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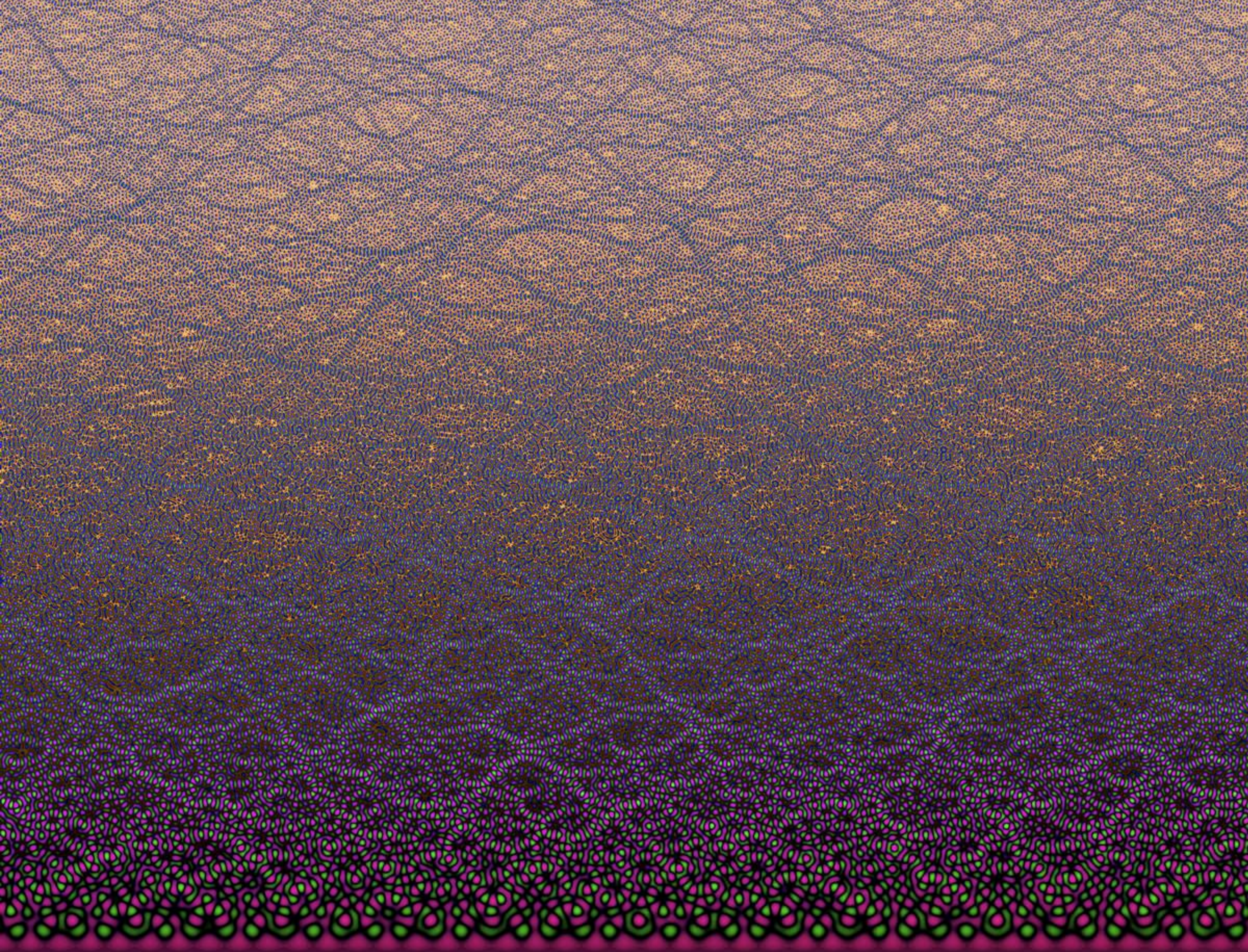

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REGULARITY ESTIMATES FOR ELLIPTIC NONLOCAL OPERATORS

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We study weak solutions to nonlocal equations governed by integrodifferential operators. Solutions are defined with the help of symmetric nonlocal bilinear forms. Throughout this work, our main emphasis is on operators with general, possibly singular, measurable kernels. We obtain regularity results which are robust with respect to the differentiability order of the equation. Furthermore, we provide a general tool for the derivation of Hölder a priori estimates from the weak Harnack inequality. This tool is applicable for several local and nonlocal, linear and nonlinear problems on metric spaces. Another aim of this work is to provide comparability results for nonlocal quadratic forms.

1. Introduction

The aim of this work is to develop a local regularity theory for general nonlocal operators. The main focus is on operators that are defined through families of measures, which might be singular. The main question that we ask is the following. Given a function $u : \mathbb{R}^d \to \mathbb{R}$ satisfying

$$\lim_{\varepsilon\to 0+} \int_{\mathbb{R}^d \setminus B_\varepsilon(x)} (u(y)-u(x))\,\mu(x,\mathrm{d}y) = f(x) \quad (x \in D), \tag{1-1}$$

which properties of u can be deduced in the interior of D? Here $D \subset \mathbb{R}^d$ is a bounded open set and the family $(\mu(x,\cdot))_{x\in D}$ of measures satisfies some assumptions to be discussed later in detail. The measures $\mu(x,\cdot)$ are assumed to have a singularity for sets $A \subset \mathbb{R}^d$ with $x \in \bar{A}$. As a result, the operators of the

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form (1-1) are not bounded integral operators but integrodifferential operators. For this reason we are able to prove regularity results which resemble results for differential operators. One aim of this work is to establish the following result:

Theorem 1.1. *Assume $\mu(x, \mathrm{d}y)$ is uniformly (with respect to the variable x) comparable on small scales to $\nu^\alpha(\mathrm{d}y - \{x\})$ for some nondegenerate α-stable measure ν^α for some $\alpha \in (0,2)$. Then solutions to* (1-1) *satisfy uniform Hölder regularity estimates in the interior of D.*

Theorem 1.1 will be proved as a special case of Theorem 1.11, which we provide with all details in Section 1E. Special cases of Theorem 1.1 have received significant attention over the last years and we give a small overview of results below. Note that it is well known how to treat functions f in (1-1). For the sake of a clear presentation, we will sometimes restrict ourselves to the case $f = 0$.

In order to approach the question raised above, we need to establish the following results:

- weak Harnack inequality,
- implications of the weak Harnack inequality,
- comparability results for nonlocal quadratic forms.

The last topic needs to be included because our concept of solutions involves quadratic forms related to $\mu(x, \mathrm{d}y)$. We present the main results in Sections 1C–1E. The following two subsections are devoted to the set-up and our main assumptions.

1A. ***Function spaces.*** Before we can formulate the first result we need to set up quadratic forms and function spaces. Let $\mu = (\mu(x, \cdot))_{x \in \mathbb{R}^d}$ be a family of measures on $\mathbb{R}^d$ which is symmetric in the sense that for every set $A \times B \subset \mathbb{R}^d \times \mathbb{R}^d \setminus \mathrm{diag}$

$$\int_A \int_B \mu(x, \mathrm{d}y)\, \mathrm{d}x = \int_B \int_A \mu(x, \mathrm{d}y)\, \mathrm{d}x. \tag{1-2}$$

We furthermore require

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \min(|x-y|^2, 1)\, \mu(x, \mathrm{d}y) < +\infty. \tag{1-3}$$

Example 1.2. An important example satisfying the above conditions is given by

$$\mu_\alpha(x, \mathrm{d}y) = (2-\alpha)|x-y|^{-d-\alpha}\, \mathrm{d}y \quad (0 < \alpha < 2). \tag{1-4}$$

The choice of the factor $(2-\alpha)$ will be discussed below in detail; see Sections 1B and 2.

For a given family μ and a real number $\alpha \in (0,2)$ we consider the following quadratic forms on $L^2(D) \times L^2(D)$, where $D \subset \mathbb{R}^d$ is some open set:

$$\mathcal{E}^\mu_D(u,u) = \int_D \int_D (u(y) - u(x))^2\, \mu(x, \mathrm{d}y)\, \mathrm{d}x. \tag{1-5}$$

We denote by $H^{\alpha/2}(\mathbb{R}^d)$ the usual Sobolev space of fractional order $\alpha/2 \in (0,1)$ with the norm

$$\|u\|_{H^{\alpha/2}(\mathbb{R}^d)} = (\|u\|^2_{L^2(\mathbb{R}^d)} + \mathcal{E}^{\mu_\alpha}_{\mathbb{R}^d}(u,u))^{\frac{1}{2}}. \tag{1-6}$$

If $D \subset \mathbb{R}^d$ is open and bounded, then by $H_D^{\alpha/2} = H_D^{\alpha/2}(\mathbb{R}^d)$ we denote the Banach space of functions from $H^{\alpha/2}(\mathbb{R}^d)$ which are zero almost everywhere on D^c. $H^{\alpha/2}(D)$ shall be the space of functions $u \in L^2(D)$ for which

$$\|u\|^2_{H^{\alpha/2}(D)} = \|u\|^2_{L^2(D)} + \int_D \int_D (u(y)-u(x))^2 \, \mu_\alpha(x, \mathrm{d}y) \, \mathrm{d}x$$

is finite. Note that for domains D with a Lipschitz boundary, $H_D^{\alpha/2}(\mathbb{R}^d)$ can be identified with the closure of $C_c^\infty(D)$ with respect to the norm of $H^{\alpha/2}(D)$. In general, these two objects might be different, though. By $V^{\alpha/2}(D \,|\, \mathbb{R}^d)$ we denote the space of all measurable functions $u : \mathbb{R}^d \to \mathbb{R}$ for which the quantity

$$\int_D \int_{\mathbb{R}^d} \frac{(u(y)-u(x))^2}{|x-y|^{d+\alpha}} \, \mathrm{d}x \, \mathrm{d}y \tag{1-7}$$

is finite, which implies finiteness of the quantity $\int_{\mathbb{R}^d} u(x)^2/(1+|x|)^{d+\alpha} \, \mathrm{d}x$. The function space $V^{\alpha/2}(D \,|\, \mathbb{R}^d)$ is a Hilbert space with the scalar product

$$(u, v) = \int_{\mathbb{R}^d} \frac{u(x)v(x)}{(1+|x|)^{d+\alpha}} \, \mathrm{d}x + \int_D \int_{\mathbb{R}^d} \frac{(u(y)-u(x))(u(v)-v(x))}{|x-y|^{d+\alpha}} \, \mathrm{d}x \, \mathrm{d}y. \tag{1-8}$$

The proof is similar to those of [Felsinger et al. 2015, Lemma 2.3] and [Dipierro et al. 2017a, Proposition 3.1]. If the scalar product (1-8) is defined with the expression $\int_{\mathbb{R}^d} u(x)v(x)/(1+|x|)^{d+\alpha}$ replaced by $\int_D u(x)v(x) \, \mathrm{d}x$, then the Hilbert space is identical. The following continuous embeddings trivially hold true:

$$H_D^{\frac{\alpha}{2}}(\mathbb{R}^d) \hookrightarrow H^{\frac{\alpha}{2}}(\mathbb{R}^d) \hookrightarrow V^{\frac{\alpha}{2}}(D \,|\, \mathbb{R}^d).$$

We make use of function spaces generated by general μ in the same way as above. Let $H^\mu(\mathbb{R}^d)$ be the vector space of functions $u \in L^2(\mathbb{R}^d)$ such that $\mathcal{E}^\mu(u,u) = \mathcal{E}^\mu_{\mathbb{R}^d}(u,u)$ is finite. If $D \subset \mathbb{R}^d$ is open and bounded, then by $H_D^\mu = H_D^\mu(\mathbb{R}^d)$ we denote the space of functions from $H^\mu(\mathbb{R}^d)$ which are zero almost everywhere on D^c. By $V_D^\mu = V^\mu(D \,|\, \mathbb{R}^d)$ we denote the space of all measurable functions $u : \mathbb{R}^d \to \mathbb{R}$ for which the quantity

$$\int_D \int_{\mathbb{R}^d} (u(y)-u(x))^2 \, \mu(x, \mathrm{d}y) \, \mathrm{d}x \tag{1-9}$$

is finite. Now we are in a position to present and discuss our main results.

1B. ***Main assumptions.*** Let us formulate our main assumptions on $(\mu(x,\cdot))_{x \in D}$. Given $\alpha \in (0,2)$ and $A \geq 1$, the following condition is an analog of (A′) for nonlocal energy forms:

$$\begin{gathered}\text{For every ball } B_\rho(x_0) \text{ with } \rho \in (0,1), x_0 \in B_1 \text{ and every } v \in H^{\alpha/2}(B_\rho(x_0)),\\ A^{-1}\, \mathcal{E}^\mu_{B_\rho(x_0)}(v,v) \leq \mathcal{E}^{\mu_\alpha}_{B_\rho(x_0)}(v,v) \leq A\, \mathcal{E}^\mu_{B_\rho(x_0)}(v,v).\end{gathered} \tag{A}$$

Condition (A) says that, locally in the unit ball, the energies $\mathcal{E}^\mu$ and $\mathcal{E}^{\mu_\alpha}$ are comparable on every scale. Note that this does not imply pointwise comparability of the densities of μ and μ_α. We also need to assume the existence of cut-off functions. Let $\alpha \in (0,2)$ and $B \geq 1$:

$$\begin{aligned} &\text{For } 0 < \rho \leq R \leq 1 \text{ and } x_0 \in B_1 \text{ there is a nonnegative measurable function } \tau : \mathbb{R}^d \to \mathbb{R} \\ &\quad \text{with } \operatorname{supp}(\tau) \subset \overline{B_{R+\rho}(x_0)},\ \tau(x) \equiv 1 \text{ on } B_R(x_0),\ \|\tau\|_\infty \leq 1, \text{ and} \\ &\quad \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} (\tau(y) - \tau(x))^2 \, \mu(x, \mathrm{d}y) \leq B\rho^{-\alpha}. \end{aligned} \tag{B}$$

In most of the cases (B) does not impose an additional restriction because the standard cut-off function $\tau(x) = \max(0, 1+\min(0, (R-|x-x_0|)/\rho))$ is an appropriate choice. It is an interesting question whether, under assumptions (1-2), (1-3) and (A), condition (B) holds or whether it holds with this standard choice. Note that condition (B) becomes $|\nabla\tau|^2 \leq B\rho^{-2}$ when $\alpha \to 2-$ and $\mu(x, \mathrm{d}y)$ is as in Example 1.2.

For every $\alpha \in (0,2)$, the family of measures μ_α given in Example 1.2 satisfies the above conditions for some constants $A, B \geq 1$. The normalizing constant $2-\alpha$ in the definition of μ_α has the effect that the constants $A, B \geq 1$ can be chosen independently of α for $\alpha \to 2-$. Since in this work we do not care about the behavior of constants for $\alpha \to 0+$, in our examples we will use factors of the form $2-\alpha$. Let us look at more examples.

Example 1.3. Assume $0 < \beta \leq \alpha < 2$. Let $f, g : \mathbb{R}^d \to [1,2]$ be measurable and symmetric functions. Set

$$\mu(x, \mathrm{d}y) = f(x,y)\, \mu_\alpha(x, \mathrm{d}y) + g(x,y)\, \mu_\beta(x, \mathrm{d}y).$$

Then μ satisfies (1-2), (1-3), (A), and (B) with exponent α. This simply follows from

$$\frac{1}{|x-y|^{d+\alpha}} \leq \frac{1}{|x-y|^{d+\beta}} + \frac{1}{|x-y|^{d+\alpha}} \leq \frac{5}{|x-y|^{d+\alpha}} \quad (x, y \in B_1(x_0),\ x_0 \in \mathbb{R}^d).$$

For the verification of (B) we may choose the standard Lipschitz-continuous cut-off function.

Here is an example with some kernels which are not rotationally symmetric.

Example 1.4. Assume $\alpha_0 \in (0,2)$, $0 < \lambda < \Lambda$, $v \in S^{d-1}$, and $\theta \in [0,1)$. Set

$$M = \left\{ h \in \mathbb{R}^d : \left| \left\langle \frac{h}{|h|}, v \right\rangle \right| \geq \theta \right\}.$$

Let $k : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty]$ be any measurable function satisfying

$$\lambda \mathbb{1}_M(x-y) \frac{(2-\alpha)}{|x-y|^{d+\alpha}} \leq k(x,y) \leq \Lambda \frac{(2-\alpha)}{|x-y|^{d+\alpha}} \tag{1-10}$$

for some $\alpha \in [\alpha_0, 2)$ and for almost every $x, y \in \mathbb{R}^d$. Set $\mu(x, \mathrm{d}y) = k(x,y)\,\mathrm{d}y$. Then, as we will prove, there are $A \geq 1$, $B \geq 1$, independent of α, such that (A) and (B) hold.

The following example of a family of measures falls into our framework. Note that the measures do not possess a density with respect to the d-dimensional Lebesgue measure.

Example 1.5. Assume $\alpha_0 \in (0,2)$, $\alpha_0 \leq \alpha < 2$. Set

$$\mu(x, dy) = (2-\alpha) \sum_{i=1}^{d} \left[|x_i - y_i|^{-1-\alpha} \mathrm{d}y_i \prod_{j \neq i} \delta_{\{x_j\}}(\mathrm{d}y_j) \right]. \tag{1-11}$$

Again, as we will prove, there are $A \geq 1$, $B \geq 1$, independent of α, such that (A) and (B) hold. Note that $\mu(x, A) = 0$ for every set A which has an empty intersection with any of the d lines $\{x + te_i : t \in \mathbb{R}\}$.

Let us now formulate our results.

1C. *The weak Harnack inequality.* Given functions $u, v : \mathbb{R}^d \to \mathbb{R}$ we define the quantity

$$\mathcal{E}^{\mu}(u, v) = \iint_{\mathbb{R}^d\,\mathbb{R}^d} (u(y) - u(x))(v(y) - v(x))\, \mu(x, \mathrm{d}y)\, \mathrm{d}x, \tag{1-12}$$

if it is finite. We write $\mathcal{E}$ instead of $\mathcal{E}^{\mu}$ when it is clear or irrelevant which measure μ is used. One aim of this work is to study properties of functions u satisfying $\mathcal{E}(u, \phi) \geq 0$ for every nonnegative test function ϕ. Note that $\mathcal{E}^{\mu}(u, \phi)$ is finite for $u \in V^{\mu}(D \mid \mathbb{R}^d)$, $\phi \in H^{\mu}_D(\mathbb{R}^d)$ for any open set $D \subset \mathbb{R}^d$. This follows from the definition of these function spaces, the Cauchy–Schwarz inequality and the decomposition

$$\mathcal{E}^{\mu}(u,\phi) = \iint_{DD} (u(y)-u(x))(\phi(y)-\phi(x))\, \mu(x,\mathrm{d}y)\, \mathrm{d}x + 2\iint_{DD^c} (u(y)-u(x))(\phi(y)-\phi(x))\, \mu(x,\mathrm{d}y)\, \mathrm{d}x.$$

Here is our first main result.

Theorem 1.6 (weak Harnack inequality). *Assume* $0 < \alpha_0 < 2$ *and* $A \geq 1$, $B \geq 1$. *Let* μ *satisfy* (A), (B) *for some* $\alpha \in [\alpha_0, 2)$. *Assume* $f \in L^{q/\alpha}(B_1)$ *for some* $q > d$. *Let* $u \in V^{\mu}(B_1 \mid \mathbb{R}^d)$, $u \geq 0$ *in* B_1, *satisfy* $\mathcal{E}^{\mu}(u, \phi) \geq (f, \phi)$ *for every nonnegative* $\phi \in H^{\mu}_{B_1}(\mathbb{R}^d)$. *Then*

$$\inf_{B_{1/4}} u \geq c \left(\fint_{B_{1/2}} u(x)^{p_0}\, \mathrm{d}x \right)^{\frac{1}{p_0}} - \sup_{x \in B_{15/16}} \int_{\mathbb{R}^d \setminus B_1} u^{-}(z)\, \mu(x, \mathrm{d}z) - \|f\|_{L^{q/\alpha}(B_{15/16})}, \tag{1-13}$$

with positive constants p_0 *and* c *depending only on* d, α_0, A, B. *In particular*, p_0 *and* c *do not depend on* α.

Note that below we explain a local counterpart to this result, which relates to the limit $\alpha \to 2-$; see Theorem 1.12.

Remark. It is remarkable that (A) and (B) do not imply a strong formulation of the Harnack inequality. Examples 1.4 and 1.5 provide cases in which the classical strong formulation fails. See the discussion in [Kassmann et al. 2014, Appendix A.1] and the concrete examples in [Bogdan and Sztonyk 2005, p. 148; Bass and Chen 2010, Section 3]. The nonlocal term, i.e., the integral of u^{-} in (1-13) is unavoidable since we do not assume nonnegativity of u on all of $\mathbb{R}^d$.

1D. *Regularity estimates.* A separate aim of our work is to provide consequences of the (weak) Harnack inequality. Before we explain this in a more abstract fashion let us formulate a regularity result, which will be derived from Theorem 1.6 and which is one of the main results of this work. We need an additional mild assumption on the decay of the kernels considered.

Given $\alpha \in (0, 2)$ we assume that for some constants $\chi > 1$, $C \geq 1$

$$\mu(x, \mathbb{R}^d \setminus B_{r2^j}(x)) \leq C r^{-\alpha} \chi^{-j} \quad (x \in B_1,\ 0 < r \leq 1,\ j \in \mathbb{N}_0). \tag{D}$$

Condition (D) rules out kernels with very heavy tails for large values of $|x-y|$. For example, μ given by $\mu(x,\mathrm{d}y)=k(x,y)\,\mathrm{d}y$ with $k(x,y)=|x-y|^{-d-1}+|x-y|^{-d}\ln(2+|x-y|)^{-2}$ does not satisfy (D).

Here is our main regularity result.

Theorem 1.7. *Let $\alpha_0\in(0,2)$, $\chi>0$, and $A\geq 1$, $B\geq 1$. Let μ satisfy* (A), (B) *and* (D) *for some $\alpha\in[\alpha_0,2)$. Assume $u\in V^{\mu}(B_1\,|\,\mathbb{R}^d)$ satisfies $\mathcal{E}(u,\phi)=0$ for every $\phi\in H^{\mu}_{B_1}(\mathbb{R}^d)$. Then the following Hölder estimate holds for almost every $x,y\in B_{1/2}$:*

$$|u(x)-u(y)|\leq c\|u\|_\infty|x-y|^{\beta}, \tag{1-14}$$

where $c\geq 1$ and $\beta\in(0,1)$ are constants which depend only on d,α_0,A,B,C,χ. In particular, c and β do not depend on α.

This result contrasts the corresponding result for differential operators; see Theorem 1.13 below.

The main tool for the proof of Theorem 1.7 is the weak Harnack inequality, Theorem 1.6. The Harnack inequality itself is an interesting object of study for nonlocal operators. In Section 2 we explain different formulations of the Harnack inequality for nonlocal operators satisfying a maximum principle. A separate aim of this article is to prove a general tool that allows us to deduce regularity estimates from the Harnack inequality for nonlocal operators. This step was subject to discussion of many recent articles in the field. We choose the set-up of a metric measure space so that this tool can be of future use in different contexts.

In the first decades after publication, the Harnack inequality itself did not attract as much attention as the resulting convergence theorems. This changed when J. Moser in 1961 showed that the inequality itself leads to a priori estimates in Hölder spaces. His result can be formulated in a metric measure space (X,d,m) as follows. For $r>0$, $x\in X$, set $B_r(x)=\{y\in X: d(y,x)<r\}$. For every $x\in X$ and $r>0$ let $\mathcal{S}_{x,r}$ denote a family of measurable functions on X satisfying the conditions

$$\begin{aligned} r>0,\ u\in\mathcal{S}_{x,r},\ a\in\mathbb{R} \quad &\Longrightarrow \quad au\in\mathcal{S}_{x,r},\ (u+1)\in\mathcal{S}_{x,r},\\ B_r(x)\subset B_s(y) \quad &\Longrightarrow \quad \mathcal{S}_{y,s}\subset\mathcal{S}_{x,r}. \end{aligned}$$

An example for $\mathcal{S}_{x,r}$ is given by the set of all functions $u:\mathbb{R}^d\to\mathbb{R}$ satisfying some (possibly nonlinear) appropriate partial differential or integrodifferential equation in a ball $B_r(x)$.

Theorem 1.8 (compare [Moser 1961]). *Assume X is separable. Let $x_0\in X$ and $\mathcal{S}_{x,r}$ be as above. Assume that there is $c\geq 1$ such that for $r>0$*

$$(u\in S_{x_0,r})\wedge(u\geq 0 \text{ in } B_r(x_0)) \quad\Longrightarrow\quad \sup_{x\in B_{r/2}(x_0)} u\leq c \inf_{x\in B_{r/2}(x_0)} u. \tag{1-15}$$

Then there exist $\beta\in(0,1)$ such that for $r>0$, $u\in\mathcal{S}_{x_0,r}$ and almost every $x\in B_r(x_0)$

$$|u(x)-u(x_0)|\leq 3\|u-u(x_0)\|_\infty\left(\frac{d(x,x_0)}{r}\right)^{\beta}.$$

Recall that "sup" denotes the essential supremum and "inf" the essential infimum. With the help of this theorem, regularity estimates can be established for various linear and nonlinear differential equations; see [Gilbarg and Trudinger 1998]. One aim of this article is to show that (1-15) can be relaxed significantly by allowing some global terms of u to show up in the Harnack inequality. Already in Section 2 we have seen that they naturally appear.

For $x \in X,\ r > 0$ let $\nu_{x,r}$ be a measure on $\mathcal{B}(X \setminus \{x\})$, which is finite on all sets M with $\operatorname{dist}(\{x\}, M) > 0$. We assume that for some $c \geq 1,\ \chi > 1$ and for every $j \in \mathbb{N}_0,\ x \in X$, and $0 < r \leq 1$

$$\nu_{x,r}(X \setminus B_{r2^j}(x)) \leq c\chi^{-j}. \tag{1-16}$$

We further assume that, given $K > 1$, there is $c \geq 1$ such that for $0 < r \leq R \leq Kr,\ x \in X,\ M \subset X \setminus B_r(x)$

$$\nu_{x,R}(M) \leq c\nu_{x,r}(M). \tag{1-17}$$

Conditions (1-16) and (1-17) will trivially hold true in the applications that are of importance to us. In Section 5 we discuss these conditions in detail. A standard case is provided in the following example.

Example 1.9. Let $\alpha \in (0, 2)$. For $x \in \mathbb{R}^d,\ r > 0$, and $A \in \mathcal{B}(\mathbb{R}^d \setminus \{x\})$ set

$$\nu_{x,r}(A) = r^\alpha \mu_\alpha(x, A) = r^\alpha \alpha(2-\alpha) \int_A |x-y|^{-d-\alpha}\, \mathrm{d}y. \tag{1-18}$$

Then $\nu_{x,r}$ satisfies conditions (1-16), (1-17).

The following result extends Theorem 1.8 to situations with nonlocal terms. It is an important tool in the theory of nonlocal operators.

Theorem 1.10. *Let $x_0 \in X,\ r_0 > 0$, and $\lambda > 1,\ \sigma > 1,\ \theta > 1$. Let $\mathcal{S}_{x,r}$ and $\nu_{x,r}$ be as above. Assume that conditions* (1-16), (1-17) *are satisfied. Assume that there is $c \geq 1$ and $p > 0$ such that for $0 < r \leq r_0$ the following holds*:

$(u \in x_0, r) \wedge (u \geq 0 \text{ in } B_r(x_0))$,

$$\implies \left(\fint_{B_{r/\lambda}(x_0)} u(x)^p\, m(\mathrm{d}x) \right)^{\frac{1}{p}} \leq c \inf_{x \in B_{r/\theta}(x_0)} u + c \sup_{x \in B_{r/\sigma}(x_0)} \int_X u^-(z)\, \nu_{x,r}(\mathrm{d}z). \tag{1-19}$$

Then there exists $\beta \in (0, 1)$ such that for $0 < r \leq r_0,\ u \in \mathcal{S}_{x_0,r}$

$$\operatorname{osc}_{B_\rho(x_0)} u \leq 2\theta^\beta \|u\|_\infty \left(\frac{\rho}{r}\right)^\beta \quad (0 < \rho \leq r), \tag{1-20}$$

where $\operatorname{osc}_M u := \sup_M u - \inf_M u$ *for* $M \subset X$.

Note that in Lemma 5.1 we provide several conditions that are equivalent to (1-16).

1E. ***Comparability of nonlocal quadratic forms.*** With regard to Theorem 1.7 one major problem is to provide conditions on μ which imply (A). Let us formulate our results in this direction.

Since $\mu = (\mu(x, \cdot))_{x \in \mathbb{R}^d}$ is a family of measures we need to impose a condition that fixes a uniform behavior of μ with respect to x. In our setup this condition implies that the integrodifferential operator from (1-1) is comparable to a translation-invariant operator — most often the generator of an α-stable process. We assume that there are measures ν_* and ν^* such that

$$\int f(x, x+z)\, \nu_*(\mathrm{d}z) \leq \int f(x, y)\, \mu(x, \mathrm{d}y) \leq \int f(x, x+z)\, \nu^*(\mathrm{d}z) \tag{T}$$

for every measurable function $f:\mathbb{R}^d \to [0,\infty]$ and every $x \in \mathbb{R}^d$. For a measure ν on $\mathbb{R}^d$ such that $\nu(\{0\}) = 0$ and a set $B \subset \mathbb{R}^d$ we define, abusing the previous notation slightly,

$$\mathcal{E}_B^\nu(u,v) = \int_B \int_{\mathbb{R}^d} (u(x)-u(x+z))(v(x)-v(x+z))\, \mathbb{1}_B(x+z)\, \nu(\mathrm{d}z)\, \mathrm{d}x. \tag{1-21}$$

Note that (T) implies for every $u \in L^2(B)$

$$\mathcal{E}_B^{\nu_*}(u,u) \le \mathcal{E}_B^{\mu}(u,u) \le \mathcal{E}_B^{\nu^*}(u,u).$$

Let $\bar\nu(A) = \nu(-A)$. It is easy to check that $\mathcal{E}^\nu = \mathcal{E}^{(\nu+\bar\nu)/2}$. Hence we may and do assume that the measures ν_*, ν^* are symmetric; i.e., $\nu_*(A) = \nu_*(-A)$ and $\nu^*(A) = \nu^*(-A)$.

We say that a measure ν on $\mathcal{B}(\mathbb{R}^d)$ satisfies the upper-bound assumption (U) if for some $C_U > 0$

$$\int_{\mathbb{R}^d} (r \wedge |z|)^2\, \nu(\mathrm{d}z) \le C_U r^{2-\alpha} \quad (0 < r \le 1). \tag{U}$$

We say that a measure ν on $\mathcal{B}(\mathbb{R}^d)$ satisfies the scaling assumption (S) if for some $a > 1$

$$\int_{\mathbb{R}^d} f(y)\, \nu(\mathrm{d}y) = a^{-\alpha} \int_{\mathbb{R}^d} f(ay)\, \nu(\mathrm{d}y) \tag{S}$$

for every measurable function $f:\mathbb{R}^d \to [0,\infty]$ with $\operatorname{supp} f \subset B_1$. For a linear subspace $E \subset \mathbb{R}^d$, let H_E denote the $\dim(E)$-dimensional Hausdorff measure supported on E.

We say that a measure ν on $\mathcal{B}(\mathbb{R}^d)$ satisfies the nondegeneracy assumption (ND) if for some $n \in \{1,\dots,d\}$

$$\begin{aligned} &\nu = \textstyle\sum_{k=1}^n f_k H_{E_k} \text{ for some linear subspaces } E_k \subset \mathbb{R}^d \text{ and densities } f_k \\ &\quad \text{with } \operatorname{lin}\big(\bigcup_k E_k\big) = \mathbb{R}^d \text{ and } \textstyle\int_{B_1} f_k\, \mathrm{d}H_{E_k} > 0 \text{ for } k = 1,\dots,n. \end{aligned} \tag{ND}$$

Here is our result on local comparability of nonlocal energy forms. It contains Theorem 1.1 as a special case.

Theorem 1.11. *Let $\mu = (\mu(x,\cdot))_{x\in\mathbb{R}^d}$ be a family of measures on $\mathcal{B}(\mathbb{R}^d)$ satisfying* (1-2). *Assume that there exist measures ν_* and ν^* for which* (T) *and* (U) *hold with $\alpha_0 \in (0,2)$ and $C_U > 0$. Assume that*

(i) *ν_* is a nondegenerate α-stable measure* (1-22), *or*

(ii) *ν_* satisfies* (ND) *and for some $a > 1$ each measure $f_k H_{E_k}$ satisfies* (S).

Then there are $A \ge 1$, $B \ge 1$ such that (A) *and* (B) *hold. One can choose $B = 4C_U$ but the constant A depends also on a, on the measure ν_* and on α_0.*

The result is robust in the following sense: *if $\mu^\alpha = (\mu^\alpha(x,\cdot))_{x\in\mathbb{R}^d}$ satisfies* (1-2) *and* (T) *with measures $(\nu_*)^\alpha$ and $(\nu^*)^\alpha$, $\alpha_0 \le \alpha < 2$, that are defined with the help of ν_* and ν^* as in Definition 6.9, then* (A) *holds with a constant A independent of $\alpha \in [\alpha_0, 2)$.*

Recall that a measure ν on $\mathcal{B}(\mathbb{R}^d)$ is a *nondegenerate α-stable measure* if for some $\alpha \in (0,2)$

$$\nu(E) = (2-\alpha) \int_{S^{d-1}} \int_0^\infty \mathbb{1}_E(r\theta) r^{-1-\alpha}\, \mathrm{d}r\, \pi(d\theta) \quad (E \in \mathcal{B}(\mathbb{R}^d)), \tag{1-22}$$

where π is some finite measure on S^{d-1} and $\operatorname{lin}(\operatorname{supp} \pi) = \mathbb{R}^d$.

1F. ***Related results.*** It is instructive to compare our results with two key results for differential operators in divergence form. Let $(A(x))_{x\in\mathbb{R}^d}$ be a family of $d\times d$-matrices. Given a subset $D\subset\mathbb{R}^d$ we introduce a bilinear form $\mathcal{A}_D$ by $\mathcal{A}_D(u,v)=\int_D(\nabla u(x), A(x)\nabla u(x))\,\mathrm{d}x$ for u and v from the Sobolev space $H^1(D)$. Instead of $\mathcal{A}_{\mathbb{R}^d}$ we write $\mathcal{A}$. The following theorem is at the heart of the theory named after E. De Giorgi, J. Moser and J. Nash; see [Gilbarg and Trudinger 1998, Chapters 8.8–8.9]:

Theorem 1.12 (weak Harnack inequality). *Let $\Lambda > 1$. Assume that for all balls $B\subset B_1$ and all functions $v\in H^1(B)$*

$$\Lambda^{-1}\mathcal{A}_B(u,u)\le\int_B|\nabla u|^2\le\Lambda\mathcal{A}_B(u,u). \tag{A$'$}$$

Assume $f\in L^{q/2}(B_1)$ for some $q>d$. Let $u\in H^1(B_1)$ satisfy $u\ge 0$ in B_1 and $\mathcal{A}_{B_1}(u,\phi)\ge(f,\phi)$ for every nonnegative $\phi\in H^1_0(B_1)$. Then

$$c\inf_{B_{1/4}}u\ge\left(\fint_{B_{1/2}}u(x)^{p_0}\,\mathrm{d}x\right)^{\frac{1}{p_0}}-\|f\|_{L^{q/2}(B_{15/16})},$$

with constants $p_0, c\in(0,1)$ depending only on d and Λ.

Remark. This by now classical result can be seen as the limit case of Theorem 1.6 for $\alpha\to 2-$. Condition (A$'$) implies that the differential operator $\div(A(\cdot)\nabla u)$ is uniformly elliptic and obviously describes a limit situation of (A). One might object that the nonlocal term in (1-13) is unnatural but in fact, it is not. In Section 2 we explain this phenomenon in detail for the fractional Laplace operator.

If u is not only a supersolution but a solution in Theorem 1.12, then one obtains a classical Harnack inequality: $\sup_{B_{1/4}}u\le c\inf_{B_{1/4}}u$. Both the Harnack inequality and the weak Harnack inequality imply Hölder a priori regularity estimates:

Theorem 1.13. *Assume condition* (A$'$) *holds true. There exist $c\ge 1$, $\beta\in(0,1)$ such that for every $u\in H^1(B_1)$ satisfying $\mathcal{A}(u,\phi)=0$ for every $\phi\in H^1_0(B_1)$ the following Hölder estimate holds for almost every $x,y\in B_{1/2}$:*

$$|u(x)-u(y)|\le c\|u\|_\infty|x-y|^\beta. \tag{1-23}$$

The constants β, c depend only on d and Λ.

After having recalled corresponding results for local differential operators, let us review some related results for nonlocal problems. Note that we restrict ourselves to nonlocal equations related to bilinear forms and distributional solutions.

Theorem 1.7 has already been proved under additional assumptions. If $\mu(x,\cdot)$ has a density $k(x,\cdot)$ which satisfies some isotropic lower bound, e.g., for some $c_0>0$, $\alpha\in(0,2)$

$$\mu(x,\mathrm{d}y)=k(x,y)\,\mathrm{d}y,\quad k(x,y)\ge c_0|x-y|^{-d-\alpha}\quad(|x-y|\le 1),$$

then Theorem 1.7 is proved in and follows from [Komatsu 1995; Bass and Levin 2002; Chen and Kumagai 2003; Caffarelli et al. 2011]. In these works the constant c in (1-14) depends on $\alpha\in(0,2)$ with $c(\alpha)\to+\infty$ for $\alpha\to 2-$. The current work follows the strategy laid out in [Kassmann 2009], which, on the one hand, allows the constants to be independent of α for $\alpha\to 2-$ and, on the other hand, allows us to treat general measures. See [Felsinger and Kassmann 2013; Kassmann and Schwab 2014] for corresponding results in the parabolic case.

The articles [Di Castro et al. 2014; 2016] study Hölder regularity estimates and Harnack inequalities for nonlinear equations. Moreover, the results therein provide boundedness of weak solutions. In [Di Castro et al. 2014; 2016] the measures $\mu(x, \mathrm{d}y)$ are assumed to be absolutely continuous with respect to the Lebesgue measure. Another difference to the present article is that our local regularity estimates require only local conditions on the data and on the operator. Note that our study of implications of (weak) Harnack inequalities in Section 5 allows for nonlinear problems in metric measure spaces and could be used to deduce the regularity results of [Di Castro et al. 2016] from results in [Di Castro et al. 2014].

To our best knowledge there has been little research addressing the question of comparability of quadratic nonlocal forms; we note here [Dyda 2006; Husseini and Kassmann 2007; Prats and Saksman 2017]. This question becomes important when studying very irregular kernels as in [Silvestre 2016, Section 4].

Theorem 1.1 has recently been established in the translation-invariant case, i.e., when $\mu(x, \mathrm{d}y) = \nu^\alpha(\mathrm{d}y - \{x\})$ for some α-stable measure ν^α; see [Ros-Oton and Serra 2016]. The methods of that paper seem not to be applicable in the general case, though. Note that anisotropic translation-invariant integrodifferential operators allow for higher interior regularity; see [Ros-Oton and Valdinoci 2016].

Related questions on nonlocal Dirichlet forms on metric measure spaces are currently investigated by several groups. We refer to the exposition in [Grigor'yan et al. 2014; Chen et al. 2019] for a discussion of results regarding the fundamental solution.

1G. *Notation.* Throughout this article, "inf" denotes the essential infimum and "sup" the essential supremum. By $S^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$ we denote the unit sphere. We define the Fourier transform as an isometry of $L^2(\mathbb{R}^d)$ determined by

$$\hat{u}(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} u(x) e^{-i\xi\cdot x}\,\mathrm{d}x, \quad u \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d).$$

1H. *Structure of the article.* The paper is organized as follows. In Section 2 we study the Harnack inequality for the Laplace and the fractional Laplace operators. We explain how one can formulate a Harnack inequality without assuming the functions under consideration to be nonnegative. In Section 3 we provide several auxiliary results and explain how the inequality $\mathcal{E}^\mu(u, \phi) \geq (f, \phi)$ is affected by rescaling the family of measures μ. In Section 4 we prove Theorem 1.6 under assumptions (A) and (B) adapting the approach by Moser to nonlocal bilinear forms. Section 5A provides the proof of Theorem 1.7. We first prove a general tool which allows us to deduce regularity results from weak Harnack inequalities; see Corollary 5.2. Then Theorem 1.7 follows immediately. Section 6 contains the proof of our main result on comparability, Theorem 1.11, in the two respective cases. We provide sufficient conditions on μ for (A) and (B) to hold true. In addition, we provide two examples of quite irregular kernels satisfying (A) and (B).

2. Harnack inequalities for the Laplace and the fractional Laplace operators

We establish a formulation of the Harnack inequality which does not require the functions to be nonnegative. This reformulation is especially interesting for nonlocal problems but our formulation seems to be new even for harmonic functions in the classical sense; see Theorem 2.5. For $\alpha \in (0,2)$ and $u \in C_c^2(\mathbb{R}^d)$ the fractional power of the Laplacian can be defined as

$$\Delta^{\frac{\alpha}{2}} u(x) = C_{\alpha,d} \lim_{\varepsilon \to 0+} \int_{|y-x|>\varepsilon} \frac{u(y)-u(x)}{|y-x|^{d+\alpha}} \,\mathrm{d}y = \frac{C_{\alpha,d}}{2} \int_{\mathbb{R}^d} \frac{u(x+h)-2u(x)+u(x-h)}{|h|^{d+\alpha}} \,\mathrm{d}h, \quad (2\text{-}1)$$

where

$$C_{\alpha,d} = \frac{\Gamma((d+\alpha)/2)}{2^{-\alpha}\pi^{\frac{d}{2}} |\Gamma(-\alpha/2)|}.$$

For later purposes we note that with some constant $c > 0$ for every $\alpha \in (0,2)$

$$c\alpha(2-\alpha) \le C_{\alpha,d} \le \frac{\alpha(2-\alpha)}{c}. \quad (2\text{-}2)$$

The use of the symbol $\Delta^{\alpha/2}$ and the term "fractional Laplacian" are justified because of $\widehat{(-\Delta)^{\alpha/2}u}(\xi) = |\xi|^\alpha \hat{u}(\xi)$ for $\xi \in \mathbb{R}^d$ and $u \in C_c^\infty(\mathbb{R}^d)$. Note that we write $\Delta^{\alpha/2}u$ instead of $-(-\Delta)^{\alpha/2}u$, which would be more appropriate. The potential theory of these operators was initiated in [Riesz 1938]. The following Harnack inequality can be easily established using the corresponding Poisson kernels.

Theorem 2.1. *There is a constant $c \ge 1$ such that for $\alpha \in (0,2)$ and $u \in C(\mathbb{R}^d)$ with*

$$\Delta^{\frac{\alpha}{2}} u(x) = 0 \quad (x \in B_1), \quad (2\text{-}3)$$

$$u(x) \ge 0 \quad (x \in \mathbb{R}^d) \quad (2\text{-}4)$$

the following inequality holds:

$$u(x) \le c u(y) \quad (x, y \in B_{\frac{1}{2}}).$$

Note that $\Delta^{\alpha/2}u(x) = 0$ at a point $x \in \mathbb{R}^d$ requires that the integral in (2-1) converges. Thus some additional regularity of $u \in C(\mathbb{R}^d)$ is assumed implicitly. Since $\Delta^{\alpha/2}$ allows for shifting and scaling, the result holds true for $B_1, B_{1/2}$ replaced by $B_R(x_0), B_{R/2}(x_0)$ with the same constant c for arbitrary $x_0 \in \mathbb{R}^d$ and $R > 0$.

Theorem 2.1 formulates the Harnack inequality in the standard way for nonlocal operators. The function u is assumed to be nonnegative on all of $\mathbb{R}^d$. In the following we discuss the necessity of this assumption and possible alternatives. The following result proves that this assumption cannot be dropped completely.

Theorem 2.2. *Assume $\alpha \in (0,2)$. Then there exists a bounded function $u \in C(\mathbb{R}^d)$ which is infinitely many times differentiable in B_1 and satisfies*

$$\Delta^{\frac{\alpha}{2}} u(x) = 0 \quad (x \in B_1),$$

$$u(x) > 0 \quad (x \in B_1 \setminus \{0\}),$$

$$u(0) = 0.$$

Therefore, the classical local formulation of the Harnack inequality as well as the local maximum principle fail for the operator $\Delta^{\alpha/2}$.

A complicated and lengthy proof can be found in [Kassmann 2007a]. An elegant way to construct such a function u would be to mollify the function $v(x) = (1-|x/2|^2)_+^{-1+\alpha/2}$ and shift it such that $u(0) = 0$. Here we provide a short proof,[1] which includes a helpful observation on radial functions. See [Bucur and Valdinoci 2016; Dipierro et al. 2017b] for further alternatives.

For an open set $D \subset \mathbb{R}^d,\ x \in D,\ 0 < \alpha \le 2$, and $v : \mathbb{R}^d \to \mathbb{R}$ $(0 < \alpha < 2)$ or $v : \overline{D} \to \mathbb{R}$ $(\alpha = 2)$ we write

$$H_\alpha(v \mid D)(x) = \int_{y \notin D} P_\alpha(x, y)\, v(y)\, \mathrm{d}y = \begin{cases} \int_{\mathbb{R}^d \setminus D} P_\alpha(x, y)\, v(y)\, \mathrm{d}y & (0 < \alpha < 2), \\ \int_{\partial D} P_2(x, y)\, v(y)\, \mathrm{d}y & (\alpha = 2). \end{cases} \tag{2-5}$$

In the case of a ball, the Poisson kernel is explicitly known; namely for $R > 0$ and $f : \mathbb{R}^d \setminus B_R(0) \to \mathbb{R}$

$$H_\alpha(f \mid B_R(0))(x) = \begin{cases} f(x) & (|x| \ge R), \\ c_\alpha (R^2 - |x|^2)^{\alpha/2} \int_{|y|>R} f(y)/((|y|^2 - R^2)^{\alpha/2} |x-y|^d)\, \mathrm{d}y & (|x| < R), \end{cases}$$

where $c_\alpha = \pi^{-d/2-1}\Gamma(d/2) \sin \pi\alpha/2$. For a function $\phi : [0, \infty) \to [0, \infty)$ we set

$$h_R^\phi := H_\alpha(\phi \circ |\cdot| B_R(0)).$$

Proposition 2.3. *For all* $0 < |x| < R$

$$h_R^\phi(x) = \frac{\sin \pi\alpha/2}{\pi} \int_0^\infty \phi(\sqrt{R^2 + s(R^2 - |x|^2)}) \frac{\mathrm{d}s}{(s+1)s^{\frac{\alpha}{2}}}.$$

Proof. Let us fix $R > 0$ and $x \in B_R(0)$. Using polar coordinates we obtain

$$h_R^\phi(x) = c_\alpha (R^2 - |x|^2)^{\frac{\alpha}{2}} \int_R^\infty \int_{\rho S^{d-1}} |x-y|^{-d} \sigma(\mathrm{d}y) \frac{\phi(\rho)\, \mathrm{d}\rho}{(\rho^2 - R^2)^{\frac{\alpha}{2}}}. \tag{2-6}$$

By the classical Poisson formula, see [Gilbarg and Trudinger 1998, formula (2.26)],

$$\int_{S^{d-1}} \frac{1 - |w|^2}{|w-y|^d} \sigma(\mathrm{d}y) = |S^{d-1}| \quad (|w| < 1);$$

hence

$$\int_{\rho S^{d-1}} |x-y|^{-d} \sigma(\mathrm{d}y) = \rho^{-1} \int_{S^{d-1}} \left| \frac{x}{\rho} - y \right|^{-d} \sigma(\mathrm{d}y) = \rho^{-1} |S^{d-1}| \left(1 - \frac{|x|^2}{\rho^2}\right)^{-1} = \frac{2\pi^{\frac{d}{2}}}{\Gamma(d/2)} \frac{\rho}{\rho^2 - |x|^2}.$$

Plugging this into (2-6) yields

$$h_R^\phi(x) = \frac{c_\alpha \pi^{\frac{d}{2}}}{\Gamma(d/2)} (R^2 - |x|^2)^{\frac{\alpha}{2}} \int_R^\infty \frac{2\rho\phi(\rho)\, \mathrm{d}\rho}{(\rho^2 - |x|^2)(\rho^2 - R^2)^{\frac{\alpha}{2}}}.$$

The simple substitution $s = (\rho^2 - R^2)/(R^2 - |x|^2)$ leads to

$$\int_R^\infty \frac{2\rho\phi(\rho)\, \mathrm{d}\rho}{(\rho^2 - |x|^2)(\rho^2 - R^2)^{\frac{\alpha}{2}}} = \frac{1}{(R^2 - |x|^2)^{\frac{\alpha}{2}}} \int_0^\infty \phi(\sqrt{R^2 + s(R^2 - |x|^2)}) \frac{ds}{(s+1)s^{\frac{\alpha}{2}}}.$$

Thus the assertion follows. □

[1] We owe the idea of this proof to Wolfhard Hansen.

Theorem 2.2 now follows directly from the following corollary.

Corollary 2.4. *Let $R > 0$ and suppose that ϕ is decreasing on $[R, \infty)$ such that $\phi(s) < \phi(r)$ for some $R < r < s$. Then*

$$h_R^\phi(x) < h_R^\phi(y), \quad \text{whenever } 0 \le x < y < R.$$

In particular, $u := h_R^\phi - h_R^\phi(0)$ is a bounded function on $\mathbb{R}^d$ which is α-harmonic on $B_R(0)$ and satisfies $0 = u(0) < u(y)$ for every $y \in B_R(0)$.

In Theorem 2.1 the function u is assumed to be nonnegative on all of $\mathbb{R}^d$. It is not plausible that the assertion should be false for functions u with small negative values at points far from the origin. A similar question can be asked for classical harmonic functions. If u is positive and large on a large part of ∂B_1, it should not matter for the Harnack inequality on $B_{1/2}$ if u is negative with small absolute values on a small part of ∂B_1. Another motivation for a different formulation of the Harnack inequality is that Theorem 2.1 does not allow us to use Moser's approach to regularity estimates, like Theorem 1.8, in a straightforward manner.

Let us give a new formulation of the Harnack[2] inequality that does not need any sign assumption on u. It is surprising that this formulation seems not to have been established since Harnack's textbook in 1887. We treat the classical local case $\alpha = 2$ together with the nonlocal case $\alpha \in (0, 2)$.

Theorem 2.5. (*Harnack inequality for $\Delta^{\alpha/2}$, $0 < \alpha \le 2$*)

(1) *There is a constant $c \ge 1$ such that for $0 < \alpha \le 2$ and $u \in C(\mathbb{R}^d)$ satisfying*

$$\Delta^{\frac{\alpha}{2}} u(x) = 0 \quad (x \in B_1) \tag{2-7}$$

the following estimate holds for every $x, y \in B_{1/2}$:

$$c\big(u(y) - H_\alpha(u^+ \mid B_1)(y)\big) \le u(x) \le c\big(u(y) + H_\alpha(u^- \mid B_1)(y)\big). \tag{2-8}$$

(2) *There is a constant $c \ge 1$ such that for $0 < \alpha \le 2$ and every function $u \in C(\mathbb{R}^d)$ which satisfies* (2-7) *and is nonnegative in B_1 the following inequality holds for every $x, y \in B_{1/2}$:*

$$u(x) \le c\left(u(y) + \alpha(2-\alpha) \int_{\mathbb{R}^d \setminus B_1} \frac{u^-(z)}{|z|^{d+\alpha}}\, dz\right). \tag{2-9}$$

Proof of Theorem 2.5. The decomposition $u = u^+ - u^-$ and an application of Theorem 2.1 give

$$\begin{aligned} u(x) &= H_\alpha(u \mid B_1)(x) \le H(u^+ \mid B_1)(x) \le c H_\alpha(u^+ \mid B_1)(y) \\ &= c H_\alpha(u \mid B_1)(y) + c H_\alpha(u^- \mid B_1)(y) = c u(y) + c H_\alpha(u^- \mid B_1)(y), \end{aligned}$$

which proves the second inequality in (2-8). The first one is proved analogously.

[2] Kassmann would like to use the opportunity to correct an error in [Kassmann 2007b] concerning the name Harnack. The correct name of the mathematician Harnack is Carl Gustav Axel Harnack. His renowned twin brother Carl Gustav Adolf carried the last name "von Harnack" after being granted the honor.

Inequality (2-9) is proved as follows. Assume u is nonnegative in B_1. Using the same strategy as above we obtain for some $c_1, c_2 > 0$ and $c = \max(c_1, c_2)$

$$\begin{aligned} u(x) &\le c_1 H_\alpha(u \mid B_{\frac{3}{4}})(y) + c_1 H_\alpha(u^- \mid B_{\frac{3}{4}})(y) \\ &\le c_1 u(y) + c_2 \alpha(2-\alpha) \int_{\mathbb{R}^d \setminus B_1} \le cu(y) + c\alpha(2-\alpha) \int_{\mathbb{R}^d \setminus B_1} \frac{u^-(z)}{|z|^{d+\alpha}} \,\mathrm{d}z. \end{aligned}$$

The proof of the theorem is complete. Note that different versions of this result have been announced in [Kassmann 2011]. ☐

Let us make some observations:

(1) There is no assumption on the sign of u needed for (2-8). Inequality (2-8) does hold in the classical case $\alpha = 2$, too.

(2) If u is nonnegative on all of $\mathbb{R}^d$ ($\alpha \in (0,2)$) or nonnegative in B_1 ($\alpha = 2$), then the second inequality in (2-8) reduces to the well-known formulation of the Harnack inequality.

(3) If u is nonnegative in B_1, then (2-9) reduces for $\alpha \to 2$ to the original Harnack inequality.

(4) For the above results, one might want to impose regularity conditions on u such that $\Delta^{\alpha/2} u(x)$ exists at every point $x \in B_1$, e.g., $u|_{B_1} \in C^2(B_1)$ and $u(x)/(1+|x|^{d+\alpha}) \in L^1(\mathbb{R}^d)$. However, the assumption that the integral in (2-1) converges is sufficient.

The proof of Theorem 2.5 does not use the special structure of $\Delta^{\alpha/2}$. The proof only uses the decomposition $u = u^+ - u^-$ and the Harnack inequality for the Poisson kernel. Roughly speaking, it holds for every linear operator that satisfies a maximum principle. One more abstract way of formulating this result in a general framework is as follows:

Lemma 2.6. *Let $(X, \mathcal{W})$ be a balayage space* (see [Bliedtner and Hansen 1986]) *such that $1 \in \mathcal{W}$. Let V, W be open sets in X with $\overline{V} \subset W$. Let $c > 0$. Suppose that, for all $x, y \in V$ and $h \in \mathcal{H}_b^+(V)$,*

$$u(x) \le cu(y). \tag{2-10}$$

Then $\varepsilon_x^{V^c} \le c\varepsilon_y^{V^c}$ and, for every $u \in \mathcal{H}_b(W)$,

$$u(x) \le cu(y) + c \int u^- \,\mathrm{d}\varepsilon_y^{V^c}. \tag{2-11}$$

Here, $\mathcal{H}_b(A)$ denotes the set of bounded functions which are harmonic in the Borel set A. Functions in $\mathcal{H}_b^+(A)$, in addition, are nonnegative.

Proof. Since, for every positive continuous function f with compact support, the mapping $f \mapsto \varepsilon_z^{V^c}(f)$ belongs to $\mathcal{H}_b^+(V)$, the first statement follows. Let $u \in \mathcal{H}_b(W)$. Then $u(x) = \varepsilon_x^{V^c}(u)$, $u(y) = \varepsilon_y^{V^c}(u)$, and hence

$$u(x) \le \varepsilon_x^{V^c}(u^+) \le c\varepsilon_y^{V^c}(u^+) = c\varepsilon_y^{V^c}(u + u^-) = cu(y) + c \int u^- \,\mathrm{d}\varepsilon_y^{V^c}.$$ ☐

3. Functional inequalities and scaling property

In this section we collect several auxiliary results. In particular, we will need some properties of the Sobolev spaces $H^{\alpha/2}(D)$. The following fact about extensions has an elementary proof; see [Di Nezza et al. 2012]. However, one has to go through it and see that the constants do not depend on α, provided one has the factor $(2-\alpha)$ in front of the Gagliardo norm; see (1-4) and (1-6).

Fact 3.1 (extension). Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain, and let $0 < \alpha < 2$. Then there exists a constant $c = c(d, D)$, which is independent of α, and an extension operator $E : H^{\alpha/2}(D) \to H^{\alpha/2}(\mathbb{R}^d)$ with norm $\|E\| \le c$.

Furthermore, we will need the following Poincaré inequality; see [Ponce 2004].

Fact 3.2 (Poincaré I). Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain, and let $0 < \alpha_0 \le \alpha < 2$. Then there exists a constant $c = c(d, \alpha_0, D)$, which is independent of α, such that

$$\left\| u - \frac{1}{|D|}\int_D u \,\mathrm{d}x \right\|^2_{L^2(D)} \le c\,\mathcal{E}^{\mu_\alpha}_D(u,u) \quad (u \in H^{\frac{\alpha}{2}}(D)). \tag{3-1}$$

The following results, Facts 3.3 and 3.4, are standard for fixed α. For $\alpha \to 2$ they follow from results in [Bourgain et al. 2001; Maz'ya and Shaposhnikova 2002; Ponce 2004]. They are established in the case when $B_r(x)$ denotes the cube of all $y \in \mathbb{R}^d$ such that $|y_i - x_i| < r$ for any $i \in \{1, \dots, d\}$. They hold true for balls likewise.

Fact 3.3 (Poincaré–Friedrichs). Assume $\alpha_0, \varepsilon > 0$ and $0 < \alpha_0 \le \alpha < 2$. There exists a constant c, which is independent of α, such that for $B_R = B_R(x_0)$

$$u \in H^{\frac{\alpha}{2}}(B_R), \quad |B_R \cap \{u = 0\}| \ge \varepsilon |B_R|$$

implies

$$\int_{B_R} (u(x))^2 \,\mathrm{d}x \le c R^\alpha \iint_{B_R B_R} \frac{(u(y)-u(x))^2}{|x-y|^{d+\alpha}} \,\mathrm{d}y\,\mathrm{d}x. \tag{3-2}$$

Fact 3.4 (Sobolev embedding). Assume $d \in \mathbb{N}$, $d \ge 2$, $R_0 > 0$, and $0 < \alpha_0 \le \alpha < 2$, $q \in [1, 2d/(d-\alpha)]$. Then there exists a constant c, which is independent of α, such that for $R \in (0, R_0)$ and $u \in H^{\alpha/2}(B_R)$

$$\left(\int_{B_R} |u(x)|^{\frac{2d}{d-\alpha}} \,\mathrm{d}x\right)^{\frac{d-\alpha}{d}} \le c \iint_{B_R B_R} \frac{(u(y)-u(x))^2}{|x-y|^{d+\alpha}} \,\mathrm{d}y\,\mathrm{d}x + c R^{-\alpha + \frac{d(q-2)}{q}} \left(\int_{B_R} |u(x)|^q \,\mathrm{d}x\right)^{\frac{2}{q}}.$$

We often make use of scaling and translations. Our main assumptions, conditions (A) and (B) assure a certain behavior of the family of measures μ with respect to the unit ball $B_1 \subset \mathbb{R}^d$. Let us formulate these conditions with respect to general balls $B_r(\xi) \subset \mathbb{R}^d$.

Given $\xi \in \mathbb{R}^d$, $r > 0$, $A \ge 1$, we say that μ satisfies (A; ξ, r) if:

For every ball $B_\rho(x_0)$ with $\rho \in (0, r)$, $x_0 \in B_r(\xi)$ and every $v \in H^{\alpha/2}(B_\rho(x_0))$,

$$A^{-1}\,\mathcal{E}^{\mu}_{B_\rho(x_0)}(v,v) \le \mathcal{E}^{\mu_\alpha}_{B_\rho(x_0)}(v,v) \le A\,\mathcal{E}^{\mu}_{B_\rho(x_0)}(v,v). \tag{A; ξ, r}$$

Given $\xi \in \mathbb{R}^d$, $r > 0$, $B \geq 1$, we say that μ satisfies (B; ξ, r) if:

$$\begin{aligned}&\text{For } 0 < \rho \leq R \leq r \text{ and } x_0 \in B_r(\xi) \text{ there is a nonnegative measurable function } \tau : \mathbb{R}^d \to \mathbb{R}\\ &\quad \text{with } \operatorname{supp}(\tau) \subset \overline{B_{R+\rho}(x_0)},\ \tau(x) \equiv 1 \text{ on } B_R(x_0),\ \|\tau\|_\infty \leq 1, \text{ and}\\ &\quad \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} (\tau(y) - \tau(x))^2\, \mu(x, \mathrm{d}y) \leq B\rho^{-\alpha}.\end{aligned} \tag{B; ξ, r}$$

Let us explain how the operator under consideration behaves with respect to rescaled functions.

Lemma 3.5 (scaling property). *Assume $\xi \in \mathbb{R}^d$ and $r \in (0, 1)$. Let $u \in V^\mu(B_r(\xi) \mid \mathbb{R}^d)$ satisfy $\mathcal{E}^\mu(u, \phi) \geq (f, \phi)$ for every nonnegative $\phi \in H^\mu_{B_r(\xi)}(\mathbb{R}^d)$. Define a diffeomorphism J by $J(x) = rx + \xi$. Define rescaled versions $\tilde{f}$, $\tilde{u}$ of u and f by $\tilde{u}(x) = u(J(x))$ and $\tilde{f}$ by $\tilde{f}(x) = r^\alpha f(J(x))$.*

(1) *Then $\tilde{u}$ satisfies for all nonnegative $\phi \in H^{\tilde{\mu}}_{B_1}(\mathbb{R}^d)$*

$$\mathcal{E}^{\tilde{\mu}}(\tilde{u}, \phi) = \iint_{\mathbb{R}^d\,\mathbb{R}^d} (\tilde{u}(y) - \tilde{u}(x))(\phi(y) - \phi(x))\, \tilde{\mu}(x, \mathrm{d}y)\, \mathrm{d}x \geq (\tilde{f}, \phi),$$

where

$$\tilde{\mu}(x, \mathrm{d}y) = r^\alpha \mu_{J^{-1}}(J(x), \mathrm{d}y) \quad \textit{and} \quad \mu_{J^{-1}}(z, A) = \mu(z, J(A)). \tag{3-3}$$

(2) *Assume μ satisfies conditions* (A; ξ, r), (B; ξ, r) *for some $\alpha \in (0, 2)$ and $A \geq 1$, $B \geq 1$, $\xi \in \mathbb{R}^d$, $r > 0$. Then the family of measures $\tilde{\mu} = \tilde{\mu}(\cdot, \mathrm{d}y)$ satisfies assumptions* (A) *and* (B) *with the same constants.*

Remark. The condition (D) is affected by scaling in a noncritical way. We deal with this phenomenon further below in Section 4 and 5A.

Proof. For the proof of the first statement, let $\phi \in H^{\tilde{\mu}}_{B_1}(\mathbb{R}^d)$ be a nonnegative test function. Define $\phi_r \in H^\mu_{B_r(\xi)}(\mathbb{R}^d)$ by $\phi_r = \phi \circ J^{-1}$. Then

$$\begin{aligned}&\iint (\tilde{u}(y) - \tilde{u}(x))(\phi(y) - \phi(x))\, \tilde{\mu}(x, \mathrm{d}y)\, \mathrm{d}x\\ &\qquad = r^\alpha \iint (u(J(y)) - u(J(x)))(\phi_r(J(y)) - \phi_r(J(x)))\, \mu_{J^{-1}}(J(x), \mathrm{d}y)\, \mathrm{d}x\\ &\qquad = r^{\alpha-d} \iint (u(J(y)) - u(x))(\phi_r(J(y)) - \phi_r(x))\, \mu_{J^{-1}}(x, \mathrm{d}y)\, \mathrm{d}x\\ &\qquad = r^{\alpha-d} \iint (u(y) - u(x))(\phi_r(y) - \phi_r(x))\, \mu(x, \mathrm{d}y)\, \mathrm{d}x\\ &\qquad \geq r^{\alpha-d} \int f(x)\, \phi_r(x)\, \mathrm{d}x = \int r^\alpha f(J(x))\, \phi(x)\, \mathrm{d}x = \int \tilde{f}(x)\phi(x)\, \mathrm{d}x,\end{aligned} \tag{3-4}$$

which is what we wanted to prove. Let us now prove that $\tilde{\mu}$ inherits properties (A), (B) from μ with the same constants A and B. Let us only consider the case $\xi = 0$. In order to verify condition (A) we need to consider an arbitrary ball $B_\rho(x_0)$ with $\rho \in (0, 1)$ and $x_0 \in B_1$. Let us simplify the situation further by assuming $x_0 = 0$. The general case can be proved analogously. Thus, we assume $r \in (0, 1)$ and $u \in H^{\alpha/2}(B_\rho)$. The estimate $\mathcal{E}^{\tilde{\mu}}_{B_\rho}(u, u) \leq A\mathcal{E}^{\mu_\alpha}_{B_\rho}(u, u)$ can be derived as follows. Define a function

$\hat{u} \in H^{\alpha/2}(B_{r\rho})$ by $\hat{u} = u \circ J^{-1}$. Then

$$\begin{aligned}
\mathcal{E}^{\tilde{\mu}}_{B_\rho}(u,u) &= \int_{B_\rho}\int_{B_\rho} (u(y)-u(x))^2\, \tilde{\mu}(x,\mathrm{d}y)\,\mathrm{d}x = r^\alpha \int_{B_\rho}\int_{B_\rho} (\hat{u}(J(y))-\hat{u}(J(x)))^2\, \mu_{J^{-1}}(J(x),\mathrm{d}y)\,\mathrm{d}x \\
&= r^{\alpha-d} \int_{B_{r\rho}}\int_{B_r} (\hat{u}(J(y))-\hat{u}(x))^2\, \mu_{J^{-1}}(x,\mathrm{d}y)\,\mathrm{d}x \\
&= r^{\alpha-d} \int_{B_{r\rho}}\int_{B_{r\rho}} (\hat{u}(y)-\hat{u}(x))^2\, \mu(x,\mathrm{d}y)\,\mathrm{d}x \le A r^{\alpha-d} \int_{B_{r\rho}}\int_{B_{r\rho}} \frac{(\hat{u}(y)-\hat{u}(x))^2}{|x-y|^{d+\alpha}}\,\mathrm{d}y\,\mathrm{d}x \\
&= A r^{-2d} \int_{B_{r\rho}}\int_{B_{r\rho}} \frac{(u(J^{-1}(y))-u(J^{-1}(x)))^2}{|J^{-1}(x)-J^{-1}(y)|^{d+\alpha}}\mathrm{d}y\,\mathrm{d}x = A \int_{B_\rho}\int_{B_\rho} \frac{(u(y)-u(x))^2}{|x-y|^{d+\alpha}}\,\mathrm{d}y\,\mathrm{d}x,
\end{aligned}$$

which proves our claim. The estimate $\mathcal{E}^{\mu_\alpha}_{B_\rho}(u,u) \le A\, \mathcal{E}^{\tilde{\mu}}_{B_\rho}(u,u)$ follows in the same way.

In order to check condition (B) for $\tilde{\mu}$ we proceed as follows. Again, we assume $x_0 = 0$, $r \in (0,1)$. The general case can be proved analogously. Assume $R, \rho \in (0,1)$. Let $\hat{\tau} : \mathbb{R}^d \to \mathbb{R}$ satisfy $\operatorname{supp}(\hat{\tau}) \subset \overline{B}_{rR+r\rho}$, $\hat{\tau} \equiv 1$ on B_{rR} and

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} (\hat{\tau}(y)-\hat{\tau}(x))^2\, \mu(x,\mathrm{d}y) \le B(r\rho)^{-\alpha} \quad\iff\quad \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} (\hat{\tau}(y)-\hat{\tau}(J(x)))^2\, \mu(J(x),\mathrm{d}y) \le B(r\rho)^{-\alpha}.$$

Such a function $\hat{\tau}$ exists because, by assumption, μ satisfies (B; ξ, r). Next, define $\tau = \hat{\tau} \circ J$. Then τ satisfies $\operatorname{supp}(\tau) \subset \overline{B}_{R+\rho}$, $\tau \equiv 1$ on B_R, and, by a change of variables,

$$\begin{aligned}
\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} (\tau(y) - \tau(x))^2\, \tilde{\mu}(x,\mathrm{d}y) &= r^\alpha \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} (\hat{\tau}(J(y)) - \hat{\tau}(J(x)))^2\, \mu_{J^{-1}}(J(x),\mathrm{d}y) \\
&= r^\alpha \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} (\hat{\tau}(y) - \hat{\tau}(J(x)))^2\, \mu(J(x),\mathrm{d}y) \le B\rho^{-\alpha},
\end{aligned}$$

which shows that $\tilde{\mu}$ satisfies (B) with the constant B. □

4. The weak Harnack inequality for nonlocal equations

The main aim of this section is to provide a proof of the weak Harnack inequality Theorem 1.6. The key result of this section is the corresponding result for supersolutions that are nonnegative in all of $\mathbb{R}^d$:

Theorem 4.1. *Assume $f : B_1 \to \mathbb{R}$ belongs to $L^{q/\alpha}(B_{15/16})$ for some $q \in (d,\infty]$, $\alpha \in [\alpha_0, 2)$. There are positive reals p_0, c such that for every $u \in V^\mu(B_1 \mid \mathbb{R}^d)$ with $u \ge 0$ in $\mathbb{R}^d$ satisfying*

$$\mathcal{E}(u,\phi) \ge (f,\phi) \quad \textit{for every nonnegative } \phi \in H^\mu_{B_1}(\mathbb{R}^d).$$

The following holds:

$$\inf_{B_{1/4}} u \ge c\left(\fint_{B_{1/2}} u(x)^{p_0}\,\mathrm{d}x\right)^{\frac{1}{p_0}} - \|f\|_{L^{q/\alpha}(B_{15/16})}.$$

The constants p_0, c depend only on d, α_0, A, B. They are independent of $\alpha \in [\alpha_0, 2)$.

Remark. All results in this section are robust with respect to $\alpha \in [\alpha_0, 2)$; i.e., constants do not depend on α.

The main application of this result is the following proof.

Proof of Theorem 1.6. Set $u = u^+ - u^-$. The assumptions imply for any nonnegative $\phi \in H^{\mu}_{B_1}(\mathbb{R}^d)$

$$\mathcal{E}(u^+, \phi) \geq \mathcal{E}(u^-, \phi) + (f, \phi) = \int_{B_1} \phi(x) \Big(f(x) - 2 \int_{\mathbb{R}^d \setminus B_1} u^-(y)\, \mu(x, \mathrm{d}y) \Big)\, \mathrm{d}x;$$

i.e., u^+ satisfies all assumptions of Theorem 4.1 with $q = +\infty$ and $\tilde{f} : B_1 \to \mathbb{R}$ defined by

$$\tilde{f}(x) = f(x) - 2 \int_{\mathbb{R}^d \setminus B_1} u^-(y)\, \mu(x, \mathrm{d}y).$$

The assertion of the theorem is true if $\sup_{x \in B_{15/16}} \int_{\mathbb{R}^d \setminus B_1} u^-(y)\, \mu(x, \mathrm{d}y)$ is infinite. Thus we can assume this quantity to be finite. Theorem 4.1 now implies

$$\inf_{B_{1/4}} u \geq c_1 \Big(\fint_{B_{1/2}} u(x)^{p_0}\, \mathrm{d}x \Big)^{\frac{1}{p_0}} - c_2 \sup_{x \in B_{15/16}} \Big(\int_{\mathbb{R}^d \setminus B_1} u^-(y)\, \mu(x, \mathrm{d}y) \Big) - \|f\|_{L^{q/\alpha}(B_{15/16})}$$

for some positive constants c_1, c_2. □

By scaling and translation, we obtain the following corollary.

Corollary 4.2. *Let $x_0 \in \mathbb{R}^d$, $R \in (0, 1)$. Assume μ is a family of measures satisfying* (A; ξ, r) *and* (B; ξ, r). *Assume $u \in V^{\mu}(B_R(x_0) \mid \mathbb{R}^d)$ satisfies $u \geq 0$ in $B_R(x_0)$ and $\mathcal{E}(u, \phi) \geq 0$ for every nonnegative $\phi \in H^{\mu}_{B_R(x_0)}(\mathbb{R}^d)$. Then*

$$\inf_{B_{R/4}(x_0)} u \geq c \Big(\fint_{B_{R/2}(x_0)} u(x)^{p_0}\, \mathrm{d}x \Big)^{\frac{1}{p_0}} - R^{\alpha} \sup_{x \in B_{15R/16}(x_0)} \int_{\mathbb{R}^d \setminus B_R(x_0)} u^-(y)\, \mu(x, \mathrm{d}y),$$

with positive constants p_0, c which depend only on d, α_0, A, B. In particular, they are independent of $\alpha \in [\alpha_0, 2)$.

Let us proceed to the proof of Theorem 4.1.

Remark. Without further mentioning we assume that μ is a family of measures that satisfies (A) and (B) for some $A \geq 1$, $B \geq 1$, and $\alpha_0 \leq \alpha < 2$. The constants in the assertions below depend, among other things, on A, B, and α_0. They do not depend on α, though.

Let us first establish several auxiliary results. Our approach is closely related to the approach in [Kassmann 2009]. Instead of Lemma 2.5 in that paper, which would be sufficient for homogeneous equations,we will use the following auxiliary result.

Lemma 4.3. *There exist positive constants $c_1, c_2 > 0$ such that for every $a, b > 0$, $p > 1$, and $0 \leq \tau_1, \tau_2 \leq 1$ the following is true*:

$$(b - a)(\tau_1^2 a^{-p} - \tau_2^2 b^{-p}) \geq c_1 (\tau_1 a^{\frac{-p+1}{2}} - \tau_2 b^{\frac{-p+1}{2}})^2 - \frac{c_2 p}{p - 1} (\tau_1 - \tau_2)^2 (b^{-p+1} + a^{-p+1}). \quad (4\text{-}1)$$

The above result is nothing but a discrete version of

$$(\nabla v, \nabla(\tau^2 v^{-p})) \geq c_1(p)|\nabla(\tau v^{\frac{-p+1}{2}})|^2 - c_2(p)|\nabla\tau|^2 v^{-p+1},$$

where v, τ are positive functions. We provide a detailed proof in the Appendix.

The next result is an extension of corresponding results in [Kassmann 2009; Barlow et al. 2009].

Lemma 4.4. *Assume $0 < \rho < r < 1$ and $z_0 \in B_1$. Set $B_r = B_r(z_0)$. Assume $f \in L^{q/\alpha}(B_{2r})$ for some $q > d$. Assume $u \in V^\mu(B_{2r} \mid \mathbb{R}^d)$ is nonnegative in $\mathbb{R}^d$ and satisfies*

$$\begin{aligned} \mathcal{E}(u,\phi) &\geq (f,\phi) \quad \text{for any nonnegative } \phi \in H^\mu_{B_{2r}}(\mathbb{R}^d), \\ u(x) &\geq \varepsilon \qquad\quad \text{for almost all } x \in B_{2r} \text{ and some } \varepsilon > 0. \end{aligned} \tag{4-2}$$

Then

$$\iint_{B_r B_r} \Bigg(\sum_{k=1}^\infty \frac{(\log u(y) - \log u(x))^{2k}}{(2k)!}\Bigg)\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \leq c\rho^{-\alpha}|B_{r+\rho}| + \varepsilon^{-1}\|f\|_{L^{q/\alpha}(B_{r+\rho})}\|\mathbb{1}\|_{L^{q/(q-\alpha)}(B_{r+\rho})}, \tag{4-3}$$

where $c > 0$ is independent of u, x_0, r, ρ, f, ε, α.

Note that for

$$\varepsilon \geq c_1(r+\rho)^\delta \|f\|_{L^{q/\alpha}(B_{r+\rho})},$$

with $\delta = \alpha((q-d)/q)$, one obtains

$$\iint_{B_r B_r} \Bigg(\sum_{k=1}^\infty \frac{(\log u(y) - \log u(x))^{2k}}{(2k)!}\Bigg)\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \leq c_2\rho^{-\alpha}|B_{r+\rho}|. \tag{4-4}$$

From the above lemma we will deduce $\log u \in \mathrm{BMO}(B_1)$, where $\mathrm{BMO}(B_1)$ contains all functions of bounded mean oscillations in B_1; see [John and Nirenberg 1961].

Proof. The proof uses several ideas developed in [Barlow et al. 2009]. Let $\tau : \mathbb{R}^d \to \mathbb{R}$ be a function according to (B); i.e., more precisely we assume

$$\operatorname{supp}(\tau) \subset \overline{B}_{r+\rho} \subset B_{2r}, \quad \|\tau\|_\infty \leq 1, \quad \tau \equiv 1 \text{ on } B_r, \quad \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} (\tau(y)-\tau(x))^2\,\mu(x,\mathrm{d}y) \leq B\rho^{-\alpha}.$$

Then

$$\begin{aligned} \iint_{\mathbb{R}^d \mathbb{R}^d} &(\tau(y)-\tau(x))^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\ &= \iint_{B_{r+\rho} B_{r+\rho}} (\tau(y)-\tau(x))^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x + 2\iint_{B_{r+\rho} B^c_{r+\rho}} (\tau(y)-\tau(x))^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\ &\leq 2\iint_{B_{r+\rho}\mathbb{R}^d} (\tau(y)-\tau(x))^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\ &\leq 2|B_{r+\rho}| \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} (\tau(y)-\tau(x))^2\,\mu(x,\mathrm{d}y) \leq 2c\rho^{-\alpha}|B_{r+\rho}|. \end{aligned} \tag{4-5}$$

We choose $\phi(x) = -\tau^2(x)\, u^{-1}(x)$ as a test function. Denote $B_{r+\rho}$ by B. We obtain

$$\begin{aligned}
(f,\phi) &\geq \iint_{\mathbb{R}^d\,\mathbb{R}^d} (u(y)-u(x))(\tau^2(x)u^{-1}(x)-\tau^2(y)u^{-1}(y))\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&= \iint_{BB} \tau(x)\,\tau(y)\Big(\frac{\tau(x)\,u(y)}{\tau(y)\,u(x)}+\frac{\tau(y)\,u(x)}{\tau(x)\,u(y)}-\frac{\tau(y)}{\tau(x)}-\frac{\tau(x)}{\tau(y)}\Big)\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\quad + 2\iint_{BB^c} (u(y)-u(x))(\tau^2(x)\,u^{-1}(x)-\tau^2(y)\,u^{-1}(y))\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\quad + \iint_{B^cB^c} (u(y)-u(x))(\tau^2(x)\,u^{-1}(x)-\tau^2(y)\,u^{-1}(y))\,\mu(x,\mathrm{d}y)\,\mathrm{d}x. \qquad (4\text{-}6)
\end{aligned}$$

Setting $A(x,y) = u(y)/u(x)$ and $B(x,y) = \tau(y)/\tau(x)$ we obtain

$$\begin{aligned}
&\iint_{BB} \tau(x)\,\tau(y)\Big(\frac{A(x,y)}{B(x,y)}+\frac{B(x,y)}{A(x,y)}-B(x,y)-\frac{1}{B(x,y)}\Big)\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\quad= \iint_{BB} \tau(x)\,\tau(y)\Big[\Big(\frac{A(x,y)}{B(x,y)}+\frac{B(x,y)}{A(x,y)}-2\Big)-\Big(\sqrt{B(x,y)}-\frac{1}{\sqrt{B(x,y)}}\Big)^2\Big]\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\quad= \iint_{BB} \tau(x)\,\tau(y)\Big(2\sum_{k=1}^{\infty}\frac{(\log A(x,y)-\log B(x,y))^{2k}}{(2k)!}\Big)\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\qquad\qquad - \iint_{BB} \tau(x)\,\tau(y)\Big(\sqrt{B(x,y)}-\frac{1}{\sqrt{B(x,y)}}\Big)^2\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\quad= \iint_{BB} \tau(x)\,\tau(y)\Big(2\sum_{k=1}^{\infty}\frac{(\log(u(y)/\tau(y))-\log(u(x)/\tau(x)))^{2k}}{(2k)!}\Big)\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\qquad\qquad - \iint_{BB} (\tau(x)-\tau(y))^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\quad\geq \int_{B_r}\int_{B_r}\Big(2\sum_{k=1}^{\infty}\frac{(\log u(y)-\log u(x))^{2k}}{(2k)!}\Big)\,\mu(x,\mathrm{d}y)\,\mathrm{d}x - \iint_{BB}(\tau(x)-\tau(y))^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x,
\end{aligned}$$

where we applied (4-5) and the fact that for positive real a, b

$$\frac{(a-b)^2}{ab} = (a-b)(b^{-1}-a^{-1}) = 2\sum_{k=1}^{\infty}\frac{(\log a-\log b)^{2k}}{(2k)!}. \qquad (4\text{-}7)$$

Altogether, we obtain

$$\begin{aligned}
(f,\phi) \geq& \int_{B_r}\int_{B_r}\Big(2\sum_{k=1}^{\infty}\frac{(\log u(y)-\log u(x))^{2k}}{(2k)!}\Big)\,\mu(x,\mathrm{d}y)\,\mathrm{d}x - \iint_{BB}(\tau(x)-\tau(y))^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&+ 2\iint_{B_{r+\rho}B^c_{r+\rho}} (u(y)-u(x))(\tau^2(x)u^{-1}(x)-\tau^2(y)u^{-1}(y))\,\mu(x,\mathrm{d}y)\,\mathrm{d}x. \qquad (4\text{-}8)
\end{aligned}$$

The third term on the right-hand side can be estimated as follows:

$$
\begin{aligned}
2 \iint_{B_{r+\rho} B^c_{r+\rho}} &(u(y)-u(x))(\tau^2(x)u^{-1}(x)-\tau^2(y)u^{-1}(y))\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&= 2 \iint_{B_{r+\rho} B^c_{r+\rho}} (u(y)-u(x))\tau^2(x)u^{-1}(x)\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&= 2 \int_{B_{r+\rho}} \int_{B^c_{r+\rho}} \frac{\tau^2(x)}{u(x)} u(y)\,\mu(x,\mathrm{d}y)\,\mathrm{d}x - 2 \int_{B_{r+\rho}} \int_{B^c_{r+\rho}} \tau^2(x)\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\geq -2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\tau(y)-\tau(x))^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x,
\end{aligned}
$$

where we used nonnegativity of u in $\mathbb{R}^d$. Therefore,

$$
\begin{aligned}
\int_{B_r} \int_{B_r} &\left(2 \sum_{k=1}^{\infty} \frac{(\log u(y)-\log u(x))^{2k}}{(2k)!}\right) \mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\leq 3 \iint_{\mathbb{R}^d \mathbb{R}^d} (\tau(x)-\tau(y))^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x + \|f\|_{L^{q/\alpha}(B_{r+\rho})} \|u^{-1}\|_{L^{q/(q-\alpha)}(B_{r+\rho})}. \qquad \text{(4-9)}
\end{aligned}
$$

The proof is complete after the trivial observation $|u^{-1}| \leq \varepsilon^{-1}$. □

Lemma 4.5. *Assume* $0 < R < 1$ *and* $f \in L^{q/\alpha}(B_{5R/4})$ *for some* $q > d$. *Assume* $u \in V^{\mu}(B_{5R/4} \,|\, \mathbb{R}^d)$ *is nonnegative in* $\mathbb{R}^d$ *and satisfies*

$$
\begin{aligned}
&\mathcal{E}(u,\phi) \geq (f,\phi) && \text{for any nonnegative } \phi \in H^{\mu}_{B_{5R/4}}(\mathbb{R}^d), \\
&u(x) \geq \varepsilon && \text{for almost all } x \in B_{\frac{5R}{4}} \text{ and some } \varepsilon > \tfrac{1}{4} R^{\delta} \|f\|_{L^{q/\alpha}(B_{9R/8})},
\end{aligned}
$$

where

$$
\delta = \alpha\left(\frac{q-d}{q}\right).
$$

Then there exist $\bar{p} \in (0,1)$ *and* $c > 0$ *such that*

$$
\left(\fint_{B_R} u(x)^{\bar{p}}\,\mathrm{d}x\right)^{\frac{1}{\bar{p}}} \mathrm{d}x \leq c \left(\fint_{B_R} u(x)^{-\bar{p}}\,\mathrm{d}x\right)^{-\frac{1}{\bar{p}}}, \qquad \text{(4-10)}
$$

where c *and* $\bar{p}$ *are independent of* x_0, R, u, ε, *and* α.

Proof. The main idea is to prove $\log u \in \mathrm{BMO}(B_R)$. Choose $z_0 \in B_R$ and $r > 0$ such that $B_{2r}(z_0) \subset B_{R/8}$. Set $\rho = r$. Lemma 4.4 and Assumption (A) imply

$$
\int_{B_r(z_0)} \int_{B_r(z_0)} \frac{(\log u(y) - \log u(x))^2}{|x-y|^{d+\alpha}}\,\mathrm{d}y\,\mathrm{d}x \leq \int_{B_r(z_0)} \int_{B_r(z_0)} (\log u(y) - \log u(x))^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \leq c_1 r^{d-\alpha}.
$$

Application of the Poincaré inequality, Fact 3.2, and the scaling property 3.3 leads to

$$
\int_{B_r(z_0)} |\log u(x) - [\log u]_{B_r(z_0)}|^2\,\mathrm{d}x \leq c_2 r^d, \qquad \text{(4-11)}
$$

where

$$[\log u]_{B_r(z_0)} = |B_r(z_0)|^{-1} \int_{B_r(z_0)} \log u = \fint_{B_r(z_0)} \log u.$$

From here

$$\int_{B_r(z_0)} |\log u(x) - [\log u]_{B_r(z_0)}| \,\mathrm{d}x \le \left(\int_{B_r(z_0)} |\log u(x) - [\log u]_{B_r(z_0)}|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} |B_r(z_0)|^{\frac{1}{2}} \le c_3 r^d.$$

An application of the John–Nirenberg embedding, see [Gilbarg and Trudinger 1998, Chapter 7.8], then gives

$$\int_{B_R} e^{\bar{p}|\log u(y) - [\log u]_{B_r}|} \,\mathrm{d}y \le c_4 R^d,$$

where $\bar{p}$ and c_4 depend only on d and c_3. One obtains

$$\left(\int_{B_R} u(y)^{\bar{p}} \,\mathrm{d}y \right) \left(\int_{B_R} u(y)^{-\bar{p}} \,\mathrm{d}y \right) = \left(\int_{B_R} e^{\bar{p}(\log u(y) - [\log u]_{B_r})} \,\mathrm{d}y \right) \left(\int_{B_R} e^{-\bar{p}(\log u(y) - [\log u]_{B_r})} \,\mathrm{d}y \right)$$
$$\le c_4^2 R^{2d}.$$

The above inequality proves assertion (4-10). □

The next result allows us to apply Moser's iteration for negative exponents. It is a purely local result although the Dirichlet form is nonlocal.

Lemma 4.6. *Assume $x_0 \in B_1$ and $0 < 8\rho < R < 1 - \rho$. Set $B_R = B_R(x_0)$. Let $f \in L^{q/\alpha}(B_{5R/4})$ for some $q > d$. Assume $u \in V^{\mu}(B_{5R/4} \,|\, \mathbb{R}^d)$ is nonnegative on all of $\mathbb{R}^d$ and satisfies*

$$\mathcal{E}(u, \phi) \ge (f, \phi) \quad \text{for any nonnegative } \phi \in H^{\mu}_{B_{5R/4}}(\mathbb{R}^d),$$
$$u(x) \ge \varepsilon \quad \text{for almost all } x \in B_{\frac{9R}{8}} \text{ and some } \varepsilon > R^{\delta} \|f\|_{L^{q/\alpha}(B_{9R/8})},$$

where

$$\delta = \alpha \left(\frac{q-d}{q} \right).$$

Then for $p > 1$

$$\|u^{-1}\|^{p-1}_{L^{(p-1)d/(d-\alpha)}(B_R)} \le c \left(\frac{p}{p-1} \right) \rho^{-\alpha} \|u^{-1}\|^{p-1}_{L^{p-1}(B_{R+\rho})}, \tag{4-12}$$

where $c > 0$ is independent of u, x_0, R, ρ, p, ε, and α.

Proof. Let $\tau : \mathbb{R}^d \to \mathbb{R}$ be a function according to assumption (B); i.e.,

$$S := \operatorname{supp}(\tau) \subset \overline{B}_{R+\rho} \subset B_{\frac{9R}{8}},$$
$$\|\tau\|_{\infty} \le 1 \quad \text{for all } x \in B_R \text{ such that } \tau(x) = 1,$$
$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} (\tau(y) - \tau(x))^2 \,\mu(x, \mathrm{d}y) \le B\rho^{-\alpha}.$$

The assumptions of the lemma imply

$$\mathcal{E}(u, -\tau^2 u^{-p}) \le (f, -\tau^2 u^{-p}).$$

Let us observe the following:

$$
\begin{aligned}
\mathcal{E}(u,-\tau^2u^{-p}) &= \iint (u(y)-u(x))(\tau^2(x)\,u(x)^{-p}-\tau^2(y)\,u(y)^{-p})\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&= \int_S\int_S (u(y)-u(x))(\tau^2(x)\,u(x)^{-p}-\tau^2(y)\,u(y)^{-p})\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\qquad + 2\int_S\int_{\mathbb{R}^d\setminus S}(u(y)-u(x))(\tau(x)-\tau(y))^2\,u(x)^{-p}\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\geq \int_S\int_S (u(y)-u(x))(\tau^2(x)\,u(x)^{-p}-\tau^2(y)\,u(y)^{-p})\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\qquad - 2\int_S u(x)^{1-p}\int_{\mathbb{R}^d\setminus S}(\tau(x)-\tau(y))^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x.
\end{aligned}
$$

The last term is finite because of our assumptions on τ. However, note that $\tau(y)=0$ for $y\in\mathbb{R}^d\setminus S$. Next, we choose $a=u(x),\ b=u(y),\ \tau_1=\tau(x),\ \tau_2=\tau(y)$, and apply Lemma 4.3 to the integrand in the first term. Then

$$
\begin{aligned}
&\iint_{S\,S}(\tau(y)\,u(y)^{\frac{-p+1}{2}}-\tau(x)\,u(x)^{\frac{-p+1}{2}})^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\quad\leq \frac{cp}{p-1}\iint_{S\,S}(\tau(y)-\tau(x))^2(u(y)^{-p+1}+u(x)^{-p+1})\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\qquad\qquad + 2\int_S u(x)^{-p+1}\int_{\mathbb{R}^d\setminus S}(\tau(x)-\tau(y))^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x + c(f,-\tau^2u^{-p}) \\
&\quad\leq \left(\frac{2cp}{p-1}+2\right)\int_S u(x)^{-p+1}\int_{\mathbb{R}^d}(\tau(x)-\tau(y))^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x + c(f,-\tau^2u^{-p}) \\
&\quad\leq c_1(p)\rho^{-\alpha}\int_{B_{R+\rho}} u(x)^{-p+1}\,\mathrm{d}x + c(f,-\tau^2u^{-p})
\end{aligned}
\tag{4-13}
$$

for some positive constant c, which is independent of p, R, f and u. It remains to estimate $|(f,-\tau^2u^{-p})|$ from above:

$$
\begin{aligned}
|(f,-\tau^2u^{-p})| &\leq \varepsilon^{-1}|(f,\tau^2u^{-p+1})| \leq \varepsilon^{-1}\|f\|_{L^{q/\alpha}(B_{9R/8})}\|\tau^2u^{-p+1}\|_{L^{q/(q-\alpha)}(B_{9R/8})} \\
&= \varepsilon^{-1}\|f\|_{L^{q/\alpha}(B_{9R/8})}\|\tau u^{\frac{-p+1}{2}}\|^2_{L^{2q/(q-\alpha)}(B_{9R/8})} \\
&\leq \varepsilon^{-1}\|f\|_{L^{q/\alpha}(B_{9R/8})}\{a\|\tau u^{\frac{-p+1}{2}}\|^2_{L^{2d/(d-\alpha)}(B_{9R/8})} + a^{-\frac{d}{q-d}}\|\tau u^{\frac{-p+1}{2}}\|^2_{L^2(B_{9R/8})}\} \\
&\leq R^{-\alpha\frac{q-d}{q}}a\|\tau^2u^{-p+1}\|_{L^{d/(d-\alpha)}(B_{9R/8})} + R^{-\alpha\frac{q-d}{q}}a^{-\frac{d}{q-d}}\|\tau^2u^{-p+1}\|_{L^1(B_{9R/8})},
\end{aligned}
$$

where $a>0$ is arbitrary. We choose $a=\omega R^{\alpha(q-d)/q}$ for some ω and obtain

$$
|(f,-\tau^2u^{-p})| \leq \omega\|\tau^2u^{-p+1}\|_{L^{d/(d-\alpha)}(B_{9R/8})} + \omega^{-\frac{d}{q-d}}R^{-\alpha}\|\tau^2u^{-p+1}\|_{L^1(B_{9R/8})}.
$$

Combining these estimates we obtain from (4-13) for every $p>1$ and every $\omega>0$

$$
\begin{aligned}
&\iint_{S\,S}\big(\tau(y)\,u(y)^{\frac{-p+1}{2}}-\tau(x)\,u(x)^{\frac{-p+1}{2}}\big)^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\qquad\qquad\leq c_3(p,\omega)\rho^{-\alpha}\int_{B_{R+\rho}} u(x)^{-p+1}\,\mathrm{d}x + c\omega\|\tau^2u^{-p+1}\|_{L^{d/(d-\alpha)}(B_{R+\rho})}.
\end{aligned}
$$

Next, we use Assumption (A) and apply the Sobolev inequality, Fact 3.4, to the left-hand side. Choosing ω small enough and subtracting the term $c\omega\|\tau^2 u^{-p+1}\|_{L^{d/(d-\alpha)}(B_{R+\rho})}$ from both sides, we prove the assertion of the lemma. ☐

Lemma 4.6 provides us with an estimate which can be iterated. As a result of this iteration we obtain the following corollary.

Corollary 4.7. *Assume $x_0 \in B_1$, $0 < R < \frac{1}{2}$, and $0 < \eta < 1 < \Theta$. Let $f \in L^{q/\alpha}(B_{\Theta R})$ for some $q > d$. Assume $u \in V^{\mu}(B_{\Theta R} \,|\, \mathbb{R}^d)$ satisfies*

$$\begin{aligned} \mathcal{E}(u,\phi) &\ge (f,\phi) \quad \text{for any nonnegative } \phi \in H^{\mu}_{B_{\Theta R}}(\mathbb{R}^d), \\ u(x) &\ge \varepsilon \quad \text{for almost all } x \in B_{\Theta R} \text{ and some } \varepsilon > (\Theta R)^{\delta} \|f\|_{L^{q/\alpha}(B_{R(1+3\Theta)/4})}, \end{aligned}$$

where

$$\delta = \alpha\left(\frac{q-d}{q}\right).$$

Then for any $p_0 > 0$

$$\inf_{x \in B_{\eta R}} u(x) \ge c \left(\fint_{B_R} u(x)^{-p_0}\, \mathrm{d}x \right)^{-\frac{1}{p_0}}, \tag{4-14}$$

where $c > 0$ is independent of u, R, ε, and α.

Proof. The idea of the proof is to apply Lemma 4.6 to radii R_k, ρ_k with $R_k \searrow \eta R$ and $\rho_k \searrow 0$ for $k \to \infty$. For each k one chooses an exponent $p_k > 1$ with $p_k \to \infty$ for $k \to \infty$. Because of Assumption (A) we can apply the Sobolev inequality, Fact 3.4, to the left-hand side in (4-12). Next, one iterates the resulting inequality as in [Moser 1961]; see also Chapter 8.6 in [Gilbarg and Trudinger 1998]. The only difference to the proof in [Moser 1961] is that the factor $d/(d-2)$ now becomes $d/(d-\alpha)$. The assertion then follows from the fact

$$\left(\fint_{B_{R_k}(x_0)} u^{-p_k} \right)^{-\frac{1}{p_k}} \to \inf_{B_{\eta R}(x_0)} u \quad \text{for } k \to \infty.$$ ☐

Let us finally prove Theorem 4.1.

Proof of Theorem 4.1. Define $\bar{u} = u + \|f\|_{L^{q/\alpha}(B_{15/16})}$ and note that $\mathcal{E}(u,\phi) = \mathcal{E}(\bar{u},\phi)$ for every ϕ. We apply Lemma 4.5 for $R = \frac{3}{4}$ and obtain that there exist $\bar{p} \in (0,1)$ and $c > 0$ such that

$$\left(\fint_{B_{3/4}} \bar{u}(x)^{\bar{p}}\, \mathrm{d}x \right)^{\frac{1}{\bar{p}}} \mathrm{d}x \le c \left(\fint_{B_{3/4}} u(x)^{-\bar{p}}\, \mathrm{d}x \right)^{-\frac{1}{\bar{p}}}.$$

Next, we apply Corollary 4.7 with $R = \frac{3}{4}$, $\eta = \frac{2}{3}$ and $\Theta = \frac{5}{4}$. Together with the estimate from above we obtain

$$\inf_{B_{1/2}} u \ge c \left(\frac{1}{|B_{\frac{3}{4}}|} \int_{B_{3/4}} \bar{u}(x)^{\bar{p}}\, \mathrm{d}x \right)^{\frac{1}{\bar{p}}}, \tag{4-15}$$

which, after recalling the definition of $\bar{u}$, proves Theorem 4.1. ☐

5. The weak Harnack inequality implies Hölder estimates

The aim of this section is to provide the proof of Theorem 1.10. As is explained in Section 1D it is well known that both the Harnack inequality and the weak Harnack inequality imply regularity estimates in Hölder spaces. Here we are going to establish such a result for quite general nonlocal operators in the framework of metric measure spaces.

We begin with a short study of condition (1-16). The standard example that we have in mind is given in Example 1.9. Let (X, d, m) be a metric measure space. For $R > r > 0,\ x \in X$, set

$$B_r(x) = \{y \in X : d(y, x) < r\}, \quad A_{r,R}(x) = B_R(x) \setminus B_r(x). \tag{5-1}$$

Lemma 5.1. *For $x \in X$, $r > 0$ let $\nu_{x,r}$ be a measure on $\mathcal{B}(X \setminus \{x\})$, which is finite on all sets M with* $\operatorname{dist}(\{x\}, M) > 0$. *Then the following conditions are equivalent*:

(1) *For some $\chi > 1,\ c \ge 1$ and all $x \in X,\ 0 < r \le 1,\ j \in \mathbb{N}_0$*

$$\nu_{x,r}(X \setminus B_{r2^j}(x)) \le c\chi^{-j}.$$

(2) *Given $\theta > 1$, there are $\chi > 1,\ c \ge 1$ such that for all $x \in X,\ 0 < r \le 1,\ j \in \mathbb{N}_0$*

$$\nu_{x,r}(X \setminus B_{r\theta^j}(x)) \le c\chi^{-j}.$$

(3) *Given $\theta > 1$, there are $\chi > 1,\ c \ge 1$ such that for all $x \in X,\ 0 < r \le 1,\ j \in \mathbb{N}_0$*

$$\nu_{x,r}(A_{r\theta^j, r\theta^{j+1}}(x)) \le c\chi^{-j}.$$

(4) *Given $\sigma > 1,\ \theta > 1$ there are $\chi > 1,\ c \ge 1$ such that for all $x \in X,\ 0 < r \le 1,\ j \in \mathbb{N}_0$ and $y \in B_{r/\sigma}(x)$*

$$\nu_{y,r'}(A_{r\theta^j, r\theta^{j+1}}(x)) \le c\chi^{-j}, \quad \textit{where } r' = r\left(1 - \frac{1}{\sigma}\right). \tag{5-2}$$

If, *in addition to any of the above conditions*, (1-17) *holds*, *then* (5-2) *can be replaced by*

$$\nu_{y,r}(A_{r\theta^j, r\theta^{j+1}}(x)) \le c\chi^{-j}. \tag{5-3}$$

Proof. If $\theta \ge 2$, the implication (1) ⇒ (2) trivially holds true. For $\theta < 2$ it can be obtained by adjusting χ appropriately. The proof of (2) ⇒ (1) is analogous. The implication (2) ⇒ (3) trivially holds true. The implication (3) ⇒ (2) follows from

$$\nu_{x,r}(X \setminus B_{r\theta^j}(x)) = \sum_{k=j}^{\infty} \nu_{x,r}(A_{r\theta^k, r\theta^{k+1}}(x)) \le c \sum_{k=j}^{\infty} \chi^{-k} = c\left(\frac{\chi}{\chi - 1}\right)\chi^{-j}.$$

The implication (4) ⇒ (3) trivially holds true. Instead of (3) ⇒ (4) we explain the proof of (2) ⇒ (4). Fix $\sigma > 1,\ \theta > 1,\ x \in X,\ r > 0,\ j \in \mathbb{N}_0$, and $y \in B_{r/\sigma}(x)$. Set $r' = r(1 - 1/\sigma)$. Then $X \setminus B_{r\theta^j}(x) \subset X \setminus B_{r'\theta^j}(y)$. Thus

$$\nu_{y,r'}(X \setminus B_{r\theta^j}(x)) \le \nu_{y,r'}(X \setminus B_{r'\theta^j}(y)) \le c\chi^{-j}. \qquad \square$$

Remark. Note that the conditions above imply that, given $j \in \mathbb{N}_0$ and $x \in X$, the quantity

$$\limsup_{r \to 0+} \nu_{x,r}(X \setminus B_{r2^j}(x))$$

is finite.

Remark. Let $x \in X$, $A \in \mathcal{B}(X \setminus \{x\})$, with $\operatorname{dist}(\{x\}, A) > 0$. In the applications that are of interest to us, the function $r \mapsto \nu_{x,r}(A)$ is strictly increasing with $\nu_{x,0}(A) = 0$.

Proof of Theorem 1.10. The proof follows closely the strategy of [Moser 1961]; see also [Silvestre 2006]. Throughout the proof, let us write B_t instead of $B_t(x_0)$ for $t > 0$. Fix $r \in (0, r_0)$ and $u \in \mathcal{S}_{x_0,r}$. Let $c_1 \geq 1$ be the constant in (1-19). Set $\kappa = (2c_1 2^{1/p})^{-1}$ and

$$\beta = \frac{\ln(2/(2-\kappa))}{\ln(\theta)} \quad \Longrightarrow \quad \left(1 - \tfrac{1}{2}\kappa\right) = \theta^{-\beta}.$$

Set $M_0 = \|u\|_\infty$, $m_0 = \inf_X u(x)$, and $M_{-n} = M_0$, $m_{-n} = m_0$ for $n \in \mathbb{N}$. We will construct an increasing sequence (m_n) and a decreasing sequence (M_n) such that for $n \in \mathbb{Z}$

$$\begin{aligned} m_n \leq u(z) \leq M_n \quad &\text{for almost all } z \in B_{r\theta^{-n}}, \\ M_n - m_n \leq K\theta^{-n\beta}&, \end{aligned} \tag{5-4}$$

where $K = M_0 - m_0 \in [0, 2\|u\|_\infty]$. Assume there is $k \in \mathbb{N}$ and there are M_n, m_n such that (5-4) holds for $n \leq k-1$. We need to choose m_k, M_k such that (5-4) still holds for $n = k$. Then the assertion of the lemma follows by complete induction. For $z \in X$ set

$$v(z) = \left(u(z) - \tfrac{1}{2}(M_{k-1} + m_{k-1})\right)\frac{2\theta^{(k-1)\beta}}{K}.$$

The definition of v implies $v \in \mathcal{S}_{x_0,r}$ and $|v(z)| \leq 1$ for almost any $z \in B_{r\theta^{-(k-1)}}$. Our next aim is to show that (1-19) implies that either $v \leq 1 - \kappa$ or $v \geq -1 + \kappa$ on $B_{r\theta^{-k}}$. Since our version of the Harnack inequality contains nonlocal terms we need to investigate the behavior of v outside of $B_{r\theta^{-(k-1)}}$. Given $z \in X$ with $d(z, x_0) \geq r\theta^{-(k-1)}$ there is $j \in \mathbb{N}$ such that

$$r\theta^{-k+j} \leq d(z, x_0) < r\theta^{-k+j+1}.$$

For such z and j we conclude

$$\begin{aligned} \frac{K}{2\theta^{(k-1)\beta}} v(z) &= \left(u(z) - \tfrac{1}{2}(M_{k-1} + m_{k-1})\right) \\ &\leq \left(M_{k-j-1} - m_{k-j-1} + m_{k-j-1} - \tfrac{1}{2}(M_{k-1} + m_{k-1})\right) \\ &\leq \left(M_{k-j-1} - m_{k-j-1} - \tfrac{1}{2}(M_{k-1} - m_{k-1})\right) \leq \left(K\theta^{-(k-j-1)\beta} - \tfrac{1}{2}K\theta^{-(k-1)\beta}\right), \end{aligned}$$

that is,

$$v(z) \leq 2\theta^{j\beta} - 1 \leq 2\left(\theta \frac{d(z, x_0)}{r\theta^{-(k-1)}}\right)^{\beta} - 1, \tag{5-5}$$

and

$$\begin{aligned}\frac{K}{2\theta^{(k-1)\beta}}v(z) &= \bigl(u(z)-\tfrac{1}{2}(M_{k-1}+m_{k-1})\bigr)\\ &\geq \bigl(m_{k-j-1}-M_{k-j-1}+M_{k-j-1}-\tfrac{1}{2}(M_{k-1}+m_{k-1})\bigr)\\ &\geq \bigl(-(M_{k-j-1}-m_{k-j-1})+\tfrac{1}{2}(M_{k-1}-m_{k-1})\bigr) \geq \bigl(-K\theta^{-(k-j-1)\beta}+\tfrac{1}{2}K\theta^{-(k-1)\beta}\bigr),\end{aligned}$$

that is,

$$v(z) \geq 1-2\theta^{j\beta} \geq 1-2\biggl(\theta\frac{d(z,x_0)}{r\theta^{-(k-1)}}\biggr)^{\beta}.$$

Now there are two cases:

Case 1: $m(\{x\in B_{r\theta^{-k+1}/\lambda} : v(x)\leq 0\}) \geq \frac{1}{2}m(B_{r\theta^{-k+1}/\lambda})$.

Case 2: $m(\{x\in B_{r\theta^{-k+1}/\lambda} : v(x)> 0\}) \geq \frac{1}{2}m(B_{r\theta^{-k+1}/\lambda})$.

We work out details for Case 1 and comment afterwards on Case 2. In Case 1 our aim is to show $v(z)\leq 1-\kappa$ for almost every $z\in B_{r\theta^{-k}}$ and some $\kappa\in(0,1)$. Because then for almost any $z\in B_{r\theta^{-k}}$

$$\begin{aligned}u(z) &\leq \tfrac{1}{2}(1-\kappa)K\theta^{-(k-1)\beta}+\tfrac{1}{2}(M_{k-1}+m_{k-1})\\ &= \tfrac{1}{2}(1-\kappa)K\theta^{-(k-1)\beta}+\tfrac{1}{2}(M_{k-1}-m_{k-1})+m_{k-1}\\ &= m_{k-1}+\tfrac{1}{2}(1-\kappa)K\theta^{-(k-1)\beta}+\tfrac{1}{2}K\theta^{-(k-1)\beta}\\ &\leq m_{k-1}+K\theta^{-k\beta}.\end{aligned}\tag{5-6}$$

We then set $m_k=m_{k-1}$ and $M_k=m_k+K\theta^{-k\beta}$ and obtain, using (5-6), $m_k\leq u(z)\leq M_k$ for almost every $z\in B_{r\theta^{-k}}$, which is what needs to be proved.

Consider $w=1-v$ and note $w\in\mathcal{S}_{x_0,r\theta^{-(k-1)}}$ and $w\geq 0$ in $B_{r\theta^{-(k-1)}}$. We apply (1-19) and obtain

$$\biggl(\fint_{B_{r\theta^{-k+1}/\lambda}(x_0)} w^p\,\mathrm{d}m\biggr)^{\frac{1}{p}} \leq c_1\inf_{B_{r\theta^{-k}}} w + c_1\sup_{x\in B_{r\theta^{-k+1}/\sigma}}\int_X w^-(z)\nu_{x,r\theta^{-(k-1)}}(\mathrm{d}z),\tag{5-7}$$

In Case 1 the left-hand side of (5-7) is bounded from below by $\bigl(\frac{1}{2}\bigr)^{1/p}$. This, together with the estimate (5-5) on v from above, leads to

$$\begin{aligned}\inf_{B_{r\theta^{-k}}} w &\geq (c_1 2^{\frac{1}{p}})^{-1} - \sup_{x\in B_{r\theta^{-k+1}/\sigma}}\int_X w^-(z)\nu_{x,r\theta^{-(k-1)}}(\mathrm{d}z)\\ &\geq (c_1 2^{\frac{1}{p}})^{-1} - \sum_{j=1}^{\infty}\sup_{x\in B_{r\theta^{-k+1}/\sigma}}\int \mathbb{1}_{A_{r\theta^{-k+j},r\theta^{-k+j+1}}(x_0)}(1-v(z))^-\nu_{x,r\theta^{-(k-1)}}(\mathrm{d}z)\\ &\geq (c_1 2^{\frac{1}{p}})^{-1} - \sum_{j=1}^{\infty}(2\theta^{j\beta}-2)\eta_{x_0,r,\theta,j,k},\end{aligned}$$

where

$$\eta_{x_0,r,\theta,j,k} = \sup_{x\in B_{r\theta^{-k+1}/\sigma}} \nu_{x,r\theta^{-(k-1)}}(A_{r\theta^{-k+j},r\theta^{-k+j+1}}(x_0)).$$

Now, (5-3) implies that $\eta_{x_0,r,\theta,j,k} \le c\chi^{-j-1}$. Thus we obtain

$$\inf_{B_{r\theta^{-k}}} w \ge (c_1 2^{\frac{1}{p}})^{-1} - 2c\sum_{j=1}^{\infty}(\theta^{j\beta}-1)\chi^{-j-1}. \tag{5-8}$$

Note that $\sum_{j=1}^{\infty}\theta^{j\beta}\chi^{-j-1} < \infty$ for $\beta > 0$ small enough; i.e., there is $l \in \mathbb{N}$ with

$$\sum_{j=l+1}^{\infty}(\theta^{j\beta}-1)\chi^{-j-1} \le \sum_{j=l+1}^{\infty}\theta^{j\beta}\chi^{-j-1} \le (16c_1)^{-1}.$$

Given l we choose $\beta > 0$ smaller (if needed) in order to ensure

$$\sum_{j=1}^{l}(\theta^{j\beta}-1)\chi^{-j-1} \le (16c_1)^{-1}.$$

The number β depends only on c_1, c, χ from (5-3) and on θ. Thus we have shown that $w \ge \kappa$ on $B_{r\theta^{-k}}$ or, equivalently, $v \le 1-\kappa$ on $B_{r\theta^{-k}}$.

In Case 2 our aim is to show $v(x) \ge -1+\kappa$. This time, set $w = 1+v$. Following the strategy above one sets $M_k = M_{k-1}$ and $m_k = M_k - K\theta^{-k\beta}$ leading to the desired result.

Let us show how (5-4) proves the assertion of the lemma. Given $\rho \le r$, there exists $j \in \mathbb{N}_0$ such that

$$r\theta^{-j-1} \le \rho \le r\theta^{-j}.$$

From (5-4) we conclude

$$\operatorname{osc}_{B_\rho} u \le \operatorname{osc}_{B_{r\theta^{-j}}} u \le M_j - m_j \le 2\theta^{\beta}\|u\|_\infty\left(\frac{\rho}{r}\right)^{\beta}. \qquad \square$$

Corollary 5.2. *Let $\Omega = B_{r_0}(x_0) \subset X$ and let $\sigma, \theta, \lambda > 1$. Let $\mathcal{S}_{x,r}$ and $\nu_{x,r}$ be as above. Assume that conditions* (1-16), (1-17) *are satisfied. Assume that there is $c \ge 1$ such that for $0 < r \le r_0$*

$$(B_r(x) \subset \Omega) \wedge (u \in x, r) \wedge (u \ge 0 \text{ in } B_r(x))$$
$$\implies \left(\fint_{B_{r/\lambda}(x)} u(\xi)^p\, m(\mathrm{d}\xi)\right)^{\frac{1}{p}} \le c \inf_{B_{r/\theta}(x)} u + c \sup_{\xi \in B_{r/\sigma}(x)} \int_X u^-(z)\, \nu_{\xi,r}(\mathrm{d}z). \tag{5-9}$$

Then there exist $\beta \in (0,1)$ such that for every $u \in \mathcal{S}_{x_0,r_0}$ and almost every $x, y \in \Omega$

$$|u(x)-u(y)| \le 16\theta^{\beta}\|u\|_\infty\left(\frac{d(x,y)}{d(x,\Omega^c) \vee d(y,\Omega^c)}\right)^{\beta}. \tag{5-10}$$

Proof. By symmetry, we may assume that $r := d(y,\Omega^c) \ge d(x,\Omega^c)$. Furthermore, it is enough to prove (5-10) for pairs x, y such that $d(x,y) < r/8$, as in the opposite case the assertion is obvious.

We fix a number $\rho \in (0, r_0/4)$ and consider all pairs of x, $y \in \Omega$ such that

$$\tfrac{1}{2}\rho \le d(x,y) \le \rho. \tag{5-11}$$

We cover the ball $B_{r_0-4\rho}(x_0)$ by a countable family of balls $\widetilde{B}_i$ with radii ρ. Without loss of generality, we may assume that $\widetilde{B}_i \cap B_{r_0-4\rho}(x_0) \ne \varnothing$. Let B_i denote the ball with the same center as the ball $\widetilde{B}_i$

and radius 2ρ and let B_i^* denote the ball with the same center as the ball $\widetilde{B}_i$ with radius the maximal radius that allows $B_i^* \subset \Omega$.

Let $x, y \in \Omega$ satisfy (5-11). From $r > 8d(x, y) \geq 4\rho$ it follows that $y \in B_{r_0-4\rho}(x_0)$; therefore $y \in \widetilde{B}_i$ for some index i. We observe that both x and y belong to B_i. We apply Theorem 1.10 to x_0 and r_0 being the center and radius of B_i^*, respectively, and obtain

$$\begin{aligned}\operatorname{osc}_{B_i} u &\leq 2\theta^\beta \|u\|_\infty \left(\frac{\text{radius}(B_i)}{\text{radius}(B_i^*)}\right)^\beta \\ &\leq 2\theta^\beta \|u\|_\infty \left(\frac{\rho}{r-\rho}\right)^\beta \leq \frac{16}{3}\theta^\beta \|u\|_\infty \left(\frac{d(x,y)}{r}\right)^\beta.\end{aligned}$$

Hence (5-10) holds, provided x and y are such that $|u(x) - u(y)| \leq \operatorname{osc}_{B_i} u$.

By considering $\rho = r_0 2^{-j}$ for $j = 3, 4, \ldots,$ we prove (5-10) for almost all x and y such that $d(x, y) \leq r_0/8$; hence the proof is finished. □

5A. *Proof of Theorem 1.7.* We are now going to use the above results and prove one of our main results.

Proof of Theorem 1.7. The proof of Theorem 1.7 follows from Corollaries 4.2 and 5.2. The proof is complete once we can apply Corollary 5.2 for $x_0 = 0$ and $r_0 = \frac{1}{2}$. Assume $0 < r \leq r_0$ and $B_r(x) \subset B_{1/2}$. Let $\mathcal{S}_{x,r}$ be the set of all functions $u \in V^\mu(B_r(x) \mid \mathbb{R}^d)$ satisfying $\mathcal{E}(u, \phi) = 0$ for every $\phi \in H^\mu_{B_r(x)}(\mathbb{R}^d)$. Assume $u \in \mathcal{S}_{x,r}$ and $u \geq 0$ in $B_r(x)$. Then Corollary 4.2 implies

$$\inf_{B_{r/4}(x)} u \geq c\left(\fint_{B_{r/2}(x)} u(x)^{p_0}\,\mathrm{d}x\right)^{\frac{1}{p_0}} - r^\alpha \sup_{y\in B_{15R/16}(x)} \int_{\mathbb{R}^d\setminus B_r(x)} u^-(z)\,\mu(y, \mathrm{d}z),$$

with positive constants p_0, c which depend only on d, α_0, A, B. Set $\theta = 4,\ \lambda = 2,\ \sigma = \frac{16}{15}$. Let $\nu_{x,r}$ be the measure on $\mathbb{R}^d \setminus B_r(x)$ defined by

$$\nu_{x,r}(A) = r^\alpha \mu(x, A).$$

The condition (1-17) obviously holds true. The condition (1-16) follows from (D). Thus we can apply Corollary 5.2 for $x_0 = 0$ and $r_0 = \frac{1}{2}$ and obtain the assertion of Theorem 1.7. □

6. Local comparability results for nonlocal quadratic forms

The aim of this section is to provide the proof of Theorem 1.11. First, we show that (T) and (U) imply (B). Then we establish the upper bound in (A) in the two cases (i) and (ii). The lower bound in (A) is more challenging. We prove it for the two cases in separate subsections. The last subsection contains two examples, which are not covered by cases (i) and (ii).

6A. (T) *and* (U) *imply* (B). It is easy to prove that (T) and (U) imply (B) with a constant $B \geq 1$ independent of $\alpha \in (\alpha_0, 2)$: Let $\tau \in C^\infty(\mathbb{R}^d)$ be a function satisfying $\operatorname{supp}(\tau) = \overline{B}_{R+\rho},\ \tau \equiv 1$ on $B_R,\ 0 \leq \tau \leq 1$ on $\mathbb{R}^d$, and $|\tau(x) - \tau(y)| \leq 2\rho^{-1}|x - y|$ for all $x, y \in \mathbb{R}^d$. In particular, we have then

$|\tau(x)-\tau(y)| \le (2\rho^{-1}|x-y|) \wedge 1$. For every $x \in \mathbb{R}^d$ we obtain

$$\int_{\mathbb{R}^d} (\tau(x)-\tau(y))^2 \, \mu(x, \mathrm{d}y) \le \int_{\mathbb{R}^d} ((4\rho^{-2}|z|^2) \wedge 1) \, \nu^*(\mathrm{d}z)$$
$$= 4\rho^{-2} \int_{\mathbb{R}^d} \left(|z|^2 \wedge \tfrac{1}{4}\rho^2\right) \nu^*(\mathrm{d}z) \le 2^\alpha C_U \rho^{-\alpha} \le 4 C_U \rho^{-\alpha}.$$

Thus we only need to concentrate on proving (A).

6B. *Upper bound in* (A). Let us formulate and prove the following comparability result.

Proposition 6.1. *Assume that ν satisfies* (U) *with the constant C_U and let $0 < \alpha_0 \le \alpha < 2$. If $D \subset \mathbb{R}^d$ is a bounded Lipschitz domain, then there exists a constant $c = c(\alpha_0, d, C_U, D)$ such that*

$$\mathcal{E}_D^{\nu}(u,u) \le c \mathcal{E}_D^{\mu_\alpha}(u,u), \quad u \in H^{\frac{\alpha}{2}}(D). \tag{6-1}$$

The constant c may be chosen such that (6-1) *holds for all balls $D = B_r$ of radius $r < 1$, and for all $\alpha \in [\alpha_0, 2)$.*

Proof. By E we denote the extension operator from $H^{\alpha/2}(D)$ to $H^{\alpha/2}(\mathbb{R}^d)$; see Fact 3.1. By subtracting a constant, we may and do assume that $\int_D u \, \mathrm{d}x = 0$. We have by Plancherel's formula and Fubini's theorem

$$\begin{aligned} \mathcal{E}_D^{\nu}(u,u) &= \int_D \int_{D-y} (u(y+z)-u(y))^2 \, \nu(\mathrm{d}z) \, \mathrm{d}y \qquad (6\text{-}2) \\ &\le \int_D \int_{B_{\mathrm{diam}\, D}(0)} (Eu(y+z)-Eu(y))^2 \, \nu(\mathrm{d}z) \, \mathrm{d}y \\ &\le \int_{B_{\mathrm{diam}\, D}(0)} \int_{\mathbb{R}^d} (Eu(y+z)-Eu(y))^2 \, \mathrm{d}y \, \nu(\mathrm{d}z) \\ &= \int_{\mathbb{R}^d} \left(\int_{B_{\mathrm{diam}\, D}(0)} |e^{i\xi\cdot z}-1|^2 \nu(\mathrm{d}z) \right) |\widehat{Eu}(\xi)|^2 \, d\xi \\ &= \int_{\mathbb{R}^d} \left(\int_{B_{\mathrm{diam}\, D}(0)} 4\sin^2\left(\frac{\xi\cdot z}{2}\right) \nu(\mathrm{d}z) \right) |\widehat{Eu}(\xi)|^2 \, d\xi. \qquad (6\text{-}3) \end{aligned}$$

For $|\xi| > 2$ we obtain, using (U)

$$\int 4\sin^2\left(\frac{\xi\cdot z}{2}\right) \nu(\mathrm{d}z) \le |\xi|^2 \int (|z|^2 \wedge 4|\xi|^{-2}) \, \nu(\mathrm{d}z) \le 4C_U |\xi|^\alpha, \tag{6-4}$$

and for $|\xi| \le 2$

$$\int 4\sin^2\left(\frac{\xi\cdot z}{2}\right) \nu(\mathrm{d}z) \le 4 \int \left(\left|\frac{\xi\cdot z}{2}\right|^2 \wedge 1 \right) \nu(\mathrm{d}z) \le 4C_U.$$

Thus

$$\begin{aligned} \mathcal{E}_D^{\nu}(u,u) &\le c' \int_{\mathbb{R}^d} (|\xi|^\alpha + 1) |\widehat{Eu}(\xi)|^2 \, \mathrm{d}\xi \\ &\le c' \|Eu\|^2_{H^{\alpha/2}(\mathbb{R}^d)} \le c\|u\|^2_{H^{\alpha/2}(D)} = c(\mathcal{E}_D^{\mu_\alpha}(u,u) + \|u\|^2_{L^2(D)}), \qquad (6\text{-}5) \end{aligned}$$

with $c = c(d, C_U, D)$. Since $\int_D u \,\mathrm{d}x = 0$, we have by Fact 3.2

$$\mathcal{E}_D^{\mu_\alpha}(u,u) \ge c(\alpha_0, d, D) \int_D u^2(x)\,\mathrm{d}x$$

and this together with (6-5) proves (6-1).

By scaling, the last assertion of the theorem is satisfied with a constant $c = c(\alpha_0, d, C_U, B_1)$. □

Proof of Theorem 1.11: upper bound in (A). The second inequality in (A) follows from Proposition 6.1. We note that the constant in this inequality is robust under the mere assumption that α is bounded away from zero. □

6C. *Lower bound in* (A)*, case* (i). The aim of this subsection is to complete the proof of Theorem 1.11 in the case (i). The strategy[3] is as follows. We will begin with a simple specific case. We set $e^k = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^d$; i.e., e^k is the k-th standard unit vector in $\mathbb{R}^d$.

Theorem 6.2. *Let $d \ge 2$, $0 < \alpha < 2$, and let μ be as in* (1-11), *i.e.,*

$$\mu(x, dy) = (2-\alpha) \sum_{i=1}^{d} \Big[|x_i - y_i|^{-1-\alpha}\,\mathrm{d}y_i \prod_{j\neq i} \delta_{\{x_j\}}(\mathrm{d}y_j) \Big] =: \sum_{i=1}^{d} \mu_i(x, \mathrm{d}y). \tag{6-6}$$

Then there exists a constant $A = A(d)$ such that

$$\begin{gathered} \textit{for every ball } B_\rho(x_0) \textit{ with } \rho \in (0,1),\ x_0 \in B_1, \textit{ and every } v \in H^{\frac{\alpha}{2}}(B_\rho(x_0)), \\ \mathcal{E}^{\mu_\alpha}_{B_\rho(x_0)}(v,v) \le A\, \mathcal{E}^{\mu}_{B_\rho(x_0)}(v,v). \end{gathered} \tag{6-7}$$

Proof. Let us fix $B = B_\rho(x_0)$ as in the theorem. We may assume that $x_0 = 0$, because the measures considered are translation invariant. For a permutation σ of $\{1, 2, \dots, d\}$ and $x, y \in B$ we define

$$p_k^\sigma(x,y) = (a_1, \dots, a_d), \quad \text{where } a_j = \begin{cases} y_j & \text{if } \sigma^{-1}(j) \le k, \\ x_j & \text{if } \sigma^{-1}(j) > k. \end{cases}$$

For example,

$$p_0^\sigma(x,y) = x, \quad p_1^\sigma(x,y) = (x_1, \dots, x_{\sigma(1)-1}, y_{\sigma(1)}, x_{\sigma(1)+1}, \dots, x_d), \quad p_d^\sigma(x,y) = y.$$

That is, $p_k(x,y)$ are vertices of a polygonal chain joining x and y whose consecutive line segments are parallel to the coordinate axes; more precisely, the j-th line segment is parallel to $\sigma(j)$-th axis. Furthermore, let

$$E^\sigma(B,x) = \{ y \in B : p_k^\sigma(x,y) \in B \text{ for each } k = 1, \dots, d \} \tag{6-8}$$

be the set of all points y which may be connected with x by such a polygonal chain lying completely in B. We obtain

$$\begin{aligned} I^\sigma &:= \int_B \int_{E^\sigma(B,x)} (u(x) - u(y))^2\, \mu_\alpha(x, \mathrm{d}y)\,\mathrm{d}x \\ &\le d \sum_{k=1}^{d} \int_B \int_{E^\sigma(B,x)} (u(p_{k-1}^\sigma(x,y)) - u(p_k^\sigma(x,y)))^2\, \mu_\alpha(x, \mathrm{d}y)\,\mathrm{d}x =: d \sum_{k=1}^{d} I_k^\sigma. \end{aligned} \tag{6-9}$$

[3]The authors thank an anonymous referee for the idea of the proof.

We will bound I_k^σ appearing on the right of (6-9), assuming for notational simplicity that σ is the identity permutation, i.e., $\sigma(k)=k$. Then

$$\begin{aligned} I_k^\sigma &= \int_B \int_{E^\sigma(B,x)} (u(p_{k-1}^\sigma(x,y)) - u(p_k^\sigma(x,y)))^2 \, \mu_\alpha(x,\mathrm{d}y)\,\mathrm{d}x \\ &= (2-\alpha)\int_B\int_{E^\sigma(B,x)} \frac{(u(y_1,\dots,y_{k-1},x_k,\dots,x_d)-u(y_1,\dots,y_k,x_{k+1},\dots,x_d))^2}{|x-y|^{d+\alpha}}\,\mathrm{d}y\,\mathrm{d}x. \end{aligned}$$

We would like to change the order of integration, so that we integrate outside with respect to

$$w := p_{k-1}^\sigma(x,y) = (y_1,\dots,y_{k-1},x_k,\dots,x_d),$$

and inside with respect to

$$z := x+y-w = (x_1,\dots,x_{k-1},y_k,\dots,y_d).$$

Then $|x-y| = |z-w|$ and $p_k^\sigma(x,y) = w + (z_k - w_k)e^k$. Let

$$\begin{aligned} F(B,w) &:= \{z\in\mathbb{R}^d : w+(z_k-w_k)e^k \in B\}, \\ F_0(B,w) &:= \{t\in\mathbb{R} : w+(t-w_k)e^k\in B\}. \end{aligned}$$

We note that if $x\in B$ and $y\in E^\sigma(B,x)$, then $w\in B$ and $p_k(x,y)\in B$; hence $z\in F(B,w)$. Therefore

$$\begin{aligned} I_k^\sigma &\le (2-\alpha)\int_B\int_{F(B,w)} \frac{(u(w)-u(w+(z_k-w_k)e^k))^2}{|w-z|^{d+\alpha}}\,\mathrm{d}z\,\mathrm{d}w \\ &= (2-\alpha)\int_B\int_{F_0(B,w)} \left[(u(w)-u(w+(z_k-w_k)e^k))^2 \int_{\mathbb{R}^{d-1}} \frac{\mathrm{d}z_1\cdots\mathrm{d}z_{k-1}\,\mathrm{d}z_{k+1}\cdots\mathrm{d}z_d}{|w-z|^{d+\alpha}}\right]\mathrm{d}z_k\,\mathrm{d}w. \end{aligned}$$

The inner integral over $\mathbb{R}^{d-1}$ is simple to calculate using scaling; it gives

$$\int_{\mathbb{R}^{d-1}} \frac{\mathrm{d}z_1\cdots\mathrm{d}z_{k-1}\,\mathrm{d}z_{k+1}\cdots\mathrm{d}z_d}{|w-z|^{d+\alpha}} = |w_k-z_k|^{-\alpha-1}c(d)\int_0^\infty (1+t^2)^{\frac{-d-\alpha}{2}}t^{d-2}\,\mathrm{d}t \le C(d)|w_k-z_k|^{-\alpha-1}.$$

Thus

$$\begin{aligned} I_k^\sigma &\le C(d)(2-\alpha)\int_B\int_{F_0(B,w)} [(u(w)-u(w+(z_k-w_k)e^k))^2|w_k-z_k|^{-\alpha-1}]\,\mathrm{d}z_k\,\mathrm{d}w \\ &= C(d)\int_B\int_B (u(w)-u(z))^2\,\mu_k(w,\mathrm{d}z)\,\mathrm{d}w. \end{aligned}$$

The same inequality as above holds for I_k^σ with an arbitrary permutation σ. We obtain

$$\sum_\sigma I^\sigma = \sum_\sigma\sum_{k=1}^d I_k^\sigma \le C(d)d!\,\mathcal{E}_B^\mu(u,u), \tag{6-10}$$

where the sum runs over the set of all permutations on $\{1,2,\dots,d\}$. On the other hand, for each pair $(x,y)\in B\times B$, there exists a permutation σ such that $y\in E^\sigma(B,x)$. Indeed, if $M = \#\{j : |y_j| < |x_j|\}$, then as σ we may take any permutation satisfying $\sigma(\{1,\dots,M\}) = \{j : |y_j| < |x_j|\}$. If $1\le j\le M$, then

$|p_j(x,y)| < |x|$, and if $j > M$, then $|p_j(x,y)| \le |y|$; therefore $p_j(x,y) \in B$ for all j; i.e., $y \in E^\sigma(B,x)$ as claimed. Thus

$$\sum_\sigma I^\sigma = \sum_\sigma \int_B \int_{E^\sigma(B,x)} (u(x)-u(y))^2 \, \mu_\alpha(x, \mathrm{d}y)\, \mathrm{d}x \ge \mathcal{E}_B^{\mu_\alpha}(u,u),$$

which together with (6-10) gives the assertion of the theorem. □

Next, we consider linear transformations of μ. Let $L : \mathbb{R}^d \to \mathbb{R}^d$ be an invertible linear transform. For a measure μ on $\mathbb{R}^d$ we define the measure $\mu \circ L$ by

$$(\mu \circ L)(E) = \mu(L(E)), \quad E \subset \mathbb{R}^d, \quad \text{where } E \text{ is a Borel set,}$$

or, equivalently, by

$$\int f(x)(\mu \circ L)(\mathrm{d}x) = \int f(L^{-1}(x))\, \mu(\mathrm{d}x), \tag{6-11}$$

for all Borel measurable functions $f : \mathbb{R}^d \to [0,\infty)$.

Lemma 6.3. *Let $0 < \alpha_0 \le \alpha < 2$ and let a measure μ on $\mathbb{R}^d$ satisfy condition* (6-7) *with some constant A, with $\mathcal{E}^\mu$ defined by* (1-21). *Let $L : \mathbb{R}^d \to \mathbb{R}^d$ be an invertible linear transform. Then $\mu \circ L$ also satisfies condition* (6-7) *with the constant depending only on A, d, α_0 and the norms $\|L\|$ and $\|L^{-1}\|$.*

Proof. Let u be a Borel measurable function on $\mathbb{R}^d$; let $B = B_\rho(x_0)$ with $x_0 \in B_1$ and $\rho \in (0,1)$. Let

$$v(x) = u(L^{-1}(x-x_0)+x_0), \quad x \in \mathbb{R}^d.$$

By a linear change of variable and (6-11) we obtain

$$\mathcal{E}_B^{\mu\circ L}(u,u) = \frac{1}{\det L} \int_{x_0+L(B(0,\rho))} \int_{\mathbb{R}^d} (v(x)-v(x+z))^2 \, \mathbb{1}_{x_0+L(B(0,\rho))}(x+z)\, \mu(\mathrm{d}z)\, \mathrm{d}t. \tag{6-12}$$

We observe that $B(0,s\rho) \subset L(B(0,\rho))$, where $s = \|L^{-1}\|^{-1} \wedge 1$; therefore

$$\begin{aligned}\mathcal{E}_B^{\mu\circ L}(u,u) &\ge \frac{1}{\det L} \int_{B(x_0,s\rho)} \int_{\mathbb{R}^d} (v(x)-v(x+z))^2 \, \mathbb{1}_{B(x_0,s\rho)}(x+z)\, \mu(\mathrm{d}z)\, \mathrm{d}t \\ &= \frac{1}{\det L} \mathcal{E}^\mu_{B(x_0,s\rho)}(v,v) \ge \frac{A^{-1}}{\det L} \mathcal{E}^{\mu_\alpha}_{B(x_0,s\rho)}(v,v),\end{aligned}$$

by the assumption and the fact that $s \le 1$. Since $L(B(0,st\rho)) \subset B(0,s\rho)$, where $t = \|L\|^{-1} \wedge 1$, we get

$$\begin{aligned}\mathcal{E}_B^{\mu\circ L}(u,u) &\ge \frac{A^{-1}}{\det L} \int_{x_0+L(B(0,st\rho))} \int_{\mathbb{R}^d} (v(x)-v(x+z))^2 \, \mathbb{1}_{x_0+L(B(0,st\rho))}(x+z)\mu_\alpha(dz)\, \mathrm{d}t \\ &= A^{-1} \mathcal{E}^{\mu_\alpha\circ L}_{B(0,st\rho)}(u,u),\end{aligned} \tag{6-13}$$

where in the last line we used (6-12) with μ_α in place of μ.

However,

$$(\mu_\alpha \circ L)(E) = (2-\alpha)\int_{L(E)} |x|^{-d-\alpha}\,\mathrm{d}x = (2-\alpha)\det L \int_E |L(x)|^{-d-\alpha}\,\mathrm{d}x$$
$$\geq (2-\alpha)\det L\|L\|^{-d-\alpha}\int_E |x|^{-d-\alpha}\,\mathrm{d}x = \det L\|L\|^{-d-\alpha}\mu_\alpha(E).$$

Plugging this into (6-13) we obtain

$$\mathcal{E}_B^{\mu\circ L}(u,u) \geq \det L\|L\|^{-d-\alpha}A^{-1}\mathcal{E}^{\mu_\alpha}_{B(0,st\rho)}(u,u).$$

The theorem follows now from Lemma 6.13; since $\det L \geq \|L^{-1}\|$ and $st = (\|L^{-1}\|^{-1}\wedge 1)(\|L\|^{-1}\wedge 1)$, the constants depend only on A, d, α_0, $\|L\|$ and $\|L^{-1}\|$. We note here that the proof of this lemma, although presented later, does not use any previous results, i.e., there is no circular reasoning. □

With the help of Lemma 6.3 we are able to prove the following generalization of Theorem 6.2.

Corollary 6.4. *Let* $0 < \alpha_0 \leq \alpha < 2$. *Let* $f^1, \ldots, f^d \in S^{d-1}$ *be linearly independent. Assume that* $L : \mathbb{R}^d \to \mathbb{R}^d$ *is the linear transform that maps* e^j *to* f^j. *Then the measure*

$$\mu(E) = (2-\alpha)\sum_{j=1}^{d}\int_0^\infty \delta_{rf_j}(E)r^{-1-\alpha}\,\mathrm{d}r \quad (E \in \mathcal{B}(\mathbb{R}^d)) \tag{6-14}$$

satisfies condition (6-7) *with* $\mathcal{E}^\mu$ *defined by* (1-21) *and the constant depending only on* d, α_0 *and the norm* $\|L^{-1}\|$.

Proof. Since $\|L\| \leq \sqrt{d}$, the result follows from Theorem 6.2 and Lemma 6.3. □

In order to prove comparability for all nondegenerate α-stable measures, we need to study combinations of measures as in (6-14). To this end, we will apply the following lemma, which essentially is contained in [Krickeberg 1968].

Lemma 6.5. *If* π *is a finite Borel measure on* S^{d-1}, *then there exists a Borel function* $\phi : [0, \pi(X)] \to S^{d-1}$ *such that* $|\phi^{-1}(A)| = \pi(A)$ *for every Borel set* $A \subset S^{d-1}$ *and* $\phi([0,\pi(X)]) \subset \operatorname{supp}\pi$.

Proof. It is enough to prove the result for measures π which are either purely atomic, or nonatomic. In the first case the construction of such ϕ is straightforward: if $\{a_j : 0 \leq j < N\}$ are all the atoms of π (where $N \in \mathbb{N}\cup\{\infty\}$), then we put

$$\phi(t) = \begin{cases} a_j & \text{for } t \in \left[\sum_{0\leq i<j}\pi(\{a_i\}), \sum_{0\leq i\leq j}\pi(\{a_i\})\right) \text{ and } 0 \leq j < N, \\ a_0 & \text{for } t = \pi(X). \end{cases}$$

In the nonatomic case, since π is Radon, the result follows from [Krickeberg 1968, Hilfssatz, page 64; Oxtoby 1970, Theorem 2]. □

We finally provide the proof of the comparability result for general α-stable measures.

Proof of Theorem 1.11(i). Assume that π is a measure on S^{d-1} as in (1-22). Let $x_1, \ldots, x_d \in \operatorname{supp}\pi$ be a basis of $\mathbb{R}^d$. Then for $\varepsilon > 0$ small enough and some $M > 0$, any $y_j \in \overline{B(x_j,\varepsilon)} \cap S^{d-1} =: B_j$ also span $\mathbb{R}^d$. If L is the linear operator mapping $e^j = (0,\ldots,0,1,0,\ldots,0)$ to y_j, then the norms $\|L\|$ and

$\|L^{-1}\|$ are bounded from above by M. The number $m = \min\{\pi(B_1), \dots, \pi(B_d)\}$ is strictly positive, because x_j belong to the support of π. Let

$$\pi_j = \frac{m}{\pi(B_j)}\pi(\,\cdot \cap B_j), \quad j = 1, \dots, d.$$

Then π_j are Borel measures on S^{d-1} with mass m and $\operatorname{supp} \pi_j \subset \bar{B}_j$. Let $\phi_j : [0, m] \to S^{d-1}$ be the Borel functions from Lemma 6.5 corresponding to π_j. For every Borel $E \subset S^{d-1}$

$$\pi(E) \geq \sum_{j=1}^{d} \pi_j(E) = \sum_{j=1}^{d} |\phi_j^{-1}(E)| = \int_0^m \sum_{j=1}^{d} \delta_{\phi_j(t)}(E)\,\mathrm{d}t.$$

Therefore

$$\nu(E) = (2-\alpha)\int_0^\infty \pi(r^{-1}E)r^{-1-\alpha}\,\mathrm{d}r \geq \int_0^m \left((2-\alpha)\sum_{j=1}^{d}\int_0^\infty \delta_{r\phi_j(t)}(E)r^{-1-\alpha}\,\mathrm{d}r\right)\mathrm{d}t.$$

By Corollary 6.4, the measure in the parentheses in the line above satisfies condition (6-7) with the constant depending on d, α_0 and M, but independent of t. Therefore also ν satisfies condition (6-7) with the constant depending on d, α_0 and M, i.e., on α_0 and π. $\square$

6D. ***Lower bound in* (A)*, case* (ii).** The aim of this subsection is to complete the proof of Theorem 1.11 in the case (ii).

The main difficulty in establishing the lower bound in (A) is that the measures might be singular. We will introduce a new convolution-type operation that, on the one hand, smooths the support of the measures and, on the other hand, interacts nicely with our quadratic forms. The main result of this subsection is Proposition 6.14.

For $\lambda < 1 \leq \eta$ and $\alpha \in (0, 2)$ let

$$g_\lambda^\eta(y, z) = \frac{1}{2-\alpha}|y+z|^\alpha\, \mathbb{1}_{A_{|y+z|}}(y)\, \mathbb{1}_{A_{|y+z|}}(z), \quad y, z \in \mathbb{R}^d, \tag{6-15}$$

where

$$A_r = B(0, \eta r) \setminus B(0, \lambda r).$$

Definition 6.6. For measures ν_1, ν_2 on $\mathcal{B}(\mathbb{R}^d)$ satisfying (U) with some $\alpha \in (0, 2)$, define a new measure $\nu_1 \heartsuit \nu_2$ on $\mathcal{B}(\mathbb{R}^d)$ by

$$\nu_1 \heartsuit \nu_2(E) = \iint \mathbb{1}_{E\cap B_2}(\eta(y+z))\, g_\lambda^\eta(y, z)\, \nu_1(\mathrm{d}y)\, \nu_2(\mathrm{d}z),$$

i.e.,

$$\int f(x)\nu_1 \heartsuit \nu_2(\mathrm{d}x) = \iint (f \cdot \mathbb{1}_{B_2})(\eta(y+z))g_\lambda^\eta(y, z)\nu_1(\mathrm{d}y)\, \nu_2(\mathrm{d}z),$$

for every measurable function $f : \mathbb{R}^d \to [0, \infty]$.

This definition is tailored for our applications and needs some explanations. We consider $\nu_1 \heartsuit \nu_2$ only for measures ν_j which satisfy (U) with some $\alpha \in (0, 2)$ for $j \in \{1, 2\}$. This α equals the exponent α in the definition of g_λ^η. The above definition does not require ν_j to satisfy (S) but most often this will be the case. Note that Definition 6.6 is valid for any choice $\lambda < 1 \le \eta$. However, it will be important to choose λ small enough and η large enough. The precise bounds depend on the number a from (S); see Proposition 6.14. Before we explain and prove the rather technical details, let us treat an example.

Let us study Example 1.5 in $\mathbb{R}^2$. Assume $\alpha \in (0,2)$ and

$$\nu_1(\mathrm{d}h) = (2-\alpha)|h_1|^{-1-\alpha}\,\mathrm{d}h_1\,\delta_{\{0\}}(\mathrm{d}h_2),$$
$$\nu_2(\mathrm{d}h) = (2-\alpha)|h_2|^{-1-\alpha}\,\mathrm{d}h_2\,\delta_{\{0\}}(\mathrm{d}h_1).$$

Both measures are one-dimensional α-stable measures which are orthogonal to each other. The factor $(2-\alpha)$ ensures that for $\alpha \to 2-$ the measures do not explode. Let us show that $\nu_1 \heartsuit \nu_2$ is already absolutely continuous with respect to the two-dimensional Lebesgue measure. For $E \subset B_2$, by the Definition 6.6 and the Fubini theorem,

$$\begin{aligned}
\nu_1 \heartsuit \nu_2(E) \\
&= (2-\alpha)\iiiint |y+z|^{\alpha}\,\mathbb{1}_E(\eta(y+z))\,\mathbb{1}_{A_{|y+z|}}(y)\,\mathbb{1}_{A_{|y+z|}}(z)|y_1|^{-1-\alpha}|z_2|^{-1-\alpha} \\
&\qquad\qquad \cdots \delta_{\{0\}}(\mathrm{d}y_2)\,\delta_{\{0\}}(\mathrm{d}z_1)\,\mathrm{d}y_1\,\mathrm{d}z_2 \\
&= (2-\alpha)\iint |(y_1,z_2)|^{\alpha}\,\mathbb{1}_E(\eta(y_1,z_2))\,\mathbb{1}_{A_{|(y_1,z_2)|}}(y_1,0)\,\mathbb{1}_{A_{|(y_1,z_2)|}}(0,z_2)|y_1|^{-1-\alpha}|z_2|^{-1-\alpha}\,\mathrm{d}y_1\,\mathrm{d}z_2 \\
&= (2-\alpha)\iint \mathbb{1}_E(\eta x)\,\mathbb{1}_{A_{|x|}}(x_1,0)\,\mathbb{1}_{A_{|x|}}(0,x_2)|x|^{\alpha}|x_1|^{-1-\alpha}|x_2|^{-1-\alpha}\,\mathrm{d}x_1\,\mathrm{d}x_2.
\end{aligned}$$

The above computation shows that the measure $\nu_1 \heartsuit \nu_2$ is absolutely continuous with respect to the two-dimensional Lebesgue measure, because $\nu_1 \heartsuit \nu_2(\mathbb{R}^d \setminus B_2) = 0$. Let us look at the density more closely.

So far, we have not specified λ and η in the definition of g_λ^η. If $\lambda < 1$ is too large (in this particular case, if $\lambda > 1/\sqrt{2}$), then $\mathbb{1}_{A_{|x|}}(x_1,0)\,\mathbb{1}_{A_{|x|}}(0,x_2) = 0$ for all $x \in \mathbb{R}^2$. If λ is sufficiently small, then the support of the function $\mathbb{1}_{A_{|x|}}(x_1,0)\,\mathbb{1}_{A_{|x|}}(0,x_2)$ is a double-cone centered around the diagonals $\{x \in \mathbb{R}^2 : |x_1| = |x_2|\}$. Let us denote this support by M. Note that on M the function $|x|^{\alpha}|x_1|^{-1-\alpha}|x_2|^{-1-\alpha}$ is comparable to $|x|^{-2-\alpha}$. Thus indeed the quantity $\nu_1 \heartsuit \nu_2$ is comparable to an α-stable measure in $\mathbb{R}^2$. If we continue the procedure and define

$$\tilde{\nu} = (\nu_1 \heartsuit \nu_2) \heartsuit (\nu_1 \heartsuit \nu_2),$$

then we can make use of the fact that $(\nu_1 \heartsuit \nu_2)$ is already absolutely continuous with respect to the two-dimensional Lebesgue measure. Note that, if $\mu_j = h_j\,\mathrm{d}x$, then $\mu_1 \heartsuit \mu_2$ has a density $h_1 \heartsuit h_2$ with respect to the Lebesgue measure given by

$$h_1 \heartsuit h_2(\eta y) = \frac{\eta^{-d}|y|^{\alpha}}{2-\alpha}\int \mathbb{1}_{A_{|y|}}(y-z)\,\mathbb{1}_{A_{|y|}}(z)\,h_1(y-z)\,h_2(z)\,\mathrm{d}z, \quad \eta y \in B_2. \tag{6-16}$$

In this way we conclude that $\tilde{\nu}$ has full support and is comparable to a rotationally symmetric α-stable measure in $\mathbb{R}^2$. With this observation we end our study of Definition 6.6 in light of Example 1.5.

Before we proceed to the proofs, let us informally explain the idea behind Definition 6.6 and our strategy. In the inner integral defining

$$\mathcal{E}_B^{\nu}(u,u) = \int_B \int_{\mathbb{R}^d} (u(x) - u(x+h))^2 \, \mathbb{1}_B(x+h) \, \nu(\mathrm{d}h) \, \mathrm{d}x$$

we take into account squared increments $(u(x) - u(x+h))^2$ in these directions h, which are charged by the measure ν and such that $x+h$ is still in B. By changing the variables, we see that we also have squared increments $(u(x+h) - u(x+h+z))^2$, again in directions z, which are charged by the measure ν and such that $x+h+z$ is still in B. This allows us to estimate the integral $\mathcal{E}_B^{\nu}(u,u)$ from below by a similar integral with ν replaced by some kind of a convolution of ν with itself. Measure $\nu \heartsuit \nu$ turns out to be the right convolution for this purpose; see Lemma 6.12.

In the definition of $\nu \heartsuit \nu$, the function g_λ^η vanishes if $|y|$ or $|z|$ is bigger than $\eta|y+z|$ or smaller than $\lambda|y+z|$. This means, in our interpretation, that we consider only those pairs of jumps which are comparable to the size of the whole two-step jump (and in particular, the jumps must be comparable to each other).

To conclude these informal remarks on the definition of $\nu_1 \heartsuit \nu_2$ let us note that if ν_1 and ν_2 have "good properties", then so has $\nu_1 \heartsuit \nu_2$ (see Lemmas 6.7 and 6.11) and that $\mathcal{E}_B^{\nu_1 \heartsuit \nu_2}(u,u)$ can be estimated from above by $\mathcal{E}_B^{\nu_j}(u,u)$ (see Lemma 6.12). This allows us to reduce the problem of estimating $\mathcal{E}_B^{\nu}(u,u)$ from below to estimating $\mathcal{E}_B^{\nu \heartsuit \nu}(u,u)$ from below, and this turns out to be easier, since the $\heartsuit$-convolution makes the measure more "smooth"; see Proposition 6.14.

Lemma 6.7. *If two measures ν_j for $j \in \{1,2\}$ satisfy the scaling assumption* (S) *for some $a > 1$, then so does the measure $\nu_1 \heartsuit \nu_2$ for the same constant a.*

Proof. If $\operatorname{supp} f \subset B_1$, then

$$\begin{aligned} \int f(ax) \nu_1 \heartsuit \nu_2(\mathrm{d}x) &= \iint f(\eta a(y+z)) \, \mathbb{1}_{B_2}(\eta(y+z)) \, g_\lambda^\eta(y,z) \, \nu_1(\mathrm{d}y) \, \nu_2(\mathrm{d}z) \\ &= a^{-\alpha} \iint f(\eta(ay+az)) \, g_\lambda^\eta(ay,az) \, \nu_1(\mathrm{d}y) \, \nu_2(\mathrm{d}z), \end{aligned}$$

because $g_\lambda^\eta(y,z) = a^{-\alpha} g_\lambda^\eta(ay,az)$. We observe that the function $(y,z) \mapsto f(\eta(y+z)) g_\lambda^\eta(y,z)$ vanishes outside $B_1 \times B_1$. Hence we may apply (S) twice to obtain

$$\int f(ax) \, \nu_1 \heartsuit \nu_2(\mathrm{d}x) = a^{\alpha} \iint f(\eta(y+z)) \, g_\lambda^\eta(y,z) \, \nu_1(\mathrm{d}y) \, \nu_2(\mathrm{d}z) = a^{\alpha} \int f(x) \, \nu_1 \heartsuit \nu_2(\mathrm{d}x). \qquad \square$$

Next, we establish conditions which are equivalent to (U). We say that a measure ν on $\mathcal{B}(\mathbb{R}^d)$ satisfies the upper-bound assumption (U0) if for some $C_0 > 0$

$$\int_{\mathbb{R}^d} (|z|^2 \wedge 1) \, \nu(\mathrm{d}z) \le C_0. \tag{U0}$$

We say that a measure ν on $\mathcal{B}(\mathbb{R}^d)$ satisfies the upper-bound assumption (U1) if there exists $C_1 > 0$ such that for every $r \in (0,1)$

$$\int_{B_r(0)} |z|^2 \nu(\mathrm{d}z) \le C_1 r^{2-\alpha}. \tag{U1}$$

Lemma 6.8. $$(\mathrm{U}) \iff (\mathrm{U0}) \wedge (\mathrm{U1}).$$

If the constants C_0, C_1 are independent of $\alpha \in [\alpha_0, 2)$, then so is C_U, and vice versa.

Proof. The implications (U) $\Rightarrow$ (U1) and (U) $\Rightarrow$ (U0) are obvious; we may take $C_0 = C_1 := C_U$. Let us now assume that (U1) and (U0) hold true. Fix $0 < r \le 1$. We consider $n = 0, 1, 2, \ldots$ such that $2^{n+1} r \le 1$ (the set of such n's is empty if $r > \frac{1}{2}$). We have by (U1)

$$\begin{aligned}\int_{2^n r \le |z| < 2^{n+1} r} \nu(\mathrm{d}z) &\le 2^{-2n} r^{-2} \int_{2^n r \le |z| < 2^{n+1} r} |z|^2 \nu(\mathrm{d}z)\\ &\le 2^{-2n} r^{-2} C_1\, 2^{(n+1)(2-\alpha)} r^{2-\alpha} = 2^{-n\alpha} 2^{2-\alpha} C_1 r^{-\alpha}.\end{aligned}$$

After summing over all such n we obtain

$$\int_{r \le |z| < \frac{1}{2}} \nu(\mathrm{d}z) \le \frac{2^{2-\alpha} C_1}{1 - 2^{-\alpha}} r^{-\alpha}.$$

Finally,

$$\int_{\frac{1}{2} \le |z|} \nu(\mathrm{d}z) \le 4 \int_{\mathbb{R}^d} (|z|^2 \wedge 1)\, \nu(\mathrm{d}z) \le 4C_0 \le 4C_0 r^{-\alpha}.$$

Combining the two inequalities above and (U1) we get (U) with

$$C_U = \left(\frac{2^{2-\alpha}}{1 - 2^{-\alpha}} + 1 \right) C_1 + 4C_0. \qquad \square$$

The following definition interpolates between measures ν which are related to different values of $\alpha \in (0, 2)$. Such a construction is important for us because we want to prove comparability results which are robust in the sense that constants stay bounded when $\alpha \to 2^-$.

Definition 6.9. Assume ν^{α_0} is a measure on $\mathcal{B}(\mathbb{R}^d)$ satisfying (U) or (S) for some $\alpha_0 \in (0, 2)$. For $\alpha_0 \le \alpha < 2$ we define a new measure ν^{α,α_0} by

$$\nu^{\alpha,\alpha_0} = \frac{2-\alpha}{2-\alpha_0} |x|^{\alpha_0 - \alpha} \nu^{\alpha_0}(dx) \quad \text{if } \alpha > \alpha_0 \qquad \text{and} \qquad \text{by } \nu^{\alpha_0,\alpha_0} = \nu^{\alpha_0}. \tag{6-17}$$

To shorten notation we write ν^α instead of ν^{α,α_0} whenever there is no ambiguity.

The above definition is consistent in the following ways. On the one hand, the first part of (6-17) holds true for $\alpha = \alpha_0$. On the other hand, for $0 < \alpha_0 < \alpha < \beta < 2$, the following is true: $\nu^{\beta,\alpha_0} = (\nu^{\alpha,\alpha_0})^{\beta,\alpha}$. This requires that ν^{α,α_0} itself satisfies (U) or (S) which is established in the following lemma.

Lemma 6.10. *Assume ν^{α_0} satisfies* (U) *with some $\alpha_0 \in (0, 2)$, $C_U > 0$ or condition* (S) *with some $\alpha_0 \in (0, 2)$, $a > 1$. Assume $\alpha_0 \le \alpha < 2$ and ν^α as in Definition 6.9.*

(a) *If ν^{α_0} satisfies* (U), *then for every* $0 < b < 1$, $0 < r \le 1$

$$\int_{br \le |z| < r} |z|^2 \nu^\alpha(\mathrm{d}z) \le \frac{2-\alpha}{2-\alpha_0} C_U b^{\alpha_0 - \alpha} r^{2-\alpha}, \tag{6-18}$$

$$\int_{B_r^c} \nu^\alpha(\mathrm{d}z) \le \frac{2-\alpha}{2-\alpha_0} C_U r^{-\alpha}. \tag{6-19}$$

(b) *If* ν^{α_0} *satisfies* (U), *then* ν^{α} *satisfies* (U) *with exponent* α *and constant* $13C_U(2-\alpha_0)^{-1}$. *In particular, the constant does not depend on* α.

(c) *If* ν^{α_0} *satisfies* (S), *then* ν^{α} *satisfies* (S) *with exponent* α.

Proof. Let $0<r\le 1$ and $0<b<1$. To prove (a)enumi, we derive,

$$\begin{aligned}\int_{br\le|z|<r}|z|^2\nu^{\alpha}(\mathrm{d}z)&=\frac{2-\alpha}{2-\alpha_0}\int_{br\le|z|<r}|z|^{2+\alpha_0-\alpha}\nu^{\alpha_0}(\mathrm{d}z)\\&\le\frac{2-\alpha}{2-\alpha_0}(br)^{\alpha_0-\alpha}\int_{B_r}|z|^2\nu^{\alpha_0}(\mathrm{d}z)\le\frac{2-\alpha}{2-\alpha_0}b^{\alpha_0-\alpha}C_Ur^{2-\alpha},\end{aligned}$$

which proves (6-18). Furthermore,

$$\int_{B_r^c}\nu^{\alpha}(\mathrm{d}z)=\frac{2-\alpha}{2-\alpha_0}\int_{B_r^c}|z|^{\alpha_0-\alpha}\nu^{\alpha_0}(\mathrm{d}z)\le\frac{2-\alpha}{2-\alpha_0}r^{\alpha_0-\alpha}C_Ur^{-\alpha_0}$$

and (6-19) follows. To prove part (b)enumi, we use (6-18) and conclude

$$\begin{aligned}\int_{B_r}|z|^2\nu^{\alpha}(\mathrm{d}z)&=\sum_{n=0}^{\infty}\int_{\frac{r}{2^{n+1}}\le|z|<\frac{r}{2^n}}|z|^2\nu^{\alpha_0}(\mathrm{d}z)\le\frac{2-\alpha}{2-\alpha_0}C_U2^{\alpha-\alpha_0}r^{2-\alpha}\sum_{n=0}^{\infty}2^{n(\alpha-2)}\\&=\frac{C_U2^{\alpha-\alpha_0}r^{2-\alpha}}{2-\alpha_0}\frac{2-\alpha}{1-2^{\alpha-2}}\le\frac{32C_U}{3(2-\alpha_0)}r^{2-\alpha},\end{aligned}\tag{6-20}$$

since the function $x\mapsto x/(1-2^{-x})$ is increasing. Furthermore, by (6-19),

$$\int_{B_r^c}r^2\nu^{\alpha}(\mathrm{d}z)\le\frac{2C_U}{2-\alpha_0}r^{2-\alpha},\tag{6-21}$$

and therefore (b)enumi follows. Finally, part (c)enumi is obvious. □

Lemma 6.11. *Assume* $\nu_j^{\alpha_0}$ *for* $j\in\{1,2\}$ *satisfies* (U) *with some* $\alpha_0\in(0,2)$, $C_U>0$. *Assume* $\alpha_0\le\alpha<2$ *and* ν_j^{α} *as in Definition 6.9. Then the measure* $\nu_1^{\alpha}\heartsuit\nu_2^{\alpha}$ *satisfies* (U) *with the same exponent* α *and a constant depending only on* α_0, C_U, λ *and* η.

Proof. By Lemma 6.8, it suffices to show that $\nu_1^{\alpha}\heartsuit\nu_2^{\alpha}$ satisfies (U0) and (U1). For $0<r\le 1$ we derive

$$\begin{aligned}\int_{B_r}|x|^2\,\nu_1^{\alpha}\heartsuit\nu_2^{\alpha}(dx)&\le\frac{1}{2-\alpha}\iint_{\lambda|y+z|\le|y|,|z|\le\eta|y+z|}|\eta(y+z)|^2\mathbb{1}_{B_r}(\eta(y+z))|y+z|^{\alpha}\,\nu_1^{\alpha}(\mathrm{d}y)\,\nu_2^{\alpha}(\mathrm{d}z)\\&\le\frac{1}{2-\alpha}\iint_{\lambda|y+z|\le|y|,|z|\le\eta|y+z|<r}\frac{\eta^2|y|^2}{\lambda^2}\frac{|z|^{\alpha}}{\lambda^{\alpha}}\,\nu_1^{\alpha}(\mathrm{d}y)\,\nu_2^{\alpha}(\mathrm{d}z)\\&\le\frac{1}{2-\alpha}\frac{\eta^2}{\lambda^{2+\alpha}}\int_{B_r}|z|^{\alpha}\int_{\frac{\lambda|z|}{\eta}\le|y|\le\frac{\eta|z|}{\lambda}}|y|^2\,\nu_1^{\alpha}(\mathrm{d}y)\,\nu_2^{\alpha}(\mathrm{d}z)\\&\le\frac{\eta^4(C_U)^2}{\lambda^4}\frac{13}{(2-\alpha_0)^2}r^{2-\alpha},\end{aligned}$$

where in the last passage we used parts (b)enumi and (a)enumi of Lemma 6.10. Furthermore, by (6-19),

$$\int_{\mathbb{R}^d\setminus B_1} \nu_1^\alpha \heartsuit \nu_2^\alpha(\mathrm{d}x) \le \frac{1}{2-\alpha} \iint_{\lambda|y+z|\le|y|,|z|<\eta|y+z|} \mathbb{1}_{B_2\setminus B_1}(\eta(y+z))|y+z|^\alpha \nu_1^\alpha(y)\nu_2^\alpha(\mathrm{d}z)$$
$$\le \frac{2^\alpha}{2-\alpha} \iint_{\frac{\lambda}{\eta}\le|y|,|z|} \nu_1^\alpha(y)\nu_2^\alpha(\mathrm{d}z) \le \frac{8(C_U)^2\eta^4}{\lambda^4(2-\alpha_0)^2}. \qquad \square$$

The following lemma shows that the quadratic form with respect to $\nu_1 \heartsuit \nu_2$ is dominated by the sum of the quadratic forms with respect to ν_1 and ν_2. Some enlargement of the domain is needed which is taken care of in Lemma 6.13 by a covering argument.

Lemma 6.12. *Assume $\nu_j^{\alpha_0}$ for $j \in \{1,2\}$ satisfies* (U) *and* (S) *with some $\alpha_0 \in (0,2)$, $a > 1$, and $C_U > 0$. Assume $\alpha_0 \le \alpha < 2$ and ν_j^α as in Definition 6.9. Let $\eta = a^k > 1$ for some $k \in \mathbb{Z}$. For $B = B_r(x_0)$ let us set $B^* = B_{3\eta r}(x_0)$. Then with $c = 4C_U\eta^6\lambda^{-4}$ it holds that*

$$\mathcal{E}_B^{\nu_1\heartsuit\nu_2}(u,u) \le c\left(\mathcal{E}_{B^*}^{\nu_1}(u,u) + \mathcal{E}_{B^*}^{\nu_2}(u,u)\right) \tag{6-22}$$

for any measurable function u on B_1 and any B such that $B^* \subset B_1$.

Proof. Let $B = B_r(x_0)$ be such that $B^* \subset B_1$. In particular, this means that $r \le 1/(3\eta)$. By definition, we obtain

$$\begin{aligned}
\mathcal{E}_B^{\nu_1\heartsuit\nu_2}(u,u) &= \iint (u(x)-u(x+z))^2\, \mathbb{1}_B(x)\, \mathbb{1}_B(x+z)\, \nu_1 \heartsuit \nu_2(\mathrm{d}z)\,\mathrm{d}x \\
&\le \iiint (u(x)-u(x+\eta(y+z)))^2\, \mathbb{1}_B(x)\, \mathbb{1}_B(x+\eta(y+z))\, g_\lambda^\eta(y,z)\, \nu_1(\mathrm{d}y)\, \nu_2(\mathrm{d}z)\,\mathrm{d}x \\
&\le 2\iiint [(u(x)-u(x+\eta y))^2 + (u(x+\eta y)-u(x+\eta(y+z)))^2] \\
&\qquad\qquad \times \mathbb{1}_B(x)\, \mathbb{1}_B(x+\eta(y+z))\, g_\lambda^\eta(y,z)\, \nu_1(\mathrm{d}y)\, \nu_2(\mathrm{d}z)\,\mathrm{d}x \\
&= 2[I_1 + I_2].
\end{aligned} \tag{6-23}$$

We may assume that

$$\lambda|y+z| \le |z| < \eta|y+z| \le 2r,$$
$$\lambda|y+z| \le |y| < \eta|y+z| \le 2r,$$

as otherwise the expression $\mathbb{1}_B(x)\, \mathbb{1}_B(x+\eta(y+z))g_\lambda^\eta(y,z)$ would be zero. Since $2r \le 1$, it follows that $\lambda|y|/\eta < |z| \le \eta|y|/\lambda \wedge 1$. Therefore, by changing the order of integration,

$$I_1 \le \int_B \int_{B_{2r}} \int_{\frac{\lambda|y|}{\eta}\vee\lambda|y+z|\le|z|\le\frac{\eta|y|}{\lambda}\wedge 1} (u(x)-u(x+\eta y))^2|y+z|^\alpha\, \nu_2(\mathrm{d}z)\, \nu_1(\mathrm{d}y)\,\mathrm{d}x.$$

We estimate the inner integral above:

$$J := \int_{\frac{\lambda|y|}{\eta}\vee\lambda|y+z|\le|z|\le\frac{\eta|y|}{\lambda}\wedge 1} |y+z|^\alpha \nu_2(\mathrm{d}z) \le \int_{|z|\le\frac{\eta|y|}{\lambda}\wedge 1} \frac{|z|^\alpha}{\lambda^\alpha}\frac{|z|^{2-\alpha}}{(\lambda|y|/\eta)^{2-\alpha}}\nu_2(\mathrm{d}z) \le \frac{\eta^4 C_U}{\lambda^4}.$$

Coming back to I_1 we obtain,

$$I_1 \le \frac{\eta^4 C_U}{\lambda^4} \int_B \int_{B_{2r}} (u(x) - u(x+\eta y))^2 \, \nu_1(\mathrm{d}y) \, \mathrm{d}x$$
$$= \frac{\eta^4 C_U}{\lambda^4} \eta^\alpha \int_B \int_{B_{2\eta r}} (u(x) - u(x+y))^2 \, \nu_1(\mathrm{d}y) \, \mathrm{d}x \le \frac{\eta^6 C_U}{\lambda^4} \mathcal{E}^{\nu_1}_{B^*}(u,u),$$

where we used (S) and the fact that $B_{2\eta r} \subset B_1$.

Finally, in order to estimate I_2, we first change variables $x = w - \eta y$,

$$I_2 \le \int_B \int_{B_{2r}} \int_{B_{2r}} (u(x+\eta y) - u(x+\eta(y+z)))^2 \, \mathbb{1}_B(x+\eta(y+z)) \, g^\eta_\lambda(y,z) \, \nu_1(\mathrm{d}y) \, \nu_2(\mathrm{d}z) \, \mathrm{d}x$$
$$\le \int_{B^*} \int_{B_{2r}} (u(w) - u(w+\eta z))^2 \, \mathbb{1}_B(w+\eta z) \int_{B_{2r}} g^\eta_\lambda(y,z) \, \nu_1(\mathrm{d}y) \, \nu_2(\mathrm{d}z) \, \mathrm{d}w$$
$$\le \int_{B^*} \int_{B_{2r}} (u(w) - u(w+\eta z))^2 \, \mathbb{1}_B(w+\eta z) \int_{\frac{\lambda|z|}{\eta} \vee \lambda|y+z| \le |y| \le \frac{\eta|z|}{\lambda} \wedge 1} |y+z|^\alpha \, \nu_1(\mathrm{d}y) \, \nu_2(\mathrm{d}z) \, \mathrm{d}w.$$

By symmetry, the following integral may be estimated exactly like J before:

$$\int_{\frac{\lambda|z|}{\eta} \vee \lambda|y+z| \le |y| \le \frac{\eta|z|}{\lambda} \wedge 1} |y+z|^\alpha \, \nu_1(\mathrm{d}y) \le \frac{\eta^4 C_U}{\lambda^4}.$$

This leads to an estimate

$$I_2 \le \frac{\eta^4 C_U}{\lambda^4} \int_{B^*} \int_{B_{2r}} (u(w) - u(w+\eta z))^2 \, \mathbb{1}_B(w+\eta z) \, \nu_2(\mathrm{d}z) \, \mathrm{d}w$$
$$= \frac{\eta^4 C_U}{\lambda^4} \eta^\alpha \int_{B^*} \int_{B_{2\eta r}} (u(w) - u(w+t))^2 \, \mathbb{1}_B(w+t) \, \nu_2(\mathrm{d}t) \, \mathrm{d}w \le \frac{\eta^6 C_U}{\lambda^4} \mathcal{E}^{\nu_2}_{B^*}(u,u),$$

where we used (S) and the fact that $B_{2\eta r} \subset B_1$. The result follows from (6-23) and the obtained estimates of I_1 and I_2. □

Lemma 6.13. *Let* $0 < \alpha_0 < \alpha < 2$, $r_0 > 0$, $\kappa \in (0,1)$, *and* ν *be a measure on* $\mathcal{B}(\mathbb{R}^d)$. *For* $B = B_r(x)$, $x \in \mathbb{R}^d$, $r > 0$, *we set* $B^* = B_{r/\kappa}(x)$. *Suppose that for some* $c_\nu > 0$

$$\mathcal{E}^\nu_{B^*}(u,u) \ge c_\nu \mathcal{E}^{\mu_\alpha}_B(u,u)$$

for every $0 < r \le r_0$, *for every* $u \in L^2(B_{r_0})$, *and for every ball* $B \subset B_{r_0}$ *of radius* κr. *Then there exists a constant* $c = c(d, \alpha_0, \kappa)$, *such that for every ball* $B \subset B_{r_0}$ *of radius* $r \le r_0$ *and every* $u \in L^2(B_{r_0})$

$$\mathcal{E}^\nu_B(u,u) \ge c c_\nu \mathcal{E}^{\mu_\alpha}_B(u,u).$$

Proof. Fix some $0 < r \le r_0$ and a ball D of radius r. We take $\mathcal{B}$ to be a family of balls with the following properties:

(i) For some $c = c(d)$ and any $x, y \in D$, if $|x-y| < c \operatorname{dist}(x, D^c)$, then there exists $B \in \mathcal{B}$ such that $x, y \in B$.

(ii) For every $B \in \mathcal{B}$, we have $B^* \subset D$.

(iii) Family $\{B^*\}_{B\in\mathcal{B}}$ has the finite overlapping property; that is, each point of D belongs to at most $M = M(d)$ balls B^*, where $B \in \mathcal{B}$.

Such a family $\mathcal{B}$ may be constructed by considering Whitney decomposition of D into cubes and then covering each Whitney cube by an appropriate family of balls.

We have

$$\begin{aligned}\mathcal{E}_D^{\nu}(u,u) &\geq \frac{1}{M^2}\sum_{B\in\mathcal{B}}\int_{B^*}\int_{B^*}(u(x)-u(x+y))^2\,\nu(\mathrm{d}y)\,\mathrm{d}x\\ &\geq \frac{c_\nu}{M^2}(2-\alpha)\sum_{B\in\mathcal{B}}\int_B\int_B(u(x)-u(y))^2|x-y|^{-d-\alpha}\,\mathrm{d}y\,\mathrm{d}x\\ &\geq \frac{c_\nu}{M^2}(2-\alpha)\int_D\int_{|x-y|<c\,\mathrm{dist}(x,D^c)}(u(x)-u(y))^2|x-y|^{-d-\alpha}\,\mathrm{d}y\,\mathrm{d}x. \end{aligned}\tag{6-24}$$

By [Dyda 2006, Proposition 5 and proof of Theorem 1], we may estimate

$$\begin{aligned}\int_D\int_{|x-y|<c\,\mathrm{dist}(x,D^c)}&(u(x)-u(y))^2|x-y|^{-d-\alpha}\,\mathrm{d}y\,\mathrm{d}x\\ &\geq c(\alpha,d)\int_D\int_D(u(x)-u(y))^2|x-y|^{-d-\alpha}\,\mathrm{d}y\,\mathrm{d}x,\end{aligned}\tag{6-25}$$

with some constant $c(\alpha,d)$. We note that in [Dyda 2006, proof of Theorem 1] the constant depends on the domain in question, but in our case, by scaling, we can take the same constant independent of the choice of the ball D. One may also check that $c(\alpha,d)$ stays bounded when $\alpha \in [\alpha_0, 2)$. By (6-24) and (6-25) the lemma follows. □

For a linear subspace $E \subset \mathbb{R}^d$, we denote by H_E the $(\dim E)$-dimensional Hausdorff measure on $\mathbb{R}^d$ with the support restricted to E. In particular, $H_{\{0\}} = \delta_{\{0\}}$, the Dirac delta measure at 0.

Proposition 6.14. *Let $E_1, E_2 \subset \mathbb{R}^d$ be two linear subspaces with $E_1, E_2 \neq \{0\}$. Assume that ν_j, $j \in \{1,2\}$, are measures on $\mathcal{B}(\mathbb{R}^d)$ of the form $\nu_j = f_j H_{E_j}$ satisfying $\nu_j(B_1) > 0$,* (U), *and* (S) *with $\alpha_0 \in (0,2)$, $C_U > 0$, and $a > 1$. Then the following are true*:

(1) *$\nu_1 \heartsuit \nu_2$ is absolutely continuous with respect to $H_{E_1+E_2}$ and satisfies* (U) *and* (S).

(2) *If $\eta \geq a^2/(a-1)$ and $\lambda \leq 1/(a^3+1)$, then $\nu_1 \heartsuit \nu_2(B_1) > 0$.*

(3) *If $\nu_j^{\alpha_0} = \nu_j$ and ν_j^{α} is defined as in Definition 6.9 for $\alpha_0 \leq \alpha < 2$, then*

$$\nu_1^{\alpha} \heartsuit \nu_2^{\alpha} \geq \eta^{-2}(\nu_1^{\alpha_0} \heartsuit \nu_2^{\alpha_0})^{\alpha}. \tag{6-26}$$

Proof. Properties (U) and (S) follow from Lemmas 6.11 and 6.7, respectively. Let $E = E_1 \cap E_2$ and let F_j be linear subspaces such that $E_j = E \oplus F_j$, where $j = 1,2$. For $y \in E_1$ let us write $y = Y + \tilde{y}$, where $Y \in E$ and $\tilde{y} \in F_1$; similarly, for $z \in E_2$ we write $z = Z + \hat{z}$, where $Z \in E$ and $\hat{z} \in F_2$. Then for $A \subset B_2$

$$\begin{aligned}\nu_1 \heartsuit \nu_2(A) &= \iiiint \mathbb{1}_A(\eta(Y+\tilde{y}+Z+\hat{z}))\, g_\lambda^\eta(Y+\tilde{y}, Z+\hat{z}) \\ &\qquad\qquad \times f_1(Y+\tilde{y})\, f_2(Z+\hat{z})\, H_E(\mathrm{d}Y)\, H_E(\mathrm{d}Z)\, H_{F_1}(\mathrm{d}\tilde{y})\, H_{F_2}(\mathrm{d}\hat{z}) \\ &= \iiint \mathbb{1}_A(\eta(W+\tilde{y}+\hat{z}))\Big(\int g_\lambda^\eta(Y+\tilde{y}, W-Y+\hat{z})\, f_1(Y+\tilde{y})\, f_2(W-Y+\hat{z})\, H_E(\mathrm{d}Y)\Big) \\ &\qquad\qquad \times H_E(\mathrm{d}W)\, H_{F_1}(\mathrm{d}\tilde{y})\, H_{F_2}(\mathrm{d}\hat{z}) \qquad (6\text{-}27)\end{aligned}$$

and since $\nu_1 \heartsuit \nu_2(\mathbb{R}^d \setminus B_2) = 0$, the desired absolute continuity follows.

To show nondegeneracy, let $G_n := B_{a^{-n}} \setminus B_{a^{-n-1}}$. By scaling property (S) it follows that $\nu_j(G_{n+1}) = a^\alpha \nu_j(G_n)$; therefore $\nu_j(G_n) > 0$ for each $n = 0, 1, \ldots$. Hence

$$\nu_1 \heartsuit \nu_2(B_1) \geq \frac{1}{2-\alpha_0} \int_{G_n} \int_{G_{n+2}} \mathbb{1}_{B_1}(\eta(y+z))\, \mathbb{1}_{A_{|y+z|}}(y)\, \mathbb{1}_{A_{|y+z|}}(z)\, |y+z|^\alpha\, \nu_1(\mathrm{d}y)\, \nu_2(\mathrm{d}z).$$

For $(y, z) \in G_{n+2} \times G_n$ it holds that

$$\frac{a-1}{a^2}(|y| \vee |z|) \leq |y+z| \leq (a^3+1)(|y| \wedge |z|)$$

and also $\eta(y+z) \in B_1$, provided n is large enough. Therefore $\nu_1 \heartsuit \nu_2(B_1) > 0$, if $\eta \geq a^2/(a-1)$ and $\lambda \leq 1/(a^3+1)$.

To prove the last part of the lemma, we calculate first the most inner integral in (6-27) corresponding to $\nu_1^\alpha \heartsuit \nu_2^\alpha$; it equals

$$\begin{aligned}L &:= \int g_\lambda^\eta(Y+\tilde{y}, W-Y+\hat{z})\, f_1^\alpha(Y+\tilde{y})\, f_2^\alpha(W-Y+\hat{z})\, H_E(\mathrm{d}Y) \\ &= \frac{2-\alpha}{(2-\alpha_0)^2} \int |W+\tilde{y}+\hat{z}|^\alpha\, |Y+\tilde{y}|^{\alpha_0-\alpha}\, |W-Y+\hat{z}|^{\alpha_0-\alpha}\, \mathbb{1}(\cdots)\, f_1^{\alpha_0}(Y+\tilde{y})\, f_2^{\alpha_0}(W-Y+\hat{z})\, H_E(\mathrm{d}Y),\end{aligned}$$

where we used an abbreviation

$$\mathbb{1}(\cdots) := \mathbb{1}_{A_{|W+\tilde{y}+\hat{z}|}}(Y+\tilde{y})\, \mathbb{1}_{A_{|W+\tilde{y}+\hat{z}|}}(W-Y+\hat{z}).$$

On the other hand, the most inner integral in (6-27) corresponding to $(\nu_1^{\alpha_0} \heartsuit \nu_2^{\alpha_0})^\alpha$ is

$$\begin{aligned}R &:= \frac{2-\alpha}{2-\alpha_0}(\eta|W+\tilde{y}+\hat{z}|)^{\alpha_0-\alpha} \int g_\lambda^\eta(Y+\tilde{y}, W-Y+\hat{z})\, f_1^{\alpha_0}(Y+\tilde{y})\, f_2^{\alpha_0}(W-Y+\hat{z})\, H_E(\mathrm{d}Y) \\ &= \frac{(2-\alpha)\eta^{\alpha_0-\alpha}}{(2-\alpha_0)^2} \int |W+\tilde{y}+\hat{z}|^{2\alpha_0-\alpha}\, \mathbb{1}(\cdots)\, f_1^{\alpha_0}(Y+\tilde{y})\, f_2^{\alpha_0}(W-Y+\hat{z})\, H_E(\mathrm{d}Y).\end{aligned}$$

Inequality (6-26) follows now from the estimate

$$|Y+\tilde{y}|^{\alpha_0-\alpha}|W-Y+\hat{z}|^{\alpha_0-\alpha}\mathbb{1}(\cdots) \geq (\eta|W+\tilde{y}+\hat{z}|)^{2(\alpha_0-\alpha)}\mathbb{1}(\cdots)$$

and the fact that both sides of (6-26) are zero on $\mathbb{R}^d \setminus B_2$. □

Proof of Theorem 1.11: lower bound in (A). We recall from Section 1E that we may and do assume that f_k are symmetric, i.e., $f_k(x) = f_k(-x)$ for all x. By Proposition 6.14 it follows that the measure

$$\nu := (f_1 H_{E_1}) \heartsuit (f_2 H_{E_2}) \heartsuit \cdots \heartsuit (f_n H_{E_n})$$

satisfies (U) and (S) and has density h with respect to the Lebesgue measure on $\mathcal{B}(\mathbb{R}^d)$ with $\int_{B_1} h(x)\,\mathrm{d}x > 0$, if η is large enough and λ small enough. We will show that the measure $\nu \heartsuit \nu$ possesses a density $h^{\heartsuit}$ with $h^{\heartsuit}(x) \geq c|x|^{-d-\alpha_0}$ for all $x \in B_1 \setminus \{0\}$ and some positive constant c to be specified. This, together with the preliminary results, will establish the assertion.

Condition (S) for ν implies that $h(ax) = a^{-d-\alpha_0} h(x)$ if $x \in B_{1/a}$. Therefore $\int_{G_0} h(x)\,\mathrm{d}x > 0$, where $G_0 = B_1 \setminus B_{1/a}$. Define $h^{G_0}(x) = h(x)\,\mathbb{1}_{G_0}(x) \wedge 1$. The function

$$x \mapsto h^{G_0} * h^{G_0}(x) = \int h^{G_0}(y-x)\, h^{G_0}(y)\,\mathrm{d}y$$

is continuous and strictly positive at 0. Thus there exists $\delta \in (0, (2a)^{-1})$ and $\varepsilon > 0$ such that

$$h^{G_0} * h^{G_0}(x) \geq \varepsilon \quad \text{for } x \in B_\delta.$$

We consider the measure $\nu \heartsuit \nu$; it has density $h^{\heartsuit}$ with respect to the Lebesgue measure on $\mathcal{B}(B_2)$ given by formula, see (6-16),

$$\begin{aligned} h^{\heartsuit}(x) &= \eta^{-2d} \int g_\lambda^\eta\Big(\frac{w}{\eta}, \frac{x-w}{\eta}\Big) h\Big(\frac{w}{\eta}\Big) h\Big(\frac{x-w}{\eta}\Big)\,\mathrm{d}w \\ &\geq \eta^{2\alpha_0} \int_{G_0} g_\lambda^\eta\Big(\frac{w}{\eta}, \frac{x-w}{\eta}\Big)\,\mathbb{1}_{G_0}(x-w)\, h(w) h(x-w)\,\mathrm{d}w \\ &= \frac{\eta^{\alpha_0}}{2-\alpha_0} \int_{G_0} |x|^{\alpha_0}\,\mathbb{1}_{A_{|x|}}(w)\,\mathbb{1}_{A_{|x|}}(x-w)\,\mathbb{1}_{G_0}(x-w)\, h(w)\, h(x-w)\,\mathrm{d}w. \end{aligned}$$

Suppose $\eta \geq a^2/\delta$ and $\lambda \leq 1/(a\delta)$. Then for $x \in B_\delta \setminus B_{\delta/a^2}$ and $w \in G_0$ such that $x - w \in G_0$ it holds that

$$\mathbb{1}_{A_{|x|}}(w)\,\mathbb{1}_{A_{|x|}}(x-w) = 1.$$

This leads to the estimate

$$h^{\heartsuit}(x) \geq \frac{\eta^{\alpha_0}\delta^{\alpha_0} a^{-2\alpha_0}}{2-\alpha_0} h^{G_0} * h^{G_0}(x) \geq \frac{\varepsilon}{2-\alpha_0} \quad \text{for } x \in B_\delta \setminus B_{\delta/a^2}.$$

For $x \in B_1 \setminus \{0\}$ let $k \in \mathbb{Z}$ be such that $\delta/a^2 < |x|a^k < \delta < |x|a^{k+1}$. Then, by scaling (S),

$$h^{\heartsuit}(x) = a^{k(d+\alpha_0)}\, h^{\heartsuit}(xa^k) \geq \frac{a^{k(d+\alpha_0)}\varepsilon}{2-\alpha_0} \geq \frac{\delta^{d+\alpha_0}\varepsilon}{a^{2d+2\alpha_0}(2-\alpha_0)} |x|^{-d-\alpha_0}.$$

Now from Lemmas 6.12 and 6.13 it follows that for any $B \subset B_1$

$$\mathcal{E}_B^{\mu_{\alpha_0}}(u,u) \leq c\,\mathcal{E}_B^{\nu_*}(u,u), \tag{6-28}$$

with $c = c((f_j), (E_j))$.

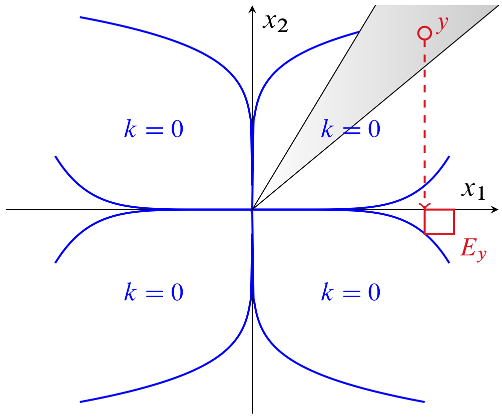

Figure 1. Support of the kernel k $\big(\text{with } b=\frac{1}{6}\big)$ consisting of four thorns. The set P from the proof below is shown, too.

Finally, to obtain a robust result, we observe that by (6-26)

$$\underbrace{(\nu_*)^\alpha\heartsuit\cdots\heartsuit(\nu_*)^\alpha}_{2n\text{ factors}} \geq \eta^{-2(2n-1)}(\underbrace{\nu_*\heartsuit\cdots\heartsuit\nu_*}_{2n\text{ factors}})^\alpha \geq \eta^{-2(2n-1)}\frac{2-\alpha}{2-\alpha_0}|x|^{\alpha_0-\alpha}\frac{\delta^{d+\alpha_0}\varepsilon}{a^{2d+2\alpha_0}}|x|^{-d-\alpha_0}\mathbb{1}_{B_1}(x)\,\mathrm{d}x.$$

This together with Lemmas 6.12 and 6.13 gives us

$$\mathcal{E}_B^\alpha(u,u) \leq c\,\mathcal{E}_B^{(\nu_*)^\alpha}(u,u),$$

with the constant c *not depending* on $\alpha \in [\alpha_0, 2)$. □

6E. *Examples.* In this subsection, we provide two examples showing that the assumptions of Theorem 1.11 are not necessary for (A) and (B). Note that condition (A) relates to integrated quantities but does not require pointwise bounds on the density of $\mu(x,\mathrm{d}y)$.

Example 6.15. Let $b \in (0,1)$ and

$$\Gamma = \{(x_1,x_2) \in \mathbb{R}^2 : |x_2| \geq |x_1|^b \text{ or } |x_1| \geq |x_2|^b\}.$$

We consider the following function

$$k(z) = (2-\alpha)\,\mathbb{1}_{\Gamma\cap B_1}(z)|z|^{-2-\beta}, \quad z \in \mathbb{R}^2, \tag{6-29}$$

where $\beta = \alpha - 1 + 1/b$; see Figure 1. Let us show that conditions (A) and (B) are satisfied in this case.

We have, for $0 < r < 1$,

$$\begin{aligned}\int_{B_r}|z|^2k(z)\,\mathrm{d}z &\leq 8(2-\alpha)\int_0^r\int_0^{x^{1/b}}(x^2+y^2)^{-\frac{\beta}{2}}\,\mathrm{d}y\,\mathrm{d}x\\ &\leq 8(2-\alpha)\int_0^r\int_0^{x^{1/b}}x^{-\beta}\,\mathrm{d}y\,\mathrm{d}x = 8r^{2-\alpha};\end{aligned}\tag{6-30}$$

hence k satisfies (U1) with $C_1 = 8$. Since (U0) is clear, from Lemma 6.8 we conclude that k satisfies (U).

Let

$$P = \{x \in B_{\frac{1}{4}} : 0 < x_1 < x_2 < 2x_1\}$$

and, for $y = (x_1, x_2) \in P$, let

$$E_y = [x_1, x_1 + x_1^{\frac{1}{b}}] \times [-x_1^{\frac{1}{b}}, 0].$$

It is easy to check that if $y \in P$ and $z \in E_y$, then

$$\frac{|y|}{3} \le |z| \le 4|y|, \quad \frac{|y|}{3} \le |y-z| \le 4|y|, \quad \text{and} \quad z, y-z \in \Gamma \cap B_1.$$

Let $\eta = 4$ and $\lambda = \frac{1}{3}$. Then for $y \in P$

$$\begin{aligned} k \heartsuit k(\eta y) &= \frac{|y|^\alpha}{2-\alpha} \int \mathbb{1}_{A_{|y|}}(z)\, \mathbb{1}_{A_{|y|}}(y-z)(2-\alpha)^2\, \mathbb{1}_{\Gamma\cap B_1}(z)\, \mathbb{1}_{\Gamma\cap B_1}(y-z)\, |z|^{-2-\beta}\, |y-z|^{-2-\beta}\, \mathrm{d}z \\ &\ge (2-\alpha)|y|^\alpha \int_{E_y} |z|^{-2-\beta}\, |y-z|^{-2-\beta}\, \mathrm{d}z \\ &\ge (2-\alpha)|y|^\alpha (4|y|)^{2(-2-\beta)} x_1^{\frac{2}{b}} \ge (2-\alpha) 3^{-\frac{2}{b}} 4^{-4-2\beta} |y|^{-2-\alpha} \ge 4^{-6} 12^{-\frac{2}{b}} (2-\alpha)|y|^{-2-\alpha}. \end{aligned}$$

In the following example, we provide a condition that cannot be handled by Theorem 1.11 but still implies comparability of corresponding quadratic forms.

Example 6.16. For a measure ν on $\mathcal{B}(\mathbb{R}^d)$ with a density k with respect to the Lebesgue measure we formulate the following condition:

$$\begin{aligned} &\text{There exist } a > 1 \text{ and } C_2, C_3 > 0 \text{ such that every annulus } B_{a^{-n+1}} \setminus B_{a^{-n}}\ (n = 0, 1, \dots) \\ &\quad \text{contains a ball } B_n \text{ with radius } C_2 a^{-n} \text{ such that } k(z) \ge C_3(2-\alpha)|z|^{-d-\alpha},\ z \in B_n. \end{aligned} \tag{6-31}$$

The following proposition provides a substitute for Theorem 1.11.

Proposition 6.17. *Let $a > 1$, $\alpha_0 \in (0,2)$, $\alpha \in [\alpha_0, 2)$, and $C_U, C_2, C_3 > 0$. Let $\mu = (\mu(x,\cdot))_{x\in\mathbb{R}^d}$ be a family of measures on $\mathbb{R}^d$ which satisfies* (1-2)*. Furthermore, we assume that there exist measures ν_* and ν^* with property* (T) *such that* (U) *and* (6-31) *hold with exponent α and the constants C_U, C_2, C_3. Then there is $A = A(a, \alpha_0, C_U, C_2, C_3) \ge 1$ not depending on α such that* (A) *hold.*

Proof. We fix $\lambda < 2/C_2 \wedge 1$ and $\eta \ge 2a^2/C_2 \vee 1$. For some $n \in \{0, 1, \dots\}$, let

$$\frac{C_2}{2} a^{-n-1} \le |y| \le \frac{C_2}{2} a^{-n},$$

and assume that $\eta y \in B_2$. By formula (6-16), we obtain

$$k \heartsuit k(\eta y) \ge \frac{\eta^{-d}|y|^\alpha}{2-\alpha} \int \mathbb{1}_{A_{|y|}}(y-z)\, \mathbb{1}_{A_{|y|}}(z)\, k(y-z)\, k(z)\, \mathrm{d}z.$$

Let us denote by B_n^o the ball concentric with B_n, but with radius $C_2 a^{-n}/2$ (that is, B_n^o is twice smaller than B_n). We observe that if $z \in B_n^o$, then $y - z \in B_n$. Furthermore, by our choice of λ and η it follows that

$$\lambda|y| \le |y-z| < \eta|y|, \quad \lambda|y| \le |z| < \eta|y| \quad \text{if } z \in B_n^o;$$

that is, $y-z,\, z \in A_{|y|}$ for $z \in B_n^o$. Hence

$$\begin{aligned} k \heartsuit k(\eta y) &\geq \frac{\eta^{-d}|y|^{\alpha}}{2-\alpha} C_3^2 (2-\alpha)^2 \int_{B_n^o} |y-z|^{-d-\alpha}|z|^{-d-\alpha}\,\mathrm{d}z \\ &\geq \frac{C_3^2\eta^{-d}(2-\alpha)C_2^{2d+2\alpha}}{2^{2d+2\alpha}a^{3d+4\alpha}}|y|^{-d-\alpha} \\ &\geq C(\alpha_0,d,C_2,C_3,\eta,a)(2-\alpha)|y|^{-d-\alpha}, \end{aligned}$$

or, equivalently, for $w \in B_2$

$$k \heartsuit k(w) \geq C'(\alpha_0,d,C_2,C_3,\eta,a)(2-\alpha)|w|^{-d-\alpha}.$$

By Lemmas 6.12 and 6.13 we conclude that the lower estimate in (A) holds. The upper estimate is in turn a consequence of Proposition 6.1. □

7. Global comparability results for nonlocal quadratic forms

In this section we provide a global comparability result; i.e., we study comparability in the whole $\mathbb{R}^d$. This result is not needed for the other results in this article; however it contains an interesting and useful observation.

Proposition 7.1. *Assume* (U) *holds. Then there exists a constant* $c = c(\alpha, d, C_U)$ *such that*

$$\mathcal{E}^{\mu}(u,u) \leq c(\mathcal{E}^{\mu_\alpha}(u,u) + \|u\|^2_{L^2(\mathbb{R}^d)}) \quad \text{for every } u \in L^2(\mathbb{R}^d). \tag{7-1}$$

Furthermore, if (U) *is satisfied for* all $r > 0$, *then for every* $u \in L^2(\mathbb{R}^d)$

$$\mathcal{E}^{\mu}(u,u) \leq c\,\mathcal{E}^{\mu_\alpha}(u,u). \tag{7-2}$$

If the constant C_U *in* (U) *is independent of* $\alpha \in (\alpha_0, 2)$, *where* $\alpha_0 > 0$, *then so are the constants in* (7-1) *and* (7-2).

Proof. By E we denote the identity operator from $H^{\alpha/2}(\mathbb{R}^d)$ to itself. One easily checks that the proof of Proposition 6.1 from (6-2) until (6-5) works also in the present case of $D = \mathbb{R}^d$. Hence (7-1) follows.

To prove (7-2) we observe that if (U) holds for all $r > 0$, then also (6-4) holds for *all* $\xi \neq 0$; we plug it into (6-3) and we are done. □

We consider the following condition.

(K2,r_0) There exists $c_0 > 0$ such that for all $h \in S^{d-1}$ and all $0 < r < r_0$

$$\int_{\mathbb{R}^d} r^2 \sin^2\left(\frac{h\cdot z}{r}\right)\nu_*(\mathrm{d}z) \geq c_0 r^{2-\alpha}. \tag{7-3}$$

Clearly (6-31) implies (K2,r_0)2,r_0 for $r_0 = 1$, and if C_3 is independent of $\alpha \in (\alpha_0, 2)$, where $\alpha_0 > 0$, then so is c_0. Condition (K2,r_0)2,r_0 is also satisfied if for all $h \in S^{d-1}$ and all $0 < r < r_0$

$$\int_{B_r(0)} |h\cdot z|^2 \nu_*(\mathrm{d}z) \geq c_2 r^{2-\alpha}. \tag{7-4}$$

We note that (7-5) under condition (7-4) has been proved in [Abels and Husseini 2010]. The following theorem extends their result by giving a *characterization* of kernels ν_* admitting comparability (7-5). We stress that $r_0 = \infty$ is allowed, and in such a case we put $1/r_0^\alpha = 0$.

Theorem 7.2. *Let* $0 < r_0 \le \infty$. *If* (K2,r_0)2,r_0 *holds*, *then*

$$\mathcal{E}^{\mu_\alpha}(u,u) \le \frac{1}{c_0}\mathcal{E}^{\mu}(u,u) + \frac{2^\alpha}{r_0^\alpha}\|u\|_{L^2}^2, \quad u \in C_c^1(\mathbb{R}^d). \tag{7-5}$$

Conversely, *if for some* $c < \infty$

$$\mathcal{E}^{\mu_\alpha}(u,u) \le c\,\mathcal{E}^{\nu_*}(u,u) + \frac{2^\alpha}{r_0^\alpha}\|u\|_{L^2}^2, \quad u \in \mathcal{S}(\mathbb{R}^d), \tag{7-6}$$

then (K2,r_0)2,r_0 *holds.*

Proof. Recalling that $(u(\cdot + z))^\wedge(\xi) = e^{i\xi\cdot z}\hat{u}(\xi)$ and using Plancherel's formula we obtain

$$\begin{aligned}
\mathcal{E}^{\mu}(u,u) &\ge \iint (u(x)-u(x+z))^2 \,\mathrm{d}x\, \nu_*(\mathrm{d}z) \\
&= \iint |e^{i\xi\cdot z}-1|^2 |\hat{u}(\xi)|^2 \,\mathrm{d}\xi\, \nu_*(\mathrm{d}z) \\
&= \int\left(\int 4\sin^2\left(\frac{\xi\cdot z}{2}\right)\nu_*(\mathrm{d}z)\right)|\hat{u}(\xi)|^2\,\mathrm{d}\xi.
\end{aligned} \tag{7-7}$$

If (K2,r_0)2,r_0 holds, then for all $|\xi| > 2/r_0$

$$\int 4\sin^2\left(\frac{\xi\cdot z}{2}\right)\nu_*(\mathrm{d}z) \ge \frac{4c_0}{2^\alpha}|\xi|^\alpha \ge c_0|\xi|^\alpha.$$

For $|\xi| \le 2/r_0$ we have $|\xi|^\alpha \le (2/r_0)^\alpha$. Inequality (7-5) follows from

$$\frac{\mathcal{A}_{d,-\alpha}}{2\alpha(2-\alpha)}\mathcal{E}^\alpha_{\mathbb{R}^d}(u,u) = \int_{\mathbb{R}^d}|\xi|^\alpha|\hat{u}(\xi)|^2\,\mathrm{d}\xi. \tag{7-8}$$

Now we prove the converse. Assume (7-6). By (7-7), the right-hand side of (7-6) equals

$$\int\left(c\int 4\sin^2\left(\frac{\xi\cdot z}{2}\right)\nu_*(\mathrm{d}z) + \frac{2^\alpha}{r_0^\alpha}\right)|\hat{u}(\xi)|^2\,\mathrm{d}\xi;$$

hence by (7-8) and (7-6) we obtain that

$$c\int 4\sin^2\left(\frac{\xi\cdot z}{2}\right)\nu_*(\mathrm{d}z) + \frac{2^\alpha}{r_0^\alpha} \ge |\xi|^\alpha \quad \text{for a.e. } \xi \in \mathbb{R}^d. \tag{7-9}$$

By continuity of the function

$$\mathbb{R}^d\setminus\{0\} \ni \xi \mapsto \int 4\sin^2\left(\frac{\xi\cdot z}{2}\right)\nu_*(\mathrm{d}z),$$

(7-9) holds for all $\xi \in \mathbb{R}^d$. For $|\xi| \ge 2^{1+1/\alpha}r_0^{-1}$ we have by (7-9)

$$c\int 4\sin^2\left(\frac{\xi\cdot z}{2}\right)\nu_*(\mathrm{d}z) \ge \frac{|\xi|^\alpha}{2},$$

and hence (K2,$2^{-1/\alpha}r_0$) holds with $c_0 = 2^{\alpha-3}c^{-1}$. Since

$$\sin^2\left(\frac{h\cdot z}{2r}\right) \geq \frac{1}{4}\sin^2\left(\frac{h\cdot z}{r}\right),$$

also (K2,r_0)2,r_0 holds with *some* constant c_0. □

Appendix

We give the proof of Lemma 4.3. It only uses basic observations.

Lemma A.1. *Assume* $\tau_1, \tau_2 \geq 0$ *and* $\tau_1/\tau_2 \in \left[\frac{1}{2}, 2\right]$. *Then*

$$\frac{\tau_1^2+\tau_2^2}{|\tau_1^2-\tau_2^2|} \geq \frac{5}{3}.$$

Proof. Note that

$$\frac{\tau_1^2+\tau_2^2}{|\tau_1^2-\tau_2^2|} = \frac{\tau_1^2/\tau_2^2+1}{|\tau_1^2/\tau_2^2-1|} = \frac{t+1}{|t-1|},$$

where $t = \tau_1^2/\tau_2^2$. There are three cases:

(1) If $t = 1$, then

$$\frac{t+1}{t-1} = +\infty$$

and the assertion is true.

(2) If $t > 1$, then

$$\frac{t+1}{|t-1|} = \frac{t+1}{t-1}.$$

Note that $(t+1)/(t-1) \geq \frac{5}{3}$ holds true if and only if

$$t+1 \geq \frac{5}{3}t - \frac{5}{3} \iff t \leq 4 \iff \frac{\tau_1}{\tau_2} \leq 2.$$

(3) If $t < 1$, then

$$\frac{t+1}{|t-1|} = \frac{t+1}{-t+1}.$$

Note that $(t+1)/(-t+1) \geq \frac{5}{3}$ holds true if and only if

$$t+1 \geq -\frac{5}{3}t + \frac{5}{3} \iff t \geq \frac{1}{4} \iff \frac{\tau_1}{\tau_2} \geq \frac{1}{2}.$$ □

Lemma A.2. *Assume* $p > 1$ *and* $\eta \in \left(1, \frac{5}{3}\right)$. *Set* $\lambda = ((\eta-1)/(1+\eta))^{1/p}$. *Assume* $a, b > 0$ *and* $b/a \notin (\lambda, 1/\lambda)$. *Then*

$$\frac{a^{-p}+b^{-p}}{|a^{-p}-b^{-p}|} \leq \eta.$$

Proof. Set $t = (b/a)^p$. Then

$$\frac{a^{-p}+b^{-p}}{|a^{-p}-b^{-p}|} = \frac{(a/b)^{-p}+1}{|(a/b)^{-p}-1|} = \frac{t+1}{|t-1|}.$$

Now there are two cases:

Case 1: $t > 1$.

$$\frac{t+1}{|t-1|} \le \eta \iff \frac{t+1}{t-1} \le \eta \iff t \ge \frac{1+\eta}{\eta-1} \iff \frac{b}{a} \ge \left(\frac{1+\eta}{\eta-1}\right)^{1/p}.$$

Case 2: $t < 1$.

$$\frac{t+1}{|t-1|} \le \eta \iff \frac{t+1}{-t+1} \le \eta \iff t \le \frac{\eta-1}{1+\eta} \iff \frac{b}{a} \le \left(\frac{\eta-1}{1+\eta}\right)^{1/p}. \qquad \square$$

Lemma A.3. *There is $c_1 > 0$ such that for $p > 1$, $\lambda = \left(\frac{1}{7}\right)^{1/p}$, and $a, b > 0$ with $b/a \in (\lambda, 1/\lambda)$ the following is true*:

$$\frac{|b-a|(a^{-p}+b^{-p})^2}{|a^{-p}-b^{-p}|} \le \frac{c_1}{p}(b^{-p+1}+a^{-p+1}).$$

Proof. Set $b/a = \xi \in (\lambda, 1/\lambda)$. Then

$$\begin{aligned}\frac{|b-a|(a^{-p}+b^{-p})^2}{|a^{-p}-b^{-p}|} \le \frac{c_1}{p}(b^{-p+1}+a^{-p+1}) &\iff \frac{|a|\,|\xi-1|a^{-2p}(1+\xi^{-p})^2}{|\xi^{-p}-1|a^{-p}} \le \frac{c_1}{p}a^{-p+1}(\xi^{-p+1}+1)\\ &\iff \frac{|\xi-1|(1+\xi^{-p})^2}{|\xi^{-p}-1|} \le \frac{c_1}{p}(\xi^{-p+1}+1)\\ &\iff \frac{|\xi-1|(1+\xi^{-p})^2}{|\xi^{-p}-1|(\xi^{-p+1}+1)} \le \frac{c_1}{p}. \qquad \text{(A-1)}\end{aligned}$$

Let us prove (A-1). Note that

$$\frac{|\xi-1|(1+\xi^{-p})^2}{|\xi^{-p}-1|(\xi^{-p+1}+1)} \le \frac{|\xi-1|(1+7)^2}{|\xi^{-p}-1|} = 64\frac{|\xi-1|}{|\xi^{-p}-1|}.$$

We want to apply the mean value theorem. Set $\xi \mapsto g(\xi) = \xi^{-p}$. Then $g'(\xi) = (-p)\xi^{-(p+1)}$. The mean value theorem implies

$$\frac{|\xi^{-p}-1|}{|\xi-1|} = \frac{|g(\xi)-1|}{|\xi-1|} = |g(x)| = px^{-(p+1)} \quad \text{for some } x \in (\xi,1)\cup(1,\xi).$$

Thus,

$$\frac{|\xi^{-p}-1|}{|\xi-1|} \ge p\left(\frac{1}{\lambda}\right)^{-(p+1)} = p(7^{1/p})^{-(p+1)} = p7^{-1-\frac{1}{p}},$$

from which we deduce

$$\frac{|\xi-1|(1+\xi^{-p})^2}{|\xi^{-p}-1|(\xi^{-p+1}+1)} \le 64\frac{7^{1+\frac{1}{p}}}{p} \le \frac{64\cdot 49}{p} = \frac{c_1}{p}. \qquad \square$$

Lemma A.4. *For $p > 1$ and $a, b > 0$ the following is true*:

$$(b-a)(a^{-p}-b^{-p}) \geq \frac{2}{p-1}(a^{\frac{-p+1}{2}} - b^{\frac{-p+1}{2}})^2.$$

The proof of the above lemma is simple and can be found in several places, e.g., in [Kassmann 2009].

Lemma A.5. *Assume $p > 1$, $a, b > 0$, and $\tau_1, \tau_2 \geq 0$. Then*

$$(\tau_1+\tau_2)^2(a^{\frac{-p+1}{2}} - b^{\frac{-p+1}{2}})^2 \geq 2(\tau_1 a^{\frac{-p+1}{2}} - \tau_2 b^{\frac{-p+1}{2}})^2 - 2(\tau_1-\tau_2)^2(a^{-p+1}+b^{-p+1}).$$

Proof. Note

$$2(\tau_1 a^{\frac{-p+1}{2}} - \tau_2 b^{\frac{-p+1}{2}}) = (\tau_1-\tau_2)(a^{\frac{-p+1}{2}} + b^{\frac{-p+1}{2}}) + (\tau_1+\tau_2)(a^{\frac{-p+1}{2}} - b^{\frac{-p+1}{2}}).$$

From this equality we obtain the assertion as follows:

$$\begin{aligned}4(\tau_1 a^{\frac{-p+1}{2}} - \tau_2 b^{\frac{-p+1}{2}})^2 &\leq 2(\tau_1-\tau_2)^2(a^{\frac{-p+1}{2}} + b^{\frac{-p+1}{2}})^2 + 2(\tau_1+\tau_2)^2(a^{\frac{-p+1}{2}} - b^{\frac{-p+1}{2}})^2 \\ &\leq 4(\tau_1-\tau_2)^2(a^{-p+1}+b^{-p+1}) + 2(\tau_1+\tau_2)^2(a^{\frac{-p+1}{2}} - b^{\frac{-p+1}{2}})^2. \quad \square\end{aligned}$$

Finally, we can give the proof of Lemma 4.3.

Proof of Lemma 4.3. Let us first consider the case $\tau_1/\tau_2 \notin \left(\frac{1}{2}, 2\right)$. Note that, in this case

$$\max\{\tau_1, \tau_2\} \leq 2|\tau_1-\tau_2| \tag{A-2}$$

and

$$-(\tau_1 a^{\frac{-p+1}{2}} - \tau_2 b^{\frac{-p+1}{2}})^2 = -\tau_1^2 a^{-p+1} - \tau_2^2 b^{-p+1} + 2\tau_1 a^{\frac{-p+1}{2}} \tau_2 b^{\frac{-p+1}{2}} \geq -\tau_1^2 a^{-p+1} - \tau_2^2 b^{-p+1}.$$

Thus, we obtain

$$\begin{aligned}(b-a)(\tau_1^2 a^{-p} - \tau_2^2 b^{-p}) &\geq -\tau_1^2 a^{-p+1} - \tau_2^2 b^{-p+1} + (\tau_1 a^{\frac{-p+1}{2}} - \tau_2 b^{\frac{-p+1}{2}})^2 - (\tau_1 a^{\frac{-p+1}{2}} - \tau_2 b^{\frac{-p+1}{2}})^2 \\ &\geq (\tau_1 a^{\frac{-p+1}{2}} - \tau_2 b^{\frac{-p+1}{2}})^2 - 2\tau_1^2 a^{-p+1} - 2\tau_2^2 b^{-p+1} \\ &\geq (\tau_1 a^{\frac{-p+1}{2}} - \tau_2 b^{\frac{-p+1}{2}})^2 - 2\max\{\tau_1,\tau_2\}^2 a^{-p+1} - 2\max\{\tau_1,\tau_2\}^2 b^{-p+1} \\ &\geq (\tau_1 a^{\frac{-p+1}{2}} - \tau_2 b^{\frac{-p+1}{2}})^2 - 8(\tau_1-\tau_2)^2(a^{-p+1}+b^{-p+1}).\end{aligned}$$

The proof in the case $\tau_1/\tau_2 \notin \left(\frac{1}{2}, 2\right)$ is complete.

Let us now assume $\tau_1/\tau_2 \in \left[\frac{1}{2}, 2\right]$. A general observation is

$$(b-a)(\tau_1^2 a^{-p} - \tau_2^2 b^{-p}) = \underbrace{\tfrac{1}{2}(b-a)(\tau_1^2-\tau_2^2)(a^{-p}+b^{-p})}_{:=P} + \underbrace{\tfrac{1}{2}(b-a)(\tau_1^2+\tau_2^2)(a^{-p}-b^{-p})}_{:=G}.$$

Recall that Lemma A.4 implies

$$\frac{1}{2}(b-a)(a^{-p}-b^{-p}) \geq \frac{1}{p-1}(a^{\frac{-p+1}{2}} - b^{\frac{-p+1}{2}})^2. \tag{A-3}$$

Choose $\eta = \frac{4}{3}$ and $\lambda = \left(\frac{1}{7}\right)^{1/p}$. Let us consider two subcases.

Case 1: $b/a \in (\lambda, 1/\lambda)$, $\tau_1/\tau_2 \in \left[\frac{1}{2}, 2\right]$. In this case

$$
\begin{aligned}
|P| &= \Big[\frac{1}{4}(\tau_1+\tau_2)|b-a|^{\frac{1}{2}}|a^{-p}-b^{-p}|^{\frac{1}{2}}\Big]\big[2|\tau_1-\tau_2|\,|a^{-p}-b^{-p}|^{-\frac{1}{2}}|b-a|^{\frac{1}{2}}(a^{-p}+b^{-p})\big] \\
&\le \frac{1}{16}(\tau_1+\tau_2)^2(b-a)(a^{-p}-b^{-p}) + 4(\tau_1-\tau_2)^2 \underbrace{\frac{(b-a)(a^{-p}+b^{-p})^2}{(a^{-p}-b^{-p})}}_{:=F}.
\end{aligned}
$$

Because of Lemma A.3, we know that there is $c_5 > 0$ such that $|F| \le (c_5/p)(b^{-p+1} + a^{-p+1})$. Altogether, we obtain

$$
\begin{aligned}
&(b-a)(\tau_1^2 a^{-p} - \tau_2^2 b^{-p}) \\
&\quad = \frac{1}{2}(b-a)(\tau_1^2-\tau_2^2)(a^{-p}+b^{-p}) + \frac{1}{2}(b-a)(\tau_1^2+\tau_2^2)(a^{-p}-b^{-p}) \\
&\quad \ge \frac{1}{2}(b-a)(\tau_1^2-\tau_2^2)(a^{-p}+b^{-p}) + \frac{1}{4}(b-a)(\tau_1+\tau_2)^2(a^{-p}-b^{-p}) \\
&\quad \ge -\frac{1}{16}(\tau_1+\tau_2)^2(b-a)(a^{-p}-b^{-p}) - 4(\tau_1-\tau_2)^2\frac{(b-a)(a^{-p}+b^{-p})^2}{(a^{-p}-b^{-p})} + \frac{1}{4}(b-a)(\tau_1+\tau_2)^2(a^{-p}-b^{-p}) \\
&\quad = \frac{3}{16}(\tau_1+\tau_2)^2(b-a)(a^{-p}-b^{-p}) - 4(\tau_1-\tau_2)^2\frac{(b-a)(a^{-p}+b^{-p})^2}{(a^{-p}-b^{-p})} \\
&\quad \ge \frac{3}{16(p-1)}(\tau_1+\tau_2)^2\big(a^{\frac{-p+1}{2}} - b^{\frac{-p+1}{2}}\big)^2 - \frac{4c_5}{p}(\tau_1-\tau_2)^2(b^{-p+1}+a^{-p+1}) \\
&\quad \ge \frac{6}{16(p-1)}\big(\tau_1 a^{\frac{-p+1}{2}} - \tau_2 b^{\frac{-p+1}{2}}\big)^2 - \left(\frac{4c_5}{p} + \frac{6}{16(p-1)}\right)(\tau_1-\tau_2)^2(b^{-p+1}+a^{-p+1}),
\end{aligned}
$$

where we applied Lemma A.5. The first case has been completed.

Case 2: $b/a \notin (\lambda, 1/\lambda)$, $\tau_1/\tau_2 \in \left[\frac{1}{2}, 2\right]$. Then Lemmas A.1 and A.2 imply

$$
\begin{aligned}
P \ge -|P| &= -\frac{1}{2}|b-a|\,|\tau_1^2-\tau_2^2|(a^{-p}+b^{-p}) \ge -\frac{3}{10}|b-a|(\tau_1^2+\tau_2^2)(a^{-p}+b^{-p}) \\
&\ge -\frac{3}{10}\cdot\frac{4}{3}|b-a|(\tau_1^2+\tau_2^2)|a^{-p}-b^{-p}| = -\frac{2}{5}(b-a)(\tau_1^2+\tau_2^2)(a^{-p}-b^{-p}) = -\frac{4}{5}G.
\end{aligned}
$$

Thus, due to Lemma A.4, we obtain

$$
\begin{aligned}
(b-a)(\tau_1^2 a^{-p} - \tau_2^2 b^{-p}) = P + G \ge \tfrac{1}{5}G &\ge \frac{1}{5(p-1)}(\tau_1^2+\tau_2^2)\big(a^{\frac{-p+1}{2}} - b^{\frac{-p+1}{2}}\big)^2 \\
&\ge \frac{1}{10(p-1)}(\tau_1+\tau_2)^2\big(a^{\frac{-p+1}{2}} - b^{\frac{-p+1}{2}}\big)^2 \\
&\ge \frac{1}{5(p-1)}\big(\tau_1 a^{\frac{-p+1}{2}} - \tau_2 b^{\frac{-p+1}{2}}\big)^2 - \frac{1}{5(p-1)}(\tau_1-\tau_2)^2(b^{-p+1}+a^{-p+1}).
\end{aligned}
$$

The proof in the case $\tau_1/\tau_2 \in \left[\frac{1}{2}, 2\right]$ is complete. The proof of Lemma 4.3 is complete if we choose c_1 and c_2 appropriately. □

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Note in proof

Since the submission of the article, interesting related contributions have appeared, e.g., [Chen et al. 2019].

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ON SOLVABILITY AND ILL-POSEDNESS OF THE COMPRESSIBLE EULER SYSTEM SUBJECT TO STOCHASTIC FORCES

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We consider the barotropic Euler system describing the motion of a compressible inviscid fluid driven by a stochastic forcing. Adapting the method of convex integration we show that the initial value problem is ill-posed in the class of weak (distributional) solutions. Specifically, we find a sequence $\tau_M \to \infty$ of positive stopping times for which the Euler system admits infinitely many solutions originating from the same initial data. The solutions are weak in the PDE sense but strong in the probabilistic sense, meaning, they are defined on an a priori given stochastic basis and adapted to the driving stochastic process.

1. Introduction

Solutions of nonlinear systems of conservation laws, including the compressible Euler system discussed in the present paper, are known to develop singularities in finite time even for smooth initial data. Weak solutions that can accommodate these singularities provide therefore a suitable framework for studying the behavior of the system in the long run. A delicate and still largely open question is well-posedness of the associated initial value problem in the class of weak solutions. More precisely, a suitable admissibility criterion is needed to select the physically relevant solution. The method of *convex integration*, developed in the context of fluid mechanics in [De Lellis and Székelyhidi 2012], gives rise to several striking results concerning well-/ill-posedness of the Cauchy problem for the Euler system and related models of inviscid fluids; see, e.g., [Chiodaroli 2014; De Lellis and Székelyhidi 2009; 2010]. In particular, the barotropic Euler system in two and three space dimensions is ill-posed in the class of *admissible* entropy solutions (solutions dissipating energy) even for rather regular initial data; see [Chiodaroli, De Lellis, and Kreml 2015; Chiodaroli and Kreml 2014]. In the context of *incompressible* fluids, the method has been used for attacking the celebrated Onsager's conjecture, finally proved in [Isett 2018], accompanied by related results obtained in [Buckmaster, De Lellis, Székelyhidi, and Vicol 2019]. Very recently, the ill-posedness in the class of weak solutions has been extended even for the incompressible Navier–Stokes system in [Buckmaster and Vicol 2019]; see also [Buckmaster, Colombo, and Vicol 2018].

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Keywords: stochastic compressible Euler system, weak solution, compressible Euler system, stochastic forcing, ill-posedness, convex integration.

In the present paper, we show that this difficulty persists even in the presence of a random forcing. As a model example, we consider the barotropic Euler system describing the time evolution of the density ϱ and the velocity $\boldsymbol{u}$ of a compressible fluid:

$$\mathrm{d}\varrho + \mathrm{div}_x(\varrho \boldsymbol{u})\,\mathrm{d}t = 0, \tag{1-1}$$

$$\mathrm{d}(\varrho \boldsymbol{u}) + \mathrm{div}_x(\varrho \boldsymbol{u} \otimes \boldsymbol{u})\,\mathrm{d}t + \nabla_x p(\varrho)\,\mathrm{d}t = \varrho \boldsymbol{G}(\varrho, \varrho \boldsymbol{u})\,\mathrm{d}W, \tag{1-2}$$

where $p = p(\varrho)$ is the pressure, and the term $\varrho \boldsymbol{G}(\varrho, \varrho \boldsymbol{u})\,\mathrm{d}W$ represents a random volume force acting on the fluid. A typical example is the so-called isentropic pressure density state equation $p(\varrho) = \varrho^\gamma$. We focus on two iconic examples of forcing, namely,

$$\varrho \boldsymbol{G}(\varrho, \varrho \boldsymbol{u})\,\mathrm{d}W = \varrho \boldsymbol{G}\,\mathrm{d}W = \varrho \sum_{i=1}^{\infty} \boldsymbol{G}^i\,\mathrm{d}\beta_i, \quad \boldsymbol{G}^i = \boldsymbol{G}^i(x), \tag{1-3}$$

or

$$\varrho \boldsymbol{G}(\varrho, \varrho \boldsymbol{u})\,\mathrm{d}W = \varrho \boldsymbol{u}\,\mathrm{d}\beta. \tag{1-4}$$

Here $\beta_i = \beta_i(t)$, $\beta = \beta(t)$ are real-valued Wiener processes, whereas the diffusion coefficients $\boldsymbol{G}^i$ are smooth functions depending only on the spatial variable x. For the sake of simplicity, we consider periodic boundary conditions, meaning the underlying spatial domain can be identified with a flat torus,

$$\mathcal{T}^N = \left([0, 1]|_{\{0,1\}}\right)^N, \quad N = 2, 3.$$

Other boundary conditions, in particular the impermeability of the boundary, could be accommodated at the expense of additional technical difficulties.

The problem of solvability of the stochastic compressible Euler system (1-1), (1-2) is very challenging with only a few results available. In space dimension 1, [Berthelin and Vovelle 2013] proved existence of entropy solutions. These solutions are also weak in the probabilistic sense; that is, the underlying stochastic elements are not known in advance and become part of the solution. The only available results in higher space dimensions concern the local well-posedness of strong solutions. To be more precise, given a sufficiently smooth initial condition

$$\varrho(0, \cdot) = \varrho_0, \quad \varrho \boldsymbol{u}(0, \cdot) = (\varrho \boldsymbol{u})_0, \tag{1-5}$$

it can be shown that the problem (1-1), (1-2), (1-5) admits a unique local strong solution taking values in the class of Sobolev spaces $W^{m,2}$ of order $m > \frac{1}{2}N + 3$. These solutions are strong in both the PDE and probabilistic sense; i.e., they are constructed on a given stochastic basis with a given Wiener process. Nevertheless, they exist (and are unique in terms of the initial data) only up to a strictly positive maximal stopping time τ. Beyond this time that may be finite, the solutions develop singularities and uniqueness is not known. We refer the reader to [Breit, Feireisl, and Hofmanová 2018], where the stochastic compressible Navier–Stokes system with periodic boundary conditions was treated, and in particular to Remark 2.10 of that work for a discussion of the inviscid case. Let us finally remark that general symmetric hyperbolic systems on the whole space R^N were studied in [Kim 2011].

For completeness, let us mention that (1-4) may be seen as a "damping" term, the regularizing effect of which in the context of *incompressible* fluids has been recognized in [Glatt-Holtz and Vicol 2014], and for a general symmetric hyperbolic system in [Kim 2011]. To be more precise, in [Kim 2011] it was shown that the probability that the strong solution never blows up can be made arbitrarily close to 1 provided the initial condition is sufficiently small. In [Glatt-Holtz and Vicol 2014] it was proved that the smallness assumption on the initial condition can be replaced by large intensity of the noise. Additionally, in the case of additive noise, which in our setting corresponds to (1-3), that work showed global existence of strong solutions to the incompressible Euler equations in two dimensions.

Our goal in the present paper is to show that the problem (1-1), (1-2) is ill-posed in the class of weak (distributional) solutions. More precisely, we show that there exists an increasing sequence of strictly positive stopping times τ_M, with $\tau_M \to \infty$ as $M \to \infty$ a.s., such that problem (1-1), (1-2), (1-3) or (1-4), (1-5) admits infinitely many weak solutions in the time interval $[0, \tau_M \wedge T)$ for any positive T. We emphasize that *weak* is meant only in the PDE sense — the spatial derivatives are understood in the distributional framework — while solutions are *strong* in the probabilistic sense. To be more precise, the stochastic basis together with a driving Wiener process W are given and we construct infinitely many solutions that are stochastic processes adapted to the given filtration. This is particularly interesting in light of the fact that uniqueness is violated. Indeed, without the knowledge of uniqueness it is typically only possible to construct probabilistically weak solutions that are not adapted to the given Wiener process. This already applies on the level of SDEs; see, for instance, the discussion in [Karatzas and Shreve 1988, Chapter 5].

Formally, both (1-3) and (1-4) represent a *multiplicative* noise. Nevertheless, under these assumptions, the system of stochastic PDEs (1-1), (1-2) may be reduced to a system of PDEs with random coefficients by means of a simple transformation. As a consequence, the stochastic integral no longer appears in the system and deterministic methods can be employed pathwise. Such a semideterministic approach was already used in many works; see for instance [Feireisl, Maslowski, and Novotný 2013; Tornatore and Fujita Yashima 1997] for the compressible setting, and the seminal paper [Bensoussan and Temam 1973] for the incompressible case. However, we point out that in all these references, the nontrivial issue of adaptedness of solutions with respect to the underlying stochastic perturbation remained unsolved. Therefore, it was not possible to return to the original formulation of the problem with a well-defined stochastic Itô integral. Even though we employ a similar semideterministic approach to (1-1), (1-2), (1-3) or (1-4), we are able to answer affirmatively the question of adaptedness and accordingly the stochastic Itô integral in the original formulation (1-1), (1-2) is well-defined.

To be more precise, for both (1-3) and (1-4), we rewrite (1-1), (1-2) as an *abstract Euler system* with variable random coefficients in the spirit of [Feireisl 2016]. This relies on the particular structure of the compressible Euler system and its interplay with stochastic perturbations satisfying (1-3) or (1-4). The resulting problem is then solved by an adaptation of the deterministic method of convex integration developed in [De Lellis and Székelyhidi 2010]. The main difficulty is to ensure that the abstract construction based on the concept of subsolutions yields a solution ϱ, $\varrho \boldsymbol{u}$ *adapted* to the noise W. This is done by a careful analysis of the oscillatory lemma of [De Lellis and Székelyhidi 2010], where adaptedness is achieved by a delicate use of the celebrated Ryll–Nardzewski theorem on the existence of a measurable selection of a multivalued mapping.

The key point is to study a certain nonpositive functional I (see Section 6D) defined on an appropriate class of subsolutions (see Section 6A) to the abstract Euler system. These subsolutions capture already all the required (probabilistic) properties expected from the solutions. Similarly to [De Lellis and Székelyhidi 2010], the existence of infinitely many solutions to the original problem is obtained by applying an abstract Baire category argument based on the possibility of augmenting a given subsolution by rapidly oscillating increments. Determining the amplitude as well as the frequency of these oscillatory components at a given time t requires knowing the behavior of a given subsolution up to the time $t+\delta,\ \delta > 0$. The specific value of δ is in general a random variable, the value of which depends on the behavior of the noise W in the interval $[t, t+\delta)$. Consequently, it is not adapted with respect to the natural filtration associated to the noise. The problem can be solved only if $\delta > 0$ is deterministic, specifically if the solution paths belong to a fixed compact set. To ensure this, we replace W by $W_M(t) = W(t \wedge \tau_M)$, where τ_M is a family of suitable stopping times defined in terms of the Hölder norm of W. It is exactly this rather technical difficulty that restricts validity of our main result to the random time interval $[0, \tau_M)$. Note, however, that τ_M can be made arbitrarily large with probability arbitrarily close to 1.

Let us stress that our results apply mutatis mutandis to situations when the driving force is given by a more general stochastic process or a deterministic signal of low regularity. Provided a suitable transformation formula to a PDE with random coefficients can be justified, the only ingredient is the one required in Section 3A for the construction of the corresponding stopping times τ_M. Namely, the trajectories of the driving stochastic process are supposed to be a.s. a-Hölder continuous for some $a \in (0, 1)$. Then existence of infinitely many weak solutions (to the transformed system) adapted to the given stochastic process follows. Whether it is possible to go back to the original formulation then depends on the particular stochastic process at hand, namely, whether a corresponding stochastic integral can be constructed. If the driving signal is a deterministic Hölder continuous path, the stopping times are not needed and we obtain infinitely many weak solutions (to the transformed system) defined on the full time interval $[0, T]$.

It is important to note that the restriction to the multidimensional case $N = 2, 3$ is absolutely essential here and the variant of the method of convex integration presented below does not work for $N = 1$. Indeed, the method leans on the property of the system to admit oscillatory solutions. As observed in the pioneering works [DiPerna 1983a; 1983b], the deterministic counterpart of (1-1), (1-2) appended by suitable admissibility conditions gives rise to a solution set that is precompact in the L^p framework if $N = 1$.

To conclude this introductory part, let us summarize the current state of understanding of a compressible flow of an inviscid fluid under stochastic perturbation. Consider a sufficiently smooth initial condition (1-5) and a fixed stochastic basis. On the one hand, it can be shown that there exists a unique local strong solution. However, in view of our result, there exist infinitely many weak solutions emanating from the same initial datum. The very natural question is therefore whether one can compare these two kinds of solutions. In fluid dynamics, it is often possible to establish a so-called weak-strong uniqueness result: strong solutions coincide with weak solutions satisfying a suitable form of energy inequality. The corresponding result for the stochastic compressible Navier–Stokes system was proved in [Breit, Feireisl, and Hofmanová 2017]. Consequently, it would be interesting to see whether our weak solutions could be constructed to satisfy an energy inequality. In analogy with the deterministic setting, we know this might be possible only for certain initial data and we leave this problem to be addressed in future work.

The paper is organized as follows. In Section 2, we introduce a proper definition of a weak solution and state our main results. In Section 3, the problem is rewritten in a semideterministic way that eliminates the explicit presence of stochastic integrals. In Section 4, we rewrite the system as an abstract Euler problem in the spirit of [Feireisl 2016]. Section 5 is the heart of the paper. Here, the apparatus of convex integration developed in [De Lellis and Székelyhidi 2010] is adapted to stochastic framework. The main result is a stochastic variant of the oscillatory lemma (Lemma 5.6) proved via the Ryll–Nardzewski theorem on measurable selection. The proof of the main result is completed in Section 6.

2. Problem formulation and main results

Let $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \mathbb{P})$ be a probability space with a complete right-continuous filtration $(\mathfrak{F}_t)_{t\geq 0}$. For the sake of simplicity, we restrict ourselves to the case of a single noise; specifically,

$$\varrho \boldsymbol{G}(\varrho, \varrho \boldsymbol{u})\,\mathrm{d}W = \varrho \boldsymbol{G}(x)\,\mathrm{d}\beta \quad \text{or} \quad \varrho \boldsymbol{G}(\varrho, \varrho \boldsymbol{u})\,\mathrm{d}W = \varrho \boldsymbol{u}\,\mathrm{d}\beta, \tag{2-1}$$

where $\beta = \beta(t)$ is a standard Wiener process relative to the filtration $(\mathfrak{F}_t)_{t\geq 0}$. In particular, we may correctly define the stochastic integral (in Itô's sense)

$$\int_0^\tau \left(\int_{\mathcal{T}^N} \varrho \boldsymbol{G}(\varrho, \varrho \boldsymbol{u}) \cdot \boldsymbol{\varphi}\,\mathrm{d}x \right) \mathrm{d}W$$

as soon as the processes

$$t \mapsto \int_{\mathcal{T}^N} \varrho \phi\,\mathrm{d}x, \quad t \mapsto \int_{\mathcal{T}^N} \varrho \boldsymbol{u} \cdot \boldsymbol{\varphi}\,\mathrm{d}x \tag{2-2}$$

are $(\mathfrak{F}_t)$-progressively measurable for any smooth (deterministic) test functions $\phi = \phi(x)$ and $\boldsymbol{\varphi} = \boldsymbol{\varphi}(x)$.

Definition 2.1. We say that $[\varrho, \boldsymbol{u}, \tau]$ is a weak solution to problem (1-1), (1-2), (1-5) with a stopping time τ provided:

(i) $\tau \geq 0$ is an $(\mathfrak{F}_t)$-stopping time.

(ii) The density ϱ is $(\mathfrak{F}_t)$-adapted and satisfies

$$\varrho \in C([0, \tau); W^{1,\infty}(\mathcal{T}^N)), \quad \varrho > 0 \ \ \mathbb{P}\text{-a.s.}$$

(iii) The momentum $\varrho \boldsymbol{u}$ satisfies $t \mapsto \int_{\mathcal{T}^N} \varrho \boldsymbol{u} \cdot \boldsymbol{\varphi}\,\mathrm{d}x \in C([0, \tau])$ for any $\boldsymbol{\varphi} \in C_c^\infty(\mathcal{T}^N; R^N)$, the stochastic process $t \mapsto \int_{\mathcal{T}^N} \varrho \boldsymbol{u} \cdot \boldsymbol{\varphi}\,\mathrm{d}x$ is $(\mathfrak{F}_t)$-adapted, and

$$\varrho \boldsymbol{u} \in C_{\mathrm{weak}}([0, \tau); L^2(\mathcal{T}^N; R^N)) \cap L^\infty((0, \tau) \times \mathcal{T}^N; R^N) \quad \mathbb{P}\text{-a.s.}.$$

(iv) For all $\phi \in C_c^\infty(\mathcal{T}^N)$ and all $t \geq 0$ the following holds $\mathbb{P}$-a.s.:

$$\int_{\mathcal{T}^N} \varrho(t \wedge \tau, \cdot)\phi\,\mathrm{d}x - \int_{\mathcal{T}^N} \varrho_0 \phi\,\mathrm{d}x = \int_0^{t\wedge\tau} \int_{\mathcal{T}^N} \varrho \boldsymbol{u} \cdot \nabla_x \phi\,\mathrm{d}x\,\mathrm{d}t. \tag{2-3}$$

(v) For all $\boldsymbol{\varphi} \in C_c^\infty(\mathcal{T}^N, R^N)$ and all $t \geq 0$ the following holds $\mathbb{P}$-a.s.:

$$\int_{\mathcal{T}^N} \varrho \boldsymbol{u}(t \wedge \tau, \cdot) \cdot \boldsymbol{\varphi} \, \mathrm{d}x - \int_{\mathcal{T}^N} (\varrho \boldsymbol{u})_0 \cdot \boldsymbol{\varphi} \, \mathrm{d}x \\ = \int_0^{t \wedge \tau} \int_{\mathcal{T}^N} [\varrho \boldsymbol{u} \otimes \boldsymbol{u} : \nabla_x \boldsymbol{\varphi} + p(\varrho) \operatorname{div}_x \boldsymbol{\varphi}] \, \mathrm{d}x \, \mathrm{d}t + \int_0^{t \wedge \tau} \left(\int_{\mathcal{T}^N} \varrho \boldsymbol{G} \cdot \boldsymbol{\varphi} \, \mathrm{d}x \right) \mathrm{d}W. \tag{2-4}$$

Remark 2.2. The processes in (2-2) are continuous and $(\mathfrak{F}_t)$-adapted, whence progressively measurable. Consequently, the stochastic integral in (2-3) is correctly defined as soon as $\boldsymbol{G} = \boldsymbol{G}(\varrho, \varrho \boldsymbol{u})$ satisfies (2-1).

We are ready to formulate our main result.

Theorem 2.3. *Let $T > 0$ and the initial data ϱ_0, $(\varrho \boldsymbol{u})_0$ be $\mathfrak{F}_0$-measurable such that*

$$\varrho_0 \in C^3(\mathcal{T}^N), \quad (\varrho \boldsymbol{u})_0 \in C^3(\mathcal{T}^N; R^N), \quad \varrho_0 > 0 \ \ \mathbb{P}\text{-a.s.} \tag{2-5}$$

Let the stochastic term satisfy (2-1), *where β is a standard Wiener process, and the coefficient $\boldsymbol{G} \in W^{1,\infty}(\mathcal{T}^N; R^N)$ is a given deterministic function. Finally, suppose that the pressure function $p = p(\varrho)$ satisfies*

$$p \in C^1[0, \infty) \cap C^2(0, \infty), \quad p(0) = 0.$$

Then there exists a family of $\mathbb{P}$-a.s. strictly positive $(\mathfrak{F}_t)$-stopping times τ_M satisfying $\tau_M \leq \tau_L$ $\mathbb{P}$-a.s. for $M \leq L$, and

$$\tau_M \to \infty \quad \text{as } M \to \infty \quad \mathbb{P}\text{-a.s.},$$

such that problem (1-1), (1-2), (1-5) *admits infinitely many weak solutions with the stopping time $\tau = \tau_M \wedge T$ in the sense of Definition 2.1.*

Remark 2.4. Solutions obtained in Theorem 2.3 are "almost global" in the sense that for any $\varepsilon > 0$, problem (1-1), (1-2), (1-5) admits infinitely many (weak) solutions living on a given time interval $(0, T)$ with probability $1 - \varepsilon$ (choosing M large enough). The necessity of considering finite stopping times is explained in detail in Remark 5.13 below.

Remark 2.5. We transform the problem to an abstract Euler system (see (4-7), (4-8) and (4-13), (4-14)) and show the existence of infinitely many solutions to the latter one. It is worth noting that our approach can be applied to other problems in fluid mechanics, in particular to the incompressible stochastic Euler equations. See also Remark 4.2.

The rest of the paper is devoted to the proof of Theorem 2.3. Let us now summarize the key points of our construction. For both (1-3) and (1-4), we rewrite (1-1), (1-2) as an *abstract Euler system* with variable random coefficients in the spirit of [Feireisl 2016]. On the set of subsolutions to this system we define the functional

$$I[\boldsymbol{v}] = \mathbb{E}\left[\int_0^T \int_{\mathcal{T}^N} \left[\frac{1}{2} \frac{|\boldsymbol{v} + \boldsymbol{h}|^2}{r} - e \right] \mathrm{d}x \, \mathrm{d}t \right].$$

Here, $\boldsymbol{h}$, r are given functions related to the density ansatz and e is the target energy. The solutions of the problem are represented by the points of continuity of I with respect to $\boldsymbol{v}$. The exact definition of

I is given in Section 6D below. It is rather standard to see that I has infinitely many continuity points and that $I[\boldsymbol{v}] = 0$ implies that $\boldsymbol{v}$ is a solution. The bulk is to show that each continuity point satisfies $I[\boldsymbol{v}] = 0$, which implies the existence of infinitely many solutions. The latter statement can be shown indirectly by augmenting a given continuity point by rapidly oscillating increments. These increments are obtained by an adaptation of the deterministic method of convex integration developed in [De Lellis and Székelyhidi 2010]. The main difficulty is to ensure progressive measurability in this construction. Following [Donatelli, Feireisl, and Marcati 2015] we proceed in three steps:

(i) Assuming the subsolution under consideration is constant in space-time (but random) we gain an oscillator sequence which is a random variable itself by the Ryll–Nardzewski theorem on measurable selection. This is first done on the unit interval with density equal to 1 (see Lemma 5.6). A more general version follows by scaling (see Lemma 5.8).

(ii) The construction from (i) can be extended to piecewise constant subsolutions which are evaluated at the first time-point of each subinterval. This ensures progressive measurability of the oscillatory sequence (see Lemma 5.10).

(iii) Finally, we consider the general case of continuous subsolutions (see Lemma 5.11). They can be approximated by piecewise constant ones and we can apply step (ii). It is important that the modulus of continuity can be controlled. This is where the stopping times in the noise come into play.

3. Transformation to a semideterministic setting

In view of the difficulties mentioned in Section 1, we are forced to replace the original Wiener process β by a suitable truncation and to rewrite the problem in a semideterministic setting.

3A. ***Stopping times.*** We start by fixing a family $(\tau_M)_{M\in\mathbb{N}}$ of stopping times enjoying the properties claimed in Theorem 2.3. For a given $0 < a < \frac{1}{2}$ and the Wiener process β, $\beta(0) = 0$ $\mathbb{P}$-a.s., we introduce

$$O(t) = \sup_{0\le s\le t} |\beta(s)| + \sup_{0\le t_1\ne t_2\le t} \frac{|\beta(t_1)-\beta(t_2)|}{|t_1-t_2|^a} \quad \text{for } t > 0,\ O(0) = 0.$$

Obviously, O is a nondecreasing stochastic process adapted to $(\mathfrak{F}_t)_{t\ge 0}$. Moreover, as β is a Wiener process, it follows from the Kolmogorov continuity criterion that

$$|\beta(t_1)-\beta(t_2)| \le B(T,b)|t_1-t_2|^b = B(T,b)|t_1-t_2|^{b-a}|t_1-t_2|^a \quad \text{whenever } 0 \le t_1, t_2 \le T,$$

for any $0 < a < b < \frac{1}{2}$, $T > 0$, where $B(T, b)$ is random and finite $\mathbb{P}$-a.s. In particular, we deduce that O is continuous in $[0, \infty)$. As a consequence, for $M \in \mathbb{N}$,

$$\tau_M = \inf_{t\ge 0}\{O(t) > M\} \wedge T$$

defines an $(\mathfrak{F}_t)$-stopping time. Moreover, $\tau_M \le \tau_L$ $\mathbb{P}$-a.s. for $M \le L$, and in particular we get

$$\tau_M \to \infty \quad \text{as } M \to \infty \quad \mathbb{P}\text{-a.s.}$$

Finally, as O is continuous and $O(0) = 0$ $\mathbb{P}$-a.s., we have that $\tau_M > 0$ $\mathbb{P}$-a.s. for all $M \in \mathbb{N}$.

Next, let us introduce the stopped stochastic process

$$W_M = \beta_M, \quad \beta_M(t) = \beta(t \wedge \tau_M) \quad \text{for } t \geq 0.$$

We recall that, for $\tau = \tau_M$, the stochastic integral in (2-4) can be rewritten as

$$\int_0^{t\wedge\tau_M} \left(\int_{\mathcal{T}^N} \varrho \boldsymbol{G} \cdot \boldsymbol{\varphi} \, \mathrm{d}x \right) \mathrm{d}W = \int_0^t \left(\int_{\mathcal{T}^N} \varrho \boldsymbol{G} \cdot \boldsymbol{\varphi} \, \mathrm{d}x \right) \mathrm{d}W_M.$$

From now on, we consider problem (1-1), (1-2), (1-5) with β replaced by β_M. Under these circumstances, our task reduces to showing Theorem 2.3 with $\beta = \beta_M$ on the *deterministic* time interval $[0, T]$. Note that the paths of β_M are *uniformly* bounded and *uniformly* Hölder continuous,

$$\|\beta_M\|_{C^a[0,T]} \leq M, \quad 0 < a < \tfrac{1}{2} \ \mathbb{P}\text{-a.s.} \tag{3-1}$$

This is the essential property we use to construct probabilistically strong solutions, that is, solutions that are adapted to the given filtration $(\mathfrak{F}_t)_{t\geq 0}$ associated to β.

3B. ***Problem with additive noise.*** If the noise is given by (1-3), we may combine Itô's calculus with the equation of continuity (2-3) to rewrite the stochastic integral in the form

$$\int_0^t \left(\int_{\mathcal{T}^N} \varrho \boldsymbol{G} \cdot \boldsymbol{\varphi} \, \mathrm{d}x \right) \mathrm{d}\beta_M = \left(\int_{\mathcal{T}^N} \varrho \boldsymbol{G} \cdot \boldsymbol{\varphi} \, \mathrm{d}x \right) \beta_M(t) - \int_0^t \beta_M(s) \int_{\mathcal{T}^N} \varrho \boldsymbol{u} \cdot \nabla_x (\boldsymbol{G} \cdot \boldsymbol{\varphi}) \, \mathrm{d}x \, \mathrm{d}s.$$

Consequently, the momentum equation (1-2) can be formally written as

$$\mathrm{d}(\varrho \boldsymbol{u} - \varrho \beta_M \boldsymbol{G}) + \mathrm{div}_x(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) \, \mathrm{d}t + \nabla_x p(\varrho) \, \mathrm{d}t = \beta_M \boldsymbol{G} \, \mathrm{div}_x(\varrho \boldsymbol{u}) \, \mathrm{d}t, \tag{3-2}$$

where no stochastic integration is necessary. Passing to the weak formulation, our task reduces to finding ϱ and $\varrho \boldsymbol{u}$ such that

$$\begin{gathered} t \mapsto \int_{\mathcal{T}^N} \varrho \phi \, \mathrm{d}x, \quad t \mapsto \int_{\mathcal{T}^N} \varrho \boldsymbol{u} \cdot \boldsymbol{\varphi} \, \mathrm{d}x \quad \text{continuous and } (\mathfrak{F}_t)\text{-adapted}, \\ \int_{\mathcal{T}^N} \varrho(0, \cdot) \phi \, \mathrm{d}x = \int_{\mathcal{T}^N} \varrho_0 \phi \, \mathrm{d}x, \quad \int_{\mathcal{T}^N} \varrho \boldsymbol{u}(0, \cdot) \cdot \boldsymbol{\varphi} \, \mathrm{d}x = \int_{\mathcal{T}^N} (\varrho \boldsymbol{u})_0 \cdot \boldsymbol{\varphi} \, \mathrm{d}x \end{gathered} \tag{3-3}$$

for any smooth test functions ϕ, $\boldsymbol{\varphi}$, satisfying

$$\int_0^T \int_{\mathcal{T}^N} [\varrho \partial_t \phi + \varrho \boldsymbol{u} \cdot \nabla_x \phi] \, \mathrm{d}x \, \mathrm{d}t = 0 \tag{3-4}$$

for any $\phi \in C_c^\infty((0, T) \times \mathcal{T}^N)$;

$$\begin{aligned} \int_0^T \int_{\mathcal{T}^N} [(\varrho \boldsymbol{u} - \varrho \beta_M \boldsymbol{G}) \cdot \partial_t \boldsymbol{\varphi} + \varrho \boldsymbol{u} \otimes \boldsymbol{u} : \nabla_x \boldsymbol{\varphi} + p(\varrho) \, \mathrm{div}_x \, \boldsymbol{\varphi}] \, \mathrm{d}x \, \mathrm{d}t \\ = \int_0^T \int_{\mathcal{T}^N} [\beta_M \varrho \boldsymbol{u} \cdot \nabla_x \boldsymbol{G} \cdot \boldsymbol{\varphi} + \beta_M \varrho \boldsymbol{u} \cdot \nabla_x \boldsymbol{\varphi} \cdot \boldsymbol{G}] \, \mathrm{d}x \, \mathrm{d}t \end{aligned} \tag{3-5}$$

for any $\boldsymbol{\varphi} \in C_c^\infty((0, T) \times \mathcal{T}^N; R^N)$.

Remark 3.1. Problem (3-4), (3-5) can be viewed as a system of partial differential equations with random coefficients. We point out that all steps leading from the original problem (2-3), (2-4) to (3-4), (3-5) are reversible as long as ϱ, $\varrho\boldsymbol{u}$ are weakly continuous $(\mathfrak{F}_t)$-adapted and Itô's calculus applies. In particular, it is enough to solve (3-3)–(3-5).

3C. ***Problem with linear multiplicative noise*** **(*stochastic "damping"*).** If the forcing is given by (1-4), we may again use Itô's calculus for $0 \le t \le \tau_M$ obtaining

$$\mathrm{d}\exp(-\beta_M) = -\exp(-\beta_M)\,\mathrm{d}\beta_M + \tfrac{1}{2}\exp(-\beta_M)\,\mathrm{d}t,$$

and

$$\exp(-\beta_M)\left[\mathrm{d}\left(\int_{\mathcal{T}^N} \varrho\boldsymbol{u}\cdot\boldsymbol{\varphi}\ \mathrm{d}x\right) - \left(\int_{\mathcal{T}^N} \varrho\boldsymbol{u}\cdot\boldsymbol{\varphi}\ \mathrm{d}x\right)\mathrm{d}\beta_M\right] = \mathrm{d}\left[\exp(-\beta_M)\int_{\mathcal{T}^N} \varrho\boldsymbol{u}\cdot\boldsymbol{\varphi}\ \mathrm{d}x\right] + \tfrac{1}{2}\exp(-\beta_M)\int_{\mathcal{T}^N} \varrho\boldsymbol{u}\cdot\boldsymbol{\varphi}\ \mathrm{d}x\,\mathrm{d}t.$$

On the other hand, in accordance with (2-4),

$$\mathrm{d}\left(\int_{\mathcal{T}^N} \varrho\boldsymbol{u}\cdot\boldsymbol{\varphi}\ \mathrm{d}x\right) - \left(\int_{\mathcal{T}^N} \varrho\boldsymbol{u}\cdot\boldsymbol{\varphi}\ \mathrm{d}x\right)\mathrm{d}\beta_M = \int_{\mathcal{T}^N} [\varrho\boldsymbol{u}\otimes\boldsymbol{u} : \nabla_x\boldsymbol{\varphi} + p(\varrho)\,\mathrm{div}_x\,\boldsymbol{\varphi}]\ \mathrm{d}x\,\mathrm{d}t.$$

We therefore conclude that

$$\mathrm{d}\left[\exp(-\beta_M)\int_{\mathcal{T}^N} \varrho\boldsymbol{u}\cdot\boldsymbol{\varphi}\ \mathrm{d}x\right] = -\tfrac{1}{2}\exp(-\beta_M)\int_{\mathcal{T}^N} \varrho\boldsymbol{u}\cdot\boldsymbol{\varphi}\ \mathrm{d}x\,\mathrm{d}t + \exp(-\beta_M)\int_{\mathcal{T}^N} [\varrho\boldsymbol{u}\otimes\boldsymbol{u} : \nabla_x\boldsymbol{\varphi} + p(\varrho)\,\mathrm{div}_x\,\boldsymbol{\varphi}]\ \mathrm{d}x\,\mathrm{d}t.$$

Similarly to the case of additive noise, we may replace (2-3), (2-4) by a system of partial differential equations with random coefficients, the weak formulation of which reads

$$\int_0^T\int_{\mathcal{T}^N} [\varrho\partial_t\phi + \varrho\boldsymbol{u}\cdot\nabla_x\phi]\ \mathrm{d}x\,\mathrm{d}t = 0 \tag{3-6}$$

for any $\varphi \in C_c^\infty((0,T)\times\mathcal{T}^N)$;

$$\begin{aligned} 0 = &\int_0^T\int_{\mathcal{T}^N} [\exp(-\beta_M)\varrho\boldsymbol{u}\cdot\partial_t\boldsymbol{\varphi} + \exp(-\beta_M)\varrho\boldsymbol{u}\otimes\boldsymbol{u} : \nabla_x\boldsymbol{\varphi} + \exp(-\beta_M)p(\varrho)\,\mathrm{div}_x\,\boldsymbol{\varphi}]\ \mathrm{d}x\,\mathrm{d}t \\ &- \frac{1}{2}\int_0^T\int_{\mathcal{T}^N} \exp(-\beta_M)\varrho\boldsymbol{u}\cdot\boldsymbol{\varphi}\ \mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{3-7}$$

for any $\boldsymbol{\varphi} \in C_c^\infty((0,T)\times\mathcal{T}^N; R^N)$, where ϱ, $\varrho\boldsymbol{u}$ are the stochastic processes satisfying (3-3).

4. Abstract Euler problem

Our next goal is to rewrite the problems (3-3), (3-4), (3-5) and (3-3), (3-6), (3-7), respectively, to fit into the abstract framework introduced in [Feireisl 2016]. In addition to (2-5) we suppose that $\mathbb{P}$-a.s.

$$\|\varrho_0\|_{C^3(\mathcal{T}^N)} + \|(\varrho\boldsymbol{u})_0\|_{C^3(\mathcal{T}^N;R^N)} + \|\varrho_0^{-1}\|_{C(\mathcal{T}^N)} \le D \tag{4-1}$$

for some deterministic constant $D > 0$. We claim that it is enough to show Theorem 2.3 for the initial data satisfying (4-1). Indeed, any initial data ϱ_0, $(\varrho \boldsymbol{u})_0$ satisfying (2-5) can be written as

$$[\varrho_0, (\varrho \boldsymbol{u})_0] = \lim_{D\to\infty} [\varrho_{0,D}, (\varrho \boldsymbol{u})_{0,D}] \quad \mathbb{P}\text{-a.s.},$$

where

$$[\varrho_{0,D}, (\varrho \boldsymbol{u})_{0,D}](\omega) = \begin{cases} [\varrho_0, (\varrho \boldsymbol{u})_0](\omega) & \text{if (4-1) holds,} \\ [1, 0] & \text{otherwise.} \end{cases}$$

Let $[\varrho_D, (\varrho \boldsymbol{u})_D]$ be the solution emanating from the data $[\varrho_{0,D}, (\varrho \boldsymbol{u})_{0,D}]$, the existence of which is guaranteed by Theorem 2.3. We set

$$\Omega_D = \{\omega \in \Omega \mid [\varrho_0, (\varrho \boldsymbol{u})_0](\omega) \text{ satisfies (4-1)}\}.$$

Note that Ω_D is $\mathfrak{F}_0$-measurable for any $D > 0$. Since

$$[\varrho_0, (\varrho \boldsymbol{u})_0] = 1_{\Omega_1}[\varrho_1, (\varrho \boldsymbol{u})_1] + \sum_{D=2}^{\infty} 1_{\Omega_D \setminus \Omega_{D-1}} [\varrho_{0,D}, (\varrho \boldsymbol{u})_{0,D}],$$

the desired solution for arbitrary initial data satisfying (2-5) can be obtained in the form

$$[\varrho, \varrho \boldsymbol{u}] = 1_{\Omega_1}[\varrho_1, (\varrho \boldsymbol{u})_1] + \sum_{D=2}^{\infty} 1_{\Omega_D \setminus \Omega_{D-1}} [\varrho_D, (\varrho \boldsymbol{u})_D].$$

4A. *Additive noise.* Going back to (3-4), (3-5) we write

$$\varrho \boldsymbol{u} - \varrho \beta_M \boldsymbol{G} = \boldsymbol{v} + \boldsymbol{V} + \nabla_x \Psi,$$

where

$$\operatorname{div}_x \boldsymbol{v} = 0, \qquad \int_{\mathcal{T}^N} \boldsymbol{v} \, \mathrm{d}x = 0, \qquad \boldsymbol{V} = \boldsymbol{V}(t) \in R^N \text{ a spatially homogeneous function.}$$

Remark 4.1. Note that $\boldsymbol{v} + \boldsymbol{V}$ represents the standard Helmholtz projection Π_H of $\varrho \boldsymbol{u} - \varrho \beta_M \boldsymbol{G}$ onto the space of solenoidal functions.

To meet the initial conditions (1-5), we fix

$$\boldsymbol{v}(0, \cdot) = \Pi_H[(\varrho \boldsymbol{u})_0] - \frac{1}{|\mathcal{T}^N|} \int_{\mathcal{T}^N} (\varrho \boldsymbol{u})_0 \, \mathrm{d}x,$$

$$\boldsymbol{V}(0) = \frac{1}{|\mathcal{T}^N|} \int_{\mathcal{T}^N} (\varrho \boldsymbol{u})_0 \, \mathrm{d}x, \qquad \nabla_x \Psi(0, \cdot) = \Pi_H^{\perp}[(\varrho \boldsymbol{u})_0].$$

Accordingly, the equation of continuity (3-4) reads

$$\partial_t \varrho + \Delta_x \Psi + \beta_M \operatorname{div}_x(\varrho \boldsymbol{G}) = 0, \qquad \varrho(0, \cdot) = \varrho_0. \tag{4-2}$$

For given Ψ, β_M, and $\boldsymbol{G}$, the density ϱ in (4-2) is uniquely determined by the method of characteristics. Moreover, as β_M satisfies (3-1) and ϱ_0 is strictly positive uniform in Ω, we may *fix* the potential Ψ and subsequently the density ϱ in such a way that

$$\Psi \in C^2([0,T]; C^3(\mathcal{T}^N)) \ \ \mathbb{P}\text{-a.s.}, \quad \Psi \ (\mathfrak{F}_t)\text{-adapted}, \quad \|\Psi\|_{C^2([0,T];C^3(\mathcal{T}^N))} \le c_M \ \ \mathbb{P}\text{-a.s.},$$
$$\varrho \in C^1([0,T]; C^1(\mathcal{T}^N)) \ \ \mathbb{P}\text{-a.s.}, \quad \varrho(0,\cdot) = \varrho_0, \quad \varrho \ (\mathfrak{F}_t)\text{-adapted},$$
$$\|\varrho\|_{C^1([0,T];C^1(\mathcal{T}^N))} \le c_M, \quad \varrho \ge \frac{1}{c_M} \ \ \mathbb{P}\text{-a.s.}, \tag{4-3}$$

where $c_M > 0$ is a deterministic constant depending on the stopping parameter M. Here, we have also used the extra hypothesis (4-1).

Remark 4.2. We would like to point out that Ψ and subsequently ϱ are not uniquely determined. As a matter of fact, there are infinitely many possibilities of how to choose Ψ and ϱ satisfying (4-2), (4-3). In particular, if

$$\mathrm{div}_x(\varrho \boldsymbol{u})_0 = 0, \quad \int_{\mathcal{T}^N} (\varrho \boldsymbol{u})_0 = 0, \quad \text{and} \quad \mathrm{div}_x \boldsymbol{G} = 0$$

we can take the ansatz

$$\varrho \equiv 1, \quad \Psi \equiv 0$$

obtaining $\varrho \boldsymbol{u} = \boldsymbol{v}$ — a solution of the *incompressible* Euler system.

Having fixed ϱ and Ψ, we compute $\boldsymbol{V}$ as the unique solution of the differential equation

$$\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = -\frac{1}{|\mathcal{T}^N|}\int_{\mathcal{T}^N} [\varrho \beta_M^2 \nabla_x \boldsymbol{G}\cdot\boldsymbol{G} + \beta_M \nabla_x \boldsymbol{G}\cdot\nabla_x \Psi]\,\mathrm{d}x, \quad \boldsymbol{V}(0) = \frac{1}{|\mathcal{T}^N|}\int_{\mathcal{T}^N} (\varrho\boldsymbol{u})_0 \,\mathrm{d}x. \tag{4-4}$$

In view of (4-3) and the assumption $\boldsymbol{G} \in W^{1,\infty}(\mathcal{T}^N; R^N)$ we easily deduce that

$$\boldsymbol{V} \in C^1([0,T]; R^N) \ \ \mathbb{P}\text{-a.s.}, \quad \boldsymbol{V} \text{ is } (\mathfrak{F}_t)\text{-adapted}, \quad \|\boldsymbol{V}\|_{C^1([0,T];R^N)} \le c_M \ \ \mathbb{P}\text{-a.s.} \tag{4-5}$$

Thus it remains to find $\boldsymbol{v}$ to satisfy (3-5). It turns out that $\boldsymbol{v}$ must be a weak solution of the abstract Euler system

$$\partial_t \boldsymbol{v} + \mathrm{div}_x\left(\frac{(\boldsymbol{v} + \varrho\beta_M\boldsymbol{G} + \boldsymbol{V} + \nabla_x\Psi)\otimes(\boldsymbol{v} + \varrho\beta_M\boldsymbol{G} + \boldsymbol{V} + \nabla_x\Psi)}{\varrho}\right)$$
$$= -\nabla_x p(\varrho) - \partial_t\nabla_x\Psi + \beta_M\,\mathrm{div}_x(\varrho\beta_M\boldsymbol{G} + \nabla_x\Psi)\boldsymbol{G} - \frac{1}{|\mathcal{T}^N|}\int_{\mathcal{T}^N} \beta_M\,\mathrm{div}_x(\varrho\beta_M\boldsymbol{G} + \nabla_x\Psi)\boldsymbol{G}\,\mathrm{d}x,$$
$$\mathrm{div}_x \boldsymbol{v} = 0, \quad \boldsymbol{v}(0,\cdot) = \boldsymbol{v}_0 = \Pi_H[(\varrho\boldsymbol{u})_0] - \frac{1}{|\mathcal{T}^N|}\int_{\mathcal{T}^N} (\varrho\boldsymbol{u})_0\,\mathrm{d}x.$$

Finally, we solve the elliptic system

$$\mathrm{div}_x\left[\nabla_x\boldsymbol{m} + \nabla_x^t\boldsymbol{m} - \frac{2}{N}\,\mathrm{div}_x\,\boldsymbol{m}\mathbb{I}\right]$$
$$= \nabla_x p(\varrho) + \partial_t\nabla_x\Psi - \beta_M\,\mathrm{div}_x(\varrho\beta_M\boldsymbol{G} + \nabla_x\Psi)\boldsymbol{G} + \frac{1}{|\mathcal{T}^N|}\int_{\mathcal{T}^N} \beta_M\,\mathrm{div}_x(\varrho\beta_M\boldsymbol{G} + \nabla_x\Psi)\boldsymbol{G}\,\mathrm{d}x. \tag{4-6}$$

Note that (4-6) admits a unique solution as the right-hand side is a function of zero mean. Consequently, setting

$$r = \varrho, \quad \boldsymbol{h} = \varrho \beta_M \boldsymbol{G} + \boldsymbol{V} + \nabla_x \Psi, \quad \mathbb{M} = \nabla_x \boldsymbol{m} + \nabla_x^t \boldsymbol{m} - \frac{2}{N} \operatorname{div}_x \boldsymbol{m} \mathbb{I}$$

we may rewrite the problem in a concise form:

$$\partial_t \boldsymbol{v} + \operatorname{div}_x \left[\frac{(\boldsymbol{v}+\boldsymbol{h}) \otimes (\boldsymbol{v}+\boldsymbol{h})}{r} + \mathbb{M} \right] = 0, \quad \operatorname{div}_x \boldsymbol{v} = 0, \quad \boldsymbol{v}(0,\cdot) = \boldsymbol{v}_0, \tag{4-7}$$

where

$$\begin{gathered} \boldsymbol{v}_0 \in C^1(\mathcal{T}^N; R^N) \ \mathbb{P}\text{-a.s.}, \ \operatorname{div}_x \boldsymbol{v}_0 = 0, \ \int_{\mathcal{T}^N} \boldsymbol{v}_0 \, \mathrm{d}x = 0, \ \boldsymbol{v}_0 \text{ is } \mathfrak{F}_0\text{-measurable}, \ \|\boldsymbol{v}_0\|_{C^1(\mathcal{T}^N; R^N)} \le c_M \ \mathbb{P}\text{-a.s.}, \\ \boldsymbol{h} \in C^a([0,T]; C^1(\mathcal{T}^N; R^N)) \ \mathbb{P}\text{-a.s.}, \ \boldsymbol{h} \text{ is } (\mathfrak{F}_t)\text{-adapted}, \ \|\boldsymbol{h}\|_{C^a([0,T]; C^1(\mathcal{T}^N; R^N))} \le c_M \ \mathbb{P}\text{-a.s.}, \\ r \in C^a([0,T]; C^1(\mathcal{T}^N)) \ \mathbb{P}\text{-a.s.}, \ r \text{ is } (\mathfrak{F}_t)\text{-adapted}, \ \|r\|_{C^a([0,T]; C^1(\mathcal{T}^N))} \le c_M, \ \frac{1}{r} \ge \frac{1}{c_M} \ \mathbb{P}\text{-a.s.}, \\ \mathbb{M} \in C^a([0,T]; C^1(\mathcal{T}^N; R^{N\times N}_{0,\mathrm{sym}})) \ \mathbb{P}\text{-a.s.}, \ \mathbb{M} \text{ is } (\mathfrak{F}_t)\text{-adapted}, \ \|\mathbb{M}\|_{C^a([0,T]; C^1(\mathcal{T}^N; R^{N\times N}_{0,sym}))} \le c_M \ \mathbb{P}\text{-a.s.} \end{gathered} \tag{4-8}$$

are given data. In the following we give a precise definition for solutions to (4-7).

Definition 4.3. Assume that the data $\boldsymbol{v}_0, \boldsymbol{h}, r, \mathbb{M}$ satisfy (4-8).[1] We say that $\boldsymbol{v}$ is a weak solution to problem (4-7) provided:

(i) We have $t \mapsto \int_{\mathcal{T}^N} \boldsymbol{v} \cdot \boldsymbol{\varphi} \, \mathrm{d}x \in C([0,T])$ for any $\boldsymbol{\varphi} \in C^\infty(\mathcal{T}^N; R^N)$, the stochastic process $t \mapsto \int_{\mathcal{T}^N} \boldsymbol{v} \cdot \boldsymbol{\varphi} \, \mathrm{d}x$ is $(\mathfrak{F}_t)$-adapted, and

$$\boldsymbol{v} \in C_{\mathrm{weak}}([0,T]; L^2(\mathcal{T}^N; R^N)) \cap L^\infty((0,T) \times \mathcal{T}^N; R^N) \ \mathbb{P}\text{-a.s.}.$$

(ii) For all $\boldsymbol{\varphi} \in C^\infty(\mathcal{T}^N, R^N)$ and all $t \in [0,T]$ the following holds $\mathbb{P}$-a.s.:

$$\int_{\mathcal{T}^N} \boldsymbol{v}(t,\cdot) \cdot \boldsymbol{\varphi} \, \mathrm{d}x - \int_{\mathcal{T}^N} \boldsymbol{v}_0 \cdot \boldsymbol{\varphi} \, \mathrm{d}x = \int_0^t \int_{\mathcal{T}^N} \left[\frac{(\boldsymbol{v}+\boldsymbol{h}) \otimes (\boldsymbol{v}+\boldsymbol{h})}{r} : \nabla_x \boldsymbol{\varphi} + \mathbb{M} : \nabla_x \boldsymbol{\varphi} \right] \mathrm{d}x \, \mathrm{d}t. \tag{4-9}$$

Let us summarize the above discussion in the following proposition.

Proposition 4.4. *Let β_M and $\boldsymbol{G} \in W^{1,\infty}(\mathcal{T}^N; R^N)$ be given. Let ϱ, Ψ belonging to the class* (4-3) *satisfy* (4-2). *Finally, let $\boldsymbol{v}$ be a weak solution of problem* (4-7) *in the sense of Definition 4.3, with*

$$r = \varrho, \quad \boldsymbol{h} = \varrho \beta_M \boldsymbol{G} + \boldsymbol{V} + \nabla_x \Psi, \quad \mathbb{M} = \nabla_x \boldsymbol{m} + \nabla_x^t \boldsymbol{m} - \frac{2}{N} \operatorname{div}_x \boldsymbol{m} \mathbb{I},$$

where $\boldsymbol{V}$, $\boldsymbol{m}$ solve (4-4), (4-6), *respectively.*

Then

$$\varrho, \varrho \boldsymbol{u} = \boldsymbol{v} + \boldsymbol{V} + \nabla_x \Psi + \varrho \beta_M \boldsymbol{G}$$

is a solution of problem (3-3)–(3-5).

Remark 4.5. In view of Proposition 4.4 and Remark 3.1, the proof of Theorem 2.3 in the case of additive noise reduces to showing the existence of infinitely many solutions to problem (4-7).

[1] A weak solution could be defined under much less restrictive assumptions on the data.

4B. ***Multiplicative noise.*** Mimicking the steps of the previous section we write

$$\exp(-\beta_M)\varrho\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{V} + \nabla_x \Psi$$

in (3-7), where

$$\operatorname{div}_x \boldsymbol{v} = 0, \quad \int_{\mathcal{T}^N} \boldsymbol{v} \,\mathrm{d}x = 0, \quad \boldsymbol{V} = \boldsymbol{V}(t) \in R^N \text{ is a spatially homogeneous function,}$$

and

$$\boldsymbol{v}(0,\cdot) = \Pi_H[(\varrho\boldsymbol{u})_0] - \frac{1}{|\mathcal{T}^N|}\int_{\mathcal{T}^N} (\varrho\boldsymbol{u})_0 \,\mathrm{d}x, \quad \boldsymbol{V}(0) = \frac{1}{|\mathcal{T}^N|}\int_{\mathcal{T}^N} (\varrho\boldsymbol{u})_0 \,\mathrm{d}x, \quad \nabla_x\Psi(0,\cdot) = \Pi_H^{\perp}[(\varrho\boldsymbol{u})_0].$$

Accordingly, the equation of continuity reads

$$\partial_t\varrho + \operatorname{div}_x(\exp(\beta_M)\nabla_x\Psi) = 0, \quad \varrho(0,\cdot) = \varrho_0. \tag{4-10}$$

Next, we fix $\boldsymbol{V}$ as the unique solution of

$$\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} + \tfrac{1}{2}\boldsymbol{V} = 0, \quad \boldsymbol{V}(0) = \frac{1}{|\mathcal{T}^N|}\int_{\mathcal{T}^N} (\varrho\boldsymbol{u})_0 \,\mathrm{d}x. \tag{4-11}$$

Accordingly, the momentum equation can be written as

$$\begin{gathered}\partial_t\boldsymbol{v} + \exp(\beta_M)\left[\operatorname{div}_x \frac{(\boldsymbol{v}+\boldsymbol{V}+\nabla_x\Psi)\otimes(\boldsymbol{v}+\boldsymbol{V}+\nabla_x\Psi)}{\varrho}\right] + \exp(-\beta_M)\nabla_x p(\varrho) + \partial_t\nabla_x\Psi + \tfrac{1}{2}\nabla_x\Psi = -\tfrac{1}{2}\boldsymbol{v},\\ \operatorname{div}_x \boldsymbol{v} = 0, \quad \boldsymbol{v}_0 = \Pi_H[(\varrho\boldsymbol{u})_0] - \frac{1}{|\mathcal{T}^N|}\int_{\mathcal{T}^N} (\varrho\boldsymbol{u})_0 \,\mathrm{d}x.\end{gathered} \tag{4-12}$$

Similarly to the above, we can fix ϱ, Ψ to satisfy (4-10) together with (4-3).

Finally, seeing that $\int_{\mathcal{T}^N} \boldsymbol{v} \,\mathrm{d}x = 0$, we may solve an analogue to the elliptic system (4-6), namely,

$$\operatorname{div}_x\left[\nabla_x\boldsymbol{m} + \nabla_x^t\boldsymbol{m} - \frac{2}{N}\operatorname{div}_x \boldsymbol{m}\mathbb{I}\right] = \exp(-\beta_M)\nabla_x p(\varrho) + \partial_t\nabla_x\Psi + \tfrac{1}{2}\nabla_x\Psi + \tfrac{1}{2}\boldsymbol{v}. \tag{4-13}$$

Note that, in contrast with (4-6), the solution $\boldsymbol{m} = \boldsymbol{m}[\boldsymbol{v}]$ depends on $\boldsymbol{v}$.

Similarly to (4-7) we can write the final problem (setting $\boldsymbol{h} = \boldsymbol{V} + \nabla_x\Psi$ and $r = \varrho$):

$$\partial_t\boldsymbol{v} + \operatorname{div}_x\left[\frac{(\boldsymbol{v}+\boldsymbol{h})\otimes(\boldsymbol{v}+\boldsymbol{h})}{r} + \mathbb{M}[\boldsymbol{v}]\right] = 0, \quad \operatorname{div}_x \boldsymbol{v} = 0, \quad \boldsymbol{v}(0,\cdot) = \boldsymbol{v}_0, \tag{4-14}$$

where

$$\boldsymbol{v}_0 \in C^1(\mathcal{T}^N; R^N) \ \mathbb{P}\text{-a.s.}, \ \operatorname{div}_x \boldsymbol{v}_0 = 0, \ \int_{\mathcal{T}^N} \boldsymbol{v}_0 \,\mathrm{d}x = 0, \ \boldsymbol{v}_0 \text{ is } \mathfrak{F}_0\text{-measurable}, \ \|\boldsymbol{v}_0\|_{C^1(\mathcal{T}^N;R^N)} \le c_M \ \mathbb{P}\text{-a.s.},$$

$$\boldsymbol{h} \in C^a([0,T]; C^1(\mathcal{T}^N; R^N)) \ \mathbb{P}\text{-a.s.}, \quad \boldsymbol{h} \text{ is } (\mathfrak{F}_t)\text{-adapted}, \quad \|\boldsymbol{h}\|_{C^a([0,T];C^1(\mathcal{T}^N;R^N))} \le c_M \ \mathbb{P}\text{-a.s.}, \tag{4-15}$$

$$r \in C^a([0,T]; C^1(\mathcal{T}^N)) \ \mathbb{P}\text{-a.s.}, \quad r \text{ is } (\mathfrak{F}_t)\text{-adapted}, \quad \|r\|_{C^a([0,T];C^1(\mathcal{T}^N))} \le c_M, \quad \frac{1}{r} \ge \frac{1}{c_M} \ \mathbb{P}\text{-a.s.}, \tag{4-16}$$

and

$$\mathbb{M} = \mathbb{M}[\boldsymbol{v}] = \nabla_x\boldsymbol{m} + \nabla_x^t\boldsymbol{m} - \frac{2}{N}\operatorname{div}_x \boldsymbol{m}\mathbb{I} \tag{4-17}$$

is the unique solution of the elliptic system (4-13).

Remark 4.6. Note that $\boldsymbol{h}$ is actually more regular than in Section 4A.

Similarly to the preceding section we have the following definition.

Definition 4.7. Assume that the data $\boldsymbol{v}_0, \boldsymbol{h}, r$ satisfy (4-15) and let $\mathbb{M}[\boldsymbol{v}]$ be given by (4-17) with ϱ, Ψ satisfying (4-3).[2] We say that $\boldsymbol{v}$ is a weak solution to problem (4-14) provided:

(i) We have $t \mapsto \int_{\mathcal{T}^N} \boldsymbol{v} \cdot \boldsymbol{\varphi} \, \mathrm{d}x \in C([0, T])$ for any $\boldsymbol{\varphi} \in C^\infty(\mathcal{T}^N; R^N)$, the stochastic process $t \mapsto \int_{\mathcal{T}^N} \boldsymbol{v} \cdot \boldsymbol{\varphi} \, \mathrm{d}x$ is $(\mathfrak{F}_t)$-adapted,

$$\boldsymbol{v} \in C_{\mathrm{weak}}([0, T]; L^2(\mathcal{T}^N; R^N)) \cap L^\infty((0, T) \times \mathcal{T}^N; R^N) \quad \mathbb{P}\text{-a.s.}.$$

(ii) For all $\boldsymbol{\varphi} \in C^\infty(\mathcal{T}^N, R^N)$ and all $t \in [0, T]$ the following holds $\mathbb{P}$-a.s.:

$$\int_{\mathcal{T}^N} \boldsymbol{v}(t, \cdot) \cdot \boldsymbol{\varphi} \, \mathrm{d}x - \int_{\mathcal{T}^N} \boldsymbol{v}_0 \cdot \boldsymbol{\varphi} \, \mathrm{d}x = \int_0^t \int_{\mathcal{T}^N} \left[\frac{(\boldsymbol{v}+\boldsymbol{h}) \otimes (\boldsymbol{v}+\boldsymbol{h})}{r} : \nabla_x \boldsymbol{\varphi} + \mathbb{M}[\boldsymbol{v}] : \nabla_x \boldsymbol{\varphi} \right] \mathrm{d}x \, \mathrm{d}t. \tag{4-18}$$

Again similarly to the preceding section, we summarize as follows.

Proposition 4.8. *Let β_M be given. Let ϱ, Ψ solve* (4-10), *and let $\boldsymbol{V}$ solve* (4-11). *Let $\boldsymbol{v}$ be a weak solution of* (4-14), *with*

$$r = \varrho, \quad \boldsymbol{h} = \boldsymbol{V} + \nabla_x \Psi, \quad \mathbb{M} = \nabla_x \boldsymbol{m} + \nabla_x^t \boldsymbol{m} - \frac{2}{N} \operatorname{div}_x \boldsymbol{m} \mathbb{I},$$

where $\boldsymbol{m} = \boldsymbol{m}[\boldsymbol{v}]$ is the unique solution of the elliptic system (4-13).

Then

$$\varrho, \varrho \boldsymbol{u} = \exp(\beta_M)(\boldsymbol{v} + \boldsymbol{V} + \nabla_x \Psi) \tag{4-19}$$

is a solution of problem (3-6), (3-7).

Remark 4.9. In view of Proposition 4.8, the proof of Theorem 2.3 in the case of the multiplicative noise reduces to showing the existence of infinitely many solutions to problem (4-14).

5. Convex integration

Problems (4-7) and (4-14) can be solved pathwise using the method of [De Lellis and Székelyhidi 2010], with the necessary modifications developed in [Feireisl 2016]. In such a way, we would obtain the existence of (infinitely many) solutions in the semideterministic spirit introduced in [Bensoussan and Temam 1973]. More specifically, solutions obtained this way would be random variables, meaning $\mathfrak{F}$-measurable but not necessarily $(\mathfrak{F}_t)$-adapted (progressively measurable). Obviously, such a semideterministic result would hold without any restriction imposed by the stopping times. Progressive measurability of ϱ, $\varrho \boldsymbol{u}$ claimed in Theorem 2.3 represents a nontrivial issue that requires a careful revisiting of the method of convex integration presented in [De Lellis and Székelyhidi 2010]. The main ingredient is a stochastic variant of the so-called *oscillatory lemma* shown in the present section.

Definition 5.1. Let $G : \Omega \to X$ be a (Borelian) random variable ranging in a topological space X. We say that G has a compact range in X if there is a (deterministic) compact set $\mathcal{K} \subset X$ such that $G \in \mathcal{K}$ a.s.

[2] A weak solution could be defined under much less restrictive assumptions on the data.

5A. *Geometric setting.* Let $R^{N\times N}_{\mathrm{sym}}$ denote the space of symmetric $N\times N$ matrices and let $R^{N\times N}_{0,\mathrm{sym}}$ be its subspace of traceless matrices. Following the ansatz of [De Lellis and Székelyhidi 2010, Lemma 3] we consider the set

$$\mathcal{S}[e] = \Big\{ [\boldsymbol{w}, \mathbb{H}] \;\Big|\; \boldsymbol{w} \in R^N,\ \mathbb{H} \in R^{N\times N}_{0,\mathrm{sym}},\ \tfrac{1}{2} N \lambda_{\max}[\boldsymbol{w}\otimes\boldsymbol{w} - \mathbb{H}] < e \Big\},$$

where $\lambda_{\max}[\mathbb{A}]$ denotes the maximal eigenvalue of a symmetric matrix $\mathbb{A}$. Thanks to the algebraic inequality

$$\tfrac{1}{2} N \lambda_{\max}[\boldsymbol{w}\otimes\boldsymbol{w} - \mathbb{H}] \geq \tfrac{1}{2}|\boldsymbol{w}|^2, \quad \mathbb{H} \in R^{N\times N}_{0,\mathrm{sym}}, \tag{5-1}$$

$\mathcal{S}[e] \neq \varnothing$ only if $e > 0$. In addition, we have

$$\tfrac{1}{2}(N-1)\lambda_{\max}[\boldsymbol{w}\otimes\boldsymbol{w} - \mathbb{H}] \geq \tfrac{1}{2}|\mathbb{H}|^2, \quad \boldsymbol{w} \in R^N; \tag{5-2}$$

see [De Lellis and Székelyhidi 2010, Lemma 3(iii)]. Thus, for given $e > 0$, $\mathcal{S}[e]$ is a convex open and bounded subset of $R^N \times R^{N\times N}_{0,\mathrm{sym}}$. Moreover, as shown in [De Lellis and Székelyhidi 2010],

$$\partial\mathcal{S}[e] = \left\{ \left[\boldsymbol{a}, \boldsymbol{a}\otimes\boldsymbol{a} - \frac{1}{N}|\boldsymbol{a}|^2\mathbb{I} \right] \;\middle|\; \tfrac{1}{2}|\boldsymbol{a}|^2 = e \right\}.$$

De Lellis and Székelyhidi [2010, Lemma 6] proved the following result. Given $e > 0$ and $[\boldsymbol{w}, \mathbb{H}] \in \mathcal{S}[e]$, there exist $\boldsymbol{a}, \boldsymbol{b} \in R^N$ enjoying the following properties:

- We have
$$\tfrac{1}{2}|\boldsymbol{a}|^2 = \tfrac{1}{2}|\boldsymbol{b}|^2 = e. \tag{5-3}$$

- There exists $L \geq 0$ such that for $\boldsymbol{s} = \boldsymbol{a} - \boldsymbol{b}$, $\mathbb{M} = \boldsymbol{a}\otimes\boldsymbol{a} - \boldsymbol{b}\otimes\boldsymbol{b}$, we have
$$\begin{gathered} [\boldsymbol{w} + \lambda\boldsymbol{s}, \mathbb{H} + \lambda\mathbb{M}] \in \mathcal{S}[e], \\ \operatorname{dist}[[\boldsymbol{w} + \lambda\boldsymbol{s}, \mathbb{H} + \lambda\mathbb{M}]; \partial\mathcal{S}[e]] \geq \tfrac{1}{2}\operatorname{dist}[[\boldsymbol{w}, \mathbb{H}]; \partial\mathcal{S}[e]] \end{gathered} \tag{5-4}$$

for all $\lambda \in [-L, L]$.

- There is a universal constant $c(N)$ depending only on the dimension such that
$$L|\boldsymbol{s}| \geq c(N)\frac{1}{\sqrt{e}}\big(e - \tfrac{1}{2}|\boldsymbol{w}|^2\big). \tag{5-5}$$

- We have
$$|\boldsymbol{a} \pm \boldsymbol{b}| \geq \chi(\operatorname{dist}[[\boldsymbol{w}, \mathbb{H}]; \partial\mathcal{S}[e]]), \tag{5-6}$$

where χ is positive for positive arguments.

Motivated by this result, we consider a set-valued mapping

$$\mathcal{F} : (0, \infty) \times R^N \times R^{N\times N}_{0,\mathrm{sym}} \to 2^{R^N \times R^N}$$

determined by the following properties:

(1) Whenever $[\boldsymbol{w}, \mathbb{H}] \notin \overline{\mathcal{S}[e]}$ we have

$$\mathcal{F}(e, \boldsymbol{w}, \mathbb{H}) = \{[\boldsymbol{w}, \boldsymbol{w}]\}. \tag{5-7}$$

(2) If $[\boldsymbol{w}, \mathbb{H}] \in \overline{\mathcal{S}[e]}$, then $[\boldsymbol{a}, \boldsymbol{b}] \in \mathcal{F}(e, \boldsymbol{w}, \mathbb{H})$ if and only if:

- We have

$$\tfrac{1}{2}|\boldsymbol{a}|^2 = \tfrac{1}{2}|\boldsymbol{b}|^2 = e. \tag{5-8}$$

- There exists $L \geq 0$ such that for $\boldsymbol{s} = \boldsymbol{a} - \boldsymbol{b}$, $\mathbb{M} = \boldsymbol{a} \otimes \boldsymbol{a} - \boldsymbol{b} \otimes \boldsymbol{b}$, we have

$$\begin{gathered}[\boldsymbol{w} + \lambda \boldsymbol{s}, \mathbb{H} + \lambda \mathbb{M}] \in \mathcal{S}[e], \\ \operatorname{dist}[[\boldsymbol{w} + \lambda \boldsymbol{s}, \mathbb{H} + \lambda \mathbb{M}]; \partial \mathcal{S}[e]] \geq \tfrac{1}{2} \operatorname{dist}[[\boldsymbol{w}, \mathbb{H}]; \partial \mathcal{S}[e]]\end{gathered} \tag{5-9}$$

  for all $\lambda \in [-L, L]$.

- We have

$$L|\boldsymbol{s}| \geq c(N) \frac{1}{\sqrt{e}} \big(e - \tfrac{1}{2}|\boldsymbol{w}|^2\big), \tag{5-10}$$

  where $c(N)$ is the universal constant from (5-5);

$$|\boldsymbol{a} \pm \boldsymbol{b}| \geq \chi(\operatorname{dist}[[\boldsymbol{w}, \mathbb{H}]; \partial \mathcal{S}[e]]), \tag{5-11}$$

  where χ has been introduced in (5-6).

Basic properties of $\mathcal{F}$ are summarized in the following lemma.

Lemma 5.2. *For any $(e, \boldsymbol{w}, \mathbb{H}) \in (0, \infty) \times R^N \times R^{N\times N}_{0,\mathrm{sym}}$ the set $\mathcal{F}(e, \boldsymbol{w}, \mathbb{H})$ is nonempty, closed, and contained in a compact set, the size of which depends only on e and $|\boldsymbol{w}|$. Moreover, the mapping*

$$\mathcal{F} : (0, \infty) \times R^N \times R^{N\times N}_{0,\mathrm{sym}} \to 2^{R^N \times R^N}$$

has closed graph with respect to the Hausdorff distance on compact sets.

Proof. As shown in [De Lellis and Székelyhidi 2010, Lemma 6], the set $\mathcal{F}(e, \boldsymbol{w}, \mathbb{H})$ is nonempty for any $[\boldsymbol{w}, \mathbb{H}] \in \mathcal{S}[e]$ for a certain universal constant $c(N)$. If $[\boldsymbol{w}, \mathbb{H}] \in \partial\mathcal{S}[e]$, then

$$\tfrac{1}{2}|\boldsymbol{w}|^2 = e,$$

and, consequently, $\mathcal{F}(e, \boldsymbol{w}, \mathbb{H})$ contains at least the trivial point $[\boldsymbol{w}, \boldsymbol{w}]$. Obviously, $\mathcal{F}(e, \boldsymbol{w}, \mathbb{H})$ is closed and bounded, whence compact.

Closedness of the graph follows by the standard compactness argument as the target space is locally compact, and conditions (5-8)–(5-11) are invariant with respect to strong convergence. ☐

Remark 5.3. The mapping assigns to any point $[\boldsymbol{w}, \mathbb{H}] \in \mathcal{S}[e]$ a segment $[\boldsymbol{w} + \lambda \boldsymbol{s}, \mathbb{H} + \lambda \mathbb{M}]$, $\lambda \in [-L, L]$, that has "maximal" length and still belongs to the set $\mathcal{S}[e]$. Solutions constructed later by the method of convex integration "oscillate" along segments of this type.

Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space endowed with a complete σ-algebra of measurable sets $\mathfrak{F}$. Suppose now that

$$[e, \boldsymbol{w}, \mathbb{H}] \text{ is an } [\mathfrak{F}, \mathfrak{B}[(0,\infty) \times R^N \times R^{N\times N}_{0,\mathrm{sym}}]]\text{-measurable random variable,}$$

where the symbol $\mathfrak{B}$ denotes the σ-algebra of Borel sets. Our goal is to show that the composed mapping $\mathcal{F}(e, \boldsymbol{w}, \mathbb{H})$, considered now as a (set-valued) random variable, admits an $\mathfrak{F}$-measurable selection. To this end, we recall the celebrated Kuratowski and Ryll–Nardzewski theorem, see e.g., the survey [Wagner 1977].

Theorem 5.4. *Let $(X, \mathcal{A}, \mu)$ be a measure space with a (complete) σ-algebra of measurable sets $\mathcal{A}$. Let*

$$\mathcal{H} : X \to 2^Y$$

be a set-valued mapping, where Y is a Polish space with the σ-algebra of Borel sets $\mathfrak{B}$. Suppose that for all $x \in X$

$$\mathcal{H}(x) \text{ is a nonempty and closed subset of } Y,$$

and that $\mathcal{H}$ is weakly measurable, meaning

$$\{x \mid \mathcal{H}(x) \cap B \neq \varnothing\} \in \mathcal{A}$$

for any open set $B \subset Y$.

Then $\mathcal{H}$ admits an $\mathcal{A}$-$\mathfrak{B}$ measurable selection, meaning a single valued $\mathcal{A}$-$\mathfrak{B}$ measurable mapping $H : X \to Y$ such that

$$H(x) \in \mathcal{H}(x), \quad x \in X.$$

As both spaces $(0,\infty) \times R^N \times R^{N\times N}_{0,\mathrm{sym}}$ and $R^N \times R^N$ are finite-dimensional, compactness of the range of $\mathcal{F}$ and closedness of its graph imply that $\mathcal{F}$ is *upper semicontinuous*; specifically,

$$\{[e, \boldsymbol{w}, \mathbb{H}] \mid \mathcal{F}(e, \boldsymbol{w}, \mathbb{H}) \cap D \neq \varnothing\} \text{ is closed whenever } D \text{ is closed in } R^N \times R^N.$$

See [Wagner 1977].

As preimages of closed sets are measurable, we get (strong) measurability of $\mathcal{F}$; specifically,

$$\{\omega \in \Omega \mid \mathcal{F}(e, \boldsymbol{w}, \mathbb{H}) \cap D \neq \varnothing\}$$

is measurable for any closed set D in $R^N \times R^{N\times N}_{0,\mathrm{sym}}$. Strong measurability implies weak measurability of $\mathcal{F}$, namely,

$$\{\omega \in \Omega \mid \mathcal{F}(e, \boldsymbol{w}, \mathbb{H}) \cap G \neq \varnothing\}$$

is measurable for any open set G in $R^N \times R^N$.

Thus applying Theorem 5.4 we obtain the following conclusion.

Proposition 5.5. *Let*

$$\mathcal{F}(e, \boldsymbol{w}, \mathbb{H}) : (0,\infty) \times R^N \times R^{N\times N}_{0,\mathrm{sym}} \to 2^{R^N \times R^N}$$

be a set-valued mapping enjoying the properties (5-7)–(5-11). *Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space endowed with a complete σ-algebra of measurable sets $\mathfrak{F}$, and let*

$$(e, \boldsymbol{w}, \mathbb{H}) \textit{ be an } [\mathfrak{F}, \mathfrak{B}[(0,\infty) \times R^N \times R^{N\times N}_{0,\mathrm{sym}}]]\textit{-measurable random variable.}$$

Then the mapping $\mathcal{F}$ admits an $[\mathfrak{F}; \mathfrak{B}[R^N \times R^N]]$-measurable selection. In particular, there exists an $[\mathfrak{F}; \mathfrak{B}[R^N \times R^N]]$-measurable mapping

$$F : \Omega \to R^N \times R^N$$

such that it holds $\mathbb{P}$-a.s.:

$$\text{if } [\boldsymbol{w}(\omega), \mathbb{H}(\omega)] \in \mathcal{S}[e(\omega)], \text{ then } F(\omega) = [\boldsymbol{a}, \boldsymbol{b}], \text{ where } [\boldsymbol{a}, \boldsymbol{b}] \text{ satisfy (5-8)–(5-11).} \tag{5-12}$$

5B. *Analytic setting.* Following [De Lellis and Székelyhidi 2010] we introduce a mapping

$$R^{N+1} \ni \xi = [\xi_0, \xi_1, \dots, \xi_n] \mapsto \mathbb{A}_{\boldsymbol{a},\boldsymbol{b}}(\xi) \in R^{(N+1)\times(N+1)}_{0,\text{sym}},$$
$$\mathbb{A}_{\boldsymbol{a},\boldsymbol{b}}(\xi) = \tfrac{1}{2}\big((\mathbb{R}\cdot\xi)\otimes(\mathbb{Q}(\xi)\cdot\xi) + (\mathbb{Q}(\xi)\cdot\xi)\otimes(\mathbb{R}\cdot\xi)\big), \tag{5-13}$$

where

$$\mathbb{Q} = \xi \otimes e_0 - e_0 \otimes \xi, \quad \mathbb{R} = ([0, \boldsymbol{a}] \otimes [0, \boldsymbol{b}]) - ([0, \boldsymbol{b}] \otimes [0, \boldsymbol{a}]),$$

and

$$e_0 = [1, 0, \dots, 0], \quad \boldsymbol{a}, \boldsymbol{b} \in R^N, \quad \tfrac{1}{2}|\boldsymbol{a}|^2 = \tfrac{1}{2}|\boldsymbol{b}|^2 = e > 0, \quad \boldsymbol{a} \neq \pm\boldsymbol{b}.$$

$\mathbb{A}_{\boldsymbol{a},\boldsymbol{b}}$ can be seen as a Fourier symbol of a pseudodifferential operator, where $\xi = (\xi_0, \xi_1, \dots, \xi_N)$ corresponds to $\partial = [\partial_t, \partial_{x_1}, \dots, \partial_{x_N}]$.

The following was shown in [De Lellis and Székelyhidi 2010, Section 4.4]:

- If $\phi \in C_c^\infty(R \times R^N)$, and if we define

$$\begin{bmatrix} 0 & \boldsymbol{w} \\ \boldsymbol{w} & \mathbb{H} \end{bmatrix} \equiv \mathbb{A}_{\boldsymbol{a},\boldsymbol{b}}(\partial)[\phi]$$

then

$$\partial_t \boldsymbol{w} + \operatorname{div}_x \mathbb{H} = 0, \quad \operatorname{div}_x \boldsymbol{w} = 0. \tag{5-14}$$

- For

$$\eta_{\boldsymbol{a},\boldsymbol{b}} = -\frac{1}{(|\boldsymbol{a}||\boldsymbol{b}| + \boldsymbol{a}\cdot\boldsymbol{b})^{2/3}}[[0, \boldsymbol{a}] + [0, \boldsymbol{b}] - (|\boldsymbol{a}||\boldsymbol{b}| + \boldsymbol{a}\cdot\boldsymbol{b})e_0], \quad \psi \in C^\infty(R), \tag{5-15}$$

we have

$$\mathbb{A}_{\boldsymbol{a},\boldsymbol{b}}(\partial)[\psi([t, \boldsymbol{x}]\cdot\eta_{\boldsymbol{a},\boldsymbol{b}})] = \psi'''([t, \boldsymbol{x}]\cdot\eta_{\boldsymbol{a},\boldsymbol{b}}) \begin{bmatrix} 0 & \boldsymbol{a}-\boldsymbol{b} \\ \boldsymbol{a}-\boldsymbol{b} & \boldsymbol{a}\otimes\boldsymbol{a}-\boldsymbol{b}\otimes\boldsymbol{b} \end{bmatrix}. \tag{5-16}$$

5C. *A stochastic version of oscillatory lemma.* Let $Q = \{(t, x) \mid t \in (0, 1),\ x \in (0, 1)^N\}$. Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space with a complete σ-algebra of measurable sets $\mathfrak{F}$. Finally, we introduce the metric on the space of weakly continuous functions $C_{\text{weak}}([0, 1]; L^2([0, 1]^N; R^N))$,

$$d[\boldsymbol{v}; \boldsymbol{w}] = \sum_{m=1}^\infty \frac{1}{2^m} \frac{\left|\int_{[0,1]^N} (\boldsymbol{v} - \boldsymbol{w})\cdot\boldsymbol{\varphi}_m \,\mathrm{d}x\right|}{1 + \left|\int_{[0,1]^N} (\boldsymbol{v} - \boldsymbol{w})\cdot\boldsymbol{\varphi}_m \,\mathrm{d}x\right|}, \tag{5-17}$$
$$\boldsymbol{\varphi}_m \in C_c^\infty((0, 1)^N; R^N), \quad m = 1, 2, \dots, \qquad \{\boldsymbol{\varphi}_m\}_{m=1}^\infty \text{ a dense set in } L^2([0, 1]^N; R^N)).$$

The following is the main result of the present section.

Lemma 5.6. *Let* $\omega \mapsto [e, \boldsymbol{w}, \mathbb{H}]$ *be a* $[\mathfrak{F}; \mathfrak{B}[(0,\infty) \times R^N \times R^{N\times N}_{0,\mathrm{sym}}]]$*-measurable mapping such that*

$$[\boldsymbol{w}, \mathbb{H}] \in \mathcal{S}[e] \quad \mathbb{P}\text{-a.s.} \tag{5-18}$$

Then there exist sequences $\boldsymbol{w}_n$ *and* $\mathbb{V}_n$ *such that* $\boldsymbol{w}_n \in C^\infty_c(Q; R^N)$ $\mathbb{P}$*-a.s. and* $\mathbb{V}_n \in C^\infty_c(Q; R^{N\times N}_{0,\mathrm{sym}})$ $\mathbb{P}$*-a.s.*, $n \in \mathbb{N}$, *enjoying the following properties*:

(i) $t \mapsto [\boldsymbol{w}_n, \mathbb{V}_n]$ *is a stochastic process, meaning*

$$[\boldsymbol{w}_n(t,\cdot); \mathbb{V}_n(t,\cdot)] \in C([0,1]^N; R^N \times R^{N\times N}_{0,\mathrm{sym}}) \ \ \mathbb{P}\text{-a.s.}$$
$$\text{is } [\mathfrak{F}; \mathfrak{B}[C([0,1]^N; R^N \times R^{N\times N}_{0,\mathrm{sym}})]]\text{-measurable for any } t \in [0,1]. \tag{5-19}$$

(ii) *In* Q *we have* $\mathbb{P}$*-a.s.*

$$\partial_t \boldsymbol{w}_n + \mathrm{div}_x \mathbb{V}_n = 0, \quad \mathrm{div}_x \boldsymbol{w}_n = 0. \tag{5-20}$$

(iii) *As* $n \to \infty$ *we have* $\mathbb{P}$*-a.s.*

$$\boldsymbol{w}_n \to 0 \quad \text{in } C_{\mathrm{weak}}([0,1]; L^2([0,1]^N; R^N)). \tag{5-21}$$

(iv) *In* Q *we have* $\mathbb{P}$*-a.s.*

$$[\boldsymbol{w} + \boldsymbol{w}_n, \mathbb{H} + \mathbb{V}_n] \in \mathcal{S}[e]. \tag{5-22}$$

(v) *The following holds* $\mathbb{P}$*-a.s.*:

$$\liminf_{n\to\infty} \frac{1}{|Q|} \int_Q |\boldsymbol{w}_n|^2 \,\mathrm{d}x\ \mathrm{d}t \ge \frac{c(N)}{e}\big(e - \tfrac{1}{2}|\boldsymbol{w}|^2\big)^2. \tag{5-23}$$

If, *in addition to* (5-18), $e \le \bar{e}_M$ $\mathbb{P}$*-a.s. for some deterministic constant* $\bar{e}_M$, *and*

$$[\boldsymbol{w}, \mathbb{H}] \in \mathcal{S}[e - \delta] \quad \text{for some deterministic constant } \delta > 0, \tag{5-24}$$

then each $\boldsymbol{w}_n, \mathbb{V}_n$ *has compact range in* $C(\overline{Q}; R^N)$, $C(\overline{Q}; R^{N\times N}_{0,\mathrm{sym}})$, *and*

$$[\boldsymbol{w} + \boldsymbol{w}_n, \mathbb{H} + \mathbb{V}_n] \in \mathcal{S}[e - \delta_n] \ \ \mathbb{P}\text{-a.s.} \tag{5-25}$$

for some deterministic constants $\delta_n > 0$. *Moreover*, *the convergence in* (5-21) *can be strengthened to*

$$\operatorname*{ess\,sup}_{\omega\in\Omega}\Big(\sup_{t\in[0,T]} d[\boldsymbol{w}_n(t,\cdot); 0]\Big) \to 0 \quad \text{as } n \to \infty. \tag{5-26}$$

Remark 5.7. Hypothesis (5-24) is equivalent to saying that

$$\operatorname*{ess\,inf}_{\Omega}\big\{e - \tfrac{1}{2}N\lambda_{\max}[\boldsymbol{w}\otimes\boldsymbol{w} - \mathbb{H}]\big\} > 0.$$

Note that if this is the case, we have $e \ge \delta > 0$, whence e is a random variable with a compact range in $(0,\infty)$.

Proof. The proof is given through several steps.

Step 1: Given $[\boldsymbol{w}, \mathbb{H}]$ and e, we use Proposition 5.5 to identify the measurable selection of vectors $[\boldsymbol{a}, \boldsymbol{b}]$ satisfying (5-12).

Step 2: For each $[\boldsymbol{a}, \boldsymbol{b}]$ we construct the operator $\mathbb{A}_{\boldsymbol{a},\boldsymbol{b}}$ and the vector $\eta_{\boldsymbol{a},\boldsymbol{b}}$ enjoying (5-14)–(5-16).

Step 3: We consider a deterministic function $\varphi \in C_c^\infty(Q)$ such that

$$0 \le \varphi \le 1, \qquad \varphi(t,x) = 1 \quad \text{whenever } -\tfrac{1}{2} \le t \le \tfrac{1}{2}, \quad |x| \le \tfrac{1}{2}.$$

Step 4: We identify the functions $\boldsymbol{w}_n$, $\mathbb{V}_n$ from the relation

$$\mathbb{A}_{\boldsymbol{a},\boldsymbol{b}}(\partial)\left[\varphi \frac{L}{n^3} \cos(n[t,x]\cdot \eta_{\boldsymbol{a},\boldsymbol{b}})\right] = \begin{bmatrix} 0 & \boldsymbol{w}_n \\ \boldsymbol{w}_n & \mathbb{V}_n \end{bmatrix}.$$

In accordance with our construction of the points $[\boldsymbol{a}, \boldsymbol{b}]$, the operator $\mathbb{A}_{\boldsymbol{a},\boldsymbol{b}}$, and the vector $\eta_{\boldsymbol{a},\boldsymbol{b}}$, it is easy to check the $\boldsymbol{w}_n$, $\mathbb{V}_n$ enjoy the required measurability properties (5-19). Moreover, by virtue of (5-14), equations (5-20) are satisfied.

Step 5: As $\mathbb{A}$ is a homogeneous differential operator of third order, we get, in agreement with (5-16),

$$\mathbb{A}_{\boldsymbol{a},\boldsymbol{b}}(\partial)\left[\varphi \frac{L}{n^3} \cos(n[t,x]\cdot \eta_{\boldsymbol{a},\boldsymbol{b}})\right] = \varphi \sin(n[t,x]\cdot \eta_{\boldsymbol{a},\boldsymbol{b}}) L \begin{bmatrix} 0 & (\boldsymbol{a}-\boldsymbol{b}) \\ (\boldsymbol{a}-\boldsymbol{b}) & \boldsymbol{a}\otimes\boldsymbol{a} - \boldsymbol{b}\otimes\boldsymbol{b} \end{bmatrix} + \frac{1}{n} R_n \tag{5-27}$$

with $|R_n|$ uniformly bounded for $n \to \infty$. As (5-9), (5-10) holds, we deduce the remaining properties (5-21)–(5-23) provided n is chosen large enough. Note that we have

$$|\varphi \sin(n[t,x]\cdot \eta_{\boldsymbol{a},\boldsymbol{b}})| \le 1$$

and

$$\begin{aligned} \liminf_{n\to\infty} \int_Q |\boldsymbol{w}_n|^2 \,\mathrm{d}x\,\mathrm{d}t &\ge \liminf_{n\to\infty} \frac{c}{e}\left(e - \tfrac{1}{2}|\boldsymbol{w}|^2\right)^2 \int_Q \varphi^2 \sin^2(n[t,x]\cdot \eta_{\boldsymbol{a},\boldsymbol{b}}) \,\mathrm{d}x\,\mathrm{d}t - \limsup_{n\to\infty} \frac{c|R_n|^2}{n^2} \\ &= \frac{c}{e}\left(e - \tfrac{1}{2}|\boldsymbol{w}|^2\right)^2 \frac{|Q|}{2} \end{aligned}$$

using [De Lellis and Székelyhidi 2010, Lemma 7] in the last step. Strictly speaking $|R_n|$ is a random variable so we need $n \ge n_0(\omega)$, where the latter is $\mathfrak{F}$-measurable. Setting $[\boldsymbol{w}_n, \mathbb{V}_n] = [0, 0]$ whenever $n \le n_0$ yields the desired inclusion (5-22).

Step 6: If $e \le \bar{e}_M$ for some deterministic constants, then $\boldsymbol{w}$, $\mathbb{H}$ have compact range in R^N, $R^{N\times N}_{0,\mathrm{sym}}$, respectively. In addition, hypothesis (5-24) implies

$$[\boldsymbol{w}, \mathbb{H}] \in \mathcal{S}[e-\varepsilon] \quad \text{for any } 0 \le \varepsilon < \delta.$$

Thus the above construction can be therefore repeated with e replaced by $e - \varepsilon$, $\varepsilon > 0$. Moreover, in view of (5-11), the remainder R_n specified in Step 5 above is now bounded uniformly by a deterministic constant depending only on ε. Since

$$\overline{\mathcal{S}[e-\delta]} \subset \mathcal{S}[e-\varepsilon] \subset \overline{\mathcal{S}[e-\varepsilon]} \subset \mathcal{S}[e],$$

compactness of the range of $\boldsymbol{w}_n$, $\mathbb{V}_n$ follows from their construction and (5-11). Notably relations (5-8) and (5-11) yield deterministic (in terms of ε) upper and lower bounds on the norm of the vector $\eta_{\boldsymbol{a},\boldsymbol{b}}$ used in the construction of $\boldsymbol{w}_n$, $\mathbb{V}_n$. More specifically,

$$0 < \underline{\eta} \le |\boldsymbol{a}|\,|\boldsymbol{b}| + \boldsymbol{a}\cdot\boldsymbol{b} \le \bar{\eta}, \quad 0 < \underline{\eta} \le |\eta_{\boldsymbol{a},\boldsymbol{b}}| \le \bar{\eta}, \tag{5-28}$$

for deterministic constants $\underline{\eta}$, $\bar{\eta}$. As $\varepsilon > 0$ can be taken arbitrarily small, the inclusion (5-25) follows.

Finally, we show the uniform convergence claimed in (5-26). As $\boldsymbol{w}_n$, $\mathbb{V}_n$ satisfy (5-20), (5-25), we observe that

$$\left| \int_{[0,1]^N} [\boldsymbol{w}_n(t_1,\cdot) - \boldsymbol{w}_n(t_2,\cdot)]\cdot\boldsymbol{\varphi}\,\mathrm{d}x \right| \le K(\bar{e}_M,\boldsymbol{\varphi})|t_1 - t_2|$$
$$\text{for any } 0 \le t_1 \le t_2 \le 1, \quad n = 1, 2, \dots, \quad \boldsymbol{\varphi} \in C_c^\infty((0,1)^N; R^N), \tag{5-29}$$

where K is a deterministic quantity.

Next we show that

$$\left| \int_Q \boldsymbol{w}_n \cdot \boldsymbol{\psi}\ \mathrm{d}x\,\mathrm{d}t \right| \le \frac{c(\bar{e}_M)}{n} \|\boldsymbol{\psi}\|_{W^{1,\infty}(R^{N+1};R^N)} \quad \text{for any } \boldsymbol{\psi} \in C_c^\infty(R^{N+1}; R^N). \tag{5-30}$$

Indeed

$$\boldsymbol{w}_n = \varphi L(\boldsymbol{a}-\boldsymbol{b}) \sin(n[t,x]\cdot\eta_{\boldsymbol{a},\boldsymbol{b}}) + \frac{1}{n} R_n,$$

where R_n is bounded in terms of the deterministic quantity $\bar{e}_M$. Next,

$$\begin{aligned}\left| \int_Q L(\boldsymbol{a}-\boldsymbol{b}) \sin(n[t,x]\cdot\eta_{\boldsymbol{a},\boldsymbol{b}})\cdot(\varphi\boldsymbol{\psi})\,\mathrm{d}x\ \mathrm{d}t \right| &= \left| \int_{R^N}\int_R L(\boldsymbol{a}-\boldsymbol{b}) \sin(n[t,x]\cdot\eta_{\boldsymbol{a},\boldsymbol{b}})\cdot(\varphi\boldsymbol{\psi})\ \mathrm{d}t\ \mathrm{d}x \right| \\ &\le \frac{1}{n|(\eta_{\boldsymbol{a},\boldsymbol{b}})_0|} \left| \int_{R^N}\int_R L(\boldsymbol{a}-\boldsymbol{b}) \cos(n[t,x]\cdot\eta_{\boldsymbol{a},\boldsymbol{b}})\cdot\partial_t(\varphi\boldsymbol{\psi})\ \mathrm{d}t\ \mathrm{d}x \right| \\ &\le \frac{c(\bar{e}_M)}{n|(\eta_{\boldsymbol{a},\boldsymbol{b}})_0|} \|\boldsymbol{\psi}\|_{W^{1,\infty}(R^{N+1};R^N)}.\end{aligned}$$

In view of (5-15),

$$(\eta_{\boldsymbol{a},\boldsymbol{b}})_0 = (|\boldsymbol{a}|\,|\boldsymbol{b}| + \boldsymbol{a}\cdot\boldsymbol{b})^{1/3},$$

whence (5-30) follows from (5-28).

It remains to observe that (5-29), (5-30) give rise to the uniform convergence claimed in (5-26). Indeed, since $\|\boldsymbol{w}_n\|_{L^\infty(Q;R^N)} \le c(\bar{e}_M)$, it is enough to show that

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \left(\sup_{t\in[0,1]} \left| \int_{[0,1]^N} \boldsymbol{w}_n(t,\cdot)\cdot\boldsymbol{\varphi}\,\mathrm{d}x \right| \right) \to 0 \quad \text{as } n \to \infty \tag{5-31}$$

for any fixed $\boldsymbol{\varphi} \in C_c^\infty((0,1)^N; R^N)$. We write,

$$\int_{[0,1]^N} \boldsymbol{w}_n(t,\cdot)\cdot\boldsymbol{\varphi}\,\mathrm{d}x = \int_R \psi_\varepsilon(t-\tau) \left(\int_{[0,1]^N} (\boldsymbol{w}_n(t,\cdot) - \boldsymbol{w}_n(\tau,\cdot))\cdot\boldsymbol{\varphi}\,\mathrm{d}x \right) \mathrm{d}\tau + \int_Q \psi_\varepsilon(t-\tau)\boldsymbol{w}_n\cdot\boldsymbol{\varphi}\ \mathrm{d}x\,\mathrm{d}\tau$$

for any

$$\psi_\varepsilon \in C_c^\infty(R), \quad \psi_\varepsilon \ge 0, \quad \operatorname{supp}[\psi_\varepsilon] \subset [-\varepsilon,\varepsilon], \quad \int_R \psi_\varepsilon(t)\ \mathrm{d}t = 1.$$

Consequently, (5-31) follows from (5-29), (5-30). □

5C.1. *Extension by scaling.* Let

$$Q = (T_1, T_2) \times (a_1, b_1) \times \cdots \times (a_N, b_N).$$

Following [Donatelli, Feireisl, and Marcati 2015, Section 4.2], we may use scaling in t and x and additivity of the integral to show the following extension of Lemma 5.6.

Lemma 5.8. *Let $\omega \mapsto [e, r, \boldsymbol{w}, \mathbb{H}]$ be a $[\mathfrak{F}; \mathfrak{B}[(0,\infty)^2, R^N, R^{N\times N}_{0,\mathrm{sym}}]]$-measurable mapping such that*

$$\left[\frac{\boldsymbol{w}}{\sqrt{r}}, \mathbb{H}\right] \in \mathcal{S}[e] \ \ \mathbb{P}\text{-a.s.}$$

Then there exist sequences $\boldsymbol{w}_n$ and $\mathbb{V}_n$ such that $\boldsymbol{w}_n \in C^\infty_c(Q; R^N)$ $\mathbb{P}$-a.s. and $\mathbb{V}_n \in C^\infty_c(Q; R^{N\times N}_{0,\mathrm{sym}})$ $\mathbb{P}$-a.s., $n \in \mathbb{N}$, enjoying the following properties:

(i) *$t \mapsto [\boldsymbol{w}_n, \mathbb{V}_n]$ is a stochastic process, meaning*

$$[\boldsymbol{w}_n(t,\cdot); \mathbb{V}_n(t,\cdot)] \in C\left(\prod_{i=1}^N [a_i, b_i]; R^N \times R^{N\times N}_{0,\mathrm{sym}}\right) \ \ \mathbb{P}\text{-a.s.}$$

$$\text{is } \left[\mathfrak{F}; \mathfrak{B}\left[C\left(\prod_{i=1}^N [a_i, b_i]; R^N \times R^{N\times N}_{0,\mathrm{sym}}\right)\right]\right]\text{-measurable for any } t \in [T_1, T_2]. \tag{5-32}$$

(ii) *In Q we have $\mathbb{P}$-a.s.*

$$\partial_t \boldsymbol{w}_n + \mathrm{div}_x \mathbb{V}_n = 0, \quad \mathrm{div}_x \boldsymbol{w}_n = 0. \tag{5-33}$$

(iii) *As $n \to \infty$ we have $\mathbb{P}$-a.s.*

$$\boldsymbol{w}_n \to 0 \quad \text{in } C_{\mathrm{weak}}([T_1, T_2]; L^2(R^N; R^N)). \tag{5-34}$$

(iv) *In Q we have $\mathbb{P}$-a.s.*

$$\left[\frac{\boldsymbol{w} + \boldsymbol{w}_n}{\sqrt{r}}, \mathbb{H} + \mathbb{V}_n\right] \in \mathcal{S}[e]. \tag{5-35}$$

(v) *The following holds $\mathbb{P}$-a.s.*:

$$\liminf_{n\to\infty} \frac{1}{|Q|} \int_Q \frac{|\boldsymbol{w}_n|^2}{r} \,\mathrm{d}x \,\mathrm{d}t \geq \frac{c(N)}{e}\left(e - \frac{1}{2}\frac{|\boldsymbol{w}|^2}{r}\right)^2. \tag{5-36}$$

If, in addition,

$$0 < \underline{r}_M \leq r \leq \bar{r}_M, \quad 0 < \underline{e}_M \leq e \leq \bar{e}_M \ \ \mathbb{P}\text{-a.s.} \tag{5-37}$$

for some deterministic constants $\underline{r}_M, \bar{r}_M, \underline{e}_M, \bar{e}_M$, and

$$\left[\frac{\boldsymbol{w}}{\sqrt{r}}, \mathbb{H}\right] \in \mathcal{S}[e - \delta] \ \ \mathbb{P}\text{-a.s. for some deterministic } \delta > 0,$$

then each $\boldsymbol{w}_n, \mathbb{V}_n$ has compact range in $C(\overline{Q}; R^N)$, $C(\overline{Q}; R^{N\times N}_{0,\mathrm{sym}})$, respectively, and

$$\left[\frac{\boldsymbol{w} + \boldsymbol{w}_n}{\sqrt{r}}, \mathbb{H} + \mathbb{V}_n\right] \in \mathcal{S}[e - \delta_n] \ \ \mathbb{P}\text{-a.s. for some deterministic } \delta_n > 0. \tag{5-38}$$

Moreover,

$$\operatorname{ess\,sup}_{\omega\in\Omega}\Big(\sup_{t\in[0,T]} d[\boldsymbol{w}_n(t,\cdot);0]\Big)\to 0 \quad \textit{as } n\to\infty.$$

Remark 5.9. Condition (5-37) can be equivalently formulated saying that the random variable $[r, e]$ has compact range in $(0,\infty)^2$.

5C.2. *Extension to piecewise constant coefficients.* Consider now a complete right-continuous filtration $(\mathfrak{F}_t)_{t\ge 0}$ of measurable sets in Ω and fix $Q=(0,T)\times(0,1)^N$. We write $[0,1]^N=\bigcup_{i\in I}\overline{K}_i$, where K_i are disjoint open cubes of the edge length $1/m$ for some $m\in\mathbb{N}$. The random variables e, r, $\boldsymbol{w}$, and $\mathbb{H}$ will be now $\mathbb{P}$-a.s. functions of the time t and the spatial variable x that are piecewise constant. More specifically, they shall $\mathbb{P}$-a.s. belong to the class of functions satisfying

$$F(t,x)=F_{j,i}\quad \text{whenever } t\in\left[\frac{jT}{m};\frac{(j+1)T}{m}\right),\ x\in K_i,\ 0\le j\le m-1,\ i\in I. \tag{5-39}$$

These functions are piecewise constant on the rectangular grid given by

$$\left[\frac{jT}{m},\frac{(j+1)T}{m}\right)\times K_i,\quad 0\le j\le m-1,\quad i\in I.$$

In addition, we suppose that $[e,r,\boldsymbol{w},\mathbb{H}]$ is $(\mathfrak{F}_t)$-adapted, meaning that

$$[e,r,\boldsymbol{w},\mathbb{H}](t,\cdot)\text{ is } \mathfrak{F}_{jT/m}\text{-measurable whenever } t\in\left[\frac{jT}{m};\frac{(j+1)T}{m}\right).$$

Keeping in mind that the oscillatory increments $[\boldsymbol{w}_n,\mathbb{V}_n]$ constructed in Lemma 5.8 are compactly supported in each cube and hence globally smooth, we get the following result when applying Lemma 5.8 with $\mathfrak{F}$ replaced by $\mathfrak{F}_{jT/m}$. Note that $\boldsymbol{w}_n$, $\mathbb{V}_n$ are even $\mathfrak{F}_{jT/m}$ adapted.

Lemma 5.10. *Let* $(\Omega,\mathfrak{F},(\mathfrak{F}_t)_{t\ge0},\mathbb{P})$ *be a probability space with a complete right continuous filtration* $(\mathfrak{F}_t)_{t\ge0}$. *Let* $[e,r,\boldsymbol{w},\mathbb{H}]$ *be an* $(\mathfrak{F}_t)$*-adapted stochastic process which is* $\mathbb{P}$*-a.s. piecewise constant and belongs to the class* (5-39). *Suppose further that* $r>0$, $e>0$ $\mathbb{P}$*-a.s. and*

$$\left[\frac{\boldsymbol{w}}{\sqrt{r}},\mathbb{H}\right]\in\mathcal{S}[e]\quad \textit{for all } (t,x)\in\overline{Q}\ \ \mathbb{P}\textit{-a.s.} \tag{5-40}$$

Then there exist sequences $\boldsymbol{w}_n$ *and* $\mathbb{V}_n$ *such that* $\boldsymbol{w}_n\in C_c^\infty(Q;R^N)$ $\mathbb{P}$*-a.s. and* $\mathbb{V}_n\in C_c^\infty(Q;R^{N\times N}_{0,\mathrm{sym}})$ $\mathbb{P}$*-a.s.*, $n\in\mathbb{N}$, *enjoying the following properties*:

(i) *The process* $[\boldsymbol{w}_n,\mathbb{V}_n]$ *is* $(\mathfrak{F}_t)$*-adapted such that*

$$[\boldsymbol{w}_n,\mathbb{V}_n]\in C(\overline{Q};R^N\times R^{N\times N}_{0,\mathrm{sym}})\ \ \mathbb{P}\textit{-a.s. with compact range}.$$

(ii) *In* Q *we have* $\mathbb{P}$*-a.s.*

$$\partial_t\boldsymbol{w}_n+\operatorname{div}_x\mathbb{V}_n=0,\quad \operatorname{div}_x\boldsymbol{w}_n=0.$$

(iii) *As* $n\to\infty$ *we have* $\mathbb{P}$*-a.s.*

$$\boldsymbol{w}_n\to 0\quad \textit{in } C_{\mathrm{weak}}([0,T];L^2(\mathcal{T}^N;R^N)).$$

(iv) *In Q we have $\mathbb{P}$-a.s.*

$$\left[\frac{\boldsymbol{w}+\boldsymbol{w}_n}{\sqrt{r}}, \mathbb{H}+\mathbb{V}_n\right] \in \mathcal{S}[e]. \tag{5-41}$$

(v) *The following holds $\mathbb{P}$-a.s.*:

$$\liminf_{n\to\infty} \int_Q \frac{|\boldsymbol{w}_n|^2}{r} \,\mathrm{d}x \,\mathrm{d}t \geq \frac{c(N)}{\sup_Q e} \int_Q \left(e - \frac{1}{2}\frac{|\boldsymbol{w}|^2}{r}\right)^2 \mathrm{d}x \,\mathrm{d}t. \tag{5-42}$$

If, in addition,

$$0 < \underline{r}_M \leq r \leq \bar{r}_M, \quad 0 < \underline{e}_M \leq e \leq \bar{e}_M \quad \mathbb{P}\text{-a.s.}$$

for some deterministic constants $\underline{r}_M, \bar{r}_M, \underline{e}_M, \bar{e}_M$, and

$$\left[\frac{\boldsymbol{w}}{r}, \mathbb{H}\right] \in \mathcal{S}[e-\delta]\ \mathbb{P}\text{-a.s. for some deterministic } \delta > 0,$$

then each $\boldsymbol{w}_n, \mathbb{V}_n$ has compact range in $C(\overline{Q}; R^N)$, $C(\overline{Q}; R^{N\times N}_{0,\mathrm{sym}})$, respectively, and

$$\left[\frac{\boldsymbol{w}+\boldsymbol{w}_n}{\sqrt{r}}, \mathbb{H}+\mathbb{V}_n\right] \in \mathcal{S}[e-\delta_n]\ \ \mathbb{P}\text{-a.s. for some deterministic } \delta_n > 0.$$

Moreover,

$$\operatorname*{ess\,sup}_{\omega\in\Omega}\Big(\sup_{t\in[0,T]} d[\boldsymbol{w}_n(t,\cdot);0]\Big) \to 0 \quad \text{as } n\to\infty.$$

5C.3. *Extension to continuous coefficients.* Using the result on the piecewise constant coefficients, we may use the approximation procedure from [Donatelli, Feireisl, and Marcati 2015, Section 4.3] to extend the oscillatory lemma to the class of continuous processes $[e, r, \boldsymbol{w}, \mathbb{H}]$. The obvious idea is to replace $[e, r, \boldsymbol{w}, \mathbb{H}]$ by piecewise constant approximations and apply Lemma 5.10. More specifically, for $e \in C([0,T]\times\mathcal{T}^N; (0,\infty))$ $\mathbb{P}$-a.s., $(\mathfrak{F}_t)$-adapted, $e > 0$ $\mathbb{P}$-a.s., we define a piecewise constant approximation

$$e_m(t,x) = \sup_{y\in K_i} e\left(\frac{jT}{m}, y\right) \quad \text{for } t \in \left[\frac{jT}{m}; \frac{(j+1)T}{m}\right),\ x\in K_i,\ 0\leq j\leq m-1,\ i\in I, \tag{5-43}$$

and, similarly, for $F \in \{r, \boldsymbol{w}, \mathbb{H}\}$,

$$F_m(t,x) = F\left(\frac{jT}{m}, y\right) \quad \text{for some } y\in K_i,\ \text{for } t\in\left[\frac{jT}{m}; \frac{(j+1)T}{m}\right),\ x\in K_i,\ 0\leq j\leq m-1,\ i\in I. \tag{5-44}$$

It is easy to check that these approximations satisfy the hypotheses of Lemma 5.10.

Now, since $\mathcal{S}[e]$ is an open set, it is possible, similarly to [Donatelli, Feireisl, and Marcati 2015, Section 4.3] to replace F_m by F as long as the approximation is uniform. Specifically, for any $\delta > 0$, there is $m = m(\delta)$ such that

$$|F_m(t,x) - F(t,x)| < \delta \quad \text{for all } (t,x) \in \overline{Q} \tag{5-45}$$

$\mathbb{P}$-a.s. For (5-45) to hold, it is necessary (and sufficient) that all random variables $F = e, r, \boldsymbol{w}, \mathbb{H}$ have compact range in the space of continuous functions on $\overline{Q}$. Repeating the arguments of [Donatelli, Feireisl, and Marcati 2015, Section 4.3] we show the final form of the oscillatory lemma.

Lemma 5.11. *Let $[\Omega, \mathfrak{F}, \mathfrak{F}_t, \mathbb{P}]$ be a probability space with a complete right continuous filtration $(\mathfrak{F}_t)_{t\geq 0}$. Let $[e, r, \boldsymbol{w}, \mathbb{H}]$ be an $(\mathfrak{F}_t)$-adapted stochastic process such that*

$$[e, r, \boldsymbol{w}, \mathbb{H}] \in C(\overline{Q}; (0,\infty)^2 \times R^N \times R^{N\times n}_{0,\mathrm{sym}}) \ \ \mathbb{P}\text{-a.s.}$$

with compact range and such that

$$\left[\frac{\boldsymbol{w}}{\sqrt{r}}, \mathbb{H}\right] \in \mathcal{S}[e-\delta] \quad \text{for all } (t,x) \in \overline{Q} \ \ \mathbb{P}\text{-a.s.} \tag{5-46}$$

for some deterministic constant $\delta > 0$.

Then there exist sequences $\boldsymbol{w}_n$ and $\mathbb{V}_n$ such that $\boldsymbol{w}_n \in C^\infty_c(Q; R^N)$ $\mathbb{P}$-a.s. and $\mathbb{V}_n \in C^\infty_c(Q; R^{N\times N}_{0,\mathrm{sym}})$ $\mathbb{P}$-a.s., $n \in \mathbb{N}$, enjoying the following properties:

(i) *The process $[\boldsymbol{w}_n, \mathbb{V}_n]$ is $(\mathfrak{F}_t)$-adapted such that*

$$[\boldsymbol{w}_n, \mathbb{V}_n] \in C(\overline{Q}; R^N \times R^{N\times N}_{0,\mathrm{sym}}) \ \mathbb{P}\text{-a.s. with compact range.}$$

(ii) *In Q we have $\mathbb{P}$-a.s.*

$$\partial_t \boldsymbol{w}_n + \mathrm{div}_x \mathbb{V}_n = 0, \quad \mathrm{div}_x \boldsymbol{w}_n = 0.$$

(iii) *We have*

$$\operatorname*{ess\,sup}_{\omega\in\Omega}\Big(\sup_{t\in[0,T]} d[\boldsymbol{w}_n(t,\cdot); 0]\Big) \to 0 \quad \text{as } n \to \infty. \tag{5-47}$$

(iv) *In Q we have $\mathbb{P}$-a.s.*

$$\left[\frac{\boldsymbol{w}+\boldsymbol{w}_n}{\sqrt{r}}, \mathbb{H} + \mathbb{V}_n\right] \in \mathcal{S}[e-\delta_n]$$

for some deterministic $\delta_n > 0$.

(v) *The following holds $\mathbb{P}$-a.s.*:

$$\liminf_{n\to\infty} \int_Q \frac{|\boldsymbol{w}_n|^2}{r}\,\mathrm{d}x\,\mathrm{d}t \geq \frac{c(N)}{\sup_Q e}\int_Q \left(e - \frac{1}{2}\frac{|\boldsymbol{w}|^2}{r}\right)^2 \mathrm{d}x\,\mathrm{d}t. \tag{5-48}$$

Remark 5.12. Observe that the assumption for a random variable $[e, r, \boldsymbol{w}, \mathbb{H}]$ to be of compact range in $C(\overline{Q}; (0,\infty)^2 \times R^N \times R^{N\times n}_{0,\mathrm{sym}})$ includes

$$0 < \underline{r}_M \leq r \leq \overline{r}_M, \quad e \leq e_M \quad \mathbb{P}\text{-a.s.}$$

for some deterministic constants $\underline{r}_M, \overline{r}_M, e_M$ as well as a positive lower bound for e already guaranteed by (5-46).

Remark 5.13. The fact that the continuous processes considered in Lemma 5.11 must have compact range is definitely restrictive but possibly unavoidable. This is also the main reason why our result holds up to a stopping time, albeit arbitrarily large with "high" probability. Otherwise, the size of the grid used to construct the approximations F_m would have to be a random variable. The oscillatory increments $\boldsymbol{w}_n$ would be then constructed on a grid determined by random points $0 < t_1 < t_2 < \cdots < t_m$ related to stopping times associated to certain norms of the random processes. Here, the length of the interval $[t_m, t_{m+1}]$ would have to be t_m predictable which seems impossible.

6. Infinitely many solutions

We are ready to show Theorem 2.3 or, equivalently, its version for the abstract "Euler" problems (4-7), (4-14), respectively. We begin with problem (4-7), where the tensor $\mathbb{M}$ is constant. Then, following [Feireisl 2016], we specify how to accommodate the dependence $\mathbb{M} = \mathbb{M}[\boldsymbol{v}]$.

6A. *Subsolutions.* We introduce the set

$$\mathcal{X}(R^N) := \{\boldsymbol{v} : \Omega \times Q \to R^N \mid \text{measureable}, \ \boldsymbol{v} \in C([0, T] \times \mathcal{T}^N; R^N) \text{ a.s. with compact range}\} \tag{6-1}$$

and analogously define $\mathcal{X}(R^{N\times N}_{0,\mathrm{sym}})$. Following [De Lellis and Székelyhidi 2010], we introduce the set of *subsolutions*. Let the functions $\boldsymbol{v}_0$, $\boldsymbol{h}$, r and $\mathbb{M}$ satisfy (4-8), and $e = e(t)$ is a real-valued $(\mathfrak{F}_t)$-adapted process specified below. In particular, the process $[\boldsymbol{h}, r, \mathbb{M}] \in C([0, T] \times \mathcal{T}^N; R^N) \times (0, \infty) \times R^{N\times N}_{0,\mathrm{sym}}$ is $(\mathfrak{F}_t)$ adapted and with compact range. We define a collection of subsolutions corresponding to $\boldsymbol{v}_0$, $\boldsymbol{h}$, r, $\mathbb{M}$ and e by

$$X_0 = \Bigg\{ \boldsymbol{v} \in \mathcal{X}(R^N) \;\Bigg|\; \boldsymbol{v} \text{ is } (\mathfrak{F}_t)\text{-adapted with } \boldsymbol{v}(0, \cdot) = \boldsymbol{v}_0, \text{ there is } \mathbb{F} \in \mathcal{X}(R^{N\times N}_{0,\mathrm{sym}})\ (\mathfrak{F}_t)\text{-adapted s.t.}$$
$$\partial_t \boldsymbol{v} + \mathrm{div}_x\, \mathbb{F} = 0, \ \mathrm{div}_x\, \boldsymbol{v} = 0 \text{ in } \mathcal{D}'((0, T)\times\mathcal{T}^N; R^N) \ \ \mathbb{P}\text{-a.s.},$$
$$\tfrac{1}{2} N \lambda_{\max}\left[\frac{(\boldsymbol{v}+\boldsymbol{h})\otimes(\boldsymbol{v}+\boldsymbol{h})}{r} - \mathbb{F} + \mathbb{M}\right] < e - \delta$$
$$\forall\, 0 \le t \le T, \ x \in \mathcal{T}^N, \ \mathbb{P}\text{-a.s. for some deterministic } \delta > 0 \Bigg\}. \tag{6-2}$$

Remark 6.1. The deterministic constant $\delta > 0$ may vary from one subsolution to another. The exact meaning of the condition

$$\tfrac{1}{2} N \lambda_{\max}\left[\frac{(\boldsymbol{v}+\boldsymbol{h})\otimes(\boldsymbol{v}+\boldsymbol{h})}{r} - \mathbb{F} + \mathbb{M}\right] < e - \delta$$

is

$$\operatorname*{ess\,sup}_{\Omega} \sup_{t\in[0,T], x\in\mathcal{T}^N} \left(\tfrac{1}{2} N \lambda_{\max}\left[\frac{(\boldsymbol{v}+\boldsymbol{h})\otimes(\boldsymbol{v}+\boldsymbol{h})}{r} - \mathbb{F} + \mathbb{M}\right] - e \right) < 0.$$

6B. *Existence of a subsolution.* Next we claim that e can be fixed in such a way that the set of subsolutions is nonempty. To this end, consider

$$\boldsymbol{v} = \boldsymbol{v}_0, \quad \mathbb{F} = 0.$$

This is obviously a subsolution provided e is taken in such a way that

$$\tfrac{1}{2} N \lambda_{\max}\left[\frac{(\boldsymbol{v}_0+\boldsymbol{h})\otimes(\boldsymbol{v}_0+\boldsymbol{h})}{r} + \mathbb{M}\right] < e - \delta.$$

In view of (4-8) this is possible, where $e = e_M$ can be taken as a sufficiently large *deterministic* constant.

6C. ***Topology on the set of subsolutions.*** The processes $\boldsymbol{v}$ belonging to X_0 are uniformly *deterministically* bounded in $L^\infty((0,T)\times\mathcal{T}^N)$; specifically, $\boldsymbol{v}(t)\in B_\infty$ for any $t\in[0,T]$ $\mathbb{P}$-a.s., where B_∞ is a ball in $L^\infty(\mathcal{T}^N)$ with a deterministic radius. Consequently, we may consider the metric d, introduced in (5-17), associated to the weak $L^2(\mathcal{T}^N)$-topology on B_∞, together with

$$D[\boldsymbol{v},\boldsymbol{w}]=\mathbb{E}[\sup_{t\in[0,T]} d[\boldsymbol{v}(t);\boldsymbol{w}(t)]].$$

Let X be the completion of X_0 with respect to the metric D. Then X is a complete metric space with infinite cardinality. Note that any element of X is $(\mathfrak{F}_t)$-adapted as the limit of measurable functions is measurable.

6D. ***Convex functional.*** Similarly to [De Lellis and Székelyhidi 2010], we introduce the functional

$$I[\boldsymbol{v}]=\mathbb{E}\left[\int_0^T\int_{\mathcal{T}^N}\left[\frac{1}{2}\frac{|\boldsymbol{v}+\boldsymbol{h}|^2}{r}-e\right]\mathrm{d}x\,\mathrm{d}t\right].$$

Here, $\boldsymbol{h}$, r are given functions related to the density ansatz and e is the target energy. Exactly as in [De Lellis and Székelyhidi 2010], it can be shown that:

- I is lower semicontinuous on the space X.
- $I[\boldsymbol{v}]\le 0$ for any $\boldsymbol{v}\in X$.
- If $I[\boldsymbol{v}]=0$ then
$$e=\frac{1}{2}\frac{|\boldsymbol{v}+\boldsymbol{h}|^2}{r}\quad\text{a.e. in } (0,T)\times\mathcal{T}^N \tag{6-3}$$
$\mathbb{P}$-a.s.

Lemma 6.2. *Under the hypotheses* (4-8), *each* $\boldsymbol{v}\in X$ *with* $I[\boldsymbol{v}]=0$ *solves the abstract Euler equation* (4-7).

Proof. Let $\boldsymbol{v}\in X$. Then there is $(\boldsymbol{v}_m)\subset X_0$ with $\boldsymbol{v}_m\to\boldsymbol{v}$ with respect to the metric D. By the definition of X_0 we can find a sequence of $(\mathfrak{F}_t)$-adapted processes $(\mathbb{F}_m)$ with $\mathbb{F}_m\in L^\infty(Q,R^{N\times N}_{0,\mathrm{sym}})$ $\mathbb{P}$-a.s. such that

$$\partial_t\boldsymbol{v}_m+\mathrm{div}_x\,\mathbb{F}_m=0\quad\text{in } \mathcal{D}'((0,T)\times R^N) \tag{6-4}$$

$\mathbb{P}$-a.s. and

$$\tfrac{1}{2}N\lambda_{\max}\left[\frac{(\boldsymbol{v}_m+\boldsymbol{h})\otimes(\boldsymbol{v}_m+\boldsymbol{h})}{r}-\mathbb{F}_m+\mathbb{M}\right]\le e.$$

Using (5-2) and the properties of $\mathbb{M}$ (recall (4-8)) we see that $\mathbb{F}_m$ is uniformly bounded in $L^\infty(\Omega\times Q,R^{N\times N}_{0,\mathrm{sym}})$. Hence, after choosing a weakly* converging subsequence, we obtain

$$\partial_t\boldsymbol{v}+\mathrm{div}_x\,\mathbb{F}=0,\quad \mathrm{div}_x\,\boldsymbol{v}=0,\quad \boldsymbol{v}(0,\cdot)=\boldsymbol{v}_0\quad\text{in } \mathcal{D}'((0,T)\times R^N), \tag{6-5}$$

for a certain $(\mathfrak{F}_t)$-adapted process $\mathbb{F}$ with $\mathbb{F}\in L^\infty(Q,R^{N\times N}_{0,\mathrm{sym}})$ $\mathbb{P}$-a.s. Due to convexity of the functional

$$[\boldsymbol{v},\mathbb{F}]\mapsto\tfrac{1}{2}N\lambda_{\max}\left[\frac{(\boldsymbol{v}+\boldsymbol{h})\otimes(\boldsymbol{v}+\boldsymbol{h})}{r}-\mathbb{F}+\mathbb{M}\right],$$

we have

$$\tfrac{1}{2} N \lambda_{\max}\left[\frac{(\boldsymbol{v}+\boldsymbol{h})\otimes(\boldsymbol{v}+\boldsymbol{h})}{r} - \mathbb{F} + \mathbb{M}\right] \le e.$$

Consequently, by virtue of (5-1), relation (6-3) implies

$$\mathbb{F} = \mathbb{M} + \frac{(\boldsymbol{v}+\boldsymbol{h})\otimes(\boldsymbol{v}+\boldsymbol{h})}{r} - \frac{1}{N}\frac{|\boldsymbol{v}+\boldsymbol{h}|^2}{r} = \mathbb{M} + \frac{(\boldsymbol{v}+\boldsymbol{h})\otimes(\boldsymbol{v}+\boldsymbol{h})}{r} - \frac{2}{N}e.$$

As e is independent of x, (6-4) yields the desired conclusion (4-7). □

Thus each zero point of I yields a weak solution of the abstract Euler problem (4-7). Our next claim is that $I[\boldsymbol{v}] = 0$ whenever $\boldsymbol{v}$ is a point of continuity of I on X. By means of the Baire category argument, the points of continuity of I, the latter being a lower-semicontinuous functional on the complete metric space X, form a residual set, and in particular are dense in X, which completes the proof of the existence of infinitely many solutions claimed in Theorem 2.3. Thus it remains to show that I vanishes at each point of continuity, which is the objective of the last section.

6E. *Points of continuity of I in X.* We show that at each point of continuity of I on X, we have $I[\boldsymbol{v}] = 0$. Let $\boldsymbol{v}$ be a point of continuity of I on X. Suppose that $I[\boldsymbol{v}] < 0$. Consequently, there is a sequence

$$\boldsymbol{v}_m \in X_0, \quad D[\boldsymbol{v}_m; \boldsymbol{v}] \to 0, \quad I[\boldsymbol{v}_m] \to I[\boldsymbol{v}], \quad I[\boldsymbol{v}_m] < -\varepsilon < 0 \quad \text{for all } m = 0, 1, \ldots.$$

Now, we use the oscillatory lemma (Lemma 5.11) with the ansatz $\boldsymbol{w} = \boldsymbol{v}_m + \boldsymbol{h}$, $\mathbb{H} = \mathbb{F}_m - \mathbb{M}$. Consequently, for each fixed m, we find a sequence $\{\boldsymbol{w}_{m,n}\}_{n=1}^{\infty} \subset X_0$ such that

$$\boldsymbol{v}_m + \boldsymbol{w}_{m,n} \in X_0, \quad D[\boldsymbol{v}_m + \boldsymbol{w}_{m,n}, \boldsymbol{v}_m] \to 0 \quad \text{as } n \to \infty.$$

The first statement follows from Lemma 5.11(iv), which also yields a uniform bound for $\boldsymbol{w}_{m,n}$ as a consequence of (5-1). The convergence with respect to the metric D follows from Lemma 5.11(iii), the uniform bounds for $\boldsymbol{w}_{m,n}$ and dominated convergence. Moreover, due to Lemma 5.11(iii), we have

$$\liminf_{n\to\infty} I[\boldsymbol{v}_m + \boldsymbol{w}_{m,n}] = I[\boldsymbol{v}_m] + \liminf_{n\to\infty} \tfrac{1}{2}\mathbb{E}\left[\int_0^T \int_{\mathcal{T}^N} \frac{|\boldsymbol{w}_{m,n}|^2}{r}\,\mathrm{d}x\,\mathrm{d}t\right].$$

Here, by virtue of (5-48), Fatou's lemma and Jensen's inequality

$$\begin{aligned}\liminf_{n\to\infty} \tfrac{1}{2}\mathbb{E}\left[\int_0^T \int_{\mathcal{T}^N} \frac{|\boldsymbol{w}_{m,n}|^2}{r}\,\mathrm{d}x\,\mathrm{d}t\right] &\ge \frac{c(N,T)}{e}\left(\mathbb{E}\left[\int_0^T \int_{\mathcal{T}^N} \left[e - \frac{1}{2}\frac{|\boldsymbol{v}_m+\boldsymbol{h}|^2}{r}\right]\mathrm{d}x\,\mathrm{d}t\right]\right)^2 \\ &= \frac{c(N,T)}{e} I^2[\boldsymbol{v}_m] \ge \varepsilon^2 \frac{c(N,T)}{e}.\end{aligned}$$

In such a way, we may construct a sequence $(\tilde{\boldsymbol{v}}_m) \subset X_0$, $\tilde{\boldsymbol{v}}_m = \boldsymbol{v}_m + \boldsymbol{w}_{m,n(m)}$, $D[\tilde{\boldsymbol{v}}_m, \boldsymbol{v}] \to 0$, and

$$\liminf_{m\to\infty} I[\tilde{\boldsymbol{v}}_m] > I[\boldsymbol{v}]. \tag{6-6}$$

Relation (6-6) contradicts the assumption that $\boldsymbol{v}$ is a point of continuity of I.

6F. ***Multiplicative noise.*** We conclude by showing how to accommodate the case of multiplicative noise, where the matrix $\mathbb{M}$ depends on the solutions $\boldsymbol{v}$; specifically,

$$\mathbb{M}[\boldsymbol{v}] = \nabla_x \boldsymbol{m} + \nabla_x \boldsymbol{m}^t - \frac{2}{N} \operatorname{div}_x \boldsymbol{m} \mathbb{I},$$

where $\boldsymbol{m}$ is the unique solution of the elliptic system (4-13). In particular, if $\boldsymbol{v} \in \mathcal{X}(R^N)$ (see (6-1)), then, in view of the standard elliptic estimates, $\mathbb{M}[\boldsymbol{v}]$ is $(\mathfrak{F}_t)_{t\geq 0}$-adapted and with compact range in $C([0,T] \times \mathcal{T}^N; R^{N\times N}_{0,\mathrm{sym}})$. Exactly as in Section 6A, we define the set of subsolutions as

$$X_0 = \Big\{ \boldsymbol{v} \in \mathcal{X}(R^N) \,\Big|\, \boldsymbol{v} \text{ is } (\mathfrak{F}_t)\text{-adapted with } \boldsymbol{v}(0,\cdot) = \boldsymbol{v}_0, \text{ there is } \mathbb{F} \in \mathcal{X}(R^{N\times N}_{0,\mathrm{sym}})\, (\mathfrak{F}_t)\text{-adapted s.t.}$$
$$\partial_t \boldsymbol{v} + \operatorname{div}_x \mathbb{F} = 0,\ \operatorname{div}_x \boldsymbol{v} = 0 \text{ in } \mathcal{D}'((0,T)\times\mathcal{T}^N; R^N)\ \mathbb{P}\text{-a.s.},$$
$$\tfrac{1}{2} N \lambda_{\max}\left[\frac{(\boldsymbol{v}+\boldsymbol{h})\otimes(\boldsymbol{v}+\boldsymbol{h})}{r} - \mathbb{F} + \mathbb{M}[\boldsymbol{v}] \right] < e - \delta$$
$$\forall\, 0 \leq t \leq T,\ x \in \mathcal{T}^N,\ \ \mathbb{P}\text{-a.s. for some deterministic } \delta > 0 \Big\}. \tag{6-7}$$

Similarly to the above, we can show that

- the set X_0 is nonempty;
- its closure with respect to the metric D is a complete metric space with infinite cardinality.

Consider the functional I on X defined in the same way as in Section 6D. We have an analogue of Lemma 6.2:

Lemma 6.3. *Under the hypotheses* (4-15), (4-16) *each* $\boldsymbol{v} \in X$ *with* $I[\boldsymbol{v}] = 0$ *solves the abstract Euler equation* (4-7).

Proof. The proof follows the same lines as that of Lemma 6.2. We have only to observe that, up to a suitable subsequence,

$$\mathbb{M}[\boldsymbol{v}_m] \to \mathbb{M}[\boldsymbol{v}] \quad \text{in } C([0,T] \times \mathcal{T}^N; R^{N\times N}_{0,\mathrm{sym}})\ \ \mathbb{P}\text{-a.s.}$$

whenever

$$\{\boldsymbol{v}_m\}_{m=1}^{\infty} \subset X_0, \qquad D[\boldsymbol{v}_m, \boldsymbol{v}] \to 0.$$

Indeed this follows from the elliptic regularity estimates as the sequence $\boldsymbol{v}_m$ is bounded by a deterministic constant in $L^\infty((0,T) \times \mathcal{T}^N; R^N)$, whence $\{\mathbb{M}[\boldsymbol{v}_m]\}_{m=1}^{\infty}$ belongs to $C_{\mathrm{weak}}([0,T]; W^{1,p}(\mathcal{T}^N; R^{N\times N}_{0,\mathrm{sym}}))$ for any $1 < p < \infty$ $\mathbb{P}$-a.s. and is bounded in the space

$$L^\infty([0,T]; W^{1,p}(\mathcal{T}^N; R^{N\times N}_{0,\mathrm{sym}}))$$

by a deterministic constant. □

Finally, we show that, necessarily, $I[\boldsymbol{v}] = 0$ at any point of continuity $\boldsymbol{v}$ of I. Following the arguments of Section 6E, we suppose $I[\boldsymbol{v}] < 0$ for some point of continuity $\boldsymbol{v} \in X$. We consider a sequence $\{\boldsymbol{v}_m\}_{m=1}^{\infty}$ satisfying

$$\boldsymbol{v}_m \in X_0, \quad D[\boldsymbol{v}_m; \boldsymbol{v}] \to 0, \quad I[\boldsymbol{v}_m] \to I[\boldsymbol{v}], \quad I[\boldsymbol{v}_m] < -\varepsilon < 0 \quad \text{for all } m = 0, 1, \ldots.$$

Next we apply the oscillatory lemma (Lemma 5.11) to

$$\boldsymbol{w} = \boldsymbol{v}_m + \boldsymbol{h}, \quad \mathbb{H} = \mathbb{F}_m - \mathbb{M}[\boldsymbol{v}_m].$$

Following the arguments of Section 6E, we find a sequence $\{\boldsymbol{w}_{m,n}\}_{n=1}^{\infty} \subset X_0$ such that

$$\boldsymbol{v}_m + \boldsymbol{w}_{m,n} \in X_0, \qquad D[\boldsymbol{v}_m + \boldsymbol{w}_{m,n}, \boldsymbol{v}_m] \to 0 \quad \text{as } n \to \infty.$$

Here, the most delicate point is to show the inclusion

$$\boldsymbol{v}_m + \boldsymbol{w}_{m,n} \in X_0,$$

as the oscillatory lemma (Lemma 5.11) asserts only

$$\tfrac{1}{2} N \lambda_{\max}\left[\frac{(\boldsymbol{v}_m + \boldsymbol{w}_{m,n} + \boldsymbol{h}) \otimes (\boldsymbol{v}_m + \boldsymbol{w}_{m,n} + \boldsymbol{h})}{r} - \mathbb{F}_m - \mathbb{F}_{m,n} + \mathbb{M}[\boldsymbol{v}_m]\right] < e - \delta_m$$

instead of the desired

$$\tfrac{1}{2} N \lambda_{\max}\left[\frac{(\boldsymbol{v}_m + \boldsymbol{w}_{m,n} + \boldsymbol{h}) \otimes (\boldsymbol{v}_m + \boldsymbol{w}_{m,n} + \boldsymbol{h})}{r} - \mathbb{F}_m - \mathbb{F}_{m,n} + \mathbb{M}[\boldsymbol{v}_m + \boldsymbol{w}_{m,n}]\right] < e - \delta_m. \tag{6-8}$$

Since we have

$$\mathbb{M}[\boldsymbol{v}_m + \boldsymbol{w}_{m,n}] - \mathbb{M}[\boldsymbol{v}_m] = \nabla_x \boldsymbol{F}_{m,n} + \nabla_x^t \boldsymbol{F}_{m,n} - \frac{2}{N} \operatorname{div}_x \boldsymbol{F}_{m,n} \mathbb{I},$$

where the field $\boldsymbol{F}_{m,n}$ is the unique solution of the elliptic system

$$\operatorname{div}_x\left[\nabla_x \boldsymbol{F}_{m,n} + \nabla_x^t \boldsymbol{F}_{m,n} - \frac{2}{N} \operatorname{div}_x \boldsymbol{F}_{m,n} \mathbb{I}\right] = \tfrac{1}{2} \boldsymbol{w}_{m,n} \quad \text{in } \mathcal{T}^N,$$

relation (6-8) follows as soon as we show

$$\operatorname*{ess\,sup}_{\omega \in \Omega} \sup_{t \in [0,T], x \in \mathcal{T}^N} \left| \nabla_x \boldsymbol{F}_{m,n} + \nabla_x^t \boldsymbol{F}_{m,n} - \frac{2}{N} \operatorname{div}_x \boldsymbol{F}_{m,n} \mathbb{I} \right| \to 0 \quad \text{as } n \to \infty. \tag{6-9}$$

To see (6-9), we use the convergence statement (5-47), namely

$$\operatorname*{ess\,sup}_{\omega \in \Omega}\Big(\sup_{t \in [0,T]} d[\boldsymbol{w}_{m,n}(t); 0]\Big) \to 0 \quad \text{as } n \to \infty. \tag{6-10}$$

On one hand, as

$$\|\boldsymbol{w}_{m,n}\|_{L^\infty((0,T) \times \mathcal{T}^N; R^N)} \le c(\bar{e}_M) \ \ \mathbb{P}\text{-a.s.}, \tag{6-11}$$

we may use the standard elliptic estimates to deduce

$$\sup_{t \in [0,T]} \|\nabla_x \boldsymbol{F}_{m,n}\|_{W^{1,q}(\mathcal{T}^N; R^{N \times N})} \le c(q, \bar{e}_M) \ \ \mathbb{P}\text{-a.s.}, \ \ 1 \le q < \infty. \tag{6-12}$$

On the other hand, by virtue of (6-10), (6-11),

$$\operatorname*{ess\,sup}_{\omega \in \Omega}\Big(\sup_{t \in [0,T]} \|\boldsymbol{w}_{m,n}\|_{W^{-1,2}(\mathcal{T}^N; R^N)}\Big) \to 0 \quad \text{as } n \to \infty,$$

whence, by the elliptic estimates,

$$\operatorname*{ess\,sup}_{\omega \in \Omega}\Big(\sup_{t \in [0,T]} \|\nabla_x \boldsymbol{F}_{m,n}\|_{L^2(\mathcal{T}^N; R^{N \times N})}\Big) \to 0 \quad \text{as } n \to \infty. \tag{6-13}$$

Seeing that $W^{1,q}(\mathcal{T}^N) \hookrightarrow\hookrightarrow C(\mathcal{T}^N)$ for $q > N$ we may interpolate (6-12), (6-13) to obtain the desired convergence (6-9).

The remaining arguments are the same as in Section 6E. Due to Lemma 5.11(iii), we have

$$\begin{aligned}\liminf_{n\to\infty} I[\boldsymbol{v}_m + \boldsymbol{w}_{m,n}] &= I[\boldsymbol{v}_m] + \liminf_{n\to\infty} \tfrac{1}{2}\mathbb{E}\left[\int_0^T \int_{\mathcal{T}^N} \frac{|\boldsymbol{w}_{m,n}|^2}{r}\, \mathrm{d}x\, \mathrm{d}t\right] \\ &\geq I[\boldsymbol{v}_m] + \varepsilon^2 \frac{c(N,T)}{e}.\end{aligned}$$

Thus for $\tilde{\boldsymbol{v}}_m = \boldsymbol{v}_m + \boldsymbol{w}_{m,n(m)}$ we get $\tilde{\boldsymbol{v}}_m \in X_0$, $D[\tilde{\boldsymbol{v}}_m, \boldsymbol{v}] \to 0$, and

$$\liminf_{m\to\infty} I[\tilde{\boldsymbol{v}}_m] > I[\boldsymbol{v}]. \tag{6-14}$$

Relation (6-14) contradicts the assumption that $\boldsymbol{v}$ is a point of continuity of I.

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VARIABLE COEFFICIENT WOLFF-TYPE INEQUALITIES AND SHARP LOCAL SMOOTHING ESTIMATES FOR WAVE EQUATIONS ON MANIFOLDS

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The sharp Wolff-type decoupling estimates of Bourgain and Demeter are extended to the variable coefficient setting. These results are applied to obtain new sharp local smoothing estimates for wave equations on compact Riemannian manifolds, away from the endpoint regularity exponent. More generally, local smoothing estimates are established for a natural class of Fourier integral operators; at this level of generality the results are sharp in odd dimensions, both in terms of the regularity exponent and the Lebesgue exponent.

1. Introduction and statement of results

1A. ***Local smoothing estimates.*** Let $n \geq 2$ and (M, g) be a smooth,[1] compact n-dimensional Riemannian manifold with associated Laplace–Beltrami operator Δ_g. Given initial data $f_0, f_1 \colon M \to \mathbb{C}$ belonging to some a priori class, consider the Cauchy problem

$$\begin{cases} (\partial_t^2 - \Delta_g)u = 0, \\ u(\,\cdot\,, 0) = f_0, \quad \partial_t u(\,\cdot\,, 0) = f_1. \end{cases} \tag{1-1}$$

It was shown, inter alia, in [Seeger, Sogge, and Stein 1991, Theorem 4.1] that for each fixed time t and $1 < p < \infty$ the solution u satisfies[2]

$$\|u(\,\cdot\,, t)\|_{L^p_{s-\bar{s}_p}(M)} \lesssim_{M,g} \|f_0\|_{L^p_s(M)} + \|f_1\|_{L^p_{s-1}(M)} \tag{1-2}$$

for all $s \in \mathbb{R}$, where $\bar{s}_p := (n-1)\left|\frac{1}{2} - \frac{1}{p}\right|$. Here $L^p_s(M)$ denotes the standard Sobolev (or Bessel potential) space on M with Lebesgue exponent p and s derivatives; the relevant definitions are recalled in Section 3

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[1] In view of the methods of the present article it is convenient to work in the C^∞ category, but the forthcoming definitions and questions certainly make sense at lower levels of regularity.

[2] Given a (possibly empty) list of objects L, for real numbers $A_{s,p}, B_{s,p} \geq 0$ depending on some Lebesgue exponent p and/or regularity exponent s the notation $A_{s,p} \lesssim_L B_{s,p}$ or $B_{s,p} \gtrsim_L A_{s,p}$ signifies that $A_{s,p} \leq C B_{s,p}$ for some constant $C = C_{L,n,p,s} \geq 0$ depending only on the objects in the list, n, p and s. In such cases it will also be useful to sometimes write $A_{s,p} = O_L(B_{s,p})$. In addition, $A_{s,p} \sim_L B_{s,p}$ is used to signify that $A_{s,p} \lesssim_L B_{s,p}$ and $A_{s,p} \gtrsim_L B_{s,p}$.

below. Moreover, provided t avoids a discrete set of times, the estimate (1-2) is sharp for all $1 < p < \infty$ in the sense that one cannot replace $\bar{s}_p$ with $\bar{s}_p - \sigma$ for any $\sigma > 0$.

The purpose of this article is to prove sharp *local smoothing* estimates for the solution u for a partial range of p, which demonstrate a gain in regularity for space-time estimates over the fixed-time case.

Theorem 1.1. *If u is the solution to the Cauchy problem* (1-1) *and $\bar{p}_n \le p < \infty$, where $\bar{p}_n := 2(n+1)/(n-1)$, then*

$$\left(\int_1^2 \|u(\,\cdot\,,t)\|^p_{L^p_{s-\bar{s}_p+\sigma}(M)}\,\mathrm{d}t\right)^{1/p} \lesssim_{M,g} \|f_0\|_{L^p_s(M)} + \|f_1\|_{L^p_{s-1}(M)} \tag{1-3}$$

holds for all $s \in \mathbb{R}$ and all $\sigma < \frac{1}{p}$.

For the given range of p, this result is sharp up to the endpoint in the sense that the inequality fails if $\sigma > \frac{1}{p}$.[3] It is likely, however, that the range of p is not optimal. For instance, Minicozzi and the third author [Minicozzi and Sogge 1997] (see also [Sogge, Xi, and Xu 2018]) found specific manifolds for which (1-3) can hold for all $\sigma < \frac{1}{p}$ only if $p \ge 2(3n+1)/(3n-3)$ for n odd or $p \ge 2(3n+2)/(3n-2)$ for n even; it is not unreasonable to speculate that these necessary conditions should, for general M, be sufficient.[4] The examples of [Minicozzi and Sogge 1997] rely on Kakeya compression phenomena for families of geodesics; the (euclidean) Kakeya conjecture, if valid, would preclude such behaviour over $\mathbb{R}^n$. Indeed, the *local smoothing conjecture* for the wave equation [Sogge 1991] asserts that in the euclidean case the estimate (1-3) should hold for all $\sigma < \frac{1}{p}$ in the larger range $2n/(n-1) \le p < \infty$. If true, this would be a remarkable result, not least because the conjecture formally implies many other major open problems in harmonic analysis (including the Bochner–Riesz, Fourier restriction and Kakeya conjectures); see [Tao 1999].

It is well known (see, for instance, [Duistermaat 1996, Chapter 5] or [Sogge 2017, Chapter 4]) that the solution u to the Cauchy problem (1-1) is given by

$$u(x,t) = \mathcal{F}_0 f_0(x,t) + \mathcal{F}_1 f_1(x,t), \tag{1-4}$$

where, using the language of [Hörmander 1971; Mockenhaupt, Seeger, and Sogge 1993], each $\mathcal{F}_j \in I^{j-1/4}(M\times\mathbb{R}, M; \mathcal{C})$ is a Fourier integral operator (FIO) with canonical relation $\mathcal{C}$ satisfying the cinematic curvature condition (the relevant definitions will be recalled below in Section 3; see also [Beltran, Hickman, and Sogge 2018] for a comprehensive Introduction to FIOs in the context of local smoothing). In local coordinates, such operators $\mathcal{F}_j$ adopt the explicit form (1-5) below with $\mu = j$. Theorem 1.1 follows from a more general result concerning Fourier integral operators.

[3] Such inequalities are also conjectured to hold at the endpoint (that is, the case $\sigma = \frac{1}{p}$) and endpoint estimates have been obtained for a further restricted range of p in high-dimensional cases: see [Heo, Nazarov, and Seeger 2011; Lee and Seeger 2013].

[4] The examples in [Minicozzi and Sogge 1997] concern certain oscillatory integral operators of Carleson–Sjölin type, defined with respect to the geodesic distance on M. Their results lead to counterexamples for local smoothing estimates via a variant of the well-known implication "local smoothing $\Rightarrow$ Bochner–Riesz". Implications of this kind will be discussed in detail in Section 4.

Theorem 1.2. *Let $n \geq 2$ and let Y and Z be precompact manifolds of dimensions n and $n+1$, respectively. Suppose that $\mathcal{F} \in I^{\mu-1/4}(Z, Y; \mathcal{C})$, where the canonical relation $\mathcal{C}$ satisfies the cinematic curvature condition. If $\bar{p}_n \leq p < \infty$, then*

$$\|\mathcal{F} f\|_{L^p_{\mathrm{loc}}(Z)} \lesssim \|f\|_{L^p_{\mathrm{comp}}(Y)}$$

holds whenever $\mu < -\bar{s}_p + \frac{1}{p}$.

An interesting feature of Theorem 1.2 is that both the restriction on μ and the restriction on p are sharp in certain cases.

Proposition 1.3. *For all odd dimensions $n \geq 3$ there exists some operator $\mathcal{F} \in I^{-(n-1)/2-1/4}(\mathbb{R}^{n+1}, \mathbb{R}^n; \mathcal{C})$ with $\mathcal{C}$ satisfying the cinematic curvature condition such that*

$$\|(I - \Delta_x)^{\gamma/2} \circ \mathcal{F} f\|_{L^p(\mathbb{R}^{n+1})} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \quad \text{for all } 0 < \gamma < \frac{n}{p}$$

fails for $p < \bar{p}_n$.

If $\mathcal{F} \in I^{-(n-1)/2-1/4}(\mathbb{R}^{n+1}, \mathbb{R}^n; \mathcal{C})$, then $(I - \Delta_x)^{\gamma/2} \circ \mathcal{F} \in I^{\mu-1/4}(\mathbb{R}^{n+1}, \mathbb{R}^n; \mathcal{C})$ for $\mu = -\frac{1}{2}(n-1) + \gamma$ by the composition theorem for Fourier integral operators (see, for instance, [Sogge 2017, Theorem 6.2.2]). The range $0 < \gamma < \frac{n}{p}$ corresponds to $-\frac{1}{2}(n-1) < \mu < -\bar{s}_p + \frac{1}{p}$ and thus Proposition 1.3 demonstrates that Theorem 1.2 is sharp in odd dimensions.

Proposition 1.3 is established by relating local smoothing estimates for Fourier integral operators to L^p estimates for oscillatory integral operators with nonhomogeneous phase (sometimes referred to as *Hörmander-type operators*) and then invoking well-known examples of [Bourgain 1991; 1995b] for the oscillatory integral problem. The details of the argument are discussed in Section 4.[5]

At this juncture some historical remarks are in order. Local smoothing estimates for the euclidean wave equation were introduced by the third author in [Sogge 1991] and then further investigated in [Mockenhaupt, Seeger, and Sogge 1992]. These early results, however, did not involve a sharp gain in regularity (that is, a sharp range of σ, at least up to the endpoint); the first sharp local smoothing estimates were established in $\mathbb{R}^2$ in the seminal work [Wolff 2000]. For this, Wolff introduced what have since become known as *decoupling inequalities* for the light cone. The results of [Wolff 2000] were improved and extended by a number of authors [Łaba and Wolff 2002; Garrigós and Seeger 2009; 2010; Bourgain 2013] before the remarkable breakthrough of [Bourgain and Demeter 2015] established essentially sharp decoupling estimates in all dimensions (see also [Bourgain 1995a; Tao and Vargas 2000; Heo, Nazarov, and Seeger 2011; Lee and Vargas 2012; Lee 2016] for alternative approaches to the local smoothing problem and [Cladek 2018] for recent work in a related direction). One of the many consequences of the theorem of [Bourgain and Demeter 2015] is the analogue of Theorem 1.1 for the wave equation in $\mathbb{R}^n$.

Local smoothing estimates were studied in the broader context of Fourier integral operators in parallel to the developments described above [Mockenhaupt, Seeger, and Sogge 1993; Lee and Seeger 2013] (see also [Sogge 2017]). Results in this vein typically follow from variable-coefficient extensions of methods

[5] It is remarked that the $\mathcal{F}$ constructed to provide sharp examples for Theorem 1.2 do not arise as solutions to wave equations of the kind discussed above. Thus, these examples *do not* show sharpness in Theorem 1.1. Indeed, it is likely that Theorem 1.1 should hold in the range suggested by [Minicozzi and Sogge 1997], as described above (see also the discussion in Section 4).

used to study wave equations on flat space. Similarly, Theorem 1.2 (and therefore Theorem 1.1) is a consequence of a natural variable-coefficient extension of the decoupling inequality of [Bourgain and Demeter 2015]. The variable-coefficient decoupling theorem is the main result of this paper and concerns certain oscillatory integral operators with homogeneous phase; the setup is described in the following subsection.

1B. ***Variable coefficient decoupling.*** Let $a = a_1 \otimes a_2 \in C_c^\infty(\mathbb{R}^{n+1} \times \mathbb{R}^n)$, where $a_1 \in C_c^\infty(\mathbb{R}^n)$ is supported in $B(0,1)$ and a_2 is supported in the domain

$$\Gamma_1 := \bigl\{\xi \in \widehat{\mathbb{R}}^n : \tfrac{1}{2} \le \xi_n \le 2 \text{ and } |\xi_j| \le |\xi_n| \text{ for } 1 \le j \le n-1\bigr\}.$$

Suppose that $\phi\colon \mathbb{R}^n \times \mathbb{R} \times \widehat{\mathbb{R}}^n \to \mathbb{R}$ is smooth away from $\mathbb{R}^n \times \mathbb{R} \times \{0\}$ and that for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ the function $\xi \mapsto \phi(x,t;\xi)$ is homogeneous of degree 1. Writing $\operatorname{supp} a \setminus 0$ for the set $(\operatorname{supp} a) \setminus (\mathbb{R}^n \times \mathbb{R} \times \{0\})$, assume, in addition, that ϕ satisfies the following geometric conditions:

(H1) $\operatorname{rank} \partial^2_{\xi z}\phi(x,t;\xi) = n$ for all $(x,t;\xi) \in \operatorname{supp} a \setminus 0$. Here and below z is used to denote a vector in $\mathbb{R}^{n+1}$ composed of the space-time variables (x,t).

(H2) Defining the generalised Gauss map by $G(z;\xi) := G_0(z;\xi)/|G_0(z;\xi)|$ for all $(z;\xi) \in \operatorname{supp} a \setminus 0$, where

$$G_0(z;\xi) := \bigwedge_{j=1}^{n} \partial_{\xi_j}\partial_z\phi(z;\xi),$$

one has

$$\operatorname{rank} \partial^2_{\eta\eta}\langle \partial_z\phi(z;\eta), G(z;\xi)\rangle|_{\eta=\xi} = n-1$$

for all $(z;\xi) \in \operatorname{supp} a \setminus 0$.

Here the wedge product of n vectors in $\mathbb{R}^{n+1}$ is associated with a vector in $\mathbb{R}^{n+1}$ in the usual manner. It is remarked that (H1)1 and (H2)2 are the natural homogeneous analogues of the [Carleson and Sjölin 1972] or [Hörmander 1973] conditions for nonhomogeneous phase functions.

The conditions (H1)1 and (H2)2 naturally arise in the study of Fourier integral operators of the type described in the previous subsection. Indeed, by standard theory (see, for instance, [Sogge 2017, Proposition 6.1.4]), any operator belonging to the class $I^{\mu-1/4}(Z,Y;\mathcal{C})$ with $\mathcal{C}$ satisfying the cinematic curvature condition can be written in local coordinates as a finite sum of operators of the form

$$\mathcal{F}f(x,t) := \int_{\widehat{\mathbb{R}}^n} e^{i\phi(x,t;\xi)} b(x,t;\xi)(1+|\xi|^2)^{\mu/2} \hat{f}(\xi)\, d\xi, \tag{1-5}$$

where b is a symbol of order 0 (with compact support in the (x,t)-variables) and ϕ satisfies the properties (H1)1 and (H2)2 (at least on the support of b).

Rather than directly studying the operators $\mathcal{F}$ as in (1-5), a decoupling inequality shall instead be formulated in terms of a certain closely related class of oscillatory integral operators.

Given $\lambda \ge 1$, define the rescaled phase and amplitude

$$\phi^\lambda(x,t;\omega) := \lambda\phi\Bigl(\frac{x}{\lambda},\frac{t}{\lambda};\omega\Bigr) \quad \text{and} \quad a^\lambda(x,t;\xi) := a_1\Bigl(\frac{x}{\lambda},\frac{t}{\lambda}\Bigr)a_2(\xi)$$

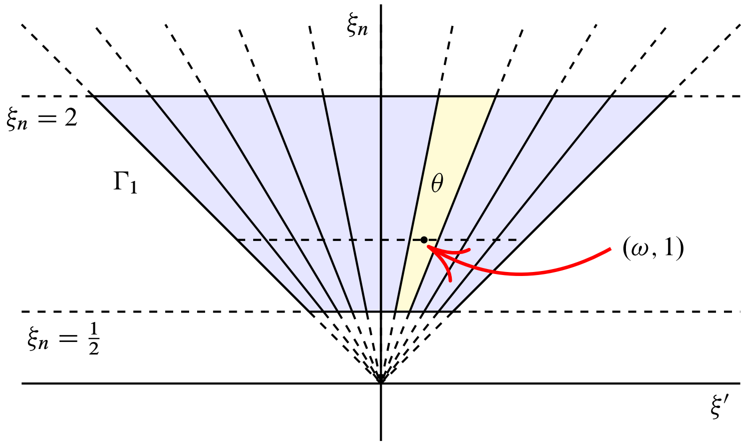

Figure 1. The decomposition of the domain Γ_1 into $R^{-1/2}$-plates. The centre $(\omega, 1)$ of one such plate θ is indicated.

and, with this data, let

$$T^\lambda f(x,t) := \int_{\widehat{\mathbb{R}}^n} e^{i\phi^\lambda(x,t;\xi)} a^\lambda(x,t;\xi) f(\xi)\,\mathrm{d}\xi.$$

The aforementioned variable-coefficient decoupling inequality compares the L^p-norm of $T^\lambda f$ with the L^p-norms of localised pieces $T^\lambda f_\theta$ which form a decomposition of the original operator. To describe this decomposition fix a second spatial parameter $1 \le R \le \lambda$ and note that the support of a_2 intersects the affine hyperplane $\xi_n = 1$ on the disc $B^{n-1}(0,1) \times \{1\}$. Fix a maximally $R^{-1/2}$-separated subset of $B^{n-1}(0,1) \times \{1\}$ and for each ω belonging to this subset define the $R^{-1/2}$-*plate*

$$\theta := \big\{(\xi', \xi_n) \in \widehat{\mathbb{R}}^n : \tfrac{1}{2} \le \xi_n \le 2 \text{ and } |\xi'/\xi_n - \omega| \le R^{-1/2}\big\}.$$

In this case $(\omega, 1) \in B^{n-1}(0,1) \times \{1\}$ is referred to as the *centre* of the $R^{-1/2}$-plate θ. Thus, the collection of all $R^{-1/2}$-plates forms a partition of the support of a_2 into finitely overlapping subsets (see Figure 1). For each θ, let $\tilde\theta$ be a subset of θ such that the family of all $\tilde\theta$ forms a partition of the support of a_2. Given any function $f \in L^1_{\mathrm{loc}}(\widehat{\mathbb{R}}^n)$ and an $R^{-1/2}$-plate θ, define $f_\theta := \chi_{\tilde\theta} f$, and for $1 \le p < \infty$ and any measurable set $E \subseteq \mathbb{R}^{n+1}$ introduce the *decoupled norm*

$$\|T^\lambda f\|_{L^{p,R}_{\mathrm{dec}}(E)} := \Big(\sum_{\theta : R^{-1/2}\text{-plate}} \|T^\lambda f_\theta\|^p_{L^p(E)}\Big)^{1/p}.$$

This definition is extended to the case $p = \infty$ and to weighted norms $\|T^\lambda f\|_{L^{p,R}_{\mathrm{dec}}(w)}$ in the obvious manner.

Finally, let $\bar p_n$ and $\bar s_p$ be as in the statement of Theorem 1.1 and given $2 \le p \le \infty$ define the exponent

$$\alpha(p) := \begin{cases} \frac{1}{2}\bar s_p & \text{if } 2 \le p \le \bar p_n, \\ \bar s_p - \frac{1}{p} & \text{if } \bar p_n \le p \le \infty. \end{cases} \tag{1-6}$$

With these definitions, the decoupling theorem reads as follows.

Theorem 1.4. *Let T^λ be an operator of the form described above and $2 \le p \le \infty$. For all $\varepsilon > 0$ and $M \in \mathbb{N}$ one has*[6]

$$\|T^\lambda f\|_{L^p(\mathbb{R}^{n+1})} \lesssim_{\varepsilon,M,\phi,a} \lambda^{\alpha(p)+\varepsilon} \|T^\lambda f\|_{L^{p,\lambda}_{\mathrm{dec}}(\mathbb{R}^{n+1})} + \lambda^{-M} \|f\|_{L^2(\widehat{\mathbb{R}}^n)}. \tag{1-7}$$

Theorem 1.4 is a natural variable-coefficient extension of (the ℓ^p variant of) Theorem 1.2 in [Bourgain and Demeter 2015], which treats the prototypical case $\phi(x,t;\xi) = \langle x, \xi\rangle + t|\xi|$. More generally, the *translation-invariant* case, where ϕ is linear in the variables x, t, can be deduced from the results of [Bourgain and Demeter 2015; 2017a] via an argument originating in [Pramanik and Seeger 2007; Garrigós and Seeger 2010]. Interestingly, it transpires that the result for general operators T^λ follows itself from the translation-invariant case. This stands in contrast with the L^p-theory of such operators (see, for instance, [Bourgain and Guth 2011; Guth, Hickman, and Iliopoulou 2019]).

Finally, it is remarked that the argument used to prove Theorem 1.4 is flexible in nature, and could equally be applied to prove natural variable-coefficient extensions of other known decoupling results, such as the ℓ^2 decoupling theorem for the paraboloid [Bourgain and Demeter 2015] or the decoupling theorem of [Bourgain, Demeter, and Guth 2016] for the moment curve (in the latter case the relevant variable-coefficient operators are those appearing in [Bak and Lee 2004; Bak, Oberlin, and Seeger 2009]).

2. A proof of the variable-coefficient decoupling inequality

2A. ***An overview of the proof.*** As indicated in the Introduction, Theorem 1.4 will be derived as a consequence of the (known) translation-invariant case; the latter result is recalled presently. Let a_2 be as in the Introduction and suppose $h\colon \widehat{\mathbb{R}}^n \to \mathbb{R}$ is smooth away from 0, homogeneous of degree 1 and satisfies rank $\partial^2_{\xi\xi} h(\xi) = n-1$ for all $\xi \in \operatorname{supp} a_2 \setminus \{0\}$. With this data, define the extension operator

$$Ef(x,t) := \int_{\widehat{\mathbb{R}}^n} e^{i(\langle x,\xi\rangle + t h(\xi))} a_2(\xi) f(\xi)\, d\xi.$$

For the exponent α defined in (1-6), the translation-invariant case of the theorem reads thus.

Theorem 2.1 [Bourgain and Demeter 2015; 2017a]. *For all $2 \le p \le \infty$ and all $\varepsilon > 0$ the estimate*

$$\|Ef\|_{L^p(w_{B_\lambda})} \lesssim_{\varepsilon,N,h,a} \lambda^{\alpha(p)+\varepsilon} \|Ef\|_{L^{p,\lambda}_{\mathrm{dec}}(w_{B_\lambda})} \tag{2-1}$$

holds for $\lambda \ge 1$.

Here B_R denotes a ball of radius R for any $R > 0$ and w_{B_R} is a rapidly decaying weight function, concentrated on B_R. In particular, if $(\bar{x}, \bar{t}) \in \mathbb{R}^n \times \mathbb{R}$ denotes the centre of B_R, then

$$w_{B_R}(x,t) := (1 + R^{-1}|x - \bar{x}| + R^{-1}|t - \bar{t}|)^{-N}, \tag{2-2}$$

[6]Strictly speaking, the proof will establish this inequality with the operator appearing on the right-hand side of (1-7) defined with respect to an amplitude with slightly larger spatial support than that appearing in the operator on the left (but both operators are defined with respect to the same phase function). This has no bearing on the applications and such slight discrepancies will be suppressed in the notation.

where N can be taken to be any sufficiently large integer (depending on n, h and p). It is remarked that the dependence on h of the implicit constant in the inequality (2-1) involves only the size of the absolute values of the nonzero eigenvalues of $\partial^2_{\xi\xi}h$ and their reciprocals, as well as upper bounds for a finite number of higher-order derivatives $\partial^\beta_\xi h$, $|\beta| \geq 3$.

As mentioned in the Introduction, Theorem 2.1 does not appear in [Bourgain and Demeter 2015; 2017a] in the stated generality, but this result may be readily deduced from the prototypical cases considered in [Bourgain and Demeter 2015; 2017a] via the arguments of [Pramanik and Seeger 2007; Garrigós and Seeger 2010] (see also [Bourgain and Demeter 2015, §7–8] and [Guo and Oh 2018]), or by using a variant of the approach developed in the present article.

The passage from Theorem 2.1 to Theorem 1.4 is, in essence, realised in the following manner. The desired decoupling inequalities have a "self-similar" structure, which is manifested in their almost-invariance under certain Lorentz rescaling (see Lemma 2.3). An implication of this self-similarity is that in order to prove the decoupling estimate, it suffices to obtain some nontrivial, but possibly very small, gain at a single spatial scale; this gain can then be propagated through all the scales via Lorentz rescaling.[7] At spatial scales K below the critical value $\lambda^{1/2}$ one can effectively approximate T^λ by an extension operator E of the form described above; this is the content of Lemma 2.6 below. Combining this approximation with Theorem 2.1 provides some gain at such scales K, and combining these observations concludes the argument.

2B. ***Basic properties of the phase.*** Before carrying out the programme described above, it is useful to note some basic properties of homogeneous phases ϕ satisfying the conditions (H1)1 and (H2)2 and to make some simple reductions.

After a localisation and a translation argument, one may assume that a is supported inside $Z \times \Xi$, where $Z := X \times T$ for $X \subseteq B(0,1) \subseteq \mathbb{R}^n$ and $T \subseteq (-1,1) \subseteq \mathbb{R}$ small open neighbourhoods of the origin and $\Xi \subseteq \Gamma_1$ is a small open sector around $e_n := (0,\dots,0,1) \in \widehat{\mathbb{R}}^n$. By choosing the size of the neighbourhoods appropriately, one may assume the phase satisfies a number of useful additional properties, described presently.

By localising, one may ensure that strengthened versions of the conditions (H1)1 and (H2)2 hold. In particular, without loss of generality one may work with phases satisfying:

(H1′) $\det \partial^2_{\xi x}\phi(z;\xi) \neq 0$ for all $(z;\xi) \in Z \times \Xi$.

(H2′) $\det \partial_{\xi'\xi'}\partial_t\phi(z;\xi) \neq 0$ for all $(z;\xi) \in Z \times \Xi$.

Indeed, by precomposing the phase with a rotation in the $z = (x,t)$-variables, one may assume that $G(0;e_n) = e_{n+1}$ and therefore $\partial_\xi\partial_t\phi(0;e_n) = 0$. Hence, by (H1)1, it follows that $\det \partial^2_{\xi x}\phi(0;e_n) \neq 0$. On the other hand, by the homogeneity of ϕ, every $(n-1)\times(n-1)$ minor of the matrix featured in the (H2)2 condition is a multiple of $\det \partial_{\eta'\eta'}\langle \partial_z\phi(z;\eta), G(z;\xi)\rangle|_{\eta=\xi}$. Thus, in order for the rank condition (H2)2 to hold, this determinant must be nonzero. In particular, as $G(0;e_n) = e_{n+1}$, it follows that $\det \partial_{\xi'\xi'}\partial_t\phi(0;e_n) \neq 0$. Choosing the neighbourhoods Z and Ξ sufficiently small now ensures both (H1′)1′ and (H2′)2′ hold.

[7] Further details and discussion of this perspective on decoupling theory can be found in the recorded lecture series given by Guth as part of the MSRI harmonic analysis programme during January 2017 [Guth 2017a; 2017b; 2017c].

By Euler's homogeneity relations,

$$\partial_x \phi(x,t;\xi) = \sum_{j=1}^{n} \xi_j \cdot \partial_{\xi_j} \partial_x \phi(x,t;\xi).$$

Thus it follows that for each $t \in (-1,1)$ and $\xi \in \hat{\mathbb{R}}^n$ the Jacobian determinant of the map $x \mapsto ((\partial_{\xi'}\phi)(x,t;\xi), \phi(x,t;\xi))$ is given by $\xi_n \cdot \det \partial^2_{\xi x}\phi(x,t;\xi)$, which is nonzero by (H1′)1′. Thus, there exists a smooth local inverse mapping $\Upsilon(\,\cdot\,,t;\xi)$ which satisfies

$$(\partial_{\xi'}\phi)(\Upsilon(y,t;\xi),t;\xi) = y' \quad \text{and} \quad \phi(\Upsilon(y,t;\xi),t;\xi) = y_n. \tag{2-3}$$

Similarly, there exists a smooth mapping $\Psi(x,t;\,\cdot\,)$ such that

$$(\partial_x \phi)(x,t;\Psi(x,t;\eta)) = \eta. \tag{2-4}$$

Given $\lambda \ge 1$, let Υ^λ and Ψ^λ denote the natural rescaled versions of these maps, so that $\Upsilon^\lambda(z;\xi) = \lambda\Upsilon(y/\lambda;\xi)$ and $\Psi^\lambda(z;\eta) := \Psi(z/\lambda;\eta)$. One may assume that Z and Ξ are such that the above mappings are everywhere defined.

2C. ***Quantitative conditions.*** Fix $\varepsilon > 0$, $M \in \mathbb{N}$, and $2 \le p < \infty$ (the $p = \infty$ case of Theorem 1.4 is trivial but nevertheless must be treated separately: see (2-7)). To facilitate certain induction arguments, it is useful to work with quantitative versions of the conditions (H1′)1′ and (H2′)2′ on the phase function. In particular, let c_{par} be a small fixed constant and assume that for some $0 \le \sigma_+ \le n-1$ and $\boldsymbol{A} = (A_1, A_2, A_3) \in [1,\infty)^3$ the phase satisfies, in addition to (H1′)1′ and (H2′)2′, the following:

(H1$_{\boldsymbol{A}}$) $|\partial^2_{\xi x}\phi(z;\xi) - I_n| \le c_{\mathrm{par}} A_1$ for all $(z;\xi) \in Z \times \Xi$.

(H2$_{\boldsymbol{A}}$) $|\partial^2_{\xi'\xi'}\partial_t\phi(z;\xi) - (1/\xi_n) I_{n-1,\sigma_+}| \le c_{\mathrm{par}} A_2$ for all $(z;\xi) \in Z \times \Xi$, where

$$I_{n-1,\sigma_+} := \operatorname{diag}(\underbrace{1,\dots,1}_{\sigma_+}, \underbrace{-1,\dots,-1}_{n-1-\sigma_+})$$

is an $(n-1)\times(n-1)$ diagonal matrix.

Some additional control on the size of various derivatives, which is of a rather technical nature, is assumed:

(D1$_{\boldsymbol{A}}$) $\|\partial^\beta_\xi \partial_{x_k}\phi\|_{L^\infty(Z\times\Xi)} \le c_{\mathrm{par}} A_1$ for all $1 \le k \le n$ and $\beta \in \mathbb{N}_0^n$ with $2 \le |\beta| \le 3$ satisfying $|\beta'| \ge 2$; $\|\partial^{\beta'}_{\xi'}\partial_t\phi\|_{L^\infty(Z\times\Xi)} \le (c_{\mathrm{par}}/(2n)) A_1$ for all $\beta' \in \mathbb{N}_0^{n-1}$ with $|\beta'| = 3$.

(D2$_{\boldsymbol{A}}$) For some large integer $N = N_{\varepsilon,M,p} \in \mathbb{N}$, depending only on the dimension n and the fixed choice of ε, M and p, one has

$$\|\partial^\beta_\xi \partial^\alpha_z \phi\|_{L^\infty(Z\times\Xi)} \le \frac{c_{\mathrm{par}}}{2n} A_3$$

for all $(\alpha,\beta) \in \mathbb{N}_0^{n+1} \times \mathbb{N}_0^n$ with $2 \le |\alpha| \le 4N$ and $1 \le |\beta| \le 4N+2$ satisfying $1 \le |\beta| \le 4N$ or $|\beta'| \ge 2$.

Finally, it is useful to assume a *margin* condition on the spatial support of the amplitude a:

(M$_{\boldsymbol{A}}$) $\operatorname{dist}(\operatorname{supp} a_1, \mathbb{R}^{n+1} \setminus Z) \ge \frac{1}{4} A_3$.

Datum (ϕ, a) satisfying $(\mathrm{H1}_{\boldsymbol{A}})1_{\boldsymbol{A}}$, $(\mathrm{H2}_{\boldsymbol{A}})2_{\boldsymbol{A}}$, $(\mathrm{D1}_{\boldsymbol{A}})1_{\boldsymbol{A}}$, $(\mathrm{D2}_{\boldsymbol{A}})2_{\boldsymbol{A}}$ and $(\mathrm{M}_{\boldsymbol{A}})_{\boldsymbol{A}}$ (in addition to $(\mathrm{H1}')1'$ and $(\mathrm{H2}')2'$) is said to be of *type* $\boldsymbol{A}$. One may easily verify that any phase function satisfying $(\mathrm{H1}')1'$ and $(\mathrm{H2}')2'$ is of type $\boldsymbol{A}$ for some $\boldsymbol{A} = (A_1, A_2, A_3) \in [1, \infty)^3$. The conditions $(\mathrm{H1}_{\boldsymbol{A}})1_{\boldsymbol{A}}$ and $(\mathrm{H2}_{\boldsymbol{A}})2_{\boldsymbol{A}}$ are quantitative substitutes for $(\mathrm{H1}')1'$ and $(\mathrm{H2}')2'$ if, say, $A_1, A_2 \le 1$; for A_1 and A_2 large, however, the conditions $(\mathrm{H1}_{\boldsymbol{A}})1_{\boldsymbol{A}}$ and $(\mathrm{H2}_{\boldsymbol{A}})2_{\boldsymbol{A}}$ are vacuous and do not imply $(\mathrm{H1}')1'$ or $(\mathrm{H2}')2'$. By various rescaling arguments, it is possible to reduce to the case where $\boldsymbol{A} = \mathbf{1} := (1, 1, 1)$, as shown in Section 2E.

2D. *Setting up the induction for* (1-7) *and reduction to* $\lambda^{1-\varepsilon/n}$*-balls.* Continuing with the fixed ε, M and p from the previous subsection, let $\boldsymbol{A} = (A_1, A_2, A_3) \in [1, \infty)^3$ and $N \in \mathbb{N}$ be as in the definition of the condition $(\mathrm{D2}_{\boldsymbol{A}})2_{\boldsymbol{A}}$. For $1 \le R \le \lambda$ let $\boldsymbol{A}^{\varepsilon}(\lambda; R)$ denote the infimum over all $C \ge 0$ for which the inequality

$$\|T^{\lambda} f\|_{L^p(B_R)} \le C R^{\alpha(p)+\varepsilon} \|T^{\lambda} f\|_{L^{p,R}_{\mathrm{dec}}(w_{B_R})} + R^{2n} \left(\frac{\lambda}{R}\right)^{-\varepsilon N/(8n)} \|f\|_{L^2(\hat{\mathbb{R}}^n)} \tag{2-5}$$

holds for all type-$\boldsymbol{A}$ data (ϕ, a)[8] and balls B_R of radius R contained in $B(0, \lambda)$. Here the weight function is understood to be defined with respect to the fixed choice of N above, as in (2-2). It is remarked that the quantity $\boldsymbol{A}^{\varepsilon}(\lambda; R)$ is always finite. To see this, note that for any $1 \le \rho \le R$ and $\rho^{-1/2}$-plate θ one may write

$$T^{\lambda} f_{\theta} = \sum_{\substack{\sigma \cap \tilde{\theta} \neq \varnothing \\ \sigma : R^{-1/2}\text{-plate}}} T^{\lambda} f_{\sigma};$$

recall that $\tilde{\theta}$ is the subset of θ upon which f_{θ} is supported. By the triangle and Hölder's inequalities, for any weight w one has

$$\|T^{\lambda} f\|_{L^{p,\rho}_{\mathrm{dec}}(w)} \le \left(\frac{R}{\rho}\right)^{(n-1)/(2p')} \|T^{\lambda} f\|_{L^{p,R}_{\mathrm{dec}}(w)}. \tag{2-6}$$

Taking $\rho = 1$, one thereby deduces the trivial bound

$$\mathfrak{D}^{\varepsilon}_{\boldsymbol{A}}(\lambda; R) \lesssim R^{(n-1)/(2p') - \alpha(p)}, \tag{2-7}$$

which, in particular, shows that $\mathfrak{D}^{\varepsilon}_{\boldsymbol{A}}(\lambda; R)$ is finite. This trivial observation also proves Theorem 1.4 in the $p = \infty$ case.

To prove Theorem 1.4 for the fixed parameters 2ε, M, and $2 \le p < \infty$ it is claimed that it suffices to show that

$$\boldsymbol{A}^{\varepsilon}(\lambda; \lambda^{1-\varepsilon/n}) \lesssim_{\boldsymbol{A},\varepsilon} 1. \tag{2-8}$$

The "(ε/n)-gain" realised by this reduction will be useful for various technical reasons. To see the above claim, observe that the support conditions on the amplitude a imply that the support of $T^{\lambda} f$ is always contained in $B(0, \lambda)$. Take a cover of $B(0, \lambda)$ by finitely overlapping $\lambda^{1-\varepsilon/n}$-balls and apply (2-8) to the relevant L^p-norm defined over each of these balls. Summing over all the contributions from the

[8] As in the statement of Theorem 1.4, a discrepancy between the amplitude functions is allowed here: the right-hand operator is understood to be defined with respect to *some* amplitude with possibly slightly larger spatial support than the original amplitude a.

collection via Minkowski's inequality, one deduces that

$$\|T^\lambda f\|_{L^p(B(0,\lambda))} \lesssim_{\boldsymbol{A},\varepsilon} \lambda^{\alpha(p)+\varepsilon} \|T^\lambda f\|_{L^{p,\lambda^{1-\varepsilon/n}}_{\mathrm{dec}}(w_{B(0,\lambda)})} + \lambda^{2n-\varepsilon N/(8n)} \|f\|_{L^2(\widehat{\mathbb{R}}^n)}.$$

Here the weight $w_{B(0,\lambda)}$ is as defined in (2-2) (with $R=\lambda$ and $\bar{x}=0$, $\bar{t}=0$). Provided N is sufficiently large, the desired estimate (1-7) now follows from (2-6).

After reducing to the case $\boldsymbol{A}=\mathbf{1}$, it will be shown in Section 2G, using induction on R, that $\mathfrak{D}^\varepsilon_{\mathbf{1}}(\lambda; R) \lesssim_\varepsilon 1$ for all $1 \le R \le \lambda^{1-\varepsilon/n}$, thus establishing (2-8). The trivial inequality (2-7) will serve as the base case for this induction.

2E. *Lorentz rescaling.* The first ingredient required in the proof of Theorem 1.4 is a standard Lorentz rescaling lemma. Before stating this result, it is useful to observe the following trivial consequence of rescaling.

Lemma 2.2. *Let $\boldsymbol{A}=(A_1,A_2,A_3)$ and $\widetilde{\boldsymbol{A}}=(A_1,A_2,1)$. Then*

$$\mathfrak{D}^\varepsilon_{\boldsymbol{A}}(\lambda;R) \lesssim_{A_3} \mathfrak{D}^\varepsilon_{\widetilde{\boldsymbol{A}}}\Big(\frac{\lambda}{A_3};\frac{R}{A_3}\Big).$$

Proof. Let (ϕ,a) be a type-$\boldsymbol{A}$ datum. Observe that $T^\lambda f = \widetilde{T}^{\lambda/A_3} f$, where $\widetilde{T}$ is defined with respect to the phase $\tilde{\phi}(z;\xi) := A_3\phi(z/A_3;\xi)$ and amplitude $\tilde{a}(z;\xi) := a(z/A_3;\xi)$. Clearly the datum $(\tilde{\phi},\tilde{a})$ satisfies (H1$_{\widetilde{\boldsymbol{A}}}$), (H2$_{\widetilde{\boldsymbol{A}}}$), (D1$_{\widetilde{\boldsymbol{A}}}$) and (D2$_{\widetilde{\boldsymbol{A}}}$). The margin of the new amplitude $\tilde{a}$ (with respect to the rescaled open set A_3Z) has been increased to size $\frac{1}{4}$ and so (M$_{\widetilde{\boldsymbol{A}}}$) holds. There is a slight issue here in that the support of the rescaled amplitude may now lie outside the unit ball, but one may decompose the amplitude via a partition of unity and translate each piece to write the operator as a sum of $O(A_3^{n+1})$ operators each associated to type-$\widetilde{\boldsymbol{A}}$ data. Finally, covering $B(0,R)$ with a union of (R/A_3)-balls and applying the definition of $\mathfrak{D}^\varepsilon_{\widetilde{\boldsymbol{A}}}(\lambda/A_3;R/A_3)$ to each of the contributions arising from these balls, the result then follows from the trivial decoupling inequality (2-6). □

Lemma 2.3 (Lorentz rescaling). *Let $1 \le \rho \le R \le \lambda$ and suppose that T^λ is defined with respect to a type-$\boldsymbol{A}=(A_1,A_2,A_3)$ datum. If g is supported on a ρ^{-1}-plate and ρ is sufficiently large depending on ϕ, then there exists a constant $\overline{C}=\overline{C}_\phi \ge 1$ such that*

$$\|T^\lambda g\|_{L^p(w_{B_R})} \lesssim_{\varepsilon,\phi,N} \mathfrak{D}^\varepsilon_{\mathbf{1}}\Big(\frac{\lambda}{\overline{C}\rho^2};\frac{R}{\overline{C}\rho^2}\Big)\Big(\frac{R}{\rho^2}\Big)^{\alpha(p)+\varepsilon} \|T^\lambda g\|_{L^{p,R}_{\mathrm{dec}}(w_{B_R})} + R^{2n}\Big(\frac{\lambda}{R}\Big)^{-\varepsilon N/(8n)} \|g\|_{L^2(\widehat{\mathbb{R}}^n)}. \tag{2-9}$$

Remark 2.4. The proof of the lemma will show, more precisely, that the lower bound for ρ and the implicit constant in (2-9) may be chosen so as to depend only on ε, $\boldsymbol{A}$ and the following quantities:

- $\inf_{(x,t;\xi)\in \operatorname{supp} a} |\det \partial^2_{x\xi}\phi(x,t;\xi)|$.
- The infimum and supremum of the magnitudes of the eigenvalues of

$$\partial^2_{\xi'\xi'}\partial_t\phi(x,t;\xi) \tag{2-10}$$

  over all $(x,t;\xi) \in \operatorname{supp} a$.

Note that the quantities appearing in the above bullet points are nonzero by the conditions (H1′)1′ and (H2′)2′.

Lemma 2.3 will be applied in two different ways:

(i) An initial application of the lemma reduces the proof of Theorem 1.4 to operators defined with respect to type-**1** data. This is achieved by introducing a partition of unity of the frequency domain Γ_1 into ρ^{-1}-plates for some sufficiently large ρ, depending on ϕ. Each of these frequency-localised pieces can be rescaled via Lemma 2.3 and then summed together to yield the desired reduction. Observe that, by the preceding remark, Lemma 2.3 is uniform for type-**1** data.

(ii) The second application of Lemma 2.3 will be to facilitate an induction argument which constitutes the proof of Theorem 1.4 proper. The uniformity afforded by the reduction to type-**1** phases is useful in order to ensure that this induction closes.

Proof of Lemma 2.3. The argument used in what follows is a generalisation of the Lorentz rescaling used to study decoupling for the light cone [Wolff 2000]; see Figure 2. Let $\omega \in B^{n-1}(0,1)$ be such that $(\omega, 1)$ is the centre of the ρ^{-1}-plate upon which g is supported, so that

$$\operatorname{supp} g \subseteq \big\{(\xi', \xi_n) \in \widehat{\mathbb{R}}^n : \tfrac{1}{2} \le \xi_n \le 2 \text{ and } |\xi'/\xi_n - \omega| \le \rho^{-1}\big\}.$$

Performing the change of variables $(\xi', \xi_n) = (\eta_n\omega + \rho^{-1}\eta', \eta_n)$, it follows that

$$T^\lambda g(z) = \int_{\widehat{\mathbb{R}}^n} e^{i\phi^\lambda(z;\eta_n\omega+\rho^{-1}\eta',\eta_n)} a^\lambda(z;\eta_n\omega+\rho^{-1}\eta',\eta_n)\tilde{g}(\eta)\,\mathrm{d}\eta,$$

where $\tilde{g}(\eta) := \rho^{-(n-1)} g(\eta_n\omega + \rho^{-1}\eta', \eta_n)$ and $\operatorname{supp}\tilde{g} \subseteq \Xi$.

By applying a Taylor series expansion and using the homogeneity, the phase function in the above oscillatory integral may be expressed as

$$\phi(z;\omega,1)\eta_n + \rho^{-1}\langle \partial_{\xi'}\phi(z;\omega,1), \eta'\rangle + \rho^{-2}\int_0^1 (1-r)\langle \partial^2_{\xi'\xi'}\phi(z;\eta_n\omega + r\rho^{-1}\eta', \eta_n)\eta', \eta'\rangle\,\mathrm{d}r.$$

Let $\Upsilon_\omega(y,t) := (\Upsilon(y,t;\omega,1), t)$ and $\Upsilon^\lambda_\omega(y,t) := \lambda\Upsilon_\omega(y/\lambda, t/\lambda)$ and introduce the anisotropic dilations $D_\rho(y', y_n, t) := (\rho y', y_n, \rho^2 t)$ and $D'_{\rho^{-1}}(y', y_n) := (\rho^{-1}y', \rho^{-2}y_n)$ on $\mathbb{R}^{n+1}$ and $\mathbb{R}^n$, respectively. Recalling (2-3), it follows that

$$T^\lambda g \circ \Upsilon^\lambda_\omega \circ D_\rho = \widetilde{T}^{\lambda/\rho^2}\tilde{g},$$

where

$$\widetilde{T}^{\lambda/\rho^2}\tilde{g}(y,t) := \int_{\widehat{\mathbb{R}}^n} e^{i\tilde{\phi}^{\lambda/\rho^2}(y,t;\eta)}\tilde{a}^\lambda(z;\eta)\tilde{g}(\eta)\,\mathrm{d}\eta$$

for the phase $\tilde{\phi}(y,t;\eta)$ given by

$$\langle y, \eta\rangle + \int_0^1 (1-r)\langle \partial^2_{\xi'\xi'}\phi(\Upsilon_\omega(D'_{\rho^{-1}}y, t); \eta_n\omega + r\rho^{-1}\eta', \eta_n)\eta', \eta'\rangle\,\mathrm{d}r \tag{2-11}$$

and the amplitude $\tilde{a}(y,t;\eta) := a(\Upsilon_\omega(D'_{\rho^{-1}}y;t);\eta_n\omega+\rho^{-1}\eta',\eta_n)$. In particular, by a change of spatial variables, it follows that

$$\|T^\lambda g\|_{L^p(B_R)} \lesssim_\phi \rho^{(n+1)/p}\|\widetilde{T}^{\lambda/\rho^2}\tilde{g}\|_{L^p((\Upsilon^\lambda_\omega\circ D_\rho)^{-1}(B_R))}.$$

Fix a collection $\mathcal{B}_{R/\rho^2}$ of finitely overlapping (R/ρ^2)-balls which cover $(\Upsilon^\lambda_\omega\circ D_\rho)^{-1}(B_R)$ and observe that

$$\|T^\lambda g\|_{L^p(B_R)} \lesssim_\phi \rho^{(n+1)/p}\Big(\sum_{B_{R/\rho^2}\in\mathcal{B}_{R/\rho^2}}\|\widetilde{T}^{\lambda/\rho^2}\tilde{g}\|^p_{L^p(B_{R/\rho^2})}\Big)^{1/p}.$$

It will be shown below that

$$\|\widetilde{T}^{\lambda/\rho^2}\tilde{g}\|_{L^p(B_{R/\rho^2})} \lesssim_{\varepsilon,\phi} \mathfrak{D}^\varepsilon_{\mathbf{1}}\Big(\frac{\lambda}{\overline{C}\rho^2};\frac{R}{\overline{C}\rho^2}\Big)\Big(\frac{R}{\rho^2}\Big)^{\alpha(p)+\varepsilon}\|\widetilde{T}^{\lambda/\rho^2}\tilde{g}\|_{L^{p,R/\rho^2}_{\mathrm{dec}}(w_{B_{R/\rho^2}})} + \Big(\frac{R}{\rho^2}\Big)^{2n}\Big(\frac{\lambda}{R}\Big)^{-\varepsilon N/(8n)}\|g\|_{L^2(\widehat{\mathbb{R}}^n)} \tag{2-12}$$

holds for each $B_{R/\rho^2}\in\mathcal{B}_{R/\rho^2}$ and a suitable constant $\overline{C}\geq 1$, depending on ϕ. Momentarily assuming this (which would follow immediately from the definitions if $(\tilde{\phi},\tilde{a})$ were a type-**1** datum), the proof of Lemma 2.3 may be completed as follows.

Since Υ_ω is a diffeomorphism, it follows that

$$\bigcup_{B_{R/\rho^2}\in\mathcal{B}_{R/\rho^2}} B_{R/\rho^2} \subseteq (\Upsilon^\lambda_\omega\circ D_\rho)^{-1}(B_{C_\phi R}),$$

where $B_{C_\phi R}$ is the ball concentric to B_R but with radius $C_\phi R$ for some suitable choice of constant $C_\phi\geq 1$ depending on ϕ. Thus, one may sum the p-th power of both sides of (2-12) over all the balls in $\mathcal{B}_{R/\rho^2}$ and reverse the changes of variables (both in spatial and frequency) to conclude that[9]

$$\|T^\lambda g\|_{L^p(B_R)} \lesssim_{\varepsilon,\phi,N} \mathfrak{D}^\varepsilon_{\mathbf{1}}\Big(\frac{\lambda}{\overline{C}\rho^2};\frac{R}{\overline{C}\rho^2}\Big)\Big(\frac{R}{\rho^2}\Big)^{\alpha(p)+\varepsilon}\Big(\sum_{\tilde{\theta}:(R/\rho^2)^{-1/2}\text{-plate}}\|T^\lambda g_\theta\|^p_{L^p(w_{B_R})}\Big)^{1/p} + R^{2n}\Big(\frac{\lambda}{R}\Big)^{-\varepsilon N/(8n)}\|g\|_{L^2(\widehat{\mathbb{R}}^n)},$$

where θ is the image of $\tilde{\theta}$ under the map $(\eta',\eta_n)\mapsto(\rho(\eta'-\eta_n\omega),\eta_n)$. In particular, if $\omega_{\tilde{\theta}}$ denotes the centre of the $(R/\rho^2)^{-1/2}$-plate $\tilde{\theta}$, then

$$\theta = \big\{(\xi',\xi_n)\in\widehat{\mathbb{R}}^n : \tfrac{1}{2}\leq\xi_n\leq 2 \text{ and } |\omega+\rho^{-1}\omega_{\tilde{\theta}}-\xi'/\xi_n| < R^{-1/2}\big\},$$

and so the θ form a cover of the support of g by $R^{-1/2}$-plates. This establishes the desired inequality (2-9) with a sharp cut-off appearing in the left-hand norm, rather than the weight function w_{B_R}. The

[9]Here one picks up $O(\rho^{n+1})$ copies of the error term $(R/\rho^2)^{2n}(\lambda/R)^{-N/8}\|g\|_{L^2(\widehat{\mathbb{R}}^n)}$ from (2-12), that is, one for each ball in the collection $\mathcal{B}_{R/\rho^2}$. This is compensated for by the factor ρ^{-4n} appearing in each of these errors; it is for this reason that the R^{2n} factor is included in the definition of $\boldsymbol{A}^\varepsilon(\lambda;R)$ in (2-5).

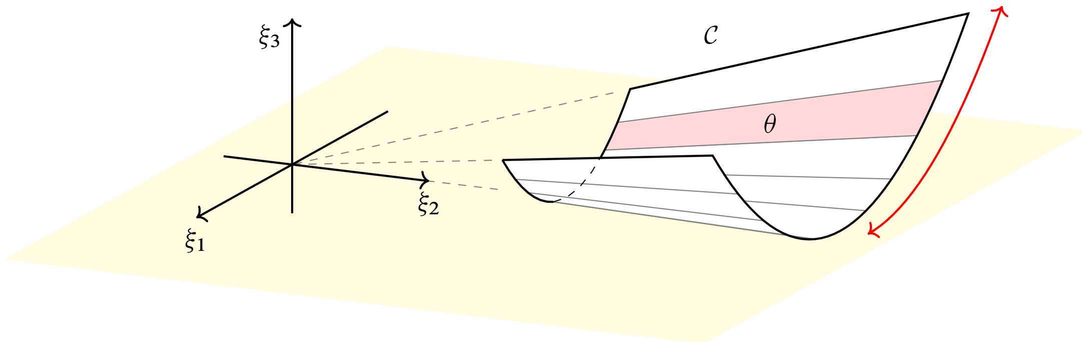

Figure 2. The simplest case of the Lorentz rescaling lemma, corresponding to the phase $\phi(x,t;\xi) := x_1\xi_1 + x_2\xi_2 + t\xi_1^2/\xi_2$. Here each plate is associated with a subset of the conic surface $\mathcal{C}$ defined by $\xi_3 = \xi_1^2/\xi_2$ for $\frac{1}{2} \le \xi_2 \le 2$. The key observation is that there exists an affine transformation of the ambient space which essentially maps θ to the whole of $\mathcal{C}$.

strengthened result, with the weight, easily follows by pointwise dominating w_{B_R} by a suitable rapidly decreasing sum of characteristic functions of R-balls.

It remains to show the validity of the inequality (2-12) for each $B_{R/\rho^2} \in \mathcal{B}_{R/\rho^2}$. Let $\mathrm{L} \in \mathrm{GL}(n,\mathbb{R})$ be such that $\mathrm{L}e_n = e_n$ and

$$\partial^2_{\eta'\eta'}\partial_t\tilde{\phi}_{\mathrm{L}}(0,0;e_n) = I_{n-1,\sigma_+} \tag{2-13}$$

for some $0 \le \sigma_+ \le n-1$, where

$$\tilde{\phi}_{\mathrm{L}}(y,t;\eta) := \tilde{\phi}(\mathrm{L}^{-1}y,t;\mathrm{L}\eta).$$

Observe that L is a composition of a rotation and an anisotropic dilation given by the matrix $\mathrm{diag}(\sqrt{|\mu_1|}, \dots, \sqrt{|\mu_{n-1}|}, 1)$, where the μ_j are the eigenvalues of (2-10) evaluated at $(0,0;e_n)$. By a linear change of both the y- and η-variables, it suffices to show that (2-12) holds with $\widetilde{T}^{\lambda/\rho^2}\tilde{g}$ replaced with $\widetilde{T}^{\lambda/\rho^2}_{\mathrm{L}}\tilde{g}_{\mathrm{L}}$, where $\widetilde{T}^{\lambda/\rho^2}_{\mathrm{L}}$ is defined with respect to the datum $(\tilde{\phi}_{\mathrm{L}}, \tilde{a}_{\mathrm{L}})$ for $\tilde{\phi}_{\mathrm{L}}$ as above, $\tilde{a}_{\mathrm{L}}(y,t;\eta) := \tilde{a}(\mathrm{L}^{-1}y,t;\mathrm{L}\eta)$, and $\tilde{g}_{\mathrm{L}} := |\det \mathrm{L}| \cdot \tilde{g} \circ \mathrm{L}$. This would follow from the definition of $\mathfrak{D}^{\varepsilon}_{\mathbf{1}}(\lambda;R)$ and Lemma 2.2 provided that the new datum $(\tilde{\phi}_{\mathrm{L}}, \tilde{a}_{\mathrm{L}})$ is of type $(1,1,\overline{C})$ for some suitable choice of constant $\overline{C} \ge 1$. Note that the amplitude $\tilde{a}_{\mathrm{L}}$ may not satisfy the required support conditions described at the beginning of Section 2B; however, by decomposing the operator, as in the proof of Lemma 2.2, this issue may easily be resolved. On the other hand, if $\overline{C}$ is suitably chosen, it is clear that $\tilde{a}_L$ satisfies the required margin condition.

To verify the remaining hypotheses in the definition of type-$(1,1,\overline{C})$ data, first note that, by retracing the steps of the argument prior to (2-11), one deduces that

$$\tilde{\phi}_{\mathrm{L}}(y,t;\eta) = \rho^2\phi(\Upsilon_\omega(D'_{\rho^{-1}} \circ \mathrm{L}^{-1}y,t),t;\eta_n\omega + \rho^{-1}\mathrm{L}'\eta',\eta_n). \tag{2-14}$$

Alternatively, using (2-11) directly, $\tilde{\phi}_{\mathrm{L}}(y,t;\eta)$ can be expressed as

$$\langle y,\eta\rangle + \int_0^1 (1-r)\langle \partial^2_{\xi'\xi'}\phi(\Upsilon_\omega(D'_{\rho^{-1}} \circ \mathrm{L}^{-1}y,t);\eta_n\omega + r\rho^{-1}\mathrm{L}'\eta',\eta_n)\mathrm{L}'\eta',\mathrm{L}'\eta'\rangle\,\mathrm{d}r, \tag{2-15}$$

where L' is the top-left $(n-1)\times(n-1)$ submatrix of L. These two formulae are used in conjunction to yield bounds on various derivatives of $\tilde{\phi}_{\mathrm{L}}$. To this end, it is also useful to note that, by the definition of Υ and the inverse function theorem,

$$\partial_y \Upsilon(y,t;\omega,1) = \partial^2_{\xi x}\phi(\Upsilon_\omega(y,t);\omega,1))^{-1},$$

so each entry $\partial_{y_j}\Upsilon^i(y,t;\omega,1)$ of the above matrix may be written as the product of

$$[\det(\partial^2_{\xi x}\phi(\Upsilon_\omega(y,t);\omega,1))]^{-1}$$

and a polynomial expression in $(\partial_{\xi_l}\partial_{x_k}\phi)(\Upsilon_\omega(y,t);\omega,1)$.

First consider the technical conditions on the derivatives. Differentiating the formula (2-14) and assuming ρ is sufficiently large, depending on ϕ, immediately implies that $(\tilde{\phi}_{\mathrm{L}},\tilde{a}_{\mathrm{L}})$ satisfies conditions (D1$_{\mathbf{1}}$) and (D2$_{\mathbf{1}}$) for $|\beta'|\geq 2$. The remaining cases of (D1$_{\mathbf{1}}$) and (D2$_{\mathbf{1}}$) can then be readily deduced by differentiating (2-15).

Concerning (H1$_{\mathbf{1}}$), by differentiating (2-15) and using the conditions (D1$_{\boldsymbol{A}}$)1$_{\boldsymbol{A}}$ and (D2$_{\boldsymbol{A}}$)2$_{\boldsymbol{A}}$ of (ϕ,a), one deduces that

$$\partial^2_{\eta y}\tilde{\phi}_{\mathrm{L}}(y,t;\eta) = I_n + O_\phi(\rho^{-1}).$$

Thus, (H1$_{\mathbf{1}}$) holds for $(\tilde{\phi}_{\mathrm{L}},\tilde{a}_{\mathrm{L}})$ provided ρ is sufficiently large depending on ϕ. Note that the conditions (D1$_{\boldsymbol{A}}$)1$_{\boldsymbol{A}}$ and (D2$_{\boldsymbol{A}}$)2$_{\boldsymbol{A}}$ are used here so as to ensure the dependence on ϕ is as described in Remark 2.4.

Concerning (H2$_{\mathbf{1}}$), the homogeneity of ϕ and (2-13) imply

$$\partial^2_{\eta'\eta'}\partial_t\tilde{\phi}_{\mathrm{L}}(z;\eta) - \frac{1}{\eta_n}I_{n-1,\sigma_+} = \frac{1}{\eta_n}\biggl(\partial^2_{\eta'\eta'}\partial_t\tilde{\phi}_{\mathrm{L}}\Bigl(z;\frac{\eta'}{\eta_n},1\Bigr) - \partial^2_{\eta'\eta'}\partial_t\tilde{\phi}_{\mathrm{L}}(0;e_n)\biggr).$$

In particular, for $1\leq i,j\leq n-1$, the (i,j)-entry of the above matrix equals

$$\int_0^1 \Bigl\langle \partial_{\eta'}\partial^2_{\eta_i\eta_j}\partial_t\tilde{\phi}_{\mathrm{L}}\Bigl(rz;\frac{r\eta'}{\eta_n},1\Bigr),\frac{\eta'}{\eta_n}\Bigr\rangle + \Bigl\langle \partial_z\partial^2_{\eta_i\eta_j}\partial_t\tilde{\phi}_{\mathrm{L}}\Bigl(rz;\frac{r\eta'}{\eta_n},1\Bigr),z\Bigr\rangle\,\mathrm{d}r.$$

Since it has been shown above that the datum $(\tilde{\phi}_{\mathrm{L}},\tilde{a}_{\mathrm{L}})$ satisfies (D1$_{\mathbf{1}}$) and (D2$_{\mathbf{1}}$), the integrand in the above expression may now be bounded above in absolute value by c_{par}. Thus, $(\tilde{\phi}_{\mathrm{L}},\tilde{a}_{\mathrm{L}})$ also satisfies (H2$_{\mathbf{1}}$) and therefore is of type $(1,1,\overline{C})$, as required. □

2F. ***Approximation by extension operators.*** This subsection deals with an approximation lemma which allows one to use Theorem 2.1 to bound variable-coefficient operators at small spatial scales.

Let T^λ be an operator associated to a type-**1** datum (ϕ,a). For each $\bar{z}\in\mathbb{R}^{n+1}$ with $\bar{z}/\lambda\in Z$ the map $\eta\mapsto(\partial_z\phi^\lambda)(\bar{z};\Psi^\lambda(\bar{z};\eta))$ is a graph parametrisation of a hypersurface $\Sigma_{\bar{z}}$ with precisely one vanishing principal curvature at each point. In particular, recalling (2-4), one has

$$\langle z,(\partial_z\phi^\lambda)(\bar{z};\Psi^\lambda(\bar{z};\eta))\rangle = \langle x,\eta\rangle + th_{\bar{z}}(\eta) \quad \text{for all } z=(x,t)\in\mathbb{R}^{n+1},$$

where $h_{\bar{z}}(\eta) := (\partial_t\phi^\lambda)(\bar{z};\Psi^\lambda(\bar{z};\eta))$. Let $E_{\bar{z}}$ denote the extension operator associated to $\Sigma_{\bar{z}}$, given by

$$E_{\bar{z}}g(x,t) := \int_{\widehat{\mathbb{R}}^n} e^{i(\langle x,\eta\rangle + th_{\bar{z}}(\eta))}a_{\bar{z}}(\eta)g(\eta)\,\mathrm{d}\eta \quad \text{for all } (x,t)\in\mathbb{R}^{n+1},$$

where $a_{\bar{z}}(\eta) := a_2 \circ \Psi^\lambda(\bar{z};\eta)|\det \partial_\eta \Psi^\lambda(\bar{z};\eta)|$. The operator T^λ is effectively approximated by $E_{\bar{z}}$ at small spatial scales. Furthermore, the conditions on the translation-invariant decoupling inequality, Theorem 2.1, are satisfied by each of the functions $h_{\bar{z}}$. In particular, the type-**1** condition implies the following uniform bound.

Lemma 2.5. *Let (ϕ, a) be a type-$\mathbf{1}$ datum. Each eigenvalue μ of $\partial_{\eta'\eta'} h_{\bar{z}}$ satisfies $|\mu| \sim 1$ on $\operatorname{supp} a_{\bar{z}}$.*

The proof of this lemma is an elementary calculus exercise, the details of which are omitted.

Concerning the approximation of T^λ by $E_{\bar{z}}$, suppose that $1 \le K \le \lambda^{1/2}$ and $z \in B(\bar{z}, K) \subseteq B(0, 3\lambda/4)$; this containment property may be assumed in view of the margin condition ($\mathrm{M}_{\mathbf{1}}$). By applying the change of variables $\xi = \Psi^\lambda(\bar{z};\eta)$ and a Taylor expansion of ϕ^λ around the point $\bar{z}$,

$$T^\lambda f(z) = \int_{\widehat{\mathbb{R}}^n} e^{i(\langle z-\bar{z}, (\partial_z \phi^\lambda)(\bar{z};\Psi^\lambda(\bar{z};\eta))\rangle + \mathcal{E}^\lambda_{\bar{z}}(z-\bar{z};\eta))} a_1^\lambda(z) a_{\bar{z}}(\eta) f_{\bar{z}}(\eta)\, \mathrm{d}\eta, \tag{2-16}$$

where $f_{\bar{z}} := e^{i\phi^\lambda(\bar{z};\Psi^\lambda(\bar{z};\,\cdot\,))} f \circ \Psi^\lambda(\bar{z};\,\cdot\,)$, and, by Taylor's theorem,

$$\mathcal{E}^\lambda_{\bar{z}}(v;\eta) = \frac{1}{\lambda}\int_0^1 (1-r)\Big\langle (\partial^2_{zz}\phi)\Big(\frac{\bar{z}+rv}{\lambda};\Psi^\lambda(\bar{z};\eta)\Big)v, v\Big\rangle\, \mathrm{d}r. \tag{2-17}$$

Since $K \le \lambda^{1/2}$ and (ϕ, a) is type-$\mathbf{1}$, so that property ($\mathrm{D2}_{\mathbf{1}}$) holds, it follows that

$$\sup_{(v;\eta)\in B(0,K)\times \operatorname{supp} a_{\bar{z}}} |\partial^\beta_\xi \mathcal{E}^\lambda_{\bar{z}}(v;\eta)| \lesssim_N 1$$

for all $\beta \in \mathbb{N}_0^n$ with $1 \le |\beta| \le 4N$. Consequently, $\mathcal{E}^\lambda_{\bar{z}}(z-\bar{z};\xi)$ does not contribute significantly to the oscillation induced by the exponential in (2-16) and it can therefore be safely removed from the phase, at the expense of some negligible error terms.

Lemma 2.6. *Let T^λ be an operator associated to a type-$\mathbf{1}$ datum (ϕ, a). Let $0 < \delta \le \frac{1}{2}$, $1 \le K \le \lambda^{1/2-\delta}$, and $\bar{z}/\lambda \in Z$ so that $B(\bar{z}, K) \subseteq B(0, 3\lambda/4)$. Then*

(i)
$$\|T^\lambda f\|_{L^p(w_{B(\bar{z},K)})} \lesssim_N \|E_{\bar{z}} f_{\bar{z}}\|_{L^p(w_{B(0,K)})} + \lambda^{-\delta N/2}\|f\|_{L^2(\widehat{\mathbb{R}}^n)} \tag{2-18}$$

holds provided N is sufficiently large depending on n, δ and p.

(ii) *Suppose that $|\bar{z}| \le \lambda^{1-\delta'}$. There exists a family of operators $\boldsymbol{T}^\lambda$ all with phase function ϕ and associated to type-$(1, 1, C)$ data such that*

$$\|E_{\bar{z}} f_{\bar{z}}\|_{L^p(w_{B(0,K)})} \lesssim_N \|T^\lambda_* f\|_{L^p(w_{B(\bar{z},K)})} + \lambda^{-\min\{\delta,\delta'\}N/2}\|f\|_{L^2(\widehat{\mathbb{R}}^n)} \tag{2-19}$$

holds for some $T^\lambda_* \in \boldsymbol{T}^\lambda$ provided N is sufficiently large depending on n, δ and p. Moreover, the family $\boldsymbol{T}^\lambda$ has cardinality $O_N(1)$ and is independent of the choice of ball $B(\bar{z}, K)$.

Remark 2.7. (i) The weights appearing in Lemma 2.6 are defined with respect to the same $N \in \mathbb{N}$ as that appearing in the λ exponent. This is also understood to be the same N as that appearing in the definition of the $(\mathrm{D2}_{\boldsymbol{A}})2_{\boldsymbol{A}}$ condition.

(ii) If one replaces $w_{B(\bar{z},K)}$ with the characteristic function $\chi_{B(\bar{z},K)}$ on the left-hand side of (2-18), then the proof of Lemma 2.6 shows that the inequality holds without the additional $\lambda^{-\delta N/2}\|f\|_{L^2(\widehat{\mathbb{R}}^n)}$ term.

Several variants of this kind of approximation (or stability) lemma have previously appeared in the literature; see, for instance, [Stein 1993, Chapter VI, §2] or [Christ 1988; Tao 1999]. In the context of decoupling, Lemma 2.6 is closely related to certain approximation arguments used to extend decoupling estimates to wider classes of surfaces in [Pramanik and Seeger 2007; Garrigós and Seeger 2010; Guo and Oh 2018] and [Bourgain and Demeter 2015, §7–8]. A variant of Lemma 2.6 (which is somewhat cleaner than the above statement) can also be applied to slightly simplify the original proof of the decoupling theorem in [Bourgain and Demeter 2015; 2017b] and, in particular, obviate the need to reformulate the problem in terms of functions with certain Fourier support conditions (the details of the original "reformulation" approach are given in [Bourgain and Demeter 2017b, §5]).

Proof of Lemma 2.6. Note that f in (2-16) may be replaced by $f\psi$, where ψ is a smooth function that equals 1 on $\operatorname{supp} a_{\bar{z}}$ and vanishes outside its double. Moreover, recalling the definition of $a_{\bar{z}}$ and that (ϕ, a) is a type-**1** datum, one may assume that the function ψ is supported in $[0, 2\pi]^n$. In view of the expression (2-16), by performing a Fourier series decomposition of $e^{i\mathcal{E}^{\lambda}_{\bar{z}}(v,\eta)}\psi(\eta)$ in the η-variable, one may write

$$e^{i\mathcal{E}^{\lambda}_{\bar{z}}(v;\eta)}\psi(\eta) = \sum_{\ell\in\mathbb{Z}^n} b_\ell(v)e^{i\langle \ell,\eta\rangle}, \tag{2-20}$$

where

$$b_\ell(v) = \int_{[0,2\pi]^n} e^{-i\langle \ell,\eta\rangle} e^{i\mathcal{E}^{\lambda}_{\bar{z}}(v;\eta)}\psi(\eta)\,\mathrm{d}\eta.$$

The formula (2-17) and property ($\mathrm{D2}_{\mathbf{1}}$) of the phase together imply

$$\sup_{\eta\in[0,2\pi]^n} |\partial_\eta^\beta \mathcal{E}^{\lambda}_{\bar{z}}(v;\eta)| \lesssim_N \frac{|v|^2}{\lambda}$$

for all multi-indices $\beta \in \mathbb{N}$ with $1 \le |\beta| \le N$. Therefore, by repeated application of integration-by-parts, one deduces that

$$|b_\ell(v)| \lesssim_N (1+|\ell|)^{-N} \quad \text{whenever } |v| \le 2\lambda^{1/2}.$$

This, (2-20) and (2-16) lead to the useful pointwise estimate

$$|T^\lambda f(\bar{z}+v)| \lesssim_N \sum_{\ell\in\mathbb{Z}^n} (1+|\ell|)^{-N} |E_{\bar{z}}(f_{\bar{z}} e^{i\langle \ell,\cdot\rangle})(v)|, \tag{2-21}$$

valid for $|v| \le 2\lambda^{1/2}$. Writing

$$\|T^\lambda f\|_{L^p(w_{B(\bar{z},K)})} \le \|(T^\lambda f)\chi_{B(\bar{z},2\lambda^{1/2})}\|_{L^p(w_{B(\bar{z},K)})} + \|(T^\lambda f)\chi_{\mathbb{R}^n\setminus B(\bar{z},2\lambda^{1/2})}\|_{L^p(w_{B(\bar{z},K)})},$$

it follows from (2-21) that

$$\|(T^\lambda f)\chi_{B(\bar{z},2\lambda^{1/2})}\|_{L^p(w_{B(\bar{z},K)})} \lesssim_N \sum_{\ell\in\mathbb{Z}^n} (1+|\ell|)^{-N} \|E_{\bar{z}}(f_{\bar{z}} e^{i\langle \ell,\cdot\rangle})\|_{L^p(w_{B(0,K)})}. \tag{2-22}$$

On the other hand, it is claimed that

$$\|(T^\lambda f)\chi_{\mathbb{R}^n\setminus B(\bar z,2\lambda^{1/2})}\|_{L^p(w_{B(\bar z,K)})} \lesssim \lambda^{n/(2p)-\delta(N-n+2)}\|f\|_{L^2(\widehat{\mathbb{R}}^n)} \tag{2-23}$$

and therefore this term can be treated as an error. Indeed, if $|v| > 2\lambda^{1/2}$ and $K \le \lambda^{1/2-\delta}$, then

$$(1+K^{-1}|v|)^{-(N-n+2)} \le (1+2\lambda^{1/2}K^{-1})^{-(N-n+2)} \le \lambda^{-\delta(N-n+2)}.$$

Combining this observation with the trivial estimate

$$\|T^\lambda f\|_{L^p(\tilde w_{B(\bar z,K)})} \lesssim K^{n/p}\|f\|_{L^2(\widehat{\mathbb{R}}^n)},$$

where $\tilde w_{B(0,K)} := (1+K^{-1}|\cdot|)^{-(n+2)}$, one readily deduces (2-23).

Observe that the operator $E_{\bar z}$ enjoys the translation-invariance property

$$E_{\bar z}[e^{i\langle \ell,\cdot\rangle}g](x,t) = E_{\bar z}g(x+\ell,t) \quad \text{for all } (x,t)\in\mathbb{R}^{n+1} \text{ and all } \ell\in\mathbb{R}^n; \tag{2-24}$$

it is for this reason that the graph parametrisation was introduced at the outset of the argument. The identity (2-24) together with (2-22) and (2-23) imply

$$\|T^\lambda f\|_{L^p(w_{B(\bar z,K)})} \lesssim_N \sum_{\ell\in\mathbb{Z}^n}(1+|\ell|)^{-N}\|E_{\bar z}f_{\bar z}\|_{L^p(w_{B((\ell,0),K)})} + \lambda^{-\delta N/2}\|f\|_{L^2(\widehat{\mathbb{R}}^n)}, \tag{2-25}$$

provided N is chosen to be suitably large. One may readily verify that

$$\sum_{\ell\in\mathbb{Z}^n}(1+|\ell|)^{-N}w_{B((\ell,0),K)} \lesssim w_{B(0,K)} \tag{2-26}$$

and combining this with (2-25) immediately yields the desired estimate (2-18).

The proof of (2-19) is similar to that of (2-18), although a slight complication arises since, in contrast with $E_{\bar z}$, the variable-coefficient operator T^λ does not necessarily satisfy the translation-invariance property described in (2-24).

One may write

$$E_{\bar z}f_{\bar z}(v) = \int_{\widehat{\mathbb{R}}^n} e^{i\phi^\lambda(\bar z+v;\Psi^\lambda(\bar z,\eta))}e^{-i\mathcal{E}^\lambda_{\bar z}(v;\eta)}a_{\bar z}(\eta)f\circ\Psi^\lambda(\bar z;\eta)\,\mathrm{d}\eta$$

and, by forming the Fourier series expansion of $e^{-i\mathcal{E}^\lambda_{\bar z}(v;\eta)}\psi(\eta)$ in η and undoing the change of variables $\xi = \Psi^\lambda(\bar z;\eta)$, thereby deduce that

$$|E_{\bar z}f_{\bar z}(v)| \lesssim_N \sum_{\ell\in\mathbb{Z}^n}(1+|\ell|)^{-4N}|T^\lambda[e^{i\langle \ell,(\partial_z\phi^\lambda)(\bar z\,\cdot)\rangle}f](\bar z+v)|$$

whenever $|v|\le 2\lambda^{1/2}$. This pointwise bound is understood to hold modulo the choice of spatial cut-off a_1 appearing in the definition of T^λ. Taking $L^p(w_{B(\bar z,K)})$-norms in z and reasoning as in the proof of (2-18), one obtains

$$\|E_{\bar z}f_{\bar z}\|_{L^p(w_{B(0,K)})} \lesssim_N \sum_{\ell\in\mathbb{Z}^n}(1+|\ell|)^{-4N}\|(T^\lambda\tilde f_\ell)\chi_{B(\bar z,2\lambda^{1/2})}\|_{L^p(w_{B(\bar z,K)})} + \lambda^{-\delta N/2}\|f\|_{L^2(\widehat{\mathbb{R}}^n)},$$

where $\tilde{f}_\ell := e^{i\langle \ell, (\partial_z \phi^\lambda)(\bar{z};\,\cdot\,)\rangle} f$. Note that the cut-off function $\chi_{B(\bar{z}, 2\lambda^{1/2})}$ can be dominated by a smooth amplitude $\tilde{a}_1^\lambda$, where $\tilde{a}_1$ is equal to 1 on $\operatorname{supp} a_1$ and has half the margin. The above sum is split into a sum over ℓ satisfying $|\ell| > C_N$ and a sum over the remaining ℓ where C_N is a constant depending on N, chosen large enough for the present purpose. To control sum over large ℓ, apply (2-18) and argue as in (2-26) to conclude that

$$\sum_{\substack{\ell \in \mathbb{Z}^n \\ |\ell| > C_N}} (1+|\ell|)^{-4N} \|T^\lambda \tilde{f}_\ell\|_{L^p(w_{B(\bar{z},K)})} \lesssim_N \sum_{\substack{\ell \in \mathbb{Z}^n \\ |\ell| > C_N}} (1+|\ell|)^{-2N} \|E_{\bar{z}} f_{\bar{z}}\|_{L^p(w_{B((\ell,0),K)})}$$
$$\lesssim C_N^{-N} \|E_{\bar{z}} f_{\bar{z}}\|_{L^p(w_{B(0,K)})}.$$

Therefore, if C_N is chosen to be sufficiently large depending on N, the above term can be absorbed into the left-hand side of the inequality and one obtains

$$\|E_{\bar{z}} f_{\bar{z}}\|_{L^p(w_{B(0,K)})} \lesssim_N \sum_{\substack{\ell \in \mathbb{Z}^n \\ |\ell| \le C_N}} \|T^\lambda \tilde{f}_\ell\|_{L^p(w_{B(\bar{z},K)})} + \lambda^{-\delta N/2} \|f\|_{L^2(\widehat{\mathbb{R}}^n)}.$$

Each $T^\lambda \tilde{f}_\ell$ can be thought of as an operator T_ℓ^λ where the latter has phase ϕ and amplitude function

$$\tilde{a}_\ell(z;\xi) := \tilde{a}_1(z;\xi) e^{i\langle \ell, (\partial_z \phi^\lambda)(\bar{z};\xi)\rangle}.$$

Unfortunately, these amplitudes depend on the choice of ball $B(\bar{z}, K)$ and therefore are unsuitable for the present purpose. To remove this undesirable dependence, one may take a Taylor series expansion to write

$$e^{i\langle \ell, (\partial_z \phi^\lambda)(\bar{z};\xi)\rangle} = \sum_{|\alpha| \le N-1} u_\alpha(\omega) \left(\frac{\bar{z}}{\lambda}\right)^\alpha + O((|\bar{z}|/\lambda)^N), \tag{2-27}$$

where each $u_\alpha \in C^\infty(\mathbb{R}^n)$ satisfies $|\partial_\xi^\beta u_\alpha(\xi)| \lesssim_N 1$ for all $|\beta| \le N$. Note that the u_α do not depend on the choice of $\bar{z}$. Furthermore, since $|\bar{z}| \le \lambda^{1-\delta'}$, it follows that the error in (2-27) is $O(\lambda^{-\delta' N})$ and the part of the operator arising from such terms can be bounded by $\lambda^{-\min\{\delta,\delta'\}N/2} \|f\|_{L^2(\widehat{\mathbb{R}}^n)}$. The family of operators $\boldsymbol{T}^\lambda$ is now given by the family of amplitudes

$$u_\alpha(\omega)\tilde{a}_\ell(z;\xi), \quad |\ell| \le C_n,\ |\alpha| \le N-1.$$

Since $|\bar{z}|/\lambda \le 1$, one concludes that

$$\|E_{\bar{z}} f_{\bar{z}}\|_{L^p(w_{B(0,K)})} \lesssim_N \sum_{T_*^\lambda \in \boldsymbol{T}^\lambda} \|T_*^\lambda \tilde{f}_\ell\|_{L^p(w_{B(\bar{z},K)})} + \lambda^{-\min\{\delta,\delta'\}N/2} \|f\|_{L^2(\widehat{\mathbb{R}}^n)}$$

and the desired inequality now holds for some choice of $T_*^\lambda \in \boldsymbol{T}^\lambda$ by pigeonholing. □

2G. ***Proof of the variable-coefficient decoupling estimates.*** By the discussion in Sections 2B–2E, to prove Theorem 1.4 for the fixed parameters 2ε, M, and p it suffices to show

$$\mathfrak{D}_1^\varepsilon(\lambda; R) \lesssim_\varepsilon 1 \quad \text{for all } 1 \le R \le \lambda^{1-\varepsilon/n}.$$

The trivial estimate (2-7) implies the above inequality if R is small (that is, $R \lesssim_\varepsilon 1$), and the proof proceeds by induction on R, using this observation as the base case. In particular, one may assume by way of induction hypothesis that the following holds.

Radial Hypothesis. *There is a constant* $\overline{C}_\varepsilon \geq 1$, *depending only on the admissible parameters* n, ε, M, *and* p, *such that*

$$\mathfrak{D}_{\mathbf{1}}^{\varepsilon}(\lambda'; R') \leq \overline{C}_\varepsilon$$

holds for all $1 \leq R' \leq R/2$ *and all* λ' *satisfying* $R' \leq (\lambda')^{1-\varepsilon/n}$.

Let $\mathcal{B}_K$ be a finitely overlapping cover of B_R by balls of radius K for some $2 \leq K \leq \lambda^{1/4}$. One may assume that any centre $\bar{z}$ of a ball in this cover satisfies $|\bar{z}| \leq \lambda^{1-\varepsilon/n}$. The estimate (2-18) from Lemma 2.6 with $\delta = \frac{1}{4}$ implies

$$\|T^\lambda f\|_{L^p(B_R)} \lesssim \biggl(\sum_{B(\bar{z},K)\in\mathcal{B}_K} \|E_{\bar{z}} f_{\bar{z}}\|_{L^p(w_{B(0,K)})}^p\biggr)^{1/p} + R^{n+1}\Bigl(\frac{\lambda}{R}\Bigr)^{-N/8} \|f\|_{L^2(\hat{\mathbb{R}}^n)}.$$

Applying the theorem of [Bourgain and Demeter 2015; 2017a] (that is, Theorem 2.1) with exponent $\varepsilon/2$ (and recalling Lemma 2.5), one deduces that the inequality

$$\|E_{\bar{z}} f_{\bar{z}}\|_{L^p(w_{B(0,K)})} \lesssim_\varepsilon K^{\alpha(p)+\varepsilon/2} \|E_{\bar{z}} f_{\bar{z}}\|_{L^{p,K}_{\mathrm{dec}}(w_{B(0,K)})}$$

holds for each of the extension operators in the previous display. Combining these observations with an application of (2-19) from Lemma 2.6 with $\delta' = \varepsilon/n$, and summing over $\mathcal{B}_K$,

$$\|T^\lambda f\|_{L^p(B_R)} \lesssim_\varepsilon K^{\alpha(p)+\varepsilon/2}\biggl(\sum_{\sigma:K^{-1/2}\text{-plate}} \|T^\lambda f_\sigma\|_{L^p(w_{B_R})}^p\biggr)^{1/p} + R^{n+1}\Bigl(\frac{\lambda}{R}\Bigr)^{-\varepsilon N/8n} \|f\|_{L^2(\hat{\mathbb{R}}^n)}.$$

The operator on the right involves a slightly different amplitude function compared with that on the left but, as in the statement of Theorem 1.4, this is suppressed in the notation.

Note that, since $K \geq 2$, $\overline{C} \geq 1$, and $R \leq \lambda^{1-\varepsilon/n}$, trivially $R/(\overline{C}K) \leq (\lambda/(\overline{C}K))^{1-\varepsilon/n}$ and $R/(\overline{C}K) \leq R/2$. Consequently, the assumptions of the radial induction hypothesis are valid for the parameters $R' := R/(\overline{C}K)$ and $\lambda' := \lambda/(\overline{C}K)$. Thus, by combining the radial induction hypothesis with (2-9) from the Lorentz rescaling lemma, one concludes that

$$\|T^\lambda f\|_{L^p(B_R)} \leq C_\varepsilon \overline{C}_\varepsilon K^{-\varepsilon/2} R^{\alpha(p)+\varepsilon} \|T^\lambda f\|_{L^{p,R}_{\mathrm{dec}}(w_{B_R})} + R^{2n}\Bigl(\frac{\lambda}{R}\Bigr)^{-\varepsilon N/(8n)} \|f\|_{L^2(\hat{\mathbb{R}}^n)}.$$

Choosing K sufficiently large (depending only on ε, n, M and p) so that $C_\varepsilon K^{-\varepsilon/2} \leq 1$, the induction closes and the desired result follows.

3. Proof of the local smoothing estimate

In this section the relationships between the theorems stated in the Introduction are established and, in particular, it is shown that

$$\text{Theorem 1.4} \implies \text{Theorem 1.2} \implies \text{Theorem 1.1}.$$

Given the formula for the solution u from (1-4), the latter implication is almost immediate. The former implication follows from a straightforward adaption of an argument due to [Wolff 2000], which treats an analogous problem for the euclidean wave equation. Nevertheless, proofs of both of the implications are included for completeness.

To begin, the definition of the cinematic curvature condition, as introduced in [Mockenhaupt, Seeger, and Sogge 1993], is recalled. As in the statement of Theorem 1.2, let Y and Z be precompact smooth manifolds of dimensions n and $n+1$, respectively. Let $\mathcal{C} \subseteq T^*Z \setminus 0 \times T^*Y \setminus 0$ be a choice of canonical relation; here $T^*M \setminus 0$ denotes the tangent bundle of a C^∞ manifold M with the 0 section removed. Thus,

$$\mathcal{C} = \{(x, t, \xi, \tau, y, \eta) : (x, t, \xi, \tau, y, -\eta) \in \Lambda\}$$

for some conic Lagrangian submanifold $\Lambda \subseteq T^*Z \setminus 0 \times T^*Y \setminus 0$; see [Hörmander 1971] or [Duistermaat 1996; Sogge 2017] for further details. Certain conditions are imposed on $\mathcal{C}$, defined in terms of the projections

$$\begin{array}{ccccc} & & \mathcal{C} & & \\ & \swarrow^{\Pi_{T^*Y}} & \downarrow{\Pi_Z} & \searrow^{\Pi_{T^*_{z_0}Z}} & \\ T^*Y \setminus 0 & & Z & & T^*_{z_0}Z \setminus 0 \end{array}$$

First there is the basic nondegeneracy hypothesis that the projections Π_{T^*Y} and Π_Z are submersions:

$$\operatorname{rank} d\Pi_{T^*Y} \equiv 2n \quad \text{and} \quad \operatorname{rank} d\Pi_Z \equiv n+1. \tag{3-1}$$

This condition implies that for each $z_0 \in Z$ the image $\Gamma_{z_0} := \Pi_{T^*_{z_0}Z}(\mathcal{C})$ of $\mathcal{C}$ under the projection onto the fibre $T^*_{z_0}Z$ is a C^∞ immersed hypersurface. Note that Γ_{z_0} is conic and therefore has everywhere vanishing Gaussian curvature. In addition to the nondegeneracy hypothesis (3-1), the following curvature condition is also assumed:

For all $z_0 \in Z$, the cone Γ_{z_0} has $n-1$ nonvanishing principal curvatures at every point. (3-2)

If both (3-1) and (3-2) hold, then $\mathcal{C}$ is said to satisfy the *cinematic curvature condition* [Mockenhaupt, Seeger, and Sogge 1993].

Remark 3.1. Using local coordinates, (3-1) and (3-2) may be expressed in terms of the conditions (H1)1 and (H2)2 introduced in Section 1B. Indeed, near any point

$$(x_0, t_0, \xi_0, \tau_0, y_0, \eta_0) \in \mathcal{C},$$

the condition (3-1) implies that there exists a phase function $\phi(z; \eta)$ satisfying (H1)1 such that $\mathcal{C}$ is given locally by

$$\{(z, \partial_z\phi(z;\eta), \partial_\eta\phi(z;\eta), \eta) : \eta \in \mathbb{R}^n \setminus \{0\} \text{ in a conic neighbourhood of } \eta_0\}.$$

Furthermore, (3-2) implies that the function ϕ satisfies (H2)2. Further details may be found in [Sogge 2017, Chapter 8].

Recall from the Introduction that the solution to the Cauchy problem (1-1) can be written as $u = \mathcal{F}_0 f_0 + \mathcal{F}_1 f_1$, where each $\mathcal{F}_j \in I^{j-1/4}(M \times \mathbb{R}, M; \mathcal{C})$ for some canonical relation $\mathcal{C}$ satisfying the cinematic curvature condition. Fix a choice of coordinate atlas $\{(\Omega_\nu, \kappa_\nu)\}_\nu$ on M and a partition of unity $\{\psi_\nu\}_\nu$ subordinate to the cover $\{\Omega_\nu\}_\nu$ of M. A choice of Bessel potential norm $\|\cdot\|_{L^p_s(M)}$ is then defined by

$$\|f\|_{L^p_s(M)} := \sum_\nu \|f_\nu\|_{L^p_s(\mathbb{R}^n)},$$

where $f_\nu := (\psi_\nu f) \circ \kappa^{-1}$ and $L^p_s(\mathbb{R}^n)$ denotes the standard Bessel potential space in $\mathbb{R}^n$. Thus, expressing everything in local coordinates and applying the composition theorem for Fourier integral operators (see, for instance, [Sogge 2017, Theorem 6.2.2]), it is clear that Theorem 1.1 is an immediate consequence of Theorem 1.2.

It remains to show that Theorem 1.2 follows from the decoupling inequality established in Theorem 1.4. Working in local coordinates (and recalling Remark 3.1 and the discussion in Section 1B), it suffices to prove an estimate for operators of the form

$$\mathcal{F} f(x,t) := \int_{\widehat{\mathbb{R}}^n} e^{i\phi(x,t;\xi)} b(x,t;\xi)(1+|\xi|^2)^{\mu/2} \hat{f}(\xi)\, d\xi, \tag{3-3}$$

where b is a symbol of order 0 with compact support in the (x,t)-variables and ϕ is a smooth homogeneous phase function satisfying (H1)1 and (H2)2 (at least on the support of b). Recall that b is a symbol of order 0 if $b \in C^\infty(\mathbb{R}^{n+1} \times \mathbb{R}^n)$ and satisfies

$$|\partial_z^\nu \partial_\xi^\gamma b(z;\xi)| \lesssim_{\gamma,\nu} (1+|\xi|)^{-|\gamma|} \quad \text{for all multi-indices } (\gamma,\nu) \in \mathbb{N}_0^{n+1} \times \mathbb{N}_0^n.$$

In particular, Theorem 1.2 is a direct consequence of the following proposition.

Proposition 3.2. *If $\bar{p}_n \le p < \infty$ and $\mathcal{F}$ is defined as in* (3-3) *with $\mu < -\alpha(p) = -\bar{s}_p + \frac{1}{p}$, then*

$$\|\mathcal{F} f\|_{L^p(\mathbb{R}^{n+1})} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. By applying a rotation and a suitable partition of unity, one may assume that b is supported in $B^n(0,\varepsilon_0) \times B^1(1,\varepsilon_0) \times \Gamma$ for a suitably small constant $0 < \varepsilon_0 \le 1$, where

$$\Gamma := \{\xi \in \widehat{\mathbb{R}}^n : |\xi_j| \le |\xi_n| \text{ for } 1 \le j \le n-1\}.$$

Further, as the symbol b has compact (x,t)-support of diameter $O(1)$, one may assume without loss of generality that it is of product type; that is, $b(x,t;\xi) = b_1(x,t) b_2(\xi)$. The latter reduction follows by taking Fourier transforms in a similar manner to that used in the proof of Lemma 2.6; the argument, which is standard, is postponed until the end of the proof.

Fix $\beta \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} \beta \in \left[\frac{1}{2}, 2\right]$ and such that $\sum_{\lambda \text{ dyadic}} \beta(r/\lambda) = 1$ for $r \neq 0$. Let $\mathcal{F}^\lambda := \mathcal{F} \circ \beta(\sqrt{-\Delta_x}/\lambda)$, so that $\mathcal{F}^\lambda f$ is given by introducing a $\beta(|\xi|/\lambda)$ factor into the symbol in (3-3),[10] and

[10] In general, $m(i^{-1}\partial_x)$ denotes the Fourier multiplier operator (defined for f belonging to a suitable a priori class) $m(i^{-1}\partial_x) f(x) := \int_{\widehat{\mathbb{R}}^n} e^{i\langle x,\xi\rangle} m(\xi) \hat{f}(\xi)\, d\xi$ for any $m \in L^\infty(\widehat{\mathbb{R}}^n)$. The operator $m(\sqrt{-\Delta_x})$ is then defined in the natural manner via the identity $-\Delta_x = i^{-1}\partial_x \cdot i^{-1}\partial_x$.

decompose $\mathcal{F}f$ as

$$\mathcal{F}f =: \mathcal{F}^{\lesssim 1}f + \sum_{\lambda\in\mathbb{N}:\text{ dyadic}} \mathcal{F}^{\lambda}f.$$

It follows that $\mathcal{F}^{\lesssim 1}$ is a pseudodifferential operator of order 0 and therefore bounded on L^p for all $1 < p < \infty$. Thus, letting $\varepsilon := -(\mu + \alpha(p))/2 > 0$, the problem is further reduced to showing that

$$\|\mathcal{F}^{\lambda}f\|_{L^p(\mathbb{R}^{n+1})} \lesssim \lambda^{\alpha(p)+\mu+\varepsilon}\|f\|_{L^p(\mathbb{R}^n)}$$

for all $\lambda \geq 1$.

By various rescaling arguments and Theorem 1.4, it follows that

$$\|\mathcal{F}^{\lambda}f\|_{L^p(\mathbb{R}^{n+1})} \lesssim_{s,p} \lambda^{\alpha(p)+\varepsilon}\bigg(\sum_{\theta:\lambda^{-1/2}\text{-plate}} \|\mathcal{F}^{\lambda}_{\theta}f\|^p_{L^p(\mathbb{R}^{n+1})}\bigg)^{1/p},$$

where $\mathcal{F}^{\lambda}_{\theta} := \mathcal{F}^{\lambda} \circ a_{\theta}(i^{-1}\partial_x)$ for a_{θ} a choice of smooth angular cut-off to θ. Thus, to conclude the proof of Proposition 3.2 (and therefore that of Theorems 1.2 and 1.1), it suffices to establish the following lemma.

Lemma 3.3. *For $\mathcal{F}^{\lambda}_{\theta}$ as defined above and $2 \leq p \leq \infty$ one has*

$$\bigg(\sum_{\theta:\lambda^{-1/2}\text{-plate}} \|\mathcal{F}^{\lambda}_{\theta}f\|^p_{L^p(\mathbb{R}^{n+1})}\bigg)^{1/p} \lesssim \lambda^{\mu}\|f\|_{L^p(\mathbb{R}^n)}.$$

This inequality essentially appears in [Seeger, Sogge, and Stein 1991] (see also [Stein 1993, Chapter IX]); a sketch of the proof is included for completeness.

Proof of Lemma 3.3. By interpolation (via Hölder's inequality) it suffices to establish the cases $p = 2$ and $p = \infty$.

To prove the $p = 2$ bound, one may use Hörmander's theorem (see, for instance, [Stein 1993, Chapter IX §1.1]) for fixed t, followed by Plancherel's theorem and the almost orthogonality of the plates θ.

To prove the $p = \infty$ bound, it suffices to show that

$$\sup_{(x,t)\in\mathbb{R}^{n+1}} \|K^{\lambda}_{\theta}(x,t;\,\cdot\,)\|_{L^1(\mathbb{R}^n)} \lesssim \lambda^{\mu},$$

where K^{λ}_{θ} is the kernel of the operator $\mathcal{F}^{\lambda}_{\theta}$. This follows from a standard stationary phase argument, which exploits heavily the homogeneity of the phase and the angular localisation; see, for instance, [Stein 1993, Chapter IX, §4.5–4.6] for further details. □

It remains to justify the initial reduction to symbols of product type. As mentioned earlier, the argument is standard and appears, for instance, in the proof of the L^2 boundedness for pseudodifferential operators of order 0 (see [Stein 1993, Chapter VI, §2]).

As b is a symbol of order 0 with compact (x,t)-support, $(n+2)$-fold integration-by-parts shows that

$$|\partial^{\gamma}_{\xi}\hat{b}(\zeta;\xi)| \lesssim_{\gamma} (1+|\zeta|)^{-(n+2)}(1+|\xi|)^{-|\gamma|} \quad \text{for all multi-indices } \gamma \in \mathbb{N}^n_0, \tag{3-4}$$

where $\hat{b}$ denotes the Fourier transform of b in the $z = (x, t)$-variable. By means of the Fourier transform one may write

$$\mathcal{F} f(x,t) = \int_{\widehat{\mathbb{R}}^{n+1}} e^{i\langle z,\zeta\rangle}(1+|\zeta|)^{-(n+2)} \int_{\widehat{\mathbb{R}}^{n}} e^{i\phi(x,t;\xi)} \frac{b_\zeta(x,t;\xi)}{(1+|\xi|^2)^{-\mu/2}} \hat{f}(\xi)\,\mathrm{d}\xi\,\mathrm{d}\zeta,$$

where $b_\zeta(x,t;\xi) := \psi(x,t)\hat{b}(\zeta;\xi)(1+|\zeta|)^{n+2}$ for ψ a smooth cut-off equal to 1 in the z-support of b and vanishing outside its double. The functions b_ζ are all of product type and, by (3-4), are symbols of order 0 uniformly in ζ. Taking L^p-norms and applying Minkowski's integral inequality, it now suffices to show the L^p boundedness of $\mathcal{F}$ under the product hypothesis. □

4. Counterexamples for local smoothing estimates for certain Fourier integral operators

To conclude the paper, the proof of Proposition 1.3 is presented. As originally observed by the third author in [Sogge 1991] and elaborated further in, for instance, [Mattila 2015; Mockenhaupt, Seeger, and Sogge 1993; Sogge 2017; Tao 1999], it is known that local smoothing estimates for Fourier integral operators imply favourable L^p estimates for a natural class of oscillatory integral operators. Indeed, this is the basis of the well-known formal implication that the local smoothing conjecture for the (euclidean) wave equation implies the Bochner–Riesz conjecture (see [Sogge 1991; 2017]). In this section a general form of this implication is combined with a counterexample of [Bourgain 1991; 1995b] to show that Theorem 1.2 is sharp when $n \geq 3$ is odd.

4A. ***Local smoothing for Fourier integrals and nonhomogeneous oscillatory integrals.*** Let $\Omega \subseteq \mathbb{R}^n$ be open and suppose that $\Phi\colon \Omega \times \Omega \to \mathbb{R}$ is smooth and satisfies

$$\partial_y \Phi(x,y) \neq 0 \quad \text{for all } (x,y) \in \Omega \times \Omega \tag{4-1}$$

and, moreover, that the Monge–Ampère matrix associated to Φ is everywhere nonsingular:

$$\det\begin{pmatrix} 0 & \partial_y \Phi(x,y) \\ \partial_x \Phi(x,y) & \partial^2_{xy}\Phi(x,y)\end{pmatrix} \neq 0 \quad \text{for all } (x,y) \in \Omega \times \Omega. \tag{4-2}$$

By (4-1), for each $(x,t) \in \Omega \times (-1,1)$ the level set

$$S_{x,t} := \{y \in \Omega : \Phi(x,y) = t\} \tag{4-3}$$

is a smooth hypersurface. The condition (4-2) implies that the smooth family of surfaces in (4-3) satisfies the *rotational curvature condition* of [Phong and Stein 1986] (see also [Stein 1993, Chapter XI]).

The above phase function can be used to construct two natural oscillatory integral operators. To describe these objects, first fix a pair of smooth cut-off functions $a \in C_c^\infty(\Omega \times \Omega)$ and $\rho \in C_c^\infty((-1,1))$.

(i) For each fixed $t \in \mathbb{R}$ the distribution

$$K(x,t;y) := \rho(t)\,a(x,y)\,\delta_0(t - \Phi(x,y)) \tag{4-4}$$

is the kernel of a conormal Fourier integral operator on $\mathbb{R}^n \times \mathbb{R}^n$ of order $-\frac{1}{2}(n-1)$. In particular, K can be written as

$$K(x,t;y) = \int_{\widehat{\mathbb{R}}} e^{i\tau(t-\Phi(x,y))} \rho(t) a(x,y)\, \mathrm{d}\tau,$$

where the right-hand side expression is understood to converge in the sense of oscillatory integrals. From this formula, one can easily deduce (using, for instance, [Sogge 2017, Theorem 0.5.1]) that the canonical relation is given by

$$\mathcal{C} = \{(x,t,-\tau\partial_x\Phi(x,y),\tau,y,\tau\partial_y\Phi(x,y)) : \Phi(x,y)=t)\}. \tag{4-5}$$

Note that the condition (4-2) ensures that each of these Fourier integrals is nondegenerate in the sense that the canonical relation is a canonical graph.

It will be useful to consider the operator

$$\mathcal{F}f(x,t) := \int_{\mathbb{R}^n} K(x,t;y) f(y)\, \mathrm{d}y, \tag{4-6}$$

which is understood to map functions on $\mathbb{R}^n$ to functions on $\mathbb{R}^{n+1}$ by taking averages over the variable hypersurfaces $S_{x,t}$.

(ii) One may also consider the nonhomogeneous oscillatory integral operator

$$S^\lambda_\Phi f(x) := \int_{\mathbb{R}^n} e^{i\lambda\Phi(x,y)} a(x,y) f(y)\, \mathrm{d}y, \tag{4-7}$$

where the amplitude $a \in C^\infty_c(\Omega\times\Omega)$ is as in (4-4) and $\lambda \geq 1$.

Assume, in addition to the condition (4-2), that

$$\rho(t)\delta_0(t-\Phi(x,y)) = \delta_0(t-\Phi(x,y)) \quad \text{for all } (x,y) \in \operatorname{supp} a \text{ and } t \in \mathbb{R}. \tag{4-8}$$

Note that this holds if, for instance, $\Phi(0,0)=0$ and $\rho(t)=1$ for all t in a neighbourhood of 0 provided that a vanishes outside of a small neighbourhood of the origin in $\mathbb{R}^n \times \mathbb{R}^n$. Under these conditions L^p bounds for the operator (4-7) can be related to Sobolev estimates for (4-6).

Proposition 4.1. *Under the conditions* (4-2) *and* (4-8), *if* $\gamma > 0$ *is fixed and* $\lambda \geq 1$, *then*

$$\|S^\lambda_\Phi\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \lesssim \lambda^{-\gamma} \|(I-\Delta_x)^{\gamma/2} \circ \mathcal{F}\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^{n+1})}. \tag{4-9}$$

Proof. Let $\beta \in C^\infty_c(\mathbb{R})$ satisfy $\beta(r)=1$ for $|r|\leq 1$ and $\beta(r)=0$ for $|r|\geq 2$. The condition (4-2) implies that $\partial_x\Phi(x,y)\neq 0$ for all $(x,y)\in\operatorname{supp} a$ and a simple integration-by-parts argument therefore shows that for some small constant $c_0>0$ the estimate

$$\left\|\beta\left(\frac{\sqrt{-\Delta_x}}{c_0\lambda}\right)\circ S^\lambda_\Phi\right\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = O_N(\lambda^{-N})$$

holds for all $N\in\mathbb{N}$. Furthermore, since $\gamma>0$, it also follows that

$$\left\|\left(1-\beta\left(\frac{\sqrt{-\Delta_x}}{c_0\lambda}\right)\right)\circ(I-\Delta_x)^{-\gamma/2}\right\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = O(\lambda^{-\gamma}).$$

Combining these observations,

$$\|S_{\Phi}^{\lambda}\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \lesssim \lambda^{-\gamma}\|(I-\Delta_x)^{\gamma/2}\circ S_{\Phi}^{\lambda}\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} + O_N(\lambda^{-N}). \tag{4-10}$$

On the other hand, the definition of K and the condition (4-8) imply

$$\int e^{i\lambda t}\, K(x,t;y)\,\mathrm{d}t = e^{i\lambda\Phi(x,y)}a(x,y).$$

One may therefore write the operator S_{Φ}^{λ} in terms of K and apply Hölder's inequality together with the estimate (4-10) to deduce the desired result. □

4B. ***Sharpness of the range of exponents $p \geq \bar{p}_n$ for optimal local smoothing bounds for odd n.*** To show that the bounds obtained in Theorem 1.2 are sharp in odd dimensions, in this section certain phase functions Φ are investigated which, in addition to (4-2), satisfy a variant of the Carleson–Sjölin condition [1972].

Note that (4-2) ensures that at each point the rank of the mixed Hessian of Φ is at least $n-1$. Assume that

$$\operatorname{rank} \partial^2_{xy}\Phi(x,y) = n-1 \quad \text{for all } (x,y)\in \operatorname{supp} a. \tag{4-11}$$

It then follows that, provided Ω is sufficiently small, for any fixed x_0 in the x-support of a the map

$$y \to \partial_x\Phi(x_0,y), \quad y\in\Omega,$$

parametrises a hypersurface $\Sigma_{x_0}\subset\mathbb{R}^n$. Suppose, in addition to (4-11), the phase also satisfies the following curvature condition:

For each $x_0\in\Omega$ the surface Σ_{x_0} has $n-1$ nonvanishing principal curvatures at every point. (4-12)

In this case, the phase function Φ is said to satisfy the $n\times n$ *Carleson–Sjölin condition* (see [Sogge 2017]). This definition should be compared with the similar conditions (H1)1 and (H2)2 for the homogeneous oscillatory integrals described in Section 1B (note, for instance, that (4-12) is equivalent to the condition that, for a suitably defined Gauss map G_Φ, the y-Hessian of $\langle \partial_x\Phi(x_0,y), G_\Phi(x_0,y_0)\rangle$ has rank $n-1$ at $y=y_0$ for every $(x_0,y_0)\in\Omega$).

If (4-11) and (4-12) are valid, then it is claimed that the Fourier integral operators $\mathcal{F}$ in (4-6) satisfy the cinematic curvature condition appearing in the hypotheses of Theorem 1.2. If $\mathcal{C}\subset T^*\mathbb{R}^n\setminus 0\times T^*\mathbb{R}^{n+1}\setminus 0$ is the canonical relation for $\mathcal{F}$, then recall that the nondegeneracy condition (3-1) is that $\operatorname{rank}\mathrm{d}\Pi_{T^*\mathbb{R}^n}\equiv 2n$ and $\operatorname{rank}\mathrm{d}\Pi_{\mathbb{R}^{n+1}}\equiv n+1$. This holds as an immediate consequence of (4-2) since, as was observed earlier, (4-2) implies that $\mathcal{C}$ is a local canonical graph. It remains to verify the cone condition (3-2). It immediately follows from the expression (4-5) that for the Fourier integral operators in (4-6) the cones Γ_{x_0,t_0} are given by

$$\Gamma_{x_0,t_0} = \{\tau(-\partial_x\Phi(x_0,y),1) : y\in\Omega, \tau\in\mathbb{R}\}.$$

Consequently, the cone condition holds if (4-11) and (4-12) are satisfied. This verifies the claim.

Recall from the discussion following Proposition 1.3 that for each fixed t the composition

$$(I-\Delta_x)^{\gamma/2}\circ(\mathcal{F}h)(\,\cdot\,,t)$$

is a Fourier integral operator of order $-\frac{1}{2}(n-1)+\gamma$. Thus, a special case of the local smoothing problem is to show that for a given exponent $2n/(n-1)\le p<\infty$ one has

$$\|(I-\Delta_x)^{\gamma/2}\circ\mathcal{F}\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^{n+1})}=O(1)\quad\text{for all } 0<\gamma<\frac{n}{p}. \tag{4-13}$$

Note that, unlike the operators in (4-6), the Fourier integrals in (4-13) do not have kernels with compact x-support; however, they are bounded and rapidly decreasing outside of any neighbourhood of the x-support of a.

Adapting a counterexample of [Bourgain 1991; 1995b], one may construct a phase Φ so that (4-13) cannot hold for $p<\bar{p}_n$ if $n\ge 3$ is odd. This establishes Proposition 1.3 and thereby shows that Theorem 1.2 is optimal in the odd-dimensional case. The details are given presently. Note that, strictly speaking, here a slight simplification of Bourgain's construction is used, which is due to [Stein 1993, Chapter IX, §6.5] (see also [Sogge 2017, pp. 67–69] for further details).

Proof of Proposition 1.3. Consider the matrix-valued function $A\colon\mathbb{R}\to\mathrm{Mat}(2,\mathbb{R})$ defined by

$$A(s):=\begin{pmatrix}1 & s\\ s & s^2\end{pmatrix}\quad\text{for all } s\in\mathbb{R}.$$

Let $n\ge 3$ be odd and $\boldsymbol{A}\colon\mathbb{R}\to\mathrm{Mat}(n-1,\mathbb{R})$ be given by

$$\boldsymbol{A}(s):=\underbrace{A(s)\oplus\cdots\oplus A(s)}_{(n-1)/2\text{-fold}}$$

so that $\boldsymbol{A}(s)$ is an $(n-1)\times(n-1)$ block-diagonal matrix. Using these matrices, define a phase function ϕ on $\mathbb{R}^n\times\mathbb{R}^{n-1}$ by

$$\phi(x,y'):=\langle x',y'\rangle+\tfrac{1}{2}\langle\boldsymbol{A}(x_n)y',y'\rangle \tag{4-14}$$

for all $(x,y')=(x',x_n,y')\in\mathbb{R}^n\times\mathbb{R}^{n-1}$. Given an amplitude function $b\in C_c^\infty(\mathbb{R}^n\times\mathbb{R}^{n-1})$ define the oscillatory integral operator

$$S_\phi^\lambda f(x):=\int_{\mathbb{R}^{n-1}}e^{i\lambda\phi(x,y')}\,b(x,y')\,f(y')\,\mathrm{d}y'.$$

A stationary phase argument (see, for instance, [Sogge 2017, pp. 68–69]) then yields

$$\lambda^{-(n-1)/4-(n-1)/(2p)}\lesssim\|S_\phi^\lambda\|_{L^p(\mathbb{R}^{n-1})\to L^p(\mathbb{R}^n)}\quad\text{if } \lambda\ge 1 \text{ and } p\ge 2, \tag{4-15}$$

provided that $b(0,0)\ne 0$.

If ϕ is as in (4-14) and

$$\Phi(x,y):=\phi(x,y')+x_n+y_n, \tag{4-16}$$

then clearly (4-2) is valid when $x = y = 0$. Since

$$y \to \partial_x \Phi(0, y) = \Big(y', \sum_{j=0}^{(n-3)/2} y_{2j+1} y_{2j+2}\Big) + e_n$$

parametrises a hyperbolic paraboloid with $\frac{1}{2}(n-1)$ positive principal curvatures and $\frac{1}{2}(n-1)$ negative principal curvatures, one concludes that for small x the Carleson–Sjölin conditions (4-11) and (4-12) must hold, provided the support of b lies in a suitably small ball about the origin.

Suppose β is as in the proof of Proposition 4.1, so that $\beta \in C_c^\infty(\mathbb{R})$ satisfies $\beta(r) = 1$ whenever $|r| \le 1$ and $\beta(r) = 0$ whenever $|r| \ge 2$. Assume $b \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^{n-1})$ satisfies $b(0, 0) \ne 0$ and is supported in a small neighbourhood of the origin. Take a in (4-7) to be equal to

$$a(x, y) = b(x, y')\beta\Big(\frac{y_n}{c_0}\Big)$$

for some suitable choice of small constant $0 < c_0 < \frac{1}{2}$. Provided the size of the support of b and the constant c_0 are suitably chosen, (4-8) holds. Furthermore, taking $F(y) := \beta(y_n)e^{-i\lambda y_n} f(y')$ in (4-7), one readily observes that

$$|S_\phi^\lambda f(x)| \sim |S_\Phi^\lambda F(x)| \quad \text{and} \quad \|F\|_{L^p(\mathbb{R}^n)} \sim \|f\|_{L^p(\mathbb{R}^{n-1})}$$

and, consequently,

$$\|S_\phi^\lambda\|_{L^p(\mathbb{R}^{n-1}) \to L^p(\mathbb{R}^n)} \lesssim \|S_\Phi^\lambda\|_{L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)}.$$

Combining this with (4-15) and (4-9), for $\gamma > 0$ and $\lambda \ge 1$ one concludes that

$$\lambda^{\gamma-(n-1)/4-(n-1)/(2p)} \lesssim \|(I - \Delta_x)^{\gamma/2} \circ \mathcal{F}\|_{L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^{n+1})},$$

where $\mathcal{F}$ is as in (4-6). Since

$$\frac{n}{p} - \frac{n-1}{4} - \frac{n-1}{2p} > 0 \quad \text{if } p < \bar{p}_n,$$

it follows that (4-13) cannot hold for any Lebesgue exponent satisfying $p < \bar{p}_n$. □

For even dimensions $n \ge 4$ one may modify the argument given in the proof of Proposition 1.3 to give a necessary condition for the local smoothing problem for the general class of Fourier integral operators under consideration. Indeed, in the even-dimensional case one simply defines

$$\boldsymbol{A}(s) := \underbrace{A(s) \oplus \cdots \oplus A(s)}_{(n-2)/2\text{-fold}} \oplus (1 + s),$$

where $(1+s)$ is a 1×1 matrix with entry $1+s$, so that once again $\boldsymbol{A}(s)$ is an $(n-1)\times(n-1)$ block-diagonal matrix. Taking the phase function ϕ as in (4-14), it follows that the resulting oscillatory integral operators satisfy

$$\lambda^{-n/4-(n-2)/(2p)} \lesssim \|S_\phi^\lambda\|_{L^\infty(\mathbb{R}^{n-1}) \to L^p(\mathbb{R}^n)}.$$

	n odd	n even
$n-1$ nonvanishing curvatures	$\frac{2(n+1)}{n-1}$	$\frac{2(n+2)}{n}$
$n-1$ positive curvatures	$\frac{2(3n+1)}{3n-3}$	$\frac{2(3n+2)}{3n-2}$

Table 1. Conjectured endpoint values for the exponent p for the sharp local smoothing estimates in Theorem 1.2 under various hypothesis on $\mathcal{F} \in I^{\mu-1/4}$. Theorem 1.2 establishes the odd-dimensional case under the hypothesis of $n-1$ nonvanishing principal curvatures.

See, for instance, [Sogge 2017, p. 69] for further details. Arguing mutatis mutandis, for even $n \geq 4$ and $\mathcal{F}$ defined as in the proof of Proposition 1.3 (with respect to the new choice of phase ϕ) the estimate (4-13) fails for $p < 2(n+2)/n$.

4C. ***Some open problems.*** The cones $\Gamma_{x_0,t_0} \subset T^*_{x_0,t_0}\mathbb{R}^{n+1}$ associated to the phase in (4-16) have principal curvatures of opposite sign (in fact, in the examples considered above the difference between the number of positive and the number of negative principal curvatures is minimal). It would be interesting to see if any improvement is possible in the range of p for which there is optimal local smoothing if the Γ_{x_0,t_0} always have $n-1$ positive principal curvatures. The model case for this is the class of Fourier integrals arising in the context of Theorem 1.1, that is, from solutions of wave equations given by a Laplace–Beltrami operator on some Riemannian manifold (M, g). In this case $\Phi(x, y)$ is given by the associated Riemannian distance function $d_g(x, y)$ minus a constant. By Proposition 4.1 and the counterexamples of [Minicozzi and Sogge 1997] (see also [Sogge, Xi, and Xu 2018]), there exist metrics for which optimal local smoothing is not possible when $p < \bar{p}_{n,+}$ where

$$\bar{p}_{n,+} := \begin{cases} 2(3n+1)/(3n-3) & \text{if } n \text{ is odd,} \\ 2(3n+2)/(3n-2) & \text{if } n \text{ is even.} \end{cases}$$

On the other hand, if $\Phi(x, y) := d_g(x, y)$, then recent results of Guth, Iliopoulou and the second author [Guth, Hickman, and Iliopoulou 2019] yield the optimal bounds for $p \geq \bar{p}_{n,+}$ for the oscillatory operators in (4-7); this suggests that one should be able to obtain optimal local smoothing bounds for $p \geq \bar{p}_{n,+}$ under the above convexity assumptions. In Table 1 the conjectured numerology for sharp local smoothing estimates for Fourier integral operators is tabulated, according to the parity of the dimension and various curvature assumptions. As mentioned in the Introduction, for the euclidean wave equation sharp local smoothing estimates are conjectured to hold for the wider range $2n/(n-1) \leq p < \infty$.

Finally, note that the conjectured numerology in Table 1 coincides with the sharp bounds to a problem posed in [Hörmander 1973] for oscillatory integral operators of the type T^λ under nonhomogeneous versions of the conditions (H1)1 and (H2)2 (and a corresponding positive-definite version of (H2)2); see [Guth, Hickman, and Iliopoulou 2019] for the details of this problem and a full historical account. In particular, the argument presented earlier in this section shows that Theorem 1.1 implies a theorem of [Stein 1986] in this context. For the remaining cases, the results of [Bourgain 1991; 1995b; Wisewell

2005; Bourgain and Guth 2011; Guth, Hickman, and Iliopoulou 2019] suggest the $p \geq 2(n+2)/n$ numerology in the general even-dimensional case and reinforce the conjectured $p \geq \bar{p}_{n,+}$ numerology in the convex case.

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ON THE HÖLDER CONTINUOUS SUBSOLUTION PROBLEM FOR THE COMPLEX MONGE–AMPÈRE EQUATION, II

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We solve the Dirichlet problem for the complex Monge–Ampère equation on a strictly pseudoconvex domain with the right-hand side being a positive Borel measure which is dominated by the Monge–Ampère measure of a Hölder continuous plurisubharmonic function. If the boundary data is continuous, then the solution is continuous. If the boundary data is Hölder continuous, then the solution is also Hölder continuous. In particular, the answer to a question of A. Zeriahi is always affirmative.

1. Introduction

The Hölder regularity of plurisubharmonic solutions to the complex Monge–Ampère equation in a strictly pseudoconvex domain has a long history. First, Bedford and Taylor [1976] obtained Hölder continuous solutions for the Dirichlet problem of the equation assuming the right-hand side is Hölder continuous. Later, this result was extended to a larger class of measures by Guedj, Kołodziej and Zeriahi [Guedj et al. 2008]; namely, the measures have L^p density with respect to the Lebesgue measure with some extra assumptions on the density near the boundary and the boundary data. The extra assumptions are removed in other subsequent works [Baracco et al. 2016; Charabati 2015]. On the other hand, the complex Monge–Ampère operator of a Hölder continuous plurisubharmonic function is not necessary absolutely continuous with respect to the Lebesgue measure. Examples of such measures are Hausdorff measures due to Charabati [2017], and the volume form of a smooth real hypersurface of codimension 1 by Pham [2010]. (See also [Vu 2018] for a generalization to generic CR manifolds of arbitrary codimension.) So far, these results give only sufficient conditions on the measures such that the solution to the equation is Hölder continuous. In [Nguyen 2018] we gave a necessary and sufficient condition for a measure whose Monge–Ampère potential is Hölder continuous. This result is partly inspired by a global result due to Dinh and Nguyên [2014]. However, to use the result in [Nguyen 2018] we require the Hölder continuous subsolution have zero value on the boundary and its total Monge–Ampère mass be finite, which cannot be true for a general Hölder continuous plurisubharmonic function (Remark 1.1). In this paper we will remove these restrictions.

There are several motivations to study the Hölder regularity of solutions. First, it is a basic question in pluripotential theory to characterize measures for which the complex Monge–Ampère equation admits bounded, continuous and Hölder continuous solutions [Kołodziej 2013]. Next, Dinh, Nguyên and Sibony [Dinh et al. 2010] showed that the Monge–Ampère measure of a Hölder continuous plurisubharmonic

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function is locally moderate, which is a very useful generalization of Skoda's theorem. We refer the reader to [Dinh et al. 2010] for its application in complex dynamics. Their work leads to the interesting open problem of whether the converse holds. The question in the toric setting has been studied recently in [Coman et al. 2018, Theorem 4.4]. In this case the problem reduces to a real Monge–Ampère equation on a convex polytope. Hölder continuity is also studied with regard to the extremal functions arising in (pluri-)potential theory. In fact the Hölder continuity of the so-called relative extremal function u_K and the Siciak–Zahariuta extremal function V_K [Siciak 1997; 2000] of a compact set $K \subset \mathbb{C}^n$ is proven to be equivalent to a Markov-type inequality in multivariate interpolation theory [Baran and Bialas-Ciez 2014; Pawłucki and Pleśniak 1986] (see [Dinh et al. 2017; Vu 2018] for analogous results in the compact Kähler manifold setting). We believe that our work will be useful to study the Hölder continuity of the above extremal functions. For example we hope the techniques developed here can be used to simplify the proof in [Vu 2018].

In this paper we continue our research, initiated in [Nguyen 2018], which focuses on the Dirichlet problem for the complex Monge–Ampère equation in a bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$, provided a Hölder continuous subsolution exists. Let $\varphi \in \mathrm{PSH}(\Omega) \cap C^{0,\alpha}(\bar{\Omega})$ for some $0 < \alpha \le 1$. Assume also that

$$\varphi = 0 \quad \text{on } \partial\Omega.$$

We consider the set

$$\mathcal{M}(\varphi, \Omega) := \{\mu \text{ is positive Borel measure} : \mu \le (dd^c\varphi)^n \text{ in } \Omega\}.$$

We also say that φ is a Hölder continuous subsolution to measures in $\mathcal{M}(\varphi, \Omega)$. Given ψ a Hölder continuous function on the boundary $\partial\Omega$ and a measure μ in $\mathcal{M}(\varphi, \Omega)$ we look for a real-valued function u satisfying

$$u \in \mathrm{PSH} \cap L^\infty(\Omega), \qquad (dd^c u)^n = \mu \quad \text{in } \Omega, \qquad \lim_{z \to x} u(z) = \psi(x) \quad \text{for } x \in \partial\Omega, \tag{1-1}$$

and

$$u \in C^{0,\alpha'}(\bar{\Omega}) \quad \text{for some } 0 < \alpha' \le 1. \tag{1-2}$$

The Dirichlet problem (1-1) was solved by Kołodziej [1995] provided that there exists a bounded plurisubharmonic subsolution. In our setting, the Hölder continuity of ψ on $\partial\Omega$ and of φ on $\bar{\Omega}$ are necessary in order to solve the Dirichlet problem (1-1), (1-2). In [Nguyen 2018] this problem is solved under the extra assumptions

$$\psi \equiv 0 \quad \text{and} \quad \int_\Omega (dd^c\varphi)^n < +\infty.$$

We will see now that these assumptions are not generic.

Remark 1.1. Let ρ be a defining function for a smoothly bounded pseudoconvex domain Ω. Then, $-|\rho|^\alpha$ for $0 < \alpha < 1$ is Hölder continuous on $\bar{\Omega}$ and its Monge–Ampère measure is

$$\alpha^n |\rho|^{n(\alpha-1)} (dd^c\rho)^n + n\alpha^n (1-\alpha) |\rho|^{n(\alpha-1)-1} d\rho \wedge d^c\rho \wedge (dd^c\rho)^{n-1}.$$

Therefore, for a neighborhood $U_x \subset \mathbb{C}^n$ of a strictly pseudoconvex point $x \in \partial\Omega$,

$$\int_{U_x \cap \Omega} [dd^c(-|\rho|^\alpha)]^n = +\infty.$$

In particular, $\int_\Omega [dd^c(-|\rho|^\alpha)]^n = +\infty$. Furthermore, let v be a plurisubharmonic function on Ω and Hölder continuous on $\bar{\Omega}$ satisfying

$$\int_\Omega (dd^c v)^n < +\infty.$$

A typical way to modify the value of v on $\partial\Omega$ is to add it to an envelope

$$h(z) = \sup\{w(z) : w \in \mathrm{PSH}(\Omega) \cap C^0(\bar{\Omega}) : w_{|\partial\Omega} \leq -v\}. \tag{1-3}$$

However, we cannot guarantee that the function $v + h$ has finite Monge–Ampère mass on Ω. Thus, removing the above assumptions is desirable for applications.

The first main result of this paper is as follows.

Theorem A. *Let $\psi \in C^0(\partial\Omega)$ and $\mu \in \mathcal{M}(\varphi, \Omega)$. Then, there exists a unique solution $u \in C^0(\bar{\Omega})$ to the Dirichlet problem* (1-1).

This theorem is closely related to a question of S. Kołodziej [Dinew et al. 2016, Question 14], where he asked if one could prove Theorem A when the subsolution φ is only *continuous*? The question is still open in general. Very recently in [Kołodziej and Nguyen 2018b] we showed that Theorem A still holds true for the subsolution φ satisfying a Dini-type continuity condition.

The next result gives a necessary and sufficient condition under which a positive Borel measure admits a Hölder continuous plurisubharmonic potential. In particular, the answer to a question of A. Zeriahi [Dinew et al. 2016, Question 17] is affirmative.

Theorem B. *Assume that ψ is Hölder continuous and $\mu \in \mathcal{M}(\varphi, \Omega)$. Then, the Dirichlet problem* (1-1), (1-2) *is solvable.*

This theorem can be seen as the local version of [Demailly et al. 2014], where the compact Kähler manifold setting was considered (see also [Kołodziej and Nguyen 2018a] for the Hermitian manifold case and the notion of subsolution in the compact manifold setting there). Now, we can say that the complex Monge–Ampère equation on a compact Kähler (Hermitian) manifold admits a Hölder continuous solution if and only if it can be written locally as Monge–Ampère operators of Hölder continuous plurisubharmonic functions. The result has been also generalized to the complex Hessian equation [Kołodziej and Nguyen 2019]. Given Hölder continuous plurisubharmonic functions $u_1, \dots, u_n$ in Ω, it follows by the theorem that there exists a Hölder continuous plurisubharmonic function such that

$$(dd^c u)^n = dd^c u_1 \wedge \cdots \wedge dd^c u_n.$$

We also obtain the convexity of the set of Monge–Ampère measures of Hölder continuous plurisubharmonic functions in Ω. Another important consequence is the so-called L^p property given in Corollary C below. In particular, our result covers the main findings in [Baracco et al. 2016, Charabati 2015; 2017], where the L^p property with respect to the Lebesgue measure was considered.

Corollary C. *Let $\mu \in \mathcal{M}(\varphi, \Omega)$ and $f \in L^p(\Omega, d\mu)$, $p > 1$, a nonnegative function. Suppose that φ is a Hölder continuous plurisubharmonic function on a neighborhood of $\bar{\Omega}$. Then, $f\mu \in \mathcal{M}(\tilde{\varphi}, \Omega)$ for a Hölder continuous plurisubharmonic function $\tilde{\varphi}$ in Ω.*

2. Preliminaries

In this section we will recall results that are needed in the proofs of Theorems A and B and Corollary C. If there is no other indication, then the notation in this section will be used for the rest of the paper.

Let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$. Let $\rho \in C^2(\bar{\Omega})$ be a strictly plurisubharmonic defining function for Ω. Namely,

$$\Omega = \{\rho < 0\} \qquad \text{and} \qquad d\rho \neq 0 \quad \text{on } \partial\Omega. \tag{2-1}$$

Let us denote by $\beta = dd^c|z|^2$ the standard Kähler form in $\mathbb{C}^n$. Without loss of generality we may assume

$$dd^c\rho \geq \beta \quad \text{on } \bar{\Omega}. \tag{2-2}$$

Throughout the paper the Hölder continuous subsolution φ and the associated set of measures $\mathcal{M}(\varphi, \Omega)$ are defined as in the Introduction.

The following estimate will be very useful for us. For simplicity we write

$$\|\cdot\|_\infty := \sup_\Omega |\cdot| \quad \text{and} \quad \|\cdot\|_p := \int_\Omega |\cdot|^p \, dV_{2n}, \tag{2-3}$$

for the Lebesgue L^p-norm for $p \geq 1$.

Lemma 2.1 [Błocki 1993]. *Let $v_1, \dots, v_n, v, h \in \mathrm{PSH} \cap L^\infty(\Omega)$ be such that $v_i \leq 0$ for $i = 1, \dots, n$, and $v \leq h$. Assume that $\lim_{z \to \partial\Omega}[h(z) - v(z)] = 0$. Then, for an integer $1 \leq k \leq n$,*

$$\int_\Omega (h - v)^k \, dd^c v_1 \wedge \cdots \wedge dd^c v_n \leq k! \, \|v_1\|_\infty \cdots \|v_k\|_\infty \int_\Omega (dd^c v)^k \wedge dd^c v_{k+1} \wedge \cdots \wedge dd^c v_n. \tag{2-4}$$

Consider also the Cegrell class

$$\mathcal{E}_0 = \left\{ v \in \mathrm{PSH} \cap L^\infty(\Omega) \;\middle|\; \lim_{x \to z} v(x) = 0 \text{ for all } z \in \partial\Omega \text{ and } \int_\Omega (dd^c v)^n < +\infty \right\}. \tag{2-5}$$

The Cegrell inequality in this class reads:

Lemma 2.2 [Cegrell 2004]. *Let $v_1, \dots, v_n \in \mathcal{E}_0$. Then,*

$$\int_\Omega dd^c v_1 \wedge \cdots \wedge dd^c v_n \leq \left(\int_\Omega (dd^c v_1)^n \right)^{1/n} \cdots \left(\int_\Omega (dd^c v_n)^n \right)^{1/n}. \tag{2-6}$$

We need also to work with a subclass of the Cegrell class:

$$\mathcal{E}_0' := \left\{ v \in \mathcal{E}_0 : \int_\Omega (dd^c v)^n \leq 1 \right\}. \tag{2-7}$$

The decay of the volume of sublevel sets of functions in the class $\mathcal{E}_0'$ is equivalent to the volume-capacity inequality. This inequality plays a crucial role in the capacity method due to Kołodziej to obtain the *a priori and stability estimates* for weak solutions of the complex Monge–Ampère equation. Here the capacity is the Bedford–Taylor capacity and it is defined as follows. For a Borel set $E \subset \Omega$,

$$\operatorname{cap}(E, \Omega) := \sup\left\{ \int_E (dd^c w)^n : w \in \operatorname{PSH}(\Omega),\ 0 \le w \le 1 \right\}. \tag{2-8}$$

In what follows we shall write $\operatorname{cap}(E)$ instead of $\operatorname{cap}(E, \Omega)$ for simplicity as the domain Ω is already fixed.

3. Proof of Theorem A

In this section we shall prove the following result.

Proposition 3.1. *Assume that $\mu \in \mathcal{M}(\varphi, \Omega)$. Then, there exist uniform constants $\alpha_0, C > 0$ depending only on φ, Ω such that, for every compact set $K \subset \Omega$,*

$$\mu(K) \le C \operatorname{cap}(K) \exp\left(\frac{-\alpha_0}{[\operatorname{cap}(K)]^{1/n}} \right). \tag{3-1}$$

Notice that under the assumption $\int_\Omega (dd^c\varphi)^n < +\infty$ a similar inequality, without the factor $\operatorname{cap}(K)$ on the right-hand side, was proven in [Nguyen 2018].

Remark 3.2. Theorem A will follow immediately from the proposition and [Kołodziej 2005, Theorem 5.9] as μ belongs to the class $\mathcal{F}(A, h)$ with $h = e^{\alpha_0 x}$ and a uniform $A > 0$.

We will need the following two lemmas. The first one tells us how fast the Monge–Ampère mass of $(dd^c\varphi)^n$ on large sublevel sets goes to infinity.

Lemma 3.3. *Let $v \in \mathcal{E}_0'$. Then, there exists a uniform constant C such that, for $s > 0$,*

$$\int_{\{v<-s\}} (dd^c\varphi)^n \le \frac{C\|\varphi\|_\infty^n}{s^n}. \tag{3-2}$$

This estimate should be compared with [Kołodziej 2005, Lemma 4.1]. If φ is a C^2-smooth function on $\overline{\Omega}$, then exponential decay as s tends to $+\infty$ has been obtained there. Although in our case, we are more interested in the situation when s tends to 0.

Proof. Set $v_s := \max\{v, -s\}$. Then, $v_s = v$ on a neighborhood of $\partial\Omega$. Moreover,

$$v_{s/2} - v \ge \frac{s}{2} \quad \text{on } \{v < -s\} \Subset \Omega. \tag{3-3}$$

Therefore,

$$\int_{\{v<-s\}} (dd^c\varphi)^n \le \left(\frac{2}{s}\right)^n \int_\Omega (v_{s/2} - v)^n (dd^c\varphi)^n \le \frac{2^n n!}{s^n} \|\varphi\|_\infty^n \int_\Omega (dd^c v)^n, \tag{3-4}$$

where the second inequality follows from Lemma 2.1. □

On the other hand the volume with respect to the measure $(dd^c\varphi)^n$ of small sublevel sets of functions in $\mathcal{E}_0'$ decays exponentially fast to zero. The Hölder continuity of φ is crucially important to prove such an estimate.

Lemma 3.4. *There exist uniform constants $\tau > 0$ and $C > 0$ such that, for $v \in \mathcal{E}_0'$ and $s \geq 2$,*

$$\int_{\{v<-s\}} (dd^c\varphi)^n \leq Ce^{-\tau s}. \tag{3-5}$$

Proof. We follow ideas of Dinh, Nguyên and Sibony [Dinh et al. 2010]. With the same notation as in the proof of Lemma 3.3 we have for $s \geq 2$

$$\int_{\{v<-s\}} (dd^c\varphi)^n \leq \frac{2}{s}\int_\Omega (v_{s/2}-v)(dd^c\varphi)^n \leq \int_\Omega (v_{s/2}-v)(dd^c\varphi)^n. \tag{3-6}$$

Let us define

$$S_k := (dd^c\varphi)^k \wedge \beta^{n-k},$$

where $\beta = dd^c|z|^2$ and $0 \leq k \leq n$ is an integer. Our first goal is to show that there exist $\alpha_k > 0$ and $C > 0$ (independent of v and s) such that, for $v \in \mathcal{E}_0'$ and $s \geq 1$,

$$\int_\Omega (v_s - v)S_k \leq C\biggl(\int_\Omega (v_s - v)\,dV_{2n}\biggr)^{\alpha_k}, \tag{3-7}$$

where $v_s = \max\{v, -s\}$. Indeed, without loss of generality we may assume that

$$0 < \|v_s - v\|_1 < \tfrac{1}{100}. \tag{3-8}$$

Otherwise, if $\|v_s - v\|_1 = 0$, then the inequality trivially holds. If $\|v_s - v\|_1 \geq \frac{1}{100}$, then we have, using $s \geq 1$, $v \leq 0$, and Lemma 2.1, that

$$\int_\Omega (v_s - v)S_k = \int_{\{v<-s\}} (-s-v)S_k \leq \int_\Omega (-v)^k S_k \leq C\|\varphi\|_\infty^k. \tag{3-9}$$

This implies the inequality.

Next, under the assumption (3-8) we prove the inequality by induction in k. The case $k = 0$ is obvious. Assume that for every integer $m \leq k$ we have

$$\int_\Omega (v_s - v)S_m \leq C\biggl(\int_\Omega (v_s - v)\,dV_{2n}\biggr)^{\alpha_m}. \tag{3-10}$$

Then, we need to show that there exists $0 < \alpha_{k+1} \leq 1$ such that

$$\int_\Omega (v_s - v)S_{k+1} \leq C\biggl(\int_\Omega (v_s - v)\,dV_{2n}\biggr)^{\alpha_{k+1}}. \tag{3-11}$$

For simplicity we write

$$S := (dd^c\varphi)^k \wedge \beta^{n-k-1}. \tag{3-12}$$

Let us still write φ to be a Hölder continuous extension of φ onto a neighborhood U of $\overline{\Omega}$. Consider the convolution of φ with the standard smoothing kernel χ, i.e., $\chi \in C_c^\infty(\mathbb{C}^n)$ a radial function such that $\chi(z) \geq 0$, $\operatorname{supp}\chi \Subset B(0,1)$ and $\int_{\mathbb{C}^n}\chi(z)\,dV_{2n} = 1$. Namely, for $z \in U$ and $\delta > 0$ small,

$$\varphi * \chi_t(z) = \int_{B(0,1)} \varphi(z - tz')\chi(z')\,dV_{2n}(z') = \frac{1}{t^{2n}}\int_{B(z,t)} \varphi(z')\chi\biggl(\frac{z-z'}{t}\biggr)\,dV_{2n}(z'). \tag{3-13}$$

Observe that

$$|\varphi * \chi_t(z) - \varphi(z)| \le \int_{B(0,1)} |\varphi(z - tz') - \varphi(z)|\chi(z')\, dV_{2n}(z') \le Ct^{\alpha} \tag{3-14}$$

and

$$\left|\frac{\partial^2 \varphi * \chi_t}{\partial z_j \partial \bar{z}_k}(z)\right| \le \frac{C\|\varphi\|_\infty}{t^2}. \tag{3-15}$$

We first have

$$\begin{aligned}\int_\Omega (v_s - v)\, dd^c\varphi \wedge S &\le \left|\int_\Omega (v_s - v)\, dd^c\varphi * \chi_t \wedge S\right| + \left|\int_\Omega (v_s - v)\, dd^c(\varphi * \chi_t - \varphi) \wedge S\right| \\ &=: I_1 + I_2. \end{aligned} \tag{3-16}$$

It follows from (3-15) that

$$I_1 \le \frac{C\|\varphi\|_\infty}{t^2}\int_\Omega (v_s - v) S \wedge \beta = \frac{C\|\varphi\|_\infty}{t^2}\int_\Omega (v_s - v) S_k. \tag{3-17}$$

Hence,

$$I_1 \le \frac{C\|\varphi\|_\infty}{t^2}\|v_s - v\|_1^{\alpha_k}. \tag{3-18}$$

We turn to the estimate of the second integral I_2. By integration by parts

$$\begin{aligned}\int_\Omega (v_s - v) dd^c(\varphi * \chi_t - \varphi) \wedge S &= \int_\Omega (\varphi * \chi_t - \varphi) dd^c(v_s - v) \wedge S \\ &= \int_{\{v < -s/2\}} (\varphi * \chi_t - \varphi)\, dd^c(v_s - v) \wedge S, \end{aligned} \tag{3-19}$$

as $v_s = v$ on $\{v \ge -s\}$. Hence,

$$I_2 \le \int_{\{v<-s/2\}} |\varphi * \chi_t - \varphi|(dd^c v + dd^c v_s) \wedge S \le Ct^{\alpha}\int_{\{v<-s/2\}} (dd^c v + dd^c v_s) \wedge S. \tag{3-20}$$

For the first term of the integral on the right-hand side we have

$$\begin{aligned}\int_{\{v<-s/2\}} dd^c v \wedge S &\le \left(\frac{4}{s}\right)^k \int_{\{v<-s/4\}} (v_{s/4} - v)^k\, dd^c v \wedge S \\ &\le \frac{C}{s^k}\int_\Omega (v_{s/4} - v)^k\, dd^c v \wedge (dd^c\varphi)^k \wedge \beta^{n-k-1}. \end{aligned} \tag{3-21}$$

Applying Lemma 2.1 we conclude that

$$\int_\Omega (v_{s/4} - v)^k\, dd^c v \wedge (dd^c\varphi)^k \wedge \beta^{n-k-1} \le C\|\varphi\|_\infty^k \int_\Omega (dd^c v)^{k+1} \wedge \beta^{n-k-1}. \tag{3-22}$$

Using $dd^c\rho \ge \beta$ (see Section 2) and Cegrell's inequality we get

$$\begin{aligned}\int_\Omega (dd^c v)^{k+1} \wedge \beta^{n-k-1} &\le \int_\Omega (dd^c v)^{k+1} \wedge (dd^c\rho)^{n-k-1} \\ &\le \left(\int_\Omega (dd^c v)^n\right)^{(k+1)/n} \left(\int_\Omega (dd^c\rho)^n\right)^{(n-k-1)/n}. \end{aligned} \tag{3-23}$$

Combining (3-21), (3-22) and (3-23) we have, for $s \geq 1$,

$$\int_{\{v<-s/2\}} dd^c v \wedge S \leq C\|\varphi\|_\infty^k. \tag{3-24}$$

Notice that $v_s \in \mathcal{E}_0'$. The same arguments as above imply that for $s \geq 1$

$$\int_{\{v<-s/2\}} dd^c v_s \wedge S \leq C\|\varphi\|_\infty^k. \tag{3-25}$$

Thus, altogether we have

$$I_1 + I_2 \leq \frac{C\|\varphi\|_\infty}{t^2}\|v_s - v\|_1^{\alpha_k} + C\|\varphi\|_\infty^k t^\alpha. \tag{3-26}$$

If we choose

$$t = \|v_s - v\|_1^{\alpha_k/3}, \quad \alpha_{k+1} = \frac{\alpha\alpha_k}{3}, \tag{3-27}$$

then the proof of (3-7) is completed.

We now conclude the proof of the lemma. It follows from [Nguyen 2018, equation (2.26)] and [Kołodziej 2005, Lemma 4.1] that

$$\int_\Omega (v_s - v)\, dV_{2n} \leq Ce^{-\tau_0 s},$$

where $\tau_0 > 0$ and $C > 0$ are uniform constants independent of v and s. Combining this with (3-6) and the inequality (3-7) for $k = n$ the lemma follows. □

Remark. The referee has suggested that the exponent 2 in the denominator of (3-15) can be improved. Thus, the inequality (3-26) can be improved too, so the final choice of α_n will be better.

We are ready to prove the main result of this section.

Proof of Proposition 3.1. Let us define $\nu := (dd^c\varphi)^n$. First, we show that for $v \in \mathcal{E}_0'$ there exist uniform constants $\alpha_1, C > 0$ such that

$$\nu(v < -s) \leq \frac{Ce^{-\alpha_1 s}}{s^n} \quad \text{for all } s > 0. \tag{3-28}$$

Indeed, there are two possibilities: either $s \geq 2$ or $s < 2$. If $s \geq 2$, then the inequality follows from Lemma 3.4 as

$$s^n e^{-\tau s/2} \leq \left(\frac{2n}{\tau}\right)^n e^{-n}.$$

(We can take $\alpha_1 = \tau/2$). Otherwise, if $0 < s < 2$, we have $e^{-\alpha_1 s} \geq C$. Then, the desired inequality follows from Lemma 3.3.

To complete the proof of the proposition we use an argument which is inspired by the proofs in [Åhag et al. 2009]. Let $K \subset \Omega$ be compact. Since ν is dominated by a Monge–Ampère measure of a bounded plurisubharmonic function, it vanishes on pluripolar sets. Hence, we may assume that K is nonpluripolar.

Let h_K^* be the relative extremal function of K with respect to Ω. Since $K \subset \Omega$ is compact, it is well known that

$$\lim_{\zeta \to \partial\Omega} h_K^*(\zeta) = 0.$$

By [Bedford and Taylor 1982, Proposition 5.3] we have

$$\tau^n := \mathrm{cap}(K, \Omega) = \int_\Omega (dd^c h_K^*)^n > 0.$$

Let $0 < x < 1$. Since the function $w := h_K^*/\tau$ satisfies assumptions of the inequality (3-28), we have

$$\nu(h_K^* < -1 + x) = \nu\left(w < \frac{-1+x}{\tau}\right) \le C \frac{\tau^n}{\alpha_1^n (1-x)^n} \exp\left(-\frac{\alpha_1(1-x)}{\tau}\right).$$

Letting $x \to 0^+$, we obtain

$$\nu(h_K^* \le -1) \le \frac{C}{\alpha_1^n} \mathrm{cap}(K, \Omega) \exp\left(\frac{-\alpha_1}{[\mathrm{cap}(K, \Omega)]^{1/n}}\right). \tag{3-29}$$

Since $h_K = h_K^*$ outside a pluripolar set, we have

$$\nu(K) \le \nu(h_K = -1) = \nu(h_K^* = -1) \le \nu(h_K^* \le -1). \tag{3-30}$$

We combine (3-29) and (3-30) to finish the proof. □

4. Proof of Theorem B

In this section we will prove the Hölder continuity of the solution obtained in Theorem A provided furthermore that the boundary data ψ is Hölder continuous. Notice that the zero boundary values of the subsolution φ are not essential. We can modify them by adding an appropriate envelope, similar to (4-5), because no condition has been imposed on the total mass of the subsolution.

By Theorem A there exists a unique continuous solution to the Dirichlet problem (1-1), namely, $u \in \mathrm{PSH}(\Omega) \cap C^0(\bar{\Omega})$ solving

$$(dd^c u)^n = \mu, \qquad u(z) = \psi(z) \quad \text{for all } z \in \partial\Omega. \tag{4-1}$$

We are going to show that $u \in C^{0,\alpha'}(\bar{\Omega})$ for some exponent $0 < \alpha' \le 1$.

Outline of the proof. Let us sketch the proof of Theorem B. Overall we follow the steps in the proof of [Nguyen 2018], which in turns followed [Guedj et al. 2008]. Though, we need to consider the problem on an increasing exhaustive sequence of relatively compact domains in Ω. Define for $\delta > 0$ small

$$\Omega_\delta := \{z \in \Omega : \mathrm{dist}(z, \partial\Omega) > \delta\}, \tag{4-2}$$

and for $z \in \Omega_\delta$ we define

$$u_\delta(z) := \sup_{|\zeta| \le \delta} u(z+\zeta), \quad \hat{u}_\delta(z) := \frac{1}{\sigma_{2n}\delta^{2n}} \int_{|\zeta|\le\delta} u(z+\zeta)\, dV_{2n}(\zeta), \tag{4-3}$$

where σ_{2n} is the volume of the unit ball.

Then, we wish to show that

$$\sup_{\Omega_\delta}(\hat{u}_\delta - u) \lesssim \delta^{\varpi}$$

for some $0 < \varpi \le 1$, where $\lesssim$ means that the inequality holds up to an absolute constant. Thanks to the Hölder continuity of the boundary data we can extend $\hat{u}_\delta$ to $\tilde{u}$ by a gluing process such that the new function is plurisubharmonic on Ω and equal to u outside Ω_ε for some (small) $\varepsilon > \delta$. Moreover, we shall still have

$$\sup_{\Omega_\delta}(\hat{u}_\delta - u) \le \sup_{\Omega}(\tilde{u} - u) + C\varepsilon^\alpha,$$

where α is the Hölder exponent of the boundary data ψ. Next, we shall show that

$$\int_{\Omega_\varepsilon} (dd^c\varphi)^n \lesssim \frac{1}{\varepsilon^n}.$$

This estimate enables us to invoke the results of [Nguyen 2018]. It gives a precise quantitative estimate $\sup_\Omega(\tilde{u} - u)$ in terms of δ and ε. Finally, we can choose $\varepsilon = \delta^{\varpi'}$ with $\varpi' > 0$ so small that our desired inequality holds.

We now proceed to give details of the argument. For the remaining part of the proof we fix a small $\delta_0 > 0$ and consider two parameters δ, ε such that

$$0 < \delta \le \varepsilon < \delta_0. \tag{4-4}$$

We may assume that $\psi \in C^{0,2\alpha}(\partial\Omega)$, where $0 < \alpha \le \frac{1}{2}$ (decreasing α if necessary) is the Hölder exponent of the subsolution φ. Then, we define

$$h(z) = \sup\{v(z) \in \mathrm{PSH}(\Omega) \cap C^0(\overline{\Omega}) : h_{|\partial\Omega} \le \psi\}. \tag{4-5}$$

It is well known [Bedford and Taylor 1976, Theorem 6.2] that $h \in \mathrm{PSH}(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$ and $h = \psi$ on $\partial\Omega$, which is also the solution of the homogeneous Monge–Ampère equation in Ω. Hence, we may assume

$$\psi \in \mathrm{PSH}(\Omega) \cap C^{0,\alpha}(\overline{\Omega}) \quad \text{and} \quad (dd^c\psi)^n \equiv 0. \tag{4-6}$$

Thanks to the comparison principle [Bedford and Taylor 1982] we get

$$\psi + \varphi \le u \le \psi \quad \text{on } \overline{\Omega}. \tag{4-7}$$

Lemma 4.1. *We have, for* $z \in \overline{\Omega}_\delta \setminus \Omega_\varepsilon$,

$$u_\delta(z) \le u(z) + C\varepsilon^\alpha. \tag{4-8}$$

In particular,

$$\sup_{\Omega_\delta}(\hat{u}_\delta - u) \le \sup_{\Omega_\varepsilon}(\hat{u}_\delta - u) + C\varepsilon^\alpha. \tag{4-9}$$

Remark 4.2. It is important to keep in mind that the uniform constants $C > 0$ which appear in the lemma, and many times below, are independent of δ and ε.

Proof. Fix a point $z \in \overline{\Omega}_\delta \setminus \Omega_\varepsilon$. Since u is continuous, there is $\zeta_1 \in \mathbb{C}^n$ with $|\zeta_1| \le \delta$ such that

$$u_\delta(z) = u(z + \zeta_1). \tag{4-10}$$

Moreover, there exists $\zeta_2 \in \mathbb{C}^n$ with $|\zeta_2| \le \varepsilon$ such that $z+\zeta_2 \in \partial\Omega$. Using this and (4-7) we get

$$\begin{aligned} u_\delta(z) - u(z) &\le \psi(z+\zeta_1) - [\psi(z)+\varphi(z)] \\ &= [\psi(z+\zeta_1)-\psi(z)] + [\varphi(z)-\varphi(z+\zeta_2)] \\ &\le C_1|\zeta_1|^\alpha + C_2|\zeta_2|^\alpha, \end{aligned} \tag{4-11}$$

where $C_1 = \|\psi\|_{C^{0,\alpha}},\ C_2 = \|\varphi\|_{C^{0,\alpha}}$. Since $\delta \le \varepsilon$, we conclude the proof of the first part.

To prove the second part, we observe that $u \le \hat{u}_\delta \le u_\delta$. Therefore,

$$\sup_{\Omega_\delta}(\hat{u}_\delta - u) \le \sup_{\Omega_\varepsilon}(\hat{u}_\delta - u) + \sup_{\Omega_\delta\setminus\Omega_\varepsilon}(u_\delta - u). \tag{4-12}$$

Combining this with the first part we get the second part. □

The lemma above tells us that to obtain the Hölder continuity of the solution u it is enough to get the estimate on the domain Ω_ε for ε a small constant, which is comparable to a small positive power of δ. To achieve our goal we will work on the domain Ω_ε and keep track of the (negative) exponent of ε.

Recall that

$$\Omega_\varepsilon = \{z \in \Omega : \operatorname{dist}(z, \partial\Omega) > \varepsilon\}. \tag{4-13}$$

We define

$$D_\varepsilon := \{\rho(z) < -\varepsilon\}, \tag{4-14}$$

where ρ is the defining function of Ω as in (2-1). The following lemma is very similar to Lemma 3.3. The main observation is that the domains D_ε and Ω_ε are comparable.

Lemma 4.3. *Let $1 \le k \le n$ be an integer. Let $v \in \mathrm{PSH} \cap L^\infty(\Omega)$. Then,*

$$\int_{\Omega_\varepsilon} (dd^c v)^k \wedge \beta^{n-k} \le \frac{C\|v\|_\infty^k}{\varepsilon^k}, \tag{4-15}$$

where C is independent of ε.

Proof. Observe that, from Hopf's lemma,

$$|\rho(z)| \ge c_0 \operatorname{dist}(z, \partial\Omega) \tag{4-16}$$

for a uniform constant $0 < c_0 \le 1$. Therefore,

$$\Omega_\varepsilon \subset \{\rho(z) < -c_0\varepsilon\}. \tag{4-17}$$

Since $\max\{\rho, -\varepsilon'/2\} - \rho \ge \varepsilon'/2$ with $\varepsilon' = c_0\varepsilon$ on the latter set, it follows that

$$\begin{aligned} \int_{\Omega_\varepsilon} (dd^c v)^k \wedge \beta^{n-k} &\le \left(\frac{2}{\varepsilon'}\right)^k \int_\Omega \left(\max\left\{\rho, -\frac{\varepsilon'}{2}\right\} - \rho\right)^k (dd^c v)^k \wedge \beta^{n-k} \\ &\le \frac{C\|v\|_\infty^k}{\varepsilon^k} \int_\Omega (dd^c\rho)^k \wedge \beta^{n-k}, \end{aligned} \tag{4-18}$$

where we used Lemma 2.1 for the second inequality. The last integral is bounded by the C^2-smoothness of ρ on $\overline{\Omega}$. □

We will now approximate the subsolution φ. Let us define

$$\varphi_\varepsilon := \max\left\{\varphi - \varepsilon, \frac{A\rho}{\varepsilon}\right\}, \tag{4-19}$$

where $A := 1 + \|\varphi\|_\infty$.

Lemma 4.4. *We have*

$$\int_\Omega (dd^c\varphi_\varepsilon)^n \le \frac{CA^n}{\varepsilon^n}. \tag{4-20}$$

Moreover,

$$\mathbf{1}_{D_\varepsilon} \cdot \mu \le (dd^c\varphi_\varepsilon)^n \tag{4-21}$$

as two measures, *where D_ε is defined in* (4-14).

Proof. To estimate the Monge–Ampère mass of φ_ε we use [Bedford and Taylor 1982, Corollary 4.3], which is a consequence of the comparison principle. Since $A\rho/\varepsilon \le \varphi_\varepsilon \le 0$ and the functions have zero values on the boundary,

$$\int_\Omega (dd^c\varphi_\varepsilon)^n \le \frac{A^n}{\varepsilon^n}\int_\Omega (dd^c\rho)^n. \tag{4-22}$$

The last integral is finite as ρ is C^2 on a neighborhood of the closure of Ω. Furthermore, since $\varphi_\varepsilon(z) = \varphi(z) - \varepsilon$ on $D_\varepsilon = \{\rho < -\varepsilon\}$ when $0 < \varepsilon < 1$, it is clear that

$$\mathbf{1}_{D_\varepsilon} \cdot \mu \le (dd^c\varphi_\varepsilon)^n. \qquad \square$$

Remark 4.5. Using the same argument, we also get that for an integer $1 \le k \le n$

$$\int_\Omega (dd^c\varphi_\varepsilon)^k \wedge \beta^{n-k} \le \frac{CA^k}{\varepsilon^k}. \tag{4-23}$$

We obtain now the volume-capacity inequality for the approximation sequence.

Corollary 4.6. *There exist uniform constants $\alpha_1 > 0$ and $C > 0$ which are independent of ε such that*, *for every compact set $K \subset \Omega$*,

$$\int_K (dd^c\varphi_\varepsilon)^n \le \frac{C}{\varepsilon^n}\cdot \operatorname{cap}(K)\cdot \exp\left(\frac{-\alpha_1}{[\operatorname{cap}(K)]^{1/n}}\right). \tag{4-24}$$

In particular, *for a fixed $\tau > 0$*, *there is a constant $C(\tau) > 0$ such that*, *for every compact set $K \subset \Omega$*,

$$\int_K (dd^c\varphi_\varepsilon)^n \le \frac{C(\tau)}{\varepsilon^n}[\operatorname{cap}(K)]^{1+\tau}. \tag{4-25}$$

Proof. This is the analogue of Proposition 3.1 with φ replaced by φ_ε, and thus the proof is the same as that of the proposition. Here we need to take into account three facts:

$$\|\varphi_\varepsilon\|_\infty \le \frac{C}{\varepsilon} \quad \text{and} \quad \|\varphi_\varepsilon\|_{C^{0,\alpha}(\bar\Omega)} \le \frac{C}{\varepsilon}, \tag{4-26}$$

and, for an integer $1 \le k \le n$ (Remark 4.5),

$$\int_\Omega (dd^c\varphi_\varepsilon)^k \wedge \beta^{n-k} \le \frac{C}{\varepsilon^k}. \tag{4-27}$$

This explains why we need an extra factor C/ε^n on the right-hand side of the inequality. $\square$

Next, we have the following stability estimate for the Monge–Ampère equation similar to [Guedj et al. 2008, Theorem 1.1]. However, it also takes into account the possibility of infinite total mass of the measure on the right-hand side.

Proposition 4.7. *Let u be the solution of* (4-1) *and Ω_ε be defined by* (4-13). *Let $v \in \mathrm{PSH} \cap L^\infty(\Omega)$ be such that $v = u$ on $\Omega \setminus \Omega_\varepsilon$. Then, there is $0 < \alpha_2 \le 1$ such that*

$$\sup_\Omega (v-u) \le \frac{C}{\varepsilon^n}\left(\int_\Omega \max\{v-u, 0\}\, d\mu\right)^{\alpha_2}. \tag{4-28}$$

Proof. Without loss of generality we may assume that $\sup_\Omega (v-u) > 0$. Set

$$s_0 := \inf_\Omega (u-v). \tag{4-29}$$

We know that for $0 < s < |s_0|$

$$U(s) := \{u < v + s_0 + s\} \Subset \Omega_\varepsilon. \tag{4-30}$$

Lemma 4.8. *Fix $\tau > 0$. Suppose $0 < s, t < |s_0|/2$. Then,*

$$t^n \operatorname{cap}(U(s)) \le \frac{C(\tau)}{\varepsilon^n}[\operatorname{cap}(U(s+t))]^{1+\tau}. \tag{4-31}$$

Proof of Lemma 4.8. Let $0 \le w \le 1$ be a plurisubharmonic function in Ω. We have the chain of inequalities

$$\begin{aligned} t^n \int_{U(s)} (dd^c w)^n &= \int_{\{u<v+s_0+s\}} [dd^c(tw)]^n \\ &\le \int_{\{u<v+s_0+s+tw\}} [dd^c(v+tw)]^n \le \int_{\{u<v+s_0+s+tw\}} (dd^c u)^n, \end{aligned} \tag{4-32}$$

where we used the comparison principle [Bedford and Taylor 1982, Theorem 4.1] for the last inequality. Since $\{u < v + s_0 + s + tw\} \subset U(s+t)$ and w is arbitrary, we get

$$t^n \operatorname{cap}(U(s)) \le \int_{U(s+t)} d\mu. \tag{4-33}$$

If we define $\varepsilon' := c_0\varepsilon$, where c_0 is the constant in (4-16), then

$$\mathbf{1}_{D_{\varepsilon'}} \cdot d\mu \le (dd^c \varphi_{\varepsilon'})^n$$

as two measures. Since $U(s+t) \subset \Omega_\varepsilon \subset D_{\varepsilon'}$, it follows that

$$\int_{U(s+t)} d\mu \le \int_{U(s+t)} (dd^c \varphi_{\varepsilon'})^n \le \frac{C(\tau)}{(c_0\varepsilon)^n}[\operatorname{cap}(U(s+t))]^{1+\tau}, \tag{4-34}$$

where the last inequality follows from Corollary 4.6. The proof of the lemma is complete. □

Now together with Lemma 4.8, the rest of the proof of the proposition is the same as in [Guedj et al. 2008, Theorem 1.1] (see also [Kołodziej and Nguyen 2016, Theorem 3.11]). □

The following result is a variation of Lemma 2.7 in [Nguyen 2018], where we considered the Hölder continuity of a measure ν on $\mathcal{E}_0'$, though the situation now is different as $\nu(\Omega)$ is no longer finite.

Theorem 4.9. *Let u be the solution of* (4-1) *and Ω_ε be defined by* (4-13). *Let $v \in \mathrm{PSH} \cap L^\infty(\Omega)$ be such that $v = u$ on $\Omega \setminus \Omega_\varepsilon$. Then, there exists $0 < \alpha_3 \le 1$ such that*

$$\int_\Omega |v-u|\, d\mu \le \frac{C}{\varepsilon^{n+1}} \left(\int_\Omega |v-u|\, dV_{2n} \right)^{\alpha_3}. \tag{4-35}$$

Proof. This is a variation of the inequality (3-7) with

$$S_{k,\varepsilon} := (dd^c \varphi_\varepsilon)^k \wedge \beta^{n-k}, \tag{4-36}$$

where $\varphi_\varepsilon = \max\{\varphi - \varepsilon, A\rho/\varepsilon\}$ and $0 \le k \le n$ is an integer. Since $\mu \le S_{n,\varepsilon}$ on Ω_ε, it is enough to show that there is $0 < \tau \le 1$ satisfying

$$\int_\Omega (v-u) S_{n,\varepsilon} \le \frac{C}{\varepsilon^{n+1}} \|v-u\|_1^\tau \tag{4-37}$$

for $v \ge u$ on Ω. (In the general case we use the identity

$$|v-u| = (\max\{v,u\} - u) + (\max\{v,u\} - v)$$

and apply twice the inequality (4-37) to get the theorem.)

Now we can repeat the inductive arguments of the proof of (3-7) with v, u and φ_ε in the places of v_s, v and φ, respectively. However, there are differences as follows. First, v, u are no longer in $\mathcal{E}_0'$. Second, if φ is extended as in the proof of Lemma 3.4, then $\varphi_\varepsilon = \max\{\varphi - \varepsilon, A\rho/\varepsilon\}$ is also defined on the neighborhood U of $\overline{\Omega}$, and

$$\|\varphi_\varepsilon\|_{C^{0,\alpha}(U)} \le \frac{C}{\varepsilon}.$$

Taking into account above differences, to pass from the k-th step to the $(k+1)$-th step we need the following inequality, corresponding to (3-16) (with $S_\varepsilon := (dd^c\varphi_\varepsilon)^k \wedge \beta^{n-k-1}$):

$$\begin{aligned} \int_\Omega (v-u)\, dd^c\varphi_\varepsilon \wedge S_\varepsilon &\le \left| \int_\Omega (v-u)\, dd^c\varphi_\varepsilon * \chi_t \wedge S_\varepsilon \right| + \left| \int_\Omega (v-u)\, dd^c(\varphi_\varepsilon * \chi_t - \varphi_\varepsilon) \wedge S_\varepsilon \right| \\ &=: I_{1,\varepsilon} + I_{2,\varepsilon}. \end{aligned} \tag{4-38}$$

Since

$$\left| \frac{\partial^2 \varphi_\varepsilon * \chi_t}{\partial z_j \partial \bar{z}_k}(z) \right| \le \frac{C\|\varphi\|_\infty}{\varepsilon t^2}, \tag{4-39}$$

and by the induction hypothesis at the k-th step, there exists $0 < \tau_k \le 1$ such that

$$\int_\Omega (v-u) S_\varepsilon \wedge \beta \le \frac{C}{\varepsilon^{k+1}} \|v-u\|_1^{\tau_k},$$

we conclude that

$$I_{1,\varepsilon} \le \frac{C\|\varphi\|_\infty}{\varepsilon\, t^2} \int_\Omega (v-u) S_\varepsilon \wedge \beta \le \frac{C\|\varphi\|_\infty}{\varepsilon^{k+2}\, t^2} \|v-u\|_1^{\tau_k}. \tag{4-40}$$

Similarly to (3-19), by integration by parts, $u = v$ on $\Omega \setminus \Omega_\varepsilon$, and

$$|\varphi_\varepsilon * \chi_t(z) - \varphi_\varepsilon(z)| \le \frac{Ct^\alpha}{\varepsilon},$$

it follows that

$$I_{2,\varepsilon} \le \frac{Ct^\alpha}{\varepsilon} \int_{\Omega_\varepsilon} (dd^c v + dd^c u) \wedge S_\varepsilon. \tag{4-41}$$

At this point as u, v do not belong to $\mathcal{E}_0'$ we need to use a different argument to bound $I_{2,\varepsilon}$. Namely, similarly to Lemma 4.3, we have

$$\int_{\Omega_\varepsilon} (dd^c u + dd^c v) \wedge S_\varepsilon \le \frac{C\|u+v\|_\infty (1+\|\varphi\|_\infty)^k}{\varepsilon^{k+1}}. \tag{4-42}$$

Indeed, we first have

$$\begin{aligned}\int_{\Omega_\varepsilon} dd^c(u+v) \wedge (dd^c\varphi_\varepsilon)^k \wedge \beta^{n-k-1} &\le \frac{2}{\varepsilon'} \int_\Omega \left(\max\left\{\rho, -\frac{\varepsilon'}{2}\right\} - \rho \right) \wedge dd^c(u+v) \wedge (dd^c\varphi_\varepsilon)^k \wedge \beta^{n-k-1} \\ &\le \frac{C}{\varepsilon} \|u+v\|_\infty \int_\Omega (dd^c\rho) \wedge (dd^c\varphi_\varepsilon)^k \wedge \beta^{n-k-1},\end{aligned}$$

where $\varepsilon' = c_0\varepsilon$ with c_0 defined by (4-16). The desired inequality (4-42) follows from Remark 4.5. Now, combining (4-41) and (4-42) we get

$$I_{2,\varepsilon} \le \frac{Ct^\alpha}{\varepsilon^{k+2}}. \tag{4-43}$$

Next, it is easy to see (from Lemma 4.4) that

$$\int_\Omega (v-u) S_n \le \frac{C\|u\|_\infty (1+\|\varphi\|_\infty)^n}{\varepsilon^n}.$$

Therefore, we can assume $0 < \|v-u\|_1 < 0.01$. Thanks to (4-40) and (4-43) we have

$$\int_\Omega (v-u)\, dd^c\varphi_\varepsilon \wedge S_\varepsilon \le \frac{C}{\varepsilon^{k+2}\, t^2} \|v-u\|_1^{\tau_k} + \frac{Ct^\alpha}{\varepsilon^{k+2}}.$$

If we choose $t = \|v-u\|_1^{\tau_k/3}$, $\tau_{k+1} = \alpha\tau_k/3$, then

$$\int_\Omega (v-u) S_\varepsilon \wedge dd^c\varphi_\varepsilon \le \frac{C}{\varepsilon^{k+2}} \|v-u\|_1^{\tau_{k+1}}.$$

Thus, the induction argument is completed, and the theorem follows. □

The last ingredient to prove Theorem B was proved first in [Baracco et al. 2016] (see also [Nguyen 2018, Lemma 2.12]). Here, the estimate is sharper and the proof is simpler too.

Lemma 4.10. *For $\delta > 0$ small we have*

$$\int_{\Omega_\delta} |\hat{u}_\delta - u|\, dV_{2n} \le C\delta. \tag{4-44}$$

Proof. First, we know from the classical Jensen formula (see, e.g., [Guedj et al. 2008, Lemma 4.3]) that

$$\int_{\Omega_{2\delta}} |\hat{u}_\delta - u| \le C\delta^2 \int_{\Omega_\delta} \Delta u(z). \tag{4-45}$$

Again, it follows from Lemma 4.3 applied for $k = 1$ and $\delta = \varepsilon$, that

$$\int_{\Omega_\delta} \Delta u(z) \le \frac{C}{\delta}. \tag{4-46}$$

Therefore,

$$\int_{\Omega_\delta} |\hat{u}_\delta - u|\, dV_{2n} \le \int_{\Omega_{2\delta}} |\hat{u}_\delta - u|\, dV_{2n} + \|u\|_\infty \int_{\Omega_\delta \setminus \Omega_{2\delta}} dV_{2n} \le C\delta. \tag{4-47}$$

This is the required inequality. □

We are ready to prove the Hölder continuity of the solution.

End of proof of Theorem B. Let us fix δ such that $0 < \delta < \delta_0$ small and let ε be such that $\delta \le \varepsilon < \delta_0$, which is to be determined later. Thanks to Lemma 4.1 and $\hat{u}_\delta \le u_\delta$ we have $\hat{u}_\delta - C\varepsilon^\alpha \le u$ on $\partial\Omega_\varepsilon$. Therefore, the function

$$\tilde{u} := \begin{cases} \max\{\hat{u}_\delta - C\varepsilon^\alpha, u\} & \text{on } \Omega_\varepsilon, \\ u & \text{on } \Omega \setminus \Omega_\varepsilon \end{cases} \tag{4-48}$$

belongs to $\mathrm{PSH}(\Omega) \cap C^0(\overline{\Omega})$. Notice that $\tilde{u} \ge u$ in Ω, and

$$\tilde{u} = u \quad \text{on } \Omega \setminus \Omega_\varepsilon. \tag{4-49}$$

Again, by the second part of Lemma 4.1 we have

$$\begin{aligned} \sup_{\Omega_\delta}(\hat{u}_\delta - u) &\le \sup_{\Omega_\varepsilon}(\hat{u}_\delta - u) + C\varepsilon^\alpha \\ &\le \sup_{\Omega}(\tilde{u} - u) + C\varepsilon^\alpha + C\varepsilon^\alpha. \end{aligned} \tag{4-50}$$

By the stability estimate (Proposition 4.7) there exists $0 < \alpha_2 \le 1$ such that

$$\begin{aligned} \sup_{\Omega}(\tilde{u} - u) &\le \frac{C}{\varepsilon^n}\left(\int_\Omega \max\{\tilde{u} - u, 0\}\, d\mu\right)^{\alpha_2} \\ &\le \frac{C}{\varepsilon^n}\left(\int_\Omega |\tilde{u} - u|\, d\mu\right)^{\alpha_2}, \end{aligned} \tag{4-51}$$

where we used the fact that $\tilde{u} = u$ outside Ω_ε. Using Theorem 4.9, there is $0 < \alpha_3 \le 1$ such that

$$\begin{aligned} \sup_{\Omega}(\tilde{u} - u) &\le \frac{C}{\varepsilon^{n+(n+1)\alpha_2}}\left(\int_\Omega |\tilde{u} - u|\, dV_{2n}\right)^{\alpha_2\alpha_3} \\ &\le \frac{C}{\varepsilon^{2n+1}}\left(\int_{\Omega_\delta} |\hat{u}_\delta - u|\, dV_{2n}\right)^{\alpha_2\alpha_3}, \end{aligned} \tag{4-52}$$

where we used $0 \le \tilde{u} - u \le \mathbf{1}_{\Omega_\varepsilon} \cdot (\hat{u}_\delta - u)$ and $\Omega_\varepsilon \subset \Omega_\delta$ for the second inequality. It follows from (4-50), (4-52), and Lemma 4.10 that

$$\sup_{\Omega_\delta}(\hat{u}_\delta - u) \le C\varepsilon^\alpha + \frac{C\delta^{\alpha_2\alpha_3}}{\varepsilon^{2n+1}}. \tag{4-53}$$

Now, we choose $\alpha_4 = \alpha\alpha_2\alpha_3/(2n + 1 + \alpha)$ and

$$\varepsilon = \delta^{\alpha_2\alpha_3/(2n+1+\alpha)}.$$

Then, $\sup_{\Omega_\delta}(\hat{u}_\delta - u) \le C\delta^{\alpha_4}$. Finally, thanks to [Guedj et al. 2008, Lemma 4.2] we infer that $\sup_{\Omega_\delta}(u_\delta - u) \le C\delta^{\alpha_4}$. The proof of the theorem is finished. □

Remark 4.11. In the above proof the Hölder exponent of the solution u is $\alpha' = \alpha_4 = \alpha\alpha_2\alpha_3/(2n+1+\alpha)$, where we can take $0 < \alpha_2 < 1/(n+1)$ and $\alpha_3 = \alpha^n/3^n$ by [Guedj et al. 2008] and Theorem 4.9 respectively. In our opinion it is far from being optimal. If we assume that the subsolution φ is merely continuous, then we do not know if the inequality (4-51) holds true. Therefore, it seems to be hard to improve the proof above to get the answer for the subsolution problem in the continuous category.

5. Proof of Corollary C

Let $\mu \in \mathcal{M}(\varphi, \Omega)$ and $0 \le f \in L^p(\Omega, d\mu)$ with $p > 1$. We wish to show that there exists $\tilde{\varphi} \in \mathrm{PSH}(\Omega) \cap C^{0,\tilde{\alpha}}(\bar{\Omega})$, with $0 < \tilde{\alpha} \le 1$, such that

$$f\, d\mu \in \mathcal{M}(\tilde{\varphi}, \Omega). \tag{5-1}$$

The proof of the corollary is similar to that of Theorem B with the aid of the following two lemmas.

Lemma 5.1. *Fix a constant $\tau > 0$. Then, there exists a uniform constant $C(\tau)$ such that, for every compact set $K \subset \Omega$,*

$$\int_K f\, d\mu \le C(\tau)[\mathrm{cap}(K)]^{1+\tau}. \tag{5-2}$$

Proof. Hölder's inequality and Proposition 3.1 give us

$$\int_K f\, d\mu \le \|f\|_{L^p(\Omega, d\mu)}[\mu(K)]^{(p-1)/p} \le C\left[\mathrm{cap}(K)\cdot \exp\left(\frac{-\alpha_0}{[\mathrm{cap}(K)]^{1/n}}\right)\right]^{(p-1)/p}. \tag{5-3}$$

Let $0 < a, b, c < 1$ be fixed. The following elementary inequality holds for $x > 0$:

$$x^a \exp\left(-\frac{c}{x^b}\right) \le C(\tau)x^{1+\tau},$$

where $C(\tau) = C(\tau, a, b, c)$ depends only on τ, a, b, c. Thus, the desired inequality follows. □

Thanks to the lemma and [Kołodziej 2005, Theorem 5.9] we can solve the Monge–Ampère equation

$$u \in \mathrm{PSH}(\Omega) \cap C^0(\bar{\Omega}), \quad (dd^c u)^n = f\, d\mu, \quad u_{|\partial\Omega} = 0. \tag{5-4}$$

Moreover, the above lemma will enable us to have the stability estimate (Proposition 4.7). The next lemma is also a consequence of the generalized Hölder inequality which was proved in [Nguyen 2018, Corollary 2.14].

Lemma 5.2. *Let $v \in \mathrm{PSH}(\Omega) \cap C^0(\bar{\Omega})$ be such that $v \ge u$ in Ω and $v = u$ near $\partial\Omega$. Then, there exist uniform constants $C > 0$ and $0 < \tilde{\alpha}_3 < 1$ such that*

$$\int_\Omega (v-u) f\, d\mu \le C\|v-u\|_{L^1(d\mu)}^{\tilde{\alpha}_3}. \tag{5-5}$$

Next, we use the extendability assumption of φ to get a result similar to Lemma 4.1 in the current setting. Namely, let $\widetilde{\Omega}$ be a strictly pseudoconvex neighborhood of $\bar{\Omega}$ such that $\varphi \in \mathrm{PSH}(\widetilde{\Omega})$ and Hölder continuous on the closure of $\widetilde{\Omega}$. Thanks to the results in [Nguyen 2018] there exists $v \in \mathrm{PSH}(\widetilde{\Omega})$ and Hölder continuous in $\widetilde{\Omega}$ satisfying

$$(dd^c v)^n = \mathbf{1}_\Omega f\, d\mu \quad \text{in } \widetilde{\Omega}, \qquad v = 0 \quad \text{on } \partial\widetilde{\Omega}.$$

Consider h to be the maximal pluriharmonic extension into Ω of $(-v)_{|\partial\Omega}$ which is Hölder continuous on $\partial\Omega$ (see (1-3)). So h is also Hölder continuous on $\bar{\Omega}$. Then, by the comparison principle,

$$v + h \le u \le 0 \quad \text{on } \bar{\Omega}.$$

From this we easily deduce the desired estimate near the boundary for u.

Now the rest of the proof goes exactly as in the proof of Theorem B. Namely, the inequality (4-51) holds for the measure $f\, d\mu$, next use Lemma 5.2 and Theorem 4.9 to get the inequality (4-52). Then we get the Hölder continuity of u. Notice that the Hölder exponent is worse by a factor of $\tilde{\alpha}_3$. Thus, $f\, d\mu \in \mathcal{M}(u, \Omega)$.

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THE CALDERÓN PROBLEM FOR THE FRACTIONAL SCHRÖDINGER EQUATION

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We show global uniqueness in an inverse problem for the fractional Schrödinger equation: an unknown potential in a bounded domain is uniquely determined by exterior measurements of solutions. We also show global uniqueness in the partial data problem where measurements are taken in arbitrary open, possibly disjoint, subsets of the exterior. The results apply in any dimension ≥ 1 and are based on a strong approximation property of the fractional equation that extends earlier work. This special feature of the nonlocal equation renders the analysis of related inverse problems radically different from the traditional Calderón problem.

1. Introduction

In this article we consider a nonlocal analogue of the inverse conductivity problem posed in [Calderón 1980]. In the standard Calderón problem, the objective is to determine the electrical conductivity of a medium from voltage and current measurements on its boundary. This problem is the mathematical model of electrical resistivity/impedance tomography in seismic, medical and industrial imaging. It serves as a model case for various inverse problems for elliptic equations, and has a rich mathematical theory with connections to many other questions. We refer to the survey [Uhlmann 2014] for more details.

In mathematical terms, if $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary (the medium of interest), after a standard reduction one often considers the Dirichlet problem for the Schrödinger equation

$$(-\Delta + q)u = 0 \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = f,$$

where $q \in L^\infty(\Omega)$ and 0 is not a Dirichlet eigenvalue for $-\Delta + q$ in Ω. The boundary measurements are given by the Dirichlet-to-Neumann map (DN map)

$$\Lambda_q : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega),$$

defined weakly in terms of the bilinear form for the equation. Here and below, we denote the standard L^2 based Sobolev spaces by H^s.

For more regular boundaries and functions f, the DN map is given by the normal derivative $\Lambda_q f = \partial_\nu u|_{\partial\Omega}$, where u is the solution with boundary value f. The inverse problem is to determine the potential q in Ω from the knowledge of the DN map Λ_q.

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We will consider an inverse problem for a nonlocal analogue of the Schrödinger equation. In fact, our equation will be the fractional Schrödinger equation $((-\Delta)^s + q)u = 0$ in Ω, where $0 < s < 1$. Here the fractional Laplacian is defined by

$$(-\Delta)^s u = \mathscr{F}^{-1}\{|\xi|^{2s}\hat{u}(\xi)\}, \quad u \in H^s(\mathbb{R}^n),$$

and $\hat{u} = \mathscr{F}u$ is the Fourier transform of u. This operator is nonlocal (it does not preserve the support of u), and one natural way to set up the Dirichlet problem is to look for solutions $u \in H^s(\mathbb{R}^n)$ satisfying

$$((-\Delta)^s + q)u = 0 \quad \text{in } \Omega, \qquad u|_{\Omega_e} = f,$$

where $f \in H^s(\Omega_e)$, and Ω_e is the exterior domain

$$\Omega_e = \mathbb{R}^n \setminus \bar{\Omega}.$$

We recall basic facts about weak solutions in Section 2. In particular, there is a countable set of Dirichlet eigenvalues, and we will assume that q is such that 0 is not an eigenvalue; that is,

$$\text{if } u \in H^s(\mathbb{R}^n) \text{ solves } ((-\Delta)^s + q)u = 0 \text{ in } \Omega \text{ and } u|_{\Omega_e} = 0, \text{ then } u \equiv 0. \tag{1-1}$$

This holds, e.g., if $q \geq 0$. Then there is a unique solution $u \in H^s(\mathbb{R}^n)$ for any $f \in H^s(\Omega_e)$, and one may define an analogue of the DN map,

$$\Lambda_q : H^s(\Omega_e) \to H^s(\Omega_e)^*,$$

that maps f to a nonlocal analogue of the Neumann boundary value of the solution u. (This discussion assumes that Ω is a bounded Lipschitz domain; see Section 2 for the case of general bounded open sets.)

We will define Λ_q in Section 2 via the bilinear form associated with the fractional Dirichlet problem, which will be sufficient for the proof of Theorem 1.1. There are other nonlocal Neumann operators that one could use, but by Theorem 1.1 any reasonable measurement operator would be determined by Λ_q; we will verify this directly for the operator $\mathcal{N}_s$ in [Dipierro et al. 2017a]. Again, if Ω has C^∞ boundary and q and f are more regular, the DN map is more explicit and is given by

$$\Lambda_q : H^{s+\beta}(\Omega_e) \to H^{-s+\beta}(\Omega_e), \quad \Lambda_q f = (-\Delta)^s u|_{\Omega_e},$$

where u is the solution of $((-\Delta)^s + q)u = 0$ in Ω with exterior value f, and $\max\{0, s - \frac{1}{2}\} < \beta < \frac{1}{2}$ (such a β exists since $0 < s < 1$). Heuristically, given an open set $W \subset \Omega_e$, one can interpret $\Lambda_q f|_W$ as measuring the cost required to maintain the exterior value f in W. For more details on these facts (which will not be needed for the proof of Theorem 1.1), we refer to the Appendix.

The following theorem is the main result in this article. It solves the fractional Schrödinger inverse problem in any dimension $n \geq 1$, and also the partial data problem with exterior Dirichlet and Neumann measurements in arbitrary open (possibly disjoint) sets $W_1, W_2 \subset \Omega_e$.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be bounded open, let $0 < s < 1$, and let $q_1, q_2 \in L^\infty(\Omega)$ satisfy* (1-1). *Let also $W_1, W_2 \subset \Omega_e$ be open. If the DN maps for the equations $((-\Delta)^s + q_j)u = 0$ in Ω satisfy*

$$\Lambda_{q_1} f|_{W_2} = \Lambda_{q_2} f|_{W_2} \quad \textit{for any } f \in C_c^\infty(W_1),$$

then $q_1 = q_2$ in Ω.

For the usual Schrödinger equation $(-\Delta+q)u=0$ and the related DN map Λ_q on the full boundary $\partial\Omega$, the corresponding result is due to [Sylvester and Uhlmann 1987] when $n \geq 3$ and to [Bukhgeim 2008] when $n=2$ for slightly more regular potentials; for the case of L^p potentials see [Blåsten et al. 2015] when $n=2$ and [Chanillo 1990; Nachman 1992] when $n \geq 3$. The partial data problem of determining q from the knowledge of $\Lambda_q f|_\Gamma$ for any f supported in Γ, when Γ is an arbitrary open subset of $\partial\Omega$, was solved in [Imanuvilov et al. 2010] when $n=2$ for $q_j \in C^{2,\alpha}$. The corresponding result in dimensions $n \geq 3$ is open, but there are several partial results including [Kenig et al. 2007; Isakov 2007; Kenig and Salo 2013]. The case of measurements on disjoint sets is even more difficult, and counterexamples may appear [Imanuvilov et al. 2011; Daudé et al. 2019a; 2019b]. See the surveys [Imanuvilov and Yamamoto 2013; Kenig and Salo 2014] for further references.

The proof of Theorem 1.1 begins by showing that if the two DN maps are equal, then (exactly as in the usual Schrödinger case) one has the integral identity

$$\int_\Omega (q_1-q_2)u_1u_2\,dx = 0$$

for any $u_j \in H^s(\mathbb{R}^n)$ that solve $((-\Delta)^s+q_j)u_j = 0$ in Ω and satisfy $\operatorname{supp}(u_j) \subset \overline{\Omega} \cup \overline{W}_j$. For the standard Schrödinger equation, one then typically uses special complex geometrical optics solutions u_j to show that the products $\{u_1u_2\}$ form a complete set in $L^1(\Omega)$. See [Uhlmann 2014] for an overview.

However, solutions of the fractional Schrödinger equation are much less rigid than those of the usual Schrödinger equation. The fractional equation enjoys stronger uniqueness and approximation properties, as demonstrated by the following theorems:

Theorem 1.2. *If $0 < s < 1$, if $u \in H^{-r}(\mathbb{R}^n)$ for some $r \in \mathbb{R}$, and if both u and $(-\Delta)^s u$ vanish in some open set, then $u \equiv 0$.*

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let $\Omega_1 \subset \mathbb{R}^n$ be any open set with $\Omega \subset \Omega_1$ and $\Omega_1 \setminus \overline{\Omega} \neq \varnothing$:*

(a) *If $q \in L^\infty(\Omega)$ satisfies* (1-1), *then any $f \in L^2(\Omega)$ can be approximated arbitrarily well in $L^2(\Omega)$ by functions $u|_\Omega$ where $u \in H^s(\mathbb{R}^n)$ satisfy*

$$((-\Delta)^s+q)u = 0 \quad \text{in } \Omega, \qquad \operatorname{supp}(u) \subset \overline{\Omega}_1.$$

(b) *If Ω has C^∞ boundary, and if $q \in C_c^\infty(\Omega)$ satisfies* (1-1), *then any $f \in C^\infty(\overline{\Omega})$ can be approximated arbitrarily well in $C^\infty(\overline{\Omega})$ by functions $d(x)^{-s}u|_\Omega$ where $u \in H^s(\mathbb{R}^n)$ satisfy*

$$((-\Delta)^s+q)u = 0 \quad \text{in } \Omega, \qquad \operatorname{supp}(u) \subset \overline{\Omega}_1.$$

(Here d is any function in $C^\infty(\overline{\Omega})$ with $d(x) = \operatorname{dist}(x, \partial\Omega)$ near Ω and $d > 0$ in Ω. Also, $v_j \to v$ in $C^\infty(\overline{\Omega})$ means that $v_j \to v$ in $C^k(\overline{\Omega})$ for all $k \geq 0$.)

Note that both of these properties fail for the usual Laplacian: if $u \in C_c^\infty(\mathbb{R}^n)$ then both u and Δu vanish in a large set but u can be nontrivial, and the set of harmonic functions in $L^2(\Omega)$ is a closed subspace of $L^2(\Omega)$ which is smaller than $L^2(\Omega)$.

Theorem 1.2 is classical [Riesz 1938] at least with stronger conditions on u, and even the strong unique continuation principle holds in this context [Fall and Felli 2014; Rüland 2015; Yu 2017]. For later applications we will give a robust proof using the Carleman estimates from [Rüland 2015] and the Caffarelli–Silvestre extension [2007].

The following version of Theorem 1.3 has been proved in [Dipierro et al. 2017b]; see also [Dipierro et al. 2019]: given $f \in C^k(\bar{B}_1)$ and $\varepsilon > 0$, there is $u \in H^s(\mathbb{R}^n)$ with $(-\Delta)^s u = 0$ in B_1 and $\operatorname{supp}(u) \subset \bar{B}_R$ for some possibly large $R = R_{\varepsilon,f} > 1$, so that

$$\|u - f\|_{C^k(\bar{B}_1)} < \varepsilon.$$

Theorem 1.3 improves this by reducing the approximation property to the uniqueness property, Theorem 1.2, using a Runge-type argument [Lax 1956; Malgrange 1956] and regularity for fractional Dirichlet problems [Hörmander 1965; Grubb 2015]. In particular, this implies that the result of [Dipierro et al. 2017b] is valid for any fixed $R > 1$. The strong approximation property replaces the method of complex geometrical optics in solving the inverse problem for the fractional Schrödinger equation.

The study of fractional and nonlocal operators is currently an active research field and the related literature is substantial. We only mention that operators of this type arise in problems involving anomalous diffusion and random processes with jumps, and they have applications in probability theory, physics, finance, and biology. See [Bucur and Valdinoci 2016; Ros-Oton 2016] for further information and references.

The mathematical study of inverse problems for fractional equations goes back at least to [Cheng et al. 2009]. By now there are a number of results, mostly for time-fractional models and including many numerical works. Here is an example of the rigorous results that are available [Sakamoto and Yamamoto 2011]: in the time-fractional heat equation

$$\partial_t^\alpha u - \Delta u = 0 \quad \text{in } \Omega \times (0, T), \qquad u|_{\partial\Omega\times(0,T)} = 0,$$

where $0 < \alpha < 1$ and ∂_t^α is the Caputo derivative, $u(0)$ is determined by $u(T)$ in a mildly ill-posed way (for $\alpha = 1$ this problem is severely ill-posed). In general, nonlocality may influence the nature of the inverse problem but there are several aspects to be taken into account. We refer to [Jin and Rundell 2015] for a detailed discussion and many further references. We are not aware of any previous rigorous works on multidimensional inverse problems for space-fractional equations.

Finally, we note that Theorem 1.1 is a global uniqueness result in the inverse problem for the fractional Schrödinger equation, both with full and partial data. This provides a starting point for further work on inverse problems for fractional equations and nonlocal models. In fact, after this article was first submitted as a preprint, several works that build upon the ideas introduced here have appeared. These include results for low regularity and stability [Rüland and Salo 2018; 2019a], matrix coefficients [Ghosh et al. 2017], semilinear equations [Lai and Lin 2019], reconstruction and shape detection [Harrach and Lin 2017; Cao et al. 2019; Ghosh et al. 2018], and quantitative Runge approximation [Rüland and Salo 2019b; 2020]. See also the survey [Salo 2017].

This paper is organized as follows. Section 1 is the introduction. In Section 2 we review weak solutions of fractional Dirichlet problems, and give a definition of the DN map. In Sections 3 and 4 we prove

Theorems 1.2 and 1.3(a). The solution of the inverse problem, Theorem 1.1, is given in Section 5. In Section 6 we invoke the regularity theory for fractional Dirichlet problems in [Grubb 2015] and prove Theorem 1.3(b). Further properties and alternative descriptions of the DN map may be found in the Appendix.

2. Fractional Laplacian

In this section we review some basic facts about Dirichlet problems for the fractional Laplacian; see, e.g., [Hoh and Jacob 1996; Felsinger et al. 2015; Grubb 2015; Ros-Oton 2016]. For simplicity, we will assume most functions to be real-valued in this paper.

2A. ***Sobolev spaces.*** We first establish the notation for Sobolev-type spaces. We write $H^s(\mathbb{R}^n) = W^{s,2}(\mathbb{R}^n)$ for the standard L^2 based Sobolev space with norm

$$\|u\|_{H^s(\mathbb{R}^n)} = \|\langle D\rangle^s u\|_{L^2(\mathbb{R}^n)},$$

where $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$, and the notation $m(D)u = \mathscr{F}^{-1}\{m(\xi)\hat{u}(\xi)\}$ is used for Fourier multipliers when $m \in C^\infty(\mathbb{R}^n)$ is polynomially bounded together with its derivatives. Our notation for the Fourier transform is

$$\hat{u}(\xi) = \mathscr{F}u(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\,dx.$$

If $U \subset \mathbb{R}^n$ is an open set (not necessarily bounded), define the spaces (we follow the notation of [McLean 2000])

$$\begin{aligned} H^s(U) &= \{u|_U : u \in H^s(\mathbb{R}^n)\}, \\ \widetilde{H}^s(U) &= \text{closure of } C_c^\infty(U) \text{ in } H^s(\mathbb{R}^n), \\ H_0^s(U) &= \text{closure of } C_c^\infty(U) \text{ in } H^s(U). \end{aligned}$$

We equip $H^s(U)$ with the quotient norm $\|u\|_{H^s(U)} = \inf\{\|w\|_{H^s} : w \in H^s(\mathbb{R}^n),\ w|_U = u\}$. Also, if $F \subset \mathbb{R}^n$ is a closed set, we define

$$H_F^s = H_F^s(\mathbb{R}^n) = \{u \in H^s(\mathbb{R}^n) : \operatorname{supp}(u) \subset F\}.$$

We say that an open set $U \subset \mathbb{R}^n$ is a Lipschitz domain if its boundary ∂U is compact and if locally near each boundary point U can be represented as the set above the graph of a Lipschitz function. Thus U could be a bounded Lipschitz domain, or U could be $\mathbb{R}^n \setminus \overline{\Omega}$, where Ω is a bounded Lipschitz domain. If U is a Lipschitz domain, then (with natural identifications, see [McLean 2000; Triebel 2002])

$$\begin{gathered} \widetilde{H}^s(U) = H^s_{\overline{U}}(\mathbb{R}^n), \quad s \in \mathbb{R}, \\ H^s_{\overline{U}}(\mathbb{R}^n)^* = H^{-s}(U) \quad \text{and} \quad H^s(U)^* = H^{-s}_{\overline{U}}(\mathbb{R}^n), \quad s \in \mathbb{R}, \\ H^s(U) = H^s_{\overline{U}}(\mathbb{R}^n) = H^s_0(U), \quad -\tfrac{1}{2} < s < \tfrac{1}{2}. \end{gathered}$$

2B. ***Fractional Laplacian.*** Let $s > -n/2$ and consider the fractional Laplacian in $\mathbb{R}^n$,

$$(-\Delta)^s u = \mathscr{F}^{-1}\{|\xi|^{2s}\hat{u}(\xi)\}, \quad u \in \mathscr{S},$$

where $\mathscr{S}$ denotes Schwartz space in $\mathbb{R}^n$. If $\psi \in C_c^\infty(\mathbb{R}^n)$ with $\psi = 1$ near 0, writing $|\xi|^{2s} = \psi(\xi)|\xi|^{2s} + (1-\psi(\xi))|\xi|^{2s}$ and using the assumption $s > -n/2$ shows that $|\xi|^{2s}$ is the sum of an L^1 function and a smooth function whose derivatives grow at most polynomially. Thus $(-\Delta)^s$ for $s > -n/2$ is a continuous map from $\mathscr{S}$ to L^∞.

There are many other definitions of the fractional Laplacian [Kwaśnicki 2017]. For instance, if $0 < s < 1$, it is given by the principal value integral

$$(-\Delta)^s u(x) = c_{n,s} \text{ p.v.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy.$$

We next extend $(-\Delta)^s$ to act on larger spaces. In particular, if $s \geq 0$, then $(-\Delta)^s$ will be well-defined on $H^r(\mathbb{R}^n)$ for any $r \in \mathbb{R}$.

Lemma 2.1. *If $s \geq 0$, the fractional Laplacian extends as a bounded map*

$$(-\Delta)^s : H^r(\mathbb{R}^n) \to H^{r-2s}(\mathbb{R}^n)$$

whenever $r \in \mathbb{R}$. If $-n/2 < s < 0$, the fractional Laplacian $(-\Delta)^s$ is the Riesz potential

$$(-\Delta)^s u = I_{2|s|} u = \frac{c_{n,s}}{|\cdot|^{n-2|s|}} * u$$

and it extends as a bounded map

$$(-\Delta)^s : L^p(\mathbb{R}^n) \to L^{np/(n-2|s|p)}(\mathbb{R}^n), \quad 1 < p < \frac{n}{2|s|}.$$

Proof. If $u \in \mathscr{S}$, then

$$\|(-\Delta)^s u\|_{H^{r-2s}} = \|\mathscr{F}^{-1}\{m(\xi)\langle\xi\rangle^r \hat{u}(\xi)\}\|_{L^2},$$

where $m(\xi) = \langle\xi\rangle^{-2s}|\xi|^{2s}$ is bounded and hence a Fourier multiplier on L^2, showing that $\|(-\Delta)^s u\|_{H^{r-2s}} \leq C\|u\|_{H^r}$. The second statement is the Hardy–Littlewood–Sobolev inequality [Hörmander 1983, Theorem 4.5.3]. □

Remark 2.2. If $s \geq 0$, the fractional Laplacian also extends as a bounded map

$$(-\Delta)^s : W^{r,p}(\mathbb{R}^n) \to W^{r-2s,p}(\mathbb{R}^n),$$
$$(-\Delta)^s : C_*^r(\mathbb{R}^n) \to C_*^{r-2s}(\mathbb{R}^n)$$

whenever $r \in \mathbb{R}$ and $1 < p < \infty$, where $W^{r,p}$ are the usual L^p Sobolev (Bessel potential) spaces and C_*^r are the Zygmund spaces; see [Taylor 1996]. An even larger domain for $(-\Delta)^s$ is obtained as in [Silvestre 2007] by considering the test function space

$$\mathscr{S}_s = \{u \in C^\infty(\mathbb{R}^n) : \langle\cdot\rangle^{n+2s}\partial^\alpha u \in L^\infty(\mathbb{R}^n) \text{ for any multi-index } \alpha\},$$

equipped with the topology induced by the seminorms $\|\langle\cdot\rangle^{n+2s}\partial^\alpha u\|_{L^\infty}$. Then $(-\Delta)^s$ is continuous from $\mathscr{S}$ to $\mathscr{S}_s$ and extends to the dual

$$\mathscr{S}_s' = \left\{u \in \mathscr{S}'(\mathbb{R}^n) : u = \textstyle\sum_{|\alpha| \leq m} \partial^\alpha u_\alpha \text{ for some } m \geq 0 \text{ and } u_\alpha \in \langle\cdot\rangle^{n+2a} L^\infty(\mathbb{R}^n)\right\}.$$

However, in this article it suffices to work with the spaces $H^s(\mathbb{R}^n)$.

2C. *Dirichlet problem.* Next we restrict our attention to nonlocal operators

$$(-\Delta)^s, \quad 0 < s < 1,$$

and consider the solvability of the Dirichlet problem

$$\begin{aligned} ((-\Delta)^s + q)u &= F \quad \text{in } \Omega, \\ u &= f \quad \text{in } \Omega_e, \end{aligned}$$

where, for a bounded open set $\Omega \subset \mathbb{R}^n$, we denote the exterior domain by $\Omega_e = \mathbb{R}^n \setminus \overline{\Omega}$. Here F may be a function in Ω, or more generally an element of $(\widetilde{H}^s(\Omega))^*$. We also denote the restriction to Ω by

$$r_\Omega u = u|_\Omega,$$

and if $U \subset \mathbb{R}^n$ is open and $u, v \in L^2(U)$ we write

$$(u, v)_U = \int_U uv\, dx.$$

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $0 < s < 1$, and let $q \in L^\infty(\Omega)$. Let B_q be the bilinear form defined for $v, w \in H^s(\mathbb{R}^n)$ by*

$$B_q(v, w) = ((-\Delta)^{s/2} v, (-\Delta)^{s/2} w)_{\mathbb{R}^n} + (q r_\Omega v, r_\Omega w)_\Omega.$$

(a) *There is a countable set $\Sigma = \{\lambda_j\}_{j=1}^\infty \subset \mathbb{R}$, $\lambda_1 \le \lambda_2 \le \cdots \to \infty$, with the following property: if $\lambda \in \mathbb{R} \setminus \Sigma$, then for any $F \in (\widetilde{H}^s(\Omega))^*$ and $f \in H^s(\mathbb{R}^n)$ there is a unique $u \in H^s(\mathbb{R}^n)$ satisfying*

$$B_q(u, w) - \lambda(u, w)_{\mathbb{R}^n} = F(w) \quad \text{for } w \in \widetilde{H}^s(\Omega), \qquad u - f \in \widetilde{H}^s(\Omega).$$

One has the norm estimate

$$\|u\|_{H^s(\mathbb{R}^n)} \le C(\|F\|_{(\widetilde{H}^s(\Omega))^*} + \|f\|_{H^s(\mathbb{R}^n)}),$$

with C independent of F and f.

(b) *The function u in* (a) *is also the unique $u \in H^s(\mathbb{R}^n)$ satisfying*

$$r_\Omega((-\Delta)^s + q - \lambda)u = F \quad \text{in the sense of distributions in } \Omega$$

and $u - f \in \widetilde{H}^s(\Omega)$.

(c) *One has $0 \notin \Sigma$ if* (1-1) *holds. If $q \ge 0$, then one has $\Sigma \subset (0, \infty)$ and* (1-1) *always holds.*

Proof. The proof is standard, but we give the details for completeness.

(a) If $u = f + v$, it is enough to find $v \in \widetilde{H}^s(\Omega)$ solving the equivalent problem

$$B_q(v, w) - \lambda(v, w)_{\mathbb{R}^n} = \widetilde{F}(w), \quad w \in \widetilde{H}^s(\Omega),$$

for a suitable $\widetilde{F} \in (\widetilde{H}^s(\Omega))^*$. Consider the bilinear form $B_q(v, w)$ for $v, w \in \widetilde{H}^s(\Omega)$. If $\mu > \|q_-\|_{L^\infty(\Omega)}$, where $q_-(x) = -\min\{0, q(x)\}$, then, for $v \in \widetilde{H}^s(\Omega)$,

$$B_q(v, v) + \mu(v, v)_{\mathbb{R}^n} \ge \|(-\Delta)^{s/2} v\|_{L^2}^2 + (\mu - \|q_-\|_{L^\infty(\Omega)})\|v\|_{L^2}^2 \ge c\|v\|_{H^s}^2.$$

By the Riesz representation theorem, there is a unique $v = G_\mu \widetilde{F}$ in $\widetilde{H}^s(\Omega)$ satisfying $B_q(v, w) + \mu(v, w)_{\mathbb{R}^n} = \widetilde{F}(w)$ for $w \in \widetilde{H}^s(\Omega)$. Now

$$B_q(v, \cdot\,) - \lambda(v, \cdot\,) = \widetilde{F}(\cdot\,) \text{ on } \widetilde{H}^s(\Omega) \quad \Longleftrightarrow \quad v = G_\mu[(\mu+\lambda)v + \widetilde{F}].$$

The operator G_μ is bounded $(\widetilde{H}^s(\Omega))^* \to \widetilde{H}^s(\Omega)$, and by compact Sobolev embedding it gives rise to a compact, self-adjoint, positive definite operator $L^2(\Omega) \to L^2(\Omega)$. The spectral theorem for compact self-adjoint operators proves (a); in particular the eigenvalues of G_μ are $\{1/(\lambda_j + \mu)\}_{j=1}^\infty$, and $\Sigma \subset [-\|q_-\|_{L^\infty}, \infty)$. In fact $\Sigma \subset (-\|q_-\|_{L^\infty}, \infty)$, since otherwise there would be a nontrivial function $u \in \widetilde{H}^s(\Omega)$ with $B_q(u, u) + \|q_-\|_{L^\infty}(u, u)_{\mathbb{R}^n} = 0$, showing that $(-\Delta)^{s/2}u = 0$ and thus $u \equiv 0$, which is a contradiction.

(b) If u is as in (a), then clearly u satisfies

$$B_q(u, v) - \lambda(u, v)_{\mathbb{R}^n} = F(v) \quad \text{for } v \in C_c^\infty(\Omega), \qquad u - f \in \widetilde{H}^s(\Omega), \tag{2-1}$$

which is equivalent to the condition in (b). Conversely, if u satisfies (2-1) for $v \in C_c^\infty(\Omega)$, then (2-1) holds for $v \in \widetilde{H}^s(\Omega)$ by density, and thus u is the unique solution in (a).

(c) Note that (1-1) states that any solution in $H^s_{\overline{\Omega}}$ is identically zero. This is stronger than stating that any solution in $\widetilde{H}^s(\Omega)$ is zero, which is equivalent to $0 \notin \Sigma$ by the Fredholm alternative. If $q \geq 0$, then it was proved in (a) that $\Sigma \subset (0, \infty)$ and thus (1-1) holds. □

DN map. By analogy with the case $s = 1$, we may define the DN map for the fractional Schrödinger equation via the bilinear form B_q for the equation given in Lemma 2.3.

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $0 < s < 1$, and let $q \in L^\infty(\Omega)$ satisfy (1-1). There is a bounded linear map*

$$\Lambda_q : X \to X^*,$$

where X is the abstract trace space $X = H^s(\mathbb{R}^n)/\widetilde{H}^s(\Omega)$, defined by

$$(\Lambda_q[f], [g]) = B_q(u_f, g), \quad f, g \in H^s(\mathbb{R}^n),$$

where $u_f \in H^s(\mathbb{R}^n)$ solves $((-\Delta)^s + q)u = 0$ in Ω with $u - f \in \widetilde{H}^s(\Omega)$. One has

$$(\Lambda_q[f], [g]) = ([f], \Lambda_q[g]), \quad f, g \in H^s(\mathbb{R}^n).$$

Proof. Let $f, g \in H^s(\mathbb{R}^n)$. Since $B_q(u_{f+\varphi}, g + \psi) = B_q(u_f, g)$ for φ, ψ in $\widetilde{H}^s(\Omega)$, the expression $(\Lambda_q[f], [g]) = B_q(u_f, g)$ is well-defined and

$$\begin{aligned} |(\Lambda_q[f], [g])| &\leq \|(-\Delta)^{s/2}u_f\|_{L^2}\|(-\Delta)^{s/2}g\|_{L^2} + \|q\|_{L^\infty}\|u_f\|_{L^2}\|g\|_{L^2} \\ &\leq C\|u_f\|_{H^s}\|g\|_{H^s} \leq C\|f\|_{H^s}\|g\|_{H^s}. \end{aligned}$$

Thus $|(\Lambda_q[f], [g])| \leq C\|[f]\|_X\|[g]\|_X$, so Λ_q is well-defined and bounded, and self-adjointness follows by taking $g = u_g$. □

If Ω has Lipschitz boundary, then $X = H^s(\Omega_e)$ and $X^* = H^{-s}_{\overline{\Omega}_e}$ with natural identifications, but functions in $H^{-s}_{\overline{\Omega}_e}$ are only uniquely determined by their restrictions to Ω_e if $s < \frac{1}{2}$. Thus, for Lipschitz domains, one should think of the DN map as an operator

$$\Lambda_q : H^s(\Omega_e) \to H^{-s}_{\overline{\Omega}_e}(\mathbb{R}^n).$$

The integral identity that allows us to solve the inverse problem is a direct consequence of Lemma 2.4. For simplicity, we will write f instead of $[f]$ for elements of X.

Lemma 2.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $0 < s < 1$, and let $q_1, q_2 \in L^\infty(\Omega)$ satisfy* (1-1). *For any $f_1, f_2 \in X$ one has*

$$((\Lambda_{q_1} - \Lambda_{q_2}) f_1, f_2) = ((q_1 - q_2) r_\Omega u_1, r_\Omega u_2)_\Omega,$$

where $u_j \in H^s(\mathbb{R}^n)$ solves $((-\Delta)^s + q_j) u_j = 0$ in Ω with $u_j|_{\Omega_e} = f_j$.

Proof. One has

$$\begin{aligned} ((\Lambda_{q_1} - \Lambda_{q_2}) f_1, f_2) &= (\Lambda_{q_1} f_1, f_2) - (f_1, \Lambda_{q_2} f_2) = B_{q_1}(u_1, u_2) - B_{q_2}(u_1, u_2) \\ &= ((q_1 - q_2) r_\Omega u_1, r_\Omega u_2)_\Omega. \end{aligned}$$

□

3. Uniqueness properties

We prove the uniqueness result for the fractional Laplacian, Theorem 1.2, which is an easy consequence of the Carleman estimates in [Rüland 2015] and the Caffarelli–Silvestre extension [2007].

Proof of Theorem 1.2. Assume first that u is a continuous bounded function in $\mathbb{R}^n$. Write $\mathbb{R}^{n+1}_+ = \{(x, y) : x \in \mathbb{R}^n,\ y > 0\}$, and denote by w the extension of u to $\mathbb{R}^{n+1}_+$ defined by

$$w(x, y) = (P_y * u)(x), \qquad P_y(x) = c_{n,s} \frac{y^{2s}}{(|x|^2 + y^2)^{(n+2s)/2}}.$$

By [Cabré and Sire 2014, Remark 3.8], w is the unique continuous bounded solution in $\overline{\mathbb{R}}^{n+1}_+$ of the Dirichlet problem

$$\operatorname{div}(y^{1-2s} \nabla w) = 0 \quad \text{in } \mathbb{R}^{n+1}_+, \qquad w|_{y=0} = u.$$

If we additionally assume that $u \in H^s(\mathbb{R}^n)$, then by [Cabré and Sire 2014, Section 3] the solution w satisfies $\int_{\mathbb{R}^{n+1}_+} y^{1-2s} |\nabla w|^2 \, dx \, dy < \infty$, and one has

$$(-\Delta)^s u = -d_s \lim_{y \to 0^+} y^{1-2s} \partial_y w(\,\cdot\,, y),$$

where the limit exists in $H^{-s}(\mathbb{R}^n)$. See [Cabré and Sire 2014] for the precise values of the constants $c_{n,s}$ and d_s.

Assume now that u is a continuous bounded function in $\mathbb{R}^n$ with $u \in H^s(\mathbb{R}^n)$, and $u|_W = (-\Delta)^s u|_W = 0$, where W is a ball in $\mathbb{R}^n$. Denote by B the ball in $\mathbb{R}^{n+1}$ with $B \cap \{y = 0\} = W$, and define $B^+ = \{(x, y) \in B : y > 0\}$. Since $u|_W = (-\Delta)^s u|_W = 0$, w satisfies

$$\operatorname{div}(y^{1-2s} \nabla w) = 0 \quad \text{in } B^+, \qquad w|_{B \cap \{y=0\}} = \lim_{y \to 0^+} y^{1-2s} \partial_y w|_{B \cap \{y=0\}} = 0.$$

The function w thus satisfies the conditions in [Rüland 2015, Proposition 2.2], and one obtains that $w|_{B^+} \equiv 0$. But w is real-analytic in $\mathbb{R}^{n+1}_+$ as the solution of an elliptic equation with real-analytic coefficients; see [Hörmander 1983, Theorem 8.6.1]. Hence $w \equiv 0$ in $\mathbb{R}^{n+1}$, which implies that $u \equiv 0$.

Finally, let $u \in H^{-r}(\mathbb{R}^n)$ for some $r > 0$, and $u|_W = (-\Delta)^s u|_W = 0$ for some ball $W \subset \mathbb{R}^n$. Consider the smooth approximations

$$u_\varepsilon = u * \varepsilon^{-n} \varphi(\,\cdot\,/\varepsilon),$$

where $\varphi \in C_c^\infty(\mathbb{R}^n)$ satisfies $\int \varphi\, dx = 1$, $\varphi \geq 0$, and $\varphi = 0$ for $|x| \geq 1$. There exist $\varepsilon_0 > 0$ and a smaller ball $W' \subset W$ such that $u_\varepsilon|_{W'} = 0$ and also $(-\Delta)^s u_\varepsilon|_{W'} = ((-\Delta)^s u) * \varepsilon^{-n}\varphi(\,\cdot\,/\varepsilon)|_{W'} = 0$ whenever $\varepsilon < \varepsilon_0$. Now each u_ε is in $H^\alpha(\mathbb{R}^n)$ for any $\alpha \in \mathbb{R}$, since $\hat{u}_\varepsilon(\xi) = m(\xi)\hat{u}(\xi)$, where $m(\xi) = \hat{\varphi}(\varepsilon\xi)$ is a Schwartz function and $\langle \xi \rangle^{-r} \hat{u}(\xi)$ is in L^2. By Sobolev embedding, each u_ε is also continuous and bounded in $\mathbb{R}^n$. The argument above implies that $u_\varepsilon \equiv 0$ whenever $\varepsilon < \varepsilon_0$, showing that $u = \lim_{\varepsilon \to 0} u_\varepsilon = 0$. □

Remark 3.1. We note that for $s = \frac{1}{2}$ the above argument simplifies: the function w in the proof is just the harmonic extension of u to $\mathbb{R}^{n+1}$, and it satisfies $w|_{W \times \{y=0\}} = \partial_y w|_{W \times \{y=0\}} = 0$. The odd extension $\tilde{w}$ of w to $W \times \mathbb{R}$ is smooth, satisfies $\Delta_{x,y} \tilde{w} = 0$, and $\tilde{w}|_{W \times \{y=0\}} = \partial_y \tilde{w}|_{W \times \{y=0\}} = 0$. Using the equation one observes that $\tilde{w}$ vanishes to infinite order on $W \times \{y = 0\}$; thus by analyticity $\tilde{w} \equiv 0$ and $u \equiv 0$.

Remark 3.2. For comparison, we recall the original argument in [Riesz 1938, Chapitre III.11] for proving a result like Theorem 1.2. There are two steps: first one uses the Kelvin transform to reduce to the case where u and $(-\Delta)^s u$ vanish outside some ball, and then one computes derivatives of u and lets $x \to \infty$ to show that all moments of $(-\Delta)^s u$ must vanish. See [Isakov 1990, Lemma 3.5.4] for another proof of the second step.

Let u be in the Sobolev space $W^{-r,q}(\mathbb{R}^n)$ for some $r \in \mathbb{R}$, where $q = 2n/(n+2s)$. By approximation, translation and scaling, we may assume that $u \in W^{t,q}(\mathbb{R}^n)$ for any $t > 0$ and $u|_B = (-\Delta)^s u|_B = 0$, where B is the unit ball. Write $f = (-\Delta)^s u$, so $f, u \in L^q \cap L^\infty$ and $u = I_{2s} f$. Define

$$v = R_{2s} u, \quad g = R_{-2s} f,$$

where $R_\alpha f(x) = |x|^{\alpha - n} f(K(x))$ and $K(x) = x/|x|^2$ is the Kelvin transform. Since $\det DK(x) = -|x|^{-2n}$ and $|K(x) - K(y)| = |x - y|/(|x||y|)$, one computes $\|R_{-2s} f\|_{L^q} = \|f\|_{L^q}$ and $R_{2s} I_{2s} f = I_{2s} R_{-2s} f$. Then $g \in L^q$, both $v = I_{2s} g$ and g vanish outside B, and

$$v(x) = c_{n,s} \int_B |x - y|^{2s-n} g(y)\, dy = 0, \quad |x| > 1.$$

In particular, letting $x \to \infty$, one gets $\int_B g(y)\, dy = 0$. Applying powers of the Laplacian to $v(x)$ we get

$$\int_B |x - y|^{2s-n-2k} g(y)\, dy = 0, \quad k \geq 0,\ |x| > 1.$$

Computing $\partial_{x_j} v(x)$ and letting $x \to \infty$ gives $\int_B y_j g(y)\, dy = 0$. Repeating this for higher-order derivatives implies that $\int_B y^\alpha g(y)\, dy = 0$ for any multi-index α; hence $g \equiv 0$. This finally gives $f \equiv 0$ and $u \equiv 0$.

The above argument seems to require that $f \in L^q$ for q close to 1 in order for $R_{-2s} f$ to be an L^p function for some p. If one starts with a solution $u \in H^{-r}$ for some r (as stated in Theorem 1.2), after approximation one gets $f \in L^2 \cap L^\infty$ and then there is an issue since $R_{-2s} f$ might have a nonintegrable singularity at 0. Thus it seems that this method is not sufficient for proving Theorem 1.2 in full generality.

4. Approximation in $L^2(\Omega)$

We will use the following Runge approximation property for solutions of the fractional Schrödinger equation. If $q \in L^\infty(\Omega)$ satisfies (1-1), we denote by P_q the Poisson operator

$$P_q : X \to H^s(\mathbb{R}^n), \quad f \mapsto u, \tag{4-1}$$

where $X = H^s(\mathbb{R}^n)/\widetilde{H}^s(\Omega)$ is the abstract space of exterior values, and $u \in H^s(\mathbb{R}^n)$ is the unique solution of $((-\Delta)^s + q)u = 0$ in Ω with $u - f \in \widetilde{H}^s(\Omega)$ given in Lemma 2.3.

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^n$ be bounded open set, let $0 < s < 1$, and let $q \in L^\infty(\Omega)$ satisfy (1-1). Let also W be any open subset of Ω_e. Consider the set*

$$\mathcal{R} = \{u|_\Omega : u = P_q f,\ f \in C_c^\infty(W)\}.$$

Then $\mathcal{R}$ is dense in $L^2(\Omega)$.

Proof. By the Hahn–Banach theorem, it is enough to show that any $v \in L^2(\Omega)$ with $(v, w)_\Omega = 0$ for all $w \in \mathcal{R}$ must satisfy $v \equiv 0$. If v is such a function, then

$$(v, r_\Omega P_q f)_\Omega = 0, \quad f \in C_c^\infty(W). \tag{4-2}$$

We claim that the formal adjoint of $r_\Omega P_q$ is given by

$$(v, r_\Omega P_q f)_\Omega = -B_q(\varphi, f), \quad f \in C_c^\infty(W), \tag{4-3}$$

where $\varphi \in H^s(\mathbb{R}^n)$ is the solution given by Lemma 2.3 of

$$((-\Delta)^s + q)\varphi = v \quad \text{in } \Omega, \qquad \varphi \in \widetilde{H}^s(\Omega).$$

In other words, $B_q(\varphi, w) = (v, r_\Omega w)_\Omega$ for any $w \in \widetilde{H}^s(\Omega)$. To prove (4-3), let $f \in C_c^\infty(W)$, and let $u_f = P_q f \in H^s(\mathbb{R}^n)$ so $u_f - f \in \widetilde{H}^s(\Omega)$. Then

$$(v, r_\Omega P_q f)_\Omega = (v, r_\Omega(u_f - f))_\Omega = B_q(\varphi, u_f - f) = -B_q(\varphi, f).$$

In the last line, we used that u_f is a solution and $\varphi \in \widetilde{H}^s(\Omega)$.

Combining (4-2) and (4-3), we have

$$B_q(\varphi, f) = 0, \quad f \in C_c^\infty(W).$$

Since $r_\Omega f = 0$, this implies

$$0 = ((-\Delta)^{s/2}\varphi, (-\Delta)^{s/2} f)_{\mathbb{R}^n} = ((-\Delta)^s \varphi, f)_{\mathbb{R}^n}, \quad f \in C_c^\infty(W).$$

In particular, $\varphi \in H^s(\mathbb{R}^n)$ satisfies

$$\varphi|_W = (-\Delta)^s \varphi|_W = 0.$$

Theorem 1.2 implies that $\varphi \equiv 0$, and thus also $v \equiv 0$. □

5. Inverse problem

It is now easy to prove the uniqueness result for the inverse problem.

Proof of Theorem 1.1. Note that if $F \in X^*$, then $F|_{W_2}$ is a distribution in W_2 with $F|_{W_2}(\varphi) = F([\varphi])$, $\varphi \in C_c^\infty(W_2)$. Now if $\Lambda_{q_1} f|_{W_2} = \Lambda_{q_2} f|_{W_2}$ for any $f \in C_c^\infty(W_1)$, the integral identity in Lemma 2.5 yields that

$$\int_\Omega (q_1 - q_2) u_1 u_2 \, dx = 0$$

whenever $u_j \in H^s(\mathbb{R}^n)$ solve $((-\Delta)^s + q_j)u_j = 0$ in Ω with exterior values in $C_c^\infty(W_j)$. Let $h \in L^2(\Omega)$, and use the approximation result, Lemma 4.1, to find sequences $(u_j^{(k)})$ of functions in $H^s(\mathbb{R}^n)$ that satisfy

$$\begin{gathered} ((-\Delta)^s + q_1)u_1^{(k)} = ((-\Delta)^s + q_2)u_2^{(k)} = 0 \quad \text{in } \Omega, \\ u_j^{(k)} \text{ have exterior values in } C_c^\infty(W_j), \\ r_\Omega u_1^{(k)} = h + r_1^{(k)}, \quad r_\Omega u_2^{(k)} = 1 + r_2^{(k)}, \end{gathered}$$

where $r_1^{(k)}, r_2^{(k)} \to 0$ in $L^2(\Omega)$ as $k \to \infty$. Inserting these solutions in the integral identity and taking the limit as $k \to \infty$ implies

$$\int_\Omega (q_1 - q_2) h \, dx = 0.$$

Since $h \in L^2(\Omega)$ was arbitrary, we conclude that $q_1 = q_2$. ☐

6. Higher-order approximation

We proceed to prove Theorem 1.3(b). The argument is similar to that in Section 4, but since the approximation is in high regularity spaces, by duality we will need to solve Dirichlet problems with data in negative-order Sobolev spaces. This follows again by duality from regularity results for the Dirichlet problem proved in [Hörmander 1965; Grubb 2015].

We will next introduce function spaces from [Grubb 2015]. Note that the smoothness indices s and $s(r)$ in this article correspond to a and $a(s)$ in [Grubb 2015]. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^∞ boundary, and let $q \in C_c^\infty(\Omega)$ satisfy the analogue of (1-1),

$$\text{if } u \in H^s(\mathbb{R}^n) \text{ solves } ((-\Delta)^s + q)u = 0 \text{ in } \Omega \text{ and } u|_{\Omega_e} = 0, \text{ then } u \equiv 0. \tag{6-1}$$

We assume q compactly supported to fit the operator theory in [Grubb 2015]. Define

$$\mathcal{E}_s(\overline{\Omega}) = e^+ d(x)^s C^\infty(\overline{\Omega}),$$

where e^+ denotes extension by zero from Ω to $\mathbb{R}^n$, and d is a C^∞ function in $\overline{\Omega}$, positive in Ω, and satisfying $d(x) = \operatorname{dist}(x, \partial\Omega)$ near $\partial\Omega$. If $r > s - \frac{1}{2}$ we will also consider the Banach space $H^{s(r)}(\overline{\Omega})$ which arises as the exact solution space of functions u satisfying

$$r_\Omega((-\Delta)^s + q)u \in H^{r-2s}(\Omega), \quad u|_{\Omega_e} = 0.$$

We will not give the actual definition, but instead we will use the following properties from [Grubb 2015].

Lemma 6.1. *For any $r > s - \frac{1}{2}$, there is a Banach space $H^{s(r)}(\bar{\Omega})$ with the following properties*:

(a) $H^{s(r)}(\bar{\Omega}) \subset H^{s-1/2}_{\bar{\Omega}}$ *with continuous inclusion.*

(b) $H^{s(r)}(\bar{\Omega}) = H^{r}_{\bar{\Omega}}$ *if* $r \in \left(s - \frac{1}{2}, s + \frac{1}{2}\right)$.

(c) *The operator* $r_\Omega((-\Delta)^s + q)$ *is a homeomorphism from* $H^{s(r)}(\bar{\Omega})$ *onto* $H^{r-2s}(\Omega)$.

(d) $H^{r}_{\bar{\Omega}} \subset H^{s(r)}(\bar{\Omega}) \subset H^{r}_{\mathrm{loc}}(\Omega)$ *with continuous inclusions*; *i.e.*, *multiplication by any* $\chi \in C^\infty_c(\Omega)$ *is bounded* $H^{s(r)}(\bar{\Omega}) \to H^r(\Omega)$.

(e) $\mathcal{E}_s(\bar{\Omega}) = \bigcap_{r>s-1/2} H^{s(r)}(\bar{\Omega})$, *and* $\mathcal{E}_s(\bar{\Omega})$ *is dense in* $H^{s(r)}(\bar{\Omega})$.

Proof. Parts (a) and (b) follow from [Grubb 2015, Section 1]. Part (c) follows since $r_\Omega((-\Delta)^s + q) : H^{s(r)}(\bar{\Omega}) \to H^{r-2s}(\Omega)$ is a Fredholm operator [Grubb 2015, Theorem 2], it has a finite-dimensional kernel and range complement independent of r [Grubb 2014, Theorem 3.5], and for $r = s$ the kernel and range complement are trivial using (6-1) and Lemma 2.3. Part (d) follows from (c) and (a), or alternatively from the definitions in [Grubb 2015, Section 1]. Part (e) is in [Grubb 2015, Proposition 4.1]. □

We next prove an approximation result in the space $\mathcal{E}_s(\bar{\Omega})$, equipped with the topology induced by the norms $\{\|\cdot\|_{H^{s(m)}(\bar{\Omega})}\}_{m=1}^{\infty}$. Then $\mathcal{E}_s(\bar{\Omega})$ is a Fréchet space.

Lemma 6.2. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain with* C^∞ *boundary*, *let* $0 < s < 1$, *let* W *be an open subset of* Ω_e, *and let* $q \in C^\infty_c(\Omega)$ *satisfy* (6-1). *If* P_q *is the Poisson operator in* (4-1), *define*

$$\mathcal{R} = \{e^+ r_\Omega P_q f : f \in C^\infty_c(W)\}.$$

Then $\mathcal{R}$ *is a dense subset of* $\mathcal{E}_s(\bar{\Omega})$.

Proof. Note that $\mathcal{R} \subset \mathcal{E}_s(\bar{\Omega})$, since for $f \in C^\infty_c(W)$ one has $P_q f = f + v$, where $r_\Omega((-\Delta)^s + q)v \in C^\infty(\bar{\Omega})$ and $v|_{\Omega_e} = 0$; hence $v \in \mathcal{E}_s(\bar{\Omega})$ by Lemma 6.1.

Let L be a continuous linear functional on $\mathcal{E}_s(\bar{\Omega})$ that satisfies

$$L(e^+ r_\Omega P_q f) = 0, \quad f \in C^\infty_c(W).$$

It is enough to show that $L \equiv 0$, since then $\mathcal{R}$ will be dense by the Hahn–Banach theorem.

By the properties of Fréchet spaces, there exists an integer r so that

$$|L(u)| \le C \sum_{m=1}^{r} \|u\|_{H^{s(m)}(\bar{\Omega})} \le C' \|u\|_{H^{s(r)}(\bar{\Omega})}, \quad u \in \mathcal{E}_s(\bar{\Omega}).$$

Since $\mathcal{E}_s(\bar{\Omega})$ is dense in $H^{s(r)}(\bar{\Omega})$, we know L has a unique bounded extension $\bar{L} \in (H^{s(r)}(\bar{\Omega}))^*$. Consider next the homeomorphism in Lemma 6.1,

$$T = r_\Omega((-\Delta)^s + q) : H^{s(r)}(\bar{\Omega}) \to H^{r-2s}(\Omega).$$

Its adjoint is a bounded map between the dual Banach spaces,

$$T^* : (H^{r-2s}(\Omega))^* \to (H^{s(r)}(\bar{\Omega}))^*.$$

The map T^* is also a homeomorphism, with inverse given by $(T^{-1})^*$. Using the identification

$$(H^{r-2s}(\Omega))^* = H^{-r+2s}_{\overline{\Omega}}$$

one has

$$T^*v(w) = (v, Tw), \quad w \in H^{s(r)}(\overline{\Omega}).$$

Now let $v \in H^{-r+2s}_{\overline{\Omega}}$ be the unique function satisfying $T^*v = \bar{L}$, and choose a sequence $(v_j)_{j=1}^{\infty} \subset C_c^\infty(\Omega)$ with $v_j \to v$ in H^{-r+2s}. If $f \in C_c^\infty(W)$, recall that $e^+ r_\Omega P_q f = P_q f - f$, and observe that

$$\begin{aligned} 0 = L(e^+ r_\Omega P_q f) = \bar{L}(P_q f - f) = T^*v(P_q f - f) = (v, T(P_q f - f)) \\ = -(v, Tf) = -\lim(v_j, ((-\Delta)^s + q)f) = -\lim(((-\Delta)^s + q)v_j, f). \end{aligned}$$

Here we used that $TP_q f = 0$ and $v_j \in C_c^\infty(\Omega)$. Since $f \in C_c^\infty(W)$, we may take the limit as $j \to \infty$ and obtain that

$$((-\Delta)^s v, f) = 0, \quad f \in C_c^\infty(W).$$

Thus $v \in H^{-r+2s}(\mathbb{R}^n)$ satisfies

$$v|_W = (-\Delta)^s v|_W = 0.$$

By Theorem 1.2 it follows that $v \equiv 0$. This implies that $\bar{L} \equiv 0$ and $L \equiv 0$ as required. □

Proof of Theorem 1.3. Let $\Omega \subset \Omega_1 \subset \mathbb{R}^n$ be open sets with $\Omega_1 \setminus \overline{\Omega} \neq \varnothing$ and Ω bounded. Since $\Omega_1 \setminus \overline{\Omega} \neq \varnothing$, we may find a small ball W with $\overline{W} \subset \Omega_1 \setminus \overline{\Omega}$. Lemma 4.1 implies that any $f \in L^2(\Omega)$ can be approximated in $L^2(\Omega)$ by functions $u|_\Omega$, where u solves $((-\Delta)^s + q)u = 0$ in Ω and $\operatorname{supp}(u) \subset \overline{\Omega} \cup \overline{W}$. Since $\overline{\Omega} \cup \overline{W} \subset \overline{\Omega}_1$, part (a) follows.

As for part (b), if $f \in C^\infty(\overline{\Omega})$ and if $g = e^+ d(x)^s f \in \mathcal{E}_s(\overline{\Omega})$, Lemma 6.2 ensures that there is a sequence $(u_j)_{j=1}^{\infty} \subset H^s(\mathbb{R}^n)$ with

$$((-\Delta)^s + q)u_j = 0 \quad \text{in } \Omega, \qquad \operatorname{supp}(u_j) \subset \overline{\Omega}_1,$$

so that $e^+ r_\Omega u_j \in \mathcal{E}_s(\overline{\Omega})$ and

$$e^+ r_\Omega u_j \to g \quad \text{in } \mathcal{E}_s(\overline{\Omega}).$$

The result will follow if we can show that

$$M : C^\infty(\overline{\Omega}) \to \mathcal{E}_s(\overline{\Omega}), \quad Mf = e^+ d(x)^s f,$$

is a homeomorphism, since then applying $M^{-1} = d(x)^{-s} r_\Omega$ gives

$$d(x)^{-s} r_\Omega u_j \to f \quad \text{in } C^\infty(\overline{\Omega}).$$

But M is a bijective linear map between Fréchet spaces and has closed graph: if $f_j \to f$ in C^∞ and $Mf_j \to h$ in $\mathcal{E}_s$, then also $Mf_j \to Mf$ in L^∞ and one obtains $Mf = h$ by uniqueness of distributional limits. Thus M is a homeomorphism by the closed graph and open mapping theorems (in other words, there is a unique Fréchet space topology on $\mathcal{E}_s(\overline{\Omega})$ stronger than the Hausdorff topology inherited from $\mathcal{D}'(\mathbb{R}^n)$). □

Remark 6.3. Let us note the following consequence of Theorem 1.3(b): if $k \geq 0$ and $R > 1$ are fixed, then for any $g \in C^k(\bar{B}_1)$ and for any $\varepsilon > 0$ there is a function $u \in H^s(\mathbb{R}^n)$ satisfying

$$(-\Delta)^s u = 0 \quad \text{in } B_1, \qquad \operatorname{supp}(u) \subset \bar{B}_R, \qquad \|u - g\|_{C^k(\bar{B}_1)} < \varepsilon.$$

This can be seen by taking $\Omega = B_r$ and $\Omega_1 = B_R$, where $1 < r < R$, and by choosing $f \in C^\infty(\bar{B}_r)$ with $\|f - d(x)^{-s} g\|_{C^k(\bar{B}_1)}$ small enough.

Appendix: The DN map

The abstract definition of the DN map Λ_q in Section 2 is sufficient for the formulation and solution of the inverse problem. However, in this appendix we will give more concrete descriptions of the DN map, valid under stronger regularity assumptions. For simplicity we assume that the boundary and the potential are C^∞.

DN map and $(-\Delta)^s$.

Lemma A.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^∞ boundary, let $0 < s < 1$, and let $q \in C_c^\infty(\Omega)$ satisfy* (1-1). *For any $\beta \geq 0$ satisfying $s - \frac{1}{2} < \beta < \frac{1}{2}$, the restriction of Λ_q to $H^{s+\beta}(\Omega_e)$ is the map*

$$\Lambda_q : H^{s+\beta}(\Omega_e) \to H^{-s+\beta}(\Omega_e), \quad \Lambda_q f = (-\Delta)^s u_f|_{\Omega_e},$$

where $u_f \in H^{s+\beta}(\mathbb{R}^n)$ solves $((-\Delta)^s + q)u = 0$ in Ω with $u|_{\Omega_e} = f$.

Proof. First we use a result from [Vishik and Eskin 1965]; see also [Grubb 2015]: if $\beta \in \left[0, \frac{1}{2}\right)$, then for any $f \in H^{s+\beta}(\Omega_e)$ there is a unique $u = u_f \in H^{s+\beta}(\mathbb{R}^n)$ satisfying

$$((-\Delta)^s + q)u = 0 \quad \text{in } \Omega, \qquad u|_{\Omega_e} = f.$$

In fact [Grubb 2015, Theorem 3.1] asserts Fredholm solvability for the inhomogeneous problem, but the result above can be reduced to this case by taking an $H^{s+\beta}$ extension of f to $\mathbb{R}^n$, and Fredholm solvability implies unique solvability since the finite-dimensional kernel and range complement are independent of β by [Grubb 2014, Theorem 3.5] and they are trivial when $\beta = 0$ by Lemma 2.3.

Now for $f, g \in H^{s+\beta}(\Omega_e)$, with $\beta \in \left[0, \frac{1}{2}\right)$, let $u_f \in H^{s+\beta}(\mathbb{R}^n)$ be the solution obtained above and let $e_g \in H^{s+\beta}(\mathbb{R}^n)$ be some extension of g. Then, by definition,

$$\begin{aligned}(\Lambda_q f, g) &= ((-\Delta)^{s/2} u_f, (-\Delta)^{s/2} e_g)_{\mathbb{R}^n} + (q r_\Omega u_f, r_\Omega e_g)_\Omega \\ &= ((-\Delta)^s u_f, e_g)_{\mathbb{R}^n} + (q r_\Omega u_f, r_\Omega e_g)_\Omega\end{aligned}$$

since $((-\Delta)^{s/2} u, (-\Delta)^{s/2} v)_{\mathbb{R}^n} = ((-\Delta)^s u, v)_{\mathbb{R}^n}$ holds first for Schwartz functions by the Parseval identity, and then also for $u, v \in H^s(\mathbb{R}^n)$ by density.

It remains to show that whenever $\alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, $u \in H^{-\alpha}(\mathbb{R}^n)$, $v \in H^\alpha(\mathbb{R}^n)$, then

$$(u, v)_{\mathbb{R}^n} = (r_\Omega u, r_\Omega v)_\Omega + (r_{\Omega_e} u, r_{\Omega_e} v)_{\Omega_e} \tag{A-1}$$

in the sense of distributional pairings. If (A-1) is true, then the assumption $\beta \in \left(s-\frac{1}{2}, \frac{1}{2}\right)$ implies $(-\Delta)^s u_f \in H^{-s+\beta}(\mathbb{R}^n)$ with $-s+\beta \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, and since u_f is a solution in Ω one has

$$(\Lambda_q f, g) = ((-\Delta)^s u_f, e_g)_{\mathbb{R}^n} + (q r_\Omega u_f, r_\Omega e_g)_\Omega = (r_{\Omega_e}(-\Delta)^s u_f, g)_{\Omega_e},$$

which concludes the proof.

To show (A-1), let χ_Ω be the characteristic function of Ω. This is a pointwise multiplier on $H^\gamma(\mathbb{R}^n)$ for $\gamma \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ [Triebel 2002], and the same is true for $1-\chi_\Omega$. We may write $u = \chi_\Omega u + (1-\chi_\Omega)u$ and similarly for v, and then

$$(u, v)_{\mathbb{R}^n} = (\chi_\Omega u, \chi_\Omega v)_{\mathbb{R}^n} + ((1-\chi_\Omega)u, (1-\chi_\Omega)v)_{\mathbb{R}^n},$$

where the cross terms vanish first for Schwartz u, v and then in general by density. Now $\chi_\Omega u$ is in $H^{-\alpha}_{\overline{\Omega}}$, and hence can be approximated by functions in $C_c^\infty(\Omega)$. Using similar approximations for the other functions and restricting to Ω and Ω_e implies (A-1). ☐

DN map and $\mathcal{N}_s$. Several nonlocal Neumann boundary operators appear in the literature; see [Dipierro et al. 2017a; Grubb 2016]. We will relate Λ_q to the nonlocal Neumann boundary operator $\mathcal{N}_s$ introduced in [Dipierro et al. 2017a], defined pointwise by

$$\mathcal{N}_s u(x) = c_{n,s} \int_\Omega \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy, \quad x \in \Omega_e. \tag{A-2}$$

The next lemma contains a definition that applies to Sobolev functions. The result states that knowing $\Lambda_q f|_W$ for $f \in C_c^\infty(W)$ is equivalent to knowing $\mathcal{N}_s u_f|_W$ for $f \in C_c^\infty(W)$, since $\Lambda_q f|_W$ and $\mathcal{N}_s u_f|_W$ only differ by quantities that do not depend on the unknown potential q.

Lemma A.2. *Assume the conditions in Lemma A.1. One has*

$$\Lambda_q f = \mathcal{N}_s u_f - mf + (-\Delta)^s (E_0 f)|_{\Omega_e}, \quad f \in H^{s+\beta}(\Omega_e),$$

where, for $\gamma > -\frac{1}{2}$, $\mathcal{N}_s$ is the map

$$\mathcal{N}_s : H^\gamma(\mathbb{R}^n) \to H^\gamma_{\mathrm{loc}}(\Omega_e), \quad \mathcal{N}_s u = mu|_{\Omega_e} + (-\Delta)^s(\chi_\Omega u)|_{\Omega_e},$$

where $m \in C^\infty(\Omega_e)$ is given by $m(x) = c_{n,s} \int_\Omega 1/|x-y|^{n+2s}\, dy$ and χ_Ω is the characteristic function of Ω. Also, E_0 is extension by zero. If $u \in L^2(\mathbb{R}^n)$, then $\mathcal{N}_s u \in L^2_{\mathrm{loc}}(\Omega_e)$ is given a.e. by the formula (A-2).

Proof. If $u \in H^\gamma(\mathbb{R}^n)$ with $\gamma > -\frac{1}{2}$, then $mu|_{\Omega_e} \in H^\gamma_{\mathrm{loc}}(\Omega_e)$. By the pointwise multiplier property of χ_Ω, we have $\chi_\Omega u \in H^\alpha(\mathbb{R}^n)$ for some $\alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ and $(-\Delta)^s(\chi_\Omega u) \in H^{\alpha-2s}(\mathbb{R}^n)$. However, if $\varphi, \psi \in C_c^\infty(\mathbb{R}^n)$ satisfy $\varphi = 1$ near $\overline{\Omega}$ and $\psi = 1$ near $\mathrm{supp}(\varphi)$, then for any $r, t \in \mathbb{R}$ one has

$$(1-\psi)(-\Delta)^s \varphi : H^{-r}(\mathbb{R}^n) \to H^t(\mathbb{R}^n)$$

by the pseudolocal property of Fourier multipliers. Thus one also has $(-\Delta)^s(\chi_\Omega u)|_{\Omega_e} \in H^t_{\mathrm{loc}}(\Omega_e)$ for any t, and $\mathcal{N}_s$ is well-defined and maps $H^\gamma(\mathbb{R}^n)$ to $H^\gamma_{\mathrm{loc}}(\Omega_e)$ for $\gamma > -\frac{1}{2}$.

Moreover, if $u \in L^2(\mathbb{R}^n)$ and if $\varphi_j \in C_c^\infty(\Omega)$ satisfy $\varphi_j \to \chi_\Omega u$ in $L^2(\mathbb{R}^n)$, then the pseudolocal property implies

$$(-\Delta)^s(\varphi_j)|_{\Omega_e} \to (-\Delta)^s(\chi_\Omega u)|_{\Omega_e} \quad \text{in } L^2_{\mathrm{loc}}(\Omega_e).$$

After extracting a subsequence (using the diagonal argument), one has convergence a.e. in Ω_e. Thus the pointwise expression (A-2) for a.e. $x \in \Omega_e$ follows from the standard formula

$$(-\Delta)^s\varphi(x) = c_{n,s}\int_{\mathbb{R}^n} \frac{\varphi(x)-\varphi(y)}{|x-y|^{n+2s}}\,dy, \quad \varphi \in C_c^\infty(\Omega),\ x \in \Omega_e.$$

Let us prove the formula for Λ_q. If $f \in H^{s+\beta}(\Omega_e)$, then $f \in H^\alpha(\Omega_e)$ for some $\alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ and hence $E_0 f, u_f \in H^\alpha(\mathbb{R}^n)$. Recall also that χ_Ω and $1-\chi_\Omega$ are pointwise multipliers on $H^\alpha(\mathbb{R}^n)$. Then

$$\begin{aligned}\Lambda_q f &= (-\Delta)^s u_f|_{\Omega_e} = (-\Delta)^s(\chi_\Omega u_f)|_{\Omega_e} + (-\Delta)^s((1-\chi_\Omega)u_f)|_{\Omega_e}\\ &= \mathcal{N}_s u_f - mf + (-\Delta)^s(E_0 f)|_{\Omega_e}.\end{aligned}$$

□

Nonlocal diffusion. Finally, we will give a heuristic interpretation of the quantity $\Lambda_q f(x)$ in terms of nonlocal diffusions [Andreu-Vaillo et al. 2010]. This discussion is mostly for illustrative purposes, so we will not give precise arguments and will restrict to the case $q = 0$.

We begin with a macroscopic description of nonlocal diffusion in $\mathbb{R}^n$. Suppose that $u(x,t)$ describes the density of particles at a point $x \in \mathbb{R}^n$ at time t. Given an initial density $u_0(x)$, we assume that $u(x,t)$ is obtained as a solution of the nonlocal diffusion equation

$$\begin{cases} \partial_t u + (-\Delta)^s u = 0 & \text{in } \mathbb{R}^n \times \{t > 0\},\\ u|_{t=0} = u_0. \end{cases} \tag{A-3}$$

Taking Fourier transforms in x, the solution at time t is given by

$$u(t,x) = (p_t * u_0)(x),$$

where $p_t(x) = \mathscr{F}^{-1}\{e^{-t|\xi|^{2s}}\}$ is the probability density function of the Lévy process X_t with infinitesimal generator $-(-\Delta)^s$. If $s = 1$, then p_t is a Gaussian, but for $0 < s < 1$ it is a heavy-tailed distribution with $p_t(x) \sim |x|^{-n-2s}$ for large $|x|$ (for $s = \frac{1}{2}$, $p_t(x) = c_n t(t^2+|x|^2)^{-(n+1)/2}$). The Lévy process X_t also gives a microscopic description of $u(x,t)$: it is obtained as the expected value

$$u(x,t) = \mathbb{E}_x[u_0(X_t)],$$

which expresses how many Lévy particles from the initial distribution u_0 have jumped to x at time t. See [Applebaum 2004; Chen et al. 2010] for Lévy processes.

Let now $\Omega \subset \mathbb{R}^n$ be a bounded open set. We consider the following Dirichlet problem for nonlocal diffusion: given $u_0 \in H^s_{\bar{\Omega}}$, find u so that

$$\begin{cases} \partial_t u + (-\Delta)^s u = 0 & \text{in } \Omega \times \{t > 0\},\\ u|_{\Omega_e \times \{t>0\}} = 0,\\ u|_{\mathbb{R}^n \times \{t=0\}} = u_0. \end{cases} \tag{A-4}$$

The solution is easily obtained in the form

$$u(x,t) = \sum_{j=1}^{\infty} e^{-\lambda_j t} c_j \rho_j(x), \tag{A-5}$$

where $u_0 = \sum_{j=1}^{\infty} c_j \rho_j$ and $\{\rho_j\}_{j=1}^{\infty} \subset H^s_{\overline{\Omega}}$ is an orthonormal basis of $L^2_{\overline{\Omega}}$ consisting of eigenfunctions for $(-\Delta)^s$ with eigenvalues λ_j, so that $(-\Delta)^s \rho_j = \lambda_j \rho_j$ in Ω, $\rho_j|_{\Omega_e} = 0$, and $0 < \lambda_1 \le \lambda_2 \le \cdots \to \infty$. The probabilistic interpretation is that we are looking at Lévy particles in Ω that are terminated when they reach the exterior. One has

$$u(x,t) = \mathbb{E}_x[u_0(X_t) 1_{\{t<\tau\}}],$$

where τ is the time when the Lévy process exits Ω.

By the Duhamel principle and a standard reduction to homogeneous Dirichlet values, given any $f \in H^s(\Omega_e)$ and any $e_f \in H^s(\mathbb{R}^n)$ with $e_f|_{\Omega_e} = f$, we can also solve the equation

$$\begin{cases} \partial_t v + (-\Delta)^s v = 0 & \text{in } \Omega \times \{t > 0\}, \\ v(\,\cdot\,,t)|_{\Omega_e} = f & \text{for } t > 0, \end{cases} \tag{A-6}$$

with initial value $v|_{\mathbb{R}^n \times \{t=0\}} = e_f$. Another solution of (A-6) is given by $v_s(x,t) = u_f(x)$, if $u_f \in H^s(\mathbb{R}^n)$ solves $(-\Delta)^s u = 0$ in Ω with $u|_{\Omega_e} = f$. The function u_f is the unique steady state of (A-6), since $v - v_s$ solves (A-4) for some u_0, and (A-5) implies

$$\|v(\,\cdot\,,t) - u_f\|_{H^s} \to 0 \quad \text{as } t \to \infty.$$

Now, given $f \in H^s(\Omega_e)$ and the solution u_f of the Dirichlet problem, we may consider two nonlocal diffusions with initial value u_f:

- the free diffusion (A-3) in $\mathbb{R}^n$ with solution $u(x,t)$,
- the diffusion (A-6) whose exterior value is fixed to be f.

If t is small and $x \in \Omega_e$, then $u(x,t)$ formally satisfies

$$\begin{aligned} u(x,t) &= u(x,0) + \partial_t u(x,0)t + O(t^2) = f(x) - (-\Delta)^s u(x,0)t + O(t^2) \\ &= f(x) - (\Lambda_0 f)(x)t + O(t^2) \end{aligned}$$

by Lemma A.1. Thus the DN map may be interpreted as follows:

- $-\Lambda_0 f(x)$ is the (infinitesimal) amount of particles migrating to x in the free diffusion that starts from the steady state u_f.
- $\Lambda_0 f(x)$ is the (infinitesimal) cost required to maintain the exterior value f at x in the steady state nonlocal diffusion.

Similar remarks apply to Λ_q at least if $q \ge 0$. We refer to [Chen et al. 2006] for some facts on the related stochastic processes, and to [Piiroinen and Simon 2017] for stochastic interpretations of the usual Calderón problem.

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SHARP STRICHARTZ INEQUALITIES FOR FRACTIONAL AND HIGHER-ORDER SCHRÖDINGER EQUATIONS

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We investigate a class of sharp Fourier extension inequalities on the planar curves $s = |y|^p,\ p > 1$. We identify the mechanism responsible for the possible loss of compactness of nonnegative extremizing sequences, and prove that extremizers exist if $1 < p < p_0$ for some $p_0 > 4$. In particular, this resolves the dichotomy of Jiang, Pausader, and Shao concerning the existence of extremizers for the Strichartz inequality for the fourth-order Schrödinger equation in one spatial dimension. One of our tools is a geometric comparison principle for n-fold convolutions of certain singular measures in $\mathbb{R}^d$, developed in the companion paper of Oliveira e Silva and Quilodrán (*Math. Proc. Cambridge Philos. Soc.*, (2019)). We further show that any extremizer exhibits fast L^2-decay in physical space, and so its Fourier transform can be extended to an entire function on the whole complex plane. Finally, we investigate the extent to which our methods apply to the case of the planar curves $s = y|y|^{p-1},\ p > 1$.

1. Introduction

Gaussians are known to extremize certain Strichartz estimates in low dimensions. Consider, for instance, the Strichartz inequality for the homogeneous Schrödinger equation in d spatial dimensions,

$$\|e^{-it\Delta} f\|_{L^{2+4/d}_{x,t}(\mathbb{R}^{d+1})} \le \boldsymbol{S}(d)\|f\|_{L^2(\mathbb{R}^d)}, \tag{1-1}$$

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with optimal constant given by

$$\boldsymbol{S}(d) := \sup_{0 \neq f \in L^2} \frac{\|e^{-it\Delta} f\|_{L^{2+4/d}_{x,t}(\mathbb{R}^{d+1})}}{\|f\|_{L^2(\mathbb{R}^d)}}. \tag{1-2}$$

That $\boldsymbol{S}(d) < \infty$ is of course due to the original work of Strichartz [1977], which in turn had precursors in [Tomas 1975; Segal 1976]. If $d \in \{1, 2\}$, then Gaussians extremize (1-1), and therefore $\boldsymbol{S}(1) = 12^{-1/12}$ and $\boldsymbol{S}(2) = 2^{-1/2}$. This was originally established in [Foschi 2007; Hundertmark and Zharnitsky 2006], and alternative proofs were subsequently given in [Bennett et al. 2009; 2015; Gonçalves 2019]. All of these approaches ultimately rely on the fact that the Strichartz exponent $2 + \frac{4}{d}$ is an even integer if $d \in \{1, 2\}$, which in turn allows us to recast inequality (1-1) in convolution form. This is a powerful technique that has proved very successful in tackling a number of problems in sharp Fourier restriction theory; see the recent survey [Foschi and Oliveira e Silva 2017].

In the recent work [Oliveira e Silva and Quilodrán 2018], we explored the convolution structure of a family of Strichartz inequalities for higher-order Schrödinger equations in two spatial dimensions in order to answer a question concerning the existence of extremizers that had appeared in the previous literature. Our purpose with the present work is three-fold. Firstly, we resolve the dichotomy from [Jiang et al. 2010] concerning the existence of extremizers for the Strichartz inequality for the fourth-order Schrödinger equation in one spatial dimension. This is related to the Fourier extension problem on the planar curve $s = y^4$. Secondly, we study similar questions in the more general setting of the Fourier extension problem on the curve $s = |y|^p$ for arbitrary $p > 1$. We also consider *odd* curves $s = y|y|^{p-1}$, $p > 1$, the case $p = 3$ relating to the Airy–Strichartz inequality [Farah and Versieux 2018; Frank and Sabin 2018; Shao 2009]. Lastly, we study superexponential decay and analyticity of the corresponding extremizers and their Fourier transform via a bootstrapping procedure.

Jiang, Pausader, and Shao [Jiang et al. 2010] considered the fourth-order Schrödinger equation with L^2 initial datum in one spatial dimension,

$$\begin{cases} i\partial_t u - \mu\partial_x^2 u + \partial_x^4 u = 0, & (x, t) \in \mathbb{R} \times \mathbb{R}, \\ u(\cdot, 0) = f \in L^2_x(\mathbb{R}), \end{cases} \tag{1-3}$$

where $u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, and $\mu \geq 0$. By scaling, one may restrict attention to $\mu \in \{0, 1\}$. The solution of the Cauchy problem (1-3) can be written in terms of the propagator

$$u(x, t) = e^{it(\partial_x^4 - \mu\partial_x^2)} f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} e^{it(\xi^4 + \mu\xi^2)} \hat{f}(\xi)\, \mathrm{d}\xi,$$

where the spatial Fourier transform is defined as[1]

$$\hat{f}(\xi) := \int_{\mathbb{R}} e^{-ix\xi} f(x)\, \mathrm{d}x.$$

[1]The Fourier transform will occasionally be denoted by $\mathcal{F}(f) = \hat{f}$.

The solution disperses as $|t| \to \infty$, and consequently the following Strichartz inequality due to Kenig, Ponce, and Vega [Kenig et al. 1991, Theorem 2.1] holds:[2]

$$\|D_\mu^{\frac{1}{3}} e^{it(\partial_x^4 - \mu\partial_x^2)} f\|_{L^6_{x,t}(\mathbb{R}^{1+1})} \lesssim \|f\|_{L^2(\mathbb{R})}. \tag{1-4}$$

The main result of [Jiang et al. 2010] is a linear profile decomposition for (1-3), which uses a refinement of the Strichartz inequality (1-4) in the scale of Besov spaces, together with improved localized Fourier restriction estimates. As a consequence, the authors of [Jiang et al. 2010] establish a dichotomy result for the existence of extremizers for (1-4) when $\mu = 0$, which can be summarized as follows: Consider the sharp inequality in multiplier form

$$\|D_0^{\frac{1}{3}} e^{it\partial_x^4} f\|_{L^6_{x,t}(\mathbb{R}^{1+1})} \le \boldsymbol{M} \|f\|_{L^2(\mathbb{R})}, \tag{1-5}$$

with optimal constant given by

$$\boldsymbol{M} := \sup_{0 \neq f \in L^2} \frac{\|D_0^{\frac{1}{3}} e^{it\partial_x^4} f\|_{L^6_{x,t}(\mathbb{R}^{1+1})}}{\|f\|_{L^2(\mathbb{R})}}. \tag{1-6}$$

Then [Jiang et al. 2010, Theorem 1.8] states that either an extremizer for (1-5) exists, or there exist a sequence $\{a_n\} \subset \mathbb{R}$ satisfying $|a_n| \to \infty$ as $n \to \infty$ and a function $f \in L^2$ such that

$$\boldsymbol{M} = \lim_{n\to\infty} \frac{\|D_0^{\frac{1}{3}} e^{it\partial_x^4} (e^{ia_n x} f)\|_{L^6_{x,t}(\mathbb{R}^{1+1})}}{\|f\|_{L^2(\mathbb{R})}}.$$

In the latter case, one necessarily has $\boldsymbol{M} = \boldsymbol{S}(1)$, where $\boldsymbol{S}(1)$ denotes the optimal constant defined in (1-2). Our first main result resolves this dichotomy.

Theorem 1.1. *There exists an extremizer for* (1-5).

Theorem 1.1 will follow from a more general result which we now introduce. As noted in [Kenig et al. 1991, §2], the operator $D_0^{1/3} e^{it\partial_x^4}$ is nothing but a constant multiple of the Fourier transform at the point $(-x, -t) \in \mathbb{R}^2$ of the singular measure

$$\mathrm{d}\sigma_4(y, s) = \boldsymbol{\delta}(s - y^4)|y|^{\frac{1}{3}}\, \mathrm{d}y\, \mathrm{d}s \tag{1-7}$$

defined on the curve $s = y^4$. As in [Oliveira e Silva and Quilodrán 2018, §6.4], one is naturally led to consider generic power curves $s = |y|^p$. The corresponding inequality is

$$\|\mathcal{M}_p(f)\|_{L^6_{x,t}(\mathbb{R}^{1+1})} \le \boldsymbol{M}_p \|f\|_{L^2(\mathbb{R})}, \tag{1-8}$$

where the multiplier operator $\mathcal{M}_p$ is defined as

$$\mathcal{M}_p(f)(x,t) = D_0^{\frac{p-2}{6}} e^{it|\partial_x|^p} f(x).$$

[2]Given $\mu \in \{0, 1\}$ and $\alpha \in \mathbb{R}$, we follow the notation from [Jiang et al. 2010] and denote by D_μ^α the differentiation operator $D_\mu^\alpha f(x) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} (\mu + 6\xi^2)^{\alpha/2} \hat{f}(\xi)\, \mathrm{d}\xi$.

Inequality (1-8) can be equivalently restated as a Fourier extension inequality,

$$\|\mathcal{E}_p(f)\|_{L^6(\mathbb{R}^2)} \le \boldsymbol{E}_p \|f\|_{L^2(\mathbb{R})}, \tag{1-9}$$

or in convolution form as

$$\|f\sigma_p * f\sigma_p * f\sigma_p\|_{L^2(\mathbb{R}^2)} \le \boldsymbol{C}_p^3 \|f\|_{L^2(\mathbb{R})}^3. \tag{1-10}$$

Here, the singular measure σ_p is defined in accordance with (1-7) by

$$\mathrm{d}\sigma_p(y,s) = \boldsymbol{\delta}(s - |y|^p)|y|^{\frac{p-2}{6}} \,\mathrm{d}y \,\mathrm{d}s, \tag{1-11}$$

and the Fourier extension operator $\mathcal{E}_p(f) = \mathcal{F}(f\sigma_p)(-\,\cdot)$ is given by

$$\mathcal{E}_p(f)(x,t) = \int_{\mathbb{R}} e^{ixy} e^{it|y|^p} |y|^{\frac{p-2}{6}} f(y) \,\mathrm{d}y, \tag{1-12}$$

so that

$$6^{\frac{p-2}{12}} \mathcal{E}_p(\hat{f}) = 2\pi \mathcal{M}_p(f).$$

If f is an extremizer for (1-9), then f is likewise an extremizer for (1-10), and $\mathcal{F}^{-1}(f)$ is an extremizer for (1-8). Thus these three existence problems are essentially equivalent. The convolution form (1-10) also shows that the search for extremizers can be restricted to the class of *nonnegative* functions. An application of Plancherel's theorem further reveals that the corresponding optimal constants satisfy

$$\boldsymbol{E}_p^6 = (2\pi)^2 \boldsymbol{C}_p^6 = (2\pi)^3 6^{1-\frac{p}{2}} \boldsymbol{M}_p^6.$$

Our next result extends the dichotomy proved in [Jiang et al. 2010, Theorem 1.8] to the case of arbitrary exponents $p > 1$. It states that one of two possible scenarios occurs, *compactness* or *concentration* at a point. We make the latter notion precise.

Definition 1.2. A sequence of functions $\{f_n\} \subset L^2(\mathbb{R})$ *concentrates at a point* $y_0 \in \mathbb{R}$ if, for every $\varepsilon, \rho > 0$, there exists $N \in \mathbb{N}$ such that, for every $n \ge N$,

$$\int_{|y-y_0|\ge\rho} |f_n(y)|^2 \,\mathrm{d}y < \varepsilon \|f_n\|_{L^2(\mathbb{R})}^2.$$

We choose to phrase our second main result in terms of the convolution inequality (1-10) because, as we shall see, condition (1-13) has a very simple geometric meaning in terms of the boundary value of the relevant 3-fold convolution measure.

Theorem 1.3. *Let* $p > 1$. *If*

$$\boldsymbol{C}_p^6 > \frac{2\pi}{\sqrt{3}\,p(p-1)}, \tag{1-13}$$

then any extremizing sequence of nonnegative functions in $L^2(\mathbb{R})$ *for* (1-10) *is precompact*, *after normalization and scaling. In this case*, *extremizers for* (1-10) *exist. If instead equality holds in* (1-13) *then*, *given any* $y_0 \in \mathbb{R}$, *there exists an extremizing sequence for* (1-10) *which concentrates at* y_0.

A few remarks may help to further orient the reader. Firstly, if $p = 1$, then the curve $s = |y|$ has no curvature, and no nontrivial Fourier extension estimate can hold. Secondly, if equality holds in (1-13), then Theorem 1.3 does *not* guarantee the nonexistence of extremizers. Indeed, $\boldsymbol{C}_2^6 = \pi/\sqrt{3}$, and Gaussians are known to extremize (1-10) when $p = 2$. Various results of a similar flavor to that of Theorem 1.3 have appeared in the recent literature. They are typically derived from a sophisticated application of concentration-compactness techniques [Christ and Shao 2012a; Shao 2016a], a full profile decomposition [Jiang et al. 2010; 2014; Shao 2009], or the missing mass method as in [Frank et al. 2016; Frank and Sabin 2018]. We introduce a new variant which follows the spirit of the celebrated works [Brézis and Lieb 1983; Lieb 1983; Lions 1984a; 1984b]. It seems more elementary and may be easier to adapt to other manifolds. The proof of Theorem 1.3 involves a variant of Lions' concentration-compactness lemma [1984a], a variant of the corollary of the Brézis–Lieb lemma from [Fanelli et al. 2011], bilinear extension estimates, and a refinement of inequality (1-9) over a suitable *cap space*.

In a range of exponents that includes the case $p = 4$, we are able to resolve the dichotomy posed by Theorem 1.3.

Theorem 1.4. *There exists $p_0 > 4$ such that, for every $p \in (1, p_0) \setminus \{2\}$, the strict inequality* (1-13) *holds. In particular, if $p \in (1, p_0)$, then there exists an extremizer for* (1-10).

Our method yields $p_0 \approx 4.803$ with three decimal places, and effectively computes arbitrarily good lower bounds for the ratio of L^2-norms in (1-10) via expansions of suitable trial functions in the orthogonal basis of Legendre polynomials. We remark that the value $p_0 \approx 4.803$ is suboptimal, in the sense that a natural refinement of our argument allows us to increase this value to ≈ 5.485; see Section 4C below.

Once the existence of extremizers has been established, their properties are typically deduced from the study of the associated Euler–Lagrange equation. Following this paradigm, we show that any extremizer of (1-9) decays superexponentially fast in L^2, which reflects the analyticity of its Fourier transform. This is the content of our next result.

Theorem 1.5. *Let $p > 1$. If f is an extremizer for* (1-9), *then there exists $\mu_0 > 0$ such that*

$$x \mapsto e^{\mu_0 |x|^p} f(x) \in L^2(\mathbb{R}).$$

In particular, its Fourier transform $\hat{f}$ can be extended to an entire function on $\mathbb{C}$.

Note that the exponent μ_0 necessarily depends on the extremizer itself; see the discussion in [Christ and Shao 2012b, p. 964]. The proof relies on a bootstrapping argument that found similar applications in [Christ and Shao 2012b; Erdoğan et al. 2011; Hundertmark and Shao 2012; Shao 2016b].

To some extent, our methods are able to handle the case of the planar *odd* curves $s = y|y|^{p-1}$, $p > 1$. Define the singular measure

$$\mathrm{d}\mu_p(y, s) = \boldsymbol{\delta}(s - y|y|^{p-1})|y|^{\frac{p-2}{6}}\, \mathrm{d}y\, \mathrm{d}s. \tag{1-14}$$

The associated Fourier extension operator $\mathcal{S}_p(f) = \mathcal{F}(f\mu_p)(-\,\cdot)$, defined in (6-2) below, satisfies the estimate $\|\mathcal{S}_p(f)\|_{L^6} \lesssim \|f\|_{L^2}$. In sharp convolution form, this can be rewritten as

$$\|f\mu_p * f\mu_p * f\mu_p\|_{L^2(\mathbb{R}^2)} \le \boldsymbol{Q}_p^3 \|f\|^3_{L^2(\mathbb{R})}, \tag{1-15}$$

where $\boldsymbol{Q}_p$ denotes the optimal constant. Odd curves are of independent interest, in particular because a new phenomenon emerges: caps centered around points with parallel tangents interact strongly, regardless of separation between the points. This mechanism was discovered in [Christ and Shao 2012a], and further explored in [Carneiro et al. 2017; Foschi 2015; Frank et al. 2016; Frank and Sabin 2018; Shao 2016a]. Some of these works include a symmetrization step which relies on the convolution structure of the underlying inequality. In the present case, we also show that the search for extremizers can be further restricted to the class of even functions, but interestingly our symmetrization argument does not depend on the convolution structure. This may be of independent interest since it applies to other Fourier extension inequalities where some additional symmetry is present, as we indicate in Section 6A below.

The following versions of Theorems 1.3 and 1.4 hold for odd curves.

Theorem 1.6. *Let $p > 1$. If*

$$\boldsymbol{Q}_p^6 > \frac{5\pi}{\sqrt{3}p(p-1)}, \tag{1-16}$$

then any extremizing sequence of nonnegative, even functions in $L^2(\mathbb{R})$ for (1-15) *is precompact, after normalization and scaling. In this case, extremizers for* (1-15) *exist. If instead equality holds in* (1-16) *then, given any $y_0 \in \mathbb{R}$, there exists an extremizing sequence for* (1-15) *which concentrates at the pair* $\{-y_0, y_0\}$.

The case $p = 3$ of Theorem 1.6 coincides with a special case of [Frank and Sabin 2018, Theorem 1], which was obtained by different methods.

Theorem 1.7. *If $p \in (1,2)$, then the strict inequality* (1-16) *holds and, in particular, there exists an extremizer for* (1-15).

We believe that extremizers do not exist if $p \geq 2$; see Conjecture 6.6 below.

Overview. The paper is organized as follows. Section 2 is devoted to the technical preliminaries for the dichotomy statement concerning the existence of extremizers: bilinear estimates and cap bounds. We then prove Theorem 1.3 in Section 3. Existence of extremizers is the subject of Section 4, where we establish Theorem 1.4. Theorem 1.5 addresses the regularity of extremizers and is established in Section 5. Odd curves are treated in Section 6, where Theorems 1.6 and 1.7 are proved. In the Appendix, we establish useful variants of Lions' concentration-compactness lemma (Proposition A.1) and of a corollary of the Brézis–Lieb lemma (Proposition B.1).

Notation. If x, y are real numbers, we write $x = O(y)$ or $x \lesssim y$ if there exists a finite absolute constant C such that $|x| \leq C|y|$. If we want to make explicit the dependence of the constant C on some parameter α, we write $x = O_\alpha(y)$ or $x \lesssim_\alpha y$. We write $x \gtrsim y$ if $y \lesssim x$, and $x \simeq y$ if $x \lesssim y$ and $x \gtrsim y$. Finally, the indicator function of a set $E \subset \mathbb{R}^d$ will be denoted by $\mathbb{1}_E$, and the complement of E will at times be denoted by $E^{\complement}$.

2. Bilinear estimates and cap refinements

In this section, we prove the bilinear extension estimates and cap refinements which will be needed in the next section. Bilinear extension estimates are usually deep [Tao 2003; Wolff 2001], but in the one-dimensional case one may rely on the classical Hausdorff–Young inequality. Throughout this section, we shall consider the dyadic regions

$$I_k := [2^k, 2^{k+1}) \quad \text{and} \quad I_k^\bullet := (-2^{k+1}, -2^k] \cup [2^k, 2^{k+1}) \quad (k \in \mathbb{Z}).$$

2A. *Bilinear estimates.* Recall the definitions (1-11) and (1-12) of the measure σ_p and the Fourier extension operator $\mathcal{E}_p$, respectively. Our first result quantifies the principle that distant caps interact weakly.

Proposition 2.1. *Let $p > 1$ and $k, k' \in \mathbb{Z}$. Then*

$$\|\mathcal{E}_p(f)\mathcal{E}_p(g)\|_{L^3(\mathbb{R}^2)} \lesssim_p 2^{-|k-k'|\frac{p-1}{6}} \|f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})} \tag{2-1}$$

for every $f, g \in L^2(\mathbb{R})$ satisfying $\operatorname{supp} f \subseteq I_k^\bullet$ *and* $\operatorname{supp} g \subseteq I_{k'}^\bullet$.

Proof. Setting $\psi = |\cdot|^p$ and $w = |\cdot|^{\frac{p-2}{3}}$, we have

$$(\mathcal{E}_p(f)\mathcal{E}_p(g))(x,t) = \int_{\mathbb{R}^2} e^{ix(y+y')} e^{it(\psi(y)+\psi(y'))} f(y) g(y') w(y)^{\frac12} w(y')^{\frac12} \, \mathrm{d}y \, \mathrm{d}y'.$$

Change variables $(y, y') \mapsto (u, v) = (y + y', \psi(y) + \psi(y'))$. Except for null sets, this is a 2-to-1 map from $\mathbb{R}^2$ onto the region $\{(u, v) \colon v \geq 2\psi(u/2)\}$. Its Jacobian is given by

$$J^{-1}(y, y') = \frac{\partial(u,v)}{\partial(y,y')} = \det \begin{pmatrix} 1 & \psi'(y) \\ 1 & \psi'(y') \end{pmatrix} = \psi'(y') - \psi'(y) = p(y'|y'|^{p-2} - y|y|^{p-2}) \tag{2-2}$$

and satisfies $|J^{-1}(y, y')| \geq p||y|^{p-1} - |y'|^{p-1}|$, with equality if and only if $yy' \geq 0$. Thus

$$(\mathcal{E}_p(f)\mathcal{E}_p(g))(x,t) = 2 \int e^{ixu} e^{itv} f(y) g(y') w(y)^{\frac12} w(y')^{\frac12} J(u, v) \, \mathrm{d}u \, \mathrm{d}v, \tag{2-3}$$

where the integral is taken over the region $\{(u, v) \colon v \geq 2\psi(u/2)\}$. Note that this implies

$$(f\sigma_p * g\sigma_p)(u, v) = 2 f(y) g(y') w(y)^{\frac12} w(y')^{\frac12} J(u, v) \tag{2-4}$$

for every (u, v) satisfying $v > 2\psi(u/2)$, where (y, y') is related to (u, v) via the change of variables described above.

By symmetry, we can and will restrict attention to $|y'| \leq |y|$. Taking the L^3-norm of (2-3), invoking the Hausdorff–Young inequality, and then changing variables back to (y, y'),

$$\begin{aligned} \|\mathcal{E}_p(f)\mathcal{E}_p(g)\|_{L^3(\mathbb{R}^2)} &\lesssim \|f(y) g(y') w(y)^{\frac12} w(y')^{\frac12} J(u, v)\|_{L^{3/2}_{u,v}(\mathbb{R}^{1+1})} \\ &= \|f(y) g(y') w(y)^{\frac12} w(y')^{\frac12} |J(y, y')|^{\frac13}\|_{L^{3/2}_{y,y'}(\mathbb{R}^{1+1})}. \end{aligned}$$

If $2^k \le |y| < 2^{k+1}$, $2^{k'} \le |y'| < 2^{k'+1}$, and $k \ge k'+2$, then

$$\frac{|yy'|^{\frac{p-2}{4}}}{p^{\frac{1}{2}}\big||y|^{p-1}-|y'|^{p-1}\big|^{\frac{1}{2}}} \lesssim \frac{2^{(k+k')\frac{p-2}{4}}}{2^{k\frac{p-1}{2}}(1-2^{-(k-k'-1)(p-1)})^{\frac{1}{2}}} \lesssim 2^{(k'-k)\frac{p}{4}-\frac{k'}{2}}. \tag{2-5}$$

It follows that

$$\begin{aligned}\|\mathcal{E}_p(f)\mathcal{E}_p(g)\|_{L^3}^{\frac{3}{2}} &\lesssim \int_{\mathbb{R}^2} |f(y)g(y')|^{\frac{3}{2}} w(y)^{\frac{3}{4}} w(y')^{\frac{3}{4}} |J(y,y')|^{\frac{1}{2}}\,\mathrm{d}y\,\mathrm{d}y' \\ &\le \int_{\mathbb{R}^2} |f(y)g(y')|^{\frac{3}{2}} \frac{|yy'|^{\frac{p-2}{4}}}{p^{\frac{1}{2}}\big||y|^{p-1}-|y'|^{p-1}\big|^{\frac{1}{2}}}\,\mathrm{d}y\,\mathrm{d}y' \\ &\lesssim 2^{(k'-k)\frac{p}{4}-\frac{k'}{2}} 2^{\frac{k}{4}} 2^{\frac{k'}{4}} \|f\|_{L^2}^{\frac{3}{2}} \|g\|_{L^2}^{\frac{3}{2}} \\ &= 2^{-|k-k'|\frac{p-1}{4}} \|f\|_{L^2}^{\frac{3}{2}} \|g\|_{L^2}^{\frac{3}{2}}. \end{aligned} \tag{2-6}$$

If $k \in \{k', k'+1\}$, then we can simply use the estimate $\|\mathcal{E}_p(f)\mathcal{E}_p(g)\|_{L^3} \lesssim \|f\|_{L^2}\|g\|_{L^2}$. □

Corollary 2.2. *Let $p > 1$ and $k, k' \in \mathbb{Z}$ be such that $k' \le k$. Then*

$$\|\mathcal{E}_p(f)\mathcal{E}_p(g)\|_{L^3(\mathbb{R}^2)} \lesssim_p 2^{-|k-k'|\frac{p-1}{6}} \|f\|_{L^2(\mathbb{R})}\|g\|_{L^2(\mathbb{R})} \tag{2-7}$$

for every $f, g \in L^2(\mathbb{R})$ satisfying $\operatorname{supp} f \subseteq \{|y| \ge 2^k\}$ *and* $\operatorname{supp} g \subseteq \{|y'| \le 2^{k'}\}$.

Proof. Write $f = \sum_{j \ge k} f_j$ and $g = \sum_{j' < k'} g_{j'}$, where $f_j := f\mathbb{1}_{I_j^\bullet}$ and $g_{j'} := g\mathbb{1}_{I_{j'}^\bullet}$. Then

$$\begin{aligned}\|\mathcal{E}_p(f)\mathcal{E}_p(g)\|_{L^3(\mathbb{R}^2)} &\le \sum_{j\ge k,\, j'<k'} \|\mathcal{E}_p(f_j)\mathcal{E}_p(g_{j'})\|_{L^3} \lesssim \sum_{j\ge k,\, j'<k'} 2^{-|j-j'|\frac{p-1}{6}} \|f_j\|_{L^2}\|g_{j'}\|_{L^2} \\ &\le \Big(\sum_{j\ge k,\, j'<k'} 2^{-|j-j'|\frac{p-1}{3}}\Big)^{\frac{1}{2}} \Big(\sum_{j\ge k,\, j'<k'} \|f_j\|_{L^2}^2 \|g_{j'}\|_{L^2}^2\Big)^{\frac{1}{2}} \\ &\simeq \Big(\sum_{j\ge k} 2^{-|j-k'|\frac{p-1}{3}}\Big)^{\frac{1}{2}} \|f\|_{L^2}\|g\|_{L^2} \\ &\simeq 2^{-|k-k'|\frac{p-1}{6}} \|f\|_{L^2}\|g\|_{L^2},\end{aligned}$$

where we used the triangle inequality, Proposition 2.1, the Cauchy–Schwarz inequality, L^2-orthogonality, and the fact that a geometric series is comparable to its largest term. □

When studying concentration at points different from the origin, it will be useful to consider dyadic decompositions of the real line with arbitrary centers. By reflection and scaling, it suffices to consider decompositions centered at 1. Define the dyadic regions

$$\mathcal{I}_k := \{2^k \le y-1 < 2^{k+1}\} \quad \text{and} \quad \mathcal{I}_k^\bullet := \{2^k \le |y-1| < 2^{k+1}\} \quad (k \in \mathbb{Z})$$

so that $\mathcal{I}_k = 1 + I_k$ and $\mathcal{I}_k^\bullet = 1 + I_k^\bullet$. The following analogue of Proposition 2.1 holds.

Proposition 2.3. *Let $p > 1$ and $k, k' \in \mathbb{Z}$. Let $\beta = \min\{\frac{1}{6}, \frac{p-1}{6}\}$. Then*

$$\|\mathcal{E}_p(f)\mathcal{E}_p(g)\|_{L^3(\mathbb{R}^2)} \lesssim_p 2^{-\beta|k-k'|}\|f\|_{L^2(\mathbb{R})}\|g\|_{L^2(\mathbb{R})} \tag{2-8}$$

for every $f, g \in L^2(\mathbb{R})$ satisfying $\operatorname{supp} f \subseteq \mathcal{I}^\bullet_k$ *and* $\operatorname{supp} g \subseteq \mathcal{I}^\bullet_{k'}$.

Before embarking on the proof, let us take a closer look at the factor $|yy'|^{(p-2)/4}|J(y,y')|^{1/2}$ that appears after applying the Hausdorff–Young inequality in (2-6). We have already seen that

$$|J^{-1}(y,y')| = p|y|y|^{p-2} - y'|y'|^{p-2}|. \tag{2-9}$$

In (2-5) we observed that, if y, y' are *separated* (say, $|y'| \le \frac{1}{2}|y|$), then

$$\frac{|yy'|^{\frac{p-2}{4}}}{|y|y|^{p-2} - y'|y'|^{p-2}|^{\frac{1}{2}}} \lesssim \frac{|yy'|^{\frac{p-2}{4}}}{|y|^{\frac{p-1}{2}}} = |y|^{-\frac{p}{4}}|y'|^{\frac{p-2}{4}}. \tag{2-10}$$

In order to obtain a useful bound in the case when both y, y' are close to 1, invoke the mean value theorem and write

$$|y|^{p-1} - |y'|^{p-1} = (p-1)s^{p-2}(|y| - |y'|)$$

for some $s \in [|y'|, |y|]$. Then, for $0 \le y' \le y$, we have

$$|y|y|^{p-2} - y'|y'|^{p-2}| = |y^{p-1} - y'^{p-1}| \gtrsim \begin{cases} |y-y'|y^{p-2} & \text{if } p \in (1,2], \\ |y-y'|y'^{p-2} & \text{if } p \in [2,\infty). \end{cases}$$

Thus the following estimate holds for every $\frac{1}{2} \le y, y' \le \frac{3}{2}$:

$$\frac{|yy'|^{\frac{p-2}{4}}}{|y|y|^{p-2} - y'|y'|^{p-2}|^{\frac{1}{2}}} \lesssim |y-y'|^{-\frac{1}{2}}. \tag{2-11}$$

Proof of Proposition 2.3. Without loss of generality, assume $|k-k'| \ge 2$. We start by considering the situation when 0 is an endpoint of $\mathcal{I}^\bullet_{k'}$, i.e., $k' \in \{-1, 0\}$. Let $k' = -1$, so that $\mathcal{I}^\bullet_{k'} = (0, \frac{1}{2}] \cup [\frac{3}{2}, 2)$, split $g = g_\ell + g_r$, with $g_\ell := g\mathbb{1}_{(0,\frac{1}{2}]}$ and $g_r := g\mathbb{1}_{[\frac{3}{2},2)}$, and take the dyadic decomposition

$$g_\ell = \sum_{j\ge1} g_j, \quad \text{with } g_j := g\mathbb{1}_{(2^{-(j+1)},2^{-j}]}.$$

If $k \le -3$, then (2-10) implies

$$\begin{aligned}\|\mathcal{E}_p(f)\mathcal{E}_p(g_\ell)\|_{L^3} &\lesssim \sum_{j\ge1}\left(\int_{\mathbb{R}^2}|f(y)g_j(y')|^{\frac{3}{2}}\frac{|yy'|^{\frac{p-2}{4}}}{||y|^{p-1} - |y'|^{p-1}|^{\frac{1}{2}}}\,\mathrm{d}y\,\mathrm{d}y'\right)^{\frac{2}{3}} \\ &\lesssim \sum_{j\ge1}\left(2^{-j\frac{p-2}{4}}\int_{\mathbb{R}^2}|f(y)g_j(y')|^{\frac{3}{2}}\,\mathrm{d}y\,\mathrm{d}y'\right)^{\frac{2}{3}} \lesssim \sum_{j\ge1}\left(2^{-j\frac{p-2}{4}}2^{\frac{k}{4}}2^{-\frac{j}{4}}\|f\|_{L^2}^{\frac{3}{2}}\|g_j\|_{L^2}^{\frac{3}{2}}\right)^{\frac{2}{3}} \\ &= 2^{\frac{k}{6}}\|f\|_{L^2}\sum_{j\ge1}2^{-j\frac{p-1}{6}}\|g_j\|_{L^2} \lesssim 2^{\frac{k}{6}}\|f\|_{L^2}\|g_\ell\|_{L^2} \lesssim 2^{-\frac{|k-k'|}{6}}\|f\|_{L^2}\|g\|_{L^2}.\end{aligned}$$

If $k \geq 1$, then Corollary 2.2 applies, and directly yields

$$\|\mathcal{E}_p(f)\mathcal{E}_p(g_\ell)\|_{L^3} \lesssim 2^{-|k-k'|\frac{p-1}{6}} \|f\|_{L^2}\|g\|_{L^2}.$$

A similar analysis applies to g_r. Setting $\beta := \min\left\{\frac{1}{6}, \frac{p-1}{6}\right\}$, we conclude that, if $k' = -1$ and $|k-k'| \geq 2$, then

$$\|\mathcal{E}_p(f)\mathcal{E}_p(g)\|_{L^3} \lesssim 2^{-\beta|k-k'|}\|f\|_{L^2}\|g\|_{L^2}.$$

The case $k' = 0$ admits a similar treatment. If $k, k' \leq -2$ and $k - k' \geq 2$, then (2-11) implies

$$\|\mathcal{E}_p(f)\mathcal{E}_p(g)\|_{L^3} \lesssim \frac{2^{\frac{k}{6}}2^{\frac{k'}{6}}}{2^{\frac{k}{3}}}\|f\|_{L^2}\|g\|_{L^2} = 2^{-\frac{|k-k'|}{6}}\|f\|_{L^2}\|g\|_{L^2}.$$

Finally, the remaining cases can be handled in a similar way by Corollary 2.2. □

Corollary 2.4. *Let $p > 1$ and $k, k' \in \mathbb{Z}$ be such that $k' \leq k$. Let $\beta = \min\{\frac{1}{6}, \frac{p-1}{6}\}$. Then*

$$\|\mathcal{E}_p(f)\mathcal{E}_p(g)\|_{L^3(\mathbb{R}^2)} \lesssim_p 2^{-\beta|k-k'|}\|f\|_{L^2(\mathbb{R})}\|g\|_{L^2(\mathbb{R})} \tag{2-12}$$

for every $f, g \in L^2(\mathbb{R})$ satisfying $\operatorname{supp} f \subseteq \{|y-1| \geq 2^k\}$ *and* $\operatorname{supp} g \subseteq \{|y'-1| \leq 2^{k'}\}$.

We finish this subsection by taking yet another look at the Jacobian factor (2-9). This will be useful in Section 2B below. Let $p \geq 2$. If $yy' \leq 0$, then $|J^{-1}(y, y')| = p(|y|^{p-1} + |y'|^{p-1})$, in which case

$$\frac{|yy'|^{\frac{p-2}{4}}}{(|y|^{p-1} + |y'|^{p-1})^{\frac{1}{2}}} \lesssim (|y| + |y'|)^{-\frac{1}{2}} = |y - y'|^{-\frac{1}{2}}$$

uniformly in y, y'. To handle the complementary case $yy' > 0$, note that, if $p \geq 2$ and $0 \leq a \leq b$, then

$$b^{p-1} - a^{p-1} \simeq (b-a)b^{p-2}. \tag{2-13}$$

It follows that, if $p \geq 2$ and $yy' > 0$, then

$$|J^{-1}(y, y')| = p\big||y|^{p-1} - |y'|^{p-1}\big| \simeq |y - y'|\max\{|y|, |y'|\}^{p-2},$$

and so if additionally $|y| \geq |y'|$, then

$$\frac{|yy'|^{\frac{p-2}{4}}}{\big||y|^{p-1} - |y'|^{p-1}\big|^{\frac{1}{2}}} \lesssim \frac{|yy'|^{\frac{p-2}{4}}}{|y|^{\frac{p-2}{2}}|y-y'|^{\frac{1}{2}}} \leq |y - y'|^{-\frac{1}{2}}.$$

Therefore the estimate

$$\|\mathcal{E}_p(f)\mathcal{E}_p(g)\|^{\frac{3}{2}}_{L^3(\mathbb{R}^2)} \lesssim \int_{\mathbb{R}^2} \frac{|f(y)g(y')|^{\frac{3}{2}}}{|y-y'|^{\frac{1}{2}}}\,\mathrm{d}y\,\mathrm{d}y' \tag{2-14}$$

holds as long as $p \geq 2$. We cannot hope for such a bound if $1 < p < 2$ since (2-13) fails in that case. However, if $|y| \simeq |y'|$, then one can check in a similar way that the estimate

$$\|\mathcal{E}_p(f_k)\mathcal{E}_p(g_k)\|^{\frac{3}{2}}_{L^3(\mathbb{R}^2)} \lesssim \int_{\mathbb{R}^2} \frac{|f_k(y)g_k(y')|^{\frac{3}{2}}}{|y-y'|^{\frac{1}{2}}}\,\mathrm{d}y\,\mathrm{d}y' \tag{2-15}$$

holds for any $p > 1$ and functions f_k, g_k which are both supported on $I_k^\bullet$.

2B. ***Cap bounds.*** An inspection of the proof of Proposition 2.1 reveals that if supp $f \subseteq I_k^\bullet$ and supp $g \subseteq I_{k'}^\bullet$ for some $k, k' \in \mathbb{Z}$ satisfying $k - k' \geq 2$, then

$$\begin{aligned}\|\mathcal{E}_p(f)\mathcal{E}_p(g)\|_{L^3(\mathbb{R}^2)} &\lesssim 2^{-|k-k'|\frac{p-1}{6}} \left(|I_k|^{-\frac{1}{4}} \int_{I_k^\bullet} |f|^{\frac{3}{2}}\right)^{\frac{2}{3}} \left(|I_{k'}|^{-\frac{1}{4}} \int_{I_{k'}^\bullet} |g|^{\frac{3}{2}}\right)^{\frac{2}{3}} \\ &\lesssim 2^{-|k-k'|\frac{p-1}{6}} \Lambda(f)^{\frac{2}{9}} \Lambda(g)^{\frac{2}{9}} \|f\|_{L^2(\mathbb{R})}^{\frac{2}{3}} \|g\|_{L^2(\mathbb{R})}^{\frac{2}{3}}, \end{aligned} \tag{2-16}$$

where the quantity $\Lambda(f)$ is defined via

$$\Lambda(f) := \sup_{k \in \mathbb{Z}} |I_k|^{-\frac{1}{4}} \int_{I_k^\bullet} |f|^{\frac{3}{2}}. \tag{2-17}$$

The purpose of this subsection is to develop on this observation. Given $f \in L^2(\mathbb{R})$, write $f = \sum_{k \in \mathbb{Z}} f_k$, with $f_k := f \mathbb{1}_{I_k^\bullet}$. Our first result is the following.

Proposition 2.5. *Let $p > 1$. Then the following estimates hold for every $f \in L^2(\mathbb{R})$:*

$$\|\mathcal{E}_p(f)\|_{L^6(\mathbb{R}^2)}^3 \lesssim_p \sum_{k \in \mathbb{Z}} \|f_k\|_{L^2(\mathbb{R})}^3, \tag{2-18}$$

$$\|\mathcal{E}_p(f)\|_{L^6(\mathbb{R}^2)}^3 \lesssim_p \sum_{k \in \mathbb{Z}} \|\mathcal{E}_p(f_k)\|_{L^6(\mathbb{R}^2)}^3 + \Lambda(f)^{\frac{4}{9}} \left(\sum_{k \in \mathbb{Z}} \|f_k\|_{L^2(\mathbb{R})}^3\right)^{\frac{1}{3}} \|f\|_{L^2(\mathbb{R})}^{\frac{4}{3}}. \tag{2-19}$$

Proof. By the triangle inequality,

$$\|\mathcal{E}_p(f)\|_{L^6}^3 \leq \sum_{(i,j,k) \in \mathbb{Z}^3} \|\mathcal{E}_p(f_i)\mathcal{E}_p(f_j)\mathcal{E}_p(f_k)\|_{L^2}.$$

For each triple (i, j, k) in the previous sum, we lose no generality in assuming that

$$|j - k| = \max\{|i' - j'| : i', j' \in \{i, j, k\}\}. \tag{2-20}$$

Hölder's inequality and Proposition 2.1 then imply

$$\|\mathcal{E}_p(f_i)\mathcal{E}_p(f_j)\mathcal{E}_p(f_k)\|_{L^2} \lesssim 2^{-|j-k|\frac{p-1}{6}} \|f_i\|_{L^2} \|f_j\|_{L^2} \|f_k\|_{L^2}.$$

By the maximality of $|j - k|$, we have $|j - k| \geq \frac{1}{3}|i - j| + \frac{1}{3}|j - k| + \frac{1}{3}|k - i|$, and hence

$$\|\mathcal{E}_p(f)\|_{L^6}^3 \lesssim \sum_{(i,j,k) \in \mathbb{Z}^3} 2^{-|i-j|\frac{p-1}{18}} 2^{-|j-k|\frac{p-1}{18}} 2^{-|k-i|\frac{p-1}{18}} \|f_i\|_{L^2} \|f_j\|_{L^2} \|f_k\|_{L^2}.$$

A final application of Hölder's inequality yields (2-18). Estimate (2-19) follows from similar considerations which we now detail. Let $S := \{(i, j, k) \in \mathbb{Z}^3 \colon \max\{|i - j|, |j - k|, |k - i|\} \leq 1\}$ and $S^\complement := \mathbb{Z}^3 \setminus S$. Split the sum into diagonal and off-diagonal contributions,

$$\|\mathcal{E}_p(f)\|_{L^6}^3 \leq \left\| \sum_{(i,j,k) \in S} \mathcal{E}_p(f_i)\mathcal{E}_p(f_j)\mathcal{E}_p(f_k) \right\|_{L^2} + \left\| \sum_{(i,j,k) \in S^\complement} \mathcal{E}_p(f_i)\mathcal{E}_p(f_j)\mathcal{E}_p(f_k) \right\|_{L^2},$$

and analyze the two terms separately. For the diagonal term, note that

$$\begin{aligned}
&\Big\| \sum_{(i,j,k)\in S} \mathcal{E}_p(f_i)\mathcal{E}_p(f_j)\mathcal{E}_p(f_k)\Big\|_{L^2} \\
&\quad \le \sum_{k\in\mathbb{Z}} \big(3\|\mathcal{E}_p(f_k)\mathcal{E}_p(f_k)\mathcal{E}_p(f_{k+1})\|_{L^2} + 3\|\mathcal{E}_p(f_{k-1})\mathcal{E}_p(f_k)\mathcal{E}_p(f_k)\|_{L^2} + \|\mathcal{E}_p(f_k)\mathcal{E}_p(f_k)\mathcal{E}_p(f_k)\|_{L^2}\big) \\
&\quad \le \sum_{k\in\mathbb{Z}} \big(3\|\mathcal{E}_p(f_k)\|_{L^6}^2\|\mathcal{E}_p(f_{k+1})\|_{L^6} + 3\|\mathcal{E}_p(f_{k-1})\|_{L^6}\|\mathcal{E}_p(f_k)\|_{L^6}^2 + \|\mathcal{E}_p(f_k)\|_{L^6}^3\big) \lesssim \sum_{k\in\mathbb{Z}} \|\mathcal{E}_p(f_k)\|_{L^6}^3.
\end{aligned}$$

To handle the off-diagonal term, note that estimate (2-16) implies

$$\begin{aligned}
\Big\| \sum_{(i,j,k)\in S^{\complement}} \mathcal{E}_p(f_i)\mathcal{E}_p(f_j)\mathcal{E}_p(f_k)\Big\|_{L^2} &\lesssim {\sum_{(i,j,k):|j-k|\ge 2}}' \|f_i\|_{L^2}\|\mathcal{E}_p(f_j)\mathcal{E}_p(f_k)\|_{L^3} \\
&\lesssim \Lambda(f)^{\frac{4}{9}} {\sum_{(i,j,k):|j-k|\ge 2}}' 2^{-|j-k|\frac{p-1}{6}} \|f_i\|_{L^2}\|f_j\|_{L^2}^{\frac{2}{3}}\|f_k\|_{L^2}^{\frac{2}{3}},
\end{aligned}$$

where the sum Σ' is taken over triples $(i,j,k)\in S^{\complement}$ for which (j,k) satisfies the maximality assumption (2-20). It follows that

$$\begin{aligned}
\Big\| \sum_{(i,j,k)\in S^{\complement}} \mathcal{E}_p(f_i)\mathcal{E}_p(f_j)\mathcal{E}_p(f_k)\Big\|_{L^2} &\lesssim \Lambda(f)^{\frac{4}{9}} \sum_{i,j,k} 2^{-(|i-j|+|j-k|+|k-i|)\frac{p-1}{18}} \|f_i\|_{L^2}\|f_j\|_{L^2}^{\frac{2}{3}}\|f_k\|_{L^2}^{\frac{2}{3}} \\
&\lesssim \Lambda(f)^{\frac{4}{9}} \Big(\sum_{k\in\mathbb{Z}} \|f_k\|_{L^2}^3\Big)^{\frac{1}{3}} \Big(\sum_{k\in\mathbb{Z}} \|f_k\|_{L^2}^2\Big)^{\frac{2}{3}}.
\end{aligned}$$

This implies (2-19) at once, and concludes the proof of the proposition. □

The following L^2 dyadic cap estimate is a direct consequence of (2-18).

Corollary 2.6. *Let $p>1$. Then, for every $f\in L^2(\mathbb{R})$,*

$$\|\mathcal{E}_p(f)\|_{L^6(\mathbb{R}^2)}^3 \lesssim_p \big(\sup_{k\in\mathbb{Z}} \|f_k\|_{L^2(\mathbb{R})}\big)\|f\|_{L^2(\mathbb{R})}^2.$$

We now derive a cap bound similar to [Jiang et al. 2010, Lemma 1.2] and [Shao 2009, Lemma 1.2].

Proposition 2.7. *Let $p>1$. Then the following estimate holds*:

$$\|\mathcal{E}_p(f)\|_{L^6(\mathbb{R}^2)}^3 \lesssim_p \big(\sup_{k\in\mathbb{Z}} \sup_{I\subseteq I_k^\bullet} |I|^{-\frac{1}{6}}\|f\|_{L^{3/2}(I)}\big)^{\frac{2}{3}} \|f\|_{L^2(\mathbb{R})}^{\frac{7}{3}} \tag{2-21}$$

for every $f\in L^2(\mathbb{R})$, where the inner supremum is taken over all subintervals $I\subseteq I_k^\bullet$.

Proof. We start by considering the case when $f=f_k(=f\mathbb{1}_{I_k^\bullet})$. From (2-15), we have

$$\|\mathcal{E}_p(f_k)\|_{L^6}^3 \lesssim \int_{\mathbb{R}^2} \frac{|f_k(y)f_k(y')|^{\frac{3}{2}}}{|y-y'|^{\frac{1}{2}}}\,\mathrm{d}y\,\mathrm{d}y'. \tag{2-22}$$

Arguing as in as in [Jiang et al. 2010; Shao 2009] we obtain, for every $q > 1$, that

$$\|\mathcal{E}_p(f_k)\|_{L^6} \lesssim \Big(\sup_{I \subseteq I_k^\bullet} |I|^{\frac{1}{2}-\frac{1}{q}} \|f_k\|_{L^q(I)}\Big)^{\frac{1}{3}} \|f_k\|_{L^2(\mathbb{R})}^{\frac{2}{3}}. \tag{2-23}$$

For the convenience of the reader, we provide the details. In light of (2-22), we may assume $f_k \geq 0$. Normalizing the supremum in (2-23) to equal 1, we may further assume that

$$\int_I f_k^q \leq |I|^{1-\frac{q}{2}} \quad \text{for every subinterval } I \subseteq I_k^\bullet. \tag{2-24}$$

Denote the collection of dyadic intervals of length 2^j by $\mathcal{D}_j := \{2^j[k, k+1) : k \in \mathbb{Z}\}$, and set $\mathcal{D} := \bigcup_{j\in\mathbb{Z}} \mathcal{D}_j$. We perform a Whitney decomposition of $\mathbb{R}^2 \setminus \{(y, y) : y \in \mathbb{R}\}$ in the following manner; see for instance [Dodson et al. 2018, Lemma 10] and [Bégout and Vargas 2007, Proof of Theorem 1.2]. Given distinct $y, y' \in \mathbb{R}$, there exists a unique pair of maximal dyadic intervals I, I' satisfying

$$(y, y') \in I \times I', \quad |I| = |I'|, \quad \text{and} \quad \operatorname{dist}(I, I') \geq 4|I|.$$

Let $\mathfrak{I}$ denote the collection of all such pairs as $y \neq y'$ ranges over $\mathbb{R} \times \mathbb{R}$. Then

$$\sum_{(I,I')\in\mathfrak{I}} \mathbb{1}_I(y)\mathbb{1}_{I'}(y') = 1 \quad \text{for every } (y, y') \in \mathbb{R}^2 \text{ with } y \neq y',$$

and therefore

$$f_k(y) f_k(y') = \sum_{(I,I')\in\mathfrak{I}} f_{k,I}(y) f_{k,I'}(y') \quad \text{for a.e. } (y, y') \in \mathbb{R}^2,$$

where $f_{k,I} := f_k \mathbb{1}_I$. Clearly, if $(y, y') \in I \times I'$ and $(I, I') \in \mathfrak{I}$, then $|y - y'| \simeq |I|$. From this and (2-22), we may choose a slightly larger dyadic interval containing $I \cup I'$ but of length comparable to $|I|$ (still denoted by I), and it suffices to show that

$$\sum_{I\in\mathcal{D}} \frac{1}{|I|^{\frac{1}{2}}} \Big(\int f_{k,I}^{\frac{3}{2}}\Big)^2 \lesssim \int f_k^2.$$

We further decompose $f_{k,I}$ as

$$f_{k,I} = \sum_{n\in\mathbb{Z}} f_{k,I,n}, \qquad \text{where } f_{k,I,n} := f_k \mathbb{1}_{\{y\in I : \frac{2^n}{|I|^{1/2}} \leq f_k(y) < \frac{2^{n+1}}{|I|^{1/2}}\}},$$

and note that it suffices to establish

$$\sum_{I\in\mathcal{D}} \frac{1}{|I|^{\frac{1}{2}}} \Big(\int f_{k,I,n}^{\frac{3}{2}}\Big)^2 \lesssim 2^{-|n|\varepsilon} \int f_k^2 \tag{2-25}$$

for some $\varepsilon > 0$ and every $n \in \mathbb{Z}$. By the Cauchy–Schwarz inequality,

$$\Big(\int f_{k,I,n}^{\frac{3}{2}}\Big)^2 \leq \Big(\int f_{k,I,n}^2\Big)\Big(\int f_{k,I,n}\Big).$$

By construction of $f_{k,I,n}$, Chebyshev's inequality, and normalization (2-24),

$$\int f_{k,I,n} \leq \frac{2^{n+1}}{|I|^{\frac{1}{2}}} \left| \left\{ y \in I : f_k(y) \geq \frac{2^n}{|I|^{\frac{1}{2}}} \right\} \right| \leq \frac{2^{n+1}}{|I|^{\frac{1}{2}}} \frac{\int_I f_k^q}{2^{nq}|I|^{-\frac{q}{2}}} \lesssim 2^{-|n|(q-1)}|I|^{\frac{1}{2}} \tag{2-26}$$

for every $q > 1$ and $n \geq 0$. If $n < 0$, then the following simpler estimate suffices:

$$\int f_{k,I,n} \lesssim \frac{2^n}{|I|^{\frac{1}{2}}}|I| = 2^{-|n|}|I|^{\frac{1}{2}}. \tag{2-27}$$

Combining (2-26) and (2-27), we conclude

$$\sum_{I \in \mathcal{D}} \frac{1}{|I|^{\frac{1}{2}}} \left(\int f_{k,I,n}^{\frac{3}{2}} \right)^2 \lesssim 2^{-|n|\varepsilon} \sum_{I \in \mathcal{D}} \int f_{k,I,n}^2$$

for some $\varepsilon > 0$, from which we get the desired (2-25) by noting that

$$\sum_{I \in \mathcal{D}} \int f_{k,I,n}^2 = \sum_{j \in \mathbb{Z}} \sum_{I \in \mathcal{D}_j} \int f_k^2 \mathbb{1}_{\{f_k \simeq 2^{n-j/2}\}} = \int_{\mathbb{R}} \Bigg(\sum_{\substack{j \in \mathbb{Z}: \\ f_k(y) \simeq 2^{n-j/2}}} f_k^2(y) \Bigg) \, \mathrm{d}y \lesssim \int f_k^2.$$

This concludes the verification of (2-23). Recalling inequality (2-19), and specializing (2-23) to $q = \frac{3}{2}$, yields

$$\begin{aligned} \|\mathcal{E}_p(f)\|_{L^6}^3 &\lesssim \Big(\sup_{k, I \subseteq I_k^\bullet} |I|^{-\frac{1}{6}} \|f_k\|_{L^{3/2}(I)} \Big) \sum_{k \in \mathbb{Z}} \|f_k\|_{L^2}^2 + \Big(\sup_{k \in \mathbb{Z}} |I_k|^{-\frac{1}{6}} \|f_k\|_{L^{3/2}} \Big)^{\frac{2}{3}} \|f\|_{L^2}^{\frac{7}{3}} \\ &\lesssim \Big(\sup_{k \in \mathbb{Z}} \sup_{I \subseteq I_k^\bullet} |I|^{-\frac{1}{6}} \|f_k\|_{L^{3/2}(I)} \Big)^{\frac{2}{3}} \|f\|_{L^2}^{\frac{7}{3}}, \end{aligned}$$

where the last line follows from Hölder's inequality. □

In the next section, it will be useful to have the L^1 version of (2-21) at our disposal, and this is the content of the following result.

Proposition 2.8. *Let* $p > 1$. *Then there exist* $\gamma \in (0,1)$ *such that*

$$\|\mathcal{E}_p(f)\|_{L^6(\mathbb{R}^2)} \lesssim_{p,\gamma} \Big(\sup_{k \in \mathbb{Z}} \sup_{I \subseteq I_k^\bullet} |I|^{-\frac{1}{2}} \|f\|_{L^1(I)} \Big)^{\gamma} \|f\|_{L^2(\mathbb{R})}^{1-\gamma} \tag{2-28}$$

for every $f \in L^2(\mathbb{R})$, *where the inner supremum is taken over all subintervals* $I \subseteq I_k^\bullet$.

The proof below yields $\gamma = \frac{2}{45}$ and is inspired by [Christ and Shao 2012a, Proposition 2.9].

Proof of Proposition 2.8. Set $\delta := \|\mathcal{E}_p(f)\|_{L^6} \|f\|_{L^2}^{-1}$. From (2-21) we have

$$\sup_{k \in \mathbb{Z}} \sup_{I \subseteq I_k^\bullet} |I|^{-\frac{1}{6}} \|f\|_{L^{3/2}(I)} \gtrsim \delta^{\frac{9}{2}} \|f\|_{L^2(\mathbb{R})}.$$

Then there exist $k \in \mathbb{Z}$ and an interval $I \subseteq I_k^\bullet$ such that

$$\int_I |f|^{\frac{3}{2}} \geq c_0 \delta^{\frac{27}{4}} |I|^{\frac{1}{4}} \|f\|_{L^2(\mathbb{R})}^{\frac{3}{2}}$$

for a universal constant c_0 (independent of f, δ). Given $R \geq 1$, define the set $E := \{y \in I : |f(y)| \leq R\}$. Set $g := f \mathbb{1}_E$ and $h := f - g$. Then g and h have disjoint supports, and $\|g\|_{L^\infty} \leq R$. Since $|h(y)| \geq R$ for almost every $y \in I$ for which $h(y) \neq 0$, we have

$$\int_I |h|^{\frac{3}{2}} \leq R^{-\frac{1}{2}} \int_I |h|^2 \leq R^{-\frac{1}{2}} \|f\|^2_{L^2(\mathbb{R})}.$$

Choose R satisfying $R^{-\frac{1}{2}} = \frac{1}{2} c_0 \delta^{\frac{27}{4}} |I|^{\frac{1}{4}} \|f\|^{-\frac{1}{2}}_{L^2(\mathbb{R})}$. Then

$$\int_I |g|^{\frac{3}{2}} = \int_I |f|^{\frac{3}{2}} - \int_I |h|^{\frac{3}{2}} \geq \frac{c_0}{2} \delta^{\frac{27}{4}} |I|^{\frac{1}{4}} \|f\|^{\frac{3}{2}}_{L^2(\mathbb{R})}.$$

Since g is supported on I, Hölder's inequality implies

$$\|g\|_{L^2} \geq |I|^{-\frac{1}{6}} \|g\|_{L^{3/2}} \geq c_1 \delta^{\frac{9}{2}} \|f\|_{L^2}, \tag{2-29}$$

where c_1 is universal. Since $\|g\|_{L^\infty} \leq R$, we have (by the definition of R) that

$$|g(y)| \leq c_2 \delta^{-\frac{27}{2}} |I|^{-\frac{1}{2}} \|f\|_{L^2(\mathbb{R})} \mathbb{1}_I(y) \quad \text{for almost every } y \in \mathbb{R},$$

where c_2 is universal. Together with (2-29), this implies the lower bound

$$\int_I |g| \geq \int_I |g| \frac{|g|}{c_2 \delta^{-\frac{27}{2}} |I|^{-\frac{1}{2}} \|f\|_{L^2}} = c_2^{-1} \delta^{\frac{27}{2}} |I|^{\frac{1}{2}} \frac{\|g\|^2_{L^2}}{\|f\|_{L^2}} \geq c_3 \delta^{\frac{45}{2}} |I|^{\frac{1}{2}} \|f\|_{L^2},$$

where c_3 is universal. Since $|g| \leq |f|$, it follows that

$$c_3 \delta^{\frac{45}{2}} |I|^{\frac{1}{2}} \|f\|_{L^2(\mathbb{R})} \leq \|g\|_{L^1(I)} \leq \|f\|_{L^1(I)}.$$

Recalling the definition of δ, we obtain (2-28) with $\gamma = \frac{2}{45}$. □

3. Existence versus concentration

This section is devoted to the proof of Theorem 1.3. Start by observing the scale invariance of (1-10), or equivalently that of (1-9). Indeed, if $f_\lambda(y) := f(\lambda y)$, then $\|f_\lambda\|_{L^2(\mathbb{R})} = \lambda^{-1/2} \|f\|_{L^2(\mathbb{R})}$. On the other hand, $\mathcal{E}_p(f_\lambda)(x,t) = \lambda^{-(p+4)/6} \mathcal{E}_p(f)(x/\lambda, t/\lambda^p)$, and so

$$\|\mathcal{E}_p(f_\lambda)\|_{L^6(\mathbb{R}^2)} = \lambda^{-\frac{p+4}{6} + \frac{p+1}{6}} \|\mathcal{E}_p(f)\|_{L^6(\mathbb{R}^2)} = \lambda^{-\frac{1}{2}} \|\mathcal{E}_p(f)\|_{L^6(\mathbb{R}^2)}.$$

In particular, given any sequence $\{a_n\} \subset \mathbb{R} \setminus \{0\}$, if $\{f_n\}$ is an L^2-normalized extremizing sequence for (1-9), then so is $\{|a_n|^{1/2} f_n(a_n \cdot)\}$.

We come to the first main result of this section.

Proposition 3.1. *Let $\{f_n\} \subset L^2(\mathbb{R})$ be an L^2-normalized extremizing sequence of nonnegative functions for* (1-9). *Then there exist a subsequence $\{f_{n_k}\}$ and a sequence $\{a_k\} \subset \mathbb{R} \setminus \{0\}$ such that the rescaled sequence $\{g_k\}$, $g_k := |a_k|^{1/2} f_{n_k}(a_k \cdot)$ satisfies one of the following conditions*:

(i) *There exists $g \in L^2(\mathbb{R})$ such that $g_k \to g$ in $L^2(\mathbb{R})$ as $k \to \infty$.*

(ii) *$\{g_k\}$ concentrates at $y_0 = 1$.*

Theorem 1.3 follows at once from Proposition 3.1 and the following result.

Lemma 3.2. *Let $p > 1$. Given $y_0 \in \mathbb{R} \setminus \{0\}$, let $\{f_n\} \subset L^2(\mathbb{R})$ be a sequence concentrating at y_0. Then*

$$\limsup_{n\to\infty} \frac{\|f_n\sigma_p * f_n\sigma_p * f_n\sigma_p\|^2_{L^2(\mathbb{R}^2)}}{\|f_n\|^6_{L^2(\mathbb{R})}} \le \frac{2\pi}{\sqrt{3}p(p-1)}. \tag{3-1}$$

If we set $f_n(y) = e^{-n(|y|^p - |y_0|^p - py_0|y_0|^{p-2}(y-y_0))}|y|^{(p-2)/6}$, then the sequence $\{f_n\|f_n\|^{-1}_{L^2}\}$ concentrates at y_0, and equality holds in (3-1).

Convolution of singular measures is treated in much greater generality in the companion paper [Oliveira e Silva and Quilodrán 2019]. Lemma 3.2 is almost contained in [Oliveira e Silva and Quilodrán 2018; 2019], and we just indicate the necessary changes.

Proof sketch of Lemma 3.2. Once the boundary value for $|\cdot|^{(p-2)/6}\sigma_p * |\cdot|^{(p-2)/6}\sigma_p * |\cdot|^{(p-2)/6}\sigma_p$ given in (4-3) below is known to equal the right-hand side of (3-1), the proof for $p \ge 2$ follows the exact same lines as that of [Oliveira e Silva and Quilodrán 2018, Lemmas 4.1 and 4.2]. We omit the details.

If $1 < p < 2$, then the function $|\cdot|^{(p-2)/6}$ fails to be continuous at the origin, and an additional argument is needed. We show how to reduce matters to the analysis of projection measure. Let $\{f_n\} \subset L^2(\mathbb{R})$ concentrate at $y_0 \neq 0$. Then

$$\limsup_{n\to\infty} \frac{\|f_n\sigma_p * f_n\sigma_p * f_n\sigma_p\|^2_{L^2}}{\|f_n\|^6_{L^2}} = |y_0|^{p-2} \limsup_{n\to\infty} \frac{\|f_n\nu_p * f_n\nu_p * f_n\nu_p\|^2_{L^2}}{\|f_n\|^6_{L^2}}, \tag{3-2}$$

where ν_p denotes the projection measure $\mathrm{d}\nu_p = \boldsymbol{\delta}(s - |y|^p)\,\mathrm{d}y\,\mathrm{d}s$. To verify (3-2), consider the interval $J := [y_0/2, 3y_0/2]$. Then

$$\begin{aligned}\limsup_{n\to\infty} \frac{\|f_n\sigma_p * f_n\sigma_p * f_n\sigma_p\|^2_{L^2}}{\|f_n\|^6_{L^2}} &= \limsup_{n\to\infty} \frac{\|f_n\mathbb{1}_J\sigma_p * f_n\mathbb{1}_J\sigma_p * f_n\mathbb{1}_J\sigma_p\|^2_{L^2}}{\|f_n\mathbb{1}_J\|^6_{L^2}} \\ &= |y_0|^{p-2} \limsup_{n\to\infty} \frac{\|f_n\nu_p * f_n\nu_p * f_n\nu_p\|^2_{L^2}}{\|f_n\|^6_{L^2}}.\end{aligned}$$

Here, to justify the first equality, invoke the continuity of the operator $\mathcal{E}_p$, and the fact that the sequence $\{f_n\}$ concentrates at y_0. For the second equality, additionally note that

$$\frac{\|f_n\mathbb{1}_J|\cdot|^{\frac{p-2}{6}} - f_n\mathbb{1}_J|y_0|^{\frac{p-2}{6}}\|_{L^2}}{\|f_n\mathbb{1}_J\|_{L^2}} \to 0 \quad \text{as } n \to \infty.$$

From [Oliveira e Silva and Quilodrán 2019, Proposition 2.1], the measure $\nu_p * \nu_p * \nu_p$ defines a continuous function in the interior of its support, with continuous extension to the boundary except at $(0, 0)$. Moreover, for any $y_0 \neq 0$,

$$(\nu_p * \nu_p * \nu_p)(3y_0, 3|y_0|^p) = \frac{2\pi}{\sqrt{3}p(p-1)|y_0|^{p-2}}.$$

The result now follows as in [Oliveira e Silva and Quilodrán 2018, Lemmas 4.1 and 4.2]. □

The proof of Proposition 3.1 relies on the bilinear extension estimates and cap bounds from Section 2, together with a suitable variant of Lions' concentration-compactness lemma, which is formulated in the Appendix as Proposition A.1. This has two important consequences for the present context, the first of which is the following.

Proposition 3.3. *Let $\{f_n\} \subset L^2(\mathbb{R})$ be an L^2-normalized extremizing sequence for* (1-9). *Let $\{r_n\}$ be a sequence of nonnegative numbers satisfying $r_n \to 0$ as $n \to \infty$ and*

$$\inf_{n\in\mathbb{N}} \int_{1-r_n}^{1+r_n} |f_n(y)|^2 \,\mathrm{d}y > 0.$$

Then the sequence $\{f_n\}$ concentrates at $y_0 = 1$.

Proof. Consider the intervals $J_n := [1-r_n, 1+r_n]$, $n \in \mathbb{N}$, and define the pseudometric

$$\varrho\colon \mathbb{R}\setminus\{1\}\times\mathbb{R}\setminus\{1\} \to [0,\infty), \qquad \varrho(x,y) := |k-k'|, \tag{3-3}$$

where k, k' are such that $|x-1| \in [2^k, 2^{k+1})$ and $|y-1| \in [2^{k'}, 2^{k'+1})$. Let R be an integer. Then the ball centered at $x \neq 1$ of radius R defined by ϱ is given by

$$B(x,R) = \{y \in \mathbb{R}\setminus\{1\} \colon 2^{k-R} \leq |y-1| < 2^{k+R+1}\}.$$

Let $\{f_n\}$ be as in the statement of the proposition. Apply Proposition A.1 to the sequence $\{|f_n|^2\}$ with $X = \mathbb{R}$ equipped with Lebesgue measure, $\bar{x} = 1$, the function ϱ defined as in (3-3), and $\lambda = 1$. Passing to a subsequence, also denoted by $\{|f_n|^2\}$, one of three cases arises.

Case 1: The sequence $\{|f_n|^2\}$ satisfies *compactness*. In this case, there exists $\{x_n\} \subset \mathbb{R}\setminus\{1\}$ with the property that for any $\varepsilon > 0$ there exists $R < \infty$ such that, for every $n \geq 1$,

$$\int_{B(x_n,R)} |f_n|^2 \geq 1-\varepsilon. \tag{3-4}$$

Suppose that $\limsup_{n\to\infty} |x_n - 1| > 0$. Then, possibly after extraction of a subsequence, $\{x_n\}$ is eventually far from 1; i.e., there exist $N_0 \in \mathbb{N}$, $\ell^* \in \mathbb{Z}$ such that $|x_n - 1| > 2^{\ell^*}$ for every $n \geq N_0$. Let $\varepsilon := \frac{1}{2}\inf_n \|f_n\|^2_{L^2(J_n)} > 0$, and choose an integer R such that (3-4) holds. Now,

$$B(x_n,R) = \{y \in \mathbb{R}\setminus\{1\} \colon 2^{k_n-R} \leq |y-1| < 2^{k_n+R+1}\},$$

where k_n is such that $|x_n - 1| \in [2^{k_n}, 2^{k_n+1})$, and hence $B(x_n,R) \subseteq \{y \neq 1 \colon |y-1| \geq 2^{\ell^*-R}\}$. Let $N_1 \geq N_0$ be such that $r_n < 2^{\ell^*-R}$ for every $n \geq N_1$. In this case, we have $J_n \cap B(x_n,R) = \varnothing$, which is impossible because our choice of ε would then force

$$1 = \int_{\mathbb{R}} |f_n|^2 \geq \int_{J_n} |f_n|^2 + \int_{B(x_n,R)} |f_n|^2 > 1.$$

It follows that $x_n \to 1$ as $n \to \infty$ and consequently the sequence $\{f_n\}$ concentrates at $y_0 = 1$. Indeed, given $\varepsilon > 0$, choose an integer R such that (3-4) holds. Then $B(x_n,R) \subseteq [1-2^{k_n+R+1}, 1+2^{k_n+R+1}]\setminus\{1\}$, where $|x_n - 1| \in [2^{k_n}, 2^{k_n+1})$ and $k_n \to -\infty$, as $n \to \infty$, so that $2^{k_n+R+1} \to 0$, as $n \to \infty$. This forces

$$\int_{1-2^{k_n+R+1}}^{1+2^{k_n+R+1}} |f_n(y)|^2 \, \mathrm{d}y \geq 1 - \varepsilon$$

for every $n \geq 1$, which implies concentration of the sequence $\{f_n\}$ at $y_0 = 1$.

Case 2: The sequence $\{|f_n|^2\}$ satisfies *dichotomy*. Let $\alpha \in (0,1)$ be as in the dichotomy condition. Given $\varepsilon > 0$, consider the corresponding data $R,\ k_0,\ \rho_{n,j} = |f_{n,j}|^2,\ j \in \{1,2\},\ \{x_n\} \subset \mathbb{R} \setminus \{1\},\ \{R_n\} \subset [0,\infty)$. In particular,

$$\operatorname{supp}(f_{n,1}) \subset B(x_n, R) \quad \text{and} \quad \operatorname{supp}(f_{n,2}) \subset B(x_n, R_n)^{\complement}.$$

Since $R_n - R \to \infty$ as $n \to \infty$, by Corollary 2.4 we obtain

$$\|\mathcal{E}_p(f_{n,1})\mathcal{E}_p(f_{n,2})\|_{L^3} \leq C_n \|f_{n,1}\|_{L^2} \|f_{n,2}\|_{L^2}, \tag{3-5}$$

where $C_n = C_n(\varepsilon) \lesssim 2^{-\beta(R_n - R)}$ for some $\beta > 0$. In particular, given $\varepsilon > 0$, we have $C_n \to 0$ as $n \to \infty$. Aiming at a contradiction, consider

$$\|\mathcal{E}_p(f_n - f_{n,1} - f_{n,2})\|_{L^6} \leq \boldsymbol{E}_p \|f_n - (f_{n,1} + f_{n,2})\|_{L^2} \leq \boldsymbol{E}_p \varepsilon^{\frac{1}{2}}. \tag{3-6}$$

The latter inequality requires a short justification which boils down to the pointwise estimate

$$(|f_n| - (|f_{n,1}| + |f_{n,2}|))^2 \leq ||f_n|^2 - (|f_{n,1}| + |f_{n,2}|)^2| = ||f_n|^2 - (|f_{n,1}|^2 + |f_{n,2}|^2)|. \tag{3-7}$$

This, in turn, follows from the disjointness of the supports of $f_{n,1}$ and $f_{n,2}$, together with the trivial estimate $||f_n| - (|f_{n,1}| + |f_{n,2}|)| \leq |f_n| + (|f_{n,1}| + |f_{n,2}|)$. In this way, (3-7) and Proposition A.1 imply

$$\|(|f_n| - (|f_{n,1}| + |f_{n,2}|))^2\|_{L^1} \leq \||f_n|^2 - (|f_{n,1}|^2 + |f_{n,2}|^2)\|_{L^1} \leq \varepsilon.$$

Coming back to (3-6), we have as an immediate consequence that

$$\|\mathcal{E}_p(f_n)\|_{L^6} \leq \boldsymbol{E}_p \varepsilon^{\frac{1}{2}} + \|\mathcal{E}_p(f_{n,1} + f_{n,2})\|_{L^6}.$$

Expanding the binomial, using $\|f_{n,1}\|_{L^2}, \|f_{n,2}\|_{L^2} \leq 1$, and Hölder's inequality together with (3-5), we find that there exists c independent of n such that, for sufficiently large n,

$$\begin{aligned} \|\mathcal{E}_p(f_{n,1} + f_{n,2})\|_{L^6}^6 &\leq \|\mathcal{E}_p(f_{n,1})\|_{L^6}^6 + \|\mathcal{E}_p(f_{n,2})\|_{L^6}^6 + cC_n \\ &\leq \boldsymbol{E}_p^6(\|f_{n,1}\|_{L^2}^6 + \|f_{n,2}\|_{L^2}^6) + cC_n \\ &\leq \boldsymbol{E}_p^6((\alpha + \varepsilon)^3 + (1 - \alpha + \varepsilon)^3) + cC_n. \end{aligned} \tag{3-8}$$

This implies, for every sufficiently large n,

$$\|\mathcal{E}_p(f_n)\|_{L^6} \leq \boldsymbol{E}_p \varepsilon^{\frac{1}{2}} + (\boldsymbol{E}_p^6((\alpha + \varepsilon)^3 + (1 - \alpha + \varepsilon)^3) + cC_n)^{\frac{1}{6}}.$$

Taking $n \to \infty$, and recalling that $\{f_n\}$ is an L^2-normalized extremizing sequence for (1-9), we find that

$$\boldsymbol{E}_p \leq \boldsymbol{E}_p \varepsilon^{\frac{1}{2}} + \boldsymbol{E}_p((\alpha + \varepsilon)^3 + (1 - \alpha + \varepsilon)^3)^{\frac{1}{6}}$$

for every $\varepsilon > 0$. Taking $\varepsilon \to 0$ yields $1 \le \alpha^3 + (1-\alpha)^3$, which is impossible since $\alpha \in (0,1)$. Hence dichotomy does not arise.

Case 3: The sequence $\{|f_n|^2\}$ satisfies *vanishing*. In this case,

$$\lim_{n\to\infty} \sup_{k\in\mathbb{Z}} \int_{2^{k-R}\le|y-1|\le 2^{k+R+1}} |f_n(y)|^2 \,\mathrm{d}y = 0$$

for every integer $R < \infty$. In particular, for fixed $k \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \int_{2^{-k}\le|y-1|\le 2^{k}} |f_n(y)|^2 \,\mathrm{d}y = 0. \tag{3-9}$$

Set $f_{n,1} := f_n \mathbb{1}_{[1-2^{-k},1+2^{-k}]}$ and $f_{n,2} := f_n \mathbb{1}_{\{|y-1|\ge 2^k\}}$. Since $\|f_n - f_{n,1} - f_{n,2}\|_{L^2} \to 0$ as $n \to \infty$ it follows that $\{f_{n,1} + f_{n,2}\}_n$ is also an extremizing sequence for (1-9) for each $k \in \mathbb{N}$. This new sequence splits the mass into two separated regions, and so we expect to reach a contradiction if $\limsup_{n\to\infty} \|f_{n,2}\|_{L^2} > 0$, just as in Case 2. Set $\alpha_k := \limsup_{n\to\infty} \|f_{n,2}\|_{L^2}^2$ (recall that $f_{n,2}$ depends on k), and note that $\{\alpha_k\}$ is a constant sequence. Indeed,

$$\int_{|y-1|\ge 2^k} |f_n(y)|^2 \,\mathrm{d}y = \int_{|y-1|\ge 2^{k+1}} |f_n(y)|^2 \,\mathrm{d}y + \int_{2^k\le|y-1|\le 2^{k+1}} |f_n(y)|^2 \,\mathrm{d}y \tag{3-10}$$

and from (3-9) with $k+1$ instead of k we have

$$\lim_{n\to\infty} \int_{2^{k}\le|y-1|\le 2^{k+1}} |f_n(y)|^2 \,\mathrm{d}y = 0.$$

Taking $\limsup_{n\to\infty}$ in (3-10) yields $\alpha_{k+1} = \alpha_k$ for every $k \in \mathbb{N}$. An argument analogous to that of Case 2 (starting from (3-8)) shows that there exist $\beta > 0$ and a sequence $\{C_k\}$, $0 \le C_k \lesssim 2^{-\beta k} \to 0$ as $k \to \infty$ such that

$$1 \le \alpha_k^3 + (1-\alpha_k)^3 + C_k \quad \text{for every } k \in \mathbb{N}.$$

Since $\alpha_k \equiv \alpha$ is constant, we may take $k \to \infty$ in the previous inequality and obtain $1 \le \alpha^3 + (1-\alpha)^3$. Since $\alpha \in [0,1]$, necessarily $\alpha \in \{0,1\}$. We claim that $\alpha = 0$. For any $k \ge 1$, the support of $f_{n,2}$ is disjoint from the interval J_n if n large enough. Thus

$$\|f_{n,2}\|_{L^2}^2 \le 1 - \int_{J_n} |f_n|^2 \le 1 - \inf_{n\in\mathbb{N}} \int_{J_n} |f_n|^2,$$

and therefore

$$\alpha \le 1 - \inf_{n\in\mathbb{N}} \int_{J_n} |f_n|^2 < 1.$$

We conclude that $\alpha = 0$, as claimed. Finally, we show that vanishing implies concentration at $y = 1$. Since

$$1 = \|f_n\|_{L^2}^2 = \|f_{n,1}\|_{L^2}^2 + \|f_{n,2}\|_{L^2}^2 + o_n(1) = \|f_{n,1}\|_{L^2}^2 + o_n(1) = \|f_n \mathbb{1}_{[1-2^{-k},1+2^{-k}]}\|_{L^2}^2 + o_n(1),$$

we find that, for every $k \in \mathbb{N}$,

$$\lim_{n\to\infty} \int_{1-2^{-k}}^{1+2^{-k}} |f_n(y)|^2 \,\mathrm{d}y = 1.$$

This implies that the sequence $\{f_n\}$ concentrates at $y_0 = 1$.

To sum up, we proved that any sequence $\{f_n\}$ as in the statement of the proposition does not satisfy dichotomy, and that if it satisfies compactness or vanishing, then it concentrates at $y_0 = 1$. Thus the proof is complete. □

As a second application of Proposition A.1, we prove dyadic localization of extremizing sequences, after rescaling. We take $X = \mathbb{R}$, $\bar{x} = 0$, and use the dyadic pseudometric

$$\varrho\colon \mathbb{R}\setminus\{0\}\times\mathbb{R}\setminus\{0\}\to[0,\infty), \quad \varrho(x,y) := |k-k'|, \tag{3-11}$$

where this time $|x| \in [2^k, 2^{k+1})$ and $|y| \in [2^{k'}, 2^{k'+1})$. In this case, if R is an integer, then

$$B(x,R) = \{y \in \mathbb{R}\setminus\{0\} \colon 2^{k-R} \le |y| < 2^{k+R+1}\}.$$

Proposition 3.4. *Let $\{f_n\} \subset L^2(\mathbb{R})$ be an L^2-normalized extremizing sequence for* (1-9). *Then there exist a subsequence $\{f_{n_k}\}$, a sequence $\{a_k\} \subset \mathbb{R}\setminus\{0\}$, and a function $\Theta : [1,\infty) \to (0,\infty)$, $\Theta(R) \to 0$ as $R \to \infty$ such that the rescaled sequence $\{g_k\}$, $g_k := |a_k|^{1/2} f_{n_k}(a_k\,\cdot)$, satisfies*

$$\|g_k\|_{L^2([-R,R]^{\complement})} \le \Theta(R) \quad \textit{for every } k \ge 1 \textit{ and } R \ge 1. \tag{3-12}$$

This proposition will provide the input for the suitable application of the Brézis–Lieb lemma, which is formulated in the Appendix as Proposition B.1.

Proof of Proposition 3.4. Let $\{f_n\}$ be as in the statement of the proposition. In view of Corollary 2.6, there exists $\ell_n \in \mathbb{Z}$ such that $\|f_n\|_{L^2(I^\bullet_{\ell_n})} \gtrsim_p 1$, if n is large enough. Setting $g_n := 2^{\ell_n/2} f_n(2^{\ell_n}\,\cdot)$, we then have

$$\|g_n\|_{L^2(I^\bullet_0)} \gtrsim_p 1 \tag{3-13}$$

for every sufficiently large n. Using Proposition A.1 with the pseudometric (3-11), we obtain a subsequence $\{|g_{n_k}|^2\}$ that satisfies one of three possibilities. Because of (3-13), vanishing does not occur. The argument given in Case 2 of the proof of Proposition 3.3 can be used in conjunction with Corollary 2.2 to show that the sequence $\{|g_{n_k}|^2\}$ does not satisfy dichotomy either. Therefore it must satisfy compactness. Thus, there exists a sequence $\{N_k\} \subset \mathbb{Z}$ such that, for every $k \ge 1$ and $\varepsilon > 0$, there exists an integer $r = r(\varepsilon)$ for which

$$\int_{2^{N_k - r} \le |y| \le 2^{N_k + r + 1}} |g_k(y)|^2 \,\mathrm{d}y \ge 1 - \varepsilon.$$

Because of (3-13), the sequence $\{N_k\}$ is bounded, $\sup_{k\ge1} |N_k| =: r_0 < \infty$. By redefining r as $r + r_0 + 1$, it follows that

$$\int_{2^{-r} \le |y| \le 2^r} |g_k(y)|^2 \,\mathrm{d}y \ge 1 - \varepsilon \quad \text{for every } k \ge 1. \tag{3-14}$$

Defining the function

$$\theta(R) := \sup_{k\ge1} \int_{\{R^{-1} \le |y| \le R\}^{\complement}} |g_k(y)|^2 \,\mathrm{d}y,$$

$R \mapsto \theta(R)$ is a nonincreasing function of R which is bounded by 1 and, in view of (3-14), satisfies $\theta(R) \to 0$ as $R \to \infty$. By construction,

$$\int_{\{R^{-1} \le |y| \le R\}^{\complement}} |g_k(y)|^2 \, \mathrm{d}y \le \theta(R) \quad \text{for every } k \ge 1, \ R \ge 1,$$

which implies (3-12) at once by taking $\Theta := \theta^{1/2}$. □

We are finally ready to prove Proposition 3.1.

Proof of Proposition 3.1. Let $\{f_n\}$ be as in the statement of the proposition. Apply Proposition 3.4 to $\{f_n\}$, and denote the resulting rescaled subsequence by $\{g_n\}$. From the L^1 cap estimate (2-28) we know that, for each sufficiently large n, there exists an interval $J_n = [s_n - r_n, s_n + r_n]$, contained in a dyadic interval[3] $[2^{k_n}, 2^{k_n+1}]$, such that

$$\int_{J_n} |g_n| \ge c|J_n|^{\frac{1}{2}}$$

for some $c > 0$ which is independent of n. By the Cauchy–Schwarz inequality,

$$\|g_n\|_{L^2(J_n)} \ge c, \tag{3-15}$$

and so estimate (3-12) implies the existence of $C > 0$ independent of n, such that $C^{-1} \le |s_n| \le C$. Rescaling again, we may assume $s_n = 1$ for every n.

If $r^* := \liminf_{n\to\infty} |J_n| > 0$, then passing to the relevant subsequence that realizes the limit inferior we have

$$\int_{1-2r^*}^{1+2r^*} g_n(y) \, \mathrm{d}y = \int_{1-2r^*}^{1+2r^*} |g_n(y)| \, \mathrm{d}y \ge \int_{J_n} |g_n| \gtrsim \sqrt{r^*},$$

provided n is large enough to ensure $J_n \subseteq [1 - 2r^*, 1 + 2r^*]$. Therefore any L^2-weak limit of the sequence $\{g_n\}$ is nonzero. Here we used the nonnegativity of the sequence $\{g_n\}$. By Proposition B.1, we conclude that there exists $0 \ne g \in L^2(\mathbb{R})$, such that possibly after a further extraction, $g_n \to g$ in $L^2(\mathbb{R})$, as $n \to \infty$. In other words, (i) holds.

It remains to consider the case when $|J_n| \to 0$, as $n \to \infty$. In view of (3-15), Proposition 3.3 applies, and the sequence $\{g_n\}$ concentrates at $y_0 = 1$, i.e., (ii) holds. This finishes the proof of Proposition 3.1 (and therefore of Theorem 1.3). □

4. Existence of extremizers

In this section, we prove Theorem 1.4. The basic strategy is to choose an appropriate trial function f for which the ratio from (1-10),

$$\Phi_p(f) := \frac{\|f\sigma_p * f\sigma_p * f\sigma_p\|^2_{L^2(\mathbb{R}^2)}}{\|f\|^6_{L^2(\mathbb{R})}}, \tag{4-1}$$

[3]Or its negative, but in that case we replace f_n by its reflection around the origin.

can be estimated via a simple lower bound. We will give different arguments depending on whether $1 < p < 2$ or $p > 2$, which rely on distinct choices of trial functions. This can be explained by the different qualitative nature of the 3-fold convolutions $w\nu_p * w\nu_p * w\nu_p$ in the two regimes of p; see Figure 1. Here, and throughout this section, $\mathrm{d}\nu_p = \boldsymbol{\delta}(s - |y|^p)\,\mathrm{d}y\,\mathrm{d}s$ denotes projection measure on the curve $s = |y|^p$, and the weight is given by $w = |\cdot|^{(p-2)/3}$. Note that $\mathrm{d}\sigma_p = \sqrt{w}\,\mathrm{d}\nu_p$.

The following analogue of [Oliveira e Silva and Quilodrán 2018, Proposition 6.4] holds for 3-fold convolutions in $\mathbb{R}^2$.

Proposition 4.1. *Given $p > 1$, the following assertions hold for $w\nu_p * w\nu_p * w\nu_p$:*

(a) *It is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^2$.*

(b) *Its support, denoted by E_p, is given by*

$$E_p = \{(\xi, \tau) \in \mathbb{R}^2 : \tau \geq 3^{1-p}|\xi|^p\}. \tag{4-2}$$

(c) *If $p \geq 2$, then its Radon–Nikodym derivative, also denoted by $w\nu_p * w\nu_p * w\nu_p$, defines a bounded, continuous function in the interior of the set E_p. If $1 < p < 2$, then $w\nu_p * w\nu_p * w\nu_p$ defines a continuous function on the set*

$$\widetilde{E}_p := \{(\xi, \tau) \in \mathbb{R}^2 : 3^{1-p}|\xi|^p < \tau < 2^{1-p}|\xi|^p\}.$$

(d) *It is even in ξ, that is,*

$$(w\nu_p * w\nu_p * w\nu_p)(-\xi, \tau) = (w\nu_p * w\nu_p * w\nu_p)(\xi, \tau)$$

for every $\xi \in \mathbb{R}$, $\tau > 0$, and is homogeneous of degree zero in the sense that

$$(w\nu_p * w\nu_p * w\nu_p)(\lambda\xi, \lambda^p\tau) = (w\nu_p * w\nu_p * w\nu_p)(\xi, \tau) \quad \textit{for every } \lambda > 0.$$

(e) *It extends continuously to the boundary of E_p, except at the point $(\xi, \tau) = (0, 0)$, with values given by*

$$(w\nu_p * w\nu_p * w\nu_p)(\xi, 3^{1-p}|\xi|^p) = \frac{2\pi}{\sqrt{3}p(p-1)} \quad \textit{if } \xi \neq 0. \tag{4-3}$$

Proof. For $p \geq 2$, the result follows from Proposition 2.1 and Remark 2.3 of [Oliveira e Silva and Quilodrán 2019]. If $1 < p < 2$, then the weight w is singular at the origin, and an additional argument is required in order to establish parts (c) and (e) (as the others follow from [loc. cit.]). Note that part (e) also follows from [loc. cit.] after we verify (c), and so it suffices to show the latter.

Let $\psi = |\cdot|^p$. From [loc. cit., Remark 2.3], the formula

$$(w\nu_p * w\nu_p * w\nu_p)(\xi, \tau) = \int_{\mathbb{S}^1} \frac{\left(\left|\frac{1}{3}\xi + \alpha(\omega_1 + \omega_2)\right|\left|\frac{1}{3}\xi - \alpha\omega_1\right|\left|\frac{1}{3}\xi - \alpha\omega_2\right|\right)^{\frac{p-2}{3}}}{\left\langle \omega_1, \frac{\mathfrak{W}_1}{\alpha}\right\rangle + \left\langle \omega_2, \frac{\mathfrak{W}_2}{\alpha}\right\rangle}\,\mathrm{d}\mu_{(\omega_1, \omega_2)}, \tag{4-4}$$

where

$$\mathfrak{W}_i(\xi, \tau, \omega_1, \omega_2) = \nabla\psi\left(\tfrac{1}{3}\xi + \alpha\omega_1 + \alpha\omega_2\right) - \nabla\psi\left(\tfrac{1}{3}\xi - \alpha\omega_i\right), \quad i = 1, 2,$$

holds on $\widetilde{E}_p$, provided that the function W defined by

$$W(\xi,\omega_1,\omega_2) := \left(\left|\tfrac{1}{3}\xi+\alpha(\omega_1+\omega_2)\right|\left|\tfrac{1}{3}\xi-\alpha\omega_1\right|\left|\tfrac{1}{3}\xi-\alpha\omega_2\right|\right)^{\frac{p-2}{3}} \tag{4-5}$$

is continuous in the domain of integration. Here $\omega_1^2+\omega_2^2=1$, arc-length measure on the unit circle $\mathbb{S}^1$ is denoted by μ, and the function $\alpha=\alpha(\xi,\tau,\omega_1,\omega_2)$ is implicitly defined by

$$\left|\tfrac{1}{3}\xi+\alpha(\omega_1+\omega_2)\right|^p+\left|\tfrac{1}{3}\xi-\alpha\omega_1\right|^p+\left|\tfrac{1}{3}\xi-\alpha\omega_2\right|^p=\tau;$$

see [Oliveira e Silva and Quilodrán 2019] for details. It follows that

$$\left|\tfrac{1}{3}\xi+\alpha(\omega_1+\omega_2)\right|^p+\left|\tfrac{1}{3}\xi-\alpha\omega_1\right|^p+\left|\tfrac{1}{3}\xi-\alpha\omega_2\right|^p<2^{1-p}|\xi|^p,$$

provided $(\xi,\tau)\in\widetilde{E}_p$. On the other hand, if $\frac{1}{3}\xi-\alpha\omega_1=0$, then convexity of ψ implies

$$\left|\tfrac{2}{3}\xi+\alpha\omega_2\right|^p+\left|\tfrac{1}{3}\xi-\alpha\omega_2\right|^p\geq 2^{1-p}|\xi|^p,$$

and similarly if $\frac{1}{3}\xi-\alpha\omega_2=0$, while if $\frac{1}{3}\xi+\alpha(\omega_1+\omega_2)=0$, then

$$\left|\tfrac{1}{3}\xi-\alpha\omega_1\right|^p+\left|\tfrac{1}{3}\xi-\alpha\omega_2\right|^p\geq 2^{1-p}\left|\tfrac{2}{3}\xi-\alpha(\omega_1+\omega_2)\right|^p=2^{1-p}|\xi|^p.$$

It follows that none of these three terms can vanish in a neighborhood of any point $(\xi,\tau)\in\widetilde{E}_p$, and therefore W is continuous there. Thus identity (4-4) holds, and this concludes the verification of part (c). ☐

The boundedness of $w\nu_p * w\nu_p * w\nu_p$ provides an alternative way towards estimate (1-10) via the usual application of the Cauchy–Schwarz inequality, at least in the restricted range $p\geq 2$. Moreover, identity (4-3) and the argument in Lemma 3.2 together imply that the corresponding optimal constant $\boldsymbol{C}_p$ satisfies

$$\boldsymbol{C}_p^6\geq\frac{2\pi}{\sqrt{3}p(p-1)},$$

which should be compared to (1-13).

4A. ***Effective lower bounds for $\boldsymbol{C}_p$.*** We start by examining a simple lower bound, which is the analogue of [Oliveira e Silva and Quilodrán 2018, Lemma 6.1] for 3-fold convolutions in $\mathbb{R}^2$.

Lemma 4.2. *Given a strictly convex function $\Psi:\mathbb{R}\to\mathbb{R}$ and a nonnegative function $w:\mathbb{R}\to[0,\infty)$, consider the measures $\mathrm{d}\nu(y,s)=\boldsymbol{\delta}(s-\Psi(y))\,\mathrm{d}y\,\mathrm{d}s$ and $\mathrm{d}\sigma=\sqrt{w}\,\mathrm{d}\nu$. Let E denote the support of the convolution measure $\nu*\nu*\nu$. Given $\lambda>0$, $a\in\mathbb{R}$, let $f_{\lambda,a}(y):=e^{-\lambda(\Psi(y)+ay)}\sqrt{w(y)}$. Then*

$$\frac{\|f_{\lambda,a}\sigma*f_{\lambda,a}\sigma*f_{\lambda,a}\sigma\|_{L^2(\mathbb{R}^2)}^2}{\|f_{\lambda,a}\|_{L^2(\mathbb{R})}^6}\geq\frac{\|f_{\lambda,a}\|_{L^2(\mathbb{R})}^6}{\int_E e^{-2\lambda(\tau+a\xi)}\,\mathrm{d}\xi\,\mathrm{d}\tau} \tag{4-6}$$

for every $f_{\lambda,a}\in L^2(\mathbb{R})$ such that $f_{\lambda,a}\sigma*f_{\lambda,a}\sigma*f_{\lambda,a}\sigma\in L^2(\mathbb{R}^2)$.

The proof is entirely parallel to that of [Oliveira e Silva and Quilodrán 2018, Lemma 6.1]. Note that (4-6) implies

$$\sup_{0\neq f\in L^2(\mathbb{R})}\frac{\|f\sigma*f\sigma*f\sigma\|_{L^2(\mathbb{R}^2)}^2}{\|f\|_{L^2(\mathbb{R})}^6}\geq\sup_{\lambda>0,\,a\in\mathbb{R}}\frac{\|f_{\lambda,a}\|_{L^2(\mathbb{R})}^6}{\int_E e^{-2\lambda(\tau+a\xi)}\,\mathrm{d}\xi\,\mathrm{d}\tau}. \tag{4-7}$$

Specializing Lemma 4.2 to the case of the measure σ_p with the natural choice of trail function $f(y) = e^{-|y|^p}|y|^{(p-2)/6}$, a quick computation yields

$$\Phi_p(f) \geq \frac{4\Gamma\left(\frac{p+1}{3p}\right)^3}{3^{1-\frac{1}{p}} p^2 \Gamma\left(\frac{1}{p}\right)}. \tag{4-8}$$

This lower bound is good enough to establish the strict inequality (1-13) in a range of p that includes the cubic case $p = 3$ but *not* the quartic case $p = 4$, so we have to refine it. For the above choice of trial function, the corresponding ratio (4-1) can be expanded as an infinite series with nonnegative terms, whose coefficients are given in terms of the Gamma function and whose first term equals the expression on the right-hand side of (4-8).

Proposition 4.3. *Let $p > 1$ and $f(y) = e^{-|y|^p}|y|^{(p-2)/6} \in L^2(\mathbb{R})$. Then*

$$\Phi_p(f) = \frac{3^{1-\frac{1}{p}} p^2 \Gamma\left(\frac{1}{p}\right)}{2^3 \Gamma\left(\frac{p+1}{3p}\right)^3} \sum_{n=0}^{\infty} (4n+1) 2^{4n-1} \left(\sum_{k=0}^{n} \binom{2n}{2k} \binom{n+k-\frac{1}{2}}{2n} I_{2k}(p) \right)^2, \tag{4-9}$$

where the coefficients $\{I_{2k}(p)\}_{k\geq 0}$ are given by expression (4-15) *below.*

The proof will make use of the classical Legendre polynomials, denoted by $\{P_n\}_{n\geq 0}$, which constitute a family of orthogonal polynomials with respect to the L^2-norm on the interval $[-1, 1]$. Explicitly, they are given by[4]

$$P_n(t) = 2^n \sum_{k=0}^{n} \binom{n}{k} \binom{\frac{n+k-1}{2}}{n} t^k, \quad -1 \leq t \leq 1, \tag{4-10}$$

from where one checks that $\langle P_m, P_n \rangle_{L^2} = (2/(2n+1))\, \boldsymbol{\delta}(n = m)$; see [Stein and Weiss 1971, Corollary 2.16, Chapter 4]. See also [Carneiro and Oliveira e Silva 2015; Christ and Shao 2012a; Foschi 2015; Gonçalves 2019; Negro 2018] for earlier appearances of Legendre and other families of orthogonal polynomials in sharp Fourier restriction theory.

Proof of Proposition 4.3. Start by noting that the function $f(y) = e^{-|y|^p}|y|^{(p-2)/6}$ coincides with $e^{-\tau}\sqrt{w(\xi)}$ on the support of σ_p. Using this together with parts (b) and (d) of Proposition 4.1, we obtain

$$\begin{aligned}
\|f\sigma_p * f\sigma_p * f\sigma_p\|_{L^2}^2 &= \|e^{-\tau}(w\nu_p * w\nu_p * w\nu_p)\|_{L^2}^2 \\
&= \int_0^\infty \int_{-3^{1-1/p}\tau^{1/p}}^{3^{1-1/p}\tau^{1/p}} e^{-2\tau} (w\nu_p * w\nu_p * w\nu_p)^2(\xi, \tau)\, \mathrm{d}\xi\, \mathrm{d}\tau \\
&= \int_0^\infty \int_{-3^{1-1/p}}^{3^{1-1/p}} \tau^{\frac{1}{p}} e^{-2\tau} (w\nu_p * w\nu_p * w\nu_p)^2(\tau^{\frac{1}{p}}\lambda, \tau)\, \mathrm{d}\lambda\, \mathrm{d}\tau \\
&= \left(\int_0^\infty \tau^{\frac{1}{p}} e^{-2\tau}\, \mathrm{d}\tau \right) \int_{-3^{1-1/p}}^{3^{1-1/p}} (w\nu_p * w\nu_p * w\nu_p)^2(\lambda, 1)\, \mathrm{d}\lambda \\
&= \frac{3^{1-\frac{1}{p}} \Gamma\left(\frac{1}{p}\right)}{p 2^{1+\frac{1}{p}}} \int_{-1}^{1} (w\nu_p * w\nu_p * w\nu_p)^2(3^{1-\frac{1}{p}} t, 1)\, \mathrm{d}t.
\end{aligned} \tag{4-11}$$

[4]Recall that the binomial coefficient $\binom{\alpha}{n} := \alpha(\alpha-1)\cdots(\alpha-n+1)/n!$ is also defined when $\alpha \notin \mathbb{Z}$.

On the other hand,

$$\|f\|_{L^2}^2 = \int_{\mathbb{R}} e^{-2|y|^p} |y|^{\frac{p-2}{3}} \, dy = 2\int_0^\infty e^{-2y^p} y^{\frac{p-2}{3}} \, dy = \frac{2^{\frac{2}{3}-\frac{1}{3p}}}{p} \Gamma\Big(\frac{p+1}{3p}\Big). \tag{4-12}$$

Given $t \in [-1,1]$, define $g_p(t) := (w\nu_p * w\nu_p * w\nu_p)(3^{1-1/p} t, 1)$. Expanding g_p in the basis of Legendre polynomials,

$$\begin{aligned}\|g_p\|^2_{L^2([-1,1],\, dt)} &= \sum_{n=0}^\infty \frac{1}{\|P_n\|^2_{L^2}} \Big(\int_{-1}^1 g_p(t) P_n(t) \, dt\Big)^2 \\ &= \sum_{n=0}^\infty (4n+1) 2^{4n-1} \Big(\sum_{k=0}^n \binom{2n}{2k}\binom{n+k-\frac{1}{2}}{2n} \int_{-1}^1 g_p(t) t^{2k} \, dt\Big)^2,\end{aligned}$$

where the last identity follows from (4-10), the normalization $\|P_n\|^2_{L^2} = 2/(2n+1)$, and the fact that g_p is an even function of t. We proceed to find an explicit expression for the moments $I_n(p) := \int_{-1}^1 g_p(t)\, t^n \, dt$. Given $b \in \mathbb{R}$, we compute

$$\begin{aligned}&\int_{\mathbb{R}^2} e^{-(\tau - b\xi)} (w\nu_p * w\nu_p * w\nu_p)(\xi,\tau) \, d\xi \, d\tau \\ &\quad = \int_0^\infty \int_{-3^{1-1/p}}^{3^{1-1/p}} \tau^{\frac{1}{p}} e^{-\tau} e^{b\tau^{1/p}\lambda} (w\nu_p * w\nu_p * w\nu_p)(\lambda, 1) \, d\lambda \, d\tau \\ &\quad = \sum_{n=0}^\infty \frac{3^{(1-\frac{1}{p})(2n+1)} b^{2n}}{(2n)!} \Big(\int_0^\infty e^{-\tau} \tau^{\frac{2n+1}{p}} \, d\tau\Big) \int_{-1}^1 t^{2n} (w\nu_p * w\nu_p * w\nu_p)(3^{1-\frac{1}{p}} t, 1) \, dt \\ &\quad = \sum_{n=0}^\infty \frac{3^{(1-\frac{1}{p})(2n+1)} b^{2n}}{(2n)!} \frac{2n+1}{p} \Gamma\Big(\frac{2n+1}{p}\Big) I_{2n}(p).\end{aligned} \tag{4-13}$$

This Laplace transform can be alternatively computed as

$$\begin{aligned}\int_{\mathbb{R}^2} e^{-(\tau-b\xi)} (w\nu_p * w\nu_p * w\nu_p)(\xi,\tau) \, d\xi \, d\tau &= \Big(\int_{\mathbb{R}} e^{-|y|^p} e^{by} |y|^{\frac{p-2}{3}} \, dy\Big)^3 \\ &= \Big(\sum_{n=0}^\infty \frac{2b^{2n}}{(2n)!} \int_0^\infty e^{-y^p} y^{\frac{p-2}{3}+2n} \, dy\Big)^3 \\ &= \Big(\sum_{n=0}^\infty \frac{2b^{2n}}{p(2n)!} \Gamma\Big(\frac{p+1+6n}{3p}\Big)\Big)^3.\end{aligned} \tag{4-14}$$

Equating coefficients of the same degree, we obtain

$$I_{2n}(p) = \frac{2^3 (2n)!}{3^{(1-\frac{1}{p})(2n+1)} p^2 (2n+1) \Gamma\big(\frac{2n+1}{p}\big)} \sum_{k=0}^n \sum_{m=0}^{n-k} \frac{\Gamma\big(\frac{p+1+6k}{3p}\big)\Gamma\big(\frac{p+1+6m}{3p}\big)\Gamma\big(\frac{p+1+6(n-k-m)}{3p}\big)}{(2k)!\,(2m)!\,(2(n-k-m))!}. \tag{4-15}$$

Identity (4-9) follows at once, and the proof is complete. □

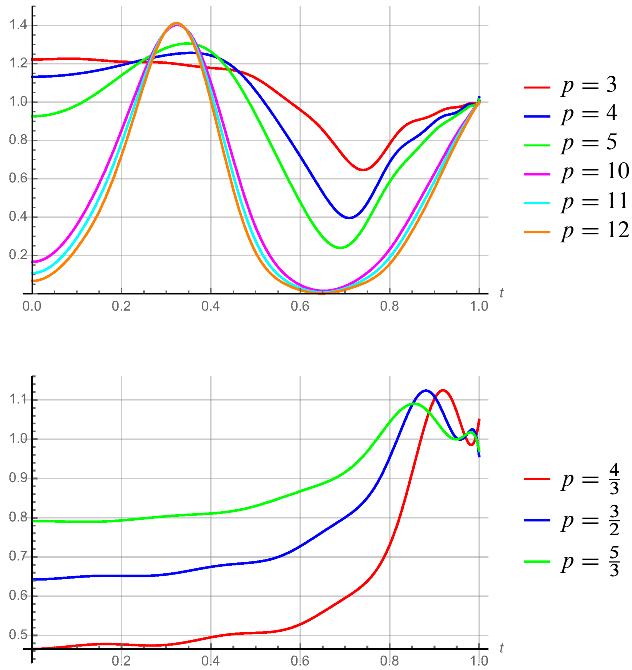

Figure 1. Plots of the functions $g_{p,N}(t)$, appropriately normalized so that they are close to 1 at $t = 1$ for $p \in \{3, 4, 5, 10, 11, 12\}$ and $p \in \{\frac{4}{3}, \frac{3}{2}, \frac{5}{3}\}$. For $p \in \{3, 4, 5\}$ and $p \in \{\frac{4}{3}, \frac{3}{2}, \frac{5}{3}\}$ we used $N = 10$, for $p \in \{10, 11, 12\}$ we used $N = 15$.

Remark 4.4. From the preceding proof, we have the following approximating sequence $\{g_{p,N}\}_{N \geq 0}$ for g_p:

$$g_{p,N}(t) := \sum_{n=0}^{N} (4n+1) 2^{2n-1} \left(\sum_{k=0}^{n} \binom{2n}{2k} \binom{n+k-\frac{1}{2}}{2n} I_{2k}(p) \right) P_{2n}(t), \quad -1 \leq t \leq 1.$$

This was used to construct Figure 1. They correspond to approximate graphs of $w\nu_p * w\nu_p * w\nu_p$ on the region $\{(\xi, 1) \colon 0 \leq \xi \leq 3^{1-1/p}\}$ for different values of p. By homogeneity, the full picture on $\mathbb{R}^2$ can be obtained from these graphs. Figure 1 (top) indicates that, for large p, the function $g_p(t)$ becomes small as $t \to 0$. The function $(w\nu_p * w\nu_p * w\nu_p)(\xi, \tau)$ should then be small near the τ-axis, unlike the case of small values of p. This suggests that extremizing sequences may concentrate at the boundary if p is large enough.

4B. ***Proof of Theorem 1.4.*** We consider the case $p > 2$ first. From Theorem 1.3 and Proposition 4.3, it suffices to show that there exists $N \in \mathbb{N}$ such that

$$\frac{3^{1-\frac{1}{p}} p^2 \Gamma(\frac{1}{p})}{2^3 \Gamma(\frac{p+1}{3p})^3} \sum_{n=0}^{N} (4n+1) 2^{4n-1} \left(\sum_{k=0}^{n} \binom{2n}{2k} \binom{n+k-\frac{1}{2}}{2n} I_{2k}(p) \right)^2 > \frac{2\pi}{\sqrt{3}\, p(p-1)}, \tag{4-16}$$

where the coefficients $I_{2k}(p)$ are given by (4-15). The range of validity of (4-16) can be estimated by performing an accurate numerical calculation. Taking $N = 15$, one checks that inequality (4-16) holds for every $p \in (2, p_0)$, where $p_0 \in [4,5]$ and can be numerically estimated by $p_0 \approx 4.803$, with three decimal places. Increasing the value of N does not seem to substantially increase p_0.

If $1 < p < 2$, then inequality (4-16) fails (for every $N \in \mathbb{N}$). Incidentally, note that if $p = 2$, then the left- and right-hand sides of (4-16) are equal (for every $N \in \mathbb{N}$) since the 3-fold convolution of projection measure on the parabola is constant inside its support; see [Foschi 2007, Lemma 4.1]. We are thus led to a different trial function. For $n \in \mathbb{N}$, define

$$f_n(y) = e^{-\frac{n}{2}(|y|^p - py)}|y|^{-\frac{2-p}{6}}. \tag{4-17}$$

In light of Lemma 3.2, the sequence $\{f_n\|f_n\|_{L^2}^{-1}\}$ concentrates at $y_0 = 1$. Passing to a continuous parameter $\lambda > 0$, Lemma 4.2 yields the lower bound

$$\Phi_p(f_\lambda) \geq \frac{\|f_\lambda\|_{L^2(\mathbb{R})}^6}{\int_{E_p} e^{-\lambda(\tau - p\xi)}\,d\xi\,d\tau} =: \phi_p(\lambda),$$

which we proceed to analyze. Since

$$\|f_\lambda\|_{L^2(\mathbb{R})}^2 = \int_{-\infty}^{\infty} e^{-\lambda(|y|^p - py)}|y|^{-\frac{2-p}{3}}\,dy,$$

$$\int_{E_p} e^{-\lambda(\tau - p\xi)}\,d\xi\,d\tau = \int_{-\infty}^{\infty} e^{\lambda p\xi}\left(\int_{3^{1-p}|\xi|^p}^{\infty} e^{-\lambda\tau}\,d\tau\right)d\xi = \frac{1}{\lambda}\int_{-\infty}^{\infty} e^{-\lambda(3^{1-p}|\xi|^p - p\xi)}\,d\xi,$$

we have

$$\phi_p(\lambda) = \lambda\frac{\left(\int_{-\infty}^{\infty} e^{-\lambda(|y|^p - py)}|y|^{-\frac{2-p}{3}}\,dy\right)^3}{\int_{-\infty}^{\infty} e^{-\lambda(3^{1-p}|\xi|^p - p\xi)}\,d\xi}.$$

In view of (4-7), we have $\boldsymbol{C}_p^6 \geq \phi_p(\lambda)$ for every $\lambda > 0$. Therefore it suffices to show that $\phi_p(\lambda) > 2\pi/(\sqrt{3}p(p-1))$, provided λ is large enough. This is the content of the following lemma, which we choose to formulate in terms of the function $\varphi_p(\lambda) := \phi_p(\lambda^{-1})$.

Lemma 4.5. *Let* $p \in (1,2)$. *Then*

$$\lim_{\lambda \to 0^+} \varphi_p(\lambda) = \frac{2\pi}{\sqrt{3}p(p-1)}, \tag{4-18}$$

$$\lim_{\lambda \to 0^+} \varphi_p'(\lambda) = \frac{\pi(2-p)(2p-1)}{9\sqrt{3}p^2(p-1)^2}, \tag{4-19}$$

In particular, if $\lambda > 0$ *is small enough, then* $\varphi_p(\lambda) > 2\pi/(\sqrt{3}p(p-1))$.

Note that (4-18) follows from Lemma 3.2, but we choose to present a unified approach that establishes both (4-18) and (4-19).

Proof of Lemma 4.5. Rewrite ϕ_p in the equivalent form

$$\phi_p(\lambda) = \lambda\frac{\left(\int_{-\infty}^{\infty} e^{-\lambda(|y|^p - 1 - p(y-1))}|y|^{-\frac{2-p}{3}}\,dy\right)^3}{\int_{-\infty}^{\infty} e^{-\lambda 3^{1-p}(|y|^p - 3^p - p3^{p-1}(y-3))}\,dy}.$$

Define real-valued functions $y \mapsto \alpha(y)$ and $y \mapsto \beta(y)$ via[5]

$$|y|^p - 1 - p(y-1) = \binom{p}{2}\big((y-1)^2 + \alpha(y-1)\big), \tag{4-20}$$
$$|y|^p - 3^p - p3^{p-1}(y-3) = 3^{p-2}\binom{p}{2}\big((y-3)^2 + \beta(y-3)\big).$$

By the binomial series expansion, if $|y| < 1$, then

$$\alpha(y) = \frac{p-2}{3}y^3 + \frac{(p-2)(p-3)}{12}y^4 + \cdots, \tag{4-21}$$
$$\beta(y) = \frac{p-2}{3\cdot 3}y^3 + \frac{(p-2)(p-3)}{12\cdot 3^2}y^4 + \cdots. \tag{4-22}$$

One easily checks that $|\alpha(y)| \to \infty$ and $|\beta(y)| \to \infty$ as $|y| \to \infty$, and

$$\lim_{\lambda\to\infty} \lambda\alpha(\lambda^{-\frac{1}{2}}y) = \lim_{\lambda\to\infty} \lambda\beta(\lambda^{-\frac{1}{2}}y) = 0 \tag{4-23}$$

for each $y \in \mathbb{R}$. We also have

$$\int_{\mathbb{R}} \exp\bigg(-\lambda\frac{|y|^p - 1 - p(y-1)}{\binom{p}{2}}\bigg)|y|^{-\frac{2-p}{3}}\,\mathrm{d}y = \lambda^{-\frac{1}{2}}\int_{\mathbb{R}} e^{-y^2}e^{-\lambda\alpha(\lambda^{-1/2}y)}|1+\lambda^{-\frac{1}{2}}y|^{-\frac{2-p}{3}}\,\mathrm{d}y,$$

$$\int_{\mathbb{R}} \exp\bigg(-\lambda\frac{3^{1-p}(|y|^p - 3^p - p3^{p-1}(y-3))}{\binom{p}{2}}\bigg)\mathrm{d}y = 3^{\frac{1}{2}}\lambda^{-\frac{1}{2}}\int_{\mathbb{R}} e^{-y^2}e^{-\frac{\lambda}{3}\beta((\frac{3}{\lambda})^{1/2}y)}\,\mathrm{d}y,$$

and consequently

$$\phi_p\bigg(\frac{2\lambda}{p(p-1)}\bigg) = \frac{2}{\sqrt{3}p(p-1)}\frac{\big(\int_{\mathbb{R}} e^{-y^2}e^{-\lambda\alpha(\lambda^{-1/2}y)}|1+\lambda^{-\frac{1}{2}}y|^{-\frac{2-p}{3}}\,\mathrm{d}y\big)^3}{\int_{\mathbb{R}} e^{-y^2}e^{-\frac{\lambda}{3}\beta((\frac{3}{\lambda})^{1/2}y)}\,\mathrm{d}y}.$$

For bookkeeping purposes, set

$$A_p(\lambda) := \bigg(\int_{\mathbb{R}} e^{-y^2}e^{-\lambda\alpha(\lambda^{-1/2}y)}|1+\lambda^{-\frac{1}{2}}y|^{-\frac{2-p}{3}}\,\mathrm{d}y\bigg)^3 \quad\text{and}\quad B_p(\lambda) := \int_{\mathbb{R}} e^{-y^2}e^{-\frac{\lambda}{3}\beta((\frac{3}{\lambda})^{1/2}y)}\,\mathrm{d}y.$$

We now analyze each expression. Recalling (4-22), the numerator $A_p(\lambda)$ is seen to satisfy

$$A_p(\lambda) = \pi^{\frac{3}{2}}\bigg(1 - \frac{(p-2)(2p-1)}{144\lambda} + O(\lambda^{-\frac{3}{2}})\bigg)^3 \quad \text{as } \lambda \to \infty. \tag{4-24}$$

Since binomial series expansions are only valid inside the unit ball, this step requires some care which we now briefly describe. Split the integral defining $A_p(\lambda)$ into three regions,

$$A_p^{\frac{1}{3}}(\lambda) = \bigg(\int_{-\infty}^{-\frac{\sqrt{\lambda}}{2}} + \int_{-\frac{\sqrt{\lambda}}{2}}^{\frac{\sqrt{\lambda}}{2}} + \int_{\frac{\sqrt{\lambda}}{2}}^{\infty}\bigg)e^{-y^2}e^{-\lambda\alpha(\lambda^{-1/2}y)}|1+\lambda^{-\frac{1}{2}}y|^{-\frac{2-p}{3}}\,\mathrm{d}y =: \mathrm{I} + \mathrm{II} + \mathrm{III},$$

[5]Note that $\alpha(y) = 3^{-2}\beta(3y)$.

and estimate each of them separately. The main contribution comes from the integral $\mathrm{II} = \mathrm{II}(\lambda)$. Appealing to (4-21) and to the binomial series expansion, we have

$$\exp(-\lambda\alpha(\lambda^{-\frac{1}{2}}y)) = 1 - \frac{p-2}{3}\lambda^{-\frac{1}{2}}y^3 - \frac{(p-2)(p-3)}{12}\lambda^{-1}y^4 + \frac{(p-2)^2}{18}\lambda^{-1}y^6 + O_y(\lambda^{-\frac{3}{2}}),$$
$$|1+\lambda^{-\frac{1}{2}}y|^{-\frac{2-p}{3}} = 1 + \frac{p-2}{3}\lambda^{-\frac{1}{2}}y + \frac{(p-2)(p-5)}{18}\lambda^{-1}y^2 + O_y(\lambda^{-\frac{3}{2}})$$

uniformly in $y \in [-\sqrt{\lambda}/2, \sqrt{\lambda}/2]$. From this one easily checks that

$$\mathrm{II}(\lambda) = \pi^{\frac{1}{2}} + \pi^{\frac{1}{2}}\frac{(p-2)(2p-1)}{144}\lambda^{-1} + O(\lambda^{-\frac{3}{2}}).$$

Matters are thus reduced to verifying that the contributions from I and III become negligible as $\lambda \to \infty$. On the region of integration of $\mathrm{I} = \mathrm{I}(\lambda)$, the factor $|1+\lambda^{-1/2}y|^{-(2-p)/3}$ has an integrable singularity at $y = -\lambda^{1/2}$. Recalling the definition (4-20) of the function α, and changing variables $\lambda^{-1/2}y \rightsquigarrow x$, we have

$$\mathrm{I}(\lambda) = \lambda^{\frac{1}{2}}\int_{-\infty}^{-\frac{1}{2}} e^{-\frac{2\lambda}{p(p-1)}(|1+x|^p-1-px)}|1+x|^{-\frac{2-p}{3}}\,\mathrm{d}x.$$

Invoking the elementary inequality $|1+x|^p - 1 - px \gtrsim_p |x|^p$, which is valid for every $x \le -\frac{1}{2}$ and $1 < p < 2$, we may use Hölder's inequality together with the local integrability of $x \mapsto |1+x|^{-(2-p)/3}$ in order to bound

$$\mathrm{I}(\lambda) = O_p(\lambda^{\frac{1}{2}}\exp(-C_p\lambda))$$

for some $C_p > 0$. The contribution of $\mathrm{III}(\lambda)$ is easier to handle because no singularity occurs on the corresponding region of integration. This concludes the verification of (4-24), which can then be differentiated term by term because there is sufficient decay. Therefore

$$\lim_{\lambda\to\infty} A_p(\lambda) = \pi^{\frac{3}{2}} \quad \text{and} \quad \lim_{\lambda\to\infty} -\lambda^2 A_p'(\lambda) = -\frac{3(p-2)(2p-1)\pi^{\frac{3}{2}}}{144}.$$

On the other hand, using the binomial series expansion (4-22) we obtain

$$\exp\left(-\frac{\lambda}{3}\beta\left(\left(\frac{3}{\lambda}\right)^{\frac{1}{2}}y\right)\right) = 1 - \frac{p-2}{3^{\frac{3}{2}}}\lambda^{-\frac{1}{2}}y^3 - \frac{(p-2)(p-3)}{36}\lambda^{-1}y^4 + \frac{(p-2)^2}{54}\lambda^{-1}y^6 + O_y(\lambda^{-\frac{3}{2}})$$

uniformly in $y \in \left[-\frac{1}{2}\left(\frac{1}{3}\lambda\right)^{1/2}, \frac{1}{2}\left(\frac{1}{3}\lambda\right)^{1/2}\right]$, so that an argument similar to that for $A_p(\lambda)$ gives

$$B_p(\lambda) = \pi^{\frac{1}{2}} + \frac{(p-2)(2p-1)\pi^{\frac{1}{2}}}{144\lambda} + O(\lambda^{-\frac{3}{2}}),$$
$$\lim_{\lambda\to\infty} B_p(\lambda) = \pi^{\frac{1}{2}} \quad \text{and} \quad \lim_{\lambda\to\infty} -\lambda^2 B_p'(\lambda) = \frac{(p-2)(2p-1)\pi^{\frac{1}{2}}}{144}.$$

We conclude

$$\lim_{\lambda\to 0+}\varphi_p(\lambda) = \lim_{\lambda\to\infty}\phi_p(\lambda) = \lim_{\lambda\to\infty}\phi_p\left(\frac{2\lambda}{p(p-1)}\right) = \frac{2\pi}{\sqrt{3}p(p-1)}.$$

To address (4-19), note that

$$\varphi_p'(\lambda) = -\lambda^{-2}\phi_p'(\lambda^{-1}), \quad \text{and so } \lim_{\lambda\to 0^+} \varphi_p'(\lambda) = \lim_{\lambda\to\infty} -\lambda^2\phi_p'(\lambda).$$

Therefore

$$\lim_{\lambda\to\infty} -\lambda^2 \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\phi_p\left(\frac{2\lambda}{p(p-1)}\right)\right) = \frac{2\pi}{\sqrt{3}p(p-1)}\left(-\frac{3(p-2)(2p-1)}{144} - \frac{(p-2)(2p-1)}{144}\right) = \frac{\pi(2-p)(2p-1)}{18\sqrt{3}p(p-1)},$$

which readily implies (4-19). This completes the proof of the lemma (and therefore of Theorem 1.4). □

4C. *Improving p_0*. In view of the results from the last subsection, it is natural to let the functional Φ_p defined on (4-1) act on trial functions $f(y) = e^{-|y|^p}|y|^{(p-2)/6+a}$ for different choices of a.[6] By doing so, the value $p_0 \approx 4.803$ can be improved. We turn to the details.

Set $\kappa := |\cdot|^{(p-2)/3+a}$, and note that

$$(\kappa\nu_p * \kappa\nu_p * \kappa\nu_p)(\lambda\xi, \lambda^p\tau) = \lambda^{3a}(\kappa\nu_p * \kappa\nu_p * \kappa\nu_p)(\xi,\tau) \quad \text{for every } \lambda > 0.$$

Reasoning as in (4-11) and (4-12), one checks that

$$\|f\sigma_p * f\sigma_p * f\sigma_p\|_{L^2(\mathbb{R}^2)}^2 = \frac{3^{1-\frac{1}{p}}\Gamma\left(\frac{1+6a}{p}\right)}{p2^{1+\frac{1+6a}{p}}}(1+6a)\int_{-1}^{1}(\kappa\nu_p * \kappa\nu_p * \kappa\nu_p)^2(3^{1-\frac{1}{p}}t, 1)\,\mathrm{d}t,$$

$$\|f\|_{L^2(\mathbb{R})}^2 = \frac{2^{\frac{2}{3}-\frac{1+6a}{3p}}}{p}\Gamma\left(\frac{p+1+6a}{3p}\right).$$

Given $t \in [-1,1]$, define $h_p(t) := (\kappa\nu_p * \kappa\nu_p * \kappa\nu_p)(3^{1-1/p}t, 1)$. Expanding h_p in the basis of Legendre polynomials,

$$\|h_p\|_{L^2([-1,1],\mathrm{d}t)}^2 = \sum_{n=0}^{\infty}(4n+1)2^{4n-1}\left(\sum_{k=0}^{n}\binom{2n}{2k}\binom{n+k-\frac{1}{2}}{2n}\int_{-1}^{1}h_p(t)t^{2k}\,\mathrm{d}t\right)^2.$$

We proceed to find explicit expressions for the moments $I_n(p,a) := \int_{-1}^{1} h_p(t)t^n\,\mathrm{d}t$. Given $b \in \mathbb{R}$, we compute as in (4-13) and (4-14)

$$\int_{\mathbb{R}^2} e^{-(\tau-b\xi)}(\kappa\nu_p*\kappa\nu_p*\kappa\nu_p)(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau = \sum_{n=0}^{\infty}\frac{3^{\left(1-\frac{1}{p}\right)(2n+1)}b^{2n}}{(2n)!}\frac{2n+1+3a}{p}\Gamma\left(\frac{2n+1+3a}{p}\right)I_{2n}(p,a)$$
$$= \left(\sum_{n=0}^{\infty}\frac{2b^{2n}}{p(2n)!}\Gamma\left(\frac{p+1+6n+3a}{3p}\right)\right)^3.$$

Equating coefficients as before, we find that the moment $I_{2n}(p,a)$ equals

$$\frac{3^{-\left(1-\frac{1}{p}\right)(2n+1)}2^3(2n)!}{p^2(2n+1+3a)\Gamma\left(\frac{2n+1+3a}{p}\right)}\sum_{k=0}^{n}\sum_{m=0}^{n-k}\frac{\Gamma\left(\frac{p+1+6k+3a}{3p}\right)\Gamma\left(\frac{p+1+6m+3a}{3p}\right)\Gamma\left(\frac{p+1+6(n-k-m)+3a}{3p}\right)}{(2k)!\,(2m)!\,(2(n-k-m))!}.$$

[6]Note that L^2-integrability forces $a > -(p+1)/6$.

This implies

$$\Phi_p(f)=\frac{3^{1-\frac{1}{p}}p^2\Gamma\big(\frac{1+6a}{p}\big)}{2^3\Gamma\big(\frac{p+1+6a}{3p}\big)^3}(1+6a)\sum_{n=0}^{\infty}(4n+1)2^{4n-1}\bigg(\sum_{k=0}^{n}\binom{2n}{2k}\binom{n+k-\frac{1}{2}}{2n}I_{2k}(p,a)\bigg)^2,$$

and consequently the following lower bound holds for every $N\geq 0$:

$$\Phi_p(f)\geq\frac{3^{1-\frac{1}{p}}p^2\Gamma\big(\frac{1+6a}{p}\big)}{2^3\Gamma\big(\frac{p+1+6a}{3p}\big)^3}(1+6a)\sum_{n=0}^{N}(4n+1)2^{4n-1}\bigg(\sum_{k=0}^{n}\binom{2n}{2k}\binom{n+k-\frac{1}{2}}{2n}I_{2k}(p,a)\bigg)^2.$$

By numerically evaluating this sum with $N=15$ and $a=\frac{7}{15}$, one can establish a lower bound that beats the critical threshold $2\pi/(\sqrt{3}p(p-1))$ for every $p\in(2,p_1)$, where $p_1\approx 5.485$ with three decimal places. One further observes that the lower bound for small values of $a>0$ is larger than that for $a=0$, strongly suggesting that the original trial function $y\mapsto e^{-|y|^p}|y|^{(p-2)/6}$ might *not* be an extremizer in that range of exponents.

5. Superexponential L^2-decay

This section is devoted to the proof of Theorem 1.5. We follow the outline of [Erdoğan et al. 2011; Hundertmark and Shao 2012], and shall sometimes be brief. The Euler–Lagrange equation associated to (1-9) is

$$\mathcal{E}_p^*\big(\mathcal{E}_p(f)(\cdot,t)|\mathcal{E}_p(f)(\cdot,t)|^4\big)=\lambda f; \tag{5-1}$$

see [Christ and Quilodrán 2014, Proposition 2.4] for the variational derivation in a related context. The following 6-linear form will play a prominent role in the analysis:

$$Q(f_1,f_2,f_3,f_4,f_5,f_6):=\int_{\mathbb{R}^2}\prod_{j=1}^{3}\mathcal{E}_p(f_j)(x,t)\overline{\mathcal{E}_p(f_{j+3})(x,t)}\,\mathrm{d}x\,\mathrm{d}t.$$

An immediate consequence of (1-9) is the basic estimate

$$|Q(f_1,f_2,f_3,f_4,f_5,f_6)|\lesssim\prod_{j=1}^{6}\|f_j\|_{L^2(\mathbb{R})}. \tag{5-2}$$

The form Q can be rewritten as

$$Q(f_1,f_2,f_3,f_4,f_5,f_6)=\int_{\mathbb{R}^6}\prod_{j=1}^{3}f_j(y_j)|y_j|^{\frac{p-2}{6}}\,\overline{f_{j+3}(y_{j+3})}|y_{j+3}|^{\frac{p-2}{6}}\,\boldsymbol{\delta}(\alpha(\boldsymbol{y}))\,\boldsymbol{\delta}(\beta(\boldsymbol{y}))\,\mathrm{d}\boldsymbol{y},$$

where $\boldsymbol{y}=(y_1,\dots,y_6)\in\mathbb{R}^6$ and

$$\alpha(\boldsymbol{y}):=|y_1|^p+|y_2|^p+|y_3|^p-|y_4|^p-|y_5|^p-|y_6|^p,$$
$$\beta(\boldsymbol{y}):=y_1+y_2+y_3-y_4-y_5-y_6.$$

We will also consider the associated form

$$K(f_1,f_2,f_3,f_4,f_5,f_6):=Q(|f_1|,|f_2|,|f_3|,|f_4|,|f_5|,|f_6|),$$

which is sublinear in each entry. Clearly,

$$|Q(f_1, f_2, f_3, f_4, f_5, f_6)| \le K(f_1, f_2, f_3, f_4, f_5, f_6), \tag{5-3}$$

$$K(f_1, f_2, f_3, f_4, f_5, f_6) \lesssim \prod_{j=1}^{6} \|f_j\|_{L^2(\mathbb{R})}. \tag{5-4}$$

Let us now introduce a parameter $s \ge 1$, which will typically be large. If there exist $j \ne k$ such that f_j is supported on $[-s, s]$ and f_k is supported outside of $[-Cs, Cs]$ for some $C > 1$, then estimate (5-4) can be improved to

$$K(f_1, f_2, f_3, f_4, f_5, f_6) \lesssim C^{-\frac{p-1}{6}} \prod_{j=1}^{6} \|f_j\|_{L^2(\mathbb{R})}, \tag{5-5}$$

in accordance with the bilinear estimates of Corollary 2.2. Introducing the weighted variant

$$K_G(f_1, f_2, f_3, f_4, f_5, f_6) := \int_{\mathbb{R}^6} e^{G(y_1) - \sum_{j=2}^{6} G(y_j)} \prod_{j=1}^{6} |f_j(y_j)| |y_j|^{\frac{p-2}{6}} \boldsymbol{\delta}(\alpha(\boldsymbol{y}))\, \boldsymbol{\delta}(\beta(\boldsymbol{y}))\, \mathrm{d}\boldsymbol{y},$$

one checks at once that

$$K(e^G f_1, e^{-G} f_2, e^{-G} f_3, e^{-G} f_4, e^{-G} f_5, e^{-G} f_6) = K_G(f_1, f_2, f_3, f_4, f_5, f_6). \tag{5-6}$$

Given $\mu, \varepsilon \ge 0$, define the function

$$G_{\mu,\varepsilon}(y) := \frac{\mu |y|^p}{1 + \varepsilon |y|^p}. \tag{5-7}$$

The same proof as [Hundertmark and Shao 2012, Proposition 4.5] yields

$$K_{G_{\mu,\varepsilon}}(f_1, f_2, f_3, f_4, f_5, f_6) \le K(f_1, f_2, f_3, f_4, f_5, f_6); \tag{5-8}$$

see also Remark 4.6 of that paper. Split $f = f_< + f_>$ with $f_> := f \mathbb{1}_{[-s^2, s^2]^\complement}$, and define

$$\|f\|_{\mu,s,\varepsilon} := \|e^{G_{\mu,\varepsilon}} f_>\|_{L^2}.$$

Definition 5.1. A function $f \in L^2(\mathbb{R})$ is said to be a *weak solution* of (5-1) if there exists $\lambda > 0$ such that

$$Q(g, f, f, f, f, f) = \lambda \langle g, f \rangle_{L^2} \quad \text{for every } g \in L^2(\mathbb{R}). \tag{5-9}$$

Note that if f extremizes (1-9), then f satisfies (5-9) with $\lambda = \boldsymbol{E}_p^6 \|f\|_{L^2}^4$. The following key step shows that for some positive μ, the quantity $\|f\|_{\mu,s,\varepsilon}$ is bounded in $\varepsilon > 0$.

Proposition 5.2. *Given $p > 1$, let f be a weak solution of the Euler–Lagrange equation* (5-1) *with $\|f\|_{L^2} = 1$. If $s \ge 1$ is sufficiently large, then there exists $C < \infty$ such that*

$$\lambda \|f\|_{s^{-2p},s,\varepsilon} \le o_1(1) \|f\|_{s^{-2p},s,\varepsilon} + C \sum_{\ell=2}^{5} \|f\|_{s^{-2p},s,\varepsilon}^{\ell} + o_2(1), \tag{5-10}$$

where for $j \in \{1, 2\}$ we have $o_j(1) \to 0$ as $s \to \infty$ uniformly in ε. Moreover the constant C is independent of s and ε.

Proof. We start by introducing some notation. Let $G := G_{\mu,\varepsilon}$ be as in (5-7). Let $h := e^G f$, $h_> := e^G f_>$, and $h_< := h - h_>$. Further split $f_< = f_\ll + f_\sim$ and $h_< = h_\ll + h_\sim$, where $f_\ll := f\mathbb{1}_{[-s,s]}$ and $h_\ll := e^G f_\ll$. Since f satisfies (5-9), we have

$$\begin{aligned}\lambda\|e^G f_>\|_{L^2}^2 &= \lambda\langle e^{2G} f_>, f_>\rangle_{L^2} = \lambda\langle e^{2G} f_>, f\rangle_{L^2} = Q(e^{2G} f_>, f, f, f, f, f)\\ &= Q(e^G h_>, f, f, f, f, f) = Q(e^G h_>, e^{-G}h, e^{-G}h, e^{-G}h, e^{-G}h, e^{-G}h) =: Q_G.\end{aligned}$$

It follows from (5-3), (5-6), and (5-8) that $|Q_G| \lesssim K(h_>, h, h, h, h, h)$. Writing $h = h_< + h_>$, the sublinearity of K implies

$$|Q_G| \lesssim K(h_>, h_<, h_<, h_<, h_<, h_<) + \Big(\sum{}' + \sum{}''\Big) K(h_>, h_{j_2}, h_{j_3}, h_{j_4}, h_{j_5}, h_{j_6}),$$

where the first sum, denoted by B_1, is taken over indices $j_2, \dots, j_6 \in \{>, <\}$ with exactly one of the j_k equal to $>$, and the second sum, denoted by B_2, is taken over indices $j_2, \dots, j_6 \in \{>, <\}$ with two or more of the j_k equal to $>$. We estimate the three terms separately. For the first one,

$$\begin{aligned}A &:= K(h_>, h_<, h_<, h_<, h_<, h_<) \le K(h_>, h_\ll, h_<, h_<, h_<, h_<) + K(h_>, h_\sim, h_<, h_<, h_<, h_<)\\ &\lesssim \|h_>\|_{L^2}(s^{-\frac{p-1}{6}}\|h_\ll\|_{L^2} + \|h_\sim\|_{L^2})\|h_<\|_{L^2}^4,\end{aligned}$$

where we made use of the support separation of $h_>$ and $h_\ll$ via (5-5). Since $\|f\|_{L^2} = 1$, the estimates

$$\|h_<\|_{L^2} \lesssim e^{\mu s^{2p}}, \quad \|h_\ll\|_{L^2} \lesssim e^{\mu s^p}, \quad \text{and} \quad \|h_\sim\|_{L^2} \lesssim e^{\mu s^{2p}}\|f_\sim\|_{L^2}$$

hold and therefore

$$A \lesssim \|h_>\|_{L^2}(s^{-\frac{p-1}{6}} e^{\mu(s^p - s^{2p})} + \|f_\sim\|_{L^2})e^{5\mu s^{2p}}.$$

The terms B_1, B_2 can be estimated in a similar way. One obtains

$$B_1 \lesssim \|h_>\|_{L^2}^2(s^{-\frac{p-1}{6}} e^{\mu(s^p - s^{2p})} + \|f_\sim\|_{L^2})e^{4\mu s^{2p}} \quad \text{and} \quad B_2 \lesssim \|h_>\|_{L^2}\Big(\sum_{\ell=2}^{5}\|h_>\|_{L^2}^\ell\Big)e^{3\mu s^{2p}}.$$

The result follows by choosing $\mu = s^{-2p}$ and noting that $\|f_\sim\|_{L^2} \to 0$, as $s \to \infty$. □

We are finally ready to prove that extremizers decay superexponentially fast.

Proof of Theorem 1.5. Let $f \in L^2$ be an extremizer of (1-9), normalized so that $\|f\|_{L^2} = 1$. Then f satisfies (5-9) with $\lambda = \boldsymbol{E}_p^6$. Note that the function $(s, \varepsilon) \mapsto \|f\|_{s^{-2p},s,\varepsilon}$ is continuous in $(s, \varepsilon) \in (0, \infty)^2$ and, for each fixed $\varepsilon > 0$,

$$\|f\|_{s^{-2p},s,\varepsilon} = \|e^{G_{s^{-2p},\varepsilon}} f\mathbb{1}_{[-s^2,s^2]^\complement}\|_{L^2} \to 0 \quad \text{as } s \to \infty. \tag{5-11}$$

Let C be the constant promised by Proposition 5.2, and consider the function

$$H(v) := \tfrac{1}{2}\lambda v - C(v^2 + v^3 + v^4 + v^5).$$

In (5-10) choose s sufficiently large so that $o_1(1) \le \frac{1}{2}\lambda$ for every $\varepsilon > 0$. This is possible since $o_1(1) \to 0$ as $s \to \infty$ uniformly in $\varepsilon > 0$. Consequently,

$$H(\|f\|_{s^{-2p},s,\varepsilon}) \le o_2(1) \quad \text{for every } \varepsilon > 0.$$

In view of (5-11), and the facts that $H(0) = 0$, $H'(0) > 0$, and H is concave on $[0, \infty)$, we may choose s sufficiently large so that $\sup_{\varepsilon>0} o_2(1) < H(v_0)$ and $\|f\|_{s^{-2p},s,1} \le v_0$, where $0 < v_0 < v_1$ are the two unique positive solutions of the equation

$$H(v_j) = \tfrac{1}{2} \max\{H(v) : v \ge 0\}.$$

By continuity, $\|f\|_{s^{-2p},s,\varepsilon} \le v_0$ for every $\varepsilon > 0$. The monotone convergence theorem then implies $\|f\|_{s^{-2p},s,0} \le v_0 < \infty$, which translates into

$$e^{s^{-2p}|\cdot|^p} f \in L^2(\mathbb{R}).$$

Letting $\mu_0 := s^{-2p}$, where s is large enough so that all of the above steps hold, we have thus proved the first part. For the second part, note that, for every $\mu \in \mathbb{R}$, the function

$$e^{\mu|x|} f(x) = e^{\mu|x| - \mu_0|x|^p} \cdot e^{\mu_0|x|^p} f(x)$$

belongs to $L^2(\mathbb{R})$, since the first factor is bounded (here we use $p > 1$) and the second factor is, as we have just seen, square integrable. The result then follows from the Paley–Wiener theorem as in [Reed and Simon 1975, Theorem IX.13]. □

We finish with two concluding remarks. Firstly, the argument can be adapted to the case of extremizers for odd curves treated in the next section. Secondly, an interesting problem is whether extremizers are smooth (and not only their Fourier transforms). This question has been addressed in the context of the Fourier extension operator on low-dimensional spheres in [Christ and Shao 2012b; Shao 2016b], but we have not investigated the extent to which their analysis can be adapted to the present case.

6. The case of odd curves

In this section we discuss the necessary modifications to establish analogues of Theorems 1.3 and 1.4 for odd curves. In general terms, the analysis is similar, but the existence of parallel tangents requires an extra symmetrization step. Estimate (1-15) can be rewritten as

$$\|\mathcal{S}_p(f)\|_{L^6(\mathbb{R}^2)} \le \boldsymbol{O}_p \|f\|_{L^2(\mathbb{R})}, \tag{6-1}$$

where the Fourier extension operator on the curve $s = y|y|^{p-1}$ is given by

$$\mathcal{S}_p(f)(x,t) = \int_{\mathbb{R}} e^{ixy} e^{ity|y|^{p-1}} |y|^{\frac{p-2}{6}} f(y)\,\mathrm{d}y. \tag{6-2}$$

Given a real-valued function $f \in L^2(\mathbb{R})$, denote the reflection of f with respect to the origin by $\tilde{f} := f(-\,\cdot)$. One easily checks that

$$\mathcal{S}_p(\tilde{f})(x,t) = \mathcal{S}_p(f)(-x,-t) = \overline{\mathcal{S}_p(f)(x,t)},$$

where the bar denotes complex conjugation. In particular,

$$\|\mathcal{S}_p(f)\mathcal{S}_p(g)\|_{L^3} = \|\mathcal{S}_p(f)\mathcal{S}_p(\tilde{g})\|_{L^3},$$

and so functions f, g supported on intervals I and $-I$, respectively, are seen to interact in the same way as if they were both supported on I, unlike the case of even curves. In this way, one is led to symmetrize with respect to reflection. This has already been observed in the case of the spheres $\mathbb{S}^1$ [Shao 2016a] and $\mathbb{S}^2$ [Christ and Shao 2012a]. Symmetrization on $\mathbb{S}^2$ has been efficiently handled via δ-calculus in [Foschi 2015]. The same method can be applied to the present case, but we choose to present a different argument which does not rely on the underlying convolution structure.

Lemma 6.1. *Let $p > 1$ and $f \in L^2(\mathbb{R})$. Then*

$$\frac{\|\mathcal{S}_p(f)\|_{L^6(\mathbb{R}^2)}}{\|f\|_{L^2(\mathbb{R})}} \le \sup_{\substack{0\neq g\in L^2(\mathbb{R})\\ g \text{ even}}} \frac{\|\mathcal{S}_p(g)\|_{L^6(\mathbb{R}^2)}}{\|g\|_{L^2(\mathbb{R})}}. \tag{6-3}$$

If equality holds in (6-3), *then f is necessarily an even function.*

Proof. Given $f \in L^2(\mathbb{R})$, $f \neq 0$, take the decomposition $f = f_e + f_o$, where f_e is an even function, $f_e = \tilde{f_e}$ a.e. in $\mathbb{R}$, and f_o is odd, $f_o = -\tilde{f_o}$ a.e. in $\mathbb{R}$. Then $\|f\|_{L^2}^2 = \|f_e\|_{L^2}^2 + \|f_o\|_{L^2}^2$, and $\mathcal{S}_p(f_e)$ is real-valued, while $\mathcal{S}_p(f_o)$ is purely imaginary. Thus

$$|\mathcal{S}_p(f)(x,t)|^2 = |\mathcal{S}_p(f_e)(x,t)|^2 + |\mathcal{S}_p(f_o)(x,t)|^2 \quad \text{for almost every } (x,t) \in \mathbb{R}^2, \tag{6-4}$$

and so, by the triangle inequality for the L^3-norm, $\|\mathcal{S}_p(f)\|_{L^6}^2 \le \|\mathcal{S}_p(f_e)\|_{L^6}^2 + \|\mathcal{S}_p(f_o)\|_{L^6}^2$. It follows that

$$\frac{\|\mathcal{S}_p(f)\|_{L^6}^2}{\|f\|_{L^2}^2} \le \frac{\|\mathcal{S}_p(f_e)\|_{L^6}^2 + \|\mathcal{S}_p(f_o)\|_{L^6}^2}{\|f_e\|_{L^2}^2 + \|f_o\|_{L^2}^2} \le \max\left\{\frac{\|\mathcal{S}_p(f_e)\|_{L^6}^2}{\|f_e\|_{L^2}^2}, \frac{\|\mathcal{S}_p(f_o)\|_{L^6}^2}{\|f_o\|_{L^2}^2}\right\},$$

where we set either ratio on the right-hand side of this chain of inequalities to zero whenever the corresponding function f_e or f_o happens to vanish identically. Therefore we may restrict attention to functions which are either even or odd. On the other hand, the equivalent convolution form (1-15) of the inequality implies $\|\mathcal{S}_p(g)\|_{L^6} \le \|\mathcal{S}_p(|g|)\|_{L^6}$, with equality if and only if $g = |g|$ a.e. in $\mathbb{R}$. Thus

$$\frac{\|\mathcal{S}_p(f)\|_{L^6}^2}{\|f\|_{L^2}^2} \le \max\left\{\frac{\|\mathcal{S}_p(f_e)\|_{L^6}^2}{\|f_e\|_{L^2}^2}, \frac{\|\mathcal{S}_p(|f_o|)\|_{L^6}^2}{\|f_o\|_{L^2}^2}\right\} \le \sup_{\substack{0\neq g\in L^2\\ g \text{ even}}} \frac{\|\mathcal{S}_p(g)\|_{L^6}}{\|g\|_{L^2}}, \tag{6-5}$$

where we used that both f_e and $|f_o|$ are even functions. In order for equality to hold in (6-3), both inequalities in (6-5) must be equalities. Inspection of the chain of inequalities leading to (6-5) shows that, if there is equality in the first inequality, then necessarily one of the following alternatives must hold:

- $\|f_o\|_{L^2} = 0$, in which case $f = f_e$, and so f is even; or
- $\|f_e\|_{L^2} = 0$ and $f_o = |f_o|$ a.e. in $\mathbb{R}$, which implies that $f_o \equiv 0$, and so $f \equiv 0$ which does not hold by assumption; or

- $\|f_e\|_{L^2}\|f_o\|_{L^2} \neq 0$ and $\|\mathcal{S}_p(f_e)\|_{L^6}\|f_e\|_{L^2}^{-1} = \|\mathcal{S}_p(f_o)\|_{L^6}\|f_o\|_{L^2}^{-1} = \|\mathcal{S}_p(|f_o|)\|_{L^6}\|f_o\|_{L^2}^{-1}$, which again forces $f_o = |f_o|$ a.e. in $\mathbb{R}$, so that $f_o = 0$ which is absurd.

Therefore equality in (6-3) forces f to be an even function, as desired. □

For the remainder of this section, we restrict attention to nonnegative, even functions f. To prove the analogue of Proposition 3.1, we need bilinear estimates as in Propositions 2.1 and 2.3, and an L^1 cap bound as in Proposition 2.8. These can be obtained in exactly the same way as for the case of even curves, since the Jacobian factor corresponding to (2-2) is now equal to $p||y'|^{p-1} - |y|^{p-1}|$, which amounts to the bound we used before. We also need an analogue of Proposition A.1 with two points removed; i.e., consider $X_{\bar{x},\bar{y}} := X \setminus \{\bar{x}, \bar{y}\}$ equipped with a pseudometric $\varrho : X_{\bar{x},\bar{y}} \times X_{\bar{x},\bar{y}} \to [0,\infty)$. The statement is analogous so we omit it. Next, defining the dyadic pseudometric centered at zero as in (3-11) and invoking the appropriate bilinear estimates, we obtain an analogue of Proposition 3.4, the statement again being identical (omitted). The analogue of Proposition 3.3 requires the pseudometric

$$\varrho\colon \mathbb{R}\setminus\{-1,1\}\times\mathbb{R}\setminus\{-1,1\} \to [0,\infty), \quad \varrho(x,y) := |k-k'|,$$

where $k, k' \in \mathbb{Z}$ are such that $||x|-1| \in [2^k, 2^{k+1})$ and $||y|-1| \in [2^{k'}, 2^{k'+1})$. It handles concentration at a pair of opposite points, which we now define.

Definition 6.2. Let $y_0 \in \mathbb{R}$. A sequence of even functions $\{f_n\} \subset L^2(\mathbb{R})$ *concentrates at the pair* $\{-y_0, y_0\}$ if, for every $\varepsilon, \rho > 0$, there exists $N \in \mathbb{N}$ such that, for every $n \geq N$,

$$\int_{\substack{|y+y_0|\geq\rho \\ |y-y_0|\geq\rho}} |f_n(y)|^2 \,\mathrm{d}y < \varepsilon \|f_n\|^2_{L^2(\mathbb{R})}.$$

The following analogue of Proposition 3.3 holds for odd curves.

Proposition 6.3. *Let $\{f_n\} \subset L^2(\mathbb{R})$ be an L^2-normalized extremizing sequence of even functions for* (6-1)*. Let $\{r_n\}$ be a sequence of nonnegative numbers satisfying $r_n \to 0$ as $n \to \infty$, and*

$$\inf_{n\in\mathbb{N}} \int_{1-r_n}^{1+r_n} |f_n(y)|^2 \,\mathrm{d}y > 0.$$

Then the sequence $\{f_n\}$ concentrates at the pair $\{-1, 1\}$.

As in the case of even curves, this can be used to prove the analogue of Proposition 3.1.

Proposition 6.4. *Let $\{f_n\} \subset L^2(\mathbb{R})$ be an L^2-normalized extremizing sequence of nonnegative, even functions for* (6-1)*. Then there exist a subsequence $\{f_{n_k}\}$ and a sequence $\{a_k\} \subset \mathbb{R}\setminus\{0\}$ such that the rescaled sequence $\{g_k\}$, $g_k := |a_k|^{1/2} f_{n_k}(a_k\,\cdot)$, satisfies one of the following conditions*:

(i) *There exists $g \in L^2(\mathbb{R})$ such that $g_k \to g$ in $L^2(\mathbb{R})$ as $k \to \infty$.*

(ii) *$\{g_k\}$ concentrates at the pair $\{-1, 1\}$.*

Let $\{f_n\} \subset L^2(\mathbb{R})$ be an L^2-normalized sequence of nonnegative, even functions concentrating at the pair $\{-1, 1\}$. Write $f_n = g_n + \tilde{g}_n$, where $g_n := f_n \mathbb{1}_{[0,\infty)}$. In particular, $\|g_n\|_{L^2} = 2^{-1/2}$, and the sequence $\{g_n\}$ concentrates at $y_0 = 1$. The left-hand side of (1-15) can be expanded into

$$\begin{aligned}\|f_n\mu_p*f_n\mu_p*f_n\mu_p\|_{L^2}^2 = {}&\|g_n\mu_p*g_n\mu_p*g_n\mu_p\|_{L^2}^2+\|\tilde{g}_n\mu_p*\tilde{g}_n\mu_p*\tilde{g}_n\mu_p\|_{L^2}^2\\ &+9\|g_n\mu_p*g_n\mu_p*\tilde{g}_n\mu_p\|_{L^2}^2+9\|g_n\mu_p*\tilde{g}_n\mu_p*\tilde{g}_n\mu_p\|_{L^2}^2\\ &+6\langle g_n\mu_p*g_n\mu_p*g_n\mu_p,\ g_n\mu_p*g_n\mu_p*\tilde{g}_n\mu_p\rangle_{L^2}\\ &+6\langle g_n\mu_p*\tilde{g}_n\mu_p*\tilde{g}_n\mu_p,\ \tilde{g}_n\mu_p*\tilde{g}_n\mu_p*\tilde{g}_n\mu_p\rangle_{L^2}\\ &+18\langle g_n\mu_p*\tilde{g}_n\mu_p*\tilde{g}_n\mu_p,\ g_n\mu_p*g_n\mu_p*\tilde{g}_n\mu_p\rangle_{L^2}\\ &+6\langle g_n\mu_p*g_n\mu_p*g_n\mu_p,\ g_n\mu_p*\tilde{g}_n\mu_p*\tilde{g}_n\mu_p\rangle_{L^2}\\ &+6\langle g_n\mu_p*g_n\mu_p*\tilde{g}_n\mu_p,\ \tilde{g}_n\mu_p*\tilde{g}_n\mu_p*\tilde{g}_n\mu_p\rangle_{L^2}\\ &+2\langle g_n\mu_p*g_n\mu_p*g_n\mu_p,\ \tilde{g}_n\mu_p*\tilde{g}_n\mu_p*\tilde{g}_n\mu_p\rangle_{L^2}. \end{aligned}\tag{6-6}$$

The last three summands vanish since the corresponding supports intersect on a Lebesgue null set. The symmetry of the inner products then implies

$$\begin{aligned}&\|f_n\mu_p * f_n\mu_p * f_n\mu_p\|_{L^2}^2\\ &\qquad = 20\|g_n\mu_p * g_n\mu_p * g_n\mu_p\|_{L^2}^2 + 30\langle g_n\mu_p * g_n\mu_p * g_n\mu_p * g_n\mu_p,\ g_n\mu_p * g_n\mu_p\rangle_{L^2}.\end{aligned}$$

Note that $\mu_p = \sigma_p$ on the support of g_n, where σ_p was defined in (1-11). It follows that

$$\begin{aligned}&\frac{\|f_n\mu_p * f_n\mu_p * f_n\mu_p\|_{L^2}^2}{\|f_n\|_{L^2}^6}\\ &\qquad = \frac{5}{2}\frac{\|g_n\sigma_p * g_n\sigma_p * g_n\sigma_p\|_{L^2}^2}{\|g_n\|_{L^2}^6} + \frac{15}{4}\frac{\langle g_n\sigma_p * g_n\sigma_p * g_n\sigma_p * g_n\sigma_p,\ g_n\sigma_p * g_n\sigma_p\rangle_{L^2}}{\|g_n\|_{L^2}^6}.\end{aligned}\tag{6-7}$$

Since the sequence $\{g_n\}$ concentrates at $y_0 = 1$, we have

$$\lim_{n\to\infty}\langle g_n\sigma_p * g_n\sigma_p * g_n\sigma_p * g_n\sigma_p,\ g_n\sigma_p * g_n\sigma_p\rangle_{L^2} = 0.$$

Heuristically, $g_n\sigma_p * g_n\sigma_p$ is supported near the point $(2, 2)$, while $(g_n\sigma_p)^{*(4)}$ is supported near the point $(4, 4)$, and so in the limit there is no contribution of the inner product. More precisely, given $\varepsilon > 0$, write $g_n = h_n + \kappa_n$, where $h_n := g_n\mathbb{1}_{[1-\varepsilon,1+\varepsilon]}$ and $\|\kappa_n\|_{L^2}^2 \to 0$ as $n \to \infty$. If ε is small enough, then support considerations force

$$\langle h_n\sigma_p * h_n\sigma_p * h_n\sigma_p * h_n\sigma_p,\ h_n\sigma_p * h_n\sigma_p\rangle_{L^2} = 0 \quad \text{for every } n,$$

whereas the cross terms involve κ_n, whose L^2-norm tends to zero as $n \to \infty$. We conclude

$$\limsup_{n\to\infty}\frac{\|f_n\mu_p * f_n\mu_p * f_n\mu_p\|_{L^2}^2}{\|f_n\|_{L^2}^6} = \frac{5}{2}\limsup_{n\to\infty}\frac{\|g_n\sigma_p * g_n\sigma_p * g_n\sigma_p\|_{L^2}^2}{\|g_n\|_{L^2}^6},\tag{6-8}$$

and similarly for the limit inferior. Lemma 3.2 applied to the sequence $\{g_n\}$ implies

$$\limsup_{n\to\infty}\frac{\|f_n\mu_p * f_n\mu_p * f_n\mu_p\|_{L^2(\mathbb{R}^2)}^2}{\|f_n\|_{L^2}^6} \le \frac{5\pi}{\sqrt{3}p(p-1)}.$$

Moreover, equality holds if we take $f_n = g_n + \tilde{g}_n$, with $g_n := 2^{-1/2} h_n \|h_n\|_{L^2}^{-1}$, and

$$h_n(y) := e^{-n(|y|^p - 1 - p(y-1))} |y|^{\frac{p-2}{6}} \mathbb{1}_{[0,\infty)}(y).$$

Theorem 1.6 is now proved.

Remark 6.5. The invariant form of condition (1-16) in Theorem 1.6 is

$$\left(\frac{\boldsymbol{Q}_p}{\boldsymbol{C}_2}\right)^6 > \frac{5}{p(p-1)}, \tag{6-9}$$

where $\boldsymbol{C}_2^6 = \pi/\sqrt{3}$ is the best constant for the parabola in convolution form. In the case $p = 3$, a similar condition appears in [Shao 2009] on the Airy–Strichartz inequality, which translates into $(\boldsymbol{Q}_3/\boldsymbol{C}_2)^6 > \frac{1}{3}$. This is of course incompatible with (6-9) but, as was recently pointed out in [Frank and Sabin 2018, Remark 2.7], there is a problem in [Shao 2009, Lemma 6.1] in the passage from equation (89) to equation (90), as the argument presented there disregards the effect of symmetrization. On the other hand, the case $p = 3$ of (6-9) agrees with [Frank and Sabin 2018, Case $p = q = 6$ of Theorem 1], once the proper normalization is considered.

We now come to the question of whether extremizers for (1-15) actually exist, and discuss the case $1 < p < 2$ first. Just as in (4-17), set $g_n(y) := e^{-(u/n)n2(|y|^p - py)} |y|^{-(2-p)/6}$. Its even extension,

$$f_n := \frac{g_n \mathbb{1}_{[0,\infty)} + \tilde{g}_n \mathbb{1}_{(-\infty,0]}}{2^{\frac{1}{2}} \|g_n\|_{L^2(0,\infty)}},$$

can be used to establish the strict inequality in (1-16). One simply uses (6-8) together with the fact that the sequence $\{g_n \|g_n\|_{L^2}^{-1}\}_{s>0}$ concentrates at $y_0 = 1$, so that an argument similar to Lemma 4.5 can be applied to the present case. Therefore, extremizers for (1-15) exist if $1 < p < 2$, and Theorem 1.7 is now proved.

The case $p \geq 2$ seems harder. In view of (6-8), it is natural to use the methods of Section 4 in order to find the series expansion for the trial functions $f = 2^{-1/2}(g + \tilde{g})$, where $g(y) = e^{-|y|^p} |y|^{(p-2)/6+a} \mathbb{1}_{[0,\infty)}(y)$ for different choices of a. By doing so, we find that we cannot reach the critical threshold $5\pi/(\sqrt{3}p(p-1))$, but that we can approach it from below by varying the value of a. We are led to the following conjecture.

Conjecture 6.6. *For every* $p \geq 2$,

$$\left(\frac{\boldsymbol{Q}_p}{\boldsymbol{C}_2}\right)^6 = \frac{5}{p(p-1)}.$$

Moreover, extremizers for (1-15) *do not exist.*

6A. ***On symmetric complex- and real-valued extremizers.*** The proof of Lemma 6.1 merits some further remarks which we attempt to insert within a broader context.

First of all, identity (6-4) holds thanks to the symmetry with respect to the origin of both the curve $s = y|y|^{p-1}$ and the measure $\mathrm{d}\mu_p = \boldsymbol{\delta}(t - y|y|^{p-1})|y|^{(p-2)/6}\,\mathrm{d}y\,\mathrm{d}s$. In fact, the proof of Lemma 6.1 immediately generalizes to the Fourier extension operator associated to any *antipodally symmetric pair* (Σ, μ). By this we mean a set $\Sigma \subseteq \mathbb{R}^d$ (usually a smooth submanifold) together with a Borel measure

μ supported on Σ, both symmetric with respect to the origin in the sense that $T(\Sigma)=\Sigma$ and $T^*\mu=\mu$, where T denotes the antipodal map $T(y)=-y$ and $T^*\mu$ denotes the pushforward measure.

Secondly, the Lebesgue exponent 6 can be replaced with any finite exponent $r\geq 2$. More precisely, in the general context of an antipodally symmetric pair (Σ,μ), if an estimate

$$\|\widehat{f\mu}\|_{L^r(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\Sigma,\mu)} \tag{6-10}$$

does hold for some $r\in[2,\infty)$, then necessarily[7]

$$\sup_{\substack{0\neq f\in L^2(\Sigma,\mu)\\ f\ \mathbb{R}\text{-valued}}} \frac{\|\widehat{f\mu}\|_{L^r(\mathbb{R}^d)}}{\|f\|_{L^2(\Sigma,\mu)}} = \sup_{\substack{0\neq g\in L^2(\Sigma,\mu)\\ g\ \mathbb{R}\text{-valued},\ g \text{ even or } g \text{ odd}}} \frac{\|\widehat{g\mu}\|_{L^r(\mathbb{R}^d)}}{\|g\|_{L^2(\Sigma,\mu)}}.$$

Thirdly, the discussion extends to the more general situation of complex-valued functions. For concreteness, let us specialize to the case of the unit sphere $\Sigma=\mathbb{S}^{d-1}\subseteq\mathbb{R}^d$, $d\geq 2$, equipped with its natural surface measure μ. Given an exponent $p\geq p_d:=2(d+1)/(d-1)$, the Tomas–Stein inequality states that

$$\|\widehat{(u\mu)}\|_{L^p(\mathbb{R}^d)} \lesssim_{p,d} \|u\|_{L^2(\mathbb{S}^{d-1})} \tag{6-11}$$

for every complex-valued function $u\in L^2(\mathbb{S}^{d-1})$. It is known [Fanelli et al. 2011; Frank et al. 2016] that complex-valued extremizers for (6-11) exist in the full range $p\geq p_d$, the endpoint existence in dimensions $d\geq 4$ being conditional on a celebrated conjecture concerning (1-2). Moreover, if $p\geq p_d$ is an even integer, then real-valued, even, nonnegative extremizers for (6-11) exist, by virtue of the equivalent convolution form; see [Christ and Shao 2012a; Foschi 2015; Shao 2016a]. Finally, if $p=\infty$, then one easily checks that the unique extremizers for (6-11) are the constant functions. For general $p\geq p_d$, $p\neq\infty$, we argue that the search for extremizers of (6-11) can be restricted to the class of complex-valued, symmetric functions. Indeed, write $u=f+ig$, with $f=\Re u$, $g=\Im u$. By reorganizing the summands, we may write $u=F+iG$, where $F=f_e+ig_o$ and $G=g_e-if_o$. The functions F,G are complex-valued and symmetric, in the sense that $F(y)=\overline{F(-y)}$ and $G(y)=\overline{G(-y)}$, for every $y\in\mathbb{S}^{d-1}$. Moreover, one easily checks that

$$F(y)=\frac{1}{2}(u(y)+\overline{u(-y)}),\quad G(y)=\frac{1}{2i}(u(y)-\overline{u(-y)}),\quad \|u\|_{L^2}^2=\|F\|_{L^2}^2+\|G\|_{L^2}^2,$$

and that, in view of the antipodal symmetry of the pair $(\mathbb{S}^{d-1},\mu)$, the functions $\widehat{F\mu}$, $\widehat{G\mu}$ are real-valued. Following the proof of Lemma 6.1, we are thus led to the following result.

Proposition 6.7. *Let $d\geq 2$ and $2(d+1)/(d-1)\leq p\leq\infty$. Then for every complex-valued $u\in L^2(\mathbb{S}^{d-1})$, $u\neq 0$, the following inequality holds*:

$$\frac{\|\widehat{(u\mu)}\|_{L^p(\mathbb{R}^d)}}{\|u\|_{L^2(\mathbb{S}^{d-1})}} \leq \sup_{0\neq F\in L^2_{\mathrm{sym}}(\mathbb{S}^{d-1})} \frac{\|\widehat{F\mu}\|_{L^p(\mathbb{R}^d)}}{\|F\|_{L^2(\mathbb{S}^{d-1})}}, \tag{6-12}$$

[7] Here, a real-valued function $g:\Sigma\to\mathbb{R}$ is naturally defined to be *even* (resp. *odd*) if $g(y)=g(-y)$ (resp. $g(y)=-g(-y)$) for μ-almost every point $y\in\Sigma$.

where $L^2_{\mathrm{sym}}(\mathbb{S}^{d-1}) := \{F \in L^2(\mathbb{S}^{d-1})\colon F(y) = \overline{F(-y)}$ *for* μ*-a.e.* $y \in \mathbb{S}^{d-1}\}$*. Moreover, if* u *realizes equality in* (6-12), *then there exist* $F \in L^2_{\mathrm{sym}}(\mathbb{S}^{d-1})$ *and a constant* $\kappa \in \mathbb{C}$ *such that* $u = \kappa F$, μ*-a.e.*

Proof. In light of the previous discussion, we can assume $p < \infty$, and only the last statement merits further justification. Suppose that u realizes equality in (6-12). In particular, u is a complex-valued extremizer for (6-11). Decompose $u = F + iG$ as before, with $F(y) = \frac{1}{2}(u(y) + \overline{u(-y)})$, $G = \frac{1}{2i}(u(y) - \overline{u(-y)})$, so that $F, G \in L^2_{\mathrm{sym}}(\mathbb{S}^{d-1})$. If either $F \equiv 0$ or $G \equiv 0$, then there is nothing to prove, and so in what follows we assume F, G not to be identically zero. Following the proof of Lemma 6.1, we note that equality occurs in the application of the triangle inequality with respect to the $L^{p/2}(\mathbb{R}^d)$-norm (recall that $p/2 > 1$ is finite) only if there exists $\lambda > 0$ such that[8]

$$|\widehat{F\mu}(\xi)| = \lambda |\widehat{G\mu}(\xi)| \quad \text{for every } \xi \in \mathbb{R}^d. \tag{6-13}$$

Subsequent cases of equality further imply

$$\frac{\|\widehat{(u\mu)}\|_{L^p(\mathbb{R}^d)}}{\|u\|_{L^2(\mathbb{S}^{d-1})}} = \frac{\|\widehat{F\mu}\|_{L^p(\mathbb{R}^d)}}{\|F\|_{L^2(\mathbb{S}^{d-1})}} = \frac{\|\widehat{G\mu}\|_{L^p(\mathbb{R}^d)}}{\|G\|_{L^2(\mathbb{S}^{d-1})}},$$

and so the functions F, G are also extremizers for (6-11). It suffices to show that $F = \kappa G$, where $\kappa \in \{-\lambda, \lambda\}$. Recall that $\widehat{F\mu}, \widehat{G\mu}$ are real-valued functions, since $F, G \in L^2_{\mathrm{sym}}(\mathbb{S}^{d-1})$. Let $\xi_0 \in \mathbb{R}^d$ be such that $|\widehat{F\mu}(\xi_0)| \neq 0$. We lose no generality in assuming that $\widehat{F\mu}(\xi_0) > 0$ and $\widehat{G\mu}(\xi_0) > 0$, for otherwise we could replace F by $-F$ or G by $-G$. By continuity, there exists $r_0 > 0$ such that

$$\widehat{F\mu}(\xi + \xi_0) = \lambda \widehat{G\mu}(\xi + \xi_0) \quad \text{for every } |\xi| < r_0. \tag{6-14}$$

On the other hand, $\widehat{F\mu}(\xi + \xi_0) = \widehat{(e^{-iy\cdot\xi_0} F\mu)}(\xi)$ and $\widehat{G\mu}(\xi + \xi_0) = \widehat{(e^{-iy\cdot\xi_0} G\mu)}(\xi)$. The functions $e^{-iy\cdot\xi_0} F$ and $e^{-iy\cdot\xi_0} G$ belong to $L^2_{\mathrm{sym}}(\mathbb{S}^{d-1})$, and may be expanded in the basis of spherical harmonics,

$$e^{-iy\cdot\xi_0} F = \sum_{n=0}^{\infty} \sum_{k=1}^{\gamma(d,n)} a_{n,k} Y_{n,k} \quad \text{and} \quad e^{-iy\cdot\xi_0} G = \sum_{n=0}^{\infty} \sum_{k=1}^{\gamma(d,n)} b_{n,k} Y_{n,k}. \tag{6-15}$$

Here, $\{Y_{n,k}\}_{k=1}^{\gamma(d,n)}$ denotes a basis for the space of spherical harmonics of degree n in the sphere $\mathbb{S}^{d-1}$, which has dimension $\gamma(d,n) := \binom{d+n-1}{n} - \binom{d+n-3}{n-2}$; see [Stein and Weiss 1971, Chapter IV]. The coefficients $a_{n,k}, b_{n,k}$ are complex numbers. Applying the Fourier transform to (6-15), we find that

$$\begin{aligned} \widehat{F\mu}(\xi + \xi_0) &= (2\pi)^{\frac{d}{2}} \sum_{n=0}^{\infty} \sum_{k=1}^{\gamma(d,n)} a_{n,k} i^{-n} |\xi|^{-\frac{d}{2}+1} J_{\frac{d}{2}-1+n}(|\xi|) Y_{n,k}\Big(\frac{\xi}{|\xi|}\Big), \\ \widehat{G\mu}(\xi + \xi_0) &= (2\pi)^{\frac{d}{2}} \sum_{n=0}^{\infty} \sum_{k=1}^{\gamma(d,n)} b_{n,k} i^{-n} |\xi|^{-\frac{d}{2}+1} J_{\frac{d}{2}-1+n}(|\xi|) Y_{n,k}\Big(\frac{\xi}{|\xi|}\Big). \end{aligned} \tag{6-16}$$

[8]As Fourier transforms of compactly supported distributions, both sides of (6-13) coincide with the absolute value of real-valued, *smooth* functions, so that the pointwise equality occurs at every point, and not just almost everywhere.

Using (6-14) and (6-16) together with the orthogonality of the functions $\{Y_{n,k}\}$ in $L^2(\mathbb{S}^{d-1})$, we obtain

$$a_{n,k} r^{-\frac{d}{2}+1} J_{\frac{d}{2}-1+n}(r) = \lambda b_{n,k} r^{-\frac{d}{2}+1} J_{\frac{d}{2}-1+n}(r) \quad \text{for every } r \in (0, r_0).$$

In particular, $a_{n,k} = \lambda b_{n,k}$. This and (6-15) together imply $F = \lambda G$. □

A similar result to Proposition 6.7 holds for a broader class of antipodally symmetric pairs (Σ, μ). Indeed, let $r \in [2, \infty)$ be such that the extension estimate (6-10) holds. Then

$$\sup_{0 \neq u \in L^2(\Sigma,\mu)} \frac{\|\widehat{(u\mu)}\|_{L^r(\mathbb{R}^d)}}{\|u\|_{L^2(\Sigma,\mu)}} = \sup_{0 \neq F \in L^2_{\text{sym}}(\Sigma,\mu)} \frac{\|\widehat{F\mu}\|_{L^r(\mathbb{R}^d)}}{\|F\|_{L^2(\Sigma,\mu)}}, \tag{6-17}$$

with the obvious definition of $L^2_{\text{sym}}(\Sigma, \mu)$. Moreover, if μ is compactly supported and finite, then any complex extremizer u for (6-10) necessarily coincides with a multiple of a symmetric extremizer $F \in L^2_{\text{sym}}(\Sigma, \mu)$. Regarding the second part of Proposition 6.7, the previous proof used the particular geometry of the sphere, but it can be modified to handle this more general situation. The crux of the matter is the fact that the Fourier transform of a compactly supported finite measure is real analytic. Indeed, if μ is a positive, compactly supported finite measure, and $F \in L^2(\Sigma, \mu)$, then, for every $\xi_0 \in \mathbb{R}^d$,

$$\begin{aligned} \widehat{F\mu}(\xi) &= \int_\Sigma e^{-i\xi\cdot y} F(y)\,\mathrm{d}\mu(y) = \int_\Sigma e^{-i(\xi-\xi_0)\cdot y} e^{-i\xi_0\cdot y} F(y)\,\mathrm{d}\mu(y) \\ &= \sum_{k=0}^{\infty} \frac{(-i)^k}{k!} \int_\Sigma ((\xi-\xi_0)\cdot y)^k e^{-i\xi_0\cdot y} F(y)\,\mathrm{d}\mu(y), \end{aligned} \tag{6-18}$$

where the convergence is locally uniform. To see this, note the tail estimate

$$\left\| \sum_{k=K}^{\infty} \frac{(-i)^k}{k!} \int_\Sigma ((\xi-\xi_0)\cdot y)^k e^{-i\xi_0\cdot y} F(y)\,\mathrm{d}\mu(y) \right\|_{L^\infty_\xi(\Omega)} \le \mu(\Sigma)^{\frac{1}{2}} \|F\|_{L^2(\Sigma,\mu)} \sum_{k=K}^{\infty} \frac{s^k}{k!},$$

which holds for every compact subset $\Omega \subseteq \mathbb{R}^d$ and every $K \in \mathbb{N}$. Here, $s = \sup_{\xi\in\Omega,\, y\in\Sigma} |\xi-\xi_0||y| < \infty$. Therefore, the analogue of (6-13) in this setting leads to the corresponding (6-14), which by analyticity of (6-18) implies $\widehat{F\mu} = \lambda\widehat{G\mu}$, and therefore $F = \lambda G$.

These observations can be of interest when combined with the main result of [Fanelli et al. 2011], which states that complex-valued extremizers exist in the nonendpoint setting, provided μ is a positive, compactly supported finite measure. Important cases of antipodally symmetric pairs (Σ, μ) which have attracted recent attention include the aforementioned case of spheres, together with ellipsoids equipped with surface measure, and the double cone, the one- and the two-sheeted hyperboloids equipped with their natural Lorentz invariant measures; see [Foschi and Oliveira e Silva 2017].

We end this section with a final remark on the multiplier form of inequality (6-1). Consider the Cauchy problem

$$\begin{cases} \partial_t u - |\partial_x|^{p-1}\partial_x u = 0, & (x,t) \in \mathbb{R}\times\mathbb{R}, \\ u(\cdot\,,0) = f \in L^2_x(\mathbb{R}), \end{cases} \tag{6-19}$$

whose solution can be written in terms of the propagator

$$u(x,t)=e^{t|\partial_x|^{p-1}\partial_x}f(x)=\frac{1}{2\pi}\int_{\mathbb{R}}e^{ix\xi}e^{it\xi|\xi|^{p-1}}\hat{f}(\xi)\,\mathrm{d}\xi. \tag{6-20}$$

In view of (6-1), and more generally of [Kenig et al. 1991, Theorem 2.1], this satisfies the mixed norm estimate

$$\||D|^{\frac{p-2}{r}}e^{t|\partial_x|^{p-1}\partial_x}f\|_{L^r_tL^s_x(\mathbb{R}^{1+1})}\lesssim_{r,s}\|f\|_{L^2(\mathbb{R})},$$

whenever the Lebesgue exponents r,s are such that $\frac{2}{r}+\frac{1}{s}=\frac{1}{2}$.

In this context, as noted in [Frank and Sabin 2018; Shao 2009] for the case $p=3$, it makes sense to distinguish between real-valued and general complex-valued L^2 initial data. This is because the evolution $e^{t|\partial_x|^{p-1}\partial_x}$ preserves real-valuedness. In other words, if f is real-valued, then so is $e^{t|\partial_x|^{p-1}\partial_x}f$ for every $t\in\mathbb{R}$. In fact, if f is real-valued, then $\hat{f}(-\xi)=\overline{\hat{f}(\xi)}$, and so taking the complex conjugate of (6-20) reveals that $\overline{u(x,t)}=u(x,t)$. The operator $|D|^{(p-2)/r}e^{t|\partial_x|^{p-1}\partial_x}$ is seen to preserve real-valuedness in a similar way.

It is then natural to consider the following family of sharp inequalities, for real- and complex-valued initial data and admissible Lebesgue exponents r,s:

$$\||D|^{\frac{p-2}{r}}e^{t|\partial_x|^{p-1}\partial_x}u\|_{L^r_tL^s_x(\mathbb{R}^{1+1})}\leq \boldsymbol{M}_{p,r,s}(\mathbb{C})\|u\|_{L^2(\mathbb{R})}, \tag{6-21}$$

$$\||D|^{\frac{p-2}{r}}e^{t|\partial_x|^{p-1}\partial_x}f\|_{L^r_tL^s_x(\mathbb{R}^{1+1})}\leq \boldsymbol{M}_{p,r,s}(\mathbb{R})\|f\|_{L^2(\mathbb{R})}, \tag{6-22}$$

where $u:\mathbb{R}\to\mathbb{C}$ is complex-valued and $f:\mathbb{R}\to\mathbb{R}$ is real-valued. The study of extremizers for (6-21)–(6-22) in the Airy–Strichartz case $p=3$ has been considered in [Farah and Versieux 2018; Frank and Sabin 2018; Hundertmark and Shao 2012; Shao 2009]. It would be interesting to determine whether the methods developed in the present paper can be adapted to the study of extremizers for (6-21)–(6-22) in the mixed norm case $r\neq s$, so as to obtain an alternative approach to profile decomposition or the missing mass method. We do not pursue these matters here. However, we would still like to point out two interesting features of this problem which are easily derived from our previous analysis, and are the content of the following result.

Proposition 6.8. *Let $p>1$, and $r,s\in(2,\infty)$ be such that $\boldsymbol{M}_{p,r,s}(\mathbb{C})$ and $\boldsymbol{M}_{p,r,s}(\mathbb{R})$ are finite. Then $\boldsymbol{M}_{p,r,s}(\mathbb{C})=\boldsymbol{M}_{p,r,s}(\mathbb{R})$. Moreover, if a complex-valued extremizer u for $\boldsymbol{M}_{p,r,s}(\mathbb{C})$ exists, then there exist $\kappa\in\mathbb{C}$ and a real-valued extremizer f for $\boldsymbol{M}_{p,r,s}(\mathbb{R})$ such that $u=\kappa f$.*

The problem of the relationship between arbitrary complex-valued extremizers and real-valued extremizers has been considered in the literature; see, e.g., [Christ and Shao 2012b] for the case of the Tomas–Stein inequality on the sphere $\mathbb{S}^2$. Note the duality with the second statement of Proposition 6.7 above.

Proof of Proposition 6.8. The equality $\boldsymbol{M}_{p,r,s}(\mathbb{C})=\boldsymbol{M}_{p,r,s}(\mathbb{R})$ follows the same lines as the proof of Lemma 6.1. To see why this is the case, let $u\in L^2(\mathbb{R})$ and write $u=f+ig$, where f and g are the real and imaginary parts of u, and hence real-valued. Therefore

$$\|u\|_{L^2}^2=\|f\|_{L^2}^2+\|g\|_{L^2}^2, \tag{6-23}$$

$$||D|^{\frac{p-2}{r}}e^{t|\partial_x|^{p-1}\partial_x}u(x)|^2=||D|^{\frac{p-2}{r}}e^{t|\partial_x|^{p-1}\partial_x}f(x)|^2+||D|^{\frac{p-2}{r}}e^{t|\partial_x|^{p-1}\partial_x}g(x)|^2 \tag{6-24}$$

for every $(x,t)\in\mathbb{R}^2$. If $r,s\ge 2$, then we can use the triangle inequality for the $L_x^{s/2}$- and the $L_t^{r/2}$-norms applied to (6-24), and obtain

$$\||D|^{\frac{p-2}{r}}e^{t|\partial_x|^{p-1}\partial_x}u\|^2_{L_t^rL_x^s}\le \||D|^{\frac{p-2}{r}}e^{t|\partial_x|^{p-1}\partial_x}f\|^2_{L_t^rL_x^s}+\||D|^{\frac{p-2}{r}}e^{t|\partial_x|^{p-1}\partial_x}g\|^2_{L_t^rL_x^s}. \tag{6-25}$$

Without loss of generality, assume that f,g are not identically zero. Reasoning as in the proof of Lemma 6.1 yields

$$\frac{\||D|^{\frac{p-2}{r}}e^{t|\partial_x|^{p-1}\partial_x}u\|^2_{L_t^rL_x^s}}{\|u\|^2_{L^2}}\le\max\left\{\frac{\||D|^{\frac{p-2}{r}}e^{t|\partial_x|^{p-1}\partial_x}f\|^2_{L_t^rL_x^s}}{\|f\|^2_{L^2}},\frac{\||D|^{\frac{p-2}{r}}e^{t|\partial_x|^{p-1}\partial_x}g\|^2_{L_t^rL_x^s}}{\|g\|^2_{L^2}}\right\} \tag{6-26}$$

and therefore $\boldsymbol{M}_{p,r,s}(\mathbb{C})\le\boldsymbol{M}_{p,r,s}(\mathbb{R})$. The reverse inequality is immediate. We gratefully acknowledge recent personal communication with R. Frank and J. Sabin [2018], who independently arrived at a similar conclusion.

We proceed to show that an arbitrary complex-valued extremizer for $\boldsymbol{M}_{p,r,s}(\mathbb{C})$ necessarily coincides with a constant multiple of a real-valued extremizer for $\boldsymbol{M}_{p,r,s}(\mathbb{R})$. Let $r,s\in(2,\infty)$, and suppose that u is a complex-valued extremizer for $\boldsymbol{M}_{p,r,s}(\mathbb{C})$, which we express as the sum of its real and imaginary parts, $u=f+ig$. An inspection of the chain of inequalities leading to (6-26) shows that one of the following alternatives must hold:

- $g=0$ and $u=f$ is a real-valued extremizer.
- $f=0$, $u=ig$, and g is a real-valued extremizer.
- f,g are both not identically zero, and

$$\frac{\||D|^{\frac{p-2}{r}}e^{t|\partial_x|^{p-1}\partial_x}f\|^2_{L_t^rL_x^s}}{\|f\|^2_{L^2}}=\frac{\||D|^{\frac{p-2}{r}}e^{t|\partial_x|^{p-1}\partial_x}g\|^2_{L_t^rL_x^s}}{\|g\|^2_{L^2}}=\boldsymbol{M}_{p,r,s}(\mathbb{R}), \tag{6-27}$$

  so that f,g are real-valued extremizers.

It suffices to analyze the latter case. An inspection of the chain of inequalities leading to (6-25) shows that equality must hold in both applications of the triangle inequality. Since $r,s\in(2,\infty)$, this implies the existence of $\lambda>0$ such that

$$||D|^{\frac{p-2}{r}}e^{t|\partial_x|^{p-1}\partial_x}f(x)|=\lambda||D|^{\frac{p-2}{r}}e^{t|\partial_x|^{p-1}\partial_x}g(x)|\quad\text{for almost every }(x,t)\in\mathbb{R}^2. \tag{6-28}$$

Equality in (6-27) then implies $\|f\|_{L^2}=\lambda\|g\|_{L^2}$. By squaring (6-28), and applying the Fourier transform, the equality of the resulting convolutions can be recast as

$$\begin{aligned}\int_{\mathbb{R}^2}&\hat f(y_1)\hat f(y_2)\,\boldsymbol{\delta}(t-\psi(y_1)-\psi(y_2))\,\boldsymbol{\delta}(x-y_1-y_2)|y_1y_2|^{\frac{p-2}{r}}\,\mathrm{d}y_1\,\mathrm{d}y_2\\&=\lambda^2\int_{\mathbb{R}^2}\hat g(y_1)\hat g(y_2)\,\boldsymbol{\delta}(t-\psi(y_1)-\psi(y_2))\,\boldsymbol{\delta}(x-y_1-y_2)|y_1y_2|^{\frac{p-2}{r}}\,\mathrm{d}y_1\,\mathrm{d}y_2,\end{aligned} \tag{6-29}$$

where $(x,t)\in\mathbb{R}^2$ and $\psi(y):=y|y|^{p-1}$. Considering points (x,t) in the interior of the support of the convolution measure $\mu_p*\mu_p$, i.e., satisfying $t>2\psi\left(\frac12x\right)$ for $x>0$, and $t<2\psi\left(\frac12x\right)$ for $x<0$, we see that there exists a unique positive solution $\alpha = \alpha(x,t) > 0$ of

$$t = \psi\big(\tfrac{1}{2}x - \alpha(x,t)\big) + \psi\big(\tfrac{1}{2}x + \alpha(x,t)\big), \tag{6-30}$$

and hence that the system of equations $t = \psi(y_1) + \psi(y_2),\ x = y_1 + y_2$ has unique solutions

$$(y_1, y_2) \in \big\{\big(\tfrac{1}{2}x - \alpha(x,t), \tfrac{1}{2}x + \alpha(x,t)\big), \big(\tfrac{1}{2}x + \alpha(x,t), \tfrac{1}{2}x - \alpha(x,t)\big)\big\}.$$

From (6-29) and a similar reasoning to that of [Oliveira e Silva and Quilodrán 2019, Proposition 2.1 and Remark 2.3], it then follows that

$$\hat{f}\big(\tfrac{1}{2}x - \alpha(x,t)\big)\hat{f}\big(\tfrac{1}{2}x + \alpha(x,t)\big) = \lambda^2 \hat{g}\big(\tfrac{1}{2}x - \alpha(x,t)\big)\hat{g}\big(\tfrac{1}{2}x + \alpha(x,t)\big)$$

for almost every $(x,t) \in \operatorname{supp}(\mu_p * \mu_p)$. Alternatively, the latter identity follows by considering the analogue of formula (2-4) obtained in the case of even curves, which by the previous discussion applies to the present scenario as well. This yields

$$\hat{f}(x)\hat{f}(x') = \lambda^2 \hat{g}(x)\hat{g}(x') \tag{6-31}$$

for almost every $(x,x') \in \mathbb{R}^2$. As $\hat{f}, \hat{g}$ belong to $L^2(\mathbb{R})$, we may integrate over any compact subset $I \subset \mathbb{R}$ in both variables x, x' and obtain

$$\bigg(\int_I \hat{f}(x)\,\mathrm{d}x\bigg)^2 = \lambda^2 \bigg(\int_I \hat{g}(x)\,\mathrm{d}x\bigg)^2. \tag{6-32}$$

Choose a compact subset $J \subset \mathbb{R}$ for which $\int_J \hat{g}(x)\,\mathrm{d}x \neq 0$. From (6-32), we have

$$\int_J \hat{f}(x)\,\mathrm{d}x = \lambda \int_J \hat{g}(x)\,\mathrm{d}x \quad \text{or} \quad \int_J \hat{f}(x)\,\mathrm{d}x = -\lambda \int_J \hat{g}(x)\,\mathrm{d}x. \tag{6-33}$$

Integrating both sides of (6-31) over $x' \in J$, one infers from (6-33) that either $\hat{f} = \lambda\hat{g}$ or $\hat{f} = -\lambda\hat{g}$, and therefore that either $f = \lambda g$ or $f = -\lambda g$. The conclusion is that there exists $\lambda > 0$ such that either $u = (\lambda + i)g$ or $u = (-\lambda + i)g$, and so u is a constant multiple of a real-valued extremizer, as desired. ☐

Appendix A: Concentration-compactness

This appendix consists of a useful observation regarding Lions' concentration-compactness lemma [1984a]. Let us start with some general considerations. Let $(X, \mathcal{B}, \mu)$ be a measure space with a *distinguished* point $\bar{x} \in X$ such that $\{\bar{x}\} \in \mathcal{B}$ and $\mu(\{\bar{x}\}) = 0$. Set $X_{\bar{x}} := X \setminus \{\bar{x}\}$. Let $\varrho : X_{\bar{x}} \times X_{\bar{x}} \to [0,\infty)$ be a pseudometric on $X_{\bar{x}}$, i.e., a measurable function on $X_{\bar{x}} \times X_{\bar{x}}$ satisfying $\varrho(x,x) = 0$, $\varrho(x,y) = \varrho(y,x)$, and $\varrho(x,y) \le \varrho(x,z) + \varrho(z,y)$ for every $x,y,z \in X_{\bar{x}}$. Define the ball of center $x \in X_{\bar{x}}$ and radius $r \ge 0$, $B(x,r) := \{y \in X_{\bar{x}} : \varrho(x,y) \le r\}$, and its complement $B(x,r)^{\complement} := X \setminus B(x,r)$. It is clear that

$$X_{\bar{x}} = \bigcup_{r \ge 0} B(x,r)$$

for every $x \neq \bar{x}$. We have the following concentration-compactness result, which should be compared to [Lions 1984a, Lemma I.1].

Proposition A.1. *Let* $(X, \mathcal{B}, \mu)$, $\bar{x} \in X$, $\varrho : X_{\bar{x}} \times X_{\bar{x}} \to [0, \infty)$ *be as above. Let* $\{\rho_n\}$ *be a sequence in* $L^1(X, \mu)$ *satisfying*

$$\rho_n \geq 0 \quad \text{in } X, \qquad \int_X \rho_n \, \mathrm{d}\mu = \lambda,$$

where $\lambda > 0$ *is fixed. Then there exists a subsequence* $\{\rho_{n_k}\}$ *satisfying one of the following three possibilities*:

(i) (*compactness*) *There exists* $\{x_k\} \subset X_{\bar{x}}$ *such that* $\rho_{n_k}(\cdot + x_k)$ *is tight*; *i.e.*,

$$\text{for all } \varepsilon > 0, \text{ there exists } R < \infty \text{ such that } \int_{B(x_k, R)} \rho_{n_k} \, \mathrm{d}\mu \geq \lambda - \varepsilon.$$

(ii) (*vanishing*) $\lim_{k\to\infty} \sup_{y \in X_{\bar{x}}} \int_{B(y,R)} \rho_{n_k} \, \mathrm{d}\mu = 0$ *for all* $R < \infty$;

(iii) (*dichotomy*) *There exists* $\alpha \in (0, \lambda)$ *with the following property. For every* $\varepsilon > 0$, *there exist* $R \in [0, \infty)$, $k_0 \geq 1$, *and nonnegative functions* $\rho_{k,1}, \rho_{k,2} \in L^1(X, \mu)$ *such that, for every* $k \geq k_0$,

$$\|\rho_{n_k} - (\rho_{k,1} + \rho_{k,2})\|_{L^1(X)} \leq \varepsilon, \quad \left| \int_X \rho_{k,1} \, \mathrm{d}\mu - \alpha \right| \leq \varepsilon, \quad \left| \int_X \rho_{k,2} \, \mathrm{d}\mu - (\lambda - \alpha) \right| \leq \varepsilon,$$

$$\operatorname{supp}(\rho_{k,1}) \subseteq B(x_k, R) \quad \text{and} \quad \operatorname{supp}(\rho_{k,2}) \subseteq B(x_k, R_k)^{\complement}$$

for certain sequences $\{x_k\} \subset X_{\bar{x}}$, $\{R_k\} \subset [0, \infty)$, *with* $R_k \to \infty$ *as* $k \to \infty$.

The proof of Proposition A.1 parallels that of [Lions 1984a, Lemma I.1] and proceeds via analysis of the sequence of *concentration functions*

$$Q_n \colon [0, \infty) \to \mathbb{R}, \quad Q_n(t) := \sup_{x \in X_{\bar{x}}} \int_{B(x,t)} \rho_n \, \mathrm{d}\mu.$$

The sequence $\{Q_n\}$ consists of nondecreasing, nonnegative, uniformly bounded functions on $[0, \infty)$ which satisfy $Q_n(t) \to \lambda$ as $t \to \infty$, since $\mu(\{\bar{x}\}) = 0$. Very briefly, the argument goes as follows. By the Helly selection principle, there exists a subsequence $\{n_k\} \subset \mathbb{N}$ and a nondecreasing, nonnegative function $Q \colon [0, \infty) \to \mathbb{R}$ such that $Q_{n_k}(t) \to Q(t)$ as $k \to \infty$ *for every* $t \geq 0$. Set $\alpha := \lim_{t\to\infty} Q(t) \in [0, \lambda]$, and note that:

- If $\alpha = 0$, then $Q \equiv 0$. This translates into the vanishing condition at once.
- If $\alpha = \lambda$, then compactness occurs.
- If $0 < \alpha < \lambda$, then dichotomy occurs. In this case, the functions $\rho_{k,1}, \rho_{k,2}$ are given by $\rho_{k,1} = \rho_{n_k} \mathbb{1}_{B(x_k,R)}$ and $\rho_{k,2} = \rho_{n_k} \mathbb{1}_{B(x_k,R_k)^{\complement}}$.

We omit further details and refer the interested reader to [Lions 1984a].

When applying Proposition A.1 to the study of extremizing sequences for (1-9), the desirable outcome (with a view towards obtaining concentration at a point under the hypotheses of Proposition 3.3) is *compactness* or *vanishing*. Therefore the possibility of *dichotomy* needs to be discarded. To this end, Lions proposes the *strict superadditivity condition* [Lions 1984a, Section I.2], which in the present setting can be recast as follows. Define

$$I_\lambda := \sup\{\|\mathcal{E}_p(f)\|^6_{L^6(\mathbb{R}^2)} : \|f\|^2_{L^2(\mathbb{R})} = \lambda\}. \tag{A-1}$$

The quantity I_λ is said to satisfy the strict superadditivity condition if, for every $\lambda > 0$,

$$I_\lambda > I_\alpha + I_{\lambda-\alpha} \quad \text{for every } \alpha \in (0, \lambda). \tag{A-2}$$

In our case, $\mathcal{E}_p$ is a linear operator, and so $I_\lambda = \lambda^3 I_1 = \lambda^3 \boldsymbol{E}_p^6$. Thus (A-2) translates into the elementary numerical inequality $\lambda^3 > \alpha^3 + (\lambda-\alpha)^3$, which holds for every $\lambda > 0$ and $\alpha \in (0, \lambda)$. As seen in the proof of Proposition 3.3, it is condition (A-2) (applied with $\lambda = 1$) which ensures that dichotomy does not occur. A similar condition in a more general context is used in [Lieb 1983, Lemma 2.7].

Appendix B: Revisiting Brézis–Lieb

In this appendix, we prove a useful variant of [Fanelli et al. 2011, Proposition 1.1], which in turn relies on the Brézis–Lieb lemma [1983]. Proposition 1.1 of [Fanelli et al. 2011] states that, in the compact setting, the only obstruction to the strong convergence of an extremizing sequence is weak convergence to zero. In the noncompact setting, it is in general nontrivial to verify condition (iv) of [Fanelli et al. 2011, Proposition 1.1]. To overcome this difficulty, various arguments using Sobolev embeddings and the Rellich–Kondrachov compactness theorem have been employed in [Carneiro et al. 2019; Fanelli et al. 2012; Quilodrán 2013]. In our case, it is not clear how such an argument would go. Instead we take a different route, and argue that condition (iv) from [Fanelli et al. 2011, Proposition 1.1] can be replaced by uniform decay of the L^2-norm, in a sense compactifying the space in question. The following is a precise formulation of this idea.

Proposition B.1. *Given $p > 1$, consider the Fourier extension operator $\mathcal{E}_p \colon L^2(\mathbb{R}) \to L^6(\mathbb{R}^2)$ defined in* (1-12). *Let $\{f_n\} \subset L^2(\mathbb{R})$, and let $\Theta : [1, \infty) \to (0, \infty)$ with $\Theta(R) \to 0$, as $R \to \infty$, be such that*

(i) $\|f_n\|_{L^2(\mathbb{R})} = 1$ *for every* $n \in \mathbb{N}$,

(ii) $\lim_{n\to\infty} \|\mathcal{E}_p(f_n)\|_{L^6(\mathbb{R}^2)} = \boldsymbol{E}_p$,

(iii) $f_n \rightharpoonup f \neq 0$ *as* $n \to \infty$,

(iv) $\|f_n\|_{L^2([-R,R]^\complement)} \leq \Theta(R)$ *for every* $n \in \mathbb{N}$ *and* $R \geq 1$.

Then $f_n \to f$ in $L^2(\mathbb{R})$, as $n \to \infty$. In particular, $\|f\|_{L^2(\mathbb{R})} = 1$ and $\|\mathcal{E}_p(f)\|_{L^6(\mathbb{R}^2)} = \boldsymbol{E}_p$, and so f is an extremizer of (1-9).

This variant was already observed in [Quilodrán 2012, Proposition 2.31] for the case of the cone, and the proof follows similar lines to that of [Fanelli et al. 2011, Proposition 1.1]. Note that the function Θ may depend on the sequence $\{f_n\}$, but *not* on n. The following proof is inspired by [Frank et al. 2016, Proposition 2.2].

Proof of Proposition B.1. Set $r_n := f_n - f$. Then $r_n \rightharpoonup 0$ as $n \to \infty$, and thus $m := \lim_{n\to\infty} \|r_n\|_{L^2}^2$ exists and satisfies $1 = \|f\|_{L^2}^2 + m$. Given $R > 0$, take the decomposition

$$r_n = r_n \mathbb{1}_{[-R,R]} + r_n \mathbb{1}_{[-R,R]^\complement} =: r_{n,1} + r_{n,2}.$$

Since the support of $r_{n,1}$ is compact and $r_{n,1} \rightharpoonup 0$ as $n \to \infty$, we know $\mathcal{E}_p(r_{n,1}) \to 0$ pointwise a.e. in $\mathbb{R}^2$ as $n \to \infty$. On the other hand, from condition (iv) we have

$$\|\mathcal{E}_p(r_{n,2})\|_{L^6} \le \boldsymbol{E}_p(\Theta(R) + \|f\|_{L^2([-R,R]^\complement)}) \tag{B-1}$$

for every $R \ge 1$. This upper bound is independent of n, and tends to 0 as $R \to \infty$. We have $\mathcal{E}_p(f_n - r_{n,2}) = \mathcal{E}_p(f) + \mathcal{E}_p(r_{n,1})$, and $\|\mathcal{E}_p(f_n - r_{n,2})\|_{L^6} \le \boldsymbol{E}_p(1 + \Theta(R) + \|f\|_{L^2([-R,R]^\complement)})$ is uniformly bounded in n. Since $\mathcal{E}_p(f_n - r_{n,2}) \to \mathcal{E}_p(f)$ pointwise a.e. in $\mathbb{R}^2$ as $n \to \infty$, we can invoke the Brézis–Lieb lemma [1983] and obtain

$$\|\mathcal{E}_p(f_n - r_{n,2})\|_{L^6}^6 = \|\mathcal{E}_p(f)\|_{L^6}^6 + \|\mathcal{E}_p(r_{n,1})\|_{L^6}^6 + o(1) \quad \text{as } n \to \infty.$$

It follows that $\mu := \limsup_{n\to\infty} \|\mathcal{E}_p(r_{n,1})\|_{L^6}^6$ and $\lambda := \limsup_{n\to\infty} \|\mathcal{E}_p(f_n - r_{n,2})\|_{L^6}^6$ satisfy

$$\lambda = \|\mathcal{E}_p(f)\|_{L^6}^6 + \mu.$$

Since $\|\mathcal{E}_p(r_{n,1})\|_{L^6}^6 \le \boldsymbol{E}_p^6 \|r_{n,1}\|_{L^2}^6 \le \boldsymbol{E}_p^6 \|r_n\|_{L^2}^6$, we have $\mu \le \boldsymbol{E}_p^6 m^3$. Therefore

$$\lambda = \|\mathcal{E}_p(f)\|_{L^6}^6 + \mu \le \|\mathcal{E}_p(f)\|_{L^6}^6 + \boldsymbol{E}_p^6 (1 - \|f\|_{L^2}^2)^3.$$

Thus, replacing the definition of λ, we have proved

$$\limsup_{n\to\infty} \|\mathcal{E}_p(f_n - r_{n,2})\|_{L^6}^6 \le \|\mathcal{E}_p(f)\|_{L^6}^6 + \boldsymbol{E}_p^6 (1 - \|f\|_{L^2}^2)^3 \tag{B-2}$$

for every $R \ge 1$. Now, $\|\mathcal{E}_p(f_n - r_{n,2})\|_{L^6} \ge \|\mathcal{E}_p(f_n)\|_{L^6} - \|\mathcal{E}_p(r_{n,2})\|_{L^6}$ and $\|\mathcal{E}_p(r_{n,2})\|_{L^6}$ is bounded above as quantified by (B-1). Thus

$$\limsup_{n\to\infty} \|\mathcal{E}_p(f_n - r_{n,2})\|_{L^6} \ge \boldsymbol{E}_p - \boldsymbol{E}_p(\Theta(R) + \|f\|_{L^2([-R,R]^\complement)})$$

for every $R \ge 1$. Using this together with (B-2), and letting $R \to \infty$, yields

$$\boldsymbol{E}_p^6 \le \|\mathcal{E}_p(f)\|_{L^6}^6 + \boldsymbol{E}_p^6 (1 - \|f\|_{L^2}^2)^3.$$

By the elementary inequality $(1-t)^3 \le 1 - t^3$, valid for every $t \in [0, 1]$, we then have

$$\boldsymbol{E}_p^6 \le \|\mathcal{E}_p(f)\|_{L^6}^6 + \boldsymbol{E}_p^6 (1 - \|f\|_{L^2}^6).$$

Since the reverse inequality holds by definition, we conclude that f is an extremizer. Moreover, since $f \neq 0$ and the elementary inequality is strict unless $t \in \{0, 1\}$, we conclude that $\|f\|_{L^2} = 1$. This completes the proof of the proposition. □

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A BOOTSTRAPPING APPROACH TO JUMP INEQUALITIES AND THEIR APPLICATIONS

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The aim of this paper is to present an abstract and general approach to jump inequalities in harmonic analysis. Our principal conclusion is the refinement of r-variational estimates, previously known for $r > 2$, to endpoint results for the jump quasiseminorm corresponding to $r = 2$. This is applied to the dimension-free results recently obtained by the first two authors in collaboration with Bourgain, and Wróbel, and also to operators of Radon type treated by Jones, Seeger, and Wright.

1. Introduction

Variational and jump inequalities in harmonic analysis, probability, and ergodic theory have been studied extensively since [Bourgain 1989], where a variational version of the Hardy–Littlewood maximal function was introduced. The purpose of this paper is to formulate general sufficient conditions that allow us to deal with variational and jump inequalities for a wide class of operators. Our approach will be based on certain bootstrap arguments. As an application we extend the known L^p estimates for r-variations for $r > 2$ (see definition (1.2)) to endpoint assertions for the jump quasiseminorm J_2^p (see definition (1.3)), which corresponds to $r = 2$. In this way our results will extend previously recently obtained assertions in [Bourgain et al. 2018; 2019] for dimension-free estimates given for $r > 2$, as well as a number of results in [Jones et al. 2008] for operators of Radon type.

We recall the notation for jump quasiseminorms from [Mirek et al. 2018b]. For any $\lambda > 0$ and $\mathbb{I} \subset \mathbb{R}$ the λ-*jump counting function* of a function $f : \mathbb{I} \to \mathbb{C}$ is defined by

$$\begin{aligned} N_\lambda(f) &:= N_\lambda(f(t) : t \in \mathbb{I}) \\ &:= \sup\{J \in \mathbb{N} : \text{there exists } t_0 < \cdots < t_J,\ t_j \in \mathbb{I},\ \text{such that } \min_{0<j\le J} |f(t_j) - f(t_{j-1})| \ge \lambda\} \end{aligned} \tag{1.1}$$

and the r-variation seminorm by

$$V^r(f) := V^r(f(t) : t \in \mathbb{I}) := \begin{cases} \sup_{J\in\mathbb{N}} \sup_{\substack{t_0<\cdots<t_J \\ t_j\in\mathbb{I}}} \big(\sum_{j=1}^{J} |f(t_j) - f(t_{j-1})|^r\big)^{\frac{1}{r}}, & 0 < r < \infty, \\ \sup_{\substack{t_0<t_1 \\ t_j\in\mathbb{I}}} |f(t_1) - f(t_0)|, & r = \infty, \end{cases} \tag{1.2}$$

where the former supremum is taken over all finite increasing sequences in $\mathbb{I}$.

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Throughout the article $(X, \mathcal{B}, \mathfrak{m})$ denotes a σ-finite measure space. For a function $f : X \times \mathbb{I} \to \mathbb{C}$ the jump quasiseminorm on $L^p(X)$ for $1 < p < \infty$ is defined by

$$\begin{aligned} J_2^p(f) &:= J_2^p(f : X \times \mathbb{I} \to \mathbb{C}) := J_2^p((f(\cdot,t))_{t\in\mathbb{I}}) := J_2^p((f(\cdot,t))_{t\in\mathbb{I}} : X \to \mathbb{C}) \\ &:= \sup_{\lambda>0} \big\| \lambda N_\lambda(f(\cdot,t) : t \in \mathbb{I})^{\frac{1}{2}} \big\|_{L^p}. \end{aligned} \tag{1.3}$$

In this connection by [Mirek et al. 2018b, Lemma 2.12] we note that

$$\|V^r(f)\|_{L^{p,\infty}} \lesssim_{p,r} J_2^p(f) \le \|V^2(f)\|_{L^p} \tag{1.4}$$

for $r > 2$, and the first inequality fails for $r = 2$.

We now briefly list our main results:

(1) The extension to the jump quasiseminorm J_2^p of dimension-free estimates for maximal averages over convex sets, as given by Theorems 1.9, 1.11 and 1.14 below.

(2) The corresponding extension to J_2^p of the previous dimension-free estimates for cubes in the discrete setting; see Theorem 1.18.

(3) The general J_2^p results for operators of Radon type (both averages and singular integrals) in Theorems 1.22 and 1.30, related to the previous results in [Jones et al. 2008].

Underlying the proofs of all these results will be the basic facts about the jump quantity J_2^p obtained in our recent paper [Mirek et al. 2018b], and the bootstrap arguments in Section 2 of the present paper. The reader might compare the methods in Section 2 with related arguments in [Bourgain et al. 2018, Section 2.2] as well as [Nagel et al. 1978; Duoandikoetxea and Rubio de Francia 1986; Carbery 1986], and Christ's observation included in [Carbery 1988]. The techniques in Section 2 will be carried out in the following framework. We assume that we are given a measure space $(X, \mathcal{B}, \mathfrak{m})$ which is endowed with a sequence of linear operators $(S_j)_{j\in\mathbb{Z}}$ acting on $L^1(X)+L^\infty(X)$ that play the role of the Littlewood–Paley operators. Namely, the following conditions are satisfied:

(1) The family $(S_j)_{j\in\mathbb{Z}}$ is a resolution of the identity on $L^2(X)$; i.e., the identity

$$\sum_{j\in\mathbb{Z}} S_j = \mathrm{Id} \tag{1.5}$$

holds in the strong operator topology on $L^2(X)$.

(2) For every $1 < p < \infty$ we have

$$\bigg\| \Big(\sum_{j\in\mathbb{Z}} |S_j f|^2 \Big)^{\frac{1}{2}} \bigg\|_{L^p} \lesssim \|f\|_{L^p}, \quad f \in L^p(X). \tag{1.6}$$

Suppose now we have a family of linear operators $(T_t)_{t\in\mathbb{I}}$ acting on $L^1(X) + L^\infty(X)$, where the index set $\mathbb{I}$ is a countable subset of $(0,\infty)$. We assume that $\mathbb{I} \subseteq (0,\infty)$ to make our exposition consistent with the results in the literature. One of our aims is to understand what kind of conditions have to be imposed on the family $(T_t)_{t\in\mathbb{I}}$, in terms of its interactions with the Littlewood–Paley operators $(S_j)_{j\in\mathbb{Z}}$

to obtain the inequality

$$J_2^p((T_t f)_{t\in\mathbb{I}} : X \to \mathbb{C}) \lesssim \|f\|_{L^p} \tag{1.7}$$

in some range of p's. We accomplish this task in Section 2 by proving Theorems 2.14 and 2.39 for positive operators[1] by certain bootstrap arguments, and Theorem 2.28 for general operators. Our approach will be based on extension of ideas from [Duoandikoetxea and Rubio de Francia 1986; Bourgain et al. 2018] to a more abstract setting.

As mentioned above it has been very well known since [Bourgain 1989] that r-variational estimates (and consequently maximal estimates, see (1.2)) can be deduced from jump inequalities. Namely, a priori jump estimates (1.7) in an open range of $p \in (1, \infty)$ imply

$$\|V^r(T_t f : t \in \mathbb{I})\|_{L^p} \lesssim_{p,r} \|f\|_{L^p}$$

in the same range of p's and for all $r \in (2, \infty]$. This follows from (1.4) and interpolation. Therefore, it is natural to say that the jump inequality in (2.2) is an endpoint for r-variations at $r = 2$. On the other hand, we also know that the range of $r \in (2, \infty]$ in r-variational estimates, for many operators in harmonic analysis, is sharp due to the sharp estimates in Lépingle's inequality for martingales; see [Mirek et al. 2018b].

Here and later we write $a \lesssim b$ if $a \le Cb$, where the constant $0 < C < \infty$ is allowed to depend on p, but not on the underlying abstract measure space X or function f. If C is allowed to depend on some additional parameters this will be indicated by adding a subscript to the symbol $\lesssim$.

1A. ***Applications to dimension-free estimates.*** An important application of the results from Section 2 will be bounds independent of the dimension in jump inequalities associated with the Hardy–Littlewood averaging operators. Let $G \subset \mathbb{R}^d$ be a symmetric convex body, that is, a nonempty symmetric convex open bounded subset of $\mathbb{R}^d$. Define for $t > 0$ and $x \in \mathbb{R}^d$ the averaging operator

$$\mathcal{A}_t^G f(x) := |G|^{-1} \int_G f(x - ty)\,\mathrm{d}y, \quad f \in L^1_{\mathrm{loc}}(\mathbb{R}^d). \tag{1.8}$$

It follows from the spherical maximal theorem that, in the case that G is the Euclidean ball, the maximal operator $\mathcal{A}_\star^G f := \sup_{t>0} |\mathcal{A}_t^G f|$ corresponding to (1.8) is bounded on $L^p(\mathbb{R}^d)$ for all $p > 1$, uniformly in $d \in \mathbb{N}$ [Stein 1982]. This result was extended to arbitrary symmetric convex bodies $G \subset \mathbb{R}^d$ in [Bourgain 1986a] (for $p = 2$) and [Bourgain 1986b; Carbery 1986] $\left(\text{for } p > \tfrac{3}{2}\right)$. For unit balls $G = B^q$ induced by ℓ^q norms in $\mathbb{R}^d$ the full range $p > 1$ of dimension-free estimates was established in [Müller 1990] (for $1 \le q < \infty$) and [Bourgain 2014] (for cubes $q = \infty$) with constants depending on q. In the latter case the product structure of the cubes is important; this result was recently extended to products of Euclidean balls of arbitrary dimensions [Sommer 2017].

Variational versions of most of the aforementioned dimension-free estimates were obtained in [Bourgain et al. 2018] for $r > 2$. In this article we give a shorter and more self-contained proof of the main results of that work and extend them to the endpoint $r = 2$ by appealing to Theorems 2.14 and 2.39. A notable

[1] A linear operator T is positive if $Tf \ge 0$ for every $f \ge 0$.

simplification is that we do not use the maximal estimates as a black box. In particular, we reprove all dimension-free estimates for the maximal function $\mathcal{A}^G_\star$.

In view of (1.4) and by real interpolation, Theorem 1.9 extends [Bourgain et al. 2018, Theorem 1.2].

Theorem 1.9. *Let $d \in \mathbb{N}$ and $G \subset \mathbb{R}^d$ be a symmetric convex body. Then for every $1 < p < \infty$ and $f \in L^p(\mathbb{R}^d)$ we have*

$$J^p_2((\mathcal{A}^G_{2^k} f)_{k\in\mathbb{Z}} : \mathbb{R}^d \to \mathbb{C}) \lesssim \|f\|_{L^p}, \tag{1.10}$$

where the implicit constant is independent of d and G.

As a consequence of Theorem 1.9 and the decomposition into long and short jumps, see (2.2), Theorems 1.11 and 1.14 below extend Theorems 1.1 and 1.3 in [Bourgain et al. 2018], respectively. Hence Theorem 1.9 can be thought of as the main result of this paper, since inequalities (1.12) and (1.15) were obtained in [Bourgain et al. 2018]. However, we shall present a different approach to establish the estimates in (1.12) and (1.15).

Theorem 1.11. *Let G be as in Theorem 1.9. Then for every $\frac{3}{2} < p < 4$ and $f \in L^p(\mathbb{R}^d)$ we have*

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} \big(V^2(\mathcal{A}^G_t f : t \in [2^k, 2^{k+1}]) \big)^2 \Big)^{\frac{1}{2}} \Big\|_{L^p} \lesssim \|f\|_{L^p}. \tag{1.12}$$

In particular,

$$J^p_2((\mathcal{A}^G_t f)_{t>0} : \mathbb{R}^d \to \mathbb{C}) \lesssim \|f\|_{L^p}, \tag{1.13}$$

where the implicit constants in (1.12) and (1.13) are independent of d and G.

Theorem 1.14. *Let $d \in \mathbb{N}$ and $G \subset \mathbb{R}^d$ be the unit ball induced by the ℓ^q norm in $\mathbb{R}^d$ for some $1 \le q \le \infty$. Then for every $1 < p < \infty$ and $f \in L^p(\mathbb{R}^d)$ we have*

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} \big(V^2(\mathcal{A}^G_t f : t \in [2^k, 2^{k+1}]) \big)^2 \Big)^{\frac{1}{2}} \Big\|_{L^p} \lesssim_q \|f\|_{L^p}. \tag{1.15}$$

In particular

$$J^p_2((\mathcal{A}^G_t f)_{t>0} : \mathbb{R}^d \to \mathbb{C}) \lesssim_q \|f\|_{L^p}, \tag{1.16}$$

where the implicit constants in (1.15) and (1.16) are independent of d.

The method of the present paper also allows us to provide estimates independent of the dimension in jump inequalities associated with the discrete averaging operator along cubes in $\mathbb{Z}^d$. For every $x \in \mathbb{Z}^d$ and $N \in \mathbb{N}$ let

$$A_N f(x) := \frac{1}{|Q_N \cap \mathbb{Z}^d|} \sum_{y \in Q_N \cap \mathbb{Z}^d} f(x-y), \quad f \in \ell^1(\mathbb{Z}^d), \tag{1.17}$$

be the discrete Hardy–Littlewood averaging operator, where $Q_N = [-N, N]^d$.

Theorem 1.18. *For every $\frac{3}{2} < p < 4$ and $f \in \ell^p(\mathbb{Z}^d)$ we have*

$$J^p_2((A_N f)_{N\in\mathbb{N}} : \mathbb{Z}^d \to \mathbb{C}) \lesssim \|f\|_{\ell^p}. \tag{1.19}$$

Moreover, if we consider only lacunary parameters, then (1.19) *remains true for all* $1 < p < \infty$ *and we have*

$$J_2^p((A_{2^k} f)_{k\geq 0} : \mathbb{Z}^d \to \mathbb{C}) \lesssim \|f\|_{\ell^p}, \tag{1.20}$$

where the implicit constants in (1.19) *and* (1.20) *are independent of* d.

Theorem 1.18 provides the endpoint estimate at $r = 2$ for the recent dimension-free estimates [Bourgain et al. 2019] for r-variations corresponding to operator (1.17).

The dimension-free results are proved in Section 3A by combining the results from Section 2 (Theorems 2.14 and 2.39) with the jump estimates for the Poisson semigroup from [Mirek et al. 2018b] and Fourier multiplier estimates from [Bourgain 1986a; 2014; Müller 1990].

1B. ***Applications to operators of Radon type.*** Another important class of operators which was extensively studied in [Jones et al. 2008] in the context of jump inequalities are operators of Radon type modeled on polynomial mappings.

Let $P = (P_1, \dots, P_d) : \mathbb{R}^k \to \mathbb{R}^d$ be a polynomial mapping, where each component $P_j : \mathbb{R}^k \to \mathbb{R}$ is a polynomial with k variables and real coefficients. We fix $\Omega \subset \mathbb{R}^k$ a convex open bounded set containing the origin (not necessarily symmetric), and for every $x \in \mathbb{R}^d$ and $t > 0$ we define the Radon averaging operator

$$\mathcal{M}_t^P f(x) := \frac{1}{|\Omega_t|} \int_{\Omega_t} f(x - P(y))\, \mathrm{d}y, \tag{1.21}$$

where $\Omega_t = \{x \in \mathbb{R}^k : t^{-1}x \in \Omega\}$. Using Theorems 2.14 and 2.39 we easily deduce Theorem 1.22; see Section 3C.

Theorem 1.22. *For every* $1 < p < \infty$ *and* $f \in L^p(\mathbb{R}^d)$ *we have*

$$J_2^p((\mathcal{M}_t^P f)_{t>0} : \mathbb{R}^d \to \mathbb{C}) \lesssim_{d,p} \|f\|_{L^p}, \tag{1.23}$$

where the implicit constant is independent of the coefficients of P.

Before we formulate a corresponding result for truncated singular integrals we need to fix some definitions and notation. A *modulus of continuity* is a function $\omega : [0, \infty) \to [0, \infty)$ with $\omega(0) = 0$ that is subadditive in the sense that

$$u \leq t + s \quad \Longrightarrow \quad \omega(u) \leq \omega(t) + \omega(s).$$

Substituting $s = 0$ one sees that $\omega(u) \leq \omega(t)$ for all $0 \leq u \leq t$. The basic example is $\omega(t) = t^\theta$, with $\theta \in (0, 1)$. Note that the composition and sum of two moduli of continuity is again a modulus of continuity. In particular, if $\omega(t)$ is a modulus of continuity and $\theta \in (0, 1)$, then $\omega(t)^\theta$ and $\omega(t^\theta)$ are also moduli of continuity.

The *Dini norm* and the *log-Dini norm* of a modulus of continuity are defined respectively by setting

$$\|\omega\|_{\mathrm{Dini}} := \int_0^1 \omega(t) \frac{\mathrm{d}t}{t} \quad \text{and} \quad \|\omega\|_{\mathrm{logDini}} := \int_0^1 \omega(t) \frac{|\log t|\, \mathrm{d}t}{t}. \tag{1.24}$$

For any $c > 0$ the integral can be equivalently (up to a c-dependent multiplicative constant) replaced by the sum over $2^{-j/c}$ with $j \in \mathbb{N}$.

Finally, for every $x \in \mathbb{R}^d$ and $t > 0$ we will consider the truncated singular Radon transform

$$\mathcal{H}_t^P f(x) := \int_{\mathbb{R}^k \setminus \Omega_t} f(x - P(y)) K(y) \, \mathrm{d}y, \tag{1.25}$$

defined for every Schwartz function f in $\mathbb{R}^d$, where $K : \mathbb{R}^k \setminus \{0\} \to \mathbb{C}$ is a kernel satisfying the following conditions:

(1) The size condition: there exists a constant $C_K > 0$ such that

$$|K(x)| \le C_K |x|^{-k} \quad \text{for all } x \in \mathbb{R}^k. \tag{1.26}$$

(2) The cancellation condition:

$$\int_{\Omega_R \setminus \Omega_r} K(y) \, \mathrm{d}y = 0 \quad \text{for } 0 < r < R < \infty. \tag{1.27}$$

(3) The smoothness condition:

$$\sup_{R>0} \sup_{|y| \le \frac{1}{2} Rt} \int_{R \le |x| \le 2R} |K(x) - K(x+y)| \, \mathrm{d}x \le \omega_K(t) \tag{1.28}$$

for every $t \in (0, 1)$ with some modulus of continuity ω_K.

In many applications it is easy to verify the somewhat stronger pointwise version of the smoothness estimate from (1.28). Namely,

$$|K(x) - K(x+y)| \le \omega_K\left(\frac{|y|}{|x|}\right) |x|^{-k}, \quad \text{provided that } |y| \le \frac{|x|}{2}, \tag{1.29}$$

for some modulus of continuity ω_K. One can immediately see that condition (1.29) implies condition (1.28). Our next result establishes an analogue of the inequality (1.23) for the operators in (1.25).

Theorem 1.30. *Suppose that* $\|\omega_K^\theta\|_{\mathrm{logDini}} + \|\omega_K^{\theta/2}\|_{\mathrm{Dini}} < \infty$ *for some* $\theta \in (0, 1]$. *Then for every* $p \in \{1 + \theta, (1+\theta)'\}$ *and* $f \in L^p(\mathbb{R}^d)$ *we have*

$$J_2^p((\mathcal{H}_t^P f)_{t>0} : \mathbb{R}^d \to \mathbb{C}) \lesssim_{d,p} \|f\|_{L^p}, \tag{1.31}$$

where the implicit constant is independent of the coefficients of P. *More precisely*:

(1) *If* $\|\omega_K^\theta\|_{\mathrm{logDini}} < \infty$, *then*

$$J_2^p((\mathcal{H}_{2^k} f)_{k \in \mathbb{Z}} : \mathbb{R}^d \to \mathbb{C}) \lesssim \|f\|_{L^p}. \tag{1.32}$$

(2) *If* $\|\omega_K^{\theta/2}\|_{\mathrm{Dini}} < \infty$, *then*

$$\left\| \left(\sum_{k \in \mathbb{Z}} V^2(\mathcal{H}_t f : t \in [2^k, 2^{k+1}])^2 \right)^{\frac{1}{2}} \right\|_{L^p} \lesssim \|f\|_{L^p}. \tag{1.33}$$

The inequality (1.23) was proved in [Jones et al. 2008] for the averages $\mathcal{M}_t^P$ over Euclidean balls. The inequality (1.31) was proved in that work for monomial curves, i.e., in the case

$$k = 1, \quad d = 2, \quad K(y) = y^{-1} \quad \text{and} \quad P(x) = (x, x^a), \quad \text{where } a > 1.$$

General polynomials were considered in [Mirek et al. 2017] (although jump estimates are not explicitly stated in that article they can also be obtained with minor modifications of the proofs). Multidimensional variants of $\mathcal{H}_t^P$ were also studied in that work under stronger regularity conditions imposed on the kernel K. Inequalities (1.23) and (1.31) will be used to establish jump inequalities for the discrete analogues of (1.21) and (1.25) in [Mirek et al. 2018a].

Finally we provide van der Corput integral estimates in Lemma B.1 and Proposition B.2, which have the feature of permitting the handling of the oscillatory integrals with nonsmooth amplitudes. Their broader scope will be needed in the proof of Theorem 1.30.

2. An abstract approach to jump inequalities

2A. ***Preliminaries.*** Let $(X, \mathcal{B}, \mathfrak{m})$ be a σ-finite measure space endowed with a sequence of linear Littlewood–Paley operators $(S_j)_{j\in\mathbb{Z}}$ satisfying (1.5), (1.6). Assume that $(T_t)_{t\in\mathbb{I}}$ is a family of linear operators acting on $L^1(X) + L^\infty(X)$, where the index set $\mathbb{I}$ is a subset of $(0, \infty)$. Under suitable conditions imposed on the family $(T_t)_{t\in\mathbb{I}}$ in terms of its interactions with the Littlewood–Paley operators $(S_j)_{j\in\mathbb{Z}}$ as in the Introduction, we will study strong uniform jump inequalities

$$J_2^p((T_t f)_{t\in\mathbb{I}} : X \to \mathbb{C}) \lesssim \|f\|_{L^p} \tag{2.1}$$

in various ranges of p's; see Theorems 2.14, 2.28, and 2.39.

To avoid further problems with measurability we will always assume that $\mathbb{I}$ is countable. Usually $\mathbb{I}$ is $\mathbb{D} := \{2^n : n \in \mathbb{Z}\}$ the set of all dyadic numbers or $\mathbb{I}$ is $\mathbb{U} := \bigcup_{n\in\mathbb{Z}} 2^{-n}\mathbb{N}$ the set of nonnegative rational numbers whose denominators in reduced form are powers of 2. In practice, the countability assumption may be removed if for every $f \in L^1(X) + L^\infty(X)$ the function $\mathbb{I} \ni t \mapsto T_t f(x)$ is continuous for $\mathfrak{m}$-almost every $x \in X$. In our applications this will always be the case.

We recall the decomposition into long and short jumps from [Jones et al. 2008, Lemma 1.3], which tells that for every $\lambda > 0$ we have

$$\lambda N_\lambda(T_t f(x) : t \in \mathbb{I})^{\frac{1}{2}} \lesssim \lambda N_{\frac{\lambda}{3}}(T_t f(x) : t \in \mathbb{D})^{\frac{1}{2}} + \Big(\sum_{k\in\mathbb{Z}} \big(\lambda N_\lambda(T_t f(x) : t \in [2^k, 2^{k+1}) \cap \mathbb{I})^{\frac{1}{2}}\big)^2\Big)^{\frac{1}{2}}. \tag{2.2}$$

In other words the λ-jump counting function can be dominated by the long jumps (the first term in (2.2) with $t \in \mathbb{D}$) and the short jumps (the square function in (2.2)). Similar inequalities hold for the maximal function and for r-variations.

We deal with L^p bounds for the long jump counting function corresponding to T_t with $t \in \mathbb{D}$ in two ways, similarly to [Duoandikoetxea and Rubio de Francia 1986]. The first approach is to find an approximating family of operators (see the family $(P_k)_{k\in\mathbb{Z}}$ in Theorem 2.14) for which the bound in question is known and control a square function that dominates the error term; see (2.15) in Theorem 2.14. In our case this method works for positive operators with martingales or related operators as the approximating family. The second approach is to express T_{2^k} as a telescoping sum

$$T_{2^k} f = \sum_{j \geq k} T_{2^j} f - T_{2^{j+1}} f = \sum_{j \geq k} B_j f \tag{2.3}$$

and try to deduce bounds in question from the behavior of $B_j = T_{2^j} - T_{2^{j+1}}$. This approach is needed if T_t is a truncated singular integral-type operator; see Theorem 2.28. Similar strategies also yield L^p bounds for maximal functions $\sup_{k \in \mathbb{Z}} |T_{2^k} f(x)|$ or r-variations $V^r(T_{2^k} f(x) : k \in \mathbb{Z})$.

In order to deal with short jumps we note that the square function on the right-hand side of (2.2) is dominated by the square function associated with 2-variations, which in turn is controlled by a series of square functions

$$\Big(\sum_{k \in \mathbb{Z}} \big(V^2(T_t f(x) : t \in [2^k, 2^{k+1}) \cap \mathbb{D})\big)^2\Big)^{\frac{1}{2}} \leq \sqrt{2} \sum_{l \geq 0} \Big(\sum_{k \in \mathbb{Z}} \sum_{m=0}^{2^l - 1} |(T_{2^k + 2^{k-l}(m+1)} - T_{2^k + 2^{k-l} m}) f(x)|^2\Big)^{\frac{1}{2}}. \tag{2.4}$$

The square function on the right-hand side of (2.4) gives rise to assumption (2.40). Inequality (2.4) follows from the next lemma with $\mathfrak{g}(t) = T_{2^k + t} f(x)$ and $r = 2$.

Lemma 2.5. *Let $r \in [1, \infty)$, $k \in \mathbb{Z}$, and a function $\mathfrak{g} : [0, 2^k] \cap \mathbb{U} \to \mathbb{C}$ be given. Then*

$$V^r(\mathfrak{g}(t) : t \in [0, 2^k] \cap \mathbb{U}) \leq 2^{\frac{r-1}{r}} \sum_{l \geq 0} \Big(\sum_{m=0}^{2^l - 1} |\mathfrak{g}(2^{k-l}(m+1)) - \mathfrak{g}(2^{k-l} m)|^r\Big)^{\frac{1}{r}}. \tag{2.6}$$

The variation norm on the left-hand side of (2.6) can be extended to all $t \in [0, 2^k]$ if $\mathfrak{g} : [0, 2^k] \to \mathbb{C}$ is continuous. Lemma 2.5 originates in [Lewko and Lewko 2012], where it was observed that the 2-variation norm of a sequence of length N can be controlled by the sum of $\log N$ square functions and this observation was used to obtain a variational version of the Rademacher–Menshov theorem. Inequality (2.6), essentially in this form, was independently proved by the first author and Trojan [Mirek and Trojan 2016] and used to estimate r-variations for discrete Radon transforms. Lemma 2.5 has been used in several recent articles on r-variations, including [Bourgain et al. 2018]. For completeness we include a proof, which is shorter than the previous proofs.

Proof of Lemma 2.5. Due to monotonicity of r-variations it suffices to prove (2.6) with $\mathbb{U}_N = \{u/2^N : u \in \mathbb{N} \text{ and } 0 \leq u \leq 2^{k+N}\}$ in place of $[0, 2^k] \cap \mathbb{U}$. Observe that

$$V^r(\mathfrak{g}(t) : t \in \mathbb{U}_N) = V^r\Big(\mathfrak{g}\Big(\frac{t}{2^N}\Big) : t \in [0, 2^{k+N}] \cap \mathbb{Z}\Big).$$

The proof will be completed if we show that

$$V^r(\mathfrak{g}(t) : t \in [0, 2^n] \cap \mathbb{Z}) \leq 2^{1 - \frac{1}{r}} \sum_{l=0}^{n} \Big(\sum_{m=0}^{2^{n-l} - 1} |\mathfrak{g}(2^l(m+1)) - \mathfrak{g}(2^l m)|^r\Big)^{\frac{1}{r}}. \tag{2.7}$$

Once (2.7) is established we apply it with $\mathfrak{g}(t/2^N)$ in place of $\mathfrak{g}(t)$ and $n = k + N$ and obtain (2.6). We prove (2.7) by induction on n. The case $n = 0$ is easy to verify. Let $n \geq 1$ and suppose that the claim is known for $n-1$. Let $0 \leq t_0 < \cdots < t_J < 2^n$ be an increasing sequence of integers. For $j \in \{0, \ldots, J\}$ let $s_j \leq t_j \leq u_j$ be the closest smaller and larger even integer, respectively. Then

$$\Big(\sum_{j=1}^{J} |\mathfrak{g}(t_j) - \mathfrak{g}(t_{j-1})|^r\Big)^{\frac{1}{r}} = \Big(\sum_{j=1}^{J} |(\mathfrak{g}(t_j) - \mathfrak{g}(s_j)) + (\mathfrak{g}(s_j) - \mathfrak{g}(u_{j-1})) + (\mathfrak{g}(u_{j-1}) - \mathfrak{g}(t_{j-1}))|^r\Big)^{\frac{1}{r}}$$
$$\leq \Big(\sum_{j=1}^{J} |\mathfrak{g}(s_j) - \mathfrak{g}(u_{j-1})|^r\Big)^{\frac{1}{r}} + \Big(\sum_{j=1}^{J} |(\mathfrak{g}(t_j) - \mathfrak{g}(s_j)) + (\mathfrak{g}(u_{j-1}) - \mathfrak{g}(t_{j-1}))|^r\Big)^{\frac{1}{r}}.$$

In the first term we notice that the sequence $u_0 \leq s_1 \leq u_1 \leq \cdots$ is monotonically increasing and takes values in $2\mathbb{N}$, so we can apply the induction hypothesis to the function $\mathfrak{g}(2\,\cdot)$. In the second term we use the elementary inequality $(a+b)^r \leq 2^{r-1}(a^r + b^r)$ and observe $|t_j - s_j| \leq 1$, $|t_{j-1} - u_{j-1}| \leq 1$, and $s_j \geq u_{j-1}$, so that this is bounded by the $l = 0$ summand in (2.7). □

2B. *Preparatory estimates.* We recall Lemma 2.8 that deduces a vector-valued inequality from a maximal one. Then we apply it to obtain Lemma 2.9.

Lemma 2.8 [Duoandikoetxea and Rubio de Francia 1986, p. 544]. *Suppose that $(X, \mathcal{B}, \mathfrak{m})$ is a σ-finite measure space and $(M_k)_{k\in\mathbb{J}}$ is a sequence of linear operators on $L^1(X) + L^\infty(X)$ indexed by a countable set $\mathbb{J}$. The corresponding maximal operator is defined by*

$$M_{*,\mathbb{J}} f := \sup_{k\in\mathbb{J}} \sup_{|g|\leq|f|} |M_k g|,$$

where the supremum is taken in the lattice sense. Let $q_0, q_1 \in [1, \infty]$ and $0 \leq \theta \leq 1$ with $\frac{1}{2} = (1-\theta)/q_0$ and $q_0 \leq q_1$. Let $q_\theta \in [q_0, q_1]$ be given by

$$\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} = \frac{1}{2} + \frac{1 - q_0/2}{q_1}.$$

Then

$$\Big\| \Big(\sum_{k\in\mathbb{J}} |M_k g_k|^2\Big)^{\frac{1}{2}} \Big\|_{L^{q_\theta}} \leq \big(\sup_{k\in\mathbb{J}} \|M_k\|_{L^{q_0}\to L^{q_0}}\big)^{1-\theta} \|M_{*,\mathbb{J}}\|_{L^{q_1}\to L^{q_1}}^{\theta} \Big\| \Big(\sum_{k\in\mathbb{J}} |g_k|^2\Big)^{\frac{1}{2}} \Big\|_{L^{q_\theta}}.$$

Proof. Consider the operator $\widetilde{M} g := (M_k g_k)_{k\in\mathbb{J}}$ acting on sequences of functions $g = (g_k)_{k\in\mathbb{J}}$ in $L^1(X) + L^\infty(X)$. By Fubini's theorem

$$\begin{aligned} \|\widetilde{M} g\|_{L^{q_0}(\ell^{q_0})} &= \| \|M_k g_k\|_{L^{q_0}} \|_{\ell^{q_0}} \\ &\leq \big(\sup_{k\in\mathbb{J}} \|M_k\|_{L^{q_0}\to L^{q_0}}\big) \| \|g_k\|_{L^{q_0}} \|_{\ell^{q_0}} \\ &= \big(\sup_{k\in\mathbb{J}} \|M_k\|_{L^{q_0}\to L^{q_0}}\big) \|g\|_{L^{q_0}(\ell^{q_0})}. \end{aligned}$$

By definition of the maximal operator

$$\begin{aligned}\|\widetilde{M}g\|_{L^{q_1}(\ell^\infty)} &= \big\|\sup_{k\in\mathbb{J}}|M_k g_k|\big\|_{L^{q_1}} \,\big\|M_{*,\mathbb{J}}\big(\sup_{k\in\mathbb{J}}|g_k|\big)\big\|_{L^{q_1}} \\ &\leq \|M_{*,\mathbb{J}}\|_{L^{q_1}\to L^{q_1}} \big\|\sup_{k\in\mathbb{J}}|g_k|\big\|_{L^{q_1}} = \|M_{*,\mathbb{J}}\|_{L^{q_1}\to L^{q_1}}\|g\|_{L^{q_1}(\ell^\infty)}.\end{aligned}$$

The claim for $q_\theta \in [q_0, q_1]$ follows by complex interpolation between $L^{q_0}(X;\ell^{q_0}(\mathbb{J}))$ and $L^{q_1}(X;\ell^\infty(\mathbb{J}))$. ☐

Lemma 2.9. *Suppose that $(X, \mathcal{B}, \mathfrak{m})$ is a σ-finite measure space with a sequence of operators $(S_k)_{k\in\mathbb{Z}}$ that satisfy the Littlewood–Paley inequality* (1.6)*. Let $1 \leq q_0 \leq q_1 \leq 2$ and $L \in \mathbb{N}$ be a positive integer and let $\mathbb{V}_L = \{(k,l) \in \mathbb{Z}^2 : 0 \leq l \leq L-1\}$. Let $(M_{k,l})_{(k,l)\in\mathbb{V}_L}$ be a sequence of operators bounded on $L^{q_1}(X)$ such that*

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\sum_{l=0}^{L-1}|M_{k,l}S_{k+j}f|^2\Big)^{\frac{1}{2}}\Big\|_{L^2} \leq a_j\|f\|_{L^2}, \quad f \in L^2(X), \tag{2.10}$$

for some positive numbers $(a_j)_{j\in\mathbb{Z}}$. Then for $p = q_1$ and for all $f \in L^p(X)$ we have

$$\begin{aligned}&\Big\|\Big(\sum_{k\in\mathbb{Z}}\sum_{l=0}^{L-1}|M_{k,l}S_{k+j}f|^2\Big)^{\frac{1}{2}}\Big\|_{L^p} \\ &\qquad\lesssim L^{\frac{1}{2}\frac{2-q_1}{2-q_0}}\Big(\sup_{(k,l)\in\mathbb{V}_L}\|M_{k,l}\|_{L^{q_0}\to L^{q_0}}^{\frac{q_0}{2}\frac{2-q_1}{2-q_0}}\Big)\|M_{*,\mathbb{V}_L}\|_{L^{q_1}\to L^{q_1}}^{\frac{2-q_1}{2}} a_j^{\frac{q_1-q_0}{2-q_0}}\|f\|_{L^p}.\end{aligned} \tag{2.11}$$

If $M_{k,l}$ are convolution operators on an abelian group $\mathbb{G}$, then (2.11) *also holds for $q_1 \leq p \leq q_1'$. The implicit constants in the conclusion do not depend on the qualitative bounds that we assume for the operators $M_{k,l}$ on $L^{q_1}(X)$.*

Proof. First we show (2.11). In the case $q_1 = 2$ this is identical to the hypothesis (2.10), so suppose $q_1 < 2$. Let θ and $q_\theta \in [q_0, q_1]$ be as in Lemma 2.8; then by that lemma and Littlewood–Paley inequality (1.6) we obtain

$$\begin{aligned}&\Big\|\Big(\sum_{k\in\mathbb{Z}}\sum_{l=0}^{L-1}|M_{k,l}S_{k+j}f|^2\Big)^{\frac{1}{2}}\Big\|_{L^{q_\theta}} \\ &\qquad\lesssim \Big(\sup_{(k,l)\in\mathbb{V}_L}\|M_{k,l}\|_{L^{q_0}\to L^{q_0}}^{1-\theta}\Big)\|M_{*,\mathbb{V}_L}\|_{L^{q_1}\to L^{q_1}}^{\theta}\Big\|\Big(\sum_{k\in\mathbb{Z}}\sum_{l=0}^{L-1}|S_{k+j}f|^2\Big)^{\frac{1}{2}}\Big\|_{L^{q_\theta}} \\ &\qquad\lesssim L^{\frac{1}{2}}\Big(\sup_{(k,l)\in\mathbb{V}_L}\|M_{k,l}\|_{L^{q_0}\to L^{q_0}}^{1-\theta}\Big)\|M_{*,\mathbb{V}_L}\|_{L^{q_1}\to L^{q_1}}^{\theta}\|f\|_{L^{q_\theta}}.\end{aligned} \tag{2.12}$$

Since $q_\theta \leq q_1 < 2$, there is a unique $\nu \in (0,1]$ such that

$$\frac{1}{q_1} = \frac{\nu}{q_\theta} + \frac{1-\nu}{2}.$$

Substituting the definition of q_θ we obtain

$$\frac{1}{q_1} = \frac{\nu\theta}{q_1} + \frac{1}{2}.$$

It follows that

$$1-\theta = \frac{q_0}{2}, \qquad \theta = \frac{2-q_0}{2}, \qquad \nu\theta = \frac{2-q_1}{2},$$
$$\nu = \frac{2-q_1}{2-q_0}, \quad \nu(1-\theta) = \frac{2-q_1}{2-q_0}\frac{q_0}{2}, \quad 1-\nu = \frac{q_1-q_0}{2-q_0}.$$

Interpolating (2.12) with the hypothesis (2.10) gives the claim (2.11) for $p = q_1$.

If $M_{k,l}$ are convolution operators, then by duality the first inequality in (2.12) also holds with q_θ replaced by q'_θ. Also,

$$\frac{1}{q'_1} = \frac{\nu}{q'_\theta} + \frac{1-\nu}{2},$$

so the same argument as before also works for $p = q'_1$. The conclusion for $q_1 < p < q'_1$ follows by complex interpolation. □

2C. *Long jumps for positive operators.* Suppose now we have a sequence of positive linear operators $(A_k)_{k\in\mathbb{Z}}$ and an approximating family of linear operators $(P_k)_{k\in\mathbb{Z}}$ both acting on $L^1(X)+L^\infty(X)$ such that for every $1 < p < \infty$ the maximal lattice operator

$$P_* f := \sup_{k\in\mathbb{Z}} \sup_{|g|\le|f|} |P_k g|$$

satisfies the maximal estimate

$$\|P_*\|_{L^p\to L^p} \lesssim 1. \tag{2.13}$$

Theorem 2.14 will be based on a variant of the bootstrap argument discussed in the context of differentiation in lacunary directions in [Nagel et al. 1978]. These ideas were also used to provide L^p bounds for maximal Radon transforms in [Duoandikoetxea and Rubio de Francia 1986]. It was observed by Christ that the argument from [Nagel et al. 1978] can be formulated as an abstract principle, which was useful in many situations [Carbery 1988] and also in the context of dimension-free estimates [Carbery 1986].

Theorem 2.14. *Assume that $(X,\mathcal{B},\mathfrak{m})$ is a σ-finite measure space endowed with a sequence of linear operators $(S_j)_{j\in\mathbb{Z}}$ satisfying* (1.5) *and* (1.6). *Given parameters $1 \le q_0 < q_1 \le 2$, let $(A_k)_{k\in\mathbb{Z}}$ be a sequence of positive linear operators such that $\sup_{k\in\mathbb{Z}}\|A_k\|_{L^{q_0}\to L^{q_0}} \lesssim 1$. Suppose that the maximal function P_* satisfies* (2.13) *with $p = q_1$ and*

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}|(A_k-P_k)S_{k+j}f|^2\Big)^{\frac12}\Big\|_{L^2} \le a_j\|f\|_{L^2}, \quad f\in L^2(X), \tag{2.15}$$

for some positive numbers $(a_j)_{j\in\mathbb{Z}}$ satisfying

$$\boldsymbol{a} := \sum_{j\in\mathbb{Z}} a_j^{\frac{q_1-q_0}{2-q_0}} < \infty.$$

Then for all $f \in L^p(X)$ *with* $p = q_1$ *we have*

$$\Big\| \Big(\sum_{k \in \mathbb{Z}} |(A_k - P_k) f|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p} \lesssim (1 + \boldsymbol{a}^{\frac{2}{q_1}}) \| f \|_{L^p}. \tag{2.16}$$

In particular

$$\| A_* \|_{L^p \to L^p} \lesssim 1 + \boldsymbol{a}^{\frac{2}{q_1}}. \tag{2.17}$$

If in addition we have the jump inequality

$$J_2^p((P_k f)_{k \in \mathbb{Z}} : X \to \mathbb{C}) \lesssim \| f \|_{L^p}, \tag{2.18}$$

then also

$$J_2^p((A_k f)_{k \in \mathbb{Z}} : X \to \mathbb{C}) \lesssim (1 + \boldsymbol{a}^{\frac{2}{q_1}}) \| f \|_{L^p}. \tag{2.19}$$

If A_k *and* P_k *are convolution operators on an abelian group* $\mathbb{G}$, *all these implications also hold for* $q_1 \le p \le q_1'$, *and we have the vector-valued estimate*

$$\Big\| \Big(\sum_{k \in \mathbb{Z}} |A_k f_k|^r \Big)^{\frac{1}{r}} \Big\|_{L^p} \lesssim \Big\| \Big(\sum_{k \in \mathbb{Z}} |f_k|^r \Big)^{\frac{1}{r}} \Big\|_{L^p} \tag{2.20}$$

in the same range $q_1 \le p \le q_1'$ *for all* $1 \le r \le \infty$.

A few remarks concerning the assumptions in Theorem 2.14 are in order. In applications it is usually not difficult to verify the assumption (2.15). For general operators the most reasonable and efficient way is to apply TT^* methods. However, for convolution operators on $\mathbb{G}$ assumption (2.15) can be verified using Fourier transform methods, which may be simpler than TT^* methods. Let us explain the second approach more precisely when $\mathbb{G} = \mathbb{R}^d$. We first have to fix some terminology.

Let A be a $d \times d$ real matrix whose eigenvalues have positive real part. We set

$$t^A := \exp(A \log t) \quad \text{for } t > 0. \tag{2.21}$$

Let $\mathfrak{q}$ be a smooth A-homogeneous quasinorm on $\mathbb{R}^d$, that is, $\mathfrak{q} : \mathbb{R}^d \to [0, \infty)$ is a continuous function, smooth on $\mathbb{R}^d \setminus \{0\}$, and such that

(1) $\mathfrak{q}(x) = 0 \iff x = 0$;

(2) there is $C \ge 1$ such that for all $x, y \in \mathbb{R}^d$ we have $\mathfrak{q}(x + y) \le C(\mathfrak{q}(x) + \mathfrak{q}(y))$;

(3) $\mathfrak{q}(t^A x) = t\mathfrak{q}(x)$ for all $t > 0$ and $x \in \mathbb{R}^d$.

Let also $\mathfrak{q}_*$ be a smooth (away from 0) A^*-homogeneous quasinorm, where A^* is the adjoint matrix to A. We only have to find a sequence of Littlewood–Paley projections associated with the quasinorm $\mathfrak{q}_*$. For this purpose let $\phi_0 : [0, \infty) \to [0, \infty)$ be a smooth function such that $0 \le \phi_0 \le \mathbf{1}_{[1/2, 2]}$ and its dilates $\phi_j(x) := \phi_0(2^j x)$ satisfy

$$\sum_{j \in \mathbb{Z}} \phi_j^2 = \mathbf{1}_{(0, \infty)}. \tag{2.22}$$

For each $j \in \mathbb{Z}$ we define the Littlewood–Paley operator $\widetilde{S}_j$ such that $\widehat{\widetilde{S}_j f} = \psi_j \hat{f}$ corresponds to a smooth function $\psi_j(\xi) := \phi_j(\mathrm{q}_*(\xi))$ on $\mathbb{R}^d$. By (2.22) we see that (1.5) holds for $S_j = \widetilde{S}_j^2$. Moreover, by [Rivière 1971, Theorem II.1.5] we obtain the Littlewood–Paley inequality (1.6) for the operators S_j and $\widetilde{S}_j$.

If $(\Phi_t : t > 0)$ is a family of Schwartz functions such that $\widehat{\Phi}_t(\xi) = \widehat{\Phi}(t\mathrm{q}_*(\xi))$, where Φ is a nonnegative Schwartz function on $\mathbb{R}^d$ with integral 1, then by [Jones et al. 2008, Theorem 1.1] we know that for every $1 < p < \infty$ we have

$$J_2^p((\Phi_{2^k} * f)_{k\in\mathbb{Z}} : \mathbb{R}^d \to \mathbb{C}) \lesssim \|f\|_{L^p}, \quad f \in L^p(\mathbb{R}^d). \tag{2.23}$$

The maximal version of inequality (2.23) has been known for a long time and follows from the Hardy–Littlewood maximal theorem [Stein 1993]. Hence taking $P_k f = \Phi_{2^k} * f$ for $k \in \mathbb{Z}$, we may assume that (2.18) is verified.

Suppose now we have a family $(A_k)_{k\in\mathbb{Z}}$ of convolution operators $A_k f = \mu_{2^k} * f$ corresponding to a family of probability measures $(\mu_t : t > 0)$ on $\mathbb{R}^d$ such that

$$|\hat{\mu}_t(\xi) - \hat{\mu}_t(0)| \le \omega(t\mathrm{q}_*(\xi)) \qquad \text{if } t\mathrm{q}_*(\xi) \le 1, \tag{2.24}$$

$$|\hat{\mu}_t(\xi)| \le \omega((t\mathrm{q}_*(\xi))^{-1}) \quad \text{if } t\mathrm{q}_*(\xi) \ge 1 \tag{2.25}$$

for some modulus of continuity ω.

Theorem 2.14, taking into account all the facts mentioned above, yields

$$J_2^p((\mu_{2^k} * f)_{k\in\mathbb{Z}} : \mathbb{R}^d \to \mathbb{C}) \lesssim \|f\|_{L^p}, \quad f \in L^p(\mathbb{R}^d), \tag{2.26}$$

for $p = q_1$ and $q_0 = 1$ as long as

$$\boldsymbol{a} = \sum_{j\in\mathbb{Z}} \omega(2^{-|j|})^{\frac{q_1-q_0}{2-q_0}} < \infty,$$

since (2.15) can be easily verified with $a_j = \omega(2^{-|j|})$ using (2.24), (2.25) and the properties of S_j and Φ.

Proof of Theorem 2.14. We begin with the proof of (2.16). If $q_1 = 2$ then we use (1.5) and (2.15) and we are done. We now assume that $q_1 < 2$. By the monotone convergence theorem it suffices to consider only finitely many $M_k := A_k - P_k$'s in (2.16), let us say those with $|k| \le K$. Restrict all summations and suprema to $|k| \le K$ and let B be the smallest implicit constant for which (2.16) holds with $p = q_1$. In view of the qualitative boundedness hypothesis we obtain $B < \infty$, but the bound may depend on K. Our aim is to show that $B \lesssim 1 + \boldsymbol{a}^{2/q_1}$. There is nothing to do if $B \lesssim 1$. Therefore, we will assume that $B \gtrsim 1$, so by (1.5), (2.13) and (2.11) with $L = 1$ and $M_{k,0} := M_k$, we obtain

$$\Big\|\Big(\sum_{|k|\le K} |M_k f|^2\Big)^{\frac12}\Big\|_{L^p} \le \sum_{j\in\mathbb{Z}} \Big\|\Big(\sum_{|k|\le K} |M_k S_{k+j} f|^2\Big)^{\frac12}\Big\|_{L^p} \lesssim (1 + \|M_*\|_{L^p\to L^p}^{\frac{2-q_1}{2}} \boldsymbol{a}) \|f\|_{L^p}.$$

By positivity we have $|A_* f| \le \sup_{|k|\le K} A_k |f|$ and consequently we obtain

$$|A_* f| \le \sup_{|k|\le K} A_k |f| \le \sup_{|k|\le K} P_k |f| + \Big(\sum_{|k|\le K} |M_k |f||^2\Big)^{\frac12}. \tag{2.27}$$

By (2.27) and (2.13) we get

$$\|M_*\|_{L^p\to L^p} \le \|P_*\|_{L^p\to L^p} + \|A_*\|_{L^p\to L^p} \le 2\|P_*\|_{L^p\to L^p} + B \lesssim 1 + B.$$

Taking into account these inequalities we have

$$\Big\|\Big(\sum_{|k|\le K}|(A_k-P_k)f|^2\Big)^{\frac{1}{2}}\Big\|_{L^p} = \Big\|\Big(\sum_{|k|\le K}|M_k f|^2\Big)^{\frac{1}{2}}\Big\|_{L^p} \lesssim (1+\boldsymbol{a}(1+B)^{\frac{2-q_1}{2}})\|f\|_{L^p}.$$

Taking the supremum over f gives

$$B \lesssim 1 + \boldsymbol{a}(1+B)^{\frac{2-q_1}{2}} \lesssim (1+\boldsymbol{a})B^{\frac{2-q_1}{2}},$$

since we have assumed $B \gtrsim 1$, and the conclusion (2.16) follows.

Once (2.16) is proven, in view of (2.27) we immediately obtain (2.17). In a similar way, if (2.18) holds, we deduce (2.19) from (2.16). Indeed,

$$\begin{aligned} J_2^p((A_k f)_{k\in\mathbb{Z}}) &\lesssim J_2^p((P_k f)_{k\in\mathbb{Z}}) + J_2^p((M_k f)_{k\in\mathbb{Z}}) \\ &\lesssim \|f\|_{L^p} + \big\|V^2(M_k f : k\in\mathbb{Z})\big\|_{L^p} \\ &\lesssim \|f\|_{L^p} + \Big\|\Big(\sum_{k\in\mathbb{Z}}|M_k f|^2\Big)^{\frac{1}{2}}\Big\|_{L^p}. \end{aligned}$$

In the case of convolution operators we can run the above proof of (2.16) with $p = q_1'$, since in this case Lemma 2.9 tells that (2.11) also holds with $p = q_1'$. Once the estimate (2.16) is known for $p = q_1, q_1'$, by interpolation we extend it to $q_1 \le p \le q_1'$, and all other inequalities follow as before. Finally, the vector-valued estimate (2.20) with $r = \infty$ is equivalent to the maximal estimate by positivity, with $r = 1$ it follows by duality, and with $1 < r < \infty$ by complex interpolation. □

2D. *Long jumps for nonpositive operators.* We now drop the positivity assumption and we will be working with general operators $(B_k)_{k\in\mathbb{Z}}$ acting on $L^1(X) + L^\infty(X)$. This will require some knowledge about the maximal lattice operator B_* defined in (2.29) and about the sum of B_k's over $k \in \mathbb{Z}$. No bootstrap argument seems to be available for nonpositive operators and therefore additional assumptions like (2.30) and (2.32) will be indispensable. The proof of Theorem 2.28 is based on the ideas from [Duoandikoetxea and Rubio de Francia 1986].

Theorem 2.28. *Assume that $(X, \mathcal{B}, \mathfrak{m})$ is a σ-finite measure space endowed with a sequence of linear operators $(S_j)_{j\in\mathbb{Z}}$ satisfying (1.5) and (1.6). Let $1 \le q_0 < q_1 \le 2$ and let $(B_k)_{k\in\mathbb{Z}}$ be a sequence of linear operators commuting with the sequence $(S_j)_{j\in\mathbb{Z}}$ such that $\sup_{k\in\mathbb{Z}}\|B_k\|_{L^{q_0}\to L^{q_0}} \lesssim 1$. Suppose that the maximal lattice operator*

$$B_* f := \sup_{k\in\mathbb{Z}}\sup_{|g|\le|f|}|B_k g| \tag{2.29}$$

satisfies

$$\|B_*\|_{L^{q_1}\to L^{q_1}} \lesssim 1. \tag{2.30}$$

We also assume

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}|B_k S_{k+j} f|^2\Big)^{\frac{1}{2}}\Big\|_{L^2} \le a_j \|f\|_{L^2}, \quad f \in L^2(X), \tag{2.31}$$

for some positive numbers $(a_j)_{j\in\mathbb{Z}}$*:*

(1) *Suppose that* $(B_k)_{k\in\mathbb{Z}}$ *additionally satisfies*

$$\Big\|\sum_{k\in\mathbb{Z}} B_k\Big\|_{L^{q_1}\to L^{q_1}} \lesssim 1. \tag{2.32}$$

Let $P_k := \sum_{j>k} S_j$ *and assume that the jump inequality* (2.18) *holds for the sequence* $(P_k)_{k\in\mathbb{Z}}$ *with* $p = q_1$*. Then for all* $f \in L^p(X)$ *with* $p = q_1$ *we have*

$$J_2^p\Big(\Big(\sum_{j\ge k} B_j f\Big)_{k\in\mathbb{Z}} : X \to \mathbb{C}\Big) \lesssim \Big(\Big\|\sum_{k\in\mathbb{Z}} B_k\Big\|_{L^{q_1}\to L^{q_1}} + \big(\sup_{k\in\mathbb{Z}}\|B_k\|_{L^{q_0}\to L^{q_0}}^{\frac{q_0}{2}\frac{2-q_1}{2-q_0}}\big)\|B_*\|_{L^{q_1}\to L^{q_1}}^{\frac{2-q_1}{2}}\tilde{\boldsymbol{a}}\Big)\|f\|_{L^p}, \tag{2.33}$$

where

$$\tilde{\boldsymbol{a}} := \sum_{j\in\mathbb{Z}}(|j|+1)a_j^{\frac{q_1-q_0}{2-q_0}} < \infty.$$

(2) *Suppose that there is a sequence of self-adjoint linear operators* $(\widetilde{S}_j)_{j\in\mathbb{Z}}$ *such that* $S_j = \widetilde{S}_j^{\,2}$ *for every* $j \in \mathbb{Z}$ *and satisfying* (1.6) *and* (2.31) *with* $\widetilde{S}_{k+j}$ *in place of* S_{k+j}*. Then for every sequence* $(\varepsilon_k)_{k\in\mathbb{Z}}$ *bounded by* 1 *and for all* $f \in L^p(X)$ *with* $p = q_1$ *we have*

$$\Big\|\sum_{k\in\mathbb{Z}} \varepsilon_k B_k f\Big\|_{L^p} \lesssim \big(\sup_{k\in\mathbb{Z}}\|B_k\|_{L^{q_0}\to L^{q_0}}^{\frac{q_0}{2}\frac{2-q_1}{2-q_0}}\big)\|B_*\|_{L^{q_1}\to L^{q_1}}^{\frac{2-q_1}{2}}\boldsymbol{a}\|f\|_{L^p}, \tag{2.34}$$

where $\boldsymbol{a}$ *is as in Theorem 2.14.*

In the case of convolution operators on an abelian group $\mathbb{G}$ *all these implications also hold for* $q_1 \le p \le q_1'$.

In applications in harmonic analysis we will take $B_k = T_{2^k} - T_{2^{k+1}}$ for $k \in \mathbb{Z}$, where T_t is a truncated singular integral operator of convolution type; see (2.3). This class of operators motivates, to a large extent, the assumptions in Theorem 2.28. In many cases they can be verified if we manage to find positive operators A_k such that $|B_k f| \lesssim A_k|f|$ for every $k \in \mathbb{Z}$ and $f \in L^1(X) + L^\infty(X)$. In practice, A_k is an averaging operator. We shall illustrate this more precisely by appealing to the discussion after Theorem 2.14.

Suppose that $(B_k)_{k\in\mathbb{Z}}$ is a family of convolution operators $B_k f = \sigma_{2^k} * f$ corresponding to a family of finite measures $(\sigma_t : t > 0)$ on $\mathbb{R}^d$ such that $\sup_{t>0}\|\sigma_t\| < \infty$ and for every $k \in \mathbb{Z}$ and $t \in [2^k, 2^{k+1}]$ we have

$$|\hat{\sigma}_t(\xi)| \le \omega(2^k \mathrm{q}_*(\xi)) \qquad \text{if } 2^k \mathrm{q}_*(\xi) \le 1, \tag{2.35}$$

$$|\hat{\sigma}_t(\xi)| \le \omega((2^k \mathrm{q}_*(\xi))^{-1}) \quad \text{if } 2^k \mathrm{q}_*(\xi) \ge 1 \tag{2.36}$$

for some modulus of continuity ω. Additionally, we assume that $|\sigma_{2^k}| \lesssim \mu_{2^k}$ for some family of finite positive measures $(\mu_t : t > 0)$ on $\mathbb{R}^d$ such that $\sup_{t>0}\|\mu_t\| < \infty$ and satisfying (2.24) and (2.25). In view of these assumptions and Theorem 2.14 we see that condition (2.30) holds, since $|B_k f| \lesssim A_k|f|$, where $A_k f = \mu_{2^k} * f$. Therefore,

$$\Big\|\sum_{k\in\mathbb{Z}} B_k f\Big\|_{L^p} \lesssim \boldsymbol{a}\|f\|_{L^p}$$

implies (2.32) with $p = q_1$ and $q_0 = 1$, provided that

$$\boldsymbol{a} = \sum_{j\in\mathbb{Z}} \omega(2^{-|j|})^{\frac{q_1-q_0}{2-q_0}} < \infty,$$

since (2.31) can be verified with $a_j = \omega(2^{-|j|})$ using (2.35), (2.36), and the properties of $\widetilde{S}_j$ associated with (2.22). Having proven (2.30) and (2.32) we see that (2.33) holds for the operators $B_k f = \sigma_{2^k} * f$ with $p = q_1$ and $q_0 = 1$ as long as

$$\tilde{\boldsymbol{a}} = \sum_{j\in\mathbb{Z}} (|j|+1)\omega(2^{-|j|})^{\frac{q_1-q_0}{2-q_0}} < \infty.$$

Proof of Theorem 2.28. In order to prove inequality (2.33) we employ the decomposition

$$\sum_{j\geq k} B_j = P_k \sum_{j\in\mathbb{Z}} B_j - \sum_{l>0}\sum_{j<0} S_{k+l}B_{k+j} + \sum_{l\leq 0}\sum_{j\geq 0} S_{k+l}B_{k+j}; \tag{2.37}$$

see [Duoandikoetxea and Rubio de Francia 1986, p. 548]. The J_2^p quasiseminorm of the first term on the right-hand side in (2.37) with $p = q_1$ is bounded, due to (2.18), and (2.32), which ensures boundedness of the operator $\sum_{j\in\mathbb{Z}} B_j$.

The estimates for the second and the third terms are similar and we only consider the last term. We take the ℓ^2 norm with respect to the parameter k and estimate

$$\begin{aligned}
J_2^p&\Big(\Big(\sum_{l\leq 0}\sum_{j\geq 0} B_{k+j}S_{k+l}f\Big)_{k\in\mathbb{Z}} : X \to \mathbb{C}\Big) \\
&\leq \Big\|\Big(\sum_{k\in\mathbb{Z}}\Big|\sum_{l\leq 0}\sum_{j\geq 0} B_{k+j}S_{k+l}f\Big|^2\Big)^{\frac{1}{2}}\Big\|_{L^p} \\
&= \Big\|\Big(\sum_{k\in\mathbb{Z}}\Big|\sum_{m\geq 0}\sum_{n=k-m}^{k} B_{n+m}S_n f\Big|^2\Big)^{\frac{1}{2}}\Big\|_{L^p} \\
&\leq \sum_{m\geq 0}\Big\|\Big(\sum_{k\in\mathbb{Z}}\Big|\sum_{n=k-m}^{k} B_{n+m}S_n f\Big|^2\Big)^{\frac{1}{2}}\Big\|_{L^p} && \text{(by the triangle inequality)} \\
&\leq \sum_{m\geq 0}(m+1)^{\frac{1}{2}}\Big\|\Big(\sum_{k\in\mathbb{Z}}\sum_{n=k-m}^{k} |B_{n+m}S_n f|^2\Big)^{\frac{1}{2}}\Big\|_{L^p} && \text{(by Hölder's inequality)} \\
&= \sum_{m\geq 0}(m+1)\Big\|\Big(\sum_{n\in\mathbb{Z}}|B_{n+m}S_n f|^2\Big)^{\frac{1}{2}}\Big\|_{L^p}.
\end{aligned}$$

By (2.11), with $L = 1$ and $M_{k,0} := B_k$, we obtain

$$\sum_{j\in\mathbb{Z}}(|j|+1)\Big\|\Big(\sum_{k\in\mathbb{Z}}|B_k S_{k+j} f|^2\Big)^{\frac{1}{2}}\Big\|_{L^p} \lesssim \big(\sup_{k\in\mathbb{Z}}\|B_k\|_{L^{q_0}\to L^{q_0}}^{\frac{q_0}{2}\frac{2-q_1}{2-q_0}}\big)\|B_*\|_{L^{q_1}\to L^{q_1}}^{\frac{2-q_1}{2}}\tilde{\boldsymbol{a}}\|f\|_{L^p}.$$

To prove the second part observe that for a sequence of functions $(f_j)_{j\in\mathbb{Z}}$ in $L^p(X;\ell^2(\mathbb{Z}))$ we have the inequality

$$\Big\|\sum_{j\in\mathbb{Z}}\widetilde{S}_j f_j\Big\|_{L^p} \lesssim \Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^2\Big)^{\frac{1}{2}}\Big\|_{L^p}, \tag{2.38}$$

which is the dual version of inequality (1.6) for the sequence $(\widetilde{S}_j)_{j\in\mathbb{Z}}$. To prove (2.34) we will use (1.5) and (2.38). Indeed,

$$\begin{aligned}
\Big\|\sum_{k\in\mathbb{Z}}\varepsilon_k B_k f\Big\|_{L^p} &\le \sum_{j\in\mathbb{Z}}\Big\|\sum_{k\in\mathbb{Z}}\varepsilon_k B_k S_{k+j} f\Big\|_{L^p} && \text{(by (1.5))}\\
&= \sum_{j\in\mathbb{Z}}\Big\|\sum_{k\in\mathbb{Z}}\widetilde{S}_{k+j}(\varepsilon_k B_k \widetilde{S}_{k+j} f)\Big\|_{L^p} && \text{(since } S_j = \widetilde{S}_j^2)\\
&\lesssim \sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|B_k \widetilde{S}_{k+j} f|^2\Big)^{\frac{1}{2}}\Big\|_{L^p} && \text{(by (2.38))}\\
&\lesssim \big(\sup_{k\in\mathbb{Z}}\|B_k\|_{L^{q_0}\to L^{q_0}}^{\frac{q_0}{2}\frac{2-q_1}{2-q_0}}\big)\|B_*\|_{L^{q_1}\to L^{q_1}}^{\frac{2-q_1}{2}}\boldsymbol{a}\|f\|_{L^p},
\end{aligned}$$

where in the last step we have used Lemma 2.9, with $L = 1$ and $M_{k,0} := B_k$. □

2E. *Short variations.* We will work with a sequence of linear operators $(A_t)_{t\in\mathbb{U}}$ (not necessarily positive) acting on $L^1(X) + L^\infty(X)$. However, positive operators will be distinguished in our proof and in this case we can also proceed as before using some bootstrap arguments.

For every $k \in \mathbb{Z}$ and $t \in [2^k, 2^{k+1}]$ we will use the notation

$$\Delta((A_s)_{s\in\mathbb{I}})_t f := \Delta(A_t) f := A_t f - A_{2^k} f.$$

Theorem 2.39. *Assume that $(X, \mathcal{B}, \mathfrak{m})$ is a σ-finite measure space endowed with a sequence of linear operators $(S_j)_{j\in\mathbb{Z}}$ satisfying* (1.5) *and* (1.6). *Let $(A_t)_{t\in\mathbb{U}}$ be a family of linear operators such that the square function estimate*

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\sum_{m=0}^{2^l-1}|(A_{2^k+2^{k-l}(m+1)} - A_{2^k+2^{k-l}m})S_{j+k} f|^2\Big)^{\frac{1}{2}}\Big\|_{L^2} \le 2^{-\frac{l}{2}}a_{j,l}\|f\|_{L^2} \tag{2.40}$$

holds for all $j \in \mathbb{Z}$ and $l \in \mathbb{N}$ with some numbers $a_{j,l} \ge 0$ such that for every $0 < \varepsilon < \rho$ we have

$$\sum_{l\ge 0}\sum_{j\in\mathbb{Z}} 2^{-\varepsilon l}a_{j,l}^{\rho} < \infty. \tag{2.41}$$

(1) *Let $1 < q_0 < 2$ and $4 < q_\infty < \infty$, and suppose that for each $q_0 \le p \le q_\infty$ the vector-valued estimate*

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} |A_{2^k(1+t)} f_k|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p} \lesssim \Big\| \Big(\sum_{k\in\mathbb{Z}} |f_k|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p} \tag{2.42}$$

holds uniformly in $t \in \mathbb{U} \cap [0,1]$. Then for each

$$\frac{3}{1+1/q_0} < p < \frac{4}{1+2/q_\infty}$$

we have

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} V^2(A_t f : t \in [2^k, 2^{k+1}] \cap \mathbb{U})^2 \Big)^{\frac{1}{2}} \Big\|_{L^p} \lesssim \|f\|_{L^p}, \tag{2.43}$$

and for each $4 \le p < q_\infty$ and

$$r > \frac{p}{2} \frac{q_\infty - 2}{q_\infty - p}$$

we have

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} V^r(A_t f : t \in [2^k, 2^{k+1}] \cap \mathbb{U})^r \Big)^{\frac{1}{r}} \Big\|_{L^p} \lesssim \|f\|_{L^p} \tag{2.44}$$

for all $f \in L^p(X)$.

(2) *Let $q_0 \in [1,2)$ and $\alpha \in [0,1]$ be such that $\alpha q_0 \le 1$. Suppose that we have the operator norm Hölder-type condition*

$$\|A_{t+h} - A_t\|_{L^{q_0} \to L^{q_0}} \lesssim \Big(\frac{h}{t}\Big)^{\alpha}, \quad t, t+h \in \mathbb{U} \text{ and } h \in (0,1]. \tag{2.45}$$

Then for every exponent q_1 satisfying

$$q_0 \le 2 - \frac{2-q_0}{2-\alpha q_0} < q_1 \le 2 \tag{2.46}$$

and such that

$$\|\Delta((A_s)_{s\in\mathbb{U}})_{*,\mathbb{U}}\|_{L^{q_1} \to L^{q_1}} \lesssim 1 \tag{2.47}$$

we have for all $f \in L^p(X)$ with $p = q_1$ that the estimate (2.43) *holds with the implicit constant which is a constant multiple of*

$$\boldsymbol{a} := \sum_{l\ge 0} \sum_{j\in\mathbb{Z}} 2^{-(\alpha \frac{2-q_1}{2-q_0} \frac{q_0}{2} + \frac{1}{2} \frac{q_1-q_0}{2-q_0} - \frac{2-q_1}{2-q_0} \frac{1}{2}) l} a_{j,l}^{\frac{q_1-q_0}{2-q_0}} < \infty.$$

(3) *Moreover, if $(A_t)_{t\in\mathbb{U}}$ is a family of positive linear operators, then the condition* (2.47) *may be replaced by a weaker condition*

$$\|A_{*,\mathbb{D}}\|_{L^{q_1} \to L^{q_1}} \lesssim 1 \tag{2.48}$$

and the estimate (2.43) *holds as well with the implicit constant which is a constant multiple of $1 + \boldsymbol{a}^{2/q_1}$.*

In the case of convolution operators on an abelian group $\mathbb{G}$ the implication from (2.48) *to* (2.43) *also holds with p replaced by p'.*

Theorem 2.39 combined with the results formulated in the previous two paragraphs for dyadic scales will allow us to control, in view of (2.2), the cases for general scales. The first part of Theorem 2.39 gives (2.43) in a restricted range of p's. If one asks for a larger range, a smoothness condition like in (2.45) must be assumed. Inequality (2.45) combined with maximal estimate (2.47) gives larger range of p's in (2.43). If we work with a family of positive operators the condition (2.47) may be relaxed to (2.48) by some bootstrap argument. In the context of discussion after Theorem 2.14 and Theorem 2.28 let us look at a particular situation of (2) and prove (2.43).

Suppose that $(A_t)_{t>0}$ is a family of convolution operators $A_t f = \sigma_t * f$ corresponding to a family of finite measures $(\sigma_t : t > 0)$ on $\mathbb{R}^d$ such that $\sup_{t>0}\|\sigma_t\| < \infty$ and satisfying (2.35) and (2.36). We assume that $|\sigma_t| \lesssim \mu_t$ for some family of finite positive measures $(\mu_t : t > 0)$ on $\mathbb{R}^d$ such that $\sup_{t>0}\|\mu_t\| < \infty$ and satisfying (2.24) and (2.25) to make sure that condition (2.47) holds. Additionally, let us assume that (2.45) holds with $\alpha = 1$ and $q_0 = 1, 2$. By Plancherel's theorem, (2.35) and (2.36) we obtain

$$\|(A_{2^k+2^{k-l}(m+1)} - A_{2^k+2^{k-l}m})S_{j+k}f\|_{L^2} \lesssim \omega(2^{-|j|})\|S_{j+k}f\|_{L^2}. \tag{2.49}$$

Thus (2.45) with $q_0 = 2,\ t = 2^k + 2^{k-l}m,\ h = 2^{k-l}$ combined with (2.49) imply

$$\|(A_{2^k+2^{k-l}(m+1)} - A_{2^k+2^{k-l}m})S_{j+k}f\|_{L^2} \lesssim \min(2^{-l}, \omega(2^{-|j|}))\|S_{j+k}f\|_{L^2}. \tag{2.50}$$

Consequently (2.40) holds with $a_{j,l} = \min\{1, 2^l\omega(2^{-|j|})\}$ and Theorem 2.39 gives the desired conclusion as long as

$$\boldsymbol{a} = \sum_{l\geq 0}\sum_{j\in\mathbb{Z}} 2^{-\frac{(q_1-1)l}{2}}(\min\{1, 2^l\omega(2^{-|j|})\})^{q_1-1} < \infty.$$

Proof of Theorem 2.39(1). By Minkowski's inequality for $2 \le s \le q_\infty < \infty$ we have

$$\begin{aligned}
&\bigg\|\bigg(\sum_{k\in\mathbb{Z}}\sum_{m=0}^{2^l-1}|(A_{2^k+2^{k-l}(m+1)} - A_{2^k+2^{k-l}m})f_k|^s\bigg)^{\frac{1}{s}}\bigg\|_{L^{q_\infty}}^s \\
&\qquad = \bigg\|\sum_{m=0}^{2^l-1}\sum_{k\in\mathbb{Z}}|(A_{2^k+2^{k-l}(m+1)} - A_{2^k+2^{k-l}m})f_k|^s\bigg\|_{L^{q_\infty/s}} \\
&\qquad \le \sum_{m=0}^{2^l-1}\bigg\|\sum_{k\in\mathbb{Z}}|(A_{2^k+2^{k-l}(m+1)} - A_{2^k+2^{k-l}m})f_k|^s\bigg\|_{L^{q_\infty/s}} \\
&\qquad \le 2^l \sup_{0\le m<2^l}\bigg\|\bigg(\sum_{k\in\mathbb{Z}}|(A_{2^k+2^{k-l}(m+1)} - A_{2^k+2^{k-l}m})f_k|^s\bigg)^{\frac{1}{s}}\bigg\|_{L^{q_\infty}}^s \\
&\qquad \le 2^{l+s} \sup_{0\le m\le 2^l}\bigg\|\bigg(\sum_{k\in\mathbb{Z}}|A_{2^k+2^{k-l}m}f_k|^2\bigg)^{\frac{1}{2}}\bigg\|_{L^{q_\infty}}^s \\
&\qquad \lesssim 2^{l+s}\bigg\|\bigg(\sum_{k\in\mathbb{Z}}|f_k|^2\bigg)^{\frac{1}{2}}\bigg\|_{L^{q_\infty}}^s,
\end{aligned}$$

where we have applied (2.42) in the last step. Using this with $f_k = S_{j+k} f$ and applying (1.6) we obtain

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} \sum_{m=0}^{2^l-1} |(A_{2^k+2^{k-l}(m+1)} - A_{2^k+2^{k-l}m}) S_{j+k} f|^s \Big)^{\frac{1}{s}} \Big\|_{L^{q_\infty}} \lesssim 2^{\frac{l}{s}} \|f\|_{L^{q_\infty}}$$

for all $2 \le s \le q_\infty < \infty$. By interpolation with (2.40) we obtain

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} \sum_{m=0}^{2^l-1} |(A_{2^k+2^{k-l}(m+1)} - A_{2^k+2^{k-l}m}) S_{j+k} f|^r \Big)^{\frac{1}{r}} \Big\|_{L^p} \lesssim 2^{-\frac{\theta l}{2} + \frac{(1-\theta)l}{s}} a_{j,l}^{\theta} \|f\|_{L^p}, \tag{2.51}$$

where $0 < \theta \le 1$ and

$$\frac{1}{r} = \frac{\theta}{2} + \frac{1-\theta}{s} \quad \text{and} \quad \frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{q_\infty},$$

so

$$\theta = \frac{2}{p} \frac{q_\infty - p}{q_\infty - 2}.$$

By Lemma 2.5, or more precisely by an analogue of inequality (2.4) with ℓ^r norm in place of ℓ^2 norm, and by (2.51) we obtain

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} V^r(A_t f : t \in [2^k, 2^{k+1}] \cap \mathbb{U})^r \Big)^{\frac{1}{r}} \Big\|_{L^p} \lesssim \sum_{l\ge 0} \sum_{j\in\mathbb{Z}} 2^{-\frac{\theta l}{2} + \frac{(1-\theta)l}{s}} a_{j,l}^{\theta} \|f\|_{L^p}. \tag{2.52}$$

In view of (2.41) with $\varepsilon = \theta/2 - (1-\theta)/s$ and $\rho = \theta$ this estimate is summable in l and j, provided that $-\theta/2 + (1-\theta)/s < 0$. In particular, for

$$2 \le p < \frac{4}{1 + 2/q_\infty}$$

we use $s = 2$. For $4 \le p < q_\infty$ we use

$$s > \frac{q_\infty(p-2)}{q_\infty - p}$$

and then

$$r > \frac{p}{2} \frac{q_\infty - 2}{q_\infty - p}.$$

For $q_0 \in (1, 2)$ by Minkowski's inequality and (2.42) we have

$$\begin{aligned} &\Big\| \Big(\sum_{k\in\mathbb{Z}} \sum_{m=0}^{2^l-1} |(A_{2^k+2^{k-l}(m+1)} - A_{2^k+2^{k-l}m}) f_k|^2 \Big)^{\frac{1}{2}} \Big\|_{L^{q_0}} \\ &\qquad \le \sum_{m=0}^{2^l-1} \Big\| \Big(\sum_{k\in\mathbb{Z}} |(A_{2^k+2^{k-l}(m+1)} - A_{2^k+2^{k-l}m}) f_k|^2 \Big)^{\frac{1}{2}} \Big\|_{L^{q_0}} \\ &\qquad \le 2^{l+1} \sup_{0\le m\le 2^l} \Big\| \Big(\sum_{k\in\mathbb{Z}} |A_{2^k+2^{k-l}m} f_k|^2 \Big)^{\frac{1}{2}} \Big\|_{L^{q_0}} \lesssim 2^l \Big\| \Big(\sum_{k\in\mathbb{Z}} |f_k|^2 \Big)^{\frac{1}{2}} \Big\|_{L^{q_0}}. \end{aligned}$$

Substituting $f_k = S_{j+k} f$, applying (1.6), and interpolating with (2.40) we obtain

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} \sum_{m=0}^{2^l-1} |(A_{2^k+2^{k-l}(m+1)} - A_{2^k+2^{k-l}m}) S_{j+k} f|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p} \lesssim 2^{-\frac{\theta l}{2}+(1-\theta)l} a_{j,l}^{\theta} \|f\|_{L^p}, \tag{2.53}$$

with

$$\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{q_0},$$

for $0<\theta<1$. Hence

$$\theta = \frac{2}{p}\frac{p-q_0}{2-q_0}$$

and in view of (2.41) with $\varepsilon = \theta/2-(1-\theta)$ and $\rho=\theta$ this estimate is summable in l and j, provided that $-\theta/2+(1-\theta)<0$. The conclusion again follows from Lemma 2.5 and (2.53) like in (2.52) with

$$\frac{3}{1+1/q_0} < p \le 2. \qquad \square$$

Proof of Theorem 2.39(2)–(3). By the monotone convergence theorem we may restrict k in (2.43) to $|k|\le K_0$ and parameters t to the set

$$\mathbb{U}_{L_0}^k := \{u/2^{L_0} : u \in \mathbb{N} \text{ and } 2^{k+L_0} \le u \le 2^{k+L_0+1}\}$$

for some $K_0\in\mathbb{N}$ and $L_0\in\mathbb{Z}$ as long as we obtain estimates independent of K_0 and L_0. Fix K_0, L_0 and let $\mathbb{I} := \bigcup_{|k|\le K_0} \mathbb{U}_{L_0}^k$. Let q_1 satisfy (2.46); then invoking (1.5) and (2.11), with $L=2^l$, we obtain

$$\begin{aligned}
&\Big\| \Big(\sum_{|k|\le K_0} \sum_{m=0}^{2^l-1} |(A_{2^k+2^{k-l}(m+1)} - A_{2^k+2^{k-l}m}) f|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p} \\
&\quad\lesssim 2^{\frac{2-q_1}{2-q_0}\frac{l}{2}} \Big(\sup_{\substack{|k|\le K_0\\ 0\le m<2^l}} \|A_{2^k+2^{k-l}(m+1)} - A_{2^k+2^{k-l}m}\|_{L^{q_0}\to L^{q_0}}^{\frac{2-q_1}{2-q_0}\frac{q_0}{2}} \Big) \|\Delta((A_s)_{s\in\mathbb{U}})_{*,\mathbb{I}}\|_{L^{q_1}\to L^{q_1}}^{\frac{2-q_1}{2}} \\
&\qquad\qquad \cdot \Big(\sum_{j\in\mathbb{Z}} (2^{-\frac{l}{2}} a_{j,l})^{\frac{q_1-q_0}{2-q_0}} \Big) \|f\|_{L^p} \\
&\quad\lesssim 2^{\frac{2-q_1}{2-q_0}\frac{l}{2}} \big((2^{-\alpha l})^{\frac{2-q_1}{2-q_0}\frac{q_0}{2}} \big) \|\Delta((A_s)_{s\in\mathbb{U}})_{*,\mathbb{I}}\|_{L^{q_1}\to L^{q_1}}^{\frac{2-q_1}{2}} 2^{-\frac{l}{2}\frac{q_1-q_0}{2-q_0}} \sum_{j\in\mathbb{Z}} a_{j,l}^{\frac{q_1-q_0}{2-q_0}} \|f\|_{L^p}.
\end{aligned}$$

In order for the right-hand side to be summable in l we need

$$\frac{1}{2}\frac{2-q_1}{2-q_0} - \alpha\frac{2-q_1}{2-q_0}\frac{q_0}{2} - \frac{1}{2}\frac{q_1-q_0}{2-q_0} < 0 \iff (2-q_1) - \alpha(2-q_1)q_0 - (q_1-q_0) < 0.$$

It suffices to ensure

$$(2-q_1)(1-\alpha q_0) - (q_1-q_0) < 0 \iff q_1 > \frac{2(1-\alpha q_0)+q_0}{2-\alpha q_0} = 2 - \frac{2-q_0}{2-\alpha q_0},$$

and this is our hypothesis (2.46). Hence under this condition by Lemma 2.5 we conclude for general operators that

$$\Big\|\Big(\sum_{|k|\le K_0} V^2(A_t f : t\in\mathbb{U}^k_{L_0})^2\Big)^{\frac12}\Big\|_{L^p} \lesssim \sum_{l=0}^{K_0+L_0}\Big\|\Big(\sum_{|k|\le K_0}\sum_{m=0}^{2^l-1}|(A_{2^k+2^{k-l}(m+1)}-A_{2^k+2^{k-l}m})f|^2\Big)^{\frac12}\Big\|_{L^p}$$
$$\lesssim \|\Delta((A_s)_{s\in\mathbb{U}})_{*,\mathbb{I}}\|^{\frac{2-q_1}{2}}_{L^{q_1}\to L^{q_1}}\boldsymbol{a}\|f\|_{L^p}, \tag{2.54}$$

as desired. For positive operators crude estimates and interpolation show that

$$B := \|A_{*,\mathbb{I}}\|_{L^p\to L^p} < \infty,$$

with $p = q_1$, since $\mathbb{I}$ is finite. Note that

$$\sup_{t\in\mathbb{I}}|A_t f(x)| \le \sup_{t\in\mathbb{D}}|A_t f(x)| + \Big(\sum_{k\in\mathbb{Z}}\sup_{t\in[2^k,2^{k+1})\cap\mathbb{I}}|(A_t - A_{2^k})f(x)|^2\Big)^{\frac12}. \tag{2.55}$$

Therefore, appealing to (2.55), (2.48) and (2.54) we obtain by a bootstrap argument that $B \lesssim 1 + B^{(2-q_1)/2}\boldsymbol{a}$, since

$$\|\Delta((A_s)_{s\in\mathbb{U}})_{*,\mathbb{I}}\|^{\frac{2-q_1}{2}}_{L^{q_1}\to L^{q_1}} \lesssim B^{\frac{2-q_1}{2}}.$$

Hence, $B \lesssim 1+\boldsymbol{a}^{2/q_1}$. In particular, the estimate (2.54) becomes uniform in $\mathbb{I}\subset\mathbb{U}$, and this simultaneously implies (2.43).

In the case of convolution operators we may replace $p = q_1$ by $p = q_1'$ in Lemma 2.9 and all subsequent arguments. □

3. Applications

3A. ***Dimension-free estimates for jumps in the continuous setting.*** We begin by providing dimension-free endpoint estimates, for $r = 2$, in the main results of [Bourgain et al. 2018]. Let $G\subset\mathbb{R}^d$ be a symmetric convex body. By the definition of the averaging operator (1.8) we have $\mathcal{A}^G_t\widetilde{U} = \widetilde{U}\mathcal{A}^{U(G)}_t$, where $\widetilde{U}f := f\circ U$ is the composition operator with an invertible linear map $U:\mathbb{R}^d\to\mathbb{R}^d$. It follows that all estimates in Section 1 are not affected if G is replaced by $U(G)$.

By [Bourgain 1986a], after replacing G by its image under a suitable invertible linear transformation, we may assume that the normalized characteristic function $\mu := |G|^{-1}\mathbf{1}_G$ satisfies

$$|\hat\mu(\xi)| \le C|\xi|^{-1}, \tag{3.1}$$
$$|\hat\mu(\xi)-1| \le C|\xi|, \tag{3.2}$$
$$|\langle\xi,\nabla\hat\mu(\xi)\rangle| \le C, \tag{3.3}$$

with the constant C independent of the dimension. In [Bourgain 1986a] these estimates were proved with $|L(G)\xi|$ in place of $|\xi|$ on the right-hand side, where $L(G)$ is the isotropic constant corresponding to G. The above form is obtained by rescaling.

Then $\mathcal{A}_t := \mathcal{A}_t^G$ is the convolution operator with μ_t and $\hat{\mu}_t(\xi) = \hat{\mu}(t\xi)$. The Poisson semigroup is defined by

$$\widehat{\mathcal{P}_t f}(\xi) := p_t(\xi)\hat{f}(\xi), \quad \text{where } p_t(\xi) := e^{-2\pi t|\xi|}.$$

The associated Littlewood–Paley operators are given by $S_k := \mathcal{P}_{2^k} - \mathcal{P}_{2^{k+1}}$. Their Fourier symbols satisfy

$$|\widehat{S}_k(\xi)| \lesssim \min\{2^k|\xi|, 2^{-k}|\xi|^{-1}\}, \tag{3.4}$$

where $\widehat{S}_k(\xi)$ is the multiplier associated with the operator S_k, i.e., $\widehat{S_k f}(\xi) = \widehat{S}_k(\xi)\hat{f}(\xi)$. From now on, for simplicity of notation, we will use this convention. The symbols associated with the Poisson semigroup $P_k := \mathcal{P}_{2^k}$ satisfy

$$|\widehat{P}_k(\xi) - 1| \lesssim |2^k\xi| \quad \text{and} \quad |\widehat{P}_k(\xi)| \lesssim 2^{-k}|\xi|^{-1}. \tag{3.5}$$

Proof of Theorem 1.9. We verify that the sequence $(A_k)_{k\in\mathbb{Z}}$, where $A_k := \mathcal{A}_{2^k}$, satisfies the hypotheses of Theorem 2.14 for every $1 = q_0 < q_1 \le 2$.

The maximal inequality (2.13) and the Littlewood–Paley inequality (1.6) for the Poisson semigroup with constants independent of the dimension are well known [Stein 1970]. The jump estimate (2.18) was recently established in [Mirek et al. 2018b, Theorem 1.5].

It remains to verify condition (2.15) for the operators $M_k := A_k - P_k$. In view of (3.1), (3.2) and (3.5), we have

$$|\widehat{M}_k(\xi)| \lesssim \min\{|2^k\xi|^{-1}, |2^k\xi|\}.$$

For $\xi \in \mathbb{R}^d \setminus \{0\}$ let $k_0 \in \mathbb{Z}$ be such that $\tilde{\xi} = 2^{k_0}\xi$ satisfies $|\tilde{\xi}| \simeq 1$. By (3.5) it follows that

$$\begin{aligned}\sum_{k\in\mathbb{Z}} |\widehat{M}_k(\xi)\widehat{S}_{k+j}(\xi)|^2 &\lesssim \sum_{k\in\mathbb{Z}} \min\{|2^k\xi|^{-1}, |2^k\xi|\}^2 \min\{|2^{k+j}\xi|^{-1}, |2^{k+j}\xi|\}^2 \\ &= \sum_{k\in\mathbb{Z}} \min\{|2^k\tilde{\xi}|^{-1}, |2^k\tilde{\xi}|\}^2 \min\{|2^{k+j}\tilde{\xi}|^{-1}, |2^{k+j}\tilde{\xi}|\}^2 \\ &\lesssim \sum_{k\in\mathbb{Z}} \min\{2^{-k}, 2^k\}^2 \min\{(2^{k+j})^{-1}, 2^{k+j}\}^2 \lesssim 2^{-\delta|j|} \end{aligned} \tag{3.6}$$

for $\delta \in (0,2)$ with the implicit constant independent of the dimension. By Plancherel's theorem this shows that (2.15) holds with $a_j \lesssim 2^{-\delta|j|/2}$. □

Proof of Theorem 1.11. We will apply Theorem 2.39 with $A_t := \mathcal{A}_t := \mathcal{A}_t^G$. By a simple scaling we have $\mathcal{A}_{2^k(1+t)} = \mathcal{A}_{2^k}^{(1+t)G}$. Hence Theorem 2.14, with $A_k = \mathcal{A}_{2^k}^{(1+t)G}$, applies and we obtain the vector-valued inequality (2.20) for all $1 < p < \infty$ and $r = 2$, which consequently guarantees (2.42). It remains to verify the hypothesis (2.40) of Theorem 2.39. We repeat the estimate [Bourgain et al. 2018, (4.23)]. By (3.3) for $t > 0$ and $h > 0$ we have

$$\bigl|\hat{\mu}((t+h)\xi) - \hat{\mu}(t\xi)\bigr| \le \int_t^{t+h} |\langle \xi, \nabla\hat{\mu}(u\xi)\rangle|\, \mathrm{d}u \lesssim \int_t^{t+h} \frac{\mathrm{d}u}{u} \lesssim \frac{h}{t}. \tag{3.7}$$

By the Plancherel theorem this implies

$$\|\mathcal{A}_{t+h} - \mathcal{A}_t\|_{L^2 \to L^2} \lesssim \frac{h}{t}. \tag{3.8}$$

This allows us to estimate the square of the left-hand side of (2.40) by

$$\begin{aligned}
\mathrm{LHS}\,(2.40)^2 &= \sum_{k\in\mathbb{Z}} \sum_{m=0}^{2^l-1} \|(\mathcal{A}_{2^k+2^{k-l}(m+1)} - \mathcal{A}_{2^k+2^{k-l}m}) S_{j+k} f\|_{L^2}^2 \\
&\lesssim \sum_{k\in\mathbb{Z}} \sum_{m=0}^{2^l-1} 2^{-2l} \|S_{j+k} f\|_{L^2}^2 \\
&= 2^{-l} \sum_{k\in\mathbb{Z}} \|S_{j+k} f\|_{L^2}^2 \lesssim 2^{-l} \|f\|_{L^2}^2.
\end{aligned}$$

Secondly, by (3.1) and (3.2) for every $0 \le m < 2^l$ we have

$$\big|\hat{\mu}((2^k + 2^{k-l}(m+1))\xi) - \hat{\mu}((2^k + 2^{k-l}m)\xi)\big| \lesssim \min\{|2^k\xi|, |2^k\xi|^{-1}\}.$$

Arguing similarly to (3.6) we obtain

$$\mathrm{LHS}\,(2.40)^2 \lesssim 2^l 2^{-\delta|j|} \|f\|_2^2.$$

Hence (2.40) holds with $a_{j,l} = \min\{1, 2^l 2^{-\delta|j|/2}\}$. □

Proof of Theorem 1.14. By Theorem 1.9 we have the hypothesis (2.48) of Theorem 2.39. The hypothesis (2.40) was verified in the proof of Theorem 1.11. The remaining hypothesis (2.45) is given by [Bourgain et al. 2018, Lemma 4.2], but we give a more direct proof.

Recall that B^q is the unit ball induced by ℓ^q norm in $\mathbb{R}^d$. From [Müller 1990] (for $1 \le q < \infty$), and [Bourgain 2014] (for $q = \infty$) we use the multiplier norm estimate

$$\|\tilde{m}\|_{M^p} \lesssim_{p,q,\alpha} 1, \quad \tilde{m} = (\xi \cdot \nabla)^\alpha \hat{\mu},$$

for $\alpha \in (0, 1)$ and $p \in (1, \infty)$ with implicit constant independent of the dimension. For a Lipschitz function $h : \left(\frac{1}{2}, \infty\right) \to \mathbb{R}$ such that $|h(t)| \lesssim |t|^{-1}$ and $|h'(t)| \lesssim |t|^{-1}$ fractional differentiation can be inverted by fractional integration:

$$h(t) = \frac{1}{\Gamma(\alpha)} \int_t^{+\infty} (u - t)^{\alpha-1} D^\alpha h(u)\, \mathrm{d}u, \quad t > \tfrac{1}{2};$$

see [Deleaval et al. 2018, Lemma 6.11]. In particular, for $t > 1$ we obtain

$$h(t) - h(1) = \frac{1}{\Gamma(\alpha)} \int_1^{+\infty} ((u - t)_+^{\alpha-1} - (u - 1)^{\alpha-1}) D^\alpha h(u)\, \mathrm{d}u,$$

where $u_+ := \max(u, 0)$ denotes the positive part. In view of (3.1) and (3.3) this result can be applied to the function $h(t) = \hat{\mu}(t\xi)$ for any $\xi \in \mathbb{R}^d \setminus \{0\}$. Observing $D^\alpha h(u) = u^{-\alpha} \tilde{m}(u\xi)$ we obtain

$$\hat{\mu}(t\xi) - \hat{\mu}(\xi) = \frac{1}{\Gamma(\alpha)} \int_1^{+\infty} ((u - t)_+^{\alpha-1} - (u - 1)_+^{\alpha-1}) u^{-\alpha} \tilde{m}(u\xi)\, \mathrm{d}u.$$

On the other hand we have

$$\int_1^{+\infty} |(u-t)_+^{\alpha-1} - (u-1)_+^{\alpha-1}| u^{-\alpha}\, \mathrm{d}u \lesssim_\alpha (t-1)^\alpha,$$

and for a Schwartz function $f \in \mathcal{S}(\mathbb{R}^d)$ this implies

$$\begin{aligned}\|\mathcal{F}_\xi^{-1}((\hat{\mu}(t\xi)-\hat{\mu}(\xi))\hat{f}(\xi))\|_{L^p} &\le \int_1^{+\infty} |(u-t)_+^{\alpha-1}-(u-1)_+^{\alpha-1}| u^{-\alpha} \cdot \|\mathcal{F}_\xi^{-1}((((u\xi\cdot\nabla)^\alpha\hat{\mu})(\xi)\hat{f}(\xi))\|_{L^p}\, \mathrm{d}u \\ &\lesssim_\alpha (t-1)^\alpha \sup_{u>0} \|\mathcal{F}_\xi^{-1}((((u\xi\cdot\nabla)^\alpha\hat{\mu})(u\xi)\hat{f}(\xi))\|_{L^p} \\ &\lesssim_\alpha (t-1)^\alpha \|((\xi\cdot\nabla)^\alpha\hat{\mu})(\xi)\|_{M^p} \|f\|_{L^p},\end{aligned}$$

where we have used the Fourier inversion formula and Fubini's theorem in the first step and scale invariance of the multiplier norm in the last step. Since the multiplier $\hat{\mu}(t\xi) - \hat{\mu}(\xi)$ is (qualitatively) bounded on L^p with norm ≤ 2, by density of Schwartz functions this implies

$$\|\hat{\mu}(t\,\cdot\,) - \hat{\mu}\|_{M^p} \lesssim_\alpha (t-1)^\alpha,$$

which by scaling implies the hypothesis (2.45). □

Finally we emphasize that once Theorem 1.9 is proved, alternative proofs of Theorems 1.11 and 1.14 follow by appealing to the short variational estimates given in [Bourgain et al. 2018].

3B. ***Dimension-free estimates for jumps in the discrete setting.*** We outline the proof of Theorem 1.18. The strategy is much the same as for the proofs of Theorems 1.9 and 1.11. Let

$$\mathfrak{m}_N(\xi) = \frac{1}{(2N+1)^d} \sum_{m\in Q_N} e^{2\pi i m\cdot\xi} \quad \text{for } \xi \in \mathbb{T}^d$$

be the multiplier corresponding to the operators A_N defined in (1.17). Here we remind the reader of the following estimates for $\mathfrak{m}_N$ established recently in [Bourgain et al. 2019]. Namely there is a constant $0 < C < \infty$ independent of the dimension such that for every $N, N_1, N_2 \in \mathbb{N}$ and for every $\xi \in \mathbb{T}^d \equiv \left[-\frac{1}{2}, \frac{1}{2}\right)^d$ we have

$$\begin{aligned}|\mathfrak{m}_N(\xi)| &\le C(N|\xi|)^{-1}, \\ |\mathfrak{m}_N(\xi) - 1| &\le CN|\xi|, \\ |\mathfrak{m}_{N_1}(\xi) - \mathfrak{m}_{N_2}(\xi)| &\le C|N_1 - N_2| \max\{N_1^{-1}, N_2^{-1}\},\end{aligned} \tag{3.9}$$

where $|\cdot|$ denotes the Euclidean norm restricted to $\mathbb{T}^d$.

The discrete Poisson semigroup is defined by

$$\widehat{\mathcal{P}_t f}(\xi) := p_t(\xi)\hat{f}(\xi), \quad \text{where } p_t(\xi) := e^{-2\pi t|\xi|_{\sin}},$$

for every $\xi \in \mathbb{T}^d$ and

$$|\xi|_{\sin} := \Big(\sum_{j=1}^d (\sin(\pi\xi_j))^2\Big)^{\frac{1}{2}}.$$

We set $P_k := \mathcal{P}_{2^k}$ and the associated Littlewood–Paley operators are given by $S_k := \mathcal{P}_{2^k} - \mathcal{P}_{2^{k+1}}$. The maximal inequality (2.13) and the Littlewood–Paley inequality (1.6) for the discrete Poisson semigroup with constants independent of the dimension follow from [Stein 1970]. The jump estimate (2.18) for discrete Poisson semigroup was recently proved in [Mirek et al. 2018b, Theorem 1.5]. Moreover, using $|\xi| \le |\xi|_{\sin} \le \pi|\xi|$ for $\xi \in \mathbb{T}^d$, we see that the corresponding Fourier symbols $\widehat{S}_k(\xi)$ and $\widehat{P}_k(\xi)$ satisfy estimates (3.4) and (3.5) as well.

In order to prove (1.20) we have to verify that the sequence $(A_k)_{k\in\mathbb{N}}$, where $A_k := \boldsymbol{A}_{2^k}$, satisfies the hypotheses of Theorem 2.14 for every $1 = q_0 < q_1 \le 2$. Taking into account (3.9), (3.4) and (3.5) (associated with the discrete Poisson semigroup) it suffices to proceed as in the proof of Theorem 1.9. To prove (1.19) we argue as in the proof of Theorem 1.11.

3C. ***Jump inequalities for the operators of Radon type.*** In this section we prove Theorems 1.22 and 1.30. By the lifting procedure for the Radon transforms described in [Stein 1993, Chapter 11, Section 2.4] we can assume without loss of generality that our polynomial mapping $P(x) := (x)^\Gamma$ is the canonical polynomial mapping for some $\Gamma \subset \mathbb{N}_0^k \setminus \{0\}$ with lexicographical order, given by

$$\mathbb{R}^k \ni x = (x_1, \dots, x_k) \mapsto (x)^\Gamma := (x_1^{\gamma_1} \cdots x_k^{\gamma_k} : \gamma \in \Gamma) \in \mathbb{R}^\Gamma,$$

where $\mathbb{R}^\Gamma := \mathbb{R}^{|\Gamma|}$ is identified with the space of all vectors whose coordinates are labeled by multi-indices $\gamma = (\gamma_1, \dots, \gamma_k) \in \Gamma$.

Throughout what follows A is the diagonal $|\Gamma| \times |\Gamma|$ matrix such that $(Ax)_\gamma = |\gamma| x_\gamma$ for every $x \in \mathbb{R}^\Gamma$ and let $\mathfrak{q}_*$ be the quasinorm associated with $A^* = A$, given by

$$\mathfrak{q}_*(\xi) = \max_{\gamma\in\Gamma}(|\xi_\gamma|^{\frac{1}{|\gamma|}}) \quad \text{for } \xi \in \mathbb{R}^\Gamma.$$

We shall later freely appeal, without explicit mention, to the discussions after Theorems 2.14, 2.28 and 2.39 with $d = |\Gamma|$, A and $\mathfrak{q}_*$ as above.

Proof of Theorem 1.22. Let $\mathcal{M}_t := \mathcal{M}_t^P$, where $P(x) = (x)^\Gamma$. Observe that $\mathcal{M}_t$ is a convolution operator with a probability measure μ_t, whose Fourier transform is defined by

$$\hat{\mu}_t(\xi) := \frac{1}{|\Omega_t|} \int_{\Omega_t} e^{-2\pi i \xi\cdot(y)^\Gamma} \, \mathrm{d}y \quad \text{for } \xi \in \mathbb{R}^\Gamma.$$

Condition (2.25) with $\omega(t) = t^{1/d}$ follows from Proposition B.2 and Lemma A.1. It is not difficult to see that (2.24) also holds.

In order to prove (1.23) it suffices, in view of (2.2), to show inequality (2.19) with $A_k := \mathcal{M}_{2^k}$ and inequality (2.43) with $A_t := \mathcal{M}_t$ for every $1 = q_0 < q_1 \le 2$. We have already seen that (2.26) holds; hence (2.19) holds and we are done. We now show (2.43). For this purpose note that (2.45) holds for all $1 \le q_0 < \infty$. This combined with (2.24) and (2.25) permits us to prove (2.49) and (2.50), which imply (2.40) and Theorem 2.39 yields the conclusion. □

Proof of Theorem 1.30. Let $\mathcal{H}_t := \mathcal{H}_t^P$, where $P(x) = (x)^\Gamma$. Denote the Fourier multiplier corresponding to the truncated singular Radon transform by

$$\Psi_t(\xi) := \int_{\mathbb{R}^k \setminus \Omega_t} e^{-2\pi i \xi \cdot (y)^\Gamma} K(y)\,\mathrm{d}y \quad \text{for } \xi \in \mathbb{R}^\Gamma. \tag{3.10}$$

For a fixed $\kappa \in (0,1)$ we claim

$$\begin{aligned}|\Psi_t(\xi) - \Psi_s(\xi)| &\lesssim_\kappa |t^A\xi|_\infty^{-\frac{1}{d}} + \omega_K(|t^A\xi|_\infty^{-\frac{1}{d}}) \\ &\lesssim (t\mathfrak{q}_*(\xi))^{-\frac{1}{d}} + \omega_K((t\mathfrak{q}_*(\xi))^{-\frac{1}{d}}) \quad \text{if } t\mathfrak{q}_*(\xi) \geq 1,\end{aligned} \tag{3.11}$$

for all $s, t \in (0,\infty)$ such that $\kappa t \leq s \leq t$. Indeed, by Proposition B.2 we obtain

$$\begin{aligned}|\Psi_t(\xi) - \Psi_s(\xi)| &= \left| \int_{\Omega_t \setminus \Omega_s} e^{-2\pi i \xi \cdot (y)^\Gamma} K(y)\,\mathrm{d}y \right| \\ &\lesssim \sup_{v \in \mathbb{R}^k : |v| \leq t\Lambda^{-\frac{1}{d}}} \int |(\mathbf{1}_{\Omega_t \setminus \Omega_s} K)(y) - (\mathbf{1}_{\Omega_t \setminus \Omega_s} K)(y - v)|\,\mathrm{d}y,\end{aligned}$$

with

$$\Lambda = \sum_{\gamma \in \Gamma} t^{|\gamma|} |\xi_\gamma|.$$

The claim (3.11) clearly holds for $\Lambda \leq 1$. If $\Lambda \geq 1$, then for a fixed v we use (1.28) and the fact that $\Omega_t \setminus \Omega_s \subseteq B(0,t) \setminus B(0, c_\Omega \kappa t)$ to estimate the contribution of y such that $y, y - v \in \Omega_t \setminus \Omega_s$. On the set of y such that exactly one of $y, y - v$ is contained in $\Omega_t \setminus \Omega_s$ we use (1.26); the measure of this set is bounded by a multiple of $t^{k-1}|v|$ due to Lemma A.1. This finishes the proof of (3.11).

Additionally, we have

$$|\Psi_t(\xi) - \Psi_s(\xi)| \lesssim |t^A\xi|_\infty^{\frac{1}{d}} \lesssim (t\mathfrak{q}_*(\xi))^{\frac{1}{d}} + \omega_K((t\mathfrak{q}_*(\xi))^{\frac{1}{d}}) \quad \text{if } t\mathfrak{q}_*(\xi) \leq 1 \tag{3.12}$$

due to the cancellation condition (1.27) and (1.26).

To prove (1.31) we fix $\theta \in (0,1]$ and $p \in \{1+\theta, (1+\theta)'\}$ and invoking (2.2) it suffices to prove inequalities (1.32) and (1.33). Inequality (1.32) will follow from (2.33) with $q_0 = 1$, $q_1 = 1+\theta$, and $B_j := \mathcal{H}_{2^j} - \mathcal{H}_{2^{j+1}}$ upon expressing $\mathcal{H}_{2^k}$ as a telescoping series like in (2.3). Inequality (1.33) will be a consequence of (2.43) with $q_0 = 1$, $q_1 = 1+\theta$, and $A_t := \mathcal{H}_t$. Let $(\sigma_t : t > 0)$ be a family of measures defined by

$$\sigma_t * f(x) = \int_{\Omega_t \setminus \Omega_{2^k}} f(x - (y)^\Gamma) K(y)\,\mathrm{d}y \quad \text{for every } t \in [2^k, 2^{k+1}],\ k \in \mathbb{Z}. \tag{3.13}$$

Estimates (3.11) and (3.12) allow us to verify (2.35) and (2.36) respectively with $\omega(t) := t^{1/d} + \omega_K(t^{1/d})$. Moreover $|\sigma_{2^k}| \lesssim \mu_{2^k}$, where μ_t is the measure associated with the averaging operator $\mathcal{M}_t$. Hence the discussion after Theorem 2.28 guarantees that inequality (2.33) holds, since $B_k f = \sigma_{2^{k+1}} * f$. To prove (2.43) it suffices to note that (2.45) holds for all $1 \leq q_0 < \infty$. Moreover inequalities (2.49) and (2.50) remain true for $A_t = \mathcal{H}_t$. Then Theorem 2.39 completes the proof. □

Appendix A: Neighborhoods of boundaries of convex sets

We will show how to control the measure of neighborhoods of the boundaries of convex sets. The proof of the lemma below is based on a simple Vitali covering argument.

Lemma A.1. *Let $\Omega \subset \mathbb{R}^k$ be a bounded and convex set and let $0 < s \lesssim \operatorname{diam}(\Omega)$. Then*

$$|\{x \in \mathbb{R}^k : \operatorname{dist}(x, \partial\Omega) < s\}| \lesssim_k s \operatorname{diam}(\Omega)^{k-1}.$$

The implicit constant depends only on the dimension k, but not on the convex set Ω.

Proof. Let $r = \operatorname{diam}\Omega$. By translation we may assume $\Omega \subseteq B(0, r)$, where $B(y, s)$ denotes an open ball centered at $y \in \mathbb{R}^k$ with radius $s > 0$. Notice

$$\{x \in \mathbb{R}^k : \operatorname{dist}(x, \partial\Omega) < s\} \subseteq \bigcup_{y \in \partial\Omega} B(y, s).$$

By the Vitali covering lemma there exists a finite subset $Y \subset \partial\Omega$ such that the balls $B(y, s)$, with $y \in Y$, are pairwise disjoint and

$$\Big|\bigcup_{y \in \partial\Omega} B(y, s)\Big| \lesssim \Big|\bigcup_{y \in Y} B(y, s)\Big|.$$

Consider the nearest-point projection $P : \mathbb{R}^k \to \operatorname{cl}\Omega$, that is, $P(x) = x'$, where $x' \in \operatorname{cl}\Omega$ is the unique point such that $|x - x'| = \operatorname{dist}(x, \operatorname{cl}\Omega)$. It is well known that P is well-defined and contractive with respect to the Euclidean metric. The restriction of P to the sphere $\partial B(0, r)$ defines a surjection $P_\partial : \partial B(0, r) \to \partial\Omega$. This follows from the fact that for every point $x \in \partial\Omega$ there exists a linear functional $\phi : \mathbb{R}^k \to \mathbb{R}$ such that $\phi(y) \le \phi(x)$ for every $y \in \operatorname{cl}\Omega$; see, e.g., [Rockafellar 1970, Corollary 11.6.1]. For each $y \in Y$ we choose $z(y) \in \partial B(0, r)$ such that $P_\partial(z(y)) = y$. Then the balls $B(z(y), s)$ are pairwise disjoint in view of the contractivity of P and contained in the set

$$\{x \in \mathbb{R}^k : r - s < |x| < r + s\},$$

which has measure $\lesssim s(r + s)^{k-1}$. But the union of the balls $B(z(y), s)$ has the same measure as $\bigcup_{y \in Y} B(y, s)$, and the conclusion follows. □

Appendix B: Estimates for oscillatory integrals

We present the following variant of van der Corput's oscillatory integral lemma with a rough amplitude function.

Lemma B.1. *Given an interval $(a, b) \subset \mathbb{R}$ suppose that $\phi : (a, b) \to \mathbb{R}$ is a smooth function such that $|\phi^{(k)}(x)| \gtrsim \lambda$ for every $x \in (a, b)$ with some $\lambda > 0$. Assume additionally that*

- *either $k \ge 2$,*
- *or $k = 1$ and ϕ' is monotonic.*

Then for every locally integrable function $\psi:\mathbb{R}\to\mathbb{C}$ *we have*

$$\left|\int_a^b e^{i\phi(x)}\psi(x)\,\mathrm{d}x\right| \lesssim_k \inf_{a\le x\le b}\int_{x-\lambda^{-1/k}}^{x+\lambda^{-1/k}} |\psi(y)|\,\mathrm{d}y + \lambda^{\frac{1}{k}}\int_{-\lambda^{-1/k}}^{\lambda^{-1/k}}\int_a^b |\psi(x)-\psi(x-y)|\,\mathrm{d}x\,\mathrm{d}y.$$

Proof. Let η be a smooth positive function with $\operatorname{supp}\eta\subseteq[-1,1]$ and $\int_{\mathbb{R}}\eta(x)\,\mathrm{d}x = 1$. Let $\rho(x) := \psi * \lambda^{1/k}\eta(\lambda^{1/k}x)$, and note that

$$|\psi(x)-\rho(x)| \le \lambda^{\frac{1}{k}}\int_{\mathbb{R}} |\psi(x)-\psi(x-y)|\,|\eta(\lambda^{\frac{1}{k}}y)|\,\mathrm{d}y.$$

Then we may replace ψ by ρ on the left-hand side of the conclusion. For every $x_0\in(a,b)$ by partial integration and the van der Corput lemma, see for example [Stein 1993, Section VIII.1.2], we have

$$\begin{aligned}\left|\int_a^b e^{i\phi(x)}\rho(x)\,\mathrm{d}x\right| &= \left|\rho(x_0)\int_a^b e^{i\phi(x)}\,\mathrm{d}x + \int_a^b e^{i\phi(x)}\int_{x_0}^x \rho'(y)\,\mathrm{d}y\,\mathrm{d}x\right| \\ &\le \left|\rho(x_0)\int_a^b e^{i\phi(x)}\,\mathrm{d}x\right| + \left|\int_a^{x_0}\rho'(y)\int_a^y e^{i\phi(x)}\,\mathrm{d}x\,\mathrm{d}y\right| + \left|\int_{x_0}^b \rho'(y)\int_y^b e^{i\phi(x)}\,\mathrm{d}x\,\mathrm{d}y\right| \\ &\lesssim \lambda^{-\frac{1}{k}}\left(|\rho(x_0)| + \int_a^b |\rho'(x)|\,\mathrm{d}x\right).\end{aligned}$$

The latter term is estimated using

$$|\rho'(x)| = |(\psi(x)-\psi) * \lambda^{\frac{1}{k}}\eta(\lambda^{\frac{1}{k}}\,\cdot)'(x)| \lesssim \lambda^{\frac{2}{k}}\int_{\mathbb{R}} |\psi(x)-\psi(x-y)|\,|\eta'(\lambda^{\frac{1}{k}}y)|\,\mathrm{d}y,$$

and the conclusion follows. □

We will also need a multidimensional version of Lemma B.1. As before $B(y,s)$ denotes an open ball centered at $y\in\mathbb{R}^k$ with radius $s>0$.

Proposition B.2 [Zorin-Kranich 2017]. *Given* $d,k\in\mathbb{N}$, *let* $P(x) = \sum_{1\le|\alpha|\le d}\lambda_\alpha x^\alpha$ *be a polynomial in* k *variables of degree at most* d *with real coefficients. Let* $R>0$ *and let* $\psi:\mathbb{R}^k\to\mathbb{C}$ *be an integrable function supported in* $B(0,R/2)$. *Then*

$$\left|\int_{\mathbb{R}^k} e^{iP(x)}\psi(x)\,\mathrm{d}x\right| \lesssim_{d,k} \sup_{v\in\mathbb{R}^k:|v|\le R\Lambda^{-\frac{1}{d}}}\int_{\mathbb{R}^k} |\psi(x)-\psi(x-v)|\,\mathrm{d}x,$$

where

$$\Lambda := \sum_{1\le|\alpha|\le d} R^{|\alpha|}|\lambda_\alpha|.$$

We include the proof for completeness.

Proof. Changing the variables we have

$$\left|\int_{\mathbb{R}^k} e^{iP(x)}\psi(x)\,\mathrm{d}x\right| = R^k\left|\int_{\mathbb{R}^k} e^{iP_R(x)}\psi_R(x)\,\mathrm{d}x\right|,$$

where

$$P_R(x) = \sum_{1 \le |\alpha| \le d} R^{|\alpha|} \lambda_\alpha x^\alpha, \quad \psi_R(x) = \psi(Rx) \quad \text{and} \quad \operatorname{supp} \psi_R \subseteq B\big(0, \tfrac{1}{2}\big).$$

Let us define

$$\beta = \sup_{v \in \mathbb{R}^k : |v| \le \Lambda^{-\frac{1}{d}}} \int_{\mathbb{R}^k} |\psi_R(x) - \psi_R(x - v)| \, \mathrm{d}x,$$

and observe that $\|\psi_R\|_{L^1} \lesssim \beta \Lambda^{1/d}$. So there is nothing to prove if $\Lambda \lesssim 1$. We assume that $\Lambda \gtrsim 1$. Let η be a nonnegative smooth bump function with integral 1, which is supported in the ball $B\big(0, \tfrac{1}{2}\big)$. Then we define $\rho(x) = \Lambda^{k/d} \eta(\Lambda^{1/d} x)$ and $\phi(x) = \psi_R * \rho(x)$ and we note

$$\int_{\mathbb{R}^k} |\psi_R(x) - \phi(x)| \, \mathrm{d}x \le \Lambda^{\frac{k}{d}} \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} |\psi_R(x) - \psi_R(x - y)| \, \mathrm{d}x \eta(\Lambda^{\frac{1}{d}} y) \mathrm{d}y \lesssim \beta.$$

The proof will be completed if we show that

$$\left| \int_{\mathbb{R}^k} e^{i P_R(x)} \phi(x) \, \mathrm{d}x \right| \lesssim_{d,k} \beta. \tag{B.3}$$

Since ϕ is a smooth function supported in $B(0, 1)$ we invoke [Stein and Wainger 2001, Lemma 2.2] to get the conclusion. Indeed, that result ensures that there exists a unit vector $\xi \in \mathbb{R}^k$ and an integer $m \in \mathbb{N}$ such that $|(\xi \cdot \nabla)^m P_R| > c_{k,d} \Lambda$ on the unit ball $B(0, 1)$ for some $c_{k,d} > 0$. We may assume, without loss of generality, that $\xi = e_1 = (1, 0, \dots, 0) \in \mathbb{R}^k$. Then by the van der Corput lemma, see for example [Stein 1993, Corollary, p. 334], we obtain

$$\begin{aligned} \left| \int_{\mathbb{R}^k} e^{i P_R(x)} \phi(x) \, \mathrm{d}x \right| &\lesssim \Lambda^{-\frac{1}{d}} \int_{\mathbb{R}^{k-1} \cap B(0,1)} \left(|\phi(1, x')| + \int_{-1}^{1} |\partial_1 \phi(x_1, x')| \, \mathrm{d}x_1 \right) \mathrm{d}x' \\ &\lesssim \Lambda^{-\frac{1}{d}} \|\nabla \phi\|_{L^1}, \end{aligned}$$

since $\operatorname{supp} \phi \subseteq B(0, 1)$ and $\phi(1, x') = 0$ for every $x' \in \mathbb{R}^{k-1} \cap B(0, 1)$.

We now show that $\|\nabla \phi\|_{L^1} \lesssim \Lambda^{1/d} \beta$. Indeed, for every $j \in \{1, \dots, k\}$ we have

$$\begin{aligned} \|\partial_j \phi\|_{L^1} &= \int_{\mathbb{R}^k} \left| \int_{\mathbb{R}^k} \psi_R(x - y) \partial_j \rho(y) \, \mathrm{d}y \right| \mathrm{d}x \\ &= \int_{\mathbb{R}^k} \left| \int_{\mathbb{R}^k} (\psi_R(x) - \psi_R(x - y)) \partial_j \rho(y) \, \mathrm{d}y \right| \mathrm{d}x \\ &\lesssim \Lambda^{\frac{k}{d} + \frac{1}{d}} \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} |\psi_R(x) - \psi_R(x - y)| |(\partial_j \eta)(\Lambda^{\frac{1}{d}} y)| \, \mathrm{d}x \, \mathrm{d}y \lesssim \Lambda^{\frac{1}{d}} \beta. \end{aligned}$$

This proves (B.3) and completes the proof of Proposition B.2. □

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ON THE TRACE OPERATOR FOR FUNCTIONS OF BOUNDED $\mathbb{A}$-VARIATION

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We consider the space $\mathrm{BV}^{\mathbb{A}}(\Omega)$ of functions of bounded $\mathbb{A}$-variation. For a given first-order linear homogeneous differential operator with constant coefficients $\mathbb{A}$, this is the space of L^1-functions $u:\Omega\to\mathbb{R}^N$ such that the distributional differential expression $\mathbb{A}u$ is a finite (vectorial) Radon measure. We show that for Lipschitz domains $\Omega\subset\mathbb{R}^n$, $\mathrm{BV}^{\mathbb{A}}(\Omega)$-functions have an $L^1(\partial\Omega)$-trace if and only if $\mathbb{A}$ is $\mathbb{C}$-elliptic (or, equivalently, if the kernel of $\mathbb{A}$ is finite-dimensional). The existence of an $L^1(\partial\Omega)$-trace was previously only known for the special cases that $\mathbb{A}u$ coincides either with the full or the symmetric gradient of the function u (and hence covered the special cases BV or BD). As a main novelty, we do not use the fundamental theorem of calculus to construct the trace operator (an approach which is only available in the BV- and BD-settings) but rather compare projections onto the nullspace of $\mathbb{A}$ as we approach the boundary. As a sample application, we study the Dirichlet problem for quasiconvex variational functionals with linear growth depending on $\mathbb{A}u$.

1. Introduction

1A. ***Aim and scope.*** Let Ω be an open, bounded Lipschitz domain in $\mathbb{R}^n$ and let $1\le p<\infty$. A key tool in the study of partial differential equations is the assignment of boundary values to elements $u\in W^{1,p}(\Omega;\mathbb{R}^N)$, often being the first step towards well-posedness results for such equations. In this respect, it is a well-established fact, see [Maz'ya 2011], that if $1<p<\infty$, then there exists a surjective, bounded linear trace embedding operator

$$\mathrm{tr}:W^{1,p}(\Omega;\mathbb{R}^N)\hookrightarrow W^{1-1/p,p}(\partial\Omega;\mathbb{R}^N) \tag{1-1}$$

which satisfies $\mathrm{tr}(u)=u|_{\partial\Omega}$ for $u\in C(\overline{\Omega};\mathbb{R}^N)\cap W^{1,p}(\Omega;\mathbb{R}^N)$. If $p=1$ instead, a result of [Gagliardo 1957] asserts that there exists a surjective, bounded linear trace embedding operator

$$\mathrm{tr}:W^{1,1}(\Omega;\mathbb{R}^N)\hookrightarrow L^1(\partial\Omega;\mathbb{R}^N). \tag{1-2}$$

The same holds true when $W^{1,1}(\Omega;\mathbb{R}^N)$ is replaced by $\mathrm{BV}(\Omega;\mathbb{R}^N)$, the $\mathbb{R}^N$-valued functions of bounded variation on Ω. Both boundary trace embeddings (1-1), (1-2) and the corresponding variant for BV hinge on inequalities

$$\begin{aligned}\|u\|_{W^{1-1/p,p}(\partial\Omega;\mathbb{R}^N)}&\le C(\|u\|_{L^p(\Omega;\mathbb{R}^N)}+\|Du\|_{L^p(\Omega;\mathbb{R}^{N\times n})}),\\ \|u\|_{L^1(\partial\Omega;\mathbb{R}^N)}&\le C(\|u\|_{L^1(\Omega;\mathbb{R}^N)}+\|Du\|_{L^1(\Omega;\mathbb{R}^{N\times n})})\end{aligned} \tag{1-3}$$

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if $1 < p < \infty$ or $p = 1$, respectively, to be satisfied for all $u \in C(\overline{\Omega}; \mathbb{R}^N) \cap C^1(\Omega; \mathbb{R}^N)$. These estimates in turn are obtained as a consequence of the fundamental theorem of calculus in conjunction with a smooth approximation argument.

As one of the fundamental achievements of 20th century harmonic analysis, Calderón and Zygmund [1956] and Mihlin [1956] established that in a wealth of inequalities, the *full gradient* can be replaced by weaker quantities only involving certain combinations of derivatives. Precisely, let $\mathbb{A}$ be a constant-coefficient, linear, homogeneous differential operator from $\mathbb{R}^N$ to $\mathbb{R}^K$; i.e., there exist fixed linear maps $\mathbb{A}_\alpha \colon \mathbb{R}^N \to \mathbb{R}^K$ with

$$\mathbb{A} = \sum_{\alpha=1}^{n} \mathbb{A}_\alpha \partial_\alpha. \tag{1-4}$$

Then for each $1 < p < \infty$ there exists $c = c(p, n, \mathbb{A}) > 0$ such that there holds

$$\|Du\|_{L^p(\mathbb{R}^n;\mathbb{R}^{N\times n})} \le c\|\mathbb{A}u\|_{L^p(\mathbb{R}^n;\mathbb{R}^K)} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^N) \tag{1-5}$$

if and only if $\mathbb{A}$ is *elliptic*. Here we say that $\mathbb{A}$ is elliptic if and only if for each $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \setminus \{0\}$ the *symbol map* $\mathbb{A}[\xi] := \sum_\alpha \xi_\alpha \mathbb{A}_\alpha \colon \mathbb{R}^N \to \mathbb{R}^K$ is an injective linear map. A special instance of (1-5) is the case of the symmetric gradient operator $\mathcal{E}u := \frac{1}{2}(Du + D^\top u)$ acting on maps $u \colon \mathbb{R}^n \to \mathbb{R}^n$ (here $N = n \ge 2$ and $K = n^2$, identifying $\mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$). In this situation, (1-5) gives the usual Korn inequalities, which play a pivotal role in elasticity or fluid mechanics; see [Fuchs and Seregin 2000] for a comprehensive overview.

Singular integrals or Fourier multiplier operators in general are not bounded on L^1. Thus one expects the exponent range $1 < p < \infty$ for (1-5) to be optimal for general elliptic operators $\mathbb{A}$. This is in fact true and manifested by Ornstein's celebrated noninequality, stating the impossibility of nontrivial L^1-estimates:

Theorem [Ornstein 1962]. *Let $\mathbb{A}$ and $\mathbb{B}$ be two constant-coefficient first-order, linear homogeneous differential operators on $\mathbb{R}^n$ from $\mathbb{R}^N$ to $\mathbb{R}^K$ and from $\mathbb{R}^N$ to $\mathbb{R}$, respectively. Suppose that there exists a constant $c > 0$ such that*

$$\|\mathbb{B}u\|_{L^1(\mathbb{R}^n)} \le c\|\mathbb{A}u\|_{L^1(\mathbb{R}^n;\mathbb{R}^K)} \quad \textit{for all } u \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^N).$$

Then there exists $T \in \mathscr{L}(\mathbb{R}^K; \mathbb{R})$ such that $\mathbb{B} = T \circ \mathbb{A}$.

This negative result — which faces contributions to date, see [Conti et al. 2005; Kirchheim and Kristensen 2016] — immediately yields that if $p = 1$, inequalities that involve the full gradients Du do not necessarily generalise to those involving only $\mathbb{A}u$. On the other hand, by [Temam and Strang 1980] it is known for the special case of $\mathbb{A}$ being the symmetric gradient operator that the second inequality in (1-3) remains valid indeed for $p = 1$ when D is replaced by $\mathcal{E}$. However, the method employed in [Temam and Strang 1980; Babadjian 2015] to arrive at this result is very specific to the symmetric gradient operator and its structural properties: again based on the fundamental theorem of calculus, $\mathcal{E}u$ then allows one to control a cone of line integrals emanating from the boundary, leading to the desired trace inequality. In particular, it is far from clear whether and if so, how, trace inequalities of the form (1-3) can be established for $p = 1$ and D being replaced by differential operators $\mathbb{A}$ of the form (1-4). As we shall see below in

Section 1C, even for general elliptic operators $\mathbb{A}$ the corresponding analogues of (1-3) break down and hence the method employed for the symmetric gradient cannot easily generalise.

This leads us to the following *classification problem*: *classify all differential operators of the form* (1-4) *such that for any open and bounded Lipschitz domain* $\Omega\subset\mathbb{R}^n$ *there exists a constant* $c>0$ *such that*

$$\|u\|_{L^1(\partial\Omega;\mathbb{R}^N)}\leq c(\|u\|_{L^1(\Omega;\mathbb{R}^N)}+\|\mathbb{A}u\|_{L^1(\Omega;\mathbb{R}^K)}) \tag{1-6}$$

holds for all $u\in C(\overline{\Omega};\mathbb{R}^N)\cap C^1(\Omega;\mathbb{R}^N)$. The overall objective of the present paper is to solve this classification problem. Before we pass on to the precise description of our results — in particular, Theorem 1.2 — we briefly pause and connect this theme to other results available in the literature first.

1B. ***Contextualisation and function spaces.*** The quest for classifying differential operators $\mathbb{A}$ of the form (1-4) such that well-known inequalities generalise to the $\mathbb{A}$-framework for $p=1$ has come up rather recently. Building on the foundational work [Bourgain and Brezis 2003; 2004; 2007], Van Schaftingen [2013] characterised all operators $\mathbb{A}$ of the form (1-4) for which a Sobolev-type inequality

$$\|u\|_{L^{n/(n-1)}(\mathbb{R}^n;\mathbb{R}^N)}\leq C\|\mathbb{A}u\|_{L^1(\mathbb{R}^n;\mathbb{R}^K)}\quad\text{for all } u\in C_c^\infty(\mathbb{R}^n;\mathbb{R}^N) \tag{1-7}$$

holds. Whereas ellipticity of $\mathbb{A}$ is easily seen to be necessary for (1-7), it is far from sufficient and needs to be augmented by the so-called *cancellation condition*. Following [Van Schaftingen 2013], we call $\mathbb{A}$ *cancelling* if and only if

$$\bigcap_{\xi\in\mathbb{R}^n\setminus\{0\}}\mathbb{A}[\xi](\mathbb{R}^N)=\{0\}.$$

Note that by ellipticity, $u\in C_c^\infty(\mathbb{R}^n;\mathbb{R}^N)$ can be represented via $u=k_{\mathbb{A}}*\mathbb{A}u$, where $k_{\mathbb{A}}:\mathbb{R}^n\setminus\{0\}\to\mathscr{L}(\mathbb{R}^K;\mathbb{R}^N)$ satisfies the growth bound $|k_{\mathbb{A}}(y)|\sim|y|^{1-n}$ for $y\in\mathbb{R}^n\setminus\{0\}$. Then the fractional integration theorem only implies that the convolution with $k_{\mathbb{A}}$ yields an operator that maps $L^1(\mathbb{R}^n;\mathbb{R}^K)\to L_{\mathrm{w}}^{n/(n-1)}(\mathbb{R}^n;\mathbb{R}^N)$ boundedly with the weak-$L^{n/(n-1)}$ space $L_{\mathrm{w}}^{n/(n-1)}(\mathbb{R}^n;\mathbb{R}^N)$, and so (1-7) implies a proper improvement based on the additional cancellation condition.

To unify this theme also in view of (1-6), we wish to interpret the above inequalities in terms of (boundary trace) embeddings and thus introduce function spaces via

$$W^{\mathbb{A},1}(\Omega):=\{v\in L^1(\Omega;\mathbb{R}^N):\mathbb{A}u\in L^1(\Omega;\mathbb{R}^K)\},$$
$$\mathrm{BV}^{\mathbb{A}}(\Omega):=\{v\in L^1(\Omega;\mathbb{R}^N):\mathbb{A}u\in\mathcal{M}(\Omega;\mathbb{R}^K)\},$$

where $\Omega\subset\mathbb{R}^n$ is open, $\mathbb{A}$ is a differential operator of the form (1-4) and $\mathcal{M}(\Omega;\mathbb{R}^K)$ denotes the $\mathbb{R}^K$-valued Radon measures of finite total variation on Ω. These spaces are normed canonically via $\|u\|_{W^{\mathbb{A},1}}=\|u\|_{L^1}+\|\mathbb{A}u\|_{L^1}$ (similarly for $\mathrm{BV}^{\mathbb{A}}$ with the obvious modifications); clearly, $W^{\mathbb{A},1}(\Omega)\subsetneq\mathrm{BV}^{\mathbb{A}}(\Omega)$ and we shall refer to $\mathrm{BV}^{\mathbb{A}}(\Omega)$ as *space of functions of bounded* $\mathbb{A}$*-variation*. In the literature, only particular instances of spaces $\mathrm{BV}^{\mathbb{A}}$ have been studied in detail, namely for $\mathbb{A}=\nabla$ or $\mathbb{A}=\mathcal{E}$, leading to the spaces BV or BD of functions of bounded variation or deformation, respectively. Precisely, we then have $W^{1,1}=W^{\nabla,1}$, $\mathrm{LD}=W^{\mathcal{E},1}$, $\mathrm{BV}=\mathrm{BV}^{\nabla}$, $\mathrm{BD}=\mathrm{BV}^{\mathcal{E}}$, and this paper is the first attempt to characterise the properties of $\mathrm{BV}^{\mathbb{A}}$-maps in terms of the properties of $\mathbb{A}$ in a unifying manner. By this, we also aim to clarify the underlying mechanisms for the corresponding trace inequalities to work in the known cases $\mathbb{A} = D$ and $\mathbb{A} = \mathcal{E}$.

Returning to the classification problem related to (1-6), we conclude this subsection by pointing out that ellipticity in itself cannot yield the required L^1-trace theory. In fact, consider the operator $\mathcal{E}^D u := \mathcal{E}u - \frac{1}{n}\operatorname{div}(u)E_n$ ($E_n \in \mathbb{R}^{n\times n}$ being the identity matrix), which is usually referred to as *trace-free symmetric gradient operator*, for $n \geq 2$. This operator enters in a variety of applications, for instance fluid mechanics or general relativity; see [Feireisl 2004; Bartnik and Isenberg 2004]. Regardless of $n \geq 2$, the operator $\mathcal{E}^D$ is elliptic; see Example 2.2(c). However, the following example from [Fuchs and Repin 2010] shows that an L^1-trace does not exist if $n = 2$. Identifying $\mathbb{R}^2 \cong \mathbb{C}$, $\ker(\mathcal{E}^D)$ essentially contains the holomorphic functions. Upon identifying $\mathbb{R}^2$ with $\mathbb{C}$ and denoting by $\mathbb{D}$ the open unit disc in $\mathbb{C}$, the map $u\colon \mathbb{D} \to \mathbb{C}$, $z \mapsto 1/(z-1)$, even belongs to $W^{\mathcal{E}^D,1}(\mathrm{B}(0,1))$, whereas it is clear that $\|\operatorname{tr}(u)\|_{L^1(\partial \mathrm{B}(0,1))} = \infty$. In view of (1-6), our main result, Theorem 1.2 below, will cover the particular case of $\mathbb{A} = \mathcal{E}^D$ as a special case and provide a positive answer for all $n \geq 3$ and a negative answer for $n = 2$.

1C. ***Main results.*** Before we state our main result, we need to provide the definitions of several important properties of our operator $\mathbb{A}$. To begin with, we write the *symbol mapping* $\mathbb{A}[\xi] : \mathbb{R}^N \to \mathbb{R}^K$ as

$$\mathbb{A}[\xi]v := v \otimes_{\mathbb{A}} \xi := \sum_{\alpha=1}^{n} \xi_\alpha \mathbb{A}_\alpha v, \quad \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n,\ v \in \mathbb{R}^N. \tag{1-8}$$

Moreover, we extend $\mathbb{A}[\xi]\eta = \eta \otimes_{\mathbb{A}} \xi$ by (1-8) also to complex-valued $\xi \in \mathbb{C}^n$ and $\eta \in \mathbb{C}^N$. We strengthen terminology and say that $\mathbb{A}$ is $\mathbb{R}$-*elliptic* if $\mathbb{A}[\xi]\colon \mathbb{R}^N \to \mathbb{R}^K$ is injective for all $\xi \in \mathbb{R}^n \setminus \{0\}$ (i.e., $\mathbb{A}$ is elliptic in the above sense), and $\mathbb{C}$-*elliptic* provided $\mathbb{A}[\xi]\colon \mathbb{C}^N \to \mathbb{C}^K$ is injective for all $\xi \in \mathbb{C}^n \setminus \{0\}$ (see Section 2C for more detail). Finally, we shall say that $\mathbb{A}$ has *finite-dimensional nullspace* if the kernel $N(\mathbb{A})$ of $\mathbb{A}$ in the distributional sense is finite-dimensional; i.e.,

$$\dim(N(\mathbb{A})) < \infty, \quad \text{with } N(\mathbb{A}) = \{v \in \mathcal{D}'(\mathbb{R}^n;\mathbb{R}^N) : \mathbb{A}v \equiv 0\}, \tag{1-9}$$

where $\mathcal{D}(\mathbb{R}^n;\mathbb{R}^N) = C_c^\infty(\mathbb{R}^n;\mathbb{R}^N)$. We will see later in Theorem 2.6 that $\mathbb{A}$ has a finite-dimensional nullspace if and only if it is $\mathbb{C}$-elliptic. It is also equivalent to the *type-*(C) *condition* in the sense of [Kałamajska 1994]; see Remark 2.1. However, the notion of $\mathbb{R}$-ellipticity is strictly weaker: For instance, $\mathcal{E}^D$ for $n = 2$ is $\mathbb{R}$-elliptic but not $\mathbb{C}$-elliptic; see Example 2.2(c). We are now in position to formulate our main result.

Theorem 1.1. *Let $\mathbb{A}$ be a differential operator of the form* (1-4)*. Then the following are equivalent*:

- (a) *For all open and bounded Lipschitz domains $\Omega \subset \mathbb{R}^n$ there exists a constant $c > 0$ such that* (1-6) *holds for all $u \in C(\overline{\Omega};\mathbb{R}^N) \cap C^1(\Omega;\mathbb{R}^N)$.*
- (b) *$\mathbb{A}$ is $\mathbb{C}$-elliptic.*

Whereas necessity of $\mathbb{C}$-ellipticity for (1-6) shall be addressed in Theorem 4.18 and essentially follows from a construction relying on the properties of the two-dimensional operator $\mathcal{E}^D$, the more involved part is the sufficiency. For future reference, we single this out and state in the following more elaborate form; the full statement can be found in Theorem 4.17:

Theorem 1.2 (trace theorem). *Let $\mathbb{A}$ be $\mathbb{C}$-elliptic* (*or equivalently*, *$\mathbb{A}$ has finite-dimensional nullspace*). *Then there exists a trace operator* $\mathrm{tr} : \mathrm{BV}^{\mathbb{A}}(\Omega) \to L^1(\partial\Omega, \mathcal{H}^{n-1})$ *such that the following holds*:

(a) $\mathrm{tr}(u)$ *coincides with the classical trace for all* $u \in \mathrm{BV}^{\mathbb{A}}(\Omega) \cap C(\bar{\Omega}; \mathbb{R}^N)$.

(b) $\mathrm{tr}(u)$ *is the unique strictly continuous extension of the classical trace on* $\mathrm{BV}^{\mathbb{A}}(\Omega) \cap C(\bar{\Omega}; \mathbb{R}^N)$. *In particular*, $\mathrm{tr}\colon \mathrm{BV}^{\mathbb{A}}(\Omega) \to L^1(\partial\Omega; \mathcal{H}^{n-1})$ *is continuous for the norm topology on* $\mathrm{BV}^{\mathbb{A}}(\Omega)$.

(c) $\mathrm{tr}(W^{\mathbb{A},1}(\Omega)) = \mathrm{tr}(\mathrm{BV}^{\mathbb{A}}(\Omega)) = L^1(\partial\Omega; \mathcal{H}^{n-1})$.

Regarding sufficiency, the core issue is how to replace the use of the fundamental theorem of calculus by that of $\mathbb{C}$-ellipticity. As a main consequence of the latter, we will employ the nullspace of $\mathbb{C}$-elliptic operators being finite-dimensional. Using local projections onto the nullspace $N(\mathbb{A})$ close to the boundary, we construct suitable approximations of $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$ that have classical traces. The limit of these traces provide us with the trace of u. In particular, the projections to the finite-dimensional nullspace replace the fundamental theorem of calculus approach as used in [Temam and Strang 1980; Babadjian 2015].

In addition to Theorem 4.17 we will show in Theorem 4.18 and Remark 4.19 that if $\mathbb{A}$ is not $\mathbb{C}$-elliptic, then in general there is no trace operator from $\mathrm{BV}^{\mathbb{A}}(\Omega)$ to $L^1(\partial\Omega; \mathcal{H}^{n-1})$. In particular, the existence of $L^1(\partial\Omega; \mathcal{H}^{n-1})$-traces on arbitrary bounded Lipschitz domains $\Omega \subset \mathbb{R}^n$ is equivalent to $\mathbb{C}$-ellipticity of $\mathbb{A}$. This conclusion also identifies the infinite-dimensional nullspace of $\mathbb{A}$ as the reason for the failure of the trace embedding of $W^{\mathcal{E}^D,1}(\Omega)$ into $L^1(\partial\Omega; \mathcal{H}^{n-1})$ for $n = 2$ (see Example 2.2(c)). As a consequence of Theorem 1.2 we also obtain a version of the Gauss–Green theorem, see Theorem 4.20, and the gluing theorem, see Corollary 4.21. Let us also remark that Theorem 1.2 includes both the trace theorems for the spaces BV and BD.

The relation between the condition of $\mathbb{C}$-ellipticity and Van Schaftingen's elliptic and cancelling condition will be investigated in detail in the follow-up [Gmeineder and Raiță 2019] to this paper by Raita and the third author; among others, it will be shown that $\mathbb{C}$-ellipticity implies Van Schaftingen's condition but in general *not* vice versa. In this sense and as might be anticipated, L^1-boundary traces require a stronger condition on $\mathbb{A}$.

1D. ***Variational problems.*** As a concluding application of the trace theorem from above, we address the Dirichlet problem for linear growth functionals involving operators $\mathbb{A}$. To be precise, we are interested in the minimisation of functionals of the form

$$\mathfrak{F}[u] := \int_\Omega f(x, \mathbb{A}u)\,\mathrm{d}x \tag{1-10}$$

over a class of maps $u\colon \Omega \to \mathbb{R}^N$ subject to Dirichlet boundary data $u = u_0$ on $\partial\Omega$. Here $f\colon \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}_{\geq 0}$ is a given variational integrand for which we suppose the linear growth assumption

$$c_1|z| \leq f(x, z) \leq c_2|z| + c_3 \quad \text{for all } x \in \Omega \text{ and } z \in \mathbb{R}^{N\times n}. \tag{1-11}$$

Additionally, we assume that our integrand f is $\mathbb{A}$-*quasiconvex* (in a sense specified in Section 5; also see [Fonseca and Müller 1999; Dacorogna 1982]). Our objective here is to minimise $\mathfrak{F}$ over the

Dirichlet class $u_0 + W_0^{\mathbb{A},1}(\Omega)$, which are the $W^{\mathbb{A},1}(\Omega)$-functions whose traces agree with the given boundary datum u_0. From the treatment of the Dirichlet problem on $\mathrm{BV}^{\mathbb{A}}$, where $\mathbb{A} = \nabla$, see [Giaquinta et al. 1979a; 1979b; Ambrosio et al. 2000], it is clear that the functional should be considered on the class of $\mathrm{BV}^{\mathbb{A}}$-maps on a larger Lipschitz domain U. More precisely, we need to consider the weak*-lower semicontinuous envelope of $\mathfrak{F}$ on $\mathrm{BV}^{\mathbb{A}}(U)$. Whereas in the convex situation one can make use of the classical results due to [Reshetnyak 1968], the quasiconvex case is substantially more involved. The sequentially weak*-lower semicontinuous envelope $\overline{\mathfrak{F}}$ of $\mathfrak{F}$ on $\mathrm{BV}(\Omega)$ (so $\mathbb{A} = \nabla$) was characterised in [Ambrosio and Dal Maso 1992; Fonseca and Müller 1993]. The corresponding issue for the symmetric-quasiconvex (so $\mathbb{A} = \mathcal{E}$) situation was resolved in [Rindler 2011]. Invoking the recent outstanding generalisation of Alberti's rank-one theorem [De Philippis and Rindler 2016], the weak*-lower semicontinuity result of [Arroyo-Rabasa et al. 2018] and the area-strict continuity of [Kristensen and Rindler 2010b], we give a precise characterization of the weak*-lower semicontinuous envelope $\overline{\mathfrak{F}}$ on $\mathrm{BV}^{\mathbb{A}}(\Omega)$; see Proposition 5.1.

Consequently, a merger with Theorem 1.2 allows us to formulate the minimisation problem with Dirichlet data u_0 purely in terms of $\mathrm{BV}^{\mathbb{A}}(\Omega)$; see Corollary 5.2. We demonstrate both the existence of minima and the absence of a Lavrentiev gap with respect to the Dirichlet class $u_0 + W_0^{\mathbb{A},1}(\Omega)$; see Theorem 5.3.

1E. ***Organisation of the paper.*** The paper is organised as follows. In Section 2 we fix notation, introduce the assumptions on the differential operators $\mathbb{A}$ and collect elementary implications for the Sobolev-type spaces $W^{\mathbb{A},1}(\Omega)$ and the spaces of functions of bounded $\mathbb{A}$-variation $\mathrm{BV}^{\mathbb{A}}(\Omega)$. In Section 3 we introduce local projection operators onto the nullspace $N(\mathbb{A})$ on balls and derive Poincaré-type inequalities. In Section 4, we construct the trace operator $\mathrm{tr} : \mathrm{BV}^{\mathbb{A}}(\Omega) \to L^1(\partial\Omega; \mathcal{H}^{n-1})$ and thereby give the proof of Theorem 1.2. Moreover, we establish a Gauss–Green formula and a gluing lemma for $\mathrm{BV}^{\mathbb{A}}$-maps. The final Section 5 is dedicated to the existence of $\mathrm{BV}^{\mathbb{A}}$-minimisers of $\mathbb{A}$-quasiconvex variational problems with linear growth subject to given Dirichlet boundary data.

2. Functions of bounded $\mathbb{A}$-variation

In this section we introduce the space of functions of bounded variation associated with a differential operator $\mathbb{A}$.

2A. ***General notation.*** To avoid too many different constants throughout, we write $a \lesssim b$ if there exists a constant c (which does not depend on the crucial quantities) with $a \le cb$. If $a \lesssim b$ and $b \lesssim a$, we also write $a \eqsim b$. By $\ell(B)$ we denote the diameter of a ball B and by $|B|$, its n-dimensional Lebesgue measure. We write $d(\,\cdot\,,\cdot\,)$ for the usual euclidean distance. For the euclidean inner product of $a, b \in \mathbb{R}^m$ we use the equivalent notations $\langle a, b\rangle$ or $a \cdot b$. Given $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^K)$ and a measurable subset $U \subset \mathbb{R}^n$ with $|U| > 0$, we use the equivalent notations

$$\fint_U f(x)\,\mathrm{d}x := \langle f\rangle_U := |U|^{-1} \int_U f(x)\,\mathrm{d}x$$

for the mean value integral. Lastly, for notational simplicity, we shall often suppress the possibly vectorial target space when dealing with function spaces and, e.g., write $L^1(\mathbb{R}^n)$ instead of $L^1(\mathbb{R}^n;\mathbb{R}^N)$, but this will be clear from the context.

2B. ***Function space setup.*** Let $\mathbb{A}$ be given by (1-4). The corresponding *dual* (*or formally adjoint*) *operator* $\mathbb{A}^*$ is the differential operator on $\mathbb{R}^n$ from $\mathbb{R}^K$ to $\mathbb{R}^N$ given by

$$\mathbb{A}^* := \sum_{\alpha=1}^{n} \mathbb{A}_\alpha^* \partial_\alpha, \tag{2-1}$$

where each $\mathbb{A}_\alpha^*$ is the adjoint matrix of $\mathbb{A}_\alpha$. For an open domain $\Omega\subset\mathbb{R}^n$ we define the *Sobolev space* $W^{\mathbb{A},1}(\Omega)$ *associated to the operator* $\mathbb{A}$ by

$$W^{\mathbb{A},1}(\Omega) = W^{\mathbb{A},1}(\Omega;\mathbb{R}^N) := \{u\in L^1(\Omega;\mathbb{R}^N) : \mathbb{A}u \in L^1(\Omega;\mathbb{R}^K)\}. \tag{2-2}$$

This is a Banach space with respect to the norm

$$\|u\|_{W^{\mathbb{A},1}(\Omega)} := \|u\|_{L^1(\Omega)} + \|\mathbb{A}u\|_{L^1(\Omega)}. \tag{2-3}$$

We moreover define the *total* $\mathbb{A}$*-variation* of $u\in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ by

$$|\mathbb{A}u|(\Omega) := \sup\left\{\int_\Omega \langle u, \mathbb{A}^*\varphi\rangle\,\mathrm{d}x : \varphi\in C_c^1(\Omega;\mathbb{R}^K),\ |\varphi|\le 1\right\} \tag{2-4}$$

and consequently say that u is of *bounded* $\mathbb{A}$*-variation* if and only if $u\in L^1(\Omega;\mathbb{R}^N)$ and $|\mathbb{A}u|(\Omega)<\infty$. Denoting by $\mathcal{M}(\Omega;\mathbb{R}^K)$ the finite $\mathbb{R}^K$-valued Radon measures on Ω, by the Riesz representation theorem this amounts to

$$\mathrm{BV}^{\mathbb{A}}(\Omega) := \{u\in L^1(\Omega;\mathbb{R}^N) : \mathbb{A}u\in\mathcal{M}(\Omega;\mathbb{R}^K)\}. \tag{2-5}$$

Here, the shorthands $\mathbb{A}u\in L^1$ or $\mathbb{A}u\in\mathcal{M}$ above have to be understood in the sense that the distributional differential expressions $\mathbb{A}u$ can be represented by L^1-functions or Radon measures, respectively. The norm

$$\|u\|_{\mathrm{BV}^{\mathbb{A}}(\Omega)} := \|u\|_{L^1(\Omega)} + |\mathbb{A}u|(\Omega) \tag{2-6}$$

makes $\mathrm{BV}^{\mathbb{A}}(\Omega)$ a Banach space. However, due to the lack of good compactness properties, the norm topology turns out not to be useful in many applications and one needs to consider weaker topologies. We now introduce the canonical generalisations of well-known convergences in the full- or symmetric-gradient cases; see [Ambrosio et al. 2000]. Let $u\in\mathrm{BV}^{\mathbb{A}}(\Omega)$ and $(u_k)\subset\mathrm{BV}^{\mathbb{A}}(\Omega)$. We say that

- (u_k) converges to u in the *weak*-sense* (in symbols $u_k\overset{*}{\rightharpoonup}u$) if and only if $u_k\to u$ strongly in $L^1(\Omega;\mathbb{R}^N)$ and $\mathbb{A}u_k\overset{*}{\rightharpoonup}\mathbb{A}u$ in the weak*-sense of $\mathbb{R}^K$-valued Radon measures on Ω as $k\to\infty$.
- (u_k) converges to u in the *strict sense* (in symbols $u_k\overset{s}{\to}u$) if and only if $d_s(u_k,u)\to 0$ as $k\to\infty$, where for $v,w\in\mathrm{BV}^{\mathbb{A}}(\Omega)$ we set

$$d_s(v,w) := \int_\Omega |v-w|\,\mathrm{d}x + \big||\mathbb{A}v|(\Omega) - |\mathbb{A}w|(\Omega)\big|.$$

- (u_k) converges to u in the *area-strict sense* (in symbols $u_k \xrightarrow{\langle\cdot\rangle} u$) if and only if

$$\int_\Omega \sqrt{1+\left|\frac{\mathrm{d}\mathbb{A}u_k}{\mathrm{d}\mathscr{L}^n}\right|^2}\,\mathrm{d}\mathscr{L}^n + |\mathbb{A}^s u_k|(\Omega) \to \int_\Omega \sqrt{1+\left|\frac{\mathrm{d}\mathbb{A}u}{\mathrm{d}\mathscr{L}^n}\right|^2}\,\mathrm{d}\mathscr{L}^n + |\mathbb{A}^s u|(\Omega), \quad k\to\infty,$$

where

$$\mathbb{A}v = \frac{\mathrm{d}\mathbb{A}v}{\mathrm{d}\mathscr{L}^n}\mathscr{L}^n + \frac{\mathrm{d}\mathbb{A}v}{\mathrm{d}|\mathbb{A}^s v|}|\mathbb{A}^s v|$$

is the Radon–Nikodym decomposition of $\mathbb{A}v \in \mathcal{M}(\Omega;\mathbb{R}^K)$ with respect to the Lebesgue measure $\mathscr{L}^n$.

Strictly speaking, these notions are reserved for the BV-versions and hence the above notions have to be read as $\mathbb{A}$-weak*, $\mathbb{A}$-strict, and $\mathbb{A}$-area-strict convergence. However, to keep terminology simple, we tacitly assume that the differential operator $\mathbb{A}$ is fixed throughout and stick to the above terminology.

Note that the $\mathbb{A}$-variation is sequentially lower semicontinuous with respect convergence in the weak*-sense; i.e., if $u_k \overset{*}{\rightharpoonup} u$, then $|\mathbb{A}u|(\Omega) \le \liminf_{k\to\infty} |\mathbb{A}u_k|(\Omega)$. Moreover, if $u_k \in \mathrm{BV}^{\mathbb{A}}(\Omega)$ is a bounded sequence with $u_k \rightharpoonup u$ in $L^1(\Omega;\mathbb{R}^N)$, then already $u_k \overset{*}{\rightharpoonup} u$. Finally, if Ω is open and bounded with Lipschitz boundary, then it is easy to conclude by the theorem of Banach and Alaoglu that if $(u_k) \subset \mathrm{BV}^{\mathbb{A}}(\Omega)$ is uniformly bounded in the $\mathrm{BV}^{\mathbb{A}}$-norm, then there exists $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$ and a subsequence $(u_{k(j)})$ of (u_k) such that $u_{k(j)} \overset{*}{\rightharpoonup} u$ as $j\to\infty$ in the sense specified above. We shall often refer to this as the *weak*-compactness principle* (*for* $\mathrm{BV}^{\mathbb{A}}$).

2C. ***Assumptions on the differential operator*** $\mathbb{A}$**.** For our trace result we need some structure on $\mathbb{A}$ which we introduce now.

Let $\mathbb{A}$ be given by (1-4). Then $\mathbb{A}$ induces a bilinear pairing $\otimes_{\mathbb{A}}: \mathbb{R}^N\times\mathbb{R}^n \to \mathbb{R}^K$ by

$$v\otimes_{\mathbb{A}} z := \sum_{\alpha=1}^n z_\alpha \mathbb{A}_\alpha v \quad \text{for } z\in\mathbb{R}^n \text{ and } v\in\mathbb{R}^N. \tag{2-7}$$

For all $\varphi \in C^1(\mathbb{R}^n)$ and $v \in C^1(\mathbb{R}^n;\mathbb{R}^N)$ we have

$$\mathbb{A}(\varphi v) = \varphi\mathbb{A}v + v\otimes_{\mathbb{A}} \nabla\varphi. \tag{2-8}$$

Note that if $\mathbb{A}$ is the usual gradient, then $\otimes_{\mathbb{A}}$ can be identified with the usual dyadic product $\otimes$, and if $\mathbb{A}$ is the symmetric gradient, then $\otimes_{\mathbb{A}}$ is given by the symmetric tensor product $\odot$.

Recalling the notions of $\mathbb{R}$- and $\mathbb{C}$-ellipticity from Section 1C, we now pass on to a more detailed discussion and begin with linking them to the *type-*(C) condition as introduced in [Kałamajska 1994].

Remark 2.1. The operator $\mathbb{A}$ is $\mathbb{C}$-elliptic if and only if it is of type (C) in the sense of [Kałamajska 1994]. More precisely, since $\mathbb{A}_\alpha[\xi]$ is a linear operator from $\mathbb{R}^N$ to $\mathbb{R}^K$ for each $\xi\in\mathbb{R}^n$, we find coefficients $\mathbb{A}_{\alpha,j,k}$ such that

$$(\mathbb{A}[\xi]\eta)_k =: \sum_{\alpha=1}^n\sum_{j=1}^N \mathbb{A}_{\alpha,j,k}\xi_\alpha\eta_j$$

for every for $\xi\in\mathbb{R}^n$ and $\eta\in\mathbb{R}^N$. Then

$$\mathbb{P}_{j,k}u := \sum_{\alpha=1}^n \mathbb{A}_{\alpha,j,k}\partial_\alpha u_j$$

for $k = 1, \dots, K$ is the family of scalar differential operators as used in [Kałamajska 1994]. The corresponding symbols are

$$\mathbb{P}_{j,k}(\xi) := \sum_{\alpha=1}^{n} \mathbb{A}_{\alpha,j,k}\xi_\alpha,$$

with $j = 1, \dots, N$ and $k = 1, \dots, K$. Now according to [Kałamajska 1994] the family $(\mathbb{P}_k)_k$ is of type (C) if and only if $(\mathbb{P}_{j,k}(\xi))_{j,k}$ has rank K for all $\eta \in \mathbb{C}^n \setminus \{0\}$. Since

$$\sum_{j=1}^{N}\sum_{k=1}^{K} \mathbb{P}_{j,k}(\xi)\eta_j = \sum_{\alpha=1}^{n}\sum_{j=1}^{N}\sum_{k=1}^{K} \mathbb{A}_{\alpha,j,k}\xi_\alpha\eta_j = \mathbb{A}[\xi]\eta,$$

this is equivalent to the injectivity of $\mathbb{A}[\xi]$ for all $\eta \in \mathbb{C}^N \setminus \{0\}$, which is exactly the $\mathbb{C}$-ellipticity of $\mathbb{A}$.

We now turn to some examples, to which we shall frequently refer.

Example 2.2. In what follows, we carefully examine the gradient, symmetric and trace-free symmetric gradient operators. As these typically map $\mathbb{R}^N$ to the matrices $\mathbb{R}^{N\times n}$ instead of a vector in $\mathbb{R}^K$, we henceforth put $K = Nn$ and identify $\mathbb{R}^K$ with $\mathbb{R}^{N\times n}$:

(a) Let $\mathbb{A}u := \nabla u$. Then $N(\mathbb{A})$ just consists of the constants and

$$(v \otimes_\nabla z)_{j,k} = v_j z_k.$$

$\mathbb{A}$ has a finite-dimensional nullspace and is $\mathbb{C}$-elliptic, since

$$|\mathbb{A}[\xi]\eta|^2 = |\xi|^2|\eta|^2.$$

(b) Let $\mathbb{A}u := \mathcal{E}(u) := \frac{1}{2}(\nabla u + (\nabla u)^T)$ with $N = n$. Then $N(\mathcal{E})$ just consists of the generators of rigid motions, i.e.,

$$N(\mathcal{E}) = \{x \mapsto Ax + b : A \in \mathbb{R}^{n\times n},\ A = -A^T,\ b \in \mathbb{R}^n\},$$

and

$$(v \otimes_{\mathcal{E}} z)_{j,k} = \frac{1}{2}(v_j z_k + v_k z_j).$$

$\mathcal{E}$ has a finite-dimensional nullspace and is $\mathbb{C}$-elliptic, since

$$|\mathbb{A}[\xi]\eta|^2 = \frac{1}{2}|\xi|^2|\eta|^2 + \frac{1}{2}|\langle \xi, \eta\rangle|^2.$$

(c) Let $\mathbb{A}u := \mathcal{E}^D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T) - \frac{1}{n}\operatorname{div}(u)E_n$ with $N = n$. Then

$$(v \otimes_{\mathcal{E}^D} z)_{j,k} = \frac{1}{2}(v_j z_k + v_k z_j) - \frac{1}{n}\delta_{j,k}\sum_{l=1}^{n} v_l z_l$$

and

$$|\mathbb{A}[\xi]\eta|^2 = \frac{1}{2}|\xi|^2|\eta|^2 + \frac{1}{2}|\langle \xi, \eta\rangle|^2 - \frac{1}{n}\langle \xi, \bar{\eta}\rangle^2.$$

If $n \geq 3$, then $\mathbb{A}$ is $\mathbb{C}$-elliptic and it has the finite-dimensional nullspace

$$N(\mathcal{E}^D) = \{x \mapsto Ax + b + (2(a\cdot x)x - |x|^2 a) : A \in \mathbb{R}^{n\times n},\ A = -A^T, a, b \in \mathbb{R}^n\}.$$

Elements of $N(\mathcal{E}^D)$ are also known as *conformal killing vectors* [Dain 2006].

If $n = 2$, then $\mathbb{A}$ is only $\mathbb{R}$-elliptic, but not $\mathbb{C}$-elliptic. Indeed, $\mathbb{A}[\xi]\eta = 0$ for $\xi = (1, i)^T$ and $\eta = (1, -i)^T$. Moreover, the nullspace $N(\mathbb{A})$ is of infinite dimension: indeed, if we identify $\mathbb{R}^2 \cong \mathbb{C}$, then the kernel of $\mathcal{E}^D$ consists of the holomorphic functions. We will substantially use this property in the proofs of Lemma 2.5 and Theorem 4.18.

We now draw some consequences of the single ellipticity conditions and link them to the finite-dimensionality of the nullspace of $\mathbb{A}$.

Lemma 2.3. *Let $\mathbb{A}$ be $\mathbb{K}$-elliptic with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Then there exist two constants $0 < \kappa_1 \le \kappa_2 < \infty$ such that*

$$\kappa_1 |v||z| \le |v \otimes_{\mathbb{A}} z| \le \kappa_2 |v||z| \quad \textit{for all } v \in \mathbb{K}^N \textit{ and } z \in \mathbb{K}^n.$$

Proof. By scaling it suffices to assume $|v| = |z| = 1$. We have $|v \otimes_{\mathbb{A}} z| > 0$, since $\mathbb{A}$ is $\mathbb{K}$-elliptic. Now the claim follows by the compactness of $\{(v, z) : |v| = |z| = 1\}$ and continuity. ☐

Lemma 2.4. *Let $\mathbb{A}$ have a finite-dimensional nullspace. Then $\mathbb{A}$ is $\mathbb{R}$-elliptic.*

Proof. We proceed by contradiction. Assume that $\mathbb{A}$ is not $\mathbb{R}$-elliptic. Then there exists $\xi \in \mathbb{R}^n \setminus \{0\}$ and $\eta \in \mathbb{R}^N \setminus \{0\}$ with $\mathbb{A}[\xi]\eta = 0$. For every $f \in C^1_c(\mathbb{R}; \mathbb{R})$ we define $u_f(x) := f(\langle \xi, x \rangle)\eta$. Then $(\mathbb{A}u_f)(x) = \mathbb{A}[\xi]\eta\, f(\langle \xi, x \rangle) = 0$. Since $\eta \ne 0$ and $\xi \ne 0$, the mapping $f \mapsto u_f$ is injective. Therefore, the set $\{u_f : f \in C^1_c(\mathbb{R})\}$ is an infinite-dimensional subspace of $N(\mathbb{A})$. This contradicts the fact that $\mathbb{A}$ has finite-dimensional nullspace. ☐

Lemma 2.5. *Let $\mathbb{A}$ have a finite-dimensional nullspace. Then $\mathbb{A}$ is $\mathbb{C}$-elliptic.*

Proof. Since $\mathbb{A}$ has finite-dimensional nullspace, it is $\mathbb{R}$-elliptic by Lemma 2.4.

We proceed by contradiction, and so assume that $\mathbb{A}$ is not $\mathbb{C}$-elliptic. Then there exists $\xi \in \mathbb{C}^n \setminus \{0\}$ and $\eta \in \mathbb{C}^N \setminus \{0\}$ with $0 = \mathbb{A}[\xi]\eta = \eta \otimes_{\mathbb{A}} \xi$. We split ξ and η into their real and imaginary parts by $\xi =: \xi_1 + i\xi_2$ and $\eta =: \eta_1 + i\eta_2$. Then $\mathbb{A}[\xi]\eta = 0$ implies

$$\mathbb{A}[\xi_1]\eta_1 - \mathbb{A}[\xi_2]\eta_2 = 0 \quad \text{and} \quad \mathbb{A}[\xi_1]\eta_2 + \mathbb{A}[\xi_2]\eta_1 = 0. \tag{2-9}$$

We will show now that ξ_1 and x_2, resp. η_1 and η_2, are linearly independent.

We begin with the linear independence of ξ_1 and ξ_2. If $\xi_1 = 0$, then $\xi_2 \ne 0$ and then the $\mathbb{R}$-ellipticity of $\mathbb{A}$ and (2-9) imply $\eta_1 = \eta_2 = 0$, which contradicts $\eta \ne 0$. By the same argument, also $\xi_2 = 0$ is not possible. Hence, we have $\xi_1 \ne 0$ and $\xi_2 \ne 0$. We now show the linear independence of ξ_1 and ξ_2 by contradiction, so let us assume that $\xi_2 = \lambda\xi_1$ with $\lambda \ne 0$. Then it follows from (2-9) that

$$\mathbb{A}[\xi_1]\eta_1 = \mathbb{A}[\xi_2]\eta_2 = \lambda\mathbb{A}[\xi_1]\eta_2 = -\lambda\mathbb{A}[\xi_2]\eta_1 = -\lambda^2\mathbb{A}[\xi_1][\eta_1].$$

This implies $\mathbb{A}[\xi_1][\eta_1] = 0$. Hence by the $\mathbb{R}$-ellipticity of $\mathbb{A}$ and $\xi_1 \ne 0$, we get $\eta_1 = 0$. Now, (2-9) implies $\mathbb{A}[\xi_2][\eta_2] = 0$, so again the $\mathbb{R}$-ellipticity of $\mathbb{A}$ gives $\eta_2 = 0$. Overall, $\eta = 0$, which is a contradiction. This proves that ξ_1 and ξ_2 are linearly independent.

The proof of the linear independence of η_1 and η_2 is completely analogous. Indeed, $\eta_1 = \gamma\eta_2$ implies $\mathbb{A}[\xi_1]\eta_1 = -\gamma^2\mathbb{A}[\xi_1]\eta_1$, so $\mathbb{A}[\xi_1][\eta_1] = 0$. As above this implies $\eta = 0$, which is a contradiction.

Let us define now $\tau : \mathbb{R}^n \to \mathbb{C}$ and $\sigma : \mathbb{C} \to \mathbb{R}^N$ by

$$\tau(x) := \langle \xi, x\rangle = \langle \xi_1, x\rangle + i\langle \xi_2, x\rangle,$$
$$\sigma(z) := \mathrm{Re}(z)\eta_1 - \mathrm{Im}(z)\eta_2.$$

Let $\mathcal{O}(\mathbb{C})$ denote the set of holomorphic functions on $\mathbb{C}$. Then $\dim(\mathcal{O}(\mathbb{C})) = \infty$. Moreover, for $f \in \mathcal{O}(\mathbb{C})$ we have $\partial_{\bar{z}} f(z) = 0$ in the sense of complex derivatives. Let us define $h_f : \mathbb{R}^n \to \mathbb{R}^N$ by $h_f := \sigma \circ f \circ \tau$. Our goal is to prove $\mathbb{A}h_f = 0$. We identify in the following $\mathbb{C}$ with $\mathbb{R}^2$. By the chain rule we conclude

$$(\mathbb{A}h_f)(x) = \mathbb{A}[\xi_1]\eta_1(\partial_1 f_1)(\tau(x)) - \mathbb{A}[\xi_1]\eta_2(\partial_1 f_2)(\tau(x)) + \mathbb{A}[\xi_2]\eta_1(\partial_2 f_1)(\tau(x)) - \mathbb{A}[\xi_2]\eta_2(\partial_2 f_2)(\tau(x)). \tag{2-10}$$

Using the Cauchy–Riemann equations $\partial_1 f_1 = \partial_2 f_2$ and $\partial_1 f_2 = -\partial_2 f_1$ and (2-9) we get

$$(\mathbb{A}h_f)(x) = (\mathbb{A}[\xi_1]\eta_1 - \mathbb{A}[\xi_2]\eta_2)(\partial_1 f_1)(\tau(x)) + (\mathbb{A}[\xi_1]\eta_2 + \mathbb{A}[\xi_2]\eta_1)(\partial_2 f_1)(\tau(x)) = 0.$$

So for each $f \in \mathcal{O}(\mathbb{C})$, we constructed an $h_f : \mathbb{R}^n \to \mathbb{R}^N$ such that $\mathbb{A}h_f = 0$. We need to show that $\dim(\{h_f : f \in \mathcal{O}(\mathbb{C})\}) = \infty$. For this, it suffices to show that the linear mapping $f \mapsto h_f$ is injective. Recall that $h_f = \sigma \circ f \circ \tau$. Hence, it suffices to show that σ is injective and that τ is surjective. This, however, follows from the fact that ξ_1 and ξ_2, resp. η_1 and η_2, are linearly independent. □

Theorem 2.6. *The following are equivalent*:

(a) $\mathbb{A}$ *has a finite-dimensional nullspace.*

(b) $\mathbb{A}$ *is* $\mathbb{C}$*-elliptic.*

(c) *There exists* $l \in \mathbb{N}$ *with* $N(\mathbb{A}) \subset \mathscr{P}_l$, *where* $\mathscr{P}_l$ *denotes the set of polynomials with degree less or equal to* l.

Proof. Lemma 2.5 proves (a)enumi ⇒ (b)enumi. Obviously, (c)enumi ⇒ (a)enumi. It remains to show (b)enumi ⇒ (c)enumi.

Since $\mathbb{A}$ is $\mathbb{C}$-elliptic, it is of type-(C) in the sense of [Kałamajska 1994]; see Remark 2.1. Fix $\omega \in C_c^\infty(B(0,1))$ with $\int_{B(0,1)} \omega \,\mathrm{d}x = 1$. Then for an arbitrary ball B, we obtain by dilation and translation a function $\omega_B \in C_c^\infty(B)$ with $\int_B \omega_B(y)\,\mathrm{d}y = 1$. For every $l \in \mathbb{N}_0$ let $\mathcal{P}_B^l$ denote the *averaged Taylor polynomial* with respect to B of order l, see [Dupont and Scott 1978]; i.e.,

$$\mathcal{P}_B^l u(x) := \int_B \sum_{|\beta|\le l} \partial_y^\beta \left(\frac{(y-x)^\beta}{\beta!}\omega_B(y)\right) u(y)\,\mathrm{d}y.$$

The formula is obtained by multiplying Taylor's polynomial of order l by the weight ω_B and integrating by parts. Note that $\mathcal{P}_B^l u \in \mathscr{P}_l$.

It follows from the representation formula of [Kałamajska 1994, Theorem 4] that for all $x \in B$

$$|u(x) - (\mathcal{P}_B^l u)(x)| \le c \int_B \frac{|(\mathbb{A}u)(y)|}{|x-y|^{n-1}}\,\mathrm{d}y \tag{2-11}$$

for some $l \in \mathbb{N}_0$ (which is fixed from now on) and all $u \in C^\infty(B)$. We do not know the exact value of l, but at least l is so large that $N(\mathbb{A}) \subset \mathscr{P}_l$ (there is, however, an upper bound for l in terms of n and N.)

Now, let $v \in N(\mathbb{A})$; i.e., $v \in \mathcal{D}'(\mathbb{R}^n; \mathbb{R}^N)$ with $\mathbb{A}v = 0$ in the distributional sense. Let ϕ_ϵ denote a standard mollifier; i.e., $\varphi_\epsilon(x) := \epsilon^{-n}\varphi(x/\varepsilon)$ with a radially symmetric function $\varphi \in C_c^\infty(\mathrm{B}; [0, 1])$ with $\int_{\mathrm{B}} \varphi \,\mathrm{d}x = 1$. Then $v * \phi_\epsilon \in C^\infty(\mathbb{R}^n)$ and $\mathbb{A}(v * \phi_\epsilon) = (\mathbb{A}v) * \phi_\epsilon = 0$. Hence, it follows from (2-11) that $v * \phi_\epsilon \in \mathscr{P}_l(\mathbb{R}^n)$. This implies $v \in \mathscr{P}_l(\mathbb{R}^n)$ as desired. □

Remark 2.7. Let us compare our conditions with the ones of [Van Schaftingen 2013], building on the fundamental work of [Bourgain and Brezis 2004; 2007]. According to [Van Schaftingen 2013] the operator $\mathbb{A}$ is *cancelling*[1] if

$$\bigcap_{\xi \neq 0} \mathbb{A}[\xi](\mathbb{R}^N) = \{0\}. \tag{2-12}$$

It has been shown in Theorem 1.4 of [Van Schaftingen 2013] that whenever $\mathbb{A}$ is $\mathbb{R}$-elliptic and cancelling, then we have the Sobolev-type inequality

$$\|u\|_{L^{n/(n-1)}(\mathbb{R}^n;\mathbb{R}^N)} \leq C\|\mathbb{A}u\|_{L^1(\mathbb{R}^n;\mathbb{R}^K)} \tag{2-13}$$

for all $u \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^N)$. Moreover, the $\mathbb{R}$-ellipticity and cancellation property of $\mathbb{A}$ are necessary for such inequality.

For our result on traces we need the $\mathbb{C}$-ellipticity of $\mathbb{A}$. So the natural question arises how $\mathbb{C}$-ellipticity compares to the canceling property. It will been shown in [Gmeineder and Raiță 2019] that $\mathbb{C}$-ellipticity implies the canceling property but not vice versa. Indeed, the operator

$$\mathbb{A}(u) := \begin{pmatrix} \frac{1}{2}\partial_1 u_1 - \frac{1}{2}\partial_2 u_2 & \frac{1}{2}\partial_1 u_2 + \frac{1}{2}\partial_2 u_1 & \partial_3 u_1 \\ \frac{1}{2}\partial_1 u_2 + \frac{1}{2}\partial_2 u_1 & \frac{1}{2}\partial_1 u_1 - \frac{1}{2}\partial_2 u_2 & \partial_3 u_2 \end{pmatrix}$$

is $\mathbb{R}$-elliptic and cancelling but it is not $\mathbb{C}$-elliptic, since it fails the finite-dimensional nullspace property (recall Theorem 2.6).

2D. ***Smooth approximations in the interior.*** In this section we show that functions from $W^{\mathbb{A},1}(\Omega)$ and $\mathrm{BV}^{\mathbb{A}}(\Omega)$ can be approximated in a certain sense by functions from $W^{\mathbb{A},1}(\Omega) \cap C^\infty(\Omega; \mathbb{R}^N)$. The proof is in the spirit of [Evans and Gariepy 1992, Chapter 5.2] and is included for the reader's convenience.

Theorem 2.8 (smooth approximation). *Let $\Omega \subset \mathbb{R}^n$ be open. Then the following hold*:

(a) *The space $(C^\infty \cap W^{\mathbb{A},1})(\Omega)$ is dense in $W^{\mathbb{A},1}(\Omega)$ with respect to the norm topology.*

(b) *The space $(C^\infty \cap \mathrm{BV}^{\mathbb{A}})(\Omega)$ is dense in $\mathrm{BV}^{\mathbb{A}}(\Omega)$ with respect to the area-strict topology.*

Proof. Fix $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$. For $k = 2, 3, \dots$ define

$$\Omega_k := \left\{ x \in \Omega : \frac{1}{k+1} < d(x, \partial\Omega) < \frac{1}{k-1} \right\}.$$

Now pick a sequence (ψ_k) such that for each $k \in \mathbb{N}$ we have $\psi_k \in C_c^\infty(\Omega_k; [0, 1])$ together with $\sum_k \psi_k = 1$ globally in Ω. Now let $\eta_\epsilon : \mathbb{R}^n \to \mathbb{R}$ be a standard mollifier (even and nonnegative).

[1] The definition of *cancelling* in [Van Schaftingen 2013] is given in terms of the annihilating operator $\mathbb{L}$ from the exact sequence in (5-6). However, it translates in our setting to (2-12).

For $j \in \mathbb{N}$ and $k \in \mathbb{N}$ we can find $\epsilon_{j,k} > 0$ such that:

(i) $\operatorname{spt}(\eta_{\varepsilon_{j,k}} * (\psi_k u)) \subset \Omega_k$.

(ii) $\|\psi_k u - \eta_{\varepsilon_{j,k}} * (\psi_k u)\|_{L^1(\Omega)} < 2^{-k-j}$.

(iii) $\|u \otimes_{\mathbb{A}} \nabla\psi_k - \eta_{\varepsilon_{j,k}} * (u \otimes_{\mathbb{A}} \nabla\psi_k)\|_{L^1(\Omega)} < 2^{-k-j}$.

(iv) If $u \in W^{\mathbb{A},1}(\Omega)$, we additionally require $\|\psi_k \mathbb{A}u - \eta_{\varepsilon_{j,k}} * (\psi_k \mathbb{A}u)\|_{L^1(\Omega)} < 2^{-k-j}$.

This allows us to define $u_j \in C^\infty(\Omega)$ by $u_j := \sum_{k\in\mathbb{N}} \eta_{\varepsilon_{j,k}} * (\psi_k u)$, which is well-defined in $L^1_{\mathrm{loc}}(\Omega)$, since the sum is locally finite. Then in $L^1_{\mathrm{loc}}(\Omega)$

$$u - u_j = \sum_k (\psi_k u - \eta_{\epsilon_{j,k}} * (\psi_k u)).$$

This and (ii)enumi imply $\|u - u_j\|_{L^1(\mathbb{R}^n)} \lesssim 2^{-j}$. If $u \in W^{\mathbb{A},1}(\Omega)$, then (iii)enumi and (iv)enumi imply $\|\mathbb{A}u - \mathbb{A}u_j\|_{L^1(\mathbb{R}^n)} \lesssim 2^{-j}$. This proves (a)enumi.

It remains to prove $u_j \xrightarrow{\langle\cdot\rangle} u$ for $j \to \infty$ for $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$. In fact, the proof is similar to the standard BV case. For simplicity of notation we just show $u_j \xrightarrow{s} u$ for $j \to \infty$. The necessary changes to pass from strict convergence to area-strict convergence are just like in [Bildhauer 2003, Lemma B.2].

Since $u_j \to u$ in $L^1(\mathbb{R}^n)$ it follows by the lower semicontinuity of the total $\mathbb{A}$-variation that $|\mathbb{A}u|(\Omega) \leq \liminf_{j\to\infty} |\mathbb{A}u_j|(\Omega)$. It remains to prove $\limsup_{j\to\infty} |\mathbb{A}u_j|(\Omega) \leq |\mathbb{A}u|(\Omega)$. For this we invoke the dual characterisation (2-4) of the total $\mathbb{A}$-variation. Let $\varphi \in C^1_c(\Omega; \mathbb{R}^K)$ with $|\varphi| \leq 1$ be arbitrary. We compute

$$\begin{aligned}\int_\Omega \langle u_j, \mathbb{A}^*\varphi\rangle \,\mathrm{d}x &= \sum_k \int_\Omega \langle \eta_{\varepsilon_{j,k}} * (\psi_k u), \mathbb{A}^*\varphi\rangle \,\mathrm{d}x = \sum_k \int_\Omega \langle \psi_k u, \mathbb{A}^*(\eta_{\varepsilon_{j,k}} * \varphi)\,\mathrm{d}x \\ &= \sum_k \int_\Omega \langle u, \mathbb{A}^*(\psi_k(\eta_{\varepsilon_{j,k}} * \varphi))\rangle \,\mathrm{d}x - \sum_k \int_\Omega \langle u, (\eta_{\varepsilon_{j,k}} * \varphi) \otimes_{\mathbb{A}^*} \nabla\psi_k\rangle \,\mathrm{d}x \\ &=: I_j + II_j.\end{aligned}$$

The sums are well-defined, since $\phi \in C^1_c(\Omega)$ and $u_j = \sum_k \eta_{\epsilon_{j,k}} * (\psi_k u)$ in $L^1_{\mathrm{loc}}(\Omega)$. Now

$$\left|\sum_k \psi_k(\eta_{\epsilon_{j,k}} * \phi)\right| \leq \sum_k \psi_k |\eta_{\epsilon_{j,k}} * \phi| \leq \sum_k \psi_k \|\phi\|_\infty = \|\phi\|_\infty \leq 1.$$

Therefore,

$$I_j = \int_\Omega \left\langle u, \mathbb{A}^*\left(\sum_k \psi_k(\eta_{\varepsilon_{j,k}} * \varphi)\right)\right\rangle \mathrm{d}x \leq |\mathbb{A}u|(\Omega).$$

Using $\sum_k \nabla\psi_k = 0$ and $\phi \in C^1_c(\Omega)$, we now rewrite II_j as

$$\begin{aligned}II_j &= \sum_k \int_\Omega \langle u, (\eta_{\varepsilon_{j,k}} * \varphi) \otimes_{\mathbb{A}^*} \nabla\psi_k\rangle \,\mathrm{d}x - \sum_k \int_\Omega \langle u, \varphi \otimes_{\mathbb{A}^*} \nabla\psi_k\rangle \,\mathrm{d}x \\ &= \sum_k \int_\Omega \langle \eta_{\varepsilon_{j,k}} * (u \otimes_{\mathbb{A}} \nabla\psi_k) - (u \otimes_{\mathbb{A}} \nabla\psi_k), \varphi\rangle \,\mathrm{d}x.\end{aligned}$$

Invoking (iii)enumi and $\|\phi\|_\infty \leq 1$ we obtain $|II_j| \lesssim 2^{-j}$. Hence, collecting estimates we obtain as desired $\limsup_{j\to\infty} |\mathbb{A}u_j|(\Omega) \leq \limsup_{j\to\infty} (|\mathbb{A}u|(\Omega) + c2^{-j}) = |\mathbb{A}u|(\Omega)$. □

3. Projections and Poincaré inequalities

In this section we derive several versions of Poincaré's inequality. We assume throughout the section that $\mathbb{A}$ is $\mathbb{C}$-elliptic (or, equivalently, $\mathbb{A}$ has a finite-dimensional nullspace; see Theorem 2.6).

3A. *Projection operator.* We begin with some projection estimates.

For every ball $B \subset \mathbb{R}^n$ and $u \in L^2(B;\mathbb{R}^N)$ we define $\Pi_B u$ as the L^2-projection of u onto $N(\mathbb{A})$. Hence,

$$\int_B |\Pi_B u|^2 \,\mathrm{d}x \le \int_B |u|^2 \,\mathrm{d}x.$$

Since $N(\mathbb{A})$ is finite-dimensional, there exists a constant $c > 0$ with

$$\|\Pi_B u\|_{L^\infty(B)} \le c \fint_B |\Pi_B u| \, dx. \tag{3-1}$$

Indeed, this is clear for the unit ball and extends to general balls by dilation and translation. It follows from this as usual that

$$\fint_B |\Pi_B u| \, dx \le c \fint_B |u| \, dx. \tag{3-2}$$

Thus, Π_B can be extended to $L^1(B;\mathbb{R}^N)$ such that (3-2) remains valid.

Lemma 3.1. *Then there exists $c \ge 1$ with*

$$\inf_{q\in N(\mathbb{A})} \|u-q\|_{L^1(B)} \le \|u-\Pi_B u\|_{L^1(B)} \le c \inf_{q\in N(\mathbb{A})} \|u-q\|_{L^1(B)}.$$

Proof. The first estimate is obvious. Now, for all $q \in N(\mathbb{A})$ we have $\Pi_B q = q$. This and (3-2) imply

$$\|u-\Pi_B u\|_{L^1(B)} \le \|u-q\|_{L^1(B)} + \|\Pi_B(u-q)\|_{L^1(B)} \le c\|u-q\|_{L^1(B)}.$$

Taking the infimum over $q \in N(\mathbb{A})$ proves the lemma. □

3B. *Poincaré inequalities.* In this subsection we derive Poincaré-type inequalities for $W^{\mathbb{A},1}$ and $\mathrm{BV}^{\mathbb{A}}$. Recall that for a ball B we denote by $\ell(B)$ its diameter.

Theorem 3.2. *There exists a constant $c > 0$ such that for all balls B and all $u \in \mathrm{BV}^{\mathbb{A}}(B)$ it holds*

$$\inf_{q\in N(\mathbb{A})} \|u-q\|_{L^1(B)} \le \|u-\Pi_B u\|_{L^1(B)} \le c\ell(B)\,|\mathbb{A}u|(B),$$

where Π_B is the L^2-orthogonal projection onto $N(\mathbb{A})$ from Section 3A.

Proof. By dilation and translation, it suffices to prove the claim for the unit ball $B = B(0,1)$. Moreover, by smooth approximation (see Theorem 2.8) it suffices to consider $u \in C^\infty(B;\mathbb{R}^N) \cap W^{\mathbb{A},1}(B)$.

We use the averaged Taylor polynomials as in the proof of Theorem 2.6. Recall that by (2-11) we have the estimate

$$|u(x) - (\mathcal{P}^l u)(x)| \le c \int_B \frac{|(\mathbb{A}u)(y)|}{|x-y|^{n-1}} \,\mathrm{d}y \quad \text{for all } x \in B. \tag{3-3}$$

Since $\mathcal{P}^l u$ is not necessarily in the kernel of $\mathbb{A}$, we wish to replace it by $\Pi_B(\mathcal{P}^l)$. Thus, we start with

$$|u(x) - \Pi_B(\mathcal{P}^l u)(x)| \le |u(x) - (\mathcal{P}^l u)(x)| + |(\mathcal{P}^l u)(x) - (\Pi_B(\mathcal{P}^l u))(x)|. \tag{3-4}$$

Now, for any $p \in \mathscr{P}_l$ there holds

$$\|p - \Pi_B p\|_{L^\infty(B)} \le c \fint_B |\mathbb{A}p|\, dx. \tag{3-5}$$

Indeed, both sides define a norm on the finite-dimensional space $\mathscr{P}_l/N(\mathbb{A})$ and vanish on $N(\mathbb{A})$. Hence, for all $x \in B$

$$|(\mathcal{P}^l u)(x) - (\Pi_B(\mathcal{P}^l u))(x)| \le \|\mathcal{P}^l u - \Pi_B(\mathcal{P}^l u)\|_{L^\infty(B)} \le c \fint_B |\mathbb{A}(\mathcal{P}^l u)|\, dx. \tag{3-6}$$

The definition of the averaged Taylor polynomial implies

$$\mathbb{A}(\mathcal{P}^l u) = \mathcal{P}^{l-1}(\mathbb{A}u), \tag{3-7}$$

where $\mathcal{P}^{-1}u := 0$ if $l = 0$. The L^1-stability of the averaged Taylor polynomial gives

$$\|\mathcal{P}^{l-1}(\mathbb{A}u)\|_{L^1(B)} \le c\|\mathbb{A}u\|_{L^1(B)}. \tag{3-8}$$

Now, (3-5) and (3-8) yield

$$|(\mathcal{P}^l u)(x) - (\Pi_B(\mathcal{P}^l u))(x)| \le c\ell(B) \fint_B |\mathbb{A}u|\, dy \le c \int_B \frac{|(\mathbb{A}u)(y)|}{|x-y|^{n-1}}\, \mathrm{d}y.$$

So, (3-3) and (3-4) imply the estimate

$$|u(x) - (\Pi_B \mathcal{P}^l u)(x)| \le c \int_B \frac{|(\mathbb{A}u)(y)|}{|x-y|^{n-1}}\, \mathrm{d}y. \tag{3-9}$$

Now, integration over $x \in B$ gives

$$\begin{aligned}\int_B |u - \Pi_B(\mathcal{P}^l u)|\, \mathrm{d}x &\le c \int_B \int_B \frac{|(\mathbb{A}u)(y)|}{|x-y|^{n-1}}\, \mathrm{d}y\, \mathrm{d}x \\ &\le c \int_B |(\mathbb{A}u)(y)| \int_B |x-y|^{1-n}\, \mathrm{d}x\, \mathrm{d}y \le c\ell(B) \int_B |\mathbb{A}u|\, \mathrm{d}y.\end{aligned}$$

We have shown

$$\|u - \Pi_B(\mathcal{P}^l u)\|_{L^1(B)} \le c\ell(B)\|\mathbb{A}u\|_{L^1(B)}. \tag{3-10}$$

The rest follows by Lemma 3.1. □

Theorem 3.3. *Let B' and B be two balls with $B' \subset B$ and $\ell(B) \lesssim \ell(B')$. Then for all $u \in \mathrm{BV}^{\mathbb{A}}(B)$ with $u = 0$ on B' there holds*

$$\|u\|_{L^1(B)} \le c\ell(B)|\mathbb{A}u|(B).$$

The constant only depends on the ratio $\ell(B)/\ell(B')$.

Proof. We use the same construction as in the proof of Theorem 3.2. However, we choose $\omega \in C_c^\infty(B)$ in the construction of the averaged Taylor polynomial additionally as $\omega \in C_c^\infty(B')$. This implies that $\mathcal{P}^l u$ only depends on the values of u on B'. Hence, we obtain $\mathcal{P}^l u = 0$. Thus, Theorem 3.2 proves the claim. □

Finally, let us remark that variants of Poincaré-type inequalities can also be established along the lines of [Adams and Hedberg 1996, Lemma 8.3.1] or [Ziemer 1989, Chapter 4]. However, this requires additional extension and compactness arguments which need to be proven first.

4. Traces

In this section we show that the space of functions of bounded $\mathbb{A}$-variation admits a continuous trace operator to $L^1(\partial\Omega)$ if and only if $\mathbb{A}$ is $\mathbb{C}$-elliptic (or, equivalently, $\mathbb{A}$ has a finite-dimensional nullspace; see Theorem 2.6).

4A. ***Assumptions on the domain.*** In order to ensure a proper trace we need to make certain regularity assumptions on Ω. Our results include all Lipschitz graph domains. However, we will consider even more general domains. Indeed, nontangentially accessible domains (NTA domains) provide a natural setting for our construction of the trace operator. We refer to [Hofmann et al. 2010] for more information on NTA domains.

We begin with the necessary conditions on our domain.

Definition 4.1 (interior/exterior corkscrew condition). Let $\Omega \subset \mathbb{R}^n$:

(a) We say that Ω satisfies the *interior corkscrew condition* if there exist $R > 0$ and $M > 2$ such that for all $x \in \partial\Omega$ and all $r \in (0, R)$ there exists a $y \in \Omega$ such that

$$\frac{1}{M} r \le |x - y| \le r \quad \text{and} \quad B\left(y, \frac{r}{M}\right) \subset \Omega.$$

(b) We say that Ω satisfies the *exterior corkscrew condition* if $\mathbb{R}^n \setminus \Omega$ satisfies the interior corkscrew condition.

Definition 4.2 (Harnack chain condition). We say that $\Omega \subset \mathbb{R}^n$ satisfies the (*interior*) *Harnack chain condition* if there exist $R > 0$ and $M \in \mathbb{N}$ such that for any $\epsilon > 0$, $r \in (0, R)$, $x \in \partial\Omega$, and $y_1, y_2 \in B(x, r) \cap \Omega$ with $|y_1 - y_2| \le \epsilon 2^k$ and $d(y_j, \partial\Omega) \ge \epsilon$ for $j = 1, 2$ there exists a chain of Mk balls $B_1, \dots, B_{Mk}$ in Ω connecting y_1 and y_2 satisfying

(a) $y_1 \in B_1$, $y_2 \in B_{Mk}$,

(b) $\frac{1}{M}\ell(B_j) \le d(B_j, \partial\Omega) \le M\ell(B_j)$ for $j = 1, \dots, Mk$,

(c) $\ell(B_j) \ge \frac{1}{M} \min\{d(y_1, B_j), d(y_2, B_j)\}$ for $j = 1, \dots, Mk$.

Definition 4.3 (NTA domain). We say that a domain $\Omega \subset \mathbb{R}^n$ is an *NTA* (nontangentially accessible) domain if Ω satisfies the interior corkscrew condition, the exterior corkscrew condition and the interior Harnack chain condition.

Definition 4.4. We say that $\Omega \subset \mathbb{R}^n$ has *Ahlfors regular boundary* if there exist $R > 0$ and $M > 0$ such that for all $r \in (0, R)$

$$\frac{1}{M} r^{n-1} \le \mathcal{H}^{n-1}(B(x, r) \cap \partial\Omega) \le M r^{n-1}. \tag{4-1}$$

In the following we tacitly require that our domains satisfy the following assumption:

Assumption 4.5. We assume that Ω satisfies the following:

(a) Ω is an NTA domain.

(b) Ω has Ahlfors regular boundary.

Note that all Lipschitz domains satisfy this assumption.

Let us now construct families of balls that we will use later in the construction of our traces:

For each $j \in \mathbb{Z}$, let $(B_{j,k})_k$ denote a (countable) cover of balls of $\mathbb{R}^n$ with diameter $\ell(B_{j,k})$ such that

(a) $\frac{1}{8} \cdot 2^{-j} \leq \ell(B_{j,k}) \leq \frac{1}{4} \cdot 2^{-j}$.

(b) The scaled balls $\left(\frac{7}{8} B_{j,k}\right)_k$ cover $\mathbb{R}^n$.

(c) Each family $(B_{j,k})_k$ is locally finite with covering constant independent of j; i.e.,

$$\sup_j \sum_k \chi_{B_{j,k}} \leq c.$$

For each j let $(\eta_{j,k})_k$ be a partition of unity with respect to the $(B_{j,k})_k$ such that for all j, k

$$\|\eta_{j,k}\|_{L^\infty} + \ell(B_{j,k}) \|\nabla \eta_{j,k}\|_{L^\infty} \leq c. \tag{4-2}$$

Now, we define the 2^{-j}-neighbourhood U_j of $\partial\Omega$ by

$$U_j := \{x \in \Omega : d(x, \partial\Omega) < 2^{-j}\}.$$

Since Ω satisfies the interior corkscrew condition, we can find for each ball $B_{j,k}$ close to the boundary a *reflected ball* $B^\sharp_{j,k}$ close by. We will use these reflected balls later to define the local projections of our functions. More precisely:

(B1) There exists $j_0 \in \mathbb{Z}$, such that the following holds: for each $B_{j,k}$ with $j \geq j_0$ and $B_{j,k} \cap U_j \neq \varnothing$, there exists a ball $B^\sharp_{j,k} \subset \Omega$ with $\ell(B^\sharp_{j,k}) \eqsim \ell(B_{j,k}) \eqsim d(B^\sharp_{j,k}, \partial\Omega)$ and $d(B_{j,k}, B^\sharp_{j,k}) \lesssim \ell(B_{j,k})$, where the hidden constants are independent of j, k.

Moreover, due to the Harnack chain condition we can connect two reflected balls of neighbouring balls by a small chain of balls. More precisely, we have the following.

(B2) If $B_{j,k} \subset \Omega$ and $j \geq j_0$, then there exists a chain of balls $W_1, \ldots, W_\gamma$, with γ uniformly bounded, such that

(a) $W_1 = B_{j,k}$ and $W_\gamma = B^\sharp_{j,k}$,

(b) $|W_\beta \cap W_{\beta+1}| \eqsim |W_\beta| \eqsim |W_{\beta+1}| \eqsim |B_{j,k}|$ for $\beta = 1, \ldots, \gamma - 1$,

(c) $\ell(W_\beta) \eqsim \ell(B_{j,k})$ for $\beta = 1, \ldots, \gamma$.

The hidden constants are independent of j, k, β.

We define $\Omega(B_{j,k}, B^\sharp_{j,k}) := \bigcup_{\beta=1}^{\gamma} W_\beta$.

(B3) If $B_{j,k} \cap B_{l,m} \neq \varnothing$ and $j, l \geq j_0$ with $|j - l| \leq 1$, then there exists a chain of balls $W_1, \ldots, W_\gamma$ with γ uniformly bounded, such that

(a) $W_1 = B^\sharp_{j,k}$ and $W_\gamma = B^\sharp_{l,m}$,

(b) $|W_\beta \cap W_{\beta+1}| \eqsim |W_\beta| \eqsim |W_{\beta+1}| \eqsim |B_{j,k}|$ for $\beta = 1, \dots, \gamma - 1$,

(c) $d(W_\beta, \partial\Omega) \eqsim \ell(W_\beta) \eqsim \ell(B_{j,k})$ for $\beta = 1, \dots, \gamma$.

The hidden constants are independent of j, k, β.

We define $\Omega(B^\sharp_{j,k}, B^\sharp_{l,m}) := \bigcup_{\beta=1}^{\gamma} W_\beta$.

By construction of the chains above, we get:

(B4) There exists $k_0 \geq 2$ such that the following hold uniformly in $j \geq j_0$:

$$\sum_{m:B_{j,m}\cap U_j \neq \varnothing} \chi_{B^\sharp_{j,m}} \leq c\chi_{U_{j-k_0}\setminus U_{j+k_0}} \quad \text{and} \quad \sum_{m:B_{j,m}\cap U_j \neq \varnothing} \;\sum_{k:B_{j+1,k}\cap B_{j,m}\neq\varnothing} \chi_{\Omega(B^\sharp_{j,m}, B^\sharp_{j+1,k})} \leq c\chi_{U_{j-k_0}\setminus U_{j+k_0}}.$$

4B. *Trace operator.* We will now construct the trace operator on $\mathrm{BV}^{\mathbb{A}}(\Omega)$. We will obtain the traces by a suitable approximation process. In particular, we will define truncations $T_j u$ which are smooth close to the boundary and admit classical traces. The limits will later provide the trace of the original function.

We define

$$\Pi_{j,k} u := \Pi_{B^\sharp_{j,k}} u.$$

Let $\rho_j \in C^\infty(\Omega)$ be such that $\chi_{U_{j+1}} \leq \rho_j \leq \chi_{U_j}$ and $\|\nabla \rho_j\|_\infty \lesssim 2^j$ and let $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$. Then for $j \geq j_0$ we define $T_j u$ in Ω by

$$T_j u := u - \rho_j \sum_k \eta_{j,k}(u - \Pi_{j,k} u) = (1 - \rho_j)u + \rho_j \sum_k \eta_{j,k} \Pi_{j,k} u. \tag{4-3}$$

Due to the support of $\eta_{j,k}$ the sum in the definition is locally finite. In particular, the sum is well-defined in $L^1_{\mathrm{loc}}(\Omega)$. The function $T_j u$ is an approximation of u that replaces the values of u in the neighbourhood of $\partial\Omega$ of distance 2^{-j} by local averages. These averages are performed slightly inside the domain on the balls $B^\sharp_{j,k}$.

We begin with an auxiliary estimate involving $\Pi_{j,k} u$.

Lemma 4.6. *We have the following estimates*:

(a) *There holds*

$$\|\Pi_{j,k} u\|_{L^\infty(B_{j,k})} \lesssim \fint_{B^\sharp_{j,k}} |u| \, dx.$$

(b) *If $B_{j,m} \cap (U_j \setminus U_{j+2}) \neq \varnothing$, then $B_{j,m} \subset \Omega$ and*

$$\|u - \Pi_{j,m} u\|_{L^1(B_{j,m})} \lesssim \ell(B_{j,m}) |\mathbb{A}u| (\Omega(B_{j,m}, B^\sharp_{j,m})).$$

(c) *If $B_{j+1,k} \cap B_{j,m} \neq \varnothing$, then*

$$|B_{j,m}| \, \|\Pi_{j+1,k} u - \Pi_{j,m} u\|_{L^\infty(B_{j,m})} \lesssim \ell(B_{j,m}) |\mathbb{A}u| (\Omega(B^\sharp_{j+1,k}, B^\sharp_{j,m})).$$

Proof. (a) Since $\Pi_{j,k}$ maps to $N(\mathbb{A})$ and $N(\mathbb{A}) \subset \mathscr{P}_l$, this is just the usual inverse estimate for polynomials of a fixed degree.

(b) The definition of U_j and $\ell(B_{j,m}) \le \frac{1}{4}2^{-j}$ imply $B_{j,m} \subset \Omega$. We compute

$$\|u - \Pi_{j,m}u\|_{L^1(B_{j,m})} = \|u - \Pi_{B^\sharp_{j,m}}u\|_{L^1(B_{j,m})} \le \|u - \Pi_{B_{j,m}}u\|_{L^1(B_{j,m})} + \|\Pi_{B_{j,m}}u - \Pi_{B^\sharp_{j,m}}u\|_{L^1(B_{j,m})}.$$

The first term can be estimated by Poincaré's inequality from Theorem 3.2 which yields immediately

$$\|u - \Pi_{B_{j,m}}u\|_{L^1(B_{j,m})} \lesssim \ell(B_{j,m})|\mathbb{A}u|(B_{j,m}).$$

For the second term we make use of the Harnack chain conditions (recall Definition 4.2) and, using (B2), connect $B_{j,m}$ and $B^\sharp_{j,m}$ by a chain

$$\Omega(B_{j,k}, B^\sharp_{j,m}) = \bigcup_{\beta=1}^{\gamma} W_\beta,$$

where $W_1, \dots, W_\gamma$ are balls of size proportional to $\ell(B_{j,m})$. In particular, we have $W_1 = B_{j,m}$ and $W_\gamma = B^\sharp_{j,m}$. Moreover, we can assume that $|W_\beta \cap W_{\beta+1}| \eqsim |W_\beta| \eqsim \ell(B_{j,m})$ for all β. Now, we gain

$$\begin{aligned}\|\Pi_{B_{j,m}}u - \Pi_{B^\sharp_{j,m}}u\|_{L^1(B_{j,m})} &\le \sum_{\beta=1}^{\gamma-1} \|\Pi_{W_{\beta+1}}u - \Pi_{W_\beta}u\|_{L^1(B_{j,m})} \\ &\lesssim \sum_{\beta=1}^{\gamma-1} \|\Pi_{W_{\beta+1}}u - \Pi_{W_\beta}u\|_{L^1(W_{\beta+1}\cap W_\beta)} \lesssim \sum_{\beta=1}^{\gamma} \|u - \Pi_{W_\beta}u\|_{L^1(W_\beta)}\end{aligned}$$

using equivalence of norms on $N(\mathbb{A})$. Finally, using again Theorem 3.2 in conjunction with (B4),

$$\|\Pi_{B_{j,m}}u - \Pi_{B^\sharp_{j,m}}u\|_{L^1(B_{j,m})} \lesssim \ell(B_{j,m}) \sum_{\beta=1}^{\gamma} |\mathbb{A}u|(W_\gamma) \lesssim \ell(B_{j,m})|\mathbb{A}u|(\Omega(B_{j,m}, B^\sharp_{j,m})).$$

Gathering estimates, we arrive at the claim.

(c) First, by the inverse estimate for polynomials, we have

$$|B_{j,m}|\,\|\Pi_{j+1,k}u - \Pi_{j,m}u\|_{L^\infty(B_{j,m})} \lesssim \|\Pi_{j+1,k}u - \Pi_{j,m}u\|_{L^1(B_{j,m})} = \|\Pi_{B^\sharp_{j+1,k}}u - \Pi_{B^\sharp_{j,m}}u\|_{L^1(B_{j,m})}.$$

Now, connecting $B^\sharp_{j+1,k}$ and $B^\sharp_{j,m}$ via the chain $\Omega(B^\sharp_{j+1,k}, B^\sharp_{j,m})$ (recall (B3)), we obtain the claim arguing exactly as in (b). □

The following lemma shows that T_j is well-defined on $L^1(\Omega)$.

Lemma 4.7. $T_j : L^1(\Omega) \to L^1(\Omega)$ *is linear and bounded.*

Proof. We estimate pointwise on Ω

$$|T_j u| \le (1 - \rho_j)|u| + \rho_j \sum_k \chi_{B_{j,k}} \|\Pi_{j,k}u\|_{L^\infty(B_{j,k})}. \tag{4-4}$$

With Lemma 4.6 we get

$$|T_j u| \lesssim \chi_{\Omega\setminus U_{j+1}}|u| + \sum_{k:B_{j,k}\cap U_j\neq\varnothing} \chi_{B_{j,k}} \fint_{B^\sharp_{j,k}} |u|\,dx.$$

This implies

$$\begin{aligned}\|T_j u\|_{L^1(\Omega)} &\lesssim \|u\|_{L^1(\Omega\setminus U_{j+1})} + \sum_{k:B_{j,k}\cap U_j\neq\varnothing} |B_{j,k}| \fint_{B^\sharp_{j,k}} |u|\,dx \\ &\lesssim \|u\|_{L^1(\Omega\setminus U_{j+1})} + \sum_{k:B_{j,k}\cap U_j\neq\varnothing} \int_{B^\sharp_{j,k}} |u|\,\mathrm{d}x.\end{aligned}$$

Since the $B^\sharp_{j,k}$ are locally finite by (B4), we get $\|T_j u\|_{L^1(\Omega)} \lesssim \|u\|_{L^1(\Omega)}$ as desired. □

The next two lemmas show now that $T_{j+1}u - T_j u$ is summable in $L^1(\Omega)$ and $\mathrm{BV}^{\mathbb{A}}(\Omega)$.

Lemma 4.8. *Let $u \in L^1(\Omega)$ and $j \geq j_0$. Then*

$$\|T_{j+1}u - T_j u\|_{L^1(\Omega)} \lesssim \|u\|_{L^1(U_{j+1-k_0}\setminus U_{j+k_0})}.$$

Proof. Let $j \geq j_0$. Then we have

$$T_{j+1}u - T_j u = (\rho_j - \rho_{j+1})u + \rho_{j+1} \sum_k \eta_{j+1,k}\Pi_{j+1,k}u - \rho_j \sum_m \eta_{j,m}\Pi_{j,m}u.$$

Now

$$\|(\rho_j - \rho_{j+1})u\|_{L^1(\Omega)} \leq \|u\|_{L^1(U_j\setminus U_{j+2})}.$$

Moreover, by Lemma 4.6 (a) it follows that

$$\|\rho_j\eta_{j,m}\Pi_{j,m}u\|_{L^1(\Omega)} \leq c|B_{j,m}|\,\|\Pi_{j,m}u\|_{L^\infty(B_{j,m})} \leq c\|u\|_{L^1(B^\sharp_{j,m})},$$

where it suffices to consider those j with $B_{j,m}\cap U_j \neq \varnothing$. Now (B4) implies

$$\sum_m \|\rho_j\eta_{j,m}\Pi_{j,m}u\|_{L^1(\Omega)} \leq c\|u\|_{L^1(U_{j-k_0}\setminus U_{j+k_0})}.$$

Analogously,

$$\sum_k \|\rho_j\eta_{j+1,k}\Pi_{j+1,k}u\|_{L^1(\Omega)} \leq c\|u\|_{L^1(U_{j+1-k_0}\setminus U_{j+1+k_0})}.$$

Combining the above estimates proves the lemma. □

Lemma 4.9. *Let $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$ and $j \geq j_0$. Then*

$$\|\mathbb{A}(T_{j+1}u - T_j u)\|_{L^1(\Omega)} \lesssim |\mathbb{A}u|(U_{j-k_0}\setminus U_{j+k_0}).$$

Proof. Using that $\sum_m \eta_{j,m} = \sum_k \eta_{j+1,k} = 1$ in Ω we get

$$T_{j+1}u - T_j u = (\rho_j - \rho_{j+1})\sum_m \eta_{j,m}(u - \Pi_{j,m}u) + \rho_{j+1}\sum_{k,m}\eta_{j+1,k}\eta_{j,m}(\Pi_{j+1,k}u - \Pi_{j,m}u) =: I + II. \quad (4\text{-}5)$$

In order to estimate $\|\mathbb{A}(T_{j+1}u - T_j u)\|_{L^1(\Omega)}$ it is crucial that $\mathbb{A}\Pi_{j+1,k}u = \mathbb{A}\Pi_{j,m}u = 0$ and the gradients of ρ_j, ρ_{j+1}, $\eta_{j,m}$, and $\eta_{j+1,k}$ are bounded by 2^j; recall (4-2). Let us consider II. We only have to estimate those summands with k, m satisfying $B_{j+1,k}\cap B_{j,m} \neq \varnothing$ since otherwise $\eta_{j+1,k}\eta_{j,m} = 0$. For each such k, m we estimate the $L^1(\Omega)$-norm of $\mathbb{A}II$ by Lemma 4.6(c)enumi. Now, in combination with (B4) we get

$$\|\mathbb{A}II\|_{L^1(\Omega)} \lesssim |\mathbb{A}u|(U_{j-k_0}\setminus U_{j+k_0}).$$

Let us consider I. We only need to estimate those summands with m satisfying $B_{j,m}\cap(U_j\setminus U_{j+2})\neq\varnothing$, since otherwise $(\rho_j-\rho_{j+1})\eta_{j,m}=0$. For each such m we estimate the $L^1(\Omega)$-norm of $\mathbb{A}I$ by Lemma 4.6(b)enumi. Now, in combination with (B4) we get

$$\|\mathbb{A}I\|_{L^1(\Omega)} \lesssim |\mathbb{A}u|(U_{j-k_0}\setminus U_{j+k_0}). \qquad \square$$

Based on the two lemmas above, we now study the convergence $T_j u \to u$.

Corollary 4.10. *If $u\in L^1(\Omega)$, then*

$$u = T_{j_0}u + \sum_{l=j_0}^{\infty}(T_{l+1}u - T_l u) = \lim_{j\to\infty} T_j u \tag{4-6}$$

in $L^1(\Omega)$. If additionally $u\in \mathrm{BV}^{\mathbb{A}}(\Omega)$, then (4-6) *also holds in* $\mathrm{BV}^{\mathbb{A}}(\Omega)$.

Proof. Since $\rho_j\to 0$ in $L^1_{\mathrm{loc}}(\Omega)$, it is clear that $T_j u\to u$ in $L^1_{\mathrm{loc}}(\Omega)$.

Note that for $j\geq j_0$

$$T_j u = T_{j_0}u + \sum_{l=j_0}^{j-1}(T_{l+1}u - T_l u). \tag{4-7}$$

It follows from Lemmas 4.8 and 4.9 that $T_{l+1}u - T_l u$ is summable in $L^1(\Omega)$, resp. in $\mathrm{BV}^{\mathbb{A}}(\Omega)$, since the $U_{j+1-k_0}\setminus U_{j+k_0}$ are locally finite with respect to j. Hence, $T_j u$ is a Cauchy sequence in $L^1(\Omega)$, resp. in $\mathrm{BV}^{\mathbb{A}}(\Omega)$. Since the limit must agree with the $L^1_{\mathrm{loc}}(\Omega)$ limit, which is u, the claim follows. $\square$

Since $T_j u$ is smooth close to the boundary $\partial\Omega$, it is possible to evaluate the classical trace $\mathrm{tr}(T_j u)$. We now show that these traces form a $L^1(\partial\Omega)$-Cauchy sequence.

Lemma 4.11. *Let $u\in \mathrm{BV}^{\mathbb{A}}(\Omega)$. Then*

$$\|\mathrm{tr}(T_{j+1}u)-\mathrm{tr}(T_j u)\|_{L^1(\partial\Omega)} \lesssim |\mathbb{A}u|(U_{j-k_0}\setminus U_{j+k_0}),$$
$$\|\mathrm{tr}(T_{j_0}u)\|_{L^1(\partial\Omega)} \lesssim 2^{j_0}\|u\|_{L^1(U_{j_0-k_0}\setminus U_{j_0+k_0})}.$$

Proof. We begin with the first estimate. It follows from (4-5) that

$$\mathrm{tr}(T_{j+1}u)-\mathrm{tr}(T_j u) = \sum_{k,m}\mathrm{tr}(\eta_{j+1,k}\eta_{j,m}(\Pi_{j+1,k}u - \Pi_{j,m}u)),$$

where the sums are locally finite sums. Hence,

$$\|\mathrm{tr}(T_{j+1}u)-\mathrm{tr}(T_j u)\|_{L^1(\partial\Omega)} \leq \sum_{k,m}\|\mathrm{tr}(\eta_{j+1,k}\eta_{j,m}(\Pi_{j+1,k}u - \Pi_{j,m}u))\|_{L^1(\partial\Omega)}.$$

We only have to consider those k,m with $B_{j+1,k}\cap B_{j,m}\neq\varnothing$. For such k,m

$$\|\mathrm{tr}(\eta_{j+1,k}\eta_{j,m}(\Pi_{j+1,k}u - \Pi_{j,m}u))\|_{L^1(\partial\Omega)} \leq \|\Pi_{j+1,k}u-\Pi_{j,m}u\|_{L^\infty(B_{j,m})}\,\mathcal{H}^{n-1}(\partial\Omega\cap B_{j+1,k}\cap B_{j,m}).$$

We estimate the first factor by Lemma 4.6(c)enumi and the second by the Ahlfors regularity of the boundary, see (4-1), and thereby obtain

$$\|\mathrm{tr}(\eta_{j+1,k}\eta_{j,m}(\Pi_{j+1,k}u - \Pi_{j,m}u))\|_{L^1(\partial\Omega)} \lesssim |\mathbb{A}u|(\Omega(B^\sharp_{j+1,k}, B^\sharp_{j,m})).$$

Summing over k and m and using (B4) implies

$$\|\mathrm{tr}(T_{j+1}u) - \mathrm{tr}(T_j u)\|_{L^1(\partial\Omega)} \lesssim |\mathbb{A}u|(U_{j-k_0} \setminus U_{j+k_0}).$$

This proves the first estimate.

Let us now estimate $\|\mathrm{tr}(T_{j_0})\|_{L^1(\partial\Omega)}$. We begin with

$$\mathrm{tr}(T_{j_0}) = \sum_k \mathrm{tr}(\eta_{j_0,k}\Pi_{j_0,k}u).$$

For each k with $B_{j_0,k} \cap \partial\Omega$ there holds

$$\|\mathrm{tr}(\eta_{j_0,k}\Pi_{j_0,k}u)\|_{L^1(\partial\Omega)} \le \|\Pi_{j_0,k}u\|_{L^\infty(B_{j_0,k})}\mathcal{H}^{n-1}(\partial\Omega \cap B_{j_0,k}).$$

We estimate the first factor by Lemma 4.6(a)enumi and the second by the Ahlfors regularity of the boundary; see (4-1). This gives

$$\|\mathrm{tr}(\eta_{j_0,k}\Pi_{j_0,k}u)\|_{L^1(\partial\Omega)} \lesssim \frac{1}{\ell(B_{j_0})}\int_{B^\sharp_{j_0,k}} |u|\,\mathrm{d}x.$$

Summing over k and m and using (B4) implies

$$\|\mathrm{tr}(T_{j_0}u)\|_{L^1(\partial\Omega)} \lesssim 2^{j_0}\|u\|_{L^1(U_{j_0-k_0}\setminus U_{j_0+k_0})}. \qquad \square$$

Recall that by Corollary 4.10 we have

$$u = T_{j_0}u + \sum_{l=j_0}^{\infty}(T_{l+1}u - T_l u) = \lim_{j\to\infty} T_j u$$

in $\mathrm{BV}^{\mathbb{A}}(\Omega)$. Moreover, Lemma 4.11 shows that

$$\mathrm{tr}(T_{j_0}u) + \sum_{j\ge j_0}(\mathrm{tr}(T_{j+1}u) - \mathrm{tr}(T_j u)) = \lim_{j\to\infty}\mathrm{tr}(T_j(u))$$

is well-defined in $L^1(\partial\Omega)$. Finally,

$$\begin{aligned}\Big\|\lim_{j\to\infty}\mathrm{tr}(T_j(u))\Big\|_{L^1(\partial\Omega)} &\le \|\mathrm{tr}(T_{j_0}(u))\|_{L^1(\partial\Omega)} + \sum_{j\ge j_0}\|\mathrm{tr}(T_{j+1}u) - \mathrm{tr}(T_j u)\|_{L^1(\partial\Omega)} \\ &\lesssim 2^{j_0}\|u\|_{L^1(U_{j_0-k_0}\setminus U_{j_0+k_0})} + \sum_{j\ge j_0}|\mathbb{A}u|(U_{j-k_0}\setminus U_{j+k_0}) \\ &\lesssim \|u\|_{L^1(\Omega)} + |\mathbb{A}u|(\Omega)\end{aligned}$$

by Lemma 4.11. This allows us to define for every $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$ a trace

$$\widetilde{\mathrm{tr}}(u) := \lim_{j\to\infty}\mathrm{tr}(T_j u), \tag{4-8}$$

the limit being understood in the $L^1(\partial\Omega)$-sense. This limit satisfies

$$\|\widetilde{\mathrm{tr}}(u)\|_{L^1(\partial\Omega)} \lesssim \|u\|_{L^1(\Omega)} + |\mathbb{A}u|(\Omega). \tag{4-9}$$

We now show that $\widetilde{\mathrm{tr}}$ coincides with tr for all smooth functions and hence start with an approximation result.

Lemma 4.12. *Let $u \in C^0(\overline{\Omega})$ be uniformly continuous. Then $T_j u \to u$ in $C^0(\overline{\Omega})$.*

Proof. We have

$$u - T_j u = \rho_j \sum_k \eta_{j,k}(u - \Pi_{j,k} u),$$

where it suffices to take the sum over those k with $B_{j,k} \cap U_j \neq \varnothing$. Let us take one of those k. We will show that $\|\eta_{j,k}(u - \Pi_{j,k}u)\|_{L^\infty(\Omega)}$ is small for large j. Since the $B_{j,k}$ are locally finite with respect to k (with a covering number independent of j), this will prove the lemma.

Since $\mathbb{A}$ maps constants to zero, the projections $\Pi_{j,k}$ map constants to themselves. Let $\langle u \rangle_{B^\sharp_{j,k}} := ⨍_{B^\sharp_{j,k}} u \, \mathrm{d}x$; then by Lemma 4.6(a)enumi

$$\begin{aligned}\|\eta_{j,k}(u - \Pi_{j,k}u)\|_{L^\infty(B_{j,k})} &\leq \|u - \langle u \rangle_{B^\sharp_{j,k}}\|_{L^\infty(B_{j,k})} + \|\Pi_{j,k}(u - \langle u \rangle_{B^\sharp_{j,k}})\|_{L^\infty(B_{j,k})} \\ &\lesssim \|u - \langle u \rangle_{B^\sharp_{j,k}}\|_{L^\infty(B_{j,k})} + ⨍_{B^\sharp_{j,k}} |u - \langle u \rangle_{B^\sharp_{j,k}}| \, dx.\end{aligned}$$

Since u is uniformly continuous, the $B_{j,k}$ and $B^\sharp_{j,k}$ are small and close to each other, see (B1), and we see that both expressions on the right-hand side are small for large j uniformly in k. □

Corollary 4.13. *Let $u \in \mathrm{BV}^{\mathbb{A}}(\Omega) \cap C^0(\overline{\Omega})$ be uniformly continuous. Then $\widetilde{\mathrm{tr}}(u) = \mathrm{tr}(u)$.*

Proof. We see from Corollary 4.10 and Lemma 4.12 that $T_j u \to u$ in $\mathrm{BV}^{\mathbb{A}}(\Omega)$ and in $C^0(\overline{\Omega})$. By the definition of $\widetilde{\mathrm{tr}}(u)$, we have $\mathrm{tr}(T_j u) \to \widetilde{\mathrm{tr}}(u)$. Since $T_j u \to u$ in $C^0(\overline{\Omega})$, we also have $\mathrm{tr}(T_j u) \to \mathrm{tr}(u)$ in $C^0(\partial\Omega)$. The limits must agree in $L^1_{\mathrm{loc}}(\partial\Omega)$, so $\widetilde{\mathrm{tr}}(u) = \mathrm{tr}(u)$. □

We have already seen that $\widetilde{\mathrm{tr}} : \mathrm{BV}^{\mathbb{A}}(\Omega) \to L^1(\partial\Omega)$ is continuous with respect to the norm topology. We wish to use this to conclude that $\widetilde{\mathrm{tr}}$ is the only extension of the classical trace to $\mathrm{BV}^{\mathbb{A}}(\Omega)$. However, as smooth functions are not dense in $\mathrm{BV}^{\mathbb{A}}$ with respect to the norm topology, we switch to strict convergence as in the BV-case.

Lemma 4.14. *The trace operator $\widetilde{\mathrm{tr}} : \mathrm{BV}^{\mathbb{A}}(\Omega) \to L^1(\partial\Omega; \mathbb{R}^N)$ is continuous with respect to the strict convergence of $\mathrm{BV}^{\mathbb{A}}(\Omega)$.*

Proof. Let $u, u_k \in \mathrm{BV}^{\mathbb{A}}(\Omega)$ with $u_k \xrightarrow{s} u$ and $m \in \mathbb{N}$.

It follows from the definition (4-3) of T_j that for $j > m + k_0$ there holds for all $v \in \mathrm{BV}^{\mathbb{A}}(\Omega)$

$$T_j(\rho_m v) = \rho_m \, T_j v.$$

Indeed, $\rho_m = 1$ on the $B_{j,k}$ and the $B^\sharp_{j,k}$ for all m that contribute to the sum in (4-3).

This implies

$$\widetilde{\mathrm{tr}}(v) = \lim_{j\to\infty} \mathrm{tr}(T_j v) = \lim_{j\to\infty} \mathrm{tr}(T_j(\rho_m v)) = \widetilde{\mathrm{tr}}(\rho_m v) \quad \text{in } L^1(\partial\Omega).$$

Now, for all $k \in \mathbb{N}$,

$$\|\widetilde{\mathrm{tr}}(u_k - u)\|_{L^1(\partial\Omega)} = \|\widetilde{\mathrm{tr}}(\rho_m(u_k - u))\|_{L^1(\partial\Omega)}.$$

Thus, by (4-9)

$$\begin{aligned}\|\widetilde{\mathrm{tr}}(u_k-u)\|_{L^1(\partial\Omega)} &\lesssim \|\rho_m(u_k-u)\|_{L^1(\Omega)}+|\mathbb{A}(\rho_m(u_k-u))|(\Omega)\\ &\lesssim \|u_k-u\|_{L^1(\Omega)}+|\mathbb{A}u_k|(U_m)+|\mathbb{A}u_k|(U_m)+2^{-m}\|u_k-u\|_{L^1(U_m)}.\end{aligned}$$

Now, let $k,l\to\infty$. Since $u_k\xrightarrow{s}u$ in $\mathrm{BV}^{\mathbb{A}}(\Omega)$ and U_m is open, we get

$$\|\widetilde{\mathrm{tr}}(u_k-u)\|_{L^1(\partial\Omega)}\lesssim|\mathbb{A}u|(U_m).$$

The right-hand side converges to zero for $m\to\infty$. Thus $\widetilde{\mathrm{tr}}(u_k)\to\widetilde{\mathrm{tr}}(u)$ in $L^1(\partial\Omega)$ for $k\to\infty$. □

In order to proceed, we need a smooth approximation result up to the boundary in the area-strict topology.

Lemma 4.15. *Let $u\in\mathrm{BV}^{\mathbb{A}}(\Omega)$. Then there exists $u_j\in C^\infty(\overline{\Omega})$ with $u_j\xrightarrow{\langle\cdot\rangle}u$ in $\mathrm{BV}^{\mathbb{A}}(\Omega)$.*

Proof. For $j\geq j_0$ consider T_ju. Then T_ju is C^∞ in $\overline{U}_{j+1}$. Indeed, for all $x\in U_{j+1}$ we have

$$(T_ju)(x)=\sum_k\eta_{j,k}\Pi_{j,k}u.$$

For each k with $B_{j,k}\cap U_{j+1}\neq 0$ we have

$$\|\nabla(\eta_{j,k}\Pi_{j,k}u)\|_\infty\lesssim\|\nabla\eta_{j,k}\|_{L^\infty(B_{j,k})}\|\Pi_{j,k}u\|_{L^\infty(B_{j,k})}+\|\nabla\Pi_{j,k}u\|_{L^\infty(B_{j,k})}.$$

Using inverse estimates for polynomials and Lemma 4.6 we get

$$\|\nabla(\eta_{j,k}\Pi_{j,k}u)\|_\infty\lesssim\ell(B_{j,k})|B_{j,k}|\,\|\Pi_{j,k}u\|_{L^1(B_{j,k})}\lesssim 2^{j(n+1)}\|u\|_{L^1(B^\sharp_{j,k})}.$$

Hence, T_ju is uniformly continuous on $\overline{U}_{j+1}$.

Now, let $\eta_\epsilon:\mathbb{R}^n\to\mathbb{R}$ be an standard mollifier (even and nonnegative). It is well known that $u_{j,\epsilon}:=\rho_{j+1}T_ju+((1-\rho_{j+1})T_ju)*\eta_\epsilon$ converges to T_ju as $\epsilon\searrow 0$ in $L^1(\Omega)$ as well as in the area-strict sense. Hence, we can find ϵ_j such that

$$\begin{aligned}\|u_{j,\epsilon_j}-T_ju\|_{L^1(\Omega)}&\leq 2^{-j},\\ \big||\mathbb{A}(T_ju)|(\Omega)-|\mathbb{A}(u_{j,\epsilon_j})|(\Omega)\big|&\leq 2^{-j}.\end{aligned}$$

Moreover, recall that $T_ju\to u$ strongly in $\mathrm{BV}^{\mathbb{A}}(\Omega)$. This implies that $u_j:=u_{j,\epsilon_j}$ has the desired property. This proves the strict convergence. The area-strict convergence follows by the same steps. □

As a consequence of Lemmas 4.14 and 4.15 we immediately obtain the following corollary.

Corollary 4.16. *The operator $\widetilde{\mathrm{tr}}:\mathrm{BV}^{\mathbb{A}}(\Omega)\to L^1(\partial\Omega;\mathcal{H}^{n-1})$ is the unique strictly continuous extension of the classical trace on $\mathrm{BV}^{\mathbb{A}}(\Omega)\cap C^0(\overline{\Omega})$.*

Due to the above results it is not anymore necessary to distinguish the classical trace and our new trace. We collect our results proven so far in the following theorem.

Theorem 4.17. *Let $\mathbb{A}$ be $\mathbb{C}$-elliptic and let Ω be an NTA domain with Ahlfors regular boundary* (*see Assumption 4.5*). *Then there exists a trace operator $\mathrm{tr}:\mathrm{BV}^{\mathbb{A}}(\Omega)\to L^1(\partial\Omega,\mathcal{H}^{n-1})$ such that the following hold*:

(a) $\operatorname{tr}(u)$ *coincides with the classical trace for all* $u \in \mathrm{BV}^{\mathbb{A}}(\Omega) \cap C^0(\Omega)$.

(b) $\operatorname{tr}(u)$ *is the unique strictly continuous extension of the classical trace on* $\mathrm{BV}^{\mathbb{A}}(\Omega) \cap C^0(\overline{\Omega})$.

(c) $\operatorname{tr}(W^{\mathbb{A},1}(\Omega)) = \operatorname{tr}(\mathrm{BV}^{\mathbb{A}}(\Omega)) = L^1(\partial\Omega, \mathcal{H}^{n-1})$.

Proof. The existence of tr is shown in Lemma 4.14. Part (a) follows from Corollary 4.13, whereas (b) is a consequence of Corollary 4.16. Finally, part (c) is a consequence of the fact that

$$\operatorname{tr}(W^{1,1}(\Omega; \mathbb{R}^N)) = L^1(\partial\Omega; \mathbb{R}^N)$$

and $W^{1,1}(\Omega; \mathbb{R}^N) \subset W^{\mathbb{A},1}(\Omega)$. In particular, the sufficiency part of Theorem 1.2 is complete. □

4C. ***Necessity of*** $\mathbb{C}$***-ellipticity.*** In this section we show that it is not possible to define an L^1-trace of $\mathrm{BV}^{\mathbb{A}}$-functions if the operator $\mathbb{A}$ is not $\mathbb{C}$-elliptic. As such, we extend the observation of [Fuchs and Repin 2010] that $\mathbb{D} \ni z \mapsto 1/(z-1) \in \mathbb{C}$ is holomorphic and belongs to $L^1(\mathbb{D}; \mathbb{C})$ but does not belong to $L^1(\partial\mathbb{D}; \mathbb{C})$; see Example 2.2(c).

Theorem 4.18 (without a trace). *Suppose that* $\mathbb{A}$ *is not* $\mathbb{C}$*-elliptic. Let* B *denote the unit ball of* $\mathbb{R}^n$*. Then there exists a vector* $\xi_1 \in \mathbb{R}^n \setminus \{0\}$ *such that for the half-ball* $B^+ := \{x \in B : \langle \xi_1, x\rangle > 0\}$ *and the hyperplane* $\mathfrak{H} := \{x \in \mathbb{R}^n : \langle \xi_1, x\rangle = 0\}$ *there exists a function* $u \in W^{\mathbb{A},1}(B^+) \cap C^\infty(B^+)$ *such that* $u \notin L^1(\mathfrak{H} \cap B, \mathcal{H}^{n-1})$.

Proof. We begin with the case that $\mathbb{A}$ is not $\mathbb{R}$-elliptic. Let us define $f(x_1, x_2) := (|x_1| + |x_2|^2)^{-3/4}$. The crucial observation now is that $f, \partial_2 f \in L^1(B)$. However, $f \notin L^1(\{x_1 = 0\}|_B, \mathcal{H}^{n-1})$. We have to adapt this example to our situation. Since $\mathbb{A}$ is not $\mathbb{R}$ elliptic, there exists $\xi_1 \in \mathbb{R}^n \setminus \{0\}$ and $\eta_1 \in \mathbb{R}^N \setminus \{0\}$ with $\mathbb{A}[\xi_1]\eta_1 = 0$. We choose $\xi_2, \dots, \xi_n$ such that $\xi_1, \dots, \xi_n$ is a basis. Now, define $\tau : \mathbb{R}^n \to \mathbb{R}^2$ and $\sigma : \mathbb{R} \to \mathbb{R}^N$ by $\tau(x) := (\langle \xi_1, x\rangle, \langle \xi_2, x\rangle)$ and $\sigma(z) := z\,\eta_1$. Moreover, we define $h_f : \mathbb{R}^n \to \mathbb{R}^N$ by $h_f := \sigma \circ f \circ \tau$. Then we obtain

$$(\mathbb{A}h_f)(x) = \sum_{j=1}^{2} \mathbb{A}[\xi_j]\eta_1(\partial_j f)(\tau(x))$$

(compare (2-10)). Since $\mathbb{A}[\xi_1]\eta_1 = 0$, this simplifies to

$$(\mathbb{A}h_f)(x) = \mathbb{A}[\xi_2]\eta_1(\partial_2 f)(\tau(x)).$$

We choose the hyperplane $\mathfrak{H} := \{x : \langle \xi_1, x\rangle = 0\}$. It follows from $f, \partial_2 f \in L^1(B)$ and $f \notin L^1(\{x_1 = 0\}|_B, \mathcal{H}^{n-1})$ that $u, \mathbb{A}u \in L^1(B)$ and so in particular $u, \mathbb{A}u \in L^1(B^+)$ with $B^+ := \{x \in B : \langle \xi_1, x\rangle > 0\}$ but $u \notin L^1(\mathfrak{H} \cap B, \mathcal{H}^{n-1})$. This concludes the proof in the case that $\mathbb{A}$ is not $\mathbb{R}$-elliptic.

Assume now that $\mathbb{A}$ is $\mathbb{R}$-elliptic but not $\mathbb{C}$-elliptic. Then as in the proof of Lemma 2.5 there exist $\xi_1, \xi_2 \in \mathbb{R}^n$ and $\eta_1, \eta_2 \in \mathbb{R}^n$, which are, resp., linearly independent such that

$$\mathbb{A}[\xi_1 + i x_2](\eta_1 + i\eta_2) = 0.$$

Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) := \frac{1}{z}$. Then $f \in L^1(B_1)$ with $B_1 := \{|z| < 1\}$ but $f \notin L^1(\{\operatorname{Re}(z) = 0\}|_{B_1}, \mathcal{H}^{n-1})$. As in Lemma 2.5 we define $\tau : \mathbb{R}^n \to \mathbb{C}$ and $\sigma : \mathbb{C} \to \mathbb{R}^N$ by $\tau(x) := \langle \xi, x\rangle = \langle \xi_1, x\rangle + i\langle \xi_2, x\rangle$ and

$\sigma(z) := \mathrm{Re}(z)\eta_1 - \mathrm{Im}(z)\eta_2$. Moreover, define $h_f\colon \mathbb{R}^n \to \mathbb{R}^N$ by $h_f := \sigma \circ f \circ \tau$. Then as in Lemma 2.5 we have $(\mathbb{A}h_f)(x) = 0$ in $\mathcal{D}'(B^+)$ with $B^+ := \{x \in B\colon \langle x_1, x\rangle > 0\}$. It follows from $f \in L^1(B^+)$ and $f \notin L^1(\{\mathrm{Re}(z) = 0\}|_{B_1}, \mathcal{H}^{n-1})$ that $h_f \in W^{\mathbb{A},1}(B)$ but $h_f \notin L^1(\mathfrak{H} \cap B, \mathcal{H}^{n-1})$ with $\mathfrak{H} := \{x : \langle \xi_1, x\rangle = 0\}$. This concludes the proof if $\mathbb{A}$ is $\mathbb{R}$-elliptic but not $\mathbb{C}$-elliptic. □

Remark 4.19. Theorem 4.18 shows the nonexistence of a trace on some particular boundary hyperplane. If Ω does not enjoy this simple geometry but is a bounded domain with C^∞-boundary, then we choose a boundary point $x_0 \in \partial\Omega$ such that a suitable translation of the hyperplanes $\mathfrak{H}$ from the preceding proof becomes tangent to $\partial\Omega$ at x_0. In this situation, flattening the boundary locally around x_0 and applying the preceding theorem directly yield the nonexistence of boundary traces in $L^1(\partial\Omega; \mathcal{H}^{n-1})$. We leave the details to the reader.

4D. ***Gauss–Green formula.*** We now deduce the Gauss–Green formula for functions from $\mathrm{BV}^{\mathbb{A}}(\Omega)$, which, with Theorem 1.2 at our disposal, is a direct consequence of the Gauss–Green formula for smooth functions. Let us note that up to here, only Assumption 4.5 is required, whereas in what follows we stick to a Lipschitz assumption[2] on $\partial\Omega$.

Theorem 4.20 (Gauss–Green formula). *Let $\Omega \subset \mathbb{R}^n$ be open and bounded with Lipschitz boundary. For all $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$ and all $\phi \in C^1(\overline{\Omega}; \mathbb{R}^N)$ we have*

$$\int_\Omega \mathbb{A}u \cdot \phi \,\mathrm{d}x = -\int_\Omega u \cdot \mathbb{A}^*\phi \,\mathrm{d}x + \int_{\partial\Omega} (\mathrm{tr}(u) \otimes_{\mathbb{A}} \nu) \cdot \phi \,\mathrm{d}\mathcal{H}^{n-1}, \tag{4-10}$$

where ν denotes the unit outer normal of Ω.

Proof. Due to Lemma 4.15 there exists a sequence $u_j \in C^\infty(\overline{\Omega})$ such that $u_j \xrightarrow{s} u$ in $\mathrm{BV}^{\mathbb{A}}(\Omega)$. Due to Lemma 4.14 we also have $u_j \to u$ in $L^1(\partial\Omega, \mathcal{H}^{n-1})$. Now, (4-10) is valid for each u_j. Passing to the limit proves the claim. □

Corollary 4.21. *Let $\Omega \Subset U \subset \mathbb{R}^n$ such that Ω and U are open and bounded and have Lipschitz boundary. For $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$ and $v \in \mathrm{BV}^{\mathbb{A}}(U \setminus \Omega)$ define $w := \chi_\Omega u + \chi_{U\setminus\Omega} v$. Then $w \in \mathrm{BV}^{\mathbb{A}}(U)$ and*

$$\mathbb{A}w = \mathbb{A}u \llcorner_\Omega + \mathbb{A}v \llcorner_{U\setminus\Omega} + (\mathrm{tr}^+(v) - \mathrm{tr}^-(u)) \otimes_{\mathbb{A}} \nu \mathcal{H}^{n-1} \llcorner_{\partial\Omega}, \tag{4-11}$$

where $\mathrm{tr}^+(u)$ denotes the interior trace of u and $\mathrm{tr}^-(v)$ denotes the exterior trace of v and ν the unit outer normal of Ω.

Proof. Let w be as given and let $\phi \in C_c^1(U)$. We split the domain U into Ω and $U \setminus \Omega$ and apply the Gauss–Green formula (4-10) first to U and then to Ω and $U \setminus \Omega$ separately. This yields

$$\begin{aligned} -\int_U w \cdot \mathbb{A}^*\phi \,\mathrm{d}x &= -\int_\Omega u \cdot \mathbb{A}^*\phi \,\mathrm{d}x - \int_{U\setminus\Omega} v \cdot \mathbb{A}^*\phi \,\mathrm{d}x \\ &= \int_\Omega \mathbb{A}u \cdot \phi \,\mathrm{d}x - \int_{\partial\Omega} (\mathrm{tr}^+(u) \otimes_{\mathbb{A}} \nu) \cdot \phi \,\mathrm{d}\mathcal{H}^{n-1} + \int_{U\setminus\Omega} \mathbb{A}v \cdot \phi \,\mathrm{d}x + \int_{\partial\Omega} (\mathrm{tr}^+(v) \otimes_{\mathbb{A}} \nu) \cdot \phi \,\mathrm{d}\mathcal{H}^{n-1}. \end{aligned}$$

This proves $w \in \mathrm{BV}^{\mathbb{A}}(U)$ and the representation formula (4-11). □

[2] In fact, this can be weakened towards more general domains, but we will not need this in the sequel.

4E. ***Sobolev spaces with zero boundary values.*** Using our trace operator, it is natural to define subspaces of functions with zero boundary values; i.e.,

$$W_0^{\mathbb{A},1}(\Omega) := \{u \in W^{\mathbb{A},1}(\Omega) : \operatorname{tr}(u) = 0\},$$
$$\mathrm{BV}_0^{\mathbb{A}}(\Omega) := \{u \in \mathrm{BV}^{\mathbb{A}}(\Omega) : \operatorname{tr}(u) = 0\}.$$

However, in the context of Sobolev spaces $W_0^{\mathbb{A},1}(\Omega)$ there are two more variants to define these spaces. One by zero extension and one by the closure of $C_c^\infty(\Omega)$. We will show below in Theorem 4.23 that all three definitions define the same spaces.

We begin with an auxiliary lemma which we need for $W_0^{\mathbb{A},1}(\Omega)$. For slightly more generality we state it for $\mathrm{BV}_0^{\mathbb{A}}(\Omega)$.

Lemma 4.22. *Let $u \in \mathrm{BV}_0^{\mathbb{A}}(\Omega)$. Then $(1-\rho_j)u \to u$ in $\mathrm{BV}^{\mathbb{A}}(\Omega)$, with ρ_j as in Section 4B.*

Proof. We can assume that $\Omega \Subset U \subset \mathbb{R}^n$ for some open, bounded U with Lipschitz boundary. By Corollary 4.21 we can extend u on $U \setminus \Omega$ by zero.

We have

$$\mathbb{A}((1-\rho_j)u - u) = -\rho_j \mathbb{A}u - u \otimes_{\mathbb{A}} \nabla \rho_j.$$

Hence,

$$|\mathbb{A}((1-\rho_j)u - u)|(\Omega) \le |\mathbb{A}u|(U_j) + c r_j^{-1} \|u\|_{L^1(U_j)}.$$

We will now show that

$$r_j^{-1} \|u\|_{L^1(U_j)} \lesssim |\mathbb{A}u|(U_{j-m})$$

for some $m \in \mathbb{N}$ (and sufficiently large, i.e., $j+m \ge j_0$). In fact, for fixed j define

$$K_j := \{k : B_{j,k} \cap U_j \neq \varnothing\}.$$

By the geometry of Ω, we can find a factor $\lambda > 0$ such that for each $k \in K_j$ the enlarged ball $\lambda B_{j,k}$ contains some ball $B'_{j,k}$ that is completely in $\mathbb{R}^n \setminus \Omega$. Now, for each $k \in K_j$, we get by Theorem 3.3

$$\|u\|_{L^1(B_{j,k})} \lesssim \|u\|_{L^1(\lambda B_{j,k})} \lesssim r_j |\mathbb{A}u|(\lambda B_{j,k}) = r_j |\mathbb{A}u|(\Omega \cap \lambda B_{j,k}).$$

Since the $(B_{j,k})_k$ are locally finite, so are the $(\lambda B_{j,k})_k$. Now, if we choose $m \in \mathbb{N}$ such that $\Omega \cap \lambda B_{j,k} \subset U_{j-m}$, then

$$r_j^{-1} \|u\|_{L^1(U_j)} \lesssim \sum_{k \in K_j} r_j^{-1} \|u\|_{L^1(B_{j,k})} \lesssim \sum_{k \in K_j} \|\mathbb{A}u\|_{L^1(\Omega \cap \lambda B_{j,k})} \lesssim |\mathbb{A}u|(U_{j-m}).$$

Overall, we obtain

$$|\mathbb{A}((1-\rho_j)u - u)|(\Omega) \le |\mathbb{A}u|(U_{j-m}).$$

Now, $|\mathbb{A}u|(U_{j-m}) \to 0$, since $U_{j-m} \searrow \varnothing$. This proves the claim by the Poincaré inequality from Theorem 3.3. □

Theorem 4.23 (zero traces). *Let $\Omega \Subset U \subset \mathbb{R}^n$ for some open, bounded U with Lipschitz boundary and let $u \in W^{\mathbb{A},1}(\Omega)$. The following are equivalent*:

(a) $u \in W_0^{\mathbb{A},1}(\Omega)$.

(b) *The extension* $\tilde{u} := \chi_\Omega u$ *by zero on* $U \setminus \Omega$ *is in* $W^{\mathbb{A},1}(U)$.

(c) *There exist* $u_k \in C_c^\infty(\Omega)$ *with* $u_k \to u$ *in* $W^{\mathbb{A},1}(\Omega)$.

Proof. (a)enumi ⇒ (b)enumi: Let $u \in W_0^{\mathbb{A},1}(\Omega)$ and let $\tilde{u} = \chi_\Omega u$ be its zero extension on U. Then by Corollary 4.21 we have $\mathbb{A}\tilde{u} = \mathbb{A}u \llcorner_\Omega \in L^1(U)$, so $\tilde{u} \in W^{\mathbb{A},1}(U)$.

(b)enumi ⇒ (a)enumi: Let $\tilde{u} = \chi_\Omega u \in W^{\mathbb{A},1}(U)$. Then by Corollary 4.21 we have $\mathbb{A}\tilde{u} = \mathbb{A}u \llcorner_\Omega + \operatorname{tr}^+(u) \otimes_{\mathbb{A}} \nu \mathcal{H}^{n-1} \llcorner_{\partial\Omega}$. Since $\mathbb{A}\tilde{u} \in L^1(U)$, the singular part must vanish; i.e., $\operatorname{tr}^+(u) \otimes_{\mathbb{A}} \nu \mathcal{H}^{n-1} \llcorner_{\partial\Omega} = 0$. So by $\mathbb{R}$-ellipticity of $\mathbb{A}$ we have $\operatorname{tr}^+(u) = 0$ on $\partial\Omega$.

(c)enumi ⇒ (a)enumi: By continuity of the trace operator we have $\operatorname{tr}(u) = \lim_{k\to\infty} \operatorname{tr}(u_k) = 0$ in $L^1(\partial\Omega)$, so $u \in W_0^{\mathbb{A},1}(\Omega)$.

(a)enumi ⇒ (c)enumi: Let $v_k := (1-\rho_k)u$ as in Lemma 4.22. Then $v_k \to u$ in $W^{\mathbb{A},1}(\Omega)$. Moreover, the v_k have compact support, since $v_k = 0$ on U_{k+1}. Now, let $\eta_\epsilon \colon \mathbb{R}^n \to \mathbb{R}$ be an standard mollifier with support on $B_\epsilon(0)$. Then we find ϵ_k such that

$$\|v_k - v_k * \phi_{\epsilon_k}\|_{L^1(\Omega)} + \|\mathbb{A}v_k - \mathbb{A}(v_k * \phi_{\epsilon_k})\|_{L^1(\Omega)} \le 2^{-k}$$

and $\operatorname{supp}(v_k * \phi_{\epsilon_k}) \Subset \Omega$. The sequence $u_k := v_k * \phi_{\epsilon_k}$ has the desired properties. □

Proposition 4.24 (trace-preserving area-strict smoothing). *Let* $\Omega \Subset U \subset \mathbb{R}^n$ *such that* Ω *and* U *are open and bounded and have Lipschitz boundary. Let* $u_0 \in W^{\mathbb{A},1}(U)$. *Further let* $u \in \mathrm{BV}^{\mathbb{A}}(U)$ *with* $u = u_0$ *on* $U \setminus \Omega$. *Then there exists* $u_j \in u_0 + C_c^\infty(\Omega)$ *such that* $u_j \xrightarrow{\langle\cdot\rangle} u$ *in* $\mathrm{BV}^{\mathbb{A}}(U)$.

Proof. The proof is a straightforward modification of the corresponding statement for BV-functions; see [Bildhauer 2003, Lemma B.2] or [Kristensen and Rindler 2010a, Lemma 1]. Let us just explain the basic idea: The usual localization argument by a partition of unity reduces the question to a local Lipschitz graph. Then split u into $u_0 + \chi_\Omega(u-u_0)$. Now the $\chi_\Omega(u-u_0)$ part is moved by translation slightly into Ω. In a second step it is mollified to get a $C_c^\infty(\Omega)$ term. □

5. The Dirichlet problem on $\mathrm{BV}^{\mathbb{A}}$

This final section is devoted to variational problems with linear growth involving $\mathbb{A}u$ subject to given boundary data.

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set with Lipschitz boundary. Our goal is to study the functional $\mathfrak{F} : W^{\mathbb{A},1}(\Omega) \to \mathbb{R}$ given by

$$\mathfrak{F}[v] := \int_\Omega f(x, \mathbb{A}v)\,\mathrm{d}x, \tag{5-1}$$

where f satisfies linear growth conditions. Given a boundary datum $u_0 \in W^{\mathbb{A},1}(\Omega)$, we wish to minimise $\mathfrak{F}$ within the Dirichlet class $u_0 + W_0^{\mathbb{A},1}(\Omega)$. The existence of a minimiser together with the precise formulation of the problem at our disposal will be given in Theorem 5.3 below.

Let us define the $\mathbb{A}$-*rank-one cone* $\mathscr{C}(\mathbb{A}) = \mathbb{R}^N \otimes_{\mathbb{A}} \mathbb{R}^n \subset \mathbb{R}^K$, with $\otimes_{\mathbb{A}}$ as given by (2-7). This cone is important to characterise the jump terms of $\mathrm{BV}^{\mathbb{A}}$ functions as in Corollary 4.21. Also in the product rule (2-8), we have $v \otimes_{\mathbb{A}} \nabla\phi \in \mathscr{C}(\mathbb{A})$ pointwise for $\phi \in C^1(\mathbb{R}^n)$ and $v \in C^1(\mathbb{R}^n; \mathbb{R}^N)$.

By use of the Fourier transform, we see that $\mathbb{A}(u) = (\mathbb{A}[\xi]\hat{u})^\vee$. Since $\mathbb{A}[\xi]\hat{u} \in \mathscr{C}(\mathbb{A})$ pointwise, we obtain $\mathbb{A}(u) \in \operatorname{span}(\mathscr{C}(\mathbb{A}))$ pointwise. Hence, we define the *effective range of* $\mathbb{A}$ as $\mathscr{R}(\mathbb{A}) := \operatorname{span}(\mathscr{C}(\mathbb{A})) \subset \mathbb{R}^K$; i.e., $\mathbb{A}u \in \mathscr{R}(\mathbb{A})$ pointwise. As a consequence, we only need to require that the second argument of f in (5-1) is from $\mathscr{R}(\mathbb{A})$. We assume that

$$f : \bar{\Omega} \times \mathscr{R}(\mathbb{A}) \to \mathbb{R} \quad \text{is continuous} \tag{5-2}$$

and satisfies the following linear growth assumption

$$c_1|z| \le f(x, z) \le c_2|z| + c_3 \tag{5-3}$$

for all $x \in \Omega$ and $z \in \mathscr{R}(\mathbb{A})$. Moreover, we require $\mathbb{A}$ to be $\mathbb{C}$-elliptic, which allows us to use the trace results of the previous sections.

Furthermore, we assume that there exists a modulus of continuity ω such that

$$|f(x, A) - f(y, A)| \le \omega(|x - y|)(1 + |A|) \tag{5-4}$$

holds for all $x, y \in \bar{\Omega}$ and all $A \in \mathscr{R}(\mathbb{A})$. In all of what follows, we tacitly stick to these assumptions.

We say that $g : \mathscr{R}(\mathbb{A}) \to \mathbb{R}$ is $\mathbb{A}$-*quasiconvex* if for all $\varphi \in W_0^{1,\infty}((0, 1)^n; \mathbb{R}^N)$ and $A \in \mathscr{R}(\mathbb{A})$ there holds

$$g(A) \le \int_{(0,1)^n} g(A + \mathbb{A}\varphi)\,\mathrm{d}x. \tag{5-5}$$

We say that $f : \bar{\Omega} \times \mathscr{R}(\mathbb{A}) \to \mathbb{R}$ is $\mathbb{A}$-quasiconvex if $f(x, \cdot)$ is $\mathbb{A}$-quasiconvex for each $x \in \bar{\Omega}$.

Let us link this notion of quasiconvexity to that of [Fonseca and Müller 1999, Definition 3.1]. Since $\mathbb{A}$ is $\mathbb{C}$-elliptic, it is also $\mathbb{R}$-elliptic. So by [Van Schaftingen 2013, Proposition 4.2], there exists $M \in \mathbb{N}$ and a linear, homogeneous constant-coefficient differential operator $\mathbb{L}$ with symbol mapping $\mathbb{L}[\xi]$ from $\mathbb{R}^K$ to $\mathbb{R}^M$ that *annihilates* $\mathbb{A}$ in the sense that the corresponding symbol complex

$$\mathbb{R}^N \xrightarrow{\mathbb{A}[\xi]} \mathbb{R}^K \xrightarrow{\mathbb{L}[\xi]} \mathbb{R}^M \tag{5-6}$$

is exact for every $\xi \in \mathbb{R}^n \setminus \{0\}$. In this situation, $\mathbb{A}$ is called a *potential* for $\mathbb{L}$, and $\mathbb{L}$ an *annihilator* for $\mathbb{A}$. Since $\mathbb{A}[\xi](\mathbb{R}^N)$ has the same dimension for all $\xi \neq 0$, the operator $\mathbb{L}$ has constant rank. Consequently, our $\mathbb{A}$-quasiconvexity equals the $\mathbb{L}$-quasiconvexity[3] of [Fonseca and Müller 1999]. By exactness of the above symbol complex (5-6), it is easy to see that the *wave cone* (or characteristic cone) $\Lambda_{\mathbb{L}} := \bigcup_{\xi \in \mathbb{R}^n \setminus \{0\}} \ker(\mathbb{L}[\xi])$ of $\mathbb{L}$ agrees with our $\mathbb{A}$-rank-one cone $\mathscr{C}(\mathbb{A})$.

We define the *strong recession function* $f^\infty : \bar{\Omega} \times \mathscr{R}(\mathbb{A}) \to \mathbb{R}$ by

$$f^\infty(x, A) := \lim_{\substack{x' \to x \\ A' \to A \\ t \to \infty}} \frac{f(x', tA')}{t}, \tag{5-7}$$

whenever the limit exists.

[3] In [Fonseca and Müller 1999], first-order annihilating operators are considered, and in general this is not the case in our situation (e.g., the symmetric gradient is annihilated by curl curl). However, the generalisation of the concept of $\mathbb{L}$-quasiconvexity extends to higher-order operators $\mathbb{L}$ in the obvious manner.

Since f is $\mathbb{A}$-quasiconvex, satisfies the linear growth condition (5-3), and satisfies the continuity condition (5-4), Lemma A.1 from the Appendix yields that f^∞ is automatically well-defined on $\overline{\Omega}\times\mathscr{C}(\mathbb{A})$.

As usual the Dirichlet class $u_0+W_0^{\mathbb{A},1}(\Omega)$ is not large enough to ensure the existence of minimisers for variational problems with linear growth. Here, the passage to $\mathrm{BV}^{\mathbb{A}}(\Omega)$ allows us to access the necessary sequential compactness. However, elements of $\mathrm{BV}^{\mathbb{A}}(\Omega)$ do not admit control over their *exterior* trace. To overcome this problem we proceed as in [Giaquinta et al. 1979a; 1979b] and pass to a larger superset U, i.e., let $\Omega \Subset U$ with ∂U Lipschitz. Now, we extend $\mathfrak{F}$ to $\mathrm{BV}^{\mathbb{A}}(U)$ and minimise over those $u\in\mathrm{BV}^{\mathbb{A}}(U)$ which agree with u_0 on $U\setminus\Omega$. For this, we further need to accomplish the following: First, we have to extend $f:\overline{\Omega}\times\mathscr{R}(\mathbb{A})\to\mathbb{R}$ to $f:\overline{U}\times\mathscr{R}(\mathbb{A})\to\mathbb{R}$, while preserving the structure of f; see Lemma A.2 in the Appendix. Second, we need to extend our boundary data to U, which is always possible, since $\mathrm{tr}(W^{\mathbb{A},1}(\Omega))=L^1(\partial\Omega,\mathcal{H}^{n-1})=\mathrm{tr}(W^{1,1}(U\setminus\Omega))$ by Theorem 4.17. In particular, we assume in the following that $u_0\in W^{\mathbb{A},1}(U)$.

We define the functional $\overline{\mathfrak{F}}_U:\mathrm{BV}^{\mathbb{A}}(U)\to\mathbb{R}$ by

$$\overline{\mathfrak{F}}_U[w]:=\int_U f\Big(x,\frac{\mathrm{d}\mathbb{A}w}{\mathrm{d}\mathscr{L}^n}\Big)\,\mathrm{d}x+\int_U f^\infty\Big(x,\frac{\mathrm{d}\mathbb{A}w}{\mathrm{d}|\mathbb{A}^s w|}\Big)\,\mathrm{d}|\mathbb{A}^s w|$$

and the Dirichlet class

$$\mathcal{D}_{u_0}=\{w\in\mathrm{BV}^{\mathbb{A}}(U):w=u_0 \text{ on } U\setminus\overline{\Omega}\}.$$

Hence, our aim is to minimise $\mathfrak{F}_U$ over $\mathcal{D}_{u_0}$. Later we will see that this minimisation can also be expressed only in terms of $\mathrm{BV}^{\mathbb{A}}(\Omega)$ with an additional term $f^\infty(\,\cdot\,,\mathrm{tr}(u-u_0)\otimes_{\mathbb{A}}\nu)$ which penalises the deviations from the correct boundary values; see Theorem 5.3.

We begin with a characterisation of the extension of $\mathfrak{F}:W^{\mathbb{A},1}(\Omega)\to\mathbb{R}$ to $\mathrm{BV}^{\mathbb{A}}(\Omega)$. For this, recall that $\Omega\subset\mathbb{R}^n$ is a bounded Lipschitz domain and that (5-2)–(5-5) are in action.

Proposition 5.1. *The functional* $\overline{\mathfrak{F}}:\mathrm{BV}^{\mathbb{A}}(\Omega)\to\mathbb{R}$ *given by*

$$\overline{\mathfrak{F}}[u]:=\int_\Omega f\Big(x,\frac{\mathrm{d}\mathbb{A}u}{\mathrm{d}\mathscr{L}^n}\Big)\,\mathrm{d}x+\int_\Omega f^\infty\Big(x,\frac{\mathrm{d}\mathbb{A}u}{\mathrm{d}|\mathbb{A}^s u|}\Big)\,\mathrm{d}|\mathbb{A}^s u|$$

is the $\mathbb{A}$*-area strict continuous extension of* $\mathfrak{F}:W^{\mathbb{A},1}(\Omega)\to\mathbb{R}$. *Moreover*, $\overline{\mathfrak{F}}[u]:\mathrm{BV}^{\mathbb{A}}(\Omega)\to\mathbb{R}$ *is sequentially weak*-lower semicontinuous on* $\mathrm{BV}^{\mathbb{A}}(\Omega)$.

Proof. We begin with the $\mathbb{A}$-area strict continuity of $\overline{\mathfrak{F}}:\mathrm{BV}^{\mathbb{A}}(\Omega)\to\mathbb{R}$. If f^∞ existed on all of $\overline{\Omega}\times\mathscr{R}(\mathbb{A})$, we could just use [Kristensen and Rindler 2010b, Theorem 4]. However, we can only rely on the existence of f^∞ on $\overline{\Omega}\times\mathscr{C}(\mathbb{A})$ due to Lemma A.1 from the Appendix. The following steps show how to overcome this technical issue and hence how the argument of [Kristensen and Rindler 2010b, Theorem 4] can be made to work.

Let us denote by $E(\overline{\Omega},\mathscr{R}(\mathbb{A}))$ those functions $g:\overline{\Omega}\times\mathscr{R}(\mathbb{A})\to\mathbb{R}$ such that

$$(x,\xi)\mapsto(1-|\xi|)g(x,(1-|\xi|)^{-1}\xi)$$

has a continuous extension to $\overline{\Omega \times \mathbb{B}_K}$; here, $\mathbb{B}_K$ denotes the unit ball in $\mathscr{R}(\mathbb{A})$. In particular, the strong recession function g^∞ exists on all of $\overline{\Omega} \times \mathscr{R}(\mathbb{A})$. Functionals with integrands from $E(\overline{\Omega}, \mathscr{R}(\mathbb{A}))$ enjoy good continuity properties.

Due to [Alibert and Bouchitté 1997, Lemma 2.3] there exists a sequence $f_k \in E(\overline{\Omega}, \mathscr{R}(\mathbb{A}))$ with

$$\sup_{k\in\mathbb{N}} f_k(x, A) = f(x, A) \quad \text{and} \quad \sup_{k\in\mathbb{N}} f_k^\infty(x, A) = f_\#(x, A) := \liminf_{\substack{x'\to x\\ A'\to A\\ t\to\infty}} \frac{f(x', tA')}{t}. \tag{5-8}$$

Let $u_j \xrightarrow{\langle\cdot\rangle} u$ in $\mathrm{BV}^{\mathbb{A}}(\Omega)$. Since $f_k \in E(\overline{\Omega}, \mathscr{R}(\mathbb{A}))$ we may apply the Reshetnyak-type continuity theorem in [Kristensen and Rindler 2010b, Theorem 5] to conclude

$$\begin{aligned}\liminf_{j\to\infty} \overline{\mathfrak{F}}[u_j] &\geq \liminf_{j\to\infty} \int_\Omega f_k\left(x, \frac{\mathrm{d}\mathbb{A}u_j}{\mathrm{d}\mathscr{L}^n}\right)\mathrm{d}x + \int_\Omega f_k^\infty\left(x, \frac{\mathrm{d}\mathbb{A}^s u_j}{\mathrm{d}|\mathbb{A}^s u_j|}\right)\mathrm{d}|\mathbb{A}^s u_j| \\ &= \int_\Omega f_k\left(x, \frac{\mathrm{d}\mathbb{A}u}{\mathrm{d}\mathscr{L}^n}\right)\mathrm{d}x + \int_\Omega f_k^\infty\left(x, \frac{\mathrm{d}\mathbb{A}^s u}{\mathrm{d}|\mathbb{A}^s u|}\right)\mathrm{d}|\mathbb{A}^s u|\end{aligned}$$

and so, by monotone convergence,

$$\int_\Omega f\left(x, \frac{\mathrm{d}\mathbb{A}u}{\mathrm{d}\mathscr{L}^n}\right)\mathrm{d}x + \int_\Omega f_\#\left(x, \frac{\mathrm{d}\mathbb{A}u}{\mathrm{d}|\mathbb{A}^s u|}\right)\mathrm{d}|\mathbb{A}^s u| \leq \liminf_{j\to\infty} \overline{\mathfrak{F}}[u_j].$$

Due to the generalisation of Alberti's celebrated rank-one theorem in [De Philippis and Rindler 2016], we know that $\mathrm{d}\mathbb{A}u/\mathrm{d}|\mathbb{A}^s u| \in \mathscr{C}(\mathbb{A})$ pointwisely $|\mathbb{A}^s u|$-a.e. Now, by Lemma A.1 from the Appendix, we find that $f_\# = f^\infty$ on $\overline{\Omega} \times \mathscr{C}(\mathbb{A})$. Hence

$$\overline{\mathfrak{F}}[u] = \int_\Omega f\left(x, \frac{\mathrm{d}\mathbb{A}u}{\mathrm{d}\mathscr{L}^n}\right)\mathrm{d}x + \int_\Omega f^\infty\left(x, \frac{\mathrm{d}\mathbb{A}u}{\mathrm{d}|\mathbb{A}^s u|}\right)\mathrm{d}|\mathbb{A}^s u| \leq \liminf_{j\to\infty} \overline{\mathfrak{F}}[u_j].$$

Since f is continuous, we may apply the same argument to $-f$ to obtain $\overline{\mathfrak{F}}[u] \geq \limsup_{j\to\infty} \overline{\mathfrak{F}}[u_j]$. Hence $\overline{\mathfrak{F}}[u] = \lim_{j\to\infty} \overline{\mathfrak{F}}[u_j]$. This proves that $\overline{\mathfrak{F}} : \mathrm{BV}^{\mathbb{A}}(\Omega) \to \mathbb{R}$ is $\mathbb{A}$-area strictly continuous.

Due to Lemma 4.15, $W^{\mathbb{A},1}(\Omega)$ is dense in $\mathrm{BV}^{\mathbb{A}}(\Omega)$ with respect to $\mathbb{A}$-area strict convergence. Since $\overline{\mathfrak{F}} = \mathfrak{F}$ on $W^{\mathbb{A},1}(\Omega)$, we see that $\overline{\mathfrak{F}} : \mathrm{BV}^{\mathbb{A}}(\Omega) \to \mathbb{R}$ is the $\mathbb{A}$-area strict extension of $\mathfrak{F} : W^{\mathbb{A},1}(\Omega) \to \mathbb{R}$.

It remains to prove the sequential weak*-lower semicontinuity of $\overline{\mathfrak{F}} : \mathrm{BV}^{\mathbb{A}}(\Omega) \to \mathbb{R}$ on $\mathrm{BV}^{\mathbb{A}}(\Omega)$. Let $\mathbb{L}$ be an $\mathbb{A}$-annihilating operator as in the exact sequence (5-6). Now, the sequential weak*-lower semicontinuity just follows from [Arroyo-Rabasa et al. 2018, Theorem 1.2] (note that f^∞ is well-defined on $\overline{\Omega} \times \mathscr{C}(\mathbb{A})$ due to Lemma A.1 from the Appendix). □

If we apply to our Dirichlet class $\mathcal{D}_{u_0}$, then we obtain the following results:

Corollary 5.2. *Let f satisfy* (5-2)–(5-5) *and let* $\overline{\mathfrak{F}}_{u_0} : \mathrm{BV}^{\mathbb{A}}(\Omega) \to \mathbb{R}$, *given by*

$$\overline{\mathfrak{F}}_{u_0}[u] := \int_\Omega f\left(x, \frac{\mathrm{d}\mathbb{A}u}{\mathrm{d}\mathscr{L}^n}\right)\mathrm{d}\mathscr{L}^n + \int_\Omega f^\infty\left(x, \frac{\mathrm{d}\mathbb{A}u}{\mathrm{d}|\mathbb{A}^s u|}\right)\mathrm{d}|\mathbb{A}^s u| + \int_{\partial\Omega} f^\infty\Big(x, \nu_{\partial\Omega} \otimes_{\mathbb{A}} \mathrm{tr}(u - u_0)\Big)\,\mathrm{d}\mathcal{H}^{n-1}, \tag{5-9}$$

be sequentially weak-lower semicontinuous on* $\mathrm{BV}^{\mathbb{A}}(\Omega)$.

Proof. Proposition 5.1 (applied with Ω replaced by U) shows that $\overline{\mathfrak{F}}_U : \mathrm{BV}^{\mathbb{A}}(U) \to \mathbb{R}$ is area-strictly continuous on $\mathrm{BV}^{\mathbb{A}}(U)$ and sequentially weak*-lower semicontinuous on $\mathrm{BV}^{\mathbb{A}}(U)$.

For $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$ let $\tilde{u} := \chi_{U\setminus\overline{\Omega}} u_0 + \chi_\Omega u$. Then due to Corollary 4.21 we have $\tilde{u} \in \mathrm{BV}^{\mathbb{A}}(U)$ and, with the outer normal ν of Ω,

$$\mathbb{A}\tilde{u} = \mathbb{A}u \llcorner \Omega + \mathbb{A}u_0 \mathscr{L}^n \llcorner (U \setminus \overline{\Omega}) + \mathrm{tr}(u - u_0) \otimes_{\mathbb{A}} \nu \mathcal{H}^{n-1} \llcorner \partial\Omega. \tag{5-10}$$

Hence,

$$\overline{\mathfrak{F}}_U[\tilde{u}] = \overline{\mathfrak{F}}_{u_0}[u] + \int_{U\setminus\overline{\Omega}} f(x, \mathbb{A}u_0)\,\mathrm{d}x. \tag{5-11}$$

If $u_k \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}^{\mathbb{A}}(\Omega)$, then $\tilde{u}_k \overset{*}{\rightharpoonup} \tilde{u}$ in $\mathrm{BV}^{\mathbb{A}}(U)$. Indeed, it is clear that $u_k \to u$ in $L^1(U)$. Moreover, since u_k is bounded in $\mathrm{BV}^{\mathbb{A}}(\Omega)$, so is $\mathbb{A}u_k \in \mathcal{M}(\Omega)$ and $\mathrm{tr}(u_k)$ in $L^1(\partial\Omega)$ (using the trace theorem, Theorem 4.17). This and (5-10) show that $\tilde{u}_k$ is bounded in $\mathrm{BV}^{\mathbb{A}}(U)$. In conjunction with $u_k \to u$ in $L^1(U)$ we obtain $\tilde{u}_k \overset{*}{\rightharpoonup} \tilde{u}$ in $\mathrm{BV}^{\mathbb{A}}(U)$.

Since $\overline{\mathfrak{F}}_U$ is sequentially weak*-lower semicontinuous on $\mathrm{BV}^{\mathbb{A}}(U)$, it follows that $\overline{\mathfrak{F}}_{u_0}$ sequentially weak*-lower semicontinuous on $\mathrm{BV}^{\mathbb{A}}(\Omega)$. ☐

Theorem 5.3. *Let f satisfy* (5-2)–(5-5). *Then the functional $\overline{\mathfrak{F}}_{u_0} : \mathrm{BV}^{\mathbb{A}}(\Omega) \to \mathbb{R}$ is coercive and has a minimiser on* $\mathrm{BV}^{\mathbb{A}}(\Omega)$. *Moreover*, *we have*

$$\min_{\mathrm{BV}^{\mathbb{A}}(\Omega)} \overline{\mathfrak{F}}_{u_0} = \inf_{u_0 + W_0^{\mathbb{A},1}(\Omega)} \mathfrak{F}. \tag{5-12}$$

Proof. We begin with the coerciveness of $\overline{\mathfrak{F}}_{u_0}$. Let $(v_k) \subset \mathrm{BV}^{\mathbb{A}}(\Omega)$ with $(\overline{\mathfrak{F}}_{u_0}(u_k))$ bounded. We have to show that (v_k) is bounded in $\mathrm{BV}^{\mathbb{A}}(\Omega)$. Let $\tilde{v}_k := \chi_{U\setminus\overline{\Omega}} u_0 + \chi_\Omega v_k$ as in Corollary 5.2. Then due to (5-11), $\overline{\mathfrak{F}}_U(\tilde{v}_k)$ is bounded. By the linear growth condition (5-3) we see that $(\mathbb{A}v_k)$ is uniformly bounded in $\mathcal{M}(U; \mathbb{R}^K)$. Now choose a ball $B' \subset \Omega$ and another ball B with $U \subset B$. Since $v_k - u_0 = 0$ on $U \setminus \overline{\Omega}$, we can extend it by zero to a function from $\mathrm{BV}^{\mathbb{A}}(B)$ due to Theorem 4.23(b). Now, we can apply Poincaré's inequality in the form of Theorem 3.3 to conclude that (v_k) is also bounded in $L^1(U)$. Hence, (v_k) is bounded on $\mathrm{BV}^{\mathbb{A}}(\Omega)$, which is the desired coerciveness.

By positivity of f and f^∞, we have $\overline{\mathfrak{F}}_{u_0}[w] \geq 0$ for all $w \in \mathrm{BV}^{\mathbb{A}}(\Omega)$, and so we may pick a minimising sequence (u_k) in $\mathrm{BV}^{\mathbb{A}}(\Omega)$. By coerciveness, this sequence is bounded in $\mathrm{BV}^{\mathbb{A}}(\Omega)$. We can pick a (nonrelabeled) subsequence such that $u_k \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}^{\mathbb{A}}(\Omega)$ for some $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$. By the sequential weak*-lower semicontinuity from Corollary 5.2, we deduce that u is a minimiser of $\overline{\mathfrak{F}}_{u_0}$.

We conclude the proof by showing (5-12). The "$\leq$"-part is obvious. Due to Proposition 4.24 we find a sequence $w_k \in \mathcal{D}_{u_0}$ such that $w_k \xrightarrow{\langle\cdot\rangle} u$ in $\mathrm{BV}^{\mathbb{A}}(U)$. By the $\mathbb{A}$-area-strict continuity of $\overline{\mathfrak{F}}_U$ on $\mathrm{BV}^{\mathbb{A}}(U)$, see Proposition 5.1, we see that $\overline{\mathfrak{F}}_U(u) = \lim_{k\to\infty} \overline{\mathfrak{F}}_U(w_k)$. This and (5-11) prove the "$\geq$"-part of (5-12). ☐

Appendix

We now collect some auxiliary results that have been used in the main part of the paper. The following lemma shows that the recession function is automatically well-defined on the $\mathbb{A}$-rank-one cone.

Lemma A.1. *Let $\mathbb{A}$ be $\mathbb{R}$-elliptic, let $f : \overline{\Omega} \times \mathscr{R}(\mathbb{A}) \to \mathbb{R}$ be $\mathbb{A}$-quasiconvex in the sense of* (5-5), *satisfy the linear growth condition* (5-3), *and satisfy the continuity condition* (5-4). *Then $f(x, \cdot)$ is Lipschitz continuous in $\mathscr{R}(\mathbb{A})$ uniformly in $x \in \overline{\Omega}$. Moreover, the strong recession function $f^\infty : \overline{\Omega} \times \mathscr{R}(\mathbb{A}) \to \mathbb{R}$ with*

$$f^\infty(x, A) := \lim_{\substack{x' \to x \\ A' \to A \\ t \to \infty}} \frac{f(x', tA')}{t}$$

is well-defined on $\overline{\Omega} \times \mathscr{C}(\mathbb{A})$. (*Note that the limit $A' \to A$ is taken in $\mathscr{R}(\mathbb{A})$.*) *Moreover,*

$$|f^\infty(x, A) - f^\infty(x', A)| \le \omega(|x' - x|)|A|$$

for all $x, x' \in \overline{\Omega}$ and $A \in \mathscr{C}(\mathbb{A})$.

Proof. We begin with the Lipschitz continuity of f on $\mathscr{R}(\mathbb{A})$.

Let $A \in \mathscr{R}(\mathbb{A})$ and $B = a \otimes_{\mathbb{A}} b \in \mathscr{C}(\mathbb{A})$. Since f is $\mathbb{A}$-quasiconvex, it is a consequence[4] of [Fonseca and Müller 1999, Proposition 3.4] that $t \mapsto f(x, A + tB)$ is convex on $\mathbb{R}$. This property is known as $\mathscr{C}(\mathbb{A})$-convexity; see [Kirchheim and Kristensen 2016].

Thus the function

$$g(t) := \frac{|f(x, A + ta \otimes_{\mathbb{A}} b) - f(x, A)|}{t}$$

is increasing. Hence, with $\lambda := (1 + |A + B| + |A|)/|B| > 1$, we obtain

$$\begin{aligned} |f(x, A + B) - f(x, A)| &= g(1) \le g(\lambda) \\ &\le |f(x, A + \lambda a \otimes_{\mathbb{A}} b) - f(x, A)| \frac{|B|}{1 + |A + B| + |A|} \\ &\le \frac{c_2(2|A| + \lambda|B|) + 2c_3}{1 + |A + B| + |A|}|B| \\ &\le \frac{c_2(1 + 3|A| + |A + B|) + 2c_3}{1 + |A + B| + |A|}|B| \\ &\le (3c_2 + 2c_3)|B| \end{aligned}$$

using (5-3). This proves the Lipschitz continuity in $\mathscr{C}(\mathbb{A})$-directions.

If $B \in \mathscr{R}(\mathbb{A})$, then by $\mathscr{R}(\mathbb{A}) = \operatorname{span}(\mathscr{C}(\mathbb{A}))$ we can decompose B into at most K summands from $\mathscr{C}(\mathbb{A})$. Now the Lipschitz continuity in $\mathscr{C}(\mathbb{A})$-directions implies

$$|f(x, A + B) - f(x, A)| \le K(3c_2 + 2c_3)|B| \tag{A-1}$$

for all $A, B \in \mathscr{R}(\mathbb{A})$. This proves the Lipschitz continuity part.

Let $A \in \mathscr{C}(\mathbb{A})$ and $x \in \overline{\Omega}$. Then $t \mapsto (f(x, tA) - f(x, 0))/t$ is increasing in t by $\mathscr{C}(\mathbb{A})$-convexity of $f(x, \cdot)$ and bounded by $c_2|A|$ due to the linear growth condition (5-3). This allows us to define

[4] As proven in [Fonseca and Müller 1999], if $\mathcal{A}$ is a first-order linear homogeneous differential operator, then $\mathcal{A}$-quasiconvex functions are $\Lambda_{\mathcal{A}}$-convex. Note that in our setting, $\mathbb{L} = \mathcal{A}$ need not be first of first order; however, their arguments extend to the case of higher-order annihilating operators $\mathbb{A}$ in a straightforward manner.

$g^\infty : \overline{\Omega} \times \mathscr{C}(\mathbb{A}) \to \mathbb{R}$ by

$$g^\infty(x, A) = \lim_{t\to\infty} \frac{f(x, tA)}{t} = \sup_{t>0} \frac{f(x, tA)}{t}.$$

Now, let $A' \in \mathscr{R}(\mathbb{A})$ and $x' \in \overline{\Omega}$; then by (A-1) and (5-4)

$$\left|\frac{f(x', tA')}{t} - \frac{f(x, tA)}{t}\right| \le \left|\frac{f(x', tA') - f(x', tA)}{t}\right| + \left|\frac{f(x', tA) - f(x, tA)}{t}\right| \\ \le K(3c_2 + 2c_3)|A - A'| + \omega(|x' - x|)\frac{1 + t|A|}{t}.$$

This proves $f^\infty(x, A) = g^\infty(x, A)$ for all $x \in \overline{\Omega}$ and $A \in \mathscr{C}(\mathbb{A})$. Consequently, we obtain the existence of f^∞ in $\overline{\Omega} \times \mathscr{C}(\mathbb{A})$.

The continuity of $f^\infty(\cdot, A)$ for $A \in \mathscr{C}(\mathbb{A})$ is a direct consequence of the continuity of $f(\cdot, A)$. □

Lemma A.2. *Let $\mathbb{A}$ be $\mathbb{R}$-elliptic, and let $f : \overline{\Omega} \times \mathscr{R}(\mathbb{A}) \to \mathbb{R}$ be $\mathbb{A}$-quasiconvex in the sense of* (5-5), *satisfy the linear growth condition* (5-3), *and satisfy the continuity condition* (5-4). *Furthermore, let $\Omega \Subset U$ with ∂U Lipschitz. Then there exists an extension $\tilde{f} : \overline{U} \times \mathscr{R}(\mathbb{A}) \to \mathbb{R}$ of f, which is $\mathbb{A}$-quasiconvex, satisfies the linear growth condition* (5-3), *and satisfies the continuity condition* (5-4). (*The modulus of continuity might change by a factor.*)

Proof. Since ∂U and $\partial\Omega$ are Lipschitz, we find a Lipschitz map $\Phi : \overline{U} \to \overline{\Omega}$, which is the identity on $\overline{\Omega}$. Now define $\tilde{f}(x, A) := f(\Phi(x), A)$. □

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OPTIMAL CONSTANTS FOR A NONLOCAL APPROXIMATION OF SOBOLEV NORMS AND TOTAL VARIATION

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We consider the family of nonlocal and nonconvex functionals proposed and investigated by J. Bourgain, H. Brezis and H.-M. Nguyen in a series of papers of the last decade. It was known that this family of functionals Gamma-converges to a suitable multiple of the Sobolev norm or the total variation, depending on the summability exponent, but the exact constants and the structure of recovery families were still unknown, even in dimension 1.

We prove a Gamma-convergence result with explicit values of the constants in any space dimension. We also show the existence of recovery families consisting of smooth functions with compact support.

The key point is reducing the problem first to dimension 1, and then to a finite combinatorial rearrangement inequality.

1. Introduction

Let $p \geq 1$ and $\delta > 0$ be real numbers, let d be a positive integer, and let $\Omega \subseteq \mathbb{R}^d$ be an open set. For every measurable function $u : \Omega \to \mathbb{R}$ we set

$$\Lambda_{\delta,p}(u, \Omega) := \iint_{I(\delta,u,\Omega)} \frac{\delta^p}{|y-x|^{d+p}} \, dx \, dy, \tag{1-1}$$

where

$$I(\delta, u, \Omega) := \{(x, y) \in \Omega^2 : |u(y) - u(x)| > \delta\}.$$

Nonconvex and nonlocal functionals of this type appeared in a paper by J. Bourgain, H. Brezis and P. Mironescu [Bourgain et al. 2005]; see Open Problem 2 of that work. Subsequently, the family (1-1) was investigated in a series of papers by H.-M. Nguyen [2006; 2007; 2008; 2011; 2014], J. Bourgain and H.-M. Nguyen [2006], and H. Brezis and H.-M. Nguyen [2018]; see also [Brezis 2015; Brezis and Nguyen 2017].

We point out that the dependence on u is just on the integration set. The fixed integrand is divergent on the diagonal $y = x$, and the integration set is closer to the diagonal where the gradient of u is large. This suggests that $\Lambda_{\delta,p}(u, \Omega)$ is proportional, in the limit as $\delta \to 0^+$, to some norm of the gradient of u,

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and more precisely to the functional

$$\Lambda_{0,p}(u,\Omega):=\begin{cases}\int_\Omega |\nabla u(x)|^p\,dx & \text{if } p>1 \text{ and } u\in W^{1,p}(\Omega),\\ \text{total variation of } u \text{ in } \Omega & \text{if } p=1 \text{ and } u\in \mathrm{BV}(\Omega),\\ +\infty & \text{otherwise.}\end{cases} \tag{1-2}$$

It is natural to compare the family (1-1) with the classical approximations of Sobolev or BV norms, based on nonlocal convex functionals such as

$$G_{\varepsilon,p}(u,\Omega):=\iint_\Omega \frac{|u(y)-u(x)|^p}{|y-x|^p}\rho_\varepsilon(|y-x|)\,dx\,dy, \tag{1-3}$$

where gradients are replaced by finite differences weighted by a suitable family ρ_ε of mollifiers. The idea of approximating integrals of the gradient with double integrals of difference quotients, where all pairs of distinct points interact, has been considered independently by many authors in different contexts. For example, E. De Giorgi proposed an approximation of this kind to the Mumford–Shah functional in any space dimension, in order to overcome the anisotropy of the discrete approximation [Chambolle 1995]. The resulting theory appears in [Gobbino 1998] and was then extended in [Gobbino and Mora 2001] to more general free discontinuity problems, and in particular to Sobolev and BV spaces. In the same years, the case of Sobolev and BV norms was considered in detail in [Bourgain et al. 2001]; see also [Ponce 2004].

The result, as expected, is that the family $G_{\varepsilon,p}(u,\mathbb{R}^d)$ converges as $\varepsilon\to 0^+$ to a suitable multiple of $\Lambda_{0,p}(u,\mathbb{R}^d)$, both in the sense of pointwise convergence, and in the sense of De Giorgi's Gamma-convergence. This provides a characterization of Sobolev functions (if $p>1$), and of bounded variation functions (if $p=1$), as those functions for which the pointwise limit or the Gamma-limit is finite.

From the heuristic point of view, the nonconvex approximating family (1-1) seems to follow a different paradigm. Indeed, it was observed by J.-M. Morel, as quoted on page 4 of the transparencies of the presentation [Brezis 2016], that this definition involves some sort of "vertical slicing" that evokes the definition of integral *à la Lebesgue*, in contrast to the definition *à la Riemann* that seems closer to the "horizontal slicing" of the finite differences in (1-3).

From the mathematical point of view, the asymptotic behavior of (1-1) exhibits some unexpected features. In order to state the precise results, let us introduce some notation. Let $\mathbb{S}^{d-1}:=\{\sigma\in\mathbb{R}^d:|\sigma|=1\}$ denote the unit sphere in $\mathbb{R}^d$. For every $p\geq 1$ we consider the geometric constant

$$G_{d,p}:=\int_{\mathbb{S}^{d-1}}|\langle v,\sigma\rangle|^p\,d\sigma, \tag{1-4}$$

where v is any element of $\mathbb{S}^{d-1}$ (of course the value of $G_{d,p}$ does not depend on the choice of v), and the integration is intended with respect to the $(d-1)$-dimensional Hausdorff measure. The value of $G_{d,p}$ can be explicitly computed in terms of special functions through Beta integrals. It turns out that $G_{d,p}=2$ for every p if $d=1$, and

$$G_{d,p}=\mathrm{meas}(\mathbb{S}^{d-2})\int_{-\pi/2}^{\pi/2}(\cos\theta)^p\cdot|\sin\theta|^{d-2}\,d\theta=\frac{2\pi^{(d-1)/2}\Gamma((p+1)/2)}{\Gamma((p+d)/2)}\quad\text{for all } d\geq 2.$$

The main convergence results obtained so far can be summed up as follows.

• *Pointwise convergence for* $p > 1$. For every $p > 1$ it turns out that

$$\lim_{\delta\to 0^+} \Lambda_{\delta,p}(u,\mathbb{R}^d) = \frac{1}{p} G_{d,p}\, \Lambda_{0,p}(u,\mathbb{R}^d) \quad \text{for all } u \in L^p(\mathbb{R}^d). \tag{1-5}$$

• *Pointwise convergence for* $p = 1$. In the case $p = 1$, equality (1-5) holds true for every $u \in C^1_c(\mathbb{R}^d)$, but there do exist functions $u \in W^{1,1}(\mathbb{R}^d)$ for which the left-hand side is infinite (while of course the right-hand side is finite). A precise characterization of equality cases is still unknown.

• *Gamma-convergence for every* $p \geq 1$. For every $p \geq 1$ there exists a constant $C_{d,p}$ such that

$$\Gamma\text{–}\lim_{\delta\to 0^+} \Lambda_{\delta,p}(u,\mathbb{R}^d) = \frac{1}{p} G_{d,p} C_{d,p}\, \Lambda_{0,p}(u,\mathbb{R}^d) \quad \text{for all } u \in L^p(\mathbb{R}^d),$$

where the Gamma-limit is intended with respect to the usual metric of $L^p(\mathbb{R}^d)$ (but the result would be the same with respect to the convergence in $L^1(\mathbb{R}^d)$ or in measure). Moreover, it was proved that $C_{d,p} \in (0,1)$; namely the Gamma-limit is always nontrivial but different from the pointwise limit.

As a consequence, again one can characterize the Sobolev space $W^{1,p}(\mathbb{R}^d)$ as the set of functions in $L^p(\mathbb{R}^d)$ for which the pointwise limit or the Gamma-limit is finite. As for $\mathrm{BV}(\mathbb{R}^d)$, in this setting it can be characterized only through the Gamma-limit.

Some problems remained open, and were stated explicitly in [Nguyen 2011; Brezis and Nguyen 2018]:

Question 1. What is the exact value of $C_{d,p}$, at least in the case $d = 1$?

Question 2. Does $C_{d,p}$ depend on d?

Question 3. Do there exist recovery families made up of continuous functions, or even of functions of class C^∞?

In this paper we answer these three questions. Concerning Questions 1 and 2, we prove that $C_{d,p}$ does not depend on d, and coincides with the value C_p conjectured in [Nguyen 2007] (see also [Nguyen 2011, Open question 2]) for the 1-dimensional case, namely

$$C_p := \begin{cases} \dfrac{1}{p-1}\left(1 - \dfrac{1}{2^{p-1}}\right) & \text{if } p > 1, \\ \log 2 & \text{if } p = 1. \end{cases} \tag{1-6}$$

Concerning the third question, we prove that smooth recovery families do exist. Our main result is the following.

Theorem 1.1 (Gamma-convergence). *Let us consider the functionals* $\Lambda_{\delta,p}$ *and* $\Lambda_{0,p}$ *defined in* (1-1) *and* (1-2), *respectively.*

Then for every positive integer d and every real number $p \geq 1$ *it turns out that*

$$\Gamma\text{–}\lim_{\delta\to 0^+} \Lambda_{\delta,p}(u,\mathbb{R}^d) = \frac{1}{p} G_{d,p} C_p\, \Lambda_{0,p}(u,\mathbb{R}^d) \quad \textit{for all } u \in L^p(\mathbb{R}^d),$$

where $G_{d,p}$ *is the geometric constant defined in* (1-4), *and* C_p *is the constant defined in* (1-6). *In particular, the following two statements hold true*:

(1) (liminf inequality) *For every family* $\{u_\delta\}_{\delta>0} \subseteq L^p(\mathbb{R}^d)$, *with* $u_\delta \to u$ *in* $L^p(\mathbb{R}^d)$ *as* $\delta \to 0^+$, *it turns out that*

$$\liminf_{\delta\to 0^+} \Lambda_{\delta,p}(u_\delta, \mathbb{R}^d) \geq \frac{1}{p} G_{d,p} C_p \, \Lambda_{0,p}(u, \mathbb{R}^d). \tag{1-7}$$

(2) (limsup inequality) *For every* $u \in L^p(\mathbb{R}^d)$ *there exists a family* $\{u_\delta\}_{\delta>0} \subseteq L^p(\mathbb{R}^d)$, *with* $u_\delta \to u$ *in* $L^p(\mathbb{R}^d)$ *as* $\delta \to 0^+$, *such that*

$$\limsup_{\delta\to 0^+} \Lambda_{\delta,p}(u_\delta, \mathbb{R}^d) \leq \frac{1}{p} G_{d,p} C_p \, \Lambda_{0,p}(u, \mathbb{R}^d).$$

We can also assume that the family $\{u_\delta\}$ *consists of functions of class* C^∞ *with compact support.*

The proof of this result requires a different approach to the problem, which we briefly sketch below. In previous literature, see [Nguyen 2011, formula (1.3)] or [Brezis and Nguyen 2018, formula (1.12)], the constant $C_{d,p}$ was defined through some sort of cell problem as

$$\frac{1}{p} G_{d,p} C_{d,p} := \inf\Bigl\{\liminf_{\delta\to 0^+} \Lambda_{\delta,p}(u_\delta, (0,1)^d) : u_\delta \to u_0 \text{ in } L^p((0,1)^d)\Bigr\},$$

where $u_0(x) = (x_1 + \cdots + x_d)/\sqrt{d}$. Unfortunately, this definition is quite implicit and provides no information on the structure of the families that approach the optimal value. This lack of structure complicates things, in such a way that just proving that $C_{d,p} > 0$ requires extremely delicate estimates; this is the content of [Bourgain and Nguyen 2006]. On the Gamma-limsup side, since $\Lambda_{\delta,p}$ is quite sensitive to jumps, what is difficult is gluing together the recovery families corresponding to different slopes, even in the case of a piecewise affine function in dimension 1. This requires a delicate surgery near the junctions; see [Nguyen 2011]. Finally, as for Question 3, difficulties originate from the lack of convexity or continuity of the functionals (1-1), which do not seem to behave well under convolution or similar smoothing techniques.

The core of our approach consists in proving that $\Lambda_{\delta,p}$ in dimension 1 behaves well under *vertical* δ*-segmentation* and *monotone rearrangement*. We refer to Section 3A for the details, but roughly speaking this means that monotone step functions whose values are consecutive integer multiples of δ are the most efficient way to fill the gap between any two given levels. The argument is purely 1-dimensional, and it is carried out in Proposition 3.2. In turn, the proof relies on a discrete combinatorial rearrangement inequality, which we investigate in Theorem 2.2 under more general assumptions.

We observe that this strategy, namely estimating the asymptotic cost of oscillations by reducing ourselves to a discrete combinatorial minimum problem, is the same as that exploited in [Gobbino 1998; Gobbino and Mora 2001], with the remarkable difference that now the reduction to the discrete setting is achieved through vertical δ-segmentation, while in [Gobbino 1998; Gobbino and Mora 2001] it was obtained through a horizontal ε-segmentation (see Figure 1).

The asymptotic estimate on the cost of oscillations opens the door to the Gamma-liminf inequality in dimension 1, which at this point follows from well-established techniques. As for the Gamma-limsup inequality, in dimension 1 we just need to exhibit a family that realizes the given explicit multiple of $\Lambda_{0,p}(u, \mathbb{R})$, and this can be achieved through a vertical δ-segmentation *à la Lebesgue* (see Proposition 3.7). This produces a recovery family made up of step functions, and it is not difficult to modify them in

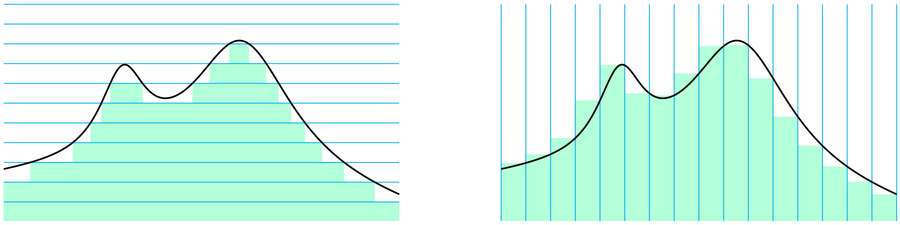

Figure 1. Vertical δ-segmentation vs. horizontal ε-segmentation (δ is the distance between the parallel lines on the left, ε is the distance between the parallel lines on the right).

order to obtain functions of class C^∞ with asymptotically the same energy (see Proposition 3.9). Finally, passing from dimension 1 to any dimension is just an application of the 1-dimensional result to all the 1-dimensional sections of a function of d variables.

At the end of the day, we have a completely self-contained proof of Theorem 1.1 above, and a clear indication that the true difficulty of the problem lies in dimension 1, and actually in the discretized combinatorial model. We hope that these ideas can be extended to the more general functionals considered in [Brezis and Nguyen 2018]. Some steps in this direction have already been done in [Antonucci et al. 2020]; see also [Antonucci et al. 2018].

This paper is organized as follows. In Section 2 we develop a theory of monotone rearrangements, first in a discrete, and then in a semidiscrete setting. In Section 3 we prove our Gamma-convergence result in dimension 1. In Section 4 we prove the Gamma-convergence result in any space dimension.

We would like to thank an anonymous referee for pointing out that the rearrangement inequality in our Theorem 2.4 is equivalent to a rearrangement inequality proved in [Garsia and Rodemich 1974]. This equivalence is not immediate (see Remark 2.5 for further details), and for this reason the proofs follow different paths. However, in both cases the basic step consists in reducing the problem to a discrete combinatorial result, namely Theorem 2.2 in this paper, and a variant of Taylor's lemma [1973] in [Garsia and Rodemich 1974].

2. An aggregation/segregation problem

In this section we study the minimum problem for two simplified versions of (1-1), which we interpret as optimizing the disposition of some objects of different types (actually dinosaurs of different species). The first problem is purely discrete, namely with a finite number of dinosaurs of a finite number of species. The second one is semidiscrete, namely with a continuum of dinosaurs belonging to a finite number of species.

2A. ***Discrete setting.*** Let us consider

- a positive integer n,
- a function $u : \{1, \dots, n\} \to \mathbb{Z}$,
- a symmetric subset $E \subseteq \mathbb{Z}^2$ (namely any subset with the property that $(i, j) \in E$ if and only if $(j, i) \in E$),
- a nonincreasing function $h : \{0, 1, \dots, n-1\} \to \mathbb{R}$.

Let us introduce the discrete interaction set

$$J(E,u) := \{(x,y) \in \{1,\ldots,n\}^2 : x \le y,\ (u(x),u(y)) \in E\}, \tag{2-1}$$

and let us finally define

$$\mathcal{H}(h,E,u) := \sum_{(x,y)\in J(E,u)} h(y-x). \tag{2-2}$$

Just to help intuition, we think of u as an arrangement of n dinosaurs placed in the points $\{1,\ldots,n\}$. There are different species of dinosaurs, indexed by integer numbers, so that $u(x)$ denotes the species of the dinosaur in position x. The subset $E \subseteq \mathbb{Z}^2$ is the list of all pairs of species that are hostile to each other. A pair of points (x,y) belongs to $J(E,u)$ if and only if $x \le y$ and the two dinosaurs placed in x and y belong to hostile species, and in this case the real number $h(y-x)$ measures the "hostility" between the two dinosaurs. As expected, the closer the dinosaurs are, the larger their hostility.

Taking this Jurassic framework into account, sometimes in the sequel we call u a "discrete arrangement of n dinosaurs", we call E an "enemy list", we call h a "discrete hostility function", and $\mathcal{H}(h,E,u)$ the "total hostility of the arrangement". At this level of generality, we admit the possibility that $(i,i) \in E$ for some integer i, namely that a dinosaur is hostile to dinosaurs of the same species, including itself. For this reason, the hostility function $h(x)$ is defined also for $x=0$. This generality turns out to be useful in the proof of the main result for discrete arrangements.

In the sequel we focus on the special case where E coincides with

$$E_k := \{(i,j) \in \mathbb{Z}^2 : |j-i| \ge k+1\} \tag{2-3}$$

for some positive integer k. In this case it is quite intuitive that the arrangements that minimize the total hostility are the "monotone" ones, namely those in which all dinosaurs of the same species are close to each other, and the groups corresponding to different species are sorted in ascending or descending order. To this end, we introduce the following notion.

Definition 2.1 (nondecreasing rearrangement: discrete setting). Let n be a positive integer, and let $u : \{1,\ldots,n\} \to \mathbb{Z}$ be a function. The *nondecreasing rearrangement* of u is the function $Mu : \{1,\ldots,n\} \to \mathbb{Z}$ defined as

$$Mu(x) := \min\{j \in \mathbb{Z} : |\{y \in \{1,\ldots,n\} : u(y) \le j\}| \ge x\},$$

where $|A|$ denotes the number of elements of the set A.

As the name suggests, Mu is the unique nondecreasing function that can be represented in the form $Mu = u \circ \pi$, where $\pi : \{1,\ldots,n\} \to \{1,\ldots,n\}$ is a suitable bijection. The nondecreasing rearrangement can also be uniquely characterized by the fact that the two level sets

$$\{x \in \{1,\ldots,n\} : u(x) = j\}, \quad \{x \in \{1,\ldots,n\} : Mu(x) = j\}$$

have the same number of elements for every $j \in \mathbb{Z}$.

As expected, the main result is that monotone arrangements minimize the total hostility with respect to the enemy list E_k.

Theorem 2.2 (total hostility minimization: discrete setting). *Let n and k be two positive integers, let $E_k \subseteq \mathbb{Z}^2$ be the subset defined by* (2-3), *and let $h : \{0, \dots, n-1\} \to \mathbb{R}$ be a nonincreasing function. Let $u : \{1, \dots, n\} \to \mathbb{Z}$ be any function, let Mu be the nondecreasing rearrangement of u introduced in Definition 2.1, and let $\mathcal{H}(h, E_k, u)$ be the total hostility defined in* (2-2).

Then it turns out that

$$\mathcal{H}(h, E_k, u) \geq \mathcal{H}(h, E_k, Mu). \tag{2-4}$$

Taylor's result [1973] is substantially equivalent to (2-4) in the special case where there are n dinosaurs of n different species indexed by n consecutive integers. It is likely that Taylor's approach based on the celebrated Hall's theorem, sometimes referred to as the "marriage theorem", could work even in the more general setting that we need here; see [Garsia and Rodemich 1974, Section 3]. The proof we present in Section 2C below follows a different path.

2B. *Semidiscrete setting.* Let us consider

- an interval $(a, b) \subseteq \mathbb{R}$,
- a measurable function $u : (a, b) \to \mathbb{Z}$ with finite image,
- a symmetric subset $E \subseteq \mathbb{Z}^2$,
- a nonincreasing function $c : (0, b-a) \to \mathbb{R}$ (note that $c(\sigma)$ might diverge as $\sigma \to 0^+$).

Let us introduce the semidiscrete interaction set

$$I(E, u) := \{(x, y) \in (a, b)^2 : (u(x), u(y)) \in E\}, \tag{2-5}$$

and let us finally define

$$\mathcal{F}(c, E, u) := \iint_{I(E,u)} c(|y-x|)\, dx\, dy. \tag{2-6}$$

In analogy with the discrete setting, we interpret $u(x)$ as a continuous arrangement of dinosaurs of a finite number of species, $c(y-x)$ as the hostility between two dinosaurs of hostile species placed in x and y, and we think of $\mathcal{F}(c, E, u)$ as the total hostility of the arrangement u with respect to the enemy list E.

Once again, we suspect that monotone arrangements minimize the total hostility with respect to the enemy list E_k. This leads to the following notion.

Definition 2.3 (nondecreasing rearrangement: semidiscrete setting). Let $u : (a, b) \to \mathbb{Z}$ be a measurable function with finite image. The *nondecreasing rearrangement* of u is the function $Mu : (a, b) \to \mathbb{Z}$ defined as

$$Mu(x) := \min\{j \in \mathbb{Z} : \text{meas}\{y \in (a, b) : u(y) \leq j\} \geq x - a\},$$

where $\text{meas}(A)$ denotes the Lebesgue measure of a subset $A \subseteq (a, b)$.

The function Mu is nondecreasing and satisfies

$$\text{meas}\{x \in (a, b) : u(x) = j\} = \text{meas}\{x \in (a, b) : Mu(x) = j\} \quad \text{for all } j \in \mathbb{Z}.$$

The following result is the semidiscrete counterpart of Theorem 2.2.

Theorem 2.4 (total hostility minimization: semidiscrete setting). *Let $(a, b) \subseteq \mathbb{R}$ be an interval, let k be a positive integer, let $E_k \subseteq \mathbb{Z}^2$ be the subset defined by* (2-3), *and let $c : (0, b-a) \to \mathbb{R}$ be a nonincreasing function. Let $u : (a, b) \to \mathbb{Z}$ be any measurable function with finite image, let Mu be the nondecreasing rearrangement of u introduced in Definition 2.3, and let $\mathcal{F}(c, E_k, u)$ be the total hostility defined in* (2-6).

Then it turns out that

$$\mathcal{F}(c, E_k, u) \geq \mathcal{F}(c, E_k, Mu). \tag{2-7}$$

Remark 2.5. Theorem 2.4 above is stated in the form that we need in the proof of Proposition 3.2. With a further approximation step in the proof, one can show that the same conclusion (2-7) holds true also without assuming that the image of u is finite and contained in $\mathbb{Z}$, and without assuming that k is a positive integer (but just a real number greater than -1).

It is interesting to compare this extended result with [Garsia and Rodemich 1974, Theorem 1.1], which states that for every nondecreasing function $\Phi : [0, +\infty) \to [0, +\infty)$, and every $t \in (0, b-a)$, it turns out that

$$\int_{D(t)} \Phi(|u(y) - u(x)|)\, dx\, dy \geq \int_{D(t)} \Phi(|Mu(y) - Mu(x)|)\, dx\, dy, \tag{2-8}$$

where $D(t) := \{(x, y) \in (a, b)^2 : |y - x| \leq t\}$. We observe that in (2-8) the integral involves only the pairs $(x, y) \in (a, b)^2$ that are close enough to the diagonal $y = x$, and the integrand Φ penalizes the pairs for which $|u(y) - u(x)|$ is large. On the contrary, in our total hostility the integral involves only the pairs with $|u(y) - u(x)|$ large enough, and the integrand c penalizes the pairs that are close to the diagonal. In this sense the two statements seem to be two sides of the same coin (again as the Riemann and the Lebesgue integral), and actually one can show that both statements are equivalent to saying that the inequality

$$\begin{aligned} &\operatorname{meas}\{(x, y) \in (a, b)^2 : |y - x| \leq t,\ |u(y) - u(x)| \geq \delta\} \\ &\qquad \geq \operatorname{meas}\{(x, y) \in (a, b)^2 : |y - x| \leq t,\ |Mu(y) - Mu(x)| \geq \delta\} \end{aligned} \tag{2-9}$$

holds true for every $t \in (0, b-a)$ and every $\delta > 0$.

The proof of (2-8) given in [Garsia and Rodemich 1974] relies on this equivalence, and establishes (2-9) through a variant of Taylor's result. The proof of (2-7) that we present in Section 2D follows a more direct path, based on our Theorem 2.2, which anyway is again discrete combinatorics.

2C. ***Proof of Theorem 2.2.*** Since the hostility function h is fixed, in the sequel we simply write $\mathcal{H}(E, u)$ instead of $\mathcal{H}(h, E, u)$.

Our idea is to proceed by induction on the number of dinosaurs. In the case $n = 1$ there is nothing to prove. Let us assume now that (2-4) holds true for all arrangements of n dinosaurs, and let u be any arrangement of $n + 1$ dinosaurs. In order to obtain an arrangement of n dinosaurs, we remove from u the rightmost dinosaur of the species indexed by the highest integer, and we shift one position to the left all subsequent dinosaurs. More formally, we set

$$\mu := \max\{u(i) : i \in \{1, \ldots, n+1\}\},$$

we consider the largest index $m \in \{1, \dots, n+1\}$ such that $u(m) = \mu$, and we define the *reduction* of u to be the new arrangement $\mathrm{Red}(u) : \{1, \dots, n\} \to \mathbb{Z}$ given by

$$[\mathrm{Red}(u)](i) := \begin{cases} u(i) & \text{if } i < m, \\ u(i+1) & \text{if } i \geq m. \end{cases}$$

When passing from u to $\mathrm{Red}(u)$, the total hostility changes by an amount that we call the *hostility gap*, defined as

$$\Delta(E, u) := \mathcal{H}(E, u) - \mathcal{H}(E, \mathrm{Red}(u)).$$

Since $M(\mathrm{Red}(u)) = \mathrm{Red}(Mu)$, the inductive hypothesis reads as

$$\mathcal{H}(E_k, \mathrm{Red}(u)) \geq \mathcal{H}(E_k, M(\mathrm{Red}(u))) = \mathcal{H}(E_k, \mathrm{Red}(Mu)),$$

and therefore

$$\begin{aligned} \mathcal{H}(E_k, u) &= \mathcal{H}(E_k, \mathrm{Red}(u)) + \Delta(E_k, u) \\ &\geq \mathcal{H}(E_k, \mathrm{Red}(Mu)) + \Delta(E_k, u) \\ &= \mathcal{H}(E_k, Mu) - \Delta(E_k, Mu) + \Delta(E_k, u). \end{aligned}$$

As a consequence, (2-4) is proved for the arrangement u if we can show that

$$\Delta(E_k, u) \geq \Delta(E_k, Mu), \tag{2-10}$$

namely that the monotone rearrangement decreases (or at least does not increase) the hostility gap.

In order to prove (2-10), we begin by deriving a formula for the hostility gap. Let us consider the removal that leads from u to $\mathrm{Red}(u)$. We observe that interactions between any two dinosaurs placed on the same side of the removed one are equal before and after the removal, and therefore they cancel out when computing the gap. On the contrary, if two hostile dinosaurs are placed within distance d on opposite sides of the removed one, their hostility changes from $h(d)$ to $h(d-1)$ after the removal. It follows that the hostility gap can be written as

$$\Delta(E, u) = \sum_{i \in J_1(E,u,m)} h(|m-i|) - \sum_{(i,j) \in J_2(E,u,m)} (h(j-i-1) - h(j-i)), \tag{2-11}$$

where

$$\begin{aligned} J_1(E, u, m) &:= \{i \in \{1, \dots, n+1\} : (u(i), u(m)) \in E\}, \\ J_2(E, u, m) &:= \{(i, j) \in \{1, \dots, n+1\}^2 : i < m < j,\ (u(i), u(j)) \in E\}. \end{aligned}$$

The first sum in (2-11) takes into account the interactions of the removed dinosaur with the rest of the world, and the second sum represents the increment of the total hostility due to the reduction of distances among the others.

Now we introduce the new enemy list

$$E_{\langle \mu \rangle} := \mathbb{Z}^2 \setminus \{\mu, \mu - 1, \dots, \mu - k\}^2,$$

and we claim that

$$\Delta(E_k, u) \geq \Delta(E_{\langle \mu \rangle}, u) \geq \Delta(E_{\langle \mu \rangle}, Mu) = \Delta(E_k, Mu), \tag{2-12}$$

which of course implies (2-10).

The equality between the last two terms of (2-12) follows from formula (2-11). Indeed, since Mu is nondecreasing, the removed dinosaur is the rightmost one, and therefore in both cases the second sum in (2-11) is void. Also the first sum in (2-11) is the same in both cases, because a dinosaur of the highest species is hostile to another dinosaur with respect to the enemy list E_k if and only if it is hostile to the same dinosaur with respect to the enemy list $E_{\langle\mu\rangle}$.

The inequality between the first two terms of (2-12) follows again from formula (2-11). Indeed, the first sum has the same terms both in the case of the enemy list E_k and in the case of the enemy list $E_{\langle\mu\rangle}$, as observed above. As for the second sum, the interactions with respect to E_k are also interactions with respect to $E_{\langle\mu\rangle}$, and therefore when passing from E_k to $E_{\langle\mu\rangle}$ the second sum cannot decrease. Since the second sum appears in (2-11) with negative sign, the hostility gap with respect to $E_{\langle\mu\rangle}$ is less than or equal to the hostility gap with respect to E_k.

It remains to prove that

$$\Delta(E_{\langle\mu\rangle}, u) \geq \Delta(E_{\langle\mu\rangle}, Mu). \tag{2-13}$$

To this end, we introduce the complement enemy list

$$E^c_{\langle\mu\rangle} := \{\mu, \mu-1, \dots, \mu-k\}^2 = \mathbb{Z}^2 \setminus E_{\langle\mu\rangle}.$$

Since $\mathbb{Z}^2$ is the disjoint union of $E_{\langle\mu\rangle}$ and $E^c_{\langle\mu\rangle}$, and the total hostility is additive with respect to the enemy list, we deduce

$$\mathcal{H}(E_{\langle\mu\rangle}, w) = \mathcal{H}(\mathbb{Z}^2, w) - \mathcal{H}(E^c_{\langle\mu\rangle}, w)$$

for every arrangement w, and for the same reason

$$\Delta(E_{\langle\mu\rangle}, w) = \Delta(\mathbb{Z}^2, w) - \Delta(E^c_{\langle\mu\rangle}, w).$$

Moreover, we observe that the total hostility with respect to $\mathbb{Z}^2$ depends only on the number of dinosaurs, and in particular

$$\Delta(\mathbb{Z}^2, u) = \Delta(\mathbb{Z}^2, Mu).$$

As a consequence, proving (2-13) is equivalent to showing that

$$\Delta(E^c_{\langle\mu\rangle}, u) \leq \Delta(E^c_{\langle\mu\rangle}, Mu). \tag{2-14}$$

The advantage of this "complement formulation" is that hostility gaps with respect to $E^c_{\langle\mu\rangle}$ depend only on the relative positions of the removed dinosaur with respect to the other dinosaurs of the species with indices between $\mu-k$ and μ.

To be more precise, let us compute the left-hand side of (2-14). Let m denote as usual the position of the dinosaur that is removed from u to $\mathrm{Red}(u)$, and let us set

$$R(u) := \{r \geq 1 : u(m+r) \in \{\mu, \mu-1, \dots, \mu-k\}\},$$
$$L(u) := \{\ell \geq 1 : u(m-\ell) \in \{\mu, \mu-1, \dots, \mu-k\}\}.$$

In other words, this means that

$$\{m-\ell : \ell \in L(u)\} \cup \{m\} \cup \{m+r : r \in R(u)\}$$

is the set of all integers $i \in \{1, \dots, n+1\}$ such that $u(i) \in \{\mu, \mu-1, \dots, \mu-k\}$, namely the set of positions where the dinosaurs of the last $k+1$ species are placed. With this notation, the first sum in (2-11) is

$$h(0) + \sum_{\ell \in L(u)} h(\ell) + \sum_{r \in R(u)} h(r)$$

(we recall that in this "complement formulation" the dinosaur in position m is also hostile to itself), while the second sum in (2-11) is

$$\sum_{(\ell,r) \in L(u) \times R(u)} (h(\ell+r-1) - h(\ell+r)).$$

Therefore, it turns out that

$$\Delta(E^c_{\langle\mu\rangle}, u) = \mathcal{G}(L(u), R(u)),$$

where the function $\mathcal{G}$ is defined by

$$\mathcal{G}(L, R) := h(0) + \sum_{\ell \in L} h(\ell) + \sum_{r \in R} h(r) - \sum_{(\ell,r) \in L \times R} (h(\ell+r-1) - h(\ell+r)) \tag{2-15}$$

for any two sets L and R of positive integers.

On the other hand, in the nondecreasing arrangement Mu the rightmost dinosaur has $|L(u)| + |R(u)|$ dinosaurs of the last $k+1$ species exactly on its left, and therefore

$$\Delta(E^c_{\langle\mu\rangle}, Mu) = \sum_{i=0}^{|L(u)|+|R(u)|} h(i).$$

As a consequence, inequality (2-14) is proved if we show that

$$\mathcal{G}(L, R) \leq \sum_{i=0}^{|L|+|R|} h(i) \tag{2-16}$$

for every choice of the sets L and R. For this final step, we argue by induction on the number of elements of R. If $R = \varnothing$, from (2-15) we deduce

$$\mathcal{G}(L, R) := h(0) + \sum_{\ell \in L} h(\ell) \leq \sum_{i=0}^{|L|} h(i) = \sum_{i=0}^{|L|+|R|} h(i),$$

where the inequality is true term-by-term because h is nonincreasing.

Let us assume now that the conclusion holds true whenever R has n elements, and let us consider any pair (L, R) with $|R| = n+1$. Let us set

$$a := \max R, \quad b := \min\{n \in \mathbb{N} \setminus \{0\} : n \notin L\},$$

and let us consider the new pair (L_1, R_1) defined as

$$L_1 := L \cup \{b\}, \quad R_1 := R \setminus \{a\}.$$

In words, we have removed the largest element of R, and added the smallest possible element to L. We observe that $|R_1| = n$ and $|L_1| + |R_1| = |L| + |R|$. Therefore, if we show that

$$\mathcal{G}(L, R) \leq \mathcal{G}(L_1, R_1), \tag{2-17}$$

then (2-16) follows from the inductive assumption.

In order to prove (2-17), we expand the left-and right-hand sides according to (2-15). After canceling out the common terms, with some algebra we obtain that inequality (2-17) holds true if and only if

$$h(a) + \sum_{r \in R_1} (h(b+r-1) - h(b+r)) \leq h(b) + \sum_{\ell \in L} (h(\ell+a-1) - h(\ell+a)). \tag{2-18}$$

All terms in the sums are nonnegative because h is nonincreasing. Let us consider the left-hand side. If $a > 1$ we know that $R_1 \subseteq \{1, \dots, a-1\}$, and hence

$$\begin{aligned} h(a) + \sum_{r \in R_1} (h(b+r-1) - h(b+r)rut) &\leq h(a) + \sum_{r=1}^{a-1} (h(b+r-1) - h(b+r)) \\ &= h(a) + h(b) - h(a+b-1). \end{aligned} \tag{2-19}$$

The same inequality is true for trivial reasons also if $a = 1$.

Let us consider now the right-hand side of (2-18). If $b > 1$ we know that $L \supseteq \{1, \dots, b-1\}$, and hence

$$\begin{aligned} h(b) + \sum_{\ell \in L} (h(\ell+a-1) - h(\ell+a)) &\geq h(b) + \sum_{\ell=1}^{b-1} (h(\ell+a-1) - h(\ell+a)) \\ &= h(b) + h(a) - h(a+b-1). \end{aligned} \tag{2-20}$$

As before, the same inequality is true for trivial reasons also if $b = 1$.

Combining (2-20) and (2-19) we obtain (2-18), which in turn is equivalent to (2-17). This completes the proof of (2-16). □

2D. ***Proof of Theorem 2.4.*** The proof relies on the following approximation result (we omit the proof, which is an exercise in basic measure theory).

Lemma 2.6. *Let m be a positive integer, and let $D_1, \dots, D_m$ be disjoint measurable subsets of $(0, 1)$ such that*

$$\bigcup_{i=1}^{m} D_i = (0, 1).$$

Then for every $\varepsilon > 0$ there exist disjoint subsets $D_{1,\varepsilon}, \dots, D_{m,\varepsilon}$ of $[0, 1]$ such that

$$\bigcup_{i=1}^{m} D_{i,\varepsilon} = (0, 1)$$

and such that for every $i = 1, \dots, m$ it turns out that

- *$D_{i,\varepsilon}$ is a finite union of intervals with rational endpoints,*
- *the Lebesgue measure of the symmetric difference between D_i and $D_{i,\varepsilon}$ is less than or equal to ε.*

We are now ready to prove Theorem 2.4. First of all, we observe that (2-7) is invariant by translations and homotheties. As a consequence, there is no loss of generality in assuming that $(a, b) = (0, 1)$ and $c : (0, 1) \to \mathbb{R}$. Then we proceed in three steps. To begin with, we prove (2-7) in the special case where the hostility function c is bounded and the arrangement u has a very rigid structure, then for general u but again bounded hostility function, and finally in the general setting.

Step 1: We prove (2-7) under the additional assumption that the hostility function $c : (0, 1) \to \mathbb{R}$ is bounded, and that there exists a positive integer d such that $u(x)$ is constant in each interval of the form $((i-1)/d, i/d)$ with $i = 1, \dots, d$.

Indeed, this is actually the discrete setting. To be more precise, we introduce the discrete arrangement $v : \{1, \dots, d\} \to \mathbb{Z}$ defined as

$$v(i) := u\left(\frac{i - \frac{1}{2}}{d}\right) \quad \text{for all } i \in \{1, \dots, d\}$$

and the discrete hostility function $h : \{0, \dots, d-1\} \to \mathbb{R}$ defined as

$$h(i) := \int_0^{1/d} dx \int_{i/d}^{(i+1)/d} c(|y - x|)\, dy \quad \text{for all } i \in \{0, \dots, d-1\},$$

which represents the contribution to the total hostility of two intervals of length $1/d$ occupied by hostile dinosaurs, and placed at distance i/d from each other. Then for every enemy list E_k it turns out that

$$\mathcal{F}(c, E_k, u) = 2\mathcal{H}(h, E_k, v),$$

where $\mathcal{H}(h, E_k, v)$ is the discrete total hostility defined in (2-2), and the factor 2 takes into account that both (x, y) and (y, x) are included in the semidiscrete interaction set $I(E_k, u)$, while only one of them is included in the discrete counterpart $J(E_k, v)$; see (2-1) and (2-5). Moreover, the monotone rearrangement Mv of v is related to the monotone rearrangement Mu of u by the formula

$$Mv(i) = Mu\left(\frac{i - \frac{1}{2}}{d}\right) \quad \text{for all } i \in \{1, \dots, d\},$$

and again it turns out that

$$\mathcal{F}(c, E_k, Mu) = 2\mathcal{H}(h, E_k, Mv)$$

for every enemy list E_k. At this point, (2-7) is equivalent to

$$\mathcal{H}(h, E_k, v) \geq \mathcal{H}(h, E_k, Mv),$$

which in turn is true because of Theorem 2.2.

Step 2: We prove (2-7) for a general arrangement $u : (0, 1) \to \mathbb{Z}$, but again under the additional assumption that the hostility function $c : (0, 1) \to \mathbb{R}$ is bounded.

To this end, let $z_1 < z_2 < \cdots < z_m$ denote the elements in the image of u, and let

$$D_i := \{x \in (0, 1) : u(x) = z_i\} \quad \text{for all } i \in \{1, \dots, m\}$$

denote the set of positions of dinosaurs of the species z_i. For every $\varepsilon > 0$, let us consider the sets $D_{1,\varepsilon}$, ..., $D_{m,\varepsilon}$ given by Lemma 2.6, and the function $u_\varepsilon : (0, 1) \to \mathbb{Z}$ defined as

$$u_\varepsilon(x) = z_i \quad \text{for all } x \in D_{i,\varepsilon}.$$

Since the hostility function c is bounded, and the symmetric difference between D_i and $D_{i,\varepsilon}$ has measure less than or equal to ε, there exists a constant Γ (depending on m and c, but independent of ε) such that

$$|\mathcal{F}(c, E_k, u) - \mathcal{F}(c, E_k, u_\varepsilon)| \le \Gamma\varepsilon \quad \text{and} \quad |\mathcal{F}(c, E_k, Mu) - \mathcal{F}(c, E_k, Mu_\varepsilon)| \le \Gamma\varepsilon.$$

On the other hand, the function u_ε satisfies the assumptions of the previous step, and therefore

$$\mathcal{F}(c, E_k, u_\varepsilon) \ge \mathcal{F}(c, E_k, Mu_\varepsilon).$$

From all these inequalities it follows that

$$\mathcal{F}(c, E_k, u) \ge \mathcal{F}(c, E_k, Mu) - 2\Gamma\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, (2-7) is proved in this case.

Step 3: We prove (2-7) without assuming that the hostility function $c(x)$ is bounded.

To this end, for every $n \in \mathbb{N}$ we consider the truncated hostility function

$$c_n(x) := \min\{c(x), n\} \quad \text{for all } x \in (0, 1).$$

We observe that

$$\mathcal{F}(c, E_k, u) \ge \mathcal{F}(c_n, E_k, u) \quad \text{for all } n \in \mathbb{N}$$

because $c(x) \ge c_n(x)$ for every $x \in (0, 1)$, and

$$\mathcal{F}(c_n, E_k, u) \ge \mathcal{F}(c_n, E_k, Mu) \quad \text{for all } n \in \mathbb{N}$$

because of the result of the previous step applied to the bounded hostility function $c_n(x)$. As a consequence, we obtain

$$\mathcal{F}(c, E_k, u) \ge \mathcal{F}(c_n, E_k, Mu) \quad \text{for all } n \in \mathbb{N}. \tag{2-21}$$

On the other hand, by monotone convergence we deduce

$$\mathcal{F}(c, E_k, Mu) = \sup_{n\in\mathbb{N}} \mathcal{F}(c_n, E_k, Mu),$$

and therefore (2-7) follows from (2-21). □

3. Gamma-convergence in dimension 1

In this section we prove Theorem 1.1 for $d = 1$, in which case

$$G_{1,p} = 2 \quad \text{for all } p \ge 1. \tag{3-1}$$

To begin with, we introduce the notion of vertical δ-segmentation, which is going to play a crucial role in many parts of the proof.

Definition 3.1 (vertical δ-segmentation). Let $\mathbb{X}$ be any set, let $w : \mathbb{X} \to \mathbb{R}$ be any function, and let $\delta > 0$. The vertical δ-segmentation of w is the function $S_\delta w : \mathbb{X} \to \mathbb{R}$ defined by

$$S_\delta w(x) := \delta \left\lfloor \frac{w(x)}{\delta} \right\rfloor \quad \text{for all } x \in \mathbb{X}. \tag{3-2}$$

The function $S_\delta w$ takes its values in $\delta\mathbb{Z}$, and it is uniquely characterized by the fact that $S_\delta w(x) = k\delta$ for some $k \in \mathbb{Z}$ if and only if $k\delta \le w(x) < (k+1)\delta$.

3A. ***Asymptotic cost of oscillations.*** Let us assume that a function $u_\delta(x)$ oscillates between two values A and B in some interval (a, b). Does this provide an estimate from below for $\Lambda_{\delta,p}(u_\delta, (a, b))$, at least when δ is small enough? The following proposition and the subsequent corollaries give a sharp quantitative answer to this question. They are the fundamental tool in the proof of the liminf inequality.

Proposition 3.2 (limit cost of vertical oscillations). *Let $p \ge 1$ be a real number, let $(a, b) \subseteq \mathbb{R}$ be an interval, and let $\{u_\delta\}_{\delta>0} \subseteq L^p((a, b))$ be a family of functions.*

Let us assume that there exist two real numbers $A \le B$ such that

$$\liminf_{\delta\to 0^+} \operatorname{meas}\{x \in (a, b) : u_\delta(x) \le A + \varepsilon\} > 0 \quad \textit{for all } \varepsilon > 0, \tag{3-3}$$

$$\liminf_{\delta\to 0^+} \operatorname{meas}\{x \in (a, b) : u_\delta(x) \ge B - \varepsilon\} > 0 \quad \textit{for all } \varepsilon > 0. \tag{3-4}$$

Then it turns out that

$$\liminf_{\delta\to 0^+} \Lambda_{\delta,p}(u_\delta, (a, b)) \ge \frac{2}{p} \cdot C_p \cdot \frac{(B - A)^p}{(b - a)^{p-1}}, \tag{3-5}$$

where C_p is the constant defined in (1-6).

Proof. To begin with, we observe that (3-5) is trivial if $A = B$, and therefore in the sequel we assume that $A < B$.

Let us fix $\varepsilon > 0$ such that $4\varepsilon < B - A$. Due to assumptions (3-3) and (3-4), there exist $\eta > 0$ and $\delta_0 > 0$ such that

$$\operatorname{meas}\{x \in (a, b) : u_\delta(x) \le A + \varepsilon\} \ge \eta \quad \text{for all } \delta \in (0, \delta_0), \tag{3-6}$$

$$\operatorname{meas}\{x \in (a, b) : u_\delta(x) \ge B - \varepsilon\} \ge \eta \quad \text{for all } \delta \in (0, \delta_0). \tag{3-7}$$

Truncation, δ-segmentation and monotone rearrangement: In this section of the proof, we replace $\{u_\delta\}$ with a new family $\{\hat{u}_\delta\}$ of monotone piecewise constant functions that still satisfies (3-3) and (3-4), without increasing the left-hand side of (3-5). To this end, we perform three operations on $u_\delta(x)$.

The first operation is a truncation between A and B. To be more precise, we define $T_{A,B}u_\delta : (a, b) \to \mathbb{R}$ by setting

$$T_{A,B}u_\delta(x) := \begin{cases} A & \text{if } u_\delta(x) < A, \\ u_\delta(x) & \text{if } A \le u_\delta(x) \le B, \\ B & \text{if } u_\delta(x) > B. \end{cases}$$

We observe that the implication

$$|T_{A,B}u_\delta(y) - T_{A,B}u_\delta(x)| > \delta \quad \Longrightarrow \quad |u_\delta(y) - u_\delta(x)| > \delta$$

holds true for every x and y in (a, b), and hence

$$\Lambda_{\delta,p}(T_{A,B}u_\delta, (a, b)) \le \Lambda_{\delta,p}(u_\delta, (a, b)) \quad \text{for all } \delta > 0.$$

We also observe that (3-6) and (3-7) remain true if we replace $u_\delta(x)$ by $T_{A,B}u_\delta(x)$.

The second operation is a vertical δ-segmentation; namely we replace $T_{A,B}u_\delta$ by the function $S_\delta T_{A,B}u_\delta$ defined according to (3-2). Again we observe that the implications

$$\begin{aligned} |S_\delta T_{A,B}u_\delta(y) - S_\delta T_{A,B}u_\delta(x)| > \delta \quad &\Longrightarrow \quad |S_\delta T_{A,B}u_\delta(y) - S_\delta T_{A,B}u_\delta(x)| \ge 2\delta \\ &\Longrightarrow \quad |T_{A,B}u_\delta(y) - T_{A,B}u_\delta(x)| > \delta \end{aligned}$$

hold true for every x and y in (a, b), and hence

$$\Lambda_{\delta,p}(S_\delta T_{A,B}u_\delta, (a, b)) \le \Lambda_{\delta,p}(T_{A,B}u_\delta, (a, b)) \quad \text{for all } \delta > 0.$$

As for (3-6) and (3-7), we set $\delta_1 := \min\{\varepsilon, \delta_0\}$, and we observe that now

$$\operatorname{meas}\{x \in (a, b) : S_\delta T_{A,B}u_\delta(x) \le A + 2\varepsilon\} \ge \eta \quad \text{for all } \delta \in (0, \delta_1), \tag{3-8}$$

$$\operatorname{meas}\{x \in (a, b) : S_\delta T_{A,B}u_\delta(x) \ge B - 2\varepsilon\} \ge \eta \quad \text{for all } \delta \in (0, \delta_1). \tag{3-9}$$

The third and last operation we perform is monotone rearrangement; namely we replace $S_\delta T_{A,B}u_\delta$ with the nondecreasing function $MS_\delta T_{A,B}u_\delta$ in (a, b) whose level sets have the same measure of the level sets of $S_\delta T_{A,B}u_\delta$ (see Definition 2.3).

From (3-8) and (3-9) we deduce that now

$$MS_\delta T_{A,B}u_\delta(x) \le A + 2\varepsilon \quad \text{for all } x \in (a, a+\eta), \text{ for all } \delta \in (0, \delta_1), \tag{3-10}$$

$$MS_\delta T_{A,B}u_\delta(x) \ge B - 2\varepsilon \quad \text{for all } x \in (b-\eta, b), \text{ for all } \delta \in (0, \delta_1). \tag{3-11}$$

Moreover, we claim that

$$\Lambda_{\delta,p}(MS_\delta T_{A,B}u_\delta, (a, b)) \le \Lambda_{\delta,p}(S_\delta T_{A,B}u_\delta, (a, b)) \quad \text{for all } \delta > 0. \tag{3-12}$$

This is a straightforward consequence of Theorem 2.4. To be more formal, let us consider the semidiscrete arrangement $v_\delta : (a, b) \to \mathbb{Z}$ defined by

$$v_\delta(x) := \frac{1}{\delta} S_\delta T_{A,B}u_\delta(x) \quad \text{for all } x \in (a, b)$$

(we recall that $S_\delta T_{A,B}u_\delta$ takes its values in $\delta\mathbb{Z}$, and hence $v_\delta(x)$ is integer-valued) and the hostility function $c : (0, b-a) \to \mathbb{R}$ defined as $c(\sigma) := \delta^p \sigma^{-1-p}$. We observe that

$$MS_\delta T_{A,B}u_\delta(x) = \delta M v_\delta(x) \quad \text{for all } x \in (a, b),$$

where Mv_δ is the nondecreasing rearrangement of v_δ according to Definition 2.3.

We observe also that for every pair of points x and y in (a,b) it turns out that

$$(x,y)\in I(\delta, S_\delta T_{A,B}u_\delta,(a,b)) \iff |v_\delta(y)-v_\delta(x)|\ge 2 \iff (x,y)\in I(E_1,v_\delta),$$

where E_1 is the enemy list defined in (2-3), and $I(E_1,v_\delta)$ is the semidiscrete interaction set defined according to (2-5). It follows that

$$\Lambda_{\delta,p}(S_\delta T_{A,B}u_\delta,(a,b))=\mathcal{F}(c,E_1,v_\delta),\qquad \Lambda_{\delta,p}(MS_\delta T_{A,B}u_\delta,(a,b))=\mathcal{F}(c,E_1,Mv_\delta),$$

and therefore (3-12) is equivalent to (2-7).

In conclusion, the three operations described so far delivered us a family

$$\hat{u}_\delta := MS_\delta T_{A,B}u_\delta$$

of nondecreasing functions such that the image of $\hat{u}_\delta$ is contained in $\delta\mathbb{Z}$. This family satisfies (3-10) and (3-11), and

$$\Lambda_{\delta,p}(u_\delta,(a,b))\ge \Lambda_{\delta,p}(\hat{u}_\delta,(a,b)) \quad \text{for all } \delta>0. \tag{3-13}$$

In the sequel we are going to show that any such family satisfies

$$\liminf_{\delta\to 0^+}\Lambda_{\delta,p}(\hat{u}_\delta,(a,b))\ge \frac{2}{p}\cdot C_p\cdot\frac{(B-A-4\varepsilon)^p}{(b-a)^{p-1}}. \tag{3-14}$$

Due to (3-13) and the arbitrariness of $\varepsilon>0$, this is enough to prove (3-5).

Extension of the integrals to a vertical strip: In this section of the proof we modify the domain of integration in order to simplify the computation of $\Lambda_{\delta,p}(\hat{u}_\delta,(a,b))$. To begin with, we observe that

$$\Lambda_{\delta,p}(\hat{u}_\delta,(a,b))=\iint_{A_\delta}\frac{\delta^p}{|y-x|^{1+p}}\,dx\,dy\ge\iint_{B_\delta}\frac{\delta^p}{|y-x|^{1+p}}\,dx\,dy,$$

where

$$A_\delta := I(\delta,\hat{u}_\delta,(a,b))=\{(x,y)\in(a,b)^2 : |\hat{u}_\delta(y)-\hat{u}_\delta(x)|>\delta\},$$
$$B_\delta := \{(x,y)\in(a+\eta,b-\eta)\times(a,b) : |\hat{u}_\delta(y)-\hat{u}_\delta(x)|>\delta\}.$$

Then we write the last integral in the form

$$\iint_{B_\delta}\frac{\delta^p}{|y-x|^{1+p}}\,dx\,dy=\iint_{B_\delta\cup C_\delta}\frac{\delta^p}{|y-x|^{1+p}}\,dx\,dy-\iint_{C_\delta}\frac{\delta^p}{|y-x|^{1+p}}\,dx\,dy,$$

where

$$C_\delta := (a+\eta,b-\eta)\times(\mathbb{R}\setminus(a,b)).$$

In other words, the set $B_\delta\cup C_\delta$ consists of the vertical strip $(a+\eta,b-\eta)\times\mathbb{R}$ minus the set of points $(x,y)\in(a+\eta,b-\eta)\times(a,b)$ such that $|\hat{u}_\delta(y)-\hat{u}_\delta(x)|\le\delta$. Now we observe that

$$\iint_{C_\delta}\frac{\delta^p}{|y-x|^{1+p}}\,dx\,dy=2\delta^p\int_{a+\eta}^{b-\eta}dx\int_b^{+\infty}\frac{1}{|y-x|^{1+p}}\,dy.$$

From the convergence of the last double integral it follows that

$$\lim_{\delta\to 0^+}\iint_{C_\delta}\frac{\delta^p}{|y-x|^{1+p}}\,dx\,dy=0,$$

and therefore

$$\liminf_{\delta\to 0^+}\Lambda_{\delta,p}(\hat{u}_\delta,(a,b))\geq\liminf_{\delta\to 0^+}\iint_{B_\delta}\frac{\delta^p}{|y-x|^{1+p}}\,dx\,dy=\liminf_{\delta\to 0^+}\iint_{B_\delta\cup C_\delta}\frac{\delta^p}{|y-x|^{1+p}}\,dx\,dy. \tag{3-15}$$

Computing the integrals: In this last part of the proof we show that

$$\liminf_{\delta\to 0^+}\iint_{B_\delta\cup C_\delta}\frac{\delta^p}{|y-x|^{1+p}}\,dx\,dy\geq\frac{2}{p}\cdot C_p\cdot\frac{(B-A-4\varepsilon)^p}{(b-a)^{p-1}}. \tag{3-16}$$

Recalling (3-15), this proves (3-14), and hence also (3-5).

To this end, we need to introduce some notation. We know that $\hat{u}_\delta$ is a nondecreasing function with finite image. Let us consider the partition

$$a=x_0<x_1<\cdots<x_n=b$$

of (a,b) with the property that $\hat{u}_\delta(x)$ is constant in each interval of the form (x_{i-1},x_i), and different intervals correspond to different constants. Let us set

$$h:=\min\{i\in\{1,\ldots,n\}:x_i\geq a+\eta\},$$
$$k:=\max\{i\in\{0,\ldots,n-1\}:x_i\leq b-\eta\}.$$

Of course n, h, k, as well as the partition, do depend on δ. Now we claim that

$$\iint_{B_\delta\cup C_\delta}\frac{\delta^p}{|y-x|^{1+p}}\,dx\,dy\geq\frac{2}{p}\cdot C_p\cdot\frac{\delta^p(k-h-1)^p}{(b-a)^{p-1}}\quad\text{for all }\delta\in(0,\delta_1). \tag{3-17}$$

To this end we can limit ourselves, without loss of generality, to the case where the values of $\hat{u}_\delta(x)$ in neighboring intervals are consecutive multiples of δ; namely if $\hat{u}_\delta(x)=m\delta$ in (x_{i-1},x_i) for some $m\in\mathbb{Z}$, then $\hat{u}_\delta(x)=(m+1)\delta$ in (x_i,x_{i+1}). Indeed, if $\hat{u}_\delta(x)\geq(m+2)\delta$ in (x_i,x_{i+1}), then it turns out that

$$\iint_{B_\delta\cup C_\delta}\frac{\delta^p}{|y-x|^{1+p}}\,dx\,dy\geq\int_{x_{i-1}}^{x_i}dx\int_{x_i}^{x_{i+1}}\frac{\delta^p}{(y-x)^{1+p}}\,dy.$$

Since the integral in the right-hand side is divergent, the left-hand side is divergent as well, and in this case (3-17) is trivially true.

Therefore, in the sequel we treat the case where the values of $\hat{u}_\delta(x)$ in neighboring intervals are consecutive multiples of δ. Under this assumption it turns out that

$$\begin{aligned}\iint_{B_\delta\cup C_\delta}\frac{\delta^p}{|y-x|^{1+p}}\,dx\,dy&\geq\sum_{i=h+1}^{k-1}\left(\int_{x_{i-1}}^{x_i}dx\int_{x_{i+1}}^{+\infty}\frac{\delta^p}{|y-x|^{1+p}}\,dy+\int_{x_i}^{x_{i+1}}dx\int_{-\infty}^{x_{i-1}}\frac{\delta^p}{|y-x|^{1+p}}\,dy\right)\\&=\frac{\delta^p}{p}\sum_{i=h+1}^{k-1}\left(\int_{x_{i-1}}^{x_i}\frac{1}{(x_{i+1}-x)^p}\,dx+\int_{x_i}^{x_{i+1}}\frac{1}{(x-x_{i-1})^p}\,dx\right).\end{aligned}$$

Now we distinguish two cases.

- If $p = 1$, computing the integrals we obtain

$$\iint_{B_\delta \cup C_\delta} \frac{\delta}{(y-x)^2}\, dx\, dy \geq \delta \sum_{i=h+1}^{k-1} \log\left(\frac{x_{i+1}-x_{i-1}}{x_{i+1}-x_i} \cdot \frac{x_{i+1}-x_{i-1}}{x_i - x_{i-1}} \right).$$

If $\ell_i := x_i - x_{i-1}$ denotes the length of the i-th interval of the partition, and we apply the inequality between arithmetic and geometric mean, we obtain

$$\iint_{B_\delta \cup C_\delta} \frac{\delta}{(y-x)^2}\, dx\, dy \geq \delta \sum_{i=h+1}^{k-1} \log \frac{(\ell_i + \ell_{i+1})^2}{\ell_i \cdot \ell_{i+1}} \geq \delta \sum_{i=h+1}^{k-1} \log 4 = 2 \log 2 \cdot \delta (k-h-1),$$

which proves (3-17) in this case.

- If $p > 1$, computing the integrals we obtain

$$\iint_{B_\delta \cup C_\delta} \frac{\delta^p}{|y-x|^{1+p}}\, dx\, dy \geq \frac{\delta^p}{p(p-1)} \sum_{i=h+1}^{k-1} \left(\frac{1}{\ell_{i+1}^{p-1}} + \frac{1}{\ell_i^{p-1}} - \frac{2}{(\ell_{i+1}+\ell_i)^{p-1}} \right),$$

where we set $\ell_i := x_i - x_{i-1}$ as before. Therefore, with two applications of Jensen's inequality to the convex function $t \to t^{1-p}$, we obtain

$$\begin{aligned} \iint_{B_\delta \cup C_\delta} \frac{\delta^p}{|y-x|^{1+p}}\, dx\, dy &\geq \frac{\delta^p}{p(p-1)} \sum_{i=h+1}^{k-1} \frac{2^p - 2}{(\ell_{i+1}+\ell_i)^{p-1}} \\ &\geq \frac{\delta^p (2^p - 2)}{p(p-1)} \cdot \frac{(k-h-1)^p}{\left(\sum_{i=h+1}^{k-1} (\ell_{i+1}+\ell_i) \right)^{p-1}} \\ &\geq \frac{\delta^p (2^p - 2)}{p(p-1)} \cdot \frac{(k-h-1)^p}{(2(b-a))^{p-1}} = \frac{2}{p} \cdot C_p \cdot \frac{\delta^p (k-h-1)^p}{(b-a)^{p-1}}, \end{aligned}$$

which proves (3-17) also in this case.

Now it remains to estimate $\delta(k-h-1)$. To this end, from (3-10) and the minimality of h we deduce

$$A + 2\varepsilon \geq \hat{u}_\delta(x) =: m_A \delta \quad \text{for all } x \in (x_{h-1}, x_h).$$

Similarly, from (3-11) and the maximality of k we deduce

$$B - 2\varepsilon \leq \hat{u}_\delta(x) =: m_B \delta \quad \text{for all } x \in (x_k, x_{k+1}).$$

Since the values of $\hat{u}_\delta$ in consecutive intervals are consecutive multiples of δ, it turns out that

$$m_B = m_A + (k-h+1),$$

and therefore

$$(k-h-1)\delta = (k-h+1)\delta - 2\delta = (m_B - m_A)\delta - 2\delta \geq B - A - 4\varepsilon - 2\delta.$$

Plugging this inequality into (3-17), and letting $\delta \to 0^+$, we obtain (3-16), which completes the proof. □

The following result is a straightforward consequence of Proposition 3.2.

Corollary 3.3. *Let us assume that $u_\delta \to u$ in $L^p(\mathbb{R})$, and let $(a, b) \subseteq \mathbb{R}$ be an interval whose endpoints a and b are Lebesgue points of u.*

Then it turns out that

$$\liminf_{\delta\to 0^+} \Lambda_{\delta,p}(u_\delta, (a, b)) \geq \frac{2}{p} \cdot C_p \cdot \frac{|u(b) - u(a)|^p}{(b - a)^{p-1}}.$$

Proof. It is enough to apply Proposition 3.2 with $A := \min\{u(a), u(b)\}$ and $B := \max\{u(a), u(b)\}$. Assumptions (3-3) and (3-4) are satisfied because a and b are Lebesgue points of the limit of the sequence u_δ. □

We conclude with another variant of Proposition 3.2. We do not need this statement in the sequel, but we think that it clarifies once more the relation between oscillations of u_δ and values of $\Lambda_{\delta,p}(u_\delta, (a, b))$.

Corollary 3.4. *Let $(a, b) \subseteq \mathbb{R}$ be an interval, let $\{u_\delta\}_{\delta>0} \subseteq L^p((a, b))$ be a family of functions, and let $\operatorname{osc}(u_\delta, (a, b))$ denote the essential oscillation of u_δ in (a, b).*

Then it turns out that

$$\liminf_{\delta\to 0^+} \Lambda_{\delta,p}(u_\delta, (a, b)) \geq \frac{2}{p} C_p \frac{1}{(b - a)^{p-1}} \Big(\liminf_{\delta\to 0^+} \operatorname{osc}(u_\delta, (a, b))\Big)^p.$$

Proof. Let i_δ and s_δ denote the essential infimum and the essential supremum of $u_\delta(x)$ in (a, b), respectively. Let us assume that i_δ and s_δ are real numbers (otherwise an analogous argument works with standard minor changes). Let us set $w_\delta(x) := u_\delta(x) - i_\delta$, and let us observe that

$$\Lambda_{\delta,p}(u_\delta, (a, b)) = \Lambda_{\delta,p}(w_\delta, (a, b)) \quad \text{for all } \delta > 0.$$

Now it is enough to apply Proposition 3.2 with $A := 0$ and

$$B := \liminf_{\delta\to 0^+} (s_\delta - i_\delta) = \liminf_{\delta\to 0^+} \operatorname{osc}(u_\delta, (a, b)).$$ □

3B. *Piecewise affine approximation.* The value of $\Lambda_{0,p}(u, \mathbb{R})$ is the supremum of $\Lambda_{0,p}(v, \mathbb{R})$ as v ranges over a sequence of piecewise affine functions that approximate u. The formal statement is the following (we omit the standard proof, based on the convexity of the norm).

Lemma 3.5 (piecewise affine horizontal segmentation). *Let $p \geq 1$ be a real number, and let $u \in L^p(\mathbb{R})$.*

Then there exists $c \in \mathbb{R}$ such that $c + q$ is a Lebesgue point of u for every $q \in \mathbb{Q}$.

Moreover, if for every positive integer k we consider the piecewise affine function $v_k : \mathbb{R} \to \mathbb{R}$ such that

$$v_k\left(c + \frac{i}{k}\right) = u\left(c + \frac{i}{k}\right) \quad \textit{for all } i \in \mathbb{Z},$$

then it turns out that

$$\Lambda_{0,p}(u, \mathbb{R}) = \lim_{k\to+\infty} \int_{\mathbb{R}} |v_k'(x)|^p \, dx = \sup_{k\geq 1} \int_{\mathbb{R}} |v_k'(x)|^p \, dx.$$

3C. ***Proof of Gamma-liminf inequality in dimension 1.*** We are now ready to prove (1-7) in the case $d = 1$. The idea is that Corollary 3.3 represents a "localized" version of the liminf inequality (1-7), which now follows from well-established techniques; see for example [Gobbino 1998; Gobbino and Mora 2001]. To this end, let $u_\delta \to u$ be any family converging in $L^p(\mathbb{R})$, and let c and v_k be as in Lemma 3.5. For every $i \in \mathbb{Z}$, we set $c_{k,i} := c + i/k$, and we apply Corollary 3.3 in the interval $(c_{k,i}, c_{k,i+1})$. We obtain

$$\liminf_{\delta\to 0^+} \Lambda_{\delta,p}(u_\delta, (c_{k,i}, c_{k,i+1})) \geq \frac{2}{p} C_p \frac{|u(c_{k,i+1}) - u(c_{k,i})|^p}{(1/k)^{p-1}} = \frac{2}{p} C_p \int_{c_{k,i}}^{c_{k,i+1}} |v_k'(x)|^p \, dx.$$

Since

$$\Lambda_{\delta,p}(u_\delta, \mathbb{R}) \geq \sum_{i\in\mathbb{Z}} \Lambda_{\delta,p}(u_\delta, (c_{k,i}, c_{k,i+1})) \quad \text{for all } \delta > 0,$$

we deduce

$$\begin{aligned} \liminf_{\delta\to 0^+} \Lambda_{\delta,p}(u_\delta, \mathbb{R}) &\geq \liminf_{\delta\to 0^+} \sum_{i\in\mathbb{Z}} \Lambda_{\delta,p}(u_\delta, (c_{k,i}, c_{k,i+1})) \\ &\geq \sum_{i\in\mathbb{Z}} \liminf_{\delta\to 0^+} \Lambda_{\delta,p}(u_\delta, (c_{k,i}, c_{k,i+1})) \\ &\geq \frac{2}{p} C_p \sum_{i\in\mathbb{Z}} \int_{c_{k,i}}^{c_{k,i+1}} |v_k'(x)|^p \, dx = \frac{2}{p} C_p \int_{\mathbb{R}} |v_k'(x)|^p \, dx. \end{aligned}$$

Letting $k \to +\infty$, and recalling (3-1), we obtain exactly (1-7). □

3D. ***Proof of Gamma-limsup inequality in dimension 1.*** This subsection is devoted to a proof of statement (2) of Theorem 1.1 in the case $d = 1$.

It is well known that we can limit ourselves to showing the existence of recovery families for every u belonging to a subset of $L^p(\mathbb{R})$ that is dense in energy with respect to $\Lambda_{0,p}(u, \mathbb{R})$. Classical examples of subsets that are dense in energy are the space $C_c^\infty(\mathbb{R})$ of functions of class C^∞ with compact support and the space of piecewise affine functions with compact support. Here for the sake of generality we consider the space $PC_c^1(\mathbb{R})$ of piecewise C^1 functions with compact support, defined as follows.

Definition 3.6. Let $u : \mathbb{R} \to \mathbb{R}$ be a function. We say that $u \in PC_c^1(\mathbb{R})$ if u has compact support, it is Lipschitz continuous, and there exists a *finite* subset $S \subseteq \mathbb{R}$ such that $u \in C^1(\mathbb{R} \setminus S)$.

We show that for every $u \in PC_c^1(\mathbb{R})$ the family $S_\delta u$ of vertical δ-segmentations of u is a recovery family. This proves the Gamma-limsup inequality in dimension 1.

Proposition 3.7 (existence of recovery families). *Let $p \geq 1$ be a real number, and let $u \in PC_c^1(\mathbb{R})$ be a piecewise C^1 function with compact support according to Definition 3.6. For every $\delta > 0$, let $S_\delta u$ denote the vertical δ-segmentation of u according to Definition 3.1.*

Then it turns out that

$$\limsup_{\delta\to 0^+} \Lambda_{\delta,p}(S_\delta u, \mathbb{R}) \leq \frac{2}{p} C_p \int_{\mathbb{R}} |u'(x)|^p \, dx. \tag{3-18}$$

Proof. To begin with, we introduce some notation. Let $R_0 \geq 1$ be any real number such that the support of u is contained in $[-R_0+1, R_0-1]$. Let L be the Lipschitz constant of u in $\mathbb{R}$, and let $S \subseteq \mathbb{R}$ be a finite set such that $u \in C^1(\mathbb{R} \setminus S)$. For every $x \in \mathbb{R}$ and every $\delta > 0$ we set

$$J(\delta, u, x) := \{y \in \mathbb{R} : |S_\delta u(y) - S_\delta u(x)| > \delta\}, \tag{3-19}$$

and

$$H_{\delta,p}(x) := \int_{J(\delta,u,x)} \frac{\delta^p}{|y-x|^{1+p}}\, dy,$$

so that

$$\Lambda_{\delta,p}(S_\delta u, \mathbb{R}) = \int_{\mathbb{R}} H_{\delta,p}(x)\, dx \quad \text{for all } \delta > 0. \tag{3-20}$$

In the sequel we call $H_{\delta,p}(x)$ the "pointwise hostility function". It represents the contribution of each point x to the double integral defining $\Lambda_{\delta,p}(S_\delta u, \mathbb{R})$.

Strategy of the proof: The outline of the proof is the following. First of all, we show that

$$\lim_{\delta \to 0^+} \int_{-\infty}^{-R_0} H_{\delta,p}(x)\, dx = \lim_{\delta \to 0^+} \int_{R_0}^{+\infty} H_{\delta,p}(x)\, dx = 0. \tag{3-21}$$

Then we define an averaged pointwise hostility function $\widehat{H}_{\delta,p}(x)$ with the property that

$$\int_{-R_0}^{R_0} H_{\delta,p}(x)\, dx = \int_{-R_0}^{R_0} \widehat{H}_{\delta,p}(x)\, dx. \tag{3-22}$$

We also show that the averaged pointwise hostility function satisfies the uniform bound

$$\widehat{H}_{\delta,p}(x) \leq \frac{2}{p} L^p \quad \text{for all } x \in [-R_0, R_0], \text{ for all } \delta > 0, \tag{3-23}$$

and the asymptotic estimate

$$\limsup_{\delta \to 0^+} \widehat{H}_{\delta,p}(x) \leq \frac{2}{p} C_p |u'(x)|^p \quad \text{for all } x \in [-R_0, R_0] \setminus S. \tag{3-24}$$

At this point, from Fatou's lemma we deduce

$$\limsup_{\delta \to 0^+} \int_{-R_0}^{R_0} H_{\delta,p}(x)\, dx = \limsup_{\delta \to 0^+} \int_{-R_0}^{R_0} \widehat{H}_{\delta,p}(x)\, dx \leq \int_{-R_0}^{R_0} \limsup_{\delta \to 0^+} \widehat{H}_{\delta,p}(x)\, dx \leq \frac{2}{p} C_p \int_{-R_0}^{R_0} |u'(x)|^p\, dx.$$

Keeping (3-20) and (3-21) into account, this estimate implies (3-18).

Reducing integration to a bounded interval: We prove (3-21).

To this end, let us consider any $x \leq -R_0$. We observe that in this case the set $J(\delta, u, x)$ defined in (3-19) is contained in the support of u, and hence

$$\int_{-\infty}^{-R_0} H_{\delta,p}(x)\, dx \leq \delta^p \int_{-\infty}^{-R_0} dx \int_{-R_0+1}^{R_0-1} \frac{1}{|y-x|^{1+p}}\, dy.$$

At this point the first limit in (3-21) follows from the convergence of the double integral. The proof of the second limit is analogous.

Uniform bound on the pointwise hostility function: We prove that

$$H_{\delta,p}(x) \le \frac{2}{p} L^p \quad \text{for all } x \in [-R_0, R_0], \text{ for all } \delta > 0. \tag{3-25}$$

To this end, we observe that the implication

$$|S_\delta u(y) - S_\delta u(x)| > \delta \quad \Longrightarrow \quad |u(y) - u(x)| > \delta$$

holds true for every $(x, y) \in \mathbb{R}^2$. Since u is Lipschitz continuous, we deduce that

$$|S_\delta u(y) - S_\delta u(x)| > \delta \quad \Longrightarrow \quad |y - x| \ge \frac{\delta}{L},$$

and hence

$$H_{\delta,p}(x) \le \int_{|y-x|\ge\delta/L} \frac{\delta^p}{|y-x|^{1+p}}\, dy = 2\int_{\delta/L}^{+\infty} \frac{\delta^p}{z^{1+p}}\, dz = \frac{2}{p} L^p,$$

as required.

Averaged pointwise hostility function: In this part of the proof we introduce the averaged pointwise hostility function. To this end, we consider the open set

$$A(u,\delta) := \{x \in (-R_0, R_0) : u(x) \notin \delta\mathbb{Z}\}.$$

A connected component (a, b) of $A(u, \delta)$ is called *monotone* if $[a, b] \cap S = \varnothing$, and $|u'(x)| \ge \delta$ for every $x \in [a, b]$. In this case there exists $k \in \mathbb{Z}$ such that $u(a) = k\delta$ and $u(b) = k\delta \pm \delta$, where the sign depends on the sign of $u'(x)$ in (a, b). From the Lipschitz continuity of u we deduce that $A(u, \delta)$ has only a finite number of monotone connected components.

The averaged pointwise hostility function $\widehat{H}_{\delta,p} : \mathbb{R} \to \mathbb{R}$ is defined as

$$\widehat{H}_{\delta,p}(x) := \frac{1}{b-a}\int_a^b H_{\delta,p}(s)\, ds$$

if $x \in [a, b)$ for some monotone connected component of $A(\delta, u)$, and $\widehat{H}_{\delta,p}(x) := H_{\delta,p}(x)$ otherwise.

At this point, inequality (3-23) follows from (3-25), while (3-22) is true because the integrals of $H_{\delta,p}(x)$ and $\widehat{H}_{\delta,p}(x)$ are the same both in all monotone connected components, and in the complement set.

Asymptotic estimate in stationary points: We prove that (3-24) holds true for every $x \in (-R_0, R_0) \setminus S$ with $|u'(x)| = 0$.

To begin with, we observe that in this case $x \notin [a, b)$ for every monotone connected component (a, b) of $A(\delta, u)$ (because $|u'(x)|$ is strictly positive in the closure of every monotone connected component), and therefore $\widehat{H}_{\delta,p}(x) = H_{\delta,p}(x)$ for every $\delta > 0$.

If $J(\delta, u, x) = \varnothing$ for every $\delta > 0$, then u is identically null, and the conclusion is trivial. Otherwise $J(\delta, u, x) \ne \varnothing$ when δ is small enough. In this case, let r_δ be the largest positive real number such that

$$(x - r_\delta, x + r_\delta) \cap J(\delta, u, x) = \varnothing,$$

so that

$$H_{\delta,p}(x) \le \int_{-\infty}^{x-r_\delta} \frac{\delta^p}{|y-x|^{1+p}}\, dy + \int_{x+r_\delta}^{+\infty} \frac{\delta^p}{|y-x|^{1+p}}\, dy = \frac{2}{p}\left(\frac{\delta}{r_\delta}\right)^p.$$

Let $\delta_k \to 0^+$ be any sequence such that

$$\limsup_{\delta\to 0^+} \frac{\delta}{r_\delta} = \lim_{k\to+\infty} \frac{\delta_k}{r_{\delta_k}}. \tag{3-26}$$

Up to subsequences, we can also assume that r_{δ_k} tends to some r_0. If $r_0 > 0$, then the limit in the right-hand side of (3-26) is 0, which proves (3-24) in this case. If $r_0 = 0$, then from the maximality of r_{δ_k} we deduce that $|u(x \pm r_{\delta_k}) - u(x)| = \delta_k$ for a suitable choice of the sign, which might depend on k. In any case, the limit in the right-hand side of (3-26) turns out to be

$$\lim_{k\to+\infty} \frac{\delta_k}{r_{\delta_k}} = \lim_{k\to+\infty} \frac{|u(x \pm r_{\delta_k}) - u(x)|}{r_{\delta_k}} = |u'(x)| = 0,$$

which proves (3-24) also in this case.

Asymptotic estimate in nonstationary points: We prove that (3-24) holds true for every $x \in (-R_0, R_0) \setminus S$ with $|u'(x)| > 0$.

Let us assume, without loss of generality, that $u'(x) > 0$ (the other case is analogous). Then for every $\delta > 0$ small enough it turns out that x lies in the closure of a monotone connected component of $A(\delta, u)$. More precisely, there exist four real numbers a_δ, b_δ, c_δ, d_δ with

$$a_\delta < b_\delta \le x < c_\delta < d_\delta,$$

and $k_\delta \in \mathbb{Z}$ such that

$$u(a_\delta) = (k_\delta - 1)\delta, \quad u(b_\delta) = k_\delta\delta, \quad u(c_\delta) = (k_\delta + 1)\delta, \quad u(d_\delta) = (k_\delta + 2)\delta,$$

and

$$u(y) \in ((k_\delta - 1)\delta, k_\delta\delta) \quad \text{for all } y \in (a_\delta, b_\delta), \tag{3-27}$$

$$u(y) \in (k_\delta\delta, (k_\delta + 1)\delta) \quad \text{for all } y \in (b_\delta, c_\delta), \tag{3-28}$$

$$u(y) \in ((k_\delta + 1)\delta, (k_\delta + 2)\delta) \quad \text{for all } y \in (c_\delta, d_\delta). \tag{3-29}$$

We observe that a_δ, b_δ, c_δ, and d_δ tend to x as $\delta \to 0^+$, and hence

$$\lim_{\delta\to 0^+} \frac{\delta}{b_\delta - a_\delta} = \lim_{\delta\to 0^+} \frac{u(b_\delta) - u(a_\delta)}{b_\delta - a_\delta} = u'(x). \tag{3-30}$$

Similarly it turns out that

$$\lim_{\delta\to 0^+} \frac{\delta}{c_\delta - b_\delta} = \lim_{\delta\to 0^+} \frac{\delta}{d_\delta - c_\delta} = u'(x), \tag{3-31}$$

$$\lim_{\delta\to 0^+} \frac{\delta}{c_\delta - a_\delta} = \lim_{\delta\to 0^+} \frac{\delta}{d_\delta - b_\delta} = \frac{u'(x)}{2}. \tag{3-32}$$

From (3-27) through (3-29) we deduce that

$$J(\delta, u, s) \subseteq (-\infty, a_\delta] \cup [d_\delta, +\infty) \quad \text{for all } s \in (b_\delta, c_\delta).$$

It follows that

$$H_{\delta,p}(s) \le \int_{\mathbb{R}\setminus(a_\delta,d_\delta)} \frac{\delta^p}{|y-s|^{1+p}}\,dy = \frac{\delta^p}{p}\left(\frac{1}{(d_\delta - s)^p} + \frac{1}{(s-a_\delta)^p}\right) \quad \text{for all } s \in [b_\delta, c_\delta),$$

and hence

$$\widehat{H}_{\delta,p}(x) = \frac{1}{c_\delta - b_\delta}\int_{b_\delta}^{c_\delta} H_{\delta,p}(s)\,ds \le \frac{\delta^p}{p}\frac{1}{c_\delta - b_\delta}\int_{b_\delta}^{c_\delta}\left(\frac{1}{(d_\delta - s)^p} + \frac{1}{(s-a_\delta)^p}\right)ds \tag{3-33}$$

for every $x \in [b_\delta, c_\delta)$. Now we distinguish two cases.

- If $p = 1$, computing the integrals in (3-33) we obtain

$$\widehat{H}_{\delta,p}(x) \le \frac{\delta}{c_\delta - b_\delta}\log\left(\frac{d_\delta - b_\delta}{\delta}\cdot\frac{\delta}{d_\delta - c_\delta}\cdot\frac{c_\delta - a_\delta}{\delta}\cdot\frac{\delta}{b_\delta - a_\delta}\right),$$

and therefore (3-24) follows from (3-30) through (3-32).

- If $p > 1$, computing the integrals in (3-33) we obtain

$$\widehat{H}_{\delta,p}(x) \le \frac{1}{p(p-1)}\frac{\delta}{c_\delta - b_\delta}\left\{\frac{\delta^{p-1}}{(d_\delta - c_\delta)^{p-1}} + \frac{\delta^{p-1}}{(b_\delta - a_\delta)^{p-1}} - \frac{\delta^{p-1}}{(d_\delta - b_\delta)^{p-1}} - \frac{\delta^{p-1}}{(c_\delta - a_\delta)^{p-1}}\right\},$$

and therefore also in this case (3-24) follows from (3-30) through (3-32). □

3E. *Smooth recovery families.* The aim of this subsection is refining the Gamma-limsup inequality by showing the existence of recovery families consisting of C^∞ functions with compact support. To this end, we introduce the following notion.

Definition 3.8 (δ-step functions). Let δ be a positive real number. A function $u : \mathbb{R} \to \mathbb{R}$ is called a *δ-step function* if there exists a positive integer n, an $(n+1)$-tuple $x_0 < x_1 < \cdots < x_n$ of real numbers, and $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ such that

- $u(x) = 0$ for every $x \in (-\infty, x_0) \cup (x_n, +\infty)$,
- $u(x) = k_i\delta$ in (x_{i-1}, x_i) for every $i = 1, \ldots, n$,
- $|k_1| = |k_n| = 1$ and $|k_i - k_{i-1}| = 1$ for every $i = 2, \ldots, n$.

The values of $u(x)$ for $x \in \{x_0, x_1, \ldots, x_n\}$ are not relevant (just to fix ideas, we can define $u(x_i)$ as the maximum between the limit of $u(x)$ as $x \to x_i^+$ and the limit of $u(x)$ as $x \to x_i^-$).

Now we show that, for every *fixed* $\delta > 0$, every δ-step function can be approximated in energy by functions of class C^∞ with compact support. Roughly speaking, this is possible because the rigid structure of δ-step functions allows us to control the effect of convolutions, which otherwise is unpredictable due to the sensitivity of the integration region in (1-1) to small perturbations.

Proposition 3.9 (smooth approximation of δ-step functions). *Let $\delta > 0$ and $p \ge 1$ be real numbers, and let $u : \mathbb{R} \to \mathbb{R}$ be a δ-step function.*

Then there exists a family $\{u_\varepsilon\}_{\varepsilon>0} \subseteq C_c^\infty(\mathbb{R})$ *such that*

$$\lim_{\varepsilon\to 0^+} u_\varepsilon = u \quad \text{in } L^p(\mathbb{R}),$$

and

$$\lim_{\varepsilon\to 0^+} \Lambda_{\delta,p}(u_\varepsilon,\mathbb{R}) = \Lambda_{\delta,p}(u,\mathbb{R}).$$

Proof. Let n, x_i and k_i be as in the definition of δ-step functions, and let

$$\tau := \min\{x_i - x_{i-1} : i = 1,\dots,n\}$$

be the length of the smallest interval of the partition. We observe that points in neighboring intervals do not contribute to the computation of $\Lambda_{\delta,p}(u,\mathbb{R})$. In particular, if we write as usual

$$\Lambda_{\delta,p}(u,\mathbb{R}) := \iint_{I(\delta,u,\mathbb{R})} \frac{\delta^p}{|y-x|^{1+p}}\,dx\,dy,$$

then it turns out that

$$|y-x| \geq \tau \quad \text{for all } (x,y) \in I(\delta,u,\mathbb{R}). \tag{3-34}$$

Let us fix a mollifier $\rho \in C_c^\infty(\mathbb{R})$ with

- $\rho(x) \geq 0$ for every $x \in \mathbb{R}$,
- $\rho(x) = 0$ for every $x \in \mathbb{R}$ with $|x| \geq 1$,
- $\int_{\mathbb{R}} \rho(x)\,dx = 1$,

and let us consider the usual regularization by convolution

$$u_\varepsilon(x) := \int_{\mathbb{R}} u(x+\varepsilon y)\rho(y)\,dy.$$

It is well known that $u_\varepsilon \in C_c^\infty(\mathbb{R})$ for every $\varepsilon > 0$, and that for every $p \geq 1$ it turns out that $u_\varepsilon \to u$ in $L^p(\mathbb{R})$ as $\varepsilon \to 0^+$.

Let us assume that $2\varepsilon < \tau$, let us consider the two open sets

$$A_\varepsilon := \bigcup_{i=0}^{n} (x_i - \varepsilon, x_i + \varepsilon) \subseteq \mathbb{R}, \qquad B_\varepsilon := (A_\varepsilon \times \mathbb{R}) \cup (\mathbb{R} \times A_\varepsilon) \subseteq \mathbb{R}^2,$$

and let us write

$$\Lambda_{\delta,p}(u_\varepsilon,\mathbb{R}) = \iint_{I(\delta,u_\varepsilon,\mathbb{R})\cap B_\varepsilon} \frac{\delta^p}{|y-x|^{1+p}}\,dx\,dy + \iint_{I(\delta,u_\varepsilon,\mathbb{R})\setminus B_\varepsilon} \frac{\delta^p}{|y-x|^{1+p}}\,dx\,dy.$$

Since the support of ρ is contained in $[-1,1]$, it turns out that $u_\varepsilon(x) = u(x)$ for every $x \in \mathbb{R} \setminus A_\varepsilon$. It follows that

$$I(\delta,u_\varepsilon,\mathbb{R}) \setminus B_\varepsilon = I(\delta,u,\mathbb{R}) \setminus B_\varepsilon,$$

and therefore

$$\lim_{\varepsilon\to 0^+} \iint_{I(\delta,u_\varepsilon,\mathbb{R})\setminus B_\varepsilon} \frac{\delta^p}{|y-x|^{1+p}}\,dx\,dy = \lim_{\varepsilon\to 0^+} \iint_{I(\delta,u,\mathbb{R})\setminus B_\varepsilon} \frac{\delta^p}{|y-x|^{1+p}}\,dx\,dy = \Lambda_{\delta,p}(u,\mathbb{R}),$$

where the last equality follows from Lebesgue's dominated convergence theorem because B_ε shrinks to a set of null measure. So it remains to show that

$$\lim_{\varepsilon\to 0^+}\iint_{I(\delta,u_\varepsilon,\mathbb{R})\cap B_\varepsilon}\frac{\delta^p}{|y-x|^{1+p}}\,dx\,dy=0. \tag{3-35}$$

To this end, from (3-34) and the properties of the support of the mollifier, we deduce that now

$$|y-x|\geq\tau-2\varepsilon\quad\text{for all }(x,y)\in I(\delta,u_\varepsilon,\mathbb{R}),$$

and therefore

$$\begin{aligned}\iint_{I(\delta,u_\varepsilon,\mathbb{R})\cap B_\varepsilon}\frac{\delta^p}{|y-x|^{1+p}}\,dx\,dy&\leq 2\sum_{i=0}^{n}\int_{x_i-\varepsilon}^{x_i+\varepsilon}dx\int_{|z|\geq\tau-2\varepsilon}\frac{\delta^p}{|z|^{1+p}}\,dz\\&\leq 2\sum_{i=0}^{n}\int_{x_i-\varepsilon}^{x_i+\varepsilon}\frac{2}{p}\frac{\delta^p}{|\tau-2\varepsilon|^p}\,dx=\frac{4}{p}\frac{\delta^p}{|\tau-2\varepsilon|^p}\cdot 2\varepsilon(n+1),\end{aligned}$$

which implies (3-35). □

We are now ready to show the existence of smooth recovery families. As usual, it is enough to show the existence of such a family for every u in a subset of $L^p(\mathbb{R})$ which is dense in energy for $\Lambda_{0,p}(u,\mathbb{R})$. In this case we consider the space $PA_c(\mathbb{R})$ of piecewise affine functions with compact support.

Since piecewise affine functions are piecewise C^1, we know from Proposition 3.7 that the family $S_\delta u$ of vertical δ-segmentations of u is a (nonsmooth) recovery family for u. The key point is that the vertical δ-segmentation of a piecewise affine function with compact support is a δ-step function according to Definition 3.8. Thus from Proposition 3.9 we deduce the existence of a function $u_\delta\in C_c^\infty(\mathbb{R})$ such that

$$\|u_\delta-S_\delta u\|_{L^p(\mathbb{R})}\leq\delta\quad\text{and}\quad\Lambda_{\delta,p}(u_\delta,\mathbb{R})\leq\Lambda_{\delta,p}(S_\delta u,\mathbb{R})+\delta$$

for every $\delta>0$. This implies that $\{u_\delta\}$ is a smooth recovery family for u. □

4. Gamma-convergence in any dimension

It remains to prove Theorem 1.1 in any space dimension. This follows from well-established sectioning techniques. For every $\sigma\in\mathbb{S}^{d-1}$, let $\langle\sigma\rangle^\perp$ denote the hyperplane orthogonal to σ, namely

$$\langle\sigma\rangle^\perp:=\{z\in\mathbb{R}^d:\langle z,\sigma\rangle=0\}.$$

Given any $u:\mathbb{R}^d\to\mathbb{R}$, for every $\sigma\in\mathbb{S}^{d-1}$ and every $z\in\langle\sigma\rangle^\perp$, we consider the 1-dimensional section $u_{\sigma,z}:\mathbb{R}\to\mathbb{R}$ defined as

$$u_{\sigma,z}(x):=u(z+\sigma x)\quad\text{for all }x\in\mathbb{R}.$$

The main idea is that Sobolev norms, total variation, and functionals such as $\Lambda_{\delta,p}$ computed in u are a sort of average of the same quantities computed on the 1-dimensional sections $u_{\sigma,z}$. The result is the following.

Proposition 4.1 (integral-geometric representation). *Let $u:\mathbb{R}^d\to\mathbb{R}$ be any measurable function. Let $\Lambda_{\delta,p}$ and $\Lambda_{0,p}$ be the functionals defined in* (1-1) *and* (1-2), *respectively.*

(1) *For every* $p \geq 1$ *it turns out that*

$$\int_{\mathbb{S}^{d-1}} d\sigma \int_{\langle\sigma\rangle^\perp} \Lambda_{0,p}(u_{\sigma,z}, \mathbb{R})\, dz = G_{d,p}\, \Lambda_{0,p}(u, \mathbb{R}^d),$$

where $G_{d,p}$ *is the geometric constant defined in* (1-4).

(2) *For every* $\delta > 0$ *and every* $p \geq 1$ *it turns out that*

$$\int_{\mathbb{S}^{d-1}} d\sigma \int_{\langle\sigma\rangle^\perp} \Lambda_{\delta,p}(u_{\sigma,z}, \mathbb{R})\, dz = 2\Lambda_{\delta,p}(u, \mathbb{R}^d). \qquad \square$$

We skip the details of the proof of Proposition 4.1, which is a simple application of variable changes in multiple integrals. More generally, for every $\sigma \in \mathbb{S}^{d-1}$ and every $g \in L^1(\mathbb{R}^d)$ it turns out that

$$\int_{\mathbb{R}^d} g(y)\, dy = \int_{\langle\sigma\rangle^\perp} dz \int_{\mathbb{R}} g(z + \sigma x)\, dx,$$

and this is the main ingredient in the proof of statement (1).

Similarly, for every $g \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$ it turns out that

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} g(u, v)\, du\, dv = \frac{1}{2} \int_{\mathbb{S}^{d-1}} d\sigma \int_{\langle\sigma\rangle^\perp} dz \iint_{\mathbb{R}\times\mathbb{R}} g(z + \sigma x, z + \sigma y) \cdot |y - x|^{d-1}\, dx\, dy,$$

and this is the main ingredient in the proof of statement (2).

We are now ready to prove Theorem 1.1.

Proof. Gamma-liminf: Let us assume that $u_\delta \to u$ in $L^1(\mathbb{R}^d)$. Then for every $\sigma \in \mathbb{S}^{d-1}$ it turns out that

$$(u_\delta)_{\sigma,z} \to u_{\sigma,z} \quad \text{in } L^1(\mathbb{R})$$

for almost every $z \in \langle\sigma\rangle^\perp$. Therefore, from the integral-geometric representations of Proposition 4.1, Fatou's lemma, and the 1-dimensional result, we obtain

$$\begin{aligned}
\liminf_{\delta\to 0^+} \Lambda_{\delta,p}(u_\delta, \mathbb{R}^d) &= \liminf_{\delta\to 0^+} \frac{1}{2} \int_{\mathbb{S}^{d-1}} d\sigma \int_{\langle\sigma\rangle^\perp} \Lambda_{\delta,p}((u_\delta)_{\sigma,z}, \mathbb{R})\, dz \\
&\geq \frac{1}{2} \int_{\mathbb{S}^{d-1}} d\sigma \int_{\langle\sigma\rangle^\perp} \liminf_{\delta\to 0^+} \Lambda_{\delta,p}((u_\delta)_{\sigma,z}, \mathbb{R})\, dz \\
&\geq \frac{1}{2} \int_{\mathbb{S}^{d-1}} d\sigma \int_{\langle\sigma\rangle^\perp} \frac{2}{p} C_p\, \Lambda_{0,p}(u_{\sigma,z}, \mathbb{R})\, dz \\
&= \frac{1}{p} G_{d,p} C_p\, \Lambda_{0,p}(u, \mathbb{R}^d).
\end{aligned}$$

Gamma-limsup: Let $u \in C_c^\infty(\mathbb{R}^d)$ be any function with compact support. For every $\delta > 0$ we consider the vertical δ-segmentation $S_\delta u$ of u, and we observe that this operation commutes with the 1-dimensional sections, in the sense that

$$(S_\delta u)_{\sigma,z} = S_\delta(u_{\sigma,z}) \quad \text{for all } \sigma \in \mathbb{S}^{d-1}, \text{ for all } z \in \langle\sigma\rangle^\perp.$$

Therefore, from the integral-geometric representations of Proposition 4.1, Fatou's lemma, and the 1-dimensional result, we obtain

$$\begin{aligned}\limsup_{\delta\to 0^+} \Lambda_{\delta,p}(S_\delta u, \mathbb{R}^d) &= \limsup_{\delta\to 0^+} \frac{1}{2}\int_{\mathbb{S}^{d-1}} d\sigma \int_{\langle\sigma\rangle^\perp} \Lambda_{\delta,p}((S_\delta u)_{\sigma,z}, \mathbb{R})\, dz \\ &\le \frac{1}{2}\int_{\mathbb{S}^{d-1}} d\sigma \int_{\langle\sigma\rangle^\perp} \limsup_{\delta\to 0^+} \Lambda_{\delta,p}((S_\delta u)_{\sigma,z}, \mathbb{R})\, dz \\ &\le \frac{1}{2}\int_{\mathbb{S}^{d-1}} d\sigma \int_{\langle\sigma\rangle^\perp} \frac{2}{p} C_p\, \Lambda_{0,p}(u_{\sigma,z}, \mathbb{R})\, dz \\ &= \frac{1}{p} G_{d,p} C_p\, \Lambda_{0,p}(u, \mathbb{R}^d).\end{aligned}$$

The δ-independent bounds on $\Lambda_{\delta,p}((S_\delta u)_{\sigma,z}, \mathbb{R})$ needed in order to apply Fatou's lemma follow from the Lipschitz continuity of u and the boundedness of its support.

Smooth recovery families: It remains to show the existence of smooth recovery families. The strategy is analogous to the 1-dimensional case, and therefore we limit ourselves to outlining the argument, sparing the reader all technicalities.

To begin with, we observe that the space $PA_c(\mathbb{R}^d)$ of piecewise affine functions with compact support is a subspace of $L^p(\mathbb{R}^d)$ that is dense in energy for $\Lambda_{0,p}(u, \mathbb{R}^d)$. This is true because $C_c^\infty(\mathbb{R}^d)$ is dense in energy, and in turn any function in $C_c^\infty(\mathbb{R}^d)$ can be approximated in $W^{1,\infty}(\mathbb{R}^d)$ by functions in $PA_c(\mathbb{R}^d)$; see for example Chapter 4 in [Brenner and Scott 1994], and in particular Corollary 4.4.24.

As a consequence, it is enough to show the existence of a recovery family for every $u \in PA_c(\mathbb{R}^d)$, in which case a nonsmooth recovery family is provided by the vertical δ-segmentations $S_\delta u$ of u. On the other hand, vertical δ-segmentations of piecewise affine functions with compact support are δ-step functions, and these functions can be approximated in energy by smooth functions. It follows that for every $\delta > 0$ there exists $u_\delta \in C_c^\infty(\mathbb{R}^d)$ such that

$$\|u_\delta - S_\delta u\|_{L^p(\mathbb{R}^d)} \le \delta \quad \text{and} \quad \Lambda_{\delta,p}(u_\delta, \mathbb{R}^d) \le \Lambda_{\delta,p}(S_\delta u, \mathbb{R}^d) + \delta,$$

and therefore $\{u_\delta\}$ is the required recovery family.

The last approximation step can be proved by convolution as we did in Proposition 3.9. To be more precise, a δ-step function in dimension d is a function $v : \mathbb{R}^d \to \mathbb{R}$ with the property that there exist a finite set $\{P_1, \ldots, P_m\}$ of disjoint open polytopes (bounded intersections of half-spaces) and integers $k_1, \ldots, k_m$ such that

- $v(x) = k_i\delta$ in P_i for every $i = 1, \ldots, m$,
- $v(x) = 0$ in the open set P_0 defined as the complement set of the closure of $P_1 \cup \cdots \cup P_m$,
- $|k_i - k_j| \le 1$ whenever the closure of P_i intersects the closure of P_j,
- $|k_i| \le 1$ whenever the closure of P_i intersects the closure of P_0.

In words, the level sets of a δ-step function are finite unions of polytopes, and values in adjacent regions differ by δ.

The key point is that for every δ-step function v there exists a positive real number τ such that

$$(x, y) \in I(\delta, v, \mathbb{R}^d) \quad \Longrightarrow \quad |y - x| \geq \tau.$$

As a consequence, when we define v_ε as the convolution of v with a mollifier whose support is contained in the ball with center in the origin and radius ε, we obtain

$$(x, y) \in I(\delta, v_\varepsilon, \mathbb{R}^d) \quad \Longrightarrow \quad |y - x| \geq \tau - 2\varepsilon,$$

and at this point the conclusion follows exactly as in the proof of Proposition 3.9. □

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