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A VECTOR FIELD METHOD FOR RADIATING BLACK HOLE SPACETIMES

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We develop a commuting vector field method for a general class of radiating spacetimes. The metrics we consider are modeled on those constructed from global nonlinear stability problems in general relativity, and our method provides sharp peeling estimates for solutions to both linear and (null form) nonlinear scalar fields.

1. Introduction

In this paper we develop a sharp variant of Klainerman's vector field method for solutions to scalar wave equations on a generic class of asymptotically flat spacetimes. These are taken to be certain long-range perturbations of Minkowski space which enjoy a standard local energy decay assumption.

The first main consideration here is to place the minimal conditions on our metrics at null infinity which are compatible with gravitational radiation, but at the same time are also strong enough to provide full peeling estimates for scalar waves. Our conditions turn out to be natural even if one is only interested in stationary long-range perturbations of Minkowski space, because they highlight certain peeling properties of Lorentzian metrics at null infinity which appear to be necessary in order to produce estimates on the order of the classical Morawetz conformal energy.

The second main consideration here is to produce a collection of norms which are natural for studying nonlinear stability problems, at least when the quadratic part of nonlinearity enjoys a certain generalized null condition. In fact, we will produce a range of norms with weights that are also capable of handling

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systems satisfying weaker "null conditions", so our setup should also be useful for a class of quasilinear stability problems where one assumes certain bounds on Ricci curvature; but we leave this to a subsequent work.

First we discuss the basic assumptions and results which are contained in this paper. Additional remarks and references with then follow.

1A. ***Basic notation and metric assumptions.*** When stating inequalities we will use $A \lesssim B$ to mean $A \leq CB$ for some fixed $C > 0$ independent of A, B. We also employ the index notation $A \lesssim_k B$ when $C = C(k)$ depends on some auxiliary parameter k (although we will not always use such notation when a dependency exists).

The setting for the paper is the following: We fix some compact $\mathcal{K} \subset \mathbb{R}^3$ with smooth boundary such that $\mathbb{R}^3 \setminus \mathcal{K}$ is connected. On the manifold $\mathcal{M} = [0, \infty) \times (\mathbb{R}^3 \setminus \mathcal{K})$ we suppose there is given a smooth Lorentzian metric $g_{\alpha\beta}$. We write (t, x) for rectangular coordinates $0 \leq t < \infty$ and $x \in \mathbb{R}^3 \setminus \mathcal{K}$. Note that we may assume $\mathcal{K} = \varnothing$. In the sequel we will also assume that the level sets $t = \text{const}$ are uniformly spacelike in the sense that $-C < g^{00} < -c$. In this paper $r = |x|$ with the Euclidean norm and $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$.

With respect to the coordinates (t, x) and Euclidean measure $dt\, dx$ we have L^p norms restricted to time slabs in $\mathcal{M}$ of the form $[0, T] \times (\mathbb{R}^3 \setminus \mathcal{K})$. These will be denoted by $L^p[0, T]$. On the time slice $t = \text{const}$ we denote by L^p_x the corresponding restricted norm, and we denote by $L^p_t L^q_x[0, T]$ the mixed norms. It will also be useful to employ various inhomogeneous Besov versions of these spaces. For example we define $\|\phi\|^p_{\ell^p_r L^q[0,T]} = \sum_{j\geq 0} \|\chi_j \phi\|^p_{L^q[0,T]}$, where χ_j is a spatial cutoff on scale $\langle x \rangle \approx 2^j$. Similar notation is used for dyadic summations with respect to the time variable t, and other auxiliary variables as well (such as the optical functions described below). We define fixed-time and spacetime Sobolev spaces which will be denoted by H^s_x and $H^s[0, T]$, with the convention that in both cases we consider *all* (t, x)-derivatives ∂^I for multiindex $|I| \leq s$.

In this work we make two basic assumptions about our spacetimes $(\mathcal{M}, g)$. The first is an asymptotic condition on the metric:

Definition 1.1 (asymptotically flat radiating spacetimes). Suppose $(\mathcal{M}, g)$ is given as above. Then we say $g_{\alpha\beta}$ is "outgoing radiating" if the following hold:

(I) (weak "optical function") On $\mathcal{M}$ there is defined a smooth $u = u(t, x)$ such that (u, x) forms a uniform set of coordinates in the sense that $C^{-1} < u_t < C$, and u has the symbol bounds

$$\|(\tau_-\partial_t)^i (\tau_x \tau_0 \partial_x)^J (\partial_t u - 1, \partial_i u + \omega^i)\|_{\ell^1_r L^\infty[0,\infty)} < \infty \quad \text{for all } (i, J) \in \mathbb{N} \times \mathbb{N}^3, \tag{1}$$

where $\tau_x = \langle x \rangle$, $\tau_- = \langle u \rangle$, $\tau_+ = \langle (t, x) \rangle$, $\tau_0 = \tau_- \tau_+^{-1}$, and where $\omega^i = x^i \tau_x^{-1}$. Henceforth we let $\partial = (\partial_t, \partial_x)$ denote the (t, x)-coordinate derivatives, and likewise $\partial^b = (\partial^b_u, \partial^b_x)$ will denote the (u, x)-coordinate derivatives.

(II) (outgoing radiation condition) First define symbol classes $\mathcal{Z}^k$ in terms of the seminorms (restricted to $t \in [0, \infty)$)

$$|||q|||_{k,N} := \sum_{i+|J|\leq N} \|\tau_0^{-k} (\tau_- \partial^b_u)^i (\tau_x \partial^b_x)^J q\|_{\ell^1_r L^\infty} + \sum_{i+|J|\leq N} \|\tau_0^{-k} (\tau_- \partial^b_u)^i (\tau_x \partial^b_x)^J q\|_{\ell^1_u \ell^1_r L^\infty(\frac{1}{2}t<r<2t)}. \tag{2}$$

Then for the inverse metric $g^{\alpha\beta}$ in (u,x)-coordinates one has the symbol bounds

$$g^{\alpha\beta}-h^{\alpha\beta}\in\mathcal{Z}^0,\quad \sqrt{|g|}g^{iu}+\omega^i\in\mathcal{Z}^{\frac{1}{2}},\quad g^{ui}-\omega^i\omega_j g^{uj}\in\mathcal{Z}^1,\quad g^{uu}\in\mathcal{Z}^2, \tag{3}$$

where

$$h^{uu}=0,\quad h^{ui}=-\omega^i,\quad h^{ij}=\delta^{ij}. \tag{4}$$

A number of further remarks are in order concerning the previous definition. Proofs are provided in Appendix A.

Remark 1.2. An immediate consequence of (1) is that one has a uniformly bounded change of frame between (∂_t,∂_x) and $(\partial_u^b,\partial_x^b)$. Specifically

$$\partial_\alpha^b=e_\alpha^\beta\partial_\beta,\quad \sum_{|I|\le k}|\partial^I e_\alpha^\beta|\lesssim_k 1,\qquad \partial_\alpha=f_\alpha^\beta\partial_\beta^b,\quad \sum_{|I|\le k}|(\partial^b)^I f_\alpha^\beta|\lesssim_k 1. \tag{5}$$

In particular one may use either (∂_t,∂_x) or $(\partial_u^b,\partial_x^b)$ to define the Sobolev spaces $H^s[0,T]$ and H_x^s, and for this purpose we will use these frames interchangeably in the sequel. Note however that the frames (∂_t,∂_x) and $(\partial_u^b,\partial_x^b)$ are not interchangeable in all contexts. See for instance Remark 1.12 below.

Remark 1.3. The condition (I) implies that $\tau_+^{-1}(u+\tau_x-t)=o_r(1)$, and together with (II) we have

$$\|(\tau_-\partial_t)^i(\tau_x\tau_0\partial_x)^J(g-\eta)\|_{\ell_r^1L^\infty[0,\infty)}<\infty\quad\text{for all } (i,J)\in\mathbb{N}\times\mathbb{N}^3,$$

where $\eta=\mathrm{diag}(-1,1,1,1)$ is the Minkowski metric in (t,x)-coordinates. In particular g is weakly asymptotically flat in the sense that $\partial^J(g-\eta)=o_r(1)$ for all $J\in\mathbb{N}^4$, and away from the region $\langle t-r\rangle\ll\langle t+r\rangle$ this can be strengthened to $(t\partial_t)^i(\tau_x\partial)^J(g-\eta)=o_r(1)$ for all $(i,J)\in\mathbb{N}\times\mathbb{N}^4$.

Remark 1.4. Definition 1.1 allows us to replace u by $\tilde{u}=\chi u+(1-\chi)(t-\tau_x)$, where χ is a cutoff supported where $\langle t-r\rangle\ll\langle t+r\rangle$ with bounds $|(\tau_+\partial)^J\chi|\lesssim 1$. Thus, in the sequel we shall always assume that $u=t-\tau_x$ away from the "wave zone" $\langle t-r\rangle\ll\langle t+r\rangle$.

Remark 1.5. For metrics which are quasistationary and quasispherical in the sense that $g=g_0+g_1$, where g_0 is spherical in (t,x)-coordinates and

$$\|\ln^2(1+\tau_x)(t\partial_t)^i(\tau_x\partial)^J(g_0-\eta)\|_{\ell_r^1L^\infty[0,\infty)}+\|\tau_x(t\partial_t)^i(\tau_x\partial)^J g_1\|_{\ell_r^1L^\infty[0,\infty)}<\infty\quad\text{for all } (i,J)\in\mathbb{N}\times\mathbb{N}^4,$$

it can be shown the conditions of Definition 1.1 hold. In particular this guarantees that our conditions hold for certain stationary long-range perturbations of the Kerr family of spacetimes. In general we leave as an open question to find natural (e.g., geometric) conditions on metrics which satisfy only

$$\|\tau_x^a(t\partial_t)^i(\tau_x\partial)^J(g-\eta)\|_{\ell_r^1L^\infty[0,\infty)}<\infty$$

for some $0\le a<1$, which would guarantee the existence of an optical function such as in Definition 1.1.

Remark 1.6. The peeling conditions (3) represent sharp streamlined versions of the asymptotic estimates obtained for geometric quantities in the nonlinear stability of Minkowski space (see [Christodoulou and Klainerman 1993; Bieri 2010]), as well as asymptotically flat dynamical black hole spacetimes (see [Dafermos et al. 2013]). While it would take a bit of work to translate the connection/curvature bounds of

[Christodoulou and Klainerman 1993; Bieri 2010; Dafermos et al. 2013] into coordinate conditions such as (3), we expect that the latter are in fact far more general. We leave this aspect of our work to further investigations.

1B. ***The local energy decay assumption.*** The second main assumption of this paper concerns the behavior of solutions to the inhomogeneous scalar wave equation on $(\mathcal{M}, g)$. Following [Marzuola et al. 2010; Tataru and Tohaneanu 2011; Tataru 2013] we define the local energy decay norms.

Definition 1.7 ("classical" local energy decay norms). First set

$$\|\phi\|_{LE[0,T]} = \|\tau_x^{-\frac{1}{2}}\phi\|_{\ell_r^\infty L^2[0,T]}, \qquad \|F\|_{LE^*[0,T]} = \|\tau_x^{\frac{1}{2}}F\|_{\ell_r^1 L^2[0,T]}.$$

Next, for a fixed $R_0 \geq 1$ sufficiently large ($r < R_0$ may be assumed to contain $\mathcal{K}$), and integer $s \geq 0$, we define

$$\|\phi\|_{LE^s_{\text{class}}[0,T]} = \sum_{|J|\leq s} (\|\tau_x^{-1}\partial^J\phi\|_{LE[0,T]} + \|\partial\partial^J\phi\|_{LE[0,T]}), \tag{6a}$$

$$\|\phi\|_{WLE^s_{\text{class}}[0,T]} = \sum_{|J|\leq s} (\|\tau_x^{-1}\partial^J\phi\|_{LE[0,T]} + \|\partial\partial^J\phi\|_{LE(r>R_0)[0,T]}), \tag{6b}$$

$$\|F\|_{LE^{*,s}[0,T]} = \sum_{|J|\leq s} \|\partial^J F\|_{LE^*[0,T]}, \tag{6c}$$

$$\|F\|_{WLE^{*,s}[0,T]} = \sum_{|J|\leq s} (\|\partial^J F\|_{LE^*[0,T]} + \|\partial\partial^J F\|_{L^2(r<R_0)[0,T]}). \tag{6d}$$

In terms of these spaces the second main assumption of the paper is:

Assumption 1.8 (weak and stationary energy boundedness/decay estimates). For R_0 as in Definition 1.7 above and any $s \geq 0$ there hold the estimates

$$\sup_{0\leq t\leq T} \|\partial\phi(t)\|_{H_x^s} + \|\phi\|_{WLE^s_{\text{class}}[0,T]} \lesssim_s \|\partial\phi(0)\|_{H_x^s} + \|\Box_g\phi\|_{(WLE^{*,s}+L_t^1 H_x^s)[0,T]}, \tag{7a}$$

$$\sup_{0\leq t\leq T} \|\partial\phi(t)\|_{H_x^s} + \|\phi\|_{LE^s_{\text{class}}[0,T]} \lesssim_s \|\partial\phi(0)\|_{H_x^s} + \|(\phi, \partial_t\phi)\|_{L^2\times H^s(r<R_0)[0,T]} + \|\Box_g\phi\|_{LE^{*,s}[0,T]}. \tag{7b}$$

In practice it is useful to have a bound which does not include the low-frequency error on the right-hand side of (7b). This is found by concatenating (7a)–(7b), which gives

$$\sup_{0\leq t\leq T} \|\partial\phi(t)\|_{H_x^s} + \|\phi\|_{LE^s_{\text{class}}[0,T]} \lesssim_s \|\partial\phi(0)\|_{H_x^s} + \|\partial_t\phi\|_{H^s(r<R_0)[0,T]} + \|\Box_g\phi\|_{WLE^{*,s}[0,T]}. \tag{8}$$

Before continuing we make a number of remarks about these assumptions.

Remark 1.9. Estimates of the form (7a) have a long history for both asymptotically flat nontrapping spacetimes, and for the Kerr family of metrics. For the case of black holes we refer the reader to [Blue and Sterbenz 2006; Dafermos and Rodnianski 2009; Marzuola et al. 2010; Tataru and Tohaneanu 2011; Dafermos et al. 2016] for more detailed discussions. In the sharp form needed for the present paper the estimate (7a) was proved by Marzuola, Metcalfe, Tataru, and Tohaneanu [Marzuola et al. 2010] for Schwarzschild space and Tataru and Tohaneanu [2011] for Kerr with $|a| \ll M$. More recently a slightly less precise version of (7a) was established for the full subextremal range of Kerr by Dafermos, Rodnianski, and Shlapentokh-Rothman [Dafermos et al. 2016]. It is expected that (7a) holds for a wide class of asymptotically flat spacetimes which satisfy certain natural structural conditions similar to those of the Kerr metric, as well as certain natural spectral assumptions. We refer the reader to [Metcalfe et al. 2017] for a definitive account in the case of nontrapping spacetimes.

Remark 1.10. The second estimate (7b) is essentially the "stationary local energy decay" estimate of [Metcalfe et al. 2012]. It can be shown that (7b) follows from estimates similar to (7a) when ∂_t is timelike on a set $\mathcal{T}$ where one loses regularity in the local energy decay norms (6). See Appendix B for a proof. We remark that such timelike conditions hold for the Kerr family of metrics when the angular momentum satisfies $0 \leq |a| < a_0$ for some $0 < a_0 < M$. On the other hand one should still expect (7b) to hold for the full subextremal range $0 \leq |a| < M$ of Kerr thanks to the (microlocal) nonvanishing of the symbol of ∂_t on the trapped set.

1C. ***Norms and vector fields.*** We now introduce the weighted local energy decay norms that will play a leading role in the remainder of the paper.

Definition 1.11 (null energy decay norms). First we define a null energy seminorm with the same scaling as LE:

$$\|\phi\|_{\mathrm{NLE}[0,T]} = \|\tau_-^{-\frac{1}{2}}\phi\|_{\ell_u^\infty L^2(\frac{1}{2}t<r<2t)[0,T]}, \qquad \|F\|_{\mathrm{NLE}^*[0,T]} = \|\tau_-^{\frac{1}{2}}F\|_{\ell_u^1 L^2(\frac{1}{2}t<r<2t)[0,T]}.$$

Next, we have the generalization of $\mathrm{LE}^s_{\mathrm{class}}$ and $\mathrm{WLE}^s_{\mathrm{class}}$ which include the seminorms

$$\|\phi\|_{\mathrm{LE}^s[0,T]} = \|\phi\|_{\mathrm{LE}^s_{\mathrm{class}}[0,T]} + \sum_{|J|\leq s} \|(\partial_x^b(\partial^b)^J\phi,\, \tau_x^{-1}\partial^J\phi)\|_{\mathrm{NLE}(r>R_0)[0,T]}, \tag{9a}$$

$$\|\phi\|_{\mathrm{WLE}^s[0,T]} = \|\phi\|_{\mathrm{WLE}^s_{\mathrm{class}}[0,T]} + \sum_{|J|\leq s} \|(\partial_x^b(\partial^b)^J\phi,\, \tau_x^{-1}\partial^J\phi)\|_{\mathrm{NLE}(r>R_0)[0,T]}. \tag{9b}$$

Remark 1.12. Notice we have specifically used Bondi coordinate derivatives $(\partial_u^b, \partial_x^b)$ inside the gradient portion of the right-hand side of (9). This appears to be necessary because condition (1) does not guarantee a good peeling estimate for $(\partial_x^b)^I\, \partial u$. In particular we don't assume improved control of the commutators $[\partial_x^b, \partial_\alpha]$ beyond the bounds (5).

Remark 1.13. We remark that our definition of NLE is essentially a sharp (dyadic) version of Alinhac's "ghost weight" energy [2001]. The authors wish to thank the anonymous referee for pointing out this connection. Note also that the NLE terms in (9a) and (9b) simply represent a weaker (averaged) version of the natural outgoing null energy on the hypersurfaces $u = \mathrm{const}$.

Next, we define LE-type norms with uniform weights in time. These will play a central role in establishing peeling estimates for solutions to the wave equation.

Definition 1.14 (weighted LE and conformal energy norms). For functions ϕ which solve the wave equation and $0 \le a \le 1$ we set

$$\|\phi(t)\|_{E^a} = \|\tau_+^a \tau_0^{\max\{a,\frac{1}{2}\}} \partial\phi(t)\|_{L^2_x} + \|\tau_+^a (\partial_x^b \phi(t),\, \tau_x^{-1}\phi(t))\|_{L^2_x}, \tag{10a}$$

$$\|\phi\|_{S^a[0,T]} = \|\tau_+^a \tau_0^{\max\{a,\frac{1}{2}\}} \partial\phi\|_{\mathrm{LE}[0,T]} + \|\tau_+^a (\partial_x^b \phi,\, \tau_x^{-1}\phi)\|_{\mathrm{LE}[0,T]} + \|\tau_+^a \tau_x^{-1} \partial_r^b (\tau_x \phi)\|_{\mathrm{NLE}[0,T]}, \tag{10b}$$

$$\|\phi\|_{S^{1,\infty}[0,T]} = \|\phi\|_{\ell_t^\infty S^1[0,T]} + \|\tau_+ \tau_x^{-1} \partial_r^b (\tau_x \phi)\|_{\mathrm{NLE}[0,T]}. \tag{10c}$$

For source terms we fix a parameter R_0 as in Definition 1.7 and set

$$\|F\|_{N^a[0,T]} = \|\tau_+^a \tau_0^{\frac{1}{2}} F\|_{\mathrm{LE}^*[0,T]} + \|\tau_+^a \partial F\|_{L^2(r<R_0)[0,T]} + \|\tau_+^a F\|_{\mathrm{NLE}^*[0,T]}, \tag{11a}$$

$$\|F\|_{N^{1,1}[0,T]} = \|\tau_+ \tau_0^{\frac{1}{2}} F\|_{\ell_t^1 \mathrm{LE}^*[0,T]} + \|\tau_+ \partial F\|_{\ell_t^1 L^2(r<R_0)[0,T]} + \|\tau_+ F\|_{\mathrm{NLE}^*[0,T]}. \tag{11b}$$

Finally, we construct higher-order versions of all the above norms.

Definition 1.15 (modified vector fields). First define the approximate Lie algebras

$$\mathbb{L}_0 = \{\partial_u^b,\, \partial_i^b - \omega^i \partial_u^b\}, \quad \mathbb{L} = \{S,\, \Omega_{ij}\} \cup \mathbb{L}_0, \quad \text{where } S = u\partial_u^b + r\partial_r^b,\ \Omega_{ij} = x^i \partial_j^b - x^j \partial_i^b. \tag{12}$$

Note that all members of $\mathbb{L}$ commute modulo $\mathbb{L}$ with the exception of

$$[\partial_i^b - \omega^i \partial_u^b,\, S] = \partial_i^b - \omega^i \partial_u^b + \tau_x^{-2} \omega^i \partial_u^b. \tag{13}$$

For a function ϕ we write

$$\phi^{(k)} = (\phi,\, \Gamma^{I_1}\phi,\, \Gamma^{I_2}\phi,\, \ldots),$$

where the right-hand side is an array of all products Γ^I of vector fields in $\mathbb{L}$ up to length $|I| \le k$. If $\|\cdot\|$ is any norm we write

$$\|\phi^{(k)}\| = \sum_{|I|\le k} \|\Gamma^I \phi\|, \qquad \Gamma \in \mathbb{L}.$$

We use a similar notation for pointwise identities; for example $|\partial\phi^{(k)}| = \sum_{|I|\le k} |\partial\Gamma^I\phi|$ etc. In the case of norms we use a subscript notation to denote higher-order derivatives by vector fields:

$$\|\phi(t)\|_{E_k^a} = \|\phi^{(k)}(t)\|_{E^a}, \qquad \|\phi\|_{S_k^a[0,T]} = \|\phi^{(k)}\|_{S^a[0,T]}, \qquad \|F\|_{N_k^a[0,T]} = \|F^{(k)}\|_{N^a[0,T]},$$

and similarly for LE-type norms. In addition we set $\|\phi\|_{H_k^s} = \|\phi^{(k)}\|_{H^s}$ and $\|\phi(t)\|_{H^s_{x,k}} = \|\phi^{(k)}(t)\|_{H^s_x}$.

1D. *Main results, I: Linear estimates.*

The main result of the paper can now be stated as follows.

Theorem 1.16 (weighted local energy decay estimates). *Assuming estimates (7a) and (7b), for $0 \le a \le 1$ and fixed $s, k \ge 0$ there exist parameters $B_a = B_a(s,k)$ such that:*

(I) *In the case $a = 0$ one has*

$$\sup_{0\le t\le T} \|\partial\phi(t)\|_{H^s_{x,k}} + \|\phi\|_{\mathrm{WLE}^s_k[0,T]} \lesssim_{s,k} \sum_{|I|\le k} \sum_{|J|\le s} \|(\tau_x \partial_x)^I \partial_x^J \partial\phi(0)\|_{L^2_x} + \|\phi\|_{H^{s+3}_{k-1}(r<B_0)[0,T]} + \|\Box_g \phi\|_{(\mathrm{WLE}^{*,s}_k + L^1_t H^s_{x,k})[0,T]}, \tag{14}$$

where in the case $k = 0$ we define $\|\phi\|_{H^{s+3}_{-1}(r<B_0)[0,T]} = 0$.

(II) *For* $0 < a < 1$ *one has*

$$\sup_{0\le t\le T} \|\phi(t)\|_{E^a_k} + \|\phi\|_{S^a_k[0,T]} \lesssim_{k,a} \sum_{|J|\le k} \|\tau_x^a(\tau_x\partial_x)^J\partial\phi(0)\|_{L^2_x} + \|\tau_+^{a-1}\phi\|_{H^1_{k+1}(r<B_a)[0,T]} + \|\Box_g\phi\|_{N^a_k[0,T]}. \quad (15)$$

(III) *Corresponding to* $a = 1$ *one has the endpoint bound*

$$\sup_{0\le t\le T} \|\phi(t)\|_{E^1_k} + \|\phi\|_{S^{1,\infty}_k[0,T]} \lesssim_k \sum_{|J|\le k} \|\tau_x(\tau_x\partial_x)^J\partial\phi(0)\|_{L^2_x} + \|\phi\|_{\ell^1_t H^1_{k+1}(r<B_1)[0,T]} + \|\Box_g\phi\|_{N^{1,1}_k[0,T]}. \quad (16)$$

Remark 1.17. In the proof of Theorem 1.16 we find that for $a \ne 0, 1$ one has $B_a \to \infty$ as $a \to 0, 1$.

In addition to the estimates above we shall also prove the following analog of Klainerman's estimate:

Theorem 1.18 (global Sobolev estimate). *For* $k \ge 1$ *one has the estimate*

$$\sum_{i+|J|\le k} \|\tau_+^{\frac32}\tau_0^{\frac12}(\tau_-\partial_u^b)^i(\tau_x\partial_x^b)^J\phi\|_{L^\infty[0,T]} \lesssim_k \sup_{0\le t\le T} \|\phi(t)\|_{E^1_{k+1}} + \|\phi\|_{S^{1,\infty}_{k+2}[0,T]} + \sum_{|J|\le k} \|\tau_x^2(\tau_x\partial)^J\Box_g\phi(0)\|_{L^2_x} + \sum_{i+|J|\le k+1} \|(\tau_-\partial_u^b)^i(\tau_x\partial_x^b)^J\Box_g\phi\|_{N^{1,1}[0,T]}. \quad (17)$$

One can combine the above two results in a straightforward way, which produces the first main conclusion of our paper:

Theorem 1.19 (peeling estimates for the inhomogeneous wave equation). *Given any* $k \ge 1$ *there exists an integer* $N = N(k)$ *depending (linearly) on* k *such that*

$$\sum_{i+|J|\le k} \|\tau_+^{\frac32}\tau_0^{\frac12}(\tau_-\partial_u^b)^i(\tau_x\partial_x^b)^J\phi\|_{L^\infty[0,T]} \lesssim_k \sum_{|J|\le N} \|\tau_x(\tau_x\partial_x)^J\partial\phi(0)\|_{L^2_x} + \sum_{|J|\le N} \|\tau_x^2(\tau_x\partial)^J\Box_g\phi(0)\|_{L^2_x} + \sum_{i+|J|\le N} \|(\tau_-\partial_u^b)^i(\tau_x\partial_x^b)^J\Box_g\phi\|_{N^{1,1}[0,T]}. \quad (18)$$

Proof that Theorems 1.16 and 1.18 imply estimate (18). We will in fact prove this for the explicit value $N = N(k) = 3k+13$. Expanding the vector fields $\mathbb{L}$ from (12) we easily see

$$\|\Box_g\phi\|_{N^{1,1}_k[0,T]} \lesssim_k \sum_{i+|J|\le k} \|(\tau_-\partial_u^b)^i(\tau_x\partial_x^b)^J\Box_g\phi\|_{N^{1,1}[0,T]}.$$

Therefore given a value of $k \ge 1$, by combining (17) and (16), it suffices to control $\|\phi\|_{\ell^1_t H^1_{k+3}(r<B_1)[0,T]}$. Note that for any $0 < a < 1$ one has

$$\|\phi\|_{\ell^1_t H^1_{k+3}(r<B_1)[0,T]} \lesssim_a \|\phi\|_{S^a_{k+3}[0,T]}.$$

In particular applying this for $a=\frac{1}{2}$ (say), and then using estimate (15) and $N^{1,1}[0,T]\subseteq N^{\frac{1}{2}}[0,T]$, it suffices to bound the right-hand side of

$$\|\tau_+^{-\frac{1}{2}}\phi\|_{H^1_{k+4}(r<B_{1/2})[0,T]}\lesssim\|\phi\|_{\mathrm{WLE}^1_{k+4}[0,T]}.$$

We now inductively use (14) in the form

$$\|\phi\|_{\mathrm{WLE}^{s_j}_{l_j}[0,T]}\lesssim_{s_j,l_j}\sum_{|I|\le l_j}\sum_{|J|\le s_j}\|(\tau_x\partial_x)^I\partial_x^J\partial\phi(0)\|_{L^2_x}+\|\phi\|_{\mathrm{WLE}^{s_j+3}_{l_j-1}[0,T]}+\|\Box_g\phi\|_{\mathrm{WLE}^{*,s_j}_{l_j}[0,T]},$$

where $l_1=k+4,\ l_2=k+3,\ldots,l_{k+4}=1$ and $s_1=1,\ s_2=4,\ldots,s_{k+4}=3k+13$. This leaves us with finally having to bound the quantity $\|\phi\|_{\mathrm{WLE}^{3k+13}_0[0,T]}$, which can be handled by an additional application of (14) with only initial and source data bounds on the right-hand side (or what amounts to the same thing by directly using our base assumption (7a)). This concludes our demonstration of (18) for the explicit value $N=N(k)=3k+13$ chosen above. □

1E. *Main results, II: Nonlinear estimates.* The estimates of Theorems 1.16 and 1.18 naturally lend themselves bounding solutions to semilinear wave equations of the form $\Box_g\phi=F(t,x,\phi,\partial\phi)$. Rather than develop a comprehensive theory we concentrate on the equations $\Box_g\phi=\mathcal{N}^{\alpha\beta}(t,x,\phi)\partial_\alpha\phi\partial_\beta\phi$, where the quadratic form $\mathcal{N}^{\alpha\beta}$ is sufficiently tame.

Definition 1.20 (generalized null forms). A 2-tensor $\mathcal{N}^{\alpha\beta}$ is called a "generalized null form" with respect to a (weak) optical function u if its Bondi coordinate components satisfy

$$|(\tau_-\partial_u^b)^i(\tau_x\partial_x^b)^J\partial_\phi^k\mathcal{N}^{\alpha\beta}|\lesssim_{i,J}c_k(|\phi|),\qquad|(\tau_-\partial_u^b)^i(\tau_x\partial_x^b)^J\partial_\phi^k\mathcal{N}^{uu}|\lesssim_{i,J}c_k(|\phi|)\tau_0.\tag{19}$$

Remark 1.21. Natural examples of $\mathcal{N}$ satisfying Definition 1.20 include multiples of the inverse metric $g^{\alpha\beta}$ by factors $\mathcal{N}(t,x,\phi)$ which satisfy derivative estimates consistent with (19). This would include the case of wave-maps from $\phi:(\mathcal{M},g)\to(\mathcal{M}',g')$ into Riemannian or Lorentzian targets where ϕ is close to a constant map, in either an intrinsic or extrinsic formulation.

Another example of such $\mathcal{N}$ would be skew symmetric forms $\mathcal{N}^{\alpha\beta}=-\mathcal{N}^{\beta\alpha}$ which obey the first condition in (19).

In order to prove a priori estimates we define the norms

$$\begin{aligned}\|\!|\phi|\!\|_{S_k[0,T]}=\sum_{j=0}^{k+4}\Big(\sup_{0\le t\le T}\|\partial\phi(t)\|_{H^{13+3(k-j)}_{x,j}}+\|\phi\|_{\mathrm{WLE}^{13+3(k-j)}_j[0,T]}\Big)+\|\phi\|_{S^{1/2}_{k+3}[0,T]}\\+\sup_{0\le t\le T}\|\phi(t)\|_{E^1_{k+2}}+\|\phi\|_{S^{1,\infty}_{k+2}[0,T]}+\sum_{i+|J|\le k}\|\tau_+^{\frac{3}{2}}\tau_0^{\frac{1}{2}}(\tau_-\partial_u^b)^i(\tau_x\partial_x^b)^J\phi\|_{L^\infty[0,T]},\end{aligned}\tag{20}$$

$$\begin{aligned}\|\!|F|\!\|_{N_k[0,T]}=\sum_{j=0}^{k+4}\|F\|_{(\mathrm{WLE}^{*,13+3(k-j)}_j+L^1_tH^{13+3(k-j)}_{x,j})[0,T]}+\|F\|_{N^{1/2}_{k+3}[0,T]}\\+\sum_{|J|\le k}\|\tau_x^2(\tau_x\partial)^JF(0)\|_{L^2_x}+\sum_{i+|J|\le k+2}\|(\tau_-\partial_u^b)^i(\tau_x\partial_x^b)^JF\|_{N^{1,1}[0,T]}.\end{aligned}\tag{21}$$

With this notation the main nonlinear theorem of our paper is the following:

Theorem 1.22. *Let ϕ be a smooth vector-valued function and let $\mathcal{N}(\phi, \partial\phi)$ denote a quadratic form obeying (19), where we are using the notation $\mathcal{N}(\phi, \partial\phi) = \mathcal{N}^{\alpha\beta}(t, x, \phi)\, \partial_\alpha\phi\, \partial_\beta\phi$. Then one has the bounds*

$$\||\phi\||_{S_k[0,T]} \lesssim_k \sum_{|J|\le 3k+13} \|\tau_x(\tau_x\partial_x)^J \partial\phi(0)\|_{L^2_x} + \||\Box_g\phi\||_{N_k[0,T]}, \tag{22}$$

$$\||\mathcal{N}(\phi, \partial\phi)\||_{N_k[0,T]} \lesssim C_k(\||\phi\||_{S_k[0,T]})\||\phi\||_{S_k[0,T]}(\||\phi\||_{S_k[0,T]} + \||\Box_g\phi\||_{N_k[0,T]}), \quad k \ge 18. \tag{23}$$

The functions $C_k(\cdot)$ are locally bounded functions determined by k as well as the functions c_k on right-hand side of (19).

From this and a standard continuity argument one has the nonlinear analog of Theorem 1.19:

Theorem 1.23 (peeling estimates for solutions to null-form systems). *Suppose quadratic forms $\mathcal{N}$ are given which satisfy (19). Let ϕ be a sufficiently smooth and well-localized solution to the system of semilinear equations*

$$\Box_g\phi = \mathcal{N}^{\alpha\beta}(t, x, \phi)\, \partial_\alpha\phi\, \partial_\beta\phi, \tag{24}$$

which is assumed to hold on a time interval $[0, T]$. Then given any $k \ge 18$ there exists an $\epsilon_0 = \epsilon_0(k) > 0$ such that one has the a priori estimate

$$\sum_{|J|\le 3k+13} \|\tau_x(\tau_x\partial_x)^J \partial\phi(0)\|_{L^2_x} = \epsilon \le \epsilon_0 \quad \Longrightarrow \quad \||\phi\||_{S_k[0,T]} \lesssim_k \epsilon.$$

In particular for sufficiently smooth, small, and well-localized initial data the solution to (24) exists globally and enjoys the peeling estimates

$$\sum_{i+|J|\le k} \|\tau_+^{\frac{3}{2}}\tau_0^{\frac{1}{2}}(\tau_-\partial_u^b)^i(\tau_x\partial_x^b)^J\phi\|_{L^\infty[0,T]} \lesssim_k \epsilon$$

for the same k as above.

1F. *Some remarks and references.* The vector field method for the wave equation on asymptotically flat spacetimes has a long and well-developed history. In the case of small perturbations of Minkowski space there are Klainerman's original works [1985; 1986], followed by the proof in [Christodoulou and Klainerman 1993] of the nonlinear stability of Minkowski space itself. In the latter work a vector field method is developed for radiating spacetimes where control is ultimately provided through certain peeling estimates for the curvature tensor of g. In a related vein there is [Bieri 2010] concerning nonlinear stability under the much less restrictive decay assumptions on the initial data. In this case the peeling properties of the metric end up being closer to the thresholds (3).

Going in a different direction there is the proof of stability of Minkowski space and its asymptotics in wave coordinates in [Lindblad and Rodnianski 2010; Lindblad 2017]. Here one proceeds more directly via Minkowski and Schwarzschild vector fields. For scalar wave equations this produces estimates with a small loss due to the divergence of radiating null hypersurfaces compared to their stationary counterparts (at least at the level of energy estimates). On the other hand strong peeling estimates such as (18) cannot hold for the (Minkowski difference of the) metric in wave coordinates due to obstructions at the level of semilinear terms. More specifically, condition (19) seems to fail when writing the Einstein equations in any reasonable way as a system of second-order equations for the metric.

Next, we turn to vector field methods on spacetimes which are locally large perturbations of Minkowski space. There are mainly two innovations here, and we make a heavy use of both in the sequel. The first innovation, due to Klainerman and Sideris [1996], allows one to replace Lorentz boosts with certain weighted identities involving the wave operator $\Box_g$ (specifically see Lemma 4.13 below). The second innovation, due initially to Keel, Smith, and Sogge [Keel et al. 2002] and used by many authors, concerns the use of local energy decay estimates such as (7) in order to control localized errors generated by commutations with vector fields. For further background and developments concerning the combination of local energy decay and vector fields on various large perturbations of Minkowski space, we refer the reader to [Metcalfe and Sogge 2005; 2007; Bony and Häfner 2010; Wang and Yu 2014; Yang 2013].

Concerning the application of vector fields to the class of black hole spacetimes there has recently been a great deal of progress. The works most closely related to the present paper are [Blue and Sterbenz 2006; Dafermos et al. 2016; Luk 2012; 2013], which proceed by way of conformal energy estimates and time-dependent weights. We also mention [Lindblad and Tohaneanu 2018] concerning certain quasilinear wave equations (in this case the vector field approach is similar to [Lindblad and Rodnianski 2010]). We remark that all of these works are written specifically to cover the case of Schwarzschild or Kerr with small angular momentum.

For more general "black hole" backgrounds there is [Moschidis 2016] building on ideas of [Dafermos and Rodnianski 2010] (see also Proposition 6.1 of [Rodnianski and Sterbenz 2010], where weighted estimates based on outgoing null vector fields were introduced). Here one produces a vector field method with time-independent weights. This shares a number of similarities with the present paper, in particular the production of a family of weighted spacetime energy estimates depending on a parameter $0 \le a \le 1$ which interpolates between the standard local energy decay estimates and the conformal energy. On the other hand our approach and the one of Dafermos and Rodnianski and Moschidis appear to diverge in many ways, especially regarding the issue of time-dependent weights. Our method also appears to be more directly applicable to studying nonlinear problems. Indeed, with the machinery of estimates (14)–(16) our proof of Theorem 1.22 occupies only a few additional pages.

Finally, one should also mention the works on Price's law for scalar waves [Tataru 2013; Metcalfe et al. 2012] which use vector fields as a launching point for much sharper estimates. An interesting open question in this regard is to find an appropriate collection of norms capable of producing interior decay rates better than $t^{-\frac{3}{2}}$, but which are still compatible with semilinear problems such as in Theorem 1.23.

We close with a few additional comments on the specific methods we employ in this work. These are a natural outgrowth of [Blue and Sterbenz 2006; Lindblad and Sterbenz 2006; Rodnianski and Sterbenz 2010; Oliver 2013; Dafermos and Rodnianski 2010]. In [Lindblad and Sterbenz 2006] we developed a conformal multiplier technique that works well for perturbations of the wave equation, and then used it to produce a collection of conformal energies with weights depending on a parameter. These ideas will again play a central role in the present work. In [Rodnianski and Sterbenz 2010] we introduced a Morawetz-type estimate based on the multiplier $f(r)(\partial_t + \partial_r)$ and used it to control null tangential derivatives for solutions to a certain wave equation with a potential (Proposition 6.1 of that paper). This idea was expanded in [Dafermos and Rodnianski 2010] to prove global decay estimates. We will use similar multipliers in this paper to produce a key portion of our estimates.[1] In [Blue and Sterbenz 2006] we produced a weighted-in-time local energy decay bound, and then used this to control the multiplier error from the conformal vector field. This technique is used again here for a wider range of weights, albeit assuming the local energy decay bound as a black box. Finally, the first author's thesis [Oliver 2013] and [Oliver 2016] developed vector fields on radiating nontrapping spacetimes with even weaker local bounds on $\partial_t g$. We will use much of the setup from [Oliver 2016] in the present work.

1G. ***Outline of the paper.*** In Section 2 we record a number of algebraic identities for energy momentum and deformation tensors. These form the basis for all the multiplier and commutator identities needed in the sequel.

In Section 3 we specialize the formulas of Section 2 to the case of Bondi coordinates satisfying Definition 1.1 and vector fields from Definition 1.15.

Section 4 is the technical heart of the paper. We begin by producing symbol bounds for the Lie derivatives of 2-tensors which satisfy certain natural asymptotic estimates. Building on this we construct a generic multiplier estimate for vector fields satisfying certain structural properties. This estimate covers all of the multiplier bounds needed in the sequel, and may be useful for other applications as well, which is one of our motivations for introducing an axiomatic setup. Following this we move on to estimates for commutators. This is done in a way that allows us to perform some delicate integration by parts later in Section 6, and is also convenient for proving pointwise bounds. We end by proving several generalized Klainerman–Sideris-type identities, and then use these to conclude the discussion of pointwise bounds for commutators.

In Section 5 we apply the abstract multiplier bound from Section 4 to produce the three main estimates of Theorem 1.16 at level $k = 0$. For the bounds (14) and (16) this is done in such a way that the source error term can be integrated by parts; this form is needed later to establish bounds for commutators.

In Section 6 we apply the formulas of the previous two sections to prove the estimates of Theorem 1.16 for an arbitrary number of vector fields. For the bound (15) this is more or less straightforward. However, for the bounds (14) and (16) the argument is more involved because one cannot proceed directly via Hölder's inequality, and several additional integrations seem necessary to close the argument.

In Section 7 we prove some sharp L^∞ decay estimates for functions in terms of the conformal energy norms (10a), (10c), and (11b). This establishes Theorem 1.18.

In Section 8 we prove Theorem 1.22, which concludes the main nonlinear application of the paper.

In Appendix A we provide proofs of a number of the remarks following Definition 1.1. In particular for large class of stationary metrics which are also spherically symmetric to highest order, we construct optical functions satisfying bounds (3). Such metrics include radial long-range perturbations of the Kerr family of metrics.

[1] For a complete the list of multipliers we use here, see Lemma 5.4 and the proofs immediately following.

In Appendix B we show that estimate (7a) implies estimate (7b), at least when the metric satisfies structural assumptions similar to the Kerr family with angular momentum in a certain range.

Finally, in Appendix C we record a number of elementary Hardy- and trace-type estimates. These will be used throughout the body of the paper.

2. Formulas for commutators and multipliers

In this section we recall some basic formulas which underlie the energy method for the wave equation on curved backgrounds. We do this first with respect to the original metric g. We then generalize such formulas cover the case of metric conformal to g. The latter will form the basis for most of the multiplier estimates in the sequel.

2A. ***Identities involving g.*** We begin with a basic definition:

Definition 2.1 (normalized deformation tensor). Let X be a vector field, and set ${}^{(X)}\pi = \mathcal{L}_X g$. Then we define ${}^{(X)}\hat{\pi} = {}^{(X)}\pi - \frac{1}{2} g \cdot \mathrm{trace}({}^{(X)}\pi)$ to be the "normalized deformation tensor" of X.

The quantity ${}^{(X)}\hat{\pi}$ underlies all of our formulas for multipliers and commutators as the following result shows:

Lemma 2.2 (formulas involving ${}^{(X)}\hat{\pi}$). *Let ϕ be a scalar field and X a vector field. As usual set $T_{\alpha\beta} = \partial_\alpha\phi\partial_\beta\phi - \frac{1}{2} g_{\alpha\beta} g^{\alpha'\beta'}\partial_{\alpha'}\phi\partial_{\beta'}\phi$ to be the energy momentum tensor of ϕ. Then one has the identities*

$$ {}^{(X)}\hat{\pi}^{\alpha\beta} = -\frac{1}{\sqrt{|g|}} X(\sqrt{|g|} g^{\alpha\beta}) - g^{\alpha\beta}\partial_\gamma X^\gamma + g^{\alpha\gamma}\partial_\gamma(X^\beta) + g^{\beta\gamma}\partial_\gamma(X^\alpha), \tag{25a} $$

$$ [\Box_g, X] = \nabla_\alpha {}^{(X)}\hat{\pi}^{\alpha\beta}\nabla_\beta - \tfrac{1}{2}\mathrm{trace}({}^{(X)}\hat{\pi})\Box_g, \tag{25b} $$

$$ \nabla^\alpha(T_{\alpha\beta}X^\beta) = \tfrac{1}{2}{}^{(X)}\hat{\pi}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \Box_g\phi X\phi. \tag{25c} $$

Finally, if q is a smooth function then one has the commutator formula

$$ {}^{(qX)}\hat{\pi}^{\alpha\beta} - q\,{}^{(X)}\hat{\pi}^{\alpha\beta} = X^\alpha\nabla^\beta q + X^\beta\nabla^\alpha q - g^{\alpha\beta}Xq. \tag{26} $$

We'll prove each of these formulas separately.

Proof of (25a) *and* (26). We have

$$ {}^{(X)}\pi^{\alpha\beta} = g^{\alpha\alpha'}g^{\beta\beta'}(\mathcal{L}_x g)_{\alpha'\beta'} = -Xg^{\alpha\beta} + g^{\alpha\gamma}\partial_\gamma(X^\beta) + g^{\beta\gamma}\partial_\gamma(X^\alpha). $$

On the other hand

$$ \tfrac{1}{2}\mathrm{trace}({}^{(X)}\pi) = \nabla_\alpha X^\alpha = \frac{1}{\sqrt{|g|}}\partial_\alpha(\sqrt{|g|}X^\alpha) = \frac{1}{\sqrt{|g|}}X(\sqrt{|g|}) + \partial_\alpha X^\alpha. $$

Subtracting the last two identities gives (25a). A direct application of (25a) shows (26). □

Proof of (25b). We begin with a formula that will also be useful in the sequel. Let $\mathcal{R}^{\alpha\beta}$ be any contravariant 2-tensor; then we claim

$$ [X, \nabla_\alpha\mathcal{R}^{\alpha\beta}\nabla_\beta] = \nabla_\alpha\widetilde{\mathcal{R}}^{\alpha\beta}\nabla_\beta + \widetilde{\mathcal{S}}^\beta\nabla_\beta, \quad \text{where } \widetilde{\mathcal{R}} = \mathcal{L}_X\mathcal{R} \text{ and } \widetilde{\mathcal{S}}^\beta = \mathcal{R}^{\alpha\beta}\nabla_\alpha(\nabla_\gamma X^\gamma). \tag{27} $$

To prove it, first note that a straightforward calculation using the coordinate-based formula for $\nabla_\alpha X^\alpha$ above shows $X(\nabla_\alpha Y^\alpha) - Y(\nabla_\alpha X^\alpha) = \nabla_\alpha [X, Y]^\alpha$ for any pair of vector fields X and Y. Applying this last formula to the vector field $Y^\alpha = \mathcal{R}^{\alpha\beta}\nabla_\beta\phi$ and using the Leibniz rule for Lie derivatives followed by $[\mathcal{L}_X, d] = 0$ for the exterior derivative d gives (27).

Now apply (27) to $\mathcal{R} = g^{-1}$. Using $(\mathcal{L}_X\mathcal{R})^{\alpha\beta} = -{}^{(X)}\pi^{\alpha\beta}$ and $\nabla_\alpha X^\alpha = -\frac{1}{2}\operatorname{trace}({}^{(X)}\hat{\pi})$ gives (25b). □

Proof of (25c). A standard calculation shows $\nabla^\alpha(T_{\alpha\beta}X^\beta) = \frac{1}{2}{}^{(X)}\pi^{\alpha\beta}T_{\alpha\beta} + \Box_g\phi X\phi$ and (25c) follows easily. □

2B. *Conformal changes for scalar fields.* In this section we recall a standard formula from geometry. Let $g_{\alpha\beta}$ be a Lorentzian metric on a (3+1)-dimensional spacetime. We consider a conformally equivalent metric $\tilde{g}_{\alpha\beta}$, where $\Omega^2\tilde{g} = g$ for some weight function $\Omega > 0$. Let $\tilde{\nabla}$ denote the Levi-Civita connection of $\tilde{g}$ and $\Box_{\tilde{g}} = \tilde{\nabla}^\alpha\tilde{\nabla}_\alpha$ the corresponding wave operator. Then we have:

Lemma 2.3 (identity for the conformal wave operator). *Let $\Box_g\phi = F$. Then one has the formula*

$$\Box_{\tilde{g}}\psi + V\psi = \Omega^3 F, \quad \textit{where } \psi = \Omega\phi \textit{ and } V = \Omega^3\Box_g\Omega^{-1}. \tag{28}$$

Proof. Start with

$$\Box_{\tilde{g}} = \Omega^4\frac{1}{\sqrt{|g|}}\partial_\alpha(\Omega^{-2}\sqrt{|g|}g^{\alpha\beta}\partial_\beta) = \Omega^2(\Box_g - 2g^{\alpha\beta}\partial_\alpha\ln(\Omega)\partial_\beta).$$

To eliminate the second term on the right-hand side we rescale ϕ via $\psi = \Omega\phi$, which gives us

$$\Omega^{-2}\Box_{\tilde{g}}\psi = \Box_g\psi - 2g^{\alpha\beta}\partial_\alpha\ln(\Omega)\partial_\beta\psi = \Omega\Box_g\phi - W\phi,$$

where

$$W = -\Box_g(\Omega) + 2\Omega\partial_\alpha\ln(\Omega)\partial^\alpha\ln(\Omega) = \Omega^2\Box_g\Omega^{-1}.$$

A straightforward manipulation of the last two equations gives (28). □

2C. *Conformal multipliers.* We now combine the identities of the last two sections to produce conjugated weighted L^2 identities. Our main result here is:

Lemma 2.4 (the conformal divergence identity). *Let $\Box_g\phi = F$. Let X be a vector field supported in the exterior region $\mathbb{R}^3\setminus\mathcal{K}$. Let $\chi(t,x)$ be a smooth function and $\Omega > 0$ a smooth weight. Then in (t,x)-coordinates one has the divergence identity*

$$\int_0^T\int_{\mathbb{R}^3\setminus\mathcal{K}} Q(X,\chi,\Omega,\phi)\sqrt{|g|}\,dx\,dt = \int_{\mathbb{R}^3\setminus\mathcal{K}} P(X,\chi,\Omega,\phi)\sqrt{|g|}\,dx\bigg|_{t=0}^{t=T}, \tag{29}$$

where

$$P(X,\chi,\Omega,\phi) = \Omega^{-2}g^{0\alpha}\partial_\alpha(\Omega\phi)X(\Omega\phi) - \tfrac{1}{2}\Omega^{-2}X^0(g^{\alpha\beta}\partial_\alpha(\Omega\phi)\partial_\beta(\Omega\phi) - \chi V\phi^2), \tag{30}$$

with $V = \Omega^3\Box_g\Omega^{-1}$, and where

$$Q(X,\chi,\Omega,\phi) = F\cdot\Omega^{-1}X(\Omega\phi) + \Omega^{-2}A^{\alpha\beta}\partial_\alpha(\Omega\phi)\partial_\beta(\Omega\phi) + B^\chi\phi^2 + C^\chi\phi,\Omega^{-1}X(\Omega\phi), \tag{31}$$

with

$$A = \tfrac{1}{2}({}^{(X)}\hat{\pi} + 2X\ln(\Omega)g^{-1}), \qquad B^\chi = \tfrac{1}{2}\Omega^{-2}(X(\chi V) - \mathrm{trace}(A)\chi V), \qquad C^\chi = \Omega^{-2}(\chi-1)V. \tag{32}$$

On the right-hand sides of the last two equations above all contractions are computed with respect to g.

Proof. First define the tensors

$$\widetilde{T}^\chi_{\alpha\beta} = \partial_\alpha\psi\partial_\beta\psi - \tfrac{1}{2}\tilde{g}_{\alpha\beta}(\tilde{g}^{\gamma\delta}\partial_\gamma\psi\partial_\delta\psi - \chi V\psi^2), \qquad {}^{(X)}\widetilde{P}^\chi_\alpha = \widetilde{T}^\chi_{\alpha\beta}X^\beta, \qquad \text{where } \psi = \Omega\phi.$$

Then by Stokes' theorem and the support property of X we have

$$\int_0^T\int_{\mathbb{R}^3\setminus\mathcal{K}} \tilde{\nabla}^{\alpha\,(X)}\widetilde{P}^\chi_\alpha\sqrt{|\tilde{g}|}\,dx\,dt = \int_{\mathbb{R}^3\setminus\mathcal{K}} \tilde{g}^{\alpha 0}\widetilde{T}^\chi_{\alpha\beta}X^\beta\sqrt{|\tilde{g}|}\,dx\Big|_{t=0}^{t=T},$$

and right-hand side of (29) follows by substituting $\Omega^{-2}g = \tilde{g}$ into the volume forms and the right-hand side contractions.

It remains to compute the $\tilde{g}$-contraction $\tilde{\nabla}^{\alpha\,(X)}\widetilde{P}^\chi_\alpha$ and show this produces the terms on left-hand side of (29). To this end suppose $(\Box_{\tilde{g}} + V)\psi = G$. Then one has

$$\tilde{\nabla}^\alpha\widetilde{T}^\chi_{\alpha\beta} = ((\chi-1)V\psi + G)\partial_\beta\psi + \tfrac{1}{2}\partial_\beta(\chi V)\psi^2.$$

Using formula (25c) we have

$$\tilde{\nabla}^{\alpha\,(X)}\widetilde{P}^\chi_\alpha = \tfrac{1}{2}(\widehat{\mathcal{L}_X\tilde{g}})^{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi - \tfrac{1}{4}\mathrm{trace}(\widehat{\mathcal{L}_X\tilde{g}})\chi V\psi^2 + ((\chi-1)V\psi + G)X\psi + \tfrac{1}{2}X(\chi V)\psi^2, \tag{33}$$

where all the contractions are computed with respect to $\tilde{g}$. To compute the first two terms on the right-hand side we use

$$\mathcal{L}_X\tilde{g} = \Omega^{-2}(\mathcal{L}_X g - 2X\ln(\Omega)g), \qquad \text{and so} \qquad \widehat{\mathcal{L}_X\tilde{g}} = \Omega^{-2}(\widehat{\mathcal{L}_X g} + 2X\ln(\Omega)g).$$

Substituting the last equation into right-hand side of (33) and using $G = \Omega^3 F$ we have (31) and (32). □

3. Algebraic formulas involving Bondi coordinates

In this section we compute the key quantities from Lemmas 2.2 and 2.4 in Bondi coordinates (u,x).

Lemma 3.1 (formulas for deformation tensors). *Let $d = |g|$ be the determinant $g_{\alpha\beta}$ in rectangular Bondi coordinates (u,x^i), and set $\Omega = \tau_x$. Then if $X = X^\alpha\partial^b_\alpha$ is any vector field in Bondi coordinates we have the formula for contravariant tensors*

$$^{(X)}\hat{\pi} + 2X\ln(\Omega)g^{-1} = -d^{-\frac{1}{2}}(\mathcal{L}_X h + (\partial^b_u X^u + \partial^b_r X^r + \partial^b_i\overline{X}^i)h) + \mathcal{R}, \tag{34}$$

where $\overline{X}^i = X^i - r^{-2}x^i x_j X^j$ denotes the angular portion of X and $X^r = r^{-1}x_i X^i$ the radial portion, and where h is given in (4). The remainder tensor $\mathcal{R}$ is given by the covariant formula

$$\mathcal{R} = -d^{-\frac{1}{2}}\mathcal{L}_X(d^{\frac{1}{2}}g^{-1} - h) - d^{-\frac{1}{2}}(\partial^b_u X^u + \partial^b_r X^r + \partial^b_i\overline{X}^i)(d^{\frac{1}{2}}g^{-1} - h) - 2r^{-1}\tau_x^{-2}X^r g^{-1}. \tag{35}$$

Proof of formula (34). Starting with formula (25a) in Bondi coordinates and then using the identity

$$\partial^b_\gamma X^\gamma = 2X\ln(\tau_x) + \partial^b_u X^u + \partial^b_r X^r + \partial^b_i \overline{X}^i + 2r^{-1}\tau_x^{-2}X^r$$

and setting $\Omega = \tau_x$, we have the formula for raised indices

$$^{(X)}\hat{\pi} + 2X\ln(\Omega)g^{-1} = -d^{-\frac{1}{2}}\mathcal{L}_X(d^{\frac{1}{2}}g^{-1}) - (\partial^b_u X^u + \partial^b_r X^r + \partial^b_i \overline{X}^i + 2r^{-1}\tau_x^{-2}X^r)g^{-1}.$$

Writing $g^{-1} = d^{-\frac{1}{2}}(d^{\frac{1}{2}}g^{-1} - h) + d^{-\frac{1}{2}}h$ and inserting into this last equation gives (34) and (35). □

Lemma 3.2 (formulas for commutators). *The following commutator formulas hold where $d = |g|$ is computed in Bondi coordinates*:

(I) *For $X \in \{\partial^b_u, \partial^b_i - \omega^i\partial^b_u, \Omega_{ij}\}$ one has the formula*

$$[\Box_g, X] = \nabla_\alpha \mathcal{R}^{\alpha\beta}\nabla_\beta + \tfrac{1}{2}(X\ln(d))\Box_g, \quad \textit{where } \mathcal{R} = -d^{-\frac{1}{2}}\mathcal{L}_X(d^{\frac{1}{2}}g^{-1} - h) + \mathcal{R}_1, \tag{36}$$

where $\mathcal{R}_1 = 0$ for $X \in \{\partial^b_u, \Omega_{ij}\}$, and $\mathcal{R}_1^{\alpha\beta} = 2d^{-\frac{1}{2}}\omega^i\tau_x^{-3}\delta^\alpha_u\delta^\beta_u$ when $X = \partial^b_i - \omega^i\partial^b_u$. Again h is as defined in (4).

(II) *For $X = S$ one has*

$$[\Box_g, S] = \nabla_\alpha \mathcal{R}^{\alpha\beta}\nabla_\beta + \tfrac{1}{2}(4 + S\ln(d))\Box_g, \quad \textit{where } \mathcal{R} = -d^{-\frac{1}{2}}\mathcal{L}_S(d^{\frac{1}{2}}g^{-1} - h) - 2d^{-\frac{1}{2}}(d^{\frac{1}{2}}g^{-1} - h) + \mathcal{R}_1, \tag{37}$$

and where $\mathcal{R}_1^{\alpha\beta} = d^{-\frac{1}{2}}\omega^i\tau_x^{-2}(\delta^\alpha_i\delta^\beta_u + \delta^\beta_i\delta^\alpha_u)$.

Proof of formulas (36) *and* (37). First note that formula (25a) above can be rewritten as

$$^{(X)}\hat{\pi} = -d^{-\frac{1}{2}}\mathcal{L}_X(d^{\frac{1}{2}}g^{-1} - h) - d^{-\frac{1}{2}}\mathcal{L}_X h - \partial^b_\alpha X^\alpha g^{-1}, \quad \mathrm{trace}(^{(X)}\hat{\pi}) = -2\partial^b_\alpha X^\alpha - X\ln(d). \tag{38}$$

Next, for each $X \in \{\partial^b_u, \Omega_{ij}, \partial^b_i - \omega^i\partial^b_u\}$ we have $\partial^b_\alpha X^\alpha = 0$, and for $X \in \{\partial^b_u, \Omega_{ij}\}$ we also have $\mathcal{L}_X h = 0$. On the other hand for $X = \partial^b_i - \omega^i\partial^b_u$ one computes $\mathcal{L}_X h^{\alpha\beta} = 0$ for all but the uu-component, and for this one has $(\mathcal{L}_X h)^{uu} = -2\omega^i\tau_x^{-3}$. Combining this information with (38) above and (25b) gives (36).

Finally, in the case when $X = S$ we compute $\mathcal{L}_S h + 2h = -\omega^i\tau_x^{-2}(\delta^\alpha_i\delta^\beta_u + \delta^\beta_i\delta^\alpha_u)$. This allows us to write for $X = S$

$$d^{-\frac{1}{2}}\mathcal{L}_X h + \partial^b_\alpha X^\alpha g^{-1} = 2d^{-\frac{1}{2}}(d^{\frac{1}{2}}g^{-1} - h) - \mathcal{R}_1 + 2g^{-1}, \quad \text{where } \mathcal{R}_1^{\alpha\beta} = d^{-\frac{1}{2}}\omega^i\tau_x^{-2}(\delta^\alpha_i\delta^\beta_u + \delta^\beta_i\delta^\alpha_u).$$

Using $\mathrm{trace}(^{(X)}\hat{\pi}) = -8 - S\ln(d)$ and combining everything with (38) above and (25b) gives (37). □

4. Asymptotic estimates involving Bondi coordinates

We now move on to the main technical calculations of the paper. We record these here in a general form that will be used throughout the sequel.

4A. ***Basic estimates for derivatives.***

Lemma 4.1 (estimates for the determinant). *Let $d = |g| = |\det(g)|$ be the absolute determinant of g computed in Bondi coordinates (u, x^i). Then for any $\mu \in \mathbb{R}$ one has the symbol bounds*

$$d^\mu - 1 \in \mathcal{Z}^0, \tag{39}$$

where the symbol spaces $\mathcal{Z}^k$ are defined in (2).

Proof of (39). We can write $d^\mu - 1 = q_\mu(d^{-1} - 1)$, where $q_\mu(s)$ is smooth for $s > -1$ and $q(0) = 0$. Thus, by Taylor expansion and the Leibniz and chain rules it suffices to consider the case $\mu = -1$. From (3) we know that

$$d^{-1} - 1 + \tau_x^{-2} = \det(h) - \det(g^{-1}) \in \mathcal{Z}^0,$$

where h is given in (4). Since $\tau_x^{-2} \in \mathcal{Z}^0$, this completes the proof. □

A useful corollary of this last lemma and the assumptions (3) is the following:

Corollary 4.2. *Let $d = |g| = |\det(g)|$ be the absolute determinant of g computed in Bondi coordinates (u, x^i), and g^{-1} the inverse metric of g in Bondi coordinates. Then if h is as in (4) and one sets $\mathcal{R}^{\alpha\beta} = (d^{\frac{1}{2}} g^{-1} - h)^{\alpha\beta}$, there holds the bounds*

$$\mathcal{R}^{ij} \in \mathcal{Z}^0, \quad \mathcal{R}^{ui} \in \mathcal{Z}^{\frac{1}{2}}, \quad \mathcal{R}^{ui} - \omega^i \omega_j \mathcal{R}^{uj} \in \mathcal{Z}^1, \quad \mathcal{R}^{uu} \in \mathcal{Z}^2. \tag{40}$$

Building on the last two results we have the following collection of symbol bounds which will underlie many of the error estimates in the sequel.

Lemma 4.3 (basic Lie derivative estimates). *Let $X = X^\alpha \partial_\alpha$ be a vector field. Then the following hold:*

(I) *Suppose that in Bondi coordinates X satisfies the symbol-type bounds for $a, b, c \in \mathbb{R}$*

$$|(\tau_- \partial_u^b)^l (\tau_x \partial_x^b)^J X^u| \lesssim_{l,J} \tau_x^a \tau_+^b \tau_-^{c+1} \left(\frac{\tau_x}{\tau_+}\right)^{\min\{|J|,1\}}, \quad |(\tau_- \partial_u^b)^l (\tau_x \partial_x^b)^J X^i| \lesssim_{l,J} \tau_x^{a+1} \tau_+^b \tau_-^c \tag{41}$$

and obeys the conditions

$$\partial_r^b X^u = \partial_u^b (X^i) = \partial_r^b r^{-1}(X^i - r^{-2} x^i x_j X^j) = 0. \tag{42}$$

Let $\mathcal{R}^{\alpha\beta}$ be any contravariant 2-tensor which satisfies (40) with similar estimates for $\mathcal{R}^{iu}$ (if it is nonsymmetric). Then its Lie derivative by X, denoted by $\mathcal{L}_X \mathcal{R} = \mathcal{R}_X$, satisfies

$$\mathcal{R}_X^{ij} \in \tau_x^a \tau_+^b \tau_-^c \cdot \mathcal{Z}^0, \quad \mathcal{R}_X^{ui} \in \tau_x^a \tau_+^b \tau_-^c \cdot \mathcal{Z}^{\frac{1}{2}}, \quad \mathcal{R}_X^{ui} - \omega^i \omega_j \mathcal{R}_X^{uj} \in \tau_x^a \tau_+^b \tau_-^c \cdot \mathcal{Z}^1, \quad \mathcal{R}_X^{uu} \in \tau_x^a \tau_+^b \tau_-^c \cdot \mathcal{Z}^2, \tag{43}$$

with similar bounds for $\mathcal{R}_X^{iu}$.

(II) *Alternatively, if one drops the condition $\partial_r^b X^u = 0$ but keeps the rest of (41) and (42), then the previous conclusion holds with the last bound in (43) replaced by*

$$\mathcal{R}_X^{uu} \in \tau_x^a \tau_+^b \tau_-^c \cdot \mathcal{Z}^{\frac{3}{2}}. \tag{44}$$

(III) *Alternatively, if one drops the condition* $\partial_u^b(X^r) = 0$ *but retains* $\partial_u^b(X^i - r^{-2}x^i x_j X^j) = 0$ *and the rest of* (41) *and* (42), *then the result of part* (I) *holds with the first bound in* (43) *replaced by the pair*

$$\mathcal{R}_X^{ij} \in \tau_x^a \tau_+^b \tau_-^c \cdot \mathcal{Z}^{-\frac{1}{2}}, \quad \mathcal{R}_X^{ij} - \omega^i \omega^j \omega_k \omega_l \mathcal{R}_X^{kl} \in \tau_x^a \tau_+^b \tau_-^c \cdot \mathcal{Z}^0. \tag{45}$$

(IV) *Likewise, if* $\mathcal{S}^\alpha$ *satisfies*

$$\mathcal{S}^i \in \tau_x^{-1} \cdot \mathcal{Z}^{-\frac{1}{2}}, \quad \mathcal{S}^u \in \tau_x^{-1} \cdot \mathcal{Z}^{\frac{1}{2}}, \tag{46}$$

then $\mathcal{L}_X \mathcal{S} = S_X$ *satisfies*

$$\mathcal{S}_X^i \in \tau_x^{a-1} \tau_+^b \tau_-^c \cdot \mathcal{Z}^{-\frac{1}{2}}, \quad \mathcal{S}_X^u \in \tau_x^{a-1} \tau_+^b \tau_-^c \cdot \mathcal{Z}^{\frac{1}{2}}, \tag{47}$$

when X *satisfies the symbol bounds* (41) (*in this case we do not need the extra conditions* (42)).

(V) *Finally, let* $X \in \mathbb{L}_0 = \{\partial_u^b, \partial_i^b - \omega^i \partial_u^b\}$, *and let* $\mathcal{R}$ *and* $\mathcal{S}$ *satisfy* (40) *and* (46) *respectively. Then* $\mathcal{L}_X \mathcal{R}$ *and* $\mathcal{L}_X \mathcal{S}$ *satisfy* (43) *and* (47) *with* $a = c = -1$ *and* $b = 1$.

Remark 4.4. As will become apparent in the proof, if one is only interested in the norms (2) at level $N = 0$ for $\mathcal{R}_X^{\alpha\beta}$ and $\mathcal{S}_X^\alpha$ (i.e., no derivatives), then one can replace the full symbol bounds (41) with first-order conditions

$$\sum_{l+|J|\le 1} |(\tau_- \partial_u^b)^l (\tau_+ \partial_x^b)^J X^u| \lesssim \tau_x^a \tau_+^b \tau_-^{c+1}, \qquad \sum_{l+|J|\le 1} |(\tau_- \partial_u^b)^l (\tau_x \partial_x^b)^J X^i| \lesssim \tau_x^{a+1} \tau_+^b \tau_-^c.$$

In this case the various implications above are true with the inclusion $\mathcal{R} \in \tau_x^a \tau_+^b \tau_-^c \cdot \mathcal{Z}^k$ replaced by the bound $|||\tau_x^{-a} \tau_+^{-b} \tau_-^{-c} \mathcal{R}|||_{k,0} < \infty$.

Proof of Lemma 4.3. We prove the various portions separately. First note that the conditions (40) are invariant with respect to dyadic cutoffs in the r-, u-, and $(t+r)$-variables. Therefore by utilizing such cutoffs and the Leibniz rule we may assume $a = b = c = 0$. As a second preliminary note the identity

$$\hat{x}^i - \omega^i = \omega^i r^{-1}(r + \tau_x)^{-1}, \quad \text{where } \hat{x}^i = r^{-1} x^i. \tag{48}$$

This allows us to trade $\hat{x}^i$ for ω^i in the region $r > 1$ as long as errors on the order of r^{-2} are acceptable.

Part 1: (*the* $\mathcal{R}$ *bounds involving condition* (41)) We begin with the proof of estimates (40) for $\mathcal{L}_X \mathcal{R}$ assuming conditions (41) and (42) or one of the alternatives listed in items (II) and (III) above. The formula for the Lie derivative is $\mathcal{L}_X \mathcal{R}^{\alpha\beta} = X(\mathcal{R}^{\alpha\beta}) - \partial_\gamma(X^\alpha)\mathcal{R}^{\gamma\beta} - \partial_\gamma(X^\beta)\mathcal{R}^{\alpha\gamma}$. We check each component separately:

Case 1a: (*the* uu*-component assuming* $\partial_r^b X^u = 0$) By assumption we have $\omega^i \partial_i^b X^u = 0$; thus

$$\mathcal{L}_X \mathcal{R}^{uu} = X(\mathcal{R}^{uu}) - 2\partial_u^b(X^u)\mathcal{R}^{uu} - \partial_i^b(X^u)(\mathcal{R}^{ui} - \omega^i \omega_j \mathcal{R}^{uj}) - \partial_i^b(X^u)(\mathcal{R}^{iu} - \omega^i \omega_j \mathcal{R}^{ju}).$$

Then the estimate in (43) for $\mathcal{R}_X^{uu}$ is immediate from the estimates (41) and (40).

Case 1b: (*the* uu*-component when* $\partial_r^b X^u \neq 0$) By the previous case we have the desired estimate modulo the additional expression $r^2 \tau_x^{-2} \partial_r^b(X^u)(\mathcal{R}^{ur} + \mathcal{R}^{ru})$, which adds a $\mathcal{Z}^{\frac{3}{2}}$-term.

Case 2: (*the ui- and iu-components*) By symmetry it suffices to treat the ui case. We have

$$\mathcal{L}_X\mathcal{R}^{ui} = X(\mathcal{R}^{ui}) - \partial_u^b(X^u)\mathcal{R}^{ui} - \partial_u^b(X^i)\mathcal{R}^{uu} - \partial_j^b(X^u)\mathcal{R}^{ji} - \partial_j^b(X^i)\mathcal{R}^{uj}.$$

Using estimates (41) and (40) we get a $\mathcal{Z}^{\frac{1}{2}}$ symbol bound for this term. In addition one sees that for all parts of the formula above, save for the expression $\mathcal{B}^i = X(\omega^i\omega_j\mathcal{R}^{uj}) - \partial_u^b(X^u)\omega^i\omega_j\mathcal{R}^{uj} - \omega^k\partial_k^b(X^i)\omega_j\mathcal{R}^{uj}$, the bound is on the order of $\mathcal{Z}^1$. To show improved bounds we only need to consider the region $r>1$. Using (48) we see that $\mathcal{B}\equiv\widetilde{\mathcal{B}} \mod r^{-2}\cdot\mathcal{Z}^{\frac{1}{2}}$, where $\widetilde{\mathcal{B}}^i = X(\hat{x}^i\mathcal{R}^{ur}) - \partial_u^b(X^u)\hat{x}^i\mathcal{R}^{ur} - \partial_r^b(X^i)\mathcal{R}^{ur}$. Again using (48), we see that in order to show $\mathcal{B}^i - \omega^i\omega_j\mathcal{B}^j\in\mathcal{Z}^1$ it suffices to prove $\chi_{r>1}(\widetilde{\mathcal{B}}^i - \hat{x}^i\widetilde{\mathcal{B}}^r)\in\mathcal{Z}^1$. This would follow immediately if $\widetilde{\mathcal{B}}$ is a radially directed vector field. Using $X(\hat{x}^i) = r^{-1}(X^i - \hat{x}^iX^r)$ we compute

$$\widetilde{\mathcal{B}}^i = \hat{x}^i(X(\mathcal{R}^{ur}) - \partial_u^b(X^u)\mathcal{R}^{ur} - \partial_r^b(X^r)\mathcal{R}^{ur}) - r\partial_r^b[r^{-1}(X^i - \hat{x}^iX^r)]\mathcal{R}^{ur},$$

which is manifestly radial thanks to the last condition in (42).

Case 3a: (*the ij-components assuming* $\partial_u^b(X^i)=0$) Here we have

$$\mathcal{L}_X\mathcal{R}^{ij} = X(\mathcal{R}^{ij}) - \partial_k^b(X^i)\mathcal{R}^{kj} - \partial_k^b(X^j)\mathcal{R}^{ik}.$$

Then the estimate in (43) for $\mathcal{R}_X^{ij}$ is immediate from the estimates (41) and (40).

Case 3b: (*the ij-components assuming* $\partial_u^b(X^r)\neq 0$) In this case we are still assuming $\partial_u^b(X^i - \hat{x}^iX^r)=0$. Therefore we have

$$\mathcal{L}_X\mathcal{R}_X^{ij} = X(\mathcal{R}^{ij}) - \hat{x}^i\partial_u^b(X^r)\mathcal{R}^{uj} - \hat{x}^i\partial_u^b(X^r)\mathcal{R}^{iu} - \partial_k^b(X^i)\mathcal{R}^{kj} - \partial_k^b(X^j)\mathcal{R}^{ik}.$$

A $\mathcal{Z}^{-\frac{1}{2}}$ symbol bound for this expression in $r>1$ is again immediate from (41) and (40). On the other hand all but the second and third terms above yield an improved $\mathcal{Z}^0$ bound. Thus, using (48) we have for $r>1$

$$\mathcal{R}_X^{ij} - \omega^i\omega^j\omega_k\omega_l\mathcal{R}_X^{kl} \equiv -\omega^i\partial_u^b(X^r)(\mathcal{R}^{uj} - \omega^j\omega_k\mathcal{R}^{uk}) - \omega^j\partial_u^b(X^r)(\mathcal{R}^{iu} - \omega^i\omega_k\mathcal{R}^{ku}) \mod r^{-2}\cdot\mathcal{Z}^{-\frac{1}{2}} + \mathcal{Z}^0.$$

By (40) and (41) we have a $\mathcal{Z}^0$ bound for this last term as well.

Part 2: (*the $\mathcal{S}$ bounds involving condition* (41)) Again we can reduce to $a=b=c=0$. Componentwise we have $\mathcal{L}_X\mathcal{S}^\alpha = X(\mathcal{S}^\alpha) - \partial_\beta(X^\alpha)\mathcal{S}^\beta$.

Case 1: (*the u-component*) Here we have

$$\mathcal{L}_X\mathcal{S}^u = X(\mathcal{S}^u) - \partial_u^b(X^u)\mathcal{S}^u - \partial_i^b(X^u)\mathcal{S}^i,$$

so the second estimate in (46) follows directly by multiplying together the bounds in (41) and (46).

Case 2: (*the i-components*) Here we have

$$\mathcal{L}_X\mathcal{S}^i = X(\mathcal{S}^i) - \partial_u^b(X^i)\mathcal{S}^u - \partial_j^b(X^i)\mathcal{S}^j,$$

and so the first estimate in (46) follows directly from (41) and (46).

Part 3: (*estimates involving* $\mathbb{L}_0$) This is largely a corollary of Parts 1 and 2 above. For any $X \in \mathbb{L}_0$ one has both conditions in (41), with $a = c = -1$ and $b = 1$. In this case one also has to deal with the fact that $\partial_r^b X^u \neq 0$ and $\partial_r^b r^{-1}(X^i - \hat{x}^i \hat{x}_j X^j) \neq 0$, save for when $X = \partial_u^b$. Recall that these two special conditions were only used in Case 1b and Case 2 of Part 1 above. So we review those cases here when $X = \partial_i^b - \omega^i \partial_u^b$.

Case 1: (*the* uu*-component of* $\mathcal{R}_X$) Recall from Case 1b of Part 1 above we only need to handle an expression of the form $\partial_r^b(X^u)(\mathcal{R}^{ur} + \mathcal{R}^{ru})$ in the region $r > 1$, where $X^u = \omega^i$. Using $\partial_r^b(\omega^i) = r^{-1}\tau_x^{-2}\omega^i$ we get an $\tau_x^{-3} \cdot \mathcal{Z}^{\frac{1}{2}} \subseteq \tau_x^{-1}\tau_-^{-1}\tau_+\mathcal{Z}^2$ bound for this expression, which suffices.

Case 2: (*the* ui*-component of* $\mathcal{R}_X$) Recall that the condition $\partial_r^b r^{-1}(X^i - \hat{x}^i \hat{x}_j X^j) = 0$ is only used to establish the improved bound $\mathcal{R}_X^{iu} - \omega^i \omega_j \mathcal{R}_X^{uj} \in \mathcal{Z}^1$. Recall further that this improved bound automatically holds modulo an expression of the form $r\partial_r^b[r^{-1}(X^i - \omega^i \omega_j X^j)]\mathcal{R}^{ur}$. When $X^i = 0, 1$ we see this expression has symbol bounds on the order of $\tau_x^{-1} \cdot \mathcal{Z}^{\frac{1}{2}} \subseteq \tau_x^{-1}\tau_-^{-1}\tau_+\mathcal{Z}^1$, which suffices. □

4B. ***A general exterior multiplier estimate.*** Next, we prove some general multiplier bounds which will be used a number of times in the sequel. To state them we first define the form of an acceptable error.

Definition 4.5 (general form of multiplier estimate errors). For a pair of parameters $0 < a < 1$ and $R > 0$, and a quantity $o_R(1) \to 0$ as $R \to \infty$, we set

$$\begin{aligned}\mathcal{E}(a, R) &= \|\tau_x^a \partial\phi(0)\|^2_{L_x^2(r>\frac{1}{2}R)} + o_R(1)\cdot\big(\sup_{0\le t\le T}\|\phi(t)\|^2_{E^a(r>\frac{1}{2}R)} + \|\phi\|^2_{S^a(r>\frac{1}{2}R)[0,T]}\big)\\ &\quad + \|\phi\|_{S^a(r>\frac{1}{2}R)[0,T]}\cdot\big(R^{-\frac{1}{2}}\|(\tau_-^a\partial_u^b\phi,\, \tau_x^a\partial_x^b\phi,\, \tau_x^{a-1}\phi)\|_{L^2(\frac{1}{2}R<r<R)[0,T]} + \|\Box_g\phi\|_{N^a(r>\frac{1}{2}R)[0,T]}\big). \qquad (49)\end{aligned}$$

Corresponding to the cases $a = 0, 1$, for parameter $R > 0$, quantity $o_R(1)$, and vector field X, we set

$$\begin{aligned}\mathcal{E}(0, R, X) &= \|\partial\phi(0)\|^2_{L_x^2(r>\frac{1}{2}R)} + o_R(1)\cdot\big(\sup_{0\le t\le T}\|\partial\phi(t)\|^2_{L_x^2(r>\frac{1}{2}R)} + \|\phi\|^2_{\mathrm{LE}^0(r>\frac{1}{2}R)[0,T]}\big)\\ &\quad + R^{-1}\|(\partial\phi,\, \tau_x^{-1}\phi)\|^2_{L^2(\frac{1}{2}R<r<R)[0,T]} + \left|\int_0^T\int_{\mathbb{R}^3\setminus\mathcal{K}} \Box_g\phi\cdot\tau_x^{-1}X(\tau_x\phi)\, dV_g\right|, \qquad (50)\end{aligned}$$

and

$$\begin{aligned}\mathcal{E}(1, R, X) &= \|\tau_x\partial\phi(0)\|^2_{L_x^2(r>\frac{1}{2}R)} + o_R(1)\cdot\big(\sup_{0\le t\le T}\|\phi(t)\|^2_{E^1(r>\frac{1}{2}R)} + \|\phi\|^2_{S^{1,\infty}(r>\frac{1}{2}R)[0,T]}\big)\\ &\quad + \|\phi\|_{S^{1,\infty}(r>\frac{1}{2}R)[0,T]}\cdot R^{-\frac{1}{2}}\|(\tau_-\partial_u^b\phi,\, \tau_x\partial_x^b\phi,\, \phi)\|_{\ell_t^1 L^2(\frac{1}{2}R<r<R)[0,T]}\\ &\quad + \left|\int_0^T\int_{\mathbb{R}^3\setminus\mathcal{K}} \Box_g\phi\cdot\tau_x^{-1}X(\tau_x\phi)\, dV_g\right|. \qquad (51)\end{aligned}$$

In the notation above the rate of $o_R(1)$ may change from line to line, but is fixed for any line on which an error of the form $\mathcal{E}$ appears. Also we define $dV_g = \sqrt{|g|}\, dx\, dt$, where $|g| = |\det g|$ is computed in (t, x)-coordinates.

With this notation in mind we have:

Proposition 4.6 (abstract multiplier estimate). *Fix $R > 0$ sufficiently large so that $\mathcal{K} \subset \{r < \frac{1}{2}R\}$, and let Y be a vector field such that $Y = Y^u \partial_u^b + Y^r \partial_r^b$, with both $Y^u \geq 0$ and $Y^r \geq 0$ depending only on the (u, r)-variables. Then the following hold:*

(I) *Assume for some $0 < a < 1$ there holds the symbol-type bounds*

$$\sum_{i+|J|\leq 1} |(\tau_- \partial_u^b)^i (\tau_+ \partial_r^b)^J Y^u| \lesssim \tau_+^{2a} \tau_0^{\max\{1,2a\}}, \qquad \sum_{i+|J|\leq 1} |(\tau_- \partial_u^b)^i (\tau_x \partial_r^b)^J Y^r| \lesssim \tau_x^{2a} + \tau_+^{2a-1}\tau_x. \tag{52}$$

Then one has the multiplier estimate

$$\int_0^T \int_{\mathbb{R}^3 \setminus \mathcal{K}} \chi_{>R} \big(\mathcal{A}^u (\partial_u^b \phi)^2 + \mathcal{A}^{ur} \partial_u^b \phi \cdot \tau_x^{-1} \partial_r^b (\tau_x \phi) + \mathcal{A}^r (\tau_x^{-1} \partial_r^b (\tau_x \phi))^2 + \not{\mathcal{A}} |\not\nabla^b \phi|^2 \big) \, dx\, dt \\ + \| \tau_x^{-1} (\sqrt{Y^u} \partial (\tau_x \phi), \sqrt{Y^r} \partial_x^b (\tau_x \phi))(T) \|^2_{L^2_x(r>R)} \lesssim \mathcal{E}(a, R), \tag{53}$$

where $|\not\nabla^b \phi|^2$ denotes the (Euclidean) angular gradient of ϕ with respect to the spheres $u = \text{const}$ and $r = \text{const}$, and where the components of $\mathcal{A}$ are given by

$$\mathcal{A}^u = -\partial_r^b Y^u, \quad \mathcal{A}^r = \tfrac{1}{2} \partial_r^b Y^r - \tfrac{1}{2} \partial_u^b Y^u - \partial_u^b Y^r, \tag{54a}$$

$$\mathcal{A}^{ur} = \partial_r^b Y^u, \qquad \not{\mathcal{A}} = r^{-1} Y^r - \tfrac{1}{2} \partial_u^b Y^u - \tfrac{1}{2} \partial_r^b Y^r. \tag{54b}$$

Here $\chi_{>R} = \chi_{>1}(R^{-1} \cdot)$ is a radial bump function with $\chi_{>R} \equiv 1$ on $r > R$ and $\chi_{>R} \equiv 0$ on $r < \frac{1}{2}R$. The implicit constant in (53) depends only on the bounds from (52), and the metric g.

(II) *In the case $a = 1$, still assuming (52), we have (53)–(54) with the right-hand side of (53) replaced by $\mathcal{E}(1, R) = \mathcal{E}(1, R, \chi_{r>R} Y)$, where $\chi_{>R}$ is as above.*

(III) *Alternatively, in the case $a = 0$ replace assumption (52) with*

$$\sum_{i+|J|\leq 1} |(\tau_- \partial_u^b)^i (\tau_+ \partial_r^b)^J Y^u| \lesssim 1, \qquad \sum_{i+|J|\leq 1} |(\tau_- \partial_u^b)^i (\tau_x \partial_r^b)^J Y^r| \lesssim 1, \qquad \partial_u^b Y^r = 0. \tag{55}$$

Then (53)–(54) hold with the right-hand side of (53) replaced by $\mathcal{E}(0, R) = \mathcal{E}(0, R, \chi_{r>R} Y)$, where $\chi_{r>R}$ is as above.

In order to prove this proposition we need a few additional supporting lemmas.

Lemma 4.7 (asymptotics of the conformal potential). *Let $\Omega = \tau_x$ and define the quantity $V = \Omega^3 \Box_g (\Omega^{-1})$. Then in Bondi coordinates (u, x^i) one has the symbol bounds*

$$V \in \mathcal{Z}^{-\frac{1}{2}}. \tag{56}$$

Proof. First write the wave operator in Bondi coordinates as $\Box_g = d^{-\frac{1}{2}} \Box_h + d^{-\frac{1}{2}} \partial_\alpha^b \mathcal{R}^{\alpha\beta} \partial_\beta^b$, where $d = |g|$ is the Bondi coordinate metric determinant, $\Box_h = \partial_\alpha^b h^{\alpha\beta} \partial_\beta^b$ where h is given in (4), and where $\mathcal{R} = d^{\frac{1}{2}} g - h$ satisfies the estimate (40). A quick calculation shows $\Box_h (\tau_x^{-1}) = -3\tau_x^{-5}$, and a little further work reveals

$$V = -d^{-\frac{1}{2}} (r \partial_\alpha^b \mathcal{R}^{\alpha r} + \tau_x^{-2} (1 - 3r^2) \mathcal{R}^{rr} + 3\tau_x^{-2}).$$

By (39) we have $d^{-\frac{1}{2}} - 1 \in \mathcal{Z}^0$, so estimate (56) follows from (40). $\square$

Lemma 4.8 (formulas for boundary terms). *Let X^r, X^u be nonnegative and set $X = X^u\partial_u^b + X^r\partial_r^b$. Then if $\Omega = \tau_x$, one has the following pointwise estimate involving the quantity $P(X, \chi, \Omega, \phi)$ defined in (30):*

$$X^u|\tau_x^{-1}\partial(\tau_x\phi)|^2 + X^r|\tau_x^{-1}\partial_x^b(\tau_x\phi)|^2$$
$$\lesssim -P(X, \chi, \Omega, \phi) + o_r(1)\cdot((X^u + \tau_0^2X^r)|\tau_x^{-1}\partial(\tau_x\phi)|^2 + \chi(X^u + X^r)\tau_0^{-\frac{1}{2}}\tau_x^{-2}\phi^2). \quad (57)$$

To prove estimate (57) we need the following elementary result:

Lemma 4.9 (approximate null frame). *Let X and Y_A, $A = 1, 2$, be approximately unit-length vectors in the Minkowski space in the sense that $\sup_\alpha |X^\alpha| \approx 1$ and $\sup_\alpha |Y_A^\alpha| \approx 1$. Suppose that there exists $\epsilon > 0$ such that $\langle X, X\rangle = O(\epsilon^2)$, $\langle X, Y_A\rangle = O(\epsilon)$, and in addition $|\langle Y_A, Y_B\rangle - \delta_{AB}| \ll 1$. Then there exists an exact null frame $\{L, \underline{L}, e_A\}$ with $\langle L, L\rangle = \langle L, e_A\rangle = \langle \underline{L}, L\rangle = 0$, $\langle L, \underline{L}\rangle = -1$, and $\langle e_A, e_B\rangle = \delta_{AB}$, and coefficients γ, c_X^A and c_A^B for $A, B = 1, 2$, such that*

$$X = L + c_X^A e_A + \gamma\underline{L}, \quad Y_A = c_A^B e_B, \quad \text{where } \gamma = O(\epsilon^2) \text{ and } c_X^A = O(\epsilon) \text{ and } |c_B^A - \delta_B^A| \ll 1.$$

Proof. Let e_A form an orthonormal basis for the space-like 2-plane spanned by Y_A, with the first e_A in the direction of one of the Y_A. Let c_A^B be the corresponding change of basis. Then $|c_B^A - \delta_B^A| \ll 1$. Let L, $\underline{L}$ generate the two null directions over the span of e_A and Y_A, chosen so that $\langle L, \underline{L}\rangle = -1$ and $X = L + c_X^A e_A + \gamma\underline{L}$ for some set of coefficients c_X^A, γ. From $\langle X, Y_A\rangle = O(\epsilon)$ we have $\langle X, e_A\rangle = O(\epsilon)$ and so $c_X^A = O(\epsilon)$. Then $\langle X, X\rangle = -2\gamma + O(\epsilon^2)$, so $\gamma = O(\epsilon^2)$ follows from $\langle X, X\rangle = O(\epsilon^2)$. □

Proof of (57). Note that it suffices to prove this bound in the region $r \gg 1$. Consider the vector fields $\{\partial_r^b, Y_A\}$, where Y_A is a (local) Euclidean orthonormal basis on the spheres $r = \text{const}$, $u = \text{const}$. Because the metric g is asymptotically Minkowskian, $|\langle Y_A, Y_B\rangle - \delta_{AB}| \ll 1$. On the other hand a quick application of the asymptotic formulas (3) and Cramer's rule shows that $\langle\partial_r^b, \partial_r^b\rangle = o_r(1)\cdot\tau_0^2$ and $\langle\partial_r^b, Y_A\rangle = o_r(1)\cdot\tau_0$. Thus, an application of the previous lemma shows that

$$\partial_r^b = L + o_r(1)\cdot\tau_0\partial\!\!\!/_x^b + o_r(1)\cdot\tau_0^2\partial,$$

where L is null, $\partial\!\!\!/_x^b$ denotes derivatives tangent to $u = \text{const}$, $r = \text{const}$ which are also orthogonal to L, and ∂ is arbitrary.

Next, let $T = T[\psi]$ denote the energy momentum tensor of ψ with respect to the metric g. Because $t = \text{const}$ are uniformly spacelike when $r \gg 1$ we have

$$T(L, -\nabla t) \approx |L\psi|^2 + |\nabla\!\!\!/_x^b\psi|^2, \quad |T(\partial\!\!\!/_x^b, -\nabla t)| \lesssim |\nabla\!\!\!/_x^b\psi|\cdot|\partial\psi| + |g^{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi|, \quad |T(\partial, -\nabla t)| \lesssim |\partial\psi|^2.$$

A quick application of (3) and the middle bound above shows that uniformly for $C > 0$

$$\tau_0|T(\partial\!\!\!/_x^b, -\nabla t)| \lesssim C^{-1}|\partial_x^b\psi|^2 + C\tau_0^2|\partial\psi|^2.$$

Combining the last three displays gives the pointwise estimate

$$|\partial_x^b\psi|^2 \lesssim T(\partial_r^b, -\nabla t) + o_r(1)\cdot\tau_0^2|\partial\psi|^2.$$

In addition to this we also have by the asymptotic flatness of g and standard properties of T

$$|\partial\psi|^2 \lesssim T(\partial_u^b, -\nabla t) + o_r(1)\cdot|\partial\psi|^2.$$

Finally, let $P(X,\chi,\Omega,\phi)$ denote the quantity defined in (30). Then

$$-P(X,\chi,\Omega,\phi) = \Omega^{-2}T[\Omega^{-1}\phi](X,-\nabla t) - \tfrac{1}{2}\Omega^{-2}X^0\chi V\phi^2, \quad \text{where } V = \Omega^3\Box_g(\Omega^{-1}),$$

and where X^0 denotes the time component of X in (t,x)-coordinates. By (1) we have $|X^0| \lesssim X^u + X^r$. Therefore estimate (57) follows from the last three displays above and estimate (56). □

We now return to the proof of the main result of this subsection. Because of the split form of the error terms (49) and (51) there are essentially two cases.

Proof of Proposition 4.6 for $0<a<1$. We use the formalism of Section 2, in particular Lemma 2.4. Let $X = X_R = \chi_{>R}Y$, where $\chi_{>R}$ is as in the statement of the proposition. We choose $\Omega = \tau_x$ and set the auxiliary cutoff to $\chi = 0$. Using the divergence identity (29) we need to estimate each spacetime term given by formulas (31) and (32), as well as the boundary term on the right-hand side of (29).

Step 1: (*output of the* $A^{\alpha\beta}$*-contraction*) We'll do this calculation by switching over to polar Bondi coordinates (u,r,x^A), where locally we can choose x^A to be two members of $\hat{x}^i = r^{-1}x^i$. From (34) and (35), and expansion of $\mathcal{L}_X h$, we may write $A^{\alpha\beta} = \chi_{>R}d^{-\frac{1}{2}}A_0^{\alpha\beta} + d^{-\frac{1}{2}}\mathcal{R}^{\alpha\beta}$, where

$$2A_0^{\alpha\beta} = \partial_\gamma^b(Y^\alpha)h^{\gamma\beta} + \partial_\gamma^b(Y^\beta)h^{\alpha\gamma} - Y(h^{\alpha\beta}) - (\partial_u^b Y^u + \partial_r^b Y^r)h^{\alpha\beta}, \tag{58}$$

and $\mathcal{R} = \mathcal{R}_0 + \chi_{>R}\mathcal{R}_1$. Here $2\mathcal{R}_0 = {}^{(X_R)}\hat{\pi} - \chi_{>R}{}^{(Y)}\hat{\pi}$, which according to formula (26) is

$$2\mathcal{R}_0^{\alpha\beta} = \tau_x^{-1}\chi_R(g^{\alpha r}Y^\beta + g^{\beta r}Y^\alpha - Y^r g^{\alpha\beta}), \tag{59}$$

and where $\chi_R = \tau_x\partial_r\chi_{>R}$ is a smooth bump function adapted to $r\approx R$. The second remainder term is

$$2\mathcal{R}_1 = -\mathcal{L}_Y(d^{\frac{1}{2}}g^{-1} - h) - (\partial_u^b Y^u + \partial_r^b Y^r)(d^{\frac{1}{2}}g^{-1} - h) - 2r^{-1}\tau_x^{-2}d^{\frac{1}{2}}X^r g^{-1}. \tag{60}$$

A little further computation shows the coefficients of the quadratic form $A_0^{\alpha\beta}$ from (58) are

$$A_0^{uu} = \mathcal{A}^u + \tau_x^{-1}(r+\tau_x)^{-1}\partial_r^b Y^u, \qquad A_0^{rr} = \mathcal{A}^r + \tau_x^{-1}(r+\tau_x)^{-1}\partial_u^b Y^r, \tag{61a}$$

$$2A_0^{ur} = \mathcal{A}^{ur} + \tau_x^{-3}Y^r, \qquad A^{AB} = r^{-2}\delta^{AB}\mathcal{A}\!\!/, \tag{61b}$$

while $A_0^{rA} = A_0^{uA} = 0$. Here the terms $\mathcal{A}$ are given in (54). Recalling the definitions of the norms (10b) and using the conditions (52) we have

$$\int_0^T\!\!\int_{\mathbb{R}^3\setminus\mathcal{K}} \chi_{>R}\big(\mathcal{A}^u(\partial_u^b\phi)^2 + \mathcal{A}^{ur}\partial_u^b\phi\cdot\tau_x^{-1}\partial_r^b(\tau_x\phi) + \mathcal{A}^r(\tau_x^{-1}\partial_r^b(\tau_x\phi))^2 + \mathcal{A}\!\!/|\nabla\!\!\!/^b\phi|^2\big)\,dx\,dt$$
$$\leq \int_0^T\!\!\int_{\mathbb{R}^3\setminus\mathcal{K}} \chi_{>R}\tau_x^{-2}d^{-\frac{1}{2}}A_0^{\alpha\beta}\partial_\alpha^b(\tau_x\phi)\partial_\beta^b(\tau_x\phi)\,dV_g + o_R(1)\|\phi\|^2_{S^a(r>\frac{1}{2}R)[0,T]}. \tag{62}$$

To estimate the remainder terms from (59), note that a straightforward calculation involving the conditions (52) and (3) gives in rectangular Bondi coordinates

$$|\mathcal{R}_0^{ij}| \lesssim (\tau_x^{2a-1} + \tau_+^{2a-1})\chi_R, \quad |\mathcal{R}_0^{ui}| \lesssim \tau_x^{-1}\tau_+^{2a}\tau_0\chi_R, \quad |\mathcal{R}_0^{uu}| \lesssim \tau_x^{-1}\tau_+^{2a}\tau_0^{\max\{1,2a\}}\chi_R, \tag{63}$$

where χ_R is supported in $\frac{1}{2}R < r < R$. This yields the estimate

$$\int_0^T\int |\tau_x^{-2}\mathcal{R}_0^{\alpha\beta}\partial_\alpha(\tau_x\phi)\partial_\beta(\tau_x\phi)|\,dx\,dt \\ \lesssim R^{-\frac{1}{2}}\|\phi\|_{S^a(\frac{1}{2}R<r<R)[0,T]}\|(\tau_-^a\partial_u^b\phi,\, \tau_x^a\partial\phi,\, \tau_x^{a-1}\phi)\|_{L^2(\frac{1}{2}R<r<R)[0,T]}. \tag{64}$$

To estimate the remainder terms from (60) we split the range into $0 < a < \frac{1}{2}$ and $\frac{1}{2} \leq a < 1$. When $0 < a < \frac{1}{2}$ the conditions (52) imply

$$\sum_{i+|J|\leq 1} |(\tau_-\partial_u^b)^i(\tau_+\partial_x^b)^J Y^u| \lesssim \tau_x^{2a-1}\cdot\tau_-, \qquad \sum_{i+|J|\leq 1} |(\tau_-\partial_u^b)^i(\tau_x\partial_x^b)^J Y^r| \lesssim \tau_x^{2a-1}\cdot\tau_x.$$

In addition we must assume $\partial_r^b Y^u \neq 0$ and $\partial_u^b Y^r \neq 0$, although we do have $\partial_u^b(Y^i - \hat{x}^i Y^r) = 0$. Therefore by simultaneously combining cases (II) and (III) of Lemma 4.3 and Remark 4.4, using $|\partial_u^b Y^u + \partial_r^b Y^r| \lesssim \tau_x^{2a-1}$ $\big($again for $0 < a < \frac{1}{2}\big)$ and (40), and directly using (3) for the last term on the right-hand side of (60) we have for $0 < a < \frac{1}{2}$

$$\|\!|w_a^{-1}\mathcal{R}_1^{rr}|\!\|_{-\frac{1}{2},0} < \infty, \qquad \|\!|w_a^{-1}\mathcal{R}_1^{uu}|\!\|_{\max\{1,2a\},0} < \infty, \quad \|\!|w_a^{-1}\mathcal{R}_1^{ru}|\!\|_{\frac{1}{2},0} < \infty, \tag{65a}$$

$$\|\!|w_a^{-1}r\mathcal{R}_1^{uA}|\!\|_{1,0} < \infty, \quad \|\!|w_a^{-1}(r\mathcal{R}_1^{rA}, r^2\mathcal{R}_1^{AB})|\!\|_{0,0} < \infty, \quad \text{where } w_a = (\tau_x^{2a-1} + \tau_+^{2a-1}). \tag{65b}$$

On the other hand in the range $\frac{1}{2} \leq a \leq 1$ the conditions (52) imply

$$\sum_{i+|J|\leq 1} \|(\tau_-\partial_u^b)^i(\tau_+\partial_x^b)^J Y^u| \lesssim \tau_-^{2a-1}\cdot\tau_-, \qquad \sum_{i+|J|\leq 1} |(\tau_-\partial_u^b)^i(\tau_x\partial_x^b)^J Y^r| \lesssim \tau_+^{2a-1}\cdot\tau_x.$$

Therefore by separately applying cases (II) and (III) of Lemma 4.3 and Remark 4.4 to the vector fields $Y^u\partial_u^b$ and $Y^r\partial_r^b$ respectively, and this time using $|\partial_u^b Y^u| + |\partial_r^b Y^r| \lesssim \tau_+^{2a-1}$, we again have (65). Finally, after several rounds of Hölder's inequality and a straightforward check of definitions (2) and (10b), the error bounds (65) yield the asymptotic estimate

$$\int_0^T\int_{\mathbb{R}^3\setminus\mathcal{K}} \chi_{>R}|\tau_x^{-2}\mathcal{R}_1^{\alpha\beta}\partial_\alpha(\tau_x\phi)\partial_\beta(\tau_x\phi)|\,dx\,dt \lesssim o_R(1)\|\phi\|^2_{S^a(r>\frac{1}{2}R)[0,T]}. \tag{66}$$

As a last step we combine estimates (62), (64), and (66), while recalling that $d^{\frac{1}{2}}A^{\alpha\beta} = \chi_{>R}A_0^{\alpha\beta} + \mathcal{R}_0 + \chi_{>R}\mathcal{R}_1$. This gives us

$$\int_0^T\int_{\mathbb{R}^3\setminus\mathcal{K}} \chi_{>R}\big(\mathcal{A}^u(\partial_u^b\phi)^2 + \mathcal{A}^{ur}\partial_u^b\phi\cdot\tau_x^{-1}\partial_r^b(\tau_x\phi) + \mathcal{A}^r(\tau_x^{-1}\partial_r^b(\tau_x\phi))^2 + \not{\mathcal{A}}|\not{\nabla}^b\phi|^2\big)\,dx\,dt \\ \leq \int_0^T\int_{\mathbb{R}^3\setminus\mathcal{K}} \tau_x^{-2}A^{\alpha\beta}\partial_\alpha(\tau_x\phi)\partial_\beta(\tau_x\phi)\,dV_g + \mathcal{E}(a,R), \tag{67}$$

where the coefficients $\mathcal{A}$ are given by (54).

Step 2: (*estimating the C^χ-term*) Using (56) we have $C^\chi \in \tau_x^{-2}\mathcal{Z}^{-\frac{1}{2}}$, while (52) give the pointwise estimate $|\tau_x^{-1}X_R(\tau_x\phi)| \lesssim \chi_{>R}\tau_+^{2a}(\tau_0^{\max\{1,2a\}}|\partial_u^b\phi| + |\tau_x^{-1}\partial_r^b(\tau_x\phi)|)$. Thus Hölder's inequality and (10b) give

$$\Big|\int_0^T\int_{\mathbb{R}^3\setminus\mathcal{K}} C^\chi\phi\tau_x^{-1}X_R(\tau_x\phi)\,dV_g\Big| \lesssim o_R(1)\|\phi\|^2_{S^a[0,T](r>\frac{1}{2}R)}. \tag{68}$$

Step 3: (*output of the boundary terms*) Using the bound from (57) we directly have

$$\begin{aligned}\|\tau_x^{-1}(\sqrt{Y^u}\partial(\tau_x\phi),\sqrt{Y^r}\partial_x^b(\tau_x\phi))(T)\|^2_{L_x^2(r>R)}\\ \lesssim \|\tau_x^a\partial\phi(0)\|^2_{L_x^2} + o_R(1)\sup_{0\le t\le T}\|\phi(t)\|^2_{E^a(r>\frac{1}{2}R)} - \int_{\mathbb{R}^3} P(X,\chi,\Omega,\phi)\sqrt{|g|}\,dx\Big|_{t=0}^{t=T}.\end{aligned} \tag{69}$$

Step 4: (*output of the source term*) Finally, another application of Hölder's inequality shows

$$\Big|\int_0^T\int_{\mathbb{R}^3\setminus\mathcal{K}} \Box_g\phi\cdot\tau_x^{-1}X_R(\tau_x\phi)\,dV_g\Big| \lesssim \|\Box_g\phi\|_{N^a[0,T]}\|\phi\|_{S^a(r>\frac{1}{2}R)[0,T]}. \tag{70}$$

Adding together formulas (67)–(70) and using (29), (31), and (32) gives (53). □

For the proof of Proposition 4.6 with $a=1$ it will help to have the following lemma:

Lemma 4.10. *Let Φ be a bounded function supported on $[0,T]$, and let $|||p^2|||_{0,0}<\infty$. Then for $R>0$ one has the estimate*

$$\|\sqrt{\tau_x/\tau_+}p\cdot\Phi\|_{L^2(r>R)} \lesssim o_R(1)\big(\|(1-\chi_{\frac{1}{2}t<r<2t})\Phi\|_{\ell_t^\infty\ell_r^\infty L^2} + \|\chi_{\frac{1}{2}t<r<2t}\Phi\|_{\ell_t^\infty\ell_u^\infty\ell_r^\infty L^2} + \|\tau_x^{\frac{1}{2}}\Phi\|_{L_t^\infty L_x^2}\big). \tag{71}$$

Proof of (71). We split the right-hand side into regions $r<\frac{1}{2}t$, $\frac{1}{2}t<r<2t$, and $r>2t$, all restricted to $r>R$ (we will largely suppress this last condition in the following notation).

In the first region we use

$$\|\sqrt{\tau_x/\tau_+}p\cdot\Phi\|_{L^2(r<\frac{1}{2}t)} \lesssim \|\sqrt{\tau_x/\tau_+}p\|_{\ell_t^2\ell_r^2L^\infty(r<\frac{1}{2}t)}\|\Phi\|_{\ell_t^\infty\ell_r^\infty L^2(r<\frac{1}{2}t)}$$

followed by Young's inequality, which gives $\|\sqrt{\tau_x/\tau_+}p\|_{\ell_t^2\ell_r^2L^\infty(r<\frac{1}{2})} \lesssim \|p\|_{\ell_r^2L^\infty(r>R)} = o_R(1)$.

In the region $\frac{1}{2}t<r<2t$ we use

$$\|\sqrt{\tau_x/\tau_+}p\cdot\Phi\|_{L^2(\frac{1}{2}t<r<2t)} \lesssim \|p\|_{\ell_u^2\ell_r^2L^\infty(\frac{1}{2}t<r<2t)}\|\Phi\|_{\ell_t^\infty\ell_u^\infty\ell_r^\infty L^2(\frac{1}{2}t<r<2t)},$$

followed by $\|p\|_{\ell_u^2\ell_r^2L^\infty(\frac{1}{2}t<r<2t)(r>R)} = o_R(1)$.

Finally, in the region $r>2t$ we use

$$\|\sqrt{\tau_x/\tau_+}p\cdot\Phi\|_{L^2(r>2t)} \lesssim \|\tau_x^{-\frac{1}{2}}p\|_{L_t^2L_x^\infty(r>2t)}\|\tau_x^{\frac{1}{2}}\Phi\|_{L_t^\infty L_x^2},$$

followed by $\|\tau_x^{-\frac{1}{2}}p\|_{L_t^2L_x^\infty(r>2t)} \lesssim \|\chi_{t<\frac{1}{2}r}\tau_x^{-\frac{1}{2}}p\|_{\ell_r^2L_t^2L_x^\infty} \lesssim \|p\|_{\ell_r^2L^\infty(r>R)} = o_R(1)$. □

Proof of Proposition 4.6 for $a=1$. The demonstration is largely similar to the previous proof, with a few key differences. We again choose $\Omega=\tau_x$, but this time set the auxiliary cutoff in Lemma 2.4 to be $\chi=\chi_{<\frac{1}{2}}(r/t)$ which vanishes in $r>\frac{3}{4}t$ with $\chi\equiv 1$ when $r<\frac{1}{2}t$.

Step 1: (*output of the $A^{\alpha\beta}$-contraction*) We again have formulas (58)–(61).

An inspection of the remainder terms on the right-hand side of (61) using condition (52) shows that we also have estimate (62) with the last term on the right-hand side replaced by $o_R(1)\|\phi\|^2_{S^{1,\infty}(r>\frac{1}{2}R)[0,T]}$.

Next, the estimates (63) are again valid except this time we use Hölder's inequality to replace (64) with

$$\int_0^T\int |\tau_x^{-2}\mathcal{R}_0^{\alpha\beta}\partial_\alpha(\tau_x\phi)\partial_\beta(\tau_x\phi)|\,dx\,dt \\ \lesssim R^{\frac{1}{2}}\|\phi\|_{S^{1,\infty}(\frac{1}{2}R<r<R)[0,T]}\|(\tau_-\partial_u^b\phi,\,\tau_x\partial\phi,\,\phi)\|_{\ell_t^1L^2(\frac{1}{2}R<r<R)[0,T]}. \tag{72}$$

To estimate the remainder term involving $\mathcal{R}_1^{\alpha\beta}$ note that (65) is still valid with $a=1$. This allows us to replace (66) in the case $a=1$ with the slight improvement

$$\int_0^T\int_{\mathbb{R}^3\setminus\mathcal{K}} \chi_{>R}|\tau_x^{-2}\mathcal{R}_1^{\alpha\beta}\partial_\alpha(\tau_x\phi)\partial_\beta(\tau_x\phi)|\,dx\,dt \\ \lesssim \|\sqrt{\tau_x/\tau_+}p\cdot\tau_x^{-\frac{1}{2}}\tau_+(\tau_0\partial\phi,\,\partial_x^b\phi,\,\tau_0^{-\frac{1}{2}}\tau_x^{-1}\partial_r^b(\tau_x\phi),\,\tau_x^{-1}\phi)\|^2_{L^2(r>\frac{1}{2}R)[0,T]}, \tag{73}$$

where $|||p^2|||_{0,0}<\infty$. Then an application of Lemma 4.10 produces

$$\text{LHS (73)} \lesssim o_R(1)\cdot\big(\sup_{0\le t\le T}\|\phi(t)\|^2_{E^1(r>\frac{1}{2}R)}+\|\phi\|^2_{S^{1,\infty}(r>\frac{1}{2}R)[0,T]}\big). \tag{74}$$

Combining (72), (74), and the analog of (62), we have estimate (67) for the case $a=1$.

Step 2: (*estimating the B^χ-term*) Inspection of (32) and estimate (56) shows the main thing is to compute trace(A). By definition trace $^{(X_R)}\hat{\pi}=-2\nabla_\alpha X_R^\alpha$, so trace$(A)=4r\tau_x^{-2}X_R^r-\nabla_\alpha X_R^\alpha$. Using the conditions (52) and the support of χ, we conclude $|||\tau_x^2\tau_+^{-1}B^\chi|||_{0,0}<\infty$. Thus

$$\Big|\int_0^T\int_{\mathbb{R}^3\setminus\mathcal{K}} B^\chi\phi^2\,dV_g\Big| \lesssim \|\sqrt{\tau_x/\tau_+}p\cdot\tau_x^{-\frac{1}{2}}\tau_+(\tau_x^{-1}\phi)\|^2_{L^2(r>\frac{1}{2}R)[0,T]}, \tag{75}$$

where $|||p^2|||_{0,0}<\infty$. From this Lemma 4.10 produces

$$\text{LHS (75)} \lesssim o_R(1)(\|\phi\|^2_{E^1(r>\frac{1}{2}R)[0,T]}+\|\phi\|^2_{S^{1,\infty}(r>\frac{1}{2}R)[0,T]}).$$

Step 3: (*estimating the C^χ-term*) Using (56) and the support property of χ we have $\tau_+^2C^\chi\in\mathcal{Z}^{-\frac{1}{2}}$, while (52) give the pointwise bound $|\tau_x^{-1}X_R(\tau_x\phi)|\lesssim\chi_{>R}\tau_+^2(\tau_0^2|\partial_u^b\phi|+|\tau_x^{-1}\partial_r^b(\tau_x\phi)|)$. Thus

$$\Big|\int_0^T\int_{\mathbb{R}^3\setminus\mathcal{K}} C^\chi\phi\tau_x^{-1}X_R(\tau_x\phi)\,dV_g\Big| \lesssim \|\sqrt{\tau_x/\tau_+}p\cdot\tau_x^{-\frac{1}{2}}\tau_+(\tau_0\partial\phi,\,\tau_0^{-\frac{1}{2}}\tau_x^{-1}\partial_r^b(\tau_x\phi),\,\tau_x^{-1}\phi)\|^2_{L^2(r>\frac{1}{2}R)[0,T]},$$

where again $p^2\in\mathcal{Z}^0$ so we conclude via Lemma 4.10.

Step 4: (*output of the boundary terms*) Here we simply note that (69) is also valid for $a=1$.

Step 5: (*output of the source term*) This term is included directly in the definition of $\mathcal{E}(1,R,X)$. □

Proof of Proposition 4.6 for $a=0$. This follows the pattern of the previous two proofs. We set $\Omega=\tau_x$ and choose $\chi=0$.

Step 1: (*output of the $A^{\alpha\beta}$-contraction*) We again have formulas (58)–(61). This time one replaces (63) with $|\mathcal{R}_0^{\alpha\beta}| \lesssim \tau_x^{-1}\chi_R$, in which case the right-hand side of (64) becomes

$$R^{-\frac{1}{2}}\|(\partial\phi, \tau_x^{-1}\phi)\|^2_{L^2(\frac{1}{2}R<r<R)[0,T]}.$$

The analog of (62) is also valid with the second term on the right-hand side replaced by

$$o_R(1)\|\phi\|^2_{\mathrm{LE}^0(r>\frac{1}{2}R)[0,T]}.$$

The main difference is that this time the conditions (55) give $|\partial_u^b X^u + \partial_r^b X^r| \lesssim \tau_x^{-1}\tau_0^{-1}$. Together with the condition $\partial_u^b Y^r = 0$ and an application of Lemma 4.3, this means we need to replace (65) with

$$|||\tau_x\mathcal{R}_1^{ij}|||_{-1,0} < \infty, \quad |||\tau_x\mathcal{R}_1^{ui}|||_{-\frac{1}{2},0} < \infty, \quad |||\tau_x\mathcal{R}_1^{uu}|||_{\frac{1}{2},0} < \infty.$$

This is enough to show the analog of (66) with right-hand side replaced by $o_R(1)\|\phi\|^2_{\mathrm{LE}^0(r>\frac{1}{2}R)[0,T]}$.
Step 2: (*estimating the C^χ-term*) Using (56) gives $C^\chi \in \tau_x^{-2}\mathcal{Z}^{-\frac{1}{2}}$, while (55) gives $|\tau_x^{-1}X_R(\tau_x\phi)| \lesssim \chi_{r>R}|(\partial\phi, \tau_x^{-1}\phi)|$. Thus (68) is valid with right-hand side replaced by $o_R(1)\|\phi\|^2_{\mathrm{LE}^0(r>\frac{1}{2}R)[0,T]}$.
Step 4: (*output of the boundary terms*) Here we note that (69) is also valid for $a = 0$.

Step 5: (*output of the source term*) This term is included directly in the definition of $\mathcal{E}(0, R, X)$. □

4C. *Abstract bounds for commutators.* We now turn to some further consequences of Lemma 4.3.

Lemma 4.11 (abstract bounds for commutators). *Let $\mathcal{R}$ and $\mathcal{S}$ be a contravariant 2-tensor and vector field respectively. From them define the operator $\mathcal{Q} = \nabla_\alpha\mathcal{R}^{\alpha\beta}\nabla_\beta + \mathcal{S}^\alpha\nabla_\alpha$. Then the following results hold*:

(I) *Suppose that $\mathcal{R}$ and $\mathcal{S}$ satisfy (40) and (46) respectively. Then if X is any vector field which satisfies (41) and (42) with $a = b = c = 0$, one has $[X, \mathcal{Q}] = \nabla_\alpha\widetilde{\mathcal{R}}^{\alpha\beta}\nabla_\beta + \widetilde{\mathcal{S}}^\alpha\nabla_\alpha$, where $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{S}}$ satisfy (40) and (46) respectively as well.*

(II) *Alternatively suppose $X \in \mathbb{L}_0 = \{\partial_u^b, \partial_i^b - \omega^i\partial_u^b\}$ with the same conditions on $\mathcal{R}$ and $\mathcal{S}$ as above. Then the previous result holds with bound (43) for $\widetilde{\mathcal{R}}$ and bound (47) for $\widetilde{\mathcal{S}}$ with $a = c = -1$ and $b = 1$.*

(III) *For any $\mathcal{Q}$ as defined above with $\mathcal{R}$ and $\mathcal{S}$ satisfying (40) and (46) we have the pointwise bound*

$$|\mathcal{Q}\phi| \lesssim q\cdot\tau_x^{-2}\sum_{1\le l+|J|\le 2}|(\tau_x\tau_0\partial_u^b)^l(\tau_x\partial_x^b)^J\phi|, \quad \text{where } q \in \mathcal{Z}^{-\frac{1}{2}}. \tag{76}$$

Alternatively, suppose $\mathcal{R}$ and $\mathcal{S}$ satisfy (43) and (47) with $a = c = -1$ and $b = 1$. Then

$$|\mathcal{Q}\phi| \lesssim q\cdot\tau_x^{-1}\sum_{|I|\le 1}(\tau_0^{\frac{1}{2}}|\partial_u^b\Gamma^I\phi| + |\partial_x^b\Gamma^I\phi|), \quad \text{where } q \in \mathcal{Z}^{-1}, \tag{77}$$

and where $\Gamma \in \mathbb{L}_0 = \{\partial_u^b, \partial_i^b - \omega^i\partial_u^b\}$.

(IV) *Finally, let $\mathcal{R}$ satisfy any combination of conditions (43), (44), and (45), and let $\mathcal{S}$ satisfy (47). Let w be a smooth weight function with*

$$|(\tau_-\partial_u^b)^l(\tau_x\partial_x^b)^J w| \lesssim_{l,J} \tau_x^{a'}\tau_+^{b'}\tau_-^{c'}. \tag{78}$$

Then with $\mathcal{Q}$ as defined above we have $w\mathcal{Q} = \nabla_\alpha \widetilde{\mathcal{R}}^{\alpha\beta}\nabla_\beta + \widetilde{\mathcal{S}}^\alpha\nabla_\alpha$, where $\widetilde{\mathcal{R}}^{\alpha\beta}$ satisfies the appropriate combination of conditions (43), (44), and (45), and $\widetilde{\mathcal{S}}^\alpha$ satisfies (47), in each case with coefficients $a+a'$, $b+b'$, and $c+c'$.

Proof. We'll show each part separately.

Part 1: (*the commutator property for X satisfying (41) and (42)*) By formula (27) and parts (I) and (IV) of Lemma 4.3, it suffices to show that $\mathcal{R}^{\alpha\beta}\nabla_\alpha(\nabla_\gamma X^\gamma)$ satisfies (46). We'll do this in a bit more generality here for use in the sequel.

Let w be any weight function which satisfies (78), and let $\mathcal{R}$ be any contravariant 2-tensor which satisfies the weaker conditions (44) and (45), and the remaining conditions in (43). Then we claim (47) holds for $\mathcal{S}^\alpha = \mathcal{R}^{\alpha\beta}\nabla_\alpha(w)$ with weights $a+a'$, $b+b'$, $c+c'$. To see this note

$$\mathcal{S}^i = \mathcal{R}^{ui}\partial_u^b w + \mathcal{R}^{ji}\partial_j^b(w), \qquad \mathcal{S}^u = \mathcal{R}^{uu}\partial_u^b w + \mathcal{R}^{ju}\partial_j^b(w),$$

so the desired estimates follow easily by multiplying together bounds (78) and the appropriate combination of (43), (44), and (45).

To show part (I) of the lemma note that if X is as stated, then (39) shows $w = \nabla_\gamma X^\gamma$ satisfies (78) with $a' = b' = c' = 0$. The desired result now follows from the discussion of the previous paragraph.

Part 2: (*the commutator property for $X \in \mathbb{L}_0$*) This follows at once from part (V) of Lemma 4.3 and the main calculation of Part 1 above. Note that if $X \in \mathbb{L}_0$ then $w = \nabla_\gamma X^\gamma$ satisfies (78) with $a' = c' = -1$ and $b' = 1$.

Part 3: (*the pointwise estimates (76) and (77)*) The bound for the $\mathcal{S}$ potion of $\mathcal{Q}\phi$ follows at once from (47). For the $\mathcal{R}$-contraction we write in Bondi coordinates

$$\nabla_\alpha \mathcal{R}^{\alpha\beta}\nabla_\beta\phi = \mathcal{R}^{\alpha\beta}\partial_\alpha^b(\ln\sqrt{|g|})\partial_\beta^b\phi + (\partial_\alpha^b\mathcal{R}^{\alpha\beta})\partial_\beta^b\phi + \mathcal{R}^{\alpha\beta}\partial_\alpha^b\partial_\beta^b\phi = \widetilde{S}^\alpha\partial_\alpha^b u + \mathcal{R}^{\alpha\beta}\partial_\alpha^b\partial_\beta^b\phi.$$

We only need to show that $\widetilde{S}$ satisfies (47) with $a=b=c=0$ in the case of estimate (76), and $a=c=-1$, $b=1$ in the case of estimate (76); then study $\mathcal{R}^{\alpha\beta}\partial_\alpha^b\partial_\beta^b\phi$.

For the first term of $\widetilde{S}$ we use the fact that $w = \ln\sqrt{|g|}$ satisfies (78) with $a' = b' = c' = 0$, which follows from (39). Then by the main calculation of Part 1 above we have (47) for $\mathcal{R}^{\alpha\beta}\partial_\alpha^b(\ln\sqrt{|g|})$.

For the expression $\partial_\alpha^b\mathcal{R}^{\alpha\beta}$ the appropriate version of (47) follows at once from (43).

For the final term note that if $\mathcal{R}$ satisfies (43) then one has the pointwise estimate

$$|\mathcal{R}^{\alpha\beta}\partial_\alpha^b\partial_\beta^b\phi| \lesssim q\cdot\tau_x^a\tau_+^b\tau_-^c(\tau_0^{\frac{5}{2}}|(\partial_u^b)^2\phi| + \tau_0|\partial_u^b\partial_x^b\phi| + \tau_0^{\frac{1}{2}}|(\partial_x^b)^2\phi|), \quad \text{where } q\in\mathcal{Z}^{-\frac{1}{2}}.$$

This is bounded by the right-hand side of (76) when $a=b=c=0$, and the right-hand side of (77) when $a=c=-1$ and $b=1$.

Part 4: (*proof of the algebra property (IV)*) It is immediate that bound (47) for $\mathcal{S}$ is stable under multiplication by w satisfying (78) with the appropriate change of weights. For the quadratic term of $\mathcal{Q}$ we write $w\nabla_\alpha\mathcal{R}^{\alpha\beta}\nabla_\beta = \nabla_\alpha w\mathcal{R}^{\alpha\beta}\nabla_\beta - \mathcal{R}^{\alpha\beta}\nabla_\alpha(w)\nabla_\beta$. For the first term we use bounds (43) which are also stable under multiplication by w. For the second term we use the main calculation of Part 1 above. □

Parts (I) and (II) of the last lemma imply the following:

Corollary 4.12 (estimates for multicommutators). *Let g be a metric which satisfies (3), and as usual set $\mathbb{L}_0 = \{\partial_u^b, \partial_i^b - \omega^i \partial_u^b\}$ and $\mathbb{L} = \{S, \Omega_{ij}\} \cup \mathbb{L}_0$. Then the following hold:*

(I) *If I is any multiindex then for products of vector fields in $\mathbb{L}$ one has the identity*

$$[\Box_g, \Gamma^I] = \sum_{I' \subsetneq I} ([\nabla_\alpha \mathcal{R}^{\alpha\beta}_{I'} \nabla_\beta + \mathcal{S}^\alpha_{I'} \nabla_\alpha]\Gamma^{I'} + w_{I'} \Gamma^{I'} \Box_g), \tag{79}$$

where the sum is taken over the collection of all multiindices I' strictly contained in I; in particular each I' satisfies $|I'| \leq |I| - 1$. Here $\mathcal{R}_{I'}$ and $\mathcal{S}_{I'}$ satisfy (40) and (46) respectively, while there exist constants $w^0_{I'} \in \mathbb{R}$ such that

$$w_{I'} - w^0_{I'} \in \mathcal{Z}^0. \tag{80}$$

(II) *If the product in the previous part is restricted to vector fields in $\mathbb{L}_0$, then one has identity (79) with estimates (43) and (47) for $\mathcal{R}_{I'}$ and $\mathcal{S}_{I'}$ with $a = c = -1$ and $b = 1$. In addition (80) in this case is replaced by $w_{I'} \in \tau_x^{-1} \cdot \mathcal{Z}^{-1}$.*

(III) *Let Γ^I be a product of vector fields in $\mathbb{L}$ and $\widetilde{\Gamma}^J$ a product of vector fields in $\mathbb{L}_0$. Then one has the identity*

$$\widetilde{\Gamma}^J [\Box_g, \Gamma^I] = \sum_{I' \subsetneq I,\, J' \subseteq J} ([\nabla_\alpha \mathcal{R}^{\alpha\beta}_{I',J'} \nabla_\beta + \mathcal{S}^\alpha_{I',J'} \nabla_\alpha]\widetilde{\Gamma}^{J'} \Gamma^{I'} + w_{I',J'} \widetilde{\Gamma}^{J'} \Gamma^{I'} \Box_g), \tag{81}$$

where $\mathcal{R}_{I',J'}$, $\mathcal{S}_{I',J'}$, and $w_{I',J'}$ satisfy respectively (40), (46), and (80).

Proof. We'll show the different parts separately.

Part 1: (*proof of* (79) *for* $\mathbb{L}$ *and* $\mathbb{L}_0$) Here we will focus only on the case of products of vector fields in $\mathbb{L}$, as the case of $\mathbb{L}_0$ is completely analogous. Using the algebra property of Part (IV) of Lemma 4.11 and an induction, it suffices to show

$$[\Box_g, \Gamma^I] = \sum_{I' \subsetneq I} [\nabla_\alpha \mathcal{R}^{\alpha\beta}_{I'} \nabla_\beta + \mathcal{S}^\alpha_{I'} \nabla_\alpha + w_{I'} \Box_g]\Gamma^{I'}, \quad \text{where } \Gamma^0 = \mathrm{Id}. \tag{82}$$

We shall prove this last bound itself by induction on the length $|I|$ of the product Γ^I.

Case 1: ($|I| = 1$) When Γ^I consists of a single vector field in $\mathbb{L}$, formula (82) follows from a combination of formulas (36) and (37), followed by estimates (39), (40), and part (I) of Lemma 4.3.

Case 2: ($|I| \geq 2$) Assume formula (82) holds for all multiindices $|I| < k$ and choose some $|I| = k$, and write $\Gamma^I = \Gamma^{I_0}\Gamma^{I_1}$ for some $|I_0| = 1$. By the Leibniz rule we have $[\Box_g, \Gamma^I] = [\Box_g, \Gamma^{I_0}]\Gamma^{I_1} + \Gamma^{I_0}[\Box_g, \Gamma^{I_1}]$. By the same calculations as in the previous step we have

$$[\Box_g, \Gamma^{I_0}]\Gamma^{I_1} = \nabla_\alpha \mathcal{R}^{\alpha\beta} \nabla_\beta \Gamma^{I_1} + w \Box_g \Gamma^{I_1},$$

where $\mathcal{R}$, w are of the desired form. On the other hand by induction we have

$$\Gamma^{I_0}[\Box_g, \Gamma^{I_1}] = \sum_{I' \subsetneq I_1} (\nabla_\alpha \mathcal{R}^{\alpha\beta}_{I'} \nabla_\beta + \mathcal{S}^\alpha_{I'} \nabla_\alpha + w_{I'} \Box_g)\Gamma^{I_0}\Gamma^{I'} + \sum_{I' \subsetneq I_1} [\Gamma^{I_0}, (\nabla_\alpha \mathcal{R}^{\alpha\beta}_{I'} \nabla_\beta + \mathcal{S}^\alpha_{I'} \nabla_\alpha + w_{I'} \Box_g)]\Gamma^{I'}.$$

By formula (27) and Part (I) of Lemma 4.11 the commutator $[\Gamma^{I_0}, (\nabla_\alpha \mathcal{R}^{\alpha\beta}_{I'} \nabla_\beta + \mathcal{S}^\alpha_{I'} \nabla_\alpha)]$ again yields an operator of the form $\mathcal{Q}_{I'} = \nabla_\alpha \widetilde{\mathcal{R}}^{\alpha\beta}_{I'} \nabla_\beta + \widetilde{\mathcal{S}}^\alpha_{I'} \nabla_\alpha$. Using (39), the same calculations of the previous step, and part (IV) of Lemma 4.11, we see the commutator $[\Gamma^{I_0}, w_{I'} \Box_g]$ yields another such operator. Combining all this yields (82).

Part 2: (*proof of* (81)) Applying $\widetilde{\Gamma}^J$ to formula (79) and then computing $[\widetilde{\Gamma}^J, (\nabla_\alpha \mathcal{R}^{\alpha\beta}_{I'} \nabla_\beta + \mathcal{S}^\alpha_{I'} \nabla_\alpha)]$ through a repeated use of part (II) of Lemma 4.11 yields the desired result. □

4D. *Klainerman–Sideris inequalities.* We now prove an analog of the Klainerman–Sideris identity [1996].

Lemma 4.13 (Klainerman–Sideris-type identity). *One has the pointwise estimates*

$$\sum_{1\leq l+|J|\leq 2} |(\tau_x \tau_0 \partial^b_u)^l (\tau_x \partial^b_x)^J \phi| \lesssim \sum_{\substack{l+|J|=1\\ |I|\leq 1}} |(\tau_x \tau_0 \partial^b_u)^l (\tau_x \partial^b_x)^J \Gamma^I \phi| + \tau_x^2 \tau_0 |\Box_g \phi|, \tag{83}$$

$$\sum_{1\leq l+|J|\leq k} |(\tau_- \partial^b_u)^l (\tau_x \partial^b_x)^J \phi| \lesssim \sum_{\substack{l+|J|=1\\ |I|\leq k-1}} |(\tau_- \partial^b_u)^l (\tau_x \partial^b_x)^J \Gamma^I \phi| + \sum_{l+|J|\leq k-2} \tau_x^2 \tau_0 |(\tau_- \partial^b_u)^l (\tau_x \partial^b_x)^J \Box_g \phi|, \tag{84}$$

where in the second bound the implicit constant depends on $k \geq 2$. Here all $\Gamma \in \mathbb{L}$.

Proof of estimates (83) *and* (84). Both estimates follow from essentially the same computation.

Step 1: (*a preliminary reduction*) First, note that in either case it suffices to restrict to $r > R \gg 1$, as estimates (83) and (84) with some implicit constant $C = C(R, k)$ are automatic in $r \leq R$ by choosing all $\Gamma \in \mathbb{L}_0 \cup \{S\}$.

Next, let $\Box_\eta$ be the Minkowski wave operator in Bondi coordinates which satisfies

$$\Box_\eta = -2\partial^b_u \partial_r + (\partial^b_r)^2 - 2r^{-1}\partial^b_u + 2r^{-1}\partial^b_r + r^{-2}\sum_{i<j}(\Omega_{ij})^2,$$

$$\Box_\eta = d^{\frac{1}{2}} \Box_g - \partial^b_\alpha \mathcal{R}^{\alpha\beta} \partial^b_\beta + O(r^{-2}) \partial^b_x \partial^b_u + O(r^{-3}) \partial^b_u,$$

where $d = |g|$ is the metric determinant in Bondi coordinates, and $\mathcal{R} = d^{\frac{1}{2}} g - h$, where h is given in (4) and $\mathcal{R}$ satisfies (40). Thanks to (76) one has

$$\tau_x^2 \tau_0 |\Box_\eta \phi| \lesssim \tau_x^2 \tau_0 |\Box_g \phi| + o_R(1) \cdot \sum_{1\leq l+|J|\leq 2} |(\tau_x \tau_0 \partial^b_u)^l (\tau_x \partial^b_x)^J \phi| \quad \text{in } r > R.$$

Therefore it suffices to replace $\Box_g$ by $\Box_\eta$ in (83), and also in (84) when proving it for $k = 2$.

Step 2: (*proof of* (83) *and* (84) *for $k = 2$ and $\Box_g$ replaced by $\Box_\eta$*) Start with the two identities

$$r^2 \tau_+^{-1} \partial^b_r S = r^2 u \tau_+^{-1} \partial^b_u \partial^b_r + r^3 \tau_+^{-1} (\partial^b_r)^2 + r^2 \tau_+^{-1} \partial^b_r,$$

$$\tfrac{1}{2} r^2 u \tau_+^{-1} \Box_\eta = -r^2 u \tau_+^{-1} \partial^b_u \partial^b_r + \tfrac{1}{2} r^2 u \tau_+^{-1} (\partial^b_r)^2 - r u \tau_+^{-1} \partial^b_u + r u \tau_+^{-1} \partial^b_r + \tfrac{1}{2} u \tau_+^{-1} \sum_{i<j} (\Omega_{ij})^2.$$

Adding the two operators on the left-hand sides above and applying to ϕ yields

$$|(\tau_x \partial^b_r)^2 \phi| \lesssim \sum_{l+|J|=1,\ |I|\leq 1} |(\tau_x \tau_0 \partial^b_u)^l (\tau_x \partial^b_x)^J \Gamma^I \phi| + \tau_x^2 \tau_0 |\Box_\eta \phi|.$$

Note that by Remark 1.4 we can assume $u + 2r \approx \tau_+$ in $r > R$.

Next, the vector fields S and ∂_u^b alone give the pair of inequalities

$$|(\tau_x\tau_0\partial_u^b)^2\phi| + |(\tau_x\tau_0\partial_u^b)(\tau_x\partial_r^b)\phi| \lesssim |(\tau_x\partial_r^b)^2\phi| + \sum_{l+|J|=1,\,|I|\leq 1} |(\tau_x\tau_0\partial_u^b)^l(\tau_x\partial_x^b)^J\Gamma^I\phi|,$$

$$|(\tau_-\partial_u^b)^2\phi| + |(\tau_-\partial_u^b)(\tau_x\partial_r^b)\phi| \lesssim |(\tau_x\partial_r^b)^2\phi| + \sum_{l+|J|=1,\,|I|\leq 1} |(\tau_-\partial_u^b)^l(\tau_x\partial_x^b)^J\Gamma^I\phi|.$$

Finally, note that all other combinations of derivatives on the left-hand side of (83) (respectively (84) when $k=2$) are automatically controlled by the first sums on right-hand side of (83) (respectively (84) when $k=2$) thanks to the rotation vector fields.

Step 3: (*estimate* (84) *when* $k>2$) It suffices to show (84) assuming it's true for $k-1$. Applying (84) with $k-1$ to $X\phi$, where $X\in\{\tau_-\partial_u^b,\,\tau_x\partial_i^b\}$, and then feeding the results back into (84) with $k=2$ applied to $\Gamma^I\phi$ where $\Gamma\in\mathbb{L}$ and $|I|\leq k-2$, we need to show the commutator estimates

$$\sum_{\substack{|I'|,|I''|=1\\ |I|\leq k-2}} |X^{I'}[\Gamma^I, X^{I''}]\phi| \lesssim \sum_{1\leq|I|\leq k-1} |X^I\phi|,$$

$$\sum_{|I|\leq k-2} \tau_x^2\tau_0|[\Box_g, \Gamma^I]\phi| \lesssim \sum_{1\leq|I|\leq k-1} |X^I\phi|,$$

$$\sum_{\substack{|I|\leq k-3\\ |I'|=1}} \tau_x^2\tau_0|X^I[\Box_g, X^{I'}]\phi| \lesssim \sum_{1\leq|I|\leq k-1} |X^I\phi|,$$

where each X^I, $X^{I'}$, and $X^{I''}$ is a product of members of $\{\tau_-\partial_u^b,\,\tau_x\partial_i^b\}$. The validity of these last three bounds is easily checked by using (79) and (76) to evaluate $[\Box_g,\Gamma^I]\phi$, and by referring to the following lemma. □

Lemma 4.14 (products of "standard" vector fields). *Let X^I and Y^J be products of vector fields whose Bondi coordinate coefficients satisfy*

$$|(\tau_-\partial_u^b)^l(\tau_x\partial_x^b)^J X^u| \lesssim_{l,J} \tau_-, \quad |(\tau_-\partial_u^b)^l(\tau_x\partial_x^b)^J X^i| \lesssim_{l,J} \tau_x,$$

with the convention a product of length zero is a scalar satisfying bounds of the form (78) *with $a=b=c=0$. Then the following hold*:

(I) *The vector field $[X^I, Y^J]$ is a sum of products of similar vector fields each with word length $|I|+|J|-1$.*

(II) *For any nonzero multiindex I we have*

$$|X^I\phi| \lesssim \sum_{1\leq l+|J|\leq|I|} |(\tau_-\partial_u^b)^l(\tau_x\partial_x^b)^J\phi|.$$

(III) *If X^I is any product of such vector fields then*

$$\tau_x^2\tau_0 X^I\tau_x^{-2}\tau_0^{-1} - X^I = \sum_{|J|\leq|I|-1} Y^J$$

for some other collection of products of similar vector fields Y^I.

(IV) *One has* $\tau_x^2\tau_0\Box_g = \sum_{1\le|I|\le 2} X^I$ *for some collection of such vector fields* X^I.

Proof. The proof of the first three parts boils down to more or less elementary calculations. The last part follows from a direct calculation involving the conditions (3) and (39) is also left to the reader. □

4E. *A pointwise bound for multicommutators.* To conclude this section, we record a combined consequence of Lemma 4.11, Corollary 4.12, and Lemma 4.13. This will be our main tool for controlling commutators in the sequel.

Lemma 4.15 (pointwise bound for commutators). *For pair of multiindices* I, J *with* $|I|\ge 1$ *one has*

$$|\widetilde{\Gamma}^J[\Box_g,\Gamma^I]\phi| \lesssim \sum_{\substack{I'\subsetneq I\\ |J'|\le|J|}}\Bigg(\sum_{\substack{l+|K|=1\\ |I''|\le 1}} q\cdot\tau_x^{-1}|(\tau_0\partial_u^b)^l(\partial_x^b)^K\widetilde{\Gamma}^{J'}\Gamma^{I'+I''}\phi| + |\widetilde{\Gamma}^{J'}\Gamma^{I'}\Box_g\phi|\Bigg), \quad \text{where } q\in\mathcal{Z}^{-\frac{1}{2}}, \tag{85}$$

where Γ^I (*etc.*) *denotes a product of vector fields in* $\mathbb{L}=\{S,\Omega_{ij},\partial_u^b,\partial_i^b-\omega^i\partial_u^b\}$, *and* $\widetilde{\Gamma}^J$ (*etc.*) *denotes a product of vector fields in* $\mathbb{L}_0=\{\partial_u^b,\partial_i^b-\omega^i\partial_u^b\}$.

Proof. Using (81) followed by (76) and then (83) we have

$$\text{LHS (85)} \lesssim \text{RHS (85)} + \sum_{I'\subsetneq I,\ J'\subseteq J}|\Box_g(\widetilde{\Gamma}^{J'}\Gamma^{I'}\phi)|.$$

Thus, modulo induction on $|I|$ we have reduced matters to estimating the commutator $|[\Box_g,\widetilde{\Gamma}^{J'}](\Gamma^{I'}\phi)|$. Applying part (II) of Corollary 4.12, again followed by (76) and then (83), and inducting on $|I|$ we have

$$|[\Box_g,\widetilde{\Gamma}^{J'}](\Gamma^{I'}\phi)| \lesssim \text{RHS (85)} + \sum_{J''\subsetneq J'}|\Box_g(\widetilde{\Gamma}^{J''}\Gamma^{I'}\phi)|,$$

so the proof concludes with an additional round of induction, this time with respect to $|J|$. □

5. Proof of the weighted L^2 estimates for $k=0$

Theorem 5.1 (generalized local energy decay estimates). *Assume estimates* (7a) *and* (7b) *for* $s=0$; *then the following are true*:

(I) *For* R *sufficiently large there exists* $C_R>0$ *and vector fields* X_j *such that one has the uniform bound*

$$\|\phi\|_{WLE^0[0,T]} \lesssim \sup_{0\le t\le T}\|\partial\phi(t)\|_{L_x^2} + C_R\|\phi\|_{WLE^0_{\text{class}}[0,T]} + \sup_j\bigg|\int_0^T\int_{\mathbb{R}^3\setminus\mathcal{K}}\Box_g\phi\cdot X_j\phi\,dV_g\bigg|^{\frac{1}{2}}, \tag{86}$$

where $X_j=\chi_{>R}q_j\partial_u^b$, *with* $q_j=q_j(u)$ *obeying the uniform bounds* $|(\tau_-\partial_u^b)^k q_j|\lesssim_k 1$. *In addition* $\chi_{>R}=\chi_{>1}(R^{-1}\cdot)$ *is a radial bump function with* $\chi_{>R}\equiv 1$ *on* $r>R$ *and* $\chi_{>R}\equiv 0$ *on* $r<\frac{1}{2}R$ *for some* R *sufficiently large so that* $\mathcal{K}\subseteq\{r<\frac{1}{2}R\}$.

(II) *For each* $0<a<1$ *there exists an* R_a *sufficiently large so that*

$$\sup_{0\le t\le T}\|\phi(t)\|_{E^a} + \|\phi\|_{S^a[0,T]} \lesssim_a \|\tau_x^a\partial\phi(0)\|_{L_x^2} + \|\tau_+^{a-1}\phi\|_{H_1^1(r<R_a)[0,T]} + \|\Box_g\phi\|_{N^a[0,T]}, \tag{87}$$

where the implicit constant depends continuously on $a\in(0,1)$.

(III) *For R sufficiently large there exists $C_R > 0$ and vector fields X_j such that one has uniformly*

$$\sup_{0\leq t\leq T} \|\phi(t)\|_{E^1} + \|\phi\|_{S^{1,\infty}[0,T]} \lesssim \|\tau_x\partial\phi(0)\|_{L^2} + C_R\|\phi\|_{\ell^1_t H^1_1(r<R)[0,T]} + \|\Box_g\phi\|_{\ell^\infty_t N^1[0,T]} + \sup_j\left|\int_0^T\int_{\mathbb{R}^3\setminus\mathcal{K}} \Box_g\phi\cdot\tau_x^{-1}X_j(\tau_x\phi)\,dV_g\right|^{\frac{1}{2}}, \quad (88)$$

where $X_j = \chi_{>R_1}q_jK_0$, with K_0 given by the formula $K_0 = (1+u^2)\partial_u^b + 2(u+r)r\partial_r^b$, where $q_j = q_j(u)$ has the uniform bounds $|(\tau_-\partial_u^b)^k q_j| \lesssim_k 1$, and where $\chi_{>R}$ is the same as in (86) above.

5A. *Splitting into interior and exterior estimates.* To control the solution in the interior we use:

Proposition 5.2 (weighted LE bounds in time-like regions). *Let $R_0\geq 1$ be as in Definition 1.7. Then for any $a\geq 0$ and $1\leq p\leq\infty$ one has the uniform bound*

$$\sup_{0\leq t\leq T} \|\tau_+^a(\partial\phi,\tau_x^{-1}\phi)(t)\|_{L^2_x(r<\frac{1}{2})} + \|\tau_+^a(\partial\phi,\tau_x^{-1}\phi)\|_{\ell^p_t\mathrm{LE}(r<\frac{1}{2})[0,T]} \lesssim \|\tau_x^a\tau_+^{-\frac{1}{2}}(\partial\phi,\tau_+^{-1}\phi)\|_{\ell^p_t L^2(r<\frac{3}{4}t)[0,T]} + \|\tau_+^{a-1}\phi\|_{\ell^p_t H^1_1(r<R_0)[0,T]} + \|\tau_+^a\tau_0\Box_g\phi\|_{\ell^p_t\mathrm{WLE}^{*,0}[0,T]}. \quad (89)$$

Here the ℓ^p_t sum is taken over a collection of dyadic regions $\langle t\rangle\approx 2^j\geq 1$.

In the exterior we use multipliers to show that:

Proposition 5.3 (weighted exterior LE bounds). *One has the following estimates uniformly for R sufficiently large that $\mathcal{K}\subseteq\{r\leq\frac{1}{2}R\}$:*

(I) *The following null energy bounds hold*:

$$\|(\partial_x^b\phi,\tau_x^{-1}\phi)\|_{\mathrm{NLE}(r>R)[0,T]} \lesssim R^{\frac{1}{2}}\|\phi\|_{\mathrm{WLE}^0_{\mathrm{class}}[0,T]} + \sqrt{\mathcal{E}(0,R)}, \quad (90)$$

where $\mathcal{E}(0,R) = \sup_j\mathcal{E}(0,R,X_j)$ is given by formula (50) with X_j as in Theorem 5.1.

(II) *For fixed $0<a<1$ there holds*

$$\sup_{0\leq t\leq T}\|\phi(t)\|_{E^a(r>\max\{R,\frac{1}{2}t\})} + \|\phi\|_{S^a(r>\max\{R,\frac{1}{2}t\})[0,T]} + \|\tau_x^a\tau_+^{-\frac{1}{2}}(\partial\phi,\tau_x^{-1}\phi)\|_{L^2(R<r<\frac{3}{4}t)[0,T]} \lesssim_a \sqrt{\mathcal{E}(a,R)}, \quad (91)$$

where $\mathcal{E}(a,R)$ is given by formula (49).

(III) *Corresponding to $a=1$ there holds the estimate*

$$\sup_{0\leq t\leq T}\|\phi(t)\|_{E^1(r>\max\{R,\frac{1}{2}t\})} + \|\phi\|_{S^{1,\infty}(r>\max\{R,\frac{1}{2}t\})[0,T]} + \|\tau_x\tau_+^{-\frac{1}{2}}(\partial\phi,\tau_+^{-1}\phi)\|_{\ell^\infty_t L^2(R<r<\frac{3}{4}t)[0,T]} \lesssim \sup_{0\leq t\leq T} R^{\frac{1}{2}}\|\tau_-^{\frac{1}{2}}(\partial\phi,\tau_x^{-1}\phi)(t)\|_{L^2_x(\frac{1}{2}R<r<R)} + \sqrt{\mathcal{E}(1,R)}, \quad (92)$$

where $\mathcal{E}(1,R) = \sup_j\mathcal{E}(1,R,X_j)$ is given in terms of formula (51) with X_j as in Theorem 5.1.

Proof that Propositions 5.2 and 5.3 imply Theorem 5.1. We do this separately for each estimate.

Case 1: ($a=0$) Here we need to show (86) follows directly from (90) and the assumed bounds (7). From inspection of $\sqrt{\mathcal{E}(0,R)}$ and taking R sufficiently large, we only need to bound $\|(\partial_x^b\phi,\tau_x^{-1}\phi)\|_{\mathrm{NLE}(R_0<r<R)[0,T]}$ in terms of C_R times $\|\phi\|_{\mathrm{WLE}^0_{\mathrm{class}}[0,T]}$. This follows by taking $C_R\approx R^{\frac{1}{2}}$.

Case 2: $(0 < a < 1)$ Adding together a suitable linear combination of estimates (89) with $p = 2$ and (91), and using the inclusions $\ell_t^2\mathrm{LE} \subseteq \mathrm{LE}$ and $\mathrm{LE}^* \subseteq \ell_t^2\mathrm{LE}^*$ (from Minkowski's inequality), we have uniformly

$$\sup_{0\le t\le T} \|\phi(t)\|_{E^a} + \|\phi\|_{S^a[0,T]} \lesssim_a \sup_{0\le t\le 2R} \|\phi(t)\|_{E^a(r<R)} + \|\phi\|_{S^a(r<R)[0,2R]} + \|\tau_x^a \tau_+^{-\frac{1}{2}}(\partial\phi,\, \tau_x^{-1}\phi)\|_{L^2(r<\min\{R,\frac{3}{4}t\})[0,T]} + \|\tau_+^{a-1}\phi\|_{H_1^1(r<R_0)[0,T]} + \|\Box_g\phi\|_{N^a[0,T]} + \sqrt{\mathcal{E}(a,\, R)},$$

Next, uniformly for $T_0 \ge 2R \ge 1$ there hold the pair of bounds

$$\|\tau_x^a \tau_+^{-\frac{1}{2}}(\partial\phi,\, \tau_x^{-1}\phi)\|_{L^2(r<\min\{R,\frac{3}{4}t\})[0,T]} \lesssim \ln(R)\|\phi\|_{S^a(r<R)[0,T_0]} + (R/T_0)^{\frac{1}{2}}\|\phi\|_{S^a[T_0,T]},$$

and

$$\mathcal{E}(a,\, R) \lesssim \|\tau_x^a\partial\phi(0)\|_{L_x^2}^2 + (o_R(1) + (R/T_0)^a)\cdot\Big(\sup_{0\le t\le T}\|\phi(t)\|_{E^a}^2 + \|\phi\|_{S^a[0,T]}^2\Big) + \|\phi\|_{S^a[0,T]}\cdot(\|\phi\|_{S^a(r<R)[0,T_0]} + R^{\frac{3}{2}}\|\tau_+^{a-1}\phi\|_{H_1^0(r<R)[0,T]} + \|\Box_g\phi\|_{N^a[0,T]}).$$

In addition for $T_0 \ge 2R$ there is the simple bound

$$\sup_{0\le t\le 2R}\|\phi(t)\|_{E^a(r<R)} + \ln(R)\|\phi\|_{S^a(r<R)[0,T_0]} \lesssim_{R,T_0} \|\tau_+^{a-1}\phi\|_{H_1^1(r<R)[0,T_0]}.$$

Therefore combining the last four inequalities with $R = R_a \ge R_0$ sufficiently large depending on a, and for some $T_0 = T_a \ge R_a$ which depends on both the size of R_a and a, we have estimate (87).

Case 3: $(a = 1)$ Here we apply (89) with $a = 1$ and $p = \infty$, and add to this a suitable linear combination of (92). Note that an application of the weighted trace estimate (188c) followed by (184) with $a = 1$ gives

$$\sup_{0\le t\le T} R^{\frac{1}{2}}\|\tau_-^{\frac{1}{2}}(\partial\phi,\, \tau_x^{-1}\phi)(t)\|_{L_x^2(\frac{1}{2}R<r<R)} \lesssim \|\tau_x\partial\phi(0)\|_{L^2} + R^{\frac{3}{2}}\|\phi\|_{\ell_t^1 H_1^1(r<R)[0,T]}.$$

The rest of the proof follows a similar pattern to Case 2 above. □

5B. *Proof of the interior estimate.* Before moving on to the proof of the exterior estimates, which are more involved, we first demonstrate Proposition 5.2.

Proof of estimate (89). Without loss of generality we work with the time interval $[1, T]$. We apply estimate (8) to $2^{ak}\chi_0(2^{-k}t)\chi_{<1}(r/t)\phi$ for $k \ge 0$, where $\chi_0(s)$ is a smooth bump function adapted to $1 \le s \le 2$, and $\chi_{<1}(s)$ is a smooth function $= 1$ for $s \le \frac{1}{2}$ and $= 0$ for $s > \frac{3}{4}$. Using the Hardy estimate (184) this yields

$$\sup_{2^k\le t\le 2^{k+1}}\|\tau_+^a(\partial\phi,\, \tau_x^{-1}\phi)(t)\|_{L_x^2(r<\frac{1}{2})} + \|\tau_+^a(\partial\phi,\, \tau_x^{-1}\phi)\|_{\mathrm{LE}(r<\frac{1}{2})[2^k,2^{k+1}]} \lesssim \|\tau_+^a(\partial_u^b\phi,\, \tau_+^{-1}\phi)\|_{H^1(r<R_0)[2^{k-1},2^{k+2}]} + \|\tau_+^{a-1}(\partial\phi,\, \tau_x^{-1}\phi)\|_{\mathrm{WLE}^{*,0}(r<\frac{3}{4})[2^{k-1},2^{k+2}]} + \|\tau_+^a\tau_0\Box_g\phi\|_{\mathrm{WLE}^{*,0}[2^{k-1},2^{k+2}]}.$$

For a fixed value of k, dyadic summation in r gives the uniform estimate

$$\|\tau_+^{a-1}(\partial\phi,\tau_x^{-1}\phi)\|_{\mathrm{WLE}^{*,0}(r<\frac{3}{4})[2^{k-1},2^{k+2}]} \lesssim \|\tau_+^{a-\frac{1}{2}}(\partial\phi,\tau_x^{-1}\phi)\|_{L^2(r<\frac{3}{4})[2^{k-1},2^{k+2}]} + \|\tau_+^{a-1}\phi\|_{H^2(r<R_0)[2^{k-1},2^{k+2}]}.$$

By splitting into regions $r < \gamma t$ and $r > \gamma t$ we have the following uniform estimate for $0 < \gamma < \frac{1}{2}$:

$$\|\tau_+^{a-\frac{1}{2}}(\partial\phi, \tau_x^{-1}\phi)\|_{L^2(r<\frac{3}{4})[2^{k-1},2^{k+2}]} \lesssim \gamma^{\frac{1}{2}}\|\tau_+^a(\partial\phi, \tau_x^{-1}\phi)\|_{LE(r<\frac{1}{2})[2^{k-1},2^{k+2}]} + \gamma^{-a-1}\|\tau_x^a\tau_+^{-\frac{1}{2}}(\partial\phi, \tau_+^{-1}\phi)\|_{L^2(r<\frac{3}{4})[2^{k-1},2^{k+2}]}.$$

On the one hand with the help of the scaling vector field we have

$$\|\tau_+^a(\partial_u^b\phi, \tau_+^{-1}\phi)\|_{H^1(r<R_0)[2^{k-1},2^{k+2}]} \lesssim \|\tau_+^{a-1}\phi\|_{H_1^1(r<R_0)[2^{k-1},2^{k+2}]}.$$

Thus (89) follows by summing the last four displays in ℓ_t^p and taking $\gamma \ll 1$. □

5C. *Proof of the exterior estimates.* At the level of multipliers there are essentially four cases here: $a = 0$, $0 < a \le \frac{1}{2}$, $\frac{1}{2} \le a < 1$, and $a = 1$. Collectively these are stated in the following lemma.

Lemma 5.4 (core multiplier bounds). *One has the following collection of estimates uniformly for R large enough that $\mathcal{K} \subseteq \{r \le \frac{1}{2}R\}$:*

(I) *Corresponding to the case $a = 0$ one has*

$$\|\tau_-^{-\frac{1}{2}}\tau_x^{-1}\partial_x^b(\tau_x\phi)\|^2_{\ell_u^\infty L^2(r>R)[0,T]} \lesssim \mathcal{E}(0, R). \quad (93)$$

(II) *In the range $0 < a \le \frac{1}{2}$ one has*

$$\sup_{0\le t\le T} \|\tau_+^a(\tau_0^{\frac{1}{2}}\partial_u^b\phi, \tau_x^{-1}\partial_x^b(\tau_x\phi))(t)\|^2_{L_x^2(r>R)} + \|\tau_x^{a-\frac{3}{2}}\partial_x^b(\tau_x\phi)\|^2_{L^2(r>R)[0,T]} + \|\tau_+^{a-\frac{1}{2}}\tau_0^2\partial_u^b\phi\|^2_{L^2(r>R)[0,T]} + \|\tau_-^{-\frac{1}{2}}\tau_x^{a-1}\partial_r^b(\tau_x\phi)\|^2_{\ell_u^\infty L^2(r>R)[0,T]} + \|\tau_+^{a-\frac{1}{2}}\tau_0^{\frac{1}{2}}\partial_u^b\phi\|^2_{\ell_{\underline{u}}^\infty L^2(r>R)[0,T]} \lesssim a^{-1}\mathcal{E}(a, R). \quad (94)$$

Here we have set $\underline{u} = u + 2r$.

(III) *When $\frac{1}{2} \le a < 1$ one has*

$$\sup_{0\le t\le T} \|\tau_+^a(\tau_0^a\partial_u^b\phi, \tau_x^{-1}\partial_x^b(\tau_x\phi))(t)\|^2_{L_x^2(r>R)} + \|\tau_x^{a-\frac{3}{2}}\partial_x^b(\tau_x\phi)\|^2_{L^2(r>R)[0,T]} + \|\tau_+^{a-\frac{1}{2}}\tau_0^2\partial_u^b\phi\|^2_{L^2(r>R)[0,T]} + \|\tau_-^{-\frac{1}{2}}\tau_x^{a-1}\partial_r^b(\tau_x\phi)\|^2_{\ell_u^\infty L^2(r>R)[0,T]} + \|\tau_+^{-\frac{1}{2}}\tau_-^a\partial_u^b\phi\|^2_{\ell_{\underline{u}}^\infty L^2(r>R)[0,T]} \lesssim (1-a)^{-1}\mathcal{E}(a, R). \quad (95)$$

Here again we have set $\underline{u} = u + 2r$.

(IV) *Finally, corresponding to the case $a = 1$ we have*

$$\sup_{0\le t\le T} \|\tau_+(\tau_0\partial_u^b\phi, \tau_x^{-1}\partial_x^b(\tau_x\phi))(t)\|^2_{L_x^2(r>R)} + \|\tau_-^{-\frac{1}{2}}\tau_+\tau_x^{-1}\partial_r^b(\tau_x\phi)\|^2_{\ell_u^\infty L^2(r>R)[0,T]} \lesssim \mathcal{E}(1, R). \quad (96)$$

We'll prove each of these estimates separately. In each case we invoke Proposition 4.6.

Proof of estimate (93). Fix a $j \in \mathbb{N}$ and define the multiplier

$$Y^u = \chi_{<j}(u), \quad Y^r = 0,$$

where $\partial_s\chi_{<j}(s) = -2^{-j}\chi_j(s)$, with $\chi_j(s)\approx 1$ when $\langle s\rangle\approx 2^j$ and vanishing away from this region. It can also be arranged that $\chi_{<j}(s)=0$ for $s\gtrsim 2^j$ by addition of a suitable constant. It is immediate these coefficients satisfy (55), so we may use case (III) of Proposition 4.6. A short calculation then shows

$$\mathcal{A}^u = \mathcal{A}^{ur} = 0, \qquad \mathcal{A}^r = \mathcal{A} = 2^{-j-1}\chi_j(u)$$

for the coefficients in (54). Repeatedly applying estimate (53) for different $j\in\mathbb{N}$, then taking $\sup_j$ of the result concludes the proof. □

Proof of estimate (94). First freeze $j,k\in\mathbb{N}$ and define the multiplier

$$Y^u = a(2^{(2a-1)k}\tau_-\chi_{<k}(\underline{u}) + \tau_+^{2a}\tau_0^4), \quad Y^r = Cr^{2a}(1+\chi_{<j}(u)), \quad \underline{u}=u+2r, \quad \tau_+ = 1+\underline{u},$$

where $\chi_{<k}$, $\chi_{<j}$ is as above and $C>0$ will be chosen shortly. It is easy to check that these coefficients satisfy the conditions on (52) (in $r>R$). Computing the formulas from (54) and choosing $C\gg 1$ we have the following uniform estimates for $0<a\le\frac{1}{2}$ in the region $r>1$:

$$\begin{aligned} &\mathcal{A}^u \gtrsim a(\tau_+^{2a-1}\tau_0\chi_k(\underline{u}) + \tau_+^{2a-1}\tau_0^4), && \mathcal{A}^r \gtrsim \tau_-^{-1}\tau_x^{2a}\chi_j(u) + a\tau_x^{2a-1},\\ &|\mathcal{A}^{ur}| \ll \sqrt{\mathcal{A}^u\mathcal{A}^r}, && \mathcal{A} \gtrsim \tau_x^{2a-1}. \end{aligned}$$

Repeatedly applying estimate (53) for different $j,k\in\mathbb{N}$ and on time intervals $[0,t]\subseteq[0,T]$ and then taking $\sup_{j,k,t}$ concludes the proof. □

Proof of estimate (95). In this case we set

$$Y^u = (1-a)(\tau_-^{2a}\chi_{<k}(\underline{u}) + \tau_+^{2a}\tau_0^4), \quad Y^r = C^2r^{2a}(1+\chi_{<j}(u)) + C(1-a)\tau_+^{2a-1}r.$$

These satisfy the conditions in (52), and the formulas from (54) yield when $C\gg 1$ the following estimates uniformly in $\frac{1}{2}\le a<1$:

$$\begin{aligned} &\mathcal{A}^u \gtrsim (1-a)(\tau_+^{-1}\tau_-^{2a}\chi_k(\underline{u}) + \tau_+^{2a-1}\tau_0^4), && \mathcal{A}^r \gtrsim \tau_-^{-1}\tau_x^{2a}\chi_j(u) + (1-a)\tau_x^{2a-1},\\ &|\mathcal{A}^{ur}| \ll \sqrt{\mathcal{A}^u\mathcal{A}^r}, && \mathcal{A} \gtrsim (1-a)\tau_x^{2a-1}. \end{aligned}$$

These suffice to give (95) through an application of (53). □

Proof of estimate (96). For this estimate we use the multiplier

$$Y^u = (1+\chi_{<j}(u))\tau_-^2, \quad Y^r = (1+\chi_{<j}(u))2(u+r)r.$$

This satisfies the conditions in (52) with $a=1$. The formulas from (54) yield

$$\begin{aligned} \mathcal{A}^u &= 0, \quad \mathcal{A}^r \gtrsim \tau_-^{-1}\tau_+^2\chi_j(u),\\ \mathcal{A}^{ur} &= 0, \quad \mathcal{A} \ge 0. \end{aligned}$$

This suffices to give (96) through an application of case (II) of Proposition 4.6. □

It remains to close the gap between Proposition 5.3 and Lemma 5.4.

Proof of (90). Applying the Hardy estimate (187) with $a = \frac{1}{2}$ to the function $2^{-\frac{1}{2}j}\chi_j(u)\phi$, followed by the Hardy estimate (184) with $a = 0$ for the term on the right-hand side of (187) at $t = 0$, one has

$$\|\tau_-^{-\frac{1}{2}}\tau_x^{-1}\phi\|_{\ell_u^\infty L^2(r>R)[0,T]} \lesssim \|\tau_-^{-\frac{1}{2}}\tau_x^{-1}\partial_x^b(\tau_x\phi)\|_{\ell_u^\infty L^2(r>R)[0,T]} + R^{\frac{1}{2}}\|\phi\|_{\mathrm{WLE}^0_{\mathrm{class}}[0,T]} + \|\partial\phi(0)\|_{L^2_x}, \quad (97)$$

where the extra factor of $R^{\frac{1}{2}}$ results from replacing $\tau_-^{-\frac{1}{2}}$ with $\tau_x^{-\frac{1}{2}}$ in the spacetime region $\frac{1}{2}R < r < R$. Combining this with estimate (93) and using the definition (50) we have (90). □

Proof of (91). Estimate (187) of Appendix C and the definition of $\mathcal{E}(a, R)$ from (49) imply

$$\|\tau_x^{a-1}\phi(T)\|^2_{L^2_x(r>R)} + \|\tau_x^{a-\frac{3}{2}}\phi\|^2_{L^2(r>R)[0,T]} \lesssim_a \|\tau_x^{a-\frac{3}{2}}\partial_r^b(\tau_x\phi)\|^2_{L^2(r>R)[0,T]} + \mathcal{E}(a, R).$$

Adding this to (94) and (95) we have (91). □

Proof of (92). Adding estimate (186) to (96) gives

$$\sup_{0\le t\le T}\|\phi(t)\|_{E^1(r>\max\{R,\frac{1}{2}t\})} + \|\tau_x\tau_+^{-\frac{1}{2}}(\partial\phi, \tau_+^{-1}\phi)\|_{\ell_t^\infty L^2(R<r<\frac{3}{4}t)[0,T]} + \|\tau_+\tau_x^{-1}\partial_r^b(\tau_x\phi)\|_{\mathrm{NLE}(r>R)[0,T]}$$
$$\lesssim \sup_{0\le t\le T} R^{\frac{1}{2}}\|\tau_-^{\frac{1}{2}}(\partial\phi, \tau_x^{-1}\phi)(t)\|_{L^2_x(\frac{1}{2}R<r<R)} + \sqrt{\mathcal{E}(1, R)}.$$

On the other hand a straightforward integration of fixed-time norms in the region $r > \frac{1}{2}t$ also gives

$$\|\phi\|_{\ell_t^\infty S^1(r>\max\{R,\frac{1}{2}t\})[0,T]} \lesssim \sup_{0\le t\le T}\|\phi(t)\|_{E^1(r>\max\{R,\frac{1}{2}t\})}.$$

Combining the previous two equations with the definition of $S^{1,\infty}$ finishes the proof. □

6. Estimates for commutators

In this section we complete the proofs of estimates (14), (15), and (16).

6A. *Splitting into interior and exterior estimates.*

In the interior we use:

Proposition 6.1 (general interior estimates for commutators). *Fix a multiindex $|I| = k \geqslant 1$ and let Γ^I denote a product in $\mathbb{L} = \{S, \Omega_{ij}, \partial_u^b, \partial_i^b - \omega^i\partial_u^b\}$. Then for R sufficiently large and any $R' > R$ one has the following estimates*:

$$\|[\Box_g, \Gamma^I]\phi\|_{\mathrm{WLE}^{*,s}(r<R)[0,T]} \lesssim_R \|\phi\|_{H^{s+3}_{k-1}(r<R)[0,T]}, \quad (98)$$

$$\|[\Box_g, \Gamma^I]\phi\|_{N^a(r<R)[0,T]} \lesssim_{R,R'} \|\tau_+^{a-1}\phi\|_{H^1_{k+1}(r<R')[0,T]} + \|\phi\|_{S^a_{k-1}[0,T]} + \|\Box_g\phi\|_{N^a_k[0,T]}, \quad (99)$$

$$\|[\Box_g, \Gamma^I]\phi\|_{\ell_t^\infty N^1(r<R)[0,T]} \lesssim_{R,R'} \|\phi\|_{\ell_t^\infty H^1_{k+1}(r<R')[0,T]} + \|\phi\|_{S^{1,\infty}_{k-1}[0,T]} + \|\Box_g\phi\|_{\ell_t^\infty N^1_k[0,T]}. \quad (100)$$

The bound (99) *is uniform in* $0 \le a \le 1$.

For the exterior we use:

Proposition 6.2 (general exterior bounds for commutators). *Fix multiindices $|I|=k\geqslant 1$ and $|J|=s\geqslant 0$, and let Γ^I denote a product of vector fields in $\mathbb{L}=\{S,\,\Omega_{ij},\,\partial_u^b,\,\partial_i^b-\omega^i\partial_u^b\}$, while $\widetilde{\Gamma}^J$ denotes a product of vector fields in $\mathbb{L}_0=\{\partial_u^b,\,\partial_i^b-\omega^i\partial_u^b\}$. Let R_0 be as in the definition of the norms (6). Then for $R\geq 2R_0>0$ sufficiently large one has the following*:

(I) *Corresponding to $a=0$ one has*

$$\|[\Box_g,\Gamma^I]\phi\|_{(\mathrm{WLE}^{*,s}+L^1_tH^s_x)(r>R)[0,T]} \lesssim o_R(1)\cdot\|\phi\|_{\mathrm{WLE}^s_k(r>R)[0,T]}+\|\Box_g\phi\|_{(\mathrm{WLE}^{*,s}_{k-1}+L^1_tH^s_{x,k-1})(r>R)[0,T]}. \quad (101)$$

In addition let $X=\chi_{>R}q\partial_u^b$, where $q=q(u)$ has the uniform bounds $|(\tau_-\partial_u^b)^lq|\lesssim_l 1$, where $\chi_{>R}$ is supported in $r>\frac{1}{2}R$ with the usual derivative bounds. Then one has the integral estimate

$$\Big|\int_0^T\!\!\int_{\mathbb{R}^3\setminus\mathcal{K}}[\Box_g,\widetilde{\Gamma}^J]\phi\cdot X\widetilde{\Gamma}^J\phi\,dV_g\Big| \lesssim o_R(1)\cdot\big(\|\phi\|^2_{\mathrm{WLE}^s[0,T]}+\sup_{0\leq t\leq T}\|\partial\phi(t)\|^2_{H^s_x}\big) +\big(\sup_{0\leq t\leq T}\|\partial\phi(t)\|_{H^s_x}+\|\phi\|_{\mathrm{WLE}^s[0,T]}\big)\cdot\|\Box_g\phi\|_{(\mathrm{LE}^{*,s}+L^1_tH^s_x)[0,T]}. \quad (102)$$

(II) *For $0<a<1$ one has uniformly the collection of estimates*

$$\|[\Box_g,\Gamma^I]\phi\|_{N^a(r>R)[0,T]}\lesssim o_R(1)\cdot\|\phi\|_{S^a_k[0,T]}+\|\Box_g\phi\|_{N^a_{k-1}[0,T]}. \quad (103)$$

(III) *Finally, corresponding to the case $a=1$ we have*

$$\|[\Box_g,\Gamma^I]\phi\|_{\ell^\infty_tN^1(r>R)[0,T]}\lesssim o_R(1)\cdot\|\phi\|_{S^{1,\infty}_k[0,T]}+\|\Box_g\phi\|_{\ell^\infty_tN^1_{k-1}[0,T]}. \quad (104)$$

In addition let $X=\chi_{>R}qK_0$, where $K_0=(1+u^2)\partial_u^b+2(u+r)r\partial_r^b$, and where q and $\chi_{>R}$ are as previously stated. Then one has

$$\Big|\int_0^T\!\!\int_{\mathbb{R}^3\setminus\mathcal{K}}[\Box_g,\Gamma^I]\phi\cdot\tau_x^{-1}X(\tau_x\Gamma^I\phi)\,dV_g\Big| \lesssim o_R(1)\cdot\big(\|\phi\|^2_{S^{1,\infty}_k[0,T]}+\sup_{0\leq t\leq T}\|\phi(t)\|^2_{E^1_k}\big)+\|\phi\|_{S^{1,\infty}_k[0,T]}\cdot\|\Box_g\phi\|_{N^{1,1}_{k-1}[0,T]}. \quad (105)$$

We will also need the following initial data bound.

Lemma 6.3 (initial data bound). *Assume that the level sets $t=\mathrm{const}$ are uniformly spacelike. Then one has the uniform estimate for $0\leqslant a\leqslant 1$*

$$\|\tau_x^a\partial\phi(0)\|_{H^s_{x,k}}\lesssim\sum_{|I|\leq k}\sum_{|J|\leq s}\|\tau_x^a(\tau_x\partial_x)^I\partial_x^J\partial\phi(0)\|_{L^2_x}+\|\tau_x^a\Box_g\phi\|_{L^1_tH^s_{x,k}[0,1]}. \quad (106)$$

Before giving the proofs of these individual components we use them to establish Theorem 1.16.

Proof of estimate (14). We need to treat separately the cases $k=0$ and $k>0$.

Case 1: ($k=0$) In light of assumption (7a) and the data bound (106) we need to show

$$\sum_{|J|\leq s}\|(\partial^b)^J\phi\|_{\mathrm{WLE}^0[0,T]}\lesssim\|\partial\phi(0)\|_{H^s_x}+\|\Box_g\phi\|_{(\mathrm{WLE}^{*,s}+L^1_tH^s_x)[0,T]}.$$

It suffices to show this bound for $(\partial^b)^J\phi$ replaced by $\widetilde{\Gamma}^J\phi$, where the product is taken over vector fields in $\mathbb{L}_0=\{\partial_u^b,\partial_i^b-\omega^i\partial_u^b\}$. Applying estimate (86) to $\widetilde{\Gamma}^J\phi$, then using assumption (7a) and (102), we find that for $|J|\le s$

$$\|(\partial^b)^J\phi\|^2_{\mathrm{WLE}^0[0,T]}\lesssim C_R(\|\partial\phi(0)\|^2_{H^s_x}+\|\Box_g\phi\|^2_{(\mathrm{WLE}^{*,s}+L^1_tH^s_x)[0,T]})+o_R(1)\cdot\|\phi\|^2_{\mathrm{WLE}^s[0,T]}.$$

Summing this in $|J|\le s$ and taking R sufficiently large finishes the proof.

Case 2: $(k>0)$ This is a straightforward application of estimate (14) with $k=0$ applied to $\Gamma^I\phi$, where Γ^I denotes a product in $\mathbb{L}=\{S,\Omega_{ij},\partial_u^b,\partial_i^b-\omega^i\partial_u^b\}$. Using estimates (98) and (101) for sufficiently large R to control the commutator, followed by the data bound (106), completes the proof. Note that the value R becomes the definition of $B_0(s,k)$ on (14). □

Proof of estimate (15). This follows by combining estimate (87) with (99), (103), and (106), and then using an induction on k. Note that the value R becomes the definition of $B_a(s,k)$ in (15). □

Proof of estimate (16). This follows from estimates (88), (100), (104), (105), and (106) in a similar pattern as the previous proof. □

6B. *Proof of the initial data bound and the interior estimates.*

Proof of (106). By Remark 1.4 we can assume $u\equiv t-\tau_x$ when $t\in[0,1]$. Therefore without loss of generality we may replace Γ^I with products of vector fields in $\mathbb{L}_{\mathrm{Minkowski}}=\{t\partial_t+r\partial_r,x^i\partial_j-x^j\partial_i,\partial_\alpha\}$. Notice that for products in $\mathbb{L}_{\mathrm{Minkowski}}$ we have the following equivalence at $t=0$:

$$\|\tau_x^a\phi^{(k)}(0)\|_{H^s_x}\approx\sum_{|I|+j\le k}\|\tau_x^a(\tau_x\partial_x)^I\partial_t^j\phi(0)\|_{H^s_x}\tag{107}$$

for a similarity constant that depends only on k. In addition to this, from the uniform spacelike condition of $t=0$ we have the identity

$$\partial_t^2=(g^{00})^{-1}(\Box_g-P(t,x;D)),\tag{108}$$

where $P(t,x;D)$ is second-order with uniformly $(r\partial)^J$ homogeneous coefficients, is at most first-order in ∂_t, and contains no zero-order term.

First we repeatedly use the $\lesssim$ direction of (107) for $\phi(0)$, followed by the substitution (108) for terms containing more than one copy of ∂_t. We then use the $\gtrsim$ direction of (107) for any terms which are produced which contain $\Box_g\phi(0)$. From this sequence of steps we have the bound

$$\|\tau_x^a\partial\phi(0)\|_{H^s_{x,k}}\lesssim\sum_{|I|\le k}\|\tau_x^a(\tau_x\partial_x)^I\partial\phi(0)\|_{H^s_x}+\|\tau_x^a(\Box_g\phi)^{(k-1)}(0)\|_{H^s_x}.$$

After a local $L^1(dt)$ trace estimate, the second term on the right-hand side above matches the right-hand side of (106).

For the first term on the right-hand side on the previous inequality we again use identity (108) to successively get rid of all additional ∂_t-derivatives from the H^s_x norm, followed by another application of the $\gtrsim$ direction of (107) for terms produced containing $\Box_g\phi(0)$. This gives

$$\sum_{|I|\le k}\|\tau_x^a(\tau_x\partial_x)^I\partial\phi(0)\|_{H_x^s}\lesssim\sum_{|I|\le k}\sum_{|J|\le s}\|\tau_x^a(\tau_x\partial_x)^I\partial_x^J\partial\phi(0)\|_{L_x^2}+\|\tau_x^a(\Box_g\phi)^{(k)}(0)\|_{H_x^{s-1}}.$$

The proof now concludes with a final local $L^1(dt)$ trace estimate the second term on the right-hand side above. $\Box$

We now move on to the proof of Proposition 6.1. Note that estimate (98) follows more or less immediately from (79). Therefore we focus attention on the last two estimates.

Proof of estimates (99) *and* (100). We will only focus here in showing (99). The proof of (100) follows from identical calculations by replacing all L_t^2 norms with $\ell_t^\infty L_t^2$ norms.

Step 1: (*inductive setup*) It suffices to show that for multiindex $|I|=k$ and R sufficiently large, for $R'>R$ we have

$$\begin{aligned}\|\tau_+^a[\Box_g,\Gamma^I]\phi\|_{H^1(r<R)[0,T]}\lesssim_{R,R'}&\ \|\tau_+^{a-1}\phi^{(k+1)}\|_{H^1(r<R')[0,T]}+\|\tau_+^a\phi^{(k-1)}\|_{L^2(r<R')[0,T]}\\&+\|\tau_+^a(\Box_g\phi)^{(k)}\|_{H^1(r<R_0)[0,T]}+\|\tau_+^a(\Box_g\phi)^{(k)}\|_{L^2(r<R')[0,T]}.\end{aligned}\tag{109}$$

This boils down to an induction. Indeed, for fixed nontrivial I and integer $s\ge 1$ it suffices to show that under the same assumptions one has

$$\begin{aligned}\|\tau_+^a[\Box_g,\Gamma^I]\phi\|_{H^s(r<R)[0,T]}\lesssim_{R,R'}&\ \|\tau_+^{a-1}\phi^{(|I|+s)}\|_{H^1(r<R')[0,T]}+\|\tau_+^a\phi^{(|I|-1)}\|_{L^2(r<R')[0,T]}\\&+\|\tau_+^a(\Box_g\phi)^{(|I|-1)}\|_{H^{s+1}(r<R_0)}+\|\tau_+^a(\Box_g\phi)^{(|I|-1)}\|_{H^s(r<R')}\\&+\sum_{I'\subsetneq I}\|\tau_+^a[\Box_g,\Gamma^{I'}]\phi\|_{H^{s+1}(r<R')[0,T]}.\end{aligned}\tag{110}$$

By repeatedly applying this last estimate for $|I|+s=k+1$, where $s=1,\dots,k$, and a sequence $R<R_s'<R_{s+1}'<R'$ we have (109).

Step 2: (*elliptic estimate in* $\frac{1}{2}R_0<r<R$) To prove (110) start with (79), which implies that

$$\|\tau_+^a[\Box_g,\Gamma^I]\phi\|_{H^s(r<R)[0,T]}\lesssim\sum_{I'\subsetneq I}\|\tau_+^a\Gamma^{I'}\phi\|_{H^{s+2}(r<R)[0,T]}.\tag{111}$$

Without loss of generality we may assume R_0 is chosen large enough that in the region $r>\frac{1}{2}R_0$ the operator $P(x,D)=\Box_g-Q_0(x,D)$ is uniformly elliptic, where Q_0 contains all terms with a $g^{0\alpha}$-factor. Standard elliptic estimates then give

$$\|\tau_+^a\Gamma^{I'}\phi\|_{H^{s+2}(\frac{1}{2}R_0<r<R)[0,T]}\lesssim_{R,R'}\|\tau_+^aP(x,D)\Gamma^{I'}\phi\|_{H^s(r<R')[0,T]}+\|\tau_+^a\Gamma^{I'}\phi\|_{L^2(r<R')[0,T]}.$$

On the other hand due to the fact that any term in $Q_0(x,D)$ contains at least one time derivative, and using the metric conditions (3), we have for $I'\subsetneq I$

$$\|\tau_+^aQ_0(x,D)\Gamma^{I'}\phi\|_{H^s(r<R')[0,T]}\lesssim_{R'}\|\tau_+^{a-1}\phi^{(|I|+s)}\|_{H^1(r<R')[0,T]}.$$

Thus, combining the last two inequalities we have

$$\sum_{I'\subsetneq I}\|\tau_+^a\Gamma^{I'}\phi\|_{H^{s+2}(\frac{1}{2}R_0<r<R)[0,T]}$$
$$\lesssim_{R,R'}\|\tau_+^{a-1}\phi^{(|I|+s)}\|_{H^1(r<R')[0,T]}+\|\tau_+^a\phi^{(|I|-1)}\|_{L^2(r<R')[0,T]}$$
$$+\|\tau_+^a(\Box_g\phi)^{(|I|-1)}\|_{H^s(r<R')}+\sum_{I'\subsetneq I}\|\tau_+^a[\Box_g,\Gamma^{I'}]\phi\|_{H^s(r<R')[0,T]}. \qquad (112)$$

Step 3: (LE *estimate in* $r<\frac{1}{2}R_0$) It remains to bound the portion of the right-hand side of (111) which is contained in $r<\frac{1}{2}R_0$. Note that we only need to focus on the region $t>1$ as the right-hand side of (111) restricted to the time slab [0, 1] is automatically bounded by $\|\tau_+^{a-1}\phi^{(|I|+s)}\|_{H^1(r<R')[0,T]}$.

We begin by applying the stationary LE bound (7b) at regularity $s+1$ to $\chi_{t>1}\chi_{r<\frac{1}{2}R_0}\tau_+^a\Gamma^{I'}\phi$, where $\chi_{r<\frac{1}{2}R_0}=1$ on $r<\frac{1}{2}R_0$ and $\chi_{r<\frac{1}{2}R_0}=0$ on $r>R_0$, and with similar properties for $\chi_{t>1}$. This results in the estimate

$$\sum_{I'\subsetneq I}\|\tau_+^a\Gamma^{I'}\phi\|_{H^{s+2}(r<\frac{1}{2}R_0)[1,T]}\lesssim_{R,R'}T_1+T_2+T_3+T_4,$$

where

$$T_1=\sum_{I'\subsetneq I}\|\Gamma^{I'}\phi\|_{H^{s+2}(r<R_0)[0,2]},\qquad T_2=\sum_{I'\subsetneq I}\|(\partial_t\tau_+^a\Gamma^{I'}\phi,\,\tau_+^a\Gamma^{I'}\phi)\|_{H^{s+1}\times L^2(r<R_0)[0,T]},$$
$$T_3=\sum_{I'\subsetneq I}\|[\Box_g,\,\chi_{r<\frac{1}{2}R_0}\tau_+^a]\Gamma^{I'}\phi\|_{H^{s+1}[0,T]},\qquad T_4=\sum_{I'\subsetneq I}\|\tau_+^a\Box_g\Gamma^{I'}\phi\|_{H^{s+1}(r<R_0)[0,T]}.$$

The term T_1 results from differentiation of $\chi_{t>1}$. The terms T_1 and T_4 are already compatible with the right-hand side of (110). It is also easy to see that

$$T_2\lesssim\|\tau_+^{a-1}\phi^{(|I|+s)}\|_{H^1(r<R_0)[0,T]}+\|\tau_+^a\phi^{(|I|-1)}\|_{L^2(r<R_0)[0,T]}.$$

It remains to bound T_3. Expanding the commutator gives

$$T_3\lesssim\|\tau_+^{a-1}\phi^{(|I|+s)}\|_{H^1(r<R_0)[0,T]}+\sum_{I'\subsetneq I}\|\tau_+^a\Gamma^{I'}\phi\|_{H^{s+2}(\frac{1}{2}R_0<r<R_0)[0,T]}.$$

The first term above is of the correct form. The second term is handled by estimate (112). □

6C. *Proof of the exterior estimates.* We first list some general calculations which take care of a large portion of the desired estimates.

Lemma 6.4. *For integers $s,k\geq 0$ define*

$$\Phi^{(s,k)}=\sum_{|J|\leq s}\tau_x^{-\frac{1}{2}}|(\tau_0\partial(\partial^b)^J\phi^{(k)},\,\partial_x^b(\partial^b)^J\phi^{(k)},\,\tau_x^{-1}(\partial^b)^J\phi^{(k)})|,\quad \not{\Phi}^{(k)}=\tau_x^{-\frac{1}{2}}\tau_0^{-\frac{1}{2}},|\tau_x^{-1}\partial_r^b(\tau_x\phi^{(k)})|.$$

We also use the shorthand $\Phi^{(0,k)}=\Phi^{(k)}$. With this notation we have

$$\|\Phi^{(s,k)}\|_{\ell_r^\infty L^2(r>R)[0,T]}+\|\tau_0^{-\frac{1}{2}}\Phi^{(s,k)}\|_{\ell_u^\infty\ell_r^\infty L^2(\frac{1}{2}t<r<2t)\cap(r>R)[0,T]}\lesssim\|\phi\|_{\mathrm{LE}_k^s(r>R)[0,T]}, \qquad (113)$$
$$\|\tau_+^a\Phi^{(k)}\|_{\ell_r^\infty L^2[0,T]}+\|\tau_+^a\not{\Phi}^{(k)}\|_{\ell_u^\infty L^2(\frac{1}{2}t<r<2t)[0,T]}\lesssim\|\phi\|_{S_k^a[0,T]}, \qquad (114)$$

$$\|\tau_+ \Phi^{(k)}\|_{\ell_t^\infty \ell_r^\infty L^2[0,T]} + \|\tau_+ \not{\Phi}^{(k)}\|_{\ell_u^\infty L^2(\frac{1}{2}t<r<2t)[0,T]} \lesssim \|\phi\|_{S_k^{1,\infty}[0,T]}, \tag{115}$$

$$\|\tau_x^{\frac{1}{2}} \tau_+ \Phi^{(k)}\|_{L_t^\infty L_x^2[0,T]} \lesssim \sup_{0\le t\le T} \|\phi(t)\|_{E_k^1}. \tag{116}$$

In addition if Γ^I *is a product of vector fields in* $\mathbb{L} = \{S, \Omega_{ij}, \partial_u^b, \partial_i^b - \omega^i \partial_u^b\}$, *and* $\widetilde{\Gamma}^J$ *a product of vector fields in* $\mathbb{L}_0 = \{\partial_u^b, \partial_i^b - \omega^i \partial_u^b\}$, *where* $|I| = k$ *and* $|J| = s$, *then we have the pointwise estimates*

$$|\widetilde{\Gamma}^J [\Box_g, \Gamma^I]\phi| \lesssim q \cdot \tau_0^{-\frac{1}{2}} \tau_x^{-\frac{1}{2}} \Phi^{(s,k)} + \sum_{|J|\le s} |\partial^J (\Box_g \phi)^{(k-1)}|, \quad \text{where } q \in \mathcal{Z}^0. \tag{117}$$

Proof of Lemma 6.4. The proof of (113)–(116) is a straightforward application of the definition of the various spaces involved. On the other hand (117) is immediate from (85). □

Note that a direct combination of (117) with either (113), (114), or (115) shows (101), (103), or (104) respectively. Thus, the remainder of the subsection is devoted to showing the integral estimates (102) and (105). In both cases the key step is to integrate by parts the bilinear operator resulting from the commutator with $\Box_g$. The relevant result here is:

Lemma 6.5. *Let* R *be sufficiently large so that* $\mathcal{K} \subseteq \{|x| < \frac{1}{2}R\}$, *and let* $\chi_{>R}$ *be a cutoff supported in* $r > \frac{1}{2}R$, *constant for* $r > R$, *with the usual derivative bounds. Then one has the following integral estimates*:

(I) *Let* $q = q(u)$ *be a smooth function such that* $|(\tau_- \partial_u^b)^l q| \lesssim_l 1$. *Furthermore let* $|J| = s \ge 1$ *and* $|J'| \le s-1$ *and let* $\widetilde{\Gamma}^J$ *and* $\widetilde{\Gamma}^{J'}$ *denote products of vector fields in* $\mathbb{L}_0 = \{\partial_u^b, \partial_i^b - \omega^i \partial_u^b\}$. *Then if* $\mathcal{R}$ *obeys the conditions* (43) *with* $a = c = -1$ *and* $b = 1$ *one has*

$$\Big| \int_0^T \int_{\mathbb{R}^3} \chi_{>R} \nabla_\alpha \mathcal{R}^{\alpha\beta} \nabla_\beta \widetilde{\Gamma}^{J'} \phi \cdot q \partial_u^b \widetilde{\Gamma}^J \phi \, dV_g \Big| \lesssim o_R(1) \cdot \big(\|\phi\|^2_{LE^s(r>\frac{1}{2}R)[0,T]} + \sup_{0\le t\le T} \|\partial\phi(t)\|^2_{H_x^s} \big). \tag{118}$$

(II) *Alternatively, suppose that* $q = q(u)$ *satisfies* $|(\tau_- \partial_u^b)^l q| \lesssim_l \tau_-$. *Let* Γ^I *and* $\Gamma^{I'}$ *be products of vector fields in* $\mathbb{L} = \{S, \Omega_{ij}, \partial_u^b, \partial_i^b - \omega^i \partial_u^b\}$ *for multiindices* $|I| = k \ge 1$ *and* $|I'| \le k-1$. *Then if* $\mathcal{R}$ *obeys the conditions* (40) *one has*

$$\Big| \int_0^T \int_{\mathbb{R}^3} \chi_{>R} \nabla_\alpha \mathcal{R}^{\alpha\beta} \nabla_\beta \Gamma^{I'} \phi \cdot q S \Gamma^I \phi \, dV_g \Big| \lesssim o_R(1) \cdot \big(\|\phi\|^2_{S_k^{1,\infty}[0,T]} + \sup_{0\le t\le T} \|\phi(t)\|^2_{E_k^1} \big). \tag{119}$$

(III) *Finally, with the same setup of estimate* (119) *let* $\mathcal{S}$ *obey the conditions* (46). *Then one has*

$$\Big| \int_0^T \int_{\mathbb{R}^3} \chi_{>R} \mathcal{S}^\alpha \partial_\alpha \Gamma^{I'} \phi \cdot q S \Gamma^I \phi \, dV_g \Big| \lesssim o_R(1) \cdot \big(\|\phi\|^2_{S_k^{1,\infty}[0,T]} + \sup_{0\le t\le T} \|\phi(t)\|^2_{E_k^1} \big). \tag{120}$$

Proof of estimate (118). For functions F and G, a vector field X, and quadratic operator $\nabla_\alpha \mathcal{R}^{\alpha\beta} \nabla_\beta$, we have the pointwise identity

$$\begin{aligned} \nabla_\alpha \mathcal{R}^{\alpha\beta} \nabla_\beta F \cdot XG &= \nabla_\alpha [\mathcal{R}^{\alpha\beta} \nabla_\beta F \cdot XG - X^\alpha \mathcal{R}^{\beta\gamma} \partial_\beta F \cdot \partial_\gamma G] \\ &\qquad + (\nabla_\gamma X^\gamma) \mathcal{R}^{\alpha\beta} \partial_\alpha F \cdot \partial_\beta G + \mathcal{R}_X^{\alpha\beta} \partial_\alpha F \cdot \partial_\beta G + \mathcal{R}^{\alpha\beta} \partial_\alpha X F \cdot \partial_\beta G, \\ &= \nabla_\alpha T_1^\alpha + T_2 + T_3 + T_4, \end{aligned} \tag{121}$$

where $\mathcal{R}_X = \mathcal{L}_X \mathcal{R}$. Setting $F = \widetilde{\Gamma}^{J'}\phi$, $G = \widetilde{\Gamma}^J \phi$, and $X = q\partial_u^b$ in the above formula we estimate the integral of each term separately.

<u>Case 1</u>: (*the T_1-term*) Using the divergence theorem gives

$$\Big|\int_0^T\int_{\mathbb{R}^3} \chi_{>R}\nabla_\alpha T_1^\alpha \, dV_g\Big| \lesssim \sup_{0\le t\le T}\int_{|x|>\frac{1}{2}R} |\nabla_\alpha t|\cdot|T_1^\alpha|\, dx + R^{-1}\Big|\int_0^T\int_{\mathbb{R}^3} \chi_R' T_1^r \, dV_g\Big|, \tag{122}$$

where $|\chi_R'| \lesssim 1$ and is supported where $\frac{1}{2}R < r < R$. Based on the fact that all components of X are uniformly bounded, and all components of $\mathcal{R}$ are $o_R(1)$ (in either Bondi or (t,x)-coordinates), we directly have the pointwise estimate

$$|\nabla_\alpha t|\cdot|T_1^\alpha| \lesssim \sup_\alpha |T_1^\alpha| \lesssim o_R(1)\sum_{|J''|\le s}|\partial\partial^{J''}\phi|^2,$$

which in turn produces a bound for the right-hand side of (122) in terms of

$$o_R(1)\cdot(\|\phi\|^2_{LE^s(r>\frac{1}{2}R)[0,T]} + \sup_{0\le t\le T}\|\partial\phi(t)\|^2_{H^s_x}).$$

<u>Case 2</u>: (*the T_2-term*) Notice that estimate (39) gives $|\nabla_\gamma X^\gamma| \lesssim 1$ for $X = q\partial_u^b$. On the other hand for $\mathcal{R}$ satisfying conditions (43) with $a = c = -1$ and $b = 1$ we have the pointwise estimate

$$|\mathcal{R}^{\alpha\beta}\partial_\alpha\widetilde{\Gamma}^{J'}\phi\partial_\beta\widetilde{\Gamma}^J\phi| \lesssim p\cdot\sum_{|J''|\le s}\tau_x^{-1}(|\partial\partial^{J''}\phi|^2 + \tau_0^{-1}|\partial_x^b\partial^{J''}\phi|^2), \quad \text{where } p\in\mathcal{Z}^0. \tag{123}$$

This suffices to give

$$\Big|\int_0^T\int_{\mathbb{R}^3} \chi_{>R}T_2\, dV_g\Big| \lesssim o_R(1)\cdot\|\phi\|^2_{LE^s(r>\frac{1}{2}R)[0,T]}.$$

<u>Case 3</u>: (*the T_3-term*) This is similar to the previous step. Notice that $X = q\partial_u^b$ obeys the symbol bounds (41) with $a = b = 0$ and $c = -1$, and satisfies all conditions in (42). Therefore, thanks to case (I) of Lemma 4.3 we have that $\mathcal{R}_X$ satisfies the bounds in (43) with $a = c = -1$ and $b = 1$. This is enough to show (123) holds for $\mathcal{R}$ replaced by $\mathcal{R}_X$.

<u>Case 4</u>: (*the T_4-term*) Modulo another bound similar to (123) it suffices to show

$$|(\partial_\alpha q)\mathcal{R}^{\alpha\beta}\partial_u^b\widetilde{\Gamma}^{J'}\phi\partial_\beta\widetilde{\Gamma}^J\phi| \lesssim \text{RHS (123)}.$$

This follows from direct inspection of various terms involved. □

Proof of estimate (119). We again use the identity (121) and estimate each term separately. As a preliminary note that with the assumptions of (119) and notation of Lemma 6.4 one has the pointwise estimate

$$\tau_-|\mathcal{R}^{\alpha\beta}\partial_\alpha\Gamma^{I'}\phi\cdot\partial_\beta\Gamma^I\phi| \lesssim p\cdot\tau_x\tau_+|\Phi^{(k)}|^2, \quad \text{where } p\in\mathcal{Z}^0. \tag{124}$$

A similar bound holds if we replace $\Gamma^{I'}\phi$ by $S\Gamma^{I'}\phi$.

Likewise, when $X = qS$ with $q = q(u)$ and bounds $|(\tau_-\partial_u^b)^l q| \lesssim_l \tau_-$ we have condition (41) with $a = b = 0$ and $c = 1$, and also the conditions in (42) save for the second identity. Therefore by (III) of Lemma 4.3 the tensor $\mathcal{R}_X = \mathcal{L}_X\mathcal{R}$ satisfies (in Bondi coordinates)

$$\mathcal{R}_X^{ij} \in \tau_+ \cdot \mathcal{Z}^0, \quad \mathcal{R}_X^{ui} \in \tau_+ \cdot \mathcal{Z}^1, \quad \mathcal{R}_X^{uu} \in \tau_+ \cdot \mathcal{Z}^2.$$

In particular we have the pointwise estimate

$$|\mathcal{R}_X^{\alpha\beta}\partial_\beta\Gamma^{I'}\phi \cdot \partial_\beta\Gamma^I\phi| \lesssim p \cdot \tau_x\tau_+|\Phi^{(k)}|^2, \quad \text{where } p \in \mathcal{Z}^0. \tag{125}$$

Case 1: (*the T_1-term*) Here we again use (122). For the first term on the right-hand side of (122) estimate (116) shows it suffices to prove

$$\sup_\alpha |T_1^\alpha| \lesssim o_R(1) \cdot \tau_x\tau_+^2|\Phi^{(k)}|^2. \tag{126}$$

For the first term in T_1^α we use

$$\sup_\alpha \tau_-|\mathcal{R}^{\alpha\beta}\partial_\beta\Gamma^{I'}\phi \cdot S\Gamma^I\phi| \lesssim o_R(1)\tau_x\tau_+^2|\Phi^{(k)}|^2,$$

which follows from $|\mathcal{R}^{\alpha\beta}| \lesssim 1$ and expanding S into ∂_u^b- and ∂_x^b-derivatives. For the second term in T_1^α we have (126) thanks to (124) and $|X^\alpha| \lesssim \tau_+\tau_-$.

For the second term on the right-hand side of (122) the pointwise bound (126) is not sufficient to recover the ℓ_t^∞ structure needed on the right-hand side of (119). However, similar calculations to those above show $T_1^r = T_{11}^r + T_{12}^r$, where

$$R^{-1}\chi_R' T_{11}^r = \chi_R'\mathcal{S}^\alpha\partial_\alpha\Gamma^{I'}\phi \cdot qS\Gamma^I\phi, \quad R^{-1}|\chi_R' T_{12}^r| \lesssim o_R(1) \cdot \chi_R(r)\tau_x\tau_+|\Phi^{(k)}|^2.$$

Here χ_R is a cutoff on a dyadic region $\approx R$, and $\mathcal{S}^\alpha = R^{-1}\chi_R(r)\mathcal{R}^{r\alpha}$ is a smooth vector field satisfying the assumptions of estimate (120) (which will be proved independently). Therefore we only need bound the second term in the display above. Using (116) in the region $t \leq R$, and (115) in the region $t > R$, we have

$$R^{-1}\int_0^T\int_{\mathbb{R}^3} |\chi_R' T_{12}^r|\,dx\,dt \lesssim o_R(1) \cdot \big(\|\phi\|^2_{S_k^{1,\infty}[0,T]} + \sup_{0\leq t\leq T}\|\phi(t)\|^2_{E_k^1}\big).$$

Case 2: (*the T_2-term*) For the remaining three terms in (121) we will set things up so as to appeal to Lemma 4.10. For $i = 2, 3, 4$ we will show

$$\int_0^T\int_{\mathbb{R}^3} \chi_{>R}|T_i|\,dV_g \lesssim \|\sqrt{\tau_x/\tau_+}p \cdot \tau_+\Phi^{(k)}\|^2_{L^2[0,T](r>R)}, \quad \text{where } p^2 \in \mathcal{Z}^0, \tag{127}$$

which by a combination of estimate (71) and estimates (115) and (116) produces

$$\int_0^T\int_{\mathbb{R}^3} \chi_{>R}|T_i|\,dV_g \lesssim o_R(1) \cdot \big(\|\phi\|^2_{S_k^{1,\infty}[0,T]} + \sup_{0\leq t\leq T}\|\phi(t)\|^2_{E_k^1}\big).$$

For the specific case of the T_2-term note that (39) shows $|\nabla_\gamma X^\gamma| \lesssim \tau_-$. Then (124) immediately gives (127).

Case 3: (*the T_3-term*) In this case (127) follows at once from (125).

Case 4: (*the T_4-term*) Modulo an application of (124) with $\Gamma^{I'}\phi$ replaced by $S\Gamma^{I'}\phi$, to produce (127) for this case it suffices to prove the pointwise estimate

$$|(\partial_\alpha q)\mathcal{R}^{\alpha\beta}S\Gamma^{I'}\phi\cdot\partial_\beta\Gamma^I\phi|\lesssim p\cdot\tau_x\tau_+|\Phi^{(k)}|^2,\quad \text{where } p\in\mathcal{Z}^0,$$

which follows from expanding S to get

$$|\mathcal{R}^{u\beta}S\Gamma^{I'}\phi\cdot\partial_\beta\Gamma^I\phi|\lesssim p\cdot\tau_+(\tau_0^2|\partial\phi^{(k)}|^2+|\partial_x^b\phi^{(k)}|^2),\quad \text{where } p\in\mathcal{Z}^0.$$

This completes the proof of (119). □

Proof of (120). For functions F and G, and vector fields $\mathcal{S}$ and X, we have the pointwise identity

$$\mathcal{S}^\alpha\partial_\alpha F\cdot XG=\nabla_\alpha(X^\alpha\mathcal{S}^\beta\partial_\beta F\cdot G)-(\nabla_\alpha X^\alpha)\mathcal{S}^\beta\partial_\beta F\cdot G-\mathcal{S}_X^\beta\partial_\beta F\cdot G-\mathcal{S}^\beta\partial_\beta XF\cdot G. \tag{128}$$

Here $\mathcal{S}_X=\mathcal{L}_X\mathcal{S}=[X,\mathcal{S}]$. We again need to estimate each term separately.

As a general first step note if $\mathcal{S}$ satisfies (46), $|I'|\le k-1$, and $|I|\le k$ then

$$\tau_-|\mathcal{S}^\alpha\partial_\alpha\Gamma^{I'}\phi\cdot\Gamma^I\phi|\lesssim p\cdot\tau_x\tau_+|\Phi^{(k)}|^2,\quad \text{where } p\in\mathcal{Z}^0. \tag{129}$$

A similar bound holds if we replace $\Gamma^{I'}\phi$ by $S\Gamma^{I'}\phi$.

Likewise, when $X=qS$, where $q=q(u)$ with bounds $|(\tau_-\partial_u^b)^l q|\lesssim_l\tau_-$, we have from part (IV) of Lemma 4.3 the estimates

$$\mathcal{S}_X^i\in\tau_x^{-1}\tau_+\cdot\mathcal{Z}^{\frac12},\quad \mathcal{S}_X^u\in\tau_x^{-1}\tau_+\cdot\mathcal{Z}^{\frac32}.$$

This gives the pointwise estimate

$$|\mathcal{S}_X^\alpha\partial_\alpha\Gamma^{I'}\phi\cdot\Gamma^I\phi|\lesssim p\cdot\tau_x\tau_+|\Phi^{(k)}|^2,\quad \text{where } p\in\mathcal{Z}^0. \tag{130}$$

Finally,

$$|(\partial_\alpha q)\mathcal{S}^\alpha S\Gamma^{I'}\phi\cdot\Gamma^I\phi|\lesssim p\cdot\tau_x\tau_+|\Phi^{(k)}|^2,\quad \text{where } p\in\mathcal{Z}^0, \tag{131}$$

which follows from $(\partial_\alpha q)\mathcal{S}^\alpha\in\tau_x^{-2}\tau_+\cdot\mathcal{Z}^{\frac12}$ and $\tau_x^{-2}\tau_+|S\Gamma^{I'}\phi\cdot\Gamma^I\phi|\lesssim\tau_x^{-2}\tau_+|\phi^{(k)}|^2$. With (128)–(131) in hand, the remainder of the proof of (120) is essentially identical to the proof of (119) above. □

Proof of estimate (102). Using part (II) of Corollary 4.12 we may write

$$\text{LHS (102)}\lesssim\Big|\int_0^T\!\!\int_{\mathbb{R}^3}\chi_{>R}T_1\,dV_g\Big|+\int_0^T\!\!\int_{\mathbb{R}^3}\chi_{>R}|T_2|\,dx\,dt$$
$$+\Big(\sup_{0\le t\le T}\|\partial\phi(t)\|_{H_x^s}+\|\phi\|_{LE^s(r>R)[0,T]}\Big)\cdot\|\Box_g\phi\|_{(LE^{*,s}+L_t^1H_x^s)(r>R)[0,T]},$$

where

$$T_1=\sum_{J'\subsetneq J}\nabla_\alpha\mathcal{R}_{J'}^{\alpha\beta}\nabla_\beta\widetilde{\Gamma}^{J'}\phi\cdot q\partial_u^b\widetilde{\Gamma}^J\phi,\quad T_2=\sum_{J'\subsetneq J}\mathcal{S}_{J'}^\alpha\partial_\alpha\widetilde{\Gamma}^{J'}\phi\cdot q\partial_u^b\widetilde{\Gamma}^J\phi,$$

and where $\mathcal{R}_{J'}$ and $\mathcal{S}_{J'}$ satisfy estimates (43) and (47) respectively with $a=c=-1$ and $b=1$. We are assuming the weight $q=q(u)$ satisfies $|(\tau_-\partial_u^b)^l q|\lesssim_l 1$. The term T_1 is therefore handled by estimate

(118). On the other hand the conditions on $\mathcal{S}_{I'}$ and inspection give the pointwise bound

$$|T_2| \lesssim p \cdot \sum_{|J''|\leq s} \tau_x^{-1}(|\partial\partial^{J''}\phi|^2 + \tau_0^{-1}|\partial_x^b(\partial^b)^{J''}\phi|^2), \quad \text{where } p \in \mathcal{Z}^0.$$

This suffices to produce $\int_0^T\int_{\mathbb{R}^3} \chi_{>R}|T_2| \lesssim o_R(1)\cdot\|\phi\|^2_{LE^s(r>R)[0,T]}$. □

Proof of estimate (105). Note that the definition of X and the notation of Lemma 6.4 give the pointwise bound

$$|\tau_x^{-1}X(\tau_x\Gamma^I\phi)| \lesssim \tau_x^{\frac{1}{2}}\tau_+^2\tau_0^{\frac{1}{2}}\cdot(\Phi^{(k)} + \slashed{\Phi}^{(k)}).$$

Next, we take the decomposition $\tau_x^{-1}X\tau_x = uS + X'$, where $X' = q(\partial_u^b + (u+2r)r\tau_x^{-1}\partial_r^b\tau_x + ur^2\tau_x^{-2})$. This leads to the following improvement of the previous inequality

$$|X'(\Gamma^I\phi)| \lesssim \tau_x^{\frac{3}{2}}\tau_+\tau_0^{\frac{1}{2}}\cdot(\Phi^{(k)} + \slashed{\Phi}^{(k)}).$$

Therefore combining (79), (117), the previous two inequalities, and (115) gives

$$\text{LHS (102)} \lesssim \sum_{i=1,2}\left|\int_0^T\int_{\mathbb{R}^3}\chi_{>R}T_i\,dV_g\right| + \|\sqrt{\tau_x/\tau_+}p\cdot\tau_+(\Phi^{(k)}+\slashed{\Phi}^{(k)})\|^2_{L^2[0,T](r>R)} + \|\phi\|_{S_k^{1,\infty}[0,T]}\cdot\|\Box_g\phi\|_{N_{k-1}^{1,1}[0,T]},$$

where $p^2 \in \mathcal{Z}^0$, where

$$T_1 = \sum_{I'\subsetneq I}\nabla_\alpha\mathcal{R}_{I'}^{\alpha\beta}\nabla_\beta\Gamma^{I'}\phi\cdot\tilde{q}S\Gamma^I\phi, \quad T_2 = \sum_{I'\subsetneq I}\mathcal{S}_{I'}^\alpha\partial_\alpha\Gamma^{I'}\phi\cdot\tilde{q}S\Gamma^I\phi, \tag{132}$$

and where $\tilde{q} = uq(u)$ satisfies the assumptions of Lemma 6.5. The proof of (105) is concluded by an application of estimate (119) to handle the contribution of T_1, estimate (120) to handle the contribution of T_2, and Lemma 4.10 followed by (115) and (116), which together show

$$\|\sqrt{\tau_x/\tau_+}p\cdot\tau_+(\Phi^{(k)}+\slashed{\Phi}^{(k)})\|^2_{L^2[0,T](r>R)} \lesssim o_R(1)\cdot\big(\|\phi\|^2_{S_k^{1,\infty}[0,T]} + \sup_{0\leq t\leq T}\|\phi(t)\|^2_{E_k^1}\big).$$ □

7. L^∞ estimates

The purpose of this section is to prove Theorem 1.18. In fact we will prove the slightly stronger bound:

Proposition 7.1 (fixed-time global Sobolev inequality). *Let $R_1 \geq 1$ be large enough so that $\mathcal{K}\subseteq\{r < R_1\}$. Then there exists $R \geq R_1$ sufficiently large such that given any $k \geq 1$ one has the following fixed-time estimate uniform in $t \geq 0$:*

$$\sum_{i+|J|\leq k}\|\tau_+^{\frac{3}{2}}\tau_0^{\frac{1}{2}}(\tau_-\partial_u^b)^i(\tau_x\partial_x^b)^J\phi(t)\|_{L_x^\infty} \lesssim \|\tau_+^{\frac{3}{2}}\phi^{(k)}(t)\|_{H_x^2(r<R)} + \|\phi(t)\|_{E_{k+1}^1} + \sum_{i+|J|\leq k}\|\tau_+^{\frac{3}{2}}\tau_x^{\frac{1}{2}}\tau_0(\tau_-\partial_u^b)^i(\tau_x\partial_x^b)^J\Box_g\phi(t)\|_{\ell_r^1L_x^2(r<t)\cap L_x^2}. \tag{133}$$

In the estimate above the implicit constant depends on R.

We first give a quick demonstration of how Proposition 7.1 produces Theorem 1.18.

Proof that (133) implies (17). By an application of $T \le 1$ and (188a) (when $T > 1$), and using the identity $\tau_- \partial_u^b = S + q\partial$ for some smooth q with $|\partial^J q| \lesssim \tau_x$, we have uniformly for $0 \le t \le T$ the bound

$$\|\tau_+^{\frac{3}{2}} \phi^{(k)}(t)\|_{H_x^2(r<R)} \lesssim_R \sup_{0\le t\le T} \|\phi(t)\|_{E^1_{k+1}} + \|\phi\|_{S^{1,\infty}_{k+2}[0,T]}.$$

Likewise, by an appropriate combination of (188a)–(188c) we have

$$\sup_{0\le t\le T} \sum_{i+|J|\le k} \|\tau_+^{\frac{3}{2}} \tau_x^{\frac{1}{2}} \tau_0 (\tau_- \partial_u^b)^i (\tau_x \partial_x^b)^J \Box_g \phi(t)\|_{\ell^1_r L^2_x(r<t)\cap L^2_x} \lesssim \sum_{|J|\le k} \|\tau_x^2 (\tau_x \partial)^J \Box_g \phi(0)\|_{L^2_x} + \sum_{i+|J|\le k+1} \|(\tau_- \partial_u^b)^i (\tau_x \partial_x^b)^J \Box_g \phi\|_{N^{1,1}[0,T]}. \qquad \square$$

7A. *Reduction of Proposition 7.1.* The proof of (133) will rest on previous material and the following three lemmas. In each of these t is a fixed parameter and $\phi = \phi(t)$ only depends on x. We also assume $R > 0$ is chosen as in Proposition 7.1.

Lemma 7.2 (basic L^∞ estimates). *Let ϕ be a test function supported in $r < \frac{3}{4}\langle t\rangle$. Then one has*

$$\|(\tau_x \partial_x \phi, \phi^{(1)})\|_{L^\infty_x} \lesssim \|\tau_x^{\frac{1}{2}} (\partial_x^2 \phi^{(1)}, \tau_x^{-1} \partial_x \phi^{(1)}, \tau_x^{-2} \phi^{(1)})\|_{\ell^\infty_r L^2_x}. \tag{134}$$

On the other hand, without any support conditions imposed on ϕ we have

$$\|\tau_x^{\frac{3}{2}} \tau_0^{\frac{1}{2}} \phi\|_{L^\infty_x} \lesssim \sum_{l+|J|\leqslant 2} \|(\tau_x \tau_0 \partial_u^b)^l (\tau_x \partial_x^b)^J \phi\|_{L^2_x}. \tag{135}$$

Lemma 7.3 (global elliptic estimate). *Let ϕ be a function supported in $r < \frac{3}{4}\langle t\rangle$. Then for R sufficiently large one has the fixed-time estimate*

$$\|\tau_x^{\frac{1}{2}} (\partial_x^2 \phi, \tau_x^{-1} \partial_x \phi, \tau_x^{-2} \phi)\|_{\ell^\infty_r L^2_x} \lesssim \|\phi\|_{H^2_x(r<R)} + \|\tau_x^{\frac{1}{2}} (\partial \partial_u^b \phi, \tau_x^{-1} \partial_u^b \phi, \tau_+^{-1} \partial \phi)\|_{\ell^1_r L^2_x} + \|\tau_x^{\frac{1}{2}} \Box_g \phi\|_{\ell^1_r L^2_x}, \tag{136}$$

where the implicit constant depends on R.

Lemma 7.4 (fixed-time commutator estimate). *Let ϕ be a function which is supported in the region $r < \frac{3}{4}\langle t\rangle$, and let Γ^I denote a product of vector fields in $\mathbb{L} = \{S, \Omega_{ij}, \partial_u^b, \partial_i^b - \omega^i \partial_u^b\}$ with $|I| = k \geqslant 1$. Then for R as in Lemma 7.3 one has the fixed-time estimate*

$$\|\tau_x^{\frac{1}{2}} [\Box_g, \Gamma^I] \phi\|_{\ell^1_r L^2_x} \lesssim \|\phi^{(k-1)}\|_{H^2_x(r<R)} + \langle t\rangle^{-\frac{3}{2}} \|\phi\|_{E^1_k} + \|\tau_x^{\frac{1}{2}} (\Box_g \phi)^{(k-1)}\|_{\ell^1_r L^2_x}, \tag{137}$$

where the implicit constant again depends on R.

We postpone the proofs of these in order to first establish Proposition 7.1.

Proof of (133). We estimate the timelike and null/spacelike regions separately.

Step 1: (*proof of* (133) *in* $r > \frac{1}{2}\langle t\rangle$) Applying estimate (135) to $\chi_{r>\frac{1}{2}\langle t\rangle}(\tau_-\partial_u^b)^l(\tau_x\partial_x^b)^J\phi$ followed by (84) we have

$$\sum_{l+|J|\leqslant k}\|\tau_+^{\frac{3}{2}}\tau_0^{\frac{1}{2}}(\tau_-\partial_u^b)^l(\tau_x\partial_x^b)^J\phi\|_{L_x^\infty(r>\frac{1}{2}\langle t\rangle)} \lesssim \sum_{l+|J|\leqslant k+2}\|(\tau_-\partial_u^b)^l(\tau_x\partial_x^b)^J\phi\|_{L_x^2},$$
$$\lesssim \|\phi\|_{E_{k+1}^1} + \sum_{l+|J|\leqslant k}\|\tau_+^{\frac{3}{2}}\tau_x^{\frac{1}{2}}\tau_0(\tau_-\partial_u^b)^l(\tau_x\partial_x^b)^J\Box_g\phi\|_{L_x^2}.$$

Step 2: (*reduction of* (133) *in* $r < \frac{1}{2}\langle t\rangle$ *to truncated functions*) To prove (133) in the region $r < \frac{1}{2}\langle t\rangle$ we claim it suffices to show the fixed-time estimate

$$\sum_{l+|J|\leqslant k}\|(\tau_-\partial_u^b)^l(\tau_x\partial_x^b)^J\psi\|_{L_x^\infty} \lesssim \|\psi^{(k)}\|_{H_x^2(r<R)} + \langle t\rangle^{-\frac{3}{2}}\|\psi\|_{E_{k+1}^1} + \sum_{l+|J|\leqslant k}\|\tau_x^{\frac{1}{2}}(\tau_-\partial_u^b)^l(\tau_x\partial_x^b)^J\Box_g\psi\|_{\ell_r^1L_x^2}, \tag{138}$$

for functions ψ supported in the region $r < \frac{3}{4}\langle t\rangle$. Indeed, applying (138) to $\psi = \chi_{r<\frac{1}{2}\langle t\rangle}\phi$ and multiplying the result by $\langle t\rangle^{\frac{3}{2}}$ we have shown (133) in $r < \frac{1}{2}\langle t\rangle$ after using $\|\chi_{r<\frac{1}{2}t}\phi\|_{E_{k+1}^1} \lesssim \|\phi\|_{E_{k+1}^1}$ as well as

$$\sum_{l+|J|\leqslant k}\|\tau_+^{\frac{3}{2}}\tau_x^{\frac{1}{2}}(\tau_-\partial_u^b)^l(\tau_x\partial_x^b)^J[\Box_g,\chi_{r<\frac{1}{2}\langle t\rangle}]\phi\|_{\ell_r^1L_x^2} \lesssim \sum_{l+|J|\leqslant k+1}\|(\tau_-\partial_u^b)^l(\tau_x\partial_x^b)^J\phi\|_{L_x^2},$$
$$\lesssim \|\phi\|_{E_k^1} + \sum_{l+|J|\leqslant k-1}\|\tau_+^{\frac{3}{2}}\tau_x^{\frac{1}{2}}\tau_0(\tau_-\partial_u^b)^l(\tau_x\partial_x^b)^J\Box_g\phi\|_{L_x^2}.$$

On the last line we have again used (84).

Step 3: (*reduction of* (138) *to the case* $k = 1$) Using (84) we have

$$\text{LHS (138)} \lesssim \sum_{l+|J|\leqslant 1}\|(\tau_-\partial_u^b)^l(\tau_x\partial_x^b)^J\psi^{(k-1)}\|_{L_x^\infty} + \sum_{l+|J|\leqslant k-2}\|\tau_x^2(\tau_-\partial_u^b)^l(\tau_x\partial_x^b)^J\Box_g\psi\|_{L_x^\infty}.$$

On the other hand for functions ψ supported in $r < \frac{3}{4}\langle t\rangle$ estimate (135) gives

$$\sum_{l+|J|\leqslant k-2}\|\tau_x^2(\tau_-\partial_u^b)^l(\tau_x\partial_x^b)^J\Box_g\psi\|_{L_x^\infty} \lesssim \sum_{l+|J|\leqslant k}\|\tau_x^{\frac{1}{2}}(\tau_-\partial_u^b)^l(\tau_x\partial_x^b)^J\Box_g\psi\|_{L_x^2}.$$

Therefore, with the help of (137) we have reduced (138) to showing

$$\sum_{l+|J|\leqslant 1}\|(\tau_-\partial_u^b)^l(\tau_x\partial_x^b)^J\psi\|_{L_x^\infty} \lesssim \|\psi^{(1)}\|_{H_x^2(r<R)} + \langle t\rangle^{-\frac{3}{2}}\|\psi\|_{E_2^1} + \|\tau_x^{\frac{1}{2}}\Box_g\psi^{(1)}\|_{\ell_r^1L_x^2} \tag{139}$$

for functions ψ supported in $r < \frac{3}{4}\langle t\rangle$.

Step 4: (*proof of* (139)) As a first step we have for ψ supported in $r < \frac{3}{4}\langle t\rangle$

$$\sum_{l+|J|\leqslant 1}\|(\tau_-\partial_u^b)^l(\tau_x\partial_x^b)^J\psi\|_{L_x^\infty} \lesssim \|(\tau_x\partial_x\psi,\psi^{(1)})\|_{L_x^\infty},$$

which follows from Remark 1.4 and by writing $S = (u - ru_r)\partial_u^b + r\partial_r$ and $\partial_i^b = \partial_i - u_i\partial_u^b$.

Next, concatenating (134) and (136) we have

$$\|(\tau_x\partial_x\psi,\,\psi^{(1)})\|_{L^\infty_x} \lesssim \|\psi^{(1)}\|_{H^2_x(r<R)} + \langle t\rangle^{-1}\|\tau_x^{\frac{1}{2}}(\partial u\partial_u^b\psi^{(1)},\,\tau_x^{-1}u\partial_u^b\psi^{(1)},\,\partial\psi^{(1)})\|_{\ell^1_r L^2_x} + \|\tau_x^{\frac{1}{2}}\Box_g\psi^{(1)}\|_{\ell^1_r L^2_x}.$$

Here we have used that $|\langle t\rangle^{-1}u| \approx 1$ on the support of ψ, as well as $|[\partial, u]| \lesssim 1$.

Finally, we use the expansion $u\partial_u^b = S + q\partial$, where $|\partial^J q| \lesssim \tau_x^{1-|J|}$ on $r < \frac{3}{4}\langle t\rangle$, as well as the estimate $\tau_x^{-\frac{1}{2}} \times$ (83) for terms involving ∂^2, which altogether gives

$$\|\tau_x^{\frac{1}{2}}(\partial u\partial_u^b\psi^{(1)},\,\tau_x^{-1}u\partial_u^b\psi^{(1)},\,\partial\psi^{(1)})\|_{\ell^1_r L^2_x} \lesssim \langle t\rangle^{-\frac{1}{2}}\|\psi\|_{E^1_2} + \langle t\rangle\|\tau_x^{\frac{1}{2}}\Box_g\psi^{(1)}\|_{L^2_x}. \tag{140}$$

This completes the proof of (139), and hence our demonstration of (133). □

7B. *Proof of the supporting lemmas.* We now prove Lemmas 7.2, 7.3, and 7.4.

Proof of estimate (134). First note that a rescaled version of the usual $H^2 \to L^\infty$ Sobolev estimates gives

$$\|\phi\|_{L^\infty_x} \lesssim \|\tau_x^{\frac{1}{2}}(\partial_x^2\phi,\,\tau_x^{-1}\partial_x\phi,\,\tau_x^{-2}\phi)\|_{\ell^\infty_r L^2_x},$$

which applied to $\phi^{(1)}$ proves half of (134).

It remains to prove (134) for $\tau_x\partial_x\phi$, and this is really only an issue where $r > 1$. In this case the result follows from Remark 1.4 and applying the following global Sobolev estimate to $\partial_x\phi$ for $R \geq 1$:

$$\|\phi\|_{L^\infty_x(\frac{1}{2}R<r<2R)} \lesssim \sum_{|J|\leq 1} R^{-\frac{1}{2}}\|(\partial_x\Omega^J\phi,\,R^{-1}\Omega^J\phi)\|_{L^2_x(\frac{1}{4}R<r<4R)}, \quad \text{where } \Omega \in \{x^i\partial_j - x^j\partial_i\}.$$

Note that this bound is scale-invariant so it suffices to prove it for $R = 1$. After using a set of angular cutoffs and a local chart on $\mathbb{S}^2$, it becomes the mixed Sobolev embedding

$$\|\phi\|_{L^\infty(\mathbb{R}^3)} \lesssim \sum_{|I|\leq 1,|J|\leq 1} \|\partial_x^I\partial_{x'}^J\phi\|_{L^2(\mathbb{R}^3)}, \tag{141}$$

where the coordinates are written as $x = (x^1, x^2, x^3) = (x^1, x') \in \mathbb{R}^3 = \mathbb{R}\times\mathbb{R}^2$. Estimate (141) follows in the usual way by combining the Fourier inversion formula, the Cauchy–Schwarz inequality, and Plancherel's theorem, and using the fact that the multiplier $\langle\xi\rangle^{-1}\langle\xi'\rangle^{-1}$ is in $L^2(\mathbb{R}^3)$. □

Proof of estimate (135). It suffices to consider the region $\frac{1}{2}\langle t\rangle < r < \frac{3}{2}\langle t\rangle$ and $r \gg 1$, as the complementary bound follows by rescaling the $H^2 \to L^\infty$ Sobolev embedding. Using dyadic cutoffs we may further assume ϕ is supported where $\tau_- \approx 2^k$ and $\tau_x \approx 2^j$, and by using angular sector cutoffs in the x-variable we may further restrict this support to a $\frac{\pi}{4}$ wedge about the x^1-axis.

Next, we introduce the following variables on $t = \text{const}$, $r \gg 1$, and the $\frac{\pi}{4}$ wedge about the x^1-axis:

$$y^1 = 2^{-k}u, \qquad y^2 = 2^{-j}x^2, \qquad y^3 = 2^{-j}x^3.$$

Changing variables we have the formulas on $t = \text{const}$

$$\partial_{y^1} = 2^k\Big(\partial_u^b + \frac{1}{u_{x^1}}\partial_{x^1}^b\Big), \qquad \partial_{y'} = 2^j\Big(\partial_{x'}^b - \frac{u_{x'}}{u_{x^1}}\partial_{x^1}^b\Big), \qquad 2^k 2^{2j}\,dy = |u_{x^1}|\,dx,$$

where the derivatives u_{x^i} are also with respect to $t = \text{const}$, and $y' = (y^2, y^3)$. In particular by condition (1) there exist coefficients c^i_α which are uniformly bounded where $\tau_- \approx 2^k$, $\tau_x \approx 2^j$, and within the $\frac{\pi}{4}$ wedge about the x^1-axis, such that $\partial_{y^i} = \sum_\alpha c^i_\alpha e_\alpha$, where $e_\alpha \in \{\tau_- \partial^b_u, \tau_x \partial^b_x\}$.

By the change of measures formula in the previous display we have

$$2^{\frac{1}{2}k+j} \sum_{|I|\leqslant 2} \|e^I \phi\|_{L^2(dy)} \approx \sum_{a+|\beta|\leqslant 2} \|(\tau_x \tau_0 \partial^b_u)^I (\tau_x \partial^b_x)^J \phi\|_{L^2(dx)}.$$

To finish the proof it suffices to establish $H^2 \to L^\infty$ Sobolev estimates in the y-coordinates in terms of the e_α vector fields. Since the coefficients c^i_α are possibly very rough with respect to the y-variable we do this by concatenating H^1 Sobolev embeddings in the following way:

$$\|\phi\|_{L^\infty(dy)} \lesssim \sum_{|I|\leq 1} \|\partial^I_y \phi\|_{L^6(dy)} \lesssim \sum_{|I|\leq 1} \|e^I \phi\|_{L^6(dy)} \lesssim \sum_{|J|\leq 1} \sum_{|I|\leq 1} \|\partial^J_y e^I \phi\|_{L^2(dy)} \lesssim \sum_{|I|\leq 2} \|e^I \phi\|_{L^2(dy)}.$$

This completes the proof of (135). □

Proof of estimate (136). Following Remark 1.4 we see that the metric in (t, x^i)-coordinates satisfies

$$\|(\langle t\rangle \partial_t)^I (\tau_x \partial_x)^J (g^{\alpha\beta} - \eta^{\alpha\beta})\|_{\ell^1_r L^\infty(r<\frac{3}{4}\langle t\rangle)} \lesssim_{I,J} 1,$$

where $\eta = \text{diag}(-1, 1, 1, 1)$ is the standard Minkowski metric. This gives the pointwise estimate

$$|\Delta\phi| \lesssim q \cdot |(\partial^2_x \phi, \tau_x^{-1} \partial_x \phi)| + |\partial \partial^b_u \phi| + \tau_x^{-1} |\partial^b_u \phi| + \tau_+^{-1} |\partial\phi| + |\Box_g \phi|, \quad \text{where } q \in \ell^1_r L^\infty,$$

and where Δ is the standard 3-dimensional Laplacian. Therefore, by choosing R sufficiently large we see that to prove (136) it suffices to show

$$\|\tau_x^{\frac{1}{2}} (\partial^2_x \phi, \tau_x^{-1} \partial_x \phi, \tau_x^{-2} \phi)\|_{\ell^\infty_r L^2_x} \lesssim \|\phi\|_{H^2_x(r<R_1)} + \|\tau_x^{\frac{1}{2}} \Delta\phi\|_{\ell^1_r L^2_x}, \tag{142}$$

where Δ is the standard 3-dimensional Laplacian and $R_1 \geq 1$ is chosen so that $\mathcal{K} \subseteq \{r < R_1\}$.

Next, using the endpoint Hardy estimate (185) and truncating ϕ smoothly so it is supported away from $\mathcal{K}$, we can reduce (142) to the following global estimate on $\mathbb{R}^3$:

$$\|\tau_x^{-\frac{1}{2}} \partial \Delta^{-1} F\|_{\ell^1_r L^2_x(\mathbb{R}^3)} + \|\tau_x^{\frac{1}{2}} \partial^2 \Delta^{-1} F\|_{\ell^1_r L^2_x(\mathbb{R}^3)} \lesssim \|\tau_x^{\frac{1}{2}} F\|_{\ell^1_r L^2_x(\mathbb{R}^3)}.$$

This last inequality follows by taking the decomposition $\partial^J \Delta^{-1} F = \sum_{k,j} \chi_j \partial^J \Delta^{-1} \chi_k F$, where χ_j is a partition of unity adapted to dyadic regions $r \approx 2^k$, $2^j \geq 1$, and using Young's inequality to sum over

$$\|\tau_x^{-\frac{1}{2}} \chi_j \partial \Delta^{-1} \chi_k F\|_{L^2_x(\mathbb{R}^3)} + \|\tau_x^{\frac{1}{2}} \chi_j \partial^2 \Delta^{-1} \chi_k F\|_{L^2_x(\mathbb{R}^3)} \lesssim 2^j 2^k 2^{-2\max\{j,k\}} \|\tau_x^{\frac{1}{2}} \chi_k F\|_{L^2_x(\mathbb{R}^3)}$$

for $j, k \geq 0$. This final estimate follows from standard L^2 fractional/singular integral bounds. □

Finally we prove the commutator estimate:

Proof of estimate (137). Using (79) in $r < \frac{3}{4}\langle t\rangle$, we have for ϕ supported in that region

$$\|\tau_x^{\frac{1}{2}} [\Box_g, \Gamma^I]\phi\|_{\ell^1_r L^2_x} \lesssim \|\tau_x^{\frac{1}{2}} (\partial^2 \phi^{(k-1)}, \tau_x^{-1} \partial \phi^{(k-1)})\|_{\ell^\infty_r L^2_x} + \|\tau_x^{\frac{1}{2}} (\Box_g \phi)^{(k-1)}\|_{\ell^1_r L^2_x}.$$

It remains to estimate the first term on right-hand side above. First, note that by the same reasoning used to establish (140) we have

$$\|\tau_x^{\frac{1}{2}}(\partial\partial_u^b\phi^{(k-1)}, \tau_x^{-1}\partial_u^b\phi^{(k-1)}, \tau_+\partial\phi^{(k-1)})\|_{\ell_r^1 L_x^2} \lesssim \langle t\rangle^{-\frac{3}{2}}\|\phi\|_{E_k^1} + \|\tau_x^{\frac{1}{2}}\Box_g\phi^{(k-1)}\|_{L_x^2}$$

for functions ϕ supported in $r < \frac{3}{4}\langle t\rangle$. In addition we have by applying (136) to $\phi^{(k-1)}$ and then using the previous bound to handle the resulting middle term on the right-hand side of (136) the bound

$$\|\tau_x^{\frac{1}{2}}(\partial_x^2\phi^{(k-1)}, r^{-1}\partial_x\phi^{(k-1)})\|_{\ell_r^\infty L_x^2} \lesssim \|\phi^{(k-1)}\|_{H_x^2(r<R)} + \langle t\rangle^{-\frac{3}{2}}\|\phi\|_{E_k^1} + \|\tau_x^{\frac{1}{2}}\Box_g\phi^{(k-1)}\|_{\ell_r^1 L_x^2}.$$

Combining the last three inequalities and using (1) to handle $[\partial_u^b, \partial]$ we see that estimate (137) follows via induction on k. □

8. Estimates for nonlinear problems

This section is devoted to the proof of Theorem 1.22.

8A. *Proof of the $N_k \to S_k$ mapping property.* This section is devoted to the first half of Theorem 1.22.

Proof of (22). We treat each component of the norm separately.

Case 1: ($L^\infty H^s$- *and* WLE-*components*) These are handled via an induction on the index j. For $j = 0$ use (14). For $j \geq 1$ we again use (14) and estimate the middle term on the right-hand side via

$$\|\phi\|_{H_{j-1}^{16+3(k-j)}(r<B_0)[0,T]} \lesssim \|\phi\|_{\mathrm{WLE}_{j-1}^{13+3(k-j+1)}[0,T]}.$$

Case 2: ($S^{\frac{1}{2}}$-*components*) This term is handled by (15) with $a = \frac{1}{2}$. The middle term on right-hand side of (15) is bounded by the output of the previous step at level $j = k+4$ as follows:

$$\|\tau_+^{-\frac{1}{2}}\phi\|_{H_{k+4}^1(r<B_{1/2})[0,T]} \lesssim \|\phi\|_{\mathrm{WLE}_{k+4}^1[0,T]}.$$

Case 3: ($S^{1,\infty}$- *and* E^1-*components*) This follows from (16). The middle term on right-hand side of (16) is bounded by the output of the previous step as follows:

$$\|\phi\|_{\ell_t^1 H_{k+3}^1(r<B_{1/2})[0,T]} \lesssim \|\phi\|_{S_{k+3}^{1/2}[0,T]}.$$

Note also that expanding vector fields via the basis ∂^b shows

$$\|F\|_{N_{k+2}^{1,1}[0,T]} \lesssim \sum_{i+|J|\leq k+2} \|(\tau_-\partial_u^b)^i(\tau_x\partial_x^b)^J F\|_{N^{1,1}[0,T]}.$$

Case 4: (L^∞-*components*) The bound for this term follows at once from (17). □

8B. *Proof of the null form estimate.* Here we prove the estimate (23). This may be broken up into a number of pieces according the constituent parts of (20) and (21).

Proposition 8.1 (constituent null form estimates). *Let $\mathcal{N} = \mathcal{N}^{\alpha\beta}(t,x,\phi)\partial_\alpha\phi\partial_\beta\phi$ denote a quadratic form satisfying the conditions of Definition 1.20. Then there exist locally bounded functions C_k, depending on the c_k from (19), such that for $k \geq 18$ one has*

$$\sum_{j=0}^{k+4} \|\mathcal{N}\|_{(\mathrm{WLE}_j^{*,13+3(k-j)} + L_t^1 H_{x,j}^{13+3(k-j)})[0,T]} \lesssim C_k(|||\phi|||_{S_k[0,T]})|||\phi|||^2_{S_k[0,T]}, \tag{143}$$

$$\|\mathcal{N}\|_{N_{k+3}^{1/2}[0,T]} \lesssim C_k(|||\phi|||_{S_k[0,T]})|||\phi|||^2_{S_k[0,T]}, \tag{144}$$

$$\sum_{i+|J|\leq k+2} \|(\tau_-\partial_u^b)^i(\tau_x\partial_x^b)^J\mathcal{N}\|_{N^{1,1}[0,T]} \lesssim C_k(|||\phi|||_{S_k[0,T]})|||\phi|||_{S_k[0,T]}(|||\phi|||_{S_k[0,T]} + |||\Box_g\phi|||_{N_k[0,T]}), \tag{145}$$

$$\sum_{|J|\leq k} \|\tau_x^2(\tau_x\partial)^J\mathcal{N}(0)\|_{L_x^2} \lesssim C_k(|||\phi|||_{S_k[0,T]})|||\phi|||_{S_k[0,T]}(|||\phi|||_{S_k[0,T]} + |||\Box_g\phi|||_{N_k[0,T]}). \tag{146}$$

Remark 8.2. In the sequel we will prove Proposition 8.1 assuming $\mathcal{N} = \mathcal{N}^{\alpha\beta}(t,x)\partial_\alpha\phi\partial_\beta\phi$. In the more general case when $\mathcal{N}^{\alpha\beta}$ depends on ϕ as well, repeated application of the Leibniz rule leads to higher-order products of the form $\prod_{i=1}^m \phi^{(k_i)}\partial\phi^{(l_1)}\partial\phi^{(l_2)}$. Such cubic and higher-order expressions are much easier to handle via the uniform norms employed in this paper (e.g., they do not require a null structure), and treating them explicitly only serves to clutter notation.

We will prove (143)–(146) with the help of the following lemma:

Lemma 8.3 (Leibniz rules). *One has*:

(I) *Let $\mathcal{N}$ be a quadratic satisfying the conditions of Definition 1.20 but not depending on ϕ, and use the notation $\mathcal{N}(\phi,\psi) = \mathcal{N}^{\alpha\beta}(t,x)\partial_\alpha\phi\partial_\beta\psi$. Then if X^I denotes a product in $\{\partial_u^b, \partial_i^b - \omega^i\partial_u^b, S, \Omega_{ij}, \tau_-\partial_u^b, \tau_x\partial_x^b\}$, we have the identity*

$$X^I\mathcal{N}(\phi,\psi) = \sum_{I'+I''\subseteq I} \mathcal{N}_{I',I''}(X^{I'}\phi, X^{I''}\psi), \tag{147}$$

where each $\mathcal{N}_{I',I''}$ satisfies the conditions of Definition 1.20 as well.

(II) *Let f, g be smooth functions compactly supported in both time and in the exterior region $\{r > R_0\}$, where R_0 is given in Definition 1.7. Let w_a be a weight satisfying $|\partial^J w_a| \lesssim_J \tau_0^a$. Then one has the balanced product estimates*

$$\|w_0 fg\|_{\mathrm{WLE}^{*,s}} \lesssim \|\tau_x^{\frac{1}{2}}\tau_-^{\frac{1}{2}} f\|_{\ell_r^1\ell_u^1 L^\infty} \|\tau_-^{-\frac{1}{2}} g\|_{\ell_r^\infty\ell_u^\infty H^s} + \|\tau_x^{-\frac{1}{2}} f\|_{\ell_r^\infty H^s} \|\tau_x g\|_{\ell_r^1 L^\infty}, \tag{148}$$

$$\|w_a fg\|_{\mathrm{WLE}^{*,s}} \lesssim \|\tau_x\tau_0^a f\|_{\ell_r^1 L^\infty} \|\tau_x^{-\frac{1}{2}} g\|_{\ell_r^\infty H^s} + \|\tau_x^{-\frac{1}{2}} f\|_{\ell_r^\infty H^s} \|\tau_x\tau_0^a g\|_{\ell_r^1 L^\infty}. \tag{149}$$

In addition there are also hold the unbalanced versions

$$\|w_0 fg\|_{\mathrm{WLE}^{*,s}} \lesssim \|\tau_x^{\frac{1}{2}}\tau_-^{\frac{1}{2}}\partial^s f\|_{\ell_r^1\ell_u^1 L^\infty} \|\tau_-^{-\frac{1}{2}} g\|_{\ell_r^\infty\ell_u^\infty H^s}, \tag{150}$$

$$\|w_a fg\|_{\mathrm{WLE}^{*,s}} \lesssim \|\tau_x\tau_0^a\partial^s f\|_{\ell_r^1 L^\infty} \|\tau_x^{-\frac{1}{2}} g\|_{\ell_r^\infty H^s}, \tag{151}$$

where we are using the shorthand $|\partial^s f| = \sum_{|J|\leq s} |\partial^J f|$.

Proof of (147). By induction it suffices to prove this identity for a single vector field X. Using the notation $\mathcal{N}_X(\phi,\psi)=X\mathcal{N}(\phi,\psi)-\mathcal{N}(X\phi,\psi)-\mathcal{N}(\phi,X\psi)$, we have $\mathcal{N}_X^{\alpha\beta}=X(\mathcal{N}^{\alpha\beta})-\partial_\gamma^b(X^\alpha)\mathcal{N}^{\gamma\beta}-\partial_\gamma^b(X^\beta)\mathcal{N}^{\alpha\gamma}$. The first bound in (19) for $\mathcal{N}_X$ follows at once from this identity and the assumption of (19) for $\mathcal{N}$. To prove the second bound in (19) for $\mathcal{N}_X$, it suffices to study the contractions $\partial_\gamma^b(X^u)\mathcal{N}^{u\gamma}$ and $\partial_\gamma^b(X^u)\mathcal{N}^{\gamma u}$. For $\gamma=u$ the bound is again immediate from assuming (19). On the other hand for $\gamma=i$ we need $|(\tau_-\partial_u^b)^i(\tau_x\partial_x^b)^J\partial_i^b(X^u)|\lesssim_{i,J}\tau_0$ for each $X\in\{\partial_u^b,\partial_i^b-\omega^i\partial_u^b,S,\Omega_{ij},\tau_-\partial_u^b,\tau_x\partial_x^b\}$. This follows from inspection. □

Proof of estimates (148)–(149). First let both f,g be not only (spacetime) compactly supported in the exterior region $\mathbb{R}^4\setminus\mathbb{R}\times\mathcal{K}$, but also supported in dyadic regions $\tau_x\approx 2^j$ and $\tau_-\approx 2^k$. Then one has the Moser estimate

$$\|fg\|_{H^s}\lesssim\|f\|_{L^\infty}\|g\|_{H^s}+\|f\|_{H^s}\|g\|_{L^\infty}.$$

Multiplying this through by the appropriate combination of 2^j and 2^k and then summing in $\ell_r^1(\ell_u^2)$, we have both (148) and (149). □

Proof of estimates (150)–(151). This follows from distributing the derivatives and Hölder's inequality. □

Proof of estimate (143). Let $\chi_\mathcal{C}$ be a smooth cutoff which is $\equiv 1$ on the cylinder $\mathcal{C}=[1,T-1]\times\{r>R_0+1\}$ and which vanishes for $r<R_0$ and $t\in\big[0,\tfrac12\big]\cup\big[T-\tfrac12,T\big]$. Our plan is to show

$$\sum_{j=0}^{k+4}\|(1-\chi_\mathcal{C})\mathcal{N}\|_{L_t^1H_{x,j}^{13+3(k-j)}[0,T]}\lesssim|||\phi|||_{S_k[0,T]}^2, \tag{152}$$

$$\sum_{j=0}^{k+4}\|\chi_\mathcal{C}\mathcal{N}\|_{\mathrm{WLE}_j^{*,13+3(k-j)}}\lesssim|||\phi|||_{S_k[0,T]}^2. \tag{153}$$

These bounds are further broken down into a number of cases.

<u>Case 1</u>: ($L_t^1H_{x,j}^s$ *bounds in* $r<R_0+1$) Expanding all derivatives into the product and discarding $\mathcal{N}^{\alpha\beta}$ it suffices to prove for $j\le k+4$ that

$$\|\partial^I\partial\phi^{(j_1)}\partial^{I'}\partial\phi^{(j_2)}\|_{L_t^1L_x^2(r<R_0+1)[0,T]}\lesssim|||\phi|||_{S_k[0,T]}^2,\quad\text{when } j_1+j_2=j,\ \ |I|+|I'|\le 13+3(k-j). \tag{154}$$

There are now two subcases:

<u>Case 1a</u>: (*evenly split derivatives*) If both $|I|+j_1\le 10+3k-2j$ and $|I'|+j_2\le 10+3k-2j$, then each factor can absorb at least two more derivatives and still go in $L^2(r<R_0+1)[0,T]$. Thus, after an $L_x^2\to L_x^\infty$ Sobolev embedding on one factor we may use a product of

$$\|\partial^I\partial\phi^{(j_1)}\|_{L_t^2L_x^\infty(r<R_0+1)[0,T]}\lesssim|||\phi|||_{S_k[0,T]},\quad \|\partial^{I'}\partial\phi^{(j_2)}\|_{L^2(r<R_0+1)[0,T]}\lesssim|||\phi|||_{S_k[0,T]}.$$

<u>Case 1b</u>: (*uneven split*) The alternative to the previous case is that one factor, say the first, is such that $|I|+j_1\ge 11+3k-2j$. But this forces $|I'|+j_2\le 2$. In particular for $k\ge 3$ we may use a product of

$$\|\partial^I\partial\phi^{(j_1)}\|_{L_t^\infty L_x^2(r<R_0+1)}\lesssim|||\phi|||_{S_k[0,T]},$$

$$\|\partial^{I'}\partial\phi^{(j_2)}\|_{L_t^1L_x^\infty(r<R_0+1)}\lesssim\|\tau_+^{\frac32}\partial^{I'}\partial\phi^{(j_2)}\|_{L^\infty(r<R_0+1)}\lesssim|||\phi|||_{S_k[0,T]}.$$

This completes the proof of (154).

Case 2: ($L^1_t H^s_{x,j}$ *bounds in* $[0,1]\cup[T-1,T]$) In this case we show the analog of (154) where the integral is restricted to $([0,1]\cup[T-1,T])\times(\mathbb{R}^3\setminus\mathcal{K})$. This follows by taking a product of two $L^\infty_t H^s_{x,j}$ bounds after an $L^2_x\to L^\infty_x$ embedding for the factor with the least number of derivatives.

Case 3: ($\mathrm{WLE}^{*,s}$ *bounds in* $\mathcal{C}$) Using (147) it suffices to show for $j\le k+4$ and $\alpha,\beta=u,1,2,3$

$$\|\tilde{\chi}^2_{\mathcal{C}}\mathcal{N}^{\alpha\beta}_{j_1,j_2}\partial^b_\alpha\phi^{(j_1)}\partial^b_\beta\phi^{(j_2)}\|_{\mathrm{WLE}^{*,13+3(k-j)}}\lesssim\|\!|\phi|\!\|^2_{S_k[0,T]},\quad\text{where } j_1+j_2=j,\tag{155}$$

where $\tilde{\chi}_{\mathcal{C}}$ is also supported in the exterior region $(0,T)\times\{r>R_0\}$. This estimate is further broken down based on the values of j_1,j_2 and α,β.

Case 3a: ($\max\{j_1,j_2\}\le k-1$) In this case we plan to use (148) and (149) to cleanly distribute the ∂^I-derivatives. To facilitate this freeze the values of j_1,j_2 and define

$$f_\alpha=\tilde{\chi}_{\mathcal{C}}\partial^b_\alpha\phi^{(j_1)},\qquad g_\beta=\tilde{\chi}_{\mathcal{C}}\partial^b_\beta\phi^{(j_2)}.\tag{156}$$

From (5) and the definition of S_k from (20) and we have

$$\|\tau_x^{\frac12}\tau_-^{\frac12}f_u\|_{\ell^1_u\ell^1_r L^\infty}+\|\tau_x\tau_0 f_u\|_{\ell^1_r L^\infty}+\|\tau_x f_i\|_{\ell^1_r L^\infty}+\|\tau_-^{-\frac12}f_i\|_{\ell^\infty_u\ell^\infty_r H^s}+\|\tau_x^{-\frac12}f_\alpha\|_{\ell^\infty_r H^s}\lesssim\|\!|\phi|\!\|_{S_k[0,T]},\tag{157}$$

where $s=13+3(k-j)$, with an identical set of estimates for the components of g.

Case 3a.1: (*uu-components*) Using (19) it suffices to show

$$\|w_1 f_u g_u\|_{\mathrm{WLE}^{*,13+3(k-j)}}\lesssim\|\!|\phi|\!\|^2_{S_k[0,T]},\tag{158}$$

where $|\partial^I w_1|\lesssim\tau_0$. This follows from (149) with $a=1$ and (157).

Case 3a.2: (*ui- and iu-components*) Here we need the analog of (158) with w_1 replaced by weight w_0 with $|\partial^I w_0|\lesssim 1$. This follows from (148) and (157).

Case 3a.3: (*ij-components*) In this case the analog of (158) follows from (149) with $a=0$ and (157).

Case 3b: ($\max\{j_1,j_2\}\ge k$) In this case from the constraint $j_1+j_2=j\le k+4$ we must have both $\min\{j_1,j_2\}\le 4$ and $13+3(k-j)\le 13$. Without loss of generality assume $\min\{j_1,j_2\}=j_1$. Then with the notation from (156) and setting $s=13+3(k-j)$, we have if $k\ge 18$ the bounds

$$\|\tau_x^{\frac12}\tau_-^{\frac12}\partial^s f_u\|_{\ell^1_u\ell^1_r L^\infty}+\|\tau_x\tau_0\partial^s f_\alpha\|_{\ell^1_r L^\infty}+\|\tau_x\partial^s f_i\|_{\ell^1_r L^\infty}+\|\tau_-^{-\frac12}g_i\|_{\ell^\infty_u\ell^\infty_r H^s}+\|\tau_x^{-\frac12}g_\alpha\|_{\ell^\infty_r H^s}\lesssim\|\!|\phi|\!\|_{S_k[0,T]}.$$

In particular (155) for this case follows from the bound above, (150), and (151). □

The proof of (144) and (145) largely boils down to (147) and the following lemma:

Lemma 8.4. *Let $\mathcal{N}$ be any quadratic form which satisfies (19), and define $\mathcal{N}(\phi,\psi)=\mathcal{N}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\psi$. Suppose X^I is a product of vector fields in $\{\partial^b_u,\partial^b_i-\omega^i\partial^b_u,S,\Omega_{ij},\tau_-\partial^b_u,\tau_x\partial^b_x\}$. Then if $|I|\le k-1$ we have the pointwise bound on $[0,T]$*

$$|\tau_0^{\frac12}\mathcal{N}(X^I\phi,\psi)|\lesssim\|\!|\phi|\!\|_{S_k[0,T]}\cdot\tau_x^{-1}\tau_+^{-\frac32}\tau_0^{-1}\sum_{i+|J|=1}|(\tau_0\partial^b_u)^i(\partial^b_x)^J\psi|.\tag{159}$$

A similar bound holds with the roles of ϕ and ψ reversed.

Proof. Expanding the $\mathcal{N}$ using condition (19) we have

$$|\tau_0^{\frac{1}{2}}\mathcal{N}(X^I\phi,\psi)| \lesssim \tau_0^{\frac{3}{2}}|\partial_u^b X^I\phi|\cdot|\partial_u^b\psi| + \tau_0^{\frac{1}{2}}|\partial X^I\phi|\cdot|\partial_x^b\psi| + \tau_0^{\frac{1}{2}}|\partial_x^b X^I\phi|\cdot|\partial\psi|.$$

On the other hand inspection of the L^∞-term from the S_k norm defined in (20) shows for $|I|\leq k-1$

$$\tau_0^{\frac{3}{2}}|\partial X^I\phi| + \tau_0^{\frac{1}{2}}|\partial_x^b X^I\phi| \lesssim \tau_x^{-1}\tau_+^{-\frac{3}{2}}|||\phi|||_{S_k[0,T]}.$$

Taking the product of the last two inequalities yields (159). □

Proof of estimate (144). Using (147) to distribute derivatives, and splitting into interior and exterior bounds, it suffices to show that when $j_1+j_2=k+3$

$$\|\tau_+\mathcal{N}_{j_1,j_2}(\phi^{(j_1)},\phi^{(j_2)})\|_{\ell_t^1 H^1(r<R_0)[0,T]} \lesssim |||\phi|||^2_{S_k[0,T]}, \tag{160}$$

$$\|\mathcal{N}_{j_1,j_2}(\phi^{(j_1)},\phi^{(j_2)})\|_{N^{1/2}(r>R_0)[0,T]} \lesssim |||\phi|||^2_{S_k[0,T]}. \tag{161}$$

Note that (160) is stronger than what we need here, and for this bound we can even assume all vector fields are in the collection $\{\partial_u^b,\ \partial_i^b-\omega^i\partial_u^b,\ S,\ \Omega_{ij},\ \tau_-\partial_u^b,\ \tau_x\partial_x^b\}$. We will use this greater generality in a moment.

<u>Case 1</u>: (*interior estimate*) From the conditions $k\geq 18$ and $j_1+j_2=k+3$ we have $\min\{j_1,j_2\}\leq k-2$. Then (160) follows by taking the product of

$$\|\tau_+\phi^{(k)}\|_{\ell_t^1 L^\infty(r<R_0)[0,T]} \lesssim |||\phi|||_{S_k[0,T]}, \qquad \|\phi^{(k+4)}\|_{H^1(r<R_0)[0,T]} \lesssim |||\phi|||_{S_k[0,T]}.$$

<u>Case 2</u>: (*exterior estimate*) Using only $\min\{j_1,j_2\}\leq k-1$ estimate (161) follows from (159) and

$$\sum_{i+|J|=1}\|\tau_x^{-\frac{1}{2}}(\tau_0\partial_u^b)^i(\partial_x^b)^J\phi^{(k+3)}\|_{\ell_r^1\ell_u^1 L^2[0,T]} \lesssim \|\tau_x^{-\frac{1}{2}}\tau_+^{\frac{1}{2}}(\tau_0\partial_u^b\phi^{(k+3)},\partial_x^b\phi^{(k+3)})\|_{\ell_r^\infty L^2[0,T]} \lesssim |||\phi|||_{S_k[0,T]}. \ \square$$

Proof of estimate (145). Combining (147), (159), and the Klainerman–Sideris identity (84) we have

$$\sum_{i+|J|\leq k+2}\tau_0^{\frac{1}{2}}|(\tau_-\partial_u^b)^i(\tau_x\partial_x^b)^J\mathcal{N}|$$

$$\lesssim |||\phi|||_{S_k[0,T]}\cdot\Big(\sum_{i+|J|=1}\tau_x^{-\frac{1}{2}}\tau_+^{-\frac{3}{2}}\tau_0^{-\frac{1}{2}}|(\tau_0\partial_u^b)^i(\partial_x^b)^J\phi^{(k+2)}| + \sum_{i+|J|\leq k+2}\tau_0^{\frac{1}{2}}|(\tau_-\partial_u^b)^i(\tau_x\partial_x^b)^J\Box_g\phi|\Big). \tag{162}$$

The proof of (145) then follows by splitting into interior and exterior estimates as in the proof of (144) above. Note that (160) already handles the interior contribution. For the exterior contribution we use

$$\sum_{i+|J|=1}\|(\tau_0\partial_u^b)^i(\partial_x^b)^J\phi^{(k+2)}\|_{\ell_u^1\ell_t^1\ell_r^1 L^2[0,T]} \lesssim \|\tau_x^{-\frac{1}{2}}\tau_+(\tau_0\partial_u^b\phi^{(k+2)},\partial_x^b\phi^{(k+2)})\|_{\ell_t^\infty\ell_r^\infty L^2[0,T]} \lesssim |||\phi|||_{S_k[0,T]}. \ \square$$

Proof of (146). This follows by applying (162) at $t=0$ with index restriction $i+|J|\leq k$. □

Appendix A: Coordinates

In this appendix we discuss some basic consequences of Definition 1.1, as well as some simple conditions which guarantee the assumptions of Definition 1.1 hold.

Bounds between (t, x)- and (u, x)-coordinates.

Lemma A.1. *Let $u(t, x)$ be a function satisfying condition (I) of Definition 1.1. Then for any smooth function q and integer $N \geq 0$ one has the following equivalence of symbol-type bounds*:

$$\sum_{i+|J|\leq N} \|(\tau_-\partial_u^b)^i(\tau_x\tau_0\partial_x^b)^J q\|_{\ell_r^1 L^\infty} \lesssim_{N,q} 1 \quad \Longleftrightarrow \quad \sum_{i+|J|\leq N} \|(\tau_-\partial_t)^i(\tau_x\tau_0\partial_x)^J q\|_{\ell_r^1 L^\infty} \lesssim_{N,q} 1. \tag{163}$$

In addition the change of frame bounds (5) also hold.

Proof. Thanks to the change of variables formula

$$\partial_u^b = \frac{1}{u_t}\partial_t, \qquad \partial_i^b = \partial_i - \frac{u_i}{u_t}\partial_t, \tag{164}$$

the implication "∂_t, ∂_x bounds" $\Rightarrow$ "$\partial_u^b, \partial_i^b$ bounds" follows easily from (assuming $i + |J| \leq N$)

$$u_t > c, \qquad |(\tau_-\partial_t)^i(\tau_x\tau_0\partial_x)^J(u_t, u_i)| \lesssim_N 1,$$

where the inequality above itself holds thanks to part (I) of Definition 1.1. Applying this to $q = \partial u$ we have

$$|(\tau_-\partial_u^b)^i(\tau_x\tau_0\partial_x^b)^J(u_t, u_i)| \lesssim_N 1.$$

Finally, using this last inequality and formula (164) the implication "$\partial_u^b, \partial_i^b$ bounds" $\Rightarrow$ "∂_t, ∂_x bounds" becomes clear.

As a last step notice that (5) follows from the formulas (164) and estimate (163) applied to $q = \partial u$. □

Lemma A.2. *Let $u(t, x)$ be a function satisfying part (I) of Definition 1.1; then $\tau_+^{-1}(u + \tau_x - t) \in \ell_r^1 L^\infty$.*

Proof. We have $\tau_+^{-1}(u + \tau_x - t) = \tau_+^{-1} \int_0^{(t,x)} \partial(u + \tau_x - t) \cdot ds + O(\tau_+^{-1})$, where ds denotes the line integral along a straight ray from the origin to (t, x). Bounding the integral in absolute value gives

$$\sup_{r\approx 2^j} |\tau_+^{-1}(u + \tau_x - t)| \lesssim 2^{-j} + \sum_{k\leq j} 2^{k-j} \sup_{r\approx 2^k} |\partial(u + \tau_x - t)|.$$

The assumption $\partial(u + \tau_x - t) \in \ell_r^1 L^\infty$ and Young's convolution inequality finishes the proof. □

Lemma A.3. *Let $u(t, x)$ be a function satisfying condition (I) of Definition 1.1, and suppose that g is a metric satisfying condition (II). Then g is weakly asymptotically flat in (t, x)-coordinates in the sense that*

$$\|(\tau_-\partial_t)^i(\tau_x\tau_0\partial_x)^J(g - \eta)^{\alpha\beta}\|_{\ell_r^1 L^\infty} < \infty \quad \textit{for all } (i, J) \in \mathbb{N} \times \mathbb{N}^4, \tag{165}$$

where $\eta = \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric.

Proof. By Lemma A.1 is suffices to prove the bound

$$\|(\tau_- \partial_u^b)^i (\tau_x \tau_0 \partial_x^b)^J (g-\eta)^{\alpha\beta}\|_{\ell_r^1 L^\infty} < \infty \quad \text{for all } (i, J) \in \mathbb{N} \times \mathbb{N}^4, \tag{166}$$

where $(g-\eta)^{\alpha\beta}$ still denotes the components in (t, x)-coordinates. Such estimates for $(g-\eta)^{ij}$ follow at once from the first inclusion in (3) because g^{ij} is the same in either (t, x)- or (u, x)-coordinates.

For remaining components we compute

$$\begin{aligned} g^{ti} &= (g^{ui} + \omega^i) + \omega_j (g^{ij} - \delta^{ij}) - g^{\alpha i} \partial_\alpha^b (u + \tau_x - t), \\ g^{tt} + 1 &= g^{uu} + 2(g^{ui} + \omega^i)\omega_i + \omega_i \omega_j (g^{ij} - \delta^{ij}) - 2(g^{u\alpha} + \omega_i g^{i\alpha}) \partial_\alpha^b (u + \tau_x - t) \\ &\qquad + g^{\alpha\beta} \partial_\alpha^b (u + \tau_x - t) \partial_\beta^b (u + \tau_x - t) + \tau_x^{-2}, \end{aligned}$$

where all metric components on the right-hand side are now computed in (u, x)-coordinates, and where we are using the notation $\omega^i = \omega_i = x^i \tau_x^{-1}$. In addition to these formulas we also have the estimate

$$\|(\tau_- \partial_u^b)^i (\tau_x \tau_0 \partial_x^b)^J \partial^b (u + \tau_x - t)\|_{\ell_r^1 L^\infty} < \infty,$$

which itself is a consequence of (1), (164), and Lemma A.1. The remaining portion of estimate (166) follows from the last three displays above combined with assumption (3). □

Lemma A.4. *Fix $\delta > 0$. Let u_1 be an approximate optical function satisfying the conditions of Definition 1.1 in the region $\langle t-r\rangle \geq \delta \langle t+r\rangle$, and let u_2 be an approximate optical function satisfying the conditions of Definition 1.1 in the region $\langle t-r\rangle \leq 2\delta\langle t+r\rangle$. Then if χ is any cutoff function with $\chi \equiv 1$ on $\langle t-r\rangle \geq 2\delta\langle t+r\rangle$, $\chi \equiv 0$ on $\langle t-r\rangle \leq \delta\langle t+r\rangle$, and $|(\tau_x\partial)^J\chi| \lesssim 1$, the function $u = \chi u_1 + (1-\chi)u_2$ satisfies the conditions of Definition 1.1 globally. In particular, in Definition 1.1 we may always assume $u = t - \tau_x$ away from the region $\langle t-r\rangle \ll \langle t+r\rangle$.*

Proof. Using Lemma A.2 we have both

$$\begin{aligned} &\|(\langle t\rangle\partial_t)^i (\tau_x \partial_x)^J \tau_+^{-1}(u_1 + \tau_x - t)\|_{\ell_r^1 L^\infty(\delta \leq \langle t+r\rangle^{-1}\langle t-r\rangle \leq 2\delta)} < \infty, \\ &\|(\langle t\rangle\partial_t)^i (\tau_x \partial_x)^J \tau_+^{-1}(u_2 + \tau_x - t)\|_{\ell_r^1 L^\infty(\delta \leq \langle t+r\rangle^{-1}\langle t-r\rangle \leq 2\delta)} < \infty. \end{aligned}$$

Thus $\tau_+^{-1}(u_1 - u_2)$ satisfies the same bound in $\delta \leq \langle t+r\rangle^{-1}\langle t-r\rangle \leq 2\delta$, and so $u = \chi u_1 + (1-\chi)u_2$ satisfies (1) globally. Note that $\tau_+^{-1}\langle u_1\rangle \approx \tau_+^{-1}\langle u\rangle \approx 1$ in $\langle t-r\rangle \geq \delta\langle t+r\rangle$ and $\tau_+^{-1}\langle u_2\rangle \approx \tau_+^{-1}\langle u\rangle \approx 1$ in $\langle t-r\rangle \leq \delta\langle t+r\rangle$ thanks to Lemma A.2, so the definition of τ_0 is not affected by splicing u_1, u_2.

It remains to show the bounds (3) hold in (u, x)-coordinates. Because $u = u_2$ when $\langle t-r\rangle \leq \delta\langle t+r\rangle$, we concentrate on the complementary region. Here it suffices to show that if u is any function satisfying (1), and g is any metric satisfying (165), then one has automatically has the first inclusion in (3) restricted to the region $\langle t-r\rangle \geq \delta\langle t+r\rangle$. Notice that by combining (165) and (1), we see that (165) also holds for all Bondi coordinate components of $(g^{\alpha\beta} - \eta^{\alpha\beta})$. Therefore, adding and subtracting the tensor $h^{\alpha\beta}$ defined in (4), our task boils down to showing

$$\|(\langle t\rangle\partial_t)^i (\tau_x \partial)^J (\eta^{uu}, \eta^{ui} + \omega^i)\|_{\ell_r^1 L^\infty(\langle t+r\rangle^{-1}\langle t-r\rangle \geq \delta)} < \infty.$$

This last inequality follows from a few simple calculations and the estimate

$$\|(\langle t\rangle\partial_t)^i(\tau_x\partial)^J\partial(u+\tau_x-t)\|_{\ell^1_r L^\infty(\tau_+^{-1}\langle t-r\rangle\geq\delta)}<\infty,$$

which is an immediate consequence of (1). □

Constructions for nearly stationary/spherically symmetric metrics. In this section we discuss a simple situation where one can construct an approximate "optical function" $u(t,x)$ satisfying conditions (3). This is given by the following definitions.

Definition A.5. Let $g_{\alpha\beta}$ be a Lorentzian metric on $[0,\infty)\times(\mathbb{R}^3\setminus\mathcal{K})$, where $\mathcal{K}$ is a compact set. Then:

(i) g is called "weakly asymptotically flat and quasistationary" if

$$\|\ln^2(1+\tau_x)(t\partial_t)^i(\tau_x\partial)^J(g_{\alpha\beta}-\eta_{\alpha\beta})\|_{\ell^1_r L^\infty}<\infty \quad \text{for all } (i,J)\in\mathbb{N}\times\mathbb{N}^4. \tag{167}$$

Here $\eta=\operatorname{diag}(-1,1,1,1)$ is the Minkowski metric in $(t,x)\in\mathbb{R}\times\mathbb{R}^3$-coordinates.

(ii) g is called "quasispherical" if one can write $g=g_0+g_1$, where g_0 is a spherically symmetric in (t,x)-coordinates, and the remainder g_1 satisfies

$$\|\tau_x(\tau_x\partial)^J(g_1)_{\alpha\beta}\|_{\ell^1_r L^\infty}<\infty \quad \text{for all } J\in\mathbb{N}^4. \tag{168}$$

Proposition A.6. *Let $g_{\alpha\beta}$ be a Lorentzian metric on $[0,\infty)\times(\mathbb{R}^3\setminus\mathcal{K})$, where $\mathcal{K}$ is a compact set. Suppose that g is weakly asymptotically flat and quasistationary/spherical in the sense that (167) and (168) both hold. Then g satisfies the assumptions of Definition 1.1 (after a possible redefinition of the x^i-coordinates).*

Proof. We'll prove this in a series of steps.

Step 1: (*preliminary reduction*) By Lemma A.4 above it suffices to construct an approximate optical function $u(t,x)$ satisfying conditions (1) and (3) in the region $\langle t-r\rangle\ll\langle t+r\rangle$. Using a partition of unity we may extend g to be the Minkowski metric in the exterior $\langle t-r\rangle\gtrsim\langle t+r\rangle$. This extension will still satisfy (167) and (168).

Next, after a possible radial change of variables which preserves both (167) and (168), we may assume that the area of $t=$ const and $r=$ const with respect to the restriction of g_0 is $4\pi r^2$. In other words we may assume the spherically symmetric part g_0 can be written in polar coordinates as

$$g_0=(g_0)_{tt}\,dt^2+2(g_0)_{tr}\,dt\,dr+(g_0)_{rr}\,dr^2+r^2\,d\sigma^2, \tag{169}$$

where $d\sigma^2$ is the standard round metric on $\mathbb{S}^2$.

The goal now is to construct u in two pieces $u=u_0+u_1$, where $u_0=u_0(t,r)$ is radially symmetric and corresponds to g_0, while the remainder u_1 takes into account $g_1=g-g_0$. The requirements for these two functions will be

$$\|\ln(r)(r\partial)^J\partial(u_0+r-t)\|_{\ell^1_r L^\infty(r\geq R)}+\|r(r\partial)^J\partial u_1\|_{\ell^1_r L^\infty(r\geq R)}\lesssim_J 1 \tag{170}$$

for sufficiently large R, and in addition

$$g_0(du_0,du_0)=2g_0(du_0,du_1)+g_1(du_0,du_0)=0. \tag{171}$$

Notice that (170) and formulas (164) allow us to freely change $(r\partial)^J$ to $(r\partial^b)^J$ in any estimate we consider.

First suppose that we have achieved both (170) and (171). By the assumption (167) and (170), we have

$$\| \ln(r)(r\partial)^J (g^{\alpha\beta} - \eta^{\alpha\beta})\|_{\ell^1_r L^\infty(r\geq R)} + \| \ln(r)(r\partial)^J (\eta^{uu}, \eta^{ui} + \omega^i)\|_{\ell^1_r L^\infty(r\geq R)} < \infty,$$

where all components of $(g^{\alpha\beta} - \eta^{\alpha\beta})$ are computed in (u, x)-coordinates. This suffices to give the first inclusion in (3) (note we only need this for $\langle t-r\rangle \ll \langle t+r\rangle$). We remark that the convergence factor $\ln(r)$ is sufficient to sum in ℓ^1_u when $r \approx t \approx 2^j$.

Next, from the explicit form (169) and the identities in (171), we have both

$$\sqrt{|g_0|}g_0^{u_0 i} + r^{-1}x^i = 0, \qquad g^{uu} = g(du_1, du_1) + 2g_1(du_0, du_1),$$

where $\sqrt{|g_0|}$ is computed in (u_0, x)-coordinates. Using (168) and (170) we have

$$\|r(r\partial)^J (\sqrt{|g_0|} - \sqrt{|g|})\|_{\ell^1_r L^\infty(r\geq R)} < \infty,$$

where $\sqrt{|g|}$ is computed in (u, x)-coordinates. Combining the last two displays and (170) again gives

$$\|r^2(r\partial)^J g^{uu}\|_{\ell^1_r L^\infty(r\geq R)} + \|r(r\partial)^J (\sqrt{|g|}g^{ui} + \omega^i, g^{ui} - \omega^i\omega_j g^{uj})\|_{\ell^1_r L^\infty(r\geq R)} < \infty,$$

which are sufficient to produce the remaining bounds in (3) (again for $\langle t-r\rangle \ll \langle t+r\rangle$).

It remains to construct u_0 and u_1 such that (170) and (171) hold.

<u>Step 2</u>: (*construction of* u_0) For g_0 we have the expression (169), where

$$\| \ln^2(r)(r\partial)^J (g_0^{tt} + 1, g_0^{rr} - 1, g_0^{tr})\|_{\ell^1_r L^\infty(r\geq 1)} < \infty \quad \text{for all } J \in \mathbb{N}^2.$$

For the remainder of the construction we only need to work in the (t, r)-coordinates.

Let $v = v(t, r)$ be any function which solves the radial eikonal equation $g_0^{\alpha\beta}\partial_\alpha v \partial_\beta v = 0$. From this we define the quantity $\zeta = \partial v - (1, -1)$, where ∂ denotes (t, r)-derivatives. As long as $|\zeta| < 1$ the coordinate change $(t, r) \mapsto (u, r)$ is well-defined and we have

$$\partial_r^b \zeta = G(t, r, \zeta), \qquad \partial^b = q(\zeta)\partial, \tag{172}$$

where both G and q are smooth universally defined functions depending only on the (t, r)-components of g_0 and not on v. Moreover $q = q_0 + q_1(\zeta)$ with q_0 a constant invertible matrix and $q_1 = O(\zeta)$ when $|\zeta| \leq \frac{1}{2}$. Finally we have the uniform symbol bounds

$$\|r \ln^2(r)(r\partial)^J (\partial_\zeta)^k G\|_{\ell^1_r L^\infty(r\geq R)} \lesssim_{J,k} o_R(1) \quad \text{for } |\zeta| \leq \tfrac{1}{2}. \tag{173}$$

Now let $\tilde{u} = v$ in the previous construction with initial normalization $\partial_r\tilde{u} < 0$ and $\tilde{u}|_{r=R} = t - R$ for sufficiently large R, and set $\tilde{\zeta} = \partial\tilde{u} - (1, -1)$. Then $\tilde{u}$ is globally defined and smooth in $r > R$ because $(1 + 1)$ Lorentzian metrics have no caustics. We also have $|\tilde{\zeta}| \ll 1$ at least initially close to $r = R$. Commuting (172) with vector fields $\partial^b = q(\zeta)\partial$, and applying a straightforward bootstrapping argument, we may extend this to uniform bounds $|\tilde{\zeta}| \ll 1$ and $|\partial^J\tilde{\zeta}| \lesssim_J 1$ for all $r > R$.

Next, define the outgoing limit

$$f(\tilde{u}) = \lim_{r\to\infty} \partial_t \tilde{u} = 1 + \int_R^\infty G_t(t(\tilde{u}, r), r, \tilde{\zeta})\, dr,$$

where G_t denotes the t-component of G. By the previous paragraph we have both $|f-1| \ll 1$ and $|\partial_{\tilde{u}}^j f| \lesssim_j 1$. Let F solve $F' = 1/f$, and finally set $u_0 = F(\tilde{u})$. Again we let $\zeta = \partial u_0 - (1, -1)$, which we remind the reader solves (172) with conditions (173).

By construction we immediately have $g_0^{\alpha\beta}\partial_\alpha u_0 \partial_\beta u_0 = 0$, $|\zeta| \ll 1$, and $|\partial^J \zeta| \lesssim_J 1$. In addition to this and the fact that $\partial_t u_0 = (1/f)\tilde{u}_t$, we have $\lim_{r\to\infty} \partial_t u_0 = 1$. Combining this last piece of information with the $r\to\infty$ limits $g_0^{tt} \to -1$, $g_0^{rr} \to 1$, $g_0^{tr} \to 0$, as well as $\partial_r u_0 < 0$, we have the normalization $\zeta \to 0$ as $r \to \infty$. In particular we may write

$$\zeta(u_0, r) = -\int_r^\infty G(t(u_0, s), s, \zeta)\, ds.$$

Differentiating this identity any number of times with respect to $\partial^b = q(\zeta)\partial$, and using the weighted estimate (173) and a straightforward bootstrapping argument, gives the first bound in (170).

Step 3: (*construction of* u_1) Using the second identity from (171), we have that the correction u_1 solves the linear equation

$$g_0^{\alpha\beta}\partial_\alpha u_0 \partial_\beta u_1 = -\tfrac{1}{2} g_1^{\alpha\beta}\partial_\alpha u_0 \partial_\beta u_0, \quad u_1 \to 0 \text{ as } r \to \infty.$$

Integrating this in (u_0, x)-coordinates we find that

$$u_1(u_0, x) = \frac{1}{2}\int_{|x|}^\infty (g_1^{u_0 u_0}/g_0^{r u_0})(u_0, sx/|x|)\, ds.$$

By the first estimate in (170) and assumptions (167)–(168) we have

$$\| r(r\partial^b)^J (g_1^{u_0 u_0}/g_0^{r u_0}) \|_{\ell_r^1 L^\infty(r\geq R)} < \infty,$$

where ∂^b are computed in (u_0, x)-coordinates. An application of Young's convolution inequality to the integral above gives (170) for u_1. □

Appendix B: Local energy decay

In this appendix we discuss how assumption (7b) relates to assumption (7a) when the metric g enjoys structural properties similar to the Kerr family of metrics with angular momentum in a moderate range.

Let $\mathcal{T} \subseteq \mathbb{R}^3$ be a compact region contained in the exterior $\mathbb{R}^3 \setminus \mathcal{K}$. We first redefine the norms (6b) and (6d) so the regularity loss occurs only on $\mathcal{T}$:

$$\|\phi\|_{WLE^s_{class}[0,T]} = \sum_{|J|\leq s} (\|\tau_x^{-1}\partial^J\phi\|_{LE[0,T]} + \|\partial\partial^J\phi\|_{LE(\mathcal{T}^c)[0,T]}), \tag{174}$$

$$\|F\|_{WLE^{*,s}[0,T]} = \sum_{|J|\leq s} (\|\partial^J F\|_{LE^*[0,T]} + \|\partial\partial^J F\|_{L^2(\mathcal{T})[0,T]}). \tag{175}$$

With respect to these modified norms we have:

Proposition B.1. *Let the norms* $\mathrm{WLE}^s_{\mathrm{class}}$ *and* $\mathrm{WLE}^{*,s}_{\mathrm{class}}$ *be defined as in* (174) *and* (175). *Suppose in addition that* ∂_t *is uniformly timelike on* $[0,\infty)\times\mathcal{T}$. *Then estimate*

$$\sup_{0\le t\le T}\|\partial\phi(t)\|_{H^s_x}+\|\phi\|_{\mathrm{WLE}^s_{\mathrm{class}}[0,T]}\lesssim\|\partial\phi(0)\|_{H^s_x}+\|\Box_g\phi\|_{\mathrm{WLE}^{*,s}[0,T]}\tag{176}$$

implies estimate

$$\sup_{0\le t\le T}\|\partial\phi(t)\|_{H^s_x}+\|\phi\|_{\mathrm{LE}^s_{\mathrm{class}}[0,T]}\lesssim\|\partial\phi(0)\|_{H^s_x}+\|\phi\|_{H^s(\widetilde{\mathcal{T}})[0,T]}+\|\partial_t\phi\|_{H^s(\widetilde{\mathcal{T}})[0,T]}+\|\Box_g\phi\|_{\mathrm{LE}^{*,s}[0,T]},\tag{177}$$

where $\widetilde{\mathcal{T}}$ *is any compact neighborhood of* $\mathcal{T}$ (*here the implicit constant depends on* $\widetilde{\mathcal{T}}$).

Remark B.2. We do not need to make other assumptions on the metric g aside from boundedness of its derivatives and ∂_t being uniformly timelike on $[0,\infty)\times\mathcal{T}$. In particular the size of the time variation of g plays no role in establishing (177) from (176).

Note that (177) implies (7b) when $s=0$. For $s>0$ a simple induction allows us to reduce the second term in the right-hand side of (177) to $\|\phi\|_{L^2(\widetilde{\mathcal{T}})[0,T]}$.

Proof. Let $\mathcal{T}\Subset\mathcal{T}'\Subset\widetilde{\mathcal{T}}$, where $\mathcal{T}'$ is an intermediate compact neighborhood. Without loss of generality we may assume $\widetilde{\mathcal{T}}$ is a small enough that ∂_t is still uniformly timelike on it. Estimate (177) will be shown by adding together the pair of bounds

$$\sup_{0\le t\le T}\|\partial\phi(t)\|_{H^s_x((\mathcal{T}')^c)}+\|\phi\|_{\mathrm{LE}^s_{\mathrm{class}}((\mathcal{T}')^c)[0,T]}\lesssim\|\partial\phi(0)\|_{H^s_x}+\|\phi\|_{H^{s+1}(\mathcal{T}')[0,T]}+\|\Box_g\phi\|_{\mathrm{LE}^{*,s}[0,T]},\tag{178}$$

$$\sup_{0\le t\le T}\|\partial\phi(t)\|_{H^s_x(\mathcal{T}')}+\|\phi\|_{H^{s+1}(\mathcal{T}')[0,T]}\lesssim\|\partial\phi(0)\|_{H^s_x}+\|\phi\|_{H^s(\widetilde{\mathcal{T}})[0,T]}+\|\partial_t\phi\|_{H^s(\widetilde{\mathcal{T}})[0,T]}+\|\Box_g\phi\|_{H^s(\widetilde{\mathcal{T}})[0,T]}.\tag{179}$$

The first estimate (178) follows directly by applying (176) to $(1-\chi_{\mathcal{T}})\phi$, where $\chi_{\mathcal{T}}=1$ on $\mathcal{T}$ and $\chi_{\mathcal{T}}=0$ on $(\mathcal{T}')^c$, and then using the Hardy estimate (184) with $a=0$ for the boundary term where all derivatives fall on the cutoff.

The second estimate (179) is slightly more involved so we do it in a series of steps. For the demonstration it will be convenient to fix an additional pair of intermediate spatial regions $\mathcal{T}'\Subset\mathcal{T}''\Subset\mathcal{T}'''\Subset\widetilde{\mathcal{T}}$, and an $\epsilon>0$ sufficiently small that the domain of dependence of $\mathcal{T}'''$ is contained in $\widetilde{\mathcal{T}}$ for each fixed time in the slab $[0,\epsilon]$, and the domain of dependence of $\mathcal{T}'$ is contained in $\mathcal{T}''$ for each fixed time in the slab $[T-\epsilon,T]$.

<u>Step 1</u>: (*elliptic bound in* $[\epsilon,T-\epsilon]\times\mathcal{T}'''$) Our assumption is that the spatial part of $\Box_g$, i.e., the operator $(1/\sqrt{|g|})\partial_i\sqrt{|g|}g^{ij}\partial_j$ for $i,j=1,2,3$, is uniformly elliptic in $\widetilde{\mathcal{T}}$. In particular the operator $P(t,x,D)=\lambda\partial_t^2+\Box_g$ is uniformly elliptic in $\widetilde{\mathcal{T}}$ for sufficiently large $\lambda\in\mathbb{R}$. It follows that for any $s\ge 0$ one has the spacetime elliptic bound

$$\|\chi\phi\|_{H^{s+1}}\lesssim\|\chi\phi\|_{H^s}+\|\partial_t(\chi\phi)\|_{H^s}+\|\Box_g(\chi\phi)\|_{H^{s-1}},$$

where $\chi\equiv 1$ on $[\epsilon,T-\epsilon]\times\mathcal{T}'''$ and $\chi\equiv 0$ outside of $[0,T]\times\widetilde{\mathcal{T}}$. Writing $\Box_g(\chi\phi)=2\nabla^\alpha(\phi\nabla_\alpha\chi)-\phi\Box_g\chi+\chi\Box_g\phi$ and removing the cutoff, the previous inequality implies

$$\|\phi\|_{H^{s+1}(\mathcal{T}''')[\epsilon,T-\epsilon]}\lesssim\|\phi\|_{H^s(\widetilde{\mathcal{T}})[0,T]}+\|\partial_t\phi\|_{H^s(\widetilde{\mathcal{T}})[0,T]}+\|\Box_g\phi\|_{H^s(\widetilde{\mathcal{T}})[0,T]}.\tag{180}$$

Step 2: (*filling in the bottom time slab*) Fixed time local energy estimates based on the domain of dependence of $\mathcal{T}'''$ in the slab $[0,\epsilon]$ give us

$$\sup_{0\le t\le\epsilon}\|\partial\phi(t)\|_{H^s_x(\mathcal{T}''')}\lesssim\|\partial\phi(0)\|_{H^s_x}+\|\Box_g\phi\|_{H^s(\tilde{\mathcal{T}})[0,T]}.$$

Integrating these with respect to L^2_t and adding them to (180) yields the improvement

$$\|\phi\|_{H^{s+1}(\mathcal{T}''')[0,T-\epsilon]}\lesssim\|\partial\phi(0)\|_{H^s_x}+\|\phi\|_{H^s(\tilde{\mathcal{T}})[0,T]}+\|\partial_t\phi\|_{H^s(\tilde{\mathcal{T}})[0,T]}+\|\Box_g\phi\|_{H^s(\tilde{\mathcal{T}})[0,T]}. \tag{181}$$

Step 3: (*fixed-time energies on the slab* $[0,T-\epsilon]\times\mathcal{T}''$) Next, we let χ be a spatial cutoff with $\chi\equiv 1$ on $\mathcal{T}''$ and $\chi\equiv 0$ outside $\mathcal{T}'''$. Using the multiplier $\chi\partial_t$ and the left-hand side of (181) to control the spacetime error we have

$$\sup_{0\le t\le T-\epsilon}\|\partial\phi(t)\|_{H^s_x(\mathcal{T}'')}\lesssim\|\partial\phi(0)\|_{H^s_x}+\|\phi\|_{H^s(\tilde{\mathcal{T}})[0,T]}+\|\partial_t\phi\|_{H^s(\tilde{\mathcal{T}})[0,T]}+\|\Box_g\phi\|_{H^s(\tilde{\mathcal{T}})[0,T]}. \tag{182}$$

Step 4: (*fixed-time energies on the slab* $[T-\epsilon,T]\times\mathcal{T}'$) Using the domain of dependence of $\mathcal{T}'$ in $[T-\epsilon,T]$ and the left-hand side of (182) at time $T-\epsilon$, we have

$$\sup_{T-\epsilon\le t\le T}\|\partial\phi(t)\|_{H^s_x(\mathcal{T}')}\lesssim\|\partial\phi(0)\|_{H^s_x}+\|\phi\|_{H^s(\tilde{\mathcal{T}})[0,T]}+\|\partial_t\phi\|_{H^s(\tilde{\mathcal{T}})[0,T]}+\|\Box_g\phi\|_{H^s(\tilde{\mathcal{T}})[0,T]}. \tag{183}$$

Finally, adding together (181), (182), (183), and the L^2_t integral of the left-hand side of (183), we have (179). □

Appendix C: Hardy and trace inequalities

Lemma C.1 (Hardy inequalities). *Let $\mathcal{K}\subseteq\mathbb{R}^3$ be compact with connected complement, and for any other $Q\subseteq\mathbb{R}^3$ define $L^2_x(Q)$ where the domain of integration is $Q\setminus\mathcal{K}$, and $L^2(Q)[0,T]$ where the domain of integration is $[0,T]\times(Q\setminus\mathcal{K})\subseteq\mathbb{R}^4$. Then for test functions ϕ one has the following:*

(I) *For all $R\ge 0$ there hold uniformly*

$$\|\tau_x^{a-1}\phi\|_{L^2_x(r>R)}\lesssim_a\|\tau_x^a\partial_x\phi\|_{L^2_x(r>R)},\qquad when\ -\tfrac12<a<\infty, \tag{184}$$

$$\|\tau_x^{-\frac32}\phi\|_{\ell^\infty_r L^2_x(r>R)}\lesssim\|\tau_x^{-\frac32}\phi\|_{L^2_x(\frac12R<r<R)}+\|\tau_x^{-\frac12}\partial_x\phi\|_{\ell^1_r L^2_x(r>\frac12R)}. \tag{185}$$

(II) *When $R\ge 1$ is large enough that $\mathcal{K}\subseteq\{r<\frac12R\}$ there is the fixed-time estimate*

$$\|\phi\|_{L^2_x(r>R)}\lesssim\|\tau_-\tau_x^{-1}\partial_x(\tau_x\phi)\|_{L^2_x(r>R)}+R^{-\frac12}\|\tau_-^{\frac12}\phi\|_{L^2_x(\frac12R<r<R)}+R^{\frac12}\|\tau_-^{\frac12}\partial\phi\|_{L^2_x(\frac12R<r<R)}. \tag{186}$$

(III) *Again for $R\ge 1$ large enough that $\mathcal{K}\subseteq\{r<\frac12R\}$ there is the spacetime estimate*

$$\|\tau_x^{a-1}\phi(T)\|_{L^2_x(r>R)}+\|\tau_x^{a-\frac32}\phi\|_{L^2(r>R)[0,T]}$$
$$\lesssim_a\|\tau_x^{a-\frac32}\partial_r^b(\tau_x\phi)\|_{L^2(r>\frac12R)[0,T]}+\|\tau_x^{a-\frac32}\phi\|_{L^2(\frac12R<r<R)[0,T]}+\|\tau_x^a\partial\phi(0)\|_{L^2_x(r>\frac12R)},\quad when\ a<1. \tag{187}$$

Lemma C.2 (weighted trace inequalities). *For $T \geq 0$ and any $R \geq 1$ one has the uniform bound*

$$\|\tau_-^{\frac{1}{2}}\phi(T)\|_{L_x^2(\frac{1}{2}R<r<R)} \lesssim \|(\tau_-\partial_u^b\phi, \phi)\|_{L^2(\frac{1}{2}R<r<R)[\frac{1}{2}T,T]}, \qquad R \lesssim T, \tag{188a}$$

$$\|\tau_-^{\frac{1}{2}}\phi(T)\|_{L_x^2(\frac{1}{2}R<r<R)} \lesssim \|(\tau_-\partial_u^b\phi, \phi)\|_{L^2(\frac{1}{2}R<r<R)[0,T]} + \|\tau_x^{\frac{1}{2}}\phi(0)\|_{L_x^2(\frac{1}{2}R<r<R)}, \quad R \gg T, \tag{188b}$$

$$\|\tau_-^{\frac{1}{2}}\phi(T)\|_{L_x^2(r<R)} \lesssim \|(\tau_-\partial_u^b\phi, \phi)\|_{L^2(r<R)[0,T]} + \|\tau_x^{\frac{1}{2}}\phi(0)\|_{L_x^2(r<R)}. \tag{188c}$$

We note that estimate (188a) also holds with the restriction $\frac{1}{2}R < r < R$ replaced by $r < R$.

Proof of (184). Let $R_1 \geq 1$ be chosen so that $\mathcal{K} \subseteq \{r < R_1\}$. First we prove the bound assuming $R \geq R_1$. We have $I = \int_R^\infty \int_{\mathbb{S}^2} \partial_r(r^{2a+1}\phi^2)\, d\omega\, dr \leq 0$, and also

$$(2a+1)\|r^{a-1}\phi\|^2_{L_x^2(r>R)} \leq I + 2\|r^{a-1}\phi\|_{L_x^2(r>R)}\|r^a\partial\phi\|_{L_x^2(r>R)},$$

which concludes the proof in this case because $2a+1 > 0$ and $r^\alpha \approx \tau_x^\alpha$ in $r > 1$.

Now suppose $0 \leq R < R_1$, where R_1 is as above. A standard compactness argument shows

$$\|\phi\|_{L_x^2(R<r<R_1)} \lesssim \|\phi\|_{L_x^2(R_1<r<2R_1)} + \|\partial\phi\|_{L^2(R<r<2R_1)}, \tag{189}$$

where the implicit constant is uniform in R. Combining this bound with estimate (184) in $r > R_1$ completes the proof. □

Proof of (185). Using estimates of the form (189) it suffices to show for $R \geq R_1$, where R_1 is as above, that

$$\|\tau_x^{-\frac{3}{2}}\phi\|^2_{\ell^\infty L_x^2(r>2R)} \lesssim \|\tau_x^{-\frac{3}{2}}\phi\|_{\ell^\infty L_x^2(r>R)}\|\tau_x^{-\frac{1}{2}}\partial\phi\|_{\ell^1 L_x^2(r>R)}.$$

To prove it choose $h_k(r)$ so that $h_k(R) = 0$ and $h_k' = 2^{-k}\chi_k$, where $\chi_k = 1$ when $R2^k \leq r \leq R2^{k+1}$ for $k \geq 1$, and $\chi_k = 0$ when either $r \leq R2^{k-1}$ or $r \geq R2^{k+2}$. Then computing the integral $I = \int_R^\infty \int_{\mathbb{S}^2} \partial_r(h_k\phi^2)\, d\omega\, dr = 0$ and taking $\sup_{k\geq 1}$ of the result yields the estimate in the display above. □

Proof of (186). Thanks to Remark 1.3 and the conditions (1) we can assume the coordinate u is chosen so that $-C \leq \partial_r u \leq -1/C$ for some fixed $C > 0$. Then (186) follows from integration of $\partial_r(\chi_{>R}u(\tau_x\phi)^2)$ with respect to $dr\, d\omega$. □

Proof of (187). This boils down to integration of the quantity $\partial_r^b(\chi_{r>R}r^{2a-2}(\tau_x\phi)^2)$ with respect to the measure $du\, dr\, d\omega$ over the slab $0 \leq t \leq T$. By Remark 1.3 and the conditions (1) we have the pair of bounds $1/C \leq \partial_r^b t,\ \partial_t u \leq C$. In particular

$$\int_0^T\int_0^\infty\int_{\mathbb{S}^2} \partial_r^b(\chi_{r>R}r^{2a-2}(\tau_x\phi)^2)\alpha\, d\omega\, dr\, dt = \int_0^\infty\int_{\mathbb{S}^2} \chi_{r>R}r^{2a-2}(\tau_x\phi)^2\beta\, d\omega\, dr\Big|_0^T \tag{190}$$

for some pair of smooth functions $\alpha, \beta \approx 1$. This yields (187) using the condition $a < 1$. □

Proof of (188). To prove (188a) let $\chi_T(t) = \chi(t/T)$, where $\chi(s) \in C_0^\infty(s > \frac{1}{2})$ with $\chi(1) = 1$. Then (188a) follows from integration of $\partial_u^b(\tau_-\chi_T\phi^2)$ with respect to $du\, dx$ on the cylinder $0 \leq t \leq T$ and $\frac{1}{2}R < r < R$. In the case of (188b) and (188c) we use a similar procedure but simply drop the cutoff χ_T. □

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