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DYNAMICS OF ONE-FOLD SYMMETRIC PATCHES FOR THE AGGREGATION EQUATION AND COLLAPSE TO SINGULAR MEASURE

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We are concerned with the dynamics of one-fold symmetric patches for the two-dimensional aggregation equation associated to the Newtonian potential. We reformulate a suitable graph model and prove a local well-posedness result in subcritical and critical spaces. The global existence is obtained only for small initial data using a weak damping property hidden in the velocity terms. This allows us to analyze the concentration phenomenon of the aggregation patches near the blow-up time. In particular, we prove that the patch collapses to a collection of disjoint segments and we provide a description of the singular measure through a careful study of the asymptotic behavior of the graph.

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1. Introduction

This paper is devoted to the study of the two-dimensional aggregation equation with the Newtonian potential:

$$\begin{cases} \partial_t \rho + \operatorname{div}(v\rho) = 0, \quad t \geq 0,\ x \in \mathbb{R}^2, \\ v(t,x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} (x-y)/|x-y|^2 \rho(t,y)\, dy, \\ \rho(0,x) = \rho_0(x). \end{cases} \tag{1-1}$$

This model with more general potential interactions, with or without dissipation, is used to explain some behavior in physics and population dynamics. As a matter of fact, it appears in vortex densities in superconductors [Ambrosio and Serfaty 2008; Du and Zhang 2003; Keller and Segel 1970], material sciences [Holm and Putkaradze 2006; Nieto et al. 2001], cooperative controls and biological swarming

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[Bernoff and Topaz 2011; Breder 1954; Boi et al. 2000; Gazi and Passino 2003; Mogilner and Edelstein-Keshet 1999; Morale et al. 2005; Topaz and Bertozzi 2004], etc. During the last few decades, a lot of intensive research activity has been devoted to exploring several mathematical and numerical aspects of this equation. It is known according to [Bertozzi et al. 2012; Nieto et al. 2001] that classical solutions can be constructed for short times. They develop a finite-time singularity if and only if the initial data is strictly positive at some points and the blow-up time is explicitly given by $T_\star = 1/\max \rho_0$. This follows from the equivalent form

$$\partial_t \rho + v \cdot \nabla \rho = \rho^2,$$

which, written with Lagrangian coordinates, gives exactly a Riccati equation. Note that similarly to Yudovich's result [1963] for Euler equations, weak unique solutions in $L^1 \cap L^\infty$ can be constructed following the same strategy; for more details see [Bertozzi et al. 2009; 2011; 2012; Bertozzi and Laurent 2007; Bertozzi and Brandman 2010; Fetecau et al. 2011; Fetecau and Huang 2013; Dong 2011; Laurent 2007; Li and Rodrigo 2009]. Since the L^1 norm is conserved at least at the formal level, a lot of effort was made to extend the classical solutions beyond the first blow-up time. Poupaud [2002] established the existence of global generalized solutions with defect measure when the initial data is a nonnegative bounded Radon measure. He also showed that when the second moment of the initial data is bounded, for such solutions the atomic part appears in finite time. This result is to some extent in contrast with what is established for Euler equations. Indeed, according to Delort's result [1991] global weak solutions without defect measure can be established when the initial vorticity is a nonnegative bounded Radon measure and the associated velocity has finite local energy. During the time, those solutions do not develop atomic part, contrary to the aggregation equation. This illustrates somehow the gap between both equations, not only at the level of classical solutions but also for the weak solutions. The literature dealing with measure-valued solutions for the aggregation equation with different potentials is very abundant and we refer the reader to [Bodnar and Velazquez 2006; Carrillo et al. 2006; 2011; Carrillo and Rosado 2010; Masmoudi and Zhang 2005].

Now we shall discuss another subject concerning the aggregation patches. Assume that the initial data takes the patch form

$$\rho_0 = \mathbf{1}_{D_0},$$

with D_0 a bounded domain; then solutions can be uniquely constructed up to the time $T^\star = 1$ and one can check that

$$\rho(t) = \frac{1}{1-t}\mathbf{1}_{D_t}, \quad \text{with } (\partial_t + v \cdot \nabla)\mathbf{1}_{D_t} = 0.$$

Note that v is computed from ρ through the Biot–Savart law. To filter the time factor in the velocity field and find an analogous equation to the Euler equations, it is more convenient to rescale the time as was done in [Bertozzi et al. 2012]. Indeed, set

$$\tau = -\ln(1-t), \quad u(\tau, x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2}\mathbf{1}_{\widetilde{D}_\tau}(y)\,dy, \quad \widetilde{D}_\tau = D_t;$$

then we get

$$(\partial_\tau + u \cdot \nabla)\mathbf{1}_{\widetilde{D}_\tau} = 0, \quad \widetilde{D}_0 = D_0.$$

We observe that with this formulation, the blow-up occurs at infinite time and so the solutions do exist globally in time. To simplify the notation we shall write this latter equation with the initial variables. Hence the vortex patch problem is reduced to understanding the evolution equation

$$\begin{cases} \partial_t \rho + v \cdot \nabla \rho = 0, \quad t \geq 0, \\ v(t,x) = -\frac{1}{2\pi} \int_{D_t} (x-y)/|x-y|^2 \, dy, \\ \rho(0) = \mathbf{1}_{D_0}. \end{cases} \tag{1-2}$$

Let us point out that the area of the domain D_t shrinks to zero exponentially; that is,

$$\text{for all } t \geq 0, \quad \|\rho(t)\|_{L^1} = e^{-t} |D_0|. \tag{1-3}$$

The solution to this problem is global in time and takes the form $\rho(t) = \mathbf{1}_{D_t}$, $D_t = \psi(t, D_0)$, where ψ denotes the flow associated to the velocity v. Similarly to the Euler equations [Bertozzi and Constantin 1993; Chemin 1993], Bertozzi, Garnett, Laurent and Verdera [Bertozzi et al. 2016] proved the global-in-time persistence of the boundary regularity in Hölder spaces C^{1+s}, $s \in (0,1)$. However the asymptotic behavior of the patches for large time is still not well understood despite some interesting numerical simulations giving some indications on the concentration dynamics. Notice first that the area of the patch shrinks to zero, which gives that the associated domains will converge in Hausdorff distance to negligible sets. The geometric structure of such sets is not well explored and hereafter we will give two pedagogic and interesting simple examples illustrating the concentration, and one can find more details in [Bertozzi et al. 2012]. The first example is the disc which shrinks to its center, leading after a normalization procedure to the convergence to Dirac mass. The second one is the ellipse patch which collapses to a segment along the big axis and the normalized patch converges weakly to Wigner's semicircle law of density

$$x_1 \mapsto \frac{2\sqrt{\boldsymbol{x_0}^2 - x_1^2}}{\pi \boldsymbol{x_0}^2} \mathbf{1}_{[-x_0, x_0]}, \quad \boldsymbol{x_0} = a - b.$$

It seems that the mechanisms governing the concentration are very complex and related in part for some special class to the initial distribution of the local mass. Indeed, the numerical experiments implemented in [Bertozzi et al. 2012] for some regular shapes indicate that generically the concentration is organized along a skeleton structure. The aim of this paper is to investigate this phenomenon and try to give a complete answer for a special class of initial data where the concentration occurs along disjoint segments lying in the same line. More precisely, we will deal with a one-fold symmetric patch, and by rotation invariance we can suppose that its axis of symmetry coincides with the real axis. We assume in addition that the boundary of the upper part is the graph of a slightly smooth function with small amplitude. Then we will show that we can track the dynamics of the graph globally in time and prove that the normalized solution converges weakly towards a probability measure supported in the union of disjoint segments lying in the real axis. The results will be formulated rigorously in Section 2. The paper is organized as follows. In next section we formulate the graph equation and state our main results. In Sections 3, 4 and 5 we shall discuss basic tools that we use frequently throughout the paper. In Section 6 we prove the local well-posedness for the graph equation. The global existence with small initial data is proved in Section 7, and Section 8 deals with the asymptotic behavior of the normalized density and its convergence towards a singular measure.

2. Graph reformulation and main results

The main purpose of this section is to describe the boundary motion of the patch associated to (1-2) under suitable symmetry structure. One of the basic properties of the aggregation equation that we shall use in a crucial way concerns its group of symmetry, which is much richer than for Euler equations. Actually, in addition to rotation and translation invariance, the aggregation equation is in fact invariant by reflection. To check this property and without loss of generality we can look for the invariance with respect to the real axis. Set

$$X=(x,y)\in\mathbb{R}^2 \quad \text{and} \quad \overline{X}=(x,-y)$$

and introduce

$$\hat{\rho}(t,X)=\rho(t,\overline{X}),\quad \hat{v}(t,X)=-\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{X-Y}{|X-Y|^2}\,\hat{\rho}(t,Y)\,dY.$$

Using straightforward change of variables, it is quite easy to get

$$v(t,X)=\overline{\hat{v}(t,\overline{X})},\quad \operatorname{div}(v\,\rho)(t,X)=\operatorname{div}(\hat{v}\hat{\rho})(t,\overline{X}).$$

Therefore we find that $\hat{\rho}$ satisfies also the aggregation equation

$$\partial_t\hat{\rho}+\operatorname{div}(\hat{v}\,\hat{\rho})=0.$$

Combining this property with the uniqueness of Yudovich's solutions, it follows that if the initial data belong to $L^1\cap L^\infty$ and admit an axis of symmetry then the solution remains invariant with respect to the same axis. In the framework of the vortex patches this result means that if the initial data are given by $\rho_0=\mathbf{1}_{D_0}$ and the domain D_0 is symmetric with respect to the real axis, the domain D_t defining the solution $\rho(t)=\mathbf{1}_{D_t}$ remains symmetric with respect to the same axis for any positive time. Recall that in the form (1-2) Yudovich-type solutions are global in time. To be precise about the terminology, here and contrary to the standard definition in topology, where "domain" means a connected open set, we mean by "domain" any measurable set of strictly positive measure. In addition, a patch whose domain is symmetric with respect to the real axis (or any axis) is called one-fold symmetric.

In the current study, we shall focus on the domains D_0 such that the boundary part lying in the upper half-plane is described by the graph of a C^1 positive function $f_0:\mathbb{R}\to\mathbb{R}_+$ with compact support. This is equivalent to

$$D_0=\{(x,y)\in\mathbb{R}^2: x\in\operatorname{supp} f_0,\ -f_0(x)\le y\le f_0(x)\}.$$

We point out that concretely we shall consider the evolution not of D_0 but of its extended set defined by

$$\widehat{D}_0=\{(x,y)\in\mathbb{R}^2: x\in\mathbb{R},\ -f_0(x)\le y\le f_0(x)\}.$$

This does not matter since the domain D_t remains symmetric with respect to the real axis and then we can simply track its evolution by knowing the dynamics of its extended domain: we just remove the extra lines located on the real axis.

One of the main objectives of this paper is to follow the dynamics of the graph and investigate local and global well-posedness issues in different function spaces. In the next lines, we shall derive the evolution equation governing the motion of the initial graph f_0. Assume that in a short time interval $[0,T]$ the part of the boundary in the upper half-plane is described by the graph of a C^1 function $f_t:\mathbb{R}\to\mathbb{R}_+$. This forces the points of the boundary ∂D_t located on the real axis to be cusp singularities. As a material point located at the boundary remains on the boundary, any parametrization $s\mapsto\gamma_t(s)$ of the boundary should satisfy

$$(\partial_t\gamma_t(s)-v(t,\gamma_t(s)))\cdot\vec{n}(\gamma_t(s))=0,$$

with $\vec{n}(\gamma_t)$ being a normal unit vector to the boundary at the point $\gamma_t(s)$. Now take the parametrization in the graph form $\gamma_t:x\mapsto(x,f(t,x))$; then the preceding equation reduces to the nonlinear transport equation

$$\begin{cases}\partial_t f(t,x)+u_1(t,x)\,\partial_x f(t,x)=u_2(t,x), & t\ge 0,\ x\in\mathbb{R},\\ f(0,x)=f_0(x),\end{cases}\tag{2-1}$$

where $(u_1,u_2)(t,x)$ is the velocity $(v_1,v_2)(t,X)$ computed at the point $X=(x,f(t,x))$. Throughout this paper we use the notation

$$f_t(x)=f(t,x)\quad\text{and}\quad f'(t,x)=\partial_x f(t,x).$$

To reformulate (2-1) in a closed form we shall recover the velocity components with respect to the graph parametrization. We start with the computation of $v_1(X)$. Here and for the sake of simplicity we drop the time parameter from the graph and the domain of the patch. One writes according to Fubini's theorem

$$-2\pi v_1(X)=\int_D\frac{x-y_1}{|X-Y|^2}\,dY=\int_{\mathbb{R}}(x-y_1)\int_{-f(y_1)}^{f(y_1)}\frac{dy_2}{(x-y_1)^2+(f(x)-y_2)^2}\,dy_1,$$

where $Y=(y_1,y_2)$. Using the change of variables $y_2-f(x)=(x-y_1)Z$ we find

$$\begin{aligned}2\pi v_1(X)&=\int_{\mathbb{R}}\left[\arctan\left(\frac{f(y)-f(x)}{y-x}\right)+\arctan\left(\frac{f(y)+f(x)}{y-x}\right)\right]dy\\&=\int_{\mathbb{R}}\left[\arctan\left(\frac{f(x+y)-f(x)}{y}\right)+\arctan\left(\frac{f(x+y)+f(x)}{y}\right)\right]dy.\end{aligned}$$

To compute v_2 in terms of f we proceed as before and we find

$$-2\pi v_2(X)=\int_D\frac{f(x)-y_2}{|X-Y|^2}\,dA(Y)=\int_{\mathbb{R}}\int_{-f(y_1)}^{f(y_1)}\frac{f(x)-y_2}{(x-y_1)^2+(f(x)-y_2)^2}\,dy_2\,dy_1.$$

Therefore we obtain the expression

$$4\pi v_2(x,f(x))=\int_{\mathbb{R}}\log\left(\frac{y^2+(f(x+y)-f(x))^2}{y^2+(f(x+y)+f(x))^2}\right)dy.$$

With the notation adopted before for (u_1,u_2) we finally get the formulas

$$\begin{aligned}u_1(t,x)&=\frac{1}{2\pi}\int_{\mathbb{R}}\left[\arctan\left(\frac{f_t(x+y)-f_t(x)}{y}\right)+\arctan\left(\frac{f_t(x+y)+f_t(x)}{y}\right)\right]dy,\\u_2(t,x)&=\frac{1}{4\pi}\int_{\mathbb{R}}\log\left(\frac{y^2+(f_t(x+y)-f_t(x))^2}{y^2+(f_t(x+y)+f_t(x))^2}\right)dy.\end{aligned}\tag{2-2}$$

We emphasize that for the coherence of the model the graph equation (2-1) is supplemented with the initial condition $f_0(x)\ge 0$. According to Proposition 6.2, the positivity is preserved for enough smooth solutions. Furthermore, and once again according to this proposition we have a maximum principle estimate:

$$\text{for all } t \geq 0, \text{ for all } x \in \mathbb{R}, \quad 0 \leq f(t,x) \leq \|f_0\|_{L^\infty}.$$

Notice that the model remains meaningful even when the function f_t changes sign. In this case the geometric domain of the patch is simply obtained by looking to the region delimited by the curve of f_t and it is symmetric with respect to the real axis. This is also equivalent to dealing with a positive function f_t but its graph will be less regular and belongs only to the Lipschitz class. Another essential element that will be analyzed later in Proposition 6.2 concerns the support of the solutions, which remains confined through the time interval. More precisely, if $\operatorname{supp} f_0 \subset [a,b]$ with $a < b$ then provided that the graph exists for $t \in [0,T]$ one has

$$\operatorname{supp} f(t) \subset [a,b].$$

This follows from the fact that the flow associated to the horizontal velocity u_1 is contractive on the boundary. It is not clear whether global weak solutions satisfying the maximum principle can be constructed. However, to deal with classical solutions one should control higher regularity of the graph and it seems from the transport structure of the equation that the optimal scaling for local well-posedness theory is Lipschitz class. Thus, in what follows we say that a function space is critical if it scales as a Lipschitz class and subcritical if it scales above like Hölder spaces C^{1+s}, $s > 0$. Denote by $g(t,x) = \partial_x f(t,x)$ the slope of the graph; then it is quite obvious from (2-1) that

$$\partial_t g + u_1 \partial_x g = -\partial_x u_1 g + \partial_x u_2. \tag{2-3}$$

For the computation of the source term we proceed in a classical way using the differentiation under the integral sign and we get successively

$$2\pi \partial_x u_1(x) = \text{p.v.} \int_{\mathbb{R}} \frac{f'(x+y) - f'(x)}{y^2 + (f(x+y) - f(x))^2} y\, dy + \text{p.v.} \int_{\mathbb{R}} \frac{f'(x+y) + f'(x)}{y^2 + (f(x+y) + f(x))^2} y\, dy \tag{2-4}$$

and

$$\begin{aligned} 2\pi \partial_x u_2(x) = \text{p.v.} &\int_{\mathbb{R}} \frac{(f(x+y) - f(x))(f'(x+y) - f'(x))}{y^2 + (f(x+y) - f(x))^2}\, dy \\ &- \text{p.v.} \int_{\mathbb{R}} \frac{(f(x+y) + f(x))(f'(x+y) + f'(x))}{y^2 + (f(x+y) + f(x))^2}\, dy, \end{aligned}$$

where the notation "p.v." is the Cauchy principal value. It is worth pointing out that the first two integrals appearing in the right-hand side of the expressions of $\partial_x u_1$ and $\partial_x u_2$ are in fact connected to the Cauchy operator associated to the curve f defined in (5-1). This operator is well-studied in the literature and some details will be given later in Section 5. Next, we shall check that the integrals appearing in the right-hand side of the preceding formulas can actually be restricted over a compact set related to the support of f. Let $[-M, M]$ be a symmetric segment containing the set $K_0 - K_0$, with K_0 being the convex hull of the support of f_0, which is denoted by $\operatorname{supp} f_0$. It is clear that the support of $\partial_x u_1 f'$ is contained in K_0 and thus for $x \in K_0$ one has

$$\text{p.v.}\int_{\mathbb{R}} \frac{f'(x+y)-f'(x)}{y^2+(f(x+y)-f(x))^2}\,y\,dy = \text{p.v.}\int_{-M}^{M} \frac{f'(x+y)-f'(x)}{y^2+(f(x+y)-f(x))^2}\,y\,dy.$$

Consequently, we obtain for $x \in \mathbb{R}$

$$\begin{aligned}2\pi f'(x)\partial_x u_1(x) = f'(x)\,\text{p.v.}\int_{-M}^{M} &\frac{f'(x+y)-f'(x)}{y^2+(f(x+y)-f(x))^2}\,y\,dy \\ &+ f'(x)\,\text{p.v.}\int_{-M}^{M} \frac{f'(x+y)+f'(x)}{y^2+(f(x+y)+f(x))^2}\,y\,dy.\end{aligned}$$

Coming back to the integral representation defining $\partial_x u_2$ one can see, using a cancellation between both integrals, that the support of $\partial_x u_2$ is contained in K_0. Furthermore, for $x \in K_0$ one may write

$$\begin{aligned}2\pi\,\partial_x u_2(x) = \text{p.v.}\int_{-M}^{M} &\frac{(f(x+y)-f(x))(f'(x+y)-f'(x))}{y^2+(f(x+y)-f(x))^2}\,dy \\ &- \text{p.v.}\int_{-M}^{M} \frac{(f(x+y)+f(x))(f'(x+y)+f'(x))}{y^2+(f(x+y)+f(x))^2}\,dy.\end{aligned}$$

Gathering the preceding identities we deduce that

$$2\pi(-\partial_x u_1 f'(x) + \partial_x u_2) = F(x) - G(x), \tag{2-5}$$

with

$$F(x) \triangleq \text{p.v.}\int_{-M}^{M} \frac{[f(x+y)-f(x)-yf'(x)](f'(x+y)-f'(x))}{y^2+(f(x+y)-f(x))^2}\,dy,$$

$$G(x) \triangleq \text{p.v.}\int_{-M}^{M} \frac{[f(x+y)+f(x)+yf'(x)](f'(x+y)+f'(x))}{y^2+(f(x+y)+f(x))^2}\,dy.$$

One should keep in mind that the integrals above can also be extended to the full real axis. Sometimes in order to reduce the size of the integral representation we use the notation

$$\Delta_y^{\pm} f(x) = f(x+y) \pm f(x). \tag{2-6}$$

Thus F and G take the form

$$F(x) = \text{p.v.}\int_{-M}^{M} \frac{[\Delta_y^- f(x) - yf'(x)]\Delta_y^- f'(x)}{y^2+(\Delta_y^- f(x))^2}\,dy, \tag{2-7}$$

$$G(x) = \text{p.v.}\int_{-M}^{M} \frac{[\Delta_y^+ f(x) + yf'(x)]\Delta_y^+ f'(x)}{y^2+(\Delta_y^+ f(x))^2}\,dy. \tag{2-8}$$

The first main result of this paper is devoted to the local well-posedness issue. We shall discuss two results related to subcritical and critical regularities. Denote by X one of the following spaces: Hölder spaces $C^s(\mathbb{R})$ with $s \in (0,1)$ or the Dini space $C^\star(\mathbb{R})$. For more details about classical properties of these spaces we refer the reader to Section 4.

Theorem 2.1. *Let f_0 be a positive compactly supported function such that $f_0' \in X$. Then, the following results hold true*:

(1) *Equation* (2-1) *admits a unique local solution such that $f' \in L^\infty([0,T], X)$, where the time existence T is related to the norm $\|f_0'\|_X$ and the size of the support of f_0. In addition, the solution satisfies the maximum principle*

$$\text{for all } t \in [0,T], \quad \|f(t)\|_{L^\infty} \le \|f_0\|_{L^\infty}.$$

(2) *There exists a constant $\varepsilon > 0$ depending only on s and the size of the support of f_0 such that if*

$$\|f_0'\|_{C^s} < \varepsilon \tag{2-9}$$

then (2-1) *admits a unique global solution $f' \in L^\infty(\mathbb{R}_+; C^s(\mathbb{R}))$. Moreover*,

$$\text{for all } t \ge 0, \quad \|\partial_x f(t)\|_{L^\infty} \le C_0 e^{-t},$$

with C_0 a constant depending only on $\|f_0'\|_{C^s}$.

Before outlining the strategy of the proof, some comments are in order.

Remarks. (1) The global existence result is only proved for the subcritical case (C^s). The critical case (Dini case) is more delicate to handle due to the lack of strong damping, which is only proved in the subcritical case (see Proposition 7.1). Roughly speaking, the damping comes from the linearization of the nonlinear term. Indeed, one finds that the equation

$$\partial_t f' + u_1\, \partial_x f' = \tfrac{1}{2\pi}(F(x) - G(x)) = -f' + L_1(x) + \text{nonlinear},$$

where (see Proposition 7.1) the term "nonlinear" has superlinear C^s-type estimates. If the term $L_1(x)$ were identically zero, then one can use the damping term $-f'$ to obtain exponentially decaying global solutions with small initial data. On the other hand, as it turns out, the almost-linear-type term $L_1(x)$ admits estimates of the form

$$\begin{aligned} \|L_1\|_s &\le (\|f'\|_s + 2\|f'\|_{L^\infty}) + C\|f'\|_{L^\infty}^s \|f'\|_s, \\ \|L_1\|_{L^\infty} &\le C \min(\|f\|_{L^\infty}^s \|f'\|_s, \|f'\|_{L^\infty}). \end{aligned}$$

The key improvement here is the first estimate in the L^∞ estimate of L_1, which is in some sense superlinear. By using Proposition 6.2 one can obtain an exponential decay estimate of $\|f\|_1$ through an area argument. This important estimate together with some interpolation estimates (and an exponential decay estimate of $\|\partial_x u_1\|_\infty$) and the strong damping term $-f'$ then yields global well-posedness for small data.

(2) Coming back to the patch domain, we see that it admits cusp-like singularities located on the axis of symmetry. This is not covered by the preceding result [Bertozzi et al. 2016] where the boundary is assumed to be more regular than C^1. From the proof of Theorem 2.1 we deduce that the graph solutions generate a Lipschitz velocity. This allows us to easily propagate a weak notion for the order of a cusp. More precisely, let $\alpha > 0$ be the order of a cusp x_0; that is, for small r, we have $|D \cap B(x_0, r)| = O(r^{2+\alpha})$, and then for the solutions constructed in Theorem 2.1 we get $|D_t \cap B(x_t, r)| = O(r^{2+\alpha})$, with x_t the image of x_0 by the flow. Notice that this problem was studied for Euler equations in [Danchin 2000].

(3) From Sobolev embeddings we deduce according to the assumption on f_0 listed in Theorem 2.1 that f_0 belongs to the space $C_c^1(\mathbb{R})$ of compactly supported C^1 functions.

(4) The maximum principle holds true globally in time; however, it is not clear whether some suitable weak global solutions could be constructed in this setting.

Now we shall give some details about the proofs. First we establish local-in-time a priori estimates based on the transport structure of the equation combined with some refined studies on modified curved Cauchy operators implemented in Section 5 and essentially based on standard arguments from singular integrals. The construction of the solutions done in Section 6C is slightly more intricate than the usual schemes used for transport equations. This is due to the fact that the establishment of the a priori estimates is not purely energetic. First, at some levels we use some nonlinear rigidity of the equation like in Theorem 2.1(3), where the factor f' behind the operator should be the derivative of the function f that appears inside the operator. Second, we use at some point the fact that the support is confined in time. Last we use at different steps the positivity of the solution. Hence it seems quite difficult to find a linear scheme taking into account all of those constraints. The idea is to implement a nonlinear scheme with two regularizing parameters ε and n. The first one is used to smooth out the singularity of the kernel and the second to smooth the solution through a nonlinear scheme. We first establish that one has uniform a priori estimates on n but on some small interval depending on ε. We are also able to pass to the limit on n and get a solution for a modified nonlinear problem. Second we check that the a priori estimates still be valid uniformly on ε. This ensures that the time existence can be in fact pushed up to the time given by the a priori estimates obtained for the initial equation (2-1). As a consequence we get a uniform time existence with respect to ε and finally we establish the convergence towards a solution of the initial value problem using standard compactness arguments.

The global existence for small initial data requires much more careful analysis because there is no apparent dissipation or damping mechanisms in the equation. Notice that the estimate of the source term G contains some linear parts as it is stated in Proposition 6.1. The basic ingredient to get rid of those linear parts is to implement a kind of linearization allowing us to capture a weak damping effect in G that can just absorb the growth of the linear part. We do not know if the damping proved for lower regularity still happens in the resolution space. As to the nonlinear terms, they are always associated with some subcritical norms and thus using an interpolation argument with the exponential decay of the L^1 norm we get a global-in-time control that leads to the global existence.

The second result that we shall discuss deals with the asymptotic behavior of the solutions to (1-2) and (2-1). We shall study the collapse of the support to a collection of disjoint segments located at the axis of symmetry. Another interesting issue that will be covered by this discussion concerns the characterization of the limit behavior of the probability measure

$$dP_t \triangleq e^t \frac{\mathbf{1}_{D_t}}{|D_0|} dA, \tag{2-10}$$

with dA being Lebesgue measure and $|D_0|$ denoting the Lebesgue measure of D_0. Our result reads as follows.

Theorem 2.2. *Let f_0 be a positive compactly supported function such that $f_0' \in C^s(\mathbb{R})$, with $s \in (0,1)$. Assume that* $\operatorname{supp} f_0$ *is the union of n disjoint segments and satisfies the smallness condition* (2-9). *Then there exists a compact set $D_\infty \subset \mathbb{R}$ composed of exactly of n disjoint segments and a constant $C > 0$ such that*

$$\text{for all } t \geq 0, \quad d_H(D_t, D_\infty) \leq Ce^{-t}, \quad |D_\infty| \geq \tfrac{1}{2}|D_0|,$$

with d_H being the Hausdorff distance and $|D_\infty|$ the one-dimensional Lebesgue measure of D_∞. In addition, the probability measures $\{dP_t\}_{t\geq 0}$ defined in (2-10) *converge weakly as t goes to ∞ to the probability measure*

$$dP_\infty := \Phi\, \delta_{D_\infty \otimes \{0\}},$$

with Φ being a compactly supported function in D_∞ belonging to $C^\alpha(\mathbb{R})$ for any $\alpha \in (0,1)$ and can be expressed in the form

$$\Phi(x) = \frac{f_0(\psi_\infty^{-1}(x))}{\|f_0\|_{L^1}} e^{g(x)}, \tag{2-11}$$

with g a function that can be implicitly recovered from the full dynamics of solution $\{f_t : t \geq 0\}$ and

$$\psi_\infty = \lim_{t\to\infty} \psi(t).$$

Note that $\psi(t)$ is the one-dimensional flow associated to u_1 defined in (6-26) *and*

$$D_t = \{(x,y) : x \in \operatorname{supp} f_t,\ -f_t(x) \leq y \leq f_t(x)\}.$$

Remark 2.3. The regularity of the profile Φ might be improved and we expect that Φ keeps the same regularity as the graph.

The proof of the collapse of the support to a disjoint union of segments can be easily derived from the formula (2-11) which ensures that the support of the limit measure is exactly the image of the support of f_0 by the limit flow ψ_∞, which is a homeomorphism of the real axis. To get the convergence with the Hausdorff distance we just use the exponential damping of the amplitude of the curve. As to the characterization of the limit measure it is based on the exponential decay of the amplitude of graph combined with the scattering as t goes to infinity of the normalized solution $e^t f(t)$. In fact, we prove that the density is nothing but the formal quantity

$$\Phi(x) = 2 \lim_{t\to\infty} e^t f(t,x)$$

whose existence is obtained using the transport structure of the equation through the method of characteristics combined with the damping effects of the nonlinear source terms.

3. Generalities on the limit shapes

In this short section we shall discuss a simple result dealing with the role of symmetry in the structure of the limit shape D_∞. Roughly speaking, we shall prove that thin initial domains along their axis of symmetry generate concentration to segments. Notice that

$$D_\infty \triangleq \big\{\lim_{t\to\infty} \psi(t,x) : x \in D_0\big\},$$

where ψ is the flow associated to the velocity v and defined through the ODE

$$\begin{cases} \partial_t \psi(t,x) = v(t,\psi(t,x)), \quad t \ge 0,\ x \in \mathbb{R}^2, \\ \psi(0,x) = x. \end{cases} \tag{3-1}$$

The existence of the set D_∞ will be proved below. We intend to prove the following.

Proposition 3.1. *The following assertions hold*:

(1) *If D_0 is a bounded domain of $\mathbb{R}^2$, then for any $x \in \mathbb{R}^2$ the quantity* $\lim_{t\to\infty} \psi(t,x)$ *exists.*

(2) *Let D_0 be a simply connected bounded domain symmetric with respect to an axis Δ. Denote by* $d_0 = \mathrm{Length}(D_0 \cap \Delta)$. *There exists an absolute constant C such that if*

$$d_0 > C|D_0|^{\frac12}$$

then the shape D_∞ contains an interval of the size $d_0 - C|D_0|^{\frac12}$.

Proof. (1) Integrating in time the flow equation (3-1) yields

$$\psi(t,x) = x + \int_0^t v(\tau,\psi(\tau,x))\,d\tau.$$

Now observe that pointwisely

$$|v(t,x)| \le \frac{1}{2\pi}\left(\frac{1}{|\cdot|^2} \star |\rho(t)|\right)(x).$$

Thus interpolation inequalities combined with (1-3) lead to

$$\|v(t)\|_{L^\infty} \le C\|\rho(t)\|_{L^1}^{\frac12}\|\rho(t)\|_{L^\infty}^{\frac12} \le Ce^{-\frac{t}{2}}|D_0|^{\frac12}, \tag{3-2}$$

with C an absolute constant. This implies that the integral $\int_0^\infty v(\tau,\psi(\tau,x))\,d\tau$ converges absolutely and therefore $\lim_{t\to\infty}\psi(t,x)$ exists in $\mathbb{R}^2$. This allows us to define the limit shape D_∞ as

$$D_\infty = \big\{\lim_{t\to\infty} \psi(t,x) : x \in D_0\big\}.$$

(2) Without loss of generality we will suppose that the straight line Δ coincides with the real axis. Since D is a simply connected bounded domain, there exist two different points $X_0^-, X_0^+ \in \mathbb{R}$ such that

$$\overline{D}_0 \cap \Delta = [X_0^-, X_0^+].$$

Then it is clear that $\mathrm{Length}(\overline{D}_0 \cap \Delta) = X_0^+ - X_0^- := d_0$. By assumption D_0 is symmetric with respect to Δ; then the domain D_t remains also symmetric with respect to the same axis and the points $X_0^\pm$ move along this axis. Set

$$X^\pm(t) = \psi(t, X_0^\pm);$$

then as the flow is a homeomorphism

$$\overline{D}_t \cap \Delta = [X^-(t), X^+(t)].$$

Now we wish to follow the evolution of the distance $d(t) := X^+(t) - X^-(t)$ and find a sufficient condition such that this distance remains away from zero up to infinity. Notice from the first point that $\lim_{t\to\infty} d(t)$ exists and is equal to some positive number d_∞. From the triangle inequality, one easily gets

$$d(t) \geq d_0 - 2\int_0^t \|v(\tau)\|_{L^\infty}\,d\tau.$$

Inequality (3-2) ensures that

$$d(t) \geq d_0 - C|D_0|^{\frac{1}{2}}$$

and therefore $d_\infty \geq d_0 - C|D_0|^{\frac{1}{2}}$. Consequently, if $d_0 > C|D_0|^{\frac{1}{2}}$ then the points $\{X^\pm(t)\}$ do not collide up to infinity and thus the set D_∞ contains a nontrivial interval as claimed. □

4. Basic properties of Dini and Hölder spaces

We now set up some function spaces that we shall use and review some of their important properties. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function; we define its modulus of continuity $\omega_f : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\omega_f(r) = \sup_{|x-y|\leq r} |f(x) - f(y)|.$$

This is a nondecreasing function satisfying $\omega_f(0) = 0$ and it is subadditive; that is, for $r_1, r_2 \geq 0$ we have

$$\omega_f(r_1 + r_2) \leq \omega_f(r_1) + \omega_f(r_2). \tag{4-1}$$

Now we intend to recall Dini and Hölder spaces. The Dini space denoted by $C^\star(\mathbb{R})$ is the set of continuous bounded functions f such that

$$\|f\|_{L^\infty} + \|f\|_D < \infty, \quad \text{with } \|f\|_D = \int_0^1 \frac{\omega_f(r)}{r}\,dr.$$

Another space that we frequently use throughout this paper is the Hölder space. Let $s \in (0, 1)$; we denote by $C^s(\mathbb{R})$ the set of functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\|f\|_{L^\infty} + \|f\|_s < \infty, \quad \text{with } \|f\|_s = \sup_{0<r<1} \frac{\omega_f(r)}{r^s}.$$

Let K be a compact set of $\mathbb{R}$; we define $C^\star_K$ as the subspace of $C^\star(\mathbb{R})$ whose elements are supported in K. Note that $C^\star_K \hookrightarrow L^\infty(\mathbb{R})$, which means that a constant C depending only on the diameter of the compact K exists such that

$$\text{for all } f \in C^\star_K, \quad \|f\|_{L^\infty} \leq C\|f\|_D. \tag{4-2}$$

This follows easily from the observation

$$\text{for all } r \in \left(0, \tfrac{1}{2}\right], \quad \omega(r)\ln 2 \leq \|f\|_D.$$

From (4-2) we deduce that for any $A \geq 1$

$$\int_0^A \frac{\omega_f(r)}{r}\,dr \leq \|f\|_D + 2\|f\|_{L^\infty}\ln A \leq C\|f\|_D(1 + \ln A). \tag{4-3}$$

Coming back to the definition of Dini seminorm one deduces the product laws: for $f, g \in C_K^\star$

$$\|fg\|_D \leq \|f\|_{L^\infty}\|g\|_D + \|g\|_{L^\infty}\|f\|_D \quad \text{and} \quad \|fg\|_D \leq C\|f\|_D\|g\|_D. \tag{4-4}$$

Another useful space is C_K^s, which is the subspace of $C^s(\mathbb{R})$ whose functions are supported on compact K. It is quite obvious that

$$C_K^s \hookrightarrow C_K^\star \hookrightarrow L^\infty. \tag{4-5}$$

We point out that all these spaces are complete. Another property which will be very useful is the following composition law. If $f \in C^s(\mathbb{R})$ with $0<s<1$ and $\psi : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function then $f \circ \psi \in C^s(\mathbb{R})$ and

$$\|f \circ \psi\|_s \leq (\|f\|_s + 2\|f\|_{L^\infty})\|\nabla\psi\|_{L^\infty}^s. \tag{4-6}$$

It is worth pointing out that in the case of the Dini space $C^\star(\mathbb{R})$ we get a more precise estimate of logarithmic type,

$$\|f \circ \psi\|_D \leq C(\|f\|_D + \|f\|_{L^\infty})(1 + \ln_+(\|\nabla\psi\|_{L^\infty})), \tag{4-7}$$

with the notation

$$\ln_+ x \triangleq \begin{cases} \ln x & \text{if } x \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Another estimate of great interest is the following product law:

$$\|fg\|_s \leq \|f\|_{L^\infty}\|g\|_s + \|g\|_{L^\infty}\|f\|_s. \tag{4-8}$$

In the next task we will be concerned with a pointwise estimate connecting a positive smooth function to its derivative and explore how this property is affected by the regularity. This kind of property will be required in Section 5 in studying Cauchy operators with special forms.

Lemma 4.1. *Let K be a compact set of $\mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}_+$ be a continuous positive function supported in K such that $f' \in C^\star(\mathbb{R})$. Then we have*

$$\text{for all } x \in \mathbb{R}, \quad |f'(x)| \leq C\frac{\|f'\|_D + \|f'\|_{L^\infty}}{1 + \ln_+(\|f'\|_D / f(x))}.$$

A weak version of this inequality is

$$\text{for all } x \in \mathbb{R}, \quad |f'(x)| \leq C\frac{(\|f'\|_D + \|f'\|_{L^\infty})(1 + \ln_+(1/\|f'\|_D))}{1 + \ln_+(1/f(x))},$$

with C an absolute constant. If in addition $f' \in C^s(\mathbb{R})$ with $s \in (0,1)$, then

$$\text{for all } x \in \mathbb{R}, \quad |f'(x)| \leq C\|f'\|_s^{\frac{1}{1+s}}[f(x)]^{\frac{s}{1+s}}$$

and the constant C depends only on s.

Proof. Let x be a given point; without any loss of generality one can assume that $f'(x) \geq 0$. Now let $h \in [0, 1]$; then using the mean value theorem, there exists $c_h \in [x-h, x)$ such that

$$\begin{aligned} f(x-h) &= f(x) - hf'(c_h) \\ &= f(x) - hf'(x) - h[f'(c_h) - f'(x)] \\ &\leq f(x) - hf'(x) + h\,\omega_{f'}(h). \end{aligned}$$

From the positivity of the function f we deduce that for any $h \in [0, 1]$ one gets

$$f(x) - hf'(x) + h\,\omega_{f'}(h) \geq 0.$$

Then dividing by h^2 and integrating in h between ε and 1, with $\varepsilon \in (0, 1]$, we get

$$f(x)\frac{1}{\varepsilon} + f'(x)\ln\varepsilon + \|f'\|_D \geq 0.$$

Multiplying by ε we obtain

$$\text{for all } \varepsilon \in (0,1), \quad f(x) + f'(x)\,\varepsilon\ln\varepsilon + \|f'\|_D\,\varepsilon \geq 0. \tag{4-9}$$

By studying the variation with respect to ε we find that the suitable value of ε is given by

$$\ln\varepsilon = -1 - \frac{\|f'\|_D}{f(x)}.$$

Inserting this choice into (4-9) we find that

$$\varepsilon f'(x) \leq f(x);$$

that is,

$$e^{-1-\|f'\|_D/f'(x)} f'(x) \leq f(x).$$

From the inequality $te^{-t} \leq e^{-1}$ we deduce that

$$e^{-1} \geq \frac{\|f'\|_D}{f'(x)} e^{-\|f'\|_D/f'(x)},$$

which implies in turn that

$$e^{-1-\|f'\|_D/f'(x)} f'(x) \geq e^{-2\|f'\|_D/f'(x)} \|f'\|_D.$$

Consequently we get

$$e^{-2\|f'\|_D/f'(x)} \|f'\|_D \leq f(x).$$

Thus when $f(x)/\|f'\|_D > 1$ this estimate does not give any useful information and then we simply write

$$f'(x) \leq \|f'\|_{L^\infty}.$$

However for $f(x)/\|f'\|_D < 1$ we get

$$f'(x) \leq C\frac{\|f'\|_D}{1 + \ln_+(\|f'\|_D/f(x))},$$

from which we deduce that

$$f'(x) \le C \frac{\|f'\|_D (1 + \ln_+(1/\|f'\|_D)}{1 + \ln_+(1/f(x))}.$$

Indeed, one may use the estimate

$$\text{for all } x > 0, \quad \frac{1 + \ln_+(1/x)}{1 + \ln_+(a/x)} \le 1 + \ln_+(1/a),$$

which can be verified easily by studying the variation of the fractional function.

Now let us move to the proof when f' is assumed to belong to the Hölder space C^s, with $s \in (0, 1)$. Following the same proof as before one deduces that under the assumption $f'(x) \ge 0$ one obtains for any $h \in \mathbb{R}_+$

$$f(x) - hf'(x) + h^{1+s}\|f'\|_s \ge 0.$$

By studying the variation of this function with respect to h we find that the best choice of h is given by

$$h^s = \frac{f'(x)}{(1+s)\|f'\|_s},$$

which implies the desired result, that is,

$$f'(x) \le C \|f'\|_s^{\frac{1}{1+s}} [f(x)]^{\frac{s}{1+s}}. \qquad \square$$

5. Modified curved Cauchy operators

This section is devoted to the study of some variants of Cauchy operators which are closely connected to the operators arising in (2-4) and (2-5). Let us first recall the classical Cauchy operator associated to the graph of a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$,

$$\mathcal{C}_f g(x) = \int_{\mathbb{R}} \frac{g(x+y) - g(x)}{y + i(f(x+y) - f(x))} \, dy, \tag{5-1}$$

which is well-defined at least for a smooth function g. According to a famous theorem of Coifman, McIntosh and Meyer [Coifman et al. 1982], this operator can be extended as a bounded operator from L^p to L^p for $1 < p < \infty$. By adapting the proof of [Wittmann 1987], this operator can also be extended continuously from C^s_K to $C^s(\mathbb{R})$ for $0 < s < 1$, provided that f belongs to $C^{1+s}(\mathbb{R})$. However this operator fails to be extended continuously from the Dini space $C^\star_K$ to itself, as can be checked from Hilbert transform. The structure of the operators that we have to deal with, as one may observe from the expression of F following (2-5), is slightly different from the Cauchy operators. It can be associated to the truncated bilinear Cauchy operator defined as follows: for given $M > 0$, $\theta \in [0, 1]$,

$$\mathcal{C}^\theta_f(g, h)(x) = \int_{-M}^{M} \frac{(g(x+\theta y) - g(x))(h(x+y) - h(x))}{y + i(f(x+y) - f(x))} \, dy.$$

The real and imaginary parts of this operator are given respectively by

$$\mathcal{C}_f^{\theta,\Re}(g,h)(x)=\int_{-M}^{M}\frac{y(g(x+\theta y)-g(x))(h(x+y)-h(x))}{y^2+[f(x+y)-f(x)]^2}\,dy \tag{5-2}$$

and

$$\mathcal{C}_f^{\theta,\Im}(g,h)(x)=-\int_{-M}^{M}\frac{(f(x+y)-f(x))(g(x+\theta y)-g(x))(h(x+y)-h(x))}{y^2+[f(x+y)-f(x)]^2}\,dy.$$

In what follows we denote by X one of the spaces C_K^s, with $0<s<1$, or $C_K^\star$. The result that we shall discuss deals with the continuity of the preceding bilinear operators on the spaces X. This may have been discussed in the literature, but as we need to control the continuity constant we shall give a detailed proof.

Proposition 5.1. *Let K be a compact set of $\mathbb{R}$ and f be a compactly supported function such that $f'\in X$. Then the following assertions hold true: The bilinear operator $\mathcal{C}_f^\theta : X\times X\to X$ is well-defined and continuous. More precisely, there exists a constant C independent of θ such that for any $g,h\in X$*

$$\|\mathcal{C}_f^{\theta,\Re}(g,h)\|_X\le C(1+\|f'\|_{L^\infty}\|f'\|_X)(\|g\|_D\|h\|_X+\|h\|_D\|g\|_X),$$
$$\|\mathcal{C}_f^{\theta,\Im}(g,h)\|_X\le C\|f'\|_X(1+\|f'\|_{L^\infty}^2)(\|g\|_D\|h\|_X+\|g\|_X\|h\|_D).$$

Proof. We shall first establish the result for the real-part operator given by (5-2). First we note that one may rewrite the expression using the notation (2-6) as follows:

$$\mathcal{C}_f^{\theta,\Re}(g,h)(x)=\int_{-M}^{M}\frac{y\Delta_{\theta y}g(x)\Delta_y h(x)}{y^2+(\Delta_y f(x))^2}\,dy,$$

where we simply replace the notation Δ_y^- by Δ_y. Using the product laws (4-4) and (4-8) one obtains

$$\|\mathcal{C}_f^{\theta,\Re}(g,h)\|_X\le\int_{-M}^{M}\|\Delta_{\theta y}g\Delta_y h\|_X\frac{dy}{|y|}+\int_{-M}^{M}|y|\,\|\Delta_{\theta y}g\Delta_y h\|_{L^\infty}\left\|\frac{1}{y^2+(\Delta_y f)^2}\right\|_X dy.$$

Using once again the product law, it becomes

$$\begin{aligned}\|\Delta_{\theta y}g\Delta_y h\|_X&\le\|\Delta_{\theta y}g\|_{L^\infty}\|\Delta_y h\|_X+\|\Delta_{\theta y}g\|_X\|\Delta_y h\|_{L^\infty}\\&\le\omega_g(|y|)\|h\|_X+2\|g\|_X\,\omega_h(|y|),\end{aligned}$$

where we have used that for $\theta\in[0,1]$, $y\in\mathbb{R}$

$$\|\Delta_y h\|_X\le 2\|h\|_X,\quad \|\Delta_{\theta y}h\|_{L^\infty}\le\omega_h(|y|). \tag{5-3}$$

Consequently

$$\int_{-M}^{M}\|\Delta_{\theta y}g\Delta_y h\|_X\frac{dy}{|y|}\le C(\|g\|_D\|h\|_X+\|h\|_D\|g\|_X). \tag{5-4}$$

By the definition it is quite easy to check that for any function $\varphi\in X\cap L^\infty(\mathbb{R})$

$$\left\|\frac{1}{y^2+\varphi^2}\right\|_X\le\frac{2\|\varphi\|_{L^\infty}}{y^4}\|\varphi\|_X.$$

Hence we get

$$\begin{aligned}\left\| \frac{1}{y^2+(\Delta_y f)^2} \right\|_X &\le 2\frac{\|\Delta_y f\|_{L^\infty}}{y^4}\|\Delta_y f\|_X \\ &\le C y^{-2}\|f'\|_{L^\infty}\|f'\|_X, \end{aligned} \tag{5-5}$$

where we have used the inequalities

$$\|\Delta_y f\|_{L^\infty} \le |y|\,\|f'\|_{L^\infty} \quad \text{and} \quad \omega_{\Delta_y f}(r) \le |y|\omega_{f'}(r).$$

Therefore we get in view of (5-3),

$$\begin{aligned}\int_{-M}^{M} |y|\,\|\Delta_{\theta y} g \Delta_y h\|_{L^\infty} \left\| \frac{1}{y^2+(\Delta_y f)^2} \right\|_X dy &\le C\|f'\|_{L^\infty}\|f'\|_X\,\|h\|_{L^\infty} \int_{-M}^{M} \frac{\omega_g(|y|)}{|y|}\,dy \\ &\le C\|f'\|_{L^\infty}\|f'\|_X\,\|h\|_{L^\infty}\,\|g\|_D.\end{aligned}$$

Combining this last estimate with (5-4) we find that

$$\|\mathcal{C}_f^{\theta,\Re}(g,h)\|_X \le C(\|g\|_D\,\|h\|_X + \|h\|_D\,\|g\|_X + \|f'\|_{L^\infty}\|f'\|_X\,\|h\|_{L^\infty}\|g\|_D).$$

To deduce the result it is enough to use (4-5).

We are left with the task of estimating the imaginary part, which takes the form

$$\mathcal{C}_f^{\theta,\Im}(g,h)(x) = \int_{-M}^{M} \frac{\Delta_y f(x)\Delta_{\theta y} g(x)\Delta_y h(x)}{y^2+(\Delta_y f(x))^2}\,dy.$$

Note that we have dropped the minus sign before the integral, which of course has no consequence on the computations. Using Taylor's formula we get

$$\Delta_y f(x) = y\int_0^1 f'(x+\tau y)\,d\tau$$

and thus

$$\mathcal{C}_f^{\theta,\Im}(g,h)(x) = \int_{-M}^{M}\int_0^1 \frac{y f'(x+\tau y)\Delta_{\theta y} g(x)\Delta_y h(x)}{y^2+(\Delta_y f(x))^2}\,dy\,d\tau.$$

It suffices to reproduce the preceding computations using in particular the estimates

$$\|f'(\cdot+\tau y)\Delta_{\theta y} g \Delta_y h\|_{L^\infty} \le \|f'\|_{L^\infty}\|h\|_{L^\infty}\omega_g(|y|)$$

and

$$\begin{aligned}\|f'(\cdot+\tau y)\Delta_{\theta y} g \Delta_y h\|_X &\le \|f'\|_{L^\infty}\,\|\Delta_{\theta y} g \Delta_y h\|_X + \|f'\|_X\,\|\Delta_{\theta y} g \Delta_y h\|_{L^\infty} \\ &\le \|f'\|_{L^\infty}\big(\omega_g(|y|)\|h\|_X + \|g\|_X\,\omega_h(|y|)\big) + 2\|f'\|_X\,\|g\|_{L^\infty}\omega_h(|y|).\end{aligned}$$

This implies, according to the Sobolev embeddings (4-5),

$$\int_{-M}^{M}\int_0^1 \|f'(\cdot+\tau y)\Delta_{\theta y} g \Delta_y h\|_X \,\frac{dy}{|y|}\,d\tau \le C\|f'\|_X(\|g\|_D\,\|h\|_X + \|g\|_X\,\|h\|_D).$$

Using (5-5) one may easily get

$$\int_{-M}^{M} |y| \, \|f'(\cdot+\tau y)\Delta_{\theta y} g \Delta_y h\|_{L^\infty} \left\| \frac{1}{y^2+(\Delta_y f)^2} \right\|_X dy \le C \|f'\|_{L^\infty}^2 \|f'\|_X \|h\|_{L^\infty} \|g\|_D,$$

which gives the desired result using the Sobolev embeddings (4-5). □

The second kind of Cauchy integrals that we have to deal with, and that are related to the integral terms in (2-4) and (2-5), are given by the linear operators

$$T_f^{\alpha,\beta} g(x) = \text{p.v.} \int_{\mathbb{R}} \frac{y g(\alpha x+\beta y)}{y^2+[f(x)+f(x+y)]^2} \, dy,$$

with α and β being two given parameters. The continuity of these operators in classical Banach spaces is not in general easy to establish and could fail for some special cases. We point out that it is not our purpose in this exposition to implement a complete study of these operators. A more complete theory may be achieved but this topic exceeds the scope of this paper and we shall restrict ourselves to some special configurations that fit with the application to the aggregation equation. Our result in this direction reads as follows.

Theorem 5.2. *Let $\alpha,\beta \in [0,1]$, K be a compact set of $\mathbb{R}$ and $f:\mathbb{R}\to\mathbb{R}_+$ be a compactly supported continuous positive function such that $f' \in C_K^\star$. Then the following assertions hold true*:

(1) *The operator $T_f^{\alpha,\beta}: C_K^\star \to L^\infty(\mathbb{R})$ is well-defined and continuous and*

$$\|T_f^{\alpha,\beta} g\|_{L^\infty} \le C(1+\|f'\|_{L^\infty}^2 + \|f'\|_{L^\infty}\|f'\|_D)\|g\|_D,$$

with C a constant depending only on K and not on α and β.

(2) *The modified operator $f'T_f^{\alpha,\beta}: C_K^\star \to C_K^\star$ is continuous. More precisely*,

$$\|f'T_f^{\alpha,\beta} g\|_D \le C\|f'\|_D (C_\beta \ln_+(1/\|f'\|_D) + \|f'\|_D^{14})\|g\|_D,$$

with C a constant depending only on K and

$$C_\beta \triangleq \begin{cases} (1-\ln\beta), & \beta\in(0,1], \\ 1, & \beta=0. \end{cases}$$

(3) *Let $s\in(0,1)$ and assume that $f'\in C_K^s$; then $f'T_f^{\alpha,\beta}: C_K^s \to C_K^s(\mathbb{R})$ is well-defined and continuous. More precisely, there exists a constant C depending only on the compact K and s such that*

$$\|f'T_f^{\alpha,\beta} g\|_s \le C(C_\beta \|f'\|_{L^\infty}^{\frac{1}{1+s}} + \|f'\|_s^{14})\|g\|_s. \tag{5-6}$$

In addition, one has the refined estimate

$$\|f'T_f^{\alpha,\beta} g\|_s \le C\|f'\|_{L^\infty}^{\frac{1}{2+s}} (\|f'\|_s^{\frac{1}{2+s}} C_\beta + \|f'\|_s^{14})\|g\|_s + C\|g\|_{L^\infty}^{\frac{1}{2+s}} \|g\|_s^{\frac{1+s}{2+s}} \|f'\|_s,$$

with

$$C_\beta \triangleq \begin{cases} \beta^{-\frac12}, & \beta\in(0,1], \\ 1, & \beta=0. \end{cases}$$

Proof. To simplify the notation we shall throughout this proof write $T_f g$ instead of $T_f^{\alpha,\beta} g$.

(1) By symmetrizing we get

$$\begin{aligned} T_f g(x) &= \int_0^\infty \frac{y\,[g(\alpha x+\beta y)-g(\alpha x-\beta y)]}{y^2+[f(x)+f(x+y)]^2}\,dy \\ &\qquad + \lim_{\varepsilon\to 0}\int_\varepsilon^\infty \frac{y\,g(\alpha x-\beta y)[f(x-y)-f(x+y)]\,[\Delta_y^+ f(x)+\Delta_{-y}^+ f(x)]}{(y^2+[\Delta_y^+ f(x)]^2)(y^2+[\Delta_{-y}^+ f(x)]^2)}\,dy \\ &\triangleq T_f^1 g(x)+T_f^2 g(x). \end{aligned} \tag{5-7}$$

Without loss of generality we can assume that $K=[-1,1]$ and $\operatorname{supp} g\subset[-1,1]$ and deal only with $x\geq 0$. We shall distinguish two cases $0\leq \alpha x\leq 2$ and $\alpha x\geq 2$. In the first case, reasoning on the support of g we simply get

$$T_f^1 g(x)=\int_{\{0\leq\beta y\leq 3\}} \frac{y[g(\alpha x+\beta y)-g(\alpha x-\beta y)]}{y^2+[f(x)+f(x+y)]^2}\,dy.$$

Hence we obtain according to the definition of the modulus of continuity, a change of variables and (4-3)

$$|T_f^1 g(x)|\leq \int_{\{0\leq\beta y\leq 3\}} \frac{\omega_g(2\beta y)}{y}\,dy\leq C\|g\|_D. \tag{5-8}$$

Coming back to the case $\alpha x\geq 2$ one may write

$$\begin{aligned} |T_f^1 g(x)| &\leq \int_{\{\alpha x-1\leq\beta y\leq 1+\alpha x\}} \frac{\omega_g(2\beta y)}{y}\,dy \\ &\leq 2\|g\|_{L^\infty}\int_{\alpha x-1}^{1+\alpha x}\frac{1}{y}\,dy \\ &\leq \|g\|_{L^\infty}\ln\biggl(\frac{1+\gamma}{-1+\gamma}\biggr), \qquad \gamma=\alpha x\geq 2 \\ &\leq C\|g\|_{L^\infty}. \end{aligned}$$

Combining this last inequality with (5-8) we deduce that

$$\|T_f^1 g\|_{L^\infty}\leq C\|g\|_D. \tag{5-9}$$

For the second term $T_f^2 g$ we split it into two parts as follows:

$$\begin{aligned} T_f^2 g(x) &= \lim_{\varepsilon\to 0} 4f(x)\int_\varepsilon^\infty \frac{y\,g(\alpha x-\beta y)[f(x-y)-f(x+y)]}{(y^2+[f(x)+f(x+y)]^2)(y^2+[f(x)+f(x-y)]^2)}\,dy \\ &\qquad + \int_0^\infty \frac{y\,g(\alpha x-\beta y)[f(x-y)-f(x+y)]\psi(x,y)}{(y^2+[f(x)+f(x+y)]^2)(y^2+[f(x)+f(x-y)]^2)}\,dy \\ &\triangleq T_f^{2,1}g(x)+T_f^{2,2}g(x), \end{aligned} \tag{5-10}$$

with

$$\psi(x,y)=f(x+y)+f(x-y)-2f(x)=y\int_0^1 [f'(x+\theta y)-f'(x-\theta y)]\,d\theta.$$

The first term $T_f^{2,1} g$ is easily estimated. Indeed, one can assume that $f(x) > 0$; otherwise the integral vanishes. Thus using the mean value theorem and a change of variables we obtain

$$\begin{aligned}|T_f^{2,1} g(x)| &\le 8\|g\|_{L^\infty}\|f'\|_{L^\infty} f(x)\int_0^\infty \frac{y^2}{(y^2+[f(x)]^2)^2}\,dy\\ &\le 8\|g\|_{L^\infty}\|f'\|_{L^\infty}\int_0^\infty \frac{y^2}{(y^2+1)^2}\,dy\\ &\le C\|g\|_{L^\infty}\|f'\|_{L^\infty}.\end{aligned} \tag{5-11}$$

As for the term $T_f^{2,2}$, straightforward arguments yield

$$\begin{aligned}|T_f^{2,2} g(x)| &\le 8\|g\|_{L^\infty}\|f\|_{L^\infty}^2 \int_{y\ge\frac12}\frac{1}{y^3}\,dy + 2\|g\|_{L^\infty}\|f'\|_{L^\infty}\int_0^{\frac12}\frac{|\psi(x,y)|}{y^2}\,dy\\ &\le C\|g\|_{L^\infty}\biggl(\|f\|_{L^\infty}^2 + C\|f'\|_{L^\infty}\int_0^{\frac12}\frac{\omega_{f'}(2y)}{y}\,dy\biggr)\\ &\le C\|g\|_{L^\infty}(\|f'\|_{L^\infty}^2 + C\|f'\|_{L^\infty}\|f'\|_D),\end{aligned}$$

where we have used the fact

$$|\psi(x,y)| \le 2y\,\omega_{f'}(2y).$$

Consequently we obtain

$$\|T_f^2 g\|_{L^\infty} \le C\|g\|_{L^\infty}(\|f'\|_{L^\infty}^2 + \|f'\|_{L^\infty}\|f'\|_D + \|f'\|_{L^\infty}). \tag{5-12}$$

Putting together this estimate with (5-11) and (4-2) we obtain the desired estimate.

(2) First, recall from part (1) of this proof the decomposition

$$T_f g(x) = T_f^1 g(x) + T_f^{2,1} g(x) + T_f^{2,2} g(x). \tag{5-13}$$

The second term is easier to deal with and one has

$$\|T_f^{2,1} g\|_D \le C\|g\|_D\,\|f'\|_D(1 + \|f'\|_{L^\infty}^{13}). \tag{5-14}$$

This implies in view of the product laws (4-4) and (5-11) that

$$\|T_f^{2,1} g\|_D \le C\|g\|_D\,\|f'\|_D(\|f'\|_{L^\infty} + \|f'\|_{L^\infty}^{14}). \tag{5-15}$$

To establish (5-14) we first note that if $f(x) = 0$ then $T_f^{2,1} g(x) = 0$. However for $f(x) > 0$, using the mean value theorem and the change of variables $y \to f(x)y$ we get

$$T_f^{2,1} g(x) = -4\int_0^\infty \frac{y^2\, g(\alpha x - \beta f(x)y)\int_0^1 [f'(x+\theta f(x)y) + f'(x-\theta f(x)y)]\,d\theta}{\varphi(x,y)\varphi(x,-y)}\,dy, \tag{5-16}$$

with

$$\varphi(x,y) = y^2 + \biggl[2 + y\int_0^1 f'(x+\theta f(x)y)\,d\theta\biggr]^2.$$

Observe that the identity (5-16) is meaningful even for $f(x)=0$ and we can check easily that it vanishes. This follows from the fact that owing to the positivity of f when $f(x)=0$ we have $f'(x)=0$. To simplify the expressions we introduce the functions

$$\mathcal{N}_1(x,y)=g(\alpha x-\beta f(x)y)\int_0^1[f'(x+\theta f(x)y)+f'(x-\theta f(x)y)]\,d\theta,$$
$$\mathcal{D}_1(x,y)=\varphi(x,y)\varphi(x,-y).$$

Then by (4-4) we obtain for fixed y

$$\|\mathcal{N}_1(\cdot,y)\|_D\le 2\|g\circ(\alpha\mathrm{Id}-\beta yf)\|_D\,\|f'\|_{L^\infty}+\|g\|_{L^\infty}\int_0^1[\|f'\circ(\mathrm{Id}+\theta yf)\|_D+\|f'\circ(\mathrm{Id}-\theta yf)\|_D]\,d\theta.$$

Using the composition law (4-7) we get successively

$$\|g\circ(\alpha\mathrm{Id}-\beta yf)\|_D\le C\|g\|_D(1+\ln_+(\alpha+\beta\|f'\|_{L^\infty}y)),$$
$$\|f'\circ(\mathrm{Id}+\theta yf)\|_D\le C\|f'\|_D(1+\ln(1+\theta\|f'\|_{L^\infty}y)).$$

This implies

$$\|\mathcal{N}_1(\cdot,y)\|_D \le C\|g\|_D(1+\ln_+(\alpha+\beta\|f'\|_{L^\infty}y))\|f'\|_{L^\infty}+C\|g\|_{L^\infty}\|f'\|_D\int_0^1(1+\ln(1+\theta\|f'\|_{L^\infty}y))\,d\theta.$$

Since

$$\ln\Big(1+\prod_{i=1}^n x_i\Big)\le\sum_{i=1}^n\ln(1+x_i)\quad\text{for all } x_i\ge 0,$$

we have

$$\|\mathcal{N}_1(\cdot,y)\|_D\le C\|g\|_D\,\|f'\|_D\,(1+\ln_+\|f'\|_{L^\infty}+\ln_+y). \tag{5-17}$$

On the other hand it is clear that

$$\|\mathcal{N}_1(\cdot,y)\|_{L^\infty}\le C\|g\|_{L^\infty}\|f'\|_{L^\infty}. \tag{5-18}$$

To estimate $1/\mathcal{D}_1(\cdot,y)$ in the Dini space $C_K^\star$ we come back to the definition, which implies

$$\|1/\mathcal{D}_1(\cdot,y)\|_D\le\|\mathcal{D}_1(\cdot,y)\|_D\,\|1/\mathcal{D}_1(\cdot,y)\|_{L^\infty}^2. \tag{5-19}$$

Now using the product law (4-4) we deduce that

$$\|\mathcal{D}_1(\cdot,y)\|_D\le\|\varphi(\cdot,y)\|_{L^\infty}\,\|\varphi(\cdot,-y)\|_D+\|\varphi(\cdot,y)\|_D\,\|\varphi(\cdot,-y)\|_{L^\infty}.$$

From simple calculations we get

$$\|\varphi(\cdot,\pm y)\|_{L^\infty}\le y^2+(2+y\|f'\|_{L^\infty})^2\le C(1+\|f'\|_{L^\infty}^2)(1+y^2).$$

Applying (4-4) and (4-7) to the expression of φ it is quite easy to check that

$$\begin{aligned}\|\varphi(\cdot,\pm y)\|_D&\le C(1+y\|f'\|_{L^\infty})y\int_0^1\|f'\circ(\mathrm{Id}\pm\theta yf)\|_D\,d\theta\\&\le C(y+y^2\|f'\|_{L^\infty})\,\|f'\|_D\,(1+\ln_+\|f'\|_{L^\infty}+\ln_+y).\end{aligned}$$

Thus combining the preceding estimates we find

$$\begin{aligned}\|\mathcal{D}_1(\cdot\,,y))\|_D &\le C(y+y^2\|f'\|_{L^\infty})\,\|f'\|_D\,(1+\ln_+\|f'\|_{L^\infty}+\ln_+ y)(1+\|f'\|_{L^\infty}^2)(1+y^2)\\ &\le C(1+y^4\ln_+ y)\,\|f'\|_D\,(1+\ln_+\|f'\|_{L^\infty})(1+\|f'\|_{L^\infty}^3).\end{aligned} \tag{5-20}$$

Now we shall use the following inequalities, which can be proved in a straightforward way: for any $y\in\mathbb{R}_+$ and for any $a,b\in\mathbb{R}$ with $|a|\le b$, one has

$$y^2+(2+ya)^2\ge y^2+(2-ya)^2\ge \frac{1+y^2}{1+a^2}\ge\frac{1+y^2}{1+b^2}. \tag{5-21}$$

It follows that

$$\|1/\varphi(\cdot\,,\pm y)\|_{L^\infty}\le\frac{1+\|f'\|_{L^\infty}^2}{1+y^2}. \tag{5-22}$$

Putting this estimate together with (5-20) and (5-19) yields

$$\begin{aligned}\|1/\mathcal{D}_1(\cdot\,,y)\|_D &\le C\,\frac{1+y^4\ln_+ y}{1+y^8}\|f'\|_D(1+\ln_+\|f'\|_{L^\infty})(1+\|f'\|_{L^\infty}^{11})\\ &\le C\,\frac{1+\ln_+ y}{1+y^4}\|f'\|_D(1+\|f'\|_{L^\infty}^{12}).\end{aligned}$$

Therefore we obtain using (5-17), (5-18) and (5-22)

$$\begin{aligned}\|(\mathcal{N}_1/\mathcal{D}_1)(\cdot\,,y)\|_D &\le \|(\mathcal{N}_1(\cdot\,,y)\|_{L^\infty}\,\|1/\mathcal{D}_1)(\cdot\,,y)\|_D+\|(\mathcal{N}_1(\cdot\,,y)\|_D\,\|1/\mathcal{D}_1)(\cdot\,,y)\|_{L^\infty}\\ &\le C\,\|g\|_{L^\infty}\,\|f'\|_{L^\infty}\frac{1+\ln_+ y}{1+y^4}\|f'\|_D(1+\|f'\|_{L^\infty}^{12})\\ &\qquad\qquad + C\,\|g\|_D\|f'\|_D\,\frac{1+\ln_+\|f'\|_{L^\infty}+\ln_+ y}{1+y^4}(1+\|f'\|_{L^\infty}^4)\\ &\le C\,\|g\|_D\,\|f'\|_D\frac{1+\ln_+ y}{1+y^4}(1+\|f'\|_{L^\infty}^{13}).\end{aligned}$$

Plugging this estimate into (5-16) we find

$$\|T_f^{2,1}g\|_D\le 4\int_0^\infty y^2\|(\mathcal{N}_1/\mathcal{D}_1)(\cdot\,,y)\|_D\,dy\le C\,\|g\|_D\,\|f'\|_D(1+\|f'\|_{L^\infty}^{13}). \tag{5-23}$$

This concludes the proof of (5-14).

Now we intend to estimate $\|T_f^1 g\|_D$, which is trickier. Let $r\in(0,1)$ and $x_1,x_2\in\mathbb{R}$ such that $|x_1-x_2|\le r$. We shall decompose $T_f^1 g$ as follows:

$$T_f^1 g=T_{f,\mathrm{int}}^{r,1}g+T_{f,\mathrm{ext}}^{r,1}g, \tag{5-24}$$

with

$$\begin{aligned}T_{f,\mathrm{int}}^{r,1}g(x)&=\int_0^r\frac{y[g(\alpha x+\beta y)-g(\alpha x-\beta y)]}{y^2+[f(x)+f(x+y)]^2}\,dy,\\ T_{f,\mathrm{ext}}^{r,1}g(x)&=\int_r^\infty\frac{y[g(\alpha x+\beta y)-g(\alpha x-\beta y)]}{y^2+[f(x)+f(x+y)]^2}dy.\end{aligned}$$

From the subadditivity of the modulus of continuity we get

$$|f'(x)T^{r,1}_{f,\mathrm{int}}\,g(x)| \le C|f'(x)|\int_0^r \frac{y\omega_g(y)}{y^2+[f(x)]^2}\,dy \le C|f'(x)|\int_0^r \frac{\omega_g(y)}{y+f(x)}\,dy.$$

Using Lemma 4.1 we find

$$|f'(x)T^{r,1}_{f,\mathrm{int}}\,g(x)| \le C\frac{\gamma(f)}{1+\ln_+(1/f(x))}\int_0^r \frac{\omega_g(y)}{y+f(x)}\,dy, \tag{5-25}$$

where

$$\gamma(f) \triangleq \|f'\|_D(1+\ln_+(1/\|f'\|_D)). \tag{5-26}$$

Now we claim that for $y\in(0,1)$

$$\sup_{\varepsilon>0}\frac{1}{1+\ln_+(1/\varepsilon)}\frac{1}{y+\varepsilon} \le \frac{C}{y(1+|\ln y|)}+\frac{1}{1+y} \tag{5-27}$$

for some universal constant $C>0$. To prove this result it is enough to get

$$\sup_{\varepsilon\in(0,1)}\frac{1}{1+\ln(1/\varepsilon)}\frac{1}{y+\varepsilon} \le \frac{C}{y(1+|\ln y|)}.$$

Indeed, we shall consider the two cases $\varepsilon\ge\sqrt{y}$ and $\varepsilon\le\sqrt{y}$. In the first case we observe

$$\frac{1}{y+\varepsilon}\le\frac{1}{\sqrt{y}} \quad\text{and}\quad \frac{1}{1+\ln(1/\varepsilon)}\le 1,$$

which implies

$$\frac{1}{1+\ln(1/\varepsilon)}\frac{1}{y+\varepsilon}\le\frac{1}{\sqrt{y}}\le\frac{C}{y(1+|\ln y|)}.$$

However in the second case $\varepsilon\le\sqrt{y}$ we write simply that

$$\frac{1}{y+\varepsilon}\le\frac{1}{y} \quad\text{and}\quad \frac{1}{1+\ln(1/\varepsilon)}\le\frac{1}{1+\frac{1}{2}\ln(1/y)},$$

which gives the desired result. Coming back to (5-25) and using (5-27) we deduce that

$$\begin{aligned}\sup_x|f'(x)T^{r,1}_{f,\mathrm{int}}\,g(x)| &\le C\gamma(f)\int_0^r \sup_x\frac{\omega_g(y)}{(1+\ln_+(1/f(x)))(y+f(x))}\,dy\\ &\le C\gamma(f)\bigg(\int_0^r\frac{\omega_g(y)}{y(1+|\ln y|)}\,dy+\int_0^r\frac{\omega_g(y)}{1+y}\,dy\bigg).\end{aligned} \tag{5-28}$$

Consequently

$$\sup_{|x_1-x_2|\le r}\big|f'(x_1)T^{r,1}_{f,\mathrm{int}}\,g(x_1)-f'(x_2)T^{r,1}_{f,\mathrm{int}}\,g(x_2)\big| \le C\gamma(f)\bigg(\int_0^r\frac{\omega_g(y)}{y(1+|\ln y|)}\,dy+\int_0^r\omega_g(y)\,dy\bigg).$$

Therefore we get by using Fubini's theorem

$$\begin{aligned}\int_0^1 \sup_{|x_1-x_2|\le r} |f'(x_1)T^{r,1}_{f,\mathrm{int}}\, g(x_1)-f'(x_2)T^{r,1}_{f,\mathrm{int}}\, g(x_2)|\,\frac{dr}{r}\\ &\le C\gamma(f)\int_0^1 \frac{\omega_g(y)}{y}\frac{|\ln y|}{(1+|\ln y|)}\,dy + C\gamma(f)\int_0^1 |\ln y|\omega_g(y)\,dy\\ &\le C\gamma(f)\|g\|_D.\end{aligned}$$

As for $T^{r,1}_{f,\mathrm{ext}} g$, we write

$$\begin{aligned}f'(x_1)T^{r,1}_{f,\mathrm{ext}} g(x_1) - f'(x_2)T^{r,1}_{f,\mathrm{ext}} g(x_2)\\ &= (f'(x_1)-f'(x_2))T^{r,1}_{f,\mathrm{ext}} g(x_2) + f'(x_1)(T^{r,1}_{f,\mathrm{ext}} g(x_1) - T^{r,1}_{f,\mathrm{ext}} g(x_2))\\ &\triangleq \mu_1(x_1,x_2)+\mu_2(x_1,x_2). \end{aligned} \tag{5-29}$$

Our current goal is to prove that for $j\in\{1,2\}$

$$\int_0^1 \sup_{|x_1-x_2|\le r} \frac{\mu_j(x_1,x_2)}{r}\,dr$$

is well-estimated. For the first term we use (5-9) leading to

$$\int_0^1 \sup_{|x_1-x_2|\le r} \frac{\mu_1(x_1,x_2)}{r}\,dr \le \|T^{r,1}_{f,\mathrm{ext}} g\|_{L^\infty}\int_0^1 \frac{\omega_{f'}(r)}{r}\,dr \le C\|g\|_D\|f'\|_D.$$

The second term is subtler. First note that if $|x_1-x_2|\le 1$ then the quantity

$$f'(x_1)T^{r,1}_{f,\mathrm{ext}} g(x_1) - f'(x_2)T^{r,1}_{f,\mathrm{ext}} g(x_2)$$

vanishes for x_1, x_2 outside a compact set related only to the support of f. Therefore the integrals defining $\mu_2(x_1,x_2)$ may be restricted to the set $\{\beta r\le \beta y\le B\}$, with B being some constant related to the size of the supports of f and g, and without loss of generality we can take $B=1$. It follows that

$$\begin{aligned}\mu_2(x_1,x_2) &= f'(x_1)\int_{\{\beta r\le\beta y\le 1\}} \frac{y\,[\hat g(x_1,y)-\hat g(x_2,y)]}{y^2+[f(x_1)+f(x_1+y)]^2}\,dy\\ &\quad + f'(x_1)\int_{\{\beta r\le\beta y\le 1\}} \frac{y\,\hat g(x_2,y)[\Delta_y^+ f(x_2)-\Delta_y^+ f(x_1)]\,[\Delta_y^+ f(x_2)+\Delta_y^+ f(x_1]}{(y^2+[\Delta_y^+ f(x_1)]^2)(y^2+[\Delta_y^+ f(x_2)]^2)}\,dy\\ &\triangleq \mu_{2,1}(x_1,x_2)+\mu_{2,2}(x_1,x_2),\end{aligned} \tag{5-30}$$

with

$$\hat g(x,y)\triangleq g(\alpha x+\beta y)-g(\alpha x-\beta y) \quad\text{and}\quad \Delta_y^+ f(x) = f(x+y)+f(x).$$

To estimate $\mu_{2,1}$ we shall use the following inequality, which is a consequence of Lemma 4.1:

$$\int_0^L \frac{|f'(x)|}{y+f(x)}\,dy = |f'(x)|\ln\Big(1+\frac{L}{f(x)}\Big)\le C\gamma(f)(1+\ln_+ L),$$

with C an absolute constant. This implies

$$\begin{aligned}\mu_{2,1}(x_1,x_2) &\le C\omega_g(\alpha|x_1-x_2|)|f'(x_1)|\int_0^{\frac{1}{\beta}}\frac{1}{y+f(x_1)}\,dy\\ &\le C\omega_g(|x_1-x_2|)\gamma(f)(1+|\ln\beta|).\end{aligned}$$

Consequently, we find that

$$\sup_{|x_1-x_2|\le r}|\mu_{2,1}(x_1,x_2)|\le C\omega_g(r)\gamma(f)(1+|\ln\beta|)$$

and therefore

$$\int_0^1\sup_{|x_1-x_2|\le r}|\mu_{2,1}(x_1,x_2)|\,\frac{dr}{r}\le C\gamma(f)(1+|\ln\beta|)\|g\|_D.$$

We emphasize that for $\beta=0$ one can still get an estimate since $\mu_{2,1}(x_1,x_2)=0$ and therefore we get the desired estimate.

Now we shall move to the estimate of $\mu_{2,2}(x_1,x_2)$. We start with using the estimate

$$\sup_{a>0}\frac{a}{y^2+a^2}\le\frac{1}{2|y|},$$

which implies

$$\frac{y|\hat{g}(x_2,y)||\Delta_y^+f(x_2)-\Delta_y^+f(x_1)||\Delta_y^+f(x_2)+\Delta_y^+f(x_1|}{(y^2+[\Delta_y^+f(x_1)]^2)(y^2+[\Delta_y^+f(x_2)]^2)}\le C|x_2-x_1|\|f'\|_{L^\infty}\frac{\omega_g(2\beta y)}{y^2}.$$

Thus

$$\sup_{|x_1-x_2|\le r}\mu_{2,2}(x_1,x_2)\le Cr\|f'\|_{L^\infty}^2\int_r^{\frac{1}{\beta}}\frac{\omega_g(2\beta y)}{y^2}\,dy,$$

which yields in view of Fubini's theorem

$$\begin{aligned}\int_0^1\sup_{|x_1-x_2|\le r}\mu_{2,2}(x_1,x_2)\,\frac{dr}{r} &\le C\|f'\|_{L^\infty}^2\int_0^1\int_{\{\beta r\le\beta y\le 1\}}\frac{\omega_g(2\beta y)}{y^2}\,dy\,dr\\ &\le C\|f'\|_{L^\infty}^2\int_{\{0\le\beta y\le 1\}}\frac{\omega_g(2\beta y)}{y}\,dy\\ &\le C\|f'\|_{L^\infty}^2\int_0^2\frac{\omega_g(y)}{y}\,dy\\ &\le C\|f'\|_{L^\infty}^2\|g\|_D.\end{aligned}$$

Note that the last constant does not depend on β. Putting together the preceding estimates we find that

$$\|f'T_f^1g\|_D\le C\|g\|_D((1+|\ln\beta|)\gamma(f)+\|f'\|_{L^\infty}^2),\tag{5-31}$$

where $\gamma(f)$ was defined in (5-26). As noted before, the case $\beta=0$ has a special structure and one gets

$$\|f'T_f^1g\|_D\le C\|g\|_D(\gamma(f)+\|f'\|_{L^\infty}^2).$$

Now let us move to the estimate of $f'(x)T_f^{2,2}g$ given by

$$T_f^{2,2}g(x)=\int_0^\infty \frac{y\,g(\alpha x-\beta y)[f(x-y)-f(x+y)]\psi(x,y)}{(y^2+[f(x)+f(x+y)]^2)(y^2+[f(x)+f(x-y)]^2)}\,dy$$
$$=T_{f,\text{int}}^{r,2,2}g(x)+T_{f,\text{ext}}^{r,2,2}g(x), \tag{5-32}$$

where

$$\psi(x,y)=y\int_0^1[f'(x+\theta y)-f'(x-\theta y)]\,d\theta$$

and the cut-off operators are given by

$$T_{f,\text{int}}^{r,2,2}g(x)\triangleq\int_0^r \frac{y\,g(\alpha x-\beta y)[f(x-y)-f(x+y)]\psi(x,y)}{(y^2+[\Delta_y^+ f(x)]^2)(y^2+[\Delta_{-y}^+ f(x)]^2)}\,dy$$

and

$$T_{f,\text{ext}}^{r,2,2}g(x)=\int_r^1 \frac{y\,g(\alpha x-\beta y)[f(x-y)-f(x+y)]\psi(x,y)}{(y^2+[\Delta_y^+ f(x)]^2)(y^2+[\Delta_{-y}^+ f(x)]^2)}\,dy\triangleq\int_r^1\frac{\mathcal{N}(x,y)}{\mathcal{D}(x,y)}\,dy. \tag{5-33}$$

We shall proceed in a similar way to $T_f^1 g$. Let us start with $f'(x)T_{f,\text{int}}^{r,2,2}g$. Since

$$|\psi(x,y)|\le 2y\omega_{f'}(y), \tag{5-34}$$

one has

$$|f'(x)T_{f,\text{int}}^{r,2,2}g(x)|\le C\|g\|_{L^\infty}\|f'\|_{L^\infty}|f'(x)|\int_0^r\frac{y^3\omega_{f'}(y)}{(y^2+[f(x)]^2)^2}\,dy$$
$$\le C\|g\|_{L^\infty}\|f'\|_{L^\infty}|f'(x)|\int_0^r\frac{\omega_{f'}(y)}{y+f(x)}\,dy.$$

Thus following the same steps as for (5-28) we obtain

$$\sup_{|x_1-x_2|\le r}|f'(x_1)T_{f,\text{int}}^{r,2,2}\,g(x_1)-f'(x_2)T_{f,\text{int}}^{r,2,2}\,g(x_2)|$$
$$\le C\|g\|_{L^\infty}\|f'\|_{L^\infty}\gamma(f)\int_0^r\frac{\omega_{f'}(y)}{y(1+|\ln y|)}\,dy+C\|g\|_{L^\infty}\|f'\|_{L^\infty}\gamma(f)\int_0^r|\omega_{f'}(y)\,dy.$$

Therefore Fubini's theorem and (4-2) imply

$$\int_0^1\sup_{|x_1-x_2|\le r}|f'(x_1)T_{f,\text{int}}^{r,2,2}\,g(x_1)-f'(x_2)T_{f,\text{int}}^{r,2,2}\,g(x_2)|\,\frac{dr}{r}\le C\|g\|_{L^\infty}\|f'\|_D^2\gamma(f).$$

What is left is to estimate the quantity $f'(x)T_{f,\text{ext}}^{r,2,2}g$. First, it is obvious that

$$f'(x_1)T_{f,\text{ext}}^{r,2,2}g(x_1)-f'(x_2)T_{f,\text{ext}}^{r,2,2}g(x_2)$$
$$=(f'(x_1)-f'(x_2))T_{f,\text{ext}}^{r,2,2}(x_2)+f'(x_1)(T_{f,\text{ext}}^{r,2,2}(x_1)-T_{f,\text{ext}}^{r,2,2}(x_2)). \tag{5-35}$$

The first term of the right-hand side is easy to estimate. Indeed,

$$|(f'(x_1)-f'(x_2))T_{f,\text{ext}}^{r,2,2}(x_2)|\le\omega_{f'}(|x_1-x_2|)\|T_{f,\text{ext}}^{r,2,2}\|_{L^\infty}.$$

It is clear that

$$|T_{f,\mathrm{ext}}^{r,2,2}g(x)| \le C\|g\|_{L^\infty}\|f'\|_{L^\infty}\int_r^1 \frac{\omega_{f'}(y)}{y}\,dy \le C\|g\|_{L^\infty}\|f'\|_D^2.$$

Hence

$$\int_0^1 \sup_{|x_1-x_2|\le r} |(f'(x_1)-f'(x_2))T_{f,\mathrm{ext}}^{r,2,2}(x_2)|\,\frac{dr}{r} \le C\|g\|_{L^\infty}\|f'\|_D^3.$$

To deal with the second term we proceed as for the term $\mu_2(x_1,x_2)$ in (5-30). From (5-33) one has

$$\begin{aligned} &f'(x_1)(T_{f,\mathrm{ext}}^{r,2,2}(x_1)-T_{f,\mathrm{ext}}^{r,2,2}(x_2)) \\ &\quad= f'(x_1)\int_r^1 \frac{\mathcal{N}(x_1,y)-\mathcal{N}(x_2,y)}{\mathcal{D}(x_1,y)}\,dy + f'(x_1)\int_r^1 \frac{\mathcal{N}(x_2,y)(\mathcal{D}(x_2,y)-\mathcal{D}(x_1,y))}{\mathcal{D}(x_1,y)\mathcal{D}(x_2,y)}\,dy. \end{aligned} \tag{5-36}$$

It is quite obvious from some straightforward computations using in particular (5-34) that for $|x_1-x_2|\le r$

$$|\mathcal{N}(x_1,y)-\mathcal{N}(x_2,y)| \le C\|f'\|_{L^\infty} y^2(\omega_g(\alpha r)\omega_{f'}(y)y + \|g\|_{L^\infty}\omega_{f'}(r)y + \|g\|_{L^\infty} r\,\omega_{f'}(y)).$$

Since

$$\frac{1}{\mathcal{D}(x,y)} \le \frac{C}{[y+f(x)]^4} \le \frac{C}{y^4},$$

we get

$$\frac{|\mathcal{N}(x_1,y)-\mathcal{N}(x_2,y)|}{\mathcal{D}(x_1,y)} \le C\|f'\|_{L^\infty}\left[\omega_g(\alpha r)\frac{\omega_{f'}(y)}{y} + \|g\|_{L^\infty}\frac{\omega_{f'}(r)}{y+f(x_1)} + \|g\|_{L^\infty} r\,\frac{\omega_{f'}(y)}{y^2}\right].$$

This gives, in view of (4-2),

$$\begin{aligned} &|f'(x_1)|\int_r^1 \frac{|\mathcal{N}(x_1,y)-\mathcal{N}(x_2,y)|}{\mathcal{D}(x_1,y)}\,dy \\ &\le C\|f'\|_D\left[\|f'\|_D^2\omega_g(\alpha r) + \|g\|_D\omega_{f'}(r)\int_0^1 \frac{|f'(x_1)|}{y+f(x_1)}\,dy\right] + \|g\|_D\|f'\|_D^2\, r\int_r^1 \frac{\omega_{f'}(y)}{y^2}\,dy, \end{aligned} \tag{5-37}$$

which implies according to (5-31)

$$\int_0^1 \sup_{|x_1-x_2|\le r} |f'(x_1)|\int_r^1 \frac{|\mathcal{N}(x_1,y)-\mathcal{N}(x_2,y)|}{\mathcal{D}(x_1,y)}\,dy\,\frac{dr}{r} \le C(\|f'\|_D^3 + \|f'\|_D^2\gamma(f))\|g\|_D.$$

Now straightforward computations show that

$$\frac{|\mathcal{N}(x_2,y)(\mathcal{D}(x_2,y)-\mathcal{D}(x_1,y))|}{\mathcal{D}(x_1,y)\mathcal{D}(x_2,y)} \le C\|g\|_{L^\infty}\|f'\|_{L^\infty}^2|x_1-x_2|\frac{\omega_{f'}(y)}{y^2}. \tag{5-38}$$

Therefore using Fubini's theorem we get

$$\int_0^1 \sup_{|x_1-x_2|\le r} |f'(x_1)|\int_r^1 \frac{|\mathcal{N}(x_2,y)(\mathcal{D}(x_2,y)-\mathcal{D}(x_1,y))|}{\mathcal{D}(x_1,y)\mathcal{D}(x_2,y)}\,dy\,\frac{dr}{r} \le C\|f'\|_D^4\|g\|_D.$$

Putting together the preceding estimates we find that

$$\begin{aligned}\|f'T_f^{2,2}g\|_D &\le C\|g\|_D(\|f'\|_D^2+\|f'\|_D^2\gamma(f)+\|f'\|_D^3)\\ &\le C\|g\|_D(\|f'\|_D^2+\|f'\|_D^4),\end{aligned}\tag{5-39}$$

with C a constant depending only on the diameter of the compact K. To get the desired estimate it suffices to put together (5-15), (5-31) and (5-39).

(3) We shall proceed as in the proof of part (2) of Theorem 5.2. We use exactly the same splitting with similar estimates and to avoid redundancy we shall only give the basic estimates with some details for the terms that require new treatment. We use the decomposition described in (5-13). To estimate $T_f^{2,1}g$ in C^s we use the expression (5-16). Then following the same lines using in particular the product law (4-8) and the composition law (4-6), one has

$$\|\mathcal{N}_1(\cdot,y)\|_s\le C\|g\|_s(\alpha^s+\beta^s\|f'\|_{L^\infty}^s y^s)\|f'\|_{L^\infty}+C\|g\|_{L^\infty}\|f'\|_s\int_0^1(1+\theta^s\|f'\|_{L^\infty}^s y^s)\,d\theta.$$

Since $\alpha,\beta\in[0,1]$ we deduce

$$\|\mathcal{N}_1(\cdot,y)\|_s\le C(\|g\|_s\|f'\|_{L^\infty}+\|g\|_{L^\infty}\|f'\|_s)(1+\|f'\|_{L^\infty}^s y^s).$$

Similarly we get

$$\begin{aligned}\|\varphi(\cdot,\pm y)\|_s &\le C(1+y\|f'\|_{L^\infty})y\int_0^1\|f'\circ(\mathrm{Id}\pm\theta yf)\|_s\,d\theta\\ &\le C(y+y^2\|f'\|_{L^\infty})\|f'\|_s(1+\|f'\|_{L^\infty}^s y^s).\end{aligned}$$

This implies

$$\|\mathcal{D}_1(\cdot,y)\|_s\le C(1+y^{4+s})(1+\|f'\|_{L^\infty}^{3+s})\|f'\|_s$$

and

$$\|1/\mathcal{D}_1(\cdot,y)\|_s\le\frac{C}{1+y^{4-s}}(1+\|f'\|_{L^\infty}^{11+s})\|f'\|_s.$$

Consequently for $s\in(0,1)$

$$\begin{aligned}\|(\mathcal{N}_1/\mathcal{D}_1)(\cdot,y)\|_s &\le \|(\mathcal{N}_1(\cdot,y)\|_{L^\infty}\|1/\mathcal{D}_1)(\cdot,y)\|_s+\|\mathcal{N}_1(\cdot,y)\|_s\|1/\mathcal{D}_1(\cdot,y)\|_{L^\infty}\\ &\le\frac{C}{1+y^{4-s}}(1+\|f'\|_{L^\infty}^{11+s})\|f'\|_s\|g\|_s.\end{aligned}$$

Therefore we get similarly to (5-23)

$$\begin{aligned}\|T_f^{2,1}g\|_s &\le C(1+\|f'\|_{L^\infty}^{11+s})\|f'\|_s\|g\|_s\int_0^\infty\frac{y^2}{1+y^{4-s}}\,ds\\ &\le C(1+\|f'\|_{L^\infty}^{11+s})\|f'\|_s\|g\|_s.\end{aligned}$$

Combining product laws with Sobolev embeddings and (5-11) we get

$$\begin{aligned}\|f'T_f^{2,1}g\|_s &\le \|f'\|_{L^\infty}\|T_f^{2,1}g\|_s+\|f'\|_s\|T_f^{2,1}g\|_{L^\infty}\\ &\le C(1+\|f'\|_{L^\infty}^{11+s})\|f'\|_s\|f'\|_{L^\infty}\|g\|_s+\|g\|_{L^\infty}\|f'\|_D\|f'\|_s\\ &\le C(1+\|f'\|_{L^\infty}^{11+s})\|f'\|_s\|f'\|_D\|g\|_s.\end{aligned}$$

Using once again Sobolev embeddings we get

$$\|f' T_f^{2,1} g\|_s \le C(\|f'\|_s + \|f'\|_s^{13})\|f'\|_D \|g\|_s. \tag{5-40}$$

Now to estimate $T_f^1 g$ we come back to the decomposition (5-24) and we easily get

$$\|T_{f,\text{int}}^{r,1} g\|_{L^\infty} \le C\|g\|_s \int_0^r y^{-1+s}\,dy \le \|g\|_s r^s.$$

Hence we obtain, since $r = |x_1 - x_2|$,

$$|T_{f,\text{int}}^{r,1} g(x_1) - T_{f,\text{int}}^{r,1} g(x_2)| \le C\|g\|_s |x_1 - x_2|^s.$$

and we also get

$$|f'(x_1) T_{f,\text{int}}^{r,1} g(x_1) - f'(x_2) T_{f,\text{int}}^{r,1} g(x_2)| \le C\|f'\|_{L^\infty} \|g\|_s |x_1 - x_2|^s.$$

To estimate the term $f' T_{f,\text{ext}}^{r,1} g$ we come back to (5-29) and (5-30) and following the same estimates one gets

$$\begin{aligned} |\mu_1(x_1, x_2)| &\le |x_1 - x_2|^s \|f'\|_s \|T_{f,\text{ext}}^{r,1} g\|_{L^\infty} \\ &\le C|x_1 - x_2|^s \|f'\|_s \|g\|_D. \end{aligned}$$

Moreover

$$|\mu_2(x_1, x_2)| \le |\mu_{2,1}(x_1, x_2)| + |\mu_{2,2}(x_1, x_2)|$$

and

$$\begin{aligned} |\mu_{2,2}(x_1, x_2)| &\le C|x_2 - x_1| \|f'\|_{L^\infty}^2 \|g\|_s \int_{\{\beta r \le \beta y \le 1\}} (\beta y)^s y^{-2}\,dy \\ &\le C\|f'\|_{L^\infty}^2 \|g\|_s |x_1 - x_2|^s. \end{aligned}$$

To deal with the term $\mu_{2,1}(x_1, x_2)$ in (5-30) one obtains in view of (5-31)

$$\begin{aligned} |\mu_{2,1}(x_1, x_2)| &\le |x_1 - x_2|^s \|g\|_s |f'(x_1)| \int_{\{\beta r \le \beta y \le 1\}} \frac{y}{y^2 + f^2(x_1)}\,dy \\ &\le |x_1 - x_2|^s \|g\|_s |f'(x_1)| \int_0^{\frac{1}{\beta}} \frac{1}{y + f(x_1)}\,dy. \end{aligned}$$

Using the second part of Lemma 4.1 one finds for $s' \in (0, s]$

$$|f'(x_1)| \int_0^{\frac{1}{\beta}} \frac{1}{y + f(x_1)}\,dy \le C\|f'\|_{s'}^{\frac{1}{1+s'}} |f(x_1)|^{\frac{s'}{1+s'}} \int_0^{\frac{1}{\beta}} \frac{1}{y + f(x_1)}\,dy.$$

Combining this inequality with

$$\sup_{a>0} \frac{a^{\frac{s'}{1+s'}}}{y + a} \le C y^{-\frac{1}{1+s'}}$$

we get

$$\sup_{x_1 \in \mathbb{R}} |f'(x_1)| \int_0^{\frac{1}{\beta}} \frac{1}{y + f(x_1)}\,dy \le C\|f'\|_{s'}^{\frac{1}{1+s'}} \beta^{-\frac{s'}{1+s'}} \tag{5-41}$$

and therefore

$$|\mu_{2,1}(x_1,x_2)| \le |x_1-x_2|^s \, \|g\|_s \, \|f'\|_{s'}^{\frac{1}{1+s'}} \beta^{-\frac{s'}{1+s'}}.$$

Hence

$$\begin{aligned} &|f'(x_1)T^{r,1}_{f,\text{ext}} g(x_1) - f'(x_2)T^{r,1}_{f,\text{ext}} g(x_2)| \\ &\qquad \le C\|g\|_D \|f'\|_s |x_1-x_2|^s + C\|f'\|^2_{L^\infty} \|g\|_s |x_1-x_2|^s + C|x_1-x_2|^s \|g\|_s \|f'\|_{s'}^{\frac{1}{1+s'}} \beta^{-\frac{s'}{1+s'}}. \end{aligned}$$

It follows that

$$\|f'T^1_f g\|_s \le C\|g\|_s (\|f'\|_{s'}^{\frac{1}{1+s'}} \beta^{-\frac{s'}{1+s'}} + \|f'\|^2_{L^\infty}) + C\|g\|_D \|f'\|_s. \tag{5-42}$$

It remains to estimate $f'T^{2,2}_f g$ described in (5-32) and (5-33). First one may write

$$\begin{aligned} |T^{r,2,2}_{f,\text{int}} g(x)| &\le C\|g\|_{L^\infty} \|f'\|_{L^\infty} \|f'\|_s \int_0^r y^{s-1}\,dy \\ &\le C\|g\|_{L^\infty} \|f'\|_{L^\infty} \|f'\|_s |x_1-x_2|^s. \end{aligned}$$

Therefore

$$|f'(x_1)T^{r,2,2}_{f,\text{int}} g(x_1) - f'(x_2)T^{r,2,2}_{f,\text{int}} g(x_2)| \le C\|g\|_{L^\infty} \|f'\|^2_{L^\infty} \|f'\|_s |x_1-x_2|^s.$$

By Sobolev embeddings we get

$$|f'(x_1)T^{r,2,2}_{f,\text{int}} g(x_1) - f'(x_2)T^{r,2,2}_{f,\text{int}} g(x_2)| \le C\|g\|_s \|f'\|_{L^\infty} \|f'\|^2_s |x_1-x_2|^s. \tag{5-43}$$

From (5-35) and the analysis following that identity one has

$$\begin{aligned} |(f'(x_1)-f'(x_2))T^{r,2,2}_{f,\text{ext}}(x_2)| &\le \|f'\|_s \|T^{r,2,2}_{f,\text{ext}} g\|_{L^\infty} |x_1-x_2|^s \\ &\le C\|g\|_{L^\infty} \|f'\|^2_s \|f'\|_{L^\infty} |x_1-x_2|^s \end{aligned}$$

Using (5-36), (5-37) and (5-41) (with $s'=s$) combined with Sobolev embeddings, one deduces

$$|f'(x_1)| \int_r^1 \frac{|\mathcal{N}(x_1,y)-\mathcal{N}(x_2,y)|}{\mathcal{D}(x_1,y)}\,dy \le C\|f'\|_{L^\infty} \|g\|_s (\|f'\|^2_s + \|f'\|_s).$$

From (5-38) we get

$$\frac{|\mathcal{N}(x_2,y)(\mathcal{D}(x_2,y)-\mathcal{D}(x_1,y))|}{\mathcal{D}(x_1,y)\mathcal{D}(x_2,y)} \le C\|g\|_{L^\infty} \|f'\|_{L^\infty} \|f'\|_s |x_1-x_2|\, y^{s-2}.$$

Therefore we get

$$|f'(x_1)| \int_r^1 \frac{|\mathcal{N}(x_2,y)(\mathcal{D}(x_2,y)-\mathcal{D}(x_1,y))|}{\mathcal{D}(x_1,y)\mathcal{D}(x_2,y)}\,dy \le C\|g\|_{L^\infty} \|f'\|^2_{L^\infty} \|f'\|_s |x_1-x_2|^s.$$

Hence plugging the preceding estimates into (5-35) and (5-36), we find

$$\begin{aligned}|f'(x_1)T^{r,2,2}_{f,\mathrm{ext}}(x_1)-f'(x_2)T^{r,2,2}_{f,\mathrm{ext}}(x_2)| &\le C\|g\|_{L^\infty}\|f'\|_s^2\|f'\|_{L^\infty}|x_1-x_2|^s\\ &\quad + C\|f'\|_{L^\infty}\|g\|_s(\|f'\|_s^2+\|f'\|_s)|x_1-x_2|^s\\ &\quad + C\|g\|_{L^\infty}\|f'\|^2_{L^\infty}\|f'\|_s|x_1-x_2|^s.\end{aligned}$$

Using standard embeddings we get

$$|f'(x_1)T^{r,2,2}_{f,\mathrm{ext}}(x_1)-f'(x_2)T^{r,2,2}_{f,\mathrm{ext}}(x_2)| \le C\|g\|_s\|f'\|_{L^\infty}|x_1-x_2|^s(\|f'\|_s+\|f'\|_s^2). \tag{5-44}$$

Putting together (5-43), (5-44) and (5-32) we obtain

$$\|f'T_f^{2,2}g\|_s \le C\|g\|_s\|f'\|_{L^\infty}(\|f'\|_s+\|f'\|_s^2). \tag{5-45}$$

Combining (5-40), (5-42) and (5-45) we get for any $s'\in(0,s]$

$$\|f'T_f g\|_s \le C\|g\|_s\|f'\|_D(\|f'\|_s+\|f'\|_s^{13}) + C\|g\|_s\|f'\|_{s'}^{\frac{1}{1+s'}}\beta^{-\frac{s'}{1+s'}} + C\|g\|_D\|f'\|_s.$$

Now using the embedding $C^s\hookrightarrow C^{s'}\hookrightarrow D$ we get

$$\|f'T_f g\|_s \le C\|g\|_s(\beta^{-\frac{s}{1+s}}\|f'\|_s^{\frac{1}{1+s}}+\|f'\|_s^{14}) \le C\|g\|_s(\beta^{-\frac12}\|f'\|_s^{\frac{1}{1+s}}+\|f'\|_s^{14}).$$

Another useful estimate that one can get from taking $s'=\frac{s}{2}$ and using the interpolation inequalities

$$\|f'\|_D \le C\|f'\|_s^{\frac{1+s}{2+s}} \le C\|f'\|_{L^\infty}^{\frac{1}{2+s}}\|f'\|_s^{\frac{1+s}{2+s}},\quad \|f'\|_{\frac{s}{2}} \le C\|f'\|_{L^\infty}^{\frac12}\|f'\|_s^{\frac12},\quad \beta^{-\frac{s}{2+s}}\le\beta^{-\frac12},$$

is the following:

$$\|f'T_f g\|_s \le C\|g\|_s\|f'\|_{L^\infty}^{\frac{1}{2+s}}(\|f'\|_s^{\frac{1}{2+s}}\beta^{-\frac12}+\|f'\|_s^{14}) + C\|g\|_{L^\infty}^{\frac{1}{2+s}}\|g\|_s^{\frac{1+s}{2+s}}\|f'\|_s.$$

This completes the proof of Theorem 5.2. ☐

6. Local well-posedness proof

The main objective of this section is to prove the local well-posedness result stated in the first part of Theorem 2.1. The approach that we shall follow is classical and will be done in several steps. We start with a priori estimates of smooth solutions in suitable Banach spaces and this will be the main concern of Sections 6A and 6B. The rigorous construction of classical solutions will be conducted in Section 6C.

6A. ***Estimates of the source terms.*** The main goal of this section is to establish the following a priori estimates for the source terms F and G described in (2-7) and (2-8).

Proposition 6.1. *Let K be a compact set of $\mathbb{R}$ and $s\in(0,1)$. We denote by X one of the spaces $C^\star_K$ or C^s_K. There exists a constant $C>0$ depending only on K such that the following estimates hold true*:

(1) *For any $f\in X$ we have*

$$\|F\|_{L^\infty} \le C\|f'\|_{L^\infty}\|f'\|_D,\quad \|F\|_X \le C\|f'\|_D(\|f'\|_X+\|f'\|_X^3).$$

(2) *For any $f \in X$ we have*

$$\|G\|_{L^\infty} \le C\|f'\|_{L^\infty}(1+\|f'\|_D^3), \quad \|G\|_X \le C(1+\|f'\|_D^{\frac{1}{3}})(\|f'\|_X+\|f'\|_X^{16}).$$

Proof. For simplicity throughout this proof we denote the operator Δ_y^- by Δ_y.

(1) The estimate of F in L^∞ is quite easy. Indeed, it is obvious according to (4-3) that

$$\begin{aligned}\|F\|_{L^\infty} &\le C\|f'\|_{L^\infty}\int_{-M}^{M}\sup_{x\in\mathbb{R}}\frac{|f'(x+y)-f'(x)|}{|y|}\,dy\\ &\le C\|f'\|_{L^\infty}\int_{-M}^{M}\frac{\omega_{f'}(|y|)}{|y|}\,dy\\ &\le C\|f'\|_{L^\infty}\|f'\|_D.\end{aligned}$$

Now let us move to the estimate of F in the function space X, which is the Dini space $C_K^\star$ or the Hölder space C_K^s. For this purpose we shall transform slightly F in order to apply Proposition 5.1. In fact from Taylor's formula one can write

$$F(x)=\int_{-M}^{M}\int_0^1\frac{y\,\Delta_{\theta y}f'(x)\Delta_y f'(x)}{y^2+(\Delta_y f(x))^2}\,dy\,d\theta.$$

From the notation (5-2) one has

$$F(x)=\int_0^1 \mathcal{C}_f^{\theta,\Re}(f',f')(x)\,d\theta.$$

At this stage it suffices to apply Proposition 5.1, which implies

$$\|F\|_X \le C(\|f'\|_D\|f'\|_X+\|f'\|_{L^\infty}\|f'\|_D\|f'\|_X^2)$$

and gives in turn the desired result according to the embedding $X \hookrightarrow L^\infty$.

(2) The expression of G is given in (2-8) and for simplicity we shall assume throughout this part that $M=1$. We shall first split G as follows:

$$\begin{aligned}G(x)&=\text{p.v.}\int_{-1}^{1}\frac{[2f(x)+\Delta_y^- f(x)+yf'(x)](f'(x+y)+f'(x))}{y^2+(f(x+y)+f(x))^2}\,dy\\ &=2f(x)\ \text{p.v.}\int_{-1}^{1}\frac{f'(x+y)+f'(x)}{y^2+(f(x+y)+f(x))^2}\,dy+\text{p.v.}\int_{-1}^{1}\frac{[\Delta_y^- f(x)+yf'(x)](2f'(x)+\Delta_y^- f'(x))}{y^2+(f(x+y)+f(x))^2}\,dy\\ &\triangleq G_1(x)+G_2(x). \qquad (6\text{-}1)\end{aligned}$$

The estimate G_1 in L^∞ is quite easy. To see this we can first assume that $f(x)>0$; otherwise the integral is vanishing. Thus by change of variables we get

$$|G_1(x)| \le 4\|f'\|_{L^\infty}\int_{-1}^{1}\frac{|f(x)|}{y^2+f^2(x)}\,dy \le C\|f'\|_{L^\infty}.$$

Note that for $x \in \operatorname{supp} f$ we have $f(x+y) = 0$ for all $y \notin [-1, 1]$. Thus

$$
\begin{aligned}
G_1(x) &= 2f(x)\int_{\mathbb{R}} \frac{f'(x+y)+f'(x)}{y^2+(f(x+y)+f(x))^2}\,dy - 4f(x)f'(x)\int_1^{\infty}\frac{1}{y^2+(f(x))^2}\,dy\\
&= 2f(x)\int_{\mathbb{R}} \frac{f'(x+y)+f'(x)}{y^2+(f(x+y)+f(x))^2}\,dy - 4f'(x)\arctan(f(x))\\
&\triangleq G_{11}+G_{12}. \qquad\qquad (6\text{-}2)
\end{aligned}
$$

The estimate of G_{12} in L^∞ is elementary:

$$\|G_{12}\|_{L^\infty} \le 4\|f'\|_{L^\infty}\|f\|_{L^\infty}. \qquad (6\text{-}3)$$

However, to estimate G_{12} in X we use the product law (4-3) leading to

$$\|f' \arctan f\|_X \le \|\arctan f\|_{L^\infty}\|f'\|_X + \|f'\|_{L^\infty}\|\arctan f\|_X.$$

It is easy to check from the mean value theorem that

$$\|\arctan f\|_{L^\infty} \le \|f\|_{L^\infty} \quad \text{and} \quad \omega_{\arctan f}(r) \le \omega_f(r),$$

which implies in view of the embedding $\mathrm{Lip} \hookrightarrow X$ that

$$\|\arctan f\|_X \le \|f\|_X \le C\|f'\|_{L^\infty}.$$

Therefore we obtain from the classical embeddings

$$\|G_{12}\|_X \le C(\|f\|_{L^\infty}\|f'\|_X + C\|f'\|_{L^\infty}^2) \le C\|f'\|_{L^\infty}\|f'\|_X. \qquad (6\text{-}4)$$

We shall now estimate the term G_{11} in the space X. First we use Taylor's formula

$$f(x+y)+f(x) = 2f(x) + y\int_0^1 f'(x+\theta y)\,d\theta,$$

which implies after the change of variables $y = f(x)z$ (assuming that $f(x) > 0$)

$$
\begin{aligned}
G_{11}(x) &= 2f(x)\int_{\mathbb{R}} \frac{f'(x)+f'(x+y)}{y^2+[2f(x)+y\int_0^1 f'(x+\theta y)\,d\theta]^2}\,dy\\
&= 2\int_{\mathbb{R}} \frac{f'(x)+f'(x+f(x)z)}{\varphi(x,z)}\,dz, \qquad\qquad (6\text{-}5)
\end{aligned}
$$

with

$$\varphi(x,z) = z^2 + \left(2 + z\int_0^1 f'(x+\theta f(x)z)\,d\theta\right)^2.$$

Note that for $f(x) = 0$ we have from the definition $G_{11}(x) = 0$, which agrees with the expression (6-5) because $f'(x) = 0$. The estimate in L^∞ is easy to get in view of (5-22):

$$\|G_{11}\|_{L^\infty} \le 4\|f'\|_{L^\infty}\int_{\mathbb{R}} \|1/\varphi(\cdot,z)\|_{L^\infty}\,dz \le C(\|f'\|_{L^\infty} + \|f'\|_{L^\infty}^3).$$

From the product laws (4-4) and (4-8) we deduce that

$$\begin{aligned}\|G_{11}\|_X &= 2\int_{\mathbb{R}} \|f'+f'\circ(\mathrm{Id}+zf)\|_X\,\|1/\varphi(\cdot,z)\|_{L^\infty}\,dz + 2\int_{\mathbb{R}} \|f'+f'\circ(\mathrm{Id}+zf)\|_{L^\infty}\,\|1/\varphi(\cdot,z)\|_X\,dz\\ &\triangleq \ell_1+\ell_2.\end{aligned}$$

According to the product laws (4-6) and (4-7), one may write

$$\|f'+f'\circ(\mathrm{Id}+zf)\|_X \le \|f'\|_X(1+\mu(1+|z|\,\|f'\|_{L^\infty})),$$

with

$$\mu(r) \triangleq \begin{cases} \ln r & \text{if } X = C_K^\star,\\ r^s & \text{if } X = C^s.\end{cases}$$

Observe that we can unify both cases through the estimate

$$\begin{aligned}\|f'+f'\circ(\mathrm{Id}+zf)\|_X &\le C\|f'\|_X(1+(1+|z|\,\|f'\|_{L^\infty})^s)\\ &\le C\|f'\|_X(1+|z|^s\,\|f'\|^s_{L^\infty}).\end{aligned}\tag{6-6}$$

Putting together (6-6) and (5-22) we find for any $s\in(0,1)$

$$\begin{aligned}\ell_1 &\le C\|f'\|_X(1+\|f'\|^2_{L^\infty})\int_{\mathbb{R}}\frac{1+|z|^s\,\|f'\|^s_{L^\infty}}{1+z^2}\,dz\\ &\le C\|f'\|_X(1+\|f'\|^3_{L^\infty}).\end{aligned}\tag{6-7}$$

To estimate ℓ_2 we use the elementary estimate

$$\|f'+f'\circ(\mathrm{Id}+zf)\|_{L^\infty} \le 2\|f'\|_{L^\infty}.$$

Notice from the definition of the spaces X and (5-22) that one can deduce

$$\|1/\varphi(\cdot,z)\|_X \le \|1/\varphi(\cdot,z)\|^2_{L^\infty}\,\|\varphi(\cdot,z)\|_X \le C\,\frac{1+\|f'\|^4_{L^\infty}}{1+z^4}\,\|\varphi(\cdot,z)\|_X.\tag{6-8}$$

Moreover by the product laws we find

$$\|\varphi(\cdot,z)\|_X \le 2|z|(2+|z|\,\|f'\|_{L^\infty})\int_0^1 \|f'\circ(\mathrm{Id}+\theta zf)\|_X\,d\theta,$$

and this implies according to (6-6)

$$\begin{aligned}\|\varphi(\cdot,z)\|_X &\le C|z|(2+|z|\,\|f'\|_{L^\infty})\|f'\|_X(1+|z|^s\,\|f'\|^s_{L^\infty})\\ &\le C(1+|z|^{2+s})(1+\|f'\|^{1+s}_{L^\infty})\|f'\|_X.\end{aligned}$$

Putting together this estimate with (6-8) we find

$$\|1/\varphi(\cdot,z)\|_X \le C\,\frac{(1+\|f'\|^{5+s}_{L^\infty})\|f'\|_X}{1+|z|^{2-s}}.\tag{6-9}$$

Therefore we deduce that

$$\ell_2 \le C\|f'\|_{L^\infty}(1+\|f'\|^{5+s}_{L^\infty})\|f'\|_X \le C(1+\|f'\|^7_{L^\infty})\|f'\|_X.$$

Combining this estimate with (6-7) we obtain

$$\|G_{11}\|_X \le C(1+\|f'\|_{L^\infty}^7)\|f'\|_X.$$

It follows from this latter estimate, (6-4) and (6-2) that

$$\|G_1\|_X \le C(1+\|f'\|_{L^\infty}^7)\|f'\|_X. \tag{6-10}$$

What is left is to estimate G_2. For this purpose we write according to Taylor's formula

$$\begin{aligned} G_2(x) &= \text{p.v.}\int_{\mathbb{R}} \frac{yf'(x)\big[f'(x)\chi(y)+2\int_0^1 f'(x+\theta y)\,d\theta+f'(x+y)\big]}{y^2+(f(x)+f(x+y))^2}\,dy \\ &\qquad + \text{p.v.}\int_{\mathbb{R}} \frac{\Delta_y f(x)\Delta_y f'(x)}{y^2+(f(x)+f(x+y))^2}\,dy + 2f(x)f'(x)\int_1^\infty \frac{dy}{y^2+f^2(x)} \\ &\triangleq G_{2,1}(x)+G_{2,2}(x)+2f'(x)\arctan(f(x)), \end{aligned}$$

where $\chi:\mathbb{R}\to\mathbb{R}$ is an even continuous compactly supported function belonging to X and taking the value 1 on the neighborhood of $[-1,1]$. Note that we have used in the first line the identity, for any $x\in K$,

$$\text{p.v.}\int_{-1}^1 \frac{y}{y^2+[f(x+y)+f(x)]^2}\,dy = \text{p.v.}\int_{\mathbb{R}} \frac{y\chi(y)}{y^2+[f(x+y)+f(x)]^2}\,dy,$$

which follows from the fact that $f(x+y)=0$ for all $y\notin[-1,1]$. Therefore we may write

$$G_{2,1}(x) = (f'(x))^2(T_f^{0,1}\chi)(x) + 2\int_0^1 f'(x)(T_f^{1,\theta}f')(x)\,d\theta + f'(x)(T_f^{1,1}f')(x),$$

where we use the notation $T_f^{\alpha,\beta}$ from Theorem 5.2. The estimate of $G_{2,1}$ in L^∞ is quite easy and follows from Theorem 5.2:

$$\|G_{2,1}\|_{L^\infty} \le C\|f'\|_{L^\infty}\|f'\|_D(1+\|f'\|_D^2).$$

However to estimate $G_{2,2}$ in L^∞ it is more convenient to write it in the form

$$G_{2,2}(x) = \text{p.v.}\int_{-1}^1 \frac{\Delta_y f(x)\Delta_y f'(x)}{y^2+(f(x)+f(x+y))^2}\,dy + 2f'(x)\arctan(f(x)).$$

Thus using the mean value theorem we find

$$\|G_{2,2}\|_{L^\infty} \le C\|f'\|_{L^\infty}\|f'\|_D.$$

Combining these estimates with (4-2) we obtain

$$\|G_2\|_{L^\infty} \le C\|f'\|_{L^\infty}(\|f'\|_D+\|f'\|_D^3). \tag{6-11}$$

We shall now implement the estimates in X and start with the term $G_{2,1}$. According to Theorem 5.2 one can unify the estimates in $C_K^\star$ and C^s and get the weak estimate

$$\|f'T_f^{\alpha,\beta}g\|_X \le C\|g\|_X(\|f'\|_X^{\frac12}\beta^{-\frac12}+\|f'\|_X^{15}). \tag{6-12}$$

From the product laws (4-4) and (4-8) one has

$$\|(f')^2(T_f^{0,1}\chi)\|_X \le \|f'\|_{L^\infty}\|f'T_f^{0,1}\chi\|_X + \|f'\|_X\,\|f'\|_{L^\infty}\|T_f^{0,1}\chi\|_{L^\infty}.$$

Hence we find

$$\begin{aligned}\|(f')^2(T_f^{0,1}\chi)\|_X &\le C\|f'\|_{L^\infty}(\|f'\|_X^{\frac12}+\|f'\|_X^{15}) + \|f'\|_X\,\|f'\|_{L^\infty}(1+\|f'\|_X^2)\\ &\le C\|f'\|_{L^\infty}(\|f'\|_X^{\frac12}+\|f'\|_X^{15}).\end{aligned}$$

Using (6-12) we get successively

$$\|f'T_f^{0,\theta}f'\|_X \le C\|f'\|_X(\|f'\|_X^{\frac12}\theta^{-\frac12}+\|f'\|_X^{15}) \tag{6-13}$$

and

$$\|f'T_f^{1,1}f'\|_X \le C\|f'\|_X(\|f'\|_X^{\frac12}+\|f'\|_X^{15}).$$

Thus using the inequalities above we deduce that

$$\begin{aligned}\|G_{2,1}\|_X &\le C\|f'\|_{L^\infty}(\|f'\|_X^{\frac12}+\|f'\|_X^{15}) + C\|f'\|_X(\|f'\|_X^{\frac12}+\|f'\|_X^{15})\\ &\le C(\|f'\|_X^{\frac32}+\|f'\|_X^{16}).\end{aligned} \tag{6-14}$$

When $X = C^s$ we can give a refined estimate for (6-13) using (5-7),

$$\|f'T_f^{0,\theta}f'\|_s \le C\|f'\|_{L^\infty}^{\frac{1}{2+s}}(\|f'\|_s^{\frac{3+2s}{2+s}}\theta^{-\frac12}+\|f'\|_s^{15}),$$

which implies

$$\begin{aligned}\|G_{2,1}\|_s &\le C\|f'\|_{L^\infty}(\|f'\|_s^{\frac12}+\|f'\|_s^{15}) + C\|f'\|_{L^\infty}^{\frac{1}{2+s}}(\|f'\|_s^{\frac{3+2s}{2+s}}+\|f'\|_s^{15})\\ &\le C\|f'\|_{L^\infty}^{\frac13}(\|f'\|_s+\|f'\|_s^{16}).\end{aligned} \tag{6-15}$$

Hence one can combine (6-14) and (6-15):

$$\|G_{2,1}\|_X \le C\|f'\|_D^{\frac13}(\|f'\|_X+\|f'\|_X^{16}). \tag{6-16}$$

As for the term $G_{2,2}$, we may write

$$\begin{aligned}G_{2,2}(x) &= 2f'(x)\arctan(f(x))\\ &\quad+\int_{-M}^{M}\frac{[\Delta_y f(x)-yf'(x)]\Delta_y f'(x)}{y^2+(f(x+y)+f(x))^2}\,dy + \text{p.v.}\int_{\mathbb{R}}\frac{yf'(x)\Delta_y f'(x)}{y^2+(f(x+y)+f(x))^2}\,dy\\ &\triangleq 2f'(x)\arctan(f(x)) + G_{2,2}^1(x)+G_{2,2}^2(x).\end{aligned}$$

The last term was treated in the preceding estimates and we obtain as in (6-16)

$$\|G_{2,2}^2\|_X \le C\|f'\|_D^{\frac13}(\|f'\|_X+\|f'\|_X^{16}). \tag{6-17}$$

It remains to estimate $G_{2,2}^1$, which can be split into two terms

$$G_{2,2}^1(x) = \widehat{G}_{\text{int},r}(x)+\widehat{G}_{\text{ext},r}(x),$$

with

$$\widehat{G}_{\mathrm{int},r}(x)=\int_{|y|\le r}\frac{[\Delta_y f(x)-yf'(x)]\Delta_y f'(x)}{y^2+(f(x+y)+f(x))^2}\,dy,$$

$$\widehat{G}_{\mathrm{ext},r}(x)=\int_{M\ge|y|\ge r}\frac{[\Delta_y f(x)-yf'(x)]\Delta_y f'(x)}{y^2+(f(x+y)+f(x))^2}\,dy.$$

Now we shall proceed as in the proof of Theorem 5.2. Let $r\in(0,1)$ and $x_1,x_2\in\mathbb{R}$ such that $|x_1-x_2|\le r$. First it is clear that

$$|\Delta_y f'(x)|\le\omega_{f'}(|y|). \tag{6-18}$$

In addition, using Taylor formula we get

$$|\Delta_y f(x)-yf'(x)|\le|y|\omega_{f'}(|y|). \tag{6-19}$$

Therefore

$$|\widehat{G}_{\mathrm{int},r}(x)|\le\int_{|y|\le r}\frac{[\omega_{f'}(|y|)]^2}{|y|}\,dy.$$

It follows that

$$\sup_{|x_1-x_2|\le r}|\widehat{G}_{\mathrm{int},r}(x_2)-\widehat{G}_{\mathrm{int},r}(x_1)|\le 4\int_0^r\frac{[\omega_{f'}(y)]^2}{y}\,dy. \tag{6-20}$$

Hence by Fubini's theorem

$$\int_0^1\sup_{|x_1-x_2|\le r}|\widehat{G}_{\mathrm{int},r}(x_1)-\widehat{G}_{\mathrm{int},r}(x_1)|\,\frac{dr}{r}\le 4\int_0^1\frac{[\omega_{f'}(y)]^2}{y}|\ln y|\,dy.$$

From the definition and the monotonicity of the modulus of continuity one deduces that for any $r\in(0,1)$

$$|\ln r|\omega_{f'}(r)\le\int_r^1\frac{\omega_{f'}(y)}{y}\,dy\le\|f'\|_D,$$

which implies

$$\int_0^1\sup_{|x_1-x_2|\le r}|\widehat{G}_{\mathrm{int},r}(x_1)-\widehat{G}_{\mathrm{int},r}(x_1)|\,\frac{dr}{r}\le 4\|f'\|_D^2. \tag{6-21}$$

To get the suitable estimate in C^s we come back to (6-20), which gives

$$\sup_{|x_1-x_2|\le r}|\widehat{G}_{\mathrm{int},r}(x_2)-\widehat{G}_{\mathrm{int},r}(x_1)|\le 4\|f'\|_s^2\int_0^r y^{2s-1}\,dy\le C\|f'\|_s^2 r^{2s},$$

and thus

$$\sup_{|x_1-x_2|\le 1}\frac{|\widehat{G}_{\mathrm{int},r}(x_2)-\widehat{G}_{\mathrm{int},r}(x_1)|}{|x_1-x_2|^s}\le C\|f'\|_s^2. \tag{6-22}$$

As for $\widehat{G}_{\mathrm{ext},r}$, one writes

$$\widehat{G}_{\mathrm{ext},r}(x_1)-\widehat{G}_{\mathrm{ext},r}(x_2)=\int_{M\ge|y|\ge r}\frac{\mathcal{N}(x_1,y)-\mathcal{N}(x_2,y)}{\mathcal{K}(x_1)}\,dy+\int_{M\ge|y|\ge r}\frac{\mathcal{N}(x_2,y)[\mathcal{K}(x_2,y)-\mathcal{K}(x_1,y)]}{\mathcal{K}(x_1,y)\mathcal{K}(x_2,y)}\,dy,$$

with

$$\mathcal{N}(x,y)=[\Delta_y f(x)-yf'(x)]\Delta_y f'(x) \quad \text{and} \quad \mathcal{K}(x,y)=y^2+(f(x)+f(x+y))^2.$$

Notice that from (6-18) and (6-19) one gets

$$|\mathcal{N}(x_1,y)-\mathcal{N}(x_2,y)|\le C|y|\omega_{f'}(r)\omega_{f'}(|y|) \quad \text{and} \quad |\mathcal{N}(x,y)|\le 2|y|\omega_{f'}(|y|)\|f'\|_{L^\infty}. \tag{6-23}$$

In addition, using straightforward calculus we obtain

$$|\mathcal{K}(x_1,y)-\mathcal{K}(x_2,y)|\le Cr\|f'\|_{L^\infty}\big(\sqrt{\mathcal{K}(x_1,y)}+\sqrt{\mathcal{K}(x_2,y)}\big).$$

Thus

$$\sup_{|x_1-x_2|\le r}\frac{|\mathcal{N}(x_2,y)|\,|\mathcal{K}(x_2,y)-\mathcal{K}(x_1,y)|}{\mathcal{K}(x_1,y)\mathcal{K}(x_2,y)}\le Cr\|f'\|_{L^\infty}^2\frac{\omega_{f'}(|y|)}{|y|^2}.$$

Hence we get by Fubini's theorem and (4-3)

$$\int_0^1\sup_{|x_1-x_2|\le r}\int_{\{M\ge|y|\ge r\}}\frac{|\mathcal{N}(x_1,y)-\mathcal{N}(x_2,y)|}{\mathcal{K}(x_1)}\,dy\,\frac{dr}{r}\le\int_0^1\int_{\{M\ge|y|\ge r\}}\omega_{f'}(r)\omega_{f'}(|y|)\,\frac{dy}{|y|}\,\frac{dr}{r}\le C\|f'\|_D^2$$

and

$$\begin{aligned}\int_0^1\sup_{|x_1-x_2|\le r}\int_{\{M\ge|y|\ge r\}}\frac{|\mathcal{N}(x_2,y)|\,|\mathcal{K}(x_2,y)-\mathcal{K}(x_1,y)|}{\mathcal{K}(x_1,y)\mathcal{K}(x_2,y)}\,dy\,\frac{dr}{r}&\le C\|f'\|_{L^\infty}^2\int_0^1\int_{\{M\ge|y|\ge r\}}\frac{\omega_{f'}(|y|)}{|y|^2}\,dy\,dr\\&\le C\|f'\|_{L^\infty}^2\|f'\|_D.\end{aligned}$$

Finally we obtain

$$\int_0^1\sup_{|x_1-x_2|\le r}|\widehat{G}_{\mathrm{ext},r}(x_1)-\widehat{G}_{\mathrm{ext},r}(x_2)|\,\frac{dr}{r}\le C\|f'\|_D^2+C\|f'\|_{L^\infty}^2\|f'\|_D.$$

As to the estimate in C^s we use (6-23) which implies

$$\begin{aligned}\int_{\{r\le|y|\le M\}}\frac{|\mathcal{N}(x_1,y)-\mathcal{N}(x_2,y)|}{\mathcal{K}(x_1)}\,dy&\le C\|f'\|_s r^s\int_{\{r\le|y|\le M\}}\frac{\omega_{f'}(|y|)}{|y|}\,dy\\&\le C\|f'\|_s\|f'\|_D r^s\end{aligned}$$

and

$$\begin{aligned}\int_{\{M\ge|y|\ge r\}}\frac{|\mathcal{N}(x_2,y)|\,|\mathcal{K}(x_2,y)-\mathcal{K}(x_1,y)|}{\mathcal{K}(x_1,y)\mathcal{K}(x_2,y)}\,dy&\le C\|f'\|_{L^\infty}^2\|f'\|_s r\int_{\{M\ge|y|\ge r\}}\frac{dy}{|y|^{2-s}}\\&\le C\|f'\|_{L^\infty}^2\|f'\|_s r^s.\end{aligned}$$

It follows from Sobolev embedding $C^s\hookrightarrow L^\infty$ that

$$\sup_{|x_1-x_2|\le r}\frac{|\widehat{G}_{\mathrm{ext},r}(x_1)-\widehat{G}_{\mathrm{ext},r}(x_2)|}{|x_1-x_2|^s}\le C\|f'\|_D\|f'\|_s+C\|f'\|_D\|f'\|_s^2.$$

Combining the estimates above with (6-21) and (6-22) we deduce

$$\|G_{2,2}^1\|_X\le C\|f'\|_D(\|f'\|_X+\|f'\|_X^2).$$

Putting together this estimate with (6-16) and (6-17) we get

$$\|G_2\|_X \le C\|f'\|_D^{\frac{1}{3}}(\|f'\|_X + \|f'\|_X^{16}). \tag{6-24}$$

Now using (6-10) and (6-24) we find

$$\begin{aligned}\|G\|_X &\le C\|f'\|_X(1+\|f'\|_D^7) + C\|f'\|_D^{\frac{1}{3}}(\|f'\|_X + \|f'\|_X^{16})\\ &\le C(1+\|f'\|_D^{\frac{1}{3}})(\|f'\|_X + \|f'\|_X^{16}),\end{aligned}$$

which ends the proof of Proposition 6.1. ☐

6B. ***A priori estimates.*** The aim of this section is to establish weak and strong a priori estimates for solutions to (2-1). This part is the cornerstone of the local well-posedness theory. The main result of this section reads as follows.

Proposition 6.2. *Let* $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ *be a smooth solution for the graph equation* (2-1). *Assume that the initial data is positive and with compact support* K_0. *Then the following assertions hold true*:

(1) *For any* $t \in [0, T]$, *the function* f_t *is positive and*

$$\text{for all } t \in [0, T], \quad \|f(t)\|_{L^\infty} \le \|f_0\|_{L^\infty}.$$

(2) *For any* $t \in [0, T]$, *we have*

$$\|f(t)\|_{L^1} = \|f_0\|_{L^1} e^{-t}.$$

(3) *The support* supp f_t *is contained in the convex hull of* K_0; *that is*,

$$\text{for all } t \in [0, T], \quad \operatorname{supp} f(t) \subset \operatorname{Conv} K_0.$$

(4) *Set* $X = C_K^\star$ *or* $X = C_K^s$, *with* $s \in (0, 1)$. *If* $f_0' \in X$ *then there exists* T *depending only on* $\|f_0'\|_X$ *such that* $f' \in L^\infty([0, T]; X)$.

Proof. (1) To get the first part about the persistence of the positivity of we shall prove that

$$\text{for all } x \in \mathbb{R}, \quad u_2(t, x) = f(t, x)U(t, x), \tag{6-25}$$

with

$$\|U(t)\|_{L^\infty} \le C(1 + \|f'(t)\|_D^6)$$

and C being a constant depending only on the size of the support of f_t. Note from part (3) of the current proposition that the support of f_t is contained in a fixed compact set and therefore the constant C can be taken independent of the time variable. Assume for a while (6-25) and let us see how to propagate the positivity. Denote by ψ the flow associated to the velocity u_1, that is, the solution of the ODE

$$\partial_t \psi(t, x) = u_1(t, \psi(t, x)), \quad \psi(0, x) = x. \tag{6-26}$$

Recall that

$$u_1(t, x) = \frac{1}{2\pi}\int_{\mathbb{R}}\left[\arctan\left(\frac{f(t, x+y) - f(t, x)}{y}\right) - \arctan\left(\frac{f(t, x+y) + f(t, x)}{y}\right)\right] dy.$$

Set

$$\eta(t,x) = f(t,\psi(t,x));$$

then

$$\partial_t \eta(t,x) = u_2(t,\psi(t,x)) = \eta(t,x)U(t,\psi(t,x)). \tag{6-27}$$

Consequently

$$\eta(t,x) = f_0(x)e^{\int_0^t U(\tau,\psi(\tau,x))\,d\tau}.$$

Since the flow $\psi(t):\mathbb{R}\to\mathbb{R}$ is a diffeomorphism we get the representation

$$f(t,x) = f_0(\psi^{-1}(t,x))e^{\int_0^t U[\tau,\psi(\tau,\psi^{-1}(t,x))]\,d\tau}. \tag{6-28}$$

As an immediate consequence we get the persistence through the time of the positivity of the solution. Let us now come back to the proof of the identity (6-25). To simplify the notation we remove the variable t from the functions. Applying Taylor's formula to the function

$$\tau\in[0,f(x)]\mapsto g(\tau)\triangleq\log\left[\frac{y^2+(\tau-f(x+y))^2}{y^2+(\tau+f(x+y))^2}\right]$$

yields

$$\begin{aligned}
&-2\pi u_2(x)\\
&\quad= f(x)\int_0^1\int_{-M}^{M}\frac{f(x+y)-\tau f(x)}{y^2+[f(x+y)-\tau f(x)]^2}\,d\tau\,dy + f(x)\int_0^1\int_{-M}^{M}\frac{f(x+y)+\tau f(x)}{y^2+[f(x+y)+\tau f(x)]^2}\,d\tau\,dy\\
&\quad\triangleq f(x)V_1(x)+f(x)V_2(x).
\end{aligned}$$

Using once again Taylor's formula we get the expressions

$$\begin{aligned}
&V_1(x)\\
&\quad=\int_0^1\int_{-M}^{M}\frac{(1-\tau)f(x)}{y^2+\big[(1-\tau)f(x)+y\int_0^1 f'(x+\theta y)\,d\theta\big]^2}\,d\tau\,dy+\text{p.v.}\int_0^1\int_{-M}^{M}\frac{y\int_0^1 f'(x+\theta y)\,d\theta}{y^2+[f(x+y)-\tau f(x)]^2}\,d\tau\,dy\\
&\quad\triangleq V_{1,1}(x)+V_{1,2}(x)
\end{aligned}$$

and

$$\begin{aligned}
&V_2(x)\\
&\quad=\int_0^1\int_{-M}^{M}\frac{(1+\tau)f(x)}{y^2+\big[(1+\tau)f(x)+y\int_0^1 f'(x+\theta y)\,d\theta\big]^2}\,d\tau\,dy+\text{p.v.}\int_0^1\int_{-M}^{M}\frac{y\int_0^1 f'(x+\theta y)\,d\theta}{y^2+[f(x+y)+\tau f(x)]^2}\,d\tau\,dy\\
&\quad\triangleq V_{2,1}(x)+V_{2,2}(x).
\end{aligned}$$

To estimate $V_{1,1}$ and $V_{2,1}$ we can assume that $f(x)>0$. Then making the change of variables $z\mapsto y=(1-\tau)f(x)z$ leads to

$$V_{1,1}(x)=\int_0^1\int_{-\frac{M}{(1-\tau)f(x)}}^{\frac{M}{(1-\tau)f(x)}}\frac{d\tau\,dz}{z^2+\big[1+z\int_0^1 f'(x+\theta(1-\tau)f(x)z)\,d\theta\big]^2}. \tag{6-29}$$

Using (5-22) we deduce that

$$\|V_{1,1}\|_{L^\infty} \le C(1+\|f'\|_{L^\infty}^2). \tag{6-30}$$

Similarly we get

$$\|V_{2,1}\|_{L^\infty} \le C(1+\|f'\|_{L^\infty}^2). \tag{6-31}$$

Let us now bound $V_{j,2}$, $j=1,2$. First by symmetry we write

$$V_{1,2}(x) = \int_0^1\int_0^M \frac{y\int_0^1 f'(x+\theta y)\,d\theta[f(x-y)-f(x+y)]\psi_\tau(x,y)}{(y^2+[f(x+y)-\tau f(x)]^2)(y^2+[f(x-y)-\tau f(x)]^2)}\,dy\,d\tau + \int_0^1\int_0^M \frac{y\int_0^1[f'(x+\theta y)-f'(x-\theta y)]\,d\theta}{y^2+[f(x-y)-\tau f(x)]^2}\,dy\,d\tau,$$

where

$$\begin{aligned}\psi_\tau(x,y) &= f(x+y)+f(x-y)-2\tau f(x)\\ &= 2(1-\tau)f(x)+y\int_0^1[f'(x+\theta y)-f'(x-\theta y)]\,d\theta.\end{aligned}$$

Thus

$$\|V_{1,2}\|_{L^\infty} \le C\int_0^1\int_0^M \frac{\|f'\|_{L^\infty}^2 y^2[(1-\tau)f(x)+y\omega_{f'}(y)]}{(y^2+[f(x+y)-\tau f(x)]^2)(y^2+[f(x-y)-\tau f(x)]^2)}\,dy\,d\tau + C\int_0^1\int_0^M \frac{\omega_{f'}(y)}{y}\,dy\,d\tau.$$

Similarly to $V_{1,1}$ one gets

$$\int_0^1\int_0^M \frac{y^2(1-\tau)f(x)\,dy\,d\tau}{(y^2+[f(x+y)-\tau f(x)]^2)(y^2+[f(x-y)-\tau f(x)]^2)} \le C(1+\|f'\|_{L^\infty}^4).$$

It follows that

$$\begin{aligned}\|V_{1,2}\|_{L^\infty} &\le C\|f'\|_{L^\infty}^2\Big(1+\|f'\|_{L^\infty}^4+\int_0^M\frac{\omega_{f'}(y)}{y}\,dy\Big)+C\|f'\|_D\\ &\le C\|f'\|_{L^\infty}^2(1+\|f'\|_{L^\infty}^4+\|f'\|_D)+C\|f'\|_D.\end{aligned} \tag{6-32}$$

The estimate of $V_{2,2}$ can be done in a similar way and one obtains

$$\|V_{2,2}\|_{L^\infty} = C\|f'\|_{L^\infty}^2(1+\|f'\|_{L^\infty}^4+\|f'\|_D)+C\|f'\|_D. \tag{6-33}$$

Combining both last estimates with (6-30) and (6-31) we finally get according to the embedding (4-2)

$$\|U\|_{L^\infty} \le C(1+\|f'\|_D^6),$$

where the constant C depends only on the size of the support of f.

Now let us establish the maximum principle. From (2-2) combined with the positivity of f_t one gets

$$\text{for all } t\in[0,T],\ \text{for all } x\in\mathbb{R},\quad u_2(t,x)\le 0.$$

Coming back to (6-27) we deduce that

$$\partial_t\eta(t,x)\le 0,$$

which implies in turn that

$$\text{for all } t \in [0,T], \text{ for all } x \in \mathbb{R}, \quad f(t,x) \le f_0(\psi^{-1}(t,x)).$$

Combined with the positivity of $f(t)$ we deduce immediately the maximum principle

$$\text{for all } t \in [0,T], \quad \|f(t)\|_{L^\infty} \le \|f_0\|_{L^\infty}.$$

Now we intend to provide a more refined identity that we shall use later in studying the asymptotic behavior of the solution. Actually we have

$$u_2(t,x) = -f(t,x)(1+R(t,x)), \tag{6-34}$$

with

$$\|R(t)\|_{L^\infty} \le C\|f'(t)\|_D(1+\|f'(t)\|_{L^\infty}^5).$$

First note that $R = \sum_{i,j=1}^2 V_{i,j}$. The estimates of $V_{1,2}$ and $V_{2,2}$ are done in (6-32) and (6-33). However to deal with $V_{1,1}$ and similarly $V_{2,1}$ we return to the expression (6-29). Set

$$\tau \mapsto K(\tau) = \frac{1}{z^2+[1+z\tau]^2}.$$

Easy computations using (5-22) show the existence of a positive constant C such that

$$\text{for all } \tau, z \in \mathbb{R}, \quad |K'(\tau)| = \frac{2|z||1+z\tau|}{(z^2+[1+zt]^2)^2} \le \frac{1}{z^2+[1+z\tau]^2} \le C\frac{1+\tau^2}{1+z^2}.$$

Applying the mean value theorem yields

$$\left| K(\tau) - \frac{1}{1+z^2} \right| \le C|\tau| \frac{1+\tau^2}{1+z^2}.$$

Therefore we get

$$\left| V_{1,1}(x) - \int_0^1 \int_{-\frac{M}{(1-\tau)f(x)}}^{\frac{M}{(1-\tau)f(x)}} \frac{dz\,d\tau}{1+z^2} \right| \le C\|f'\|_{L^\infty}(1+\|f'\|_{L^\infty}^2),$$

which implies that

$$|V_{1,1}(x) - \pi| \le C\|f'\|_{L^\infty}(1+\|f'\|_{L^\infty}^2) + C\|f\|_{L^\infty}. \tag{6-35}$$

Similarly we obtain

$$|V_{2,1}(x) - \pi| \le C\|f'\|_{L^\infty}(1+\|f'\|_{L^\infty}^2) + C\|f\|_{L^\infty}. \tag{6-36}$$

Putting together (6-32), (6-33), (6-35), (6-36) we get (6-34).

(2) Integrating (1-2) in the space variable we get after integration by parts

$$\frac{d}{dt}\int_{\mathbb{R}} \rho(t,x)\,dx = \int_{\mathbb{R}} \operatorname{div}\, v(t,x)\rho(t,x)\,dx = -\int_{\mathbb{R}} \rho^2(t,x)\,dx = -\int_{\mathbb{R}} \rho(t,x)\,dx,$$

where in the last line we have used that for the characteristic function one has $\rho^2 = \rho$. The time decay follows then easily.

(3) According to the representation of the solution given by (6-28) we have easily that the support of $f(t)$ is the image by the flow $\psi(t)$ of the initial support, that is,

$$K_t = \psi(t, K_0). \tag{6-37}$$

We have to check that if $K_0 \subset [a, b]$, with $a < b$, then $K_t \subset [a, b]$. To do so it is enough to prove that

$$\psi(t, [a, b]) \subset [a, b].$$

This means that the flow is contractive on the boundary of the support. As $\psi(t)$ is a homeomorphism, we have $\psi(t, [a, b]) = [\psi(t, a), \psi(t, b)]$. Hence to get the desired inclusion it suffices to establish that

$$a_t \triangleq \psi(t, a) \geq a \quad \text{and} \quad b_t \triangleq \psi(t, b) \leq b.$$

This reduces to studying the derivative in time of a_t and b_t. First one has

$$\dot{a}_t = u_1(t, a_t) \quad \text{and} \quad \dot{b}_t = u_1(t, b_t).$$

Since $f(t, y) = 0$, for all $y \notin (a_t, b_t)$ and f_t positive everywhere,

$$u_1(t, a_t) = \frac{1}{\pi} \int_0^{b_t - a_t} \arctan\left(\frac{f_t(a_t + y)}{y}\right) dy \geq 0.$$

Hence $\dot{a}_t \geq 0$ and therefore $a_t \geq a$, for any $t \in [0, T]$.

Similarly we get

$$u_1(t, b_t) = -\frac{1}{\pi} \int_0^{b_t - a_t} \arctan\left(\frac{f_t(b_t - y)}{y}\right) dy \leq 0,$$

which implies that $b_t \leq b$ for any $t \in [0, T]$. This ends the proof of part (3).

(4) Recall from (2-3) and (2-5) that $g \triangleq f'$ satisfies the equation

$$\partial_t g + u_1 \partial_1 g = \frac{1}{2\pi}(F - G).$$

Set $h(t, x) = g(t, \psi(t, x))$, where ψ is the flow defined in (6-26). Then

$$\partial_t h(t, x) = \frac{1}{2\pi}(F(t, \psi(t, x)) - G(t, \psi(t, x))).$$

Thus

$$g(t, x) = g_0(\psi^{-1}(t, x)) + \frac{1}{2\pi} \int_0^t (F - G)\big(\tau, \psi(\tau, \psi^{-1}(t, x))\big)\, d\tau.$$

Recall the classical estimate

$$\|\partial_x[\psi(\tau, \psi^{-1}(t, \cdot))]\|_{L^\infty} \leq e^{\int_\tau^t \|\partial_x u_1(t', \cdot)\|_{L^\infty}\, dt'}, \tag{6-38}$$

which we may combine with the composition laws (4-6) and (4-7) to get

$$\|g(t)\|_X \leq C e^{V(t)} \left[\|g_0\|_X + \int_0^t \|(F - G)(\tau)\|_X \, d\tau \right], \quad V(t) \triangleq \int_0^t \|\partial_x u_1(\tau)\|_{L^\infty} \, d\tau. \tag{6-39}$$

To estimate $\|\partial_x u_1(t)\|_{L^\infty}$ we come back to (2-4). The first integral term can be restricted to a compact set $[-M, M]$ and thus

$$\left|\text{p.v.} \int_{-M}^{M} \frac{f'(x+y)-f'(x)}{y^2+(f(x+y)-f(x))^2} y\, dy\right| \le 2\int_0^M \frac{\omega_{f'}(y)}{y}\, dy \le C\|f'\|_D.$$

As for the second term, the integral can be restricted to $[-M, M]$ and we simply write

$$\begin{aligned}\text{p.v.} \int_{\mathbb{R}} &\frac{f'(x+y)+f'(x)}{y^2+(f(x+y)+f(x))^2} y\, dy \\ &= \text{p.v.} \int_{-M}^{M} \frac{f'(x+y)-f'(x)}{y^2+(f(x+y)+f(x))^2} y\, dy + \text{p.v.} \int_{\mathbb{R}} \frac{2f'(x)}{y^2+(f(x+y)+f(x))^2} y\, dy.\end{aligned}$$

The first term of the right-hand side is controlled as before:

$$\left|\text{p.v.} \int_{-M}^{M} \frac{f'(x+y)-f'(x)}{y^2+(f(x+y)+f(x))^2} y\, dy\right| \le C\|f'\|_D.$$

However, the last term can be estimated as in the proof of Theorem 5.2(1). One gets in view of (5-7), (5-10) and (5-11)

$$\left|\text{p.v.} \int_{\mathbb{R}} \frac{y}{y^2+(f(x+y)+f(x))^2}\, dy\right| \le C(\|f'\|_{L^\infty}^2 + \|f'\|_{L^\infty}\|f'\|_D + \|f'\|_{L^\infty}).$$

Hence using the embedding $X \hookrightarrow C_K^\star \hookrightarrow L^\infty$ we find

$$\begin{aligned}\|\partial_x u_1(t)\|_{L^\infty} &\le C(\|f'\|_D + \|f'\|_{L^\infty}\|f'\|_D) \\ &\le C(\|f'(t)\|_X + \|f'(t)\|_X^2),\end{aligned} \tag{6-40}$$

which implies

$$V(t) \le Ct(\|f'\|_{L_t^\infty X} + \|f'\|_{L_t^\infty X}^2). \tag{6-41}$$

Using Proposition 6.1 we obtain

$$\|(F-G)(t)\|_X \le C(\|f'(t)\|_X + \|f'(t)\|_X^{17}). \tag{6-42}$$

Plugging (6-41) and (6-42) into (6-39) we obtain

$$\|f'\|_{L_T^\infty X} \le e^{CT(\|f'\|_{L_T^\infty X}+\|f'\|_{L_T^\infty X}^2)}(\|f_0'\|_X + T(\|f'\|_{L_T^\infty X} + \|f'\|_{L_T^\infty X}^{17})).$$

This shows the existence of small T depending only on $\|f_0'\|_X$ and such that

$$\|f'\|_{L_T^\infty X} \le 2\|f_0'\|_X,$$

which ends the proof of the proposition. □

6C. ***Scheme construction of the solutions.*** This section is devoted to the construction of the solutions to (2-3) in short time. Before giving a precise description about the method used here and based on a double regularization, let us explain the main ideas of the strategy. The a priori estimates developed in the previous sections require some rigid properties like the confinement of the support, the positivity of the solution and some nonlinear effects in order to control some singular terms, as was mentioned in Theorem 5.2. So it appears hard to find a linear scheme that respects all of those constraints. The idea is to proceed with a nonlinear double regularization scheme. First, we fix a small parameter $\varepsilon > 0$ used to regularize the singularity of the kernels around the origin, and second we elaborate an iterative nonlinear scheme giving rise to a family of solutions $(f_n^\varepsilon)_n$ that may violate some of the mentioned constraints. With this scheme we are able to derive a priori estimates uniformly with respect to n during a short time $T_\varepsilon > 0$, but this time may shrink to zero as ε goes to zero. By compactness arguments we prove that these approximate solutions $(f_n^\varepsilon)_n$ converge as n goes to infinity to a solution f^ε living in our function space during the time interval $[0, T_\varepsilon]$. Now the function f^ε satisfies a modified nonlinear problem but the important fact is that all the a priori estimates developed in the preceding sections hold uniformly on ε. This allows us by a classical procedure to implement the bootstrap argument and prove that the family $(f^\varepsilon)_\varepsilon$ is actually defined on some time interval $[0, T]$ independently on ε. To conclude it remains to pass to the limit when ε goes to zero and this allows us to construct a solution for our initial problem.

Let us now give more details about this double scheme regularization. Consider the iterative scheme

$$\begin{cases} \partial_t f_{n+1}^\varepsilon + u_1^\varepsilon(f_n^\varepsilon)\partial_x f_{n+1}^\varepsilon = u_2^\varepsilon(f_{n+1}^\varepsilon), \quad n \in \mathbb{N}, \\ f_0^\varepsilon(t,x) = f_0(x), \\ f_{n+1}^\varepsilon(0,x) = f_0(x), \end{cases} \tag{6-43}$$

with

$$\begin{aligned} u_1^\varepsilon(g)(t,x) &\triangleq \frac{1}{2\pi}\chi(x)\int_{|y|\geq\varepsilon}\chi(y)\left[\arctan\left(\frac{g(t,x+y)-g(t,x)}{y}\right)+\arctan\left(\frac{g(t,x+y)+g(t,x)}{y}\right)\right]dy, \\ u_2^\varepsilon(g)(t,x) &\triangleq \frac{1}{4\pi}\int_{|y|\geq\varepsilon}\chi(y)\log\left(\frac{y^2+(g(t,x+y)-g(t,x))^2}{y^2+(g(t,x+y)+g(t,x))^2}\right)dy. \end{aligned} \tag{6-44}$$

The function χ is a positive smooth cut-off function taking the value 1 on some interval $[-M, M]$ such that

$$K_0,\, K_0 - K_0 \subset [-M, M],$$

with K_0 being the convex hull of $\operatorname{supp} f_0$. The function χ is introduced in order to guarantee the convergence of the integrals. We shall see later by using the support structure of the solutions that one can in fact remove this cut-off function. Define

$$\mathcal{E}_T = \{f : f \in L^\infty([0,T]\times\mathbb{R}),\ f' \in L^\infty([0,T], X)\}$$

equipped with the norm

$$\|f\|_{\mathcal{E}_T} = \|f\|_{L^\infty([0,T]\times\mathbb{R})} + \|\partial_x f\|_{L^\infty([0,T],X)},$$

where X denotes the Dini space $C^\star$ or Hölder space $C^s(\mathbb{R})$, $0 < s < 1$, and for simplicity we shall during this part work only with the Hölder space. We intend to explain the approach without giving all the details, because some of them are classical. Using the method of characteristics, one can transform

(6-43) into a fixed-point problem

$$f_{n+1}=\mathcal{N}_n^\varepsilon(f_{n+1}),\quad \text{with } \mathcal{N}(f)(t,x)=f_0(\psi_{n,\varepsilon}^{-1}(t,x))+\int_0^t u_2^\varepsilon(f)(\tau,\psi_{n,\varepsilon}(\tau,\psi_{n,\varepsilon}^{-1}(t,x)))\,d\tau,$$

with $\psi_{n,\varepsilon}$ being the one-dimensional flow associated to $u_1^n(f_n^\varepsilon)$, that is, the solution of the ODE

$$\psi_{n,\varepsilon}(t,x)=x+\int_0^t u_1^n(f_n^\varepsilon)(\tau,\psi_{n,\varepsilon}(\tau,x))\,d\tau. \tag{6-45}$$

It is clear that

$$\|\mathcal{N}(f)(t)\|_{L^\infty}\le\|f_0\|_{L^\infty}+\int_0^t\|u_2^\varepsilon(f)(\tau)\|_{L^\infty}\,d\tau.$$

Applying the elementary inequality, for $a>0,\ b,c\in\mathbb{R}_+$,

$$\left|\log\left(\frac{a+b}{a+c}\right)\right|\le\frac{b+c}{a},$$

we get from (6-44) that

$$|u_2^\varepsilon(f)(t,x)|\le\frac{1}{4\pi}\int_{|y|\ge\varepsilon}\chi(y)\frac{f^2(t,x+y)+f^2(t,x)}{y^2}\,dy\le C\varepsilon^{-2}\|f(t)\|_{L^\infty}^2.$$

It follows that

$$\|\mathcal{N}(f)\|_{L_T^\infty L^\infty}\le\|f_0\|_{L^\infty}+C\varepsilon^{-2}T\|f\|_{L_T^\infty L^\infty}^2. \tag{6-46}$$

We shall move to the estimate of $\|\partial_x\mathcal{N}(f)\|_{L_T^\infty X}$. Let us first start with the estimate of $\|\partial_x\{f_0(\psi_{n,\varepsilon}^{-1})\}\|_{L_T^\infty X}$. By straightforward computations using product laws (4-8), composition laws (4-6) in Hölder spaces and the classical estimate on the flow

$$\|\partial_x\psi_{n,\varepsilon}^{\pm1}\|_{L_T^\infty X}\le Ce^{C\|\partial_x(u_1^\varepsilon(f_n^\varepsilon))\|_{L_T^1L^\infty}}(1+\|\partial_x(u_1^\varepsilon(f_n^\varepsilon))\|_{L_T^1X}),$$

one gets

$$\begin{aligned}\|\partial_x\{f_0(\psi_{n,\varepsilon}^{-1})\}\|_{L_T^\infty X}&\le\|\{\partial_xf_0\}(\psi_{n,\varepsilon}^{-1})\|_{L_T^\infty X}\,\|\partial_x\psi_{n,\varepsilon}^{-1}\|_{L_T^\infty X}\\&\le C\|\partial_xf_0\|_Xe^{C\|\partial_x(u_1^\varepsilon(f_n^\varepsilon))\|_{L_T^1L^\infty}}(1+\|\partial_x(u_1^\varepsilon(f_n^\varepsilon))\|_{L_T^1X}).\end{aligned}$$

Differentiating the expression of $u_1^\varepsilon(f_n^\varepsilon)$ in (6-44) and making standard estimates we get easily

$$\begin{aligned}\|\partial_x\{u_1^\varepsilon(f_n^\varepsilon)(t)\}\|_X&\le C+C\varepsilon^{-1}\|\partial_xf_n^\varepsilon(t)\|_X+C\varepsilon^{-3}\|\partial_xf_n^\varepsilon(t)\|_{L^\infty}\|f_n^\varepsilon(t)\|_X^2\\&\le C+C\varepsilon^{-1}\|f_n^\varepsilon\|_{\mathcal{E}_T}+C\varepsilon^{-3}\|f_n^\varepsilon\|_{\mathcal{E}_T}^3,\end{aligned}$$

where we have used

$$\left\|\frac{1}{y^2+f^2}\right\|_X\le C\|f\|_X^2y^{-4}.$$

Therefore

$$\|\partial_x\{f_0(\psi_{n,\varepsilon}^{-1})\}\|_{L_T^\infty X}\le C\|\partial_xf_0\|_Xe^{CT+C\varepsilon^{-1}T\|f_n^\varepsilon\|_{\mathcal{E}_T}+C\varepsilon^{-3}T\|f_n^\varepsilon\|_{\mathcal{E}_T}^3}, \tag{6-47}$$

$$\|\partial_x\psi_{n,\varepsilon}^{\pm1}\|_{L_T^\infty X}\le Ce^{CT+CT\varepsilon^{-1}\|f_n^\varepsilon\|_{\mathcal{E}_T}+CT\varepsilon^{-3}\|f_n^\varepsilon\|_{\mathcal{E}_T}^3}. \tag{6-48}$$

Similarly we get

$$\begin{aligned}\|\partial_x\{u_2^\varepsilon(f)\}\|_{L^\infty_T X} &\le C\varepsilon^{-2}\|\partial_x f\|_{L^\infty_T X}\|f\|_{L^\infty_T X} + C\varepsilon^{-4}\|\partial_x f\|_{L^\infty_T L^\infty}\|f\|_{L^\infty_T L^\infty}\|f\|^2_{L^\infty_T X}\\ &\le C\varepsilon^{-2}\|f\|^2_{\mathcal{E}_T} + C\varepsilon^{-4}\|f\|^4_{\mathcal{E}_T}.\end{aligned}$$

Combining this estimate with product laws and (6-48) we deduce that

$$\|\partial_x\{u_2^\varepsilon(f)(\tau,\psi_{n,\varepsilon}(\tau,\psi_{n,\varepsilon}^{-1}))\}\|_X \le C(\varepsilon^{-2}\|f\|^2_{\mathcal{E}_T} + \varepsilon^{-4}\|f\|^4_{\mathcal{E}_T})e^{CT+CT\varepsilon^{-1}\|f_n^\varepsilon\|_{\mathcal{E}_T}+CT\varepsilon^{-3}\|f_n^\varepsilon\|^3_{\mathcal{E}_T}}.$$

Putting together this estimate with (6-47) we find that

$$\|\partial_x\mathcal{N}(f)\|_{L^\infty_T X} \le C(\|\partial_x f_0\|_X + T\varepsilon^{-2}\|f\|^2_{\mathcal{E}_T} + T\varepsilon^{-4}\|f\|^4_{\mathcal{E}_T})e^{CT+CT\varepsilon^{-1}\|f_n^\varepsilon\|_{\mathcal{E}_T}+CT\varepsilon^{-3}\|f_n^\varepsilon\|^3_{\mathcal{E}_T}},$$

which yields in view of (6-46)

$$\|\mathcal{N}(f)\|_{\mathcal{E}_T} \le C(\|f_0\|_{L^\infty} + \|\partial_x f_0\|_X + T\varepsilon^{-2}\|f\|^2_{\mathcal{E}_T} + T\varepsilon^{-4}\|f\|^4_{\mathcal{E}_T})e^{CT+CT\varepsilon^{-1}\|f_n^\varepsilon\|_{\mathcal{E}_T}+CT\varepsilon^{-3}\|f_n^\varepsilon\|^3_{\mathcal{E}_T}}.$$

We can assume that $0 < T \le 1$ and then

$$\|\mathcal{N}(f)\|_{\mathcal{E}_T} \le C(\|f_0\|_{L^\infty} + \|\partial_x f_0\|_X + T\varepsilon^{-4}\|f\|^4_{\mathcal{E}_T})e^{CT\varepsilon^{-3}\|f_n^\varepsilon\|^3_{\mathcal{E}_T}}.$$

Consider now the closed ball

$$B = \{f \in \mathcal{E}_T : \|f\|_{\mathcal{E}_T} \le 2C(\|f_0\|_{L^\infty} + \|\partial_x f_0\|_X)e^{CT\varepsilon^{-3}\|f_n^\varepsilon\|^3_{\mathcal{E}_T}}\};$$

if we choose T such that

$$16C^3\varepsilon^{-4}T(\|f_0\|_{L^\infty} + \|\partial_x f_0\|_X)^3 e^{5CT\varepsilon^{-3}\|f_n^\varepsilon\|^3_{\mathcal{E}_T}} \le 1 \tag{6-49}$$

then $\mathcal{N} : B \to B$ is well-defined and proceeding as before we can show under this condition that it is also a contraction. This implies the existence in this ball of a unique solution to the fixed-point problem and so one can construct a solution $f_{n+1}^\varepsilon \in \mathcal{E}_T$ to (6-43) and we have the estimates

$$\text{for all } n \in \mathbb{N}, \quad \|f_{n+1}^\varepsilon\|_{\mathcal{E}_T} \le 2C(\|f_0\|_{L^\infty} + \|\partial_x f_0\|_X)e^{CT\varepsilon^{-3}\|f_n^\varepsilon\|^3_{\mathcal{E}_T}}.$$

Now we select T such that it satisfies also

$$64C^4(\|f_0\|_{L^\infty} + \|\partial_x f_0\|_X)^3 T\varepsilon^{-3} \le \ln 2; \tag{6-50}$$

then we get the uniform estimates

$$\text{for all } n \in \mathbb{N}, \quad \|f_n\|_{\mathcal{E}_T} \le 4C(\|f_0\|_{L^\infty} + \|\partial_x f_0\|_X).$$

In order to satisfy mutually the conditions (6-49) and (6-50) it suffices to take

$$T_\varepsilon := C_0\varepsilon^2, \tag{6-51}$$

with C_0 depending only on $\|f_0\|_{L^\infty} + \|\partial_x f_0\|_X$ such that

$$\text{for all } n \in \mathbb{N}, \quad \|f_n\|_{\mathcal{E}_T} \le 4C(\|f_0\|_{L^\infty} + \|\partial_x f_0\|_X). \tag{6-52}$$

Now we shall check that we can remove the localization in space through the cut-off function χ. To do so, it suffices to get suitable information on the support of (f_n^ε). We shall prove that

$$\text{for all } n\in\mathbb{N},\quad \operatorname{supp} f_n^\varepsilon(t)\subset K_0, \tag{6-53}$$

where K_0 is the convex hull of the support of f_0. Before giving the proof let us assume for a while this property and see how to get rid of the localizations in the velocity fields. From the expression of $u_2^\varepsilon(f_{n+1}^\varepsilon)$ one has

$$\begin{aligned} u_2^\varepsilon(f_{n+1}^\varepsilon)(t,x)=\frac{1}{4\pi}\int_{|y|\geq\varepsilon}\log\biggl(\frac{y^2+(f_{n+1}^\varepsilon(t,x+y)-f_{n+1}^\varepsilon(t,x))^2}{y^2+(f_{n+1}^\varepsilon(t,x+y)+f_{n+1}^\varepsilon(t,x))^2}\biggr)\,dy\\ -\frac{1}{4\pi}\int_{|y|\geq\varepsilon}[1-\chi(y)]\log\biggl(\frac{y^2+(f_{n+1}^\varepsilon(t,x+y)-f_{n+1}^\varepsilon(t,x))^2}{y^2+(f_{n+1}^\varepsilon(t,x+y)+f_{n+1}^\varepsilon(t,x))^2}\biggr)\,dy. \end{aligned}$$

Since for all $x\notin K_0$ we have $f_{n+1}(t,x)=0$, it follows that $u_2^\varepsilon(f_{n+1}^\varepsilon)(t,x)=0$; hence $\operatorname{supp} u_2^\varepsilon(f_{n+1}^\varepsilon)(t)\subset K_0$. Thus for all $x\in K_0$

$$\begin{aligned} &\int_{|y|\geq\varepsilon}[1-\chi(y)]\log\biggl(\frac{y^2+(f_{n+1}^\varepsilon(t,x+y)-f_{n+1}^\varepsilon(t,x))^2}{y^2+(f_{n+1}^\varepsilon(t,x+y)+f_{n+1}^\varepsilon(t,x))^2}\biggr)\,dy\\ &\qquad=\int_{\{|y|\geq|\varepsilon\}\cap K_0-K_0}[1-\chi(y)]\log\biggl(\frac{y^2+(f_{n+1}^\varepsilon(t,x+y)-f_{n+1}^\varepsilon(t,x))^2}{y^2+(f_{n+1}^\varepsilon(t,x+y)+f_{n+1}^\varepsilon(t,x))^2}\biggr)\,dy=0 \end{aligned}$$

because $\chi=1$ on K_0-K_0. Now we claim that in the advection term $u_1^\varepsilon(f_{n+1}^\varepsilon)(t,x)\partial_x f_{n+1}^\varepsilon$ of (6-43) one can remove the cut-off function. Since $\partial_x f_{n+1}^\varepsilon=0$ outside K_0, one gets immediately $\chi(x)\partial_x f_{n+1}^\varepsilon=\partial_x f_{n+1}^\varepsilon$. Similarly one has

$$\begin{aligned} u_1^\varepsilon(g)(t,x)\triangleq\frac{1}{2\pi}\int_{|y|\geq\varepsilon}\biggl[\arctan\Bigl(\frac{g(t,x+y)-g(t,x)}{y}\Bigr)+\arctan\Bigl(\frac{g(t,x+y)+g(t,x)}{y}\Bigr)\biggr]\,dy\\ -\frac{1}{2\pi}\int_{|y|\geq\varepsilon}(1-\chi(y))\biggl[\arctan\Bigl(\frac{g(t,x+y)-g(t,x)}{y}\Bigr)+\arctan\Bigl(\frac{g(t,x+y)+g(t,x)}{y}\Bigr)\biggr]\,dy, \end{aligned}$$

and for $x\in K_0$ it is clear that

$$\begin{aligned} &\int_{|y|\geq\varepsilon}(1-\chi(y))\biggl[\arctan\Bigl(\frac{g(t,x+y)-g(t,x)}{y}\Bigr)+\arctan\Bigl(\frac{g(t,x+y)+g(t,x)}{y}\Bigr)\biggr]\,dy\\ &\quad=\int_{\{y|\geq\varepsilon\}\cap K_0-K_0}(1-\chi(y))\biggl[\arctan\Bigl(\frac{g(t,x+y)-g(t,x)}{y}\Bigr)+\arctan\Bigl(\frac{g(t,x+y)+g(t,x)}{y}\Bigr)\biggr]\,dy=0. \end{aligned}$$

Now let us come back to the proof of (6-53) and provide further qualitative properties. Similarly to the identity (6-25) one obtains

$$u_2^{n+1,\varepsilon}(t,x)=f_{n+1}^\varepsilon(t,x)(1+U_{n+1,\varepsilon}(t,x)),\quad \|U_{n+1,\varepsilon}(t)\|_{L^\infty}\leq C(1+\|f_{n+1}(t)\|_D^6).$$

So following the same lines as in the proof of Proposition 6.2 we get a similar formula to (6-28) which implies the positivity result

$$f_{n+1}(t,x)\geq 0,$$

where we have used in particular that the initial data satisfies $f^{\varepsilon}_{n+1}(0,x)=f_0(x)\geq 0$. Thus we obtain

$$\text{for all } n\in\mathbb{N},\quad f_n(t,x)\geq 0.$$

As $u^2_{n,\varepsilon}(t,x)\leq 0$, following the same lines as the proof of Proposition 6.2 we get the maximum principle

$$\text{for all } n\in\mathbb{N},\quad \|f^{\varepsilon}_n(t)\|_{L^\infty}\leq \|f_0\|_{L^\infty}.$$

The proof of the confinement of the support (6-53) follows exactly the same lines as the proof of Proposition 6.2(3). Now we shall study the strong convergence of the sequence $(f^{\varepsilon}_n)_n$. Set

$$\theta^{\varepsilon}_n(t,x):=f_{n+1}(t,x)-f_n(t,x).$$

Then

$$\partial_t\theta^{\varepsilon}_{n+1}+u^{\varepsilon}_1(f^{\varepsilon}_{n+1})\partial_x\theta^{\varepsilon}_{n+1}=-(u^{\varepsilon}_1(f^{\varepsilon}_{n+1})-u^{\varepsilon}_1(f^{\varepsilon}_n))\partial_x f^{\varepsilon}_{n+1}+u^{\varepsilon}_2(f^{\varepsilon}_{n+2})-u^{\varepsilon}_2(f^{\varepsilon}_{n+1}).$$

According to the mean value theorem one has for $a>0,\ x,y\in\mathbb{R}$,

$$|\arctan(x)-\arctan(y)|\leq|x-y|\quad\text{and}\quad|\log(a+|x|)-\log(a+|y|)|\leq|x-y|a^{-1},$$

which imply

$$\begin{aligned}\|u^{\varepsilon}_1(f^{\varepsilon}_{n+1})(t)-u^{\varepsilon}_1(f^{\varepsilon}_n)(t)\|_{L^\infty}&\leq C\|f^{\varepsilon}_{n+1}(t)-f^{\varepsilon}_n(t)\|_{L^\infty}\int_{|y|\geq\varepsilon}\frac{\chi(y)}{|y|}\,dy\\&\leq C\varepsilon^{-1}\|f^{\varepsilon}_{n+1}(t)-f^{\varepsilon}_n(t)\|_{L^\infty}.\end{aligned}$$

Similarly, we obtain

$$\|u^{\varepsilon}_2(f^{\varepsilon}_{n+1})(t)-u^{\varepsilon}_2(f^{\varepsilon}_n)(t)\|_{L^\infty}\leq C\varepsilon^{-2}(\|f^{\varepsilon}_{n+1}(t)\|_{L^\infty}+\|f^{\varepsilon}_n(t)\|_{L^\infty})\|f^{\varepsilon}_{n+1}(t)-f^{\varepsilon}_n(t)\|_{L^\infty}.$$

Using the uniform estimates (6-52) we get for any $t\in[0,T_\varepsilon]$

$$\|u^{\varepsilon}_2(f^{\varepsilon}_{n+1})(t)-u^{\varepsilon}_2(f^{\varepsilon}_n)(t)\|_{L^\infty}\leq C\|f_0'\|_X\varepsilon^{-2}\|f^{\varepsilon}_{n+1}(t)-f^{\varepsilon}_n(t)\|_{L^\infty},\quad\|\partial_x f^{\varepsilon}_{n+1}(t)\|_{L^\infty}\leq C\|f_0'\|_X.$$

Using the maximum principle for the transport equation allows us to get for any $t\in[0,T_\varepsilon]$

$$\|\theta_{n+1}(t)\|_{L^\infty}\leq C\varepsilon^{-2}\|f_0'\|_X\|\int_0^t[\|\theta_{n+1}(\tau)\|_{L^\infty}+\|\theta_n(\tau)\|_{L^\infty}]\,d\tau.$$

By virtue of Gronwall's lemma one finds that for any $t\in[0,T_\varepsilon]$

$$\|\theta_{n+1}(t)\|_{L^\infty}\leq e^{C\varepsilon^{-2}\|f_0'\|_X T_\varepsilon}\int_0^t\|\theta_n(\tau)\|_{L^\infty}\,d\tau.$$

Hence we obtain in view of (6-51)

$$\|\theta_{n+1}(t)\|_{L^\infty}\leq C_0\int_0^t\|\theta_n(\tau)\|_{L^\infty}\,d\tau.$$

By induction we find

$$\text{for all } n\in\mathbb{N},\text{ for all } t\in[0,T_\varepsilon],\quad\|\theta_n\|_{L^\infty_t L^\infty}\leq C_0^n\frac{t^n}{n!}\|\theta_0\|_{L^\infty_t L^\infty}.$$

This implies the convergence of the series

$$\sum_{n\in\mathbb{N}} \|\theta_{n+1}\|_{L^\infty_{T_\varepsilon}L^\infty} < \infty.$$

Therefore $(f_n^\varepsilon)_n$ converges strongly in $L^\infty_{T_\varepsilon}L^\infty$ to an element $f^\varepsilon \in L^\infty_{T_\varepsilon}L^\infty$. From the uniform estimates (6-52) we deduce that $f^\varepsilon \in \mathcal{E}_{T_\varepsilon}$. This allows us to pass to the limit in (6-43) and obtain that f^ε is a solution to

$$\begin{cases} \partial_t f^\varepsilon + u_1^\varepsilon(f^\varepsilon)\partial_x f^\varepsilon = u_2^\varepsilon(f^\varepsilon), \\ f_0^\varepsilon(t,x) = f_0(x), \end{cases} \tag{6-54}$$

with

$$\begin{aligned} u_1^\varepsilon(f^\varepsilon)(t,x) &\triangleq \frac{1}{2\pi}\int_{|y|\geq\varepsilon}\left[\arctan\left(\frac{f^\varepsilon(t,x+y)-f^\varepsilon(t,x)}{y}\right)+\arctan\left(\frac{f^\varepsilon(t,x+y)+f^\varepsilon(t,x)}{y}\right)\right]dy, \\ u_2^\varepsilon(f^\varepsilon)(t,x) &\triangleq \frac{1}{4\pi}\int_{|y|\geq\varepsilon}\log\left(\frac{y^2+(f^\varepsilon(t,x+y)-f^\varepsilon(t,x))^2}{y^2+(f^\varepsilon(t,x+y)+f^\varepsilon(t,x))^2}\right)dy. \end{aligned} \tag{6-55}$$

Now, the proofs used to get the a priori estimates can be adapted to (6-54) supplemented with (6-55). For instance the a priori estimates obtained in Proposition 6.2 hold for the modified equation (6-54) independently on vanishing ε. In particular one can bound uniformly in ε the solution f^ε in the space X_{T_ε} and therefore T_ε is not maximal and by a standard bootstrap argument we can continue the solution up to the local time T constructed in Proposition 6.2. It follows that f^ε belongs to $\mathcal{E}_T$ uniformly with respect to small ε. This yields according once again to Proposition 6.2 and the inequalities (6-25) and (6-40)

$$\sup_{\varepsilon\in[0,1]} \|\partial_t f^\varepsilon\|_{L^\infty_T L^\infty} \leq \|u_1^\varepsilon(f^\varepsilon)\|_{L^\infty_T L^\infty}\|\partial_x f^\varepsilon\|_{L^\infty_T L^\infty} + \|u_2^\varepsilon(f^\varepsilon)\|_{L^\infty_T L^\infty} \leq C_0,$$

and C_0 is a constant depending on the size of the initial data. Now from the compact embedding $C_K^s \to C_b$ and Ascoli's lemma we deduce that up to a subsequence (f^ε) converges strongly in $L^\infty_T L^\infty$ to some element f which belongs in turn to $\mathcal{E}_T$. This allows us to pass to the limit in (6-54) and (6-55) and find a solution to the initial value problem (6-43). We point out that by working more one may obtain the strong convergence of the full sequence (f^ε) to f. Note finally that the uniqueness follows easily from the arguments used to prove that (θ_n) is a Cauchy sequence.

7. Global well-posedness

We are concerned here with the global existence of strong solutions already constructed in Theorem 2.1. This will be established under a smallness condition on the initial data and it is probable that for arbitrary large initial data the graph structure might be destroyed in finite time. The basic ingredient which allows us to balance the energy amplification during the time evolution is a damping effect generated by the source terms. Note that this damping effect is plausible from the graph equation (2-1) according to the identity (6-34). However, as we shall see in the next section, it is quite complicated to extend this behavior for higher regularity at the level of the resolution space due to the existence of a linear part in the source term governing the motion of the slope (2-3). This part could in general bring an amplification in time of the energy. To circumvent this difficulty we establish a weakly damping property of the linearized operator associated to the source term that we combine with the time decay of the solution for weak regularity using an interpolation argument.

7A. ***Weak and strong damping behavior of the source term.*** Note from Proposition 6.1 that F does not contribute at the linear level, which is not the case of the functional G. We shall prove that actually there is no linear contribution for G. This will be done by establishing a damping property that occurs at least at the linear level. This is described by the following proposition.

Proposition 7.1. *Let K be a compact set of $\mathbb{R}$ and $s\in(0,1)$; then for any $f\in C^s_K$ we have the decomposition*

$$G(x)=2\pi f'(x)+L(x)+N(x),$$

with

$$\|L\|_s\le 2\pi(\|f'\|_s+2\|f'\|_{L^\infty})+C\|f'\|^s_{L^\infty}\|f'\|_s \quad \textit{and} \quad \|N\|_s\le C\|f'\|_D^{\frac{1}{3}}(\|f'\|_s+\|f'\|_s^{16}),$$

where $C>0$ is a constant depending only on K and s. Moreover,

$$\|L\|_{L^\infty}\le C\min(\|f\|^s_{L^\infty}\|f'\|_s,\|f'\|_{L^\infty}) \quad \textit{and} \quad \|N\|_{L^\infty}\le C\|f'\|_{L^\infty}(\|f'\|_D+\|f'\|_D^3).$$

Proof. In view of (6-1), (6-2), (6-4), (6-16), (6-17) and (6-24) one gets

$$G(x)=G_{11}(x)+H(x),\quad H=G_{12}+G_2,$$

with

$$\|H\|_s\le C\|f'\|_D^{\frac{1}{3}}(\|f'\|_s+\|f'\|_s^{16}). \tag{7-1}$$

Note also that from (6-3) and (6-11) we get

$$\|H\|_{L^\infty}\le C\|f'\|_{L^\infty}(\|f'\|_s+\|f'\|_s^3). \tag{7-2}$$

Now from (6-5) we get

$$G_{11}(x)=2\int_{\mathbb{R}}\frac{f'(x)+f'(x+f(x)z)}{\varphi(x,z)}\,dz,$$

with

$$\varphi(x,z)=z^2+\left(2+z\int_0^1 f'(x+\theta f(x)z)\,d\theta\right)^2.$$

We shall split again G_{11} as follows:

$$\begin{aligned}G_{11}(x)&=2\int_{\mathbb{R}}\frac{f'(x)+f'(x+f(x)z)}{z^2+4}\,dz-2\int_{\mathbb{R}}\frac{[f'(x)+f'(x+f(x)z)]\psi(x,z)}{\varphi(x,z)(z^2+4)}\,dz\\&\triangleq\mathcal{L}(x)+\mathcal{N}(x),\end{aligned}$$

with

$$\psi(x,z)\triangleq 4z\int_0^1 f'(x+\theta f(x)z)\,d\theta+z^2\left(\int_0^1 f'(x+\theta f(x)z)\,d\theta\right)^2.$$

From (5-22) one gets

$$\|\mathcal{N}\|_{L^\infty}\le C\|f'\|^2_{L^\infty}(1+\|f'\|^3_{L^\infty}). \tag{7-3}$$

Using the product law (4-8) we get

$$\begin{aligned}\left\|\frac{[f'+f'\circ(\mathrm{Id}+zf)]\psi(\cdot,z)}{\varphi(\cdot,z)}\right\|_s &\le 2\|f'\|_{L^\infty}\|\psi(\cdot,z)\|_{L^\infty}\|1/\varphi(\cdot,z)\|_s\\ &\quad+2\|f'\|_{L^\infty}\|\psi(\cdot,z)\|_s\|1/\varphi(\cdot,z)\|_{L^\infty}\\ &\quad\quad+(\|f'\|_s+\|f'\circ(\mathrm{Id}+zf)\|_s)\|\psi(\cdot,z)\|_{L^\infty}\|1/\varphi(\cdot,z)\|_{L^\infty}.\end{aligned}$$

In addition, it is clear that

$$\|\psi(\cdot,z)\|_{L^\infty}\le 4|z|\,\|f'\|_{L^\infty}+|z|^2\,\|f'\|_{L^\infty}^2.$$

Performing the composition law (4-6) we deduce that

$$\|\psi(\cdot,z)\|_s\le C|z|\,\|f'\|_s(1+|z|^s\,\|f'\|_{L^\infty}^s)+C|z|^2\,\|f'\|_{L^\infty}\,\|f'\|_s(1+|z|^s\,\|f'\|_{L^\infty}^s).$$

Combining this latter estimate with (6-9) and (5-22) yields

$$\left\|\frac{[f'+f'\circ(\mathrm{Id}+zf)]\psi(\cdot,z)}{\varphi(\cdot,z)}\right\|_s\le C\,\|f'\|_{L^\infty}\,\|f'\|_s(1+\|f'\|_{L^\infty}^{7+s})(1+|z|^s).$$

Hence we get according to the embedding $C^s_K\hookrightarrow L^\infty$

$$\begin{aligned}\|\mathcal{N}\|_s&\le C\,\|f'\|_{L^\infty}\,\|f'\|_s(1+\|f'\|_{L^\infty}^{7+s})\\ &\le C\,\|f'\|_{L^\infty}^{\frac13}(\|f'\|_s^{\frac53}+\|f'\|_s^{\frac{26}{3}+s})\\ &\le C\,\|f'\|_{L^\infty}^{\frac13}(\|f'\|_s+\|f'\|_s^{10}).\end{aligned}$$

Setting $N=\mathcal{N}+H$ and combining the latter estimate with (7-1) we find the desired estimate for N stated in the proposition. Putting together (7-2) and (7-3) combined with Sobolev embedding we find

$$\|N\|_{L^\infty}\le C\,\|f'\|_{L^\infty}(\|f'\|_s+\|f'\|_s^4).$$

Coming back to $\mathcal{L}$ one may write

$$\mathcal{L}(x)=4f'(x)\int_{\mathbb{R}}\frac{1}{z^2+4}\,dz+2\int_{\mathbb{R}}\frac{f'(x+f(x)z)-f'(x)}{z^2+4}\,dz\triangleq 2\pi f'(x)+L(x). \tag{7-4}$$

To estimate L in C^s we simply write

$$\|L\|_s\le 2\int_{\mathbb{R}}\frac{\|f'\circ(\mathrm{Id}+zf)\|_s+\|f'\|_s}{z^2+4}\,dz.$$

Combined with (4-6) we find

$$\begin{aligned}\|f'\circ(\mathrm{Id}+zf)\|_s&\le(\|f'\|_s+2\|f'\|_{L^\infty})(1+|z|\|f'\|_{L^\infty})^s\\ &\le(\|f'\|_s+2\|f'\|_{L^\infty})(1+|z|^s\|f'\|_{L^\infty}^s),\end{aligned}$$

where in the last line we have used the inequality, for all $s\in(0,1)$, for all $x,y\ge 0$ one has

$$(x+y)^s\le x^s+y^s.$$

Using (4-2), it follows that

$$\|L\|_s \le 2\pi(\|f'\|_s + 2\|f'\|_{L^\infty}) + C\|f'\|_s\|f'\|_{L^\infty}^s.$$

The estimate of L in L^∞ is easier and one gets according to (7-4),

$$|L(x)| \le 2|f(x)|^s\|f'\|_s \int_{\mathbb{R}} \frac{|z|^s}{z^2+4}\,dz \le C|f(x)|^s\|f'\|_s.$$

Therefore we obtain

$$\|L\|_{L^\infty} \le C\|f\|_{L^\infty}^s\|f'\|_s.$$

We point out that we have obviously

$$\|L\|_{L^\infty} \le 2\pi\|f'\|_{L^\infty}.$$

It follows that

$$\|L\|_{L^\infty} \le C\min(\|f\|_{L^\infty}^s\|f'\|_s, \|f'\|_{L^\infty}). \tag{7-5}$$

This completes the proof of Proposition 7.1. ☐

7B. *Global a priori estimates.* The main goal of this section is to show how we may use the weakly damping effect of the source terms stated in Proposition 7.1 in order to get global a priori estimates when the initial data is small enough. The basic result reads as follows.

Proposition 7.2. *Let K be a compact set of $\mathbb{R}$ and $s \in (0,1)$. There exists a constant $\varepsilon > 0$ such that if $\|f_0'\|_s \le \varepsilon$ then* (2-1) *admits a unique global solution*

$$f' \in L^\infty(\mathbb{R}_+; C_K^s).$$

Moreover, there exists a constant C_0 depending on the initial data such that

$$\textit{for all } t \ge 0, \quad \|f'(t)\|_{L^\infty} \le C_0 e^{-t}.$$

Proof. According to the decomposition of Proposition 7.1 combined with (2-3) and (2-5) we get that $g = \partial_x f$ satisfies the equation

$$\partial_t g(t,x) + u_1(t,x)\,\partial_1 g(t,x) + g(t,x) = \mathcal{R}(t,x), \quad \mathcal{R} \triangleq \tfrac{1}{2\pi}(F - L - N). \tag{7-6}$$

Using Propositions 6.1 and 7.1 combined with the (4-2) we find

$$\|\mathcal{R}\|_s \le \|f'\|_s + 2\|f'\|_{L^\infty} + C\|f'\|_D(\|f'\|_s + \|f'\|_s^3) + C\|f'\|_{L^\infty}^s\|f'\|_s + C\|f'\|_{L^\infty}^{\frac{1}{3}}(\|f'\|_s + \|f'\|_s^{16}).$$

The embedding $C_K^{\frac{s}{2}} \subset C_K^\star$ combined with interpolation inequalities in Hölder spaces yields

$$\|f'\|_D \le C\|f'\|_{L^\infty}^{\frac{1}{2}}\|f'\|_s^{\frac{1}{2}}. \tag{7-7}$$

Set $s_0 = \min\big(s, \frac{1}{3}\big)$; then it is easy to get

$$\|\mathcal{R}\|_s \le \|f'\|_s + 2\|f'\|_{L^\infty} + C\|f'\|_{L^\infty}^{s_0}(\|f'\|_s + \|f'\|_s^{16}). \tag{7-8}$$

Let $h(t,x) \triangleq g(t,\psi(t,x))$, where ψ is the flow introduced in (6-26). Then it is obvious that

$$\partial_t h(t,x) + h(t,x) = \mathcal{R}(t,\psi(t,x)).$$

This allows us to deduce the Duhamel integral representation

$$e^t g(t,x) = g_0(\psi^{-1}(t,x)) + \int_0^t e^\tau \mathcal{R}(\tau,\psi(\tau,\psi^{-1}(t,x)))\,d\tau.$$

Thus

$$e^t\|g(t)\|_s \le \|g_0(\psi^{-1}(t))\|_s + \int_0^t e^\tau \|\mathcal{R}(\tau,\psi(\tau,\psi^{-1}(t)))\|_s\,d\tau.$$

According to (6-38) and (4-6) we obtain

$$\|g_0(\psi^{-1}(t))\|_s \le C\|g_0\|_s e^{V(t)}, \quad V(t) = \int_0^t \|\partial_x u_1(\tau)\|_{L^\infty}\,d\tau$$

and

$$\|\mathcal{R}(\tau,\psi(\tau,\psi^{-1}(t)))\|_s \le (\|\mathcal{R}(\tau)\|_s + 2\|\mathcal{R}(\tau)\|_{L^\infty})e^{V(t)-V(\tau)}.$$

Note that the estimate of $\mathcal{R}$ in C^s has been already stated in (7-8). However to get a suitable estimate in L^∞ we use Propositions 6.1 and 7.1 combined with Sobolev embedding,

$$\begin{aligned}\|\mathcal{R}(t)\|_{L^\infty} &\le C\|f'(t)\|_{L^\infty}(\|f'(t)\|_D + \|f'(t)\|_D^3) + C\min(\|f(t)\|_{L^\infty}^s\|f'(t)\|_s, \|f'(t)\|_{L^\infty})\\ &\le C(\|f'(t)\|_{L^\infty} + \|f'(t)\|_{L^\infty}^s)(\|f'(t)\|_s + \|f'(t)\|_s^3)\\ &\le C\|f'(t)\|_{L^\infty}^{s_0}(\|f'(t)\|_s + \|f'(t)\|_s^4).\end{aligned} \tag{7-9}$$

It follows that

$$\begin{aligned}&\|\mathcal{R}(\tau,\psi(\tau,\psi^{-1}(t)))\|_s\\ &\qquad\le (\|f'(\tau)\|_s + 2\|f'(\tau)\|_{L^\infty})e^{V(t)-V(\tau)} + C\|f'(\tau)\|_{L^\infty}^{s_0}(\|f'(\tau)\|_s + \|f'(\tau)\|_s^{16})e^{V(t)-V(\tau)}.\end{aligned}$$

Set $K(t) = e^{-V(t)}e^t\|f'(t)\|_s$ and

$$S(t) = Ce^te^{-V(t)}(\|f'(t)\|_{L^\infty} + \|f'(t)\|_{L^\infty}^{s_0}(\|f'(t)\|_s + \|f'(t)\|_s^{16})).$$

Then

$$K(t) \le CK(0) + \int_0^t K(\tau)\,d\tau + \int_0^t S(\tau)\,d\tau.$$

By virtue of Gronwall's lemma we deduce that

$$K(t) \le Ce^tK(0) + \int_0^t e^{t-\tau}S(\tau)\,d\tau.$$

This implies

$$\begin{aligned}&\|f'(t)\|_s \le Ce^{V(t)}\|f_0'\|_s\\ &\qquad + Ce^{V(t)}\int_0^t \|f'(\tau)\|_{L^\infty}\,d\tau + e^{V(t)}\int_0^t \|f'(\tau)\|_{L^\infty}^{s_0}(\|f'(\tau)\|_s + \|f'(\tau)\|_s^{16})\,d\tau.\end{aligned} \tag{7-10}$$

Combining the interpolation inequality

$$\|f_t'\|_{L^\infty} \le C\|f_t\|_{L^1}^{\frac{s}{2+s}}\|f_t'\|_s^{\frac{2}{2+s}},$$

with Proposition 6.2(2) we obtain

$$\|f'(t)\|_{L^\infty} \le Ce^{-\frac{s}{2+s}t}\|f_0\|_{L^1}^{\frac{s}{2+s}}\|f'(t)\|_s^{\frac{2}{2+s}}. \tag{7-11}$$

Plugging this estimate into (6-40) we find

$$\|\partial_x u_1(t)\|_{L^\infty} \le Ce^{-\frac{s}{2+s}t}\|f_0\|_{L^1}^{\frac{s}{2+s}}\big(\|f'(t)\|_s^{\frac{2}{2+s}} + \|f'(t)\|_s^{\frac{4+s}{2+s}}\big). \tag{7-12}$$

It is quite obvious from (4-2) and the compactness of the support that

$$\|f_0\|_{L^1} \le C\|f_0'\|_s,$$

with C a constant depending on the size of the support of f_0. Set

$$\rho(T) = \sup_{t\in[0,T]}\|f'(t)\|_s.$$

Then combining (7-10) with (7-11) and (7-12) yields

$$\rho(T) \le Ce^{C\|f_0'\|_s^{\frac{s}{2+s}}([\rho(T)]^{\frac{2}{2+s}}+[\rho(T)]^{\frac{4+s}{2+s}})}\mu(T),$$

with

$$\mu(T) = \|f_0'\|_s + \|f_0'\|_s^{\frac{s}{2+s}}[\rho(T)]^{\frac{2}{2+s}} + \|f_0'\|_s^{\frac{ss_0}{2+s}}[\rho(T)]^{\frac{2s_0}{2+s}}(\rho(T) + [\rho(T)]^{16}).$$

This implies the existence of small number $\varepsilon > 0$ depending only on C, and thus on the size of the support of f_0, such that

$$\|f_0'\|_s \le \varepsilon \quad\Longrightarrow\quad \text{for all } T > 0,\quad \rho(T) \le \delta(\|f_0'\|_s), \tag{7-13}$$

with $\lim_{x\to 0}\delta(x) = 0$. This gives the global a priori estimates.

What is left is to establish the precise time decay of $\|f'(t)\|_{L^\infty}$ stated in Proposition 7.2. From (7-6) it is easy to establish the following estimate using the method of characteristics:

$$\|g(t)\|_{L^\infty} \le e^{-t}\|g_0\|_{L^\infty} + \int_0^t e^{-(t-\tau)}\|\mathcal{R}(\tau)\|_{L^\infty}\,d\tau. \tag{7-14}$$

According to (7-9) we obtain

$$e^t\|f'(t)\|_{L^\infty} \le \|f_0'\|_{L^\infty} + C\int_0^t e^\tau\|f'(\tau)\|_{L^\infty}(\|f'(\tau)\|_D + \|f'\|_D^3)\,d\tau.$$

Using Gronwall's lemma we obtain

$$e^t\|f'(t)\|_{L^\infty} \le \|f_0'\|_{L^\infty}e^{W(t)}, \quad W(t) = C\int_0^t(\|f'(\tau)\|_D + \|f'\|_D^3)\,d\tau.$$

Putting together (7-7) with (7-11) we obtain

$$\|f'(t)\|_D \le Ce^{-\frac{s}{4+2s}t}\|f_0\|_{L^1}^{\frac{s}{4+2s}}\|f'(t)\|_s^{\frac{4+s}{4+2s}}.$$

Hence we deduce from (7-13) that

$$\text{for all } t \geq 0, \quad W(t) \leq C_0,$$

and therefore

$$\text{for all } t \geq 0, \quad \|f'(t)\|_{L^\infty} \leq C_0 e^{-t}, \quad \|f'(t)\|_D \leq C_0 e^{-\frac{s}{4+2s}t} \tag{7-15}$$

for a suitable constant C_0 depending on the initial data. Inserting these estimates into (7-9) we obtain

$$\text{for all } t \geq 0, \quad \|\mathcal{R}(t)\|_{L^\infty} \leq C_0 e^{-t}. \tag{7-16}$$

Since f_t is compactly supported in a fixed compact set

$$\text{for all } t \geq 0, \quad \|f(t)\|_{L^\infty} \leq C_1 e^{-t}. \tag{7-17}$$

Finally, we point out that all the constants involved in the preceding estimates are time independent. Indeed, they are related to the support of f_t which is confined in the convex hull of the support of the initial data, as has been stated in Proposition 6.2(3). □

8. Scattering and collapse to singular measure

The aim of the last section is to analyze and identify the long time behavior of the global solutions stated in Theorem 2.2. It attempts to investigate the time evolution of the probability measure

$$dP_t(x) \triangleq \frac{\rho(t,x)}{\|\rho_t\|_{L^1}}\, dA(x) = e^t \mathbf{1}_{D_t}(x)\, dA(x),$$

where dA denotes the usual Lebesgue measure. Note that without loss of generality we have assumed in the last line that $\|\rho_0\|_{L^1} = 1$. As we shall see, this measure converges weakly as t goes to infinity to a probability measure concentrated on the real line and absolutely continuous with respect to Lebesgue measure on the real line. The description of the density and the support of this limiting measure will be the subject of the next two sections.

8A. ***Structure of the singular measure.*** In this section we shall prove the part of Theorem 2.2 dealing with the weak convergence of the measure dP_t when t goes to ∞. First, it is obvious that the probability measure dP_t is absolutely continuous with respect to the Lebesgue measure. The convergence of the family $\{dP_t : t \geq 0\}$ will be done in a weak sense as follows. Let $\varphi \in \mathcal{D}(\mathbb{R}^2)$ be a test function; one can write using Fubini's theorem

$$I_t \triangleq \int_{\mathbb{R}^2} \varphi(x,y)\, dP_t = e^t \int_{\mathbb{R}} \int_{-f_t(x)}^{f_t(x)} \varphi(x,y)\, dy.$$

According to Taylor expansion in the second variable one gets

$$\text{for all } (x,y) \in \mathbb{R}^2, \quad \varphi(x,y) = \varphi(x,0) + y\psi(x,y) \quad \text{and} \quad \|\psi\|_{L^\infty} \leq C.$$

This implies

$$I_t = 2e^t \int_{\mathbb{R}} f_t(x)\varphi(x,0)\, dx + I_t^1, \quad I_t^1 \triangleq e^t \int_{\mathbb{R}} \int_{-f_t(x)}^{f_t(x)} y\psi(x,y)\, dy. \tag{8-1}$$

We shall check that the term I_t^1 does not contribute in the limiting behavior. Actually it vanishes for t going to infinity. Indeed,

$$|I_t^1| \le e^t \|\psi\|_{L^\infty} \int_{\mathbb{R}} [f_t(x)]^2 \, dx.$$

Using (7-17) and the localization of the support of f_t in the convex hull of the initial support, we deduce that

$$|I_t^1| \le Ce^{-t},$$

and thus

$$\lim_{t\to\infty} I_t^1 = 0.$$

Combining (2-1), (6-34), (7-15), (7-17) and (7-13) we deduce that

$$\partial_t f(t,x) + u_1 \partial_x f(t,x) + f(t,x) = -f(t,x)R(t,x), \tag{8-2}$$

with

$$\|R(t)\|_{L^\infty} \le \|f'(t)\|_D (1 + \|f'(t)\|_\infty^5) \le Ce^{-\frac{s}{4+2s}t}. \tag{8-3}$$

From the method of characteristics developed in studying (7-6) we get the representation

$$e^t f(t,\psi(t,x)) = f_0(x) e^{\int_0^t R(\tau,\psi(\tau,x))\, d\tau}. \tag{8-4}$$

From the integrability property (8-3) we deduce that $\{e^t f(t,\psi(t))\}$ converges uniformly as t goes to ∞ to the positive function

$$x \mapsto f_0(x) e^{\int_0^\infty R(\tau,\psi(\tau,x))\, d\tau} \triangleq R_2(x). \tag{8-5}$$

More precisely, using straightforward computations we easily get

$$\|e^t f_t \circ \psi(t) - R_2\|_{L^\infty} \le \|R_2\|_{L^\infty} \int_t^\infty \|R(\tau)\|_{L^\infty}\, d\tau \le Ce^{-\frac{s}{4+2s}t}. \tag{8-6}$$

The next goal is prove that the flow $\psi(t)$ converges uniformly as t goes to infinity to some homeomorphism $\psi_\infty : \mathbb{R} \to \mathbb{R}$ which belongs to the bi-Lipschitz class. First, recall from the definition (6-26) that

$$\psi(t,x) = x + \int_0^t u_1(\tau, \psi(\tau,x))\, d\tau.$$

Recall from Section 2 that $u_1(x) = v_1(x, f(x))$ and the velocity is computed from the density ρ according to the second equation of (1-2). Hence we get

$$\|u_1(t)\|_{L^\infty} \le \|\Delta^{-1}\nabla\rho\|_{L^\infty}.$$

Now using the classical interpolation inequality

$$\|\Delta^{-1}\nabla\rho\|_{L^\infty} \le C\|\rho\|_{L^1}^{\frac{1}{2}} \|\rho\|_{L^\infty}^{\frac{1}{2}}$$

combined with the decay rate stated in Proposition 6.2(2) we deduce that

$$\|u_1(t)\|_{L^\infty} \le Ce^{-\frac{t}{2}}. \tag{8-7}$$

Consequently, it follows that $\psi(t)$ converges uniformly to the function

$$\psi_\infty(x) \triangleq x + \int_0^\infty u_1(\tau, \psi(\tau, x))\, d\tau.$$

More precisely, we have

$$\|\psi(t) - \psi_\infty\|_{L^\infty} \le \int_t^\infty \|u_1(\tau)\|_{L^\infty}\, d\tau \le Ce^{-\frac{t}{2}}. \tag{8-8}$$

It remains to check that ψ_∞ is bi-Lipschitz. First we know that

$$\|\partial_x \psi(t)\|_{L^\infty} \le e^{V(t)}, \quad V(t) = \int_0^t \|\partial_x u_1(\tau)\|_{L^\infty}\, d\tau.$$

Using (7-12) and (7-13) we deduce that

$$\text{for all } t \ge 0,\ \|\partial_x \psi(\tau)\|_{L^\infty} \le C, \quad \|\partial_x u_1(t)\|_{L^\infty} \le C\varepsilon^{\frac{s}{2+s}} e^{-\frac{s}{2+s}t}. \tag{8-9}$$

Differentiating ψ_∞ and using the triangle inequality we get

$$1 - \int_0^\infty \|\partial_x u_1(\tau)\|_{L^\infty} \|\partial_x \psi(\tau)\|_{L^\infty}\, d\tau \le \psi'_\infty(x) \le 1 + \int_0^\infty \|\partial_x u_1(\tau)\|_{L^\infty} \|\partial_x \psi(\tau)\|_{L^\infty}\, d\tau.$$

Therefore we obtain

$$\text{for all } x \in \mathbb{R}, \quad 1 - C\varepsilon^{\frac{s}{2+s}} \le \psi'_\infty(x) \le 1 + C\varepsilon^{\frac{s}{2+s}}.$$

Taking ε small enough, meaning that the initial data is very small, we get

$$\text{for all } x \in \mathbb{R}, \quad \tfrac{1}{2} \le \psi'_\infty(x) \le \tfrac{3}{2}. \tag{8-10}$$

This shows that ψ_∞ is a bi-Lipschitz function from $\mathbb{R}$ to $\mathbb{R}$. Furthermore, it is obvious that

$$\psi_\infty(x) = \psi(t, x) + \int_t^\infty u_1(\tau, \psi(\tau, x))\, d\tau,$$

and hence

$$\psi_\infty(\psi^{-1}(t, x)) = x + \int_t^\infty u_1(\tau, \psi(\tau, \psi^{-1}(t, x)))\, d\tau.$$

Combining this identity with $\psi_\infty \circ \psi_\infty^{-1} = \mathrm{Id}$ and (8-7) yields

$$|\psi_\infty(\psi^{-1}(t, x)) - \psi_\infty(\psi_\infty^{-1} x)| \le \int_t^\infty \|u_1(\tau)\|_{L^\infty}\, d\tau \le Ce^{-\frac{t}{2}}.$$

Applying (8-10) we deduce that

$$\|\psi^{-1}(t) - \psi_\infty^{-1}\|_{L^\infty} \le Ce^{-\frac{t}{2}}.$$

This shows that $\psi^{-1}(t)$ converges uniformly to ψ_∞^{-1} with an exponential rate. Set

$$\Phi = R_2 \circ \psi_\infty^{-1} \tag{8-11}$$

and assume for a while that R_2 belongs to C^α for any $\alpha \in (0,1)$; then we deduce from the preceding estimates, especially (8-6) and (8-4), that

$$\begin{aligned}\|e^t f(t)-\Phi\|_{L^\infty} &\le \|e^t f(t)-R_2\circ\psi^{-1}(t)\|_{L^\infty}+\|R_2\circ\psi^{-1}(t)-R_2\circ\psi_\infty^{-1}\|_{L^\infty}\\ &\le Ce^{-\frac{s}{4+2s}t}+\|R_2\|_\alpha\|\psi^{-1}(t)-\psi_\infty^{-1}\|_{L^\infty}^\alpha\\ &\le Ce^{-\frac{s}{4+2s}t}+Ce^{-\alpha\frac{t}{2}}.\end{aligned}$$

Taking $\alpha=\frac{2s}{4+2s}$ we get

$$\|e^t f(t)-\Phi\|_{L^\infty}\le Ce^{-\frac{s}{4+2s}t}. \tag{8-12}$$

Let us now check that R_2 belongs to C^α for any $\alpha\in(0,1)$. For this goal we shall express differently the function R_2. Set $R_1(t,x)=-f(t,x)R(t,x)$; then from the method of characteristics the solution to (8-2) may be recovered as follows:

$$e^t f(t,\psi(t,x))=f_0(x)+\int_0^t e^\tau R_1(\tau,\psi(\tau,x))\,d\tau.$$

Putting together (8-3) and (7-17) we deduce that

$$\|R_1(\tau,\psi(\tau))\|_{L^\infty}\le Ce^{-\frac{4+3s}{4+2s}t}. \tag{8-13}$$

Therefore we find the identity

$$R_2(x)=f_0(x)+\int_0^\infty e^\tau R_1(\tau,\psi(\tau,x))\,d\tau. \tag{8-14}$$

We shall now study the regularity of R_2 through the use of this representation.

Differentiating (8-2) in x and comparing it to (7-6) we get the identity

$$\partial_x R_1(t,x)=\mathcal{R}(t,x)+\partial_x u_1(t,x)\,\partial_x f(t,x).$$

Using (7-15), (7-16) and (8-9) we find

$$\text{for all } t\ge 0,\quad \|\partial_x R_1(t)\|_{L^\infty}\le Ce^{-t}.$$

Combining this latter estimate with the Leibniz formula and (8-9) implies

$$\text{for all } t\ge 0,\quad \|\partial_x(R_1(t,\psi(t,\cdot)))\|_{L^\infty}\le Ce^{-t}. \tag{8-15}$$

It suffices now to apply the following classical interpolation inequality: for any $\alpha\in(0,1)$ there exists $C>0$ such that

$$\|h\|_\alpha\le C\|h\|_{L^\infty}^{1-\alpha}\|h'\|_{L^\infty}^\alpha,$$

which implies that according to (8-13) and (8-15)

$$\text{for all } t\ge 0,\quad \|R_1(t,\psi(t,\cdot))\|_\alpha\le Ce^{-t}e^{-t(1-\alpha)\frac{s}{4+2s}}. \tag{8-16}$$

Returning to the identity (8-14), one obtains in view of (8-16)

$$\|R_2\|_\alpha\le\|f_0\|_\alpha+\int_0^\infty e^\tau\|R_1(\tau,\psi(\tau,\cdot))\|_\alpha\,d\tau\le C$$

for any $\alpha \in (0,1)$. As an immediate consequence of (8-11), (8-10) and (4-6) we find that Φ belongs to C^α for any $\alpha \in (0,1)$. We guess the profile Φ to keep the same regularity as f_0, that is, in C^{1+s}, but this could require much more refined analysis.

Now coming back to (8-1) we find in view of (8-12) and the Lebesgue theorem

$$\lim_{t\to\infty} I(t) = 2\int_{\mathbb{R}} \Phi(x)\varphi(x,0)\,dx.$$

This is equivalent to writing in the weak sense

$$\lim_{t\to\infty} dP_t = 2\Phi\,\delta_{\mathbb{R}\otimes\{0\}} \triangleq dP_\infty. \tag{8-17}$$

Now we shall discuss some properties of Φ. From (8-5) and (8-11) we have

$$\operatorname{supp}\Phi = \psi_\infty(K_0), \quad K_0 = \operatorname{supp} f_0. \tag{8-18}$$

According to (8-10), the measure of $\operatorname{supp}\Phi$ is strictly positive with

$$|\operatorname{supp}\Phi| \ge \tfrac{1}{2}|K_0|. \tag{8-19}$$

It remains to check that dP_∞ is a probability measure on the real axis, which reduces to verifying that

$$2\int_{\mathbb{R}} \Phi(x)\,dx = 1.$$

First note that using Proposition 6.2(2) one obtains for any $t \ge 0$

$$1 = 2\int_{\mathbb{R}} e^t f(t,x)\,dx.$$

To exchange the limit and integral it suffices to apply the Lebesgue theorem thanks to the condition (8-12) and the fact that $\operatorname{supp} f_t \in \operatorname{Conv} K_0$ (recall that for simplicity we have assumed that $\|\rho_0\|_{L^1} = 1$):

$$\lim_{t\to\infty} \int_{\mathbb{R}} e^t f(t,x)\,dx = \int_{\mathbb{R}} \Phi(x)\,dx.$$

This provides the desired result. We point out that with the normalization $\|\rho_0\|_{L^1} = 1$ one gets instead of (8-17)

$$dP_\infty = \frac{\Phi}{\|f_0\|_{L^1}}\delta_{\mathbb{R}\otimes\{0\}},$$

which gives the structure of the limiting measure stated in Theorem 2.2 thanks to (8-5) and (8-11).

8B. ***Concentration of the support.*** In this section we shall complete the study of the limiting measure dP_∞ and identify its support, denoted by K_∞. What is left to conclude the proof of Theorem 2.2 is just to check that the support D_t of the solution ρ_t converges in the Hausdorff sense to K_∞. Recall that K_0 is the support of f_0 and is assumed to be a finite collection of increasing segments $[a_i; b_i]$, $i = 1, \dots, n$, such that $a_i < b_i < a_{i+1}$. According to (8-18) one has

$$\operatorname{supp}\Phi = \psi_\infty(K_0) \triangleq K_\infty.$$

Since Ψ_∞ is strictly increasing due to (8-10) one deduces easily that

$$\operatorname{supp}\Phi = \cup_{i=1}^n [a_i^\infty, b_i^\infty], \quad a_i^\infty \triangleq \psi_\infty(a_i), \quad b_i^\infty \triangleq \psi_\infty(b_i).$$

Using once again (8-10) one may easily obtain that

$$\text{for all } i, \quad |a_i^\infty - b_i^\infty| \geq \tfrac{1}{2}|a_i - b_i|.$$

Now to establish the convergence in the Hausdorff sense of D_t towards K_∞ it suffices to prove the result for each connected component, that is,

$$\text{for all } i = 1, \dots, n, \quad d_H(\Gamma_t^i, [a_i^\infty, b_i^\infty]) \leq Ce^{-t},$$

with

$$\Gamma_t^i \triangleq \{(x, f_t(x)) : x \in [a_i^t, b_i^t]\}.$$

By straightforward analysis using (7-17) one obtains

$$d_H(\Gamma_t^i, [a_i^\infty, b_i^\infty]) \leq Ce^{-t} + C \max(|a_i^t - a_i^\infty|, |b_i^t - b_i^\infty|).$$

From (8-8) one gets

$$\max(|a_i^t - a_i^\infty|, |b_i^t - b_i^\infty|) \leq Ce^{-t}$$

and therefore

$$\text{for all } t \geq 0, \quad d_H(D_t, K_\infty) \leq Ce^{-t}.$$

The proof of Theorem 2.1 is now complete.

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