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We prove a necessary and sufficient condition in terms of the barycenters of a collection of polytopes for existence of coupled Kähler–Einstein metrics on toric Fano manifolds. This confirms the toric case of a coupled version of the Yau–Tian–Donaldson conjecture and as a corollary we obtain an example of a coupled Kähler–Einstein metric on a manifold which does not admit Kähler–Einstein metrics. We also obtain a necessary and sufficient condition for existence of torus-invariant solutions to a system of soliton-type equations on toric Fano manifolds.

1. Introduction

Given a compact Kähler manifold (X, ω) , an important question in complex geometry is the problem of finding a metric of constant scalar curvature in the Kähler class $[\omega]$. It has been known for a long time that there are deep obstructions to existence of these metrics. In the case when $[\omega] = \pm c_1(X)$, constant scalar curvature metrics coincide with Kähler–Einstein metrics, i.e., metrics that are proportional to their Ricci tensor. It was recently shown [Chen et al. 2015a; 2015b; 2015c] that existence of such metrics is equivalent to a certain algebraic stability condition: K-polystability (see also [Tian 2015]). A similar stability condition for general Kähler classes is conjectured to be equivalent to existence of constant scalar curvature metrics. However, except for some special classes of manifolds (see [Donaldson 2009]) this is open. It should also be pointed out that even in light of [Chen et al. 2015a; 2015b; 2015c], determining if a given manifold admits a Kähler–Einstein metric is not a straightforward task. The condition of K-polystability is not readily checkable. On the other hand, a large class of manifolds where existence of Kähler–Einstein metrics reduces to a simple criterion is given by toric Fano manifolds. Here, as was originally proved in [Wang and Zhu 2004], existence of Kähler–Einstein metrics is equivalent to the condition that the barycenter of the polytope associated to the anticanonical polarization is the origin. In addition, Wang and Zhu [2004] proved that any toric Fano manifold admits a Kähler–Ricci soliton, in other words a metric ω such that

$$\text{Ric } \omega = L_V(\omega) + \omega \tag{1}$$

for a holomorphic vector field V . Here L_V denotes Lie derivative along V . These appear as natural long-time solutions to the Kähler–Ricci flow and have attracted great interest over the years; see for example [Hamilton 1993; 1995; Cao 1997; Tian 1997].

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In a recent paper Witt Nyström together with the present author introduced the concept of coupled Kähler–Einstein metrics [Hultgren and Nyström 2018]. These are k -tuples of Kähler metrics $(\omega_1, \dots, \omega_k)$ on a compact Kähler manifold X satisfying

$$\operatorname{Ric} \omega_1 = \dots = \operatorname{Ric} \omega_k = \pm \sum_i \omega_i. \quad (2)$$

These generalize Kähler–Einstein metrics in the sense that for $k = 1$ this equation reduces to the classical equation

$$\operatorname{Ric} \omega_1 = \pm \omega_1$$

defining Kähler–Einstein metrics. Moreover, (2) implies a cohomological condition on $\omega_1, \dots, \omega_k$, namely

$$\sum_i [\omega_i] = \pm c_1(X). \quad (3)$$

We see that, much as for Kähler–Einstein metrics, the theory splits into two cases: $c_1(X) < 0$ and $c_1(X) > 0$. Now, as in [Hultgren and Nyström 2018] we will say that a k -tuple of Kähler classes $(\alpha_1, \dots, \alpha_k)$ such that $\sum_i \alpha_i = \pm c_1(X)$ is a *decomposition of $\pm c_1(X)$* and given a decomposition of $c_1(X)$ we will say that it admits a coupled Kähler–Einstein metric if there is a coupled Kähler–Einstein metric $(\omega_1, \dots, \omega_k)$ such that $[\omega_i] = \alpha_i$ for all i . In [Hultgren and Nyström 2018] it was shown that fixing a decomposition of $c_1(X)$ imposes the right boundary conditions on (2) in the sense that:

- If $c_1(X) < 0$, then any decomposition of $-c_1(X)$ admits a unique coupled Kähler–Einstein metric.
- If $c_1(X) > 0$, then any coupled Kähler–Einstein metric admitted by a given decomposition of $c_1(X)$ is unique up to the flow of holomorphic vector fields.

Moreover, it was shown that if $c_1(X) > 0$ and $(\omega_1, \dots, \omega_k)$ is a coupled Kähler–Einstein metric, then the associated k -tuple of Kähler classes $([\omega_1], \dots, [\omega_k])$ satisfies a certain algebraic stability condition which, by analogy, was called K-polystability. It was also conjectured that the converse of this holds, providing a “coupled” Yau–Tian–Donaldson conjecture:

Conjecture 1 [Hultgren and Nyström 2018]. *Assume $c_1(X) > 0$. Then a decomposition of $c_1(X)$ admits a coupled Kähler–Einstein metric if and only if it is K-polystable.*

Our main theorem confirms this conjecture in the toric case and provides a simple condition for K-polystability in terms of the barycenters of a collection of polytopes associated to $(\alpha_1, \dots, \alpha_k)$. More precisely, consider the anticanonical line bundle $-K_X$ over a toric Fano manifold X . Fixing the action of $(\mathbb{C}^*)^n$ on X , this defines a polytope P_{-K_X} in the vector space $M \otimes \mathbb{R}$, where M is the character lattice of $(\mathbb{C}^*)^n$. For a general Kähler class that arises as the curvature of a toric line bundle, this correspondence is well-defined up to translation of the polytope (or equivalently, up to choice of action on the toric line bundle). Moreover, the correspondence trivially extends to all Kähler classes that can be written as linear combinations with positive real coefficients of Kähler classes of this type. By general facts (see Lemma 23 and the discussion following it) this holds for any Kähler class on a toric Fano manifold. This means that

a decomposition of $c_1(X)$ determines (up to translations) a set of polytopes P_1, \dots, P_k in \mathbb{R}^n . Moreover, the condition $\sum_i \alpha_i = c_1(X)$ means the polytopes can be chosen so that the Minkowski sum satisfies

$$\sum_i P_i = P_{-K_X}. \tag{4}$$

Enforcing this, we note that the polytopes associated to a decomposition of $c_1(X)$ are well-defined up to translations

$$(P_1, \dots, P_k) \mapsto (P_1 + c_1, \dots, P_k + c_k),$$

where $c_1, \dots, c_k \in \mathbb{R}^n$ satisfies $\sum_i c_i = 0$.

Now, given a polytope P in \mathbb{R}^n we will let $b(P)$ be the (normalized) barycenter of P ,

$$b(P) = \frac{1}{\text{Vol}(P)} \int_P p \, dp,$$

where dp is the uniform measure on P and $\text{Vol}(P) = \int_P dp$. Note that $b(P + c) = b(P) + c$; hence, assuming (4), the quantity $\sum_i b(P_i)$ is independent of the choices of translation of P_1, \dots, P_k . Our main theorem is:

Theorem 2. *Let X be a toric Fano manifold. Assume (α_i) is a decomposition of $c_1(X)$ and P_1, \dots, P_k are the associated polytopes. Then the following are equivalent:*

- (i) (α_i) admits a coupled Kähler–Einstein tuple.
- (ii) (α_i) is K -polystable in the sense of [Hultgren and Nyström 2018].
- (iii) $\sum_i b(P_i) = 0$.

Remark 3. One important point is $\sum_i b(P_i)$ is not in general equal to

$$b\left(\sum_i P_i\right) = b(P_{-K_X});$$

hence the condition on P_1, \dots, P_k in Theorem 2 is not (a priori) equivalent to existence of a classical Kähler–Einstein metric. In fact, none of these conditions imply the others. By Corollary 4 below, there is an example of a manifold that doesn’t admit Kähler–Einstein metrics but does admit coupled Kähler–Einstein metrics. Moreover, by Remark 6 there is an example of a Kähler–Einstein manifold with decompositions of $c_1(X)$ that doesn’t admit coupled Kähler–Einstein metrics.

Corollary 4. *Let E be the rank-2 vector bundle*

$$E = \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$$

over $\mathbb{P}^2 \times \mathbb{P}^1$ and consider the toric four-manifold $X = \mathbb{P}(E)$. Then X does not admit a Kähler–Einstein metric. On the other hand, let $\pi : X \rightarrow \mathbb{P}^1$ be the natural projection onto \mathbb{P}^1 and $\beta_1, \beta_2 \in H^{(1,1)}(X)$ be the classes corresponding to the divisors given by $\pi^{-1}(0)$ and $\pi^{-1}(\infty)$, respectively. Then

$$\alpha_1 = \frac{1}{2}c_1(X) - \frac{1}{4}\sqrt{\frac{5}{7}}(\beta_1 + \beta_2), \quad \alpha_2 = \frac{1}{2}c_1(X) + \frac{1}{4}\sqrt{\frac{5}{7}}(\beta_1 + \beta_2) \tag{5}$$

are Kähler and the decomposition of $c_1(X)$ given by (α_1, α_2) admits a coupled Kähler–Einstein metric.

Remark 5. It would be interesting to see if there are simpler examples than the one given in Corollary 4 of manifolds which admit coupled Kähler–Einstein metrics but no Kähler–Einstein metrics. However, by Corollary 1.6 in [Hultgren and Nyström 2018], the automorphism group of any manifold that admits a coupled Kähler–Einstein metric is reductive. Among other things, this rules out \mathbb{P}^2 blown up in one or two points.

Remark 6. The following is an example of a decomposition of $c_1(X)$ on an Einstein manifold that does not admit a coupled Kähler–Einstein metric. Let X be the toric Fano manifold acquired by blowing up \mathbb{P}^2 in three points and D be the $(S^1)^n$ -invariant divisor in X that corresponds to the ray generated by $(1, 1)$ in the fan of X . Let $D_t = -\frac{1}{2}K_X + tD$. We have $D_t + D_{-t} = -K_X$. Computer calculations show that

$$b(P_{D_t}) + b(P_{D_{-t}}) \neq 0$$

for small t ; in other words the decomposition of $c_1(X)$ given by $(c_1(D_t), c_1(D_{-t}))$ does not admit a coupled Kähler–Einstein metric for small t .

Remark 7. As discussed in [Hultgren and Nyström 2018], fixing a Kähler class α on X we get a family of decompositions of $c_1(X)$

$$\{(t\alpha, c_1(X) - t\alpha) : t \in (0, t_\alpha)\},$$

where $t_\alpha = \sup\{t : c_1(X) - t\alpha > 0\}$. Assuming they admit coupled Kähler–Einstein metrics (η_1^t, η_2^t) we get a canonical family of metrics $\{\omega_t := \eta_1^t/t\}$ in α . Now, let X be a toric Fano surface. By Theorem 2, $(t\alpha, \alpha - c_1(X))$ admits a coupled Kähler–Einstein metric if and only if

$$tb(P_{L_\alpha}) + b(P_{-K_X - tL_\alpha}) = 0, \tag{6}$$

where L_α is a toric (\mathbb{R}) -line bundle such that $c_1(L_\alpha) = \alpha$. On the other hand, it was proven in [Donaldson 2009] that α admits a constant scalar curvature metric if and only if

$$\frac{\int_{\partial P_{L_\alpha}} f d\sigma}{\int_{\partial P_{L_\alpha}} d\sigma} - \frac{\int_{P_{L_\alpha}} f dp}{\int_{P_{L_\alpha}} dp} \geq 0 \tag{7}$$

for every convex function f on the closure of P_{L_α} , with equality if and only if f is affine linear. Here $d\sigma$ is the measure on ∂P_{L_α} defined by the identity

$$\frac{d}{dt} \left(\int_{P_{L_\alpha + tP_{-K_X}}} h dp \right) \Big|_{t=0} = \int_{\partial P_{L_\alpha}} h d\sigma$$

for all functions h continuous in a neighborhood of P . In particular, for affine linear functions f , (7) reduces to the barycenter condition

$$b(P_{L_\alpha}) = b(d\sigma) = \frac{\int_{\partial P_{L_\alpha}} \sigma d\sigma}{\int_{\partial P_{L_\alpha}} d\sigma}. \tag{8}$$

It would be interesting to understand the relationship of (6) with the conditions (7) and (8).

Our second result considers a more general (soliton-type) version of (2), namely, given holomorphic vector fields V_1, \dots, V_k

$$\text{Ric } \omega_1 - L_{V_1}(\omega_1) = \dots = \text{Ric } \omega_k - L_{V_k}(\omega_k) = \sum_i \omega_i. \tag{9}$$

We will say that a k -tuple of Kähler metrics satisfying (2) is a *coupled Kähler–Ricci soliton*. When $k = 1$, (9) reduces to (1) and defines classical Kähler–Ricci solitons. As mentioned above these appear as natural solutions to the Kähler–Ricci flow. In fact, a similar interpretation in terms of natural solutions to a geometric flow can be given for (9). Given k Kähler metrics $\omega_1^0, \dots, \omega_k^0$ we may consider the flow defined by

$$\frac{d}{dt} \omega_1^t = \text{Ric } \omega_1^t - \sum_i \omega_i^t, \quad \dots, \quad \frac{d}{dt} \omega_k^t = \text{Ric } \omega_k^t - \sum_i \omega_i^t \tag{10}$$

for $t \in [0, \infty)$. Stationary solutions to (10) are given by coupled Kähler–Einstein metrics, i.e., solutions to (2). On the other hand, putting $V_1 = \dots = V_k = V$ and letting (ω_i^t) be the flow along V of a k -tuple (ω_i^0) satisfying (9) means (ω_i^t) will satisfy (9) for each t . Plugging this into the right-hand side of (10) gives

$$\text{Ric } \omega_j^t - \sum_i \omega_i^t = L_V(\omega_j^t)$$

for all j . By definition $(d/dt)\omega_j^t = L_V(\omega_j^t)$ for all j ; hence (ω_i^t) satisfies (10).

To state our second result we need some terminology. Note that a point in the vector space that is dual to $M \otimes \mathbb{R}$, namely $N \otimes \mathbb{R}$ where N is the lattice consisting of one-parameter subgroups in $(\mathbb{C}^*)^n$, determines a holomorphic vector field on X . We will call any holomorphic vector field on X that arises in this manner a *toric vector field*. These can be given a concrete description in the following way: By definition, the action of $(\mathbb{C}^*)^n$ on X admits an open, dense and free orbit. Identifying $(\mathbb{C}^*)^n$ with this orbit and letting $\sigma_1, \dots, \sigma_n$ be the standard logarithmic coordinates on $(\mathbb{C}^*)^n$ the toric vector fields are simply the vector fields that arise as linear combinations of the coordinate vector fields $\partial/\partial\sigma_1, \dots, \partial/\partial\sigma_k$. We will often identify a toric vector field with its associated point in $N \otimes \mathbb{R}$.

In this context there is a natural vector-valued invariant $\mathcal{A}_V(P)$ determined by a polytope P in $\mathbb{R}^n = M \otimes \mathbb{R}$ and a point V in the dual vector space $N \otimes \mathbb{R}$. To define it we first introduce the V -weighted volume of P

$$\text{Vol}_V(P) = \int_P e^{\langle V, p \rangle} dp.$$

Then $\mathcal{A}_V(P)$ is given by

$$\mathcal{A}_P(V) = \frac{1}{\text{Vol}_V(P)} \int_P p e^{\langle V, p \rangle} dp. \tag{11}$$

With respect to this we have:

Theorem 8. *Let V_1, \dots, V_k be toric vector fields on a toric Fano manifold X . Assume $(\alpha_1, \dots, \alpha_k)$ is a decomposition of $c_1(X)$ and P_1, \dots, P_k are the associated polytopes. Then there is an $(S^1)^n$ -invariant solution $(\omega_1, \dots, \omega_k)$ to (9) such that $\omega_i \in \alpha_i$ for each i if and only if*

$$\sum_i \mathcal{A}_{P_i}(V_i) = 0. \tag{12}$$

Remark 9. Much as in Theorem 2, the polytopes P_1, \dots, P_k associated to $(\alpha_1, \dots, \alpha_k)$ are only well-defined up to translations $P_i \rightarrow P_i + c_i$ for $c_i \in \mathbb{R}^n$ satisfying $\sum_i c_i = 0$. On the other hand, much as the barycenter, $\mathcal{A}_V(P)$ satisfies

$$\mathcal{A}_{P+c}(V) = \mathcal{A}_P(V) + c,$$

and hence the left-hand side of (12) is invariant under such translations.

Remark 10. Theorem 8 is a generalization of Wang and Zhu’s theorem [2004] on the existence of Kähler–Ricci solitons on toric manifolds. See also [Berman and Berndtsson 2013; Delcroix 2017] for generalizations in other directions.

A straightforward corollary of Theorem 8, using that (11) is the gradient of a strictly convex and proper function on \mathbb{R}^n , is:

Corollary 11. *Let (α_i) be a decomposition of $c_1(X)$ on a toric Fano manifold. Then there is a unique toric vector field V such that (α_i) admits an $(S^1)^n$ -invariant coupled Kähler–Ricci soliton where $V_1 = \dots = V_k = V$.*

Remark 12. Naturally, we expect solutions of the flow (10) to converge to the Kähler–Ricci solitons in Corollary 11. This parallels the theory in the case $k = 1$ (see [Tian and Zhu 2007]). On the other hand, it is interesting to note that by Theorem 8 there exists a large class of solitons that do not appear as natural solutions to (10) in the sense discussed above (this happens whenever $V_i \neq V_j$ for some i and j). This suggests that there is a more general flow, which includes (10) as a special case, and where the solitons of Theorem 8 appear as natural solutions.

A second corollary of Theorem 8 is related to the corresponding real Monge–Ampère equation. Let f_1, \dots, f_k be twice differentiable convex functions on \mathbb{R}^n . Let ∇f_i denote the gradient of f_i . Then, given a decomposition $(\alpha_1, \dots, \alpha_k)$ and associated polytopes P_1, \dots, P_k , existence of coupled Kähler–Ricci solitons is equivalent to the solvability of the equation

$$\frac{e^{\langle V_1, \nabla f_1 \rangle}}{\text{Vol}_{V_1}(P_1)} \det\left(\frac{d^2 f_1}{dx_l dx_m}\right) = \dots = \frac{e^{\langle V_k, \nabla f_k \rangle}}{\text{Vol}_{V_k}(P_k)} \det\left(\frac{d^2 f_k}{dx_l dx_m}\right) = e^{-\sum_i f_i} \tag{13}$$

under the boundary conditions

$$\overline{\nabla f_i(\mathbb{R}^n)} = P_i, \tag{14}$$

where the left-hand side of (14) denotes the closure of the image of ∇f_i in \mathbb{R}^n . We will say that a k -tuple of polytopes in \mathbb{R}^n is *toric Fano* if it is defined by a decomposition of $c_1(X)$ on a toric Fano manifold.

Corollary 13. *Assume P_1, \dots, P_k is a toric Fano k -tuple of polytopes and $V_1, \dots, V_k \in \mathbb{R}^n$. Then (13) admits a solution satisfying (14) if and only if*

$$\sum_i \mathcal{A}_{P_i}(V_i) = 0.$$

In particular, if $V_1 = \dots = V_k = 0$ then (13) admits a solution satisfying (14) if and only if

$$\sum_i b(P_i) = 0.$$

Theorem 2 essentially follows from considering the case $V_1 = \dots = V_k = 0$ in Theorem 8. Doing this gives that (iii) in Theorem 2 implies (i). As mentioned above, by a previous result [Hultgren and Nyström 2018, Theorem 1.15] (i) implies (ii). Finally, an explicit formula for the (coupled) Donaldson–Futaki invariant of test configurations induced by toric vector fields shows that (ii) implies (iii). To be more precise, if V is a toric vector field and (α_i) is a decomposition of $c_1(X)$ with associated polytopes P_1, \dots, P_k , then the test configuration for (α_i) induced by V has Donaldson–Futaki invariant

$$\left\langle V, \sum_i b(P_i) \right\rangle.$$

It follows that if $\sum_i b(P_i) \neq 0$, then there is a test configuration for (α_i) with negative Donaldson–Futaki invariant. By definition, this means (α_i) is not K-polystable (see Section 3.2 for a detailed argument).

The main point in the proof of Theorem 8 is to establish a priori C^0 -estimates along an associated continuity path. More precisely, let $\theta_1, \dots, \theta_k$ be Kähler metrics such that $[\theta_i] = \alpha_i$. Assume, using the Calabi–Yau theorem, that ω_0 is a Kähler form such that $\text{Ric } \omega_0 = \sum_i \theta_i$ and $\int_X \omega_0^n = 1$. For each i , let $g_i = g_{\theta_i, V_i}$ be a θ_i -plurisubharmonic function on X such that

$$dd^c g_i = L_{V_i}(\theta_i)$$

and $\int_X e^{g_i} \theta_i^n = 1$ (see Lemma 17). For $t \in [0, 1]$ we will consider the equation

$$e^{g_1 + V_1(\phi_1)}(\theta_1 + dd^c \phi_1)^n = \dots = e^{g_k + V_k(\phi_k)}(\theta_k + dd^c \phi_k)^n = e^{-t \sum_i \phi_i} \omega_0^n. \tag{15}$$

Moreover, fixing a point $x_0 \in X$ we will assume solutions to (15) are normalized according to

$$\phi_1(x_0) = \dots = \phi_k(x_0). \tag{16}$$

The significance of these equations is that for $t = 1$, a k -tuple of functions ϕ_1, \dots, ϕ_k such that each ϕ_i is θ_i -plurisubharmonic solves (15) if and only if the k -tuple of Kähler metrics $(\theta_i + dd^c \phi_i)$ is a coupled Kähler–Ricci soliton. We prove:

Theorem 14. *Let V_i, α_i and P_i be as in Theorem 8 and assume (12) holds. Let x_0 be the point in X that, under the identification of $(\mathbb{C}^*)^n$ with its open, dense and free orbit, corresponds to the identity element in $(\mathbb{C}^*)^n$. Then, for any $t_0 > 0$ there is a constant C such that any solution (ϕ_1, \dots, ϕ_k) of (15) for $t \geq t_0$, normalized according to (16), satisfies*

$$\sup_X |\phi_i| < C$$

for all i .

Pingali [2018] reduced existence of coupled Kähler–Einstein metrics to a priori C^0 -estimates. This means that Theorem 8 in the special case when $V_1 = \dots = V_k = 0$, and thus Theorem 2, follows from Theorem 14 above and Pingali’s work. For the general case we adapt the argument of Pingali to the soliton setting, essentially following the computations in [Tian and Zhu 2000]. Letting $\text{Aut}(X)$ be the automorphism group of X we prove:

Theorem 15. *Let X be a Fano manifold and V_1, \dots, V_k be holomorphic vector fields in the reductive part of the Lie algebra of $\text{Aut}(X)$ such that $\text{Im } V_i$ generate a compact one-parameter subgroup in $\text{Aut}(X)$ for each i . Let (α_i) be a decomposition of $c_1(X)$ with representatives $\theta_1, \dots, \theta_k$ such that $\text{Im } L_{V_i}\theta_i = 0$ for all i . Assume also C^0 -estimates hold for (15); in other words, for each $t_0 > 0$, there is a constant C such that any solution (ϕ_i) to (15) at $t > t_0$ satisfies*

$$\sup_X |\phi_i| < C$$

for all i . Then (α_i) admits a solution to (9).

We get that the positive part of Theorem 8 follows directly from Theorems 14 and 15. The negative part of Theorem 8 follows directly from a change of variables in (13) (see Lemma 28).

Remark 16. Berman and Berndtsson [2013] used a variational approach to prove existence of Kähler–Ricci solitons on toric log Fano varieties. They give a direct argument for coercivity of the associated Ding functional on $(S^1)^n$ -invariant metrics. It would be interesting if this coercivity estimate could be extended to the coupled setting. This would provide a stronger result than this paper in two respects: First of all, it would cover the singular setting of log Fano varieties. Secondly, since this bypasses the higher-order a priori estimates from complex geometry it would provide a version of Corollary 13 that is valid for all k -tuples of polytopes, not only the ones that are defined by decompositions of $c_1(X)$ on toric Fano manifolds.

This paper is organized in the following way: Sections 2.1 and 2.2 are devoted to the proof of Theorem 15. In Section 2.1 we prove openness along the continuity path and solvability at $t = 0$. In Section 2.2 we prove $C^{2,\alpha}$ -estimates assuming C^0 -estimates, thus finishing the proof of Theorem 15. In Section 3 we set up the real convex geometric framework and in Section 3.1 we use this to prove the C^0 -estimate of Theorem 14. Finally, at the end of Section 3.1 we prove Theorem 8, Corollary 11 and Corollary 13 and in Section 3.2 we prove Theorem 2.

2. Openness and higher-order estimates

This section is devoted to proving Theorem 15.

The following lemma is well known. However, as a courtesy to the reader we include a proof of it.

Lemma 17. *Assume X is a Fano manifold, V a holomorphic vector field on X and θ a Kähler form on X such that the imaginary part of $L_V(\theta)$ vanishes. Then there is a smooth real-valued function g on X such that*

$$dd^c g = L_V(\theta).$$

Proof. Since V is a holomorphic vector field, the contraction operator i_V anticommutes with $\bar{\partial}$; hence $i_V\theta$ is a $\bar{\partial}$ -closed $(0, 1)$ -form. By the Kodaira vanishing theorem, since X is Fano, the sheaf cohomology group satisfies

$$H^1(X, \mathcal{O}) = H^1(X, -K_X + K_X) = 0.$$

This means the Dolbeault cohomology group satisfies

$$H^{(0,1)}(X) \cong H^1(X, \mathcal{O}) = 0;$$

hence $i_V\theta_i$ is also $\bar{\partial}$ -exact. Let g be a smooth function such that $\sqrt{-1}\bar{\partial}g = i_V\theta$. As $L_V(\theta)$ is real, so is g . Moreover,

$$dd^c g = i\bar{\partial}\bar{\partial}g = \partial i_V\theta = L_V(\theta). \quad \square$$

For each i , let $\text{PSH}(X, \theta_i)$ be the space of θ_i -plurisubharmonic functions on X , in other words the space of upper semicontinuous and locally integrable functions ϕ_i satisfying $dd^c\phi_i + \theta_i \geq 0$. Note that if ϕ_i is a smooth function in $\text{PSH}(X, \theta_i)$, then

$$\begin{aligned} L_{V_i}(dd^c\phi) &= \partial i_{V_i}\sqrt{-1}\bar{\partial}\bar{\partial}\phi_i \\ &= \sqrt{-1}\bar{\partial}\bar{\partial}i_{V_i}\partial\phi_i = dd^c V_i(\phi_i); \end{aligned}$$

hence $dd^c(g_i + V_i(\phi_i)) = L_V(\theta_i + dd^c\phi_i)$. This means that, much as in [Hultgren and Nyström 2018], we get:

Lemma 18. *Let X be a Fano manifold, V_1, \dots, V_k holomorphic vector fields on X and (α_i) a k -tuple of Kähler classes on X such that $\sum \alpha_i = c_1(X)$. Assume each class α_i has a representative θ_i such that $\text{Im } L_V(\theta_i) = 0$ and, for each i , let ϕ_i be a smooth function in $\text{PSH}(X, \theta_i)$. Then (ϕ_1, \dots, ϕ_k) is a solution to (15) at $t = 1$ if and only if the k -tuple of Kähler metrics $(\theta_i + dd^c\phi_i)$ is a coupled Kähler–Ricci soliton.*

2.1. Openness. Here we will prove the first part of Theorem 15, namely that the set of t such that (15) is solvable is open.

We will use the Banach spaces

$$\begin{aligned} A &= \{(\phi_1, \dots, \phi_k) : \phi_i \in C^{4,\alpha}(X)\}, \\ B &= \{(v_1, \dots, v_k) : v_i \in C^{2,\alpha}(X)\}. \end{aligned}$$

Moreover, let A_{PSH} be the open subset of A given by

$$A_{\text{PSH}} = \{(\phi_1, \dots, \phi_k) : \phi_i \in C^{4,\alpha}(X) \cap \text{PSH}(X, \theta_i)\}.$$

Let $F : \mathbb{R} \times A_{\text{PSH}} \rightarrow B$ be defined by

$$F(t, (\phi_i)) = \begin{pmatrix} \log((\theta_1 + dd^c\phi_1)^n / \omega_0^n) + g_1 + V_1(\phi_1) + t \sum \phi_i \\ \vdots \\ \log((\theta_k + dd^c\phi_k)^n / \omega_0^n) + g_k + V_k(\phi_k) + t \sum \phi_i \end{pmatrix}.$$

Note that $F(t, (\phi_i)) = 0$ if and only if (ϕ_i) defines a solution to (15) at t . Moreover, in this case the measure

$$\mu := (\theta_i + dd^c\phi_i)^n e^{g_i + V_i(\phi_i)}$$

is independent of i .

Lemma 19. *The Fréchet derivative of F at $(t, (\phi_i))$ with respect to the second argument is given by $H : A \rightarrow B$ defined by*

$$H(v_1, \dots, v_k) = \begin{pmatrix} -\Delta_{\omega_1} v_1 + V_1(v_1) + t \sum v_i \\ \vdots \\ -\Delta_{\omega_k} v_k + V_k(v_k) + t \sum v_i \end{pmatrix}, \tag{17}$$

where $\omega_i = \theta_i + dd^c \phi_i$ and Δ_{ω_i} is the associated Laplace–Beltrami operator. Moreover, H is elliptic. Finally, assume $F(t, \phi) = 0$ and let $\langle \cdot, \cdot \rangle$ be the inner product on B given by

$$\langle (u_i), (v_i) \rangle = \sum_i \int_X u_i v_i \mu.$$

Then

$$\langle H(u_1, \dots, u_k), (v_i) \rangle = \langle (u_i), H(v_1, \dots, v_k) \rangle$$

for any $(u_i), (v_i) \in B$.

Proof. Equation (17) follows from straightforward differentiation and the well-known identity

$$\frac{d}{ds} \log \frac{(\theta_i + dd^c(\phi_i + s v_i))^n}{\theta_i^n} \Big|_{s=0} = n \frac{dd^c v(\theta_i + dd^c \phi_i)^{n-1}}{(\theta_i + dd^c \phi_i)^n} = \Delta_{\omega_i} v_i.$$

Now, H takes the following form in local coordinates:

$$(u_i) \mapsto (v_j) = \left(\sum_{i,l,m} a_{ij}^{lm}(x) \frac{\partial^2 u_i}{\partial x_l \partial x_m} + \text{lower-order terms} \right),$$

where $a_{ij}^{lm} = 0$ if $i \neq j$ and $\{a_{ii}^{lm}\}_{l,m}$ are the coefficients for the Laplace operator Δ_{ω_i} . Recall that H is elliptic if the matrix

$$\left(\sum_{l,m} a_{ij}^{lm}(x) \xi_l \xi_m \right) \tag{18}$$

is invertible for all $p \in X$ and all nonzero $\xi = \sum \xi_l (\partial/\partial x_l) \in T_p X$, but this follows immediately. To see this note that

$$\sum_{l,m} a_{ij}^{lm}(x) \xi_l \xi_m$$

is 0 if $i \neq j$ and, by ellipticity of Δ_{ω_i} , positive if $i = j$. This means (18) is a diagonal matrix with positive entries on the diagonal; hence it is invertible.

We will now prove the last statement in the lemma. It is a consequence of the following identity for functions $u, v \in C^{2,\alpha}(X)$ (see Lemma 2.2 in [Tian and Zhu 2000]):

$$\int_X (\Delta_{\omega_i} v + V_i(v)) u \mu = - \int_X \langle dv, du \rangle_{\omega_i} \mu. \tag{19}$$

We get

$$\begin{aligned} \sum_i \int_X \left(\Delta_{\omega_i} v_i + V_i(v_i) + \sum_j v_j \right) u_i \mu &= - \sum_i \int_X \langle dv_i, du_i \rangle \mu + \sum_{i,j} \int_X v_j u_i \mu \\ &= \sum_i \int_X v_i \left(\Delta_{\omega_i} u_i + V_i(u_i) + \sum_j u_j \right) \mu, \end{aligned}$$

and the last statement in the lemma follows. □

Lemma 20. *Assume $t \in [0, 1)$ and $(v_i) \in A$ are not all constant and satisfy*

$$\Delta_{\omega_1} v_1 + V_1(v_1) = \dots = \Delta_{\omega_k} v_k + V_k(v_k) = \lambda \sum_i v_i \tag{20}$$

for a k -tuple $\omega_1, \dots, \omega_k$ satisfying

$$\text{Ric } \omega_1 - L_{V_1}(\omega_1) = \dots = \text{Ric } \omega_k - L_{V_k}(\omega_k) = t \sum_i \omega_i + (1-t) \sum_i \theta_i. \tag{21}$$

Then $\lambda > t$.

Proof. Let $\partial_{\omega_i} v$ denote the gradient of v with respect to the metric ω_i . Moreover, we will use the notation $\text{Ric}_{\omega_i} = \text{Ric}(\omega_i)$. The proof is based on the following Weitzenböck identity (see [Tian and Zhu 2000], equation 2.7, page 277):

$$- \int_X \langle d(\Delta_{\omega_i} v + V_i(v)), dv \rangle_{\omega_i} \mu \geq \int_X (\text{Ric}_{\omega_i} - L_V(\omega_i))(\partial_{\omega_i} v, \overline{\partial_{\omega_i} v}) \mu.$$

Combining this with (21) and (20) gives

$$\begin{aligned} \lambda^2 \int_X \left(\sum_j v_j \right)^2 \mu &= \int_X (\Delta_{\omega_i} v_i + V_i(v_i))^2 \mu = - \int_X \langle d(\Delta_{\omega_i} v_i + V_i(v_i)), dv \rangle_{\omega_i} \mu \\ &\geq \int_X (\text{Ric}_{\omega_i} + L_V(\omega_i))(\partial_{\omega_i} v_i, \overline{\partial_{\omega_i} v_i}) \mu \geq t \int_X \sum_j |\partial_{\omega_j} v_i|_{\omega_j}^2 \mu. \end{aligned} \tag{22}$$

Moreover, we claim that (20) implies

$$\int_X |\partial_{\omega_i} v_i|_{\omega_j}^2 \mu \geq \int_X |dv_j|_{\omega_j}^2 \mu \tag{23}$$

for any i and j . Assuming that this is true we see that (22) implies

$$\begin{aligned} \lambda^2 \int_X \left(\sum_j v_j \right)^2 \mu &\geq t \int_X \sum_j |\partial_{\omega_j} v_j|_{\omega_j}^2 \mu = t \int_X \sum_j |dv_j|_{\omega_j}^2 \mu \\ &= t \int_X \sum_j (\Delta_{\omega_j} v_j + V_j(v_j)) v_j \mu \\ &= t \lambda \int_X \sum_j \left(\sum_i v_i \right) v_j \mu = t \lambda \int_X \left(\sum_j v_j \right)^2 \mu. \end{aligned}$$

We conclude that $\lambda \geq t$. Moreover, if $\lambda = t$ then equality holds in all inequalities above. In particular, equality holds in the last inequality of (22); hence, by (21),

$$\begin{aligned} 0 &= \int_X (\text{Ric}_{\omega_i} - L_V(\omega_i))(\partial_{\omega_i} v_i, \overline{\partial_{\omega_i} v_i}) \mu - t \int_X \sum_j |\partial_{\omega_i} v_i|_{\omega_j}^2 \mu \\ &= (1 - t) \int_X \sum_j |\partial_{\omega_i} v_i|_{\theta_j}^2 \mu \end{aligned}$$

from which it follows that v_i is constant for every i . This means that to finish the proof of the lemma it suffices to prove (23). To do this, note that for any i and j , by (20)

$$\begin{aligned} \int_X |dv_j|_{\omega_j}^2 \mu &= \int_X (\Delta_{\omega_j} v_j + V_i(v_j)) v_j \mu \\ &= \int_X (\Delta_{\omega_i} v_i + V_i(v_i)) v_j \mu = \int_X \langle dv_i, dv_j \rangle_{\omega_i} \mu. \end{aligned}$$

Moreover, choosing coordinates (z_1, \dots, z_n) that are normal with respect to ω_j and such that ω_i is diagonal with eigenvalues β_1, \dots, β_n at a point p we get

$$|\langle dv_i, dv_j \rangle_{\omega_i}| = \left| \sum_l \frac{1}{\beta_l} \frac{\partial v_i}{\partial z_l} \overline{\frac{\partial v_j}{\partial z_l}} \right| \leq \sqrt{\sum_l \left| \frac{1}{\beta_l} \frac{\partial v_i}{\partial z_l} \right|^2} \sqrt{\sum_l \left| \frac{\partial v_j}{\partial z_l} \right|^2} = |\partial_{\omega_i} v_i|_{\omega_j} |dv_j|_{\omega_j}.$$

Combining this with the Cauchy–Schwarz inequality we get

$$\int_X |dv_j|_{\omega_j}^2 \mu = \int_X \langle dv_i, dv_j \rangle_{\omega_i} \mu \leq \int_X |\partial_{\omega_i} v_i|_{\omega_j} |dv_j|_{\omega_j} \mu \leq \sqrt{\int_X |\partial_{\omega_i} v_i|_{\omega_j}^2 \mu} \sqrt{\int_X |dv_j|_{\omega_j}^2 \mu},$$

and (23) follows. □

We can now prove the first part of Theorem 15.

Proof of Theorem 15. First part: openness and the case $t = 0$. The theorem is proved using the continuity method along the path defined by (15). Here we will prove that the set of t such that (15) is solvable is nonempty and open in $[0, 1]$. At the end of Section 2.2 we will prove that it is also closed in $[0, 1]$, hence that (15) is solvable for all $t \in [0, 1]$.

First of all, to see that the set of t such that (15) is solvable is nonempty, note that for $t = 0$, (15) reduces to the collection of equations

$$(\theta_j + dd^c \phi_j)^n e^{g_j + V_j(\phi_j)} = \omega_0^n. \tag{24}$$

This means that for each i we can apply the Main Theorem in [Zhu 2000] to get ϕ_i such that

$$(\theta_j + dd^c \phi_j)^n e^{g_j + V_j(\phi_j) + c_j} = \omega_0^n \tag{25}$$

for some $c_j \in \mathbb{R}$. Integrating both sides of this and using the fact that

$$\int_X e^{g_i + V_i(\phi_i)} (\theta_i + dd^c \phi_i)^n = \int_X e^{g_i} \theta_i^n = 1 = \int_X \omega_0^n \tag{26}$$

for all smooth $\phi_i \in \text{PSH}(X, \theta_i)$ we see that $c_j = 0$ for all j ; in other words (ϕ_1, \dots, ϕ_k) provides a solution to (15) at $t = 0$.

Now, (26) is well known but for completeness we provide an argument for it here. Consider the variation of the left-hand side of (26) with respect to ϕ_i

$$\int_X (\Delta_{\omega_i} \dot{\phi}_i + V(\dot{\phi}_i)) \mu_i, \tag{27}$$

where we use the notation $\mu_i = e^{g_i + V_i(\phi_i)}(\theta_i + dd^c \phi_i)^n$. By (19),

$$\int_X (\Delta_{\omega_i} \dot{\phi}_i + V(\dot{\phi}_i))(\dot{\phi}_i + 1) \mu_i = \int_X |d\dot{\phi}|_{\omega_i}^2 \mu_i = \int_X (\Delta_{\omega_i} \dot{\phi}_i + V(\dot{\phi}_i)) \dot{\phi}_i \mu_i, \tag{28}$$

and hence (27) vanishes. This proves (26).

The fact that the set of t such that (15) is solvable is open follows from Lemmas 19 and 20 and a standard application of the implicit function theorem. More precisely, H is elliptic by Lemma 19. This means the image of $H : (W^{2,2}(X))^k \rightarrow (L^2(X))^k$ is closed (see for example Theorem 10.4.7 in [Nicolaeescu 1996]). Taking (v_i) in the orthogonal complement of the image of H gives

$$\langle (v_i), H(u_i) \rangle = 0$$

for all $(u_i) \in (W^{2,2}(X))^k$. In particular, it holds for all $(u_i) \in (C^\infty(X))^k$. By the last point in Lemma 19 this means $H(v_i) = 0$ as a distribution. By elliptic regularity (see for example Corollary 10.3.10 in [Nicolaeescu 1996]) $(v_i) \in (C^\infty(X))^k$ and hence, by Lemma 20, $(v_i) = (C_i)$ for constants C_1, \dots, C_k . As $H(C_i) = 0$ we get $\sum C_i = 0$. Using this and elliptic regularity again (see for example Theorem 10.3.11(b) in [Nicolaeescu 1996]), we may conclude that the kernel of H is $\{C_1, \dots, C_k : \sum C_i = 0\}$ and the image of H is

$$\widehat{B} = \left\{ (v_i) \in B : \int_X v_1 \mu = \dots = \int_X v_k \mu \right\}. \tag{29}$$

It follows that H is invertible as a map from

$$\widehat{A} = \{(v_i) \in A : v_1(x_0) = \dots = v_k(x_0)\}$$

to \widehat{B} . Moreover, the derivative of F with respect to t , $(t, (\phi_i)) \mapsto (\sum \phi_i, \dots, \sum \phi_i)$, trivially maps to \widehat{B} . Thus, applying the implicit function theorem to F restricted to $\widehat{A} \cap A_{\text{PSH}}$ completes the proof of the theorem. □

2.2. Higher-order estimates. We begin with:

Lemma 21. *Assume (ϕ_i) satisfies (15) for some $t \in [0, 1]$. Then*

$$\sup_X |\Delta_{\theta_j} \phi_j| \leq C,$$

where C depends only on $\sup_i \|\phi_i\|_{C^0(X)}$.

We will use the following lemma from [Zhu 2000] (page 768, Corollary 5.3):

Lemma 22. *Let X be a compact Kähler manifold, ω a Kähler form on X and V a holomorphic vector field on X . Assume $\phi \in \text{PSH}(X, \omega)$ is smooth and $X(\phi)$ is a real-valued function. Then*

$$\sup_X |V(\phi)| < C$$

for a constant C that is independent of ϕ .

Proof of Lemma 21. We start with the following inequality originating in [Yau 1978] (see for example equation 2.3 on page 1587 in [Chen and He 2012]): assume ω is a Kähler form and v is a smooth function satisfying

$$(\omega + dd^c v)^n = e^F \omega^n.$$

Then there are constants C_1, C_2 and C_3 , independent of v , such that

$$\Delta_{\omega+dd^c v}(e^{-C_1 v}(n + \Delta_{\omega} v)) \geq e^{-C_1 v} \Delta_{\omega} F + C_2(n + \Delta_{\omega} v)^{n/(n-1)} - C_3. \tag{30}$$

For each j , we have that ϕ_j satisfies the equation

$$(\theta_j + dd^c \phi_j) = e^{-g_j - V_j(\phi_j) - t \sum_i \phi_i + \log(\omega_0^n / \theta_j^n)} \theta_j^n. \tag{31}$$

Applying (30) to this and letting

$$u_j = e^{-C_1 \phi_j}(n + \Delta_{\theta_j} \phi_j),$$

for all j we get

$$\Delta_{\omega_j} u_j \geq e^{-C_1 \phi_j} \Delta_{\theta_j} \left(-g_j - V_j(\phi_j) - t \sum_i \phi_i + \log(\omega_0^n / \theta_j^n) \right) + C_2(n + \Delta_{\theta_j} \phi_j)^{n/(n-1)} - C_3. \tag{32}$$

Note that $dd^c \phi_i > -\theta_i$; hence

$$\Delta_{\theta_j} \phi_j = n \frac{(dd^c \phi_j) \wedge \theta_j^{n-1}}{\theta_j^n} > -n.$$

This means $u_j > 0$ for all j . Moreover, $u_j - e^{-C_1 \phi_j} \Delta_{\theta_j} \phi_j = ne^{-C_1 \phi_j}$. Hence, adjusting C_2 and C_3 in a way which only depends on $\sup_i \|\phi_i\|_{C^0(X)}$, we get

$$\Delta_{\omega_j} u_j \geq -e^{-C_1 \phi_j} \Delta_{\theta_j} (g_j + V_j(\phi_j)) - t \sum_i u_i + C_2 u_j^{n/(n-1)} - C_3. \tag{33}$$

Now, let

$$V_j = \sum V_m^j \frac{\partial}{\partial z_m} \quad \text{and} \quad \theta_j = \sum \theta_{m\bar{l}}^j dz_m d\bar{z}_l.$$

As in [Tian and Zhu 2000], we compute

$$\begin{aligned} \Delta_{\theta_j} (g_j + V_j(\phi_j)) &= \sum_{m,l} \frac{\partial}{\partial z_l} \left(V_m^j \left(\theta_{m\bar{l}}^j + \frac{\partial \phi_j}{\partial z_m \partial \bar{z}_l} \right) \right) \\ &= \sum_{m,l} \frac{\partial V_m^j}{\partial z_l} \left(\theta_{m\bar{l}}^j + \frac{\partial^2 \phi_j}{\partial z_m \partial \bar{z}_l} \right) + V_m^j \left(\frac{\partial \theta_{m\bar{l}}^j}{\partial z_l} + \frac{\partial^3 \phi_j}{\partial z_m \partial z_l \partial \bar{z}_l} \right). \end{aligned} \tag{34}$$

We will be interested in this at a point, p , where u_j attains its maximum. Choosing coordinates around p that are normal with respect to θ_j and such that $\omega_j = \theta_j + dd^c\phi_j$ is diagonal, (34) reduces to

$$\sum_m \frac{\partial V_m^j}{\partial z_m} \left(1 + \frac{\partial^2 \phi_j}{\partial z_m \partial \bar{z}_m} \right) + V_j(\Delta\phi_j).$$

The first term of this can be bounded by

$$\sup_m \left| \frac{\partial V_m^j}{\partial z_m} \right| (1 + \Delta_{\theta_j} \phi_j).$$

Moreover, as u_j is stationary at p we get that

$$V_j(u_j) = C_1 V_j(\phi_j) u_j - e^{-C_1 \phi_j} V_j(\Delta_{\theta_j} \phi_j)$$

vanishes at p ; hence

$$(e^{-C_1 \phi_j} V_j(\Delta_{\theta_j} \phi_j))|_p = (C_1 V_j(\phi_j) u_j)|_p.$$

We conclude that

$$e^{-C_1 \phi_j} \Delta_{\theta_j} (g_j + V_j(\phi_j)) \leq \left(\sup_m \left| \frac{\partial V_m^j}{\partial z_m} \right| + C_1 V_j(\phi_j) \right) u_j.$$

By Lemma 22 this is bounded by Cu_j for a uniform constant C .

We will now plug this into (33). By the maximum principle $\Delta_{\omega_j} u_j \leq 0$ at p . Letting $M_i = \max_X u_i \geq 0$ we get

$$0 \geq -Cu_j - t \sum_i M_i + C_2 u_j^{n/(n-1)} - C_3$$

at p . Summing over j and using Young’s inequality $a \leq \epsilon a^{n/(n-1)} + C(n, \epsilon)$ we get, after adjusting C_3 ,

$$\begin{aligned} 0 &\geq -C \sum M_i - kt \sum M_i + \frac{C_2}{\epsilon} \sum M_i - C_3 \\ &= \left(-C - kt + \frac{C_2}{\epsilon} \right) \sum M_i - C_3. \end{aligned}$$

Choosing ϵ small enough that the expression in the parentheses is positive gives an upper bound on $\sum M_j$. Since $M_i \geq 0$ for all i , this implies a bound on $\sup M_i = \sup |u_i|$. This proves the lemma. \square

Proof of Theorem 15. Second part: $C^{2,\alpha}$ -estimates. Here we will prove that the set of t such that (15) is solvable is closed.

By Lemma 21, $|\Delta_{\theta_i} \phi_i|$ is bounded by a constant that depends only on $\|\phi_i\|_{C^0(X)}$ for all i . We wish to apply Theorem 1 in [Wang 2012]. To do this we need uniform bounds on the Hölder norms of ϕ_i and $V_i(\phi_i)$. These are implied by the uniform bounds on $\Delta_{\theta_i} \phi_i$. To see this, choose coordinates that are normal with respect to θ_i and such that $\theta_i + dd^c\phi_i$ is diagonal at a point p . Since

$$\theta_i + dd^c\phi_i = \sum \left(1 + \frac{\partial^2 \phi_i}{\partial z_m \partial \bar{z}_m} \right) dz_m d\bar{z}_m > 0$$

we get that $\partial^2\phi_i/(\partial z_m\partial\bar{z}_m) > -1$ for all m . Together with the bound

$$\Delta_{\theta_i}\phi_i = \sum_m \frac{\partial^2\phi_i}{\partial z_m\partial\bar{z}_m} \leq C$$

this gives uniform bounds on $|\partial^2\phi_i/(\partial z_m\partial\bar{z}_m)|$ for all m and l and the bounds on the Hölder norms follow.

Combining this with the argument at the end of Section 2.1, we conclude that the set of t such that (15) is solvable is nonempty, open and closed in $[0, 1]$. It follows that (15) has a solution (ϕ_i) at $t = 1$. Consequently, by Lemma 18 $(\theta_i + dd^c\phi_i)$ solves (9). □

3. C^0 -estimates

In this section X will always be a toric Fano manifold. In other words $c_1(X) > 0$ and, letting $n = \dim X$, there is an n -dimensional complex torus $(\mathbb{C}^*)^n$ acting on X by biholomorphisms such that the action admits an open, dense and free orbit. The purpose of the section is to prove Theorem 14. We will begin by recalling the well-known correspondence between metrics on line bundles over toric varieties and convex functions in \mathbb{R}^n . As in the Introduction we fix an action of $(\mathbb{C}^*)^n$ on X and identify $(\mathbb{C}^*)^n$ with its open, dense and free orbit. Let θ be an $(S^1)^n$ -invariant Kähler form on X that arises as the curvature of a metric $\|\cdot\|$ on a toric line bundle over X . Let P be the polytope associated to this toric line bundle. Assume s_0 is the $(\mathbb{C}^*)^n$ -invariant section corresponding to the point $0 \in P$. By the invariance s_0 is nonvanishing on $(\mathbb{C}^*)^n$ and the metric can be represented by a plurisubharmonic function ψ on $(\mathbb{C}^*)^n$ by

$$\psi = -\log \|s_0\|^2.$$

Then ψ satisfies $dd^c\psi = \theta$. Using toric coordinates

$$(x_1, \dots, x_n) = (\log |z_1|, \dots, \log |z_n|) \in \mathbb{R}^n$$

ψ defines a convex function on \mathbb{R}^n

$$f(x_1, \dots, x_n) := \psi(e^{x_1}, \dots, e^{x_n})$$

which will have the property $\overline{\nabla f(\mathbb{R}^n)} = P$. Moreover, in logarithmic coordinates $\sigma_i = \log z_i$ we have

$$\sum_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} d\sigma_i d\bar{\sigma}_j = dd^c\psi = \theta. \tag{35}$$

Now, for a convex polytope P , let $E(P)$ be the space of smooth, strictly convex functions f such that

$$\overline{\nabla f(\mathbb{R}^n)} = P.$$

Then it is well known (see for example Proposition 3.3, page 687 in [Berman and Berndtsson 2013]) that (35) gives a one-to-one correspondence between the $(S^1)^n$ -invariant elements in $[\theta]$ and $E(P)$.

As noted in the Introduction, the correspondence above extends trivially to any θ such that $[\theta]$ can be written as a linear combination with positive real coefficients of Kähler classes that arise as the curvature

of toric line bundles. On the other hand, we have the following general principle which we record for the convenience of the reader:

Lemma 23. *Let α be a Kähler class on a Fano manifold X . Then there are some ample line bundles L_1, \dots, L_m over X and positive real coefficients $\lambda_1, \dots, \lambda_m$ such that*

$$\alpha = \sum_i \lambda_i c_1(L_i). \tag{36}$$

Proof. First of all, any Kähler class α can be written as (36) where the line bundles L_i are not necessarily ample and the constants λ_i are not necessarily positive. To see this, recall that the map

$$c_1 : H^1(X, \mathcal{O}^*) \rightarrow H^2(X, \mathbb{Z})$$

is part of the exact sequence

$$H^1(X, \mathcal{O}^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathcal{O}).$$

By the Kodaira vanishing theorem, since X is Fano,

$$H^2(X, \mathcal{O}) = H^2(X, K_X - K_X) = 0.$$

It follows that c_1 is surjective, and hence any element in $H^2_{DR}(X) \cong H^2(X, \mathbb{R})$ can be written as a linear combination over \mathbb{R} of elements in the image of c_1 . Note that this means the set of rational classes, in other words the set of classes of the form $q c_1(L)$ for some rational number q and some line bundle L , is dense in $H^{(1,1)}(X)$.

Now, the cone of Kähler classes K is open in $H^{(1,1)}(X)$. This means we can take a set of rational classes η_1, \dots, η_j in K that span $H^{(1,1)}(X)$ over \mathbb{R} . Moreover, these classes define an open subcone of K ,

$$C = \left\{ \sum_i \lambda_i \eta_i : \lambda_i \in \mathbb{R}_+ \right\}.$$

For any $\alpha \in K$ we may take a rational class η_0 in the open set $(\alpha - C) \cap K$ which is nonempty since α is in the interior of K . This means $\alpha = \eta_0 + \kappa$, where $\kappa \in C$ and (36) follows. □

Noting that any divisor on a toric manifold is linearly equivalent to an $(S^1)^n$ -invariant divisor, Lemma 23 and the discussion preceding it gives:

Lemma 24. *Let α be a Kähler class on X and P be the polytope corresponding to α . Then (35) gives a one-to-one correspondence between the $(S^1)^n$ -invariant elements in α and $E(P)$. Moreover, if $\alpha = c_1(L)$, where L is a toric line bundle over X , then this correspondence is given by $\theta \mapsto f$, where*

$$f(\log |z_1|, \dots, \log |z_n|) := -\log \|s_0\|^2,$$

where s_0 is the $(S^1)^n$ -invariant (meromorphic) section corresponding to the point $0 \in \mathcal{M} \otimes \mathbb{R}$ and $\|\cdot\|$ is the metric on L with curvature θ .

For each i , let $h_i : \mathbb{R}^n \rightarrow \mathbb{R}$ be defined by

$$h_i(x) = \log \frac{1}{N_P} \sum_y e^{\langle y, x \rangle},$$

where the sum is taken over all vertices of the polytope P_i and N_P is the number of vertices of the polytope P_i . These functions are smooth, strictly convex and satisfy $\overline{\nabla h_i(\mathbb{R}^n)} = P_i$; hence $h_i \in E(P_i)$. For each i , let θ_i be the element in α_i corresponding to h_i . Then there is a one-to-one correspondence between $E(P_i)$ and the smooth $(S^1)^n$ -invariant elements of $\text{PSH}(X, \theta_i)$ given by

$$f_i(x) - h_i(x) = \phi_i(e^x). \tag{37}$$

Moreover, $h_i(0) = 0$ for each i . This means the normalization (16) is equivalent to

$$f_1(0) = \dots = f_k(0). \tag{38}$$

Using the correspondence in (37), it is possible to rewrite (15) as a real Monge–Ampère equation.

Lemma 25. *Assume (ϕ_i) and (f_i) are related as in (37). Then, for $t \in [0, 1]$, (ϕ_i) satisfies (15) if and only if (f_i) satisfies*

$$\frac{e^{\langle V_1, \nabla f_1 \rangle}}{\text{Vol}_{V_1}(P_1)} \det\left(\frac{\partial^2 f_1}{\partial x_m \partial x_l}\right) = \dots = \frac{e^{\langle V_k, \nabla f_k \rangle}}{\text{Vol}_{V_k}(P_k)} \det\left(\frac{\partial^2 f_k}{\partial x_m \partial x_l}\right) = e^{-t \sum_i f_i - (1-t) \sum_i h_i}. \tag{39}$$

Proof. First of all, using (35) we see that

$$(\theta_i + dd^c \phi_i)^n = \left(\sum_{m,l} \frac{\partial^2 f_i}{\partial x_m \partial x_l} d\sigma_j d\bar{\sigma}_l \right)^n = \det\left(\frac{\partial^2 f_i}{\partial x_m \partial x_l}\right) d\sigma d\bar{\sigma}, \tag{40}$$

where $d\sigma d\bar{\sigma} = d\sigma_1 \dots d\sigma_n d\bar{\sigma}_1 \dots d\bar{\sigma}_n$.

Abusing notation, we may think of f_i and h_i as $(S^1)^n$ -invariant plurisubharmonic functions on $(\mathbb{C}^*)^n \subset X$. We will show that

$$e^{-t \sum_i \phi_i} \omega_0^n = e^{-t \sum_i (f_i - h_i)} \omega_0^n = e^{-t \sum_i f_i - (1-t) \sum_i h_i} d\sigma d\bar{\sigma}. \tag{41}$$

This will follow if we show that

$$e^{\sum h_i} \omega_0^n = d\sigma d\bar{\sigma}. \tag{42}$$

To do this, we note that by convexity

$$\overline{\left(\sum_i h_i \right) (\mathbb{R}^n)} = \overline{\sum_i \nabla h_i (\mathbb{R}^n)} = \sum P_i = P_{-K_X}.$$

By Lemma 24, $\sum h_i$ defines a metric on $-K_X$ of curvature $\sum \theta_i$ by the relation

$$\|s_0\|_{\sum h_i}^2 = e^{-\sum h_i},$$

where s_0 is the unique $(\mathbb{C}^*)^n$ -invariant section of $-K_X$, in other words

$$s_0 = \frac{\partial}{\partial \sigma_1} \wedge \dots \wedge \frac{\partial}{\partial \sigma_k} = d\sigma^{-1}.$$

Moreover, the volume form ω_0^n defines a metric on $-K_X$ by the relation

$$\|d\sigma^{-1}\|_{\omega_0^n}^2 = \frac{\omega_0^n}{d\sigma d\bar{\sigma}}.$$

The curvature of $\|\cdot\|_{\omega_0^n}$ is $\text{Ric } \omega_0 = \sum \theta_i$. By uniqueness in the Calabi–Yau theorem $\|\cdot\|_{\sum h_k} = \|\cdot\|_{\omega_0^n}$ and (42) follows.

It remains to show that

$$\frac{e^{\langle V_i, \nabla f_i \rangle}}{\text{Vol}_{V_i}(P_i)} = e^{g_i + V_i(\phi_i)}. \tag{43}$$

We will first show that

$$\langle V_i, \nabla f_i \rangle + C_i = g_i + V_i(\phi_i) \tag{44}$$

for some $C_i \in \mathbb{R}$. Abusing notation again, and thinking of f_i as an $(S^1)^n$ -invariant plurisubharmonic function on $(\mathbb{C}^*)^n \subset X$, we compute

$$\begin{aligned} dd^c \langle V_i, \nabla f_i \rangle &= dd^c \left(\sum_m \frac{\partial f_i}{\partial x_m} a_m \right) = \sum_{m,j,l} \frac{\partial^3 f_i}{\partial x_j \partial x_l \partial x_m} a_m d\sigma_j d\bar{\sigma}_l \\ &= \partial i_V \left(\sum_{m,l} \frac{\partial^2 f_i}{\partial x_m \partial x_l} d\sigma_m d\bar{\sigma}_l \right) \\ &= \partial i_V (\theta_i + dd^c \phi_i) = L_V(\theta_i) = dd^c (g_i + V_i(\phi_i)) \end{aligned}$$

and (44) follows by the maximum principle. To get (43), note that the push forward of $d\sigma d\bar{\sigma}$ under the map $(z_1, \dots, z_n) \mapsto (\log |z_1|, \dots, \log |z_n|)$ is the Euclidean measure dx on \mathbb{R}^n . This means, by (40) and (44),

$$\int_X e^{g_i + V_i(\phi_i)} (\theta_i + dd^c \phi_i)^n = \int_{\mathbb{R}^n} \det \left(\frac{\partial^2 f_i}{\partial x_m \partial x_l} \right) e^{\langle V_i, \nabla f_i \rangle + C_i} dx. \tag{45}$$

Performing the change of variables $\nabla f_i = p$ we get

$$(45) = e^{C_i} \int_{P_i} e^{\langle V_i, p \rangle} dp.$$

By (26)

$$\int_X e^{g_i + V_i(\phi_i)} (\theta_i + dd^c \phi_i)^n = \int_X e^{g_i} \theta_i^n = 1.$$

This means $C = \log \text{Vol}_{V_i}(P_i)$ and (43) follows.

Using (40), (41) and (43) we conclude that (f_i) satisfies (13) if and only if (ϕ_i) satisfies (15) on $(\mathbb{C}^*)^n$. As (ϕ_i) is assumed to be smooth, the lemma follows. \square

3.1. Estimates. To prove Theorem 14 we need to prove that for all $t_0 > 0$ there is a constant C such that any solution (f_i) to (39) at $t > t_0$, normalized according to (38), satisfies

$$\sup_X |f_i - h_i| \leq C \tag{46}$$

for all i .

For each i , let u_i be the Legendre transform of f_i . Recall that f_i is a smooth, strictly convex function on \mathbb{R}^n such that $\overline{\nabla f_i(\mathbb{R}^n)} = P_i$. This means each u_i is a smooth, strictly convex function on P_i . Moreover, a standard property of the Legendre transform is that

$$\sup_{\mathbb{R}^n} |f_i - h_i| = \sup_{P_i} |u_i - h_i^*|,$$

where h_i^* is the Legendre transform of h_i . Since h_i^* is bounded on P_i (this is easy to verify) we have that (46) is equivalent to a uniform bound on $\sup_{P_i} |u_i|$.

We will use a variant of the method of [Wang and Zhu 2004] (see also [Donaldson 2008]). The first step is to establish bounds on the function

$$w = w_t = \sum_i (t f_i + (1 - t) h_i).$$

Since w is strictly convex and 0 is in the interior of $P_{-K_X} = \overline{\nabla w(\mathbb{R}^n)}$ we have that w is bounded from below and attains its minimal value at a unique point. Let $m = \inf w$ and let x_w be the minimal point of w .

Lemma 26. *Assume $t_0 > 0$ and (12) holds. Then there are constants C and ϵ such that if (f_i) is a solution to (39) at $t > t_0$, then*

$$w \geq \epsilon |x - x_w| - C \tag{47}$$

and

$$|x_w| \leq C. \tag{48}$$

The proof of Lemma 26 follows one of the arguments in [Donaldson 2008], which is based on [Wang and Zhu 2004]. The main point is the following convex geometric fact (see Proposition 2 in [Donaldson 2008]).

Lemma 27. *Assume f is a convex function on \mathbb{R}^n attaining minimal value 0 , and suppose*

$$\det\left(\frac{\partial^2 f}{\partial x_m \partial x_l}\right) \geq \lambda$$

on $K = \{f \leq 1\}$. Then

$$\text{Vol}(K) \leq C \lambda^{-1/2}$$

for some constant C depending only on the dimension n .

Using Lemma 27 we can prove Lemma 26.

Proof of Lemma 26. The proof proceeds in four steps:

Step 1: m is bounded from below. Let ρ_{-K_X} be the support function of P_{-K_X} defined by

$$\rho_{-K_X}(x) = \sup_{p \in P_{-K_X}} \langle x, p \rangle.$$

Since $\nabla w(\mathbb{R}^n) = P_{-K_X}$ we have $w \leq m + \rho_{-K_X}$. Moreover, by the change of variables $p = \nabla f_i$

$$1 = \frac{\int_{P_i} e^{\langle V_i, p \rangle} dp}{\text{Vol}_{V_i}(P_i)} = \int_{\mathbb{R}^n} \frac{e^{\langle V_i, \nabla f_i \rangle}}{\text{Vol}_{V_i}(P_i)} \det\left(\frac{\partial^2 f_i}{\partial x_m \partial x_l}\right) dx = \int_{\mathbb{R}^n} e^{-w} dx \geq C e^{-m} \int_{\mathbb{R}^n} e^{-\rho_{-K_X}} dx \geq C e^{-m},$$

possibly changing C in the last inequality. This means m is bounded from below by a uniform constant.

Step 2: m is bounded from above. By monotonicity of the determinant function and convexity we have

$$\begin{aligned} \det\left(\frac{\partial^2 w}{\partial x_m \partial x_l}\right) &= \det\left[t \sum_i \left(\frac{\partial^2 f_i}{\partial x_m \partial x_l}\right) + (1-t) \sum_i \left(\frac{\partial^2 h_i}{\partial x_m \partial x_l}\right)\right] \\ &\geq t_0^n \det\left(\frac{\partial^2 f_j}{\partial x_m \partial x_l}\right) = t_0^n \text{Vol}_{V_j}(P_j) e^{-(V_j, \nabla f_j) - w} \geq C e^{-w} dx, \end{aligned}$$

where the last inequality follows from the fact that $\overline{\nabla f_j(\mathbb{R}^n)} = P_j$ is bounded. This means

$$\det\left(\frac{\partial^2 w}{\partial x_m \partial x_l}\right) \geq C e^{-m-1}$$

on $K = \{w \leq m + 1\}$. By Lemma 27, possibly redefining C ,

$$\text{Vol}(K) \leq C e^{m/2}. \tag{49}$$

Convexity of w and the coarea formula give

$$1 = \int_{\mathbb{R}^n} e^{-w} dx \leq C e^{-m/2}.$$

This means m is bounded from above.

Step 3: $w \geq \epsilon |\cdot - x_w| - m + 1$ for uniform constants ϵ and C . Since $\overline{\nabla w(\mathbb{R}^n)} = P_{-K_x}$ and P_{-K_x} is bounded we have that there is a uniform constant $r > 0$ such that K contains a small ball centered at x_w of radius r . If there was a point in K far from x_w then the volume of K would be very big, contradicting (49). This means K is contained in a ball centered at x_w of radius R for some uniform constant R . Convexity of w gives

$$w(x) \geq \begin{cases} R^{-1}|x - x_w| + m & \text{if } x \notin K, \\ m & \text{if } x \in K. \end{cases}$$

Moreover, $R^{-1}|x - x_w| \leq 1$ on K . This means putting $\epsilon = 1/R$ finishes Step 3.

Step 4: $|x_w|$ is bounded. In this step we will use the assumption (12). By the divergence theorem, since $e^{-w} \rightarrow 0$ exponentially as $|x| \rightarrow \infty$,

$$\int_{\mathbb{R}^n} \nabla w e^{-w} dx = \int_{\mathbb{R}^n} \text{div} \nabla(e^{-w}) dx = 0.$$

Moreover,

$$\begin{aligned} \int_{\mathbb{R}^n} \nabla \left(\sum_i f_i \right) e^{-w} dx &= \sum_i \int_{\mathbb{R}^n} \nabla f_i e^{-w} dx \\ &= \sum_i \frac{1}{\text{Vol}_{V_i}(P_i)} \int_{\mathbb{R}^n} \nabla f_i e^{\langle V_i, \nabla f_i \rangle} \det\left(\frac{\partial^2 f_i}{\partial x_m \partial x_l}\right) dx \\ &= \sum_i \frac{1}{\text{Vol}_{V_i}(P_i)} \int_{P_i} p e^{\langle V_i, p \rangle} dp = 0, \end{aligned}$$

where the last two equalities are given by performing the change of variables $p = \nabla f_i(x)$ in each summand and (12). This means

$$\int_{\mathbb{R}^n} \nabla \left(\sum_i h_i \right) e^{-w} dx = 0. \tag{50}$$

Recall that $\sum h_i$ is convex and hence $\nabla(\sum_i h_i)$ is monotone. Hence, if $|x_w|$ is large then, putting $v = x_w/|x_w|$, we get that $\langle x, v \rangle$ is positive and bounded away from 0 on some large ball centered at x_w . By (47) the mass of $e^{-w} dx$ is concentrated around x_w . This contradicts (50). \square

We can now prove Theorem 14.

Proof of Theorem 14. First of all, by the change of variables $x = \nabla u_i(p)$ and (15) we have

$$\begin{aligned} \int_{P_i} |\nabla u_i|^q dp &= \int_{\mathbb{R}^n} |x|^q \det \left(\frac{\partial^2 f_i}{\partial x_m \partial x_l} \right) dx \\ &\leq \text{Vol}_{V_i}(P_i) \int_{\mathbb{R}^n} |x|^q e^{-(V_i, \nabla f_i) - w} dx \\ &\leq C \int_{\mathbb{R}^n} |x|^q e^{-w} dx \leq C_q, \end{aligned} \tag{51}$$

where the second inequality follows from the fact that $\overline{\nabla f_i(\mathbb{R}^n)} = P_i$ is bounded and the last inequality follows from Lemma 26. Put $q = n + 1$ and

$$\hat{u}_i = \frac{1}{\text{Vol}(P_i)} \int_{P_i} u_i dp.$$

By Morrey’s inequality (see [Hařkovec and Schmeiser 2009]) we have

$$\begin{aligned} \|u_i - \hat{u}_i\|_{C^{0,\gamma}(P_i)} &\leq C \|u_i - \hat{u}_i\|_{W^{1,q}(P_i)} \\ &= C \|u_i - \hat{u}_i\|_{L^q(P_i)} + C \|\nabla u_i\|_{L^q(P_i)}, \end{aligned} \tag{52}$$

where $\gamma = 1 - n/q$. By the Poincaré–Wirtinger inequality this can be bounded by

$$C \|\nabla u_i\|_{L^q(P_i)}$$

for some C . This is bounded by (51). Since P_i is bounded we may conclude from this that

$$\sup_{p_1, p_2 \in P_i} |u_i(p_1) - u_i(p_2)| \leq C \|u_i - \hat{u}_i\|_{C^{0,\gamma}(P_i)} \leq C. \tag{53}$$

This means it suffices to bound each u_i in some point.

To bound each u_i in some point, note that by general properties of the Legendre transform $f_i(0) = -u_i(\nabla f_i(0))$. This means

$$|u_i(\nabla f_i(0))| = |f_i(0)| = \frac{1}{k} \left| \sum_j f_j(0) \right| = \frac{1}{k} |w(0)|,$$

where the last two equalities follow from (38) and the fact that $h_i(0) = 0$ for all i . Since $|x_w|$ is bounded and $\nabla w \in P_{-K_X}$ is bounded we have that $|w(0) - w(x_w)|$ is bounded. By Lemma 26, $|w(x_w)| = |m|$ is

bounded. This means $|u_i(\nabla f_i(0))|$ and hence, by (53), $\sup_{P_i} |u_i|$ is bounded for each i . By the discussion following (46) this proves the theorem. \square

Proof of Theorem 8. Assuming (12) holds, existence of coupled Kähler–Ricci solitons follow directly from Theorem 14 and Theorem 15. Indeed, any toric holomorphic vector field V_i is in the reductive part of the Lie algebra of $\text{Aut}(X)$. Moreover, $\text{Im } V_i$ generates a compact one-parameter subgroup of $\text{Aut}(X)$ and, since θ_i is $(S^1)^n$ -invariant, $\text{Im } L_{V_i}(\theta_i) = 0$.

Assume that (α_i) admits a coupled Kähler–Ricci soliton. By Lemmas 18 and 25, (13) admits a solution. Then (12) follows from Lemma 28 below. \square

Lemma 28. *Assume (13) admits a solution. Then*

$$\sum_i \mathcal{A}_{P_i}(V_i) = 0.$$

Proof. Let (f_i) be a solution to (13). As in the proof of Lemma 26, by the divergence theorem, since $e^{-\sum f_i} \rightarrow 0$ exponentially as $|x| \rightarrow \infty$,

$$\int_{\mathbb{R}^n} \left(\sum_i \nabla f_i \right) e^{-\sum_i f_i} dx = \int_{\mathbb{R}^n} \text{div } \nabla (e^{-\sum_i f_i}) dx = 0. \tag{54}$$

On the other hand, by (13)

$$(54) = \sum_i \int_{\mathbb{R}^n} \nabla f_i e^{-\sum_i f_i} dx = \sum_i \int_{\mathbb{R}^n} \nabla f_i \frac{e^{\langle V_i, \nabla f_i \rangle}}{\text{Vol}_{V_i}(P_i)} \det \left(\frac{\partial^2 f_i}{\partial x_m \partial x_l} \right) dx.$$

Performing the change of variables $\nabla f_i = p$ in each summand gives that the right-hand side of this equals

$$\sum_i \frac{1}{\text{Vol}_{V_i}(P_i)} \int_{P_i} p e^{\langle V_i, p \rangle} dp = \sum_i \mathcal{A}_{P_i}(V_i). \tag{55}$$

Proof of Corollary 11. Note that

$$\sum_i \mathcal{A}_{P_i}(V) \tag{55}$$

is the gradient of the function on \mathbb{R}^n defined by

$$V \mapsto \sum_i \log \int_{P_i} e^{\langle V, p \rangle} dp.$$

This is strictly convex and proper (in fact, its gradient image is $\sum_i P_i = P_{-K_X}$, which contains zero as an interior point); hence it admits a unique minimum. Letting V be this minimum means (12) is fulfilled. The corollary then follows from Theorem 8. \square

Proof of Corollary 13. The corollary follows from Theorem 8 and Lemma 25. \square

3.2. Toric test configurations and proof of Theorem 2. Theorem 2 will follow from Theorem 8 combined with Theorem 1.15 in [Hultgren and Nyström 2018] and an explicit calculation of the Donaldson–Futaki invariant of test configurations induced by toric vector fields.

In [Hultgren and Nyström 2018] a type of test configuration for decompositions of $c_1(X)$ was defined. The data defining them is essentially given by k test configurations $(\mathcal{X}_1, \mathcal{L}_1), \dots, (\mathcal{X}_k, \mathcal{L}_k)$, where $\mathcal{X}_1 = \dots = \mathcal{X}_k =: \mathcal{X}$, such that $(\mathcal{X}, \sum_i \mathcal{L}_i)$ defines a test configuration for $(X, -K_X)$. The Donaldson–Futaki invariant associated to this data is defined as the intersection number

$$\text{DF}(\mathcal{X}, (\mathcal{L}_i)) = - \sum_i \frac{\mathcal{L}_i^{n+1}}{|\alpha_i|} - (n + 1) \frac{(-K_{\mathcal{X}/\mathbb{P}^1} - \sum_i \mathcal{L}_i) \cdot (\sum_i \mathcal{L}_i)^n}{(-K_X)^n}, \tag{56}$$

where $|\alpha_i| = \int_X \theta^n$ for any θ such that $[\theta] = \alpha$. We point out that the notation here differs from [Hultgren and Nyström 2018] in that here $(\mathcal{X}, \mathcal{L}_i)$ are the $(\mathbb{C}^*$ -invariantly) compactified test configurations over \mathbb{P}^1 .

Now, recall that if L is a toric line bundle over a toric manifold X , then a toric vector field V induces a test configuration $(\mathcal{X}^V, \mathcal{L}^V)$ for (X, L) . This can be described in the following way: Let $d_1, \dots, d_k \in N \otimes \mathbb{R}$ and $c_1, \dots, c_k \in \mathbb{R}$ be the data defining the polytope P_L , i.e.,

$$P_L = \{ \langle d_i, \cdot \rangle \geq -c_i \}.$$

Then, the polytope of \mathcal{L}^V can be arranged to be

$$P_{\mathcal{L}^V} = \{ \langle d_i, \cdot \rangle \geq -c_i \} \cap \{ \langle d_0 + V, \cdot \rangle \geq 0 \} \cap \{ \langle -d_0, \cdot \rangle \geq -C_{\mathcal{L}^V} \},$$

where d_0 corresponds to the divisor given by the central fiber of \mathcal{X} and $C_{\mathcal{L}^V}$ is a number that can be modified without changing the Donaldson–Futaki invariant by adding a multiple $\mathcal{O}_{\mathbb{P}^1}(1)$ to \mathcal{L}^V . In particular, as long as $C_{\mathcal{L}^V}$ is big enough for \mathcal{L}^V to be ample,

$$(\mathcal{L}^V)^{n+1} = \text{Vol}(P_{\mathcal{L}^V}) = \text{Vol}(P_L)(C_{\mathcal{L}^V} + \langle V, b(P_L) \rangle).$$

This also gives

$$\begin{aligned} (n + 1)\mathcal{O}_{\mathbb{P}^1}(1) \cdot (\mathcal{L}^V)^n &= \frac{d}{dt} (\mathcal{L}^V + t\mathcal{O}_{\mathbb{P}^1}(1))^{n+1} \\ &= \frac{d}{dt} \text{Vol}(P_{\mathcal{L}^V + t\mathcal{O}_{\mathbb{P}^1}(1)}) = \text{Vol}(P_L). \end{aligned} \tag{57}$$

Finally, we note that if $L = -K_X$ then \mathcal{L}^V is the relative canonical bundle of \mathcal{X}^V up to a twist determined by $C_{\mathcal{L}^V}$.

$$\mathcal{L}^V = -K_{\mathcal{X}^V/\mathbb{P}^1} + C_{\mathcal{L}^V}\mathcal{O}_{\mathbb{P}^1}(1). \tag{58}$$

Proof of Theorem 2. Putting $V_1 = \dots = V_k = 0$ gives

$$\sum_i \mathcal{A}_{P_i}(V_i) = \sum_i b(P_i);$$

hence it follows from Theorem 8 that part (iii) of the theorem implies part (i). Moreover, (i) implies (ii) by Theorem 1.15 in [Hultgren and Nyström 2018]. Thus, to finish the proof of Theorem 2, it suffices to prove that (ii) implies (iii).

We will prove the contrapositive. Assume $\sum_i b(P_i) \neq 0$; in other words $\sum_i \langle V, b(P_i) \rangle < 0$ for some toric vector field V . Let $(\mathcal{X}^V, (\mathcal{L}_i^V))$ be the associated test configuration. As $(\mathcal{X}^V, \sum_i \mathcal{L}_i^V)$ is a test configuration for $-K_X$ we get, using (58) and $|\alpha_i| = \text{Vol}(P_i)$,

$$\begin{aligned} \text{DF}(\mathcal{X}^V, (\mathcal{L}_i^V)) &= \sum_i \frac{(\mathcal{L}_i^V)^{n+1}}{\text{Vol}(P_i)} - (n+1) \frac{(\sum_i C_{\mathcal{L}_i^V}) \mathcal{O}_{\mathbb{P}^1}(1) \cdot (\sum_i \mathcal{L}_i^V)^n}{\text{Vol}(P_{-K_X})} \\ &= \sum_i (C_{\mathcal{L}_i^V} + \langle V, b(P_i) \rangle) - \sum_i C_{\mathcal{L}_i^V} \\ &= \sum_i \langle V, b(P_i) \rangle < 0, \end{aligned} \tag{59}$$

and hence (α_i) is not K-polystable. □

3.3. Proof of Corollary 4.

Proof of Corollary 4. First of all, by [Futaki et al. 1990] (see also [Futaki 1983; Wang 1991]) the Futaki invariant of X is nonzero; hence X does not admit a Kähler–Einstein metric. To prove the rest of the corollary, we fix a $(\mathbb{C}^*)^4$ -action on X in the following way: Consider the standard embeddings of $\mathcal{O}_{\mathbb{P}^2}(-1)$ and $\mathcal{O}_{\mathbb{P}^1}(-1)$ in to $\mathbb{C}^3 \times \mathbb{P}^2$ and $\mathbb{C}^2 \times \mathbb{P}^1$ respectively:

$$\begin{aligned} \mathcal{O}_{\mathbb{P}^2}(-1) &= \{((z_0, z_1, z_2), (a_0 : a_1 : a_2)) : z_0 a_1 = z_1 a_0, z_1 a_2 = z_2 a_1\}, \\ \mathcal{O}_{\mathbb{P}^1}(-1) &= \{((w_0, w_1), (b_0 : b_1)) : w_0 b_1 = w_1 b_0\}. \end{aligned}$$

We get an embedding of $X = \mathbb{P}(E)$ into $\mathbb{P}^4 \times \mathbb{P}^2 \times \mathbb{P}^1$ as

$$X = \{((z_0 : z_1 : z_2 : w_0 : w_1), (a_0 : a_1 : a_2), (b_0 : b_1)) : z_0 a_1 = z_1 a_0, z_1 a_2 = z_2 a_1, w_0 b_1 = w_1 b_0\}.$$

We define a $(\mathbb{C}^*)^4$ -action by letting an element $(t_1, t_2, t_3, t_4) \in (\mathbb{C}^*)^4$ act on X by

$$((z_0 : z_1 : z_2 : w_0 : w_1), (a_0 : a_1 : a_2), (b_0 : b_1)) \mapsto ((z_0 : t_1 z_1 : t_2 z_2 : t_4 w_0 : t_3 w_1), (a_0 : t_1 a_1 : t_2 a_2), (b_0 : t_3 b_1)).$$

The invariant divisors are

$$\begin{aligned} D_1 &= \{z_0 = a_0 = 0\}, & D_2 &= \{z_1 = a_1 = 0\}, & D_3 &= \{z_2 = a_2 = 0\}, & D_4 &= \{w_0 = b_0 = 0\}, \\ D_5 &= \{w_1 = b_1 = 0\}, & D_6 &= \{z_0 = z_1 = z_2 = 0\}, & D_7 &= \{w_0 = w_1 = 0\} \end{aligned}$$

corresponding to the following elements in the lattice $N \cong \mathbb{Z}^4$ of one-parameter subgroups of $(\mathbb{C}^*)^4$:

$$\begin{aligned} d_1 &= (-1, -1, 0, -1), & d_2 &= (1, 0, 0, 0), & d_3 &= (0, 1, 0, 0), & d_4 &= (0, 0, -1, 1), \\ d_5 &= (0, 0, 1, 0), & d_6 &= (0, 0, 0, -1), & d_7 &= (0, 0, 0, 1). \end{aligned}$$

The divisor corresponding to $-K_X$ is $\sum_{i=1}^7 D_i$. For $c \in (\frac{1}{4}, \frac{3}{4})$, we will be interested in divisors of the form

$$D(c) = c(D_4 + D_5) + \sum_{i \neq 4,5} \frac{1}{2} D_i$$

corresponding to polytopes

$$P(c) = \{y \in \mathbb{R}^4 : \langle y, d_i \rangle \leq \frac{1}{2}, i \neq 4, 5, \langle y, d_i \rangle \leq c, i = 4, 5\}. \tag{60}$$

Note that the two classes in (5) are given by $D(c)$ and $D(1 - c)$ for

$$c = \frac{1}{2} + \frac{1}{4}\sqrt{\frac{5}{7}} \in \left(\frac{1}{4}, \frac{3}{4}\right). \tag{61}$$

To prove the corollary we will verify the following two facts:

- As long as $c \in \left(\frac{1}{4}, \frac{3}{4}\right)$, none of the conditions in (60) are redundant. (By standard theory for toric varieties this implies $D(c)$ and $D(-c)$ are ample and hence β_1 and β_2 are Kähler.)
- We have

$$\frac{\int_{P(c)} y \, dy}{\int_{P(c)} dy} + \frac{\int_{P(1-c)} y \, dy}{\int_{P(1-c)} dy} = 0$$

when c is given by (61).

Note that both these conditions are invariant under linear transformations of \mathbb{R}^n . Applying to the generators d_1, \dots, d_7 the linear transformation

$$A = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

gives new generators

$$\begin{aligned} d'_1 &= (-1, -1, 0, -2), & d'_2 &= (1, 0, 0, -2), & d'_3 &= (0, 1, 0, -2), & d'_4 &= (0, 0, -1, 3), \\ d'_5 &= (0, 0, 1, 3), & d'_6 &= (0, 0, 0, 6), & d'_7 &= (0, 0, 0, -6), \end{aligned}$$

and a new polytope

$$P'(c) = \{y \in \mathbb{R}^4 : \langle y, d'_i \rangle \leq \frac{1}{2}, i \neq 4, 5, \langle y, d'_i \rangle \leq c, i = 4, 5\}. \tag{62}$$

It is straightforward to check that as long as $c \in \left(\frac{1}{4}, \frac{3}{4}\right)$, none of the conditions in (62) are redundant; hence $D(c)$ is ample for any $c \in \left(\frac{1}{4}, \frac{3}{4}\right)$. Moreover, the sets $\{d'_1, d'_2, d'_3, d'_6, d'_7\}$ and $\{d'_4, d'_5\}$ are both invariant under the linear transformation

$$B = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It follows that $P'(c)$ and hence the barycenter of $P'(c)$ is invariant under B . As any fixed point of B is parallel to $(0, 0, 0, 1)$ we conclude that

$$\int_{P'(c)} y_1 \, dy = \int_{P'(c)} y_2 \, dy = \int_{P'(c)} y_3 \, dy = 0.$$

Moreover, we denote by S_2 the two-dimensional simplex corresponding to the anticanonical bundle of \mathbb{P}^2

$$S_2 = \{y \in \mathbb{R}^2 : y_1 \leq 1, y_2 \leq 1, -y_1 - y_2 \leq 1\}$$

and note that $(y_1, \dots, y_4) \in P'(c)$ if and only if $y_4 \in (-\frac{1}{12}, \frac{1}{12})$, $|y_3| \leq c - 3y_4$ and $(y_1, y_2) \in (\frac{1}{2} + 2y_4)S_2$. We get

$$\begin{aligned} \int_{P'(c)} y_4 dy &= \int_{\frac{1}{12}[-1,1]} y_4 \left(\int_{(\frac{1}{2}+2y_4)S_2} dy_1 dy_2 \right) \left(\int_{(c-3y_4)[-1,1]} dy_3 \right) dy_4 \\ &= 2 \text{Vol}(S_2) \int_{\frac{1}{12}[-1,1]} y_4 \left(\frac{1}{2} + 2y_4\right)^2 (c - 3y_4) dy_4 = \frac{5c - 2}{720} \end{aligned}$$

and similarly

$$\int_{P'(c)} dy = 2 \text{Vol}(S_2) \int_{\frac{1}{12}[-1,1]} \left(\frac{1}{2} + 2y_4\right)^2 (c - 3y_4) dy_4 = \frac{56c - 3}{144}.$$

It follows that

$$\frac{\int_{P'(c)} y_4 dy}{\int_{P'(c)} dy} + \frac{\int_{P'(1-c)} y_4 dy}{\int_{P'(1-c)} dy} = \frac{1}{5} \left(\frac{5c - 2}{56c - 3} + \frac{5(1 - c) - 2}{56(1 - c) - 3} \right) = \frac{(112c^2 - 112c + 23)}{(56c - 53)(56c - 3)}, \tag{63}$$

which vanishes as

$$c = \frac{1}{2} \pm \frac{1}{4} \sqrt{\frac{5}{7}} \in \left(\frac{1}{4}, \frac{3}{4}\right). \quad \square$$

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