

# ANALYSIS & PDE

Volume 13      No. 1      2020

MANUEL DEL PINO, MONICA MUSSO AND JUNCHENG WEI

**INFINITE-TIME BLOW-UP FOR  
THE 3-DIMENSIONAL ENERGY-CRITICAL HEAT EQUATION**



## INFINITE-TIME BLOW-UP FOR THE 3-DIMENSIONAL ENERGY-CRITICAL HEAT EQUATION

MANUEL DEL PINO, MONICA MUSSO AND JUNCHENG WEI

We construct globally defined in time, unbounded positive solutions to the energy-critical heat equation in dimension 3

$$u_t = \Delta u + u^5 \quad \text{in } \mathbb{R}^3 \times (0, \infty), \quad u(x, 0) = u_0(x) \quad \text{in } \mathbb{R}^3.$$

For each  $\gamma > 1$  we find initial data (not necessarily radially symmetric) with  $\lim_{|x| \rightarrow \infty} |x|^\gamma u_0(x) > 0$  such that as  $t \rightarrow \infty$

$$\|u(\cdot, t)\|_\infty \sim t^{\gamma - \frac{1}{2}} \quad \text{if } 1 < \gamma < 2, \quad \|u(\cdot, t)\|_\infty \sim \sqrt{t} \quad \text{if } \gamma > 2, \quad \|u(\cdot, t)\|_\infty \sim \sqrt{t} (\ln t)^{-1} \quad \text{if } \gamma = 2.$$

Furthermore we show that this infinite-time blow-up is codimensional-1 stable. The existence of such solutions was conjectured by Fila and King (*Netw. Heterog. Media* **7:4** (2012), 661–671).

### 1. Introduction

Let  $n \geq 3$ . The *energy-critical heat equation* in  $\mathbb{R}^n$  is the parabolic Cauchy problem

$$\begin{cases} u_t = \Delta u + |u|^{\frac{4}{n-2}} u & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^n. \end{cases} \quad (1-1)$$

The *energy*

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 - \frac{n-2}{2n} \int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}}$$

defines a Lyapunov functional for problem (1-1). In fact for classical solutions  $u(x, t)$  with sufficient decay in space variable we have that

$$\frac{d}{dt} E(u(\cdot, t)) = - \int_{\mathbb{R}^n} |u_t|^2.$$

Classical parabolic theory yields that the Cauchy problem (1-1) is well-posed in its natural finite-energy space for short time intervals.

In this paper we are interested in *positive finite-energy solutions* of (1-1) which are global in time, namely defined and smooth in the entire time interval  $(0, \infty)$ . The presence of the Lyapunov functional implies that limits of bounded solutions along sequences  $t = t_n \rightarrow +\infty$  can only be steady states, namely solutions of the Yamabe equation

$$\Delta u + |u|^{\frac{4}{n-2}} u = 0 \quad \text{in } \mathbb{R}^n. \quad (1-2)$$

*MSC2010:* 35B33, 35B40, 35K58.

*Keywords:* blow-up, critical exponents, nonlinear parabolic equations.

All positive solutions of (1-2) are given by the *Aubin–Talenti bubbles*

$$U_{\mu,\xi}(x) = \mu^{-\frac{n-2}{2}} w\left(\frac{x-\xi}{\mu}\right),$$

where  $\mu > 0$ ,  $\xi \in \mathbb{R}^n$  and

$$w(x) = (n(n-2))^{\frac{n-2}{4}} \left(\frac{1}{1+|x|^2}\right)^{\frac{n-2}{2}}.$$

They are precisely the extremals of Sobolev’s embedding. The *criticality* of problem (1-1) refers to the presence of this continuum of steady states which become singular as  $\mu \rightarrow 0$ , in addition to energy invariance. In fact we immediately see that

$$E(U_{\mu,\xi}) = E(U) \quad \text{for all } \xi \in \mathbb{R}^n, \mu > 0.$$

A solution  $u(x, t)$  of (1-1) which around one or more points of space looks like  $u(x, t) \approx U_{\mu(t),\xi(t)}(x)$  with  $\mu(t) \rightarrow 0$  is called a bubbling blow-up solution. Bubbling phenomena is present in many important time-dependent and stationary settings, usually carrying deep meaning in the global structure of their solutions. Notable examples include the Yamabe and harmonic map flows and the Keller–Segel chemotaxis system; see [Ciraolo et al. 2018; Daskalopoulos et al. 2018; Raphaël and Schweyer 2013; Davila et al. 2017; Ghoul and Masmoudi 2016]. In the last decade or so it has been extensively studied in energy-critical wave equations, Schrödinger maps and other dispersive settings.

Problem (1-1) is a simple-looking model which contains much of the complexity of the bubbling blow-up issue. Basic questions have remain unanswered until today. Existence or nonexistence of infinite-time bubbling positive solutions in problem (1-1) is not known. This question has been explicitly stated for instance in [Poláčik and Yanagida 2014; Quittner and Souplet 2007, Remark 22.10]. Detecting such solutions rigorously is not easy. Usual behaviors in the flow (1-1) are either asymptotic vanishing or blow-up in finite time. Global solutions with nontrivial asymptotic patterns are typically unstable objects and hence harder to be detected.

In a very interesting paper Fila and King [2012] provided insight on the question in the case of a radially symmetric, positive initial condition with an exact power decay rate. Using formal matching asymptotic analysis, they demonstrated that the power decay determines the blow-up rate in a precise manner. Intriguingly enough, their analysis leads them to conjecture that infinite-time blow-up *should only happen* in low dimensions 3 and 4; see Conjecture 1.1 in [Fila and King 2012].

In this paper we rigorously establish the existence of solutions with infinite-time blow-up in dimension 3, confirming the conjecture in [Fila and King 2012]. Thus we consider the Cauchy problem

$$\begin{cases} u_t = \Delta u + u^5 & \text{in } \mathbb{R}^3 \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^3 \end{cases} \quad (1-3)$$

for an initial datum  $u_0$  which we assume first radially symmetric with an exact power decay of the form

$$\lim_{|x| \rightarrow \infty} |x|^\gamma u_0(x) =: A > 0. \quad (1-4)$$

As in [Fila and King 2012] we assume that  $\gamma > 1$ , which means that  $u_0$  decays faster than the bubble

$$w(x) = 3^{\frac{1}{4}} \left( \frac{1}{1 + |x|^2} \right)^{\frac{1}{2}}. \quad (1-5)$$

**Theorem 1.1.** *Given  $\gamma > 1$ , there exists a positive, radially symmetric global solution  $u(x, t)$  to problem (1-3) whose initial condition  $u_0(|x|)$  satisfies (1-4) and as  $t \rightarrow +\infty$*

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^3)} \sim \begin{cases} t^{\frac{\gamma-1}{2}} & \text{if } 1 < \gamma < 2, \\ \sqrt{t}/\ln t & \text{if } \gamma = 2, \\ \sqrt{t} & \text{if } \gamma > 2. \end{cases} \quad (1-6)$$

More precisely, the blow-up takes place by bubbling near the origin. The solution of Theorem 1.1 is in the inner self-similar region,  $|x| \ll \sqrt{t}$ , in leading order of the *bubbling blow-up form*

$$u(x, t) \sim \frac{1}{\mu(t)^{\frac{1}{2}}} w\left(\frac{x}{\mu(t)}\right),$$

where

$$\mu(t) \sim \begin{cases} t^{1-\gamma} & \text{if } 1 < \gamma < 2, \\ t^{-1} \ln^2 t & \text{if } \gamma = 2, \\ t^{-1} & \text{if } \gamma > 2 \end{cases} \quad (1-7)$$

and  $w$  is given by (1-5). In the outer self-similar region  $|x| \gg \sqrt{t}$ , the solution dissipates in the form of a self-similar solution of heat equation  $u_t = \Delta u$  in  $\mathbb{R}^3 \times (0, \infty)$ .

A surprising feature of the construction is the dynamics discovered for the scaling parameter  $\mu(t)$ . It has a highly nonlocal character governed by an equation involving a perturbation of the fractional  $\frac{1}{2}$ -Caputo derivative. In fact, in order to find the precise lower-order corrections needed for the scaling parameter  $\mu(t)$  we will need to solve linear equations of the type

$$\int_0^t \frac{\beta'(s)}{\sqrt{t-s}} (1 - e^{-\frac{M^2}{(t-s)}}) ds = h(t)$$

for suitably decaying right-hand sides  $h(t)$ . See (6-8) and (6-13) below.

Problem (1-1) is a special case of the *Fujita equation*

$$\begin{cases} u_t = \Delta u + u^p & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^n, \end{cases} \quad (1-8)$$

with  $p > 1$ . Blow-up phenomena in problem (1-8) is extremely sensitive to the values of the exponent  $p$ . A vast literature has been devoted to this problem after the seminal work [Fujita 1966]. We refer the reader for instance to the book [Quittner and Souplet 2007] for background and a comprehensive account of results until 2007 and to the more recent works [Matano and Merle 2004; 2009; 2011]. The case  $p = (n+2)/(n-2)$  is special in many ways. Positive steady states do not exist when  $p < (n+2)/(n-2)$ . Positive radial global solutions must be bounded and go to zero; see [Poláčik and Quittner 2006; Poláčik et al. 2007; Quittner and Souplet 2007]. They exist when  $p > (n+2)/(n-2)$  but they have infinite

energy; see [Gui et al. 1992]. Infinite-time blow-up exists in that case but it has an entirely different nature; see [Poláčik and Yanagida 2003; 2014].

The study of energy-critical problems has attracted much attention in the last decade. For energy-critical wave equations, blow-up solutions have been characterized and constructed in [Duyckaerts et al. 2012; 2013; 2016a; 2016b; Krieger et al. 2009]. In [Schweyer 2012] Type-II sign changing, finite time blow-up for (1-1) is constructed, first formally predicted in [Filippas et al. 2000]. Threshold dynamics around the steady states of (1-1) has been characterized in large dimensions  $n \geq 7$  in [Collot et al. 2017]. Also in large dimensions  $n \geq 5$  in [Cortázar et al. 2016] infinite-time bubbling solutions of (1-1) in a bounded domain under Dirichlet boundary conditions are constructed for  $n \geq 5$ . The cases  $n = 3, 4$  are indeed considerably more delicate and not treated there. The solutions in Theorem 1.1 are specially meaningful for the full dynamics since they are *threshold solutions* in the sense that the solution of (1-3) with initial condition  $\lambda u_0$  goes to zero as  $t \rightarrow \infty$  if  $\lambda < 1$ , while it blows-up in finite time if  $\lambda > 1$ . Radial threshold solutions for various ranges of exponents in (1-3) are analyzed in [Quittner and Souplet 2007].

We recall that from [Fila and King 2012], it is not expected to have this blow-up in entire space in dimensions  $n \geq 5$ . Our approach is entirely different from that in [Schweyer 2012] for  $n = 4$  in which a finite-time type-II blow-up solution of (1-1) is constructed on the basis of the modulation equation methods developed for critical dispersive equations in [Donninger and Krieger 2013; Ortoleva and Perelman 2013; Merle et al. 2013; Raphaël and Rodnianski 2012; Raphaël and Schweyer 2013].

Our approach has a parabolic-elliptic flavor, in line with the recent works [Cortázar et al. 2016; Davila et al. 2017]. Since our proofs only rely on elliptic and parabolic estimates, we can easily modify the proof to deal with nonradial and general initial data, in particular establishing *codimension-1 stability* of the solution built. This is concordant with a result of [Krieger et al. 2015] on the corresponding wave analogue. In Section 10 we prove the following:

**Theorem 1.2.** *Let  $\bar{v}_0 = \bar{v}_0(x)$  be a positive continuous function, uniformly bounded for  $x \in \mathbb{R}^3$ . Let  $\gamma > 1$  and  $\kappa > \max\{(\gamma + 3)/2, \gamma\}$ . Then, there exists a positive global solution  $u(x, t)$  to problem (1-3) with initial condition*

$$u(x, 0) = u_0(|x|) + \frac{\bar{v}_0(x)}{|x|^\kappa} \left[ 1 - \eta\left(\frac{|x|}{t_0}\right) \right],$$

where  $u_0$  is positive, radially symmetric, satisfies (1-4),  $t_0 > 0$  is a fixed large number and  $\eta$  is a smooth cut-off function with  $\eta(s) = 1$  for  $s < 1$  and  $\eta(s) = 0$  for  $s > 2$ . As  $t \rightarrow +\infty$ ,  $u(x, t)$  satisfies (1-6).

Furthermore, there exists a codimension-1 manifold of functions in  $C^1(\mathbb{R}^3)$  converging to 0 at infinity, with a sufficiently fast decay, that contains  $u_0(|x|) + (\bar{v}_0(x)/|x|^\kappa)(1 - \eta(|x|/t_0))$  such that if  $\bar{u}_0$  lies in that manifold and it is sufficiently close to  $u_0(|x|) + (\bar{v}_0(x)/|x|^\kappa)(1 - \eta(|x|/t_0))$  in the sense that  $\bar{u}_0 = u_0(|x|) + (\bar{v}_0(x)/|x|^\kappa)(1 - \eta(|x|/t_0)) + \mathcal{O}(|x|e^{-b|x|})$  for some  $b > 0$ , then the solution  $\bar{u}(x, t)$  to (1-3) with  $\bar{u}(x, 0) = \bar{u}_0(x)$  is global in time and satisfies (1-6).

In the nonradial setting, the profile of the solution in the inner self-similar regime is

$$u(x, t) \sim \frac{1}{\mu(t)^{\frac{1}{2}}} w\left(\frac{x - p(t)}{\mu(t)}\right), \quad \frac{|p(t)|}{\mu(t)} \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

where  $w$  is given by (1-5) and  $\mu$  satisfies the asymptotics (1-7). Precise description of the dynamics of the center  $p = p(t)$  is provided.

A surprising feature of the construction is the dynamics discovered for the scaling parameter  $\mu(t)$ . It has a highly nonlocal character governed by an equation involving a perturbation of the fractional  $\frac{1}{2}$ -Caputo derivative. In fact, in order to find the precise lower-order corrections needed for the scaling parameter  $\mu(t)$  we will need to solve linear equations of the type

$$\int_0^t \frac{\beta'(s)}{\sqrt{t-s}} (1 - e^{-\frac{M^2}{(t-s)}}) ds = h(t)$$

for suitably decaying right-hand sides  $h(t)$ . See (6-8) and (6-13) below.

We believe that an approach similar to that in this paper could be used to prove the existence of a global unbounded solution when  $N = 4$ ,  $p = 3$  as conjectured in [Fila and King 2012]. We will undertake that issue in a future work.

The proof of Theorem 1.1 starts with the construction of an approximate solution to problem (1-3) with the asymptotic behavior described in (1-6). This is done in full detail in Section 2. We then show the existence of an actual solution to problem (1-3) deforming the approximation, by means of a *inner-outer gluing* procedure. This scheme is described in Section 3, and its proof is addressed in Sections 4–9. In Section 10 we prove Theorem 1.2. Appendices A–C gather some technical results needed to prove the theorems.

In the rest of the paper, we shall denote by  $C$  a generic positive constant, whose value may change from line to line, and within the same line. We shall use the notation  $c$  to indicate a positive constant, with  $c < 1$ , whose explicit value may change from line to line. Furthermore,  $t_0$  will denote a large fixed positive number and

$$\eta : \mathbb{R} \rightarrow \mathbb{R} \tag{1-9}$$

a smooth cut-off function with  $\eta(s) = 1$  for  $s < 1$  and  $= 0$  for  $s > 2$ .

## 2. Construction of an approximate solution and estimate of the associated error

After shifting the initial time to  $t_0 > 0$ , problem (1-3) takes the form

$$u_t = \Delta u + u^5 \quad \text{in } \mathbb{R}^3 \times (t_0, \infty), \tag{2-1}$$

with initial condition  $u_0(r) = u(r, t_0)$  satisfying

$$\lim_{r \rightarrow \infty} r^\gamma u_0(r) = A > 0 \quad \text{for some } \gamma > 1. \tag{2-2}$$

This section is devoted to the construction of a first approximation for a solution to (2-1)–(2-2), and to the description of the associated error.

The first approximation is built by matching an inner profile, made upon solving the elliptic problem

$$\Delta u + u^5 = 0 \quad \text{in } \mathbb{R}^3 \tag{2-3}$$

and an outer profile, made upon solving the heat equation in the whole space

$$u_t = \Delta u \quad \text{in } \mathbb{R}^3, \quad (2-4)$$

in the set of functions satisfying the decaying conditions (2-2). It is constructed in Section 2A (for the inner profile) and Section 2B (for the outer profile), and in Section 2C we derive a precise description of the *error* of approximation. In [Fila and King 2012], this approximate solution was already derived. We realize though that, for our rigorous construction to work, we need a further improvement of the approximation. This is done in Section 2D, where we introduce a next correction term, and describe the associated error. It turns out that this next correction term gives the right dynamics for the blow-up rate which turns out to be governed by a nonlocal differential equation with a fractional time-derivative closely related to the so-called  $\frac{1}{2}$ -Caputo derivative. See (6-13).

**2A. Construction of the first inner profile.** We recall that all positive radially symmetric solutions to (2-3) constitute a one-parameter family of functions, which are given explicitly by

$$w(r) = 3^{\frac{1}{4}} \left( \frac{1}{1+r^2} \right)^{\frac{1}{2}}, \quad w_\mu(r) = \mu^{-\frac{1}{2}} w\left(\frac{r}{\mu}\right), \quad (2-5)$$

for any positive number  $\mu > 0$ ; see [Aubin 1976; Caffarelli et al. 1989]. We denote by  $Z_0$  the only bounded and radial function belonging to the kernel of the linear operator

$$L_0(\phi) = \Delta\phi + 5w^4\phi. \quad (2-6)$$

See [Rey 1990]. The function  $Z_0$  is explicitly defined by

$$Z_0(r) = -\left[ \frac{w}{2} + w'(r)r \right] = \frac{3^{\frac{1}{4}}}{2} \frac{r^2 - 1}{(1+r^2)^{\frac{3}{2}}}. \quad (2-7)$$

Given  $Z_0$ , we denote by  $\Phi_1(r)$  the solution to

$$\Delta\Phi_1 + 5w^4\Phi_1 = Z_0, \quad (2-8)$$

defined as

$$\begin{aligned} \Phi_1(r) &= \Phi_0(r) + \pi_0 + \bar{\Phi}_1(r), \quad \text{where } \Phi_0(r) = \frac{3^{\frac{1}{4}}}{4}r, \\ \left( 5 \int_0^\infty w^4 Z_0 r^2 dr \right) \pi_0 &= \int_0^\infty \left( Z_0 - \frac{3^{\frac{1}{4}}}{2r} \right) Z_0 r^2 dr - 5 \int_0^\infty w^4 \Phi_0 Z_0 r^2 dr, \end{aligned} \quad (2-9)$$

and  $\bar{\Phi}_1$  is the unique solution to

$$\Delta\phi + 5w^{p-1}\phi = \underbrace{\left( Z_0 - \frac{3^{\frac{1}{4}}}{2r} \right) - 5w^4(\Phi_0 + \pi_0)}_{:=\Pi_0(r)},$$

explicitly given by

$$\bar{\Phi}_1(r) = \tilde{Z}(r) \int_0^r \Pi_0(s) Z_0(s) s^2 ds - Z_0(r) \int_0^r \Pi_0(s) \tilde{Z}(s) s^2 ds.$$

In the above expression,  $\tilde{Z}$  denotes another solution to  $\Delta\phi + 5w^4\phi = 0$ , linearly independent to  $Z_0$ .  $\tilde{Z}$  satisfies the asymptotic behavior  $\tilde{Z}(s) \sim s^{-1}$  as  $s \rightarrow 0$ , and  $\tilde{Z}(s) \sim 1$  as  $s \rightarrow \infty$ .

A closer look at the expression of  $\bar{\Phi}_1$  gives

$$\|r^{2-\sigma}\bar{\Phi}_1(r)\|_\infty < C$$

for some fixed positive constant  $C$  and any  $\sigma > 0$  small.

**Remark 2.1.** The solution to (2-8) is not unique. (In fact one can add any multiple of  $Z_0$ .) The choice we made in (2-9) is used to match the outer solution in the next section.

We have now the elements to define the first inner profile. We introduce a smooth positive function  $\mu(t)$  of the form

$$\mu(t) = \mu_0(t)(1 + \Lambda(t))^2, \quad \text{where } \mu_0(t) > 0, \quad \lim_{t \rightarrow \infty} \mu_0(t) = 0. \quad (2-10)$$

The function  $\mu_0$  will be defined below, see (2-23), (2-32), (2-36), as an explicit function of  $t$  depending on the decay rate  $\gamma$ . On the other hand, the function  $\Lambda = \Lambda(t)$  will be left as a parameter in the construction, and it will be determined in the final argument to get an actual solution to the problem. In the meanwhile, we shall assume that  $\Lambda = \Lambda(t)$  is a smooth function in  $(t_0, \infty)$ , defined by

$$\Lambda(t) := \int_t^\infty \lambda(s) ds,$$

where  $\lambda$  satisfies

$$\|\lambda\|_\# := \sup_{t > t_0} \mu_0(t)^{-1} t [\|\lambda\|_{\infty, [t, t+1]} + [\lambda]_{0, \sigma, [t, t+1]}] \leq \ell \quad (2-11)$$

for  $\sigma = \frac{1}{2} + \sigma'$ , with  $\sigma' > 0$  small, and for some fixed constant  $\ell$ . Here we intend

$$\begin{aligned} \|f\|_{\infty, [t, t+1]} &= \sup_{s \in [t, t+1]} |f(s)|, \\ [f]_{0, \sigma, [t, t+1]} &= \sup_{s_1 \neq s_2 \in [t, t+1]} \frac{|f(s_1) - f(s_2)|}{|s_1 - s_2|^\sigma}. \end{aligned}$$

For a later purpose we introduce the space

$$X_\# = \{\lambda \in C(t_0, \infty) : \|\lambda\|_\# \text{ is bounded}\}. \quad (2-12)$$

With this in mind, we define the inner approximation to be

$$u_{\text{in}}(r, t) = w_\mu(r) + \mu'_0 \psi_1(r, t), \quad \psi_1(r, t) = \mu^{\frac{1}{2}} \Phi_1\left(\frac{r}{\mu}\right). \quad (2-13)$$

A direct computation gives

$$\Delta\psi_1 + 5w_\mu^4\psi_1 = -\mu^{-\frac{3}{2}}Z_0\left(\frac{r}{\mu}\right) = \frac{\partial w_\mu}{\partial \mu}(r).$$

In the region  $\{r : r > R\mu_0\}$ , where  $R$  is any large but fixed positive number, the inner approximation looks like

$$u_{\text{in}}(r, t) = 3^{\frac{1}{4}} \frac{\mu^{\frac{1}{2}}}{r} - \frac{3^{\frac{1}{4}}}{4} \mu'_0 \mu^{-\frac{1}{2}} r + \mu_0^{\frac{1}{2}} \mu'_0 \Theta[\mu](r, t) + \frac{\mu_0^{\frac{1}{2}}}{r} \left( \frac{\mu_0}{r} \right)^2 \Theta[\mu](r, t), \quad (2-14)$$

where  $\Theta[\mu](r, t)$  denotes a generic function, which depends smoothly on  $\mu$ , and on  $(r, t)$ , and which is uniformly bounded, for parameters  $\mu$  satisfying (2-10), for  $r$  in the considered region, and any  $t$  large.

**2B. Construction of the first outer profile and choice of  $\mu_0(t)$ .** The outer profile is chosen to satisfy the heat equation  $u_t = \Delta u$ , in the whole space  $\mathbb{R}^3$ , and to fit the requested decaying property for the initial condition (2-2). Its properties and exact definitions change depending on the value of the decay rate  $\gamma$  of the initial condition  $u_0$ ; see (2-2). We consider three different situations:  $1 < \gamma < 2$ ,  $\gamma = 2$ , and  $\gamma > 2$ .

Case 1:  $1 < \gamma < 2$ . In this case we define  $u_{\text{out}}$  as

$$u_{\text{out}}(r, t) = t^{-\frac{\gamma}{2}} g\left(\frac{r}{\sqrt{t}}\right), \quad (2-15)$$

with  $g$  the positive solution to

$$g''(s) + \left(\frac{2}{s} + \frac{s}{2}\right) g'(s) + \frac{\gamma}{2} g(s) = 0, \quad s \in (0, \infty), \quad (2-16)$$

that satisfies the properties

- (1)  $\lim_{s \rightarrow \infty} s^\gamma g(s) = A$ ,
- (2)  $\lim_{s \rightarrow 0^+} s g(s) = d$  for a certain positive constant  $d$  for which  $\lim_{s \rightarrow 0^+} [g(s) - d/s] = 0$ .

Such a function  $g$  indeed exists. Let

$$L_\nu(g) = g'' + \left(\frac{2}{s} + \frac{s}{2}\right) g' + \nu g, \quad s \in (0, \infty).$$

In Appendix A, we prove the following:

**Lemma 2.2.** *If  $\frac{1}{2} < \nu < 1$ , there exist two positive linearly independent solutions  $y_1 = y_1(s)$  and  $y_2 = y_2(s)$  to*

$$L_\nu(g) = 0, \quad s \in (0, \infty), \quad (2-17)$$

that satisfy respectively

$$y_1(s) = \frac{1}{s} + (\nu - 1) \left( \int_0^\infty s y_1(s) ds \right) + \frac{1 - 2\nu}{4} s + O(s^2) \quad \text{if } s \rightarrow 0^+, \quad (2-18)$$

$$y_2(s) = c_2 + o(s) \quad \text{if } s \rightarrow 0^+, \quad (2-19)$$

$$y_1(s) = c_1 e^{-\frac{s^2}{4}} s^{4\nu-3}, \quad y_2(s) = \frac{1}{s^{2\nu}} \left( 1 + o\left(\frac{1}{s}\right) \right) \quad \text{if } s \rightarrow \infty \quad (2-20)$$

for some positive constants  $c_1, c_2$ .

Thanks to the lemma, which we apply to solve (2-16) when  $\nu = \gamma/2$ , we get that the function  $g$  we are looking for in (2-15) is thus given by

$$g(s) = dy_1(s) + Ay_2(s), \quad \text{with } d = \frac{2Ay_2(0)}{(2-\gamma)\left(\int_0^\infty sy_1(s) ds\right)} > 0. \quad (2-21)$$

We observe that, in a region like  $r < R^{-1}\sqrt{t}$ , for some large but fixed  $R$ , we get

$$u_{\text{out}}(r, t) = d \frac{t^{-\frac{\gamma-1}{2}}}{r} + t^{-\frac{\gamma+1}{2}} A \frac{(1-\gamma)y_2(0)}{2(2-\gamma)\int_0^\infty zy_1(z) dz} r + t^{-\frac{\gamma}{2}} O\left(\frac{r^2}{t}\right). \quad (2-22)$$

We next choose the function  $\mu_0(t)$  in the definition of  $\mu(t)$ , (2-10), in such a way that the functions  $u_{\text{in}}$  and  $u_{\text{out}}$  automatically match in the whole region  $R\mu_0 < r < R^{-1}\sqrt{t}$ , for some  $R$  large, but fixed independent of  $t$ . This is possible if

$$\mu_0(t) = \frac{d^2}{\sqrt{3}} t^{1-\gamma}. \quad (2-23)$$

Indeed, with this choice for  $\mu_0(t)$ , and given the bound (2-11), there exists a constant  $C$  so that

$$|u_{\text{in}}(r, t) - u_{\text{out}}(r, t)| \leq C \frac{\mu_0^{\frac{1}{2}}}{r}, \quad |\nabla u_{\text{in}}(r, t) - \nabla u_{\text{out}}(r, t)| \leq C \frac{\mu_0^{\frac{1}{2}}}{r^2} \quad (2-24)$$

for any  $R\mu_0 < r < R^{-1}\sqrt{t}$ , and  $t$  large enough.

Case 2:  $\gamma = 2$ . In this case, we define  $u_{\text{out}}$  as

$$u_{\text{out}}(r, t) = t^{-1}(\log t)kAg_0\left(\frac{r}{\sqrt{t}}\right) + t^{-1}h\left(\frac{r}{\sqrt{t}}\right), \quad (2-25)$$

where  $g_0(s) = s^{-1}e^{-\frac{s^2}{4}}$  is a solution to

$$g''(s) + \left(\frac{2}{s} + \frac{s}{2}\right)g'(s) + g(s) = 0 \quad (2-26)$$

and  $h$  solves

$$h''(s) + \left(\frac{2}{s} + \frac{s}{2}\right)h'(s) + h(s) = kAg_0(s), \quad (2-27)$$

with  $\lim_{s \rightarrow \infty} s^\gamma h(s) = A$ , and  $\lim_{s \rightarrow 0^+} sh(s) = d$ , so that  $\lim_{s \rightarrow 0^+} [h(s) - d/s] = 0$ . The function  $h$  can be described explicitly. Let  $g_1(s) = s^{-1}e^{-\frac{s^2}{4}} \int_0^s e^{\frac{z^2}{4}} dz$ . This function solves (2-26). Since  $g_1$  and  $g_0$  are linearly independent, the variation of parameters formula gives that, for any constants  $d$  and  $b$ ,

$$h(s) = g_0(s) \left[ d - kA \int_0^s zg_1(z) dz \right] + g_1(s) \left[ b + kA \int_0^s zg_0(z) dz \right] \quad (2-28)$$

solves (2-27). In order to have  $\lim_{s \rightarrow \infty} s^\gamma h(s) = A$ , we need  $2[b + kA \int_0^\infty zg_0(z) dz] = A$ . Furthermore, to have  $\lim_{s \rightarrow 0^+} [h(s) - d/s] = 0$ , we need  $b = 0$ . Thus we select

$$b = 0, \quad k = \frac{1}{2 \int_0^\infty zg_0(z) dz}. \quad (2-29)$$

Observe that, up to this moment, the constant  $d$  is arbitrary. Nevertheless, recall that we want  $u_{\text{out}}$  to be a solution to  $u_t = \Delta u = u_{rr} + (2/r)u_r$ . Multiplying this equation by  $r$ , and integrating in  $(0, R)$ , for some fixed, large  $R$ , we get

$$\frac{d}{dt} \left( \int_0^R r u(r, t) dr \right) = R u_r(R, t) + u(R, t),$$

where we use the fact that  $\lim_{r \rightarrow 0} [r u_r(r, t) + u(r, t)] = 0$ . Next, we integrate the above equation in  $t$ , from 0 to  $\infty$ , and using the fact that  $\lim_{t \rightarrow \infty} \int_0^R r u(r, t) dt = 0$ , we get

$$- \int_0^R r u(r, 0) dr = \int_0^\infty [R u_r(R, t) + u(R, t)] dt. \quad (2-30)$$

Take now  $u = u_{\text{out}}$  and compute the right-hand side of (2-30):

$$\begin{aligned} & \int_0^\infty [R u_r(R, t) + u(R, t)] dt \\ &= Ak \int_0^\infty t^{-1} (\log t) \left[ \frac{R}{\sqrt{t}} g_0' \left( \frac{R}{\sqrt{t}} \right) + g_0 \left( \frac{R}{\sqrt{t}} \right) \right] dt + \int_0^\infty t^{-1} \left[ \frac{R}{\sqrt{t}} h' \left( \frac{R}{\sqrt{t}} \right) + h \left( \frac{R}{\sqrt{t}} \right) \right] dt \\ &= \left( 4Ak \int_0^\infty s^{-1} [s g_0'(s) + g_0(s)] ds \right) \log R + \bar{d} + \left( 2 \int_0^\infty s^{-1} [s h'(s) + h(s)] ds \right), \end{aligned}$$

where  $s := R/\sqrt{t}$  and  $\bar{d}$  is the constant defined by

$$\bar{d} = - \left( 4Ak \int_0^\infty s^{-1} (\log s) [s g_0'(s) + g_0(s)] ds \right).$$

We can simplify the expression of the constant in front of  $\log R$ . Indeed, multiplying (2-26) against  $s$ , we get that  $(s g'(s) + g + (s^2/2)g)' = 0$ . For  $g = g_0$ , and using the fact that  $g_0$  decays very fast as  $s \rightarrow \infty$ , we get that  $s g_0'(s) + g_0(s) = -(s^2/2)g_0(s)$  for any  $s$ ; thus

$$4Ak \int_0^\infty s^{-1} [s g_0'(s) + g_0(s)] ds = Ak \left( -2 \int_0^\infty s g_0(s) ds \right) = -A$$

since (2-29). On the other hand, the decaying condition  $\lim_{r \rightarrow \infty} r^2 u(r, 0) = A$  gives

$$- \int_0^R r u(r, 0) dr = -A \log R + B(R),$$

with  $\lim_{R \rightarrow \infty} B(R) = B$ , where  $B$  is a real constant. Plugging this information in (2-30), we get

$$\bar{d} + \left( 2 \int_0^\infty s^{-1} [s h'(s) + h(s)] ds \right) = B.$$

This last relation defines in a unique way the constant  $d > 0$  in the definition of  $h$ , (2-28). Indeed, a direct computation gives

$$\int_0^\infty s^{-1} [s h'(s) + h(s)] ds = -\frac{d}{2} \left( \int_0^\infty s g_0(s) ds \right) + \omega,$$

with

$$\omega = \frac{kA}{2} \int_0^\infty s g_0(s) \left( \int_0^s z g_1(z) dz \right) ds + \int_0^s s^{-1} [s g_1' + g_1] \left( kA \int_0^s z g_0(z) dz \right) ds,$$

from which we deduce that

$$d = \frac{\bar{d} - 2\omega - B}{\int_0^\infty s g_0(s) ds}.$$

With this choice for the function  $h$  in (2-25), we get

$$h(s) = \frac{d}{s} - \frac{s}{4} [d + 10kA] + O(s^3) \quad \text{as } s \rightarrow 0^+$$

and

$$u_{\text{out}}(r, t) = \frac{t^{-\frac{1}{2}}}{r} [kA(\log t) + d] + t^{-1} \left[ -\frac{kA(\log t)}{4} - \frac{d + 10kA}{4} \right] \frac{r}{\sqrt{t}} + O\left( (\log t) \frac{r^3}{t^3 \sqrt{t}} \right) \quad (2-31)$$

in the region  $r < R^{-1} \sqrt{t}$ , for some large but fixed  $R$ , as  $t \rightarrow \infty$ .

In this case, namely when  $\gamma = 2$ , we choose  $\mu_0$  in (2-10) as

$$\mu_0(t) = \frac{[d + kA(\log t)]^2}{\sqrt{3}} t^{-1}, \quad (2-32)$$

and thanks to this choice, and to the bound (2-11) on  $\lambda$ , we find a constant  $C$  so that

$$|u_{\text{in}}(r, t) - u_{\text{out}}(r, t)| \leq C \frac{\mu_0^{\frac{1}{2}}}{r}, \quad |\nabla u_{\text{in}}(r, t) - \nabla u_{\text{out}}(r, t)| \leq C \frac{\mu_0^{\frac{1}{2}}}{r^2} \quad (2-33)$$

for any  $R\mu_0 < r < R^{-1} \sqrt{t}$ , for some fixed and large  $R$ , and for all  $t$  large enough.

Case 3:  $\gamma > 2$ . In this case, we define  $u_{\text{out}}^1$  as

$$u_{\text{out}}^1(r, t) = t^{-1} d g_0\left(\frac{r}{\sqrt{t}}\right), \quad d = \left( \frac{\int_0^\infty r u_0(r) dr}{\int_0^\infty s g_0(s) ds} \right),$$

where  $g_0(s) = s^{-1} e^{-\frac{s^2}{4}}$  solves (2-26), and  $u_0(r)$  is the initial condition for (2-1)–(2-2). Observe that, in a region like  $r < R^{-1} \sqrt{t}$ , for some large but fixed  $R$ , we get

$$u_{\text{out}}^1(r, t) = d \frac{t^{-\frac{1}{2}}}{r} - t^{-1} \frac{d}{4} \frac{r}{\sqrt{t}} + t^{-1} O\left(\frac{r^2}{t^{\frac{3}{2}}}\right). \quad (2-34)$$

For a given time  $t$ , the function  $u_{\text{out}}^1$  is decaying very fast as  $r \rightarrow \infty$ . For this reason, we modify  $u_{\text{out}}^1$  with a function that has the right decay to match the initial condition  $u_0(r)$ , for  $r$  large. Define

$$u_{\text{out}}(r, t) = \eta\left(\frac{r}{t}\right) u_{\text{out}}^1(r, t) + \left(1 - \eta\left(\frac{r}{t}\right)\right) u_{\text{out}}^2(r), \quad \text{with } u_{\text{out}}^2(r) = \frac{A}{r^\gamma}, \quad (2-35)$$

where  $\eta$  is the cut-off function defined in (1-9).

In this case,  $\gamma > 2$ , we choose  $\mu_0$  in (2-10) as

$$\mu_0(t) = \frac{d^2}{\sqrt{3}} t^{-1}. \quad (2-36)$$

With this choice for  $\mu_0(t)$ , and thanks to (2-11), given any large but fixed number  $R > 0$ , there exists a constant  $C$  so that

$$|u_{\text{in}}(r, t) - u_{\text{out}}(r, t)| \leq C \frac{\mu_0^{\frac{1}{2}}}{r}, \quad |\nabla u_{\text{in}}(r, t) - \nabla u_{\text{out}}(r, t)| \leq C \frac{\mu_0^{\frac{1}{2}}}{r^2} \quad (2-37)$$

for any  $R\mu_0 < r < R^{-1}\sqrt{t}$ , and for all  $t$  large.

**2C. Construction of the first global approximation and estimate of the error.** Let  $r_0 > 0$  be a small and fixed number, and define

$$U_1(r, t) = \eta\left(\frac{r}{r_0\sqrt{t}}\right)u_{\text{in}}(r, t) + \left(1 - \eta\left(\frac{r}{r_0\sqrt{t}}\right)\right)u_{\text{out}}(r, t), \quad (2-38)$$

where  $\eta$  is given by (1-9). For any smooth function  $u = u(r, t)$ , we define the error function as

$$\mathcal{E}[u](r, t) = \Delta u + u^5 - u_t. \quad (2-39)$$

Our next purpose is to describe

$$\mathcal{E}_1(r, t) = \mathcal{E}[U_1](r, t), \quad (2-40)$$

with  $U_1$  given by (2-38). To this end, we introduce the function  $\alpha = \alpha(t)$ ,  $t > t_0$ ,

$$\alpha(t) = 3^{\frac{1}{4}} \mu_0^{-\frac{1}{2}} (\mu_0 \Lambda)'. \quad (2-41)$$

Since  $\Lambda$  satisfies (2-11), definition (2-41) defines a linear homeomorphism  $\mathcal{A} : X_{\#} \rightarrow X_b$ ,  $\mathcal{A}(\lambda) = \alpha$ , where

$$X_b = \{\alpha \in C(t_0, \infty) : \|\alpha\|_b \text{ is bounded}\}, \quad (2-42)$$

and

$$\|\alpha\|_b := \sup_{t > t_0} \mu_0^{-\frac{3}{2}}(t) t [\|\alpha\|_{\infty, [t, t+1]} + |\alpha|_{0, \sigma, [t, t+1]}]. \quad (2-43)$$

Here  $\sigma$  is the number introduced in (2-11). Let us denote by  $h_0 : (0, \infty) \rightarrow (0, \infty)$  a smooth function with the property that

$$h_0(s) = \begin{cases} 1/s & \text{for } s \rightarrow 0, \\ 1/s^3 & \text{for } s \rightarrow \infty, \end{cases} \quad (2-44)$$

and define the following norm for any function  $f : \mathbb{R}^3 \times (t_0, \infty) \rightarrow \mathbb{R}$ :

$$\|f\|_* := \sup_{x \in \mathbb{R}^3, t > t_0} \mu_0^{-\frac{1}{2}} t^{\frac{3}{2}} h_0^{-1}\left(\frac{r}{\sqrt{t}}\right) [\|f\|_{\infty, B(x, 1) \times [t, t+1]} + [f]_{0, \sigma, B(x, 1) \times [t, t+1]}], \quad r = |x|. \quad (2-45)$$

Here  $\sigma$  is defined in (2-11),

$$\|f\|_{\infty, B(x, 1) \times [t, t+1]} = \sup_{y \in B(x, 1), s \in [t, t+1]} |f(y, s)| \quad (2-46)$$

and

$$[f]_{0,\sigma,B(x,1)\times[t,t+1]} = \sup_{y_1 \neq y_2 \in B(x,1), s_1 \neq s_2 \in [t,t+1]} \frac{|f(y_1, s_1) - f(y_2, s_2)|}{|y_1 - y_2|^{2\sigma} + |s_1 - s_2|^\sigma}. \quad (2-47)$$

The following estimates are valid, and we delay the proof to Appendix B since it is quite technical.

**Lemma 2.3.** *Assume  $\lambda = \lambda(t)$  satisfies (2-11). The error function defined in (2-40) can be described as*

$$\mathcal{E}_1(r, t) = \frac{\alpha(t)}{\mu + r} \eta\left(\frac{r}{r_0 \sqrt{t}}\right) + \mathcal{E}_{1,*}[\lambda](r, t), \quad (2-48)$$

where  $\eta$  is the smooth cut-off function defined in (1-9),  $\alpha$  is the function defined in (2-41), and  $r_0$  is a given fixed small number. The function  $\mathcal{E}_{1,*}[\lambda](r, t)$  depends smoothly on  $\lambda$ . Furthermore, there exists  $C > 0$  such that

$$\|\mathcal{E}_{1,*}\|_* \leq C. \quad (2-49)$$

If the initial time  $t_0$  in problem (2-1) is large enough, there exist  $c \in (0, 1)$  so that, for any  $\lambda_1, \lambda_2$  satisfying (2-11), we have

$$\|\mathcal{E}_{1,*}[\lambda_1] - \mathcal{E}_{1,*}[\lambda_2]\|_{\infty, B(x,1)\times[t,t+1]} \leq c \mu_0^{\frac{1}{2}} t^{-\frac{3}{2}} h_0\left(\frac{r}{\sqrt{t}}\right) \|\lambda_1 - \lambda_2\|_{\#}, \quad (2-50)$$

$$[\mathcal{E}_{1,*}[\lambda_1] - \mathcal{E}_{1,*}[\lambda_2]]_{0,\sigma, B(x,1)\times[t,t+1]} \leq c \mu_0^{\frac{1}{2}} t^{-\frac{3}{2}} h_0\left(\frac{r}{\sqrt{t}}\right) \|\lambda_1 - \lambda_2\|_{\#} \quad (2-51)$$

for any  $r = |x|$  and any  $t$ . The definitions of the function  $h_0$  and of the norm  $\|\cdot\|_*$  are given respectively in (2-44) and in (2-45). Furthermore the constant  $c$  in (2-50) and (2-51) can be made as small as one needs, provided that the initial time  $t_0$  is chosen large enough.

**2D. Construction of the second global approximation and estimate of the new error.** Taking into account the expression of the error function given in (2-48), we introduce a correction function  $\phi_0$  to partially get rid of the term  $\alpha(t)/(\mu + r)$ . More precisely, let

$$\bar{\alpha}(t) = \begin{cases} \alpha(t_0) & \text{for } t < t_0, \\ \alpha(t) & \text{for } t \geq t_0, \end{cases} \quad (2-52)$$

and introduce the function  $\phi_0$ , a solution to

$$\partial_t \phi_0 = \Delta \phi_0 + \frac{\bar{\alpha}(t)}{\mu + r} \mathbf{1}_{\{r < M\}} \quad \text{in } \mathbb{R}^3 \times (0, \infty), \quad \phi_0(x, t_0 - 1) = 0 \quad \text{in } \mathbb{R}^3, \quad M^2 = t_0. \quad (2-53)$$

Here, for a set  $K$ , we mean

$$\mathbf{1}_K(x) = \begin{cases} 1 & \text{if } x \in K, \\ 0 & \text{if } x \notin K. \end{cases}$$

Duhamel's formula provides an explicit expression for  $\phi_0$ :

$$\phi_0(x, t) = \int_{t_0-1}^t \frac{1}{(4\pi(t-s))^{\frac{3}{2}}} \int_{\mathbb{R}^3} e^{-\frac{|x-y|^2}{4(t-s)}} \frac{\bar{\alpha}(s)}{\mu + |y|} \mathbf{1}_{\{|y| < M\}} dy ds. \quad (2-54)$$

Since  $\lambda$  satisfies (2-11), classical parabolic estimates give that  $\phi_0$  is locally  $C^{2+2\sigma, 1+\sigma}$ , where  $\sigma$  is the Hölder exponent in (2-11). In the interval  $(t_0, \infty)$ , the function  $\phi_0$  solves

$$\partial_t \phi_0 = \Delta \phi_0 + \frac{\alpha(t)}{\mu + r} \mathbf{1}_{\{r < M\}} \quad \text{in } \mathbb{R}^3 \times (t_0, \infty), \quad (2-55)$$

and at time  $t = t_0$ , the function  $\phi_0(x, t_0)$  is radial in  $x$  and decays fast as  $|x| \rightarrow \infty$ ; that is,

$$|\phi_0(x, t_0)| \leq c e^{-a|x|^2} \quad \text{as } |x| \rightarrow \infty \quad (2-56)$$

for some positive, fixed constants  $a$  and  $c$ . Indeed, let  $x = \ell e$ , with  $\|e\| = 1$ , and assume that  $\ell > \max\{1, 2M\}$ . Thus  $|x - y|^2 > \ell^2/4$  for any  $|y| < M$ , and

$$|\phi_0(x, t_0)| \leq C |\alpha(t_0)| \left( \int_{t_0-1}^{t_0} \frac{e^{-\frac{\ell^2}{16(t_0-s)}}}{(t_0-s)^{\frac{3}{2}}} ds \right) \left( \int_{|y| < M} \frac{dy}{|y|} \right) \leq C |\alpha(t_0)| M^2 e^{-\frac{\ell^2}{16}}.$$

Taking  $\ell \rightarrow \infty$ , estimate (2-56) thus follows from (2-41).

The second approximation is given by

$$U_2[\lambda](r, t) = U_1(r, t) + \phi_0(r, t), \quad (2-57)$$

where  $U_1$  is in (2-38). Observe that  $U_2$  satisfies the decaying conditions (2-2) at the initial time  $t_0$  as consequence of (2-56). The new error function

$$\mathcal{E}_2[\lambda](r, t) = \mathcal{E}[U_2](r, t)$$

is thus

$$\mathcal{E}_2[\lambda](r, t) = \underbrace{\mathcal{E}_{1,*} + \frac{\alpha(t)}{r} \left( \eta \left( \frac{r}{r_0 \sqrt{t}} \right) - \mathbf{1}_{\{r < 2M\}} \right)}_{:= \mathcal{E}_{21}} + \underbrace{(U_1 + \phi_0)^5 - U_1^5}_{\mathcal{E}_{22}}. \quad (2-58)$$

The function  $\mathcal{E}_{1,*}$  is defined in (2-48). For a later purpose, it is useful to estimate, in the  $\|\cdot\|_*$ -norm introduced in (2-45), the function

$$\bar{\mathcal{E}}_2 := \mathcal{E}_{21} + (1 - \eta_R(x, t)) \mathcal{E}_{22}, \quad \text{where } \eta_R(x, t) = \eta \left( \frac{x}{R\mu_0} \right). \quad (2-59)$$

Here  $\eta(s)$  is given by (1-9), while the number  $R$  is a large number, whose definition will depend on  $t_0$  but will not depend on  $t$ .

We have the validity of the following lemma, whose proof is given in Appendix C.

**Lemma 2.4.** *Assume  $\lambda = \lambda(t)$  satisfies (2-11). The error function defined in (2-58) depends smoothly on  $\lambda$  and it satisfies the following estimates: There exists  $C > 0$  such that*

$$\|\bar{\mathcal{E}}_2\|_* \leq C. \quad (2-60)$$

If the initial time  $t_0$  is large enough, there exists small positive number  $c \in (0, 1)$  such that, for any  $\lambda_1, \lambda_2$  satisfying (2-11), we have

$$\|\bar{\mathcal{E}}_2[\lambda_1] - \bar{\mathcal{E}}_2[\lambda_2]\|_{\infty, B(x,1) \times [t, t+1]} \leq c \mu_0^{\frac{1}{2}} t^{-\frac{3}{2}} h_0 \left( \frac{r}{\sqrt{t}} \right) \|\lambda_1 - \lambda_2\|_{\#}, \quad r = |x|, \quad (2-61)$$

$$[\bar{\mathcal{E}}_2[\lambda_1](r, t) - \bar{\mathcal{E}}_2[\lambda_2](r, t)]_{0, \sigma, [t, t+1]} \leq c \mu_0^{\frac{1}{2}} t^{-\frac{3}{2}} h_0 \left( \frac{r}{\sqrt{t}} \right) \|\lambda_1 - \lambda_2\|_{\#} \quad (2-62)$$

for any  $x$  and  $t > t_0$ , provided the initial time  $t_0$  in problem (2-1) is chosen large enough. The definition of the function  $h_0$  is given in (2-44), and the definition of the  $\|\cdot\|_{\#}$ -norm is given in (2-45).

**Remark 2.5.** From the proof of the result, we also get that the constant  $c$  in (2-61) and (2-62) can be made as small as one needs, provided that the initial time  $t_0$  is chosen large enough.

### 3. The inner-outer gluing

Recall that our ultimate purpose is to construct a global unbounded solution  $u$  to (2-1)–(2-2) of the form

$$u = U_2[\lambda](r, t) + \tilde{\phi}, \quad t > t_0, \quad (3-1)$$

where  $U_2$  is defined in (2-57), while  $\tilde{\phi}(x, t)$  is a smaller perturbation. The rest of the paper is thus devoted to finding  $\tilde{\phi}(x, t)$ . The construction of  $\tilde{\phi}(x, t)$  is done by means of an *inner-outer gluing* procedure. This procedure consists in writing

$$\tilde{\phi}(x, t) = \psi(x, t) + \phi^{\text{in}}(x, t), \quad \text{where } \phi^{\text{in}}(x, t) := \eta_R(x, t) \hat{\phi}(x, t), \quad (3-2)$$

with

$$\hat{\phi}(x, t) := \mu_0^{-\frac{1}{2}} \phi \left( \frac{x}{\mu_0}, t \right), \quad \eta_R(x, t) = \eta \left( \frac{x}{R\mu_0} \right), \quad (3-3)$$

where  $\eta(s)$  is given in (1-9).

In terms of  $\tilde{\phi}$ , problem (2-1)–(2-2) reads as

$$\partial_t \tilde{\phi} = \Delta \tilde{\phi} + 5U_2^4 \tilde{\phi} + N(\tilde{\phi}) + \mathcal{E}_2 \quad \text{in } \mathbb{R}^3 \times [t_0, \infty), \quad (3-4)$$

where  $\mathcal{E}_2$  is defined in (2-58) and

$$N(\tilde{\phi}) = (U_2 + \tilde{\phi})^5 - U_2^5 - 5U_2^4 \tilde{\phi}.$$

Recalling that  $w_\mu = \mu^{-\frac{1}{2}} w(r/\mu)$ , we let

$$V[\lambda](r, t) = 5(U_2^4 - w_\mu^4) \eta_R + 5U_2^4(1 - \eta_R) \quad (3-5)$$

and write  $5U_2^4 = 5w_\mu^4 \eta_R + V[\lambda](r, t)$ . A main observation we make is that  $\tilde{\phi}$  solves problem (3-4) if the tuple  $(\psi, \phi)$  solves the following coupled system of nonlinear equations:

$$\partial_t \psi = \Delta \psi + V[\lambda] \psi + [2\nabla \eta_R \nabla_x \hat{\phi} + \hat{\phi}(\Delta_x - \partial_t) \eta_R] + N[\lambda](\tilde{\phi}) + \mathcal{E}_{21} + \mathcal{E}_{22}(1 - \eta_R) \quad \text{in } \mathbb{R}^3 \times [t_0, \infty), \quad (3-6)$$

and

$$\partial_t \hat{\phi} = \Delta \hat{\phi} + 5w_\mu^4 \hat{\phi} + 5w_\mu^4 \psi + \mathcal{E}_{22} \quad \text{in } B_{2R\mu_0}(0) \times [t_0, \infty). \quad (3-7)$$

We refer to (2-58) for the definition of  $\mathcal{E}_{21}$  and  $\mathcal{E}_{22}$ . In terms of  $\phi$ , see (3-3), (3-7) becomes

$$\begin{aligned} \mu_0^2 \partial_t \phi &= \Delta_y \phi + 5w^4 \phi + \mu_0^{\frac{5}{2}} \mathcal{E}_{22}(\mu_0 y, t) + 5 \frac{\mu_0^{\frac{1}{2}}}{(1+\Lambda)^4} w^4 \left( \frac{y}{(1+\Lambda)^2} \right) \psi(\mu_0 y, t) \\ &\quad + B[\phi] + B^0[\phi] \quad \text{in } B_{2R}(0) \times [t_0, \infty), \end{aligned} \quad (3-8)$$

where

$$B[\phi] := \mu_0 (\partial_t \mu_0) \left( \frac{\phi}{2} + y \cdot \nabla_y \phi \right), \quad (3-9)$$

$$B^0[\phi] := 5 \left[ w^4 \left( \frac{y}{(1+\Lambda)^2} \right) - w^4(y) \right] \phi + 5 \left( \frac{1 - (1+\Lambda)^4}{(1+\Lambda)^4} \right) w^4 \left( \frac{y}{(1+\Lambda)^2} \right) \phi. \quad (3-10)$$

We call (3-6) the *outer problem* and (3-8) the *inner problem(s)*.

We next describe precisely our strategy to solve (3-6)–(3-8). For given parameter  $\lambda$  satisfying (2-11), and function  $\phi$  fixed in a suitable range, we first solve for  $\psi$  the outer problem (3-6), in the form of a (nonlocal) nonlinear operator  $\psi = \Psi(\lambda, \phi)$ . This is done in full detail in Section 4.

We then replace this  $\psi$  in (3-8). At this point we consider the change of variable

$$t = t(\tau), \quad \frac{dt}{d\tau} = \mu_0^2(t),$$

which reduces (3-8) to

$$\partial_\tau \phi = \Delta_y \phi + 5w^4 \phi + H[\psi, \lambda, \phi](y, t(\tau)), \quad y \in B_{2R}(0), \quad \tau \geq \tau_0, \quad (3-11)$$

where  $\tau_0$  is such that  $t(\tau_0) = t_0$ , and

$$H[\psi, \lambda, \phi](y, t(\tau)) = \mu_0^{\frac{5}{2}} \mathcal{E}_{22}(\mu_0 y, t) + 5 \frac{\mu_0^{\frac{1}{2}}}{(1+\Lambda)^4} w^4 \left( \frac{y}{(1+\Lambda)^2} \right) \psi(\mu_0 y, t) + B[\phi] + B^0[\phi]. \quad (3-12)$$

Next step is to construct a solution  $\phi$  to problem (3-11). We can do this for functions  $\phi$  which furthermore satisfy

$$\phi(y, \tau_0) = e_0 Z(y), \quad y \in B_{2R}(0), \quad (3-13)$$

for some constant  $e_0$ . Here  $Z$  is the positive radially symmetric bounded eigenfunction associated to the only negative eigenvalue  $\lambda_0$  to the problem

$$L_0(\phi) + \lambda \phi = 0, \quad \phi \in L^\infty(\mathbb{R}^3). \quad (3-14)$$

Here  $L_0$  is the linear operator around the standard bubble  $w$  in  $\mathbb{R}^3$ . We refer to (2-6) for the definition of  $L_0$ . Furthermore, it is known that  $\lambda_0$  is simple and  $Z$  decays like

$$Z(y) \sim |y|^{-1} e^{-\sqrt{|\lambda_0|}|y|} \quad \text{as } |y| \rightarrow \infty.$$

To be more precise, we prove that problem (3-11)–(3-13) is solvable in  $\phi$ , provided that in addition the parameter  $\lambda$  is chosen so that  $H[\psi, \lambda, \phi](y, t(\tau))$  satisfies the orthogonality condition

$$\int_{B_{2R}} H[\psi, \lambda, \phi](y, t(\tau)) Z_0(y) dy = 0 \quad \text{for all } t > t_0. \quad (3-15)$$

We recall that  $Z_0(y)$ , defined in (2-7), is the only bounded radial element in the kernel of the linear elliptic operator  $L_0$ .

Equation (3-15) becomes a nonlinear, nonlocal problem in  $\lambda$  for any fixed  $\phi$ . We attack this problem in Sections 5, 6, and 7. In Section 5, we get the precise form of (3-15) as a nonlocal nonlinear operator in  $\lambda$ . The principal part of the operator in  $\lambda$  defined by (3-15) is a linear nonlocal operator which turns out to be a perturbation of the  $\frac{1}{2}$ -Caputo derivative. We refer to [Caputo 1967] for the original definition of Caputo derivatives. In Section 6 we develop an invertibility theory for such a linear operator. In Section 7 we fully solve (3-15) in  $\lambda$ , by means of a Banach fixed-point argument. The solution  $\lambda = \lambda[\phi]$  is a nonlinear operator in  $\phi$ , and we also describe the Lipschitz dependence of  $\lambda$  with respect to  $\phi$ , which is a key property for our final argument.

At this point, one realizes that a central point of our complete proof is to design a linear theory that allows us to solve in  $\phi$  problem (3-11)–(3-13). For this purpose, we shall construct a solution to an initial value problem of the form

$$\phi_\tau = \Delta\phi + 5w^4\phi + h(y, \tau) \quad \text{in } B_{2R} \times (\tau_0, \infty), \quad \phi(y, \tau_0) = e_0 Z(y) \quad \text{in } B_{2R}. \quad (3-16)$$

And then we solve problem (3-11)–(3-13) by means of a contraction mapping argument.

Let  $a$  be a fixed number with  $a \in (0, 2)$ , and let  $\nu > 0$  so that, for  $t$  large,

$$\tau^{-\nu} \sim \mu_0^{\frac{3}{2}} t^{-1} \quad \text{if } \gamma \neq 2 \quad \text{and} \quad \tau^{-\nu} \sim \mu_0^{\frac{3}{2}} t^{-1+\nu'} \quad \text{if } \gamma = 2$$

for some  $\nu' > 0$  that can be fixed arbitrarily small. We solve (3-16) for functions  $h$  with  $\|h\|_{\nu, 2+a}$ -norm bounded, where

$$\|h\|_{\nu, 2+a} := \sup_{\tau > \tau_0, y \in \mathbb{R}^3} \tau^\nu (1 + |y|^{2+a}) [\|h\|_{\infty, B(y, 1) \times [\tau, \tau+1]} + [h]_{0, \sigma, B(y, 1) \times [\tau, \tau+1]}], \quad (3-17)$$

and we construct solutions  $\phi$  in the class of functions with  $\|\phi\|_{\nu, a}$ -norm bounded, where

$$\begin{aligned} \|\phi\|_{\nu, a} := & \sup_{\tau > \tau_0, y \in \mathbb{R}^3} \tau^\tau (1 + |y|^a) [\|\phi\|_{\infty, B(y, 1) \times [\tau, \tau+1]} + [\phi]_{0, \sigma, B(y, 1) \times [\tau, \tau+1]}] \\ & + \sup_{\tau > \tau_0, y \in \mathbb{R}^3} \tau^\nu (1 + |y|^{1+a}) [\|\nabla\phi\|_{\infty, B(y, 1) \times [\tau, \tau+1]} + [\nabla\phi]_{0, \sigma, B(y, 1) \times [\tau, \tau+1]}]. \end{aligned} \quad (3-18)$$

We have the validity of the following result:

**Proposition 3.1.** *Let  $\nu, a$  be given positive numbers with  $0 < a < 2$ . Then, for all sufficiently large  $R > 0$  and function  $h = h(y, \tau)$ , with  $h(y, \tau) = h(|y|, \tau)$  and  $\|h\|_{\nu, 2+a} < +\infty$ , that satisfies*

$$\int_{B_{2R}} h(y, \tau) Z_0(y) dy = 0 \quad \text{for all } \tau \in (\tau_0, \infty) \quad (3-19)$$

*there exist  $\phi \in C^{2+2\sigma, 1+\sigma}$ -loc., which is radial in  $y$ , and  $e_0$  which solve problem (3-16). Moreover,  $\phi = \phi[h]$  and  $e_0 = e_0[h]$  define linear operators of  $h$  that satisfy the estimates*

$$|\phi(y, \tau)| \leq C \tau^{-\nu} \frac{R^{4-a}}{1 + |y|^3} \|h\|_{\nu, 2+a}, \quad |\nabla_y \phi(y, \tau)| \leq C \tau^{-\nu} \frac{R^{4-a}}{1 + |y|^4} \|h\|_{\nu, 2+a}, \quad (3-20)$$

and

$$|e_0[h]| \leq C \|h\|_{v,2+a}$$

for some fixed constant  $C$ .

We postpone the proof of this proposition to Section 9. Section 8 is devoted to solving problem (3-11)–(3-13) and this concludes the proof of Theorem 1.1.

#### 4. Solving the outer problem

The aim of this section is to solve the *outer problem* (3-6) for given parameter  $\lambda$  satisfying (2-11), and for given small functions  $\phi$ , in the form of a nonlinear nonlocal operator

$$\psi(x, t) = \Psi[\lambda, \phi](x, t).$$

We recall that  $\phi^{\text{in}}(x, t) = \eta_R(x, t)\hat{\phi}(x, t)$ , with

$$\hat{\phi}(x, t) := \mu_0^{-\frac{1}{2}}\phi\left(\frac{x}{\mu_0}, t\right) \quad \text{and} \quad \eta_R(x, t) = \eta\left(\frac{x}{R\mu_0}\right).$$

Here  $\eta(s)$  is defined in (1-9), and  $R$  is a sufficiently large number, independent of  $t$ . We assume that

$$\|\phi\|_{v,a} \text{ is bounded.} \quad (4-1)$$

Let  $\varphi_0 : (0, \infty) \rightarrow (0, \infty)$  be a smooth and bounded given function with the property that

$$\varphi_0(s) = \begin{cases} s & \text{for } s \rightarrow 0^+, \\ 1/s^3 & \text{for } s \rightarrow \infty. \end{cases} \quad (4-2)$$

We introduce the following  $L^\infty$ -weighted norms for functions  $f = f(r, t)$ :

$$\|f\|_{**} := \|f\|_1 + \|Df\|_2, \quad (4-3)$$

$$\|f\|_1 := \sup_{x \in \mathbb{R}^3, t > t_0} \mu_0^{-\frac{1}{2}} t^{\frac{1}{2}} \varphi_0^{-1}\left(\frac{r}{\sqrt{t}}\right) [\|f\|_{\infty, B(x,1) \times [t, t+1]} + [f]_{0, \sigma, B(x,1) \times [t, t+1]}], \quad r = |x|, \quad (4-4)$$

$$\|f\|_2 := \sup_{x \in \mathbb{R}^3, t > t_0} \mu_0^{-\frac{1}{2}} t (\varphi_0')^{-1}\left(\frac{r}{\sqrt{t}}\right) [\|f\|_{\infty, B(x,1) \times [t, t+1]} + [f]_{0, \sigma, B(x,1) \times [t, t+1]}], \quad r = |x|. \quad (4-5)$$

Refer to (2-46) and (2-47) for the definitions of  $\|f\|_{\infty, B(x,1) \times [t, t+1]}$  and  $[f]_{0, \sigma, B(x,1) \times [t, t+1]}$ .

**Proposition 4.1.** *Assume that  $\lambda$  satisfies (2-11), and that the function  $\phi$  satisfies the bound (4-1). Let  $\psi_0 \in C^2(\mathbb{R}^3)$ , radially symmetric so that*

$$|y| |\psi_0(y)| + |y| |\nabla \psi_0(y)| \leq t_0^{-a} e^{-b|y|} \quad (4-6)$$

for some positive constants  $a$  and  $b$ . There exists  $t_0$  large so that problem (3-6) has a unique solution  $\psi = \Psi[\lambda, \phi]$  so that

$$\psi(r, t_0) = \psi_0(r), \quad \|\psi\|_1 + \|D\psi\|_2 \leq C. \quad (4-7)$$

*Proof.* Let  $f$  be a given function with  $\|f\|_*$ -norm bounded. Classical parabolic estimates give that any solution to  $\partial_t \psi = \Delta \psi + f$  is locally  $C^{2+2\sigma, 1+\sigma}$ . Furthermore, a consequence of Lemma A.1 is that the function  $\bar{\varphi}_0(r, t) = \mu_0^{\frac{1}{2}} t^{-\frac{1}{2}} \varphi_0(r/\sqrt{t})$  is a positive supersolution for  $\partial_t \psi \geq \Delta \psi + f(r, t)$ . Observe also that  $\bar{\varphi}_0(r, t_0) \geq \psi_0(r)$ . Combining these facts with the maximum principle, we see that, for a function  $f$  with  $\|f\|_*$ -norm bounded, the unique solution to  $\partial_t \psi = \Delta \psi + f$ , with  $\psi(r, t_0) = \psi_0$ , has  $\|\psi\|_{**}$ -norm bounded. We claim that a possibly large multiple of  $\bar{\varphi}_0$  works as a supersolution also for the problem

$$\partial_t \psi \geq \Delta \psi + V(r, t)\psi + f(r, t). \tag{4-8}$$

Indeed, recalling the definition of  $V$  in (3-5), we write

$$V = V_1 + V_2, \quad V_1 = 5(U_2^4 - w_\mu^4)\eta_R, \quad V_2 = 5U_2^4(1 - \eta_R).$$

In the region where  $\eta_R \neq 0$ , namely when  $r < 2R\mu_0$ , we expand in Taylor the function  $V_1$  and we find  $s^* \in (0, 1)$  so that

$$V_1(r, t) = 20(w_\mu + s^*(\mu_0' \Psi_1(r, t) + \phi_0(r, t)))^3 [\mu_0' \Psi_1(r, t) + \phi_0(r, t)] \eta_R.$$

From here, we see that, in this region,  $|V_1(r, t)| \lesssim R t^{-1} \eta_R$ , so that

$$|V_1(r, t)\psi_0(r, t)| \lesssim \mu_0^{\frac{1}{2}} t^{-\frac{3}{2}} h_0 \left( \frac{r}{\sqrt{t}} \right). \tag{4-9}$$

Let us now consider  $V_2$ . This function is not zero only when  $r > R\mu_0$ , and in this region we have that  $|V_2(r, t)| \lesssim (\mu_0^2/r^4)(1 - \eta_R)$ , so that

$$|V_2(r, t)\psi_0(r, t)| \lesssim \frac{\mu^2}{r^4} \mu_0^{\frac{1}{2}} t^{-\frac{1}{2}} \varphi_0 \left( \frac{r}{\sqrt{t}} \right) (1 - \eta_R) \lesssim R^{-2} \mu_0^{\frac{1}{2}} t^{-\frac{3}{2}} h_0 \left( \frac{r}{\sqrt{t}} \right). \tag{4-10}$$

Choosing  $R$  large, but independent of  $t$ , we thus find that a multiple of  $\bar{\varphi}_0$  is a supersolution for (4-8).

We call  $T_o : (f, \psi_0) \rightarrow \psi$  the linear operator that to any  $f$  with  $\|f\|_*$ -norm bounded and any initial condition  $\psi_0$  satisfying (4-6) associates the unique solution to

$$\partial_t \psi = \Delta \psi + V[\lambda](r, t)\psi + f(r, t), \quad \psi(r, t_0) = \psi_0(r), \tag{4-11}$$

which has bounded  $\|\psi\|_{**}$ -norm. Define  $\bar{\psi} = T_o(0, \psi_0)$ . We observe that  $\psi + \bar{\psi}$  is a solution to (3-6) if  $\psi$  is a fixed point for the operator

$$\mathcal{A}_o(\psi) = T_o([2\nabla \eta_R \nabla_x \hat{\phi} + \hat{\phi}(\Delta_x - \partial_t)\eta_R] + N[\lambda](\tilde{\phi} + \bar{\psi}) + \mathcal{E}_{21} + \mathcal{E}_{22}(1 - \eta_R)). \tag{4-12}$$

We shall show the existence and uniqueness of such a fixed point as consequence of the contraction mapping theorem. We perform a fixed-point argument in the set of functions  $\psi$  in

$$B_o = \{\psi \in L^\infty : \|\psi\|_{**} < r\} \tag{4-13}$$

for some  $r > 0$ .

From Lemma 2.3 we have that there exists a constant  $c_1$  so that

$$\|\mathcal{E}_{21} + \mathcal{E}_{22}(1 - \eta_R)\|_* \leq c_1. \tag{4-14}$$

We now claim that there exists a constant  $c_2$  such that, if the parameter  $\lambda$  satisfies (2-11), and if the function  $\phi$  satisfies the bound (4-1), then

$$\|2\nabla\eta_R\nabla_x\hat{\phi} + \hat{\phi}(\Delta_x - \partial_t)\eta_R\|_* + \|N(\tilde{\phi} + \bar{\psi})\|_* \leq c_2. \quad (4-15)$$

Furthermore, we claim that there exists a constant  $c \in (0, 1)$  so that, for any  $\psi_1, \psi_2 \in B_0$ ,

$$\|\mathcal{A}_o(\psi_1) - \mathcal{A}_o(\psi_2)\|_{**} \leq c\|\psi_1 - \psi_2\|_{**}. \quad (4-16)$$

If we assume, for the moment, the validity of (4-14), (4-15) and (4-16), we get the existence of a fixed point for problem (4-12) in the set (4-13), provided  $r$  is chosen large enough.

*Proof of (4-15):* We start with the estimate of the first term in (4-15). Since we assume the validity of the bound (4-1) on  $\phi$ , we write

$$|\hat{\phi}\Delta_x\eta_R| \lesssim \frac{|\eta''(|x|/R\mu_0)|}{R^2\mu_0^2}|\hat{\phi}| \lesssim \frac{|\eta''(|x|/R\mu_0)|}{R^2\mu_0^2} \frac{\mu_0^{\frac{3}{2}}t^{-1}}{(1+|x|/\mu_0)^a} \|\phi\|_{v,a};$$

see (3-18) for the notation  $\|\phi\|_{v,a}$ . Thus, we get

$$\begin{aligned} |\hat{\phi}\Delta_x\eta_R| &\lesssim \frac{|\eta''(|x|/R\mu_0)|}{R^{2+a}} \mu_0^{-\frac{1}{2}}t^{-1} \frac{r}{\sqrt{t}} h_0\left(\frac{r}{\sqrt{t}}\right) \|\phi\|_{v,a} \\ &\lesssim \frac{|\eta''(|x|/R\mu_0)|}{R^{1+a}} \mu_0^{\frac{1}{2}}t^{-\frac{3}{2}} h_0\left(\frac{r}{\sqrt{t}}\right) \|\phi\|_{v,a} \lesssim \mu_0^{\frac{1}{2}}t^{-\frac{3}{2}} h_0\left(\frac{r}{\sqrt{t}}\right) \frac{\|\phi\|_{v,a}}{R^{1+a}}. \end{aligned}$$

Arguing similarly, we get

$$|\hat{\phi}\partial_x\eta_R| \lesssim \mu_0^{\frac{1}{2}}t^{-\frac{3}{2}} h_0\left(\frac{r}{\sqrt{t}}\right) \frac{\|\phi\|_{v,a}}{R^{1+a}} \quad \text{and} \quad |\nabla\hat{\phi}\nabla\eta_R| \lesssim \mu_0^{\frac{1}{2}}t^{-\frac{3}{2}} h_0\left(\frac{r}{\sqrt{t}}\right) \frac{\|\phi\|_{v,a}}{R^{1+a}},$$

which proves the  $L^\infty$  bound in the first estimate in (4-15). To check the Hölder bound for this term, we focus the analysis on the term  $g(x, t) := \hat{\phi}\Delta_x\eta_R$ . The other terms can be treated in a similar way. We write

$$\begin{aligned} &\frac{|g(x_1, t_1) - g(x_2, t_2)|}{|x_1 - x_2|^{2\sigma} + |t_1 - t_2|^\sigma} \\ &= |\Delta_x\eta_R(x_1, t_1)| \frac{|\hat{\phi}(x_1, t_1) - \hat{\phi}(x_2, t_2)|}{|x_1 - x_2|^{2\sigma} + |t_1 - t_2|^\sigma} + |\hat{\phi}(x_2, t_2)| \frac{|\Delta_x\eta_R(x_1, t_1) - \Delta_x\eta_R(x_2, t_2)|}{|x_1 - x_2|^{2\sigma} + |t_1 - t_2|^\sigma}. \end{aligned}$$

In order to control the first term, we use the definition in (3-18) of  $\|\phi\|_{v,a}$  and we argue as before. The second term can be easily treated using the  $L^\infty$ -bound on  $\hat{\phi}$  and the smoothness of the function  $\Delta_x\eta_R$ . This completes the analysis of the first estimate in (4-15).

We continue with the proof of the second estimate in (4-15). We recall that

$$N(\tilde{\phi}) = (U_2 + \tilde{\phi})^5 - U_2^5 - 5U_2^4\tilde{\phi}.$$

It is convenient to estimate this function in three different regions: where  $r < \bar{M}^{-1}\mu_0$ , where  $\bar{M}^{-1}\mu_0 < r < \bar{M}\sqrt{t}$  and where  $r > \bar{M}\sqrt{t}$ , with  $\bar{M}$  a large positive number.

From the definition of  $U_2$  in (2-57), we see that, if  $r < \bar{M}^{-1}\mu_0$ , then

$$|N(\tilde{\phi})| \lesssim \mu_0^{-\frac{3}{2}} |\tilde{\phi}|^2 \lesssim \mu_0^{-\frac{3}{2}} [|\psi|^2 + |\eta_R \hat{\phi}|^2].$$

We recall that

$$|\psi| \lesssim \|\psi\|_{**} \mu_0^{\frac{1}{2}} t^{-\frac{1}{2}} \varphi_0\left(\frac{r}{\sqrt{t}}\right), \quad |\eta_R \hat{\phi}| \lesssim \mu_0^{\frac{3}{2}} t^{-1} |\eta_R| \|\phi\|_{\mu,a}, \tag{4-17}$$

so that we get, for  $r < \bar{M}^{-1}\mu_0$ ,

$$|N(\tilde{\phi} + \bar{\psi})| \lesssim \mu_0^2 t^{-1} [\|\psi + \bar{\psi}\|_{**}^2 + \|\phi\|_{\mu,a}^2] \left( \mu_0^{\frac{1}{2}} t^{-\frac{3}{2}} h_0\left(\frac{r}{\sqrt{t}}\right) \right). \tag{4-18}$$

Let us now consider the region  $\bar{M}^{-1}\mu_0 < r < \bar{M}\sqrt{t}$ . Here, after a Taylor expansion, we get

$$|N(\tilde{\phi} + \bar{\psi})| \lesssim w_\mu^3 [|\psi + \bar{\psi}|^2 + |\eta_R \hat{\phi}|^2] \lesssim \frac{\mu_0^{\frac{3}{2}}}{r^3} [|\psi|^2 + |\eta_R \hat{\phi}|^2].$$

Using again (4-17), we obtain, for  $\bar{M}^{-1}\mu_0 < r < \bar{M}\sqrt{t}$ ,

$$|N(\tilde{\phi} + \bar{\psi})| \lesssim \mu_0^2 t^{-1} [\|\psi + \bar{\psi}\|_{**}^2 + \|\phi\|_{\mu,a}^2] \left( \mu_0^{\frac{1}{2}} t^{-\frac{3}{2}} h_0\left(\frac{r}{\sqrt{t}}\right) \right). \tag{4-19}$$

Let us now consider  $r > \bar{M}\sqrt{t}$ . Observe that in this region  $\eta_R = 0$ ,  $|(\psi + \bar{\psi})(r, t)| \lesssim \mu_0^{\frac{1}{2}} t^{-\frac{1}{2}} \varphi_0(r/\sqrt{t})$  and, from (C-3), also  $|U_2(r, t)| \lesssim \mu_0/r$ . Thus we have

$$|N(\tilde{\phi} + \bar{\psi})| \lesssim \left(\frac{\mu_0}{r}\right)^5 \lesssim \mu_0^{\frac{9}{2}} t^{-\frac{1}{2}} \left( \mu_0^{\frac{1}{2}} t^{-\frac{3}{2}} h_0\left(\frac{r}{\sqrt{t}}\right) \right). \tag{4-20}$$

From (4-18)–(4-20), we get the  $L^\infty$  bound for the second estimate in (4-15).

*Proof of (4-16):* For any  $\psi_1, \psi_2 \in B_o$ , we have

$$\mathcal{A}_o(\psi_1) - \mathcal{A}_o(\psi_2) = T_0(N(\psi_1 + \bar{\psi} + \phi^{\text{in}}) - N(\psi_2 + \bar{\psi} + \phi^{\text{in}}));$$

thus

$$\|\mathcal{A}_o(\psi_1) - \mathcal{A}_o(\psi_2)\|_{**} \leq C \|N(\psi_1 + \bar{\psi} + \phi^{\text{in}}) - N(\psi_2 + \bar{\psi} + \phi^{\text{in}})\|_{**}.$$

We write

$$\begin{aligned} & N(\psi_1 + \phi^{\text{in}}) - N(\psi_2 + \phi^{\text{in}}) \\ &= (U_2 + \psi_1 + g)^5 - (U_2 + \psi_2 + g)^5 - 5U_2^4(\psi_1 - \psi_2) \\ &= \underbrace{(U_2 + \psi_1 + g)^5 - (U_2 + \psi_2 + g)^5 - 5(U_2 + g)^4(\psi_1 - \psi_2)}_{:=N_1} + \underbrace{5[(U_2 + g)^4 - U_2^4](\psi_1 - \psi_2)}_{:=N_2}, \end{aligned}$$

where  $g := \phi^{\text{in}} + \bar{\psi}$ . In the region where  $r < \bar{M}\sqrt{t}$ , we have

$$|N_1(x, t)| \lesssim w_\mu^3 |\psi_1 - \psi_2|^2,$$

which yields

$$|N_1(x, t)| \lesssim \mu_0^2 t^{-1} [\|\psi_1 - \psi_2\|_{**}^2] \left( \mu_0^{\frac{1}{2}} t^{-\frac{3}{2}} h_0\left(\frac{r}{\sqrt{t}}\right) \right),$$

while  $N_2$  can be estimated as

$$|N_2(x, t)| \lesssim [\mu_0^2 t^{-1} \|\bar{\psi}\|_{**} + \mu_0^2 t^{-1} \|\phi^{\text{in}}\|_{v,a}] \|\psi_1 - \psi_2\|_{**} \left( \mu_0^{\frac{1}{2}} t^{-\frac{3}{2}} h_0 \left( \frac{r}{\sqrt{t}} \right) \right).$$

On the other hand, if  $r > \bar{M} \sqrt{t}$ , we have  $\phi^{\text{in}} \equiv 0$ , so that

$$|N_2(x, t)| \lesssim \mu_0^2 \|\bar{\psi}\|_{**} \|\psi_1 - \psi_2\|_{**} \left( \mu_0^{\frac{1}{2}} t^{-\frac{3}{2}} h_0 \left( \frac{r}{\sqrt{t}} \right) \right).$$

On the other hand  $N_1$  can be estimated as follows:

$$|N_1(x, t)| \lesssim |\psi_1 - \psi_2|^5, \quad \text{from which } |N_1(x, t)| \lesssim \mu_0^2 \mu_0^{\frac{1}{2}} t^{-\frac{1}{2}} h_0 \left( \frac{r}{\sqrt{t}} \right) \|\psi_1 - \psi_2\|_{**}.$$

In summary, we get

$$\|N(\psi_1 + \phi^{\text{in}} + \bar{\psi}) - N(\psi_2 + \phi^{\text{in}} + \bar{\psi})\|_{*,\beta} \leq C \mu_0^2 \|\psi_1 - \psi_2\|_{**},$$

where  $C = \max\{\|\psi_1 - \psi_2\|_{**}, \|\phi^{\text{in}}\|_{v,a}\}$ . Thus we get the validity of (4-16) provided that  $t_0$  is large enough.  $\square$

**Remark 4.2.** Proposition 4.1 defines the solution to problem (3-6) as a function of the initial condition  $\psi_0$ , in the form of an operator  $\psi = \bar{\Psi}[\psi_0]$ , from a small neighborhood of 0 in the Banach space  $L^\infty(\Omega)$  equipped with the norm

$$\sup_{y \in \mathbb{R}^3} [|y| |e^{b|y|} \psi_0(y)| + |y| |e^{b|y|} \nabla \psi_0(y)|] \quad (4-21)$$

into the Banach space of functions  $\psi \in L^\infty(\Omega)$  equipped with the norm  $\|\psi\|_{**}$ , defined in (4-3). A closer look to the proof of Proposition 4.1, and the implicit function theorem give that  $\psi_0 \rightarrow \bar{\Psi}[\psi_0]$  is a diffeomorphism, and that

$$\|\bar{\Psi}[\psi_0^1] - \bar{\Psi}[\psi_0^2]\|_{**} \leq c \left[ \sup_{y \in \mathbb{R}^3} |y| e^{b|y|} |\psi_0^1 - \psi_0^2| + \sup_{y \in \mathbb{R}^3} |y| e^{b|y|} |\nabla \psi_0^1 - \nabla \psi_0^2| \right]$$

for some positive constant  $c$ .

**Proposition 4.3.** *Assume the validity of the assumptions of Proposition 4.1. Then the function  $\psi = \Psi(\lambda, \phi)$  depends smoothly on  $\lambda$  and  $\phi$ , and we have the validity of the following estimates: for any initial time  $t_0$  in problem (2-1) sufficiently large, and any sufficiently large radius  $R$  in the cut-off function  $\eta_R$  introduced in (3-2), there exists  $c$  such that, given  $\lambda_1, \lambda_2$  satisfying (2-11), one has*

$$\|\Psi[\lambda_1, \phi] - \Psi[\lambda_2, \phi]\|_{**} \leq c \|\lambda_1 - \lambda_2\|_{\#} \quad (4-22)$$

for any  $\phi$  satisfying (4-1). Moreover, given  $\phi_1, \phi_2$  satisfying (4-1), one has

$$\|\Psi[\lambda, \phi_1] - \Psi[\lambda, \phi_2]\|_{**} \leq c \|\phi_1 - \phi_2\|_{v,a} \quad (4-23)$$

for any  $\lambda$  satisfying (2-11).

*Proof.* Fix  $\phi$  and define  $\bar{\psi} = \psi[\lambda_1, \phi] - \psi[\lambda_2, \phi]$  for  $\lambda_1$  and  $\lambda_2$  satisfying (2-11). Then  $\bar{\psi}$  solves

$$\partial_t \bar{\psi} = \Delta \bar{\psi} + (V[\lambda_1] + N'[\lambda])(\bar{\psi}) + F, \quad \mathbb{R}^3 \times (t_0, \infty), \quad \bar{\psi}(r, t_0) = 0,$$

for  $\lambda = s\lambda_1 + (1-s)\lambda_2$ ,  $s \in (0, 1)$ , where

$$F = \mathcal{E}_{21}[\lambda_1] - \mathcal{E}_{21}[\lambda_2] + (1 - \eta_R)[\mathcal{E}_{22}[\lambda_1] - \mathcal{E}_{22}[\lambda_2]] + [V[\lambda_1] - V[\lambda_2]]\psi_2 + [N[\lambda_1] - N[\lambda_2]](\psi_2 + \phi^{\text{in}}),$$

where  $\psi_j = \psi[\lambda_j, \phi]$ ,  $j = 1, 2$ . From Lemma 2.3 and estimates (2-61)–(2-62), we get

$$\begin{aligned} \|\mathcal{E}_{21}[\lambda_1] - \mathcal{E}_{21}[\lambda_2]\|_* &\leq c \|\lambda_1 - \lambda_2\|_{\#}, \\ \|(1 - \eta_R)[\mathcal{E}_{22}[\lambda_1] - \mathcal{E}_{22}[\lambda_2]]\|_* &\leq c \|\lambda_1 - \lambda_2\|_{\#}, \end{aligned}$$

provided  $t_0$  is large enough. One also checks that, for some  $c \in (0, 1)$ ,

$$\|[V[\lambda_1] - V[\lambda_2]]\psi_2\|_* \leq c \|\lambda_1 - \lambda_2\|_{\#}, \quad \|[N[\lambda_1] - N[\lambda_2]](\psi_2 + \phi^{\text{in}})\|_* \leq c \|\lambda_1 - \lambda_2\|_{\#}.$$

The constant  $c_1$  can be made arbitrarily small provided  $t_0$  is large. Arguing as in (4-9) and (4-10), one can show that a certain multiple of the function  $\|\lambda_1 - \lambda_2\|_{\#} \bar{\varphi}_0(r, t)$ , where  $\bar{\varphi}_0 = \mu_0^{\frac{1}{2}} t^{-\frac{1}{2}} \varphi_0(r/\sqrt{t})$ , serves as supersolution for  $\bar{\psi}$ . This proves (4-22).

Let us now fix  $\lambda$ , and take  $\phi_1, \phi_2$  satisfying (4-1). Define  $\phi_j^{\text{in}} = \eta_R \hat{\phi}_j$  and  $\hat{\phi}_j(x, t) = \mu_0^{-\frac{1}{2}} \phi_j(x/\mu_0, t)$ , for  $j = 1, 2$ , as natural. Let  $\bar{\psi} = \psi(\lambda, \phi_1) - \psi(\lambda, \phi_2)$ . We have  $\bar{\psi}(r, t_0) = 0$  and

$$\begin{aligned} \partial_t \bar{\psi} = \Delta \bar{\psi} + V[\lambda] \bar{\psi} + (\psi_1 + \phi_1^{\text{in}})^5 - (\psi_2 + \phi_1^{\text{in}})^5 + [2\nabla \eta_R \nabla_x (\hat{\phi}_1 - \hat{\phi}_2) + (\hat{\phi}_1 - \hat{\phi}_2)(\Delta_x - \partial_t) \eta_R] \\ + (\psi_2 + \phi_1^{\text{in}})^5 - (\psi_2 + \phi_2^{\text{in}})^5 - 5U_2^4 (\phi_1^{\text{in}} - \phi_2^{\text{in}}). \end{aligned}$$

Arguing as in (4-6)–(4-21), we get

$$\begin{aligned} |[2\nabla \eta_R \nabla_x (\hat{\phi}_1 - \hat{\phi}_2) + (\hat{\phi}_1 - \hat{\phi}_2)(\Delta_x - \partial_t) \eta_R]| &\leq \mu_0^{\frac{1}{2}} t^{-\frac{3}{2}} h_0 \left( \frac{r}{\sqrt{t}} \right) \frac{\|\phi_1 - \phi_2\|_{v,a}}{R^{1+a}} \\ &\leq c \mu_0^{\frac{1}{2}} t^{-\frac{3}{2}} h_0 \left( \frac{r}{\sqrt{t}} \right) \|\phi_1 - \phi_2\|_{v,a} \end{aligned}$$

and also

$$\begin{aligned} |(\psi_2 + \phi_1^{\text{in}})^5 - (\psi_2 + \phi_2^{\text{in}})^5 - 5U_2^4 (\phi_1^{\text{in}} - \phi_2^{\text{in}})| &\leq \mu_0^{\frac{1}{2}} t^{-\frac{3}{2}} h_0 \left( \frac{r}{\sqrt{t}} \right) \frac{\|\phi_1 - \phi_2\|_{v,a}}{R^{1+a}} \\ &\leq c \mu_0^{\frac{1}{2}} t^{-\frac{3}{2}} h_0 \left( \frac{r}{\sqrt{t}} \right) \|\phi_1 - \phi_2\|_{v,a}. \end{aligned}$$

The constant  $c_1$  in the last two formulas can be made arbitrarily small provided  $R$  is chosen large enough.  $\square$

## 5. Choice of $\lambda$ : part I

Let  $\psi = \Psi[\lambda, \phi]$  be the solution to problem (3-6) predicted by Proposition 4.1, and satisfying the properties described in Proposition 4.3. We substitute  $\psi$  in (3-11) and (3-12), and we want to solve, in  $\phi$ , problem

(3-11), satisfying the initial condition (3-13). As we stated in Proposition 3.1, problem (3-11)–(3-13) can be solved for functions  $\phi$  satisfying (4-1), provided that

$$\int_{B_{2R}} H[\psi, \lambda, \phi](y, t(\tau)) Z_0(y) dy = 0 \quad \text{for all } t > t_0, \tag{5-1}$$

where  $H[\psi, \lambda, \phi]$  is defined in (3-12).

Next lemma states that (5-1) is a nonlinear, nonlocal equation in  $\lambda$ , at any fixed  $\phi$ .

**Lemma 5.1.** *Assume that  $\lambda$  satisfies (2-11), and that the function  $\phi$  satisfies the bound (4-1). Let  $\psi = \Psi[\lambda, \phi]$  be the solution to problem (3-6) predicted by Proposition 4.1. Then (5-1) is equivalent to*

$$[1 + \mu_0 \mu'_0 b(t) + q_1(\lambda)] \phi_0(0, t) = g(t) + G[\lambda, \phi](t). \tag{5-2}$$

Here  $\phi_0$  is the function defined in (2-53) and also in (2-54); thus

$$\phi_0(0, t) = \int_{t_0-1}^t \frac{1}{(4\pi(t-s))^{\frac{3}{2}}} \int_{\mathbb{R}^3} e^{-\frac{|y|^2}{4(t-s)}} \frac{\bar{\alpha}(s)}{\mu + |y|} \mathbf{1}_{\{r < M\}} dy ds. \tag{5-3}$$

The function  $b = b(t)$  is a smooth function in  $(t_0, \infty)$ . By  $q_1(s)$  we denote a smooth function such that  $q_1(0) = 0$  and  $q'_1(0) \neq 0$ . Moreover,

$$\|b\|_\infty < C, \quad \|g\|_b \leq C, \quad \|G[\lambda, \phi]\|_b \leq C. \tag{5-4}$$

Furthermore, if the initial time  $t_0$  in problem (2-1) is chosen large enough, there exists  $R$  in the definition of the cut-off function in (3-2) sufficiently large and there exists a constant  $c \in (0, 1)$  so that, for any  $\phi$ ,

$$\|G[\lambda_1, \phi] - G[\lambda_2, \phi]\|_b \leq c \|\lambda_1 - \lambda_2\|_\# \tag{5-5}$$

and, for any  $\lambda$ ,

$$\|G[\lambda, \phi_1] - G[\lambda, \phi_2]\|_b \leq c \|\phi_1 - \phi_2\|_{v,a}. \tag{5-6}$$

The constants  $c$  in (5-5) and (5-6) can be made as small as one needs, provided that the initial time  $t_0$  is chosen large enough. We refer to (2-43) and (3-18) for the definitions of  $\|\cdot\|_b$  and  $\|\cdot\|_{v,a}$  respectively.

*Proof.* Throughout the proof, we denote by  $q_i = q_i(s)$ , for any integer  $i$ , a smooth real function, with the property that

$$\frac{d}{(ds)^j} q_i(0) = 0 \quad \text{for } j < i \quad \text{and} \quad \frac{d}{(ds)^i} q_i(0) \neq 0.$$

We have the decomposition

$$\begin{aligned} & \int_{B_{2R}} H[\psi, \lambda, \phi](y, t(\tau)) Z_0(y) dy \\ &= \mu_0^{\frac{5}{2}} \int_{B_{2R}} \mathcal{E}_{22}(\mu_0 y, t) Z_0(y) dy + 5 \int_{B_{2R}} \frac{\mu_0^{\frac{1}{2}}}{(1+\lambda)^2} w^4\left(\frac{y}{1+\lambda}\right) \psi(\mu_0 y, t) Z_0(y) dy \\ & \quad + \int_{B_{2R}} B[\phi] Z_0(y) dy + \int_{B_{2R}} B^0[\phi] Z_0(y) dy \\ &= i_1 + i_2 + i_3 + i_4. \end{aligned}$$

For any  $j = 1, \dots, 4$ ,  $i_j$  is a function of  $t$  and depends also on  $\lambda$  and  $\phi$ . To emphasize this dependence, we write  $i_j = i_j[\lambda, \phi](t)$ .

We claim that

$$\begin{aligned} & \mu_0^{-\frac{1}{2}} i_1[\lambda, \phi](t) \\ &= \mu_0^2 \mu^{-2} \left[ \left( 5 \int_{B_{2R}} w^4(y) Z_0(y) dy \right) \phi_0(0, t) + (q_1(\lambda) + \mu_0 \mu'_0 q_0(\lambda)) \phi_0(0, t) + \mu_0^\sigma \alpha(t) b(t) \right], \end{aligned} \quad (5-7)$$

where  $b(t)$  is a smooth function in  $(t_0, \infty)$  which is uniformly bounded as  $t \rightarrow \infty$ .

Observe that  $i_1$  does not depend on  $\phi$ . From (2-53) satisfied by  $\phi_0$ , and Lemma A.1, we get the existence of a positive constant  $c$  so that  $|\phi_0(\mu_0 y, t)| \leq c \alpha(t) \mu_0(t)$  for any  $y \in B_{2R}$ . Thus, we Taylor expand  $\mathcal{E}_{22}$  in the region  $y \in B_{2R}$  as follows:

$$\mathcal{E}_{22}(\mu_0 y, t) = 5U_1^4 \phi_0 + 4(U_1 + s\phi_0)^3 \phi_0^2 = a + b$$

for some  $s \in (0, 1)$ . Let us first analyze  $a$ . We write

$$a = 5\mu^{-2} w^4(y) \phi_0(0, t) + \underbrace{5[U_1^4(\mu_0 y) - \mu^{-2} w^4(y)] \phi_0(0, t)}_{:=a_1} + \underbrace{5U_1^4[\phi_0(\mu_0 y, t) - \phi_0(0, t)]}_{:=a_2}.$$

Observe that, by the definition of  $U_1$  in (2-38), and (2-13), we have

$$\begin{aligned} U_1^4(\mu_0 y) - \mu^{-2} w^4(y) &= \left[ w_\mu(\mu_0 y) + \mu'_0 \mu^{\frac{1}{2}} \Phi_1\left(\frac{\mu_0 r}{\mu}\right) \right]^4 - \mu^{-2} w^4(y) \\ &= \mu^{-2} \left[ w(y) + \left( w\left(\frac{y}{(1+\Lambda)^2}\right) - w(y) \right) + \mu'_0 \mu \Phi_1\left(\frac{\mu_0 r}{\mu}\right) \right]^4 - \mu^{-2} w^4(y) \\ &= 4\mu^{-2} w^3(y) s \left[ \left( w\left(\frac{y}{(1+\Lambda)^2}\right) - w(y) \right) + \mu'_0 \mu \Phi_1\left(\frac{\mu_0 r}{\mu}\right) \right] \end{aligned}$$

for some  $s \in (0, 1)$ . Observe that

$$w\left(\frac{y}{(1+\Lambda)^2}\right) - w(y) = \nabla w(y) \cdot y + \nabla w(y) \cdot y z [-2\Lambda - \Lambda^2] \quad (5-8)$$

for some  $z \in (0, 1)$ . Taking into account also the description of  $\Phi_1$  in (2-9), we get

$$\int_{B_{2R}} a_1 Z_0 dy = \mu^{-2} [q_1(\Lambda) + \mu_0 \mu'_0 q_0(\Lambda)] \phi_0(0, t). \quad (5-9)$$

We next claim that, for  $y \in B_{2R}$ , we have

$$\phi_0(\mu_0 y, t) - \phi_0(0, t) = \alpha(t) |\mu_0 y|^\sigma \Pi(t) \Theta(|y|) \quad (5-10)$$

for some  $\sigma \in (0, 1)$ . We postpone the proof of (5-10) to the Appendix. We thus get

$$\int_{B_{2R}} a_2 Z_0 dy = \mu^{-2} \mu_0^\sigma \alpha(t) b(t). \quad (5-11)$$

Collecting estimates (5-9)–(5-11) we get (5-7).

We claim that

$$\mu_0^{-\frac{1}{2}} i_2[\lambda, \phi](t) = g(t) + G[\lambda, \phi](t), \quad (5-12)$$

with

$$\|g\|_{\mathfrak{b}} \leq c, \quad \|G[\lambda, \phi]\|_{\mathfrak{b}} \leq c$$

for some constant  $c$ . We refer to (2-43) for the definition of  $\|\cdot\|_{\mathfrak{b}}$ . Furthermore, we claim that  $G$  satisfies estimates (5-5) and (5-6) for some constant  $c_1 \in (0, 1)$ . To prove the above assertion, we write

$$\begin{aligned} \mu_0^{-\frac{1}{2}} i_2[\lambda, \phi](t) &= 5 \int_{B_{2R}} w^4(y) \psi[0, 0](\mu_0 y, t) Z_0(y) dy \\ &+ 5 \int_{B_{2R}} w^4(y) [\psi[\lambda, 0] - \psi[0, 0]](\mu_0 y, t) Z_0(y) dy \\ &+ 5 \int_{B_{2R}} w^4(y) [\psi[\lambda, \phi] - \psi[\lambda, 0]](\mu_0 y, t) Z_0(y) dy \\ &+ 5 \int_{B_{2R}} \left[ w^4\left(\frac{y}{(1+\Lambda)^2}\right) - w^4(y) \right] \psi[\lambda, \phi](\mu_0 y, t) Z_0(y) dy \\ &+ 5 \left[ \frac{1}{(1+\Lambda)^4} - 1 \right] \int_{B_{2R}} w^4\left(\frac{y}{(1+\Lambda)^2}\right) \psi[\lambda, \phi](\mu_0 y, t) Z_0(y) dy = \sum_{j=1}^5 g_j. \end{aligned}$$

The first term,

$$g_1(t) = 5 \int_{B_{2R}} w^4(y) \psi(\mu_0 y, t) [0, 0] Z_0(y) dy,$$

is an explicit smooth function, globally defined in  $(t_0, \infty)$ , which satisfies the bound

$$\|g_1\|_{\mathfrak{b}} \leq c \left( 5 \int_{B_{2R}} w^4(y) |y| Z_0(y) dy \right) \quad (5-13)$$

for some constant  $c > 0$ , as direct consequence of (4-7). Let us analyze the term  $g_5$ . We see that  $g_5 = g_5[\lambda, \phi](t)$ . Let us first assume that  $\lambda$  and  $\phi$  are fixed. From (4-7), we get

$$|g_5(t)| \leq c q_1(\lambda) \int_{B_{2R}} |w^4(y) \psi[\lambda, \phi](\mu_0 y, t) Z_0(y)| dy \leq c \mu_0^{\frac{3}{2}} t^{-1} q_1(\lambda) \int \frac{|y|}{(1+|y|^5)} dy.$$

Using again (4-7) and the assumptions on  $\lambda$  and on  $\phi$ , we get  $[g_5]_{0, \sigma, [t, t+1]} \leq c \mu_0^{\frac{3}{2}} t^{-1}$ , from which we conclude that  $\|g_5\|_{\mathfrak{b}} \leq c$ , for some constant  $c > 0$ . Let us now fix  $\phi$  and take  $\lambda_1, \lambda_2$  satisfying (2-11).

We write

$$\begin{aligned} &g_5[\lambda_1, \phi] - g_5[\lambda_2, \phi] \\ &= 5 \left[ \frac{1}{(1+\Lambda_1)^4} - \frac{1}{(1+\Lambda_2)^4} \right] \int_{B_{2R}} w^4\left(\frac{y}{(1+\Lambda_1)^2}\right) \psi[\lambda_1, \phi](\mu_0 y, t) Z_0(y) dy \\ &+ 5 \left[ \frac{1}{(1+\Lambda_2)^4} - 1 \right] \int_{B_{2R}} \left[ w^4\left(\frac{y}{(1+\Lambda_1)^2}\right) - w^4\left(\frac{y}{(1+\Lambda_2)^2}\right) \right] \psi[\lambda_1, \phi](\mu_0 y, t) Z_0(y) dy \\ &+ 5 \left[ \frac{1}{(1+\Lambda_2)^4} - 1 \right] \int_{B_{2R}} w^4\left(\frac{y}{(1+\Lambda_2)^2}\right) [\psi[\lambda_1, \phi] - \psi[\lambda_2, \phi]](\mu_0 y, t) Z_0(y) dy = e_1 + e_2 + e_3. \end{aligned}$$

Thanks to (2-11), and arguing as before, we see that

$$\begin{aligned} |e_1(t)| &\leq c|\Lambda_1(t) - \Lambda_2(t)| \int_{B_{2R}} |w^4(y)\psi[\lambda_1, \phi](\mu_0 y, t)Z_0(y)| dy \\ &\leq c\mu_0(t)^{\frac{3}{2}}t^{-1} \left( \int_t^\infty s^{-1}\mu_0(s) ds \right) \|\lambda_1 - \lambda_2\|_{\#} \\ &\leq [\mu_0(t_0)]\mu_0(t)^{\frac{3}{2}}t^{-1} \|\lambda_1 - \lambda_2\|_{\#} \leq c_1\mu_0(t)^{\frac{3}{2}}t^{-1} \|\lambda_1 - \lambda_2\|_{\#}, \end{aligned}$$

where  $c_1$  is a positive number, which can be chosen arbitrarily small, in particular  $c_1 < 1$ , provided  $t_0$  is chosen large enough. Similarly one can show that, thanks to (2-11),

$$[e_1]_{0,\sigma,[t,t+1]} \leq c_1\mu_0(t)^{\frac{3}{2}}t^{-1} \|\lambda_1 - \lambda_2\|_{\#}.$$

We thus can conclude that there exists a positive small number  $c_1 < 1$  so that

$$\|e_1\|_b \leq c_1 \|\lambda_1 - \lambda_2\|_{\#}.$$

A similar argument allow us to say that also  $\|e_2\|_b \leq c_1 \|\lambda_1 - \lambda_2\|_{\#}$ . We next analyze  $e_3$ . From (4-22) we get

$$\begin{aligned} |e_3(t)| &\leq \mu_0^{\frac{3}{2}}t^{-1} \left( \int w^4(y) \frac{|y|}{1+|y|} dy \right) \|\psi[\lambda_1, \phi] - \psi[\lambda_2, \phi]\|_{**} \\ &\leq c_1\mu_0^{\frac{3}{2}}t^{-1} \|\lambda_1 - \lambda_2\|_{\#}, \end{aligned}$$

and also

$$[e_3]_{0,\sigma,[t,t+1]} \leq c_1\mu_0^{\frac{3}{2}}t^{-1} \|\lambda_1 - \lambda_2\|_{\#}$$

for some constant  $c_1 \in (0, 1)$ . We can conclude that

$$\|g_5[\lambda_1, \phi] - g_5[\lambda_2, \phi]\|_b \leq c_1 \|\lambda_1 - \lambda_2\|_{\#}.$$

The same estimate can be obtained for  $g_4$ , arguing in a similar way.

Let us now consider  $g_2$ . This term does not depend on  $\phi$ ; namely  $g_2[\lambda, \phi](t) = g_2[\lambda](t)$ . From Proposition 4.3, we get

$$|g_2(t)| \leq \mu_0^{\frac{3}{2}}t^{-2} \left( \int w^4 \frac{|y|}{1+|y|} dy \right) \|\lambda\|_{\#} \leq c\mu_0^{\frac{3}{2}}t^{-2} \|\lambda\|_{\#},$$

and similarly

$$[g_2(t)]_{0,\sigma,[t,t+1]} \leq c\mu_0^{\frac{3}{2}}t^{-2} \|\lambda\|_{\#}.$$

Furthermore, if  $t_0$  is large enough, there exists  $c_1 \in (0, 1)$  so that

$$\begin{aligned} |g_2[\lambda_1](t) - g_2[\lambda_2](t)| &\leq 5 \int_{\mathbb{R}^3} w^4(y) \|[\psi[\lambda_1, 0] - \psi[\lambda_2, 0]](\mu_0 y, t)\| Z_0 dy \\ &\leq Ct_0^{-1} \mu_0^{\frac{3}{2}}t^{-2} \|\lambda_1 - \lambda_2\|_{\#} \leq c_1\mu_0^{\frac{3}{2}}t^{-2} \|\lambda_1 - \lambda_2\|_{\#} \end{aligned}$$

and also

$$[g_1[\lambda_2] - g_2[\lambda_2]]_{0,\sigma,[t,t+1]} \leq c_1\mu_0^{\frac{3}{2}}t^{-2} \|\lambda_1 - \lambda_2\|_{\#}$$

thanks to the results of Proposition 4.3. Arguing in the same way, one gets similar estimates for  $g_3$ .

Collecting all the above arguments, we conclude that  $\mu_0^{-\frac{1}{2}}i_2[\lambda, \phi](t)$  can be written as in (5-12), with  $g$  and  $G$  satisfying (5-4), (5-5) and (5-6).

Next we claim that

$$\mu_0^{-\frac{1}{2}}i_j[\lambda, \phi](t) = G[\lambda, \phi](t), \quad j = 3, 4, \quad (5-14)$$

and  $G$  satisfies (5-4), (5-5) and (5-6). We start with  $j = 3$ . First, we see that  $i_3$  does not depend on  $\lambda$ , and it is linear in  $\phi$ . Since we are assuming that  $\phi$  satisfies (4-1), we have

$$\begin{aligned} |\mu_0^{-\frac{1}{2}}i_3(t)| &\leq (\mu_0\mu'_0R^{2-a})\mu_0^{\frac{3}{2}}(t)t^{-1}\|\phi\|_{v,a} \leq c\mu_0^{\frac{3}{2}}(t)t^{-1}\|\phi\|_{v,a}, \\ [\mu_0^{-\frac{1}{2}}i_3(t)]_{0,\sigma,[t,t+1]} &\leq c\mu_0^{\frac{3}{2}}(t)t^{-1}\|\phi\|_{v,a} \end{aligned}$$

for some constant  $c > 0$ . Let us now take  $\phi_1$ , and  $\phi_2$ , and we get that, if  $\mu_0(t_0)\mu'_0(t_0)R^{2-a}$  is small enough,

$$\begin{aligned} |\mu_0^{-\frac{1}{2}}(i_3[\phi_1] - i_3[\phi_2])(t)| &\leq c_1\mu_0^{\frac{3}{2}}(t)t^{-1}\|\phi_1 - \phi_2\|_{v,a}, \\ [\mu_0^{-\frac{1}{2}}(i_3[\phi_1] - i_3[\phi_2])(t)]_{0,\sigma,[t,t+1]} &\leq c_1\mu_0^{\frac{3}{2}}(t)t^{-1}\|\phi_1 - \phi_2\|_{v,a} \end{aligned}$$

for some  $c_1 \in (0, 1)$ . Estimate (5-14) for  $j = 4$  can be proved in a very similar way. We leave the details to the interested reader. Combining (5-7), (5-12) and (5-14), we complete the proof of (5-2).  $\square$

## 6. Solving a nonlocal linear problem

Let  $\phi_0$  be the function introduced in (2-53). Later in our argument we will need to solve in  $\lambda$ , a nonlocal equation of the form

$$\phi_0(0, t) = h(t), \quad t \in (t_0, \infty), \quad (6-1)$$

for a certain right-hand side  $h$ . We see from (5-3) that  $\phi_0(0, t)$ , defined as

$$\phi_0(0, t) = \int_{t_0-1}^t \int_{\mathbb{R}^3} \frac{\bar{\alpha}(s)}{(4\pi(t-s))^{\frac{3}{2}}} \frac{e^{-\frac{|y|^2}{4(t-s)}}}{\mu + |y|} \mathbf{1}_{\{|y| < M\}} dy ds,$$

defines a nonlocal nonlinear operator in  $\lambda$ . For convenience we recall that

$$\alpha(t) = 3^{\frac{1}{4}}\mu_0^{-\frac{1}{2}}(\mu_0\Lambda)', \quad \bar{\alpha}(t) = \begin{cases} \alpha(t_0) & \text{for } t < t_0, \\ \alpha(t) & \text{for } t \geq t_0, \end{cases} \quad \Lambda(t) = \int_t^\infty \lambda(s) ds.$$

We write

$$\phi_0(0, t) = T[\lambda](t) + \hat{T}[\lambda](t), \quad (6-2)$$

where  $T$  is

$$T[\lambda](t) = \int_{t_0-1}^t \int_{\mathbb{R}^3} \frac{\bar{\alpha}(s)}{(4\pi(t-s))^{\frac{3}{2}}} \frac{e^{-\frac{|y|^2}{4(t-s)}}}{|y|} \mathbf{1}_{\{|y| < M\}} dz ds. \quad (6-3)$$

We shall see that  $\hat{T}$  is a small perturbation of  $T$ , in a sense we will make precise later. In this section, we start with the analysis of problem

$$T[\lambda](t) = h(t), \quad t > t_0. \quad (6-4)$$

Straightforward computations give

$$T[\lambda](t) = -\frac{\bar{\omega}_3}{4} \int_{t_0-1}^t \frac{\bar{\alpha}(s)}{\sqrt{t-s}} (1 - e^{-\frac{M^2}{(t-s)}}) ds. \tag{6-5}$$

Indeed, letting  $z = y/(2\sqrt{t-s})$ , one gets

$$\begin{aligned} T[\lambda](t) &= \int_{t_0-1}^t \int_{\mathbb{R}^3} \frac{\bar{\alpha}(s)}{2\sqrt{t-s}} \frac{e^{-|z|^2}}{|z|} \mathbf{1}_{\{|z| < \frac{M}{\sqrt{t-s}}\}} dz ds \\ &= \frac{\bar{\omega}_3}{2} \int_{t_0-1}^t \int_0^\infty \frac{\bar{\alpha}(s)}{\sqrt{t-s}} e^{-\rho^2} \rho \mathbf{1}_{\{\rho < \frac{M}{\sqrt{t-s}}\}} d\rho ds = \frac{\bar{\omega}_3}{4} \int_{t_0-1}^t \frac{\bar{\alpha}(s)}{\sqrt{t-s}} \int_0^{\frac{M}{\sqrt{t-s}}} e^{-\rho^2} 2\rho d\rho \\ &= -\frac{\bar{\omega}_3}{4} \int_{t_0-1}^t \frac{\bar{\alpha}(s)}{\sqrt{t-s}} (1 - e^{-\frac{M^2}{(t-s)}}) ds. \end{aligned} \tag{6-6}$$

Introduce the function  $\beta = \beta(t)$  as

$$\beta(t) = \frac{\bar{\omega}_3}{4} \int_t^\infty \bar{\alpha}(s) ds. \tag{6-7}$$

If  $\beta = \beta(t)$  solves

$$\int_{t_0-1}^t \frac{\beta'(s)}{\sqrt{t-s}} (1 - e^{-\frac{M^2}{(t-s)}}) ds = h(t), \tag{6-8}$$

then the function  $\Lambda(t) = \int_t^\infty \lambda(s) ds$ , defined as

$$\bar{\omega} \Lambda(t) = \mu_0^{-\frac{1}{2}}(t) \beta(t) + \frac{\mu_0^{-1}(t)}{2} \int_t^\infty \beta(s) \mu_0^{-\frac{1}{2}} \mu_0'(s) ds, \quad \bar{\omega} = \frac{\bar{\omega}_3}{4} 3^{\frac{1}{4}}, \tag{6-9}$$

solves (6-4).

The next lemma constructs a solution to (6-8). If we formally let  $M \rightarrow \infty$  in (6-8), we get that the left-hand side of (6-8) is nothing but the  $\frac{1}{2}$ -Caputo derivative of  $\beta$ . This fact inspires the proof of the following:

**Lemma 6.1.** *Let  $\sigma = \frac{1}{2} + \sigma'$ , with  $\sigma' > 0$  small, be the number fixed in (2-11), and  $h : (t_0, \infty) \rightarrow \mathbb{R}$  a smooth function satisfying*

$$\sup_{t > t_0} \mu_0^{-\frac{3}{2}} t [ \|h\|_{0,[t,t+1]} + [h]_{0,\sigma,[t,t+1]} ] \leq C \tag{6-10}$$

for some constant  $C$ . Then there exist a constant  $C_1$  and a unique smooth function  $\beta : (t_0 - 1, \infty) \rightarrow \mathbb{R}$  which solves (6-8),  $\beta \in C^1$ , that satisfy the bound

$$\sup_{t > t_0} \mu_0^{-\frac{3}{2}} t [ \|\beta'\|_{0,[t,t+1]} + [\beta']_{0,\sigma,[t,t+1]} ] \leq C_1 M^{-1}. \tag{6-11}$$

We recall that  $M^2 = t_0$  was first introduced in (2-53).

Observe that a direct consequence of this lemma, together with (6-9) and (2-41), is the invertibility theory for problem (6-4) that will be used in next section to solve (5-1). This is contained in the following:

**Proposition 6.2.** *The function  $T : X_{\#} \rightarrow X_b$ , defined in (6-3) is a linear, nonlocal, homeomorphism such that*

$$\|T^{-1}(h)\|_{\#} \leq CM^{-1}\|h\|_b \quad \text{for any } h \in X_b, \quad (6-12)$$

for some fixed positive constant  $C$ . We refer to (2-11) and to (2-12) for the definition of the  $\|\cdot\|_{\#}$ -norm and of the set  $X_{\#}$ , and to (2-43) and (2-42) for the definition of the norm  $\|\cdot\|_b$  and of the space  $X_b$ .

We devote the rest of the section to the following:

*Proof of Lemma 6.1.* We start performing a change of variables, to transform problem (6-8) into an equivalent one with simpler form: Let

$$s = t_0 - 1 + M^2a, \quad t = t_0 - 1 + M^2b, \quad \tilde{\beta}(a) = \beta(s), \quad \tilde{h}(b) = h(t).$$

After this change of variables, problem (6-8) takes the form

$$\int_0^b \frac{\tilde{\beta}'(a)}{\sqrt{b-a}} (1 - e^{-\frac{1}{b-a}}) da = M\tilde{h}(b). \quad (6-13)$$

Let

$$K(\eta) = \frac{1 - e^{-\frac{1}{\sqrt{\eta}}}}{\sqrt{\eta}}$$

and take the Laplace transform of both sides in (6-13), thus getting

$$\mathcal{L}(\tilde{\beta}')(\xi)\mathcal{L}(K)(\xi) = M\mathcal{L}(\tilde{h})(\xi).$$

Since  $\mathcal{L}(\tilde{\beta}') = \xi\mathcal{L}(\tilde{\beta}) - \tilde{\beta}(0)$ , we get

$$\mathcal{L}(\tilde{\beta})(\xi) = \frac{\tilde{\beta}(0)}{\xi} + M \frac{\mathcal{L}(\tilde{h})(\xi)}{\xi\mathcal{L}(K)(\xi)}. \quad (6-14)$$

Observe now that

$$\mathcal{L}(K)(\xi) = \int_0^{\infty} e^{-\xi\eta} \left( \frac{1 - e^{-\frac{1}{\sqrt{\eta}}}}{\sqrt{\eta}} \right) d\eta = \frac{2}{\sqrt{\xi}} \int_0^{\infty} e^{-p^2} (1 - e^{-\frac{\xi}{p^2}}) dp.$$

We readily get that

$$\mathcal{L}(K)(\xi) = \frac{1}{\sqrt{\xi}} \left( 2 \int_0^{\infty} e^{-p^2} dp \right) (1 + o(1)) \quad \text{as } \xi \rightarrow \infty. \quad (6-15)$$

To describe the behavior of  $\mathcal{L}(K)(\xi)$ , for  $\xi \rightarrow 0$ , we first notice that

$$\int_0^{\frac{1}{\xi}} e^{-\xi\eta} \left( \frac{1 - e^{-\frac{1}{\sqrt{\eta}}}}{\sqrt{\eta}} \right) d\eta = \int_0^{\infty} \frac{1 - e^{-\frac{1}{\sqrt{\eta}}}}{\sqrt{\eta}} d\eta + O(\sqrt{\xi}).$$

On the other hand,

$$\int_{\frac{1}{\xi}}^{\infty} e^{-\xi\eta} \left( \frac{1 - e^{-\frac{1}{\sqrt{\eta}}}}{\sqrt{\eta}} \right) d\eta = \int_{\frac{1}{\xi}}^{\infty} e^{-\xi\eta} \left( \frac{1 - \frac{1}{\eta} - e^{-\frac{1}{\sqrt{\eta}}}}{\sqrt{\eta}} \right) d\eta + \int_{\frac{1}{\xi}}^{\infty} \frac{e^{-\xi\eta}}{\eta\sqrt{\eta}} d\eta = O(\sqrt{\xi}).$$

Thus we conclude that

$$\mathcal{L}(K)(\xi) = \int_0^\infty \frac{1 - e^{-\frac{1}{\eta}}}{\sqrt{\eta}} d\eta + O(\sqrt{\xi}) \quad \text{as } \xi \rightarrow 0. \tag{6-16}$$

From (6-15) and (6-16), we conclude that

$$\frac{1}{\xi \mathcal{L}(K)(\xi)} = \begin{cases} c_1/\xi + c_2/\sqrt{\xi} + O(1) & \text{if } \xi \rightarrow 0, \\ c_3/\sqrt{\xi}(1 + o(1)) & \text{if } \xi \rightarrow \infty. \end{cases}$$

Let now  $G = G(t)$  be so that  $\mathcal{L}(G)(\xi) = 1/(\xi \mathcal{L}(K)(\xi))$ . Standard arguments on Laplace transformation imply that

$$G(t) = \begin{cases} \tilde{c}_1 + \tilde{c}_2/\sqrt{t} + O(1/t) & \text{if } t \rightarrow \infty, \\ \tilde{c}_3/\sqrt{t}(1 + o(1)) & \text{if } t \rightarrow 0 \end{cases}$$

for certain constants  $\tilde{c}_1, \tilde{c}_2$  and  $\tilde{c}_3$ . From (6-14), taking the anti-Laplace transform of both sides, we get

$$\begin{aligned} \tilde{\beta}(b) &= \tilde{\beta}(0) + M \int_0^b \tilde{h}(a)G(b-a) da \\ &= \tilde{\beta}(0) + M\tilde{c}_1 \int_0^\infty \tilde{h}(a) da + M\tilde{c}_1 \int_b^\infty \tilde{h}(a) da + M \int_0^b \tilde{h}(a)[G(b-a) - \tilde{c}_1] da. \end{aligned}$$

We select the solution to problem (6-13) so that

$$\tilde{\beta}(0) + M\tilde{c}_1 \int_0^\infty \tilde{h}(a) da = 0.$$

In the original variables, we thus obtain an explicit solution to (6-8):

$$\beta(t) = \underbrace{\frac{\tilde{c}_1}{M} \int_t^\infty h(s) ds}_{:=\beta_1(t)} + \underbrace{\frac{1}{M} \int_{t_0-1}^t h(s) \left[ G\left(\frac{t-s}{M^2}\right) - \tilde{c}_1 \right] ds}_{:=\beta_2(t)}. \tag{6-17}$$

Let us now check (6-11). Since (6-10) holds, we easily get

$$\sup_{t > t_0} \mu_0^{-\frac{3}{2}} |\beta_1(t)| \lesssim M^{-1}.$$

To control the second term in (6-17), we change variable  $t = M^2\bar{t}, s = M^2\bar{s}$ , so that

$$\beta_2(t) = M \int_{\frac{t_0-1}{M^2}}^{\bar{t}} h(M^2\bar{s}) [G(\bar{t} - \bar{s}) - \tilde{c}_1] d\bar{s}.$$

Since  $t_0 = M^2$  and since (6-10) holds, we get

$$|\beta_2(t)| \lesssim \frac{1}{M} \int_{1-\frac{1}{t_0}}^{\bar{t}} \frac{\mu_0^{\frac{3}{2}}(\bar{s})}{\bar{s}} [G(\bar{t} - \bar{s}) - \tilde{c}_1] d\bar{s} \lesssim \frac{1}{M} \mu_0^{\frac{3}{2}}(\bar{t}) \lesssim M^{-1} \mu_0^{\frac{3}{2}}(t),$$

from which we get the validity of (6-11).

The assumption that  $\mu_0^{-\frac{3}{2}}t[h]_{0,\sigma,[t,t+1]}$  is bounded guarantees that the function  $\beta$  defined in (6-17) is differentiable. Indeed, trivially one has  $\beta'_1(t) = -(\tilde{c}_1/M)h(t)$ . Let us write  $\beta_2$  in the following way:

$$\beta_2(t) = \frac{1}{M} \int_{t_0}^t (h(s) - h(t)) \left[ G\left(\frac{t-s}{M^2}\right) - \tilde{c}_1 \right] ds + \frac{h(t)}{M} \int_{t_0}^t \left[ G\left(\frac{t-s}{M^2}\right) - \tilde{c}_1 \right] ds.$$

Thus we have

$$\begin{aligned} \beta'_2(t) &= \frac{1}{M} \lim_{s \rightarrow t} \underbrace{\left[ (h(s) - h(t)) \left[ G\left(\frac{t-s}{M^2}\right) - \tilde{c}_1 \right] \right]}_{=0} + \frac{1}{M^3} \int_{t_0}^t (h(s) - h(t)) G'\left(\frac{t-s}{M^2}\right) ds \\ &\quad + \underbrace{\left[ \frac{h'(t)}{M} \int_{t_0}^t \left[ G\left(\frac{t-s}{M^2}\right) - \tilde{c}_1 \right] ds - \frac{h'(t)}{M} \int_{t_0}^t \left[ G\left(\frac{t-s}{M^2}\right) - \tilde{c}_1 \right] ds \right]}_{=0} \\ &\quad + \frac{h(t)}{M} \frac{d}{dt} \left( \int_{t_0}^t \left[ G\left(\frac{t-s}{M^2}\right) - \tilde{c}_1 \right] ds \right) \\ &= \frac{1}{M^3} \int_{t_0}^t (h(s) - h(t)) G'\left(\frac{t-s}{M^2}\right) ds + \frac{h(t)}{M} \frac{d}{dt} \left( \int_{t_0}^t \left[ G\left(\frac{t-s}{M^2}\right) - \tilde{c}_1 \right] ds \right). \end{aligned}$$

Both the last two integrals are well-defined, as consequence of the behavior of  $G(\eta)$ , as  $\eta \rightarrow 0$ , and the assumption that  $\mu_0^{-\frac{3}{2}}t[h]_{0,\sigma,[t,t+1]}$  is bounded. Since  $G(\eta) \sim \eta^{-\frac{1}{2}}$  as  $\eta \rightarrow 0$ , direct computations give the bounds in (6-11) for  $\beta'(t)$ .  $\square$

## 7. Choice of $\lambda$ : part II

This section is devoted to solving (5-1) in  $\lambda$  for fixed  $\phi$  satisfying (4-1). We have the validity of the following:

**Proposition 7.1.** *For any  $\phi$  satisfying (4-1), there exists  $L > 0$  and a unique solution  $\lambda = \lambda[\phi]$  to (5-1), with*

$$\|\lambda\|_{\#} \leq LM^{-1}, \quad (7-1)$$

where  $M = \sqrt{t_0}$ , provided the initial time  $t_0$  in problem (2-1) is chosen large enough. Furthermore, there exists a constant  $c \in (0, 1)$  such that, for any  $\phi_1, \phi_2$  satisfying (4-1), we have

$$\|\lambda[\phi_1] - \lambda[\phi_2]\|_{\#} \leq c \|\phi_1 - \phi_2\|_{v,a}. \quad (7-2)$$

*Proof of Proposition 7.1.* Lemma 5.1 states that solving (5-1) is equivalent to solving (5-2). We write (5-2) as

$$T[\lambda](t) + \hat{T}[\lambda](t) = (1 + \mu_0 \mu'_0 b(t) + q_1(\lambda))^{-1} [g(t) + G[\lambda, \phi](t)], \quad (7-3)$$

where  $T$  and  $\hat{T}$  are defined in (6-2) and (6-3), while  $b, g$  and  $G$  satisfy the bounds in (5-4), (5-5) and (5-6). Here  $q_1 = q_1(s)$  denotes a smooth function such that  $q_1(0) = 0$  and  $q'_1(0) \neq 0$ . We observe first that

$$(1 + \mu_0 \mu'_0 b(t) + q_1(\lambda))^{-1} [g(t) + G[\lambda, \phi](t)] = g_1(t) + G_1[\lambda, \phi](t)$$

for some new functions  $g_1$  and  $G_1$  that also satisfy (5-4), (5-5), and (5-6).

Thanks to the result of Proposition 6.2, solving (7-3) in  $\lambda$  reduces to solving the fixed-point problem

$$\lambda(t) = \mathcal{F}(\lambda)(t), \quad \mathcal{F}(\lambda) := T^{-1}(g_1 + G_1[\lambda, \phi] - \hat{T}[\lambda]), \quad (7-4)$$

where  $T^{-1}$  is the operator introduced in Proposition 6.2.

Step 1: First we show that, for any fixed  $\phi$  satisfying (4-1), there exists a unique fixed point  $\lambda = \lambda[\phi]$  of contraction type for  $\mathcal{F}$  in the set

$$B = \{\lambda \in X_{\#} : \|\lambda\|_{\#} \leq LM^{-1}\}$$

for some  $L > 0$  large.

In order to prove this fact, we claim that, if the initial time  $t_0$  in problem (2-1) is large enough, there are positive constants  $\bar{c}_1, \bar{c}_2 \in (0, 1)$  so that, for any  $\lambda \in B$ ,

$$\|\hat{T}[\lambda]\|_b \leq \bar{c}_1 M \|\lambda\|_{\#}, \quad \text{with } \bar{c}_1 C < 1, \quad (7-5)$$

$$\|\hat{T}[\lambda_1] - \hat{T}[\lambda_2]\|_b \leq \bar{c}_2 \|\lambda_1 - \lambda_2\|_{\#}, \quad \text{with } CM^{-1}(c + \bar{c}_2) < 1, \quad (7-6)$$

for any  $\lambda_1, \lambda_2 \in B$ . The constant  $C$  is the constant appearing in (6-12), while  $c$  is the one appearing in (5-5).

Assume for the moment the validity of (7-5) and (7-6). For any  $\lambda \in B$ , we have

$$\begin{aligned} \|\mathcal{F}(\lambda)\|_{\#} &\leq CM^{-1} \|g_1 + G_1[\lambda, \phi] - \hat{T}[\lambda]\|_b \\ &\leq CM^{-1} (\|g_1\|_b + \|G_1[\lambda, \phi]\|_b + \|\hat{T}[\lambda]\|_b) \\ &\leq CM^{-1} (2c + \bar{c}_1 L) \leq LM^{-1}, \end{aligned}$$

provided  $L > (2cC)/(1 - \bar{c}_1 C)$ , where  $C$  is the constant in (6-12),  $c$  is the constant in (5-4), and  $\bar{c}_1$  is the constant in (7-5), which satisfies  $\bar{c}_1 C < 1$ .

Let us take now  $\lambda_1, \lambda_2 \in B$ . We have

$$\begin{aligned} \|\mathcal{F}(\lambda_1) - \mathcal{F}(\lambda_2)\|_{\#} &= \|T^{-1}(G_1[\lambda_1, \phi] - G_1[\lambda_2, \phi]) - T^{-1}(\hat{T}[\lambda_1] - \hat{T}[\lambda_2])\|_{\#} \\ &\leq CM^{-1} (\|G_1[\lambda_1, \phi] - G_1[\lambda_2, \phi]\|_b + \|\hat{T}[\lambda_1] - \hat{T}[\lambda_2]\|_b) \\ &\leq CM^{-1} (c_1 + \bar{c}_2) \|\lambda_1 - \lambda_2\|_{\#} < \varepsilon \|\lambda_1 - \lambda_2\|_{\#} \end{aligned}$$

for some  $\varepsilon < 1$ , thanks to the choice of  $\bar{c}_2$  in (7-6).

A direct application of the Banach fixed-point theorem gives the existence and uniqueness of a solution  $\lambda$  to (5-1), satisfying (7-1). We complete the first part of the proof of the proposition with the proofs of (7-5) and (7-6).

*Proof of (7-5).* Let  $\lambda \in B$ . From (6-2) and (6-3), we get

$$\begin{aligned} \hat{T}[\lambda](t) &= - \int_{t_0-1}^t \int_{\mathbb{R}^3} \frac{\bar{\alpha}(s)}{(4\pi(t-s))^{\frac{3}{2}}} \frac{e^{-\frac{|y|^2}{4(t-s)}}}{|y|} \frac{\mu(s)}{\mu(s) + |y|} \mathbf{1}_{\{|y| < M\}} dz ds \\ &= \bar{c} \int_{t_0-1}^t \frac{\bar{\alpha}(s)\mu(s)}{\sqrt{t-s}} \int_0^{\frac{M}{\sqrt{t-s}}} e^{-\rho^2} \frac{\rho}{\mu + \rho} d\rho ds \end{aligned}$$

for some explicit constant  $\bar{c}$ . Since

$$\left| \int_0^{\frac{M}{\sqrt{t-s}}} e^{-\rho^2} \frac{\rho}{\mu+\rho} d\rho \right| \leq c \frac{M}{\sqrt{t}}$$

for any  $t$  large, we observe that

$$|\widehat{T}[\lambda](t)| \leq A \frac{M}{\sqrt{t}} \left| \int_{t_0-1}^t \frac{\bar{\alpha}(s)\mu(s)}{\sqrt{t-s}} ds \right| \quad (7-7)$$

for some fixed constant  $A$ . We claim that

$$\int_{t_0-1}^t \frac{\bar{\alpha}(s)\mu(s)}{\sqrt{t-s}} ds = \bar{\alpha}(t)\mu(t)\sqrt{t-t_0+1}\Pi(t), \quad t > t_0, \quad (7-8)$$

for some smooth and uniformly bounded function  $\Pi(t)$ . Indeed, we write, for  $\beta_*(s) = \bar{\alpha}(s)\mu(s)$ ,

$$\int_{t_0-1}^t \frac{\beta_*(s)}{\sqrt{t-s}} ds = \int_{t_0-1}^t \frac{\beta_*(s) - \beta_*(t)}{\sqrt{t-s}} ds + 2\beta_*(t)\sqrt{t-t_0+1} = i + 2\beta_*(t)\sqrt{t-t_0+1}. \quad (7-9)$$

Using the change of variables  $x = \sqrt{t-s}$ ,

$$i = -2 \int_0^{\sqrt{t-t_0+1}} [\beta_*(t) - \beta_*(t-x^2)] dx = -2\beta_*(t) \int_0^{\sqrt{t-t_0+1}} \frac{[\beta_*(t) - \beta_*(t-x^2)]}{\beta_*(t)} dx.$$

We now observe that the function  $x \rightarrow [\beta_*(t) - \beta_*(t-x^2)]/\beta_*(t)$  is uniformly bounded in  $x \in [0, \sqrt{t-t_0+1}]$ , since

$$\frac{[\beta_*(t) - \beta_*(t-x^2)]}{\beta_*(t)} = \begin{cases} 1 - (1-x^2/t)^{-1}(1-x^2/t)^{-\frac{3}{2}\bar{\gamma}-1} & \text{for } \gamma \neq 2, \\ 1 - (1-x^2/t)^{-\frac{5}{2}}[1 + \log(1-x^2/t)]^3 & \text{for } \gamma = 2, \end{cases}$$

where  $\bar{\gamma} = 1$  if  $\gamma > 2$ , and  $\bar{\gamma} = \gamma - 1$  if  $1 < \gamma < 2$ . With this in mind, we conclude that

$$i = \beta_*(t)\sqrt{t-t_0+1}\Pi(t) \quad (7-10)$$

for some smooth and bounded function  $\Pi$ . Inserting (7-10) into (7-9), we get (7-8).

Using (7-8) in (7-7), we conclude that

$$|\widehat{T}[\lambda](t)| \leq A\mu_0(t)M \|\lambda\|_{\#} [\mu_0^{\frac{3}{2}}(t)t^{-1}]$$

for some fixed constant  $A$ , independent of  $t$  and of  $M$ . Thus, for  $t$  large, if we choose  $t_0$  sufficiently large, there exists a constant  $c_1 \in (0, 1)$  such that

$$|\widehat{T}[\lambda](t)| \leq c_1 M \|\lambda\|_{\#} [\mu_0^{\frac{3}{2}}(t)t^{-1}].$$

Let now consider  $t_1$  and  $t_2 \in [t, t + 1]$ . We write

$$\begin{aligned} \widehat{T}[\lambda](t_1) - \widehat{T}[\lambda](t_2) &= \bar{c} \int_{t_0-1}^{t_1} \left[ \frac{\bar{\alpha}(s)}{\sqrt{t_1-s}} - \frac{\bar{\alpha}(s)}{\sqrt{t_2-s}} \right] \int_0^{\frac{M}{\sqrt{t_1-s}}} e^{-\rho^2} \frac{\rho\mu}{\mu+\rho} d\rho ds \\ &\quad - \bar{c} \int_{t_0-1}^{t_1} \frac{\bar{\alpha}(s)}{\sqrt{t_2-s}} \int_{\frac{M}{\sqrt{t_2-s}}}^{\frac{M}{\sqrt{t_1-s}}} e^{-\rho^2} \frac{\rho\mu}{\mu+\rho} d\rho ds \\ &\quad - \bar{c} \int_{t_1}^{t_2} \frac{\bar{\alpha}(s)}{\sqrt{t_2-s}} \int_0^{\frac{M}{\sqrt{t_2-s}}} e^{-\rho^2} \frac{\rho\mu}{\mu+\rho} d\rho ds = \sum_{j=1}^3 i_j. \end{aligned}$$

Observe that, for  $t_1, t_2 \in [t, t + 1]$ , for  $t$  large, we have

$$\sup_{t_1, t_2 \in [t, t+1]} \frac{|\mu(t_1) - \mu(t_2)|}{|t_1 - t_2|^\sigma} \leq C\mu_0(t) \quad \sup_{t_1, t_2 \in [t, t+1]} \frac{|\Lambda(t_1) - \Lambda(t_2)|}{|t_1 - t_2|^\sigma} \leq CM^{-1}\mu_0(t)(\mu_0^\frac{3}{2}(t)t^{-1}) \quad (7-11)$$

for some constant  $C$ . With this, we can estimate  $i_1$  and  $i_5$  as follows:

$$[i_j]_{0, \sigma, [t, t+1]} \leq CM^{-1}\mu_0(t)(\mu_0^\frac{3}{2}(t)t^{-1}) \quad \text{for } j = 1, 5.$$

Straightforward computation gives

$$[i_j]_{0, \sigma, [t, t+1]} \leq CM^{-1}\mu_0(t)t^{-\sigma} \|\lambda\|_{\#} (\mu_0^\frac{3}{2}(t)t^{-1}) \quad \text{for } j = 1, 2, 3.$$

These estimates, together with the ones we obtained before, constitute the proof of (7-5).

*Proof of (7-6).* Let  $\lambda_1, \lambda_2 \in B$ . From (6-2) and (6-3),

$$\widehat{T}[\lambda_1](t) - \widehat{T}[\lambda_2](t) = \bar{c} \int_{t_0-1}^t \frac{\bar{\alpha}(s)}{\sqrt{t-s}} \int_0^{\frac{M}{\sqrt{t-s}}} e^{-\rho^2} \left[ \frac{\rho\mu[\lambda_1]}{\mu[\lambda_1] + \rho} - \frac{\rho\mu[\lambda_2]}{\mu[\lambda_2] + \rho} \right] d\rho ds.$$

Observe that

$$\begin{aligned} |(\mu[\lambda_1] - \mu[\lambda_2])(s)| &\leq A\mu_0(s)|\Lambda_1(s) - \Lambda_2(s)| \\ &\leq A\mu_0(s) \int_s^\infty |\lambda_1 - \lambda_2|(x) dx \leq A\mu_0^2(s) \|\lambda_1 - \lambda_2\|_{\#} \end{aligned}$$

for some constant  $A$ , whose value may change from one line to the other, and which is independent of  $t$  and  $t_0$ . A Taylor expansion gives

$$|\widehat{T}[\lambda_1](t) - \widehat{T}[\lambda_2](t)| \leq \int_{t_0-1}^t \frac{|\bar{\alpha}(s)|}{\sqrt{t-s}} \int_0^{\frac{M}{\sqrt{t-s}}} e^{-\rho^2} \frac{\rho}{(\tilde{\mu} + \rho)^2} |\mu[\lambda_1](s) - \mu[\lambda_2](s)| d\rho ds$$

for some  $\tilde{\mu}$  between  $\mu[\lambda_1]$  and  $\mu[\lambda_2]$ . Thus we get

$$|\widehat{T}[\lambda_1](t) - \widehat{T}[\lambda_2](t)| \leq A\mu_0^2(t)M[\mu_0^\frac{3}{2}(t)t^{-1}] \|\lambda_1 - \lambda_2\|_{\#},$$

where  $A$  is a constant independent of  $t_0$  and  $t$ . Using again (7-11), we can show that

$$|\widehat{T}[\lambda_1] - \widehat{T}[\lambda_2]|_{0, \sigma, [t, t+1]} \leq A\mu_0^2(t)M[\mu_0^\frac{3}{2}(t)t^{-1}] \|\lambda_1 - \lambda_2\|_{\#},$$

where  $A$  is a constant independent of  $t_0$  and  $t$ . Choosing  $t_0$  large enough, we can find  $\bar{c}_2$  small enough so that (7-6) holds true.

**Step 2:** In the second part of the proof, we show the validity of (7-2). For this purpose, we fix  $\phi_1$  and  $\phi_2$  satisfying (4-1), and we let  $\lambda_j = \lambda[\phi_j]$ ,  $j = 1, 2$ . If  $\bar{\lambda} = \lambda_1 - \lambda_2$ , then we see that  $\bar{\lambda}$  solves

$$\begin{aligned}\bar{\lambda} &= T^{-1}(G_1[\lambda_1, \phi_1] - G_1[\lambda_2, \phi_2]) \\ &= T^{-1}(G_1[\bar{\lambda}_1, \phi_1] - G[\bar{\lambda}_1, \phi_2]) + T^{-1}(G_1[\lambda_1, \phi_2] - G[\lambda_2, \phi_2]).\end{aligned}$$

Thus

$$\begin{aligned}\|\bar{\lambda}\|_{\#} &\leq CM^{-1}(\|G_1[\bar{\lambda}_1, \phi_1] - G[\bar{\lambda}_1, \phi_2]\|_{\mathfrak{b}} + \|G_1[\lambda_1, \phi_2] - G[\lambda_2, \phi_2]\|_{\mathfrak{b}}) \\ &\leq CM^{-1}(c\|\phi_1 - \phi_2\|_{v,a} + c\|\lambda_1 - \lambda_2\|_{\#}),\end{aligned}$$

where  $C$  is the constant in (6-12),  $M^2 = t_0$ ,  $c$  are the constants defined respectively in (5-5) and (5-6). We now observe that the proof of Lemma 5.1 also gives that the constants  $c$  in (5-5) and (5-6) can be such that  $CM^{-1}c < 1$ . Thus the proof of (7-2) readily follows.  $\square$

**Remark 7.2.** Recall that the function  $\psi = \bar{\Psi}[\psi_0]$ , the solution to problem (3-6), depends smoothly on the initial condition  $\psi_0$ , provided  $\psi_0$  belongs to a small neighborhood of 0 in the Banach space  $L^\infty(\Omega)$  equipped with the norm defined in (4-21), as observed in Remark 4.2. This fact implies that also  $\lambda = \lambda[\psi_0]$ , the solution to (5-1), depends on  $\psi_0$ . A closer look at the definitions of  $\lambda = \lambda[\psi_0]$  gives that

$$\|\lambda[\psi_0^{(1)}] - \lambda[\psi_0^{(2)}]\|_{\#} \lesssim \|e^{b|y|}[\psi_0^{(1)} - \psi_0^{(2)}]\|_{L^\infty(\mathbb{R}^3)} + \|e^{b|y|}[\nabla\psi_0^{(1)} - \nabla\psi_0^{(2)}]\|_{L^\infty(\mathbb{R}^3)}.$$

This fact will be useful in the final argument of finding  $\phi$ , the solution to (3-13).

## 8. Final argument: solving (3-8)

We are constructing a global unbounded solution to problem (2-1)–(2-2) of the form (3-1)

$$u = U_2[\lambda](r, t) + \tilde{\phi}.$$

The function  $U_2$  is defined in (2-57), while  $\tilde{\phi}$  is given in (3-2). The function  $\psi$  which enters in the definition of  $\tilde{\phi}$  solves the *outer problem* (3-6), and its properties are contained in Propositions 4.1 and 4.3. The parameter  $\lambda = \lambda(t)$  belongs to the space  $X_{\#}$ , (2-12), and has been chosen to solve (5-1). The properties of this  $\lambda = \lambda(t)$  are collected in Proposition 7.1. What is left is to solve the *inner problem* (3-8) in  $\phi$ . Thanks to the choice of  $\lambda = \lambda(t)$ , the orthogonality condition (3-19) is satisfied, so that we can use the result of Proposition 3.1 to solve problem (3-8) in  $\phi$ .

In other words, we want to find  $\phi$ , with its  $\|\phi\|_{v,a}$ -bounded, solution to problem (3-8). The function  $\psi = \Psi[\lambda[\phi], \phi]$  solves (3-6), while  $\lambda = \lambda[\phi]$  solves (5-1).

At this point, we fix  $a$  in the definition of  $\|\cdot\|_{v,a}$  to be equal to 1. Proposition 3.1 defines a linear operator  $\phi = \mathcal{T}(h)$ , where  $\phi$  is the solution to (3-16) so that

$$\|\phi\|_{v,1} \leq C_0 R^4 \|h\|_{v,3}$$

for some fixed constant  $C_0$ . We refer to (3-17) for  $\|h\|_{v,2+a}$  and to (3-18) for  $\|\phi\|_{v,a}$ , for  $a = 1$ . Thus we can say that  $\phi$  solves (3-11)–(3-13) if and only if  $\phi$  is a fixed point for the problem

$$\phi = \mathcal{T}(H[\phi]), \quad \text{where } H[\phi] = H(\psi[\phi], \lambda[\phi], \phi), \tag{8-1}$$

and  $H$  is defined in (3-12). Choose the number  $R$  in the cut-off function  $\eta_R$ , defined in (2-59) and appearing in the ansatz (3-2), to be sufficiently large in terms of  $t_0$ , say  $R^6 \mu_0^{\frac{1}{2}}(t_0) = 1$ . We claim that there exists a unique  $\phi$  solution to (8-1) in the set

$$B_1 = \{\phi : \|\phi\|_{v,1} \leq L_1\}$$

for some  $L_1 > 0$ , fixed.

From (2-59) and (4-7), we see that

$$|\mu_0^{\frac{5}{2}} \mathcal{E}_{22}(\mu_0 y, t)| \lesssim \mu_0^{\frac{1}{2}} \frac{\mu_0^{\frac{3}{2}} t^{-1}}{(1 + |y|^2)^2}, \quad \left| 5 \frac{\mu_0^{\frac{1}{2}}}{(1 + \Lambda)^4} w^4 \left( \frac{y}{(1 + \Lambda)^2} \right) \psi(\mu_0 y, t) \right| \lesssim \frac{\mu_0^2(t) t^{-1}}{(1 + |y|^3)}.$$

Furthermore,

$$|B[\phi](t)| \leq CR^2 \mu_0 \mu_0' \frac{\mu_0^{\frac{3}{2}} t^{-1}}{(1 + |y|^{2+a})}, \quad |B^0[\phi](t)| \leq C\Lambda(t) \frac{\mu_0^{\frac{3}{2}} t^{-1}}{(1 + |y|^{4+a})}.$$

In fact, one can prove that

$$\|H[\phi]\|_{v,2+a} \leq C_1 R^{-4}$$

for some fixed number  $C_1$ , independent from  $t$  and of  $t_0$ . This implies that, if  $\phi \in B_1$ , then  $\mathcal{T}(\phi) \in B_1$ , provided  $L_1$  is chosen large. Furthermore, combining (2-61), the result of Proposition 4.3, and the result of Proposition 7.1, we get the existence of a number  $c \in (0, 1)$  so that

$$\|\mathcal{T}[\phi_1] - \mathcal{T}[\phi_2]\|_{v,a} \leq c \|\phi_1 - \phi_2\|_{v,a}$$

for any  $\phi_1$  and  $\phi_2 \in B_1$ . We apply Banach fixed-point theorem to get the existence of a unique solution to (8-1) with  $\|\cdot\|_{v,a}$ -bounded.

This concludes the proof of the existence of the solution to problem (2-1)–(2-2), or equivalently problem (1-3)–(1-4), as predicted by Theorem 1.1.  $\square$

### 9. Basic linear theory for the inner problem

Let  $R > 0$  be a fixed large number. This section is devoted to constructing a solution to the initial value problem

$$\phi_\tau = \Delta\phi + 5w^4\phi + h(y, \tau) \quad \text{in } B_{2R} \times (\tau_0, \infty), \quad \phi(y, \tau_0) = e_0 Z(y) \quad \text{in } B_{2R}, \tag{9-1}$$

for any given function  $h$  with  $\|h\|_{v,2+a} < +\infty$ , not necessarily radial in the  $y$ -variable. We refer to (3-17) for the explicit definition of the  $\|\cdot\|_{v,2+a}$  norm. The corresponding problem in dimension  $n \geq 5$  has already been treated in [Cortázar et al. 2016, Section 7]. We follow the same strategy in the procedure to

construct the solution to (9-1), but in dimension 3 we get a decay estimate for the solution different from the one valid for dimensions  $n \geq 5$ .

We recall that the operator  $L_0(\phi) = \Delta\phi + 5w^4\phi$  has a 4-dimensional kernel generated by the bounded functions  $Z_0$  defined in (2-7) and also by

$$Z_i(y) = \frac{\partial w}{\partial y_i}, \quad i = 1, 2, 3. \quad (9-2)$$

In the class of radially symmetric functions, the only element in the kernel of  $L_0$  is  $Z_0$ . To describe our construction, we consider an orthonormal basis  $\vartheta_m$ ,  $m = 0, 1, \dots$ , in  $L^2(S^2)$  of spherical harmonics, namely eigenfunctions of the problem

$$\Delta_{S^2}\vartheta_m + \lambda_m\vartheta_m = 0 \quad \text{in } S^2$$

so that  $0 = \lambda_0 < \lambda_1 = \dots = \lambda_3 = 2 < \lambda_4 \leq \dots$ . Let  $h(\cdot, \tau) \in L^2(B_{2R})$  for any  $\tau \in [\tau_0, \infty)$ . We decompose it into the form

$$h(y, \tau) = \sum_{j=0}^{\infty} h_j(r, \tau) \vartheta_j\left(\frac{y}{r}\right), \quad r = |y|, \quad h_j(r, \tau) = \int_{S^2} h(r\theta, \tau) \vartheta_j(\theta) d\theta.$$

In addition, we write  $h = h^0 + h^1 + h^\perp$ , where

$$h^0 = h_0(r, \tau), \quad h^1 = \sum_{j=1}^3 h_j(r, \tau) \vartheta_j, \quad h^\perp = \sum_{j=4}^{\infty} h_j(r, \tau) \vartheta_j.$$

Observe that  $h^1 = h^\perp = 0$  if  $h$  is radially symmetric in the  $y$ -variable. Consider also the analogous decomposition for  $\phi$  into  $\phi = \phi^0 + \phi^1 + \phi^\perp$ . We build the solution  $\phi$  of problem (9-1) by doing so separately for the pairs  $(\phi^0, h^0)$ ,  $(\phi^1, h^1)$  and  $(\phi^\perp, h^\perp)$ .

Our main result in this section is the following proposition.

**Proposition 9.1.** *Let  $\nu, a$  be given positive numbers with  $0 < a < 2$ . Then, for all sufficiently large  $R > 0$  and any  $h = h(y, \tau)$  with  $\|h\|_{\nu, 2+a} < +\infty$  that satisfies for all  $j = 0, 1, \dots, 3$*

$$\int_{B_{2R}} h(y, \tau) Z_j(y) dy = 0 \quad \text{for all } \tau \in (\tau_0, \infty), \quad (9-3)$$

*there exist  $\phi = \phi[h]$  and  $e_0 = e_0[h]$  which solve problem (9-1). They define linear operators of  $h$  that satisfy the estimates*

$$|\phi(y, \tau)| \lesssim \tau^{-\nu} \left[ \frac{R^{4-a}}{1+|y|^3} \|h^0\|_{\nu, 2+a} + \frac{R^{4-a}}{1+|y|^4} \|h^1\|_{\nu, 2+a} + \frac{\|h\|_{\nu, 2+a}}{1+|y|^a} \right], \quad (9-4)$$

$$|\nabla_y \phi(y, \tau)| \lesssim \tau^{-\nu} \left[ \frac{R^{4-a}}{1+|y|^4} \|h^0\|_{\nu, 2+a} + \frac{R^{4-a}}{1+|y|^5} \|h^1\|_{\nu, 2+a} + \frac{\|h\|_{\nu, 2+a}}{1+|y|^{a+1}} \right], \quad (9-5)$$

and

$$|e_0[h]| \lesssim \|h\|_{\nu, 2+a}. \quad (9-6)$$

Proposition 3.1 is a direct consequence of Proposition 9.1. Indeed, if  $h$  is radially symmetric in the  $y$ -variable, (9-3) is automatically satisfied for  $j = 1, \dots, 3$ , and  $h \equiv h^0$ .

The result contained in Proposition 9.1 follows from the next proposition, which refers to the problem

$$\phi_\tau = \Delta\phi + 5w(y)^4\phi + h(y, \tau) - c(\tau)Z \quad \text{in } B_{2R} \times (\tau_0, \infty), \quad \phi(y, \tau_0) = 0 \quad \text{in } B_{2R}. \quad (9-7)$$

**Proposition 9.2.** *Let  $\nu, a$  be given positive numbers with  $0 < a < 2$ . Then, for all sufficiently large  $R > 0$  and any  $h$  with  $\|h\|_{\nu, 2+a} < +\infty$  and satisfying the orthogonality conditions (3-19), there exist  $\phi = \phi[h]$  and  $c = c[h]$  which solve problem (9-7), and define linear operators of  $h$ . The function  $\phi[h]$  satisfies estimate (9-4), (9-5) and for some  $\Gamma > 0$*

$$\left| c(\tau) - \int_{B_{2R}} hZ \right| \lesssim \tau^{-\nu} \left[ R^{2-a} \left\| h - Z \int_{B_{2R}} hZ \right\|_{\nu, 2+a} + e^{-\Gamma R} \|h\|_{\nu, 2+a} \right]. \quad (9-8)$$

Assuming the validity of Proposition 9.2, we proceed with:

*Proof of Proposition 9.1.* Let  $\phi_1$  be the solution of problem (9-7) predicted by Proposition 9.2. Let us write

$$\phi(y, \tau) = \phi_1(y, \tau) + e(\tau)Z(y) \quad (9-9)$$

for some  $e \in C^1([\tau_0, \infty))$ . We find

$$\partial_\tau \phi = \Delta\phi + 5w^4\phi + h(y, \tau) + [e'(\tau) - \lambda_0 e(\tau) - c(\tau)]Z(y).$$

We choose  $e(\tau)$  to be the unique bounded solution of the equation

$$e'(\tau) - \lambda_0 e(\tau) = c(\tau), \quad \tau \in (\tau_0, \infty),$$

which is explicitly given by

$$e(\tau) = \int_\tau^\infty \exp(\sqrt{\lambda_0}(\tau - s))c(s) ds.$$

The function  $e$  depends linearly on  $h$ . Besides, we clearly have from (9-8),  $|e(\tau)| \lesssim \tau^{-\nu} \|h\|_{\nu, 2+a}$ , and thus, from the fact that  $\phi_1$  satisfies estimates (9-4), (9-5), so does  $\phi$  given by (9-9). Thus  $\phi$  satisfies problem (9-1) with initial condition  $\phi(y, \tau_0) = e(\tau_0)Z(y)$ .  $\square$

The rest of the section is devoted to the following:

*Proof of Proposition 9.2.* The proof is divided in two steps. In the first step, we construct a solution to (9-7) which has value zero on the boundary  $\partial B_{2R}$ , at any time  $\tau$ , for a right-hand side  $h$  not necessarily satisfying the orthogonality conditions (9-3). In the second step, we make use of this construction to solve (9-7), for a right-hand side satisfying (9-3), and to obtain estimates (9-4), (9-5) and (9-6).

**Step 1:** We claim that for all sufficiently large  $R > 0$  and any  $H$  with  $\|H\|_{\nu, a} < +\infty$  there exist  $\phi = \phi(y, \tau)$  and  $c = c(\tau)$  which solve problem

$$\begin{aligned} \phi_\tau &= \Delta\phi + 5w^4\phi + H(y, \tau) - c(\tau)Z(y) \quad \text{in } B_{2R} \times (\tau_0, \infty), \\ \phi &= 0 \quad \text{on } \partial B_{2R} \times (\tau_0, \infty), \quad \phi(\cdot, \tau_0) = 0 \quad \text{in } B_{2R}. \end{aligned} \quad (9-10)$$

The functions  $\phi$  and  $c$  are linear operators of  $h$  and satisfy the estimates

$$(1 + |y|)|\nabla\phi(y, \tau)| + |\phi(y, \tau)| \lesssim \tau^{-\nu} \left[ \frac{R^{4-a}\|H^0\|_{\nu,a}}{1 + |y|} + \frac{R^{4-a}\|H^1\|_{\nu,a}}{1 + |y|^2} + R^2 \frac{\|H\|_{\nu,a}}{1 + |y|^a} \right] \quad (9-11)$$

and for some  $\Gamma > 0$

$$\left| c(\tau) - \int_{B_{2R}} HZ \right| \lesssim \tau^{-\nu} \left[ R^2 \left\| H - Z \int_{B_{2R}} HZ \right\|_{\nu,a} + e^{-\Gamma R} \|H\|_{\nu,a} \right]. \quad (9-12)$$

We construct the solution  $\phi$  mode by mode, considering first mode 0, then modes 1, 2, 3 and finally modes greater or equal to 4. For each mode, we get the corresponding estimates.

*Construction at mode 0.* Consider problem (9-10) for a right-hand side  $H = H_0(r, \tau)$  radially symmetric. Let  $\eta(s)$  be the smooth cut-off function in (1-9), and consider  $\eta_\ell(y) = \eta(|y| - \ell)$  for a large but fixed number  $\ell$  independent of  $R$ . By standard parabolic theory, there exists a unique solution  $\phi_*[\bar{h}_0]$  to

$$\begin{aligned} \phi_\tau &= \Delta\phi + 5w(r)^4(1 - \eta_\ell)\phi + \bar{H}_0(y, \tau) \quad \text{in } B_{2R} \times (\tau_0, \infty), \\ \phi &= 0 \quad \text{on } \partial B_{2R} \times (\tau_0, \infty), \quad \phi(\cdot, \tau_0) = 0 \quad \text{in } B_{2R}, \end{aligned} \quad (9-13)$$

where

$$\bar{H}_0 = H_0 - c_0(\tau)Z, \quad c_0(\tau) = \int_{B_{2R}} H_0(y, \tau)Z(y) dy.$$

The function  $\phi_*[\bar{h}_0]$  is radial and satisfies the bound

$$|\phi_*[\bar{H}_0]| \lesssim \tau^{-\nu} R^{2-a} \|H\|_{\nu,a}.$$

This can be proven with the use of a special supersolution arguing as in Lemma 7.3 in [Cortázar et al. 2016]. Setting  $\phi = \phi_*[\bar{H}_0] + \tilde{\phi}$  and  $c(\tau) = c_0(\tau) + \tilde{c}(\tau)$ , problem (9-10) gets reduced to

$$\begin{aligned} \tilde{\phi}_\tau &= \Delta\tilde{\phi} + 5w(r)^4\tilde{\phi} + \tilde{H}_0(r, \tau) - \tilde{c}(\tau)Z \quad \text{in } B_{2R} \times (\tau_0, \infty), \\ \tilde{\phi} &= 0 \quad \text{on } \partial B_{2R} \times (\tau_0, \infty), \quad \tilde{\phi}(\cdot, \tau_0) = 0 \quad \text{in } B_{2R}. \end{aligned} \quad (9-14)$$

where  $\tilde{H}_0 = 5w^4\eta_\ell\phi_*[\bar{H}_0]$ . Observe that  $\tilde{H}_0$  is radial, compactly supported and with size controlled by that of  $\bar{H}_0$ . In particular we have that for any  $m > 0$

$$|\tilde{H}_0(r, \tau)| \lesssim \frac{\tau^{-\nu}}{1 + r^m} \left[ \sup_{\tau > \tau_0} \tau^\nu \|\phi_*[\bar{H}_0](\cdot, \tau)\|_{L^\infty} \right] \lesssim \frac{\tau^{-\nu}}{1 + r^m} R^{2-a} \|H\|_{\nu,a}. \quad (9-15)$$

We shall next solve problem (9-14) under the additional orthogonality constraint

$$\int_{B_{2R}} \tilde{\phi}(\cdot, \tau)Z = 0 \quad \text{for all } \tau \in (\tau_0, \infty). \quad (9-16)$$

Problem (9-14)–(9-16) is equivalent to solving just (9-14) for  $\tilde{c}$  given by the explicit linear functional  $\tilde{c} := \tilde{c}[\tilde{\phi}, \tilde{H}_0]$  determined by the relation

$$\tilde{c}(\tau) \int_{B_{2R}} Z^2 = \int_{B_{2R}} \tilde{H}_0(\cdot, \tau)Z + \int_{\partial B_{2R}} \partial_r \tilde{\phi}(\cdot, \tau)Z. \quad (9-17)$$

If the function  $\tilde{c} = \tilde{c}(\tau)$  defined by (9-17) were independent of  $\phi$ , standard linear parabolic theory would give the existence of a unique solution. On the other hand, a close look at (9-17) shows that the dependence of  $\tilde{c} = \tilde{c}(\tau)$  on  $\phi$  is small in an  $L^\infty\text{-}C^{1+\alpha, \frac{1+\alpha}{2}}$  setting, since  $Z(R) = O(e^{-\Gamma R})$  for some  $\Gamma > 0$ . A contraction argument applies to yield existence of a unique solution to (9-14)–(9-16) defined at all times. To get the estimates, we assume smoothness of the data so that integrations by parts and differentiations can be carried over, and then argue by approximations. Testing (9-14)–(9-16) against  $\tilde{\phi}$  and integrating in space, we obtain the relation

$$\partial_\tau \int_{B_{2R}} \tilde{\phi}^2 + Q(\tilde{\phi}, \tilde{\phi}) = \int_{B_{2R}} g \tilde{\phi}, \quad g = \tilde{H}_0 - \tilde{c}(\tau)Z_0,$$

where  $Q$  is the quadratic form defined by

$$Q(\phi, \phi) := \int [|\nabla\phi|^2 - 5w^4|\phi|^2]. \tag{9-18}$$

Since dimension is 3, there exists  $\beta > 0$  such that, for any  $\phi$  with  $\int \phi Z = 0$ , the following inequality holds:

$$Q(\phi, \phi) \geq \frac{\beta}{R^2} \int \phi^2.$$

The proof of this inequality is a slight modification of the proof for the corresponding inequality in dimensions  $n \geq 5$  that can be found in [Cortázar et al. 2016, Lemma 7.2], considering that  $\int_{B_R} Z_0^2 = O(R)$  as  $R \rightarrow \infty$ , when dimension is 3. Thus we have, for some  $\beta' > 0$ ,

$$\partial_\tau \int_{B_{2R}} \tilde{\phi}^2 + \frac{\beta'}{R^2} \int_{B_{2R}} \tilde{\phi}^2 \lesssim R^2 \int_{B_{2R}} g^2. \tag{9-19}$$

We observe that from (9-17) and (9-15) for  $m = 0$  we get

$$|\tilde{c}(\tau)| \leq \tau^{-\nu} K, \quad K := \left[ \sup_{\tau > \tau_0} \tau^\nu \|\phi_*[\bar{H}_0](\cdot, \tau)\|_{L^\infty} \right] + e^{-\Gamma R} \left[ \sup_{\tau > \tau_0} \tau^\nu \|\nabla\phi_*[\bar{H}_0](\cdot, \tau)\|_{L^\infty} \right].$$

Besides, using again estimate (9-15) for a sufficiently large  $m$ , we get

$$\int_{B_{2R}} g^2 \lesssim \tau^{-2\nu} K^2.$$

Using that  $\tilde{\phi}(\cdot, \tau_0) = 0$  and Gronwall’s inequality, we readily get from (9-19) the  $L^2$ -estimate

$$\|\tilde{\phi}(\cdot, \tau)\|_{L^2(B_{2R})} \lesssim \tau^{-\nu} R^2 K \tag{9-20}$$

for all  $\tau > \tau_0$ . Now, using standard parabolic estimates in the equation satisfied by  $\tilde{\phi}$  we obtain then that on any large fixed radius  $\ell > 0$ ,

$$\|\tilde{\phi}(\cdot, \tau)\|_{L^\infty(B_M)} \lesssim \tau^{-\nu} R^2 K \quad \text{for all } \tau > \tau_0.$$

Since the right-hand side has a fast decay at infinity and taking into account that we are in dimension 3, outside  $B_\ell$  we can dominate the solution by a barrier of the order  $\tau^{-\nu}|y|^{-1}$ . As a conclusion, also using

local parabolic estimates for the gradient, we find that

$$(1 + |y|)|\nabla_y \tilde{\phi}(y, \tau)| + |\tilde{\phi}(y, \tau)| \lesssim \tau^{-\nu} \frac{R^2}{1 + |y|} \left[ \sup_{\tau > \tau_0} \tau^\nu \|\phi_*[\bar{H}_0](\cdot, \tau)\|_{L^\infty} \right]. \quad (9-21)$$

It clearly follows from this estimate and inequality (9-15) that the function

$$\phi_0[h_0] := \tilde{\phi} + \phi_*[\bar{H}_0] \quad (9-22)$$

solves problem (9-10) for  $H = H_0$  and satisfies

$$(1 + |y|)|\nabla_y \phi_0(y, \tau)| + |\phi_0(y, \tau)| \lesssim \tau^{-\nu} \frac{R^{4-a}}{1 + |y|} \|H\|_{v,a}.$$

Finally, from (9-17) we see that

$$c(\tau) = \int_{B_{2R}} HZ + \int_{B_{2R}} 5w^4 \eta_\ell \phi_*[\bar{H}_0]Z + O(e^{-\Gamma R}) \|H\|_{v,a}.$$

From here we find the validity of the estimate

$$\left| c(\tau) - \int_{B_{2R}} H_0 Z \right| \lesssim \tau^{-\nu} \left[ R^2 \left\| H_0 - Z \int_{B_{2R}} H_0 Z \right\|_{v,a} + e^{-\Gamma R} \|H_0\|_{v,a} \right].$$

Hence estimates (9-11) and (9-12) hold. The construction of the solution at mode 0 is concluded.

*Construction at modes 1 to 3.* Here we consider the case  $H = H^1$ , where  $H^1(y, \tau) = \sum_{j=1}^3 H_j(r, \tau) \vartheta_j$ .

The function

$$\phi^1[H^1] := \sum_{j=1}^n \phi_j(r, \tau) \vartheta_j \quad (9-23)$$

solves the initial-boundary value problem

$$\begin{aligned} \phi_\tau &= \Delta \phi + 5w^4 \phi + H^1(y, \tau) \quad \text{in } B_{2R} \times (\tau_0, \infty), \\ \phi &= 0 \quad \text{on } \partial B_{2R} \times (\tau_0, \infty), \quad \phi(\cdot, \tau_0) = 0 \quad \text{in } B_{2R}, \end{aligned} \quad (9-24)$$

if the function  $\phi_j(r, \tau)$  solves

$$\begin{aligned} \partial_\tau \phi_j &= \mathcal{L}_1[\phi_j] + H_j(r, \tau) \quad \text{in } (0, 2R) \times (\tau_0, \infty), \\ \partial_r \phi_j(0, \tau) &= 0 = \phi_j(R, \tau) \quad \text{for all } \tau \in (\tau_0, \infty), \quad \phi_j(r, \tau_0) = 0 \quad \text{for all } r \in (0, R), \end{aligned} \quad (9-25)$$

where

$$\mathcal{L}_1[\phi_j] := \partial_{rr} \phi_j + 2 \frac{\partial_r \phi_j}{r} - 2 \frac{\phi_j}{r^2} + 5w^4 \phi_j. \quad (9-26)$$

Let us consider the solution of the stationary problem  $\mathcal{L}_1[\phi] + (1+r)^{-a} = 0$  given by the variation of parameters formula

$$\bar{\phi}(r) = Z(r) \int_r^{2R} \frac{1}{\rho^2 Z(\rho)^2} \int_0^\rho (1+s)^{-a} Z(s) s^2 ds,$$

where  $Z(r) = w_r(r)$ . Since  $w_r(r) \sim r^{-2}$  for large  $r$ , we find the estimate  $|\bar{\phi}(r)| \lesssim (R^{4-a})/(1+r^2)$ . Then, provided that  $\tau_0$  was chosen sufficiently large, the function  $2\|H_j\|_{v,a} \tau^{-\nu} \bar{\phi}(r)$  is a positive supersolution

of problem (9-25) and thus we find  $|\phi_j(r, \tau)| \lesssim \tau^{-\nu} ((R^{4-a})/(1+r^2)) \|H_j\|_{v,a}$ . Hence  $\phi^1[H^1]$  given by (9-23) satisfies

$$|\phi^1[H^1](y, \tau)| \lesssim \frac{R^{4-a}}{1+|y|^2} \|H^1\|_{v,a}.$$

A corresponding estimate for the gradient follows.

*Construction at higher modes.* We consider now the case of higher modes,

$$\begin{aligned} \phi_\tau &= \Delta\phi + 5w^4\phi + H^\perp && \text{in } B_{2R} \times (\tau_0, \infty), \\ \phi &= 0 && \text{on } \partial B_{2R} \times (\tau_0, \infty), \quad \phi(\cdot, \tau_0) = 0 && \text{in } B_{2R}, \end{aligned} \tag{9-27}$$

where  $H = H^\perp = \sum_{j=4}^\infty H_j(r)\Theta_j$  whose solution has the form  $\phi^\perp = \sum_{j=4}^\infty \phi_j(r, \tau)\Theta_j$ . Given the quadratic form in (9-18), for  $\phi^\perp \in H_0^1(B_{2R})$ ,

$$\int_{B_{2R}} \frac{|\phi^\perp|^2}{r^2} \lesssim Q(\phi^\perp, \phi^\perp). \tag{9-28}$$

The proof of this fact is elementary. The interested reader can find it in [Cortázar et al. 2016]. Let  $\phi_*[H^\perp]$  be the solution to

$$\begin{aligned} \phi_\tau &= \Delta\phi + 5w(r)^4(1-\eta_\ell)\phi + \bar{H}^\perp(y, \tau) && \text{in } B_{2R} \times (\tau_0, \infty), \\ \phi &= 0 && \text{on } \partial B_{2R} \times (\tau_0, \infty), \quad \phi(\cdot, 0) = 0 && \text{in } B_{2R}, \end{aligned}$$

where  $\bar{H}^\perp = H^\perp - c^\perp Z$ , and  $c^\perp = \int_{B_{2R}} H^\perp Z$ . By writing  $\phi = \phi_*[H^\perp] + \tilde{\phi}$ , problem (9-27) reduces to solving

$$\begin{aligned} \tilde{\phi}_\tau &= \Delta\tilde{\phi} + 5w(y)^4\tilde{\phi} + \tilde{H} && \text{in } B_{2R} \times (\tau_0, \infty), \\ \tilde{\phi} &= 0 && \text{on } \partial B_{2R} \times (\tau_0, \infty), \quad \tilde{\phi}(\cdot, \tau_0) = 0 && \text{in } B_{2R}, \end{aligned} \tag{9-29}$$

where  $\tilde{H} = 5w(y)^4\eta_\ell\phi_*[H^\perp]$  for a sufficiently large  $\ell$ . Arguing as in (9-19) we now get

$$\partial_\tau \int_{B_{2R}} \tilde{\phi}^2 + c \int_{B_{2R}} \frac{|\tilde{\phi}|^2}{|y|^2} \lesssim \int_{B_{2R}} |y|^2 |\tilde{H}|^2. \tag{9-30}$$

Similarly to (9-20) we get

$$\| |y|^{-1} \tilde{\phi}(\cdot, \tau) \|_{L^2(B_{2R})} \lesssim \tau^{-\nu} R^{2-a} \|H\|_{v,a}. \tag{9-31}$$

From elliptic estimates we then get

$$\| \tilde{\phi}(\cdot, \tau) \|_{L^\infty(B_{2R})} \lesssim \tau^{-\nu} R^{2-a} \|H^\perp\|_{v,a} \quad \text{for all } \tau > \tau_0,$$

so that with the aid of a barrier we obtain

$$|\tilde{\phi}(y, \tau)| \lesssim \tau^{-\nu} R^{2-a} \|H^\perp\|_{v,a} (1+|y|)^{-1}.$$

It follows that the function

$$\phi^\perp[H^\perp] := \tilde{\phi} + \phi_*[H^\perp] \tag{9-32}$$

satisfies

$$|\phi^\perp[H^\perp](y, \tau)| \lesssim \tau^{-\nu} R^2 [(1 + |y|)^{-1} + (1 + |y|)^{-a}] \|H^\perp\|_{\nu, a} \quad \text{in } B_{2R}.$$

Similar estimates for the gradient follow.

*Conclusion.* Let

$$\phi[h] := \phi^0[h^0] + \phi^1[h^1] + \phi^\perp[h^\perp]$$

for the functions defined in (9-22), (9-23), (9-32). By construction,  $\phi[h]$  solves (9-10). It defines a linear operator of  $h$  and satisfies (9-11). The proof of Step 1 is concluded.

Step 2: To complete the proof of Proposition 9.2, we decompose the right-hand side  $h$  in (9-7) in modes,  $h = h^0 + h^1 + h^\perp$  as before, and define separately associated solutions of (9-7) in a decomposition  $\phi = \phi^0 + \phi^1 + \phi^\perp$ .

*Construction at mode 0.* For a bounded radial  $h = h(|y|)$  defined in  $B_{2R}$  with  $\int_{B_{2R}} h Z_0 = 0$ , let  $\tilde{h}$  designate the extension of  $h$  as zero outside  $B_{2R}$ . The equation

$$\Delta H + 5w^4(y)H + \tilde{h}(|y|) = 0 \quad \text{in } \mathbb{R}^3, \quad H(y) \rightarrow 0 \quad \text{as } |y| \rightarrow \infty,$$

has a solution  $H =: L_0^{-1}[h]$  represented by the variation of parameters formula

$$H(r) = \tilde{Z}(r) \int_r^\infty \tilde{h}(s) Z_0(s) s^2 ds + Z_0(r) \int_r^\infty \tilde{h}(s) \tilde{Z}(s) s^2 ds, \quad (9-33)$$

where  $\tilde{Z}(r)$  is a suitable second radial solution of  $L_0[\tilde{Z}] = 0$ , linearly independent with  $Z_0$ . The mode-0 function  $h_0 = h_0(|y|, \tau)$  is defined in  $B_{2R}$  and satisfies  $\|h_0\|_{\nu, 2+a} < +\infty$  and  $\int_{B_{2R}} h_0 Z_0 = 0$  for all  $\tau$ . Then  $H_0 := L_0^{-1}[h_0(\cdot, \tau)]$  satisfies the estimate

$$|H_0(r, \tau)| \lesssim \frac{\tau^{-\nu}}{(1+r)^a} \|h_0\|_{\nu, 2+a}.$$

Let  $\Phi_0[h_0]$  be the radial solution in  $B_{3R}$  to

$$\begin{aligned} \Phi_\tau &= \Delta \Phi + 5w^4(y)\Phi + H_0(|y|, \tau) - c_0(\tau)Z \quad \text{in } B_{3R} \times (\tau_0, \infty), \\ \Phi &= 0 \quad \text{on } \partial B_{3R} \times (\tau_0, \infty), \quad \Phi(\cdot, \tau_0) = 0 \quad \text{in } B_{3R} \end{aligned} \quad (9-34)$$

that we discussed in Step 1. The solution  $\Phi_0[h_0]$  defines a linear operator of  $h_0$  and satisfies the estimates

$$|\Phi_0(y, \tau)| \lesssim \frac{\tau^{-\nu} R^{4-a}}{(1+|y|)} \|H_0\|_{\nu, a}, \quad (9-35)$$

where for some  $\Gamma > 0$

$$\left| c_0(\tau) - \int_{B_{2R}} H_0 Z \right| \lesssim \tau^{-\nu} \left[ R^2 \left\| H_0 - Z \int_{B_{2R}} H_0 Z \right\|_{\nu, a} + e^{-\Gamma R} \|H_0\|_{\nu, a} \right]. \quad (9-36)$$

Since  $L_0[Z] = \lambda_0 Z$ ,

$$\lambda_0 \int_{B_{2R}} H_0 Z = \int_{B_{2R}} H_0 L_0[Z] = \int_{B_{2R}} L_0[H_0]Z + \int_{\partial B_{2R}} (Z \partial_\nu H_0 - H_0 \partial_\nu Z),$$

and hence

$$\int_{B_{2R}} H_0 Z = \lambda_0^{-1} \int_{B_{2R}} h_0 Z + O(e^{-\Gamma R}) \tau^{-\nu} \|h_0\|_{v,2+a}.$$

Also, from the definition of the operator  $L_0^{-1}$  we see that  $Z = \lambda_0 L_0^{-1}[Z]$ . Thus

$$\begin{aligned} \left\| H_0 - Z \int_{B_{2R}} H_0 Z \right\|_{v,a} &= \left\| L_0^{-1} \left[ h_0 - \lambda_0 Z \int_{B_{2R}} H_0 Z \right] \right\|_{v,a} \\ &\lesssim \left\| h_0 - Z \int_{B_{2R}} h_0 Z \right\|_{v,2+a} + e^{-\Gamma R} \|h_0\|_{v,2+a}. \end{aligned}$$

Next, we discuss estimates on the first and second derivatives of  $\Phi_0$ . Let us fix now a vector  $e$  with  $|e| = 1$ , a large number  $\rho > 0$  with  $\rho \leq 2R$  and a number  $\tau_1 \geq \tau_0$ . Consider the change of variables

$$\Phi_\rho(z, t) := \Phi_0(\rho e + \rho z, \tau_1 + \rho^2 t), \quad H_\rho(z, t) := \rho^2 [H_0(\rho e + \rho z, \tau_1 + \rho^2 t) - c_0(\tau_1 + \rho^2 t) Z(\rho e + \rho z)].$$

Then  $\Phi_\rho(z, t)$  satisfies an equation of the form

$$\partial_t \Phi_\rho = \Delta_z \Phi_\rho + B_\rho(z, t) \Phi_\rho + H_\rho(z, t) \quad \text{in } B_1(0) \times (0, 2),$$

where  $B_\rho = O(\rho^{-2})$  uniformly in  $B_2(0) \times (0, \infty)$ . Standard parabolic estimates yield that for any  $0 < \alpha < 1$

$$\|\nabla_z \Phi_\rho\|_{L^\infty(B_{1/2}(0) \times (1, 2))} \lesssim \|\Phi_\rho\|_{L^\infty(B_1(0) \times (0, 2))} + \|H_\rho\|_{L^\infty(B_1(0) \times (0, 2))}.$$

Moreover

$$\|H_\rho\|_{L^\infty(B_1(0) \times (0, 2))} \lesssim \rho^{2-a} \tau_1^{-\nu} \|H_0\|_{v,a}, \quad \|\Phi_\rho\|_{L^\infty(B_1(0) \times (0, 2))} \lesssim \tau_1^{-1} K(\rho),$$

where

$$K(\rho) = \frac{R^{2-a}}{1+\rho} R^2 \|h^0\|_{v,2+a}. \tag{9-37}$$

This yields in particular that

$$\rho |\nabla_y \Phi(\rho e, \tau_1 + \rho^2)| = |\nabla \tilde{\phi}(0, 1)| \lesssim \tau_1^{-\nu} K(\rho).$$

Hence if we choose  $\tau_0 \geq R^2$ , we get that for any  $\tau > 2\tau_0$  and  $|y| \leq 3R$

$$(1 + |y|) |\nabla_y \Phi(y, \tau)| \lesssim \tau^{-\nu} K(|y|). \tag{9-38}$$

We obtain that these bounds are as well valid for  $\tau < 2\tau_0$  by the use of similar parabolic estimates up to the initial time (with condition 0).

Now, we observe that the function  $H_0$  is of class  $C^1$  in the variable  $y$  and  $\|\nabla_y H_0\|_{v,1+a} \leq \|h^0\|_{v,2+a}$ . It follows that we have the estimate

$$(1 + |y|^2) |D_y^2 \Phi(y, \tau)| \lesssim \tau^{-\nu} K(|y|)$$

for all  $\tau > \tau_0$ ,  $|y| \leq 2R$ , where  $K$  is the function in (9-37). The proof follows simply by differentiating the equation satisfied by  $\Phi$ , rescaling in the same way we did to get the gradient estimate, and applying

the bound already proven for  $\nabla_y \Phi$ . Thus we have in  $B_{2R}$

$$(1 + |y|^2)|D^2\Phi(y, \tau)| + (1 + |y|)|\nabla\Phi(y, \tau)| + |\Phi(y, \tau)| \lesssim \tau^{-\nu} \|h^0\|_{v,2+a} \frac{R^{4-a}}{1 + |y|}.$$

This yields in particular

$$|L_0[\Phi](\cdot, \tau)| \lesssim \tau^{-\nu} \|h^0\|_{v,2+a} \frac{R^{4-a}}{1 + |y|^3} \quad \text{in } B_{2R}.$$

We define

$$\phi^0[h_0] := L_0[\Phi]|_{B_{2R}}.$$

Then  $\phi^0[h_0]$  solves problem (9-7) with

$$c(\tau) := \lambda_0 c_0(\tau), \quad (9-39)$$

$\phi^0[h_0]$  satisfies the estimate

$$|\phi^0[h_0](y, \tau)| \lesssim \tau^{-\nu} \|h_0\|_{v,2+a} \frac{R^{4-a}}{1 + |y|^3} \quad \text{in } B_{2R}, \quad (9-40)$$

and from (9-36), estimate (9-8) holds too.

*Construction for modes 1 to 3.* We consider now  $h^1(y, \tau) = \sum_{j=1}^3 h_j(r, \tau) \vartheta_j$ , with  $\|h^1\|_{v,2+a} < +\infty$ , that satisfies, for all  $i = 1, \dots, 3$ ,  $\int_{B_{2R}} h^1 Z_i = 0$  for all  $\tau \in (\tau_0, \infty)$ . We will show that there is a solution

$$\phi^1[h^1] = \sum_{j=1}^3 \phi_j(r, \tau) \vartheta_j \left( \frac{y}{r} \right)$$

to problem (9-7) for  $h = h^1$  which defines a linear operator of  $h^1$  and satisfies the estimate

$$|\phi^1(y, \tau)| \lesssim \frac{R^4}{1 + |y|^4} R^{-a} \|h\|_{v,2+a}. \quad (9-41)$$

Let us fix  $1 \leq j \leq 3$ . For a function  $h = h_j(r) \vartheta_j(y/r)$  defined in  $B_{2R}$ , we let  $H = L_0^{-1}[h] := H_j(r) \vartheta_j(y/r)$  be the solution of the equation

$$\Delta H + pU^{p-1}H + \tilde{h}_j \vartheta_j = 0 \quad \text{in } \mathbb{R}^n, \quad H(y) \rightarrow 0 \quad \text{as } |y| \rightarrow \infty,$$

where  $\tilde{h}_j$  designates the extension of  $h_j$  as zero outside  $B_{2R}$ , represented by the variation of parameters formula

$$H_j(r) = w_r(r) \int_r^{2R} \frac{1}{\rho^{n-1} w_r(\rho)^2} \int_\rho^\infty \tilde{h}_j(s) w_r(s) s^{n-1} ds.$$

If we consider a function  $h^j = h_j(r, \tau) \vartheta_j$  defined in  $B_{2R}$  with  $\|h^j\|_{2+a,v} < +\infty$  and  $\int_{B_{2R}} h^j Z_j = 0$  for all  $\tau$ , then  $H_j = L_0^{-1}[h^j(\cdot, \tau)]$  satisfies the estimate  $\|H_j\|_{v,a} \lesssim \|h_j\|_{v,2+a}$ . Let us consider the boundary value problem in  $B_{3R}$

$$\begin{aligned} \Phi_\tau &= \Delta\Phi + pU(y)^{p-1}\Phi + H_j(r) \vartheta_j(y) \quad \text{in } B_{3R} \times (\tau_0, \infty), \\ \Phi &= 0 \quad \text{on } \partial B_{3R} \times (\tau_0, \infty), \quad \Phi(\cdot, \tau_0) = 0 \quad \text{in } B_{3R}. \end{aligned} \quad (9-42)$$

As consequence of Step 1, we find a solution  $\Phi_j[h]$  to this problem, which defines a linear operator of  $h_j$  and satisfies the estimates

$$|\Phi_j(y, \tau)| \lesssim \frac{\tau^{-\nu} R^{3-a}}{1 + |y|^2} R^1 \|h_j\|_{v, 2+a}. \quad (9-43)$$

Arguing by scaling and parabolic estimates, we find as in the construction for mode 0,

$$|L[\Phi_j](\cdot, \tau)| \lesssim \tau^{-\nu} \|h\|_{v, 2+a} \frac{R^{4-a}}{1 + |y|^4} \quad \text{in } B_{2R}.$$

We define  $\phi_j[h_j] := L[\Phi_j]|_{B_{2R}}$  and  $\phi^1[h^1] := \sum_{j=1}^3 \phi_j[h_j] \vartheta_j$ . This function solves (9-7) for  $h = h^1$  and satisfies

$$|\phi^1[h^1](y, \tau)| \lesssim \tau^{-\nu} \|h_j\|_{2+a, v} \frac{R^{4-a}}{1 + |y|^4} \quad \text{in } B_{2R}. \quad (9-44)$$

*Construction at higher modes.* In order to deal with the higher modes, for  $h = h^\perp = \sum_{j=4}^\infty h_j(r) \Theta_j$  we let  $\phi^\perp[h^\perp]$  be just the unique solution of the problem

$$\begin{aligned} \phi_\tau &= \Delta \phi + pU(y)^{p-1} \phi + h^\perp \quad \text{in } B_{2R} \times (\tau_0, \infty), \\ \phi &= 0 \quad \text{on } \partial B_{2R} \times (\tau_0, \infty), \quad \phi(\cdot, \tau_0) = 0 \quad \text{in } B_{2R}, \end{aligned} \quad (9-45)$$

which is estimated as

$$|\phi^\perp[h^\perp](y, \tau)| \lesssim \tau^{-\nu} \frac{\|h^\perp\|_{v, 2+a}}{1 + |y|^a} \quad \text{in } B_{2R}. \quad (9-46)$$

We just let

$$\phi[h] := \phi^0[h^0] + \phi^1[h^1] + \phi^\perp[h^\perp]$$

be the function constructed above. According to estimates (9-40) and (9-46) we find that this function solves problem (9-7) for  $c(\tau)$  given by (9-17), with bounds (9-4), (9-5), (9-8) as required.  $\square$

## 10. Nonradially symmetric case

In this section, we discuss the existence of solutions for problem (2-1) when the initial condition is not radially symmetric, and we discuss the codimension-1 stability. Let  $\bar{v}_0$  be a positive, uniformly bounded smooth function, not radially symmetric and define

$$v_0(x) = \frac{\bar{v}_0(x)}{|x|^\kappa}, \quad \text{with } \kappa > \max\left\{\frac{\gamma + 3}{2}, \gamma\right\}. \quad (10-1)$$

We construct a solution to the initial value problem

$$\begin{cases} u_t = \Delta u + u^5 & \text{in } \mathbb{R}^3 \times (t_0, \infty), \\ u(x, t_0) = u_0(|x|) + v_0(x), \end{cases} \quad (10-2)$$

where  $u_0$  is radial and satisfies the decay condition (2-2), while  $v_0$  is a nonradial function of the form (10-1).

Since the strategy of the proof is similar to the one already performed in detail for  $\bar{v}_0(x) \equiv 0$ , we shall indicate the changes in the argument that are required when the initial condition is not radially symmetric.

We start with a slightly different first approximation. Let  $p = p(t) : [t_0, \infty) \rightarrow \mathbb{R}^3$  be a smooth function so that

$$p(t_0) = \mathbf{0}, \quad p(t) = \int_{t_0}^t P(s) ds,$$

where  $P$  satisfies

$$\|P\|_{\diamond} := \sup_{t > t_0} \mu_0(t)^{-\frac{1}{2}} t^{\kappa-1} [\|P(s)\|_{\infty, [t, t+1]} + [P]_{0, \sigma, [t, t+1]}] \leq \ell, \quad (10-3)$$

with  $\sigma$  the number fixed in (2-11), and  $\ell$  a positive fixed number. Observe that, under these assumptions, and the bound on  $\kappa$  in (10-1), we have  $|p(t)|/\mu_0(t) \rightarrow 0$  as  $t \rightarrow \infty$ . Define

$$U[\lambda, P](x, t) = \widehat{U}_2(x, t) + U_3(x, t), \quad \widehat{U}_2(x, t) := U_2(|x - p(t)|, t), \quad (10-4)$$

where  $U_2$  is given by (2-57) and

$$U_3(x, t) = \left(1 - \eta\left(\frac{|x|}{t}\right)\right) v_0(x). \quad (10-5)$$

If we call  $\mathcal{E}[\lambda, P](x, t) := \Delta U + U^5 - U_t$ , we can write

$$\mathcal{E}[\lambda, P](x, t) = \mathcal{E}_2[\lambda](|x - p|, t) - \nabla U_2(|x - p|, t) \cdot \dot{p}(t) + \underbrace{\Delta U_3 - \frac{\partial U_3}{\partial t} + (\widehat{U}_2 + U_3)^5 - (\widehat{U}_2)^5}_{:= \mathcal{E}_3}.$$

Define

$$\bar{\mathcal{E}}(x, t) = \mathcal{E}_{21}(|x - p|, t) + \left(1 - \eta_R\left(\frac{|x|}{R\mu_0}\right)\right) [\mathcal{E}_{22}(|x - p|, t) - \nabla U_2(|x - p|, t) \cdot \dot{p}(t)], \quad (10-6)$$

where  $\eta_R$  is defined in (2-59). We have

$$|\bar{\mathcal{E}}(x, t)| \leq C\mu_0^{\frac{1}{2}} t^{-\frac{3}{2}} h_0 \left(\frac{|x|}{\sqrt{t}}\right), \quad |\mathcal{E}_3(x, t)| \leq C t^{-\kappa + \frac{1}{2}} h_0 \left(\frac{|x|}{\sqrt{t}}\right). \quad (10-7)$$

A solution to (10-2) does exist and has the form

$$u = U[\lambda, P](r, t) + \tilde{\phi}, \quad t > t_0, \quad (10-8)$$

where  $U$  is defined in (10-4), while  $\tilde{\phi}(x, t)$  is given as in (3-2),

$$\tilde{\phi}(x, t) = \psi(x, t) + \phi^{\text{in}}(x, t), \quad \text{where } \phi^{\text{in}}(x, t) := \eta_R(x, t) \hat{\phi}(x, t),$$

and  $\hat{\phi}(x, t) := \mu_0^{-\frac{1}{2}} \phi(x/\mu_0, t)$ . For any  $\psi_0 \in C^2(\mathbb{R}^3)$  so that

$$|y| |\psi_0(y)| + |y| |\nabla \psi_0(y)| \leq t_0^{-a} e^{-b|y|} \quad (10-9)$$

for some positive constants  $a$  and  $b$ , the function  $\psi$  is the solution to

$$\begin{aligned} \partial_t \psi &= \Delta \psi + V \psi + [2\nabla \eta_R \nabla_x \hat{\phi} + \hat{\phi}(\Delta_x - \partial_t) \eta_R] + N[\lambda](\tilde{\phi}) + \bar{\mathcal{E}} + \mathcal{E}_3 \quad \text{in } \mathbb{R}^3 \times [t_0, \infty), \\ \psi(x, t_0) &= \psi_0, \end{aligned} \quad (10-10)$$

where  $V$  is defined as in (3-5) with  $U$  instead of  $U_2$ , and  $N(\tilde{\phi}) = (U + \tilde{\phi})^5 - U^5 - 5U^4\tilde{\phi}$ . This solution  $\psi$  can be described as

$$\psi(x, t) = \psi_r(x, t) + \psi_{nr}(x, t), \tag{10-11}$$

where  $\psi_r$  is a radial function in  $|x - p(t)|$ , for any  $t$ , and

$$|\psi_r(x, t)| \leq C\mu_0^{\frac{1}{2}}t^{-\frac{1}{2}}\varphi_0\left(\frac{|x|}{\sqrt{t}}\right), \quad |\psi_{nr}(x, t)| \leq Ct^{-\eta+\frac{3}{2}}\varphi_0\left(\frac{|x|}{\sqrt{t}}\right). \tag{10-12}$$

We refer to (4-2) for the definition of  $\varphi_0$ .

On the other hand, the function  $\hat{\phi}$  satisfies

$$\partial_t \hat{\phi} = \Delta \hat{\phi} + 5w_\mu^4 \hat{\phi} + 5w_\mu^4 \psi + \mathcal{E}_{22}(|x - p(t)|, t) - \nabla U_2(|x - p(t)|, t) \cdot \dot{p}(t) \quad \text{in } B_{2R\mu_0}(0) \times [t_0, \infty),$$

with  $\hat{\phi}(x, t_0) = \mu_0^{-\frac{1}{2}}(t_0)e_0Z(x/\mu_0(t_0))$ . In terms of  $\phi$ , this equation becomes

$$\begin{aligned} \mu_0^2 \partial_t \phi &= \Delta_y \phi + 5w^4 \phi + f(y, t) \quad \text{in } B_{2R}(0) \times [t_0, \infty), \\ \phi(y, t_0) &= e_0 Z(y), \end{aligned} \tag{10-13}$$

where

$$\begin{aligned} f(y, t) &= \mu_0^{\frac{5}{2}} \mathcal{E}_{22}(|\mu_0 y - p(t)|, t) - \nabla U_2(|\mu_0 y - p(t)|, t) \cdot \dot{p}(t) \\ &\quad + 5 \frac{\mu_0^{\frac{1}{2}}}{(1 + \Lambda)^4} w^4 \left( \frac{y}{(1 + \Lambda)^2} \right) \psi(\mu_0 y, t) + B[\phi] + B^0[\phi]. \end{aligned}$$

In the above expression,  $\psi$  is the solution to (10-10), while  $B$  and  $B^0$  are defined respectively in (3-9) and (3-10). The solution  $\phi$  exists in the class of functions with  $\|\cdot\|_{v,a}$ -norm bounded, see (4-1), as consequence of Proposition 9.1, and a contraction-type argument, provided the parameter functions  $\lambda$  and  $P$  can be chosen so that

$$\int_{B_R} f(y, t) Z_j(y) dy = 0 \quad \text{for all } t > t_0, \quad j = 0, 1, \dots, n. \tag{10-14}$$

The system of  $(n+1)$  nonlinear, nonlocal equations in  $\lambda$  and  $P$  is solvable for  $\lambda$  and  $P$  satisfying (2-11) and (10-3). Indeed, (10-14), for  $j = 0$ , can be treated as we did for (5-1) in Sections 5, 6, 7. On the other hand, when  $j = 1, \dots, n$ , (10-14) are perturbations of

$$\dot{p}(t) = \mu_0^{\frac{1}{2}} t^{-\kappa+1} \bar{u}$$

for some fixed vector  $\bar{u} \in \mathbb{R}^3$ . Thus it can be solved for parameters  $p(t) = \int_{t_0}^t P(s) ds$  satisfying (10-3). This concludes the proof of existence of a positive global solution to (10-2).

Next we discuss the codimension-1 stability. Let us observe that the construction of  $\phi$  and  $e_0$ , the solution to (10-13), is possible for any initial condition  $\psi_0$  to the outer problem (10-10). We have the validity of Lipschitz dependence of  $\phi = \phi[\psi_0]$ , and  $e_0 = e_0[\psi_0]$  in the  $C^1$ -topology described in (10-9). As a consequence of the implicit function theorem, the maps  $\phi[\psi_0]$  and  $e_0[\psi_0]$  depend in the  $C^1$ -sense on  $\psi_0$  in our  $C^1$ -topology (10-9), thanks to the corresponding dependence for  $\psi$ ,  $\lambda$  and  $p$ .

Let us consider the following map defined in a small neighborhood of 0 in  $X = C^1(\bar{\Omega})$ :

$$F(\psi_0) = \psi_0 - (e_0[\psi_0] - e_0[t_0])Z_0$$

so that  $F[0] = 0$ ,  $F$  is differentiable, and

$$D_{\psi_0}F(0)[h] = h - \langle D_{\psi_0}e_0[0], h \rangle Z_0, \quad h \in X.$$

We have a solution which blows-up as  $t \rightarrow +\infty$  provided that

$$u(\cdot, t_0) = u^*(\cdot, t_0) - e_0[0]Z_0 + g, \quad (10-15)$$

where  $u^*$  is the solution corresponding to  $\psi_0 = 0$ , and  $g = F[\psi_0]$  for any small  $\psi_0$ .

The vector space of the functionals in  $X$  given by  $D_{\psi_0}e_0[0]$  has dimension 1. We write  $W := \text{Ker}(D_{\psi_0}e_0[0])$  is a space with codimension 1. Indeed, we can find a nonzero function  $u$  such that

$$X = W \oplus \langle u \rangle.$$

We consider the operator in a neighborhood of 0 in  $X$  given by

$$G(w + \alpha u) = \alpha u + F(w), \quad \alpha_j \in \mathbb{R}, \quad w \in W.$$

Then  $G$  is of class  $C^1$  near the origin,  $G(0) = 0$  and  $D_{\psi_0}G(0)[h] = h$ . By the local inverse theorem,  $G$  defines a local  $C^1$  diffeomorphism onto a neighborhood of the origin. For all small  $g$  we can find smooth functions  $\alpha(g)$ ,  $w(g)$  with

$$\alpha(g)u + F(w(g)) = g.$$

Thus the set  $\mathcal{M}$  of functions  $F[w]$ ,  $w \in W$ , can be described in a neighborhood of 0 exactly as those  $g \in X$  such that

$$\alpha(g) = 0.$$

This says precisely that  $\mathcal{M}$  is locally a codimension-1  $C^1$ -manifold such that if  $g$  in (10-15) is selected there, then the desired phenomenon takes place.  $\square$

## Appendix A

*Proof of Lemma 2.2.* We denote by  $y_2(s)$  the solution to (2-17) with  $\lim_{s \rightarrow \infty} s^{2\nu} y_2(s) = 1$ , and by  $y_1(s)$  another solution, linearly independent from  $y_2$ , defined explicitly by

$$y_1(s) = c y_2(s) \int_s^\infty \frac{e^{-\frac{z^2}{4}}}{y_2(z)^2 z^2} dz \quad (A-1)$$

for some positive constant  $c$  we fix later. The function  $y_1(s)$  decays fast at infinity, since

$$y_1(s) = c_1 e^{-\frac{s^2}{4}} s^{4\nu-3} (1 + o(s^{-1})),$$

as  $s \rightarrow \infty$ , for some positive constant  $c_1$ , as a direct consequence from (A-1). The function  $y_2(s)$  is definite for any  $s \in (0, \infty)$ , and it is positive. Indeed, we first observe that the operator  $L_\nu$  satisfies the

maximum principle. This is consequence of the fact that the positive function  $g_0(s) = e^{-\frac{s^2}{4}}/s$ , which solves  $L_1(g_0) = 0$ , satisfies  $L_\nu(g_0) < 0$  in  $(0, \infty)$ . With this in mind, we define  $\bar{g}_0(s) = \int_s^\infty e^{-\frac{z^2}{4}}/z^2 dz$ . This is a positive function which satisfies  $L_\nu(\bar{g}_0) = \nu\bar{g}_0 > 0$  in  $(0, \infty)$ . Thus  $\bar{g}_0$  is a subsolution. Moreover, it is easy to see that  $\bar{g}_0(R) < y_2(R)$  for any  $R$  large enough. A standard application of the maximum principle thus gives that  $y_2$  is positive in  $(0, \infty)$ .

We now claim that  $\lim_{s \rightarrow 0^+} sy_1(s)$  exists and it is positive. Write  $y_1(s) = \phi(s^2/4)$ ,  $x = s^2/4$ , from which we get

$$x\phi'' + \left(\frac{3}{2} + x\right)\phi' + \nu\phi = 0, \quad x \in (0, \infty).$$

Performing the further change of variables  $\phi(x) = e^{-x}\varphi(x)$ , we get that  $\varphi$  satisfies

$$x\varphi'' + \left(\frac{3}{2} - x\right)\varphi' - \left(\frac{3}{2} - \nu\right)\varphi = 0, \quad x \in (0, \infty). \tag{A-2}$$

In [Filippas et al. 2000, Appendix A], it is proven that (A-2) admits polynomial solutions if and only if  $\frac{3}{2} - \nu = -k$ ,  $k = 0, 1, 2, \dots$ . Since  $\frac{1}{2} < \nu < 1$ , this never happens; thus  $\varphi$  cannot be bounded, as  $x \rightarrow 0^+$ . On the other hand, the behavior of the solutions to (A-2), as  $x \rightarrow 0^+$ , is determined by  $x\varphi'' + \frac{3}{2}\varphi' = 0$ , which implies that the solutions to (A-2) are bounded around  $x = 0$ , or they behave like  $x^{-\frac{1}{2}}$  as  $x \rightarrow 0^+$ . Combining all the above information, for a proper choice of the constant  $c$  in (A-1), we get

$$y_1(s) = \frac{1}{s}(1 + o(1)) \quad \text{as } s \rightarrow 0.$$

To understand further the behavior of  $y_1$  around  $s = 0$ , we write  $sy_1(s) = f(s)$ , so that

$$f'' + \frac{1}{2}sf' + \left(\nu - \frac{1}{2}\right)f = 0, \quad s \in (0, \infty). \tag{A-3}$$

Integrating (A-3) between 0 and  $\infty$ , and using the fast decay of  $y_1$  to 0 as  $s \rightarrow \infty$ , we compute

$$f'(0) = (\nu - 1) \int_0^\infty f(s) ds < 0, \quad f''(0) = \frac{1}{2} - \nu. \tag{A-4}$$

With this information, we get the estimates (2-18) and (2-20) for  $y_1(s)$ .

Since the Wronskian associated to problem (2-17) is given by a multiple of  $e^{-\frac{s^2}{4}}/s^2$ , we conclude that, since  $y_1$  is unbounded as  $s \rightarrow 0^+$ , we have  $y_2(s)$  is bounded as  $s \rightarrow 0^+$ .  $\square$

**Lemma A.1.** *Let  $h = h(s)$  be a smooth function defined for  $s \geq 0$  so that*

$$h(s) = \begin{cases} 1/s & \text{for } s \rightarrow 0, \\ 1/s^3 & \text{for } s \rightarrow \infty. \end{cases}$$

*Then there exists a solution to*

$$\partial_t \psi = \Delta \psi + t^{-\beta} h\left(\frac{r}{\sqrt{t}}\right), \tag{A-5}$$

*of the form*

$$\psi(r, t) = t^{-\beta+1} \varphi\left(\frac{r}{\sqrt{t}}\right), \quad \text{with } \varphi(s) = \begin{cases} s & \text{for } s \rightarrow 0, \\ 1/s^3 & \text{for } s \rightarrow \infty. \end{cases} \tag{A-6}$$

*Proof.* We look for a solution to (A-5) of the form  $\psi(r, t) = t^{-(\beta-1)}\varphi(r/\sqrt{t})$ . Thus  $\varphi$  satisfies

$$\varphi'' + \left(\frac{2}{s} + \frac{s}{2}\right)\varphi' + (\beta-1)\varphi + h(s) = 0.$$

We look for a solution of the above equation of the form

$$\varphi(s) = z(s)y_1(s),$$

where  $y_1$  solves  $y_1'' + (2/s + s/2)y_1' + (\beta-1)y_1 = 0$ , and

$$y_1(s) \sim \begin{cases} 1/s & \text{as } s \rightarrow 0, \\ e^{-\frac{s^2}{4}} s^{4(\beta-1)-3} & \text{as } s \rightarrow \infty. \end{cases}$$

The existence of  $y_1$  is consequence of Lemma 2.2. A direct computation gives

$$z(s) = - \int_0^s \frac{e^{-\frac{\eta^2}{4}}}{y_1(\eta)^2 \eta^2} \left( \int_0^\eta h(x)y_1(x)x^2 e^{\frac{x^2}{4}} dx \right) d\eta.$$

One can easily see that

$$z(s) \sim \begin{cases} s^2 & \text{as } s \rightarrow 0, \\ e^{\frac{s^2}{4}} s^{-4(\beta-1)} & \text{as } s \rightarrow \infty. \end{cases}$$

This fact gives (A-6), and concludes the proof of the lemma.  $\square$

*Proof of (5-10).* For  $x \in B_{2R}$ , we shall prove

$$\phi_0(\mu_0 x, t) - \phi_0(0, t) = \alpha(t)|\mu_0 x|^\sigma \Pi(t)\Theta(|x|) \quad (\text{A-7})$$

for some  $\sigma \in (0, 1)$ . Here  $\Pi = \Pi(t)$  denotes a smooth and bounded function of  $t$ , and  $\Theta$  a smooth and bounded function of  $x$ .

We have

$$\begin{aligned} \phi_0(\mu_0 x, t) - \phi_0(0, t) &= \int_{t_0}^t \frac{1}{(4\pi(t-s))^{\frac{3}{2}}} \int_{\mathbb{R}^3} [e^{-\frac{|x-y|^2}{4(t-s)}} - e^{-\frac{|y|^2}{4(t-s)}}] \frac{\alpha(s)}{|y|} \mathbf{1}_{\{r < M\}} dy ds \\ &= \frac{1}{2} \int_{t_0}^t \int \frac{\beta'(s)}{(t-s)^{\frac{1}{2}}} [e^{-|z - \frac{\mu_0 x}{2\sqrt{t-s}}|^2} - e^{-|z|^2}] \frac{1}{|z|} \mathbf{1}_{\{|z| < \frac{M}{2\sqrt{t-s}}\}} dy ds \\ &= I + II, \end{aligned}$$

where

$$I = \int_{t_0}^{t - (\frac{\mu_0 x}{2m})} \int \frac{\beta'(s)}{(t-s)^{\frac{1}{2}}} [e^{-|z - \frac{\mu_0 x}{2\sqrt{t-s}}|^2} - e^{-|z|^2}] \frac{1}{|z|} \mathbf{1}_{\{|z| < \frac{M}{2\sqrt{t-s}}\}} dy ds.$$

We start estimating  $II$ . We observe that, if  $t - (\mu_0 x / (2m)) < s < t$ , then  $\mu_0 |x| / (2\sqrt{t-s}) > m$ . We write

$$II = II_1 + II_2 + II_3,$$

where

$$II_j = \int_{t - (\frac{\mu_0 x}{2m})}^t \int_{D_j} \frac{\beta'(s)}{(t-s)^{\frac{1}{2}}} [e^{-|z - \frac{\mu_0 x}{2\sqrt{t-s}}|^2} - e^{-|z|^2}] \frac{1}{|z|} \mathbf{1}_{\{|z| < \frac{M}{2\sqrt{t-s}}\}} dy ds,$$

with

$$D_1 = \left\{ z : \left| z - \frac{\mu_0 x}{2\sqrt{t-s}} \right| < \frac{1}{4} \frac{\mu_0 |y|}{2\sqrt{t-s}} \right\}, \quad D_2 = \left\{ z : |z| < \frac{1}{4} \frac{\mu_0 |y|}{2\sqrt{t-s}} \right\},$$

and  $D_3$  is the complement of the two above regions.

We start estimating  $II_1$ . We see that

$$\int_{D_1} e^{-|z - \frac{\mu_0 x}{2\sqrt{t-s}}|^2} \frac{1}{|z|} \mathbf{1}_{\{|z| < \frac{M}{2\sqrt{t-s}}\}} dy = \int e^{-|\bar{z}|} \frac{1}{|\bar{z} + \frac{\mu_0 x}{2\sqrt{t-s}}|} d\bar{z} = c \frac{2\sqrt{t-s}}{\mu_0 |x|}$$

for some constant  $c$ , as a direct application of dominated convergence theorem. Thus

$$\int_{t - (\frac{\mu_0 x}{2m})}^t \int_{D_1} e^{-|z - \frac{\mu_0 x}{2\sqrt{t-s}}|^2} \frac{1}{|z|} \mathbf{1}_{\{|z| < \frac{M}{2\sqrt{t-s}}\}} dy ds = \frac{2c}{\mu_0 |x|} \int_{t - (\frac{\mu_0 x}{2m})}^t \sqrt{t-s} ds = c' (\mu_0 |x|)^{\frac{1}{2}}.$$

On the other hand, for any  $z$  in  $D_1$ , one has

$$|z| > \frac{1}{4} \frac{\mu_0 |x|}{2\sqrt{t-s}},$$

and hence we can bound

$$\left| \int_{D_1} e^{-|z|^2} \frac{1}{|z|} dz \right| \leq c \left[ \frac{\sqrt{t-s}}{\mu_0 |x|} \right]^\sigma$$

for any  $\sigma > 0$ . We take  $\sigma > 1$ , so that

$$\left| \int_{t_0}^{t - (\frac{\mu_0 x}{2m})} \int_{D_1} e^{-|z|^2} \frac{1}{|z|} \mathbf{1}_{\{|z| < \frac{M}{2\sqrt{t-s}}\}} dy ds \right| \leq \frac{1}{(\mu_0 |x|)^\sigma} \left| \int_{t_0}^{t - (\frac{\mu_0 x}{2m})} (t-s)^{\frac{\sigma}{2} - \frac{1}{2}} ds \right| \leq c' \mu_0 |x|.$$

Thus we conclude that

$$|II_1| \lesssim \beta'(t) (\mu_0 |x|)^{\frac{1}{2}}.$$

Arguing in a similar way, one finds the same type of estimate for  $II_2$ . In the third region  $D_3$ , we have

$$|z| > \frac{1}{4} \frac{\mu_0 |x|}{2\sqrt{t-2}}, \quad \left| z - \frac{\mu_0 x}{2\sqrt{t-s}} \right| > \frac{1}{4} \frac{\mu_0 |x|}{2\sqrt{t-s}},$$

so that again one gets the estimate

$$|II_3| \lesssim \beta'(t) \mu_0 |x|.$$

Let us now consider the interval of time  $t_0 < s < t - (\mu_0 |x| / (2m\sqrt{t-s}))^2$ , that is, region where one has  $\mu_0 |x| / (2\sqrt{t-s}) < m$ . We have the decomposition

$$I = III + IV,$$

where

$$III = \int_{t_0}^{t-1} \int \frac{\beta'(s)}{(t-s)^{\frac{1}{2}}} [e^{-|z - \frac{\mu_0 x}{2\sqrt{t-s}}|^2} - e^{-|z|^2}] \frac{1}{|z|} \mathbf{1}_{\{|z| < \frac{M}{2\sqrt{t-s}}\}} dy ds.$$

We start with  $IV$ , where we expand in Taylor:

$$\begin{aligned}
IV &= \int_{t-1}^{t-\left(\frac{\mu_0|x|}{2m\sqrt{t-s}}\right)^2} \int \frac{\beta'(s)}{(t-s)^{\frac{1}{2}}} [e^{-|z-\frac{\mu_0 x}{2\sqrt{t-s}}|^2} - e^{-|z|^2}] \frac{1}{|z|} \mathbf{1}_{\{|z| < \frac{M}{2\sqrt{t-s}}\}} dy ds \\
&= \beta'(t) \int_{t-1}^{t-\left(\frac{\mu_0|x|}{2m\sqrt{t-s}}\right)^2} \frac{\mu_0|x|}{t-s} \left( \int \frac{e^{-|z|^2}}{|z|} dz \right) ds = \beta'(t) \log \left( \frac{1}{t} \left( t - \left( \frac{\mu_0|x|}{2m\sqrt{t-s}} \right)^2 \right) \right) \mu_0|x| \\
&= \beta'(t) \mu_0|x| [\log(\mu_0|x|)] = \beta'(t) (\mu_0|x|)^\sigma
\end{aligned}$$

for some positive  $\sigma < 1$ . Finally, we consider  $III$ . Again, after a Taylor expansion, we have

$$III = \mu_0|x| \int_{t_0}^{t-1} \frac{\beta'(s)}{(t-s)} ds = \mu_0|x| \int_{t_0}^{t-1} \frac{\beta'(s)}{t-s} ds.$$

Collecting the previous estimates, we conclude with the validity of (A-7).  $\square$

## Appendix B

*Proof of Lemma 2.3.* Throughout the proof of the lemma, we denote by  $q_i = q_i(s)$ , for any integer  $i$ , a smooth real function with the property that

$$\frac{d}{(ds)^j} q_i(0) = 0 \quad \text{for } j < i \quad \text{and} \quad \frac{d}{(ds)^i} q_i(0) \neq 0.$$

With  $\Theta = \Theta(r)$  we intend a smooth function of the space variable, which is uniformly bounded. Also,  $\Pi = \Pi(t)$  stands for a smooth function of the time variable, which is uniformly bounded in  $t \in (0, \infty)$ . The explicit expressions of these functions change from line to line, and also within the same line.

Let  $R_0 = r_0\sqrt{t}$ . A simple computation gives the explicit expression of the error  $\mathcal{E}_1$  in (2-40):

$$\begin{aligned}
\mathcal{E}_1(r, t) &= \mathcal{E}_{\text{in}}^1 \eta \left( \frac{r}{R_0} \right) + \mathcal{E}_{\text{out}}^1 \left( 1 - \eta \left( \frac{r}{R_0} \right) \right) \\
&\quad + \underbrace{R_0^{-2} (u_{\text{in}} - u_{\text{out}}) \Delta \eta \left( \frac{r}{R_0} \right) + 2R_0^{-1} \nabla (u_{\text{in}} - u_{\text{out}}) \cdot \nabla \eta \left( \frac{r}{R_0} \right)}_{:= \tilde{\mathcal{E}}_1} + \underbrace{(u_{\text{in}} - u_{\text{out}}) \frac{R_0'}{R_0^2} \eta' \left( \frac{r}{R_0} \right)}_{:= \hat{\mathcal{E}}_1}, \quad (\text{B-1})
\end{aligned}$$

where

$$\mathcal{E}_{\text{in}}^1 = \Delta u_{\text{in}} + u_{\text{in}}^5 - \partial_t u_{\text{in}} \quad \text{and} \quad \mathcal{E}_{\text{out}}^1 = \Delta u_{\text{out}} + u_{\text{out}}^5 - \partial_t u_{\text{out}}. \quad (\text{B-2})$$

We start analyzing  $\mathcal{E}_{\text{in}}^1$ , getting

$$\begin{aligned}
\mathcal{E}_{\text{in}}^1(r, t) &= \mu_0' [\Delta \psi_1 + 5w_\mu^4 \psi_1] - \mu_0' \frac{\partial w_\mu}{\partial \mu} + (w_\mu + \mu_0' \psi_1)^5 - w_\mu^4 - 5w_\mu^4 \mu_0' \psi_1 - \mu_0'' \psi_1 - \mu_0' \mu_0' \frac{\partial \psi_1}{\partial \mu} \\
&= (\mu' - \mu_0') \mu^{-\frac{3}{2}} Z_0 \left( \frac{r}{\mu} \right) + [(w_\mu + \mu_0' \psi_1)^5 - w_\mu^5 - 5w_\mu^4 \mu_0' \psi_1] - \mu_0'' \psi_1 - \mu_0' \mu_0' \frac{\partial \psi_1}{\partial \mu}. \quad (\text{B-3})
\end{aligned}$$

Now we write

$$(\mu' - \mu'_0)\mu^{-\frac{3}{2}}Z_0\left(\frac{r}{\mu}\right) = [2(\mu^{\frac{1}{2}} - \mu_0^{\frac{1}{2}})' + (\mu^{\frac{1}{2}} - \mu_0^{\frac{1}{2}})\mu_0^{-1}\mu'_0]\mu^{-1}Z_0\left(\frac{r}{\mu}\right) - \frac{(\mu^{\frac{1}{2}} - \mu_0^{\frac{1}{2}})^2}{\mu^{\frac{1}{2}}}\mu_0^{-1}\mu'_0\mu^{-1}Z_0\left(\frac{r}{\mu}\right).$$

Taking into account that

$$Z_0(s) = \frac{3^{\frac{1}{4}}}{2} \frac{1}{s} + O\left(\frac{1}{s^3}\right)$$

as  $s \rightarrow \infty$ , it is convenient to write

$$\begin{aligned} [2(\mu^{\frac{1}{2}} - \mu_0^{\frac{1}{2}})' + (\mu^{\frac{1}{2}} - \mu_0^{\frac{1}{2}})\mu_0^{-1}\mu'_0]\mu^{-1}Z_0\left(\frac{r}{\mu}\right) \\ = \frac{\alpha(t)}{\mu + r} + [2(\mu^{\frac{1}{2}} - \mu_0^{\frac{1}{2}})' + (\mu^{\frac{1}{2}} - \mu_0^{\frac{1}{2}})\mu_0^{-1}\mu'_0]\mu^{-1}\left[Z_0\left(\frac{r}{\mu}\right) - \frac{3^{\frac{1}{4}}}{2} \frac{\mu}{\mu + r}\right], \end{aligned}$$

where  $\alpha$  is defined in (2-41). We decompose (B-3) as

$$\mathcal{E}_{\text{in}}^1(r, t) = \frac{\alpha(t)}{\mu + r} + \bar{\mathcal{E}}_{\text{in}}^1(r, t), \quad (\text{B-4})$$

where  $\bar{\mathcal{E}}_{\text{in}}^1$  is explicitly given by

$$\begin{aligned} \bar{\mathcal{E}}_{\text{in}}^1(r, t) = -\frac{(\mu^{\frac{1}{2}} - \mu_0^{\frac{1}{2}})^2}{\mu^{\frac{1}{2}}}\mu_0^{-1}\mu'_0\mu^{-1}Z_0\left(\frac{r}{\mu}\right) - \mu_0''\psi_1 + [(w_\mu + \mu'_0\psi_1)^5 - w_\mu^5 - 5w_\mu^4\mu'_0\psi_1] \\ + [2(\mu^{\frac{1}{2}} - \mu_0^{\frac{1}{2}})' + (\mu^{\frac{1}{2}} - \mu_0^{\frac{1}{2}})\mu_0^{-1}\mu'_0]\mu^{-1}\left[Z_0\left(\frac{r}{\mu}\right) - \frac{3^{\frac{1}{4}}}{2} \frac{\mu}{\mu + r}\right] - \mu'_0\mu' \frac{\partial \psi_1}{\partial \mu} = \sum_{j=1}^5 e_j. \quad (\text{B-5}) \end{aligned}$$

We observe now that  $(e_1 + e_2 + e_3)\eta(r/R_0)$  can be described as sum of functions of the form

$$\frac{\mu_0^{\frac{1}{2}}t^{-2}R_0^2}{\mu_0 + r}q_0(\Lambda)\Pi(t)\Theta(r), \quad \frac{\mu_0^{-\frac{1}{2}}t^{-1}}{\mu_0 + r}q_2(\Lambda)\Pi(t)\Theta(r), \quad (\text{B-6})$$

where  $q_0$  is a smooth function with  $q_0(0) \neq 0$ , while  $q_2$  is a smooth function with  $q_2(0) = q_2'(0) = 0$ , and  $q_2''(0) \neq 0$ . On the other hand, we see that

$$e_4 = \frac{\alpha(t)\mu_0^2}{\mu_0^3 + r^3}\Pi(t)\Theta(r), \quad e_5 = \frac{\mu_0^{\frac{1}{2}}t^{-1}}{\mu_0 + r}[R_0^2\Lambda' + R_0^2t^{-1}q_1(\Lambda)]\Pi(t)\Theta(r), \quad (\text{B-7})$$

where  $q_1$  is a smooth function with  $q_1(0) = 0$ ,  $q_1'(0) \neq 0$ . Under assumption (2-11) and combining (B-4), (B-6) and (B-7) we find that

$$|\bar{\mathcal{E}}_{\text{in}}^1\eta|_{\infty, \mathcal{B}(x,1) \times [t, t+1]} \lesssim \mu_0^{\frac{1}{2}}t^{-\frac{3}{2}}h_0\left(\frac{r}{\sqrt{t}}\right), \quad r = |x|. \quad (\text{B-8})$$

Since (2-41), we observe that

$$\left| \frac{\alpha(t)}{\mu + r} \left(1 - \eta\left(\frac{r}{R_0}\right)\right) \right| \lesssim \mu_0^{\frac{3}{2}}t^{-\frac{3}{2}}h_0\left(\frac{r}{\sqrt{t}}\right), \quad r = |x|.$$

Let us fix  $\lambda_1$  and  $\lambda_2$  satisfying (2-11). We write, for some  $\bar{\lambda} = s\lambda_1 + (1-s)\lambda_2$ ,  $s \in (0, 1)$ ,

$$(\bar{\mathcal{E}}_{\text{in}}^1[\lambda_1] - \bar{\mathcal{E}}_{\text{in}}^1[\lambda_2])\eta\left(\frac{r}{R_0}\right) = (D_\lambda \bar{\mathcal{E}}_{\text{in}}^1[\bar{\lambda}] [\lambda_1 - \lambda_2])\eta\left(\frac{r}{R_0}\right), \quad \text{with } D_\lambda \bar{\mathcal{E}}_{\text{in}}^1[\bar{\lambda}] = \sum_{j=1}^5 (D_\lambda e_j)[\bar{\lambda}],$$

where the  $e_j$  are defined in (B-5). Let us consider  $e_1$ . We have

$$(D_\lambda e_1)[\bar{\lambda}] = 2\mu_0(1 + \Lambda)D_\mu(e_1)[\bar{\lambda}].$$

Direct computation gives

$$|D_\mu(e_1)[\bar{\lambda}](r, t)| \lesssim \frac{\mu_0^{-\frac{1}{2}}t^{-1}}{\mu_0 + r} q_0(\bar{\lambda})\Pi(t)\Theta(r).$$

We combine the above estimates to get

$$\begin{aligned} |e_1[\lambda_1] - e_1[\lambda_2]|\eta\left(\frac{r}{R_0}\right) &\leq \mu_0 \frac{\mu_0^{-\frac{1}{2}}t^{-1}}{\mu_0 + r} |\lambda_1 - \lambda_2| \eta\left(\frac{r}{R_0}\right) \\ &\leq C(\mu_0(t)t^{-1})\mu_0^{\frac{3}{2}}(t)t^{-\frac{3}{2}}h_0\left(\frac{r}{\sqrt{t}}\right) \|\lambda_1 - \lambda_2\|_{\#} \\ &\leq C(\mu_0(t_0)t_0^{-1})\mu_0^{\frac{3}{2}}(t)t^{-\frac{3}{2}}h_0\left(\frac{r}{\sqrt{t}}\right) \|\lambda_1 - \lambda_2\|_{\#}. \end{aligned}$$

Choosing  $t_0$  large if necessary, we get  $C(\mu_0(t_0)t_0^{-1}) < 1$ . Similar estimates can be obtained for the other terms  $e_2, \dots, e_5$ . Thus we get

$$|(\bar{\mathcal{E}}_{\text{in}}^1[\lambda_1] - \bar{\mathcal{E}}_{\text{in}}^1[\lambda_2])\chi|_{\infty, \mathcal{B}(x,1) \times [t, t+1]} \leq c_1^o \mu_0^{\frac{1}{2}} t^{-\frac{3}{2}} h_0\left(\frac{r}{\sqrt{t}}\right) \|\lambda_1 - \lambda_2\|_{\#}$$

for some constant  $c_1^o$  which can be made arbitrarily small, if  $t_0$  is chosen large. Also, we have

$$[\bar{\mathcal{E}}_{\text{in}}^1[\lambda_1] - \bar{\mathcal{E}}_{\text{in}}^1[\lambda_2]]_{0, \sigma, [t, t+1]} \leq c_1^o \mu_0^{\frac{1}{2}} t^{-\frac{3}{2}} h_0\left(\frac{r}{\sqrt{t}}\right) [\lambda_1 - \lambda_2]_{0, \sigma, [t, t+1]}.$$

Let us now describe  $\mathcal{E}_{\text{out}}^1$ . A first observation is that, for any value of  $\gamma$ , we immediately see that  $\mathcal{E}_{\text{out}}^1$  does not depend on  $\lambda$ . On the other hand, if  $1 < \gamma \leq 2$  the expression for  $\mathcal{E}_{\text{out}}^1$  becomes

$$\mathcal{E}_{\text{out}}^1(r, t) = u_{\text{out}}^5,$$

so that we directly get

$$\left| \mathcal{E}_{\text{out}}^1\left(1 - \chi\left(\frac{r}{R_0}\right)\right) \right| \leq C \frac{\mu_0^{\frac{5}{2}}}{r^5} \mathbf{1}_{\{r > R_0^{-1}\}}. \quad (\text{B-9})$$

Let us consider now  $\gamma > 2$ . In this case, the expression of  $\mathcal{E}_{\text{out}}^1$  is a bit more involved

$$\begin{aligned} \mathcal{E}_{\text{out}}^1(r, t) &= \eta\left(\frac{r}{t}\right) (u_{\text{out}}^1)^5 + \left(1 - \eta\left(\frac{r}{t}\right)\right) A \left[ \frac{\gamma(\gamma-1)}{r^{\gamma+2}} + \frac{A^4}{r^{5\gamma}} \right] \\ &\quad + \underbrace{t^{-2}(u_{\text{out}}^1 - u_{\text{out}}^2)\Delta\eta\left(\frac{r}{t}\right) + 2t^{-\nabla}(u_{\text{out}}^1 - u_{\text{out}}^2) \cdot \nabla\eta\left(\frac{r}{t}\right)}_{:= \widehat{\mathcal{E}}_1^{\text{out}}} + \underbrace{(u_{\text{out}}^1 - u_{\text{out}}^2)t^{-2}\eta'\left(\frac{r}{t}\right)}_{:= \widehat{\mathcal{E}}_1^{\text{out}}}. \quad (\text{B-10}) \end{aligned}$$

A close analysis of each one of the terms appearing in (B-10) gives

$$\left| \mathcal{E}_{\text{out}}^1 \left( 1 - \eta \left( \frac{r}{R_0} \right) \right) \right| \leq C \left\{ \frac{t^{-(\gamma-1)}}{r^3} \mathbf{1}_{\{r>t\}} + \frac{t^{-2} \mu_0^{\frac{1}{2}}}{r} \mathbf{1}_{\{t<r<2t\}} + \frac{t^{-\frac{5}{2}}}{r^5} \mathbf{1}_{\{r_0 \sqrt{t} < r < t\}} \right\}. \quad (\text{B-11})$$

From (B-9)–(B-10) and (B-11), we obtain

$$\left| \mathcal{E}_{\text{out}}^1 \left( 1 - \chi \left( \frac{r}{R_0} \right) \right) \right| \lesssim \begin{cases} \mu_0^{\frac{1}{2}} t^{-\frac{3}{2}} h_0(r/\sqrt{t}) & \text{if } 1 < \gamma \leq 2, \\ t^{-2} h_0(r/\sqrt{t}) & \text{if } \gamma > 2. \end{cases}$$

Going back to (B-1), we are left with the description of  $\bar{\mathcal{E}}_1 = \bar{\mathcal{E}}_1[\lambda]$  and  $\hat{\mathcal{E}}_1[\lambda]$ . Directly we check

$$|\bar{\mathcal{E}}_1(r, t)|, |\hat{\mathcal{E}}_1(r, t)| \leq C R_0^{-2} \frac{\mu_0^{\frac{1}{2}}}{r} \mathbf{1}_{\{R_0 < r < 2R_0\}} \quad (\text{B-12})$$

for some positive constant  $C$ . This gives right away

$$|\bar{\mathcal{E}}_1 + \hat{\mathcal{E}}_1| \lesssim \mu_0^{\frac{1}{2}} t^{-\frac{3}{2}} h_0 \left( \frac{r}{\sqrt{t}} \right).$$

Let us fix  $\lambda_1$  and  $\lambda_2$  satisfying (2-11). We write, for some  $\bar{\lambda} = s\lambda_1 + (1-s)\lambda_2$ ,  $s \in (0, 1)$ ,

$$\bar{\mathcal{E}}_1[\lambda_1](r, t) - \bar{\mathcal{E}}_1[\lambda_2](r, t) = D_\lambda \bar{\mathcal{E}}_1[\bar{\lambda}][\lambda_1 - \lambda_2](r, t),$$

where

$$D_\lambda \bar{\mathcal{E}}_1[\bar{\lambda}] = R_0^{-2} (\partial_\lambda u_{\text{in}}[\bar{\lambda}]) \Delta \eta \left( \frac{r}{R_0} \right) + 2R_0^{-1} \nabla((\partial_\lambda u_{\text{in}})[\bar{\lambda}]) \cdot \nabla \eta \left( \frac{r}{R_0} \right).$$

Since in the region we are considering

$$\partial_\lambda u_{\text{in}}[\bar{\lambda}] = 2\mu_0(1 + \Lambda)(\partial_\mu u_{\text{in}})[\bar{\lambda}], \quad |(\partial_\mu u_{\text{in}})| \leq c \frac{\mu_0^{-\frac{1}{2}}}{r},$$

we have

$$\begin{aligned} |\bar{\mathcal{E}}_1[\lambda_1](r, t) - \bar{\mathcal{E}}_1[\lambda_2](r, t)|_{\infty, B(x, 1) \times [t, t+1]} &\leq (\mu_0(t_0) t_0^{-1}) \mu_0^{\frac{1}{2}} t^{-\frac{3}{2}} h_0 \left( \frac{r}{\sqrt{t}} \right) \|\lambda_1 - \lambda_2\|_{\#} \\ &\leq c_1 \mu_0^{\frac{1}{2}} t^{-\frac{3}{2}} h_0 \left( \frac{r}{\sqrt{t}} \right) \|\lambda_1 - \lambda_2\|_{\#} \end{aligned}$$

for some constant  $c_1 \in (0, 1)$ , provided  $t_0$  is large enough. Furthermore, we also have, for any  $t > t_0$ ,

$$|\bar{\mathcal{E}}_1[\lambda_1] - \bar{\mathcal{E}}_1[\lambda_2]|_{0, \sigma, [t, t+1]} \leq c_1 \mu_0^{\frac{1}{2}} t^{-\frac{3}{2}} h_0 \left( \frac{r}{\sqrt{t}} \right) (\|\lambda_1 - \lambda_2\|_{0, \sigma, [t, t+1]}),$$

with again  $c_1 \in (0, 1)$ . Collecting all the previous estimates, we get the proof of the lemma.  $\square$

**Remark B.1.** From the proof of the result, we also get that the constants  $c$  in (2-50) and (2-51) can be made as small as one needs, provided that the initial time  $t_0$  is chosen large enough.

### Appendix C

*Proof of Lemma 2.4.* Under the assumptions (2-11) on  $\lambda$ , we get that, for any  $r > 0$  and  $t > t_0$ ,

$$|\mathcal{E}_{2,1}(r, t)| + [\mathcal{E}_{2,1}(r, t)]_{0, \sigma, [t, t+1]} \lesssim \mu_0^{\frac{1}{2}} t^{-\frac{3}{2}} h_0 \left( \frac{r}{\sqrt{t}} \right), \quad (\text{C-1})$$

where  $h_0$  is given by (2-44), and also estimates similar to (2-50) and (2-51) for  $\partial_\lambda \mathcal{E}_{2,1}$ . These estimates follow from (2-49)–(2-50), (2-41) and from

$$\left| \frac{\alpha(t)}{\mu + r} \left( \eta \left( \frac{r}{R_0} \right) - \mathbf{1}_{\{r < 2M\}} \right) \right| \leq |\alpha(t)| t^{-\frac{1}{2}} h_0 \left( \frac{r}{\sqrt{t}} \right).$$

Here we use again  $R_0 = r_0 \sqrt{t}$ . Furthermore, in the region where  $\eta(r/R_0) - \mathbf{1}_{\{r < 2M\}} \neq 0$ , the above function is regular enough to have

$$\left[ \frac{\alpha(t)}{\mu + r} \left( \eta \left( \frac{r}{R_0} \right) - \mathbf{1}_{\{r < 2M\}} \right) \right]_{0, \sigma, B(x, 1) \times [t, t+1]} \leq |\alpha(t)| t^{-\frac{1}{2}} h_0 \left( \frac{r}{\sqrt{t}} \right), \quad r = |x|.$$

Using (2-43), we get (C-1). Let us consider now  $\mathcal{E}_{22}(1 - \eta_R)(r, t)$ . We claim that

$$\|\mathcal{E}_{22}(1 - \eta_R)(r, t)\|_* \leq c_2. \quad (\text{C-2})$$

Given  $d > 1$ , define

$$h_*(s) = \begin{cases} 1/s & \text{for } s \rightarrow 0, \\ 1/s^d & \text{for } s \rightarrow \infty. \end{cases}$$

Arguing as in the proof of Lemma A.1, we get the existence of  $\psi_*$  so that

$$\partial_t \psi_* = \Delta \psi_* + \mu_0^{\frac{1}{2}} t^{-\frac{3}{2}} h_* \left( \frac{r}{\sqrt{t}} \right), \quad \text{with } \psi_*(r, t) = \mu_0^{\frac{1}{2}} t^{-\frac{1}{2}} \varphi_* \left( \frac{r}{\sqrt{t}} \right), \quad \varphi(s) = \begin{cases} s & \text{for } s \rightarrow 0, \\ 1/s^d & \text{for } s \rightarrow \infty. \end{cases}$$

Comparing the above equation and the equation satisfied by  $\phi_0$ , and using the maximum principle, we obtain that, in the region where  $(1 - \eta_R) \neq 0$ ,

$$|\phi_0(x, t)| \leq \|\lambda\|_{\#} \mu_0^{\frac{1}{2}} t^{-\frac{1}{2}} \varphi_* \left( \frac{r}{\sqrt{t}} \right). \quad (\text{C-3})$$

We proceed now with the estimate of  $(1 - \eta_R) \mathcal{E}_{22}$ . A Taylor expansion gives the existence of  $s^* \in (0, 1)$ , so that

$$\mathcal{E}_{22}(r, t) = 5(U_1 + s^* \phi_0)^4 \phi_0.$$

Let  $\bar{M}$  be a large fixed number. From (2-38) and (2-13), we see that, if  $r < \bar{M} \sqrt{t}$ ,

$$|(1 - \eta_R) \mathcal{E}_{22}| \lesssim w_\mu^4 \phi_0 \lesssim R^{-2} \mu_0^{\frac{1}{2}} t^{-\frac{3}{2}} h_0 \left( \frac{r}{\sqrt{t}} \right).$$

On the other hand, thanks to (C-3) we see that, for  $r > \bar{M} \sqrt{t}$ , we get

$$|(1 - \eta_R) \mathcal{E}_{22}| \lesssim (\phi_0)^5 \lesssim \mu_0^{\frac{5}{2}} t^{-\frac{5}{2}} h_0 \left( \frac{r}{\sqrt{t}} \right).$$

Thus we get the  $L^\infty$  bound in estimate (C-2). The control on the Hölder norm contained in (2-61) and (2-62) follows arguing as in the proof of (2-50)–(2-51) in the proof of Lemma 2.3, and from the assumption on  $\lambda$  in (2-11). We leave the details to the reader.  $\square$

### Acknowledgements

We are indebted to Marek Fila for introducing this problem to us and for many useful discussions. Del Pino was supported by a UK Royal Society Research Professorship and Grant PAI AFB-170001, Chile. Musso was partly supported by FONDECYT grant 1160135, Chile. The research of Wei is partially supported by the NSERC of Canada.

### References

- [Aubin 1976] T. Aubin, “Problèmes isopérimétriques et espaces de Sobolev”, *J. Differential Geom.* **11**:4 (1976), 573–598. MR Zbl
- [Caffarelli et al. 1989] L. A. Caffarelli, B. Gidas, and J. Spruck, “Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth”, *Comm. Pure Appl. Math.* **42**:3 (1989), 271–297. MR Zbl
- [Caputo 1967] M. Caputo, “Linear models of dissipation whose  $Q$  is almost frequency independent, II”, *Geophys. J. R. Astr. Soc.* **13**:5 (1967), 529–539.
- [Ciraolo et al. 2018] G. Ciraolo, A. Figalli, and F. Maggi, “A quantitative analysis of metrics on  $\mathbb{R}^n$  with almost constant positive scalar curvature, with applications to fast diffusion flows”, *Int. Math. Res. Not.* **2018**:21 (2018), 6780–6797. MR Zbl
- [Collot et al. 2017] C. Collot, F. Merle, and P. Raphaël, “Dynamics near the ground state for the energy critical nonlinear heat equation in large dimensions”, *Comm. Math. Phys.* **352**:1 (2017), 215–285. MR Zbl
- [Cortázar et al. 2016] C. Cortázar, M. del Pino, and M. Musso, “Green’s function and infinite-time bubbling in the critical nonlinear heat equation”, preprint, 2016. To appear in *J. Eur. Math. Soc.* arXiv
- [Daskalopoulos et al. 2018] P. Daskalopoulos, M. del Pino, and N. Sesum, “Type II ancient compact solutions to the Yamabe flow”, *J. Reine Angew. Math.* **738** (2018), 1–71. MR Zbl
- [Davila et al. 2017] J. Davila, M. del Pino, and J. Wei, “Singularity formation for the two-dimensional harmonic map flow into  $S^2$ ”, preprint, 2017. arXiv
- [Donninger and Krieger 2013] R. Donninger and J. Krieger, “Nonscattering solutions and blowup at infinity for the critical wave equation”, *Math. Ann.* **357**:1 (2013), 89–163. MR Zbl
- [Duyckaerts et al. 2012] T. Duyckaerts, C. Kenig, and F. Merle, “Profiles of bounded radial solutions of the focusing, energy-critical wave equation”, *Geom. Funct. Anal.* **22**:3 (2012), 639–698. MR Zbl
- [Duyckaerts et al. 2013] T. Duyckaerts, C. Kenig, and F. Merle, “Classification of radial solutions of the focusing, energy-critical wave equation”, *Cambridge J. Math.* **1**:1 (2013), 75–144. MR Zbl
- [Duyckaerts et al. 2016a] T. Duyckaerts, C. Kenig, and F. Merle, “Concentration-compactness and universal profiles for the non-radial energy critical wave equation”, *Nonlinear Anal.* **138** (2016), 44–82. MR Zbl
- [Duyckaerts et al. 2016b] T. Duyckaerts, C. Kenig, and F. Merle, “Solutions of the focusing nonradial critical wave equation with the compactness property”, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **15** (2016), 731–808. MR Zbl
- [Fila and King 2012] M. Fila and J. R. King, “Grow up and slow decay in the critical Sobolev case”, *Netw. Heterog. Media* **7**:4 (2012), 661–671. MR Zbl
- [Filippas et al. 2000] S. Filippas, M. A. Herrero, and J. J. L. Velázquez, “Fast blow-up mechanisms for sign-changing solutions of a semilinear parabolic equation with critical nonlinearity”, *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **456**:2004 (2000), 2957–2982. MR Zbl
- [Fujita 1966] H. Fujita, “On the blowing up of solutions of the Cauchy problem for  $u_t = \Delta u + u^{1+\alpha}$ ”, *J. Fac. Sci. Univ. Tokyo Sect. I* **13** (1966), 109–124. MR Zbl
- [Ghoul and Masmoudi 2016] T.-E. Ghoul and N. Masmoudi, “Stability of infinite time blow up for the Patlak–Keller–Segel system”, preprint, 2016. arXiv

- [Gui et al. 1992] C. Gui, W.-M. Ni, and X. Wang, “On the stability and instability of positive steady states of a semilinear heat equation in  $\mathbb{R}^n$ ”, *Comm. Pure Appl. Math.* **45**:9 (1992), 1153–1181. MR Zbl
- [Krieger et al. 2009] J. Krieger, W. Schlag, and D. Tataru, “Slow blow-up solutions for the  $H^1(\mathbb{R}^3)$  critical focusing semilinear wave equation”, *Duke Math. J.* **147**:1 (2009), 1–53. MR Zbl
- [Krieger et al. 2015] J. Krieger, K. Nakanishi, and W. Schlag, “Center-stable manifold of the ground state in the energy space for the critical wave equation”, *Math. Ann.* **361**:1-2 (2015), 1–50. MR Zbl
- [Matano and Merle 2004] H. Matano and F. Merle, “On nonexistence of type II blowup for a supercritical nonlinear heat equation”, *Comm. Pure Appl. Math.* **57**:11 (2004), 1494–1541. MR Zbl
- [Matano and Merle 2009] H. Matano and F. Merle, “Classification of type I and type II behaviors for a supercritical nonlinear heat equation”, *J. Funct. Anal.* **256**:4 (2009), 992–1064. MR Zbl
- [Matano and Merle 2011] H. Matano and F. Merle, “Threshold and generic type I behaviors for a supercritical nonlinear heat equation”, *J. Funct. Anal.* **261**:3 (2011), 716–748. MR Zbl
- [Merle et al. 2013] F. Merle, P. Raphaël, and I. Rodnianski, “Blowup dynamics for smooth data equivariant solutions to the critical Schrödinger map problem”, *Invent. Math.* **193**:2 (2013), 249–365. MR Zbl
- [Ortoleva and Perelman 2013] C. Ortoleva and G. Perelman, “Nondispersive vanishing and blow up at infinity for the energy critical nonlinear Schrödinger equation in  $\mathbb{R}^3$ ”, *Algebra i Analiz* **25**:2 (2013), 162–192. In Russian; translated in *St. Petersburg Math. J.* **25**:2 (2014), 271–294. MR Zbl
- [Poláčik and Quittner 2006] P. Poláčik and P. Quittner, “A Liouville-type theorem and the decay of radial solutions of a semilinear heat equation”, *Nonlinear Anal.* **64**:8 (2006), 1679–1689. MR Zbl
- [Poláčik and Yanagida 2003] P. Poláčik and E. Yanagida, “On bounded and unbounded global solutions of a supercritical semilinear heat equation”, *Math. Ann.* **327**:4 (2003), 745–771. MR Zbl
- [Poláčik and Yanagida 2014] P. Poláčik and E. Yanagida, “Global unbounded solutions of the Fujita equation in the intermediate range”, *Math. Ann.* **360**:1-2 (2014), 255–266. MR Zbl
- [Poláčik et al. 2007] P. Poláčik, P. Quittner, and P. Souplet, “Singularity and decay estimates in superlinear problems via Liouville-type theorems, II: Parabolic equations”, *Indiana Univ. Math. J.* **56**:2 (2007), 879–908. MR Zbl
- [Quittner and Souplet 2007] P. Quittner and P. Souplet, *Superlinear parabolic problems: blow-up, global existence and steady states*, Birkhäuser, Basel, 2007. MR Zbl
- [Raphaël and Rodnianski 2012] P. Raphaël and I. Rodnianski, “Stable blow up dynamics for the critical co-rotational wave maps and equivariant Yang–Mills problems”, *Publ. Math. Inst. Hautes Études Sci.* **115** (2012), 1–122. MR Zbl
- [Raphaël and Schweyer 2013] P. Raphaël and R. Schweyer, “Stable blowup dynamics for the 1-corotational energy critical harmonic heat flow”, *Comm. Pure Appl. Math.* **66**:3 (2013), 414–480. MR Zbl
- [Rey 1990] O. Rey, “The role of the Green’s function in a nonlinear elliptic equation involving the critical Sobolev exponent”, *J. Funct. Anal.* **89**:1 (1990), 1–52. MR Zbl
- [Schweyer 2012] R. Schweyer, “Type II blow-up for the four dimensional energy critical semi linear [sic] heat equation”, *J. Funct. Anal.* **263**:12 (2012), 3922–3983. MR Zbl

Received 22 Apr 2018. Accepted 29 Dec 2018.

MANUEL DEL PINO: m.delpino@bath.ac.uk  
 Department of Mathematical Sciences, University of Bath, Bath, United Kingdom

MONICA MUSSO: m.musso@bath.ac.uk  
 Department of Mathematical Sciences, University of Bath, Bath, United Kingdom

JUNCHENG WEI: jcwei@math.ubc.ca  
 Department of Mathematics, University of British Columbia, Vancouver, BC, Canada

# Analysis & PDE

msp.org/apde

## EDITORS

EDITOR-IN-CHIEF

Patrick Gérard  
patrick.gerard@math.u-psud.fr  
Université Paris Sud XI  
Orsay, France

## BOARD OF EDITORS

Massimiliano Berti	Scuola Intern. Sup. di Studi Avanzati, Italy berti@sissa.it	Gilles Pisier	Texas A&M University, and Paris 6 pisier@math.tamu.edu
Michael Christ	University of California, Berkeley, USA mchrist@math.berkeley.edu	Tristan Rivière	ETH, Switzerland riviere@math.ethz.ch
Charles Fefferman	Princeton University, USA cf@math.princeton.edu	Igor Rodnianski	Princeton University, USA irod@math.princeton.edu
Ursula Hamenstaedt	Universität Bonn, Germany ursula@math.uni-bonn.de	Yum-Tong Siu	Harvard University, USA siu@math.harvard.edu
Vadim Kaloshin	University of Maryland, USA vadim.kaloshin@gmail.com	Terence Tao	University of California, Los Angeles, USA tao@math.ucla.edu
Herbert Koch	Universität Bonn, Germany koch@math.uni-bonn.de	Michael E. Taylor	Univ. of North Carolina, Chapel Hill, USA met@math.unc.edu
Izabella Laba	University of British Columbia, Canada ilaba@math.ubc.ca	Gunther Uhlmann	University of Washington, USA gunther@math.washington.edu
Richard B. Melrose	Massachusetts Inst. of Tech., USA rbm@math.mit.edu	András Vasy	Stanford University, USA andras@math.stanford.edu
Frank Merle	Université de Cergy-Pontoise, France Frank.Merle@u-cergy.fr	Dan Virgil Voiculescu	University of California, Berkeley, USA dvv@math.berkeley.edu
William Minicozzi II	Johns Hopkins University, USA minicozz@math.jhu.edu	Steven Zelditch	Northwestern University, USA zelditch@math.northwestern.edu
Clément Mouhot	Cambridge University, UK c.mouhot@dpms.cam.ac.uk	Maciej Zworski	University of California, Berkeley, USA zworski@math.berkeley.edu
Werner Müller	Universität Bonn, Germany mueller@math.uni-bonn.de		

## PRODUCTION

production@msp.org  
Silvio Levy, Scientific Editor

---

See inside back cover or [msp.org/apde](http://msp.org/apde) for submission instructions.

---

The subscription price for 2020 is US \$340/year for the electronic version, and \$550/year (+\$60, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

---

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

---

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

 **mathematical sciences publishers**  
nonprofit scientific publishing

<http://msp.org/>

© 2020 Mathematical Sciences Publishers

# ANALYSIS & PDE

Volume 13 No. 1 2020

---

Absence of Cartan subalgebras for right-angled Hecke von Neumann algebras MARTIJN CASPERS	1
A vector field method for radiating black hole spacetimes JESÚS OLIVER and JACOB STERBENZ	29
Stable ODE-type blowup for some quasilinear wave equations with derivative-quadratic nonlinearities JARED SPECK	93
Asymptotic expansions of fundamental solutions in parabolic homogenization JUN GENG and ZHONGWEI SHEN	147
Capillary surfaces arising in singular perturbation problems ARAM L. KARAKHANYAN	171
A spiral interface with positive Alt–Caffarelli–Friedman limit at the origin MARK ALLEN and DENNIS KRIVENTSOV	201
Infinite-time blow-up for the 3-dimensional energy-critical heat equation MANUEL DEL PINO, MONICA MUSSO and JUNCHENG WEI	215
A well-posedness result for viscous compressible fluids with only bounded density RAPHAËL DANCHIN, FRANCESCO FANELLI and MARIUS PAICU	275



2157-5045(2020)13:1;1-D