterization the *qualitative* continuity and discontinuity of φ_* can be investigated solely by studying (2-4).

In the following, Sections 3 and 4, we will explain background regularity results for linear systems and nonlinear divergence form equations and the well-posedness of Dirichlet problems in half-spaces. In particular we will prove that properties we used in the heuristic arguments above do hold for the type of equations/systems we consider. In Section 5 we will go into more detail about the boundary layer equation (1-1) in rational and irrational half-spaces. In Section 6 we will make rigorous the above outline obtaining intermediate-scale asymptotics which lead to the second cell equation (2-4). In Section 7 we show how to derive continuity of φ_* from the second cell problem for linear problems, and in Section 8 we show how nonlinearity can cause discontinuity of φ_*.

3. Linear systems background results

In this section we will recall some results about divergence form linear systems. Let Ω be a domain of $\mathbb{R}^d$ and $N \geq 1$; we consider solutions of the elliptic linear system

$$-\nabla\cdot(A(x)\nabla u) = 0 \quad \text{in } \Omega,$$

where $u \in H^1(\Omega;\mathbb{R}^N)$ is at least a weak solution. Here we use the notation $A = (A_{ij}^{\alpha\beta}(x))$ for $1 \leq \alpha, \beta \leq d$ and $1 \leq i, j \leq N$ defined for $x \in \mathbb{R}^d$, where we mean, using the summation convention,

$$(\nabla\cdot(A(x)\nabla u^\varepsilon))_i = \partial_{x_\alpha}(A_{ij}^{\alpha\beta}(x)\partial_{x_\beta}u_j^\varepsilon).$$

We assume that A satisfies the following hypotheses:

(i) Periodicity:

$$A(x+z) = A(x) \quad \text{for all } x \in \mathbb{R}^d,\ z \in \mathbb{Z}^d. \tag{3-1}$$

(ii) Ellipticity: for some $\lambda > 0$ and all $\xi \in \mathbb{R}^{d\times N}$,

$$\lambda \xi_\alpha^i \xi_\alpha^i \leq A_{ij}^{\alpha\beta}\xi_\alpha^i\xi_\beta^j \leq \xi_\alpha^i\xi_\alpha^i. \tag{3-2}$$

(iii) Regularity: for some $M > 0$,

$$\|A\|_{C^5(\mathbb{R}^d)} \leq M. \tag{3-3}$$

We remark that the regularity on A is far more than is necessary for most of the results below. When we say that C is a universal constant below we mean that it depends only on the parameters, d, N, λ, M.

3A. ***Integral representation.*** Consider the following boundary layer problem, which will be the main object of our study:

$$\begin{cases} -\nabla\cdot(A(x)\nabla u) = \nabla\cdot f + g & \text{in } P_n, \\ u(x) = \varphi(x) & \text{on } \partial P_n \end{cases} \tag{3-4}$$

for f, g smooth vector-valued functions with compact support and φ continuous and bounded. A solution is given by the Green's function formula

$$u(x) = \int_{P_n} \nabla G(x,y)\cdot f(y)\,dy + \int_{P_n} G(x,y)g(y)dy + \int_{\partial P_n} P(x,y)\varphi(y)\,dy.$$

Here G, P are the Green matrix and Poisson kernel corresponding to our operator. For $y \in P_n$, G solves

$$\begin{cases} -\nabla_x\cdot(A(x)\nabla_x G(x,y)) = \delta(x-y)I_N & \text{in } P_n, \\ G(x,y) = 0 & \text{on } \partial P_n, \end{cases} \tag{3-5}$$

and the Poisson kernel is given, for $x \in P_n$ and $y \in \partial P_n$, by

$$P(x,y) = -n\cdot(A^t(y)\nabla_y G(x,y)),$$

that is,

$$P_{ij}(x,y) = -n_\alpha A^{\beta\alpha}_{ki}(y)\partial_{y_\beta} G_{kj}(x,y).$$

Following from [Avellaneda and Lin 1991], and exactly stated in [Gérard-Varet and Masmoudi 2012, Proposition 5], G and P satisfy the same bounds as for a constant coefficient operator:

Theorem 3.1. *Call $\delta(y) := \operatorname{dist}(y, \partial P_n)$. For all $x \neq y$ in P_n, one has*

$$|G(x,y)| \le \frac{C}{|x-y|^{d-2}} \qquad \textit{for } d \ge 3,$$

$$|G(x,y)| \le C(|\log|x-y|| + 1) \qquad \textit{for } d = 2,$$

$$|G(x,y)| \le \frac{C\delta(x)\delta(y)}{|x-y|^{d}} \qquad \textit{for all } d,$$

$$|\nabla_x G(x,y)| \le \frac{C}{|x-y|^{d-1}} \qquad \textit{for all } d,$$

$$|\nabla_x G(x,y)| \le C\left(\frac{\delta(y)}{|x-y|^{d}} + \frac{\delta(x)\delta(y)}{|x-y|^{d+1}}\right) \qquad \textit{for all } d.$$

For all $x \in P_n$ and $y \in \partial P_n$, one has

$$|P(x,y)| \le \frac{C\delta(x)}{|x-y|^{d}},$$

$$|\nabla P(x,y)| \le C\left(\frac{1}{|x-y|^{d}} + \frac{\delta(x)}{|x-y|^{d+1}}\right).$$

Although it is not precisely stated there, the methods of [Avellaneda and Lin 1991] also can achieve the same bounds for the Green's function and Poisson kernel associated with the operator $-\nabla\cdot(A(x)\nabla)$

in the strip-type domains

$$\Pi_n(0, R) := \{0 < x \cdot n < R\},$$

with constants independent of R. This will be useful later.

From the Poisson kernel bounds we can derive the L^∞ estimate which replaces the maximum principle for linear systems.

Lemma 3.2. *Suppose that u_1, u_2 are bounded solutions of* (3-4) *with respective boundary data φ_1, φ_2 and zero right-hand side. Then,*

$$\sup_{P_n} |u_1 - u_2| \le C \|\varphi_1 - \varphi_2\|_{L^\infty(\partial P_n)},$$

where C is a universal constant. The same holds for solutions in $\Pi_n(0, R)$.

For the solutions given by the Poisson kernel representation formula, the result of Lemma 3.2 follows from a standard calculation using Theorem 3.1. There is some subtlety in showing uniqueness; see [Gérard-Varet and Masmoudi 2012, Section 2.2] for a proof.

3B. ***Large-scale boundary regularity.*** In this section we consider the large-scale boundary regularity used in the heuristic argument of Section 2 for linear elliptic systems. We will need a boundary regularity result [Avellaneda and Lin 1987, Theorem 1]. For the following we assume Ω is some domain with $0 \in \partial\Omega$ and that u^ε solves

$$-\nabla \cdot \Big(A\Big(\frac{x}{\varepsilon}\Big)\nabla u^\varepsilon\Big) = 0 \quad \text{in } \Omega \cap B_1 \qquad \text{and} \qquad u^\varepsilon = g \quad \text{on } \partial\Omega \cap B_1.$$

Lemma 3.3. *For every $0 < \alpha < 1$ there is a constant C depending on α and universal quantities such that, if $\Omega = \{x_d > 0\} \cap B_1 =: B_1^+$,*

$$[u^\varepsilon]_{C^\alpha(B_{1/2}^+)} \le C(\|\nabla g\|_{L^\infty(\{x_d=0\}\cap B_1)} + \|u^\varepsilon - g(0)\|_{L^2(B_1^+)}),$$

and for every $\nu > 0$

$$\|\nabla u^\varepsilon\|_{L^\infty(B_{1/2}^+)} \le C(\|\nabla g\|_{C^{0,\nu}(\{x_d=0\}\cap B_1)} + \|u^\varepsilon - g(0)\|_{L^2(B_1^+)}).$$

We need the Hölder regularity result in cone-type domains which are the intersection of two half-spaces with normal directions n_1, n_2 very close to each other. We will consider the more general class of domains Ω which are a Lipschitz graph over $\mathbb{R}^{d-1}$ with small Lipschitz constant. In particular we assume that there is an $f : \mathbb{R}^{d-1} \to \mathbb{R}$ Lipschitz with $f(0) = 0$ such that

$$\Omega \cap B_1 = \{(x', x_d) : x_N > f(x')\} \cap B_1.$$

Lemma 3.4. *For every $0 < \alpha < 1$ there is a $\delta(\alpha) > 0$ universal such that, if Ω as above with $\|\nabla f\|_\infty \le \delta$, then*

$$[u^\varepsilon]_{C^\alpha(\Omega\cap B_{1/2})} \le C(\|\nabla g\|_{L^\infty(\partial\Omega\cap B_1)} + \|u^\varepsilon - g(0)\|_{L^2(\Omega\cap B_1)}).$$

The proof is by compactness; we postpone it to Appendix A.

3C. ***Poisson kernel in half-space intersection.*** From the regularity estimates of the previous subsection we can derive estimates on the Poisson kernel in the intersection of nearby half-space domains. Consider two unit vectors n_1, n_2 with $|n_1 - n_2| \sim \varepsilon$ small. For simplicity we suppose that

$$n_j = (\cos\varepsilon)e_d + (-1)^j(\sin\varepsilon)e_1.$$

Set

$$K = P_{n_1} \cap P_{n_2}.$$

Define $G_K(x, y)$ to be the Green's matrix. Although the domain is Lipschitz, G_K still satisfies the bound (via [Avellaneda and Lin 1987]), in $d \geq 3$,

$$|G_K(x, y)| \lesssim \frac{1}{|x-y|^{d-2}}.$$

We set $P_K(x, y)$, for $x \in K$ and $y \in \partial K$, to be the Poisson kernel for K, which is well-defined as long as $y_1 \neq 0$. Call $\delta(x) = \operatorname{dist}(x, \partial K)$.

Lemma 3.5. *For any $\alpha \in (0, 1)$ and ε sufficiently small depending on α and universal quantities,*

$$|P_K(x, y)| \lesssim_\alpha \begin{cases} \dfrac{\delta(x)^\alpha}{|x-y|^{d-1+\alpha}} & \text{for } |y_1| \geq \frac{1}{2}|x-y|, \\ \dfrac{1}{|y_1|}\dfrac{\delta(x)^\alpha}{|x-y|^{d-2+\alpha}} & \text{for } |y_1| \leq \frac{1}{2}|x-y|. \end{cases}$$

The proof is postponed to Appendix A; we show how the estimates are used. Suppose $\psi : \partial K \to \mathbb{R}^N$ satisfies

$$|\psi(x)| \leq \min\{|x_1|, 1\}.$$

We consider the Poisson kernel solution of the Dirichlet problem,

$$u(x) = \int_{\partial K} P_K(x, y)\psi(y)\, dy.$$

In particular we are interested in the continuity at 0; we only consider really $x = te_d$ for some $t > 0$ (or $x = tn_1$ or tn_2 but this is basically the same) so we restrict to that case. Now for $y \in \partial K$, $|x-y| \sim t + |y|$ and so $|x-y| \gtrsim |y_1|$ and the first bound in Lemma 3.5 implies the second. Thus we can compute

$$\begin{aligned} |u(te_d)| &\lesssim \int_{\partial K} \frac{1}{|y_1|}\frac{t^\alpha}{(t+|y|)^{d-2+\alpha}} \min\{|y_1|, 1\}\, dy \\ &\lesssim \int_{\partial K} \frac{t^\alpha}{(t+|y|)^{d-2+\alpha}} \min\left\{1, \frac{1}{|y_1|}\right\} dy \\ &\lesssim \int_{\mathbb{R}} \int_{\mathbb{R}^{d-2}} \min\left\{1, \frac{1}{|y_1|}\right\} \frac{t^\alpha}{(t+|y_1|+|z|)^{d-2+\alpha}}\, dz\, dy_1. \end{aligned}$$

Computing the inner integrals, we have

$$\int_{\mathbb{R}^{d-2}} \frac{1}{(t+|y_1|+|z|)^{d-2+\alpha}}\, dz = \frac{1}{(t+|y_1|)^\alpha} \int_{\mathbb{R}^{d-2}} \frac{1}{(1+|w|)^{d-2+\alpha}}\, dw \lesssim \frac{1}{(t+|y_1|)^\alpha}.$$

Then

$$|u(te_d)| \lesssim \int_{\mathbb{R}} \min\left\{1, \frac{1}{|y_1|}\right\} \frac{t^\alpha}{(t+|y_1|)^\alpha}\, dy_1 \lesssim t^\alpha \quad \text{for } t \le 1.$$

We state the result of a slight generalization of this calculation as a lemma.

Lemma 3.6. *Suppose that $K = P_{n_1} \cap P_{n_2}$, $\alpha \in (0,1)$ and $\varepsilon = |n_1 - n_2|$ is sufficiently small so that the estimates of Lemma 3.5 hold, $\psi : \partial K \to \mathbb{R}$ smooth and satisfies the bound*

$$|\psi(x)| \le \min\{\delta^\beta |x \cdot (n_1 - n_2)|^\beta, 1\}$$

for some $\delta > 0$ and $1 \ge \beta > \alpha$, Then for any bounded solution u of

$$-\nabla \cdot (A(x)\nabla u) = 0 \quad \text{in } K \text{ with } u = \psi \text{ on } \partial K,$$

it holds

$$|u(te_d)| \lesssim \delta^\alpha t^\alpha \quad \text{for } t \le \frac{1}{\delta}.$$

There is an additional subtlety which is the uniqueness of the bounded solution of the Dirichlet problem in K; the argument is the same as in the half-space case; see [Gérard-Varet and Masmoudi 2012]. To derive Lemma 3.6 from the previous calculation just do a rescaling to $u\big(\frac{\cdot}{\delta}\big)$; the domain K is scaling-invariant and the Poisson kernel associated with $A\big(\frac{\cdot}{\delta}\big)$ satisfies the same bounds as for A.

4. Nonlinear equations background results

In this section we consider the boundary layer problem for nonlinear operators. To explain the assumptions we write out the problem in a general domain

$$\begin{cases} -\nabla \cdot a\big(\frac{x}{\varepsilon}, \nabla u^\varepsilon\big) = 0 & \text{in } \Omega, \\ u^\varepsilon(x) = g\big(x, \frac{x}{\varepsilon}\big) & \text{on } \partial\Omega. \end{cases} \tag{4-1}$$

This type of equation would arise as the Euler–Lagrange equation of a variational problem,

$$\text{minimize } \; E(u) = \int_\Omega F\Big(\frac{x}{\varepsilon}, \nabla u\Big)\, dx \; \text{ over } \; u \in H^1_0(\Omega) + g\Big(\cdot, \frac{\cdot}{\varepsilon}\Big).$$

A natural uniform ellipticity assumption on the functional F is

$$F \text{ is convex with } 1 \ge D^2 F \ge \lambda > 0.$$

Then $a = DF$ is 1-Lipschitz continuous in p and has the monotonicity property

$$(a(x,p) - a(x,q)) \cdot (p-q) \ge \lambda |p-q|^2 \quad \text{for all } p, q \in \mathbb{R}^d.$$

Now we consider how to determine the effective boundary conditions for the homogenization equation (4-1). We zoom in at a boundary point $x_0 \in \partial\Omega$ defining

$$v^\varepsilon(y) = u^\varepsilon(x_0 + \varepsilon y), \quad \text{which solves } \begin{cases} -\nabla \cdot a\big(y + \frac{x_0}{\varepsilon}, \frac{1}{\varepsilon}\nabla v^\varepsilon\big) = 0 & \text{in } \frac{1}{\varepsilon}(\Omega - x_0), \\ v^\varepsilon(y) = g\big(x_0 + \varepsilon y, y + \frac{x_0}{\varepsilon}\big) & \text{on } \frac{1}{\varepsilon}\partial(\Omega - x_0). \end{cases}$$

Now in order to have a unique equation in the limit $\varepsilon \to 0$ the following limit needs to exist:

$$a_*(y,p) = \lim_{t\to 0} t a(y, t^{-1}p).$$

Note that, if said limit exists, it is always 1-homogeneous in p,

$$a_*(y,\lambda p) = \lim_{t\to 0} t a(y,(\lambda^{-1}t)^{-1}p) = \lambda a_*(y,p).$$

In other words we need a to be 1-homogeneous in p at ∞; then the operator a_* is this limiting homogeneous profile of a at x_0.

The above discussion motivates our assumption on the operators we study in the half-space problem:

(i) Periodicity:

$$a(x+z,p) = a(x,p) \quad \text{for all } x \in \mathbb{R}^d, z \in \mathbb{Z}^d, p \in \mathbb{R}^d. \tag{4-2}$$

(ii) Ellipticity: for some $\lambda > 0$ and all $p, q \in \mathbb{R}^d$

$$(a(x,p)-a(x,q))\cdot(p-q) \geq \lambda|p-q|^2 \quad \text{and} \quad |a(x,p-a(x,q)| \leq |p-q|. \tag{4-3}$$

(iii) Positive homogeneity: for all x, p and $t > 0$,

$$a(x,tp) = ta(x,p). \tag{4-4}$$

For convenience will also assume $a(x,p)$ is C^1 in x so that, by the De Giorgi regularity theorem, solutions are locally $C^{1,\alpha}$ for some universal $\alpha > 0$.

4A. ***Regularity estimates for nonlinear equations.*** In this section we explain the regularity estimates which we use to obtain (1) existence of boundary layer limits and (2) the characterization of limits at rational directions. For both results we need the De Giorgi estimates respectively for the interior and boundary. As is the usual approach for regularity of nonlinear equations, we can reduce to considering actually the regularity of linear equations but with only bounded measurable coefficients.

For what follows we will take $A : \mathbb{R}^d \to M_{d\times d}$ to be measurable and elliptic,

$$\lambda \leq A(x) \leq 1.$$

Recall that results for bounded measurable coefficients imply results for solutions of nonlinear uniformly elliptic equations and for the difference of two solutions. If $u_1, u_2 \in H^1_{\mathrm{loc}}(\Omega)$ solve

$$-\nabla\cdot a(x,\nabla u_j) = 0 \quad \text{in } \Omega$$

then $w = u_1 - u_2$ solves

$$-\nabla\cdot(A(x)\nabla w) = 0 \quad \text{in } \Omega \quad \text{with } A(x) = \int_0^1 D_p a(x, s\nabla u_1 + (1-s)\nabla u_2)\,ds, \tag{4-5}$$

and one can easily check that $\lambda \leq A(x) \leq 1$.

We remind that, despite the overlap of notation, the results in this section apply to solutions of scalar equations not systems.

Theorem 4.1 (De Giorgi–Nash–Moser). *There is an $\alpha \in (0,1)$ and $C > 0$ depending on d, λ so that if u solves*

$$-\nabla \cdot (A(x)\nabla u) = 0 \quad \text{in } B_1$$

then,

$$[u]_{C^\alpha(B_{1/2})} \le C \inf_{c\in\mathbb{R}} \|u - c\|_{L^2(B_1)}.$$

A similar result holds up to the boundary for regular domains. We say that Ω is a regular domain of $\mathbb{R}^d$ if there are $r_0, \mu > 0$ so that for every $x \in \partial\Omega$ and every $0 < r < r_0$,

$$|\Omega^C \cap B_r(x)| \ge \mu |B_r|.$$

Lemma 4.2. *Suppose that Ω is a regular domain, $r_0 \ge 1$ and $0 \in \partial\Omega$, and $\varphi \in C^\beta$. There is an $\alpha_0(d,\lambda,\mu) \in (0,1)$ such that for $0 < \alpha < \min\{\alpha_0, \beta\}$ there is $C(d,\lambda,\mu,\alpha) > 0$ so that if u solves*

$$-\nabla \cdot (A(x)\nabla u) = 0 \quad \text{in } B_1 \cap \Omega, \quad \text{with } u = \varphi \text{ on } \partial\Omega,$$

then for every $r \le 1$,

$$\operatorname*{osc}_{B_r} u \le C\big([\varphi]_{C^\beta(B_1)} + \inf_{c\in\mathbb{R}} \|u - c\|_{L^2(B_1)}\big) r^\alpha.$$

The proof is postponed to Appendix A. We make a remark on the optimality of this estimate. Using these results one can show local $C^{1,\alpha}$ estimates for solutions of nonlinear uniformly elliptic equations. Large-scale $C^{1,\alpha}$ estimates are not possible due to the x-dependence, but in the spirit of [Avellaneda and Lin 1991] one can prove large-scale Lipschitz estimates; this was done in [Moser and Struwe 1992]. See also [Armstrong and Smart 2016] for the stochastic case. These estimates however are for *solutions*, we seem to require the result of Lemma 4.2 for *differences* of solutions (i.e., basically it is a C^α estimate of a derivative). It is not clear, therefore, whether we can do better than Lemma 4.2.

4B. *Half-space problem.* We consider the basic well-posedness results for nonlinear problems set in half-spaces. Consider

$$\begin{cases} -\nabla \cdot a(x, \nabla u) = 0 & \text{in } P_n, \\ u = \varphi(x) & \text{on } \partial P_n. \end{cases} \tag{4-6}$$

Then the maximum principle holds.

Lemma 4.3. *Suppose u_1 and u_2 are respectively bounded subsolutions and supersolutions of* (4-6) *with boundary data $\varphi_1 \le \varphi_2$ on ∂P_n; then,*

$$u_1 \le u_2 \quad \text{in } P_n.$$

The result follows from Lemma 4.2 or, more precisely, its proof. The proof is postponed to Appendix A.

4C. *Homogenization of nonoscillatory Dirichlet problem.* In this section we recall quantitative homogenization results for nonlinear divergence form problems in bounded domains with regular Dirichlet boundary condition. We will refer mainly to [Armstrong and Smart 2016]; they considered the stochastic case but their arguments also apply to the periodic case. The problem has also been studied in [Cardone, Pastukhova, and Zhikov 2005; Pastukhova 2008].

More precisely we study the limit

$$\begin{cases} -\nabla\cdot a\big(\frac{x}{\varepsilon},\nabla u^\varepsilon\big)=0 & \text{in } \Omega,\\ u^\varepsilon(x)=g(x) & \text{on } \partial\Omega \end{cases} \quad \text{to} \quad \begin{cases} -\nabla\cdot a^0(\nabla u^0)=0 & \text{in } \Omega,\\ u^0(x)=g(x) & \text{on } \partial\Omega, \end{cases} \tag{4-7}$$

where the boundary data g is a trace of $g\in W^{1,p}(\Omega)$ for some $p>2$. The following result is a combination of Proposition 4.1 and Corollary 4.2 in [Armstrong and Smart 2016] adapted to the periodic setting.

Theorem 4.4 [Armstrong and Smart 2016]. *Let $\Omega\subseteq\mathbb{R}^d$ be a bounded Lipschitz domain and $p>2$. Fix $\varepsilon\in(0,1]$ and let $u^\varepsilon, u^0\in g+H_0^1(\Omega)$ satisfying* (4-7). *There exist constants $C(d,\lambda,p,\Omega)\geq 1$ and $\beta(d,\lambda,p)\in(0,1]$ such that*

$$\|u^\varepsilon-u^0\|_{L^2_{\mathrm{avg}}(\Omega)}\leq C\varepsilon^\beta\|\nabla g\|_{L^p_{\mathrm{avg}}(\Omega)}.$$

By interpolating the L^2 estimate with the interior and boundary regularity, Theorem 4.1 and Lemma 4.2, there exist constants $C'(d,\lambda,\Omega)\geq 1$ and $\beta'(d,\lambda)\in(0,1]$ such that

$$\sup_\Omega |u^\varepsilon-u^0|\leq C'\varepsilon^{\beta'}\|\nabla g\|_{L^\infty(\partial\Omega)}.$$

Actually Corollary 4.2 of [Armstrong and Smart 2016] only does the interpolation argument for interior points; adding in the boundary regularity Lemma 4.2 to get the uniform estimate up to $\partial\Omega$ is an elementary argument. There are additional error terms in [Armstrong and Smart 2016] but these can be made zero in the periodic setting using the existence of periodic correctors.

5. Boundary layers limits

In this section we will discuss the boundary layer problem for divergence form elliptic problems in rational and irrational half-spaces. The results that we need for this paper are valid for both nonlinear scalar equations and linear systems and the proofs have only minor differences. For that reason, in this section and the next, we will discuss both types of equations in a unified way. We use the nonlinear notation for the PDE. We consider the cell problem

$$\begin{cases} -\nabla\cdot a(y,\nabla v_n^s)=0 & \text{in } P_n^s,\\ v_n^s=\varphi(y) & \text{on } \partial P_n^s. \end{cases} \tag{5-1}$$

We will first consider the case when $n\in S^{d-1}\setminus\mathbb{R}\mathbb{Z}^d$ is irrational.

5A. ***Irrational half-spaces.*** For linear systems, (5-1) in irrational half-spaces has been much studied [Gérard-Varet and Masmoudi 2011; 2012; Aleksanyan, Shahgholian, and Sjölin 2015; Aleksanyan 2017; Armstrong, Kuusi, Mourrat, and Prange 2017; Prange 2013; Shen and Zhuge 2017]. Typically the focus has been on the Diophantine irrational directions. We do not give the definition, since it is not needed for our work, but basically the Diophantine condition is a quantification of the irrationality. Under this assumption strong quantitative results can be derived for the convergence to the boundary layer limit.

For the purposes of this paper we are only interested in the qualitative result, the existence of a boundary layer limit for (5-1) in a generic irrational half-space (no Diophantine assumption). The existence of a boundary layer tail in general irrational half-spaces was originally proven by Prange [2013] for divergence form linear systems, and for nonlinear nondivergence form equations by the first author in [Feldman and Kim 2017] (following [Choi and Kim 2014] on the Neumann problem). To our knowledge the case of nonlinear divergence form equations has not been studied yet.

What we would like to explain here is that the proof of [Feldman and Kim 2017] applies also to the problems we consider in this paper, careful inspection shows that the proof of [Feldman and Kim 2017] only required the interior regularity, continuity up to the boundary (small-scale), and the L^∞ estimate (or maximum principle) with respect to the boundary data.

Theorem 5.1. *Suppose that* $n \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$. *Then there exists* $\varphi_*(n)$ *such that*

$$\sup_s \sup_{y \in \partial P_n} |v_n^s(y + Rn) - \varphi_*(n)| \to 0 \quad \textit{as } R \to \infty.$$

One consequence of this theorem is that, for irrational directions, we can just study $v_n = v_n^0$. We give a sketch of the proof following [Feldman 2014].

Proof (*sketch*). The boundary data, and hence the solution v_n^s as well by uniqueness (L^∞ estimate Lemma 3.2 or maximum principle Lemma 4.3), satisfies an almost periodicity property in the directions parallel to ∂P_n. More precisely, given $N \geq 1$ there is a modulus $\omega_n(N) \to 0$ as $N \to \infty$ (uses n irrational) so that for any $y \in \partial P_n$ there is a lattice vector $z \in \mathbb{Z}^d$ with $|z - y| \leq N$ and $|z \cdot n - s| \leq \omega(N)$; see [Feldman and Kim 2017, Lemma 2.3]. Define z' to be the projection of z onto ∂P_n^s; then

$$|\varphi(x + z') - \varphi(x)| \leq |\varphi(x + z) - \varphi(x + z')| \leq \|\nabla\varphi\|_\infty \omega(N).$$

The same estimate, up to a universal constant, holds for $|v_n^s(x + z') - v_n^s(x)|$ by Lemma 3.2 or Lemma 4.3.

Since $v_n^0(\,\cdot + z)$ solves the same equation in $P_n^{z \cdot n}$, we can use the up-to-the-boundary Hölder continuity and the L^∞ estimate (or maximum principle) to see that

$$\|v_n^s(\cdot) - v_n^0(\,\cdot\, + z)\|_{L^\infty(P_n^s \cap P_n^{z \cdot n})} \lesssim \|\nabla\varphi\|_\infty \omega_n(N)^\alpha.$$

Sending N to ∞ we see that if v_n^0 has a boundary layer limit then so does v_n^s and they have the same value.

Then we just need to argue for v_n^0. Given $y \in \partial P_n$ the same argument as above shows there is $\bar{z} \in \partial P_n$ with $|\bar{z} - y| \leq N$ and

$$|v_n^0(\cdot) - v_n^0(\,\cdot\, + \bar{z})| \lesssim \|\nabla\varphi\|_\infty \omega_n(N)^\alpha.$$

Then using the L^∞ estimate Lemma 3.2 (or the maximum principle) and the large-scale interior regularity estimates, Theorem 4.1 above for the nonlinear case or Lemma 9 in [Avellaneda and Lin 1987] for the linear systems case,

$$\begin{aligned}\operatorname*{osc}_{y \cdot n \geq R} v_n^s(y) \lesssim \operatorname*{osc}_{y \cdot n = R} v_n^s(y) &\leq \operatorname*{osc}_{y \in B_N(0) \cap \partial P_n} v_n^s(y + Rn) + C\|\nabla\varphi\|_\infty \omega_n(N)^\alpha \\ &\lesssim \|\nabla\varphi\|_\infty \Big(\Big(\frac{N}{R}\Big)^\alpha + \omega_n(N)^\alpha\Big).\end{aligned}$$

Choosing N large first to make $\omega_n(N)$ small and then $R \gg N$ gets the existence of a boundary layer limit. □

5B. ***Rational half-spaces.*** Next we consider the case of a rational half-space. Let $\xi \in \mathbb{Z}^d \setminus \{0\}$ be an irreducible lattice direction, and v^s_ξ be the corresponding half-space problem solution. In this case φ is periodic with respect to a $(d-1)$-dimensional lattice parallel to ∂P_ξ. There exist $\ell_1, \dots, \ell_{d-1}$ with $\ell_j \perp \xi$ and $|\ell_j| \le |\xi|$ which are periods of φ. Then by uniqueness ℓ_j are also periods of v^s_ξ. In this special situation it is possible to show that there is a boundary layer limit with an exponential rate of convergence.

We give a general set-up. We consider the half-space problem

$$\begin{cases} -\nabla \cdot a(x, \nabla v) = \nabla \cdot f & \text{in } \mathbb{R}^d_+, \\ v = \psi(x') & \text{on } \partial\mathbb{R}^d_+, \end{cases} \tag{5-2}$$

where $\psi : \partial\mathbb{R}^d_+ \to \mathbb{R}$ and f are smooth, and ψ, f, and $a(\,\cdot\,, p)$ all share $d-1$ linearly independent periods $\ell_1, \dots, \ell_{d-1} \in \partial\mathbb{R}^d_+$ such that

$$\max_{1 \le j \le d-1} |\ell_j| \le M.$$

The operators a, as always, will also satisfy the assumptions of either Section 3 or Section 4. For now we will take $f = 0$; this covers most of the situations we will run into in this paper. Then v has a boundary layer limit with exponential rate of convergence.

Lemma 5.2. *There exists a value $c_*(\psi)$ such that*

$$\sup_{y \in \partial\mathbb{R}^d_+} |v(y + Re_d) - c_*| \le C(\operatorname{osc} \psi) e^{-cR/M},$$

with $C, c > 0$ depending only on λ, d.

The proof of this result is the same as the proof of the analogous result, [Feldman and Kim 2017, Lemma 3.1], so we only include a sketch. The only tools necessary are the maximum principle (or L^∞ estimate Lemma 3.2) and the large-scale interior Hölder estimates via De Giorgi–Nash–Moser for nonlinear equations (Theorem 4.1) or [Avellaneda and Lin 1987, Lemma 9] for linear systems.

Proof (*sketch*). Let $L \ge 1$ to be chosen, call Q to be the unit periodicity cell of ψ which has diameter at most $\sim M$. Apply the De Giorgi interior Hölder estimates or the Avellaneda–Lin large-scale Hölder estimates to find

$$\operatorname*{osc}_{\partial P_n + LMn} v = \operatorname*{osc}_{y \in Q} v(y + LMn) \le CL^{-\alpha} \operatorname*{osc}_{P_n} u \le CL^{-\alpha} \operatorname{osc} \psi \le \tfrac{1}{2} \operatorname{osc} \psi.$$

The second inequality is by the maximum principle or the L^∞ estimate Lemma 3.2; for the third inequality we have chosen $L \ge 1$ universal to make $CL^{-\alpha} \le \frac{1}{2}$. Then iterate the argument with the new boundary data on $\partial P_n + LMn$ with oscillation decayed by a factor of $\frac{1}{2}$. □

We will also need a slight variant of the above result when the operator a does not share the same periodicity as the boundary data, but instead has oscillations at a much smaller scale. We assume that ψ has periods $\ell_1, \dots, \ell_{d-1}$ as before, and now we also assume that there are $e_1, \dots, e_d$ which are periods of a and

$$\max_{1 \le j \le d} |e_j| \le \varepsilon.$$

For example this is the case with $a\big(\frac{x}{\varepsilon}, p\big)$ when $a(\cdot, p)$ is $\mathbb{Z}^d$-periodic. In this situation we do not quite have a boundary layer limit with exponential rate, but at least there is an exponential decay of the oscillation down to a scale $\sim \varepsilon^\alpha$.

Lemma 5.3. *There exists a value $c_*(\psi)$ such that for any $\beta \in (0,1]$*

$$\sup_{y \in \partial \mathbb{R}^d_+} |v(y + Re_d) - c_*| \le C(\operatorname{osc} \psi) e^{-cR/M} + C\|\psi\|_{C^\beta} \varepsilon^\alpha$$

for some universal $\alpha(\beta) \in (0,1)$ (nonlinear case) or for every $\alpha \in (0, \beta)$ (linear case), with $c, C > 0$ universal and C depending on α as well.

Again the proof of this result mirrors the proof of Lemma 3.2 in [Feldman and Kim 2017] and we do not include it. Briefly, the idea is the same as Lemma 5.2 except that the lattice vectors generated by $\ell_1, \dots, \ell_{d-1}$ are no longer periods of v; instead for each lattice vector there is a nearby vector (distance at most ε) which is a period of the operator. This vector will almost be a period of v, with error of ε^α which comes from the boundary continuity estimate Lemma 4.2 (nonlinear) or Lemma 3.4 (linear system).

Finally we discuss the boundary layer equation (5-1) with nonzero right-hand side f. We will restrict to the case of linear systems. We need to put a decay assumption on f to guarantee even the existence of a solution. We will assume that there are $K, b > 0$ so that

$$\sup_{y_d \ge R} |f(y)| \le \frac{K}{R} e^{-bR/M}. \tag{5-3}$$

Such assumption arises naturally; it is exactly the decay obtained for ∇v when v solves (5-1) with $f = 0$. The $\frac{1}{R}$ polynomial decay is important since we will care about the dependence on $M \gg 1$; the exponential does not take effect until $R \gg M$, while the $\frac{1}{R}$ decay begins at the unit scale.

Lemma 5.4. *Suppose that f satisfies the bound* (5-3) *and v is the solution of the half-space equation* (5-1) *for a linear system satisfying the standard assumptions of Section 3. Then there exists $c_*(\psi, f)$ such that*

$$\sup_{y \in \partial \mathbb{R}^d_+} |v(y + Re_d) - c_*| \le C((\operatorname{osc} \psi) + K \log M) e^{-b_0 R/M},$$

where the constants C and b_0 depend on universal parameters as well as b from (5-3).

See the Appendix and [Feldman and Kim 2017, Lemma A.4] for more details.

5C. ***Interior homogenization of a boundary layer problem.*** In this section we will consider the *interior* homogenization of half-space problems with periodic boundary data; as explained in Section 2 such a problem arises in the course of computing the directional limits of φ_* at a rational direction:

$$\begin{cases} -\nabla \cdot a\big(\frac{x}{\varepsilon}, \nabla u^\varepsilon\big) = 0 & \text{in } P_n, \\ u^\varepsilon = \psi(x) & \text{on } \partial P_n \end{cases} \tag{5-4}$$

homogenizing to

$$\begin{cases} -\nabla \cdot a^0(\nabla u^0) = 0 & \text{in } P_n, \\ u^0 = \psi(x) & \text{on } \partial P_n. \end{cases} \tag{5-5}$$

Here $\psi : \partial P_n \to \mathbb{R}^N$, as in the previous section, will be smooth and periodic with respect to $d-1$ linearly independent translations parallel to ∂P_n, which we call $\ell_1, \dots, \ell_{d-1} \in \partial P_n$. As before we call $M = \max_j |\ell_j|$ and assume that $M \gg \varepsilon$. For convenience we can assume that $M = 1$; general results can be derived by scaling.

This problem is quite similar to the standard homogenization problem for Dirichlet boundary data, the unboundedness of the domain is compensated by the periodicity of the boundary data and by the existence of a boundary layer limit which is a kind of (free) boundary condition at infinity. The main result of this section is the *uniform* convergence of u^ε to u^0, and hence also (importantly for us) the convergence of the boundary layer limits.

Proposition 5.5. *Homogenization holds for* (5-4) *with estimates*:

(i) (nonlinear equations) *For every* $\beta \in (0,1)$, *there exists* $0 < \alpha(\beta, \lambda, d) \le \beta$ *such that, for all* $\varepsilon \le \frac{1}{2}$,

$$\sup_{P_n} |u^\varepsilon - u^0| \lesssim_\beta [\psi]_{C^\beta} \varepsilon^\alpha.$$

(ii) (linear systems) *For every* $\varepsilon \le \frac{1}{2}$,

$$\sup_{P_n} |u^\varepsilon - u^0| \lesssim [\psi]_{C^4} \varepsilon \Big(\log \frac{1}{\varepsilon}\Big)^3.$$

We will follow the idea of [Feldman and Kim 2017, Lemma 4.5]; there is a slight additional difficulty since for divergence form nonlinear problems it is not possible to add a linear function $n \cdot x$ and preserve the solution property, even for the homogenized problem. The C^4 norm we require for ψ in the linear systems case is more than necessary.

For convenience we will make some additional assumptions so that u^ε shares the periods of the boundary data ψ. Assume that $n = \xi/|\xi|$ for an irreducible lattice direction $\xi \in \mathbb{Z}^d \setminus \{0\}$. In that case $a(\cdot, p)$ is periodic with respect to the lattice $\xi^\perp \cap \mathbb{Z}^d = \{k \in \mathbb{Z}^d : k \cdot \xi = 0\}$. Then we assume that the periods of ψ are also periods of $a\big(\frac{\cdot}{\varepsilon}, p\big)$,

$$\ell_1, \dots, \ell_{d-1} \in \varepsilon \xi^\perp \cap \mathbb{Z}^d. \tag{5-6}$$

Then by the uniqueness of bounded solutions to (5-4) the solution u^ε also has $\ell_1, \dots, \ell_{d-1}$ as periods. The result of Proposition 5.5 should hold without this assumption, as was proven in the nondivergence form case in [Feldman and Kim 2017, Lemma 4.5].

The proof will use known results about homogenization of Dirichlet boundary value problems in bounded domains; specifically we consider the problem in a strip-type domain,

$$\begin{cases} -\nabla \cdot a\big(\frac{x}{\varepsilon}, \nabla u_R^\varepsilon\big) = 0 & \text{in } \Pi_n(0,R) = \{0 < x \cdot n < R\}, \\ u_R^\varepsilon = \psi(x) & \text{on } \partial \Pi_n(0,R) = \{x \cdot n \in \{0, R\}\}, \end{cases} \tag{5-7}$$

where we make some choice to extend ψ to $x \cdot n = R$, preserving the regularity and periodic structure. The solution of the homogenized problem u_R^0 is defined analogously. Because of (5-6), u_R^ε and u_R^0 have periods $\ell_1, \dots, \ell_{d-1}$, so although the domain $\Pi_n(0,R)$ is unbounded, actually we can consider (5-7) as a homogenization problem on the bounded domain $\mathbb{T}^{d-1} \times [0, R]$, or rather a rotation/rescaling of this domain.

For linear systems we have, for $R \geq 1$, the rate for convergence

$$\sup_{\Pi_n(0,R)} |u^\varepsilon_R - u^0_R| \leq C R^4 \|\psi\|_{C^4} (R^{-1}\varepsilon), \tag{5-8}$$

which can be derived from the rate of convergence proved in [Avellaneda and Lin 1991] by scaling. The C^4 regularity on ψ is sufficient; we did not state the precise regularity requirement on ψ which can be found in [Avellaneda and Lin 1991]. With less regularity on ψ one can also obtain an algebraic rate of convergence $O(\varepsilon^\alpha)$.

For nonlinear equations there is an algebraic rate of convergence, for any $\beta \in (0, 1)$,

$$\sup_{\Pi_n(0,R)} |u^\varepsilon_R - u^0_R| \leq C R^\beta \|\psi\|_{C^{0,\beta}} (R^{-1}\varepsilon)^\alpha, \tag{5-9}$$

with some $\alpha = \alpha(\beta) \in (0, 1)$ universal. This result was recounted above in Section 4C, and can be found in [Armstrong and Smart 2016; Pastukhova 2008].

Proof of Proposition 5.5. We define the boundary layer limits of, respectively, the ε-problem and the homogenized problem in (5-4). We have not proven that the ε-problem has a boundary layer limit; however Lemma 5.3 gives that the limit values are concentrated in a set of diameter $o_\varepsilon(1)$. So we define,

$$\mu^\varepsilon \in \lim_{R\to\infty} u^\varepsilon(Rn) \quad \text{and} \quad \mu^0 = \lim_{R\to\infty} u^0(Rn),$$

where μ^ε can be any subsequential limit and satisfies, again via Lemma 5.3,

$$|\mu^\varepsilon - u^\varepsilon(Rn)| \leq C\|\nabla\psi\|_\infty(\varepsilon^\alpha + e^{-cR}) \quad \text{(nonlinear case)}, \tag{5-10}$$

$$|\mu^\varepsilon - u^\varepsilon(Rn)| \leq C\|\nabla\psi\|_{C^{0,\nu}}(\varepsilon + e^{-cR}) \quad \text{(linear system case)}. \tag{5-11}$$

Instead of arguing directly with u^ε and u^0 we consider

$$\begin{cases} -\nabla\cdot a\big(\frac{x}{\varepsilon}, \nabla u^\varepsilon_R\big) = 0 & \text{in } \Pi_n(0, R), \\ u^\varepsilon_R = \psi(x) & \text{on } x\cdot n = 0, \\ u^\varepsilon_R = \mu^\varepsilon & \text{on } x\cdot n = R \end{cases} \tag{5-12}$$

and, for $j \in \{0, \varepsilon\}$

$$\begin{cases} -\nabla\cdot a^0(\nabla u^0_{R,j}) = 0 & \text{in } \Pi_n(0, R), \\ u^0_{R,j} = \psi(x) & \text{on } x\cdot n = 0, \\ u^0_{R,j} = \mu^j & \text{on } x\cdot n = R. \end{cases} \tag{5-13}$$

We will choose $R = R(\varepsilon)$ below to balance the various errors. The error in replacing u^ε by u^ε_R is given by

$$|u^\varepsilon(x) - u^\varepsilon_R(x)| \leq C\|\nabla\psi\|_\infty(\varepsilon^\alpha + e^{-cR}) \quad \text{for } x \in \Pi_n(0, R),$$

and replacing u^0 by $u^0_{R,0}$ by

$$|u^0(x) - u^0_{R,0}(x)| \leq C(\operatorname{osc}\psi)e^{-cR} \quad \text{for } x \in \Pi_n(0, R);$$

the estimates hold on $\partial\Pi_n(0, R)$ by (5-10) (or for linear we use (5-11) instead), and therefore by the maximum principle (or by Lemma 3.2 for linear systems) they hold on the interior as well. To estimate the error in replacing $u^0_{R,0}$ by $u^0_{R,\varepsilon}$ we need to estimate the difference $\mu^\varepsilon - \mu^0$, which is basically the goal of the proof; this will be achieved below.

By Lemma 4.2 (or Lemma 3.3 in the linear systems case) there exists a universal $\delta_0(\lambda, d) > 0$ so that if B is uniformly elliptic and q solves

$$\begin{cases} -\nabla\cdot(B(x)\nabla q) = 0 & \text{in } \Pi_n(0,1), \\ q = 0 & \text{on } x\cdot n = 0, \\ |q| = 1 & \text{on } x\cdot n = 1; \end{cases} \tag{5-14}$$

then $|q(x)| \le \frac{1}{2}$ for $x\cdot n \le \delta_0$. Now set

$$q^\varepsilon = u^0_{R,0} - u^0_{R,\varepsilon}, \quad \text{which solves } \begin{cases} -\nabla\cdot(B(x)\nabla q^\varepsilon) = 0 & \text{in } 0 < x\cdot n < R, \\ q^\varepsilon = 0 & \text{on } x\cdot n = 0, \\ q^\varepsilon = \mu^0 - \mu^\varepsilon & \text{on } x\cdot n = R, \end{cases}$$

with $B(x) = A^0$ in the linear case, or

$$B(x) = \int_0^1 Da^0(t\nabla u^0_{R,0}(x) + (1-t)\nabla u^0_{R,\varepsilon}(x))\,dt \quad \text{uniformly elliptic,}$$

in the nonlinear case. Now $(1/|\mu^0 - \mu^\varepsilon|)q(Rx)$ solves an equation of the type (5-14) and so,

$$|q(\delta_0 Rn)| \le \tfrac{1}{2}|\mu^0 - \mu^\varepsilon|.$$

Now we apply the homogenization error estimates (5-9) and (5-8) for the domain $\Pi_n(0, R)$ to (5-12)

$$|u^0_{R,\varepsilon} - u^\varepsilon_R| \le CR\|\nabla\psi\|_\infty (R^{-1}\varepsilon)^\gamma$$

or respectively in the linear system case

$$|u^0_{R,\varepsilon} - u^\varepsilon_R| \le CR^4\|\psi\|_{C^4}(R^{-1}\varepsilon).$$

Now we estimate the error in $\mu^\varepsilon - \mu^0$ for the nonlinear case

$$\begin{aligned} |\mu^\varepsilon - \mu^0| &\le |u^\varepsilon(\delta_0 Rn) - u^0(\delta_0 Rn)| + C\|\nabla\psi\|_\infty(\varepsilon^\alpha + e^{-cR}) \\ &\le |u^\varepsilon_R(\delta_0 Rn) - u^0_{R,\varepsilon}(\delta_0 Rn)| + |q^\varepsilon(\delta_0 Rn)| + C\|\nabla\psi\|_\infty(\varepsilon^\alpha + e^{-cR}) \\ &\le CR\|\nabla\psi\|_\infty(R^{-1}\varepsilon)^\gamma + \tfrac{1}{2}|\mu^\varepsilon - \mu^0| + C\|\nabla\psi\|_\infty(\varepsilon^\alpha + e^{-cR}). \end{aligned}$$

Moving the middle term above to the left-hand side we find,

$$|\mu^\varepsilon - \mu^0| \le C\|\nabla\psi\|_\infty(R(R^{-1}\varepsilon)^\gamma + \varepsilon^\alpha + e^{-cR}) \le C\|\nabla\psi\|_\infty \varepsilon^{\alpha'},$$

where finally we have chosen $R = C\log\frac{1}{\varepsilon}$ and $\alpha' < \min\{\alpha, \gamma\}$. The same argument in the linear case yields,

$$|\mu^\varepsilon - \mu^0| \le C[\psi]_{C^4}(R^4(R^{-1}\varepsilon) + \varepsilon + e^{-cR}) \le C[\psi]_{C^4}\varepsilon\Big(\log\frac{1}{\varepsilon}\Big)^3. \qquad \square$$

6. Asymptotics near a rational direction

We study asymptotic behavior of the cell problems as $n \in S^{d-1}$ approaches a rational direction $\xi \in \mathbb{Z}^d \setminus \{0\}$. We call v_ξ^s the solution of the cell problem

$$\begin{cases} -\nabla \cdot a(x + s\xi, \nabla v_\xi^s) = 0 & \text{in } P_\xi, \\ v_\xi^s(x) = \varphi(x + s\xi) & \text{on } \partial P_\xi. \end{cases} \tag{6-1}$$

The boundary layer limit of the above cell problem depends on the parameter s and we define

$$\varphi_*(\xi, s) := \lim_{R \to \infty} v_\xi^s(x + R\xi), \tag{6-2}$$

which is well-defined and the limit is independent of x; see Lemma 5.2. It follows from Bézout's identity that φ_* is a $1/|\xi|$-periodic function on $\mathbb{R}$; see [Feldman and Kim 2017, Lemma 2.9]. As long as we can we will combine the arguments for linear systems and nonlinear equations.

6A. *Regularity of* $\varphi_*(\xi, \cdot)$. To begin we need to establish some regularity of $\varphi_*(\xi, \cdot)$. For quantitative purposes it is important to control the dependence of the regularity on $|\xi|$. We just state the results, postponing the proofs until the end of the section. A modulus of continuity for $\varphi_*(\xi, \cdot)$ which is uniform in $|\xi|$ is not difficult to establish. This follows from the continuity up to the boundary Lemma 4.2 (or Lemma 3.3) and the maximum principle Lemma 4.3 (or the L^∞ estimate Lemma 3.2).

Lemma 6.1. *The boundary layer limits $\varphi_*(\xi, s)$ are continuous in s:*

(i) (*nonlinear equations*)

$$[\varphi_*(\xi, \cdot)]_{C^\alpha} \le C \|\nabla \varphi\|_\infty,$$

which holds for some universal $C \ge 1$ and $\alpha \in (0, 1)$.

(ii) (*linear systems*) *Hölder estimates as above hold for all $\alpha \in (0, 1)$ and moreover,*

$$\left\| \frac{d}{ds} \varphi_*(\xi, \cdot) \right\|_\infty \le C \|\nabla \varphi\|_{C^{0,\nu}} \quad \textit{for any } 0 < \nu \le 1.$$

To optimize our estimates, in the linear case we will also need higher regularity of φ_* which is (almost) uniform in $|\xi|$; this is somewhat harder to establish.

Lemma 6.2 (linear systems). *For any $\xi \in \mathbb{Z}^d \setminus \{0\}$, suppose $\varphi_*(\xi, s)$ is defined as above. Then for all $j \in \mathbb{N}^d$ and any $\nu > 0$ there exists some constant C_j universal such that*

$$\sup_s \left| \frac{d^j}{ds^j} \varphi_*(\xi, s) \right| \le C_j \|\varphi\|_{C^{j,\nu}} \log^j(1 + |\xi|).$$

Note that Lemma 6.2 is a bit weaker than Lemma 6.1 in the case $j = 1$; this is because we take a different approach which is suboptimal in the $j = 1$ case; it is not clear if the logarithmic terms are necessary when $j > 1$. The proof is similar to [Feldman and Kim 2017, Lemma 7.2], taking the derivative of v_ξ^s with respect to s and estimating based on the PDE. Probably more precise Sobolev estimates are possible but we did not pursue this.

6B. ***Intermediate-scale asymptotics.*** Consider an irrational direction n close to a lattice direction $\xi \in \mathbb{Z}^d \setminus \{0\}$. Let $\varepsilon > 0$ small and we write

$$n = (\cos\varepsilon)\hat{\xi} - (\sin\varepsilon)\eta \quad \text{for some } \xi \in \mathbb{Z}^d \setminus \{0\} \text{ and a unit vector } \eta \perp \xi.$$

We will assume below that $|\varepsilon| \le \frac{\pi}{6}$. We consider the cell problem in P_n

$$\begin{cases} -\nabla \cdot a(y, \nabla v_n) = 0 & \text{in } P_n, \\ v_n = \varphi(y) & \text{on } \partial P_n. \end{cases} \tag{6-3}$$

The first step of the argument is to show, with error estimate, that the boundary layer limit of v_n is close to the boundary layer limit of the problem

$$\begin{cases} -\nabla \cdot a(y/\tan\varepsilon, \nabla v_n^{\text{int}}) = 0 & \text{in } P_n, \\ v_n^{\text{int}} = \varphi_*(\xi, y \cdot \eta) & \text{on } \partial P_n. \end{cases} \tag{6-4}$$

The solution v_n^{int} approximates v_n, asymptotically as $\varepsilon \to 0$, starting at an intermediate scale $1 \ll R \ll \frac{1}{\varepsilon}$ away from ∂P_n. The argument is by direct comparison of v_n with v_ξ^s in their common domain.

Since (6-4) has a boundary layer of size uniform in ε we can replace, again with small error, by a problem in a fixed domain

$$\begin{cases} -\nabla \cdot a(y/\tan\varepsilon, \nabla w_{\xi,\eta}^{\varepsilon}) = 0 & \text{in } P_\xi, \\ w_{\xi,\eta}^{\varepsilon} = \varphi_*(\xi, y \cdot \eta) & \text{on } \partial P_\xi. \end{cases} \tag{6-5}$$

We note that there may be some confusion due to similarities in the notation between v_ξ^s and $w_{\xi,\eta}^{\varepsilon}$. The boundary value problem for $w_{\xi,\eta}^{\varepsilon}$, or its homogenized version introduced later, will always be set in P_ξ, so there will be no need for the translation parameter s.

We remark that for both (6-4) and (6-5) we have not proven the existence of a boundary layer limit; rather we use Lemma 5.3. For convenience we will state estimates on $\lim_{R\to\infty} v_n^{\text{int}}(Rn)$ or on $\lim_{R\to\infty} w_{\xi,\eta}^{\varepsilon}(R\hat{\xi})$, but technically we will mean that the estimate holds for every subsequential limit.

Proposition 6.3. *Let $\xi \in \mathbb{Z}^d \setminus \{0\}$ and $n = (\cos\varepsilon)\hat{\xi} - (\sin\varepsilon)\eta$ with $\varepsilon > 0$ small and a unit vector $\eta \perp \xi$:*

(i) (*nonlinear equations*) *There is universal $\alpha \in (0,1)$ such that*

$$\Big|\varphi_*(n) - \lim_{R\to\infty} w_{\xi,\eta}^{\varepsilon}(R\hat{\xi})\Big| \lesssim \|\nabla\varphi\|_\infty |\xi|^\alpha \varepsilon^\alpha,$$

where we mean that the estimate holds for any subsequential limit of $w_{\xi,\eta}^{\varepsilon}(R\hat{\xi})$ as $R \to \infty$.

(ii) (*linear systems*) *For every $\alpha \in (0,1)$ and any $\nu > 0$*

$$\Big|\varphi_*(n) - \lim_{R\to\infty} w_{\xi,\eta}^{\varepsilon}(R\hat{\xi})\Big| \lesssim_{\alpha,\nu} [\varphi]_{C^{1,\nu}} |\xi|^\alpha \varepsilon^\alpha,$$

where again we mean that the estimate holds for any subsequential limit of $w_{\xi,\eta}^{\varepsilon}(R\hat{\xi})$ as $R \to \infty$.

The first step is to compare the boundary layer limits of (6-3) and (6-4).

Lemma 6.4. *Fix any* $x \in \partial P_n$, $1 \leq R \leq \frac{1}{\varepsilon}$ *and let* $s = x \cdot \eta \tan \varepsilon$:

(i) (*nonlinear equations*) *There is a universal* $\alpha \in (0,1)$ *such that*

$$|v_n - v^s_\xi|(x + Rn) \lesssim \|\nabla \varphi\|_\infty (R\varepsilon)^\alpha.$$

(ii) (*linear systems*) *For every* $\alpha \in (0,1)$

$$|v_n - v^s_\xi|(x + Rn) \lesssim_\alpha \|\nabla \varphi\|_\infty (R\varepsilon)^\alpha.$$

Proof. Let us define the cone domains

$$K(x) := (P_\xi + x) \cap P_n \quad \text{and} \quad K_R(x) = K(x) \cap B_R(x);$$

we may simply write K, K_R if $x = 0$. Let $x_0 \in \partial P_n$; we compute using $n \cdot x_0 = 0$ and $n = (\cos \varepsilon)\hat{\xi} - (\sin \varepsilon)\eta$ that

$$x_0 \cdot \hat{\xi} = (x_0 \cdot \eta) \tan \varepsilon.$$

Let $x \in \partial K(x_0)$; then $x \in \partial P_n$ (or $x \in \partial P_\xi + x_0$) and there exists $y \in \partial P_\xi + x_0$ (or respectively ∂P_n) with

$$|x - y| \leq |x - x_0| \sin \varepsilon \leq \varepsilon |x - x_0|.$$

Nonlinear equations: Applying the De Giorgi boundary continuity estimates Lemma 4.2 for small enough $\alpha \in (0,1)$ universal, for all $x \in \partial K(x_0)$,

$$|v^s_\xi(x) - v_n(x)| \leq |v^s_\xi(x) - \varphi(y)| + |\varphi(y) - v_n(x)| \lesssim \|\nabla \varphi\|_\infty \varepsilon^\alpha |x - x_0|^\alpha.$$

Now since $v^s_\xi(x) - v_n(x)$ is a difference of solutions we can apply the boundary continuity estimate from Lemma 4.2 again,

$$|v^s_\xi(x) - v_n(x)| \lesssim \|\nabla \varphi\|_\infty \varepsilon^\alpha |x - x_0|^\alpha \quad \text{for } x \in K(x_0),$$

with perhaps a slightly smaller $\alpha(d, \lambda)$.

Linear systems: We have, by almost the same argument as above now using instead Lemma 3.3, for any $\alpha \in (0,1)$

$$|v^s_\xi(x) - v_n(x)| \lesssim \|\nabla \varphi\|_\infty \varepsilon^\alpha |x - x_0|^\alpha \quad \text{on } \partial K(x_0).$$

Now by the Poisson kernel bounds in $K(x_0)$, Lemmas 3.5 and 3.6, for a slightly smaller α and ε sufficiently small depending on α

$$|v^s_\xi(x) - v_n(x)| \lesssim \|\nabla \varphi\|_\infty \varepsilon^\alpha |x - x_0|^\alpha \quad \text{for } x \in K(x_0).$$

The remainder of the proof is the same as the case of scalar equations. □

Now we derive some consequences of Lemma 6.4. Let's assume that $\|\nabla \varphi\|_\infty \leq 1$ to simplify the exposition; the general inequalities can of course be derived by rescaling. Combining Lemma 5.2 with Lemma 6.4 we find that for any $R \geq 1$

$$|v_n(x + Rn) - \varphi_*(\xi, x \cdot \eta \tan \varepsilon)| \lesssim [(R\varepsilon)^\alpha + e^{-cR/|\xi|}] \quad \text{for } x \in \partial P_n.$$

Choosing $R = |\xi| \log \frac{1}{\varepsilon}$ we obtain,

$$|v_n(x + Rn) - \varphi_*(\xi, x \cdot \eta \tan \varepsilon)| \lesssim |\xi|^\alpha \varepsilon^\alpha \quad \text{for } x \in \partial P_n, \tag{6-6}$$

either for a slightly smaller universal α in the nonlinear case, or again for every $\alpha \in (0, 1)$ in the case of linear systems.

Now consider the rescaling

$$\tilde{v}_n^{\text{int}}(y) = v_n\left([Rn] + \frac{y}{\tan \varepsilon}\right) \quad \text{defined for } y \in P_n, \tag{6-7}$$

where $[Rn] \in \mathbb{Z}^d$ is the lattice point such that $Rn - [Rn] \in [0, 1)^d$.

Lemma 6.5. *Let* $R = |\xi| \log \frac{1}{\varepsilon}$ *and* $\tilde{v}_n^{\text{int}}$ *be defined as above in* (6-7). *Then*:

(i) (*nonlinear equations*) *There is universal* $\alpha \in (0, 1)$ *such that*

$$\sup_{P_n} |\tilde{v}_n^{\text{int}} - v_n^{\text{int}}| \lesssim \|\nabla \varphi\|_\infty |\xi|^\alpha \varepsilon^\alpha.$$

(ii) (*linear systems*) *For every* $\alpha \in (0, 1)$

$$\sup_{P_n} |\tilde{v}_n^{\text{int}} - v_n^{\text{int}}| \lesssim_\alpha \|\nabla \varphi\|_\infty |\xi|^\alpha \varepsilon^\alpha.$$

Proof. Again assume that $\|\nabla \varphi\|_\infty \le 1$ to simplify the exposition. Note that

$$\tilde{v}_n^{\text{int}}(y) = v_n\left(Rn + \frac{1}{\tan \varepsilon}(y + ([Rn] - Rn) \tan \varepsilon)\right)$$

so by (6-6)

$$|\tilde{v}_n^{\text{int}}(y) - \varphi_*(\xi, (y + ([Rn] - Rn) \tan \varepsilon) \cdot \eta)| \lesssim_\alpha |\xi|^\alpha \varepsilon^\alpha.$$

Then applying the regularity of φ_* from Lemma 6.1

$$|\tilde{v}_n^{\text{int}}(y) - \varphi_*(\xi, y \cdot \eta)| \lesssim_\alpha |\xi|^\alpha \varepsilon^\alpha.$$

Thus $\tilde{v}_n^{\text{int}}$ solves

$$\begin{cases} -\nabla \cdot a(y/\tan \varepsilon, \nabla \tilde{v}_n^{\text{int}}) = 0 & \text{in } P_n, \\ |\tilde{v}_n^{\text{int}}(y) - \varphi_*(\xi, y \cdot \eta)| \le C |\xi|^\alpha \varepsilon^\alpha & \text{on } \partial P_n. \end{cases} \tag{6-8}$$

This is almost the same as (6-4) solved by v_n^{int}. The L^∞-estimate Lemma 3.2 (or the maximum principle) implies

$$\sup_{P_n} |v_n^{\text{int}} - \tilde{v}_n^{\text{int}}| \lesssim_\alpha |\xi|^\alpha \varepsilon^\alpha \tag{6-9}$$

either for every $\alpha \in (0, 1)$ in the linear systems case, or for some universal α in the nonlinear case. □

To complete the proof of Proposition 6.3 we just need to compare the solutions v_n^{int} of (6-4) and $w_{\xi,\eta}^\varepsilon$ of (6-5). The width of the boundary layer is now of uniform size in ε so this is not a problem; we will just need to use the boundary continuity estimate (Lemmas 3.4 and 4.2) and the continuity estimate of $\varphi_*(\xi, \cdot)$ Lemma 6.1.

Lemma 6.6. *The following estimates hold for the boundary layers of* v_n^{int} *and* $w^{\varepsilon}_{\xi,\eta}$:

(i) (*nonlinear equations*) *There is* $\alpha \in (0,1)$ *universal such that*

$$\Big|\lim_{R\to\infty} v_n^{\text{int}}(Rn) - \lim_{R\to\infty} w^{\varepsilon}_{\xi,\eta}(R\hat{\xi})\Big| \lesssim \|\nabla\varphi\|_\infty |\xi|^\alpha \varepsilon^\alpha,$$

where technically we mean that the estimate holds for any pair of subsequential limits.

(ii) (*linear systems*) *For every* $\alpha \in (0,1)$ *and any* $\nu > 0$

$$\Big|\lim_{R\to\infty} v_n^{\text{int}}(Rn) - \lim_{R\to\infty} w^{\varepsilon}_{\xi,\eta}(R\hat{\xi})\Big| \lesssim_{\alpha,\nu} [\varphi]_{C^{1,\nu}} |\xi|^\alpha \varepsilon^\alpha,$$

where technically we mean that the estimate holds for any pair of subsequential limits.

Proof. We compare the two solutions in their common domain. As before let $K = P_n \cap P_\xi$ and

$$u = v_n^{\text{int}} - w^{\varepsilon}_{\xi,\eta}.$$

Nonlinear equations: We have

$$-\nabla\cdot(A(x)\nabla u) = 0 \quad \text{in } K \text{ with some } \lambda \le A(x) \le 1 \text{ as in (4-5)}.$$

We compute the error on ∂K in the same way that we did in Lemma 6.4. Using Lemma 4.2 we find for $x \in \partial K$,

$$|u(x)| = |v_n^{\text{int}}(x) - w^{\varepsilon}_{\xi,\eta}(x)| \lesssim \|\varphi_*(\xi,\cdot)\|_{C^{\alpha'}} \varepsilon^\alpha |x|^\alpha \lesssim \|\nabla\varphi\|_\infty \varepsilon^\alpha |x|^\alpha,$$

where α' is the universal, continuity modulus from Lemma 6.1 and $\alpha < \alpha'$. Next we use the De Giorgi boundary continuity estimate, Lemma 4.2 to obtain, again with a slightly smaller α,

$$|u(x)| \lesssim \|\nabla\varphi\|_\infty \varepsilon^\alpha |x|^\alpha \quad \text{for } x \in K. \tag{6-10}$$

Next we use that the size of the boundary layers for v_n^{int} and $w^{\varepsilon}_{\xi,\eta}$ are uniformly bounded in ε, via Lemma 5.3, to find for all $R_0 \ge 1$,

$$\sup_{y\in\partial P_n} \Big|v_n^{\text{int}}(y + R_0 n) - \lim_{R\to\infty} v_n^{\text{int}}(Rn)\Big| \lesssim \|\varphi_*(\xi,\cdot)\|_{C^{\alpha'}} \varepsilon^\alpha + (\operatorname{osc}\varphi_*) e^{-R_0/|\xi|},$$

where again we mean that the estimate holds for any subsequential limit of $v_n^{\text{int}}(Rn)$. An analogous estimate holds for $w^{\varepsilon}_{\xi,\eta}$ replacing Rn with $R\hat{\xi}$. Using our assumption that $\varepsilon \le \frac{\pi}{4}$ we have $n\cdot\hat{\xi} \ge \frac{1}{\sqrt{2}}$ and so we have

$$\max\Big\{\Big|v_n^{\text{int}}(R_0\hat{\xi}) - \lim_{R\to\infty} v_n^{\text{int}}(Rn)\Big|, \Big|w^{\varepsilon}_{\xi,\eta}(R_0\hat{\xi}) - \lim_{R\to\infty} w^{\varepsilon}_{\xi,\eta}(R\hat{\xi})\Big|\Big\} \\ \lesssim \|\varphi_*(\xi,\cdot)\|_{C^{\alpha'}} \varepsilon^\alpha + (\operatorname{osc}\varphi_*) e^{-R_0/|\xi|}. \tag{6-11}$$

Finally we combine (6-10) with (6-11), choosing $R_0 = |\xi| \log 1/(|\xi|\varepsilon)$, to find

$$\begin{aligned}\Big|\lim_{R\to\infty} v_n^{\text{int}}(Rn) - \lim_{R\to\infty} w^{\varepsilon}_{\xi,\eta}(R\hat{\xi})\Big| &\le |v_n^{\text{int}}(R_0\hat{\xi}) - w^{\varepsilon}_{\xi,\eta}(R_0\hat{\xi})| + C\|\nabla\varphi\|_\infty |\xi|^\alpha \varepsilon^\alpha \\ &\lesssim \|\nabla\varphi\|_\infty \varepsilon^\alpha R_0^\alpha \\ &\lesssim \|\nabla\varphi\|_\infty |\xi|^\alpha \varepsilon^\alpha \Big(\log \frac{1}{|\xi|\varepsilon}\Big)^\alpha.\end{aligned}$$

Making α slightly smaller we can remove the logarithmic term.

Linear systems: We have

$$-\nabla\cdot(A(x)\nabla u)=0 \quad \text{in } K.$$

Using Lemma 3.3 we find, for $x\in\partial K$ and any $\nu>0$,

$$|u(x)|=|v_n^{\text{int}}(x)-w^{\varepsilon}_{\xi,\eta}(x)|\lesssim_\alpha \|\nabla\varphi_*(\xi,\cdot)\|_\infty\varepsilon^\alpha|x|^\alpha\lesssim_\nu \|\nabla\varphi\|_{C^{0,\nu}}\varepsilon^\alpha|x|^\alpha.$$

By the Poisson kernel bounds in K, Lemmas 3.5 and 3.6, we have for a slightly smaller $\alpha\in(0,1)$ and ε sufficiently small depending on α

$$|u(x)|\lesssim_\alpha [\varphi]_{C^{1,\nu}}\varepsilon^\alpha|x|^\alpha \quad \text{for } x\in K.$$

The remainder of the proof is the same as the case of scalar equations. □

Proposition 6.3 follows combining Lemmas 6.5 and 6.6.

6C. ***Interior homogenization of the intermediate-scale problem.*** We take $\varepsilon\to 0$ in (6-5) and derive the second cell problem

$$\begin{cases}-\nabla\cdot a(x/\tan\varepsilon,\nabla w^{\varepsilon}_{\xi,\eta})=0 & \text{in } P_\xi,\\ w^{\varepsilon}_{\xi,\eta}(x)=\varphi_*(\xi,x\cdot\eta) & \text{on } \partial P_\xi,\end{cases} \tag{6-12}$$

which homogenizes to

$$\begin{cases}-\nabla\cdot a^0(\nabla w_{\xi,\eta})=0 & \text{in } P_\xi,\\ w_{\xi,\eta}(x)=\varphi_*(\xi,x\cdot\eta) & \text{on } \partial P_\xi,\end{cases} \tag{6-13}$$

where a^0 is the homogenized operator associated with $a\big(\frac{x}{\varepsilon},\cdot\big)$.

We make the definition

$$L(\xi,\eta)=\lim_{R\to\infty} w_{\xi,\eta}(x+R\xi).$$

As we will show below $L(\xi,\cdot)$ is the limiting 0-homogeneous profile of φ_* at the direction ξ,

$$\lim_{j\to\infty}\varphi_*(n_j)=L(\xi,\eta)$$

for any sequence of n_j irrational with $n_j\to\xi$ and $(\hat{\xi}-n_j)/|\hat{\xi}-n_j|\to\eta$. This characterization is the first main result of the paper Theorem 1.1.

We make a further remark about the second cell problem in (1-2). It is straightforward to see that $w_{\xi,\eta}$ is actually a function only of two variables $x\cdot\xi$ and $x\cdot\eta$. The boundary data $\varphi_*(\xi,x\cdot\eta)$ is invariant with respect to translations which are perpendicular to both ξ and η, and so by uniqueness the solution $w_{\xi,\eta}$ is invariant in those directions as well. Note that we are using the spatial homogeneity of the operator here; the same is not true of $w^{\varepsilon}_{\xi,\eta}$. This property was useful in [Feldman and Kim 2017] since solutions of nonlinear nondivergence form elliptic problems in dimension $d=2$ have better regularity properties. Although we do not use this in a significant way here, we point it out anyway since it could be potentially useful in the future.

Now we state the quantitative version of Theorem 1.1:

Theorem 6.7. *Let $\xi \in \mathbb{Z}^d \setminus \{0\}$ be irreducible and $n = (\cos\varepsilon)\hat{\xi} - (\sin\varepsilon)\eta$ be an irrational direction. Then*:

(i) (*nonlinear equations*) *There is a universal $\alpha \in (0,1)$ such that*

$$|\varphi_*(n) - L(\xi,\eta)| \lesssim \|\nabla\varphi\|_\infty |\xi|^\alpha \varepsilon^\alpha.$$

(ii) (*linear systems*) *For every $\alpha \in (0,1)$*

$$|\varphi_*(n) - L(\xi,\eta)| \lesssim_\alpha [\varphi]_{C^5} |\xi|^\alpha \varepsilon^\alpha.$$

We will need one more lemma in the proof of Theorem 6.7, which is independently interesting since it gives the continuity of $L(\xi,\eta)$ in η.

Lemma 6.8. *Let $\xi \in \mathbb{Z}^d \setminus \{0\}$ be irreducible and $\eta, \eta' \perp \xi$. Then*

$$\Big|\lim_{R\to\infty} w^\varepsilon_{\xi,\eta}(R\xi) - \lim_{R\to\infty} w^\varepsilon_{\xi,\eta'}(R\xi)\Big| \lesssim_\alpha \|\varphi\|_{C^k}(|\xi|^{-\alpha}|\eta-\eta'|^\alpha + \varepsilon^\alpha)$$

and

$$|L(\xi,\eta) - L(\xi,\eta')| \lesssim_\alpha \|\varphi\|_{C^k}|\xi|^{-\alpha}|\eta-\eta'|^\alpha$$

either for a universal $\alpha \in (0,1)$ and $k = 1$ (nonlinear case), or for every $\alpha \in (0,1)$ and $k = 3$ (linear systems case). For the first estimate we mean that the inequality holds for any pair of subsequential limits of $w^\varepsilon_{\xi,\eta}(R\xi)$, $w^\varepsilon_{\xi,\eta'}(R\xi)$ as $R \to \infty$.

Proof of Theorem 6.7. The ingredients have all been established elsewhere, we just need to combine them.

There is some set up to use Proposition 5.5 since the (5-6) does not necessarily hold for (6-12). Recall that $\xi^\perp \cap \mathbb{Z}^d$ is spanned by $d-1$ linearly independent vectors $\ell_1, \dots, \ell_{d-1}$ with norms $|\ell_j| \le |\xi|$. Then for each $\varepsilon > 0$ we can choose a vector $\eta_\varepsilon \in \varepsilon\xi^\perp \cap \mathbb{Z}^d$, i.e., a period of $a(\cdot/\tan\varepsilon, p)$ with

$$|\eta_\varepsilon - \eta| \le C\varepsilon|\xi|. \tag{6-14}$$

Now Proposition 5.5 will apply to get a quantitative estimate of the difference $w^\varepsilon_{\xi,\eta_\varepsilon} - w_{\xi,\eta_\varepsilon}$; we will use this below.

Nonlinear equations: By Proposition 6.3, there is universal $\alpha \in (0,1)$ such that

$$\Big|\varphi_*(n) - \lim_{R\to\infty} w^\varepsilon_{\xi,\eta}(R\hat{\xi})\Big| \lesssim \|\nabla\varphi\|_\infty |\xi|^\alpha \varepsilon^\alpha,$$

where we mean that the estimate holds for any subsequential limit of $w^\varepsilon_{\xi,\eta}(R\hat{\xi})$ as $R \to \infty$. Proposition 5.5, homogenization of problems in half-space-type domains, applies to $w^\varepsilon_{\xi,\eta_\varepsilon}$

$$\sup_{P_\xi} |w^\varepsilon_{\xi,\eta_\varepsilon} - w_{\xi,\eta_\varepsilon}| \lesssim [\varphi_*(\xi,\cdot)]_{C^\beta}|\xi|^{\alpha-\beta}\varepsilon^\alpha \lesssim \|\nabla\varphi\|_\infty \varepsilon^\alpha$$

for some universal $\beta > \alpha \in (0,1)$. We have used Lemma 6.1 to estimate the Hölder norm of $\varphi_*(\xi,\cdot)$. Then Lemma 6.8 and (6-14) implies

$$\Big|\lim_{R\to\infty} w^\varepsilon_{\xi,\eta}(R\hat{\xi}) - \lim_{R\to\infty} w^\varepsilon_{\xi,\eta_\varepsilon}(R\hat{\xi})\Big| + |L(\xi,\eta_\varepsilon) - L(\xi,\eta)| \lesssim \|\nabla\varphi\|_\infty(|\xi|^{-\alpha}|\eta_\varepsilon-\eta|^\alpha + \varepsilon^\alpha) \lesssim \|\nabla\varphi\|_\infty \varepsilon^\alpha.$$

Combining these

$$|\varphi_*(n) - L(\xi,\eta)| \lesssim \|\nabla\varphi\|_\infty |\xi|^\alpha \varepsilon^\alpha.$$

Linear systems: By Proposition 6.3, for every $\alpha \in (0,1)$ and any $\nu > 0$

$$\big|\varphi_*(n) - \lim_{R\to\infty} w^\varepsilon_{\xi,\eta}(R\hat{\xi})\big| \lesssim_{\alpha,\nu} [\varphi]_{C^{1,\nu}} |\xi|^\alpha \varepsilon^\alpha,$$

where again we mean that the estimate holds for any subsequential limit of $w^\varepsilon_{\xi,\eta}(R\hat{\xi})$ as $R\to\infty$. Now Proposition 5.5 (properly rescaled) applies to $w^\varepsilon_{\xi,\eta_\varepsilon}$,

$$\begin{aligned}\sup_{P_\xi} |w_{\xi,\eta_\varepsilon} - w^\varepsilon_{\xi,\eta_\varepsilon}| &\lesssim_\alpha [\varphi_*(\xi,\cdot)]_{C^4} |\xi|^{\alpha-1}\varepsilon^\alpha \\ &\lesssim \|\varphi\|_{C^5} |\xi|^{\alpha-1} \log^4(1+|\xi|)\varepsilon^\alpha\end{aligned}$$

for every $\alpha \in (0,1)$. We have used Lemma 6.2 to obtain the C^4 regularity of $\varphi_*(\xi,\cdot)$. We also have $|\xi|^{\alpha-1}\log^4(1+|\xi|) \lesssim_\alpha 1$. Then Lemma 6.8 and (6-14) imply

$$\big|\lim_{R\to\infty} w^\varepsilon_{\xi,\eta}(R\hat{\xi}) - \lim_{R\to\infty} w^\varepsilon_{\xi,\eta_\varepsilon}(R\hat{\xi})\big| + |L(\xi,\eta_\varepsilon) - L(\xi,\eta)| \lesssim_\alpha [\varphi]_{C^3}(|\xi|^{-\alpha}|\eta_\varepsilon - \eta|^\alpha + \varepsilon^\alpha) \lesssim [\varphi]_{C^3}\varepsilon^\alpha.$$

Combining these, for any $\alpha \in (0,1)$,

$$|\varphi_*(n) - L(\xi,\eta)| \lesssim_\alpha [\varphi]_{C^5} |\xi|^\alpha \varepsilon^\alpha. \qquad \square$$

Proof of Lemma 6.8. We just argue for $W = w^\varepsilon_{\xi,\eta} - w^\varepsilon_{\xi,\eta'}$; the argument for $w_{\xi,\eta} - w_{\xi,\eta'}$ is almost the same and slightly simpler.

Nonlinear equations: Note that $W(0) = 0$ and the boundary data for W on ∂P_ξ has

$$|\varphi_*(x\cdot\eta) - \varphi_*(x\cdot\eta')| \le \|\varphi_*\|_{C^\alpha} |\eta-\eta'|^\alpha |x|^\alpha \lesssim \|\nabla\varphi\|_\infty |\eta-\eta'|^\alpha |x|^\alpha$$

for a universal $\alpha \in (0,1)$ by Lemma 6.1. By the boundary regularity Lemma 4.2 and the maximum principle,

$$|W(x)| \lesssim \|\nabla\varphi\|_\infty |\eta-\eta'|^\alpha |x|^\alpha \quad \text{for } x \in P_\xi \cap B_R,$$

for a, possibly smaller, universal α. Now by Lemma 5.3 applied to $w^\varepsilon_{\xi,\eta}, w^\varepsilon_{\xi,\eta'}$ separately, there is $c_* \in \mathbb{R}$ such that for all $R \ge 1$

$$\sup_{x\cdot\hat{\xi}\ge R} |W(x) - c_*| \lesssim [\varphi_*]_{C^\alpha}\left(\frac{1}{|\xi|^\alpha}\exp(-c|\xi|R) + \varepsilon^{\alpha'}\right),$$

where α universal is from Lemma 6.1, and $\alpha' < \alpha$ universal. Combining the two estimates with $R = c^{-1}|\xi|^{-1}|\log|\eta-\eta'||$ we get

$$|c_*| \lesssim_\alpha \|\nabla\varphi_*\|_\infty (|\xi|^{-\alpha}|\eta-\eta'|^\alpha + \varepsilon^\alpha),$$

again with a possibly different universal $\alpha \in (0,1)$.

Linear systems: Note that $W(0)=0$ and the boundary data for W on ∂P_ξ has

$$|\nabla(\varphi_*(x\cdot\eta)-\varphi_*(x\cdot\eta'))| \le \left\|\frac{d}{ds}\varphi_*\right\|_\infty |\eta-\eta'| + \left\|\frac{d^2}{ds^2}\varphi_*\right\|_\infty |\eta-\eta'|\,|x|.$$

By the boundary regularity Lemma 3.3, for any $\alpha\in(0,1)$,

$$|W(x)| \lesssim_\alpha \left\|\frac{d}{ds}\varphi_*\right\|_{C^2}(1+R)|\eta-\eta'|\,|x|^\alpha \quad \text{for } x\in P_\xi\cap B_R.$$

Now by Lemma 5.3 applied to $w^\varepsilon_{\xi,\eta}, w^\varepsilon_{\xi,\eta'}$ separately, there is $c_*\in\mathbb{R}$ such that for all $R\ge 1$

$$\sup_{x\cdot\hat\xi\ge R} |W(x)-c_*| \lesssim \left\|\frac{d}{ds}\varphi_*\right\|_\infty \Big(\frac{1}{|\xi|}\exp(-c|\xi|R)+\varepsilon^\alpha\Big).$$

Combining the two estimates with $R=c^{-1}|\xi|^{-1}|\log|\eta-\eta'||$, we get

$$|c_*| \lesssim_\alpha \|\varphi_*\|_{C^2}(|\xi|^{-\alpha}|\eta-\eta'|^\alpha+\varepsilon^\alpha).$$

We are ignoring some negative powers of $|\xi|$ since they are ≤ 1. □

6D. ***Proofs of regularity estimates of*** φ_*. We return to prove the regularity estimates of φ_* Lemma 6.1 and Lemma 6.2. The Hölder regularity Lemma 6.1 is relatively straightforward, while the higher regularity Lemma 6.2 requires some more careful estimates.

Proof of Lemma 6.1. We will show an upper bound for $|\varphi_*(\xi,h)-\varphi_*(\xi,0)|$ with $h<0$; the proof works also for nonzero s and $h\in\mathbb{R}$. Consider v^0_ξ a solution in P_ξ and v^h_ξ a solution in $P_\xi+h\hat\xi\supset P_\xi$. By the boundary continuity estimates for v^h_ξ, for every $y\in\partial P_\xi$,

$$|v^h_\xi(y)-v^0_\xi(y)| = |v^h_\xi(y)-\varphi(y)| \le |v^h_\xi(y)-\varphi(y-h\hat\xi)|+\|\nabla\varphi\|_\infty h \le C\|\nabla\varphi\|_\infty h^\alpha$$

for some $\alpha\in(0,1)$ by Lemma 4.2. For the case of linear systems we have similarly,

$$|v^h_\xi(y)-v^0_\xi(y)| = |v^h_\xi(y)-\varphi(y)| \le C[\varphi]_{C^{1,\nu}}h$$

for any $\nu>0$ by the boundary gradient estimates for smooth coefficient linear systems. Then the maximum principle, or respectively the L^∞ estimate for systems Lemma 3.2, implies the same bound holds in all of P_ξ and therefore also for the boundary layer limits. □

Proof of Lemma 6.2. In order to get estimates on higher derivatives of v^s_ξ in s, the method for Lemma 6.1 doesn't work; we need to differentiate in the equation. Since we only consider one normal direction $\xi\in\mathbb{Z}^d\setminus\{0\}$ we drop the dependence $v^s=v^s_\xi$ on ξ. We denote derivatives with respect to s by ∂ and then

$$\begin{cases} -\nabla\cdot(A(x+s\hat\xi)\nabla\partial^k v^s)=\nabla\cdot f & \text{in } P_\xi,\\ \partial^k v^s=(\hat\xi\cdot\nabla)^k\varphi(x+s\hat\xi) & \text{on } \partial P_\xi, \end{cases} \tag{6-15}$$

where f involves derivatives $\partial^j v$ for $j<k$,

$$f=\sum_{j=0}^{k-1}\binom{k}{j}(\hat\xi\cdot\nabla)^{k-j}A(x+s\hat\xi)\nabla\partial^j v^s.$$

Let $p > d$ arbitrary but fixed. We will suppose, inductively, that we can prove for any $R \geq 0$ and every $j < k$,

$$\sup_{y \in \partial P_\xi, R' \geq R} \|\nabla \partial^j v^s\|_{L^p_{\mathrm{avg}}(B_{R'/2}(y+R'\hat{\xi}))} \leq C_j [\varphi]_{C^{j+1,\nu}} \frac{1}{R} \log^j(1+|\xi|) e^{-c_j R/|\xi|},$$

where the constants depend on j, $[A]_{C^j}$ and universal parameters. The case $R \leq 1$ corresponds basically to an L^∞ bound on P_ξ.

Then by Lemma B.1

$$\|\partial^k v^s\|_{L^\infty(P_\xi)} \leq C \|(\hat{\xi} \cdot \nabla)^k \varphi\|_\infty + C \log^k(1+|\xi|)[\varphi]_{C^{k,\nu}}. \tag{6-16}$$

Furthermore, by Lemma B.2, $\partial^k v^s$ has a boundary layer limit

$$\mu_k = \frac{d^k}{ds^k} \varphi_*(\xi, s),$$

with

$$|\partial^k v^s - \mu_k| \leq C \log^k(1+|\xi|)[\varphi]_{C^{k,\nu}} e^{-cR/|\xi|}.$$

Now we aim to establish the inductive hypothesis. The following argument will also establish the base case when $j = 0$. First we consider the case $R \leq 1$. This follows from (6-16) and the up-to-the-boundary gradient estimates (Lemma 3.3),

$$\|\nabla \partial^k v^s\|_{L^\infty(P_\xi)} \leq C \|(\hat{\xi} \cdot \nabla)^k \varphi\|_{C^{1,\nu}} + C \log^k(1+|\xi|)[\varphi]_{C^{k,\nu}} \leq C \log^k(1+|\xi|)[\varphi]_{C^{k+1,\nu}}.$$

In the case $R \geq 1$, by the Avellaneda–Lin large-scale interior $W^{1,p}$ estimates and the inductive hypothesis,

$$\begin{aligned}\|\nabla \partial^k v^s\|_{L^p_{\mathrm{avg}}(B_{R/2}(y+R\hat{\xi}))} &\leq C \frac{1}{R} \operatorname*{osc}_{B_{3R/4}(y+R\hat{\xi})} \partial^k v^s + \|f\|_{L^p_{\mathrm{avg}}(B_{3R/4}(y+R\hat{\xi}))} \\ &\leq C \frac{1}{R} \log^k(1+|\xi|)[\varphi]_{C^{k,\nu}} e^{-cR/|\xi|}.\end{aligned}$$

Combining the cases $R \leq 1$ and $R \geq 1$ establishes the inductive hypothesis for $j = k$. The bound on $\|\partial^k v^s\|_{L^\infty}$ and hence on the boundary layer limit μ_k, which is also a consequence of the induction, is the desired result. □

7. Continuity estimate for homogenized boundary data associated with linear systems

In this section we use the limiting structure at rational directions established above to prove that the homogenized boundary condition associated with a linear system is continuous. We recall the second cell problem; let $\xi \in \mathbb{Z}^d \setminus \{0\}$ a rational direction and suppose that we have a sequence of directions $n_k \to \hat{\xi}$ such that

$$\frac{\hat{\xi} - n_k}{|\hat{\xi} - n_k|} \to \eta, \quad \text{where } \eta \text{ is a unit vector with } \eta \perp \xi.$$

Then the limit of $\varphi_*(n_k)$ is determined by the second cell problem

$$\begin{cases} -\nabla \cdot (A^0 \nabla w_{\xi,\eta}) = 0 & \text{in } P_\xi, \\ w_{\xi,\eta} = \varphi_*(\xi, x \cdot \eta) & \text{on } \partial P_\xi, \end{cases} \tag{7-1}$$

and thus $\lim_{k\to\infty} \varphi_*(n_k) = \lim_{R\to\infty} w_{\xi,\eta}(R\xi)$, where A^0, constant, is the homogenized matrix associated with $A\big(\frac{\cdot}{\varepsilon}\big)$ and $\varphi_*(\xi,\cdot)$ defined in (6-2) is a $1/|\xi|$ periodic function on $\mathbb{R}$ (see [Feldman and Kim 2017, Lemma 2.9] where the period of φ_* is explained).

First we state the qualitative result, identifying the limit and showing continuity at rational directions. Continuity of φ_* at the irrational directions has been established, for example in [Prange 2013]. Combining those results shows that φ_* extends to a continuous function on S^{d-1}.

Lemma 7.1. *Let $\xi \in \mathbb{Z}^d \setminus \{0\}$; then for any sequence $n_k \to \hat{\xi}$,*

$$\lim_{k\to\infty} \varphi_*(n_k) = |\xi| \int_0^{1/|\xi|} \varphi_*(\xi,t)\,dt.$$

From this we know that $L(\xi,\eta)$, defined in Section 6C, is independent of η in the linear case. And we will simply write $L(\xi) = L(\xi,\eta)$.

Proof. By rotation and rescaling we can reduce to proving that the boundary layer limit associated with the half-space problem

$$\begin{cases} -\nabla\cdot(A^0\nabla v) = 0 & \text{in } \mathbb{R}^d_+, \\ v = g(x_1,\dots,x_{d-1}) & \text{on } \partial\mathbb{R}^d_+, \end{cases} \tag{7-2}$$

where A^0 is a constant and uniformly elliptic and g is a $\mathbb{Z}^{d-1}$-periodic continuous function $\mathbb{R}^{d-1}\to\mathbb{R}^N$, is

$$\lim_{R\to\infty} v(Re_d) = \int_{[0,1]^{d-1}} g(x')\,dx'.$$

We will actually give two proofs of this result, especially since it plays a key role in our main results.

Riesz representation: Consider the (linear) map $T : C(\mathbb{R}^{d-1}/\mathbb{Z}^{d-1}) \to \mathbb{R}^N$ mapping $g \mapsto \lim_{R\to\infty} v(Re_d)$. The L^∞ estimates Lemma 3.2 imply that T is continuous. Since A^0 is constant, translating g parallel to $\partial\mathbb{R}^d_+$ just translates the solution v and so we also get translation invariance, for any $y' \in \mathbb{T}^{d-1}$,

$$Tg(\cdot - y') = Tg.$$

The Riesz representation theorem implies that $Tg = \int_{\mathbb{R}^{d-1}/\mathbb{Z}^{d-1}} g(x')\,d\mu(x')$ for some (vector-valued) measure μ. The translation invariance of T implies translation invariance of μ which means it is a constant multiple of the Haar measure, Lebesgue measure in this case. Then $T1 = 1$ implies that $d\mu = dx'$.

Direct method: Consider

$$\mu(t) = \int_{[0,1]^{d-1}} v(x',t)\,dx'.$$

If we can show that μ is constant we are done. Compute, using a summation convention,

$$\begin{aligned} A^{0,ij}_{dd}\,\mu_j''(t) &= \int_{[0,1]^{d-1}} A^{0,ij}_{dd}\,\partial_d^2 v_j(x',t)\,dx' \\ &= -\int_{[0,1]^{d-1}} \sum_{\alpha\beta\neq dd} A^{0,ij}_{\alpha\beta}\,\partial^2_{\alpha\beta} v_j(x',t)\,dx' = 0. \end{aligned}$$

Now note that for each derivative $\partial^2_{\alpha\beta}$ appearing in the sum either α or β is $\neq d$ and so we are integrating the derivative of a periodic function over its unit cell. Thus

$$A^{0,ij}_{dd}\mu_j''(t) = 0 \quad \text{for all } 1 \le i \le N.$$

Let $\bar\xi \in \mathbb{R}^N$; applying (3-2) with the vector $\xi^i_\alpha = \bar\xi^i \delta_{\alpha d}$ gives

$$\lambda|\bar\xi|^2 = \lambda\xi^i_\alpha\xi^i_\alpha \le A^{0,ij}_{\alpha\beta}\xi^i_\alpha\xi^j_\beta = A^{0,ij}_{dd}\bar\xi^i\bar\xi^j.$$

In particular the $N \times N$ matrix with coefficients $A^{0,ij}_{dd}$ is invertible and therefore

$$\mu''(t) = 0 \quad \text{for all } t \ge 0.$$

Thus μ is linear, since μ is bounded it must be constant. □

The next result is quantitative; the argument, which is the same as in [Feldman and Kim 2017], uses the Dirichlet approximation theorem. We recall that number-theoretic result here.

Theorem 7.2 (Dirichlet approximation). *For given real numbers $\alpha_1, \dots, \alpha_n$ and $N \in \mathbb{N}$, there are integers $p_1, \dots, p_n, q \in \mathbb{Z}$ with $1 \le q \le N$ such that*

$$|q\alpha_i - p_i| \le \frac{1}{N^{1/n}}.$$

This is proved by the pigeonhole principle.

Theorem 7.3. *Let $\varphi_*(\cdot)$ be defined the boundary layer limit associated with* (1-1) *defined for $n \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$. Then for every $\alpha < \frac{1}{d}$ and all $n_1, n_2 \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$,*

$$|\varphi_*(n_1) - \varphi_*(n_2)| \lesssim_\alpha \|\varphi\|_{C^5}|n_1 - n_2|^\alpha.$$

Proof. Let n_1, n_2 be a pair of irrational unit vectors and set $\delta = |n_1 - n_2|$. Assume $\delta \le 2^{-d}d^{-d/2}$. Let $M = \delta^{-s/(s+1)}$ with $s = d-1$. By Dirichlet's approximation theorem, there exists $\xi \in \mathbb{Z}^d \setminus \{0\}$ and $k \in \mathbb{Z}$ with $1 \le k \le M$ such that

$$\left|\frac{n_1}{|n_1|_\infty} - k^{-1}\xi\right| \le k^{-1}M^{-1/s}.$$

Now $k^{-1}|n_1|_\infty\xi$ is not a unit vector, but by the above inequality

$$k^{-1}|n_1|_\infty|\xi| \ge 1 - \sqrt{d}\delta^{1/d} \ge \tfrac{1}{2}.$$

Then, since the map $x \mapsto x/|x|$ is Lipschitz on $\mathbb{R}^d \setminus B_{1/2}$,

$$\left|n_1 - \frac{\xi}{|\xi|}\right| \lesssim_d k^{-1}M^{-1/s}, \quad \left|n_2 - \frac{\xi}{|\xi|}\right| \le \delta + Ck^{-1}M^{-1/s}.$$

Note also that

$$|\xi| \le \frac{k}{|n_1|_\infty} + M^{-1/s} \le \sqrt{d}k + 1 \lesssim k.$$

Thus, for $j = 1, 2$,

$$|\xi|\left|n_j - \frac{\xi}{|\xi|}\right| \le |\xi||\delta + Ck^{-1}M^{-1/s}| \lesssim M\delta + M^{-1/s} \sim \delta^{1/(s+1)} = \delta^{1/d},$$

where we chose M at the beginning so that the two terms are of the same size.

For appropriate choices of $\eta_j \perp \xi$,

$$n_j = (\cos\varepsilon_j)\hat{\xi} - (\sin\varepsilon_j)\eta_j,$$

with $\varepsilon_j \sim |n_j - \xi/|\xi||$.

Now apply Theorem 6.7, noting that $L(\xi, \eta_1) = L(\xi, \eta_2) = L(\xi)$ by Lemma 7.1. For any $0 < \alpha < 1$ we have

$$\begin{aligned} |\varphi_*(n_1) - \varphi_*(n_2)| &\le |\varphi_*(n_1) - L(\xi)| + |L(\xi) - \varphi_*(n_2)| \\ &\lesssim_\alpha \|\varphi\|_{C^5}\left(|\xi|^\alpha\left|n_1 - \frac{\xi}{|\xi|}\right|^\alpha + |\xi|^\alpha\left|n_2 - \frac{\xi}{|\xi|}\right|^\alpha\right) \\ &\lesssim \|\varphi\|_{C^5}\delta^{\alpha/d}. \end{aligned}$$

This completes the proof for $|n_1 - n_2|$ small; for general $n_1, n_2 \in S^{d-1}$ just use the boundedness of φ_*. □

8. A nonlinear equation with discontinuous homogenized boundary data

In this final section we study the second cell equation (1-2) for nonlinear equations. We give an example of a nonlinear divergence form equation, with smooth boundary condition, for which the boundary layer limit of (1-2) depends on the approach direction η.

We consider the nonlinear operator

$$a(p_1, p_2, p_3) = (p_1, p_2, p_3 + f(p_1, p_3))^t,$$

where

$$f(p_1, p_3) := \tfrac{1}{8}\big(\sqrt{8p_1^2 + 9p_3^2} + p_3\big).$$

Here f is a solution of

$$8f^2 - 2p_3 f - (p_1^2 + p_3^2) = 0.$$

It is easy to check that f is positively 1-homogeneous and uniformly elliptic.

We will take $\xi = e_3$ and $\eta = e_1$ or e_2 and we will set $(x_1, x_2, x_3) = (x, y, z)$. For the boundary condition we choose

$$\varphi(y) = \tfrac{1}{3} + \cos(y\cdot\xi) \quad \text{so that} \quad \varphi_*(\xi, s) = \tfrac{1}{3} + \cos(s).$$

It is worthwhile to note that arbitrary $\varphi_*(\xi, s)$ can be achieved by choosing $\varphi(y) = \varphi_*(\xi, y\cdot\xi)$. We aim to compute $L(\xi, \eta)$.

If $\eta = e_1$, (1-2) becomes

$$\begin{cases} -\nabla\cdot(u_x, u_y, u_z + f(u_x, u_z)) = 0 & \text{in } \mathbb{R}^3_+, \\ u(x, y, 0) = \tfrac{1}{3} + \cos x & \text{in } \mathbb{R}^3_+. \end{cases} \tag{8-1}$$

The operator and boundary data were chosen to make the solution

$$u(x,y,z) = \left(\tfrac{1}{3} + \cos x\right)e^{-z}.$$

Note that

$$f(u_x,u_z) = \tfrac{1}{3}e^{-z}$$

and so

$$(u_x, u_y, u_z + f(u_x,u_z)) = (-\sin x\, e^{-z},\ 0,\ -\cos x\, e^{-z}),$$

from which it is easy to verify that u solves (8-1). The boundary layer limit in this case is 0 and so, by its definition, $L(\xi, e_1) = 0$.

If $\eta = e_2$ then the equation becomes

$$\begin{cases} -\nabla\cdot(u_x, u_y, u_z + f(u_x,u_z)) = 0 & \text{in } \mathbb{R}^3_+, \\ u(x,y,0) = \frac{1}{3} + \cos y & \text{in } \mathbb{R}^3_+. \end{cases} \tag{8-2}$$

This reduces to the following two-dimensional problem for $v(y,z) = u(x,y,z)$:

$$\begin{cases} -\nabla\cdot\left(v_y, \frac{9}{8}v_z + \frac{3}{8}|v_z|\right) = 0 & \text{in } \mathbb{R}^2_+, \\ v(y,0) = \frac{1}{3} + \cos y & \text{on } \partial\mathbb{R}^2_+. \end{cases} \tag{8-3}$$

Let v be the solution of (8-3). Consider $w(y,z) := \left(\frac{1}{3} + \cos y\right)e^{-z}$, the solution from before,

$$-\nabla\cdot\left(w_y, \tfrac{9}{8}w_z + \tfrac{3}{8}|w_z|\right) = \left[\left(-\tfrac{4}{9} - \tfrac{1}{3}\cos y\right)\mathbf{1}_{\{\cos y<0\}} + \tfrac{1}{4}(\cos y - 1)\mathbf{1}_{\{\cos y>0\}}\right]e^{-z} \le 0. \tag{8-4}$$

Thus w is a subsolution of (8-3); from Lemma 4.3 we have $w \le v$.

The operator $\left(v_y, \frac{9}{8}v_z + \frac{3}{8}|v_z|\right)$ is uniformly elliptic and Lipschitz continuous. We use a strong maximum principle [Serrin 1970, Theorem 1′]; in any bounded domain, we either have $w \equiv v$ or $w < v$. Since the inequality in (8-4) is strict, except when $y = 0 \bmod 2\pi$, the case must be $w < v$. Since both w, v are 1-periodic in the y-direction, restricting to the set $z = 1$, $w(y,1) \le v(y,1) - \delta$ for some $\delta > 0$. Then by comparing w and $v - \delta$ on $z \ge 1$, again using Lemma 4.3, we deduce that $w \le v - \delta$; in particular

$$\lim_{z\to\infty} v \ge \lim_{z\to\infty} w + \delta = \delta.$$

Thus $L(\xi, e_2) < 0 = L(\xi, e_1)$ and therefore $\varphi_*(n)$ is discontinuous at the direction e_3.

Appendix A

Hölder estimate in cone domain. We complete the proof of Lemma 3.4, the Hölder estimate in the flat cone domain which we used above.

Proof of Lemma 3.4. Suppose that

$$\|\nabla g\|_{L^\infty(\partial\Omega\cap B_1)} \le 1 \quad \text{and} \quad ⨍_{B_1\cap\Omega} |u^\varepsilon - g(0)|^2 \le 1.$$

Let some $\alpha < \alpha' < 1$; by Lemma 3.3 there is a $1 > \theta > 0$ so that if $K_\Sigma = P_n$ for some $n \in S^{d-1}$ then

$$\sup_{B_\theta \cap P_n} |u^\varepsilon - g(0)| \le \theta^{\alpha'}.$$

We prove by compactness that there exists $\delta > 0$ sufficiently small such that for any solution u^ε as above

$$\left(\fint_{B_\theta \cap \Omega} |u^\varepsilon - g(0)|^2 \right)^{1/2} \le \theta^\alpha. \tag{A-1}$$

To achieve the Hölder estimate from (A-1) is the standard iteration argument.

Suppose that the previous statement fails; that is, there exists f_k and corresponding Ω_k with $\delta_k = \|\nabla f_k\|_\infty \to 0$, A_k satisfying the standard assumptions, $\varepsilon_k > 0$, g_k with Lipschitz norm at most 1 and corresponding u_k solving the equation with boundary data g_k on $\partial\Omega_k \cap B_1$ and

$$\left(\fint_{B_\theta \cap \Omega_k} |u_k - g_k(0)|^2 \right)^{1/2} > \theta^\alpha.$$

By taking subsequences we can assume that $A_k \to A$ uniformly, $g_k \to g$ uniformly and the u_k converge to some u weakly in H^1 and strongly in L^2. Then, assuming that $\varepsilon_k \to \varepsilon > 0$, we claim u solves

$$-\nabla \cdot A\left(\frac{x}{\varepsilon}\right)\nabla u = 0 \quad \text{in } \Omega \cap B_1 \text{ with } u = g \text{ on } \{x_d = 0\} \cap B_1. \tag{A-2}$$

If $\varepsilon_k \to 0$ or $\varepsilon_k \to \infty$ then we replace $A(x/\varepsilon)$ by A^0 or $A(0)$ respectively.

The only part which is not the same as in [Avellaneda and Lin 1987] is to check the boundary condition. Consider the transformations

$$\Phi_k(x) = (x', x_d + f_k(x')) \;\text{ mapping }\; \Phi_k : \{x_d > 0\} \to \{x_d > f_k(x')\}.$$

Define $v_k = u_k \circ \Phi_k$. Note that $|\Phi_k - x| \le \delta_k$, $\nabla v_k = \nabla \Phi_k \nabla u_k$ and $\|\nabla \Phi_k - I\|_{L^\infty} \le \delta_k$. Therefore, up to taking a subsequence, the v_k converge weakly in $H^1(B_1^+)$ and strongly in $H^{1/2}(B_1^+)$ to the same limit u. Since the trace operator is continuous $T : H^{1/2}(B_1^+) \to L^2(\{x_d = 0\} \cap B_1)$, we have that the trace of v is the limit of the traces g_k of the v_k.

Then, once we have established the limit (A-2), from the regularity estimate in the flat domain

$$\theta^\alpha \le \left(\fint_{B_\theta \cap P_n} |u - g(0)|^2 \right)^{1/2} \le \theta^{\alpha'},$$

which is a contradiction since $\alpha < \alpha'$ and $\theta < 1$. □

Poisson kernel bounds in half-space intersection. We return to prove the Poisson kernel bounds in the intersection of nearby half-spaces, Lemma 3.5.

Proof of Lemma 3.5. The proof basically follows the proof of the Poisson kernel bounds in a smooth domain in [Avellaneda and Lin 1987, Lemma 21] except we need to be careful to deal with the singularity of the boundary. We do the case $d \ge 3$; the $d = 2$ case is a similar modification of the arguments in

[Avellaneda and Lin 1987, Lemma 21]. Let $x, y \in K$ and call $r = |y - x|$. We have the Green's function bound holding for $x, y \in K$ (see Theorem 13 in [Avellaneda and Lin 1987] and the remark below)

$$|G_K(x, y)| \lesssim \frac{1}{r^{d-2}}.$$

We will first improve the Green's function bound; the bound on the Poisson kernel will follow.

If $\delta(x) > \frac{1}{3}r$ then $|G_K(x, y)| \lesssim \delta(x)/r^{d-1}$. Consider the case $\delta(x) < \frac{1}{3}r$. Let $\bar{x} \in \partial K$ with $|x-\bar{x}| = \delta(x)$. Then $G_K(\,\cdot\,, y)$ is a solution of the system in $B\big(\bar{x}, \frac{1}{2}r\big) \cap K$. For ε sufficiently small depending on α the boundary Hölder estimates Lemma 3.4 apply and

$$G(z, y) \lesssim \frac{\delta(z)^\alpha}{r^{d-2+\alpha}} \quad \text{for all } z \in B\big(\bar{x}, \tfrac{1}{3}r\big) \cap K; \tag{A-3}$$

in particular the bound holds at $z = x$.

Now we make a similar argument in the y-variable starting from (A-3); however Hölder regularity is not sufficient anymore so we need to deal more directly with the singularity. Since we will send $y \to \partial K \setminus \{y_1 = 0\}$ we can just consider the case $\delta(y) \le \min\big\{\frac{1}{3}r, \frac{1}{2}|y_1|\big\}$. Let $\bar{y} \in \partial K$ with $|y - \bar{y}| = \delta(y)$. Then $G_K(x, \cdot)$ is a solution of the adjoint equation in $B\big(\bar{y}, \frac{1}{2}r\big) \cap K$. If $|y_1| \ge \frac{1}{2}r$ then $|\bar{y}_1| \ge |y_1| \ge \frac{1}{2}r$ and $B\big(\bar{y}, \frac{1}{2}r\big) \cap K$ is the intersection of a half-space with the ball $B\big(\bar{y}, \frac{1}{2}r\big)$. The boundary Lipschitz estimate of [Avellaneda and Lin 1987] applies and

$$|G(x, z)| \lesssim \frac{\delta(z)\delta(x)^\alpha}{r^{d-1+\alpha}} \quad \text{for all } z \in B\big(\bar{y}, \tfrac{1}{3}r\big) \cap K;$$

since $\delta(y) \le \frac{1}{3}r$ we get the bound at $z = y$. If $|y_1| \le \frac{1}{2}r$ then we instead apply the boundary Lipschitz estimate in $B(\bar{y}, |y_1|)$ to find

$$|G(x, z)| \lesssim \frac{\delta(z)\delta(x)^\alpha}{|y_1| r^{d-2+\alpha}} \quad \text{for all } z \in B(\bar{y}, |y_1|/2) \cap K;$$

since $\delta(y) \le \frac{1}{2}|y_1|$ we get the bound at $z = y$. The bounds for the Poisson kernel follow by taking appropriate difference quotients. □

Large-scale boundary regularity nonlinear equations. We return to prove the De Giorgi boundary Hölder estimates, Lemma 4.2, for scalar equations with bounded uniformly elliptic coefficients.

Proof of Lemma 4.2. Without loss we can assume that $\operatorname{osc}_{\Omega \cap B_1} u = 1$ and $0 \le u \le 1$ in $\Omega \cap B_1$. Call $M = \max_{\partial\Omega \cap B_1} \varphi$ and consider

$$v = (u - M)_+, \quad \text{which is a subsolution of } -\nabla \cdot (A(x)\nabla v) \le 0 \text{ in } B_1.$$

Now since

$$|\{v \le 0\} \cap B_1| \ge \mu,$$

we apply the De Giorgi weak Harnack inequality to find

$$v \le (1 - \delta)\big(\max_{\Omega \cap B_1} u - M\big) \quad \text{in } B_{1/2}$$

for some $\delta > 0$ depending on μ, d, λ. Making the same argument for $-u$ we find

$$\underset{\Omega\cap B_{1/2}}{\operatorname{osc}} u \leq (1-\delta) \underset{\Omega\cap B_1}{\operatorname{osc}} u + \delta \underset{\partial\Omega\cap B_1}{\operatorname{osc}} \varphi.$$

Iterating this argument we obtain,

$$\underset{\Omega\cap B_{1/2^k}}{\operatorname{osc}} u \leq (1-\delta)^k \underset{\Omega\cap B_1}{\operatorname{osc}} u + \sum_{j=0}^{k-1} \delta(1-\delta)^{k-j-1} \underset{\partial\Omega\cap B_{1/2^j}}{\operatorname{osc}} \varphi.$$

Using the Hölder continuity of φ,

$$\underset{\Omega\cap B_{1/2^k}}{\operatorname{osc}} u \leq (1-\delta)^k \biggl(\underset{\Omega\cap B_1}{\operatorname{osc}} u + [\varphi]_{C^\beta} \sum_{j=0}^{k-1} \delta(1-\delta)^{-j-1} 2^{-j\beta} \biggr).$$

Choosing δ smaller if necessary so that

$$2^{-\beta} < (1-\delta),$$

the summation is bounded independent of k and

$$\underset{\Omega\cap B_{1/2^k}}{\operatorname{osc}} u \leq C(\alpha) \bigl(\underset{\Omega\cap B_1}{\operatorname{osc}} u + [\varphi]_{C^\beta} \bigr) 2^{-\alpha k}, \quad \text{with } \alpha = -\frac{\log(1-\delta)}{\log 2} < \beta.$$ □

Proof of Lemma 4.3. Set $w = u_1 - u_2$; then by (4-5) w solves a uniformly elliptic equation in P_n:

$$-\nabla \cdot (A(x)\nabla w) = 0 \quad \text{in } \Omega \quad \text{with } A(x) = \int_0^1 D_p a(x, s\nabla u_1 + (1-s)\nabla u_2)\, ds,$$

with $w \leq 0$ on ∂P_n, and $w \leq M$ for some $M > 0$. Define

$$v = w_+ = \max\{w, 0\}, \quad \text{which is a subsolution of } -\nabla \cdot (A(x)\nabla v) \leq 0 \text{ in } \mathbb{R}^d.$$

Now since,

$$|\{v \leq 0\} \cap B_r| \geq \tfrac{1}{2}|B_r|$$

for any $r \geq 1$ we apply the De Giorgi weak Harnack inequality to find

$$\max_{B_{r/2}\cap P_n} w_+ \leq \max_{B_{r/2}} v \leq (1-\delta) \max_{B_r\cap P_n} w \quad \text{in } B_{r/2}.$$

Iterating this argument we find

$$\max_{B_r\cap P_n} w_+ \leq (1-\delta)^k \max_{B_{2^k r}\cap P_n} w_+ \leq (1-\delta)^k M.$$

Sending $k \to \infty$ and then $r \to \infty$ we find $w_+ \equiv 0$. □

Appendix B

In this section we complete the proof of Lemma 5.4. Recall that we are considering the boundary layer problem with

$$\begin{cases} -\nabla \cdot (A(x)\nabla v) = \nabla \cdot f & \text{in } \mathbb{R}^d_+, \\ v = \psi(x') & \text{on } \partial\mathbb{R}^d_+, \end{cases} \tag{B-1}$$

where $\psi : \partial\mathbb{R}^d_+ \to \mathbb{R}$ and f are smooth, A satisfies the usual assumptions from Section 3 and, furthermore, ψ, f, and A all share $d-1$ linearly independent periods $\ell_1, \dots, \ell_{d-1} \in \partial\mathbb{R}^d_+$ such that, for some $M > 2$,

$$\max_{1\le j\le d-1} |\ell_j| \le M.$$

The following maximal function-type norms turn out to be useful:

$$M_p(f, R) := \sup_{y\cdot e_d=0, R'\ge R} \|f\|_{L^p_{\mathrm{avg}}(B_{R'/2}(y+R'e_d))}, \tag{B-2}$$

$$I_p(f) := M_p(f,0) + \sum_{N\in 2^{\mathbb{N}}} N M_p(f,N). \tag{B-3}$$

Note that $M_p(f,0) = \|f\|_{L^\infty(\mathbb{R}^d_+)}$.

We write v by the Green's function formula,

$$v(x) = \int_{\partial\mathbb{R}^d_+} P(x,y)\psi(y)\,dy + \int_{\mathbb{R}^d_+} \nabla_x G(x,y) f(y)\,dy.$$

The first result is an L^∞ estimate:

Lemma B.1. *For any $p > d$,*

$$\operatorname*{osc}_{\mathbb{R}^d_+} v \lesssim_p \operatorname*{osc}_{\partial\mathbb{R}^d_+} \psi + I_p(f).$$

Proof. The bound for the Poisson integral is already done in Lemma 3.2. For the Green's function term we use the Avellaneda–Lin bounds in Theorem 3.1 along with a Whitney-type decomposition.

Let $x\in\mathbb{R}^d_+$; without loss of generality $x=(0,x_d)$. If $x_d\ge 1$, let $N_x\in 2^{\mathbb{N}}$ be the unique dyadic such that $N_x\le x_d<2N_x$. Then define $\alpha\in[1,2)$ such that $\alpha N_x = x_d$. If $x_d\le 1$ define $\alpha=2$. Now we make a cube decomposition $Q_{N,j} := \alpha N(j,1)+\alpha\big[-\tfrac14 N, \tfrac12 N\big)^d$ for $2\le N\in 2^{\mathbb{N}}$ and $j\in\mathbb{Z}^{d-1}$, with side length comparable to the distance to $x_d=0$. For $N=1$ we define $Q_{1,j}=\alpha(j,1)+\alpha\big[-1,\tfrac12\big)^d$. In this set up $(0,x_d)\in Q_{N_x,0}$.

Now we bound the Green's function integral by

$$\begin{aligned}\int_{\mathbb{R}^d_+} |\nabla_x G(x,y)||f(y)|\,dy &= \sum_Q \int_Q |\nabla_x G(x,y)||f(y)|\,dy \\ &\le \sum_Q |Q|\,\|\nabla_x G(x,\cdot)\|_{L^p_{\mathrm{avg}}(Q)} \|f\|_{L^p_{\mathrm{avg}}(Q)} \\ &\le \sum_N\sum_j N^d \|\nabla_x G(x,\cdot)\|_{L^{p'}_{\mathrm{avg}}(Q)} M_p(f,N).\end{aligned}$$

We claim that, for any $p>d$ and any $N\in 2^{\mathbb{N}}$, $j\in\mathbb{Z}^{d-1}$,

$$\|\nabla_x G(x,\cdot)\|_{L^{p'}_{\mathrm{avg}}(Q)} \lesssim_p N^{1-d}(1+|j|)^{-d}. \tag{B-4}$$

Taking the bound for granted we can complete the computation,

$$\int_{\mathbb{R}^d_+} |\nabla_x G(x,y)||f(y)|\,dy \lesssim \sum_N\sum_j N(1+|j|)^{-d} M_p(f,N) \lesssim I_p(f),$$

where for the last inequality we used that $(1+|j|)^{-d}$ is summable on $\mathbb{Z}^{d-1}$.

Now we finish by proving (B-4) using the Avellaneda–Lin bounds, Theorem 3.1. When $j = 0$ and $N = N_x$ we bound

$$\begin{aligned}\left(\frac{1}{|Q|}\int_Q |\nabla_x G(x,y)|^{p'}\,dy\right)^{1/p'} &\lesssim \left(\frac{1}{|Q|}\int_Q |x-y|^{(1-d)p'}\,dy\right)^{1/p'}\\ &\lesssim N^{-d/p'}\left(\int_0^{CN} r^{(1-d)p'}r^{d-1}\,dr\right)^{1/p'}\\ &\lesssim N^{-d/p'}N^{((1-p')(d-1)+1)/p'} = N^{1-d},\end{aligned}$$

where we have used $p > d$ so that $(p'-1)(d-1) < 1$ and the integral in the second line converges.

When $j \neq 0$ and/or $N \neq N_x$ we have that $|x-y| \gtrsim \max\{N(1+|j|), N_x\}$ for $y \in Q_{N,j}$. In this case,

$$\begin{aligned}|\nabla_x G(x,y)| &\lesssim \frac{y_d}{|x-y|^d} + \frac{x_d y_d}{|x-y|^{d+1}}\\ &\lesssim N^{1-d}(1+|j|)^{-d} + NN_x \max\{N(1+|j|), N_x\}^{-(d+1)}\\ &\lesssim N^{1-d}(1+|j|)^{-d},\end{aligned}$$

which was the desired estimate. □

Next we prove the existence of a boundary layer limit with convergence rate. We assume the following exponential-type bounds on f, which are well suited to the boundary layer problem: there are $K, b > 0$ so that, for all $R > 0$,

$$M_p(f,R) \leq \frac{K}{1+R}e^{-bR/M}. \tag{B-5}$$

From (B-5) one can compute,

$$I_p(f) \lesssim_b K \log M,$$

and also

$$I_p(f,R) := \sum_{2^{\mathbb{N}} \ni N \geq R} NM_p(f,N) \lesssim_b K \log M e^{-bR/M}.$$

Lemma B.2. *Let v, f, ψ and A as above in* (B-1) *with f satisfying the exponential bound* (B-5). *There exists $c_* \in \mathbb{R}^m$ such that*

$$\sup_{y \cdot e_d \geq R} |v(y) - c_*| \lesssim_b ((\operatorname{osc}\psi) + K\log M)e^{-c_0 R/M},$$

where the rate c_0 depends on b and universal constants.

The proof is almost the same as [Feldman and Kim 2017, Lemma A.4] so we omit it.

Acknowledgements

Feldman was partially supported by the National Science Foundation RTG grant DMS-1246999. Zhang was partially supported by National Science Foundation grant DMS-1566578.

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Received 7 Feb 2018. Revised 28 Sep 2018. Accepted 30 Nov 2018.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

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ANALYSIS & PDE

Volume 12 No. 8 2019

2157-5045(2019)12:8;1-I