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We prove some Bernstein-type theorems for a class of stationary points of the Alt–Caffarelli functional in $\mathbb{R}^2$ and $\mathbb{R}^3$ arising as limits of the singular perturbation problem

$$\begin{cases} \triangle u_\varepsilon(x) = \beta_\varepsilon(u_\varepsilon) & \text{in } B_1, \\ |u_\varepsilon| \le 1 & \text{in } B_1, \end{cases}$$

in the unit ball B_1 as $\varepsilon \to 0$. Here $\beta_\varepsilon(t) = (1/\varepsilon)\beta(t/\varepsilon) \ge 0$, $\beta \in C_0^\infty[0,1]$, $\int_0^1 \beta(t)\,dt = M > 0$, is an approximation of the Dirac measure and $\varepsilon > 0$. The limit functions $u = \lim_{\varepsilon_j \to 0} u_{\varepsilon_j}$ of uniformly converging sequences $\{u_{\varepsilon_j}\}$ solve a Bernoulli-type free boundary problem in some weak sense. Our approach has two novelties: First we develop a hybrid method for stratification of the free boundary $\partial\{u_0 > 0\}$ of blow-up solutions which combines some ideas and techniques of viscosity and variational theory. An important tool we use is a new monotonicity formula for the solutions u_ε based on a computation of J. Spruck. It implies that any blow-up u_0 of u either vanishes identically or is a homogeneous function of degree 1, that is, $u_0 = rg(\sigma)$, $\sigma \in \mathbb{S}^{N-1}$, in spherical coordinates (r, θ). In particular, this implies that in two dimensions the singular set is empty at the nondegenerate points, and in three dimensions the singular set of u_0 is at most a singleton. Second, we show that the spherical part g is the support function (in Minkowski's sense) of some capillary surface contained in the sphere of radius $\sqrt{2M}$. In particular, we show that $\nabla u_0 : \mathbb{S}^2 \to \mathbb{R}^3$ is an almost conformal and minimal immersion and the singular Alt–Caffarelli example corresponds to a piece of catenoid which is a unique ring-type stationary minimal surface determined by the support function g.

1. Introduction

In this paper we study the singular perturbation problem

$$\begin{cases} \triangle u_\varepsilon(x) = \beta_\varepsilon(u_\varepsilon) & \text{in } B_1, \\ |u_\varepsilon| \le 1 & \text{in } B_1, \end{cases} \tag{$\mathcal{P}_\varepsilon$}$$

where $\varepsilon > 0$ is a small parameter,

$$\begin{cases} \beta_\varepsilon(t) = (1/\varepsilon)\beta(t/\varepsilon), \\ \beta(t) \ge 0, \quad \operatorname{supp}\beta \subset [0,1], \\ \int_0^1 \beta(t)\,dt = M > 0 \end{cases} \tag{1-1}$$

is an approximation of the Dirac measure, and $B_1 \subset \mathbb{R}^N$ is the unit ball centered at the origin. It is well known that $(\mathcal{P}_\varepsilon)$ models propagation of equidiffusional premixed flames with high activation of energy

[Caffarelli 1995]. Heuristically, the limit $u_0 = \lim_{\varepsilon_j \to 0} u_{\varepsilon_j}$ (for a suitable sequence $\varepsilon_j \to 0$) solves a Bernoulli-type free boundary problem with the free boundary condition

$$|\nabla u^+|^2 - |\nabla u^-|^2 = 2M.$$

If the functions $\{u_\varepsilon\}$ are also minimizers of

$$J_\varepsilon[u_\varepsilon] = \int_\Omega \frac{|\nabla u_\varepsilon|^2}{2} + \mathcal{B}\left(\frac{u_\varepsilon}{\varepsilon}\right), \quad \mathcal{B}(t) = \int_0^t \beta(s)\, ds, \qquad (1\text{-}2)$$

then the limits of $\{u_\varepsilon\}$ inherit the generic features of minimizers (e.g., nondegeneracy, rectifiability of $\partial\{u > 0\}$, etc.). Consequently, the limits of uniformly converging sequences $\{u_{\varepsilon_j}\}$ as $\varepsilon_j \to 0$ are minimizers of the Alt–Caffarelli functional $J[u] = \int_{B_1} |\nabla u|^2 + 2M\chi_{\{u>0\}}$. It is known that the singular set of minimizers is empty in dimensions 2, 3 and 4; see [Alt and Caffarelli 1981; Caffarelli et al. 2004; Jerison and Savin 2015]. However, if u_ε is not a minimizer then the analysis of the limits of u presents a more delicate problem. The main difficulty in carrying out such analysis is that the free boundary may contain degenerate points [Weiss 2003].

This paper is devoted to the study of the blow-ups of the limits of the singular perturbation problem $(\mathcal{P}_\varepsilon)$ and establishes a new and direct connection with *minimal surfaces*. In particular, we show that every blow-up of a limit function $u = \lim_{\varepsilon_j \to 0} u_{\varepsilon_j}$ in $\mathbb{R}^3$ (for an appropriate sequence ε_j) defines an almost conformal and minimal immersion which is perpendicular to the sphere of radius $\sqrt{2M}$, where $M = \int_0^1 \beta(t)\, dt$. In other words, one obtains a capillary surface inside the sphere of radius $\sqrt{2M}$.

Our first result is:

Theorem A. *Let $u_{\varepsilon_j} \to u$ locally uniformly in B_1 for some subsequence ε_j. Then any blow-up of u at a free boundary point $x_0 \in \partial\{u > 0\}$ is either identically zero or a homogeneous function of degree 1. In particular, if $N = 2$ and u is not degenerate at $x_0 \in \partial\{u > 0\}$ then every blow-up of u at x_0 must be one of the following functions (after some rotation of coordinates):*

(1) $\sqrt{2M}\, x_1^+$, *a half-plane solution provided that there is a measure-theoretic normal at x_0,*

(2) *a wedge* $\alpha|x_1|$, $0 < \alpha \le \sqrt{2M}$,

(3) *a two-plane solution* $\alpha x_1^+ - \beta x_1^-$, $\alpha^2 - \beta^2 = 2M$, $\alpha, \beta > 0$.

In order to prove Theorem A we will introduce a monotone quantity based on a computation of Joel Spruck [1983]. From Theorem A it follows that in $\mathbb{R}^2$ the blow-up limits at nondegenerate free boundary points can be explicitly computed. It is worthwhile to note that the minimizers of

$$J[u] = \int_{B_1} |\nabla u|^2 + 2M\chi_{\{u>0\}} \qquad (1\text{-}3)$$

are nondegenerate; i.e., for each subdomain $\Omega' \Subset B_1$ there is a constant $c_0 > 0$ depending on $\mathrm{dist}(\partial B_1, \partial\Omega')$, N, M, such that

$$\sup_{B_r(x_0)} u^+ \ge c_0 r \quad \text{for all } x_0 \in \partial\{u > 0\} \cap \Omega', \ B_r(x_0) \subset B_1. \qquad (1\text{-}4)$$

However, if u_ε is *any* solution of ($\mathcal{P}_\varepsilon$) then nondegeneracy may not be true. There is a sufficient condition [Caffarelli et al. 1997, Theorem 6.3] that implies (1-4).

Some well-known examples demonstrate rather strikingly that for the stationary case there are wedge-like global solutions for which the measure-theoretic boundary of $\{u > 0\}$ is empty. This is impossible for minimizers. In fact, the zero set of a minimizer has uniformly positive Lebesgue density. In this respect Theorem A only states that if u is nondegenerate at x_0 then the blow-up is a nontrivial cone.

The existence of wedge solutions, see [Caffarelli et al. 1997, Remark 5.1], suggests that some further assumptions are needed to formulate the free boundary condition. For instance, one may assume that the upper Lebesgue density at $x \in \partial\{u > 0\}$ satisfies $\Theta^*(x, \{u > 0\}) < 1$; i.e., the upper density measure is not covering the full ball. We emphasize that for some solutions the topological and measure-theoretic boundaries may not coincide. Our next result addresses the degeneracy and wedge-formation in $\mathbb{R}^3$ of blow-ups at free boundary points.

Theorem B. *Suppose $N = 3$. Let $u \geq 0$ be a limit of some uniformly converging sequence $\{u_{\varepsilon_j}\}$ solving ($\mathcal{P}_\varepsilon$) such that u is nondegenerate at $y_0 \in \partial\{u > 0\}$. Let u_0 be a blow-up of u at y_0. If $\mathfrak{C}$ is a component of $\partial\{u_0 > 0\}$ such that the measure-theoretic boundary of $\{u_0 > 0\}$ in $\mathfrak{C}$ is nonempty then*

 (1) *all points of $\mathfrak{C}$ are nondegenerate,*

 (2) *$\mathfrak{C}$ is a subset of the measure-theoretic boundary of $\{u_0 > 0\}$,*

 (3) *$\mathfrak{C} \setminus \{0\}$ is smooth.*

In particular in $\mathbb{R}^3$ the singular set of $\partial\{u_0 > 0\}$ is at most a singleton.

Theorem B implies that the reduced boundary propagates instantaneously in the components of $\partial\{u_0 > 0\}$. Our last result sheds some new light on the characterization of the blow-ups as minimal surfaces inside spheres with contact angle $\frac{\pi}{2}$.

Theorem C. *Let u_0 be as in Theorem B and $u_0 = rg(\sigma)$, $\sigma \in \mathbb{S}^2$, in spherical coordinates. Then the parametrization $\mathfrak{X}(\sigma) = \sigma g(\sigma) + \nabla_{\mathbb{S}^2} g(\sigma)$ defines an almost conformal and minimal immersion. If $\{g > 0\}$ is homeomorphic to a disk then u_0 is a half-plane solution $\sqrt{2M}\, x_1^+$. If $\{g > 0\}$ is homeomorphic to a ring then the only singular cone is the Alt–Caffarelli catenoid.*

Observe that $\triangle u_0 = 0$ implies that the spherical part g satisfies the following equation on the sphere:

$$\triangle_{\mathbb{S}^{N-1}} g + (N-1)g = 0,$$

where $\triangle_{\mathbb{S}^{N-1}}$ is the Laplace–Beltrami operator. If we regard g as the support function of some embedded hypersurface $\mathcal{M}$ then the matrix $[\nabla_{ij} g + \delta_{ij} g]^{-1}$ gives the Weingarten mapping and its eigenvalues are the principal curvatures $k_1, \ldots, k_{N-1}$ of $\mathcal{M}$. If $N = 3$ then we have

$$0 = \triangle_{\mathbb{S}^2} g + 2g = \operatorname{trace}[\nabla_{ij} g + \delta_{ij} g] = \frac{1}{k_1} + \frac{1}{k_2} = \frac{k_1 + k_2}{k_1 k_2},$$

implying that the mean curvature is zero at the points where the Gauss curvature $k_1 k_2$ does not vanish. This is how the minimal surfaces enter into the game. One of the main obstacles is to show that the

surface parametrized by $\mathcal{X}(\sigma) = \nabla u_0(\sigma)$ is embedded. Then the classification for the disk-type domains $\{g > 0\}$ follows from a result of [Nitsche 1985]. To prove the last statement of Theorem C we will use the moving plane method. It is worthwhile to point out that the results of this paper can be extended to other classes of stationary points. For instance, the weak solutions introduced in [Alt and Caffarelli 1981] can be analyzed in similar way provided that the zero set has uniformly positive Lebesgue density at free boundary points in order to guarantee that the class of weak solution is closed with respect to blow-ups; see [Alt and Caffarelli 1981, Example 5.8].

Related works. In [Hauswirth et al. 2011] L. Hauswirth, F. Hélein, and F. Pacard considered the overdetermined problem

$$\begin{cases} \triangle u(x) = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u(x) = 0, \quad |\nabla u| = 1 & \text{on } \partial\Omega, \end{cases} \tag{1-5}$$

where Ω is a smooth domain and the boundary conditions are satisfied in the classical sense. A domain Ω admitting a solution u to (1-5) is called exceptional. Note that every nonnegative smooth solution of the limiting singular perturbation problem solves (1-5) with $M = \frac{1}{2}$. In [Hauswirth et al. 2011] the authors constructed a number of examples of exceptional domains and proposed to classify them. In particular, they proved that if $\Omega \subset \mathbb{R}^2$ is conformal to a half-plane such that u is strictly monotone in one fixed direction then Ω is a half-space [Hauswirth et al. 2011, Proposition 6.1]. However the general problem remained open.

Later M. Traizet [2014, Proposition 1] showed that the smoothness assumption can be relaxed, namely if $\Omega \subset \mathbb{R}^2$ has C^0 boundary and the boundary conditions are still satisfied in the classical sense then Ω is real-analytic. Under various topological conditions on the two-dimensional domain $\Omega \subset \mathbb{R}^2$ (such as finite connectivity and periodicity), M. Traizet classified the possible exceptional domains. One of his remarkable results is that from Ω one can construct a *complete* minimal surface using the Weierstrass representation formula [Traizet 2014, Theorem 9]. Another classification result in $\mathbb{R}^2$, under stronger topological hypotheses than in [Traizet 2014], was given by D. Khavinson, E. Lundberg and R. Teodorescu [Khavinson et al. 2013]. Moreover, their results in the simply connected case are stronger because unlike M. Traizet they do not assume the finite connectivity (i.e., $\partial\Omega$ has finite number of components). As opposed to these results (1) we do not assume any regularity of the free boundary (which plays the role of $\partial\Omega$ in (1-5)), (2) the Neumann condition is not satisfied in the classical sense, (3) the minimal surface we construct in Theorem C is *not complete* and it is a capillary surface inside sphere, and (4) our techniques do not impose any restriction on the dimension. Note that in [Hauswirth et al. 2011] the authors suggested to study more general classes of exceptional domains: if (M, g) is an m-dimensional Riemannian manifold admitting a harmonic function with zero Dirichlet and constant Neumann boundary data then M is called exceptional and u a roof function. In this context Theorem C provides a way of constructing a roof function on the sphere from the blow-ups of stationary points of the Alt–Caffarelli functional.

One may consider higher-order critical points as well, such as mountain passes (which are, in fact, minimizers over some subspace of admissible functions), for which one has nondegeneracy and nontrivial

Lebesgue density properties [Jerison and Perera 2018, Propositions 1.7–5.1]. Observe that neither of these properties is available for our solutions as Theorem 6.3 and Remark 5.1 in [Caffarelli et al. 1997] indicate, and in the present work we do not impose any additional assumptions on our stationary points of this kind.

It seems that the only result in high dimensions that appears in [Hauswirth et al. 2011; Khavinson et al. 2013; Traizet 2014] states that if the complement of Ω is connected and has $C^{2,\alpha}$ boundary, then Ω is the exterior of a ball [Khavinson et al. 2013, Theorem 7.1]. The restriction $\Omega \subset \mathbb{R}^2$ is because the authors have mainly used the techniques from complex analysis. Our approach does not have this restriction since our main tool is the representation of the solution in terms of the Minkowski support function. We remark that using our method in high dimensions we can construct a surface $\mathcal{M}$ inside the sphere of radius $\sqrt{2M}$ such that the sum of its principal radii of curvature is zero, and $\mathcal{M}$ is transversal to the sphere.

Finally, we point out that our approach may lead to a new characterization of global minimizers in $\mathbb{R}^3$ [Caffarelli et al. 2004]. Indeed, [Ros and Vergasta 1995, Theorem 6] implies that the capillary surface $\mathcal{M}$ in Theorem C associated with the blow-up limit must be totally geodesic (i.e., the second fundamental form is identically zero). Consequently, the blow-up must be the half-plane solution.

The paper is organized as follows: In Section 2 we set up some basic notation which will be used throughout the paper. Section 3 is devoted to the study of a new monotone quantity $s(x_0, u, r)$. This interesting quantity is derived from a computation of Spruck [1983]. Among other things, properties of s imply that every blow-up of u is either a homogeneous function of degree 1 or identically zero. Section 4 contains the proof of Theorem A. In Section 5 we develop a new method of stratification of the free boundary points and prove Theorem B. Section 6 contains the proof of Theorem C. For the convenience of the reader, in the Appendix we repeat the relevant material from [Caffarelli et al. 1997] without proofs.

2. Notation

Throughout the paper N will denote the spatial dimension. $B_r(x_0) = \{x \in \mathbb{R}^N : |x - x_0| < r\}$ denotes the open ball of radius $r > 0$ centered at $x_0 \in \mathbb{R}^N$. The s-dimensional Hausdorff measure is denoted by $\mathcal{H}^s$, the unit sphere by $\mathbb{S}^{N-1} \subset \mathbb{R}^N$, and the characteristic function of the set D by χ_D. We also let

$$M = \int_0^1 \beta(t)\, dt.$$

Sometimes we will set $x = (x_1, x')$, where $x' \in \mathbb{R}^{N-1}$. For a given function v, we will define $v^+ = \max(0, v)$ and $v^- = \max(0, -v)$. Finally, we say that $v \in C^{0,1}_{\text{loc}}(\mathcal{D})$ if for every $\mathcal{D}' \Subset \mathcal{D}$, there is a constant $L(\mathcal{D}')$ such that

$$|v(x) - v(y)| \le L(\mathcal{D}')|x - y| \quad \text{for all } x, y \in \mathcal{D}.$$

If $v \in C^{0,1}_{\text{loc}}(\mathcal{D})$ then we say that v is locally Lipschitz continuous in $\mathcal{D}$. For $x = (x_1, \ldots, x_N)$ and fixed $x_0 \in \mathbb{R}^N$ we denote by $(x - x_0)_1^+$ the positive part of the first coordinate of $x - x_0$. If $u(x_0) = 0$ then $(u(x))_r = u(x_0 + rx)/r$, $r > 0$, denotes the scaled function at x_0. For given $r_j \to 0$ the sequence $(u(x))_{r_j}$ is called a blow-up sequence and its limit u_0 a blow-up of u at x_0.

3. Monotonicity formula of Spruck

It is convenient to work with a weaker definition of nondegeneracy which only ensures that the blow-up does not vanish identically.

Definition 3.1. We say that u is degenerate at $x_0 \in \partial\{u > 0\}$ if $\liminf_{r \to 0} (1/r) \fint_{B_r(x_0)} u^+ = 0$.

Observe that $u^+(x) = o(|x - x_0|)$ near the degenerate point x_0 because u^+ is subharmonic.

It is known that the solutions of $(\mathcal{P}_\varepsilon)$ are locally Lipschitz continuous; see the Appendix, Proposition A.1. Consequently, there is a subsequence $\varepsilon_j \to 0$ such that $u_{\varepsilon_j} \to u$ locally uniformly. Furthermore, u is a stationary point of the Alt–Caffarelli problem in some weak sense and the blow-up of u can be approximated by some scaled family of solutions to $(\mathcal{P}_\varepsilon)$; see the Appendix, Propositions A.5 and A.6.

Proposition 3.1. *Let u be a limit of some sequence u_{ε_j} as in Proposition A.2. Then any blow-up of u at a nondegenerate point is a homogeneous function of degree* 1.

Proof. To fix the ideas we assume that $0 \in \partial\{u > 0\}$ is a nondegenerate point. We begin with writing the Laplacian in polar coordinates

$$\triangle u = u_{rr} + \frac{N-1}{r} u_r + \frac{1}{r^2} \triangle_{\mathbb{S}^{N-1}} u \tag{3-1}$$

and then introducing the auxiliary function

$$v(t, \sigma) = \frac{u(r, \sigma)}{r}, \quad r = e^{-t}. \tag{3-2}$$

A straightforward computation yields

$$v_t = -u_r + v, \quad v_\sigma = \frac{u_\sigma}{r}, \quad v_{tt} = u_{rr} r + v_t, \quad \triangle_{\mathbb{S}^{N-1}} v = \frac{1}{r} \triangle_{\mathbb{S}^{N-1}} u,$$

where, with some abuse of notation, v_σ denotes the gradient of v computed on the sphere. Rewriting the equation $\triangle u_\varepsilon = \beta_\varepsilon(u_\varepsilon)$ in t- and σ-derivatives we obtain

$$\frac{1}{r}[(N-1)(v - \partial_t v_\varepsilon) + \partial_t^2 v_\varepsilon - \partial_t v_\varepsilon + \triangle_{\mathbb{S}^{N-1}} v_\varepsilon] = \frac{1}{\varepsilon} \beta\left(\frac{r}{\varepsilon} v_\varepsilon\right).$$

Next, we multiply both sides of the last equation by $\partial_t v_\varepsilon$ to get

$$\partial_t v_\varepsilon[(N-1)(v - \partial_t v_\varepsilon) + \partial_t^2 v_\varepsilon - \partial_t v_\varepsilon + \triangle_{\mathbb{S}^{N-1}} v_\varepsilon] = \partial_t v_\varepsilon \frac{r}{\varepsilon} \beta\left(\frac{r}{\varepsilon} v_\varepsilon\right). \tag{3-3}$$

The right-hand side of (3-3) can be further transformed as follows:

$$\frac{r}{\varepsilon} \beta\left(\frac{e^{-t}}{\varepsilon} v_\varepsilon\right) \partial_t v_\varepsilon = \beta\left(\frac{e^{-t}}{\varepsilon} v_\varepsilon\right)\left[\frac{e^{-t}}{\varepsilon} \partial_t v_\varepsilon - \frac{e^{-t}}{\varepsilon} v_\varepsilon\right] + \beta\left(\frac{e^{-t}}{\varepsilon} v_\varepsilon\right) \frac{e^{-t}}{\varepsilon} v_\varepsilon$$

$$= \partial_t \int_0^{(e^{-t}/\varepsilon)v_\varepsilon} \beta(s)\, ds + \beta\left(\frac{e^{-t}}{\varepsilon} v_\varepsilon\right) \frac{e^{-t}}{\varepsilon} v_\varepsilon$$

$$= \partial_t \mathcal{B}\left(\frac{e^{-t}}{\varepsilon} v_\varepsilon\right) + \beta\left(\frac{e^{-t}}{\varepsilon} v_\varepsilon\right) \frac{e^{-t}}{\varepsilon} v_\varepsilon \equiv I_1.$$

It is important to note that by our assumption (1-1) the last term is nonnegative; in other words

$$\beta\left(\frac{e^{-t}}{\varepsilon}v_\varepsilon\right)\frac{e^{-t}}{\varepsilon}v_\varepsilon \geq 0. \tag{3-4}$$

Moreover, we have

$$I_2 \equiv [(N-1)v_\varepsilon - N\partial_t v_\varepsilon + \partial_t^2 v_\varepsilon + \triangle_{\mathbb{S}^{N-1}}v_\varepsilon]\partial_t v_\varepsilon$$

$$= (N-1)\partial_t\left(\frac{v_\varepsilon^2}{2}\right) - N(\partial_t v_\varepsilon)^2 + \partial_t\left(\frac{(\partial_t v_\varepsilon)^2}{2}\right) + \partial_t v_\varepsilon \triangle_{\mathbb{S}^{N-1}}v_\varepsilon.$$

Next we integrate the identity

$$I_2 = rI_1$$

over $\mathbb{S}^{N-1}$ and then over $[T_0, T]$ in order to get

$$(N-1)\int_{\mathbb{S}^{N-1}}\frac{v_\varepsilon^2}{2}\Big|_{T_0}^T - N\int_{T_0}^T\int_{\mathbb{S}^{N-1}}(\partial_t v_\varepsilon)^2 + \int_{\mathbb{S}^{N-1}}\frac{(\partial_t v_\varepsilon)^2}{2}\Big|_{T_0}^T + \int_{T_0}^T\int_{\mathbb{S}^{N-1}}\partial_t v_\varepsilon\triangle_{\mathbb{S}^{N-1}}v_\varepsilon$$

$$= \int_{\mathbb{S}^{N-1}}\mathcal{B}\left(\frac{e^{-t}}{\varepsilon}v_\varepsilon\right)\Big|_{T_0}^T + \int_{T_0}^T\int_{\mathbb{S}^{N-1}}\beta\left(\frac{r}{\varepsilon}v_\varepsilon\right)\frac{r}{\varepsilon}v_\varepsilon.$$

Note that

$$\int_{T_0}^T\int_{\mathbb{S}^{N-1}}\partial_t v_\varepsilon\triangle_{\mathbb{S}^{N-1}}v_\varepsilon = -\frac{1}{2}\int_{\mathbb{S}^{N-1}}|\nabla_\sigma v_\varepsilon|^2\Big|_{T_0}^T. \tag{3-5}$$

Rearranging the terms and utilizing (3-4) we get the identity

$$N\int_{T_0}^T\int_{\mathbb{S}^{N-1}}(\partial_t v_\varepsilon)^2 + \int_{T_0}^T\int_{\mathbb{S}^{N-1}}\beta\left(\frac{r}{\varepsilon}v_\varepsilon\right)\frac{r}{\varepsilon}v_\varepsilon$$

$$= (N-1)\int_{\mathbb{S}^{N-1}}\frac{v_\varepsilon^2}{2}\Big|_{T_0}^T + \int_{\mathbb{S}^{N-1}}\frac{(\partial_t v_\varepsilon)^2}{2}\Big|_{T_0}^T - \frac{1}{2}\int_{\mathbb{S}^{N-1}}|\nabla_\sigma v_\varepsilon|^2\Big|_{T_0}^T - \int_{\mathbb{S}^{N-1}}\mathcal{B}\left(\frac{e^{-t}}{\varepsilon}v_\varepsilon\right)\Big|_{T_0}^T. \tag{3-6}$$

From here it follows that

$$\int_{T_0}^T\int_{\mathbb{S}^{N-1}}(\partial_t v_\varepsilon)^2 \leq C, \tag{3-7}$$

where C depends on $\|\nabla u_\varepsilon\|_\infty, M, N$ but not on ε, T_0 or T.

Letting $\varepsilon \to 0$ we conclude

$$\int_{T_0}^T\int_{\mathbb{S}^{N-1}}(\partial_t v)^2 \leq C, \tag{3-8}$$

where $v(t, \sigma) = u(r, \sigma)/r$. But $\partial_t v = -u_r + u/r$, implying that

$$\int_{T_0}^\infty\int_{\mathbb{S}^{N-1}}\left(u_r - \frac{u}{r}\right)^2 dt\, d\sigma \leq C. \tag{3-9}$$

The proof of Theorem A follows if we note that $-u_r + u/r = 0$ is the Euler equation for the homogeneous functions of degree 1. $\qquad\square$

In the proof of Proposition 3.1 we used Spruck's original computation [1983]. The identity (3-6) can be interpreted as a local energy balance for u_ε. Moreover, using (3-6) we can construct a monotone quantity which has some remarkable properties.

Corollary 3.2. *Suppose* $0 \in \partial\{u > 0\}$ *and let* (r, σ), $\sigma \in \mathbb{S}^{N-1}$, *be the spherical coordinates. Introduce*

$$S_\varepsilon(r) = \int_{\mathbb{S}^{N-1}} \left[2\mathcal{B}\left(\frac{u_\varepsilon(r,\sigma)}{\varepsilon}\right) + \frac{1}{r^2}|\nabla_\sigma u_\varepsilon|^2 - (N-1)\frac{u_\varepsilon^2(r,\sigma)}{r^2} - \left(\partial_r u_\varepsilon(r,\sigma) - \frac{u_\varepsilon(r,\sigma)}{r}\right)^2 \right] d\sigma. \quad (3\text{-}10)$$

- *Then* $S_\varepsilon(r)$ *is nondecreasing in* r.
- *Moreover, if* $u_{\varepsilon_j} \to u$ *for some subsequence* $\varepsilon_j \to 0$, *then* $S_{\varepsilon_j}(r) \to S(r)$ *for a.e.* r, *where*

$$S(r) = \int_{\mathbb{S}^{N-1}} \left[2M\chi_{\{u>0\}} + \frac{1}{r^2}|\nabla_\sigma u|^2 - (N-1)\frac{u^2(r,\sigma)}{r^2} - \left(\partial_r u(r,\sigma) - \frac{u(r,\sigma)}{r}\right)^2 \right] d\sigma. \quad (3\text{-}11)$$

  In particular, $S(r)$ *is a nondecreasing function of* r.

- $S(r)$ *is constant if and only if* u *is a homogeneous function of degree* 1.

Proof. By setting $r_1 = e^{-T}$, $r_2 = e^{-T_0}$ and noting that $r_1 < r_2$ if $T > T_0$ we obtain from (3-6)

$$S_\varepsilon(r_2) - S_\varepsilon(r_1) = 2N \int_{T_0}^{T} \int_{\mathbb{S}^{N-1}} (\partial_t v_\varepsilon)^2 + 2 \int_{T_0}^{T} \int_{\mathbb{S}^{N-1}} \beta\left(\frac{r}{\varepsilon}v_\varepsilon\right)\frac{r}{\varepsilon}v_\varepsilon \geq 0,$$

where we applied (3-4) and hence the first claim follows. The second part follows from Propositions A.1 and A.2. Indeed, integrating $S_\varepsilon(r) \leq S_\varepsilon(r+t)$, $t \geq 0$, over $[r_1 - \delta, r_1 + \delta]$ we infer

$$\frac{1}{2\delta} \int_{r_1-\delta}^{r_1+\delta} S_\varepsilon(r)\, dr \leq \frac{1}{2\delta} \int_{r_1-\delta}^{r_1+\delta} S_\varepsilon(r+t)\, dr.$$

Then first letting $\varepsilon \to 0$ and utilizing Proposition A.2 together with (A-1) and then sending $\delta \to 0$ we infer that $S(r)$ is nondecreasing for a.e. r. Finally the last part follows as in the proof of Proposition 3.1. $\square$

As one can see we did not use the Pohozhaev identity, as opposed to the monotonicity formula in [Weiss 2003]. Spruck's monotonicity formula enjoys a remarkable property.

Lemma 3.3. *Let* u *be as in Proposition 3.1. Set* $S(x_0, r, u)$ *for* $S(r)$ *defined by the sphere centered at* $x_0 \in \partial\{u > 0\}$. *Suppose* $x_k \in \partial\{u > 0\}$ *such that* $x_k \to x_0$. *Then*

$$\limsup_{x_k \to x_0} S(x_k, 0, u) \leq S(x_0, 0, u).$$

Proof. For given $\delta > 0$ there is $\rho_0 > 0$ such that $S(x_0, \rho, u) \leq S(x_0, 0, u) + \delta$ whenever $\rho < \rho_0$. Fix such ρ and choose k so large that $S(x_k, \rho, u) < \delta + S(x_0, \rho, u)$. From the monotonicity of $S(x_k, \rho, u)$ it follows that

$$S(x_k, 0, u) \leq S(x_k, \rho, u) \leq \delta + S(x_0, \rho, u)$$
$$\leq 2\delta + S(x_0, 0, u).$$

First letting $x_k \to x_0$ and then $\delta \to 0$ the result follows. $\square$

Lemma 3.4. *Let S be the monotone quantity in (3-11). Then the following hold:*

(i) $s(x_0, R, u) = (1/R^N) \int_0^R r^{N-1} S(x_0, r, u) \, dr$ *is monotone nondecreasing and*

$$\frac{d}{dR} s(x_0, u, R) = \frac{1}{R^{N+1}} \int_0^R r^N S'(x_0, u, r) \, dr \geq 0.$$

(ii) *If the solution $u \geq 0$ is degenerate at $x_0 \in \partial\{u > 0\}$ then the set $\{u > 0\}$ has well-defined Lebesgue density $\Theta(x_0, \{u > 0\})$ equal to*

$$\frac{1}{2M|B_1|} s(x_0, 0, u) = \frac{1}{2M|B_1|} \lim_{R \to 0} s(x_0, R, u).$$

(iii) *Suppose $x_k \in \partial\{u > 0\}$ such that $x_k \to x_0$. Then*

$$\limsup_{x_k \to x_0} s(x_k, 0, u) \leq s(x_0, 0, u).$$

Proof. It is easy to compute

$$s'(x_0, R, u) = -\frac{N}{R^{N+1}} \int_0^R r^{N-1} S(x_0, r, u) \, dr + \frac{S(x_0, R, u)}{R}$$

$$= -\frac{S(x_0, R, u)}{R} + \frac{1}{R^{N+1}} \int_0^R r^N S'(x_0, u, r) \, dr + \frac{S(x_0, R, u)}{R}$$

$$= \frac{1}{R^{N+1}} \int_0^R r^N S'(x_0, r, u) \, dr.$$

To prove the second claim notice that at the degenerate point x_0 we have $u(x) = o(|x - x_0|)$ by virtue of the subharmonicity of u. Consequently $\fint_{B_R(x_0)} |\nabla u|^2 = o(1)$ as $r \to 0$ by virtue of the Caccioppoli inequality. Therefore the only surviving term in S comes from $2M\chi_{\{u>0\}}$. The proof of the last claim is analogous to that of Lemma 3.3. $\qquad\square$

Lemma 3.5. *Let $0 \in \partial\{u > 0\}$ and assume that $u_0 = r g(\sigma)$, $\sigma \in \mathbb{S}^{N-1}$, is a blow-up limit of u at 0 which is homogeneous function of degree 1. Then*

$$\int_{\mathbb{S}^{N-1}} |\nabla_\sigma g|^2 - (N-1) \int_{\mathbb{S}^{N-1}} g^2 \begin{cases} = 0 & \text{if } \partial\{u > 0\} \text{ is flat at } 0, \\ \leq 0 & \text{otherwise.} \end{cases}$$

Proof. Let (r, σ) be the spherical coordinates; then the Laplacian takes the form

$$\triangle u_\varepsilon = \partial_{rr}^2 u_\varepsilon + \frac{N-1}{r} \partial_r u_\varepsilon + \frac{1}{r^2} \triangle_{\mathbb{S}^{N-1}} u_\varepsilon.$$

Multiply both sides of $\triangle u_\varepsilon$ by $r^{N-1} u_\varepsilon$ and integrate over $[0, R] \times \mathbb{S}^{N-1}$ to get

$$I_1(u_{\varepsilon_j}) := \int_0^R \int_{\mathbb{S}^{N-1}} u_\varepsilon \partial_{rr}^2 u_\varepsilon r^{N-1} \, d\sigma \, dr$$

$$= R^{N-1} \int_{\mathbb{S}^{N-1}} u_\varepsilon \partial_r u_\varepsilon - \int_0^R \int_{\mathbb{S}^{N-1}} \left[(\partial_r^2 u_\varepsilon)^2 r^{N-1} + (N-1) \partial_r u_\varepsilon u_\varepsilon r^{N-1}\right] d\sigma \, dr,$$

$$I_2(u_{\varepsilon_j}) := \int_0^R \int_{\mathbb{S}^{N-1}} u_\varepsilon \partial_r^2 u_\varepsilon r^{N-2} \, d\sigma \, dr$$

$$= R^{N-2} \int_{\mathbb{S}^{N-1}} \frac{(u_\varepsilon)^2}{2} - (N-2) \int_0^R \int_{\mathbb{S}^{N-1}} \left[\frac{(u_\varepsilon)^2}{2} r^{N-3} \right] d\sigma \, dr,$$

$$I_3(u_{\varepsilon_j}) := \int_0^R \int_{\mathbb{S}^{N-1}} \triangle_{\mathbb{S}^{N-1}} u_\varepsilon u_\varepsilon r^{N-3} \, d\sigma \, dr = - \int_0^R \int_{\mathbb{S}^{N-1}} |\nabla_\sigma u_\varepsilon|^2 r^{N-3} \, d\sigma \, dr.$$

Choosing a converging sequence u_{ε_j} and letting $\varepsilon_j \to 0$ we get by virtue of Proposition A.2

$$\lim_{\varepsilon_j \to 0} \int_{B_R} \beta_{\varepsilon_j} u_{\varepsilon_j} = \lim_{\varepsilon_j \to 0} [I_1(u_{\varepsilon_j}) + (N-1)I_2(u_{\varepsilon_j}) + I_3(u_{\varepsilon_j})] \to I_1(u) + (N-1)I_2(u) + I_3(u).$$

Suppose that u_{R_k} is a blow-up sequence at the origin and $u_{R_k} \to u_0 = rg(\sigma)$; then

$$I_1(u_0) = R^N \int_{\mathbb{S}^{N-1}} g^2 - \frac{R^N}{N} \int_{\mathbb{S}^{N-1}} g^2 - \frac{N-1}{N} R^N \int_{\mathbb{S}^{N-1}} g^2 = 0,$$

$$I_2(u_0) = R^N \int_{\mathbb{S}^{N-1}} \frac{g^2}{2} - \frac{N-2}{N} R^N \int_{\mathbb{S}^{N-1}} \frac{g^2}{2} = \frac{R^N}{N} \int_{\mathbb{S}^{N-1}} g^2.$$

By Proposition A.5 and (A-2) there is a sequence $\delta_j \to 0$ such that $u_{\delta_j} \to u_0$ and

$$\lim_{\delta_j \to 0} \int_{B_1} \beta_{\delta_j} u_{\delta_j} \le \|\beta\|_\infty |\{x \in B_1 : 0 < u_{\delta_j} < \delta_j\}| \to 0,$$

provided that u is flat at 0. Hence we have

$$\lim_{\delta_j \to 0} \int_{B_1} \beta_{\delta_j} u_{\delta_j} = \frac{R^N}{N} \left[(N-1) \int_{\mathbb{S}^{N-1}} g^2 - \int_{\mathbb{S}^{N-1}} |\nabla_\sigma g|^2 \right] \begin{cases} = 0 & \text{if } \partial\{u > 0\} \text{ is flat at 0,} \\ \ge 0 & \text{otherwise.} \end{cases} \qquad \square$$

4. Proof of Theorem A

The first part of the theorem follows from Proposition 3.1. Since u is not degenerate at the origin, by Propositions A.2 and A.5 $u_{\rho_k}(x) \to u_0(x)$ locally uniformly and by Proposition 3.1 u_0 is homogeneous of degree 1. Write $\triangle$ in polar coordinates (r, θ) to obtain

$$\triangle w = \frac{1}{r} \frac{\partial}{\partial r}(r w_r) + \frac{1}{r^2} \frac{\partial}{\partial \theta}(w_\theta).$$

In particular, writing $u_0 = rg(\theta)$, this yields a second-order ODE for g,

$$g + \ddot{g} = 0. \tag{4-1}$$

Suppose $g(0) = g(\theta_0) = 0$, $\theta_0 \in [0, 2\pi)$; then (4-1) implies that $g(\theta) = A \sin \theta$ for some constant A, consequently forcing $\theta_0 = \pi$. Hence, since $N = 2$, we obtain that u_0 must be linear; in other words the free boundary $\partial\{u_0 > 0\}$ is everywhere flat. This in turn implies that in two dimensions the singular set of the free boundary $\partial\{u_0 > 0\}$ is empty. Consequently, u_0 is linear in $\{u_0 > 0\}$ and $\{u_0 < 0\}$. From here parts (2) and (3) of Theorem A follow from [Caffarelli et al. 1997, Propositions 5.3 and 5.1].

So it remains to check (1). For the elliptic problem the only difference is that the limit function $M(x) = \lim_{\delta_j \to 0} \mathcal{B}_{\delta_j}(u_{\delta_j})$ cannot have nontrivial concentration on the free boundary coming from $\{x_1 < 0\}$, as opposed to the parabolic case studied in [Caffarelli et al. 1997]. Observe that $\nabla \mathcal{B}(u_{\delta_j}/\delta_j) = \nabla u_{\delta_j} \beta_{\delta_j}(u_{\delta_j}) = 0$ in $B_1 \setminus \{0 < u_{\delta_j} < \delta_j\}$. By Proposition A.5 and (A-2) there is sequence $0 < \lambda_j \to 0$ such that $(u_{\varepsilon_j})_{\lambda_j} \to u_0, \varepsilon_j/\lambda_j \to 0$ and $M(x) = M\chi_{\{x_1>0\}} + M_0\chi_{\{x_1<0\}}$. It follows from (A-3) that

$$\int_{\{x_1>0\}} M \, \partial_1 \phi + \int_{\{x_1<0\}} M_0 \, \partial_1 \phi = \int_{\{x_1>0\}} \frac{\alpha^2}{2} \partial_1 \phi \quad \text{for all } \phi \in C_0^\infty(B_1). \tag{4-2}$$

After integration by parts we obtain $M_0 \int_{-1}^{1} \phi(0, x_2) \, dx_2 = \left(M - \frac{1}{2}\alpha^2\right) \int_{-1}^{1} \phi(0, x_2) \, dx_2$. This yields

$$M_0 = M - \frac{\alpha^2}{2}.$$

Next we claim that $M_0 = 0$. Suppose $M_0 > 0$; then $I_0 := \{t \in \mathbb{R} : \mathcal{B}(t) = M_0\} \neq \varnothing$ and there is $a \in (0, 1)$ such that $I_0 \subset [a, 1]$. Since $\mathcal{B}(t)$ is continuous and nondecreasing, it follows that there is $0 < a_0 < a$ such that $u_{\delta_j}(x)/\delta_j \in [a_0, 1]$ provided that j is sufficiently large.

Let

$$\mathcal{C} = \left\{ x : \frac{u_{\delta_j}(x)}{\delta_j} \in [a_0, 1] \right\} \cap \{x_1 < 0\} \cap B_1.$$

Then,

$$\mathcal{C} \subset \{x \in B_1 : a_0\delta_j \leq u_{\delta_j}(x) \leq \delta_j\} \subset \{0 < u_{\delta_j} < 2\delta_j\} \cap B_1.$$

But $|\{0 < u_{\delta_j} < 2\delta_j\} \cap B_1| \to 0$, which implies that M_0 cannot be positive. $\qquad \square$

5. The structure of the free boundary of blow-ups in $\mathbb{R}^3$

In this section we assume that $u \geq 0$ is a limit of u_{ε_j} solving $(\mathcal{P}_\varepsilon)$ for some sequence $\varepsilon_j \to 0$, u is nondegenerate at some $y_0 \in \partial\{u > 0\}$ and u_0 is a blow-up of u at y_0. Note that by Corollary 3.2 u_0 is a homogeneous function of degree 1. If u_0 is not a minimizer then it is natural to expect that the solutions of $(\mathcal{P}_\varepsilon)$ develop singularities in $\mathbb{R}^N$, $N \geq 3$.

We first prove a nondegeneracy result.

Lemma 5.1. *Let $x_0 \in \partial\{u_0 > 0\}$ be a free boundary point such that there is a ball $B \subset \{u_0 = 0\}$ touching $\partial\{u_0 > 0\}$ at x_0 and $\Theta(x_0, \{u_0 > 0\}) \geq \frac{1}{2}$. Then u_0 is nondegenerate at x_0 and*
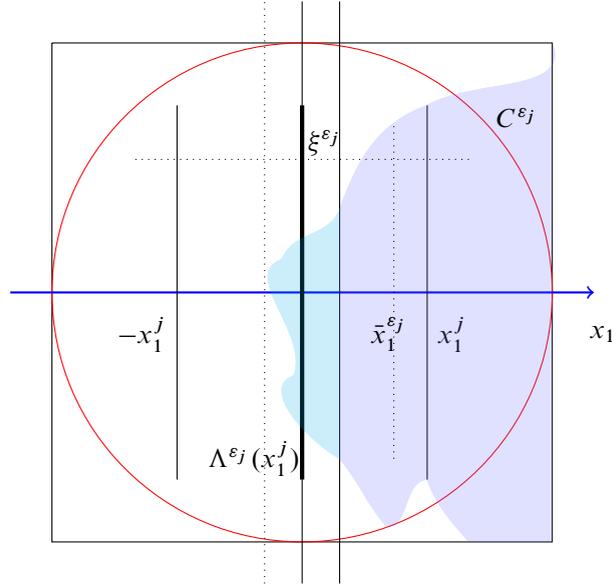
$$u_0(x) = \sqrt{2M}(x - x_0)^+ + o(x - x_0).$$

Proof. Let $(u_0)_r = u_0(x_0 + rx)/r$. There is r_0 such that

$$(u_0)_r = 0 \quad \text{in } \{x_1 < -\delta\} \cap Q_1, \quad \text{for all } r \leq r_0, \tag{5-1}$$

for some small $\delta > 0$, where $Q_1 = (-1, 1)^3$ is the unit cube. Moreover, there is $\hat{r}_0 > 0$ such that

$$\frac{|\{(u_0)_r > 0\} \cap \{x_1 > 0\} \cap B_1|}{|B_1|} > \frac{1}{2} - \delta \quad \text{for all } r \leq \hat{r}_0. \tag{5-2}$$

Figure 1. The construction of the point $(\bar{x}_1^{\varepsilon_j}, \xi^{\varepsilon_j})$. The purple region is C^{ε_j}.

Fix r with these two properties (5-1) and (5-2). There exists $\gamma > 0$ such that

$$\frac{|\{(u_0)_r > \gamma\} \cap \{x_1 > 0\} \cap B_1|}{|B_1|} > \frac{1}{2} - 2\delta. \tag{5-3}$$

Let $v^{\varepsilon_j} = (u_{\varepsilon_j})_r$, where $u_{\varepsilon_j} \to u_0$ (see Proposition A.5) and $A^{\varepsilon_j} = \{v^{\varepsilon_j} > \gamma/2\} \cap \{x_1 > 0\} \cap B_1$. Since $v^{\varepsilon_j} \to (u_0)_r$ uniformly (see Proposition A.2), it follows that there is $j_0(r)$ such that for $j \geq j_0(r)$ we have

$$|A^{\varepsilon_j}| > |B_1|\left(\tfrac{1}{2} - 2\delta\right). \tag{5-4}$$

Let $B^{\varepsilon_j} = \{x_1 \in (-1, -\delta)\} \cap Q_1$ and $-B^{\varepsilon_j} = \{x_1 \in (\delta, 1)\} \cap Q_1$. Let $C^{\varepsilon_j} = A^{\varepsilon_j} \cap (-B^{\varepsilon_j})$. Then we have

$$|C^{\varepsilon_j}| \geq |B_1|\left(\tfrac{1}{2} - 2\delta\right) > 0.$$

Define $\Lambda^{\varepsilon_j}(x_1) = \{x' : (x_1, x') \in C^{\varepsilon_j}\}$ and $f^{\varepsilon_j}(x_1) = |\Lambda^{\varepsilon_j}(x_1)|$. We claim that

$$\left|\left\{x_1 : f^{\varepsilon_j}(x_1) > |B_1|\left(\tfrac{1}{2} - 3\delta\right)\right\}\right| > 0.$$

Indeed, if the claim fails then we have

$$|B_1|\left(\tfrac{1}{2} - 2\delta\right) \leq |C^{\varepsilon_j}| = \int_\delta^1 f^{\varepsilon_j}(x_1)\, dx_1 \leq |B_1|\left(\tfrac{1}{2} - 3\delta\right),$$

which is a contradiction.

Hence there is $x_1^{\varepsilon_j} \in (\delta, 1)$ such that $f^{\varepsilon_j}(x_1^{\varepsilon_j}) > |B_1|\left(\tfrac{1}{2} - 3\delta\right)$. Now choose $0 < a' < a < b < b' < 1$ such that

$$\beta(s) > \kappa \quad \text{for all } s \in [a', b'].$$

Let $\varepsilon'_j = \varepsilon_j/r$. We claim that there is $\xi^{\varepsilon_j} \in \Lambda^{\varepsilon_j}$ and $\bar{x}_1^{\varepsilon_j}$ such that

$$\frac{v^{\varepsilon_j}}{\varepsilon'_j}(x_1^{\varepsilon_j}, \xi^{\varepsilon_j}) \in (a, b).$$

Indeed, for sufficiently large j we have

$$a > \frac{v^{\varepsilon_j}}{\varepsilon'_j}(-x_1^{\varepsilon_j}, x') = 0 \quad \text{for all } x' \in \Lambda^{\varepsilon_j}(x_1^{\varepsilon_j}),$$

$$\frac{v^{\varepsilon_j}}{\varepsilon'_j}(x_1^{\varepsilon_j}, x') > \frac{\gamma}{2\varepsilon'_j} > b \quad \text{for all } x' \in \Lambda^{\varepsilon_j}(x_1^{\varepsilon_j}),$$

provided that $j > j_1(r)$; see Figure 1. Hence from the mean value theorem we see that the claim is true. From the uniform Lipschitz continuity of the functions v^{ε_j} it follows that there is a constant $c_0 > 0$ such that

$$\frac{v^{\varepsilon_j}}{\varepsilon'_j}(x_1, x') \in (a', b') \quad \text{if } |x_1 - \bar{x}_1^{\varepsilon_j}| < \varepsilon'_j c_0, \ x' \in \Lambda^{\varepsilon_j}(x_1^{\varepsilon_j}).$$

Consequently we have

$$\int_{B_1} \beta_{\varepsilon'_j}(v^{\varepsilon_j}) \geq \frac{\kappa}{\varepsilon'_j} \int_{|x_1 - \bar{x}_1^{\varepsilon_j}| < \varepsilon'_j c_0} |\Lambda^{\varepsilon_j}(x_1)| dx_1 \geq \frac{\kappa}{\varepsilon'_j}(1 - 3\delta)2c_0\varepsilon'_j = 2\kappa |B_1|(1 - 3\delta)c_0 := \tilde{C}.$$

Now the nondegeneracy follows from the proof of Part II of Theorem 6.3 in [Caffarelli et al. 1997]. The asymptotic expansion follows from Theorem A and Proposition 3.1. □

Remark 5.2. Note that under the weaker assumption $\Theta(x_0, \{u_0 > 0\}) > 0$ the argument in the proof of Lemma 5.1 still works. However for a self-crossing free boundary [Weiss 2003] (see Figure 2) the assumptions of Lemma 5.1 may not be satisfied.

As an immediate corollary we have:

Corollary 5.3. *Let $x_0 \in \partial\{u_0 > 0\}$ be a point of reduced boundary. Then u_0 is nondegenerate at x_0.*

Proof. Suppose that $0 \in \partial\{u_0 > 0\}$ and $\partial\{(u_0)_r > 0\} \subset B_2 \cap \{|x \cdot e| < \varepsilon\}$ for some unit vector e and small $\varepsilon > 0$. Here $(u_0)_r = u_0(rx)/r$. Consider the family of balls $B_{1/2}(et), t \in [-\varepsilon, \varepsilon]$. Then there

$u > 0$ $u > 0$

O
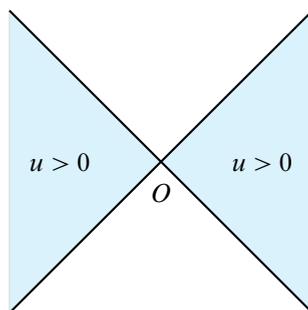
Figure 2. Possible self-crossing free boundary which fails to satisfy the conditions of Lemma 5.1.

is $t_\varepsilon \in [-\varepsilon, \varepsilon]$ such that $B_{1/2}(et_\varepsilon)$ touches the free boundary at some point $z_0 \in B_1$ provided that ε is sufficiently small. Let $\nu_0 = t_\varepsilon e$. Introduce the barrier function

$$w(x) = \frac{\varphi(\frac{1}{2}) - \varphi(|x - \nu_0|)}{\varphi(\frac{1}{2}) - \varphi(1)} \sup_{B_1(\nu_0)} u_0,$$

where $\varphi(|x|) = 1/|x|^{N-2}$. We have $\triangle(u_0 - w) = \triangle u_0 \geq 0$ in $D = B_1(\nu_0) \setminus B_{1/2}(\nu_0)$ and $u_0 - w \leq 0$ on ∂D. From the maximum principle we infer that $u_0 \leq w$ in D. But we have that the maximum of $u_0 - w$ is realized at z_0. Hence from the Hopf lemma we get

$$-\sqrt{2M} = \partial_{\nu_0} u_0(z_0) > \partial_{\nu_0} w(z_0) = -\left|\varphi'\left(\tfrac{1}{2}\right)\right| \frac{\sup_{B_1(\nu_0)} u_0}{\varphi(\frac{1}{2}) - \varphi(1)}$$

or

$$\sup_{B_1(\nu_0)} (u_0)_r \geq \sqrt{2M} \frac{\varphi(\frac{1}{2}) - \varphi(1)}{\left|\varphi'\left(\tfrac{1}{2}\right)\right|}. \qquad \square$$

In the following definition we let $\Omega^+(u) = \{u > 0\}$ and $\Omega^-(u) = \{u < 0\}$. Moreover, let

$$G(u_\nu^+, u_\nu^-) := (u_\nu^+)^2 - (u_\nu^-)^2 - 2M, \tag{5-5}$$

where u_ν^+ and u_ν^- are the normal derivatives in the inward direction ν to $\partial\Omega^+(u)$ and $\partial\Omega^-(u)$, respectively. For more details see [Caffarelli and Salsa 2005, Definition 2.4].

Definition 5.1. Let Ω be a bounded domain of $\mathbb{R}^N$ and let u be a continuous function in Ω. We say that u is a viscosity solution in Ω if:

(i) $\Delta u = 0$ in $\Omega^+(u)$ and $\Omega^-(u)$.

(ii) Along the free boundary $\partial\{u > 0\}$, the function u satisfies the free boundary condition in the sense that:

(a) If at $x_0 \in \partial\{u > 0\}$ there exists a ball $B \subset \Omega^+(u)$ such that $x_0 \in \partial B$ and

$$u^+(x) \geq \alpha\langle x - x_0, \nu\rangle^+ + o(|x - x_0|) \quad \text{for } x \in B, \tag{5-6}$$
$$u^-(x) \leq \beta\langle x - x_0, \nu\rangle^- + o(|x - x_0|) \quad \text{for } x \in B^c, \tag{5-7}$$

for some $\alpha > 0$ and $\beta \geq 0$, with equality along every nontangential domain, then the free boundary condition is satisfied:

$$G(\alpha, \beta) \leq 0.$$

(b) If at $x_0 \in \partial\{u > 0\}$ there exists a ball $B \subset \Omega^-(u)$ such that $x_0 \in \partial B$ and

$$u^-(x) \geq \beta\langle x - x_0, \nu\rangle^- + o(|x - x_0|) \quad \text{for } x \in B,$$
$$u^+(x) \leq \alpha\langle x - x_0, \nu\rangle^+ + o(|x - x_0|) \quad \text{for } x \in \partial B,$$

for some $\alpha \geq 0$ and $\beta > 0$, with equality along every nontangential domain, then

$$G(\alpha, \beta) \geq 0.$$

In our case $\beta = 0$ and we have only u^+. However, one has to check that the free boundary conditions (a) and (b) in Definition 5.1 are satisfied.

Lemma 5.4. *Let u_0 be a blow-up of u at some nondegenerate point such that $\Theta(x, \{u_0 > 0\}) \geq \frac{1}{2}$ for every $x \in \partial\{u_0 > 0\}$. Then u_0 is a viscosity solution in the sense of Definition 5.1.*

Proof. We have to show that the properties (a), (b) in Definition 5.1 hold. Suppose that $B \subset \{u_0 > 0\}$ touches $\partial\{u_0 > 0\}$ at some point x_0. Then it follows from Hopf's lemma that u_0 is nondegenerate at x_0. Consequently, if u_{00} is a blow-up at x_0 then by Theorem A $u_{00}(x) = \alpha x_1^+$ after some rotation of coordinate system. Moreover $0 < \alpha \leq \sqrt{2M}$. Hence $G(\alpha, 0) \leq 0$.

Now suppose that $B \subset \{u_0 = 0\}$ and B touches $\partial\{u_0 > 0\}$ at z_0. By Lemma 5.1 u_0 is nondegenerate at z_0. Theorem A implies that any blow-up u_{00} of u_0 at z_0 must be $u_{00}(x) = \sqrt{2M} x_1^+$ after some rotation of coordinates. Hence $G(\sqrt{2M}, 0) \geq 0$. $\square$

5A. *Properties of $\partial\{u_0 > 0\}$.* We want to study the properties of g. We first prove a Bernstein-type result which is a simple consequence of a refinement of the Alt–Caffarelli–Friedman monotonicity formula [Alt et al. 1984b; Caffarelli et al. 2000].

Lemma 5.5. *Let $u \geq 0$ be a limit of solutions to $(\mathcal{P}_\varepsilon)$. Let $u_0 = rg(\sigma)$, $\sigma \in \mathbb{S}^{N-1}$, be a nontrivial blow-up of u at some free boundary point. If there is a hemisphere containing $\mathrm{supp}\, g$ then the graph of u_0 is a half-plane.*

Proof. Without loss of generality we assume $\mathrm{supp}\, g \subset \mathbb{S}_+^{N-1} = \{X \in \mathbb{S}^{N-1} : x_N \geq 0\}$. Let $v(x_1, \ldots, x_N) = u(x_1, \ldots, -x_N)$ be the reflection of u with respect to the hyperplane $x_N = 0$. Then v is a nonnegative subharmonic function satisfying the requirements of [Caffarelli et al. 2000, Lemma 2.3]. Thus

$$\Phi(r) = \frac{1}{r^4} \int_{B_r} \frac{|\nabla u_0|^2}{|x|^{N-2}} \int_{B_r} \frac{|\nabla v|^2}{|x|^{N-2}}$$

is nondecreasing in r. Moreover

$$\Phi'(r) \geq \frac{2\Phi(r)}{r} A_r, \quad A_r = \frac{C_N}{r^{N-1}} \mathrm{Area}(\partial B_r \setminus (\mathrm{supp}\, u_0 \cup \mathrm{supp}\, v)).$$

Thus, if $\mathrm{supp}\, g$ digresses from the hemisphere by size $\delta > 0$ then $A_r \geq c(\delta) > 0$. Hence integrating the differential inequality for Φ we see that Φ grows exponentially, which is a contradiction since in view of Proposition A.3 u_0 is Lipschitz and hence Φ must be bounded. $\square$

It is convenient to define the following subsets of the free boundary:

$$\Gamma_{1/2} = \left\{ x \in \partial\{u_0 > 0\} \text{ such that } \Theta(x, \{u_0 > 0\}) = \tfrac{1}{2} \right\}, \tag{5-8}$$

$$\Gamma_1 = \{ x \in \partial\{u_0 > 0\} \text{ such that } \Theta(x, \{u_0 > 0\}) = 1 \}, \tag{5-9}$$

where $\Theta(x, D)$ denotes the Lebesgue density of D at x. We will see that $\Theta(x, \{u_0 > 0\})$ exists at every nondegenerate point and equals either 1 or $\frac{1}{2}$.

Lemma 5.6. *Assume $N = 3$. Let $x_0 \in \partial\{u_0 > 0\} \setminus \{0\}$ be a nondegenerate free boundary point such that the lower Lebesgue density satisfies $\Theta_*(x_0, \{u_0 \equiv 0\}) > 0$. Then there is a unit vector ν_0 such that*

$$u_0(x) = \sqrt{2M}[(x - x_0) \cdot \nu_0]^+ + o(x - x_0). \tag{5-10}$$

In particular, $x_0 \in \Gamma_{1/2}$.

Proof. Set $v_k = u_0(x_0 + \rho_k x)/\rho_k$. Since u_0 is nondegenerate at x_0 it follows from a customary compactness argument that $v_k \to v$ and by virtue of Corollary 3.2 v is a homogeneous function of degree 1. We have

$$\frac{u_0(x_0 + \rho_k x)}{\rho_k} = u_0(\rho_k^{-1} x_0 + x) = \nabla u_0(\rho_k^{-1} x_0 + x)(\rho_k^{-1} x_0 + x)$$

$$= \frac{1}{\rho_k} \nabla u_0(x_0 + \rho_k x)(x_0 + \rho_k x), \tag{5-11}$$

where the last line follows from the zero-degree homogeneity of the gradient; hence

$$\rho_k v_k(x) = \nabla u_0(x_0 + \rho_k x)(x_0 + \rho_k x) = \nabla v_k(x)(x_0 + \rho_k x). \tag{5-12}$$

By the Lipschitz continuity of u_0 it follows that v_k is locally bounded. Consequently, for a suitable subsequence of ρ_k we have $v_{k_j} \to v$ and $\nabla v(x)x_0 = 0$. Without loss of generality we may assume that x_0 is on the x_3-axis, implying that v depends only on x_1 and x_2. Applying Proposition A.6 and Corollary 3.2 we conclude that $S(x_0, r, u_0)$ is nondecreasing and thus v must be homogeneous of degree 1. Indeed, there is a sequence $\delta_j \to 0$ such that $(u_{\varepsilon_j})_{\lambda_j} \to v$, $\delta_j = \varepsilon_j/\lambda_j$ by Proposition A.6.

Finally, applying Theorem A and the assumption $\Theta_*(x_0, \{u_0 = 0\}) > 0$ we see that v must be a half-plane solution. It remains to note that the approximate tangent of $\partial\{u_0 > 0\}$ at x_0 is unique and this completes the proof.   □

Lemma 5.7. *We want to show that $\Theta(x, \{u_0 > 0\}) \geq \frac{1}{2}$ in some neighborhood of x_0. Let $x_0 \in \Gamma_{1/2}$. Then there exists $r_0 > 0$ such that $B_{r_0}(x_0) \cap \partial\{u_0 > 0\}$ is a $C^{1,\alpha}$ surface.*

Proof. Let $y_0 \in \partial\{u_0 > 0\}$ be a degenerate point. Suppose there is $\rho > 0$ such that u_0 is degenerate at every point of $B_\rho(y_0) \cap \partial\{u_0 > 0\}$. Since $\mathrm{supp}\, \Delta u_0 \subset \partial\{u_0 > 0\}$, it follows that $u_0 \equiv 0$ in $B_\rho(y_0)$. Consequently, there is a sequence of nondegenerate points $y_k \to y_0$. Note that if y_k is a nondegenerate point then by Theorem A the Lebesgue density satisfies $\Theta(y_k, \{u_0 > 0\}) \geq \frac{1}{2}$.

Let u_{00}^k be a blow-up of u_0 at y_k. By Proposition A.6 for fixed k there are $\delta_j^k \to 0$ such that $(u_{\varepsilon_j^k})_{\lambda_j^k} \to u_{00}^k$, $\delta_j^k = \varepsilon_j^k/\lambda_j^k$. Thus applying Theorem A it follows that u_{00}^k is a half-plane solution or a wedge.

From scaling properties of Spruck's monotonicity formula and Lemma 3.4 we get

$$s(0, y_k, u_0) = s(1, 0, u_{00}^k) = 2M \, \mathrm{vol}(B_1 \cap \{u_{00}^k > 0\}) = \begin{cases} 2\pi M & \text{if } y_k \text{ is a wedge point,} \\ \pi M & \text{otherwise.} \end{cases} \tag{5-13}$$

Then applying Corollary 3.2 to $u_{\delta_j^k}$ and using the semicontinuity of S, Lemma 3.3 together with Lemma 3.5, we have

$$2M \, \mathrm{vol}(B_1 \cap \{u_{00}^k > 0\}) = \limsup_{y_k \to y_0} s(0, y_k, u_0) \leq s(0, y_0, u_0) = 2M\pi\Theta(x_0, \{u_0 > 0\}). \tag{5-14}$$

Therefore we conclude that $\Theta(x, \{u_0 > 0\}) \geq \frac{1}{2}$ for every free boundary point x in some neighborhood of x_0. By virtue of Lemma 5.4 u_0 is a viscosity solution which is flat x_0. Applying the "flatness-implies-$C^{1,\alpha}$" regularity results from [Caffarelli 1987; 1989] the lemma follows. $\qquad\square$

Next we prove a representation formula for $\triangle u_0$.

Lemma 5.8. *Let u_0 be as in Lemma 5.5. Then*:

(i) $\mathcal{H}^2(\Gamma_{1/2} \cap B_R) < \infty$ *for any $R > 0$.*

(ii) *Away from Γ_1 the following representation formula holds*:

$$\triangle u_0 = \sqrt{2M}\, \mathcal{H}^2 \llcorner \Gamma_{1/2}.$$

Proof. (i) For given $x \in \Gamma_{1/2}$ there is a $\tilde{\rho}_x > 0$ such that

$$\sup_{B_r(x)} u_0 \geq \sqrt{M}\, r, \quad r \in (0, \tilde{\rho}_x). \tag{5-15}$$

This follows from the asymptotic expansion in Lemma 5.6. Consequently, there is $\rho'_x > 0$ such that

$$\int_{B_r(x)} \triangle u_0 \geq \sqrt{M}\, r^2, \quad r \in (0, \rho'_x). \tag{5-16}$$

Indeed, if this inequality is false then there is a sequence $r_j \searrow 0$ such that

$$\int_{B_{r_j}(x)} \triangle u_0 < \sqrt{M}\, r_j^2.$$

Set $v_j(x) = u_0(x + r_j x)/r_j$. By (5-15) $\sup_{B_1} v_j(x) \geq \sqrt{M}$. Moreover, it follows from Lemma 5.6 that $v_j(x) \to \sqrt{2M}\, x_1^+$ in a suitable coordinate system, while $\int_{B_1} \triangle v \leq \sqrt{M}$. However, $\int_{B_1} \triangle x_1^+ = \sqrt{2M}\, \frac{\pi}{2}$ and this is in contradiction with the former inequality. Putting $\bar{\rho}_x = \min(\rho'_x, \tilde{\rho}_x)$ we see that the collection of balls $\mathcal{F} = \bigcup B_{\rho_x}(x)$, $x \in \Gamma_{1/2} \cap B_R$, $\rho_x < \bar{\rho}_x$, is a Besicovitch-type covering of $\Gamma_{1/2} \cap B_R$. Consequently, there is a positive integer $m > 0$ and subcoverings $\mathcal{F}_1, \ldots, \mathcal{F}_m$ such that the balls in each $\mathcal{F}_i$, $1 \leq i \leq m$, are disjoint and $\Gamma_{1/2} \cap B_R \subset \bigcup_{i=1}^m \mathcal{F}_i$. We have from (5-16)

$$4\pi R^2 \|\nabla u_0\|_\infty \geq \int_{\partial B_R} \partial_\nu u_0 \geq \int_{B_{\rho_x}(x) \in \mathcal{F}_i} \triangle u_0 = \sum_{B_{\rho_x}(x) \in \mathcal{F}_i} \int_{B_{\rho_x}(x)} \triangle u_0 \geq m\sqrt{M} \sum_{B_{\rho_x}(x) \in \mathcal{F}_i} \rho_x^2.$$

This yields

$$\sum_{B_{\rho_x}(x) \in \bigcup_{i=1}^m \mathcal{F}_i} \rho_x^2 \leq \frac{4\|\nabla u_0\|_\infty \pi R^2}{m\sqrt{M}}. \tag{5-17}$$

Given $\delta > 0$ small, suppose there is $x \in \Gamma_{1/2}$ such that $\bar{\rho}_x \geq \delta$. Then we choose $\rho_x < \delta$. Thus, in any case we can assume that $\rho_x < \delta$. In view of (5-17) this implies that the δ-Hausdorff premeasure is bounded independently of δ. This proves (i).

(ii) From the estimate

$$\sqrt{M}r^2 \leq \int_{B_r(x)} \triangle u_0 \leq 4\pi r^2 \|\nabla u_0\|, \quad r \in (0, \bar{\rho}_x), \ B_r(x) \cap \Gamma_{1/2} \subset \Gamma_{1/2},$$

we see that there is a positive bounded function q such that $\triangle u_0 = q\mathcal{H}^2 \llcorner \Gamma_{1/2}$. Using Lemma 5.6 we conclude that $q = \sqrt{2M}$. $\qquad\square$

Next we prove the full nondegeneracy of u_0 near $\Gamma_{1/2}$.

Lemma 5.9. *Let u_0 be as above and $x_0 \in \Gamma_{1/2}$. Then for any $B_r(x)$ such that $x \in \partial\{u_0 > 0\}$, $B_r(x) \cap \partial\{u_0 > 0\} \subset \Gamma_{1/2}$, we have*

$$\sup_{B_r(x)} u_0 \geq \sqrt{2M}\pi r.$$

Proof. By a direct computation we have

$$r^{-2} \int_{\partial B_r(x)} u_0 = \int_0^r \frac{dt}{t^2} \int_{B_t(x)} \triangle u_0 \geq \int_0^r \frac{1}{t^2} \sqrt{2M}\pi t^2 = \sqrt{2M}\pi r,$$

where the inequality follows from the representation formula and the fact that $\partial\{u_0 > 0\}$ is a cone; hence for all $t \in (0, r)$ we have $\mathcal{H}^2(B_r(x) \cap \Gamma_{1/2}) \geq \pi t^2$. It remains to note that $r^{-2} \int_{\partial B_r(x)} u_0 \leq \sup_{B_r(x)} u_0$. $\quad\square$

5B. *Weak solutions.*

Combining Lemmas 5.8 and 5.9 as well as Propositions A.2(iii) and A.3(i) we see that u_0 is a weak solution near $\Gamma_{1/2}$ in the sense of [Alt and Caffarelli 1981, Definition 5.1]. Furthermore, $\partial\{u_0 > 0\} \setminus \{0\}$ is flat at each point.

Lemma 5.10. *The blow-up u_0 is a weak solution in the Alt–Caffarelli sense away from Γ_1. Furthermore, $\Gamma_{1/2}$ is smooth.*

Proof. All conditions in [Alt and Caffarelli 1981, Definition 5.1] are satisfied and u_0 is flat at every point $z_0 \in \partial\{u_0 > 0\} \setminus \{0\}$ thanks to (5-10). Applying Theorem 8.1 of the same paper we infer that $\Gamma_{1/2}$ is smooth at every $z_0 \in \partial\{u_0 > 0\} \setminus \{0\}$. $\qquad\square$

5C. *Minimal perimeter.*

In this section we prove that the local perturbations $S' \subset \{u_0 > 0\}$ of a portion $S \subset \Gamma_{1/2}$ have larger $\mathcal{H}^2$ measure than S. This can be seen from the estimate $|\nabla u_0(x)| \leq \sqrt{2M}$, which follows from Lemma A.7. Since by Lemma 5.10 on $\Gamma_{1/2}$ the free boundary condition $|\nabla u_0| = \sqrt{2M}$ is satisfied in the classical sense, it follows that

$$0 = \int_D \triangle u_0 = \int_S \partial_\nu u_0 + \int_{S'} \partial_\nu u_0 = \sqrt{2M}\mathcal{H}^2(S) + \int_{S'} \partial_\nu u_0,$$

where $D \subset \{u_0 > 0\}$ such that $\partial D = S \cup S'$. But $\left|\int_{S'} \partial_\nu u_0\right| \leq \sqrt{2M}\mathcal{H}^2(S')$ and thereby

$$\mathcal{H}^2(S) \leq \mathcal{H}^2(S'). \tag{5-18}$$

The estimate for the perimeter can be reformulated as follows:

Theorem 5.11. *Let $N = 3$. Then the components of $\Gamma_{1/2}$ are surfaces of nonpositive outward mean curvature. In particular, $\Gamma_{1/2}$ is a union of smooth convex surfaces.*

Proof. Since u_0 is a weak solution, by Lemma 5.10 $\Gamma_{1/2}$ is smooth. If $z_0 \in \Gamma_{1/2}$ then choosing the coordinate system in $\mathbb{R}^3$ so that x_3-axis has the direction of the inward normal of $\{u_0 > 0\}$ at z_0 and considering the free boundary near z_0 as a graph $x_3 = h(x_1, x_2)$, we can consider the one-sided variations of the surface area functional. Indeed, let $\mathcal{D} \subset \mathbb{R}^2$ be a open bounded domain in the $x_1 x_2$-plane containing z_0 and assume $t > 0$, $0 \le \psi \in C_0^\infty(\mathcal{D})$. Then from (5-18) we have

$$
\begin{aligned}
0 &\ge \frac{1}{t} \int_{\mathcal{D}} \left[\sqrt{1 + |\nabla h|^2} - \sqrt{1 + |\nabla(h - t\psi)|^2} \right] \\
&= \int_{\mathcal{D}} \frac{2\nabla h \nabla \psi - t|\nabla \psi|^2}{\sqrt{1 + |\nabla h|^2} + \sqrt{1 + |\nabla(h - t\psi)|^2}} \\
&\to \int_{\mathcal{D}} \frac{\nabla h \nabla \psi}{\sqrt{1 + |\nabla h|^2}} \quad \text{as } t \to 0.
\end{aligned}
\tag{5-19}
$$

Therefore

$$
\operatorname{div}\left(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) \ge 0
$$

and, noting that $\Gamma_{1/2}$ is a cone, the result follows. $\qquad\square$

5D. *Full nondegeneracy.*

Lemma 5.12. *Assume that $N = 3$ and let u_0 be a nontrivial blow-up of u such that the measure-theoretic boundary of $\{u_0 > 0\}$ is nonempty. Then $\partial\{u_0 > 0\} \setminus \{0\} \subset \Gamma_{1/2}$. In particular the set of degenerate points of $\partial\{u_0 > 0\}$ is empty.*

Proof. Let u_0 be a blow-up of u at 0. Since u is nondegenerate at 0, it follows that u_0 does not vanish identically. Hence there is a ball $B \subset \{u_0 > 0\}$ touching $\partial\{u_0 > 0\}$ at some point $z_0 \in \partial\{u_0 > 0\} \cap B$. By Hopf's lemma, the Lipschitz estimate Proposition A.3(i) and asymptotic expansion [Caffarelli 1989, Lemma A1] it follows that u_0 is not degenerate at z_0. Consequently, the set of nondegenerate points of u_0 is not empty.

Suppose that S is a component of $\partial\{u_0 > 0\}$ containing a point of measure-theoretic boundary of $\{u_0 > 0\}$. Note that by Lemma 5.7 and Theorem 5.11 S is a smooth convex surface. Let $x_0 \in \partial S$, $x_0 \ne 0$. Then either (a) $x_0 \in \Gamma_1$ or (b) u_0 is degenerate at x_0.

We first analyze the case (a). Let ℓ be the ray passing through x_0 and Π the tangent half-plane to S along ℓ. First note that u_0 is nondegenerate at x_0 because

$$
\int_{B_r(x_0)} \Delta u_0 \ge \int_{B_r(x_0) \cap S} \Delta u_0 \ge \sqrt{2M} \mathcal{H}^2(S \cap B_r(x_0)) \ge \sqrt{2M} \frac{\pi r^2}{2}
$$

for sufficiently small r. Consequently

$$
\frac{1}{R^2} \int_{\partial B_R} u_0 = \int_0^R \frac{1}{r^2} \int_{B_r(x_0)} \Delta u_0 \ge \sqrt{2M} \frac{\pi r^2}{2} R.
\tag{5-20}
$$

Let u_{00} be a blow-up of u_0 at x_0. Then from Theorem A it follows that u_{00} is two-dimensional. Moreover $\Pi \subset \partial\{u_{00} > 0\}$, $\{u_{00} > 0\}$ has unit density at 0, and the interior of $\{u_{00} = 0\}$ near Π is not empty. Note
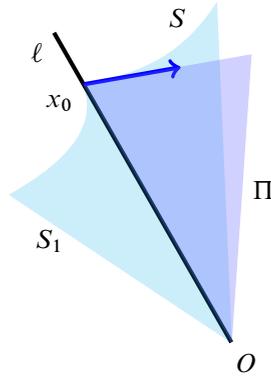
Figure 3. The structure of the free boundary near the point x_0.

that the interior of the set $\{u_0 = 0\}$ propagates to x_0 along another component S_1 of measure-theoretic boundary; see Figure 3. Consequently, near Π, we have $u_{00}(x) = \sqrt{2M} x_1^+$ after some rotation of coordinates. From the unique continuation theorem it follows that $u_{00}(x) = \sqrt{2M} x_1^+$ everywhere, which is in contradiction with the fact that $\{u_{00} > 0\}$ has unit density at 0.

As for the case (b), (5-20) shows that u_0 is nondegenerate at x_0 as long as x_0 is on the boundary of S. $\quad\square$

5E. *Properties of* $\Gamma_{1/2}$.

Lemma 5.13. *Suppose u_0 is not degenerate at $x_0 \in \partial\{u_0 > 0\} \setminus \{0\}$, such that $\Theta_*(x_0, \{u_0 = 0\}) > 0$. Then there is a unique component $\mathfrak{C}$ of $\partial\{u_0 > 0\}$ containing x_0 such that $\mathfrak{C} \subset \Gamma_{1/2}$.*

Proof. We only have to show the uniqueness of $\mathfrak{C}$; the rest follows from Lemmas 5.6 and 5.7. Suppose there are two components of $\partial\{u_0 > 0\} \setminus \{0\}$, $\mathfrak{C}_1$ and $\mathfrak{C}_2$, containing x_0. From the dimension-reduction argument as in the proof of Lemma 5.6, it follows that $\mathfrak{C}_1$ and $\mathfrak{C}_2$ have the same approximate tangent plane at x_0. This is in contradiction with our assumption $\Theta_*(x_0, \{u_0 = 0\}) > 0$. $\quad\square$

Lemma 5.14. *Let $\mathfrak{C}$ be a component of $\partial\{u_0 > 0\}$ such that $\mathfrak{C} \cap \Gamma_{1/2} \neq \varnothing$. Then $\mathfrak{C} \setminus \Gamma_{1/2} = \varnothing$; in other words all points of $\mathfrak{C}$ are in $\Gamma_{1/2}$.*

Proof. By Lemma 5.12 $\mathfrak{C}$ cannot have degenerate points; thus we have to show that $\Gamma_{1/2}$ cannot have limit points in Γ_1. Note that $\Gamma_{1/2}$ is of locally finite perimeter (see Lemma 5.8(i)) and hence locally it is a countable union of convex surfaces. Let $x_0 \in \Gamma_1 \cap \mathfrak{C}$ be a limit point of $\Gamma_{1/2} \cap \mathfrak{C}$. The generatrix of the cone $\partial\{u_0 > 0\}$ passing through x_0 splits $\mathfrak{C}$ into two parts, one of which must be convex near x_0 because by assumption x_0 is a limit point of $\Gamma_{1/2}$; see Theorem 5.11. The set $\{u_0 = 0\}^\circ$ propagates to x_0 because $\Gamma_{1/2}$ is a subset of reduced boundary. Thus, there is another subset of $\Gamma_{1/2}$ approaching x_0, and it is a part of the topological boundary of $\{u_0 = 0\}^\circ$. Therefore, the ray passing through x_0 is on the boundaries of two convex pieces of $\partial\{u_0 > 0\}$ (near x_0). Note that if these pieces of $\Gamma_{1/2}$ contain flat parts then from the unique continuation theorem we infer that $\partial\{u_0 > 0\}$ cannot have singularity at 0. Thus, they cannot contain flat parts and consequently the density of $\{u_0 > 0\}$ at x_0 cannot be 1, because by convexity of $\Gamma_{1/2}$ it follows that $\{u_0 \equiv 0\}^\circ$ has positive density at x_0. But this is in contradiction with the assumption $x_0 \in \Gamma_1$. $\quad\square$

Summarizing we have:

Proposition 5.15. *Let u_0 be as above and $N = 3$. Then $\partial\{u_0 > 0\}\backslash\{0\}$ is a union of smooth convex cones.*

5F. *Proof of Theorem B.* The first part of Theorem B follows from Lemma 5.12, while the second part is a corollary of Lemma 5.14 since $\Gamma_{1/2}$ coincides with the reduced boundary. Finally, the last part follows from Lemma 5.10, because by Lemma 5.14 the reduced boundary propagates instantaneously in $\partial\{u_0 > 0\}$.

6. Proof of Theorem C

6A. *Inverse Gauss map and the support function.* Suppose $N = 3$ and $u = rg(\theta, \phi)$, where

$$x = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

Then

$$\triangle u_0 = \frac{1}{r}\left(g_{\theta\theta} + \frac{\cos\theta}{\sin\theta}g_\theta + \frac{g_{\phi\phi}}{\sin^2\theta} + 2g\right).$$

Note that

$$\triangle_{\mathbb{S}^2} g = g_{\theta\theta} + \frac{\cos\theta}{\sin\theta}g_\theta + \frac{g_{\phi\phi}}{\sin^2\theta}$$

is the Laplace–Beltrami operator. Thus we get

$$\triangle_{\mathbb{S}^2} g + 2g = 0. \tag{6-1}$$

Let $H(n)$, $n \in \mathbb{S}^{N-1}$, be the Minkowski support function of some hypersurface $\mathcal{M}$. $H(n)$ is the distance between the point on $\mathcal{M}$ with normal n and the origin. It is known [Alexandroff 1939] that the eigenvalues of the matrix

$$\nabla^2_{ij} H(n) + \delta_{ij} H(n)$$

are the principal radii of curvature of the surface determined by H, where the second-order derivatives are taken with respect to an orthonormal frame at $n \in \mathbb{S}^{N-1}$. The support function uses the inverse of the Gauss map to parametrize the surface as

$$H(n) = G^{-1}(n) \cdot n.$$

Furthermore, we have the following formula for the Gauss curvature K [Alexandroff 1939]:

$$\frac{1}{K} = \det(\nabla^2_{ij} H(n) + \delta_{ij} H(n)). \tag{6-2}$$

The Gauss map is a local diffeomorphism whenever $K \neq 0$ [Langevin and Rosenberg 1988]. Since $u_0 = rg$ is harmonic in $\{u_0 > 0\}$, we infer that g is smooth on $\mathbb{S}^2 \cap \{g > 0\}$.

Remark 6.1. In higher dimensions (6-1) becomes

$$\triangle_{\mathbb{S}^{N-1}} g + (N-1)g = \sum_{i=1}^{N-1} \frac{1}{k_i} = \frac{\sigma_{N-2}(k)}{\sigma_{N-1}(k)} = 0, \quad \sigma_m = \sum_{i_1 < i_2 < \cdots < i_m} k_{i_1} k_{i_2} \cdots k_{i_m}, \tag{6-3}$$

where $\sigma_m(k)$ is the m-th elementary symmetric function and k_i, $i = 1, \ldots, N-1$, are the principal curvatures. Observe that any positive function $g > 0$ satisfying the equation $\Delta_{\mathbb{S}^{N-1}} g + (N-1)g = 0$ defines an $(N-2)$-minimal surface (i.e., $\sigma_{N-2}(k) = 0$) provided that the Gauss curvature satisfies $\sigma_{N-1} \neq 0$. From here we infer that the spherical parts of the homogeneous stationary points of the Alt–Caffarelli functional are support functions of an $(N-2)$-capillary surface in $\mathbb{S}^{N-1}$, because they are solutions to (6-3).

6B. *Catenoid is a solution.* Alt and Caffarelli [1981, page 110] constructed a weak solution which is not a minimizer. Their solution can be given explicitly as follows: let

$$x = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

and take

$$u(x) = r \max\left(\frac{f(\theta)}{f'(\theta_0)}, 0\right),$$

where

$$f(\theta) = 2 + \cos\theta \log\left(\frac{1-\cos\theta}{1+\cos\theta}\right) = 2 + \cos\theta \log\left(\tan^2\frac{\theta}{2}\right)$$

and θ_0 is the unique zero of f between 0 and $\frac{\pi}{2}$. The aim of this section is to show that f is the support function of catenoid. Recall that the principal radii of curvature of a smooth surface are the eigenvalues of the matrix $\nabla^2_{\mathbb{S}^{N-1}} H + \delta_{ij} H$, where the Hessian is taken with respect to the sphere $\mathbb{S}^{N-1}$ [Alexandroff 1939]. At each point where the Gauss curvature does not vanish, the zero mean curvature condition for $N = 3$ can be written as

$$\triangle_{\mathbb{S}^2} H + 2H = 0,$$

where $\triangle_{\mathbb{S}^2}$ is the Laplace–Beltrami operator and $H(n)$ is the value of Minkowski's support function corresponding to the normal $n \in \mathbb{S}^2$. From now on let us consider the (x, y)-variables on $\mathbb{R}^2$. Recall that by rotating the graph of $y(x) = a\cosh(x/a)$ around the x-axis one obtains a catenoid for some constant a. Thus it is enough to compute the support function for the graph of y. Let α be the angle the tangent line of y at $(x, y(x))$ forms with the x-axis. If n is the unit normal to the graph of y then $n = (-\sin\alpha, \cos\alpha)$ and

$$H(n) = (x, y(x)) \cdot n = -x\sin\alpha + a\cos\alpha \cosh\frac{x}{a}.$$

Noting that the unit tangent at $(x, y(x))$ is $(\cos\alpha, \sin\alpha)$ and equating with the slope of tangent line, which is $(\sinh(x/a), -1)$, we obtain

$$\cos\alpha = \frac{\sinh(x/a)}{\sqrt{1+\sinh^2(x/a)}}, \quad \sin\alpha = -\frac{1}{\sqrt{1+\sinh^2(x/a)}}.$$

From second equation we get that $\sinh(x/a) = \tan\alpha$ and solving the quadratic equation $e^{2(x/a)} - 1 = 2e^{(x/a)}\tan\alpha$ we find that

$$x = a\log\frac{1+\sin\alpha}{\cos\alpha}, \quad \cosh\frac{x}{a} = \frac{1}{\cos\alpha}.$$

Consequently,

$$H(n) = -\frac{a}{2}\sin\alpha \log\left(\frac{1+\sin\alpha}{\cos\alpha}\right)^2 + a.$$

Taking $\alpha = \theta + \frac{\pi}{2}$ we have

$$\frac{1+\sin\alpha}{\cos\alpha} = \frac{1+\cos\theta}{-\sin\theta} = \frac{2\cos^2\frac{\theta}{2}}{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = -\cot\frac{\theta}{2}$$

and thus choosing $a = 2$ the result follows.

6C. *Almost minimal immersions.* Consider the parametrization $\mathcal{X} : U_g \to \mathbb{R}^3$, where

$$\mathcal{X}(n) = ng(n) + \nabla_{\mathbb{S}^2}g, \quad U_g = \{g > 0\} \subset \mathbb{S}^2. \tag{6-4}$$

Let $\mathcal{M}$ be the hypersurface determined by $\mathcal{X}$. The spherical part g of u_0 solves (6-1) and by [Reznikov 1992, Theorem 1] $\mathcal{X}$ determines a smooth map which is either constant or a conformal minimal immersion outside a locally finite set of isolated singularities (branch points). Recall that if at some point p

$$\mathcal{X}_{\xi_1} \times \mathcal{X}_{\xi_2} = 0, \quad \mathcal{X} = \mathcal{X}(\xi_1, \xi_2) \text{ in local coordinates } \xi_1, \xi_2, \tag{6-5}$$

then p is called branch point; see [Nitsche 1989, page 314].

Observe that $\mathcal{X}(n)$ is the gradient of the blow-up u_0 at $n = x/|x|$. Indeed,

$$\begin{aligned}
\mathcal{X}(n) &= \frac{n}{r}rg + \frac{1}{r}\nabla_{\mathbb{S}^2}(rg) = \frac{n}{r}u_0(x) + \frac{1}{r}\nabla_{\mathbb{S}^2}u_0 \\
&= \frac{n}{r}(\nabla u_0(x) \cdot x) + \frac{1}{r}\nabla_{\mathbb{S}^2}u_0 = n\left(\nabla u_0(x) \cdot \frac{x}{|x|}\right) + \frac{1}{r}\nabla_{\mathbb{S}^2}u_0 = \nabla u_0(x). \tag{6-6}
\end{aligned}$$

In particular, the computation above shows that

$$\nabla u_0(x) = \nabla u_0\left(\frac{x}{|x|}\right), \quad \nabla_{\mathbb{S}^2}g(n) \perp n; \tag{6-7}$$

in other words the gradient is homogeneous of degree 0.

The absence of branch points does not rule out the possibility of self-intersection. Therefore we need to prove that under conditions of Theorem C $\mathcal{M}$ is embedded.

6D. *Dual cones and center of mass.* If u_0 is a blow-up and the assumptions in Theorem C are satisfied, then by virtue of Proposition 5.15 the free boundary $\partial\{u_0 > 0\} \setminus \{0\}$ is a union of smooth convex cones $\mathcal{C}_1$ and $\mathcal{C}_2$. We define the dual cones as

$$\mathcal{C}_i^\star = \partial\{y \in \mathbb{R}^3 : x \cdot y \leq 0, \ x \in \mathcal{C}_i\}, \quad i = 1, 2. \tag{6-8}$$

It is well known that the dual of a convex cone is also convex [Schneider 2014, page 35].

Lemma 6.2. *The largest principal curvature of $\mathcal{C}_i \setminus \{0\}$ is strictly positive.*

Proof. To fix the ideas, we prove the statement for $\mathcal{C}_1$. Note that one of the principal curvatures of $\mathcal{C}_1 \setminus \{0\}$ is zero because $\mathcal{C}_1$ is a cone and $\mathcal{C}_1 \setminus \{0\}$ is smooth; see Theorem B. Let $\kappa(p)$ be the largest principal curvature at $p \in \mathcal{C}_1 \setminus \{0\}$. Suppose there is p such that $\kappa(p) = 0$. Choose the coordinate system at p so that x_1 points in the outward normal direction at p (into $\{u_0 \equiv 0\}$), the x_2-axis is tangential at p and is the principal direction corresponding to $\kappa(p)$. Then we have $\nabla u_0(p) = e_1$, where e_1 is the unit direction of the x_1-axis, and the mean curvature of $\mathcal{C}_1$ at p vanishes because we assumed that $\kappa(p) = 0$. Writing the mean curvature at p in terms of the derivatives of u_0 we have

$$0 = \frac{\nabla u_0 D^2 u_0 (\nabla u_0)^T - |\nabla u_0|^2 \triangle u_0}{|\nabla u_0|^3} = \frac{\partial_{11} u_0}{\sqrt{2M}},$$

implying that $\partial_{11} u_0 = 0$. Moreover, since u_0 is homogeneous of degree 1, $\nabla u_0 = e_1$ along the x_1-axis. This yields $\partial_{13} u_0 = \partial_{23} u_0 = \partial_{33} u_0 = 0$ along the x_1-axis. From the harmonicity of u_0 it follows that $\partial_{22} u_0 = 0$ along the x_3-axis. Summarizing, we have that along the points of the x_3-axis the Hessian of u_0 has the form

$$\begin{pmatrix} 0 & \partial_{12} u_0 & 0 \\ \partial_{12} u_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Finally, letting $\sigma(t)$, $t \in (-\delta, \delta)$, be the parametrization of the curve along which the $x_1 x_2$-plane intersects $\mathcal{C}_1$ and differentiating $|\nabla u_0(\sigma(t))| = 1$ in t we get that at p one must have

$$0 = e_1 \begin{pmatrix} 0 & \partial_{12} u_0 & 0 \\ \partial_{12} u_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} e_2 = \partial_{12} u_0(p).$$

Thus, the Hessian $D^2 u_0$ vanishes along the x_1-axis. The function $w = \sqrt{2M} - \partial_1 u_0$ is harmonic in $\{u_0 > 0\}$ and $w \geq 0$ thanks to Lemma A.7. Moreover, $w(e_1) = 0 = \min w$. Since at e_1 the free boundary is regular, by Hopf's lemma $\partial_1 w = -\partial_{11} u_0 \neq 0$. However, $D^2 u_0(te_1) = 0$ for every $t > 0$ and hence $\partial_{11} w(e_1) = 0$, which is a contradiction. $\qquad \square$

Remark 6.3. It follows from Lemma 6.2 and Theorem B that there are two positive constants κ_0, κ_1 such that

$$0 < \kappa_0 \leq \kappa(p) \leq \kappa_1, \quad p \in (\partial\{u_0 > 0\} \setminus \{0\}) \cap \partial B_{\sqrt{2M}},$$

where $\kappa(p)$ is the largest curvature of $\partial\{u_0 > 0\}$ at $p \in (\partial\{u_0 > 0\} \setminus \{0\}) \cap \partial B_{\sqrt{2M}}$.

Let us put $\gamma_i = \mathbb{S}^2 \cap \mathcal{C}_i^\star$.

Lemma 6.4. *Let $\mathcal{C}_1^\star, \mathcal{C}_2^\star$ be the dual cones* (6-8). *Then we have:*

(i) *$\partial \mathcal{M}$ is differentiable and there are two positive constants $\kappa_0^\star, \kappa_1^\star$ such that the largest curvature $\kappa^\star(p)$ of $(\mathcal{C}_i^\star \setminus \{0\}) \cap \mathbb{S}^2$ satisfies $\kappa_0^\star \leq \kappa^\star(p) \leq \kappa_1^\star$.*

(ii) *There is $\delta > 0$ small such that every component E_δ of $\partial B_{1-\delta} \cap \mathcal{M}$ defines a convex cone $K_\delta = \{\sigma t : \sigma \in E_\delta, t > 0\}$,*

(iii) *$\mathcal{M}$ is star-shaped with respect to the origin and hence embedded.*

Proof. Suppose that $\mathcal{C}_1^{\star}$ is not differentiable at some $z \neq 0$. Then $\mathcal{C}_i$ must have a flat piece. Indeed, if n_1, n_2 are the normals of two supporting hyperplanes of $\mathcal{C}_i^{\star}$ at z then the unit vectors

$$n_t = \frac{tn_1 + (1-t)n_2}{|tn_1 + (1-t)n_2|}$$

define a support function at z for every $t \in (0, 1)$. Since the vectors n_t lie on the same plane, $\mathcal{C}_1$ must have a flat piece. The unique continuation theorem implies that the free boundary is a hyperplane and cannot have singularities. Now the desired estimate follows from Remark 6.3 and the definition of dual cone. The first claim is proved.

Let k_1, k_2 be the principal curvatures of $\mathcal{M}$. Then $k_1 + k_2 = 0$ and the Gauss curvature is $K = -k_1^2 = -k_2^2$. Since $\mathcal{M}$ is a smooth immersion, from (6-2) and the smoothness of $\mathcal{X} = \nabla u_0$ in U_g we see that $K \neq 0$. Furthermore, there is a tame constant $c_0 > 0$ such that $k_i^2 \geq c_0$, $i = 1, 2$, at every point of $\mathcal{M}$. Thus by virtue of the part (i) $\mathcal{M}$ is fibered by $\partial B_{1-\delta}$ for $\delta > 0$ small. We claim that $|\mathcal{X}(n)| > 0$, $n \in \overline{U}_g$. Clearly this is true if $n \in \partial U_g$, where $|\mathcal{X}(n)| = 1$. Suppose there is $n \in U_g$ such that $\mathcal{X}(n) = 0$. Since $\mathcal{X}(n) = ng + \nabla_{\mathbb{S}^2} g$, it follows that $g(n) = 0$, but this is impossible since $n \in \{g > 0\} = U_g$. From $g(n) = \mathcal{X}(n) \cdot n > 0$, $n \in U_g$, it follows that $\mathcal{M}$ is star-shaped with respect to the origin. Consequently, $\mathcal{M}$ is fibered by ∂B_t for every $t \in (0, 1)$ and hence embedded. $\qquad\square$

Let $n \in U_g$. Then from $\mathcal{X}(n) = \nabla u_0(n)$ it follows that

$$|\mathcal{X}(n)|_{\partial\{u_0 > 0\}}| = |\nabla u_0|_{\partial\{u_0 > 0\}}| = \sqrt{2M}.$$

Since by Lemma 6.4 $\mathcal{M}$ is differentiable along γ_i, we see that the contact angle α between $\mathcal{M}$ and $\mathbb{S}^2$ is

$$\cos \alpha = n \cdot \frac{\mathcal{X}(n)|_{\partial\{u_0 > 0\}}}{\sqrt{2M}} = g(n)|_{\partial\{u_0 > 0\}} = 0.$$

Thus, the minimal surface defined by g is inside of the sphere of radius $\sqrt{2M}$ because in view of Lemma A.7 $|\nabla u_0|^2 = g^2 + |\nabla g|^2 \leq 2M$. Moreover, $\mathcal{M}$ is tangential to $\mathcal{C}_1^{\star}$ and $\mathcal{C}_2^{\star}$ along $\mathbb{S}^2$ since $n \perp \nabla_{\mathbb{S}^2} g$ by (6-7).

We recall the definition of topological type $[\varepsilon, r, \chi]$ of hypersurface $\mathcal{M} \subset \mathbb{R}^3$ from [Nitsche 1985, page 47].

Definition 6.1. We say that $\mathcal{M}$ is of topological type $[\varepsilon, r, \chi]$ if it has orientation ε, Euler characteristic χ, and r boundary curves. Here $\varepsilon = \pm 1$, where $+1$ means that $\mathcal{M}$ is orientable and $\varepsilon = -1$ is nonorientable. For an orientable surface the Euler characteristic is defined by the relation $\chi = 2 - 2g - r$, where g is the genus of $\mathcal{M}$.

Now the first part of Theorem C follows from Nitsche's theorem [1985, page 2]. Moreover, the only stationary surfaces of disk type are the totally geodesic disks and the spherical cups. From Lemma 5.5 it follows that if $u_0 = rg$ and $\mathrm{supp}\, g$ is a disk then u_0 is a half-plane.

In view of Lemma 6.4(iii) the proof of Theorem C can be deduced from the result of [Nitsche 1962] but we will sketch a shorter proof based on Aleksandrov's moving plane method and Serrin's boundary lemma. We reformulate Theorem C as follows:

Lemma 6.5. *Let $\mathcal{M}$ be of topological type $[1, 2, 0]$, i.e., a ring-type minimal surface. Then $\mathcal{M}$ is a part of a catenoid.*

Proof. By Lemma 6.4(iii) $\mathcal{M}$ is embedded. In particular, $\mathcal{X}$ is a conformal minimal immersion (see the discussion in Section 6C).

Let $\partial\mathcal{M} = \gamma_1 \cup \gamma_2$. Then applying Stokes' formula we have

$$\int_{\mathcal{M}} \triangle_{\mathcal{M}}\mathcal{X} = \int_{\partial\mathcal{M}} n^* \, ds = \int_{\gamma_1} n^* + \int_{\gamma_2} n^* \, ds, \tag{6-9}$$

where n^* is the outward conormal, i.e., n^* is tangent to $\mathcal{M}$ and normal to $\partial\mathcal{M}$; see [Fang 1996, page 81]. Since $\mathcal{X}$ is minimal, $\triangle_{\mathcal{M}}\mathcal{X} = 0$. Thus

$$\int_{\gamma_1} n^* \, ds + \int_{\gamma_2} n^* \, ds = 0. \tag{6-10}$$

Since $\mathcal{M}$ is tangential to $\mathcal{C}_i^\star$ it follows that the conormal n^* on γ_i points in the direction of the generatrix of the dual cone $\mathcal{C}_i^\star$. Observe that if we use the arc-length parametrization of γ_i and let $s_k \in [0, |\gamma_i|]$ be some partition points then the sums $S_m = \sum_{k=0}^{m} n_k^{*i}(s_{k+1} - s_k)$, $n_k^{*i} \in \mathcal{C}_i^\star$, approximate the boundary integrals in (6-9). Consequently the vector S_m is strictly inside of the cone $\mathcal{C}_i^\star$ and in the limit converges to the center of mass of γ_i computed with respect to the origin (the vertex of the cone). In view of (6-10) there is a diameter of $\mathbb{S}^2$ strictly inside of both dual cones $\mathcal{C}_1^\star$ and $\mathcal{C}_2^\star$.

Without loss of generality we assume that the diameter passes through the north and south poles. Now we can apply Aleksandrov's moving plane method and Serrin's boundary point lemma to finish the proof. Let Π_t be the family of planes containing the x_1-axis where t measures the angle between Π_t and x_3-axis.

Now start rotating Π_t about the x_1-axis starting from a position when Π_t is a support hyperplane to either of the cones $\mathcal{C}_1^\star, \mathcal{C}_2^\star$ and $\Pi_t \cap \mathcal{C}_i^\star \neq \varnothing$, $i = 1, 2$.

<u>Case 1</u>: If the first touch of $\mathcal{M}$ and its reflection $\widetilde{\mathcal{M}}$ with respect to the plane Π_t occurs at some interior point of $\mathcal{M}$, then from the maximum principle it follows that $\mathcal{M} = \widetilde{\mathcal{M}}$.

By Lemma 6.4, both dual cones are strictly convex. Moreover, we claim that for δ small the cones generated by $\mathcal{M} \cap \partial B_{1-\delta}$ are convex, otherwise the inflection point would propagate to $\mathcal{C}_i^\star$.

The two remaining possibilities are:

<u>Case 2</u>: The first touch of $\mathcal{M}$ and its reflection $\widetilde{\mathcal{M}}$ occurs at some boundary point where $\partial\mathcal{M}$ is perpendicular to Π_t.

<u>Case 3</u>: The first touch of $\mathcal{M}$ and its reflection $\widetilde{\mathcal{M}}$ occurs at some boundary point where $\partial\mathcal{M}$ is not lying on Π_t.

We cannot directly apply Serrin's boundary point lemma [1971] because $\partial\mathcal{M}$ is only $C^{1,1}$ by virtue of Lemma 6.4. However, from the fibering of $\mathcal{M}$ near $\partial\mathcal{M}$ we conclude that $\tilde{g} \leq g$ near the contact point, where $\tilde{g}$ is the support function of $\widetilde{M}$. Thus $\tilde{u} = r\tilde{g} \leq rg = u$. Hence applying Serrin's boundary point lemma to the harmonic functions $\tilde{u}$ and u we conclude that $\mathcal{M} = \widetilde{\mathcal{M}}$.

Choosing Π_t to be an arbitrary family passing through a line perpendicular to the diameter it follows that γ_1, γ_2 are circles and (6-10) forces them to lie on parallel planes. Applying [Schoen 1983, Corollary 2] we infer that $\mathcal{M}$ is a part of catenoid. $\qquad\square$

Appendix

This section contains some well-known results about the solutions of the singular perturbation problem $(\mathcal{P}_\varepsilon)$. We begin with the uniform Lipschitz estimates of Luis Caffarelli; see [Caffarelli 1995] for the proof.

Proposition A.1. *Let $\{u_\varepsilon\}$ be a family of solutions of $(\mathcal{P}_\varepsilon)$. Then there is a constant C depending only on $N, \|\beta\|_\infty$ and independent of ε such that*

$$\|\nabla u_\varepsilon\|_{L^\infty(B_{1/2})} \leq C. \tag{A-1}$$

As a consequence we get that one can extract converging sequences $\{u_{\varepsilon_n}\}$ of solutions of $(\mathcal{P}_\varepsilon)$ such that the limit functions are stationary points of the Alt–Caffarelli problem.

Proposition A.2. *Let u_ε be a family of solutions to $(\mathcal{P}_\varepsilon)$ in a domain $\mathcal{D} \subset \mathbb{R}^N$. Let us assume that $\|u_\varepsilon\|_{L^\infty(\mathcal{D})} \leq A$ for some constant $A > 0$ independent of ε. For every $\varepsilon_n \to 0$ there exists a subsequence $\varepsilon_{n'} \to 0$ and $u \in C^{0,1}_{\mathrm{loc}}(\mathcal{D})$ such that*

(i) *$u_{\varepsilon_{n'}} \to u$ uniformly on compact subsets of $\mathcal{D}$,*

(ii) *$\nabla u_{\varepsilon_{n'}} \to \nabla u$ in $L^2_{\mathrm{loc}}(\mathcal{D})$,*

(iii) *u is harmonic in $\mathcal{D} \setminus \partial\{u > 0\}$.*

Proof. See [Caffarelli et al. 1997, Lemma 3.1]. $\qquad\square$

Next, we recall the estimates for the slopes of some global solutions.

Proposition A.3. *Let u be as in Proposition A.2. Then the following statements hold true*:

(i) *u is Lipschitz.*

(ii) *If $u_{\varepsilon_j} \to u = \alpha x_1^+$ locally uniformly, then $0 \leq \alpha \leq \sqrt{2M}$ (see [Caffarelli et al. 1997, Proposition 5.2]).*

(iii) *If $u_{\varepsilon_j} \to u = \alpha x_1^+ - \gamma x_1^- + o(|x|)$ and $\gamma > 0$ then $\alpha^2 - \gamma^2 = \sqrt{2M}$ (see [Caffarelli et al. 1997, Proposition 5.1]). In this lemma the essential assumption is that $\gamma > 0$.*

Remark A.4. Observe that if $u(x) = \alpha x_1^+ + \bar\alpha x_1^-$ then we must necessarily have that $\alpha = \bar\alpha \leq \sqrt{2M}$; see [Caffarelli et al. 1997, Proposition 5.3]. In this case the interior of the zero set of u is empty. Thus one might have a wedge-like solution.

Using Proposition A.1 we can extract a sequence u_{ε_j} for some sequence ε_j such that $u_{\varepsilon_j} \to u$ uniformly in $B_{1/2}$; see Proposition A.2. Let u be a limit and $0 < \rho_j \downarrow 0$ and $u_j(x) = u(x_0 + \rho_j x)/\rho_j$, $x_0 \in \partial\{u > 0\}$. Thanks to Proposition A.3(i) we can extract a subsequence, still labeled ρ_j, such that u_j converges to some function u_0 defined in $\mathbb{R}^N$. The function u_0 is called a blow-up limit of u at the free boundary point x_0 and it depends on $\{\rho_j\}$.

The two propositions to follow establish an important property of the blow-up limits, namely that the first and second blow-ups of u can be obtained from $(\mathcal{P}_\varepsilon)$ for a suitable choice of parameter ε. Observe that the scaled function $\nabla(u_{\varepsilon_j})_{\lambda_n}$ satisfies the equation

$$\triangle(u_{\varepsilon_j})_{\lambda_j} = \frac{\lambda_j}{\varepsilon_j}\beta\left(\frac{\lambda_j}{\varepsilon_j}(u_{\varepsilon_j})_{\lambda_j}\right). \tag{A-2}$$

Taking $\delta_j = \varepsilon_j/\lambda_j \to 0$ we see that $(u_{\varepsilon_j})_{\lambda_j}$ is solution to $\triangle u_{\delta_j} = \beta_{\delta_j}(u_{\delta_j})$.

Proposition A.5. *Let u_{ε_j} be a family of solutions to $(\mathcal{P}_\varepsilon)$ in a domain $\mathcal{D} \subset \mathbb{R}^N$ such that $u_{\varepsilon_j} \to u$ uniformly on $\mathcal{D}$ and $\varepsilon_j \to 0$. Let $x_0 \in \mathcal{D} \cap \partial\{u > 0\}$ and let $x_n \in \partial\{u > 0\}$ be such that $x_n \to x_0$ as $n \to \infty$. Let $\lambda_n \to 0$, $u_{\lambda_n}(x) = (1/\lambda_n)u(x_n + \lambda_n x)$ and $(u_{\varepsilon_j})_{\lambda_n} = (1/\lambda_n)u_{\varepsilon_j}(x_n + \lambda_n x)$. Assume that $u_{\lambda_n} \to U$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}^N$. Then there exists $j(n) \to \infty$ such that for every $j_n \geq j(n)$ it holds that $\varepsilon_j/\lambda_n \to 0$ and*

- *$(u_{\varepsilon_{j_n}})_{\lambda_n} \to U$ uniformly on compact subsets of $\mathbb{R}^N$,*

- *$\nabla(u_{\varepsilon_{j_n}})_{\lambda_n} \to \nabla U$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$,*

- *$\nabla u_{\lambda_n} \to \nabla U$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$.*

Proof. See [Caffarelli et al. 1997, Lemma 3.2]. $\qquad\square$

Finally, recall that the result of the previous proposition extends to the second blow-up.

Proposition A.6. *Let u_{ε_j} be a solution to $(\mathcal{P}_\varepsilon)$ in a domain $\mathcal{D}_j \subset \mathcal{D}_{j+1}$ and $\bigcup_j \mathcal{D}_j = \mathbb{R}^N$ such that $u_{\varepsilon_j} \to U$ uniformly on compact sets of $\mathbb{R}^N$ and $\varepsilon_j \to 0$. Let us assume that for some choice of positive numbers d_n and points $x_n \in \partial\{U > 0\}$, the sequence*

$$U_{d_n}(x) = \frac{1}{d_n}U(x_n + d_n x)$$

converges uniformly on compact sets of $\mathbb{R}^N$ to a function U_0. Let

$$(u_{\varepsilon_j})_{d_n} = \frac{1}{d_n}u_{\varepsilon_j}(x_n + d_n x).$$

Then there exists $j(n) \to \infty$ such that for every $j_n \geq j(n)$, it holds that $\varepsilon_{j_n}/d_n \to 0$ and

- *$(u_{\varepsilon_{j_n}})_{d_n} \to U_0$ uniformly on compact subsets of $\mathbb{R}^N$,*

- *$\nabla(u_{\varepsilon_j})_{d_n} \to \nabla U_0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$.*

Proof. See [Caffarelli et al. 1997, Lemma 3.3]. $\qquad\square$

The next lemma contains one of the crucial estimates needed for the proof of Proposition 5.15.

Lemma A.7. *Let $u \geq 0$ be as in Proposition A.2. Then*

$$\limsup_{x \to x_0, u(x) > 0} |\nabla u(x)| \leq \sqrt{2M}.$$

Proof. To fix the ideas we let $x_0 = 0$ and $l = \limsup_{x \to 0, u(x) > 0} |\nabla u(x)|$. Suppose $l > 0$, otherwise we are done. Choose a sequence $z_k \to 0$ such that $u(z_k) > 0$ and $|\nabla u(z_k)| \to l$. Setting $\rho_k = |y_k - z_k|$, where $y_k \in \partial\{u > 0\}$ is the nearest point to z_k on the free boundary and proceeding as in the proof of [Alt et al. 1984a, Lemma 3.4] we can conclude that the blow-up sequence $u_k(x) = \rho_k^{-1} u(z_k + \rho_k x)$ has a limit u_0 (at least for a subsequence, thanks to Proposition A.1) such that $u_0(x) = l x_1$, $x_1 > 0$, in a suitable coordinate system. Moreover, by Proposition A.5 it follows that u_0 is a limit of some u_{λ_j} solving $\triangle u_{\lambda_j} = \beta_{\lambda_j}(u_{\lambda_j})$ in B_{r_j}, $r_j \to \infty$. If there is a point $z \in \{x_1 = 0\}$ and $r > 0$ such that $u_0 > 0$ in $B_r(z) \cap \{x_1 < 0\}$ then near z we must have $u_0(x) = l(x - z)_1^+ + l(x - z)_1^- + o(x - z)$; see Remark A.4. Applying the unique continuation theorem to $u_0(x) - u_0(-x_1, x_2, \ldots, x_n)$ we see that $u_0 = l(-x_1)^+$, $x_1 < 0$. Thus recalling Remark A.4 again we infer that $l \le \sqrt{2M}$. $\qquad\square$

Finally, we mention a useful identity for the solutions u_ε; see [Caffarelli et al. 1997, equation (5.2)]: Let u_ε be a solution of $(\mathcal{P}_\varepsilon)$. Then for any $\phi \in C_0^\infty(B_1)$ there holds

$$\int \left(\frac{|\nabla u_\varepsilon|^2}{2} + \mathcal{B}\left(\frac{u_\varepsilon}{\varepsilon}\right) \right) \partial_1 \phi = \int \sum_k \partial_k u_\varepsilon \, \partial_1 u_\varepsilon \partial_k \phi. \tag{A-3}$$

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