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## REGULARITY OF THE FREE BOUNDARY FOR THE VECTORIAL BERNOLLI PROBLEM

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We study the regularity of the free boundary for a vector-valued Bernoulli problem, with no sign assumptions on the boundary data. More precisely, given an open, smooth set of finite measure  $D \subset \mathbb{R}^d$ ,  $\Lambda > 0$ , and  $\phi_i \in H^{1/2}(\partial D)$ , we deal with

$$\min \left\{ \sum_{i=1}^k \int_D |\nabla v_i|^2 + \Lambda \left| \bigcup_{i=1}^k \{v_i \neq 0\} \right| : v_i = \phi_i \text{ on } \partial D \right\}.$$

We prove that, for any optimal vector  $U = (u_1, \dots, u_k)$ , the free boundary  $\partial(\bigcup_{i=1}^k \{u_i \neq 0\}) \cap D$  is made of a regular part, which is relatively open and locally the graph of a  $C^\infty$  function, a (one-phase) singular part, of Hausdorff dimension at most  $d - d^*$ , for a  $d^* \in \{5, 6, 7\}$ , and by a set of branching (two-phase) points, which is relatively closed and of finite  $\mathcal{H}^{d-1}$  measure. For this purpose we shall exploit the NTA property of the regular part to reduce ourselves to a scalar one-phase Bernoulli problem.

### 1. Introduction

Dirichlet problems with free boundary arise in several models describing relevant physical phenomena, such as for example thermal insulation, or flows with two liquids with jets and cavities, and have been the object of an extensive mathematical study in the last decades, starting from the seminal work [Alt and Caffarelli 1981]. The wide literature on these topics has provided many new techniques which have turned out to be extremely useful also in very different fields; see, e.g., [Caffarelli and Salsa 2005]. In two recent papers [Caffarelli et al. 2018; Mazzoleni et al. 2017] the authors dealt with problems connected with vector-valued Bernoulli-type problems, under the assumption that at least one of the components has constant sign, leaving as a main open question the opposite case (we refer to [Spolaor and Velichkov 2019] for the complete analysis in dimension 2). In this paper we give a partial answer to this question, proving some regularity of the free boundary in any dimension and without any assumption on the sign of the components. A key step is to show that in a neighborhood of any flat point, that is, a point of Lebesgue density  $\frac{1}{2}$ , at least one of the components has in fact a constant sign.

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To start with, given a smooth open set  $D \subset \mathbb{R}^d$ ,  $\Lambda > 0$ , and  $\Phi = (\phi_1, \dots, \phi_k) \in H^{1/2}(\partial D; \mathbb{R}^k)$ , that is,  $\phi_i \in H^{1/2}(\partial D)$  for  $i = 1, \dots, k$ , we consider the vectorial free boundary problem

$$\min \left\{ \int_D |\nabla U|^2 dx + \Lambda |\Omega_U| : U \in H^1(D; \mathbb{R}^k), U = \Phi \text{ on } \partial D \right\}, \quad (1-1)$$

where, for a vector-valued function  $U = (u_1, \dots, u_k) : D \rightarrow \mathbb{R}^k$ , we use the notation

$$|U| := \sqrt{u_1^2 + \dots + u_k^2}, \quad |\nabla U|^2 := \sum_{i=1}^k |\nabla u_i|^2, \quad \text{and} \quad \Omega_U := \{|U| > 0\} = \bigcup_{i=1}^k \{u_i \neq 0\} \subset D.$$

We will refer to the set  $\partial\Omega_U \cap D$  as the *free boundary* given by  $U$ . More precisely, for any minimizer  $U$  we shall prove that  $\Omega_U$  is open and that any boundary point has a definite density. We denote by  $\Omega_U^{(\gamma)}$  the set of all points having density  $\gamma \in [0, 1]$ . Hence we can divide the topological boundary into three parts:

$$\begin{aligned} \text{Reg}(\partial\Omega_U) &:= \Omega_U^{(1/2)} \cap D, \quad \text{Sing}_2(\partial\Omega_U) := \Omega_U^{(1)} \cap \partial\Omega_U \cap D, \\ \text{Sing}_1(\partial\Omega_U) &:= (\partial\Omega_U \cap D) \setminus (\text{Sing}_2(\partial\Omega_U) \cup \text{Reg}(\partial\Omega_U)). \end{aligned}$$

The main result of this paper is the following:

**Theorem 1.1.** *There exists a solution to problem (1-1). Any solution  $U \in H^1(D; \mathbb{R}^k)$  is Lipschitz continuous in  $D \subset \mathbb{R}^d$  and the set  $\Omega_U$  has a locally finite perimeter in  $D$ . The free boundary  $\partial\Omega_U \cap D$  is the union of three disjoint sets: a regular part  $\text{Reg}(\partial\Omega_U)$ , a (one-phase) singular set  $\text{Sing}_1(\partial\Omega_U)$  and a set of branching points  $\text{Sing}_2(\partial\Omega_U)$ . Moreover, we have:*

- (1) *The regular part  $\text{Reg}(\partial\Omega_U)$  is an open subset of  $\partial\Omega_U$  and is locally the graph of a  $C^\infty$  function.*
- (2) *The one-phase singular set  $\text{Sing}_1(\partial\Omega_U)$  consists only of points in which the Lebesgue density of  $\Omega_U$  is strictly between  $\frac{1}{2}$  and 1. Moreover, there is a  $d^* \in \{5, 6, 7\}$  such that*
  - *if  $d < d^*$ , then  $\text{Sing}_1(\partial\Omega_U)$  is empty;*
  - *if  $d = d^*$ , then the singular set  $\text{Sing}_1(\partial\Omega_U)$  contains at most a finite number of isolated points;*
  - *if  $d > d^*$ , then the  $(d-d^*)$ -dimensional Hausdorff measure of  $\text{Sing}_1(\partial\Omega_U)$  is locally finite in  $D$ .*
- (3) *The set of branching points  $\text{Sing}_2(\partial\Omega_U)$  is a closed set of locally finite  $(d-1)$ -Hausdorff measure in  $D$  and consists only of points in which the Lebesgue density of  $\Omega_U$  is 1 and the blow-up limits are linear functions.*

Our main improvement with respect to the results of the quoted papers [Caffarelli et al. 2018; Kriventsov and Lin 2018; Mazzoleni et al. 2017] is the smoothness of  $\text{Reg}(\partial\Omega_U)$  regardless of the sign of the boundary data. The dimension estimates of the singular set  $\text{Sing}_1(\partial\Omega_U)$  from claim (2) are already known (see Section 1B below) as they do not depend on the constant-sign assumption on the solution. The analysis of the blow-up limits and the dimension of  $\text{Sing}_2(\partial\Omega_U)$  from point (3) are, to our knowledge, new for the vectorial problem, as is the stratification result Theorem 4.3 that we prove in Section 4.

**1A. Further results on the set of branching points  $\text{Sing}_2(\partial\Omega_U)$ .** Under the assumption that one of the components of the optimal vector has constant sign [Caffarelli et al. 2018; Mazzoleni et al. 2017], all 1-homogeneous singular solutions are multiples of the same global solution for the one-phase scalar problem. In this case, the singular set of  $\partial\Omega_U$  is given precisely by  $\text{Sing}_1(\partial\Omega_U)$ . Without the constant-sign assumption, the structure of the singular set changes drastically. A set  $\text{Sing}_2(\partial\Omega_U)$  of branching points, in which the free boundary may form cusps pointing inwards, might appear. This is natural since the scalar case corresponds to the two-phase Bernoulli problem, for which this is a well-known, though not completely understood, phenomenon. In particular, the dimension of this set of branching points can be as large as the dimension of the regular free boundary.<sup>1</sup> This is somehow natural since for the two-phase case the branching points are contact points of the two level sets  $\{u > 0\}$  and  $\{u < 0\}$ .

The free boundaries around branching points for the vectorial problem have more complex structure. Indeed, even in dimension 2, true cusps may appear on the free boundary, that is, around a branching point  $x_0$ , the set  $B_r(x_0) \cap \Omega_U$  might stay connected, while the Lebesgue density  $|B_r \setminus \Omega_U|/|B_r|$  might decay as  $r$  goes to zero (see [Spolaor and Velichkov 2019] for an example of such a free boundary). On the other hand, the nodal set may also degenerate into linear subspace of codimension higher than 1 (see Lemma 2.7 for an example of homogeneous solution with a thin nodal set). In Section 2D we classify the blow-up limits. In Section 4, using a Federer reduction principle, we prove a stratification result, Theorem 4.3, for the branching points, which in particular shows that the only significant (in terms of Hausdorff measure) set of branching points is the one for which the nodal set degenerates into a  $(d-1)$ -dimensional plane.

**1B. Remarks on the one-phase singular set  $\text{Sing}_1(\partial\Omega_U)$ .** The critical dimension  $d^*$  is the lowest dimension at which the free boundaries of the one-phase scalar Alt–Caffarelli problem [1981] admit singularities. Caffarelli, Jerison and Kenig proved in [Caffarelli et al. 2004] that  $d^* \geq 4$ , Jerison and Savin [2015] showed that  $d^* \geq 5$ , while De Silva and Jerison [2009] gave an example of a singular minimal cone in dimension 7, so  $d^* \in \{5, 6, 7\}$ . The first claim of Theorem 1.1(2) follows by the fact that at points of the one-phase singular set  $\text{Sing}_1(\partial\Omega_U)$  the blow-up limits of the minimizers of (1-1) are multiples of a solution of the one-phase scalar Alt–Caffarelli problem (Section 2D). The second claim of Theorem 1.1(2) was proved in [Mazzoleni et al. 2017, Section 5.5] together with the Hausdorff dimension bound

$$\dim_{\mathcal{H}}(\text{Sing}_1(\partial\Omega_U)) \leq d - d^* \quad \text{for } d > d^*,$$

which follows by a dimension-reduction argument based on the Weiss' monotonicity formula [1999]. The last claim of Theorem 1.1(2) was proved in [Edelen and Engelstein 2019, Theorem 1.15] by a finer argument based on the quantitative dimension reduction of [Naber and Valtorta 2017; 2018]. We notice that [Edelen and Engelstein 2019] contains also a stratification result on  $\text{Sing}_1(\partial\Omega_U)$ .

**1C. Connection with shape optimization problems for the eigenvalues of the Dirichlet Laplacian.** The vectorial Bernoulli problem is strictly related to a whole class of shape optimization problems involving the eigenvalues of the Dirichlet Laplacian. In particular, suppose that  $U^* = (u_1^*, \dots, u_k^*)$  is the vector whose

<sup>1</sup> After this paper was accepted, the preprint [De Philippis et al. 2019] appeared, and in it a complete characterization of the two-phase free boundaries is achieved.

components are the Dirichlet eigenfunctions on the set  $\Omega^*$ , a solution of the shape optimization problem

$$\min \left\{ \sum_{j=1}^k \lambda_j(\Omega) : \Omega \subset \mathbb{R}^d, \Omega \text{ open}, |\Omega| = 1 \right\}.$$

It was proved in [Mazzoleni et al. 2017] that  $U^*$  is a quasiminimizer of (1-1). Thus, the regularity of the optimal set  $\Omega^*$  is strongly related to (not to say a consequence of) the regularity of the free boundaries of the solutions of (1-1). A result for more general functionals was proved in [Kriventsov and Lin 2018], still under some structural assumption on the free boundary. We highlight that, having the results of Sections 3A and 3B of the present paper in our hands, a possible way to obtain the  $C^{1,\alpha}$  regularity of  $\text{Reg}(\partial\Omega_U)$  is to apply [Kriventsov and Lin 2018, Theorem 1.4]. Our approach is different and relies on the application of [Mazzoleni et al. 2017, Theorem 1.4] after having proved that there is locally always a constant-sign component. The result of [Kriventsov and Lin 2018] was then extended by the same authors to general spectral functionals in [Kriventsov and Lin 2019]. The shape optimization problem considered in the latter paper corresponds to (1-1) with sign-changing components. On the other hand the nature of the spectral functionals forces the authors to take a very different road and use an approximation with functionals for which the constant-sign assumption is automatically satisfied. A delicate point concerns the selection of a special representative of the optimal set, which, roughly speaking, corresponds to the largest *quasiopen* set which solves the problem. The problem (1-1) allows a more direct approach and in particular our regularity result holds for the free boundary of *any* optimal vector and allows for a finer classification of the branching points.

**1D. Plan of the paper and sketch of the proof of Theorem 1.1.** Since the existence of an optimal vector is nowadays standard, we start Section 2A by proving the Lipschitz continuity of  $U$ , which follows by the fact that each component is quasiminimizer for the scalar Alt–Caffarelli functional and so, by [Bucur et al. 2015], is Lipschitz continuous. In Section 2B we prove that the positivity set  $\Omega_U$  has finite perimeter in  $D$  and that the  $(d-1)$ -Hausdorff measure of  $\partial\Omega_U$  is finite. Our argument is different from the classical approach of Alt and Caffarelli and is based on a comparison of the energy of the different level sets of  $|U|$ .

In Section 2C we summarize the convergence results on the blow-up sequences and Section 2D is dedicated to the classification of the blow-up limits, which are 1-homogeneous global minimizers (that is, globally defined local minimizers) of (1-1) (see Remark 2.9). In Lemma 2.7 we show that a new class of global minimizers appears with respect to the problem considered in [Caffarelli et al. 2018; Mazzoleni et al. 2017]. In Lemma 2.11 we classify the possible blow-up limits according to the Lebesgue density; this is the main result of the section. Finally, in Definition 2.12, we define the sets  $\text{Reg}(\partial\Omega_U)$ ,  $\text{Sing}_1(\partial\Omega_U)$ , and  $\text{Sing}_2(\partial\Omega_U)$ .

In Section 3 we prove the smoothness of  $\text{Reg}(\partial\Omega_U)$ . In Section 3A we prove that on the one-phase free boundary  $\text{Reg}(\partial\Omega_U) \cup \text{Sing}_1(\partial\Omega_U)$ ,  $U$  satisfies the extremality condition  $|\nabla|U|| = \sqrt{\Lambda}$  in a viscosity sense. In Section 3B we prove that  $\text{Reg}(\partial\Omega_U)$  is Reifenberg flat and NTA domain.

Section 3C deals with the proof that in a neighborhood of a point  $x_0 \in \text{Reg}(\partial\Omega_U)$  at least one of the components of  $U$  remains strictly positive and (up to a multiplicative constant) controls  $|U|$  (see Lemma 3.10). This is the main result of this section and the proof is based on the geometric properties of

NTA domains and on the boundary Harnack principle. In Sections 3D and 3E we prove that  $\text{Reg}(\partial\Omega_U)$  is, respectively,  $C^{1,\alpha}$  and  $C^\infty$ . The result of Lemma 3.10 allows us to apply the results from [Mazzoleni et al. 2017]. We give the main steps of the proof for the sake of completeness.

Section 4 is dedicated to the study of the set  $\text{Sing}_2(\partial\Omega_U)$  of points  $x_0 \in \partial\Omega_U$  in which all the blow-up limits  $U_0 \in \mathcal{BU}_U(x_0)$  are linear functions of the form  $U_0(x) = Ax$ . In Section 4A we prove that the rank of the linear map  $U_0$  depends only on the point  $x_0$  and we define the  $j$ -th stratum  $\mathcal{S}_j$  as the set of points for which this rank is precisely  $j$ . In Section 4B we use a dimension-reduction argument in the spirit of Federer to prove that the Hausdorff dimension of each stratum  $\mathcal{S}_j$  is  $d - j$ . Finally, in Section 4C we give a criterion for the uniqueness of the blow-up limits in terms of the Lebesgue density of  $\Omega_U$ .

## 2. Boundary behavior of the solutions

The existence of an optimal vector  $U = (u_1, \dots, u_k)$  is standard and follows by the direct method of the calculus of variations; for more details we refer to [Alt and Caffarelli 1981].

**2A. Lipschitz continuity and nondegeneracy.** Any minimizer  $U$  has the following properties:

- (i) The vector-valued function  $U : D \rightarrow \mathbb{R}^k$  is locally Lipschitz continuous in  $D$ .
- (ii) The real-valued function  $|U|$  is nondegenerate; i.e., there are constants  $c_0 > 0$  and  $r_0 > 0$  such that for every  $x_0 \in \partial\Omega_U \cap D$  and  $r \in (0, r_0]$  we have

$$\int_{B_r(x_0)} |U| dx < c_0 r \quad \Rightarrow \quad U \equiv 0 \quad \text{in } B_{r/2}(x_0). \quad (2-1)$$

- (iii) There are constants  $\varepsilon_0, r_0$  such that the *lower density estimate* holds:

$$\varepsilon_0 |B_r| \leq |\Omega_U \cap B_r(x_0)| \quad \text{for every } x_0 \in \partial\Omega_U \cap D \text{ and } r \leq r_0. \quad (2-2)$$

**Remark 2.1.** Claim (i) in particular implies that, for every minimizer  $U$  of (1-1), the set  $\Omega_U$  is open.

**Remark 2.2.** It is important to highlight that, unlike the case treated in [Mazzoleni et al. 2017; Caffarelli et al. 2018] where it was assumed at least one component  $u_i$  was positive, we cannot hope to have a density estimate *from above* on  $\partial\Omega_U \cap D$ . Actually, we expect a set of branching points (cusps) will come out. Indeed, the case  $k = 1$  corresponds to a scalar *two-phase* problem for which (at least in dimension 2) the set  $\Omega_U$  is composed of two  $C^{1,\alpha}$  sets; see [Spolaor and Velichkov 2019]. At the points of the common boundary of these two sets, the Lebesgue density of  $\Omega_U$  is 1.

*Proof of (i).* The Lipschitz continuity of each component  $u_i$ ,  $i = 1, \dots, k$ , descends from a quasiminimality property. Indeed, reasoning as in [Mazzoleni et al. 2017, Section 6.2], for every  $\tilde{u}_i : D \rightarrow \mathbb{R}$  such that  $\tilde{u}_i - u_i \in H_0^1(D)$  we consider the competitor  $\tilde{U} := (u_1, \dots, \tilde{u}_i, \dots, u_k)$ . By the optimality of  $U$  we have

$$\int_D |\nabla u_i|^2 dx + \Lambda |\{|U| > 0\}| \leq \int_D |\nabla \tilde{u}_i|^2 dx + \Lambda |\{|\tilde{U}| > 0\}|,$$

which implies that each component  $u_i$  is a quasiminimizer of the Dirichlet energy; that is,

$$\int |\nabla u_i|^2 dx \leq \int |\nabla \tilde{u}_i|^2 dx + \Lambda |B_r| \quad \text{for every } \tilde{u}_i \text{ such that } \tilde{u}_i - u_i \in H_0^1(B_r). \quad (2-3)$$

Applying [Bucur et al. 2015, Theorem 3.3] we get that  $u_i$  is Lipschitz continuous in  $D$ , and since  $i = 1, \dots, k$  is arbitrary, so is  $U$ .  $\square$

*Proof of (ii) and (iii).* The nondegeneracy of  $|U|$  follows by [Mazzoleni et al. 2017, Lemma 2.6], which can be applied since  $U$  satisfies the condition (2.9) therein with  $K = 0$  and  $\varepsilon > 0$ . Finally, we notice that the density estimate *from below* (2-2) holds for every Lipschitz function satisfying the nondegeneracy condition (2-1); see for example [Mazzoleni et al. 2017, Lemma 2.11; Alt and Caffarelli 1981].  $\square$

**2B. Finiteness of the perimeter.** For any  $U \in H^1(D; \mathbb{R}^k)$  solution of (1-1), the set  $\Omega_U$  has locally finite perimeter in  $D$  and, moreover,

$$\mathcal{H}^{d-1}(\partial\Omega_U \cap K) < \infty \quad \text{for every compact set } K \subset D. \quad (2-4)$$

**Remark 2.3.** We notice that the condition (2-4) is more general than the finiteness of the perimeter since  $\partial\Omega_U$  may contain points  $x_0$  which are in the measure-theoretic interior of  $\Omega_U$ , that is,  $|B_r(x_0) \setminus \Omega_U| = 0$ .

In order to prove the claim of this subsection, we will use the following lemma, which holds in general.

**Lemma 2.4.** Suppose that  $D \subset \mathbb{R}^d$  is an open set and that  $\phi : D \rightarrow [0, +\infty]$  is a function in  $H^1(D)$  for which there exist  $\bar{\varepsilon} > 0$  and  $C > 0$  such that

$$\int_{\{0 < \phi \leq \varepsilon\} \cap D} |\nabla \phi|^2 dx + \Lambda |\{0 < \phi \leq \varepsilon\} \cap D| \leq C\varepsilon \quad \text{for every } 0 < \varepsilon \leq \bar{\varepsilon}. \quad (2-5)$$

Then  $P(\{\phi > 0\}; D) \leq C\sqrt{\Lambda}$ .

*Proof.* By the coarea formula, the Cauchy–Schwarz inequality and (2-5), we have that, for every  $\varepsilon \leq \bar{\varepsilon}$ ,

$$\int_0^\varepsilon \mathcal{H}^{d-1}(\{\phi = t\} \cap D) dt = \int_{\{0 < \phi \leq \varepsilon\}} |\nabla \phi| dx \leq \left( \int_{\{0 < \phi \leq \varepsilon\}} |\nabla \phi|^2 dx \right)^{1/2} |\{0 < \phi \leq \varepsilon\}|^{1/2} \leq \varepsilon C\sqrt{\Lambda}.$$

Taking  $\varepsilon = 1/n$ , we get that there is  $\delta_n \in [0, 1/n]$  such that

$$\mathcal{H}^{d-1}(\partial^* \{\phi > \delta_n\} \cap D) \leq n \int_0^{1/n} \mathcal{H}^{d-1}(\{\phi = t\} \cap D) dt \leq C\sqrt{\Lambda}.$$

Passing to the limit as  $n \rightarrow \infty$ , we obtain  $\mathcal{H}^{d-1}(\partial^* \{\phi > 0\} \cap D) \leq C\sqrt{\Lambda}$ , which concludes the proof of the lemma.  $\square$

**Lemma 2.5.** Let  $D \subset \mathbb{R}^d$  be an open set and  $\phi : D \rightarrow [0, +\infty)$  be a Lipschitz function such that:

(a)  $\phi$  is nondegenerate, that is, there is a constant  $c > 0$  such that

$$\sup_{B_r(x_0)} \phi \geq cr \quad \text{for every } x_0 \in \partial\Omega_\phi \cap D \text{ and every } 0 < r < \text{dist}(x_0, \partial D),$$

where we have set  $\Omega_\phi = \{\phi > 0\}$ .

(b) There is a constant  $C > 0$  such that  $\phi$  satisfies the estimate

$$|\{0 < \phi \leq \varepsilon\} \cap D| \leq C\varepsilon \quad \text{for every } \varepsilon > 0.$$

Then, for every compact set  $K \subset \Omega$ , we have  $\mathcal{H}^{d-1}(K \cap \partial\Omega_\phi) < \infty$ .

*Proof.* Let us first recall that the  $(d-1)$ -dimensional Hausdorff measure  $\mathcal{H}^{d-1}(A)$  of a set  $A \subset \mathbb{R}^d$  is given by  $\mathcal{H}^{d-1}(A) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^{d-1}(A)$ , where, for every  $\delta > 0$ , we set

$$\mathcal{H}_\delta^{d-1}(A) = \omega_{d-1} \inf \left\{ \sum_{j=1}^{\infty} r_j^{d-1} : \text{for every } B_{r_j}(x_j) \text{ such that } \bigcup_{j=1}^{\infty} B_{r_j}(x_j) \supset A \text{ and } r_j \leq \delta \right\}.$$

Let  $\delta > 0$  be fixed. By the Vitali covering lemma, there is a family of balls  $\{B_\delta(x_j)\}_{j=1}^N$  which is a covering of  $K \cap \partial\Omega_\phi$ ,  $x_j \in \partial\Omega_\phi$  for every  $j = 1, \dots, N$ , and the balls  $B_{\delta/5}(x_j)$  are disjoint. The nondegeneracy of  $\phi$  implies that in every ball  $B_{\delta/10}(x_j)$  there is a point  $y_j$  such that  $\phi(y_j) \geq c\delta/10$ . The Lipschitz continuity of  $u$  implies that  $B_{c\delta/10L}(y_j) \subset \Omega_\phi$ , where  $L = \max\{1, \|\nabla\phi\|_{L^\infty}\}$ . On the other hand, since  $\phi(x_j) = 0$ , we have

$$\phi < L \left( \frac{c\delta}{10L} + \frac{c\delta}{10} \right) = (L+1) \frac{c\delta}{10} \quad \text{on } B_{c\delta/10L}(y_j).$$

This implies that the balls  $B_{c\delta/10L}(y_j)$ ,  $j = 1, \dots, N$ , are pairwise disjoint and contained in the set

$$\left\{ 0 < \phi < (L+1) \frac{c\delta}{10} \right\}.$$

Now, the estimate from point (b) implies

$$C(L+1) \frac{c\delta}{10} \geq \sum_{j=1}^N |B_{c\delta/10L}(y_j)| \geq N \omega_d \frac{c^d \delta^d}{L^d 10^d},$$

which implies

$$N d \omega_d \delta^{d-1} \leq dC \frac{10^{d-1}}{c^{d-1}} L^d (L+1).$$

Since, the right-hand side does not depend on  $\delta$ , we get

$$\mathcal{H}^{d-1}(K \cap \partial\Omega_\phi) \leq dC \frac{10^{d-1}}{c^{d-1}} L^d (L+1). \quad \square$$

*Proof of the claim (2-4) of Section 2B.* We aim to prove an estimate of the form (2-5) for  $\phi = |U|$  by constructing a suitable competitor, which is inspired by the approach of Dorin Bucur [2012]. Since we aim to prove a local result, we take  $x_0 \in \partial\Omega \cap D$  and  $B_r(x_0) \subset D$ ; moreover, we can assume without loss of generality that  $x_0 = 0$  and  $r = 1$ . Setting  $\rho := |U|$ , for every  $\varepsilon > 0$ , we define

$$\tilde{U} = (\tilde{u}_1, \dots, \tilde{u}_k) := \frac{(\rho - \varepsilon)_+}{\rho} U, \quad \text{where } \tilde{u}_i = \left(1 - \frac{\varepsilon}{\rho}\right)_+ u_i \text{ for every } i = 1, \dots, k,$$

and, for a smooth function  $\phi \in C^\infty(D)$  such that  $0 \leq \phi \leq 1$  in  $D$ ,  $\phi = 1$  in  $B_{1/2}$ , and  $\phi = 0$  on  $D \setminus B_1$ ,

$$V = (v_1, \dots, v_k) := (1 - \phi)U + \phi \tilde{U} = \begin{cases} (1 - \phi)U & \text{if } |U| = \rho < \varepsilon, \\ (1 - \varepsilon\phi/\rho)U & \text{if } |U| = \rho \geq \varepsilon. \end{cases}$$

Thus, clearly  $V$  is an admissible competitor in problem (1-1).

We observe that the following relations, which we will use in the rest of the proof, hold true:

$$\begin{aligned} |\nabla \rho| &\leq |\nabla U| \quad \text{and} \quad \sum_{i=1}^k u_i \nabla u_i = \rho \nabla \rho \quad \text{in } D, \\ \frac{\varepsilon \phi}{\rho} &\leq 1 \quad \text{in } \{|U| \geq \varepsilon\}. \end{aligned}$$

We can now compute on  $\{|U| \geq \varepsilon\}$

$$\begin{aligned} |\nabla V|^2 - |\nabla U|^2 &= \sum_i \left| \nabla \left( 1 - \frac{\varepsilon \phi}{\rho} \right) u_i \right|^2 - |\nabla u_i|^2 \\ &= \left( -\frac{2\varepsilon \phi}{\rho} + \frac{\varepsilon^2 \phi^2}{\rho^2} \right) |\nabla U|^2 + \rho^2 \left| \nabla \frac{\varepsilon \phi}{\rho} \right|^2 - 2(\rho - \varepsilon \phi) \nabla \rho \cdot \nabla \frac{\varepsilon \phi}{\rho} \\ &= \varepsilon^2 |\nabla \phi|^2 - 2\varepsilon \nabla \phi \cdot \nabla \rho + (|\nabla \rho|^2 - |\nabla U|^2) \left( 2\varepsilon \frac{\phi}{\rho} - \varepsilon^2 \frac{\phi^2}{\rho^2} \right) \leq \varepsilon^2 |\nabla \phi|^2 - 2\varepsilon \nabla \phi \cdot \nabla \rho \leq C_1 \varepsilon, \end{aligned}$$

where  $C_1$  depends only on  $\|\nabla \phi\|_{L^\infty}$  and  $\|\nabla U\|_{L^\infty}$ . Next, on the set  $\{|U| < \varepsilon\}$ , we compute

$$\begin{aligned} |\nabla U|^2 - |\nabla V|^2 &= |\nabla U|^2 - |\nabla(1 - \phi)U|^2 \\ &= (2\phi - \phi^2) |\nabla U|^2 + 2(1 - \phi) U \nabla \phi \cdot \nabla U + |U|^2 |\nabla \phi|^2 \geq |\nabla U|^2 \mathbb{1}_{B_{1/2}} - C_2 \varepsilon, \end{aligned}$$

where again  $C_2$  depends only on  $\|\nabla \phi\|_{L^\infty}$  and  $\|\nabla U\|_{L^\infty}$ . By testing the optimality of  $U$  with  $V$  we get

$$\int_{B_1} |\nabla U|^2 + \Lambda \{0 < |U| \leq \varepsilon\} \cap B_1 \leq \int_{B_1} |\nabla V|^2 + \Lambda \{|V| > 0\} \cap B_1,$$

so we deduce

$$\int_{\{0 < |U| \leq \varepsilon\}} (|\nabla U|^2 - |\nabla V|^2) + \Lambda \{0 < |U| \leq \varepsilon\} \cap B_{1/2} \leq \int_{\{|U| \geq \varepsilon\}} (|\nabla V|^2 - |\nabla U|^2) \leq C_1 \varepsilon,$$

and finally, since  $V = 0$  on the set  $\{0 < |U| \leq \varepsilon\} \cap \{\phi = 1\}$ , we get

$$\int_{\{0 < |U| \leq \varepsilon\} \cap B_{1/2}} |\nabla U|^2 + \Lambda \{0 < |U| \leq \varepsilon\} \cap B_{1/2} \leq (C_1 + C_2) \varepsilon,$$

and, since  $|\nabla \rho| \leq |\nabla U|$  we obtain the estimate (2-5) for  $\rho = |U|$  in the ball  $B_{1/2}$ . By Lemmas 2.4 and 2.5, we get that  $\Omega_U$  has locally finite perimeter in  $D$  and that (2-4) holds.  $\square$

**2C. Compactness and convergence of the blow-up sequences.** Let  $U : D \rightarrow \mathbb{R}^k$  be a solution of (1-1) or, more generally, a Lipschitz function. For  $r \in (0, 1)$  and  $x \in \mathbb{R}^d$  such that  $U(x) = 0$ , we define

$$U_{r,x}(y) := \frac{1}{r} U(x + ry).$$

When  $x = 0$  we will use the notation  $U_r := U_{r,0}$ .

Suppose now that  $(r_n)_{n \geq 0} \subset \mathbb{R}^+$  and  $(x_n)_{n \geq 0} \subset D$  are two sequences such that

$$\lim_{n \rightarrow \infty} r_n = 0, \quad \lim_{n \rightarrow \infty} x_n = x_0 \in D, \quad B_{r_n}(x_n) \subset D \quad \text{and} \quad x_n \in \partial \{|U| > 0\} \quad \text{for every } n \geq 0. \quad (2-6)$$

Then the sequence  $\{U_{r_n, x_n}\}_{n \in \mathbb{N}}$  is uniformly Lipschitz and locally uniformly bounded in  $\mathbb{R}^d$ . Thus, up to a subsequence,  $U_{r_n, x_n}$  converges, as  $n \rightarrow \infty$ , locally uniformly to a Lipschitz continuous function  $U_0 : \mathbb{R}^d \rightarrow \mathbb{R}^k$ . Moreover, if  $U$  is a minimizer of (1-1), then for every  $R > 0$  the following properties hold; see [Mazzoleni et al. 2017, Proposition 4.5]:

- (i)  $U_{r_n, x_n}$  converges to  $U_0$  strongly in  $H^1(B_R; \mathbb{R}^k)$ .
- (ii) The sequence of characteristic functions  $\mathbb{1}_{\Omega_n}$  converges in  $L^1(B_R)$  to  $\mathbb{1}_{\Omega_0}$ , where

$$\Omega_n := \{|U_{r_n}| > 0\} \quad \text{and} \quad \Omega_0 := \{|U_0| > 0\}.$$

- (iii) The sequences of closed sets  $\bar{\Omega}_n$  and  $\Omega_n^c$  Hausdorff converge in  $B_R$  respectively to  $\bar{\Omega}_0$  and  $\Omega_0^c$ .
- (iv)  $U_0$  is nondegenerate at zero; that is, there is a dimensional constant  $c_d > 0$  such that

$$\|U_0\|_{L^\infty(B_r)} \geq c_d r \quad \text{for every } r > 0.$$

**Definition 2.6.** Let  $U : \mathbb{R}^d \rightarrow \mathbb{R}^k$  be a Lipschitz function, and  $r_n$  and  $x_n$  be two sequences satisfying (2-6). We say that the sequence  $U_{r_n, x_n}$  is a *blow-up sequence with variable center* (or a *pseudo-blow-up*). If the sequence  $x_n$  is constant,  $x_n = x_0$  for every  $n \geq 0$ , we say that  $U_{r_n, x_0}$  is a *blow-up sequence with fixed center*. We denote by  $\mathcal{BU}_U(x_0)$  the space of all the limits of blow-up sequences with fixed center  $x_0$ .

**2D. Classification of the blow-up limits.** In this section we prove that for any  $x_0 \in \partial\Omega_U \cap D$  the blow-up limits  $U_0 \in \mathcal{BU}_U(x_0)$  have one of the following forms:

- *Multiples of a scalar solution of the one-phase problem*, that is, there is a 1-homogeneous nonnegative global minimizer  $u : \mathbb{R}^d \rightarrow \mathbb{R}^+$  of the one-phase Alt–Caffarelli functional

$$\mathcal{F}(u) = \int |\nabla u|^2 dx + \Lambda |\{u > 0\}|$$

such that

$$U_0(x) = \xi u(x), \quad \text{where } \xi \in \mathbb{R}^k \text{ and } |\xi| = 1. \quad (2-7)$$

- *Linear functions*, that is, there is a matrix  $A = (a_{ij})_{ij} \in \mathcal{M}_{d \times k}(\mathbb{R})$  such that

$$U_0(x) = Ax. \quad (2-8)$$

It was shown in [Mazzoleni et al. 2017] that every function of the form (2-7) is a global solution of (1-1). In the following lemma we classify the linear solutions.

**Lemma 2.7.** *Let  $U : \mathbb{R}^d \rightarrow \mathbb{R}^k$  be a linear function,  $U(x) = Ax$  with  $A = (a_{ij})_{ij} \in \mathcal{M}_{d \times k}(\mathbb{R})$ . If*

$$\|A\|^2 := \sum_{i=1}^k \sum_{j=1}^d a_{ij}^2 \geq \Lambda,$$

*then  $U$  is a solution of (1-1) in the unit ball  $B_1$ . Moreover, if  $\text{rank } A = 1$ , then the condition  $\|A\|^2 \geq \Lambda$  is also necessary.*

*Proof.* We first prove that if  $\|A\|^2 \geq \Lambda$ , then  $U(x) = Ax$  is a solution of (1-1). Let  $U =: (u_1, \dots, u_k)$  and let the vector-valued function  $\tilde{U} = (\tilde{u}_1, \dots, \tilde{u}_k) : B_1 \rightarrow \mathbb{R}^d$  be such that  $\tilde{U} = U$  on  $\partial B_1$ . We will show that  $\tilde{U}$  has a higher energy than  $U$ . Notice that each component  $u_j$ ,  $j = 1, \dots, k$ , can be written as  $u_j(x) = \alpha_j v_j(x)$ , where  $\alpha_j \in \mathbb{R}$  and  $v_j(x) = x \cdot v_j$  for some  $v_j \in \partial B_1$ . We will also write  $\tilde{u}_j(x) = \alpha_j \tilde{v}_j(x)$  and we notice that  $\tilde{v}_j = v_j$  on  $\partial B_1$ . Now since  $(v_j)_+$  and  $(v_j)_-$  are solutions of the one-phase scalar Alt–Caffarelli problem we have

$$\begin{aligned} \int_{B_1} |\nabla v_j|^2 dx + |B_1| &= \int_{B_1} |\nabla(v_j)_+|^2 dx + |\{v_j > 0\} \cap B_1| + \int_{B_1} |\nabla(v_j)_-|^2 dx + |\{v_j < 0\} \cap B_1| \\ &\leq \int_{B_1} |\nabla(\tilde{v}_j)_+|^2 dx + |\{\tilde{v}_j > 0\} \cap B_1| + \int_{B_1} |\nabla(\tilde{v}_j)_-|^2 dx + |\{\tilde{v}_j < 0\} \cap B_1| \\ &\leq \int_{B_1} |\nabla \tilde{v}_j|^2 dx + |\Omega_{\tilde{u}} \cap B_1|. \end{aligned}$$

Multiplying by  $\alpha_j^2$ , taking the sum over  $j$ , and using that  $\|A\|^2 = \sum_{j=1}^k \alpha_j^2$ , we obtain

$$\begin{aligned} \int_{B_1} |\nabla U|^2 dx + \|A\|^2 |B_1| &= \sum_{j=1}^k \alpha_j^2 \left( \int_{B_1} |\nabla v_j|^2 dx + |B_1| \right) \\ &\leq \sum_{j=1}^k \alpha_j^2 \left( \int_{B_1} |\nabla \tilde{v}_j|^2 dx + |\Omega_{\tilde{U}} \cap B_1| \right) = \int_{B_1} |\nabla \tilde{U}|^2 dx + \|A\|^2 |\Omega_{\tilde{U}} \cap B_1|. \end{aligned}$$

Now since  $\Lambda \leq \|A\|^2$ , we have

$$\int_{B_1} |\nabla U|^2 dx + \Lambda |B_1| \leq \int_{B_1} |\nabla \tilde{U}|^2 dx + \Lambda |\Omega_{\tilde{U}} \cap B_1|.$$

We will now prove that if  $\text{rank } A = 1$  and  $U(x) = Ax$  is a solution to (1-1), then  $\|A\|^2 \geq \Lambda$ . Indeed, let  $U = (u_1, \dots, u_k)$ . The rank-1 condition implies that there is a vector  $v \in \partial B_1$  such that  $u_j(x) = \alpha_j x \cdot v_j$  for some  $v_j \in \partial B_1$ . Let

$$u_j^\varepsilon := u_j \circ \Phi_\varepsilon^{-1}, \quad \text{where } \Phi_\varepsilon(x) = x - \varepsilon \phi(x)v,$$

$\phi \in C^\infty(B_1)$  being a nonnegative function with compact support in  $B_1$ . Setting  $U_\varepsilon = (u_1^\varepsilon, \dots, u_k^\varepsilon)$ , the optimality of  $U$  gives

$$\int_{B_1} |\nabla U_\varepsilon|^2 dx + |\{|U_\varepsilon| > 0\} \cap B_1| \geq \int_{B_1} |\nabla U|^2 dx + |\{|U| > 0\} \cap B_1|.$$

Taking the derivative in  $\varepsilon = 0$ , see [Alt and Caffarelli 1981, Section 2], we get

$$\sum_{j=1}^k \int_{B_1 \cap \partial \Omega_U} |\nabla u_j|^2 \phi d\mathcal{H}^{d-1} - \Lambda \int_{B_1 \cap \partial \Omega_U} \phi d\mathcal{H}^{d-1} \geq 0.$$

Since  $|\nabla u_j|^2 = \alpha_j^2$ , we obtain  $\sum_{j=1}^k \alpha_j^2 \geq \Lambda$ , which proves the claim.  $\square$

The classification of the blow-up limits strongly relies on the monotonicity of the vectorial Weiss' boundary-adjusted energy introduced in [Mazzoleni et al. 2017]

$$W(U, x_0, r) := \frac{1}{r^d} \left( \int_{B_r(x_0)} |\nabla U|^2 dx + \Lambda |\{|U| > 0\} \cap B_r(x_0)| \right) - \frac{1}{r^{d+1}} \int_{\partial B_r(x_0)} |U|^2 d\mathcal{H}^{d-1}, \quad (2-9)$$

which turns out to be monotone in  $r$ . Precisely, by Proposition 3.1 of that paper we have the following estimate.

**Lemma 2.8** (Weiss monotonicity formula). *Let  $U = (u_1, \dots, u_k)$  be a minimizer for problem (1-1) and  $x_0 \in \partial\Omega_U \cap D$ . Then, the function  $r \mapsto W(U, x_0, r)$  is nondecreasing and*

$$\frac{d}{dr} W(U, x_0, r) \geq \frac{1}{r^{d+2}} \sum_{i=1}^k \int_{\partial B_r(x_0)} |(x - x_0) \cdot \nabla u_i - u_i|^2 d\mathcal{H}^{d-1}(x); \quad (2-10)$$

in particular, the limit  $\lim_{r \rightarrow 0^+} W(U, x_0, r)$  exists and is finite.

**Remark 2.9** (homogeneity and minimality of the blow-up limits). As a consequence of the monotonicity formula, we obtain that if  $U$  is a solution of (1-1),  $x_0 \in \partial\Omega_U \cap D$ , and  $U_0 \in \mathcal{BU}_U(x_0)$ , then  $U_0$  is a 1-homogeneous global solution of the vectorial Bernoulli problem. Precisely, the fact that  $U_0$  is a global solution follows by [Mazzoleni et al. 2017, Proposition 4.2], while for the homogeneity of  $U_0$  we use the fact that  $U_0$  is a blow-up limit,  $U_0 = \lim_{n \rightarrow \infty} U_{r_n, x_0}$ , and the scaling property of the Weiss energy

$$W(U, x_0, rs) = W(U_{r, x_0}, s, 0) \quad \text{for every } r, s > 0,$$

which gives that the function  $s \mapsto W(U_0, s, 0)$  is constant. In fact, for every  $s > 0$ , we have

$$W(U_0, s, 0) = \lim_{n \rightarrow \infty} W(U_{r_n, x_0}, s, 0) = \lim_{n \rightarrow \infty} W(U, r_n s, x_0) = \lim_{r \rightarrow 0} W(U, r, x_0).$$

Now, the homogeneity of  $U_0$  follows by (2-10) applied to  $U_0$  and its components.

**Remark 2.10** (Lebesgue and energy density). Keeping the notation from Remark 2.9, we notice that the homogeneity of the blow-up limits and the strong convergence of the blow-up sequences give

$$W(U_0, 1, 0) = \Lambda |\{|U_0| > 0\} \cap B_1| = \lim_{r \rightarrow 0} W(U, r, x_0) = \Lambda \omega_d \lim_{r \rightarrow 0} \frac{|\Omega_U \cap B_r(x_0)|}{|B_r|}$$

for every  $U_0 \in \mathcal{BU}_U(x_0)$ . That is, the energy density  $\lim_{r \rightarrow 0} W(U, r, x_0)$  coincides, up to a multiplicative constant, with the Lebesgue density, which (as a consequence) exists in every point  $x_0$  of the free boundary. In particular, we get

$$\Omega_U^{(\gamma)} = \left\{ x \in \partial\Omega_U : \lim_{r \rightarrow 0} \frac{|\Omega_U \cap B_r(x)|}{|B_r|} = \gamma \right\} = \left\{ x_0 \in \partial\Omega_U : \lim_{r \rightarrow 0} W(U, x_0, r) = \Lambda \omega_d \gamma \right\}.$$

**Lemma 2.11** (structure of the blow-up limits). *Let  $U$  be a solution of (1-1),  $x_0 \in \partial\Omega_U \cap D$ . Then, there is a dimensional constant  $0 < \delta < \frac{1}{2}$  such that precisely one of the following holds:*

(i) *The Lebesgue density of  $\Omega_U$  at  $x_0$  is  $\frac{1}{2}$  and every blow-up  $U_0 \in \mathcal{BU}_U(x_0)$  is of the form*

$$U_0(x) = \xi(x \cdot \nu)_+, \quad \text{where } \xi \in \mathbb{R}^k, |\xi| = \sqrt{\Lambda}, \nu \in \mathbb{R}^d, |\nu| = 1. \quad (2-11)$$

(ii) *The Lebesgue density of  $\Omega_U$  at  $x_0$  satisfies*

$$\frac{1}{2} + \delta \leq \lim_{r \rightarrow 0} \frac{|\Omega_U \cap B_r(x_0)|}{|B_r|} \leq 1 - \delta,$$

*and every blow-up in  $\mathcal{BU}_U(x_0)$  is a one-phase blow-up of the form (2-7) with singularity in zero.*

(iii) *The Lebesgue density of  $\Omega_U$  at  $x_0$  is 1 and every blow-up in  $\mathcal{BU}_U(x_0)$  is of the form (2-8).*

*Proof.* Let  $x_0 \in \partial\Omega_U \cap D$ .

Step 1: The following claim holds true:

$$\begin{aligned} x_0 \in \Omega_U^{(1/2)} &\iff \text{there is } U_0 \in \mathcal{BU}_U(x_0) \text{ of the form (2-11)} \\ &\iff \text{every } U_0 \in \mathcal{BU}_U(x_0) \text{ is of the form (2-11).} \end{aligned}$$

Indeed, if one blow-up is of the form (2-11), then by [Remark 2.10](#)  $x_0 \in \Omega^{(1/2)}$ . On the other hand, if  $x_0 \in \Omega^{(1/2)}$  and  $U_0 \in \mathcal{BU}_U(x_0)$ , then again by [Remark 2.10](#)  $|\Omega_{U_0} \cap B_1| = \frac{1}{2}|B_1|$ . The homogeneity of  $U_0$  and the fact that  $\Delta U_0 = 0$  on  $\Omega_{U_0}$  imply that each component of  $U_0$  is an eigenfunction on the sphere corresponding to the eigenvalue  $d - 1$ . By the Faber–Krahn inequality on the sphere we get that, up to a rotation,  $\Omega_{U_0} = \{x_d > 0\}$  and all the components of  $U_0$  are multiples of  $x_d^+$ , that is,  $U_0(x) = \xi x_d^+$  for some  $\xi \in \mathbb{R}^k$ . Let  $\phi$  be a compactly supported function and let  $\tilde{U}_0 = \xi(x_d^+ + \phi)$ . Testing the optimality of  $U_0$  against  $\tilde{U}_0$ , it is immediate to check, see [\[Mazzoleni et al. 2017\]](#), that  $|\xi|x_d^+|$  is a global minimizer of the one-phase Alt–Caffarelli functional. Thus, an internal perturbation, see [\[Alt and Caffarelli 1981\]](#), gives  $|\xi| = \sqrt{\Lambda}$  and concludes Step 1.

Step 2: The following claim holds true:

$$\begin{aligned} x_0 \in \Omega_U^{(1)} &\iff \text{there is } U_0 \in \mathcal{BU}_U(x_0) \text{ of the form (2-8)} \\ &\iff \text{every } U_0 \in \mathcal{BU}_U(x_0) \text{ is of the form (2-8).} \end{aligned}$$

Indeed, if one blow-up  $U_0 \in \mathcal{BU}_U(x_0)$  is of the form (2-8), then by [Remark 2.10](#)  $x_0 \in \Omega_U^{(1)}$ . On the other hand, if  $x_0 \in \Omega_U^{(1)}$ , then still by [Remark 2.10](#)  $|U_0 \cap B_1| = |B_1|$  and so, the minimality of  $U_0$  implies that  $U_0$  is harmonic in  $B_1$ . Now the homogeneity of  $U_0$  implies that it is a linear function,  $U_0(x) = Ax$ , for some matrix  $A = (a_{ij})_{ij}$ .

Step 3: Finally, suppose that  $x_0 \in (\partial\Omega_U \cap D) \setminus (\Omega_U^{(1/2)} \cup \Omega_U^{(1)})$  and let  $x_0 \in \Omega_U^{(\gamma)}$  for some  $\gamma \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ . Let  $U_0 = (u_1, \dots, u_k) \in \mathcal{BU}_U(x_0)$ . Then each component  $u_i$  is 1-homogeneous and the functions  $u_i^+$  and  $u_i^-$  are eigenfunctions corresponding to the eigenvalue  $d - 1$  on the spherical sets  $\{u_i > 0\} \cap \partial B_1$  and  $\{u_i < 0\} \cap \partial B_1$ . Now since the density  $\gamma$  is at most 1, we get that at least one of the sets is empty. Thus, none of the components  $u_i$  change sign and they are all multiples of the first eigenfunction on the set  $\Omega_{U_0} \cap \partial B_1$ ; that is,  $U_0 = \xi|U_0|$  for some  $\xi \in \mathbb{R}^k$ . Now, reasoning as in [\[Mazzoleni et al. 2017, Section 5.2\]](#), we get that  $|\xi| = \Lambda$  and that  $|U_0|$  is a global solution of the one-phase scalar functional  $u \mapsto \int |\nabla u|^2 dx + |\{u > 0\}|$ . In particular, the density estimate for the one-phase Alt–Caffarelli functional implies that  $\gamma < 1 - \delta$  for some dimensional constant  $\delta > 0$ . Now, the fact that the first eigenvalue on  $\Omega_{U_0} \cap \partial B_1$  is  $d - 1$  implies  $\gamma \geq \frac{1}{2}$ . As in [\[Mazzoleni et al. 2017, Section 5.2\]](#), the improvement of flatness for the scalar problem now implies  $\gamma > \frac{1}{2} + \delta$ , which concludes the proof.  $\square$

**Definition 2.12.** Let  $x_0 \in \partial\Omega_U$ . We say that:

- $x_0$  is a regular point,  $x_0 \in \text{Reg}(\partial\Omega_U)$ , if [Lemma 2.11\(i\)](#) holds.
- $x_0$  is a (one-phase) singular point,  $x_0 \in \text{Sing}_1(\partial\Omega_U)$ , if [Lemma 2.11\(ii\)](#) holds.
- $x_0$  is a branching point,  $x_0 \in \text{Sing}_2(\partial\Omega_U)$ , if [Lemma 2.11\(iii\)](#) holds.

In view of [Lemma 2.11](#) we have

$$\begin{aligned}\text{Reg}(\partial\Omega_U) &= \Omega_U^{(1/2)} \cap D, \quad \text{Sing}_2(\partial\Omega_U) = \Omega_U^{(1)} \cap \partial\Omega_U \cap D, \\ \text{Sing}_1(\partial\Omega_U) &= (\partial\Omega_U \cap D) \setminus (\text{Sing}_2(\partial\Omega_U) \cup \text{Reg}(\partial\Omega_U)).\end{aligned}$$

**Lemma 2.13.**  *$\text{Sing}_2(\partial\Omega_U)$  is a closed set and  $\text{Reg}(\partial\Omega_U)$  is an open subset of  $\partial\Omega_U$ .*

*Proof.* We first notice that the function  $W(U, x_0, 0) := \lim_{r \rightarrow 0^+} W(U, x_0, r)$  is upper semicontinuous in  $x_0$ . This follows by the fact that  $(x_0, r) \mapsto W(U, x_0, r)$  is increasing in  $r > 0$  and continuous in  $x_0$ . Thus, the first part of the claim follows since in the points  $x_0 \in \text{Sing}_2(\partial\Omega_U)$  the density  $W(U, x_0, 0)$  is maximal. The second part of the claim follows by the lower density gap from [Lemma 2.11\(ii\)](#) and the argument of [\[Mazzoleni et al. 2017, Proposition 5.6\]](#).  $\square$

### 3. Regularity of the one-phase free boundary

Following the argument from [\[Mazzoleni et al. 2017\]](#), we first deduce the optimality condition on the free boundary in a viscosity sense; then we notice that  $\text{Reg}(\partial\Omega_U)$  is open and Reifenberg flat. Next we show that around every point of  $\text{Reg}(\partial\Omega_U)$  at least one of the components of the optimal vector  $U$  has a constant sign. Thus we fall into the framework of [\[loc. cit.\]](#) and can conclude the proof by using the boundary Harnack principle in NTA domains and the regularity of the one-phase free boundaries for the scalar problem. Finally, thanks to [Lemma 3.10](#), we can apply the arguments of [\[loc. cit., Section 5\]](#) in order to obtain the  $C^\infty$  regularity of  $\text{Reg}(\partial\Omega_U)$ , using the component of locally constant sign provided by [Lemma 3.10](#) instead of  $u_1$  in the boundary Harnack principle [\[loc. cit., Lemma 5.12\]](#). We recall here the updated statements for the reader's sake.

**3A. The stationarity condition on the free boundary.** It is well known, see for example [\[Alt and Caffarelli 1981\]](#), that if  $u$  is a local minimizer of the Alt–Caffarelli functional

$$H_{\text{loc}}^1(\mathbb{R}^d) \ni u \mapsto \mathcal{F}(u) = \int |\nabla u|^2 dx + \Lambda |\{u > 0\}|$$

and the boundary  $\partial\{u > 0\}$  is smooth, then  $|\nabla u| = \sqrt{\Lambda}$  on  $\partial\{u > 0\}$ . There are various ways to state this optimality for free boundaries that are not a priori smooth; see for example [\[Alt and Caffarelli 1981; De Silva 2011\]](#). In the case of vector-valued functionals, we use the notion of viscosity solution from [\[Mazzoleni et al. 2017\]](#).

**Definition 3.1.** Let  $\Omega \subset \mathbb{R}^d$  be an open set. We say that the continuous function  $U = (u_1, \dots, u_k) : \bar{\Omega} \rightarrow \mathbb{R}^k$  is a viscosity solution of the problem

$$-\Delta U = 0 \quad \text{in } \Omega, \quad U = 0 \quad \text{on } \partial\Omega \cap D, \quad |\nabla|U|| = \sqrt{\Lambda} \quad \text{on } \partial\Omega \cap D$$

if for every  $i = 1, \dots, k$  the component  $u_i$  is a solution of the PDE

$$-\Delta u_i = 0 \quad \text{in } \Omega, \quad u_i = 0 \quad \text{on } \partial\Omega \cap D,$$

and the boundary condition  $|\nabla|U|| = \sqrt{\Lambda}$  on  $\partial\Omega \cap D$  holds in viscosity sense, that is:

- For every continuous  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ , differentiable in  $x_0 \in \partial\Omega \cap D$  and such that “ $\phi$  touches  $|U|$  from below in  $x_0$ ” (that is,  $|U| - \phi : \bar{\Omega} \rightarrow \mathbb{R}$  has a local minimum equal to zero in  $x_0$ ), we have  $|\nabla\phi|(x_0) \leq \sqrt{\Lambda}$ .
- For every function  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ , differentiable in  $x_0 \in \partial\Omega \cap D$  and such that “ $\phi$  touches  $|U|$  from above in  $x_0$ ” (that is  $|U| - \phi : \bar{\Omega} \rightarrow \mathbb{R}$  has a local maximum equal to zero in  $x_0$ ), we have  $|\nabla\phi|(x_0) \geq \sqrt{\Lambda}$ .

**Lemma 3.2.** *Let  $U$  be a minimizer for (1-1) and  $x_0 \in \text{Reg}(\partial\Omega_U) \cup \text{Sing}_1(\partial\Omega_U)$ . Then, there is  $r > 0$  such that  $U$  is a viscosity solution of*

$$-\Delta U = 0 \quad \text{in } \Omega_U \cap B_r(x_0), \quad U = 0 \quad \text{on } \partial\Omega_U \cap B_r(x_0), \quad |\nabla|U|| = \sqrt{\Lambda} \quad \text{on } \partial\Omega_U \cap B_r(x_0). \quad (3-1)$$

*Proof.* Suppose that  $\phi$  touches  $|U|$  from above in  $y_0 \in B_r(x_0)$ . Then  $|\phi(y_0)| \geq \Lambda$  precisely as in [Mazzoleni et al. 2017, Lemma 5.2]. If  $\phi$  touches  $|U|$  from below in  $y_0$ , then every blow-up  $U_0 \in \mathcal{B}\mathcal{U}_U(y_0)$  is a 1-homogeneous global minimizer of (1-1) such that  $\Omega_{U_0}$  contains the half-space  $\{x : \nabla\phi(y_0) \cdot x < 0\}$ . Now since the Lebesgue density of  $\Omega_{U_0}$  is strictly smaller than 1, the argument of [loc. cit., Lemma 5.2] gives that all the components of  $U_0$  must be multiples of the same global minimizer of the scalar one-phase Alt–Caffarelli problem. Thus  $\Omega_{U_0} = \{x : \nabla\phi(y_0) \cdot x < 0\}$  and the conclusion follows as in [loc. cit., Lemma 5.2].  $\square$

**3B. Reifenberg flat and NTA domains.** In this section we briefly recall the basic geometric properties of the Reifenberg flat and NTA domains. The Reifenberg flatness of  $\text{Reg}(\partial\Omega_U)$  follows precisely as in [Mazzoleni et al. 2017]. Then a result of [Kenig and Toro 1997] shows that it is also NTA. In the next section we will use the NTA property to prove regularity. For more details on the properties and the structure of the Reifenberg flat domains we refer to [Kenig and Toro 1997], while NTA domains were studied in [Kenig and Toro 1997; Jerison and Kenig 1982].

**Definition 3.3** (Reifenberg flat domains). Let  $\Omega \subset \mathbb{R}^d$  be an open set and let  $0 < \delta < \frac{1}{2}$ ,  $R > 0$ . We say that  $\Omega$  is a  $(\delta, R)$ -Reifenberg flat domain if:

- (1) For every  $x \in \partial\Omega$  and every  $0 < r \leq R$  there is a hyperplane  $H = H_{x,r}$  containing  $x$  such that

$$\text{dist}_H(B_r(x) \cap H, B_r(x) \cap \partial\Omega) < r\delta.$$

- (2) For every  $x \in \partial\Omega$ , one of the connected components of the open set  $B_R(x) \cap \{x : \text{dist}(x, H_{x,R}) > 2\delta R\}$  is contained in  $\Omega$ , while the other one is contained in  $\mathbb{R}^d \setminus \bar{\Omega}$ .

**Theorem 3.4** (Reifenberg flat implies NTA, [Kenig and Toro 1997, Theorem 3.1]). *There exists a  $\delta_0 > 0$  such that if  $\Omega \subset \mathbb{R}^d$  is a  $(\delta, R)$ -Reifenberg flat domain for  $\delta < \delta_0$ , then it is NTA; that is, there exist constants  $M > 0$  and  $r_0 > 0$  (called NTA constants) such that:*

(1)  $\Omega$  satisfies the **corkscrew condition**; that is, given  $x \in \partial\Omega$  and  $r \in (0, r_0)$ , there exists  $x_0 \in \Omega$  such that

$$M^{-1}r < \text{dist}(x_0, \partial\Omega) < |x - x_0| < r.$$

(2)  $\mathbb{R}^d \setminus \Omega$  satisfies the corkscrew condition.

(3) If  $w \in \partial\Omega$  and  $w_1, w_2 \in B(w, r_0) \cap \Omega$ , then there is a rectifiable curve  $\gamma : [0, 1] \rightarrow \Omega$  with  $\gamma(0) = w_1$  and  $\gamma(1) = w_2$  such that  $\mathcal{H}^1(\gamma([0, 1])) \leq M|w_1 - w_2|$  and

$$\min \{\mathcal{H}^1(\gamma([0, t])), \mathcal{H}^1(\gamma([t, 1]))\} \leq M \text{dist}(\gamma(t), \partial\Omega) \quad \text{for every } t \in [0, 1].$$

**Remark 3.5.** We note that an NTA domain  $\Omega \subset \mathbb{R}^d$  is obviously connected, while its intersection with a ball is not necessarily so. This is due to the fact that an arc, contained in  $\Omega$  and connecting two points inside the ball, may go out and then back in. On the other hand the NTA condition implies that the two points can be connected with an arc of length comparable to the length of the radius of the ball. Precisely, there exists a constant  $M > 0$  such that the following property holds:

*For every  $x \in \partial\Omega$  and every  $r > 0$ , there is exactly one connected component of  $B_r(x) \cap \Omega$  that intersects  $B_{r/M}(x) \cap \Omega$ .*

**Lemma 3.6.** *Let  $U$  be a solution of (1-1) and  $x_0 \in \text{Reg}(\partial\Omega_U)$ . Then  $\Omega_U$  is Reifenberg flat and NTA in a neighborhood of  $x_0$ .*

*Proof.* The proof follows by the same contradiction argument as in [Mazzoleni et al. 2017, Proposition 5.9]. Indeed, suppose that  $\text{Reg}(\partial\Omega_U) \ni x_n \rightarrow x_0$  and let  $r_n \rightarrow 0$  be such that  $\partial\Omega_U$  is not  $(\delta, r_n)$  flat in  $B_{r_n}(x_n)$ . Let  $U_n := U_{2r_n, x_n}$ . Up to a subsequence  $U_n$  converges to  $U_0 \in H^1(B_1; \mathbb{R}^k)$ , which is a solution of (1-1) in  $B_1$ . We will prove that  $U_0$  is of the form (2-7); then the conclusion will follow by the Hausdorff convergence of  $\partial\Omega_{U_n}$  to  $\partial\Omega_{U_0}$ . Now, for fixed  $0 < r < 1$  we have  $W(U_n, 0, r) = W(U, x_n, rr_n) \rightarrow W(U_0, x, r)$  as  $n \rightarrow \infty$ . Let now  $\varepsilon > 0$  be fixed. Since  $x_0 \in \text{Reg}(\partial\Omega_U)$ , there is some  $R > 0$  such that  $W(U, x_0, R) - \Lambda\omega_d/2 \leq \varepsilon/2$ . By the continuity of  $W$  in  $x$  we get that for  $n$  large enough,  $W(U, x_n, R) - \Lambda\omega_d/2 \leq \varepsilon$  and, by the monotonicity of  $W$ , we have  $W(U, x_n, rr_n) - \Lambda\omega_d/2 \leq \varepsilon$ . Passing to the limit in  $n$  we obtain  $W(U_0, x, r) - \Lambda\omega_d/2 \leq \varepsilon$ . Since  $\varepsilon$  is arbitrary, we get  $W(U_0, x, r) = \Lambda\omega_d/2$ . Finally, Lemma 2.8 implies that  $U_0$  is 1-homogeneous and  $|B_1 \cap \Omega_{U_0}| = \omega_d/2$ . Thus,  $U_0$  is necessarily of the form (2-7), which concludes the proof.  $\square$

**3C. Existence of a constant sign component.** After showing in the previous section that the regular part of the free boundary is an NTA domain, we aim now to apply a boundary Harnack principle on it. It was proved in [Jerison and Kenig 1982] that in any NTA domain  $\Omega \subset \mathbb{R}^d$  the boundary Harnack principle does hold; that is, if  $u$  and  $v$  are positive harmonic functions in  $\Omega$ , vanishing on the boundary  $\partial\Omega \cap B_r$ , then

$$\frac{v}{u} \text{ is Hölder continuous on } \bar{\Omega} \cap B_r.$$

The precise statement of the boundary Harnack property for harmonic functions which we will use in Lemma 3.10 is the following [Jerison and Kenig 1982, Theorems 5.1 and 7.9].

**Theorem 3.7** (boundary Harnack Principle for NTA Domains). *Let  $\Omega \subset \mathbb{R}^d$  be an NTA domain and  $A \subset \mathbb{R}^d$  an open set. For any compact  $K \subset A$  there exists a constant  $C > 0$  such that for all positive harmonic functions  $u, v$  vanishing continuously on  $\partial\Omega \cap A$  we have*

$$C^{-1} \frac{v(y)}{u(y)} \leq \frac{v(x)}{u(x)} \leq C \frac{v(y)}{u(y)} \quad \text{for all } x, y \in K \cap \bar{\Omega}.$$

Moreover, there exists  $\beta > 0$ , depending only on the NTA constants, such that the function  $v/u$  is Hölder continuous of order  $\beta$  in  $K \cap \bar{\Omega}$ . In particular, for any  $y \in \partial\Omega \cap K$ , the limit  $\lim_{x \rightarrow y, x \in \Omega} v(x)/u(x)$  exists.

**Remark 3.8** (boundary Harnack principle for sign-changing  $v$ ). **Theorem 3.7** still holds in the case when  $u > 0$  on the NTA domain  $\Omega$  and  $v$  is a harmonic function on  $\Omega$  that may change sign. Indeed, if  $v : B_1 \cap \Omega \rightarrow \mathbb{R}$  is a harmonic function that changes sign in  $B_1 \cap \Omega$  and vanishes on  $\partial\Omega \cap B_1$ , then we consider the harmonic extensions  $h_+$  and  $h_-$  solutions of the positive and negative parts of  $v$ :

$$\Delta h_{\pm} = 0 \quad \text{in } \Omega \cap B_1, \quad h_{\pm} = 0 \quad \text{on } \partial\Omega \cap B_1, \quad h_{\pm} = v_{\pm} \quad \text{on } \partial B_1 \cap \Omega.$$

Now, by [Remark 3.5](#), each of the functions  $h_{\pm}$  is strictly positive or vanishes identically in  $\Omega \cap B_{1/M}$ . Thus, the claim follows by the boundary Harnack principle for positive functions applied to  $h_+$  and  $u$  (and  $h_-$  and  $u$ ), the fact that  $v = h_+ - h_-$ , and a standard covering argument.

**Remark 3.9.** The constants  $C$  and  $\beta$  in the boundary Harnack principle do not change under blow-up. That is, given  $x_0 = 0 \in \partial\Omega$ , there is  $r_0 > 0$  such that for all harmonic functions  $u, v$ , solutions of

$$\Delta u = \Delta v = 0 \quad \text{in } \Omega_r \cap B_1, \quad u = v = 0 \quad \text{on } \partial\Omega_r \cap B_1, \quad \Omega_r := \frac{1}{r} \Omega, \quad 0 < r < r_0,$$

we have

$$C^{-1} \frac{v(y)}{u(y)} \leq \frac{v(x)}{u(x)} \leq C \frac{v(y)}{u(y)} \quad \text{for all } x, y \in B_{1/2} \cap \bar{\Omega}_r. \quad (3-2)$$

Following [\[Mazzoleni et al. 2017\]](#) we aim to apply the boundary Harnack principle to the components of the vector  $U$  in order to obtain that, for some  $i \in \{1, \dots, k\}$ ,  $|\nabla u_i|$  is Hölder continuous on  $\partial\Omega_U$  and to apply the known regularity results for the one-phase Bernoulli problem to deduce that  $\partial\Omega_U$  is  $C^{1,\alpha}$ . In our setting the functions  $u_i$ ,  $i = 1, \dots, k$ , may change sign, which is a major obstruction since (3-2) can be applied only in the case when the denominator  $u$  is strictly positive. In order to overcome this issue, we first show that, at every point  $x_0$  of the regular free boundary  $\text{Reg}(\partial\Omega_U)$ , there is a neighborhood of  $x_0$  and a component  $u_i$  which has constant sign in it.

**Lemma 3.10.** *Let  $U = (u_1, \dots, u_k)$  be a solution for (1-1). For all  $x_0 \in \text{Reg}(\partial\Omega_U)$ , there is  $r > 0$  and  $i \in \{1, \dots, k\}$  such that the component  $u_i$  has constant sign in  $B_r(x_0) \cap \Omega_U$ . Moreover, there is a constant  $C_{\text{sign}} > 0$  such that  $C_{\text{sign}} u_i \geq |U|$  in  $B_r(x_0) \cap \Omega_U$ .*

*Proof.* Without loss of generality  $x_0 = 0$ . Let  $U_0 \in \mathcal{BU}_U(x_0)$  and  $U_n := U_{r_n}$  be a blow-up sequence converging to  $U_0$ . By [Lemma 2.11](#) there is a vector  $\xi = (\xi_1, \dots, \xi_k) \in \mathbb{R}^k$  such that  $|\xi| = \sqrt{\Lambda}$  and  $U_0(x) = \xi x_d^+$  up to a rotation of  $\mathbb{R}^d$ . Now since  $|\xi| = \sqrt{\Lambda}$ , there is at least one component  $\xi_i$  such that  $|\xi_i| \geq \sqrt{\Lambda/k}$ . Without loss of generality we can assume that  $i = 1$  and  $\xi_1 \geq \sqrt{\Lambda/k}$ .

Let  $\Omega_n = \Omega_{U_n}$  and  $U_n = (u_{n1}, \dots, u_{nk}) : \Omega_n \cap B_1 \rightarrow \mathbb{R}^k$ , let  $u_{n1}^+$  and  $u_{n1}^-$  be the positive and the negative parts of  $u_{n1}$ , and let  $\tilde{u}_n^+$  and  $\tilde{u}_n^-$  be the solutions of

$$\Delta \tilde{u}_n^\pm = 0 \quad \text{in } \Omega_n \cap B_1, \quad \tilde{u}_{n1}^\pm = 0 \quad \text{on } \partial \Omega_n \cap B_1, \quad \tilde{u}_n^\pm = u_{n1}^\pm \quad \text{on } \Omega_n \cap \partial B_1.$$

Now, notice that both  $u_{n1}^+$  and  $u_{n1}^-$  are subharmonic on  $\Omega_n \cap B_1$ . Thus,

$$\tilde{u}_n^+ - \tilde{u}_n^- = u_{n1}^+ - u_{n1}^- = u_{n1}, \quad \tilde{u}_n^+ \geq u_{n1}^+ \quad \text{and} \quad \tilde{u}_n^- \geq u_{n1}^- \quad \text{in } \Omega_n \cap B_1.$$

Let  $M$  be the constant from [Remark 3.5](#). By the fact that the blow-up limit  $U_0$  has a positive first component, for a fixed  $n$ , in the ball  $B_{1/M}$  exactly one of the following situations can happen:

- (i)  $\tilde{u}_n^+ > 0$  and  $\tilde{u}_n^- > 0$  in  $\Omega_n \cap B_{1/M}$ .
- (ii)  $\tilde{u}_n^+ > 0$  and  $\tilde{u}_n^- = 0$  in  $\Omega_n \cap B_{1/M}$ .

Moreover, again by [Remark 3.5](#) we obtain that in both cases we have  $\Omega_n \cap B_{1/M} = \{\tilde{u}_n^+ > 0\} \cap B_{1/M}$ , while if (i) holds, then also  $\Omega_n \cap B_{1/M} = \{\tilde{u}_n^- > 0\} \cap B_{1/M}$ . Now, notice that in the case (ii) the first part of the claim of the lemma is trivial, so we concentrate our attention on the case (i). Let  $x_M := e_d/(2M)$  and  $r_M := 1/(4M)$ . Recall that  $U_n$  converges uniformly to  $U_0$  and  $\partial \Omega_n$  converges to  $\partial \Omega_{U_0} = \{x_d = 0\}$  in the Hausdorff distance. Then, for every  $\varepsilon > 0$ , there is  $n_0 > 0$  such that for every  $n \geq n_0$  we have

$$B_{r_M}(x_M) \subset \Omega_n, \quad u_{n1}^+(x_M) \geq \sqrt{\frac{\Lambda}{k}} \frac{r_M}{2}, \quad \text{and} \quad |u_{n1}^-| \leq \varepsilon \quad \text{in } B_1.$$

Now, by the definition of  $\tilde{u}_n^+$  and  $\tilde{u}_n^-$  and the maximum principle (applied to  $\tilde{u}_n^-$ ), we have

$$\tilde{u}_n^+(x_M) \geq \sqrt{\frac{\Lambda}{k}} \frac{r_M}{2} \quad \text{and} \quad \tilde{u}_n^-(x_M) \leq \varepsilon.$$

Finally, by [\(3-2\)](#), we obtain

$$\frac{\tilde{u}_n^-(x)}{\tilde{u}_n^+(x)} \leq C \frac{\tilde{u}_n^-(x_M)}{\tilde{u}_n^+(x_M)} \leq \varepsilon C \sqrt{\frac{\Lambda}{k}} \frac{r_M}{2} \quad \text{for every } x \in \Omega_n \cap B_{1/2}.$$

Choosing  $\varepsilon$  such that the right-hand side is smaller than 1, we get

$$u_{n1}(x) = \tilde{u}_n^+(x) - \tilde{u}_n^-(x) > 0 \quad \text{for every } x \in \Omega_n \cap B_{1/2},$$

which proves the first claim. The second part of the statement follows by the boundary Harnack principle applied to  $u_{n1}$  and every component  $u_{ni}$  for  $i = 2, \dots, k$ .  $\square$

**3D. The regular part of the free boundary is  $C^{1,\alpha}$ .** In the following lemma we show that the positive optimal component is locally a solution of a one-phase scalar free boundary problem with Hölder condition on the free boundary. The  $C^{1,\alpha}$  regularity of  $\text{Reg}(\partial \Omega_U)$  then follows by known results on the regularity of the one-phase free boundaries; see [\[De Silva 2011, Theorem 1.1\]](#).

**Lemma 3.11.** *Let  $U = (u_1, \dots, u_k)$  be a minimizer for [\(1-1\)](#) and  $0 \in \text{Reg}(\partial \Omega_U)$  and let the first component be of constant sign in a neighborhood of 0; that is,  $u_1 > 0$  in  $B_{r_0} \cap \Omega_U$ . Then there is a constant  $0 < c_0 \leq 1$ ,*

$0 < r \leq r_0$ , and a Hölder continuous function  $g : B_r \cap \partial\Omega_U \rightarrow [c_0, 1]$  such that  $u_1$  is a viscosity solution to the problem

$$-\Delta u_1 = 0 \quad \text{in } \Omega_U \cap B_r, \quad u_1 = 0 \quad \text{on } \partial\Omega_U \cap B_r, \quad |\nabla u_1| = g\sqrt{\Lambda} \quad \text{on } \partial\Omega_U \cap B_r.$$

*Proof.* First notice that, by [Lemma 3.6](#),  $\Omega_U$  is an NTA domain in a neighborhood of 0 and there exists  $\beta > 0$ , depending only on the NTA constants, such that for  $i = 2, \dots, k$ ,  $u_i/u_1$  is Hölder continuous of order  $\beta$  on  $\bar{\Omega}_U \cap B_r$  for some  $r \leq r_0$ . In particular, for every  $x_0 \in \Omega^{(1/2)} \cap B_r$ , the limit  $g_i(x_0) := \lim_{\Omega \ni x \rightarrow x_0} u_i(x)/u_1(x)$ , exists and  $g_i : B_r \cap \partial\Omega \rightarrow \mathbb{R}$  is an  $\beta$ -Hölder continuous function. Then we have

$$u_i = g_i u_1 \quad \text{on } B_r \cap \bar{\Omega} \quad \text{and} \quad u_1 = g|U| \quad \text{on } B_r \cap \bar{\Omega}, \quad \text{where } g := (1 + g_2^2 + \dots + g_k^2)^{-1/2}.$$

We notice that  $g$  is a  $\beta$ -Hölder continuous function on  $\bar{\Omega} \cap B_r$  for some  $\beta > 0$  and is such that  $c_0 \leq g \leq 1$ , where  $c_0 = 1/C_{\text{sign}}$  and  $C_{\text{sign}}$  is the constant from [Lemma 3.10](#). Suppose now that the function  $\phi \in C^1(\mathbb{R}^d)$  is touching  $u_1$  from below (see [Definition 3.1](#), note that it is local) in a point  $x_0 \in \partial\Omega \cap B_r$ . For  $\rho$  small enough, there is a constant  $C > 0$  such that

$$\frac{1}{g(x)} \geq \frac{1}{g(x_0)} - C|x - x_0|^\gamma \geq 0 \quad \text{for every } x \in \bar{\Omega} \cap B_\rho(x_0),$$

and so, setting  $\psi(x) = \phi(x)(1/g(x_0) - C|x - x_0|^\gamma)$ , we get that  $\psi(x_0) = |U|(x_0)$  and

$$\psi(x) \leq u_1(x) \left( \frac{1}{g(x_0)} - C|x - x_0|^\gamma \right) \leq |U|(x) \quad \text{for every } x \in \bar{\Omega} \cap B_\rho(x_0);$$

that is, in the ball  $B_\rho(x_0)$  we have that  $\psi$  touches  $|U|$  from below in  $x_0$ . On the other hand,  $\psi$  is differentiable in  $x_0$  and

$$|\nabla \psi(x_0)| = \frac{1}{g(x_0)} |\nabla \phi(x_0)|.$$

Since  $U$  is a viscosity solution of [\(3-1\)](#) we obtain

$$\sqrt{\Lambda} \geq |\nabla \psi(x_0)| = \frac{1}{g(x_0)} |\nabla \phi(x_0)|,$$

which gives the claim, the case when  $\phi$  touches  $u_1$  from below being analogous.  $\square$

**3E. Higher regularity: the regular part of the free boundary is  $C^\infty$ .** Thanks to [Lemma 3.10](#), we can apply the arguments of [\[Mazzoleni et al. 2017, Section 5\]](#) in order to obtain the  $C^\infty$  regularity of  $\text{Reg}(\partial\Omega_U)$ , using the component of locally constant sign provided by [Lemma 3.10](#) instead of  $u_1$  in the boundary Harnack principle [\[loc. cit., Lemma 5.12\]](#). We recall here the updated statements for the reader's sake.

In order to pass from  $C^{1,\alpha}$  to  $C^\infty$  we need an improved boundary Harnack principle, as was proved in [\[De Silva and Savin 2015\]](#) for harmonic functions.

**Theorem 3.12** (improved boundary Harnack principle). *Let  $U = (u_1, \dots, u_k)$  be a minimizer for [\(1-1\)](#),  $0 \in \text{Reg}(\partial\Omega_U)$ , and let the first component be of constant sign in a neighborhood of 0, that is,  $u_1 > 0$  in  $B_{r_0} \cap \Omega_U$ . There exists  $R_0 < \frac{1}{2}$  such that, if for  $r < \min\{R_0, r_0\}$  we have  $\text{Reg}(\partial\Omega_U) \cap B_r$  is of class  $C^{k,\alpha}$*

for  $k \geq 1$ , then for all  $i = 2, \dots, k$  we have

$$\frac{u_i}{u_1} \text{ is of class } C^{k,\alpha} \text{ on } \bar{\Omega}_U \cap B_r.$$

In particular, for every  $x_0 \in \text{Reg}(\partial\Omega_U) \cap B_r$ , the limit  $g_i(x_0) := \lim_{\Omega_U \ni x \rightarrow x_0} u_i(x)/u_1(x)$ , exists and  $g_i : B_r \cap \partial\Omega_U \rightarrow \mathbb{R}$  is a  $C^{k,\alpha}$  function.

*Proof.* In order to show the claim, it is enough to apply [De Silva and Savin 2015, Theorem 2.4] for the case  $k = 1$  and Theorem 3.1 of the same work for the case  $k \geq 2$ .  $\square$

At this point we are in position to prove the full regularity of  $\text{Reg}(\partial\Omega_U)$ .

**Lemma 3.13.** *Let  $U = (u_1, \dots, u_k)$  be a minimizer for (1-1),  $0 \in \text{Reg}(\partial\Omega_U)$ , and let the first component be of constant sign in a neighborhood of 0, that is,  $u_1 > 0$  in  $B_{r_0} \cap \Omega_U$ . Then  $\text{Reg}(\partial\Omega_U)$  is locally a graph of a  $C^\infty$  function.*

*Proof.* The smoothness of the free boundary follows by a bootstrap argument as in [Kinderlehrer and Nirenberg 1977]. Let us assume that  $\text{Reg}(\partial\Omega_U)$  is locally  $C^{k,\alpha}$  regular for some  $k \geq 1$ , the case  $k = 1$  being true thanks to Section 3D. We will prove that  $\text{Reg}(\partial\Omega_U)$  is locally  $C^{k+1,\alpha}$ . By Lemma 3.11 the first component  $u_1$  is locally a (classical) solution to the problem

$$\Delta u_1 = 0 \quad \text{in } \Omega_U, \quad u_1 = 0 \quad \text{on } \text{Reg}(\partial\Omega_U), \quad |\nabla u_1| = g\sqrt{\Lambda} \quad \text{on } \text{Reg}(\partial\Omega_U).$$

Now thanks to Theorem 3.12 and the definition of  $g$  we have that  $g$  is a  $C^{k,\alpha}$  function. Now by [Kinderlehrer and Nirenberg 1977, Theorem 2] we have that  $\text{Reg}(\partial\Omega_U)$  is locally a graph of a  $C^{k+1,\alpha}$  function, and this concludes the proof.  $\square$

#### 4. Structure of the branching free boundary

In this section we study in more detail the set of branching points  $\text{Sing}_2(\partial\Omega_U)$ . By the results of Section 2D we know that for  $x_0 \in \partial\Omega_U$  we have

$$x_0 \in \text{Sing}_2(\partial\Omega_U) \iff x_0 \in \Omega_U^{(1)} \iff \text{Every blow-up } U_0 \in \mathcal{BU}_U(x_0) \text{ is a linear function.}$$

In Section 4A we prove that the rank of  $U_0$  depends only on  $x_0$ . Then, in Section 4B we stratify the singular set according to the rank at each point and finally, in the last subsection, we give some measure-theoretical criterion for the uniqueness of the blow-up.

##### 4A. Definition of the strata and decomposition of $\text{Sing}_2(\partial\Omega_U)$ .

**Lemma 4.1.** *Let  $U = (u_1, \dots, u_k)$  be a solution of (1-1) and  $Q \in \mathcal{O}(k)$  be an orthogonal matrix. Then  $V := QU$  is also a solution of (1-1) corresponding to the boundary datum  $Q\Phi$ .*

*Proof.* It is sufficient to notice that for every  $U : D \rightarrow \mathbb{R}^k$  we have  $|QU| = |U|$  and  $|\nabla(QU)|^2 = |\nabla U|^2$ .  $\square$

**Lemma 4.2.** *Let  $U = (u_1, \dots, u_k)$  be a solution of (1-1) and  $x_0 \in \Omega_U^{(1)} \cap \partial\Omega_U$ . Then every blow-up  $U_0 \in \mathcal{BU}_U(x_0)$  is a linear function given by a matrix  $A \in M^{d \times k}(\mathbb{R})$ , whose rank does not depend on  $U_0$  but only on  $x_0$  and  $U$ .*

*Proof.* Without loss of generality we may assume that  $x_0 = 0$ . Let  $U_0 \in \mathcal{BU}_U(0)$ ,  $U_0(x) = Ax$ , be a blow-up such that  $\text{rank } A = j$  for some  $j \in \{1, \dots, k\}$ . We claim that all the blow-ups in  $\mathcal{BU}_U(x_0)$  are of rank  $j$ .

We first prove the claim in the case  $j = 1$ . Indeed, consider a matrix  $Q \in \mathcal{O}(k)$  such that  $QAx = (\nu \cdot x, 0, \dots, 0)$  for some  $\nu \in \mathbb{R}^d$  and consider the vector-valued function  $V = (v_1, \dots, v_k) := QU$ , which is also a solution of (1-1) by Lemma 4.1. Now, since each of the components  $v_i$  is a harmonic function on the set  $\{v_i \neq 0\}$ , the Alt–Caffarelli–Friedman monotonicity formula (see [Alt et al. 1984]) gives that the function

$$r \mapsto \Phi(r, v_i) := \left( \frac{1}{r^2} \int_{B_r} \frac{|\nabla v_i^+|^2}{|x|^{d-2}} dx \right) \left( \frac{1}{r^2} \int_{B_r} \frac{|\nabla v_i^-|^2}{|x|^{d-2}} dx \right) = \int_{B_1} \frac{|\nabla(v_i)_r^+|^2}{|x|^{d-2}} dx \int_{B_1} \frac{|\nabla(v_i)_r^-|^2}{|x|^{d-2}} dx, \quad (4-1)$$

is increasing in  $r$ , where as usual  $(v_i)_r(x) := (1/r)v_i(rx)$ . Now, since for  $i \in \{2, \dots, k\}$  the  $i$ -th component of the blow-up  $QA \in \mathcal{BU}_V(0)$  constantly vanishes, we have  $\Phi(0, v_i) := \lim_{r \rightarrow 0} \Phi(r, v_i) = 0$ . In particular, the  $i$ -th component of any blow-up  $V_0 \in \mathcal{BU}_V(0)$  should vanish and so, the only nonvanishing component of  $V_0$  is the first one (recall that the blow-ups are nontrivial by the nondegeneracy of the solutions of (1-1)). Now since  $\mathcal{BU}_V(0) = Q(\mathcal{BU}_U(0))$  we obtain that the rank of any blow-up  $\mathcal{BU}_U(0)$  is precisely 1, which proves our claim.

Let us now suppose that  $2 \leq j \leq k$  and that the claim holds for all  $i \in \{1, \dots, j-1\}$ . We will now prove the claim for  $j$ . Reasoning as above, we first find a matrix  $Q \in \mathcal{O}(k)$  such that the last  $k-j$  components of  $QA$  vanish; that is,  $(QA)_{j+1} = \dots = (QA)_k = 0 \in \mathbb{R}^k$ . Then, we consider the vector-valued function  $V = (v_1, \dots, v_k) := QU$  and notice that, for all  $i = 1, \dots, k$ , the function  $r \mapsto \Phi(r, v_i)$  is increasing in  $r$ . As above, the strong  $H^1$  convergence of the blow-up sequences implies that  $\Phi(0, v_{j+1}) = \dots = \Phi(0, v_k) = 0$  and that the components  $j+1, \dots, k$  of any blow-up  $V_0 \in \mathcal{BU}_V(0)$  do vanish identically. Thus, the rank of  $V_0$  is at most  $j$ . On the other hand, since the claim does hold for every  $i \in \{1, \dots, j-1\}$ , the rank of  $V_0$  is precisely  $j$ , which concludes the proof.  $\square$

Lemma 4.2 allows us to define, for every  $j \in \{1, \dots, d\}$ , the stratum

$$\mathcal{S}_j := \{x_0 \in \Omega_U^{(1)} \cap \partial\Omega_U : \text{every blow-up } U_0 \in \mathcal{BU}_U(x_0) \text{ has rank } j\}. \quad (4-2)$$

Again, by Lemma 4.2, the singular set  $\partial\Omega_U \cap \Omega_U^{(1)}$  can be decomposed as a disjoint union

$$\Omega_U^{(1)} \cap \partial\Omega_U = \bigcup_{j=1}^d \mathcal{S}_j. \quad (4-3)$$

**4B. Dimension of the strata.** In this subsection we give an estimate on the Hausdorff dimension,  $\dim_{\mathcal{H}}$  of the stratum  $\mathcal{S}_j$ . The proof is based on a well-known technique in geometric measure theory known as the Federer reduction principle.

Given  $A \in \mathbb{R}^d$ ,  $0 \leq s < \infty$ , and  $0 < \delta \leq \infty$ , we recall the notation

$$\mathcal{H}_\delta^s(A) = \frac{\omega_s}{2^s} \inf \left\{ \sum_{i=1}^{\infty} (\text{diam } C_i)^s : A \subset \bigcup_{i=1}^{\infty} C_i, \text{diam } C_i < \delta \right\}, \quad \mathcal{H}^s(A) = \sup_{\delta \geq 0} \mathcal{H}_\delta^s(A), \quad (4-4)$$

$$\dim_{\mathcal{H}} A = \inf\{s \geq 0 : \mathcal{H}^s(A) = 0\}.$$

It is well known that  $\mathcal{H}^s(A) = 0$  if and only if  $\mathcal{H}_\infty^s(A) = 0$ . The other fact (for a proof we refer to [Giusti 1984, Proposition 11.3]) that we will use is

$$\limsup_{r \rightarrow 0} \frac{\mathcal{H}_\infty^s(A \cap B_r(x))}{2^{-s} \omega_s r^s} \geq 1 \quad \text{for } \mathcal{H}^s\text{-almost every } x \in A. \quad (4-5)$$

**Theorem 4.3.** *Let  $U : \mathbb{R}^d \supset D \rightarrow \mathbb{R}^k$  be a solution of (1-1) and  $\mathcal{S}_j$  be as in (4-2). If  $j = d$ , then  $\mathcal{S}_d$  is a discrete subset of  $D$ . More precisely each point of  $\mathcal{S}_d$  is isolated in  $\partial\Omega_U$ . If  $1 \leq j < d$ , then  $\mathcal{S}_j$  is a set of Hausdorff dimension  $\dim_{\mathcal{H}} \mathcal{S}_j \leq d - j$ .*

*Proof.* We start with the first claim. Suppose that  $x_0 \in \mathcal{S}_d$  and there is a sequence  $\partial\Omega_U \ni x_n \rightarrow x_0$ . Taking  $r_n = |x_n - x_0|$ ,  $\xi_n := (x_n - x_0)/r_n$ ,  $\xi_0 = \lim_{n \rightarrow \infty} \xi_n$ , and a blow-up limit  $U_0 \in \mathcal{B}\mathcal{U}_U(x_0)$  of the sequence  $U_{r_n, x_0}$  we obtain that  $\xi_0 \in \partial B_1$ ,  $U_0(\xi_0) = 0$  and so  $\dim \ker U_0 \geq 1$ , which is a contradiction with the definition of  $\mathcal{S}_d$ .

Let now  $j < d$ . Suppose by contradiction that there is  $\varepsilon > 0$  and a solution  $U$  of (1-1) such that  $\mathcal{H}^{d-j+\varepsilon}(\mathcal{S}_j) > 0$ . Then, by (4-5), we get that there is a point  $x_0 \in \mathcal{S}_j$  such that

$$\limsup_{r \rightarrow 0} \frac{\mathcal{H}_\infty^{d-j+\varepsilon}(\partial\Omega_U \cap B_r(x_0))}{r^{d-j+\varepsilon}} \geq \limsup_{r \rightarrow 0} \frac{\mathcal{H}_\infty^{d-j+\varepsilon}(\mathcal{S}_j \cap B_r(x_0))}{r^{d-j+\varepsilon}} \geq 2^{-(d-j+\varepsilon)} \omega_{d-j+\varepsilon}. \quad (4-6)$$

Now let  $r_n \rightarrow 0$  be a sequence realizing the first limsup above and  $U_n = U_{r_n, x_0}$  be a blow-up sequence converging to some  $U_0 \in \mathcal{B}\mathcal{U}_U(x_0)$ . In particular,  $\partial\Omega_{U_n}$  converges in the Hausdorff distance to  $\partial\Omega_{U_0}$ . Now, since  $\mathcal{H}_\infty^s$  is upper semicontinuous with respect to the Hausdorff convergence of sets, (4-6) gives

$$\mathcal{H}_\infty^{d-j+\varepsilon}(\partial\Omega_{U_0} \cap B_1) \geq \lim_{n \rightarrow \infty} \mathcal{H}_\infty^{d-j+\varepsilon}(\partial\Omega_{U_n} \cap B_1) \geq 2^{-(d-j+\varepsilon)} \omega_{d-j+\varepsilon},$$

which is in contradiction with the fact that  $\mathcal{H}^{d-j+\varepsilon}(\partial\Omega_{U_0} \cap B_1) = 0$ . □

**Remark 4.4.** A more refined argument in the spirit of Naber and Valtorta, essentially based on the Weiss monotonicity formula and the structure of the blow-up limits, can be used to deduce that the set  $\mathcal{S}_j$  has finite  $(d-j)$ -dimensional Hausdorff measure. For more details on this technique in the context of the free-boundary problems considered in this paper we refer the reader to [Edelen and Engelstein 2019].

**4C. A density criterion for the uniqueness of the blow-up limit.** The uniqueness of the blow-up limit is a central question in free boundary problems and is strictly related to the  $C^1$ -rectifiability of the singular set. It remains a major open question even in the case of the two-phase problem corresponding to the case  $k = 1$ . In this last subsection we give a general criterion for the uniqueness of the blow-up at the singular points, which depends only on the Lebesgue density of the positivity set  $\Omega_U$  (see Proposition 4.5). Now, even if at this point this criterion by itself is not sufficient for the conclusion, it provides a proof of the fact that the regularity of the singular set only reduces to a control over the measure of the nodal set  $B_r \setminus \Omega_U$ . We prove the lemma by choosing a power rate of convergence, but the argument can be carried out under more general assumptions. For example, a logarithmic decay of the density still translates into a decay of the Weiss energy. This, again implies a blow-up uniqueness and a logarithmic rate of convergence; see [Engelstein et al. 2018]. In this subsection we use the notation  $W(U, r) := W(U, 0, r)$

and  $W_0(U, r) := W_0(U, 0, r)$ , where

$$W_0(U, x_0, r) = \frac{1}{r^d} \int_{B_r(x_0)} |\nabla U|^2 dx - \frac{1}{r^{d+1}} \int_{\partial B_r(x_0)} |U|^2 d\mathcal{H}^{d-1}.$$

**Proposition 4.5.** *Suppose that  $U$  is a solution of (1-1) and  $x_0 \in \partial\Omega_U$ . If there are constants  $C > 0$  and  $\alpha > 0$  such that*

$$\frac{|B_r(x_0) \setminus \Omega_U|}{r^d} \leq Cr^\alpha \quad \text{for every } 0 < r < \text{dist}(x_0, \partial D),$$

*then there is a unique blow-up  $U_0 \in \mathcal{BU}_U(x_0)$  and we have the estimate  $\|U_{r,x_0} - U_0\|_{L^2(\partial B_1)} \leq Cr^\beta$  for some  $\beta = \beta(\alpha, d)$ .*

*Proof.* Let  $x_0 = 0$  and  $r > 0$  be fixed. Let  $H : B_r \rightarrow \mathbb{R}^k$  be the harmonic extension of  $U$  in the ball  $B_r$ . A classical estimate for harmonic functions, see [Spolaor and Velichkov 2019, Lemma 2.5], states that there is a dimensional constant  $\bar{\varepsilon} > 0$  such that

$$(1 + \varepsilon)W_0(H, r) \leq W_0(Z, r) \quad \text{for every } \varepsilon \in [0, \bar{\varepsilon}], \quad (4-7)$$

where  $Z$  is the 1-homogeneous extension of  $U$  in the ball  $B_r(x_0)$ . On the other hand,  $|B_r \setminus \Omega_H| = 0$  and so, the optimality of  $U$  gives

$$W_0(U, r) \leq W_0(H, r) + r^{-d}\Lambda|B_r \setminus \Omega_U|. \quad (4-8)$$

Finally, we notice that for every function  $U$  we have the formula

$$\frac{\partial}{\partial r} W_0(U, r) = \frac{d}{r}(W_0(Z, r) - W_0(U, r)) + \frac{1}{r^{d+2}} \sum_{i=1}^k \int_{\partial B_r} |x \cdot \nabla u_i - u_i|^2 d\mathcal{H}^{d-1}. \quad (4-9)$$

Now, using (4-9), (4-7) and (4-8), we have

$$\begin{aligned} \frac{\partial}{\partial r} W_0(U, r) &\geq \frac{d}{r}(W_0(Z, r) - W_0(U, r)) \geq \frac{d}{r}(W_0(Z, r) - W_0(H, r) - r^{-d}\Lambda|B_r \setminus \Omega_U|) \\ &\geq \frac{d}{r}(\varepsilon W_0(H, r) - r^{-d}\Lambda|B_r \setminus \Omega_U|) \geq \frac{d}{r}(\varepsilon W_0(U, r) - (1 + \varepsilon)r^{-d}\Lambda|B_r \setminus \Omega_U|) \\ &\geq \frac{d\varepsilon}{r}W_0(U, r) - 2d\Lambda Cr^{\alpha-1}. \end{aligned}$$

In particular, this implies that the function

$$r \mapsto \frac{W_0(U, r)}{r^{\varepsilon d}} + \frac{2d\Lambda C}{\alpha - d\varepsilon} r^{\alpha - d\varepsilon}$$

is increasing in  $r$  and so, choosing  $\varepsilon = \alpha/(2d)$ , we get that there is a constant  $C_{U,x_0}$  depending on  $U$  and the point  $x_0 = 0 \in D$  such that

$$W_0(U, r) \leq C_{U,x_0}r^{\alpha/2} \quad \text{and} \quad W(U, r) - \Lambda\omega_d = W_0(U, r) - \Lambda \frac{|B_r \setminus \Omega_U|}{r^d} \leq W_0(U, r) \leq C_{U,x_0}r^{\alpha/2}.$$

Now, the uniqueness of the blow-up and the convergence rate follow by a standard argument; see [Spolaor and Velichkov 2019].  $\square$

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