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A HIGHER-DIMENSIONAL BOURGAIN–DYATLOV FRACTAL UNCERTAINTY PRINCIPLE

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We establish a version of the fractal uncertainty principle, obtained by Bourgain and Dyatlov in 2016, in higher dimensions. The Fourier support is limited to sets $Y \subset \mathbb{R}^d$ which can be covered by finitely many products of δ-regular sets in one dimension, but relative to arbitrary axes. Our results remain true if Y is distorted by diffeomorphisms. Our method combines the original approach by Bourgain and Dyatlov, in the more quantitative 2017 rendition by Jin and Zhang, with Cartan set techniques.

1. Introduction

Bourgain and Dyatlov [2018] proved the following result.

Theorem 1.1. *Let $X, Y \subset \mathbb{R}$ and $N \geq 1$ be such that $X \subset [-1, 1]$ is δ-regular with constant C_R on scales N^{-1} to 1 and $Y \subset [-N, N]$ is δ-regular with constant C_R on scales 1 to N. Then there exist constants $\beta > 0$ and C depending on δ, C_R so that*

$$\|f\|_{L^2(X)} \leq CN^{-\beta}\|f\|_{L^2(\mathbb{R})}$$

for all $f \in L^2(\mathbb{R})$ with $\operatorname{supp}(\hat{f}) \subset Y$.

The δ-regularity condition is akin to asking for a Frostman measure at dimension δ; see Definition 6.1 below for the precise statement. Theorem 1.1 is most interesting for δ close to 1. For $\delta < \frac{1}{2}$, Cauchy–Schwarz and measure estimates in phase space suffice. The β was made effective later by Jin and Zhang [2017]. Combining this fractal uncertainty principle with earlier results by Dyatlov and Zahl [2016] led to a breakthrough on the existence for an essential spectral gap for convex cocompact hyperbolic surfaces. This refers to a strip to the left of the $\frac{1}{2}$ line in the complex plane in which the Selberg zeta function has only finitely many zeros. This result can be reformulated in terms of strips below the real axis in which the meromorphic continuation of the resolvent of the Laplacian of the hyperbolic surface exhibits only finitely many resonances. This in turn can be rephrased as a decay rate of the resolvent for large energies within such a strip.

For other applications see [Bourgain and Dyatlov 2017; Dyatlov and Jin 2017; 2018], and for a survey [Dyatlov 2017].

It remained an open problem to establish an analogue of Theorem 1.1 in higher dimensions. This is the main goal of this paper.

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We now present our main results. Let $X \subset [-1,1]^d$ be a δ-regular set in the sense of Bourgain and Dyatlov with $\delta \in (0,d)$ and constant C_R, on scales N^{-1} to 1. In [Bourgain and Dyatlov 2018] this concept is defined only on the line, but the definition, together with its main properties, carries over to higher dimensions. Strictly speaking, we do not need the regularity condition per se, but rather the porosity property of such sets as stated precisely in Definition 5.1 below. Second, let $Y \subset [-N,N]^d$ be of the form

$$Y = \left\{ \sum_{i=1}^{d} \xi_i \vec{e}_i : \xi_i \in Y_i \right\}, \tag{1-1}$$

where $\vec{e}_i$ are unit vectors with $|\det(\vec{e}_1, \dots, \vec{e}_d)| \geq \varepsilon_0$, a positive constant (possibly small), and $Y_i \subset [-2N, 2N]$ is a δ_1-regular set with $\delta_1 \in (0,1)$ and constant C_R, on scales 1 to N.

Theorem 1.2. *Let X, Y be as in the previous paragraph in dimension $d \geq 2$. Then there exists a constant $C = C(d, \varepsilon_0, \delta, \delta_1, C_R) > 0$ such that for*

$$\beta = \exp\left\{ -\exp\left[\left(\frac{(C_R^2/\iota)^{\frac{2d-2\delta+2}{d-\delta}}}{\delta_1(1-\delta_1)} \right)^{\frac{2}{1-\delta_1}} \right] \right\},$$

where $\iota > 0$ is a small constant depending on d and ε_0, and for any $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp}(\widehat{f}) \subset Y$ one has

$$\|f\|_{L^2(X)} \leq C N^{-\beta} \|f\|_{L^2(\mathbb{R}^d)} \tag{1-2}$$

for sufficiently large $N \geq N_0(d, \varepsilon_0, \delta, \delta_1, C_R)$.

As a corollary of our main theorem, we allow Y to be covered by the union of a finite number of Y_j's, each satisfying (1-1) but with a uniform ε_0:

$$Y \subset \bigcup_{j=1}^{m} Y_j, \quad \text{where } Y_j = \left\{ \sum_{i=1}^{d} \xi_{j,i} \vec{e}_{j,i} : \xi_{j,i} \in Y_{j,i} \right\}. \tag{1-3}$$

Furthermore, the number m of covers can grow in N. To be specific, we prove:

Corollary 1.3. *Let X be as above and Y be as in (1-3). Suppose m grows with N as follows:*

$$m = \lfloor N^{\gamma} \rfloor,$$

in which $0 \leq \gamma < \beta$. Then for any $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp}(\widehat{f}) \subset Y$, and constants C, β in Theorem 1.2, one has

$$\|f\|_{L^2(X)} \leq C N^{\gamma-\beta} \|f\|_2 \tag{1-4}$$

for sufficiently large $N \geq N_0(d, \varepsilon_0, \delta, \delta_1, C_R)$.

Theorem 1.2 and Corollary 1.3 require that the Fourier support Y may be covered by products of regular sets in one dimension *along lines*; see (1-3). Our third result asserts that one may distort these lines by means of diffeomorphisms which are obtained as follows. Let $\Psi_N : [-N,N]^d \to [-N,N]^d$ be a diffeomorphism such that

$$\|D\Phi_N\|_\infty + \|D\Phi_N^{-1}\|_\infty + N\|D^2\Phi_N\|_\infty \leq C(d, D_0), \tag{1-5}$$

where the supremum norm is taken over the cube $[-N, N]^d$. One example of a diffeomorphism satisfying (1-5) is $\Psi_N(x) = N\Psi_0(x/N)$, where Ψ_0 is a diffeomorphism from $[-1, 1]^d$ to $[-1, 1]^d$ such that

$$\|D\Psi_0\|_\infty + \|D\Psi_0^{-1}\|_\infty + \|D^2\Psi_0\|_\infty \le D_0, \tag{1-6}$$

where the supremum norm is taken over the cube.

Theorem 1.4. *Theorem 1.2 remains correct with $\Phi_N(Y)$ in place of Y. Constants depend on D_0, but not on Ψ_0.*

In the following section we demonstrate the Cartan techniques by reproving a certain step in [Bourgain and Dyatlov 2018] which was proved there by means of harmonic measure of the strip with a real line-segment removed. In Section 3 we go beyond the one-dimensional setting via these Cartan methods. The subsequent sections implement the argument in analogy with [Bourgain and Dyatlov 2018] albeit in dimensions and higher. We haven striven to present the argument in a modular fashion. In particular, the delicate Beurling–Malliavin step appears only in Section 6 in order to prove the existence of *damping functions*. We do not use a higher-dimensional version of the Beurling–Malliavin theorem, which appears to be unknown. Rather, we reduce ourselves in that step to the aforementioned product structure of Y (or covers of finitely many of such products) precisely so as to be able to still use the one-dimensional construction of such damping functions. Moreover, as in [Jin and Zhang 2017] it is important for us to use the weaker form of the Beurling–Malliavin theorem obtained via outer functions; see [Mashregi et al. 2005]. Any other construction of damping functions in Section 6 would lead to different formulations of our main theorems in terms of the conditions on Y without needing to change anything in the other sections. Theorem 1.4 is proved in Section 6D. An FUP for Fourier integral operators is presented in Section 6E.

2. L^2 localization in one dimension

Let us first introduce notation. For $\xi = (\xi_1, \xi_2, \dots, \xi_d) \in \mathbb{R}^d$, let

$$|\xi|_1 := \sum_{j=1}^{d} |\xi_k|, \quad |\xi|_2 := \sum_{j=1}^{d} |\xi_k|^2, \quad \text{and} \quad \langle\xi\rangle := (1 + |\xi|_2^2)^{\frac{1}{2}}.$$

Let $e(\theta) := e^{2\pi i\theta}$. For $x \in \mathbb{R}$, let $\lceil x\rceil := \min\{n \in \mathbb{N} : n \ge x\}$, and $\lfloor x\rfloor := \max\{n \in \mathbb{N} : n \le x\}$.

Throughout, we let $\mathcal{R}(q)$ be the rectangle with vertices $\pm iq$, $1 \pm iq$. We begin with quantitative bounds on the Schwarz–Christoffel map from the disk onto a rectangle. The goal is to control this conformal mapping as the eccentricity of $\mathcal{R}(q)$ tends to 0.

Lemma 2.1. *Let $0 < q \le 1$ and define Φ_q to be the unique conformal map, continuous up to the boundary, which takes the unit disk $\mathbb{D}$ onto the rectangle $\mathcal{R}(q)$ and so that $\Phi_q(-1) = 0$ and $\Phi_q(\pm i) = \pm iq$; see Figure 1. Then $\Phi_q(1) = 1$ and $\Phi_q(e^{\pm i\theta(q)}) = 1 \pm iq$, where*

$$\theta(q) = 8\exp\Big(-\frac{\pi}{2q}\Big)(1 + O(q)), \quad q \to 0.$$

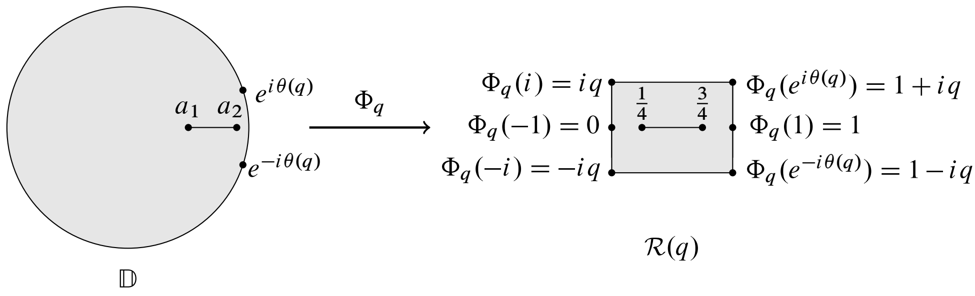

Figure 1. Conformal map Φ_q.

Moreover,

$$\Phi_q([a_1(q), a_2(q)]) = \left[\tfrac{1}{4}, \tfrac{3}{4}\right], \quad a_j(q) = 1 - \delta_j(q),$$

with

$$\delta_1(q) = 4\exp\Big(-\frac{\pi}{8q}\Big)(1 + O(q)), \quad \delta_2(q) = 4\exp\Big(-\frac{3\pi}{8q}\Big)(1 + O(q))$$

as $q \to 0$. Let $E \subset [a_1(q), a_2(q)]$ be a measurable set. Then for sufficiently small q one has $|\Phi_q(E)| \le 2\delta_2(q)^{-2}|E|$, where $|\cdot|$ denotes Lebesgue measure.

Proof. Let $0 < k < 1$ and consider the elliptic integral of the first kind

$$\operatorname{arcsn}(z, k) = \int_0^z \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad \operatorname{Im} z > 0,$$

which maps the upper half-plane onto the rectangle with vertices $\pm L(k)$, $\pm L(k) + iH(k)$; see Figure 2. Here $2L(k)$ and $iH(k)$ are the periods of the elliptic function $\operatorname{sn}(z, k)$ and satisfy, as $k \to 0$,

$$L(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \tfrac{\pi}{2} + O(k^2),$$
$$H(k) = \int_1^{k^{-1}} \frac{dt}{\sqrt{(t^2-1)(1-k^2t^2)}} = \int_0^\infty \frac{ds}{\sqrt{(1+s^2)(1+k^2s^2)}} = \log 4 - \log k + O(k).$$

The latter expansion is a standard fact; see for example [Abramowitz and Stegun 1966, Section 17.3.26]. Let $q := L(k)/H(k)$ and set

$$F_q(z) = -\frac{i}{H(k)} \operatorname{arcsn}(z, k), \tag{2-1}$$

which maps the upper half-plane onto the rectangle with vertices $\pm iq, 1 \pm iq$. With $k = e^{-(\pi/2)\ell}$,

$$q = \frac{\frac{\pi}{2} + O(k^2)}{\log 4 + \frac{\pi}{2}\ell + O(k)} = \ell^{-1}\Big(1 - \frac{\log 16}{\pi \ell} + O(k)\Big),$$

and thus

$$\ell = q^{-1}\Big(1 - \frac{2\log 4}{\pi} q + O(q^2)\Big), \quad k = 4\exp\Big(-\frac{\pi}{2q}\Big)(1 + O(q)).$$

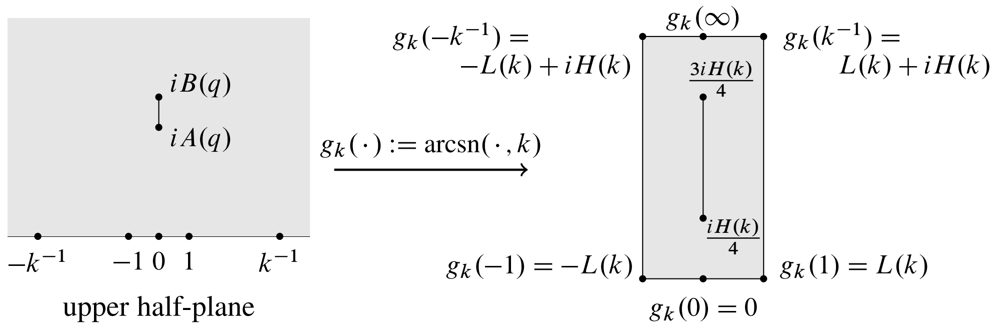

Figure 2. Elliptic integral $\operatorname{arcsn}(z, k)$.

Define $A(q), B(q)$ by $F_q(iA(q)) = \frac{1}{4}$ and $F_q(iB(q)) = \frac{3}{4}$. Thus,

$$\int_0^{A(q)} \frac{ds}{\sqrt{(1+s^2)(1+k^2s^2)}} = \tfrac{1}{4}H(k),$$
$$\int_0^{B(q)} \frac{ds}{\sqrt{(1+s^2)(1+k^2s^2)}} = \tfrac{3}{4}H(k).$$

We make the ansatz $A(q) = ck^{-1/4}(1+\varepsilon(q))$. Then

$$\begin{aligned}\int_0^{A(q)} \frac{ds}{\sqrt{(1+s^2)(1+k^2s^2)}} &= (1+O(k^{\frac{3}{2}}))\int_0^{A(q)} \frac{ds}{\sqrt{1+s^2}}\\ &= \operatorname{arcsinh}(ck^{-\frac{1}{4}}(1+\varepsilon(q)))(1+O(k^{\frac{3}{2}}))\\ &= \log(2ck^{-\frac{1}{4}}(1+\varepsilon(q)))(1+O(k^{\frac{3}{2}}))\\ &= \tfrac{1}{4}(\log 4 - \log k + O(k)).\end{aligned}$$

Hence,

$$\log(2c) - \tfrac{1}{4}\log k + \log(1+\varepsilon(q)) = \tfrac{1}{4}(\log 4 - \log k + O(k)),$$
$$c = \tfrac{1}{2}\sqrt{2}, \quad \varepsilon(q) = O(k),$$
$$A(q) = \tfrac{1}{2}\sqrt{2}\,k^{-\frac{1}{4}}(1+O(k)).$$

Similarly, with $B(q) = \tilde{c}k^{-3/4}(1+\tilde{\varepsilon}(q))$,

$$\log(2\tilde{c}) - \tfrac{3}{4}\log k + \log(1+\tilde{\varepsilon}(q)) = \tfrac{3}{4}(\log 4 - \log k + O(k))(1+O(k^{\frac{1}{2}})),$$
$$\tilde{c} = \sqrt{2}, \quad \tilde{\varepsilon}(q) = O(k^{\frac{1}{2}}\log k),$$

and so

$$B(q) = \sqrt{2}\,k^{-\frac{3}{4}}(1+O(k^{\frac{1}{2}}\log k)).$$

Expressing k in terms of q we obtain

$$A(q) = \tfrac{1}{2}\exp\Big(\frac{\pi}{8q}\Big)(1+O(q)), \quad B(q) = \tfrac{1}{2}\exp\Big(\frac{3\pi}{8q}\Big)(1+O(q)).$$

Next, we conformally map the upper half-plane $\operatorname{Im} z > 0$ onto the unit disk $|w| < 1$ via

$$z = \varphi(w) = i\,\frac{w+1}{1-w}, \quad w = \frac{z-i}{z+i}.$$

One has $\varphi(-1)=0$, $\varphi(\pm i)=\mp 1$, $\varphi(e^{i\theta})=-k^{-1}$ with $\theta=2k+O(k^3)$. Furthermore, $\varphi([a_1(q), a_2(q)]) = i[A(q), B(q)]$, where

$$a_1(q) = \frac{A(q)-1}{A(q)+1} = 1-2A(q)^{-1} + O(A(q)^{-2}),$$
$$a_2(q) = \frac{B(q)-1}{B(q)+1} = 1-2B(q)^{-1} + O(B(q)^{-2}).$$

Setting $a_j(q) = 1-\delta_j(q)$ we have

$$\delta_1(q) = 4\exp\Bigl(-\frac{\pi}{8q}\Bigr)(1+O(q)), \quad \delta_2(q) = 4\exp\Bigl(-\frac{3\pi}{8q}\Bigr)(1+O(q)),$$

as claimed. The final claim of the lemma follows from

$$|(F_q\circ\varphi)'(w)| \le |F_q'(z)|\,|\varphi'(w)| \le 2(1-|w|)^{-2},$$

where $\varphi(w) = z,\ w \in (0,1)$. We used here that for $z = is,\ s > 0$,

$$|F_q'(z)| = H(k)^{-1}(1+|z|^2)^{-\frac{1}{2}}(1+k^2|z|^2)^{-\frac{1}{2}} \le H(k)^{-1}(1+|z|^2)^{-\frac{1}{2}} \le 1$$

for small q. □

By a subharmonic function v on a domain $\Omega \subset \mathbb{C}$ we mean a function $v : \Omega \to [-\infty, \infty)$, which is upper semicontinuous and satisfies the submean-value property. We recall the basic Riesz representation of a subharmonic function on the disk, albeit with precise quantitative control on the Riesz mass and the harmonic part. In view of Lemma 2.1 we need to consider the case where the lower bound on the subharmonic function is attained arbitrarily close to the boundary of the unit disk.

Lemma 2.2. *Let v be subharmonic on a neighborhood of $\mathbb{D}$, with $v \le M$ on $\mathbb{D}$, and assume $\sup_{\rho\mathbb{D}} v \ge m$ for some $0 < \rho < 1$. Let $\rho < r_1 < r < 1$. Then there exists a nonnegative measure μ on $\mathbb{D}$, called the Riesz measure, with the property that for all $w \in r\mathbb{D}$*

$$v(w) = \int_{r\mathbb{D}} \log|z-w|\,\mu(dz) + h(w), \tag{2-2}$$

with h harmonic on $r\mathbb{D}$. We have the quantitative bounds on the Riesz mass

$$\mu(r\mathbb{D}) \le \frac{M-m}{\log((1+\rho r)/(\rho+r))} \tag{2-3}$$

and on the deviations of the harmonic function

$$\min_{c\in\mathbb{R}} \max_{|w|\le r_1} |h(w)-c| \le \tfrac{1}{2}(M-m)\frac{r+r_1}{r-r_1}\,\frac{\log((1+\rho r)/(1-r^2))}{\log((1+\rho r)/(\rho+r))} =: \varepsilon. \tag{2-4}$$

The constant c which minimizes the left-hand side satisfies

$$c \geq m - \varepsilon - \log(r+\rho)\mu(r\mathbb{D}). \tag{2-5}$$

Proof. We will assume that v is smooth, the general case following by approximation. The Green's function $G : \mathbb{D} \times \mathbb{D} \to \mathbb{R}$ given by

$$G(z,w) := \frac{1}{2\pi} \log\left|\frac{z-w}{1-z\bar{w}}\right|$$

satisfies $\Delta_z G(z,w) = \delta_w$ and $G(z,w) = 0$ when $|z| = 1$.

Let $w \in \mathbb{D}$. By Green's second identity for the domain $\mathbb{D}$, we have

$$v(w) - \int_{\mathbb{D}} G(z,w)\Delta v(z)\ \mathrm{Vol}(dz) = \int_{\partial\mathbb{D}} v(z)\frac{\partial G}{\partial n_z}(z,w)\,\sigma(dz),$$

where Vol is the standard volume measure and σ is the (unnormalized) arc-length measure on the circle $\partial\mathbb{D}$. Since v is smooth and subharmonic, Δv is a nonnegative, continuous function, call it $2\pi\mu$. Therefore

$$v(w) = \int_{\mathbb{D}} 2\pi G(z,w)\,\mu(dz) + h_0(w), \tag{2-6}$$

where

$$h_0(w) := \int_{\partial\mathbb{D}} v(z)\frac{\partial G}{\partial n_z}(z,w)\,\sigma(dz). \tag{2-7}$$

Let $0 < r < 1$. On the disk $r\mathbb{D}$ we have the Riesz representation

$$v(w) = \int_{r\mathbb{D}} \log|z-w|\,\mu(dz) + h(w), \tag{2-8}$$

where

$$h(w) := \int_{\mathbb{D}\setminus r\mathbb{D}} \log\left|\frac{z-w}{1-z\bar{w}}\right|\mu(dz) - \int_{r\mathbb{D}} \log|1-z\bar{w}|\,\mu(dz) + h_0(w) \tag{2-9}$$

is harmonic in $r\mathbb{D}$. Note that $(\partial G/\partial n_z)(z,w)$ is the Poisson kernel, whence

$$h_0(w) = \int_0^1 v(e(\theta))P_{|w|}(\varphi-\theta)\,d\theta, \quad w = |w|e(\varphi). \tag{2-10}$$

We now set out to bound the Riesz measure μ. Without loss of generality, assume $m = v(\rho)$. Then setting $w = \rho$ in (2-6) yields

$$\int_{\mathbb{D}} \log\frac{|1-\rho z|}{|z-\rho|}\,\mu(dz) = h_0(\rho) - v(\rho) \leq M - m, \tag{2-11}$$

in which we used

$$h_0(\rho) \leq M. \tag{2-12}$$

This follows from the maximum principle and the fact that h_0 is the harmonic function on $\mathbb{D}$ with boundary values v by (2-10). By an elementary calculation,

$$\min_{|z|\leq r} \frac{|1-\rho z|}{|z-\rho|} = \frac{1+\rho r}{\rho+r} > 1$$

for all $0<\rho, r<1$. Inserting this bound into (2-11) implies

$$\mu(r\,\mathbb{D}) \le \frac{M-m}{\log((1+\rho r)/(\rho+r))}. \tag{2-13}$$

Let $\rho<r_1<r<1$. For all $w\in r\mathbb{D}$ we have

$$\begin{aligned} h(w) &= \int_{\mathbb{D}\setminus r\mathbb{D}} 2\pi G(w,z)\,\mu(dz) - \int_{r\mathbb{D}} \log|1-z\bar{w}|\,\mu(dz) + h_0(w) \\ &\le -\log(1-r^2)\mu(r\mathbb{D}) + M =: h^*. \end{aligned} \tag{2-14}$$

By Harnack's inequality on $r_1\mathbb{D}$ we conclude from this that for any $w\in r_1\mathbb{D}$

$$(h^*-h(w)) \le \frac{r+r_1}{r-r_1}(h^*-h(\rho)),$$

whence

$$h(w) \ge \frac{r+r_1}{r-r_1}h(\rho) - \frac{2r_1}{r-r_1}h^*.$$

By (2-8),

$$h(\rho) = v(\rho) - \int_{r\mathbb{D}} \log|z-\rho|\,\mu(dz) \ge m - \log(r+\rho)\mu(r\mathbb{D}) \tag{2-15}$$

and thus

$$h(w) \ge \frac{r+r_1}{r-r_1}(m-\log(r+\rho)\mu(r\mathbb{D})) - \frac{2r_1}{r-r_1}h^* =: h_*.$$

In summary,

$$\begin{aligned} \min_{c\in\mathbb{R}} \max_{|w|\le r_1} |h(w)-c| &\le \frac{1}{2}(h^*-h_*) \\ &= \frac{1}{2}\frac{r+r_1}{r-r_1}(h^*-m+\log(r+\rho)\mu(r\mathbb{D})) \\ &= \frac{1}{2}\frac{r+r_1}{r-r_1}\left(M-m+\log\left(\frac{r+\rho}{1-r^2}\right)\mu(r\mathbb{D})\right). \end{aligned} \tag{2-16}$$

Finally, bounding the μ-mass by (2-13) finally implies

$$\min_{c\in\mathbb{R}} \max_{|w|\le r_1} |h(w)-c| \le \tfrac{1}{2}(M-m)\frac{r+r_1}{r-r_1}\frac{\log((1+\rho r)/(1-r^2))}{\log((1+\rho r)/(\rho+r))} =: \varepsilon,$$

as claimed. Finally, to establish (2-5), we return to (2-15) and note that the left-hand side is at most $c+\varepsilon$ for c the minimizer in the previous line. Then

$$c \ge m - \log(r+\rho)\mu(r\mathbb{D}) - \varepsilon.$$

Note that one may insert (2-13) on the right-hand side to control the mass. □

We now apply the Cartan estimate for logarithmic potentials to the Riesz representation (2-2) in order to derive lower bounds on v up to a small measure of exceptions.

Corollary 2.3. *Let v be as in Lemma 2.2 with $\rho = 1-3\delta$, $0<\delta<\frac{1}{3}$. Then for all $0<H\le 1$ there exist disks $D(z_j, s_j)$ so that*

$$v(z) \ge m-(M-m)\Big[2\delta^{-3}\log\Big(\frac{2}{\delta}\Big)+\delta^{-2}\log\Big(\frac{2e}{H}\Big)\Big]$$

for all $z\in r_1\mathbb{D}\setminus\bigcup_j D(z_j,s_j)$ with $\sum_j s_j\le 5H$ and $r_1 = 1-2\delta$.

Proof. By Cartan's estimate, for any $H>0$ there exist disks $D(z_j,s_j)$ such that $\sum_j s_j\le 5H$ and

$$\int_{r\mathbb{D}}\log|w-z|\,\mu(dw)\ge \mu(r\mathbb{D})\log\Big(\frac{H}{e}\Big)\quad \text{for all } z\in r_1\mathbb{D}\setminus\bigcup_j D(z_j,s_j). \tag{2-17}$$

See [Levin 1996, Theorem 3, Section 11.2]. To invoke the measure bound (2-3) we estimate

$$\log\Big(\frac{1+\rho r}{\rho+r}\Big)=\log\Big(\frac{2-4\delta+3\delta^2}{2-4\delta}\Big)=\log\Big(1+\frac{3\delta^2}{2-4\delta}\Big)\ge\log\Big(1+\frac{3\delta^2}{2}\Big)\ge\delta^2$$

since $\delta^2\le\frac{1}{2}$ and $\log\big(1+\frac{3}{2}x\big)\ge x$ for $0\le x\le\frac{1}{2}$. Consequently,

$$\mu(r\mathbb{D})\le\delta^{-2}(M-m).$$

Next,

$$\frac{1+\rho r}{1-r^2}\le\frac{2}{2\delta-\delta^2}\le\delta^{-1}(1+\delta),$$

as well as

$$\frac{r+r_1}{r-r_1}=\frac{2-3\delta}{\delta}\le 2\delta^{-1},$$

whence (2-4) implies

$$\min_{c\in\mathbb{R}}\max_{|w|\le r_1}|h(w)-c|\le\varepsilon\le(M-m)\delta^{-3}\log\Big(\frac{2}{\delta}\Big)=:\tilde{\varepsilon}.$$

Finally, by (2-5), one has

$$c\ge m-\varepsilon-\log(r+\rho)\mu(r\mathbb{D})\ge m-\varepsilon-\log(2)\mu(r\mathbb{D}).$$

In view of (2-2) and the preceding estimates we obtain

$$\begin{aligned} v(z)&\ge c+\mu(r\mathbb{D})\log\Big(\frac{H}{e}\Big)-\varepsilon\ge m-2\varepsilon+\log\Big(\frac{H}{2e}\Big)\mu(r\mathbb{D})\\ &\ge m-(M-m)\Big[2\delta^{-3}\log\Big(\frac{2}{\delta}\Big)-\delta^{-2}\log\Big(\frac{H}{2e}\Big)\Big] \end{aligned} \tag{2-18}$$

for all z as in (2-17). □

By means of the conformal transformation Φ_q from Lemma 2.1 we can obtain a version of the Riesz representation theorem on thin rectangles $\mathcal{R}(q)$.

Corollary 2.4. *There exists $q_* \in (0,1]$ with the following property: Let u be subharmonic on $\mathcal{R}(q)$ for some $0 < q \le q_*$, continuous up to the boundary. Assume that $u \le M$ on $\mathcal{R}(q)$ and $\max_{x\in[\frac14,\frac34]} u(x) \ge m$. Then*

$$u(x) \ge m - (M-m)\exp\Big(\frac{9\pi}{8q}\Big)\Big[\log(4) + \frac{9\pi}{4q} + \exp\Big(-\frac{3\pi}{8q}\Big)\log\Big(\frac{2e}{H}\Big)\Big] \tag{2-19}$$

for all $x \in [\frac14,\frac34] \setminus \bigcup_j I_j$, where $\sum_j |I_j| \le 3H\exp(\frac{3\pi}{4q})$.

Proof. Let $v = u \circ \Phi_q$, with Φ_q as in Lemma 2.1. Then v satisfies the assumptions of Corollary 2.3 with $\rho \ge 1 - \delta_2(q)$, and

$$\begin{aligned} \delta_2(q) &= 4\exp\Big(-\frac{3\pi}{8q}\Big)(1+O(q)) \ge 3\delta, \\ \delta &:= \exp\Big(-\frac{3\pi}{8q}\Big) < \frac13, \end{aligned} \tag{2-20}$$

provided q_* is small enough. By Corollary 2.3 we have

$$\begin{aligned} v(z) &\ge m - (M-m)\exp\Big(\frac{9\pi}{8q}\Big)\Big[2\log\Big(\frac{2}{\delta}\Big) + \delta\log\Big(\frac{2e}{H}\Big)\Big] \\ &= m - (M-m)\exp\Big(\frac{9\pi}{8q}\Big)\Big[\log(4) + \frac{9\pi}{4q} + \exp\Big(-\frac{3\pi}{8q}\Big)\log\Big(\frac{2e}{H}\Big)\Big] \end{aligned}$$

for all $z \in r_1\mathbb{D} \setminus \bigcup_j D(z_j, s_j)$, $\sum_j s_j \le 5H$, where $r_1 = 1 - 2\delta$. The inverse image of $[\frac14,\frac34]$ under Φ_q is $[a_1(q), a_2(q)]$. Define $\tilde I_j := \mathbb{R} \cap D(z_j, s_j)$, $I_j = \Phi_q(\tilde I_j)$, and $E := \bigcup_j \tilde I_j$ so that $\sum_j |\tilde I_j| \le 10H$. By Lemma 2.1 we have

$$|\Phi_q(E)| \le 20H\delta_2(q)^{-2} < 3H\exp\Big(\frac{3\pi}{4q}\Big),$$

as claimed. □

Next, we apply the previous results on subharmonic functions to $\log|F|$, where F is analytic.

Corollary 2.5. *Let F be an analytic function on a neighborhood of $\mathcal{R}(q)$ with $0 < q \le q^*$, and F not identically equal to zero. Define*

$$B_1 := \|F\|_{L^2([\frac14,\frac34])}, \quad B_2 := \|F\|_{L^2(\partial\mathcal{R}(q))}.$$

Then for some absolute constant C_0, and all $H > 0$,

$$\begin{aligned} &B_1^{K+1} \le e^{\frac{C_0 K}{q}} B_2^K |F(x)| \\ \text{holds for any}\quad &K \ge \exp\Big(\frac{9\pi}{8q}\Big)\Big[\log(4) + \frac{9\pi}{4q} + \exp\Big(-\frac{3\pi}{8q}\Big)\log\Big(\frac{2e}{H}\Big)\Big] \end{aligned} \tag{2-21}$$

for all $x \in [\frac14,\frac34] \setminus \bigcup_j I_j$, where $\sum_j |I_j| \le 3H\exp(\frac{3\pi}{4q})$.

Proof. We apply our previous results to $u(z) := \log|F(z)|$, which is subharmonic on a neighborhood of $\mathcal{R}(q)$. However, Corollary 2.4 does not apply directly since we do not have a pointwise upper bound on u. Returning to the subharmonic function $v = u \circ \Phi_q$ on the unit disk $\mathbb{D}$, we note that the pointwise upper bound M on v only entered through the estimate $h_0 \le M$; see (2-12), (2-14). The analytic function

$\widetilde{F} = F \circ \Phi_q$ satisfies $\log|\widetilde{F}| = v$. Denoting by

$$P_w(d\theta) = P_{|w|}(d(\theta-\varphi)) = \frac{1-|w|^2}{1-2|w|\cos(2\pi(\theta-\varphi))+|w|^2}$$

the Poisson kernel centered at $w = |w|e(\varphi)$, we estimate h_0 from (2-11) as follows:

$$\begin{aligned} h_0(w) &= \int_0^1 v(e(\theta))\, P_w(d\theta) = \int_0^1 \log|\widetilde{F}(e(\theta))|\, P_w(d\theta) \\ &\le \log\Big(\int_0^1 |\widetilde{F}(e(\theta))|\, P_w(d\theta)\Big) \\ &\le \log\Big(\int_0^1 |\widetilde{F}(e(\theta))|\, d\theta \Big\|\frac{P_w(d\theta)}{d\theta}\Big\|_\infty\Big) \\ &\le \log(B_2) + \log\Big(\Big\|\frac{d\theta}{d\sigma}\Big\|_{L^2(\partial\mathcal{R}(q))}\Big) + \log\Big\|\frac{P_w(d\theta)}{d\theta}\Big\|_\infty, \end{aligned} \tag{2-22}$$

where $d\sigma$ denotes arc-length measure on $\partial\mathcal{R}(q)$, and the correspondence between $\partial\mathbb{D}$ and $\partial\mathcal{R}(q)$ is given by $\xi \mapsto \Phi_q(e(\xi))$. On the one hand,

$$\Big\|\frac{P_w(d\theta)}{d\theta}\Big\|_\infty \le 2(1-|w|)^{-1},$$

and on the other hand,

$$\Big\|\frac{d\theta}{d\sigma}\Big\|^2_{L^2(\partial\mathcal{R}(q))} = \int_{\partial\mathcal{R}(q)} \Big|\frac{d\theta}{d\sigma}\Big|^2 d\sigma = \int_0^1 \Big|\frac{d\sigma}{d\theta}(\xi)\Big|^{-1} d\xi. \tag{2-23}$$

Using the notation of Lemma 2.1, the boundary map $\partial\mathbb{D} \to \partial\mathcal{R}(q)$ induced by Φ_q is

$$\begin{gathered} \xi \mapsto \zeta(\xi) := iH(k)^{-1}\operatorname{arcsn}(x(\xi), k), \\ x(\xi) := \varphi(e(\xi)) = -\cot(\pi\xi), \quad x'(\xi) = \pi(1+x(\xi)^2) \end{gathered}$$

where $\varphi(w) = i(w+1)/(1-w)$ takes the disk to the upper half-plane. If $0 < 2\pi\xi < \theta(q)$, then $\zeta(\xi) = 1 + iy(\xi)$, where

$$\frac{dy}{d\xi} = \frac{\pi}{H(k)} \frac{1+x^2}{\sqrt{(x^2-1)(k^2x^2-1)}} \ge \frac{\pi}{kH(k)}, \quad x(\xi) < -k^{-1}.$$

Therefore, this region contributes at most

$$\tfrac{1}{2}kH(k)\theta(q) \lesssim 1 \quad \text{uniformly in } q$$

to the integral in (2-23). Next, if $\theta(q) < 2\pi\xi < \frac{\pi}{2}$, then $\zeta = u + iq$, with

$$\Big|\frac{du}{d\xi}\Big| = \frac{\pi}{H(k)} \frac{1+x^2}{\sqrt{(x^2-1)(1-k^2x^2)}} \ge \frac{\pi}{H(k)}, \quad -k^{-1} < x(\xi) < -1,$$

and so this case contributes $\lesssim H(k)$ to (2-23). Finally, the region $\frac{\pi}{2} < 2\pi\xi < 2\pi$ similarly adds $\lesssim H(k)$ to (2-23).

Combining these estimates with (2-22) yields

$$h_0(w) \le \log(B_2) + \log(CH(k)) + \log\left(\frac{2}{\pi(1-r)}\right) \le \log(B_2) + C_0 q^{-1} =: M \tag{2-24}$$

for all $|w| < r = 1-\delta$ with some absolute constant C_0; see (2-20). This bound replaces (2-12) and (2-14) above.

As for the lower bound m on u, one has $m \ge \log(B_1)$ and thus (2-19) holds with

$$M - m \le \log\left(\frac{B_2}{B_1}\right) + C_0 q^{-1}.$$

Finally, (2-21) follows from (2-19) by exponentiating. □

Integrating the previous result over a small set of x yields the following localization estimate for the L^2 norm of F.

Proposition 2.6. *There exists an absolute constant $C_1 > 0$ with the following property: Let F be an analytic function on a neighborhood of $\mathcal{R}(q)$ with $0 < q \le q^*$, and F not identically equal to zero. Define*

$$B_1 := \|F\|_{L^2([\frac{1}{4},\frac{3}{4}])}, \quad B_2 := \|F\|_{L^2(\partial\mathcal{R}(q))}.$$

For any $J \subset [\frac{1}{4}, \frac{3}{4}]$ some Borel set of positive measure,

$$B_1 \le e^{\frac{C_1}{q}} B_2^{1-\kappa} \|F\|^{\kappa}_{L^2(J)},$$

with $0 < \kappa \le e^{-C_1/q}(\log(1/|J|))^{-1}$.

Proof. We apply Corollary 2.5 with $3H \exp\left(\frac{3\pi}{4q}\right) = |J|/2$. Thus,

$$\begin{aligned} B_1^{K+1}\left(\frac{|J|}{2}\right)^{\frac{1}{2}} &\le e^{\frac{C_0 K}{q}} B_2^K \|F\|_{L^2(J)}, \\ K &:= \exp\left(\frac{9\pi}{8q}\right)\left[\log(4) + \frac{9\pi}{4q} + \exp\left(-\frac{3\pi}{8q}\right)\left(\log\left(\frac{12e}{|J|}\right) + \frac{3\pi}{4q}\right)\right] \end{aligned} \tag{2-25}$$

or

$$B_1 \le e^{\frac{C_0}{q}} \left(\frac{|J|}{2}\right)^{-\frac{\kappa}{2}} B_2^{1-\kappa} \|F\|^{\kappa}_{L^2(J)}, \quad \kappa \le (1+K)^{-1}. \tag{2-26}$$

We write $\kappa \le (1+K)^{-1}$ instead of $\kappa = (1+K)^{-1}$, since we may increase the value of K. One checks that

$$\log\left(\left(\frac{|J|}{2}\right)^{-\frac{\kappa}{2}}\right) \le \frac{\log(2/|J|)}{\exp\left(\frac{9\pi}{8q}\right)\left[\log(4) + \frac{9\pi}{4q} + \exp\left(-\frac{3\pi}{8q}\right)\left(\log(12e/|J|) + \frac{3\pi}{4q}\right)\right]} \le \exp\left(-\frac{3\pi}{4}\right) < 0.1, \tag{2-27}$$

uniformly in $0 < q < 1$ and in $|J|$. Note that

$$K \le \begin{cases} \exp\bigl(\frac{9\pi}{8q}\bigr)\bigl[\log(4) + \frac{9\pi}{4q} + \exp\bigl(-\frac{3\pi}{8q}\bigr)\bigl(\log(12e) + \frac{3\pi}{2q}\bigr)\bigr] & \text{if } \log 2 \le \log(1/|J|) < \frac{3\pi}{4q}, \\ 8\exp\bigl(\frac{9\pi}{8q}\bigr)\bigl[1 + \exp\bigl(-\frac{3\pi}{8q}\bigr)\bigr]\log(1/|J|) & \text{if } \max\bigl(\log 2, \frac{3\pi}{4q}\bigr) \le \log(1/|J|) \end{cases}$$
$$\le e^{\frac{C_2}{q}} \log(1/|J|) - 1$$

for some absolute constant $C_2 > 0$. Taking $C_1 := \max(2C_0, C_2)$ and

$$K_0 := e^{\frac{C_1}{q}} \log\Bigl(\frac{1}{|J|}\Bigr),$$

we conclude from (2-25), (2-26) and (2-27) with the estimate $K \le K_0 - 1$ that

$$B_1 \le e^{\frac{C_0}{q}+0.1} B_2^{1-\kappa} \|F\|^{\kappa}_{L^2(J)} \le e^{\frac{C_1}{q}} B_2^{1-\kappa} \|F\|^{\kappa}_{L^2(J)}, \quad \kappa \le K_0^{-1},$$

as claimed. □

We next apply Proposition 2.6 to a band-limited L^2 function in order to obtain the main result of this section.

Proposition 2.7. *Fix $\lambda \in \bigl(0, \frac{1}{2}\bigr]$ and for each integer n let $I_n \subset [n, n+1]$ be some Borel set with $|I_n| = \lambda$. Let $f \in L^2(\mathbb{R})$ be band-limited; i.e., $\hat{f}$ is of compact support. Then for each $0 < q \le q^*$*

$$\|f\|^2_{L^2(\mathbb{R})} \le 12 e^{\frac{10C_1}{q}} \Bigl(\sum_n \|f\|^2_{L^2(I_n)}\Bigr)^{\kappa} \|e^{2\pi q|\xi|} \hat{f}(\xi)\|^{2(1-\kappa)}_{L^2(\mathbb{R})}, \tag{2-28}$$

with $0 < \kappa \le e^{-5C_1/q}(-\log\lambda)^{-1}$, and C_1, q^* are as in Proposition 2.6.

Proof. Let F be the entire function with $F = f$ on the real line. Fix $0 \le t \le 1$ and define $\mathcal{R}_{n,t}(q)$ to be the rectangle with vertices $n - 1 - t \pm iq$, $n + 2 + t \pm iq$. We claim that by Proposition 2.6 we have

$$\|f\|_{L^2([n,n+1])} \le e^{\frac{5C_1}{q}} \|F\|^{1-\kappa}_{L^2(\partial\mathcal{R}_{n,t}(q))} \|f\|^{\kappa}_{L^2(I_n)}, \tag{2-29}$$

with $\kappa \le e^{-5C_1/q}\bigl(\log((3+2t)/|I_n|)\bigr)^{-1}$. To see this, we set $n = 0$ without loss of generality, translate $\mathcal{R}_{n,t}(q) \to \mathcal{R}_{n,t}(q) + 1 + t$, and dilate $z \mapsto z/(3+2t)$. After these operations, the transformed interval I_0 lies in

$$\Bigl[\frac{1+t}{3+2t}, \frac{2+t}{3+2t}\Bigr] \subset \Bigl[\frac{1}{4}, \frac{3}{4}\Bigr],$$

and the height q becomes $q/(3+2t) \ge q/5$, whence the claim.

Squaring, summing, and applying Hölder's inequality yields

$$\|f\|^2_{L^2(\mathbb{R})} \le e^{\frac{10C_1}{q}} \Bigl(\sum_n \|F\|^2_{L^2(\partial\mathcal{R}_{n,t}(q))}\Bigr)^{1-\kappa} \Bigl(\sum_n \|f\|^2_{L^2(I_n)}\Bigr)^{\kappa}.$$

Let $\mathbb{E}$ denote the expected value with respect to $0 \le t \le 1$, uniformly distributed. On the one hand, taking expectations of the previous line yields

$$\|f\|^2_{L^2(\mathbb{R})} \le e^{\frac{10C_1}{q}} \Bigl(\sum_n \mathbb{E}\|F\|^2_{L^2(\partial\mathcal{R}_{n,t}(q))}\Bigr)^{1-\kappa} \Bigl(\sum_n \|f\|^2_{L^2(I_n)}\Bigr)^{\kappa}. \tag{2-30}$$

On the other hand, since

$$\sup_{0\le t\le 1}\sum_n \mathbb{1}_{[n-1-t,n+2+t)}\le 5, \tag{2-31}$$

we have

$$\begin{aligned}\sum_n \mathbb{E}\|F\|^2_{L^2(\partial\mathcal{R}_{n,t}(q))} \\ \le 5\|F(\cdot+iq)\|^2_{L^2(\mathbb{R})}+5\|F(\cdot-iq)\|^2_{L^2(\mathbb{R})}+2\sum_n\int_0^1\int_{-q}^q |F(n-t+is)|^2\,ds\,dt.\end{aligned} \tag{2-32}$$

Since $\|F(\cdot\pm iq)\|_{L^2(\mathbb{R})}=\|e^{\pm 2\pi q\xi}\hat f(\xi)\|_{L^2(\mathbb{R})}$ and

$$\begin{aligned}\sum_n\int_0^1\int_{-q}^q |F(n-t+is)|^2\,ds\,dt &= \int_{\mathbb{R}}\int_{-q}^q |F(x+is)|^2\,ds\,dx \\ &= \int_{-q}^q\int_{\mathbb{R}} e^{4\pi s\xi}|\hat f(\xi)|^2\,d\xi\,ds \le 2q\|e^{2\pi q|\xi|}\hat f(\xi)\|^2_{L^2(\mathbb{R})},\end{aligned}$$

assuming as we may that $q^*\le\frac12$, we infer from (2-32) that

$$\sum_n \mathbb{E}\|F\|^2_{L^2(\partial\mathcal{R}_{n,t}(q))}\le 12\|e^{2\pi q|\xi|}\hat f(\xi)\|^2_{L^2(\mathbb{R})}.$$

Inserting this into (2-30) concludes the proof. □

3. L^2 localization in higher dimensions

Our goal is to prove a version of Proposition 2.7 for band-limited functions $f\in L^2(\mathbb{R}^d)$, $d\ge 2$. For the sake of simplicity, we first limit ourselves to $d=2$ and begin with a Cartan-type estimate for functions on $\mathbb{D}\times\mathbb{D}$ which are subharmonic relative to each variable.

We begin with the definition of a Cartan-2 set; see [Goldstein and Schlag 2001, Definition 8.1; 2008, Definition 2.12].

Definition 3.1. We say that $\mathcal{B}\subset\mathbb{C}^2$ is a Cartan-2 set with parameter $H>0$ if for all $(z_1,z_2)\in\mathcal{B}$ one has either

- $z_1\in\bigcup_j D(\zeta_j,s_j)$ with $\sum_j s_j\le 5H$,
- or for all other z_1, one has $z_2\in\bigcup_k D(w_k,t_k)$ with $\sum_k t_k\le 5H$ and (w_k,t_k) depend on z_1.

Of particular relevance to us with be the fact that a Cartan-2 set has a real "trace" of small measure.

Lemma 3.1. *Let $\mathcal{B}\subset\prod_{j=1}^2 D(z_{j,0},1)$ be a Cartan-2 set with parameter $H>0$. Then*

$$|\mathcal{B}\cap\mathbb{R}^2|\le 40H.$$

Proof. This follows from Fubini and $|D(\zeta,s)\cap\mathbb{R}|\le 2s$ for all $\zeta\in\mathbb{C}$. □

We can now formulate a Cartan-type bound for plurisubharmonic functions.

Lemma 3.2. *Let* $v : \overline{\mathbb{D}\times\mathbb{D}} \to [-\infty,\infty)$ *be continuous so that* $v = v(z_1,z_2)$ *is separately subharmonic in each variable. Suppose for* $0<\rho<r<1$

$$\max_{|z_1|\le r,|z_2|\le r} \int_{\mathbb{S}^1\times\mathbb{S}^1} v(e(\theta_1),e(\theta_2))\, P_{z_1}(d\theta_1)\, P_{z_2}(d\theta_2) \le M \tag{3-1}$$

and

$$\max_{|z_1|\le \rho,|z_2|\le \rho} v(z_1,z_2) \ge m. \tag{3-2}$$

Let $\rho = r(1-3\delta)$ *with* $0<\delta<\frac{1}{3}$. *Then for any* $0<H\le 1$ *one has*

$$v(z_1,z_2) \ge m-(M-m)(L+1)^2, \quad \text{where } L := 2\delta^{-3}\log\Big(\frac{2}{\delta}\Big)+\delta^{-2}\log\Big(\frac{2e}{H}\Big), \tag{3-3}$$

for all $(z_1,z_2)\in r_1\mathbb{D}\times r_1\mathbb{D}\setminus\mathcal{B}$ *where* $\mathcal{B}$ *is a Cartan-2 set with parameter* rH, *and* $r_1 = r(1-2\delta)$.

Proof. The function

$$h(z_1,z_2) := \int_{\mathbb{S}^1\times\mathbb{S}^1} v(e(\theta_1),e(\theta_2))\, P_{z_1}(d\theta_1)\, P_{z_2}(d\theta_2) \tag{3-4}$$

is separately harmonic in each variable, is continuous up to $\partial(\mathbb{D}\times\mathbb{D})$, and satisfies $v\le h$ pointwise. The latter property follows from the pointwise inequalities

$$v(z_1,z_2) \le \int_{\mathbb{S}^1} v(z_1,e(\theta_2))\, P_{z_2}(d\theta_2),$$

which hold due to harmonicity of the right-hand side in z_2, whence

$$v(z_1,z_2) \le \int_{\mathbb{S}^1} v(e(\theta_1),z_2)\, P_{z_1}(d\theta_1) \le \int_{\mathbb{S}^1\times\mathbb{S}^1} v(e(\theta_1),e(\theta_2))\, P_{z_1}(d\theta_1)\, P_{z_2}(d\theta_2) = h(z_1,z_2) \tag{3-5}$$

as claimed. Define

$$\tilde{v}(z_1) := \max_{|z_2|\le\rho} v(z_1,z_2). \tag{3-6}$$

Then $\tilde{v}$ is continuous (by uniform continuity) and subharmonic (as the supremum of a family of subharmonic functions). It satisfies $\tilde{v}(z_1)\le M$ for all $|z_1|\le r$ by (3-1) and (3-5), and $\max_{|z_1|\le\rho}\tilde{v}(z_1)\ge m$. The latter follows from

$$v(z_1,z_2) \le \tilde{v}(z_1) \quad \text{for all } |z_1|\le r,\ |z_2|\le\rho,$$

and (3-2).

We apply Corollary 2.3 to $\tilde{v}$, which requires rescaling from $\mathbb{D}$ to $r\mathbb{D}$. Thus, with $\rho = r(1-3\delta)$, and $r_1 = r(1-2\delta)$,

$$\tilde{v}(z_1) \ge m-(M-m)L =: m^* \tag{3-7}$$

for all $z_1\in r_1\mathbb{D}\setminus\bigcup_j D(\zeta_j,s_j)$ with $\sum_j s_j\le 5rH$. Fix such a *good* z_1. By definition, there exists z_2^* with $|z_2^*|\le\rho$ and $v(z_1,z_2^*)\ge m^*$. On the other hand, $v(z_1,z_2)\le M$ for all $|z_2|\le r$.

Once again, by Corollary 2.3 rescaled from $\mathbb{D}$ to $r\mathbb{D}$, it follows that

$$v(z_1,z_2) \ge m^*-(M-m^*)L \ge m-(M-m)L(2+L) \tag{3-8}$$

for all $z_2\in r_1\mathbb{D}\setminus\bigcup_j D(w_j,t_j)$ with $\sum_j t_j\le 5rH$. These disks depend on z_1. ∎

By means of Lemma 3.2 we establish a two-dimensional analogue of Proposition 2.6.

Proposition 3.3. *Let F be an analytic function of two variables on a neighborhood of $\mathcal{R}(q)\times\mathcal{R}(q)$ with $0<q\le q^*$, and F not identically equal to zero. Define*

$$B_1 := \|F\|_{L^2([\frac14,\frac34]\times[\frac14,\frac34])},\quad B_2 := \|F\|_{L^2(\partial\mathcal{R}(q)\times\partial\mathcal{R}(q))}.$$

For any $J\subset\big[\frac14,\frac34\big]\times\big[\frac14,\frac34\big]$ some Borel set of positive measure,

$$B_1\le e^{\frac{C}{q}}B_2^{1-\kappa}\|F\|_{L^2(J)}^{\kappa},$$

with $0<\kappa\le e^{-C/q}(\log(1/|J|))^{-2}$ with some absolute constant C.

Proof. Set $u(z_1,z_2):=\log|F(z_1,z_2)|$, which is plurisubharmonic on a neighborhood of $\mathcal{R}(q)\times\mathcal{R}(q)$. We pull u back to the polydisk $\mathbb{D}\times\mathbb{D}$, and define

$$v(z_1,z_2)=u(\Phi_q(z_1),\Phi_q(z_2))=\log|\widetilde{F}(z_1,z_2)|,\quad \widetilde{F}(z_1,z_2)=F(\Phi_q(z_1),\Phi_q(z_2)).$$

With h defined as in (3-4), for all $|z_1|,|z_2|\le r$,

$$\begin{aligned}
h(z_1,z_2)&=\int_0^1\int_0^1 v(e(\theta_1),e(\theta_2))\,P_{z_1}(d\theta_1)\,P_{z_2}(d\theta_2)\\
&=\int_0^1\int_0^1 \log|\widetilde{F}(e(\theta_1),e(\theta_2))|\,P_{z_1}(d\theta_1)\,P_{z_2}(d\theta_2)\\
&\le\log\bigg(\int_0^1\int_0^1 |\widetilde{F}(e(\theta_1),e(\theta_2))|\,P_{z_1}(d\theta_1)\,P_{z_2}(d\theta_2)\bigg)\\
&\le\log\bigg(\int_0^1\int_0^1 |\widetilde{F}(e(\theta_1),e(\theta_2))|\,d\theta_1\,d\theta_2\bigg\|\frac{P_{z_1}(d\theta)}{d\theta}\bigg\|_\infty\bigg\|\frac{P_{z_2}(d\theta)}{d\theta}\bigg\|_\infty\bigg)\\
&\le\log(B_2)+2\log\bigg(\bigg\|\frac{d\theta}{d\sigma}\bigg\|_{L^2(\partial\mathcal{R}(q))}\bigg)+2\sup_{|w|\le r}\log\bigg\|\frac{P_w(d\theta)}{d\theta}\bigg\|_\infty\\
&\le\log(B_2)+\log(Cq^{-1})+2\log\Big(\frac{2}{1-r}\Big),
\end{aligned}\tag{3-9}$$

where $d\sigma$ denotes arc-length measure on $\partial\mathcal{R}(q)$; see (2-24). By Lemma 2.1, we can apply Lemma 3.2 to v with $\rho=1-\exp(-A/q)$ with some absolute constant A,

$$m=\log B_1,\quad M=\log(B_2)+3Aq^{-1},\quad \delta=\exp\Big(-\frac{2A}{q}\Big),\quad r=\rho(1-3\delta)^{-1},$$

and $0<q\le q^*\ll 1$. Thus, for any $H>0$ there exists a Cartan-2 set $\mathcal{B}$ with parameter H such that for

$$r_1=1-\exp\Big(-\frac{A}{q}\Big)<r(1-2\delta),$$

and any $(z_1,z_2)\in r_1\mathbb{D}\times r_1\mathbb{D}\setminus\mathcal{B}$, we have

$$v(z_1,z_2)\ge m-(M-m)(L+1)^2,$$

where

$$L = 2e^{\frac{6A}{q}} \log(2e^{\frac{2A}{q}}) + e^{\frac{4A}{q}} \log\left(\frac{2e}{H}\right) < e^{\frac{8A}{q}} + e^{\frac{4A}{q}} \log\left(\frac{2e}{H}\right) - 1.$$

Returning to the original geometry, and analytic function F, we conclude the following via Lemmas 2.1 and 3.1: with $K := (e^{8A/q} + e^{4A/q} \log(2e/H))^2$,

$$B_1^{K+1} \le e^{\frac{3AK}{q}} |F(x_1, x_2)| B_2^K$$

for all $(x_1, x_2) \in \left[\frac{1}{4}, \frac{3}{4}\right] \times \left[\frac{1}{4}, \frac{3}{4}\right] \setminus \mathcal{E}$, where $\mathcal{E} \subset \mathbb{R}^2$ and $|\mathcal{E}| \le e^{5A/q} H$.

We now pick H so that $e^{5A/q} H = |J|/2$, and integrate over J, and we obtain

$$B_1^{K+1} \left(\frac{|J|}{2}\right)^{\frac{1}{2}} \le e^{\frac{3AK}{q}} B_2^K \|F\|_{L^2(J)}$$

or

$$B_1 \le e^{\frac{3A}{q}} \left(\frac{|J|}{2}\right)^{-\frac{\kappa}{2}} B_2^{1-\kappa} \|F\|_{L^2(J)}^{\kappa}, \quad \kappa \le (1+K)^{-1}. \tag{3-10}$$

We write $\kappa \le (1+K)^{-1}$ instead of $\kappa = (1+K)^{-1}$ since we could increase K. One easily checks that $(|J|/2)^{-\kappa/2} \lesssim 1$, and

$$K \le e^{\frac{C_1}{q}} \left(\log\left(\frac{1}{|J|}\right)\right)^2 - 1,$$

with some absolute constant C_1. Taking $C := \max(4A, C_1)$, and

$$K_0 := e^{\frac{C}{q}} \left(\log\left(\frac{1}{|J|}\right)\right)^2.$$

We conclude from (3-10) with the estimate $K \le K_0 - 1$ that

$$B_1 \le e^{\frac{C}{q}} B_2^{1-\kappa} \|F\|_{L^2(J)}^{\kappa}, \quad \kappa \le K_0^{-1},$$

as claimed. □

In analogy with the one-dimensional case in Proposition 2.7, we can deduce the following L^2 localization result.

Proposition 3.4. *Fix $\lambda \in \left(0, \frac{1}{2}\right]$ and for each integers n_1, n_2 let*

$$I_{n_1,n_2} \subset [n_1, n_1+1] \times [n_2, n_2+1]$$

be some Borel set with $|I_{n_1,n_2}| = \lambda$. Let $f \in L^2(\mathbb{R}^2)$ be band-limited; i.e., $\hat{f}$ is of compact support. Then for each $0 < q \le q^*$

$$\|f\|_{L^2(\mathbb{R}^2)}^2 \le e^{\frac{2C}{q}} \left(\sum_{(n_1,n_2)\in\mathbb{Z}^2} \|f\|_{L^2(I_{n_1,n_2})}^2 \right)^{\kappa} \|e^{2\pi q(|\xi_1|+|\xi_2|)} \hat{f}(\xi)\|_{L^2(\mathbb{R}^2)}^{2(1-\kappa)}, \tag{3-11}$$

with $0 < \kappa \le e^{-C/q}(-\log \lambda)^{-2}$, and C some absolute constant.

Proof. Let F be the entire function with $F = f$ on $\mathbb{R}^2$. Fix $0 \le t_1, t_2 \le 1$ and for $j = 1, 2$ define $\mathcal{R}_{n,t_j}(q)$ to be the rectangle with vertices $n-1-t_j \pm iq$, $n+2+t_j \pm iq$. We obtain from Proposition 3.3 that for any $n_1, n_2 \in \mathbb{Z}$

$$\|f\|_{L^2([n_1,n_1+1]\times[n_2,n_2+1])} \le e^{\frac{5C}{q}} \|F\|^{1-\kappa}_{L^2(\partial\mathcal{R}_{n_1,t_1}(q)\times\partial\mathcal{R}_{n_2,t_2}(q))} \|f\|^{\kappa}_{L^2(I_{n_1,n_2})},$$

with

$$\kappa \le e^{-\frac{5C}{q}} \left(\log\left(\frac{(3+2t_1)(3+2t_2)}{|I_{n_1,n_2}|}\right)\right)^{-2},$$

and C being the absolute constant in Proposition 3.3. Squaring, summing, and applying Hölder's inequality, we have

$$\|f\|^2_{L^2(\mathbb{R}^2)} \le e^{\frac{10C}{q}} \left(\sum_{(n_1,n_2)\in\mathbb{Z}^2} \|F\|^2_{L^2(\partial\mathcal{R}_{n_1,t_1}(q)\times\partial\mathcal{R}_{n_2,t_2}(q))}\right)^{1-\kappa} \left(\sum_{(n_1,n_2)\in\mathbb{Z}^2} \|f\|^2_{L^2(I_{n_1,n_2})}\right)^{\kappa}.$$

Taking expectation of the previous line with respect to $0 \le t_1, t_2 \le 1$, we obtain

$$\|f\|^2_{L^2(\mathbb{R}^2)} \le e^{\frac{10C}{q}} \left(\sum_{(n_1,n_2)\in\mathbb{Z}^2} \mathbb{E}_{t_1}\mathbb{E}_{t_2} \|F\|^2_{L^2(\partial\mathcal{R}_{n_1,t_1}(q)\times\partial\mathcal{R}_{n_2,t_2}(q))}\right)^{1-\kappa} \left(\sum_{(n_1,n_2)\in\mathbb{Z}^2} \|f\|^2_{L^2(I_{n_1,n_2})}\right)^{\kappa}. \tag{3-12}$$

By decomposing each $\partial\mathcal{R}_{n,t}(q)$ into its four sides, we decompose

$$\sum_{(n_1,n_2)\in\mathbb{Z}^2} \mathbb{E}_{t_1}\mathbb{E}_{t_2} \|F\|^2_{L^2(\partial\mathcal{R}_{n_1,t_1}(q)\times\partial\mathcal{R}_{n_2,t_2}(q))} \tag{3-13}$$

into the following three parts:

Part 1: *vertical and horizontal mixed terms.* This part contains eight terms; each can be bounded in the same way. Taking the left vertical side of $\mathcal{R}_{n_1,t_1}(q)$ and upper horizontal side of $\mathcal{R}_{n_2,t_2}(q)$ for example, we have

$$\begin{aligned}
\sum_{(n_1,n_2)\in\mathbb{Z}^2} \mathbb{E}_{t_2} \int_{\mathbb{R}} \mathbb{1}_{[n_2-1-t_2,n_2+2+t_2)} \mathbb{E}_{t_1} \int_{-q}^{q} |F(n_1-1-t_1+is, x_2+iq)|^2 \, ds \, dx_2 \\
&\le 5 \sum_{n_1\in\mathbb{Z}} \mathbb{E}_{t_1} \int_{\mathbb{R}} \int_{-q}^{q} |F(n_1-1-t_1+is, x_2+iq)|^2 \, ds \, dx_2 \\
&= 5 \int_{-q}^{q} \int_{\mathbb{R}^2} |F(x_1+is, x_2+iq)|^2 \, dx_1 \, dx_2 \, ds \\
&\le 5 \int_{-q}^{q} \int_{\mathbb{R}^2} e^{4\pi(s\xi_1+q\xi_2)} |\hat{f}(\xi_1,\xi_2)|^2 \, d\xi_1 \, d\xi_2 \, ds \\
&\le 10q \|e^{2\pi q(|\xi_1|+|\xi_2|)} \hat{f}(\xi)\|^2_{L^2(\mathbb{R}^2)},
\end{aligned}$$

in which we used (2-31) in the first step. Hence, Part 1 contributes in total at most

$$80q \|e^{2\pi q(|\xi_1|+|\xi_2|)} \hat{f}(\xi)\|^2_{L^2(\mathbb{R}^2)}. \tag{3-14}$$

Part 2: *vertical+vertical sides.* This part contains four terms. Taking the left vertical sides of $\mathcal{R}_{n_1,t_1}(q)$ and $\mathcal{R}_{n_2,t_2}(q)$ for example, we have

$$\sum_{(n_1,n_2)\in\mathbb{Z}^2} \mathbb{E}_{t_1}\mathbb{E}_{t_2} \int_{-q}^{q}\int_{-q}^{q} |F(n_1-1-t_1+is_1, n_2-1-t_2+is_2)|^2\, ds_1\, ds_2$$
$$= \int_{-q}^{q}\int_{-q}^{q}\int_{\mathbb{R}^2} |F(x_1+is_1, x_2+is_2)|^2\, dx_1\, dx_2\, ds_1\, ds_2$$
$$\le 4q^2 \|e^{2\pi q(|\xi_1|+|\xi_2|)}\hat{f}(\xi)\|^2_{L^2(\mathbb{R}^2)}.$$

Hence, Part 2 contributes in total at most

$$16q^2 \|e^{2\pi q(|\xi_1|+|\xi_2|)}\hat{f}(\xi)\|^2_{L^2(\mathbb{R}^2)}. \tag{3-15}$$

Part 3: *horizontal+horizontal sides.* This part also contains four terms. Taking the upper horizontal sides of $\mathcal{R}_{n_1,t_1}(q)$ and $\mathcal{R}_{n_2,t_2}(q)$ for example, we have

$$\sum_{(n_1,n_2)\in\mathbb{Z}^2} \mathbb{E}_{t_1}\mathbb{E}_{t_2} \int_{\mathbb{R}^2} \mathbb{1}_{[n_1-1-t_1,n_1+2+t_1)}\mathbb{1}_{[n_2-1-t_2,n_2+2+t_2)} |F(x_1+iq, x_2+iq)|^2\, dx_1\, dx_2$$
$$\le 25 \int_{\mathbb{R}^2} |F(x_1+iq, x_2+iq)|^2\, dx_1\, dx_2$$
$$\le 25 \|e^{2\pi q(|\xi_1|+|\xi_2|)}\hat{f}(\xi)\|^2_{L^2(\mathbb{R}^2)},$$

in which we used (2-31) in the first step. Hence, the contribution of Part 3 is at most

$$100 \|e^{2\pi q(|\xi_1|+|\xi_2|)}\hat{f}(\xi)\|^2_{L^2(\mathbb{R}^2)}. \tag{3-16}$$

Plugging the estimates in (3-14), (3-15) and (3-16) into (3-13), we obtain

$$\sum_{(n_1,n_2)\in\mathbb{Z}^2} \mathbb{E}_{t_1}\mathbb{E}_{t_2} \|F\|^2_{L^2(\partial\mathcal{R}_{n_1,t_1}(q)\times\partial\mathcal{R}_{n_2,t_2}(q))} \le (4q+10)^2 \|e^{2\pi q(|\xi_1|+|\xi_2|)}\hat{f}(\xi)\|^2_{L^2(\mathbb{R}^2)}$$
$$\le 144 \|e^{2\pi q(|\xi_1|+|\xi_2|)}\hat{f}(\xi)\|^2_{L^2(\mathbb{R}^2)} \tag{3-17}$$

for $q \le \frac{1}{2}$. Plugging (3-17) into (3-12) yields

$$\|f\|^2_{L^2(\mathbb{R}^2)} \le 144 e^{\frac{10C}{q}} \Bigg(\sum_{(n_1,n_2)\in\mathbb{Z}^2} \|f\|^2_{L^2(I_{n_1,n_2})}\Bigg)^{\kappa} \|e^{2\pi q(|\xi_1|+|\xi_2|)}\hat{f}(\xi)\|^{2(1-\kappa)}_{L^2(\mathbb{R}^2)},$$

as claimed. □

In general dimensions, one can proceed similarly. First, we inductively define Cartan sets in higher dimensions.

Definition 3.2. We say that $\mathcal{B}\subset\mathbb{C}^2$ is a Cartan-d set with parameter $H>0$ if for all $(z_1, z_2, \dots, z_d)\in\mathcal{B}$ one has either

- $z_1 \in \bigcup_j D(\zeta_j, s_j)$ with $\sum_j s_j \le 5H$ or for all other z_1 one has
- $(z_2, \dots, z_d)$ belongs to a Cartan-$(d-1)$ set with parameter $H>0$ depending on z_1.

By arguments analogous to those used above for $d=2$, one can exploit these Cartan sets in higher dimensions to obtain the following result. We leave the details to the reader. Throughout, we let $C(d)\ge 1$ be a constant depending only on the dimension d. It is allowed to change its values from line to line.

Proposition 3.5. *Fix $\lambda\in\left(0,\frac12\right]$ and for each integer vector $n=(n_1,\dots,n_d)\in\mathbb{Z}^d,\ d\ge 2$, let*

$$I_n\subset\prod_{j=1}^d[n_j,n_j+1)$$

be some Borel set with $|I_n|=\lambda$. Let $f\in L^2(\mathbb{R}^d)$ be band-limited; i.e., $\hat f$ is of compact support. Then for each $0<q\le q^*=q^*(d)\ll 1$

$$\|f\|_{L^2(\mathbb{R}^d)}^2\le e^{\frac{2C(d)}{q}}\Bigl(\sum_{n\in\mathbb{Z}^d}\|f\|_{L^2(I_n)}^2\Bigr)^{\kappa}\|e^{2\pi q|\xi|_1}\hat f(\xi)\|_{L^2(\mathbb{R}^d)}^{2(1-\kappa)},\tag{3-18}$$

with $0<\kappa\le e^{-C(d)/q}(-\log\lambda)^{-d},\ C(d)\ge 1$ some absolute constant depending on d.

As a precursor to the results of the next section, which involve L^2 functions with Fourier support in thin sets, we now establish an uncertainty principle for $L^2(\mathbb{R}^d)$ functions under a quantitative decay assumption on their Fourier transforms.

Corollary 3.6. *Let $\Theta(\xi)=\Theta(|\xi|_1)=(\log(2+|\xi|_1))^{-\alpha},\ 0<\alpha<1$. Let $\mathcal{S}:=\bigcup_{n\in\mathbb{Z}^d}I_n$ be as in Proposition 3.5. Then*

$$\|f\|_2\le C(d,\alpha,A,\lambda)\|f\|_{L^2(\mathcal{S})}\tag{3-19}$$

for all $f\in L^2(\mathbb{R}^d)$ with $\|e^{\Theta(\xi)|\xi|_1}\hat f\|_{L^2(\mathbb{R}^d)}\le A\|f\|_{L^2(\mathbb{R}^d)}$.

Proof. With $0<q$ small to be determined, we fix $R\ge 1$ so that $2\pi q=\Theta(R)$. Split $f=f_1+f_2$, $\hat f_1(\xi)=\hat f(\xi)\mathbb{1}_{[|\xi|_1\le R]}$. Then by (3-18), and since $2\pi q\le\Theta(\xi)$ for $|\xi|_1\le R$,

$$\|f_1\|_2^2\le e^{\frac{2C(d)}{q}}\|f_1\|_{L^2(\mathcal{S})}^{2\kappa}\|e^{\Theta(\xi)|\xi|_1}\hat f_1\|_2^{2(1-\kappa)}\le e^{\frac{2C(d)}{q}}\|f_1\|_{L^2(\mathcal{S})}^{2\kappa}(A\|f\|_2)^{2(1-\kappa)},$$

with

$$\kappa=e^{-\frac{C(d)}{q}}(-\log\lambda)^{-d}=e^{-\frac{2\pi C(d)}{\Theta(R)}}(-\log\lambda)^{-d}.$$

Moreover, since

$$\|f\|_2^2=\|f_1\|_2^2+\|f_2\|_2^2\le e^{\frac{2C(d)}{q}}(\|f\|_{L^2(\mathcal{S})}+\|f_2\|_2)^{2\kappa}(A\|f\|_2)^{2(1-\kappa)}+\|f_2\|_2^2$$

and

$$\|f_2\|_2\le e^{-\Theta(R)R}\|e^{\Theta(\xi)|\xi|_1}\hat f\|_2\le Ae^{-\Theta(R)R}\|f\|_2\le\tfrac12\|f\|_2,$$

where we chose R large enough depending on $A\ge 1$, it follows that

$$\|f\|_2^2\le 2e^{\frac{2C(d)}{q}}(\|f\|_{L^2(\mathcal{S})}+Ae^{-\Theta(R)R}\|f\|_2)^{2\kappa}(A\|f\|_2)^{2(1-\kappa)},$$

whence

$$\begin{aligned}\|f\|_2&\le 2^{\frac{1}{2\kappa}}A^{\frac{1-\kappa}{\kappa}}e^{\frac{C(d)}{\kappa q}}(\|f\|_{L^2(\mathcal{S})}+Ae^{-\Theta(R)R}\|f\|_2)\\&=2^{\frac{1}{2\kappa}}A^{\frac{1-\kappa}{\kappa}}e^{\frac{C(d)}{\kappa q}}\|f\|_{L^2(\mathcal{S})}+\exp\bigl(-T(R)\bigr)\|f\|_2,\end{aligned}$$

with

$$\begin{aligned} T(R) &= \Theta(R)R - \frac{C(d)}{\kappa q} - \kappa^{-1}\log(\sqrt{2}A) \\ &= \Theta(R)R - \left(\frac{2\pi C(d)}{\Theta(R)} + \log(\sqrt{2}A)\right) e^{\frac{2\pi C(d)}{\Theta(R)}}(-\log\lambda)^d. \end{aligned}$$

In addition to $2A \le e^{\Theta(R)R}$ we require that $T(R) \ge 1$. These conditions hold for sufficiently large R. □

The proof of the corollary gives an explicit and effective dependence of the constant $C(d,\alpha,A,\lambda)$ on A,λ, but we have no need for it. Corollary 3.6 follows (perhaps with a different dependence on the constants) from a quantitative version of the Logvinenko–Sereda theorem; see, e.g., [Kovrijkine 2001; Muscalu and Schlag 2013]. The results in the next section, however, do not.

4. Uncertainty principle with thin Fourier support

We begin with the concept of a damping function.

Definition 4.1. Let Θ be as in Corollary 3.6, with $\alpha \in (0,1)$ fixed. Let $Y \subset \mathbb{R}^d$. We say that Y *admits a damping function* with parameters c_1, c_2, c_3, all falling into the interval $(0,1)$, if there exists a function $\psi \in L^2(\mathbb{R}^d)$ satisfying

- $\operatorname{supp}(\psi) \subset [-c_1, c_1]^d$,
- $\|\hat{\psi}\|_{L^2([-1,1]^d)} \ge c_2$,
- $|\hat{\psi}(\xi)| \le \langle\xi\rangle^{-d}$ for all $\xi \in \mathbb{R}^d$,
- $|\hat{\psi}(\xi)| \le \exp\big(-c_3\Theta(|\xi|_1)|\xi|_1\big)$ for all $\xi \in Y$.

Lemma 4.1. *Fix $c_1 \in \big(0, \tfrac{1}{2}\big]$ and for each integer vector $n = (n_1, \dots, n_d) \in \mathbb{Z}^d$, $d \ge 2$, let*

$$I_n \subset \prod_{j=1}^{d} [n_j, n_j + 1)$$

be a square with side length $2c_1$. Define $\mathcal{S} := \bigcup_{n \in \mathbb{Z}^d} I_n$. Suppose $Y \subset \mathbb{R}^d$ is such that $Y + [-2,2]^d$ admits a damping function with parameters c_1, and $c_2, c_3 \in (0,1)$. Then every $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp}(\hat{f}) \subset Y$ satisfies

$$\|\hat{f}\|^2_{L^2([-1,1]^d)} \le C(d) c_2^{-2} \langle R\rangle^{2d} e^{\frac{4\pi C(d)}{c_3\Theta(R)}} \left(\|\mathbb{1}_{\mathcal{S}} f\|^{2\kappa}_{H^{-d}} \|f\|^{2(1-\kappa)}_{H^{-d}} + \exp(-2c_3\kappa\Theta(R)R)\, \|f\|^2_{H^{-d}}\right) \tag{4-1}$$

and $\kappa = e^{-2\pi C(d)/(c_3\Theta(R))}(-d\log c_1)^{-d}$, provided $R \ge (2d/c_3)^2$ and $0 < c_3 \le c_3^*(d) := 2\pi q_*$, where q_* is as in Proposition 3.5.

Proof. Let $\eta \in [-2,2]^d$. Set $f_\eta(x) := e^{2\pi i x\cdot\eta} f(x)$, and $g_\eta := f_\eta * \psi$, where ψ is the damping function as in Definition 4.1 associated with $Y + [-2,2]^d$. Split g_η into

$$\begin{gathered} g_\eta = g_1 + g_2, \\ \operatorname{supp}(\widehat{g_1}) \subset \{\xi \in \mathbb{R}^d : |\xi|_1 \le R\}, \quad \operatorname{supp}(\widehat{g_2}) \subset \{\xi \in \mathbb{R}^d : |\xi|_1 > R\}, \end{gathered} \tag{4-2}$$

where $2\pi q = c_3\Theta(R)$. Note that our assumption $c_3 \le 2\pi q_*$ guarantees that $q \le q_*$ holds for any $R \ge 1$. Note also that since $\mathrm{supp}(\psi) \subset [-c_1, c_1]^d$, we have $\mathbb{1}_{\mathcal{S}'} g_\eta = \mathbb{1}_{\mathcal{S}'}(\mathbb{1}_{\mathcal{S}} f_\eta * \psi)$, where $\mathcal{S}' := \bigcup_{n\in\mathbb{Z}^d} I_n'$, with I_n' a square with the same center as I_n, but half the side length. By Proposition 3.5 with $\lambda = c_1^d$ one has

$$\|g_\eta\|_2^2 = \|g_1\|_2^2 + \|g_2\|_2^2 \le e^{\frac{2C(d)}{q}} \big(\|g_\eta\|_{L^2(\mathcal{S}')} + \|g_2\|_2\big)^{2\kappa} \|e^{2\pi q|\xi|_1}\widehat{g_1}\|_2^{2(1-\kappa)} + \|g_2\|_2^2, \tag{4-3}$$

with

$$0 < \kappa \le e^{-\frac{C(d)}{q}} (-d \log c_1)^{-d} = e^{-\frac{2\pi C(d)}{c_3\Theta(R)}} (-d \log c_1)^{-d},$$

$C(d)$ some absolute constant. By construction, $\mathrm{supp}(\widehat{f_\eta}) \subset Y + \eta \subset Y + [-2,2]^d$; hence

$$|\widehat{g_\eta}(\xi)| \le |\widehat{f_\eta}(\xi)| \exp\big(-c_3\Theta(|\xi|_1)|\xi|_1\big) \quad \text{for all } \xi \in \mathbb{R}^d,$$

whence

$$\|e^{2\pi q|\xi|_1}\widehat{g_1}\|_2 = \|e^{c_3\Theta(R)|\xi|_1}\widehat{g_1}\|_2 \le \sup_{|\xi|_1 \le R} \langle\xi\rangle^d \|f_\eta\|_{H^{-d}} \le \langle R\rangle^d \, \|f_\eta\|_{H^{-d}},$$

$$\|g_2\|_2 \le \sup_{|\xi|_1 \ge R} \exp(-c_3\Theta(|\xi|_1)|\xi|_1)\langle\xi\rangle^d \, \|f_\eta\|_{H^{-d}} \le \exp(-c_3\Theta(R)R)\langle R\rangle^d \, \|f_\eta\|_{H^{-d}},$$

where we used that $|\xi|_2 \le |\xi|_1$, and that $r \mapsto \exp(-c_3\Theta(r)r)\langle r\rangle^d$ is decreasing for large r. To be specific,

$$\exp(-c_3\Theta(r)r)\langle r\rangle^d = \exp(-h(r)), \quad h(r) = c_3(\log(2+r))^{-\alpha}\, r - \frac{d}{2}\log(1+r^2).$$

Differentiating, we obtain

$$\begin{aligned} h'(r) &= c_3(\log(2+r))^{-\alpha}\left[1 - \frac{\alpha r}{2+r}(\log(2+r))^{-1}\right] - \frac{dr}{1+r^2} \\ &\ge \frac{c_3}{2}(\log(2+r))^{-\alpha} - dr^{-1} \ge \frac{c_3}{2}(\log(2+r))^{-1} - dr^{-1}, \end{aligned}$$

where we used that

$$\frac{\alpha r}{2+r}(\log(2+r))^{-1} \le \frac{1}{2}$$

for all $r \ge 0$. One has $u > \log(2+u^2)$ for $u \ge 2$, say. Hence, if $r \ge (2d/c_3)^2$, then

$$\frac{c_3}{2}(\log(2+r))^{-1} - dr^{-1} > 0$$

and thus $h'(r) > 0$. So it suffices to assume that $R \ge (2d/c_3)^2$.

Inserting these bounds into (4-3) yields

$$\begin{aligned} \|g_\eta\|_2^2 \le e^{\frac{2C(d)}{q}} \big(\|\mathbb{1}_{\mathcal{S}} f_\eta\|_{H^{-d}} + \exp(-c_3\Theta(R)R)\langle R\rangle^d \, \|f_\eta\|_{H^{-d}}\big)^{2\kappa} (\langle R\rangle^d \, \|f_\eta\|_{H^{-d}})^{2(1-\kappa)} \\ + \exp(-2c_3\Theta(R)R)\langle R\rangle^{2d} \, \|f_\eta\|_{H^{-d}}^2. \end{aligned}$$

Since $\sup_{\eta\in[-2,2]^d} \|f_\eta\|_{H^{-d}} \le C(d)\|f\|_{H^{-d}}$, we can simplify this further:

$$\|g_\eta\|_2^2 \le C(d)\langle R\rangle^{2d} \, e^{\frac{4\pi C(d)}{c_3\Theta(R)}} \big(\|\mathbb{1}_{\mathcal{S}} f\|_{H^{-d}}^{2\kappa} \|f\|_{H^{-d}}^{2(1-\kappa)} + \exp(-2c_3\kappa\Theta(R)R)\, \|f\|_{H^{-d}}^2\big). \tag{4-4}$$

Finally,

$$\begin{aligned}\|\widehat{f}\|^2_{L^2([-1,1]^d)} &\le c_2^{-2}\int_{[-1,1]^d}|\widehat{f}(\zeta)|^2\,d\zeta\int_{[-1,1]^d}|\widehat{\psi}(\xi)|^2\,d\xi\\ &\le c_2^{-2}\int_{[-1,1]^d}\int_{[-2,2]^d}|\widehat{f}(\xi-\eta)|^2|\widehat{\psi}(\xi)|^2\,d\eta\,d\xi\\ &\le c_2^{-2}\int_{[-2,2]^d}\int_{\mathbb{R}^d}|\widehat{f}(\xi-\eta)|^2|\widehat{\psi}(\xi)|^2\,d\xi\,d\eta = c_2^{-2}\int_{[-2,2]^d}\|g_\eta\|_2^2\,d\eta,\end{aligned}$$

and we are done. □

We now remove the localization in Fourier space on the left-hand side of (4-1) in order to obtain the main result of this section.

Corollary 4.2. *Fix $c_1 \in \left(0, \frac{1}{2}\right]$ and for each integer vector $n = (n_1, \dots, n_d) \in \mathbb{Z}^d$, $d \ge 2$, let*

$$I_n \subset \prod_{j=1}^{d}[n_j, n_j+1)$$

be a square with side length $2c_1$. Define $\mathcal{S} := \bigcup_{n\in\mathbb{Z}^d} I_n$. Suppose $Y \subset [-\alpha_1, \alpha_1]^d \subset \mathbb{R}^d$ with $\alpha_1 \ge 1$ is such that $Y + [-2,2]^d + \eta$ admits a damping function with parameters c_1, and $c_2, c_3 \in (0,1)$ for each $\eta \in [-\alpha_1 - 1, \alpha_1 + 1]^d$. Assume further that $0 < c_3 < c_3^*(d) \ll 1$, with $c_3^*(d)$ as in Lemma 4.1. Then every $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp}(\hat{f}) \subset Y$ satisfies

$$\|f\|_2 \le C_* \|f\|_{L^2(\mathcal{S})}, \tag{4-5}$$

with constant C_* depending only on d, c_1, c_2, c_3, α explicitly as in (4-15).

Proof. Let $\ell \in (2\mathbb{Z})^d$ be such that $\ell + [-1,1]^d \cap [-\alpha_1, \alpha_1]^d \ne \varnothing$ and define $f_\ell(x) := e^{2\pi i x\cdot\ell} f(x)$ so that $\widehat{f_\ell}(\xi) = \widehat{f}(\xi - \ell)$ and $\operatorname{supp}(\widehat{f_\ell}) \subset Y + \ell$. In order to apply Lemma 4.1, we also need to ensure that $Y + [-2,2]^d + \ell$ admits a damping function. This, however, follows from our assumptions. Hence, for each such ℓ,

$$\|\hat{f}\|^2_{L^2([-1,1]^d+\ell)} \le C(d)c_2^{-2}\,\langle R\rangle^{2d}\,e^{\frac{4\pi C(d)}{c_3\Theta(R)}}\left(\|\mathbb{1}_{\mathcal{S}} f_\ell\|^{2\kappa}_{H^{-d}}\|f_\ell\|^{2(1-\kappa)}_{H^{-d}} + \exp(-2c_3\kappa\Theta(R)R)\,\|f_\ell\|^2_{H^{-d}}\right) \tag{4-6}$$

and $\kappa = e^{-2\pi C(d)/(c_3\Theta(R))}(-d\log c_1)^{-d}$, provided $R \ge (2d/c_3)^2$. Summing over $\ell \in (2\mathbb{Z})^d$, and using Hölder's inequality yields

$$\begin{aligned}\|f\|_2^2 &\le C(d)c_2^{-2}\langle R\rangle^{2d}\,e^{\frac{4\pi C(d)}{c_3\Theta(R)}}\left(\|\mathbb{1}_{\mathcal{S}} f\|_2^{2\kappa}\|f\|_2^{2(1-\kappa)} + \exp(-2c_3\kappa\Theta(R)R)\|f\|_2^2\right)\\ &= C(d)c_2^{-2}\langle R\rangle^{2d}\,e^{\frac{4\pi C(d)}{c_3\Theta(R)}}\|\mathbb{1}_{\mathcal{S}} f\|_2^{2\kappa}\|f\|_2^{2(1-\kappa)} + C(d)c_2^{-2}\langle R\rangle^{2d}\,e^{\frac{4\pi C(d)}{c_3\Theta(R)}}e^{-2c_3\kappa\Theta(R)R}\|f\|_2^2.\end{aligned} \tag{4-7}$$

Suppose further that R satisfies

$$R \geq R_0(d,c_1,c_2,c_3,\alpha) := \max \begin{cases} \text{(i)} & \exp\Big[\Big(\frac{16\pi C(d)}{c_3}\Big)^{\frac{1}{1-\alpha}}\Big], \\ \text{(ii)} & \exp(4^{\frac{1}{1-\alpha}}), \\ \text{(iii)} & \Big(\frac{(-d\log c_1)^d}{c_3}\Big)^8, \\ \text{(iv)} & \Big(4\log\frac{2C(d)}{c_2^2}\Big)^2, \\ \text{(v)} & (8d)^4. \end{cases} \tag{4-8}$$

Note that (i), (ii), (iii) of (4-8) imply

$$e^{-\frac{2\pi C(d)}{c_3\Theta(R)}}(R+2)^{\frac14} \geq 1, \quad \Theta(R)(R+2)^{\frac18} \geq 1, \quad \text{and} \quad \frac{c_3}{(-d\log c_1)^d}(R+2)^{\frac18} \geq 1, \tag{4-9}$$

respectively. Hence multiplying the three inequalities of (4-9) yields

$$c_3\kappa\Theta(R)(R+2) \geq \sqrt{R+2} \quad \text{or} \quad \kappa \geq (c_3\Theta(R)\sqrt{R+2})^{-1}, \tag{4-10}$$

and thus

$$e^{2c_3\kappa\theta(R)R} \geq e^{c_3\kappa\theta(R)(R+2)} \geq e^{\sqrt{R+2}}. \tag{4-11}$$

One also derives from (iv), (v) and (i) that

$$\tfrac14\sqrt{R+2} \geq \log\frac{2C(d)}{c_2^2}, \quad \tfrac12\sqrt{R+2} \geq 2d\log(R+2) \geq \log\langle R\rangle^{2d}, \quad \text{and} \quad \tfrac14\sqrt{R+2} \geq \frac{4\pi C(d)}{c_3\Theta(R)}, \tag{4-12}$$

respectively. Hence by summing up the three inequalities of (4-12), and exponentiating, we obtain

$$e^{\sqrt{R+2}} \geq 2C(d)c_2^{-2}\langle R\rangle^{2d} e^{\frac{4\pi C(d)}{c_3\Theta(R)}}. \tag{4-13}$$

Combining (4-11) with (4-13), we arrive at

$$C(d)c_2^{-2}\langle R\rangle^{2d} e^{\frac{4\pi C(d)}{c_3\Theta(R)}} e^{-2c_3\kappa\Theta(R)R} \leq \tfrac12.$$

Thus (4-7) yields

$$\|f\|_2 \leq \big(2C(d)c_2^{-2}\langle R\rangle^{2d} e^{\frac{4\pi C(d)}{c_3\Theta(R)}}\big)^{\frac{1}{2\kappa}} \|\mathbb{1}_{\mathcal{S}} f\|_2.$$

Combining the estimate of κ in (4-10) with (4-13), we obtain

$$\big(2C(d)c_2^{-2}\langle R\rangle^{2d} e^{\frac{4\pi C(d)}{c_3\Theta(R)}}\big)^{\frac{1}{2\kappa}} \leq e^{\frac{c_3\Theta(R)(R+2)}{2}}.$$

Now we take R_0 as in (4-8) and define R_1 as

$$R_1(d,c_1,c_2,c_3,\alpha) := \max\bigg(\Big(\frac{2d}{c_3}\Big)^2, R_0(d,c_1,c_2,c_3,\alpha)\bigg). \tag{4-14}$$

Then

$$\|f\|_2 \leq C_*(d,c_1,c_2,c_3,\alpha)\|\mathbb{1}_{\mathcal{S}} f\|_2,$$

with

$$C_*(d,c_1,c_2,c_3,\alpha) = e^{\frac{c_3\Theta(R_1)(R_1+2)}{2}}, \tag{4-15}$$

as claimed. □

5. FUP assuming damping functions on Y

In this section we prove, by the same iteration as in [Bourgain and Dyatlov 2018], the fractal uncertainty principle for sets $X \subset [-1,1]^d$ and $Y \subset [-N,N]^d$. On Y we do not impose a geometric condition. Rather, in this section we still restrict ourselves to assuming the existence of damping functions living on Y, as well as on sets derived from Y through translations and dilations; see Definition 4.1. On X we impose a certain tree structure "with gaps"; see [Bourgain and Dyatlov 2018, Lemma 2.10].

Definition 5.1. We say that $X \subset [-1,1]^d \subset \mathbb{R}^d$ *is porous at scale* $L \geq 3$ *with depth* n, where L is an integer, if the following holds: denote by $\mathcal{C}_n$ the cubes obtained from $[-1,1]^d$ by partitioning it into congruent cubes of side length L^{-n}. Thus, $\#\mathcal{C}_n = 2^d L^{nd}$. The condition on X is that for all $Q \in \mathcal{C}_n$ with $Q \cap X \neq \varnothing$, there exists $Q' \in \mathcal{C}_{n+1}$ so that $Q' \subset Q$ and $Q' \cap X = \varnothing$.

It is shown in [Bourgain and Dyatlov 2018] that sets $X \subset \mathbb{R}$ obeying the δ-regularity condition on scales N^{-1} to 1 (see Definition 6.1) satisfy this porosity property at depth n for all $n \geq 0$ with $L^{n+1} \leq N$. We include a d-dimensional analogy in Appendix A; see Lemma A.7. We can now formulate the fractal uncertainty principle, conditionally on the existence of damping functions in Y. As in [Bourgain and Dyatlov 2018] the argument is based on an induction on scales, where at each step a small gain is achieved by means of Corollary 4.2. Recall that $\alpha \in (0,1)$ is the parameter from the damping function.

Theorem 5.1. *Let $X \subset [-1,1]^d \subset \mathbb{R}^d$ be porous at scale $L \geq 3$ with depth n for all $n \geq 0$ with $L^{n+1} \leq N$. Suppose $Y \subset [-N,N]^d$ is such that for all $n \geq 0$ with $L^{n+1} \leq N$ one has that for all*

$$\eta \in [-NL^{-n} - 3, NL^{-n} + 3]^d$$

the set

$$L^{-n} Y + [-4,4]^d + \eta \tag{5-1}$$

admits a damping function with parameters $c_1 = (2L)^{-1} \in \left(0, \tfrac{1}{2}\right]$, and $c_2, c_3 \in (0,1)$. Assume $0 < c_3 < c_3^*(d)$ as in Corollary 4.2. Then there exists $\beta = \beta(L, c_2, c_3, d, \alpha) > 0$ and $\widetilde{C} = \widetilde{C}(L, c_2, c_3, d, \alpha) > 0$ so that any $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp}(\hat{f}) \subset Y$ satisfies

$$\|f\|_{L^2(X)} \leq \widetilde{C} N^{-\beta} \|f\|_{L^2(\mathbb{R}^d)} \tag{5-2}$$

for all $N \geq N_0(L, c_2, c_3, d, \alpha)$.

Proof. We pick a nonnegative Schwartz function φ in $\mathbb{R}^d$ with $\operatorname{supp}(\hat{\varphi}) \subset [-1,1]^d$ and $\hat{\varphi}(0) = 1$. With $T \in \mathbb{N}$ to be determined, we set $\psi(x) := L^{Td} \varphi(L^T x)$ so that $\operatorname{supp}(\hat{\psi}) \subset [-L^T, L^T]^d$. Let

$$\mathcal{S}_n := \bigcup_{\substack{Q \in \mathcal{C}_n \\ Q \cap X \neq \varnothing}} Q \quad \text{and} \quad \mathcal{S}_n^* := \mathcal{S}_n + \left[-\frac{L^{-n}}{10}, \frac{L^{-n}}{10}\right]^d, \tag{5-3}$$

and define $\Psi_n := \psi_n * \mathbb{1}_{\mathcal{S}_{n+1}^*}$, where $\psi_k(x) := L^{kd} \psi(L^k x)$. There exists a constant C_φ depending only on φ such that for any $n \geq 0$

$$\Psi_n \geq \left(1 - \frac{C_\varphi}{L^{T-1}}\right) \mathbb{1}_X.$$

Thus, for all $m \geq 1$,

$$\prod_{n=0}^{m-1} \Psi_n \geq \left(1 - \frac{C_\varphi}{L^{T-1}}\right)^m \mathbb{1}_X. \tag{5-4}$$

Moreover, if $Q \in \mathcal{C}_{n+1}$ with $n \geq 0$ satisfies $Q \cap X = \varnothing$, denote by Q^* the cube with the same center as Q, but half the side length, i.e., of side length $L^{-(n+1)}/2$. Denote the collection of all such cubes Q^* by U_{n+1}. By the definitions of S^*_{n+1} and Q^*, we clearly have

$$S^*_{n+1} \cap \left(U_{n+1} + \left[-\tfrac{1}{10} L^{-(n+1)}, \tfrac{1}{10} L^{-(n+1)}\right]^d\right) = \varnothing.$$

Then for $x \in U_{n+1}$, and a constant c_φ that depends on φ only, we have

$$\begin{aligned} \Psi_n(x) &= \int_{\mathbb{R}^d} \varphi_{n+T}(x) \mathbb{1}_{S^*_{n+1}}(x-y)\, dy \\ &= \int_{\mathbb{R}^d} \varphi(y) \mathbb{1}_{S^*_{n+1}}(x - L^{-(n+T)} y)\, dy \\ &\leq \int_{\mathbb{R}^d \setminus [-\frac{1}{10} L^{T-1},\, \frac{1}{10} L^{T-1}]^d} \varphi(y)\, dy \leq \frac{c_\varphi}{L^{T-1}}, \end{aligned} \tag{5-5}$$

uniformly in n.

Let $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp}(\hat{f}) \subset Y$. Then for $m \geq 1$,

$$f_m := \prod_{n=0}^{m-1} \Psi_{nT} \cdot f$$

satisfies

$$\begin{aligned} \operatorname{supp}(\widehat{f_m}) &\subset Y + \sum_{n=0}^{m-1} \operatorname{supp}(\widehat{\psi_{nT}}) \\ &\subset Y + \sum_{n=0}^{m-1} [-L^{(n+1)T}, L^{(n+1)T}]^d = Y + \ell_m [-1,1]^d, \end{aligned} \tag{5-6}$$

where

$$\ell_m := L^T \frac{L^{mT} - 1}{L^T - 1}.$$

One has $f_{m+1} = \Psi_{mT} f_m$ for all $m \geq 0$ with $f_0 = f$. We claim that there exists $\gamma_0 = \gamma_0(L, d, c_1, c_2, c_3) \in (0,1)$ with

$$\| f_{m+1} \|_{L^2([-1,1]^d)} \leq (1 - \gamma_0) \| f_m \|_{L^2([-1,1]^d)}. \tag{5-7}$$

Define $g_m(x) := f_m(L^{mT} x)$. Then

$$\operatorname{supp}(\widehat{g_m}) \subset L^{-mT} Y + \ell_m L^{-mT} [-1,1]^d \subset L^{-mT} Y + [-2,2]^d, \tag{5-8}$$

where we used

$$\ell_m L^{-mT} \leq \frac{L^T}{L^T - 1} \leq 2.$$

In particular, assuming also that $L^{mT} \le N$,

$$\operatorname{supp}(\widehat{g_m}) \subset [-NL^{-mT}, NL^{-mT}]^d + [-2,2]^d = [-NL^{-mT}-2, NL^{-mT}+2]^d,$$

where $NL^{-mT}+2$ will be our parameter α_1 in Corollary 4.2.

Under this rescaling, the cubes in $\mathcal{C}_{mT}$ turn into unit cubes. Assuming further $L^{mT+1} \le N$, the porosity condition at scale L with depth mT ensures that we always have a "missing cube" of side length L^{-1} inside. In view of our definition of Q^*, we only use the concentric cube of half that side length. In view of the conditions on Y in the theorem we can apply Corollary 4.2 to g_m to obtain the following: with all norms being taken locally on $[-1,1]^d$, and with U_{mT+1} the missing cubes of the next generation as above,

$$\begin{aligned}
\|\Psi_{mT} f_m\|_2^2 &\le \|\Psi_{mT}\|_\infty^2 \|f_m\|_{L^2([-1,1]^d\setminus U_{mT+1})}^2 + \|\Psi_{mT}\|_{L^\infty(U_{mT+1})}^2 \|f_m\|_{L^2(U_{mT+1})}^2 \\
&\le \|f_m\|_{L^2([-1,1]^d\setminus U_{mT+1})}^2 + \|\Psi_{mT}\|_{L^\infty(U_{mT+1})}^2 \|f_m\|_{L^2(U_{mT+1})}^2 \\
&= \|f_m\|_{L^2([-1,1]^d)}^2 - (1-\|\Psi_{mT}\|_{L^\infty(U_{mT+1})}^2)\|f_m\|_{L^2(U_{mT+1})}^2 \\
&\le \left(1 - C_*^{-2}\left(1 - \frac{c_\varphi^2}{L^{2(T-1)}}\right)\right)\|f_m\|_2^2.
\end{aligned} \tag{5-9}$$

To obtain this estimate, we used that

$$\|\Psi_{mT}\|_\infty \le 1, \quad \|\Psi_{mT}\|_{L^\infty(U_{mT+1})} \le \frac{c_\varphi}{L^{T-1}},$$

and

$$\|f_m\|_{L^2(U_{mT+1})} \ge C_*^{-1}\|f_m\|_2^2,$$

with $C_* = C_*(d, L, c_2, c_3, \alpha)$ by Corollary 4.2. Choosing

$$\gamma_0(T) := \frac{1 - c_\varphi^2/L^{2(T-1)}}{2C_*^2}, \tag{5-10}$$

and using $(1-x)^{1/2} \le 1 - x/2$ for $0 \le x \le 1$, we have

$$\left(1 - C_*^{-2}\left(1 - \frac{c_\varphi^2}{L^{2(T-1)}}\right)\right)^{\frac{1}{2}} \le 1 - \gamma_0(T).$$

This establishes the claim (5-7).

Applying (5-7) iteratively and using (5-4), we obtain

$$\begin{aligned}
\|f\|_{L^2(X)} &\le \left(1 - \frac{C_\varphi}{L^{T-1}}\right)^{-(m+1)} \left\| \prod_{n=0}^{m} \Psi_n f \right\|_{L^2(X)} \\
&\le \left[\left(1 - \frac{C_\varphi}{L^{T-1}}\right)^{-1}(1-\gamma_0(T))\right]^{m+1} \|f\|_2 \le \left(1 - \frac{\gamma_0(T)}{2}\right)^{m+1}\|f\|_2.
\end{aligned} \tag{5-11}$$

In the last inequality we used

$$1 - \gamma_0(T) \le 1 - \frac{\gamma_0(T)}{2} - \frac{C_\varphi}{L^{T-1}} \le \left(1 - \frac{\gamma_0(T)}{2}\right)\left(1 - \frac{C_\varphi}{L^{T-1}}\right),$$

which requires

$$L^{T-1} - \frac{c_\varphi^2}{L^{T-1}} \geq 4C_\varphi C_*^2 \quad \text{or} \quad T \geq T_0(d, L, c_2, c_3, \alpha) := \left\lceil \frac{\log(2C_\varphi C_*^2 + \sqrt{4C_\varphi^2 C_*^4 + c_\varphi^2})}{\log L} \right\rceil. \tag{5-12}$$

Finally, for any $T \geq T_0$, taking $m \in \mathbb{N}$ be such that $L^{mT+1} \leq N < L^{(m+1)T+1}$, (5-11) yields (5-2) with

$$\beta = -\frac{\log(1 - \gamma_0(T)/2)}{T \log L}, \tag{5-13}$$

and

$$\widetilde{C} = \left(1 - \frac{\gamma_0(T)}{2}\right)^{-\frac{1}{T}}, \tag{5-14}$$

as claimed. In the current theorem, we could simply choose $T = T_0$. The flexibility of choosing T will simplify our computations in our proof of Theorem 1.2. ☐

6. Geometry of Y and damping functions

6A. *Regular sets.* We will call a set $I = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d]$ of equal side lengths a *d-dimensional cube* in $\mathbb{R}^d$; we denote its side length by r_I.

Recall the notion of δ-regularity from [Bourgain and Dyatlov 2018, Definition 1.1]; below is a d-dimensional analogy.

Definition 6.1. Suppose $X \subset \mathbb{R}^d$, $X \neq \varnothing$ is closed, and $0 < \delta < d$, $C_R \geq 1$, $0 \leq \alpha_0 \leq \alpha_1 \leq \infty$. Then X is δ-regular on scales α_0 to α_1, with constant C_R, if there exists a Borel measure μ_X with the following properties:

- μ_X is supported on X.
- $\mu_X(I) \leq C_R r_I^\delta$ for each d-dimensional cube I of side length $\alpha_0 \leq r_I \leq \alpha_1$.
- $\mu_X(I) \geq C_R^{-1} r_I^\delta$ for each d-dimensional cube $I \subset \mathbb{R}^d$, centered at a point in X and of side length $\alpha_0 \leq r_I \leq \alpha_1$.

See [Bourgain and Dyatlov 2018, Section 2.2] for the geometry of such sets in $\mathbb{R}$. Loosely speaking, they behave like δ-dimensional fractal sets. The properties of δ-regular sets carry over to higher dimensions. We include some properties in Appendix A.

6B. *Geometry of Y and damping functions.* Bourgain and Dyatlov observed that δ-regular sets on $\mathbb{R}$ admit damping functions as in Definition 4.1 above with $\alpha = (1 + \delta)/2$. They obtained these functions as a consequence of the Beurling–Malliavin theorem [1962]. However, one does not need the full strength of this theorem. To be more precise, in place of the original Beurling–Malliavin condition $\|(\log \omega)'\|_\infty < \infty$, with ω the weight, a much easier proof is possible (via outer functions) if we assume instead that $\|(H \log \omega)'\|_\infty \ll 1$ where H is the Hilbert transform on $\mathbb{R}$; see [Mashregi et al. 2005, Section 1.14, Theorem 1]. By means of this technique, Jin and Zhang [2017, Lemma 4.1] proved the following quantitative result on damping functions.

Lemma 6.1. *Let $S \geq 1$ be a constant. Let $Y \subset [-SN, SN]$ be δ_1-regular on scales 2 to N, with constant C_R, $0 < \delta_1 < 1$. For any $0 < c_1 < 1$, Y admits a damping function with $\alpha = (1+\delta_1)/2$ and parameters c_1,*

$$c_2 = \iota c_1^6, \quad c_3 = \iota c_1 C_R^{-2} \delta_1 (1-\delta_1), \tag{6-1}$$

where $\iota > 0$ is some small constant that depends on S. Instead of the pointwise global decay of $\langle \xi \rangle^{-1}$ in Definition 4.1, we have

$$|\widehat{\psi}(\xi)| \leq \exp(-c_3 \langle \xi \rangle^{\frac{1}{2}}) \quad \textit{for all } \xi \in \mathbb{R}. \tag{6-2}$$

In this paper we need a slightly different version, where we have pointwise lower bound of $|\widehat{\psi}(\xi)|$ on $\left[-\frac{3}{4}, \frac{3}{4}\right]$. The advantage of a pointwise lower bound over an L^2 bound is that it leads to a lower bound of the product of several $\widehat{\psi}$'s. Let us also note that in Lemma 4.1 of [Jin and Zhang 2017], $S = 1$. But it is clear from their proof that it works for any $S \geq 1$. We will briefly discuss the changes of constants caused by S in Appendix B. We need the extra factor S in our proof of Lemma 6.3.

Lemma 6.2. *Let $S \geq 1$ be a constant. Assume that $Y \subset [-SN, SN]$ is a δ_1-regular set with constant C_R on scales 2 to N and $\delta_1 \in (0,1)$. Fix $0 < c_1 < 1$; then there exists a function $\psi \in L^2(\mathbb{R})$ such that*

$$\begin{aligned} &\operatorname{supp} \psi \subset \left[-\tfrac{1}{10}c_1, \tfrac{1}{10}c_1\right], \\ &|\widehat{\psi}(\xi)| \leq \exp(-c_3 \langle \xi \rangle^{\frac{1}{2}}) \qquad \textit{for all } \xi \in \mathbb{R}, \\ &|\widehat{\psi}(\xi)| \leq \exp(-c_3 \Theta(|\xi|)|\xi|) \quad \textit{for all } \xi \in Y,\ |\xi| \geq 10, \end{aligned}$$

and

$$|\widehat{\psi}(\xi)| \geq c_2 \quad \textit{for all } \xi \in \left[-\tfrac{3}{4}, \tfrac{3}{4}\right], \tag{6-3}$$

with

$$\alpha = \frac{1+\delta_1}{2}, \quad c_2 = \iota c_1^{10}, \quad c_3 = \iota c_1 C_R^{-2} \delta_1 (1-\delta_1),$$

where $\iota > 0$ is some small constant that depends on S.

We include the proof of Lemma 6.2 in Appendix B.

In higher dimensions, we reduce ourselves to this one-dimensional setting by taking finite unions of products. For simplicity, we restrict ourselves to two dimensions, although the exact analogue can be done in any finite dimension.

Definition 6.2. Pick some $\varepsilon_0 \in (0,1)$ and let $Y \subset \mathbb{R}^2$ be of the form

$$Y \subset \bigcup_{j=1}^{m} Y_j, \quad \text{where } Y_j = \{\xi_1 \vec{e}_{j,1} + \xi_2 \vec{e}_{j,2} : \xi_i \in Y_{j,i},\ i = 1,2\}. \tag{6-4}$$

Here $\vec{e}_{j,i} \in \mathbb{S}^1$ with $|\vec{e}_{j,1} \cdot \vec{e}_{j,2}| < 1 - \varepsilon_0$ for all $1 \leq j \leq m$, and $Y_{j,i}$ are δ_1-regular on scales α_0 to α_1 with constant C_R, where $0 < \delta_1 < 1$. In that case Y is called *admissible on scales α_0 to α_1* with parameters δ_1, C_R, ε_0, m. In general dimensions, we require that $\vec{e}_{j,i}$ are unit vectors with $|\det(\vec{e}_{j,1}, \dots, \vec{e}_{j,d})| \geq \varepsilon_0$; see (1-3).

Throughout, we will freeze ε_0 and constants are allowed to depend on it. The admissible sets on scale 2 to N that are contained in $[-N,N]^d$ carry damping functions.

We note that for our proof of Theorem 1.2, we only need $m=1$. We give a construction with arbitrary $m\geq 1$ here, since the construction itself may be of independent interest.

Lemma 6.3. *Let $Y\subset[-N,N]^2$ be admissible on scales 2 to N as in Definition 6.2. Then Y admits a damping function with parameters c_1,*

$$c_2=\iota^{2m+4}c_1^{20m+4}\,m^{-20m}\,C_R^{-8}(\delta_1(1-\delta_1))^4,$$
$$c_3=\iota c_1\,m^{-1}C_R^{-2}\delta_1(1-\delta_1),$$

where $\iota>0$ is a small constant that depends on ε_0.

Remark 6.4. For general dimension d, we can take

$$c_2=\iota^m c_1^{(10m+2)d}\,m^{-10md}\,C_R^{-4d}(\delta_1(1-\delta_1))^{2d},$$
$$c_3=\iota c_1\,m^{-1}C_R^{-2}\delta_1(1-\delta_1),$$

where $\iota>0$ is a small constant that depends on ε_0 and d.

Proof. Let $\psi_{j,i}$ be the damping function associated with $Y_{j,i}\subset[-SN,SN]$, with $S=S(\varepsilon_0)\geq 1$, via Lemma 6.2 with parameters $\tilde{c}_1:=\varepsilon_1 c_1 m^{-1}$, where ε_1 is a small parameter depending on ε_0, and c_2,c_3 are as given by Lemma 6.2, but in terms of $\tilde{c}_1$; i.e.,

$$c_2=\iota\varepsilon_1^{10}\,c_1^{10}m^{-10},$$
$$c_3=c_1\,m^{-1}\iota\varepsilon_1\,C_R^{-2}\delta_1(1-\delta_1),$$

where ι depends ε_0. We will absorb the constant ε_1 into ι. In the following we will also allow ι to change its value from line to line, as long as it only depends on ε_0.

Denote the coordinates associated with the basis $\vec{e}_{j,1}$, $\vec{e}_{j,2}$ by $(\xi_{j,1},\xi_{j,2})$. We set, with $\xi\in\mathbb{R}^2$,

$$\widehat{\psi}(\xi):=\prod_{j=1}^m\widehat{\psi}_j(\xi),\quad \widehat{\psi}_j(\xi):=\widehat{\psi_{j,1}}(\xi_{j,1})\widehat{\psi_{j,2}}(\xi_{j,2}).$$

Then

$$|\widehat{\psi}_j(\xi)|\leq\exp(-c_3\langle\xi_{j,1}\rangle^{\frac12})\exp(-c_3\langle\xi_{j,2}\rangle^{\frac12})\leq\exp(-c_3\langle\xi\rangle^{\frac12}),\tag{6-5}$$

where c_3, more precisely, ι, can change its value in the last line depending on ε_0. Taking products gives

$$|\widehat{\psi}(\xi)|\leq\exp(-mc_3\langle\xi\rangle^{\frac12})=\exp(-c_1\nu\langle\xi\rangle^{\frac12}),\quad \nu=\iota C_R^{-2}\delta_1(1-\delta_1).\tag{6-6}$$

In particular, $\psi\in L^2(\mathbb{R}^2)$ as well as $\psi_j\in L^2(\mathbb{R}^2)$. Since ψ_j are also compactly supported functions, $\psi_j\in L^1(\mathbb{R}^2)$. Hence in the sense of L^1 functions,

$$\psi=\mathop{\huge *}_{j=1}^m\psi_j,$$

whence

$$\operatorname{supp}(\psi) \subset \sum_{j=1}^{m} \operatorname{supp}(\psi_j) \subset \sum_{j=1}^{m} [-c_1 m^{-1}, c_1 m^{-1}]^2 \subset [-c_1, c_1]^2,$$

where we used that each $\psi_{j,i}$ is a damping function with $\tilde{c}_1 = \varepsilon_1 c_1 m^{-1}$. Next, if $\xi \in Y_j$, then

$$|\widehat{\psi}_j(\xi)| \le \exp(-c_3\Theta(|\xi_{j,1}|)|\xi_{j,1}|)\exp(-c_3\Theta(|\xi_{j,2}|)|\xi_{j,2}|) \le \exp(-c_3\Theta(|\xi|_1)|\xi|_1),$$

where again ι is allowed to change in the second line. Since Y is covered by the union of Y_j, we have

$$|\widehat{\psi}(\xi)| \le \exp(-c_3\Theta(|\xi|_1)|\xi|_1) \quad \text{for all } \xi \in Y. \tag{6-7}$$

Finally, from (6-3), for each $1 \le j \le m$,

$$|\widehat{\psi}_j(\xi)| \ge c_2^2 \quad \text{for all } \xi_{j,1}, \xi_{j,2} \in \left[-\tfrac{3}{4}, \tfrac{3}{4}\right].$$

Hence,

$$\|\widehat{\psi}\|_{L^2([-1,1]^2)} \ge c_2^{2m}|E|^{\frac{1}{2}},$$

where E is the subset of $[-1,1]^2$ where all conditions $\xi_{j,i} \in \left[-\frac{3}{4}, \frac{3}{4}\right]$, $i = 1, 2$, $1 \le j \le m$, are met. Clearly, $|E|^{1/2}$ is some number depending on ε_0. It follows that

$$\|\widehat{\psi}\|_{L^2([-1,1]^2)} \ge \iota^{2m} c_1^{20m} m^{-20m}, \tag{6-8}$$

where ι depends on ε_0.

We required $|\widehat{\psi}(\xi)| \le \langle\xi\rangle^{-2}$ in our definition of damping function; see Definition 4.1. Since for any $0 < \rho < 1$

$$\exp(-\rho\langle\xi\rangle^{\frac{1}{2}}) \le 5\rho^{-4}\langle\xi\rangle^{-2},$$

it follows from (6-6) that $\tilde{\psi} := \frac{1}{5}(c_1\nu)^4\psi$ is a damping function in the sense of the definition. Since $\frac{1}{5}(c_1\nu)^4 \le 1$, the decay (6-7) remains intact, as does the support condition. However, (6-8) needs to be modified:

$$\|\widehat{\tilde{\psi}}\|_{L^2([-1,1]^2)} \ge \tfrac{1}{5}(c_1\nu)^4\iota^{2m} c_1^{20m} m^{-20m} = \tfrac{1}{5}\iota^{2m+4} c_1^{20m+4} m^{-20m} C_R^{-8}(\delta_1(1-\delta_1))^4.$$

Absorbing the $\frac{1}{5}$ into ι, the lemma is proved. □

Finally, we need to check that Y remains admissible if it is transformed by the similarities in (5-1).

Lemma 6.5. *Let $Y \subset [-N, N]^d$ with $N \ge 10$ be admissible on scales 2 to N with parameters δ_1, C_R, ε_0, m. Let $L \ge 4$ be an integer. Then for all integers $n \ge 0$ with $L^{n+1} \le N$ and for all*

$$\eta \in [-NL^{-n} - 3, NL^{-n} + 3]^d,$$

the set

$$L^{-n}Y + [-4, 4]^d + \eta \subset [-(2NL^{-n} + 7), 2NL^{-n} + 7]^d$$

is admissible at scale $S(2NL^{-n} + 7)$ with parameters δ_1, $576S^2C_R$, ε_0, m, where $S = S(\varepsilon_0, d) \ge 1$.

Proof. First,

$$L^{-n}Y+[-4,4]^d+\eta\subset[-2NL^{-n}-7,2NL^{-n}+7]^d$$

for all η as above. Second, by (6-4),

$$L^{-n}Y+[-4,4]^d+\eta\subset\bigcup_{j=1}^{m}(L^{-n}Y_j+[-4,4]^d+\eta),$$

where

$$L^{-n}Y_j=\left\{\sum_{k=1}^{d}\xi_k\vec{e}_{j,k}:\xi_k\in L^{-n}Y_{j,k},\ k=1,2,\dots,d\right\},$$

and

$$L^{-n}Y_j+[-4,4]^d+\eta\subset\left\{\sum_{k=1}^{d}\xi_k\vec{e}_{j,k}:\xi_k\in L^{-n}Y_{j,k}+[-4S,4S]+\eta_{j,k},\ k=1,2,\dots,d\right\},$$

where $S=S(\varepsilon_0,d)\ge 1$ and $|\eta_{j,k}|\le S(NL^{-n}+3)$. By Lemmas 2.1, 2.2, and 2.3 in [Bourgain and Dyatlov 2018], see also Lemmas A.2, A.3, and A.4 with $d=1$, the sets

$$L^{-n}Y_{j,k}+[-4S,4S]+\eta_{j,k}\subset[-S(2NL^{-n}+7),S(2NL^{-n}+7)]$$

are δ_1-regular with constant $576S^2C_R$ on scales 2 to $S(2NL^{-n}+7)$. Indeed, for $n\ge 1$, Lemma A.2 implies that $L^{-n}Y_{j,k}$ is δ_1-regular on scales $2L^{-n}\le\frac{1}{2}$ to $L^{-n}N$ with constant C_R. Lemma A.4 implies that

$$L^{-n}Y_{j,k}+[-4S,4S]=L^{-n}Y_{j,k}+8S\left[-\tfrac{1}{2},\tfrac{1}{2}\right]$$

is δ_1-regular on scales 1 to $L^{-n}N$ with constant $32SC_R$. Lemma A.3 allows us to increase the upper scale from $L^{-n}N$ to $9SL^{-n}N\ge S(2L^{-n}N+7)$, with changing the constant from $32SC_R$ to $576S^2C_R$. Note that shifting a set does not change its δ_1-regularity; hence $L^{-n}Y_{j,k}+[-4S,4S]+\eta_{j,k}$ is δ_1-regular with constant $576S^2C_R$. The proof for $n=0$ is similar.

The lemma now follows from Definition 6.2. □

6C. *Proof of Theorem 1.2.*

Proof. The proof of Theorem 1.2 is now a corollary to Theorem 5.1 and the considerations in this section, with $m=1$. We will keep track of various constants in order to obtain the effective exponent β.

First, let

$$L:=\lceil(2^{\frac{d}{2}}\sqrt{2d+1}C_R)^{\frac{2}{d-\delta}}\rceil\ge 4$$

be as in (A-3). Lemma A.7 implies that for all $n\ge 0$ with $L^{n+1}\le N$, X is porous at scale L with depth n. This verifies the porosity condition on X in Theorem 5.1.

Combining Lemma 6.3, more specifically Remark 6.4, with Lemma 6.5, we obtain that for any $n\in\mathbb{N}$ such that $L^{n+1}\le N$, and for all $\eta\in[-L^{-n}N-3,L^{-n}N+3]^d$, the set

$$L^{-n}Y+[-4,4]^d+\eta$$

admits a damping function with parameters c_1,

$$c_2 = \iota c_1^{12d}(576S^2C_R)^{-4d}(\delta_1(1-\delta_1))^{2d},$$
$$c_3 = \iota c_1(576S^2C_R)^{-2}\delta_1(1-\delta_1),$$

where ι and S are constants depending on ε_0. We absorb the constant S into ι, and allow ι to depend on d as well. Hence we can simply write

$$c_2 = \iota c_1^{12d}C_R^{-4d}(\delta_1(1-\delta_1))^{2d},$$
$$c_3 = \iota c_1 C_R^{-2}\delta_1(1-\delta_1).$$

Note that this verifies the condition on Y in Theorem 5.1.

Before applying Theorem 5.1, let us first determine the constant C_* in Corollary 4.2 with c_1, c_2, c_3 defined above. Recall that

$$C_* = e^{\frac{c_3\Theta(R_1)(R_1+2)}{2}},$$

with $\alpha = (1+\delta_1)/2$ and let

$$R_1 = \max\begin{cases}\exp\Big[\Big(\dfrac{C_R^2}{\iota c_1\delta_1(1-\delta_1)}\Big)^{\frac{2}{1-\delta_1}}\Big]\\ \exp(4^{\frac{2}{1-\delta_1}})\\ \Big(\dfrac{C_R^2(-\log c_1)^d}{\iota c_1\delta_1(1-\delta_1)}\Big)^8\\ \Big[4\log\Big(\dfrac{C_R^{8d}}{\iota c_1^{24d}(\delta_1(1-\delta_1))^{4d}}\Big)\Big]^2\\ (8d)^4\\ \dfrac{C_R^4}{\iota c_1^2(\delta_1(1-\delta_1))^2}\end{cases} \tag{6-9}$$

be as in (4-14), in which we absorb all the d-dependent constants into ι.

Now we can apply Theorem 5.1 with

$$c_1 = (2L)^{-1} = (2\lceil(2^{\frac{d}{2}}\sqrt{2d+1}C_R)^{\frac{2}{d-\delta}}\rceil)^{-1}.$$

We need to trace out the constant β.

Plugging c_1 into (6-9), and making ι smaller if necessary (depending only on d and ε_0), we have

$$R_1 \le \exp\Big[\Big(\frac{(C_R^2/\iota)^{\frac{2d-2\delta+2}{d-\delta}}}{\delta_1(1-\delta_1)}\Big)^{\frac{2}{1-\delta_1}}\Big] =: R_2.$$

This implies

$$C_* = \exp\big(c_1C_R^{-2}\delta_1(1-\delta_1)\Theta(R_1)(R_1+2)\big) \le \exp(R_2).$$

Recall T_0 as in (5-12) and γ_0 as in (5-10). We compute that

$$T_0 = \Big\lceil\frac{\log(2C_\varphi C_*^2+\sqrt{4C_\varphi^2C_*^4+c_\varphi^2})}{\log L}\Big\rceil \le \frac{2\log C_* + \log(5C_\varphi)}{\log L} \le \frac{2R_2+\log(5C_\varphi)}{\log L} =: T_1, \tag{6-10}$$

and

$$\gamma_0(T_1) = \frac{1 - c_\varphi^2 / L^{2(T_1-1)}}{2C_*^2} \geq \frac{1}{4C_*^2} \geq \frac{1}{4}\exp(-2R_2). \tag{6-11}$$

In both inequalities above, we used $C_* \leq \exp(R_2)$.

Recall β as in (5-13). Using that $-\log(1-x) \geq x$ for $x < 1$, we have

$$\beta = -\frac{\log(1 - \gamma_0(T_1)/2)}{T_1 \log L} \geq \frac{\gamma_0(T_1)}{2T_1 \log L}.$$

Combining this with the estimates of T_1 and $\gamma_0(T_1)$ as in (6-10) and (6-11), we have

$$\beta \geq \exp\biggl\{-\exp\biggl[\biggl(\frac{(C_R^2/\iota)^{\frac{2d-2\delta+2}{d-\delta}}}{\delta_1(1-\delta_1)}\biggr)^{\frac{2}{1-\delta_1}}\biggr]\biggr\},$$

with ι being a small constant depending on ε_0 and d. □

Corollary 1.3 follows from Theorem 1.2 by the triangle inequality.

Remark 6.6. If we try to combine the construction of a damping function for m covers as in Lemma 6.3, with Theorem 5.1, we could allow m to grow in N like $\log\log\log N$. This is worse than the power-law growth obtained via the triangle inequality.

6D. *Distortion of Y by diffeomorphisms.* Let $\mathcal{F}_\hbar$ be the unitary semiclassical Fourier transform on $L^2(\mathbb{R}^d)$ defined by

$$\mathcal{F}_\hbar f(\xi) = \hbar^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{2\pi i x\cdot\xi}{\hbar}} f(x)\,dx = \hbar^{-\frac{d}{2}} \hat{f}\Bigl(\frac{\xi}{\hbar}\Bigr).$$

We will use the following proposition which roughly says that the intersection of an admissible set with a cube is still admissible. We only work with admissible sets with $m = 1$ throughout this section.

Proposition 6.7. *Let $Y \subset \mathbb{R}^d$ be an admissible set on scales N^{-1} to 1 with parameters $\delta_1, C_R, \varepsilon_0$. Let $Q \subset \mathbb{R}^d$ be a cube of side length $r_Q \leq r_0$. Then*

$$Y \cap Q \subseteq \bigcup_{j=1}^{C(\varepsilon_0,d,r_0)} W_j,$$

where each W_j is contained in a cube of side length $C(\varepsilon_0, d)$, and is admissible on scales N^{-1} to 1 with parameters δ_1, $(4C_R)^{2/(1-\delta_1)}C_R, \varepsilon_0$.

Proof. Let $Y = \bigl\{\sum_{k=1}^d \xi_k \vec{e}_k : \xi_k \in Y_k\bigr\}$, where $\vec{e}_k \in \mathbb{S}^1$ and $|\det(\vec{e}_1, \dots, \vec{e}_d)| \geq \varepsilon_0$. We cover Q by the smallest parallelepiped $\widetilde{Q}$, whose edges are determined by $\vec{e}_1, \dots, \vec{e}_d$, that contains Q. We can write $\widetilde{Q} = \bigl\{\sum_{k=1}^d \xi_k \vec{e}_k : \xi_k \in \widetilde{Q}_k\bigr\}$.

By Lemma A.1, there exist disjoint intervals $\mathcal{J}_k$ such that

$$Y_k = \bigcup_{J_{k,\ell} \in \mathcal{J}_k} (Y_k \cap J_{k,\ell}), \qquad \text{with } (4C_R)^{-\frac{2}{1-\delta_1}} \leq |J_{k,\ell}| \leq 1 \quad \text{for all } J_{k,\ell} \in \mathcal{J}_k,$$

where the $(Y_k \cap J_{k,\ell})$'s are δ_1-regular sets with constant $\widetilde{C}_R = (4C_R)^{2/(1-\delta_1)} C_R$ on scales N^{-1} to 1. For any $\underline{\ell} \in \mathbb{N}^d$, let $Y_{\underline{\ell}} := \{\sum_{k=1}^d \xi_k \vec{e}_k : \xi_k \in Y_k \cap J_{k,\ell_k}\}$. Hence $Y_{\underline{\ell}}$ is admissible on scales N^{-1} to 1 with parameters $\delta_1, \widetilde{C}_R, \varepsilon_0$. Furthermore, $Y_{\underline{\ell}}$ is contained in a cube of side length $C(\varepsilon_0, d)$. Finally note that $\widetilde{Q}_k$ intersects at most finitely many $J_{k,\ell}$'s, and this number depends only on ε_0, d and r_0. □

In this section we prove Theorem 1.4. We need to show that Theorem 1.2 remains valid if an admissible set Y is distorted by a diffeomorphism $\Phi_N(x)$ from the cube $[-N, N]^d \to [-N, N]^d$; see (1-5). The argument is related to Section 4 of [Bourgain and Dyatlov 2018]. Thus, let $Y = \Phi_N(\widetilde{Y})$, where $\widetilde{Y} \subset [-N, N]^d$ is an admissible set with constants C_R, ε_0 on scales 1 to N. Suppose $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp}(\hat{f}) \subset Y$ and set $\hat{g} := \hat{f} \circ \Phi_N$ so that $\operatorname{supp}(\hat{g}) \subset \widetilde{Y}$. Furthermore,

$$\begin{aligned} f(x) &= \int_{[-N,N]^d} e^{2\pi i x\cdot\xi}\, \hat{f}(\xi)\, d\xi = \int_{[-N,N]^d} e^{2\pi i x\cdot\xi}\, \hat{g}(\Phi_N^{-1}(\xi))\, d\xi \\ &= \int_{[-N,N]^d} e^{2\pi i x\cdot\Phi_N(\eta)}\, \hat{g}(\eta)\, |\det(D\Phi_N(\eta))|\, d\eta. \end{aligned} \tag{6-12}$$

We claim that for some $\beta > 0$ and $C > 0$ depending on all the same parameters in Theorem 1.2 as well as on D_0

$$\left\| \int_{[-N,N]^d} e^{2\pi i x\cdot\Phi_N(\eta)}\, \hat{h}(\eta)\, d\eta \right\|_{L^2(X)} \le C N^{-\beta} \|h\|_2 \tag{6-13}$$

for all $h \in L^2$ with $\operatorname{supp}(\hat{h}) \subset \widetilde{Y}$, in which $\widetilde{Y} \subset [-N, N]^d$ is an admissible set with constants C_R, ε_0 on scales 1 to N. Setting $\hat{h}(\eta) := \hat{g}(\eta)\, |\det(D\Phi_N(\eta))|$, we conclude from (6-13) that

$$\|f\|_{L^2(X)} \le C N^{-\beta} \|\hat{h}\|_2 \le C N^{-\beta} \|\hat{f}\|_2 = C N^{-\beta} \|f\|_2,$$

with possibly a different constant. So it remains to prove the claim (6-13). We will prove it from another statement, namely

$$\left\| \int_{[-N,N]^d} e^{2\pi i x\cdot\Phi_N(\eta)}\, \mathbb{1}_{\widetilde{Y}}(\eta)\, h(\eta)\, d\eta \right\|_{L^2(X)} \le C N^{-\beta} \|h\|_2 \tag{6-14}$$

for all $h \in L^2$. Notice that by Plancherel we could remove the Fourier transform from h.

To prove (6-14), divide $[-N, N]^d = \bigcup_k Q_k$ into congruent cubes of side length L_N with $\frac{1}{2}\sqrt{N} \le L_N \le \sqrt{N}$. Let $\{\chi_k\}_k$ be a partition of unity adapted to these cubes. With η_k being the center of Q_k,

$$\begin{aligned} \int_{[-N,N]^d} e^{2\pi i x\cdot\Phi_N(\eta)}\, \mathbb{1}_{\widetilde{Y}}(\eta)\, h(\eta)\, d\eta &= \sum_k \int_{\mathbb{R}^d} e^{2\pi i x\cdot\Phi_N(\eta)}\, \chi_k(\eta)\, \mathbb{1}_{\widetilde{Y}}(\eta)\, h(\eta)\, d\eta \\ &= \sum_k \int_{\mathbb{R}^d} e^{2\pi i x\cdot(\Phi_N(\eta_k) + D\Phi_N(\eta_k)(\eta-\eta_k))}\, a_k(x,\eta)\, \mathbb{1}_{\widetilde{Y}}(\eta)\, h(\eta)\, d\eta \\ &=: \sum_k (T_k h)(x), \end{aligned} \tag{6-15}$$

where

$$\begin{aligned} a_k(x,\eta) &:= e^{2\pi i x\cdot R_k(\eta)}\,\chi_k(\eta),\\ R_k(\eta) &:= \int_0^1 (1-t)\langle D^2\Phi_N(\eta_k+t(\eta-\eta_k))(\eta-\eta_k),\eta-\eta_k\rangle\,dt, \end{aligned} \tag{6-16}$$

the latter being the error in the second-order Taylor expansion (we are suppressing the parameter N here). Then

$$\begin{gathered} \|R_k\|_{L^\infty(\operatorname{supp}\chi_k)} \le C = C(d,D_0),\\ \|\partial_x^\alpha a_k(x,\eta)\|_{L^\infty([-1,1]^d\times \operatorname{supp}\chi_k)} \le C(d,D_0,\alpha), \quad \operatorname{diam}\operatorname{supp}\chi_k \le C\sqrt{N}, \end{gathered} \tag{6-17}$$

for every multi-index α. By Hörmander's variable-coefficient Plancherel theorem,

$$\max_k \|T_k\|_{2\to 2} \le C(d,D_0). \tag{6-18}$$

This follows by the usual T^*T argument:

$$\begin{aligned} \|T_k h\|_2^2 &= \langle T_k^*T_k h,h\rangle,\\ (T_k^*T_k h)(\eta') &= \int_{\mathbb{R}^d} K_k(\eta',\eta)\,h(\eta)\,d\eta,\\ K_k(\eta',\eta) &= \int_{\mathbb{R}^d} e^{2\pi i x\cdot(\Phi_N(\eta)-\Phi_N(\eta'))}\,\mathbb{1}_{\widetilde{Y}}(\eta)\,\mathbb{1}_{\widetilde{Y}}(\eta')\,\chi_k(\eta)\chi_k(\eta')\,dx. \end{aligned} \tag{6-19}$$

Since $\|\Phi_N(\eta)-\Phi_N(\eta')\| \ge D_0^{-1}\|\eta-\eta'\|$ in the sense of Euclidean lengths, repeated integrations by parts yield the decay

$$|K_k(\eta',\eta)| \le C(d,D_0)\langle \eta-\eta'\rangle^{-d-1},$$

whence (6-18) follows by Schur's test. In particular, $\|\mathbb{1}_X T_k\|_{2\to2} \le C$ with the same constant as in (6-18).

Next, we would like to show that $\mathbb{1}_X T_k$ and $\mathbb{1}_X T_\ell$ do not interact much for all cubes Q_k, Q_ℓ which are not nearest neighbors. In order to integrate by parts in x, see (6-19), we need to smooth out $\mathbb{1}_X$ at the correct scale. Define

$$X(N^{-\frac12}) := X + [-N^{-\frac12}, N^{-\frac12}]^d.$$

By [Dyatlov and Zahl 2016, Lemma 3.3] there exists a smooth ψ taking values in [0, 1] with $\psi = 1$ on X and with $\operatorname{supp}(\psi) \subset X(N^{-1/2})$, as well as so that

$$\|\partial_x^\alpha \psi\|_\infty \le C(\alpha) N^{\frac{|\alpha|}{2}} \tag{6-20}$$

for all multi-indices. Define $S_k := \psi\, T_k$. On the one hand, S_k still obeys (6-18). On the other hand, for any cubes Q_k, Q_ℓ which are not nearest neighbors one has

$$\|S_k^* S_\ell\|_{2\to2} \le C(d,D_0,p) N^{\frac{p}{2}} \operatorname{dist}(Q_k,Q_\ell)^{-p} \tag{6-21}$$

for every positive integer p. This follows from the fact that the kernel of $S_k^* S_\ell$ equals

$$K_{k,\ell}(\eta',\eta) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot(\Phi_N(\eta)-\Phi_N(\eta'))}\,\mathbb{1}_{\widetilde{Y}}(\eta)\,\mathbb{1}_{\widetilde{Y}}(\eta')\,\chi_k(\eta)\chi_\ell(\eta')\,\psi(x)^2\,dx.$$

Using the differential operator

$$\mathcal{L} = \frac{1}{2\pi i} \frac{\Phi_N(\eta) - \Phi_N(\eta')}{\|\Phi_N(\eta) - \Phi_N(\eta')\|^2} \cdot \nabla_x,$$

which obeys

$$\mathcal{L}\, e^{2\pi i x\cdot(\Phi_N(\eta)-\Phi_N(\eta'))} = e^{2\pi i x\cdot(\Phi_N(\eta)-\Phi_N(\eta'))},$$

repeated integration by parts now yields (6-21). Finally, given any k, only a uniformly bounded number of choices of ℓ will satisfy

$$S_k S_\ell^* = \psi\, T_k T_\ell^*\, \psi \neq 0.$$

This is due to the fact that $\chi_k(\eta)\chi_\ell(\eta) = 0$ up to a bounded number of choices of ℓ given k. If we label the cubes by lattice points $\underline{k} \in \mathbb{Z}^d$, then $\eta_{\underline{k}} = L_N \underline{k}$, whence

$$N^{\frac{p}{2}} \operatorname{dist}(Q_{\underline{k}}, Q_{\underline{\ell}})^{-p} \lesssim N^{\frac{p}{2}} (L_N |\underline{k} - \underline{\ell}|)^{-p} \lesssim |\underline{k} - \underline{\ell}|^{-p},$$

which is summable over $\mathbb{Z}^d$ provided $p > d$. On the other hand, we also have

$$\|S_k^* S_\ell\|_{2\to 2} \le \|S_k\|_{2\to 2} \|S_\ell\|_{2\to 2} \le B^2, \quad B := \sup_j \|S_j\|_{2\to 2}.$$

Combining these two estimates we infer that for any $0 < \varepsilon < 1$

$$\|S_{\underline{k}} S_{\underline{\ell}}^*\|_{2\to 2} + \|S_{\underline{k}}^* S_{\underline{\ell}}\|_{2\to 2} \le C(d, D_0, \varepsilon)\, B^{2(1-\varepsilon)} \langle \underline{k} - \underline{\ell} \rangle^{-2(d+1)}$$

for all $\underline{k}, \underline{\ell} \in \mathbb{Z}^d$. Note that $B \le C(d, D_0)$ by Hörmander's bound (6-18). Hence by Cotlar's lemma,

$$\left\| \int_{[-N,N]^d} e^{2\pi i x\cdot \Phi_N(\eta)}\, \mathbb{1}_{\tilde{Y}}(\eta)\, h(\eta)\, d\eta \right\|_{L^2(X)} \le C(\varepsilon, d, D_0) \max_k \|S_k\|_{2\to 2}^{1-\varepsilon}. \tag{6-22}$$

The claim (6-14) will now follow from (6-22) by applying the fractal uncertainty principle of Theorem 1.2 to each S_k. For this we need to linearize the phase as in (6-15), which in turn makes the localization to scales $\sqrt{N}$ necessary.

To be specific, we reduce (6-14) to the following estimate. Let ψ_0 be compactly supported functions satisfying the bounds

$$\|\partial_x^\alpha \psi_0\|_\infty \le C_s N^s \quad \text{for all } |\alpha| = s \ge 0, \tag{6-23}$$

where $N \ge 1$ is arbitrary and all constant are independent of N. We assume that ψ_0 is supported in a δ-regular set in $[-1, 1]^d$ on scales $1/N$ to 1, and with $0 < \delta < d$. Let

$$Z = N^{-1} Y_1$$

be a rescaled version of an admissible set Y_1 with constants C_R, δ_1, ε_0 on scales 1 to N. The point is Y_1 is not assumed to be contained in $[-N, N]^d$; hence Theorem 1.2 does not apply directly. Hence we need to use Proposition 6.7 instead, for which we need to make assumptions on supp a. Suppose that the symbol a is smooth and compactly supported with the bounds

$$\|\partial_x^\alpha a(x, \xi)\|_\infty \le C(\alpha) \quad \text{for all } \alpha, \qquad \text{and} \qquad \operatorname{supp} a(x, \cdot) \subset Q, \tag{6-24}$$

where Q is a cube in $\mathbb{R}^d$ that is independent of x, and is of side length $r_Q \le r_0$. Then for some $\beta > 0$ and C as above,

$$\|\psi_0\, A\, \mathbb{1}_Z\, h\|_2 \le C N^{-\beta} \|h\|_2, \tag{6-25}$$

where

$$(Ah)(x) := N^{\frac{d}{2}} \int_{\mathbb{R}^d} e^{2\pi i N x\cdot\xi}\, a(x,\xi) h(\xi)\, d\xi.$$

Indeed,

$$\begin{aligned}
&\left\| \int_{\mathbb{R}^d} e^{2\pi i x\cdot(\Phi_N(\eta_k)+D\Phi_N(\eta_k)(\eta-\eta_k))}\, \psi(x)\, a_k(x,\eta)\, \mathbb{1}_{\widetilde{Y}}(\eta)\, h(\eta)\, d\eta \right\|_2 \\
&\lesssim \left\| \int_{\mathbb{R}^d} e^{2\pi i x\cdot\zeta}\, \psi(x)\, a_k(x, D\Phi_N(\eta_k)^{-1}\zeta + \eta_k)\, \mathbb{1}_{\widetilde{Y}-\eta_k}(D\Phi_N(\eta_k)^{-1}\zeta) h(D\Phi_N(\eta_k)^{-1}\zeta + \eta_k)\, d\zeta \right\|_2 \\
&= N^{\frac{d}{4}} \left\| \int_{\mathbb{R}^d} e^{2\pi i N^{1/2} x\cdot\xi}\, \psi(x)\, \tilde{a}_k(x, N^{\frac{1}{2}}\xi)\, N^{\frac{d}{4}}\, \mathbb{1}_{Y_1}(N^{\frac{1}{2}}\xi)\, \tilde{h}(N^{\frac{1}{2}}\xi)\, d\xi \right\|_2 .
\end{aligned}$$

Here $\tilde{a}$, $\tilde{h}$ signify the functions on the second line but with the linear isomorphism $D\Phi_N(\eta_k)^{-1}$ and the shift η_k included, and $Y_1 = D\Phi_N(\eta_k)(\widetilde{Y} - \eta_k)$ is an admissible set on scales 1 to N with constants that depend on D_0. Note that $\mathbb{1}_{Y_1}(N^{1/2}\xi) = \mathbb{1}_Z(\xi)$, with $Z = N^{-1/2} Y_1$, which is an admissible set on scales $N^{-1/2}$ to 1. By (6-20), $\psi_0(x) := \psi(x)$ satisfies the required bound, and furthermore ψ_0 is supported on $X(N^{-1/2})$, which is a δ-regular set on scales $N^{-1/2}$ to 1; see Lemma A.4. As for the amplitude, ignoring the distinction between $\tilde{a}_k$ and a_k,

$$\begin{aligned}
a_k(x, N^{\frac{1}{2}}\xi) &:= e^{2\pi i x\cdot R_k(N^{1/2}\xi)}\, \chi_k(N^{\frac{1}{2}}\xi), \\
R_k(N^{\frac{1}{2}}\xi) &:= N \int_0^1 (1-t)\langle D^2\Phi_N(\eta_k + t(N^{\frac{1}{2}}\xi - \eta_k))(\xi - \eta_k'), \xi - \eta_k' \rangle\, dt,
\end{aligned}$$

where $\eta_k' = N^{-1/2}\eta_k$. Setting $a(x,\xi) = a_k(x, N^{1/2}\xi)$, we conclude from (6-17) that a satisfies (6-24) with constant $r_0 = C$, which is an absolute constant. Finally,

$$\|N^{\frac{d}{4}} \tilde{h}(N^{\frac{1}{2}}\xi)\|_2 \simeq \|h\|_2.$$

Thus, we can apply (6-25) with N replaced by $N^{1/2}$ to obtain a gain of $N^{-\beta/2}$, and we are done.

It remains to prove (6-25). Note that this is equivalent to proving

$$\|\psi_0\, A\, \mathbb{1}_{Z\cap Q}\, h\|_2 \le C N^{-\beta} \|h\|_2. \tag{6-26}$$

By Proposition 6.7, we can cover $Z \cap Q$ by $C(\varepsilon_0, d, r_0)$ many admissible sets W_j with constants δ_1, $\widetilde{C}_R := (4C_R)^{2/(1-\delta_1)} C_R$, $\tilde{\varepsilon}_0 = \tilde{\varepsilon}_0(\varepsilon_0, D_0)$. Hence, via triangle inequality, it suffices to prove (6-26) with $Z \cap Q$ replaced by W_j.

If $a = 1$ on the $\mathrm{supp}(\psi_0) \times W_j$, then this follows immediately from Theorem 1.2 by a rescaling. Indeed, one has by that theorem

$$\begin{aligned}
\left\| N^{\frac{d}{2}} \int_{\mathbb{R}^d} e^{2\pi i N x\cdot\xi}\, \psi_0(x)\, \mathbb{1}_{W_j}(\xi)\, h(\xi)\, d\xi \right\|_2 &= \left\| \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi}\, \psi_0(x)\, \mathbb{1}_{W_j}\Big(\frac{\xi}{N}\Big)\, N^{-\frac{d}{2}} h\Big(\frac{\xi}{N}\Big)\, d\xi \right\|_2 \\
&\lesssim N^{-\beta} \left\| N^{-\frac{d}{2}} h\Big(\frac{\xi}{N}\Big) \right\|_2 = N^{-\beta} \|h\|_2.
\end{aligned}$$

Let us now consider general a satisfying (6-24). Let $\rho \in (0,1)$ with its value determined later.

Let us note that by the usual A^*A argument, we have Hörmander's bound,

$$\|A\|_{2\to 2} \le C. \tag{6-27}$$

Next we decompose $\psi_0 A \mathbb{1}_{W_j}$ into

$$\psi_0 A \mathbb{1}_{W_j} = \psi_0 \mathcal{F}_\hbar^{-1} A_1 + A_2 \mathcal{F}_\hbar A \mathbb{1}_{W_j},$$
$$A_1 := \mathbb{1}_{\mathbb{R}^d \setminus W_j(N^{-\rho})} \mathcal{F}_\hbar A \mathbb{1}_{W_j}, \quad A_2 := \psi_0 \mathcal{F}_\hbar^{-1} \mathbb{1}_{W_j(N^{-\rho})},$$

where $\hbar = N^{-1}$. Clearly, by (6-27), we have

$$\|\psi_0 A \mathbb{1}_{W_j}\|_{2\to 2} \lesssim \|A_1\|_{2\to 2} + \|A_2\|_{2\to 2}. \tag{6-28}$$

Thus it suffices to bound $\|A_1\|_{2\to 2}$ and $\|A_2\|_{2\to 2}$.

We compute the integral kernel of A_1:

$$K_{A_1}(\xi,\eta) = \mathbb{1}_{\mathbb{R}^d \setminus W_j(N^{-\rho})}(\xi)\, \mathbb{1}_{W_j}(\eta)\, N^d \int_{\mathbb{R}^d} e^{2\pi i N x\cdot(\eta-\xi)} a(x,\eta)\, dx.$$

Note that the Euclidean distance satisfies $\|\eta-\xi\| \ge N^{-\rho}$ on the support of K_{A_1}. Hence by repeated integration by parts in x, we obtain that

$$|K_{A_1}(\xi,\eta)| \le C_{d,\rho} N^{d-\lceil \frac{d+10}{1-\rho}\rceil} \langle \eta-\xi\rangle^{-\lceil \frac{d+10}{1-\rho}\rceil} \le C_{d,\rho} N^{-10}.$$

By Schur's test, we arrive at

$$\|A_1\|_{2\to 2} \le C N^{-10}. \tag{6-29}$$

In view of A_2. Note that

$$W_j(N^{-\rho}) \subset \bigcup_{\substack{\|k\|_\infty \le N^{1-\rho} \\ k\in\mathbb{Z}^d}} (W_j(N^{-1}) + k),$$

and

$$W_j(N^{-1}) \subset \widehat{W}_j := \left\{ \sum_{\ell=1}^d \xi_\ell \vec{e}_\ell : \xi_\ell \in N^{-1}\cdot W_{j,\ell}(2) \right\},$$

which is an admissible set on scales $2N^{-1}$ to 1. Thus by Theorem 1.2 and triangle inequality, we have for $f \in L^2(\mathbb{R}^d)$

$$\begin{aligned}\|A_2 f\| &\le \sum_{\|k\|\le N^{1-\rho}} \|\psi_0 \mathcal{F}_\hbar^{-1} \mathbb{1}_{\widehat{W}_j+k} f\|_2 \\ &\lesssim \sum_{\|k\|\le N^{1-\rho}} \|\mathbb{1}_{\operatorname{supp}\psi_0} \mathcal{F}_\hbar^{-1} \mathbb{1}_{\widehat{W}_j+k} f\|_2 \le C N^{-\beta+d(1-\rho)} \|f\|_2,\end{aligned}$$

where $\hbar = N^{-1}$. Hence for $\rho = 1-\beta/2d$,

$$\|A_2\|_{2\to 2} \le C N^{-\frac{\beta}{2}}. \tag{6-30}$$

Combining (6-28), (6-29) with (6-30), we obtain (6-25). This concludes the proof of Theorem 1.4.

6E. *Fourier integral operator.* In this section, we prove a fractal uncertainty principle for Fourier integral operators on $\mathbb{R}^d$. The proof follows that of the one-dimensional case in [Bourgain and Dyatlov 2018, Section 4]; thus we shall be very brief.

Let

$$(B(\hbar)f)(x) := \hbar^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{2\pi i \Phi(x,y)}{\hbar}} b(x,y) f(y)\, dy, \tag{6-31}$$

where for some open set $U \subset \mathbb{R}^{2d}$

$$\begin{gathered}\Phi \in C^\infty(U;\mathbb{R}), \quad b \in C_0^\infty(U), \quad \det\left(\frac{\partial^2 \Phi}{\partial x_j \partial y_k}\right) \neq 0 \quad \text{on } U,\\ \left(\sup_U \left\| \left(\frac{\partial^2 \Phi}{\partial x_j \partial y_k}\right)\right\|\right) \cdot \left(\sup_U \left\| \left(\frac{\partial^2 \Phi}{\partial x_j \partial y_k}\right)^{-1}\right\|\right) \le C_\Phi\end{gathered} \tag{6-32}$$

for some constant $C_\Phi \geq 1$, in which $\|\cdot\|$ is the matrix norm.

Proposition 6.8. *Let $X, Y \subset [-1,1]^d$. Assume that X is a δ-regular set on scales 0 to 1 with constant C_R, and Y is an admissible set on scales 0 to 1 with parameters $\delta_1, C_R, \varepsilon_0$. Assume (6-32) holds. Then there exist $\beta > 0$, $\rho \in (0,1)$ depending only on $\delta, \delta_1, C_R, \varepsilon_0, d, C_\Phi$, and $C > 0$ depending only on $\delta, \delta_1, C_R, \varepsilon_0, d, \Phi, b$ such that for $0 < \hbar < h_0(\Phi) < 1$,*

$$\|\mathbb{1}_{X(\hbar^{\rho/2})} B(\hbar) \mathbb{1}_{Y(\hbar^\rho)}\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \le C \hbar^\beta.$$

Proof. As was pointed out in [Bourgain and Dyatlov 2018], it is enough to prove Proposition 6.8 under the assumption that

$$1 < \left| \det\left(\frac{\partial^2 \Phi}{\partial x_j \partial y_k}\right)\right| < 2 \quad \text{on } U. \tag{6-33}$$

Let $\tilde{h} := \hbar^{1/2}$. Divide $[-2,2]^d = \bigcup_k Q_k$ into congruent cubes of side length L with $\tilde{h}/2 \le L < \tilde{h}$. Let $\{\chi_k\}_k$ be a partition of unity adapted to these cubes. With y_k being the center of Q_k, we have

$$\begin{aligned}\hbar^{-\frac{d}{2}} &\int_{\mathbb{R}^d} e^{-\frac{2\pi i \Phi(x,y)}{\hbar}} b(x,y) \mathbb{1}_{Y(\hbar^\rho)}(y) f(y)\, dy\\ &= \sum_k \hbar^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{2\pi i \Phi(x,y)}{\hbar}} b(x,y) \chi_k(y) \mathbb{1}_{Y(\hbar^\rho)}(y) f(y)\, dy\\ &= \sum_k e^{-\frac{2\pi i \Phi(x,y_k)}{\hbar}} \hbar^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{2\pi i \nabla_y \Phi(x,y_k)\cdot(y-y_k)}{\hbar}} \tilde{b}_k(x,y) \mathbb{1}_{Y(\hbar^\rho)}(y) f(y)\, dy\\ &=: \sum_k (T_k f)(x),\end{aligned}$$

where

$$\begin{aligned}\tilde{b}_k(x,y) &= e^{-\frac{2\pi i \Psi_k(x,y)}{\hbar}} \chi_k(y) b(x,y),\\ \Psi_k(x,y) &= \int_0^1 (1-t)\langle (y-y_k), \mathrm{H}\Phi(x, y_k + t(y-y_k))(y-y_k)\rangle\, dt,\end{aligned} \tag{6-34}$$

in which $\mathrm{H}\Phi(x,\cdot)$ is the Hessian of $\Phi(x,\cdot)$ in the y-variable.

We will prove

$$\|\mathbb{1}_{X(\tilde{h}^\rho)} T_k\|_{L^2\to L^2} \le C\tilde{h}^\beta, \tag{6-35}$$

and the estimate for $\sum_k \mathbb{1}_{X(\tilde{h}^\rho)} T_k$ follows from almost orthogonality and Cotlar's lemma; see the proof of Proposition 4.3 in [Bourgain and Dyatlov 2018].

Let

$$\varphi(x) := \nabla_y \Phi(x, y_k).$$

By (6-33), the Jacobian matrix $\mathrm{J}\varphi$ satisfies $1 < |\det(\mathrm{J}\varphi(x))| < 2$; hence φ admits an inverse function.

We have, by a change variable $x \to \varphi^{-1}(x)$,

$$\begin{aligned}
&\|\mathbb{1}_{X(\tilde{h}^\rho)}(x)(T_k f)(x)\|_{L^2}\\
&\quad= \Big\|\mathbb{1}_{\varphi(X(\tilde{h}^\rho))}(x)|\det(\mathrm{J}\varphi^{-1}(x))|^{\frac12}\hbar^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{-\frac{2\pi i x\cdot y}{\hbar}}\tilde{b}_k(\varphi^{-1}(x), y+y_k)\mathbb{1}_{Y(\hbar^\rho)-y_k}(y) f(y+y_k)\,dy\Big\|_{L^2}\\
&\quad= \Big\|\mathbb{1}_{\varphi(X(\tilde{h}^\rho))}(x)|\det(\mathrm{J}\varphi^{-1}(x))|^{\frac12}\int_{\mathbb{R}^d} e^{-\frac{2\pi i x\cdot y}{\tilde{h}}}\tilde{b}_k(\varphi^{-1}(x), \tilde{h}y+y_k)\mathbb{1}_{Y(\hbar^\rho)-y_k}(\tilde{h}y) f(\tilde{h}y+y_k)\,dy\Big\|_{L^2}\\
&\quad\le \|\mathbb{1}_{\varphi(X(\tilde{h}^\rho))}\,A(\hbar)\,\mathbb{1}_{\tilde{h}^{-1}(Y(\tilde{h}^{2\rho})-y_k)}\|_{L^2\to L^2}\cdot\|\tilde{h}^{\frac{d}{2}} f(\tilde{h}y+y_k)\|_{L^2}\\
&\quad= \|\mathbb{1}_{\varphi(X(\tilde{h}^\rho))}\,A(\hbar)\,\mathbb{1}_{Y(\tilde{h}^{2\rho-1})-\tilde{h}^{-1}y_k}\|_{L^2\to L^2}\cdot\|f\|_{L^2},
\end{aligned}$$

where

$$\begin{aligned}
(A(\hbar)f)(x) &= \tilde{h}^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{-\frac{2\pi i x\cdot y}{\tilde{h}}}\hat{b}_k(x,y)f(y)\,dy,\\
\tilde{b}(x,y) &= |\det(\mathrm{J}\varphi^{-1}(x))|^{\frac12}\,\tilde{b}_k(\varphi^{-1}(x), \tilde{h}y+y_k).
\end{aligned} \tag{6-36}$$

Now it suffices to bound

$$\|\mathbb{1}_{\varphi(X(\tilde{h}^\rho))}\,A(\hbar)\,\mathbb{1}_{Y(\tilde{h}^{2\rho-1})-\tilde{h}^{-1}y_k}\|_{L^2\to L^2}. \tag{6-37}$$

Let $\widetilde{X} := \varphi(X)$. By (6-32),

$$(\sup\|\mathrm{J}\varphi\|)\cdot(\sup\|(\mathrm{J}\varphi)^{-1}\|) \le C_\Phi.$$

Note (6-33) implies $C_1 := \sup\|\mathrm{J}\varphi\| \ge 1$ and hence $C_2 := \sup\|(\mathrm{J}\varphi)^{-1}\| \le C_\Phi$. By Lemma A.5, $\widetilde{X}$ is δ-regular with constant $C_R(d\,C_\Phi)^{\delta/2}$ on scales 0 to $d^{-1/2}C_2^{-1}$.

If $d^{-1/2}C_2^{-1} < 1$, Lemma A.3 implies $\widetilde{X}$ is δ-regular with constant

$$2(d^{\frac12}C_2)^d C_R(d\,C_\Phi)^{\frac{\delta}{2}} \le 2d^{\frac{d+\delta}{2}}C_R C_\Phi^{d+\frac{\delta}{2}} =: \widetilde{C}_R$$

on scales 0 to 1. If $d^{-1/2}C_2^{-1} \ge 1$, let $\widetilde{C}_R := C_R(d\,C_\Phi)^{\delta/2}$. Hence $\widetilde{X}$ is always δ-regular with constant $\widetilde{C}_R$ on scales 0 to 1.

It is also easy to see that $\varphi(X(\tilde{h}^\rho)) \subseteq \widetilde{X}(C(\Phi)\tilde{h}^\rho)$, where $C(\Phi)$ is a constant depending on Φ. For $0 < \hbar < h_0(\Phi)$, we have $C(\Phi)\tilde{h}^\rho < \tilde{h}^{2\rho-1}$; hence

$$\|\mathbb{1}_{\varphi(X(\tilde{h}^\rho))}A(\hbar)\mathbb{1}_{(Y(\tilde{h}^{2\rho-1})-\tilde{h}^{-1}y_k)}\|_{L^2\to L^2} \le \|\mathbb{1}_{\widetilde{X}(\tilde{h}^{2\rho-1})}A(\hbar)\mathbb{1}_{(Y(\tilde{h}^{2\rho-1})-\tilde{h}^{-1}y_k)}\|_{L^2\to L^2}.$$

Next note that

$$\begin{aligned}\tilde{X}(\tilde{h}^{2\rho-1}) &\subseteq \bigcup_{\substack{j\in\mathbb{Z}\\ \|j\|\le\tilde{h}^{2\rho-2}}} (\tilde{X}(\tilde{h})+\tilde{h}j) =: \bigcup_{\substack{j\in\mathbb{Z}\\ \|j\|\le\tilde{h}^{2\rho-2}}} \tilde{X}_j,\\ Y(\tilde{h}^{2\rho-1})-\tilde{h}^{-1}y &\subseteq \bigcup_{\substack{k\in\mathbb{Z}\\ \|p\|\le\tilde{h}^{2\rho-2}}} (Y(\tilde{h})-\tilde{h}^{-1}y+\tilde{h}p) =: \bigcup_{\substack{k\in\mathbb{Z}\\ \|p\|\le\tilde{h}^{2\rho-2}}} Y_p.\end{aligned} \tag{6-38}$$

Hence, it is eventually reduced to estimating each $\|\mathbb{1}_{\tilde{X}_j}A(\hbar)\mathbb{1}_{Y_p}\|_{L^2\to L^2}$.

It is easy to check that $\hat{b}_k(x,y)$ satisfy (6-24); hence by (6-25), we have

$$\|\mathbb{1}_{\tilde{X}_j}A(\hbar)\mathbb{1}_{Y_p}\|_{L^2\to L^2} \le C\tilde{h}^{\beta}$$

for some $\beta>0$. Choosing $2d(\rho-1)<\beta/2$, we conclude that

$$\|\mathbb{1}_{\tilde{X}(\tilde{h}^{2\rho-1})}A(\hbar)\mathbb{1}_{(Y(\tilde{h}^{2\rho-1})-\tilde{h}^{-1}y_k)}\|_{L^2\to L^2} \le C\tilde{h}^{\frac{\beta}{2}}$$

by the triangle inequality. □

Appendix A: Regular sets

We show that certain operations preserve the class of δ-regular sets if we allow one to increase the regularity constant and shrink the scales.

The first lemma is from [Bourgain and Dyatlov 2018]. It shows a δ-regular set in $\mathbb{R}^1$, $0<\delta<1$, can be split into smaller δ-regular sets.

Lemma A.1. *Let $X\subset\mathbb{R}^1$ be a δ-regular set with constant C_R on scales α_0 to α_1, and assume that $0<\delta<1$ and $(4C_R)^{2/(1-\delta)}\alpha_0\le\rho\le\alpha_1$. Then there exists a collection of disjoint intervals $\mathcal{J}$ such that*

$$X=\bigcup_{J\in\mathcal{J}}(X\cap J), \qquad (4C_R)^{-\frac{2}{1-\delta}}\rho\le|J|\le\rho \quad \text{for all } J\in\mathcal{J},$$

and each $X\cap J$ is δ-regular with constant $\tilde{C}_R:=(4C_R)^{2/(1-\delta)}C_R$ on scales α_0 to ρ.

The rest of this section concerns δ-regular sets in $\mathbb{R}^d$. We show that certain operations preserve the class of δ-regular sets if we allow one to increase the regularity constant and shrink the scales.

Lemma A.2. *Let X be a δ-regular set with $\delta\in(0,d)$ and constant C_R on scales α_0 to α_1. Fix $\lambda>0$ and $y\in\mathbb{R}^d$. Then the set $\tilde{X}:=y+\lambda X$ is a δ-regular set with constant C_R on scales $\lambda\alpha_0$ to $\lambda\alpha_1$.*

Proof. Taking the measure

$$\mu_{\tilde{X}}(A):=\lambda^{\delta}\mu_X(\lambda^{-1}(A-y)),$$

it is easy to verify. □

Lemma A.3. *Let X be a δ-regular set with constant C_R on scales α_0 to α_1. Fix $T>1$. Then X is δ-regular with constant $\tilde{C}_R:=2T^dC_R$ on scales α_0 to $T\alpha_1$.*

Proof. Let I be a cube such that $\alpha_0 \le r_I \le T\alpha_1$. For $\alpha_0 \le r_I \le \alpha_1$, the upper bound is immediate. For $\alpha_1 < r_I \le T\alpha_1$, I can be covered by $\lceil T\rceil^d \le 2T^d$ cubes of side length α_1 each; therefore

$$\mu_X(I) \le 2T^d C_R \alpha_1^\delta \le \widetilde{C}_R r_I^\delta.$$

In view of the lower bound estimate, we assume I is centered at a point in X. As before, we may assume $\alpha_1 < r_I \le T\alpha_1$. Let $I' \subset I$ be the cube with the same center and $r_{I'} = \alpha_1$. Then

$$\mu_X(I) \ge \mu_X(I') \ge C_R^{-1}\alpha_1^\delta \ge \widetilde{C}_R^{-1} r_I^\delta,$$

as claimed. □

Lemma A.4. *Let X be a δ-regular set with constant C_R on scales α_0 to α_1. Fix $T \ge 1$:*

(1) *Suppose $\alpha_1 \ge 2\alpha_0$. Then the neighborhood $X + [-T\alpha_0, T\alpha_0]^d$ is δ-regular with constant $\widetilde{C}_R := 4^d T^d C_R$ on scales $2\alpha_0$ to α_1.*

(2) *Suppose that $\alpha_1 \ge T\alpha_0$. Then $X + [-T\alpha_0, T\alpha_0]^d$ is δ-regular with constant $C'_R = 4^d C_R$ on scales $T\alpha_0$ to α_1.*

Proof. Let $\widetilde{X} := X + [-T\alpha_0, T\alpha_0]^d$ and define $\mu_{\widetilde{X}}$ supported on $\widetilde{X}$ by convolution

$$\mu_{\widetilde{X}}(A) := \frac{1}{(T\alpha_0)^d}\int_{[-T\alpha_0,T\alpha_0]^d} \mu_X(A+y)\,dy.$$

Let I be a cube such that $M\alpha_0 \le r_I \le \alpha_1$ with $M \ge 1$. Then

$$\mu_{\widetilde{X}}(I) \le 2^d C_R r_I^\delta,$$

which proves the upper bound estimates for both cases.

Now assume that I is centered at a point $x_1 \in \widetilde{X}$. Take $x_0 \in X$ such that $x_0 \in x_1 + [-T\alpha_0, T\alpha_0]^d$, and let I' be the cube centered at x_0 with side length $r_{I'} = r_I/2$. Then

$$\mu_X(I') \ge C_R^{-1}\left(\frac{r_I}{2}\right)^\delta \ge 2^{-d} C_R^{-1} r_I^\delta.$$

Let $J = x_0 - x_1 + [-\alpha_0/2, \alpha_0/2]^d$; then $J \cap [-T\alpha_0, T\alpha_0]^d$ contains a cube with side length at least $\alpha_0/2$. Clearly, $I' \subset I + y$ for any $y \in J$. Hence

$$\mu_{\widetilde{X}}(I) \ge \frac{1}{(2T)^d}\mu_X(I') \ge \widetilde{C}_R^{-1} r_I^\delta,$$

which proves the lower bound estimate for (1).

Let $J = x_0 - x_1 + [-T\alpha_0/2, T\alpha_0/2]^d$; then $J \cap [-T\alpha_0, T\alpha_0]^d$ contains a cube with side length at least $T\alpha_0/2$. Clearly, $I' \subset I + y$ for any $y \in J$. Hence

$$\mu_{\widetilde{X}}(I) \ge \frac{1}{2^d}\mu_X(I') \ge (C'_R)^{-1} r_I^\delta,$$

which proves the lower bound estimate for (2). □

Lemma A.5. *Assume $F:\mathbb{R}^d\to\mathbb{R}^d$ is C^1 diffeomorphism. Let $C_1:=\sup_{x\in\mathbb{R}^d}\|\mathrm{J}F(x)\|$ and $C_2:=\sup_{x\in\mathbb{R}^d}\|\mathrm{J}F^{-1}(x)\|$, where $\mathrm{J}F$ is the Jacobian matrix and $\|\cdot\|$ is the matrix norm. Assume that for some constant $C_F\geq 1$, we have*

$$C_1C_2\leq C_F. \tag{A-1}$$

Let X be a δ-regular set with constant C_R on scales α_0 to $\alpha_1\geq C_F^2\alpha_0$. Then $F(X)$ is a δ-regular set with constant $\widetilde{C}_R:=C_R(d\,C_F)^{\delta/2}$ on scales $d^{1/2}C_1\alpha_0$ to $d^{-1/2}C_2^{-1}\alpha_1$.

Proof. Let $\widetilde{X}:=F(X)$ and define the measure $\mu_{\widetilde{X}}$ supported on $\widetilde{X}$ as

$$\mu_{\widetilde{X}}(A):=C_F^{-\frac{\delta}{2}}C_1^{\delta}\,\mu_X(F^{-1}(A)).$$

Let $\tilde{I}$ be a cube with side length $r_{\tilde{I}}$ with

$$d^{\frac{1}{2}}C_1\alpha_0\leq r_{\tilde{I}}\leq d^{-\frac{1}{2}}C_2^{-1}\alpha_1. \tag{A-2}$$

Clearly, $F^{-1}\tilde{I}$ is contained in a cube of side length r, where $r\leq\sqrt{d}\,C_2r_{\tilde{I}}$. Indeed, let y be the center of $\tilde{I}$. Then for any $x\in\tilde{I}$, we have

$$\|F^{-1}(x)-F^{-1}(y)\|\leq C_2\|x-y\|\leq\frac{\sqrt{d}}{2}C_2r_{\tilde{I}}.$$

Let I be the cube centered at $F^{-1}y$ of side length $\sqrt{d}\,C_2r_{\tilde{I}}\leq\alpha_1$. Then

$$\mu_{\widetilde{X}}(\tilde{I})\leq\mu_X(I)\leq C_F^{-\frac{\delta}{2}}C_1^{\delta}C_R(\sqrt{d}\,C_2r_{\tilde{I}})^{\delta}=C_R(d\,C_F)^{\frac{\delta}{2}}r_{\tilde{I}}^{\delta}.$$

If, in addition, $y\in\widetilde{X}$, let $y=F(z)$, where $z\in X$. Then the cube Q centered at z of side length $r=d^{-1/2}C_1^{-1}r_{\tilde{I}}\geq\alpha_0$ is contained in $F^{-1}(\tilde{I})$. Indeed, for any $x\in Q$, we have

$$\|F(x)-F(z)\|\leq\frac{\sqrt{d}}{2}C_1r=\frac{r_{\tilde{I}}}{2}.$$

Hence

$$\mu_{\widetilde{X}}(\tilde{I})=C_F^{-\frac{\delta}{2}}C_1^{\delta}\mu_X(F^{-1}(\tilde{I}))\geq C_F^{-\frac{\delta}{2}}C_1^{\delta}C_R^{-1}(d^{-\frac{1}{2}}C_1^{-1}r_{\tilde{I}})^{\delta}=C_R^{-1}(d\,C_F)^{-\frac{\delta}{2}}r_{\tilde{I}}^{\delta}.$$

This proves the claim. □

Lemma A.6. *Let X be a δ-regular set with constant C_R on scales α_0 to α_1, and $0<\delta<d$. Fix an integer*

$$L\geq(2^{\frac{d}{2}}\sqrt{2d+1}C_R)^{\frac{2}{d-\delta}}. \tag{A-3}$$

Assume that I is a cube with $\alpha_0\leq r_I/L\leq r_I\leq\alpha_1$ and $I_1,\dots,I_{L^d}$ is the partition of I into cubes of side length r_I/L. Then there exists ℓ such that $X\cap I_\ell=\varnothing$.

Proof. Using Lemma A.2, it suffices to consider $I=[0,L]^d$, $\alpha_0\leq 1\leq L\leq\alpha_1$. We argue by contradiction. Assume that each I_ℓ intersects X. Then $I'_\ell:=I_\ell+\left[-\frac{1}{2},\frac{1}{2}\right]^d$ contains a unit cube centered at a point in X and thus

$$\mu_X(I'_\ell)\geq C_R^{-1}\quad\text{for all } 1\leq\ell\leq L^d.$$

On the other hand,

$$\bigcup_{\ell=1}^{L^d} I'_\ell = \left[-\tfrac{1}{2}, L+\tfrac{1}{2}\right]^d,$$

and each point in $\left[-\frac{1}{2}, L+\frac{1}{2}\right]^d$ can be covered by at most $2d+1$ of the cubes I'_ℓ. Therefore

$$C_R^{-1} L^d \le \sum_{\ell=1}^{L^d} \mu_X(I'_\ell) \le (2d+1)\mu_X\left(\left[-\tfrac{1}{2}, L+\tfrac{1}{2}\right]^d\right) \le (2d+1) C_R (L+1)^\delta,$$

which contradicts (A-3). □

Recall our definition of $\mathcal{C}_n$ and porosity in Definition 5.1.

Lemma A.7. *Let $X \subset [-1,1]^d$ be a δ-regular set with constant C_R on scales α_0 to α_1. Let L satisfy (A-3), and take $n \in \mathbb{Z}$ such that $\alpha_0 \le L^{-n-1} \le L^{-n} \le \alpha_1$. Then X is porous at scale L with depth n.*

Lemma A.8. *Let X be a δ-regular set with constant C_R on scales α_0 to α_1. Let $C \ge 1$ be a constant. Let I be a cube of side length r_I satisfying $\alpha_0 \le r_I \le C\alpha_1$. Let $\rho > 0$ satisfy $\alpha_0 \le \rho \le \min(r_I, \alpha_1)$. Then there exists a nonoverlapping[1] collection $\mathcal{J}$ of $N_{\mathcal{J}}$ cubes of side length ρ each such that*

$$X \cap I \subset \bigcup_{J \in \mathcal{J}} J, \quad N_{\mathcal{J}} \le \left(6\left\lceil \frac{3+C}{2} \right\rceil\right)^d C_R^2 \left(\frac{r_I}{\rho}\right)^\delta.$$

We will only use this lemma in dimension 1. Note that in [Bourgain and Dyatlov 2018], this is formulated with $C = 1$. We use this form with a constant C in the proof of Lemma 6.2.

Proof. Let $\mathcal{J}$ consist of all cubes of the form $\times_{k=1}^d \rho[j_k, j_k+1]$, $(j_1, j_2, \dots, j_d) \in \mathbb{Z}^d$, which intersect $X \cap I$. Then $X \cap I \subset \bigcup_{J \in \mathcal{J}} J$. Next, we will prove the upper bound on $N_{\mathcal{J}}$.

For each $J \in \mathcal{J}$, let $J' \supset J$ be the cube with the same center and with side length 2ρ. Since J intersects X, J' contains a cube of side length ρ centered at a point in X. Therefore

$$\mu_X(J') \ge C_R^{-1} \rho^\delta.$$

It is also clear that $\bigcup_{J \in \mathcal{J}} J' \subset I\left(\frac{3}{2}\rho\right)$, and each point lies in at most 3^d of the cubes J'.

If $r_I \le \alpha_1$, $I\left(\frac{3}{2}\rho\right)$ can be covered by 4^d cubes of side length r_I. If $\alpha_1 < r_I \le C\alpha_1$, $I\left(\frac{3}{2}\rho\right)$ can be covered by $2^d \lceil (3+C)/2 \rceil^d$ cubes of side length α_1. Therefore, we always have

$$N_{\mathcal{J}} \cdot C_R^{-1} \rho^\delta \le \sum_{J \in \mathcal{J}} \mu_X(J') \le 3^d \mu_X\left(\bigcup_{J \in \mathcal{J}} J'\right) \le \left(6\left\lceil \frac{3+C}{2} \right\rceil\right)^d C_R r_I^\delta,$$

and this proves the upper bound on $N_{\mathcal{J}}$. □

Appendix B: Proof of Lemma 6.2

We follow the proofs of Theorem 3.2 and Lemma 4.1 in [Jin and Zhang 2017]. Let us start with introducing some notation.

[1] A collection of cubes is nonoverlapping if the intersection of every two distinct cubes has empty interior.

Hilbert transform. Let $\mathcal{H}_0$ be the standard Hilbert transform defined as convolution with p.v. $\frac{1}{\pi x}$: for $f \in C_0^\infty(\mathbb{R})$ (or more generally, $f \in L^1(\mathbb{R}, \langle x\rangle^{-1}\,dx)$)

$$\mathcal{H}_0(f)(x) := \frac{1}{\pi}\lim_{\varepsilon\to 0+}\int_{|x-t|\geq\varepsilon}\frac{f(t)}{x-t}\,dt.$$

Let $\mathcal{H}$ be the modified Hilbert transform with integral kernel that decays like $|x|^{-2}$ as $|x|\to\infty$:

$$\mathcal{H}(f)(x) := \frac{1}{\pi}\lim_{\varepsilon\to 0+}\int_{|x-t|\geq\varepsilon} f(t)\Big(\frac{1}{x-t}+\frac{t}{t^2+1}\Big)\,dt, \quad f\in L^1(\mathbb{R},\langle x\rangle^{-2}\,dx).$$

The advantage of $\mathcal{H}$ is that it applies to a larger space that contains $L^\infty(\mathbb{R})$ as well as functions that grow like $|x|^{1-\varepsilon}$ as $|x|\to\infty$.

If $f\in L^1(\mathbb{R},\langle x\rangle^{-1}\,dx)$, then $\mathcal{H}(f)$ differs from $\mathcal{H}_0(f)$ by a constant. Moreover, we have the inversion formula for all $f\in L^1(\mathbb{R},\langle x\rangle^{-2}\,dx)$ with $\mathcal{H}(f)\in L^1(\mathbb{R},\langle x\rangle^{-2}\,dx)$:

$$\mathcal{H}(\mathcal{H}(f)) = -f + c(f), \tag{B-1}$$

where $c(f)$ is a real constant depending on f.

We will use the following example later in the proof.

Example B.1 [Jin and Zhang 2017, Example 2.3]. Let $f(x)=\log(x^2+1)$, then we can compute

$$\mathcal{H}(f)'(x) = \mathcal{H}_0(f')(x) = -\frac{2}{x^2+1}. \tag{B-2}$$

Hardy space and outer functions. We recall the definition of Hardy space on the real line

$$H^2 = H^2(\mathbb{R}) = \{f\in L^2(\mathbb{R}) : \operatorname{supp}\hat{f}\subset[0,\infty)\}.$$

If $f\in L^2(\mathbb{R})$, then $f+i\mathcal{H}_0(f)\in H^2(\mathbb{R})$.

The space of modulus of functions in H^2 can be characterized by the logarithmic integral: for $\omega\in L^2$, $\omega\geq 0$, we define

$$\mathcal{L}(\omega) := \int_{\mathbb{R}}\frac{\log\omega(x)}{1+x^2}\,dx.$$

Theorem B.2 [Havin and Jöricke 1994, Section 1.5]. *If $f\in H^2$ and $\mathcal{L}(|f|)=-\infty$, then $f\equiv 0$. On the other hand, if $\omega\in L^2$ and $\mathcal{L}(\omega)>-\infty$, then there exists a function $f\in H^2$ with $|f|=\omega$, unique up to a multiplication by a complex constant with unit modulus.*

If $\mathcal{L}(\omega)>-\infty$. Let $\Omega=-\log\omega$, then $\Omega\in L^1(\mathbb{R},\langle x\rangle^{-2}\,dx)$. Therefore we can define $\widetilde{\Omega}=\mathcal{H}(\Omega)$ and take

$$f = ae^{-(\Omega+i\widetilde{\Omega})}, \quad |a|=1. \tag{B-3}$$

We call functions of the form (B-3) for general $\Omega\in L^1(\mathbb{R},\langle x\rangle^{-2}\,dx)$ *outer functions*. The class of outer functions is closed under multiplications. Moreover if two outer functions have the same modulus, then they differ by a complex constant with unit modulus.

The following lemma gives a sufficient condition of a function to be the modulus of the Fourier transform of a function supported in $[0, \sigma]$.

Lemma B.3 [Mashregi et al. 2005, Theorem 1]. *Assume that $\omega = e^{-\Omega} \in L^2$ and $\mathcal{L}(\omega) > -\infty$. In addition, we assume that $\omega^2 e^{2\pi i \sigma x}$ is an outer function. Then there exists $\psi \in L^2$ with $\operatorname{supp} \psi \subset [0, \sigma]$ and $|\widehat{\psi}| = \omega$.*

An effective multiplier theorem. We prove an effective multiplier theorem. This proof is essentially in [Jin and Zhang 2017, Section 3], the only change we make lies in the definition of $k(x)$ below. Our modified definition makes sure that $k(x)$ is a constant function in a neighborhood of 0, which leads to a pointwise lower bound of $\widehat{\psi}(x)$ on the whole interval $\left[-\frac{3}{4}, \frac{3}{4}\right]$.

Theorem B.4. *Assume that $0 < \omega \le 1$ satisfies $\mathcal{L}(\omega) > -\infty$, and*

$$\|\mathcal{H}(\Omega)'\|_{L^\infty} \le \tfrac{\pi}{2}\sigma,$$

where $0 < \sigma < \frac{1}{10}$, $\Omega = -\log \omega$. Then there exists $\psi \in L^2(\mathbb{R})$ with

$$\operatorname{supp} \psi \subset [0, \sigma], \quad |\widehat{\psi}| \le \omega,$$

and

$$|\widehat{\psi}| \ge \frac{\sigma^{10}}{4 \times 10^{11}} \omega \quad \text{on } \left[-\tfrac{3}{4}, \tfrac{3}{4}\right].$$

Proof. We first set

$$\omega_0(x) = \frac{\omega(x)}{(x^2 + T^2)^5}, \qquad \Omega_0(x) = -\log(\omega_0(x)),$$

with constant T that will be specified later. We then have

$$\Omega_0 = \Omega + 5 \log(x^2 + T^2).$$

We compute

$$\mathcal{H}(\log(x^2 + T^2))(0) = \lim_{\varepsilon \to 0+} \int_{|t| \ge \varepsilon} \log(t^2 + T^2) \left(\frac{1}{-t} - \frac{t}{t^2 + 1} \right) dt,$$

in which the integrand is an odd function. Hence the integration is zero. Therefore we have

$$\mathcal{H}(\Omega_0)(0) = \mathcal{H}(\Omega)(0) + 5\mathcal{H}(\log(x^2 + T^2)) = \mathcal{H}(\Omega)(0). \tag{B-4}$$

By (B-2), we compute

$$\mathcal{H}(\log(x^2 + T^2))' = T^{-1} \mathcal{H}(\log(x^2 + 1))' \left(\frac{\cdot}{T} \right) = -\frac{2T}{x^2 + T^2}.$$

Thus if we choose $T = \frac{20}{\pi\sigma} \ge \frac{200}{\pi} \ge 60$, we have

$$\|\mathcal{H}(\Omega_0)'\|_{L^\infty} \le \|\mathcal{H}(\Omega)'\|_{L^\infty} + 5\|\mathcal{H}(\log(x^2 + T^2))'\|_{L^\infty} \le \pi\sigma. \tag{B-5}$$

Let us define

$$s_0(x) = \pi\sigma x + \mathcal{H}(\Omega_0)(x).$$

Hence by (B-4),

$$s_0(0) = \mathcal{H}(\Omega)(0),$$

depending only on ω.

Let $s(x)$ be defined as

$$s(x) = s_0(x) - \pi k(x) - \tfrac{\pi}{2},$$

in which

$$k(x) = \begin{cases} \lfloor \frac{1}{\pi} s_0(x) \rfloor & \text{if } \frac{1}{\pi} s_0(0) \in \left[\frac{1}{4}, \frac{3}{4}\right] \bmod 1, \\ \lfloor \frac{1}{\pi} s_0(x) - \frac{1}{2} \rfloor & \text{if } \frac{1}{\pi} s_0(0) \in \left[0, \frac{1}{4}\right) \bigcup \left(\frac{3}{4}, 1\right) \bmod 1. \end{cases} \tag{B-6}$$

Note that our definition of $k(x)$ is different from that in [Jin and Zhang 2017]. We modify the definition in order to make sure $k(x)$ is a constant near $x = 0$. This will be explained and used later in the proof.

By (B-5), $s_0(x)$ is a nondecreasing function and so is k. Note also that by our definition of $s(x)$, we have

$$\|s\|_{L^\infty} \le \pi. \tag{B-7}$$

Let $m = e^{-M}$, where $M = \mathcal{H}(s)$. Next, we will estimate $M(x) = \mathcal{H}(s)(x)$. We split the integral into three parts $M(x) = J_1(x) + J_2(x) + J_3(x)$, where

$$J_1(x) = \frac{1}{\pi} \int_{|x-t|<\frac{1}{2}} \frac{s(t) - s(x)}{x - t}\, dt,$$

$$J_2(x) = \frac{1}{\pi} \int_{|x-t|<\frac{1}{2}} s(t) \frac{t}{t^2+1}\, dt,$$

$$J_3(x) = \frac{1}{\pi} \int_{|x-t|\ge\frac{1}{2}} s(t) \left(\frac{1}{x-t} + \frac{t}{t^2+1} \right) dt.$$

We estimate J_2 and J_3 in the same way as in [Jin and Zhang 2017]. By (B-7), we have

$$|J_2(x)| \le \tfrac{1}{\pi} \cdot \|s\|_{L^\infty} \cdot \tfrac{1}{2} \le \tfrac{1}{2}. \tag{B-8}$$

Also, we have

$$|J_3(x)| \le \tfrac{1}{\pi} \cdot \|s\|_{L^\infty} \int_{|x-t|\ge\frac{1}{2}} \left| \frac{1}{x-t} + \frac{t}{t^2+1} \right| dt \le 6 \log(|x| + 2). \tag{B-9}$$

Finally, we need to bound $|J_1|$. By (B-5), we know $s_0(x) = \pi\sigma x + \mathcal{H}(\Omega_0)(x)$ is nondecreasing with $\|s_0'\|_{L^\infty} \le 2\pi\sigma$. Since we assume $0 < \sigma < \frac{1}{10}$, we have

$$\|\pi^{-1} s_0'\|_{L^\infty} < \tfrac{1}{5}.$$

This leads to the following:

- If $\pi^{-1} s_0(0) \in \left[\frac{1}{4}, \frac{3}{4}\right] \bmod 1$,

$$\tfrac{1}{\pi} s_0(x) \in (0, 1) \bmod 1 \quad \text{for all } x \in \left[-\tfrac{5}{4}, \tfrac{5}{4}\right].$$

- If $\pi^{-1} s_0(0) \in \left[0, \frac{1}{4}\right) \bigcup \left(\frac{3}{4}, 1\right) \bmod 1$,

$$\tfrac{1}{\pi} s_0(x) - \tfrac{1}{2} \in (0, 1) \bmod 1 \quad \text{for all } x \in \left[-\tfrac{5}{4}, \tfrac{5}{4}\right].$$

Recalling our definition of $k(x)$ in (B-6), we know in each case $k(x)$ is a constant function on the interval $\left[-\frac{5}{4}, \frac{5}{4}\right]$.

Thus for $x \in \left[-\frac{3}{4}, \frac{3}{4}\right]$, we have

$$|J_1(x)| \leq \frac{1}{\pi} \int_{|x-t|<\frac{1}{2}} \left| \frac{s_0(t) - s_0(x)}{x-t} \right| dt \leq \frac{1}{\pi} \|s_0'\|_{L^\infty} \leq 2\sigma. \tag{B-10}$$

For all x, we only have a lower bound of J_1. Since k is nondecreasing, we have

$$J_1(x) \geq \frac{1}{\pi} \int_{|x-t|<\frac{1}{2}} \frac{s_0(t) - s_0(x)}{x-t}\, dt \geq -2\sigma. \tag{B-11}$$

Now combining (B-8), (B-9) with (B-10), we have the following estimate of M on $\left[-\frac{3}{4}, \frac{3}{4}\right]$:

$$|M(x)| \leq 2\sigma + \tfrac{1}{2} + 6 \log \tfrac{11}{4} < 7. \tag{B-12}$$

Using (B-11) instead of (B-10), we obtain that for all x,

$$M(x) \geq -2\sigma - \tfrac{1}{2} - 6 \log\left(|x| + 2\right) > -1 - 6\log(|x|+2). \tag{B-13}$$

Next we will apply Lemma B.3 to $\tilde{\omega} = \frac{1}{3} m \omega_0$. We check that $\tilde{\omega}$ satisfies all the assumptions. First, by (B-13), we have

$$0 \leq \tilde{\omega} \leq \tfrac{1}{3} e (|x|+2)^6 \omega_0 \leq \frac{\omega}{x^2 + T^2}.$$

Hence $0 \leq \tilde{\omega} \leq \omega$ and $\tilde{\omega} \in L^2$. Moreover

$$\mathcal{L}(\tilde{\omega}) = \mathcal{L}\left(\tfrac{1}{3} m\right) + \mathcal{L}(\omega_0) > -\infty.$$

By the construction $M = \mathcal{H}(s)$ and the inversion formula (B-1), we have

$$\mathcal{H}(-2M - 2\Omega_0) = 2s - 2\mathcal{H}(\Omega_0) - 2c(M) = 2\pi\sigma x - 2\pi k(x) - \pi - 2c(M),$$

where $k(x) \in \mathbb{Z}$ and $c(M)$ is a real constant. Therefore for some constant a with $|a| = 1$, we have

$$\tilde{\omega}^2 e^{2\pi i \sigma x} = \tfrac{1}{9} e^{-2M - 2\Omega_0 + 2\pi i \sigma x} = \tfrac{1}{9} a e^{-2M - 2\Omega_0 + i\mathcal{H}(-2M - 2\Omega_0)},$$

which shows $\tilde{\omega}^2 e^{2\pi i \sigma x}$ is an outer function.

By Lemma B.3, there exists $\psi \in L^2$ with $\mathrm{supp}(\psi) \subset [0, \sigma]$ and $|\hat{\psi}| \leq \tilde{\omega} \leq \omega$. Furthermore, on $\left[-\frac{3}{4}, \frac{3}{4}\right]$, by (B-12), and since $T = \frac{20}{\pi\sigma}$, we have

$$|\hat{\psi}(x)| = \tilde{\omega}(x) \geq \tfrac{1}{3}(1 + T^2)^{-5} e^{-7} \omega(x) \geq \frac{\sigma^{10}}{4 \times 10^{11}} \omega(x),$$

as claimed. □

Multiplier adapted to the regular sets. Now we are in the place to finish the proof of Lemma 6.2.

Proof. The proof is the essentially same as that of Lemma 4.1 of [Jin and Zhang 2017]. We briefly go through the various constants below.

We define $n_1 \in \mathbb{N}$ by $2^{n_1} < S\alpha_1 \le 2^{n_1+1}$. For $1 \le n \le n_1$, let $A_n := [-2^{n+1}, -2^n] \bigcup [2^n, 2^{n+1}]$. Then by Lemma A.8, we have a collection $\mathcal{J}_n$ of N_n intervals of size $\rho_n := n^{-(1+\delta)/2} 2^n$ such that each element is of the form $[j, j+1]$, $j \in \mathbb{Z}$, intersects A_n, and

$$Y \cap A_n \subset \bigcup_{J \in \mathcal{J}_n} J.$$

Moreover, the number N_n satisfies

$$N_n \le 6\left\lceil \frac{3+S}{2} \right\rceil C_R^2 \left(\frac{2^n}{\rho_n}\right)^{\delta} = 6\left\lceil \frac{3+S}{2} \right\rceil C_R^2 n^{\frac{\delta(1+\delta)}{2}}. \tag{B-14}$$

Following the proof of [Jin and Zhang 2017], we define a weight function ω such that

$$\begin{aligned}
&\omega(\xi) = \exp(-\langle \xi \rangle^{\frac{1}{2}}) \ge 0.3 && \text{for all } \xi \in [-1, 1], \\
&\omega(\xi) \le \exp(-\langle \xi \rangle^{\frac{1}{2}}) && \text{for all } \xi \in \mathbb{R}, \\
&\omega(\xi) \le \exp(-\Theta(|\xi|)|\xi|) && \text{for all } \xi \in Y,\ |\xi| \ge 10,
\end{aligned}$$

$$\|\mathcal{H}(\omega)'\|_{L^\infty} \le \frac{\iota^{-1} C_R^2}{\delta_1(1-\delta_1)},$$

where $0 < \iota < 1$ is a constant depending only on S. The dependence comes from the upper bound of N_n in (B-14).

Applying Theorem B.4 to ω^{c_3} with

$$\sigma = \tfrac{1}{5} c_1, \quad c_3 = \tfrac{\pi}{10} \iota c_1 C_R^{-2} \delta_1 (1-\delta_1) < 1.$$

We obtain ψ with

$$\operatorname{supp} \psi \subset \left[0, \tfrac{1}{5} c_1\right],$$

$$\begin{aligned}
&|\widehat{\psi}(\xi)| \ge \frac{c_1^{10}}{4 \times 10^{18}} \omega(\xi)^{c_3} \ge \frac{3}{4 \times 10^{19}} c_1^{10} && \text{for all } \xi \in \left[-\tfrac{3}{4}, \tfrac{3}{4}\right], \\
&|\widehat{\psi}(\xi)| \le \exp(-c_3 \langle \xi \rangle^{\frac{1}{2}}) && \text{for all } \xi \in \mathbb{R}, \\
&|\widehat{\psi}(\xi)| \le \exp(-c_3 \Theta(|\xi|)|\xi|) && \text{for all } \xi \in Y,\ |\xi| \ge 10.
\end{aligned}$$

Finally, shifting ψ by $\frac{1}{10} c_1$ yields the desired function. □

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