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# ASYMPTOTIC EXPANSIONS OF FUNDAMENTAL SOLUTIONS IN PARABOLIC HOMOGENIZATION

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For a family of second-order parabolic systems with rapidly oscillating and time-dependent periodic coefficients, we investigate the asymptotic behavior of fundamental solutions and establish sharp estimates for the remainders.

## 1. Introduction

In this paper we study the asymptotic behavior of fundamental solutions  $\Gamma_\varepsilon(x, t; y, s)$  for a family of second-order parabolic operators  $\partial_t + \mathcal{L}_\varepsilon$  with rapidly oscillating and time-dependent periodic coefficients. Specifically, we consider

$$\mathcal{L}_\varepsilon = -\operatorname{div}(A(x/\varepsilon, t/\varepsilon^2)\nabla) \tag{1-1}$$

in  $\mathbb{R}^d \times \mathbb{R}$ , where  $\varepsilon > 0$  and  $A(y, s) = (a_{ij}^{\alpha\beta}(y, s))$  with  $1 \leq i, j \leq d$  and  $1 \leq \alpha, \beta \leq m$ . Throughout the paper we will assume that the coefficient matrix  $A = A(y, s)$  is real, bounded measurable and satisfies the ellipticity condition

$$\|A\|_\infty \leq \mu^{-1} \quad \text{and} \quad \mu|\xi|^2 \leq a_{ij}^{\alpha\beta}(y, s)\xi_i^\alpha\xi_j^\beta \tag{1-2}$$

for any  $\xi = (\xi_i^\alpha) \in \mathbb{R}^{m \times d}$  and a.e.  $(y, s) \in \mathbb{R}^{d+1}$ , where  $\mu > 0$ . We also assume that  $A$  is 1-periodic; i.e.,

$$A(y+z, s+t) = A(y, s) \quad \text{for } (z, t) \in \mathbb{Z}^{d+1} \text{ and a.e. } (y, s) \in \mathbb{R}^{d+1}. \tag{1-3}$$

Under these assumptions it is known that as  $\varepsilon \rightarrow 0$ , the operator  $\partial_t + \mathcal{L}_\varepsilon$  G-converges to a parabolic operator  $\partial_t + \mathcal{L}_0$  with constant coefficients [Bensoussan et al. 1978].

In the scalar case  $m = 1$ , it follows from a celebrated theorem of John Nash [1958] that local solutions of  $(\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = 0$  are Hölder continuous. More precisely, if  $(\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = 0$  in  $Q_{2r} = Q_{2r}(x_0, t_0)$  for some  $(x_0, t_0) \in \mathbb{R}^{d+1}$  and  $0 < r < \infty$ , where

$$Q_r(x_0, t_0) = B(x_0, r) \times (t_0 - r^2, t_0), \tag{1-4}$$

then there exists some  $\sigma \in (0, 1)$ , depending only on  $d$  and  $\mu$ , such that

$$\|u_\varepsilon\|_{C^{\sigma, \sigma/2}(Q_r)} \leq Cr^{-\sigma} \left( \frac{1}{|Q_{2r}|} \int_{Q_{2r}} |u_\varepsilon|^2 \right)^{1/2}, \tag{1-5}$$

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where  $C > 0$  depends only on  $d$  and  $\mu$ . In particular,  $C$  and  $\sigma$  are independent of  $\varepsilon > 0$ . The periodicity assumption (1-3) is not needed here. It follows that the fundamental solution  $\Gamma_\varepsilon(x, t; y, s)$  for  $\partial_t + \mathcal{L}_\varepsilon$  exists and satisfies the Gaussian estimate

$$|\Gamma_\varepsilon(x, t; y, s)| \leq \frac{C}{(t-s)^{d/2}} \exp\left\{-\frac{\kappa|x-y|^2}{t-s}\right\} \tag{1-6}$$

for any  $x, y \in \mathbb{R}^d$  and  $-\infty < s < t < \infty$ , where  $\kappa > 0$  depends only on  $\mu$  and  $C > 0$  depends on  $d$  and  $\mu$  (also see [Aronson 1967; Fabes and Stroock 1986] for lower bounds).

If  $m \geq 2$ , the global Hölder estimate (1-5) for  $1 < r < \infty$  was established recently in [Geng and Shen 2015] for any  $\sigma \in (0, 1)$  under the assumptions that  $A$  is elliptic, periodic, and  $A \in \text{VMO}_x$  (see (2-4) for the definition of  $\text{VMO}_x$ ). We mention that the local Hölder estimate for  $0 < r < \varepsilon$  without the periodicity condition was obtained earlier in [Byun 2007; Krylov 2007]. Consequently, by [Hofmann and Kim 2004; Cho et al. 2008], the matrix of fundamental solutions  $\Gamma_\varepsilon(x, t; y, s) = (\Gamma_\varepsilon^{\alpha\beta}(x, t; y, s))$ , with  $1 \leq \alpha, \beta \leq m$ , exists and satisfies the estimate (1-6), where  $\kappa > 0$  depends only on  $\mu$ . The constant  $C > 0$  in (1-6) depends on  $d, m, \mu$  and the function  $A^\#(r)$  in (2-5), but not on  $\varepsilon > 0$ .

The primary purpose of this paper is to study the asymptotic behavior, as  $\varepsilon \rightarrow 0$ , of  $\Gamma_\varepsilon(x, t; y, s)$ ,  $\nabla_x \Gamma_\varepsilon(x, t; y, s)$ ,  $\nabla_y \Gamma_\varepsilon(x, t; y, s)$ , and  $\nabla_x \nabla_y \Gamma_\varepsilon(x, t; y, s)$ . Our main results extend the analogous estimates for elliptic operators  $-\text{div}(A(x/\varepsilon)\nabla)$  in [Avellaneda and Lin 1991; Kenig et al. 2014] to the parabolic setting. As demonstrated in the elliptic case [Kenig and Shen 2011], the estimates in this paper open the doors for the use of layer potentials in solving initial-boundary value problems for the parabolic operators  $\partial_t + \mathcal{L}_\varepsilon$  with sharp estimates that are uniform in  $\varepsilon > 0$ .

Let  $\Gamma_0(x, t; y, s)$  denote the matrix of fundamental solutions for the homogenized operator  $\partial_t + \mathcal{L}_0$ , where  $\mathcal{L}_0 = -\text{div}(\hat{A}\nabla)$  and  $\hat{A} = (\hat{a}_{ij}^{\alpha\beta})$  is given by (2-7). Since  $\hat{A}$  is constant and satisfies the ellipticity condition (2-8), it is well known that  $\Gamma_0(x, t; y, s) = \Gamma_0(x - y, t - s; 0, 0)$  and for any  $x, y \in \mathbb{R}^d$  and  $-\infty < s < t < \infty$ ,

$$|\nabla_x^M \partial_t^N \Gamma_0(x, t; y, s)| \leq \frac{C}{(t-s)^{(d+M+2N)/2}} \exp\left\{-\frac{\kappa|x-y|^2}{t-s}\right\} \tag{1-7}$$

for any  $M, N \geq 0$ , where  $\kappa > 0$  depends only on  $\mu$ , and  $C$  depends on  $d, m, M, N$ , and  $\mu$ .

Our first result provides the sharp estimate for  $\Gamma_\varepsilon - \Gamma_0$ .

**Theorem 1.1.** *Suppose that the coefficient matrix  $A$  satisfies conditions (1-2) and (1-3). If  $m \geq 2$ , we also assume that  $A \in \text{VMO}_x$ . Then*

$$|\Gamma_\varepsilon(x, t; y, s) - \Gamma_0(x, t; y, s)| \leq \frac{C\varepsilon}{(t-s)^{(d+1)/2}} \exp\left\{-\frac{\kappa|x-y|^2}{t-s}\right\} \tag{1-8}$$

for any  $x, y \in \mathbb{R}^d$  and  $-\infty < s < t < \infty$ , where  $\kappa > 0$  depends only on  $\mu$ . The constant  $C$  depends on  $d, m, \mu$ , and  $A^\#$  (if  $m \geq 2$ ).

Let  $\chi(y, s) = (\chi_j^{\alpha\beta}(y, s))$ , where  $1 \leq j \leq d$  and  $1 \leq \alpha, \beta \leq m$ , denote the matrix of correctors for  $\partial_t + \mathcal{L}_\varepsilon$  (see Section 2 for its definition). The next theorem gives an asymptotic expansion for  $\nabla_x \Gamma_\varepsilon(x, t; y, s)$ .

**Theorem 1.2.** *Suppose that the coefficient matrix  $A$  satisfies conditions (1-2) and (1-3). Also assume that  $A$  is Hölder continuous,*

$$|A(x, t) - A(y, s)| \leq \tau(|x - y| + |t - s|^{1/2})^\lambda \quad (1-9)$$

for any  $(x, t), (y, s) \in \mathbb{R}^{d+1}$ , where  $\tau \geq 0$  and  $\lambda \in (0, 1)$ . Then

$$\begin{aligned} & |\nabla_x \Gamma_\varepsilon(x, t; y, s) - (I + \nabla \chi(x/\varepsilon, t/\varepsilon^2)) \nabla_x \Gamma_0(x, t; y, s)| \\ & \leq \frac{C\varepsilon}{(t-s)^{(d+2)/2}} \log(2 + \varepsilon^{-1}|t-s|^{1/2}) \exp\left\{-\frac{\kappa|x-y|^2}{t-s}\right\} \end{aligned} \quad (1-10)$$

for any  $x, y \in \mathbb{R}^d$  and  $-\infty < s < t < \infty$ , where  $\kappa > 0$  depends only on  $\mu$ . The constant  $C$  depends on  $d, m, \mu$ , and  $(\lambda, \tau)$  in (1-9).

With the summation convention this means that for  $1 \leq i \leq d$  and  $1 \leq \alpha, \beta \leq m$

$$\left| \frac{\partial \Gamma_\varepsilon^{\alpha\beta}}{\partial x_i}(x, t; y, s) - \frac{\partial \Gamma_0^{\alpha\beta}}{\partial x_i}(x, t; y, s) - \frac{\partial \chi_j^{\alpha\gamma}}{\partial x_i}(x/\varepsilon, t/\varepsilon^2) \frac{\partial \Gamma_0^{\gamma\beta}}{\partial x_j}(x, t; y, s) \right| \quad (1-11)$$

is bounded by the right-hand side of (1-10). Let  $\tilde{A}(y, s) = (\tilde{a}_{ij}^{\alpha\beta}(y, s))$ , where  $\tilde{a}_{ij}^{\alpha\beta}(y, s) = a_{ji}^{\beta\alpha}(y, -s)$ . Let  $\tilde{\Gamma}_\varepsilon(x, t; y, s) = (\tilde{\Gamma}_\varepsilon^{\alpha\beta}(x, t; y, s))$  denote the matrix of fundamental solutions for the operator  $\partial_t + \tilde{\mathcal{L}}_\varepsilon$ , where  $\tilde{\mathcal{L}}_\varepsilon = -\operatorname{div}(\tilde{A}(x/\varepsilon, t/\varepsilon^2)\nabla)$ . Then

$$\Gamma_\varepsilon^{\alpha\beta}(x, t; y, s) = \tilde{\Gamma}_\varepsilon^{\beta\alpha}(y, -s; x, -t). \quad (1-12)$$

Since  $\tilde{A}$  satisfies the same conditions as  $A$ , it follows from (1-10), (1-11) and (1-12) that

$$\left| \frac{\partial \Gamma_\varepsilon^{\beta\alpha}}{\partial y_i}(x, t; y, s) - \frac{\partial \Gamma_0^{\beta\alpha}}{\partial y_i}(x, t; y, s) - \frac{\partial \tilde{\chi}_j^{\alpha\gamma}}{\partial y_i}(y/\varepsilon, -s/\varepsilon^2) \frac{\partial \Gamma_0^{\beta\gamma}}{\partial y_j}(x, t; y, s) \right| \quad (1-13)$$

is bounded by the right-hand side of (1-10), where  $\tilde{\chi}(y, s) = (\tilde{\chi}_j^{\alpha\beta}(y, s))$  denotes the correctors for  $\partial_t + \tilde{\mathcal{L}}_\varepsilon$ . That is,

$$\begin{aligned} & |\nabla_y \Gamma_\varepsilon^T(x, t; y, s) - (I + \nabla \tilde{\chi}(y/\varepsilon, -s/\varepsilon^2)) \nabla_y \Gamma_0^T(x, t; y, s)| \\ & \leq \frac{C\varepsilon}{(t-s)^{(d+2)/2}} \log(2 + \varepsilon^{-1}|t-s|^{1/2}) \exp\left\{-\frac{\kappa|x-y|^2}{t-s}\right\}, \end{aligned} \quad (1-14)$$

where  $\Gamma_\varepsilon^T$  denotes the transpose of the matrix  $\Gamma_\varepsilon$ .

We also obtain an asymptotic expansion for  $\nabla_x \nabla_y \Gamma_\varepsilon(x, t; y, s)$ .

**Theorem 1.3.** *Under the same assumptions on  $A$  as in Theorem 1.2, the estimate*

$$\begin{aligned} & \left| \frac{\partial}{\partial x_i \partial y_j} \{\Gamma_\varepsilon^{\alpha\beta}(x, t; y, s)\} \right. \\ & \quad \left. - \frac{\partial}{\partial x_i} \{\delta^{\alpha\gamma} x_k + \varepsilon \chi_k^{\alpha\gamma}(x/\varepsilon, t/\varepsilon^2)\} \frac{\partial^2}{\partial x_k \partial y_\ell} \{\Gamma_0^{\gamma\sigma}(x, t; y, s)\} \frac{\partial}{\partial y_j} \{\delta^{\beta\sigma} y_\ell + \varepsilon \tilde{\chi}_\ell^{\beta\sigma}(y/\varepsilon, -s/\varepsilon^2)\} \right| \\ & \leq \frac{C\varepsilon}{(t-s)^{(d+3)/2}} \log(2 + \varepsilon^{-1}|t-s|^{1/2}) \exp\left\{-\frac{\kappa|x-y|^2}{t-s}\right\} \end{aligned} \quad (1-15)$$

holds for  $x, y \in \mathbb{R}^d$  and  $-\infty < s < t < \infty$ , where  $\kappa$  depends only on  $\mu$ . The constant  $C$  depends on  $d, m, \mu$ , and  $(\lambda, \tau)$  in (1-9).

**Remark 1.4.** The estimates (1-10), (1-14) and (1-15) are sharp, up to the logarithmic factor  $\log(2 + \varepsilon^{-1}|t - s|^{1/2})$ , which is probably not necessary. It may be possible to remove the logarithmic factor by using higher-order correctors in the proof. However, we will not pursue this idea in the present paper.

In the scale case  $m = 1$ , the estimate (1-8), *without* the exponential factor, is known under the conditions that  $A$  is elliptic, periodic, symmetric, and time-independent; see [Jikov et al. 1994, p. 77]. This was proved by using the Floquet–Bloch decomposition of the fundamental solutions and by studying the spectral properties of elliptic operators

$$-(\nabla + ik) \cdot A(\nabla + ik)$$

in a periodic cell, where  $i = \sqrt{-1}$  and  $k \in \mathbb{R}^d$ . Such an approach is not available when the coefficient matrix  $A$  is time-dependent. To the best of authors’ knowledge, the Gaussian bound in Theorem 1.1 as well as our estimates in Theorems 1.2 and 1.3 are new even in the case that  $m = 1$  and  $A$  is time-independent.

As a corollary of Theorems 1.1 and 1.2, we establish an interesting result on equistabilization for time-dependent coefficients; cf. [Jikov et al. 1994, p. 77].

**Corollary 1.5.** *Assume that  $A$  satisfies the same conditions as in Theorem 1.1. Let  $f \in L^\infty(\mathbb{R}^d)$  and  $u_\varepsilon$  be the bounded solution of the Cauchy problem,*

$$\begin{cases} (\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = 0 & \text{in } \mathbb{R}^d \times (0, \infty), \\ u_\varepsilon = f & \text{on } \mathbb{R}^d \times \{t = 0\}, \end{cases} \tag{1-16}$$

with  $\varepsilon = 1$  or  $0$ . Then for any  $x \in \mathbb{R}^d$  and  $t \geq 1$ ,

$$|u_1(x, t) - u_0(x, t)| \leq Ct^{-1/2} \|f\|_\infty. \tag{1-17}$$

Furthermore, if  $A$  is Hölder continuous,

$$\left| \nabla u_1^\alpha(x, t) - \nabla u_0^\alpha(x, t) - \nabla \chi_j^{\alpha\beta}(x, t) \frac{\partial u_0^\beta}{\partial x_j}(x, t) \right| \leq Ct^{-1} \log(2 + t) \|f\|_\infty \tag{1-18}$$

for any  $x \in \mathbb{R}^d$  and  $t \geq 1$ .

We now describe some of the key ideas in the proof of Theorems 1.1, 1.2, and 1.3. As indicated earlier, our main results extend the analogous results in [Avellaneda and Lin 1991; Kenig et al. 2014] for the elliptic operators  $-\operatorname{div}(A(x/\varepsilon)\nabla)$ , where  $A = A(y)$  is elliptic and periodic. Our general approach is inspired by [Kenig et al. 2014], which uses a two-scale expansion and relies on regularity estimates that are uniform in  $\varepsilon > 0$ . Following [Geng and Shen 2017], we consider the two-scale expansion

$$w_\varepsilon = u_\varepsilon(x, t) - u_0(x, t) - \varepsilon \chi(x/\varepsilon, t/\varepsilon^2) \mathcal{S}_\varepsilon(\nabla u_0) - \varepsilon^2 \phi(x/\varepsilon, t/\varepsilon^2) \nabla \mathcal{S}_\varepsilon(\nabla u_0), \tag{1-19}$$

where  $\chi(y, s)$  and  $\phi(y, s)$  are correctors and dual correctors respectively for  $\partial_t + \mathcal{L}_\varepsilon$  (see [Section 2](#) for their definitions). In (1-19) the operator  $S_\varepsilon$  is a parabolic smoothing operator at scale  $\varepsilon$ . In comparison with the elliptic case, an extra term is added in the right-hand side of (1-19). This modification allows us to show that if  $(\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = (\partial_t + \mathcal{L}_0)u_0$ , then

$$(\partial_t + \mathcal{L}_\varepsilon)w_\varepsilon = \varepsilon \operatorname{div}(F_\varepsilon) \quad (1-20)$$

for some function  $F_\varepsilon$ , which depends only on  $u_0$ . As a consequence, we may apply the uniform interior  $L^\infty$  estimates established in [\[Geng and Shen 2015\]](#) to the function  $w_\varepsilon$ . To fully utilize the ideas above, we will consider the functions

$$\begin{aligned} u_\varepsilon(x, t) &= \int_{-\infty}^t \int_{\mathbb{R}^d} \Gamma_\varepsilon(x, t; y, s) f(y, s) e^{-\psi(y)} dy ds, \\ u_0(x, t) &= \int_{-\infty}^t \int_{\mathbb{R}^d} \Gamma_0(x, t; y, s) f(y, s) e^{-\psi(y)} dy ds, \end{aligned} \quad (1-21)$$

where  $\psi$  is a Lipschitz function in  $\mathbb{R}^d$  and  $f \in C_0^\infty(Q_r(y_0, s_0); \mathbb{R}^m)$ . The main technical step in proving [Theorem 1.1](#) involves bounding the  $L^\infty$  norm

$$\|e^\psi(u_\varepsilon - u_0)\|_{L^\infty(Q_r(x_0, t_0))} \quad (1-22)$$

by  $\|f\|_{L^2(Q_r(y_0, s_0))}$ , where  $0 < \varepsilon < r = c\sqrt{t_0 - s_0}$ . We remark that the use of weighted inequalities with weight  $e^\psi$  to generate the exponential factor in the Gaussian bound is more or less well known. Our approach may be regarded as a variation of the standard one found in [\[Hofmann and Kim 2004; Cho et al. 2008\]](#); also see earlier work in [\[Fabes and Stroock 1986; Davies 1987a; Davies 1987b\]](#).

The proof of [Theorem 1.2](#) uses the estimate in [Theorem 1.1](#). The stronger assumption that  $A$  is Hölder continuous allows us to apply the uniform interior Lipschitz estimate obtained in [\[Geng and Shen 2015\]](#) to the function  $w_\varepsilon$  in (1-19). To see [Theorem 1.3](#), one uses the fact that as a function of  $(x, t)$ ,  $\nabla_y \Gamma_\varepsilon(x, t; y, s)$  is a solution of  $(\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = 0$ , away from the pole  $(y, s)$ .

We end this section with some notation that will be used throughout the paper. A function  $h = h(y, s)$  in  $\mathbb{R}^{d+1}$  is said to be 1-periodic if  $h$  is periodic with respect to  $\mathbb{Z}^{d+1}$ . We will use the notation

$$\int_E f = \frac{1}{|E|} \int_E f \quad \text{and} \quad h^\varepsilon(x, t) = h(x/\varepsilon, t/\varepsilon^2)$$

for  $\varepsilon > 0$ , as well as the summation convention that the repeated indices are summed. Finally, we shall use  $\kappa$  to denote positive constants that depend only on  $\mu$ , and  $C$  constants that depend at most on  $d, m, \mu$  and the smoothness of  $A$ , but never on  $\varepsilon$ .

## 2. Preliminaries

Let  $\mathcal{L}_\varepsilon = -\operatorname{div}(A^\varepsilon(x, t)\nabla)$ , where  $A^\varepsilon(x, t) = A(x/\varepsilon, t/\varepsilon^2)$ . Assume that  $A(y, s)$  is 1-periodic in  $(y, s)$  and satisfies the ellipticity condition (1-2). For  $1 \leq j \leq d$  and  $1 \leq \beta \leq m$ , the corrector  $\chi_j^\beta = \chi_j^\beta(y, s) =$

$(\chi_j^{\alpha\beta}(y, s))$  is defined as the weak solution of the cell problem

$$\begin{cases} (\partial_s + \mathcal{L}_1)(\chi_j^\beta) = -\mathcal{L}_1(P_j^\beta) & \text{in } Y, \\ \chi_j^\beta = \chi_j^\beta(y, s) \text{ is 1-periodic in } (y, s), \\ \int_Y \chi_j^\beta = 0, \end{cases} \quad (2-1)$$

where  $Y = [0, 1)^{d+1}$ ,  $P_j^\beta(y) = y_j e^\beta$ , and  $e^\beta = (0, \dots, 1, \dots, 0)$  with 1 in the  $\beta$ -th position. Note that

$$(\partial_s + \mathcal{L}_1)(\chi_j^\beta + P_j^\beta) = 0 \quad \text{in } \mathbb{R}^{d+1}. \quad (2-2)$$

By the rescaling property of  $\partial_t + \mathcal{L}_\varepsilon$ , one obtains

$$(\partial_t + \mathcal{L}_\varepsilon)\{\varepsilon \chi_j^\beta(x/\varepsilon, t/\varepsilon^2) + P_j^\beta(x)\} = 0 \quad \text{in } \mathbb{R}^{d+1}. \quad (2-3)$$

We say  $A \in \text{VMO}_x$  if

$$\lim_{r \rightarrow 0} A^\#(r) = 0, \quad (2-4)$$

where

$$A^\#(r) = \sup_{\substack{0 < \rho < r \\ (x, t) \in \mathbb{R}^{d+1}}} \int_{t-\rho^2}^t \int_{y \in B(x, \rho)} \int_{z \in B(x, \rho)} |A(y, s) - A(z, s)| dz dy ds. \quad (2-5)$$

Observe that if  $A(y, s)$  is continuous in the variable  $y$ , uniformly in  $(y, s)$ , then  $A \in \text{VMO}_x$ .

**Lemma 2.1.** *Assume that  $A(y, s)$  is 1-periodic in  $(y, s)$  and satisfies (1-2). If  $m \geq 2$ , we also assume  $A \in \text{VMO}_x$ . Then  $\chi_j^\beta \in L^\infty(Y; \mathbb{R}^m)$ .*

*Proof.* In the scalar case  $m = 1$ , this follows from (2-2) by Nash's classical estimate. Moreover, the estimate

$$\left( \int_{Q_r(x, t)} |\nabla \chi_j^\beta|^2 \right)^{1/2} \leq Cr^{\sigma-1} \quad (2-6)$$

holds for any  $0 < r < 1$  and  $(x, t) \in \mathbb{R}^{d+1}$ , where  $Q_r(x, t) = B(x, r) \times (t-r^2, t)$ , and  $C > 0$  and  $\sigma \in (0, 1)$  depend on  $d$  and  $\mu$ . If  $m \geq 2$  and  $A \in \text{VMO}_x$ , the boundedness of  $\chi_j^\beta$  follows from the interior  $W^{1,p}$  estimates for local solutions of  $(\partial_t + \mathcal{L}_1)(u) = \text{div}(f)$  [Byun 2007; Krylov 2007]. In this case the estimate (2-6) holds for any  $\sigma \in (0, 1)$ .  $\square$

Let  $\hat{A} = (\hat{a}_{ij}^{\alpha\beta})$ , where  $1 \leq i, j \leq d$ ,  $1 \leq \alpha, \beta \leq m$ , and

$$\hat{a}_{ij}^{\alpha\beta} = \int_Y \left[ a_{ij}^{\alpha\beta} + a_{ik}^{\alpha\gamma} \frac{\partial}{\partial y_k} \chi_j^{\gamma\beta} \right]; \quad (2-7)$$

that is

$$\hat{A} = \int_Y \{A + A \nabla \chi\}.$$

It is known that the constant matrix  $\hat{A}$  satisfies the ellipticity condition

$$\mu |\xi|^2 \leq \hat{a}_{ij}^{\alpha\beta} \xi_i^\alpha \xi_j^\beta \quad \text{for any } \xi = (\xi_j^\beta) \in \mathbb{R}^{m \times d}, \quad (2-8)$$

and  $|\hat{a}_{ij}^{\alpha\beta}| \leq \mu_1$ , where  $\mu_1 > 0$  depends only on  $d, m$  and  $\mu$  [Bensoussan et al. 1978]. Define

$$\mathcal{L}_0 = -\operatorname{div}(\hat{A}\nabla).$$

Then  $\partial_t + \mathcal{L}_0$  is the homogenized operator for the family of parabolic operators  $\partial_t + \mathcal{L}_\varepsilon$ ,  $\varepsilon > 0$ .

To introduce the dual correctors, we consider the 1-periodic matrix-valued function

$$B = A + A\nabla\chi - \hat{A}. \quad (2-9)$$

More precisely,  $B = B(y, s) = (b_{ij}^{\alpha\beta})$ , where  $1 \leq i, j \leq d$ ,  $1 \leq \alpha, \beta \leq m$ , and

$$b_{ij}^{\alpha\beta} = a_{ij}^{\alpha\beta} + a_{ik}^{\alpha\gamma} \frac{\partial \chi_j^{\gamma\beta}}{\partial y_k} - \hat{a}_{ij}^{\alpha\beta}. \quad (2-10)$$

**Lemma 2.2.** *Let  $1 \leq j \leq d$  and  $1 \leq \alpha, \beta \leq m$ . Then there exist 1-periodic functions  $\phi_{kij}^{\alpha\beta}(y, s)$  in  $\mathbb{R}^{d+1}$  such that  $\phi_{kij}^{\alpha\beta} \in H^1(Y)$ ,*

$$b_{ij}^{\alpha\beta} = \frac{\partial}{\partial y_k}(\phi_{kij}^{\alpha\beta}) \quad \text{and} \quad \phi_{kij}^{\alpha\beta} = -\phi_{ikj}^{\alpha\beta}, \quad (2-11)$$

where  $1 \leq k, i \leq d+1$ ,  $b_{ij}^{\alpha\beta}$  is defined by (2-10) for  $1 \leq i \leq d$ ,  $b_{(d+1)j}^{\alpha\beta} = -\chi_j^{\alpha\beta}$ , and we have used the notation  $y_{d+1} = s$ .

*Proof.* This lemma was proved in [Geng and Shen 2015]. We give a proof here for reader's convenience.

By (2-1) and (2-7),  $b_{ij}^{\alpha\beta} \in L^2(Y)$  and

$$\int_Y b_{ij}^{\alpha\beta} = 0 \quad (2-12)$$

for  $1 \leq i \leq d+1$ . It follows that there exist  $f_{ij}^{\alpha\beta} \in H^2(Y)$  such that

$$\begin{aligned} \Delta_{d+1} f_{ij}^{\alpha\beta} &= b_{ij}^{\alpha\beta} \quad \text{in } \mathbb{R}^{d+1}, \\ f_{ij}^{\alpha\beta} &\text{ is 1-periodic in } \mathbb{R}^{d+1}, \end{aligned} \quad (2-13)$$

where  $\Delta_{d+1}$  denotes the Laplacian in  $\mathbb{R}^{d+1}$ . Write

$$b_{ij}^{\alpha\beta} = \frac{\partial}{\partial y_k} \left\{ \frac{\partial}{\partial y_k} f_{ij}^{\alpha\beta} - \frac{\partial}{\partial y_i} f_{kj}^{\alpha\beta} \right\} + \frac{\partial}{\partial y_i} \left\{ \frac{\partial}{\partial y_k} f_{kj}^{\alpha\beta} \right\}, \quad (2-14)$$

where the index  $k$  is summed from 1 to  $d+1$ . Note that by (2-1),

$$\sum_{i=1}^{d+1} \frac{\partial b_{ij}^{\alpha\beta}}{\partial y_i} = \sum_{i=1}^d \frac{\partial}{\partial y_i} b_{ij}^{\alpha\beta} - \frac{\partial}{\partial s} \chi_j^{\alpha\beta} = 0. \quad (2-15)$$

In view of (2-13) this implies

$$\sum_{i=1}^{d+1} \frac{\partial}{\partial y_i} f_{ij}^{\alpha\beta}$$

is harmonic in  $\mathbb{R}^{d+1}$ . Since it is 1-periodic, it must be constant. Consequently, by (2-14), we obtain

$$b_{ij}^{\alpha\beta} = \frac{\partial}{\partial y_k}(\phi_{kij}^{\alpha\beta}), \quad (2-16)$$

where

$$\phi_{kij}^{\alpha\beta} = \frac{\partial}{\partial y_k} f_{ij}^{\alpha\beta} - \frac{\partial}{\partial y_i} f_{kj}^{\alpha\beta} \tag{2-17}$$

is 1-periodic and belongs to  $H^1(Y)$ . It is easy to see that  $\phi_{kij}^{\alpha\beta} = -\phi_{ikj}^{\alpha\beta}$ . □

The 1-periodic functions  $(\phi_{kij}^{\alpha\beta})$ , given by Lemma 2.2, are called dual correctors for the family of parabolic operators  $\partial_t + \mathcal{L}_\varepsilon$ ,  $\varepsilon > 0$ .

**Lemma 2.3.** *Let  $\phi = (\phi_{kij}^{\alpha\beta})$  be the dual correctors, given by Lemma 2.2. Under the same assumptions as in Lemma 2.1, one has  $\phi_{kij}^{\alpha\beta} \in L^\infty(Y)$ .*

*Proof.* It follows from (2-6) that if  $(x, t) \in \mathbb{R}^{d+1}$  and  $0 < r < 1$ ,

$$\int_{Q_r(x,t)} |b_{ij}^{\alpha\beta}|^2 \leq Cr^{d+2\sigma} \tag{2-18}$$

for some  $\sigma \in (0, 1)$ . By covering the interval  $(t - r, t)$  with intervals of length  $r^2$ , we obtain

$$\int_{B_r(x,t)} |b_{ij}^{\alpha\beta}|^2 \leq Cr^{d-1+2\sigma},$$

where  $B_r(x, t) = B(x, r) \times (t - r, t)$ . Hence, by Hölder's inequality,

$$\int_{B_r(x,t)} |b_{ij}^{\alpha\beta}| \leq Cr^{d+\sigma}.$$

Thus, for any  $(x, t) \in Y$ ,

$$\int_Y \frac{|b_{ij}^{\alpha\beta}(y, s)|}{(|x - y| + |t - s|)^d} dy ds \leq C \sum_{j=1}^\infty 2^{jd} \int_{|y-x|+|t-s|\sim 2^{-j}} |b_{ij}^{\alpha\beta}(y, s)| dy ds \leq C. \tag{2-19}$$

In view of (2-13), by using the fundamental solution for  $\Delta_{d+1}$  in  $\mathbb{R}^{d+1}$ , we may show that

$$\|\nabla_{y,s} f_{ij}^{\alpha\beta}\|_{L^\infty(Y)} \leq C \|\nabla_{y,s} f_{ij}^{\alpha\beta}\|_{L^2(Y)} + \sup_{(x,t) \in Y} \int_Y \frac{|b_{ij}^{\alpha\beta}(y, s)|}{(|x - y| + |t - s|)^d} dy ds,$$

where  $\nabla_{y,s}$  denotes the gradient in  $\mathbb{R}^{d+1}$ . This, together with (2-19), shows that  $|\nabla_{y,s} f_{ij}^{\alpha\beta}| \in L^\infty(Y)$ . By (2-17) we obtain  $\phi_{kij}^{\alpha\beta} \in L^\infty(Y)$ . □

**Remark 2.4.** Suppose  $A = A(y, s)$  is Hölder continuous in  $(y, s)$ . By (2-2) and the standard regularity theory for  $\partial_s + \mathcal{L}_1$ , we have  $\nabla \chi(y, s)$  is Hölder continuous in  $(y, s)$ . It follows that  $b_{ij}^{\alpha\beta}(y, s)$  is Hölder continuous in  $(y, s)$ . In view of (2-13) and (2-17) one may deduce that  $\nabla_{y,s} \phi_{kij}^{\alpha\beta}$  is Hölder continuous in  $(y, s)$ . This will be used in the proof of Theorems 1.2 and 1.3.

**Theorem 2.5.** *Suppose that  $A$  satisfies the conditions (1-2) and (1-3). If  $m \geq 2$ , we also assume  $A \in \text{VMO}_x$ . Let  $u_\varepsilon$  be a weak solution of  $(\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = \text{div}(f)$  in  $Q_{2r} = Q_{2r}(x_0, t_0)$  for some  $0 < r < \infty$ , where  $f = (f_i^\alpha) \in L^p(Q_{2r}; \mathbb{R}^{m \times d})$  for some  $p > d + 2$ . Then*

$$\|u_\varepsilon\|_{L^\infty(Q_r)} \leq C \left\{ \left( \int_{Q_{2r}} |u_\varepsilon|^2 \right)^{1/2} + r \left( \int_{Q_{2r}} |f|^p \right)^{1/p} \right\}, \tag{2-20}$$

where  $C$  depends only on  $d, m, p, \mu$ , and  $A^\#$  in (2-5) (if  $m \geq 2$ ).

*Proof.* If  $m = 1$ , this follows from the well-known Nash's estimate. The periodicity is not needed. If  $m \geq 2$ , (2-20) follows from the uniform interior Hölder estimate in [Geng and Shen 2015, Theorem 1.1].  $\square$

Under the assumptions on  $A$  in Theorem 2.5, the matrix of fundamental solutions for  $\partial_t + \mathcal{L}_\varepsilon$  in  $\mathbb{R}^{d+1}$  exists and satisfies the Gaussian estimate (1-6). This follows from the  $L^\infty$  estimate (2-20) by a general result in [Hofmann and Kim 2004]; also see [Auscher 1996; Cho et al. 2008].

**Theorem 2.6.** *Suppose that  $A$  satisfies conditions (1-2) and (1-3). Also assume that  $A$  satisfies the Hölder condition (1-9). Let  $u_\varepsilon$  be a weak solution of  $(\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = F$  in  $Q_{2r} = Q_{2r}(x_0, t_0)$  for some  $0 < r < \infty$ , where  $F \in L^p(Q_{2r}; \mathbb{R}^m)$  for some  $p > d + 2$ . Then*

$$\|\nabla u_\varepsilon\|_{L^\infty(Q_r)} \leq C \left\{ \frac{1}{r} \left( \int_{Q_{2r}} |u_\varepsilon|^2 \right)^{1/2} + r \left( \int_{Q_{2r}} |F|^p \right)^{1/p} \right\}, \quad (2-21)$$

where  $C$  depends only on  $d, m, p, \mu$ , and  $(\lambda, \tau)$  in (1-9).

*Proof.* This was proved in [Geng and Shen 2015, Theorem 1.2].  $\square$

The Lipschitz estimate (2-21) allows us to bound  $\nabla_x \Gamma_\varepsilon(x, t; y, s)$ ,  $\nabla_y \Gamma_\varepsilon(x, t; y, s)$  and  $\nabla_x \nabla_y \Gamma_\varepsilon(x, t; y, s)$ .

**Theorem 2.7.** *Assume that  $A$  satisfies the same conditions as in Theorem 2.6. Then*

$$|\nabla_x \Gamma_\varepsilon(x, t; y, s)| + |\nabla_y \Gamma_\varepsilon(x, t; y, s)| \leq \frac{C}{(t-s)^{(d+1)/2}} \exp\left\{-\frac{\kappa|x-y|^2}{t-s}\right\}, \quad (2-22)$$

$$|\nabla_x \nabla_y \Gamma_\varepsilon(x, t; y, s)| \leq \frac{C}{(t-s)^{(d+2)/2}} \exp\left\{-\frac{\kappa|x-y|^2}{t-s}\right\} \quad (2-23)$$

for any  $x, y \in \mathbb{R}^d$  and  $-\infty < s < t < \infty$ , where  $\kappa > 0$  depends only on  $\mu$ . The constant  $C$  depends on  $d, m, \mu$ , and  $(\lambda, \tau)$  in (1-9).

*Proof.* Fix  $x_0, y_0 \in \mathbb{R}^d$  and  $s_0 < t_0$ . Let  $u_\varepsilon(x, t) = \Gamma_\varepsilon(x, t; y_0, s_0)$ . Then  $(\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = 0$  in  $Q_{2r}(x_0, t_0)$ , where  $r = \sqrt{t_0 - s_0}/8$ . The estimate for  $|\nabla_x \Gamma_\varepsilon(x_0, t_0; y_0, s_0)|$  now follows from (2-21) and (1-6) (with a different  $\kappa$ ). In view of (1-12) this also gives the estimate for  $|\nabla_y \Gamma_\varepsilon(x_0, t_0; y_0, s_0)|$ . Finally, to see (2-23), we let  $v_\varepsilon(x, t) = \nabla_y \Gamma_\varepsilon(x, t; y_0, s_0)$ . Then  $(\partial_t + \mathcal{L}_\varepsilon)v_\varepsilon = 0$  in  $Q_{2r}(x_0, t_0)$ . By applying (2-21) to  $v_\varepsilon$  and using the estimate in (2-22) for  $\nabla_y \Gamma_\varepsilon(x, t; y, s)$ , we obtain the desired estimate for  $|\nabla_x \nabla_y \Gamma_\varepsilon(x_0, t_0; y_0, s_0)|$ .  $\square$

### 3. A two-scale expansion

Suppose that

$$(\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = (\partial_t + \mathcal{L}_0)u_0 \quad (3-1)$$

in  $\Omega \times (T_0, T_1)$ , where  $\Omega \subset \mathbb{R}^d$ . Let  $S_\varepsilon$  be a linear operator to be chosen later. Following [Geng and Shen 2017], we consider the two-scale expansion  $w_\varepsilon = (w_\varepsilon^\alpha)$ , where

$$w_\varepsilon^\alpha(x, t) = u_\varepsilon^\alpha(x, t) - u_0^\alpha(x, t) - \varepsilon \chi_j^{\alpha\beta}(x/\varepsilon, t/\varepsilon^2) S_\varepsilon \left( \frac{\partial u_0^\beta}{\partial x_j} \right) - \varepsilon^2 \phi_{(d+1)ij}^{\alpha\beta}(x/\varepsilon, t/\varepsilon^2) \frac{\partial}{\partial x_i} S_\varepsilon \left( \frac{\partial u_0^\beta}{\partial x_j} \right), \quad (3-2)$$

and  $\chi_j^{\alpha\beta}, \phi_{(d+1)ij}^{\alpha\beta}$  are the correctors and dual correctors introduced in the last section. The repeated indices  $i, j$  in (3-2) are summed from 1 to  $d$ .

**Proposition 3.1.** *Let  $u_\varepsilon \in L^2(T_0, T_1; H^1(\Omega))$  and  $u_0 \in L^2(T_0, T_1; H^2(\Omega))$ . Let  $w_\varepsilon$  be defined by (3-2). Assume (3-1) holds in  $\Omega \times (T_0, T_1)$ . Then*

$$(\partial_t + \mathcal{L}_\varepsilon)w_\varepsilon = \varepsilon \operatorname{div}(F_\varepsilon) \tag{3-3}$$

in  $\Omega \times (T_0, T_1)$ , where  $F_\varepsilon = (F_{\varepsilon,i}^\alpha)$  and

$$\begin{aligned} F_{\varepsilon,i}^\alpha(x, t) &= \varepsilon^{-1}(a_{ij}^{\alpha\beta}(x/\varepsilon, t/\varepsilon^2) - \hat{a}_{ij}^{\alpha\beta}) \left( \frac{\partial u_0^\beta}{\partial x_j} - S_\varepsilon \left( \frac{\partial u_0^\beta}{\partial x_j} \right) \right) \\ &\quad + a_{ij}^{\alpha\beta}(x/\varepsilon, t/\varepsilon^2) \chi_k^{\beta\gamma}(x/\varepsilon, t/\varepsilon^2) \frac{\partial}{\partial x_j} S_\varepsilon \left( \frac{\partial u_0^\gamma}{\partial x_k} \right) \\ &\quad + \phi_{ikj}^{\alpha\beta}(x/\varepsilon, t/\varepsilon^2) \frac{\partial}{\partial x_k} S_\varepsilon \left( \frac{\partial u_0^\beta}{\partial x_j} \right) + \varepsilon \phi_{i(d+1)j}^{\alpha\beta}(x/\varepsilon, t/\varepsilon^2) \partial_t S_\varepsilon \left( \frac{\partial u_0^\beta}{\partial x_j} \right) \\ &\quad - a_{ij}^{\alpha\beta}(x/\varepsilon, t/\varepsilon^2) \left( \frac{\partial}{\partial x_j} (\phi_{(d+1)\ell k}^{\beta\gamma}) \right) (x/\varepsilon, t/\varepsilon^2) \frac{\partial}{\partial x_\ell} S_\varepsilon \left( \frac{\partial u_0^\gamma}{\partial x_k} \right) \\ &\quad - \varepsilon a_{ij}^{\alpha\beta}(x/\varepsilon, t/\varepsilon^2) \phi_{(d+1)\ell k}^{\beta\gamma}(x/\varepsilon, t/\varepsilon^2) \frac{\partial^2}{\partial x_j \partial x_\ell} S_\varepsilon \left( \frac{\partial u_0^\gamma}{\partial x_k} \right). \end{aligned} \tag{3-4}$$

The repeated indices  $i, j, k, \ell$  are summed from 1 to  $d$ .

*Proof.* This proposition was proved in [Geng and Shen 2017, Theorem 2.2]. □

We now introduce a parabolic smoothing operator. Let

$$\mathcal{O} = \{(x, t) \in \mathbb{R}^{d+1} : |x|^2 + |t| \leq 1\}.$$

Fix a nonnegative function  $\theta = \theta(x, t) \in C_0^\infty(\mathcal{O})$  such that  $\int_{\mathbb{R}^{d+1}} \theta = 1$ . Let  $\theta_\varepsilon(x, t) = \varepsilon^{-d-2} \theta(x/\varepsilon, t/\varepsilon^2)$ .

Define

$$S_\varepsilon(f)(x, t) = f * \theta_\varepsilon(x, t) = \int_{\mathbb{R}^{d+1}} f(x - y, t - s) \theta_\varepsilon(y, s) dy ds. \tag{3-5}$$

**Lemma 3.2.** *Let  $g = g(x, t)$  be a 1-periodic function in  $(x, t)$  and  $\psi = \psi(x)$  a bounded Lipschitz function in  $\mathbb{R}^d$ . Then*

$$\|e^\psi g^\varepsilon S_\varepsilon(f)\|_{L^p(\mathbb{R}^{d+1})} \leq C e^{\varepsilon \|\nabla \psi\|_\infty} \|g\|_{L^p(Y)} \|e^\psi f\|_{L^p(\mathbb{R}^{d+1})} \tag{3-6}$$

for any  $1 \leq p < \infty$ , where  $g^\varepsilon(x, t) = g(x/\varepsilon, t/\varepsilon^2)$  and  $C$  depends only on  $d$  and  $p$ .

*Proof.* Using  $\int_{\mathbb{R}^{d+1}} \theta_\varepsilon = 1$  and Hölder's inequality, we obtain

$$|S_\varepsilon(e^{-\psi} f)(x, t)|^p \leq \int_{\mathbb{R}^{d+1}} |e^{-\psi(y)} f(y, s)|^p \theta_\varepsilon(x - y, t - s) dy ds.$$

It follows that

$$\begin{aligned} |e^{\psi(x)} S_\varepsilon(e^{-\psi} f)(x, t)|^p &\leq \int_{\mathbb{R}^{d+1}} |e^{\psi(x) - \psi(y)} f(y, s)|^p \theta_\varepsilon(x - y, t - s) dy ds \\ &\leq e^{\varepsilon p \|\nabla \psi\|_\infty} \int_{\mathbb{R}^{d+1}} |f(y, s)|^p \theta_\varepsilon(x - y, t - s) dy ds, \end{aligned}$$

where we have used the facts that  $|\psi(x) - \psi(y)| \leq \|\nabla\psi\|_\infty|x - y|$  and  $\theta_\varepsilon(x - y, t - s) = 0$  if  $|x - y| > \varepsilon$ , for the last step. Hence, by Fubini's theorem,

$$\begin{aligned} \int_{\mathbb{R}^{d+1}} |g^\varepsilon(x, t)|^p |e^\psi S_\varepsilon(e^{-\psi} f)(x, t)|^p dx dt \\ \leq e^{\varepsilon p \|\nabla\psi\|_\infty} \sup_{(y, s) \in \mathbb{R}^{d+1}} \int_{\mathbb{R}^{d+1}} |g^\varepsilon(x, t)|^p \theta_\varepsilon(x - y, t - s) dx dt \int_{\mathbb{R}^{d+1}} |f(y, s)|^p dy ds \\ \leq C e^{\varepsilon p \|\nabla\psi\|_\infty} \|g\|_{L^p(Y)}^p \|f\|_{L^p(\mathbb{R}^{d+1})}^p, \end{aligned}$$

where  $C$  depends only on  $d$ . This gives (3-6).  $\square$

**Remark 3.3.** Let  $\Omega \subset \mathbb{R}^d$  and  $(T_0, T_1) \subset \mathbb{R}$ . Define

$$\Omega_\varepsilon = \{x \in \mathbb{R}^d : \text{dist}(x, \Omega) < \varepsilon\}. \quad (3-7)$$

Observe that for  $(x, t) \in \Omega \times (T_0, T_1)$ , we have  $S_\varepsilon(f)(x, t) = S_\varepsilon(f\eta_\varepsilon)(x, t)$ , where  $\eta_\varepsilon = \eta_\varepsilon(x, t)$  is the characteristic function of  $\Omega_\varepsilon \times (T_0 - \varepsilon^2, T_1 + \varepsilon^2)$ . By applying (3-6) to the function  $f\eta_\varepsilon$ , one may deduce that

$$\int_{T_0}^{T_1} \int_{\Omega} |e^\psi g^\varepsilon S_\varepsilon(f)|^p dx dt \leq C e^{\varepsilon p \|\nabla\psi\|_\infty} \|g\|_{L^p(Y)}^p \int_{T_0 - \varepsilon^2}^{T_1 + \varepsilon^2} \int_{\Omega_\varepsilon} |e^\psi f|^p dx dt. \quad (3-8)$$

Using  $\int_{\mathbb{R}^{d+1}} |\nabla\theta_\varepsilon| dx dt \leq C\varepsilon^{-1}$ , the same argument as in the proof of Lemma 3.2 also shows that

$$\int_{T_0}^{T_1} \int_{\Omega} |e^\psi g^\varepsilon \nabla S_\varepsilon(f)|^p dx dt \leq C \varepsilon^{-p} e^{\varepsilon p \|\nabla\psi\|_\infty} \|g\|_{L^p(Y)}^p \int_{T_0 - \varepsilon^2}^{T_1 + \varepsilon^2} \int_{\Omega_\varepsilon} |e^\psi f|^p dx dt \quad (3-9)$$

for  $1 \leq p < \infty$ , where  $C$  depends only on  $d$  and  $p$ .

**Lemma 3.4.** Let  $S_\varepsilon$  be defined as in (3-5). Let  $1 \leq p < \infty$  and  $\psi$  be a bounded Lipschitz function in  $\mathbb{R}^d$ . Then for  $\Omega \subset \mathbb{R}^d$  and  $(T_0, T_1) \subset \mathbb{R}$ ,

$$\int_{T_0}^{T_1} \int_{\Omega} |e^\psi (S_\varepsilon(\nabla f) - \nabla f)|^p dx dt \leq C \varepsilon^p e^{\varepsilon p \|\nabla\psi\|_\infty} \int_{T_0 - \varepsilon^2}^{T_1 + \varepsilon^2} \int_{\Omega_\varepsilon} |e^\psi (|\nabla^2 f| + |\partial_t f|)|^p dx dt, \quad (3-10)$$

where  $\Omega_\varepsilon$  is given by (3-7) and  $C$  depends only on  $d$  and  $p$ .

*Proof.* Write

$$S_\varepsilon(\nabla f)(x, t) - \nabla f(x, t) = J_1(x, t) + J_2(x, t),$$

where

$$\begin{aligned} J_1(x, t) &= \int_{\mathbb{R}^{d+1}} \theta_\varepsilon(y, s) (\nabla f(x - y, t - s) - \nabla f(x - y, t)) dy ds, \\ J_2(x, t) &= \int_{\mathbb{R}^{d+1}} \theta_\varepsilon(y, s) (\nabla f(x - y, t) - \nabla f(x, t)) dy ds. \end{aligned}$$

To estimate  $J_2$ , we observe that by Hölder's inequality and the fact  $\int_{\mathbb{R}^{d+1}} \theta_\varepsilon dy ds = 1$ ,

$$|J_2(x, t)|^p \leq \int_{\mathbb{R}^{d+1}} \theta_\varepsilon(y, s) |\nabla f(x - y, t) - \nabla f(x, t)|^p dy ds,$$

and

$$\begin{aligned} |\nabla f(x-y, t) - \nabla f(x, t)| &= \left| \int_0^1 \frac{\partial}{\partial \tau} \nabla f(x - \tau y, t) d\tau \right| \\ &\leq |y| \int_0^1 |\nabla^2 f(x - \tau y, t)| d\tau \leq |y| \left( \int_0^1 |\nabla^2 f(x - \tau y, t)|^p d\tau \right)^{1/p}. \end{aligned}$$

It follows by Fubini's theorem that

$$\begin{aligned} &\int_{T_0}^{T_1} \int_{\Omega} |e^{\psi(x)} J_2(x, t)|^p dx dt \\ &\leq \int_{T_0}^{T_1} \int_{\Omega} \int_{\mathbb{R}^{d+1}} \int_0^1 e^{p\psi(x)} \theta_{\varepsilon}(y, s) |y|^p |\nabla^2 f(x - \tau y, t)|^p d\tau dy ds dx dt \\ &\leq \varepsilon^p e^{\varepsilon p \|\nabla \psi\|_{\infty}} \int_{T_0}^{T_1} \int_{\Omega} \int_{\mathbb{R}^{d+1}} \int_0^1 e^{p\psi(x-\tau y)} \theta_{\varepsilon}(y, s) |\nabla^2 f(x - \tau y, t)|^p d\tau dy ds dx dt \\ &\leq \varepsilon^p e^{\varepsilon p \|\nabla \psi\|_{\infty}} \int_{T_0}^{T_1} \int_{\Omega_{\varepsilon}} |e^{\psi} \nabla^2 f|^p dx dt, \end{aligned}$$

where we have used the facts that  $|\psi(x) - \psi(x - \tau y)| \leq |\tau| |y| \|\nabla \psi\|_{\infty}$  and  $\theta_{\varepsilon}(y, s) = 0$  if  $|y| > \varepsilon$ .

Finally, to estimate  $J_1$ , we first use integration by parts to obtain

$$|J_1(x, t)| \leq \int_{\mathbb{R}^{d+1}} |\nabla \theta_{\varepsilon}(y, s)| |f(x-y, t-s) - f(x-y, t)| dy ds.$$

By Hölder's inequality,

$$|J_1(x, t)|^p \leq C \varepsilon^{1-p} \int_{\mathbb{R}^{d+1}} |\nabla \theta_{\varepsilon}(y, s)| |f(x-y, t-s) - f(x-y, t)|^p dy ds,$$

where we have also used the fact  $\int_{\mathbb{R}^{d+1}} |\nabla \theta_{\varepsilon}| dy ds \leq C \varepsilon^{-1}$ . Using

$$|f(x-y, t-s) - f(x-y, t)| \leq \left| \int_0^1 \frac{\partial}{\partial \tau} f(x-y, t-\tau s) d\tau \right| \leq |s| \left( \int_0^1 |\partial_t f(x-y, t-\tau s)|^p d\tau \right)^{1/p},$$

we see that by Fubini's theorem,

$$\begin{aligned} &\int_{T_0}^{T_1} \int_{\Omega} |e^{\psi(x)} J_1(x, t)|^p dx dt \\ &\leq C \varepsilon^{1-p} \int_{T_0}^{T_1} \int_{\Omega} \int_{\mathbb{R}^{d+1}} \int_0^1 e^{p\psi(x)} |\nabla \theta_{\varepsilon}(y, s)| |s|^p |\partial_t f(x-y, t-\tau s)|^p d\tau dy ds dx dt \\ &\leq C \varepsilon^{1+p} e^{\varepsilon p \|\nabla \psi\|_{\infty}} \int_{T_0}^{T_1} \int_{\Omega} \int_{\mathbb{R}^{d+1}} \int_0^1 e^{p\psi(x-y)} |\nabla \theta_{\varepsilon}(y, s)| |\partial_t f(x-y, t-\tau s)|^p d\tau dy ds dx dt \\ &\leq C \varepsilon^p e^{\varepsilon p \|\nabla \psi\|_{\infty}} \int_{T_0-\varepsilon^2}^{T_1+\varepsilon^2} \int_{\Omega_{\varepsilon}} |e^{\psi} \partial_t f|^p dx dt, \end{aligned}$$

where we have used the facts that  $|\psi(x) - \psi(x-y)| \leq \|\nabla \psi\|_{\infty} |y|$  and  $\theta_{\varepsilon}(y, s) = 0$  if  $|y| > \varepsilon$  or  $|s| > \varepsilon^2$ . This, together with the estimate for  $J_2$ , completes the proof.  $\square$

**Theorem 3.5.** Let  $F_\varepsilon = (F_{\varepsilon,i}^\alpha)$  be given by (3-4) and  $1 \leq p < \infty$ . Then for any  $\Omega \subset \mathbb{R}^d$  and  $(T_0, T_1) \subset \mathbb{R}$ ,

$$\int_{T_0}^{T_1} \int_{\Omega} |e^\psi F_\varepsilon|^p dx dt \leq C e^{\varepsilon p \|\nabla \psi\|_\infty} \int_{T_0 - \varepsilon^2}^{T_1 + \varepsilon^2} \int_{\Omega_\varepsilon} \{|e^\psi \nabla^2 u_0|^p + |e^\psi \partial_t u_0|^p\} dx dt, \quad (3-11)$$

where  $\Omega_\varepsilon$  is given by (3-7) and  $C$  depends only on  $d, m, p$  and  $\mu$ .

*Proof.* Observe that

$$\begin{aligned} & \int_{T_0}^{T_1} \int_{\Omega} |e^\psi F_\varepsilon|^p dx dt \\ & \leq C \varepsilon^{-p} \int_{T_0}^{T_1} \int_{\Omega} |\nabla u_0 - S_\varepsilon(\nabla u_0)|^p e^{p\psi} dx dt + C \int_{T_0}^{T_1} \int_{\Omega} |\chi^\varepsilon|^p |S_\varepsilon(\nabla^2 u_0)|^p e^{p\psi} dx dt \\ & \quad + C \int_{T_0}^{T_1} \int_{\Omega} |\phi^\varepsilon|^p |S_\varepsilon(\nabla^2 u_0)|^p e^{p\psi} dx dt + C \varepsilon^p \int_{T_0}^{T_1} \int_{\Omega} |\phi^\varepsilon|^p |\nabla S_\varepsilon(\partial_t u_0)|^p e^{p\psi} dx dt \\ & \quad + C \int_{T_0}^{T_1} \int_{\Omega} |(\nabla \phi)^\varepsilon|^p |S_\varepsilon(\nabla^2 u_0)|^p e^{p\psi} dx dt + C \varepsilon^p \int_{T_0}^{T_1} \int_{\Omega} |\phi^\varepsilon|^p |\nabla S_\varepsilon(\nabla^2 u_0)|^p e^{p\psi} dx dt, \end{aligned} \quad (3-12)$$

where  $C$  depends only on  $d$  and  $\mu$ . In (3-12) we have also used the observation that  $\partial_t S_\varepsilon(\nabla u_0) = \nabla S_\varepsilon(\partial_t u_0)$  and  $\nabla S_\varepsilon(\nabla u_0) = S_\varepsilon(\nabla^2 u_0)$ .

We now proceed to bound each term in the right-hand side of (3-12), using Lemma 3.4 and Remark 3.3. By Lemma 3.4, the first term in the right-hand side of (3-12) is bounded by

$$C e^{p\varepsilon \|\nabla \psi\|_\infty} \int_{T_0 - \varepsilon^2}^{T_1 + \varepsilon^2} \int_{\Omega_\varepsilon} |e^\psi (|\nabla^2 u_0| + |\partial_t u_0|)|^p dx dt. \quad (3-13)$$

Using (3-8) we may bound the second, third, fifth terms in the right-hand side of (3-12) by

$$C e^{p\varepsilon \|\nabla \psi\|_\infty} \int_{T_0 - \varepsilon^2}^{T_1 + \varepsilon^2} \int_{\Omega_\varepsilon} |e^\psi \nabla^2 u_0|^p dx dt. \quad (3-14)$$

Finally, by (3-9), the fourth and sixth terms in the right-hand side of (3-12) are bounded by (3-13).  $\square$

#### 4. Weighted estimates for $\partial_t + \mathcal{L}_0$

Recall that  $\Gamma_0(x, t; y, s)$  denotes the matrix of fundamental solutions for the homogenized operator  $\partial_t + \mathcal{L}_0$  in  $\mathbb{R}^{d+1}$ . Let  $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$  be a bounded Lipschitz function and

$$u_0(x, t) = \int_{-\infty}^t \int_{\mathbb{R}^d} \Gamma_0(x, t; y, s) f(y, s) e^{-\psi(y)} dy ds, \quad (4-1)$$

where  $f \in C_0^\infty(\mathbb{R}^{d+1}; \mathbb{R}^m)$ . Then

$$(\partial_t + \mathcal{L}_0)u_0 = e^{-\psi} f \quad \text{in } \mathbb{R}^{d+1}. \quad (4-2)$$

The goal of this section is to prove the following.

**Theorem 4.1.** *Let  $u_0$  be defined by (4-1). Suppose that  $f(x, t) = 0$  for  $t \leq s_0$ . Then*

$$\int_{s_0}^t \int_{\mathbb{R}^d} |e^\psi (|\nabla^2 u_0| + |\partial_t u_0|)^2 dx dt \leq C e^{\kappa(t-s_0)\|\nabla\psi\|_\infty^2} \int_{s_0}^t \int_{\mathbb{R}^d} |f|^2 dx dt \quad (4-3)$$

for any  $s_0 < t < \infty$ , where  $\kappa > 0$  depends only on  $\mu$  and  $C$  depends only on  $d$  and  $\mu$ .

We start with an estimate on a lower-order term.

**Lemma 4.2.** *Let  $u_0$  be defined by (4-1). Suppose that  $f(x, t) = 0$  for  $t < s_0$ . Then*

$$\int_{s_0}^t \int_{\mathbb{R}^d} |e^\psi \nabla u_0|^2 dx dt \leq C(t-s_0) e^{\kappa_1(t-s_0)\|\nabla\psi\|_\infty^2} \int_{s_0}^t \int_{\mathbb{R}^d} |f|^2 dx dt \quad (4-4)$$

for any  $s_0 < t < \infty$ , where  $\kappa_1 > 0$  depends only on  $\mu$  and  $C$  depends only on  $d$  and  $\mu$ .

*Proof.* It follows from (1-7) that for  $x, y \in \mathbb{R}^d$  and  $t > s$ ,

$$\begin{aligned} e^{\psi(x)-\psi(y)} |\nabla_x \Gamma_0(x, t; y, s)| &\leq \frac{C}{(t-s)^{(d+1)/2}} \exp\left\{ \psi(x) - \psi(y) - \frac{\kappa|x-y|^2}{t-s} \right\} \\ &\leq \frac{C}{(t-s)^{(d+1)/2}} \exp\left\{ \|\nabla\psi\|_\infty |x-y| - \frac{\kappa|x-y|^2}{t-s} \right\}. \end{aligned}$$

This, together with the inequality

$$\|\nabla\psi\|_\infty |x-y| \leq \frac{(t-s)\|\nabla\psi\|_\infty^2}{2\kappa} + \frac{\kappa|x-y|^2}{2(t-s)}, \quad (4-5)$$

yields

$$e^{\psi(x)-\psi(y)} |\nabla_x \Gamma_0(x, t; y, s)| \leq C e^{(t-s)\|\nabla\psi\|_\infty^2/(2\kappa)} \cdot \frac{1}{(t-s)^{(d+1)/2}} e^{-\kappa|x-y|^2/(2(t-s))}. \quad (4-6)$$

It follows that

$$\begin{aligned} |e^{\psi(x)} \nabla u_0(x, t)| &\leq \int_{s_0}^t \int_{\mathbb{R}^d} e^{\psi(x)-\psi(y)} |\nabla_x \Gamma_0(x, t; y, s)| |f(y, s)| dy ds \\ &\leq C e^{(t-s_0)\|\nabla\psi\|_\infty^2/(2\kappa)} \int_{s_0}^t \int_{\mathbb{R}^d} \frac{1}{(t-s)^{(d+1)/2}} e^{-\kappa|x-y|^2/(2(t-s))} |f(y, s)| dy ds \\ &\leq C e^{(t-s_0)\|\nabla\psi\|_\infty^2/(2\kappa)} (t-s_0)^{1/4} \left( \int_{s_0}^t \int_{\mathbb{R}^d} \frac{1}{(t-s)^{(d+1)/2}} e^{-\kappa|x-y|^2/(2(t-s))} |f(y, s)|^2 dy ds \right)^{1/2}, \end{aligned}$$

where we have used Hölder's inequality for the last step. The estimate (4-4) now follows by Fubini's theorem.  $\square$

*Proof of Theorem 4.1.* In view of (4-2) we have

$$(\partial_t + \mathcal{L}_0) \frac{\partial u_0}{\partial x_k} = \frac{\partial}{\partial x_k} (e^{-\psi} f)$$

in  $\mathbb{R}^{d+1}$ . It follows that

$$\int_{\mathbb{R}^d} \partial_t \nabla u_0 \cdot (\nabla u_0) e^{2\psi} dx - \int_{\mathbb{R}^d} \hat{a}_{ij}^{\alpha\beta} \frac{\partial^3 u_0^\beta}{\partial x_i \partial x_j \partial x_k} \cdot \frac{\partial u_0^\alpha}{\partial x_k} e^{2\psi} dx = \int_{\mathbb{R}^d} \frac{\partial}{\partial x_k} (e^{-\psi} f^\alpha) \frac{\partial u_0^\alpha}{\partial x_k} e^{2\psi} dx.$$

Using integration by parts, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} |\nabla u_0|^2 e^{2\psi} dx + \int_{\mathbb{R}^d} \hat{a}_{ij}^{\alpha\beta} \frac{\partial^2 u_0^\beta}{\partial x_j \partial x_k} \cdot \frac{\partial^2 u_0^\alpha}{\partial x_i \partial x_k} e^{2\psi} dx \\ = - \int_{\mathbb{R}^d} f \cdot (\Delta u_0) e^\psi dx - \int_{\mathbb{R}^d} e^{-\psi} f^\alpha \frac{\partial u_0^\alpha}{\partial x_k} \frac{\partial e^{2\psi}}{\partial x_k} dx - \int_{\mathbb{R}^d} \hat{a}_{ij}^{\alpha\beta} \frac{\partial^2 u_0^\beta}{\partial x_j \partial x_k} \cdot \frac{\partial u_0^\alpha}{\partial x_k} \frac{\partial e^{2\psi}}{\partial x_i} dx. \end{aligned}$$

By the ellipticity of  $\mathcal{L}_0$ , this yields

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} |\nabla u_0|^2 e^{2\psi} dx + \mu \int_{\mathbb{R}^d} |\nabla^2 u_0|^2 e^{2\psi} dx \\ \leq C \int_{\mathbb{R}^d} |f| |\nabla^2 u_0| e^\psi dx + C \int_{\mathbb{R}^d} |f|^2 dx + C \|\nabla \psi\|_\infty^2 \int_{\mathbb{R}^d} |\nabla u_0|^2 e^{2\psi} dx + C \|\nabla \psi\|_\infty \int_{\mathbb{R}^d} |\nabla^2 u_0| |\nabla u_0| e^{2\psi} dx, \end{aligned}$$

where  $C$  depends only on  $d$  and  $\mu$ . Using the Cauchy inequality, we may further deduce that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} |\nabla u_0|^2 e^{2\psi} dx + \frac{\mu}{2} \int_{\mathbb{R}^d} |\nabla^2 u_0|^2 e^{2\psi} dx \leq C \int_{\mathbb{R}^d} |f|^2 dx + C \|\nabla \psi\|_\infty^2 \int_{\mathbb{R}^d} |\nabla u_0|^2 e^{2\psi} dx.$$

We now integrate the inequality above in  $t$  over the interval  $(s_0, s_1)$ . This leads to

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u_0(x, s_1)|^2 e^{2\psi} dx + \frac{\mu}{2} \int_{s_0}^{s_1} \int_{\mathbb{R}^d} |\nabla^2 u_0|^2 e^{2\psi} dx dt \\ \leq C \int_{s_0}^{s_1} \int_{\mathbb{R}^d} |f|^2 dx dt + C \|\nabla \psi\|_\infty^2 \int_{s_0}^{s_1} \int_{\mathbb{R}^d} |\nabla u_0|^2 e^{2\psi} dx dt \\ \leq C e^{\kappa(s_1-s_0) \|\nabla \psi\|_\infty^2} \int_{s_0}^{s_1} \int_{\mathbb{R}^d} |f|^2 dx dt, \end{aligned} \quad (4-7)$$

where we have used (4-4) for the last inequality. Estimate (4-3) follows readily from (4-7).  $\square$

## 5. Proof of Theorem 1.1

We start with some weighted estimates.

**Lemma 5.1.** *Suppose that*

$$\begin{cases} (\partial_t + \mathcal{L}_\varepsilon) w_\varepsilon = \varepsilon \operatorname{div}(F_\varepsilon) & \text{in } \mathbb{R}^d \times (s_0, \infty), \\ w_\varepsilon = 0 & \text{on } \mathbb{R}^d \times \{t = s_0\}. \end{cases} \quad (5-1)$$

Let  $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$  be a bounded Lipschitz function. Then for any  $t > s_0$

$$\int_{\mathbb{R}^d} |w_\varepsilon(x, t)|^2 e^{2\psi(x)} dx \leq C \varepsilon^2 e^{\kappa(t-s_0) \|\nabla \psi\|_\infty^2} \int_{s_0}^t \int_{\mathbb{R}^d} |F_\varepsilon(x, s)|^2 e^{2\psi(x)} dx ds, \quad (5-2)$$

where  $\kappa > 0$  and  $C > 0$  depends only on  $\mu$ .

*Proof.* Let

$$I(t) = \int_{\mathbb{R}^d} |w_\varepsilon(x, t)|^2 e^{2\psi(x)} dx. \quad (5-3)$$

Note that

$$\begin{aligned} I'(t) &= 2 \int_{\mathbb{R}^d} \langle \partial_t w_\varepsilon, e^{2\psi} w_\varepsilon \rangle dx \\ &= -2 \int_{\mathbb{R}^d} \langle \mathcal{L}_\varepsilon(w_\varepsilon), e^{2\psi} w_\varepsilon \rangle dx + 2\varepsilon \int_{\mathbb{R}^d} \langle \operatorname{div}(F_\varepsilon), e^{2\psi} w_\varepsilon \rangle dx \\ &= -2 \int_{\mathbb{R}^d} A^\varepsilon \nabla w_\varepsilon \cdot \nabla(e^{2\psi} w_\varepsilon) dx - 2\varepsilon \int_{\mathbb{R}^d} F_\varepsilon \cdot \nabla(e^{2\psi} w_\varepsilon) dx \\ &= -2 \int_{\mathbb{R}^d} A^\varepsilon \nabla w_\varepsilon \cdot (\nabla w_\varepsilon) e^{2\psi} dx - 2 \int_{\mathbb{R}^d} A^\varepsilon \nabla w_\varepsilon \cdot \nabla(e^{2\psi}) w_\varepsilon dx \\ &\quad - 2\varepsilon \int_{\mathbb{R}^d} F_\varepsilon \cdot (\nabla w_\varepsilon) e^{2\psi} dx - 2\varepsilon \int_{\mathbb{R}^d} F_\varepsilon \cdot \nabla(e^{2\psi}) w_\varepsilon dx, \end{aligned}$$

where  $\langle \cdot, \cdot \rangle$  denotes the pairing in  $H^{-1}(\mathbb{R}^d; \mathbb{R}^m) \times H^1(\mathbb{R}^d; \mathbb{R}^m)$ . It follows that

$$\begin{aligned} I'(t) &\leq -2\mu \int_{\mathbb{R}^d} |\nabla w_\varepsilon|^2 e^{2\psi} dx + \kappa \|\nabla \psi\|_\infty \int_{\mathbb{R}^d} |\nabla w_\varepsilon| |w_\varepsilon| e^{2\psi} dx \\ &\quad + 2\varepsilon \int_{\mathbb{R}^d} |\nabla w_\varepsilon| |F_\varepsilon| e^{2\psi} dx + 4\varepsilon \|\nabla \psi\|_\infty \int_{\mathbb{R}^d} |w_\varepsilon| |F_\varepsilon| e^{2\psi} dx, \end{aligned}$$

where  $\kappa > 0$  depends only on  $\mu$ . By the Cauchy inequality this implies

$$I'(t) \leq \kappa \|\nabla \psi\|_\infty^2 I(t) + \kappa \varepsilon^2 \int_{\mathbb{R}^d} |F_\varepsilon(x, t)|^2 e^{2\psi} dx, \quad (5-4)$$

where  $\kappa > 0$  depends only on  $\mu$ . Hence,

$$\frac{d}{dt} \{I(t) e^{-\kappa(t-s_0)\|\nabla \psi\|_\infty^2}\} \leq C \varepsilon^2 e^{-\kappa(t-s_0)\|\nabla \psi\|_\infty^2} \int_{\mathbb{R}^d} |F_\varepsilon(x, t)|^2 e^{2\psi} dx.$$

Since  $I(s_0) = 0$ , it follows that

$$\begin{aligned} I(t) &\leq C \varepsilon^2 \int_{s_0}^t \int_{\mathbb{R}^d} e^{\kappa(t-s)\|\nabla \psi\|_\infty^2} |F_\varepsilon(x, s)|^2 e^{2\psi} dx ds \\ &\leq C \varepsilon^2 e^{\kappa(t-s_0)\|\nabla \psi\|_\infty^2} \int_{s_0}^t \int_{\mathbb{R}^d} |F_\varepsilon(x, s)|^2 e^{2\psi} dx ds. \end{aligned} \quad \square$$

**Lemma 5.2.** *Suppose that  $u_\varepsilon \in L^2((-\infty, T); H^1(\mathbb{R}^d))$  and  $u_0 \in L^2((-\infty, T); H^2(\mathbb{R}^d))$  for any  $T \in \mathbb{R}$ , and that*

$$\begin{cases} (\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = (\partial_t + \mathcal{L}_0)u_0 & \text{in } \mathbb{R}^{d+1}, \\ u_\varepsilon(x, t) = u_0(x, t) = 0 & \text{for } t \leq s_0. \end{cases}$$

Let  $w_\varepsilon$  be defined by (3-2), where the operator  $S_\varepsilon$  is given by (3-5). Then for any  $t > s_0$ ,

$$\begin{aligned} \int_{\mathbb{R}^d} |w_\varepsilon(x, t)|^2 e^{2\psi(x)} dx \\ \leq C\varepsilon^2 e^{2\varepsilon\|\nabla\psi\|_\infty + \kappa(t-s_0)\|\nabla\psi\|_\infty^2} \int_{s_0}^{t+\varepsilon^2} \int_{\mathbb{R}^d} \{|\nabla^2 u_0(x, s)|^2 + |\partial_s u_0(x, s)|^2\} e^{2\psi(x)} dx ds, \end{aligned} \quad (5-5)$$

where  $\psi$  is a bounded Lipschitz function in  $\mathbb{R}^d$ ,  $\kappa$  depends only on  $\mu$ , and  $C$  depends only on  $d, m$  and  $\mu$ .

*Proof.* This follows readily from Lemma 5.1 and Theorem 3.5 with  $p = 2$ .  $\square$

The next theorem gives a weighted  $L^\infty$  estimate.

**Theorem 5.3.** Assume that  $A$  is 1-periodic and satisfies (1-2). If  $m \geq 2$ , we also assume that  $A \in \text{VMO}_x$ . Suppose that  $(\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = (\partial_t + \mathcal{L}_0)u_0$  in  $B(x_0, 3r) \times (t_0 - 5r^2, t_0 + r^2)$  for some  $(x_0, t_0) \in \mathbb{R}^{d+1}$  and  $\varepsilon \leq r < \infty$ . Then

$$\begin{aligned} \|e^\psi(u_\varepsilon - u_0)\|_{L^\infty(Q_r(x_0, t_0))} &\leq C e^{3r\|\nabla\psi\|_\infty} \left( \int_{Q_{2r}(x_0, t_0)} |e^\psi(u_\varepsilon - u_0)|^2 \right)^{1/2} \\ &\quad + C\varepsilon r e^{3r\|\nabla\psi\|_\infty} \|e^\psi(|\nabla^2 u_0| + |\partial_t u_0|)\|_{L^\infty(B(x_0, 3r) \times (t_0 - 5r^2, t_0 + r^2))} \\ &\quad + C\varepsilon e^{3r\|\nabla\psi\|_\infty} \|e^\psi \nabla u_0\|_{L^\infty(B(x_0, 3r) \times (t_0 - 5r^2, t_0 + r^2))}, \end{aligned} \quad (5-6)$$

where  $\psi$  is a Lipschitz function in  $\mathbb{R}^d$  and  $C$  depends only on  $d, m, \mu$  and  $A^\#$  (if  $m \geq 2$ ).

*Proof.* Let  $w_\varepsilon$  be defined by (3-2). Then  $(\partial_t + \mathcal{L}_\varepsilon)w_\varepsilon = \varepsilon \operatorname{div}(F_\varepsilon)$  in  $Q_{2r}(x_0, t_0)$ , where  $F_\varepsilon$  is given by (3-4). It follows by Theorem 2.5 that

$$\|w_\varepsilon\|_{L^\infty(Q_r(x_0, t_0))} \leq C \left\{ \left( \int_{Q_{2r}(x_0, t_0)} |w_\varepsilon|^2 \right)^{1/2} + \varepsilon r \left( \int_{Q_{2r}(x_0, t_0)} |F_\varepsilon|^p \right)^{1/p} \right\}, \quad (5-7)$$

where  $p > d + 2$ . This leads to

$$\begin{aligned} \|u_\varepsilon - u_0\|_{L^\infty(Q_r(x_0, t_0))} &\leq C \left( \int_{Q_{2r}(x_0, t_0)} |u_\varepsilon - u_0|^2 \right)^{1/2} + C\varepsilon r \left( \int_{Q_{2r}(x_0, t_0)} |F_\varepsilon|^p \right)^{1/p} \\ &\quad + C\varepsilon \|S_\varepsilon(\nabla u_0)\|_{L^\infty(Q_{2r}(x_0, t_0))} + C\varepsilon^2 \|S_\varepsilon(\nabla^2 u_0)\|_{L^\infty(Q_{2r}(x_0, t_0))}, \end{aligned}$$

where we have used the boundedness of  $\chi$  and  $\phi$  in Lemmas 2.1 and 2.3. Hence, using  $|\psi(x) - \psi(y)| \leq 2r\|\nabla\psi\|_\infty$  for  $x, y \in B(x_0, 2r)$ , we obtain

$$\begin{aligned} \|e^\psi(u_\varepsilon - u_0)\|_{L^\infty(Q_r(x_0, t_0))} \\ \leq C e^{2r\|\nabla\psi\|_\infty} \left( \int_{Q_{2r}(x_0, t_0)} |e^\psi(u_\varepsilon - u_0)|^2 \right)^{1/2} + C\varepsilon r e^{2r\|\nabla\psi\|_\infty} \left( \int_{Q_{2r}(x_0, t_0)} |e^\psi F_\varepsilon|^p \right)^{1/p} \\ + C\varepsilon e^{2r\|\nabla\psi\|_\infty} \|e^\psi S_\varepsilon(\nabla u_0)\|_{L^\infty(Q_{2r}(x_0, t_0))} + C\varepsilon^2 e^{2r\|\nabla\psi\|_\infty} \|e^\psi S_\varepsilon(\nabla^2 u_0)\|_{L^\infty(Q_{2r}(x_0, t_0))}. \end{aligned} \quad (5-8)$$

Finally, we use [Theorem 3.5](#) to bound the second term in the right-hand side of (5-8). This yields

$$\begin{aligned} & \|e^\psi(u_\varepsilon - u_0)\|_{L^\infty(Q_r(x_0, t_0))} \\ & \leq C e^{2r\|\nabla\psi\|_\infty} \left( \int_{Q_{2r}(x_0, t_0)} |e^\psi(u_\varepsilon - u_0)|^2 \right)^{1/2} + C \varepsilon r e^{3r\|\nabla\psi\|_\infty} \left( \int_{t_0-5r^2}^{t_0+r^2} \int_{B(x_0, 3r)} \{|e^\psi \nabla^2 u_0|^p + |e^\psi \partial_t u_0|^p\} \right)^{1/p} \\ & \quad + C \varepsilon e^{2r\|\nabla\psi\|_\infty} \|e^\psi \mathcal{S}_\varepsilon(\nabla u_0)\|_{L^\infty(Q_{2r}(x_0, t_0))} + C \varepsilon^2 e^{2r\|\nabla\psi\|_\infty} \|e^\psi \mathcal{S}_\varepsilon(\nabla^2 u_0)\|_{L^\infty(Q_{2r}(x_0, t_0))}, \end{aligned}$$

where  $p > d + 2$  and we also used the assumption  $\varepsilon \leq r$ . Estimate (5-6) now follows.  $\square$

We are now in a position to give the proof of [Theorem 1.1](#).

*Proof of Theorem 1.1.* We begin by fixing  $x_0, y_0 \in \mathbb{R}^{d+1}$  and  $s_0 < t_0$ . We may assume that

$$\varepsilon < r = (t_0 - s_0)^{1/2}/100.$$

For otherwise the desired estimate (1-8) follows directly from (1-6).

For  $f \in C_0^\infty(Q_r(y_0, s_0); \mathbb{R}^m)$ , define

$$\begin{aligned} u_\varepsilon(x, t) &= \int_{-\infty}^t \int_{\mathbb{R}^d} e^{-\psi(y)} \Gamma_\varepsilon(x, t; y, s) f(y, s) dy ds, \\ u_0(x, t) &= \int_{-\infty}^t \int_{\mathbb{R}^d} e^{-\psi(y)} \Gamma_0(x, t; y, s) f(y, s) dy ds, \end{aligned}$$

where  $\psi$  is a bounded Lipschitz function in  $\mathbb{R}^d$  to be chosen later. Then

$$(\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = (\partial_t + \mathcal{L}_0)u_0 = e^{-\psi} f \quad \text{in } \mathbb{R}^{d+1}$$

and  $u_\varepsilon(x, t) = u_0(x, t) = 0$  if  $t \leq s_0$ . Let  $w_\varepsilon$  be defined by (3-2). It follows from [Lemma 5.2](#) and [Theorem 4.1](#) that

$$\int_{\mathbb{R}^d} |w_\varepsilon(x, t)|^2 e^{2\psi(x)} dx \leq C \varepsilon^2 e^{2\varepsilon\|\nabla\psi\|_\infty + \kappa(t-s_0+\varepsilon^2)\|\nabla\psi\|_\infty} \int_{s_0}^{t+\varepsilon^2} \int_{\mathbb{R}^d} |f|^2 dx ds \quad (5-9)$$

for any  $t > s_0$ .

Next, we use (5-6) to obtain

$$\begin{aligned} \|e^\psi(u_\varepsilon - u_0)\|_{L^\infty(Q_r(x_0, t_0))} & \leq C e^{3r\|\nabla\psi\|_\infty} \left( \int_{Q_{2r}(x_0, t_0)} |e^\psi w_\varepsilon|^2 \right)^{1/2} \\ & \quad + C \varepsilon r e^{3r\|\nabla\psi\|_\infty} \|e^\psi (|\nabla^2 u_0| + |\partial_t u_0|)\|_{L^\infty(B(x_0, 3r) \times (t_0-5r^2, t_0+r^2))} \\ & \quad + C \varepsilon e^{3r\|\nabla\psi\|_\infty} \|e^\psi \nabla u_0\|_{L^\infty(B(x_0, 3r) \times (t_0-5r^2, t_0+r^2))}. \end{aligned} \quad (5-10)$$

Since  $\text{supp}(f) \subset Q_r(y_0, s_0)$ , it follows from the estimate (1-7) for  $\Gamma_0(x, t; y, s)$  that

$$|\nabla^2 u_0(x, t)| + |\partial_t u_0(x, t)| + r^{-1} |\nabla u_0(x, t)| \leq C \exp\left\{-\frac{\kappa|x_0 - y_0|^2}{t_0 - s_0}\right\} \int_{Q_r(y_0, s_0)} |f e^{-\psi}| dy ds \quad (5-11)$$

for any  $x \in B(x_0, 3r)$  and  $|t - t_0| \leq 5r^2$ , where  $\kappa > 0$  depends only on  $\mu$ . Thus, by (5-10), we obtain

$$\begin{aligned} & \|e^\psi(u_\varepsilon - u_0)\|_{L^\infty(Q_r(x_0, t_0))} \\ & \leq C e^{3r\|\nabla\psi\|_\infty} \left( \int_{Q_{2r}(x_0, t_0)} |e^\psi w_\varepsilon|^2 \right)^{1/2} + \varepsilon r e^{c(|x_0 - y_0| + r)\|\nabla\psi\|_\infty} \exp\left\{-\frac{\kappa|x_0 - y_0|^2}{t_0 - s_0}\right\} \int_{Q_r(y_0, s_0)} |f| dy ds \\ & \leq C \varepsilon r e^{cr\|\nabla\psi\|_\infty} \left\{ e^{cr^2\|\nabla\psi\|_\infty^2} + e^{c|x_0 - y_0|\|\nabla\psi\|_\infty} \exp\left\{-\frac{\kappa|x_0 - y_0|^2}{t_0 - s_0}\right\} \right\} \cdot \left( \int_{Q_r(y_0, s_0)} |f|^2 \right)^{1/2}, \end{aligned} \quad (5-12)$$

where we have used (5-9) for the last step. By duality this implies

$$\begin{aligned} & \left( \int_{Q_r(y_0, s_0)} |e^{\psi(x) - \psi(y)} (\Gamma_\varepsilon(x, t; y, s) - \Gamma_0(x, t; y, s))|^2 dy ds \right)^{1/2} \\ & \leq C \varepsilon r^{-d-1} e^{cr\|\nabla\psi\|_\infty} \left\{ e^{cr^2\|\nabla\psi\|_\infty^2} + e^{c|x_0 - y_0|\|\nabla\psi\|_\infty} \exp\left\{-\frac{\kappa|x_0 - y_0|^2}{t_0 - s_0}\right\} \right\} \end{aligned} \quad (5-13)$$

for any  $(x, t) \in Q_r(x_0, t_0)$ .

To deduce the  $L^\infty$  bound for

$$e^{\psi(x) - \psi(y)} (\Gamma_\varepsilon(x, t; y, s) - \Gamma_0(x, t; y, s))$$

from its  $L^2$  bound in (5-13), we apply [Theorem 5.3](#) (with  $\psi$  replaced by  $-\psi$  and  $A$  replaced by  $\tilde{A} = \tilde{A}(y, s) = A^*(y, -s)$ ) to the functions

$$v_\varepsilon(y, s) = \Gamma_\varepsilon(x_0, t_0; y, -s) \quad \text{and} \quad v_0(y, s) = \Gamma_0(x_0, t_0; y, -s).$$

Note that  $(\partial_t + \tilde{\mathcal{L}}_\varepsilon)v_\varepsilon = (\partial_t + \tilde{\mathcal{L}}_0)v_0 = 0$  in  $B(y_0, 3r) \times (-s_0 - 5r^2, -s_0 + r^2)$ . Since  $\tilde{A}$  satisfies the same conditions as  $A$ , we obtain

$$\begin{aligned} & |e^{\psi(x_0) - \psi(y_0)} (v_\varepsilon(y_0, -s_0) - v_0(y_0, -s_0))| \\ & \leq C e^{3r\|\nabla\psi\|_\infty} \left( \int_{Q_r(y_0, -s_0)} |e^{\psi(x_0) - \psi(y)} (v_\varepsilon - v_0)|^2 dy ds \right)^{1/2} \\ & \quad + C \varepsilon r^{-d-1} e^{cr\|\nabla\psi\|_\infty} e^{\psi(x_0) - \psi(y_0)} \exp\left\{-\frac{\kappa|x_0 - y_0|^2}{t_0 - s_0}\right\} \\ & = C e^{3r\|\nabla\psi\|_\infty} \left( \int_{Q_r(y_0, s_0 + r^2)} |e^{\psi(x_0) - \psi(y)} (\Gamma_\varepsilon(x_0, t_0; y, s) - \Gamma_0(x_0, t_0; y, s))|^2 dy ds \right)^{1/2} \\ & \quad + C \varepsilon r^{-d-1} e^{cr\|\nabla\psi\|_\infty} e^{\psi(x_0) - \psi(y_0)} \exp\left\{-\frac{\kappa|x_0 - y_0|^2}{t_0 - s_0}\right\} \\ & \leq C \varepsilon r^{-d-1} e^{cr\|\nabla\psi\|_\infty} \left\{ e^{cr^2\|\nabla\psi\|_\infty^2} + e^{c|x_0 - y_0|\|\nabla\psi\|_\infty} \exp\left\{-\frac{\kappa|x_0 - y_0|^2}{t_0 - s_0}\right\} \right\}, \end{aligned} \quad (5-14)$$

where we have used (5-13) for the last inequality.

Finally, as in [[Hofmann and Kim 2004](#); [Cho et al. 2008](#)], we let  $\psi(y) = \gamma\psi_0(|y - y_0|)$ , where  $\gamma \geq 0$  is to be chosen,  $\psi_0(\rho) = \rho$  if  $\rho \leq |x_0 - y_0|$ , and  $\psi_0(\rho) = |x_0 - y_0|$  if  $\rho > |x_0 - y_0|$ . Note that  $\|\nabla\psi\|_\infty = \gamma$

and  $\psi(x_0) - \psi(y_0) = \gamma|x_0 - y_0|$ . It follows from (5-14) that

$$|\Gamma_\varepsilon(x_0, t_0; y_0, s_0) - \Gamma_0(x_0, t_0; y_0, s_0)| \leq C\varepsilon r^{-d-1} e^{-\gamma|x_0 - y_0| + c\gamma\sqrt{t_0 - s_0}} \left\{ e^{c\gamma^2(t_0 - s_0)} + e^{c\gamma|x_0 - y_0|} \exp\left\{-\frac{\kappa|x_0 - y_0|^2}{t_0 - s_0}\right\} \right\}, \quad (5-15)$$

where  $c > 0$  depends at most on  $\mu$ . If  $|x_0 - y_0| \leq 2c\sqrt{t_0 - s_0}$ , we may simply choose  $\gamma = 0$ . This gives

$$|\Gamma_\varepsilon(x_0, t_0; y_0, s_0) - \Gamma_0(x_0, t_0; y_0, s_0)| \leq C\varepsilon r^{-d-1} \leq C\varepsilon (t_0 - s_0)^{-(d+1)/2} \exp\left\{-\frac{\kappa|x_0 - y_0|^2}{t_0 - s_0}\right\}.$$

If  $|x_0 - y_0| > 2c\sqrt{t_0 - s_0}$ , we choose

$$\gamma = \frac{\delta|x_0 - y_0|}{t_0 - s_0}.$$

Note that

$$\begin{aligned} -\gamma|x_0 - y_0| + c\gamma\sqrt{t_0 - s_0} + c\gamma^2(t_0 - s_0) &= -\delta(1 - c\delta)\frac{|x_0 - y_0|^2}{t_0 - s_0} + c\delta\frac{|x_0 - y_0|}{\sqrt{t_0 - s_0}} \\ &\leq \left\{-\delta(1 - c\delta) + \frac{1}{2}\delta\right\}\frac{|x_0 - y_0|^2}{t_0 - s_0} \leq \frac{-\delta|x_0 - y_0|^2}{4(t_0 - s_0)} \end{aligned}$$

if  $\delta \leq \frac{1}{4}c^{-1}$ . Also, observe that

$$c\gamma\sqrt{t_0 - s_0} + c\gamma|x_0 - y_0| - \frac{\kappa|x_0 - y_0|^2}{t_0 - s_0} \leq \left\{\frac{1}{2}\delta + c\delta - \kappa\right\}\frac{|x_0 - y_0|^2}{t_0 - s_0} \leq -\frac{\kappa|x_0 - y_0|^2}{2(t_0 - s_0)},$$

if  $\delta \leq \frac{1}{2}(c + \frac{1}{2})^{-1}\kappa$ . Recall that  $r = (100)^{-1}\sqrt{t_0 - s_0}$ . As a result, we have proved that there exists  $\kappa_1 > 0$ , depending only on  $\mu$ , such that

$$|\Gamma_\varepsilon(x_0, t_0; y_0, s_0) - \Gamma_0(x_0, t_0; y_0, s_0)| \leq \frac{C\varepsilon}{(t_0 - s_0)^{(d+1)/2}} \exp\left\{-\frac{\kappa_1|x_0 - y_0|^2}{t_0 - s_0}\right\}.$$

This completes the proof of [Theorem 1.1](#). □

## 6. Proof of [Theorem 1.2](#)

Define

$$\|F\|_{C^{\lambda,0}(K)} = \sup\left\{\frac{|F(x, t) - F(y, t)|}{|x - y|^\lambda} : (x, t), (y, t) \in K \text{ and } x \neq y\right\}.$$

The proof of [Theorem 1.2](#) relies on the following Lipschitz estimate.

**Theorem 6.1.** *Assume that  $A$  satisfies conditions (1-2), (1-3) and (1-9). Suppose that*

$$(\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = (\partial_t + \mathcal{L}_0)u_0$$

in  $Q_{2r}(x_0, t_0)$  for some  $(x_0, t_0) \in \mathbb{R}^{d+1}$  and  $\varepsilon \leq r < \infty$ . Then

$$\begin{aligned} \|\nabla u_\varepsilon - \nabla u_0 - (\nabla \chi)^\varepsilon \nabla u_0\|_{L^\infty(Q_r(x_0, t_0))} &\leq Cr^{-1} \left( \int_{Q_{2r}(x_0, t_0)} |u_\varepsilon - u_0|^2 \right)^{1/2} + C\varepsilon r^{-1} \|\nabla u_0\|_{L^\infty(Q_{2r}(x_0, t_0))} \\ &\quad + C\varepsilon \ln[\varepsilon^{-1}r + 2] \|\nabla^2 u_0\| + \varepsilon \|\partial_t \nabla u_0\| + \varepsilon \|\nabla^3 u_0\|_{L^\infty(Q_{2r}(x_0, t_0))} \\ &\quad + C\varepsilon^{1+\lambda} \|\nabla^2 u_0\| + \varepsilon \|\partial_t \nabla u_0\| + \varepsilon \|\nabla^3 u_0\|_{C^{\lambda,0}(Q_{2r}(x_0, t_0))}, \end{aligned} \quad (6-1)$$

where  $C$  depends only on  $d, m, \mu$  and  $(\lambda, \tau)$  in (1-9).

*Proof.* Let

$$w_\varepsilon = u_\varepsilon - u_0 - \varepsilon \chi_j^\varepsilon \frac{\partial u_0}{\partial x_j} - \varepsilon^2 \phi_{(d+1)ij}^\varepsilon \frac{\partial^2 u_0}{\partial x_i \partial x_j}, \quad (6-2)$$

where  $\chi_j^\varepsilon(x, t) = \chi_j(x/\varepsilon, t/\varepsilon^2)$  and  $\phi_{(d+1)ij}^\varepsilon(x, t) = \phi_{(d+1)ij}(x/\varepsilon, t/\varepsilon^2)$ . It follows by [Proposition 3.1](#) that  $(\partial_t + \mathcal{L}_\varepsilon)w_\varepsilon = \varepsilon \operatorname{div}(F_\varepsilon)$  in  $Q_{2r}(x_0, t_0)$ , where  $F_\varepsilon$  is given by (3-4) with  $S_\varepsilon$  being the identity operator. Choose a cut-off function  $\varphi \in C_0^\infty(\mathbb{R}^{d+1})$  such that

$$\begin{aligned} 0 &\leq \varphi \leq 1, \quad \varphi = 1 \quad \text{in } Q_{3r/2}(x_0, t_0), \\ \varphi(x, t) &= 0 \quad \text{if } |x - x_0| \geq \frac{7}{4}r \text{ or } t < t_0 - \left(\frac{7}{4}r\right)^2, \\ |\nabla \varphi| &\leq Cr^{-1}, \quad |\nabla^2 \varphi| + |\partial_t \varphi| \leq Cr^{-2}. \end{aligned}$$

Using

$$(\partial_t + \mathcal{L}_\varepsilon)(\varphi w_\varepsilon) = (\partial_t \varphi)w_\varepsilon + \varepsilon \operatorname{div}(\varphi F_\varepsilon) - \varepsilon F_\varepsilon(\nabla \varphi) - \operatorname{div}(A^\varepsilon(\nabla \varphi)w_\varepsilon) - A^\varepsilon \nabla w_\varepsilon(\nabla \varphi),$$

where  $A^\varepsilon(x, t) = A(x/\varepsilon, t/\varepsilon^2)$ , we may deduce that for any  $(x, t) \in Q_r(x_0, t_0)$ ,

$$\begin{aligned} w_\varepsilon(x, t) &= \int_{-\infty}^t \int_{\mathbb{R}^d} \Gamma_\varepsilon(x, t; y, s) \{(\partial_s \varphi)w_\varepsilon - \varepsilon F_\varepsilon(\nabla \varphi) - A^\varepsilon \nabla w_\varepsilon(\nabla \varphi)\} dy ds \\ &\quad - \int_{-\infty}^t \int_{\mathbb{R}^d} \nabla_y \Gamma_\varepsilon(x, t; y, s) \{\varepsilon \varphi F_\varepsilon - A^\varepsilon(\nabla \varphi)w_\varepsilon\} dy ds \\ &= I(x, t) + J(x, t), \end{aligned}$$

where

$$J(x, t) = -\varepsilon \int_{-\infty}^t \int_{\mathbb{R}^d} \nabla_y \Gamma_\varepsilon(x, t; y, s) \varphi(y, s) F_\varepsilon(y, s) dy ds.$$

Since  $\varphi = 1$  in  $Q_{3r/2}(x_0, t_0)$ , we see that for  $(x, t) \in Q_r(x_0, t_0)$ ,

$$\begin{aligned} |\nabla I(x, t)| &\leq C \int_{-\infty}^t \int_{\mathbb{R}^d} |\nabla_x \Gamma_\varepsilon(x, t; y, s)| \{|\partial_s \varphi| |w_\varepsilon| + \varepsilon |F_\varepsilon| |\nabla \varphi| + |\nabla w_\varepsilon| |\nabla \varphi|\} dy ds \\ &\quad + C \int_{-\infty}^t \int_{\mathbb{R}^d} |\nabla_x \nabla_y \Gamma_\varepsilon(x, t; y, s)| |\nabla \varphi| |w_\varepsilon| dy ds \\ &\leq C \left\{ \frac{1}{r} \int_{Q_{2r}(x_0, t_0)} |w_\varepsilon| + \varepsilon \int_{Q_{2r}(x_0, t_0)} |F_\varepsilon| + \int_{Q_{7r/4}(x_0, t_0)} |\nabla w_\varepsilon| \right\} \\ &\leq C \left\{ \frac{1}{r} \left( \int_{Q_{2r}(x_0, t_0)} |w_\varepsilon|^2 \right)^{1/2} + \varepsilon \left( \int_{Q_{2r}(x_0, t_0)} |F_\varepsilon|^2 \right)^{1/2} \right\}, \end{aligned}$$

where we have used (parabolic) Caccioppoli's inequality for the last step. In view of (3-4) with  $S_\varepsilon$  being the identity operator,

$$|F_\varepsilon| \leq C\{|\nabla^2 u_0| + \varepsilon|\partial_t \nabla u_0| + \varepsilon|\nabla^3 u_0|\},$$

where we have used the boundedness of  $\nabla\phi$  (see Remark 2.4). It follows that  $\|\nabla I\|_{L^\infty(Q_r(x_0, t_0))}$  is bounded by the right-hand side of (6-1).

Finally, to estimate  $J(x, t)$ , we write

$$J(x, t) = -\varepsilon \int_{-\infty}^t \int_{\mathbb{R}^d} \nabla_y \{\Gamma_\varepsilon(x, t; y, s)\varphi(y, s)\} (F_\varepsilon(y, s) - F_\varepsilon(x, s)) dy ds + \varepsilon \int_{-\infty}^t \int_{\mathbb{R}^d} \Gamma_\varepsilon(x, t; y, s) (\nabla\varphi)(y, s) F_\varepsilon(y, s) dy ds.$$

It follows that for  $(x, t) \in Q_r(x_0, t_0)$

$$\begin{aligned} |\nabla J(x, t)| &\leq \varepsilon \int_{Q_{2r}(x_0, t_0)} |\nabla_x \nabla_y \{\Gamma_\varepsilon(x, t; y, s)\varphi(y, s)\}| |F_\varepsilon(y, s) - F_\varepsilon(x, s)| dy ds \\ &\quad + \varepsilon \int_{Q_{2r}(x_0, t_0)} |\nabla_x \Gamma_\varepsilon(x, t; y, s)| |\nabla\varphi(y, s)| |F_\varepsilon(y, s)| dy ds \\ &\leq C\varepsilon \int_{Q_{2r}(x_0, t_0)} \frac{|F_\varepsilon(y, s) - F_\varepsilon(x, s)|}{(|x - y| + |t - s|^{1/2})^{d+2}} dy ds + C\varepsilon \int_{Q_{2r}(x_0, t_0)} |F_\varepsilon|. \end{aligned} \tag{6-3}$$

To bound the first integral in the right-hand side of (6-3), we subdivide the domain  $Q_{2r}(x_0, t_0)$  into  $Q_\varepsilon(x, t)$  and  $Q_{2r}(x_0, t_0) \setminus Q_\varepsilon(x, t)$ . On  $Q_{2r}(x_0, t_0) \setminus Q_\varepsilon(x, t)$ , we use the bound

$$|F_\varepsilon(y, s) - F_\varepsilon(x, s)| \leq 2\|F_\varepsilon\|_{L^\infty(Q_{2r}(x_0, t_0))},$$

while for  $Q_\varepsilon(x, t)$ , we use

$$|F_\varepsilon(y, s) - F_\varepsilon(x, s)| \leq |x - y|^\lambda \|F\|_{C^{\lambda,0}(Q_{2r}(x_0, t_0))}.$$

This leads to

$$\begin{aligned} |\nabla J(x, t)| &\leq C\varepsilon \ln[\varepsilon^{-1}r + 1] \|F_\varepsilon\|_{L^\infty(Q_{2r}(x_0, t_0))} + C\varepsilon^{1+\lambda} \|F_\varepsilon\|_{C^{\lambda,0}(Q_{2r}(x_0, t_0))} \\ &\leq C\varepsilon \ln[\varepsilon^{-1}r + 1] \{ \|\nabla^2 u_0\| + \varepsilon|\partial_t \nabla u_0| + \varepsilon|\nabla^3 u_0\| \}_{L^\infty(Q_{2r}(x_0, t_0))} \\ &\quad + C\varepsilon^{1+\lambda} \{ \|\nabla^2 u_0\| + \varepsilon|\partial_t \nabla u_0| + \varepsilon|\nabla^3 u_0\| \}_{C^{\lambda,0}(Q_{2r}(x_0, t_0))}. \end{aligned}$$

Thus, in view of the estimate for  $\nabla I(x, t)$ , we have proved that  $\|\nabla w_\varepsilon\|_{L^\infty(Q_r(x_0, t_0))}$  is bounded by the right-hand side of (6-1). Since

$$\|\nabla w_\varepsilon - \{\nabla u_\varepsilon - \nabla u_0 - (\nabla\chi)^\varepsilon \nabla u_0\}\|_{L^\infty(Q_r(x_0, t_0))} \leq C\varepsilon \{ \|\nabla^2 u_0\| + \varepsilon|\nabla^3 u_0\| \}_{L^\infty(Q_r(x_0, t_0))},$$

the estimate (6-1) follows. □

To prove Theorem 1.2, we fix  $x_0, y_0 \in \mathbb{R}^d$  and  $s_0 < t_0$ . We may assume that  $\varepsilon < (t_0 - s_0)/8$ . For otherwise the estimate (1-10) follows directly from (2-22). We apply Theorem 6.1 to the functions

$$u_\varepsilon(x, t) = \Gamma_\varepsilon(x, t; y_0, s_0) \quad \text{and} \quad u_0(x, t) = \Gamma_0(x, t; y_0, s_0)$$

in  $Q_{2r}(x_0, t_0)$ , where  $r = (t_0 - s_0)/8$ . Note that  $(\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = (\partial_t + \mathcal{L}_0)u_0 = 0$  in  $Q_{4r}(x_0, t_0)$ . To bound the first term in the right-hand side of (6-1), we use the estimate (1-8) in Theorem 1.1. All other terms in the right-hand side of (6-1) may be handled easily by using the estimates (1-7) for  $\Gamma_0(x, t; y, s)$ . We leave the details to the reader.

### 7. Proof of Theorem 1.3

To prove Theorem 1.3, we fix  $x_0, y_0 \in \mathbb{R}^d$  and  $s_0 < t_0$ . As before, we may assume that  $\varepsilon < (t_0 - s_0)/8$ , for otherwise the estimate (1-15) follows directly from (2-23).

Let  $r = (t_0 - s_0)/8$ . Fix  $1 \leq j \leq d$  and  $1 \leq \beta \leq m$ . We apply Theorem 6.1 to the functions  $u_\varepsilon = (u_\varepsilon^\alpha)$  and  $u_0 = (u_0^\alpha)$  in  $Q_{2r}(x_0, t_0)$ , where

$$u_\varepsilon^\alpha(x, t) = \frac{\partial}{\partial y_j} \{ \Gamma_\varepsilon^{\alpha\beta} \}(x, t; y_0, s_0),$$

$$u_0^\alpha(x, t) = \frac{\partial}{\partial y_\ell} \{ \Gamma_0^{\alpha\sigma} \}(x, t; y_0, s_0) \cdot \left\{ \delta^{\beta\sigma} \delta_{j\ell} + \frac{\partial}{\partial y_j} (\tilde{\chi}_\ell^{\beta\sigma})(y_0/\varepsilon, -s_0/\varepsilon^2) \right\},$$

where  $\tilde{\chi}$  denotes the correctors for  $\partial_t + \tilde{\mathcal{L}}_\varepsilon$ . Observe that  $(\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = (\partial_t + \mathcal{L}_0)u_0 = 0$  in  $Q_{4r}(x_0, t_0)$ . To bound the first term in the right-hand side of (6-1), we use the estimate (1-14). As in the proof of Theorem 1.1, all other terms in the right-hand side of (6-1) may be handled readily by using estimate (1-7) for  $\Gamma_0(x, t; y, s)$ .

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