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ON THE TRACE OPERATOR FOR FUNCTIONS OF BOUNDED $\mathbb{A}$-VARIATION

DOMINIC BREIT, LARS DIENING AND FRANZ GMEINEDER

We consider the space $\mathrm{BV}^{\mathbb{A}}(\Omega)$ of functions of bounded $\mathbb{A}$-variation. For a given first-order linear homogeneous differential operator with constant coefficients $\mathbb{A}$, this is the space of L^1-functions $u : \Omega \to \mathbb{R}^N$ such that the distributional differential expression $\mathbb{A}u$ is a finite (vectorial) Radon measure. We show that for Lipschitz domains $\Omega \subset \mathbb{R}^n$, $\mathrm{BV}^{\mathbb{A}}(\Omega)$-functions have an $L^1(\partial\Omega)$-trace if and only if $\mathbb{A}$ is $\mathbb{C}$-elliptic (or, equivalently, if the kernel of $\mathbb{A}$ is finite-dimensional). The existence of an $L^1(\partial\Omega)$-trace was previously only known for the special cases that $\mathbb{A}u$ coincides either with the full or the symmetric gradient of the function u (and hence covered the special cases BV or BD). As a main novelty, we do not use the fundamental theorem of calculus to construct the trace operator (an approach which is only available in the BV- and BD-settings) but rather compare projections onto the nullspace of $\mathbb{A}$ as we approach the boundary. As a sample application, we study the Dirichlet problem for quasiconvex variational functionals with linear growth depending on $\mathbb{A}u$.

1. Introduction

1A. *Aim and scope.* Let Ω be an open, bounded Lipschitz domain in $\mathbb{R}^n$ and let $1 \le p < \infty$. A key tool in the study of partial differential equations is the assignment of boundary values to elements $u \in W^{1,p}(\Omega; \mathbb{R}^N)$, often being the first step towards well-posedness results for such equations. In this respect, it is a well-established fact, see [Maz'ya 2011], that if $1 < p < \infty$, then there exists a surjective, bounded linear trace embedding operator

$$\mathrm{tr} : W^{1,p}(\Omega; \mathbb{R}^N) \hookrightarrow W^{1-1/p,p}(\partial\Omega; \mathbb{R}^N) \tag{1-1}$$

which satisfies $\mathrm{tr}(u) = u|_{\partial\Omega}$ for $u \in C(\overline{\Omega}; \mathbb{R}^N) \cap W^{1,p}(\Omega; \mathbb{R}^N)$. If $p = 1$ instead, a result of [Gagliardo 1957] asserts that there exists a surjective, bounded linear trace embedding operator

$$\mathrm{tr} : W^{1,1}(\Omega; \mathbb{R}^N) \hookrightarrow L^1(\partial\Omega; \mathbb{R}^N). \tag{1-2}$$

The same holds true when $W^{1,1}(\Omega; \mathbb{R}^N)$ is replaced by $\mathrm{BV}(\Omega; \mathbb{R}^N)$, the $\mathbb{R}^N$-valued functions of bounded variation on Ω. Both boundary trace embeddings (1-1), (1-2) and the corresponding variant for BV hinge on inequalities

$$\begin{aligned}
\|u\|_{W^{1-1/p,p}(\partial\Omega;\mathbb{R}^N)} &\le C(\|u\|_{L^p(\Omega;\mathbb{R}^N)} + \|Du\|_{L^p(\Omega;\mathbb{R}^{N\times n})}), \\
\|u\|_{L^1(\partial\Omega;\mathbb{R}^N)} &\le C(\|u\|_{L^1(\Omega;\mathbb{R}^N)} + \|Du\|_{L^1(\Omega;\mathbb{R}^{N\times n})})
\end{aligned} \tag{1-3}$$

if $1 < p < \infty$ or $p = 1$, respectively, to be satisfied for all $u \in C(\overline{\Omega}; \mathbb{R}^N) \cap C^1(\Omega; \mathbb{R}^N)$. These estimates in turn are obtained as a consequence of the fundamental theorem of calculus in conjunction with a smooth approximation argument.

As one of the fundamental achievements of 20th century harmonic analysis, Calderón and Zygmund [1956] and Mihlin [1956] established that in a wealth of inequalities, the *full gradient* can be replaced by weaker quantities only involving certain combinations of derivatives. Precisely, let $\mathbb{A}$ be a constant-coefficient, linear, homogeneous differential operator from $\mathbb{R}^N$ to $\mathbb{R}^K$; i.e., there exist fixed linear maps $\mathbb{A}_\alpha \colon \mathbb{R}^N \to \mathbb{R}^K$ with

$$\mathbb{A} = \sum_{\alpha=1}^{n} \mathbb{A}_\alpha \partial_\alpha. \tag{1-4}$$

Then for each $1 < p < \infty$ there exists $c = c(p, n, \mathbb{A}) > 0$ such that there holds

$$\|Du\|_{L^p(\mathbb{R}^n; \mathbb{R}^{N \times n})} \leq c\|\mathbb{A}u\|_{L^p(\mathbb{R}^n; \mathbb{R}^K)} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^N) \tag{1-5}$$

if and only if $\mathbb{A}$ is *elliptic*. Here we say that $\mathbb{A}$ is elliptic if and only if for each $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \setminus \{0\}$ the *symbol map* $\mathbb{A}[\xi] := \sum_\alpha \xi_\alpha \mathbb{A}_\alpha \colon \mathbb{R}^N \to \mathbb{R}^K$ is an injective linear map. A special instance of (1-5) is the case of the symmetric gradient operator $\mathcal{E}u := \frac{1}{2}(Du + D^\top u)$ acting on maps $u \colon \mathbb{R}^n \to \mathbb{R}^n$ (here $N = n \geq 2$ and $K = n^2$, identifying $\mathbb{R}^{n \times n} \cong \mathbb{R}^{n^2}$). In this situation, (1-5) gives the usual Korn inequalities, which play a pivotal role in elasticity or fluid mechanics; see [Fuchs and Seregin 2000] for a comprehensive overview.

Singular integrals or Fourier multiplier operators in general are not bounded on L^1. Thus one expects the exponent range $1 < p < \infty$ for (1-5) to be optimal for general elliptic operators $\mathbb{A}$. This is in fact true and manifested by Ornstein's celebrated noninequality, stating the impossibility of nontrivial L^1-estimates:

Theorem [Ornstein 1962]. *Let $\mathbb{A}$ and $\mathbb{B}$ be two constant-coefficient first-order, linear homogeneous differential operators on $\mathbb{R}^n$ from $\mathbb{R}^N$ to $\mathbb{R}^K$ and from $\mathbb{R}^N$ to $\mathbb{R}$, respectively. Suppose that there exists a constant $c > 0$ such that*

$$\|\mathbb{B}u\|_{L^1(\mathbb{R}^n)} \leq c\|\mathbb{A}u\|_{L^1(\mathbb{R}^n; \mathbb{R}^K)} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^N).$$

Then there exists $T \in \mathscr{L}(\mathbb{R}^K; \mathbb{R})$ such that $\mathbb{B} = T \circ \mathbb{A}$.

This negative result—which faces contributions to date, see [Conti et al. 2005; Kirchheim and Kristensen 2016]—immediately yields that if $p = 1$, inequalities that involve the full gradients Du do not necessarily generalise to those involving only $\mathbb{A}u$. On the other hand, by [Temam and Strang 1980] it is known for the special case of $\mathbb{A}$ being the symmetric gradient operator that the second inequality in (1-3) remains valid indeed for $p = 1$ when D is replaced by $\mathcal{E}$. However, the method employed in [Temam and Strang 1980; Babadjian 2015] to arrive at this result is very specific to the symmetric gradient operator and its structural properties: again based on the fundamental theorem of calculus, $\mathcal{E}u$ then allows one to control a cone of line integrals emanating from the boundary, leading to the desired trace inequality. In particular, it is far from clear whether and if so, how, trace inequalities of the form (1-3) can be established for $p = 1$ and D being replaced by differential operators $\mathbb{A}$ of the form (1-4). As we shall see below in

Section 1C, even for general elliptic operators $\mathbb{A}$ the corresponding analogues of (1-3) break down and hence the method employed for the symmetric gradient cannot easily generalise.

This leads us to the following *classification problem*: *classify all differential operators of the form* (1-4) *such that for any open and bounded Lipschitz domain* $\Omega \subset \mathbb{R}^n$ *there exists a constant* $c > 0$ *such that*

$$\|u\|_{L^1(\partial\Omega;\mathbb{R}^N)} \leq c(\|u\|_{L^1(\Omega;\mathbb{R}^N)} + \|\mathbb{A}u\|_{L^1(\Omega;\mathbb{R}^K)}) \tag{1-6}$$

holds for all $u \in C(\overline{\Omega}; \mathbb{R}^N) \cap C^1(\Omega; \mathbb{R}^N)$.   The overall objective of the present paper is to solve this classification problem. Before we pass on to the precise description of our results — in particular, Theorem 1.2 — we briefly pause and connect this theme to other results available in the literature first.

1B. *Contextualisation and function spaces.* The quest for classifying differential operators $\mathbb{A}$ of the form (1-4) such that well-known inequalities generalise to the $\mathbb{A}$-framework for $p = 1$ has come up rather recently. Building on the foundational work [Bourgain and Brezis 2003; 2004; 2007], Van Schaftingen [2013] characterised all operators $\mathbb{A}$ of the form (1-4) for which a Sobolev-type inequality

$$\|u\|_{L^{n/(n-1)}(\mathbb{R}^n;\mathbb{R}^N)} \leq C\|\mathbb{A}u\|_{L^1(\mathbb{R}^n;\mathbb{R}^K)} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^N) \tag{1-7}$$

holds. Whereas ellipticity of $\mathbb{A}$ is easily seen to be necessary for (1-7), it is far from sufficient and needs to be augmented by the so-called *cancellation condition*. Following [Van Schaftingen 2013], we call $\mathbb{A}$ *cancelling* if and only if

$$\bigcap_{\xi \in \mathbb{R}^n \setminus \{0\}} \mathbb{A}[\xi](\mathbb{R}^N) = \{0\}.$$

Note that by ellipticity, $u \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^N)$ can be represented via $u = k_{\mathbb{A}} * \mathbb{A}u$, where $k_{\mathbb{A}} \colon \mathbb{R}^n \setminus \{0\} \to \mathscr{L}(\mathbb{R}^K; \mathbb{R}^N)$ satisfies the growth bound $|k_{\mathbb{A}}(y)| \sim |y|^{1-n}$ for $y \in \mathbb{R}^n \setminus \{0\}$. Then the fractional integration theorem only implies that the convolution with $k_{\mathbb{A}}$ yields an operator that maps $L^1(\mathbb{R}^n; \mathbb{R}^K) \to L_w^{n/(n-1)}(\mathbb{R}^n; \mathbb{R}^N)$ boundedly with the weak-$L^{n/(n-1)}$ space $L_w^{n/(n-1)}(\mathbb{R}^n; \mathbb{R}^N)$, and so (1-7) implies a proper improvement based on the additional cancellation condition.

To unify this theme also in view of (1-6), we wish to interpret the above inequalities in terms of (boundary trace) embeddings and thus introduce function spaces via

$$W^{\mathbb{A},1}(\Omega) := \{v \in L^1(\Omega; \mathbb{R}^N) : \mathbb{A}u \in L^1(\Omega; \mathbb{R}^K)\},$$
$$\mathrm{BV}^{\mathbb{A}}(\Omega) := \{v \in L^1(\Omega; \mathbb{R}^N) : \mathbb{A}u \in \mathcal{M}(\Omega; \mathbb{R}^K)\},$$

where $\Omega \subset \mathbb{R}^n$ is open, $\mathbb{A}$ is a differential operator of the form (1-4) and $\mathcal{M}(\Omega; \mathbb{R}^K)$ denotes the $\mathbb{R}^K$-valued Radon measures of finite total variation on Ω. These spaces are normed canonically via $\|u\|_{W^{\mathbb{A},1}} = \|u\|_{L^1} + \|\mathbb{A}u\|_{L^1}$ (similarly for $\mathrm{BV}^{\mathbb{A}}$ with the obvious modifications); clearly, $W^{\mathbb{A},1}(\Omega) \subsetneq \mathrm{BV}^{\mathbb{A}}(\Omega)$ and we shall refer to $\mathrm{BV}^{\mathbb{A}}(\Omega)$ as *space of functions of bounded $\mathbb{A}$-variation*. In the literature, only particular instances of spaces $\mathrm{BV}^{\mathbb{A}}$ have been studied in detail, namely for $\mathbb{A} = \nabla$ or $\mathbb{A} = \mathcal{E}$, leading to the spaces BV or BD of functions of bounded variation or deformation, respectively. Precisely, we then have $W^{1,1} = W^{\nabla,1}$, $\mathrm{LD} = W^{\mathcal{E},1}$, $\mathrm{BV} = \mathrm{BV}^\nabla$, $\mathrm{BD} = \mathrm{BV}^{\mathcal{E}}$, and this paper is the first attempt to characterise the properties of $\mathrm{BV}^{\mathbb{A}}$-maps in terms of the properties of $\mathbb{A}$ in a unifying manner. By this, we also aim to

clarify the underlying mechanisms for the corresponding trace inequalities to work in the known cases $\mathbb{A} = D$ and $\mathbb{A} = \mathcal{E}$.

Returning to the classification problem related to (1-6), we conclude this subsection by pointing out that ellipticity in itself cannot yield the required L^1-trace theory. In fact, consider the operator $\mathcal{E}^D u := \mathcal{E}u - \frac{1}{n} \operatorname{div}(u) E_n$ ($E_n \in \mathbb{R}^{n \times n}$ being the identity matrix), which is usually referred to as *trace-free symmetric gradient operator*, for $n \geq 2$. This operator enters in a variety of applications, for instance fluid mechanics or general relativity; see [Feireisl 2004; Bartnik and Isenberg 2004]. Regardless of $n \geq 2$, the operator $\mathcal{E}^D$ is elliptic; see Example 2.2(c). However, the following example from [Fuchs and Repin 2010] shows that an L^1-trace does not exist if $n = 2$. Identifying $\mathbb{R}^2 \cong \mathbb{C}$, $\ker(\mathcal{E}^D)$ essentially contains the holomorphic functions. Upon identifying $\mathbb{R}^2$ with $\mathbb{C}$ and denoting by $\mathbb{D}$ the open unit disc in $\mathbb{C}$, the map $u \colon \mathbb{D} \to \mathbb{C}$, $z \mapsto 1/(z-1)$, even belongs to $W^{\mathcal{E}^D,1}(B(0,1))$, whereas it is clear that $\|\operatorname{tr}(u)\|_{L^1(\partial B(0,1))} = \infty$. In view of (1-6), our main result, Theorem 1.2 below, will cover the particular case of $\mathbb{A} = \mathcal{E}^D$ as a special case and provide a positive answer for all $n \geq 3$ and a negative answer for $n = 2$.

1C. Main results. Before we state our main result, we need to provide the definitions of several important properties of our operator $\mathbb{A}$. To begin with, we write the *symbol mapping* $\mathbb{A}[\xi] \colon \mathbb{R}^N \to \mathbb{R}^K$ as

$$\mathbb{A}[\xi]v := v \otimes_{\mathbb{A}} \xi := \sum_{\alpha=1}^{n} \xi_\alpha \mathbb{A}_\alpha v, \quad \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n, \ v \in \mathbb{R}^N. \tag{1-8}$$

Moreover, we extend $\mathbb{A}[\xi]\eta = \eta \otimes_{\mathbb{A}} \xi$ by (1-8) also to complex-valued $\xi \in \mathbb{C}^n$ and $\eta \in \mathbb{C}^N$. We strengthen terminology and say that $\mathbb{A}$ is $\mathbb{R}$-*elliptic* if $\mathbb{A}[\xi] \colon \mathbb{R}^N \to \mathbb{R}^K$ is injective for all $\xi \in \mathbb{R}^n \setminus \{0\}$ (i.e., $\mathbb{A}$ is elliptic in the above sense), and $\mathbb{C}$-*elliptic* provided $\mathbb{A}[\xi] \colon \mathbb{C}^N \to \mathbb{C}^K$ is injective for all $\xi \in \mathbb{C}^n \setminus \{0\}$ (see Section 2C for more detail). Finally, we shall say that $\mathbb{A}$ has *finite-dimensional nullspace* if the kernel $N(\mathbb{A})$ of $\mathbb{A}$ in the distributional sense is finite-dimensional; i.e.,

$$\dim(N(\mathbb{A})) < \infty, \quad \text{with } N(\mathbb{A}) = \{v \in \mathcal{D}'(\mathbb{R}^n; \mathbb{R}^N) : \mathbb{A}v \equiv 0\}, \tag{1-9}$$

where $\mathcal{D}(\mathbb{R}^n; \mathbb{R}^N) = C_c^\infty(\mathbb{R}^n; \mathbb{R}^N)$. We will see later in Theorem 2.6 that $\mathbb{A}$ has a finite-dimensional nullspace if and only if it is $\mathbb{C}$-elliptic. It is also equivalent to the *type-(C) condition* in the sense of [Kałamajska 1994]; see Remark 2.1. However, the notion of $\mathbb{R}$-ellipticity is strictly weaker: For instance, $\mathcal{E}^D$ for $n = 2$ is $\mathbb{R}$-elliptic but not $\mathbb{C}$-elliptic; see Example 2.2(c). We are now in position to formulate our main result.

Theorem 1.1. *Let $\mathbb{A}$ be a differential operator of the form (1-4). Then the following are equivalent*:

(a) *For all open and bounded Lipschitz domains $\Omega \subset \mathbb{R}^n$ there exists a constant $c > 0$ such that (1-6) holds for all $u \in C(\overline{\Omega}; \mathbb{R}^N) \cap C^1(\Omega; \mathbb{R}^N)$.*

(b) *$\mathbb{A}$ is $\mathbb{C}$-elliptic.*

Whereas necessity of $\mathbb{C}$-ellipticity for (1-6) shall be addressed in Theorem 4.18 and essentially follows from a construction relying on the properties of the two-dimensional operator $\mathcal{E}^D$, the more involved part is the sufficiency. For future reference, we single this out and state in the following more elaborate form; the full statement can be found in Theorem 4.17:

Theorem 1.2 (trace theorem). *Let $\mathbb{A}$ be $\mathbb{C}$-elliptic (or equivalently, $\mathbb{A}$ has finite-dimensional nullspace). Then there exists a trace operator* $\mathrm{tr} : \mathrm{BV}^{\mathbb{A}}(\Omega) \to L^1(\partial\Omega, \mathcal{H}^{n-1})$ *such that the following holds*:

(a) $\mathrm{tr}(u)$ *coincides with the classical trace for all* $u \in \mathrm{BV}^{\mathbb{A}}(\Omega) \cap C(\bar{\Omega}; \mathbb{R}^N)$.

(b) $\mathrm{tr}(u)$ *is the unique strictly continuous extension of the classical trace on* $\mathrm{BV}^{\mathbb{A}}(\Omega) \cap C(\bar{\Omega}; \mathbb{R}^N)$. *In particular,* $\mathrm{tr} : \mathrm{BV}^{\mathbb{A}}(\Omega) \to L^1(\partial\Omega; \mathcal{H}^{n-1})$ *is continuous for the norm topology on* $\mathrm{BV}^{\mathbb{A}}(\Omega)$.

(c) $\mathrm{tr}(W^{\mathbb{A},1}(\Omega)) = \mathrm{tr}(\mathrm{BV}^{\mathbb{A}}(\Omega)) = L^1(\partial\Omega; \mathcal{H}^{n-1})$.

Regarding sufficiency, the core issue is how to replace the use of the fundamental theorem of calculus by that of $\mathbb{C}$-ellipticity. As a main consequence of the latter, we will employ the nullspace of $\mathbb{C}$-elliptic operators being finite-dimensional. Using local projections onto the nullspace $N(\mathbb{A})$ close to the boundary, we construct suitable approximations of $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$ that have classical traces. The limit of these traces provide us with the trace of u. In particular, the projections to the finite-dimensional nullspace replace the fundamental theorem of calculus approach as used in [Temam and Strang 1980; Babadjian 2015].

In addition to Theorem 4.17 we will show in Theorem 4.18 and Remark 4.19 that if $\mathbb{A}$ is not $\mathbb{C}$-elliptic, then in general there is no trace operator from $\mathrm{BV}^{\mathbb{A}}(\Omega)$ to $L^1(\partial\Omega; \mathcal{H}^{n-1})$. In particular, the existence of $L^1(\partial\Omega; \mathcal{H}^{n-1})$-traces on arbitrary bounded Lipschitz domains $\Omega \subset \mathbb{R}^n$ is equivalent to $\mathbb{C}$-ellipticity of $\mathbb{A}$. This conclusion also identifies the infinite-dimensional nullspace of $\mathbb{A}$ as the reason for the failure of the trace embedding of $W^{\mathcal{E}^D,1}(\Omega)$ into $L^1(\partial\Omega; \mathcal{H}^{n-1})$ for $n = 2$ (see Example 2.2(c)). As a consequence of Theorem 1.2 we also obtain a version of the Gauss–Green theorem, see Theorem 4.20, and the gluing theorem, see Corollary 4.21. Let us also remark that Theorem 1.2 includes both the trace theorems for the spaces BV and BD.

The relation between the condition of $\mathbb{C}$-ellipticity and Van Schaftingen's elliptic and cancelling condition will be investigated in detail in the follow-up [Gmeineder and Raiță 2019] to this paper by Raita and the third author; among others, it will be shown that $\mathbb{C}$-ellipticity implies Van Schaftingen's condition but in general *not* vice versa. In this sense and as might be anticipated, L^1-boundary traces require a stronger condition on $\mathbb{A}$.

1D. *Variational problems.* As a concluding application of the trace theorem from above, we address the Dirichlet problem for linear growth functionals involving operators $\mathbb{A}$. To be precise, we are interested in the minimisation of functionals of the form

$$\mathfrak{F}[u] := \int_{\Omega} f(x, \mathbb{A}u) \, \mathrm{d}x \tag{1-10}$$

over a class of maps $u : \Omega \to \mathbb{R}^N$ subject to Dirichlet boundary data $u = u_0$ on $\partial\Omega$. Here $f : \Omega \times \mathbb{R}^{N \times n} \to \mathbb{R}_{\geq 0}$ is a given variational integrand for which we suppose the linear growth assumption

$$c_1|z| \leq f(x, z) \leq c_2|z| + c_3 \quad \text{for all } x \in \Omega \text{ and } z \in \mathbb{R}^{N \times n}. \tag{1-11}$$

Additionally, we assume that our integrand f is $\mathbb{A}$-*quasiconvex* (in a sense specified in Section 5; also see [Fonseca and Müller 1999; Dacorogna 1982]). Our objective here is to minimise $\mathfrak{F}$ over the

Dirichlet class $u_0 + W_0^{\mathbb{A},1}(\Omega)$, which are the $W^{\mathbb{A},1}(\Omega)$-functions whose traces agree with the given boundary datum u_0. From the treatment of the Dirichlet problem on $\mathrm{BV}^{\mathbb{A}}$, where $\mathbb{A} = \nabla$, see [Giaquinta et al. 1979a; 1979b; Ambrosio et al. 2000], it is clear that the functional should be considered on the class of $\mathrm{BV}^{\mathbb{A}}$-maps on a larger Lipschitz domain U. More precisely, we need to consider the weak*-lower semicontinuous envelope of $\mathfrak{F}$ on $\mathrm{BV}^{\mathbb{A}}(U)$. Whereas in the convex situation one can make use of the classical results due to [Reshetnyak 1968], the quasiconvex case is substantially more involved. The sequentially weak*-lower semicontinuous envelope $\overline{\mathfrak{F}}$ of $\mathfrak{F}$ on $\mathrm{BV}(\Omega)$ (so $\mathbb{A} = \nabla$) was characterised in [Ambrosio and Dal Maso 1992; Fonseca and Müller 1993]. The corresponding issue for the symmetric-quasiconvex (so $\mathbb{A} = \mathcal{E}$) situation was resolved in [Rindler 2011]. Invoking the recent outstanding generalisation of Alberti's rank-one theorem [De Philippis and Rindler 2016], the weak*-lower semicontinuity result of [Arroyo-Rabasa et al. 2018] and the area-strict continuity of [Kristensen and Rindler 2010b], we give a precise characterization of the weak*-lower semicontinuous envelope $\overline{\mathfrak{F}}$ on $\mathrm{BV}^{\mathbb{A}}(\Omega)$; see Proposition 5.1.

Consequently, a merger with Theorem 1.2 allows us to formulate the minimisation problem with Dirichlet data u_0 purely in terms of $\mathrm{BV}^{\mathbb{A}}(\Omega)$; see Corollary 5.2. We demonstrate both the existence of minima and the absence of a Lavrentiev gap with respect to the Dirichlet class $u_0 + W_0^{\mathbb{A},1}(\Omega)$; see Theorem 5.3.

1E. *Organisation of the paper.* The paper is organised as follows. In Section 2 we fix notation, introduce the assumptions on the differential operators $\mathbb{A}$ and collect elementary implications for the Sobolev-type spaces $W^{\mathbb{A},1}(\Omega)$ and the spaces of functions of bounded $\mathbb{A}$-variation $\mathrm{BV}^{\mathbb{A}}(\Omega)$. In Section 3 we introduce local projection operators onto the nullspace $N(\mathbb{A})$ on balls and derive Poincaré-type inequalities. In Section 4, we construct the trace operator $\mathrm{tr} : \mathrm{BV}^{\mathbb{A}}(\Omega) \to L^1(\partial\Omega; \mathcal{H}^{n-1})$ and thereby give the proof of Theorem 1.2. Moreover, we establish a Gauss–Green formula and a gluing lemma for $\mathrm{BV}^{\mathbb{A}}$-maps. The final Section 5 is dedicated to the existence of $\mathrm{BV}^{\mathbb{A}}$-minimisers of $\mathbb{A}$-quasiconvex variational problems with linear growth subject to given Dirichlet boundary data.

2. Functions of bounded $\mathbb{A}$-variation

In this section we introduce the space of functions of bounded variation associated with a differential operator $\mathbb{A}$.

2A. *General notation.* To avoid too many different constants throughout, we write $a \lesssim b$ if there exists a constant c (which does not depend on the crucial quantities) with $a \leq cb$. If $a \lesssim b$ and $b \lesssim a$, we also write $a \approx b$. By $\ell(B)$ we denote the diameter of a ball B and by $|B|$, its n-dimensional Lebesgue measure. We write $d(\cdot, \cdot)$ for the usual euclidean distance. For the euclidean inner product of $a, b \in \mathbb{R}^m$ we use the equivalent notations $\langle a, b \rangle$ or $a \cdot b$. Given $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^K)$ and a measurable subset $U \subset \mathbb{R}^n$ with $|U| > 0$, we use the equivalent notations

$$\fint_U f(x)\,\mathrm{d}x := \langle f \rangle_U := |U|^{-1} \int_U f(x)\,\mathrm{d}x$$

for the mean value integral. Lastly, for notational simplicity, we shall often suppress the possibly vectorial target space when dealing with function spaces and, e.g., write $L^1(\mathbb{R}^n)$ instead of $L^1(\mathbb{R}^n; \mathbb{R}^N)$, but this will be clear from the context.

2B. *Function space setup.* Let $\mathbb{A}$ be given by (1-4). The corresponding *dual* (*or formally adjoint*) *operator* $\mathbb{A}^*$ is the differential operator on $\mathbb{R}^n$ from $\mathbb{R}^K$ to $\mathbb{R}^N$ given by

$$\mathbb{A}^* := \sum_{\alpha=1}^{n} \mathbb{A}_\alpha^* \partial_\alpha, \tag{2-1}$$

where each $\mathbb{A}_\alpha^*$ is the adjoint matrix of $\mathbb{A}_\alpha$. For an open domain $\Omega \subset \mathbb{R}^n$ we define the *Sobolev space* $W^{\mathbb{A},1}(\Omega)$ *associated to the operator* $\mathbb{A}$ by

$$W^{\mathbb{A},1}(\Omega) = W^{\mathbb{A},1}(\Omega; \mathbb{R}^N) := \{u \in L^1(\Omega; \mathbb{R}^N) : \mathbb{A}u \in L^1(\Omega; \mathbb{R}^K)\}. \tag{2-2}$$

This is a Banach space with respect to the norm

$$\|u\|_{W^{\mathbb{A},1}(\Omega)} := \|u\|_{L^1(\Omega)} + \|\mathbb{A}u\|_{L^1(\Omega)}. \tag{2-3}$$

We moreover define the *total $\mathbb{A}$-variation* of $u \in L^1_{\text{loc}}(\Omega; \mathbb{R}^N)$ by

$$|\mathbb{A}u|(\Omega) := \sup\left\{ \int_\Omega \langle u, \mathbb{A}^*\varphi \rangle \, dx : \varphi \in C_c^1(\Omega; \mathbb{R}^K), \ |\varphi| \le 1 \right\} \tag{2-4}$$

and consequently say that u is of *bounded $\mathbb{A}$-variation* if and only if $u \in L^1(\Omega; \mathbb{R}^N)$ and $|\mathbb{A}u|(\Omega) < \infty$. Denoting by $\mathcal{M}(\Omega; \mathbb{R}^K)$ the finite $\mathbb{R}^K$-valued Radon measures on Ω, by the Riesz representation theorem this amounts to

$$\text{BV}^{\mathbb{A}}(\Omega) := \{u \in L^1(\Omega; \mathbb{R}^N) : \mathbb{A}u \in \mathcal{M}(\Omega; \mathbb{R}^K)\}. \tag{2-5}$$

Here, the shorthands $\mathbb{A}u \in L^1$ or $\mathbb{A}u \in \mathcal{M}$ above have to be understood in the sense that the distributional differential expressions $\mathbb{A}u$ can be represented by L^1-functions or Radon measures, respectively. The norm

$$\|u\|_{\text{BV}^{\mathbb{A}}(\Omega)} := \|u\|_{L^1(\Omega)} + |\mathbb{A}u|(\Omega) \tag{2-6}$$

makes $\text{BV}^{\mathbb{A}}(\Omega)$ a Banach space. However, due to the lack of good compactness properties, the norm topology turns out not to be useful in many applications and one needs to consider weaker topologies. We now introduce the canonical generalisations of well-known convergences in the full- or symmetric-gradient cases; see [Ambrosio et al. 2000]. Let $u \in \text{BV}^{\mathbb{A}}(\Omega)$ and $(u_k) \subset \text{BV}^{\mathbb{A}}(\Omega)$. We say that

- (u_k) converges to u in the *weak*-sense* (in symbols $u_k \overset{*}{\rightharpoonup} u$) if and only if $u_k \to u$ strongly in $L^1(\Omega; \mathbb{R}^N)$ and $\mathbb{A}u_k \overset{*}{\rightharpoonup} \mathbb{A}u$ in the weak*-sense of $\mathbb{R}^K$-valued Radon measures on Ω as $k \to \infty$.

- (u_k) converges to u in the *strict sense* (in symbols $u_k \overset{s}{\to} u$) if and only if $d_s(u_k, u) \to 0$ as $k \to \infty$, where for $v, w \in \text{BV}^{\mathbb{A}}(\Omega)$ we set

$$d_s(v, w) := \int_\Omega |v - w| \, dx + \big| |\mathbb{A}v|(\Omega) - |\mathbb{A}w|(\Omega) \big|.$$

• (u_k) converges to u in the *area-strict sense* (in symbols $u_k \xrightarrow{\langle \cdot \rangle} u$) if and only if

$$\int_\Omega \sqrt{1 + \left| \frac{d\mathbb{A}u_k}{d\mathscr{L}^n} \right|^2} \, d\mathscr{L}^n + |\mathbb{A}^s u_k|(\Omega) \to \int_\Omega \sqrt{1 + \left| \frac{d\mathbb{A}u}{d\mathscr{L}^n} \right|^2} \, d\mathscr{L}^n + |\mathbb{A}^s u|(\Omega), \quad k \to \infty,$$

where

$$\mathbb{A}v = \frac{d\mathbb{A}v}{d\mathscr{L}^n} \mathscr{L}^n + \frac{d\mathbb{A}v}{d|\mathbb{A}^s v|} |\mathbb{A}^s v|$$

is the Radon–Nikodym decomposition of $\mathbb{A}v \in \mathcal{M}(\Omega; \mathbb{R}^K)$ with respect to the Lebesgue measure $\mathscr{L}^n$.

Strictly speaking, these notions are reserved for the BV-versions and hence the above notions have to be read as $\mathbb{A}$-weak*, $\mathbb{A}$-strict, and $\mathbb{A}$-area-strict convergence. However, to keep terminology simple, we tacitly assume that the differential operator $\mathbb{A}$ is fixed throughout and stick to the above terminology.

Note that the $\mathbb{A}$-variation is sequentially lower semicontinuous with respect convergence in the weak*-sense; i.e., if $u_k \xrightarrow{*} u$, then $|\mathbb{A}u|(\Omega) \le \liminf_{k \to \infty} |\mathbb{A}u_k|(\Omega)$. Moreover, if $u_k \in \mathrm{BV}^{\mathbb{A}}(\Omega)$ is a bounded sequence with $u_k \rightharpoonup u$ in $L^1(\Omega; \mathbb{R}^N)$, then already $u_k \xrightarrow{*} u$. Finally, if Ω is open and bounded with Lipschitz boundary, then it is easy to conclude by the theorem of Banach and Alaoglu that if $(u_k) \subset \mathrm{BV}^{\mathbb{A}}(\Omega)$ is uniformly bounded in the $\mathrm{BV}^{\mathbb{A}}$-norm, then there exists $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$ and a subsequence $(u_{k(j)})$ of (u_k) such that $u_{k(j)} \xrightarrow{*} u$ as $j \to \infty$ in the sense specified above. We shall often refer to this as the *weak*-compactness principle* (*for* $\mathrm{BV}^{\mathbb{A}}$).

2C. *Assumptions on the differential operator* $\mathbb{A}$. For our trace result we need some structure on $\mathbb{A}$ which we introduce now.

Let $\mathbb{A}$ be given by (1-4). Then $\mathbb{A}$ induces a bilinear pairing $\otimes_{\mathbb{A}} \colon \mathbb{R}^N \times \mathbb{R}^n \to \mathbb{R}^K$ by

$$v \otimes_{\mathbb{A}} z := \sum_{\alpha=1}^n z_\alpha \mathbb{A}_\alpha v \quad \text{for } z \in \mathbb{R}^n \text{ and } v \in \mathbb{R}^N. \tag{2-7}$$

For all $\varphi \in C^1(\mathbb{R}^n)$ and $v \in C^1(\mathbb{R}^n; \mathbb{R}^N)$ we have

$$\mathbb{A}(\varphi v) = \varphi \mathbb{A}v + v \otimes_{\mathbb{A}} \nabla\varphi. \tag{2-8}$$

Note that if $\mathbb{A}$ is the usual gradient, then $\otimes_{\mathbb{A}}$ can be identified with the usual dyadic product $\otimes$, and if $\mathbb{A}$ is the symmetric gradient, then $\otimes_{\mathbb{A}}$ is given by the symmetric tensor product $\odot$.

Recalling the notions of $\mathbb{R}$- and $\mathbb{C}$-ellipticity from Section 1C, we now pass on to a more detailed discussion and begin with linking them to the *type-*(C) condition as introduced in [Kałamajska 1994].

Remark 2.1. The operator $\mathbb{A}$ is $\mathbb{C}$-elliptic if and only if it is of type (C) in the sense of [Kałamajska 1994]. More precisely, since $\mathbb{A}_\alpha[\xi]$ is a linear operator from $\mathbb{R}^N$ to $\mathbb{R}^K$ for each $\xi \in \mathbb{R}^n$, we find coefficients $\mathbb{A}_{\alpha,j,k}$ such that

$$(\mathbb{A}[\xi]\eta)_k =: \sum_{\alpha=1}^n \sum_{j=1}^N \mathbb{A}_{\alpha,j,k} \xi_\alpha \eta_j$$

for every for $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^N$. Then

$$\mathbb{P}_{j,k} u := \sum_{\alpha=1}^n \mathbb{A}_{\alpha,j,k} \partial_\alpha u_j$$

for $k = 1, \ldots, K$ is the family of scalar differential operators as used in [Kałamajska 1994]. The corresponding symbols are

$$\mathbb{P}_{j,k}(\xi) := \sum_{\alpha=1}^{n} \mathbb{A}_{\alpha,j,k} \xi_\alpha,$$

with $j = 1, \ldots, N$ and $k = 1, \ldots, K$. Now according to [Kałamajska 1994] the family $(\mathbb{P}_k)_k$ is of type (C) if and only if $(\mathbb{P}_{j,k}(\xi))_{j,k}$ has rank K for all $\eta \in \mathbb{C}^n \setminus \{0\}$. Since

$$\sum_{j=1}^{N} \sum_{k=1}^{K} \mathbb{P}_{j,k}(\xi) \eta_j = \sum_{\alpha=1}^{n} \sum_{j=1}^{N} \sum_{k=1}^{K} \mathbb{A}_{\alpha,j,k} \xi_\alpha \eta_j = \mathbb{A}[\xi]\eta,$$

this is equivalent to the injectivity of $\mathbb{A}[\xi]$ for all $\eta \in \mathbb{C}^N \setminus \{0\}$, which is exactly the $\mathbb{C}$-ellipticity of $\mathbb{A}$.

We now turn to some examples, to which we shall frequently refer.

Example 2.2. In what follows, we carefully examine the gradient, symmetric and trace-free symmetric gradient operators. As these typically map $\mathbb{R}^N$ to the matrices $\mathbb{R}^{N \times n}$ instead of a vector in $\mathbb{R}^K$, we henceforth put $K = Nn$ and identify $\mathbb{R}^K$ with $\mathbb{R}^{N \times n}$:

(a) Let $\mathbb{A}u := \nabla u$. Then $N(\mathbb{A})$ just consists of the constants and

$$(v \otimes_\nabla z)_{j,k} = v_j z_k.$$

$\mathbb{A}$ has a finite-dimensional nullspace and is $\mathbb{C}$-elliptic, since

$$|\mathbb{A}[\xi]\eta|^2 = |\xi|^2 |\eta|^2.$$

(b) Let $\mathbb{A}u := \mathcal{E}(u) := \frac{1}{2}(\nabla u + (\nabla u)^T)$ with $N = n$. Then $N(\mathcal{E})$ just consists of the generators of rigid motions, i.e.,

$$N(\mathcal{E}) = \{x \mapsto Ax + b : A \in \mathbb{R}^{n \times n}, \ A = -A^T, \ b \in \mathbb{R}^n\},$$

and

$$(v \otimes_\mathcal{E} z)_{j,k} = \frac{1}{2}(v_j z_k + v_k z_j).$$

$\mathcal{E}$ has a finite-dimensional nullspace and is $\mathbb{C}$-elliptic, since

$$|\mathbb{A}[\xi]\eta|^2 = \frac{1}{2}|\xi|^2 |\eta|^2 + \frac{1}{2}|\langle \xi, \eta \rangle|^2.$$

(c) Let $\mathbb{A}u := \mathcal{E}^D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T) - \frac{1}{n} \operatorname{div}(u) E_n$ with $N = n$. Then

$$(v \otimes_{\mathcal{E}^D} z)_{j,k} = \frac{1}{2}(v_j z_k + v_k z_j) - \frac{1}{n} \delta_{j,k} \sum_{l=1}^{n} v_l z_l$$

and

$$|\mathbb{A}[\xi]\eta|^2 = \frac{1}{2}|\xi|^2 |\eta|^2 + \frac{1}{2}|\langle \xi, \eta \rangle|^2 - \frac{1}{n}\langle \xi, \bar{\eta} \rangle^2.$$

If $n \geq 3$, then $\mathbb{A}$ is $\mathbb{C}$-elliptic and it has the finite-dimensional nullspace

$$N(\mathcal{E}^D) = \{x \mapsto Ax + b + (2(a \cdot x)x - |x|^2 a) : A \in \mathbb{R}^{n \times n}, \ A = -A^T, a, b \in \mathbb{R}^n\}.$$

Elements of $N(\mathcal{E}^D)$ are also known as *conformal killing vectors* [Dain 2006].

If $n = 2$, then $\mathbb{A}$ is only $\mathbb{R}$-elliptic, but not $\mathbb{C}$-elliptic. Indeed, $\mathbb{A}[\xi]\eta = 0$ for $\xi = (1, i)^T$ and $\eta = (1, -i)^T$. Moreover, the nullspace $N(\mathbb{A})$ is of infinite dimension: indeed, if we identify $\mathbb{R}^2 \cong \mathbb{C}$, then the kernel of $\mathcal{E}^D$ consists of the holomorphic functions. We will substantially use this property in the proofs of Lemma 2.5 and Theorem 4.18.

We now draw some consequences of the single ellipticity conditions and link them to the finite-dimensionality of the nullspace of $\mathbb{A}$.

Lemma 2.3. *Let $\mathbb{A}$ be $\mathbb{K}$-elliptic with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Then there exist two constants $0 < \kappa_1 \leq \kappa_2 < \infty$ such that*

$$\kappa_1 |v| |z| \leq |v \otimes_{\mathbb{A}} z| \leq \kappa_2 |v| |z| \quad \text{for all } v \in \mathbb{K}^N \text{ and } z \in \mathbb{K}^n.$$

Proof. By scaling it suffices to assume $|v| = |z| = 1$. We have $|v \otimes_{\mathbb{A}} z| > 0$, since $\mathbb{A}$ is $\mathbb{K}$-elliptic. Now the claim follows by the compactness of $\{(v, z) : |v| = |z| = 1\}$ and continuity. $\square$

Lemma 2.4. *Let $\mathbb{A}$ have a finite-dimensional nullspace. Then $\mathbb{A}$ is $\mathbb{R}$-elliptic.*

Proof. We proceed by contradiction. Assume that $\mathbb{A}$ is not $\mathbb{R}$-elliptic. Then there exists $\xi \in \mathbb{R}^n \setminus \{0\}$ and $\eta \in \mathbb{R}^N \setminus \{0\}$ with $\mathbb{A}[\xi]\eta = 0$. For every $f \in C_c^1(\mathbb{R}; \mathbb{R})$ we define $u_f(x) := f(\langle \xi, x \rangle)\eta$. Then $(\mathbb{A}u_f)(x) = \mathbb{A}[\xi]\eta \, f'(\langle \xi, x \rangle) = 0$. Since $\eta \neq 0$ and $\xi \neq 0$, the mapping $f \mapsto u_f$ is injective. Therefore, the set $\{u_f : f \in C_c^1(\mathbb{R})\}$ is an infinite-dimensional subspace of $N(\mathbb{A})$. This contradicts the fact that $\mathbb{A}$ has finite-dimensional nullspace. $\square$

Lemma 2.5. *Let $\mathbb{A}$ have a finite-dimensional nullspace. Then $\mathbb{A}$ is $\mathbb{C}$-elliptic.*

Proof. Since $\mathbb{A}$ has finite-dimensional nullspace, it is $\mathbb{R}$-elliptic by Lemma 2.4.

We proceed by contradiction, and so assume that $\mathbb{A}$ is not $\mathbb{C}$-elliptic. Then there exists $\xi \in \mathbb{C}^n \setminus \{0\}$ and $\eta \in \mathbb{C}^N \setminus \{0\}$ with $0 = \mathbb{A}[\xi]\eta = \eta \otimes_{\mathbb{A}} \xi$. We split ξ and η into their real and imaginary parts by $\xi =: \xi_1 + i\xi_2$ and $\eta =: \eta_1 + i\eta_2$. Then $\mathbb{A}[\xi]\eta = 0$ implies

$$\mathbb{A}[\xi_1]\eta_1 - \mathbb{A}[\xi_2]\eta_2 = 0 \quad \text{and} \quad \mathbb{A}[\xi_1]\eta_2 + \mathbb{A}[\xi_2]\eta_1 = 0. \tag{2-9}$$

We will show now that ξ_1 and x_2, resp. η_1 and η_2, are linearly independent.

We begin with the linear independence of ξ_1 and ξ_2. If $\xi_1 = 0$, then $\xi_2 \neq 0$ and then the $\mathbb{R}$-ellipticity of $\mathbb{A}$ and (2-9) imply $\eta_1 = \eta_2 = 0$, which contradicts $\eta \neq 0$. By the same argument, also $\xi_2 = 0$ is not possible. Hence, we have $\xi_1 \neq 0$ and $\xi_2 \neq 0$. We now show the linear independence of ξ_1 and ξ_2 by contradiction, so let us assume that $\xi_2 = \lambda \xi_1$ with $\lambda \neq 0$. Then it follows from (2-9) that

$$\mathbb{A}[\xi_1]\eta_1 = \mathbb{A}[\xi_2]\eta_2 = \lambda \mathbb{A}[\xi_1]\eta_2 = -\lambda \mathbb{A}[\xi_2]\eta_1 = -\lambda^2 \mathbb{A}[\xi_1][\eta_1].$$

This implies $\mathbb{A}[\xi_1][\eta_1] = 0$. Hence by the $\mathbb{R}$-ellipticity of $\mathbb{A}$ and $\xi_1 \neq 0$, we get $\eta_1 = 0$. Now, (2-9) implies $\mathbb{A}[\xi_2][\eta_2] = 0$, so again the $\mathbb{R}$-ellipticity of $\mathbb{A}$ gives $\eta_2 = 0$. Overall, $\eta = 0$, which is a contradiction. This proves that ξ_1 and ξ_2 are linearly independent.

The proof of the linear independence of η_1 and η_2 is completely analogous. Indeed, $\eta_1 = \gamma \eta_2$ implies $\mathbb{A}[\xi_1]\eta_1 = -\gamma^2 \mathbb{A}[\xi_1]\eta_1$, so $\mathbb{A}[\xi_1][\eta_1] = 0$. As above this implies $\eta = 0$, which is a contradiction.

Let us define now $\tau : \mathbb{R}^n \to \mathbb{C}$ and $\sigma : \mathbb{C} \to \mathbb{R}^N$ by

$$\tau(x) := \langle \xi, x \rangle = \langle \xi_1, x \rangle + i \langle \xi_2, x \rangle,$$

$$\sigma(z) := \operatorname{Re}(z)\eta_1 - \operatorname{Im}(z)\eta_2.$$

Let $\mathcal{O}(\mathbb{C})$ denote the set of holomorphic functions on $\mathbb{C}$. Then $\dim(\mathcal{O}(\mathbb{C})) = \infty$. Moreover, for $f \in \mathcal{O}(\mathbb{C})$ we have $\partial_{\bar{z}} f(z) = 0$ in the sense of complex derivatives. Let us define $h_f : \mathbb{R}^n \to \mathbb{R}^N$ by $h_f := \sigma \circ f \circ \tau$. Our goal is to prove $\mathbb{A}h_f = 0$. We identify in the following $\mathbb{C}$ with $\mathbb{R}^2$. By the chain rule we conclude

$$(\mathbb{A}h_f)(x) = \mathbb{A}[\xi_1]\eta_1(\partial_1 f_1)(\tau(x)) - \mathbb{A}[\xi_1]\eta_2(\partial_1 f_2)(\tau(x))$$
$$+ \mathbb{A}[\xi_2]\eta_1(\partial_2 f_1)(\tau(x)) - \mathbb{A}[\xi_2]\eta_2(\partial_2 f_2)(\tau(x)). \quad (2\text{-}10)$$

Using the Cauchy–Riemann equations $\partial_1 f_1 = \partial_2 f_2$ and $\partial_1 f_2 = -\partial_2 f_1$ and (2-9) we get

$$(\mathbb{A}h_f)(x) = (\mathbb{A}[\xi_1]\eta_1 - \mathbb{A}[\xi_2]\eta_2)(\partial_1 f_1)(\tau(x)) + (\mathbb{A}[\xi_1]\eta_2 + \mathbb{A}[\xi_2]\eta_1)(\partial_2 f_1)(\tau(x)) = 0.$$

So for each $f \in \mathcal{O}(\mathbb{C})$, we constructed an $h_f : \mathbb{R}^n \to \mathbb{R}^N$ such that $\mathbb{A}h_f = 0$. We need to show that $\dim(\{h_f : f \in \mathcal{O}(\mathbb{C})\}) = \infty$. For this, it suffices to show that the linear mapping $f \mapsto h_f$ is injective. Recall that $h_f = \sigma \circ f \circ \tau$. Hence, it suffices to show that σ is injective and that τ is surjective. This, however, follows from the fact that ξ_1 and ξ_2, resp. η_1 and η_2, are linearly independent. $\qquad\square$

Theorem 2.6. *The following are equivalent*:

(a) $\mathbb{A}$ *has a finite-dimensional nullspace.*

(b) $\mathbb{A}$ *is $\mathbb{C}$-elliptic.*

(c) *There exists $l \in \mathbb{N}$ with $N(\mathbb{A}) \subset \mathscr{P}_l$, where $\mathscr{P}_l$ denotes the set of polynomials with degree less or equal to l.*

Proof. Lemma 2.5 proves (a)enumi $\Rightarrow$ (b)enumi. Obviously, (c)enumi $\Rightarrow$ (a)enumi. It remains to show (b)enumi $\Rightarrow$ (c)enumi.

Since $\mathbb{A}$ is $\mathbb{C}$-elliptic, it is of type-(C) in the sense of [Kałamajska 1994]; see Remark 2.1. Fix $\omega \in C_c^\infty(B(0,1))$ with $\int_{B(0,1)} \omega \, dx = 1$. Then for an arbitrary ball B, we obtain by dilation and translation a function $\omega_B \in C_c^\infty(B)$ with $\int_B \omega_B(y) \, dy = 1$. For every $l \in \mathbb{N}_0$ let $\mathcal{P}_B^l$ denote the *averaged Taylor polynomial* with respect to B of order l, see [Dupont and Scott 1978]; i.e.,

$$\mathcal{P}_B^l u(x) := \int_B \sum_{|\beta| \le l} \partial_y^\beta \left(\frac{(y-x)^\beta}{\beta!} \omega_B(y) \right) u(y) \, dy.$$

The formula is obtained by multiplying Taylor's polynomial of order l by the weight ω_B and integrating by parts. Note that $\mathcal{P}_B^l u \in \mathscr{P}_l$.

It follows from the representation formula of [Kałamajska 1994, Theorem 4] that for all $x \in B$

$$|u(x) - (\mathcal{P}_B^l u)(x)| \le c \int_B \frac{|(\mathbb{A}u)(y)|}{|x-y|^{n-1}} \, dy \quad (2\text{-}11)$$

for some $l \in \mathbb{N}_0$ (which is fixed from now on) and all $u \in C^\infty(B)$. We do not know the exact value of l, but at least l is so large that $N(\mathbb{A}) \subset \mathscr{P}_l$ (there is, however, an upper bound for l in terms of n and N.)

Now, let $v \in N(\mathbb{A})$; i.e., $v \in \mathcal{D}'(\mathbb{R}^n; \mathbb{R}^N)$ with $\mathbb{A}v = 0$ in the distributional sense. Let ϕ_ϵ denote a standard mollifier; i.e., $\varphi_\epsilon(x) := \epsilon^{-n} \varphi(x/\varepsilon)$ with a radially symmetric function $\varphi \in C_c^\infty(\mathrm{B}; [0, 1])$ with $\int_{\mathrm{B}} \varphi \, \mathrm{d}x = 1$. Then $v * \phi_\epsilon \in C^\infty(\mathbb{R}^n)$ and $\mathbb{A}(v * \phi_\epsilon) = (\mathbb{A}v) * \phi_\epsilon = 0$. Hence, it follows from (2-11) that $v * \phi_\epsilon \in \mathscr{P}_l(\mathbb{R}^n)$. This implies $v \in \mathscr{P}_l(\mathbb{R}^n)$ as desired. $\qquad\square$

Remark 2.7. Let us compare our conditions with the ones of [Van Schaftingen 2013], building on the fundamental work of [Bourgain and Brezis 2004; 2007]. According to [Van Schaftingen 2013] the operator $\mathbb{A}$ is *cancelling*[1] if

$$\bigcap_{\xi \neq 0} \mathbb{A}[\xi](\mathbb{R}^N) = \{0\}. \tag{2-12}$$

It has been shown in Theorem 1.4 of [Van Schaftingen 2013] that whenever $\mathbb{A}$ is $\mathbb{R}$-elliptic and cancelling, then we have the Sobolev-type inequality

$$\|u\|_{L^{n/(n-1)}(\mathbb{R}^n; \mathbb{R}^N)} \leq C \|\mathbb{A}u\|_{L^1(\mathbb{R}^n; \mathbb{R}^K)} \tag{2-13}$$

for all $u \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^N)$. Moreover, the $\mathbb{R}$-ellipticity and cancellation property of $\mathbb{A}$ are necessary for such inequality.

For our result on traces we need the $\mathbb{C}$-ellipticity of $\mathbb{A}$. So the natural question arises how $\mathbb{C}$-ellipticity compares to the canceling property. It will been shown in [Gmeineder and Raiţă 2019] that $\mathbb{C}$-ellipticity implies the canceling property but not vice versa. Indeed, the operator

$$\mathbb{A}(u) := \begin{pmatrix} \frac{1}{2}\partial_1 u_1 - \frac{1}{2}\partial_2 u_2 & \frac{1}{2}\partial_1 u_2 + \frac{1}{2}\partial_2 u_1 & \partial_3 u_1 \\ \frac{1}{2}\partial_1 u_2 + \frac{1}{2}\partial_2 u_1 & \frac{1}{2}\partial_1 u_1 - \frac{1}{2}\partial_2 u_2 & \partial_3 u_2 \end{pmatrix}$$

is $\mathbb{R}$-elliptic and cancelling but it is not $\mathbb{C}$-elliptic, since it fails the finite-dimensional nullspace property (recall Theorem 2.6).

2D. *Smooth approximations in the interior.* In this section we show that functions from $W^{\mathbb{A},1}(\Omega)$ and $\mathrm{BV}^{\mathbb{A}}(\Omega)$ can be approximated in a certain sense by functions from $W^{\mathbb{A},1}(\Omega) \cap C^\infty(\Omega; \mathbb{R}^N)$. The proof is in the spirit of [Evans and Gariepy 1992, Chapter 5.2] and is included for the reader's convenience.

Theorem 2.8 (smooth approximation). *Let $\Omega \subset \mathbb{R}^n$ be open. Then the following hold*:

(a) *The space $(C^\infty \cap W^{\mathbb{A},1})(\Omega)$ is dense in $W^{\mathbb{A},1}(\Omega)$ with respect to the norm topology.*

(b) *The space $(C^\infty \cap \mathrm{BV}^{\mathbb{A}})(\Omega)$ is dense in $\mathrm{BV}^{\mathbb{A}}(\Omega)$ with respect to the area-strict topology.*

Proof. Fix $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$. For $k = 2, 3, \dots$ define

$$\Omega_k := \left\{ x \in \Omega : \frac{1}{k+1} < d(x, \partial\Omega) < \frac{1}{k-1} \right\}.$$

Now pick a sequence (ψ_k) such that for each $k \in \mathbb{N}$ we have $\psi_k \in C_c^\infty(\Omega_k; [0, 1])$ together with $\sum_k \psi_k = 1$ globally in Ω. Now let $\eta_\epsilon : \mathbb{R}^n \to \mathbb{R}$ be a standard mollifier (even and nonnegative).

---

[1]The definition of *cancelling* in [Van Schaftingen 2013] is given in terms of the annihilating operator $\mathbb{L}$ from the exact sequence in (5-6). However, it translates in our setting to (2-12).

For $j \in \mathbb{N}$ and $k \in \mathbb{N}$ we can find $\epsilon_{j,k} > 0$ such that:

(i) $\operatorname{spt}(\eta_{\varepsilon_{j,k}} * (\psi_k u)) \subset \Omega_k$.

(ii) $\|\psi_k u - \eta_{\varepsilon_{j,k}} * (\psi_k u)\|_{L^1(\Omega)} < 2^{-k-j}$.

(iii) $\|u \otimes_{\mathbb{A}} \nabla \psi_k - \eta_{\varepsilon_{j,k}} * (u \otimes_{\mathbb{A}} \nabla \psi_k)\|_{L^1(\Omega)} < 2^{-k-j}$.

(iv) If $u \in W^{\mathbb{A},1}(\Omega)$, we additionally require $\|\psi_k \mathbb{A} u - \eta_{\varepsilon_{j,k}} * (\psi_k \mathbb{A} u)\|_{L^1(\Omega)} < 2^{-k-j}$.

This allows us to define $u_j \in C^\infty(\Omega)$ by $u_j := \sum_{k \in \mathbb{N}} \eta_{\varepsilon_{j,k}} * (\psi_k u)$, which is well-defined in $L^1_{\mathrm{loc}}(\Omega)$, since the sum is locally finite. Then in $L^1_{\mathrm{loc}}(\Omega)$

$$u - u_j = \sum_k (\psi_k u - \eta_{\epsilon_{j,k}} * (\psi_k u)).$$

This and (ii)enumi imply $\|u - u_j\|_{L^1(\mathbb{R}^n)} \lesssim 2^{-j}$. If $u \in W^{\mathbb{A},1}(\Omega)$, then (iii)enumi and (iv)enumi imply $\|\mathbb{A} u - \mathbb{A} u_j\|_{L^1(\mathbb{R}^n)} \lesssim 2^{-j}$. This proves (a)enumi.

It remains to prove $u_j \xrightarrow{\langle \cdot \rangle} u$ for $j \to \infty$ for $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$. In fact, the proof is similar to the standard BV case. For simplicity of notation we just show $u_j \xrightarrow{s} u$ for $j \to \infty$. The necessary changes to pass from strict convergence to area-strict convergence are just like in [Bildhauer 2003, Lemma B.2].

Since $u_j \to u$ in $L^1(\mathbb{R}^n)$ it follows by the lower semicontinuity of the total $\mathbb{A}$-variation that $|\mathbb{A} u|(\Omega) \le \liminf_{j \to \infty} |\mathbb{A} u_j|(\Omega)$. It remains to prove $\limsup_{j \to \infty} |\mathbb{A} u_j|(\Omega) \le |\mathbb{A} u|(\Omega)$. For this we invoke the dual characterisation (2-4) of the total $\mathbb{A}$-variation. Let $\varphi \in C^1_c(\Omega; \mathbb{R}^K)$ with $|\varphi| \le 1$ be arbitrary. We compute

$$\int_\Omega \langle u_j, \mathbb{A}^* \varphi \rangle \, \mathrm{d}x = \sum_k \int_\Omega \langle \eta_{\varepsilon_{j,k}} * (\psi_k u), \mathbb{A}^* \varphi \rangle \, \mathrm{d}x = \sum_k \int_\Omega \langle \psi_k u, \mathbb{A}^* (\eta_{\varepsilon_{j,k}} * \varphi) \, \mathrm{d}x$$

$$= \sum_k \int_\Omega \langle u, \mathbb{A}^* (\psi_k (\eta_{\varepsilon_{j,k}} * \varphi)) \rangle \, \mathrm{d}x - \sum_k \int_\Omega \langle u, (\eta_{\varepsilon_{j,k}} * \varphi) \otimes_{\mathbb{A}^*} \nabla \psi_k \rangle \, \mathrm{d}x$$

$$=: I_j + II_j.$$

The sums are well-defined, since $\phi \in C^1_c(\Omega)$ and $u_j = \sum_k \eta_{\epsilon_{j,k}} * (\psi_k u)$ in $L^1_{\mathrm{loc}}(\Omega)$. Now

$$\left| \sum_k \psi_k (\eta_{\epsilon_{j,k}} * \phi) \right| \le \sum_k \psi_k |\eta_{\epsilon_{j,k}} * \phi| \le \sum_k \psi_k \|\phi\|_\infty = \|\phi\|_\infty \le 1.$$

Therefore,

$$I_j = \int_\Omega \left\langle u, \mathbb{A}^* \left(\sum_k \psi_k (\eta_{\varepsilon_{j,k}} * \varphi) \right) \right\rangle \mathrm{d}x \le |\mathbb{A} u|(\Omega).$$

Using $\sum_k \nabla \psi_k = 0$ and $\phi \in C^1_c(\Omega)$, we now rewrite II_j as

$$II_j = \sum_k \int_\Omega \langle u, (\eta_{\varepsilon_{j,k}} * \varphi) \otimes_{\mathbb{A}^*} \nabla \psi_k \rangle \, \mathrm{d}x - \sum_k \int_\Omega \langle u, \varphi \otimes_{\mathbb{A}^*} \nabla \psi_k \rangle \, \mathrm{d}x$$

$$= \sum_k \int_\Omega \langle \eta_{\varepsilon_{j,k}} * (u \otimes_{\mathbb{A}} \nabla \psi_k) - (u \otimes_{\mathbb{A}} \nabla \psi_k), \varphi \rangle \, \mathrm{d}x.$$

Invoking (iii)enumi and $\|\phi\|_\infty \le 1$ we obtain $|II_j| \lesssim 2^{-j}$. Hence, collecting estimates we obtain as desired $\limsup_{j \to \infty} |\mathbb{A} u_j|(\Omega) \le \limsup_{j \to \infty} (|\mathbb{A} u|(\Omega) + c 2^{-j}) = |\mathbb{A} u|(\Omega)$. $\square$

3. Projections and Poincaré inequalities

In this section we derive several versions of Poincaré's inequality. We assume throughout the section that $\mathbb{A}$ is $\mathbb{C}$-elliptic (or, equivalently, $\mathbb{A}$ has a finite-dimensional nullspace; see Theorem 2.6).

3A. *Projection operator.* We begin with some projection estimates.

For every ball $B \subset \mathbb{R}^n$ and $u \in L^2(B; \mathbb{R}^N)$ we define $\Pi_B u$ as the L^2-projection of u onto $N(\mathbb{A})$. Hence,

$$\int_B |\Pi_B u|^2 \, dx \le \int_B |u|^2 \, dx.$$

Since $N(\mathbb{A})$ is finite-dimensional, there exists a constant $c > 0$ with

$$\|\Pi_B u\|_{L^\infty(B)} \le c \fint_B |\Pi_B u| \, dx. \tag{3-1}$$

Indeed, this is clear for the unit ball and extends to general balls by dilation and translation. It follows from this as usual that

$$\fint_B |\Pi_B u| \, dx \le c \fint_B |u| \, dx. \tag{3-2}$$

Thus, Π_B can be extended to $L^1(B; \mathbb{R}^N)$ such that (3-2) remains valid.

Lemma 3.1. *Then there exists $c \ge 1$ with*

$$\inf_{q \in N(\mathbb{A})} \|u - q\|_{L^1(B)} \le \|u - \Pi_B u\|_{L^1(B)} \le c \inf_{q \in N(\mathbb{A})} \|u - q\|_{L^1(B)}.$$

Proof. The first estimate is obvious. Now, for all $q \in N(\mathbb{A})$ we have $\Pi_B q = q$. This and (3-2) imply

$$\|u - \Pi_B u\|_{L^1(B)} \le \|u - q\|_{L^1(B)} + \|\Pi_B(u - q)\|_{L^1(B)} \le c\|u - q\|_{L^1(B)}.$$

Taking the infimum over $q \in N(\mathbb{A})$ proves the lemma. $\qquad\square$

3B. *Poincaré inequalities.* In this subsection we derive Poincaré-type inequalities for $W^{\mathbb{A},1}$ and $BV^{\mathbb{A}}$. Recall that for a ball B we denote by $\ell(B)$ its diameter.

Theorem 3.2. *There exists a constant $c > 0$ such that for all balls B and all $u \in BV^{\mathbb{A}}(B)$ it holds*

$$\inf_{q \in N(\mathbb{A})} \|u - q\|_{L^1(B)} \le \|u - \Pi_B u\|_{L^1(B)} \le c\ell(B) \, |\mathbb{A}u|(B),$$

where Π_B is the L^2-orthogonal projection onto $N(\mathbb{A})$ from Section 3A.

Proof. By dilation and translation, it suffices to prove the claim for the unit ball $B = B(0, 1)$. Moreover, by smooth approximation (see Theorem 2.8) it suffices to consider $u \in C^\infty(B; \mathbb{R}^N) \cap W^{\mathbb{A},1}(B)$.

We use the averaged Taylor polynomials as in the proof of Theorem 2.6. Recall that by (2-11) we have the estimate

$$|u(x) - (\mathcal{P}^l u)(x)| \le c \int_B \frac{|(\mathbb{A}u)(y)|}{|x - y|^{n-1}} \, dy \quad \text{for all } x \in B. \tag{3-3}$$

Since $\mathcal{P}^l u$ is not necessarily in the kernel of $\mathbb{A}$, we wish to replace it by $\Pi_B(\mathcal{P}^l)$. Thus, we start with

$$|u(x) - \Pi_B(\mathcal{P}^l u)(x)| \leq |u(x) - (\mathcal{P}^l u)(x)| + |(\mathcal{P}^l u)(x) - (\Pi_B(\mathcal{P}^l u))(x)|. \tag{3-4}$$

Now, for any $p \in \mathscr{P}_l$ there holds

$$\|p - \Pi_B p\|_{L^\infty(B)} \leq c \fint_B |\mathbb{A}p| \, dx. \tag{3-5}$$

Indeed, both sides define a norm on the finite-dimensional space $\mathscr{P}_l / N(\mathbb{A})$ and vanish on $N(\mathbb{A})$. Hence, for all $x \in B$

$$|(\mathcal{P}^l u)(x) - (\Pi_B(\mathcal{P}^l u))(x)| \leq \|\mathcal{P}^l u - \Pi_B(\mathcal{P}^l u)\|_{L^\infty(B)} \leq c \fint_B |\mathbb{A}(\mathcal{P}^l u)| \, dx. \tag{3-6}$$

The definition of the averaged Taylor polynomial implies

$$\mathbb{A}(\mathcal{P}^l u) = \mathcal{P}^{l-1}(\mathbb{A}u), \tag{3-7}$$

where $\mathcal{P}^{-1} u := 0$ if $l = 0$. The L^1-stability of the averaged Taylor polynomial gives

$$\|\mathcal{P}^{l-1}(\mathbb{A}u)\|_{L^1(B)} \leq c\|\mathbb{A}u\|_{L^1(B)}. \tag{3-8}$$

Now, (3-5) and (3-8) yield

$$|(\mathcal{P}^l u)(x) - (\Pi_B(\mathcal{P}^l u))(x)| \leq c\ell(B) \fint_B |\mathbb{A}u| \, dy \leq c \int_B \frac{|(\mathbb{A}u)(y)|}{|x - y|^{n-1}} \, dy.$$

So, (3-3) and (3-4) imply the estimate

$$|u(x) - (\Pi_B \mathcal{P}^l u)(x)| \leq c \int_B \frac{|(\mathbb{A}u)(y)|}{|x - y|^{n-1}} \, dy. \tag{3-9}$$

Now, integration over $x \in B$ gives

$$\int_B |u - \Pi_B(\mathcal{P}^l u)| \, dx \leq c \int_B \int_B \frac{|(\mathbb{A}u)(y)|}{|x - y|^{n-1}} \, dy \, dx$$
$$\leq c \int_B |(\mathbb{A}u)(y)| \int_B |x - y|^{1-n} \, dx \, dy \leq c\ell(B) \int_B |\mathbb{A}u| \, dy.$$

We have shown

$$\|u - \Pi_B(\mathcal{P}^l u)\|_{L^1(B)} \leq c\ell(B)\|\mathbb{A}u\|_{L^1(B)}. \tag{3-10}$$

The rest follows by Lemma 3.1. □

Theorem 3.3. *Let B' and B be two balls with $B' \subset B$ and $\ell(B) \lesssim \ell(B')$. Then for all $u \in \mathrm{BV}^{\mathbb{A}}(B)$ with $u = 0$ on B' there holds*

$$\|u\|_{L^1(B)} \leq c\ell(B)|\mathbb{A}u|(B).$$

The constant only depends on the ratio $\ell(B)/\ell(B')$.

Proof. We use the same construction as in the proof of Theorem 3.2. However, we choose $\omega \in C_c^\infty(B)$ in the construction of the averaged Taylor polynomial additionally as $\omega \in C_c^\infty(B')$. This implies that $\mathcal{P}^l u$ only depends on the values of u on B'. Hence, we obtain $\mathcal{P}^l u = 0$. Thus, Theorem 3.2 proves the claim. □

Finally, let us remark that variants of Poincaré-type inequalities can also be established along the lines of [Adams and Hedberg 1996, Lemma 8.3.1] or [Ziemer 1989, Chapter 4]. However, this requires additional extension and compactness arguments which need to be proven first.

4. Traces

In this section we show that the space of functions of bounded $\mathbb{A}$-variation admits a continuous trace operator to $L^1(\partial\Omega)$ if and only if $\mathbb{A}$ is $\mathbb{C}$-elliptic (or, equivalently, $\mathbb{A}$ has a finite-dimensional nullspace; see Theorem 2.6).

4A. *Assumptions on the domain.* In order to ensure a proper trace we need to make certain regularity assumptions on Ω. Our results include all Lipschitz graph domains. However, we will consider even more general domains. Indeed, nontangentially accessible domains (NTA domains) provide a natural setting for our construction of the trace operator. We refer to [Hofmann et al. 2010] for more information on NTA domains.

We begin with the necessary conditions on our domain.

Definition 4.1 (interior/exterior corkscrew condition). Let $\Omega \subset \mathbb{R}^n$:

(a) We say that Ω satisfies the *interior corkscrew condition* if there exist $R > 0$ and $M > 2$ such that for all $x \in \partial\Omega$ and all $r \in (0, R)$ there exists a $y \in \Omega$ such that

$$\frac{1}{M}r \le |x - y| \le r \quad \text{and} \quad B\left(y, \frac{r}{M}\right) \subset \Omega.$$

(b) We say that Ω satisfies the *exterior corkscrew condition* if $\mathbb{R}^n \setminus \Omega$ satisfies the interior corkscrew condition.

Definition 4.2 (Harnack chain condition). We say that $\Omega \subset \mathbb{R}^n$ satisfies the (*interior*) *Harnack chain condition* if there exist $R > 0$ and $M \in \mathbb{N}$ such that for any $\epsilon > 0$, $r \in (0, R)$, $x \in \partial\Omega$, and $y_1, y_2 \in B(x, r) \cap \Omega$ with $|y_1 - y_2| \le \epsilon 2^k$ and $d(y_j, \partial\Omega) \ge \epsilon$ for $j = 1, 2$ there exists a chain of Mk balls $B_1, \ldots, B_{Mk}$ in Ω connecting y_1 and y_2 satisfying

(a) $y_1 \in B_1$, $y_2 \in B_{Mk}$,

(b) $\frac{1}{M}\ell(B_j) \le d(B_j, \partial\Omega) \le M\ell(B_j)$ for $j = 1, \ldots, Mk$,

(c) $\ell(B_j) \ge \frac{1}{M} \min \{d(y_1, B_j), d(y_2, B_j)\}$ for $j = 1, \ldots, Mk$.

Definition 4.3 (NTA domain). We say that a domain $\Omega \subset \mathbb{R}^n$ is an *NTA* (nontangentially accessible) domain if Ω satisfies the interior corkscrew condition, the exterior corkscrew condition and the interior Harnack chain condition.

Definition 4.4. We say that $\Omega \subset \mathbb{R}^n$ has *Ahlfors regular boundary* if there exist $R > 0$ and $M > 0$ such that for all $r \in (0, R)$

$$\frac{1}{M}r^{n-1} \le \mathcal{H}^{n-1}(B(x, r) \cap \partial\Omega) \le Mr^{n-1}. \tag{4-1}$$

In the following we tacitly require that our domains satisfy the following assumption:

Assumption 4.5. We assume that Ω satisfies the following:

(a) Ω is an NTA domain.

(b) Ω has Ahlfors regular boundary.

Note that all Lipschitz domains satisfy this assumption.

Let us now construct families of balls that we will use later in the construction of our traces:

For each $j \in \mathbb{Z}$, let $(B_{j,k})_k$ denote a (countable) cover of balls of $\mathbb{R}^n$ with diameter $\ell(B_{j,k})$ such that

(a) $\frac{1}{8} \cdot 2^{-j} \le \ell(B_{j,k}) \le \frac{1}{4} \cdot 2^{-j}$.

(b) The scaled balls $\left(\frac{7}{8} B_{j,k}\right)_k$ cover $\mathbb{R}^n$.

(c) Each family $(B_{j,k})_k$ is locally finite with covering constant independent of j; i.e.,

$$\sup_j \sum_k \chi_{B_{j,k}} \le c.$$

For each j let $(\eta_{j,k})_k$ be a partition of unity with respect to the $(B_{j,k})_k$ such that for all j, k

$$\|\eta_{j,k}\|_{L^\infty} + \ell(B_{j,k})\|\nabla \eta_{j,k}\|_{L^\infty} \le c. \tag{4-2}$$

Now, we define the 2^{-j}-neighbourhood U_j of $\partial\Omega$ by

$$U_j := \{x \in \Omega : d(x, \partial\Omega) < 2^{-j}\}.$$

Since Ω satisfies the interior corkscrew condition, we can find for each ball $B_{j,k}$ close to the boundary a *reflected ball* $B_{j,k}^\sharp$ close by. We will use these reflected balls later to define the local projections of our functions. More precisely:

(B1)   There exists $j_0 \in \mathbb{Z}$, such that the following holds: for each $B_{j,k}$ with $j \ge j_0$ and $B_{j,k} \cap U_j \ne \varnothing$, there exists a ball $B_{j,k}^\sharp \subset \Omega$ with $\ell(B_{j,k}^\sharp) \approx \ell(B_{j,k}) \approx d(B_{j,k}^\sharp, \partial\Omega)$ and $d(B_{j,k}, B_{j,k}^\sharp) \lesssim \ell(B_{j,k})$, where the hidden constants are independent of j, k.

Moreover, due to the Harnack chain condition we can connect two reflected balls of neighbouring balls by a small chain of balls. More precisely, we have the following.

(B2)   If $B_{j,k} \subset \Omega$ and $j \ge j_0$, then there exists a chain of balls $W_1, \ldots, W_\gamma$, with γ uniformly bounded, such that

(a) $W_1 = B_{j,k}$ and $W_\gamma = B_{j,k}^\sharp$,

(b) $|W_\beta \cap W_{\beta+1}| \approx |W_\beta| \approx |W_{\beta+1}| \approx |B_{j,k}|$ for $\beta = 1, \ldots, \gamma - 1$,

(c) $\ell(W_\beta) \approx \ell(B_{j,k})$ for $\beta = 1, \ldots, \gamma$.

The hidden constants are independent of j, k, β.

We define $\Omega(B_{j,k}, B_{j,k}^\sharp) := \bigcup_{\beta=1}^\gamma W_\beta$.

(B3)   If $B_{j,k} \cap B_{l,m} \ne \varnothing$ and $j, l \ge j_0$ with $|j - l| \le 1$, then there exists a chain of balls $W_1, \ldots, W_\gamma$ with γ uniformly bounded, such that

(a) $W_1 = B_{j,k}^\sharp$ and $W_\gamma = B_{l,m}^\sharp$,

(b) $|W_\beta \cap W_{\beta+1}| \approx |W_\beta| \approx |W_{\beta+1}| \approx |B_{j,k}|$ for $\beta = 1, \ldots, \gamma - 1$,

(c) $d(W_\beta, \partial\Omega) \approx \ell(W_\beta) \approx \ell(B_{j,k})$ for $\beta = 1, \ldots, \gamma$.

The hidden constants are independent of j, k, β.

We define $\Omega(B_{j,k}^\sharp, B_{l,m}^\sharp) := \bigcup_{\beta=1}^\gamma W_\beta$.

By construction of the chains above, we get:

(B4) There exists $k_0 \geq 2$ such that the following hold uniformly in $j \geq j_0$:

$$\sum_{m: B_{j,m} \cap U_j \neq \varnothing} \chi_{B_{j,m}^\sharp} \leq c\chi_{U_{j-k_0} \setminus U_{j+k_0}} \quad \text{and} \quad \sum_{m: B_{j,m} \cap U_j \neq \varnothing} \sum_{k: B_{j+1,k} \cap B_{j,m} \neq \varnothing} \chi_{\Omega(B_{j,m}^\sharp, B_{j+1,k}^\sharp)} \leq c\chi_{U_{j-k_0} \setminus U_{j+k_0}}.$$

4B. *Trace operator.* We will now construct the trace operator on $\mathrm{BV}^\mathbb{A}(\Omega)$. We will obtain the traces by a suitable approximation process. In particular, we will define truncations $T_j u$ which are smooth close to the boundary and admit classical traces. The limits will later provide the trace of the original function.

We define

$$\Pi_{j,k} u := \Pi_{B_{j,k}^\sharp} u.$$

Let $\rho_j \in C^\infty(\Omega)$ be such that $\chi_{U_{j+1}} \leq \rho_j \leq \chi_{U_j}$ and $\|\nabla\rho_j\|_\infty \lesssim 2^j$ and let $u \in \mathrm{BV}^\mathbb{A}(\Omega)$. Then for $j \geq j_0$ we define $T_j u$ in Ω by

$$T_j u := u - \rho_j \sum_k \eta_{j,k}(u - \Pi_{j,k}u) = (1 - \rho_j)u + \rho_j \sum_k \eta_{j,k}\Pi_{j,k}u. \tag{4-3}$$

Due to the support of $\eta_{j,k}$ the sum in the definition is locally finite. In particular, the sum is well-defined in $L_{\mathrm{loc}}^1(\Omega)$. The function $T_j u$ is an approximation of u that replaces the values of u in the neighbourhood of $\partial\Omega$ of distance 2^{-j} by local averages. These averages are performed slightly inside the domain on the balls $B_{j,k}^\sharp$.

We begin with an auxiliary estimate involving $\Pi_{j,k}u$.

Lemma 4.6. *We have the following estimates*:

(a) *There holds*

$$\|\Pi_{j,k}u\|_{L^\infty(B_{j,k})} \lesssim \fint_{B_{j,k}^\sharp} |u|\, dx.$$

(b) *If* $B_{j,m} \cap (U_j \setminus U_{j+2}) \neq \varnothing$, *then* $B_{j,m} \subset \Omega$ *and*

$$\|u - \Pi_{j,m}u\|_{L^1(B_{j,m})} \lesssim \ell(B_{j,m})|\mathbb{A}u|(\Omega(B_{j,m}, B_{j,m}^\sharp)).$$

(c) *If* $B_{j+1,k} \cap B_{j,m} \neq \varnothing$, *then*

$$|B_{j,m}|\, \|\Pi_{j+1,k}u - \Pi_{j,m}u\|_{L^\infty(B_{j,m})} \lesssim \ell(B_{j,m})|\mathbb{A}u|(\Omega(B_{j+1,k}^\sharp, B_{j,m}^\sharp)).$$

Proof. (a) Since $\Pi_{j,k}$ maps to $N(\mathbb{A})$ and $N(\mathbb{A}) \subset \mathscr{P}_l$, this is just the usual inverse estimate for polynomials of a fixed degree.

(b) The definition of U_j and $\ell(B_{j,m}) \leq \frac{1}{4} 2^{-j}$ imply $B_{j,m} \subset \Omega$. We compute

$$\|u - \Pi_{j,m} u\|_{L^1(B_{j,m})} = \|u - \Pi_{B_{j,m}^\sharp} u\|_{L^1(B_{j,m})} \leq \|u - \Pi_{B_{j,m}} u\|_{L^1(B_{j,m})} + \|\Pi_{B_{j,m}} u - \Pi_{B_{j,m}^\sharp} u\|_{L^1(B_{j,m})}.$$

The first term can be estimated by Poincaré's inequality from Theorem 3.2 which yields immediately

$$\|u - \Pi_{B_{j,m}} u\|_{L^1(B_{j,m})} \lesssim \ell(B_{j,m}) |\mathbb{A}u|(B_{j,m}).$$

For the second term we make use of the Harnack chain conditions (recall Definition 4.2) and, using (B2), connect $B_{j,m}$ and $B_{j,m}^\sharp$ by a chain

$$\Omega(B_{j,k}, B_{j,m}^\sharp) = \bigcup_{\beta=1}^{\gamma} W_\beta,$$

where $W_1, \ldots, W_\gamma$ are balls of size proportional to $\ell(B_{j,m})$. In particular, we have $W_1 = B_{j,m}$ and $W_\gamma = B_{j,m}^\sharp$. Moreover, we can assume that $|W_\beta \cap W_{\beta+1}| \approx |W_\beta| \approx \ell(B_{j,m})$ for all β. Now, we gain

$$\|\Pi_{B_{j,m}} u - \Pi_{B_{j,m}^\sharp} u\|_{L^1(B_{j,m})} \leq \sum_{\beta=1}^{\gamma-1} \|\Pi_{W_{\beta+1}} u - \Pi_{W_\beta} u\|_{L^1(B_{j,m})}$$

$$\lesssim \sum_{\beta=1}^{\gamma-1} \|\Pi_{W_{\beta+1}} u - \Pi_{W_\beta} u\|_{L^1(W_{\beta+1} \cap W_\beta)} \lesssim \sum_{\beta=1}^{\gamma} \|u - \Pi_{W_\beta} u\|_{L^1(W_\beta)}$$

using equivalence of norms on $N(\mathbb{A})$. Finally, using again Theorem 3.2 in conjunction with (B4),

$$\|\Pi_{B_{j,m}} u - \Pi_{B_{j,m}^\sharp} u\|_{L^1(B_{j,m})} \lesssim \ell(B_{j,m}) \sum_{\beta=1}^{\gamma} |\mathbb{A}u|(W_\gamma) \lesssim \ell(B_{j,m}) |\mathbb{A}u|(\Omega(B_{j,m}, B_{j,m}^\sharp)).$$

Gathering estimates, we arrive at the claim.

(c) First, by the inverse estimate for polynomials, we have

$$|B_{j,m}| \, \|\Pi_{j+1,k} u - \Pi_{j,m} u\|_{L^\infty(B_{j,m})} \lesssim \|\Pi_{j+1,k} u - \Pi_{j,m} u\|_{L^1(B_{j,m})} = \|\Pi_{B_{j+1,k}^\sharp} u - \Pi_{B_{j,m}^\sharp} u\|_{L^1(B_{j,m})}.$$

Now, connecting $B_{j+1,k}^\sharp$ and $B_{j,m}^\sharp$ via the chain $\Omega(B_{j+1,k}^\sharp, B_{j,m}^\sharp)$ (recall (B3)), we obtain the claim arguing exactly as in (b). $\qquad \square$

The following lemma shows that T_j is well-defined on $L^1(\Omega)$.

Lemma 4.7. $T_j : L^1(\Omega) \to L^1(\Omega)$ *is linear and bounded.*

Proof. We estimate pointwise on Ω

$$|T_j u| \leq (1 - \rho_j) |u| + \rho_j \sum_k \chi_{B_{j,k}} \|\Pi_{j,k} u\|_{L^\infty(B_{j,k})}. \tag{4-4}$$

With Lemma 4.6 we get

$$|T_j u| \lesssim \chi_{\Omega \setminus U_{j+1}} |u| + \sum_{k : B_{j,k} \cap U_j \neq \varnothing} \chi_{B_{j,k}} \fint_{B_{j,k}^\sharp} |u| \, dx.$$

This implies

$$\|T_j u\|_{L^1(\Omega)} \lesssim \|u\|_{L^1(\Omega \setminus U_{j+1})} + \sum_{k: B_{j,k} \cap U_j \neq \varnothing} |B_{j,k}| \fint_{B_{j,k}^\sharp} |u| \, dx$$

$$\lesssim \|u\|_{L^1(\Omega \setminus U_{j+1})} + \sum_{k: B_{j,k} \cap U_j \neq \varnothing} \int_{B_{j,k}^\sharp} |u| \, \mathrm{d}x.$$

Since the $B_{j,k}^\sharp$ are locally finite by (B4), we get $\|T_j u\|_{L^1(\Omega)} \lesssim \|u\|_{L^1(\Omega)}$ as desired. $\qquad \square$

The next two lemmas show now that $T_{j+1}u - T_j u$ is summable in $L^1(\Omega)$ and $\mathrm{BV}^{\mathbb{A}}(\Omega)$.

Lemma 4.8. *Let* $u \in L^1(\Omega)$ *and* $j \geq j_0$. *Then*

$$\|T_{j+1}u - T_j u\|_{L^1(\Omega)} \lesssim \|u\|_{L^1(U_{j+1-k_0} \setminus U_{j+k_0})}.$$

Proof. Let $j \geq j_0$. Then we have

$$T_{j+1}u - T_j u = (\rho_j - \rho_{j+1})u + \rho_{j+1} \sum_k \eta_{j+1,k} \Pi_{j+1,k} u - \rho_j \sum_m \eta_{j,m} \Pi_{j,m} u.$$

Now

$$\|(\rho_j - \rho_{j+1})u\|_{L^1(\Omega)} \leq \|u\|_{L^1(U_j \setminus U_{j+2})}.$$

Moreover, by Lemma 4.6 (a) it follows that

$$\|\rho_j \eta_{j,m} \Pi_{j,m} u\|_{L^1(\Omega)} \leq c |B_{j,m}| \, \|\Pi_{j,m} u\|_{L^\infty(B_{j,m})} \leq c \|u\|_{L^1(B_{j,m}^\sharp)},$$

where it suffices to consider those j with $B_{j,m} \cap U_j \neq \varnothing$. Now (B4) implies

$$\sum_m \|\rho_j \eta_{j,m} \Pi_{j,m} u\|_{L^1(\Omega)} \leq c \|u\|_{L^1(U_{j-k_0} \setminus U_{j+k_0})}.$$

Analogously,

$$\sum_k \|\rho_j \eta_{j+1,k} \Pi_{j+1,k} u\|_{L^1(\Omega)} \leq c \|u\|_{L^1(U_{j+1-k_0} \setminus U_{j+1+k_0})}.$$

Combining the above estimates proves the lemma. $\qquad \square$

Lemma 4.9. *Let* $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$ *and* $j \geq j_0$. *Then*

$$\|\mathbb{A}(T_{j+1}u - T_j u)\|_{L^1(\Omega)} \lesssim |\mathbb{A}u|(U_{j-k_0} \setminus U_{j+k_0}).$$

Proof. Using that $\sum_m \eta_{j,m} = \sum_k \eta_{j+1,k} = 1$ in Ω we get

$$T_{j+1}u - T_j u = (\rho_j - \rho_{j+1}) \sum_m \eta_{j,m} (u - \Pi_{j,m} u) + \rho_{j+1} \sum_{k,m} \eta_{j+1,k} \eta_{j,m} (\Pi_{j+1,k} u - \Pi_{j,m} u) =: I + II. \quad (4\text{-}5)$$

In order to estimate $\|\mathbb{A}(T_{j+1}u - T_j u)\|_{L^1(\Omega)}$ it is crucial that $\mathbb{A}\Pi_{j+1,k} u = \mathbb{A}\Pi_{j,m} u = 0$ and the gradients of $\rho_j, \rho_{j+1}, \eta_{j,m}$, and $\eta_{j+1,k}$ are bounded by 2^j; recall (4-2). Let us consider II. We only have to estimate those summands with k, m satisfying $B_{j+1,k} \cap B_{j,m} \neq \varnothing$ since otherwise $\eta_{j+1,k} \eta_{j,m} = 0$. For each such k, m we estimate the $L^1(\Omega)$-norm of $\mathbb{A}II$ by Lemma 4.6(c)enumi. Now, in combination with (B4) we get

$$\|\mathbb{A}II\|_{L^1(\Omega)} \lesssim |\mathbb{A}u|(U_{j-k_0} \setminus U_{j+k_0}).$$

Let us consider I. We only need to estimate those summands with m satisfying $B_{j,m} \cap (U_j \setminus U_{j+2}) \neq \varnothing$, since otherwise $(\rho_j - \rho_{j+1})\eta_{j,m} = 0$. For each such m we estimate the $L^1(\Omega)$-norm of $\mathbb{A}I$ by Lemma 4.6(b)enumi. Now, in combination with (B4) we get

$$\|\mathbb{A}I\|_{L^1(\Omega)} \lesssim |\mathbb{A}u|(U_{j-k_0} \setminus U_{j+k_0}). \qquad \square$$

Based on the two lemmas above, we now study the convergence $T_j u \to u$.

Corollary 4.10. *If* $u \in L^1(\Omega)$, *then*

$$u = T_{j_0}u + \sum_{l=j_0}^{\infty}(T_{l+1}u - T_l u) = \lim_{j \to \infty} T_j u \qquad (4\text{-}6)$$

in $L^1(\Omega)$. *If additionally* $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$, *then* (4-6) *also holds in* $\mathrm{BV}^{\mathbb{A}}(\Omega)$.

Proof. Since $\rho_j \to 0$ in $L^1_{\mathrm{loc}}(\Omega)$, it is clear that $T_j u \to u$ in $L^1_{\mathrm{loc}}(\Omega)$.

Note that for $j \geq j_0$

$$T_j u = T_{j_0}u + \sum_{l=j_0}^{j-1}(T_{l+1}u - T_l u). \qquad (4\text{-}7)$$

It follows from Lemmas 4.8 and 4.9 that $T_{l+1}u - T_l u$ is summable in $L^1(\Omega)$, resp. in $\mathrm{BV}^{\mathbb{A}}(\Omega)$, since the $U_{j+1-k_0} \setminus U_{j+k_0}$ are locally finite with respect to j. Hence, $T_j u$ is a Cauchy sequence in $L^1(\Omega)$, resp. in $\mathrm{BV}^{\mathbb{A}}(\Omega)$. Since the limit must agree with the $L^1_{\mathrm{loc}}(\Omega)$ limit, which is u, the claim follows. $\square$

Since $T_j u$ is smooth close to the boundary $\partial\Omega$, it is possible to evaluate the classical trace $\mathrm{tr}(T_j u)$. We now show that these traces form a $L^1(\partial\Omega)$-Cauchy sequence.

Lemma 4.11. *Let* $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$. *Then*

$$\|\mathrm{tr}(T_{j+1}u) - \mathrm{tr}(T_j u)\|_{L^1(\partial\Omega)} \lesssim |\mathbb{A}u|(U_{j-k_0} \setminus U_{j+k_0}),$$
$$\|\mathrm{tr}(T_{j_0}u)\|_{L^1(\partial\Omega)} \lesssim 2^{j_0}\|u\|_{L^1(U_{j_0-k_0} \setminus U_{j_0+k_0})}.$$

Proof. We begin with the first estimate. It follows from (4-5) that

$$\mathrm{tr}(T_{j+1}u) - \mathrm{tr}(T_j u) = \sum_{k,m}\mathrm{tr}(\eta_{j+1,k}\eta_{j,m}(\Pi_{j+1,k}u - \Pi_{j,m}u)),$$

where the sums are locally finite sums. Hence,

$$\|\mathrm{tr}(T_{j+1}u) - \mathrm{tr}(T_j u)\|_{L^1(\partial\Omega)} \leq \sum_{k,m}\|\mathrm{tr}(\eta_{j+1,k}\eta_{j,m}(\Pi_{j+1,k}u - \Pi_{j,m}u))\|_{L^1(\partial\Omega)}.$$

We only have to consider those k, m with $B_{j+1,k} \cap B_{j,m} \neq \varnothing$. For such k, m

$$\|\mathrm{tr}(\eta_{j+1,k}\eta_{j,m}(\Pi_{j+1,k}u - \Pi_{j,m}u))\|_{L^1(\partial\Omega)} \leq \|\Pi_{j+1,k}u - \Pi_{j,m}u\|_{L^\infty(B_{j,m})}\mathcal{H}^{n-1}(\partial\Omega \cap B_{j+1,k} \cap B_{j,m}).$$

We estimate the first factor by Lemma 4.6(c)enumi and the second by the Ahlfors regularity of the boundary, see (4-1), and thereby obtain

$$\|\mathrm{tr}(\eta_{j+1,k}\eta_{j,m}(\Pi_{j+1,k}u - \Pi_{j,m}u))\|_{L^1(\partial\Omega)} \lesssim |\mathbb{A}u|(\Omega(B_{j+1,k}^\sharp, B_{j,m}^\sharp)).$$

Summing over k and m and using (B4) implies

$$\|\mathrm{tr}(T_{j+1}u) - \mathrm{tr}(T_j u)\|_{L^1(\partial\Omega)} \lesssim |\mathbb{A}u|(U_{j-k_0} \setminus U_{j+k_0}).$$

This proves the first estimate.

Let us now estimate $\|\mathrm{tr}(T_{j_0})\|_{L^1(\partial\Omega)}$. We begin with

$$\mathrm{tr}(T_{j_0}) = \sum_k \mathrm{tr}(\eta_{j_0,k} \Pi_{j_0,k} u).$$

For each k with $B_{j_0,k} \cap \partial\Omega$ there holds

$$\|\mathrm{tr}(\eta_{j_0,k} \Pi_{j_0,k} u)\|_{L^1(\partial\Omega)} \leq \|\Pi_{j_0,k} u\|_{L^\infty(B_{j_0,k})} \mathcal{H}^{n-1}(\partial\Omega \cap B_{j_0,k}).$$

We estimate the first factor by Lemma 4.6(a)enumi and the second by the Ahlfors regularity of the boundary; see (4-1). This gives

$$\|\mathrm{tr}(\eta_{j_0,k} \Pi_{j_0,k} u)\|_{L^1(\partial\Omega)} \lesssim \frac{1}{\ell(B_{j_0})} \int_{B_{j_0,k}^\sharp} |u| \, \mathrm{d}x.$$

Summing over k and m and using (B4) implies

$$\|\mathrm{tr}(T_{j_0}u)\|_{L^1(\partial\Omega)} \lesssim 2^{j_0} \|u\|_{L^1(U_{j_0-k_0} \setminus U_{j_0+k_0})}. \qquad \square$$

Recall that by Corollary 4.10 we have

$$u = T_{j_0}u + \sum_{l=j_0}^{\infty} (T_{l+1}u - T_l u) = \lim_{j\to\infty} T_j u$$

in $\mathrm{BV}^{\mathbb{A}}(\Omega)$. Moreover, Lemma 4.11 shows that

$$\mathrm{tr}(T_{j_0}u) + \sum_{j \geq j_0} (\mathrm{tr}(T_{j+1}u) - \mathrm{tr}(T_j u)) = \lim_{j\to\infty} \mathrm{tr}(T_j(u))$$

is well-defined in $L^1(\partial\Omega)$. Finally,

$$\begin{aligned}
\left\| \lim_{j\to\infty} \mathrm{tr}(T_j(u)) \right\|_{L^1(\partial\Omega)} &\leq \|\mathrm{tr}(T_{j_0}(u))\|_{L^1(\partial\Omega)} + \sum_{j\geq j_0} \|\mathrm{tr}(T_{j+1}u) - \mathrm{tr}(T_j u)\|_{L^1(\partial\Omega)} \\
&\lesssim 2^{j_0} \|u\|_{L^1(U_{j_0-k_0} \setminus U_{j_0+k_0})} + \sum_{j\geq j_0} |\mathbb{A}u|(U_{j-k_0} \setminus U_{j+k_0}) \\
&\lesssim \|u\|_{L^1(\Omega)} + |\mathbb{A}u|(\Omega)
\end{aligned}$$

by Lemma 4.11. This allows us to define for every $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$ a trace

$$\widetilde{\mathrm{tr}}(u) := \lim_{j\to\infty} \mathrm{tr}(T_j u), \tag{4-8}$$

the limit being understood in the $L^1(\partial\Omega)$-sense. This limit satisfies

$$\|\widetilde{\mathrm{tr}}(u)\|_{L^1(\partial\Omega)} \lesssim \|u\|_{L^1(\Omega)} + |\mathbb{A}u|(\Omega). \tag{4-9}$$

We now show that $\widetilde{\mathrm{tr}}$ coincides with tr for all smooth functions and hence start with an approximation result.

Lemma 4.12. *Let* $u \in C^0(\bar{\Omega})$ *be uniformly continuous. Then* $T_j u \to u$ *in* $C^0(\bar{\Omega})$.

Proof. We have

$$u - T_j u = \rho_j \sum_k \eta_{j,k}(u - \Pi_{j,k}u),$$

where it suffices to take the sum over those k with $B_{j,k} \cap U_j \neq \varnothing$. Let us take one of those k. We will show that $\|\eta_{j,k}(u - \Pi_{j,k}u)\|_{L^\infty(\Omega)}$ is small for large j. Since the $B_{j,k}$ are locally finite with respect to k (with a covering number independent of j), this will prove the lemma.

Since $\mathbb{A}$ maps constants to zero, the projections $\Pi_{j,k}$ map constants to themselves. Let $\langle u \rangle_{B_{j,k}^\sharp} := \fint_{B_{j,k}^\sharp} u \, dx$; then by Lemma 4.6(a)enumi

$$\|\eta_{j,k}(u - \Pi_{j,k}u)\|_{L^\infty(B_{j,k})} \leq \|u - \langle u \rangle_{B_{j,k}^\sharp}\|_{L^\infty(B_{j,k})} + \|\Pi_{j,k}(u - \langle u \rangle_{B_{j,k}^\sharp})\|_{L^\infty(B_{j,k})}$$

$$\lesssim \|u - \langle u \rangle_{B_{j,k}^\sharp}\|_{L^\infty(B_{j,k})} + \fint_{B_{j,k}^\sharp} |u - \langle u \rangle_{B_{j,k}^\sharp}| \, dx.$$

Since u is uniformly continuous, the $B_{j,k}$ and $B_{j,k}^\sharp$ are small and close to each other, see (B1), and we see that both expressions on the right-hand side are small for large j uniformly in k. $\square$

Corollary 4.13. *Let* $u \in \mathrm{BV}^{\mathbb{A}}(\Omega) \cap C^0(\bar{\Omega})$ *be uniformly continuous. Then* $\widetilde{\mathrm{tr}}(u) = \mathrm{tr}(u)$.

Proof. We see from Corollary 4.10 and Lemma 4.12 that $T_j u \to u$ in $\mathrm{BV}^{\mathbb{A}}(\Omega)$ and in $C^0(\bar{\Omega})$. By the definition of $\widetilde{\mathrm{tr}}(u)$, we have $\mathrm{tr}(T_j u) \to \widetilde{\mathrm{tr}}(u)$. Since $T_j u \to u$ in $C^0(\bar{\Omega})$, we also have $\mathrm{tr}(T_j u) \to \mathrm{tr}(u)$ in $C^0(\partial\Omega)$. The limits must agree in $L^1_{\mathrm{loc}}(\partial\Omega)$, so $\widetilde{\mathrm{tr}}(u) = \mathrm{tr}(u)$. $\square$

We have already seen that $\widetilde{\mathrm{tr}} : \mathrm{BV}^{\mathbb{A}}(\Omega) \to L^1(\partial\Omega)$ is continuous with respect to the norm topology. We wish to use this to conclude that $\widetilde{\mathrm{tr}}$ is the only extension of the classical trace to $\mathrm{BV}^{\mathbb{A}}(\Omega)$. However, as smooth functions are not dense in $\mathrm{BV}^{\mathbb{A}}$ with respect to the norm topology, we switch to strict convergence as in the BV-case.

Lemma 4.14. *The trace operator* $\widetilde{\mathrm{tr}} : \mathrm{BV}^{\mathbb{A}}(\Omega) \to L^1(\partial\Omega; \mathbb{R}^N)$ *is continuous with respect to the strict convergence of* $\mathrm{BV}^{\mathbb{A}}(\Omega)$.

Proof. Let $u, u_k \in \mathrm{BV}^{\mathbb{A}}(\Omega)$ with $u_k \xrightarrow{s} u$ and $m \in \mathbb{N}$.

It follows from the definition (4-3) of T_j that for $j > m + k_0$ there holds for all $v \in \mathrm{BV}^{\mathbb{A}}(\Omega)$

$$T_j(\rho_m v) = \rho_m T_j v.$$

Indeed, $\rho_m = 1$ on the $B_{j,k}$ and the $B_{j,k}^\sharp$ for all m that contribute to the sum in (4-3).

This implies

$$\widetilde{\mathrm{tr}}(v) = \lim_{j \to \infty} \mathrm{tr}(T_j v) = \lim_{j \to \infty} \mathrm{tr}(T_j(\rho_m v)) = \widetilde{\mathrm{tr}}(\rho_m v) \quad \text{in } L^1(\partial\Omega).$$

Now, for all $k \in \mathbb{N}$,

$$\|\widetilde{\mathrm{tr}}(u_k - u)\|_{L^1(\partial\Omega)} = \|\widetilde{\mathrm{tr}}(\rho_m(u_k - u))\|_{L^1(\partial\Omega)}.$$

Thus, by (4-9)

$$\|\widetilde{\mathrm{tr}}(u_k - u)\|_{L^1(\partial\Omega)} \lesssim \|\rho_m(u_k - u)\|_{L^1(\Omega)} + |\mathbb{A}(\rho_m(u_k - u))|(\Omega)$$
$$\lesssim \|u_k - u\|_{L^1(\Omega)} + |\mathbb{A}u_k|(U_m) + |\mathbb{A}u_k|(U_m) + 2^{-m}\|u_k - u\|_{L^1(U_m)}.$$

Now, let $k, l \to \infty$. Since $u_k \xrightarrow{s} u$ in $\mathrm{BV}^{\mathbb{A}}(\Omega)$ and U_m is open, we get

$$\|\widetilde{\mathrm{tr}}(u_k - u)\|_{L^1(\partial\Omega)} \lesssim |\mathbb{A}u|(U_m).$$

The right-hand side converges to zero for $m \to \infty$. Thus $\widetilde{\mathrm{tr}}(u_k) \to \widetilde{\mathrm{tr}}(u)$ in $L^1(\partial\Omega)$ for $k \to \infty$. $\qquad\square$

In order to proceed, we need a smooth approximation result up to the boundary in the area-strict topology.

Lemma 4.15. *Let $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$. Then there exists $u_j \in C^{\infty}(\overline{\Omega})$ with $u_j \xrightarrow{\langle\cdot\rangle} u$ in $\mathrm{BV}^{\mathbb{A}}(\Omega)$.*

Proof. For $j \geq j_0$ consider $T_j u$. Then $T_j u$ is C^{∞} in $\overline{U}_{j+1}$. Indeed, for all $x \in U_{j+1}$ we have

$$(T_j u)(x) = \sum_k \eta_{j,k} \Pi_{j,k} u.$$

For each k with $B_{j,k} \cap U_{j+1} \neq 0$ we have

$$\|\nabla(\eta_{j,k}\Pi_{j,k}u)\|_{\infty} \lesssim \|\nabla\eta_{j,k}\|_{L^{\infty}(B_{j,k})}\|\Pi_{j,k}u\|_{L^{\infty}(B_{j,k})} + \|\nabla\Pi_{j,k}u\|_{L^{\infty}(B_{j,k})}.$$

Using inverse estimates for polynomials and Lemma 4.6 we get

$$\|\nabla(\eta_{j,k}\Pi_{j,k}u)\|_{\infty} \lesssim \ell(B_{j,k})|B_{j,k}|\|\Pi_{j,k}u\|_{L^1(B_{j,k})} \lesssim 2^{j(n+1)}\|u\|_{L^1(B_{j,k}^{\sharp})}.$$

Hence, $T_j u$ is uniformly continuous on $\overline{U}_{j+1}$.

Now, let $\eta_{\epsilon} : \mathbb{R}^n \to \mathbb{R}$ be an standard mollifier (even and nonnegative). It is well known that $u_{j,\epsilon} := \rho_{j+1}T_j u + ((1 - \rho_{j+1})T_j u) * \eta_{\epsilon}$ converges to $T_j u$ as $\epsilon \searrow 0$ in $L^1(\Omega)$ as well as in the area-strict sense. Hence, we can find ϵ_j such that

$$\|u_{j,\epsilon_j} - T_j u\|_{L^1(\Omega)} \leq 2^{-j},$$
$$\left||\mathbb{A}(T_j u)|(\Omega) - |\mathbb{A}(u_{j,\epsilon_j})|(\Omega)\right| \leq 2^{-j}.$$

Moreover, recall that $T_j u \to u$ strongly in $\mathrm{BV}^{\mathbb{A}}(\Omega)$. This implies that $u_j := u_{j,\epsilon_j}$ has the desired property. This proves the strict convergence. The area-strict convergence follows by the same steps. $\qquad\square$

As a consequence of Lemmas 4.14 and 4.15 we immediately obtain the following corollary.

Corollary 4.16. *The operator $\widetilde{\mathrm{tr}} : \mathrm{BV}^{\mathbb{A}}(\Omega) \to L^1(\partial\Omega; \mathcal{H}^{n-1})$ is the unique strictly continuous extension of the classical trace on $\mathrm{BV}^{\mathbb{A}}(\Omega) \cap C^0(\overline{\Omega})$.*

Due to the above results it is not anymore necessary to distinguish the classical trace and our new trace. We collect our results proven so far in the following theorem.

Theorem 4.17. *Let $\mathbb{A}$ be $\mathbb{C}$-elliptic and let Ω be an NTA domain with Ahlfors regular boundary (see Assumption 4.5). Then there exists a trace operator $\mathrm{tr} : \mathrm{BV}^{\mathbb{A}}(\Omega) \to L^1(\partial\Omega, \mathcal{H}^{n-1})$ such that the following hold*:

(a) $\mathrm{tr}(u)$ *coincides with the classical trace for all* $u \in \mathrm{BV}^{\mathbb{A}}(\Omega) \cap C^0(\Omega)$.

(b) $\mathrm{tr}(u)$ *is the unique strictly continuous extension of the classical trace on* $\mathrm{BV}^{\mathbb{A}}(\Omega) \cap C^0(\overline{\Omega})$.

(c) $\mathrm{tr}(W^{\mathbb{A},1}(\Omega)) = \mathrm{tr}(\mathrm{BV}^{\mathbb{A}}(\Omega)) = L^1(\partial\Omega, \mathcal{H}^{n-1})$.

Proof. The existence of tr is shown in Lemma 4.14. Part (a) follows from Corollary 4.13, whereas (b) is a consequence of Corollary 4.16. Finally, part (c) is a consequence of the fact that

$$\mathrm{tr}(W^{1,1}(\Omega; \mathbb{R}^N)) = L^1(\partial\Omega; \mathbb{R}^N)$$

and $W^{1,1}(\Omega; \mathbb{R}^N) \subset W^{\mathbb{A},1}(\Omega)$. In particular, the sufficiency part of Theorem 1.2 is complete. $\qquad\square$

4C. *Necessity of* $\mathbb{C}$-*ellipticity.* In this section we show that it is not possible to define an L^1-trace of $\mathrm{BV}^{\mathbb{A}}$-functions if the operator $\mathbb{A}$ is not $\mathbb{C}$-elliptic. As such, we extend the observation of [Fuchs and Repin 2010] that $\mathbb{D} \ni z \mapsto 1/(z-1) \in \mathbb{C}$ is holomorphic and belongs to $L^1(\mathbb{D}; \mathbb{C})$ but does not belong to $L^1(\partial\mathbb{D}; \mathbb{C})$; see Example 2.2(c).

Theorem 4.18 (without a trace). *Suppose that* $\mathbb{A}$ *is not* $\mathbb{C}$-*elliptic. Let B denote the unit ball of* $\mathbb{R}^n$. *Then there exists a vector* $\xi_1 \in \mathbb{R}^n \setminus \{0\}$ *such that for the half-ball* $B^+ := \{x \in B : \langle \xi_1, x \rangle > 0\}$ *and the hyperplane* $\mathfrak{H} := \{x \in \mathbb{R}^n : \langle \xi_1, x \rangle = 0\}$ *there exists a function* $u \in W^{\mathbb{A},1}(B^+) \cap C^\infty(B^+)$ *such that* $u \notin L^1(\mathfrak{H} \cap B, \mathcal{H}^{n-1})$.

Proof. We begin with the case that $\mathbb{A}$ is not $\mathbb{R}$-elliptic. Let us define $f(x_1, x_2) := (|x_1| + |x_2|^2)^{-3/4}$. The crucial observation now is that $f, \partial_2 f \in L^1(B)$. However, $f \notin L^1(\{x_1 = 0\}|_B, \mathcal{H}^{n-1})$. We have to adapt this example to our situation. Since $\mathbb{A}$ is not $\mathbb{R}$ elliptic, there exists $\xi_1 \in \mathbb{R}^n \setminus \{0\}$ and $\eta_1 \in \mathbb{R}^N \setminus \{0\}$ with $\mathbb{A}[\xi_1]\eta_1 = 0$. We choose $\xi_2, \dots, \xi_n$ such that $\xi_1, \dots, \xi_n$ is a basis. Now, define $\tau : \mathbb{R}^n \to \mathbb{R}^2$ and $\sigma : \mathbb{R} \to \mathbb{R}^N$ by $\tau(x) := (\langle \xi_1, x \rangle, \langle \xi_2, x \rangle)$ and $\sigma(z) := z\eta_1$. Moreover, we define $h_f : \mathbb{R}^n \to \mathbb{R}^N$ by $h_f := \sigma \circ f \circ \tau$. Then we obtain

$$(\mathbb{A}h_f)(x) = \sum_{j=1}^2 \mathbb{A}[\xi_j]\eta_1(\partial_j f)(\tau(x))$$

(compare (2-10)). Since $\mathbb{A}[\xi_1]\eta_1 = 0$, this simplifies to

$$(\mathbb{A}h_f)(x) = \mathbb{A}[\xi_2]\eta_1(\partial_2 f)(\tau(x)).$$

We choose the hyperplane $\mathfrak{H} := \{x : \langle \xi_1, x \rangle = 0\}$. It follows from $f, \partial_2 f \in L^1(B)$ and $f \notin L^1(\{x_1 = 0\}|_B, \mathcal{H}^{n-1})$ that $u, \mathbb{A}u \in L^1(B)$ and so in particular $u, \mathbb{A}u \in L^1(B^+)$ with $B^+ := \{x \in B : \langle \xi_1, x \rangle > 0\}$ but $u \notin L^1(\mathfrak{H} \cap B, \mathcal{H}^{n-1})$. This concludes the proof in the case that $\mathbb{A}$ is not $\mathbb{R}$-elliptic.

Assume now that $\mathbb{A}$ is $\mathbb{R}$-elliptic but not $\mathbb{C}$-elliptic. Then as in the proof of Lemma 2.5 there exist $\xi_1, \xi_2 \in \mathbb{R}^n$ and $\eta_1, \eta_2 \in \mathbb{R}^n$, which are, resp., linearly independent such that

$$\mathbb{A}[\xi_1 + ix_2](\eta_1 + i\eta_2) = 0.$$

Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) := \frac{1}{z}$. Then $f \in L^1(B_1)$ with $B_1 := \{|z| < 1\}$ but $f \notin L^1(\{\mathrm{Re}(z) = 0\}|_{B_1}, \mathcal{H}^{n-1})$. As in Lemma 2.5 we define $\tau : \mathbb{R}^n \to \mathbb{C}$ and $\sigma : \mathbb{C} \to \mathbb{R}^N$ by $\tau(x) := \langle \xi, x \rangle = \langle \xi_1, x \rangle + i \langle \xi_2, x \rangle$ and

$\sigma(z) := \mathrm{Re}(z)\eta_1 - \mathrm{Im}(z)\eta_2$. Moreover, define $h_f : \mathbb{R}^n \to \mathbb{R}^N$ by $h_f := \sigma \circ f \circ \tau$. Then as in Lemma 2.5 we have $(\mathbb{A}h_f)(x) = 0$ in $\mathcal{D}'(B^+)$ with $B^+ := \{x \in B : \langle x_1, x \rangle > 0\}$. It follows from $f \in L^1(B^+)$ and $f \notin L^1(\{\mathrm{Re}(z) = 0\}|_{B_1}, \mathcal{H}^{n-1})$ that $h_f \in W^{\mathbb{A},1}(B)$ but $h_f \notin L^1(\mathfrak{H} \cap B, \mathcal{H}^{n-1})$ with $\mathfrak{H} := \{x : \langle \xi_1, x \rangle = 0\}$. This concludes the proof if $\mathbb{A}$ is $\mathbb{R}$-elliptic but not $\mathbb{C}$-elliptic. $\qquad\square$

Remark 4.19. Theorem 4.18 shows the nonexistence of a trace on some particular boundary hyperplane. If Ω does not enjoy this simple geometry but is a bounded domain with C^∞-boundary, then we choose a boundary point $x_0 \in \partial\Omega$ such that a suitable translation of the hyperplanes $\mathfrak{H}$ from the preceding proof becomes tangent to $\partial\Omega$ at x_0. In this situation, flattening the boundary locally around x_0 and applying the preceding theorem directly yield the nonexistence of boundary traces in $L^1(\partial\Omega; \mathcal{H}^{n-1})$. We leave the details to the reader.

4D. *Gauss–Green formula.* We now deduce the Gauss–Green formula for functions from $\mathrm{BV}^{\mathbb{A}}(\Omega)$, which, with Theorem 1.2 at our disposal, is a direct consequence of the Gauss–Green formula for smooth functions. Let us note that up to here, only Assumption 4.5 is required, whereas in what follows we stick to a Lipschitz assumption[2] on $\partial\Omega$.

Theorem 4.20 (Gauss–Green formula). *Let $\Omega \subset \mathbb{R}^n$ be open and bounded with Lipschitz boundary. For all $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$ and all $\phi \in C^1(\overline{\Omega}; \mathbb{R}^N)$ we have*

$$\int_{\Omega} \mathbb{A}u \cdot \phi \, \mathrm{d}x = -\int_{\Omega} u \cdot \mathbb{A}^*\phi \, \mathrm{d}x + \int_{\partial\Omega} (\mathrm{tr}(u) \otimes_{\mathbb{A}} \nu) \cdot \phi \, \mathrm{d}\mathcal{H}^{n-1}, \qquad (4\text{-}10)$$

where ν denotes the unit outer normal of Ω.

Proof. Due to Lemma 4.15 there exists a sequence $u_j \in C^\infty(\overline{\Omega})$ such that $u_j \xrightarrow{s} u$ in $\mathrm{BV}^{\mathbb{A}}(\Omega)$. Due to Lemma 4.14 we also have $u_j \to u$ in $L^1(\partial\Omega, \mathcal{H}^{n-1})$. Now, (4-10) is valid for each u_j. Passing to the limit proves the claim. $\qquad\square$

Corollary 4.21. *Let $\Omega \Subset U \subset \mathbb{R}^n$ such that Ω and U are open and bounded and have Lipschitz boundary. For $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$ and $v \in \mathrm{BV}^{\mathbb{A}}(U \setminus \Omega)$ define $w := \chi_{\Omega} u + \chi_{U \setminus \Omega} v$. Then $w \in \mathrm{BV}^{\mathbb{A}}(U)$ and*

$$\mathbb{A}w = \mathbb{A}u \llcorner_{\Omega} + \mathbb{A}v \llcorner_{U \setminus \Omega} + (\mathrm{tr}^+(v) - \mathrm{tr}^-(u)) \otimes_{\mathbb{A}} \nu \mathcal{H}^{n-1} \llcorner_{\partial\Omega}, \qquad (4\text{-}11)$$

where $\mathrm{tr}^+(u)$ denotes the interior trace of u and $\mathrm{tr}^-(v)$ denotes the exterior trace of v and ν the unit outer normal of Ω.

Proof. Let w be as given and let $\phi \in C_c^1(U)$. We split the domain U into Ω and $U \setminus \Omega$ and apply the Gauss–Green formula (4-10) first to U and then to Ω and $U \setminus \Omega$ separately. This yields

$$-\int_U w \cdot \mathbb{A}^*\phi \, \mathrm{d}x = -\int_{\Omega} u \cdot \mathbb{A}^*\phi \, \mathrm{d}x - \int_{U \setminus \Omega} v \cdot \mathbb{A}^*\phi \, \mathrm{d}x$$

$$= \int_{\Omega} \mathbb{A}u \cdot \phi \, \mathrm{d}x - \int_{\partial\Omega} (\mathrm{tr}^+(u) \otimes_{\mathbb{A}} \nu) \cdot \phi \, \mathrm{d}\mathcal{H}^{n-1} + \int_{U \setminus \Omega} \mathbb{A}v \cdot \phi \, \mathrm{d}x + \int_{\partial\Omega} (\mathrm{tr}^+(v) \otimes_{\mathbb{A}} \nu) \cdot \phi \, \mathrm{d}\mathcal{H}^{n-1}.$$

This proves $w \in \mathrm{BV}^{\mathbb{A}}(U)$ and the representation formula (4-11). $\qquad\square$

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[2]In fact, this can be weakened towards more general domains, but we will not need this in the sequel.

4E. *Sobolev spaces with zero boundary values.* Using our trace operator, it is natural to define subspaces of functions with zero boundary values; i.e.,

$$W_0^{\mathbb{A},1}(\Omega) := \{u \in W^{\mathbb{A},1}(\Omega) : \mathrm{tr}(u) = 0\},$$
$$\mathrm{BV}_0^{\mathbb{A}}(\Omega) := \{u \in \mathrm{BV}^{\mathbb{A}}(\Omega) : \mathrm{tr}(u) = 0\}.$$

However, in the context of Sobolev spaces $W_0^{\mathbb{A},1}(\Omega)$ there are two more variants to define these spaces. One by zero extension and one by the closure of $C_c^\infty(\Omega)$. We will show below in Theorem 4.23 that all three definitions define the same spaces.

We begin with an auxiliary lemma which we need for $W_0^{\mathbb{A},1}(\Omega)$. For slightly more generality we state it for $\mathrm{BV}_0^{\mathbb{A}}(\Omega)$.

Lemma 4.22. *Let* $u \in \mathrm{BV}_0^{\mathbb{A}}(\Omega)$. *Then* $(1 - \rho_j)u \to u$ *in* $\mathrm{BV}^{\mathbb{A}}(\Omega)$, *with* ρ_j *as in Section 4B.*

Proof. We can assume that $\Omega \Subset U \subset \mathbb{R}^n$ for some open, bounded U with Lipschitz boundary. By Corollary 4.21 we can extend u on $U \setminus \Omega$ by zero.

We have

$$\mathbb{A}((1 - \rho_j)u - u) = -\rho_j \mathbb{A}u - u \otimes_{\mathbb{A}} \nabla \rho_j.$$

Hence,

$$|\mathbb{A}((1 - \rho_j)u - u)|(\Omega) \le |\mathbb{A}u|(U_j) + cr_j^{-1}\|u\|_{L^1(U_j)}.$$

We will now show that

$$r_j^{-1}\|u\|_{L^1(U_j)} \lesssim |\mathbb{A}u|(U_{j-m})$$

for some $m \in \mathbb{N}$ (and sufficiently large, i.e., $j + m \ge j_0$). In fact, for fixed j define

$$K_j := \{k : B_{j,k} \cap U_j \ne \varnothing\}.$$

By the geometry of Ω, we can find a factor $\lambda > 0$ such that for each $k \in K_j$ the enlarged ball $\lambda B_{j,k}$ contains some ball $B'_{j,k}$ that is completely in $\mathbb{R}^n \setminus \Omega$. Now, for each $k \in K_j$, we get by Theorem 3.3

$$\|u\|_{L^1(B_{j,k})} \lesssim \|u\|_{L^1(\lambda B_{j,k})} \lesssim r_j |\mathbb{A}u|(\lambda B_{j,k}) = r_j |\mathbb{A}u|(\Omega \cap \lambda B_{j,k}).$$

Since the $(B_{j,k})_k$ are locally finite, so are the $(\lambda B_{j,k})_k$. Now, if we choose $m \in \mathbb{N}$ such that $\Omega \cap \lambda B_{j,k} \subset U_{j-m}$, then

$$r_j^{-1}\|u\|_{L^1(U_j)} \lesssim \sum_{k \in K_j} r_j^{-1}\|u\|_{L^1(B_{j,k})} \lesssim \sum_{k \in K_j} \|\mathbb{A}u\|_{L^1(\Omega \cap \lambda B_{j,k})} \lesssim |\mathbb{A}u|(U_{j-m}).$$

Overall, we obtain

$$|\mathbb{A}((1 - \rho_j)u - u)|(\Omega) \le |\mathbb{A}u|(U_{j-m}).$$

Now, $|\mathbb{A}u|(U_{j-m}) \to 0$, since $U_{j-m} \searrow \varnothing$. This proves the claim by the Poincaré inequality from Theorem 3.3. $\qquad\square$

Theorem 4.23 (zero traces). *Let* $\Omega \Subset U \subset \mathbb{R}^n$ *for some open, bounded* U *with Lipschitz boundary and let* $u \in W^{\mathbb{A},1}(\Omega)$. *The following are equivalent*:

(a) $u \in W_0^{\mathbb{A},1}(\Omega)$.

(b) *The extension $\tilde{u} := \chi_\Omega u$ by zero on $U \setminus \Omega$ is in $W^{\mathbb{A},1}(U)$.*

(c) *There exist $u_k \in C_c^\infty(\Omega)$ with $u_k \to u$ in $W^{\mathbb{A},1}(\Omega)$.*

Proof. (a)enumi $\Rightarrow$ (b)enumi: Let $u \in W_0^{\mathbb{A},1}(\Omega)$ and let $\tilde{u} = \chi_\Omega u$ be its zero extension on U. Then by Corollary 4.21 we have $\mathbb{A}\tilde{u} = \mathbb{A}u \llcorner_\Omega \in L^1(U)$, so $\tilde{u} \in W^{\mathbb{A},1}(U)$.

(b)enumi $\Rightarrow$ (a)enumi: Let $\tilde{u} = \chi_\Omega u \in W^{\mathbb{A},1}(U)$. Then by Corollary 4.21 we have $\mathbb{A}\tilde{u} = \mathbb{A}u \llcorner_\Omega + \mathrm{tr}^+(u) \otimes_\mathbb{A} \nu \mathcal{H}^{n-1} \llcorner_{\partial\Omega}$. Since $\mathbb{A}\tilde{u} \in L^1(U)$, the singular part must vanish; i.e., $\mathrm{tr}^+(u) \otimes_\mathbb{A} \nu \mathcal{H}^{n-1} \llcorner_{\partial\Omega} = 0$. So by $\mathbb{R}$-ellipticity of $\mathbb{A}$ we have $\mathrm{tr}^+(u) = 0$ on $\partial\Omega$.

(c)enumi $\Rightarrow$ (a)enumi: By continuity of the trace operator we have $\mathrm{tr}(u) = \lim_{k\to\infty} \mathrm{tr}(u_k) = 0$ in $L^1(\partial\Omega)$, so $u \in W_0^{\mathbb{A},1}(\Omega)$.

(a)enumi $\Rightarrow$ (c)enumi: Let $v_k := (1 - \rho_k)u$ as in Lemma 4.22. Then $v_k \to u$ in $W^{\mathbb{A},1}(\Omega)$. Moreover, the v_k have compact support, since $v_k = 0$ on U_{k+1}. Now, let $\eta_\epsilon \colon \mathbb{R}^n \to \mathbb{R}$ be an standard mollifier with support on $B_\epsilon(0)$. Then we find ϵ_k such that

$$\|v_k - v_k * \phi_{\epsilon_k}\|_{L^1(\Omega)} + \|\mathbb{A}v_k - \mathbb{A}(v_k * \phi_{\epsilon_k})\|_{L^1(\Omega)} \le 2^{-k}$$

and $\mathrm{supp}(v_k * \phi_{\epsilon_k}) \Subset \Omega$. The sequence $u_k := v_k * \phi_{\epsilon_k}$ has the desired properties. □

Proposition 4.24 (trace-preserving area-strict smoothing). *Let $\Omega \Subset U \subset \mathbb{R}^n$ such that Ω and U are open and bounded and have Lipschitz boundary. Let $u_0 \in W^{\mathbb{A},1}(U)$. Further let $u \in \mathrm{BV}^\mathbb{A}(U)$ with $u = u_0$ on $U \setminus \Omega$. Then there exists $u_j \in u_0 + C_c^\infty(\Omega)$ such that $u_j \xrightarrow{\langle \cdot \rangle} u$ in $\mathrm{BV}^\mathbb{A}(U)$.*

Proof. The proof is a straightforward modification of the corresponding statement for BV-functions; see [Bildhauer 2003, Lemma B.2] or [Kristensen and Rindler 2010a, Lemma 1]. Let us just explain the basic idea: The usual localization argument by a partition of unity reduces the question to a local Lipschitz graph. Then split u into $u_0 + \chi_\Omega(u - u_0)$. Now the $\chi_\Omega(u - u_0)$ part is moved by translation slightly into Ω. In a second step it is mollified to get a $C_c^\infty(\Omega)$ term. □

5. The Dirichlet problem on $\mathrm{BV}^\mathbb{A}$

This final section is devoted to variational problems with linear growth involving $\mathbb{A}u$ subject to given boundary data.

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set with Lipschitz boundary. Our goal is to study the functional $\mathfrak{F} \colon W^{\mathbb{A},1}(\Omega) \to \mathbb{R}$ given by

$$\mathfrak{F}[v] := \int_\Omega f(x, \mathbb{A}v) \, \mathrm{d}x, \tag{5-1}$$

where f satisfies linear growth conditions. Given a boundary datum $u_0 \in W^{\mathbb{A},1}(\Omega)$, we wish to minimise $\mathfrak{F}$ within the Dirichlet class $u_0 + W_0^{\mathbb{A},1}(\Omega)$. The existence of a minimiser together with the precise formulation of the problem at our disposal will be given in Theorem 5.3 below.

Let us define the $\mathbb{A}$-*rank-one cone* $\mathscr{C}(\mathbb{A}) = \mathbb{R}^N \otimes_\mathbb{A} \mathbb{R}^n \subset \mathbb{R}^K$, with $\otimes_\mathbb{A}$ as given by (2-7). This cone is important to characterise the jump terms of $\mathrm{BV}^\mathbb{A}$ functions as in Corollary 4.21. Also in the product rule (2-8), we have $v \otimes_\mathbb{A} \nabla\phi \in \mathscr{C}(\mathbb{A})$ pointwise for $\phi \in C^1(\mathbb{R}^n)$ and $v \in C^1(\mathbb{R}^n; \mathbb{R}^N)$.

By use of the Fourier transform, we see that $\mathbb{A}(u) = (\mathbb{A}[\xi]\hat{u})^{\vee}$. Since $\mathbb{A}[\xi]\hat{u} \in \mathscr{C}(\mathbb{A})$ pointwise, we obtain $\mathbb{A}(u) \in \text{span}(\mathscr{C}(\mathbb{A}))$ pointwise. Hence, we define the *effective range of $\mathbb{A}$* as $\mathscr{R}(\mathbb{A}) := \text{span}(\mathscr{C}(\mathbb{A})) \subset \mathbb{R}^K$; i.e., $\mathbb{A}u \in \mathscr{R}(\mathbb{A})$ pointwise. As a consequence, we only need to require that the second argument of f in (5-1) is from $\mathscr{R}(\mathbb{A})$. We assume that

$$f : \overline{\Omega} \times \mathscr{R}(\mathbb{A}) \to \mathbb{R} \quad \text{is continuous} \tag{5-2}$$

and satisfies the following linear growth assumption

$$c_1 |z| \leq f(x, z) \leq c_2 |z| + c_3 \tag{5-3}$$

for all $x \in \Omega$ and $z \in \mathscr{R}(\mathbb{A})$. Moreover, we require $\mathbb{A}$ to be $\mathbb{C}$-elliptic, which allows us to use the trace results of the previous sections.

Furthermore, we assume that there exists a modulus of continuity ω such that

$$|f(x, A) - f(y, A)| \leq \omega(|x - y|)(1 + |A|) \tag{5-4}$$

holds for all $x, y \in \overline{\Omega}$ and all $A \in \mathscr{R}(\mathbb{A})$. In all of what follows, we tacitly stick to these assumptions.

We say that $g : \mathscr{R}(\mathbb{A}) \to \mathbb{R}$ is $\mathbb{A}$-*quasiconvex* if for all $\varphi \in W_0^{1,\infty}((0,1)^n; \mathbb{R}^N)$ and $A \in \mathscr{R}(\mathbb{A})$ there holds

$$g(A) \leq \int_{(0,1)^n} g(A + \mathbb{A}\varphi) \, dx. \tag{5-5}$$

We say that $f : \overline{\Omega} \times \mathscr{R}(\mathbb{A}) \to \mathbb{R}$ is $\mathbb{A}$-quasiconvex if $f(x, \cdot)$ is $\mathbb{A}$-quasiconvex for each $x \in \overline{\Omega}$.

Let us link this notion of quasiconvexity to that of [Fonseca and Müller 1999, Definition 3.1]. Since $\mathbb{A}$ is $\mathbb{C}$-elliptic, it is also $\mathbb{R}$-elliptic. So by [Van Schaftingen 2013, Proposition 4.2], there exists $M \in \mathbb{N}$ and a linear, homogeneous constant-coefficient differential operator $\mathbb{L}$ with symbol mapping $\mathbb{L}[\xi]$ from $\mathbb{R}^K$ to $\mathbb{R}^M$ that *annihilates* $\mathbb{A}$ in the sense that the corresponding symbol complex

$$\mathbb{R}^N \xrightarrow{\mathbb{A}[\xi]} \mathbb{R}^K \xrightarrow{\mathbb{L}[\xi]} \mathbb{R}^M \tag{5-6}$$

is exact for every $\xi \in \mathbb{R}^n \setminus \{0\}$. In this situation, $\mathbb{A}$ is called a *potential* for $\mathbb{L}$, and $\mathbb{L}$ an *annihilator* for $\mathbb{A}$. Since $\mathbb{A}[\xi](\mathbb{R}^N)$ has the same dimension for all $\xi \neq 0$, the operator $\mathbb{L}$ has constant rank. Consequently, our $\mathbb{A}$-quasiconvexity equals the $\mathbb{L}$-quasiconvexity[3] of [Fonseca and Müller 1999]. By exactness of the above symbol complex (5-6), it is easy to see that the *wave cone* (or characteristic cone) $\Lambda_{\mathbb{L}} := \bigcup_{\xi \in \mathbb{R}^n \setminus \{0\}} \ker(\mathbb{L}[\xi])$ of $\mathbb{L}$ agrees with our $\mathbb{A}$-rank-one cone $\mathscr{C}(\mathbb{A})$.

We define the *strong recession function* $f^{\infty} : \overline{\Omega} \times \mathscr{R}(\mathbb{A}) \to \mathbb{R}$ by

$$f^{\infty}(x, A) := \lim_{\substack{x' \to x \\ A' \to A \\ t \to \infty}} \frac{f(x', tA')}{t}, \tag{5-7}$$

whenever the limit exists.

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[3]In [Fonseca and Müller 1999], first-order annihilating operators are considered, and in general this is not the case in our situation (e.g., the symmetric gradient is annihilated by curl curl). However, the generalisation of the concept of $\mathbb{L}$-quasiconvexity extends to higher-order operators $\mathbb{L}$ in the obvious manner.

Since f is $\mathbb{A}$-quasiconvex, satisfies the linear growth condition (5-3), and satisfies the continuity condition (5-4), Lemma A.1 from the Appendix yields that f^∞ is automatically well-defined on $\overline{\Omega} \times \mathscr{C}(\mathbb{A})$.

As usual the Dirichlet class $u_0 + W_0^{\mathbb{A},1}(\Omega)$ is not large enough to ensure the existence of minimisers for variational problems with linear growth. Here, the passage to $\mathrm{BV}^{\mathbb{A}}(\Omega)$ allows us to access the necessary sequential compactness. However, elements of $\mathrm{BV}^{\mathbb{A}}(\Omega)$ do not admit control over their *exterior* trace. To overcome this problem we proceed as in [Giaquinta et al. 1979a; 1979b] and pass to a larger superset U, i.e., let $\Omega \Subset U$ with ∂U Lipschitz. Now, we extend $\mathfrak{F}$ to $\mathrm{BV}^{\mathbb{A}}(U)$ and minimise over those $u \in \mathrm{BV}^{\mathbb{A}}(U)$ which agree with u_0 on $U \setminus \Omega$. For this, we further need to accomplish the following: First, we have to extend $f : \overline{\Omega} \times \mathscr{R}(\mathbb{A}) \to \mathbb{R}$ to $f : \overline{U} \times \mathscr{R}(\mathbb{A}) \to \mathbb{R}$, while preserving the structure of f; see Lemma A.2 in the Appendix. Second, we need to extend our boundary data to U, which is always possible, since $\mathrm{tr}(W^{\mathbb{A},1}(\Omega)) = L^1(\partial\Omega, \mathcal{H}^{n-1}) = \mathrm{tr}(W^{1,1}(U \setminus \Omega))$ by Theorem 4.17. In particular, we assume in the following that $u_0 \in W^{\mathbb{A},1}(U)$.

We define the functional $\overline{\mathfrak{F}}_U : \mathrm{BV}^{\mathbb{A}}(U) \to \mathbb{R}$ by

$$\overline{\mathfrak{F}}_U[w] := \int_U f\left(x, \frac{\mathrm{d}\mathbb{A}w}{\mathrm{d}\mathscr{L}^n}\right) \mathrm{d}x + \int_U f^\infty\left(x, \frac{\mathrm{d}\mathbb{A}w}{\mathrm{d}|\mathbb{A}^s w|}\right) \mathrm{d}|\mathbb{A}^s w|$$

and the Dirichlet class

$$\mathcal{D}_{u_0} = \{w \in \mathrm{BV}^{\mathbb{A}}(U) : w = u_0 \text{ on } U \setminus \overline{\Omega}\}.$$

Hence, our aim is to minimise $\mathfrak{F}_U$ over $\mathcal{D}_{u_0}$. Later we will see that this minimisation can also be expressed only in terms of $\mathrm{BV}^{\mathbb{A}}(\Omega)$ with an additional term $f^\infty(\cdot, \mathrm{tr}(u - u_0) \otimes_{\mathbb{A}} \nu)$ which penalises the deviations from the correct boundary values; see Theorem 5.3.

We begin with a characterisation of the extension of $\mathfrak{F} : W^{\mathbb{A},1}(\Omega) \to \mathbb{R}$ to $\mathrm{BV}^{\mathbb{A}}(\Omega)$. For this, recall that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and that (5-2)–(5-5) are in action.

Proposition 5.1. *The functional* $\overline{\mathfrak{F}} : \mathrm{BV}^{\mathbb{A}}(\Omega) \to \mathbb{R}$ *given by*

$$\overline{\mathfrak{F}}[u] := \int_\Omega f\left(x, \frac{\mathrm{d}\mathbb{A}u}{\mathrm{d}\mathscr{L}^n}\right) \mathrm{d}x + \int_\Omega f^\infty\left(x, \frac{\mathrm{d}\mathbb{A}u}{\mathrm{d}|\mathbb{A}^s u|}\right) \mathrm{d}|\mathbb{A}^s u|$$

is the $\mathbb{A}$-area strict continuous extension of $\mathfrak{F} : W^{\mathbb{A},1}(\Omega) \to \mathbb{R}$. Moreover, $\overline{\mathfrak{F}}[u] : \mathrm{BV}^{\mathbb{A}}(\Omega) \to \mathbb{R}$ is sequentially weak*-lower semicontinuous on $\mathrm{BV}^{\mathbb{A}}(\Omega)$.

Proof. We begin with the $\mathbb{A}$-area strict continuity of $\overline{\mathfrak{F}} : \mathrm{BV}^{\mathbb{A}}(\Omega) \to \mathbb{R}$. If f^∞ existed on all of $\overline{\Omega} \times \mathscr{R}(\mathbb{A})$, we could just use [Kristensen and Rindler 2010b, Theorem 4]. However, we can only rely on the existence of f^∞ on $\overline{\Omega} \times \mathscr{C}(\mathbb{A})$ due to Lemma A.1 from the Appendix. The following steps show how to overcome this technical issue and hence how the argument of [Kristensen and Rindler 2010b, Theorem 4] can be made to work.

Let us denote by $E(\overline{\Omega}, \mathscr{R}(\mathbb{A}))$ those functions $g : \overline{\Omega} \times \mathscr{R}(\mathbb{A}) \to \mathbb{R}$ such that

$$(x, \xi) \mapsto (1 - |\xi|)g(x, (1 - |\xi|)^{-1}\xi)$$

has a continuous extension to $\overline{\Omega \times \mathbb{B}_K}$; here, $\mathbb{B}_K$ denotes the unit ball in $\mathscr{R}(\mathbb{A})$. In particular, the strong recession function g^∞ exists on all of $\overline{\Omega} \times \mathscr{R}(\mathbb{A})$. Functionals with integrands from $E(\overline{\Omega}, \mathscr{R}(\mathbb{A}))$ enjoy good continuity properties.

Due to [Alibert and Bouchitté 1997, Lemma 2.3] there exists a sequence $f_k \in E(\overline{\Omega}, \mathscr{R}(\mathbb{A}))$ with

$$\sup_{k \in \mathbb{N}} f_k(x, A) = f(x, A) \quad \text{and} \quad \sup_{k \in \mathbb{N}} f_k^\infty(x, A) = f_\#(x, A) := \liminf_{\substack{x' \to x \\ A' \to A \\ t \to \infty}} \frac{f(x', tA')}{t}. \tag{5-8}$$

Let $u_j \xrightarrow{\langle \cdot \rangle} u$ in $BV^{\mathbb{A}}(\Omega)$. Since $f_k \in E(\overline{\Omega}, \mathscr{R}(\mathbb{A}))$ we may apply the Reshetnyak-type continuity theorem in [Kristensen and Rindler 2010b, Theorem 5] to conclude

$$\liminf_{j \to \infty} \overline{\mathfrak{F}}[u_j] \geq \liminf_{j \to \infty} \int_\Omega f_k\left(x, \frac{d\mathbb{A}u_j}{d\mathscr{L}^n}\right) dx + \int_\Omega f_k^\infty\left(x, \frac{d\mathbb{A}^s u_j}{d|\mathbb{A}^s u_j|}\right) d|\mathbb{A}^s u_j|$$

$$= \int_\Omega f_k\left(x, \frac{d\mathbb{A}u}{d\mathscr{L}^n}\right) dx + \int_\Omega f_k^\infty\left(x, \frac{d\mathbb{A}^s u}{d|\mathbb{A}^s u|}\right) d|\mathbb{A}^s u|$$

and so, by monotone convergence,

$$\int_\Omega f\left(x, \frac{d\mathbb{A}u}{d\mathscr{L}^n}\right) dx + \int_\Omega f_\#\left(x, \frac{d\mathbb{A}u}{d|\mathbb{A}^s u|}\right) d|\mathbb{A}^s u| \leq \liminf_{j \to \infty} \overline{\mathfrak{F}}[u_j].$$

Due to the generalisation of Alberti's celebrated rank-one theorem in [De Philippis and Rindler 2016], we know that $d\mathbb{A}u/d|\mathbb{A}^s u| \in \mathscr{C}(\mathbb{A})$ pointwisely $|\mathbb{A}^s u|$-a.e. Now, by Lemma A.1 from the Appendix, we find that $f_\# = f^\infty$ on $\overline{\Omega} \times \mathscr{C}(\mathbb{A})$. Hence

$$\overline{\mathfrak{F}}[u] = \int_\Omega f\left(x, \frac{d\mathbb{A}u}{d\mathscr{L}^n}\right) dx + \int_\Omega f^\infty\left(x, \frac{d\mathbb{A}u}{d|\mathbb{A}^s u|}\right) d|\mathbb{A}^s u| \leq \liminf_{j \to \infty} \overline{\mathfrak{F}}[u_j].$$

Since f is continuous, we may apply the same argument to $-f$ to obtain $\overline{\mathfrak{F}}[u] \geq \limsup_{j \to \infty} \overline{\mathfrak{F}}[u_j]$. Hence $\overline{\mathfrak{F}}[u] = \lim_{j \to \infty} \overline{\mathfrak{F}}[u_j]$. This proves that $\overline{\mathfrak{F}} : BV^{\mathbb{A}}(\Omega) \to \mathbb{R}$ is $\mathbb{A}$-area strictly continuous.

Due to Lemma 4.15, $W^{\mathbb{A},1}(\Omega)$ is dense in $BV^{\mathbb{A}}(\Omega)$ with respect to $\mathbb{A}$-area strict convergence. Since $\overline{\mathfrak{F}} = \mathfrak{F}$ on $W^{\mathbb{A},1}(\Omega)$, we see that $\overline{\mathfrak{F}} : BV^{\mathbb{A}}(\Omega) \to \mathbb{R}$ is the $\mathbb{A}$-area strict extension of $\mathfrak{F} : W^{\mathbb{A},1}(\Omega) \to \mathbb{R}$.

It remains to prove the sequential weak*-lower semicontinuity of $\overline{\mathfrak{F}} : BV^{\mathbb{A}}(\Omega) \to \mathbb{R}$ on $BV^{\mathbb{A}}(\Omega)$. Let $\mathbb{L}$ be an $\mathbb{A}$-annihilating operator as in the exact sequence (5-6). Now, the sequential weak*-lower semicontinuity just follows from [Arroyo-Rabasa et al. 2018, Theorem 1.2] (note that f^∞ is well-defined on $\overline{\Omega} \times \mathscr{C}(\mathbb{A})$ due to Lemma A.1 from the Appendix). $\qquad \square$

If we apply to our Dirichlet class $\mathcal{D}_{u_0}$, then we obtain the following results:

Corollary 5.2. *Let f satisfy (5-2)–(5-5) and let $\overline{\mathfrak{F}}_{u_0} : BV^{\mathbb{A}}(\Omega) \to \mathbb{R}$, given by*

$$\overline{\mathfrak{F}}_{u_0}[u] := \int_\Omega f\left(x, \frac{d\mathbb{A}u}{d\mathscr{L}^n}\right) d\mathscr{L}^n + \int_\Omega f^\infty\left(x, \frac{d\mathbb{A}u}{d|\mathbb{A}^s u|}\right) d|\mathbb{A}^s u| + \int_{\partial\Omega} f^\infty\left(x, \nu_{\partial\Omega} \otimes_{\mathbb{A}} \text{tr}(u - u_0)\right) d\mathcal{H}^{n-1}, \tag{5-9}$$

be sequentially weak*-lower semicontinuous on $BV^{\mathbb{A}}(\Omega)$.

Proof. Proposition 5.1 (applied with Ω replaced by U) shows that $\overline{\mathfrak{F}}_U : \mathrm{BV}^{\mathbb{A}}(U) \to \mathbb{R}$ is area-strictly continuous on $\mathrm{BV}^{\mathbb{A}}(U)$ and sequentially weak*-lower semicontinuous on $\mathrm{BV}^{\mathbb{A}}(U)$.

For $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$ let $\tilde{u} := \chi_{U \setminus \overline{\Omega}} u_0 + \chi_{\Omega} u$. Then due to Corollary 4.21 we have $\tilde{u} \in \mathrm{BV}^{\mathbb{A}}(U)$ and, with the outer normal ν of Ω,

$$\mathbb{A}\tilde{u} = \mathbb{A}u \llcorner \Omega + \mathbb{A}u_0 \mathscr{L}^n \llcorner (U \setminus \overline{\Omega}) + \mathrm{tr}(u - u_0) \otimes_{\mathbb{A}} \nu \mathcal{H}^{n-1} \llcorner \partial\Omega. \tag{5-10}$$

Hence,

$$\overline{\mathfrak{F}}_U[\tilde{u}] = \overline{\mathfrak{F}}_{u_0}[u] + \int_{U \setminus \overline{\Omega}} f(x, \mathbb{A}u_0) \, \mathrm{d}x. \tag{5-11}$$

If $u_k \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}^{\mathbb{A}}(\Omega)$, then $\tilde{u}_k \overset{*}{\rightharpoonup} \tilde{u}$ in $\mathrm{BV}^{\mathbb{A}}(U)$. Indeed, it is clear that $u_k \to u$ in $L^1(U)$. Moreover, since u_k is bounded in $\mathrm{BV}^{\mathbb{A}}(\Omega)$, so is $\mathbb{A}u_k \in \mathcal{M}(\Omega)$ and $\mathrm{tr}(u_k)$ in $L^1(\partial\Omega)$ (using the trace theorem, Theorem 4.17). This and (5-10) show that $\tilde{u}_k$ is bounded in $\mathrm{BV}^{\mathbb{A}}(U)$. In conjunction with $u_k \to u$ in $L^1(U)$ we obtain $\tilde{u}_k \overset{*}{\rightharpoonup} \tilde{u}$ in $\mathrm{BV}^{\mathbb{A}}(U)$.

Since $\overline{\mathfrak{F}}_U$ is sequentially weak*-lower semicontinuous on $\mathrm{BV}^{\mathbb{A}}(U)$, it follows that $\overline{\mathfrak{F}}_{u_0}$ sequentially weak*-lower semicontinuous on $\mathrm{BV}^{\mathbb{A}}(\Omega)$. $\qquad\square$

Theorem 5.3. *Let f satisfy* (5-2)–(5-5). *Then the functional* $\overline{\mathfrak{F}}_{u_0} : \mathrm{BV}^{\mathbb{A}}(\Omega) \to \mathbb{R}$ *is coercive and has a minimiser on* $\mathrm{BV}^{\mathbb{A}}(\Omega)$. *Moreover, we have*

$$\min_{\mathrm{BV}^{\mathbb{A}}(\Omega)} \overline{\mathfrak{F}}_{u_0} = \inf_{u_0 + W_0^{\mathbb{A},1}(\Omega)} \mathfrak{F}. \tag{5-12}$$

Proof. We begin with the coerciveness of $\overline{\mathfrak{F}}_{u_0}$. Let $(v_k) \subset \mathrm{BV}^{\mathbb{A}}(\Omega)$ with $(\overline{\mathfrak{F}}_{u_0}(u_k))$ bounded. We have to show that (v_k) is bounded in $\mathrm{BV}^{\mathbb{A}}(\Omega)$. Let $\tilde{v}_k := \chi_{U \setminus \overline{\Omega}} u_0 + \chi_{\Omega} v_k$ as in Corollary 5.2. Then due to (5-11), $\overline{\mathfrak{F}}_U(\tilde{v}_k)$ is bounded. By the linear growth condition (5-3) we see that $(\mathbb{A}v_k)$ is uniformly bounded in $\mathcal{M}(U; \mathbb{R}^K)$. Now choose a ball $B' \subset \Omega$ and another ball B with $U \subset B$. Since $v_k - u_0 = 0$ on $U \setminus \overline{\Omega}$, we can extend it by zero to a function from $\mathrm{BV}^{\mathbb{A}}(B)$ due to Theorem 4.23(b). Now, we can apply Poincaré's inequality in the form of Theorem 3.3 to conclude that (v_k) is also bounded in $L^1(U)$. Hence, (v_k) is bounded on $\mathrm{BV}^{\mathbb{A}}(\Omega)$, which is the desired coerciveness.

By positivity of f and f^{∞}, we have $\overline{\mathfrak{F}}_{u_0}[w] \geq 0$ for all $w \in \mathrm{BV}^{\mathbb{A}}(\Omega)$, and so we may pick a minimising sequence (u_k) in $\mathrm{BV}^{\mathbb{A}}(\Omega)$. By coerciveness, this sequence is bounded in $\mathrm{BV}^{\mathbb{A}}(\Omega)$. We can pick a (nonrelabeled) subsequence such that $u_k \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}^{\mathbb{A}}(\Omega)$ for some $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$. By the sequential weak*-lower semicontinuity from Corollary 5.2, we deduce that u is a minimiser of $\overline{\mathfrak{F}}_{u_0}$.

We conclude the proof by showing (5-12). The "$\leq$"-part is obvious. Due to Proposition 4.24 we find a sequence $w_k \in \mathcal{D}_{u_0}$ such that $w_k \xrightarrow{\langle\cdot\rangle} u$ in $\mathrm{BV}^{\mathbb{A}}(U)$. By the $\mathbb{A}$-area-strict continuity of $\overline{\mathfrak{F}}_U$ on $\mathrm{BV}^{\mathbb{A}}(U)$, see Proposition 5.1, we see that $\overline{\mathfrak{F}}_U(u) = \lim_{k\to\infty} \overline{\mathfrak{F}}_U(w_k)$. This and (5-11) prove the "$\geq$"-part of (5-12). $\quad\square$

Appendix

We now collect some auxiliary results that have been used in the main part of the paper. The following lemma shows that the recession function is automatically well-defined on the $\mathbb{A}$-rank-one cone.

Lemma A.1. *Let $\mathbb{A}$ be $\mathbb{R}$-elliptic, let $f : \overline{\Omega} \times \mathscr{R}(\mathbb{A}) \to \mathbb{R}$ be $\mathbb{A}$-quasiconvex in the sense of (5-5), satisfy the linear growth condition (5-3), and satisfy the continuity condition (5-4). Then $f(x, \cdot)$ is Lipschitz continuous in $\mathscr{R}(\mathbb{A})$ uniformly in $x \in \overline{\Omega}$. Moreover, the strong recession function $f^\infty : \overline{\Omega} \times \mathscr{R}(\mathbb{A}) \to \mathbb{R}$ with*

$$f^\infty(x, A) := \lim_{\substack{x' \to x \\ A' \to A \\ t \to \infty}} \frac{f(x', t A')}{t}$$

is well-defined on $\overline{\Omega} \times \mathscr{C}(\mathbb{A})$. (Note that the limit $A' \to A$ is taken in $\mathscr{R}(\mathbb{A})$.) Moreover,

$$|f^\infty(x, A) - f^\infty(x', A)| \le \omega(|x' - x|)|A|$$

for all $x, x' \in \overline{\Omega}$ and $A \in \mathscr{C}(\mathbb{A})$.

Proof. We begin with the Lipschitz continuity of f on $\mathscr{R}(\mathbb{A})$.

Let $A \in \mathscr{R}(\mathbb{A})$ and $B = a \otimes_\mathbb{A} b \in \mathscr{C}(\mathbb{A})$. Since f is $\mathbb{A}$-quasiconvex, it is a consequence[4] of [Fonseca and Müller 1999, Proposition 3.4] that $t \mapsto f(x, A + t B)$ is convex on $\mathbb{R}$. This property is known as $\mathscr{C}(\mathbb{A})$-convexity; see [Kirchheim and Kristensen 2016].

Thus the function

$$g(t) := \frac{|f(x, A + t a \otimes_\mathbb{A} b) - f(x, A)|}{t}$$

is increasing. Hence, with $\lambda := (1 + |A + B| + |A|)/|B| > 1$, we obtain

$$|f(x, A + B) - f(x, A)| = g(1) \le g(\lambda)$$

$$\le |f(x, A + \lambda a \otimes_\mathbb{A} b) - f(x, A)| \frac{|B|}{1 + |A + B| + |A|}$$

$$\le \frac{c_2(2|A| + \lambda|B|) + 2c_3}{1 + |A + B| + |A|} |B|$$

$$\le \frac{c_2(1 + 3|A| + |A + B|) + 2c_3}{1 + |A + B| + |A|} |B|$$

$$\le (3c_2 + 2c_3)|B|$$

using (5-3). This proves the Lipschitz continuity in $\mathscr{C}(\mathbb{A})$-directions.

If $B \in \mathscr{R}(\mathbb{A})$, then by $\mathscr{R}(\mathbb{A}) = \operatorname{span}(\mathscr{C}(\mathbb{A}))$ we can decompose B into at most K summands from $\mathscr{C}(\mathbb{A})$. Now the Lipschitz continuity in $\mathscr{C}(\mathbb{A})$-directions implies

$$|f(x, A + B) - f(x, A)| \le K(3c_2 + 2c_3)|B| \tag{A-1}$$

for all $A, B \in \mathscr{R}(\mathbb{A})$. This proves the Lipschitz continuity part.

Let $A \in \mathscr{C}(\mathbb{A})$ and $x \in \overline{\Omega}$. Then $t \mapsto (f(x, t A) - f(x, 0))/t$ is increasing in t by $\mathscr{C}(\mathbb{A})$-convexity of $f(x, \cdot)$ and bounded by $c_2|A|$ due to the linear growth condition (5-3). This allows us to define

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[4]As proven in [Fonseca and Müller 1999], if $\mathcal{A}$ is a first-order linear homogeneous differential operator, then $\mathcal{A}$-quasiconvex functions are $\Lambda_\mathcal{A}$-convex. Note that in our setting, $\mathbb{L} = \mathcal{A}$ need not be first of first order; however, their arguments extend to the case of higher-order annihilating operators $\mathbb{A}$ in a straightforward manner.

$g^\infty : \overline{\Omega} \times \mathscr{C}(\mathbb{A}) \to \mathbb{R}$ by

$$g^\infty(x, A) = \lim_{t \to \infty} \frac{f(x, tA)}{t} = \sup_{t>0} \frac{f(x, tA)}{t}.$$

Now, let $A' \in \mathscr{R}(\mathbb{A})$ and $x' \in \overline{\Omega}$; then by (A-1) and (5-4)

$$\left| \frac{f(x', tA')}{t} - \frac{f(x, tA)}{t} \right| \leq \left| \frac{f(x', tA') - f(x', tA)}{t} \right| + \left| \frac{f(x', tA) - f(x, tA)}{t} \right|$$

$$\leq K(3c_2 + 2c_3)|A - A'| + \omega(|x' - x|)\frac{1 + t|A|}{t}.$$

This proves $f^\infty(x, A) = g^\infty(x, A)$ for all $x \in \overline{\Omega}$ and $A \in \mathscr{C}(\mathbb{A})$. Consequently, we obtain the existence of f^∞ in $\overline{\Omega} \times \mathscr{C}(\mathbb{A})$.

The continuity of $f^\infty(\,\cdot\,, A)$ for $A \in \mathscr{C}(\mathbb{A})$ is a direct consequence of the continuity of $f(\,\cdot\,, A)$.   □

Lemma A.2. *Let $\mathbb{A}$ be $\mathbb{R}$-elliptic, and let $f : \overline{\Omega} \times \mathscr{R}(\mathbb{A}) \to \mathbb{R}$ be $\mathbb{A}$-quasiconvex in the sense of* (5-5), *satisfy the linear growth condition* (5-3), *and satisfy the continuity condition* (5-4). *Furthermore, let $\Omega \Subset U$ with ∂U Lipschitz. Then there exists an extension $\tilde{f} : \overline{U} \times \mathscr{R}(\mathbb{A}) \to \mathbb{R}$ of f, which is $\mathbb{A}$-quasiconvex, satisfies the linear growth condition* (5-3), *and satisfies the continuity condition* (5-4). (*The modulus of continuity might change by a factor.*)

Proof. Since ∂U and $\partial \Omega$ are Lipschitz, we find a Lipschitz map $\Phi : \overline{U} \to \overline{\Omega}$, which is the identity on $\overline{\Omega}$. Now define $\tilde{f}(x, A) := f(\Phi(x), A)$.   □

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