ANALYSIS & PDE

Volume 13 No. 4 2020

msp

Analysis & PDE

msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2020 is US $340/year for the electronic version, and $550/year (+$60, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ANALYSIS AND PDE
Vol. 13, No. 4, 2020
dx.doi.org/10.2140/apde.2020.13.945

msp

ESTIMATES FOR THE NAVIER–STOKES EQUATIONS IN THE HALF-SPACE FOR NONLOCALIZED DATA

YASUNORI MAEKAWA, HIDEYUKI MIURA AND CHRISTOPHE PRANGE

This paper is devoted to the study of the Stokes and Navier–Stokes equations, in a half-space, for initial data in a class of locally uniform Lebesgue integrable functions, namely $L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$. We prove the analyticity of the Stokes semigroup e^{-tA} in $L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$ for $1<q\le\infty$. This follows from the analysis of the Stokes resolvent problem for data in $L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$, $1<q\le\infty$. We then prove bilinear estimates for the Oseen kernel, which enables us to prove the existence of mild solutions. The three main original aspects of our contribution are: the proof of Liouville theorems for the resolvent problem and the time-dependent Stokes system under weak integrability conditions, the proof of pressure estimates in the half-space, and the proof of a concentration result for blow-up solutions of the Navier–Stokes equations. This concentration result improves a recent result by Li, Ozawa and Wang and provides a new proof.

1. Introduction

This paper is devoted to the study of fluid equations in the half-space $\mathbb{R}^d_+$. Our goal is two-fold. First we show the analyticity of the Stokes semigroup for data belonging to the locally uniform Lebesgue space $L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$ for $1<q\le\infty$ (uniformly locally in L^p, divergence-free and normal component zero on the boundary; a precise definition is given below in Section 2B). Second we prove optimal bounds for the Oseen kernel $e^{-tA}\mathbb{P}\nabla\cdot$ and get as a by-product the short-time existence of mild solutions to the Navier–Stokes system with no-slip boundary conditions

$$\begin{cases}\partial_t u-\Delta u+\nabla p=-\nabla\cdot(u\otimes u), \quad \nabla\cdot u=0 \quad \text{in } (0,T)\times\mathbb{R}^d_+,\\ u=0 \quad \text{on } (0,T)\times\partial\mathbb{R}^d_+, \qquad u|_{t=0}=u_0 \quad \text{in } \partial\mathbb{R}^d_+\end{cases} \tag{1-1}$$

for nonlocalized initial data $u_0\in L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$, $q\ge d$. Our results directly yield the concentration of the scale-critical L^d norm for blow-up solutions of the Navier–Stokes equations.

1A. ***Outline of our results.*** Our analysis relies on the study of the Stokes resolvent problem. The first subsection below contains our main result in this direction. This enables (see second subsection) to prove the analyticity of the Stokes semigroup in $L^q_{\mathrm{uloc},\sigma}$ for $q\in(1,\infty]$. We then (third subsection below) state the bilinear estimates for the Oseen kernel, which allow the study of mild solutions in a way that is standard since [Fujita and Kato 1964]. We state the concentration result. The fourth subsection is devoted to the Liouville theorems proved in Appendices A and B.

MSC2010: primary 35A01, 35A02, 35B44, 35B53, 35Q30; secondary 35C99, 76D03, 76D05.
Keywords: Navier–Stokes equations, resolvent estimates, analyticity, Stokes semigroup, local uniform Lebesgue spaces, mild solutions, concentration, Liouville theorems, pressure, half-space.

Let us emphasize three aspects of our results, which we believe are the most original. First, our Liouville theorems for the resolvent problem and the time-dependent Stokes system hold under weak integrability conditions. Second, we prove pressure estimates in the half-space, which are key to our analysis of local energy weak solutions in [Maekawa et al. 2019]. Third, we show a concentration phenomenon for blow-up solutions of the Navier–Stokes equations. Our result improves a recent result by Li, Ozawa and Wang [Li et al. 2018] and provides a new proof. These aspects are discussed more extensively in [Prange 2018].

Stokes resolvent problem. The following statements are the main tools of the rest of the paper. A considerable part of our work is concerned with estimates for the resolvent problem for the (stationary) Stokes system

$$\begin{cases} \lambda u - \Delta u + \nabla p = f, \quad \nabla \cdot u = 0 \quad \text{in } \mathbb{R}^d_+, \\ u = 0 \quad \text{on } \partial\mathbb{R}^d_+ \end{cases} \tag{1-2}$$

for nonlocalized data f in the class $L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$ for $1 < q \leq \infty$.

Theorem 1. *Let $1 < q \leq \infty$, $\varepsilon > 0$. Let λ be a complex number in the sector $S_{\pi-\varepsilon}$. Let $f \in L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$. Then there exist $C(d, \varepsilon, q) < \infty$ (independent of λ) and a unique solution $(u, \nabla p) \in L^q_{\mathrm{uloc}}(\mathbb{R}^d_+)^d \times L^1_{\mathrm{uloc}}(\mathbb{R}^d_+)^d$ to (1-2) in the sense of distributions. Moreover, this unique solution is such that $u \in W^{1,q}_{0,\mathrm{uloc}}(\mathbb{R}^d_+)$ and*

$$|\lambda| \|u\|_{L^q_{\mathrm{uloc}}} + |\lambda|^{1/2} \|\nabla u\|_{L^q_{\mathrm{uloc}}} \leq C \|f\|_{L^q_{\mathrm{uloc}}}, \tag{1-3}$$

$$\|\nabla^2 u\|_{L^q_{\mathrm{uloc}}} + \|\nabla p\|_{L^q_{\mathrm{uloc}}} \leq C(1 + e^{-c|\lambda|^{1/2}} \log|\lambda|) \|f\|_{L^q_{\mathrm{uloc}}} \quad \textit{for } q \neq \infty, \tag{1-4}$$

and

$$\lim_{R\to\infty} \|\nabla' p\|_{L^1(|x'|<1,\, R<x_d<R+1)} = 0. \tag{1-5}$$

Moreover, for $1 < q = p \leq \infty$ or $1 \leq q < p \leq \infty$ satisfying $1/q - 1/p < 1/d$, there exists a constant $C(d, \varepsilon, q, p) < \infty$ (independent of λ) such that

$$\|u\|_{L^p_{\mathrm{uloc}}} \leq C|\lambda|^{-1}(1 + |\lambda|^{(d/2)(1/q-1/p)}) \|f\|_{L^q_{\mathrm{uloc}}}, \tag{1-6}$$

$$\|\nabla u\|_{L^p_{\mathrm{uloc}}} \leq C|\lambda|^{-1/2}(1 + |\lambda|^{(d/2)(1/q-1/p)}) \|f\|_{L^q_{\mathrm{uloc}}}. \tag{1-7}$$

Theorem 1 is proved in Section 4.

Uniqueness comes from the condition (1-5) which eliminates the parasitic solutions of our Liouville-type result, Theorem 4, which is proved in Appendix A. Condition (1-5) is easily verified for the pressure represented via the integral formulas of Section 2C. With Theorem 1, one can define the resolvent operator $R(\lambda) = (\lambda + \boldsymbol{A})^{-1}$ on the sector $S_{\pi-\varepsilon}$ for given $\varepsilon > 0$. As is classical, the bounds on the solution to the resolvent problem are crucial to estimate the semigroup. The mixed p, q bounds (1-6) and (1-7) are particularly important in view of the study of the nonlinear term in the Navier–Stokes equations. Let us comment on two points. First, we are not able to remove the $|\log(\lambda)|$ loss for small λ in (1-4). We are ignorant as to whether there is a real obstruction or if this is just a technical issue. Second, the estimate (1-3) fails for $q = 1$. This is a fundamental point, which was already observed in the case of $L^1(\mathbb{R}^d_+)$

by Desch, Hieber and Prüss [Desch et al. 2001]. It is specific to the case of $\mathbb{R}^d_+$ as opposed to $\mathbb{R}^d$. We comment more on this below.

Stokes semigroup. Let $\boldsymbol{A}$ be the Stokes operator realized in $L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$ (for precise definitions, see Section 5). Time-dependent estimates for the semigroup are derived from the resolvent bounds using classical techniques from complex analysis.

Theorem 2. *Let $1 < q \leq \infty$. Then $-\boldsymbol{A}$ generates a bounded analytic semigroup in $L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$.*

More precise statements (and their proofs) along with global-in-time estimates for the linear Stokes dynamic are given in Section 5; see in particular Propositions 5.2 and 5.3. Again, because of the failure of (1-3) when $q = 1$, $-\boldsymbol{A}$ fails to generate an analytic semigroup in $L^1_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$. This is due to the presence of the boundary.

Bilinear estimates, mild solutions and concentration for blow-up solutions. Our main result is the following bilinear estimate, from which the short-time existence of mild solutions follows as a corollary.

Theorem 3. *Let $1 < q \leq p \leq \infty$ or $1 \leq q < p \leq \infty$ and let $\boldsymbol{A}$ be the Stokes operator realized in $L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$. Then for $\alpha = 0, 1$ and $t > 0$*

$$\|\nabla^\alpha e^{-t\boldsymbol{A}}\mathbb{P}\nabla\cdot(u\otimes v)\|_{L^p_{\mathrm{uloc}}} \leq Ct^{-(1+\alpha)/2}(t^{-(d/2)(1/q-1/p)}+1)\|u\otimes v\|_{L^q_{\mathrm{uloc}}}, \tag{1-8}$$

$$\|\nabla e^{-t\boldsymbol{A}}\mathbb{P}\nabla\cdot(u\otimes v)\|_{L^q_{\mathrm{uloc}}} \leq Ct^{-1/2}(\|u\nabla v\|_{L^q_{\mathrm{uloc}}}+\|v\nabla u\|_{L^q_{\mathrm{uloc}}}). \tag{1-9}$$

Estimates (1-8) *and* (1-9) *are valid also for $q = 1$.*

The proof of Theorem 3 is given in Section 6. The function $e^{-t\boldsymbol{A}}\mathbb{P}\nabla\cdot(u\otimes v)$ is expressed in terms of the integral over $\mathbb{R}^3_+$ with some kernels satisfying suitable pointwise estimates, and such a representation itself is well-defined and satisfies (1-8) and (1-9) even for the case $q = 1$, though we do not have the analyticity of the semigroup $\{e^{-t\boldsymbol{A}}\}_{t\geq 0}$ in $L^1_{\mathrm{uloc},\sigma}(\mathbb{R}^3_+)$.

There is quite some work to go from the semigroup bounds on $e^{-t\boldsymbol{A}}$ to the bounds on $e^{-t\boldsymbol{A}}\mathbb{P}\nabla\cdot$ of Theorem 3, which are needed to solve the Navier–Stokes equations. Two of the key difficulties are making sense of the action of $e^{-t\boldsymbol{A}}\mathbb{P}\nabla\cdot$ on $L^q_{\mathrm{uloc},\sigma}$ functions, since the Helmholtz–Leray projection does not act as a bounded operator on L^q_{uloc}; and the noncommutativity of vertical derivatives and the symbol (in tangential Fourier) for the projection $\mathbb{P}$.

The existence of mild solutions for initial data $u_0 \in L^q_{\mathrm{uloc},\sigma}$, $q \geq d$, is stated in Propositions 7.1 and 7.2 in Section 7. Once the bilinear estimate of Theorem 3 is established, the local-in-time existence of mild solutions can be shown by the standard arguments. It is not our objective to include a deeper discussion of the mild solutions here. We only note that the pressure p associated to the mild solution u can be constructed so that the pair (u, p) satisfies (1-1) in the sense of distributions; see [Maekawa et al. 2019] for the details. Many other issues, such as the convergence to the initial data, are similar in the half-space and the whole space. Hence we simply refer to [Maekawa and Terasawa 2006] where a thorough discussion is carried out.

As an application of the well-posedness in $L^q_{\mathrm{uloc},\sigma}$, $q \geq d$, we study the behavior of the blow-up solution. Here a solution $u \in C((0, T_*); L^\infty_\sigma(\mathbb{R}^d_+))$ blows up at $t = T_*$ if $\limsup_{t\uparrow T_*}\|u(t)\|_{L^\infty} = \infty$. Leray

[1934] proved a lower bound for the blow-up solutions in $\mathbb{R}^d$ of the form

$$\|u(t)\|_{L^\infty} \geq \frac{\varepsilon}{\sqrt{T_* - t}} \quad \text{for } t < T_*. \tag{1-10}$$

More recently lower bounds for scale-critical norms of the blow-up solution have been also studied extensively. It was shown in [Barker and Seregin 2015] that $\lim_{t\to T_*} \|u(t)\|_{L^3(\mathbb{R}^3_+)} = \infty$. On the other hand, in [Li et al. 2018] a lower bound for the local L^d norm along the parabolic cone for the blow-up solutions is shown. More precisely, it is proved that there exists a sequence $(x_n, t_n) \in \mathbb{R}^3 \times (0, T_*)$ such that $\|u(t_n)\|_{L^d(x_n+(0,c_0\sqrt{T_*-t_n})^d)} \geq \varepsilon$. This can be seen as a concentration phenomenon of the critical norm at the blow-up time. In the same work, concentration results along the parabolic cone are also proved for L^q norms, $q \geq d$, along a discrete sequence of times. To the best of our knowledge, this type of concentration result is new for the Navier–Stokes equations, even in the whole space $\mathbb{R}^d$. While in [Li et al. 2018] the concentration results are proved via a clever use of frequency decomposition techniques, it occurred to us that they are simple consequences of the existence of mild solutions in L^q_{uloc}. As a direct consequence of the well-posedness result, we thus have the following corollary.

Corollary 1.1. *For all $q \geq d$, there exists a positive constant $\gamma < \infty$ such that for all $0 < T_* < \infty$, for all $u \in C((0, T_*); L^\infty_\sigma(\mathbb{R}^d_+))$ mild solutions to* (1-1), *if u blows up at $t = T_*$, then for all $t \in (0, T_*)$, there exists $x(t) \in \mathbb{R}^d_+$ with the estimate*

$$\|u(t)\|_{L^q(|x(t)-\,\cdot\,|\leq\sqrt{T_*-t})} \geq \frac{\gamma}{(T_* - t)^{(1/2)(1-d/q)}}.$$

This corollary is proved in Section 7C. Our result also holds for $\mathbb{R}^d$ instead of $\mathbb{R}^d_+$ and seems to be new even in that case. It obviously includes the concentration of the scale-critical norm L^d. It improves Theorem 1.2 in [Li et al. 2018] in the sense that the concentration holds not only along a sequence of times $t_n \to T_*$, but for all times $t \in (0, T_*)$. Moreover, our method gives a new and simple proof of this type of result, which appears to be a nice application of the existence of mild solutions in the L^q_{uloc} setting.

Liouville theorems. Here we give a uniqueness result to (1-2) in our functional framework. This Liouville theorem is the counterpart for the resolvent system to the Liouville theorem for the unsteady Stokes system in the half-space proved in [Jia et al. 2012] and crucial for the uniqueness part of Theorem 1.

Theorem 4. (i) *Let $\lambda \in S_{\pi-\varepsilon}$ with $\varepsilon \in (0, \pi)$. Let $(u, \nabla p) \in L^1_{\mathrm{uloc}}(\mathbb{R}^d_+)^d \times L^1_{\mathrm{uloc}}(\mathbb{R}^d_+)^d$ with $p \in L^1_{\mathrm{loc}}(\mathbb{R}^d_+)$ be a solution to* (1-2) *with $f = 0$ in the sense of distributions. Then $(u, \nabla p)$ is a parasitic solution; i.e., $u = (a'(x_d), 0)^\top$ and $p = D \cdot x' + c$. Here $a'(x_d) = (a_1(x_d), \dots, a_{d-1}(x_d))^\top$ is smooth and bounded and its trace at $x_d = 0$ is zero, while $D \in \mathbb{C}^{d-1}$ is a constant vector and $c \in \mathbb{C}$ is a constant. If either $\lim_{R\to\infty} \|\nabla' p\|_{L^1(|x'|<1, R<x_d<R+1)} = 0$ or $\lim_{|y'|\to\infty} \|u\|_{L^1(|x'-y'|<1, 1<x_d<2)} = 0$ in addition, then p is a constant and $u = 0$.*

(ii) *Let $(u, \nabla p) \in L^1_{\mathrm{uloc}}(\mathbb{R}^d_+)^d \times L^1_{\mathrm{uloc}}(\mathbb{R}^d_+)^d$ with $p \in L^1_{\mathrm{loc}}(\mathbb{R}^d_+)$ be a solution to* (1-2) *with $\lambda = 0$ and $f = 0$ in the sense of distributions. Then $u = 0$ and p is a constant.*

The proof of this theorem is given in Appendix A. By using a similar argument we shall show a uniqueness result for the time-dependent Stokes problem

$$\begin{cases} \partial_t u - \Delta u + \nabla p = f, \quad \nabla \cdot u = 0 \quad \text{in } (0,T)\times\mathbb{R}^d_+, \\ u = 0 \quad \text{on } (0,T)\times\partial\mathbb{R}^d_+, \qquad u|_{t=0} = u_0 \quad \text{in } \partial\mathbb{R}^d_+. \end{cases} \tag{1-11}$$

Compared with the known Liouville theorem by [Jia et al. 2012] for bounded solutions, our result imposes the condition on the pressure, while the regularity condition on the velocity is weaker than in [Jia et al. 2012]. This framework will be useful in the study of the local energy weak solutions.

Theorem 5. *Let* $(u, \nabla p)$ *be a solution to* (1-11) *in the sense of distributions with* $u_0 = f = 0$. *Then* $(u, \nabla p)$ *is a parasitic solution*; *i.e.*, $u(t,x) = (a'(t,x_d), 0)^\top$ *and* $p(t,x) = D(t)\cdot x' + c(t)$. *Here* $a'(t,x_d) = (a_1(t,x_d), \dots, a_{d-1}(t,x_d))^\top$ *with* $a_j \in L^1_{\mathrm{loc}}([0,T)\times\overline{\mathbb{R}}^d_+)$, *while* $D \in L^1_{\mathrm{loc}}(0,T)^{d-1}$ *and* $c \in L^1_{\mathrm{loc}}(0,T)$. *If either*

$$\lim_{R\to\infty}\int_\delta^T \|\nabla' p(t)\|_{L^1(|x'|<1,\,R<x_d<R+1)}\,dt = 0 \quad \textit{for all } \delta\in(0,T)$$

or

$$\lim_{|y'|\to\infty}\int_0^T \|u(t)\|_{L^1(|x'-y'|<1,\,1<x_d<2)}\,dt = 0$$

in addition, *then* p *is a constant and* $u = 0$.

This theorem is proved in Appendix B. There we state precisely the notion of solutions to (1-11). To our knowledge, these two Liouville-type results, Theorems 4 and 5, are new under these assumptions.

1B. ***Comparison to other works.*** We give an overview of some works related to our result. Although we try to give a faithful account of the state of the art of the study of fluid equations (mainly Stokes and Navier–Stokes systems) with infinite energy or nonlocalized data, we are very far from being exhaustive in our description. We divide the description into three parts: first we deal with the class of bounded functions, second we handle the class of locally uniform Lebesgue spaces and finally we describe some differences between the whole space and the half-space.

A common feature of the analysis in L^∞_σ and $L^q_{\mathrm{uloc},\sigma}$ is the failure of classical techniques used in works on Stokes and Navier–Stokes equations. This appears at several levels. Firstly, there is of course no global energy inequality. The substitute is a local energy inequality, which involves the pressure. Hence one has to obtain precise information on the pressure. Secondly, there is obviously no uniqueness for flows with infinite energy. This is due to flows driven by the pressure (solving the Stokes and Navier–Stokes equations), such as in the whole space $\mathbb{R}^d$

$$u(x,t) := f(t) \quad \text{and} \quad p(x,t) := -f'(t)\cdot x,$$

or in the half-space $\mathbb{R}^d_+$

$$u(x,t) := (v_1(x_d,t), \dots, v_{d-1}(x_d,t), 0) \quad \text{and} \quad p(x,t) := -f(t)\cdot x',$$

where $f \in C_0^\infty((0,\infty);\mathbb{R}^{d-1})$ and $v(x_d,t)$ solves the heat equation $\partial_t v - \partial_d^2 v = f$ with $v(0,t) = 0$. Hence, one has to handle or eliminate these parasitic solutions. Thirdly, even in the instance where flows driven by the pressure are ruled out, one needs to make sense of a representation formula for the pressure. Indeed the source term in the elliptic equation for the pressure,

$$-\Delta p = \nabla \cdot (\nabla \cdot (u \otimes u)),$$

is nonlocalized, and thus a priori nondecaying at infinity. Fourthly, the Helmholtz–Leray projection is not bounded on L^∞ or on L^q_{uloc}, basically because the Riesz transform does not map L^∞ into itself. This makes the study of the mapping properties of the Stokes operator $\boldsymbol{A}$, which is usually defined as $\boldsymbol{A} = -\mathbb{P}\Delta_D$, where $\mathbb{P}$ is the Helmholtz–Leray projection and Δ_D the Dirichlet Laplacian, particularly delicate.

Bounded functions. From the point of view of both the results and the techniques, the main source of inspiration of the linear Stokes estimates is the paper by Desch, Hieber and Prüss [Desch et al. 2001]. This paper is concerned with the study of the Stokes semigroup in the half-space, in particular in the class of bounded functions. The authors prove the analyticity of the Stokes semigroup in $L^q_\sigma(\mathbb{R}^d_+)$ for $1 < q \le \infty$. The case of $1 < q < \infty$ was previously known. Their approach is based on the study of the Stokes resolvent problem (1-2). In order to circumvent the use of the Helmholtz–Leray projection, one of the key ideas is to decompose the resolvent operator into a part corresponding to the Dirichlet–Laplace part and another part associated with the nonlocal pressure term

$$(\lambda + \boldsymbol{A})^{-1} = \boldsymbol{R}_{\mathrm{D.L.}}(\lambda) + \boldsymbol{R}_{\mathrm{n.l.}}(\lambda).$$

Our work uses the same idea, but we need more precise estimates on the kernels than the mere L^1 bounds proved in [Desch et al. 2001, Section 3], which are not enough for our purposes.

The L^∞ theory for the Stokes equations has recently been advanced thanks to a series of works by Abe, Giga and Hieber. In [Abe and Giga 2013], the Stokes semigroup is proved to be analytic via an original (in this context) compactness (or blow-up) method in admissible domains, which include bounded domains and the half-space. In these domains a bound on the pressure holds, which excludes the parasitic solutions previously mentioned. However, an intrinsic drawback of the compactness argument is that it only gives an L^∞ bound on the solution for times $0 < t < T_0$, with T_0 depending only on the domain. The papers [Abe 2016; Abe and Giga 2014] build on the same method. Concerning the resolvent problem, it was considered in [Abe et al. 2015a] by a localization argument, which boils down to applying locally the L^p theory and interpolating to get a control in L^∞. These developments enabled the investigation of blow-up rates (1-10) in L^∞ for potential singularities in the solutions of the Navier–Stokes equations [Abe 2015].

Locally uniform Lebesgue spaces. The locally uniform Lebesgue spaces L^q_{uloc} form a wider class of functions than L^∞. They include a richer spectrum of behaviors, obviously allowing for some singular behavior (homogeneous functions slowly decaying at ∞) or nondecaying functions such as locally L^p periodic or almost-periodic functions. The main advantage of this class is that it is easy to define and visualize, while it includes various class of functions as mentioned above. In the operator-theoretical point of view, another advantage is that we can characterize the domain of the Stokes operator in the L^q_{uloc} spaces if $1 < q < \infty$ (see Section 5), which is hard to expect in the L^∞ framework even for the Laplace operator.

On the other hand, the main and major drawback is that the L^q_{uloc} functions are difficult to handle in the context of pseudodifferential calculus. Indeed, there is no obvious characterization in Fourier space, which makes it difficult to straightly analyze the action of Fourier symbols. Most of the time, one has to first derive kernel bounds for the symbols in physical space, before estimating the L^q_{uloc} norms. Many equations have been studied in the framework of loc-uniform spaces. Without aiming at exhaustivity, let us mention some works parabolic-type equations: on linear parabolic equations [Arrieta et al. 2004], on Ginzburg–Landau equations [Mielke and Schneider 1995; Ginibre and Velo 1997] and on reaction-diffusion equations [Cholewa and Dlotko 2004]. We refer to these works for basic properties of the space L^q_{uloc}.

The study of the Stokes semigroup and the application to the existence of mild solutions to the Navier–Stokes equations with initial data $u_0 \in L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d)$, $q \geq d$, was carried out in [Maekawa and Terasawa 2006]. Regarding the existence of weak solutions for initial data in $L^2_{\mathrm{uloc},\sigma}(\mathbb{R}^d)$ satisfying the local energy inequality, the so-called suitable weak solutions it was handled in [Lemarié-Rieusset 2002, Chapter 32 and 33]. This result was also worked out in [Kikuchi and Seregin 2007]. Existence of these local energy solutions is the key to the blow-up of the L^3 norm criteria at the blow-up time proved in [Seregin 2012] for the three-dimensional Navier–Stokes equations and also to the construction of the forward self-similar solutions in [Jia and Šverák 2014].

Whole space vs. half-space. Fewer results are proved for the half-space and more generally for unbounded domains with (unbounded) boundaries. Let us emphasize some phenomena related to the presence of boundaries.

A striking feature of the half-space case is the failure of $L^1(\mathbb{R}^d_+)$ estimates, for the resolvent problem as well as the semigroup. This fact was proved in [Desch et al. 2001, Section 5]. In the whole space, the Stokes semigroup is known to be analytic even for $q = 1$; see [Maekawa and Terasawa 2006]. As underlined in [Desch et al. 2001], one should relate this lack of analyticity in L^1 to the nonexistence of local mild solutions to the Navier–Stokes equations in L^1 for an exterior domain [Kozono 1998]. Existence of such solutions would imply that the total force acting on the boundary is zero.

On a different note, the Helmholtz decomposition may fail even in L^q for some $1 < q < \infty$ for smooth sector-like domains with sufficiently large opening; see [Galdi 2011, Remark III.1.3]. On the contrary, the decomposition is known to hold for any $1 < q < \infty$, for any smooth domain with compact boundary, for the half-space and the whole space; see [Galdi 2011, Theorem III.1.2]. The definition of the Stokes semigroup in L^q spaces for finite q in non-Helmholtz sector-like domains was recently addressed in [Abe et al. 2015b].

The works of Abe and Giga, notably [Abe and Giga 2013], aim at extending results known for the Stokes semigroup in $L^\infty_\sigma(\mathbb{R}^d_+)$ to more general domains with boundaries. They introduce a class of admissible domains (which includes smooth bounded domains and $\mathbb{R}^d_+$) in which the analyticity of the Stokes semigroup holds in L^∞_σ. This work however says nothing in general about the long-time behavior of the linear Stokes dynamics. Indeed the L^∞ bound for the Stokes dynamics is true only on a time interval $(0, T_0)$, with T_0 depending only on the domain. Notice that $T_0 = \infty$ for smooth bounded domains and for the half-space. Regarding the existence of mild solutions to the Navier–Stokes equations in the half-space for bounded data, let us mention [Solonnikov 2003] (initial data bounded and continuous) and [Bae and Jin 2012] (initial data in just bounded). These works are based on direct estimates on the kernels of the nonstationary Stokes system. Our approach for the solvability of the Navier–Stokes equations in the nonlocalized class $L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$ is based on the analysis of the bilinear operator

$$(u, v) \mapsto (\lambda + \boldsymbol{A})^{-1}\mathbb{P}\nabla \cdot (u \otimes v), \tag{1-12}$$

from which we derive bounds for the unsteady problem. A key issue of the half-space as opposed to the whole space is the noncommutativity of $\mathbb{P}$ and vertical derivatives ∂_d (the commutation with tangential derivatives does work). This prompts the need to integrate by parts in the vertical direction so as to analyze (1-12) (see Section 6B below).

To finish this overview, let us mention that stationary Stokes, Stokes–Coriolis and Navier–Stokes–Coriolis systems with infinite-energy Dirichlet boundary condition were also considered in the context of boundary layer theory. The domain is usually a perturbed half-space with a highly oscillating boundary $x_d \geq \omega(x')$. The results of [Dalibard and Prange 2014; Dalibard and Gérard-Varet 2017] are well-posedness results in the class of Sobolev functions with locally uniform L^2 integrability in the tangential variable and L^2 integrability in the vertical variable. The main challenges are first the bumpiness of the boundary, which prevents from using the Fourier transform close to the boundary and second the lack of a priori bounds on the function itself, which requires reliance on Poincaré-type inequalities. The reader is also referred to [Geissert and Giga 2008], where the Stokes resolvent equations in the exterior domain are analyzed in the L^p_{uloc} space. In [Geissert and Giga 2008] the compactness of the boundary is essentially used.

1C. ***Overview of the paper.*** In Section 2, the reader can find standard notations used throughout the paper, the definitions of the functional spaces as well as the computation of the Fourier symbols for the resolvent problem. As stated above, we rely on the decomposition of the solution to the resolvent problem into a part corresponding to the solution of the Dirichlet–Laplace problem and a part associated with the nonlocal pressure. Section 3 is devoted to getting pointwise bounds on the kernels for the resolvent problem defined in the physical space. In this regard, Lemma 3.1 is the basic tool so as to get the optimal pointwise estimates. These bounds on the kernels stated in Proposition 3.2 (local Dirichlet–Laplace part), Proposition 3.5 (nonlocal pressure part) and Proposition 3.7 (pressure) form an essential part of our work. They are indispensable for the estimates in L^q_{uloc} obtained in Section 4. In this section, Theorem 1 is proved. The next section, Section 5 establishes the analyticity of the Stokes semigroup (Theorem 2) along with the bounds on the longtime dynamic of the linear Stokes equation stated in Proposition 5.3. Section 6 contains the crucial bilinear estimates (Proposition 6.4 and Theorem 3) needed to prove the existence of mild solutions to the Navier–Stokes equations (1-1). The proofs of Proposition 7.1, Proposition 7.2 and Corollary 1.1 are given in Section 7. Appendix A is concerned with the proof of the Liouville-type result of Theorem 4 for the resolvent problem (1-2). The Liouville theorem for the nonsteady Stokes system, Theorem 5, is proved in Appendix B.

2. Preliminaries

2A. ***Notation.*** Throughout the paper (unless stated otherwise), the small Greek letters α, β, γ, ι, η usually denote integers or multi-indices, and ε, δ, $\kappa > 0$ denote small positive real numbers. When it is clear from the context, we also sometimes use Einstein's summation convention for repeated indices. For $(x', x_d) \in \mathbb{R}^d_+$, $x' \in \mathbb{R}^{d-1}$ is the tangential component, while x_d is the vertical one. The complex scalar number $\lambda \in \mathbb{C}$ belongs to the sector

$$S_{\pi-\varepsilon} := \{\rho e^{i\theta} : \rho > 0,\ \theta \in [-\pi+\varepsilon, \pi-\varepsilon]\} \subseteq \mathbb{C},$$

with ε fixed in $(0, \pi)$. For $\xi \in \mathbb{R}^{d-1}$, we define

$$\omega_\lambda(\xi) := \sqrt{\lambda + |\xi|^2}. \tag{2-1}$$

The following inequality is used repeatedly in the paper: there exists a constant $C(\varepsilon) < \infty$ such that, for all $\lambda \in S_{\pi-\varepsilon}$, for all $\xi \in \mathbb{R}^{d-1}$,

$$|\omega_\lambda(\xi)| \geq \mathrm{Re}(\omega_\lambda(\xi)) \geq C(|\lambda|^{1/2} + |\xi|).$$

Finally, let us fix our convention for the Fourier transform: for $\xi \in \mathbb{R}^{d-1}$,

$$\hat{u}(\xi) := \int_{\mathbb{R}^{d-1}} e^{-i\xi\cdot x'} u(x')\, dx'$$

for $u \in \mathcal{S}(\mathbb{R}^{d-1})$. The inverse Fourier transform is defined by

$$u(x') := \frac{1}{(2\pi)^{d-1}} \int_{\mathbb{R}^{d-1}} e^{i\xi\cdot x'} \hat{u}(\xi)\, d\xi$$

for $\hat{u} \in \mathcal{S}(\mathbb{R}^{d-1})$. Both the Fourier transform and its inverse are naturally extended to $\mathcal{S}'(\mathbb{R}^{d-1})$ by duality. The definitions of the functional spaces are given in the next paragraph.

2B. ***Functional setting and notion of solutions.*** The results of the paper take place in the class $L^p_{\mathrm{uloc}}(\mathbb{R}^d_+)$ of uniformly locally L^p functions. More precisely,

$$L^p_{\mathrm{uloc}}(\mathbb{R}^d_+) := \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^d_+) \;\middle|\; \sup_{\eta \in \mathbb{Z}^{d-1}\times\mathbb{Z}_{\geq 0}} \|f\|_{L^p(\eta+(0,1)^d)} < \infty \right\}.$$

Let us define the space $L^p_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$ of solenoidal vector fields in L^p_{uloc} as

$$L^p_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+) := \left\{ f \in L^p_{\mathrm{uloc}}(\mathbb{R}^d_+)^d \;\middle|\; \int_{\mathbb{R}^d_+} f \cdot \nabla\varphi\, dx = 0 \text{ for any } \varphi \in C^\infty_0(\bar{\mathbb{R}}^d_+) \right\}. \tag{2-2}$$

Notice that this definition encodes both the fact that f is divergence-free in the sense of distributions (take test functions $\varphi \in C^\infty_0(\mathbb{R}^d_+)$) and the fact that f_d vanishes on $\partial\mathbb{R}^d_+$. As usual, $\mathrm{WL}^{p'}_{\mathrm{uloc}}(\mathbb{R}^d_+)$ for $1 < p' \leq \infty$ denotes the dual space of $L^p_{\mathrm{uloc}}(\mathbb{R}^d_+)$, where p' is the Hölder conjugate of $1 \leq p < \infty$, $1 = 1/p + 1/p'$. It is defined as

$$\mathrm{WL}^{p'}_{\mathrm{uloc}}(\mathbb{R}^d_+) := \left\{ g \in L^1_{\mathrm{loc}}(\mathbb{R}^d_+) \;\middle|\; \sum_{\eta \in \mathbb{Z}^{d-1}\times\mathbb{Z}_{\geq 0}} \|g\|_{L^{p'}(\eta+(0,1)^d)} < \infty \right\}.$$

As is usual $\mathrm{BUC}(\mathbb{R}^d_+)$ denotes the space of bounded uniformly continuous functions, and

$$\mathrm{BUC}_\sigma(\mathbb{R}^d_+) = \{ f \in \mathrm{BUC}(\mathbb{R}^d_+)^d \mid \mathrm{div}\, f = 0,\ f|_{x_d=0} = 0 \}.$$

Note that any function in $\mathrm{BUC}(\mathbb{R}^d_+)$ is uniquely extended as a bounded uniformly continuous function on $\bar{\mathbb{R}}^d_+$, and thus the trace is defined as a restriction on $\partial\mathbb{R}^d_+$ of this extended continuous function on $\bar{\mathbb{R}}^d_+$.

Let us also fix the notion of solutions to (1-2). Let $f \in L^1_{\mathrm{uloc}}(\mathbb{R}^d_+)^d$. We say that $(u, \nabla p) \in L^1_{\mathrm{uloc}}(\mathbb{R}^d_+)^d \times L^1_{\mathrm{uloc}}(\mathbb{R}^d_+)^d$ with $p \in L^1_{\mathrm{loc}}(\mathbb{R}^d_+)$ is a solution to (1-2) in the sense of distributions if

$$\int_{\mathbb{R}^d_+} u \cdot (\lambda\varphi - \Delta\varphi) + \nabla p \cdot \varphi \, dx = \int_{\mathbb{R}^d_+} f \cdot \varphi \, dx, \quad \varphi \in C^\infty_0(\bar{\mathbb{R}}^d_+)^d \text{ with } \varphi|_{x_d=0} = 0, \tag{2-3}$$

and

$$\int_{\mathbb{R}^d_+} u \cdot \nabla\phi \, dx = 0, \quad \phi \in C^\infty_0(\bar{\mathbb{R}}^d_+). \tag{2-4}$$

Let us notice that the notion of solutions defined by (2-3) and (2-4) is enough to apply our uniqueness result, Theorem 4. Moreover, we emphasize that the solution u of the resolvent problem (1-2) given by Theorem 1 is a strong solution, thanks to the estimates (1-3) and (1-4). Hence the trace of u is well-defined in the sense of the trace of $W^{1,q}_{\mathrm{loc}}(\mathbb{R}^d_+)$ functions and must be zero; roughly speaking, the trace of the normal component is zero due to (2-4), and the trace of the tangential component is shown to be zero due to (2-3).

Remark 2.1. In the definition of the solution in the sense of distributions, since $(u, \nabla p) \in L^1_{\mathrm{uloc}}(\mathbb{R}^d_+)^d \times L^1_{\mathrm{uloc}}(\mathbb{R}^d_+)^d$, the class of test functions is easily relaxed as follows: $\varphi \in C^2(\bar{\mathbb{R}}^d_+)^d$ with $\varphi|_{x_d=0} = 0$ and $\phi \in C^1(\bar{\mathbb{R}}^d_+)$ such that, for $\alpha = 0, 1, 2$ and $\beta = 0, 1$,

$$\nabla^\alpha \varphi(x), \quad \nabla^\beta \phi(x) \sim \mathcal{O}(|x|^{-d-\kappa}), \quad |x| \gg 1,$$

for some $\kappa > 0$.

2C. ***Integral representation formulas for the resolvent system.*** The solution to the resolvent problem (1-2) can be computed in Fourier space. We build on the formulas for the symbols, which were derived in [Desch et al. 2001]. In particular in that paper, the authors showed that the symbol associated with the Dirichlet–Stokes resolvent problem can be decomposed into one part corresponding to the symbol of the Dirichlet–Laplace resolvent problem and a remainder term due to the pressure. Hence, the solution in the Fourier side about the tangential variables can be decomposed into $\hat{u} = \hat{v} + \hat{w}$, with, for all $\xi \in \mathbb{R}^{d-1}$, $y_d > 0$,

$$\hat{v}(\xi, y_d) = \frac{1}{2\omega_\lambda(\xi)} \int_0^\infty (e^{-\omega_\lambda(\xi)|y_d - z_d|} - e^{-\omega_\lambda(\xi)(y_d + z_d)}) \hat{f}(\xi, z_d) \, dz_d, \tag{2-5a}$$

$$\hat{w}'(\xi, y_d) = \int_0^\infty \frac{\xi}{|\xi|} \frac{e^{-|\xi| y_d} - e^{-\omega_\lambda(\xi) y_d}}{\omega_\lambda(\xi)(\omega_\lambda(\xi) - |\xi|)} e^{-\omega_\lambda(\xi) z_d} \xi \cdot \hat{f}'(\xi, z_d) \, dz_d, \tag{2-5b}$$

$$\hat{w}_d(\xi, y_d) = i \int_0^\infty \frac{e^{-|\xi| y_d} - e^{-\omega_\lambda(\xi) y_d}}{\omega_\lambda(\xi)(\omega_\lambda(\xi) - |\xi|)} e^{-\omega_\lambda(\xi) z_d} \xi \cdot \hat{f}'(\xi, z_d) \, dz_d \tag{2-5c}$$

The solution of the form $u = v + w$ is then obtained by taking the inverse Fourier transform. Notice that v is the solution to the Dirichlet–Laplace resolvent problem, while the remainder term w comes from the contribution of the nonlocal pressure term. The above formulas are derived for $f \in C^\infty_0(\mathbb{R}^d_+)^d$ satisfying $\nabla \cdot f = 0$ in $\mathbb{R}^d_+$ and $f_d = 0$ on $\partial\mathbb{R}^d_+$. But as is seen below, these formulas are also well-defined for any

L^p_{uloc} function f. If moreover f is solenoidal, i.e., $\nabla \cdot f = 0$ in the sense of distributions and $f_d = 0$ on $\partial\mathbb{R}^d_+$ in the sense of the generalized trace, see [Galdi 2011, (III.2.14) p. 159], the velocity u, together with the pressure p defined below, is a solution to (1-2).

The formula for the pressure in the Fourier variables is written as follows: for all $\xi \in \mathbb{R}^{d-1}$, $y_d > 0$,

$$\begin{aligned}\hat{p}(\xi, y_d) &= -\int_0^\infty e^{-|\xi| y_d} e^{-\omega_\lambda(\xi) z_d} \frac{\omega_\lambda(\xi) + |\xi|}{|\xi|} \hat{f}_d(\xi, z_d)\, dz_d \\ &= \int_0^\infty e^{-|\xi| y_d} e^{-\omega_\lambda(\xi) z_d} \left(\frac{1}{|\xi|} + \frac{1}{\omega_\lambda(\xi)}\right) i\xi \cdot \hat{f}'(\xi, z_d)\, dz_d. \end{aligned} \tag{2-5d}$$

Another useful representation of $\hat{p}$ is

$$\hat{p}(\xi, y_d) = -\frac{i\xi}{|\xi|} e^{-|\xi| y_d} \cdot \partial_{y_d} \hat{u}'(\xi, 0),$$

which in particular leads to

$$i\xi_j \hat{p}(\xi, y_d) = \frac{\xi_j \xi}{|\xi|} e^{-|\xi| y_d} \cdot \partial_{y_d} \hat{u}'(\xi, 0), \tag{2-6a}$$

$$\partial_{y_d} \hat{p}(\xi, y_d) = i\xi e^{-|\xi| y_d} \cdot \partial_{y_d} \hat{u}'(\xi, 0) \tag{2-6b}$$

for all $y_d > 0$. Formula (2-6) is important when one deals with the nondecaying solutions. Indeed, it excludes the flow driven by the pressure; that is, the pressure is completely determined by the velocity. Notice that such a formula rules out the parasitic linearly growing solutions to the pressure equation (see the Liouville theorem, Theorem 4, proved in Appendix A). By using integration by parts the formula (2-6a) is also written as

$$\begin{aligned} i\xi_j \hat{p}(\xi, y_d) &= \frac{\xi_j \xi}{|\xi|} \cdot \int_0^\infty e^{-|\xi| y_d} e^{-\omega_\lambda(\xi) z_d} \big(\omega_\lambda(\xi) \partial_{z_d} \hat{u}'(\xi, z_d) - \partial_{z_d}^2 \hat{u}'(\xi, z_d)\big)\, dz_d \\ &= \frac{\xi_j \xi}{|\xi|} \cdot \int_0^\infty e^{-|\xi| y_d} e^{-\omega_\lambda(\xi) z_d} \big(\omega_\lambda(\xi)^2 \hat{u}'(\xi, z_d) - \partial_{z_d}^2 \hat{u}'(\xi, z_d)\big)\, dz_d, \end{aligned} \tag{2-7a}$$

$$\partial_{y_d} \hat{p}(\xi, y_d) = i\xi \cdot \int_0^\infty e^{-|\xi| y_d} e^{-\omega_\lambda(\xi) z_d} \big(\omega_\lambda(\xi)^2 \hat{u}'(\xi, 0) - \partial_{z_d}^2 \hat{u}'(\xi, z_d)\big)\, dz_d. \tag{2-7b}$$

The expression (2-7a) is useful in obtaining the characterization of the domain of the Stokes operator in L^q_{uloc} spaces; see Proposition 5.1.

We now define the kernels $k_{1,\lambda}\colon \mathbb{R}^d \to \mathbb{C}$ and $k_{2,\lambda}\colon \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{C}$ associated with the Dirichlet–Laplace part by, for all $y' \in \mathbb{R}^{d-1}$ and $y_d \in \mathbb{R}$,

$$k_{1,\lambda}(y', y_d) := \int_{\mathbb{R}^{d-1}} e^{iy' \cdot \xi} \frac{1}{2\omega_\lambda(\xi)} e^{-\omega_\lambda(\xi)|y_d|}\, d\xi, \tag{2-8a}$$

and, for all $y' \in \mathbb{R}^{d-1}$ and $y_d, z_d > 0$,

$$k_{2,\lambda}(y', y_d, z_d) := \int_{\mathbb{R}^{d-1}} e^{iy' \cdot \xi} \frac{1}{2\omega_\lambda(\xi)} e^{-\omega_\lambda(\xi)(y_d + z_d)}\, d\xi. \tag{2-8b}$$

We also define the kernels $r'_\lambda\colon \mathbb{R}^d_+ \times \mathbb{R}_+ \to \mathbb{C}^{d-1}$ and $r_{d,\lambda}\colon \mathbb{R}^d_+ \times \mathbb{R}_+ \to \mathbb{C}$ associated with the nonlocal part by, for all $y' \in \mathbb{R}^{d-1}$ and $y_d, z_d > 0$,

$$r'_\lambda(y', y_d, z_d) := \int_{\mathbb{R}^{d-1}} e^{iy'\cdot\xi} \frac{e^{-|\xi|y_d} - e^{-\omega_\lambda(\xi)y_d}}{\omega_\lambda(\xi)(\omega_\lambda(\xi)-|\xi|)} e^{-\omega_\lambda(\xi)z_d} \frac{\xi\otimes\xi}{|\xi|}\, d\xi, \tag{2-8c}$$

$$r_{d,\lambda}(y', y_d, z_d) := \int_{\mathbb{R}^{d-1}} i e^{iy'\cdot\xi} \frac{e^{-|\xi|y_d} - e^{-\omega_\lambda(\xi)y_d}}{\omega_\lambda(\xi)(\omega_\lambda(\xi)-|\xi|)} e^{-\omega_\lambda(\xi)z_d} \xi\, d\xi. \tag{2-8d}$$

Moreover, the kernel associated with the pressure is defined by, for all $y' \in \mathbb{R}^{d-1}$ and $y_d, z_d > 0$,

$$q_\lambda(y', y_d, z_d) := i \int_{\mathbb{R}^{d-1}} e^{iy'\cdot\xi} e^{-|\xi|y_d} e^{-\omega_\lambda(\xi)z_d} \left(\frac{\xi}{|\xi|} + \frac{\xi}{\omega_\lambda(\xi)}\right) d\xi. \tag{2-8e}$$

Notice that, for all $y' \in \mathbb{R}^{d-1}$, for all $y_d > 0$,

$$v(y', y_d) = \int_{\mathbb{R}^{d-1}} \int_0^\infty k_{1,\lambda}(y'-z', y_d - z_d) f(z', z_d)\, dz_d\, dz' + \int_{\mathbb{R}^{d-1}} \int_0^\infty k_{2,\lambda}(y'-z', y_d, z_d) f(z', z_d)\, dz_d\, dz', \tag{2-9a}$$

$$w'(y', y_d) = \int_{\mathbb{R}^{d-1}} \int_0^\infty r'_\lambda(y'-z', y_d, z_d) f'(z', z_d)\, dz_d\, dz', \tag{2-9b}$$

$$w_d(y', y_d) = \int_{\mathbb{R}^{d-1}} \int_0^\infty r_{d,\lambda}(y'-z', y_d, z_d) \cdot f'(z', z_d)\, dz_d\, dz', \tag{2-9c}$$

$$p(y', y_d) = \int_{\mathbb{R}^{d-1}} \int_0^\infty q_\lambda(y'-z', y_d, z_d) \cdot f'(z', z_d)\, dz_d\, dz'. \tag{2-9d}$$

These integral representation formulas, together with pointwise estimates on the kernels, are the basis for estimates in L^p_{uloc} spaces.

Note that, in view of (2-6), the pressure ∇p is also written as

$$\nabla' p(\,\cdot\,, y_d) = -(\nabla'\nabla'(-\Delta')^{-1/2}) \cdot P(y_d)\gamma\partial_{y_d} u', \quad y_d > 0, \tag{2-10a}$$

$$\partial_{y_d} p(\,\cdot\,, y_d) = \nabla' \cdot P(y_d)\gamma\partial_{y_d} u', \quad y_d > 0. \tag{2-10b}$$

Here γ is the trace operator on $\partial\mathbb{R}^d_+$ and $P(t)$ is the Poisson semigroup whose kernel is the Poisson kernel defined by $\mathcal{F}^{-1}[e^{-|\xi|t}]$. We note that, when $\gamma\partial_{y_d} u'$ belongs to $L^q_{\mathrm{uloc}}(\mathbb{R}^{d-1})^{d-1}$ for some $q \in [1, \infty]$, the function $P(y_d)\gamma\partial_{y_d} u'$ is smooth and bounded including its derivatives in $\mathbb{R}^d_{+,\delta} = \{(y', y_d) \in \mathbb{R}^d \mid y_d > \delta\}$ for each $\delta > 0$. This can be proved from the pointwise estimate of the Poisson kernel (and its derivatives) and we omit the details here. Then, the action of $\nabla'\nabla'(-\Delta')^{-1/2}$ or ∇' on $P(y_d)\gamma\partial_{y_d} u'$ makes sense for each $y_d > 0$, when $\gamma\partial_{y_d} u' \in L^q_{\mathrm{uloc}}(\mathbb{R}^{d-1})^{d-1}$. Indeed, one natural way to realize the action of $\nabla'\nabla'(-\Delta')^{-1/2}$ is to define it as

$$\nabla'\nabla'(-\Delta')^{-1/2} f = \int_0^\infty \nabla'\nabla' P(t) f\, dt,$$

which is well-defined for any bounded C^2 function f (or more sharply, for any bounded $C^{1+\varepsilon}$ function f with $\varepsilon > 0$). The formula (2-10) will be used in Section 5.

We end this section with the following scaling properties of the kernels, which will be used to work with $|\lambda| = 1$: for all $y' \in \mathbb{R}^{d-1}$, $y_d \in \mathbb{R}$,

$$k_{1,\lambda}(y', y_d) = |\lambda|^{d/2-1} k_{1,\lambda/|\lambda|}(|\lambda|^{1/2} y', |\lambda|^{1/2} y_d) \tag{2-11a}$$

and, for all $y' \in \mathbb{R}^{d-1}$, $y_d, z_d \in \mathbb{R}_+$,

$$k_{2,\lambda}(y', y_d, z_d) = |\lambda|^{d/2-1} k_{2,\lambda/|\lambda|}(|\lambda|^{1/2} y', |\lambda|^{1/2} y_d, |\lambda|^{1/2} z_d), \tag{2-11b}$$

$$r'_{\lambda}(y', y_d, z_d) = |\lambda|^{d/2-1} r'_{\lambda/|\lambda|}(|\lambda|^{1/2} y', |\lambda|^{1/2} y_d, |\lambda|^{1/2} z_d), \tag{2-11c}$$

$$r_{d,\lambda}(y', y_d, z_d) = |\lambda|^{d/2-1} r_{d,\lambda/|\lambda|}(|\lambda|^{1/2} y', |\lambda|^{1/2} y_d, |\lambda|^{1/2} z_d), \tag{2-11d}$$

$$q_{\lambda}(y', y_d, z_d) = |\lambda|^{(d-1)/2} q_{\lambda/|\lambda|}(|\lambda|^{1/2} y', |\lambda|^{1/2} y_d, |\lambda|^{1/2} z_d). \tag{2-11e}$$

There is no evident characterization of L^p_{uloc} spaces in Fourier space. These spaces are easily defined in physical space. Therefore, a prominent task is to derive pointwise estimates on the kernels. The goal of the next section is to address this task.

3. Pointwise kernel estimates

Deriving pointwise bounds for the Dirichlet–Laplace part is rather classical. The nonlocal part requires a more refined analysis.

3A. ***General ideas for the estimates.*** Before starting the estimates of the kernels derived in Section 2, we give some general remarks, which serve as guidelines for this section. First, we always start by using the formulas (2-11) in order to make $|\lambda| = 1$. Second, integrability of derivatives of the Fourier multipliers are traded in decay of the kernels in physical space in the tangential direction. This is the role of Lemma 3.1 below, which is central in our approach. Third, the analysis of the integrability of derivatives of the Fourier multipliers sometimes requires us to analyze separately the low frequencies and the high frequencies, or small y_d and large y_d. More heuristic explanations are given in [Prange 2018].

The following lemma is standard. Since we use it repeatedly, we state and prove it here.

Lemma 3.1. *Let $m \in C^\infty(\mathbb{R}^{d-1} \setminus \{0\})$ be a smooth Fourier multiplier. Let K be the kernel associated with m. For all $y' \in \mathbb{R}^{d-1}$,*

$$K(y') := \int_{\mathbb{R}^{d-1}} e^{iy'\cdot\xi} m(\xi)\, d\xi.$$

Assume that there exists $n > -d+1$, and positive constants $c_0(d, n, m)$, $C_0(d, n, m) < \infty$ such that, for all $\alpha \in \mathbb{N}$, $0 \le \alpha \le n+d$, for all $\xi \in \mathbb{R}^{d-1} \setminus \{0\}$,

$$|\nabla^\alpha m(\xi)| \le C_0 |\xi|^{n-\alpha} e^{-c_0|\xi|} \quad \text{if } n > 1-d. \tag{3-1}$$

Then, there exists a constant $C(d, n, c_0, C_0) < \infty$ such that for all $y' \in \mathbb{R}^{d-1} \setminus \{0\}$

$$|K(y')| \le \frac{C}{|y'|^{n+d-1}}. \tag{3-2}$$

In the definition of K the integral is, as usual, considered as the oscillatory integral. Note that n does not need to be an integer. The lemma will be typically used to get bounds on the kernel when y' is large, say $|y'| \geq 1$. Let us now give the proof of the lemma.

Proof. The proof is by integration by parts. There are two steps. For the first step, due to the singularity of the multiplier m at 0, we can integrate by parts $[n+d-2]$ times, which yields the decay $|K(y')| \leq C/|y'|^{n+d-2}$. Here $[a]$ denotes the Gauss symbol; i.e., $[a]$ is the integer such that $a = [a] + \delta$ with $\delta \in [0, 1)$. In the second step, we cut off the singularity around 0 at an ad hoc frequency R, and integrate by parts two more times in high frequencies. This makes it possible to get the optimal decay stated in (3-2). Let $y' \in \mathbb{R}^{d-1} \setminus \{0\}$ be fixed.

Step 1: There is $j \in \{1, \dots, d-1\}$ such that $|y_j| \geq |y'|/(d-1)$. For such j we have from integration by parts

$$
\begin{aligned}
(-iy_j)^{[n+d-2]} K(y') &= \int_{\mathbb{R}^{d-1}} e^{iy'\cdot\xi} \partial_{\xi_j}^{[n+d-2]} m(\xi)\, d\xi \\
&= \int_{\mathbb{R}^{d-1}} \chi_R e^{iy'\cdot\xi} \partial_{\xi_j}^{[n+d-2]} m(\xi)\, d\xi + \int_{\mathbb{R}^{d-1}} (1-\chi_R) e^{iy'\cdot\xi} \partial_{\xi_j}^{[n+d-2]} m(\xi)\, d\xi \\
&=: I_R + II_R.
\end{aligned}
$$

Here $R \in (0, \infty)$ is fixed (and will be chosen below) and $\chi_R \in C_0^\infty(\mathbb{R}^{d-1})$ is a smooth radial cut-off function such that $\chi_R = 1$ for $|\xi| \leq R$ and $\chi_R = 0$ for $|\xi| \geq 2R$. Notice that using the bound (3-1) we get

$$
|(-iy_j)^{[n+d-2]} K(y')| \leq C_0 \int_{\mathbb{R}^{d-1}} |\xi|^{n-[n+d-2]} e^{-c_0|\xi|}\, d\xi \leq C,
$$

which is not optimal.

Step 2: We have

$$
|I_R| \leq C_0 \int_{|\xi|\leq 2R} |\xi|^{n-[n+d-2]}\, d\xi \leq C_0 R^{1+\delta}, \quad n+d-2 = [n+d-2]+\delta,
$$

while

$$
\begin{aligned}
|(-iy_j)^2 II_R| &= \left| \int_{\mathbb{R}^{d-1}} e^{iy'\cdot\xi} \partial_{\xi_j}^2 ((1-\chi_R) \partial_{\xi_j}^{[n+d-2]} m(\xi))\, d\xi \right| \\
&\leq C_0 \int_{|\xi|\geq R} |\xi|^{n-[n+d-2]-2} e^{-c_0|\xi|}\, d\xi \leq C_0 R^{-1+\delta}.
\end{aligned}
$$

Now we take $R = |y'|^{-1}$, which yields from $|y_j| \geq |y'|/(d-1)$,

$$
|K(y')| \leq |y_j|^{-[n+d-2]} (|I_R| + |II_R|) \leq C|y'|^{-n-d+1}. \qquad \square
$$

3B. ***Kernel estimates for the Dirichlet–Laplace part.*** The Dirichlet–Laplace part of the operator is nothing but the part corresponding to the resolvent problem for the scalar Laplace equation in $\mathbb{R}^d_+$ with Dirichlet boundary conditions. The kernels $k_{1,\lambda}$ and $k_{2,\lambda}$ are given by the expressions (2-8a) and (2-8b) respectively. We recall and prove the following classical pointwise bounds.

Proposition 3.2. *Let $\lambda \in S_{\pi-\varepsilon}$. There exist constants $c(d,\varepsilon)$, $C(d,\varepsilon) < \infty$ such that, for $y' \in \mathbb{R}^{d-1}$, $y_d \in \mathbb{R}$,*

$$|k_{1,\lambda}(y', y_d)| \le \begin{cases} Ce^{-c|\lambda|^{1/2}|y_d|} \min\left\{\log\left(e + \dfrac{1}{|\lambda|^{1/2}(|y_d|+|y'|)}\right), \dfrac{1}{|\lambda|(|y_d|+|y'|)^2}\right\}, & d=2, \\ \dfrac{Ce^{-c|\lambda|^{1/2}|y_d|}}{(|y_d|+|y'|)^{d-2}(1+|\lambda|^{1/2}(|y_d|+|y'|))^2}, & d \ge 3, \end{cases} \tag{3-3}$$

and, for $\alpha \in \mathbb{N}$,

$$|\nabla k_{1,\lambda}(y', y_d)| \le \frac{Ce^{-c|\lambda|^{1/2}|y_d|}}{(|y'|+|y_d|)^{d-1}(1+|\lambda|^{1/2}(|y_d|+|y'|))^2}, \tag{3-4}$$

$$|\nabla^\alpha k_{1,\lambda}(y', y_d)| \le \frac{Ce^{-c|\lambda|^{1/2}|y_d|}}{(|y_d|+|y'|)^{d-2+\alpha}(1+|\lambda|^{1/2}(|y_d|+|y'|))}. \tag{3-5}$$

Moreover we have, for $y' \in \mathbb{R}^{d-1}$, $y_d, z_d \in \mathbb{R}_+$, $\alpha \in \mathbb{N}$,

$$|k_{2,\lambda}(y', y_d, z_d)| \le \begin{cases} Ce^{-c|\lambda|^{1/2}(y_d+z_d)} \min\left\{\log\left(e + \dfrac{1}{|\lambda|^{1/2}(y_d+z_d+|y'|)}\right), \dfrac{1}{|\lambda|(y_d+z_d+|y'|)^2}\right\}, & d=2, \\ \dfrac{Ce^{-c|\lambda|^{1/2}(y_d+z_d)}}{(y_d+z_d+|y'|)^{d-2}(1+|\lambda|^{1/2}(y_d+z_d+|y'|))^2}, & d \ge 3, \end{cases} \tag{3-6}$$

and, for $\alpha \in \mathbb{N}$,

$$|\nabla k_{2,\lambda}(y', y_d, z_d)| \le \frac{Ce^{-c|\lambda|^{1/2}(y_d+z_d)}}{(y_d+z_d+|y'|)^{d-1}(1+|\lambda|^{1/2}(y_d+z_d+|y'|))^2}, \tag{3-7}$$

$$|\nabla^\alpha k_{2,\lambda}(y', y_d, z_d)| \le \frac{Ce^{-c|\lambda|^{1/2}(y_d+z_d)}}{(y_d+z_d+|y'|)^{d-2+\alpha}(1+|\lambda|^{1/2}(y_d+z_d+|y'|))}. \tag{3-8}$$

Remark 3.3. From (3-6), it is clear that

$$|k_{2,\lambda}(y', y_d, z_d)| \le \begin{cases} Ce^{-c|\lambda|^{1/2}|y_d-z_d|} \min\left\{\log\left(e + \dfrac{1}{|\lambda|^{1/2}(|y_d-z_d|+|y'|)}\right), \dfrac{1}{|\lambda|(|y_d-z_d|+|y'|)^2}\right\}, & d=2, \\ \dfrac{Ce^{-c|\lambda|^{1/2}|y_d-z_d|}}{(|y_d-z_d|+|y'|)^{d-2}(1+|\lambda|^{1/2}(|y_d-z_d|+|y'|))^2}, & d \ge 3, \end{cases} \tag{3-9}$$

and similar estimates hold for the derivatives. Hence the integral operator associated with $k_{2,\lambda}$ can be estimated as a convolution kernel in $\mathbb{R}^d$ as $k_{1,\lambda}$.

Proof. Since these bounds can be estimated in a similar way, we only deal with (3-3). The scaling property (2-11) allows us to assume $|\lambda| = 1$ in the following argument.

We begin with the case $d \ge 3$. First observe that

$$|k_{1,\lambda}(y', y_d)| \le C \int_{\mathbb{R}^{d-1}} \frac{e^{-c(1+|\xi|)y_d}}{|\xi|}\, d\xi \le Ce^{-cy_d} y_d^{-(d-2)}. \tag{3-10}$$

Secondly, for all $\alpha \in \mathbb{N}$ we have the pointwise bound

$$\left|\nabla_\xi^\alpha\left(\frac{1}{\omega_\lambda(\xi)}e^{-\omega_\lambda(\xi)|y_d|}\right)\right| \le \frac{Ce^{-cy_d}e^{-c_0|\xi|y_d}}{(1+|\xi|)^{\alpha+1}}. \tag{3-11}$$

Therefore, applying Lemma 3.1 with

$$m(\xi) := \frac{1}{\omega_\lambda(\xi)}e^{-\omega_\lambda(\xi)|y_d|},$$

$n=-1$ and $C_0 := Ce^{-cy_d}$ (remember that in this computation y_d is a parameter), we get

$$|k_{1,\lambda}(y', y_d)| \le Ce^{-cy_d}|y'|^{-(d-2)}.$$

Combining the previous estimate with (3-10) yields

$$|k_{1,\lambda}(y', y_d)| \le Ce^{-cy_d}(|y'|+y_d)^{-(d-2)}. \tag{3-12}$$

In the same way as above, integration by parts and (3-11) give

$$\begin{aligned}|(-iy_j)^d k_{1,\lambda}(y', y_d)| &= \left|\int_{\mathbb{R}^{d-1}} e^{iy'\cdot\xi}\partial_{\xi_j}^d\left(\frac{1}{2\omega_\lambda(\xi)}e^{-\omega_\lambda(\xi)|y_d|}\right)d\xi\right| \\ &\le Ce^{-cy_d}\int_{\mathbb{R}^{d-1}}\frac{1}{(1+|\xi|)^{d+1}}\,d\xi \le Ce^{-cy_d},\end{aligned}$$

which together with (3-10) implies

$$|k_{1,\lambda}(y', y_d)| \le Ce^{-cy_d}(|y'|+y_d)^{-d}. \tag{3-13}$$

Combining this with (3-12), we obtain the desired estimate (3-3) for $d \ge 3$.

For the case $d=2$, it suffices to show

$$|k_{1,\lambda}(y', y_d)| \le Ce^{-c|\lambda|^{1/2}|y_d|}\log(e+e|\lambda|^{-1/2}(|y'|+|y_d|)^{-1}), \tag{3-14}$$

since the other case in (3-3) for $d=2$ can be shown in the same manner as in (3-13). Splitting the integral as in the proof of Lemma 3.1, we have

$$\begin{aligned}k_{1,\lambda}(y', y_d) &= \int_{\mathbb{R}}\frac{1}{2\omega_\lambda(\xi)}e^{-\omega_\lambda(\xi)|y_d|}\,d\xi \\ &= \int_{\mathbb{R}} e^{iy'\cdot\xi}\frac{1}{2\omega_\lambda(\xi)}e^{-\omega_\lambda(\xi)|y_d|}\chi_R(\xi)\,d\xi + \int_{\mathbb{R}} e^{iy'\cdot\xi}\frac{1}{2\omega_\lambda(\xi)}e^{-\omega_\lambda(\xi)|y_d|}(1-\chi_R(\xi))\,d\xi \\ &= I + II.\end{aligned}$$

On the one hand (3-11) with $\alpha = 0$ gives

$$\begin{aligned}|I| &\le C\int_{|\xi'|\le 2R}\frac{1}{1+|\xi|}e^{-(1+|\xi|)y_d}\,d\xi \\ &\le Ce^{-cy_d}\int_{|\xi'|\le 2R}\frac{1}{1+|\xi|}\,d\xi \le Ce^{-cy_d}\log(1+R),\end{aligned}$$

and on the other hand, using the identity $iy_je^{iy'\cdot\xi} = \partial_{\xi_j}e^{iy'\cdot\xi}$, integration by parts and (3-11) we obtain

$$|II| \le \frac{C}{|y'|}\left|\int_{\mathbb{R}} e^{iy'\cdot\xi}\partial_{\xi'}\left(\frac{1}{2\omega_\lambda(\xi)}e^{-\omega_\lambda(\xi)|y_d|}(1-\chi_R(\xi))\right)d\xi\right| \le Ce^{-cy_d}(R|y'|)^{-1},$$

from which we have

$$|k_{1,\lambda}(y', y_d)| \le Ce^{-cy_d}(\log(1+R) + (|y'|R)^{-1}).$$

Hence taking $R = |y'|^{-1}$ we obtain

$$|k_{1,\lambda}(y', y_d)| \le Ce^{-cy_d}(\log(1+|y'|^{-1}) + 1).$$

Moreover, we have

$$\begin{aligned}|k_{1,\lambda}(y', y_d)| &\le \left|\int_0^{y_d^{-1}} e^{iy'\cdot\xi}\frac{1}{2\omega_\lambda(\xi)}e^{-\omega_\lambda(\xi)|y_d|}\,d\xi\right| + \left|\int_{y_d^{-1}}^{\infty} e^{iy'\cdot\xi}\frac{1}{2\omega_\lambda(\xi)}e^{-\omega_\lambda(\xi)|y_d|}\,d\xi\right| \\ &\le Ce^{-cy_d}\left(\int_0^{y_d^{-1}}\frac{1}{1+|\xi|}\,d\xi + \int_{y_d^{-1}}^{\infty}\frac{e^{-|\xi|y_d}}{|\xi|}\,d\xi\right) \\ &\le Ce^{-cy_d}(\log(1+y_d^{-1})+1).\end{aligned}$$

Combining both cases, we obtain

$$|k_{1,\lambda}(y', y_d)| \le Ce^{-cy_d}(\log(1+(|y'|+y_d)^{-1})+1), \tag{3-15}$$

which immediately implies the desired estimate (3-14). This completes the proof of (3-3). □

Remark 3.4 (on the estimate of the tangential derivatives). The tangential derivatives of k_1 or k_2 should a priori be better behaved than the vertical derivatives in y_d or z_d, since differentiating in y' brings a ξ in the symbol. We were however unable to get an estimate of the type

$$|\nabla_{y'}^2 k_{1,\lambda}(y', y_d)| \le \frac{C|y_d|e^{-c|\lambda|^{1/2}|y_d|}}{(|y_d|+|y'|)^{d+1}},$$

contrary to $\nabla_{y'}^2 r'_\lambda$ for which this is true (see (3-18)). A pointwise bound such as (3-18) makes it possible to prove uniform bounds in λ on second-order tangential derivatives in L^q_{uloc}, without loss of a factor $\log|\lambda|$ for small $|\lambda|$ (compare (4-32) to (4-30)). On a different note, the argument above also provides the estimate for the fractional derivative in the tangential variables. Indeed, if $m_\alpha(D')$ is any Fourier multiplier, homogeneous of order $\alpha > 0$, then we have, for $\beta = 0, 1$,

$$\begin{aligned}|m_\alpha(D')\nabla^\beta k_{1,\lambda}(y', y_d)| &\le \frac{Ce^{-c|\lambda|^{1/2}|y_d|}}{(|y_d|+|y'|)^{d-2+\alpha+\beta}(1+|\lambda|^{1/2}(|y_d|+|y'|))}, \\ |m_\alpha(D')\nabla^\beta k_{2,\lambda}(y', y_d, z_d)| &\le \frac{Ce^{-c|\lambda|^{1/2}(y_d+z_d)}}{(y_d+z_d+|y'|)^{d-2+\alpha+\beta}(1+|\lambda|^{1/2}(y_d+z_d+|y'|))}.\end{aligned} \tag{3-16}$$

3C. ***Kernel estimates for the nonlocal part.*** We now consider the nonlocal part w. We estimate the kernels r'_λ and $r_{d,\lambda}$ defined by (2-8c) and (2-8d) respectively. The nonlocal effects are due to the pressure of the Stokes equations. This part is the most difficult one. As above, our aim is to get pointwise estimates on the kernels following the general guidelines of Section 3A. We summarize our results in the following proposition.

Proposition 3.5. *Let $\lambda \in S_{\pi-\varepsilon}$. There exist positive constants $c(d,\varepsilon)$, $C(d,\varepsilon) < \infty$ such that, for all $y' \in \mathbb{R}^{d-1}$, $y_d, z_d > 0$,*

$$|r'_\lambda(y', y_d, z_d)| + |r_{d,\lambda}(y', y_d, z_d)| \leq \frac{C y_d}{(y_d + z_d + |y'|)^{d-1}} \frac{e^{-c|\lambda|^{1/2} z_d}}{(1 + |\lambda|^{1/2}(y_d + z_d + |y'|))(1 + |\lambda|^{1/2}(y_d + z_d))}. \tag{3-17}$$

Moreover, for $\alpha = 1, 2$,

$$|\nabla^\alpha_{y'} r'_\lambda(y', y_d, z_d)| + |\nabla^\alpha_{y'} r_{d,\lambda}(y', y_d, z_d)| \leq \frac{C y_d}{(y_d + z_d + |y'|)^{d-1+\alpha}} \frac{e^{-c|\lambda|^{1/2} z_d}}{(1 + |\lambda|^{1/2}(y_d + z_d + |y'|))(1 + |\lambda|^{1/2}(y_d + z_d))}, \tag{3-18}$$

and, for $\beta = 0, 1$,

$$|\nabla^\beta_{y'} \partial_{y_d} r'_\lambda(y', y_d, z_d)| + |\nabla^\beta_{y'} \partial_{y_d} r_{d,\lambda}(y', y_d, z_d)| \leq \frac{C}{(y_d + z_d + |y'|)^{d-1+\beta}} \frac{e^{-c|\lambda|^{1/2} z_d}}{(1 + |\lambda|^{1/2}(y_d + z_d + |y'|))(1 + |\lambda|^{1/2}(y_d + z_d))}, \tag{3-19}$$

and

$$|\partial^2_{y_d} r'_\lambda(y', y_d, z_d)| + |\partial^2_{y_d} r_{d,\lambda}(y', y_d, z_d)| \leq \frac{C e^{-c|\lambda|^{1/2} z_d}}{(y_d + z_d + |y'|)^d (1 + |\lambda|^{1/2}(y_d + z_d))}. \tag{3-20}$$

Finally, for $\beta = 0, 1$,

$$|\nabla^\beta_{y'} \partial_{z_d} r'_\lambda(y', y_d, z_d)| + |\nabla^\beta_{y'} \partial_{z_d} r_{d,\lambda}(y', y_d, z_d)| \leq \frac{C y_d}{(y_d + z_d + |y'|)^{d+\beta}} \frac{e^{-c|\lambda|^{1/2} z_d}}{(1 + |\lambda|^{1/2}(y_d + z_d))}, \tag{3-21}$$

and

$$|\partial_{y_d} \partial_{z_d} r'_\lambda(y', y_d, z_d)| + |\partial_{y_d} \partial_{z_d} r_{d,\lambda}(y', y_d, z_d)| \leq \frac{C e^{-c|\lambda|^{1/2} z_d}}{(y_d + z_d + |y'|)^d (1 + |\lambda|^{1/2}(y_d + z_d))}. \tag{3-22}$$

Remark 3.6. Related to (3-18), as in case of the Dirichlet–Laplace kernel, we also have the estimate for the fractional derivative in the tangential variables. Let $m_\alpha(D')$ be any Fourier multiplier, homogeneous of order $\alpha > 0$. Then we have

$$\begin{aligned} &|m_\alpha(D') r'_\lambda(y', y_d, z_d)| + |m_\alpha(D') r_{d,\lambda}(y', y_d, z_d)| \\ &\qquad \leq \frac{C y_d}{(y_d + z_d + |y'|)^{d-1+\alpha}} \frac{e^{-c|\lambda|^{1/2} z_d}}{(1 + |\lambda|^{1/2}(y_d + z_d + |y'|))(1 + |\lambda|^{1/2}(y_d + z_d))}, \\ &|m_\alpha(D') \nabla r'_\lambda(y', y_d, z_d)| + |m_\alpha(D') \nabla r_{d,\lambda}(y', y_d, z_d)| \\ &\qquad \leq \frac{C}{(y_d + z_d + |y'|)^{d-1+\alpha}} \frac{e^{-c|\lambda|^{1/2} z_d}}{(1 + |\lambda|^{1/2}(y_d + z_d + |y'|))}. \end{aligned} \tag{3-23}$$

Estimate (3-23) is proved similarly to (3-18), and thus the proof of (3-23) is omitted in this paper.

Proof of Proposition 3.5. Using the scaling (2-11), we assume $|\lambda| = 1$ for the remainder of this section. We give the proof only for r'_λ. Indeed from the representations (2-8c) and (2-8d) it is clear that the estimate of $r_{d,\lambda}$ is obtained in the similar manner. By using the identity

$$\frac{1}{\omega_\lambda(\xi) - |\xi|} = \frac{\omega_\lambda(\xi) + |\xi|}{\lambda},$$

we rewrite r'_λ as

$$\begin{aligned} r'_\lambda(y', y_d, z_d) &= \frac{1}{\lambda} \int_{\mathbb{R}^{d-1}} e^{iy'\cdot\xi} (e^{-|\xi| y_d} - e^{-\omega_\lambda(\xi) y_d})\, e^{-\omega_\lambda(\xi) z_d} \frac{\xi \otimes \xi}{|\xi|}\, d\xi \\ &\qquad + \frac{1}{\lambda} \int_{\mathbb{R}^{d-1}} e^{iy'\cdot\xi} (e^{-|\xi| y_d} - e^{-\omega_\lambda(\xi) y_d})\, e^{-\omega_\lambda(\xi) z_d} \frac{\xi \otimes \xi}{\omega_\lambda(\xi)}\, d\xi \\ &= r'_{\lambda,1}(y', y_d, z_d) + r'_{\lambda,2}(y', y_d, z_d). \end{aligned} \tag{3-24}$$

Since $\lambda \in S_{\pi-\varepsilon}$ and $|\lambda| = 1$, the factor $1/\omega_\lambda(\xi)$ is more regular than $1/|\xi|$. Therefore we focus on the pointwise estimate of $r'_{\lambda,1}$, which is automatically satisfied by $r'_{\lambda,2}$ as well. Again from $|\lambda| = 1$ it suffices to consider the estimate of

$$s_\lambda(y', y_d, z_d) = \int_{\mathbb{R}^{d-1}} e^{iy'\cdot\xi} (e^{-|\xi| y_d} - e^{-\omega_\lambda(\xi) y_d})\, e^{-\omega_\lambda(\xi) z_d} \frac{\xi \otimes \xi}{|\xi|}\, d\xi. \tag{3-25}$$

<u>Step 1</u>: Case $y_d \geq 1$. In this case, by virtue of the factors $e^{-|\xi| y_d}$ and $e^{-\omega_\lambda(\xi) y_d}$, the kernel s_λ becomes smooth. Moreover, the factor $e^{-\omega_\lambda(\xi) z_d}$ gives exponential decay like e^{-cz_d} since $\lambda \in S_{\pi-\varepsilon}$ and $|\lambda| = 1$. Thus the main issue is the decay in y' and y_d. By the change of the variables $\eta = \xi y_d$ we see

$$\begin{aligned} s_\lambda(y', y_d, z_d) &= y_d^{-d} \int_{\mathbb{R}^{d-1}} e^{i\tilde{y}'\cdot\eta} (e^{-|\eta|} - e^{-\sqrt{\lambda y_d^2 + |\eta|^2}})\, e^{-\omega_\lambda(\eta/y_d) z_d} \frac{\eta \otimes \eta}{|\eta|}\, d\eta \\ &=: y_d^{-d}\, \tilde{s}_\lambda(\tilde{y}', y_d, z_d), \end{aligned}$$

where $\tilde{y}' = y'/y_d$. We will show that

$$|\tilde{s}_\lambda(\tilde{y}', y_d, z_d)| \leq \frac{Ce^{-cz_d}}{(1 + |\tilde{y}'|)^d}, \tag{3-26}$$

from which we can derive the desired bound of s_λ for $y_d \geq 1$, since

$$|s_\lambda(y', y_d, z_d)| \leq \frac{Ce^{-cz_d}}{(y_d + |y'|)^d} \leq \frac{Cy_d\, e^{-cz_d}}{(1 + y_d + z_d + |y'|)^d (1 + y_d + z_d)} \tag{3-27}$$

by changing the constants C and c suitably. To show (3-26) we first observe that

$$|\tilde{s}_\lambda(\tilde{y}, y_d, z_d)| \leq \int_{\mathbb{R}^{d-1}} (e^{-|\eta|} + e^{-c|\eta|}) e^{-cz_d} |\eta|\, d\eta \leq Ce^{-cz_d},$$

which gives the estimate (3-26) for the case $|\tilde{y}'| \leq 1$. Next we consider the case $|\tilde{y}'| \geq 1$. In this case, we notice that, for $y_d \geq 1$ and $\alpha \in \mathbb{N},\ 0 \leq \alpha \leq d + 1$, for all $\eta \in \mathbb{R}^{d-1} \setminus \{0\}$,

$$\left| \nabla_\eta^\alpha \left\{ (e^{-|\eta|} - e^{-\sqrt{\lambda y_d^2 + |\eta|^2}}) e^{-\omega_\lambda(\eta/y_d) z_d} \frac{\eta \otimes \eta}{|\eta|} \right\} \right| \leq Ce^{-cz_d} e^{-c_0|\eta|} |\eta|^{-\alpha+1}.$$

Therefore, we apply Lemma 3.1 with

$$m(\eta) := (e^{-|\eta|} - e^{-\sqrt{\lambda y_d^2+|\eta|^2}})e^{-\omega_\lambda(\eta/y_d)z_d}\,\frac{\eta\otimes\eta}{|\eta|}$$

for all $\eta\in\mathbb{R}^{d-1}\setminus\{0\}$ and $K(\tilde{y}') := \tilde{s}_\lambda(\tilde{y}', y_d, z_d)$, where λ, y_d and z_d are parameters, $n=1$ and $C_0 := Ce^{-cz_d}$. This gives the bound

$$|\tilde{s}_\lambda(\tilde{y}', y_d, z_d)| \le C|\tilde{y}'|^{-d}e^{-cz_d}.$$

Hence, estimate (3-26) holds also for $|\tilde{y}'|\ge 1$.

Step 2: Case $0<y_d\le 1$. In this case we have to be careful about both the decay in y' and the singularity in y' near $y'=0$. Set $R_0=2$ and we decompose s_λ by using the cut-off χ_{R_0} as

$$s_\lambda = \int_{\mathbb{R}^{d-1}} \chi_{R_0}(\xi)\cdots\,d\xi + \int_{\mathbb{R}^{d-1}}(1-\chi_{R_0}(\xi))\cdots\,d\xi =: s_{\lambda,\mathrm{low}} + s_{\lambda,\mathrm{high}}.$$

As for the term $s_{\lambda,\mathrm{low}}$, we have from $|e^{-|\xi|y_d} - e^{-\omega_\lambda(\xi)y_d}| \le Cy_d$ for $|\xi|\le 3$,

$$|s_{\lambda,\mathrm{low}}(y', y_d, z_d)| \le C\int_{|\xi|\le 3} y_d e^{-cz_d}|\xi|\,d\xi \le Cy_d e^{-cz_d} \le \frac{Cy_d\,e^{-cz_d}}{(1+y_d+z_d)^{d+1}}.$$

Here the condition $0<y_d\le 1$ is used. This estimate gives the desired bound of $s_{\lambda,\mathrm{low}}$ for the case $|y'|\le 1$. Next we consider the case $|y'|\ge 1$. A direct computation implies that, for $0<y_d\le 1$, $\alpha\in N$, $0\le\alpha\le d+1$, for all $\xi\in\mathbb{R}^{d-1}\setminus\{0\}$,

$$\left|\nabla_\xi^\alpha\left\{\chi_{R_0}(\xi)(e^{-|\xi|y_d} - e^{-\omega_\lambda(\xi)y_d})\,e^{-\omega_\lambda(\xi)z_d}\,\frac{\xi\otimes\xi}{|\xi|}\right\}\right| \le Cy_d e^{-c_0z_d}|\xi|^{-\alpha+1}\chi_{R_0}(\xi) \le Cy_d e^{-cz_d}e^{-c_0|\xi|}|\xi|^{-\alpha+1}.$$

Hence, we can apply Lemma 3.1 with

$$m(\xi) := \chi_{R_0}(\xi)(e^{-|\xi|y_d} - e^{-\omega_\lambda(\xi)y_d})e^{-\omega_\lambda(\xi)z_d}\,\frac{\xi\otimes\xi}{|\xi|}$$

for all $\eta\in\mathbb{R}^{d-1}\setminus\{0\}$ and $K(y') := s_{\lambda,\mathrm{low}}(y', y_d, z_d)$, where λ, y_d and z_d are parameters, and $n=1$. This yields

$$|s_{\lambda,\mathrm{low}}| \le Cy_d|y'|^{-d}e^{-cz_d}$$

for $|y'|\ge 1$. Combining this with the estimate in the case $|y'|\le 1$, we have

$$|s_{\lambda,\mathrm{low}}(y', y_d, z_d)| \le \frac{Cy_d\,e^{-cz_d}}{(1+y_d+z_d+|y'|)^d(1+y_d+z_d)} \tag{3-28}$$

for $0<y_d\le 1$, $z_d\ge 0$, and $y'\in\mathbb{R}^{d-1}$.

Finally, let us estimate $s_{\lambda,\mathrm{high}}$. Since the associated symbol is smooth, the singularity around $y'=0$ is the main issue. The key point is to use the smoothing effect from the symbol

$$e^{-|\xi|y_d} - e^{-\omega_\lambda(\xi)y_d} = (1-e^{(|\xi|-\omega_\lambda(\xi))y_d})e^{-|\xi|y_d}.$$

Indeed,

$$|\xi|-\omega_\lambda(\xi)=|\xi|\biggl(1-\sqrt{1+\frac{\lambda}{|\xi|^2}}\biggr)=-\frac{\lambda}{2|\xi|}\int_0^1\frac{1}{\sqrt{1+\lambda/|\xi|^2t}}\,dt=-\frac{\lambda}{2|\xi|}+\mathcal{O}\biggl(\frac{\lambda^2}{|\xi|^3}\biggr)$$

for $|\xi|\geq 2$ and $|\lambda|=1$. Hence, we have for $|\xi|\geq 2,\ 0<y_d\leq 1$, and for all $\alpha\in\mathbb{N},\ 0\leq\alpha\leq d+1$,

$$\biggl|\nabla_\xi^\alpha\biggl((e^{-|\xi|y_d}-e^{-\omega_\lambda(\xi)y_d})e^{-\omega_\lambda(\xi)z_d}\frac{\xi\otimes\xi}{|\xi|}\biggr)\biggr|\leq Cy_de^{-cz_d}e^{-c|\xi|(y_d+z_d)}|\xi|^{-\alpha}. \tag{3-29}$$

If $|y'|\geq\frac{1}{4}$ then (3-29) implies, for $j=1,\dots,d-1$,

$$\begin{aligned}|(-iy_j)^{d+1}s_{\lambda,\mathrm{high}}(y',y_d,z_d)|&=\biggl|\int_{\mathbb{R}^{d-1}}e^{iy'\cdot\xi}\partial_{\xi_j}^{d+1}\biggl((1-\chi_{R_0})(e^{-|\xi|y_d}-e^{-\omega_\lambda(\xi)y_d})e^{-\omega_\lambda(\xi)z_d}\frac{\xi\otimes\xi}{|\xi|}\biggr)\,d\xi\biggr|\\&\leq Cy_de^{-cz_d}\int_{|\xi|\geq R_0}|\xi|^{-d-1}\,d\xi\leq Cy_de^{-cz_d},\end{aligned}$$

which gives

$$|s_{\lambda,\mathrm{high}}(y',y_d,z_d)|\leq Cy_de^{-cz_d}|y'|^{-d-1}\leq\frac{Cy_d\,e^{-cz_d}}{(1+y_d+z_d+|y'|)^{d+1}},$$

since $0<y_d\leq 1$ and $|y'|\geq\frac{1}{4}$. It remains to consider the case $|y'|\leq\frac{1}{4}$. If $|y'|\leq y_d+z_d$ and $|y'|\leq\frac{1}{4}$, then estimate (3-29) with $\alpha=0$ yields

$$\begin{aligned}|s_{\lambda,\mathrm{high}}(y',y_d,z_d)|&\leq C\int_{|\xi|\geq R_0}y_de^{-cz_d}e^{-c|\xi|(y_d+z_d)}\,d\xi\\&\leq Cy_d\,e^{-cz_d}\,e^{-c'(y_d+z_d)}\\&\leq\frac{Cy_d\,e^{-cz_d}}{(y_d+z_d+|y'|)^{d-1}(1+y_d+z_d+|y'|)^2}.\end{aligned}$$

On the other hand, if $0<y_d+z_d\leq|y'|\leq\frac{1}{4}$ then we take $R\geq 4$ and the cut-off χ_R, and decompose $s_{\lambda,\mathrm{high}}$ into

$$s_{\lambda,\mathrm{high}}=\int_{\mathbb{R}^{d-1}}\chi_R(\xi)(1-\chi_{R_0}(\xi))\cdots\,d\xi+\int_{\mathbb{R}^{d-1}}(1-\chi_R(\xi))\cdots\,d\xi=:I_R+II_R.$$

The term I_R, on the one hand, is estimated from (3-29) with $\alpha=0$ as

$$|I_R|\leq C\int_{R_0\leq|\xi|\leq 2R}y_de^{-cz_d}\,d\xi\leq Cy_de^{-cz_d}R^{d-1},$$

and the term II_R, on the other hand, is estimated by integration by parts,

$$|(-iy_j)^dII_R|=\biggl|\int_{\mathbb{R}^{d-1}}e^{iy'\cdot\xi}\partial_{\xi_j}^d((1-\chi_R)\cdots)\,d\xi\biggr|\leq C\int_{|\xi|\geq R}y_de^{-cz_d}|\xi|^{-d}\,d\xi\leq Cy_de^{-cz_d}R^{-1}$$

for each $j=1,\dots,d-1$. Therefore, by taking $R=|y'|^{-1}$ we have

$$|s_{\lambda,\mathrm{high}}(y',y_d,z_d)|\leq\frac{Cy_d\,e^{-cz_d}}{|y'|^{d-1}}\leq\frac{Cy_d\,e^{-cz_d}}{(y_d+z_d+|y'|)^{d-1}(1+y_d+z_d+|y'|)^2}$$

for $0 < y_d + z_d \le |y'| \le \frac{1}{4}$. Thus, we have arrived at the following estimate for s_λ when $0 < y_d \le 1$:

$$\begin{aligned} |s_\lambda(y', y_d, z_d)| &\le |s_{\lambda,\mathrm{low}}(y', y_d, z_d)| + |s_{\lambda,\mathrm{high}}(y', y_d, z_d)| \\ &\le \frac{C y_d\, e^{-c z_d}}{(y_d + z_d + |y'|)^{d-1}(1 + y_d + z_d + |y'|)(1 + y_d + z_d)}. \end{aligned} \tag{3-30}$$

From (3-27) for $y_d \ge 1$ and (3-30) for $0 < y_d \le 1$ we conclude that (3-30) holds for all $y_d > 0$. The same bound is also valid for r'_λ by the identity (3-24) and $|\lambda| = 1$. By scaling back to general λ, we complete the proof of (3-17).

Step 3: Next we consider the estimates for derivatives of the kernel. Again we assume that $\lambda \in S_{\pi-\varepsilon}$ and $|\lambda| = 1$, and it suffices to focus on the estimate of s_λ in view of (3-24) and (3-25). The estimate for the derivative in y' is obtained from the same argument as above for s_λ itself, for the symbol of $\partial^\alpha_{y'} s_\lambda$ is just the multiplication by $(i\xi)^\alpha$ of the symbol of s_λ. Hence, the argument for the proof of (3-17) gives the bound

$$|\nabla^\alpha_{y'} s_\lambda(y', y_d, z_d)| \le \frac{C y_d\, e^{-c z_d}}{(y_d + z_d + |y'|)^{d-1+\alpha}(1 + y_d + z_d + |y'|)(1 + y_d + z_d)} \tag{3-31}$$

for $|\lambda| = 1$ and $\alpha = 1, 2$. Thus, estimate (3-18) holds.

As for the derivative in y_d, we observe the identity

$$\begin{aligned} \partial_{y_d} s_\lambda(y', y_d, z_d) = &-\int_{\mathbb{R}^{d-1}} e^{-iy'\cdot\xi} |\xi| (e^{-|\xi| y_d} - e^{-\omega_\lambda(\xi) y_d}) e^{-\omega_\lambda(\xi) z_d} \frac{\xi \otimes \xi}{|\xi|}\, d\xi \\ &+ \int_{\mathbb{R}^{d-1}} e^{-iy'\cdot\xi} (\omega_\lambda(\xi) - |\xi|) e^{-\omega_\lambda(\xi)(y_d + z_d)} \frac{\xi \otimes \xi}{|\xi|}\, d\xi. \end{aligned}$$

Then the first term of this right-hand side satisfies the estimate (3-31) with $\alpha = 1$. As for the second term, we see that the symbol $(\omega_\lambda(\xi) - |\xi|)(\xi \otimes \xi)/|\xi|$ behaves like

$$(\omega_\lambda(\xi) - |\xi|) \frac{\xi \otimes \xi}{|\xi|} \sim \begin{cases} \mathcal{O}(|\xi|) & \text{for } |\xi| \ll 1, \\ \mathcal{O}(1) & \text{for } |\xi| \gg 1. \end{cases} \tag{3-32}$$

Thus, we decompose the integral into the low-frequency part $|\xi| \ll 1$ and the high-frequency part $|\xi| \gg 1$ using the cut-off χ_{R_0} as in the proof for s_λ. We can show that the contribution from the low-frequency part is bounded by

$$\frac{C e^{-c(y_d + z_d)}}{(1 + y_d + z_d + |y'|)^d} \le \frac{C e^{-c(y_d + z_d)}}{(y_d + z_d + |y'|)^{d-1}(1 + y_d + z_d + |y'|)(1 + y_d + z_d)},$$

while the contribution from the high-frequency part is bounded by

$$\frac{C e^{-c(y_d + z_d)}}{(y_d + z_d + |y'|)^{d-1}(1 + y_d + z_d + |y'|)} \le \frac{C e^{-c(y_d + z_d)}}{(y_d + z_d + |y'|)^{d-1}(1 + y_d + z_d + |y'|)(1 + y_d + z_d)}.$$

Here we have replaced the constant $c > 0$ suitably. Collecting these bounds, we conclude that

$$|\partial_{y_d} s_\lambda(y', y_d, z_d)| \le \frac{C e^{-c z_d}}{(y_d + z_d + |y'|)^{d-1}(1 + y_d + z_d + |y'|)(1 + y_d + z_d)} \tag{3-33}$$

for $|\lambda| = 1$, which implies (3-19) with $\beta = 0$. A similar observation yields the estimate (3-19) with $\beta = 1$ and also (3-20). The details are omitted here. Finally we consider the estimate for the derivative in z_d. Again it suffices to consider the estimate of s_λ with $|\lambda| = 1$. We observe from (3-25) that

$$\begin{aligned}\partial_{z_d} s_\lambda(y', y_d, z_d) &= -\int_{\mathbb{R}^{d-1}} e^{iy'\cdot\xi}(e^{-|\xi|y_d} - e^{-\omega_\lambda(\xi)y_d})\, e^{-\omega_\lambda(\xi)z_d}\omega_\lambda(\xi)\frac{\xi\otimes\xi}{|\xi|}\,d\xi \\ &= -\int_{\mathbb{R}^{d-1}} e^{iy'\cdot\xi}(e^{-|\xi|y_d} - e^{-\omega_\lambda(\xi)y_d})\, e^{-\omega_\lambda(\xi)z_d}|\xi|\frac{\xi\otimes\xi}{|\xi|}\,d\xi \\ &\quad + \int_{\mathbb{R}^{d-1}} e^{iy'\cdot\xi}(e^{-|\xi|y_d} - e^{-\omega_\lambda(\xi)y_d})\, e^{-\omega_\lambda(\xi)z_d}(|\xi| - \omega_\lambda(\xi))\frac{\xi\otimes\xi}{|\xi|}\,d\xi.\end{aligned}$$

Then the first term of the right-hand side has a pointwise estimate similar to that of $\nabla_{y'} s_\lambda$, which was already obtained, while the symbol of the second term has behavior to that of s_λ for $|\xi| \ll 1$ and also decays faster for $|\xi| \gg 1$. Hence the second term satisfies at least the same estimate as s_λ. From these observations we conclude that

$$|\partial_{z_d} s_\lambda(y', y_d, z_d)| \le \frac{C y_d\, e^{-cz_d}}{(y_d + z_d + |y'|)^d(1 + y_d + z_d)}, \qquad |\lambda| = 1.$$

This proves (3-21) with $\beta = 0$. Estimate (3-21) with $\beta = 1$ and estimate (3-22) are proved in the same manner, and the details are omitted here. The proof of estimates (3-17)–(3-22) is complete. □

3D. ***Kernel bounds for the pressure.*** The goal of this section is to prove the following bounds on the pressure kernel q_λ defined by (2-8e). These bounds are crucial to the estimate of the pressure in [Maekawa et al. 2019, Sections 2–5].

Proposition 3.7. *Let $\lambda \in S_{\pi-\varepsilon}$. There exist positive constants $c(d, \varepsilon)$, $C(d, \varepsilon) < \infty$ such that, for all $y' \in \mathbb{R}^{d-1}$, $y_d, z_d > 0$,*

$$|q_\lambda(y', y_d, z_d)| \le \frac{Ce^{-c|\lambda|^{1/2}z_d}}{(y_d + z_d + |y'|)^{d-1}}. \tag{3-34}$$

Moreover, for $\alpha = 1, \dots, 3$,

$$|\nabla^\alpha_{y'} q_\lambda(y', y_d, z_d)| + |\partial^\alpha_{y_d} q_\lambda(y', y_d, z_d)| \le \frac{Ce^{-c|\lambda|^{1/2}z_d}}{(y_d + z_d + |y'|)^{d-1+\alpha}}, \tag{3-35}$$

$$|\nabla_{y'}\partial_{y_d} q_\lambda(y', y_d, z_d)| \le \frac{Ce^{-c|\lambda|^{1/2}z_d}}{(y_d + z_d + |y'|)^{d+1}}, \tag{3-36}$$

and, for $\beta = 0, 1, 2$,

$$|\nabla^\beta_y \partial_{z_d} q_\lambda(y', y_d, z_d)| \le \frac{Ce^{-c|\lambda|^{1/2}z_d}}{(y_d + z_d + |y'|)^{d-1+\beta}}\left(|\lambda|^{1/2} + \frac{1}{y_d + z_d + |y'|}\right). \tag{3-37}$$

The general scheme of the proof is the same as for the kernels corresponding to the nonlocal part (see Section 3C). Again, using the scaling (2-11), we assume without loss of generality that $|\lambda| = 1$.

Proof. Step 1: We assume $y_d \geq 1$. By the change of variable $\eta = \xi y_d$, we get

$$q_\lambda(y', y_d, z_d) = \frac{i}{y_d^{d-1}} \int_{\mathbb{R}^{d-1}} e^{i\tilde{y}'\cdot\eta} e^{-|\eta|} e^{-\omega_\lambda(\eta/y_d)z_d} \left(\frac{\eta}{|\eta|} + \frac{\eta}{\omega_{\lambda y_d^2}(\eta)}\right) d\eta = \frac{1}{y_d^{d-1}} \tilde{q}_\lambda(\tilde{y}', y_d, z_d),$$

with $\tilde{y}' = y'/y_d$. We aim at proving

$$|\tilde{q}_\lambda(\tilde{y}', y_d, z_d)| \leq \frac{Ce^{-cz_d}}{(1+|\tilde{y}'|)^{d-1}}, \tag{3-38}$$

from which we can derive the desired bound of q_λ for $y_d \geq 1$,

$$|q_\lambda(y', y_d, z_d)| \leq \frac{Ce^{-cz_d}}{(y_d+|y'|)^{d-1}} \leq \frac{Ce^{-cz_d}}{(y_d+z_d+|y'|)^{d-1}}, \tag{3-39}$$

by changing the constants C and c suitably. For $|\tilde{y}| \leq 1$, we simply bound the integrand by its modulus and get

$$|\tilde{q}_\lambda(\tilde{y}', y_d, z_d)| \leq e^{-cz_d} \int_{\mathbb{R}^{d-1}} e^{-|\eta|}\, d\eta \leq e^{-cz_d},$$

hence (3-38). For $|\tilde{y}| \geq 1$, we rely on Lemma 3.1. It follows from the bound

$$\left|\nabla_\eta^\alpha \left\{ e^{-|\eta|} e^{-\omega_\lambda(\eta/y_d)z_d} \left(\frac{\eta}{|\eta|} + \frac{\eta}{\omega_{\lambda y_d^2}(\eta)}\right)\right\}\right| \leq Ce^{-cz_d} e^{-c_0|\eta|} |\eta|^{-\alpha},$$

valid for all $\eta \in \mathbb{R}^{d-1} \setminus \{0\}$, and the lemma that there exists $C > 0$ such that for all $\tilde{y}' \in \mathbb{R}^{d-1}$, $y_d, z_d > 0$,

$$|\tilde{q}_\lambda(\tilde{y}', y_d, z_d)| \leq \frac{Ce^{-cz_d}}{|\tilde{y}'|^{d-1}}.$$

This implies (3-38).

Step 2: We now deal with the case $y_d \leq 1$. We split the kernel between low and high frequencies:

$$q_\lambda = \int_{\mathbb{R}^{d-1}} \chi_{R_0}(\xi) \cdots d\xi + \int_{\mathbb{R}^{d-1}} (1-\chi_{R_0}(\xi)) \cdots d\xi =: q_{\lambda,\mathrm{low}} + q_{\lambda,\mathrm{high}}.$$

We first deal with $q_{\lambda,\mathrm{low}}$. Our goal is to show that

$$|q_{\lambda,\mathrm{low}}(y', y_d, z_d)| \leq \frac{Ce^{-cz_d}}{(y_d+z_d+|y'|)^{d-1}}. \tag{3-40}$$

If $|y'| \leq 1$, we bound straightforwardly and get

$$|q_{\lambda,\mathrm{low}}(y', y_d, z_d)| \leq Ce^{-cz_d} \leq \frac{Ce^{-cz_d}}{(y_d+|y'|)^{d-1}},$$

from which (3-40) follows up to changing the constants c and C. If $|y'| \geq 1$, we apply Lemma 3.1 and get

$$|q_{\lambda,\mathrm{low}}(y', y_d, z_d)| \leq \frac{Ce^{-cz_d}}{|y'|^{d-1}} \leq \frac{Ce^{-cz_d}}{(y_d+|y'|)^{d-1}},$$

from which (3-40) follows up to changing the constants c and C. We now handle $q_{\lambda,\mathrm{high}}$. We aim at proving that

$$|q_{\lambda,\mathrm{high}}(y', y_d, z_d)| \le \frac{Ce^{-cz_d}}{(y_d + z_d + |y'|)^{d-1}}. \tag{3-41}$$

If $|y'| \ge \frac{1}{4}$, we integrate by parts d times and obtain for $j = 1, \dots, d-1$,

$$\begin{aligned} |(-iy_j)^d q_{\lambda,\mathrm{high}}(y', y_d, z_d)| &= \left| \int_{\mathbb{R}^{d-1}} e^{iy'\cdot\xi} \partial_{\xi_j}^d \left((1-\chi_{R_0}) e^{-|\xi| y_d} e^{-\omega_\lambda(\xi) z_d} \left(\frac{\xi}{|\xi|} + \frac{\xi}{\omega(\xi)} \right) \right) d\xi \right| \\ &\le Ce^{-cz_d} \int_{|\xi| \ge R_0} |\xi|^{-d}\, d\xi \le Ce^{-cz_d}, \end{aligned}$$

which gives

$$|q_{\lambda,\mathrm{high}}(y', y_d, z_d)| \le \frac{Ce^{-cz_d}}{|y'|^d} \le \frac{Ce^{-cz_d}}{|y'|^{d-1}},$$

from which (3-41) follows. If $|y'| \le \frac{1}{4}$, we directly bound the kernel by

$$|q_{\lambda,\mathrm{high}}(y', y_d, z_d)| \le Ce^{-c(y_d+z_d)},$$

which implies (3-41) in the case when $y_d + z_d \ge |y'|$. If $|y'| \le \frac{1}{4}$ and $y_d + z_d \le |y'|$, then we take $R \ge 4$ and the cut-off χ_R, and decompose $q_{\lambda,\mathrm{high}}$ into

$$q_{\lambda,\mathrm{high}} = \int_{\mathbb{R}^{d-1}} \chi_R(\xi)(1-\chi_{R_0}(\xi)) \cdots d\xi + \int_{\mathbb{R}^{d-1}} (1-\chi_R(\xi)) \cdots d\xi =: I_R + II_R.$$

The term I_R, on the one hand, is estimated directly,

$$|I_R| \le C \int_{R_0 \le |\xi| \le 2R} e^{-cz_d}\, d\xi \le Ce^{-cz_d} R^{d-1},$$

and the term II_R, on the other hand, is estimated by integration by parts,

$$|(-iy_j)^d II_R| =\le Ce^{-cz_d} \int_{|\xi| \ge R} |\xi|^{-d}\, d\xi \le Ce^{-cz_d} R^{-1}$$

for each $j = 1, \dots, d-1$. Therefore, by taking $R = |y'|^{-1}$ we have

$$|q_{\lambda,\mathrm{high}}(y', y_d, z_d)| \le \frac{Ce^{-cz_d}}{|y'|^{d-1}}$$

for $0 < y_d + z_d \le |y'| \le \frac{1}{4}$, which yields (3-41). Consequently, we have proved (3-34).

The bounds for the derivatives (3-35)–(3-37) are obtained in a rigorously similar way. Therefore, we do not repeat the argument. □

4. Resolvent estimates

This section is devoted to the proof of Theorem 1. In particular, the resolvent estimates (1-3)–(1-7) for the Dirichlet–Laplace part and the nonlocal part are shown in Sections 4A and 4B, respectively. Note that since we work on the space including the nondecaying functions, an assumption on the behavior of the pressure p itself, rather than ∇p, is needed to ensure the uniqueness; see Theorem 4. Indeed, if one allows the linear growth for p, the uniqueness is proved only "modulo shear flows" in general. The proof of Theorem 1 including the uniqueness part is completed in the end of this section.

The general principles to estimate the integral formulas (2-9) are to localize the integrals on small cubes and to use convolution estimates in the tangential direction. Integrals in the vertical direction on $z_d \in (0, 1)$ may require relying on singular integral estimates. Further insights are given in [Prange 2018].

4A. *Estimates for the Dirichlet–Laplace part.* In this subsection, we prove the L^p_{uloc}-L^q_{uloc} estimate for the resolvent problem for the Laplacian. The following lemma plays a crucial role for our purpose.

Lemma 4.1. *Assume that*

$$1 \le q \le p \le \infty, \quad 0 \le \frac{1}{q} - \frac{1}{p} < \frac{1}{d}. \tag{4-1}$$

Define the functions $K = K_\lambda(y', y_d)$ and $K' = K'_\lambda(y', y_d)$ by

$$K_\lambda(y', y_d) = \begin{cases} Ce^{-c|\lambda|^{1/2}|y_d|} \min\left\{\log\left(e + \dfrac{1}{|\lambda|^{1/2}(|y'| + |y_d|)}\right), \dfrac{1}{|\lambda|(|y'| + |y_d|)^2}\right\}, & d = 2, \\ \dfrac{Ce^{-c|\lambda|^{1/2}|y_d|}}{(|y'| + |y_d|)^{d-2}(1 + |\lambda|^{1/2}(|y'| + |y_d|))^2}, & d \ge 3, \end{cases} \tag{4-2}$$

$$K'_\lambda(y', y_d) = \frac{Ce^{-c|\lambda|^{1/2}|y_d|}}{(|y'| + |y_d|)^{d-1}(1 + |\lambda|^{1/2}(|y'| + |y_d|))^2} \tag{4-3}$$

for $\lambda \in S_{\pi-\varepsilon}$. Then there exists a constant $C = C(d, \varepsilon, q, p) > 0$ (independent of λ) such that

$$\|K_\lambda *_y f\|_{L^p_{\mathrm{uloc}}} \le \frac{C}{|\lambda|}(1 + |\lambda|^{(d/2)(1/q-1/p)})\|f\|_{L^q_{\mathrm{uloc}}}, \tag{4-4}$$

$$\|K'_\lambda *_y f\|_{L^p_{\mathrm{uloc}}} \le \frac{C}{|\lambda|^{1/2}}(1 + |\lambda|^{(d/2)(1/q-1/p)})\|f\|_{L^q_{\mathrm{uloc}}}, \tag{4-5}$$

where $*_y$ denotes the convolution in $\mathbb{R}^d$.

Proof. Since both estimates can be proved in the same way, we will only deal with (4-4). For $\eta = (\eta', \eta_d) \in \mathbb{Z}^{d-1} \times \mathbb{Z}_{\ge 0}$ we estimate the L^p norm of $K_\lambda *_y f$ in the cube of the form $B_\eta = B'_{\eta'} \times [\eta_d, \eta_d + 1]$, where $B'_{\eta'} = \eta' + [0, 1]^{d-1}$. We first consider the case when $d \ge 3$. Without loss of generality we may assume that $\eta = 0$. Let χ_α be the characteristic function on the cube B_α for $\alpha \in \mathbb{Z}^d$. Then we have

$$(K_\lambda) *_y f = \left(\sum_{\beta \in \mathbb{Z}^d} \chi_\beta K_\lambda\right) *_y \left(\sum_{\alpha \in \mathbb{Z}^d} \chi_\alpha f\right) = \sum_{\substack{\alpha, \beta \in \mathbb{Z}^d \\ \max|\alpha_i + \beta_i| \le 2}} (\chi_\beta K_\lambda) *_y (\chi_\alpha f),$$

due to the support of χ_β and χ_α. Thus, the Young inequality for convolution yields, for $1/p = 1/s + 1/q - 1$,

$$\begin{aligned} \|K_\lambda *_y f\|_{L^p(B_0)} &\le \sum_{\substack{\alpha, \beta \in \mathbb{Z}^d \\ \max|\alpha_i + \beta_i| \le 2}} \|\chi_\beta K_\lambda\|_{L^s(\mathbb{R}^d)} \|\chi_\alpha f\|_{L^q(\mathbb{R}^d)} \\ &= \sum_{\substack{\max|\beta_i| \le 2 \\ \max|\alpha_i + \beta_i| \le 2}} \|\chi_\beta K_\lambda\|_{L^s(\mathbb{R}^d)} \|\chi_\alpha f\|_{L^q(\mathbb{R}^d)} + \sum_{\substack{\max|\beta_i| \ge 3 \\ \max|\alpha_i + \beta_i| \le 2}} \|\chi_\beta K_\lambda\|_{L^s(\mathbb{R}^d)} \|\chi_\alpha f\|_{L^q(\mathbb{R}^d)} =: I_1 + I_2. \end{aligned}$$

For the estimate of I_1 we have

$$\begin{aligned}\|K_\lambda\|^s_{L^s(\mathbb{R}^d)} &\le C\int_{\mathbb{R}^d}|y|^{-(d-2)s}(1+|\lambda|^{1/2}|y|)^{-2s}\,dy\\ &= C\int_{\mathbb{R}^d}|\lambda|^{(d-2)s/2}|z|^{-(d-2)s}(1+|z|)^{-2s}\,dz|\lambda|^{-d/2}\\ &\le C|\lambda|^{(d-2)s/2-d/2}\biggl(\int_{|z|\le 1}|z|^{-(d-2)s}\,dz+\int_{|z|\ge 1}|z|^{-ds}\,dz\biggr)\\ &\le C|\lambda|^{((d-2)s)/2-d/2},\end{aligned}$$

where we have used the assumption (4-1) in the last line. Therefore

$$\begin{aligned}I_1 &\le C\sum_{\max|\beta_i|\le 2}\|\chi_\beta K_\lambda\|_{L^s(\mathbb{R}^d)}\|f\|_{L^q_{\mathrm{uloc}}(\mathbb{R}^d)}\\ &\le C|\lambda|^{(d-2)/2-d/(2s)}\|f\|_{L^q_{\mathrm{uloc}}(\mathbb{R}^d)} \le C|\lambda|^{-1+(d/2)(1/q-1/p)}\|f\|_{L^q_{\mathrm{uloc}}(\mathbb{R}^d)}.\end{aligned}$$

In order to estimate I_2 we further decompose the sum in β as

$$I_2 \le \sum_{\substack{\max|\beta_i'|\ge 3\\ \beta_d\in\mathbb{Z}}}\|\chi_\beta K_\lambda\|_{L^s(\mathbb{R}^d)}\|f\|_{L^q_{\mathrm{uloc}}(\mathbb{R}^d)}+\sum_{\substack{\max|\beta_i'|\le 3\\ |\beta_d|\ge 3}}\|\chi_\beta K_\lambda\|_{L^s(\mathbb{R}^d)}\|f\|_{L^q_{\mathrm{uloc}}(\mathbb{R}^d)}.$$

Using (4-2), we have

$$\begin{aligned}\sum_{\substack{\max|\beta_i'|\ge 3\\ \beta_d\in\mathbb{Z}}}\|\chi_\beta K_\lambda\|_{L^s(\mathbb{R}^d)} &\le C\sum_{\substack{\beta_d\in\mathbb{Z}\\ \max|\beta_i'|\ge 3}}\biggl(\int_{\beta_d}^{\beta_d+1}e^{-cs|\lambda|^{1/2}|y_d|}\int_{B'_{\beta'}}|\lambda|^{-3s/4}(|y'|+|y_d|)^{-(d-2)s-3s/2}\,dy'\,dy_d\biggr)^{1/s}\\ &\le C|\lambda|^{-3/4}\sum_{\beta_d\in\mathbb{Z}}e^{-c|\lambda|^{1/2}|\beta_d|}\sum_{\max|\beta_i'|\ge 3}(|\beta'|+|\beta_d|)^{-(d-1/2)}\\ &\le C|\lambda|^{-3/4}\sum_{\beta_d\in\mathbb{Z}}e^{-c|\lambda|^{1/2}|\beta_d|}(3+|\beta_d|)^{-1/2}\\ &\le C|\lambda|^{-3/4}\int_{\mathbb{R}}e^{-c|\lambda|^{1/2}t}t^{-1/2}\,dt \le C|\lambda|^{-1}.\end{aligned}$$

On the other hand, from (4-2) we also have

$$\begin{aligned}\sum_{\substack{|\beta_d|\ge 3\\ \max|\beta_i'|\le 3}}\|\chi_\beta K_\lambda\|_{L^s(\mathbb{R}^d)} &\le \sum_{|\beta_d|\ge 3}\sum_{\max|\beta_i'|\le 3}\biggl(\int_{\beta_d}^{\beta_d+1}\int_{B'_{\beta'}}|y_d|^{-ds}|\lambda|^{-s}\,dy'\,dy_d\biggr)^{1/s}\\ &\le C|\lambda|^{-1}\sum_{|\beta_d|\ge 3}\beta_d^{-d} \le C|\lambda|^{-1}.\end{aligned}$$

Therefore we obtain

$$I_2 \le C|\lambda|^{-1}\|f\|_{L^q_{\mathrm{uloc}}}.$$

Thus we obtain (4-4) for $d\ge 3$.

For the case when $d = 2$, from (4-2) we easily see that

$$|K_\lambda(y', y_d)| \le |\lambda|^{1/4}(|y'| + |y_d|)^{-1/2}(1 + |\lambda|^{1/2}(|y'| + |y_d|))^{-1}.$$

By using this bound, the same argument as for the case $d \ge 3$ applies to prove (4-4) for $d = 2$. So we omit the details. □

Proposition 4.2. *Let $\lambda \in S_{\pi-\varepsilon}$ and let $m_\alpha(D')$ be any Fourier multiplier (in the tangential variables), homogeneous of order $\alpha > 0$. Assume that $p, q \in [1, \infty]$ fulfill the condition* (4-1)*. Then for the function v defined in* (2-9), *i.e.*,

$$v(y', y_d) = \int_{\mathbb{R}^{d-1}} \int_0^\infty k_{1,\lambda}(y'-z', y_d-z_d) f(z', z_d)\, dz_d\, dz' + \int_{\mathbb{R}^{d-1}} \int_0^\infty k_{2,\lambda}(y'-z', y_d, z_d) f(z', z_d)\, dz_d\, dz',$$

with the kernels $k_{1,\lambda}$ and $k_{2,\lambda}$ given in (2-8a) *and* (2-8b) *respectively, the following estimates hold: there exist positive constants $C(d, \varepsilon, q, p) < \infty$ and $C_\alpha = C(\alpha, m_\alpha, d, \varepsilon, q) < \infty$ (independent of λ) such that*

$$\|v\|_{L^p_{\mathrm{uloc}}} \le \frac{C}{|\lambda|}(1 + |\lambda|^{(d/2)(1/q-1/p)})\|f\|_{L^q_{\mathrm{uloc}}}, \tag{4-6}$$

$$\|\nabla v\|_{L^p_{\mathrm{uloc}}} \le \frac{C}{|\lambda|^{1/2}}(1 + |\lambda|^{(d/2)(1/q-1/p)})\|f\|_{L^q_{\mathrm{uloc}}}, \tag{4-7}$$

$$\begin{aligned} \|m_\alpha(D')v\|_{L^q_{\mathrm{uloc}}} &\le \frac{C_\alpha}{|\lambda|^{(2-\alpha)/2}}\|f\|_{L^q_{\mathrm{uloc}}}, \quad \alpha \in (0, 2), \\ \|m_\alpha(D')\nabla v\|_{L^q_{\mathrm{uloc}}} &\le \frac{C_\alpha}{|\lambda|^{(1-\alpha)/2}}\|f\|_{L^q_{\mathrm{uloc}}}, \quad \alpha \in (0, 1). \end{aligned} \tag{4-8}$$

Moreover we have, for $1 < q < \infty$,

$$\|\nabla^2 v\|_{L^q_{\mathrm{uloc}}} \le C(1 + e^{-c|\lambda|^{1/2}} \log|\lambda|)\|f\|_{L^q_{\mathrm{uloc}}}. \tag{4-9}$$

Proof. We extend f by zero in $\mathbb{R}^d$ and still denote the extension by f. By Proposition 3.2 we have $|k_{1,\lambda}(y', y_d)| \le \mathrm{C}K_\lambda(y', y_d)$ for $y' \in \mathbb{R}^{d-1}$ and $y_d \in R$, and $|k_{2,\lambda}(y', y_d, z_d)| \le \mathrm{C}K_\lambda(y', y_d - z_d)$ for $y' \in \mathbb{R}^{d-1}$ and $y_d, z_d \ge 0$, where $K \ge 0$ is the function defined in (4-2). This shows that

$$\begin{aligned} \|v\|_{L^p_{\mathrm{uloc}}(\mathbb{R}^d_+)} &= C\|K * |f|\|_{L^p_{\mathrm{uloc}}(\mathbb{R}^d_+)} \le C\|K * |f|\|_{L^p_{\mathrm{uloc}}(\mathbb{R}^d)} \\ &\le \frac{C}{|\lambda|}(1 + |\lambda|^{(d/2)(1/q-1/p)})\|f\|_{L^p_{\mathrm{uloc}}(\mathbb{R}^d)} = \frac{C}{|\lambda|}(1 + |\lambda|^{(d/2)(1/q-1/p)})\|f\|_{L^p_{\mathrm{uloc}}(\mathbb{R}^d_+)}, \end{aligned}$$

which yields the desired estimate (4-6). Since the estimates (4-7) and (4-8) can be proved in the same way (for (4-8) with $\alpha \in (0, 2)$ we use the pointwise bound (3-16) and then apply the calculation as in Lemma 4.1), the details will be omitted.

In order to prove (4-9), we focus on the estimate for $(\nabla^2 k_1) *_y f$, since the term associated with the kernel k_2 is easier to handle. As in the proof of Lemma 4.1, it suffices to consider the L^p norm in B_0. We take the decomposition

$$\begin{aligned}\|(\nabla^2 k_{1,\lambda})*_y f\|_{L^q(B_0)} &= \Big\| \Big(\sum_{\beta\in\mathbb{Z}^d} \chi_\beta \nabla^2 k_{1,\lambda}\Big) *_y \Big(\sum_{\alpha\in\mathbb{Z}^d} \chi_\alpha f\Big)\Big\|_{L^q(\mathbb{R}^d)} \\ &\le \sum_{\substack{\alpha,\beta\in\mathbb{Z}^d\\ \max|\alpha_i+\beta_i|\le 2}} \|(\chi_\beta \nabla^2 k_{1,\lambda})*_y(\chi_\alpha f)\|_{L^q(\mathbb{R}^d)} \\ &= \sum_{\substack{\max|\beta_i|\le 2\\ \max|\alpha_i+\beta_i|\le 2}} \|(\chi_\beta \nabla^2 k_{1,\lambda})*_y(\chi_\alpha f)\|_{L^q(\mathbb{R}^d)} + \sum_{\substack{\max|\beta_i|\ge 3\\ \max|\alpha_i+\beta_i|\le 2}} \|(\chi_\beta \nabla^2 k_{1,\lambda})*_y(\chi_\alpha f)\|_{L^q(\mathbb{R}^d)} \\ &=: I_1 + I_2.\end{aligned}$$

By (3-5), the Hörmander–Mihlin theorem applies for $\nabla^2 k_{1,\lambda}$ and therefore

$$I_1 \le C \sum_{\max|\alpha_i|\le 4} \|\chi_\alpha f\|_{L^q(\mathbb{R}^d)} \le C\|f\|_{L^q_{\mathrm{uloc}}(\mathbb{R}^d_+)}.$$

We further decompose the sum in β as

$$I_2 \le \sum_{\substack{\max|\beta_i'|\ge 3\\ \beta_d\in\mathbb{Z}}} \|\chi_\beta \nabla^2 k_{1,\lambda}\|_{L^1(\mathbb{R}^d)} \|f\|_{L^q_{\mathrm{uloc}}(\mathbb{R}^d)} + \sum_{\substack{\max|\beta_i'|\le 3\\ |\beta_d|\ge 3}} \|\chi_\beta \nabla^2 k_{1,\lambda}\|_{L^1(\mathbb{R}^d)} \|f\|_{L^q_{\mathrm{uloc}}(\mathbb{R}^d)} =: I_{2,1} + I_{2,2}.$$

Using (3-5), we have

$$\begin{aligned}\sum_{\substack{\max|\beta_i'|\ge 3\\ \beta_d\in\mathbb{Z}}} \|\chi_\beta \nabla^2 k_{1,\lambda}\|_{L^1(\mathbb{R}^d)} &\le C \sum_{\beta_d\in\mathbb{Z}} \sum_{\max|\beta_i'|\ge 3} \int_{\beta_d}^{\beta_d+1} e^{-c|\lambda|^{1/2}|y_d|} \int_{B'_{\beta'}} (|y'|+y_d)^{-d}\, dy'\, dy_d \\ &\le C \sum_{\beta_d\in\mathbb{Z}} e^{-c|\lambda|^{1/2}|\beta_d|} \sum_{\max|\beta_i'|\ge 3} (|\beta'|+|\beta_d|)^{-d} \\ &\le C \sum_{\beta_d\in\mathbb{Z}} e^{-c|\lambda|^{1/2}|\beta_d|} (1+|\beta_d|)^{-1} \\ &\le C \int_{\mathbb{R}} e^{-c|\lambda|^{1/2}t} (1+t)^{-1}\, dt \le C(1+e^{-c|\lambda|^{1/2}} \log|\lambda|).\end{aligned}$$

On the other hand, from (4-2) we also have

$$\begin{aligned}\sum_{\substack{|\beta_d|\ge 3\\ \max|\beta_i'|\le 3}} \|\chi_\beta \nabla^2 k_{1,\lambda}\|_{L^1(\mathbb{R}^d)} &\le \sum_{|\beta_d|\ge 3} \sum_{\max|\beta_i'|\le 3} \int_{\beta_d}^{\beta_d+1} \int_{B'_{\beta'}} y_d^{-d}\, dy'\, dy_d \\ &\le C \sum_{|\beta_d|\ge 3} \beta_d^{-d} \le C.\end{aligned}$$

Therefore we obtain

$$I_2 \le C(1+e^{-c|\lambda|^{1/2}} \log|\lambda|)\|f\|_{L^q_{\mathrm{uloc}}}. \qquad \square$$

4B. ***Estimates for the nonlocal part.*** In this subsection we give the L^p_{uloc}-L^q_{uloc} estimates of

$$\begin{aligned} w'(y', y_d) = I'[f'](y', y_d) &= \int_{\mathbb{R}^{d-1}} \int_0^\infty r'_\lambda(y'-z', y_d, z_d) f'(z', z_d)\, dz_d\, dz', \\ w_d(y', y_d) = I_d[f'](y', y_d) &= \int_{\mathbb{R}^{d-1}} \int_0^\infty r_{d,\lambda}(y'-z', y_d, z_d)\cdot f'(z', z_d)\, dz_d\, dz', \end{aligned} \tag{4-10}$$

where the kernels are defined by (2-8c) and (2-8d).

Proposition 4.3. *Let $\lambda \in S_{\pi-\varepsilon}$ and let $m_\alpha(D')$ be any Fourier multiplier, homogeneous of order $\alpha > 0$. Assume that*

$$1 < q = p \le \infty \quad \textit{or} \quad 1 \le q < p \le \infty \quad \textit{with } 0 \le \frac{1}{q} - \frac{1}{p} < \frac{1}{d}. \tag{4-11}$$

Then for the function w defined in (4-10) *the following estimates hold: there exist positive constants $C(d, \varepsilon, q, p)$ and $C_\alpha = C(\alpha, m_\alpha, d, \varepsilon, q)$ (independent of λ) such that*

$$\|w\|_{L^p_{\text{uloc}}} \le \frac{C}{|\lambda|}(1 + |\lambda|^{(d/2)(1/q-1/p)})\|f\|_{L^q_{\text{uloc}}}, \tag{4-12}$$

$$\|\nabla w\|_{L^p_{\text{uloc}}} \le \frac{C}{|\lambda|^{1/2}}(1 + |\lambda|^{(d/2)(1/q-1/p)})\|f\|_{L^q_{\text{uloc}}}, \tag{4-13}$$

$$\begin{aligned} \|m_\alpha(D')w\|_{L^q_{\text{uloc}}} &\le \frac{C_\alpha}{|\lambda|^{(2-\alpha)/2}}\|f\|_{L^q_{\text{uloc}}}, \quad \alpha \in (0, 2), \\ \|m_\alpha(D')\nabla w\|_{L^q_{\text{uloc}}} &\le \frac{C_\alpha}{|\lambda|^{(1-\alpha)/2}}\|f\|_{L^q_{\text{uloc}}}, \quad \alpha \in (0, 1). \end{aligned} \tag{4-14}$$

Moreover we have, for $1 < q < \infty$,

$$\|\nabla^2 w\|_{L^q_{\text{uloc}}} \le C(1 + e^{-c|\lambda|^{1/2}} \log|\lambda|)\|f\|_{L^q_{\text{uloc}}}. \tag{4-15}$$

Remark 4.4. Estimate (4-13) holds even for the case $p = q = 1$. Similarly, if $\alpha \in (0, 2)$ then (4-14) holds also for the case $p = q = 1$. It is not difficult to check these facts from the proof below, and we do not give the details here.

Proof of Proposition 4.3. We focus on the estimate of $w' = I'[f']$, for the estimate of $w_d = I_d[f]$ is obtained in the same manner.

Step 1: We first focus on the estimate of $I'[f']$ itself. The next steps will be devoted to derivative estimates. Our estimate is based on the pointwise estimate (3-17) of the kernel r_λ. In particular, we often use the estimate

$$|r'_\lambda(y', y_d, z_d)| \le \frac{C y_d\, e^{-c|\lambda|^{1/2} z_d}}{|\lambda|^{1/2}(y_d + z_d + |y'|)^d (1 + |\lambda|^{1/2}(y_d + z_d))}, \tag{4-16}$$

which easily follows from (3-17). Notice that the variables y_d and z_d are not interchangeable with each other. In particular, we do not have exponential decay in y_d. Hence the trick used in the previous section, which transforms the action of the kernel into a convolution, does not work here.

Let $\eta = (\eta', \eta_d) \in \mathbb{Z}^{d-1} \times \mathbb{Z}_{\geq 0}$. Let us estimate the L^p norm of I' in the cube of the form $B_\eta = B'_{\eta'} \times [\eta_d, \eta_d + 1]$, where $B'_{\eta'} = \eta' + [0, 1]^{d-1}$. Without loss of generality we may assume that $\eta' = 0$. Let $\chi_{\eta'}$ be the characteristic function on the cube $B'_{\eta'}$. Then we have for $y = (y', y_d) \in B'_{0'} \times [\eta_d, \eta_d + 1]$,

$$\begin{aligned} r'_\lambda(\,\cdot\,, y_d, z_d) *_{y'} f'(\,\cdot\,, z_d) &= \biggl(\sum_{\alpha' \in \mathbb{Z}^{d-1}} \chi_{\alpha'} r'_\lambda(\,\cdot\,, y_d, z_d)\biggr) *_{y'} \biggl(\sum_{\beta' \in \mathbb{Z}^{d-1}} \chi_{\beta'} f'(\,\cdot\,, z_d)\biggr) \\ &= \sum_{\substack{\alpha', \beta' \in \mathbb{Z}^{d-1} \\ \max |\alpha'_i + \beta'_i| \leq 2}} (\chi_{\alpha'} r'_\lambda(\,\cdot\,, y_d, z_d)) *_{y'} (\chi_{\beta'} f'(\,\cdot\,, z_d)), \end{aligned}$$

due to the support of $\chi_{\alpha'}$ and $\chi_{\beta'}$. Thus, the Young inequality for convolution yields, for $1/p = 1/s + 1/q - 1$,

$$\begin{aligned} \|I'[f'](\,\cdot\,, y_d)\|_{L^p(B'_{0'})} &\leq \sum_{\substack{\alpha', \beta' \in \mathbb{Z}^{d-1} \\ \max |\alpha'_i + \beta'_i| \leq 2}} \int_0^\infty \|\chi_{\alpha'} r'_\lambda(\,\cdot\,, y_d, z_d)\|_{L^s(\mathbb{R}^{d-1})} \|\chi_{\beta'} f'(\,\cdot\,, z_d)\|_{L^q(\mathbb{R}^{d-1})}\, dz_d \\ &\leq \sum_{\substack{\max |\alpha'_i| \leq 2 \\ \max |\alpha'_i + \beta'_i| \leq 2}} \int_0^\infty \|\chi_{\alpha'} r'_\lambda(\,\cdot\,, y_d, z_d)\|_{L^s(\mathbb{R}^{d-1})} \|\chi_{\beta'} f'(\,\cdot\,, z_d)\|_{L^q(\mathbb{R}^{d-1})}\, dz_d \\ &\qquad + \sum_{\substack{\max |\alpha'_i| \geq 3 \\ \max |\alpha'_i + \beta'_i| \leq 2}} \int_0^\infty \|\chi_{\alpha'} r'_\lambda(\,\cdot\,, y_d, z_d)\|_{L^s(\mathbb{R}^{d-1})} \|\chi_{\beta'} f'(\,\cdot\,, z_d)\|_{L^q(\mathbb{R}^{d-1})}\, dz_d \\ &=: I_1 + I_2. \end{aligned}$$

For the term I_1 the data is localized and $\max |\beta'| \leq 4$ holds, and therefore,

$$\begin{aligned} I_1 &\leq C \sum_{n=0}^\infty \int_n^{n+1} \|r'_\lambda(\,\cdot\,, y_d, z_d)\|_{L^s(\mathbb{R}^{d-1})} \|f'(\,\cdot\,, z_d)\|_{L^q(\{|z'| \leq 8\})}\, dz_d \\ &\leq C \int_0^1 \|r'_\lambda(\,\cdot\,, y_d, z_d)\|_{L^s(\mathbb{R}^{d-1})} \|f'(\,\cdot\,, z_d)\|_{L^q(\{|z'| \leq 8\})}\, dz_d \\ &\qquad + C \sum_{n=1}^\infty \biggl(\int_n^{n+1} \|r'_\lambda(\,\cdot\,, y_d, z_d)\|^{q'}_{L^s(\mathbb{R}^{d-1})}\, dz_d\biggr)^{1/q'} \|f'\|_{L^q_{\mathrm{uloc}}} \\ &=: I_{1,1} + I_{1,2}. \end{aligned}$$

From $1/p = 1/s + 1/q - 1$ the pointwise estimate (4-16) implies

$$\begin{aligned} \|r'_\lambda(\,\cdot\,, y_d, z_d)\|_{L^s(\mathbb{R}^{d-1})} &\leq \frac{C y_d\, e^{-c|\lambda|^{1/2} z_d}}{|\lambda|^{1/2}(1 + |\lambda|^{1/2}(y_d + z_d))(y_d + z_d)^{1 + (d-1)(1/q - 1/p)}} \\ &\leq \frac{C\, e^{-c|\lambda|^{1/2} z_d}}{|\lambda|^{1/2}(1 + |\lambda|^{1/2}(y_d + z_d))(y_d + z_d)^{(d-1)(1/q - 1/p)}}. \end{aligned} \tag{4-17}$$

To estimate $I_{1,1}$ for the case $p = q$ we introduce the operator $T_{1,1}$ given by

$$(T_{1,1} h)(y_d) = \int_0^1 \frac{e^{-c|\lambda|^{1/2} z_d}}{|\lambda|^{1/2}(1 + |\lambda|^{1/2}(y_d + z_d))} h(z_d)\, dz_d.$$

It is straightforward to see

$$\|T_{1,1}h\|_{L^\infty_{y_d}} \le \frac{C}{|\lambda|}\|h\|_{L^\infty_{z_d}(0,1)}.$$

Moreover, we have

$$|T_{1,1}h(y_d)| \le \frac{1}{|\lambda| y_d}\|h\|_{L^1_{z_d}(0,1)},$$

which implies

$$\|T_{1,1}h\|_{L^{1,\infty}_{y_d}} \le \frac{C}{|\lambda|}\|h\|_{L^1_{z_d}(0,1)}.$$

Thus, $T_{1,1}$ is bounded from $L^1(0,1)$ to $L^{1,\infty}(\mathbb{R}_+)$, where $L^{1,\infty}(\mathbb{R}_+)$ is the weak L^1 space on $\mathbb{R}_+$. By the Marcinkiewicz interpolation theorem, $T_{1,1}$ is bounded from $L^q(0,1)$ to $L^q(\mathbb{R}_+)$ for any $1 < q < \infty$, and we have

$$\|I_{1,1}\|_{L^q(\mathbb{R}_+)} \le \frac{C}{|\lambda|}\|f'\|_{L^q_{\mathrm{uloc}}}, \quad 1 < q \le \infty. \tag{4-18}$$

Next we estimate $I_{1,1}$ for the case $q < p$. Note that (4-17) implies, for $y_d, z_d > 0$,

$$\|r'_\lambda(\,\cdot\,, y_d, z_d)\|_{L^s(\mathbb{R}^{d-1})} \le \frac{C}{|\lambda|^{1/2}(1+|\lambda|^{1/2}|y_d - z_d|)|y_d - z_d|^{(d-1)(1/q-1/p)}}.$$

Then, recall that $0 < 1/q - 1/p < 1/d$, which implies $0 < s(d-1)(1/q-1/p) < 1$ for $1/p = 1/s + 1/q - 1$. By the Young inequality for convolution, the term $I_{1,1}$ is estimated as

$$\begin{aligned}\|I_{1,1}\|_{L^p_{y_d}} &\le \frac{C}{|\lambda|^{1/2}}\biggl(\int_{\mathbb{R}} \frac{1}{(1+|\lambda|^{1/2}|y_d|)^s |y_d|^{s(d-1)(1/q-1/p)}}\,dy_d\biggr)^{1/s}\|f'\|_{L^q_{\mathrm{uloc}}} \\ &\le \frac{C}{|\lambda|^{1-(d/2)(1/q-1/p)}}\|f'\|_{L^q_{\mathrm{uloc}}}. \end{aligned}\tag{4-19}$$

Here we have used the fact $s > 1$ since $q < p$. It is also not difficult to see $I_{1,1} \in L^1_{\mathrm{loc}}(\mathbb{R}_+; L^q_{\mathrm{uloc}}(\mathbb{R}^{d-1}))$ when $f' \in L^q_{\mathrm{uloc}}(\mathbb{R}^d_+)$ for $q \in [1,\infty]$ (e.g., it is shown from the expression of $T_{1,1}$), and the details are omitted here. To estimate $I_{1,2}$ we observe from (4-16),

$$\begin{aligned}\sum_{n=1}^{\infty}\biggl(\int_n^{n+1}\|r'_\lambda(\,\cdot\,, y_d, z_d)\|^{q'}_{L^s(\mathbb{R}^{d-1})}\,dz_d\biggr)^{1/q'} &\le \frac{C}{|\lambda|^{1/2}}\sum_{n=1}^{\infty}\frac{e^{-c|\lambda|^{1/2}n}}{(1+|\lambda|^{1/2}(y_d+n))(y_d+n)^{(d-1)(1/q-1/p)}} \\ &\le \frac{C}{|\lambda|^{1/2}}\int_1^\infty \frac{e^{-c|\lambda|^{1/2}z_d}}{(1+|\lambda|^{1/2}(y_d+z_d))(y_d+z_d)^{(d-1)(1/q-1/p)}}\,dz_d \\ &\le \frac{C}{|\lambda|},\end{aligned}$$

which shows

$$\|I_{1,2}\|_{L^\infty_{y_d}} \le \frac{C}{|\lambda|}\|f'\|_{L^q_{\mathrm{uloc}}}.$$

Hence we have

$$\|I_1\|_{L^p_{y_d}} \le \frac{C}{|\lambda|}(1+|\lambda|^{(d/2)(1/q-1/p)})\|f'\|_{L^q_{\mathrm{uloc}}}, \quad p, q \text{ satisfy (4-11)}. \tag{4-20}$$

Next we estimate I_2. First we observe that, when $\max |\alpha_i'| \geq 3$,

$$\begin{aligned}\|\chi_{\alpha'} r_\lambda'(\,\cdot\,, y_d, z_d)\|_{L^s(\mathbb{R}^{d-1})} &\leq \frac{C y_d\, e^{-c|\lambda|^{1/2} z_d}}{(1+y_d+z_d+|\alpha'|)^{d-1}(1+|\lambda|^{1/2}(1+y_d+z_d+|\alpha'|))(1+|\lambda|^{1/2}(y_d+z_d))}\\ &\leq \frac{C y_d\, e^{-c|\lambda|^{1/2} z_d}}{|\lambda|^{1/2}(1+y_d+z_d+|\alpha'|)^{d}(1+|\lambda|^{1/2}(y_d+z_d))}.\end{aligned}$$

Thus we have

$$\begin{aligned}I_2 &\leq \sum_{\alpha'\in\mathbb{Z}^{d-1}} \sum_{n=0}^{\infty} \biggl(\int_n^{n+1} \frac{C y_d^{q'}\, e^{-cq'|\lambda|^{1/2} z_d}}{|\lambda|^{q'/2}(1+y_d+z_d+|\alpha'|)^{dq'}(1+|\lambda|^{1/2}(y_d+z_d))^{q'}}\, dz_d\biggr)^{1/q'} \|f'\|_{L^q_{\mathrm{uloc}}}\\ &\leq \sum_{\alpha'\in\mathbb{Z}^{d-1}} \sum_{n=0}^{\infty} \frac{C y_d\, e^{-c|\lambda|^{1/2} n}}{|\lambda|^{1/2}(1+y_d+n+|\alpha'|)^{d}(1+|\lambda|^{1/2}(y_d+n))} \|f'\|_{L^q_{\mathrm{uloc}}}\\ &\leq \sum_{\alpha'\in\mathbb{Z}^{d-1}} \int_0^{\infty} \frac{C y_d\, e^{-c|\lambda|^{1/2} z_d}}{|\lambda|^{1/2}(1+y_d+z_d+|\alpha'|)^{d}(1+|\lambda|^{1/2}(y_d+z_d))}\, dz_d \|f'\|_{L^q_{\mathrm{uloc}}}.\end{aligned}$$

Then

$$\begin{aligned}I_2 &\leq \sum_{\alpha'\in\mathbb{Z}^{d-1}} \frac{C y_d}{|\lambda|^{1/2}} \int_0^{\infty} \frac{e^{-c|\lambda|^{1/2} z_d}}{(1+y_d+z_d+|\alpha'|)^{d}(1+|\lambda|^{1/2}(y_d+z_d))}\, dz_d \|f'\|_{L^q_{\mathrm{uloc}}}\\ &\leq \frac{C y_d}{|\lambda|^{1/2}} \int_0^{\infty} \frac{e^{-c|\lambda|^{1/2} z_d}}{(1+y_d+z_d)(1+|\lambda|^{1/2}(y_d+z_d))}\, dz_d \|f'\|_{L^q_{\mathrm{uloc}}}\\ &\leq \frac{C}{|\lambda|^{1/2}} \int_0^{\infty} \frac{e^{-c|\lambda|^{1/2} z_d}}{(1+|\lambda|^{1/2} z_d)}\, dz_d \|f'\|_{L^q_{\mathrm{uloc}}},\end{aligned} \tag{4-21}$$

which implies

$$\|I_2\|_{L^\infty_{y_d}} \leq \frac{C}{|\lambda|} \|f'\|_{L^q_{\mathrm{uloc}}}. \tag{4-22}$$

Combining (4-20) with (4-22), we obtain, for p, q satisfying (4-11),

$$\|I'[f']\|_{L^p_{\mathrm{uloc}}} \leq \frac{C}{|\lambda|}(1+|\lambda|^{(d/2)(1/q-1/p)})\|f'\|_{L^q_{\mathrm{uloc}}}.$$

Note that the above proof also shows that $I'[f'] \in L^1_{\mathrm{loc}}(\mathbb{R}_+; L^q_{\mathrm{uloc}}(\mathbb{R}^{d-1}))$ if $f' \in L^q_{\mathrm{uloc}}(\mathbb{R}^d_+)$ for some $q \in [1, \infty]$.

Step 2: Next we consider the estimate for the derivatives. We will use

$$|\nabla^{1+\alpha} r_\lambda'(y', y_d, z_d)| \leq \frac{C e^{-c|\lambda|^{1/2} z_d}}{(y_d+z_d+|y'|)^{d-1+\alpha}(1+\delta_{0\alpha}|\lambda|^{1/2}(y_d+z_d+|y'|))} \tag{4-23}$$

for $\alpha = 0, 1$, which follows from (3-18), (3-19), and (3-20). Here $\delta_{0\alpha}$ is the Kronecker delta. From (4-23) we observe that, for $\delta \in (0, 1)$,

$$|\nabla r_\lambda'(y', y_d, z_d)| \leq \frac{C e^{-c|\lambda|^{1/2} z_d}}{|\lambda|^{\delta/2}(y_d+z_d+|y'|)^{d-1+\delta}(1+|\lambda|^{1/2}(y_d+z_d))^{1-\delta}}. \tag{4-24}$$

By arguing as above, we see

$$\begin{aligned}\|\nabla I'[f'](\,\cdot\,,y_d)\|_{L^p(B'_{0'})} &\le C\int_0^1 \|\nabla r'_\lambda(\,\cdot\,,y_d,z_d)\|_{L^s(\mathbb{R}^{d-1})}\|f'(\,\cdot\,,z_d)\|_{L^q(\{|z'|\le 8\})}\,dz_d \\ &\quad + C\sum_{n=1}^{\infty}\biggl(\int_n^{n+1}\|\nabla r'_\lambda(\,\cdot\,,y_d,z_d)\|^{q'}_{L^s(\mathbb{R}^{d-1})}\,dz_d\biggr)^{1/q'}\|f'\|_{L^q_{\mathrm{uloc}}} \\ &\quad + \sum_{\substack{\max|\alpha'_i|\ge 3\\ \max|\alpha'_i+\beta'_i|\le 2}}\int_0^\infty \|\chi_{\alpha'}\nabla r'_\lambda(\,\cdot\,,y_d,z_d)\|_{L^s(\mathbb{R}^{d-1})}\|\chi_{\beta'}f'(\,\cdot\,,z_d)\|_{L^q(\mathbb{R}^{d-1})}\,dz_d \\ &=: II_{1,1}+II_{1,2}+II_2.\end{aligned}$$

The last term II_2 is computed as in the derivation of (4-21) and (4-22), and one can show

$$\begin{aligned}II_2 &\le \sum_{\alpha'\in\mathbb{Z}^{d-1}}\frac{C}{|\lambda|^{1/4}}\int_0^\infty \frac{e^{-c|\lambda|^{1/2}z_d}}{(1+y_d+z_d+|y'|)^{d-1/2}(1+|\lambda|^{1/2}(y_d+z_d))^{1/2}}\,dz_d\|f'\|_{L^q_{\mathrm{uloc}}} \\ &\le \frac{C}{|\lambda|^{1/4}}\int_0^\infty \frac{e^{-c|\lambda|^{1/2}z_d}}{(1+z_d)^{1/2}}\,dz_d\|f'\|_{L^q_{\mathrm{uloc}}} \le \frac{C}{|\lambda|^{1/2}}\|f'\|_{L^q_{\mathrm{uloc}}}. \end{aligned} \tag{4-25}$$

As for $II_{1,1}$ and $II_{1,2}$, it follows from estimate (4-24) that

$$\|\nabla r'_\lambda(\,\cdot\,,y_d,z_d)\|_{L^s(\mathbb{R}^{d-1})} \le \frac{Ce^{-c|\lambda|^{1/2}z_d}}{|\lambda|^{\delta/2}(1+|\lambda|^{1/2}(y_d+z_d))^{1-\delta}(y_d+z_d)^{\delta+(d-1)(1/q-1/p)}}. \tag{4-26}$$

Take $\delta\in(0,1)$ small so that $s(\delta+(d-1)(1/q-1/p))<1$. Then, the Young inequality as in the derivation of (4-19) implies

$$\begin{aligned}\|II_{1,1}\|_{L^p_{y_d}} &\le \frac{C}{|\lambda|^{\delta/2}}\biggl(\int_{\mathbb{R}}\frac{1}{(1+|\lambda|^{1/2}|y_d|)^{s(1-\delta)}|y_d|^{s(\delta+(d-1)(1/q-1/p))}}\,dy_d\biggr)^{1/s}\|f'\|_{L^q_{\mathrm{uloc}}} \\ &\le \frac{C}{|\lambda|^{1/2-(d/2)(1/q-1/p)}}\|f'\|_{L^q_{\mathrm{uloc}}}. \end{aligned} \tag{4-27}$$

On the other hand, the term $II_{1,2}$ is estimated as in the proof for $I_{1,2}$ by using (4-26), and we have

$$\begin{aligned}\|II_{1,2}\|_{L^\infty_{y_d}} &\le \sup_{y_d}\frac{C}{|\lambda|^{\delta/2}}\int_1^\infty \frac{e^{-c|\lambda|^{1/2}z_d}}{(1+|\lambda|^{1/2}(y_d+z_d))^{1-\delta}(y_d+z_d)^{\delta+(d-1)(1/q-1/p)}}\,dz_d\|f'\|_{L^q_{\mathrm{uloc}}} \\ &\le \sup_{y_d}\frac{C}{|\lambda|^{\delta/2}}\int_1^\infty \frac{e^{-c|\lambda|^{1/2}z_d}}{(1+|\lambda|^{1/2}(y_d+z_d))^{1-\delta}(y_d+z_d)^{\delta}}\,dz_d\|f'\|_{L^q_{\mathrm{uloc}}} \\ &\le \frac{C}{|\lambda|^{1/2}}\|f'\|_{L^q_{\mathrm{uloc}}}. \end{aligned} \tag{4-28}$$

Thus, we have from (4-25), (4-27), and (4-28),

$$\|\nabla I'[f']\|_{L^p_{\mathrm{uloc}}} \le \frac{C}{|\lambda|^{1/2}}(1+|\lambda|^{(d/2)(1/q-1/p)})\|f'\|_{L^q_{\mathrm{uloc}}}, \quad 0\le \frac{1}{q}-\frac{1}{p}<\frac{1}{d}. \tag{4-29}$$

Note that the case $p=q=1$ is allowed in (4-29). The proof of (4-14) is the same as above (it suffices to use the bound (3-23)), and we omit the details.

Step 3: Finally we give the estimate for $\nabla^2 I'[f']$. Our aim is to show

$$\|\nabla^2 I'[f']\|_{L^q_{\mathrm{uloc}}} \le C(1+e^{-c|\lambda|^{1/2}} \log|\lambda|)\|f'\|_{L^q_{\mathrm{uloc}}}, \quad 1<q<\infty. \tag{4-30}$$

The key pointwise estimate reads

$$|\nabla^2 r'_\lambda(y', y_d, z_d)| \le \frac{Ce^{-c|\lambda|^{1/2} z_d}}{(y_d+z_d+|y'|)^d},$$

which follows from (4-23). This bound implies

$$\|\nabla^2 r'_\lambda(\,\cdot\,, y_d, z_d)\|_{L^1(\mathbb{R}^{d-1})} \le \frac{Ce^{-c|\lambda|^{1/2} z_d}}{y_d+z_d}. \tag{4-31}$$

As in the proof for $I'[f']$ and $\nabla I'[f']$ above, we start from

$$\begin{aligned}\|\nabla^2 I'[f'](\,\cdot\,, y_d)\|_{L^q(B'_{0'})} &\le C\int_0^1 \|\nabla^2 r'_\lambda(\,\cdot\,, y_d, z_d)\|_{L^1(\mathbb{R}^{d-1})}\|f'(\,\cdot\,, z_d)\|_{L^q(\{|z'|\le 8\})}\,dz_d\\ &\quad + C\sum_{n=1}^{\infty}\biggl(\int_n^{n+1}\|\nabla^2 r'_\lambda(\,\cdot\,, y_d, z_d)\|^{q'}_{L^1(\mathbb{R}^{d-1})}\,dz_d\biggr)^{1/q'}\|f'\|_{L^q_{\mathrm{uloc}}}\\ &\quad + \sum_{\substack{\max|\alpha'_i|\ge 3\\ \max|\alpha'_i+\beta'_i|\le 2}}\int_0^\infty \|\chi_{\alpha'}\nabla^2 r'_\lambda(\,\cdot\,, y_d, z_d)\|_{L^1(\mathbb{R}^{d-1})}\|\chi_{\beta'} f'(\,\cdot\,, z_d)\|_{L^q(\mathbb{R}^{d-1})}\,dz_d\\ &=: III_{1,1}+III_{1,2}+III_2.\end{aligned}$$

To estimate $III_{1,1}$ we introduce the operator T given by

$$(Th)(y_d) = \int_0^1 \frac{e^{-c|\lambda|^{1/2} z_d}}{y_d+z_d} h(z_d)\,dz_d.$$

It is straightforward to see

$$|(Th)(y_d)| \le \frac{C}{y_d}\|h\|_{L^1_{z_d}}, \quad |(Th)(y_d)| \le \frac{C}{y_d^{1/q}}\|h\|_{L^q_{y_d}}$$

for any $1<q<\infty$. Thus, T is bounded from $L^q(\mathbb{R}_+)$ to $L^{q,\infty}(\mathbb{R}_+)$ for any $1\le q<\infty$, where $L^{q,\infty}(\mathbb{R}_+)$ is the weak L^q space on $\mathbb{R}_+$. By the Marcinkiewicz interpolation theorem, T is bounded from $L^q(\mathbb{R}_+)$ to $L^q(\mathbb{R}_+)$ for any $1<q<\infty$. This implies

$$\|III_{1,1}\|_{L^q(\mathbb{R}_+)} \le C\|f'\|_{L^q_{\mathrm{uloc}}}, \quad 1<q<\infty.$$

The terms $III_{1,2}$ and III_2 are estimated much as $I_{1,2}$ and I_2 above and we see

$$\begin{aligned}\|III_{1,2}\|_{L^\infty_{y_d}} + \|III_2\|_{L^\infty_{y_d}} &\le C\int_1^\infty \frac{e^{-c|\lambda|^{1/2} z_d}}{z_d}\,dz_d\|f'\|_{L^q_{\mathrm{uloc}}} + C\int_0^\infty \frac{e^{-c|\lambda|^{1/2} z_d}}{1+z_d}\,dz_d\|f'\|_{L^q_{\mathrm{uloc}}}\\ &\le C(1+e^{-c|\lambda|^{1/2}}\log|\lambda|)\|f'\|_{L^q_{\mathrm{uloc}}}.\end{aligned}$$

□

To conclude, let us notice that it is easy to get the uniform estimate

$$\|\nabla^2_{y'} I'[f']\|_{L^q_{\mathrm{uloc}}} \le C\|f'\|_{L^q_{\mathrm{uloc}}}, \quad \text{and thus} \quad \|\nabla^2_{y'} w\|_{L^q_{\mathrm{uloc}}} \le C\|f'\|_{L^q_{\mathrm{uloc}}}, \quad 1 < q < \infty. \tag{4-32}$$

Indeed, for these tangential derivatives, we can rely on the kernel bound (3-18), which yields

$$\|\nabla^2_{y'} r'_\lambda(\,\cdot\,, y_d, z_d)\|_{L^1(\mathbb{R}^{d-1})} \le \frac{y_d}{(y_d+z_d)^2},$$

instead of (4-31). This enables us to get estimates uniform in λ following the same strategy as above. We do not know whether the difficulty we encounter here to show a similar uniform estimate on $\partial^2_{y_d} I'[f']$ or $\partial^2_{z_d} I'[f']$ is a technical one or reveals an essential obstruction.

Remark 4.5 (estimates for the pressure). We are only concerned with gradient estimates on the pressure. From the pointwise estimate (3-35), it is clear that ∇p is estimated in $L^q_{\mathrm{uloc}}(\mathbb{R}^d_+)$ in the exact same way as we estimated $\nabla^2 I'[f']$ stated in (4-30); i.e.,

$$\|\nabla p\|_{L^q_{\mathrm{uloc}}} \le C(1+e^{-c|\lambda|^{1/2}} \log|\lambda|)\|f'\|_{L^q_{\mathrm{uloc}}}, \quad 1 < q < \infty. \tag{4-33}$$

On the other hand, recalling the identity $\omega_\lambda(\xi)^2 = \lambda + |\xi|^2$, we also have from the formula (2-7a)

$$\|\nabla p\|_{L^q_{\mathrm{uloc}}} \le C(1+e^{-c|\lambda|^{1/2}} \log|\lambda|)\|(\lambda-\Delta)u'\|_{L^q_{\mathrm{uloc}}}, \quad 1 < q < \infty. \tag{4-34}$$

The estimate (4-34) is crucial in obtaining the characterization of the domain of the Stokes operator in the L^q_{uloc} spaces with $1 < q < \infty$.

Proof of Theorem 1. The estimates (1-3), (1-4), (1-6) and (1-7) are proved in Propositions 4.2 and 4.3 and (4-33). Hence taking into account Theorem 4, proved in Appendix A, it suffices to show the pressure gradient given by the formula (2-6) satisfies (1-5). Consider only the tangential gradient $\nabla' p$, for the normal gradient can be estimated in the same manner. By the formula (2-6a), the tangential gradient is written

$$\nabla' p(y_d) = R'\nabla' P(y_d) \cdot \partial_{y_d} u'(0),$$

where $R' = (R_1, R_2, \dots, R_{d-1})$ is the vector-valued Riesz transform in $\mathbb{R}^{d-1}$ and $\partial_{y_d} u'(0)$ makes sense in $L^q_{\mathrm{uloc}}(\mathbb{R}^{d-1})$ by the trace theorem and the regularity $\nabla^\alpha u' \in L^q_{\mathrm{uloc}}(\mathbb{R}^d_+)$ for $\alpha = 0, 1, 2$ and $1 < q < \infty$. By the property of the Poisson kernel $P_{y_d}(y')$, it is easy to see that $\nabla' P_{y_d}$ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^{d-1})$ for $y_d > 0$. Therefore from the boundedness of the Riesz transform from $\mathcal{H}^1(\mathbb{R}^{d-1})$ to $L^1(\mathbb{R}^{d-1})$ we have

$$\begin{aligned}
\|\nabla' p(y_d)\|_{L^1_{\mathrm{uloc}}(\mathbb{R}^{d-1})} &= \|R'\nabla' P(y_d) * \partial_{y_d} u'(0)\|_{L^1_{\mathrm{uloc}}(\mathbb{R}^{d-1})} \\
&\le \|R'\nabla' P_{y_d}\|_{L^1(\mathbb{R}^{d-1})} \|\partial_{y_d} u'(0)\|_{L^1_{\mathrm{uloc}}(\mathbb{R}^{d-1})} \\
&\le \|\nabla' P_{y_d}\|_{\mathcal{H}^1(\mathbb{R}^{d-1})} \|\partial_{y_d} u'(0)\|_{L^1_{\mathrm{uloc}}(\mathbb{R}^{d-1})} \\
&\le C y_d^{-1} \|\partial_{y_d} u'(0)\|_{L^1_{\mathrm{uloc}}(\mathbb{R}^{d-1})},
\end{aligned}$$

which proves the desired bound (1-5). □

5. The Stokes semigroup in L^p_{uloc} spaces

In this section we construct the Stokes operator in the L^q_{uloc} spaces and the associated semigroup. Usually the Stokes operator $\boldsymbol{A}$ is written as $\boldsymbol{A} = -\mathbb{P}\Delta_D$, where $\mathbb{P}$ is the Helmholtz–Leray projection and Δ_D is the realization of the Laplace operator under the Dirichlet boundary condition. However, the action of $\mathbb{P}$ does not make sense in general for nondecaying data, so we need to define the Stokes operator in a different way. In principle, we follow the argument of [Desch et al. 2001] to define the Stokes operator but with a slight change of some technical details.

Notice that [Ukai 1987; Cannone et al. 2000; Danchin and Zhang 2014] provide representation formulas for the solution of the unsteady Stokes problem. However, these formulas involve singular integral operators, which are unbounded on spaces of nonintegrable functions. Our approach which relies on the Dunford formula and the Stokes resolvent problem takes advantage of the fundamental insight of Desch, Hieber and Prüss, which circumvents the unboundedness of the Helmholtz–Leray transform.

Let $\lambda \in S_{\pi-\varepsilon}$ with $\varepsilon \in (0, \pi)$. Let $1 < q \le \infty$ and $f \in L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$. Then there exists a unique solution $(u, \nabla p)$ to (1-2) in the class stated in Theorem 1. We denote this linear map from $L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$ to $L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$ as $R(\lambda)$. For convenience we also write the associated pressure ∇p as ∇p_u to emphasize that ∇p_u is determined from u by the formula (2-10). Note that $\gamma\partial_{y_d}u'$ makes sense in $L^q_{\mathrm{uloc}}(\mathbb{R}^{d-1})$ as is stated in the proof of Theorem 1. What we need to show is that

(i) the null space of $R(\lambda)$ is trivial, and

(ii) the resolvent identity $R(\lambda) - R(\mu) = -(\lambda - \mu)R(\lambda)R(\mu)$ holds for any $\lambda, \mu \in S_{\pi-\varepsilon}$.

Note that (ii) implies in particular that $R(\lambda)$ commutes with $R(\mu)$. To prove (i), we assume that $u := R(\lambda)f = 0$ for some $f \in L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$. Then the associated pressure ∇p_u is zero by the formula (2-10). Hence, we must have $f = 0$ since $(u, \nabla p_u)$ solves (1-2). Thus, $R(\lambda)$ is injective. Next we prove the resolvent identity. Fix any $f \in L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$ and set $u = R(\lambda)f$ and $v = R(\mu)f$. Then $(u - v, \nabla p_u - \nabla p_v)$ solves (1-2) with $f = -(\lambda - \mu)v$. By Theorem 1 there exists a solution $(w, \nabla p_w)$ to (1-2) with $f = -(\lambda - \mu)v$, which is unique in the class stated in Theorem 1, and $w = R(\lambda)(\mu - \lambda)v = -(\lambda - \mu)R(\lambda)v$ by the definition of $R(\lambda)$. Since $(w, \nabla p_w)$ and $(u - v, \nabla p_u - \nabla p_v)$ belong to the same class as stated in Theorem 1 (in particular, both satisfy the decay condition on the derivative of the pressure as $y_d \to \infty$), by the uniqueness result of Theorem 1, we have $w = u - v$. This implies $R(\lambda)f - R(\mu)f = -(\lambda - \mu)R(\lambda)R(\mu)f$ for any $f \in L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$, and hence the resolvent identity is proved.

From (i) and (ii) we conclude that there exists a closed linear operator $\boldsymbol{A} : D(\boldsymbol{A}) \subseteq L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+) \to L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$ such that the domain $D(\boldsymbol{A})$ of $\boldsymbol{A}$ is the range of $R(\lambda)$ which is independent of λ, and the resolvent set of $-\boldsymbol{A}$ includes $S_{\pi-\varepsilon}$ for any $\varepsilon \in (0, \pi)$, and $(\lambda + \boldsymbol{A})^{-1} = R(\lambda)$ for any $\lambda \in S_{\pi-\varepsilon}$. We say that $\boldsymbol{A}$ is the Stokes operator realized in $L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$.

Proposition 5.1. *Let $1 < q < \infty$ and let $\boldsymbol{A}$ be the Stokes operator realized in $L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$. Then*

$$D(\boldsymbol{A}) = \{u \in L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+) \mid \nabla^\alpha u \in L^q_{\mathrm{uloc}}(\mathbb{R}^d_+),\ \alpha = 0, 1, 2,\ u = 0 \text{ on } \partial\mathbb{R}^d_+\}. \tag{5-1}$$

Proof. Theorem 1 implies that for any $f \in L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$, the function $R(\lambda)f$ belongs to $L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$, $\nabla^\alpha R(\lambda)f \in L^q_{\mathrm{uloc}}(\mathbb{R}^d_+)$ for $\alpha = 0, 1, 2$, and $R(\lambda)f = 0$ on $\partial\mathbb{R}^d_+$. Thus, the domain $D(\boldsymbol{A})$, which is the range of $R(\lambda)$, belongs to the set defined in the right-hand side of (5-1). Conversely, let u be any function belonging to the right-hand side of (5-1). Then the pair $(u, \nabla p_u)$ with ∇p_u defined by (2-10) solves (1-2) with $f = \lambda u - \Delta u + \nabla p_u$, and f belongs to $L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$ by the definition of ∇p_u and the estimate (4-34). This implies that u belongs to the range of $R(\lambda)$, and thus to $D(\boldsymbol{A})$. □

Note that we do not have the characterization of the domain of $\boldsymbol{A}$ in the space $L^\infty_\sigma(\mathbb{R}^d_+)$. Theorem 1 and the definition of $R(\lambda)$ immediately yield the following:

Proposition 5.2. *Let $1 < q \le \infty$ and let $\boldsymbol{A}$ be the Stokes operator realized in $L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$. Then for any $\varepsilon \in (0, \pi)$ the sector $S_{\pi-\varepsilon}$ belongs to the resolvent of $-\boldsymbol{A}$ and*

$$|\lambda| \|(\lambda + \boldsymbol{A})^{-1} f\|_{L^q_{\mathrm{uloc}}} \le C_\varepsilon \|f\|_{L^q_{\mathrm{uloc}}}, \quad \lambda \in S_{\pi-\varepsilon},\ f \in L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+).$$

Therefore, $-\boldsymbol{A}$ generates a bounded analytic semigroup in $L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$.

Notice that $\boldsymbol{A}$ is not known to be strongly continuous, because $D(\boldsymbol{A})$ is not dense in $L^q_{\mathrm{uloc},\sigma}$ (this is seen easily, see for instance [Mielke and Schneider 1995, Lemma 3.1(d)]). Applying Theorem 1, we also have the L^p_{uloc}-L^q_{uloc} estimates for $e^{-t\boldsymbol{A}}$ as follows.

Proposition 5.3. *Let $1 < q \le \infty$ and let $\boldsymbol{A}$ be the Stokes operator realized in $L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$. Then there exists a constant $C(d, q) < \infty$, for $\alpha = 0, 1$,*

$$t^{\alpha/2} \|\nabla^\alpha e^{-t\boldsymbol{A}} f\|_{L^q_{\mathrm{uloc}}} + t \left\| \frac{d}{dt} e^{-t\boldsymbol{A}} f \right\|_{L^q_{\mathrm{uloc}}} \le C \|f\|_{L^q_{\mathrm{uloc}}}, \quad t > 0,\ f \in L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+), \tag{5-2}$$

and when $1 < q < \infty$,

$$\frac{t}{\log(e+t)} \|\nabla^2 e^{-t\boldsymbol{A}} f\|_{L^q_{\mathrm{uloc}}} \le C \|f\|_{L^q_{\mathrm{uloc}}}, \quad t > 0,\ f \in L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+). \tag{5-3}$$

Moreover, for $1 < q \le p \le \infty$ or $1 \le q < p \le \infty$, there exists a constant $C(d, p, q) < \infty$ such that

$$\|e^{-t\boldsymbol{A}} f\|_{L^p_{\mathrm{uloc}}} \le C(t^{-(d/2)(1/q-1/p)} + 1) \|f\|_{L^q_{\mathrm{uloc}}}, \qquad t > 0,\ f \in L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+), \tag{5-4}$$

$$\|\nabla e^{-t\boldsymbol{A}} f\|_{L^p_{\mathrm{uloc}}} \le C t^{-1/2} (t^{-(d/2)(1/q-1/p)} + 1) \|f\|_{L^q_{\mathrm{uloc}}}, \quad t > 0,\ f \in L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+). \tag{5-5}$$

Remark 5.4. In (5-4) and (5-5) the estimates are stated, in particular, for the exponents $1 = q < p \le \infty$, while the generation of the analytic semigroup in $L^1_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$ seems to fail, by a reason similar to the case of L^1 observed in [Desch et al. 2001]. The estimate for the case $p = q = \infty$ is also well known. In (5-3) the logarithmic growth factor appears due to the logarithmic factor in the resolvent estimate (1-4). This additional growth does not seem to be optimal at least for the semigroup bound, and it is possible to remove it if one obtains the resolvent estimate such as

$$\|\nabla^2 (\lambda + \boldsymbol{A})^{-1} f\|_{L^q_{\mathrm{uloc}}} \le C(\|f\|_{L^q_{\mathrm{uloc}}} + |\lambda|^{-1/2} \|\nabla f\|_{L^q_{\mathrm{uloc}}}), \tag{5-6}$$

for one can then use the identity

$$e^{-tA}f = \frac{1}{2\pi i}\int_\Gamma e^{t\lambda}(\lambda + A)^{-1} f\,d\lambda = \frac{1}{2\pi i t}\int_\Gamma e^{t\lambda}(\lambda + A)^{-2} f\,d\lambda$$

in estimating $\nabla^2 e^{-tA} f$, where the integration by parts is used. Estimate (5-6) seems to be valid, though we do not give the detailed proof in this paper. We also note that the estimates for the higher-order derivatives can be shown by our method, but we do not go into the details here.

Proof. The estimate $\|e^{-tA} f\|_{L^q_{\mathrm{uloc}}} \le C\|f\|_{L^q_{\mathrm{uloc}}}$ for $t > 0$ was already shown in Proposition 5.2, and we focus on the other estimates. Let us recall the standard representation formula of e^{-tA} in terms of the Dunford integral

$$e^{-tA} mf = \frac{1}{2\pi i}\int_\Gamma e^{t\lambda}(\lambda + A)^{-1} f\,d\lambda. \tag{5-7}$$

Here $\Gamma = \Gamma_\kappa$ with $\kappa \in (0, 1)$ is the curve $\{\lambda \in \mathbb{C} \mid |\arg\lambda| = \eta,\ |\lambda| \ge \kappa\} \cup \{\lambda \in \mathbb{C} \mid |\arg\lambda| \le \eta,\ |\lambda| = \kappa\}$ for some $\eta \in \left(\frac{\pi}{2}, \pi\right)$. Then, estimate (1-3) with $\alpha = 1$ yields

$$\|\nabla e^{-tA} f\|_{L^q_{\mathrm{uloc}}} \le C\int_\Gamma e^{t\Re(\lambda)}|\lambda|^{-1/2}\,|d\lambda|\,\|f\|_{L^q_{\mathrm{uloc}}}.$$

Since $\kappa \in (0, 1)$ is arbitrary, we may take the limit $\kappa \to 0$ and obtain

$$\|\nabla e^{-tA} f\|_{L^q_{\mathrm{uloc}}} \le C\int_0^\infty e^{-tr\cos\eta} r^{-1/2}\,dr\,\|f\|_{L^q_{\mathrm{uloc}}} \le Ct^{-1/2}\|f\|_{L^q_{\mathrm{uloc}}}, \quad t > 0.$$

The estimates of $(d/dt)e^{-tA} f$ and $\nabla^2 e^{-tA} f$ are obtained in the same manner. Note that, as for the estimate of $\nabla^2 e^{-tA} f$, we have for $t > 0$

$$\|\nabla^2 e^{-tA} f\|_{L^q_{\mathrm{uloc}}} \le C\int_0^\infty e^{-tr\cos\eta} e^{-r^{1/2}}\log r\,dr\,\|f\|_{L^q_{\mathrm{uloc}}} \le C\log(e + t)\|f\|_{L^q_{\mathrm{uloc}}}.$$

Let $1 < q < p \le \infty$. To prove (5-4) we first observe that the following the formula holds for each $m \in \mathbb{N}$ by virtue of the integration by parts in (5-7):

$$e^{-tA} f = \frac{m!}{2\pi i t^m}\int_\Gamma e^{t\lambda}(\lambda + A)^{-m-1} f\,d\lambda. \tag{5-8}$$

By taking m large enough, we can choose $\{q_j\}_{j=0}^m$ such that $q_0 = q,\ q_j < q_{j+1},\ q_m = p$, and $1/q_j - 1/q_{j+1} < 1/d$. Then, estimate (1-6) is applied for each pair (q_j, q_{j+1}), and we obtain

$$\begin{aligned}\|e^{-tA} f\|_{L^p_{\mathrm{uloc}}} &\le Ct^{-m}\int_\Gamma e^{-t\Re(\lambda)}\|(\lambda + A)^{-m-1} f\|_{L^p_{\mathrm{uloc}}}\,|d\lambda| \\ &\le Ct^{-m}\int_\Gamma e^{-t\Re(\lambda)}|\lambda|^{-m-1}(1+|\lambda|^{(d/2)(1/q_m - 1/q_{m-1})})\cdots(1+|\lambda|^{(d/2)(1/q_1 - 1/q_0)})\|f\|_{L^p_{\mathrm{uloc}}}\,|d\lambda| \\ &\le Ct^{-m}\int_\Gamma e^{-t\Re(\lambda)}|\lambda|^{-m-1}(1+|\lambda|^{(d/2)(1/p - 1/q)})\,|d\lambda|\,\|f\|_{L^q_{\mathrm{uloc}}}.\end{aligned} \tag{5-9}$$

Thus, again by taking the limit $\kappa \to 0$, we have

$$\begin{aligned}\|e^{-tA} f\|_{L^p_{\mathrm{uloc}}} &\le Ct^{-m}\int_0^\infty e^{-tr\cos\eta} r^{-m-1}(1+r^{(d/2)(1/p-1/q)})\,dr\,\|f\|_{L^q_{\mathrm{uloc}}} \\ &\le C(1+t^{-(d/2)(1/q-1/p)})\|f\|_{L^q_{\mathrm{uloc}}}. \end{aligned} \tag{5-10}$$

This proves (5-4). Estimate (5-5) is shown in the same manner by using the formula (5-8) and the resolvent estimate (1-7), we omit the details. □

6. Bilinear estimates for the Navier–Stokes equations

6A. ***The symbol of the Helmholtz–Leray projector.*** The formulas derived for the resolvent problem in Section 2C are valid for a right-hand side f in the class $L^p_{\mathrm{uloc},\sigma}$, i.e., solenoidal vector fields such that f_d vanishes on $\partial\mathbb{R}^d_+$. When dealing with the Navier–Stokes system, the nonlinear term

$$u\cdot\nabla u = \nabla\cdot(u\otimes u)$$

is such that for any $z'\in\mathbb{R}^{d-1}$

$$(u\cdot\nabla u)_d(z',0) = u(z',0)\cdot\nabla u_d(z',0) = 0$$

by the no-slip boundary condition, but it is not divergence-free. Hence, for $f\in C^\infty(\mathbb{R}^d_+)$ and $f_d=0$ on $\partial\mathbb{R}^d_+$, we have to compute the symbol of the Helmholtz–Leray projector $\mathbb{P}$ on the divergence-free fields.

In order to compute the Helmholtz–Leray projection we look for a formal decomposition of f into

$$f = \mathbb{P}f + \nabla g,$$

with

$$\nabla\cdot\mathbb{P}f = 0, \qquad (\mathbb{P}f)_d(z',0) = 0 \quad \text{for any } z'\in\mathbb{R}^{d-1}.$$

For the moment f is assumed to be smooth and decay fast enough at spatial infinity. We have to solve the problem elliptic problem for g with Neumann boundary condition

$$\begin{cases}\Delta g = \nabla\cdot f & \text{in } \mathbb{R}^d_+,\\ \nabla g\cdot e_d = f_d & \text{on } \partial\mathbb{R}^d_+,\end{cases} \tag{6-1a}$$

and such that

$$\nabla g(z',z_d)\to 0, \quad \text{when } z_d\to\infty. \tag{6-1b}$$

The solution g to (6-1) is expressed in Fourier space by, for all $\xi\in\mathbb{R}^{d-1}\setminus\{0\}$, for all $z_d>0$,

$$\begin{aligned}\hat g(\xi,z_d) &= -\frac{e^{-z_d|\xi|}}{|\xi|} f_d(\xi,0) - \int_0^\infty \frac{i\xi\cdot\hat f'(\xi,s)+\partial_d\hat f_d(\xi,s)}{2|\xi|}[e^{-|z_d-s||\xi|}+e^{-(z_d+s)|\xi|}]\,ds \\ &= -\int_0^\infty \frac{i\xi\cdot\hat f'(\xi,s)}{2|\xi|}[e^{-|z_d-s||\xi|}+e^{-(z_d+s)|\xi|}]\,ds + \frac12\int_0^{z_d}\hat f_d(\xi,s)e^{-(z_d-s)|\xi|}\,d\xi \\ &\qquad - \frac12\int_{z_d}^\infty \hat f_d(\xi,s)e^{-(s-z_d)|\xi|}\,d\xi - \frac12\int_0^\infty \hat f_d(\xi,s)e^{-(z_d+s)|\xi|}\,d\xi.\end{aligned}$$

Here we have used integration by parts. As a consequence, we obtain the formulas for the Helmholtz–Leray projection: for all $\xi \in \mathbb{R}^{d-1} \setminus \{0\}$, for all $z_d > 0$,

$$\begin{aligned}\widehat{(\mathbb{P}f)}'(\xi, z_d) = \hat{f}'(\xi, z_d) + \frac{i\xi}{2|\xi|}\int_0^\infty i\xi\cdot\hat{f}'(\xi, s)[e^{-|z_d-s||\xi|}+e^{-(z_d+s)|\xi|}]\,ds \\ -\frac{i\xi}{2}\int_0^{z_d}\hat{f}_d(\xi, s)e^{-(z_d-s)|\xi|}\,d\xi \\ +\frac{i\xi}{2}\int_{z_d}^\infty \hat{f}_d(\xi, s)e^{-(s-z_d)|\xi|}\,d\xi \\ +\frac{i\xi}{2}\int_0^\infty \hat{f}_d(\xi, s)e^{-(z_d+s)|\xi|}\,d\xi \end{aligned} \tag{6-2a}$$

and

$$\begin{aligned}\widehat{(\mathbb{P}f)}_d(\xi, z_d) = -\frac{1}{2}\int_0^{z_d} i\xi\cdot\hat{f}'(\xi, s)[e^{-(z_d-s)|\xi|}+e^{-(z_d+s)|\xi|}]\,ds \\ +\frac{1}{2}\int_{z_d}^\infty i\xi\cdot\hat{f}'(\xi, s)[e^{(z_d-s)|\xi|}-e^{-(z_d+s)|\xi|}]\,ds \\ +\frac{1}{2}\int_0^\infty \hat{f}_d(\xi, s)[e^{-|z_d-s||\xi|}-e^{-(z_d+s)|\xi|}]\,d\xi. \end{aligned} \tag{6-2b}$$

6B. ***The Helmholtz–Leray projector and the divergence.*** In view of the application to the Navier–Stokes system, we need to analyze the operator

$$F \in L^q_{\mathrm{uloc}} \mapsto \mathbb{P}\nabla\cdot F = (\mathbb{P}_{\beta\gamma}(\partial_\alpha F_{\alpha\gamma}))_{\beta=1,\dots,d},$$

rather than the Helmholtz–Leray projector $\mathbb{P}$ itself. Here we develop an approach similar to the one of [Lemarié-Rieusset 2002, Chapter 11]. An analogous method has also been used in other works concerned with nonlocalized solutions of fluid equations, such as for instance [Taniuchi et al. 2010; Ambrose et al. 2015], reminiscent of [Serfati 1995]. In the setting of the whole space $\mathbb{R}^d$, let $\chi \in C^\infty_c(\mathbb{R}^d)$ be a cut-off in physical space which is supported in $B(0, 2)$ and equal to 1 on $B(0, 1)$. The operator $\mathbb{P}\nabla\cdot$ is equal to $\nabla\cdot + D\otimes D\nabla\cdot/-\Delta$. The kernel $T_{\alpha\beta\gamma}$ of the operator $D_\alpha D_\beta D_\gamma/-\Delta$ is decomposed into

$$T_{\alpha\beta\gamma} = \partial_\alpha\partial_\beta((1-\chi)T_\gamma) + \partial_\alpha\partial_\beta(\chi T_\gamma) =: A_{\alpha\beta\gamma} + \partial_\alpha\partial_\beta B_\gamma,$$

where T_γ is the kernel associated with the operator $D_\gamma/-\Delta$. We have $A_{\alpha\beta\gamma} \in \mathrm{WL}^\infty(\mathbb{R}^d)$; i.e.,

$$\sum_{\eta\in\mathbb{Z}^{d-1}\times\mathbb{Z}_{\geq 0}} \sup_{\eta+(0,1)^d} |A_{\alpha\beta\gamma}| < \infty$$

and $B_\gamma \in L^1_c(\mathbb{R}^d)$. Hence, for any $1 \leq p < \infty$, for all $f \in L^p_{\mathrm{uloc}}(\mathbb{R}^d)$, $A_{\alpha\beta\gamma} * f \in L^p_{\mathrm{uloc}}(\mathbb{R}^d)$ and $B_\gamma * f \in L^p_{\mathrm{uloc}}(\mathbb{R}^d)$.

We now return to the case of $\mathbb{R}^d_+$. We first compute the action of $\mathbb{P}\nabla\cdot$ on $F \in C^\infty_c(\overline{\mathbb{R}}^d_+)^{d^2}$. Assume that, for all $\alpha, \gamma \in \{1, \dots, d\}$, for all $z' \in \mathbb{R}^{d-1}$,

$$F_{\alpha d}(z', 0) = F_{d\gamma}(z', 0) = \partial_d F_{dd}(z', 0) = 0. \tag{6-3}$$

Notice that we will later take F in product form, i.e., $F = u \otimes u$, but for now we stick to the general F just described. For the tangential component, we have for all $\xi \in \mathbb{R}^{d-1}$, for all $z_d > 0$, for all $\beta \in \{1, \dots, d-1\}$,

$$\begin{aligned}
\{\widehat{\mathbb{P}(\partial_\alpha(F_{\alpha\cdot}))}\}_\beta(\xi, z_d) &= \widehat{\mathbb{P}_{\beta\gamma}(\partial_\alpha(F_{\alpha\gamma}))}(\xi, z_d) \\
&= i\xi_\alpha \widehat{F}_{\alpha\beta}(\xi, z_d) + \partial_d(\widehat{F}_{d\beta})(\xi, z_d) \\
&\quad + \frac{i\xi_\beta}{2|\xi|}\int_0^\infty (-\xi_\gamma\xi_\alpha \widehat{F}_{\alpha\gamma} + i\xi_\gamma \partial_d \widehat{F}_{d\gamma})(\xi, s)[e^{-|z_d-s||\xi|} + e^{-(z_d+s)|\xi|}]\, ds \\
&\quad - \frac{i\xi_\beta}{2}\int_0^{z_d} (i\xi_\alpha \widehat{F}_{\alpha d} + \partial_d \widehat{F}_{dd})(\xi, s)e^{-(z_d-s)|\xi|}\, d\xi \\
&\quad + \frac{i\xi_\beta}{2}\int_{z_d}^\infty (i\xi_\alpha \widehat{F}_{\alpha d} + \partial_d \widehat{F}_{dd})(\xi, s)e^{-(s-z_d)|\xi|}\, d\xi \\
&\quad + \frac{i\xi_\beta}{2}\int_0^\infty (i\xi_\alpha \widehat{F}_{\alpha d} + \partial_d \widehat{F}_{dd})(\xi, s)e^{-(z_d+s)|\xi|}\, d\xi
\end{aligned}$$

Hence, integrating by parts we get

$$\begin{aligned}
\{\widehat{\mathbb{P}(\partial_\alpha(F_{\alpha\cdot}))}\}_\beta(\xi, z_d) &= i\xi_\alpha \widehat{F}_{\alpha\beta}(\xi, z_d) + \partial_d \widehat{F}_{d\beta}(\xi, z_d) - i\xi_\beta \widehat{F}_{dd}(\xi, z_d) \\
&\quad - \frac{i\xi_\alpha\xi_\beta\xi_\gamma}{2|\xi|}\int_0^\infty \widehat{F}_{\alpha\gamma}(\xi, s)[e^{-|z_d-s||\xi|} + e^{-(z_d+s)|\xi|}]\, ds \\
&\quad + \frac{i\xi_\beta|\xi|}{2}\int_0^\infty \widehat{F}_{dd}(\xi, s)[e^{-|z_d-s||\xi|} + e^{-(z_d+s)|\xi|}]\, ds \\
&\quad - \frac{i\xi_\beta}{2}\int_0^{z_d} (i\xi_\gamma \widehat{F}_{d\gamma} + i\xi_\alpha \widehat{F}_{\alpha d})(\xi, s)e^{-(z_d-s)|\xi|}\, ds \\
&\quad + \frac{i\xi_\beta}{2}\int_{z_d}^\infty (i\xi_\gamma \widehat{F}_{d\gamma} + i\xi_\alpha \widehat{F}_{\alpha d})(\xi, s)e^{-(s-z_d)|\xi|}\, ds \\
&\quad + \frac{i\xi_\beta}{2}\int_0^\infty (i\xi_\gamma \widehat{F}_{d\gamma} + i\xi_\alpha \widehat{F}_{\alpha d})(\xi, s)e^{-(z_d+s)|\xi|}\, ds. \qquad (6\text{-}4)
\end{aligned}$$

As for the vertical component, we have for all $\xi \in \mathbb{R}^{d-1}$, for all $z_d > 0$,

$$\begin{aligned}
\{\widehat{\mathbb{P}(\partial_\alpha(F_{\alpha\cdot}))}\}_d(\xi, z_d) &= \widehat{\mathbb{P}_{d\gamma}(\partial_\alpha(F_{\alpha\gamma}))}(\xi, z_d) \\
&= -\frac{1}{2}\int_0^{z_d} (-\xi_\gamma\xi_\alpha \widehat{F}_{\alpha\gamma}(\xi, s) + i\xi_\gamma \partial_d \widehat{F}_{d\gamma})(\xi, s)[e^{-(z_d-s)|\xi|} + e^{-(z_d+s)|\xi|}]\, ds \\
&\quad + \frac{1}{2}\int_{z_d}^\infty (-\xi_\gamma\xi_\alpha \widehat{F}_{\alpha\gamma}(\xi, s) + i\xi_\gamma \partial_d \widehat{F}_{d\gamma})(\xi, s)[e^{-(z_d-s)|\xi|} - e^{-(z_d+s)|\xi|}]\, ds \\
&\quad + \frac{1}{2}\int_0^\infty (i\xi_\alpha \widehat{F}_{\alpha d} + \partial_d \widehat{F}_{dd})(\xi, s)[e^{-(z_d-s)|\xi|} - e^{-(z_d+s)|\xi|}]\, ds.
\end{aligned}$$

Thus, again integrating by parts we have

$$\begin{aligned}\{\widehat{\mathbb{P}(\partial_\alpha(F_{\alpha\,\cdot}))}\}_d(\xi,z_d) = -i\xi_\gamma \widehat{F}_{d\gamma}(\xi,z_d) &+ \frac{1}{2}\int_0^{z_d}(\xi_\alpha\xi_\gamma\widehat{F}_{\alpha\gamma} - |\xi|^2\widehat{F}_{dd})(\xi,s)[e^{-(z_d-s)|\xi|}+e^{-(z_d+s)|\xi|}]\,ds\\ &- \frac{1}{2}\int_{z_d}^{\infty}(\xi_\alpha\xi_\gamma\widehat{F}_{\alpha\gamma} - |\xi|^2\widehat{F}_{dd})(\xi,s)[e^{-(s-z_d)|\xi|}-e^{-(z_d+s)|\xi|}]\,ds\\ &+ \frac{|\xi|}{2}\int_0^{z_d}(i\xi_\gamma\widehat{F}_{d\gamma}+i\xi_\alpha\widehat{F}_{\alpha d})(\xi,s)[e^{-(z_d-s)|\xi|}-e^{-(z_d+s)|\xi|}]\,ds\\ &+ \frac{|\xi|}{2}\int_{z_d}^{\infty}(i\xi_\gamma\widehat{F}_{d\gamma}+i\xi_\alpha\widehat{F}_{\alpha d})(\xi,s)[e^{-(s-z_d)|\xi|}-e^{-(z_d+s)|\xi|}]\,ds. \qquad (6\text{-}5)\end{aligned}$$

Notice that the integrations by parts carried out above are in the same vein as the decomposition of the multiplier $R(\lambda)$ of the resolvent problem (1-2) into a local part associated with the Dirichlet–Laplace operator and a nonlocal part coming from the pressure. This technique was introduced in [Desch et al. 2001]. In both situations, the goal is to get around the direct use of the Helmholtz–Leray projector $\mathbb{P}$.

We have to deal with several types of multipliers: for $\alpha,\beta,\gamma,\delta,\iota\in\{1,\dots,d-1\}$, for $\xi\in\mathbb{R}^{d-1}$,

$$\frac{\xi_\alpha\xi_\beta\xi_\gamma}{|\xi|}e^{-(z_d-s)|\xi|}\widehat{F}_{\delta\iota},\quad \xi_\alpha\xi_\beta e^{-(z_d-s)|\xi|}\widehat{F}_{\gamma\delta},\quad \xi_\alpha|\xi|e^{-(z_d-s)|\xi|}\widehat{F}_{\beta\gamma}\quad \text{for all } \xi\in\mathbb{R}^{d-1},\ z_d>s, \qquad \text{(type A)}$$

$$\frac{\xi_\alpha\xi_\beta\xi_\gamma}{|\xi|}e^{-(s-z_d)|\xi|}\widehat{F}_{\delta\iota},\quad \xi_\alpha\xi_\beta e^{-(s-z_d)|\xi|}\widehat{F}_{\gamma\delta},\quad \xi_\alpha|\xi|e^{-(s-z_d)|\xi|}\widehat{F}_{\beta\gamma}\quad \text{for all } \xi\in\mathbb{R}^{d-1},\ s>z_d, \qquad \text{(type B)}$$

$$\frac{\xi_\alpha\xi_\beta\xi_\gamma}{|\xi|}e^{-(s+z_d)|\xi|}\widehat{F}_{\delta\iota},\quad \xi_\alpha\xi_\beta e^{-(s+z_d)|\xi|}\widehat{F}_{\gamma\delta},\quad \xi_\alpha|\xi|e^{-(s+z_d)|\xi|}\widehat{F}_{\beta\gamma}\quad \text{for all } \xi\in\mathbb{R}^{d-1},\ s,z_d>0. \qquad \text{(type C)}$$

All the terms associated with the multipliers (type A), (type B) and (type C) can be handled via Lemma 6.1 below, which will allow us in Section 6C to combine the operator $\mathbb{P}\nabla\cdot$ with the operator $(\lambda+\boldsymbol{A})^{-1}$.

We develop an idea similar to the one of Lemarié-Rieusset explained above, except that rather than cutting-off in physical space, we cut-off on the Fourier side. This appears to be more convenient for us, since we will have a decomposition based on the nonlocal operator $(-\Delta')^{(2-\theta)/2}$ instead of the local derivatives $\partial_\alpha\partial_\beta$.

Lemma 6.1. *Let $\chi\in C_0^\infty(\mathbb{R}^{d-1})$ be such that $\chi=1$ for $|\xi|\le 2$ and $\chi=0$ for $|\xi|\ge 3$. Let $m_2\in C^\infty(\mathbb{R}^{d-1}\setminus\{0\})$ a multiplier in Fourier space homogeneous of order* 2, *i.e., such that, for all $t>0$, $\eta\in\mathbb{R}^{d-1}$, we have $m_2(t\eta)=t^2m_2(\eta)$, and such that for all $\xi\in\mathbb{R}^{d-1}$, for all $\alpha\in\mathbb{N}^{d-1}$,*

$$|\partial^l m_2(\xi)|\le|\xi|^{2-|\alpha|}.$$

Let $\theta\in[0,2]$ and let $K_2\in C^\infty(\mathbb{R}^d\setminus\{0\})$ (resp. $K_\theta\in C^\infty(\mathbb{R}^d\setminus\{0\})$) be the kernel associated with $m_2(\xi)e^{-t|\xi|}$ (resp. $(m_2(\xi)/|\xi|^{2-\theta})e^{-t|\xi|}$); i.e., for all $y'\in\mathbb{R}^{d-1}$ and $t>0$,

$$K_2(y',t):=\int_{\mathbb{R}^{d-1}}e^{iy'\cdot\xi}m_2(\xi)e^{-t|\xi|}\,d\xi\quad \textit{and}\quad K_\theta(y',t):=\int_{\mathbb{R}^{d-1}}e^{iy'\cdot\xi}\frac{m_2(\xi)}{|\xi|^{2-\theta}}e^{-t|\xi|}\,d\xi.$$

Then, for each $\lambda\in\mathbb{C}\setminus\{0\}$ we can decompose K_2 into

$$K_2=(-\Delta')^{(2-\theta)/2}K_{\theta,\ge|\lambda|^{1/2}}+K_{2,\le|\lambda|^{1/2}}, \qquad (6\text{-}6)$$

where the symbol of $K_{\theta,\geq|\lambda|^{1/2}}$ *is*

$$(1-\chi(|\lambda|^{-1/2}\xi))\frac{m_2(\xi)}{|\xi|^{2-\theta}}e^{-t|\xi|},$$

while the symbol of $K_{2,\leq|\lambda|^{1/2}}$ *is* $\chi(|\lambda|^{-1/2}\xi)m_2(\xi)e^{-t|\xi|}$. *Moreover, there exists* $C=C(d)>0$ *such that, for all* $y'\in\mathbb{R}^{d-1}$ *and* $t>0$,

$$|K_{\theta,\geq|\lambda|^{1/2}}(y',t)|\leq\frac{Ce^{-t|\lambda|^{1/2}}}{(|y'|+t)^{d-1+\theta}},\tag{6-7}$$

$$|K_{2,\leq|\lambda|^{1/2}}(y',t)|\leq\frac{C}{(|\lambda|^{-1/2}+|y'|+t)^{d+1}}.\tag{6-8}$$

Proof. The decomposition (6-6) simply follows from

$$1=(1-\chi(|\lambda|^{-1/2}\xi))+\chi(|\lambda|^{-1/2}\xi)$$

and

$$m_2(\xi)=|\xi|^{2-\theta}\frac{m_2(\xi)}{|\xi|^{2-\theta}}$$

for the "high"-frequency part, and thus, we focus on the proof of (6-7) and (6-8). Set

$$m_{\theta,\geq|\lambda|^{1/2}}(\xi,t):=(1-\chi(|\lambda|^{-1/2}\xi))\frac{m_2(\xi)}{|\xi|^{2-\theta}}e^{-t|\xi|},$$
$$m_{2,\leq|\lambda|^{1/2}}(\xi,t):=\chi(|\lambda|^{-1/2}\xi)m_2(\xi)e^{-t|\xi|}.$$

Then it is straightforward to show

$$|\partial_\xi^\alpha m_{\theta,\geq|\lambda|^{1/2}}(\xi,t)|\leq C|\xi|^{\theta-\alpha}e^{-(t/2)|\xi|}e^{-t|\lambda|^{1/2}},$$
$$|\partial_\xi^\alpha m_{2,\leq|\lambda|^{1/2}}(\xi,t)|\leq C|\xi|^{2-\alpha}e^{-(t/2)|\xi|}.$$

Hence, as for (6-7), if $|y'|\geq t$, then the argument in Lemma 3.1 (i.e., introduce the cut-off in the Fourier side with the radius R and optimize R later as $R=|y'|^{-1}$) gives the bound

$$|K_{\theta,\geq|\lambda|^{1/2}}(y',t)|\leq\frac{Ce^{-t|\lambda|^{1/2}}}{|y'|^{d-1+\theta}},$$

while if $t>|y'|$, we simply estimate

$$|K_{\theta,\geq|\lambda|^{1/2}}(y',t)|\leq C\int_{|\xi|\geq2|\lambda|^{1/2}}|\xi|^\theta e^{-t|\xi|}\,d\xi\leq Ct^{-d+1-\theta}e^{-t|\lambda|^{1/2}}\tag{6-9}$$

by changing the variables $t\xi=\eta$. Thus, estimate (6-7) holds. As for (6-8), let us first consider the case $|y'|+t\geq|\lambda|^{-1/2}$. If $t>|y'|$ in addition, then the simple calculation as in (6-9) gives the bound

$$|K_{2,\leq|\lambda|^{1/2}}(y',t)|\leq Ct^{-d-1}.$$

While if $|y'| > t$, then the argument as in Lemma 3.1 yields

$$|K_{2,\le|\lambda|^{1/2}}(y',t)| \le \frac{C}{|y'|^{d+1}}.$$

Finally, if $|\lambda|^{-1/2} > |y'| + t$, then we have

$$|K_{2,\le|\lambda|^{1/2}}(y',t)| \le C \int_{|\xi|\le 3|\lambda|^{1/2}} |\xi|^2 \, d\xi \le C|\lambda|^{(d+1)/2}.$$

Collecting these, we obtain (6-8). □

We now estimate the action of $K_{\theta,\ge|\lambda|^{1/2}}$ and $K_{2,\le|\lambda|^{1/2}}$ on functions in $L^q_{\mathrm{uloc}}(\mathbb{R}^d_+)$.

Lemma 6.2. *Let $p, q \in [1,\infty]$ satisfy*

$$1 \le q \le p \le \infty, \quad 0 \le \frac{1}{q} - \frac{1}{p} < \frac{1}{d}.$$

Then there exist $\theta \in (0,1)$ and $C = C(d,p,q,\theta) > 0$ such that, for all $\lambda \in \mathbb{C} \setminus \{0\}$, $f \in L^q_{\mathrm{uloc}}(\mathbb{R}^d_+)$, and $y_d > 0$, we have

$$\left\| \int_0^{y_d} \int_{\mathbb{R}^{d-1}} K_{\theta,\ge|\lambda|^{1/2}}(y'-z', y_d \pm z_d) f(z',z_d)\, dz'\, dz_d \right\|_{L^p_{\mathrm{uloc}}} \le C|\lambda|^{-(1-\theta)/2}(1+|\lambda|^{(d/2)(1/q-1/p)}) \|f\|_{L^q_{\mathrm{uloc}}},$$

$$\left\| \int_0^{y_d} \int_{\mathbb{R}^{d-1}} K_{2,\le|\lambda|^{1/2}}(y'-z', y_d \pm z_d) f(z',z_d)\, dz'\, dz_d \right\|_{L^q_{\mathrm{uloc}}} \le C|\lambda|^{1/2} \|f\|_{L^q_{\mathrm{uloc}}},$$

and

$$\left\| \int_{y_d}^{\infty} \int_{\mathbb{R}^{d-1}} K_{\theta,\ge|\lambda|^{1/2}}(y'-z', y_d \pm z_d) f(z',z_d)\, dz'\, dz_d \right\|_{L^p_{\mathrm{uloc}}} \le C|\lambda|^{-(1-\theta)/2}(1+|\lambda|^{(d/2)(1/q-1/p)}) \|f\|_{L^q_{\mathrm{uloc}}},$$

$$\left\| \int_{y_d}^{\infty} \int_{\mathbb{R}^{d-1}} K_{2,\le|\lambda|^{1/2}}(y'-z', y_d \pm z_d) f(z',z_d)\, dz'\, dz_d \right\|_{L^q_{\mathrm{uloc}}} \le C|\lambda|^{1/2} \|f\|_{L^q_{\mathrm{uloc}}}.$$

Proof. Let $s \in [1,\infty]$ such that $1/p = 1/q + 1/s - 1$. By the condition $1/q - 1/p < 1/d$ we can take $\theta \in (0,1)$ so that $(d-1+\theta)s < d$. We fix such a $\theta \in (0,1)$ below. To show the estimates stated in the lemma it suffices to consider the case with the variable $y_d - z_d$. Then, by virtue of the bounds (6-7) and (6-8) all terms are reduced to the estimate for the convolution $K_\lambda * f$ in $\mathbb{R}^d$ with K_λ either

$$K_\lambda(y', y_d) = \frac{Ce^{-|y_d||\lambda|^{1/2}}}{(|y'| + |y_d|)^{d-1+\theta}} \quad \text{(for the terms involving } K_{\theta,\ge|\lambda|^{1/2}}) \tag{6-10}$$

or

$$K_\lambda(y', y_d) = \frac{C}{(|\lambda|^{-1/2} + |y'| + |y_d|)^{d+1}} \quad \text{(for the terms involving } K_{2,\le|\lambda|^{1/2}}). \tag{6-11}$$

Note that f is extended by zero to $\mathbb{R}^d$. Then, the proof is parallel to that of Lemma 4.1. Without loss of generality it suffices to estimate the L^p norm on the cube $B_0 = (0,1)^d$. The case when K_λ is given by

(6-11) is easily estimated. Indeed,

$$\begin{aligned}\biggl(\int_0^1\int_{(0,1)^{d-1}}\biggl|\int_0^{z_d}\int_{\mathbb{R}^{d-1}} K_\lambda(y'-z',y_d-z_d)f(z',z_d)\,dz'\,dz_d\biggr|^q dy'\,dy_d\biggr)^{1/q}& \\ \le C\sum_{\eta\in\mathbb{Z}^d}\biggl(\int_0^1\int_{(0,1)^{d-1}}\biggl|\frac{1}{\bigl(|\lambda|^{-1/2}+|\cdot|\bigr)^{d+1}} *_y \chi_\eta f\biggr|^q dy'\,dy_d\biggr)^{1/q}& \\ \le C\sum_{\eta\in\mathbb{Z}^d}\frac{1}{(|\lambda|^{-1/2}+|\eta'|+|\eta_d|)^{d+1}}\|f\|_{L^q_{\mathrm{uloc}}}& \\ \le C\int_0^\infty \frac{r^{d-1}}{(|\lambda|^{-1/2}+r)^{d+1}}\,dr \le C|\lambda|^{1/2}\|f\|_{L^q_{\mathrm{uloc}}}.&\end{aligned}$$

Next we consider the case when K_λ is given by (6-10). Let $1/p=1/s+1/q-1$. Then, arguing as in the proof of Lemma 4.1, we have from the Young inequality for convolution

$$\|K_\lambda *_y f\|_{L^p(B_0)} \le C\biggl(\|K_\lambda\|_{L^s}+\sum_{\substack{\max|\beta_i'|\ge 3\\ \beta_d\in\mathbb{Z}}}\|\chi_\beta K_\lambda\|_{L^s}+\sum_{\substack{\max|\beta_i'|\le 3\\ |\beta_d|\ge 3}}\|\chi_\beta K_\lambda\|_{L^s}\biggr)\|f\|_{L^q_{\mathrm{uloc}}}.$$

Here χ_β is the characteristic function on the cube B_β (see the proof of Lemma 4.1). By virtue of the choice of $\theta\in(0,1)$ above, we see that $K_\lambda\in L^s(\mathbb{R}^d)$ and

$$\|K_\lambda\|_{L^s}\le C|\lambda|^{-(1-\theta)/2+(d/2)(1/q-1/p)}.$$

Similarly, the direct computation yields

$$\begin{aligned}\sum_{\substack{\max|\beta_i'|\ge 3\\ \beta_d\in\mathbb{Z}}}\|\chi_\beta K_\lambda\|_{L^s}&\le C\int_0^\infty e^{-ct|\lambda|^{1/2}}(1+t)^{-\theta}\,dt\le C|\lambda|^{-(1-\theta)/2}\\ \sum_{\substack{\max|\beta_i'|\le 3\\ |\beta_d|\ge 3}}\|\chi_\beta K_\lambda\|_{L^s}&\le Ce^{-c|\lambda|^{1/2}}\le C|\lambda|^{-(1-\theta)/2}.\end{aligned}$$

□

6C. ***The bilinear operator.*** The goal of this section is to study the bilinear operator

$$(u,v)\mapsto(\lambda+\boldsymbol{A})^{-1}\mathbb{P}\nabla\cdot(u\otimes v). \tag{6-12}$$

The idea is to combine the results on the operator $(\lambda+\boldsymbol{A})^{-1}$ obtained in Section 4 with the decomposition and estimates for $\mathbb{P}\nabla\cdot$ obtained in Section 6B.

Let $F:=u\otimes v$ with $u,\ v\in L^p_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$. The outcome of the computations (6-4) and (6-5) as well as Lemmas 6.1, 6.2 is the following proposition.

Proposition 6.3. *Let $\lambda\in S_{\pi-\varepsilon}$ and let $u,\ v\in L^p_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$. Assume that $p,q\in[1,\infty]$ satisfy*

$$1<q\le p\le\infty \quad or \quad 1\le q<p\le\infty,\quad 0\le\frac1q-\frac1p<\frac1d.$$

Then there exist $\theta \in (0,1)$ and $G_{\theta,\geq|\lambda|^{1/2}}(u\otimes v)$, $G_{\leq|\lambda|^{1/2}}(u\otimes v) \in L^1_{\mathrm{loc}}(\mathbb{R}^d_+;\mathbb{R}^d)$ such that, for all $\beta \in \{1,\dots,d-1\}$,

$$\{\mathbb{P}\partial_\alpha(u_\alpha v)\}_\beta = \partial_\gamma(u_\gamma v_\beta)+\partial_d(u_d v_\beta)-\partial_\beta(u_d v_d)+(-\Delta')^{(2-\theta)/2}G^\beta_{\theta,\geq|\lambda|^{1/2}}(u\otimes v)+G^\beta_{\leq|\lambda|^{1/2}}(u\otimes v), \quad (6\text{-}13a)$$

$$\{\mathbb{P}\partial_\alpha(u_\alpha v)\}_d = -\partial_\gamma(u_d v_\gamma)+(-\Delta')^{(2-\theta)/2}G^d_{\theta,\geq|\lambda|^{1/2}}(u\otimes v)+G^d_{\leq|\lambda|^{1/2}}(u\otimes v), \quad (6\text{-}13b)$$

where Einstein's convention is used (the sums run over $\alpha \in \{1,\dots,d\}$ and $\gamma \in \{1,\dots,d-1\}$), and such that

$$\|G_{\theta,\geq|\lambda|^{1/2}}(u\otimes v)\|_{L^p_{\mathrm{uloc}}} \leq C|\lambda|^{-(1-\theta)/2}(1+|\lambda|^{(d/2)(1/q-1/p)})\|u\otimes v\|_{L^q_{\mathrm{uloc}}}, \quad (6\text{-}14)$$

$$\|G_{\leq|\lambda|^{1/2}}(u\otimes v)\|_{L^q_{\mathrm{uloc}}} \leq C|\lambda|^{1/2}\|u\otimes v\|_{L^q_{\mathrm{uloc}}}. \quad (6\text{-}15)$$

Here $C = C(d,\varepsilon,p,q) > 0$ is independent of $\lambda \in S_{\pi-\varepsilon}$.

The only thing which remains to be done so as to estimate the bilinear operator (6-12) is to combine the result of Proposition 6.3 with the kernel bounds of Section 3 and the estimates of Section 4. Doing so we obtain the important result stated below.

Proposition 6.4. *Let $\lambda \in S_{\pi-\varepsilon}$. For all $p, q \in [1,\infty]$ satisfying*

$$1 < q \leq p \leq \infty \quad \text{or} \quad 1 \leq q < p \leq \infty, \qquad 0 \leq \frac{1}{q} - \frac{1}{p} < \frac{1}{d},$$

there exists $C = C(d,\varepsilon,p,q) > 0$ (independent of λ) such that, for all u, $v \in L^p_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+;\mathbb{R}^d)$,

$$\|(\lambda+\boldsymbol{A})^{-1}\mathbb{P}\nabla\cdot(u\otimes v)\|_{L^p_{\mathrm{uloc}}} \leq C|\lambda|^{-1/2}(1+|\lambda|^{(d/2)(1/q-1/p)})\|u\otimes v\|_{L^q_{\mathrm{uloc}}}. \quad (6\text{-}16)$$

Moreover, if ∇u, $\nabla v \in L^q_{\mathrm{uloc}}(\mathbb{R}^d)$ in addition, then

$$\|\nabla(\lambda+\boldsymbol{A})^{-1}\mathbb{P}\nabla\cdot(u\otimes v)\|_{L^q_{\mathrm{uloc}}} \leq C|\lambda|^{-1/2}(\|u\nabla v\|_{L^q_{\mathrm{uloc}}} + \|v\nabla u\|_{L^q_{\mathrm{uloc}}}). \quad (6\text{-}17)$$

Remark 6.5. As for (6-17), we can also show

$$\|\nabla(\lambda+\boldsymbol{A})^{-1}\mathbb{P}\nabla\cdot(u\otimes v)\|_{L^q_{\mathrm{uloc}}} \leq C|\lambda|^{-1/2}(1+|\lambda|^{(d/2)(1/q-1/p)})(\|u\nabla v\|_{L^q_{\mathrm{uloc}}} + \|v\nabla u\|_{L^q_{\mathrm{uloc}}}) \quad (6\text{-}18)$$

for $1 < q \leq p \leq \infty$ or $1 \leq q < p \leq \infty$ satisfying in addition $0 \leq 1/q - 1/p < 1/d$. The proof is the same as in the case $p = q$, but we state the proof only for the simplest case (6-17) in this paper.

Proof of Proposition 6.4. Proof of (6-16): All the ingredients are already proved, we just have to indicate how they fit together. The key point is the formulas (6-13). The idea is that we integrate by parts in the formulas (2-9) and then we use the estimates of Propositions 3.2, 3.5, 4.2, 4.3 and 6.3. The estimates of the nonlocalized Lebesgue norms follow exactly from the bounds in Section 4. In particular, we recall the resolvent $(\lambda+\boldsymbol{A})^{-1}$ consists of the Dirichlet–Laplace part (Section 4A), and the nonlocal part (Section 4B), that is,

$$(\lambda+\boldsymbol{A})^{-1} = \boldsymbol{R}_{\mathrm{D.L.}}(\lambda) + \boldsymbol{R}_{\mathrm{n.l.}}(\lambda).$$

The operators $\boldsymbol{R}_{\text{D.L.}}(\lambda)$ and $\boldsymbol{R}_{\text{n.l.}}(\lambda)$ respectively satisfy the estimates in Propositions 4.2 and 4.3, that is, for $\alpha = 0, 1$,

$$\|\nabla^\alpha \boldsymbol{R}_{\text{D.L.}}(\lambda) f\|_{L^p_{\text{uloc}}} + \|\nabla^\alpha \boldsymbol{R}_{\text{n.l.}}(\lambda) f\|_{L^p_{\text{uloc}}} \le C|\lambda|^{-(2-\alpha)/2}(1+|\lambda|^{(d/2)(1/q-1/p)})\|f\|_{L^q_{\text{uloc}}},$$

and

$$\|m_\alpha(D')\boldsymbol{R}_{\text{D.L.}}(\lambda) f\|_{L^q_{\text{uloc}}} + \|m_\alpha(D')\boldsymbol{R}_{\text{n.l.}}(\lambda) f\|_{L^q_{\text{uloc}}} \le C|\lambda|^{-(2-\alpha)/2}\|f\|_{L^q_{\text{uloc}}}, \quad \alpha \in (0,2),$$

$$\|m_\alpha(D')\nabla\boldsymbol{R}_{\text{D.L.}}(\lambda) f\|_{L^q_{\text{uloc}}} + \|m_\alpha(D')\nabla\boldsymbol{R}_{\text{n.l.}}(\lambda) f\|_{L^q_{\text{uloc}}} \le C|\lambda|^{-(1-\alpha)/2}\|f\|_{L^q_{\text{uloc}}}, \quad \alpha \in (0,1).$$

Here $m_\alpha(D')$ is any Fourier multiplier, homogeneous of order α. Moreover, we also have

$$\|\boldsymbol{R}_{\text{D.L.}}(\lambda)\partial_d f\|_{L^p_{\text{uloc}}} + \|\boldsymbol{R}_{\text{n.l.}}(\lambda)\partial_d f\|_{L^p_{\text{uloc}}} \le C|\lambda|^{-1/2}(1+|\lambda|^{(d/2)(1/q-1/p)})\|f\|_{L^q_{\text{uloc}}}, \tag{6-19}$$

and

$$\|m_\alpha(D')\boldsymbol{R}_{\text{D.L.}}(\lambda)\partial_d f\|_{L^q_{\text{uloc}}} + \|m_\alpha(D')\boldsymbol{R}_{\text{n.l.}}(\lambda)\partial_d f\|_{L^q_{\text{uloc}}} \le C|\lambda|^{-(1-\alpha)/2}\|f\|_{L^q_{\text{uloc}}}, \quad \alpha \in (0,1),$$

if $f \in C_0^\infty(\mathbb{R}^d_+;\mathbb{R}^d)$, by the integration by parts in (2-9) and applying the derivative estimates of the associated kernels. Thus, $\boldsymbol{R}_{\text{D.L.}}(\lambda)\partial_d$ and $\boldsymbol{R}_{\text{n.l.}}(\lambda)\partial_d$ are extended to bounded operators from $L^q_{\text{uloc}}(\mathbb{R}^d_+;\mathbb{R}^d)$ to $L^p_{\text{uloc}}(\mathbb{R}^d_+;\mathbb{R}^d)$ with p, q satisfying $0 \le 1/q - 1/p < 1/d$ together with the bound (6-19). Indeed any function in $L^q_{\text{uloc}}(\mathbb{R}^d_+)$ is approximated by a sequence of functions in $C_0^\infty(\mathbb{R}^d_+)$ in the topology of $L^q_{\text{loc}}(\mathbb{R}^d_+)$ with a uniform bound in the norm of $L^q_{\text{uloc}}(\mathbb{R}^d_+)$. This extension with the estimate (6-19) is applied to the term $\partial_d(u_d v_\beta)$ in the formula (6-13). This concludes the proof of (6-16).

Proof of (6-17): We first observe that the proof of (6-16) in fact ensures the existence of the number $\delta_0 \in (0,1)$ such that, for any $\delta \in (0, \delta_0]$,

$$\|m_\delta(D')(\lambda+\boldsymbol{A})^{-1}\mathbb{P}\nabla\cdot(u\otimes v)\|_{L^q_{\text{uloc}}} \le C|\lambda|^{-(1-\delta)/2}\|u\otimes v\|_{L^q_{\text{uloc}}}, \tag{6-20}$$

where $m_\delta(D')$ is any Fourier multiplier, homogeneous of order δ. Indeed, for $\theta \in (0,1)$ in Proposition 6.3 we can take δ_0 such that $\delta_0 \in (0,\theta)$. We will use (6-20) later.

Since the tangential derivatives commute with $(\lambda+\boldsymbol{A})^{-1}$ and $\mathbb{P}$, estimate (6-16) implies

$$\|\nabla'(\lambda+\boldsymbol{A})^{-1}\mathbb{P}\nabla\cdot(u\otimes v)\|_{L^q_{\text{uloc}}} \le C|\lambda|^{-1/2}\|\nabla'(u\otimes v)\|_{L^q_{\text{uloc}}},$$

and thus, combining with (6-20), we also have

$$\|m_\delta(D')\nabla'(\lambda+\boldsymbol{A})^{-1}\mathbb{P}\nabla\cdot(u\otimes v)\|_{L^q_{\text{uloc}}} \le C|\lambda|^{-(1-\delta)/2}\|\nabla'(u\otimes v)\|_{L^q_{\text{uloc}}} \tag{6-21}$$

for $\delta \in (0,\delta_0]$. Next we consider the estimate of $\partial_d(\lambda+\boldsymbol{A})^{-1}\mathbb{P}\nabla\cdot(u\otimes v)$. Set $U = (\lambda+\boldsymbol{A})^{-1}\mathbb{P}\nabla\cdot(u\otimes v)$. Then the divergence-free condition implies $\partial_d U_d = -\nabla'\cdot U'$, and hence, the estimate of $\partial_d U_d$ follows from the estimate for the tangential derivatives which are already shown. It remains to estimate $\partial_d U'$. To this end, we note that, by regarding the associated pressure $\nabla' p$ as the source term, the vector U' is also written as

$$U_\beta = (\lambda+\Delta_D)^{-1}(\partial_\beta p + \nabla\cdot(u v_\beta))$$

for $\beta \in \{1, \dots, d-1\}$. Here $(\lambda + \Delta_D)^{-1}$ denotes the resolvent for the Dirichlet–Laplace operator (hence, $\boldsymbol{R}_{\text{D.L.}}(\lambda)$ in the proof of (6-16) above), for which we have already established the estimates in L^p_{uloc} spaces in Section 4A. In particular, we have

$$\|\partial_d(\lambda + \Delta_D)^{-1}\nabla \cdot (uv_\beta)\|_{L^q_{\text{uloc}}} \leq \frac{C}{|\lambda|^{1/2}}\|\nabla \cdot (uv_\beta)\|_{L^q_{\text{uloc}}}.$$

As for the term $\partial_d(\lambda + \Delta_D)^{-1}\partial_\beta p$ we have from the integration by parts in the kernel representation,

$$\partial_d(\lambda + \Delta_D)^{-1}\partial_\beta p = (\lambda + \Delta_N)^{-1}\partial_\beta \partial_d p = \partial_\beta(\lambda + \Delta_N)^{-1}(-\lambda U_d + \Delta U_d + \nabla \cdot (uv_d)). \tag{6-22}$$

Here $(\lambda + \Delta_N)^{-1}$ denotes the resolvent of the Neumann–Laplace operator, for which we have clearly the same estimates as for the resolvent of the Dirichlet–Laplace operator, since the argument in Section 4A is based only on the pointwise estimates of the kernel function. Hence the first term of the right-hand side of (6-22) is estimated as

$$\|\lambda\partial_\beta(\lambda + \Delta_N)^{-1}U_d\|_{L^q_{\text{uloc}}} \leq C\|\partial_\beta U_d\|_{L^q_{\text{uloc}}} \leq C|\lambda|^{-1/2}\|\partial_\beta(u \otimes v)\|_{L^q_{\text{uloc}}},$$

and the third term is estimated as

$$\|\partial_\beta(\lambda + \Delta_N)^{-1}\nabla \cdot (uv_d)\|_{L^q_{\text{uloc}}} \leq C|\lambda|^{-1/2}\|\nabla \cdot (uv_d)\|_{L^q_{\text{uloc}}}.$$

Finally, we note that $\Delta U_d = \Delta' U_d - \partial_d \nabla' \cdot U'$. Then

$$\begin{aligned}\|\partial_\beta(\lambda + \Delta_N)^{-1}\Delta' U_d\|_{L^q_{\text{uloc}}} &= \|(-\Delta')^{(2-\delta)/2}(\lambda + \Delta_N)^{-1}(-\Delta')^{\delta/2}\partial_\beta U_d\|_{L^q_{\text{uloc}}} \\ &\leq C|\lambda|^{-\delta/2}\|(-\Delta')^{\delta/2}\partial_\beta U_d\|_{L^q_{\text{uloc}}} \\ &\leq C\|\lambda|^{-1/2}\|\partial_\beta(u \otimes v)\|_{L^q_{\text{uloc}}},\end{aligned}$$

and similarly,

$$\begin{aligned}\|\partial_\beta(\lambda + \Delta_N)^{-1}\partial_d \nabla' \cdot U'\|_{L^q_{\text{uloc}}} &= \|(-\Delta')^{(1-\delta)/2}\partial_d(\lambda + \Delta_D)^{-1}\partial_\beta(-\Delta')^{-(1-\delta)/2}\nabla' \cdot U'\|_{L^q_{\text{uloc}}} \\ &\leq C|\lambda|^{-\delta/2}\|\partial_\beta(-\Delta')^{-(1-\delta)/2}\nabla' \cdot U'\|_{L^q_{\text{uloc}}} \\ &\leq C|\lambda|^{-1/2}\|\nabla'(uv)\|_{L^q_{\text{uloc}}}.\end{aligned}$$

Here we have regarded $\partial_\beta(-\Delta')^{-(1-\delta)/2}$ as the Fourier multiplier, homogeneous of order δ and applied the estimate (6-21). The proof of (6-17) is complete. □

It remains to transfer the stationary bounds of Proposition 6.4 to the nonstationary operator $e^{-t\boldsymbol{A}}\mathbb{P}\nabla\cdot$. These bounds are stated in Theorem 3 in the Introduction. We now prove this theorem.

Proof of Theorem 3. Proof of (1-8): By the semigroup property $e^{-t\boldsymbol{A}} = e^{-(t/2)\boldsymbol{A}}e^{-(t/2)\boldsymbol{A}}$ and

$$\|\nabla e^{-(t/2)\boldsymbol{A}} f\|_{L^p_{\text{uloc}}} \leq Ct^{-1/2}\|f\|_{L^p_{\text{uloc}}},$$

it suffices to consider the case $\alpha = 0$. As in the estimate of $e^{-t\boldsymbol{A}}$, we use the formula

$$e^{-t\boldsymbol{A}}\mathbb{P}\nabla \cdot (u \otimes v) = \frac{1}{2\pi i}\int_\Gamma e^{t\lambda}(\lambda + \boldsymbol{A})^{-1}\mathbb{P}\nabla \cdot (u \otimes v)\, d\lambda.$$

Here the curve Γ is taken as in the proof of Proposition 5.3. Assume that $p, q \in [1, \infty]$ satisfy $0 \le 1/q - 1/p < 1/d$. Then we have from (6-16),

$$\begin{aligned}\|e^{-tA}\mathbb{P}\nabla\cdot(u\otimes v)\|_{L^p_{\mathrm{uloc}}} &\le C\int_\Gamma |e^{t\lambda}|\,|\lambda|^{-1/2}(1+|\lambda|^{(d/2)(1/q-1/p)})\,|d\lambda|\,\|u\otimes v\|_{L^q_{\mathrm{uloc}}}\\ &\le Ct^{-1/2}(1+t^{-(d/2)(1/q-1/p)})\|u\otimes v\|_{L^q_{\mathrm{uloc}}},\end{aligned}$$

as in the computation of (5-10). For general p, q we use the same trick as in (5-10), and then combine the estimate of $(\lambda + A)^{-1}$ and $(\lambda + A)^{-1}\mathbb{P}\nabla\cdot$. The details are omitted here. The proof of (1-8) is complete.

Proof of (1-9): In this case it suffices to use (6-17) instead of (6-16). Thus we omit the details. □

7. Mild solutions in $L^q_{\mathrm{uloc},\sigma}$, $q \ge d$

In this section we consider the Navier–Stokes equations in $\mathbb{R}^d_+$

$$\begin{cases}\partial_t u - \Delta u + \nabla p = -\nabla\cdot(u\otimes u), \quad \nabla\cdot u = 0 \quad \text{in } (0,T)\times\mathbb{R}^d_+,\\ u = 0 \quad \text{on } (0,T)\times\partial\mathbb{R}^d_+, \qquad u|_{t=0} = u_0 \quad \text{in } \partial\mathbb{R}^d_+.\end{cases} \tag{7-1}$$

The corresponding integral equation is

$$u(t) = e^{-tA}u_0 - \int_0^t e^{-(t-s)A}\mathbb{P}\nabla\cdot(u\otimes u)\,ds, \qquad t > 0, \tag{7-2}$$

and the solution to this integral equation is called the mild solution. The existence of such solutions was pioneered in [Fujita and Kato 1964]. Our objective is to prove the short-time existence of the mild solution for nondecaying data. In view of the scaling of the equation, a natural class for the initial data is $L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$ with $q \ge d$. In principle, one can prove the existence in a short time for initial data of arbitrary size if $q > d$, and locally in time for small data if $q = d$. We note that, contrary to the L^d space, the global existence for small data in L^d_{uloc} is not expected in this functional framework since L^d_{uloc} contains nondecaying functions. Another issue in the framework of L^q_{uloc} spaces is the meaning of the initial condition, for the domain of A is not dense in $L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$. The convergence to the initial data as $t \to 0$ in the topology of $L^q_{\mathrm{uloc}}(\mathbb{R}^d_+)$ is achieved if and only if the initial data is taken from $\overline{D(A)}^{L^q_{\mathrm{uloc}}}$, where $D(A)$ denotes the domain of the Stokes operator in $L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$. Thanks to the result of [Desch et al. 2001, Theorem 4.3] on the L^∞ theory of the Stokes operator in the half-space, the embedding $L^\infty(\mathbb{R}^d_+) \subseteq L^q_{\mathrm{uloc}}(\mathbb{R}^d_+)$ implies that

$$\overline{\mathrm{BUC}_\sigma(\mathbb{R}^d_+)}^{L^q_{\mathrm{uloc}}} \subseteq \overline{D(A)}^{L^q_{\mathrm{uloc}}}, \qquad \overline{C^\infty_{0,\sigma}(\mathbb{R}^d_+)}^{L^q_{\mathrm{uloc}}} \subseteq \overline{D(A)}^{L^q_{\mathrm{uloc}}}.$$

If the initial data is taken from these spaces with $q \ge d$ (the case $q = d$ is allowed), then by using the density argument one can show the short-time existence of the mild solution which satisfies the initial condition in the topology of $L^q_{\mathrm{uloc}}(\mathbb{R}^d_+)$. These facts are now quite standard. In this paper we state the local existence results for (7-2) without going into the details on the meaning of the initial condition.

7A. *Existence of mild solutions for initial data in $L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$, $q > d$.*

Proposition 7.1. *For any $q > d$ and $u_0 \in L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$ there exists a unique mild solution*

$$u \in L^\infty\big(0,T; L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)\big) \cap C\big((0,T); W^{1,q}_{\mathrm{uloc},0}(\mathbb{R}^d_+)^d \cap \mathrm{BUC}_\sigma(\mathbb{R}^d_+)\big)$$

such that

$$\sup_{0<t<T} (\|u(t)\|_{L^q_{\mathrm{uloc}}} + t^{d/(2q)}\|u(t)\|_{L^\infty} + t^{1/2}\|\nabla u(t)\|_{L^q_{\mathrm{uloc}}}) \le C_* \|u_0\|_{L^q_{\mathrm{uloc}}}.$$

Moreover there exists a constant $\gamma > 0$ depending only on d and q such that T can taken be as

$$T^{1/2+d/(2q)} + T^{1/2-d/(2q)} \ge \frac{\gamma}{\|u_0\|_{L^q_{\mathrm{uloc}}}}.$$

Proof. The proof is based on the standard Banach fixed-point theorem. Set $\|f\|_T$ as

$$\|f\|_T = \sup_{0<t<T} (\|f(t)\|_{L^q_{\mathrm{uloc}}} + t^{d/(2q)}\|f(t)\|_{L^\infty} + t^{1/2}\|\nabla f(t)\|_{L^q_{\mathrm{uloc}}}).$$

Let $C_0 > 0$ be a constant such that

$$\|e^{-\cdot\, \boldsymbol{A}} f\|_T \le C_0(1 + T^{d/(2q)})\|f\|_{L^q_{\mathrm{uloc}}}, \quad f \in L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+),$$

which is well-defined by virtue of Proposition 5.3. Then let us introduce the set

$$X_T = \bigl\{ f \in L^\infty\bigl(0, T; L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)\bigr) \cap C\bigl((0, T); W^{1,q}_{\mathrm{uloc},0}(\mathbb{R}^d_+; \mathbb{R}^d) \cap \mathrm{BUC}_\sigma(\mathbb{R}^d_+)\bigr) \\ \mid \|f\|_T \le 2C_0(1 + T^{d/(2q)})\|u_0\|_{L^q_{\mathrm{uloc}}} \bigr\}.$$

For each $f \in X_T$ we define the map $\Phi[f](t) = e^{-t\boldsymbol{A}}u_0 + B[f, f](t)$, where

$$B[f, g](t) = -\int_0^t e^{-(t-s)\boldsymbol{A}} \mathbb{P}\nabla \cdot (f \otimes g)\, ds, \quad t > 0, \ f, g \in X_T.$$

We will show that if T is sufficiently small, then Φ defines a contraction map in X_T. Theorem 3 yields for $f, g \in X_T$,

$$\begin{aligned}\|B[f, g](t)\|_{L^q_{\mathrm{uloc}}} &\le C \int_0^t (t-s)^{-1/2}\|f \otimes g\|_{L^q_{\mathrm{uloc}}}\, ds \\ &\le C \int_0^t (t-s)^{-1/2}s^{-d/(2q)}\, ds \sup_{0<s<t} s^{d/(2q)}\|f(s)\|_{L^\infty} \sup_{0<s<t} \|g(s)\|_{L^q_{\mathrm{uloc}}}.\end{aligned}$$

Similarly, we have, for $f, g \in X_T$,

$$\begin{aligned}\|B[f, g](t)\|_{L^\infty} &\le C \int_0^t (t-s)^{-1/2}((t-s)^{-d/(2q)}+1)\|f\otimes g(s)\|_{L^q_{\mathrm{uloc}}}\, ds \\ &\le C(t^{1/2-d/q}+t^{1/2-d/(2q)}) \sup_{0<s<t} s^{d/(2q)}\|f(s)\|_{L^\infty} \sup_{0<s<t} (\|g(s)\|_{L^q_{\mathrm{uloc}}}+s^{d/(2q)}\|g(s)\|_{L^\infty}),\end{aligned}$$

and

$$\begin{aligned}&\|\nabla B[f, g](t)\|_{L^q_{\mathrm{uloc}}} \\ &\le C\int_0^{t/2} (t-s)^{-1}\|f\otimes g(s)\|_{L^q_{\mathrm{uloc}}}\, ds + C\int_{t/2}^t (t-s)^{-1/2}(\|f\nabla g(s)\|_{L^q_{\mathrm{uloc}}} + \|g\nabla f(s)\|_{L^q_{\mathrm{uloc}}})\, ds \\ &\le Ct^{-d/(2q)}\bigl(\sup_{0<s<t} s^{d/(2q)}\|f(s)\|_{L^\infty} \sup_{0<s<t} \|g(s)\|_{L^q_{\mathrm{uloc}}} + \sup_{0<s<t} s^{d/(2q)}\|f(s)\|_{L^\infty} \sup_{0<s<t} s^{1/2}\|\nabla g(s)\|_{L^q_{\mathrm{uloc}}} \\ &\qquad + \sup_{0<s<t} s^{1/2}\|\nabla f(s)\|_{L^q_{\mathrm{uloc}}} \sup_{0<s<t} s^{d/(2q)}\|g(s)\|_{L^\infty}\bigr).\end{aligned}$$

Thus we obtain

$$\|B[f,g]\|_T \le C_1(T^{1/2-d/(2q)} + T^{1/2})\|f\|_T\|g\|_T, \quad f,g \in X_T.$$

The continuity in time for $t \in (0,T)$ also follows from that of f, g in $(0,T)$, and we skip the details here. If T is small so that

$$C_1(T^{1/2-d/(2q)} + T^{1/2})2C_0(1+T^{d/(2q)})\|u_0\|_{L^q_{\mathrm{uloc}}} \le \tfrac{1}{4}, \tag{7-3}$$

then (7-3) and the definition of C_0 imply that Φ defines a contraction map from X_T into X_T. Hence, there exists a unique fixed point u of Φ in X_T, which is the unique mild solution to (7-1) in X_T. □

7B. *Existence of mild solutions for initial data in $L^d_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$.* The first result is stated in any dimension $d \ge 2$. Below we define $W^{1,d}_{\mathrm{uloc},0}(\mathbb{R}^d_+)$ as

$$W^{1,d}_{\mathrm{uloc},0}(\mathbb{R}^d_+) = \{f \in L^d_{\mathrm{uloc}}(\mathbb{R}^d_+) \mid \nabla f \in L^d_{\mathrm{uloc}}(\mathbb{R}^d_+),\ f|_{x_d=0} = 0\}.$$

Proposition 7.2. *For any $T > 0$ there exist ε, $C_* > 0$ such that the following statement holds. For any $u_0 \in L^d_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$ satisfying $\|u_0\|_{L^d_{\mathrm{uloc}}} \le \varepsilon$ there exists a unique mild solution*

$$u \in L^\infty\big(0,T; L^d_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)\big) \cap C\big((0,T); W^{1,d}_{\mathrm{uloc},0}(\mathbb{R}^d_+)^d \cap \mathrm{BUC}_\sigma(\mathbb{R}^d_+)\big)$$

such that

$$\sup_{0<t<T}(\|u(t)\|_{L^d_{\mathrm{uloc}}} + t^{1/2}\|u(t)\|_{L^\infty} + t^{1/2}\|\nabla u(t)\|_{L^d_{\mathrm{uloc}}}) \le C_*\|u_0\|_{L^d_{\mathrm{uloc}}}.$$

If $u_0 \in \overline{D(A)}^{L^d_{\mathrm{uloc}}}$ in addition, then $\lim_{t\to 0} u(t) = u_0$ in $L^d_{\mathrm{uloc}}(\mathbb{R}^d_+)^d$.

Remark 7.3. As usual, by using the density argument, we do not need to assume the smallness of $\|u_0\|_{L^d_{\mathrm{uloc}}}$ to show the short-time existence of the mild solution if u_0 belongs of $\overline{D(A)}^{L^d_{\mathrm{uloc}}}$.

Proof. The proof is based on the standard Banach fixed-point theorem. We fix $T > 0$. Set $\|f\|_T$ as

$$\|f\|_T := \sup_{0<t<T}(\|f(t)\|_{L^d_{\mathrm{uloc}}} + t^{1/2}\|f(t)\|_{L^\infty} + t^{1/2}\|\nabla f(t)\|_{L^d_{\mathrm{uloc}}}).$$

Let $C_0 > 0$ be a constant such that

$$\|e^{-\cdot A}f\|_T \le C_0(1+T^{1/2})\|f\|_{L^d_{\mathrm{uloc}}}, \quad f \in L^d_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+),$$

which is well-defined by virtue of Proposition 5.3. Then let us introduce the set

$$X_T := \big\{f \in L^\infty\big(0,T; L^d_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)\big) \cap C\big((0,T); W^{1,d}_{\mathrm{uloc},0}(\mathbb{R}^d_+;\mathbb{R}^d) \cap \mathrm{BUC}_\sigma(\mathbb{R}^d_+)\big) \;\big|\; \|f\|_T \le 2C_0(1+T^{1/2})\|u_0\|_{L^d_{\mathrm{uloc}}}\big\}.$$

For each $f \in X_T$ we define the map $\Phi[f](t) = e^{-tA}u_0 + B[f,f](t)$, where

$$B[f,g](t) = -\int_0^t e^{-(t-s)A}\mathbb{P}\nabla\cdot(f\otimes g)\,ds, \quad t>0,\ f,g \in X_T.$$

We will show that if $\|u_0\|_{L^d_{\mathrm{uloc}}} \le \varepsilon$ and $\varepsilon > 0$ is sufficiently small then Φ defines a contraction map in X_T. Theorem 3 yields for $f, g \in X_T$,

$$\begin{aligned} \|B[f,g](t)\|_{L^d_{\mathrm{uloc}}} &\le C\int_0^t (t-s)^{-1/2}\|f\otimes g\|_{L^d_{\mathrm{uloc}}}\,ds \\ &\le C\int_0^t (t-s)^{-1/2}s^{-1/2}\,ds \sup_{0<s<t} s^{1/2}\|f(s)\|_{L^\infty} \sup_{0<s<t}\|g(s)\|_{L^d_{\mathrm{uloc}}}. \end{aligned} \tag{7-4}$$

Similarly, we have for $f, g \in X_T$,

$$\begin{aligned} \|B[f,g](t)\|_{L^\infty} &\le C\int_0^{t/2}(t-s)^{-1/2}((t-s)^{-1/2}+1)\|f\otimes g(s)\|_{L^d_{\mathrm{uloc}}}\,ds + C\int_{t/2}^t (t-s)^{-1/2}\|f\otimes g(s)\|_{L^\infty}\,ds \\ &\le C(t^{-1/2}+1)\sup_{0<s<t} s^{1/2}\|f(s)\|_{L^\infty}\sup_{0<s<t}(\|g(s)\|_{L^d_{\mathrm{uloc}}} + s^{1/2}\|g(s)\|_{L^\infty}), \end{aligned}$$

and

$$\begin{aligned} &\|\nabla B[f,g](t)\|_{L^d_{\mathrm{uloc}}} \\ &\le C\int_0^{t/2}(t-s)^{-1}\|f\otimes g(s)\|_{L^d_{\mathrm{uloc}}}\,ds + C\int_{t/2}^t (t-s)^{-1/2}(\|f\nabla g(s)\|_{L^d_{\mathrm{uloc}}} + \|g\nabla f(s)\|_{L^d_{\mathrm{uloc}}})\,ds \\ &\le C(t^{-1/2}+1)\Big(\sup_{0<s<t} s^{1/2}\|f(s)\|_{L^\infty}\sup_{0<s<t}\|g(s)\|_{L^d_{\mathrm{uloc}}} + \sup_{0<s<t} s^{1/2}\|f(s)\|_{L^\infty}\sup_{0<s<t} s^{1/2}\|\nabla g(s)\|_{L^d_{\mathrm{uloc}}} \\ &\qquad + \sup_{0<s<t} s^{1/2}\|\nabla f(s)\|_{L^d_{\mathrm{uloc}}}\sup_{0<s<t} s^{1/2}\|g(s)\|_{L^\infty}\Big). \end{aligned} \tag{7-5}$$

Thus we obtain

$$\|B[f,g]\|_T \le C_1(1+T^{1/2})\|f\|_T\|g\|_T, \quad f, g \in X_T.$$

The continuity in time for $t \in (0, T)$ also follows from that of f, g in $(0, T)$, and we skip the details here. If ε is small so that

$$C_1(1+T^{1/2})2C_0(1+T^{1/2})\varepsilon \le \tfrac{1}{4}, \tag{7-6}$$

then (7-6) and the definition of C_0 imply that Φ defines a contraction map from X_T into X_T. Hence, there exists a unique fixed point u of Φ in X_T, which is the unique mild solution to (7-2) in X_T. If $u_0 \in \overline{D(\boldsymbol{A})}^{L^d_{\mathrm{uloc}}}$ then we just modify the set X_T as

$$\begin{aligned} \widetilde{X}_T = \big\{ f \in C\big([0,T); L^d_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)\big) \cap C\big((0,T); W^{1,d}_{\mathrm{uloc},0}(\mathbb{R}^d_+;\mathbb{R}^d) \cap \mathrm{BUC}_\sigma(\mathbb{R}^d_+)\big) \\ \big|\ \|f\|_T \le 2C_0(1+T^{1/2})\|u_0\|_{L^d_{\mathrm{uloc}}},\ \lim_{t\to 0} t^{1/2}\|f(t)\|_{L^\infty} = 0\big\}. \end{aligned}$$

Then the estimates (7-4)–(7-5) yield

$$\lim_{t\to 0} t^{1/2}\|B[f,g](t)\|_{L^\infty} = \lim_{t\to 0}\|B[f,g](t)\|_{L^d_{\mathrm{uloc}}} = 0,$$

when $f, g \in \widetilde{X}_T$, and we can construct the unique mild solution in $\widetilde{X}_T$. The details are omitted here. □

For the next result, we specialize to $d = 3$.

Proposition 7.4. *For any $u_0 \in \mathcal{L}^3_{\mathrm{uloc},\sigma}(\mathbb{R}^3_+)$ there exist $T > 0$ and a unique mild solution*

$$u \in C\big([0,T); \mathcal{L}^3_{\mathrm{uloc}}(\mathbb{R}^3_+)\big) \cap C\big((0,T); W^{1,5}_{\mathrm{uloc},0}(\mathbb{R}^3_+) \cap \mathrm{BUC}_\sigma(\mathbb{R}^3_+)\big)$$

such that

$$\sup_{0<t<T} (\|u(t)\|_{L^3_{\mathrm{uloc}}} + t^{1/5}\|u(t)\|_{L^5} + t^{7/10}\|\nabla u(t)\|_{L^5_{\mathrm{uloc}}}) \le C_* \|u_0\|_{L^3_{\mathrm{uloc}}}.$$

Proof. The proof is based on the argument in [Kato 1984]. Set $\|f\|_T$ as

$$\|f\|_T := \sup_{0<t<T} (t^{1/5}\|f(t)\|_{L^5_{\mathrm{uloc}}} + t^{7/10}\|\nabla f(t)\|_{L^5_{\mathrm{uloc}}}).$$

For any $\varepsilon > 0$ there exists $\tilde{u}_0 \in C^\infty_{0,\sigma}(\mathbb{R}^3_+)$ such that $\|u_0 - \tilde{u}_0\|_{L^3_{\mathrm{uloc}}} < \varepsilon$. Therefore

$$\begin{aligned} t^{1/5}\|e^{-tA}u_0\|_{L^5_{\mathrm{uloc}}} &\le t^{1/5}(\|e^{-tA}(u_0 - \tilde{u}_0)\|_{L^5_{\mathrm{uloc}}} + \|e^{-tA}\tilde{u}_0\|_{L^5_{\mathrm{uloc}}}) \\ &\le C(1 + t^{1/5})\|u_0 - \tilde{u}_0\|_{L^3_{\mathrm{uloc}}} + t^{1/5}\|\tilde{u}_0\|_{L^5_{\mathrm{uloc}}}. \end{aligned}$$

Similarly

$$t^{7/10}\|\nabla e^{-tA}u_0\|_{L^5_{\mathrm{uloc}}} \le C(1 + t^{1/5})\|u_0 - \tilde{u}_0\|_{L^3_{\mathrm{uloc}}} + t^{1/5}\|\tilde{u}_0\|_{L^5_{\mathrm{uloc}}}.$$

Therefore there exist C_0 and $T_0 > 0$ such that for $T < T_0$

$$\|e^{-\cdot A}u_0\|_T \le C_0 \varepsilon.$$

Let us introduce the set

$$Y_T = \big\{ f \in L^\infty\big(0,T; L^3_{\mathrm{uloc},\sigma}(\mathbb{R}^3_+)\big) \cap C\big((0,T); W^{1,3}_{\mathrm{uloc},0}(\mathbb{R}^d_+; \mathbb{R}^3) \cap \mathrm{BUC}_\sigma(\mathbb{R}^3_+)\big) \;\big|\; \|f\|_T \le 2C_0\varepsilon \big\}.$$

A similar argument to that in the proof of Proposition 7.1 shows

$$\|B[f,g]\|_T \le C_1(1 + T^{1/5})\|f\|_T\|g\|_T, \quad f, g \in X_T. \tag{7-7}$$

Therefore if T and ε are small so that

$$C_1(1 + T^{1/5})2C_0\varepsilon \le \tfrac{1}{2},$$

Φ defines a contraction map from Y_T into itself. Hence, there exists a unique fixed point u of Φ in Y_T, which is the unique mild solution to (7-1) in Y_T. We also easily see $u \in L^\infty(0,T; L^3_{\mathrm{uloc}})$ as follows:

$$\begin{aligned} \|u(t)\|_{L^3_{\mathrm{uloc}}} &\le \|u_0\|_{L^3_{\mathrm{uloc}}} + \|B[u,u](t)\|_{L^3_{\mathrm{uloc}}} \\ &\le \|u_0\|_{L^3_{\mathrm{uloc}}} + C(1 + T^{1/10})\|u\|_T^2. \end{aligned}$$

We will show that u belongs to $C([0,T); \mathcal{L}^3_{\mathrm{uloc},\sigma}(\mathbb{R}^3_+))$. It is enough to show u is strongly continuous on $[0,T)$ in L^3_{uloc}. Since the continuity away from $t = 0$ can be shown as stated in the proof of Proposition 7.1, we focus on the case when $t = 0$. We have

$$\begin{aligned} \|u(t) - u_0\|_{L^3_{\mathrm{uloc}}} &\le \|e^{-tA}u_0 - u_0\|_{L^3_{\mathrm{uloc}}} + C\|B[u,u](t)\|_{L^3_{\mathrm{uloc}}} \\ &\le \|e^{-tA}u_0 - u_0\|_{L^3_{\mathrm{uloc}}} + C\|u\|_T^2. \end{aligned}$$

The standard density argument yields that the first term converges to 0 as $t \to 0$, while in the second term, (7-7) implies $\lim_{T\to 0} \|u\|_T = 0$. □

7C. *Concentration of L^q norm, $q \geq d$, near the blow up time.* This subsection is devoted to the proof of Corollary 1.1. We introduce the space $L^q_{\mathrm{uloc},(\rho)}(\mathbb{R}^d_+)$ for $\rho > 0$ which is defined as

$$L^q_{\mathrm{uloc},(\rho)}(\mathbb{R}^d_+) := \Big\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^d_+) \;\Big|\; \sup_{\eta \in \mathbb{Z}^{d-1}\times\mathbb{Z}_{\geq 0}} \|f\|_{L^q(\rho\eta+(0,\rho)^d)} < \infty \Big\}.$$

The following variant of Proposition 7.1 and 7.2 plays a crucial role in the proof.

Proposition 7.5. *Let $q \geq d$. There exist constants $\gamma,\ C_* > 0$ such that the following statement holds. For any $u_0 \in L^q_{\mathrm{uloc},(\rho),\sigma}(\mathbb{R}^d_+)$ satisfying $\|u_0\|_{L^q_{\mathrm{uloc},(\rho)}} \leq \gamma \rho^{d/q-1}$ for some $\rho > 0$, there exist $T \geq \rho^2$ and a unique mild solution $u \in L^\infty(0, T; L^q_{\mathrm{uloc},(\rho),\sigma}(\mathbb{R}^d_+))$ such that*

$$\sup_{0<t<T} \big(\|u(t)\|_{L^q_{\mathrm{uloc},(\rho)}} + t^{d/(2q)}\|u(t)\|_{L^\infty}\big) \leq C_*\|u_0\|_{L^q_{\mathrm{uloc},(\rho)}}. \tag{7-8}$$

This can be proved by a simple rescaling argument from Propositions 7.1 and 7.2, and so we omit the details here. This results enables to control the existence time in terms of the smallness of the initial data in $L^q_{\mathrm{uloc},(\rho)}$.

Proof of Corollary 1.1. Let $q \geq d$ and $\gamma = \gamma(q) > 0$ be given by Proposition 7.5. We define $\rho_* = \rho_*(t)$ for $t \in (0, T_*)$ by

$$\rho_*(t) = \inf\{\rho > 0 \mid \|u(t)\|_{L^q_{\mathrm{uloc},(\rho)}} \rho^{1-d/q} > \gamma\}.$$

Note that ρ_* is finite since $u \not\equiv 0$, and $\rho_* > 0$ since $u(t)$ is bounded for all $t \in (0, T_*)$. For $t \in (0, T_*)$ fixed (our new initial time) let $\rho > 0$ be a constant such that $\rho < \rho_*(t)$. It suffices to show that $\rho \leq \sqrt{T_* - t}$. Since (by the definition of ρ) $\|u(t)\|_{L^q_{\mathrm{uloc},(\rho)}} \leq \gamma\rho^{d/q-1}$, Proposition 7.5 shows existence of the solution v in $[t, t+T]$ such that at initial time $v(t) = u(t)$ and $T \geq \rho^2$. Assume for a moment that u agrees with v in $[t, T']$ for $T' = \min(t+T, T_* - \varepsilon)$ for small $\varepsilon > 0$. Then by the definition of T_* we must have $t + T < T_* - \varepsilon$. Since ε is arbitrary, this yields the desired estimate for ρ. As for the uniqueness, we see from the continuity of u on $(0, T_*)$ that there exists a constant $\delta > 0$ such that

$$\sup_{t<s<t+\delta} \big(\|u(s)\|_{L^q_{\mathrm{uloc},(\rho)}} + s^{d/(2q)}\|u(s)\|_{L^\infty}\big) \leq C_*\|u(t)\|_{L^q_{\mathrm{uloc},(\rho)}}.$$

Hence the uniqueness in $[t, t+\delta]$ follows from Proposition 7.5. To show the uniqueness up to time T', notice that u and v are both bounded in $\mathbb{R}^d_+ \times [t+\delta, T']$. Then the bilinear estimate shows that the difference $w := u - v$ satisfies

$$\begin{aligned}\|w\|_{L^\infty(t_1,t_2;L^\infty)} &\leq C(t_2-t_1)^{1/2}\big(\|u\|_{L^\infty(t_1,t_2;L^\infty)} + \|v\|_{L^\infty(t_1,t_2;L^\infty)}\big)\|w\|_{L^\infty(t_1,t_2;L^\infty)} \\ &\leq C(t_2-t_1)^{1/2}\big(\|u\|_{L^\infty(t+\delta,T';L^\infty)} + \|v\|_{L^\infty(t+\delta,T';L^\infty)}\big)\|w\|_{L^\infty(t_1,t_2;L^\infty)}\end{aligned}$$

for $t+\delta \leq t_1 < t_2 \leq T'$. Thus taking $t_2 - t_1$ sufficiently small shows $w = 0$ on $[t_1, t_2]$. Using this argument a finite number of times, we have the uniqueness in $[t, T']$. □

Appendix A: A Liouville theorem for the resolvent problem in L^1_{uloc} spaces

This appendix is devoted to the proof of the Liouville theorem, Theorem 4 for the Stokes resolvent problem.

Proof of Theorem 4. (i) First we introduce the regularization of (u, p) in x' as follows:

$$u^\kappa(x', x_d) = \int_{\mathbb{R}^{d-1}} \kappa^{-(d-1)} \eta\left(\frac{x'-y'}{\kappa}\right) u(y', x_d)\, dy',$$
$$p^\kappa(x', x_d) = \int_{\mathbb{R}^{d-1}} \kappa^{-(d-1)} \eta\left(\frac{x'-y'}{\kappa}\right) p(y', x_d)\, dy'.$$

Here $\kappa \in (0, 1)$ and $\eta \in C_0^\infty(\mathbb{R}^{d-1})$ satisfies $\int_{\mathbb{R}^{d-1}} \eta(x')\, dx' = 1$ and $\operatorname{supp} \eta \subseteq B_1(0')$. Then, by the symmetry of $\mathbb{R}^d_+$, (u^κ, p^κ) is also a solution to (1-2) with $f = 0$ in the sense of distributions. Moreover, $(u^\kappa(\,\cdot\,, x_d), \nabla p^\kappa(\,\cdot\,, x_d))$ is smooth in x' and satisfies the estimate

$$\|\nabla'^\alpha u^\kappa\|_{L^\infty_{x'}(\mathbb{R}^{d-1}; L^1_{\mathrm{uloc}}(\mathbb{R}_+))} \le C\kappa^{-d+1-|\alpha|} \|u\|_{L^1_{\mathrm{uloc}}},$$
$$\|\nabla'^\alpha \nabla p^\kappa\|_{L^\infty_{x'}(\mathbb{R}^{d-1}; L^1_{\mathrm{uloc}}(\mathbb{R}_+))} \le C\kappa^{-d+1-|\alpha|} \|\nabla p\|_{L^1_{\mathrm{uloc}}}$$

for any multi-index α. We can also check that $p^\kappa \in L^1_{\mathrm{loc}}(\mathbb{R}^d_+)$ without difficulty. The divergence-free condition on u^κ then implies

$$\partial_{x_d} u^\kappa_d = -\nabla' \cdot (u^\kappa)' \in L^\infty(\mathbb{R}^{d-1}; L^1_{\mathrm{uloc}}(\mathbb{R}_+))$$

in the sense of distributions, which implies that u^κ_d is continuous up to the boundary. Then, again from the divergence-free condition, $\int_{\mathbb{R}^d_+} u^\kappa \cdot \nabla\phi\, dx = 0$ for all $\phi \in C_0^\infty(\bar{\mathbb{R}}^d_+)$, we have $\lim_{x_d \to 0} u^\kappa_d(x', x_d) = 0$ for all $x' \in \mathbb{R}^{d-1}$. Next we take an arbitrary $g \in C_0^\infty(\mathbb{R}^d_+)^d$, and let $(v, \nabla q)$ be the smooth and decaying solution to (1-2) with $f = \Delta' g$. By virtue of the presence of Δ' one can show that $(v, \nabla q)$ is constructed so that

$$\nabla' v(x), \nabla^\alpha \Delta' v(x) = \mathcal{O}(|x|^{-d-1/2}), \quad \nabla^{\tilde\alpha} \Delta' q(x) = \mathcal{O}(|x|^{-d-1/2}), \quad |x| \gg 1, \tag{A-1}$$

for any multi-indexes α and $\tilde\alpha$ with $|\alpha| \le 2$ and $|\tilde\alpha| \le 1$; see Proposition A.1 below. Then, by Theorem 4 we see

$$\begin{aligned}
\int_{\mathbb{R}^d_+} u^\kappa \cdot (\Delta')^2 g\, dx &= \int_{\mathbb{R}^d_+} u^\kappa \cdot \Delta'(\lambda v - \Delta v + \nabla q)\, dx \\
&= -\int_{\mathbb{R}^d_+} \nabla p^\kappa \cdot \Delta' v\, dx + \int_{\mathbb{R}^d_+} u^\kappa \cdot \nabla \Delta' q\, dx \\
&= \sum_{j=1}^{d-1} \int_{\mathbb{R}^d_+} \nabla \partial_j p^\kappa \cdot \partial_j v\, dx = -\sum_{j=1}^{d-1} \int_{\mathbb{R}^d_+} \partial_j p^\kappa \partial_j \operatorname{div} v\, dx = 0
\end{aligned} \tag{A-2}$$

from the definition of the solution in the sense of distributions. Note that the above integration by parts is justified from (A-1) and from the fact that ∇p^κ and $\partial_j \nabla p^\kappa$ with $j = 1, \dots, d-1$ belong to

$L^\infty_{x'}(\mathbb{R}^{d-1}; L^1_{\mathrm{uloc}}(\mathbb{R}_+))$. Since $g \in C_0^\infty(\mathbb{R}^d_+)^d$ is arbitrary, this identity implies that

$$(\Delta')^2 u^\kappa = 0 \quad \text{a.e. } x \in \mathbb{R}^d_+.$$

Set $U^\kappa(x', x_d) = \int_0^{x_d} u^\kappa(x', y_d)\, dy_d$. Recall that $\nabla'^\alpha u^\kappa(\,\cdot\,, x_d) \in L^\infty_{x'}(\mathbb{R}^{d-1}; L^1_{\mathrm{uloc}}(\mathbb{R}_+))$, which shows that for each fixed $x_d \geq 0$, $U^\kappa(x', x_d)$ is smooth and bounded including its derivatives in x', while it is absolutely continuous in x_d for each fixed x'. Moreover, for each $x_d \geq 0$, $U^\kappa(\,\cdot\,, x_d)$ satisfies $(\Delta')^2 U^\kappa(\,\cdot\,, x_d) = 0$. By the Liouville theorem of the bi-Laplace equation in $\mathbb{R}^{d-1}$, we conclude that $U^\kappa(\,\cdot\,, x_d)$ is constant in x' for each $x_d \geq 0$, that is, $U^\kappa(x', x_d) = A(x_d)$. Since the left-hand side is absolutely continuous in x_d, so is A, and we have

$$u^\kappa(x', x_d) = \partial_{x_d} U^\kappa(x', x_d) = \partial_{x_d} A(x_d) =: a^\kappa(x_d) \in L^1_{\mathrm{uloc}}(\mathbb{R}_+)^d, \quad a^\kappa_d \in C(\overline{\mathbb{R}}_+). \tag{A-3}$$

Then, the divergence-free condition implies that $\partial_{x_d} a^\kappa_d = 0$, and thus, together with the fact

$$\lim_{x_d \to 0} u^\kappa_d(x', x_d) = 0,$$

we have $a^\kappa_d = 0$. Next we take $\phi \in C_0^\infty(\mathbb{R}^d_+)$ and set $\varphi = (0, \ldots, 0, \phi)^\top \in C_0^\infty(\mathbb{R}^d_+)^d$ in (2-3), which yields from $u^\kappa_d = 0$,

$$\int_{\mathbb{R}^d_+} \partial_{x_d} p^\kappa \phi\, dx = 0.$$

Thus, p^κ does not depend on x_d. On the other hand, by taking $\varphi = \nabla\phi$ with $\phi \in C_0^\infty(\mathbb{R}^d_+)$ in (2-3), it follows that

$$\int_{\mathbb{R}^d_+} \nabla p^\kappa \cdot \nabla\phi\, dx = 0, \quad \phi \in C_0^\infty(\mathbb{R}^d_+).$$

Hence, p^κ is harmonic in $\mathbb{R}^d_+$, and moreover, since p^κ is independent of x_d, we have $\Delta' p^\kappa(x') = 0$ for all $x' \in \mathbb{R}^{d-1}$. The Liouville theorem implies that p^κ is a harmonic polynomial. Then, going back to (2-3) and using (A-3), we have, for each $j = 1, \ldots, d-1$,

$$\int_{\mathbb{R}_+} a^\kappa_j(x_d)(\lambda\phi - \partial^2_{x_d}\phi)(x_d)\, dx_d \int_{\mathbb{R}^{d-1}} \psi(x')\, dx' \\ = -\int_{\mathbb{R}_+} \phi(x_d)\, dx_d \int_{\mathbb{R}^{d-1}} \partial_j p^\kappa(x')\psi(x')\, dx \quad \text{for all } \psi \in C_0^\infty(\mathbb{R}^{d-1}),\ \phi \in C_0^\infty(\mathbb{R}_+). \tag{A-4}$$

We first fix ϕ such that $\int_{\mathbb{R}_+} \phi\, dx_d \neq 0$. Since $\psi \in C_0^\infty(\mathbb{R}^{d-1})$ is arbitrary, $\partial_j p^\kappa(x')$ is constant for all $j = 1, \ldots, n-1$. Hence, $p^\kappa(x')$ is polynomial at most first order about x'. Thus, (u^κ, p^κ) is a parasitic solution. Since (u^κ, p^κ) converges to (u, p) in $L^1_{\mathrm{loc}}(\mathbb{R}^d_+)$, the limit (u, p) must be a parasitic solution. Note that the limit (u, p) with $u = (a'(x_d), 0)^\top$ and $p = D \cdot x' + c$ satisfies the reduced version of (A-4):

$$\int_{\mathbb{R}_+} a_j(x_d)(\lambda\phi - \partial^2_{x_d}\phi)(x_d)\, dx_d = -D_j \int_{\mathbb{R}} \phi\, dx_d \quad \text{for all } \phi \in C_0^\infty(\overline{\mathbb{R}}_+) \text{ with } \phi|_{x_d=0} = 0. \tag{A-5}$$

In particular, each a_j is smooth and bounded and has a zero boundary trace, for λ which belongs to the resolvent set of the Dirichlet Laplacian in $L^1(\mathbb{R}_+)$. Moreover, if

$$\lim_{R\to\infty} \|\nabla' p\|_{L^1(|x'|<1,\, R<x_d<R+1)} = 0$$

then the vector D must be 0, and thus the pressure p is a constant. Then (A-5) is reduced to

$$\int_{\mathbb{R}_+} a_j(x_d)(\lambda\phi - \partial_{x_d}^2\phi)(x_d)\,dx_d = 0 \quad \text{for all } \phi \in C_0^\infty(\overline{\mathbb{R}}_+) \text{ with } \phi|_{x_d=0} = 0. \tag{A-6}$$

The uniqueness of this very weak solution is standard and also follows from the fact that λ belongs to the resolvent set of the Dirichlet Laplacian in $L^1(\mathbb{R}_+)$. Thus we have arrived at $a_j = 0$ for each $j = 1, \dots, d-1$; that is, $u = 0$. On the other hand, if $\lim_{|y'|\to\infty} \|u\|_{L^1(|x'-y'|<1,1<x_d<2)} = 0$ then $u = 0$ in $1 < x_d < 2$ since $u = (a'(x_d), 0)^\top$ is independent of x', which also gives $D = 0$ by (A-5). Thus p is a constant. Hence a_j satisfies (A-6) also in this case, which gives $u = 0$ for $x_d > 0$. The proof of (i) is complete.

(ii) The proof is similar to that of (i). Again it suffices to consider the mollified solution (u^κ, p^κ) as in (i). Fix an arbitrary $\mu \in (0, 1)$ and let $(v_\mu, \nabla q_\mu)$ be the smooth and decaying solution to (1-2) with $\lambda = \mu$ and $f = \Delta' g$, where $g \in C_0^\infty(\mathbb{R}^d_+)^d$ is arbitrarily taken. Then (A-2) is replaced by

$$\int_{\mathbb{R}^d_+} u^\kappa \cdot (\Delta')^2 g\,dx = \mu \int_{\mathbb{R}^d_+} u^\kappa \cdot \Delta' v_\mu\,dx. \tag{A-7}$$

We observe from Proposition A.1 that $|\Delta' v_\mu(x)| \le C|\mu|^{-3/4}(1+|x|)^{-d-1/2}$ with C independent of $\mu \in (0, 1)$, and thus, we can take the limit $\mu \downarrow 0$, which leads to $(\Delta')^2 u^\kappa = 0$. Then the same argument as in (i) shows that $u^\kappa = (a'(x_d), 0)^\top$ and $p^\kappa = p^\kappa(x')$ satisfy (A-4) with $\lambda = 0$, which implies that p^κ is a first-order polynomial, and we have (A-5) with $\lambda = 0$. Then each $a_j(x_d)$ is smooth and satisfies $\partial_{x_d}^2 a_j = D_j$ with the Dirichlet boundary condition $a_j(0) = 0$. Since $u \in L^1_{\mathrm{uloc}}(\mathbb{R}^d_+)^d$ such an a_j must be zero, and thus $u = 0$ and then we also see from the equation that p is a constant. The proof of (ii) is complete. □

Proposition A.1. *Let $\lambda \in S_{\pi-\varepsilon}$ with $\varepsilon \in (0, \pi)$. Let $g \in C_0^\infty(\mathbb{R}^d_+)^d$. Then there exists a unique solution*

$$(u, \nabla p) \in \big(W^{2,2}(\mathbb{R}^d_+)^d \cap W_0^{1,2}(\mathbb{R}^d_+)^d \cap L^2_\sigma(\mathbb{R}^d_+)\big) \times L^2(\mathbb{R}^d_+)^d$$

to (1-2) *with $f = \Delta' g$ such that u and p are smooth and satisfy*

$$\begin{aligned} |\nabla'^\alpha u(x)| + |\nabla'^\alpha \nabla u(x)| &\le C_\varepsilon \left(\frac{1}{|\lambda|^{1/4}} + \frac{1}{|\lambda|^{3/4}}\right) \frac{1}{(1+|x|)^{d+1/2}}, \\ |\nabla'^\alpha \nabla^2 u(x)| + |\nabla^{\tilde\alpha} \nabla'^2 p(x)| &\le \frac{C_{\varepsilon,\lambda}}{(1+|x|)^{d+1/2}} \end{aligned} \tag{A-8}$$

for any multi-indices α and $\tilde\alpha$ with $|\alpha| \le 2$ and $|\tilde\alpha| \le 1$. Here the constant C_ε is taken uniformly in $\lambda \in S_{\pi-\varepsilon}$, while $C_{\varepsilon,\lambda}$ depends on ε and $\lambda \in S_{\pi-\varepsilon}$.

Notice that the uniform estimates in $|\lambda|$ is used in the proof of (ii) of Theorem 4.

Proof. The uniqueness is well known and we focus on the estimate (A-8). The Helmholtz decomposition implies $g = h + \nabla p_g$, where $h \in L^2_\sigma(\mathbb{R}^d_+)$ and $\nabla p_g \in L^2(\mathbb{R}^d_+)^d$ with $p_g \in L^2_{\mathrm{loc}}(\mathbb{R}^d_+)$. Since $g \in C_0^\infty(\mathbb{R}^d_+)^d$, ∇p_g and $h = g - \nabla p_g$ are also smooth and bounded in $\mathbb{R}^d_+$ including their derivatives. Then the pressure ∇p is constructed in the form $\nabla p = \Delta' \nabla p_g + \nabla p_u$, where $(u, \nabla p_u)$ is the unique solution to (1-2) with $f = \Delta' h$. We first show that

$$\nabla^\beta \nabla' \nabla p_g(x), \quad \nabla^\beta \nabla' h(x) = \mathcal{O}(|x|^{-d-1}), \quad |x| \gg 1, \tag{A-9}$$

for any multi-index β. The estimate of h follows from the one of ∇p_g by the relation $h = g - \nabla p_g$. To show (A-9) for ∇p_g, we recall that p_g is constructed as the solution to the Neumann problem $\Delta p_g = \operatorname{div} g$ in $\mathbb{R}^d_+$ and $\partial_{x_d} p_g = g_d = 0$ on $\partial \mathbb{R}^d_+$, which is given by the formula

$$p_g(x) = -\int_{\mathbb{R}^d_+} (E(x-y) + E(x-y^*)) \operatorname{div} g(y)\, dy,$$

where $y^* = (y', -y_d)$, and $E(x)$ is the Newton potential in $\mathbb{R}^d$. Then, the integration by parts and the condition $g \in C_0^\infty(\mathbb{R}^d_+)^d$ yield

$$p_g(x) = -\nabla' \cdot \int_{\mathbb{R}^d_+} (E(x-y) + E(x-y^*)) g'(y)\, dy - \partial_{x_d} \int_{\mathbb{R}^d_+} (E(x-y) - E(x-y^*)) g_d(y)\, dy.$$

Hence, we obtain the formula

$$\nabla' \nabla p_g(x) = -\nabla' \nabla \nabla' \cdot \int_{\mathbb{R}^d_+} (E(x-y) + E(x-y^*)) g'(y)\, dy - \nabla' \nabla \partial_{x_d} \int_{\mathbb{R}^d_+} (E(x-y) - E(x-y^*)) g_d(y)\, dy.$$

Since $|\nabla^\beta \nabla^3 E(x)| \le C|x|^{-d-1-|\beta|}$, we verify the estimate (A-9) for $|x| \gg 1$ when $g \in C_0^\infty(\mathbb{R}^d_+)^d$. Next we consider the estimate of $(u, \nabla p_u)$. We can now apply the results of Section 3, in particular the integral representation formulas and the kernel estimates given there. That is, u is written as

$$u(x) = \int_{\mathbb{R}^d_+} K_\lambda(x' - y', x_d, y_d) \Delta' h(y)\, dy,$$

with the kernel K_λ whose pointwise estimates are given in Proposition 3.2 for $k_{i,\lambda}$ with $i = 1, 2$ and in Proposition 3.5 for r_λ. Since ∂_j with $j = 1, \dots, d-1$ commutes with the Stokes operator in $\mathbb{R}^d_+$, we verify the bound

$$|u(x)| \le \frac{C}{|\lambda|^{1/4}} \int_{\mathbb{R}^d_+} \frac{1}{|x-y|^{d-1}(1 + |\lambda|^{1/2}|x-y|)} |\nabla' h(y)|\, dy \tag{A-10}$$

with C independent of $\lambda \in S_{\pi-\varepsilon}$. Indeed, by virtue of (3-18), the kernel $|\nabla' r_\lambda(y, z_d)|$ is bounded from above by

$$\frac{C}{|\lambda|^{1/4}(y_d + z_d + |y'|)^{d-1/2}(1 + |\lambda|^{1/2}(y_d + z_d + |y'|))(1 + |\lambda|^{1/2}(y_d + z_d))^{1/2}}.$$

A similar bound is valid also for the kernel $\nabla' k_{i,\lambda}$, $i = 1, 2$, by Proposition 3.2. Estimate (A-10) implies from (A-9) that

$$|u(x)| \le C\left(\frac{1}{|\lambda|^{1/4}} + \frac{1}{|\lambda|^{3/4}}\right) \frac{1}{(1 + |x|)^{d+1/2}}. \tag{A-11}$$

Since the tangential derivatives commute with the kernel, we have the same estimate for $|\nabla'^{\alpha} u(x)|$ as in (A-11). The estimate for the x_d derivative is a bit more complicated. We decompose the kernel K_λ as $K_\lambda(y', y_d, z_d) = \chi(y')K_\lambda(y', y_d, z_d) + (1-\chi(y'))K_\lambda(y', y_d, z_d)$, where $\chi(y')$ is a smooth cut-off such that $\chi(y') = 1$ for $|y'| < 1$ and $\chi(y') = 2$ for $|y'| \geq 2$. Then we compute

$$\partial_{x_d} u(x) = \int_{\mathbb{R}^d_+} \chi \partial_{x_d} K_\lambda(x'-y', x_d, y_d)\Delta' h(y)\, dy + \int_{\mathbb{R}^d_+} \nabla'_x((1-\chi)\partial_{x_d} K_\lambda(x'-y', x_d, y_d)) \cdot \nabla' h(y)\, dy,$$

and then the former term is estimated from above by

$$\frac{C}{|\lambda|^{1/4}} \int_{|x'-y'|<1} \frac{1}{|x-y|^{d-1/2}(1+|\lambda|^{1/2}|x_d-y_d|)^{3/2}} |\Delta' h(y)|\, dy,$$

and the latter is bounded by

$$C \int_{|x'-y'|\geq 1} \frac{1}{(1+|x-y|)^d(1+|\lambda|^{1/2}(x_d+y_d+|x'-y'|))} |\nabla' h(y)|\, dy.$$

These bounds follow again from Propositions 3.2 and 3.5. Then it is straightforward to see

$$|\nabla u(x)| \leq C\left(\frac{1}{|\lambda|^{1/4}} + \frac{1}{|\lambda|^{3/4}}\right)\frac{1}{(1+|x|)^{d+1/2}}, \tag{A-12}$$

with C depending on h. The same bound holds also for $|\nabla'^{\alpha} \nabla u(x)|$. Since

$$\partial_{x_d} p_u = -\lambda u_d + \Delta u_d + \Delta' h_d = -\lambda u_d + \Delta' u_d - \partial_{x_d} \nabla' \cdot u' + \Delta' h_d, \tag{A-13}$$

we obtain $|\partial_{x_d} p_u(x)| \leq C_{\varepsilon,\lambda}(1+|x|)^{-d-1/2}$ from the above results. Thus, the similar bound is valid also for $|\nabla'^{\alpha} \partial_{x_d} p_u(x)|$ since ∇' commutes with the kernel. Next we consider the estimate of $\nabla' p$. We apply the argument as in the estimate of ∇u; that is, with the cut-off χ, we write $\nabla' p_u$ as

$$\nabla' p_u(x) = \int_{\mathbb{R}^d_+} (\chi q_\lambda(x'-y', x_d, y_d))\nabla'_y \Delta' h(y)\, dy + \int_{\mathbb{R}^d_+} \nabla'^2_x((1-\chi)q_\lambda(x'-y', x_d, y_d)) \cdot \nabla' h(y)\, dy.$$

Then Proposition 3.7 yields

$$\begin{aligned} |\nabla' p_u(x)| \leq C \int_{|x'-y'|<1} &\frac{e^{-c|\lambda|^{1/2} y_d}}{(|x'-y'|+x_d+y_d)^{d-1}} |\Delta' h(y)|\, dy \\ &+ C \int_{|x'-y'|\geq 1} \frac{e^{-c|\lambda|^{1/2} y_d}}{(1+|x'-y'|+x_d+y_d)^{d-1}(1+|x'-y'|)^2} |\nabla' h(y)|\, dy. \end{aligned}$$

From this bound it is not difficult to derive the estimate

$$|\nabla' p_u(x)| \leq C\left(\frac{1}{|\lambda|^{1/4}} + \frac{1}{|\lambda|^{3/4}}\right)\frac{1}{(1+|x|)^{d+1/2}}. \tag{A-14}$$

The details are omitted. Then we also obtain the same estimate for $|\nabla'^{\alpha} \nabla' p(x)|$. It remains to estimate $\partial^2_{x_d} u$, but from the divergence-free condition we have already obtained the estimate for $\partial^2_{x_d} u_d$, and thus, it suffices to consider $\partial^2_{x_d} u'$. But the decay estimate immediately follows from the equation $\partial^2_{x_d} u' = \lambda u' - \Delta' u' + \nabla' p_u - \Delta' h'$. □

Remark A.2. Let $(u, \nabla p_u)$ be the solution to the resolvent problem (1-2) with $f = \Delta' h$ as in the proof of Proposition A.1. From (A-14) we have shown that

$$|\nabla'^{\alpha}\nabla' p_u(x)| \le C\biggl(\frac{1}{|\lambda|^{1/4}} + \frac{1}{|\lambda|^{3/4}}\biggr)\frac{1}{(1+|x|)^{d+1/2}} \tag{A-15}$$

for any $\lambda \in S_{\pi-\varepsilon}$ and $|\alpha| \le 2$, where C depends only on ε and h. On the other hand, from (A-12) and (A-8), we have

$$|\nabla'^{\alpha}\partial_{x_d} p_u(x)| \le C\biggl(|\lambda|^{3/4} + \frac{1}{|\lambda|^{1/4}} + \frac{1}{|\lambda|^{3/4}}\biggr)\frac{1}{(1+|x|)^{d+1/2}} \tag{A-16}$$

for any $\lambda \in S_{\pi-\varepsilon}$ and $|\alpha| \le 2$, where C depends only on ε and h.

Appendix B: A Liouville theorem for the nonstationary problem in L^1_{uloc} spaces

B

The class of weak solutions for the nonsteady Stokes system (1-11) is stated as follows. Let $u_0 \in L^1_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+)$ and $f \in L^1_{\mathrm{loc}}((0,T)\times\overline{\mathbb{R}}^d_+))^d$. We say that $(u, \nabla p)$ is a solution to (1-11) in the sense of distributions if

(i) $u \in L^\infty(0,T; L^1_{\mathrm{uloc},\sigma}(\mathbb{R}^d_+))$, $\nabla p \in L^1_{\mathrm{loc}}((0,T)\times\mathbb{R}^d_+)^d$ with $p \in L^1_{\mathrm{loc}}((0,T)\times\mathbb{R}^d_+)$, and

$$\sup_{x\in\mathbb{R}^d_+}\int_\delta^T \|\nabla p(t)\|_{L^1(B(x)\cap\mathbb{R}^d_+)}\,dt < \infty \quad \text{for any } 0<\delta<T. \tag{B-1}$$

Here $B(x)$ is the ball of radius 1 centered at x.

(ii) The map $t \mapsto \int_{\mathbb{R}^d_+} u(t,x)\cdot\varphi(x)\,dx$ belongs to $C([0,T))$ for any $\varphi \in C_0^\infty(\overline{\mathbb{R}}^d_+)^d$. In particular, the initial condition is satisfied in this sense.

(iii) For all $t', t \in (0,T)$ with $t > t'$ and for all $\varphi \in C_0^\infty((0,T)\times\overline{\mathbb{R}}^d)^d$ with $\varphi|_{x_d=0} = 0$,

$$\int_{\mathbb{R}^d_+} u(t,x)\cdot\varphi(t,x)\,dx - \int_{t'}^t\int_{\mathbb{R}^d_+} u(t,x)\cdot(\partial_s\varphi+\Delta\varphi)(t,x) - \nabla p(t,x)\cdot\varphi(t,x)\,dx\,ds$$
$$= \int_{\mathbb{R}^d_+} u(t',x)\cdot\varphi(t',x)dx + \int_{t'}^t f(t,x)\cdot\varphi(t,x)\,dx\,ds. \tag{B-2}$$

Proof of Theorem 5. By considering the mollification $(u^\kappa, \nabla p^\kappa)$ instead of $(u, \nabla p)$ as in the proof of Theorem 4, we may assume in addition that $(u, \nabla p)$ is smooth in x' and ∇u_d is bounded. We denote by $\langle\,\cdot\,,\cdot\,\rangle$ the usual inner product of $L^2(\mathbb{R}^d_+)^d$. Take arbitrary $t, t' \in (0,T)$ with $t > t'$ and $g \in C_0^\infty(\mathbb{R}^d_+)$. We introduce a mollification in time, and set $u^\rho = j_\rho * u$ and $p^\rho = j_\rho * p$, where $*$ here is the convolution in time and $j_\rho(\tau) \in C_0^\infty((-\rho,\rho))$ is the mollifier with a small parameter $\rho > 0$. The parameter $\rho > 0$ is taken small enough so that $t' > 2\rho$ and $t < T - 2\rho$. Then, we have the bound such as $\nabla p^\rho \in L^\infty(t', t; L^1_{\mathrm{uloc}}(\mathbb{R}^d_+))$. Note also that $(u^\rho, \nabla p^\rho)$ satisfies (B-2) for t, t'. Fix arbitrary $\varepsilon \in (0,1)$. Let $R \ge 1$ and χ_R be a smooth cut-off such that $\chi_R = 1$ for $|x| \le R$ and $\chi_R = 0$ for $|x| \ge 2R$. Let $\mathbb{P} = I - \mathbb{Q}$ be the Helmholtz projection in $L^2(\mathbb{R}^d_+)$, where $\mathbb{Q}g = \nabla p_g$ is defined as in the proof of Proposition A.1. Note that $\Delta'^2 p_g$ and $\Delta'^2\nabla p_g$ are smooth and decay fast enough so that $O(|x|^{-d-2})$ as $|x| \to \infty$. Then one can verify that $\langle u^\rho(t), \Delta'^2\mathbb{Q}g\rangle = 0$, and thus, $\langle u^\rho(t), \chi_R\Delta'^2\mathbb{Q}g\rangle = -\langle u^\rho(t), (1-\chi_R)\Delta'^2\mathbb{Q}g\rangle$. Hence, we can take $R = R_\varepsilon$ large enough so that $|\langle u^\rho(t), \chi_{R_\varepsilon}\Delta'^2\mathbb{Q}g\rangle| \le \varepsilon$. We may also assume that $\operatorname{supp} g \subseteq \{|x| < R_\varepsilon\}$. Next, since $u^\rho(t)\chi_{R_\varepsilon} \in L^1(\mathbb{R}^d_+)$, there exists $u^{\rho,\varepsilon}(t) \in C_0^\infty(\mathbb{R}^d_+)^d$ such that $\|u^\rho(t)\chi_{R_\varepsilon} - u^{\rho,\varepsilon}(t)\|_{L^1(\mathbb{R}^d_+)} \le \varepsilon$. We take $\tilde{t} > t$ which will be chosen later. Let v be the velocity field defined by

$$v(s) = e^{-(\tilde{t}-s)\boldsymbol{A}}\Delta'\mathbb{P}g = \frac{1}{2\pi i}\int_\Gamma e^{(\tilde{t}-s)\lambda}(\lambda+\boldsymbol{A})^{-1}\Delta'\mathbb{P}g\, d\lambda$$

for $0 \le s < \tilde{t}$, where Γ is the curve as in the proof of Proposition 5.3. Then v satisfies $\partial_s v + \Delta v - \nabla q = 0$ for $0 \le s < \tilde{t}$, where the associated pressure $\nabla q(s)$ is given by the formula

$$\nabla q(s) = \frac{1}{2\pi i}\int_\Gamma e^{(\tilde{t}-s)\lambda}\nabla q_\lambda\, d\lambda.$$

Here ∇q_λ is the pressure for each resolvent problem. Note that for ∇q_λ we can apply the pointwise estimate stated in Remark A.2, which gives the bounds

$$|\nabla'^\alpha\nabla' q(s,x)| \le \frac{C_{\tilde{t}-t'}}{(\tilde{t}-s)^{3/4}(1+|x|)^{d+1/2}}, \tag{B-3}$$

$$|\nabla'^\alpha\partial_{x_d} q(s,x)| \le \frac{C_{\tilde{t}-t'}}{(\tilde{t}-s)^{7/4}(1+|x|)^{d+1/2}} \tag{B-4}$$

for $t' \le s < \tilde{t}$ and $|\alpha| \le 2$. We see

$$\begin{aligned}\langle u^\rho(t), \Delta'^2 g\rangle &= \langle u^\rho(t), \chi_{R_\varepsilon}\Delta'^2 g\rangle \\ &= \langle u^\rho(t), \chi_{R_\varepsilon}\Delta' v(t)\rangle - \langle u^\rho(t), \chi_{R_\varepsilon}(e^{-(\tilde{t}-t)\boldsymbol{A}}\Delta'^2\mathbb{P}g - \Delta'^2\mathbb{P}g)\rangle + \langle u^\rho(t), \chi_{R_\varepsilon}\Delta'\mathbb{Q}\Delta' g\rangle.\end{aligned}$$

Then, from the identity

$$\begin{aligned}&\langle u^\rho(t), \chi_{R_\varepsilon}(e^{-(\tilde{t}-t)\boldsymbol{A}}\Delta'^2\mathbb{P}g - \Delta'^2\mathbb{P}g)\rangle \\ &\quad= \langle u^\rho(t)\chi_{R_\varepsilon} - u^{\rho,\varepsilon}(t), (e^{-(\tilde{t}-t)\boldsymbol{A}}\Delta'^2\mathbb{P}g - \Delta'^2\mathbb{P}g)\rangle + \langle u^{\rho,\varepsilon}(t), (e^{-(\tilde{t}-t)\boldsymbol{A}}\Delta'^2\mathbb{P}g - \Delta'^2\mathbb{P}g)\rangle,\end{aligned}$$

we have

$$\begin{aligned}&|\langle u^\rho(t), \chi_{R_\varepsilon}(e^{-(\tilde{t}-t)\boldsymbol{A}}\Delta'^2\mathbb{P}g - \Delta'^2\mathbb{P}g)\rangle| \\ &\quad\le \varepsilon\|e^{-(\tilde{t}-t)\boldsymbol{A}}\Delta'^2\mathbb{P}g - \Delta'^2\mathbb{P}g\|_{L^\infty} + \|u^{\rho,\varepsilon}(t)\|_{L^2}\|e^{-(\tilde{t}-t)\boldsymbol{A}}\Delta'^2\mathbb{P}g - \Delta'^2\mathbb{P}g\|_{L^2} \\ &\quad\le C\varepsilon\|\Delta'^2\mathbb{P}g\|_{L^\infty} + \|u^{\rho,\varepsilon}(t)\|_{L^2}\|e^{-(\tilde{t}-t)\boldsymbol{A}}\Delta'^2\mathbb{P}g - \Delta'^2\mathbb{P}g\|_{L^2}.\end{aligned}$$

Here we have used the fact that the Stokes semigroup is a bounded semigroup in $L^\infty_\sigma(\mathbb{R}^d_+)$. Note that $\|\Delta'^2\mathbb{P}g\|_{L^\infty}$ is finite since $g \in C_0^\infty(\mathbb{R}^d_+)^d$, and there exists $\tilde{t}_\varepsilon > t$ such that

$$\|u^{\rho,\varepsilon}(t)\|_{L^2}\|e^{-(\tilde{t}-t)\boldsymbol{A}}\Delta'^2\mathbb{P}g - \Delta'^2\mathbb{P}g\|_{L^2} \le \varepsilon$$

for any $\tilde{t} \in (t, \tilde{t}_\varepsilon)$. Hence we have

$$|\langle u^\rho(t), \chi_{R_\varepsilon}(e^{-(\tilde{t}-t)\boldsymbol{A}}\Delta'^2\mathbb{P}g - \Delta'^2\mathbb{P}g)\rangle| \le C\varepsilon.$$

The term $\langle u^\rho(t), \chi_{R_\varepsilon}\Delta' v(t)\rangle$ is estimated by using the definition of the solution in the sense of distributions, for $\chi_{R_\varepsilon}\Delta' v$ is admissible as a test function on the time interval $[t', t]$. Then we observe that

$$\begin{aligned}\langle u^\rho(t), \chi_{R_\varepsilon}\Delta' v(t)\rangle &= \int_{t'}^{t} \langle u^\rho, (\partial_s+\Delta)\chi_{R_\varepsilon}\Delta' v(s)\rangle - \langle \nabla p^\rho, \chi_{R_\varepsilon}\Delta' v(s)\rangle\, ds + \langle u^\rho(t'), \chi_{R_\varepsilon}\Delta' v(t')\rangle \\ &= \int_{t'}^{t} \langle u^\rho, (\Delta\chi_R + 2\nabla\chi_{R_\varepsilon}\cdot\nabla)\Delta' v(s) - \chi_{R_\varepsilon}\Delta'\nabla q(s)\rangle\, ds \\ &\quad - \int_{t'}^{t} \langle \nabla p^\rho, (\chi_{R_\varepsilon}-1)\Delta' v(s)\rangle\, ds + \langle u^\rho(t'), \chi_{R_\varepsilon}\Delta' v(t')\rangle \\ &= \int_{t'}^{t} \langle u^\rho, (\Delta\chi_{R_\varepsilon} + 2\nabla\chi_{R_\varepsilon}\cdot\nabla)\Delta' v(s)\rangle\, ds \\ &\quad + \int_{t'}^{t} \langle u^\rho, (\nabla\chi_{R_\varepsilon})\Delta' q(s)\rangle\, ds - \int_{t'}^{t} \langle \nabla p^\rho, (\chi_{R_\varepsilon}-1)\Delta' v(s)\rangle\, ds \\ &\quad + \langle u^\rho(t'), (\chi_{R_\varepsilon}-1)\Delta' v(t')\rangle + \langle u^\rho(t'), \Delta' v(t')\rangle.\end{aligned}$$

Here we have used the fact $\langle u^\rho(s), \nabla(\chi_{R_\varepsilon}\Delta' q(s))\rangle = 0$ and $\langle \nabla p^\rho(s), \Delta' v(s)\rangle = 0$ for each $s \in (t', t)$, where the latter is verified from $\nabla'^\alpha \nabla p^\rho(s) \in L^1_{\mathrm{uloc}}(\mathbb{R}^d_+)$ with $|\alpha| \le 2$ and the pointwise estimate such as

$$|\nabla'^\alpha v(s,x)| + |\nabla'^\alpha \nabla v(s,x)| \le C_{\tilde t - t'}(\tilde t - s)^{-3/4}(1+|x|)^{-d-1/2}, \tag{B-5}$$

which are obtained from Proposition A.1 for the resolvent problem and the representation

$$v(s) = \frac{1}{2\pi i}\int_\Gamma e^{(\tilde t - s)\lambda}(\lambda + \boldsymbol{A})^{-1}\Delta'\mathbb{P}g\, d\lambda.$$

From (B-3) and (B-5), we also observe that

$$\left|\int_{t'}^{t} \langle u^\rho, (\Delta\chi_{R_\varepsilon} + 2\nabla\chi_{R_\varepsilon}\cdot\nabla)\Delta' v(s)\rangle\, ds\right| \le C R_\varepsilon^{-1}\|u^\rho\|_{L^\infty(t',t;L^1_{\mathrm{uloc}}(\mathbb{R}^d_+))}\int_{t'}^{t}(\tilde t - s)^{-3/4}\, ds \le C R_\varepsilon^{-1},$$

$$\left|\int_{t'}^{t} \langle u^\rho, (\nabla\chi_{R_\varepsilon})\Delta' q(s)\rangle\, ds\right| \le C R_\varepsilon^{-1}\|u^\rho\|_{L^\infty(t',t;L^1_{\mathrm{uloc}}(\mathbb{R}^d_+))}\int_{t'}^{t}(\tilde t - s)^{-3/4}\, ds \le C R_\varepsilon^{-1},$$

$$\left|\int_{t'}^{t} \langle \nabla p^\rho, (\chi_{R_\varepsilon}-1)\Delta' v(s)\rangle\, ds\right| \le C R_\varepsilon^{-1/4}\|\nabla p^\rho\|_{L^\infty(t',t;L^1_{\mathrm{uloc}}(\mathbb{R}^d_+))}\int_{t'}^{t}(\tilde t - s)^{-3/4}\, ds \le C R_\varepsilon^{-1/4},$$

and similarly,

$$|\langle u^\rho(t'), (\chi_{R_\varepsilon}-1)\Delta' v(t')\rangle| \le C(\tilde t - t)^{-3/4} R_\varepsilon^{-1/4}.$$

Therefore, we can take the limit $\varepsilon \to 0$, which leads to $R_\varepsilon \to \infty$ and $\tilde t \to t$, resulting in the identity

$$\langle u^\rho(t), \Delta'^2 g\rangle = \langle u^\rho(t'), \Delta' e^{-(t-t')\boldsymbol{A}}\Delta'\mathbb{P}g\rangle.$$

Then we take the limit $\rho \to 0$, which gives

$$\langle u(t), \Delta'^2 g\rangle = \langle u(t'), \Delta' e^{-(t-t')\boldsymbol{A}}\Delta'\mathbb{P}g\rangle. \tag{B-6}$$

Finally, we take the limit $t' \to 0$ in (B-6). Then the time continuity in the weak sense, which is assumed in the definition of solutions, together with the pointwise estimate for $e^{-(t-t')\boldsymbol{A}}\Delta'\mathbb{P}g$ similar to (B-5) implies that

$$\langle u(t), \Delta'^2 g \rangle = 0. \tag{B-7}$$

Since $g \in C_0^\infty(\mathbb{R}_+^d)^d$ is arbitrary, we conclude that for a.e. $t \in (0, T)$, $x_d > 0$, we have $u(t, x', x_d) = (a'(t, x_d), 0)^\top$ by arguing as in the proof of Theorem 4. Once this is shown, the argument is parallel to the proof of Theorem 4; we can show from (B-2) that p is independent of x_d and also $\Delta' p = 0$ for a.e. $t \in (0, T)$, which implies $p(t, x) = D(t) \cdot x' + c(t)$ for some $D \in L^1_{\mathrm{loc}}(0, T)^{d-1}$ and $c \in L^1_{\mathrm{loc}}(0, T)$. The last statement for the conclusion $u = 0$ is proved in the same manner as in Theorem 4, so the details are omitted. □

Acknowledgements

Maekawa was partially supported by JSPS Program for Advancing Strategic International Networks to Accelerate the Circulation of Talented Researchers, "Development of Concentrated Mathematical Center Linking to Wisdom of the Next Generation", which is organized by Mathematical Institute of Tohoku University. Miura was partially supported by JSPS grant 25707005. Prange acknowledges financial support from the French Agence Nationale de la Recherche under grant ANR-16-CE40-0027-01, as well as from the IDEX of the University of Bordeaux for the BOLIDE project.

References

[Abe 2015] K. Abe, "The Navier–Stokes equations in a space of bounded functions", *Comm. Math. Phys.* **338**:2 (2015), 849–865. MR Zbl

[Abe 2016] K. Abe, "On estimates for the Stokes flow in a space of bounded functions", *J. Differential Equations* **261**:3 (2016), 1756–1795. MR Zbl

[Abe and Giga 2013] K. Abe and Y. Giga, "Analyticity of the Stokes semigroup in spaces of bounded functions", *Acta Math.* **211**:1 (2013), 1–46. MR Zbl

[Abe and Giga 2014] K. Abe and Y. Giga, "The L^∞-Stokes semigroup in exterior domains", *J. Evol. Equ.* **14**:1 (2014), 1–28. MR Zbl

[Abe et al. 2015a] K. Abe, Y. Giga, and M. Hieber, "Stokes resolvent estimates in spaces of bounded functions", *Ann. Sci. Éc. Norm. Supér.* (4) **48**:3 (2015), 537–559. MR Zbl

[Abe et al. 2015b] K. Abe, Y. Giga, K. Schade, and T. Suzuki, "On the Stokes semigroup in some non-Helmholtz domains", *Arch. Math.* (*Basel*) **104**:2 (2015), 177–187. MR Zbl

[Ambrose et al. 2015] D. M. Ambrose, J. P. Kelliher, M. C. Lopes Filho, and H. J. Nussenzveig Lopes, "Serfati solutions to the 2D Euler equations on exterior domains", *J. Differential Equations* **259**:9 (2015), 4509–4560. MR Zbl

[Arrieta et al. 2004] J. M. Arrieta, A. Rodriguez-Bernal, J. W. Cholewa, and T. Dlotko, "Linear parabolic equations in locally uniform spaces", *Math. Models Methods Appl. Sci.* **14**:2 (2004), 253–293. MR Zbl

[Bae and Jin 2012] H.-O. Bae and B. J. Jin, "Existence of strong mild solution of the Navier–Stokes equations in the half space with nondecaying initial data", *J. Korean Math. Soc.* **49**:1 (2012), 113–138. MR Zbl

[Barker and Seregin 2015] T. Barker and G. Seregin, "A necessary condition of possible blowup for the Navier–Stokes system in half-space", preprint, 2015. arXiv

[Cannone et al. 2000] M. Cannone, F. Planchon, and M. Schonbek, "Strong solutions to the incompressible Navier–Stokes equations in the half-space", *Comm. Partial Differential Equations* **25**:5-6 (2000), 903–924. MR Zbl

[Cholewa and Dlotko 2004] J. W. Cholewa and T. Dlotko, "Cauchy problems in weighted Lebesgue spaces", *Czechoslovak Math. J.* **54**:4 (2004), 991–1013. MR Zbl

[Dalibard and Gérard-Varet 2017] A.-L. Dalibard and D. Gérard-Varet, "Nonlinear boundary layers for rotating fluids", *Anal. PDE* **10**:1 (2017), 1–42. MR Zbl

[Dalibard and Prange 2014] A.-L. Dalibard and C. Prange, "Well-posedness of the Stokes–Coriolis system in the half-space over a rough surface", *Anal. PDE* **7**:6 (2014), 1253–1315. MR Zbl

[Danchin and Zhang 2014] R. Danchin and P. Zhang, "Inhomogeneous Navier–Stokes equations in the half-space, with only bounded density", *J. Funct. Anal.* **267**:7 (2014), 2371–2436. MR Zbl

[Desch et al. 2001] W. Desch, M. Hieber, and J. Prüss, "L^p-theory of the Stokes equation in a half space", *J. Evol. Equ.* **1**:1 (2001), 115–142. MR Zbl

[Fujita and Kato 1964] H. Fujita and T. Kato, "On the Navier–Stokes initial value problem, I", *Arch. Ration. Mech. Anal.* **16** (1964), 269–315. MR Zbl

[Galdi 2011] G. P. Galdi, *An introduction to the mathematical theory of the Navier–Stokes equations: steady-state problems*, 2nd ed., Springer, 2011. MR Zbl

[Geissert and Giga 2008] M. Geissert and Y. Giga, "On the Stokes resolvent equations in locally uniform L^p spaces in exterior domains", pp. 307–314 in *Functional analysis and evolution equations*, edited by H. Amann et al., Birkhäuser, Basel, 2008. MR Zbl

[Ginibre and Velo 1997] J. Ginibre and G. Velo, "The Cauchy problem in local spaces for the complex Ginzburg–Landau equation, II: Contraction methods", *Comm. Math. Phys.* **187**:1 (1997), 45–79. MR Zbl

[Jia and Šverák 2014] H. Jia and V. Šverák, "Local-in-space estimates near initial time for weak solutions of the Navier–Stokes equations and forward self-similar solutions", *Invent. Math.* **196**:1 (2014), 233–265. MR Zbl

[Jia et al. 2012] H. Jia, G. Seregin, and V. Šverák, "Liouville theorems in unbounded domains for the time-dependent Stokes system", *J. Math. Phys.* **53**:11 (2012), art. id. 115604. MR Zbl

[Kato 1984] T. Kato, "Strong L^p-solutions of the Navier–Stokes equation in $\mathbb{R}^m$, with applications to weak solutions", *Math. Z.* **187**:4 (1984), 471–480. MR Zbl

[Kikuchi and Seregin 2007] N. Kikuchi and G. Seregin, "Weak solutions to the Cauchy problem for the Navier–Stokes equations satisfying the local energy inequality", pp. 141–164 in *Nonlinear equations and spectral theory*, edited by M. S. Birman and N. N. Uraltseva, Amer. Math. Soc. Transl. Ser. 2 **220**, Amer. Math. Soc., Providence, RI, 2007. MR Zbl

[Kozono 1998] H. Kozono, "L^1-solutions of the Navier–Stokes equations in exterior domains", *Math. Ann.* **312**:2 (1998), 319–340. MR Zbl

[Lemarié-Rieusset 2002] P. G. Lemarié-Rieusset, *Recent developments in the Navier–Stokes problem*, Chapman & Hall/CRC Res. Notes in Math. **431**, Chapman & Hall/CRC, Boca Raton, FL, 2002. MR Zbl

[Leray 1934] J. Leray, "Sur le mouvement d'un liquide visqueux emplissant l'espace", *Acta Math.* **63**:1 (1934), 193–248. MR Zbl

[Li et al. 2018] K. Li, T. Ozawa, and B. Wang, "Dynamical behavior for the solutions of the Navier–Stokes equation", *Commun. Pure Appl. Anal.* **17**:4 (2018), 1511–1560. MR Zbl

[Maekawa and Terasawa 2006] Y. Maekawa and Y. Terasawa, "The Navier–Stokes equations with initial data in uniformly local L^p spaces", *Differential Integral Equations* **19**:4 (2006), 369–400. MR Zbl

[Maekawa et al. 2019] Y. Maekawa, H. Miura, and C. Prange, "Local energy weak solutions for the Navier–Stokes equations in the half-space", *Comm. Math. Phys.* **367**:2 (2019), 517–580. MR Zbl

[Mielke and Schneider 1995] A. Mielke and G. Schneider, "Attractors for modulation equations on unbounded domains: existence and comparison", *Nonlinearity* **8**:5 (1995), 743–768. MR Zbl

[Prange 2018] C. Prange, "Infinite energy solutions to the Navier–Stokes equations in the half-space and applications", exposé II in *Séminaire Laurent Schwartz: EDP et applications* (Bures-sur-Yvette, France, 2017-2018), CEDRAM, Grenoble, 2018.

[Seregin 2012] G. Seregin, "A certain necessary condition of potential blow up for Navier–Stokes equations", *Comm. Math. Phys.* **312**:3 (2012), 833–845. MR Zbl

[Serfati 1995] P. Serfati, "Structures holomorphes à faible régularité spatiale en mécanique des fluides", *J. Math. Pures Appl.* (9) **74**:2 (1995), 95–104. MR Zbl

[Solonnikov 2003] V. A. Solonnikov, "On nonstationary Stokes problem and Navier–Stokes problem in a half-space with initial data nondecreasing at infinity", *J. Math. Sci.* (*N. Y.*) **114**:5 (2003), 1726–1740. MR Zbl

[Taniuchi et al. 2010] Y. Taniuchi, T. Tashiro, and T. Yoneda, "On the two-dimensional Euler equations with spatially almost periodic initial data", *J. Math. Fluid Mech.* **12**:4 (2010), 594–612. MR Zbl

[Ukai 1987] S. Ukai, "A solution formula for the Stokes equation in $\mathbb{R}^n_+$", *Comm. Pure Appl. Math.* **40**:5 (1987), 611–621. MR Zbl

Received 16 Nov 2017. Revised 4 Mar 2019. Accepted 18 Apr 2019.

YASUNORI MAEKAWA: maekawa@math.kyoto-u.ac.jp
Department of Mathematics, Kyoto University, Kyoto, Japan

HIDEYUKI MIURA: miura@is.titech.ac.jp
Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Tokyo, Japan

CHRISTOPHE PRANGE: christophe.prange@math.u-bordeaux.fr
CNRS, UMR 5251, IMB, Université de Bordeaux, Bordeaux, France

ANALYSIS AND PDE
Vol. 13, No. 4, 2020
dx.doi.org/10.2140/apde.2020.13.1011

ALMOST-SURE SCATTERING FOR THE RADIAL ENERGY-CRITICAL NONLINEAR WAVE EQUATION IN THREE DIMENSIONS

BJOERN BRINGMANN

We study the Cauchy problem for the radial energy-critical nonlinear wave equation in three dimensions. Our main result proves almost-sure scattering for radial initial data below the energy space. In order to preserve the spherical symmetry of the initial data, we construct a radial randomization that is based on annular Fourier multipliers. We then use a refined radial Strichartz estimate to prove probabilistic Strichartz estimates for the random linear evolution. The main new ingredient in the analysis of the nonlinear evolution is an interaction flux estimate between the linear and nonlinear components of the solution. We then control the energy of the nonlinear component by a triple bootstrap argument involving the energy, the Morawetz term, and the interaction flux estimate.

1. Introduction

We consider the defocusing nonlinear wave equation (NLW) in three dimensions

$$\begin{cases} -\partial_{tt} u + \Delta u = u^5, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^3, \\ u(0,x) = f(x) \in \dot{H}^s_x(\mathbb{R}^3), \quad \partial_t u(0,x) = g(x) \in \dot{H}^{s-1}_x(\mathbb{R}^3). \end{cases} \tag{1}$$

The flow of nonlinear wave equation (1) conserves the energy

$$E[u](t) := \int_{\mathbb{R}^3} \tfrac{1}{2}|\nabla u(t,x)|^2 + \tfrac{1}{2}|\partial_t u(t,x)|^2 + \tfrac{1}{6} u(t,x)^6 \, \mathrm{d}x. \tag{2}$$

MSC2010: 35L05, 35L15, 35L71.
Keywords: nonlinear wave equation, probabilistic well-posedness, scattering, spherical symmetry.

Since the scaling-symmetry $u(t,x) \mapsto u_\lambda(t,x) = \lambda^{-1/2} u(t/\lambda, x/\lambda)$ of (1) leaves the energy invariant, we call (1) energy critical. Using Sobolev embedding, it follows that the energy of the initial data is finite if and only if $(f,g) \in \dot{H}^1_x(\mathbb{R}^3) \times L^2_x(\mathbb{R}^3)$. Therefore, we refer to $\dot{H}^1_x(\mathbb{R}^3) \times L^2_x(\mathbb{R}^3)$ as the energy space.

If the initial data has finite energy, the nonlinear wave equation (1) is now well-understood. In a series of seminal papers by several authors [Bahouri and Gérard 1999; Grillakis 1990; 1992; Rauch 1981; Shatah and Struwe 1993; 1994; Strauss 1968; Struwe 1988; Tao 2006b], it was proven that solutions to (1) exist globally, obey global space-time bounds, and scatter as $t \mapsto \pm\infty$. In contrast, the equation is ill-posed if the initial data only lies in $H^s_x(\mathbb{R}^3) \times H^{s-1}_x(\mathbb{R}^3)$ for some $0 < s < 1$. For instance, it has been shown in [Christ et al. 2003] that solutions to (1) exhibit norm-inflation with respect to the $H^s_x \times H^{s-1}_x$-norm. Consequently, this shows that we cannot construct local solutions of (1) with initial data in $H^s_x \times H^{s-1}_x$ by a contraction mapping argument.

In recent years, there has been much interest in determining whether bad behavior such as norm inflation is generic or only occurs for exceptional initial data. To answer this question, multiple authors have studied solutions to dispersive equations with randomized initial data. In the following discussion, we will focus on the Wiener randomization, and we refer the reader to the introduction of [Pocovnicu 2017], as well as [Bourgain 1994; 1996; Burq and Tzvetkov 2008a; 2008b; Nahmod et al. 2012; Thomann and Tzvetkov 2010].

Let us first recall the definition of the Wiener randomization from [Bényi et al. 2015b; Lührmann and Mendelson 2014]. We denote by $Q = \left[-\frac{1}{2}, \frac{1}{2}\right)^d$ the unit cube centered at the origin. The family of translates $\{Q - k\}_{k \in \mathbb{Z}^d}$ forms a partition of $\mathbb{R}^d$ (see Figure 1). By convolving the indicator function χ_Q with a smooth and compactly supported kernel, we can construct a function $\psi \in C^\infty_c(\mathbb{R}^d)$ such that

$$\psi|_{[-\frac{1}{4},\frac{1}{4})^d} \equiv 1, \quad \psi|_{\mathbb{R}^d \setminus [-1,1)^d} \equiv 0, \quad \text{and} \quad \sum_{k \in \mathbb{Z}^d} \psi(\xi - k) = 1.$$

Then, any function $f \in L^2_x(\mathbb{R}^d)$ can be decomposed in frequency space as

$$\hat{f}(\xi) = \sum_{k \in \mathbb{Z}^d} \psi(\xi - k) \hat{f}(\xi).$$

If $\{g_k\}_{k \in \mathbb{Z}^d}$ is a family of independent standard complex-valued Gaussians, the Wiener randomization f^ω_W of f defined as

$$\hat{f}^\omega_W(\xi) := \sum_{k \in \mathbb{Z}^d} g_k(\omega) \psi(\xi - k) \hat{f}(\xi).$$

Thus, f^ω_W is a random linear combination of functions whose Fourier transform is supported in unit-scale cubes. The Wiener randomization has been used to prove almost-sure local and global well-posedness of nonlinear wave equations below the scaling-critical regularity. Lührmann and Mendelson [2014; 2016] proved the almost-sure global well-posedness of energy-subcritical nonlinear wave equations in $\mathbb{R}^3$. The first probabilistic result on the energy-critical NLW was obtained in [Pocovnicu 2017], which treated the dimensions $d = 4, 5$. This method was extended in [Oh and Pocovnicu 2016] to the three-dimensional case. In addition to nonlinear wave equations, the Wiener randomization has also been applied to nonlinear Schrödinger equations (NLS). Bényi, Oh, and Pocovnicu [Bényi et al. 2015a; 2015b; 2019] proved the

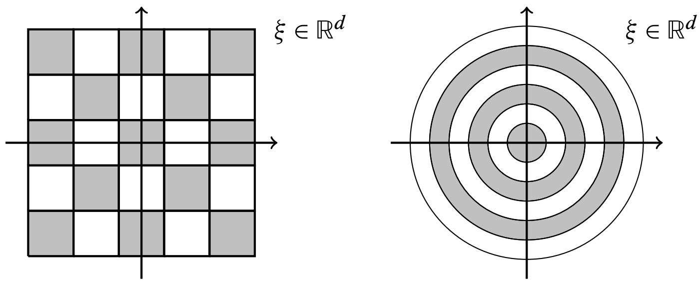

Figure 1. In the left image, we display a partition of $\mathbb{R}^d$ into unit-scale cubes, which forms the basis of the Wiener randomization. In the right image, we display a partition of $\mathbb{R}^d$ into annuli, which forms the basis of the radial randomization.

almost-sure local well-posedness of the cubic NLS in $\mathbb{R}^d$. This method was then extended in [Brereton 2019] to the quintic NLS in $\mathbb{R}^d$. In [Oh et al. 2017], the authors proved the almost-sure global well-posedness of the energy-critical NLS in dimensions $d = 5, 6$. However, the global well-posedness results above do not give any information on the asymptotic behavior of the solutions.

In contrast, Dodson, Lührmann, and Mendelson [Dodson et al. 2017; 2019] proved almost-sure scattering for the energy-critical NLW. Their result holds in dimension $d = 4$ and requires that the original initial data (before the randomization) is spherically symmetric. The main idea is to control the energy-increment of the nonlinear component of u by a bootstrap argument involving both the energy and a Morawetz term. The spherical symmetry is needed since the Morawetz estimate is centered around the origin. However, the Wiener randomization breaks the spherical symmetry, so that f_W^ω is no longer radial. This method was subsequently extended to the energy-critical NLS in dimension $d = 4$ in [Dodson et al. 2019; Killip et al. 2019].

In this work, we introduce a radial randomization that preserves the spherical symmetry of the initial data. To this end, let us first define a family of annular Fourier multipliers.

Definition 1.1 (annular multiplier). Let $f \in L_x^2(\mathbb{R}^d)$, $a > 0$, and $\delta \in (0, 1)$. Then, we define the operator $A_{a,\delta}$ by setting

$$\widehat{A_{a,\delta} f}\,(\xi) := \chi_{[a,(1+\delta)a)}(\|\xi\|_2)\hat{f}(\xi). \tag{3}$$

In addition, for any $0 < a_1 < a_2 \le \infty$, we also define the operator $A_{[a_1,a_2)}$ by setting

$$\widehat{A_{[a_1,a_2)} f}\,(\xi) := \chi_{[a_1,a_2)}(\|\xi\|_2)\hat{f}(\xi).$$

Instead of partitioning $\mathbb{R}^d$ into unit-scale cubes, the idea of the radial randomization is to decompose $\mathbb{R}^d$ into thin annuli (see Figure 1).

Definition 1.2 (radial randomization). Fix a parameter $\gamma > 0$ and let $\{g_k\}_{k=0}^\infty$ be a sequence of independent standard real-valued Gaussians. For any $f \in L_{\text{rad}}^2(\mathbb{R}^d)$, we define its radial symmetrization by

$$f^\omega(x) := \sum_{k=0}^\infty g_k(w) A_{[k^\gamma,(k+1)^\gamma)} f(x). \tag{4}$$

There exist two natural choices of γ: choosing $\gamma = 1$ leads to annuli of unit width, whereas choosing $\gamma = 1/d$ leads to annuli of approximately unit volume.

We now make a few remarks on the properties of f^ω. First, since the Fourier transform of f^ω is radial, it follows that f^ω is radial. Using the same argument as for the Wiener randomization [Oh 2017, Lemma 43], it is easy to see that the radial randomization does not improve the regularity of f. More precisely, if $s \in \mathbb{R}$ is such that $f \notin H_x^s(\mathbb{R}^d)$, then $f^\omega \notin H_x^s(\mathbb{R}^d)$ almost surely. In light of the unboundedness of the ball-multiplier, see [Chanillo 1984; Fefferman 1971], it is much harder to prove L^p-improving properties for the radial randomization than for the Wiener randomization. The probabilistic Strichartz estimates for the random linear evolution $\exp(\pm it|\nabla|)f^\omega$ will be derived from a refined (deterministic) radial Strichartz estimate. In contrast to the Wiener randomization, the radial randomization does not lead to a probabilistic gain of integrability in every nonsharp admissible Strichartz space. Thus, we see a relationship between the geometric structure of the linear evolution and the effects of the randomization, which was also discussed in [Chanillo et al. 2017].

Let us now formulate the main result of this work. In the following, we restrict the discussion to the dimension $d = 3$. Let $(f, g) \in H^s_{\mathrm{rad}}(\mathbb{R}^3) \times H^{s-1}_{\mathrm{rad}}(\mathbb{R}^3)$ be the given (deterministic) initial data. For technical reasons, we split the randomized initial data (f^ω, g^ω) into low- and high-frequency components. For the high-frequency component, we let

$$F^\omega(t,x) = \cos(t|\nabla|)P_{>2^6} f^\omega(x) + \frac{\sin(t|\nabla|)}{|\nabla|} P_{>2^6} g^\omega(x) \tag{5}$$

be the random and rough linear evolution. Next, we decompose the solution u of the energy-critical NLW into the linear component F^ω and a nonlinear component v; i.e., $u = F^\omega + v$. Then, the nonlinear component solves the initial value problem

$$\begin{cases} -\partial_{tt} v + \Delta v = (v + F^\omega)^5, \\ v(0,x) = P_{\leq 2^6} f^\omega, \quad \partial_t v(0,x) = P_{\leq 2^6} g^\omega. \end{cases} \tag{6}$$

Note that the initial data in (6) almost surely lies in the energy-space $\dot{H}^1_x(\mathbb{R}^3) \times L^2_x(\mathbb{R}^3)$. The above decomposition into a linear and nonlinear part is often called the Da Prato–Debussche trick [2002]. In the following, we analyze the solution v of the forced equation (6). Since $u = F^\omega + v$, any statement about v can easily be translated into a statement about u.

Theorem 1.3 (almost-sure scattering). *Let $(f, g) \in H^s_{\mathrm{rad}}(\mathbb{R}^3) \times H^{s-1}_{\mathrm{rad}}(\mathbb{R}^3)$, let $0 < \gamma \leq 1$, and let $\max\big(1 - \frac{1}{12\gamma}, 0\big) < s < 1$. Then, almost surely there exists a global solution v of (6) such that*

$$v \in C_t^0 \dot{H}^1_x(\mathbb{R} \times \mathbb{R}^3) \cap L_t^5 L_x^{10}(\mathbb{R} \times \mathbb{R}^3), \quad \partial_t v \in C_t^0 L_x^2(\mathbb{R} \times \mathbb{R}^3).$$

Furthermore, there exist scattering states $(v_0^\pm, v_1^\pm) \in \dot{H}^1_x(\mathbb{R}^3) \times L^2_x(\mathbb{R}^3)$ such that, if $w^\pm(t)$ are the solutions to the linear wave equation with initial data $(v_0^\pm, v_1^\pm)$, we have

$$\|(v(t) - w^\pm(t), \partial_t v(t) - \partial_t w^\pm(t))\|_{\dot{H}^1_x(\mathbb{R}^3) \times L^2_x(\mathbb{R}^3)} \to 0 \quad \text{as } t \to \pm\infty.$$

We remark that the restriction on s and the range for γ are not optimal; see, e.g., Lemma 7.3 and Remark 8.5. For any $(u_0, u_1) \in \dot{H}^1_{\mathrm{rad}}(\mathbb{R}^3) \times L^2_{\mathrm{rad}}(\mathbb{R}^3)$, we can also replace the initial data in (6) by

$(u_0 + P_{\le 2^6} f^\omega, u_1 + P_{\le 2^6} g^\omega)$. This implies the stability of the scattering mechanism of (1) under random radial perturbations.

By using the deterministic theory and a perturbation theorem, the proof of Theorem 1.3 reduces to an a priori energy bound on v; see [Bényi et al. 2015a; Dodson et al. 2017; Pocovnicu 2017]. We will discuss this reduction in Section 5. For now, let us simply state the a priori energy bound as a separate theorem.

Theorem 1.4 (a priori energy bound). *Let $(f, g) \in H^s_{\mathrm{rad}}(\mathbb{R}^3) \times H^{s-1}_{\mathrm{rad}}(\mathbb{R}^3)$, let $0 < \gamma \le 1$, and let $\max\bigl(1 - \frac{1}{12\gamma}, 0\bigr) < s < 1$. Assume that almost surely there exists a solution v of* (6) *with some maximal time interval of existence I. Then, we have that almost surely*

$$\sup_{t\in I} E[v](t) < \infty. \tag{7}$$

We now sketch the idea behind the proof of the a priori energy bound, which relies on a bootstrap argument. Let us fix a time interval $I = [a, b] \subseteq \mathbb{R}$. We want to bound the energy increment $E[v](b) - E[v](a)$ by the maximal energy $\mathcal{E}$ of v on I. We will see that the main error term in the energy increment is given by

$$\int_I \int_{\mathbb{R}^3} F^\omega v^4\, \partial_t v \,\mathrm{d}x\,\mathrm{d}t. \tag{8}$$

In the following discussion, we argue heuristically and ignore all other error terms. Using a Littlewood–Paley decomposition, we may assume that the linear evolution F^ω is localized to frequency $\sim N$. In dimension $d = 4$, Dodson, Lührmann and Mendelson [Dodson et al. 2017] used the Morawetz estimate to control the energy increment. Following their idea, we may assume under a bootstrap hypothesis that

$$\| |x|^{-\frac{1}{6}} v\|^6_{L^6_{t,x}(I\times\mathbb{R}^3)} \lesssim \mathcal{E}.$$

After directly applying the Morawetz estimate to (8), the best possible bound is $\sim (E^{1/6})^4 E^{1/2} \sim E^{7/6}$. However, this cannot prevent the finite-time blowup of the energy. Following [Oh and Pocovnicu 2016], we move the time derivative onto the linear evolution F^ω_N. First, we write $\partial_t F^\omega_N = |\nabla| \widetilde{F}^\omega_N$, where $\widetilde{F}^\omega_N$ is a different solution to the linear wave equation. After neglecting boundary terms, we heuristically rewrite the main error term as

$$\int_I \int_{\mathbb{R}^3} (\partial_t F^\omega_N) v^5 \,\mathrm{d}x\,\mathrm{d}t = \int_I \int_{\mathbb{R}^3} (|\nabla| \widetilde{F}^\omega_N) v^5 \,\mathrm{d}x\,\mathrm{d}t \sim \int_I \int_{\mathbb{R}^3} (|\nabla|^{\frac{1}{2}} \widetilde{F}^\omega_N) v^4 (|\nabla|^{\frac{1}{2}} v) \,\mathrm{d}x\,\mathrm{d}t. \tag{9}$$

By using the Morawetz term and the energy, we estimate

$$\begin{aligned}\left|\int_I \int_{\mathbb{R}^3} (|\nabla|^{\frac{1}{2}} \widetilde{F}^\omega_N) v^4 (|\nabla|^{\frac{1}{2}} v) \,\mathrm{d}x\,\mathrm{d}t\right| &\lesssim \| |x|^{\frac{3}{4}} |\nabla|^{\frac{1}{2}} \widetilde{F}^\omega_N \|_{L^4_t L^\infty_x(I\times\mathbb{R}^3)} \| |x|^{-\frac{1}{6}} v\|^{\frac{9}{2}}_{L^6_{t,x}(I\times\mathbb{R}^3)} \|\nabla v\|^{\frac{1}{2}}_{L^\infty_t L^2_x(I\times\mathbb{R}^3)} \\ &\lesssim \| |x|^{\frac{3}{4}} |\nabla|^{\frac{1}{2}} \widetilde{F}^\omega_N \|_{L^4_t L^\infty_x(I\times\mathbb{R}^3)} \mathcal{E}.\end{aligned}$$

In this bound, the power of $\mathcal{E}$ allows us to use a Gronwall-type argument. However, even for smooth and localized initial data, the linear evolution $|\nabla|^{1/2} \widetilde{F}^\omega_N$ only decays like $(1 + |t|)^{-1}$ and is morally supported near the light cone $|x| = |t|$. Thus, the norm $\| |x|^{3/4} |\nabla|^{1/2} \widetilde{F}^\omega_N \|_{L^4_t L^\infty_x(I\times\mathbb{R}^3)}$ diverges logarithmically as the time interval I increases. Since the energy yields better decay for ∇v than for v itself, the logarithmic divergence cannot be avoided by placing fewer derivatives on v. Consequently, this argument does not yield global bounds on the energy of v.

To overcome the logarithmic divergence, we introduce two additional ingredients. First, since the radial randomization preserves the spherical symmetry of the initial data, the linear evolution $|\nabla|^{1/2}\widetilde{F}_N^{\omega}$ is spherically symmetric. Using this, we can decompose the linear evolution into an incoming and outgoing wave; i.e.,

$$|\nabla|^{\frac{1}{2}}\widetilde{F}_N^{\omega} = \frac{1}{|x|}\big(W_{\mathrm{in}}[|\nabla|^{\frac{1}{2}}\widetilde{F}_n^{\omega}](t+|x|) + W_{\mathrm{out}}[|\nabla|^{\frac{1}{2}}\widetilde{F}_n^{\omega}](t-|x|)\big).$$

Second, we use a flux estimate to control the integral of the potential v^6 on shifted light cones by the energy. We now combine both of these tools by integrating the profiles $|W_{\mathrm{in}}[|\nabla|^{1/2}\widetilde{F}_n^{\omega}]|^2(\tau)$ and $|W_{\mathrm{out}}[|\nabla|^{1/2}\widetilde{F}_n^{\omega}]|^2(\tau)$ against the flux estimate on $t \pm |x| = \tau$. Under a bootstrap hypothesis, we obtain the interaction flux estimate

$$\begin{aligned}\int_I\int_{\mathbb{R}^3}|x|^2\,\big||\nabla|^{\frac{1}{2}}\widetilde{F}_N^{\omega}\big|^2 v^6\,\mathrm{d}x\,\mathrm{d}t &\lesssim \big(\|W_{\mathrm{in}}[|\nabla|^{\frac{1}{2}}\widetilde{F}_n^{\omega}](\tau)\|^2_{L^2_\tau(\mathbb{R})} + \|W_{\mathrm{out}}[|\nabla|^{\frac{1}{2}}\widetilde{F}_n^{\omega}](\tau)\|^2_{L^2_\tau(\mathbb{R})}\big)\mathcal{E}\\ &\lesssim \|(f_N^{\omega}, g_N^{\omega})\|^2_{\dot{H}^{1/2}_x\times\dot{H}^{-1/2}_x}\mathcal{E}.\end{aligned}$$

We have not seen this estimate in the previous literature. It is reminiscent of the interaction Morawetz estimate for the NLS [Colliander et al. 2004], but it controls an interaction between the linear and nonlinear evolution rather than the interaction of the nonlinear evolution with itself. We believe that similar interaction estimates may be of interest beyond this work. Using the interaction flux estimate, we bound

$$\begin{aligned}&\left|\int_I\int_{\mathbb{R}^3}(|\nabla|^{\frac{1}{2}}\widetilde{F}_N)v^4(|\nabla|^{\frac{1}{2}}v)\,\mathrm{d}x\,\mathrm{d}t\right|\\ &\lesssim \||x|^{\frac{3}{8}}|\nabla|^{\frac{1}{2}}\widetilde{F}_N^{\omega}\|^{\frac{2}{3}}_{L_t^{8/3}L_x^{\infty}(I\times\mathbb{R}^3)}\||x|^{\frac{1}{3}}(|\nabla|^{\frac{1}{2}}\widetilde{F}_N^{\omega})^{\frac{1}{3}}v\|_{L^6_{t,x}(I\times\mathbb{R}^3)}\||x|^{-\frac{1}{6}}v\|^{\frac{7}{2}}_{L^6_{t,x}(I\times\mathbb{R}^3)}\|\nabla v\|^{\frac{1}{2}}_{L_t^{\infty}L_x^2(I\times\mathbb{R}^3)}\\ &\lesssim \||x|^{\frac{3}{8}}|\nabla|^{\frac{1}{2}}\widetilde{F}_N^{\omega}\|^{\frac{2}{3}}_{L_t^{8/3}L_x^{\infty}(I\times\mathbb{R}^3)}\|(f_N^{\omega}, g_N^{\omega})\|^{\frac{1}{3}}_{\dot{H}^{1/2}_x\times\dot{H}^{-1/2}_x}\mathcal{E}.\end{aligned}$$

From the probabilistic Strichartz estimates, we will see that the seminorm of $\widetilde{F}^{\omega}$ scales like $\dot{H}_x^{5/4}\times\dot{H}_x^{1/4}$ and has a probabilistic gain of $\frac{1}{8\gamma}$-derivatives. Thus, we expect the regularity restriction

$$s > \tfrac{2}{3}\cdot\big(\tfrac{5}{4} - \tfrac{1}{8\gamma}\big) + \tfrac{1}{3}\cdot\tfrac{1}{2} = 1 - \tfrac{1}{12\gamma}.$$

Outline. In Section 2, we review basic facts from harmonic analysis. In Sections 3 and 4, we study solutions to the radial linear wave equation. First, we prove a refined radial Strichartz estimate which is based on [Sterbenz 2005]. As a consequence, we obtain probabilistic Strichartz estimates for the radial randomization. Then, we discuss the in/out decomposition mentioned above in detail. In Sections 5 and 6, we study solutions to the forced nonlinear wave equation (6). We prove an almost energy conservation law and an approximate Morawetz estimate. Here, we also introduce the novel interaction flux estimate between the linear and nonlinear evolution. In Sections 7 and 8, we set up a bootstrap argument to bound the energy and estimate the error terms. Finally, we prove the main theorem in Section 9.

2. Notation and preliminaries

In this section, we introduce the notation that will be used throughout the rest of this paper. We also recall some basic results from harmonic analysis and prove certain auxiliary lemmas.

If A and B are two nonnegative quantities, we write $A \lesssim B$ if there exists an absolute constant $C > 0$ such that $A \leq CB$. We write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. For a vector $x \in \mathbb{R}^d$, we write $|x| := \left(\sum_{i=1}^d x_i^2\right)^{1/2}$. We define the Fourier transform of a Schwartz function f by setting

$$\hat{f}(\xi) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \exp(-ix\xi) f(x) \,\mathrm{d}x.$$

We denote by $J_\nu(x)$ the Bessel functions of the first kind. Recall that for a spherically symmetric function f we have

$$\hat{f}(\xi) = |\xi|^{-\frac{d-2}{2}} \int_0^\infty J_{\frac{d-2}{2}}(|\xi| r) f(r) r^{\frac{d}{2}} \,\mathrm{d}r.$$

With a slight abuse of notation, we identify a spherically symmetric function $f : \mathbb{R}^d \to \mathbb{R}$ with a function $f : \mathbb{R}_{>0} \to \mathbb{R}$.

2A. ***Littlewood–Paley theory and Sobolev embeddings.*** We start this section by defining the Littlewood–Paley operators P_L. Let $\phi \in C_c^\infty(\mathbb{R}^d)$ be a nonnegative radial bump function such that $\phi|_{B(0,1)} \equiv 1$ and $\phi_{\mathbb{R}^d \setminus B(0,2)} \equiv 0$. We set $\Psi_1(\xi) = \phi(\xi)$ and, for a dyadic $L > 1$, we set

$$\Psi_L(\xi) = \phi\left(\frac{\xi}{L}\right) - \phi\left(\frac{\xi}{2L}\right).$$

Then, we define the Littlewood–Paley operators P_L by

$$\widehat{P_L f}(\xi) = \Psi_L(\xi) \hat{f}(\xi).$$

To simplify the notation, we also write $f_L := P_L f$.

Lemma 2.1 (Bernstein estimate). *For any* $1 < p_1 \leq p_2 < \infty$ *and* $s \geq 0$, *we have the Bernstein inequalities*

$$\begin{aligned}
&\text{for all } L \geq 1, && \|f_L\|_{L_x^{p_2}(\mathbb{R}^d)} \lesssim L^{\frac{d}{p_1} - \frac{d}{p_2}} \|f_L\|_{L_x^{p_1}(\mathbb{R}^d)}, \\
&\text{for all } L > 1, && \||\nabla|^{\pm s} f_L\|_{L_x^{p_1}(\mathbb{R}^d)} \sim L^{\pm s} \|f_L\|_{L_x^{p_1}(\mathbb{R}^d)}, \\
&\text{for all } L > 1, && \|\nabla f_L\|_{L_x^{p_1}(\mathbb{R}^d)} \sim L \|f_L\|_{L_x^{p_1}(\mathbb{R}^d)}.
\end{aligned}$$

Lemma 2.2 (square-function estimate, see [Muscalu and Schlag 2013, Theorem 8.3]). *Let* $1 < p < \infty$. *Then, we have for all* $f \in L_x^p(\mathbb{R}^d)$ *that*

$$\|f\|_{L_x^p(\mathbb{R}^d)} \sim_{d,p} \|f_L\|_{L_x^p \ell_L^2(\mathbb{R}^d \times 2^{\mathbb{N}})}. \tag{10}$$

For notational convenience, we use a different function to define a dyadic decomposition in physical space. As before, we let $\chi \in C_c^\infty(\mathbb{R}^d)$ be a nonnegative, radial bump function such that $\chi|_{B(0,1)} \equiv 1$ and

$\chi|_{\mathbb{R}^d \setminus B(0,2)} \equiv 0$. We also assume that χ is radially nonincreasing. We set $\chi_1 := \chi$, and for any dyadic $J > 1$, we set

$$\chi_J(x) := \chi\left(\frac{x}{J}\right) - \chi\left(\frac{2x}{J}\right).$$

Thus, the family $\{\chi_J\}_{J\geq 1}$defines a partition of unity adapted to dyadic annuli. Furthermore, we let $\tilde{\chi}_J$ be a slightly fattened version of χ_J.

Lemma 2.3 (mismatch estimate). *Let $L, J, K \in 2^{\mathbb{N}_0}$. Furthermore, we assume that the separation condition $J/K + K/J \geq 2^5$ holds. Then, we have for all $1 \leq r \leq \infty$ that*

$$\|\chi_J P_L \chi_K\|_{L^r_x(\mathbb{R}^d)\to L^r_x(\mathbb{R}^d)} \lesssim_M (LJK)^{-M} \quad \textit{for all } M > 0. \tag{11}$$

We follow the argument in [Dodson et al. 2019, Lemma 5.10], which treats the case $L = 1$.

Proof. Let $f \in L^r_x(\mathbb{R}^d)$ be arbitrary. Let ϕ be a suitable bump function on the annulus $|x| \sim 1$. Using the separation condition, it holds that

$$\begin{aligned}\chi_J P_L \chi_K f(x) &= \chi_J(x) L^d \int_{\mathbb{R}^d} \check{\Psi}(L(x-y))\chi_K(y) f(y)\,\mathrm{d}y \\ &= \chi_J(x) L^d \int_{\mathbb{R}^d} \check{\Psi}(L(x-y))\phi(\max(J,K)^{-1}(x-y))\chi_K(y) f(y)\,\mathrm{d}y.\end{aligned}$$

From Young's inequality, it follows that

$$\|\chi_J P_L \chi_K f\|_{L^r_x(\mathbb{R}^d)} \leq \|L^d \check{\Psi}(Lx)\phi(\max(J,K)^{-1}x)\|_{L^1_x(\mathbb{R}^d)} \|f\|_{L^r_x(\mathbb{R}^d)}.$$

Next, we estimate

$$\begin{aligned}\|L^d \check{\Psi}(Lx)\phi(\max(J,K)^{-1}x)\|_{L^1_x} &= L^d \int_{\mathbb{R}^d} |\check{\Psi}(Lx)|\phi(\max(J,K)^{-1}x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} |\check{\Psi}(x)|\phi(L^{-1}\max(J,K)^{-1}x)\,\mathrm{d}x \\ &= \int_{|x|\sim L\max(J,K)} |\check{\Psi}(x)|\,\mathrm{d}x \lesssim_M (L\max(J,K))^{-M}. \qquad \square\end{aligned}$$

Lemma 2.4 (Bernstein-type estimate). *Let $L \in 2^{\mathbb{N}_0}$, $1 < p < \infty$, and $\alpha > 0$. Then, we have*

$$\|\langle x\rangle^{-\alpha} P_L f\|_{L^p_x(\mathbb{R}^d)} \lesssim L^{-1}\|\langle x\rangle^{-\alpha}\nabla f\|_{L^p_x(\mathbb{R}^d)} + L^{-1}\|\langle x\rangle^{-\alpha-1} f\|_{L^p_x(\mathbb{R}^d)}. \tag{12}$$

By iterating this inequality, we could further decrease the weight in the term $\|\langle x\rangle^{-\alpha-1} f\|_{L^p_x}$.

Proof. The proof is based on a dyadic decomposition, the localized kernel estimate (11), and the standard Bernstein estimate. We have

$$\begin{aligned}\|\langle x\rangle^{-\alpha} P_L f\|^p_{L^p_x(\mathbb{R}^d)} &\lesssim \sum_{J\geq 1}^{\infty} J^{-\alpha p}\|\chi_J P_L f\|^p_{L^p_x(\mathbb{R}^d)} \\ &\lesssim \sum_{J\geq 1}^{\infty} a J^{-\alpha p}\|\chi_J P_L \tilde{\chi}_J f\|^p_{L^p_x(\mathbb{R}^d)} + \sum_{J\geq 1}^{\infty} J^{-\alpha p}\Big(\sum_{K:K\not\sim J}\|\chi_J P_L \chi_K f\|_{L^p_x(\mathbb{R}^d)}\Big)^p. \end{aligned}\tag{13}$$

We now estimate the first summand in (13). Using the Bernstein estimate, we have

$$
\begin{aligned}
\sum_{J\geq 1}^{\infty} J^{-\alpha p}\|\chi_J P_L \tilde{\chi}_J f\|^p_{L^p_x(\mathbb{R}^d)} &\leq \sum_{J\geq 1}^{\infty} J^{-\alpha p}\|P_L \tilde{\chi}_J f\|^p_{L^p_x(\mathbb{R}^d)} \\
&\lesssim \sum_{J\geq 1}^{\infty} J^{-\alpha p} L^{-p}\|\nabla(\tilde{\chi}_J f)\|^p_{L^p_x(\mathbb{R}^d)} \\
&\lesssim \sum_{J\geq 1}^{\infty} J^{-\alpha p} L^{-p}\|\tilde{\chi}_J \nabla f\|^p_{L^p_x(\mathbb{R}^d)} + \sum_{J\geq 1}^{\infty} J^{-\alpha p} L^{-p}\|\nabla(\tilde{\chi}_J) f\|^p_{L^p_x(\mathbb{R}^d)} \\
&\lesssim \sum_{J\geq 1}^{\infty} J^{-\alpha p} L^{-p}\|\nabla f\|^p_{L^p_x(|x|\sim J)} + \sum_{J\geq 1}^{\infty} J^{-(\alpha+1)p} L^{-p}\|f\|^p_{L^p_x(|x|\sim J)} \\
&\lesssim L^{-p}\|\langle x\rangle^{-\alpha}\nabla f\|^p_{L^p_x(\mathbb{R}^d)} + L^{-p}\|\langle x\rangle^{-\alpha-1} f\|^p_{L^p_x(\mathbb{R}^d)}.
\end{aligned}
$$

Thus, it remains to estimate the second summand in (13). Using (11) and choosing $M > 0$ large, we have

$$
\begin{aligned}
&\sum_{J\geq 1}^{\infty} J^{-\alpha p}\Big(\sum_{K:K\not\sim J}\|\chi_J P_L \chi_K f\|_{L^p_x(\mathbb{R}^d)}\Big)^p \\
&\qquad\lesssim \sum_{J\geq 1}^{\infty} J^{-\alpha p}\Big(\sum_{K:K\not\sim J}(JKL)^{-(M+\alpha+1)}\|\tilde{\chi}_K f\|_{L^p_x(\mathbb{R}^d)}\Big)^p \\
&\qquad\lesssim L^{-(M+\alpha+1)p}\sum_{J\geq 1}^{\infty} J^{-(M+2\alpha+1)p}\Big(\sum_{K\geq 1} K^{-M}\Big)^p \|\langle x\rangle^{-\alpha-1} f\|^p_{L^p_x(\mathbb{R}^d)} \\
&\qquad\lesssim L^{-p}\|\langle x\rangle^{-\alpha-1} f\|^p_{L^p_x(\mathbb{R}^d)}. \qquad \square
\end{aligned}
$$

In Section 8B, we will use a Littlewood–Paley decomposition in an error term coming from the Morawetz estimate. To control this error, we will need the following estimate for the Morawetz weight $x/|x|$.

Lemma 2.5. *Let $L > 1$ and let $d \geq 2$. Then, we have*

$$
\left|P_L\Big(\frac{x}{|x|}\Big)\right| \lesssim \frac{1}{L|x|}. \tag{14}
$$

Proof. Let $j = 1, \dots, d$. It holds that

$$
\begin{aligned}
\left|P_L\Big(\frac{x_j}{|x|}\Big)\right| &= L^d\left|\int_{\mathbb{R}^d}\check{\Psi}(Ly)\frac{x_j - y_j}{|x-y|}\,\mathrm{d}y\right| \\
&= L^d\left|\int_{\mathbb{R}^d}\check{\Psi}(Ly)\Big(\frac{x_j - y_j}{|x-y|} - \frac{x_j}{|x|}\Big)\,\mathrm{d}y\right| \\
&\leq L^d\int_{\mathbb{R}^d}|\check{\Psi}(Ly)|\left|\frac{x_j(|x| - |x-y|) - y_j|x|}{|x-y|\,|x|}\right|\mathrm{d}y \\
&\leq L^d\int_{\mathbb{R}^d}|\check{\Psi}(Ly)|\frac{|y|}{|x-y|}\,\mathrm{d}y \leq \int_{\mathbb{R}^d}|\check{\Psi}(y)|\frac{|y|}{|Lx-y|}\,\mathrm{d}y.
\end{aligned}
$$

Using the rapid decay of $\check{\Psi}$, the estimate then follows by splitting the integral into the regions $|y| \leq \frac{1}{2}L|x|$, $|y| \sim L|x|$, and $|y| \geq 2L|x|$. □

In addition to the standard Sobolev embedding, we will also rely on the following weighted Sobolev embedding for radial functions.

Proposition 2.6 (radial Sobolev embedding, see [De Nápoli and Drelichman 2016, Remark 2.1] and [De Nápoli et al. 2011]). *Let $d \geq 1$, $0 < s < d$, $1 < p < \infty$,*

$$\alpha < \frac{d}{p'}, \quad \beta > -\frac{d}{q}, \quad \alpha - \beta \geq (d-1)\left(\frac{1}{q} - \frac{1}{p}\right), \quad \text{and} \quad \frac{1}{q} = \frac{1}{p} + \frac{\alpha - \beta - s}{d}.$$

If $p \leq q < \infty$, then the inequality

$$\| |x|^\beta f \|_{L^q_x} \lesssim \| |x|^\alpha |\nabla|^s f \|_{L^p_x} \tag{15}$$

holds for all radially symmetric f. If $q = \infty$, the result holds provided that

$$\alpha - \beta > (d-1)\left(\frac{1}{q} - \frac{1}{p}\right).$$

2B. *Calderón–Zygmund theory.* In order to use weighted estimates, we introduce some basic Calderón–Zygmund theory.

Definition 2.7 [Stein 1993, Section V]. Let $w \in L^1_{\text{loc}}(\mathbb{R}^d)$ be nonnegative. For $1 < p < \infty$, we say that w satisfies the A_p-condition if

$$\sup_{B = B_r(x)} \left(\frac{1}{|B|} \int_B w \, \mathrm{d}y \right) \left(\frac{1}{|B|} \int_B w^{-\frac{p'}{p}} \, \mathrm{d}y \right)^{\frac{p}{p'}} < \infty. \tag{16}$$

The following well-known criterion for power weights can be proven by a simple computation.

Lemma 2.8 [Stein 1993, Section V.6]. *Let $w = |x|^\alpha$ and let $1 < p < \infty$. Then w satisfies the A_p-condition if and only if*

$$-d < \alpha < d(p-1).$$

The following proposition is a consequence of Theorem 7.21 and the proof of Theorem 8.2 in [Muscalu and Schlag 2013]. We also refer the reader to [Stein 1993, p. 205].

Proposition 2.9 (Mikhlin-multiplier theorem). *Let $m : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ be a smooth function. Assume that m satisfies for any multi-index γ of length $|\gamma| \leq d+2$*

$$|\partial^\gamma m(\xi)| \leq B|\xi|^{-|\gamma|}.$$

Let $m(\nabla/i)$ be the associated Fourier multiplier and let $1 < p < \infty$. For any A_p-weight w, there exists a constant C, depending only on d, p, and the supremum in (16), *such that*

$$\|m(\nabla/i) f\|_{L^p(w\mathrm{d}x)} \leq CB \|f\|_{L^p(w\mathrm{d}x)} \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^d).$$

Remark 2.10. We will apply Proposition 2.9 to the Riesz multipliers $m_j(\xi) = \xi_j/|\xi|$ and to the Littlewood–Paley multipliers $\Psi_L(\xi)$.

3. Probabilistic Strichartz estimates

In this section, we derive probabilistic Strichartz estimates for the radial randomization. For the Wiener randomization, there exist two different methods for proving probabilistic Strichartz estimates.

The first method relies on Bernstein-type inequalities for the multipliers $f \mapsto \psi(\nabla/i - k) f$. After using Khintchine's inequality to decouple the individual atoms of the randomization, the L_x^p-improving properties of the multiplier are used to move from a space $L_t^q L_x^{p_{\text{hi}}}$ into a space $L_t^q L_x^{p_{\text{lo}}}$. Then, one applies the usual Strichartz estimate to control the evolution in $L_t^q L_x^{p_{\text{lo}}}$, which depends more favorably on the regularity of the initial data. For example, this method has been used in [Bényi et al. 2015a; 2015b; 2019; Oh et al. 2017; Killip et al. 2019; Lührmann and Mendelson 2014].

The second method relies on refined Strichartz inequalities. Here, the frequency localization is used explicitly to derive improved Strichartz estimates. To mention one example, the refined Strichartz estimate in [Klainerman and Tataru 1999] is based on a new $L_x^1 \to L_x^\infty$-dispersive decay estimate. In the probabilistic context, this approach was first used in [Dodson et al. 2017].

For the radial randomization, the multipliers are of the form $f \mapsto A_{a,\delta} f$. In a celebrated paper, Fefferman [1971] proved that the annular Fourier multipliers in dimension $d \geq 2$ are bounded on L^p if and only if $p = 2$. However, if we restrict to radial functions, then the annular Fourier multipliers are bounded on L^p for all $2d/(d+1) < p < 2d/(d-1)$; see [Chanillo 1984]. Using Young's inequality, it is also possible to prove $L_x^1 \to L_x^p$ bounds for $p > 2d/(d+1)$. From interpolation and duality, one can then obtain the strong-type diagram for the annular Fourier-multipliers on radial functions. However, the dependence of the operator norm on the normalized width δ is rather complicated, and the resulting Strichartz estimates are nonoptimal. Instead of using the Bernstein-based method, we therefore prove a new refined Strichartz estimate for radial initial data. As in previous works, we can then use Khintchine's inequality to obtain probabilistic Strichartz estimates.

Proposition 3.1 (refined radial Strichartz estimate). *Let* $f \in L^2_{\text{rad}}(\mathbb{R}^d)$. *Let* $0 < \delta \leq 1$ *and assume that there exists an interval* $I \subseteq \left[\frac{1}{2}, 2\right]$ *such that* $|I| \leq \delta$ *and* $\operatorname{supp} \hat{f} \subseteq \{\xi : \|\xi\|_2 \in I\}$. *Then, we have*

$$\| |x|^\alpha \exp(\pm it|\nabla|) f \|_{L_t^q L_x^p(\mathbb{R}\times\mathbb{R}^d)} \lesssim_{\alpha,q,p} \delta^{\frac{1}{2} - \frac{1}{\min(p,q)}} \|f\|_{L_x^2(\mathbb{R}^d)} \tag{17}$$

as long as

$$-\frac{d}{p} < \alpha < (d-1)\Big(\frac{1}{2} - \frac{1}{p}\Big) - \frac{1}{q} \quad \text{if } 2 \leq q, p < \infty, \tag{18}$$

$$-\frac{d}{p} < \alpha \leq (d-1)\Big(\frac{1}{2} - \frac{1}{p}\Big) \quad \text{if } q = \infty,\ 2 \leq p < \infty, \tag{19}$$

$$0 \leq \alpha < \frac{d-1}{2} - \frac{1}{q} \quad \text{if } 2 \leq q < \infty,\ p = \infty, \tag{20}$$

$$0 \leq \alpha \leq \frac{d-1}{2} \quad \text{if } q = p = \infty. \tag{21}$$

The estimates of Proposition 3.1 can be visualized using a "Strichartz game room"; see Figure 2. Proposition 3.1 is a refinement of [Jiang et al. 2012, Theorem 1.5] and [Sterbenz 2005, Proposition 1.2],

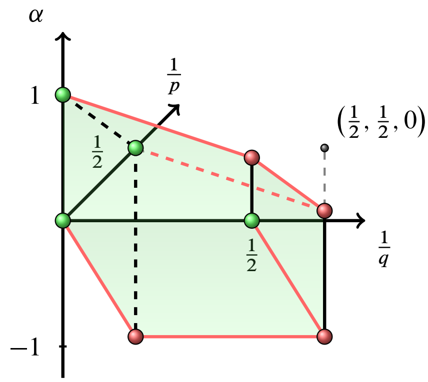

Figure 2. We display the weighted radial Strichartz estimate from Proposition 3.1 in $d = 3$. The true endpoint estimates correspond to either green spheres or black lines, whereas the false endpoint estimates correspond to either red spheres or red lines. The black sphere at $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$ serves as a visual aid.

and we follow their argument closely. We remark that the corresponding Strichartz estimate for non-frequency-localized functions [Jiang et al. 2012, Theorem 1.5] may fail for some of the endpoints above.

Proof. By time-reflection symmetry, it suffices to treat the operator $\exp(it|\nabla|)$. Recall that we denote by J_ν the Bessel functions of the first kind. For any radial function $f \in L^2_{\mathrm{rad}}(\mathbb{R}^d)$, we identify $\hat{f}$ with a function $\hat{f} : \mathbb{R}_{>0} \to \mathbb{R}$. Then, it holds that

$$\exp(it|\nabla|) f(r) = r^{-\frac{d-2}{2}} \int_0^\infty \exp(it\rho) J_{\frac{d-2}{2}}(r\rho) \hat{f}(\rho) \rho^{\frac{d}{2}} \, \mathrm{d}\rho. \tag{22}$$

Inserting the known asymptotics for Bessel functions, see [Jiang et al. 2012], we may estimate

$$(1+r)^{-\frac{d-1}{2}} \int_0^{2\pi} \exp(i(t \pm r)\rho) m(r;\rho) \phi_{(\frac{1}{4},4)}(\rho) \hat{f}(\rho) \, \mathrm{d}\rho. \tag{23}$$

Here, $\phi_{(1/4,4)}$ is a smooth cutoff-function that equals 1 on $\left[\frac{1}{2}, 2\right]$ and is supported on $\left[\frac{1}{4}, 4\right]$, and $m(r;\rho)$ is a smooth function that satisfies $|\partial_\rho^j m(r;\rho)| \lesssim_j 1$ for all $j \geq 0$. Since $\operatorname{supp} \hat{f} \subseteq \left[\frac{1}{2}, 2\right]$, we may write

$$\hat{f}(\rho) = \sum_{k \in \mathbb{Z}} c_k \exp(ik\rho), \quad \text{where } c_k = \frac{1}{2\pi} \int_0^{2\pi} \exp(-ik\rho) \hat{f}(\rho) \, \mathrm{d}\rho. \tag{24}$$

Inserting (24) into (23), we have to bound

$$\sum_{k \in \mathbb{Z}} (1+r)^{-\frac{d-1}{2}} c_k \int_0^{2\pi} \exp(i(t \pm r + k)\rho) m(r;\rho) \phi_{(\frac{1}{4},4)}(\rho) \, \mathrm{d}\rho. \tag{25}$$

Integrating by parts $2M$ times, we have

$$\left| \int_0^{2\pi} \exp(i(t \pm r + k)) m(r;\rho) \phi_{(\frac{1}{4},4)}(\rho) \, \mathrm{d}\rho \right| \lesssim_M (1 + |t \pm r + k|)^{-2M}.$$

Therefore, we obtain

$$\begin{aligned}\||x|^\alpha \exp(it|\nabla|)f\|_{L^p_x(\mathbb{R}^d)} &\lesssim \|(1+r)^{-\frac{d-1}{2}} r^\alpha r^{\frac{d-1}{p}}(1+|t+k\pm r|)^{-2M} c_k\|_{L^p_r \ell^1_k(\mathbb{R}_{>0}\times\mathbb{Z})}\\ &\lesssim \|(1+r)^{-\frac{d-1}{2}} r^{\alpha+\frac{d-1}{p}}(1+|t+k\pm r|)^{-M} c_k\|_{L^p_r \ell^p_k(\mathbb{R}_{>0}\times\mathbb{Z})}, \qquad (26)\end{aligned}$$

where we have used Hölder's inequality in the k-variable. Since

$$\alpha+\frac{d-1}{p} > -\frac{1}{p} \quad \text{if } 2\le p<\infty,$$

and $\alpha \ge 0$ if $p=\infty$, we obtain for sufficiently large M that

$$\|(1+r)^{-\frac{d-1}{2}} r^{\alpha+\frac{d-1}{p}}(1+|t+k\pm r|)^{-M}\|_{L^p_r(\mathbb{R}_{>0})} \lesssim (1+|t+k|)^{-\frac{d-1}{2}}|t+k|^{\alpha+\frac{d-1}{p}}.$$

From the embedding $\ell_k^{\min(p,q)} \hookrightarrow \ell_k^p$ and Minkowski's integral inequality, we obtain

$$\begin{aligned}\||x|^\alpha \exp(it|\nabla|)f\|_{L^q_t L^p_x(\mathbb{R}\times\mathbb{R}^d)} &\lesssim \|(1+|t+k|)^{-\frac{d-1}{2}}|t+k|^{\alpha+\frac{d-1}{p}} c_k\|_{L^q_t \ell^p_k(\mathbb{R}\times\mathbb{Z})}\\ &\lesssim \|(1+|t+k|)^{-\frac{d-1}{2}}|t+k|^{\alpha+\frac{d-1}{p}} c_k\|_{\ell_k^{\min(p,q)} L^q_t(\mathbb{Z}\times\mathbb{R})}\\ &\lesssim \|c_k\|_{\ell_k^{\min(p,q)}(\mathbb{Z})}. \qquad (27)\end{aligned}$$

From Plancherel's theorem and the support condition on $\hat{f}$, we have

$$\|c_k\|^2_{\ell^2_k(\mathbb{Z})} = \frac{1}{2\pi}\int_0^{2\pi} |\hat{f}(\rho)|^2 \,\mathrm{d}\rho \sim \|f\|^2_{L^2_x(\mathbb{R}^d)}.$$

Furthermore, since supp $\hat{f}$ is contained in an interval of size $\le\delta$, we have

$$\|c_k\|_{\ell^\infty_k(\mathbb{Z})} \le \frac{1}{2\pi}\int_I |\hat{f}(\rho)|\,\mathrm{d}\rho \lesssim \delta^{\frac12}\|f\|_{L^2_x(\mathbb{R}^d)}.$$

Then (17) follows from (27) and Hölder's inequality. □

Remark 3.2. We note that there is no δ-gain for $q=2$. For instance, this follows from a nonstationary phase argument by choosing f as the inverse Fourier transform of $\chi_{[1,1+\delta]}(|\xi|)$. As a consequence, we obtain no probabilistic gain for Strichartz estimates with parameter $q=2$; see Lemma 3.4. This indicates that the spherical symmetry imposes restrictions on the randomized linear evolutions. We therefore view the radial randomization as a modest step towards probabilistic treatments of the geometric equations discussed in [Chanillo et al. 2017].

Corollary 3.3. *Let $f\in L^2_{\mathrm{rad}}(\mathbb{R}^d)$ and $A_{a,\delta}$ as in* (3) *with $a\sim N$. If α, p, and q satisfy* (18)–(21), *then*

$$\||x|^\alpha \exp(\pm it|\nabla|)A_{a,\delta}f\|_{L^q_t L^p_x} \lesssim N^{\frac{d}{2}-\alpha-\frac1q-\frac{d}{p}}\delta^{\frac12-\frac{1}{\min(p,q)}}\|A_{a,\delta}f\|_{L^2_x}. \qquad (28)$$

Proof. For any $g\in L^2_{\mathrm{rad}}(\mathbb{R}^d)$, we have

$$A_{a,\delta}g(x) = \Big(A_{\frac{a}{N},\delta}\Big(g\Big(\frac{\cdot}{N}\Big)\Big)\Big)(Nx).$$

From scaling and (17), it then follows that

$$\| |x|^{\alpha} \exp(\pm it|\nabla|) A_{a,\delta} f \|_{L_t^q L_x^p} \lesssim N^{\frac{d}{2}-\alpha-\frac{1}{q}-\frac{d}{p}} \delta^{\frac{1}{2}-\frac{1}{\min(p,q)}} \|f\|_{L_x^2}.$$

Finally, replacing f by $A_{a,\delta} f$ above, we arrive at (28). □

Lemma 3.4 (probabilistic Strichartz estimates). *Let $f \in H^s_{\mathrm{rad}}(\mathbb{R}^d)$ with*

$$s \geq \frac{d}{2} - \frac{1}{q} - \frac{d}{p} - \alpha - \frac{1}{\gamma}\Big(\frac{1}{2} - \frac{1}{\min(p,q)}\Big), \tag{29}$$

where γ is as in Definition 1.1. Let α and $2 \leq p, q < \infty$ satisfy (18). *Then, we have for all $1 \leq \sigma < \infty$ that*

$$\| |x|^{\alpha} \exp(\pm it|\nabla|) f^{\omega} \|_{L^{\sigma}_{\omega} L_t^q L_x^p} \lesssim_{p,q,\alpha,s} \sqrt{\sigma} \|f\|_{H^s_x(\mathbb{R}^4)}. \tag{30}$$

Proof. We prove (30) only for $\sigma \geq \max(p,q)$. The general case then follows by Hölder in the ω-variable. From the square-function estimate (Lemma 2.2), Minkowski's integral inequality, Khintchine's inequality, and Corollary 3.3, it follows that

$$\begin{aligned}
\| |x|^{\alpha} \exp(\pm it|\nabla|) f^{\omega} \|_{L^{\sigma}_{\omega} L_t^q L_x^p} &\leq \| |x|^{\alpha} \exp(\pm it|\nabla|) f_N^{\omega} \|_{L^{\sigma}_{\omega} L_t^q L_x^p \ell_N^2} \\
&\leq \| |x|^{\alpha} \exp(\pm it|\nabla|) f_N^{\omega} \|_{\ell_N^2 L_t^q L_x^p L^{\sigma}_{\omega}} \\
&\lesssim \sqrt{\sigma} \| |x|^{\alpha} \exp(\pm it|\nabla|) A_k f_N \|_{\ell_N^2 L_t^q L_x^p \ell_k^2} \\
&\leq \sqrt{\sigma} \| |x|^{\alpha} \exp(\pm it|\nabla|) A_k f_N \|_{\ell_N^2 \ell_k^2 L_t^q L_x^p} \\
&\leq \sqrt{\sigma} \| N^{\frac{d}{2}-\alpha-\frac{1}{q}-\frac{d}{p}} (N^{-\frac{1}{\gamma}})^{\frac{1}{2}-\frac{1}{\min(p,q)}} A_k f_N \|_{\ell_N^2 \ell_k^2 L_x^2} \\
&\leq \sqrt{\sigma} \| N^s f_N \|_{\ell_N^2 L_x^2} \\
&\leq \sqrt{\sigma} \|f\|_{H^s_x}.
\end{aligned}$$

We remark that f_1 is only localized to frequencies $\lesssim 1$, so that the inhomogeneous Sobolev norm above is necessary. □

Lemma 3.5 (probabilistic L^{∞}_x-Strichartz estimates). *Let $f_N \in L^2_{\mathrm{rad}}(\mathbb{R}^3)$ and let f_N^{ω} be its radial randomization. Then, we have*

$$\| |x|^{\frac{3}{8}} \exp(\pm it|\nabla|) f_N^{\omega} \|_{L^{\sigma}_{\omega} L_t^{8/3} L_x^{\infty}} \lesssim \sqrt{\sigma} N^{\frac{3}{4}-\frac{1}{8\gamma}} \|f_N\|_{L_x^2},$$

$$\| |x|^{\frac{1}{4}} \exp(\pm it|\nabla|) f_N^{\omega} \|_{L^{\sigma}_{\omega} L_t^{4} L_x^{\infty}} \lesssim \sqrt{\sigma} N^{1-\frac{1}{4\gamma}} \|f_N\|_{L_x^2}.$$

Remark 3.6. Since $p = \infty$, we can no longer use the usual combination of Minkowski's integral inequality and Khintchine's inequality. We resolve this by using a radial Sobolev embedding.

Proof. Let $1 \leq p < \infty$ be a sufficiently large exponent. Using Proposition 2.6 and Lemma 3.4, we have for all $p \leq \sigma < \infty$ that

$$\| |x|^{\frac{3}{8}} \exp(\pm it|\nabla|) f_N^{\omega} \|_{L^{\sigma}_{\omega} L_t^{8/3} L_x^{\infty}} \lesssim \| |x|^{\frac{3}{8}} \exp(\pm it|\nabla|) |\nabla|^{\frac{3}{p}} f_N^{\omega} \|_{L^{\sigma}_{\omega} L_t^{8/3} L_x^{p}} \lesssim \sqrt{\sigma} N^{\frac{3}{4}-\frac{1}{8\gamma}} \|f_N\|_{L_x^2}.$$

Note that, due to scaling, the parameter p does not appear in the final estimate. Similarly, we have

$$\| |x|^{\frac{1}{4}} \exp(\pm it|\nabla|) f_N^\omega \|_{L^\sigma_\omega L^4_t L^\infty_x} \lesssim \| |x|^{\frac{1}{4}} \exp(\pm it|\nabla|) |\nabla|^{\frac{3}{p}} f_N^\omega \|_{L^\sigma_\omega L^4_t L^p_x} \lesssim \sqrt{\sigma} N^{1-\frac{1}{4\gamma}} \| f_N \|_{L^2_x}. \qquad \square$$

Lemma 3.7 (probabilistic L^∞_t-Strichartz estimates). *Let $f \in L^2_{\mathrm{rad}}(\mathbb{R}^3)$ and let $\delta > 0$. Then, we have for all $1 \le \sigma < \infty$ and all $N \in 2^{\mathbb{Z}}$ that*

$$\| \exp(\pm it|\nabla|) f_N^\omega \|_{L^\sigma_\omega L^\infty_t L^6_x} \lesssim \sqrt{\sigma} N^{1-\frac{1}{3\gamma}} \| f_N \|_{L^2_x}, \tag{31}$$

$$\| |x|^{\frac{1}{2}} \exp(\pm it|\nabla|) f_N^\omega \|_{L^\sigma_\omega L^\infty_t L^\infty_x} \lesssim_\delta \sqrt{\sigma} N^{1-\frac{1-\delta}{2\gamma}} \| f_N \|_{L^2_x}. \tag{32}$$

Remark 3.8. Since $q = \infty$, we can no longer use the same combination of Minkowski's integral inequality and Khintchine's inequality as in the proof of Lemma 3.4. The same problem was encountered in previous works using the Wiener randomization. In [Oh and Pocovnicu 2016, Proposition 3.3], a chaining-type method was used to bound L^∞_t-norms on compact time intervals. In [Killip et al. 2019, Proposition 2.10], the authors obtain global control on an L^∞_t-norm via the fundamental theorem of calculus. Here we present a slight modification of their argument. An alternative approach consists of using a fractional Sobolev embedding in time [Dodson et al. 2019].

Proof. Let $1 < q < \infty$ be sufficiently large and assume that $\sigma \ge q$. We fix $t_0, t_1 \in \mathbb{R}$. By the fundamental theorem of calculus, it holds that

$$\begin{aligned} \| \exp(it_1|\nabla|) f_N^\omega \|_{L^6_x} &\le \| \exp(it_0|\nabla|) f_N^\omega \|_{L^6_x} + \int_{[t_0,t_1]} \| \partial_t (\exp(it|\nabla|) f_N^\omega) \|_{L^6_x} \,\mathrm{d}t \\ &\lesssim \| \exp(it_0|\nabla|) f_N^\omega \|_{L^6_x} + N \int_{[t_0,t_1]} \| \exp(it|\nabla|) f_N^\omega \|_{L^6_x} \,\mathrm{d}t \\ &\lesssim \| \exp(it_0|\nabla|) f_N^\omega \|_{L^6_x} + N (t_1 - t_0)^{\frac{1}{q'}} \| \exp(it|\nabla|) f_N^\omega \|_{L^q_t L^6_x(\mathbb{R}\times\mathbb{R}^3)}. \end{aligned}$$

By taking the q-th power of this inequality and integrating over $t_0 \in [t_1 - N^{-1}, t_1 + N^{-1}]$, we obtain

$$\| \exp(it_1|\nabla|) f_N^\omega \|^q_{L^6_x} \lesssim N \| \exp(it|\nabla|) f_N^\omega \|^q_{L^q_t L^6_x(\mathbb{R}\times\mathbb{R}^3)}.$$

Taking the supremum in t_1 and using Lemma 3.4, it follows that

$$\| \exp(\pm it|\nabla|) f_N^\omega \|_{L^\sigma_\omega L^\infty_t L^6_x} \lesssim N^{\frac{1}{q}} \| \exp(it|\nabla|) f_N^\omega \|_{L^\sigma_\omega L^q_t L^6_x(\mathbb{R}\times\mathbb{R}^3)} \lesssim \sqrt{\sigma} N^{1-\frac{1}{3\gamma}} \| f_N \|_{L^2_x}.$$

Using the radial Sobolev embedding (Proposition 2.6), Proposition 2.9, and the same argument as before, we obtain

$$\begin{aligned} \| |x|^{\frac{1}{2}} \exp(\pm it|\nabla|) f_N^\omega \|_{L^\sigma_\omega L^\infty_t L^\infty_x} &\lesssim \| |x|^{\frac{1}{2}} \exp(\pm it|\nabla|) |\nabla|^{\frac{3}{q}} f_N^\omega \|_{L^\sigma_\omega L^\infty_t L^q_x} \\ &\lesssim N^{\frac{1}{q}+\frac{3}{q}} \| |x|^{\frac{1}{2}} \exp(\pm it|\nabla|) f_N^\omega \|_{L^\sigma_\omega L^q_t L^q_x} \\ &\lesssim \sqrt{\sigma} N^{1-\frac{1}{\gamma}(\frac{1}{2}-\frac{1}{q})} \| f_N \|_{L^2_x}. \end{aligned}$$

This completes the proof of the second estimate. $\square$

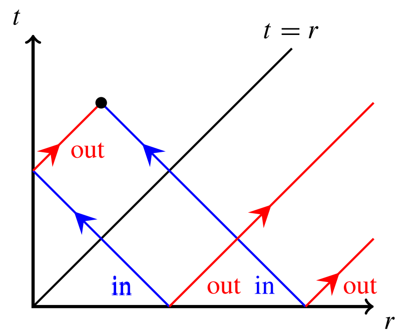

Figure 3. We display the in/out-decomposition for radial solutions of the linear wave equation in $d = 3$. The blue lines correspond to incoming waves and the red lines correspond to outgoing waves. The incoming wave will be reflected at the origin and transformed into an outgoing wave.

4. An in/out decomposition

In this section, we describe a decomposition of solutions to the linear wave equation into incoming and outgoing components (see Figure 3). This decomposition relies heavily on the spherical symmetry of the initial data. The in/out-decomposition can be derived in physical space by using spherical means; see, e.g., [Sogge 1995]. However, for our purposes it is more convenient to derive the decomposition in frequency space. A similar method has been used for the mass-critical NLS in [Killip et al. 2009].

Let $f \in L^2_{\mathrm{rad}}(\mathbb{R}^3)$ be spherically symmetric. Using the explicit expression

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin(x),$$

see [Bell 1968], it follows that

$$\begin{aligned}\cos(t|\nabla|)f(r) &= r^{-\frac{1}{2}}\int_0^\infty \cos(t\rho)J_{\frac{1}{2}}(r\rho)\hat{f}(\rho)\rho^{\frac{3}{2}}\,\mathrm{d}\rho\\ &= \sqrt{\frac{2}{\pi}}\frac{1}{r}\int_0^\infty \cos(t\rho)\sin(r\rho)\hat{f}(\rho)\rho\,\mathrm{d}\rho\\ &= \frac{1}{\sqrt{2\pi}}\frac{1}{r}\int_0^\infty (\sin((t+r)\rho)-\sin((t-r)\rho))\hat{f}(\rho)\rho\,\mathrm{d}\rho.\end{aligned}$$

By defining

$$W_s[h](\tau) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \sin(\tau\rho)h(\rho)\rho\,\mathrm{d}\rho, \tag{33}$$

it follows that

$$\cos(t|\nabla|)f = \frac{1}{r}(W_s[\hat{f}](t+r) - W_s[\hat{f}](t-r)).$$

Next, let us derive the corresponding decomposition for the operator $\sin(t|\nabla|)/|\nabla|$. Let $g \in \dot{H}_x^{-1}(\mathbb{R}^3)$ be spherically symmetric. Then,

$$\begin{aligned}\frac{\sin(t|\nabla|)}{|\nabla|}g(r) &= r^{-\frac{1}{2}}\int_0^\infty \sin(t\rho)J_{\frac{1}{2}}(r\rho)\hat{g}(\rho)\rho^{\frac{1}{2}}\,\mathrm{d}\rho\\ &= \sqrt{\frac{2}{\pi}}\frac{1}{r}\int_0^\infty \sin(t\rho)\sin(r\rho)\hat{g}(\rho)\,\mathrm{d}\rho\\ &= \frac{1}{\sqrt{2\pi}}\frac{1}{r}\int_0^\infty (\cos((t-r)\rho)-\cos((t+r)\rho))\hat{g}(\rho)\,\mathrm{d}\rho.\end{aligned}$$

By defining

$$W_c[h](\tau) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \cos(\tau\rho)h(\rho)\rho\,\mathrm{d}\rho,$$

it follows that

$$\frac{\sin(t|\nabla|)}{|\nabla|}g = r^{-1}\big(-W_c[\rho^{-1}\hat{g}](t+r) + W_c[\rho^{-1}\hat{g}](t-r)\big).$$

Thus, the solution F of the linear wave equation with initial data $(f, g) \in L^2_{\mathrm{rad}}(\mathbb{R}^3) \times \dot{H}^{-1}_{\mathrm{rad}}(\mathbb{R}^3)$ is given by

$$F(t,x) = \frac{1}{r}\big(W_s[\hat{f}](t+r) - W_c[\rho^{-1}\hat{g}](t+r) - W_s[\hat{f}](t-r) + W_c[\rho^{-1}\hat{g}](t-r)\big).$$

Definition 4.1 (in/out-decomposition). Let $(f, g) \in L^2_{\mathrm{rad}}(\mathbb{R}^3) \times \dot{H}^{-1}_{\mathrm{rad}}(\mathbb{R}^3)$ and let F be the corresponding solution to the linear wave equation. Then, we define

$$\begin{aligned}W_{\mathrm{in}}[F](\tau) &= W_s[\hat{f}](\tau) - W_c[\rho^{-1}\hat{g}](\tau),\\ W_{\mathrm{out}}[F](\tau) &= -W_s[\hat{f}](\tau) + W_c[\rho^{-1}\hat{g}](\tau).\end{aligned}$$

As a consequence, we have

$$F(t,x) = \frac{1}{r}\big(W_{\mathrm{in}}[F](t+r) + W_{\mathrm{out}}[F](t-r)\big). \tag{34}$$

Even though $W_{\mathrm{in}}[F]$ equals $-W_{\mathrm{out}}[F]$ we introduced two different notations to serve as a visual aid. This also allows us to safely leave out the arguments $t+r$ and $t-r$ in subsequent computations.

From Plancherel's theorem, it follows that

$$\|W_s[h](\tau)\|_{L^2_\tau(\mathbb{R})} + \|W_c[h]\|_{L^2_\tau(\mathbb{R})} \lesssim \|\rho h\|_{L^2_\rho(\mathbb{R}_{>0})}. \tag{35}$$

As a consequence, we have

$$\|W_{\mathrm{in}}[F](\tau)\|_{L^2_\tau(\mathbb{R})} + \|W_{\mathrm{out}}[F](\tau)\|_{L^2_\tau(\mathbb{R})} \lesssim \|f\|_{L^2_x(\mathbb{R}^3)} + \|g\|_{\dot{H}_x^{-1}(\mathbb{R}^3)}. \tag{36}$$

In the analysis of the Morawetz error term (see Section 8B), we will need to control an interaction between ∇F and the nonlinear part v. However, the individual components of ∇F are not radial. To overcome this technical problem, we write

$$\partial_{x_j} F(t,x) = \frac{x_j}{r}\,\partial_r F(t,r)$$
$$= -\frac{x_j}{r^3}\big(W_{\mathrm{out}}[F](t-r) + W_{\mathrm{in}}[F](t+r)\big) + \frac{x_j}{r^2}\big(-(\partial_\tau W_{\mathrm{out}}[F])(t-r) + (\partial_\tau W_{\mathrm{in}}[F])(t+r)\big).$$

After a short calculation, we see that

$$\partial_\tau W_s[\hat{f}](\tau) = W_c[\rho \hat{f}](\tau) \quad \text{and} \quad \partial_\tau W_c[\rho^{-1}\hat{g}](\tau) = -W_s[\hat{g}](\tau).$$

Then, we define

$$W_{\mathrm{in},\nabla}[F](\tau) := W_c[\rho \hat{f}](\tau) + W_s[\hat{g}](\tau), \tag{37}$$

$$W_{\mathrm{out},\nabla}[F](\tau) := W_c[\rho \hat{f}](\tau) + W_s[\hat{g}](\tau). \tag{38}$$

Using these definitions, it follows that

$$\partial_{x_j} F(t,x) = -\frac{x_j}{r^2} F(t,x) + \frac{x_j}{r^2}\big(W_{\mathrm{out},\nabla}[F](t-r) + W_{\mathrm{in},\nabla}[F](t+r)\big). \tag{39}$$

Using the same argument as above, we have

$$\|W_{\mathrm{out},\nabla}[F](\tau)\|_{L^2_\tau(\mathbb{R})} + \|W_{\mathrm{in},\nabla}[F](\tau)\|_{L^2_\tau(\mathbb{R})} \lesssim \|f\|_{\dot{H}^1_x(\mathbb{R}^3)} + \|g\|_{L^2_x(\mathbb{R}^3)}.$$

Lemma 4.2. *Let $f \in L^2_{\mathrm{rad}}(\mathbb{R}^3)$ be such that*

$$\mathrm{supp}(\hat{f}) \subseteq \{\xi : |\xi| \in [a, (1+\delta)a]\}.$$

Then, we have for all $2 \le q \le \infty$ that

$$\|W_s[f](\tau)\|_{L^q_\tau(\mathbb{R})} + \|W_c[f](\tau)\|_{L^q_\tau(\mathbb{R})} \lesssim (a\delta)^{\frac{1}{2}-\frac{1}{q}} \|f\|_{L^2_x(\mathbb{R}^3)}. \tag{40}$$

Proof. Using Hölder's inequality, we have

$$|W_s[f](\tau)| + |W_c[f](\tau)| \lesssim \int_a^{(1+\delta)a} |\hat{f}(\rho)|\rho \,\mathrm{d}\rho \le (a\delta)^{\frac{1}{2}} \left(\int_0^\infty |\hat{f}(\rho)|^2 \rho^2 \,\mathrm{d}\rho\right)^{\frac{1}{2}} = (a\delta)^{\frac{1}{2}} \|f\|_{L^2_x(\mathbb{R}^3)}.$$

This proves (40) for $q = \infty$. Together with (35), the general case follows by interpolation. □

Lemma 4.2 is the analog of the square-function estimate [Dodson et al. 2017, Lemma 2.2] for the Wiener randomization. However, since f is radial, it is much easier to prove.

Corollary 4.3 (improved integrability for the in/out decomposition). *Let $f \in L^2_{\mathrm{rad}}(\mathbb{R}^3)$. Then, we have for all $2 \le q < \infty$ that*

$$\|W_s[f_N^\omega](\tau)\|_{L^\sigma_\omega L^q_\tau} + \|W_c[f_N^\omega](\tau)\|_{L^\sigma_\omega L^q_\tau} \lesssim N^{\left(1-\frac{1}{\gamma}\right)\left(\frac{1}{2}-\frac{1}{q}\right)} \|f_N\|_{L^2_x(\mathbb{R}^3)}.$$

Proof. As in Section 3, we restrict to the case $q \le \sigma < \infty$. Using a combination of Khintchine's inequality, Minkowski's integral inequality, and Lemma 4.2, we have

$$\begin{aligned}
\|W_s[f_N^\omega](\tau)\|_{L^\sigma_\omega L^q_\tau} &\le \|W_s[f_N^\omega](\tau)\|_{L^q_\tau L^\sigma_\omega} \\
&\lesssim \sqrt{\sigma}\|W_s[A_k f_N](\tau)\|_{L^q_\tau \ell^2_k} \\
&\le \sqrt{\sigma}\|W_s[A_k f_N](\tau)\|_{\ell^2_k L^q_\tau} \\
&\lesssim N^{(1-\frac{1}{\gamma})(\frac{1}{2}-\frac{1}{q})}\|A_k f_N\|_{\ell^2_k L^2_x} \\
&\lesssim N^{(1-\frac{1}{\gamma})(\frac{1}{2}-\frac{1}{q})}\|f_N\|_{L^2_x}.
\end{aligned}$$

The same argument also works for $W_c[f_N^\omega](\tau)$. □

Remark 4.4. For $\gamma = 1$, Corollary 4.3 shows that $W_s[f_N^\omega](\tau) \in \bigcap_{2\le q<\infty} L^q_\tau(\mathbb{R})$ almost surely for all $f \in H^{0+}_{\text{rad}}(\mathbb{R}^3)$. This holds because the radial randomization is similar to a Wiener randomization of the function $f(r)r$.

5. Local well-posedness and conditional scattering

Recall that the forced nonlinear wave equation is given by

$$\begin{cases} -\partial_{tt}v + \Delta v = (v+F)^5, & (t,x) \in \mathbb{R}\times\mathbb{R}^3, \\ v(t_0,x) = v_0 \in \dot{H}^1_x(\mathbb{R}^3), & \partial_t v(t_0,x) = v_1 \in L^2_x(\mathbb{R}^3). \end{cases} \tag{41}$$

In this section, it is not important that F solves a linear wave equation. However, this will be essential in Sections 6–9.

Lemma 5.1 (local well-posedness). *Let $(v_0, v_1) \in \dot{H}^1_x(\mathbb{R}^3)\times L^2_x(\mathbb{R}^3)$ and assume that $F \in L^5_t L^{10}_x(\mathbb{R}\times\mathbb{R}^3)$. Then, there exists a maximal time interval of existence I and a corresponding unique solution v of* (41) *satisfying*

$$(v, \partial_t v) \in \big(C^0_t \dot{H}^1_x(I\times\mathbb{R}^3) \cap L^5_{t,\text{loc}} L^{10}_x(I\times\mathbb{R}^3)\big) \times C^0_t L^2_x(I\times\mathbb{R}^3).$$

Moreover, if both the initial data (v_0, v_1) and the forcing term F are radial, then v is also radial.

The proof consists of a standard application of Strichartz estimates, and we omit the details. We refer the reader to [Dodson et al. 2017, Lemma 3.1] and [Pocovnicu 2017, Theorem 1.1] for related results. In [Pocovnicu 2017] the stability theory for energy-critical equations was used to reduce to the proof of almost-sure global well-posedness to an a priori energy bound. Similar methods have also been used in [Bényi et al. 2015a; Dodson et al. 2017; 2019; Killip et al. 2019; Oh and Pocovnicu 2016].

Proposition 5.2 [Dodson et al. 2017, Theorem 1.3]. *Suppose $(v_0, v_1) \in \dot{H}^1_x(\mathbb{R}^3) \times L^2_x(\mathbb{R}^3)$ and $F \in L^5_t L^{10}_x(\mathbb{R}\times\mathbb{R}^3)$. Let $v(t)$ be a solution* (41) *and let I be its maximal time interval of existence. Furthermore, we assume that v satisfies the a priori bound*

$$M := \sup_{t\in I} E[v](t) < \infty. \tag{42}$$

Then v is a global solution, it obeys the global space-time bound

$$\|v\|_{L_t^5 L_x^{10}(\mathbb{R}\times\mathbb{R}^3)} \le C(M, \|F\|_{L_t^5 L_x^{10}(\mathbb{R}\times\mathbb{R}^3)}) < \infty,$$

and it scatters as $t \to \pm\infty$.

Theorem 1.3 in [Dodson et al. 2017] is stated for the energy-critical NLW in $d = 4$. However, the same argument also yields Proposition 5.2. We point out that the proof crucially relies on the deterministic theory for the energy-critical NLW [Bahouri and Gérard 1999; Tao 2006b].

6. Almost energy conservation and decay estimates

In this section, we prove new estimates for the solution to the forced NLW

$$\begin{cases} -\partial_{tt} v + \Delta v = (v+F)^5, & (t,x) \in \mathbb{R}\times\mathbb{R}^3, \\ v(t_0,x) = v_0 \in \dot{H}_x^1(\mathbb{R}^3), & \partial_t v(t_0,x) = v_1 \in L_x^2(\mathbb{R}^3). \end{cases} \tag{43}$$

In contrast to Section 5, we now assume that F is a solution to the linear wave equation. Recall that the stress-energy tensor of the energy-critical NLW is given by

$$\begin{aligned} T^{00} &:= \tfrac{1}{2}((\partial_t v)^2 + |\nabla v|^2) + \tfrac{1}{6} v^6, \\ T^{j0} &:= -\partial_t v\, \partial_{x_j} v, \\ T^{jk} &:= \partial_{x_j} v\, \partial_{x_k} v - \tfrac{1}{4}\delta_{jk}(-\partial_{tt} + \Delta)(v^2) + \tfrac{1}{3}\delta_{jk} v^6. \end{aligned}$$

In the above tensor, we have $j,k = 1,2,3$. If v solves the energy-critical NLW (1), then the stress-energy tensor is divergence-free. This leads to energy conservation, momentum conservation, and several decay estimates, such as Morawetz estimates, flux estimates, or potential energy decay; see [Sogge 1995; Tao 2006a]. If v solves the forced nonlinear wave equation (43), then the stress-energy tensor is no longer divergence-free. However, the error terms in the divergence are of lower order, so we can still hope for almost conservation laws and some decay estimates. More precisely, with $\mathcal{N} := (v+F)^5 - v^5$, it follows from a standard computation that

$$\partial_t T^{00} + \partial_{x_k} T^{0k} = -\mathcal{N}\, \partial_t v, \tag{44}$$

$$\partial_t T^{j0} + \partial_{x_k} T^{jk} = \mathcal{N}\, \partial_{x_j} v - \tfrac{1}{2}\partial_{x_j}(\mathcal{N} v). \tag{45}$$

For our purposes, the most important quantity measuring the size and regularity of v is its energy

$$E[v](t) = \int \tfrac{1}{2}|\nabla v|^2 + \tfrac{1}{2}|\partial_t v|^2 + \tfrac{1}{6}|v|^6 \,\mathrm{d}x.$$

For future use, we also define the local energy as

$$e[v](t) := \int_{|x|\le|t|} \tfrac{1}{2}|\nabla v|^2 + \tfrac{1}{2}|\partial_t v|^2 + \tfrac{1}{6}|v|^6 \,\mathrm{d}x.$$

Next, we determine the error terms in the almost energy conservation law.

Proposition 6.1 (energy increment). *Let $I=[a,b]$ be a time interval and $v : I\times\mathbb{R}^3\to\mathbb{R}$ be a solution to the forced nonlinear wave equation* (43). *Then, we have*

$$|E[v](b)-E[v](a)| \lesssim \|F\|_{L_t^\infty L_x^6(I\times\mathbb{R}^3)} \sup_{t\in I} E[v](t)^{\frac{5}{6}} + \left|\int_I\int_{\mathbb{R}^3}(\partial_t F)v^5\,\mathrm{d}x\,\mathrm{d}t\right| + \int_I\int_{\mathbb{R}^3}|F|^2(|F|+|v|)^3|\partial_t v|\,\mathrm{d}x\,\mathrm{d}t. \quad (46)$$

The first summand on the right-hand side of (46) has a lower power in the energy. After placing the random linear evolution in $L_t^\infty L_x^6(\mathbb{R}\times\mathbb{R}^3)$, it can easily be controlled via a bootstrap argument. The second summand is the main error term in this almost energy conservation law, and we will control it in Section 8A. Finally, the third summand in (46) only includes lower-order error terms, and they are controlled in Section 8D.

The idea to integrate by parts in the energy increment has previously been used in [Dodson et al. 2019; Killip et al. 2019; Oh and Pocovnicu 2016].

Proof. From the divergence formula (44), it follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E[v](t) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} T^{00}(t,x)\,\mathrm{d}x\\ &= -\int_{\mathbb{R}^3}\mathcal{N}\,\partial_t v\,\mathrm{d}x\\ &= -5\int_{\mathbb{R}^3} Fv^4\,\partial_t v\,\mathrm{d}x - \int_{\mathbb{R}^3}(10F^2v^3+10F^3v^2+5F^4v+F^5)\,\partial_t v\,\mathrm{d}x.\end{aligned}$$

Integrating in time, we obtain

$$|E[v](b)-E[v](a)| \lesssim \left|\int_I\int_{\mathbb{R}}^3 Fv^4\,\partial_t v\,\mathrm{d}x\,\mathrm{d}t\right| + \int_I\int_{\mathbb{R}^3}|F|^2(|F|+|v|)^3|\partial_t v|\,\mathrm{d}x\,\mathrm{d}t. \quad (47)$$

The second summand in (47) is already acceptable; thus, we now turn to the first summand. Using integration by parts, we have

$$\begin{aligned}5\left|\int_I\int_{\mathbb{R}^3} Fv^4\,\partial_t v\,\mathrm{d}x\,\mathrm{d}t\right| &= \left|\int_I\int_{\mathbb{R}^3} F\,\partial_t(v^5)\,\mathrm{d}x\,\mathrm{d}t\right|\\ &\le \left|\int_I\int_{\mathbb{R}^3}\partial_t(F)\,v^5\,\mathrm{d}x\,\mathrm{d}t\right| + \int_{\mathbb{R}^3}|F|(b,x)|v|^5(b,x)\,\mathrm{d}x + \int_{\mathbb{R}^3}|F|(a,x)|v|^5(a,x)\,\mathrm{d}x\\ &\lesssim \left|\int_I\int_{\mathbb{R}^3}\partial_t(F)\,v^5\,\mathrm{d}x\,\mathrm{d}t\right| + \|F\|_{L_t^\infty L_x^6(I\times\mathbb{R}^3)}\sup_{t\in I}E[v](t)^{\frac{5}{6}}.\end{aligned}$$

Thus, the contribution of the first summand in (47) is also acceptable. □

By contracting the stress-energy tensor against different vector fields, one sees that solutions to the energy-critical NLW obey a range of decay estimates. One of the most important decay estimates in the study of dispersive equations is the Morawetz estimate, and it has been used to prove almost-sure scattering in [Dodson et al. 2017; 2019; Killip et al. 2019]. For the reader's convenience, we recall a classical Morawetz identity.

Lemma 6.2 (Morawetz identity). *Let $I=[a,b]$ be a given time interval, and let $v: I\times\mathbb{R}^3\to\mathbb{R}$ be a solution of* (43). *Then, we have the Morawetz identity*

$$\frac{2}{3}\int_I\int_{\mathbb{R}^3}\frac{v^6}{|x|}\,\mathrm{d}x\,\mathrm{d}t+\pi\int_I |v|^2(t,0)\,\mathrm{d}t+\int_I\int_{\mathbb{R}^3}|\nabla_{\mathrm{ang}}v|^2\,\mathrm{d}x\,\mathrm{d}t \\ =\int_{\mathbb{R}^3}\partial_t v\,\frac{x}{|x|}\cdot\nabla v-4\frac{v}{|x|}\,\partial_t v\,\mathrm{d}x\bigg|_{t=a}^{b}-\int_I\int_{\mathbb{R}^3}\mathcal{N}\frac{x}{|x|}\cdot\nabla v\,\mathrm{d}x\,\mathrm{d}t-\int_I\int_{\mathbb{R}^3}\frac{1}{|x|}\mathcal{N}v\,\mathrm{d}x\,\mathrm{d}t. \tag{48}$$

Here, $\nabla_{\mathrm{ang}}v:=\nabla v-(x/|x|)\cdot\nabla v$ denotes the angular component of the gradient of v.

The lemma follows along a line of standard computations using (44) and (45); see, e.g., [Tao 2006a]. We now rewrite the error terms in (48) more explicitly in terms of F, and group similar terms together.

Proposition 6.3 (Morawetz estimate). *Let $I=[a,b]$ be a given time interval, and let $v: I\times\mathbb{R}^3\to\mathbb{R}$ be a solution of* (43). *Then, we have the Morawetz estimate*

$$\int_I\int_{\mathbb{R}^3}\frac{v^6}{|x|}\,\mathrm{d}x\,\mathrm{d}t\lesssim\sup_{t\in I}E[v](t)+\left|\int_I\int_{\mathbb{R}^3}\frac{x}{|x|}\cdot\nabla_x(F)v^5\,\mathrm{d}x\,\mathrm{d}t\right| \tag{49}$$

$$+\int_I\int_{\mathbb{R}^3}\frac{1}{|x|}|F|(|v|^5+|F|^5)\,\mathrm{d}x\,\mathrm{d}t+\int_I\int_{\mathbb{R}^3}|F|^2(|F|+|v|)^3\left(\frac{|v|}{|x|}+|\nabla v|\right)\mathrm{d}x\,\mathrm{d}t. \tag{50}$$

The second summand in (49) is the main error term in this estimate, and we will control it in Section 8B. In contrast, the error terms in (50) are easier to control, and they will be handled in Section 8D.

Proof. To prove the proposition, we have to control the terms on the right-hand side of (48). First, using Hardy's inequality, we have

$$\left|\int_{\mathbb{R}^3}\partial_t v\,\frac{x}{|x|}\cdot\nabla v-4\frac{v}{|x|}\,\partial_t v\,\mathrm{d}x\bigg|_{t=a}^{b}\right| \\ \lesssim\|\partial_t v(t)\|_{L_t^\infty L_x^2(I\times\mathbb{R}^3)}\|\nabla v\|_{L_t^\infty L_x^2(I\times\mathbb{R}^3)}+\|\partial_t v(t)\|_{L_t^\infty L_x^2(I\times\mathbb{R}^3)}\left\|\frac{v}{|x|}\right\|_{L_t^\infty L_x^2(I\times\mathbb{R}^3)} \\ \lesssim\sup_{t\in I}E[v](t).$$

Thus, the contribution is acceptable. Second, we have

$$\left|\int_I\int_{\mathbb{R}^3}\mathcal{N}\frac{x}{|x|}\cdot\nabla v\,\mathrm{d}x\,\mathrm{d}t\right| \\ \lesssim\left|\int_I\int_{\mathbb{R}^3}Fv^4\frac{x}{|x|}\cdot\nabla v\,\mathrm{d}x\,\mathrm{d}t\right|+\int_I\int_{\mathbb{R}^3}|F|^2(|F|+|v|)^3|\nabla v|\,\mathrm{d}x\,\mathrm{d}t \\ \lesssim\left|\int_I\int_{\mathbb{R}^3}F\frac{x}{|x|}\cdot\nabla(v^5)\,\mathrm{d}x\,\mathrm{d}t\right|+\int_I\int_{\mathbb{R}^3}|F|^2(|F|+|v|)^3|\nabla v|\,\mathrm{d}x\,\mathrm{d}t \\ \lesssim\left|\int_I\int_{\mathbb{R}^3}\nabla\cdot\left(F\frac{x}{|x|}\right)v^5\,\mathrm{d}x\,\mathrm{d}t\right|+\int_I\int_{\mathbb{R}^3}|F|^2(|F|+|v|)^3|\nabla v|\,\mathrm{d}x\,\mathrm{d}t \\ \lesssim\left|\int_I\int_{\mathbb{R}^3}\frac{x}{|x|}\cdot\nabla(F)v^5\,\mathrm{d}x\,\mathrm{d}t\right|+\int_I\int_{\mathbb{R}^3}\frac{|F|}{|x|}|v|^5\mathrm{d}x\mathrm{d}t+\int_I\int_{\mathbb{R}^3}|F|^2(|F|+|v|)^3|\nabla v|\,\mathrm{d}x\,\mathrm{d}t.$$

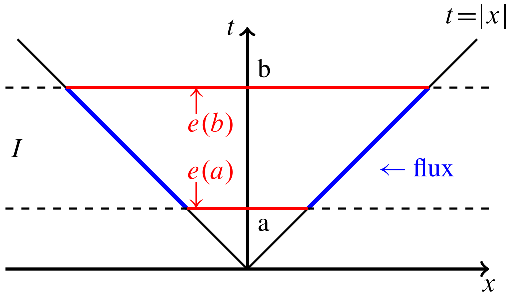

Figure 4. This figure displays the quantities involved in the forward flux estimate. The local energy at times $t = a, b$ is the integral of the energy density over the red regions. The flux is the integral of v^6 over the blue region in space-time. Using the stress-energy tensor, we can control the flux by the increment of the local energy.

Thus, the contribution is acceptable. Finally, we have

$$\left|\int_I \int_{\mathbb{R}^3} \frac{1}{|x|} \mathcal{N} v \, \mathrm{d}x \, \mathrm{d}t\right| \lesssim \int_I \int_{\mathbb{R}^3} \frac{1}{|x|} |F|(|F|+|v|)^4 |v| \, \mathrm{d}x \, \mathrm{d}t \lesssim \int_I \int_{\mathbb{R}^3} \frac{1}{|x|} |F|(|F|+|v|)^5 \, \mathrm{d}x \, \mathrm{d}t. \quad \square$$

In contrast to the case $d = 4$ as in [Dodson et al. 2017; 2019], the energy and the Morawetz term are not strong enough to control the main error terms. In addition, we will rely on the following flux estimates on light cones.

Lemma 6.4 (forward flux estimate). *Let v be a solution of* (43) *on a compact time interval $I = [a, b] \subseteq [0, \infty)$. Then, we have*

$$\frac{1}{6} \int_{|x|=t,\, t \in I} v^6(t, x) \, \mathrm{d}\sigma(t, x) \le e[v](b) - e[v](a) + \int_{|x| \le t,\, t \in I} \partial_t v((v+F)^5 - v^5) \, \mathrm{d}x \, \mathrm{d}t. \tag{51}$$

Remark 6.5. The flux estimate is a monotonicity formula based on the increment of the local energy. See Figure 4. The term on the left-hand side of (51) describes the inflow of potential energy through the light cone.

Proof. We have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} e[v](t) &= \int_{|x|=t} \tfrac{1}{2}|\nabla v|^2 + \tfrac{1}{2}|\partial_t v|^2 + \tfrac{1}{6}|v|^6 \, \mathrm{d}\sigma(t,x) + \int_{|x| \le t} \partial_t \nabla v \nabla v + \partial_{tt} v \, \partial_t v + v^5 \, \partial_t v \, \mathrm{d}x \\
&= \int_{|x|=t} \tfrac{1}{2}|\nabla v|^2 + \tfrac{1}{2}|\partial_t v|^2 - \partial_t v \, \nabla v \cdot \vec{n} + \tfrac{1}{6}|v|^6 \, \mathrm{d}\sigma(t,x) + \int_{|x| \le t} \partial_t v(\partial_{tt} v - \Delta v + v^5) \, \mathrm{d}x \\
&\ge \int_{|x|=t} \tfrac{1}{6}|v|^6 \, \mathrm{d}\sigma(t,x) + \int_{|x| \le t} \partial_t v(-(v+F)^5 + v^5) \, \mathrm{d}x.
\end{aligned}$$

Integrating over $t \in I$, we arrive at (51). $\square$

The estimate (51) by itself is not useful. Indeed, it only controls the size of v on a lower-dimensional surface in space-time. We will now use time-translation invariance to integrate it against a weight $w \in L^1_\tau(\mathbb{R})$.

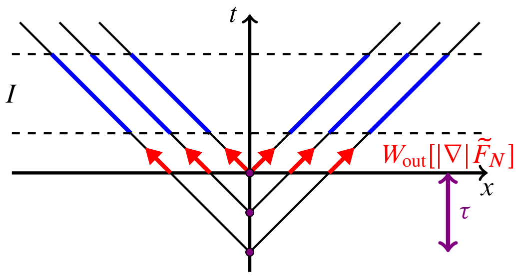

Figure 5. We display the idea behind the interaction flux estimate. By using the time-translation invariance of the equation, we can control v^6 on the blue region of each shifted light cone. Then, we integrate the forward flux estimate against a weight w depending only on the shift τ. Since the outgoing component $W_{\text{out}}[|\nabla|\widetilde{F}_N](t-|x|)$ is constant on forward light cones, we choose $w = |W_{\text{out}}[|\nabla|\widetilde{F}_N]|^2$.

Proposition 6.6 (forward interaction flux estimate). *Let v be a solution to the forced NLW* (43) *on a compact time interval $I=[a,b]\subseteq[0,\infty)$. Also, let $w\in L^1_\tau(\mathbb{R})$ be nonnegative. Then, we have*

$$\int_I\int_{\mathbb{R}^3} w(t-|x|)|v|^6(t,x)\,\mathrm{d}x\,\mathrm{d}t$$

$$\lesssim \|w\|_{L^1_\tau(\mathbb{R})}\sup_{t\in I}E[v](t)+\|w\|_{L^1_\tau(\mathbb{R})}\|F\|_{L^\infty_t L^6_x(I\times\mathbb{R}^3)}\sup_{t\in I}E[v](t)^{\frac{5}{6}} \tag{52}$$

$$+\left|\int_I\int_{\mathbb{R}^3}\left(\int_{-\infty}^{t-|x|}w(\tau)\,\mathrm{d}\tau\right)\partial_t(F)v^5\,\mathrm{d}x\,\mathrm{d}t\right|+\left|\int_I\int_{\mathbb{R}^3}w(t-|x|)Fv^5\,\mathrm{d}x\,\mathrm{d}t\right| \tag{53}$$

$$+\|w\|_{L^1_\tau(\mathbb{R})}\int_I\int_{\mathbb{R}^3}|F|^2(|F|+|v|)^3|\partial_t v|\,\mathrm{d}x\,\mathrm{d}t. \tag{54}$$

In order to control the energy, we essentially choose w as the outgoing component of the linear wave F (see Section 7 and Figure 5).

The terms in (52) correspond to boundary terms, and they can easily be controlled by a bootstrap argument. The main error terms are in (53), and they will be controlled in Section 8C. In contrast, the errors in (54) are of lower order, and they will be controlled in Section 8D.

To remember that the weight w in (53) should be integrated over $(-\infty, t-|x|]$, note that the contribution of the error $\partial_t(F)v^5$ should be weighted less as $t\to-\infty$ and $|x|\to\infty$.

Proof. By time-translation invariance and Lemma 6.4, we obtain for any $\tau\in\mathbb{R}$ that

$$\int_{|x|=t-\tau,\,t\in I}\tfrac{1}{6}|v|^6(t,x)\,\mathrm{d}\sigma(t,x)$$

$$\leq \int_{|x|\leq b-\tau}\tfrac12|\nabla v|^2+\tfrac12|\partial_t v|^2+\tfrac16|v|^6\,\mathrm{d}x\Big|_{t=b}$$

$$-\int_{|x|\leq a-\tau}\tfrac12|\nabla v|^2+\tfrac12|\partial_t v|^2+\tfrac16|v|^6\,\mathrm{d}x\Big|_{t=a}+\int_{|x|\leq t-\tau,\,t\in I}\partial_t v((v+F)^5-v^5)\mathrm{d}x\,\mathrm{d}t$$

$$\leq 2\sup_{t\in I}E[v](t)+\int_{|x|\leq t-\tau,\,t\in I}\partial_t v((v+F)^5-v^5)\,\mathrm{d}x\,\mathrm{d}t. \tag{55}$$

Integrating (55) against the function $w(\tau)$, we obtain

$$\begin{aligned}
&\frac{1}{6}\int_I\int_{\mathbb{R}^3} w(t-|x|)|v(t,x)|^6\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \frac{1}{6}\iint_{|x|=t-\tau} w(\tau)|v|^6(t,x)\,\mathrm{d}\sigma(t,x)\,\mathrm{d}\tau \\
&\quad \le 2\|w\|_{L^1_\tau(\mathbb{R})}\sup_{t\in I} E[v](t) + \iint_{|x|\le t-\tau,\, t\in I} w(\tau)\,\partial_t v(t)((v+F)^5-v^5)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\tau \\
&\quad \lesssim \|w\|_{L^1_\tau(\mathbb{R})}\sup_{t\in I} E[v](t) + \left|\int_{\mathbb{R}}\int_{|x|\le t-\tau,\, t\in I} w(\tau)Fv^4\,\partial_t v\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\tau\right| \\
&\qquad\qquad + \|w\|_{L^1_\tau(\mathbb{R})}\int_I\int_{\mathbb{R}^3}|F|^2(|F|+|v|)^3|\partial_t v|\,\mathrm{d}x\,\mathrm{d}t. \qquad (56)
\end{aligned}$$

The first and third summand in the last line of (56) are acceptable contributions. Thus, we turn to the second summand in the last line of (56). Using integration by parts, we have

$$\begin{aligned}
&5\left|\int_I\int_{|x|\le t-\tau,\, t\in I} w(\tau)Fv^4\,\partial_t v\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\tau\right| \\
&\quad = \left|\int_I\int_{\mathbb{R}^3}\left(\int_{-\infty}^{t-|x|} w(\tau)\,\mathrm{d}\tau\right)F\,\partial_t(v^5)\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad = \left|\int_{\mathbb{R}^3}\left(\int_{-\infty}^{t-|x|} w(\tau)\,\mathrm{d}\tau\right)Fv^5\,\mathrm{d}x\Big|_{t=b}\right| + \left|\int_{\mathbb{R}^3}\left(\int_{-\infty}^{t-|x|} w(\tau)\,\mathrm{d}\tau\right)Fv^5\,\mathrm{d}x\Big|_{t=a}\right| \\
&\qquad + \left|\int_I\int_{\mathbb{R}^3}\left(\int_{-\infty}^{t-|x|} w(\tau)\,\mathrm{d}\tau\right)\partial_t(F)v^5\,\mathrm{d}x\,\mathrm{d}t\right| + \left|\int_I\int_{\mathbb{R}^3} w(t-|x|)Fv^5\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad \lesssim \|w\|_{L^1_\tau(\mathbb{R})}\|F\|_{L^\infty_t L^6_x(I\times\mathbb{R}^3)}\sup_{t\in I} E[v](t)^{\frac{5}{6}} + \left|\int_I\int_{\mathbb{R}^3}\left(\int_{-\infty}^{t-|x|} w(\tau)\,\mathrm{d}\tau\right)\partial_t(F)v^5\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\qquad + \left|\int_I\int_{\mathbb{R}^3} w(t-|x|)Fv^5\,\mathrm{d}x\,\mathrm{d}t\right|. \qquad \square
\end{aligned}$$

By replacing the forward light-cones in the derivation of Proposition 6.6 by backward light-cones, one easily derives the following proposition.

Proposition 6.7 (backward interaction flux estimate). *Let v be a solution of* (43) *on a compact time interval $I=[a,b]\subseteq[0,\infty)$. Also, let $w\in L^1_\tau(\mathbb{R})$ be nonnegative. Then, we have*

$$\begin{aligned}
&\int_I\int_{\mathbb{R}^3} w(t+|x|)|v|^6(t,x)\,\mathrm{d}x\,\mathrm{d}t \qquad (57) \\
&\quad \lesssim \|w\|_{L^1_\tau(\mathbb{R})}\sup_{t\in I} E[v](t) + \|w\|_{L^1_\tau(\mathbb{R})}\|F\|_{L^\infty_t L^6_x(I\times\mathbb{R}^3)}\sup_{t\in I} E[v](t)^{\frac{5}{6}} \\
&\qquad + \left|\int_I\int_{\mathbb{R}^3}\left(\int_{t+|x|}^{\infty} w(\tau)\,\mathrm{d}\tau\right)\partial_t(F)v^5\,\mathrm{d}x\,\mathrm{d}t\right| + \left|\int_I\int_{\mathbb{R}^3} w(t+|x|)Fv^5\,\mathrm{d}x\,\mathrm{d}t\right| \qquad (58) \\
&\qquad + \|w\|_{L^1_\tau(\mathbb{R})}\int_I\int_{\mathbb{R}^3}|F|^2(|F|+|v|)^3|\partial_t v|\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

To remember that the weight w in (58) should be integrated over $[t+|x|,\infty)$, note that the contribution of the error $\partial_t(F)v^5$ should be weighted less as $t,|x|\to\infty$.

7. Bootstrap argument

In this section, we introduce the quantities in the bootstrap argument to control the energy. For a given time interval $I\subseteq\mathbb{R}$, we define the energy

$$\mathcal{E}_I := \sup_{t\in I} E[v](t) = \sup_{t\in I}\int_{\mathbb{R}^3} \tfrac{1}{2}(\partial_t v(t,x))^2 + \tfrac{1}{2}|\nabla v(t,x)|^2 + \tfrac{1}{6}|v(t,x)|^6\,\mathrm{d}x \tag{59}$$

and the Morawetz term

$$\mathcal{A}_I := \||x|^{-\frac{1}{6}} v\|^6_{L^6_{t,x}(I\times\mathbb{R}^3)}. \tag{60}$$

Before we can define the interaction flux term, we need to introduce some further notation. Let F be a solution to the linear wave equation with initial data $F|_{t=0}=f_0\in L^2_{\mathrm{rad}}(\mathbb{R}^3)$ and $\partial_t F|_{t=0}=g_0\in \dot{H}^{-1}_{\mathrm{rad}}(\mathbb{R}^3)$. As in the definition of F^ω in (5) , we assume that $P_{\leq 2^5} f_0 = P_{\leq 2^5} g_0 = 0$. We recall from (6) that the low-frequency component of (f^ω, g^ω) will be treated as the initial data of the nonlinear component v. In order to use Littlewood–Paley theory in the spatial variables, it is convenient to introduce a second solution $\widetilde{F}$ to the linear wave equation. A short computation shows that

$$\partial_t F = |\nabla|\Big(\cos(t|\nabla|)|\nabla|^{-1} g + \frac{\sin(t|\nabla|)}{|\nabla|}(-|\nabla| f)\Big).$$

Then,

$$\widetilde{F} := \cos(t|\nabla|)|\nabla|^{-1} g + \frac{\sin(t|\nabla|)}{|\nabla|}(-|\nabla| f) \tag{61}$$

satisfies $\partial_t F = |\nabla|\widetilde{F}$ and has initial data $\widetilde{F}|_{t=0} = |\nabla|^{-1} g \in L^2_{\mathrm{rad}}(\mathbb{R}^3)$ and $\partial_t \widetilde{F}|_{t=0} = -|\nabla| f \in \dot{H}^{-1}_{\mathrm{rad}}(\mathbb{R}^3)$. After localizing in frequency space, we write

$$|\nabla|\widetilde{F}_N(t,x) = \frac{1}{|x|}\big(W_{\mathrm{out}}[|\nabla|\widetilde{F}_N](t-r) + W_{\mathrm{in}}[|\nabla|\widetilde{F}_N](t+r)\big). \tag{62}$$

In the bootstrap argument, we want to apply the interaction flux estimate to the Littlewood–Paley pieces $P_K v$ of v. In order to deal with the operators P_K, we need to slightly modify the weights. Unfortunately, we cannot use the Hardy–Littlewood maximal function, since it is unbounded in L^1. Instead, we define for each $K\in 2^{\mathbb{N}}$ the operator

$$S_K w = K\langle K\tau\rangle^{-2} * w. \tag{63}$$

Definition 7.1 (interaction flux term). Let $(f_0,g_0)\in L^2_{\mathrm{rad}}(\mathbb{R}^3)\times \dot{H}^{-1}_{\mathrm{rad}}(\mathbb{R}^3)$ and assume that $P_{\leq 2^5} f_0 = P_{\leq 2^5} g_0 = 0$. Let F be the solution of the linear wave equation with data (f_0,g_0), let $\widetilde{F}$ be as in (61), let

v be a solution to (43), and let $I \subseteq \mathbb{R}$. For $* \in \{\text{out}, \text{in}\}$, we define

$$\mathcal{F}_{I,*} := \sum_{N \geq 1} (N^{-\frac{1}{6\gamma}+2\delta} + N^{-2+2\delta}) \sup_{K \in 2^{\mathbb{Z}}} \| w_{*,K,N} (t-|x|)^{\frac{1}{6}} v(t,x) \|^6_{L^6_{t,x}(I \times \mathbb{R}^3)} \tag{64}$$

$$+ \sum_{N \geq 1} (N^{-\frac{1}{6\gamma}+2\delta} + N^{-2+2\delta}) \sup_{K \in 2^{\mathbb{Z}}} \| w_{*,\nabla,K,N} (t-|x|)^{\frac{1}{6}} v(t,x) \|^6_{L^6_{t,x}(I \times \mathbb{R}^3)} \tag{65}$$

$$+ \| W_*[F](t-|x|)^{\frac{1}{3}} v \|^6_{L^6_{t,x}(I \times \mathbb{R}^3)}, \tag{66}$$

where $w_{*,K,N} = S_K(|W_*[|\nabla|\widetilde{F}_N]|^2)$ and $w_{*,\nabla,K,N} = S_K(|W_{*,\nabla}[F_N]|^2)$; see Section 4. For notational convenience, we also set

$$\mathcal{F}_I := \mathcal{F}_{I,\text{out}} + \mathcal{F}_{I,\text{in}}.$$

In the following definition, we introduce two auxiliary norms on F that will be used in the rest of this paper.

Definition 7.2 (Y_I and Z-norms). Let $(f_0, g_0) \in L^2_{\text{rad}}(\mathbb{R}^3) \times \dot{H}^{-1}_{\text{rad}}(\mathbb{R}^3)$ and assume that $P_{\leq 2^5} f_0 = P_{\leq 2^5} g_0 = 0$. Let F be the solution of the linear wave equation with data (f_0, g_0), let $\widetilde{F}$ be as in (61), and let $I \subseteq \mathbb{R}$. Then, we define

$$\begin{aligned}\|F\|_{Y(I)} &:= \|N^{-\frac{3}{4}+\frac{1}{24\gamma}+\delta} |x|^{\frac{3}{8}} |\nabla| \widetilde{F}_N\|_{\ell_N^{8/3} L_t^{8/3} L_x^\infty (2^{\mathbb{N}} \times I \times \mathbb{R}^3)} \\ &+ \|N^{-\frac{3}{4}+\frac{1}{24\gamma}+\frac{5\delta}{2}} |x|^{\frac{3+2\delta}{8}} |\nabla| F_N\|_{\ell_N^{8/(3-2\delta)} L_t^{8/(3-2\delta)} L_x^{2/\delta} (2^{\mathbb{N}} \times I \times \mathbb{R}^3)} \\ &+ \|N^{-1+\delta} |x|^{-\frac{1}{6}} |\nabla| F_N\|_{\ell^6_N L^6_t L^6_x (2^{\mathbb{N}} \times I \times \mathbb{R}^3)} + \|N^{-1+\delta} |x|^{\frac{2}{3}} |\nabla| \widetilde{F}_N\|_{\ell^{12}_N L^{12}_t L^{12}_x (2^{\mathbb{N}} \times I \times \mathbb{R}^3)} \\ &+ \||x|^{\frac{1}{4}} F\|_{L^4_t L^\infty_x (I \times \mathbb{R}^3)} + \|F\|_{L^5_t L^{10}_x (I \times \mathbb{R}^3)} + \||x|^{-\frac{1}{6}} F\|_{L^6_{t,x}(I \times \mathbb{R}^3)} + \||x|^{\frac{2}{3}} F\|_{L^{12}_{t,x}(I \times \mathbb{R}^3)}.\end{aligned}$$

Furthermore, we also define

$$\begin{aligned}\|F\|_Z &:= \sum_{* \in \{\text{out},\text{in}\}} \sum_{p \in \{2,4,24\}} \|(N^{-\frac{1}{12\gamma}+2\delta} + N^{-1+\delta}) W_*[|\nabla|\widetilde{F}_N]\|_{\ell^1_N L^p_\tau (2^{\mathbb{N}} \times \mathbb{R})} \\ &+ \sum_{* \in \{\text{out},\text{in}\}} \sum_{p \in \{2,4,24\}} \|(N^{-\frac{1}{12\gamma}+2\delta} + N^{-1+\delta}) W_{*,\nabla}[F_N]\|_{\ell^1_N L^p_\tau (2^{\mathbb{N}} \times \mathbb{R})} \\ &+ \sum_{* \in \{\text{out},\text{in}\}} \sum_{p \in \{2,4,24\}} \|W_*[F]\|_{L^p_\tau(\mathbb{R})} + \|N^\delta |x|^{\frac{1}{2}} F_N\|_{\ell^1_N L^\infty_t L^\infty_x (2^{\mathbb{N}} \times \mathbb{R} \times \mathbb{R}^3)} + \|F\|_{L^\infty_t L^6_x (\mathbb{R} \times \mathbb{R}^3)}.\end{aligned}$$

We remark that $\|F\|_{Y_I}$ is divisible in space-time. More precisely, let $\eta > 0$ be given and assume that $\|F\|_{Y(\mathbb{R})} < \infty$. Then, there exists a finite number $J = J(\eta, \|F\|_{Y(\mathbb{R})})$ and a partition of $\mathbb{R}$ into finitely many intervals $I_1, \ldots, I_J$ such that $\|F\|_{I_j} < \eta$ for all $j = 1, \ldots, J$.

Lemma 7.3 (almost-sure finiteness of Y and Z-norms). *Let $(f, g) \in H^s_{\text{rad}}(\mathbb{R}^3) \times H^{s-1}_{\text{rad}}(\mathbb{R}^3)$, let $0 < \gamma \leq 1$, let $s > \max\big(0, 1 - \frac{1}{12\gamma}\big)$, and let F^ω be as in (5). If $\delta = \delta(s, \gamma) > 0$ is chosen sufficiently small, we have*

$$\|F^\omega\|_{Y(\mathbb{R})} < \infty \quad \textit{and} \quad \|F^\omega\|_Z < \infty \quad \textit{a.s.}$$

Proof. In the following, we assume that $\delta = \delta(\gamma, s) > 0$ is sufficiently small. In the computations below, we have $N \geq 2^6$ and $(t, x) \in \mathbb{R} \times \mathbb{R}^3$. For $\sigma \geq \frac{8}{3}$, it follows from Minkowski's integral inequality and Lemma 3.5 that

$$\begin{aligned}\|N^{-\frac{3}{4}+\frac{1}{24\gamma}+\delta}|x|^{\frac{3}{8}}|\nabla|\widetilde{F}_N^{\omega}\|_{L_\omega^\sigma \ell_N^{8/3} L_t^{8/3} L_x^\infty} &\leq \|N^{-\frac{3}{4}+\frac{1}{24\gamma}+\delta}|x|^{\frac{3}{8}}|\nabla|\widetilde{F}_N^{\omega}\|_{\ell_N^{8/3} L_\omega^\sigma L_t^{8/3} L_x^\infty} \\ &\lesssim \sqrt{\sigma}\|N^{1-\frac{1}{12\gamma}+\delta}(f_N, g_N)\|_{\ell_N^{8/3}(L_x^2 \times \dot{H}_x^{-1})} \\ &\lesssim \sqrt{\sigma}\|(f, g)\|_{H_x^s \times H_x^{s-1}}.\end{aligned}$$

In particular, we have

$$\|N^{-\frac{3}{4}+\frac{1}{24\gamma}+\delta}|x|^{\frac{3}{8}}|\nabla|\widetilde{F}_N^{\omega}\|_{\ell_N^{8/3} L_t^{8/3} L_x^\infty} < \infty$$

almost surely. A similar argument for the remaining terms in the $Y_{\mathbb{R}}$-norm leads to the regularity restrictions

$$s > \max\bigl(1 - \tfrac{1}{12\gamma}, 1 - \tfrac{1}{3\gamma}, \tfrac{1}{2} - \tfrac{5}{12\gamma}, 1 - \tfrac{1}{4\gamma}, 1 - \tfrac{3}{10\gamma}, 1 - \tfrac{1}{3\gamma}, \tfrac{1}{2} - \tfrac{5}{12\gamma}\bigr),$$

which have been listed in the same order as the terms in the definition of $\|F^\omega\|_{Y_{\mathbb{R}}}$. Next, we estimate $\|F^\omega\|_Z$. Using Corollary 4.3, the terms involving $\|W_*[|\nabla|\widetilde{F}_N]\|_{L_\tau^p}$ lead to the restriction

$$s > \max\bigl((1 - \tfrac{1}{\gamma})(\tfrac{1}{2} - \tfrac{1}{24}), 0\bigr) + \max\bigl(1 - \tfrac{1}{12\gamma}, 0\bigr).$$

Since $0 < \gamma \leq 1$, this leads to $s > \max\bigl(1 - \frac{1}{12\gamma}, 0\bigr)$. Using Lemma 3.7, the fourth and fifth summand in the Z-norm lead to the restriction

$$s > \max\bigl(1 - \tfrac{1}{3\gamma}, 1 - \tfrac{1}{2\gamma}\bigr). \qquad \square$$

In this paper, the condition $\gamma \leq 1$ is only used in the proof of Lemma 7.3. By changing the restriction on s, we could also treat a slightly larger range of parameters γ.

8. Control of error terms

In this section, we estimate the error terms in Propositions 6.1, 6.3, and 6.6. Before we begin with our main estimates we prove an auxiliary lemma.

Lemma 8.1. *Let $w \in L_\tau^1(\mathbb{R})$ be nonnegative. Let $K \in 2^{\mathbb{N}}$ be arbitrary, and let S_K be defined by*

$$S_K w = K\langle K\rho\rangle^{-2} * w.$$

Then, we have for all $v \in L_{\mathrm{loc}}^1(\mathbb{R}^3)$ that

$$\int_{\mathbb{R}^3} |P_K v(x)|^6 w(t - |x|)\,\mathrm{d}x \lesssim \int_{\mathbb{R}^3} |v(x)|^6 ((S_K w)(t - |x|) + |x|^{-1}\|w\|_{L_\tau^1})\,\mathrm{d}x. \tag{67}$$

Proof. We prove (67) by interpolation. The $L^\infty \to L^\infty$ estimate is trivial. Thus, it suffices to prove the $L^1 \to L^1$ estimate

$$\int_{\mathbb{R}^3} |P_K v(x)| w(t - |x|)\,\mathrm{d}x \lesssim \int_{\mathbb{R}^3} |v(x)| ((S_K w)(t - |x|) + |x|^{-1}\|w\|_{L_\tau^1})\,\mathrm{d}x. \tag{68}$$

Let $\Psi \in \{\phi, \psi\}$ be as in the definition of the Littlewood–Paley projection. Then,

$$\begin{aligned}\int_{\mathbb{R}^3} |P_K v(x)| w(t-|x|)\,\mathrm{d}x &\le \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |v(y)| K^3 |\check{\Psi}(K(x-y))| w(t-|x|)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{\mathbb{R}^3} |v(y)| \biggl(K^3 \int_{\mathbb{R}^3} |\check{\Psi}(K(y-x))| w(t-|x|)\,\mathrm{d}x \biggr)\,\mathrm{d}y.\end{aligned}$$

Hence, it remains to establish the pointwise bound

$$K^3 \int_{\mathbb{R}^3} |\check{\Psi}(K(y-x))| w(t-|x|)\,\mathrm{d}x \lesssim (S_K * w)(t-|y|) + |y|^{-1} \|w\|_{L^1_\tau}.$$

Now, the main task consists of converting the left-hand side into a one-dimensional integral. Using an integral formula from [Sogge 1995, p. 8], we have

$$\begin{aligned}K^3 \int_{\mathbb{R}^3} |\check{\Psi}(K(y-x))| w(t-|x|)\,\mathrm{d}x &= K^3 \int_{\mathbb{R}^3} |\check{\Psi}(Kx)| w(t-|y-x|)\,\mathrm{d}x \\ &\lesssim K^3 \int_0^\infty |\check{\Psi}(Kr)| \biggl(\int_{|x|=r} w(t-|y-x|)\,\mathrm{d}\sigma(t,x) \biggr)\,\mathrm{d}r \\ &= K^3 \int_0^\infty |\check{\Psi}(Kr)| \biggl(\int_{|y-x|=r} w(t-|x|)\,\mathrm{d}\sigma(t,x) \biggr)\,\mathrm{d}r \\ &= K^3 \int_0^\infty |\check{\Psi}(Kr)| \frac{2\pi r}{|y|} \int_{||y|-r|}^{|y|+r} w(t-\rho)\rho\,\mathrm{d}\rho\,\mathrm{d}r \\ &\lesssim \frac{K^3}{|y|} \int_0^{4|y|} \int_{|y|-r}^{|y|+r} r |\check{\Psi}(Kr)| w(t-\rho) |\rho|\,\mathrm{d}\rho\,\mathrm{d}r \\ &\qquad + \frac{K^3}{|y|} \int_{4|y|}^\infty \int_{r-|y|}^{r+|y|} r |\check{\Psi}(Kr)| w(t-\rho)\rho\,\mathrm{d}\rho\,\mathrm{d}r. \end{aligned} \tag{69}$$

Let us now estimate the first summand in the last line of (69). We have

$$\begin{aligned}\frac{K^3}{|y|} \int_0^{4|y|} \int_{|y|-r}^{|y|+r} r |\check{\Psi}(Kr)| w(t-\rho) |\rho|\,\mathrm{d}\rho\,\mathrm{d}r &= \frac{K^3}{|y|} \int_0^{4|y|} \int_{-r}^{r} r |\check{\Psi}(Kr) w(t-|y|-\rho) |(|y|+\rho)|\,\mathrm{d}\rho\,\mathrm{d}r \\ &\lesssim K^3 \int_0^{4|y|} \int_{-r}^{r} r |\check{\Psi}(Kr)| w(t-|y|-\rho)\,\mathrm{d}\rho\,\mathrm{d}r \\ &\le K^3 \int_{-\infty}^{\infty} \biggl(\int_{|\rho|}^\infty |\check{\Psi}(Kr)| r\,\mathrm{d}r \biggr) w(t-|y|-\rho)\,\mathrm{d}\rho \\ &\le K \int_{-\infty}^{\infty} \biggl(\int_{K|\rho|}^\infty |\check{\Psi}(r)| r\,\mathrm{d}r \biggr) w(t-|y|-\rho)\,\mathrm{d}\rho \\ &\lesssim K \int_{-\infty}^{\infty} \langle K|\rho| \rangle^{-2} w(t-|y|-\rho)\,\mathrm{d}\rho \\ &= (S_K w)(t-|y|).\end{aligned}$$

Thus, it remains to estimate the second integral in the last line of (69). We have

$$\begin{aligned}\frac{K^3}{|y|}\int_{4|y|}^{\infty}\int_{r-|y|}^{r+|y|} r|\check{\Psi}(Kr)|w(t-\rho)\rho\,\mathrm{d}\rho\,\mathrm{d}r &\lesssim \frac{K^3}{|y|}\int_{4|y|}^{\infty}\int_{r-|y|}^{r+|y|} r^2|\check{\Psi}(Kr)|w(t-\rho)\,\mathrm{d}\rho\,\mathrm{d}r\\ &\leq \frac{K^3}{|y|}\|w\|_{L^1_\tau(\mathbb{R})}\int_0^\infty |\check{\Psi}(Kr)|r^2\,\mathrm{d}r\\ &\leq \frac{1}{|y|}\|w\|_{L^1_\tau(\mathbb{R})}\int_0^\infty |\check{\Psi}(r)|r^2\,\mathrm{d}r\\ &\lesssim \frac{1}{|y|}\|w\|_{L^1_\tau(\mathbb{R})}.\end{aligned}$$

□

Corollary 8.2 (frequency-localized interaction flux estimate). *Let F be as in Definition 7.2 and let $v : I\times\mathbb{R}^3\to\mathbb{R}$ be a solution of* (43). *Then, we have*

$$\sup_{K\in 2^{\mathbb{N}}}\||x|^{\frac{1}{3}}(|\nabla|\widetilde{F}_N)^{\frac{1}{3}}P_K v\|^6_{L^6_{t,x}(I\times\mathbb{R}^3)} \lesssim \min(N^{\frac{1}{6\gamma}-2\delta}, N^{2-2\delta})(\mathcal{F}_I+\|F\|^2_Z\mathcal{A}_I). \tag{70}$$

Remark 8.3. The flux estimate yields much better integrability in the spatial variable x than the Morawetz estimate. To see this, note that (70) cannot be controlled by the Morawetz term. For instance, one might try to estimate

$$\||x|\,|\nabla|\widetilde{F}_N v^3\|_{L^2_{t,x}(I\times\mathbb{R}^3)} \lesssim \||x|^{\frac{3}{2}}|\nabla|\widetilde{F}_N\|_{L^\infty_{t,x}(I\times\mathbb{R}^3)}\||x|^{-\frac{1}{6}}v\|^3_{L^6_{t,x}(I\times\mathbb{R}^3)}.$$

Even for smooth and compactly supported initial data, $|\nabla|\widetilde{F}_N$ only decays like $\sim(1+|t|)^{-1}$ and is morally supported around the light cone $|x|=|t|$. Thus, the term $\||x|^{\frac{3}{2}}|\nabla|\widetilde{F}_N\|_{L^\infty_{t,x}(I\times\mathbb{R}^3)}$ grows like $\sim(1+|t|)^{1/2}$ as I increases.

Proof. Using the in/out-decomposition and Lemma 8.1, it follows that

$$\begin{aligned}&\||x|^{\frac{1}{3}}(|\nabla|\widetilde{F}_N)^{\frac{1}{3}}P_K v\|^6_{L^6_{t,x}(I\times\mathbb{R}^3)}\\ &\quad\lesssim \|(W_{\mathrm{out}}[|\nabla|\widetilde{F}_N])^{\frac{1}{3}}P_K v\|^6_{L^6_{t,x}(I\times\mathbb{R}^3)} + \|(W_{\mathrm{in}}[|\nabla|\widetilde{F}_N])^{\frac{1}{3}}P_K v\|^6_{L^6_{t,x}(I\times\mathbb{R}^3)}\\ &\quad\lesssim \|S_K(|W_{\mathrm{out}}[|\nabla|\widetilde{F}_N]|^2)^{\frac{1}{6}}v\|^6_{L^6_{t,x}(I\times\mathbb{R}^3)} + \||W_{\mathrm{out}}[|\nabla|\widetilde{F}_N]|^2\|_{L^1_\tau}\||x|^{-\frac{1}{6}}v\|^6_{L^6_{t,x}(I\times\mathbb{R}^3)}\\ &\qquad + \|S_K(|W_{\mathrm{in}}[|\nabla|\widetilde{F}_N]|^2)^{\frac{1}{6}}v\|^6_{L^6_{t,x}(I\times\mathbb{R}^3)} + \||W_{\mathrm{in}}[|\nabla|\widetilde{F}_N]|^2\|_{L^1_\tau}\||x|^{-\frac{1}{6}}v\|^6_{L^6_{t,x}(I\times\mathbb{R}^3)}\\ &\quad\lesssim \min(N^{\frac{1}{6\gamma}-2\delta}, N^{2-2\delta})(\mathcal{F}_I+\|F\|^2_Z\mathcal{A}_I).\end{aligned}$$

By taking the supremum over $K\in 2^{\mathbb{N}}$, we arrive at (70). □

8A. *Energy increment.* In this section, we control the main error term in the energy increment.

Proposition 8.4 (main error term in energy increment). *Let F be as in Definition 7.2 and let $v : I\times\mathbb{R}^3\to\mathbb{R}$ be a solution of* (43). *Then, it holds that*

$$\left|\int_I\int_{\mathbb{R}^3}(|\nabla|\widetilde{F})v^5\,\mathrm{d}x\,\mathrm{d}t\right| \lesssim (\mathcal{F}_I+\|F\|^2_Z\mathcal{A}_I)^{\frac{1}{6}}\mathcal{A}_I^{\frac{7}{12}}\mathcal{E}_I^{\frac{1}{4}}\|F\|^{\frac{2}{3}}_{Y_I}. \tag{71}$$

Remark 8.5. Instead of using $\mathcal{F}_I^{1/6}$ to overcome the logarithmic divergence, we could also just use $\mathcal{F}_I^{\epsilon}$. Then, $\| |x|^{3/8} |\nabla| \widetilde{F}_N \|_{L_t^{8/3} L_x^\infty(I\times\mathbb{R}^3)}$ changes into a (nonendpoint) term $\| |x|^{1/4-} |\nabla| \widetilde{F}_N \|_{L_t^{4-} L_x^\infty(I\times\mathbb{R}^3)}$. The probabilistic gain should then increase from $\frac{2}{3}\cdot\frac{1}{8\gamma}$ to $\frac{1}{4\gamma}$ derivatives, which should lead to the restriction $s > \max\bigl(1-\frac{1}{4\gamma}, 0\bigr)$. For expository purposes, we do not present this argument here.

Proof. Using a Littlewood–Paley decomposition, we write

$$v = \sum_{K\geq 1} P_K v \quad\text{and}\quad \widetilde{F} = \sum_{N\geq 2^6} \widetilde{F}_N.$$

Thus,

$$\begin{aligned}\left|\int_I\int_{\mathbb{R}^3} (|\nabla|\widetilde{F}) v^5 \,\mathrm{d}x\,\mathrm{d}t\right| &\lesssim \sum_{N\geq 2^6}\ \sum_{K_1\geq K_2\geq\cdots\geq K_5\geq 1}\left|\int_I\int_{\mathbb{R}^3}(|\nabla|\widetilde{F}_N)\prod_{j=1}^5 P_{K_j} v\,\mathrm{d}x\,\mathrm{d}t\right| \\ &= \sum_{N\geq 2^6}\ \sum_{\substack{K_1\geq K_2\geq\cdots\geq K_5\geq 1\\ K_1\geq 2^{-4}N}}\left|\int_I\int_{\mathbb{R}^3}(|\nabla|\widetilde{F}_N)\prod_{j=1}^5 P_{K_j} v\,\mathrm{d}x\,\mathrm{d}t\right|.\end{aligned}$$

Note that, for all summands above, we have $K_1 > 1$. Using Proposition 2.9 and Corollary 8.2, it follows that

$$\begin{aligned}&\left|\int_I\int_{\mathbb{R}^3}(|\nabla|\widetilde{F}_N)\prod_{j=1}^5 P_{K_j} v\,\mathrm{d}x\,\mathrm{d}t\right| \\ &\quad\leq \| |x|^{\frac{3}{8}} |\nabla|\widetilde{F}_N\|_{L_t^{8/3}L_x^\infty(I\times\mathbb{R}^3)}^{\frac{2}{3}} \| |x|^{\frac{1}{3}} (|\nabla|\widetilde{F}_N)^{\frac{1}{3}} P_{K_5} v\|_{L^6_{t,x}(I\times\mathbb{R}^3)} \prod_{j=2}^4 \| |x|^{-\frac{1}{6}} P_{K_j} v\|_{L^6_{t,x}(I\times\mathbb{R}^3)} \\ &\qquad\qquad \cdot \| |x|^{-\frac{1}{6}} P_{K_1} v\|_{L^6_{t,x}(I\times\mathbb{R}^3)}^{\frac{1}{2}} \|P_{K_1} v\|_{L_t^\infty L_x^2(I\times\mathbb{R}^3)}^{\frac{1}{2}} \\ &\quad\lesssim N^{\frac{2}{3}\left(\frac{3}{4}-\delta-\frac{1}{24\gamma}\right)} \|F\|_{Y_I}^{\frac{2}{3}} N^{\frac{1}{36\gamma}-\frac{\delta}{3}} (\mathcal{F}_I + \|F\|_Z^2 \mathcal{A}_I)^{\frac{1}{6}} \mathcal{A}_I^{\frac{7}{12}} K_1^{-\frac{1}{2}} \mathcal{E}_I^{\frac{1}{4}} \\ &\quad= \left(\frac{N}{K_1}\right)^{\frac{1}{2}-\delta} K_1^{-\delta} \|F\|_{Y_I}^{\frac{2}{3}} (\mathcal{F}_I + \|F\|_Z^2 \mathcal{A}_I)^{\frac{1}{6}} \mathcal{A}_I^{\frac{7}{12}} \mathcal{E}_I^{\frac{1}{4}}.\end{aligned}$$

Using that $K_1 \gtrsim N$ and $K_1,\ldots,K_5 \geq 1$, we obtain (71) after summing. □

8B. *Morawetz estimate.* In this section, we control the main error term in the Morawetz estimate. The main new difficulty is the weight $x/|x|$.

Proposition 8.6 (main error term in Morawetz estimate). *Let F be as in Definition 7.2 and let v be a solution of* (43). *Then,*

$$\left|\int_I\int_{\mathbb{R}^3}\frac{x}{|x|}\cdot\nabla_x(F)v^5\,\mathrm{d}x\,\mathrm{d}t\right| \lesssim (\mathcal{F}_I + \|F\|_Z^2\mathcal{A}_I)^{\frac{1}{6}}\mathcal{A}_I^{\frac{7}{12}+\frac{\delta}{6}}\mathcal{E}_I^{\frac{1}{4}-\frac{\delta}{2}}\|F\|_{Y_I}^{\frac{2}{3}} + \|F\|_{Y_I}\mathcal{A}_I^{\frac{5}{6}}.$$

Proof. As before, we use a Littlewood–Paley decomposition and write

$$\left|\int_I\int_{\mathbb{R}^3}\frac{x}{|x|}\cdot\nabla_x(F)v^5\,\mathrm{d}x\,\mathrm{d}t\right| \lesssim \sum_{N\geq 2^5}\ \sum_{\substack{L\geq 1, K_1\geq\cdots\geq K_5\geq 1\\ \max(L,K_1)\geq 2^{-4}N}}\left|\int_I\int_{\mathbb{R}^3} P_L\left(\frac{x}{|x|}\right)\cdot\nabla_x(F_N)\prod_{j=1}^5 P_{K_j} v\,\mathrm{d}x\,\mathrm{d}t\right|.$$

Case 1: $K_1 \geq L$. From the conditions $K_1 \geq 2^{-4}N$ and $N \geq 2^5$, it follows that $K_1 > 1$. Thus, we can place $P_{K_1} v$ in $L_t^\infty L_x^2(I \times \mathbb{R}^3)$. Using (39), we estimate

$$\begin{aligned}
&\left| \int_I \int_{\mathbb{R}^3} P_L\Big(\frac{x}{|x|}\Big) \cdot \nabla_x (F_N) \prod_{j=1}^5 P_{K_j} v \,\mathrm{d}x\,\mathrm{d}t \right| \\
&\quad \leq \int_I \int_{\mathbb{R}^3} \Big| P_L\Big(\frac{x}{|x|}\Big)\Big| \frac{1}{|x|^{\frac{1}{3}}} (|W_{\mathrm{out},\nabla}[F_N]| + |W_{\mathrm{in},\nabla}[F]|)^{\frac{1}{3}} |\nabla_x F_N|^{\frac{2}{3}} \prod_{j=1}^5 |P_{K_j} v| \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_I \int_{\mathbb{R}^3} \Big| P_L\Big(\frac{x}{|x|}\Big)\Big| \frac{1}{|x|^{\frac{1}{3}}} |F_N|^{\frac{1}{3}} |\nabla_x F_N|^{\frac{2}{3}} \prod_{j=1}^5 |P_{K_j} v| \,\mathrm{d}x\,\mathrm{d}t. \qquad (72)
\end{aligned}$$

To control the first term in the right-hand side above, we estimate

$$\begin{aligned}
&\int_I \int_{\mathbb{R}^3} \Big| P_L\Big(\frac{x}{|x|}\Big)\Big| \frac{1}{|x|^{\frac{1}{3}}} (|W_{\mathrm{out},\nabla}[F_N]| + |W_{\mathrm{in},\nabla}[F]|)^{\frac{1}{3}} |\nabla_x F_N|^{\frac{2}{3}} \prod_{j=1}^5 |P_{K_j} v| \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \lesssim \Big\| P_L\Big(\frac{x}{|x|}\Big)\Big\|_{L^\infty_{t,x}(I\times\mathbb{R}^3)} (\| |W_{\mathrm{out},\nabla}[F_N]|^{\frac{1}{3}} P_{K_5} v\|_{L^6_{t,x}(I\times\mathbb{R}^3)} + \| |W_{\mathrm{in},\nabla}[F]|^{\frac{1}{3}} P_{K_5} v\|_{L^6_{t,x}(I\times\mathbb{R}^3)}) \\
&\qquad \cdot \prod_{j=2}^4 \| |x|^{-\frac{1}{6}} P_{K_j} v\|_{L^6_{t,x}(I\times\mathbb{R}^3)} \\
&\qquad \cdot \| |x|^{-\frac{1}{6}} P_{K_1} v\|^{\frac{1}{2}+\delta}_{L^6_{t,x}(I\times\mathbb{R}^3)} \|P_{K_1} v\|^{\frac{1}{2}-\delta}_{L_t^\infty L_x^2(I\times\mathbb{R}^3)} \| |x|^{\frac{3+2\delta}{8}} \nabla_x F_N\|^{\frac{2}{3}}_{L_t^{8/(3-2\delta)} L_x^{2/\delta}(I\times\mathbb{R}^3)}.
\end{aligned}$$

The first factor is estimated by

$$\Big\| P_L\Big(\frac{x}{|x|}\Big)\Big\|_{L^\infty_{t,x}(I\times\mathbb{R}^3)} \lesssim \Big\| \frac{x}{|x|}\Big\|_{L^\infty_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \lesssim 1.$$

Using Lemma 8.1 and arguing as in the proof of Corollary 8.2, we estimate the second factor by

$$\begin{aligned}
&\| |W_{\mathrm{out},\nabla}[F_N]|^{\frac{1}{3}} P_{K_5} v\|_{L^6_{t,x}(I\times\mathbb{R}^3)} + \| |W_{\mathrm{in},\nabla}[F]|^{\frac{1}{3}} P_{K_5} v\|_{L^6_{t,x}(I\times\mathbb{R}^3)} \\
&\quad \lesssim \|S_{K_5}(|W_{\mathrm{out},\nabla}[F_N]|^2)^{\frac{1}{6}} v\|_{L^6_{t,x}(I\times\mathbb{R}^3)} + \|S_{K_5}(|W_{\mathrm{in},\nabla}[F]|^2)^{\frac{1}{6}} v\|_{L^6_{t,x}(I\times\mathbb{R}^3)} \\
&\qquad + (\|W_{\mathrm{out},\nabla}[F_N]\|^{\frac{1}{3}}_{L^2_\tau(\mathbb{R})} + \|W_{\mathrm{in},\nabla}[F]\|^{\frac{1}{3}}_{L^2_\tau(\mathbb{R})}) \| |x|^{-\frac{1}{6}} v\|_{L^6_{t,x}(I\times\mathbb{R}^3)} \\
&\quad \lesssim N^{\frac{1}{36\gamma}-\frac{\delta}{3}} (\mathcal{F}_I + \|F\|_Z^2 \mathcal{A}_I)^{\frac{1}{6}}.
\end{aligned}$$

From Proposition 2.9, we have

$$\| |x|^{-\frac{1}{6}} P_{K_j} v\|_{L^6_{t,x}(I\times\mathbb{R}^3)} \lesssim \| |x|^{-\frac{1}{6}} v\|_{L^6_{t,x}(I\times\mathbb{R}^3)} \lesssim \mathcal{A}_I^{\frac{1}{6}}.$$

Furthermore, since $K_1 > 1$, we have

$$\|P_{K_1} v\|_{L_t^\infty L_x^2(I\times\mathbb{R}^3)} \lesssim K_1^{-1} \mathcal{E}_I^{\frac{1}{2}}.$$

Finally, applying Proposition 2.9 to the Riesz multipliers, we have

$$\||x|^{\frac{3+2\delta}{8}}\nabla_x F_N\|_{L_t^{8/(3-2\delta)(I\times\mathbb{R}^3)}L_x^{2/\delta}(I\times\mathbb{R}^3)} \lesssim \||x|^{\frac{3+2\delta}{8}(I\times\mathbb{R}^3)}|\nabla|F_N\|_{L_t^{8/(3-2\delta)}L_x^{2/\delta}(I\times\mathbb{R}^3)} \lesssim N^{\frac{3}{4}-\frac{1}{24\gamma}-\frac{5\delta}{2}}\|F\|_{Y_I}.$$

Putting everything together, it follows that

$$\int_I\int_{\mathbb{R}^3}\left|P_L\left(\frac{x}{|x|}\right)\right|\frac{1}{|x|^{\frac{1}{3}}}(|W_{\mathrm{out},\nabla}[F_N]|+|W_{\mathrm{in},\nabla}[F]|)^{\frac{1}{3}}|\nabla_x F_N|^{\frac{2}{3}}\prod_{j=1}^{5}|P_{K_j}v|\,\mathrm{d}x\,\mathrm{d}t \lesssim \left(\frac{N}{K_1}\right)^{\frac{1}{2}-2\delta}K_1^{-\delta}(\mathcal{F}_I+\|F\|_Z^2\mathcal{A}_I)^{\frac{1}{6}}\mathcal{A}_I^{\frac{7}{12}+\frac{\delta}{6}}\mathcal{E}_I^{\frac{1}{2}-\frac{\delta}{2}}\|F\|_{Y_I}^{\frac{2}{3}}.$$

Using the decay $K_1^{-\delta}$ in the highest frequency, we may sum $N, L, K_1, \dots, K_5$.

Next, we estimate the second term in the right-hand side of (72). We have

$$\begin{aligned}&\int_I\int_{\mathbb{R}^3}\left|P_L\left(\frac{x}{|x|}\right)\right|\frac{1}{|x|^{\frac{1}{3}}}|F_N|^{\frac{1}{3}}|\nabla_x F_N|^{\frac{2}{3}}\prod_{j=1}^{5}|P_{K_j}v|\,\mathrm{d}x\,\mathrm{d}t\\ &\lesssim \left\|P_L\left(\frac{x}{|x|}\right)\right\|_{L^\infty_{t,x}(I\times\mathbb{R}^3)}\prod_{j=2}^{5}\||x|^{-\frac{1}{6}}P_{K_j}v\|_{L^6_{t,x}(I\times\mathbb{R}^3)}\cdot\||x|^{-\frac{1}{6}}P_{K_1}v\|^{\frac{1}{2}+\delta}_{L^6_{t,x}(I\times\mathbb{R}^3)}\|P_{K_1}v\|^{\frac{1}{2}-\delta}_{L_t^\infty L_x^2(I\times\mathbb{R}^3)}\\ &\quad\cdot\||x|^{\frac{1}{2}}F_N\|^{\frac{1}{3}}_{L^\infty_{t,x}(I\times\mathbb{R}^3)}\||x|^{\frac{3+2\delta}{8}}\nabla_x F_N\|^{\frac{2}{3}}_{L_t^{8/(3-2\delta)}L_x^{2/\delta}(I\times\mathbb{R}^3)}.\end{aligned}$$

Arguing as above, together with $\||x|^{(1+\delta)/2}F_N\|_{L^\infty_{t,x}(I\times\mathbb{R}^3)} \le N^{-\delta}\|F\|_Z$, we get

$$\begin{aligned}\int_I\int_{\mathbb{R}^3}\left|P_L\left(\frac{x}{|x|}\right)\right|\frac{1}{|x|^{\frac{1}{3}}}|F_N|^{\frac{1}{3}}|\nabla_x F_N|^{\frac{2}{3}}\prod_{j=1}^{5}|P_{K_j}v|\,\mathrm{d}x\,\mathrm{d}t &\lesssim N^{\frac{1}{2}-\frac{1}{36\gamma}-2\delta}K_1^{-\frac{1}{2}+\delta}\mathcal{A}_I^{\frac{3}{4}+\frac{\delta}{6}}\mathcal{E}_I^{\frac{1}{4}-\delta}\|F\|_{Y_I}^{\frac{2}{3}}\|F\|_Z^{\frac{1}{3}}\\ &\lesssim \left(\frac{N}{K_1}\right)^{\frac{1}{2}-2\delta}K_1^{-\delta}\mathcal{A}_I^{\frac{3}{4}+\frac{\delta}{6}}\mathcal{E}_I^{\frac{1}{4}-\frac{\delta}{2}}\|F\|_{Y_I}^{\frac{2}{3}}\|F\|_Z^{\frac{1}{3}}.\end{aligned}$$

Summing over the appropriate range, this contribution is acceptable.

Case 2: $L \ge K_1$. Consequently, we have $L \ge 2^{-4}N > 1$. Using Lemma 2.5, it follows that

$$\left|P_L\left(\frac{x}{|x|}\right)\right| \lesssim (L|x|)^{-1}.$$

This yields

$$\begin{aligned}\left|\int_I\int_{\mathbb{R}^3}P_L\left(\frac{x}{|x|}\right)\cdot\nabla_x(F_N)\prod_{j=1}^{5}P_{K_j}v\,\mathrm{d}x\,\mathrm{d}t\right| &\lesssim L^{-1}\int_I\int_{\mathbb{R}^3}\frac{1}{|x|}|\nabla_x(F_N)|\prod_{j=1}^{5}|P_{K_j}v|\,\mathrm{d}x\,\mathrm{d}t\\ &\lesssim L^{-1}\||x|^{-\frac{1}{6}}|\nabla|F_N\|_{L^6_{t,x}}\prod_{j=1}^{5}\||x|^{-\frac{1}{6}}P_{K_j}v\|_{L^6_{t,x}}\\ &\lesssim \left(\frac{N}{L}\right)^{1-\delta}L^{-\delta}\|F\|_{Y_I}\mathcal{A}_I^{\frac{5}{6}}\end{aligned}$$

Using the decay $L^{-\delta}$ in the highest frequency, we may sum $N, L, K_1, \dots, K_5$. □

8C. ***Interaction flux estimate.*** In this section, we control the main error terms in the interaction flux estimate. The main difficulty is the weight $\int_{-\infty}^{t-|x|} w(\tau)\,\mathrm{d}\tau$. First, we recall a radial Sobolev embedding.

Lemma 8.7. *For any $v \in L_t^\infty \dot{H}^1_{\mathrm{rad}}(I \times \mathbb{R}^3)$, we have*

$$\sup_{K \in 2^{\mathbb{N}}} \||x|^{\frac{1}{2}} P_K v\|_{L^\infty_{t,x}(I\times\mathbb{R}^3)} \lesssim \|v\|^{\frac{3}{4}}_{L_t^\infty L_x^6(I\times\mathbb{R}^3)} \|\nabla v\|^{\frac{1}{4}}_{L_t^\infty L_x^2(I\times\mathbb{R}^3)} \lesssim \sup_{t\in I} E[v](t)^{\frac{1}{4}}.$$

Proof. Let $r \in \mathbb{R}_{>0}$. Then, we have

$$\begin{aligned}(P_K v)^4(t,r) &= 4\int_r^\infty (P_K v)^3(t,\rho)\,(\partial_r P_K v)(t,\rho)\,\mathrm{d}\rho \\ &\leq 4r^{-2}\int_r^\infty |(P_K v)^3(t,\rho)|\,|\partial_r P_K v(t,\rho)|\rho^2\,\mathrm{d}\rho \\ &\leq 4r^{-2}\|P_K v(t,x)\|^3_{L_x^6(\mathbb{R}^3)}\|\nabla P_K v(t,x)\|_{L_x^2(\mathbb{R}^3)} \\ &\leq 4r^{-2}\|v(t,x)\|^3_{L_x^6(\mathbb{R}^3)}\|\nabla v(t,x)\|_{L_x^2(\mathbb{R}^3)}.\end{aligned}$$

The first inequality then follows by taking the supremum in r and t. The second inequality follows from the definition of $E[v]$. □

Proposition 8.8 (first main error term in interaction flux estimate). *Let $w \in L^1_\tau(\mathbb{R}) \cap L^{12}_\tau(\mathbb{R})$ be a nonnegative weight. Let F be as in Definition 7.2 and let $v : I \times \mathbb{R}^3 \to \mathbb{R}$ be a solution of* (43). *Then, it holds that*

$$\left|\int_I\int_{\mathbb{R}^3}\left(\int_{-\infty}^{t-|x|} w\,\mathrm{d}\tau\right)(|\nabla|\widetilde{F})v^5\,\mathrm{d}x\,\mathrm{d}t\right| \lesssim \|w\|_{L^1_\tau(\mathbb{R})}\|F\|^{\frac{2}{3}}_{Y_I}(\mathcal{F}_I + \|F\|^2_Z\mathcal{A}_I)^{\frac{1}{6}}\mathcal{A}_I^{\frac{7}{12}}\mathcal{E}_I^{\frac{1}{4}} + \|w\|_{L^2_\tau(\mathbb{R})}(\mathcal{F}_I + \|F\|^2_Z\mathcal{A}_I)^{\frac{1}{2}}\mathcal{E}_I^{\frac{1}{2}} + \|w\|_{L^{12}_\tau(\mathbb{R})}\|F\|_{Y_I}\mathcal{A}_I^{\frac{5}{6}}.$$

The same argument also controls the main error term in the backward interaction flux estimate.

Proof. As before, we use Littlewood–Paley theory to decompose into frequency-localized functions. Then, it remains to control

$$\sum_{N\geq 2^6}\sum_{\substack{L\geq 1, K_1\geq\cdots\geq K_5\geq 1\\ \max(L,K_1)\geq 2^{-4}N}}\left|\int_I\int_{\mathbb{R}^3} P_L\left(\int_{-\infty}^{t-|x|} w\,\mathrm{d}\tau\right)(|\nabla|\widetilde{F}_N)\prod_{j=1}^5 P_{K_j}v\,\mathrm{d}x\,\mathrm{d}t\right|.$$

We distinguish several different cases.

<u>Case 1</u>: $K_1 \geq L$. We have

$$\begin{aligned}&\left|\int_I\int_{\mathbb{R}^3} P_L\left(\int_{-\infty}^{t-|x|} w\,\mathrm{d}\tau\right)(|\nabla|\widetilde{F}_N)\prod_{j=1}^5 P_{K_j}v\,\mathrm{d}x\,\mathrm{d}t\right| \\ &\quad\lesssim \left\|P_L\left(\int_{-\infty}^{t-|x|} w\,\mathrm{d}\tau\right)\right\|_{L^\infty_{t,x}(I\times\mathbb{R}^3)} \||x|^{\frac{3}{8}}|\nabla|\widetilde{F}_N\|^{\frac{2}{3}}_{L_t^{8/3}L_x^\infty(I\times\mathbb{R}^3)} \||x|^{\frac{1}{3}}(|\nabla|\widetilde{F}_N)^{\frac{1}{3}}P_{K_5}v\|_{L^6_{t,x}(I\times\mathbb{R}^3)} \\ &\qquad\cdot\prod_{j=2}^4\||x|^{-\frac{1}{6}}P_{K_j}v\|_{L^6_{t,x}(I\times\mathbb{R}^3)}\||x|^{-\frac{1}{6}}P_{K_1}v\|^{\frac{1}{2}}_{L^6_{t,x}(I\times\mathbb{R}^3)}\|P_{K_1}v\|^{\frac{1}{2}}_{L_t^\infty L_x^2(I\times\mathbb{R}^3)}.\end{aligned}$$

The first factor is controlled by

$$\Big\| P_L\Big(\int_{-\infty}^{t-|x|} w \,\mathrm{d}\tau\Big)\Big\|_{L^\infty_{t,x}(I\times\mathbb{R}^3)} \lesssim \Big\| \int_{-\infty}^{t-|x|} w \,\mathrm{d}\tau\Big\|_{L^\infty_{t,x}(\mathbb{R}\times\mathbb{R}^3} \leq \|w\|_{L^1_\tau(\mathbb{R})}.$$

Arguing as in the proof of Proposition 8.4, this leads to the total contribution

$$\lesssim \|w\|_{L^1_\tau} \|F\|_{Y_I}^{\frac{2}{3}} (\mathcal{F}_I + \|F\|_Z^2 \mathcal{A}_I)^{\frac{1}{6}} \mathcal{A}_I^{\frac{7}{12}} \mathcal{E}_I^{\frac{1}{4}}.$$

Case 2: $L \geq K_1$. In this case, the most severe term is the low-frequency scenario $K_1 = \cdots = K_5 = 1$. Then, we can no longer place $P_{K_1} v$ in $L^\infty_t L^2_x(I\times\mathbb{R}^3)$ and therefore lack space-integrability. To resolve this, we make use of the integrability of $w(t-|x|)$ in time.

Subcase 2(a): $L \geq K_1, |x| \geq 1$. Using Proposition 2.9, Corollary 8.2 and Lemma 8.7, we obtain

$$\begin{aligned}
&\Big|\int_I\int_{|x|\geq 1} P_L\Big(\int_{-\infty}^{t-|x|} w\,\mathrm{d}\tau\Big)(|\nabla|\widetilde{F}_N)\prod_{j=1}^5 P_{K_j} v \,\mathrm{d}x\,\mathrm{d}t\Big| \\
&\leq \Big\|\langle x\rangle^{-2} P_L\Big(\int_{-\infty}^{t-|x|} w\,\mathrm{d}\tau\Big)\Big\|_{L^2_{t,x}(I\times\mathbb{R}^3)} \prod_{j=3}^5 (\| |x|^{\frac{1}{3}} (|\nabla|\widetilde{F}_N)^{\frac{1}{3}} P_{K_j} v\|_{L^6_{t,x}(I\times\mathbb{R}^3)}) \prod_{j=1}^2 \| |x|^{\frac{1}{2}} P_{K_j} v\|_{L^\infty_{t,x}(I\times\mathbb{R}^3)} \\
&\lesssim N^{1-\delta} \Big\|\langle x\rangle^{-2} P_L\Big(\int_{-\infty}^{t-|x|} w\,\mathrm{d}\tau\Big)\Big\|_{L^2_{t,x}(I\times\mathbb{R}^3)} (\mathcal{F}_I + \|F\|_Z^2\mathcal{A}_I)^{\frac{1}{2}} \mathcal{E}_I^{\frac{1}{2}}.
\end{aligned}$$

It remains to control the weighted $L^2_{t,x}$-norm. We recall that the kernel of P_L has zero mean. Using Lemma 2.4 and the boundedness of the Hardy–Littlewood maximal function M, we obtain

$$\begin{aligned}
&\Big\|\langle x\rangle^{-2} P_L\Big(\int_{-\infty}^{t-|x|} w\,\mathrm{d}\tau\Big)\Big\|_{L^2_{t,x}(I\times\mathbb{R}^3)} \\
&\qquad = \|\langle x\rangle^{-2} P_L\Big(\int_{t-|x|}^{t} w\,\mathrm{d}\tau\Big)\|_{L^2_{t,x}(I\times\mathbb{R}^3)} \\
&\qquad \lesssim L^{-1}\|\langle x\rangle^{-2} w(t-|x|)\|_{L^2_{t,x}(I\times\mathbb{R}^3)} + L^{-1}\|\langle x\rangle^{-3}\int_{t-|x|}^t w(\tau)\,\mathrm{d}\tau\|_{L^2_{t,x}(I\times\mathbb{R}^3)} \\
&\qquad \lesssim L^{-1}\|\langle x\rangle^{-2} w(t-|x|)\|_{L^2_{t,x}(I\times\mathbb{R}^3)} + L^{-1}\|\langle x\rangle^{-3}|x|(Mw)(t-|x|)\|_{L^2_{t,x}(I\times\mathbb{R}^3)} \\
&\qquad \lesssim L^{-1}\|\langle x\rangle^{-2}\|_{L^2_x(\mathbb{R}^3)}(\|w(t)\|_{L^2_t(\mathbb{R})} + \|Mw(t)\|_{L^2_t(\mathbb{R})}) \\
&\qquad \lesssim L^{-1}\|w\|_{L^2_t(\mathbb{R})}.
\end{aligned}$$

Putting everything together, it follows that

$$\Big|\int_I\int_{|x|\geq 1} P_L\Big(\int_{-\infty}^{t-|x|} w\,\mathrm{d}\tau\Big)(|\nabla|\widetilde{F}_N)\prod_{j=1}^5 P_{K_j} v \,\mathrm{d}x\,\mathrm{d}t\Big| \lesssim \Big(\frac{N}{L}\Big)^{1-\delta} L^{-\delta}\|w\|_{L^2_\tau(\mathbb{R})}(\mathcal{F}_I + \|F\|_Z^2\mathcal{A}_I)^{\frac{1}{2}}\mathcal{E}_I^{\frac{1}{2}}.$$

Using the decay $L^{-\delta}$ in the highest frequency, we may sum $N, L, K_1, \ldots, K_5$.

Subcase 2(b): $L \geq K_1,\ |x| \leq 1$. Near the origin, our strongest tool is the Morawetz estimate. Thus, we write

$$\begin{aligned}
&\left| \int_I \int_{|x|\leq 1} P_L\Big(\int_{-\infty}^{t-|x|} w \,\mathrm{d}\tau\Big)(|\nabla|\widetilde{F}_N) \prod_{j=1}^{5} P_{K_j} v \,\mathrm{d}x\,\mathrm{d}t \right| \\
&\quad \lesssim \Big\| |x|^{\frac{1}{6}} P_L\Big(\int_{-\infty}^{t-|x|} w \,\mathrm{d}\tau\Big)\Big\|_{L^{12}_{t,x}(I\times\{|x|\leq 1\})} \| |x|^{\frac{2}{3}} |\nabla| \widetilde{F}_N \|_{L^{12}_{t,x}(I\times\mathbb{R}^3)} \prod_{j=1}^{5} \| |x|^{-\frac{1}{6}} P_{K_j} v\|_{L^6_{t,x}(I\times\mathbb{R}^3)} \\
&\quad \lesssim N^{1-\delta} \| \langle x\rangle^{-1} P_L\Big(\int_{-\infty}^{t-|x|} w \,\mathrm{d}\tau\Big)\|_{L^{12}_{t,x}(I\times\mathbb{R}^3)} \|\widetilde{F}\|_{Y_I} \mathcal{A}_I^{\frac{5}{6}}.
\end{aligned}$$

Using Lemma 2.4, we have

$$\begin{aligned}
&\Big\| \langle x\rangle^{-1} P_L\Big(\int_{-\infty}^{t-|x|} w \,\mathrm{d}\tau\Big)\Big\|_{L^{12}_{t,x}(I\times\mathbb{R}^3)} \\
&\qquad \leq \Big\| \langle x\rangle^{-1} P_L\Big(\int_{t-|x|}^{t} w \,\mathrm{d}\tau\Big)\Big\|_{L^{12}_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \\
&\qquad \lesssim L^{-1} \| \langle x\rangle^{-1} w(t-|x|)\|_{L^{12}_{t,x}(\mathbb{R}\times\mathbb{R}^3)} + L^{-1} \Big\| \langle x\rangle^{-2} \Big(\int_{t-|x|}^{t} w \,\mathrm{d}\tau\Big)\Big\|_{L^{12}_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \\
&\qquad \lesssim L^{-1} \| \langle x\rangle^{-1} w(t-|x|)\|_{L^{12}_{t,x}(\mathbb{R}\times\mathbb{R}^3)} + L^{-1} \| \langle x\rangle^{-1} (Mw)(t-|x|)\|_{L^{12}_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \\
&\qquad = L^{-1} \| \langle x\rangle^{-1}\|_{L^{12}_x(\mathbb{R}^3)} (\|w\|_{L^{12}_\tau(\mathbb{R})} + \|Mw\|_{L^{12}_\tau(\mathbb{R})}) \\
&\qquad \lesssim L^{-1} \|w\|_{L^{12}_\tau(\mathbb{R})}.
\end{aligned}$$

Putting everything together, it follows that

$$\left| \int_I \int_{|x|\leq 1} P_L\Big(\int_{-\infty}^{t-|x|} w \,\mathrm{d}\tau\Big)(|\nabla|\widetilde{F}_N) \prod_{j=1}^{5} P_{K_j} v \,\mathrm{d}x\,\mathrm{d}t \right| \lesssim \Big(\frac{N}{L}\Big)^{1-\delta} L^{-\delta} \|w\|_{L^{12}_\tau(\mathbb{R})} \|\widetilde{F}\|_{Y_I} \mathcal{A}_I^{\frac{5}{6}}.$$

Using the decay $L^{-\delta}$ in the highest frequency, we may sum $N, L, K_1, \dots, K_5$. □

Proposition 8.9 (second main error term in interaction flux estimate). *Let $w \in L^1_\tau(\mathbb{R}) \cap L^{12}_\tau(\mathbb{R})$ be a nonnegative weight. Let F be as in Definition 7.2 and let $v : I \times \mathbb{R}^3 \to \mathbb{R}$ be a solution of* (43). *Then, it holds that*

$$\left| \int_I \int_{\mathbb{R}^3} w(t-|x|) F v^5 \,\mathrm{d}x\,\mathrm{d}t \right| \lesssim \|w\|_{L^2_\tau(\mathbb{R})} \mathcal{F}_I^{\frac{1}{2}} \mathcal{E}_I^{\frac{1}{2}} + \|w\|_{L^{12}_\tau(\mathbb{R})} \|F\|_{Y_I} \mathcal{A}_I^{\frac{5}{6}}.$$

Proof. We follow an easier version of the arguments in the proof of Proposition 8.8. As before, we distinguish the two cases $|x| \geq 1$ and $|x| \leq 1$. First, we have

$$\begin{aligned}
&\left| \int_I \int_{|x|\geq 1} w(t-|x|) F v^5 \,\mathrm{d}x\,\mathrm{d}t \right| \\
&\quad \leq \| |x|^{-2} w(t-|x|)\|_{L^2_{t,x}(|x|\geq 1)} (\|W_{\mathrm{out}}[F]^{\frac{1}{3}} v\|_{L^6_{t,x}(I\times\mathbb{R}^3)} + \|W_{\mathrm{in}}[F]^{\frac{1}{3}} v\|_{L^6_{t,x}(I\times\mathbb{R}^3)})^3 \| |x|^{\frac{1}{2}} v\|^2_{L^\infty_{t,x}} \\
&\quad \lesssim \|w\|_{L^2_\tau} \mathcal{F}_I^{\frac{1}{2}} \mathcal{E}_I^{\frac{1}{2}}.
\end{aligned}$$

Second, we have

$$\left|\int_I \int_{|x|\le 1} w(t-|x|) F v^5 \,\mathrm{d}x\,\mathrm{d}t\right| \le \||x|^{\frac{1}{6}} w(t-|x|)\|_{L^{12}_{t,x}(|x|\le 1)} \||x|^{-\frac{1}{6}} v\|^5_{L^6_{t,x}(I\times\mathbb{R}^3)} \||x|^{\frac{2}{3}} F\|_{L^{12}_{t,x}(I\times\mathbb{R}^3)}$$
$$\lesssim \|w\|_{L^{12}_\tau} \|F\|_{Y_I} \mathcal{A}_I^{\frac{5}{6}}. \qquad \square$$

8D. *Lower-order error terms.*

Lemma 8.10 (control of lower-order error terms). *Let F be as in Definition 7.2 and let v be a solution of* (43). *Then*, *it holds that*

$$\int_I \int_{\mathbb{R}^3} |F|^5 \Big(|\partial_t v| + \frac{|v|}{|x|} + |\nabla v| \Big) \,\mathrm{d}x\,\mathrm{d}t \lesssim \|F\|^5_{Y_I} \mathcal{E}_I^{\frac{1}{2}},$$
$$\int_I \int_{\mathbb{R}^3} |F|^2 |v|^3 \Big(|\partial_t v| + \frac{|v|}{|x|} + |\nabla v| \Big) \,\mathrm{d}x\,\mathrm{d}t \lesssim \|F\|^2_{Y_I} \mathcal{A}_I^{\frac{1}{2}} \mathcal{E}_I^{\frac{1}{2}},$$
$$\int_I \int_{\mathbb{R}^3} \frac{1}{|x|} |F| |v|^5 \,\mathrm{d}x\,\mathrm{d}t \lesssim \|F\|_{Y_I} \mathcal{A}_I^{\frac{5}{6}},$$
$$\int_I \int_{\mathbb{R}^3} \frac{1}{|x|} |F|^6 \,\mathrm{d}x\,\mathrm{d}t \lesssim \|F\|^6_{Y_I}.$$

Proof. Using Hardy's inequality, the first inequality follows from

$$\begin{aligned}\int_I \int_{\mathbb{R}^3} |F|^5 \Big(|\partial_t v| + \frac{|v|}{|x|} + |\nabla v| \Big) \,\mathrm{d}x\,\mathrm{d}t &\le \|F\|^5_{L^5_t L^{10}_x(I\times\mathbb{R}^3)} \Big(\|\partial_t v\|_{L^\infty_t L^2_x(I\times\mathbb{R}^3)} + \Big\| \frac{v}{|x|} \Big\|_{L^\infty_t L^2_x(I\times\mathbb{R}^3)} + \|\nabla v\|_{L^\infty_t L^2_x(I\times\mathbb{R}^3)} \Big) \\ &\lesssim \|F\|^5_{Y_I} \mathcal{E}_I^{\frac{1}{2}}.\end{aligned}$$

A similar argument yields that

$$\int_I \int_{\mathbb{R}^3} |F|^2 |v|^3 \Big(|\partial_t v| + \frac{|v|}{|x|} + |\nabla v| \Big) \,\mathrm{d}x\,\mathrm{d}t \lesssim \||x|^{\frac{1}{4}} F\|^2_{L^4_t L^\infty_x(I\times\mathbb{R}^3)} \||x|^{-\frac{1}{6}} v\|^3_{L^6_{t,x}(I\times\mathbb{R}^3)} \sup_{t\in I} E[v](t)^{\frac{1}{2}}.$$

Finally, the third and fourth inequality follow from Hölder's inequality and

$$\||x|^{-\frac{1}{6}} F\|_{L^6_{t,x}(I\times\mathbb{R}^3)} \le \|F\|_{Y_I}. \qquad \square$$

9. Proof of the main theorem

In this section, we collect all previous estimates to prove the a priori energy bound (Theorem 1.4). Using the conditional scattering result of [Dodson et al. 2017], we finish the proof of Theorem 1.3.

Proof of Theorem 1.4. By time-reversal symmetry, it suffices to prove that $\sup_{t\in[0,\infty)} E[v](t) < \infty$. Let $\frac{1}{2} \ge \eta_0 > 0$ be a sufficiently small absolute constant, and let $\frac{1}{2} \ge \eta > 0$ be sufficiently small depending on η_0. In the following, $C = C(\|F\|_Z) > 0$ denotes a large positive constant that depends only on $\|F\|_Z$. By Lemma 7.3 and space-time divisibility, we can choose a finite partition $I_1, \dots, I_J$ of $[0,\infty)$ such that $\|F\|_{Y_{I_j}} < \eta$ for all $j = 1, \dots, J$. With a slight abuse of notation, we write $\mathcal{E}_j := \mathcal{E}_{I_j}$, $\mathcal{A}_j := \mathcal{A}_{I_j}$ and $\mathcal{F}_j := \mathcal{F}_{I_j}$. We also set $\mathcal{E}_0 := E[v](0)$.

First, we estimate the energy increment. Combining Propositions 6.1 and 8.4, and Lemma 8.10, we have

$$\begin{aligned}\mathcal{E}_{j+1} &\le \mathcal{E}_j + C\|F\|_{Y_{I_{j+1}}}^{\frac{2}{3}}(\mathcal{F}_{j+1}+\mathcal{A}_{j+1}\|F\|_Z^2)^{\frac{1}{6}}\mathcal{A}_{j+1}^{\frac{7}{12}}\mathcal{E}_{j+1}^{\frac{1}{4}} + C\|F\|_{Y_{I_{j+1}}}^2\mathcal{A}_{j+1}^{\frac{1}{2}}\mathcal{E}_{j+1}^{\frac{1}{2}} + C\|F\|_{Y_{I_{j+1}}}^5\mathcal{E}_{j+1}^{\frac{1}{2}} \\ &\le C(\mathcal{E}_j+1)+\eta_0\mathcal{E}_{j+1}+\eta_0(\mathcal{F}_{j+1}+\mathcal{A}_{j+1}). \qquad (73)\end{aligned}$$

Next, we estimate the Morawetz term. By combining Propositions 6.3 and 8.6, and Lemma 8.10, we have

$$\begin{aligned}\mathcal{A}_{j+1} &\le C\mathcal{E}_{j+1} + C\|F\|_{Y_{I_{j+1}}}^{\frac{2}{3}}(\mathcal{F}_{j+1}+\mathcal{A}_{j+1}\|F\|_Z^2)^{\frac{1}{6}}\mathcal{A}_{j+1}^{\frac{7}{12}+\frac{\delta}{6}}\mathcal{E}_{j+1}^{\frac{1}{4}-\frac{\delta}{2}} + C\|F\|_{Y_{I_{j+1}}}\mathcal{A}_{j+1}^{\frac{5}{6}} + C\|F\|_{Y_{I_{j+1}}}^6 \\ &\le C(\mathcal{E}_{j+1}+1)+\tfrac{1}{4}(\mathcal{F}_{j+1}+\mathcal{A}_{j+1}). \qquad (74)\end{aligned}$$

Finally, we control the interaction flux term. First, recall that from the definition of $\|F\|_Z$ and the embedding $\ell_1 \hookrightarrow \ell_2$, we have

$$\sum_{*\in\{\text{out},\text{in}\}}\ \sum_{p\in\{2,4,24\}}\left(\sum_{N\ge 2^5}(N^{-\frac{1}{6\gamma}+2\delta}+N^{-2+2\delta})(\|W_*[|\nabla|F_N]\|_{L_\tau^p}^2+\|W_{*,\nabla}[F_N]\|_{L_\tau^p}^2)+\|W_*[F]\|_{L_\tau^p}^2\right) \lesssim \|F\|_Z^2.$$

We now apply our estimates to each of the terms in (64), (65), and (66) separately. By Young's inequality, the estimate $\|S_K w\|_{L_\tau^p} \lesssim_p \|w\|_{L_\tau^p}$ holds uniformly in K. Using the control on the main and lower-order error terms, i.e., Propositions 6.6, 6.7, 8.8, and 8.9 and Lemma 8.10, we obtain

$$\begin{aligned}\mathcal{F}_{j+1} &\le C\|F\|_Z^2\mathcal{E}_{j+1} + C\|F\|_Z^2\|F\|_{Y_{I_{j+1}}}^{\frac{2}{3}}(\mathcal{F}_{j+1}+\|F\|_Z^2\mathcal{A}_{j+1})^{\frac{1}{6}}\mathcal{A}_{j+1}^{\frac{7}{12}}\mathcal{E}_{j+1}^{\frac{1}{4}} \\ &\quad + C\|F\|_Z^2(\mathcal{F}_{j+1}+\|F\|_Z^2\mathcal{A}_{j+1})^{\frac{1}{2}}\mathcal{E}_{j+1}^{\frac{1}{2}} \\ &\quad + C\|F\|_Z^2\|F\|_{Y_{I_{j+1}}}\mathcal{A}_{j+1}^{\frac{5}{6}} + C\|F\|_Z^2\mathcal{F}_{j+1}^{\frac{1}{2}}\mathcal{E}_{j+1}^{\frac{1}{2}} \\ &\quad + C\|F\|_Z^2\|F\|_{Y_{I_{j+1}}}^2(\|F\|_{Y_{I_{j+1}}}^3+\mathcal{A}_{j+1}^{\frac{1}{2}})\mathcal{E}_{j+1}^{\frac{1}{2}} \\ &\le C(\mathcal{E}_{j+1}+1)+\tfrac{1}{4}(\mathcal{F}_{j+1}+\mathcal{A}_{j+1}). \qquad (75)\end{aligned}$$

We briefly note that, as long as $C > 0$ remains independent of η_0 and η, terms such as $C\|F\|_Z^2\mathcal{F}_{j+1}^{1/2}\mathcal{E}_{j+1}^{1/2}$ prevent us from placing an η_0 in front of $\mathcal{F}_{j+1}+\mathcal{A}_{j+1}$. Combining (73), (74), and (75), we arrive at

$$\begin{aligned}\mathcal{E}_{j+1} &\le C(\mathcal{E}_j+1)+\eta_0\mathcal{E}_{j+1}+\eta_0(\mathcal{A}_{j+1}+\mathcal{F}_{j+1}), \\ \mathcal{A}_{j+1}+\mathcal{F}_{j+1} &\le C(\mathcal{E}_{j+1}+1)+\tfrac{1}{2}(\mathcal{A}_{j+1}+\mathcal{F}_{j+1}).\end{aligned}$$

Finally, choosing $\eta_0 > 0$ sufficiently small depending on $C = C(\|F\|_Z)$, we obtain

$$\mathcal{E}_{j+1}+1 \le \widetilde{C}(\mathcal{E}_j+1). \qquad (76)$$

By iterating this inequality finitely many times, we obtain

$$\sup_{t\in[0,\infty)} E[v](t) = \max_{j=1,\dots,J}\mathcal{E}_j < \infty. \qquad \square$$

Proof of Theorem 1.3. Using Lemmas 5.1 and 7.3, it follows that the forced nonlinear wave equation (6) is almost surely locally well-posed. Then, Theorem 1.3 follows from Theorem 1.4 and Proposition 5.2. $\square$

Acknowledgements

The author thanks his advisor Terence Tao for his invaluable guidance and support. The author also thanks Laura Cladek, Rowan Killip, and Monica Visan for many insightful discussions.

References

[Bahouri and Gérard 1999] H. Bahouri and P. Gérard, "High frequency approximation of solutions to critical nonlinear wave equations", *Amer. J. Math.* **121**:1 (1999), 131–175. MR Zbl

[Bell 1968] W. W. Bell, *Special functions for scientists and engineers*, Van Nostrand, London, 1968. MR Zbl

[Bényi et al. 2015a] Á. Bényi, T. Oh, and O. Pocovnicu, "On the probabilistic Cauchy theory of the cubic nonlinear Schrödinger equation on $\mathbb{R}^d$, $d \geq 3$", *Trans. Amer. Math. Soc. Ser. B* **2** (2015), 1–50. MR Zbl

[Bényi et al. 2015b] Á. Bényi, T. Oh, and O. Pocovnicu, "Wiener randomization on unbounded domains and an application to almost sure well-posedness of NLS", pp. 3–25 in *Excursions in harmonic analysis, IV* (College Park, MD, 2002-2013), edited by R. Balan et al., Birkhäuser, Cham, 2015. MR Zbl

[Bényi et al. 2019] Á. Bényi, T. Oh, and O. Pocovnicu, "Higher order expansions for the probabilistic local Cauchy theory of the cubic nonlinear Schrödinger equation on $\mathbb{R}^3$", *Trans. Amer. Math. Soc. Ser. B* **6** (2019), 114–160. MR Zbl

[Bourgain 1994] J. Bourgain, "Periodic nonlinear Schrödinger equation and invariant measures", *Comm. Math. Phys.* **166**:1 (1994), 1–26. MR Zbl

[Bourgain 1996] J. Bourgain, "Invariant measures for the 2D-defocusing nonlinear Schrödinger equation", *Comm. Math. Phys.* **176**:2 (1996), 421–445. MR Zbl

[Brereton 2019] J. T. Brereton, "Almost sure local well-posedness for the supercritical quintic NLS", *Tunis. J. Math.* **1**:3 (2019), 427–453. MR Zbl

[Burq and Tzvetkov 2008a] N. Burq and N. Tzvetkov, "Random data Cauchy theory for supercritical wave equations, I: Local theory", *Invent. Math.* **173**:3 (2008), 449–475. MR Zbl

[Burq and Tzvetkov 2008b] N. Burq and N. Tzvetkov, "Random data Cauchy theory for supercritical wave equations, II: A global existence result", *Invent. Math.* **173**:3 (2008), 477–496. MR Zbl

[Chanillo 1984] S. Chanillo, "The multiplier for the ball and radial functions", *J. Funct. Anal.* **55**:1 (1984), 18–24. MR Zbl

[Chanillo et al. 2017] S. Chanillo, M. Czubak, D. Mendelson, A. Nahmod, and G. Staffilani, "Almost sure boundedness of iterates for derivative nonlinear wave equations", preprint, 2017. arXiv

[Christ et al. 2003] M. Christ, J. Colliander, and T. Tao, "Ill-posedness for nonlinear Schrödinger and wave equations", preprint, 2003. arXiv

[Colliander et al. 2004] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, "Global existence and scattering for rough solutions of a nonlinear Schrödinger equation on $\mathbb{R}^3$", *Comm. Pure Appl. Math.* **57**:8 (2004), 987–1014. MR Zbl

[Da Prato and Debussche 2002] G. Da Prato and A. Debussche, "Two-dimensional Navier–Stokes equations driven by a space-time white noise", *J. Funct. Anal.* **196**:1 (2002), 180–210. MR Zbl

[De Nápoli and Drelichman 2016] P. L. De Nápoli and I. Drelichman, "Elementary proofs of embedding theorems for potential spaces of radial functions", pp. 115–138 in *Methods of Fourier analysis and approximation theory*, edited by M. Ruzhansky and S. Tikhonov, Birkhäuser, Cham, 2016. MR Zbl

[De Nápoli et al. 2011] P. L. De Nápoli, I. Drelichman, and R. G. Durán, "On weighted inequalities for fractional integrals of radial functions", *Illinois J. Math.* **55**:2 (2011), 575–587. MR Zbl

[Dodson et al. 2017] B. Dodson, J. Lührmann, and D. Mendelson, "Almost sure scattering for the 4D energy-critical defocusing nonlinear wave equation with radial data", preprint, 2017. To appear in *Amer. J. Math.* arXiv

[Dodson et al. 2019] B. Dodson, J. Lührmann, and D. Mendelson, "Almost sure local well-posedness and scattering for the 4D cubic nonlinear Schrödinger equation", *Adv. Math.* **347** (2019), 619–676. MR Zbl

[Fefferman 1971] C. Fefferman, "The multiplier problem for the ball", *Ann. of Math.* (2) **94** (1971), 330–336. MR Zbl

[Grillakis 1990] M. G. Grillakis, "Regularity and asymptotic behaviour of the wave equation with a critical nonlinearity", *Ann. of Math.* (2) **132**:3 (1990), 485–509. MR Zbl

[Grillakis 1992] M. G. Grillakis, "Regularity for the wave equation with a critical nonlinearity", *Comm. Pure Appl. Math.* **45**:6 (1992), 749–774. MR Zbl

[Jiang et al. 2012] J.-C. Jiang, C. Wang, and X. Yu, "Generalized and weighted Strichartz estimates", *Commun. Pure Appl. Anal.* **11**:5 (2012), 1723–1752. MR Zbl

[Killip et al. 2009] R. Killip, T. Tao, and M. Visan, "The cubic nonlinear Schrödinger equation in two dimensions with radial data", *J. Eur. Math. Soc.* **11**:6 (2009), 1203–1258. MR Zbl

[Killip et al. 2019] R. Killip, J. Murphy, and M. Visan, "Almost sure scattering for the energy-critical NLS with radial data below $H^1(\mathbb{R}^4)$", *Comm. Partial Differential Equations* **44**:1 (2019), 51–71. MR Zbl

[Klainerman and Tataru 1999] S. Klainerman and D. Tataru, "On the optimal local regularity for Yang–Mills equations in $\mathbb{R}^{4+1}$", *J. Amer. Math. Soc.* **12**:1 (1999), 93–116. MR Zbl

[Lührmann and Mendelson 2014] J. Lührmann and D. Mendelson, "Random data Cauchy theory for nonlinear wave equations of power-type on $\mathbb{R}^3$", *Comm. Partial Differential Equations* **39**:12 (2014), 2262–2283. MR Zbl

[Lührmann and Mendelson 2016] J. Lührmann and D. Mendelson, "On the almost sure global well-posedness of energy sub-critical nonlinear wave equations on $\mathbb{R}^3$", *New York J. Math.* **22** (2016), 209–227. MR Zbl

[Muscalu and Schlag 2013] C. Muscalu and W. Schlag, *Classical and multilinear harmonic analysis, I*, Cambridge Studies in Adv. Math. **137**, Cambridge Univ. Press, 2013. MR Zbl

[Nahmod et al. 2012] A. R. Nahmod, T. Oh, L. Rey-Bellet, and G. Staffilani, "Invariant weighted Wiener measures and almost sure global well-posedness for the periodic derivative NLS", *J. Eur. Math. Soc.* **14**:4 (2012), 1275–1330. MR Zbl

[Oh 2017] T. Oh, "Probabilistic perspectives in nonlinear dispersive PDEs", lecture notes, 2017, available at https://tinyurl.com/tohlecture.

[Oh and Pocovnicu 2016] T. Oh and O. Pocovnicu, "Probabilistic global well-posedness of the energy-critical defocusing quintic nonlinear wave equation on $\mathbb{R}^3$", *J. Math. Pures Appl.* (9) **105**:3 (2016), 342–366. MR Zbl

[Oh et al. 2017] T. Oh, M. Okamoto, and O. Pocovnicu, "On the probabilistic well-posedness of the nonlinear Schrödinger equations with non-algebraic nonlinearities", preprint, 2017. arXiv

[Pocovnicu 2017] O. Pocovnicu, "Almost sure global well-posedness for the energy-critical defocusing nonlinear wave equation on $\mathbb{R}^d$, $d = 4$ and 5", *J. Eur. Math. Soc.* **19**:8 (2017), 2521–2575. MR Zbl

[Rauch 1981] J. Rauch, "I: The u^5 Klein–Gordon equation; II: Anomalous singularities for semilinear wave equations", pp. 335–364 in *Nonlinear partial differential equations and their applications* (Paris, 1978/1979), edited by H. Brézis and J.-L. Lions, Res. Notes in Math. **53**, Pitman, Boston, 1981. MR Zbl

[Shatah and Struwe 1993] J. Shatah and M. Struwe, "Regularity results for nonlinear wave equations", *Ann. of Math.* (2) **138**:3 (1993), 503–518. MR Zbl

[Shatah and Struwe 1994] J. Shatah and M. Struwe, "Well-posedness in the energy space for semilinear wave equations with critical growth", *Int. Math. Res. Not.* **1994**:7 (1994), 303–309. MR Zbl

[Sogge 1995] C. D. Sogge, *Lectures on nonlinear wave equations*, Monographs in Analysis **2**, Int. Press, Boston, 1995. MR Zbl

[Stein 1993] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Math. Series **43**, Princeton Univ. Press, 1993. MR Zbl

[Sterbenz 2005] J. Sterbenz, "Angular regularity and Strichartz estimates for the wave equation", *Int. Math. Res. Not.* **2005**:4 (2005), 187–231. MR Zbl

[Strauss 1968] W. A. Strauss, "Decay and asymptotics for $cmu = F(u)$", *J. Funct. Anal.* **2** (1968), 409–457. MR Zbl

[Struwe 1988] M. Struwe, "Globally regular solutions to the u^5 Klein–Gordon equation", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **15**:3 (1988), 495–513. MR Zbl

[Tao 2006a] T. Tao, *Nonlinear dispersive equations: local and global analysis*, CBMS Regional Conf. Series in Math. **106**, Amer. Math. Soc., Providence, RI, 2006. MR Zbl

[Tao 2006b] T. Tao, "Spacetime bounds for the energy-critical nonlinear wave equation in three spatial dimensions", *Dyn. Partial Differ. Equ.* **3**:2 (2006), 93–110. MR Zbl

[Thomann and Tzvetkov 2010] L. Thomann and N. Tzvetkov, "Gibbs measure for the periodic derivative nonlinear Schrödinger equation", *Nonlinearity* **23**:11 (2010), 2771–2791. MR Zbl

Received 24 Apr 2018. Accepted 18 Apr 2019.

BJOERN BRINGMANN: bringmann@math.ucla.edu
Department of Mathematics, University of California, Los Angeles, CA, United States

ANALYSIS AND PDE
Vol. 13, No. 4, 2020
dx.doi.org/10.2140/apde.2020.13.1051

ON THE EXISTENCE OF TRANSLATING SOLUTIONS OF MEAN CURVATURE FLOW IN SLAB REGIONS

THEODORA BOURNI, MAT LANGFORD AND GIUSEPPE TINAGLIA

We prove, in all dimensions $n \geq 2$, that there exists a convex translator lying in a slab of width $\pi \sec\theta$ in $\mathbb{R}^{n+1}$ (and in no smaller slab) if and only if $\theta \in \left[0, \frac{\pi}{2}\right]$. We also obtain convexity and regularity results for translators which admit appropriate symmetries and study the asymptotics and reflection symmetry of translators lying in slab regions.

1. Introduction

A solution of mean curvature flow is a smooth one-parameter family $\{\Sigma_t\}_{t\in\mathbb{R}}$ of hypersurfaces Σ_t in $\mathbb{R}^{n+1}$ with normal velocity equal to the mean curvature vector. A translating solution of mean curvature flow is one which evolves purely by translation: $\Sigma_{t+s} = \Sigma_t + se$ for some $e \in \mathbb{R}^{n+1}\setminus\{0\}$ and each $s, t \in (-\infty, \infty)$. In that case, the time slices are all congruent and satisfy

$$H = -\langle \nu, e\rangle, \tag{1}$$

where ν is a choice of unit normal field and $H = \operatorname{div}\nu$ is the corresponding mean curvature. Conversely, if a hypersurface satisfies (1) then the one-parameter family of translated hypersurfaces $\Sigma_t := \Sigma + te$ satisfies mean curvature flow. We shall eliminate the scaling invariance and isotropy of (1) by restricting attention to translating solutions which move with unit speed in the "upwards" direction. That is, we henceforth assume that $e = e_{n+1}$. We will refer to a hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$ satisfying (1) with $e = e_{n+1}$ as a *translator*.

The most prominent example of a translator is the Grim Reaper curve, $\Gamma^1 \subset \mathbb{R}^2$, defined by

$$\Gamma^1 := \left\{(x, -\log\cos x) : |x| < \tfrac{\pi}{2}\right\}.$$

Taking products with lines then yields the Grim hyperplanes

$$\Gamma^n := \left\{(x_1, \ldots, x_n, -\log\cos x_1) : |x_1| < \tfrac{\pi}{2}\right\}.$$

The Grim hyperplane Γ^n lies in the slab $\left\{(x_1, \ldots, x_n) : |x_1| < \frac{\pi}{2}\right\}$ (and in no smaller slab). More generally, if Σ^{n-k} is a translator in $\mathbb{R}^{n-k+1}$ then $\Sigma^{n-k} \times \mathbb{R}^k$ is a translator in $\mathbb{R}^{n-k+1} \times \mathbb{R}^k \cong \mathbb{R}^{n+1}$.

There is also a family of "oblique" Grim planes $\Gamma^n_{\theta,\phi}$ parametrized by $(\theta, \phi) \in \left[0, \frac{\pi}{2}\right) \times S^{n-2}$. These are obtained by rotating the "standard" Grim plane Γ^n through the angle $\theta \in \left[0, \frac{\pi}{2}\right)$ in the plane $\operatorname{span}\{\phi, e_{n+1}\}$

MSC2010: 53A10.
Keywords: mean curvature flow, translators, ancient solutions.

for some unit vector $\phi \in \operatorname{span}\{e_2, \dots, e_n\}$ and then scaling by the factor $\sec\theta$. To see that the result is indeed a translator, we need only check that

$$-H_\theta = -\cos\theta H = \cos\theta\langle\nu, e_{n+1}\rangle = \langle\cos\theta\nu + \sin\theta\phi, e_{n+1}\rangle = \langle\nu_\theta, e_{n+1}\rangle,$$

where H_θ and ν_θ are the mean curvature and outward unit normal of $\Gamma^n_{\theta,\phi}$ respectively. The oblique Grim hyperplane $\Gamma^n_{\theta,\phi}$ lies in the slab $S^{n+1}_\theta := \left\{(x_1, \dots, x_n) : |x_1| < \frac{\pi}{2}\sec\theta\right\}$ (and in no smaller slab). More generally, if Σ^{n-k} is a translator in $\mathbb{R}^{n-k+1}$ then the hypersurface $\Sigma^n_{\theta,\phi}$ obtained by rotating $\Sigma^{n-k}\times\mathbb{R}^k$ counterclockwise through angle θ in the plane $\phi\wedge e_{n+1}$ and then scaling by $\sec\theta$ is a translator in $\mathbb{R}^{n+1}$, so long as ϕ is a nonzero vector in $\operatorname{span}\{e_{n-k+1}, \dots, e_n\}$. The oblique Grim hyperplanes will play an important role in our analysis.

A convex entire translator asymptotic to a paraboloid was constructed in [Altschuler and Wu 1994]; see also [Clutterbuck et al. 2007]. White conjectured [2003, Conjecture 2] that the bowl is the only strictly convex translator of dimension $n \ge 2$. X.-J. Wang [2011] proved that it is the only convex entire translator in $\mathbb{R}^3$ and constructed further convex entire examples in higher dimensions. This disproves White's conjecture; however, White [2003, unnumbered remark on page 133] also stated that, even if the conjecture is false, it may be true for translating limit flows to an embedded mean-convex flow. Since limit flows to mean convex, embedded flows are noncollapsing (and hence entire) [Andrews 2012; Sheng and Wang 2009; White 2003], Wang's result proves the modified conjecture when the dimension is 2. More recently, Haslhofer [2015] proved that the bowl is the only noncollapsing translator of dimension $n \ge 2$ which is uniformly two-convex, confirming White's modified conjecture for two-convex, embedded mean curvature flows. The first two authors removed the embeddedness requirement when $n \ge 3$ [Bourni and Langford 2016].

Wang also proved the existence of strictly convex translating solutions which lie in slab regions in $\mathbb{R}^{n+1}$ for all $n \ge 2$. Since convexity of solutions of the Dirichlet problem for the graphical translator equation remains open,[1] this was achieved by exploiting the Legendre transform and the existence of convex solutions of certain fully nonlinear equations [Wang 2011]. Unfortunately, this method loses track of the precise geometry of the domain on which the solution is defined and so it remained unclear exactly which slabs admit translators; see [Spruck and Xiao 2017, Remark 1.6]. Our main result resolves this problem.

Recall that the slab region $S^{n+1}_\theta \subset \mathbb{R}^{n+1}$ is defined by

$$S^{n+1}_\theta := \left\{(x, y, z) \in \mathbb{R}\times\mathbb{R}^{n-1}\times\mathbb{R} : |x| < \tfrac{\pi}{2}\sec\theta\right\} \subset \mathbb{R}^{n+1}.$$

Theorem 1 (existence of convex translators in slab regions). *For every $n \ge 2$ and every $\theta \in \left(0, \frac{\pi}{2}\right)$ there exists a strictly convex translator Σ^n_θ which lies in S^{n+1}_θ and in no smaller slab.*

The solutions we construct are reflection symmetric across the midplane of the slab, rotationally symmetric with respect to the subspace $\mathbb{E}^{n-1} := \operatorname{span}\{e_2, \dots, e_n\}$ and asymptotic to the "correct" oblique Grim

[1] Recently, Spruck and Xiao [2017] proved that complete mean convex translators in $\mathbb{R}^3$ are necessarily convex. We extend their result to higher dimensions in Section 3, assuming the translator has at most two distinct principal curvatures.

hyperplanes $\Gamma^n_{\theta,\phi}$ in the following sense: if ϕ is any unit vector in $\mathbb{E}^{n-1}$ then the curve $\{\sin\omega\phi - \cos\omega e_{n+1} : \omega \in (0,\theta)\}$ lies in the normal image of Σ^n_θ and the translators

$$\Sigma^n_{\theta,\omega} := \Sigma^n_\theta - P(\sin\omega\phi - \cos\omega e_{n+1})$$

converge locally uniformly in the smooth topology to the oblique Grim hyperplane $\Gamma^n_{\theta,\phi}$ as $\omega \to \theta$, where $P : S^n \to \Sigma^n_\theta$ is the inverse of the Gauss map.

Spruck and Xiao [2017, Theorem 1.1] recently proved that every mean convex translator is actually convex and Wang [2011, Corollary 2.2] proved that any convex translator which is not an entire graph must lie in a slab region. The bowl translator of Altschuler and Wu and the Grim hyperplane provide examples in the limiting cases $\theta \in \{0, \frac{\pi}{2}\}$ and there can exist no convex translator inside a slab of width less than π (the Grim hyperplane is a barrier); Theorem 1 provides the existence of a convex translator in all remaining cases, so we obtain the following corollary.

Corollary 2. *Let Ω be an open subset of $\mathbb{R}^n$ for some $n \geq 2$. There exists a convex translator in the cylinder $\Omega \times \mathbb{R}$ (and in no smaller cylinder) if and only if Ω is a slab of width $\pi \sec\theta$ for some $\theta \in [0, \frac{\pi}{2}]$.*

A systematic classification of translators lying in slab regions remains an open problem. As a first step towards addressing it, we show that the asymptotics of the solutions described in Theorem 1 are universal.

Theorem 3 (unique asymptotics modulo translation). *Given $n \geq 2$ and $\theta \in (0, \frac{\pi}{2})$ let Σ^n_θ be a convex translator which lies in S^{n+1}_θ and in no smaller slab. If $n \geq 3$, assume in addition that Σ^n_θ is rotationally symmetric with respect to the subspace $\mathbb{E}^{n-1} := \operatorname{span}\{e_2, \dots, e_n\}$. Given any unit vector $\phi \in \mathbb{E}^{n-1}$ the curve $\{\sin\omega\phi - \cos\omega e_{n+1} : \omega \in [0,\theta)\}$ lies in the normal image of Σ^n_θ and the translators*

$$\Sigma^n_{\theta,\omega} := \Sigma^n_\theta - P(\sin\omega\phi - \cos\omega e_{n+1})$$

converge locally uniformly in the smooth topology to the oblique Grim hyperplane $\Gamma^n_{\theta,\phi}$ as $\omega \to \theta$, where $P : S^n \to \Sigma^n_\theta$ is the inverse of the Gauss map.

We note that, in the important special case $n = 2$, this result was already obtained in [Spruck and Xiao 2017] using different methods.

The rotational symmetry hypothesis — which is not required when $n = 2$ — may be necessary in higher dimensions: it is conceivable that there exist convex translators in the slab $S^4_\theta \subset \mathbb{R}^4$, for example, which are asymptotic to an "oblique" $\Sigma^2_\theta \times \mathbb{R}$, where $\Sigma^2_\theta \subset \mathbb{R}^3$ is the translator from Theorem 1.

Using the Alexandrov reflection principle, we deduce that such solutions are reflection symmetric.

Corollary 4. *Given $\theta \in (0, \frac{\pi}{2})$, let Σ be a strictly convex translator which lies in S^{n+1}_θ and in no smaller slab. If $n \geq 3$ assume in addition that Σ is rotationally symmetric with respect to $\mathbb{E}^{n-1}$. Then Σ is reflection symmetric across the hyperplane $\{0\} \times \mathbb{R}^n$.*

This result was also obtained in [Spruck and Xiao 2017] when $n = 2$.

Remark. After this work was completed, Hoffman, Ilmanen, Martín and White [Hoffman et al. 2019] provided a different approach to the problem of existence of graphical translators over strip regions in $\mathbb{R}^3$ and moreover proved uniqueness of such translators.

2. Compactness

Recall that, given a smooth function u over a domain $\Omega \subset \mathbb{R}^n$, the downward-pointing unit normal ν and the mean curvature $H[u]$ of graph u are given by

$$\nu = \frac{(Du, -1)}{\sqrt{1+|Du|^2}} \quad \text{and} \quad H[u] = \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right),$$

respectively. So graph u is a translator (possibly with boundary) when

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \frac{1}{\sqrt{1+|Du|^2}}. \tag{2}$$

We will derive uniform $C^{1,\alpha}$ estimates for hypersurfaces that are given by the graphs of rotationally symmetric solutions of the Dirichlet problem

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \frac{1}{\sqrt{1+|Du|^2}} \quad \text{in } \Omega, \qquad u = \psi \quad \text{on } \partial\Omega, \tag{3}$$

where Ω is a bounded open subset of $\mathbb{R}^{n+1}$ with $C^{1,\alpha}$ boundary and $\psi : \partial\Omega \to \mathbb{R}$ is a $C^{1,\alpha}$ function for some $\alpha \in (0, 1]$.

By Allard's regularity theorems [1972; 1975], see also [Bourni 2016], the desired estimates are a consequence of the following lemma. We remark that the usual dimension restriction is circumvented here due to the rotational symmetry of the solutions; see Remark 2.4 below.

Lemma 2.1. *Given any $\varepsilon, K > 0$ there exists $\lambda_0 = \lambda_0(\varepsilon, K)$ with the following property: Let u be a solution of* (3), *with $\partial\Omega$ and ψ being rotationally symmetric with respect to the subspace $\mathbb{E}^{n-1} := \operatorname{span}\{e_2, \dots, e_n\}$ and having $C^{1,\alpha}$ norms bounded by K. For any $p \in \operatorname{graph} u$ and $\lambda \le \lambda_0$*

$$\lambda^{-1} \sup_{y \in \operatorname{graph} u \cap B_\lambda^{n+1}(p)} \operatorname{dist}(y-p, P) < \varepsilon \tag{4}$$

for some n-dimensional linear subspace $P = P(p, \varepsilon, \lambda)$. If $B_\lambda^{n+1}(p) \cap \operatorname{graph} \psi = \varnothing$ then

$$\omega_n^{-1} \lambda^{-n} \mathcal{H}^n(\operatorname{graph} u \cap B_\lambda^{n+1}(p)) \le 1 + \varepsilon. \tag{5}$$

If $p \in \operatorname{graph} \psi$ then (4) *holds with P replaced by an n-dimensional half-hyperplane $P_+ = P_+(p, \varepsilon, \lambda)$ such that $0 \in \partial P_+$,*

$$\lambda^{-1} \sup_{\operatorname{graph} \psi \cap B_\lambda^{n+1}(p)} \operatorname{dist}(y-p, \partial P_+) < \varepsilon \tag{6}$$

and (5) *holds with the bound $1+\varepsilon$ replaced by $\frac{1}{2}+\varepsilon$.*

Proof. We assume that the conclusion is not true. Then there exist $\varepsilon_0 > 0$ and $K_0 > 0$, sequences of rotationally symmetric domains Ω_i and boundary data $\psi_i : \partial\Omega_i \to \mathbb{R}$ bounded in $C^{1,\alpha}$ by K_0, corresponding solutions u_i of the Dirichlet problem (3), points $p_i \in \operatorname{graph} u_i$ and scales $\lambda_i \downarrow 0$ such that either (4) or (5) (or (6) in the case $p_i \in \operatorname{graph} \psi_i$), with this ε_0 and with $u = u_i$, $p = p_i$ and $\lambda = \lambda_i$, fails for all i.

Set $\widetilde{\Omega}_i = \eta_{p_i,\lambda_i}(\Omega_i)$, $\Psi_i :=$ graph ψ_i and $\widetilde{\Psi}_i = \eta_{p_i,\lambda_i}(\Psi_i)$, where $\eta_{p,\lambda}(y) = \lambda^{-1}(y-p)$. We define the current $\widetilde{T}_i = \eta_{p_i,\lambda_i\#}(T_i)$, where $T_i = [\![\mathrm{graph}\, u_i]\!]$ and note that $\widetilde{T}_i = [\![\mathrm{graph}\, \tilde{u}_i]\!]$, where $\tilde{u}_i \in C^{1,\alpha}(\widetilde{\Omega}_i)$ is defined by $\tilde{u}_i(p) = \eta_{p_i,\lambda_i}(u_i(\lambda_i p + p_i))$ with mean curvature satisfying

$$\widetilde{H}_i(p) = \lambda_i H_i(x_i + \lambda_i p) \le \lambda_i \quad \Longrightarrow \quad \|\widetilde{H}_i\|_{0,\widetilde{\Omega}_i} \xrightarrow{i\to\infty} 0.$$

It follows, after passing to a subsequence, that [Bourni 2011, Lemma 2.15], see also [Simon 1983, Theorem 34.5]:

(i) $\widetilde{T}_i \to T$ in the weak sense of currents, where T is area-minimizing.

(ii) $\mu_{\widetilde{T}_i} \to \mu_T$ as Radon measures, where $\mu_{\widetilde{T}_i}$ and μ_T denote the total variation measures of $\widetilde{T}_i$ and T respectively.

(iii) For any $\varepsilon > 0$ and any compact subset $W \subset \mathbb{R}^{n+1}$ such that $W \cap \mathrm{spt}\, T \neq \varnothing$ there exists i_0 such that, for all $i \ge i_0$,

$$\mathrm{spt}\, T_i \cap W \subset \varepsilon\text{-neighborhood of } \mathrm{spt}\, T.$$

By the measure convergence (ii), for every $\varepsilon > 0$ there exists i_0 such that, for all $i \ge i_0$,

$$\lambda_i^{-n} \mu_{T_i}(B^{n+1}_{\lambda_i}(p_i)) = \mu_{\widetilde{T}_i}(B^{n+1}_1(0)) \le |\mathrm{spt}\, T \cap B^{n+1}_1(0)| + \varepsilon.$$

By the Hausdorff convergence (3), for any $\varepsilon > 0$ there exists i_0 such that, for all $i \ge i_0$,

$$\frac{1}{\lambda_i} \sup_{y \in B^{n+1}_{\lambda_i}(p_i) \cap \mathrm{spt}\, T_i} \mathrm{dist}(y - x_i, \mathrm{spt}\, T) = \sup_{y \in B^{n+1}_1(0) \cap \mathrm{spt}\, \widetilde{T}_i} \mathrm{dist}(y, \mathrm{spt}\, T) < \varepsilon.$$

So it remains to prove that spt T is either a hyperplane or a half-hyperplane.

It suffices to consider the following three cases for the sequence of points p_i:

<u>Case 1</u>: $p_i \in \Psi_i = \partial\, \mathrm{graph}\, u_i$.

<u>Case 2a</u>: $p_i = (x_i, y_i, u(x_i, y_i)) \notin \Psi_i$, $y_i \in \mathbb{R}^{n-1}$ with $|y_i| = 0$ for all i and $\liminf_i \mathrm{dist}(p_i, \Psi_i) \neq 0$.

<u>Case 2b</u>: $p_i = (x_i, y_i, u(x_i, y_i)) \notin \Psi_i$, $y_i \in \mathbb{R}^{n-1}$ with $\liminf_i |y_i| \neq 0$ and $\liminf_i \mathrm{dist}(p_i, \Psi_i) \neq 0$.

We will show that in Case 1 spt T is half-hyperplane and in Cases 2a and 2b it is a hyperplane.

We need the following fact, which is a consequence of the divergence theorem applied to the normals of the graphs (extended to be independent of the e_{n+1}-direction) in two appropriately chosen domains. For a proof see [Bourni 2011, Lemmas 2.10, 2.12].

Claim 2.1.1. *There exists a constant c such that for any i, $p \in \overline{\Omega}_i \times \mathbb{R}$ and $\rho > 0$ the following hold*:

(i) *Let H_i denote the mean curvature of* graph u_i; *then*

$$\mathcal{H}^n(\mathrm{graph}\, u_i \cap B^{n+1}_\rho(p)) \le c(1 + \rho \|H_i\|_0)\omega_n \rho^n.$$

(ii) *Let $\sigma \in (0, \rho)$, $Q_{\rho,\sigma} = [-\sigma\rho, \sigma\rho] \times B^n_\rho(0)$ and q be an orthogonal transformation of $\mathbb{R}^{n+1}$ such that $q(0) = p$. Then*

$$\mathcal{H}^n(\mathrm{graph}\, u_i \cap q(Q_{\rho,\sigma})) \le \omega_n \rho^n (1 + c\sigma(n + \rho\|H_i\|_0)).$$

In Case 1, by [Bourni 2011, Lemma 2.15], $\operatorname{spt} T$ is an n-dimensional half-space and $\partial T = [\![\Psi]\!]$ with Ψ being the limit of $\widetilde{\Psi}_i$ and where the convergence $\widetilde{\Psi}_i \to \Psi$ is with respect to the $C^{1,\beta}$ topology for any $\beta < \alpha$, which implies that

$$\frac{1}{\lambda_i} \sup_{y \in B^{n+1}_{\lambda_i}(x_i) \cap \Psi_i} \operatorname{dist}(y - x_i, \operatorname{spt} \partial T) = \sup_{y \in B^{n+1}_1(0) \cap \widetilde{\Psi}_i} \operatorname{dist}(y, \Psi) < \varepsilon.$$

Hence taking $P_+ = \operatorname{spt} T$ we get a contradiction for Case 1.

Having proven the boundary case, we will now proceed with the interior. We will first consider Case 2a, that is when $p_i = (x_i, y_i, u(x_i, y_i)) \notin \Psi_i$ with $y_i = 0 \in \mathbb{R}^{n-1}$ and $\liminf_i \operatorname{dist}(p_i, \Psi_i) \neq 0$. In this case the support of the area-minimizing current T is rotationally symmetric in the y-space. Using the uniform area ratio bounds, Claim 2.1.1 and the interior monotonicity formula [Allard 1972], see also [Simon 1983, Section 17], we have

$$\begin{aligned} 1 \le \omega_n^{-1} r^{-n} \mu_T(B^{n+1}_r(p)) &= \omega_n^{-1} r^{-n} \lim_i \mu_{\widetilde{T}_i}(B^{n+1}_r(p)) \\ &= \omega_n^{-1} (\lambda_i r)^{-n} \lim_i \mu_{T_i}(B^{n+1}_{\lambda_i r}(p)) \le c \end{aligned} \tag{7}$$

for all $p \in \operatorname{spt} T$ and any $r > 0$, where c is a constant which is independent of i. Thus, for a sequence $\{\Lambda_k\} \uparrow \infty$, we can apply the Federer–Fleming compactness theorem [1960], see also [Simon 1983, Theorem 32.2], to the sequence $T_{0,\Lambda_k} = \eta_{0,\Lambda_k \#} T$; after passing to a subsequence, this yields $T_{0,\Lambda_k} \to C$ in the weak sense of currents, where C is an area-minimizing cone, and $\mu_{T_{p,\Lambda_k}} \to \mu_C$ as radon measures. Note that C is rotationally symmetric in the y-space, $\mathbb{E}^{n-1}$. Since $\operatorname{spt} C \cap S^n$ is an embedded minimal surface in S^n which is rotationally symmetric with respect to $\mathbb{E}^{n-1}$, it must be congruent to either the equator S^{n-1} or the Clifford torus $S^1_{\sqrt{1/(n-1)}} \times S^{n-2}_{\sqrt{(n-2)/(n-1)}}$ [Brito and Leite 1990; Ôtsuki 1970; 1972]. Since the cone over the Clifford torus cannot arise as a limit of graphs, we conclude that $C = m[\![\mathbb{R}^n \times \{0\}]\!]$. We claim that in fact $m = 1$.

For $\sigma \in (0, 1)$, let $Q_{1,\sigma} = B^n_1(0) \times [-\sigma, \sigma]$. Then $\mu_C(Q_{1,\sigma}) = m\omega_n$. By the measure convergence $\mu_{T_{0,\Lambda_k}} \to \mu_C$ and $\mu_{\widetilde{T}_i} \to \mu_T$, we have that for any $\sigma_0 \in (0, 1)$ and any $\delta > 0$ there exists some $\Lambda > 0$ and k_0 such that for all $k \ge k_0$ and $\sigma \le \sigma_0$

$$m - \delta \le \frac{1}{\Lambda^n \omega_n} \mu_{\widetilde{T}_i}(p + \Lambda Q_{1,\sigma}).$$

Using Claim 2.1.1, the right-hand side of the above inequality is less than $1 + c\sigma\Lambda$ and hence taking σ small enough we conclude that m has to be 1. Hence, recalling (7), we obtain

$$\omega_n^{-1} r^{-n} \mu_T(B^{n+1}_r(0)) = 1 \quad \text{for all } r > 0,$$

which implies that $\operatorname{spt} T$ itself is a hyperplane and the multiplicity is 1. This provides a contradiction for Case 2a.

We are left with Case 2b. So suppose that $\liminf_i |y_i| \neq 0$ and $\liminf_i \operatorname{dist}(p_i, \Psi_i) \neq 0$. After passing to a subsequence we can assume that $\lim |y_i| = |y_\infty|$ exists, with $|y_\infty| \in (0, \infty]$. Rotational symmetry of $\operatorname{graph} u_i$ in the y-space then implies that $T = [\![\mathbb{R}^{n-2}]\!] \times T_0$, where T_0 is an area-minimizing 2-current in $\mathbb{R}^3$.

Since any such current is regular, T_0, and hence also T, is regular. We conclude that spt T_0 must be a plane [do Carmo and Peng 1979; Pogorelov 1981; Schoen 1983] with (arguing as in Case 2a) multiplicity 1. This provides a contradiction for Case 2b. □

Lemma 2.1 allows us to apply Allard's interior and boundary regularity theorems [1972; 1975] to obtain uniform $C^{1,\alpha}$ estimates for the graphs of solutions u to (3) with boundary data that satisfy the hypotheses of Lemma 2.1. Assuming higher-regularity of the boundary data we can apply Schauder theory to obtain higher-regularity estimates for these graphs.

Corollary 2.2. *Given any $K > 0$ and $\ell_0 \in \mathbb{N}$, there exists a constant C with the following property: Let u be a solution of* (3) *with $\partial\Omega$ and ψ bounded in $C^{\ell_0+2,\alpha}$ by K for some $\alpha \in (0, 1]$ and rotationally symmetric with respect to the subspace $\mathbb{E}^{n-1} := \operatorname{span}\{e_2, \dots, e_n\}$. Then*

$$\sup_{p\in \operatorname{graph} u} |\nabla^\ell A(p)| \le C \quad \text{for all } \ell \in \{0, \dots, \ell_0\},$$

where A is the second fundamental form of graph u *and $\nabla^0 A := A$.*

Remark 2.3. If we allow $\ell_0 = -1$ in the hypotheses of Corollary 2.2 then we obtain uniform $C^{1,\alpha}$ estimates for the graphs of solutions u to (3) with boundary data that satisfy the hypotheses of Lemma 2.1 using the results of [Bourni 2016].

Remark 2.4. If $n \le 6$ then Lemma 2.1, and hence Corollary 2.2 and Remark 2.3, still hold without the rotational symmetry hypothesis on the boundary data. To see this, note that the proof of the boundary case (Case 1) of Lemma 2.1 does not make use of the rotational symmetry hypothesis and hence holds in all dimensions without this restriction. To show interior regularity in the case $n+1 \le 7$ we can refer to known results on regularity of almost-minimizing surfaces; see for example [Duzaar and Steffen 1993; Massari and Miranda 1984]. One can alternatively see this from Cases 2a and 2b in the proof of Lemma 2.1, since there are no stable nonplanar minimal cones in low dimensions [Simons 1968]; see also [Schoen et al. 1975] or [Simon 1983, Appendix B].

3. Convexity

We need to extend the convexity result [Spruck and Xiao 2017, Theorem 1.1] to higher dimensions. Our proof is a straightforward modification of theirs.

We make use of the following lemma.

Lemma 3.1. *Let Σ^n be a connected translator in $\mathbb{R}^{n+1}$. Suppose that Σ^n has constant mean curvature H_0. Then $H_0 = 0$ and Σ^n lies in a vertical minimal cylinder. In particular, if $n = 2$ or, more generally, if Σ^n has at most two principal curvatures at each point, then Σ^n lies in a vertical hyperplane.*

Proof. The mean curvature of Σ^n satisfies

$$-(\Delta + \nabla_V)H = |A|^2 H,$$

where $V := e_{n+1}^\top$. Thus,

$$\langle \nu, e_{n+1}\rangle = -H \equiv 0,$$

so e_{n+1} is tangential and hence $V \equiv e_{n+1}$. It follows that the integral curves of V are vertical lines, which completes the proof. □

Theorem 3.2. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a strictly mean convex translator with at most two distinct principal curvatures at each point and bounded second fundamental form. Then Σ is convex.*

Proof. Denote the principal curvatures of Σ by $\kappa \le \mu$. Note that κ is smooth and has constant multiplicity $m \in \{1, \dots, n-1\}$ in the open set $U := \{X \in \Sigma : \kappa(X) < 0\}$. Recall that

$$-(\nabla_V + \Delta)A = |A|^2 A,$$

where $V := e_{n+1}^{\top}$ is the tangential part of e_{n+1}. Computing locally in a principal frame $\{\tau_1, \dots, \tau_n\}$ with $\kappa_i = A_{ii} = \kappa$ when $i \le m$ and $\kappa_i = A_{ii} = \mu$ when $i \ge m+1$, we obtain

$$-(\nabla_V + \Delta)\kappa = |A|^2\kappa + 2\sum_{\ell=1}^{n}\sum_{p=m+1}^{n}\frac{(\nabla_\ell A_{1p})^2}{\mu-\kappa} \quad \text{in } U.$$

Since the mean curvature satisfies

$$-(\nabla_V + \Delta)H = |A|^2 H,$$

straightforward manipulations then yield

$$-(\nabla_V + \Delta)\frac{\kappa}{\mu} = -(\nabla_V + \Delta)\frac{(n-m)\kappa}{H-m\kappa} = \frac{2}{n-m}\frac{H}{\mu^2}\sum_{\ell=1}^{n}\sum_{p=m+1}^{n}\frac{(\nabla_\ell A_{1p})^2}{\mu-\kappa} + 2\left\langle\nabla\frac{\kappa}{\mu}, \frac{\nabla\mu}{\mu}\right\rangle. \tag{8}$$

Suppose that

$$-\varepsilon_0 := \inf_{\Sigma}\frac{\kappa}{\mu} < 0.$$

If the infimum is attained at some point $X_0 \in \Sigma$ then $\kappa(X_0) < 0$ and the strong maximum principle yields $\kappa/\mu \equiv -\varepsilon_0 < 0$. In particular,

$$0 \equiv \nabla_\ell\frac{\kappa}{\mu} = \frac{\nabla_\ell A_{pp}}{\kappa} - \frac{\kappa}{\mu}\frac{\nabla_\ell A_{qq}}{\mu}$$

when $p \le m < q$. It is a general observation that

$$0 = \tau_\ell A_{ij} = \nabla_\ell A_{ij} + (\kappa_j - \kappa_i)\Gamma_{\ell ij} = \nabla_\ell A_{ij} \tag{9}$$

for each ℓ whenever $\kappa_i = \kappa_j$ and $i \ne j$, where $\Gamma_{\ell ij} := \langle\nabla_\ell\tau_i, \tau_j\rangle$. Thus,[2]

$$0 = \nabla_\ell A_{11} \quad \text{when } \ell = 2, \dots, m,$$
$$0 = \nabla_\ell A_{nn} \quad \text{when } \ell = m+1, \dots, n-1.$$

Recalling (8), we also find that

$$0 \equiv \sum_{\ell=1}^{n}\sum_{p=m+1}^{n}(\nabla_\ell A_{1p})^2.$$

[2]Here, and elsewhere, we freely make use of the Codazzi identity.

It follows that the components $\nabla_1 A_{nn}$, $\nabla_1 A_{11}$, $\nabla_n A_{11}$ and $\nabla_n A_{nn}$ are all identically zero and hence, by the translator equation (1),

$$0 \equiv m\nabla_\ell A_{11} + (n-m)\nabla_\ell A_{nn} = \nabla_\ell H$$

for each $\ell = 1, \ldots, n$. Lemma 3.1 now implies that Σ^n is a vertical hyperplane, contradicting strict mean convexity.

Suppose then that the infimum is not attained. Since

$$\frac{\kappa}{\mu} \geq -\frac{n-m}{m}$$

and the sectional curvatures of Σ are bounded, the Omori–Yau maximum principle may be applied. This yields a sequence of points $X_i \to \infty$ such that

$$\frac{\kappa}{\mu}(X_i) \to -\varepsilon_0, \qquad \left|\nabla\frac{\kappa}{\mu}(X_i)\right| \leq \frac{1}{i} \quad \text{and} \quad -\Delta\frac{\kappa}{\mu}(X_i) \leq \frac{1}{i}. \tag{10}$$

Consider the sequence of translators $\Sigma_i := \Sigma - X_i$. By Corollary 2.2, the translators Σ_i converge locally uniformly in C^∞, after passing to a subsequence, to a limit translator Σ_∞. Note that, whenever $\kappa < 0 < \mu$,

$$\nabla_\ell\frac{m\kappa}{\mu} = \frac{m\nabla_\ell A_{11}}{\mu} - \frac{m\kappa}{\mu^2}\nabla_\ell A_{nn} = \frac{\nabla_\ell H}{\mu} - m\left(\frac{n-m}{m} + \frac{\kappa}{\mu}\right)\frac{\nabla_\ell A_{nn}}{\mu}. \tag{11}$$

We claim that

$$\left(\frac{n-m}{m} + \frac{\kappa}{\mu}(X_i)\right)\frac{\nabla_k A_{nn}}{\mu}(X_i) \to 0 \quad \text{as } i \to \infty \tag{12}$$

for each $\ell = 1, \ldots, n$. Suppose that this is not the case. Then there exists $i_0 \in \mathbb{N}$ and $\delta_0 > 0$ such that

$$\left(\frac{n-m}{m} + \frac{\kappa}{\mu}(X_i)\right)\frac{|\nabla_k A_{nn}|}{\mu}(X_i) > \delta_0 \tag{13}$$

for all $i > i_0$ and some $k \in \{1, \ldots, n\}$. By (10),

$$\left(\frac{\nabla_\ell A_{11}}{\mu} - \frac{\kappa}{\mu}\frac{\nabla_\ell A_{nn}}{\mu}\right)(X_i) \to 0$$

for each $\ell = 1, \ldots, n$ as $i \to \infty$ so that, replacing δ_0 and i_0 if necessary,

$$\left(\frac{n-m}{m} + \frac{\kappa}{\mu}(X_i)\right)\frac{|\nabla_k A_{11}|}{\mu}(X_i) > \delta_0 \tag{14}$$

for all $i > i_0$. Moreover, by (9),

$$\frac{\nabla_\ell A_{nn}}{\mu}(X_i) \to 0$$

as $i\to\infty$ for all $\ell=2,\dots,n-1$. So (13) (and hence also (14)) holds with $k\in\{1,n\}$. Combining (10) and (8) we obtain, at the point X_i,

$$
\begin{aligned}
\frac{1}{i} &\geq \frac{1}{n-m}\frac{H}{\mu-\kappa}\sum_{\ell=1}^{n}\sum_{p=m+1}^{n}\frac{(\nabla_\ell A_{1p})^2}{\mu^2}+\left\langle\nabla\frac{\kappa}{\mu},\frac{\nabla\mu}{\mu}\right\rangle\\
&=\frac{1}{n-m}\frac{H}{\mu-\kappa}\left(\sum_{p=m+1}^{n}\frac{(\nabla_p\kappa)^2}{\mu^2}+(n-m)\frac{(\nabla_1\mu)^2}{\mu^2}\right)+\nabla_1\frac{\kappa}{\mu}\frac{\nabla_1\mu}{\mu}\\
&\qquad\qquad+\sum_{\ell=2}^{m}\nabla_\ell\frac{\kappa}{\mu}\frac{\nabla_\ell\mu}{\mu}+\sum_{\ell=m+1}^{n}\nabla_\ell\frac{\kappa}{\mu}\left(\frac{\nabla_\ell\kappa}{\kappa}-\frac{\mu}{\kappa}\nabla_\ell\frac{\kappa}{\mu}\right)\\
&=-\frac{\mu}{\kappa}\sum_{\ell=m+1}^{n}\left(\nabla_\ell\frac{\kappa}{\mu}\right)^2+\sum_{\ell=2}^{m}\nabla_\ell\frac{\kappa}{\mu}\frac{\nabla_\ell\mu}{\mu}+\nabla_1\frac{\kappa}{\mu}\frac{\nabla_1\mu}{\mu}+\frac{H}{\mu-\kappa}\frac{(\nabla_1\mu)^2}{\mu^2}\\
&\qquad\qquad+\frac{\mu}{\kappa}\sum_{\ell=m+1}^{n}\nabla_\ell\frac{\kappa}{\mu}\frac{\nabla_\ell\kappa}{\mu}+\frac{1}{n-m}\frac{H}{\mu-\kappa}\sum_{\ell=m+1}^{n}\frac{(\nabla_\ell\kappa)^2}{\mu^2}\\
&\geq-\frac{\mu}{\kappa}\sum_{\ell=m+1}^{n}\left(\nabla_\ell\frac{\kappa}{\mu}\right)^2+\sum_{\ell=2}^{m}\nabla_\ell\frac{\kappa}{\mu}\frac{\nabla_\ell\mu}{\mu}+\left(m\frac{(n-m)/m+\kappa/\mu}{1-\kappa/\mu}\frac{|\nabla_1\mu|}{\mu}-\left|\nabla_1\frac{\kappa}{\mu}\right|\right)\frac{|\nabla_1\mu|}{\mu}\\
&\qquad\qquad+\sum_{\ell=m+1}^{n}\left(\frac{m}{n-m}\frac{(n-m)/m+\kappa/\mu}{1-\kappa/\mu}\frac{|\nabla_\ell\kappa|}{\mu}+\frac{\mu}{\kappa}\left|\nabla_\ell\frac{\kappa}{\mu}\right|\right)\frac{|\nabla_\ell\kappa|}{\mu}.
\end{aligned}
$$

Suppose that $k=1$ in (13). If

$$\left(\frac{n-m}{m}+\frac{\kappa}{\mu}(X_i)\right)\frac{|\nabla_n\kappa|}{\mu}(X_i)\not\to 0\quad\text{as } i\to\infty$$

then, taking $i\to\infty$, we find $(|\nabla_1\mu|/\mu)(X_i)\to 0$ as $i\to\infty$, contradicting (13). Else,

$$\frac{|\nabla_n\kappa|}{\mu}(X_i)\leq\frac{|\nabla_1\mu|}{\mu}(X_i)$$

for i sufficiently large and we again obtain $(|\nabla_1\mu|/\mu)(X_i)\to 0$ as $i\to\infty$, contradicting (13). If $k=n$ in (13) we may argue similarly, using (14).

So (12) does indeed hold. Applying (10) and (12) to (11) yields

$$\frac{\nabla_\ell H}{\mu}(X_i)\to 0$$

for each $\ell=1,\dots,n$. On the other hand, by the translator equation,

$$\frac{\nabla_\ell H}{\mu}=-\frac{\kappa_\ell\langle\tau_\ell,e_{n+1}\rangle}{\mu}.$$

Since $(\kappa/\mu)(X_i)\to-\varepsilon_0\neq 0$, we conclude that $\nu(X_i)\to -e_{n+1}$ and hence $H(X_i)\to 1$. Since the infimum of κ/μ is attained at the origin on Σ_∞, we deduce as before that Σ_∞ has constant mean curvature, which must be 1 since $H(X_i)\to 1$. But this contradicts Lemma 3.1. $\square$

4. Barriers

Next, we introduce appropriate barriers. When $n = 2$, the outer barrier is obtained by (nonisotropically) "stretching" the level set function corresponding to the Angenent oval so that it lies in the correct slab and is asymptotic to the correct oblique Grim planes. The higher-dimensional barrier is then obtained by rotating in the $(n-1)$-dimensional complimentary subspace.

Lemma 4.1. *The function* $\underline{u} : \left\{(x, y) \in \mathbb{R} \times \mathbb{R}^{n-1} : |x| < \frac{\pi}{2} \sec\theta\right\} \to \mathbb{R}$ *defined by*

$$\underline{u}(x, y) := -\sec^2\theta \log\cos\left(\frac{x}{\sec\theta}\right) + \tan^2\theta \log\cosh\left(\frac{|y|}{\tan\theta}\right)$$

is a subsolution of the graphical translator equation (2).

In particular, *given any* $R > 0$, *the surface*

$$\underline{\Sigma}_R := \operatorname{graph} \underline{u}_R$$

is a subsolution of the translator equation (1), *where*

$$\underline{u}_R := \underline{u} - \tan^2\theta \log\cosh\left(\frac{R}{\tan\theta}\right).$$

Proof. The relevant derivatives of $\underline{u}$ are given by

$$D\underline{u} = \left(\sec\theta \tan\left(\frac{x}{\sec\theta}\right), \tan\theta \tanh\left(\frac{|y|}{\tan\theta}\right)\frac{y}{|y|}\right)$$

and

$$D^2\underline{u} = \begin{pmatrix} \sec^2\left(\frac{x}{\sec\theta}\right) & \cdots \;\; 0 \;\; \cdots \\ \vdots & \\ 0 & \operatorname{sech}^2\left(\frac{|y|}{\tan\theta}\right)\frac{y_i y_j}{|y|^2} + \tan\theta \tanh\left(\frac{|y|}{\tan\theta}\right)\left(\frac{\delta_{ij}}{|y|} - \frac{y_i y_j}{|y|^3}\right) \\ \vdots & \end{pmatrix}.$$

So

$$\begin{aligned} 1 + |D\underline{u}|^2 &= 1 + \sec^2\theta \tan^2\left(\frac{x}{\sec\theta}\right) + \tan^2\theta \tanh^2\left(\frac{|y|}{\tan\theta}\right) \\ &= \sec^2\theta \sec^2\left(\frac{x}{\sec\theta}\right) - \tan^2\theta \operatorname{sech}^2\left(\frac{|y|}{\tan\theta}\right). \end{aligned}$$

Estimating

$$\Delta\underline{u} \geq \sec^2\left(\frac{x}{\sec\theta}\right) + \operatorname{sech}^2\left(\frac{|y|}{\tan\theta}\right),$$

we find

$$\begin{aligned} (1 + |D\underline{u}|^2)^{3/2} H[\underline{u}] &= (1 + |D\underline{u}|^2)\Delta\underline{u} - D^2\underline{u}(D\underline{u}, D\underline{u}) \\ &\geq 1 + |D\underline{u}|^2 + \sec^2\left(\frac{x}{\sec\theta}\right)\operatorname{sech}^2\left(\frac{|y|}{\tan\theta}\right) \geq 1 + |D\underline{u}|^2. \end{aligned}$$

□

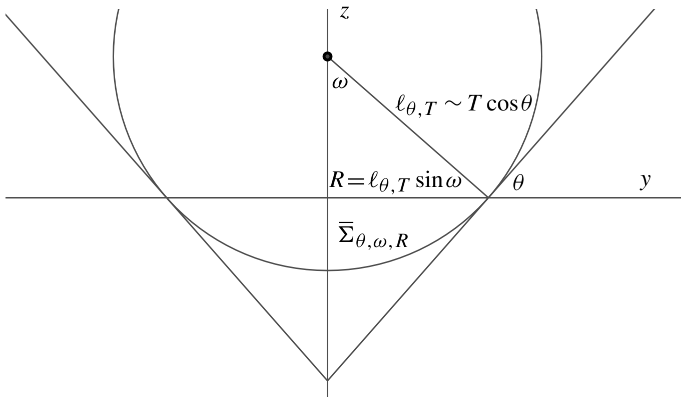

Figure 1. Given any $\varepsilon \in (0, \varepsilon_0(n,\theta))$, the portion of Π_R (the rotated time-T-slice of the Angenent oval of width $\pi\sec\theta$, where $T = \sec^2\theta\cosh(R/\tan\theta)$) lying below height $z = -R\cos(\theta-\varepsilon)/\sin\theta$ is a supersolution of the translator equation when $R > R_\varepsilon := 2(n-1)/\varepsilon$. The surface $\overline{\Sigma}_{R,\varepsilon}$ is obtained by translating this piece upward so that its boundary lies in $\mathbb{R}^n \times \{0\}$.

Consider the "outer" domain

$$\underline{\Omega}_R := \{(x,y) \in S^{n-1}_\theta : \underline{u}_R(x,y) < 0\} = \left\{(x,y) \in S^{n-1}_\theta : \cos\left(\frac{x}{\sec\theta}\right) < \left[\frac{\cosh(|y|/\tan\theta)}{\cosh(R/\tan\theta)}\right]^{\sin^2\theta}\right\},$$

where $S^n_\theta := \left(-\frac{\pi}{2}\sec\theta, \frac{\pi}{2}\sec\theta\right) \times \mathbb{R}^{n-1}$. Note that

$$\partial\underline{\Omega}_R = \partial(\underline{\Sigma}_R \cap \mathbb{R}^n \times (-\infty, 0]).$$

The inner barrier is obtained by rotating the Angenent oval of width $\pi\sec\theta$ and cutting off at an appropriate height (see Figure 1).

Lemma 4.2. *Given $R > 0$, let $\Pi_R \subset \mathbb{R}^{n+1}$ be the surface formed by rotating about the x-axis the time-T-slice of the Angenent oval of width $\pi\sec\theta$, where*

$$T := -\sec^2\theta\cosh\left(\frac{R}{\tan\theta}\right).$$

That is,

$$\Pi_R := \{(x,y,z) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} : v(x,y,z) = T\},$$

where

$$v := \sec^2\theta\left[\log\left(\cosh\left(\frac{\sqrt{|y|^2+z^2}}{\sec\theta}\right)\right) - \log\left(\cos\left(\frac{x}{\sec\theta}\right)\right)\right].$$

There exists $\varepsilon_0 = \varepsilon_0(n,\theta) > 0$ such that the sublevel set

$$\overline{\Sigma}_{R,\varepsilon} := \Pi_R \cap \left\{z \le -R\frac{\cos(\theta-\varepsilon)}{\sin\theta}\right\} + R\frac{\cos(\theta-\varepsilon)}{\sin\theta}e_{n+1}$$

is a supersolution of the translator equation (1) *whenever $\varepsilon < \varepsilon_0$ and $R > R_\varepsilon := 2(n-1)/\varepsilon$.*

Proof. Set $w = (y, z)$. Then

$$Dv = \sec\theta\left(\tan\left(\frac{x}{\sec\theta}\right), \tanh\left(\frac{|w|}{\sec\theta}\right)\frac{w}{|w|}\right)$$

and

$$D^2v = \begin{pmatrix} \sec^2\left(\frac{x}{\sec\theta}\right) & \cdots \quad 0 \quad \cdots \\ \vdots & \\ 0 & \operatorname{sech}^2\left(\frac{|w|}{\sec\theta}\right)\frac{w_iw_j}{|w|^2} + \sec\theta\tanh\left(\frac{|w|}{\sec\theta}\right)\left(\frac{\delta_{ij}}{|w|} - \frac{w_iw_j}{|w|^3}\right) \\ \vdots & \end{pmatrix}.$$

Tedious computations then yield, on the one hand,

$$-\langle \nu, e_{n+1}\rangle = \left\langle \frac{Dv}{|Dv|}, e_{n+1}\right\rangle = \frac{\tanh(\sqrt{|y|^2+z^2}/\sec\theta)(|z|/\sqrt{|y|^2+z^2})}{\sqrt{\tan^2(x/\sec\theta)+\tanh^2(\sqrt{|y|^2+z^2}/\sec\theta)}}$$

and, on the other hand,

$$H = \operatorname{div}\left(\frac{Dv}{|Dv|}\right) = \frac{1/\sec\theta + ((n-1)/|w|)\tanh(|w|/\sec\theta)}{\sqrt{\tan^2(x/\sec\theta)+\tanh^2(|w|/\sec\theta)}}.$$

It follows that Π_R is a supersolution in the region where

$$\frac{|z|-(n-1)}{\sqrt{|y|^2+z^2}}\tanh\left(\frac{\sqrt{|y|^2+z^2}}{\sec\theta}\right) \ge \cos\theta.$$

Note that

$$|y| \le \frac{\sin(\theta-\varepsilon)}{\sin\theta}R$$

wherever

$$|z| \ge \frac{\cos(\theta-\varepsilon)}{\sin\theta}R.$$

Thus, whenever

$$R > R_\varepsilon := \frac{2(n-1)}{\varepsilon} \quad \text{and} \quad z \le -\frac{\cos(\theta-\varepsilon)}{\sin\theta}R,$$

we have

$$\begin{aligned}\frac{|z|-(n-1)}{\sqrt{|y|^2+z^2}}\tanh\left(\frac{\sqrt{|y|^2+z^2}}{\sec\theta}\right) &\ge \left(\cos(\theta-\varepsilon) - \frac{(n-1)}{R}\sin\theta\right)\tanh\left(\frac{\cos(\theta-\varepsilon)}{\tan\theta}R\right) \\ &\ge \cos\theta\left(1+\tfrac{1}{2}\varepsilon\tan\theta + o(\varepsilon)\right)\sqrt{1-4\mathrm{e}^{-2(n-1)\cos^2\theta\sin\theta/\varepsilon}}.\end{aligned}$$

This is no less than $\cos\theta$ when $\varepsilon < \varepsilon_0(n, \theta)$. □

Consider the "inner" domain

$$\overline{\Omega}_{R,\varepsilon} := \left\{(x, y) \in S^n_\theta : \cos\left(\frac{x}{\sec\theta}\right) < \frac{\cosh(\sqrt{|y|^2\sin^2\theta + R^2\cos^2(\theta-\varepsilon)}/\tan\theta)}{\cosh(R/\tan\theta)}\right\}.$$

Note that $\partial\overline{\Omega}_{R,\varepsilon} = \partial\overline{\Sigma}_{R,\varepsilon}$.

The following lemma implies that the inner barrier which touches the outer barrier at Re_2 lies above it, so long as R is sufficiently large.

Lemma 4.3. *Given any $R>0$, we have $\overline{\Omega}_{\rho_\varepsilon,\varepsilon}\subset\underline{\Omega}_R$, where*

$$\rho_\varepsilon := \frac{\sin\theta}{\sin(\theta-\varepsilon)}R.$$

Proof. It suffices to show that the function $f:\mathbb{R}_+\to\mathbb{R}$ defined by

$$f(\zeta) := \frac{\cosh(\sqrt{\zeta^2\sin^2\theta+\rho_\varepsilon^2\cos^2(\theta-\varepsilon)}/\tan\theta)}{\cosh(\rho_\varepsilon/\tan\theta)} - \left[\frac{\cosh(\zeta/\tan\theta)}{\cosh(R/\tan\theta)}\right]^{\sin^2\theta}$$

is nonpositive. This follows from log-concavity of the function

$$g(w) := \cosh\left(\frac{\sqrt{w}}{\tan\theta}\right).$$

Indeed, given any $s\in(0,1)$ and $w>0$, log-concavity of g implies that the function

$$G(z) := \frac{g(sz+(1-s)w)}{g(z)^s}$$

is monotone nondecreasing for $z<w$. Since

$$\zeta < R < \frac{\tan\theta}{\tan(\theta-\varepsilon)}R = \frac{\cos(\theta-\varepsilon)}{\cos\theta}\rho_\varepsilon,$$

this implies

$$\frac{g(\zeta^2\sin^2\theta+\rho_\varepsilon^2\cos^2(\theta-\varepsilon))}{g(\zeta^2)^{\sin^2\theta}} \le \frac{g(R^2\sin^2\theta+\rho_\varepsilon^2\cos^2(\theta-\varepsilon))}{g(R^2)^{\sin^2\theta}} = \frac{g(\rho_\varepsilon^2)}{g(R^2)^{\sin^2\theta}}.$$

The claim follows. □

Corollary 4.4. *Set $\varepsilon_R := 2(n-1)/R$, $\overline{\Sigma}_R := \overline{\Sigma}_{\rho_{\varepsilon_R},\varepsilon_R}$ and $\overline{\Omega}_R := \overline{\Omega}_{\rho_{\varepsilon_R},\varepsilon_R}$. Then, for $R>R_0:=2(n-1)/\varepsilon_0$, $\overline{\Sigma}_R$ is a supersolution of the translator equation with boundary $\partial\overline{\Sigma}_R=\partial\overline{\Omega}_R$.*

5. Existence

We are ready to prove the existence theorem, which we now recall.

Theorem (existence of convex translators in slab regions). *For every $n\geq 2$ and every $\theta\in\left(0,\frac{\pi}{2}\right)$ there exists a strictly convex translator Σ_θ^n which lies in S_θ^{n+1} and in no smaller slab.*

Proof. Given $R>0$, let u_R be the solution of

$$H[u_R] = \frac{1}{\sqrt{1+|Du_R|^2}} \quad \text{in } \Omega_R, \qquad u_R = 0 \quad \text{on } \partial\Omega_R,$$

where $\Omega_R := \underline{\Omega}_R$. Since the equation admits upper and lower barriers (0 and $\underline{u}_R$, respectively), existence and uniqueness of a smooth solution follows from well-known methods; see, for example, [Gilbarg and Trudinger 1983, Chapter 15]. Uniqueness implies that u_R is rotationally symmetric with respect to the subspace $\mathbb{E}^{n-1} = \operatorname{span}\{e_2, \dots, e_n\}$. Since $\underline{u}_R$ is a subsolution, its graph lies below graph u_R. Since the two surfaces coincide on the boundary $\partial\Omega_R$,

$$H[u_R] = -\langle \nu_R, e_{n+1}\rangle \geq -\langle \underline{\nu}_R, e_{n+1}\rangle \geq \cos\theta\cos(x\cos\theta) \geq \cos\theta\left(1 - \frac{x}{\frac{\pi}{2}\sec\theta}\right) \tag{15}$$

on $\partial\Omega_R$, where $\underline{\nu}_R$ is the downward-pointing unit normal to graph $\underline{u}_R$. By Corollary 4.4, we also find, for $R > R_0$, that

$$-u_R(0) \geq \frac{1-\cos\theta}{\sin\theta} R \to \infty \quad \text{as } R \to \infty. \tag{16}$$

Let $R_i \to \infty$ be a diverging sequence and consider the translators-with-boundary

$$\Sigma_i := \operatorname{graph} u_{R_i} - u_{R_i}(0) e_{n+1}.$$

By Corollary 2.2 and the height estimate (16) some subsequence converges locally uniformly in the smooth topology to some limiting translator, Σ, with bounded second fundamental form. By Theorem 3.2, Σ is convex.

Certainly Σ lies in the slab S_θ, so it remains only to prove that it lies in no smaller slab (strict convexity will then follow from the splitting theorem and uniqueness of the Grim Reaper). Set

$$v := 1 - \frac{x}{\frac{\pi}{2}\sec\theta},$$

where $x(X) := \langle X, e_1\rangle$. We claim that

$$\inf_{\Sigma\cap\{x>0\}} \frac{H}{v} > 0. \tag{17}$$

Since $\inf_\Sigma H = 0$, we conclude that $\sup_\Sigma x = \frac{\pi}{2}\sec\theta$ as desired. To prove (17), first observe that

$$-(\Delta + \nabla_V)v = 0$$

and hence

$$-(\Delta + \nabla_V)\frac{H}{v} = |A|^2\frac{H}{v} + 2\left\langle \nabla\frac{H}{v}, \frac{\nabla v}{v}\right\rangle,$$

where V is the tangential projection of e_{n+1}. The maximum principle then yields

$$\min_{\Sigma_i\cap\{x>0\}} \frac{H}{v} \geq \min\left\{\min_{\partial\Sigma_i\cap\{x>0\}} \frac{H}{v}, \min_{\Sigma_i\cap\{x=0\}} \frac{H}{v}\right\} = \min\left\{\cos\theta, \min_{\Sigma_i\cap\{x=0\}} H\right\}.$$

If $\liminf_{i\to\infty} \min_{\Sigma_i\cap\{x=0\}} H > 0$ then we are done. So suppose that $\liminf_{i\to\infty} H(X_i) = 0$ along some sequence of points $X_i \in \Sigma_i \cap \{x = 0\}$. Then, by Corollary 2.2, after passing to a subsequence, the translators-with-boundary

$$\widehat{\Sigma}_i := \Sigma_i - X_i$$

converge locally uniformly in C^∞ to a translator (possibly with boundary) $\widehat{\Sigma}$ which lies in S_θ and satisfies $H \geq 0$ with equality at the origin. By Corollary 2.2 the origin must be an interior point since, recalling (15), $H > \cos\theta$ on $\partial\Sigma_i \cap \{x = 0\}$ for all i. The strong maximum principle then implies that $H \equiv 0$ on $\widehat{\Sigma}$ and we conclude that $\widehat{\Sigma}$ is either a hyperplane or half-hyperplane. Since, by the reflection symmetry, the limit cannot be parallel to $\{0\} \times \mathbb{R}^{n-1} \times \mathbb{R}$, neither option can be reconciled with the fact that $\widehat{\Sigma}$ lies in S_θ. $\square$

6. Asymptotics and reflection symmetry

We next prove that, after translation, our translators have the correct asymptotics (Theorem 3).

Theorem (unique asymptotics modulo translation). *Given $n \geq 2$ and $\theta \in \left(0, \frac{\pi}{2}\right)$ let Σ^n_θ be a convex translator which lies in S^{n+1}_θ and in no smaller slab. If $n \geq 3$, assume in addition that Σ^n_θ is rotationally symmetric with respect to the subspace $\mathbb{E}^{n-1} := \operatorname{span}\{e_1, \dots, e_n\}$. Given any unit vector $\phi \in \mathbb{E}^{n-1}$ the curve $\{\sin\omega\phi - \cos\omega e_{n+1} : \omega \in [0, \theta)\}$ lies in the normal image of Σ^n_θ and the translators*

$$\Sigma^n_{\theta,\omega} := \Sigma^n_\theta - P(\sin\omega\phi - \cos\omega e_{n+1})$$

converge locally uniformly in the smooth topology to the oblique Grim hyperplane $\Gamma^n_{\theta,\phi}$ as $\omega \to \theta$, where $P : S^n \to \Sigma^n_\theta$ is the inverse of the Gauss map.

Fix a unit vector $\phi \in \operatorname{span}\{e_2, \dots, e_n\}$ and define

$$\bar\omega := \sup\{\omega \in [0, \infty) : \sin\omega\phi - \cos\omega e_{n+1} \in \nu(\Sigma)\}.$$

Let ω_i be a sequence of points converging to $\bar\omega$. Then the translators

$$\Sigma_{i,\phi} := \Sigma - P_\phi(\omega_i)$$

have uniformly bounded curvature and pass through the origin. After passing to a subsequence, they must therefore converge locally uniformly to a limit translator. The limit must be the oblique Grim hyperplane $\Gamma^n_{\bar\omega,\phi}$ since it contains the ray $\{r(\cos\bar\omega\phi + \sin\bar\omega e_{n+1}) : r > 0\}$ and lies in a slab parallel to S_θ (and, when $n \geq 3$, splits off an additional $n-2$ lines due to the rotational symmetry). In fact, since the components of the normal are monotone along the curve $\gamma(\omega) := P(\sin\omega\phi - \cos\omega e_{n+1})$, the normal must actually converge (to $\sin\bar\omega\phi - \cos\bar\omega e_{n+1}$) along γ. It follows that the limit is independent of the subsequence and we conclude that the translators

$$\Sigma_{\omega,\phi} := \Sigma - P_\phi(\omega)$$

converge locally uniformly in C^∞ to $\Gamma^n_{\bar\omega,\phi}$ as $\omega \to \bar\omega$. Note that $\bar\omega \leq \theta$ since the limit $\Gamma^n_{\bar\omega,\phi}$ must lie in S_θ. It remains to show that $\bar\omega \geq \theta$.

Suppose, to the contrary, that $\bar\omega < \theta$. Given $\omega \in \left[0, \frac{\pi}{2}\right)$, let $\Pi^\omega_t = \sec\omega\Pi_{\cos^2\omega t}$ be the rotationally symmetric ancient pancake which lies in the slab Ω_ω (and no smaller slab) and becomes extinct at the origin at time zero. The "radius" $\ell_\omega(t)$ of the pancake satisfies [Bourni et al. 2017]

$$\ell_\omega(t) := \max_{p \in \Pi^\omega_t} \langle p, e_2\rangle = \sec\omega\,\ell_0(\cos^2\omega t) = -t\cos\omega + (n-1)\sec\omega\log(-t) + c + o(1) \tag{18}$$

as $t \to -\infty$, where the constant c and the remainder term depend on ω and n. Observe that the ray $L_\omega = \{r(\cos\omega\phi + \sin\omega e_{n+1}) : r > 0\}$ is tangent to the circle in the plane $\operatorname{span}\{\phi, e_{n+1}\}$ of radius $-\cos\omega t$ centered at $-te_{n+1}$. Indeed, a point $r(\cos\omega\phi + \sin\omega e_{n+1})$ lies on this circle if and only if

$$|r\cos\omega\phi + (r\sin\omega + t)e_{n+1}|^2 = \cos^2\omega t^2 \quad\Longleftrightarrow\quad (r - \sin\omega(-t))^2 = 0.$$

So there exists a unique point with this property, as claimed. Since, by hypothesis, $\theta < \bar\omega$, we conclude from (18) that the circle of radius $\ell_\theta(-t)$ lies above the line $L_{\bar\omega}$ for $-t$ sufficiently large (see Figure 2).

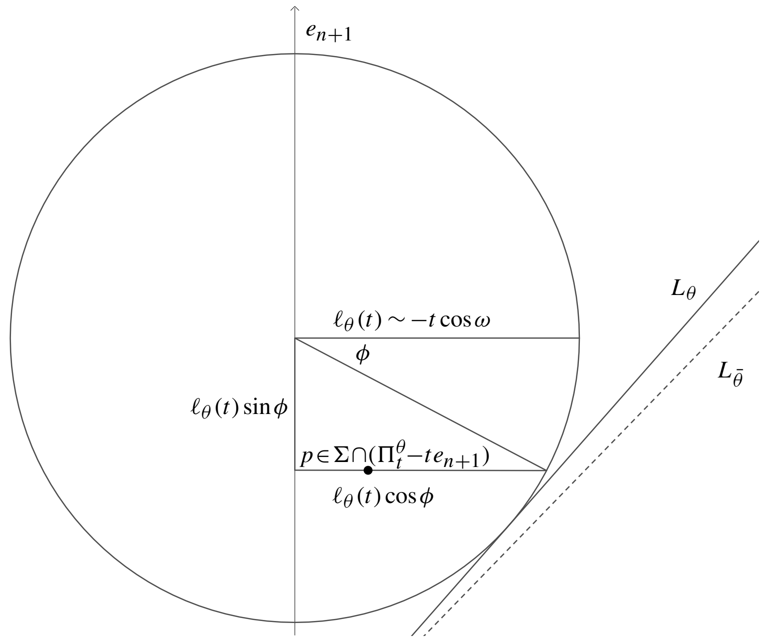

Figure 2. If $\bar{\omega} < \theta$ then the pancake lies above the translator for $-t$ sufficiently large.

We will show that, in fact, $\Pi_t^\theta - te_{n+1}$ lies above Σ for $-t$ sufficiently large (and hence Π_t^θ lies above $\Sigma_t := \Sigma + te_{n+1}$ for $-t$ sufficiently large). But Π_t^θ and Σ_t both reach the origin at time zero, so this contradicts the avoidance principle.

We will need an estimate for the "width" of Σ. Given $p \in \Sigma$ set

$$x(p) := \langle p, e_1 \rangle, \quad y(p) := \langle p, \phi \rangle \quad \text{and} \quad z(p) := \langle p, e_{n+1} \rangle$$

and, given $h > 0$, set

$$\ell(h) := \max_{p \in \Sigma_h} y(p),$$

where Σ_h is the level set $\Sigma_h := \{p \in \Sigma : z(p) = h\}$. We know that, near its "edge region", Σ looks like a Grim hyperplane of width $\sec\bar{\omega}$, whereas, in its "middle region", it looks like two parallel planes of width $\sec\theta$. By convexity, it must lie outside the linearly interpolating region in between (see Figure 3). The following estimate quantifies this elementary observation.

Lemma 6.1 (width estimate). *Set*

$$\beta := \sec\theta - \sec\bar{\omega} > 0 \quad \textit{and} \quad x_0 := \lim_{\omega \to \bar{\omega}} x(P_\phi(\omega)).$$

For any $\varepsilon > 0$ there exist $K_\varepsilon < \infty$ and $h_\varepsilon < \infty$ with the following property: Given $h > h_\varepsilon$, $p \in \Sigma_h$ and $s \in [0, 1]$, suppose that

$$0 \le y(p) \le s(\ell(h) - K_\varepsilon).$$

Then

$$|x(p) - x_0| \ge \tfrac{\pi}{2}\left(\sec\bar{\omega} + (1-s)\left(\beta - \tfrac{2}{\pi}x_0\right) - \varepsilon\right).$$

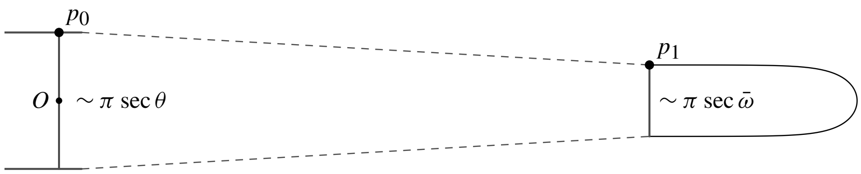

Figure 3. Linearly interpolating between the "middle" and "edge" regions in the level set Σ_h. The horizontal axis is compressed.

Proof of Lemma 6.1. Choose $\varepsilon > 0$. Because Σ converges to the oblique Grim hyperplane $\Gamma_{\bar{\omega},\phi}$ after translating the "tips", $P_\phi(\omega)$, we can find some h_ε and K_ε such that

$$|x(p) - x_0| \geq \tfrac{\pi}{2}\sec\bar{\omega} - \varepsilon$$

for all $p \in \Sigma_h$ satisfying

$$0 \leq y(p) \leq \ell(h) - K_\varepsilon$$

so long as $h \geq h_\varepsilon$. Choose some $h \geq h_\varepsilon$ and consider the point $p_1 \in \Sigma_h \cap \{e_1, \phi, e_{n+1}\}$ satisfying $x(p_1) \geq x_0$ and $0 \leq y(p_1) = \ell(h) - K_\varepsilon$. (If there is no such point then the claim is vacuously true, else p_1 is uniquely determined.) Then

$$x(p_1) - x_0 \geq \tfrac{\pi}{2}\sec\omega - \varepsilon.$$

On the other hand, because Σ converges to the boundary of S_θ after translating vertically, we can assume that h_ε is so large that

$$x(p_0) \geq \tfrac{\pi}{2}\sec\theta - \varepsilon$$

at the point $p_0 \in \Sigma_h \cap \operatorname{span}\{e_1, \phi, e_{n+1}\}$ satisfying $y(p_0) = 0$ and $x(p_0) \geq x_0$. Since Σ_h is convex, we conclude that any point $p \in \Sigma_h \cap \operatorname{span}\{e_1, \phi, e_{n+1}\}$ satisfying $0 \leq y(p) \leq \ell(h) - K_\varepsilon$ and $x(p) \geq x_0$ lies beyond the segment joining p_0 and p_1. In particular, if $y(p) \leq s(\ell(h) - K_\varepsilon)$ then

$$\begin{aligned} x(p) &\geq s x(p_1) + (1-s) x(p_0) \\ &\geq s\big(x_0 + \tfrac{\pi}{2}\sec\bar{\omega} - \varepsilon\big) + (1-s)\big(\tfrac{\pi}{2}\sec\theta - \varepsilon\big) \\ &= x_0 + \tfrac{\pi}{2}\big(\sec\bar{\omega} + (1-s)\big(\beta - \tfrac{2}{\pi}x_0\big)\big) - \varepsilon. \end{aligned}$$

The other inequality is proved in much the same way (simply choose the points p_0 and p_1 on the other side of the $\{x = x_0\}$-plane). □

Reflecting Σ^n through the $\{x = 0\}$-hyperplane if necessary, we may assume in what follows that $x_0 \geq 0$.

Given $\varepsilon > 0$, choose h_ε and K_ε as in Lemma 6.1 and consider $h \geq h_\varepsilon$. Then, given any $p \in \Sigma$ satisfying $0 \leq y(p) < (\ell(h) - K_\varepsilon)$ and $x(p) \geq x_0$, we can choose

$$s = s(p) := \frac{|y(p)|}{\ell(h) - K_\varepsilon} \in [0, 1]$$

and hence estimate

$$x(p) \geq \frac{\pi}{2}\left(\sec\theta - \beta\frac{|y(p)|}{\ell(h) - K_\varepsilon} - \varepsilon\right).$$

Choosing h_ε larger if necessary, we may assume that $\ell(h_\varepsilon) \geq 2K_\varepsilon$ and hence

$$x(p) \geq \frac{\pi}{2}\left(\sec\theta - \beta\frac{|y(p)|}{\ell(h)}\left(1+\frac{2K_\varepsilon}{\ell(h)}\right) - \varepsilon\right). \tag{19}$$

Note also that, by convexity,

$$\tan\bar{\omega} \geq \frac{h}{\ell(h)} \to \tan\bar{\omega} \quad \text{as } h \to \infty.$$

Assume now that, given $t < 0$ and $\omega \in (\bar{\omega}, \theta)$, there is some point $p \in (\Pi_t^\omega - te_{n+1}) \cap \Sigma \cap \{X : \langle X, \phi\rangle > 0\}$. Then there is some $\phi \in \left[0, \frac{\pi}{2}\right]$ such that

$$h := z(p) = -t - \ell_\omega(t)\sin\phi, \quad |y(p)| \leq \ell_\omega(t)\cos\phi \quad \text{and} \quad |x(p)| < \tfrac{\pi}{2}\sec\omega,$$

where ℓ_ω is defined by (18) (see Figure 2). Suppose further that $h \geq h_\varepsilon$. Recalling (19), we find

$$\begin{aligned}\sec\omega &\geq \sec\theta - \beta\frac{|y(p)|}{h}\frac{h}{\ell(h)}\left(1+\frac{2K_\varepsilon}{h}\frac{h}{\ell(h)}\right) - \varepsilon \\ &\geq \sec\theta - \beta\frac{\ell_\omega(t)\cos\phi}{h}\tan\bar{\omega}\left(1+\frac{2K_\varepsilon}{h}\tan\bar{\omega}\right) - \varepsilon.\end{aligned}$$

That is,

$$\begin{aligned}\frac{\sec\theta - \sec\omega}{\sec\theta - \sec\bar{\omega}} &\leq \frac{\ell_\omega(t)\cos\phi}{h}\tan\bar{\omega}\left(1+\frac{2K_\varepsilon}{h}\tan\bar{\omega}\right) + \frac{\varepsilon}{\beta} \\ &= \frac{\ell_\omega(t)\cos\phi\tan\bar{\omega}}{-t-\ell_\omega(t)\sin\phi}\left(1+\frac{2K_\varepsilon\tan\bar{\omega}}{-t-\ell_\omega(t)\sin\phi}\right) + \frac{\varepsilon}{\beta}.\end{aligned}$$

Since the right-hand side is nonincreasing with respect to ϕ for $\phi \in \left[0, \frac{\pi}{2}\right]$, we may estimate

$$\frac{\sec\theta - \sec\omega}{\sec\theta - \sec\bar{\omega}} \leq \frac{\ell_\omega(t)}{-t}\tan\bar{\omega}\left(1+\frac{2K_\varepsilon}{-t}\tan\bar{\omega}\right) + \frac{\varepsilon}{\beta}.$$

But $\ell_\omega(t)/-t \to \cos\omega$ as $t \to -\infty$, so we conclude, for $-t \geq -t_\varepsilon$ sufficiently large, that

$$\frac{\sec\theta - \sec\omega}{\sec\theta - \sec\bar{\omega}} \leq \cos\omega\tan\bar{\omega} + \frac{2\varepsilon}{\beta} \leq \sin\bar{\omega} + \frac{2\varepsilon}{\beta}.$$

Choosing ω sufficiently close to $\bar{\omega}$ and ε sufficiently small results in a contradiction. This completes the proof of Theorem 3 in the case $n \geq 3$. It remains to consider the case that $n = 2$ and Σ is asymptotic to the correct oblique Grim plane in one direction, say $-e_2$, but not the other, e_2. This can be achieved with a similar argument by centering the ancient pancake not on the z-axis but rather on the axis bisecting the two asymptotic lines, i.e., the ray

$$\left\{r\left(\cos\frac{\theta-\bar{\omega}}{2}e_3 + \sin\frac{\theta-\bar{\omega}}{2}e_2\right) : r > 0\right\}.$$

We omit the details since the result in this case was already proved in [Spruck and Xiao 2017].

Combining the unique asymptotics with the Alexandrov reflection principle, we may now prove Corollary 4.

Corollary. *Given $\theta \in \left(0, \frac{\pi}{2}\right)$, let Σ be a strictly convex translator which lies in S_θ^{n+1} and in no smaller slab. If $n \geq 3$, assume in addition that Σ is rotationally symmetric with respect to $\mathbb{E}^{n-1}$. Then Σ is reflection symmetric across the hyperplane $\{0\} \times \mathbb{R}^n$.*

We proceed much as in [Bourni et al. 2017, Theorem 6.2]. Let us begin by introducing some notation. Given a unit vector $e \in S^n$ and some $\alpha \in \mathbb{R}$, denote by $\mathrm{H}_{e,\alpha}$ the half-space $\{p \in \mathbb{R}^{n+1} : \langle p, e\rangle < \alpha\}$ and by $R_{e,\alpha} \cdot \Sigma := \{p - 2(\langle p, e\rangle - \alpha)e : p \in \Sigma\}$ the reflection of Σ across the hyperplane $\partial \mathrm{H}_{e,\alpha}$. We say that Σ *can be reflected strictly about* $\mathrm{H}_{e,\alpha}$ if $(R_{e,\alpha} \cdot \Sigma) \cap \mathrm{H}_{e,\alpha} \subset \Omega \cap \mathrm{H}_{e,\alpha}$.

Lemma 6.2 (Alexandrov reflection principle). *Let Σ be a convex translator. If*

$$\Sigma_h := \Sigma \cap \{(x, y, z) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} : z > h\}$$

can be reflected strictly about $\mathrm{H}_{e,\alpha}$ for some $e \in \{e_{n+1}\}^\perp$ then Σ can be reflected strictly about $\mathrm{H}_{e,\alpha}$.

Proof. This is a consequence of the strong maximum principle and the boundary point lemma; see [Gilbarg and Trudinger 1983, Chapter 10]. □

Claim 6.2.1. *For every $\alpha \in \left(0, \frac{\pi}{4}\right)$ there exists $h_\alpha < \infty$ such that*

$$\Sigma_{h_\alpha} := \Sigma \cap \{(x, y, z) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} : z > h_\alpha\}$$

can be reflected strictly about $\mathrm{H}_\alpha := \mathrm{H}_{e_1,\alpha}$.

Proof. Suppose that the claim does not hold. Then there must be some $\alpha \in \left(0, \frac{\pi}{4}\right)$ and a sequence of heights $h_i \to \infty$ such that $(R_\alpha \cdot \Sigma_{h_i}) \cap \mathrm{H}_\alpha \cap \Sigma_{h_i} \neq \varnothing$. Choose a sequence of points $p_i = x_i e_1 + y_i e_2 \in \Sigma_{h_i}$ whose reflection about the hyperplane H_α satisfies

$$(2\alpha - x_i)e_1 + y_i e_2 \in (R_\alpha \cdot \Sigma_{h_i}) \cap \Sigma_{h_i} \cap \mathrm{H}_\alpha$$

and set $p_i' = x_i' e_1 + y_i' e_2 := (2\alpha - x_i)e_1 + y_i e_2$. Without loss of generality, we may assume that $y_i' = y_i \geq 0$. Since $\alpha \leq x_i < \frac{\pi}{2}$, the point p_i' satisfies $\alpha \geq x_i' > -\frac{\pi}{2} + 2\alpha$ so that, after passing to a subsequence, $\lim_{i\to\infty} x_i' \in \left[-\frac{\pi}{2} + 2\alpha, \alpha\right]$. But since Σ is convex and converges, after translating in the plane $\mathrm{span}\{e_2, e_{n+1}\}$, to the Grim hyperplane $\Gamma_{e_2,\theta}$, we conclude that

$$0 = \lim_{i\to\infty} (x_i + x_i') = 2\alpha.$$

So $\alpha = 0$, a contradiction. □

It now follows from Lemma 6.2 that Σ can be reflected across H_α for all $\alpha \in \left(0, \frac{\pi}{2}\right)$. The same argument applies when the half-space H_α is replaced by $-\mathrm{H}_\alpha = \{(x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-1} : x > -\alpha\}$. Now take $\alpha \to 0$.

Acknowledgements

We would like to thank Joel Spruck and Ling Xiao for providing detailed explanations of their results [Spruck and Xiao 2017].

References

[Allard 1972] W. K. Allard, "On the first variation of a varifold", *Ann. of Math.* (2) **95** (1972), 417–491. MR Zbl

[Allard 1975] W. K. Allard, "On the first variation of a varifold: boundary behavior", *Ann. of Math.* (2) **101** (1975), 418–446. MR Zbl

[Altschuler and Wu 1994] S. J. Altschuler and L. F. Wu, "Translating surfaces of the non-parametric mean curvature flow with prescribed contact angle", *Calc. Var. Partial Differential Equations* **2**:1 (1994), 101–111. MR Zbl

[Andrews 2012] B. Andrews, "Noncollapsing in mean-convex mean curvature flow", *Geom. Topol.* **16**:3 (2012), 1413–1418. MR Zbl

[Bourni 2011] T. Bourni, "$C^{1,\alpha}$ theory for the prescribed mean curvature equation with Dirichlet data", *J. Geom. Anal.* **21**:4 (2011), 982–1035. MR Zbl

[Bourni 2016] T. Bourni, "Allard-type boundary regularity for $C^{1,\alpha}$ boundaries", *Adv. Calc. Var.* **9**:2 (2016), 143–161. MR Zbl

[Bourni and Langford 2016] T. Bourni and M. Langford, "Type-II singularities of two-convex immersed mean curvature flow", *Geom. Flows* **2**:1 (2016), 1–17. MR Zbl

[Bourni et al. 2017] T. Bourni, M. Langford, and G. Tinaglia, "Collapsing ancient solutions of mean curvature flow", preprint, 2017. To appear in *J. Diff. Geom.* arXiv

[Brito and Leite 1990] F. Brito and M. L. Leite, "A remark on rotational hypersurfaces of S^n", *Bull. Soc. Math. Belg. Sér. B* **42**:3 (1990), 303–318. MR Zbl

[do Carmo and Peng 1979] M. do Carmo and C. K. Peng, "Stable complete minimal surfaces in $\mathbb{R}^3$ are planes", *Bull. Amer. Math. Soc.* (*N.S.*) **1**:6 (1979), 903–906. MR Zbl

[Clutterbuck et al. 2007] J. Clutterbuck, O. C. Schnürer, and F. Schulze, "Stability of translating solutions to mean curvature flow", *Calc. Var. Partial Differential Equations* **29**:3 (2007), 281–293. MR Zbl

[Duzaar and Steffen 1993] F. Duzaar and K. Steffen, "λ minimizing currents", *Manuscripta Math.* **80**:4 (1993), 403–447. MR Zbl

[Federer and Fleming 1960] H. Federer and W. H. Fleming, "Normal and integral currents", *Ann. of Math.* (2) **72** (1960), 458–520. MR Zbl

[Gilbarg and Trudinger 1983] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Grundlehren der Math. Wissenschaften **224**, Springer, 1983. MR Zbl

[Haslhofer 2015] R. Haslhofer, "Uniqueness of the bowl soliton", *Geom. Topol.* **19**:4 (2015), 2393–2406. MR Zbl

[Hoffman et al. 2019] D. Hoffman, T. Ilmanen, F. Martín, and B. White, "Graphical translators for mean curvature flow", *Calc. Var. Partial Differential Equations* **58**:4 (2019), art. id. 117. MR Zbl

[Massari and Miranda 1984] U. Massari and M. Miranda, *Minimal surfaces of codimension one*, North-Holland Math. Studies **91**, North-Holland, Amsterdam, 1984. MR Zbl

[Ôtsuki 1970] T. Ôtsuki, "Minimal hypersurfaces in a Riemannian manifold of constant curvature", *Amer. J. Math.* **92** (1970), 145–173. MR Zbl

[Ôtsuki 1972] T. Ôtsuki, "On integral inequalities related with a certain nonlinear differential equation", *Proc. Japan Acad.* **48** (1972), 9–12. MR Zbl

[Pogorelov 1981] A. V. Pogorelov, "On the stability of minimal surfaces", *Dokl. Akad. Nauk SSSR* **260**:2 (1981), 293–295. In Russian; translated in *Soviet Math. Dokl.* **24**:2 (1981), 274–276. MR Zbl

[Schoen 1983] R. Schoen, "Estimates for stable minimal surfaces in three-dimensional manifolds", pp. 111–126 in *Seminar on minimal submanifolds*, edited by E. Bombieri, Ann. of Math. Stud. **103**, Princeton Univ. Press, 1983. MR Zbl

[Schoen et al. 1975] R. Schoen, L. Simon, and S. T. Yau, "Curvature estimates for minimal hypersurfaces", *Acta Math.* **134**:3-4 (1975), 275–288. MR Zbl

[Sheng and Wang 2009] W. Sheng and X.-J. Wang, "Singularity profile in the mean curvature flow", *Methods Appl. Anal.* **16**:2 (2009), 139–155. MR Zbl

[Simon 1983] L. Simon, *Lectures on geometric measure theory*, Proc. Cent. Math. Anal. Aust. Nat. Univ. **3**, Australian Nat. Univ., Canberra, 1983. MR Zbl

[Simons 1968] J. Simons, "Minimal varieties in Riemannian manifolds", *Ann. of Math.* (2) **88** (1968), 62–105. MR Zbl

[Spruck and Xiao 2017] J. Spruck and L. Xiao, "Complete translating solitons to the mean curvature flow in $\mathbb{R}^3$ with nonnegative mean curvature", preprint, 2017. arXiv

[Wang 2011] X.-J. Wang, "Convex solutions to the mean curvature flow", *Ann. of Math.* (2) **173**:3 (2011), 1185–1239. MR Zbl

[White 2003] B. White, "The nature of singularities in mean curvature flow of mean-convex sets", *J. Amer. Math. Soc.* **16**:1 (2003), 123–138. MR Zbl

Received 12 Jun 2018. Revised 3 Apr 2019. Accepted 18 Apr 2019.

THEODORA BOURNI: tbourni@utk.edu
Department of Mathematics, University of Tennessee, Knoxville, TN, United States

MAT LANGFORD: mlangford@utk.edu
Department of Mathematics, University of Tennessee, Knoxville, TN, United States

GIUSEPPE TINAGLIA: giuseppe.tinaglia@kcl.ac.uk
Department of Mathematics, King's College London, London, United Kingdom

ANALYSIS AND PDE
Vol. 13, No. 4, 2020
dx.doi.org/10.2140/apde.2020.13.1073

CONVEX PROJECTIVE SURFACES WITH COMPATIBLE WEYL CONNECTION ARE HYPERBOLIC

THOMAS METTLER AND GABRIEL P. PATERNAIN

We show that a properly convex projective structure $\mathfrak{p}$ on a closed oriented surface of negative Euler characteristic arises from a Weyl connection if and only if $\mathfrak{p}$ is hyperbolic. We phrase the problem as a nonlinear PDE for a Beltrami differential by using that $\mathfrak{p}$ admits a compatible Weyl connection if and only if a certain holomorphic curve exists. Turning this nonlinear PDE into a transport equation, we obtain our result by applying methods from geometric inverse problems. In particular, we use an extension of a remarkable L^2-energy identity known as Pestov's identity to prove a vanishing theorem for the relevant transport equation.

1. Introduction

A *projective structure* on a smooth manifold M is an equivalence class $\mathfrak{p}$ of torsion-free connections on its tangent bundle TM, where two such connections are declared to be projectively equivalent if they share the same unparametrised geodesics. The set of torsion-free connections on TM is an affine space modelled on the sections of $S^2(T^*M) \otimes TM$. By a classical result of Cartan, Eisenhart, Weyl (see [Spivak 1999] for a modern reference), two connections are projectively equivalent if and only if their difference is pure trace. In particular, it follows from the representation theory of $\mathrm{GL}(2, \mathbb{R})$ that a projective structure on a surface M is a section of a natural affine bundle of rank 4 whose associated vector bundle is canonically isomorphic to $V = S^3(T^*M) \otimes \Lambda^2(TM)$. Choosing an orientation and Riemannian metric g on M, the bundle V decomposes into irreducible $\mathrm{SO}(2)$-bundles $V \simeq T^*M \oplus S^3_0(T^*M)$, where the latter summand denotes the totally symmetric $(0,3)$-tensors on M that are trace-free with respect to g, or equivalently, the cubic differentials with respect to the complex structure J induced by g and the orientation. In other words, fixing an orientation and Riemannian metric g on M, a projective structure $\mathfrak{p}$ may be encoded in terms of a unique triple (g, A, θ), where A is a cubic differential — and θ a 1-form on M. A conformal change of the metric $g \mapsto \mathrm{e}^{2u} g$ corresponds to a change

$$(g, A, \theta) \mapsto (\mathrm{e}^{2u} g, \mathrm{e}^{2u} A, \theta + du).$$

Consequently, the section $\Phi = A/d\sigma$ of $K^2 \otimes \overline{K^*}$ does only depend on the complex structure J. Here $d\sigma$ denotes the area form of g and K the canonical bundle of M. In addition, we obtain a connection D on the anticanonical bundle K^* inducing the complex structure by taking the Chern connection with respect

MSC2010: primary 32W50, 53A20; secondary 30F30, 37D40.
Keywords: convex projective structures, Weyl connections, transport equations, energy identity.

to g and by subtracting twice the $(1,0)$-part of θ. Again, the connection D does only depend on J. Fixing a complex structure J on M thus encodes a given projective structure $\mathfrak{p}$ in terms of a unique pair (D, Φ).

There are two special cases of particular interest. Firstly, we can find a complex structure J so that D is the Chern connection of a metric in the conformal class determined by J. This amounts to finding a complex structure for which θ is exact. Secondly, we can find a complex structure J so that Φ vanishes identically. This turns out to be equivalent to $\mathfrak{p}$ containing a *Weyl connection* for the conformal structure $[g]$ determined by J, that is, a torsion-free connection on TM whose parallel transport maps are angle-preserving with respect to $[g]$.

In [Mettler 2014], it is shown that a two-dimensional projective structure $\mathfrak{p}$ does locally always contain a Weyl connection and moreover, finding the Weyl connection turns out to be equivalent to finding a holomorphic curve into a certain complex surface Z fibering over M. Here we use this observation to rephrase the problem in terms of a nonlinear PDE for a Beltrami differential. More precisely, we think of $\mathfrak{p}$ as being given on a Riemann surface (M, J) in terms of (D, Φ). We show (see Proposition 4.4) that $\mathfrak{p}$ contains a Weyl connection with respect to the complex structure defined by the Beltrami differential μ on (M, J) if and only if

$$\mathrm{D}''\mu - \mu\,\mathrm{D}'\mu = \Phi\mu^3 + \overline{\Phi}, \tag{1-1}$$

where D' and D'' denote the (1,0)- and (0,1)-part of D. Since every two-dimensional projective structure locally contains a Weyl connection, the above PDE for the Beltrami differential μ can locally always be solved. Moreover, on the 2-sphere every solution μ lies in a complex 5-manifold of solutions, whereas on a closed surface of negative Euler characteristic the solution is unique, provided it exists; see [Mettler 2015b] (and Corollary 4.6 below).

Here we address the problem of finding a projective structure $\mathfrak{p}$ for which the above PDE has no global solution. Naturally, one might start by looking at projective structures $\mathfrak{p}$ at "the other end", that is, those that arise from pairs (D, Φ) where D is the Chern connection of a conformal metric, or equivalently, those for which there exists a metric g so that $\mathfrak{p}$ is encoded in terms of the triple $(g, A, 0)$. This class of projective structures includes the so-called *properly convex projective structures*. A projective surface $(M, \mathfrak{p})$ is called properly convex if it arises as a quotient of a properly convex open set $\Omega \subset \mathbb{RP}^2$ by a free and cocompact action of a group $\Gamma \subset \mathrm{SL}(3, \mathbb{R})$ of projective transformations. In particular, using the Beltrami–Klein model of two-dimensional hyperbolic geometry, it follows that every closed hyperbolic Riemann surface is a properly convex projective surface. Motivated by Hitchin's generalisation of Teichmüller space [1992], Labourie [2007] and Loftin [2001] have shown independently that on a closed oriented surface M of negative Euler characteristic every properly convex projective structure arises from a unique pair $(g, A, 0)$, where g and A are subject to the equations

$$K_g = -1 + 2|A|_g^2 \quad \text{and} \quad \bar{\partial} A = 0.$$

Using quasilinear elliptic PDE techniques, C. P. Wang [1991] previously showed (see also [Dumas and Wolf 2015]) that the metric g is uniquely determined in terms of $([g], A)$ by the equation for the Gauss curvature K_g of g. Consequently, Labourie and Loftin concluded that on M the properly convex projective structures are in bijective correspondence with pairs $([g], A)$ consisting of a conformal structure and a cubic holomorphic differential.

Naturally one might speculate that (1-1) does not admit a global solution for a properly convex projective structure $\mathfrak{p}$ unless A vanishes identically, in which case $\mathfrak{p}$ is hyperbolic. This is indeed the case:

Corollary 6.2. *Let $(M, \mathfrak{p})$ be a closed oriented properly convex projective surface with $\chi(M) < 0$ and with $\mathfrak{p}$ containing a Weyl connection D. Then $\mathfrak{p}$ is hyperbolic and moreover D is the Levi-Civita connection of the hyperbolic metric.*

This corollary is an application of the more general vanishing theorem, Theorem 6.1 (see below), whose proof makes use of a remarkable L^2-energy identity. This energy identity — known for geodesic flows as *Pestov's identity* — is ubiquitous when solving uniqueness problems for X-ray transforms, including tensor tomography. To make the bridge between (1-1) and this circle of ideas, it is necessary to recast the nonlinear PDE in dynamical terms as a transport problem. Given a projective structure $\mathfrak{p}$ captured by the triple (g, A, θ) we associate a dynamical system on the unit tangent bundle $\pi : SM \to M$ of g as follows. We consider a vector field of the form $F = X + (a - V\theta)V$, where X, V denote the geodesic and vertical vector fields of SM, $a \in C^\infty(SM, \mathbb{R})$ represents the cubic differential A (essentially its imaginary part) and where we think of θ as a function on SM. The flow of the vector field F is a *thermostat* (see Section 3 below for more details) and it has the property that its orbits project to M as unparametrised geodesics of $\mathfrak{p}$. We show that (1-1) is equivalent to the transport equation (see Corollary 5.6)

$$Fu = Va + \beta \tag{1-2}$$

on SM, where the real-valued function u encodes a conformal metric of the sought-after complex structure $\hat{J}$ and β is a 1-form on M, again thought of as a function on SM. Explicitly

$$u = \frac{3}{2}\log\left(\frac{p}{(pq - r^2)^{2/3}}\right),$$

where p, q, r are given in terms of a $\hat{J}$-conformal metric $\hat{g}$ and the complex structure J of (M, g) by

$$p(x, v) = \hat{g}(v, v), \quad r(x, v) = \hat{g}(v, Jv) \quad \text{and} \quad q(x, v) = \hat{g}(Jv, Jv).$$

The right-hand side in (1-2) has degree 3 in the velocities and the dynamics of F is Anosov when $\mathfrak{p}$ is a properly convex projective structure [Mettler and Paternain 2019]; hence it is natural to think that techniques from tensor tomography might work. Regular tensor tomography involves the geodesic vector field X and the typical question at the level of the transport equation is the following: if $Xu = f$ where f has degree m in the velocities, is it true that u has degree $m - 1$ in the velocities? The case $m = 2$ is perhaps the most important and it is at the core of spectral rigidity of negatively curved manifolds and Anosov surfaces [Croke and Sharafutdinov 1998; Guillemin and Kazhdan 1980; Paternain et al. 2014]. Thermostats introduce new challenges; however we are able to successfully use a general L^2 energy identity developed in [Jane and Paternain 2009] (following earlier results for geodesic flows in [Sharafutdinov and Uhlmann 2000]) together with ideas in [Mettler and Paternain 2019] to show that if (1-2) holds then $a = 0$ and β is exact. Our vanishing theorem, Theorem 6.1, is actually rather general and it applies to a class of projective structures considerably larger than properly convex projective structures; see Corollary 6.4 below.

For the case of surfaces with boundary, a full solution to the tensor tomography problem was given in [Paternain et al. 2013]; the solution was inspired by the proof of the Kodaira vanishing theorem in complex geometry. In the present paper, we go in the opposite direction; we import ideas from geometric inverse problems to solve an existence question for a nonlinear PDE in complex geometry. These connections were not anticipated, and it is natural to wonder if they are manifestations of something deeper.

2. Preliminaries

Here we collect some standard facts about Riemann surfaces and the unit tangent bundle that will be needed throughout the paper.

2A. ***The frame bundle.*** Throughout the article M will denote a connected oriented smooth surface with empty boundary. Unless stated otherwise, all maps are assumed to be smooth, i.e., C^∞. Let $\pi : P \to M$ denote the *oriented frame bundle* of M whose fibre at a point $x \in M$ consists of the linear isomorphisms $f : \mathbb{R}^2 \to T_x M$ that are orientation-preserving, where we equip $\mathbb{R}^2$ with its standard orientation. The Lie group $\mathrm{GL}^+(2, \mathbb{R})$ acts transitively from the right on each fibre by the rule $R_h(f) = f \circ h$ and this action turns $\pi : P \to M$ into a principal right $\mathrm{GL}^+(2, \mathbb{R})$-bundle. The bundle P is equipped with a tautological $\mathbb{R}^2$-valued 1-form $\omega = (\omega^i)$ defined by $\omega_f = f^{-1} \circ d\pi_f$ and which satisfies the equivariance property $R_h^*\omega = h^{-1}\omega$. The components of ω are a basis for the 1-forms on P that are semibasic for the projection $\pi : P \to M$, i.e., those 1-forms that vanish when evaluated on a vector field that is tangent to the fibres of $\pi : P \to M$. Therefore, if g is a Riemannian metric on M, there exist unique real-valued functions $g_{ij} = g_{ji}$ on P so that $\pi^* g = g_{ij}\omega^i \otimes \omega^j$. The Levi-Civita connection ${}^g\nabla$ of g corresponds to the unique connection form $\psi = (\psi^i_j) \in \Omega^1(P, \mathfrak{gl}(2, \mathbb{R}))$ satisfying the structure equations

$$\begin{aligned} d\omega^i &= -\psi^i_j \wedge \omega^j, \\ dg_{ij} &= g_{ik}\psi^k_j + g_{kj}\psi^k_i. \end{aligned} \tag{2-1}$$

The curvature $\Psi = (\Psi^i_j)$ of ψ is the 2-form

$$\Psi^i_j = d\psi^i_j + \psi^i_k \wedge \psi^k_j = K_g g_{jk}\omega^i \wedge \omega^k,$$

where K_g denotes (the pullback to P of) the Gauss curvature of g.

2B. ***Conformal connections.*** The *conformal frame bundle* of the conformal equivalence class $[g]$ of g is the principal right $\mathrm{CO}(2)$-subbundle $\pi : P_{[g]} \to M$ defined by

$$P_{[g]} = \{f \in P : g_{11}(f) = g_{22}(f) \text{ and } g_{12}(f) = 0\}.$$

Here $\mathrm{CO}(2) = \mathbb{R}^+ \times \mathrm{SO}(2)$ denotes the linear conformal group whose Lie algebra $\mathfrak{co}(2)$ is spanned by the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

A *conformal connection* for $[g]$ is principal CO(2) connection

$$\kappa = \begin{pmatrix} \kappa_1 & -\kappa_2 \\ \kappa_2 & \kappa_1 \end{pmatrix}, \quad \kappa_i \in \Omega^1(P_{[g]}),$$

on $P_{[g]}$ which is *torsion-free*, that is, satisfies

$$d\begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} = -\begin{pmatrix} \kappa_1 & -\kappa_2 \\ \kappa_2 & \kappa_1 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}. \tag{2-2}$$

The standard identification $\mathbb{R}^2 \simeq \mathbb{C}$ gives an identification $\mathrm{CO}(2) \simeq \mathrm{GL}(1, \mathbb{C})$ and consequently, $\mathfrak{co}(2) \simeq \mathbb{C}$. In particular, (2-2) takes the form $d\omega = -\kappa \wedge \omega$, where we think of κ and ω as being complex-valued. Writing $r\mathrm{e}^{i\phi}$ for the elements of CO(2), the equivariance property for ω implies $(R_{r\mathrm{e}^{i\phi}})^*\omega = \frac{1}{r}\mathrm{e}^{-i\phi}\omega$. In particular, we see that the π-semibasic complex-valued 1-form ω is well-defined on M up to complex scale. It follows that there exists a unique complex-structure J on M whose (1,0)-forms are represented by smooth complex-valued functions u on $P_{[g]}$ satisfying the equivariance property $(R_{r\mathrm{e}^{i\phi}})^*u = r\mathrm{e}^{i\phi}u$, that is, so that $u\omega$ is invariant under the CO(2)-right action. Of course, this is the standard complex structure on M obtained by rotation of a tangent vector v counterclockwise by $\frac{\pi}{2}$ with respect to $[g]$. Denoting the canonical bundle of M with respect to J by K, it follows that the sections of $L_{m,\ell} := K^m \otimes \overline{K^\ell}$ are in one-to-one correspondence with the smooth complex-valued functions u on $P_{[g]}$ satisfying the equivariance property $(R_{r\mathrm{e}^{i\phi}})^*u = r^{m+\ell}\mathrm{e}^{i(m-\ell)\phi}u$. Infinitesimally, this translates to the existence of unique smooth complex-valued functions u' and u'' on $P_{[g]}$ so that

$$du = u'\omega + u''\bar{\omega} + mu\kappa + \ell u\bar{\kappa}. \tag{2-3}$$

Recall, if α is a 1-form on M taking values in some complex vector bundle over M, the decomposition $\alpha = \alpha' + \alpha''$ of α into its (1,0)-part α' and (0,1)-part α'' is given by

$$\alpha' = \tfrac{1}{2}(\alpha - iJ\alpha) \quad \text{and} \quad \alpha'' = \tfrac{1}{2}(\alpha + iJ\alpha),$$

where we define $(J\alpha)(v) := \alpha(Jv)$ for all tangent vectors $v \in TM$. The principal CO(2)-connection κ induces a connection on all (real or complex) vector bundles associated to $P_{[g]}$ and — by standard abuse of notation — we use the same letter D to denote the induced connection on the various bundles. If s is the section of $L_{m,\ell}$ represented by the function u satisfying (2-3), then $\mathrm{D}'s := (\mathrm{D}s)'$ is represented by u' and $\mathrm{D}''s := (\mathrm{D}s)''$ is represented by u''.

Since $dg_{11} = dg_{22}$ and $dg_{12} = 0$ on $P_{[g]}$, it follows from (2-1) that the pullback of the Levi-Civita connection ψ of g to $P_{[g]}$ is a conformal connection. The difference of any two principal CO(2)-connections is π-semibasic. Therefore, any other torsion-free principal CO(2)-connection κ on $P_{[g]}$ is of the form $\kappa = \psi - 2\theta_1\omega$ for a unique complex-valued function θ_1 on $P_{[g]}$. Since κ is a connection, it satisfies the equivariance property $(R_{r\mathrm{e}^{i\phi}})^*\kappa = \frac{1}{r}\mathrm{e}^{-i\phi}\kappa r\mathrm{e}^{i\phi} = \kappa$ and so does ψ. Therefore, $2\theta_1\omega$ is invariant under the CO(2)-right action as well and hence twice the pullback of a (1,0)-form on M, which we denote by θ'. From (2-3) we see that we may think of κ as being the connection form of the induced connection on the anticanonical bundle K^*. In particular, ψ may be thought of as being the connection form of the Chern connection induced by g on K^*. By the definition of the Chern connection, it induces the complex structure of K^*. Since ψ and κ differ by a (1,0)-form, κ also induces the complex structure of K^*. Consequently, the conformal connections on $P_{[g]}$ are in one-to-one correspondence with the connections D on K^* inducing the complex structure, that is, $\mathrm{D}'' = \bar{\partial}_{K^*}$.

2C. *The unit tangent bundle.* For what follows it will be necessary to further reduce $P_{[g]}$. The unit tangent bundle

$$SM = \{(x, v) \in TM : g(v, v) = 1\}$$

of g may be interpreted as the principal right SO(2)-subbundle of P defined by

$$SM = \{f \in P : g_{ij}(f) = \delta_{ij}\}.$$

On SM the identities $dg_{ij} \equiv 0$ imply the identities $\psi^1_1 \equiv \psi^2_2 \equiv 0$ and $\psi^1_2 \equiv -\psi^2_1$, so that ψ is purely imaginary.

Abusing notation by henceforth writing ψ instead of ψ^2_1, the structure equations thus take the form

$$d\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = -\begin{pmatrix}0 & -\psi\\ \psi & 0\end{pmatrix}\wedge\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} \quad\text{and}\quad d\psi = -K_g\,\omega_1\wedge\omega_2, \tag{2-4}$$

where we write $\omega_i = \delta_{ij}\omega^j$. Note that on SM the 1-forms ω_1, ω_2 take the explicit form

$$\omega_1(\xi) = g(v, d\pi(\xi)) \quad\text{and}\quad \omega_2(\xi) = g(Jv, d\pi(\xi)), \quad \xi \in T_{(x,v)}SM. \tag{2-5}$$

Furthermore, the 1-form ψ becomes

$$\psi(\xi) = g(\gamma''(0), Jv), \tag{2-6}$$

where $\xi \in T_{(x,v)}SM$ and $\gamma : (-\varepsilon, \varepsilon) \to SM$ is any curve with $\gamma(0) = (x, v)$, $\dot{\gamma}(0) = \xi$ and γ'' denotes the covariant derivative of γ along $\pi \circ \gamma$.

The three 1-forms $(\omega_1, \omega_2, \psi)$ trivialise the cotangent bundle of SM and we denote by (X, H, V) the corresponding dual vector fields. The vector field X is the geodesic vector field of g, V is the infinitesimal generator of the SO(2)-action and H is the horizontal vector field satisfying $H = [V, X]$. The structure equations (2-4) imply the additional commutation relations

$$[V, H] = -X \quad\text{and}\quad [X, H] = K_g V.$$

Following [Guillemin and Kazhdan 1980], we use the volume form $\Theta = \omega_1 \wedge \omega_2 \wedge \psi$ on SM to define an inner product

$$\langle u, v\rangle = \int_{SM} u\bar{v}\,\Theta$$

for complex-valued functions u, v on SM and we denote by $L^2(SM)$ the corresponding space of square integrable complex-valued functions on SM. The structure equations (2-4) and Cartan's formula imply that all vector fields X, H, V preserve Θ. In particular, $-iV$ is densely defined and self-adjoint with respect to $\langle\cdot,\cdot\rangle$. Consequently, we have an orthogonal direct sum decomposition into the kernels $\mathcal{H}_m$ of the operators $m\mathrm{Id} + iV$

$$L^2(SM) = \bigoplus_{m\in\mathbb{Z}} \mathcal{H}_m. \tag{2-7}$$

2D. *Weyl connections.* If θ is a 1-form on M, we may write $\pi^*\theta = \theta\omega_1 + V(\theta)\omega_2$, where on the right-hand side we think of θ as being a real-valued function on SM. Therefore, $\pi^*\theta' = \theta_1\omega$, where $\theta_1 = \frac{1}{2}(\theta - iV\theta)$ and likewise $\pi^*\theta'' = \theta_{-1}\bar{\omega}$, where $\theta_{-1} = \frac{1}{2}(\theta + iV\theta)$. On SM the connection form κ of a conformal connection thus becomes $\kappa = i\psi - 2\theta_1\omega$, or in matrix notation

$$\kappa = \begin{pmatrix} 0 & -\psi \\ \psi & 0 \end{pmatrix} + \begin{pmatrix} -\theta\omega_1 - V(\theta)\omega_2 & -V(\theta)\omega_1 + \theta\omega_2 \\ V(\theta)\omega_1 - \theta\omega_2 & -\theta\omega_1 - V(\theta)\omega_2 \end{pmatrix}. \tag{2-8}$$

Finally, without the identification $\mathbb{R}^2 \simeq \mathbb{C}$, we may equivalently think of the connection form κ as the connection form of a torsion-free connection on TM. Writing κ as

$$\kappa = \begin{pmatrix} 0 & -\psi \\ \psi & 0 \end{pmatrix} + \begin{pmatrix} \theta\omega_1 & \theta\omega_2 \\ V(\theta)\omega_1 & V(\theta)\omega_2 \end{pmatrix} - \begin{pmatrix} 2\theta\omega_1 + V(\theta)\omega_2 & V(\theta)\omega_1 \\ \theta\omega_2 & \theta\omega_1 + 2V(\theta)\omega_2 \end{pmatrix},$$

the reader may easily check that κ is the connection form of

$$\mathrm{D} = {}^g\nabla + g \otimes \theta^\sharp - \mathrm{Sym}(\theta), \tag{2-9}$$

where the section $\mathrm{Sym}(\theta)$ of $S^2(T^*M) \otimes TM$ is defined by the rule

$$\mathrm{Sym}(\theta)(v_1, v_2) = \theta(v_1)v_2 + \theta(v_2)v_1$$

for all tangent vectors $v_1, v_2 \in T_xM$ and all $x \in M$. Connections of the form (2-9) for $g \in [g]$ and $\theta \in \Omega^1(M)$ are known as *Weyl connections* for the conformal structure $[g]$. By construction, they preserve $[g]$; that is, the parallel transport maps are angle-preserving with respect to $[g]$. Conversely, every torsion-free connection on TM preserving $[g]$ is of the form (2-9) for some $g \in [g]$ and 1-form θ. Summarising, we have the following folklore result:

Proposition 2.1. *On a Riemann surface M with conformal structure $[g]$ the following sets are in one-to-one correspondence*:

(i) *the conformal connections on $P_{[g]}$*,

(ii) *the connections on K^* inducing the complex structure*,

(iii) *the Weyl connections for $[g]$*.

3. Projective thermostats

In this section we show how to associate the triple (g, A, θ) to a given projective structure $\mathfrak{p}$. As mentioned in the Introduction, the existence of such a triple is a consequence of some elementary facts about SO(2)-representation theory and a description of projective structures as sections of a certain affine bundle over M (see [Mettler 2015a] for a construction of (g, A, θ) in that spirit); here instead we obtain the triple as a by-product of a characterisation of *projective thermostats*.

A (generalised) *thermostat* is a flow ϕ on SM which is generated by a vector field of the form $F = X + \lambda V$, where λ is a smooth real-valued function on SM. In this article we are mainly interested in the case where the generalised thermostat is *projective*. By this we mean that there exists a torsion-free connection ∇ on TM having the property that for every ϕ-orbit $\gamma : I \to SM$ there exists a reparametrisation $\varphi : I' \to I$ so that $\pi \circ \gamma \circ \varphi : I' \to M$ is a geodesic of ∇.

Phrased more loosely, the orbit projections to M agree with the geodesics of a projective structure $\mathfrak{p}$ on M. By a classical result of Cartan, Eisenhart, Weyl (see for instance [Spivak 1999, Chapter 6, Addendum 1, Proposition 17] for a modern reference), two torsion-free connections ∇ and ∇' on TM are projectively equivalent if and only if there exists a 1-form α on M so that

$$\nabla' - \nabla = \mathrm{Sym}(\alpha).$$

3A. ***A characterisation of projective thermostats.*** It turns out that projective thermostats admit a simple characterisation in terms of the vertical Fourier decomposition (2-7) of λ. Towards this end we first show:

Lemma 3.1. *Let ∇ be a torsion-free connection on the tangent bundle TM and $\varphi=(\varphi^i_j)\in\Omega^1(SM, \mathfrak{gl}(2,\mathbb{R}))$ its connection form. Then, up to reparametrisation, the leaves of the foliation $\mathcal{F}$ defined by $\varphi^2_1 = \omega_2 = 0$ project to M to become the geodesics of ∇. Conversely, every geodesic of ∇, parametrised with respect to g-arc length, lifts to become a leaf of $\mathcal{F}$.*

Proof. Recall that the set of torsion-free connections on TM is an affine space modelled on the sections of $S^2(T^*M) \otimes TM$. It follows that there exists a 1-form $\widetilde{B}$ on M with values in the endomorphisms of TM so that $\nabla = {}^g\nabla + \widetilde{B}$. As we have seen, the connection form of the Levi-Civita connection of g on TM is

$$\kappa = \begin{pmatrix} 0 & -\psi \\ \psi & 0 \end{pmatrix}.$$

Hence there exist unique real-valued function $b^i_{jk} = b^i_{kj}$ on SM so that

$$\varphi = \begin{pmatrix} 0 & -\psi \\ \psi & 0 \end{pmatrix} + \begin{pmatrix} b^1_{11}\omega_1 + b^1_{12}\omega_2 & b^1_{21}\omega_1 + b^1_{22}\omega_2 \\ b^2_{11}\omega_1 + b^2_{12}\omega_2 & b^2_{21}\omega_1 + b^2_{22}\omega_2 \end{pmatrix}.$$

Explicitly, $b^i_{jk}(v) = g(\widetilde{B}(e_j)e_k, e_i)$, where we write $e_1 = v$ and $e_2 = Jv$ for $v \in SM$.

Let $\delta : I \to SM$ be a leaf of $\mathcal{F}$, so that $\delta^*\omega_2 = 0$. Writing $\gamma := \pi \circ \delta$ and evaluating $\delta^*\omega_2$ on the standard vector field ∂_t of $\mathbb{R}$, we obtain

$$0 = \partial_t \lrcorner\, \delta^*\omega_2 = g(d(\pi \circ \delta)(\partial_t), J\delta(t)) = g(\dot\gamma(t), J\delta(t)),$$

so that $\delta = f\dot\gamma$ for some unique $f \in C^\infty(I)$. Hence without losing generality, we may assume that the leaves of $\mathcal{F}$ are of the form $\dot\gamma$ for some smooth curve $\gamma : I \to M$ having unit-length velocity vector with respect to g.

By the construction of ψ, see (2-6), the pullback 1-form $\dot\gamma^*\psi$ evaluated on ∂_t gives the function $g({}^g\nabla_{\dot\gamma}\dot\gamma, J\dot\gamma)$; hence $\dot\gamma^*\varphi^2_1 = 0$ if and only if

$$0 = g({}^g\nabla_{\dot\gamma}\dot\gamma, J\dot\gamma) + b^2_{11}(\dot\gamma) = g({}^g\nabla_{\dot\gamma}\dot\gamma + \widetilde{B}(\dot\gamma)\dot\gamma, J\dot\gamma).$$

It follows that there exists a function $f \in C^\infty(I)$ so that

$${}^g\nabla_{\dot\gamma}\dot\gamma + \widetilde{B}(\dot\gamma)\dot\gamma = \nabla_{\dot\gamma}\dot\gamma = f\dot\gamma.$$

By a standard lemma in projective differential geometry [Spivak 1999, Chapter 6, Addendum 1, Proposition 17] a smooth immersed curve $\gamma : I \to M$ can be reparametrised to become a geodesic of the torsion-free connection ∇ on TM if and only if there exists a smooth function $f : I \to \mathbb{R}$ so that $\nabla_{\dot\gamma}\dot\gamma = f\dot\gamma$. The claim follows by applying this lemma. □

Lemma 3.2. *Suppose the thermostat $F = X + \lambda V$ is projective. Then*

$$0 = \tfrac{3}{2}\lambda + \tfrac{5}{3}VV\lambda + \tfrac{1}{6}VVVV\lambda.$$

Proof. Using notation as in the proof of Lemma 3.1, we must have $F \lrcorner \varphi^2_1 = 0$ and $F \lrcorner \omega_2 = 0$. The latter condition is trivially satisfied, but the former gives

$$F \lrcorner \varphi^2_1 = (X + \lambda V) \lrcorner (\psi + b^2_{11}\omega_1 + b^2_{12}\omega_2) = \lambda + b^2_{11} = 0,$$

so that $\lambda = -b^2_{11}$. Since the functions b^i_{jk} represent a section of $S^2(T^*M) \otimes TM$, they satisfy the structure equations

$$db^i_{jk} = b^i_{jl}\kappa^l_k + b^i_{lk}\kappa^l_j - b^l_{jk}\kappa^i_l, \qquad \operatorname{mod} \omega_i.$$

In particular, from this we compute

$$Vb^2_{11} = V \lrcorner db^2_{11} = V \lrcorner (2b^2_{12} - b^1_{11})\psi = 2b^2_{12} - b^1_{11}.$$

Applying V again we obtain

$$VVb^2_{11} = 2b^2_{22} - 3b^2_{11} - 4b^1_{12},$$

and likewise

$$VVVVb^2_{11} = 40b^1_{12} + 21b^2_{11} - 20b^2_{22},$$

so that the claim follows from an elementary calculation. □

Lemma 3.3. *For $\lambda \in C^\infty(SM)$ the following statements are equivalent*:

(i) $0 = \tfrac{3}{2}\lambda + \tfrac{5}{3}VV\lambda + \tfrac{1}{6}VVVV\lambda$.

(ii) $\lambda \in \mathcal{H}_{-1} \oplus \mathcal{H}_1 \oplus \mathcal{H}_{-3} \oplus \mathcal{H}_3$.

Proof. Let $\lambda \in \mathcal{H}_{-3} \oplus \mathcal{H}_{-1} \oplus \mathcal{H}_1 \oplus \mathcal{H}_3$ so that we may write $\lambda = \lambda_{-3} + \lambda_{-1} + \lambda_1 + \lambda_3$ with $\lambda_m \in \mathcal{H}_m$. Since λ is real-valued we have $\lambda_{-1} = \bar\lambda_1$ and $\lambda_{-3} = \bar\lambda_3$. Hence setting $\nu_1 = \lambda_{-1} + \lambda_1$ and $\nu_3 = \lambda_{-3} + \lambda_3$, we obtain $VV\nu_1 = -\nu_1$ and $VV\nu_3 = -9\nu_3$ so that

$$\tfrac{3}{2}\lambda + \tfrac{5}{3}VV\lambda + \tfrac{1}{6}VVVV\lambda = \tfrac{3}{2}(\nu_3 + \nu_1) + \tfrac{5}{3}(-9\nu_3 - \nu_1) + \tfrac{1}{6}(81\nu_3 + \nu_1) = 0.$$

Conversely, suppose $\lambda \in C^\infty(SM)$ satisfies $0 = \tfrac{3}{2}\lambda + \tfrac{5}{3}VV\lambda + \tfrac{1}{6}VVVV\lambda$ and write $\lambda = \sum_m \lambda_m$, with $\lambda_m \in \mathcal{H}_m$. Hence we obtain

$$0 = \tfrac{3}{2}\lambda + \tfrac{5}{3}VV\lambda + \tfrac{1}{6}VVVV\lambda = \sum_m \bigl(\tfrac{3}{2} - \tfrac{5}{3}m^2 + \tfrac{1}{6}m^4\bigr)\lambda_m,$$

so that $\lambda_m = 0$ unless

$$0 = \tfrac{3}{2} - \tfrac{5}{3}m^2 + \tfrac{1}{6}m^4 = \tfrac{1}{6}(m-3)(m-1)(m+1)(m+3).$$

The claim follows. □

Finally, we obtain:

Proposition 3.4. *A thermostat $F = X + \lambda V$ is projective if and only if $\lambda \in \mathcal{H}_{-1} \oplus \mathcal{H}_1 \oplus \mathcal{H}_{-3} \oplus \mathcal{H}_3$.*

Proof. It remains to show that if $\lambda \in \mathcal{H}_{-1} \oplus \mathcal{H}_1 \oplus \mathcal{H}_{-3} \oplus \mathcal{H}_3$, then there exists a torsion-free connection ∇ on TM so that $F \lrcorner \varphi^2_1$ vanishes identically, where $\varphi = (\varphi^i_j)$ denotes the connection form of ∇. We may write

$$\lambda = a - V\theta,$$

where $a \in C^\infty(SM)$ satisfies $9a + VVa = 0$ and θ is a smooth 1-form on M, thought of as a real-valued function on SM. Since $9a + VVa = 0$, there exists a unique cubic differential A on M so that

$$\pi^* A = \big(\tfrac{1}{3} Va + ia\big)\omega^3.$$

Hence simple computations show that

$$\begin{aligned} a(v) &= \operatorname{Re} A(Jv, Jv, Jv) = -\operatorname{Re} A(Jv, v, v), \\ \tfrac{1}{3} Va(v) &= \operatorname{Re} A(v, v, v) = -\operatorname{Re} A(v, Jv, Jv) \end{aligned} \tag{3-1}$$

for all $v \in SM$. Let B be the unique 1-form on M with values in the endomorphisms of TM satisfying

$$g(B(v_1)v_2, v_3) = \operatorname{Re} A(v_1, v_2, v_3) \tag{3-2}$$

for all tangent vectors $v_1, v_2, v_3 \in T_xM$ and all $x \in M$. On TM consider the torsion-free connection $\nabla = \mathrm{D} + B$, where D is the Weyl connection

$$\mathrm{D} = {}^g\nabla + g \otimes \theta^\sharp - \operatorname{Sym}(\theta).$$

Using (2-8) and (3-1), we compute that the connection form of ∇ is

$$\varphi = \begin{pmatrix} -\theta\omega_1 - V(\theta)\omega_2 & -V(\theta)\omega_1 + \theta\omega_2 - \psi \\ \psi + V(\theta)\omega_1 - \theta\omega_2 & -\theta\omega_1 - V(\theta)\omega_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{3}V(a)\omega_1 - a\omega_2 & -a\omega_1 - \frac{1}{3}V(a)\omega_2 \\ -a\omega_1 - \frac{1}{3}V(a)\omega_2 & -\frac{1}{3}V(a)\omega_1 + a\omega_2 \end{pmatrix}. \tag{3-3}$$

In particular, we have

$$\varphi^2_1 = \psi + (V(\theta) - a)\omega_1 - \big(\theta + \tfrac{1}{3}V(a)\big)\omega_2,$$

so that $F \lrcorner \varphi^2_1 = 0$. □

3B. ***The effect of a conformal change.*** Summarising the previous subsection, we have seen that if ∇ is a torsion-free connection on TM and we fix a Riemannian metric g on M, then we may write $\nabla = {}^g\nabla + \widetilde{B}$ for some endomorphism-valued 1-form $\widetilde{B}$ on M. The thermostat on SM defined by $\lambda = -b^2_{11}$ has the property that its orbits project to M to become the geodesics of ∇ up to parametrisation. Moreover, we obtain a 1-form $\theta \in \Omega^1(M)$ as well as a cubic differential $A \in \Gamma(K^3)$, so that the connection ∇ shares its geodesics — up to parametrisation — with the projections to M of the orbits of the projective thermostat defined by $\lambda = a - V\theta$, where a represents the imaginary part of A.

Next we compute how θ and A transform under conformal change of the metric. As a consequence, we obtain:

Proposition 3.5. *Let* ∇ *be a torsion-free connection on TM. Then the choice of a conformal structure* $[g]$ *on* M *determines a unique Weyl connection* D *for* $[g]$ *and a unique section* Φ *of* $K^2 \otimes \overline{K^*}$ *so that* $\mathrm{D} + \operatorname{Re}\Phi$ *is projectively equivalent to* ∇.

Proof. Let $g \mapsto \hat{g} = \mathrm{e}^{2u} g$ be a conformal change of the metric, where $u \in C^\infty(M)$. For the new metric $\hat{g}$ there exists a 1-form $\hat{\theta}$ and a cubic differential $\hat{A}$ on M so that $\mathrm{D} + B$ and $\widehat{\mathrm{D}} + \widehat{B}$ are projectively equivalent. Here $\widehat{B}$ denotes the 1-form constructed from $\hat{A}$ by using the metric $\hat{g}$. Projective equivalence corresponds to the existence of a 1-form α on M so that

$$\mathrm{D} + B = \widehat{\mathrm{D}} + \widehat{B} + \operatorname{Sym}(\alpha).$$

Using (2-9) as well as (see [Besse 1987, Theorem 1.159])

$$^{\exp(2u)g}\nabla = {}^g\nabla - g \otimes {}^g\nabla u + \operatorname{Sym}(du), \tag{3-4}$$

this is equivalent to

$$^g\nabla + g \otimes \theta^\sharp - \operatorname{Sym}(\theta) + B = {}^g\nabla - g \otimes {}^g\nabla u + \operatorname{Sym}(du) + \mathrm{e}^{2u} g \otimes \hat{\theta}^{\hat{\sharp}} - \operatorname{Sym}(\hat{\theta}) + \widehat{B} + \operatorname{Sym}(\alpha)$$

or

$$g \otimes (\theta^\sharp + {}^g\nabla u - \hat{\theta}^\sharp) + B - \widehat{B} = \operatorname{Sym}(\beta),$$

where $\beta = \alpha + \theta + du - \hat{\theta}$. Evaluating this equation on the pair (v, Jv) with v a unit tangent vector with respect to g gives

$$B(v)Jv - \widehat{B}(v)Jv = \operatorname{Sym}(\beta)(v, Jv).$$

Computing the inner product with the tangent vector v yields

$$\operatorname{Re} A(v, Jv, v) - \mathrm{e}^{-2u} \operatorname{Re} \hat{A}(v, Jv, v) = \beta(Jv).$$

Thought of as an identity for functions on SM, the left-hand side lies in $\mathcal{H}_{-3} \oplus \mathcal{H}_3$, whereas the right-hand side lies in $\mathcal{H}_{-1} \oplus \mathcal{H}_1$, and hence they can only be equal if both sides vanish identically. Consequently, it follows that $\beta = 0$ and that

$$\hat{A} = \mathrm{e}^{2u} A.$$

Therefore, $B = \widehat{B}$ and

$$\hat{\theta} = \theta + du, \tag{3-5}$$

so that $\alpha = 0$ as well as $\mathrm{D} = \widehat{\mathrm{D}}$.

In particular, we see that both D and B do only depend on the conformal equivalence class of g. We may define a section Φ of $K^2 \otimes \overline{K^*}$ by $\Phi d\sigma = A$, where $d\sigma$ denotes the area form of g. Comparing with (3-2), we see that B is the real part of Φ. □

4. Holomorphic curves

It is natural to ask whether for a given torsion-free connection ∇ on TM one can always (at least locally) choose a conformal structure $[g]$ on M so that Φ vanishes identically, or equivalently, whether every torsion-free connection ∇ on TM is locally projectively equivalent to a Weyl connection D. This question was answered in the affirmative in [Mettler 2014], where it is also observed that the problem is equivalent to finding a suitable holomorphic curve into a complex surface fibering over M. Here we will briefly review this observation and use it to derive a nonlinear PDE for the Beltrami differential of the sought-after conformal structure.

Remark 4.1. Given that one can locally always find a conformal structure so that Φ vanishes identically, one might wonder whether it is possible to simultaneously pick a conformal metric so that the 1-form θ is closed. Indeed, (3-4) and (3-5) imply that the additional closedness condition corresponds to ∇ being locally projectively equivalent to a Levi-Civita connection of some metric. However, this is not always possible; see [Bryant et al. 2009].

4A. ***A complex surface.*** Inspired by the twistorial construction of holomorphic projective structures in [Hitchin 1982], it was shown in [Dubois-Violette 1983; O'Brian and Rawnsley 1985] how to construct a "twistor space" for smooth projective structures. Let ∇ be a torsion-free connection on TM and $\varphi = (\varphi^i_j) \in \Omega^1(P, \mathfrak{gl}(2, \mathbb{R}))$ its connection form on the frame bundle P. We can use φ to construct a complex structure on the quotient $P/\operatorname{CO}(2)$. By definition, an element of $P/\operatorname{CO}(2)$ gives a frame in some tangent space of M, well defined up to rotation and scaling. Therefore, the conformal structures on M are in one-to-one correspondence with the sections of the fibre bundle $P/\operatorname{CO}(2) \to M$ whose fibre is $\mathrm{GL}^+(2, \mathbb{R})/\operatorname{CO}(2)$, that is, the open disk. We will construct a complex structure on $P/\operatorname{CO}(2)$ in terms of its (1,0)-forms, or more precisely, the pullbacks of the (1,0)-forms to P. Recall that the Lie algebra $\mathfrak{co}(2)$ of $\operatorname{CO}(2)$ is spanned by the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Consequently, the complex-valued 1-forms on P that are semibasic for the quotient projection $P \to P/\operatorname{CO}(2)$ are spanned by the form ω and

$$\zeta = (\varphi^1_1 - \varphi^2_2) + i(\varphi^1_2 + \varphi^2_1),$$

as well as their complex conjugates. Recall that we have $(R_{re^{i\phi}})^*\omega = \frac{1}{r}e^{-i\phi}$ and using that φ satisfies the equivariance property $R_h^*\varphi = h^{-1}\varphi h$ for all $h \in \mathrm{GL}^+(2, \mathbb{R})$, we compute $(R_{re^{i\phi}})^*\zeta = e^{-2i\phi}\zeta$. It follows that there exists a unique almost-complex structure J on $P/\operatorname{CO}(2)$ whose (1,0)-forms pull back to P to become linear combinations of the forms ω, ζ. The almost-complex structure J can be shown to only depend on the projective equivalence class of ∇, and moreover, an application of the Newlander–Nirenberg theorem shows that J is always integrable; see [Mettler 2014] for details.

4B. ***Möbius action.*** In our setting it is convenient to reduce the frame bundle P to the unit tangent bundle SM of some fixed metric g. In order to get a handle on the complex surface $P/\operatorname{CO}(2)$ after having carried out this reduction, we interpret the disk bundle $P/\operatorname{CO}(2) \to M$ as an associated bundle to the frame bundle P. This requires an action of the structure group $\mathrm{GL}^+(2, \mathbb{R})$ on the open disk and this is what we compute next.

The group $\mathrm{GL}^+(2,\mathbb{R})$ acts from the left on the lower half plane

$$-\mathbb{H} := \{w \in \mathbb{C} : \Im(w) < 0\}$$

by Möbius transformations, where w denotes the standard coordinate on $\mathbb{C}$. We let $\mathbb{D} \subset \mathbb{C}$ denote the open unit disk. Identifying $-\mathbb{H}$ with $\mathbb{D}$ via the Möbius transformation

$$-\mathbb{H} \to \mathbb{D}, \quad w \mapsto -\left(\frac{w+i}{w-i}\right),$$

we obtain an induced action of $\mathrm{GL}^+(2,\mathbb{R})$ on $\mathbb{D}$ making this transformation equivariant:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{iz(a+d)+z(b-c)-i(a-d)+(b+c)}{-iz(a-d)-z(b+c)+i(a+d)-(b-c)}. \tag{4-1}$$

The stabiliser subgroup of the point $z=0$ consists of elements in $\mathrm{GL}^+(2,\mathbb{R})$ satisfying $a=d$ and $b+c=0$, i.e., the linear conformal group $\mathrm{CO}(2)$. Consequently, we have

$$\mathbb{D} \simeq \mathrm{GL}^+(2,\mathbb{R})/\,\mathrm{CO}(2)$$

and we obtain a projection

$$\lambda : \mathrm{GL}^+(2,\mathbb{R}) \to \mathbb{D}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot 0 = \frac{-i(a-d)+(b+c)}{i(a+d)-(b-c)}.$$

In particular, a mapping $z : N \to \mathbb{D}$ from a smooth manifold N into $\mathbb{D}$ is covered by a map

$$\tilde{z} = \begin{pmatrix} \frac{1-|z|^2}{(1+z)(1+\bar{z})} & \frac{i(z-\bar{z})}{(1+z)(1+\bar{z})} \\ 0 & 1 \end{pmatrix}$$

into $\mathrm{GL}^+(2,\mathbb{R})$. Equivalently, we have $\tilde{z} \cdot 0 = z$ or $z \cdot \tilde{z} = 0$, where as usual we turn the left action into a right action by the definition $z \cdot \tilde{z} := \tilde{z}^{-1} \cdot z$.

Let $\rho : Z \to M$ denote the disk-bundle associated to the above $\mathrm{GL}^+(2,\mathbb{R})$ action on $\mathbb{D}$. Suppose $z : P \to \mathbb{D}$ represents a section of $Z \to M$ so that z is a $\mathrm{GL}^+(2,\mathbb{R})$-equivariant map. For every coframe $u \in P$ the pair $(u, z(u)) \in P \times \mathbb{D}$ lies in the same $\mathrm{GL}^+(2,\mathbb{R})$ orbit as

$$(u \cdot \tilde{z}(u), z(u) \cdot \tilde{z}(u)) = (u \cdot \tilde{z}(u), 0). \tag{4-2}$$

Therefore, the map z gives for every point $p \in M$ a coframe $u \cdot \tilde{z}(u)$ which is unique up to the action of $\mathrm{CO}(2)$. It follows that the bundle $Z \to M$ is isomorphic to $P/\,\mathrm{CO}(2) \to M$, as desired.

Let $\Upsilon : P \times \mathbb{D} \to P$ be the map defined by (4-2). We will next compute the pullback of ω, ζ under Υ. Note that we may write $\Upsilon = R \circ (\mathrm{Id}_P \times \tilde{z})$, where $R : P \times \mathrm{GL}^+(2,\mathbb{R}) \to P$ denotes the $\mathrm{GL}^+(2,\mathbb{R})$ right action of P. Recall the standard identities

$$R^*\varphi = h^{-1}\varphi h + h^{-1} dh \quad \text{and} \quad R^*\omega = h^{-1}\omega,$$

where $h : P \times \mathrm{GL}^+(2, \mathbb{R}) \to \mathrm{GL}^+(2, \mathbb{R})$ denotes the projection onto the latter factor. From this we compute

$$\omega_\Upsilon := \Upsilon^*\omega = \tilde{z}^{-1}\omega = \left(\frac{1+\bar{z}}{1-|z|^2}\right)(\omega + z\bar{\omega}), \tag{4-3}$$

$$\varphi_\Upsilon := \Upsilon^*\varphi = \tilde{z}^{-1}\varphi\tilde{z} + \tilde{z}^{-1}d\tilde{z}. \tag{4-4}$$

We also obtain $\zeta_\Upsilon = \Upsilon^*\zeta = (\varphi_\Upsilon)^1_1 - (\varphi_\Upsilon)^2_2 + i((\varphi_\Upsilon)^1_2 + (\varphi_\Upsilon)^2_1)$. Writing

$$\chi = \tfrac{1}{2}(3(\varphi^1_1 + \varphi^2_2) + i(\varphi^2_1 - \varphi^1_2)),$$

and using (4-4), a tedious but straightforward calculation gives

$$\zeta_\Upsilon = \frac{2(1+\bar{z})}{(|z|^2-1)(z+1)}\big(dz - \tfrac{1}{2}\zeta + \tfrac{1}{2}z^2\bar{\zeta} + z\chi - z\bar{\chi}\big). \tag{4-5}$$

Remark 4.2. The complex-valued 1-form χ is chosen so that χ, $\bar{\chi}$, ω, $\bar{\omega}$, ζ, $\bar{\zeta}$ span the complex-valued 1-forms on P. Clearly, this condition does not pin down χ uniquely. The particular choice is so that in the absence of θ the form χ becomes the connection form of the Chern connection on K^* upon reducing to SM; see (4-6) below.

The complex structure on Z only depends on the projective equivalence class of ∇. Thus, after possibly replacing φ with a projectively equivalent connection, we can assume that the torsion-free connection on TM corresponding to φ is of the form $\mathrm{D} + B$ for some 1-form θ and some cubic differential A on M. On the unit tangent bundle SM of g the connection form of $\mathrm{D} + B$ takes the form (3-3). Using this equation and reducing to $SM \subset P$ yields the following identities on SM:

$$\begin{aligned} \zeta &= 2a_{-3}\bar{\omega}, \\ \chi &= i\psi - 4\theta_1\omega - 2\theta_{-1}\bar{\omega}. \end{aligned} \tag{4-6}$$

Recall, we write $a_3 = \frac{1}{3}Va + ia$ and $a_{-3} = \bar{a}_3$ as well as $\theta_1 = \frac{1}{2}(\theta - iV\theta)$ and $\theta_{-1} = \bar{\theta}_1$. Also, the connection form κ of the induced Weyl connection is $\kappa = i\psi - 2\theta_1\omega$; see (2-8). Therefore, we have

$$\chi = 2\kappa + \bar{\kappa}.$$

The SO(2)-action induced by (4-1) is

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \cdot z = \frac{2iz\cos\phi - 2z\sin\phi}{2i\cos\phi + 2\sin\phi} = \mathrm{e}^{2i\phi}z,$$

and hence the equivariance property of a function $z : SM \to \mathbb{D}$ representing a section of $Z \to M$ becomes $(R_{\mathrm{e}^{i\phi}})^*z = \mathrm{e}^{-2i\phi}z$; that is, z represents a section of K^{-2}. Since we have a metric, we have an identification $K^* \simeq \overline{K}$ and hence $K^{-2} \simeq K^* \otimes \overline{K}$. In particular, we may write

$$dz = z'\omega + z''\bar{\omega} + \bar{\kappa}z - \kappa z \tag{4-7}$$

for unique complex-valued functions z' and z'' on SM. Consequently, using (4-5), (4-6) and (4-7) we obtain

$$\left(\frac{(|z|^2-1)(z+1)}{2(1+\bar{z})}\right)\zeta_\Upsilon = z'\omega + z''\bar{\omega} + \bar{\kappa}z - \kappa z - a_{-3}\bar{\omega} + z^2 a_3\omega + z(2\kappa+\bar{\kappa}) - z(2\bar{\kappa}+\kappa)$$
$$= (z'+z^2a_3)\omega + (z''-a_{-3})\bar{\omega}. \tag{4-8}$$

In order to connect the expressions for ω_Υ and ζ_Υ to the condition of z representing a conformal structure $[\hat{g}]$ that defines a holomorphic curve into Z, we use the following elementary lemma:

Lemma 4.3. *Let Z be a complex surface and $\omega, \zeta \in \Omega^1(Z, \mathbb{C})$ a basis for the (1,0)-forms of Z. Suppose $M \subset Z$ is a smooth surface on which $\omega\wedge\bar{\omega}$ is nonvanishing. Then M admits the structure of a holomorphic curve — that is, a complex one-dimensional submanifold of Z — if and only if $\omega\wedge\zeta$ vanishes identically on M.*

Proof. Since $\omega\wedge\bar{\omega}$ is nonvanishing on M, the forms ω and $\bar{\omega}$ span the complex-valued 1-forms on M. Since M is a complex submanifold of Z if and only if the pullback of a (1,0)-form on Z is a (1,0)-form on M, the claim follows. □

The reduction of P to SM identifies Z with $SM \times_{\mathrm{SO}(2)} \mathbb{D}$. Now suppose the conformal structure $[\hat{g}] : M \to Z$ is represented by the map $z : SM \to \mathbb{D}$. If $v : U \to SM$ is a local section of $\pi : SM \to M$, then $[\hat{g}]|_U : U \to Z$ is covered by the map $(\mathrm{Id}_{SM}\times z)\circ v : U \to SM \times \mathbb{D}$. Recall that the complex structure on Z has the property that its (1,0)-forms pull-back to become linear combinations of ω_Υ and ζ_Υ. Using the expressions (4-3) and (4-8) for the pullbacks of ω_Υ and ζ_Υ to SM we obtain

$$\omega_\Upsilon\wedge\zeta_\Upsilon = -\frac{2(1+\bar{z})^2}{(|z|^2-1)^2(z+1)}(z''-zz'-z^3a_3-a_{-3})\omega\wedge\bar{\omega}.$$

In particular, since $v : U \to SM$ is a π-section and ω and $\bar{\omega}$ are π-semibasic, the pullback $v^*(\omega_\Upsilon\wedge\zeta_\Upsilon)$ vanishes if and only if $\omega_\Upsilon\wedge\zeta_\Upsilon$ vanishes on $\pi^{-1}(U)$. Thus, Lemma 4.3 implies that z represents a holomorphic curve if and only if

$$z''-zz' = z^3a_3 + a_{-3}. \tag{4-9}$$

4C. ***The Beltrami differential.*** So far we have not explicitly tied the conformal structure $[\hat{g}]$ to the function $z : SM \to \mathbb{D}$ representing it. In order to do this we first recall the Beltrami differential. The choice of a metric $\hat{g}$ on M allows us to define the functions

$$p(x,v) = \hat{g}(v,v), \quad r(x,v) = \hat{g}(v,Jv) \quad \text{and} \quad q(x,v) = \hat{g}(Jv,Jv)$$

on SM. The orientation-compatible complex structure $\hat{J}$ on M induced by the conformal equivalence class of $\hat{g}$ has matrix representation

$$\hat{J} = \frac{1}{\sqrt{pq-r^2}}\begin{pmatrix} -r & -q \\ p & r \end{pmatrix}.$$

In particular, we compute that the (1,0)-forms with respect to $\hat{J}$ pull-back to SM to become complex multiples of

$$\omega_{\hat{J}} := \tfrac{1}{2}(\omega - i\hat{J}\omega) = \left(\frac{p+q+2\sqrt{pq-r^2}}{4\sqrt{pq-r^2}}\right)(\omega + \mu\bar{\omega}), \tag{4-10}$$

where

$$\mu = \frac{(p-q)+2ir}{p+q+2\sqrt{pq-r^2}}$$

is the *Beltrami coefficient* of $\hat{J}$. Clearly, μ does only depend on the conformal equivalence class $[\hat{g}]$ of $\hat{g}$. Moreover, the function μ represents a (0,1)-form on M with values in K^* called the *Beltrami differential* of $[\hat{g}]$, which — by abuse of language — we denote by μ as well.

The reduction of P to the unit tangent bundle SM of g turns ω into a basis for the (1,0)-forms with respect to the complex structure induced by g and the orientation. The mapping z represents a conformal structure $[\hat{g}]$ and consequently, induces an orientation-compatible complex structure $\hat{J}$ whose (1,0)-forms we computed in (4-3). Comparing this expression with the formula (4-10) for the Beltrami coefficient shows that we obtain the same (1,0)-forms if and only if $z = \mu$. Remember, z' and z'' represent the (1,0)- and (0,1)-part of the derivative of z with respect to the connection D induced by the Weyl connection D. Furthermore, the function a_3 represents the cubic differential A or equivalently, the form Φ, since $\Phi d\sigma = A$ and $d\sigma$ is represented by the constant function 1 on SM. Using (4-9) and the fact that $\mathfrak{p}$ contains a Weyl connection with respect to $[\hat{g}]$ if and only if $[\hat{g}] : M \to Z$ is a holomorphic curve [Mettler 2014, Theorem 3], we have thus shown:

Proposition 4.4. *Let* $(M, [g])$ *be a Riemann surface equipped with a projective structure* $\mathfrak{p}$ *given in terms of* (D, Φ). *Then* $\mathfrak{p}$ *contains a Weyl connection with respect to the conformal structure defined by the Beltrami differential* μ *if and only if*

$$\mathrm{D}''\mu - \mu\,\mathrm{D}'\mu = \Phi\mu^3 + \overline{\Phi}. \tag{4-11}$$

Remark 4.5. In the special case where $\mathfrak{p}$ is a properly convex projective structure, an equation equivalent to (4-11) was previously obtained by N. Hitchin using the Higgs bundle description of $\mathfrak{p}$.[1]

As a corollary, we obtain:

Corollary 4.6. *Let* M *be a closed oriented surface with* $\chi(M) < 0$. *Suppose the Weyl connections* D *and* $\widehat{\mathrm{D}}$ *on* TM *are projectively equivalent. Then* $\mathrm{D} = \widehat{\mathrm{D}}$ *and they preserve the same conformal structure.*

Proof. Equip M with the Riemann surface structure defined by $[g]$ and the orientation. Let $\mathfrak{p}$ be the projective structure defined by D (or $\widehat{\mathrm{D}}$). The projective structure $\mathfrak{p}$ is encoded in terms of the pair (D, 0). Moreover, the Beltrami differential μ defined by $[\hat{g}]$ solves (4-11); that is,

$$\mathrm{D}''\mu - \mu\mathrm{D}'\mu = 0.$$

[1] Private communication, August 2014.

Now observe that $\bar{\partial}_\mu = \mathrm{D}'' - \mathrm{D}'\mu$ defines a del-bar operator on $\overline{K} \otimes K^*$ and hence (4-11) can be written as $\bar{\partial}_\mu \mu = 0$. Therefore, μ is holomorphic with respect to the holomorphic line bundle structure defined by $\bar{\partial}_\mu$ on $\overline{K} \otimes K^*$. However, since $\chi(M) < 0$, the line bundle $\overline{K} \otimes K^*$ has negative degree, so that its only holomorphic section is the zero-section. It follows that $\mu = 0$ and hence $[g] = [\hat{g}]$. Since D and $\widehat{\mathrm{D}}$ are projectively equivalent and preserve the same conformal structure $[g]$, we conclude exactly as in the proof of Proposition 3.5 that $\mathrm{D} = \widehat{\mathrm{D}}$. □

Remark 4.7. The above corollary was first proved in [Mettler 2015b]. In particular, as a special case, it also shows that on a closed surface with $\chi(M) < 0$, the unparametrised geodesics of a Riemannian metric determine the metric up to rescaling by a positive constant. This was first observed in [Matveev and Topalov 2000].

5. The transport equation

While the PDE (4-11) for the Beltrami differential μ is natural from a complex geometry point of view, it turns out to be advantageous to rephrase it as a transport equation on SM. The relevant transport equation on SM can be derived using (4-11) — see the Appendix — but here we will instead take a different approach, as it leads to a more general result about thermostats having the same unparametrised geodesics; see Proposition 5.2.

Let g, $\hat{g}$ be Riemannian metrics on M. In what follows all objects defined in terms of the metric $\hat{g}$ will be decorated with a hat symbol. There is an obvious scaling map

$$\ell : SM \to \widehat{SM}, \quad (x, v) \mapsto \left(x, \frac{v}{\sqrt{\hat{g}(v, v)}}\right),$$

which is a fibre-bundle isomorphism covering the identity on M. As before we define

$$p(x, v) = \hat{g}(v, v), \quad r(x, v) = \hat{g}(v, Jv) \quad \text{and} \quad q(x, v) = \hat{g}(Jv, Jv).$$

Lemma 5.1. *The pullback of the volume form $\widehat{\Theta}$ on $\widehat{SM}$ is*

$$\ell^*\widehat{\Theta} = \left(\frac{pq - r^2}{p}\right)\Theta.$$

Proof. Since

$$d\pi(X(x, v)) = v \quad \text{and} \quad d\pi(H(x, v)) = Jv,$$

we obtain

$$\pi^*\hat{g} = p\omega_1 \otimes \omega_1 + 2r\omega_1 \circ \omega_2 + q\omega_2 \otimes \omega_2,$$

where we write $\omega_1 \circ \omega_2 := \frac{1}{2}(\omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1)$. We first compute

$$\begin{aligned}(X \lrcorner\, \ell^*\hat{\omega}_1)(x, v) &= \hat{\omega}_1(d\ell(X(x, v))) = \hat{g}\big(\ell(x, v), (d\hat{\pi} \circ d\ell)(X(x, v))\big) \\ &= \frac{1}{\sqrt{\hat{g}(v, v)}}\hat{g}(v, d\pi(X(x, v))) = \sqrt{\hat{g}(v, v)} = \sqrt{p},\end{aligned}$$

where we have used that $\hat{\pi}\circ\ell=\pi$. Likewise, we obtain

$$\begin{aligned}(H \lrcorner \ell^{*}{}_1)(x,v) &= \hat{\omega}_1(d\ell(H(x,v))) = \hat{g}\big(\ell(x,v),(d\hat{\pi}\circ d\ell)(H(x,v))\big)\\ &= \frac{1}{\sqrt{\hat{g}(v,v)}}\hat{g}(v,d\pi(H(x,v))) = \frac{\hat{g}(v,Jv)}{\sqrt{\hat{g}(v,v)}} = \frac{r}{\sqrt{p}}.\end{aligned}$$

Since $\hat{\omega}_1$ is semibasic for the projection $\hat{\pi}$, the pullback $\ell^*\hat{\omega}_1$ is semibasic for the projection π; hence $V \lrcorner \ell^*\hat{\omega}_1=0$, so that we have

$$\ell^*\hat{\omega}_1=\sqrt{p}\,\omega_1+\frac{r}{\sqrt{p}}\omega_2. \tag{5-1}$$

The pullback $\ell^*\hat{\omega}_2$ must be a multiple of ω_2. Indeed, $\ell^*\hat{\omega}_2$ is π-semibasic and we obtain

$$\begin{aligned}(X \lrcorner \ell^*\hat{\omega}_2)(x,v) &= \hat{\omega}_2(d\ell(X(x,v))) = \hat{g}\big(\hat{J}\ell(x,v),(d\hat{\pi}\circ d\ell)(X(x,v))\big)\\ &= \frac{1}{\sqrt{\hat{g}(v,v)}}\hat{g}(\hat{J}v,d\pi(X(x,v))) = \frac{\hat{g}(\hat{J}v,v)}{\sqrt{\hat{g}(v,v)}}=0.\end{aligned}$$

Recall that the area form $d\hat{\sigma}$ of $\hat{g}$ satisfies $\hat{\pi}^*d\hat{\sigma}=\hat{\omega}_1\wedge\hat{\omega}_2$; hence

$$\ell^*(\hat{\omega}_1\wedge\hat{\omega}_2)=\pi^*d\hat{\sigma}=\sqrt{pq-r^2}\,\omega_1\wedge\omega_2.$$

Thus we must have

$$\ell^*\hat{\omega}_2=\frac{\sqrt{pq-r^2}}{\sqrt{p}}\omega_2. \tag{5-2}$$

Since the Lie derivative of $\pi^*\hat{g}$ with respect to V vanishes identically, we compute that $V\sqrt{p}=r/\sqrt{p}$. Moreover, since $\sqrt{pq-r^2}$ is the π-pullback of a function on M, we obtain

$$V\left(\frac{\sqrt{pq-r^2}}{\sqrt{p}}\right)=-\frac{r\sqrt{pq-r^2}}{p^{3/2}}.$$

Pulling back the structure equation $d\hat{\omega}_2=-\hat{\psi}\wedge\hat{\omega}_1$ whilst using (5-1) and (5-2) gives

$$\begin{aligned}\ell^*(d\hat{\omega}_2)=d(\ell^*\hat{\omega}_2)=d\left(\frac{\sqrt{pq-r^2}}{\sqrt{p}}\omega_2\right)&=\left(\hat{a}\omega_1-\frac{r\sqrt{pq-r^2}}{p^{3/2}}\psi\right)\wedge\omega_2-\frac{\sqrt{pq-r^2}}{\sqrt{p}}\psi\wedge\omega_1\\ &=-\ell^*\hat{\psi}\wedge\ell^*\hat{\omega}_1=-\ell^*\hat{\psi}\wedge\left(\sqrt{p}\,\omega_1+\frac{r}{\sqrt{p}}\omega_2\right)\end{aligned}$$

for some unique real-valued function $\hat{a}$ on SM. Comparing the coefficients in the above equations, it follows that

$$\ell^*\hat{\psi}=a\omega_1+b\omega_2+\frac{\sqrt{pq-r^2}}{p}\psi \tag{5-3}$$

for some unique real-valued functions a, b on SM. In particular, we obtain

$$\ell^*\widehat{\Theta}=\ell^*(\hat{\omega}_1\wedge\hat{\omega}_2\wedge\hat{\psi})=\left(\frac{pq-r^2}{p}\right)\omega_1\wedge\omega_2\wedge\psi,$$

as claimed. □

We use this lemma to derive the following observation about general thermostats:

Proposition 5.2. *If two thermostats determined by pairs (g, λ) and $(\hat{g}, \hat{\lambda})$ have the same unparametrised geodesics, then*

$$\sqrt{p}\,(\widehat{V}\hat{\lambda} \circ \ell) = F \log\left(\frac{pq - r^2}{p^{3/2}}\right) + V\lambda.$$

As an immediate application we obtain the following classical fact:

Corollary 5.3. *Let g and $\hat{g}$ be two Riemannian metrics on M having the same unparametrised geodesics. Then $p/(pq - r^2)^{2/3}$ is an integral for the geodesic flow of g.*

Proof. This special case corresponds to $\lambda = \hat{\lambda} = 0$ and hence Proposition 5.2 implies

$$0 = X \log\left(\frac{pq - r^2}{p^{3/2}}\right) = -\frac{3}{2} X \log\left(\frac{p}{(pq - r^2)^{2/3}}\right) = -\frac{3}{2}\frac{(pq - r^2)^{2/3}}{p} X\left(\frac{p}{(pq - r^2)^{2/3}}\right). \qquad \square$$

In order to prove Proposition 5.2 we also recall a general lemma whose proof is elementary and thus omitted.

Lemma 5.4. *Let X be a vector field on a manifold M with volume form Ω. Let f and $s > 0$ be smooth functions. Then*

$$\mathrm{Div}_\Omega(fX) = Xf + f\,\mathrm{Div}_\Omega X \quad \textit{and} \quad \mathrm{Div}_{s\Omega}(X) = X \log s + \mathrm{Div}_\Omega X.$$

Proof of Proposition 5.2. This follows from Lemmas 5.1 and 5.4 and the key fact that if the thermostats have the same unparametrised geodesics then

$$\ell^* \widehat{F} = \frac{1}{\sqrt{p}}\,F. \tag{5-4}$$

To see the last equality, note that we can rephrase the hypothesis as follows. There is a smooth function $\tau : SM \times \mathbb{R} \to \mathbb{R}$ implementing the time change so that

$$\ell \circ \phi_{\tau(x,v,t)}(x, v) = \hat{\phi}_t \circ \ell(x, v).$$

Differentiating this with respect to t and setting $t = 0$ gives

$$d\ell(fF) = \widehat{F} \circ \ell,$$

where

$$f(x, v) := \frac{d}{dt}\tau(x, v, t)|_{t=0}.$$

To check that f has the desired form, apply $d\hat{\pi}$ to the last equation to get $fv = v/\sqrt{\hat{g}(v, v)}$.

Writing $s := (pq - r^2)/p$ and taking the divergence of (5-4) with respect to $\ell^*\widehat{\Theta} = s\Theta$ gives

$$\begin{aligned}\mathrm{Div}_{s\Theta}(\sqrt{p}\,\ell^*\widehat{F}) &= (\ell^*\widehat{F})\sqrt{p} + \sqrt{p}\ \mathrm{Div}_{s\Theta}(\ell^*\widehat{F})\\ &= (1/\sqrt{p})F\sqrt{p} + \sqrt{p}\ \mathrm{Div}_{\ell^*\widehat{\Theta}}(\ell^*\widehat{F})\\ &= F(\log\sqrt{p}) + \sqrt{p}(\mathrm{Div}_{\widehat{\Theta}}\,\widehat{F}) \circ \ell\\ &= \mathrm{Div}_{s\Theta} F = F \log s + \mathrm{Div}_\Theta F,\end{aligned}$$

where we have used Lemma 5.4. Since $\mathrm{Div}_\Theta F = V\lambda$ and $\mathrm{Div}_{\widehat{\Theta}} \widehat{F} = \widehat{V}\hat{\lambda}$, this last equation is equivalent to

$$\sqrt{p}(\widehat{V}\hat{\lambda}\circ\ell) = F\log\left(\frac{s}{\sqrt{p}}\right) + V\lambda,$$

which proves the claim. □

Remark 5.5. Note that the crucial identity (5-4) also follows from a different argument. Since the orbits of F and $\widehat{F}$ project onto the same unparametrised curves, there must exist a smooth function w on SM, so that $\ell^*\widehat{F} = wF$. From (5-1), (5-2) and (5-3), we compute

$$\ell^*\widehat{X} = \frac{1}{\sqrt{p}}X - \frac{a\sqrt{p}}{\sqrt{pq-r^2}}V \quad\text{and}\quad \ell^*\widehat{V} = \frac{p}{\sqrt{pq-r^2}}V,$$

from which one immediately obtains $w = 1/\sqrt{p}$.

A special case of Proposition 5.2 is the following:

Corollary 5.6. *Suppose the projective thermostat associated to the pair* $(g,\lambda) = (g, a - V\theta)$ *has the same unparametrised geodesics as the Weyl connection* D *defined by* $(\hat{g},\alpha)$. *Then*

$$u = \frac{3}{2}\log\left(\frac{p}{(pq-r^2)^{2/3}}\right)$$

satisfies the transport equation

$$Fu = Va + \beta, \tag{5-5}$$

where $\beta = \theta - \alpha$.

Proof. Applying Proposition 5.2 in the special case $\lambda = a - V\theta$ and $\hat{\lambda} = -\widehat{V}\alpha$ gives

$$-\sqrt{p}(\widehat{V}\widehat{V}\alpha\circ\ell) = \sqrt{p}(\alpha\circ\ell) = F\log\left(\frac{pq-r^2}{p^{3/2}}\right) + V(a - V\theta),$$

the left-hand side of which is simply α, thought of as a function on SM. Hence we obtain

$$-(Va+\theta-\alpha) = F\log\left(\frac{pq-r^2}{p^{3/2}}\right) = F\left(-\tfrac{3}{2}\left(\log p - \tfrac{2}{3}\log(pq-r^2)\right)\right) = -Fu,$$

as claimed. □

6. The tensor tomography result

In this final section we prove a vanishing theorem for the transport equation $Fu = Va + \beta$, provided the triple (g, A, θ) defining F satisfies certain conditions. Recall that every properly convex projective structure $\mathfrak{p}$ arises from a triple $(g, A, 0)$ satisfying

$$K_g = -1 + 2|A|_g^2 \quad\text{and}\quad \bar{\partial}A = 0.$$

In particular, we would like to conclude that if such a $\mathfrak{p}$ contains a Weyl connection, then A must vanish identically and hence $\mathfrak{p}$ is hyperbolic. It turns out that one can prove a more general vanishing theorem for a class of thermostats arising from a triple (g, A, θ) where A is a differential of degree $m \geqslant 3$ on M, that is, a section of K^m. Suppose $A \in \Gamma(K^m)$. Like in the case $m = 3$ there exists a unique smooth real-valued function a on SM lying in $\mathcal{H}_{-m} \oplus \mathcal{H}_m$, so that $\pi^* A = (V(a)/m + ia)\omega^m$. In particular, to a triple (g, A, θ) we may associate the thermostat $F = X + (a - V\theta)V$. We now have:

Theorem 6.1. *Let M be a closed oriented surface and (g, A, θ) be a triple satisfying*

$$\bar{\partial} A = \left(\tfrac{1}{2}(m-1)\right)(\theta - i \star_g \theta) \otimes A \quad \text{and} \quad K_g - \delta_g \theta + (2-m)|A|_g^2 \leqslant 0.$$

Let F denote the vector field of the thermostat determined by (g, A, θ). Suppose there is a 1-form $\beta \in \Omega^1(M)$ and a function $u \in C^\infty(SM)$ such that

$$Fu = Va + \beta.$$

Then $A = 0$ and β is exact.

Let us first verify that this gives the desired statement.

Corollary 6.2. *Let $(M, \mathfrak{p})$ be a closed oriented properly convex projective surface with $\chi(M) < 0$ and with $\mathfrak{p}$ containing a Weyl connection* D. *Then $\mathfrak{p}$ is hyperbolic and moreover* D *is the Levi-Civita connection of the hyperbolic metric.*

Proof. By a result of [Calabi 1972], if $m = 3$ and (g, A) satisfy

$$K_g = -1 + 2|A|_g^2 \quad \text{and} \quad \bar{\partial} A = 0,$$

then $K_g \leqslant 0$. In particular, the triple $(g, A, 0)$ satisfies the assumptions of Theorem 6.1, and Corollary 5.6 implies that we have a solution u to the transport equation $Fu = Va + \beta$. Hence the theorem gives right away that A vanishes identically and hence $\mathfrak{p}$ is hyperbolic. In particular, the Levi-Civita connection ${}^g\nabla$ of the hyperbolic metric and the connection D both lie in $\mathfrak{p}$ and hence are projectively equivalent, but this can happen if and only if ${}^g\nabla = \mathrm{D}$, by Corollary 4.6. □

Remark 6.3. In [Mettler 2019] the notion of a minimal Lagrangian connection is introduced. These are torsion-free connections on TM of the form $\nabla = \mathrm{D} + B$, where (g, A, θ) defining D and B are subject to the equations

$$K_g - \delta_g \theta = -1 + 2|A|_g^2, \quad \bar{\partial} A = (\theta - i \star_g \theta) \otimes A, \quad d\theta = 0.$$

In particular, on a closed oriented surface of negative Euler characteristic every properly convex projective structure arises from a minimal Lagrangian connection. Another immediate consequence of Theorem 6.1 and Corollary 4.6 thus is:

Corollary 6.4. *Let M be a closed oriented surface of negative Euler characteristic and ∇ a minimal Lagrangian connection arising from the triple (g, A, θ). Suppose $|A|_g^2 \leqslant 1$ and that ∇ is projectively equivalent to a Weyl connection* D. *Then A vanishes identically and hence $\nabla = \mathrm{D}$.*

In order to show the theorem we use the following L^2 identity proved in [Jane and Paternain 2009, equation (5)], which is in turn an extension of an identity in [Sharafutdinov and Uhlmann 2000] for geodesic flows. The identity holds for arbitrary thermostats $F = X + \lambda V$. If we let $H_c := H + cV$, where $c : SM \to \mathbb{R}$ is any smooth function, then

$$2\langle H_c u, VFu\rangle = \|Fu\|^2 + \|H_c u\|^2 - \langle Fc + c^2 + K_g - H_c\lambda + \lambda^2, (Vu)^2\rangle, \tag{6-1}$$

where u is any smooth function. All norms and inner products are L^2 with respect to the volume form Θ.

We also need the following lemma whose proof is a straightforward calculation (see [Mettler and Paternain 2019, Lemma 4.1] for a proof).

Lemma 6.5. *We have*

$$\bar\partial A = \left(\tfrac{1}{2}(m-1)\right)(\theta - i \star_g \theta) \otimes A$$

if and only if

$$XVa - mHa - (m-1)(\theta Va - maV\theta) = 0.$$

Proof of Theorem 6.1. Without loss of generality we may assume that β has zero divergence. Indeed if not, a standard application of scalar elliptic PDE theory shows that we can always find a smooth function h on M such that $\beta + dh$ has zero divergence. Now note that $F(u+h) = Va + \beta + dh$.

A calculation shows that if we pick $c = \theta + V(a)/m$, then

$$Fc + c^2 + K_g - H_c\lambda + \lambda^2 = K_g - \delta_g\theta + (1-m)|A|_g^2,$$

where we use that

$$\pi^*|A|_g^2 = (Va)^2/m^2 + a^2 \quad \text{and} \quad \pi^*\delta_g\theta = -(X\theta + HV\theta);$$

hence for this choice of c, (6-1) simplifies to

$$2\langle H_c u, VFu\rangle - \||A|_g Vu\|^2 = \|Fu\|^2 + \|H_c u\|^2 - \langle K_g - \delta_g\theta + (2-m)|A|_g^2, (Vu)^2\rangle. \tag{6-2}$$

If $Fu = Va + \beta$, then $VFu = -m^2 a + V\beta$. Using that X and H preserve Θ and that $XVa - mHa - (m-1)(\theta Va - maV\theta) = 0$ we compute

$$\begin{aligned}2\langle H_c u, -m^2 a\rangle &= -2m^2\langle Hu, a\rangle - 2m^2\langle cVu, a\rangle\\ &= 2m^2\langle u, Ha\rangle - 2m^2\langle cVu, a\rangle\\ &= -2m^2\langle Xu, V(a)/m\rangle - 2m(m-1)\langle u, \theta Va - maV\theta\rangle - 2m^2\langle cVu, a\rangle\\ &= -2m\|Va\|^2 = -2m^3\|a\|^2,\end{aligned}$$

where the last equation is obtained using that $Xu = \beta + Va - (a - V\theta)Vu$, $\langle \beta, Va\rangle = 0$ and $c = \theta + V(a)/m$.

Using that X and H preserve Θ and that $X\beta + HV\beta = 0$ (β is assumed to have zero divergence) we compute

$$\begin{aligned}2\langle H_c u, V\beta\rangle &= 2\langle Hu, V\beta\rangle + 2\langle cVu, V\beta\rangle\\ &= -2\langle u, HV\beta\rangle + 2\langle cVu, V\beta\rangle\\ &= -2\langle Xu, \beta\rangle + 2\langle cVu, V\beta\rangle\\ &= -2\|\beta\|^2 + 2\langle (a - V\theta)Vu, \beta\rangle + 2\langle cVu, V\beta\rangle\\ &= -2\|\beta\|^2 + 2\langle aVu, \beta\rangle + 2\langle (VaVu)/m, V\beta\rangle,\end{aligned}$$

where the penultimate equation is obtained using that $Xu = \beta + Va - (a - V\theta)Vu$ and $\langle \beta, Va\rangle = 0$. The last equation uses that $c = \theta + V(a)/m$ and

$$V(\theta V\beta - V\theta\beta) = 0.$$

Inserting these calculations back into (6-2), we derive

$$-2m^3\|a\|^2 - 2\|\beta\|^2 + 2\langle aVu, \beta\rangle + 2\langle (VaVu)/m, V\beta\rangle - \||A|_g Vu\|^2 \\ = \|Fu\|^2 + \|H_c u\|^2 - \langle K_g - \delta_g\theta + (2-m)|A|_g^2, (Vu)^2\rangle.$$

Since $|A|_g^2 = a^2 + (Va)^2/m^2$ this can be rewritten as

$$-2m^3\|a\|^2 - \|\beta - aVu\|^2 - \|V\beta - VaVu/m\|^2 = \|Fu\|^2 + \|H_c u\|^2 - \langle K_g - \delta_g\theta + (2-m)|A|_g^2, (Vu)^2\rangle,$$

where we have used that $\|\beta\|^2 = \|V\beta\|^2$. By hypothesis the right-hand side is $\geqslant 0$, which gives right away that $a = \beta = 0$. □

Appendix: Deriving the transport equation

Here we sketch how to derive the transport equation for the function u starting from the PDE

$$\mathrm{D}''\mu - \mu\,\mathrm{D}'\mu = \Phi\mu^3 + \overline{\Phi}$$

for the Beltrami differential μ. Let (g, A, θ) be the triple encoding $\mathfrak{p}$ so that the connection form of D on SM is (see (2-8)) $\kappa = i\psi - 2\theta_1\omega$, where we write $\theta_1 = \frac{1}{2}(\theta - iV\theta)$. Moreover, on SM the section Φ of $K^2 \otimes \overline{K^*}$ is represented by $a_3 = \frac{1}{3}Va + ia$, where $a(v) = \mathrm{Re}\, A(Jv, Jv, Jv)$, $v \in SM$. Writing μ_{-2} for the complex-valued function on SM representing the Beltrami differential μ and $\mu_2 = \bar{\mu}_{-2}$, the PDE for μ is equivalent to

$$d\mu_{-2} = \mu'_{-2}\omega + (\mu_{-2}\mu'_{-2} + a_3\mu_{-2}^3 + \bar{a}_3)\bar{\omega} + \bar{\kappa}\mu_{-2} - \kappa\mu_{-2},$$

where μ'_{-2} is a complex-valued function on SM. Since μ_{-2} represents a section of $\overline{K} \otimes K^* \simeq K^{-2}$, writing $\eta_\pm = \frac{1}{2}(X \mp iH)$ we also have

$$d\mu_{-2} = \eta_+(\mu_{-2})\omega + \eta_-(\mu_{-2})\bar{\omega} - 2i\mu_{-2}\psi.$$

Thus the PDE is equivalent to the system

$$\eta_-\mu_{-2} - \mu_{-2}\eta_+\mu_{-2} = a_3\mu_{-2}^3 - 2\mu_{-2}^2\theta_1 - 2\mu_{-2}\bar{\theta}_1 + \bar{a}_3 \tag{A-1}$$

and $V\mu_{-2} = -2i\mu_{-2}$. The Beltrami differential only defines a conformal equivalence class $[\hat{g}]$. We may fix a metric $\hat{g} \in [\hat{g}]$ by requiring

$$\tfrac{1}{2}(p+q) = \frac{1 + |\mu_2|^2}{(1 - |\mu_2|^2)^4},$$

where again we specify the metric $\hat{g}$ in terms of the functions p, q, r. Explicitly, we have

$$\tfrac{1}{2}(p-q) = \frac{\mu_{-2} + \mu_2}{(1 - |\mu_2|^2)^4} \quad \text{and} \quad r = \frac{i(\mu_2 - \mu_{-2})}{(1 - |\mu_2|^2)^4}.$$

In particular, this yields

$$h := \frac{p}{(pq - r^2)^{2/3}} = (\mu_{-2} + 1)(\mu_2 + 1).$$

Writing $F = X + (a - V\theta)V$ and using (A-1), a lengthy but straightforward calculation shows that

$$Fh = \tfrac{2}{3} hVa + 2h \operatorname{Re}(a_3 \mu_{-2}^2 - \mu_2 a_{-3} - 2\mu_2 \theta_{-1} + \eta_+ \mu_{-2}).$$

Hence if we define $u = \frac{3}{2} \log h$, then we obtain

$$Fu - Va = 3 \operatorname{Re}(a_3 \mu_{-2}^2 - \mu_2 a_{-3} - 2\mu_2 \theta_{-1} + \eta_+ \mu_{-2}).$$

Note that the right-hand side of the last equation lies in $\mathcal{H}_{-1} \oplus \mathcal{H}_1$; hence there exists a 1-form β on M so that

$$Fu = Va + \beta,$$

which is the transport equation (5-5).

Acknowledgements

The authors are grateful to Nigel Hitchin for helpful conversations. A part of the research for this article was carried out while Mettler was visiting FIM at ETH Zürich. Mettler thanks FIM for its hospitality. Mettler was partially funded by the priority programme SPP 2026 "Geometry at Infinity" of DFG. Paternain was partially supported by EPSRC grant EP/R001898/1.

References

[Besse 1987] A. L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik (3) **10**, Springer, 1987. MR Zbl

[Bryant et al. 2009] R. Bryant, M. Dunajski, and M. Eastwood, "Metrisability of two-dimensional projective structures", *J. Differential Geom.* **83**:3 (2009), 465–499. MR Zbl

[Calabi 1972] E. Calabi, "Complete affine hyperspheres, I", pp. 19–38 in *Symposia Mathematica, X* (Rome, 1971), Academic Press, London, 1972. MR Zbl

[Croke and Sharafutdinov 1998] C. B. Croke and V. A. Sharafutdinov, "Spectral rigidity of a compact negatively curved manifold", *Topology* **37**:6 (1998), 1265–1273. MR Zbl

[Dubois-Violette 1983] M. Dubois-Violette, "Structures complexes au-dessus des variétés, applications", pp. 1–42 in *Mathématique et physique* (Paris, 1979/1982), edited by L. Boutet de Monvel et al., Progr. Math. **37**, Birkhäuser, Boston, 1983. MR Zbl

[Dumas and Wolf 2015] D. Dumas and M. Wolf, "Polynomial cubic differentials and convex polygons in the projective plane", *Geom. Funct. Anal.* **25**:6 (2015), 1734–1798. MR Zbl

[Guillemin and Kazhdan 1980] V. Guillemin and D. Kazhdan, "Some inverse spectral results for negatively curved 2-manifolds", *Topology* **19**:3 (1980), 301–312. MR Zbl

[Hitchin 1982] N. J. Hitchin, "Complex manifolds and Einstein's equations", pp. 73–99 in *Twistor geometry and nonlinear systems* (Primorsko, Bulgaria, 1980), edited by H.-D. Doebner and T. D. Palev, Lecture Notes in Math. **970**, Springer, 1982. MR Zbl

[Hitchin 1992] N. J. Hitchin, "Lie groups and Teichmüller space", *Topology* **31**:3 (1992), 449–473. MR Zbl

[Jane and Paternain 2009] D. Jane and G. P. Paternain, "On the injectivity of the X-ray transform for Anosov thermostats", *Discrete Contin. Dyn. Syst.* **24**:2 (2009), 471–487. MR Zbl

[Labourie 2007] F. Labourie, “Flat projective structures on surfaces and cubic holomorphic differentials”, *Pure Appl. Math. Q.* **3**:4 (2007), 1057–1099. MR Zbl

[Loftin 2001] J. C. Loftin, “Affine spheres and convex $\mathbb{RP}^n$-manifolds”, *Amer. J. Math.* **123**:2 (2001), 255–274. MR Zbl

[Matveev and Topalov 2000] V. S. Matveev and P. J. Topalov, “Metric with ergodic geodesic flow is completely determined by unparameterized geodesics”, *Electron. Res. Announc. Amer. Math. Soc.* **6** (2000), 98–104. MR Zbl

[Mettler 2014] T. Mettler, “Weyl metrisability of two-dimensional projective structures”, *Math. Proc. Cambridge Philos. Soc.* **156**:1 (2014), 99–113. MR Zbl

[Mettler 2015a] T. Mettler, “Extremal conformal structures on projective surfaces”, preprint, 2015. To appear in *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* arXiv

[Mettler 2015b] T. Mettler, “Geodesic rigidity of conformal connections on surfaces”, *Math. Z.* **281**:1-2 (2015), 379–393. MR Zbl

[Mettler 2019] T. Mettler, “Minimal Lagrangian connections on compact surfaces”, *Adv. Math.* **354** (2019), art. id. 106747. MR Zbl

[Mettler and Paternain 2019] T. Mettler and G. P. Paternain, “Holomorphic differentials, thermostats and Anosov flows”, *Math. Ann.* **373**:1-2 (2019), 553–580. MR Zbl

[O’Brian and Rawnsley 1985] N. R. O’Brian and J. H. Rawnsley, “Twistor spaces”, *Ann. Global Anal. Geom.* **3**:1 (1985), 29–58. MR Zbl

[Paternain et al. 2013] G. P. Paternain, M. Salo, and G. Uhlmann, “Tensor tomography on surfaces”, *Invent. Math.* **193**:1 (2013), 229–247. MR Zbl

[Paternain et al. 2014] G. P. Paternain, M. Salo, and G. Uhlmann, “Spectral rigidity and invariant distributions on Anosov surfaces”, *J. Differential Geom.* **98**:1 (2014), 147–181. MR Zbl

[Sharafutdinov and Uhlmann 2000] V. Sharafutdinov and G. Uhlmann, “On deformation boundary rigidity and spectral rigidity of Riemannian surfaces with no focal points”, *J. Differential Geom.* **56**:1 (2000), 93–110. MR Zbl

[Spivak 1999] M. Spivak, *A comprehensive introduction to differential geometry, II*, 3rd ed., Publish or Perish, Houston, 1999. Zbl

[Wang 1991] C. P. Wang, “Some examples of complete hyperbolic affine 2-spheres in $\mathbb{R}^3$”, pp. 271–280 in *Global differential geometry and global analysis* (Berlin, 1990), edited by D. Ferus et al., Lecture Notes in Math. **1481**, Springer, 1991. MR Zbl

Received 14 Jun 2018. Revised 20 Apr 2019. Accepted 1 Jun 2019.

THOMAS METTLER: mettler@math.uni-frankfurt.de
Institut für Mathematik, Goethe-Universität Frankfurt, Frankfurt am Main, Germany

GABRIEL P. PATERNAIN: g.p.paternain@dpmms.cam.ac.uk
Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Cambridge, United Kingdom

ANALYSIS AND PDE
Vol. 13, No. 4, 2020
dx.doi.org/10.2140/apde.2020.13.1099

STABILITY OF SMALL SOLITARY WAVES FOR THE ONE-DIMENSIONAL NLS WITH AN ATTRACTIVE DELTA POTENTIAL

SATOSHI MASAKI, JASON MURPHY AND JUN-ICHI SEGATA

We consider the initial-value problem for the one-dimensional nonlinear Schrödinger equation in the presence of an attractive delta potential. We show that for sufficiently small initial data, the corresponding global solution decomposes into a small solitary wave plus a radiation term that decays and scatters as $t \to \infty$. In particular, we establish the asymptotic stability of the family of small solitary waves.

1. Introduction

We study the one-dimensional nonlinear Schrödinger equation (NLS) with an attractive delta potential. This equation takes the form

$$\begin{cases} i\,\partial_t u = Hu + \mu|u|^p u, \\ u(0) = u_0. \end{cases} \tag{1-1}$$

Here we take $u : \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{C}$, $\mu \in \mathbb{R}\backslash\{0\}$, and H is the Schrödinger operator

$$H = -\tfrac{1}{2}\partial_x^2 + q\delta(x),$$

where $q < 0$ (the attractive case) and δ is the Dirac delta distribution. Equation (1-1) provides a simple model describing the resonant nonlinear propagation of light through optical wave guides with localized defects [Goodman et al. 2004]. For reasons to be detailed below, we consider the L^2-supercritical case, namely, $p \geq 4$. For technical simplicity we also assume p is an even integer.

In the repulsive case ($q > 0$), (1-1) is studied from the point of view of scattering. Banica and Visciglia [2016] proved global well-posedness and scattering in the energy space for the defocusing mass-supercritical case. Ikeda and Inui [2017] considered the focusing mass-supercritical regime and proved scattering below the ground-state threshold. In our previous work [Masaki et al. 2019], we considered (1-1) with a cubic nonlinearity and proved decay and (modified) scattering for small initial data in a weighted space; see also [Segata 2015].

Such results are not expected in the attractive case. Indeed, in the attractive case the operator H has a single eigenvalue $-\frac{1}{2}q^2$, with a one-dimensional eigenspace spanned by the L^2-normalized eigenfunction

$$\phi_0(x) := |q|^{1/2}e^{q|x|}.$$

MSC2010: 35Q55.
Keywords: NLS, solitary waves, asymptotic stability, delta potential.

One can then prove that there exists a family of small nonlinear bound states Q, parametrized by small $z \in \mathbb{C}$, which satisfy

$$HQ + \mu |Q|^p Q = EQ, \tag{1-2}$$

with $Q = Q[z] = z\phi_0 + \mathcal{O}(z^2)$ and $E = E[|z|] = -\frac{1}{2}q^2 + \mathcal{O}(z)$. The functions $u(t) = e^{-iEt}Q$ are then small solitary wave solutions to (1-1). In particular, one does not expect small solutions simply to decay and scatter in general. Instead, we will show that for small initial data, the corresponding solution decouples into a small solitary wave plus radiation. The existence and properties of $Q[z]$ are discussed in Section 2C. In fact, in the special case of the delta potential, one can find explicit formulas for the nonlinear ground states.

Our main result is the following theorem. We write P_c for the projection onto the continuous spectral subspace of H. The notation D_j denotes the derivative with respect to z_j, where we identify $z \in \mathbb{C}$ with the real vector (z_1, z_2). Finally, $\langle \cdot , \cdot \rangle$ denotes the standard L^2 inner product.

Theorem 1.1. *Let $\|u_0\|_{H^1} = \delta$, $q < 0$, and let $p \geq 4$ be an even integer. For δ sufficiently small, there exists a unique global solution u to* (1-1) *and $z(t) \in \mathbb{C}$ such that writing*

$$u(t) = Q[z(t)] + v(t), \tag{1-3}$$

where $Q[z(t)]$ is the solution to (1-2), *we have the following*:

- *v satisfies the orthogonality conditions*

$$\operatorname{Im}\langle v(t), D_j Q[z(t)]\rangle \equiv 0 \quad \textit{for } j \in \{1, 2\}. \tag{1-4}$$

- *v obeys the global space-time bounds*

$$\|v\|_{L_t^\infty H_x^1 \cap L_t^4 L_x^\infty} + \|\langle x\rangle^{-3/2} v\|_{L_x^\infty L_t^2} + \|\partial_x v\|_{L_x^\infty L_t^2} \lesssim \delta,$$

and there exists unique $v_+ \in P_c H^1$ such that

$$\lim_{t\to\infty} \|v(t) - e^{-itH} v_+\|_{H^1} = 0.$$

- *$\|z\|_{L_t^\infty} \lesssim \delta$ and there exists $z_+ \in \mathbb{C}$ satisfying $\big||z_+| - |z(0)|\big| \lesssim \delta^2$ and*

$$\lim_{t\to\infty} z(t) \exp\left\{ i \int_0^t E[z(s)]\, ds \right\} = z_+. \tag{1-5}$$

Theorem 1.1 shows that any small solution decomposes into a nonlinear bound state plus a radiation term. In particular, we have the asymptotic stability of the family of small solitary waves. The condition (1-4) makes $v(t)$ orthogonal to the nondecaying solutions of the linearization of (1-1) about the solitary wave at $z(t)$; this is an essential ingredient for establishing decay and scattering for v (see Section 3 for further discussion).

Theorem 1.1 fits in the context of the stability of small solitary waves for nonlinear Schrödinger equations with potential, for which there are many results available. An even more extensive literature exists concerning other notions of stability, stability of large solitary waves, and so on. We refer the interested reader to [Buslaev and Perelman 1992; Gustafson et al. 2004; Mizumachi 2008; Kirr and

Mızrak 2009; Kirr and Zarnescu 2009; Soffer and Weinstein 1990; 1992; 2004; Weder 2000; Tsai and Yau 2002a; 2002b; 2002c; 2002d; Cuccagna 2011; 2014; Cuccagna and Pelinovsky 2014] for a sample of the many relevant results that are available. See in particular [Datchev and Holmer 2009; Deift and Park 2011; Holmer et al. 2007a; 2007b; Holmer and Zworski 2007; 2009; Kaminaga and Ohta 2009; Fukuizumi et al. 2008] for related results in the setting of NLS with a delta potential. We will keep our focus on the discussion of small solitary waves.

Our result is closely related to those appearing in [Gustafson et al. 2004; Mizumachi 2008], both of which prove asymptotic stability of small solitary waves for NLS with a potential that supports a single negative eigenvalue, with data in H^1 and mass-supercritical nonlinearities. In [Gustafson et al. 2004], the authors relied crucially on the endpoint Strichartz estimate in three dimensions. Mizumachi [2008] addressed the one-dimensional case, in which case the usual endpoint Strichartz estimate is unavailable. His approach was to establish suitable linear estimates in "reversed" Strichartz spaces, in which case the L_t^2 endpoint comes back into play.

Theorem 1.1 is an analogue of the main result appearing in [Mizumachi 2008], which treats a class of potentials that does not include the attractive delta potential. The key to extending this type of result to the delta potential is to observe that by relying on exact identities related to the Schrödinger operator with a delta potential, one can recover the full range of linear estimates that played such an essential role in [Mizumachi 2008]. We carry this out in Section 2B. Once the requisite linear estimates are in place, one could then follow many of the remaining arguments in [Mizumachi 2008] rather directly, although this is not the route that we take. Instead, we set up the problem and prove the main result in a way that is inspired by the presentation in [Gustafson et al. 2004], which we found to be rather conceptually clear.

Our result is also closely tied to the work of Fukuizumi, Ohta, and Ozawa [Fukuizumi et al. 2008] and Kaminaga and Ohta [2009]; see also [Goodman et al. 2004]. These authors considered the problem of stability and instability of nonlinear bound states for NLS with an attractive delta potential, relying in particular on explicit formulas that they derived for the nonlinear bound states (see Section 2C below). For focusing nonlinearities, one finds that in the mass-subcritical and mass-critical cases, nonlinear bound states are all orbitally stable. In the mass-supercritical case, that there exists $E_1 < -\frac{1}{2}q^2$ such that ground states corresponding to $E \in \left(E_1, -\frac{1}{2}q^2\right)$ are orbitally stable, while those corresponding to $E \in (-\infty, E_1)$ are unstable. For defocusing nonlinearities, all nonlinear ground states are orbitally stable. Thus our main result, Theorem 1.1, extends the results of [Fukuizumi et al. 2008; Kaminaga and Ohta 2009] in the mass-supercritical case to asymptotic stability for E in a neighborhood of $-\frac{1}{2}q^2$.

Finally, we would also like to mention the result of [Deift and Park 2011], which establishes the asymptotic stability of solitons for the focusing cubic NLS with a delta potential and even initial data by making use of complete integrability and the method of nonlinear steepest descent. This result in particular extended the results appearing [Datchev and Holmer 2009; Holmer et al. 2007a; 2007b; Holmer and Zworski 2007].

As mentioned above, our previous work on the one-dimensional NLS with a repulsive delta potential [Masaki et al. 2019] considered the case of a cubic nonlinearity. It is an interesting question whether one also has asymptotic stability in the setting of an attractive potential and L^2-subcritical nonlinearities; recall that orbital stability was proven by [Fukuizumi et al. 2008; Kaminaga and Ohta 2009; Goodman et al. 2004]. Proving asymptotic stability would most likely require the introduction of stronger integrability conditions on the initial data; for example, this is the case in [Kirr and Mızrak 2009; Kirr and Zarnescu 2009], which proved stability of small solitary waves for NLS with potential for some mass-subcritical nonlinearities in dimensions $d \in \{2, 3\}$. In our case, we start only with H^1 data and are therefore restricted to $p \geq 4$; this is completely analogous to the situation of trying to prove small-data scattering for the standard power-type NLS. To see specific the technical points that lead to this restriction, see the estimates of the $|v|^p v$ term in the proofs of Lemmas 4.6, 4.7, and 4.8 (as well as the $\mathcal{O}(v^p Q)$-term in Lemma 4.7).

Briefly, the proof of Theorem 1.1 goes as follows. One shows that as long as the u remains small in H^1, there exists a unique decomposition (1-3) such that (1-4) holds. Using (1-1) and differentiating (1-4) leads to a coupled system of equations for $v(t)$ and $z(t)$. Relying largely on estimates for the linear propagator e^{-itH} and estimates on the bound states $Q[z]$ for small z, one can use these equations to close a bootstrap argument, proving that the smallness of u in H^1 (as well as the smallness of v and z in various norms) persists. Thus, one can extend the decomposition for all times; furthermore, the bounds proved on v and z suffice to establish the asymptotics claimed in Theorem 1.1. The particular choice of the orthogonality condition (1-4) guarantees that the ODE involving $z[t]$ is at least quadratic in v, which is essential for proving the necessary bootstrap estimates; see Remark 3.2 for further discussion of this point.

Outline of the paper. In Section 2 we introduce notation and gather some preliminary results. We introduce the linear operator H in Section 2A. In Section 2B, we prove a range of Strichartz and local smoothing estimates for $e^{-itH} P_c$. These match the form of the estimates of [Mizumachi 2008], who considered a class of potentials that did not include the delta potential. We are able to give rather direct proofs using the explicit formula for the resolvent. We also prove a technical result related to the comparison of the $\dot{H}^1$ inner product to the bilinear form given by HP_c. In Section 2C we discuss the existence and properties of small nonlinear bound states, and in Section 2D we record a local well-posedness result for (1-1). In Section 3 we set up the problem, describing in detail how to find the decomposition (1-3) satisfying (1-4). Finally, in Section 4 we carry out the main bootstrap argument and complete the proof of Theorem 1.1.

2. Preliminaries

We begin by recording some notation. We write

$$\langle f, g\rangle = \int \bar{f} g \, dx$$

for the usual L^2 inner product. Throughout the paper we will write $F(u) = \mu |u|^p u$ for the nonlinearity. We write $\mathcal{F} f$ or $\hat{f}$ for the Fourier transform. We write $A \lesssim B$ to denote $A \leq CB$ for some $A, B, C > 0$.

Constants below may depend on the parameter q (the strength of the potential), but we will not make explicit reference to this dependence. We would like to point out that some of the implicit constants in the estimates for $e^{-itH} P_c$ below would blow up as $|q| \to 0$ (for example, when the proof relies on the fact that $|q - i\mu| \gtrsim |q|$ for $\mu \in \mathbb{R}$). In particular, the small parameter δ appearing in the statement of the main result (Theorem 1.1) depends on q and would degenerate to zero as $|q| \to 0$.

2A. ***Linear theory.*** The linear Schrödinger equation with a delta potential is a classical model in quantum mechanics that is covered extensively in [Albeverio et al. 1988]. We consider in this paper the case of an attractive delta potential of the form

$$H = -\tfrac{1}{2}\partial_x^2 + q\delta(x), \quad q < 0.$$

More precisely, the operator H is defined by $-\frac{1}{2}\partial_x^2$ on its domain

$$D(H) = \{u \in H^1(\mathbb{R}) \cap H^2(\mathbb{R}\backslash\{0\}) : \partial_x u(0+) - \partial_x u(0-) = 2qu(0)\}$$

and extends to a self-adjoint operator on L^2 with purely absolutely continuous essential spectrum equal to $[0, \infty)$. If $q > 0$ (the repulsive case), then H has no eigenvalues. In $q < 0$ (the attractive case), H has a single negative eigenvalue $-\frac{1}{2}q^2$ with a one-dimensional eigenspace spanned by the L^2-normalized eigenfunction

$$\phi_0(x) := |q|^{1/2} e^{q|x|}.$$

In this paper we restrict attention to the attractive case.

2B. ***Local smoothing and Strichartz estimates.*** In this section we prove several local smoothing and Strichartz estimates for e^{-itH}. We write P_c to denote the projection onto the absolutely continuous spectrum.

The starting point for the estimates we will prove is the following spectral resolution of the free propagator:

$$e^{-itH} P_c = \int_0^\infty e^{-it\lambda} E(\lambda)\, d\lambda, \quad E(\lambda) := \frac{1}{2\pi i}[R(\lambda + i0) - R(\lambda - i0)]. \tag{2-1}$$

Here $R(z) = (H - z)^{-1}$ is the resolvent, and $R(\lambda \pm i0)$ denotes the analytic continuation onto the real line from the upper/lower half plane. In fact, working with the resolvent is not the only way to carry out the requisite linear analysis for this problem. As an alternative, one could also proceed through the development of the distorted Fourier transform; e.g., this is how we proceeded in our previous work involving the delta potential [Masaki et al. 2019].

For the case of the delta potential, we have explicit formulas for the integral kernel of the resolvent, namely

$$R(\lambda + i0; x, y) = \frac{i}{2\sqrt{\lambda}}\left[e^{i|x-y|\sqrt{\lambda}} - \frac{q}{q - i\sqrt{\lambda}} e^{i(|x|+|y|)\sqrt{\lambda}}\right],$$
$$R(\lambda - i0; x, y) = \frac{i}{2\sqrt{\lambda}}\left[e^{-i|x-y|\sqrt{\lambda}} - \frac{q}{q + i\sqrt{\lambda}} e^{-i(|x|+|y|)\sqrt{\lambda}}\right]$$

for $\lambda > 0$. We similarly write $E(\lambda; x, y)$ for the kernel of $E(\lambda)$. These identities can be found, for example, in [Albeverio et al. 1988, Chapter I.3], but they are also readily derived by hand. In particular, one can recognize the first term as the free resolvent, while the second term (representing the contribution of the potential) simply fixes the boundary condition.

Typically we will focus on estimating $R(\lambda+i0)$, as the other term is similar. We write the kernel in two pieces, namely

$$R(\lambda+i0;x,y)=R_1(\lambda;x,y)+R_2(\lambda;x,y),$$

where

$$R_1(\lambda;x,y)=\frac{i}{2\sqrt{\lambda}}[e^{i|x-y|\sqrt{\lambda}}-e^{i(|x|+|y|)\sqrt{\lambda}}], \tag{2-2}$$

$$R_2(\lambda;x,y)=\frac{1}{2(q-i\sqrt{\lambda})}e^{i(|x|+|y|)\sqrt{\lambda}}. \tag{2-3}$$

We note that

$$R_1(\lambda;x,y)=\frac{i}{2\sqrt{\lambda}}\begin{cases}[e^{-ix\sqrt{\lambda}}-e^{ix\sqrt{\lambda}}]e^{iy\sqrt{\lambda}}, & y\geq x\geq 0,\\ 0, & y\geq 0\geq x,\\ e^{-ix\sqrt{\lambda}}[e^{iy\sqrt{\lambda}}-e^{-iy\sqrt{\lambda}}], & 0\geq y\geq x.\end{cases} \tag{2-4}$$

There are analogous formulas in the cases $x\geq y\geq 0$, $x\geq 0\geq y$, and $0\geq x\geq y$. We will focus on treating the three cases appearing in (2-4).

To simplify the presentation below, we will use $\widetilde{\mathcal{F}}f$ to denote quantities that are similar (but not identical) to the Fourier transform of f; in particular, we use this notation for quantities that obey the bounds

$$\|\widetilde{\mathcal{F}}f\|_{L^2}\lesssim\|f\|_{L^2} \quad\text{and}\quad \|\widetilde{\mathcal{F}}f\|_{L^\infty}\lesssim\|f\|_{L^1}. \tag{2-5}$$

As a typical example, we could apply this notation to a term like

$$\int_x^\infty e^{i\xi y}f(y)\,dy=(2\pi)^{1/2}\mathcal{F}[\chi_{(x,\infty)}f](-\xi).$$

Indeed, for this example the two bounds in (2-5) can be easily checked (and are uniform in x).

We begin with the standard one-dimensional Strichartz estimates.

Proposition 2.1 (Strichartz estimates). *The following estimates hold on any space-time slab $I\times\mathbb{R}$ with $0\in I$:*

$$\|e^{-itH}P_cf\|_{(L_t^4L_x^\infty\cap L_t^\infty L_x^2)(I\times\mathbb{R})}\lesssim\|f\|_{L^2},$$

$$\left\|\int_0^t e^{-i(t-s)H}P_cF(s)\,ds\right\|_{(L_t^4L_x^\infty\cap L_t^\infty L_x^2)(I\times\mathbb{R})}\lesssim\|F\|_{L_t^\alpha L_x^\beta(I\times\mathbb{R})}$$

for any $(\alpha,\beta)\in\left[1,\frac{4}{3}\right]\times[1,2]$ satisfying $\frac{2}{\alpha}+\frac{1}{\beta}=\frac{5}{2}$.

As is well known, the proof boils down to the following dispersive estimates.

Lemma 2.2 (dispersive estimates). *The following estimates hold*:

$$\|e^{-itH}P_cf\|_{L^2}\lesssim\|f\|_{L^2} \quad\textit{and}\quad \|e^{-itH}P_cf\|_{L^\infty}\lesssim|t|^{-1/2}\|f\|_{L^1}.$$

Proof of Lemma 2.2. It is clear that $e^{-itH}P_c$ maps L^2 to L^2 boundedly. For the $L^1 \to L^\infty$ estimate, we start from (2-1). The desired estimate is well known for the case of the free Schrödinger equation, and hence we consider only the contribution of the potential. After a change of variables, we are left to prove

$$\sup_x \left| \int f(y) \int e^{-it\lambda^2/2 - i(|x|+|y|)\lambda} \frac{q}{q - i\lambda}\, d\lambda\, dy \right| \le |t|^{-1/2} \|f\|_{L^1}.$$

We apply Plancherel in the $d\lambda$ integral and observe (by explicit computation) that

$$\sup_{\theta \in \mathbb{R}} \|\mathcal{F}(e^{-it\lambda^2/2 + i\theta\lambda})\|_{L^\infty} \lesssim |t|^{-1/2}.$$

Therefore the proof boils down to showing that $\mathcal{F}((q - i\lambda)^{-1}) \in L^1$. In fact, by Cauchy–Schwarz and Plancherel,

$$\|\mathcal{F}((q - i\lambda)^{-1})\|_{L^1} \lesssim \|(1 - \partial_\lambda^2)(q - i\lambda)^{-1}\|_{L^2} \lesssim 1. \tag{2-6}$$

The result follows. □

We turn to the following weighted estimates for the linear propagator.

Proposition 2.3 (local smoothing estimates). *The following estimates hold*:

$$\|\langle x\rangle^{-3/2} e^{-itH} P_c f\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L^2}, \tag{2-7}$$

$$\|\partial_x e^{-itH} P_c f\|_{L_x^\infty L_t^2} \lesssim \|f\|_{H^1}. \tag{2-8}$$

Remark 2.4. The H^1-norm in (2-8) may be replaced by the $H^{1/2}$-norm by utilizing the analogous estimate for the free propagator, see, e.g., [Kenig et al. 1991, Lemma 2.1], and estimating the contribution of the potential separately. We proceed with (2-8), as it admits a simple self-contained proof and is sufficient for our applications.

Proof of Proposition 2.3. We begin by reducing each estimate to one given purely in terms of the resolvent. Let

$$m = m(x, \partial_x) \in \{\langle x\rangle^{-3/2}, \partial_x\}$$

and $X = L^2$or H^1. We will show

$$\|m e^{-itH} P_c\|_{X \to L_x^\infty L_t^2} \lesssim \|m E(\lambda)\|_{X \to L_x^\infty L_\lambda^2}. \tag{2-9}$$

To see this, we let $G \in L_x^1 L_t^2$ and use Plancherel to estimate

$$\begin{aligned} |\langle m e^{-itH} P_c f, G\rangle_{L_{t,x}^2}| &= \left| \int e^{-it\lambda} \overline{G(t,x)} m(x, \partial_x) E(\lambda; x, y) f(y)\, d\lambda\, dy\, dx\, dt \right| \\ &= \left| \int \overline{[\mathcal{F}_t^{-1} G](\lambda, x)} [m E(\lambda) f](x)\, dx\, d\lambda \right| \\ &\lesssim \|\mathcal{F}_t^{-1} G\|_{L_x^1 L_\lambda^2} \|m E(\lambda) f\|_{L_x^\infty L_\lambda^2} \\ &\lesssim \|G\|_{L_x^1 L_t^2} \|m E(\lambda)\|_{X \to L_x^\infty L_\lambda^2} \|f\|_X. \end{aligned}$$

Thus (2-9) follows.

Using (2-9), we see that (2-7) will follow from

$$\|\langle x\rangle^{-3/2}R(\lambda\pm i0)f\|_{L_x^\infty L_\lambda^2}\lesssim\|f\|_{L^2}. \tag{2-10}$$

We focus on $R(\lambda+i0)$ and write $R=R_1+R_2$ as in (2-2) and (2-3). The contribution of (2-3) is easily handled. In fact, by a change of variables,

$$\begin{aligned}\left\|\frac{1}{q-i\sqrt{\lambda}}\int e^{i|y|\sqrt{\lambda}}f(y)\,dy\right\|_{L_\lambda^2}^2&=\left\|\frac{1}{q-i\sqrt{\lambda}}\widetilde{\mathcal{F}}f(\sqrt{\lambda})\right\|_{L_\lambda^2}^2\\&\lesssim\int\frac{|\mu|}{q^2+\mu^2}|\widetilde{\mathcal{F}}f(\mu)|^2\,d\mu\lesssim\|f\|_{L^2}^2.\end{aligned}$$

To estimate the contribution of (2-2), we split into low and high energies. We let $\chi(\lambda)$ denote a smooth cutoff to $|\lambda|\leq 1$ and write $\chi^c=1-\chi$. On the support of χ^c, we can argue as we did for (2-3), changing variables and estimating the contribution via

$$\int_{|\mu|\geq 1}\frac{1}{|\mu|}|\widetilde{\mathcal{F}}f(\mu)|^2\,d\mu\lesssim\|f\|_{L^2}^2,$$

which is acceptable.

We turn to the low-energy contribution of (2-2). Here we use (2-4); in particular, we will consider the cases $y\geq x\geq 0$ and $0\geq y\geq x$. In the first case, we use the bound

$$|e^{ix\sqrt{\lambda}}-e^{-ix\sqrt{\lambda}}|\lesssim|x|\sqrt{\lambda}$$

and estimate

$$\begin{aligned}\left\|\langle x\rangle^{-1}\chi(\lambda)\frac{1}{\sqrt{\lambda}}[e^{ix\sqrt{\lambda}}-e^{-ix\sqrt{\lambda}}]\int_x^\infty e^{i\sqrt{\lambda}y}f(y)\,dy\right\|_{L_x^\infty L_\lambda^2}&\lesssim\|\chi(\lambda)\widetilde{\mathcal{F}}f(\sqrt{\lambda})\|_{L_\lambda^2}\\&\lesssim\|\sqrt{\mu}\widetilde{\mathcal{F}}f(\mu)\|_{L_\mu^2(|\mu|\leq 1)}\lesssim\|f\|_{L^2},\end{aligned}$$

which is acceptable. In the remaining case, we use Cauchy–Schwarz to estimate

$$\begin{aligned}\left\|\langle x\rangle^{-3/2}\chi(\lambda)\frac{1}{\sqrt{\lambda}}\int_x^0[e^{i\sqrt{\lambda}y}-e^{-i\sqrt{\lambda}y}]f(y)\,dy\right\|_{L_x^\infty L_\lambda^2}&\lesssim\left\|\langle x\rangle^{-3/2}\chi(\lambda)\int_0^x|y|\,|f(y)|\,dy\right\|_{L_x^\infty L_\lambda^2}\\&\lesssim\left\|\chi(\lambda)|x|^{3/2}\langle x\rangle^{-3/2}\|f\|_{L^2}\right\|_{L_x^\infty L_\lambda^2}\lesssim\|f\|_{L^2},\end{aligned}$$

which is acceptable. This completes the proof of (2-7).

We turn to (2-8). Using (2-9), it suffices to prove

$$\|\partial_x R(\lambda\pm i0)f\|_{L_x^\infty L_\lambda^2}\lesssim\|f\|_{H^1}. \tag{2-11}$$

Let us again focus on the contribution of $R(\lambda+i0)$. This time we go back to the original form of the resolvent and estimate the two pieces separately. Let us first establish

$$\left\|\lambda^{-1/2}\,\partial_x\int e^{i|x-y|\sqrt{\lambda}}f(y)\,dy\right\|_{L_x^\infty L_\lambda^2}\lesssim\|f\|_{H^1}.$$

To see this, we firstly observe that

$$\partial_x e^{i|x-y|\sqrt{\lambda}} = -\partial_y e^{i|x-y|\sqrt{\lambda}} = \pm i\sqrt{\lambda} e^{i|x-y|\sqrt{\lambda}} \tag{2-12}$$

almost everywhere, where the sign depends on the ordering of x and y. Using the final equality in (2-12) and writing χ for a cutoff to $\lambda \leq 1$, we can change variables and use (2-5) to estimate

$$\begin{aligned}\left\| \lambda^{-1/2} \chi\, \partial_x \int e^{i|x-y|\sqrt{\lambda}} f(y)\, dy \right\|_{L_x^\infty L_\lambda^2} &\lesssim \|\chi \widetilde{\mathcal{F}} f(\sqrt{\lambda})\|_{L_\lambda^2} \\ &\lesssim \| |\xi|^{1/2} \widetilde{\mathcal{F}} f \|_{L_\xi^2(\{|\xi|\leq 1\})} \lesssim \|f\|_{L^2},\end{aligned}$$

which is acceptable. Writing $\chi^c = 1 - \chi$, we can use the second equality in (2-12), integrate by parts, change variables, and use (2-5) to estimate

$$\begin{aligned}\left\| \lambda^{-1/2} \chi^c\, \partial_x \int e^{i|x-y|\sqrt{\lambda}} f(y)\, dy \right\|_{L_x^\infty L_\lambda^2} &\lesssim \|\lambda^{-1/2} \chi^c \widetilde{\mathcal{F}}[f'](\sqrt{\lambda})\|_{L_\lambda^2} \\ &\lesssim \| |\xi|^{-1/2} \widetilde{\mathcal{F}}[f'] \|_{L_\xi^2(\{|\xi|>1\})} \lesssim \|f'\|_{L^2},\end{aligned}$$

which is also acceptable.

It remains to consider the piece coming from the potential. We need to show

$$\left\| \frac{1}{\sqrt{\lambda}[q - i\sqrt{\lambda}]} \partial_x [e^{i|x|\sqrt{\lambda}}] \int e^{i|y|\sqrt{\lambda}} f(y)\, dy \right\|_{L_x^\infty L_\lambda^2} \lesssim \|f\|_{H^1}.$$

In fact, by a change of variables and (2-5), we can control this term by

$$\left\| \frac{1}{q - i\sqrt{\lambda}} \widetilde{\mathcal{F}} f(\sqrt{\lambda}) \right\|_{L_\lambda^2} \lesssim \left\| \frac{|\xi|^{1/2}}{\langle \xi \rangle} \widetilde{\mathcal{F}} f \right\|_{L_\xi^2} \lesssim \|f\|_{L^2},$$

which is acceptable.

This completes the proof of (2-11) and hence the proof of Proposition 2.3. □

Combining the usual Strichartz estimates (Proposition 2.1) with the weighted local smoothing estimate in Proposition 2.3 yields the following corollary:

Corollary 2.5. *The following estimate holds*:

$$\left\| \int_0^t e^{-i(t-s)H} P_c F(s)\, ds \right\|_{L_t^4 L_x^\infty \cap L_t^\infty L_x^2} \lesssim \|\langle x \rangle^{5/2} F\|_{L_{t,x}^2}.$$

Proof. Using the Strichartz estimate Proposition 2.1, the dual estimate to (2-7), and Cauchy–Schwarz, we have

$$\begin{aligned}\left\| \int_{\mathbb{R}} e^{-i(t-s)H} P_c F(s)\, ds \right\|_{L_t^4 L_x^\infty \cap L_t^\infty L_x^2} &\lesssim \left\| \int_{\mathbb{R}} e^{isH} P_c F(s)\, ds \right\|_{L^2} \\ &\lesssim \|\langle x \rangle^{3/2} P_c F\|_{L_x^1 L_t^2} \lesssim \|\langle x \rangle^{5/2} P_c F\|_{L_{t,x}^2}.\end{aligned}$$

The desired estimate now follows from the Christ–Kiselev lemma [2001]. □

We will also need the following inhomogeneous local smoothing estimates.

Proposition 2.6. *For any $t \geq 0$, we have*

$$\left\| \langle x\rangle^{-1} \int_0^t e^{-i(t-s)H} P_c F(s)\, ds \right\|_{L_x^\infty L_t^2} \lesssim \|\langle x\rangle F\|_{L_x^1 L_t^2}, \tag{2-13}$$

$$\left\| \int_0^t \partial_x e^{-i(t-s)H} P_c F(s)\, ds \right\|_{L_x^\infty L_t^2} \lesssim \|F\|_{L_x^1 L_t^2}. \tag{2-14}$$

Proof of Proposition 2.6. We begin with the identity

$$2\int_0^t e^{-i(t-s)H} P_c F(s)\, ds \\ = \int_{\mathbb{R}} e^{-i(t-s)H} P_c F(s)\, ds + \int_0^\infty e^{-i(t-s)H} P_c F(s)\, ds - \int_{-\infty}^0 e^{-i(t-s)H} P_c F(s)\, ds. \tag{2-15}$$

In fact, this is a consequence of

$$\int_{\mathbb{R}} e^{-i(t-s)H} P_c F(s)\, ds = \int_{-\infty}^t e^{-i(t-s)H} P_c F(s)\, ds - \int_t^\infty e^{-i(t-s)H} P_c F(s)\, ds,$$

which follows from the fact that both sides solve

$$i\partial_t u = Hu, \quad \text{with } u(0) = \int_{\mathbb{R}} e^{isH} P_c F(s)\, ds.$$

In light of (2-15), it therefore suffices to estimate

$$\int_{\mathbb{R}} e^{-i(t-s)H} P_c \chi(s) F(s)\, ds,$$

where $\chi \in \{1, \chi_{(0,\infty)}, \chi_{(-\infty,0)}\}$.

Similar to the proof of Proposition 2.3, we will use (2-1) and Plancherel to reduce the desired bounds to an estimate given in terms of the resolvent. In particular, we write

$$\begin{aligned} \int e^{-i(t-s)H} P_c \chi F(s)\, ds &= \int e^{-it\lambda} E(\lambda) \int e^{is\lambda} \chi(s) F(s)\, ds\, d\lambda \\ &= \mathcal{F}_\lambda[E(\lambda)\mathcal{F}_s^{-1}(\chi F))](t), \end{aligned}$$

where we use subscripts to denote the variable of integration in the definition of the Fourier transform. Thus, writing $m = m(x, \partial_x) \in \{\langle x\rangle^{-1}, \partial_x\}$ and $X = \langle x\rangle^{-1} L^1$ or $X = L^1$, we use Plancherel and Minkowski's inequality to estimate

$$\begin{aligned} \left\| m \int e^{-i(t-s)H} P_c \chi F(s)\, ds \right\|_{L_x^\infty L_t^2} &\lesssim \|m E(\lambda)\mathcal{F}_s^{-1}(\chi F)\|_{L_x^\infty L_\lambda^2} \lesssim \|m E(\lambda)\mathcal{F}_s^{-1}(\chi F)\|_{L_\lambda^2 L_x^\infty} \\ &\lesssim \left\| \|m E(\lambda)\|_{X \to L_x^\infty} \|\mathcal{F}_s^{-1}(\chi F)(\lambda)\|_X \right\|_{L_\lambda^2} \\ &\lesssim \Big[\sup_\lambda \|m E(\lambda)\|_{X \to L_x^\infty}\Big] \|F\|_{L_t^2 X} \\ &\lesssim \Big[\sup_\lambda \|m E(\lambda)\|_{X \to L_x^\infty}\Big] \|F\|_{X L_t^2}. \end{aligned}$$

The proof of (2-13) and (2-14) therefore reduces to the two estimates

$$\sup_\lambda \|\langle x\rangle^{-1} R(\lambda \pm i0) f\|_{L_x^\infty} \lesssim \|\langle x\rangle f\|_{L_x^1}, \tag{2-16}$$

$$\sup_\lambda \|\partial_x R(\lambda \pm i0) f\|_{L_x^\infty} \lesssim \|f\|_{L_x^1}. \tag{2-17}$$

We consider $R(\lambda + i0)$, the other case being similar. We decompose the kernel as $R_1 + R_2$, as in (2-2) and (2-3). The contribution of R_2 to both (2-16) and (2-17) is handled easily. In fact, since $|q - i\sqrt{\lambda}| \geq |q|$, we have

$$\|R_2(\lambda) f\|_{L_x^\infty} + \|\partial_x R_2(\lambda) f\|_{L_x^\infty} \lesssim \|\widetilde{\mathcal{F}} f\|_{L_x^\infty} \lesssim \|f\|_{L_x^1}$$

uniformly in λ.

We turn to the contribution of R_1. The contribution to (2-17) is straightforward, as we can estimate

$$\|\partial_x R_1(\lambda) f\|_{L_x^\infty} \lesssim |\widetilde{\mathcal{F}} f(\sqrt{\lambda})| \lesssim \|f\|_{L^1},$$

uniformly in λ. For the contribution to (2-16), we recall (2-4). In particular, we need only consider the cases $y \geq x \geq 0$ and $0 \geq y \geq x$. In the first case, we estimate

$$\left| \frac{1}{\sqrt{\lambda}} (e^{-ix\sqrt{\lambda}} - e^{ix\sqrt{\lambda}}) \int_x^\infty e^{i\sqrt{\lambda} y} f(y)\, dy \right| \lesssim |x| \widetilde{\mathcal{F}} f(\sqrt{\lambda}),$$

and hence the desired bound holds in this regime (see (2-5)). Finally, if $0 \geq y \geq x$, we estimate

$$\left| \frac{1}{\sqrt{\lambda}} \int_x^0 [e^{i\sqrt{\lambda} y} - e^{-i\sqrt{\lambda} y}] f(y)\, dy \right| \lesssim \|y f(y)\|_{L^1}$$

uniformly in x and λ. Thus the desired bound holds in this regime as well. This completes the proof of Proposition 2.6. □

Finally, let us record one additional corollary of Proposition 2.3.

Corollary 2.7. *The following estimates hold*:

$$\left\| \langle x\rangle^{-3/2} \int_0^t e^{-i(t-s)H} P_c F(s)\, ds \right\|_{L_x^\infty L_t^2(\mathbb{R}\times[0,T])} \lesssim \|F\|_{L_t^1 L_x^2([0,T]\times\mathbb{R})},$$

$$\left\| \partial_x \int_0^t e^{-i(t-s)H} P_c F(s)\, ds \right\|_{L_x^\infty L_t^2(\mathbb{R}\times[0,T])} \lesssim \|F\|_{L_t^1 H_x^1([0,T]\times\mathbb{R})}.$$

Proof. To rid ourselves of the integral over $[0, t]$ we again use the decomposition (2-15) as in Proposition 2.6 and endeavor to estimate χF, with

$$\chi \in \{1, \chi_{(0,\infty)}, \chi_{(-\infty,0)}\}.$$

Let $m = m(x, \partial_x) \in \{\langle x\rangle^{-3/2}, \partial_x\}$ and write $X = L^2$ if $m = \langle x\rangle^{-3/2}$ and $X = H^1$ if $m = \partial_x$. Then using Proposition 2.3, the boundedness of e^{isH} on X, and Minkowski's inequality, we may estimate

$$\left\| m \int_{\mathbb{R}} e^{-i(t-s)H} P_c \chi(s) F(s)\, ds \right\|_{L_x^\infty L_t^2} \lesssim \left\| \int_{\mathbb{R}} e^{-isH} \chi(s) F(s)\, ds \right\|_X \lesssim \|F\|_{L_t^1 X}.$$

The result follows. □

We close this section with a technical result relating the usual Sobolev spaces with those defined in terms of H. We state the result we need as follows. In the following, we let $m(\partial_x)$ denote the Fourier multiplier operator with symbol $m(\mu)$.

Lemma 2.8. *We have*

$$\langle f, HP_c g\rangle = \langle f, -\tfrac{1}{2}\partial_x^2 g\rangle + B(f, g), \tag{2-18}$$

where $B(f, g)$ is a linear combination of terms of the form

$$\langle m(\partial_x)\partial_x f, \partial_x g\rangle, \quad \text{where } m(\mu) = (q - i\mu)^{-1}.$$

Consequently, for $f = P_c f$,

$$\|\sqrt{H} f\|_{L^2} \lesssim \|f\|_{\dot H^1} \quad \text{and} \quad \|f\|_{\dot H^1} \lesssim \|\sqrt{H} f\|_{L^2} + \|f\|_{L^2}. \tag{2-19}$$

Although (2-19) has already been shown in [Duchêne et al. 2011, Section VIII D] via the $W^{1,p}$ boundedness of wave operators for H, we give a simpler proof of (2-19) by using the explicit representation of $\sqrt{H}$.

Proof of Lemma 2.8. By the spectral theorem and the explicit form of the resolvent, we have the identity

$$\langle f, HP_c g\rangle = \langle f, -\tfrac{1}{2}\partial_x^2 g\rangle + B(f, g),$$

where $B(f, g)$ is a linear combination of terms like

$$\iiint \frac{\lambda}{\sqrt{\lambda}(q - i\sqrt{\lambda})} e^{i(|x|+|y|)\sqrt{\lambda}} f(x) g(y)\, dx\, dy\, d\lambda = \int \frac{\mu^2}{2(q - i\mu)} \widetilde{\mathcal{F}} f(\mu) \widetilde{\mathcal{F}} g(\mu)\, d\mu.$$

Here we use the notation

$$\widetilde{\mathcal{F}} f(\mu) = \int e^{i|x|\mu} f(x)\, dx.$$

This is consistent with the usage above, and in fact in this case $\widetilde{\mathcal{F}} f$ can be written exactly as the sum of Fourier transforms of f and its reflection. Thus (2-18) follows from Plancherel.

We turn to (2-19). For the first estimate we simply observe that $m(\partial_x)$ maps $L^2 \to L^2$ boundedly. For the second estimate, we observe in fact that $m(\partial_x)\partial_x$ maps $L^2 \to L^2$ boundedly, and hence by Young's inequality

$$\begin{aligned} \|\partial_x f\|_{L^2}^2 &\lesssim \|\sqrt{H} f\|_{L^2}^2 + \|f\|_{L^2}\|\partial_x f\|_{L^2} \\ &\lesssim \|\sqrt{H} f\|_{L^2}^2 + \varepsilon\|\partial_x f\|_{L^2}^2 + \varepsilon^{-1}\|f\|_{L^2}^2 \end{aligned}$$

for any $\varepsilon > 0$. Choosing $\varepsilon \ll 1$ implies the desired bound. □

Remark 2.9. The multiplier $m(\partial_x)$ appearing in (2-18) actually maps $L^r \to L^r$ boundedly for any $1 \le r \le \infty$. Indeed, it was already proven in (2-6) that $\mathcal{F}^{-1} m \in L^1$, and hence this is a consequence of Young's inequality. In particular, we are not using any multiplier theorems and are able to access the L^1, L^∞ endpoints. In a similar way, we see that $m(\partial_x)$ is bounded on $L_x^p L_t^q$ for all $1 \le p, q \le \infty$. These will be useful in the proof of Lemma 4.7 below.

2C. ***Existence of small solitary waves.*** In this section we discuss the existence and properties of solutions to (1-2).

In [Fukuizumi et al. 2008], the authors considered (1-1) with a focusing nonlinearity and provided an explicit formula for the family of nonlinear bound states. Using our notation, these solutions are given by

$$Q(x)=\left(\frac{(p+2)|E|}{2|\mu|}\right)^{1/p}\cosh^{-2/p}\left(p\sqrt{\frac{|E|}{2}}|x|+\operatorname{arctanh}\left(\frac{|q|}{\sqrt{2|E|}}\right)\right),$$

where $E<-\frac{1}{2}q^2$ and $\mu<0$. This formula is obtained by solving the relevant ODE on each side of $x=0$ and then gluing them together at $x=0$ to impose the jump condition $Q'(0+)-Q'(0-)=2qQ(0)$. This approach also works in the defocusing case $\mu>0$ (see [Kaminaga and Ohta 2009]); the resulting formula is

$$Q(x)=\left(\frac{(p+2)|E|}{2\mu}\right)^{1/p}\sinh^{-2/p}\left(p\sqrt{\frac{|E|}{2}}|x|+\operatorname{arctanh}\left(\frac{\sqrt{2|E|}}{|q|}\right)\right)$$

for $-\frac{1}{2}q^2<E<0$. When $E=0$, one has the solution

$$Q(x)=\left(\frac{(p+2)|q|^2}{\mu(p|q||x|+2)^2}\right)^{1/p},$$

which belongs to L^2 provided $p<4$.

From the explicit formulas for Q, one can observe that as E approaches $-\frac{1}{2}q^2$, the functions Q behave like a small multiple of the linear eigenfunction. It will be convenient to describe this behavior in Proposition 2.10 below. In particular, we find it convenient to follow the approach of [Gustafson et al. 2004] and parametrize the family of ground states by small $z\in\mathbb{C}$.

In the following, we write

$$D_jQ[z]=\frac{\partial}{\partial z_j}Q[z],$$

where we identify $z\in\mathbb{C}$ with the real vector (z_1,z_2). We write $DQ[z]$ for the Jacobian $DQ[z]:\mathbb{C}\to\mathbb{C}$ with

$$DQ[z]w=D_1Q[z]\operatorname{Re}w+D_2Q[z]\operatorname{Im}w\quad\text{for } w\in\mathbb{C}.$$

We will prove the following.

Proposition 2.10. *There exists small enough $\delta>0$ such that for $z\in\mathbb{C}$ with $|z|<\delta$, we have the following.*

- *There exists a unique solution $Q=Q[z]$ to (1-2) with $E=E[|z|]\in\mathbb{R}$.*
- *We may write $Q[z]=z\phi_0+h$, where*

$$\|h\|_{H^{1,k}(\mathbb{R})\cap H^2(\mathbb{R}\setminus\{0\})}\lesssim|z|^2,\quad \|Dh\|_{H^{1,k}}\lesssim|z|,\quad\textit{and}\quad \|D^2h\|_{L^2}\lesssim 1$$

for any $k\geq 0$.

- $E[|z|]=-\frac{1}{2}q^2+\mathcal{O}(z)$.
- *$Q[ze^{i\theta}]=Q[z]e^{i\theta}$ and $Q[|z|]$ is real-valued.*

Remark 2.11. In fact, the proof will show that $h = \mathcal{O}(z^{p+1})$ and $E[|z|] = -\frac{1}{2}q^2 + \mathcal{O}(z^p)$, but we will not need this refinement in what follows. Similarly, we can control Dh and $D^2 h$ in the same norms as h, but we will not need this.

Remark 2.12. Using gauge invariance (i.e., differentiating the identity $Q[ze^{i\theta}] = Q[z]e^{i\theta}$) leads to the useful identity

$$Q[z] = -i\,DQ[z]iz. \tag{2-20}$$

Results similar to Proposition 2.10 are proved in [Soffer and Weinstein 1990; 1992]; we will sketch a proof that follows the presentation given in the appendix of [Gustafson et al. 2004]. The key ingredient is the following estimate for the resolvent at the linear eigenvalue.

Lemma 2.13. *For any integer $k \geq 0$, $\left(H + \frac{1}{2}q^2\right)^{-1} P_c$ is bounded from L^2 to $H^2(\mathbb{R}\backslash\{0\})$ and from $H^{0,k}$ to $H^{1,k}$.*

Proof. Evaluating the resolvent at $-\frac{1}{2}q^2$, we see that the integral kernel of $\left(H + \frac{1}{2}q^2\right)^{-1} P_c$ is a linear combination of terms of the form

$$e^{|x-y|q} \quad \text{and} \quad e^{q(|x|+|y|)}.$$

Terms of the second type are straightforward to handle; one needs only observe that

$$\left| \int e^{q|y|} f(y)\, dy \right| \lesssim \|f\|_{L^2}$$

and that $e^{q|x|} \in H^2(\mathbb{R}\backslash\{0\}) \cap H^{1,k}$ for any k. It remains to verify that convolution with $e^{q|x|}$ sends L^2 to $H^2(\mathbb{R}\backslash\{0\})$ and $H^{0,k}$ to $H^{1,k}$ for any k. Mapping to $H^2(\mathbb{R}\backslash\{0\})$ is clear, so let us consider a weighted norm. As the derivative of $e^{q|x|}$ still decays exponentially, it is enough to work with $H^{0,k}$. The desired estimate therefore reduces to the fact that the operator with kernel $\langle x\rangle^k e^{q|x-y|}\langle y\rangle^{-k}$ maps $L^2 \to L^2$ for any k (a consequence of Schur's test, for example). □

With Lemma 2.13 in place, we turn to the proof of Proposition 2.10.

Proof of Proposition 2.10. We wish to solve

$$(H - E)Q + F(Q) = 0, \quad \text{with} \quad Q = z\phi_0 + h \quad \text{and} \quad E = -\tfrac{1}{2}q^2 + e$$

for small enough (nonzero) z, where $h = \mathcal{O}(z^2)$ is orthogonal to ϕ_0 and $e = \mathcal{O}(z)$ is real. Expanding the equation and projecting onto and away from ϕ_0 leads to the following system for (e, h):

$$e = z^{-1}\langle \phi_0, F(z\phi_0 + h)\rangle, \tag{2-21}$$

$$h = \left(H + \tfrac{1}{2}q^2\right)^{-1}\{-P_c F(z\phi_0 + h) + eh\}, \tag{2-22}$$

where z is to be small. To solve this system, let us construct (e, h) as a fixed point of the operator

$$\Phi(e, h) = (\text{RHS (2-21)}, \text{RHS (2-22)}).$$

Let us prove that Φ is a contraction on the set

$$A = \{(e, h) \in \mathbb{R} \times P_c H^1 : |e| \leq |z|,\ \|h\|_{H^1} \leq |z|^2\},$$

where z will be chosen sufficiently small. We will then prove the desired estimates for h and e as a priori estimates using (2-21) and (2-22).

It is straightforward to show that $\Phi : A \to A$; indeed, writing $(e_1, h_1) = \Phi(e_0, h_0)$ for some $(e_0, h_0) \in A$, we can use Lemma 2.13 to estimate

$$|e_1| \lesssim |z|^{-1} \|z^{p+1}\phi_0^{p+1} + h_0^{p+1}\|_{L^2} \lesssim |z|^p \ll |z|$$

and

$$\|h_1\|_{H^1} \lesssim \|z^{p+1}\phi_0^{p+1} + h_0^{p+1}\|_{L^2} + \|e_0 h_0\|_{L^2} \lesssim |z|^{p+1} + |z|^3 \ll |z|^2.$$

Similarly, writing $(e_1, h_1) = \Phi(e_0, h_0)$ and $(\tilde{e}_1, \tilde{h}_1) = \Phi(\tilde{e}_0, \tilde{h}_0)$, we can estimate

$$\begin{aligned} |e_1 - \tilde{e}_1| &\lesssim |z|^{-1} \|(h_0 - \tilde{h}_0)(z^p \phi_0^p + h_0^p + \tilde{h}_0^p)\|_{L^2} \\ &\lesssim |z|^{p-1} \|h_0 - \tilde{h}_0\|_{H^1} \ll \|h_0 - \tilde{h}_0\|_{H^1} \end{aligned}$$

and

$$\begin{aligned} \|h_1 - \tilde{h}_1\|_{H^1} &\lesssim |z|^p \|h_0 - \tilde{h}_0\|_{H^1} + |z|^2 |e_0 - \tilde{e}_0| + |z| \|h_0 - \tilde{h}_0\|_{H^1} \\ &\ll \|h_0 - \tilde{h}_0\|_{H^1} + |e_0 - \tilde{e}_0|. \end{aligned}$$

Thus Φ defines a contraction on A (for z small enough) and hence has a unique fixed point.

Using uniqueness and gauge invariance of the nonlinearity, we can deduce that $Q[ze^{i\theta}] = e^{i\theta} Q[z]$ and $E = E[|z|]$. Similarly, by uniqueness we can guarantee that $Q[|z|]$ is real-valued.

Next, let us estimate h in $H^2(\mathbb{R}\backslash\{0\})$ and $H^{1,k}$. Using (2-22), Lemma 2.13, and Sobolev embedding, we first estimate

$$\begin{aligned} \|h\|_{H^2(\mathbb{R}\backslash\{0\})} &\lesssim \|F(z\phi_0 + h) + eh\|_{L^2} \\ &\lesssim \|z^{p+1}\phi_0^{p+1} + h^{p+1}\|_{L^2} + |z| \|h\|_{L^2} \\ &\lesssim |z|^{p+1} + \{|z|^{2p} + |z|\} \|h\|_{L^2}, \end{aligned}$$

which (for small z) implies $\|h\|_{H^2(\mathbb{R}\backslash\{0\})} \lesssim |z|^2$. Similarly,

$$\|h\|_{H^{1,k}} \lesssim |z|^{p+1} + \{|z|^{2p} + |z|\} \|h\|_{H^{0,k}},$$

which again implies $\|h\|_{H^{1,k}} \lesssim |z|^2$.

To prove bounds for Dh, we differentiate (2-21) and (2-22). This leads to

$$De = -z^{-2} Dz \langle \phi_0, F(z\phi_0 + h) \rangle + z^{-1} \langle \phi_0, D[F(z\phi_0 + h)] \rangle, \tag{2-23}$$

$$Dh = \left(H + \tfrac{1}{2}q^2\right)^{-1} \{-P_c D[F(z\phi_0 + h)] + [De]h + e[Dh]\}. \tag{2-24}$$

Using (2-23), we can readily deduce that $|De| \lesssim 1$. Feeding this into (2-24) and estimating as above using Lemma 2.13, we find

$$\|Dh\|_{H^{1,k}} \lesssim |z|^p + |z| \|h\|_{H^{0,k}} + |z| \|Dh\|_{H^{0,k}} \lesssim |z|^p + |z|^3 + |z| \|Dh\|_{H^{0,k}},$$

which implies

$$\|Dh\|_{H^{1,k}} \lesssim |z|,$$

as desired. Differentiating (2-23) and (2-24) once more and arguing similarly yields the final estimate, namely,

$$\|D^2 h\|_{L^2} \lesssim 1. \qquad \square$$

2D. *Local well-posedness.* In this section we record a local well-posedness result for (1-1). Such results have appeared previously in the literature (e.g., in [Fukuizumi et al. 2008, Proposition 1]); we provide a proof here for the sake of completeness.

Proposition 2.14 (local well-posedness). *For any $u_0 \in H^1$, there exists a local-in-time solution to* (1-1). *The solution may be extended as long as the H^1-norm does not blow up.*

Proof. We will look for u decomposed as

$$u(t) = v(t) + a(t)\phi_0 := P_c u(t) + \langle \phi_0, u(t)\rangle \phi_0.$$

Equation (1-1) then becomes a coupled system for $(v(t), a(t))$, namely,

$$\begin{aligned} i\partial_t v(t) &= Hv + P_c F(v(s) + a(s)\phi_0), \\ i\partial_t a(t) &= -\tfrac{1}{2}q^2 a(t) + \langle \phi_0, F(v(t) + a(t)\phi_0)\rangle. \end{aligned} \tag{2-25}$$

Using an integrating factor in (2-25), we may rewrite these as

$$v(t) = e^{-itH} P_c u_0 - i \int_0^t e^{-i(t-s)H} P_c F(v(s) + a(s)\phi_0)\, ds, \tag{2-26}$$

$$a(t) = e^{iq^2 t/2} a(0) - i \int_0^t e^{iq^2(t-s)/2} \langle \phi_0, F(v(s) + a(s)\phi_0)\rangle\, ds. \tag{2-27}$$

Defining $\Phi(v, a) = (\text{RHS (2-26)}, \text{RHS (2-27)})$, we will prove that Φ defines a contraction on a suitable complete metric space. Writing $M = \|u_0\|_{H^1}$ and letting $T > 0$ to be chosen below, we define

$$B_T = \{(v, a) : \|v\|_{(L_t^\infty H_x^1 \cap L_t^4 L_x^\infty)([0,T]\times\mathbb{R})} \le 2CM,\ \|a\|_{L_t^\infty([0,T])} \le 2CM\},$$

where C encodes constants appearing in Strichartz estimates. In light of (2-19), we can freely exchange $\langle \sqrt{H}\rangle$ and $\langle \partial_x \rangle$ in what follows.

Writing $(w, b) = \Phi(v, a)$ for some $(v, a) \in B_T$, we first use Proposition 2.1 to estimate

$$\begin{aligned} \|w\|_{(L_t^\infty H_x^1 \cap L_t^4 L_x^\infty)([0,T]\times\mathbb{R})} &\lesssim \|u_0\|_{H^1} + \|F(v + a\phi_0)\|_{L_t^1 H_x^1([0,T]\times\mathbb{R})} \\ &\lesssim M + T\{\|v\|^p_{L^\infty_{t,x}([0,T]\times\mathbb{R})} \|v\|_{L_t^\infty H_x^1([0,T]\times\mathbb{R})} + \|a\|^{p+1}_{L_t^\infty([0,T])} \|\phi_0\|^p_{L_x^\infty} \|\phi_0\|_{H_x^1}\} \\ &\lesssim M + TM^{p+1}, \end{aligned}$$

while

$$\|b\|_{L_t^\infty([0,T])} \le |a(0)| + \|F(v + a\phi_0)\|_{L_t^1 L_x^2([0,T]\times\mathbb{R})} \|\phi_0\|_{L_x^2} \lesssim M + TM^{p+1}.$$

Thus, for $T = T(M)$ sufficiently small, Φ maps B_T to B_T. Similar estimates show that Φ is a contraction in the norm

$$d((v, a), (\tilde{v}, \tilde{a})) = \|v - \tilde{v}\|_{L_t^\infty L_x^2([0,T]\times\mathbb{R})} + \|a - \tilde{a}\|_{L_t^\infty([0,T])}$$

for T sufficiently small. The result follows. $\square$

3. Setting up the problem

Suppose $u : [0, T] \times \mathbb{R} \to \mathbb{C}$ is a (small) solution to (1-1). We will look for a decomposition of u of the form

$$u(t) = Q[z(t)] + v(t). \tag{3-1}$$

We view $z(t)$ as a small unknown to be specified, with Q a solution to (1-2) (see Proposition 2.10) and $v(t)$ defined through (3-1).

Using (1-1), (1-2), and (2-20), any such decomposition would lead to an evolution equation for v, namely,

$$\begin{aligned} i\partial_t v &= Hv + \mathcal{N}, \\ \mathcal{N} &:= F(Q+v) - F(Q) - iDQ(\dot{z} + iEz), \end{aligned} \tag{3-2}$$

where we have written $Q = Q[z(t)]$, $E = E[|z(t)|]$, and $\dot{z}$ denotes the time derivative. We wish to choose $z(t)$ in such a way that the solution to (3-2) is well-behaved (and such that $z(t)$ remains small).

To choose $z(t)$ and thereby fix the decomposition (3-1), we will impose the orthogonality conditions

$$\operatorname{Im}\langle u - Q[z], D_j Q[z]\rangle = 0 \quad \text{for } j \in \{1, 2\}, \tag{3-3}$$

for all $t \in [0, T]$. This condition makes $v = u - Q[z]$ orthogonal to the nondecaying solutions to the linearization of (1-1) around $e^{-iEt}Q[z]$ and agrees with the condition appearing in [Gustafson et al. 2004]. We discuss the motivation for this choice in Remark 3.2 below.

The following lemma tells us that as long as the solution $u(t)$ remains small, it is always possible to choose $z(t)$ such that (3-3) holds; moreover, this choice is unique.

Lemma 3.1. *There exists $\delta > 0$ small enough such that if $\|u\|_{H^1} \le \delta$, then there exists unique $z \in \mathbb{C}$ such that* (3-3) *holds*, *with $|z| + \|u - Q[z]\|_{H^1} \lesssim \|u\|_{H^1}$.*

Proof of Lemma 3.1. The proof is the same as [Gustafson et al. 2004, Lemma 2.3]. The idea is that if we were to choose $v = u - \langle\phi_0, u\rangle\phi_0 = P_c u$, then we would not be too far off from satisfying (3-3). We can therefore use the inverse function theorem to find z exactly satisfying (3-3). This is made precise using Proposition 2.10. We sketch the details.

Set $\varepsilon = \|u\|_{H^1}$. Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ via

$$f_j(z) = \operatorname{Im}\langle u - Q[z], D_j Q[z]\rangle$$

for $j = 1, 2$, and set $z_0 = \langle\phi_0, u\rangle$. Note that $|z_0| \le \varepsilon$. A computation using the expansion of $Q[z]$ in Proposition 2.10 yields

$$f(z_0) = \mathcal{O}(\varepsilon^2).$$

Similarly (using Proposition 2.10), the Jacobian of the map $z \mapsto f(z)$ is computed by

$$D_j f_k(z) = \operatorname{Im}\langle u - Q[z], D_j D_k Q[z]\rangle + \operatorname{Im}\langle D_j Q, D_k Q\rangle = j - k + \mathcal{O}(\varepsilon + |z|). \tag{3-4}$$

Therefore, by the inverse function theorem, for ε small enough we may find unique z such that $f(z) = 0$. The result follows. □

Under the (bootstrap) assumption that $\sup_{t\in[0,T]} \|u(t)\|_{H^1} \le \delta$ for δ small enough, we can therefore uniquely decompose $u(t)$ in the form (3-1) such that (3-3) holds for each $t \in [0, T]$.

The evolution equation for v is given by (3-2). To derive the evolution equation for z, we differentiate the orthogonality conditions (3-3). Recalling (3-1), (3-2), and self-adjointness of H, this firstly leads to

$$0 = \operatorname{Im}\bigl[i\langle v, HD_j Q\rangle + i\langle F(Q+v) - F(Q), D_j Q\rangle - \langle DQ(\dot z + iEz), D_j Q\rangle + \langle v, D_j DQ\dot z\rangle\bigr].$$

Differentiating (1-2) and observing that (2-20) and (3-3) imply $\operatorname{Im} i\langle v, Q\rangle = 0$, we may write

$$\begin{aligned}\operatorname{Im} i\langle v, HD_j Q\rangle &= \operatorname{Im}[i\langle v, ED_j Q\rangle - i\mu\langle v, D_j(|Q|^p Q)\rangle] \\ &= \operatorname{Im}\langle v, D_j DQiEz\rangle - \operatorname{Im} i\mu\langle v, D_j(|Q|^p Q)\rangle,\end{aligned}$$

where we have used (2-20) again in the final line. Continuing from above, we arrive at the system

$$\begin{aligned}&\operatorname{Im}\langle v, D_j DQ(\dot z + iEz)\rangle + \operatorname{Im}\langle D_j Q, DQ(\dot z + iEz)\rangle \\ &\qquad = -\operatorname{Im} i[\langle F(Q+v) - F(Q), D_j Q\rangle - \langle v, D_j(|Q|^p Q)\rangle]. \end{aligned}\tag{3-5}$$

The inner product on the right-hand side of (3-5) is of the form $\langle G(v, Q), D_j Q\rangle$, where G is at least quadratic in v (see Section 4A). Identifying $\mathbb{C}$ with $\mathbb{R}^2$, we may write this system in the more compact form

$$A(\dot z + iEz) = b, \tag{3-6}$$

where A is the 2×2 real matrix with entries

$$A_{jk} = \operatorname{Im}\langle v, D_j D_k Q\rangle + \operatorname{Im}\langle D_j Q, D_k Q\rangle$$

and $b \in \mathbb{R}^2$ satisfies $b_j = \text{RHS (3-5)}$. Note that A coincides with the Jacobian matrix appearing in (3-4), and hence $A_{jk} = j - k + \mathcal{O}(\delta + |z|)$.

3A. *Summary.* We have set up the problem as follows: assuming that we have a sufficiently small solution u to (1-1) on a time interval $[0, T]$, we choose $z(t)$ uniquely such that (3-3) holds for each t (using Lemma 3.1). Defining $v(t) = u(t) - Q[z(t)]$ (where Q is the solution to (1-2) as in Proposition 2.10), we find that v and z solve the coupled system (3-2) and (3-6).

In the next section we will use these equations to prove bounds for v and z. In particular, this will show that u remains small, which implies that the decomposition for u can be continued for all time. Furthermore, the bounds we obtain will allow us to complete the proof of the main result, Theorem 1.1.

Remark 3.2. Let us discuss in some more detail the orthogonality condition (3-3). We begin by considering the linearization of (1-1) around a fixed solitary wave $e^{-iEt}Q$. Identifying v with the real vector $v = (\operatorname{Re} v, \operatorname{Im} v)^t$, we can write the linearized equation the form $v_t = Lv$ for an explicit real matrix of operators L. Recalling that Q solves (1-2) and employing the identity (2-20), we can connect the functions $D_j Q$ to this linearized equation. In particular (recalling the identification of $\mathbb{C}$ and $\mathbb{R}^2$), one can compute

$$L^t i D_j Q = -\widetilde{E} z_j [z_2(iD_1 Q) - z_1(iD_2 Q)],$$

where L^t denotes the transpose of L and we write $D_j E[|z|] = \widetilde{E} z_j$. One therefore finds that the pair $\{iD_j Q\}$ spans the generalized null space of L^t. The orthogonality condition (3-3) is equivalent to the orthogonality of v (identified with the real vector $(\operatorname{Re} v, \operatorname{Im} v)^t$) to $i D_j Q$ (identified with $(-\operatorname{Im} D_j Q, \operatorname{Re} D_j Q)^t$)); here we use the usual inner product for vectors of $\mathbb{R}$-valued functions, i.e.,

$$(f_1, f_2)^t \cdot (g_1, g_2)^t = \int f_1 g_1 + \int f_2 g_2.$$

This condition projects v away from the nondecaying solutions to $\partial_t v = Lv$, as we now explain. We let $\{w_1, w_2\}$ be a basis for the generalized null space of L^t (denoted by N) satisfying $L^t w_1 = 0$ and $L^t w_2 = w_1$. It is not difficult to check that $N^\perp$ is invariant under the flow $\partial_t v = Lv$. Similarly, for $v(0) \in N$, we can find a solution to $\partial_t v = Lv$ of the form $v(t) = q_1(t) w_1 + q_2(t) w_2$. In fact, explicit computation reveals that q_1 and q_2 are linear functions in t. Thus, (3-3) exactly projects v away from the nondecaying solutions of $\partial_t v = Lv$, and hence we expect that the component v should decay.

At a technical level, the key benefit of imposing (3-3) arises in the computation of the ODE (3-5) for $\dot{z} + iEz$. In particular, imposing (3-3) leads to an ODE for $\dot{z} + iEz$ that contains only quadratic and higher terms in v. This is crucial because to describe the asymptotics of z will require that we estimate $\dot{z} + iEz$ in L^1_t, while we can only hope to estimate v in spaces as low as L^2_t (through reversed Strichartz estimates).

In contrast, suppose that we were to impose the natural condition

$$\langle v(t), \phi_0 \rangle = 0, \tag{3-7}$$

so that $v = P_c v$. This type of condition appears in [Pillet and Wayne 1997; Weder 2000] and has the advantage of allowing for Strichartz estimates for $e^{-itH} P_c$ to be applied directly to v. In this case, one would find that the ODE for z contains a term that is *linear* in v, and hence we would have no hope of estimating in L^1_t.

On the other hand, as $v \neq P_c v$ under the assumption (3-3), we cannot apply Strichartz estimates for e^{-itH} directly to v. However, if we recall the decomposition $Q[z] = z\phi_0 + \mathcal{O}(z^2)$, then we can see that the condition (3-3) implies $\langle v(t), \phi_0 \rangle = \mathcal{O}(z^2)$, which suggests that the portion of v parallel to ϕ_0 should be small compared to v. In fact, in Lemma 4.4 we will prove that we can control v by $P_c v$ in all relevant norms, and hence we will be able to utilize the estimates for $e^{-itH} P_c$ after all.

4. Proof of the main result

We suppose u is a solution to (1-1) satisfying

$$\sup_{t \in [0,T]} \|u(t)\|_{H^1} \le \delta \tag{4-1}$$

for δ sufficiently small, so that we may take the decomposition

$$u(t) = Q[z(t)] + v(t), \quad \text{where } \operatorname{Im}\langle v(t), D_j Q[z(t)] \rangle \equiv 0 \text{ for } j \in \{1, 2\},$$

as outlined in the previous section. By Lemma 3.1, we also have

$$\sup_{t \in [0,T]} \{|z(t)| + \|v(t)\|_{H^1}\} \lesssim \sup_{t \in [0,T]} \|u(t)\|_{H^1} \lesssim \delta.$$

Our goal is to extend these bounds to $[0, \infty)$ and to describe the asymptotics of $z(t)$ and $v(t)$ as $t \to \infty$. To accomplish this, we will prove a bootstrap estimate using the following norms, which should all be taken over $[0, T]\times\mathbb{R}$ or $[0, T]$. We first define

$$\|v\|_X := \|v\|_{L_t^\infty H_x^1\cap L_t^4 L_x^\infty} + \|\langle x\rangle^{-3/2} v\|_{L_x^\infty L_t^2} + \|\partial_x v\|_{L_x^\infty L_t^2}, \tag{4-2}$$

$$\|z\|_Y := \|\dot z + iEz\|_{L_t^1\cap L_t^2}. \tag{4-3}$$

Noting that

$$|z(t)| = \left| z(t)\exp\left\{ i\int_0^t E[z(s)]\,ds\right\}\right|,$$

we observe that

$$\|z\|_{L_t^\infty} \le |z(0)| + \|z\|_Y. \tag{4-4}$$

As the equation for v involves $Q[z(t)]$, it will be convenient to introduce notation for norms of Q as well. In particular, we define

$$\|Q\|_Z := \|\langle x\rangle^{5/2} Q\|_{L_x^1 L_t^\infty \cap L_{t,x}^\infty} + \|\partial_x Q\|_{L_{t,x}^\infty\cap L_t^\infty L_x^2}, \tag{4-5}$$

$$\|DQ\|_W := \|\langle x\rangle DQ\|_{L_x^1 L_t^\infty} + \|\langle x\rangle DQ\|_{L_t^\infty L_x^2} + \|\partial_x DQ\|_{L_t^\infty L_x^2}, \tag{4-6}$$

where $Q = Q[z(t)]$. Using Proposition 2.10, we can control these norms as long as $z(t)$ remains sufficiently small.

Lemma 4.1. *If $\|z\|_{L_t^\infty}$ is sufficiently small, then*

$$\|Q\|_Z \lesssim \|z\|_{L_t^\infty} \quad \textit{and} \quad \|DQ\|_W \lesssim 1.$$

Proof. We begin with the estimate

$$\|\langle x\rangle^\ell G\|_{L_x^r L_t^\infty} \lesssim \|G\|_{L_t^\infty H_x^{1,k}} \quad \text{for any } 1\le r\le\infty \text{ and } k > \ell + \frac{1}{r}, \tag{4-7}$$

which follows from Hölder's inequality and the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$. In particular,

$$\|Q\|_Z \lesssim \|Q[z(t)]\|_{L_t^\infty H_x^{1,k}\cap L_t^\infty H_x^2(\mathbb{R}\setminus\{0\})} \quad \text{and} \quad \|DQ\|_W \lesssim \|DQ[z(t)]\|_{L_t^\infty H_x^{1,k}}$$

for large enough k. Here we only use $H^2(\mathbb{R}\setminus\{0\})$ to control $\partial_x Q$ in L^∞.

The result now follows from Proposition 2.10; indeed, for $\sup_{t\in[0,T]}|z(t)|$ small enough, we can write

$$Q[z(t)] = z(t)\phi_0 + h(z(t)),$$

where $h(z(t)) = \mathcal{O}(|z(t)|^2)$ and $Dh(z(t)) = \mathcal{O}(|z(t)|)$ in the norms detailed in Proposition 2.10. □

4A. ***Estimates for the ODE.*** We first consider the ODE (3-6) for z, which we recall has the form

$$A(\dot z + iEz) = b,$$

with $A_{jk} = j - k + \mathcal{O}(\delta + |z|)$ and

$$b_j = -\operatorname{Im} i[\langle F(Q+v) - F(Q), D_j Q\rangle - \mu\langle v, D_j(|Q|^p Q)\rangle].$$

To get the error bound on A_{jk}, we use Proposition 2.10 (similar to the proof of Lemma 4.1). In particular, A is invertible with uniformly bounded inverse.

Lemma 4.2. *The following estimate holds*:

$$\|z\|_Y \lesssim \|DQ\|_W\{\|v\|_X^2\|Q\|_Z^{p-1} + \|v\|_X^{p+1}\}.$$

Proof. We examine the right-hand side of the ODE (3-6) in a more detail. First,

$$D_j(|Q|^p Q) = \frac{p}{2}|Q|^{p-2}Q^2 D_j\bar{Q} + \frac{p+2}{2}|Q|^p D_j Q,$$

while

$$F(Q+v) - F(Q) = \frac{p+2}{2}\mu v\int_0^1 |Q+\theta v|^p\,d\theta + \frac{p}{2}\mu\bar{v}\int_0^1 |Q+\theta v|^{p-2}(Q+\theta v)^2\,d\theta. \tag{4-8}$$

Thus, we may write

$$b_j = -\operatorname{Im} i\langle G(v, Q), D_j Q\rangle,$$

where

$$G(v, Q) := \frac{p+2}{2}\mu v\int_0^1 [|Q+\theta v|^p - |Q|^p]\,d\theta + \frac{p}{2}\mu\bar{v}\int_0^1 [|Q+\theta v|^{p-2}(Q+\theta v)^2 - |Q|^{p-2}Q^2]\,d\theta.$$

In particular,

$$|G(v, Q)| = \mathcal{O}(v^2 Q^{p-1} + v^{p+1}). \tag{4-9}$$

Using the above together with Proposition 2.10 and Sobolev embedding, we may now estimate

$$\begin{aligned}
\|\dot{z} + iEz\|_{L_t^2} &\lesssim \big\| \|v\|_{L_x^\infty}^2\|Q\|_{L_x^\infty}^{p-1} + \|v\|_{L_x^\infty}^{p+1}\big\|_{L_t^2}\|DQ\|_{L_t^\infty L_x^1} \\
&\lesssim \|DQ\|_W\{\|v\|_{L_t^4 L_x^\infty}^2\|Q\|_{L_{t,x}^\infty}^{p-1} + \|v\|_{L_t^4 L_x^\infty}^2\|v\|_{L_t^\infty H_x^1}^{p-1}\} \\
&\lesssim \|DQ\|_W\{\|v\|_X^2\|Q\|_Z^{p-1} + \|v\|_X^{p+1}\},
\end{aligned} \tag{4-10}$$

which is acceptable. We next estimate the L_t^1-norm. Using (4-9), we estimate

$$\begin{aligned}
\|\dot{z} + iEz\|_{L_t^1} &\lesssim \int |A^{-1}\operatorname{Im}\langle G(v, Q), DQ\rangle|\,dt \\
&\lesssim \|G(v, Q)DQ\|_{L_{t,x}^1} \\
&\lesssim \|\langle x\rangle^{-3/2}v\|_{L_x^\infty L_t^2}^2\|\langle x\rangle^{3/(p-1)}Q\|_{L_{t,x}^\infty}^{p-1}\|DQ\|_{L_x^1 L_t^\infty} + \|v\|_{L_t^4 L_x^\infty}^4\|v\|_{L_{t,x}^\infty}^{p-3}\|DQ\|_{L_t^\infty L_x^1} \\
&\lesssim \|DQ\|_W\{\|v\|_X^2\|Q\|_Z^{p-1} + \|v\|_X^{p+1}\},
\end{aligned}$$

which is acceptable. □

4B. ***Estimates for the PDE.*** We next consider the PDE (3-2) for v.

We will prove the following.

Proposition 4.3. *The following estimate holds*:

$$\|v\|_X \lesssim \|v(0)\|_{H^1} + \|z\|_Y\|DQ\|_W + \|v\|_X\|Q\|_Z^p + \|v\|_X^{p+1}.$$

The plan is to use Strichartz and local smoothing estimates for e^{-itH}. However, we cannot apply these estimates directly to v because the orthogonality conditions (3-3) do *not* imply that v belongs to the continuous spectral subspace of H. Nonetheless, using Proposition 2.10 and (3-3), we can prove that v can be controlled by $P_c v$.

Lemma 4.4. *There exists $\delta > 0$ small enough that the following holds: If $\|z\|_{L_t^\infty} \le \delta$ and $v \in X$ satisfies the orthogonality condition*

$$\mathrm{Im}\langle v(t), D_j Q[z(t)]\rangle \equiv 0 \quad \textit{for } j \in \{1, 2\} \tag{4-11}$$

(*where $Q[z]$ is as in Proposition 2.10*), *then*

$$\|v\|_X \lesssim \|P_c v\|_X.$$

Here X is as in (4-2) *and P_c denotes the projection onto the continuous spectral subspace of H.*

Proof. Writing $v = P_c v + \langle \phi_0, v\rangle \phi_0$, we see that it suffices to prove

$$\|\langle \phi_0, v\rangle \phi_0\|_X \ll \|v\|_X.$$

To this end, we use Proposition 2.10 to write $Q[z(t)] = z(t)\phi_0 + h(z(t))$, with $h(z) = \mathcal{O}(z^2)$ and $Dh(z) = \mathcal{O}(z)$ in the norms detailed in Proposition 2.10. As (4-11) yields

$$|\langle \phi_0, v(t)\rangle| \lesssim |\langle Dh, v(t)\rangle|,$$

we can estimate

$$\|\langle \phi_0, v\rangle \phi_0\|_X \lesssim \|\langle Dh, v(t)\rangle\|_{L_t^2 \cap L_t^\infty}.$$

We now claim that

$$\|\langle Dh, v(t)\rangle\|_{L_t^2 \cap L_t^\infty} \lesssim \|z\|_{L_t^\infty} \|v\|_X, \tag{4-12}$$

from which the result follows. To see this, first note that by the triangle inequality and Minkowski's inequality, we have

$$\|\langle Dh, v(t)\rangle\|_{L_t^2} \lesssim \|Dh\, v(t)\|_{L_t^2 L_x^1} \lesssim \|Dh\, v(t)\|_{L_x^1 L_t^2} \lesssim \|\langle x\rangle^{3/2} Dh\|_{L_x^1 L_t^\infty} \|\langle x\rangle^{-3/2} v\|_{L_x^\infty L_t^2}.$$

Using (4-7), we see that this term is acceptable. Next,

$$\|\langle Dh, v(t)\rangle\|_{L_t^\infty} \lesssim \|Dh\|_{L_t^\infty L_x^2} \|v\|_{L_t^\infty L_x^2},$$

which is acceptable as well. The result follows. □

Using Lemma 4.4, we see that it suffices to estimate the X-norm of $P_c v$. Applying P_c to (3-2), we have

$$i\partial_t P_c v = H P_c v + P_c \mathcal{N},$$

where we recall

$$\mathcal{N} = F(Q + v) - F(Q) - i DQ(\dot{z} + iEz).$$

In particular,

$$P_c v(t) = e^{-itH} P_c v(0) - i \int_0^t e^{-i(t-s)H} P_c \mathcal{N}\, ds. \tag{4-13}$$

We begin with the linear evolution term.

Lemma 4.5. *The following bound holds*:

$$\|e^{-itH} P_c v(0)\|_X \lesssim \|v(0)\|_{H^1}.$$

Proof. Recalling the definition of the X-norm in (4-2), we find that the lemma follows from Proposition 2.1, Proposition 2.3, and (2-19). □

We turn to the Strichartz norms for the inhomogeneous term.

Lemma 4.6. *The following bound holds*:

$$\left\| \int_0^t e^{-i(t-s)H} P_c \mathcal{N} \, ds \right\|_{L_t^\infty L_x^2 \cap L_t^4 L_x^\infty} \lesssim \|z\|_Y \|DQ\|_W + \|v\|_X \|Q\|_Z^p + \|v\|_X^{p+1}.$$

Proof. Using Corollary 2.5 we first estimate

$$\begin{aligned}\left\| \int_0^t e^{-i(t-s)H} P_c[DQ(\dot{z} + iEz)] \, ds \right\|_{L_t^\infty L_x^2 \cap L_t^4 L_x^\infty} &\lesssim \|\langle x\rangle^{5/2} DQ(\dot{z} + iEz)\|_{L^2_{t,x}} \\ &\lesssim \|\langle x\rangle^{5/2} DQ\|_{L_t^\infty L_x^2} \|\dot{z} + iEz\|_{L_t^2} \\ &\lesssim \|DQ\|_W \|z\|_Y,\end{aligned}$$

which is acceptable.

Next we write nonlinear term in the form

$$F(Q+v) - F(Q) = F_1 + F_2 + F_3,$$

where

$$F_1 = \mathcal{O}(vQ^p), \quad F_2 = \mathcal{O}(v^2 Q^{p-1} + v^p Q), \quad \text{and} \quad F_3 = \mu |v|^p v. \tag{4-14}$$

Such a decomposition is easily achieved under the assumption that $F(u) = \mu|u|^p u$ with p equal to an even integer greater than or equal to 4.

The linear term is handled as follows. Using Corollary 2.5, we have

$$\begin{aligned}\left\| \int_0^t e^{-i(t-s)H} P_c F_1 \, ds \right\|_{L_t^\infty L_x^2 \cap L_t^4 L_x^\infty} &\lesssim \|\langle x\rangle^{5/2} Q^p v\|_{L^2_{t,x}} \\ &\lesssim \|\langle x\rangle^{4/p} Q\|^p_{L_x^{2p} L_t^\infty} \|\langle x\rangle^{-3/2} v\|_{L_x^\infty L_t^2} \lesssim \|Q\|_Z^p \|v\|_X,\end{aligned}$$

which is acceptable.

Next, we use Proposition 2.1 to estimate

$$\left\| \int_0^t e^{-i(t-s)H} P_c F_2 \, ds \right\|_{L_t^\infty L_x^2 \cap L_t^4 L_x^\infty} \lesssim \|v^2 Q^{p-1}\|_{L^{6/5}_{t,x}} + \|v^p Q^2\|_{L^{6/5}_{t,x}}.$$

Using Minkowski's inequality to control $L_x^\infty L_t^4$ by $L_t^4 L_x^\infty$, we firstly estimate

$$\begin{aligned}\|v^2 Q^{p-1}\|_{L^{6/5}_{t,x}} &\lesssim \|\langle x\rangle^{-3/2} v\|^{4/3}_{L_x^\infty L_t^2} \|v\|^{2/3}_{L_x^\infty L_t^4} \|\langle x\rangle^{2/(p-1)} Q\|^{p-1}_{L_x^{6(p-1)/5} L_t^\infty} \\ &\lesssim \|v\|_X^2 \|Q\|_Z^{p-1},\end{aligned}$$

which (after an application of Young's inequality) is acceptable. The other term is treated similarly:

$$\begin{aligned}\|v^pQ\|_{L^{6/5}_{t,x}} &\lesssim \|\langle x\rangle^{-3/2}v\|^{4/3}_{L^\infty_x L^2_t}\|v\|^{2/3}_{L^4_tL^\infty_x}\|v\|^{p-2}_{L^\infty_{t,x}}\|\langle x\rangle^2 Q\|_{L^{6/5}_x L^\infty_t}\\ &\lesssim \|v\|^p_X\|Q\|_Z,\end{aligned}$$

which is again acceptable after applying Young's inequality.

Finally, the contribution of the F_3 term containing only v is estimated as follows: using Proposition 2.1,

$$\begin{aligned}\left\|\int_0^t e^{-i(t-s)H}P_c(|v|^pv)\,ds\right\|_{L^\infty_tL^2_x\cap L^4_tL^\infty_x} &\lesssim \||v|^pv\|_{L^{4/3}_tL^1_x}\\ &\lesssim \|v\|^3_{L^4_tL^\infty_x}\|v\|^{p-2}_{L^\infty_tL^{p-2}_x} \lesssim \|v\|^3_{L^4_tL^\infty_x}\|v\|^{p-2}_{L^\infty_tH^1_x} \lesssim \|v\|^{p+1}_X,\end{aligned}$$

which is acceptable. This completes the proof of Lemma 4.6. □

We next consider the $L^\infty_t\dot H^1_x$ norm of v. We treat this term by an energy estimate. We will make use of Lemma 2.8.

Lemma 4.7. *The following estimate holds uniformly over* $t\in[0,T]$:

$$\|P_cv(t)\|^2_{\dot H^1} \le \|v(0)\|^2_{\dot H^1} + \|v\|_X\|z\|_Y\|DQ\|_W + \|v\|^2_X\|Q\|^p_Z + \|v\|^{p+2}_X,$$

where norms are taken over $[0,t]\times\mathbb{R}$.

Proof. By (2-19), we have

$$\|P_cv(t)\|_{\dot H^1} \lesssim \|\sqrt{H}P_cv(t)\|^2_{L^2} + \|v(t)\|^2_{L^2}.$$

As the $L^\infty_tL^2_x$ norm is controlled via Lemma 4.6, it suffices to estimate $\sqrt{H}P_cv$.

To this end, we use the self-adjointness of H and (3-2) to write

$$\|\sqrt{H}P_cv(t)\|^2_{L^2_x} = \|\sqrt{H}P_cv(0)\|^2_{L^2} + \operatorname{Im}\int_0^t\langle\sqrt{H}P_cv(s),\sqrt{H}P_c\mathcal{N}\rangle\,ds,$$

where

$$\mathcal{N} = DQ(\dot z + iEz) + F_1 + F_2 + F_3$$

as in (4-14). In fact, we will split the term F_2 (which collects the terms of orders v^2Q^{p-1} through v^pQ) further by writing

$$F_2 = F_2^1 + F_2^2,$$

where F_2^1 collects terms that are linear in Q. We do this so that we can group this term with those appearing in (4-17) below (rather than (4-16)). This is necessary because when the derivative lands on Q we cannot additionally absorb weights in order to produce a $\langle x\rangle^{-3/2}v$ term in $L^\infty_xL^2_t$; indeed, we only control ∂_xQ in $L^\infty_{t,x}$. Thus we must put the whole term in $L^1_tL^2_x$; see (4-18) below.

We first observe that by (2-19), we have

$$\|\sqrt{H}P_cv(0)\|^2_{L^2} \lesssim \|v(0)\|^2_{\dot H^1},$$

which is acceptable.

We next use Lemma 2.8 to write

$$\int_0^t \langle \sqrt{H} P_c v(s) \sqrt{H} P_c \mathcal{N} \rangle \, ds = \int_0^t \langle \partial_x v(s), \partial_x \mathcal{N} \rangle \, ds + \int_0^t \langle m(\partial_x) \partial_x v(s), \partial_x \mathcal{N} \rangle \, ds, \tag{4-15}$$

where $m(\mu) = (q - i\mu)^{-1}$ (up to the addition of similar terms). We claim that both terms in (4-15) may be controlled by

$$\|\partial_x v\|_{L_x^\infty L_t^2} \|\partial_x (F_1 + F_2^2)\|_{L_x^1 L_t^2} \tag{4-16}$$

$$+ \|\partial_x v\|_{L_t^\infty L_x^2} \|\partial_x [DQ(\dot{z} + iEz) + F_2^1 + F_3]\|_{L_t^1 L_x^2}. \tag{4-17}$$

For the first term in (4-15), this follows directly from Hölder's inequality. For the second term in (4-15), we use Hölder's inequality and the fact that $m(\partial_x)$ maps $L_x^\infty L_t^2 \to L_x^\infty L_t^2$ and $L^2 \to L^2$ boundedly (see Remark 2.9).

We turn to estimating the terms in (4-16) and (4-17).

We begin with (4-16). First, by the chain rule:

$$\begin{aligned}\|\partial_x F_1\|_{L_x^1 L_t^2} &\lesssim \|\partial_x v\|_{L_x^\infty L_t^2} \|Q\|_{L_x^p L_t^\infty}^p + \|\langle x\rangle^{-3/2} v\|_{L_x^\infty L_t^2} \|\langle x\rangle^{3/(2(p-1))} Q\|_{L_x^{p-1} L_t^\infty}^{p-1} \|\partial_x Q\|_{L_{t,x}^\infty} \\ &\lesssim \|v\|_X \|Q\|_Z^p,\end{aligned}$$

which is acceptable.

We turn to the intermediate terms in F_2^2, which contains terms of the order $v^2 Q^{p-1}$ through $v^{p-1} Q^2$. Applying the chain and product rule and Young's inequality, we are led to estimate four types of terms in $L_x^1 L_t^2$ corresponding to these two extreme cases. When the derivative lands on a copy of v, we estimate

$$\|(\partial_x v) v Q^{p-1}\|_{L_x^1 L_t^2} \lesssim \|\partial_x v\|_{L_x^\infty L_t^2} \|v\|_{L_{t,x}^\infty} \|Q\|_{L_x^{p-1} L_t^\infty}^{p-1},$$

$$\|(\partial_x v) v^{p-2} Q^2\|_{L_x^1 L_t^2} \lesssim \|\partial_x v\|_{L_x^\infty L_t^2} \|v\|_{L_{t,x}^\infty}^{p-1} \|Q\|_{L_x^2 L_t^\infty}^2,$$

which are acceptable. When the derivative lands on a copy of Q, we instead estimate

$$\|v^2 Q^{p-2} \partial_x Q\|_{L_x^1 L_t^2} \lesssim \|\langle x\rangle^{-3/2} v\|_{L_x^\infty L_t^2} \|v\|_{L_{t,x}^\infty} \|\langle x\rangle^{3/(2(p-2))} Q\|_{L_x^{p-2} L_t^\infty}^{p-2} \|\partial_x Q\|_{L_{t,x}^\infty},$$

$$\|v^{p-1} Q \partial_x Q\|_{L_x^1 L_t^2} \lesssim \|\langle x\rangle^{-3/2} v\|_{L_x^\infty L_t^2} \|v\|_{L_{t,x}^\infty}^{p-2} \|\langle x\rangle^{3/2} Q\|_{L_x^1 L_t^\infty} \|\partial_x Q\|_{L_{t,x}^\infty},$$

which are acceptable.

We turn to (4-17). We first have

$$\begin{aligned}\|\partial_x v\|_{L_t^\infty L_x^2} \|\partial_x [DQ(\dot{z} + iEz)]\|_{L_t^1 L_x^2} &\lesssim \|\partial_x v\|_{L_t^\infty L_x^2} \|\partial_x DQ\|_{L_t^\infty L_x^2} \|\dot{z} + iEz\|_{L_t^1} \\ &\lesssim \|v\|_X \|z\|_Y \|DQ\|_W,\end{aligned}$$

which is acceptable.

Next, we estimate the contribution of F_2^1 in (4-17), which contains terms that are linear in Q. Distributing the derivative, we are led to estimate the following terms. First,

$$\|v^p \partial_x Q\|_{L_t^1 L_x^2} \lesssim \|v\|_{L_t^4 L_x^\infty}^4 \|v\|_{L_{t,x}^\infty}^{p-4} \|\partial_x Q\|_{L_t^\infty L_x^2} \lesssim \|v\|_X^p \|Q\|_Z, \tag{4-18}$$

which is acceptable. Next,

$$\|v^{p-1}(\partial_x v)Q\|_{L_t^1 L_x^2} \lesssim \|v\|_{L_t^4 L_x^\infty}^2 \|v^{p-3}(\partial_x v)Q\|_{L_{t,x}^2}$$
$$\lesssim \|v\|_{L_t^4 L_x^\infty}^2 \|\partial_x v\|_{L_x^\infty L_t^2} \|v\|_{L_{t,x}^\infty}^{p-3} \|Q\|_{L_x^2 L_t^\infty} \lesssim \|v\|_X^p \|Q\|_Z,$$

which is acceptable.

It remains to estimate the contribution of F_3 in (4-17). The purely nonlinear term $F_3 = \mu |v|^p v$ is estimated as

$$\|\partial_x v\|_{L_t^\infty L_x^2} \|\partial_x(|v|^p v)\|_{L_t^1 L_x^2} \lesssim \|v\|_{L_t^4 L_x^\infty}^4 \|\partial_x v\|_{L_t^\infty L_x^2}^2 \|v\|_{L_{t,x}^\infty}^{p-4} \lesssim \|v\|_X^{p+2},$$

which is acceptable. □

It remains to estimate the contribution of the inhomogeneous Duhamel term to the $L_x^\infty L_t^2$ components of the X-norm (see (4-2)). The key ingredients will be Proposition 2.6 and Corollary 2.7.

Lemma 4.8. *The following estimates hold: For $m \in \{\langle x\rangle^{-3/2}, \partial_x\}$,*

$$\left\| m \int_0^t e^{-i(t-s)H} P_c \mathcal{N}\, ds \right\|_{L_x^\infty L_t^2} \lesssim \|DQ\|_W \|z\|_Y + \|v\|_X \|Q\|_Z^p + \|v\|_X^{p+1}.$$

Proof. We recall that

$$\mathcal{N} = DQ(\dot z + iEz) + F_1 + F_2 + F_3,$$

where F_j are as in (4-14).

We first use Proposition 2.6 to estimate

$$\left\| m \int_0^t e^{-i(t-s)H} P_c [DQ(\dot z + iEz)]\, ds \right\|_{L_x^\infty L_t^2} \lesssim \|\langle x\rangle DQ(\dot z + iEz)\|_{L_x^1 L_t^2}$$
$$\lesssim \|\langle x\rangle DQ\|_{L_x^1 L_t^\infty} \|\dot z + iEz\|_{L_t^2} \lesssim \|DQ\|_W \|z\|_Y,$$

which is acceptable.

Next, we estimate

$$\left\| m \int_0^t e^{-i(t-s)H} P_c F_1\, ds \right\|_{L_x^\infty L_t^2} \lesssim \|\langle x\rangle Q^p v\|_{L_x^1 L_t^2}$$
$$\lesssim \|\langle x\rangle^{5/(2p)} Q\|_{L_x^p L_t^\infty}^p \|\langle x\rangle^{-3/2} v\|_{L_x^\infty L_t^2} \lesssim \|Q\|_Z^p \|v\|_X,$$

which is acceptable.

The contribution of F_2 is estimated by

$$\left\| m \int_0^t e^{-i(t-s)H} P_c F_2\, ds \right\|_{L_x^\infty L_t^2}$$
$$\lesssim \|\langle x\rangle v^2 Q^{p-1}\|_{L_x^1 L_t^2} + \|\langle x\rangle v^p Q\|_{L_x^1 L_t^2}$$
$$\lesssim \|\langle x\rangle^{-3/2} v\|_{L_x^\infty L_t^2} \|v\|_{L_{t,x}^\infty} \|\langle x\rangle^{5/(2(p-1))} Q\|_{L_x^{p-1} L_t^\infty}^{p-1} + \|\langle x\rangle^{-3/2} v\|_{L_x^\infty L_t^2} \|v\|_{L_{t,x}^\infty}^{p-1} \|\langle x\rangle^{5/2} Q\|_{L_x^1 L_t^\infty}$$
$$\lesssim \|v\|_X^2 \|Q\|_Z^{p-1} + \|v\|_X^p \|Q\|_Z,$$

which is acceptable (after an application of Young's inequality).

Finally, we use Corollary 2.7 to estimate

$$\left\| m \int_0^t e^{-i(t-s)H} P_c F_3 \, ds \right\|_{L_x^\infty L_t^2} \lesssim \| |v|^p v \|_{L_t^1 H_x^{1/2}} \lesssim \|v\|_{L_t^4 L_x^\infty}^4 \|v\|_{L_{t,x}^\infty}^{p-4} \|v\|_{L_t^\infty H_x^1} \lesssim \|v\|_X^{p+1},$$

which is acceptable. This completes the proof of Lemma 4.8. □

Finally, using Lemmas 4.4, 4.5, 4.6, 4.7, and 4.8 we complete the proof of Proposition 4.3.

4C. ***Completing the proof.*** In this section, we first use the estimates of the previous two sections in order to close a bootstrap estimate, which allows us to continue the decomposition of u for all time, as well as to prove the desired properties for $z(t)$ and $v(t)$ and hence complete the proof of Theorem 1.1.

We let $u(t)$ be the solution to (1-1) with initial data u_0, where $\|u_0\|_{H^1} = \delta$ for some small $\delta > 0$. By local well-posedness and Lemma 3.1, we can uniquely decompose u as

$$u(t) = Q[z(t)] + v(t), \quad \text{with } \operatorname{Im}\langle v(t), D_j Q[z(t)]\rangle \equiv 0 \text{ for } j \in \{1, 2\}, \tag{4-19}$$

at least on some time interval, with $|z(t)| + \|v(t)\|_{H^1} \lesssim \|u(t)\|_{H^1} \lesssim \delta$. On such an interval, we can now collect the estimates from the previous section. Collecting Lemma 4.1, (4-4), Lemma 4.2, and Proposition 4.3, we have

$$\|z\|_{L_t^\infty} \lesssim \delta \quad \Longrightarrow \quad \|Q\|_Z \lesssim \|z\|_{L_t^\infty} \text{ and } \|DQ\|_W \lesssim 1, \tag{4-20}$$

$$\|z\|_{L_t^\infty} \le |z(0)| + \|z\|_Y,$$

$$\|z\|_Y \lesssim \|DQ\|_W \{ \|v\|_X^2 \|Q\|_Z^{p-1} + \|v\|_X^{p+1} \}, \tag{4-21}$$

$$\|v\|_X \lesssim \|v(0)\|_{H^1} + \|z\|_Y \|DQ\|_W + \|v\|_X \|Q\|_Z^p + \|v\|_X^{p+1}. \tag{4-22}$$

By a standard bootstrap argument (choosing δ small), it follows that the bounds

$$\|u(t)\|_{H^1} \lesssim \delta, \quad \|v\|_X \lesssim \delta, \quad \|z\|_{L_t^\infty} \lesssim \delta, \quad \text{and} \quad \|z\|_Y \lesssim \delta^2,$$

as well as the decomposition (4-19), persist for all time.

We turn to establishing the asymptotics $v(t)$ and $z(t)$.

First, we prove scattering in H^1 for $v(t)$. We claim that it suffices to prove scattering for $P_c v(t)$. Writing $v = P_c v + \langle \phi_0, v\rangle \phi_0$, the claim reduces to proving

$$\lim_{t\to\infty} \|\langle \phi_0, v(t)\rangle \phi_0\|_{H^1} = 0. \tag{4-23}$$

Proof of (4-23). Using the orthogonality conditions in (4-19) and using Proposition 2.10 to write $Q[z(t)] = z(t)\phi_0 + h(z(t))$ (as in the proof of Lemma 4.4), we find

$$\|\langle \phi_0, v(t)\rangle \phi_0\|_{H^1} \lesssim \|Dh(z(t))\|_{L_x^{4/3}} \|v(t)\|_{L_x^4}.$$

As

$$\|Dh\|_{L_t^\infty L_x^{4/3}} \lesssim \|z\|_{L_t^\infty},$$

it suffices to prove that $\|v(t)\|_{L_x^4} \to 0$ as $t \to \infty$. To see this, we firstly observe (by interpolation of $L_t^\infty L_x^2$ and $L_t^4 L_x^\infty$) that $\|v(t)\|_{L_x^4}^4 \in L_t^2$. We will now show that $\partial_t \|v(t)\|_{L_x^4}^4$ is bounded, which implies the desired result. Using (3-2) for v and Lemma 2.8 (writing $Hv = HP_c v - q^2 \phi_0 \langle \phi_0, v\rangle$), we can firstly estimate

$$\partial_t \|v(t)\|_{L^4}^4 \lesssim \|v\|_{L_{t,x}^\infty}^2 \|\partial_x v\|_{L_t^\infty L_x^2}^2 + \|v\|_{L_t^\infty L_x^3}^3 \|v\|_{L_t^\infty L_x^2} \|\phi_0\|_{H^1}^2 \\ + \|v\|_{L_t^\infty L_x^6}^3 \|F(Q+v) - F(Q)\|_{L_t^\infty L_x^2} + \|v\|_{L_t^\infty L_x^6}^3 \|DQ\|_{L_t^\infty L_x^2} \|\dot z + iEz\|_{L_t^\infty}$$

uniformly in t. Using the bounds on v and $Q[z]$, we see the proof boils down to controlling $\dot z + iEz$ in L_t^∞. For this, we go back to the ODE (3-6) and use the computations at the beginning of Lemma 4.2 to bound

$$\begin{aligned}\|\dot z + iEz\|_{L_t^\infty} &\lesssim \|(v^2 Q^{p-1} + v^{p+1}) DQ\|_{L_t^\infty L_x^1} \\ &\lesssim \|DQ\|_{L_t^\infty L_x^2} \|v\|_{L_t^\infty L_x^4}^2 \{\|Q\|_{L_{t,x}^\infty}^{p-1} + \|v\|_{L_{t,x}^\infty}^{p-1}\}.\end{aligned}$$

This completes the proof of (4-23). □

It finally remains to prove scattering for $P_c v(t)$. For this we use the Duhamel formula (4-13) to show that $\{e^{itH} P_c v(t)\}$ is Cauchy in H^1. Indeed, using the estimates from Lemmas 4.6 and 4.7, we can deduce

$$\|e^{itH} P_c v(t) - e^{isH} P_c v(s)\|_{H^1} \lesssim \|z\|_Y \|DQ\|_W + \|v\|_X \|Q\|_Z^p + \|v\|_X^{p+1},$$

where now the norms on the right-hand side are restricted to (s, t) (and not all of the components of the X-norm are L^∞ in time). Sending $s, t \to \infty$ yields the claim.

Finally, we note that $\|\dot z + iEz\|_{L_t^1} \lesssim \delta^2$ yields the desired bounds and asymptotics for z. This completes the proof of Theorem 1.1.

Acknowledgements

Masaki was supported by JSPS KAKENHI, grant numbers 17K14219, 17H02854, and 17H02851. Segata is partially supported by JSPS KAKENHI, grant number 17H02851. We are grateful to T. Inui for pointing out an error in the original proof of Proposition 2.3, as well as to the anonymous referee for helpful comments.

References

[Albeverio et al. 1988] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, *Solvable models in quantum mechanics*, Springer, 1988. MR Zbl

[Banica and Visciglia 2016] V. Banica and N. Visciglia, “Scattering for NLS with a delta potential”, *J. Differential Equations* **260**:5 (2016), 4410–4439. MR Zbl

[Buslaev and Perelman 1992] V. S. Buslaev and G. S. Perelman, “Scattering for the nonlinear Schrödinger equation: states that are close to a soliton”, *Algebra i Analiz* **4**:6 (1992), 63–102. In Russian; translated in *St. Petersburg Math. J.* **4**:6 (1993), 1111–1142. MR Zbl

[Christ and Kiselev 2001] M. Christ and A. Kiselev, “Maximal functions associated to filtrations”, *J. Funct. Anal.* **179**:2 (2001), 409–425. MR Zbl

[Cuccagna 2011] S. Cuccagna, “The Hamiltonian structure of the nonlinear Schrödinger equation and the asymptotic stability of its ground states”, *Comm. Math. Phys.* **305**:2 (2011), 279–331. MR Zbl

[Cuccagna 2014] S. Cuccagna, “On asymptotic stability of moving ground states of the nonlinear Schrödinger equation”, *Trans. Amer. Math. Soc.* **366**:6 (2014), 2827–2888. MR Zbl

[Cuccagna and Pelinovsky 2014] S. Cuccagna and D. E. Pelinovsky, “The asymptotic stability of solitons in the cubic NLS equation on the line”, *Appl. Anal.* **93**:4 (2014), 791–822. MR Zbl

[Datchev and Holmer 2009] K. Datchev and J. Holmer, “Fast soliton scattering by attractive delta impurities”, *Comm. Partial Differential Equations* **34**:7-9 (2009), 1074–1113. MR Zbl

[Deift and Park 2011] P. Deift and J. Park, “Long-time asymptotics for solutions of the NLS equation with a delta potential and even initial data”, *Int. Math. Res. Not.* **2011**:24 (2011), 5505–5624. MR Zbl

[Duchêne et al. 2011] V. Duchêne, J. L. Marzuola, and M. I. Weinstein, “Wave operator bounds for one-dimensional Schrödinger operators with singular potentials and applications”, *J. Math. Phys.* **52**:1 (2011), art. id. 013505. MR Zbl

[Fukuizumi et al. 2008] R. Fukuizumi, M. Ohta, and T. Ozawa, “Nonlinear Schrödinger equation with a point defect”, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **25**:5 (2008), 837–845. MR Zbl

[Goodman et al. 2004] R. H. Goodman, P. J. Holmes, and M. I. Weinstein, “Strong NLS soliton-defect interactions”, *Phys. D* **192**:3-4 (2004), 215–248. MR Zbl

[Gustafson et al. 2004] S. Gustafson, K. Nakanishi, and T.-P. Tsai, “Asymptotic stability and completeness in the energy space for nonlinear Schrödinger equations with small solitary waves”, *Int. Math. Res. Not.* **2004**:66 (2004), 3559–3584. MR Zbl

[Holmer and Zworski 2007] J. Holmer and M. Zworski, “Slow soliton interaction with delta impurities”, *J. Mod. Dyn.* **1**:4 (2007), 689–718. MR Zbl

[Holmer and Zworski 2009] J. Holmer and M. Zworski, “Breathing patterns in nonlinear relaxation”, *Nonlinearity* **22**:6 (2009), 1259–1301. MR Zbl

[Holmer et al. 2007a] J. Holmer, J. Marzuola, and M. Zworski, “Fast soliton scattering by delta impurities”, *Comm. Math. Phys.* **274**:1 (2007), 187–216. MR Zbl

[Holmer et al. 2007b] J. Holmer, J. Marzuola, and M. Zworski, “Soliton splitting by external delta potentials”, *J. Nonlinear Sci.* **17**:4 (2007), 349–367. MR Zbl

[Ikeda and Inui 2017] M. Ikeda and T. Inui, “Global dynamics below the standing waves for the focusing semilinear Schrödinger equation with a repulsive Dirac delta potential”, *Anal. PDE* **10**:2 (2017), 481–512. MR Zbl

[Kaminaga and Ohta 2009] M. Kaminaga and M. Ohta, “Stability of standing waves for nonlinear Schrödinger equation with attractive delta potential and repulsive nonlinearity”, *Saitama Math. J.* **26** (2009), 39–48. MR Zbl

[Kenig et al. 1991] C. E. Kenig, G. Ponce, and L. Vega, “Well-posedness of the initial value problem for the Korteweg–de Vries equation”, *J. Amer. Math. Soc.* **4**:2 (1991), 323–347. MR Zbl

[Kirr and Mızrak 2009] E. Kirr and O. Mızrak, “Asymptotic stability of ground states in 3D nonlinear Schrödinger equation including subcritical cases”, *J. Funct. Anal.* **257**:12 (2009), 3691–3747. MR Zbl

[Kirr and Zarnescu 2009] E. Kirr and A. Zarnescu, “Asymptotic stability of ground states in 2D nonlinear Schrödinger equation including subcritical cases”, *J. Differential Equations* **247**:3 (2009), 710–735. MR Zbl

[Masaki et al. 2019] S. Masaki, J. Murphy, and J.-I. Segata, “Modified scattering for the one-dimensional cubic NLS with a repulsive delta potential”, *Int. Math. Res. Not. IMRN* **2019**:24 (2019), 7577–7603. MR Zbl

[Mizumachi 2008] T. Mizumachi, “Asymptotic stability of small solitary waves to 1D nonlinear Schrödinger equations with potential”, *J. Math. Kyoto Univ.* **48**:3 (2008), 471–497. MR Zbl

[Pillet and Wayne 1997] C.-A. Pillet and C. E. Wayne, “Invariant manifolds for a class of dispersive, Hamiltonian, partial differential equations”, *J. Differential Equations* **141**:2 (1997), 310–326. MR Zbl

[Segata 2015] J.-I. Segata, “Final state problem for the cubic nonlinear Schrödinger equation with repulsive delta potential”, *Comm. Partial Differential Equations* **40**:2 (2015), 309–328. MR Zbl

[Soffer and Weinstein 1990] A. Soffer and M. I. Weinstein, “Multichannel nonlinear scattering for nonintegrable equations”, *Comm. Math. Phys.* **133**:1 (1990), 119–146. MR Zbl

[Soffer and Weinstein 1992] A. Soffer and M. I. Weinstein, “Multichannel nonlinear scattering for nonintegrable equations, II: The case of anisotropic potentials and data”, *J. Differential Equations* **98**:2 (1992), 376–390. MR Zbl

[Soffer and Weinstein 2004] A. Soffer and M. I. Weinstein, "Selection of the ground state for nonlinear Schrödinger equations", *Rev. Math. Phys.* **16**:8 (2004), 977–1071. MR Zbl

[Tsai and Yau 2002a] T.-P. Tsai and H.-T. Yau, "Asymptotic dynamics of nonlinear Schrödinger equations: resonance-dominated and dispersion-dominated solutions", *Comm. Pure Appl. Math.* **55**:2 (2002), 153–216. MR Zbl

[Tsai and Yau 2002b] T.-P. Tsai and H.-T. Yau, "Classification of asymptotic profiles for nonlinear Schrödinger equations with small initial data", *Adv. Theor. Math. Phys.* **6**:1 (2002), 107–139. MR Zbl

[Tsai and Yau 2002c] T.-P. Tsai and H.-T. Yau, "Relaxation of excited states in nonlinear Schrödinger equations", *Int. Math. Res. Not.* **2002**:31 (2002), 1629–1673. MR Zbl

[Tsai and Yau 2002d] T.-P. Tsai and H.-T. Yau, "Stable directions for excited states of nonlinear Schrödinger equations", *Comm. Partial Differential Equations* **27**:11-12 (2002), 2363–2402. MR Zbl

[Weder 2000] R. Weder, "Center manifold for nonintegrable nonlinear Schrödinger equations on the line", *Comm. Math. Phys.* **215**:2 (2000), 343–356. MR Zbl

Received 31 Jul 2018. Revised 28 Jan 2019. Accepted 18 Apr 2019.

SATOSHI MASAKI: masaki@sigmath.es.osaka-u.ac.jp
Department of Systems Innovation, Graduate School of Engineering Science, Osaka University, Osaka, Japan

JASON MURPHY: jason.murphy@mst.edu
Department of Mathematics and Statistics, Missouri University of Science and Technology, Rolla, MO, United States

JUN-ICHI SEGATA: segata@m.tohoku.ac.jp
Mathematical Institute, Tohoku University, Sendai, Japan

mathematical sciences publishers msp

ANALYSIS AND PDE
Vol. 13, No. 4, 2020
dx.doi.org/10.2140/apde.2020.13.1129

GEOMETRIC REGULARITY FOR ELLIPTIC EQUATIONS IN DOUBLE-DIVERGENCE FORM

RAIMUNDO LEITÃO, EDGARD A. PIMENTEL AND MAKSON S. SANTOS

We examine the regularity of the solutions to the double-divergence equation. We establish improved Hölder continuity as solutions approach their zero level-sets. In fact, we prove that α-Hölder continuous coefficients lead to solutions of class $\mathcal{C}^{1^-}$, locally. Under the assumption of Sobolev-differentiable coefficients, we establish regularity in the class $\mathcal{C}^{1,1^-}$. Our results unveil improved continuity along a nonphysical free boundary, where the weak formulation of the problem vanishes. We argue through a geometric set of techniques, implemented by approximation methods. Such methods connect our problem of interest with a target profile. An iteration procedure imports information from this limiting configuration to the solutions of the double-divergence equation.

1. Introduction

In the present paper we study the regularity theory for solutions to the double-divergence partial differential equation (PDE)

$$\partial^2_{x_i x_j}(a^{ij}(x)u(x)) = 0 \quad \text{in } B_1, \tag{1}$$

where $(a^{ij})_{i,j=1}^d \in \mathcal{S}(d)$ is uniformly (λ, Λ)-elliptic. We produce new (sharp) regularity results for the solutions to (1). In particular, we are concerned with gains of regularity as solutions approach their zero level-sets. We argue through a genuinely geometric class of methods, inspired by the ideas introduced by L. Caffarelli [1989].

Introduced in [Littman 1959], equations in the double-divergence form have been the object of important advances. See [Sjögren 1973; Bogachev and Shaposhnikov 2017; Hervé 1962; Littman 1963; Fabes and Stroock 1984; Bogachev et al. 2015]. The interest in (1) is due to its own mathematical merits, as well as to its varied set of applications.

The primary motivation for the study of (1) is in the realm of stochastic analysis. In fact, (1) is the Kolmogorov–Fokker–Planck equation associated with the stochastic process whose infinitesimal generator is given by

$$Lv(x) := a^{ij}(x)\,\partial^2_{x_i x_j} v(x).$$

Therefore, one can derive information on the stochastic process through the understanding of (1).

MSC2010: 35B65, 35J15.
Keywords: double-divergence equations, geometric regularity, improved regularity at zero level-sets.

A further instance where double-divergence equations play a role is the fully nonlinear mean-field games theory. The model-problem here is

$$\begin{cases} F(D^2V) = g(u) & \text{in } B_1, \\ \partial^2_{x_i x_j}(F^{ij}(D^2V)u(x)) = 0 & \text{in } B_1, \end{cases} \tag{2}$$

where $F : \mathcal{S}(d) \to \mathbb{R}$ is a (λ, Λ)-elliptic operator, $F^{ij}(M)$ stands for the derivative of F with respect to the entry $m_{i,j}$ of M and $g : \mathbb{R} \to \mathbb{R}$ is a given function. In this case, the first equation in (2) is a Hamilton–Jacobi, associated with an optimal control problem. Its solution V accounts for the value function of the game. On the other hand, the population of players, whose density is denoted by u, solves a double-divergence (Fokker–Planck) equation. The mean-field coupling g encodes the preferences of the players with respect to the density of the entire population. Therefore, the solution u describes the equilibrium distribution of a population of rational players facing a scenario of strategic interaction. Through this framework, double-divergence equations are relevant in the modeling of several phenomena in the life and social sciences. As regards the mean-field games theory, we refer the reader to [Gomes et al. 2016].

A further application of equations in double-divergence form occurs in the theory of Hamiltonian stationary Lagrangian manifolds [Chen and Warren 2019]. Let $\Omega \subset \mathbb{R}^d$ be a domain and consider $u \in C^\infty(\Omega)$. The gradient graph of u is the set

$$\Gamma_u := \{(x, Du(x)) : x \in \Omega\},$$

whereas the volume of Γ_u is given by

$$F_\Omega(u) = \int_\Omega \sqrt{\det(I + (D^2u)^T D^2u)}\, \mathrm{d}x.$$

Given $\Omega \subset \mathbb{R}$, the study of critical points/minimizers for $F_\Omega(u)$ yields the compactly supported first variation

$$\int_\Omega \sqrt{\det g}\, g^{ij} \delta^{kl} u_{x_i x_k} \phi_{x_j x_l}\, \mathrm{d}x = 0 \tag{3}$$

for all $\phi \in C^\infty_c(\Omega)$, where

$$g := I + (D^2u)^T D^2u$$

is the induced metric. It is easy to check that (3) is the weak (distributional) formulation of

$$\partial^2_{x_j x_l}(\sqrt{\det g}\, g^{ij} \delta^{kl} u_{x_i x_k}) = 0 \quad \text{in } \Omega.$$

Hence, given a domain, the minimizers of the volume of the gradient graph relate to the solutions of a PDE in the double-divergence form.

As mentioned before, the study of (1) starts in [Littman 1959]. In that paper, the author considers weak solutions to the inequality

$$\partial^2_{x_i x_j}(a^{ij}(x)u(x)) \geq 0 \quad \text{in } B_1,$$

and establishes a strong maximum principle. In [Hervé 1962], the author develops a potential theory associated with (1). This theory is shown to satisfy the same axioms as the potential theory for the elliptic operator

$$a^{ij}(\,\cdot\,)\,\partial^2_{x_i x_j}.$$

Hence, the study of the former provides information on the latter. An improved maximum principle, as well as a preliminary approximation scheme for (1), are the subject of [Littman 1963].

It was only in [Sjögren 1973] that the regularity for the solutions to (1) was first investigated. In that paper, the author proves that solutions coincide with a continuous function, except in a set of measure zero. Together with its converse — and under further conditions — this is called the *fundamental equivalence*. In addition, a result on the α-Hölder continuity of the solutions is presented. Namely, solutions are proven to be locally α-Hölder continuous provided the coefficients satisfy $a^{ij} \in \mathcal{C}^{\alpha}_{\text{loc}}(B_1)$.

In [Fabes and Stroock 1984], the authors examine properties of the Green's function associated with the operator driving (1). One of the results in that paper regards gains of integrability for the solutions. In fact, it is reported that locally integrable, nonnegative solutions are in $L^{d/(d-1)}_{\text{loc}}(B_1)$.

A distinct approach to (1) regards the study of the *densities* of solutions, that is, their Radon–Nikodym derivatives with respect to the Lebesgue measure. In this realm, several developments have been produced; see [Bogachev et al. 2015]. For example it is widely known that, if $(a^{ij})^d_{i,j=1}$ is nondegenerate in B_1, every solution to (1) has a density; see [loc. cit.].

In [Bogachev et al. 2001] the authors prove that $\det[(a^{ij})^d_{i,j=1}]u$ has a density in $L^{d/(d-1)}_{\text{loc}}(B_1)$, provided $u \geq 0$. If, in addition, $(a^{ij})^d_{i,j=1}$ is Hölder continuous and uniformly elliptic, u is proven to have a density in $L^{d/(d-1)}_{\text{loc}}(B_1)$. Regularity in Sobolev spaces is also studied in [loc. cit.]. Under the assumptions that $(a^{ij})^d_{i,j=1}$ is in $W^{1,p}_{\text{loc}}(B_1)$ and $\det[(a^{ij})^d_{i,j=1}]$ is bounded away from zero, the authors prove that solutions have a density in $W^{1,p}_{\text{loc}}(B_1)$. It is worth noticing that [loc. cit.] addresses differential *inequalities* of the form

$$\int_{B_1} a^{ij}(x)\,u(x)\,\phi_{x_i x_j}(x)\,\mathrm{d}x \leq C\|\phi\|_{W^{1,\infty}(B_1)}$$

for some $C > 0$. The corpus of results reported in [loc. cit.] refines important previous developments; see, for instance [Bogachev et al. 1997; Krylov 1986].

In the recent paper [Bogachev and Shaposhnikov 2017], the authors consider densities of the solutions to (1) and investigate their regularity in Hölder and Lebesgue spaces. In addition, they prove a Harnack inequality for nonnegative solutions; see [loc. cit., Corollary 3.6]. Among other things, this result is relevant as it answers in the positive an open question raised in [Mamedov 1992]. In fact, it is shown that densities are in $L^p_{\text{loc}}(B_1)$, for every $p \geq 1$, if $(a^{ij})^d_{i,j=1} \in \text{VMO}(B_1)$. Moreover, the authors examine the regularity of densities in Hölder spaces, provided the coefficients are in the same class.

A remarkable feature of PDEs in the double-divergence form is the following: the regularity of $(a^{ij})^d_{i,j=1}$ acts as an upper bound for the regularity of the solutions. It means that gains of regularity are not (universally) available for the solutions, vis-a-vis the data of the problem. To see this phenomenon in a (very) simple setting, we detail an example presented in [Bogachev and Shaposhnikov 2017]. Set $d = 1$ and consider the homogeneous problem

$$(a(x)v(x))_{xx} = 0 \quad \text{in }]-1, 1[. \tag{4}$$

Take an arbitrary affine function $\ell : B_1 \to \mathbb{R}$ and let $u(x) := \ell(x)/a(x)$. Notice that

$$\int_{B_1} a(x) \frac{\ell(x)}{a(x)} \phi_{xx} \, \mathrm{d}x = 0$$

for every $\phi \in \mathcal{C}_0^2(]-1, 1[)$. Therefore, u is a solution to (4). It is clear that, if $a(x)$ is discontinuous, then u is as well.

Although solutions lack gains of regularity in the entire domain, a natural question regards the conditions under which improvements on the Hölder continuity can be established. Let $S \subset B_1$ be a fixed subset of the domain and suppose that further, natural conditions are placed on $(a^{ij})_{i,j=1}^d \in \mathcal{C}^\beta_{\mathrm{loc}}(B_1)$. The regularity of the solutions along S will be important. Even more relevant in some settings is the regularity of the solutions as they *approach* $S \subset B_1$.

In this paper, we consider the zero level-set of the solutions to (1). That is,

$$S_0[u] := \{x \in B_1 : u(x) = 0\}.$$

We prove that, along S_0, solutions to (1) are of class $\mathcal{C}^\alpha$ for every $\alpha \in (0, 1)$, provided $(a^{ij})_{i,j=1}^d$ is Hölder continuous and satisfies a proximity regime of the form

$$\|a^{ij} - a^{ij}(0)\|_{L^\infty(B_1)} \ll \tfrac{1}{2}.$$

The precise statement of our first main result is the following:

Theorem 1. *Let* $u \in L^1_{\mathrm{loc}}(B_1)$ *be a weak solution to* (1)*. Suppose assumptions A1–A2, to be set forth in Section 2.1, are in force. Let* $x_0 \in S_0(u)$*. Then* u *is of class* $\mathcal{C}^{1-}$ *at* x_0 *and there exists a constant* $C > 0$ *such that*

$$\sup_{B_r(x_0)} |u(x_0) - u(x)| \le C r^{\alpha^*}$$

for every $\alpha^* \in (0, 1)$.

The contribution of Theorem 1 is to ensure gains of regularity for the solutions to (1) *as they approach the zero level-set*, though estimates in the whole domain are constrained by the regularity of the coefficients a^{ij}. From a heuristic viewpoint, whichever level of ε-Hölder continuity is available for the coefficients — with $0 < \varepsilon \ll \frac{1}{2}$ — suffices to produce $\mathcal{C}^{1^-}$ regularity for the solutions along $S_0[u]$. See Figure 1.

The choice for S_0 is two-fold. Indeed, along this set, the weak formulation of (1) vanishes. Hence, at least intuitively, the weak formulation of the problem fails to provide information on the original equation along $S_0[u]$. A remarkable feature of (1) is related to this apparent lack of information across the zero level-set. As a matter of fact, the structure of the equation is capable of enforcing higher regularity of the solutions along the set where the weak formulation vanishes.

A second instance of motivation for the choice of S_0 falls within the scope of the *nonphysical free boundaries*. Introduced as a technology inspired by free boundary problems in the regularity theory of (nonlinear) partial differential equations, this class of methods has advanced the understanding of fine properties of solutions to a number of important examples. We refer the reader to [Teixeira 2014].

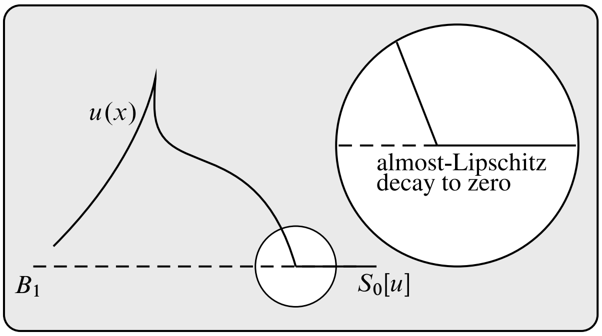

Figure 1. Almost-Lipschitz decay to zero: although the graph of the solutions to (1) admits cusps in the presence of merely Hölder continuous coefficients, they approach their zero level-sets with $\mathcal{C}^{\alpha}$-regularity for every $\alpha \in (0, 1)$. It means that solutions reach the nonphysical free boundary in an almost-Lipschitz manner.

In addition to the study of (1) in the presence of Hölder continuous coefficients, we also consider the case $(a^{ij})_{i,j=1}^{d} \in W^{2,p}_{\text{loc}}(B_1)$ for $p > d$. In this setting, (1) becomes

$$\partial_{x_i}\big(a^{ij}(x)\,\partial_{x_j}u(x) + \partial_{x_j}a^{ij}(x)u(x)\big) = 0 \quad \text{in } B_1. \tag{5}$$

Here, two new layers of information are unveiled. First, it is known that solutions to (5) are in $\mathcal{C}^{1,1-d/p}_{\text{loc}}(B_1)$ — see [Ladyzhenskaya and Uraltseva 1968, Chapter 3, Theorem 15.1]; i.e., the gradient of the solutions exists in classical sense. Second, the weak formulation of the problem vanishes at a different subset of the domain, namely

$$S_1[u] := \{x \in B_1 : u(x) = 0 \text{ and } Du(x) = \mathbf{0}\}.$$

Under the assumption $(a^{ij})_{i,j=1}^{d} \in W^{2,p}_{\text{loc}}(B_1)$, and the appropriate proximity regime, we prove that solutions to (1) are locally of class $\mathcal{C}^{1,1^-}$ along $S_1[u]$. This is the content of our second main result:

Theorem 2 (Hölder regularity of the gradient). *Let $u \in L^1_{\text{loc}}(B_1)$ be a weak solution to* (1). *Suppose A1 and A3, to be introduced in Section 2.1, hold true. Let $x_0 \in S_1[u]$. Then u is of class $\mathcal{C}^{1,1^-}$ at x_0 and there exists a constant $C > 0$ such that*

$$\sup_{x \in B_r(x_0)} |Du(x) - Du(x_0)| \le Cr^{\alpha^*}$$

for every $\alpha^* \in (0, 1)$.

The regularity of the coefficients in Sobolev spaces is pivotal in establishing Theorem 2. Here, Sobolev-differentiable coefficients switch the regularity regime of (1) allowing for an alternative weak formulation of the problem.

We remark that our methods accommodate equations with explicit dependence on lower-order terms; i.e.,

$$\partial^2_{x_i x_j}(a^{ij}(x)u(x)) - \partial_{x_i}(b^i(x)u(x)) + c(x)u(x) = 0 \quad \text{in } B_1,$$

provided the vector field $b : B_1 \to \mathbb{R}^d$ and the function $c : B_1 \to \mathbb{R}$ are well-prepared. See Remarks 13 and 18.

Our arguments are intrinsically geometric. We approximate weak solutions to (1) by solutions to a homogeneous, fixed-coefficient equation of the form

$$a^{ij}(0)\,\partial^2_{x_i x_j} v(x) = 0 \quad \text{in } B_1. \tag{6}$$

Among such solutions, we select v such that $S_0[u] \subset S_0[v]$, and $S_1[v] \subset S_1[u]$, when appropriate. An approximation routine builds upon the regularity theory available for the solutions to (6). This is achieved through a geometric strategy, which produces a preliminary oscillation control. To turn this initial information into an oscillation control in every scale, an iterative method takes place. This line of reasoning is inspired by trail-blazing ideas first introduced in [Caffarelli 1989]. See also [Caffarelli and Cabré 1995].

The remainder of this paper is organized as follows: Section 2.1 details our main assumption, whereas Section 2.2 collects a few elementary facts and notions, together with auxiliary results. In Section 3 we put forward a zero level-set approximation lemma and present the proof of Theorem 1. A finer approximation result appears in Section 4, where we conclude the proof of Theorem 2.

2. Preliminary material and main assumptions

In this section we introduce the main elements used in our arguments throughout the paper. Firstly we discuss our assumptions on the structure of the problem. Then, we collect a few definitions and results.

2.1. ***Main assumptions.*** In what follows, we detail the main hypotheses under which we work in the present paper. We start with an assumption on the uniform ellipticity of the coefficients matrix $(a^{ij})_{i,j=1}^d$.

A1 (uniform ellipticity). *We assume the symmetric matrix* $(a^{ij}(x))_{i,j=1}^d$ *satisfies a* (λ, Λ)*-ellipticity condition of the form*

$$\lambda\,\mathrm{Id} \le (a^{ij}(x))_{i,j=1}^d \le \Lambda\,\mathrm{Id}$$

for some $0 < \lambda \le \Lambda$, *uniformly in* $x \in B_1$.

The next assumption concerns the regularity requirements on the coefficients to ensure Hölder continuity of the solutions to (1). This fact is central in the proof of Theorem 1.

A2 (α-Hölder continuity). *The map* $(a^{ij}(x))_{i,j=1}^d : B_1 \to \mathcal{S}(d)$ *is locally uniformly* α*-Hölder continuous. That is, we have*

$$a^{ij} \in \mathcal{C}^{\alpha}_{\mathrm{loc}}(B_1)$$

for every $1 \le i \le d$ *and* $1 \le j \le d$.

We conclude this section with a further set of conditions on the coefficients a^{ij}. Such an assumption unlocks the study of the gradient-regularity for the solutions to (1), along $S_1[u]$.

A3 (Sobolev differentiability of the coefficients). *Let* $p > d$. *The map*

$$(a^{ij})_{i,j=1}^d : B_1 \to \mathcal{S}(d)$$

is in $W^{2,p}_{\mathrm{loc}}(B_1)$. *That is, we have*

$$a^{ij} \in W^{2,p}_{\mathrm{loc}}(B_1)$$

for every $1 \le i \le d$ *and* $1 \le j \le d$.

In the next section we gather elementary notions and basic facts used further in the paper.

2.2. *Preliminary notions and results.* We start with a result first proven in [Sjögren 1973]. It concerns the existence of a continuous version to the weak solutions to (1).

Proposition 3 (continuous version of weak solutions). *Let $u \in L^1_{\mathrm{loc}}(B_1)$ be a weak solution to* (1). *Then, there exists a null set $\Omega \subset B_1$ and $v \in \mathcal{C}(B_1)$ such that*

$$u \equiv v \quad \text{in } B_1 \setminus \Omega.$$

Proof. For the proof of the proposition, we refer the reader to [Sjögren 1973, Lemma 1]; see also [Sjögren 1975]. □

Remark 4. Hereafter, we suppose that every locally integrable function solving (1) in the weak sense is continuous.

Before proceeding we recall the fundamental solution of the operator

$$a^{ij}(y)\,\partial^2_{x_i x_j};$$

such a function will be denoted by $H(x, y)$. In the case $d > 2$, H is defined as

$$H(x, y) := \frac{[a_{ij}(y)(x_i - y_i)(x_j - y_j)]^{(2-d)/2}}{(d-2)\alpha(d)\sqrt{\det[(a^{ij})^d_{i,j=1}]}}, \tag{7}$$

where $(a_{ij})^d_{i,j=1} := [(a^{ij})^d_{i,j=1}]^{-1}$ and $\alpha(d)$ stands for the volume of the unit ball in dimension d.

A fundamental result in the context of this paper regards initial levels of compactness for the solutions to (1). This is the subject of the next proposition, which we recall here for the sake of completeness.

Proposition 5 (compactness of the solutions). *Let $u \in L^1_{\mathrm{loc}}(B_1)$ be a weak solution to* (1). *Suppose A1–A2 are in force. Then, $u \in \mathcal{C}^\alpha_{\mathrm{loc}}(B_1)$ and there exists a constant $C > 0$ such that*

$$\|u\|_{\mathcal{C}^\alpha(B_{1/2})} \leq C, \tag{8}$$

with $C = C(d, \lambda, \Lambda, \|a^{ij}\|_{\mathcal{C}^\alpha(B_1)}, \|u\|_{L^\infty(B_1)})$.

Proof. The inclusion $u \in \mathcal{C}^\alpha_{\mathrm{loc}}(B_1)$ is a well-known result; see, for instance [Sjögren 1973, Theorem 2]. As for the estimate in (8), it follows from considerations on the oscillation of the fundamental solution H, defined in (7), and its derivatives; see the proof of [loc. cit., Theorem 2]. □

We proceed with a proposition on the sequential stability of the solutions to (1). It will be used later to establish two approximation lemmas.

Proposition 6 (sequential stability of weak solutions). *Suppose that*

$$([a^{ij}_n]^d_{i,j=1})_{n\in\mathbb{N}} \subset \mathcal{C}^\alpha_{\mathrm{loc}}(B_1; \mathcal{S}(d))$$

is a sequence of matrices such that

$$\|a^{ij}_n - a^{ij}(x_0)\|_{L^\infty(B_1)} \to 0$$

as $n \to \infty$. Suppose further that $(f_n)_{n\in\mathbb{N}} \subset L^p(B_1)$ is such that

$$\|f_n\|_{L^p(B_1)} \to 0$$

as $n \to \infty$. Let $(u_n)_{n\in\mathbb{N}} \subset L^1_{\mathrm{loc}}(B_1)$ satisfy

$$\partial^2_{x_i x_j}(a_n^{ij}(x)u_n(x)) = f_n \quad \text{in } B_1.$$

If there exists $u_\infty \in \mathcal{C}(B_1)$ such that

$$\|u_n - u_\infty\|_{L^\infty(B_1)} \to 0$$

as $n \to \infty$, then u_∞ satisfies

$$\int_{B_1} a^{ij}(x_0)u_\infty(x)\phi_{x_i x_j}(x)\,\mathrm{d}x = 0$$

for every $\phi \in \mathcal{C}^2_c(B_1)$.

Proof. First, notice that we have $a_n^{ij}(x_0) \to a^{ij}(x_0)$ as $n \to \infty$. Now, for every $\phi \in \mathcal{C}^2_c(B_1)$ we have

$$\left|\int_{B_1} \phi_{x_i x_j} a^{ij}(x_0)u_\infty(x)\,\mathrm{d}x\right| \le \int_{B_1} |\phi_{x_i x_j}|\,|a^{ij}(x_0) - a_n^{ij}(x)|\,|u_\infty(x)|\,\mathrm{d}x + \int_{B_1} |\phi_{x_i x_j}|\,|a_n^{ij}(x)|\,|u_n(x) - u_\infty(x)|\,\mathrm{d}x + \int_{B_1} |\phi|\,|f_n|\,\mathrm{d}x.$$

Notice that the right-hand side of this inequality converges to zero as $n \to \infty$. Therefore,

$$\int_{B_1} \phi_{x_i x_j} a^{ij}(x_0)u_\infty(x)\,\mathrm{d}x = 0. \qquad \square$$

In addition to the sequential stability, our arguments require an initial degree of compactness for the solutions to (1). When it comes to the proof of Theorem 1, uniform compactness comes from Proposition 5. In the case of Theorem 2, we turn to a well-known result on the regularity of the (weak) solutions to equations in the divergence form. We start with an observation.

In case A3 is in force, we claim that (1) can be written as

$$\partial_{x_i}\big(a^{ij}(x)\,\partial_{x_j}u(x) + \partial_{x_j}a^{ij}(x)u(x)\big) = 0 \quad \text{in } B_1. \tag{9}$$

Indeed, if a^{ij} is weakly differentiable, we have

$$\int_{B_1} a^{ij}u\,\partial_{x_i x_j}\phi\,\mathrm{d}x = -\int_{B_1} (a^{ij}\,\partial_{x_j}u + \partial x_j a^{ij}u)\,\partial_{x_i}\phi\,\mathrm{d}x$$

for every $\phi \in \mathcal{C}^2_c(B_1)$. Hence, under A3, the homogeneous version of (1) is equivalent to (9). Now we are in position to state the following:

Proposition 7. *Let $v \in W^{1,p}(B_1)$ be a weak solution to* (9)*. Suppose A1 and A3 are in force. Then, $v \in \mathcal{C}^{1,\alpha}_{\mathrm{loc}}(B_1)$, where*

$$\alpha := \frac{p-d}{p}.$$

Moreover, there exists a universal constant $C > 0$ such that

$$\|v\|_{\mathcal{C}^{1,\alpha}(B_{1/2})} \le C\|v\|_{L^\infty(B_1)}.$$

For the proof of Proposition 7, we refer the reader to [Ladyzhenskaya and Uraltseva 1968, Chapter 3, Theorem 15.1]. The former proposition is paramount in establishing Theorem 2. Apart from compactness, it produces gradient-continuity for the solutions to (9). This information plays a critical role in the treatment of fine regularity properties of the solutions to the homogeneous version of (1) along $S_1[u]$. In particular, it unlocks a first zero level-set approximation result.

We conclude this section with a comment on the scaling properties of (1). Indeed, we consider weak solutions satisfying $\|u\|_{L^\infty(B_1)} \leq 1$. Let $\bar{u} \in \mathcal{C}(B_1)$ be defined as

$$\bar{u}(x) := \frac{u(x)}{\max\{1, \|u\|_{L^\infty(B_1)}\}},$$

where u is a weak solution to (1). It is clear that $\bar{u}$ is a weak solution to

$$\partial^2_{x_i x_j}(a^{ij}(x)\bar{u}(x)) = 0 \quad \text{in } B_1.$$

Notice that $\|\bar{u}\|_{L^\infty(B_1)} \leq 1$. Then, hereinafter we consider, without loss of generality, normalized solutions to (1). In the sequel, we set forth the proof of Theorem 1.

3. Improved regularity of the solutions

In this section we detail the proof of Theorem 1. As mentioned before, we reason through an approximation/geometric method. At the core of our argument lies a zero level-set approximation lemma. It reads as follows:

Proposition 8 (zero level-set approximation lemma). *Let $u \in L^1_{\text{loc}}(B_1)$ be a weak solution to* (1), *$x_0 \in S_0[u] \cap B_{9/10}$ and suppose A1–A2 are in force. Given $\delta > 0$, there exists $\varepsilon = \varepsilon(\delta) > 0$ such that, if*

$$\sup_{x \in B_1} |a^{ij}(x) - a^{ij}(0)| < \varepsilon,$$

there exists $h \in \mathcal{C}^{1,1}(B_{9/10})$ satisfying

$$\|u - h\|_{L^\infty(B_{9/10})} < \delta,$$

with

$$h(x_0) = 0.$$

Proof. The proof follows from a contradiction argument. We start by supposing that the statement of the proposition is false. Therefore, there exist $\delta_0 > 0$ and sequences $([a_n^{ij}]_{i,j=1}^d)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}} \subset L^\infty(B_1)$ such that

$$\sup_{x\in B_1} |a_n^{ij}(x) - a_n^{ij}(0)| \sim \frac{1}{n},$$
$$x_0 \in S_0[u_n] \cap B_{9/10},$$
$$\partial^2_{x_i x_j}(a_n^{ij}(x)u_n(x)) = 0 \quad \text{in } B_1,$$

but

$$|u_n(x) - h(x)| > \delta_0 \quad \text{or} \quad h(x_0) \neq 0$$

for every $h \in \mathcal{C}^{1,1}(B_{9/10})$ and every $n \in \mathbb{N}$.

Notice that $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $\mathcal{C}^\alpha(B_1)$. Therefore, there exists u_∞ such that

$$\|u_n - u_\infty\|_{\mathcal{C}^\beta(B_1)} \to 0$$

for every $0 < \beta < \alpha$, through a subsequence, if necessary. On the other hand, we have $a_n^{ij}(0) \to \bar{a}^{ij}(0)$ as $n \to \infty$; hence

$$|a_n^{ij}(x) - \bar{a}^{ij}(0)| \le |a_n^{ij}(x) - a_n^{ij}(0)| + |a_n^{ij}(0) - \bar{a}^{ij}(0)|.$$

Therefore

$$\|a_n^{ij} - \bar{a}^{ij}(0)\|_{L^\infty(B_1)} \to 0$$

as $n \to \infty$. Hence, the sequential stability of weak solutions (Proposition 6) leads to

$$\partial^2_{x_i x_j}(\bar{a}^{ij}(0) u_\infty(x)) = 0 \quad \text{in } B_{9/10}.$$

The regularity theory for constant-coefficient equations implies that $u_\infty \in \mathcal{C}^{1,1}(B_{9/10})$ and, moreover, $u_\infty(x_0) = 0$. Finally, there exists $N \in \mathbb{N}$ such that

$$|u_n(x) - u_\infty(x)| < \delta_0,$$

provided $n > N$. By taking $h \equiv u_\infty$, we produce a contradiction and conclude the proof. □

Remark 9. The proof of Proposition 8 shows that the approximating function h solves the problem

$$\begin{cases} \partial^2_{x_i x_j}(\bar{a}^{ij}(0) h(x)) = 0 & \text{in } B_{9/10}, \\ h = h_0 & \text{on } \partial B_{9/10}, \end{cases} \tag{10}$$

where

$$\|h_0\|_{L^\infty(\partial B_{9/10})} \le \delta + \|u\|_{L^\infty(B_1)}.$$

Therefore, it follows from standard results in elliptic regularity theory that

$$\|h\|_{\mathcal{C}^{1,1}(B_{9/10})} \le C(1 + \|u\|_{L^\infty(B_1)}),$$

where $C > 0$ depends on the dimension d, the ellipticity constants λ and Λ and $\bar{a}^{ij}(0)$. We notice the constant C does not depend on u.

Remark 10. A priori, the parameter $\varepsilon > 0$ depends only on $\delta > 0$. We notice however that (a universal) choice of δ, made further in the paper, implies that ε will depend on the exponent α, the dimension d, λ, Λ and $\|u\|_{L^\infty(B_1)}$. Therefore, we have

$$\varepsilon = \varepsilon(\alpha, d, \lambda, \Lambda, \|u\|_{L^\infty(B_1)}).$$

Next, we control the oscillation of the solutions to (1) within a ball of radius $0 < \rho \ll \frac{1}{2}$, to be determined later.

Proposition 11. *Let $u \in L^1(B_1)$ be a weak solution to* (1). *Suppose A1–A2 are in force. Then, for every $\alpha \in (0, 1)$, there exists $\varepsilon > 0$ such that, if $x_0 \in S_0[u] \cap B_{9/10}$ and*

$$\sup_{x\in B_1} |a^{ij}(x) - a^{ij}(0)| < \varepsilon,$$

we can find $0 < \rho \ll \frac{1}{2}$ *for which*

$$\sup_{B_\rho(x_0)} |u(x)| \leq \rho^\alpha.$$

Proof. We start by taking a function $h \in \mathcal{C}^{1,1}_{\text{loc}}(B_{9/10})$ satisfying

$$\|u - h\|_{L^\infty(B_{9/10})} < \delta,$$

with

$$h(x_0) = 0.$$

The existence of such a function is guaranteed by Proposition 8. We have

$$\sup_{x \in B_\rho(x_0)} |h(x) - h(x_0)| \leq C\rho$$

for some constant $C > 0$; see Remark 9. Therefore,

$$\sup_{x \in B_\rho(x_0)} |u(x) - h(x_0)| \leq \sup_{x \in B_\rho(x_0)} |u(x) - h(x)| + \sup_{x \in B_\rho(x_0)} |h(x) - h(x_0)| \leq \delta + C\rho. \tag{11}$$

In the sequel, we make universal choices for ρ and δ; in fact, for a given $\alpha \in (0, 1)$, we set

$$\rho := \left(\frac{1}{2C}\right)^{1/(1-\alpha)} \quad \text{and} \quad \delta := \frac{\rho^\alpha}{2}. \tag{12}$$

Finally, we combine (11) with (12) to obtain

$$\sup_{B_\rho(x_0)} |u(x)| \leq \rho^\alpha. \qquad \square$$

Proposition 12. *Let* $u \in L^1_{\text{loc}}(B_1)$ *be a weak solution to* (1). *Suppose assumptions A1–A2 are in force. Then, there exists* $\varepsilon > 0$ *so that, if* $x_0 \in S_0[u] \cap B_{9/10}$ *and*

$$\sup_{x \in B_1} |a^{ij}(x) - a^{ij}(0)| < \varepsilon,$$

we can find $0 < \rho \ll \frac{1}{2}$ *for which*

$$\sup_{B_{\rho^n}(x_0)} |u(x)| \leq \rho^{n\alpha}$$

for every $n \in \mathbb{N}$.

Proof. We resort to an induction argument. First, we make the same choices as in (12); this (universally) determines the parameter ε. The first step of induction — the case $n = 1$ — follows from Proposition 11. The induction hypothesis refers to the case $n = k$; i.e.,

$$\sup_{B_{\rho^k}(x_0)} |u(x)| \leq \rho^{k\alpha}$$

for some $k \in \mathbb{N}$.

In the sequel we address the case $n = k+1$. To that end, we introduce an auxiliary function $v_k : B_1 \to \mathbb{R}$, defined as

$$v_k(x) := \frac{u(x_0 + \rho^k x)}{\rho^{k\alpha}}.$$

We observe that $v_k(0) = 0$. In addition v_k solves

$$\partial^2_{x_i x_j}(a_k^{ij}(x) v_k(x)) = 0 \quad \text{in } B_1, \tag{13}$$

where

$$a_k^{ij}(x) := a^{ij}(x_0 + \rho^k x).$$

Now, notice that

$$|a_k^{ij}(x) - a^{ij}(0)| = |a^{ij}(x_0 + \rho^k x) - a^{ij}(0)| \leq \varepsilon.$$

Finally, the matrix $(a_k^{ij})_{i,j=1}^d$ inherits the Hölder continuity and the (λ, Λ)-ellipticity of $(a^{ij})_{i,j=1}^d$. Therefore, (13) falls within the scope of Proposition 11. Hence,

$$\sup_{B_{\rho^k}} |v_k(x)| \leq \rho^\alpha;$$

by rescaling back to the unitary setting, we get

$$\sup_{B_{\rho^{k+1}}(x_0)} |u(x)| \leq \rho^{(k+1)\alpha}$$

and complete the proof. □

Proof of Theorem 1. Let $0 < r \ll \frac{1}{2}$ be fixed and take $x_0 \in S_0[u]$. We must verify that

$$\sup_{B_r(x_0)} |u(x) - u(x_0)| \leq C r^\alpha,$$

where $C > 0$ is universal. Fix $n \in \mathbb{N}$ such that $\rho^{n+1} \leq r \leq \rho^n$. Observe that

$$\begin{aligned}\sup_{B_r(x_0)} |u(x) - u(x_0)| &\leq \sup_{B_{\rho^n}(x_0)} |u(x) - u(x_0)| \\ &\leq \rho^{-\alpha} \rho^{(n+1)\alpha} \leq C r^\alpha.\end{aligned}$$

□

We conclude this section with a remark on double divergence equations with explicit dependence on lower-order terms.

Remark 13. To extend our result to model-problems of the form

$$\partial^2_{x_i x_j}(a^{ij}(x) u(x)) + \partial_{x_i}(b^i(x) u(x)) + c(x) u(x) = 0 \quad \text{in } B_1,$$

it suffices to impose two conditions on $b : B_1 \to \mathbb{R}^d$ and $c : B_1 \to \mathbb{R}$. Indeed, these maps must be Hölder continuous; such a requirement unlocks the uniform compactness of the solutions. Secondly, a proximity regime must be in force; that is, there must be $\bar{b} \in \mathbb{R}^d$ and $\bar{c} \in \mathbb{R}$ so that

$$\|b^i - \bar{b}^i\|_{L^\infty(B_1)} + \|c - \bar{c}|_{L^\infty(B_1)} \ll \tfrac{1}{2}.$$

In what follows we focus on the proof of Theorem 2.

4. Hölder continuity of the gradient

This section sets forth the proof of Theorem 2. As before, the main ingredient is a first level-set approximation lemma.

Proposition 14 (first level-set approximation lemma). *Let $u \in L^1_{\text{loc}}(B_1)$ be a weak solution to* (1) *and suppose A1 and A3 are in force. Given $\delta > 0$, there exists $\varepsilon > 0$ such that, if $x_0 \in S_1[u] \cap B_{9/10}$ and*

$$\sup_{x \in B_1} |a^{ij}(x) - a^{ij}(x_0)| <, \varepsilon,$$

there exists $h \in C^{1,1}(B_{9/10})$ satisfying

$$\|u - h\|_{C^{1,\beta}(B_{9/10})} < \delta$$

for some $\beta \in (0, 1)$, with

$$h(x_0) = 0 \quad \textit{and} \quad Dh(x_0) = \mathbf{0}.$$

Proof. We argue by contradiction. Suppose the statement of the proposition is false, in this case there exists $\delta_0 > 0$ and sequences $([a_n^{ij}]_{i,j=1}^d)_{n\in\mathbb{N}}$, $(u_n)_{n\in\mathbb{N}}$ such that

$$\|a_n^{ij}(x) - a_n^{ij}(x_0)\|_{L^\infty(B_1)} \sim \frac{1}{n},$$
$$x_0 \in S_1[u_n] \cap B_{9/10},$$
$$\partial^2_{x_i x_j}(a_n^{ij}(x) u_n(x)) = 0 \quad \text{in } B_1,$$

with

$$|u_n(x) - h(x)| > \delta_0,$$

and either $h(x_0) \neq 0$ or $Dh(x_0) \neq \mathbf{0}$ for every $h \in C^{1,1}(B_{9/10})$ and $n \in \mathbb{N}$. By Proposition 7 we have that $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $C^{1,\alpha}(B_1)$. Then, through a subsequence, if necessary, there exists a function u_∞ such that

$$\|u_n - u_\infty\|_{C^{1,\gamma}(B_1)} \to 0$$

for every $0 < \gamma < \beta$. In particular

$$u_n(x_0) \to u_\infty(x_0) \quad \text{and} \quad Du_n(x_0) \to Du_\infty(x_0).$$

Then $u_\infty(x_0) = 0$ and $Du_\infty(x_0) = \mathbf{0}$. Furthermore, $a_n^{ij}(x_0) \to \bar{a}^{ij}(x_0)$ as $n \to \infty$; hence, as before, $a_n^{ij}(x) \to \bar{a}^{ij}(x_0)$ as $n \to \infty$.

Here, we evoke once again the sequential stability of the weak solutions, Proposition 6, to conclude that u_∞ solves

$$\partial^2_{x_i x_j}(\bar{a}^{ij}(x_0) u_\infty(x)) = 0 \quad \text{in } B_{9/10}.$$

The regularity theory for constant coefficients implies that $u_\infty \in C^{1,1}(B_{9/10})$. By taking $h \equiv u_\infty$, we produce a contradiction and establish the result. □

Remark 15. As in Remark 9, we notice that the norm of h in $\mathcal{C}^2$ depends on the solution u only through its L^∞-norm.

Proposition 16. *Let $u \in L^1_{\mathrm{loc}}(B_1)$ be a weak solution to* (1) *and suppose A1 and A3 are in force. Then, for every $\alpha \in (0,1)$, there exists $\varepsilon > 0$ such that, if $x_0 \in S_1[u] \cap B_{9/10}$ and*

$$\sup_{x \in B_1} |a^{ij}(x) - a^{ij}(x_0)| < \varepsilon,$$

we can find $0 < \rho \ll \frac{1}{2}$ such that

$$\sup_{B_\rho(x_0)} |Du(x) - Du(x_0)| \leq \rho^\alpha.$$

Proof. By Proposition 14, there exists $h \in C^{1,1}(B_1)$ such that

$$\|u - h\|_{C^{1,\beta}(B_{9/10})} < \delta,$$

with $x_0 \in S_1[u] \cap B_{9/10}$. We have

$$\begin{aligned}\sup_{B_\rho(x_0)} |Du(x)-Du(x_0)| &\leq \sup_{B_\rho(x_0)} |Du(x)-Dh(x)| + \sup_{B_\rho(x_0)} |Dh(x)-Dh(x_0)| + \sup_{B_\rho(x_0)} |Dh(x_0)-Du(x_0)| \\ &\leq \delta + C\rho.\end{aligned}$$

Now, by choosing

$$\rho := \left(\frac{1}{2C}\right)^{1/(1-\alpha)} \quad \text{and} \quad \delta := \frac{\rho^\alpha}{2},$$

we obtain

$$\sup_{B_\rho(x_0)} |Du(x) - Du(x_0)| \leq \rho^\alpha$$

and finish the proof. □

Proposition 17. *Let $u \in L^1_{\mathrm{loc}}(B_1)$ be a weak solution to* (1) *and suppose A1 and A3 are in force. Then, there exists $\varepsilon > 0$ such that, if $x_0 \in S_1[u] \cap B_{9/10}$ and*

$$\sup_{x \in B_1} |a^{ij}(x) - a^{ij}(x_0)| < \varepsilon,$$

we can find $0 < \rho \ll \frac{1}{2}$ for which

$$\sup_{B_{\rho^n}(x_0)} |Du(x) - Du(x_0)| \leq \rho^{n\alpha}$$

for every $n \in \mathbb{N}$ and every $\alpha \in (0,1)$.

Proof. We shall verify the proposition by induction. Notice that Proposition 16 amounts to the first step in the induction argument. Suppose we have verified the statement for $n = k$. It remains to verify it in the case $n = k+1$. Define the function

$$v_k(x) := \frac{u(x_0 + \rho^k x)}{\rho^{k(1+\alpha)}}.$$

We start by noting that $0 \in S_1[v_k]$. Additionally, v_k solves

$$\partial^2_{x_i x_j}(a_k^{ij}(x) v_k(x)) = 0 \quad \text{in } B_1, \tag{14}$$

where

$$a_k^{ij}(x) := a^{ij}(x_0 + \rho^k x).$$

It is clear that

$$\int_{B_1} |a^{ij}(x_0+\rho^k x)|^p \,\mathrm{d}x = \frac{1}{\rho^{dk}} \int_{B_{\rho^k}(x_0)} |a^{ij}(y)|^p \,\mathrm{d}y < C,$$

where the inequality follows from A1. Also,

$$\int_{B_1} |D(a^{ij}(x_0+\rho^k x))|^p \,\mathrm{d}x = \rho^{k(p-d)} \int_{B_{\rho^k}(x_0)} |Da^{ij}(y)|^p \,\mathrm{d}y < C,$$

since $p > d$, by hypothesis. Similarly

$$\int_{B_1} |D^2(a^{ij}(x_0+\rho^k x))|^p \,\mathrm{d}x = \rho^{k(2p-d)} \int_{B_{\rho^k}(x_0)} |D^2a^{ij}(y)|^p \,\mathrm{d}y < C.$$

Hence, (14) falls within the scope of Proposition 16. Therefore

$$\sup_{B_\rho} |Dv_k(x) - Dv_k(0)| \leq \rho^\alpha.$$

Rescaling back to the unit ball, the former inequality implies

$$\sup_{B_{\rho^{k+1}}(x_0)} |Du(x) - Du(x_0)| \leq \rho^{(k+1)\alpha}. \qquad \square$$

Proof of Theorem 2. The proof follows the general lines of the proof of Theorem 1 and will be omitted. □

Remark 18. As in the previous case, it is possible to extend this result to model-problems of the form

$$\partial^2_{x_i x_j}(a^{ij}(x)u(x)) + \partial_{x_i}(b^i(x)u(x)) + c(x)u(x) = f(x) \quad \text{in } B_1.$$

As before, it suffices to impose two conditions on $b : B_1 \to \mathbb{R}^d$ and $c : B_1 \to \mathbb{R}$. Indeed, the map b must be $W^{1,p}(B_1)$, and the map c must be $L^p(B_1),\ p > d$; such a requirement unlocks the uniform compactness of the solutions. Secondly, a proximity regime must be in force; that is, there must be $\bar{b} \in \mathbb{R}^d$ and $\bar{c} \in \mathbb{R}$ so that

$$\|b^i - \bar{b}^i\|_{W^{1,p}(B_1)} + \|c - \bar{c}\|_{L^\infty(B_1)} \ll \tfrac{1}{2}.$$

Acknowledgements

The authors are very grateful to an anonymous referee for comments and suggestions, which led to a substantial improvement in the material in the paper. Pimentel is partially funded by CNPq-Brazil (grants #433623/2018-7 and # 307500/2017-9) and FAPERJ (grant #E26/200.002/2018). Santos is supported by CAPES-Brazil. Part of this work was developed during a visit of Pimentel and Santos to the International Centre for Theoretical Physics (ICTP, Trieste); the authors are grateful for the Centre's support and hospitality. This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brazil (CAPES) - Finance Code 001.

References

[Bogachev and Shaposhnikov 2017] V. I. Bogachev and S. V. Shaposhnikov, "Integrability and continuity of solutions to double divergence form equations", *Ann. Mat. Pura Appl.* (4) **196**:5 (2017), 1609–1635. MR Zbl

[Bogachev et al. 1997] V. I. Bogachev, N. V. Krylov, and M. Röckner, "Elliptic regularity and essential self-adjointness of Dirichlet operators on $\mathbb{R}^n$", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **24**:3 (1997), 451–461. MR Zbl

[Bogachev et al. 2001] V. I. Bogachev, N. V. Krylov, and M. Röckner, "On regularity of transition probabilities and invariant measures of singular diffusions under minimal conditions", *Comm. Partial Differential Equations* **26**:11-12 (2001), 2037–2080. MR Zbl

[Bogachev et al. 2015] V. I. Bogachev, N. V. Krylov, M. Röckner, and S. V. Shaposhnikov, *Fokker–Planck–Kolmogorov equations*, Math. Surveys and Monographs **207**, Amer. Math. Soc., Providence, RI, 2015. MR Zbl

[Caffarelli 1989] L. A. Caffarelli, "Interior a priori estimates for solutions of fully nonlinear equations", *Ann. of Math.* (2) **130**:1 (1989), 189–213. MR Zbl

[Caffarelli and Cabré 1995] L. A. Caffarelli and X. Cabré, *Fully nonlinear elliptic equations*, Amer. Math. Soc. Colloquium Publ. **43**, Amer. Math. Soc., Providence, RI, 1995. MR Zbl

[Chen and Warren 2019] J. Chen and M. Warren, "On the regularity of Hamiltonian stationary Lagrangian submanifolds", *Adv. Math.* **343** (2019), 316–352. MR Zbl

[Fabes and Stroock 1984] E. B. Fabes and D. W. Stroock, "The L^p-integrability of Green's functions and fundamental solutions for elliptic and parabolic equations", *Duke Math. J.* **51**:4 (1984), 997–1016. MR Zbl

[Gomes et al. 2016] D. A. Gomes, E. A. Pimentel, and V. Voskanyan, *Regularity theory for mean-field game systems*, Springer, 2016. MR Zbl

[Hervé 1962] R.-M. Hervé, "Recherches axiomatiques sur la théorie des fonctions surharmoniques et du potentiel", *Ann. Inst. Fourier* (*Grenoble*) **12** (1962), 415–571. MR Zbl

[Krylov 1986] N. V. Krylov, "An approach to controlled diffusion processes", *Teor. Veroyatnost. i Primenen.* **31**:4 (1986), 685–709. In Russian; translated in *Theory Probab. Appl.* **31**:4 (1987), 604–626. MR Zbl

[Ladyzhenskaya and Uraltseva 1968] O. A. Ladyzhenskaya and N. N. Uraltseva, *Linear and quasilinear elliptic equations*, Math. Sci. Engineering **46**, Academic Press, New York, 1968. MR Zbl

[Littman 1959] W. Littman, "A strong maximum principle for weakly L-subharmonic functions", *J. Math. Mech.* **8** (1959), 761–770. MR Zbl

[Littman 1963] W. Littman, "Generalized subharmonic functions: monotonic approximations and an improved maximum principle", *Ann. Scuola Norm. Sup. Pisa* (3) **17** (1963), 207–222. MR Zbl

[Mamedov 1992] F. I. Mamedov, "On the Harnack inequality for an equation that is formally conjugate to a linear elliptic differential equation", *Sibirsk. Mat. Zh.* **33**:5 (1992), 100–106. In Russian; translated in *Siberian Math. J.* **33**:5 (1992), 835–841. MR Zbl

[Sjögren 1973] P. Sjögren, "On the adjoint of an elliptic linear differential operator and its potential theory", *Ark. Mat.* **11**:1-2 (1973), 153–165. MR Zbl

[Sjögren 1975] P. Sjögren, "Harmonic spaces associated with adjoints of linear elliptic operators", *Ann. Inst. Fourier* (*Grenoble*) **25**:3-4 (1975), 509–518. MR Zbl

[Teixeira 2014] E. V. Teixeira, "Regularity for quasilinear equations on degenerate singular sets", *Math. Ann.* **358**:1-2 (2014), 241–256. MR Zbl

Received 8 Aug 2018. Revised 18 Mar 2019. Accepted 18 Apr 2019.

RAIMUNDO LEITÃO: rleitao@mat.ufc.br
Department of Mathematics, Universidade Federal do Ceara, Fortaleza, Brazil

EDGARD A. PIMENTEL: pimentel@puc-rio.br
Department of Mathematics, Pontifical Catholic University of Rio de Janeiro, Rio de Janeiro, Brazil

MAKSON S. SANTOS: makson@mat.puc-rio.br
Department of Mathematics, Pontifical Catholic University of Rio de Janeiro, Rio de Janeiro, Brazil

mathematical sciences publishers msp

ANALYSIS AND PDE
Vol. 13, No. 4, 2020
dx.doi.org/10.2140/apde.2020.13.1145

NONEXISTENCE OF GLOBAL CHARACTERISTICS FOR VISCOSITY SOLUTIONS

VALENTINE ROOS

Two different types of generalized solutions, namely viscosity and variational solutions, were introduced to solve the first-order evolutionary Hamilton–Jacobi equation. They coincide if the Hamiltonian is convex in the momentum variable. We prove that there exists no other class of integrable Hamiltonians sharing this property. To do so, we build for any nonconvex, nonconcave integrable Hamiltonian a smooth initial condition such that the graph of the viscosity solution is not contained in the wavefront associated with the Cauchy problem. The construction is based on a new example for a saddle Hamiltonian and a precise analysis of the one-dimensional case, coupled with reduction and approximation arguments.

1. Introduction

Let $H : \mathbb{R} \times T^\star\mathbb{R}^d \to \mathbb{R}$ be a $\mathcal{C}^2$ Hamiltonian. We study the Cauchy problem associated with the evolutionary Hamilton–Jacobi equation

$$\partial_t u(t, q) + H(t, q, \partial_q u(t, q)) = 0, \tag{HJ}$$

where $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is the unknown function, with a Lipschitz initial datum $u(0, \cdot\,) = u_0$.

The method of characteristics shows that a classical solution of this equation is given by characteristics (see Section 1A). If the projections of characteristics associated with u_0 cross, the method gives rise to a multivalued solution, with a multigraph called a *wavefront* and denoted by $\mathcal{F}_{u_0}$ (see (F)). This implies in particular that for some u_0 and H, even if if smooth, the evolutionary Hamilton–Jacobi equation does not admit classical solutions in large time.

A first type of generalized solution, called a *viscosity solution* (see Section 1B), was introduced by Lions, Crandall and Evans in the early 80s for Hamilton–Jacobi equations. It possesses multiple assets: it is well-defined, unique and stable in a large range of assumptions on the Hamiltonian and the initial condition. It has a local definition avoiding the delicate question of how to choose a solution amongst the multivalued solution and its associated characteristics. This local definition can be extended effortlessly to larger classes of elliptic PDEs, which is another major asset of viscosity solutions. Also, the operator giving the viscosity solution satisfies a convenient semigroup property.

When the Hamiltonian is convex in the fiber (more precisely when it is Tonelli), this viscosity operator is given by the Lax–Oleinik semigroup, which by definition gives a section of the wavefront. The main

MSC2010: 49L25.
Keywords: Hamilton–Jacobi equation, nonconvex Hamiltonian dynamics, viscosity solution, variational solution, wavefronts, characteristics.

result of this article addresses the converse question, in the case of integrable (i.e., depending only on the fiber variable) Hamiltonians.

Theorem 1. *If $p \mapsto H(p)$ is a neither convex nor concave, integrable Hamiltonian with bounded second derivative, there exists a smooth Lipschitz initial condition u_0 such that the graph of the viscosity solution associated with u_0 is not included in the wavefront $\mathcal{F}_{u_0}$.*

The term of *variational solution* (see Section 1C) does not appear in this statement but the idea of this other generalized solution is essential in the whole article: roughly speaking, they can be defined as continuous functions whose graph is included in the wavefront. The notion was introduced in the early 90s by Sikorav and Chaperon, who found a way to choose a continuous section of the wavefront by selecting the min-max value of the generating family for the Lagrangian geometrical solution. Joukovskaia [1994] showed that their construction coincides with the Lax–Oleinik semigroup in the fiberwise convex case. The study of the variational operator given by this Chaperon–Sikorav method gives local estimates on the variational solutions. These estimates can be used regardless of the construction of the variational solution thanks to Proposition 1.9, which gives an elementary characterization of the variational solution for semiconcave initial data. This fact makes the whole article accessible to a reader with no specific background on symplectic geometry.

To show Theorem 1, we reduce the problem to the study of two key situations in dimensions 1 and 2; see Propositions 3.1 and 2.4. The example for the dimension 1 was already well-studied. It appears in [Chenciner 1975]; see also [Izumiya and Kossioris 1996]. The creation of the example for the saddle Hamiltonian in dimension 2 is the main contribution of this article. Special care was then taken to state the reduction and approximation arguments finishing the demonstration.

Recent breakthroughs have been made in the study of the singularities of the viscosity solution of (HJ) for convex Hamiltonians; see [Cannarsa et al. 2015; 2017; Cannarsa and Cheng 2018] for a survey. A natural question following from Theorem 1 is to compare these singularities for viscosity and variational solutions when the Hamiltonian is not convex anymore. On the close topic of multitime Hamilton–Jacobi equations, let us also highlight a recent discussion about the nonexistence of viscosity solutions when convexity assumptions are dropped; see [Davini and Zavidovique 2015]. This gives another point of comparison with variational solutions, which are well-defined for this framework; see [Cardin and Viterbo 2008].

Since Proposition 1.9 holds for nonintegrable Hamiltonians, we present the different objects in the nonintegrable framework. We will underline how they simplify in the integrable case. In that purpose, we introduce a second Hypothesis on H, automatically satisfied by integrable Hamiltonians with bounded second derivative, that provides the existence of both viscosity and variational solutions in the nonintegrable case.

Hypothesis 1.1. *There is a $C > 0$ such that for each (t, q, p) in $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$,*

$$\|\partial^2_{(q,p)} H(t, q, p)\| < C, \quad \|\partial_{(q,p)} H(t, q, p)\| < C(1 + \|p\|),$$

where $\partial_{(q,p)} H$ and $\partial^2_{(q,p)} H$ denote the first- and second-order spatial derivatives of H.

1A. ***Classical solutions*: *the method of characteristics.*** In this section we only assume that d^2H is bounded by C. The *Hamiltonian system*

$$\begin{cases} \dot{q}(t) = \partial_p H(t, q(t), p(t)), \\ \dot{p}(t) = -\partial_q H(t, q(t), p(t)) \end{cases} \tag{HS}$$

hence admits a complete *Hamiltonian flow* ϕ_s^t, meaning that $t \mapsto \phi_s^t(q, p)$ is the unique solution of (HS) with initial conditions $(q(s), p(s)) = (q, p)$. We denote by (Q_s^t, P_s^t) the coordinates of ϕ_s^t. We call a function $t \mapsto (q(t), p(t))$ solving the Hamiltonian system (HS) a *Hamiltonian trajectory*. The *Hamiltonian action* of a $\mathcal{C}^1$ path $\gamma(t) = (q(t), p(t)) \in T^\star\mathbb{R}^d$ is denoted by

$$\mathcal{A}_s^t(\gamma) = \int_s^t p(\tau)\cdot\dot{q}(\tau) - H(\tau, q(\tau), p(\tau))\,d\tau.$$

Note that in the case of an integrable Hamiltonian (that depends only on p), the flow is given by $\phi_s^t(q, p) = (q + (t - s)\nabla H(p), p)$ and the action of a Hamiltonian path is reduced to $\mathcal{A}_s^t(\gamma) = (t-s)(p\cdot\nabla H(p) - H(p))$.

The method of characteristics states that if u_0 is a $\mathcal{C}^2$ function with second derivative bounded by $B > 0$, there exists T depending only on C and B (for example $T = 1/(BC)$ for an integrable Hamiltonian) such that the Cauchy problem (HJ) with initial condition u_0 has a unique $\mathcal{C}^2$ solution on $[0, T] \times \mathbb{R}^d \to \mathbb{R}$. Furthermore, if u is a $\mathcal{C}^2$ solution on $[0, T] \times \mathbb{R}^d$, for all (t, q) in $[0, T] \times \mathbb{R}^d$, there exists a unique q_0 in $\mathbb{R}^d$ such that $Q_0^t(q_0, du_0(q_0)) = q$ and if γ denotes the Hamiltonian trajectory issued from $(q_0, du_0(q_0))$, the $\mathcal{C}^2$ solution is given by the Hamiltonian action as

$$u(t, q) = u_0(q_0) + \mathcal{A}_0^t(\gamma),$$

and its derivative satisfies $\partial_q u(t, q) = P_0^t(q_0, du_0(q))$ at the point $q = Q_0^t(q_0, du_0(q_0))$. As a consequence, if the image $\phi_0^t(\mathrm{graph}(du_0))$ of the graph of du_0 by the Hamiltonian flow is not a graph for some t, there is no classical solution on $[0, t] \times \mathbb{R}^d$, whence the necessity to introduce generalized solutions.

1B. ***Viscosity solutions.*** The viscosity solutions were introduced in the framework of Hamilton–Jacobi equations by Lions, Evans and Crandall in the early 80's; see [Crandall and Lions 1983]. We will use the following definition.

Definition 1.2. A continuous function u is a *subsolution* of (HJ) on the set $(0, T) \times \mathbb{R}^d$ if for each $\mathcal{C}^1$ function $\phi : (0, T)\times\mathbb{R}^d \to \mathbb{R}$ such that $u - \phi$ admits a (strict) local maximum at a point $(t, q) \in (0, T)\times\mathbb{R}^d$, we have

$$\partial_t\phi(t, q) + H(t, q, \partial_q\phi(t, q)) \le 0.$$

A continuous function u is a *supersolution* of (HJ) on the set $(0, T) \times \mathbb{R}^d \to \mathbb{R}$ if for each $\mathcal{C}^1$ function $\phi : (0, T) \times \mathbb{R}^d$ such that $u - \phi$ admits a (strict) local minimum at a point $(t, q) \in (0, T) \times \mathbb{R}^d$, we have

$$\partial_t\phi(t, q) + H(t, q, \partial_q\phi(t, q)) \ge 0.$$

A viscosity solution is both a sub- and supersolution of (HJ).

The set of assumptions of this paper is well-adapted to the theory of viscosity solutions developed by Crandall, Lions and Ishii [Crandall et al. 1992], from which one can deduce the following well-posedness property.

Proposition 1.3. *If H satisfies Hypothesis 1.1, the Cauchy problem associated with the* (HJ) *equation and a Lipschitz initial condition admits a unique Lipschitz solution. This defines a viscosity operator $(V_s^t)_{s\le t}$ on the set of Lipschitz functions $\mathcal{C}^{0,1}(\mathbb{R}^d)$ which is monotonic*:

$$V_s^t u \le V_s^t v \quad \text{if } u \le v.$$

Furthermore, if u and v are Lipschitz with bounded difference,

$$\|V_s^t u - V_s^t v\|_\infty \le \|u-v\|_\infty \quad \text{for all } s \le t.$$

In dimension 1, the theory of viscosity solutions of the (HJ) equation is the counterpart of the theory of entropy solutions for conservation laws: if $p(t,q) = \partial_q u(t,q)$ and u satisfies (HJ),

$$\partial_t p(t,q) + \partial_q(H(t,q,p(t,q))) = 0.$$

The following entropy condition, first proposed by O. Oleinik [1959] for conservation laws, gives a geometric criterion to decide if a function solves the (HJ) equation in the viscosity sense at a point of shock. It is proved for example in [Kossioris 1993, Theorem 2.2] in the modern viscosity terms, as a direct application of Theorem 1.3 in [Crandall et al. 1984]. We give the statement for H integrable, i.e., depending only on p.

Definition 1.4 (Oleinik's entropy condition). Let $H : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^2$ Hamiltonian. If $(p_1, p_2) \in \mathbb{R}^2$, we say that *Oleinik's entropy condition* is (strictly) satisfied between p_1 and p_2 if

$$H(\mu p_1 + (1-\mu)p_2) \overset{(<)}{\le} \mu H(p_1) + (1-\mu)H(p_2) \quad \text{for all } \mu \in (0,1),$$

i.e., if and only if the graph of H lies (strictly) under the cord joining $(p_1, H(p_1))$ and $(p_2, H(p_2))$.

We say that the *Lax condition* is (strictly) satisfied if

$$H'(p_1)(p_2-p_1) \overset{(<)}{\le} H(p_2) - H(p_1) \overset{(<)}{\le} H'(p_2)(p_2-p_1),$$

which is implied by the entropy condition.

Proposition 1.5. *Let $u = \min(f_1, f_2)$ on an open neighborhood U of (t,q) in $\mathbb{R}_+ \times \mathbb{R}$, with f_1 and f_2 $\mathcal{C}^1$ solutions on U of the Hamilton–Jacobi equation* (HJ). *Let p_1 and p_2 denote respectively $\partial_q f_1(t,q)$ and $\partial_q f_2(t,q)$. If $f_1(t,q) = f_2(t,q)$, then u is a viscosity solution at (t,q) if and only if the entropy condition is satisfied between p_1 and p_2.*

Oleinik's entropy condition is also valid in higher dimensions (for shock along a smooth hypersurface); see Theorem 3.1 in [Izumiya and Kossioris 1996], and can be generalized when u is the minimum of more than two functions: see [Bernard 2013].

1C. ***Variational solutions.*** If u_0 is a $\mathcal{C}^1$ initial condition, the *wavefront* associated with the Cauchy problem for u_0 is denoted by $\mathcal{F}_{u_0}$ and defined by

$$\mathcal{F}_{u_0} = \{(t, q, u_0(q_0) + \mathcal{A}_0^t(\gamma)) \mid t \geq 0,\ q \in \mathbb{R}^d,\ p_0 = du_0(q_0),\ Q_0^t(q_0, p_0) = q\}. \tag{F}$$

With this definition, the method of characteristics explained in Section 1A states that if u is a $\mathcal{C}^2$ solution on $[0, T] \times \mathbb{R}^d$, the restrictions on $[0, T] \times \mathbb{R}^d$ of the graph of u and of the wavefront coincide.

If u_0 is $\mathcal{C}^1$, we will call a *variational solution* of the Cauchy problem associated with u_0 a continuous function whose graph is included in the wavefront $\mathcal{F}_{u_0}$, i.e., a continuous function $g : [0, T] \times \mathbb{R}^d$ such that for all (t, q) in $[0, \infty) \times \mathbb{R}^d$ there exists (q_0, p_0) such that $p_0 = d_{q_0} u_0,\ Q_0^t(q_0, p_0) = q$ and

$$g(t, q) = u_0(q_0) + \mathcal{A}_0^t(\gamma),$$

where γ denotes the Hamiltonian trajectory issued from (q_0, p_0).

A family of operators $(R_s^t)_{s \leq t}$ mapping $\mathcal{C}^{0,1}(\mathbb{R}^d)$ into itself is called a *variational operator* if it satisfies the following conditions:

(1) Monotonicity: if $u \leq v$ are Lipschitz on $\mathbb{R}^d$, then $R_s^t u \leq R_s^t v$ on $\mathbb{R}^d$ for each $s \leq t$.

(2) Additivity: if u is Lipschitz on $\mathbb{R}^d$ and $c \in \mathbb{R}$, then $R_s^t(c + u) = c + R_s^t u$.

(3) Variational property: for each $\mathcal{C}^1$ Lipschitz function u_s, q in $\mathbb{R}^d$ and $s \leq t$, there exists (q_s, p_s) such that $p_s = d_{q_s} u_s,\ Q_s^t(q_s, p_s) = q$ and

$$R_s^t u_s(q) = u_s(q_s) + \mathcal{A}_s^t(\gamma),$$

where γ denotes the Hamiltonian trajectory issued from $(q(s), p(s)) = (q_s, p_s)$.

In the case of a compactly supported Hamiltonian, the existence of such a variational operator was introduced by Sikorav to his peers in 1990 and reported in [Chaperon 1991]. The author proceeded to its construction without compactness assumptions in [Roos 2019]; see Proposition 1.7.

The third property means that the variational operator maps initial data in variational solutions. There may be more than one variational solution associated with a Cauchy problem, and Proposition 1.9 states that some of them cannot be given by a variational operator. Example 1.10 presents such a situation with a nonsmooth initial value.

Remark 1.6. If a family of operators R satisfies (1) and (2), and if u and v are two Lipschitz functions on $\mathbb{R}^d$ with bounded difference, then

$$\|R_s^t u - R_s^t v\|_\infty \leq \|u - v\|_\infty.$$

As a consequence, for all $s \leq t$, R_s^t is a weak contraction, and it is continuous for the uniform norm.

Existence and local estimates. The existence of such a variational operator is given by the method of Sikorav and Chaperon; see [Viterbo 1996]. It is possible to obtain localized estimates on this family of variational operators that are also valid for the viscosity operator (in fact, they are obtained for the viscosity operator by a limit iterating process, see [Wei 2014]). They are stated explicitly for integrable Hamiltonians in [Roos 2019, Addendum 2.26].

Proposition 1.7. *There exists a family of variational operators* $(R^t_{s,H})_H$ *such that if* $H(p)$ *and* $\widetilde{H}(p)$ *are two integrable Hamiltonians with bounded second derivatives, then for* $0 \le s \le t$ *and* u *L-Lipschitz*

- $\|R^t_{s,\widetilde{H}}u - R^t_{s,H}u\|_\infty \le (t-s)\|\widetilde{H} - H\|_{\bar{B}(0,L)}$,
- $\|V^t_{s,\widetilde{H}}u - V^t_{s,H}u\|_\infty \le (t-s)\|\widetilde{H} - H\|_{\bar{B}(0,L)}$,

where $\bar{B}(0, L)$ *denotes the closed ball of radius* L *centered in* 0 *and* $\|f\|_K := \sup_K |f|$.

1D. *Extension to nonsmooth initial data.*

Lipschitz initial data. We will denote by $\partial u(q)$ the *Clarke derivative* of a function $u : \mathbb{R}^d \to \mathbb{R}$ at a point $q \in \mathbb{R}^d$. If u is Lipschitz, it is the convex envelope of the set of reachable derivatives:

$$\partial u(q) = \mathrm{co}\Big(\big\{ \lim_{n\to\infty} du(q_n) \;\big|\; q_n \to q \text{ as } n \to \infty,\ q_n \in \mathrm{dom}(du)\big\}\Big).$$

It is the singleton $\{du(q)\}$ if u is $\mathcal{C}^1$ on a neighborhood of q. Variational property (3) can be extended to include a Lipschitz initial condition with the help of this generalized derivative.

Proposition 1.8. *If* R^t_s *is a variational operator, for each Lipschitz function* u_s, q *in* $\mathbb{R}^d$ *and* $s \le t$, *there exists* (q_s, p_s) *such that* $p_s \in \partial_{q_s} u_s$, $Q^t_s(q_s, p_s) = q$ *and if* γ *denotes the Hamiltonian trajectory issued from* $(q(s), p(s)) = (q_s, p_s)$,

$$R^t_s u_s(q) = u_s(q_s) + \mathcal{A}^t_s(\gamma).$$

The proof of this proposition can be found in [Roos 2017, Proposition 1.22].

If u_0 is a Lipschitz initial condition, the generalized wavefront associated with the Cauchy problem for u_0 is still denoted by $\mathcal{F}_{u_0}$ and defined by

$$\mathcal{F}_{u_0} = \{(t, q, u_0(q_0) + \mathcal{A}^t_0(\gamma)) \mid t \ge 0,\ q \in \mathbb{R}^d,\ p_0 \in \partial u_0(q_0),\ Q^t_0(q_0, p_0) = q\}. \tag{F′}$$

Proposition 1.8 implies that a variational operator applied to u_0 gives a continuous section of the wavefront $\mathcal{F}_{u_0}$. We will still call a variational solution a Lipschitz function whose graph is contained in the generalized wavefront.

Semiconcave initial data. A function $u : \mathbb{R}^d \to \mathbb{R}$ is B*-semiconcave* if $q \mapsto u(q) - \frac{1}{2}B\|q\|^2$ is concave. The function u is *semiconcave* if there exists B for which u is B-semiconcave.

The following theorem states that every variational operator maps semiconcave initial data onto the minimal section of the wavefront $\mathcal{F}_{u_0}$, at least for $[0, T]$, where T depends only on the semiconcavity constant and on the constant C given by Hypothesis 1.1.

Proposition 1.9. *If* R^t_s *is a variational operator and if* u_0 *is a Lipschitz* B*-semiconcave initial condition for some* $B > 0$, *then there exists* $T > 0$ *depending only on* C *and* B *such that, for all* (t, q) *in* $[0, T] \times \mathbb{R}^d$,

$$\begin{aligned} R^t_0 u_0(q) &= \inf\{S \mid (t, q, S) \in \mathcal{F}_{u_0}\} \\ &= \inf\{u_0(q_0) + \mathcal{A}^t_0(\gamma) \mid (q_0, p_0) \in \mathbb{R}^d \times \mathbb{R}^d,\ p_0 \in \partial u_0(q_0),\ Q^t_0(q_0, p_0) = q\}, \end{aligned} \tag{1}$$

where γ *denotes the Hamiltonian trajectory issued from* $(q(0), p(0)) = (q_0, p_0)$.

Moreover if H *is integrable* (*i.e., depends only on* p), *we can choose* $T = 1/(BC)$.

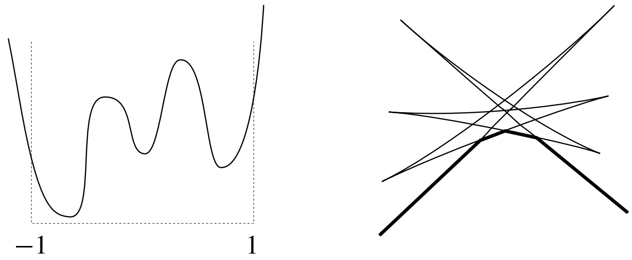

Figure 1. Left: graph of H. Right: cross-section of the wavefront $\mathcal{F}_{u_0}$ at time t.

This theorem implies on one hand that for a semiconcave initial condition, the minimal section of the wavefront is continuous for small times. On the other hand, it yields that the variational operator gives in that case a variational solution which is pointwise less than or equal to all other variational solutions on $[0, T] \times \mathbb{R}^d$.

Example 1.10. In dimension 1, if $u_0(q) = -|q|$ and if the Hamiltonian is integrable and has the shape of Figure 1, left, the wavefront at time t has the shape of Figure 1, right, and its minimal section, thickened in the figure, gives the value of $R_0^t u_0$ above each point q. In this example, there are five different variational solutions, but only the minimal one is given by a variational operator.

An analogous argument to the one proving Proposition 1.9 gives a first element of comparison between viscosity and variational solutions in the semiconcave framework. It is originally due to P. Bernard [2013].

Proposition 1.11. *Let H be a Hamiltonian satisfying Hypothesis 1.1 with constant C. If R_s^t is a variational operator and u_0 is a Lipschitz B-semiconcave initial condition for some $B > 0$, then there exists $T > 0$ depending only on C and B such that*

$$V_0^t u_0 \leq R_0^t u_0$$

for all $0 \leq t \leq T$. Consequently, the viscosity solution is smaller than any variational solution on $[0, T] \times \mathbb{R}^d$.

Moreover if H is integrable, we can choose $T = 1/(BC)$.

The article is organized as follows: Section 6 is independent from the rest; in it we prove Propositions 1.9 and 1.11 for any Hamiltonian satisfying Hypothesis 1.1. The rest of the article deals with integrable Hamiltonians: In Section 2 we prove Corollary 2.2, which is a Lipschitz version of Theorem 1. It is a corollary of Proposition 2.1, stated in terms of semiconcave initial conditions, which is proved by reduction to one- or two-dimensional considerations, contained in Propositions 2.3 and 2.4. In Section 3 we study the case of dimension 1 and prove Proposition 2.3. In Section 4 we study an example for the saddle Hamiltonian in dimension 2 in order to prove Proposition 2.4. In Section 5 we deduce Theorem 1 from its Lipschitz counterpart Corollary 2.2 by approximation.

2. Nonsmooth version of Theorem 1

Nonsmooth refers here to the initial condition. In this section we prove the following proposition, from which we deduce Corollary 2.2, which is the counterpart of Theorem 1 for a nonsmooth initial condition.

Proposition 2.1. *If $p \mapsto H(p)$ is a neither convex nor concave, integrable Hamiltonian with second derivative bounded by C, there exist $B > 0$ and a Lipschitz B-semiconcave initial condition u_0 such that the variational solution given by the minimal section of the wavefront does not solve* (HJ) *in the viscosity sense at some point (t, q) of* $[0, 1/(BC)] \times \mathbb{R}^d$.

Corollary 2.2. *If $p \mapsto H(p)$ is a neither convex nor concave, integrable Hamiltonian with bounded second derivative, there exists a Lipschitz initial condition u_0 such that the graph of the viscosity solution associated with u_0 is not included in the wavefront $\mathcal{F}_{u_0}$.*

To be more precise, the initial condition can be chosen so that the graph of the viscosity solution is below the minimal section of the wavefront for small times:

Proof of Corollary 2.2. Take a B-semiconcave initial condition u_0 as in Proposition 2.1. If C is a bound on d^2H, Proposition 1.9 states on one hand that the minimal section of the wavefront coincides with a variational solution on $[0, 1/(BC)] \times \mathbb{R}^d$, and on the other hand Proposition 1.11 gives that on the same set, the viscosity solution associated with u_0 is pointwise less than or equal to any variational solution. As a consequence the graph of the viscosity solution lies below the wavefront, and cannot coincide with the minimal section by Proposition 2.1. Hence there is a point of $[0, 1/(BC)] \times \mathbb{R}^d$ above which the graph of the viscosity solution lies strictly below the wavefront. □

The outline of the proof of Proposition 2.1 is the following: we give the statements in dimension 1 (Proposition 2.3) and for $H(p_1, p_2) = p_1 p_2$ (Proposition 2.4), and then reduce the situation to the first case or to an approximation of the second case. Proposition 2.5 gives for that purpose a characterization of neither convex nor concave functions, and Proposition 2.7 deals with the effect on the variational and viscosity operators of an affine transformation or dimensional reduction of the Hamiltonian.

Proposition 2.3 (one-dimensional case). *If $H : \mathbb{R} \to \mathbb{R}$ is a neither convex nor concave, integrable Hamiltonian with bounded second derivative, there exists $\delta > 0$ and a semiconcave Lipschitz initial condition u_0 such that*

$$R^t_{0,H} u_0 \neq V^t_{0,H} u_0 \quad \textit{for all } t < \delta.$$

Note that δ will be small enough so that $R^t_{0,H} u_0$ is uniquely defined, by Proposition 1.9. This proposition is proved in Section 3A, and is really based on the example in dimension 1 known at least since [Chenciner 1975]. In contrast, the following two-dimensional example is the main novelty of this work.

Proposition 2.4 (saddle Hamiltonian). *If $H(p_1, p_2) = p_1 p_2$, for all $L > 0$ there exists an L-Lipschitz, L-semiconcave initial condition u_0 such that*

$$R^t_{0,H} u_0 \neq V^t_{0,H} u_0 \quad \textit{for all } t < \frac{1}{2L}.$$

Note that $R^t_{0,H} u_0$ is uniquely defined when $t < 1/(2L)$ by Proposition 1.9. This proposition is proved in Section 4, where we explicitly state a suitable initial condition for which the wavefront has a single continuous section with a shock denying the entropy condition.

The following proposition makes precise the idea that a nonconvex, nonconcave function is either a wave or a saddle. We will proceed further with the reduction of a one-dimensional nonconvex, nonconcave function in Lemma 3.4.

Proposition 2.5. *A $\mathcal{C}^2$ function $f : \mathbb{R}^n \to \mathbb{R}$ is neither convex nor concave if and only if it is neither convex nor concave along a straight line, or there exists x in $\mathbb{R}^n$ such that the Hessian $\mathcal{H}f(x)$ admits both positive and negative eigenvalues.*

Proof. We denote by $S_n^+(\mathbb{R})$ and $S_n^-(\mathbb{R})$ respectively the sets of nonnegative and nonpositive symmetric matrices.

Since a $\mathcal{C}^2$ function is convex (resp. concave) if and only if its Hessian admits only nonnegative (resp. nonpositive) eigenvalues, it is enough to prove the following statement: if f is a nonconvex and nonconcave $\mathcal{C}^2$ function with $\mathcal{H}f(x) \in S_n^+(\mathbb{R}) \cup S_n^-(\mathbb{R})$ for all x, there exists a straight line along which f is neither concave nor convex.

Under the assumptions of this statement, the sets $U_1 = \{x \in \mathbb{R}^n \mid \mathcal{H}f(x) \in S_n^-(\mathbb{R}) \setminus \{0\}\}$ and $U_2 = \{x \in \mathbb{R}^n \mid \mathcal{H}f(x) \in S_n^+(\mathbb{R}) \setminus \{0\}\}$ are open and nonempty: if U_1 is empty, f is necessarily convex. If x_1 is in U_1, then $\mathcal{H}f(x_1)$ admits a negative eigenvalue. Hence for x close enough to x_1, $\mathcal{H}f(x)$ admits a negative eigenvalue and since $\mathcal{H}f(x) \in S_n^+(\mathbb{R}) \cup S_n^-(\mathbb{R})$ by hypothesis, necessarily $\mathcal{H}f(x)$ is in U_1. We are going to apply the following lemma to the continuous function $A = \mathcal{H}f$ and the sets U_1 and U_2.

Lemma 2.6. *If $A : \mathbb{R}^n \to M_n(\mathbb{R})$ is a continuous function and U_1 and U_2 are two disjoint open sets on which A does not vanish, there exists $(x_1, x_2) \in U_1 \times U_2$ such that*

$$x_1 - x_2 \notin \operatorname{Ker} A(x_1) \cup \operatorname{Ker} A(x_2).$$

Now, let us take (x_1, x_2) in $U_1 \times U_2$ such that $x_1 - x_2 \notin \operatorname{Ker} \mathcal{H}f(x_1) \cup \operatorname{Ker} \mathcal{H}f(x_2)$ and define $g(t) = f(tx_1 + (1-t)x_2)$. To show that the $\mathcal{C}^2$ function g is neither concave nor convex, we evaluate its second derivative

$$g''(t) = \mathcal{H}f(tx_1 + (1-t)x_2)(x_1 - x_2) \cdot (x_1 - x_2).$$

If A is in $S_n^+(\mathbb{R}) \cup S_n^-(\mathbb{R})$, $Ax \cdot x = 0$ if and only if $Ax = 0$. Since $\mathcal{H}f(x_1)$ is in $S_n^-(\mathbb{R})$ and $\mathcal{H}f(x_2)$ is in $S_n^+(\mathbb{R})$, and $x_1 - x_2 \notin \operatorname{Ker} \mathcal{H}f(x_1) \cup \operatorname{Ker} \mathcal{H}f(x_2)$, we obtain $g''(1) = \mathcal{H}f(x_1)(x_1 - x_2) \cdot (x_1 - x_2) < 0$ since $x_1 - x_2$ is not in $\operatorname{Ker} \mathcal{H}f(x_1)$, and $g''(0) = \mathcal{H}f(x_2)(x_1 - x_2) \cdot (x_1 - x_2) > 0$ since $x_1 - x_2$ is not in $\operatorname{Ker} \mathcal{H}f(x_2)$. Thus, g is neither concave nor convex. □

Proof of Lemma 2.6. For each $x_1^\circ \in U_1$, since $A(x_1^\circ)$ is a nonzero matrix, there exists x_2° in the open set U_2 such that $A(x_1^\circ)(x_1^\circ - x_2^\circ) \neq 0$. Since $(x_1, x_2) \mapsto A(x_1)(x_1 - x_2)$ is continuous, we may assume up to a reduction of U_1 and U_2 that $A(x_1)(x_1 - x_2) \neq 0$ for all $(x_1, x_2) \in U_1 \times U_2$.

Now let us fix x_2° in U_2. Again, since $A(x_2^\circ)$ is nonzero, there exists x_1° in the open set U_1 such that $A(x_2^\circ)(x_1^\circ - x_2^\circ) \neq 0$, and the previous argument gives that $A(x_1^\circ)(x_1^\circ - x_2^\circ) \neq 0$, hence the conclusion. □

The next proposition deals with the behavior of the variational and viscosity operators when reducing or transforming the Hamiltonian. Let us first describe formally the effect of such transformations on the classical solutions.

Affine transformations. Let H be a Hamiltonian on $\mathbb{R}^d$. Let A be an invertible matrix, b and n be vectors of $\mathbb{R}^d$, α a real value and λ a nonzero real value, and define $\overline{H}(p) = \frac{1}{\lambda} H(Ap+b) + p \cdot n + \alpha$. If $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is $\mathcal{C}^1$ and $v(t,q) = u(\lambda t, {}^tAq + \lambda tn) + b \cdot q + \alpha\lambda t$, then for all (t,q)

$$\partial_t u(\tilde{t},\tilde{q}) + \overline{H}(\partial_q u(\tilde{t},\tilde{q})) = 0 \quad \iff \quad \partial_t v(t,q) + H(\partial_q v(t,q)) = 0,$$

with $(\tilde{t},\tilde{q}) = (\lambda t, {}^tAq + \lambda tn)$.

Reduction. Assume that H is defined on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Let us fix p_2 in $\mathbb{R}^{d_2}$ and define $\overline{H}(p_1) = H(p_1, p_2)$. If $u : \mathbb{R} \times \mathbb{R}^{d_1} \to \mathbb{R}$ is $\mathcal{C}^1$ and $v(t,q_1,q_2) = u(t,q_1) + p_2 \cdot q_2$, then for all (t,q_1,q_2)

$$\partial_t u(t,q_1) + \overline{H}(\partial_{q_1} u(t,q_1)) = 0 \quad \iff \quad \partial_t v(t,q_1,q_2) + H(\partial_{q_1} v(t,q_1,q_2), \partial_{q_2} v(t,q_1,q_2)) = 0.$$

Let us translate this in terms of operators.

Proposition 2.7. *Let H be a $\mathcal{C}^2$ Hamiltonian with second derivative bounded by C:*

(1) *Affine transformations: Let u_0 be a Lipschitz B-semiconcave initial condition. If $\overline{H}(p) = \frac{1}{\lambda} H(Ap+b) + p \cdot n + \alpha$ and $v_0(q) = u_0({}^tAq) + b \cdot q$, then*

$$V^t_{0,H} v_0(q) = V^{\lambda t}_{0,\overline{H}} u_0({}^tAq + \lambda tn) + b \cdot q + \alpha\lambda t$$

for all (t,q) and

$$R^t_{0,H} v_0(q) = R^{\lambda t}_{0,\overline{H}} u_0({}^tAq + \lambda tn) + b \cdot q + \alpha\lambda t$$

as long as $t < 1/(\|A\|^2 BC)$.

(2) *Reduction: Assume that H is defined on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, fix p_2 in $\mathbb{R}^{d_2}$ and define $\overline{H}(p_1) = H(p_1,p_2)$. If u_0 is a Lipschitz B-semiconcave function on $\mathbb{R}^{d_1}$, and $v_0(q_1,q_2) = u_0(q_1) + p_2 \cdot q_2$, then*

$$V^t_{0,H} v_0(q_1,q_2) = V^t_{0,\overline{H}} u_0(q_1) + p_2 \cdot q_2$$

for all (t,q_1,q_2) and

$$R^t_{0,H} v_0(q_1,q_2) = R^t_{0,\overline{H}} u_0(q_1) + p_2 \cdot q_2,$$

as long as $t < 1/(BC)$.

Proof. The viscosity equality is obtained by applying the formal transformation or reduction on the test functions (see Definition 1.2), and the variational equality is obtained for small times by applying Proposition 1.9 with the domain of validity given for integrable Hamiltonians, which is the same for $(\overline{H}, u_0)$ and (H, v_0) in both cases:

Affine transformations: Since v_0 is $B\|A\|^2$-semiconcave, the domain of validity for (H, v_0) is at least $[0, 1/(\|A\|^2 BC))$. But $\|d^2\overline{H}\| \leq C\|A\|^2/\lambda$, and hence the domain of validity for $(\overline{H}, u_0)$ is at least $[0, \lambda/(\|A\|^2 BC))$ and λt is in this domain if $t < 1/(\|A\|^2 BC)$.

Reduction: Since $\|d^2\overline{H}\| \leq C$ and v_0 is B-semiconcave, the domain of validity for both $(\overline{H}, u_0)$ and (H, v_0) is at least $[0, 1/(BC)]$. □

Proof of Proposition 2.1. If H is a neither convex nor concave, integrable Hamiltonian, Proposition 2.5 states that there is either a straight line along which H is neither convex nor concave, or a point p_0 such that the Hessian matrix $\mathcal{H}H(p_0)$ has both a positive and a negative eigenvalue.

In the first case, applying an affine transformation on the vector space we may assume without loss of generality (see Proposition 2.7(1)) that $p \in \mathbb{R} \mapsto H(p, 0, \dots, 0)$ is neither convex nor concave, and we denote by $\overline{H}(p) = H(p, 0, \dots, 0)$ the reduced Hamiltonian. Applied to $\overline{H}$, Proposition 2.3 gives a semiconcave initial condition u_0 such that $R^t_{0,\overline{H}}u_0 \neq V^t_{0,\overline{H}}u_0$ for all $t < T$. With Proposition 2.7(2), we get from u_0 a semiconcave Lipschitz initial condition $v_0 : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ for which $R^t_{0,H}v_0 \neq V^t_{0,H}v_0$ for all $t < T$.

In the second case, we may assume that the point of interest is a (strict) saddle point at 0: if p_0 denotes the point for which $\mathcal{H}H(p_0)$ has both a positive and a negative eigenvalue, take $\widetilde{H}(p) = H(p_0 - p) + p \cdot \nabla H(p_0) - H(p_0)$ and apply Proposition 2.7(1).

Then, up to another linear transformation on the vector space, the Hamiltonian may even be taken as

$$H(p_1, p_2, \dots, p_d) = p_1 p_2 + K(p_1, p_2, \dots, p_d),$$

where K is a $\mathcal{C}^2$ Hamiltonian with partial derivatives with respect to p_1 and p_2 vanishing at the second order:

$$K(0, \dots, 0) = 0, \quad \partial_{p_1} K(0, \dots, 0) = 0, \quad \partial_{p_2} K(0, \dots, 0) = 0, \quad \partial^2_{(p_1, p_2)} K(0, \dots, 0) = 0.$$

We denote by $\overline{H}$ and $\overline{K}$ the reduced Hamiltonians such that

$$\overline{H}(p_1, p_2) = H(p_1, p_2, 0, \dots, 0) = p_1 p_2 + \overline{K}(p_1, p_2).$$

We still denote by C a bound on the second derivatives of H and $\overline{H}$.

Now, we define

$$\overline{H}_\varepsilon(p_1, p_2) = \frac{1}{\varepsilon^2}\overline{H}(\varepsilon p_1, \varepsilon p_2) = p_1 p_2 + \frac{1}{\varepsilon^2}\overline{K}(\varepsilon p_1, \varepsilon p_2)$$

and

$$\overline{H}_0(p_1, p_2) = p_1 p_2.$$

We fix $L > 0$ and take u_0 as in Proposition 2.4: for all $0 < t < 1/(2L)$, there exists a point q_t such that $R^t_{0,\overline{H}_0}u_0(q_t) \neq V^t_{0,\overline{H}_0}u_0(q_t)$. Let us now fix t in $(0, 1/(2L))$.

Proposition 1.7 gives

$$\|R^t_{0,\overline{H}_\varepsilon}u_0(q_t) - R^t_{0,\overline{H}_0}u_0(q_t)\| \leq t \sup_{\|p\|\leq L} \frac{1}{\varepsilon^2}\overline{K}(\varepsilon p),$$

$$\|V^t_{0,\overline{H}_\varepsilon}u_0(q_t) - V^t_{0,\overline{H}_0}u_0(q_t)\| \leq t \sup_{\|p\|\leq L} \frac{1}{\varepsilon^2}\overline{K}(\varepsilon p).$$

Since $\overline{K}$ is zero until second order at 0, we know that $(1/\varepsilon^2)\overline{K}(\varepsilon p) = \mathrm{o}(\|p\|^2)$ and $\sup_{\|p\|\leq L}(1/\varepsilon^2)\overline{K}(\varepsilon p)$ tends to 0 when ε tends to 0. Thus, there exists $\varepsilon > 0$ (depending on t) such that

$$\sup_{\|p\|\leq L} \frac{1}{\varepsilon^2}\overline{K}(\varepsilon p) < \frac{1}{3t}|R^t_{0,\overline{H}_0}u_0(q_t) - V^t_{0,\overline{H}_0}u_0(q_t)|,$$

and for such an ε, we then have $R^t_{0,\overline{H}_\varepsilon}u_0(q_t) \neq V^t_{0,\overline{H}_\varepsilon}u_0(q_t)$.

Let us go back to $\overline{H}$, using Proposition 2.7(1) with $\lambda = \varepsilon^2$, $A = \varepsilon\mathrm{id}$ and n, b and α equal to zero. Defining $v_0(q) = u_0(\varepsilon q)$, we get

$$V^{t/\varepsilon^2}_{0,\overline{H}} v_0\left(\frac{q_t}{\varepsilon}\right) = V^t_{0,\overline{H}_\varepsilon} u_0(q_t),$$
$$R^{t/\varepsilon^2}_{0,\overline{H}} v_0\left(\frac{q_t}{\varepsilon}\right) = R^t_{0,\overline{H}_\varepsilon} u_0(q_t),$$

as long as

$$\frac{t}{\varepsilon^2} < \frac{1}{\varepsilon^2 LC}$$

(which is the case since $C > 2$and $t < 1/(2L)$), and as a consequence

$$V^{t/\varepsilon^2}_{0,\overline{H}} v_0\left(\frac{q_t}{\varepsilon}\right) \neq R^{t/\varepsilon^2}_{0,\overline{H}} v_0\left(\frac{q_t}{\varepsilon}\right).$$

Note that since v_0 is $\varepsilon^2 L$-semiconcave, t/ε^2 belongs to the domain of validity of Proposition 1.9, which is here $(0, 1/(\varepsilon^2 LC))$. As in the previous case we get the semiconcave initial condition suiting the nonreduced Hamiltonian H via Proposition 2.7(2). □

3. One-dimensional integrable Hamiltonian

With the help of Lemma 3.4, stated and proved at the end of this section, we reduce Proposition 2.3, the one-dimensional counterpart of Proposition 2.1 (see Section 2), to the following statement, giving a situation where there is only one variational solution, which does not match with the viscosity solution.

Proposition 3.1. *Let H be a C^2 Hamiltonian with bounded second derivative such that $H(-1) = H(1) = H'(1) = 0$, $H'(-1) < 0$, $H''(1) < 0$, and $H < 0$ on $(-1, 1)$.*

Then if f is a C^2 Lipschitz function with $f(0) = f'(0) = 0$, with bounded second derivative and strictly convex on $\mathbb{R}_+$, and $u_0(q) = -|q| + f(q)$, the unique variational solution $(t, q) \mapsto R^t_0 u_0(q)$ does not solve the Hamilton–Jacobi equation (HJ) *in the viscosity sense for all t small enough.*

With the vocabulary of Definition 1.4, we work here on a specific case where the entropy condition is strictly satisfied between the derivatives at 0 of the initial condition, and the Lax condition is strictly satisfied on one side with equality on the other side; see Figure 2, left.

The proof consists in showing that under the assumptions of Proposition 3.1, when t is small enough, the wavefront at time t presents a unique continuous section, with a shock that does not satisfy Oleinik's entropy condition (see Proposition 1.5).

3A. ***Proof of Proposition 3.1.*** Let us fix the notation for the parametrization that follows directly from the wavefront definition (see (F′)). Since u_0 is differentiable on $\mathbb{R} \setminus \{0\}$, its Clarke derivative is reduced to a point outside zero and is the segment $[-1, 1]$ at zero. The wavefront is hence the union of three pieces $\mathcal{F}^\ell_t$, $\mathcal{F}^r_t$ and $\mathcal{F}^0_t$ respectively issued from the left part, the right part, and the singularity of the

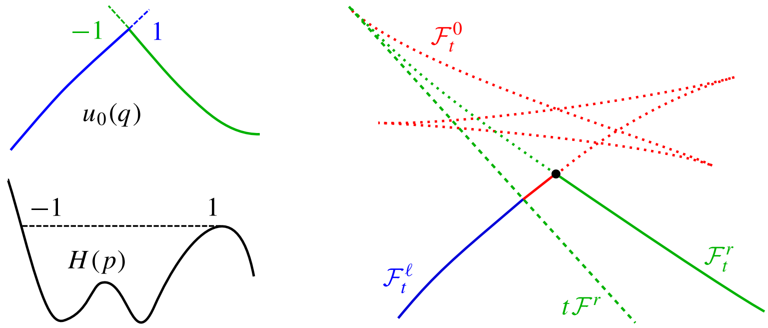

Figure 2. The variational solution, given by the unique continuous section of the wavefront, does not solve the (HJ) equation in the viscosity sense at the dot.

initial condition, with the following parametrizations:

$$\mathcal{F}_t^\ell : \begin{cases} q + tH'(u_0'(q)), \\ u_0(q) + tu_0'(q)H'(u_0'(q)) - tH(u_0'(q)), \end{cases} \qquad q < 0,$$

$$\mathcal{F}_t^r : \begin{cases} q + tH'(u_0'(q)), \\ u_0(q) + tu_0'(q)H'(u_0'(q)) - tH(u_0'(q)), \end{cases} \qquad q > 0,$$

$$\mathcal{F}_t^0 : \begin{cases} tH'(p), \\ t(pH'(p) - H(p)), \end{cases} \qquad p \in [-1, 1].$$

The structure of the wavefront for small times is addressed by Lemma 3.2. Figure 2 presents an example of the situation.

Lemma 3.2. *Under the assumptions of Proposition 3.1, there exists $\delta > 0$ such that for all $0 < t < \delta$, the wavefront $\mathcal{F}_t$ has a unique continuous section, presenting a shock between $\mathcal{F}_t^0$ and $\mathcal{F}_t^r$.*

With the previous parametrization, we may easily compute the slopes and convexity of the wavefront. We still denote by C and B the bounds on the second derivatives of H and u_0.

Proposition 3.3. (1) *Slopes on the wavefront: If $H''(p) \neq 0$ and $t > 0$, the slope of $\mathcal{F}_t^0$ at the point of parameter p is p. If $t < 1/(BC)$, the slope of $\mathcal{F}_t^r$ at the point of parameter q is $u_0'(q)$.*

(2) *Convexity of the right arm: If u_0 is convex (resp. concave) on $[0, \delta]$, then for $t < 1/(BC)$, the portion of $\mathcal{F}_t^r$ parametrized by $q \in (0, \delta]$ is convex (resp. concave).*

Proof. (1) If $(x(u), y(u))$ is the parametrization of a curve, the slope at the point of parameter u is given by $y'(u)/x'(u)$ when $x'(u)$ is nonzero. For $\mathcal{F}_t^0$, we have $x'(p) = tH''(p)$ and $y'(p) = px'(p)$, which proves the statement. For $\mathcal{F}_t^r$, if $t < 1/(BC)$, then $x'(q) = 1 + tu_0''(q)H''(u_0'(q)) > 0$ since u_0'' and H'' are respectively bounded by B and C, and the statement results from $y'(q) = u_0'(q)x'(q)$.

(2) The convexity of $\mathcal{F}_t^r$ at a point of parameter q is given by the sign of the ratio

$$\frac{x'(q)y''(q) - x''(q)y'(q)}{x'(q)^3}.$$

For $t < 1/(BC)$, we have $x'(q) > 0$ and as $y'(q) = u_0'(q)x'(q)$,

$$\frac{x'(q)y''(q) - x''(q)y'(q)}{x'(q)^3} = \frac{x'(u_0''x' + u_0'x'') - x''u_0'x'}{x'^3} = \frac{u_0''(q)}{x'(q)},$$

which proves the statement. □

The fact that $\mathcal{F}_t^0$ depends homothetically on t suggests to look for each $t > 0$ at the homothetic reduction of the wavefront at time t, where both coordinates are divided by t. We call it *reduced wavefront* and denote it by $\widetilde{\mathcal{F}}_t$. It admits the following parametrizations, where $q = tx$:

$$\widetilde{\mathcal{F}}_t^\ell : \begin{cases} x + H'(u_0'(tx)), \\ \dfrac{u_0(tx)}{t} + u_0'(tx)H'(u_0'(tx)) - H(u_0'(tx)), \end{cases} \qquad x < 0,$$

$$\widetilde{\mathcal{F}}_t^r : \begin{cases} x + H'(u_0'(tx)), \\ \dfrac{u_0(tx)}{t} + u_0'(tx)H'(u_0'(tx)) - H(u_0'(tx)), \end{cases} \qquad x > 0,$$

$$\widetilde{\mathcal{F}}_t^0 : \begin{cases} H'(p), \\ pH'(p) - H(p), \end{cases} \qquad p \in [-1, 1].$$

The asset of the reduced wavefront is that it admits a nontrivial limit when t tends to 0. The piece issued from the singularity $\mathcal{F}^0 = \widetilde{\mathcal{F}}_t^0$ does not depend on t, while $\widetilde{\mathcal{F}}_t^r$ and $\widetilde{\mathcal{F}}_t^\ell$ converge to straight half-lines denoted by $\mathcal{F}^r$ and $\mathcal{F}^\ell$. These half-lines coincide respectively with $\widetilde{\mathcal{F}}_t^r$ and $\widetilde{\mathcal{F}}_t^\ell$ at their fixed endpoints; see $t\mathcal{F}^r$ and $\mathcal{F}_t^r$ in Figure 2. A consequence of Proposition 3.3 is that $\widetilde{\mathcal{F}}_t^\ell$ is a graph as long as $t < 1/(BC)$, and the same applies to $\widetilde{\mathcal{F}}_t^r$.

Proof of Lemma 3.2. It is enough to prove the result for the reduced wavefront $\widetilde{\mathcal{F}}_t$, where both coordinates are divided by t. Using the left and right derivatives of u_0 and the fact that $H(1) = H(-1) = H'(1) = 0$, we write the parametrization of the limit of the reduced wavefront:

$$\mathcal{F}^\ell : \begin{cases} x, \\ x, \end{cases} \qquad x < 0,$$

$$\mathcal{F}^r : \begin{cases} x + H'(-1), \\ -x - H'(-1), \end{cases} \qquad x > 0,$$

$$\mathcal{F}^0 : \begin{cases} H'(p), \\ pH'(p) - H(p), \end{cases} \qquad p \in [-1, 1].$$

The left and right arms of the limit front are respectively the graph of $-\mathrm{id}$ and id on $(-\infty, 0)$ and on $(H'(-1), \infty)$, where $H'(-1) < 0$. The assumption $H < 0$ on $(-1, 1)$ implies that for all p in $(-1, 1)$,

$$pH'(p) - H(p) > -|H'(p)|, \tag{2}$$

and this inequality is also satisfied for $p = -1$ since $H(-1) = 0$ and $H'(-1) < 0$. The unique continuous section of the limit front is hence the graph of $x \mapsto -|x|$. It presents a shock at $(0, 0)$, which belongs to $\mathcal{F}^r$ and $\mathcal{F}^0$ respectively with parameters $x = -H'(-1) > 0$ and $p = 1$. Furthermore, (2) implies that this shock is not a double point of $\mathcal{F}^0$.

Proposition 3.3 states that since f is strictly convex on $\mathbb{R}_+$, $\mathcal{F}_t^r$ and hence $\widetilde{\mathcal{F}}_t^r$ are strictly convex curves for all $t > 0$. Looking at the slope for a parameter $x \to 0$ shows that $\widetilde{\mathcal{F}}_t^r$ admits the right arm of the limit front, $\mathcal{F}^r$, as a tangent at its endpoint. Since $\widetilde{\mathcal{F}}_t^r$ is convex, it is hence positioned strictly above $\mathcal{F}^r$. Since $\widetilde{\mathcal{F}}_t^\ell$ is for all $t < 1/(BC)$ a graph with fixed endpoint at $(0, 0)$, we may focus on what happens on the half-plane situated over the second diagonal.

As $H''(1) < 0$, there exists $\eta > 0$ such that $H'' < 0$ on $(1-\eta, 1]$, and the piece of $\mathcal{F}^0$ parametrized by $p \in (1-\eta, 1]$, denoted by $\mathcal{F}^0_{(1-\eta,1]}$, is immersed. Since $\mathcal{F}^0$ is compact, we may assume up to taking a smaller η that $\mathcal{F}^0_{(1-\eta,1]}$ does not contain any double point either. With this choice of η, the intersection $\mathcal{F}^r \cap \mathcal{F}^0_{(1-\eta,1]}$ is exactly the point $(0, 0)$ and is transverse, since the slopes at the shock are -1 and 1.

Let us denote the family of parametrizations of $\widetilde{\mathcal{F}}_r^t \cup \mathcal{F}^r$ by

$$g^r(t, x) = \begin{cases} (x + H'(u_0'(tx)), u_0(tx)/t + u_0'(tx)H'(u_0'(tx)) - H(u_0'(tx))) & \text{if } t \neq 0, \\ (x + H'(-1), -x - H'(-1)) & \text{if } t = 0. \end{cases}$$

The function $t \mapsto g^r(t, \cdot)$ is continuous on $[0, \infty)$ in the $\mathcal{C}^1$-topology since the function

$$(t, x) \mapsto \begin{cases} u_0(tx)/t & \text{if } t > 0, \\ -x & \text{if } t = 0 \end{cases}$$

is $\mathcal{C}^1$ on $[0, \infty) \times [0, \infty)$. The transverse intersection hence persists by the implicit function theorem in an intersection between $\widetilde{\mathcal{F}}_t^r$ and $\mathcal{F}^0_{(1-\eta,1]}$, since $\widetilde{\mathcal{F}}_t^r$ is contained in the half-plane situated over the second diagonal.

There is no other continuous section in $\widetilde{\mathcal{F}}_t$: for small times t, $\widetilde{\mathcal{F}}_t^r$ and $\widetilde{\mathcal{F}}_t^\ell$ do not cross and do not present double points; the existence of a second continuous section would then imply the existence of an intersection between $\mathcal{F}^0$ and the part of $\widetilde{\mathcal{F}}_t^r$ at the right of the shock, or an intersection between $\mathcal{F}^0$ and $\widetilde{\mathcal{F}}_t^\ell$, and neither can exist, by continuity. □

It is now enough to prove that the obtained shock denies the Lax condition.

Proof of Proposition 3.1. For all t, the graph of a variational solution is included in the wavefront $\mathcal{F}_t$. Lemma 3.2 gives a $\delta > 0$ for which every $\mathcal{F}_t$ has a unique continuous section if $t \leq \delta$, which implies that the variational solution is given by this section for small t. Lemma 3.2 states also that this section presents a shock between $\mathcal{F}_t^0$ and $\mathcal{F}_t^r$.

Let us prove that Lax condition is violated at this shock. A fortiori, Oleinik's entropy condition is violated, which by Proposition 1.5 will imply that the variational solution is not a viscosity solution. For all t in $(0, \delta)$, the shock is given by parameters (q_t, p_t) such that $q_t > 0$, $p_t \in [-1, 1]$ and

$$\begin{cases} q_t + tH'(u_0'(q_t)) = tH'(p_t), \\ u_0(q_t) + tu_0'(q_t)H'(u_0'(q_t)) - tH(u_0'(q_t)) = tp_tH'(p_t) - tH(p_t). \end{cases}$$

Substituting the first equation multiplied by $u_0'(q_t)$ into the second gives, after reorganization,

$$t\big(H(p_t) - H(u_0'(q_t)) - (p_t - u_0'(q_t))H'(p_t)\big) = q_t u_0'(q_t) - u_0(q_t).$$

The linear part of u_0 cancels in the right-hand side, which equals $q_t f'(q_t) - f(q_t)$. The strict convexity of f implies that $f'(h) > f(h)/h$ for all $h > 0$; hence the right-hand side is positive for $t > 0$. As a

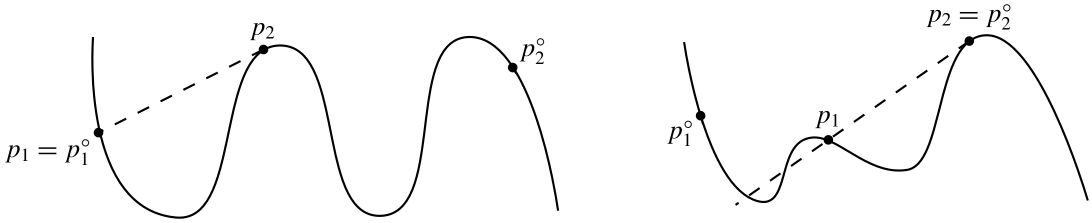

Figure 3. Both figures present a graph of H with a dashed tangent at p_2. Left: entropy condition not satisfied between p_1° and p_2°. Right: entropy condition satisfied between p_1° and p_2°.

consequence, for t in $(0, \delta)$,

$$H(p_t) - H(u_0'(q_t)) > (p_t - u_0'(q_t))H'(p_t).$$

By Proposition 3.3, the slopes at the shock are $u_0'(q_t)$ and p_t. Comparing with Definition 1.4, this inequality reads as the opposite of the Lax condition; hence Oleinik's entropy condition is violated for the shock presented by the variational solution for $t < \delta$, and the conclusion holds. □

3B. ***Proof of Proposition 2.3.*** The idea behind Lemma 3.4 is that for any nonconvex nonconcave Hamiltonian in dimension 1, there is a frame of variables over which the Hamiltonian looks like Figure 2, left.

Lemma 3.4. *If $H : \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^2$, neither convex nor concave Hamiltonian, up to a change of function $p \mapsto H(-p)$, there exist $p_1 < p_2$ such that $H''(p_2) < 0$, and,*

$$\text{for all } p \in (p_1, p_2), \quad \frac{H(p) - H(p_1)}{p - p_1} < \frac{H(p_2) - H(p_1)}{p_2 - p_1}, \tag{3}$$

$$H'(p_1) < \frac{H(p_2) - H(p_1)}{p_2 - p_1} = H'(p_2). \tag{4}$$

In terms of Definition 1.4, (3) means that the entropy condition is strictly satisfied between p_1 and p_2, and (4) that the Lax condition is an equality at p_2 and an inequality at p_1. We are now just one affine step away from the hypotheses of Proposition 3.1.

Proof. If $H : \mathbb{R} \to \mathbb{R}$ is neither convex nor concave, there exist in particular p_1° and p_2° such that $H''(p_1^\circ) > 0$ and $H''(p_2^\circ) < 0$, and we may assume up to the change of Hamiltonian $p \mapsto H(-p)$ that $p_1^\circ < p_2^\circ$.

Sketch of proof. The proof consists in choosing adequate p_1 and p_2, which will be done differently depending on the entropy condition between p_1° and p_2° being satisfied or not. An impatient reader could be satisfied by the choice of p_1 and p_2 suggested in Figure 3. If the entropy condition is not satisfied, we take $p_1 = p_1^\circ$ and p_2 such that the slope of the cord joining p_1 and p_2 is maximal. We then need to slightly perturb p_1 in order to get the condition $H''(p_2) < 0$. If the entropy condition is satisfied, we take $p_2 = p_2^\circ$ and p_1 is given by the last (before p_2) intersection between the tangent at p_2 and the graph of H. Again, a perturbation will be done to ensure that $H'(p_1) < H'(p_2)$.

- If the entropy condition is not satisfied between p_1° and p_2°, we define $p_1 = p_1^\circ$ and

$$p_2 = \inf\left\{ p \in (p_1, p_2^\circ) \;\middle|\; \frac{H(p) - H(p_1)}{p - p_1} = \sup_{\tilde{p} \in (p_1, p_2^\circ]} \frac{H(\tilde{p}) - H(p_1)}{\tilde{p} - p_1} \right\}.$$

Let us show that these quantities are well-defined, and prove (3) and (4).

The function

$$f : p \mapsto \frac{H(p) - H(p_1)}{p - p_1}$$

may be extended continuously at p_1 by $H'(p_1)$; hence it reaches a maximum M on $[p_1, p_2^\circ]$. It cannot be attained at p_1, or else the Taylor expansion of

$$\frac{H(p) - H(p_1)}{p - p_1} \le H'(p_1)$$

gives $H''(p_1) \le 0$, which is excluded. As a consequence $M > H'(p_1)$. It cannot be attained at p_2° because

$$\frac{H(p) - H(p_1)}{p - p_1} \le \frac{H(p_2^\circ) - H(p_1)}{p_2^\circ - p_1}$$

for all p in $[p_1, p_2^\circ)$ if and only if the entropy condition is satisfied between p_1 and p_2°, which is excluded. We hence proved that the supremum is attained on (p_1, p_2°). The infimum thus exists and belongs to $[p_1, p_2^\circ)$. By the continuity of f, we have $f(p_2) = M$. This implies that $p_2 > p_1$ since $f(p_1) = H'(p_1) < M$; hence the infimum is a minimum. The equality (3) follows directly from the definition of p_2.

Since p_2 is in (p_1, p_2°) and maximizes f, it is a critical point of f, which gives

$$H'(p_2) = \frac{H(p_2) - H(p_1)}{p_2 - p_1} = M.$$

Since $H'(p_1) < M$, (4) is proved.

Since p_2 maximizes f, we have $f''(p_2) \le 0$ and as a consequence $H''(p_2) \le 0$. In order to get $H''(p_2) < 0$, let us prove that if p_2° is fixed, $p_1 \mapsto H'(p_2)$ is increasing in a neighborhood of p_1.

For $\varepsilon > 0$ small enough, $p_1 + \varepsilon < p_2$, $H''(p_1 + \varepsilon) > 0$ and the entropy condition is not satisfied between $p_1 + \varepsilon$ and p_2°. We denote by $p_{2,\varepsilon}$ the quantity associated with $p_1 + \varepsilon$ and p_2° as before.

On one hand, by the definition of p_2, the entropy condition is strictly satisfied between p_1 and p_2, and in particular since $p_1 + \varepsilon$ is in (p_1, p_2),

$$\frac{H(p_2) - H(p_1 + \varepsilon)}{p_2 - (p_1 + \varepsilon)} > \frac{H(p_2) - H(p_1)}{p_2 - p_1} = H'(p_2).$$

On the other hand, the previous work applied to $p_{2,\varepsilon}$ gives

$$H'(p_{2,\varepsilon}) = \max_{p \in (p_1 + \varepsilon, p_2^\circ]} \frac{H(p) - H(p_1 + \varepsilon)}{p - (p_1 + \varepsilon)} \ge \frac{H(p_2) - H(p_1 + \varepsilon)}{p_2 - (p_1 + \varepsilon)},$$

and the two inequalities combined give $H'(p_{2,\varepsilon}) > H'(p_2)$.

Since $p_1 \mapsto H'(p_2)$ is increasing in a neighborhood of p_1, using Sard's theorem, we may assume without loss of generality that $H'(p_2)$ is a regular value of H', up to a perturbation of p_1 within the open set $\{H'' > 0\}$. As a consequence, $H''(p_2) < 0$, and the pair (p_1, p_2) satisfies Lemma 3.4.

- If the entropy condition is satisfied between p_1° and p_2°, we define $p_2 = p_2^\circ$ and

$$p_1 = \sup\left\{ p_1^\circ \le p \le p_2 \,\middle|\, \frac{H(p_2) - H(p)}{p_2 - p} = H'(p_2) \right\}. \tag{5}$$

As $H''(p_2)$ is negative, the graph of H is situated strictly under its tangent at p_2 over a neighborhood of p_2; hence

$$\frac{H(p_2) - H(p)}{p_2 - p} > H'(p_2)$$

on this neighborhood. The entropy condition satisfied between p_1° and p_2 implies the Lax condition

$$\frac{H(p_2) - H(p_1^\circ)}{p_2 - p_1^\circ} \ge H'(p_2).$$

By the mean value theorem, the considered set is nonempty and its supremum belongs to $[p_1^\circ, p_2)$, and by continuity of

$$p \mapsto \frac{H(p_2) - H(p)}{p_2 - p},$$

we have

$$\frac{H(p_2) - H(p_1)}{p_2 - p_1} = H'(p_2).$$

The entropy condition is strictly satisfied between p_1 and p_2 by the maximality of p_1. The mean value theorem and the maximality of p_1 make it clear that $H'(p_1) \le H'(p_2)$ and that if $H'(p_1) = H'(p_2)$, $H''(p_1) \le 0$. Let us prove that up to a perturbation we can assume $H'(p_1) < H'(p_2)$.

Let us hence assume that $H'(p_1) = H'(p_2)$. First, by Sard's theorem, up to a perturbation of p_2°, we may assume that $H'(p_2^\circ)$ is not a critical value of H', which ensures, since $H'(p_1) = H'(p_2^\circ)$, that $H''(p_1)$ is nonzero, and hence negative (note that the sign of $H''(p_1^\circ)$ had no influence in the previous paragraph). We set $p_1^\circ = p_1$ and look at the previous construction for this p_1° fixed and for a new p_2 close to p_2°. Without loss of generality we suppose that $H'(p_2^\circ) = H'(p_1^\circ) = H(p_2^\circ) = H(p_1^\circ) = 0$. Since $H''(p_1^\circ)$ and $H''(p_2^\circ)$ are negative, there exists δ such that H'' is negative on $[p_1^\circ, p_1^\circ + \delta] \cup [p_2^\circ - \delta, p_2^\circ]$. By compactness, H admits a maximum on $[p_1^\circ + \delta, p_2^\circ - \delta]$ which is negative, since the entropy condition is strictly satisfied between p_1° and p_2°.

Since H' is decreasing on $[p_2^\circ - \delta, p_2^\circ]$, there exists $p_2 \in [p_2^\circ - \delta, p_2^\circ]$ such that

$$0 < H'(p_2) < -\frac{\frac{1}{2}m}{p_2^\circ - p_1^\circ}.$$

For such a p_2, the tangent of the graph of H at p_2 lies strictly below the graph of H over $[p_1^\circ + \delta, p_2^\circ - \delta]$ by definition of m, and also over $[p_2^\circ - \delta, p_2^\circ]$ by the concavity of H. Equation (5) then defines a p_1 which is necessarily in $(p_1^\circ, p_1^\circ + \delta]$: as $H(p_1^\circ) = 0$ and $H(p_2) < 0$, the point $(p_1^\circ, H(p_1^\circ))$ is situated over the tangent of the graph of H at p_2 which has a positive slope $H'(p_2)$. By concavity of H on $[p_1^\circ, p_1^\circ+\delta]$, $H'(p_1) < H'(p_1^\circ) = 0$, and as a consequence $H'(p_1) < H'(p_2)$. The previous work proves that all the conditions of the proposition are then gathered for p_1 and p_2. □

We may now prove Proposition 2.3, joining Lemma 3.4 and Proposition 3.1.

Proof of Proposition 2.3. Let H be a nonconvex, nonconcave Hamiltonian with bounded second derivative. Using Proposition 2.7(1) with $A = -\mathrm{id}$, we may apply Lemma 3.4 up to the change of function $p \mapsto H(-p)$. It gives $p_1 < p_2$ such that $H''(p_2) < 0$, and

$$H'(p_1) < \frac{H(p) - H(p_1)}{p - p_1} < H'(p_2) = \frac{H(p_2) - H(p_1)}{p_2 - p_1}$$

for all p in (p_1, p_2). We define

$$\widetilde{H}(p) = H(p) - H(p_2) - (p - p_2)H'(p_2),$$

so that $\widetilde{H}(p_2) = \widetilde{H}'(p_2) = \widetilde{H}(p_1) = 0$. Note also that $\widetilde{H}'(p_1) = H'(p_1) - H'(p_2) < 0$. The second-order derivatives as well as the entropy condition are preserved by this transformation: $\widetilde{H}''(p_2) = H''(p_2) < 0$, and $\widetilde{H} < 0$ on (p_1, p_2).

At last, we take the affine transformation $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi(-1) = p_1$ and $\phi(1) = p_2$ and define

$$\overline{H}(p) = \widetilde{H}(\phi(p)),$$

so that $\overline{H}$ satisfies the assumptions of Proposition 3.1: $\overline{H}(-1) = \overline{H}(1) = \overline{H}'(1) = 0$ and $\overline{H}'(-1) < 0$ since $\phi' > 0$, $\overline{H}''(1) < 0$ and $\overline{H} < 0$ on $(-1, 1)$. Proposition 3.1 then gives a Lipschitz semiconcave initial condition $\bar{u}_0$ such that the variational solution denies the (HJ) equation associated with $\overline{H}$ for all t small enough. Proposition 2.7(1) applied to the two successive transformations gives then a Lipschitz semiconcave initial condition u_0, with right and left derivatives at 0 respectively equal to p_1 and p_2, such that the variational solution denies the (HJ) equation associated with H for all t small enough. □

4. Example for the saddle Hamiltonian: proof of Proposition 2.4

In this section we assume that $H(p_1, p_2) = p_1 p_2$, with $(p_1, p_2) \in \mathbb{R}^2$, and prove Proposition 2.4 by presenting a suitable initial condition. For a convex-concave Hamiltonian, [Bernardi and Cardin 2011; Wei 2013] proved that the variational solution coincides with the viscosity solution for initial conditions with separated variables; hence the wanted initial condition cannot be elementary.

We choose an initial condition that coincides with the piecewise quadratic function $u(q_1, q_2) = \min(a(q_1^2 - q_2), b(q_1^2 - q_2))$ on a large enough subset while being Lipschitz and semiconcave. We make explicit the value of the variational solution for this initial condition on a large enough subset.

Proposition 4.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a compactly supported C^2 function coinciding with $x \mapsto x^2$ on $[-1, 1]$.*
Let $u(q_1, q_2) = \min(a(f(q_1) - q_2), b(f(q_1) - q_2))$ with $b > a > 0$.
Then if $-1 \le q_1 \le -\frac{3b}{2}t$,

$$R_0^t u(q_1, q_2) = \min(a((q_1 + at)^2 - q_2), b((q_1 + bt)^2 - q_2)).$$

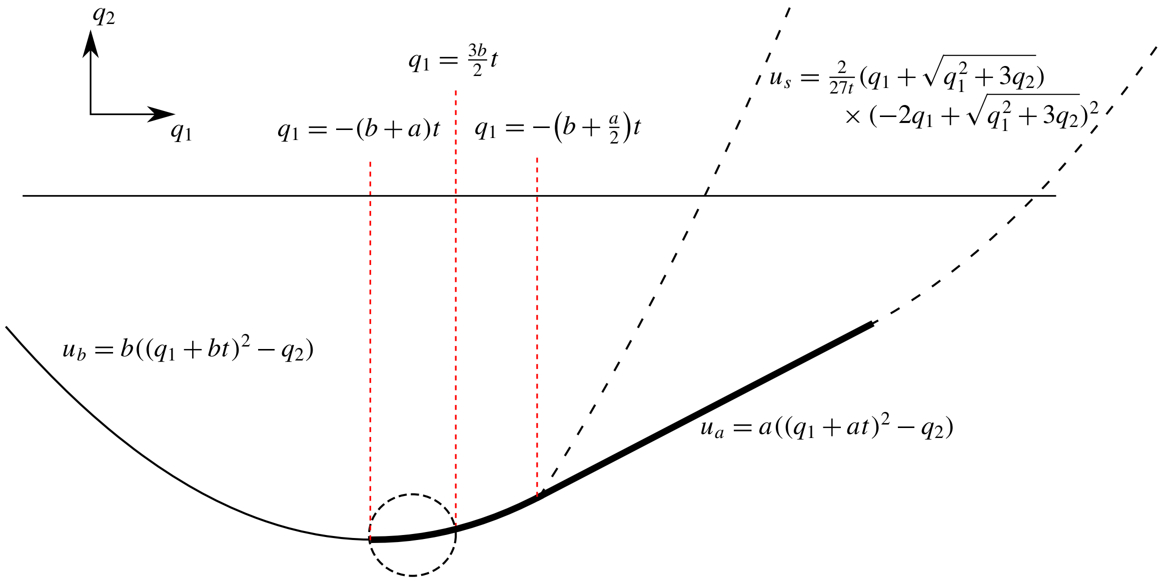

Figure 4. Value of the variational solution associated with u at time t, here for $a = 1$, $b = 2$ and $t = \frac{1}{10}$.

Figure 4 shows the explicit value of the variational solution for small times t, which is given by the unique continuous section in the wavefront. The plain curve represents a shock of the variational solution, whereas the different expressions coincide $\mathcal{C}^1$-continuously along the dotted curves. One can show that the variational solution does not satisfy the Hamilton–Jacobi equation in the viscosity sense along the thick portion of the shock, and also that it does satisfy the Hamilton–Jacobi equation in the viscosity sense everywhere except on this portion. For the purpose of this article, it is enough to show that the variational solution does not satisfy the Hamilton–Jacobi equation in the viscosity sense along the parabola circled in Figure 4. This is included in the domain concerned by Proposition 4.1, which can be proved by using an efficient convexity argument that spares us many computations.

Proof of Proposition 4.1. Using general arguments stated in Section 6, we are first going to prove that

$$R_0^t u(q_1, q_2) = \min_{c\in[a,b]} u_c(t, q_1, q_2) \quad \text{for all } t \geq 0,\ (q_1, q_2) \in \mathbb{R}^2,$$

where $u_c(t, q_1, q_2) = c(f(q_1 + ct) - q_2)$ is the unique $\mathcal{C}^2$ solution of the Cauchy problem associated with $H(p_1, p_2) = p_1 p_2$ and the initial condition $u_c^0 : (q_1, q_2) \mapsto c(f(q_1) - q_2)$.

We want to apply Proposition 6.2, observing that $u = \min_{c\in[a,b]} u_c^0$. To do so, we only need to check that the family $\{u_c^0, c \in [a, b]\}$ satisfies the conditions of Lemma 6.1, i.e., that for all (q, p) in the graph of the Clarke derivative ∂u, there exists $c \in [a, b]$ such that $u_c^0(q) = u(q)$ and $du_c^0(q) = p$.

Let us compute the Clarke derivative of u. If $f(q_1) > q_2$, then $u(q_1, q_2) = a(f(q_1) - q_2)$ on a neighborhood of (q_1, q_2); hence $\partial u(q_1, q_2)$ is reduced to the point $a\binom{f'(q_1)}{-1}$, which is also the derivative of u_a^0 at (q_1, q_2). If $f(q_1) < q_2$, then $\partial u(q_1, q_2)$ is reduced to the point $b\binom{f'(q_1)}{-1}$ which is also the derivative of u_b^0 at (q_1, q_2). If $f(q_1) = q_2$, then $\partial u(q_1, q_2)$ is the segment $\left\{c\binom{f'(q_1)}{-1} \mid c \in [a, b]\right\}$. For all $c \in [a, b]$, $c\binom{f'(q_1)}{-1}$ is the derivative of u_c^0 at the point $(q_1, q_2 = f(q_1))$.

We hence proved that the family $\{u_c^0, c \in [a, b]\}$ satisfies the condition of Lemma 6.1; thus by Proposition 6.2

$$R_0^t u(q_1, q_2) = \min_{c\in[a,b]} u_c(t, q_1, q_2) \quad \text{for all } t \geq 0,\ (q_1, q_2) \in \mathbb{R}^2.$$

Now, for all $-1 < q_1 < -bt$, we have $f(q_1 + ct) = (q_1 + ct)^2$ since $f(x) = x^2$ for x in $[-1, 1]$, and $c \in (0, b]$. Hence if $-1 < q_1 < -bt$,

$$R_0^t u(q_1, q_2) = \min_{c\in[a,b]} c((q_1 + ct)^2 - q_2).$$

The second derivative of $g : c \mapsto c((q_1 + ct)^2 - q_2)$ is $g''(c) = 2t(2q_1 + 3ct)$. Hence if $q_1 < -\frac{3b}{2}t$, then g is concave on $[a, b]$ and the minimum defining $R_0^t u(q_1, q_2)$ is attained at an endpoint of $[a, b]$.

Thus, we proved that for $-1 < q_1 < -\frac{3b}{2}t$,

$$R_0^t u(q_1, q_2) = \min(a((q_1 + at)^2 - q_2), b((q_1 + bt)^2 - q_2)). \qquad \square$$

Proof of Proposition 2.4. Let $b > 0$, $a \in \left(\frac{b}{2}, b\right)$ and u be defined as in Proposition 4.1: f is a compactly supported $\mathcal{C}^2$ function coinciding with $x \mapsto x^2$ on $[-1, 1]$ and

$$u(q_1, q_2) = \min(a(f(q_1) - q_2), b(f(q_1) - q_2)).$$

We define $u_a : (t, q_1, q_2) \mapsto a((q_1 + at)^2 - q_2)$ and $u_b : (t, q_1, q_2) \mapsto b((q_1 + bt)^2 - q_2)$ (note that the notations slightly differ from the previous proof), so that Proposition 4.1 gives that for $-1 < q_1 < -\frac{3b}{2}t$

$$R_0^t u(q_1, q_2) = \min(u_a(t, q_1, q_2), u_b(t, q_1, q_2)).$$

Let us prove that this variational solution does not satisfy the Hamilton–Jacobi equation at the point (t, q_1, q_2) if

$$\begin{cases} q_2 = q_1^2 + 2(a + b)tq_1 + t^2(a^2 + ab + b^2), \\ -1 < q_1 < -\frac{3b}{2}t, \\ -(a + b)t < q_1. \end{cases}$$

This corresponds to the piece of parabola circled in Figure 4, which exists only if $a > \frac{b}{2}$ and $t < \frac{2}{3b}$. Note that the first line is just an equation of this parabola, which is obtained by solving $u_a = u_b$.

Let us exhibit a test function denying the viscosity equation: we define the mean function $\phi = \frac{1}{2}(u_a + u_b)$, which is $\mathcal{C}^1$, larger than $\min(u_a, u_b)$ on a neighborhood of (t, q_1, q_2) and equal to it at (t, q_1, q_2) since $u_a(t, q_1, q_2) = u_b(t, q_1, q_2)$, so that $R_0^t u - \phi$ attains a local maximum at (t, q_1, q_2). The derivatives of ϕ are given by

$$\begin{aligned} \partial_t \phi(t, q_1, q_2) &= a^2(q_1 + at) + b^2(q_1 + bt), \\ \partial_{q_1} \phi(t, q_1, q_2) &= a(q_1 + at) + b(q_1 + bt), \\ \partial_{q_2} \phi(t, q_1, q_2) &= -\tfrac{1}{2}(a + b). \end{aligned}$$

We compute

$$\partial_t\phi(t,q_1,q_2)+H(\partial_q\phi(t,q_1,q_2)) = a^2(q_1+at)+b^2(q_1+bt)-\tfrac{1}{2}(a+b)(a(q_1+at)+b(q_1+bt))$$
$$= \tfrac{1}{2}(a-b)^2(at+bt+q_1)>0$$

when $q_1 > -(a+b)t$, and as a consequence the variational solution is not a viscosity subsolution at the point (t, q_1, q_2).

Note that b can be chosen as small as needed, and hence for all L we are able to take the initial condition u L-Lipschitz and L-semiconcave, with $b \le L$. The previous work shows that for all $t < \frac{2}{3b}$, the variational solution does not satisfy the Hamilton–Jacobi solution in the viscosity sense at some point (t, q). But since $L \ge b$, we have $\frac{L}{2} \le \frac{2}{3b}$ and we hence have proved Proposition 2.4. □

5. Proof of Theorem 1

In this section we will deduce Theorem 1 from Corollary 2.2. To do so, we approach the Lipschitz initial condition of Corollary 2.2 by a smooth initial condition, keeping the Hausdorff distance between the (Clarke) derivatives small. We will use elementary properties of the Hausdorff distance, stated in Lemmas 5.1 and 5.2 and proved for completeness.

The Hausdorff distance d_{Haus} is defined (though not necessarily finite) by

$$d_{\text{Haus}}(X,Y)=\sup\bigl(\sup_{x\in X} d(x,Y),\ \sup_{y\in Y} d(y,X)\bigr)$$

for X and Y closed subsets of a metric space (E, d) (d being the euclidean distance on $\mathbb{R}^d$ in our context). The following approximation result is proved in [Czarnecki and Rifford 2006, Theorem 2.2] and its Corollary 2.1:

Theorem 2. *If $u : \mathbb{R}^d \to \mathbb{R}$ is locally Lipschitz, there exists a sequence of smooth functions u_n such that*

$$\lim_{n\to\infty} \|u_n - u\|_\infty = 0,$$
$$\lim_{n\to\infty} d_{\text{Haus}}(\text{graph}(du_n), \text{graph}(\partial u)) = 0,$$

where ∂ denotes the Clarke derivative.

Here is a sketch of the proof: for H an integrable nonconvex, nonconcave Hamiltonian with bounded second derivative, Corollary 2.2 gives a Lipschitz initial condition u_L such that the graph of the viscosity solution is not included in the wavefront $\mathcal{F}_{u_L}$ for some time $t > 0$. We are going to approach u_L by a Lipschitz smooth function u such that both the viscosity solutions at time t are close, and the Hausdorff distance between the wavefronts at time t is small. The following enhanced triangle inequality will conclude that the graph of the viscosity solution associated with u is not included in the wavefront $\mathcal{F}_u$.

Lemma 5.1 (enhanced triangle inequality). *If (E, d) is a metric space and X and Y are subsets of E, then for all x and y in E*

$$d(x,X) \le d(x,y)+d(y,Y)+d_{\text{Haus}}(X,Y).$$

Proof. The triangle inequality for d gives that, for all x, $\tilde{x}$ and y, we have $d(x, \tilde{x}) \le d(x, y) + d(y, \tilde{x})$, and taking the infimum for $\tilde{x}$ on X gives

$$d(x, X) \le d(x, y) + d(y, X) \quad \text{for all } x, y \in E. \tag{6}$$

We change the variables in (6): for all y and $\tilde{y}$,

$$d(y, X) \le d(y, \tilde{y}) + d(\tilde{y}, X).$$

If $\tilde{y}$ is in Y, by definition of the Hausdorff distance we get

$$d(y, X) \le d(y, \tilde{y}) + d_{\text{Haus}}(X, Y)$$

and taking the infimum for $\tilde{y}$ on Y gives

$$d(y, X) \le d(y, Y) + d_{\text{Haus}}(X, Y).$$

We conclude by injecting this last inequality into (6). □

To bound the Hausdorff distance between the wavefronts, we will describe the wavefront at time t as the image of the (Clarke) derivative of the initial condition by a suitable function ψ depending on the initial condition, which will allow us to apply the following elementary continuity result for the Hausdorff distance.

Lemma 5.2 (continuity for the Hausdorff distance). *Let $f, g : (F, \tilde{d}) \mapsto (E, d)$ be two functions between two topological spaces, and X and Y be two subsets of F:*

(1) *If $d(f(x), g(x)) \le a$ for all x in X, then $d_{\text{Haus}}(f(X), g(X)) \le a$.*

(2) *If f is uniformly continuous on X, i.e., for all $\alpha > 0$, there exists $\varepsilon > 0$ such that for all $(x, y) \in X$ $\tilde{d}(x, y) < \varepsilon$ implies $d(f(x), f(y)) < \alpha$, then*

$$\tilde{d}_{\text{Haus}}(X, Y) < \varepsilon \quad \Longrightarrow \quad d_{\text{Haus}}(f(X), f(Y)) < \alpha.$$

Proof of Lemma 5.2. (1) By the definition of the Hausdorff distance, it is enough to observe that $d(f(x), g(X)) \le a$ for all x in X, since this quantity is smaller than $d(f(x), g(x))$.

(2) Using the symmetry of the definition of d_{Haus}, it is enough to prove that if $\tilde{d}_{\text{Haus}}(X, Y) < \varepsilon$, $d(f(x), f(Y)) < \alpha$ for all x in X. For all x in X, there exists a sequence y_n in Y such that $\tilde{d}(x, y_n) \to \tilde{d}(x, Y)$ as $n \to \infty$. Since $\tilde{d}(x, Y) \le \tilde{d}_{\text{Haus}}(X, Y)$, this implies that $\tilde{d}(x, y_n) < \varepsilon$ for n large enough, and the uniform continuity of f gives that $d(f(x), f(y_n)) < \alpha$ for n large enough; hence $d(f(x), f(Y)) < \alpha$. □

Proof of Theorem 1. Let H be an integrable nonconvex, nonconcave Hamiltonian with bounded second derivative. Corollary 2.2 gives a Lipschitz initial condition u_L for which there exist $t > 0$ and q such that

$$d((q, V_0^t u_L(q)), \mathcal{F}_{u_L}^t) > 0,$$

where $\mathcal{F}_{u_L}^t$ denotes the section of $\mathcal{F}_{u_L}$ at time t. We denote by α this positive quantity.

Let us denote by L the Lipschitz constant of u_L.

We propose another description of the wavefront at time t: if v is a Lipschitz function, we define

$$\psi_v^t : T^\star\mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}, \qquad (q, p) \mapsto (q + t\nabla H(p), v(q) + t(p\cdot\nabla H(p) - H(p))),$$

in such a way that $\mathcal{F}_v^t = \psi_v^t(\mathrm{graph}(\partial v))$ (see (F′) for a comparison).

Note that ψ_v^t is Lipschitz, and hence uniformly continuous on every $\mathbb{R}^d \times \{\|p\| \le R\}$ for $R > 0$: it is Lipschitz with respect to q because v is, and its derivative with respect to p, $(td^2H(p), tp\cdot d^2H(p))$, is bounded on this set since d^2H is bounded.

The uniform continuity of $\psi_{u_L}^t$ on $\mathbb{R}^d \times \{p \le L+1\}$ gives an $\varepsilon \in (0, 1)$ such that

$$\begin{cases} \|(q, p) - (\tilde{q}, \tilde{p})\| < \varepsilon, \\ \|p\|, \|\tilde{p}\| \le L+1, \end{cases} \implies \|\psi_{u_L}^t(q, p) - \psi_{u_L}^t(\tilde{q}, \tilde{p})\| < \tfrac{1}{4}\alpha.$$

By Theorem 2, there exists a smooth function u such that

$$\|u - u_L\|_\infty < \tfrac{1}{4}\alpha, \tag{7}$$

$$d_{\mathrm{Haus}}(\mathrm{graph}(du), \mathrm{graph}(\partial u_L)) < \varepsilon. \tag{8}$$

Note that since $\varepsilon \in (0, 1)$, u is $(L+1)$-Lipschitz.

On the one hand, Proposition 1.3 gives the comparison between the viscosity solutions:

$$\|V_0^t u - V_0^t u_L\|_\infty \le \|u - u_L\|_\infty \le \tfrac{1}{4}\alpha.$$

On the other hand, we estimate the Hausdorff distance between the wavefronts, using the definition of ψ:

$$\begin{aligned} d_{\mathrm{Haus}}(\mathcal{F}_u^t, \mathcal{F}_{u_L}^t) &= d_{\mathrm{Haus}}(\psi_u^t(\mathrm{graph}(du)), \psi_{u_L}^t(\mathrm{graph}(\partial u_L))) \\ &\le d_{\mathrm{Haus}}(\psi_u^t(\mathrm{graph}(du)), \psi_{u_L}^t(\mathrm{graph}(du))) + d_{\mathrm{Haus}}(\psi_{u_L}^t(\mathrm{graph}(du)), \psi_{u_L}^t(\mathrm{graph}(\partial u_L))). \end{aligned}$$

The first part of Lemma 5.2 applied with $f = \psi_{u_L}^t,\ g = \psi_u^t,\ X = \mathrm{graph}(du)$ gives that the first term of the right-hand side is bounded by $\|\psi_u^t - \psi_{u_L}^t\|_\infty = \|u - u_L\|_\infty \le \frac{1}{4}\alpha$.

The second part of Lemma 5.2 applied with $f = \psi_{u_L}^t,\ X = \mathrm{graph}(\partial u_L)$ and $Y = \mathrm{graph}(du)$ gives that the second term of the right-hand side is smaller than $\frac{1}{4}\alpha$, by uniform continuity of $\psi_{u_L}^t$, since $\mathrm{graph}(du)$ and $\mathrm{graph}(\partial u_L)$ are both contained in $\mathbb{R}^d \times \{p \le L+1\}$ and are ε-close for the Hausdorff distance; see (8). We hence proved that

$$d_{\mathrm{Haus}}(\mathcal{F}_u^t, \mathcal{F}_{u_L}^t) \le \tfrac{1}{2}\alpha.$$

Let us now apply Lemma 5.1 with $x = (q, V_0^t u_L(q)),\ y = (q, V_0^t u_L(q)),\ X = \mathcal{F}_{u_L}^t$ and $Y = \mathcal{F}_u^t$:

$$\begin{aligned} \alpha &= d((q, V_0^t u_L(q)), \mathcal{F}_{u_L}^t) \\ &\le \underbrace{d((q, V_0^t u_L(q)), (q, V_0^t u(q)))}_{\le \|V_0^t u_L - V_0^t u\|_\infty \le \frac{1}{4}\alpha} + d((q, V_0^t u(q)), \mathcal{F}_u^t) + \underbrace{d_{\mathrm{Haus}}(\mathcal{F}_{u_L}^t, \mathcal{F}_u^t)}_{\le \frac{1}{2}\alpha}. \end{aligned}$$

As a consequence, $d((q, V_0^t u(q)), \mathcal{F}_u^t) \ge \frac{1}{4}\alpha > 0$ and the graph of the viscosity solution associated with the smooth initial condition u is not contained in the wavefront $\mathcal{F}_u$. □

6. Semiconcavity arguments

This section contains the proofs of Propositions 1.9 and 1.11, as well as an additional Proposition 6.2 used in the proof of the two-dimensional case (see Section 4). The three proofs rely on the following lemma, proved in [Bernard 2013, Lemma 6]:

Lemma 6.1. *If u is a Lipschitz and B-semiconcave function on $\mathbb{R}^d$, there exists a family F of $\mathcal{C}^2$ equi-Lipschitz functions with second derivatives bounded by B such that*

- $u(q) = \min_{f \in F} f(q)$ *for any q,*
- *for each q in $\mathbb{R}^d$ and p in $\partial u(q)$, there exists f in F such that*

$$\begin{cases} f(q) = u(q), \\ df(q) = p. \end{cases}$$

Proof of Proposition 1.9. Proposition 1.8 states that the variational solution gives a section of the generalized wavefront. As a consequence

$$R_0^t u_0(q) \geq \inf\{u_0(q_0) + \mathcal{A}_0^t(\gamma) \mid (q_0, p_0) \in \mathbb{R}^d \times \mathbb{R}^d,\ p_0 \in \partial u_0(q_0),\ Q_0^t(q_0, p_0) = q\}.$$

If u_0 is L-Lipschitz and B-semiconcave, take T such that the method of characteristics is valid ($T = 1/(BC)$ if H is integrable). Let us fix definitively q, q_0, $p_0 \in \partial u_0(q_0)$ and $0 \leq t \leq T$ such that $Q_0^t(q_0, p_0) = q$ and show that $R_0^t u_0(q) \leq u_0(q_0) + \mathcal{A}_0^t(\gamma)$, where γ is the Hamiltonian trajectory issued from (q_0, p_0).

Lemma 6.1 gives a $\mathcal{C}^2$ function f_0 of F such that $f_0(q_0) = u_0(q_0)$ and $df_0(q_0) = p_0$. Since this function is $\mathcal{C}^2$ with second derivative bounded by B, the method of characteristics gives that q_0 is the only point such that $Q_0^t(q_0, df_0(q_0)) = q$, and the variational operator applied to the initial condition f_0 gives necessarily the $\mathcal{C}^2$ solution:

$$R_0^t f_0(t, q) = f_0(q_0) + \mathcal{A}_0^t(\gamma).$$

But by the definition of F, f_0 is larger than u_0 on $\mathbb{R}^d$, and the monotonicity of the variational operator brings the conclusion

$$R_0^t u_0(q) \leq R_0^t f_0(q) = f_0(q_0) + \mathcal{A}_0^t(\gamma) = u_0(q_0) + \mathcal{A}_0^t(\gamma). \qquad \square$$

Proof of Proposition 1.11. Take T such that the method of characteristics is valid (for example $T = 1/(BC)$ if H is integrable).

If t and q are fixed, Proposition 1.8 gives the existence of (q_0, p_0) in $gr(\partial u_0)$ such that $Q_0^t(q_0, p_0) = q$ and that $R_0^t u_0(q) = u_0(q_0) + \mathcal{A}_0^t(\gamma)$, where γ is the Hamiltonian trajectory issued from (q_0, p_0).

Lemma 6.1 gives a $\mathcal{C}^2$ function f_0 of F such that $f_0(q_0) = u_0(q_0)$ and $df_0(q_0) = p_0$. The method of characteristics states that there exists on $[0, T] \times \mathbb{R}^d$ a unique $\mathcal{C}^2$ solution of the (HJ) equation with initial condition f_0, which satisfies in particular

$$f(t, q) = f_0(q_0) + \mathcal{A}_0^t(\gamma).$$

Since a $\mathcal{C}^1$ solution is a viscosity solution, the uniqueness of viscosity solutions hence gives that $V_0^t f = f(t,\cdot)$ for all t in $(0,T)$, and in particular

$$V_0^t f_0(q) = f(t,q) = f_0(q_0) + \mathcal{A}_0^t(\gamma).$$

But by the definition of F, f_0 is larger than u_0 on $\mathbb{R}^d$, and the monotonicity of the viscosity operator V_0^t brings the conclusion

$$V_0^t u_0(q) \le V_0^t f_0(t,q) = f_0(q_0) + \mathcal{A}_0^t(\gamma) = R_0^t u_0(q).$$

Since $(t,q) \mapsto R_0^t u_0(q)$ is pointwise less than or equal to any variational solution as long as $t < T$ (Proposition 1.9), this implies that for all variational solutions g, $V_0^t u_0(q) \le g(t,q)$ on $[0,T]\times\mathbb{R}^d$. □

We end this section with another result of the same flavor, used in the proof of Proposition 2.4.

Proposition 6.2. *Let F be as in Lemma 6.1 and $u = \min_{f\in F} f$. If $T > 0$ denotes a time of shared existence of $\mathcal{C}^2$ solutions for initial conditions in F, and u_f denotes the $\mathcal{C}^2$ solution of the Hamilton–Jacobi equation associated with the $\mathcal{C}^2$ initial condition f, then for all $0 \le t \le T$*

$$R_0^t u(q) = \min_{f\in F} u_f(t,q).$$

Proof. Since $u \le f$ for all f in F, the monotonicity of the variational operator guarantees that $R_0^t u(q) \le \min_{f\in F} R_0^t f(q)$. The method of characteristics implies that the variational operator is given by the classical solution if it exists; hence $R_0^t f(q) = u_f(t,q)$ for all t in $[0,T]$ and thus

$$R_0^t u(q) \le \min_{f\in F} u_f(t,q). \tag{9}$$

Now, for all (t,q), the variational property gives the existence of a (q_0,p_0) in the graph of ∂u such that

$$R_0^t u(q) = u(q_0) + \mathcal{A}_0^t(\gamma),$$

where γ denotes the Hamiltonian trajectory issued from (q_0,p_0). Since F is as in Lemma 6.1, there exists f in F such that $f(q_0) = u(q_0)$ and $df(q_0) = p_0$. The method of characteristics implies furthermore that $u_f(t,q) = f(q_0) + \mathcal{A}_0^t(\gamma)$. Summing all this up, we get

$$R_0^t u(q) = u(q_0) + \mathcal{A}_0^t(\gamma) = f(q_0) + \mathcal{A}_0^t(\gamma) = u_f(t,q)$$

and the inequality (9) is an equality. □

Acknowledgement

The author is very grateful to P. Bernard and J.-C. Sikorav who helped organize this paper and proposed many improvements in the argumentation, as well as to M. Zavidovique for a fruitful discussion. The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP/2007-2013)/ERC Grant Agreement 307062 and from the French National Research Agency via ANR-12-BLAN-WKBHJ.

References

[Bernard 2013] P. Bernard, "Semi-concave singularities and the Hamilton–Jacobi equation", *Regul. Chaotic Dyn.* **18**:6 (2013), 674–685. MR Zbl

[Bernardi and Cardin 2011] O. Bernardi and F. Cardin, "On C^0-variational solutions for Hamilton–Jacobi equations", *Discrete Contin. Dyn. Syst.* **31**:2 (2011), 385–406. MR Zbl

[Cannarsa and Cheng 2018] P. Cannarsa and W. Cheng, "On and beyond propagation of singularities of viscosity solutions", preprint, 2018. arXiv

[Cannarsa et al. 2015] P. Cannarsa, M. Mazzola, and C. Sinestrari, "Global propagation of singularities for time dependent Hamilton–Jacobi equations", *Discrete Contin. Dyn. Syst.* **35**:9 (2015), 4225–4239. MR Zbl

[Cannarsa et al. 2017] P. Cannarsa, W. Cheng, and A. Fathi, "On the topology of the set of singularities of a solution to the Hamilton–Jacobi equation", *C. R. Math. Acad. Sci. Paris* **355**:2 (2017), 176–180. MR Zbl

[Cardin and Viterbo 2008] F. Cardin and C. Viterbo, "Commuting Hamiltonians and Hamilton–Jacobi multi-time equations", *Duke Math. J.* **144**:2 (2008), 235–284. MR Zbl

[Chaperon 1991] M. Chaperon, "Lois de conservation et géométrie symplectique", *C. R. Acad. Sci. Paris Sér. I Math.* **312**:4 (1991), 345–348. MR Zbl

[Chenciner 1975] A. Chenciner, "Aspects géométriques de l'étude des chocs dans les lois de conservation", pp. 1–37 in *Problèmes d'évolution non linéaires: séminaire de Nice*, 1975.

[Crandall and Lions 1983] M. G. Crandall and P.-L. Lions, "Viscosity solutions of Hamilton–Jacobi equations", *Trans. Amer. Math. Soc.* **277**:1 (1983), 1–42. MR Zbl

[Crandall et al. 1984] M. G. Crandall, L. C. Evans, and P.-L. Lions, "Some properties of viscosity solutions of Hamilton–Jacobi equations", *Trans. Amer. Math. Soc.* **282**:2 (1984), 487–502. MR Zbl

[Crandall et al. 1992] M. G. Crandall, H. Ishii, and P.-L. Lions, "User's guide to viscosity solutions of second order partial differential equations", *Bull. Amer. Math. Soc.* (*N.S.*) **27**:1 (1992), 1–67. MR Zbl

[Czarnecki and Rifford 2006] M.-O. Czarnecki and L. Rifford, "Approximation and regularization of Lipschitz functions: convergence of the gradients", *Trans. Amer. Math. Soc.* **358**:10 (2006), 4467–4520. MR Zbl

[Davini and Zavidovique 2015] A. Davini and M. Zavidovique, "On the (non) existence of viscosity solutions of multi-time Hamilton–Jacobi equations", *J. Differential Equations* **258**:2 (2015), 362–378. MR Zbl

[Izumiya and Kossioris 1996] S. Izumiya and G. T. Kossioris, "Formation of singularities for viscosity solutions of Hamilton–Jacobi equations", pp. 127–148 in *Singularities and differential equations* (Warsaw, 1993), edited by S. Janeczko et al., Banach Center Publ. **33**, Polish Acad. Sci. Inst. Math., Warsaw, 1996. MR Zbl

[Joukovskaia 1994] T. Joukovskaia, *Singularités de minimax et solutions faibles d'équations aux dérivées partielles*, thèse de doctorat, Université Paris Diderot, 1994.

[Kossioris 1993] G. T. Kossioris, "Formation of singularities for viscosity solutions of Hamilton–Jacobi equations in one space variable", *Comm. Partial Differential Equations* **18**:5-6 (1993), 747–770. MR Zbl

[Oleinik 1959] O. A. Oleinik, "Uniqueness and stability of the generalized solution of the Cauchy problem for a quasi-linear equation", *Uspehi Mat. Nauk* **14**:2 (86) (1959), 165–170. In Russian; translated in *Amer. Math. Soc. Transl.* (2) **33** (1963), 285–290. MR Zbl

[Roos 2017] V. Roos, *Variational and viscosity solutions of the Hamilton–Jacobi equation*, thèse de doctorat, PSL Université, 2017, available at http://perso.ens-lyon.fr/valentine.roos/these.pdf.

[Roos 2019] V. Roos, "Variational and viscosity operators for the evolutionary Hamilton–Jacobi equation", *Commun. Contemp. Math.* **21**:4 (2019), art. id. 1850018, 76 pp. MR Zbl

[Viterbo 1996] C. Viterbo, "Solutions of Hamilton–Jacobi equations and symplectic geometry. Addendum to: *Séminaire sur les Équations aux Dérivées Partielles 1994–1995*", in *Séminaire sur les Équations aux Dérivées Partielles, 1995–1996*, École Polytech., Palaiseau, 1996. MR Zbl

[Wei 2013] Q. Wei, *Solutions de viscosité des équations de Hamilton–Jacobi et minmax itérés*, thèse de doctorat, Université Paris Diderot, 2013.

[Wei 2014] Q. Wei, "Viscosity solution of the Hamilton–Jacobi equation by a limiting minimax method", *Nonlinearity* **27**:1 (2014), 17–41. MR Zbl

Received 22 Aug 2018. Revised 23 Mar 2019. Accepted 18 Apr 2019.

VALENTINE ROOS: valentine.roos@ens-lyon.fr
UMPA, ENS Lyon, Lyon, France

ANALYSIS AND PDE
Vol. 13, No. 4, 2020
dx.doi.org/10.2140/apde.2020.13.1173

STRICHARTZ ESTIMATES FOR THE SCHRÖDINGER FLOW ON COMPACT LIE GROUPS

YUNFENG ZHANG

We establish scale-invariant Strichartz estimates for the Schrödinger flow on any compact Lie group equipped with canonical rational metrics. In particular, full Strichartz estimates without loss for some nonrectangular tori are given. The highlights of this paper include estimates for some Weyl-type sums defined on rational lattices, different decompositions of the Schrödinger kernel that accommodate different positions of the variable inside the maximal torus relative to the cell walls, and an application of the BGG-Demazure operators or Harish-Chandra's integral formula to the estimate of the difference between characters.

1. Introduction

We start with a complete Riemannian manifold (M, g) of dimension d, associated to which are the Laplace–Beltrami operator Δ_g and the volume-form measure μ_g. Then it is well known that Δ_g is essentially self-adjoint on $L^2(M) := L^2(M, d\mu_g)$; see [Strichartz 1983] for a proof. This gives the functional calculus of Δ_g, and in particular gives the one-parameter unitary operator $e^{it\Delta_g}$, which provides the solution to the linear Schrödinger equation on (M, g). We refer to $e^{it\Delta_g}$ as the *Schrödinger flow*. The functional calculus of Δ_g also gives the definition of the Bessel potentials, and thus the definition of the Sobolev space

$$H^s(M) := \{u \in L^2(M) \mid \|u\|_{H^s(M)} := \|(I-\Delta)^{\frac{s}{2}} u\|_{L^2(M)} < \infty\}.$$

We are interested in obtaining estimates of the form

$$\|e^{it\Delta_g} f\|_{L^p L^r(I\times M)} \le C\|f\|_{H^s(M)}, \tag{1-1}$$

MSC2010: primary 42B37; secondary 22E30.
Keywords: compact Lie groups, Schrödinger equation, circle method, Strichartz estimates, BGG-Demazure operators, Harish-Chandra's integral formula.

where $I \subset \mathbb{R}$ is a fixed time interval, and $L^p L^q(I \times M)$ is the space of L^p functions on I with values in $L^q(M)$. Such estimates are often called Strichartz estimates (for the Schrödinger flow), in honor of Robert Strichartz [1977] who first derived such estimates for the wave equation on Euclidean spaces.

The significance of Strichartz estimates is evident in many ways. Strichartz estimates have important applications in the field of nonlinear Schrödinger equations, in the sense that many perturbative results often require good control on the linear solution, which is exactly provided by Strichartz estimates. Strichartz estimates can also be interpreted as Fourier restriction estimates, which play a fundamental rule in the field of classical harmonic analysis. Furthermore, the relevance of the distribution of eigenvalues and the norm of eigenfunctions of Δ_g in deriving the estimates makes Strichartz estimates also a subject in the field of spectral geometry.

Many cases of Strichartz estimates for the Schrödinger flow are known in the literature. For noncompact manifolds, first we have the sharp Strichartz estimates on the Euclidean spaces obtained in [Ginibre and Velo 1995; Keel and Tao 1998]:

$$\|e^{it\Delta} f\|_{L^p L^q(\mathbb{R}\times\mathbb{R}^d)} \le C\|f\|_{L^2(\mathbb{R}^d)}, \tag{1-2}$$

where $\frac{2}{p}+\frac{d}{q}=\frac{d}{2}$, $p,q\ge 2$, $(p,q,d)\ne(2,\infty,2)$. Such pairs (p,q) are called *admissible*. This implies by Sobolev embedding that

$$\|e^{it\Delta} f\|_{L^p L^r(\mathbb{R}\times\mathbb{R}^d)} \le C\|f\|_{H^s(\mathbb{R}^d)}, \tag{1-3}$$

where

$$s=\frac{d}{2}-\frac{2}{p}-\frac{d}{r}\ge 0, \tag{1-4}$$

$p,q\ge 2$, $(p,r,d)\ne(2,\infty,2)$. Note that the equality in (1-4) can be derived from a standard scaling argument, and we call exponent triples (p,r,s) that satisfy (1-4) as well as the corresponding Strichartz estimates *scale-invariant*. Similar Strichartz estimates hold on many noncompact manifolds. For example, see [Anker and Pierfelice 2009; Banica 2007; Ionescu and Staffilani 2009; Pierfelice 2006] for Strichartz estimates on the real hyperbolic spaces, [Anker et al. 2011; Pierfelice 2008; Banica and Duyckaerts 2007] for Damek–Ricci spaces which include all rank-1 symmetric spaces of noncompact type, [Bouclet 2011] for asymptotically hyperbolic manifolds, [Hassell et al. 2006] for asymptotically conic manifolds, [Bouclet and Tzvetkov 2008; Staffilani and Tataru 2002] for some perturbed Schrödinger equations on Euclidean spaces, and [Fotiadis et al. 2018] for symmetric spaces G/K, where G is complex.

For compact manifolds, however, Strichartz estimates such as (1-2) are expected to fail. The Sobolev exponent s in (1-1) is expected to be positive for (1-1) to possibly hold. And we also expect sharp Strichartz estimates that are *non-scale-invariant*, in the sense that the exponents (p,r,s) in (1-1) satisfy

$$s>\frac{d}{2}-\frac{2}{p}-\frac{d}{r}.$$

For example, from the results in [Staffilani and Tataru 2002; Burq et al. 2004], we know that on a general compact Riemannian manifold (M,g) it holds that, for any finite interval I,

$$\|e^{it\Delta_g} f\|_{L^p L^r(I\times M)} \le C\|f\|_{H^{1/p}(M)} \tag{1-5}$$

for all admissible pairs (p, r). These estimates are non-scale-invariant, and the special case of which when $(p, r, s) = \big(2, \frac{2d}{d-2}, \frac{1}{2}\big)$ can be shown to be sharp on spheres of dimension $d \geq 3$ equipped with canonical Riemannian metrics. On the other hand, scale-invariant estimates are out of reach of the local methods employed in [Staffilani and Tataru 2002; Burq et al. 2004], and they are not well explored yet in the literature. To my best knowledge, the only known results in the literature in this direction are on Zoll manifolds, which include all compact symmetric spaces of rank 1, the standard sphere being a typical example, and on rectangular tori. We summarize the results here. Consider the scale-invariant estimates

$$\|e^{it\Delta_g} f\|_{L^p(I\times M)} \leq C\|f\|_{H^{d/2-(d+2)/p}(M)}. \tag{1-6}$$

In the direction of Zoll manifolds, (1-6) is first proved in [Burq et al. 2007] for the standard three-sphere for $p = 6$. Then in [Herr 2013], (1-6) is proved for all $p > 4$ for any three-dimensional Zoll manifold, but the methods employed in that paper in fact prove (1-6) for $p > 4$ for any Zoll manifold with dimension $d \geq 3$ and for $p \geq 6$ for any Zoll surface ($d = 2$). The paper crucially uses the property of Zoll manifolds that the spectrum of the Laplace–Beltrami operator is clustered around a sequence of squares, and the spectral cluster estimates [Sogge 1988] which are optimal on spheres. In the direction of tori, (1-6) was first proved in [Bourgain 1993] for $p \geq \frac{2(d+4)}{d}$ on square tori, by interpolating the distributional Strichartz estimate

$$\begin{aligned}\lambda\cdot\mu\{(t,x)\in I\times\mathbb{T}^d \mid |e^{it\Delta_g}\varphi(N^{-2}\Delta_g)f(x)|>\lambda\}^{\frac{1}{p}} &\leq CN^{\frac{d}{2}-\frac{d+2}{p}}\|f\|_{L^2(\mathbb{T}^d)}\\ &\leq C\|f\|_{H^{d/2-(d+2)/p}(\mathbb{T}^d)}\end{aligned} \tag{1-7}$$

for $\lambda > N^{d/4}$, $p > \frac{2(d+2)}{d}$, $N \geq 1$, with the trivial subcritical Strichartz estimate

$$\|e^{it\Delta_g} f\|_{L^2(I\times\mathbb{T}^d)} \leq C\|f\|_{L^2(\mathbb{T}^d)}. \tag{1-8}$$

The estimate (1-7) is a consequence of an arithmetic version of dispersive estimates:

$$\|e^{it\Delta_g}\varphi(N^{-2}\Delta_g)\|_{L^\infty(\mathbb{T}^d)} \leq C\left(\frac{N}{\sqrt{q}\big(1+N\big\|\frac{t}{T}-\frac{a}{q}\big\|^{1/2}\big)}\right)^d \|f\|_{L^1(\mathbb{T}^d)}, \tag{1-9}$$

where $\|\cdot\|$ stands for the distance from 0 on the standard circle with length 1, $\big\|\frac{t}{T}-\frac{a}{q}\big\| < \frac{1}{qN}$, a, q are nonnegative integers with $a < q$ and $(a, q) = 1$, and $q < N$. Here T is the period for the Schrödinger flow $e^{it\Delta_g}$. Then in [Bourgain 2013], the author improved (1-8) into a stronger subcritical Strichartz estimate

$$\|e^{it\Delta_g} f\|_{L^{2(d+1)/d}(I\times\mathbb{T}^d)} \leq C\|f\|_{L^2(\mathbb{T}^d)}, \tag{1-10}$$

which yields (1-6) for $p \geq \frac{2(d+3)}{d}$. Eventually, (1-6) with an ε-loss is proved for the full range $p > \frac{2(d+2)}{d}$ in [Bourgain and Demeter 2015], and (1-7) can be used to remove this ε-loss. Then authors in [Guo et al. 2014; Killip and Vişan 2016] extended the results to all rectangular tori. We will see in this paper that by a slight adaptation of the methods in [Bourgain 1993], we may generalize (1-7) to all rational (not necessarily rectangular) tori $\mathbb{T}^d = \mathbb{R}^d/\Gamma$, where $\Gamma \cong \mathbb{Z}^d$ is a lattice such that there exists some $D \neq 0$

for which $\langle \lambda, \mu \rangle \in D^{-1}\mathbb{Z}$ for all $\lambda, \mu \in \Gamma$, which can also be used for the removal of the ε-loss of the results in [Bourgain and Demeter 2015] to yield (1-6) for the full range $p > \frac{2(d+2)}{d}$ on such rational tori.

The understanding of Strichartz estimates on compact manifolds is far from complete. It is not known in general how the exponents (p, r, s) in the sharp Strichartz estimates are related to the geometry and topology of the underlying manifold. Also, there still are important classes of compact manifolds on which Strichartz estimates have not been explored yet. Note that both standard tori and spheres on which Strichartz estimates are known are special cases of compact globally symmetric spaces, and since all compact globally symmetric spaces share the same behavior of geodesic dynamics as tori, from a semiclassical point of view, it's natural to conjecture that similar Strichartz estimates should hold on general compact globally symmetric spaces. An important class of such spaces is the class of compact Lie groups. The goal of this paper is to prove scale-invariant Strichartz estimates of the form (1-6) for $M = G$ being any connected compact Lie group equipped with a canonical *rational metric* in the sense that is described below, for all $p \geq \frac{2(r+4)}{r}$, r being the *rank* of G. In particular, full Strichartz estimates without loss for some nonrectangular tori will be given.

2. Statement of the main theorem

2A. *Rational metric.* Let G be a connected compact Lie group and $\mathfrak{g}$ be its Lie algebra. By the classification theorem of connected compact Lie groups, see [Procesi 2007, Chapter 10, Section 7.2, Theorem 4], there exists an exact sequence of Lie group homomorphisms

$$1 \to A \to \widetilde{G} \cong \mathbb{T}^n \times K \to G \to 1,$$

where $\mathbb{T}^n$ is the n-dimensional torus, K is a compact simply connected semisimple Lie group, and A is a finite and central subgroup of the *covering group* $\widetilde{G}$. As a compact simply connected semisimple Lie group, K is a direct product $K_1 \times K_2 \times \cdots \times K_m$ of compact simply connected simple Lie groups.

Now each K_i is equipped with the canonical bi-invariant Riemannian metric g_i that is induced from the negative of the Cartan–Killing form. We use $\langle \cdot, \cdot \rangle$ to denote the Cartan–Killing form. Then we equip the torus factor $\mathbb{T}^n$ with a flat metric g_0 inherited from its representation as the quotient $\mathbb{R}^n / 2\pi\Gamma$ and require that there exists some $D \in \mathbb{N}$ such that $\langle \lambda, \mu \rangle \in D^{-1}\mathbb{Z}$ for all $\lambda, \mu \in \Gamma$. Then we equip $\widetilde{G} \cong \mathbb{T}^n \times K_1 \times \cdots \times K_m$ with the bi-invariant metric

$$\tilde{g} = \bigotimes_{j=0}^{m} \beta_j g_j, \tag{2-1}$$

$\beta_j > 0$, $j = 0, \ldots, m$. Then $\tilde{g}$ induces a bi-invariant metric g on G.

Definition 2.1. Let g be the bi-invariant metric induced from $\tilde{g}$ in (2-1) as described above. We call g a *rational metric* provided the numbers $\beta_0, \ldots, \beta_m$ are rational multiples of each other. If not, we call it an *irrational metric*.

Provided the numbers $\beta_0, \ldots, \beta_m$ are rational multiples of each other, the periods of the Schrödinger flow $e^{it\Delta_{\tilde{g}}}$ on each factor of $\widetilde{G}$ are rational multiples of each other, which implies that the Schrödinger flow on $\widetilde{G}$, as well as on G, is also periodic (see Section 5).

2B. *Main theorem.* We define the *rank* of G to be the dimension of any of its maximal torus. This paper mainly proves the following theorem.

Theorem 2.2. *Let G be a connected compact Lie group equipped with a rational metric g. Let d be the dimension of G and r the rank of G. Let $I \subset \mathbb{R}$ be a finite time interval. Consider the scale-invariant Strichartz estimate*

$$\|e^{it\Delta_g} f\|_{L^p(I\times G)} \le C\|f\|_{H^{d/2-(d+2)/p}(G)}. \tag{2-2}$$

Then the following statements hold true:

(i) (2-2) *holds for all* $p \ge 2 + \frac{8}{r}$.

(ii) *Let $G = \mathbb{T}^d$ be a flat torus equipped with a rational metric*; *that is, we can write $\mathbb{T}^d = \mathbb{R}^d/2\pi\Gamma$ such that there exists some $D \in \mathbb{R}$ for which $\langle \lambda, \mu\rangle \in D^{-1}\mathbb{Z}$ for all $\lambda, \mu \in \Gamma$. Then* (2-2) *holds for all $p > 2 + \frac{4}{d}$.*

The framework for the proof of this theorem will be based on [Bourgain 1993], in which the author proves some Strichartz estimates for the case of square tori, based on the Hardy–Littlewood circle method. We also refer to [Bourgain 1989] for applications of the circle method to Fourier restriction problems on tori. Note that part (ii) of the above theorem provides full expected Strichartz estimates without loss for some nonrectangular tori. We then have the following immediate corollary.

Corollary 2.3. *Let $d = 3, 4$ and let $\mathbb{T}^d$ be the flat torus equipped with a rational metric* (*not necessarily rectangular*). *Then the nonlinear Schrödinger equation $i\partial u_t = -\Delta u \pm |u|^{4/(d-2)}u$ is locally well-posed for initial data in $H^1(\mathbb{T}^d)$. Furthermore, for $d = 3$, we have $i\partial u_t = -\Delta u \pm |u|^2 u$ is locally well-posed for initial data in $H^{1/2}(\mathbb{T}^d)$.*

We refer to [Herr et al. 2011; Killip and Vişan 2016] for the definition of local well-posedness and a proof of this corollary.

Remark 2.4. To the best of my knowledge, the only known optimal range of p for (2-2) to hold is on square tori $\mathbb{T}^d$, with $p > 2 + \frac{4}{d}$ [Bourgain 1993], and on spheres $\mathbb{S}^d$ ($d \ge 3$), with $p > 4$ [Burq et al. 2004; Herr 2013]. For a general compact Lie group, we do not yet have a conjecture about the optimal range. We will prove (Theorem 6.2) the following distributional estimate: for any $p > 2 + \frac{4}{r}$,

$$\lambda \cdot \mu\{(t,x) \in I \times G \mid |e^{it\Delta_g}\varphi(N^{-2}\Delta_g) f(x)| > \lambda\}^{\frac{1}{p}} \le CN^{\frac{d}{2}-\frac{d+2}{p}}\|f\|_{L^2(G)} \tag{2-3}$$

for all $\lambda \gtrsim N^{d/2-r/4}$. It seems reasonable to conjecture that the above distributional estimate could be upgraded to the estimate (2-2) for all $p > 2 + \frac{4}{r}$ (which is the case for the tori). But this still will not be the optimal range for a general compact Lie group, by looking at the example of the three-sphere $\mathbb{S}^3$, which is isomorphic to the group SU(2). The optimal range for $\mathbb{S}^3$ is $p > 4$, while Theorem 2.2 proves the range $p \ge 10$, and the above conjecture indicates the range $p > 6$. Estimate (2-2) for $\mathbb{S}^3$ on the optimal range $p > 4$ is proved in [Herr 2013] by crucially using the L^p-estimates of the spectral clusters for the Laplace–Beltrami operator [Sogge 1988], which are optimal on spheres. On tori and more generally compact Lie groups with rank higher than 1, such spectral cluster estimates fail to be optimal and do not help provide the desired Strichartz estimates. On the other hand, the Stein–Tomas argument in our proof of Theorem 2.2 seems only sensitive to the L^∞-estimate of the Schrödinger kernel (Theorem 6.1) but not to the L^p-estimate (as in Proposition 7.28). This failure of incorporating L^p-estimates for either the spectral clusters or the Schrödinger kernel may be one of the reasons why Theorem 2.2 is still a step away from the optimal range.

2C. ***Organization of the paper.*** The organization of the paper is as follows. In Section 3, we will first reduce the Strichartz estimates on $G \cong \widetilde{G}/A$ to the spectrally localized Strichartz estimates with respect Littlewood–Paley projections of product type on the covering group $\widetilde{G}$. In Section 4, we will review the basic facts of structures and harmonic analysis on compact Lie groups, including the Fourier transform, root systems, structure of maximal tori, Weyl's character and dimension formulas, and the functional calculus of the Laplace–Beltrami operator. In Section 5 we will explicitly write down the Schrödinger kernel and interpret the Strichartz estimates as Fourier restriction estimates on the space-time, which then makes applicable the argument of Stein–Tomas type in Section 6. Then comes the core of the paper, Section 7, in which we will derive dispersive estimates for the Schrödinger kernel as the time variable lies in major arcs. In Section 7A, we will estimate some Weyl-type exponential sums over the so-called rational lattices, which in particular will imply the desired bound on the Schrödinger kernel for the nonrectangular rational tori. In Section 7B, we will rewrite the Schrödinger kernel for compact Lie groups into an exponential sum over the whole weight lattice instead of just one chamber of the lattice, and will prove the desired bound on the kernel for the case when the variable in the maximal torus stays away from all the cell walls by an application of the Weyl-type sum estimate established in Section 7A. In Section 7C, we will record two approaches to the pseudopolynomial behavior of characters, which will be applied to proving the desired bound on the Schrödinger kernel when the variable in the maximal torus stays close to the identity. In Section 7D, we further extend the result to the case when the variable in the maximal torus stays close to some corner. Section 7E will finally deal with the case when the variable in the maximal torus stays away from all the corners but close to some cell walls. These cell walls will be identified as those of a root subsystem, and we will then decompose the Schrödinger kernel into exponential sums over the root lattice of this root subsystem, thus reducing the problem into one similar to those already discussed in previous sections. This will finish the proof of the main theorem. In Section 7F, we will derive $L^p(G)$ estimates on the Schrödinger kernel as an upgrade of the $L^\infty(G)$-estimate.

Throughout the paper:

- $A \lesssim B$ means $A \le CB$ for some constant C.
- $A \lesssim_{a,b,\dots} B$ means $A \le CB$ for some constant C that depends on $a, b, \dots$.
- Δ, μ are short for the Laplace–Beltrami operator Δ_g and the associated volume-form measure μ_g respectively when the underlying Riemannian metric g is clear from context.
- L^p_x, H^s_x, L^p_t, $L^p_t L^q_x$, $L^p_{t,x}$ are short for $L^p(M)$, $H^s(M)$, $L^p(I)$, $L^p L^q(I \times M)$, $L^p(I \times M)$ respectively when the underlying manifold M and time interval I are clear from context.
- p' denotes the number such that $\frac{1}{p} + \frac{1}{p'} = 1$.

3. First reductions

3A. *Littlewood–Paley theory.* Let (M, g) be a compact Riemannian manifold and Δ be the Laplace–Beltrami operator. Let φ be a bump function on $\mathbb{R}$. Then for $N \geq 1$, $P_N := \varphi(N^{-2}\Delta)$ defines a bounded operator on $L^2(M)$ through the functional calculus of Δ. These operators P_N are often called the *Littlewood–Paley projections*. We reduce the problem of obtaining Strichartz estimates for $e^{it\Delta}$ to those for $P_N e^{it\Delta}$.

Proposition 3.1. *Fix $p, q \geq 2$, $s \geq 0$. Then the Strichartz estimate* (1-1) *is equivalent to the following statement: given any bump function φ,*

$$\|P_N e^{it\Delta} f\|_{L^p L^q(I\times M)} \lesssim N^s \|f\|_{L^2(M)}$$

holds for all dyadic natural numbers N (that is, for $N = 2^m$, $m \in \mathbb{Z}_{\geq 0}$). In particular, (2-2) *reduces to*

$$\|P_N e^{it\Delta} f\|_{L^p(I\times G)} \leq N^{\frac{d}{2}-\frac{d+2}{p}} \|f\|_{L^2(G)}. \tag{3-1}$$

This reduction is classical. We refer to [Burq et al. 2004] for a proof.

We also record here the Bernstein-type inequalities that will be useful in the sequel.

Proposition 3.2 [Burq et al. 2004, Corollary 2.2]. *Let d be the dimension of M. Then for all $1 \leq p \leq r \leq \infty$,*

$$\|P_N f\|_{L^r(M)} \lesssim N^{d(\frac{1}{p}-\frac{1}{r})} \|f\|_{L^p(M)}. \tag{3-2}$$

Note that the above proposition in particular implies that (3-1) holds for $N \lesssim 1$ or $p = \infty$.

3B. *Reduction to a finite cover.*

Proposition 3.3. *Let $\pi : (\widetilde{M}, \tilde{g}) \to (M, g)$ be a Riemannian covering map between compact Riemannian manifolds (then automatically with finite fibers). Let $\Delta_{\tilde{g}}$, Δ_g be the Laplace–Beltrami operators on $(\widetilde{M}, \tilde{g})$ and (M, g) respectively and let $\tilde{\mu}$ and μ be the normalized volume-form measures respectively, which define the L^p spaces. Let π^* be the pull-back map. Define*

$$C_\pi^\infty(\widetilde{M}) := \pi^*(C^\infty(M)),$$

and similarly define $C_\pi(\widetilde{M})$, $L_\pi^p(\widetilde{M})$ and $H_\pi^s(\widetilde{M})$. Then the following statements hold:

(i) *$\pi^* : C(M) \to C_\pi(\widetilde{M})$ and $\pi^* : C^\infty(M) \to C_\pi^\infty(\widetilde{M})$ are well-defined and are linear isomorphisms.*

(ii) *$\pi^* : L^p(M) \to L_\pi^p(\widetilde{M})$ is well-defined and is an isometry.*

(iii) *$\Delta_{\tilde{g}}$ maps $C_\pi^\infty(\widetilde{M})$ into $C_\pi^\infty(\widetilde{M})$ and the diagram*

$$\begin{array}{ccc} C^\infty(M)_{\Delta_g} & \xrightarrow{\pi^*} & C_\pi^\infty(\widetilde{M}) \\ \big\downarrow & & \big\downarrow \Delta_{\tilde{g}} \\ C^\infty(M) & \xrightarrow{\pi^*} & C_\pi^\infty(\widetilde{M}) \end{array}$$

commutes.

(iv) $e^{it\Delta_{\tilde g}}$ *maps* $L^2_\pi(\widetilde M)$ *into* $L^2_\pi(\widetilde M)$ *and is an isometry, and the diagrams*

$$\begin{array}{ccc} L^2(M) & \xrightarrow{\ \pi^*\ } & L^2_\pi(\widetilde M) \\ \big\downarrow {\scriptstyle e^{it\Delta_g}} & & \big\downarrow {\scriptstyle e^{it\Delta_{\tilde g}}} \\ L^2(M) & \xrightarrow{\ \pi^*\ } & L^2_\pi(\widetilde M) \end{array} \qquad \begin{array}{ccc} L^2(M) & \xrightarrow{\ \pi^*\ } & L^2_\pi(\widetilde M) \\ \big\downarrow {\scriptstyle P_N} & & \big\downarrow {\scriptstyle P_N} \\ L^2(M) & \xrightarrow{\ \pi^*\ } & L^2_\pi(\widetilde M) \end{array} \tag{3-3}$$

commute, where P_N *stands for both* $\varphi(N^{-2}\Delta_g)$ *and* $\varphi(N^{-2}\Delta_{\tilde g})$.

(v) $\pi^* : H^s(M) \to H^s_\pi(\widetilde M)$ *is well-defined and is an isometry.*

Proof. Parts (i), (ii) and (iii) are direct consequences of the definition of a Riemannian covering map. For part (iv), note that (i), (ii) and (iii) together imply that the triples $(L^2(M), C^\infty(M), \Delta_g)$ and $(L^2_\pi(\widetilde M), C^\infty_\pi(\widetilde M), \Delta_{\tilde g})$ are isometric as systems of essentially self-adjoint operators on Hilbert spaces, and thus have isometric functional calculus. This implies (iv). Note that the $H^s(M)$ and $H^s_\pi(\widetilde M)$ norms are also defined in terms of the isometric functional calculus of $(L^2(M), C^\infty(M), \Delta_g)$ and $(L^2_\pi(\widetilde M), C^\infty_\pi(\widetilde M), \Delta_{\tilde g})$ respectively, which implies (v). □

Combining Proposition 3.1 and 3.3, Theorem 2.2 is reduced to the following.

Theorem 3.4. *Let* K_i*'s be simply connected simple Lie groups and let* $G = \mathbb{T}^n \times K_1 \times \cdots \times K_m$ *be equipped with a rational metric as in Definition 2.1. Then*

$$\|P_N e^{it\Delta} f\|_{L^p(I\times G)} \lesssim N^{\frac{d}{2}-\frac{d+2}{p}} \|f\|_{L^2(G)} \tag{3-4}$$

holds for $p \geq 2 + \frac{8}{r}$ *and* $N \gtrsim 1$.

3C. *Littlewood–Paley projections of product type.* Let (M, g) be the Riemannian product of the compact Riemannian manifolds (M_j, g_j), $j = 0, \dots, m$. Any eigenfunction of the Laplace–Beltrami operator Δ on M with the eigenvalue $\lambda \leq 0$ is of the form $\prod_{j=0}^m \psi_{\lambda_j}$, where each ψ_{λ_j} is an eigenfunction of Δ_j on M_i with eigenvalue $\lambda_j \leq 0$, $j = 0, \dots, m$, such that $\lambda = \lambda_0 + \cdots + \lambda_n$.

Given any bump function φ on $\mathbb{R}$, there always exist bump functions φ_j, $j = 0, \dots, m$, such that for all $(x_0, \dots, x_m) \in \mathbb{R}^{m+1}_{\leq 0}$ with $\varphi(x_0 + \cdots + x_m) \neq 0$, we have $\prod_{j=0}^m \varphi_j(x_j) = 1$. In particular,

$$\varphi \cdot \prod_{j=0}^m \varphi_j(x_j) = \varphi.$$

For $N \geq 1$, define

$$P_N := \varphi(N^{-2}\Delta),$$
$$\boldsymbol{P}_N := \varphi_0(N^{-2}\Delta_0) \otimes \cdots \otimes \varphi_m(N^{-2}\Delta_m)$$

as bounded operators on $L^2(M)$. We call $\boldsymbol{P}_N$ a *Littlewood–Paley projection of product type*. We have

$$\boldsymbol{P}_N \circ P_N = P_N.$$

This implies that we can further reduce Theorem 3.4 into the following.

Theorem 3.5. *Let $G = \mathbb{T}^n \times K_1 \times \cdots \times K_m$ be equipped with a rational metric. Let $\Delta_0, \Delta_1, \ldots, \Delta_m$ be respectively the Laplace–Beltrami operators on $\mathbb{T}^n, K_1, \ldots, K_m$. Let φ_j be any bump function for each $j = 0, \ldots, m$. For $N \geq 1$, let $\boldsymbol{P}_N = \bigotimes_{j=0}^m \varphi_j(N^{-2}\Delta_j)$. Then*

$$\|\boldsymbol{P}_N e^{it\Delta} f\|_{L^p(I\times G)} \lesssim N^{\frac{d}{2}-\frac{d+2}{p}} \|f\|_{L^2(G)} \tag{3-5}$$

holds for $p \geq 2 + \frac{8}{r}$ and $N \gtrsim 1$.

On the other hand, similarly, for each Littlewood–Paley projection $\boldsymbol{P}_N$ of product type, there exists a bump function φ such that $P_N = \varphi(N^{-2}\Delta)$ satisfies $P_N \circ \boldsymbol{P}_N = \boldsymbol{P}_N$. Noting that $\|\boldsymbol{P}_N f\|_{L^2} \lesssim \|f\|_{L^2}$, (3-2) then implies

$$\|\boldsymbol{P}_N f\|_{L^r(M)} \lesssim N^{d\left(\frac{1}{2}-\frac{1}{r}\right)} \|f\|_{L^2(M)} \tag{3-6}$$

for all $2 \leq r \leq \infty$.

4. Preliminaries on harmonic analysis on compact Lie groups

4A. ***Fourier transform.*** Let G be a compact group and $\widehat{G}$ be its Fourier dual, i.e., the set of equivalent classes of irreducible unitary representations of G. For $\lambda \in \widehat{G}$, let $\pi_\lambda : V_\lambda \to V_\lambda$ be the irreducible unitary representation in the class λ, and let $d_\lambda = \dim(V_\lambda)$. Let μ be the normalized Haar measure on G. Then for $f \in L^2(G)$, define the Fourier transform

$$\hat{f}(\lambda) = \int_G f(x)\pi_\lambda(x^{-1})\, d\mu.$$

Then the inverse Fourier transform

$$f(x) = \sum_{\lambda\in\widehat{G}} d_\lambda \operatorname{tr}(\hat{f}(\lambda)\pi_\lambda(x))$$

converges in $L^2(G)$. We have the Plancherel identities

$$\|f\|_{L^2(G)} = \Big(\sum_{\lambda\in\widehat{G}} d_\lambda \|\hat{f}(\lambda)\|_{\mathrm{HS}}^2\Big)^{\frac{1}{2}}, \tag{4-1}$$

$$\langle f, g\rangle_{L^2(G)} = \sum_{\lambda\in\widehat{G}} d_\lambda \operatorname{tr}(\hat{f}(\lambda)\hat{g}(\lambda)^*). \tag{4-2}$$

Here $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert–Schmidt norm of endomorphisms.

For the convolution

$$(f * g)(x) = \int_G f(xy^{-1})g(y)\, d\mu(y),$$

we have

$$(f * g)^\wedge(j) = \hat{f}(j)\hat{g}(j). \tag{4-3}$$

If $\hat{g}(\lambda) = c_\lambda \cdot \mathrm{Id}_{d_\lambda\times d_\lambda}$, where c_λ is a scalar, then

$$\|f * g\|_{L^2(G)} \leq \sup_\lambda |c_\lambda| \cdot \|f\|_{L^2(G)}. \tag{4-4}$$

We also have the Hausdorff–Young inequality

$$\|\hat{f}(\lambda)\|_{\mathrm{HS}} \le d_\lambda^{\frac{1}{2}} \|f\|_{L^1(G)} \quad \text{for all } \lambda \in \widehat{G}. \tag{4-5}$$

4B. ***Root system and the Laplace–Beltrami operator.*** Let G be a compact simply connected semisimple Lie group of dimension d and $\mathfrak{g}$ be its Lie algebra, and let $\mathfrak{g}_{\mathbb{C}}$ denote the complexification of $\mathfrak{g}$. Choose a *maximal torus* $B \subset G$ and let r be the dimension of B. Let $\mathfrak{b}$ be the Lie algebra of B, which is a *Cartan subalgebra* of $\mathfrak{g}$, and let $\mathfrak{b}_{\mathbb{C}}$ denote its complexification. The Fourier dual $\widehat{B}$ of B is isomorphic to a lattice $\Lambda \subset i\,\mathfrak{b}^*$, which is the *weight lattice*, under the isomorphism

$$\Lambda \xrightarrow{\sim} \widehat{B}, \quad \lambda \mapsto e^\lambda. \tag{4-6}$$

We have the *root space decomposition* $\mathfrak{g}_{\mathbb{C}} = \mathfrak{b}_{\mathbb{C}} \oplus \big(\bigoplus_{\alpha\in\Phi} \mathfrak{g}_{\mathbb{C}}^\alpha\big)$. Here $\Phi \subset i\,\mathfrak{b}^*$,

$$\mathfrak{g}_{\mathbb{C}}^\alpha = \{X \in \mathfrak{g}_{\mathbb{C}} \mid \mathrm{Ad}_b(X) = e^\alpha(b)X \text{ for all } b \in B\},$$

and $\dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}}^\alpha = 1$. This implies

$$|\Phi| + r = d. \tag{4-7}$$

The Cartan–Killing form $\langle\cdot,\cdot\rangle$ on $i\,\mathfrak{b}^*$ becomes a real inner product, and $(\Psi, \langle\cdot,\cdot\rangle)$ becomes an *integral root system*, that is, a finite set Φ in a finite-dimensional real inner product space with the following requirements:

(i) $\Phi = -\Phi$.

(ii) $\alpha \in \Phi,\ k \in \mathbb{R},\ k\alpha \in \Phi \Rightarrow k = \pm 1$.

(iii) $s_\alpha \Phi = \Phi$ for all $\alpha \in \Phi$.

(iv) $2\langle\alpha,\beta\rangle/\langle\alpha,\alpha\rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.

Here s_α is the reflection about the hyperplane $\alpha^\perp$ orthogonal to α; that is,

$$s_\alpha(x) := x - 2\frac{\langle x,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha.$$

Let P be a system of positive roots such that $\Phi = P \sqcup -P$. Then by (4-7), we have

$$|P| = \frac{d-r}{2}. \tag{4-8}$$

We can describe the weight lattice Λ purely in terms of the root system

$$\Lambda = \left\{\lambda \in i\,\mathfrak{b}^* \;\middle|\; \frac{2\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z} \text{ for all } \alpha \in \Phi\right\}. \tag{4-9}$$

The set Φ of roots generate the *root lattice* Γ and we have $\Gamma \subset \Lambda$ and Λ/Γ is finite.

Let

$$\Lambda^+ := \left\{\lambda \in i\,\mathfrak{b}^* \;\middle|\; \frac{2\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}_{\ge 0} \text{ for all } \alpha \in P\right\}$$

be the set of dominant weights. We describe Λ, Λ^+ in terms of a basis. Let $\{\alpha_1, \dots, \alpha_r\}$ be the set of simple roots in P. Let $\{w_1, \dots, w_r\}$ be the corresponding fundamental weights, i.e., the dual basis to the coroot basis $\{2\alpha_1/\langle\alpha_1, \alpha_1\rangle, \dots, 2\alpha_r/\langle\alpha_r, \alpha_r\rangle\}$. Then

$$\Lambda = \mathbb{Z}w_1 + \cdots + \mathbb{Z}w_r,$$
$$\Lambda^+ = \mathbb{Z}_{\geq 0}w_1 + \cdots + \mathbb{Z}_{\geq 0}w_r.$$

Let

$$C = \mathbb{R}_{>0}w_1 + \cdots + \mathbb{R}_{>0}w_r \tag{4-10}$$

be the *fundamental Weyl chamber*, and we have the decomposition

$$i\mathfrak{b}^* = \Big(\bigsqcup_{s\in W} sC\Big) \sqcup \Big(\bigcup_{\alpha\in\Phi} \{\lambda \in i\mathfrak{b}^* \mid \langle\lambda, \alpha\rangle = 0\}\Big), \tag{4-11}$$

where W is the Weyl group. Here $\sqcup$ stands for disjoint union.

Define

$$\rho := \frac{1}{2}\sum_{\alpha\in P} \alpha = \sum_{i=1}^{r} w_i. \tag{4-12}$$

Then we have

$$\widehat{G} \cong \Lambda^+$$

such that the irreducible representation π_λ corresponding to $\lambda \in \Lambda^+$ has the character χ_λ and dimension d_λ given by Weyl's formulas

$$\chi_\lambda|_B = \frac{\sum_{s\in W}(\det s)e^{s(\lambda+\rho)}}{\sum_{s\in W}(\det s)e^{s\rho}}, \tag{4-13}$$

$$d_\lambda = \frac{\prod_{\alpha\in P}\langle\alpha, \lambda+\rho\rangle}{\prod_{\alpha\in P}\langle\alpha, \rho\rangle}. \tag{4-14}$$

Let $H \in \mathfrak{b}$. We can think of $-iH$ as a real linear functional on $i\mathfrak{b}^*$, and by the Cartan–Killing inner product on $i\mathfrak{b}^*$, we thus get a correspondence between $H \in \mathfrak{b}$ and an element in $i\mathfrak{b}^*$, still denoted as H. Under this correspondence, $e^{\lambda(H)} = e^{i\langle\lambda, H\rangle}$ and we rewrite Weyl's character formula as

$$\chi_\lambda(\exp H) = \frac{\sum_{s\in W}(\det s)e^{i\langle s(\lambda+\rho), H\rangle}}{\sum_{s\in W}(\det s)e^{i\langle\rho, H\rangle}}. \tag{4-15}$$

Also under this correspondence between $\mathfrak{b}$ and $i\mathfrak{b}^*$, we have

$$B \cong i\mathfrak{b}^*/2\pi\Gamma^\vee,$$

where

$$\Gamma^\vee = \mathbb{Z}\frac{2\alpha_1}{\langle\alpha_1, \alpha_1\rangle} + \cdots + \mathbb{Z}\frac{2\alpha_r}{\langle\alpha_r, \alpha_r\rangle}$$

is the *coroot lattice*.

We define the *cells* to be the connected components of $\{H \in i\mathfrak{b}^*/2\pi\Gamma^\vee \mid \langle\alpha, H\rangle \notin 2\pi\mathbb{Z}\}$ and call $\{H \in i\mathfrak{b}^*/2\pi\Gamma^\vee \mid \langle\alpha, H\rangle \in 2\pi\mathbb{Z}\}$ the *cell walls*.

We also record here Weyl's integral formula. Let $f \in L^1(G)$ be invariant under the adjoint action of G. Then

$$\int_G f\, d\mu = \frac{1}{|W|} \int_B f(b) |D_P(b)|^2 \, db. \tag{4-16}$$

Here $d\mu, db$ are respectively the normalized Haar measures of G and B, and

$$D_P(H) = \sum_{s \in W} (\det s) e^{i\langle \rho, H\rangle}$$

is the *Weyl denominator*.

Finally we describe the functional calculus of the Laplace–Beltrami operator Δ. Given any irreducible unitary representation (π_λ, V_λ) of G in the class $\lambda \in \widehat{G} \cong \Lambda^+$, the operator Δ acts on the space $\mathcal{M}_\lambda = \{\mathrm{tr}(\pi_\lambda T) \mid T \in \mathrm{End}(V_\lambda)\}$ of matrix coefficients by

$$\Delta f = -k_\lambda f \quad \text{for all } f \in \mathcal{M}_\lambda,\ \lambda \in \widehat{G},$$

where

$$k_\lambda = |\lambda + \rho|^2 - |\rho|^2. \tag{4-17}$$

Let $f \in L^2(G)$ and consider the inverse Fourier transform $f(x) = \sum_{\lambda \in \Lambda^+} d_\lambda \,\mathrm{tr}(\pi_\lambda(x) \hat{f}(\lambda))$; then for any bounded Borel function $F : \mathbb{R} \to \mathbb{C}$, we have

$$F(\Delta) f = \sum_{\lambda \in \Lambda^+} F(-k_\lambda) d_\lambda \,\mathrm{tr}(\pi_\lambda(x) \hat{f}(\lambda)).$$

In particular, we have

$$e^{it\Delta} f = \sum_{\lambda \in \Lambda^+} e^{-itk_\lambda} d_\lambda \,\mathrm{tr}(\pi_\lambda(x) \hat{f}(\lambda)), \tag{4-18}$$

$$P_N e^{it\Delta} f = \sum_{\lambda \in \Lambda^+} \varphi\Big(-\frac{k_\lambda}{N^2}\Big) e^{-itk_\lambda} d_\lambda \,\mathrm{tr}(\pi_\lambda(x) \hat{f}(\lambda)). \tag{4-19}$$

Example 4.1. Let $M = \mathrm{SU}(2)$, which is of dimension 3 and rank 1. Let $\mathfrak{a} \cong \mathbb{R}$ be the Cartan subalgebra and $A \cong \mathbb{R}/2\pi\mathbb{Z}$ be the maximal torus. The root system is $\{\pm\alpha\}$, where α acts on $\mathfrak{a}$ by $\alpha(\theta) = 2\theta$. The fundamental weight is $w = \frac{1}{2}\alpha$. We normalize the Cartan–Killing form so that $|w| = 1$. The Weyl group W is of order 2, and acts on $\mathfrak{a}$ as well as $\mathfrak{a}^*$ through multiplication by ± 1. For $m \in \mathbb{Z}_{\geq 0} \cong \mathbb{Z}_{\geq 0} w = \Lambda^+$, we have

$$d_m = m + 1, \tag{4-20}$$

$$\chi_m(\theta) = \frac{e^{i(m+1)\theta} - e^{-i(m+1)\theta}}{e^{i\theta} - e^{-i\theta}} = \frac{\sin(m+1)\theta}{\sin\theta}, \quad \theta \in \mathbb{R}/2\pi\mathbb{Z}, \tag{4-21}$$

$$k_m = (m+1)^2 - 1. \tag{4-22}$$

5. The Schrödinger kernel

Let $f \in L^2(G)$. Then (4-19) implies

$$(P_N e^{it\Delta} f)^\wedge(\lambda) = \varphi\Big(\frac{k_\lambda}{N^2}\Big) e^{-itk_\lambda} \hat{f}(\lambda).$$

Define

$$(K_N(t,\cdot))^\wedge(\lambda)=\varphi\left(\frac{k_\lambda}{N^2}\right)e^{-itk_\lambda}\mathrm{Id}_{d_\lambda\times d_\lambda},$$

which implies

$$K_N(t,x)=\sum_{\lambda\in\Lambda^+}\varphi\left(\frac{k_\lambda}{N^2}\right)e^{-itk_\lambda}d_\lambda\chi_\lambda(x). \tag{5-1}$$

Then we can write

$$P_Ne^{it\Delta}f=K_N(t,\cdot)*f=f*K_N(t,\cdot),$$

and we call $K_N(t,x)$ the *Schrödinger kernel*. Incorporating (4-14), (4-15) and (4-17) into (5-1), we get

$$K_N(t,x)=\sum_{\lambda\in\Lambda^+}e^{-it(|\lambda+\rho|^2-|\rho|^2)}\varphi\left(\frac{|\lambda+\rho|^2-|\rho|^2}{N^2}\right)\frac{\prod_{\alpha\in P}\langle\alpha,\lambda+\rho\rangle}{\prod_{\alpha\in P}\langle\alpha,\rho\rangle}\frac{\sum_{s\in W}(\det s)e^{i\langle s(\lambda+\rho),H\rangle}}{\sum_{s\in W}(\det s)e^{i\langle s(\rho),H\rangle}}. \tag{5-2}$$

Example 5.1. Specializing the Schrödinger kernel (5-2) to $G=\mathrm{SU}(2)$, using (4-20), (4-21), and (4-22), we have

$$K_N(t,\theta)=\sum_{m=0}^{\infty}\varphi\left(\frac{(m+1)^2-1}{N^2}\right)(m+1)e^{-i((m+1)^2-1)t}\frac{e^{i(m+1)\theta}-e^{-i(m+1)\theta}}{e^{i\theta}-e^{-i\theta}},\quad \theta\in\mathbb{R}/2\pi\mathbb{Z}. \tag{5-3}$$

More generally, let $G=\mathbb{R}^n/2\pi\Gamma_0\times K_1\times\cdots\times K_m$ be equipped with a rational metric g as in Definition 2.1. Let Λ_0 be the dual lattice of Γ_0 and Λ_j be the weight lattice for K_j, $j=1,\dots,m$. Let $\boldsymbol{P}_N=\bigotimes_{j=0}^m\varphi_j(N^{-2}\Delta_j)$ be a Littlewood–Paley projection of product type as described in Section 3C. Define the *Schrödinger kernel* $\boldsymbol{K}_N$ on G by

$$\boldsymbol{P}_Ne^{it\Delta}f=f*\boldsymbol{K}_N(t,\cdot)=\boldsymbol{K}_N(t,\cdot)*f. \tag{5-4}$$

Then

$$\boldsymbol{K}_N=\prod_{j=0}^m K_{N,j}, \tag{5-5}$$

where the $K_{N,j}$'s are respectively the Schrödinger kernels on each component of G

$$K_{N,0}=\sum_{\lambda_0\in\Lambda_0}\varphi_0\left(\frac{-|\lambda_0|^2}{\beta_0N^2}\right)e^{-it\beta_0^{-1}|\lambda_0|^2}e^{i\langle\lambda_0,H_0\rangle},$$

$$K_{N,j}=\sum_{\lambda_j\in\Lambda_j^+}\varphi_j\left(\frac{-|\lambda_j+\rho_j|^2+|\rho_j|^2}{\beta_jN^2}\right)e^{it\beta_j^{-1}(-|\lambda_j+\rho_j|^2+|\rho_j|^2)}d_{\lambda_j}\chi_{\lambda_j},$$

$j=1,\dots,m$. Here the ρ_j's are defined in terms of (4-12). We also write

$$\boldsymbol{K}_N=\sum_{\lambda\in\widehat{G}}\varphi(\lambda,N)e^{-itk_\lambda}d_\lambda\chi_\lambda,$$

where

$$\lambda=(\lambda_0,\dots,\lambda_m)\in\widehat{G}=\Lambda_0\times\Lambda_1^+\times\cdots\times\Lambda_m^+,$$

$$-k_\lambda=-\beta_0^{-1}|\lambda_0|^2+\sum_{j=1}^m\beta_j^{-1}(-|\lambda_j+\rho_j|^2+|\rho_j|^2), \tag{5-6}$$

$$\varphi(\lambda, N) = \varphi_0\left(\frac{-|\lambda_0|^2}{\beta_0 N^2}\right) \cdot \prod_{j=1}^{n} \varphi_j\left(\frac{-|\lambda_j + \rho_j|^2 + |\rho_j|^2}{\beta_j N^2}\right), \tag{5-7}$$

$$d_\lambda = \prod_{j=1}^{m} d_{\lambda_j}, \;\; \chi_\lambda = e^{i\langle \lambda_0, H_0\rangle} \prod_{j=1}^{m} \chi_{\lambda_j}.$$

Tracking all the definitions, we get the following lemma.

Lemma 5.2. *Let d, r be respectively the dimension and rank of G:*

(i) $|\{\lambda \in \widehat{G} \mid k_\lambda \lesssim N^2\}| \lesssim N^r$.

(ii) $d_\lambda \lesssim N^{(d-r)/2}$ *uniformly for all $\lambda \in \widehat{G}$ such that $k_\lambda \lesssim N^2$.*

Now we interpret the Strichartz estimates on G as *Fourier restriction estimates*.

Lemma 5.3. *For a compact simply connected semisimple Lie group G and its weight lattice Λ, there exists $D \in \mathbb{N}$ such that $\langle \lambda_1, \lambda_2\rangle \in D^{-1}\mathbb{Z}$ for all $\lambda_1, \lambda_2 \in \Lambda$.*

Proof. Let Φ be the set of roots for G. Then by Lemma 4.3.5 in [Varadarajan 1974], $\langle \alpha, \beta\rangle$ are rational numbers for all $\alpha, \beta \in \Phi$. Let $S = \{\alpha_1, \dots, \alpha_r\} \subset \Phi$ be a system of simple roots. Since the set of fundamental weights $\{w_1, \dots, w_n\}$ forms a dual basis to $\{2\alpha_1/\langle \alpha_1, \alpha_1\rangle, \dots, 2\alpha_r/\langle \alpha_r, \alpha_r\rangle\}$ with respect to the Cartan–Killing form $\langle \cdot, \cdot\rangle$, and $\langle \alpha_i, \alpha_j\rangle$ are rational numbers for all $i, j = 1, \dots, r$, we have that the w_j's can be expressed as linear combinations of the α_j's with rational coefficients. This implies that $\langle w_i, w_j\rangle$ are rational numbers for all $i, j = 1, \dots, r$. Since there are only finitely many such numbers as $\langle w_i, w_j\rangle$, there exists $D \in \mathbb{N}$ so that $\langle w_i, w_j\rangle \in D^{-1}\mathbb{Z}$ for all $i, j = 1, \dots, r$. Thus $\langle \lambda_1, \lambda_2\rangle \in D^{-1}\mathbb{Z}$ for all $\lambda_1, \lambda_2 \in \Lambda$, since $\Lambda = \mathbb{Z}w_1 + \dots + \mathbb{Z}w_n$. □

For $G = \mathbb{R}^n/2\pi\Gamma_0 \times K_1 \times \dots \times K_m$, by the previous lemma, there exists for each $j = 1, \dots, m$ some $D_j \in \mathbb{N}$ such that $\langle \lambda, \mu\rangle \in D_j^{-1}\mathbb{Z}$ for all $\lambda, \mu \in \Lambda_j^+$, which implies by (4-12) that

$$-|\lambda_j + \rho_j|^2 + |\rho_j|^2 = -|\lambda_j|^2 - \langle \lambda_j, 2\rho_j\rangle \in D_j^{-1}\mathbb{Z}$$

for all $\lambda_j \in \Lambda_j$. Also recall that we require that there exists some $D \in \mathbb{N}$ such that $\langle u, v\rangle \in D^{-1}\mathbb{Z}$ for all $u, v \in \Gamma_0$. This implies that there also exists some $D_0 \in \mathbb{N}$ such that $\langle \lambda, \mu\rangle \in D_0^{-1}\mathbb{Z}$ for all $\lambda, \mu \in \Lambda_0$. By Definition 2.1 of a rational metric, there exists some $D_* > 0$ such that

$$\beta_0^{-1}, \dots, \beta_m^{-1} \in D_*^{-1}\mathbb{N}.$$

Define

$$T = 2\pi D_* \cdot \prod_{j=0}^{m} D_j. \tag{5-8}$$

Then (5-6) implies that $Tk_\lambda \in 2\pi\mathbb{Z}$, which then implies that the Schrödinger kernel as in (5-5) is periodic in t with a period of T. Thus we may view the time variable t as living on the circle $\mathbb{T} = \mathbb{R}/T\mathbb{Z}$. Now the formal dual to the operator

$$\boldsymbol{T} : L^2(G) \to L^p(\mathbb{T} \times G), \quad f \mapsto \boldsymbol{P}_N e^{it\Delta}, \tag{5-9}$$

is computed to be

$$\boldsymbol{T}^*: L^{p'}(\mathbb{T}\times G) \to L^2(G), \quad F \mapsto \int_{\mathbb{T}} \boldsymbol{P}_N e^{-is\Delta} F(s,\cdot)\,\frac{ds}{T}, \tag{5-10}$$

and thus

$$\boldsymbol{T}\boldsymbol{T}^*: L^{p'}(\mathbb{T}\times G) \to L^p(\mathbb{T}\times G), \quad F \mapsto \int_{\mathbb{T}} \boldsymbol{P}_N^2 e^{i(t-s)\Delta} F(s,\cdot)\,\frac{ds}{T} = \widetilde{\boldsymbol{K}}_N * F, \tag{5-11}$$

where

$$\widetilde{\boldsymbol{K}}_N = \sum_{\lambda\in\widehat{G}} \varphi^2(\lambda,N) e^{-itk_\lambda} d_\lambda \chi_\lambda = \boldsymbol{K}_N * \boldsymbol{K}_N.$$

Note that the cutoff function $\varphi^2(\lambda,N)$ still defines a Littlewood–Paley projection of product type and $\widetilde{\boldsymbol{K}}_N$ is the associated Schrödinger kernel. Now the argument of $\boldsymbol{T}\boldsymbol{T}^*$ says that the boundedness of the operators (5-9), (5-10) and (5-11) are all equivalent; thus the Strichartz estimate in (3-1) is equivalent to the *space-time Strichartz estimate*

$$\|\widetilde{\boldsymbol{K}}_N * F\|_{L^p(\mathbb{T}\times G)} \lesssim N^{d-\frac{2(d+2)}{p}} \|F\|_{L^{p'}(\mathbb{T}\times G)}. \tag{5-12}$$

We have the *space-time Fourier transform* on $\mathbb{T}\times G$ as follows. For $(n,\lambda)\in\frac{2\pi}{T}\mathbb{Z}\times\widehat{G}$, we have

$$\widehat{\boldsymbol{K}}_N(n,\lambda) = \begin{cases} \varphi(\lambda,N)\cdot \mathrm{Id}_{d_\lambda\times d_\lambda} & \text{if } n=-k_\lambda,\\ 0 & \text{otherwise.}\end{cases} \tag{5-13}$$

Similarly, for $f\in L^2(G)$, we have

$$(\boldsymbol{P}_N e^{it\Delta} f(x))^\wedge(n,\lambda) = \begin{cases} \varphi(\lambda,N)\cdot \hat{f}(\lambda) & \text{if } n=-k_\lambda,\\ 0 & \text{otherwise.}\end{cases} \tag{5-14}$$

For $m(t)=\sum_{n\in(2\pi/T)\mathbb{Z}} \hat{m}(n)e^{itn}$, we compute

$$(m\boldsymbol{K}_N)^\wedge(n,\lambda) = \hat{m}(n+k_\lambda)\varphi(\lambda,N)\mathrm{Id}_{d_\lambda\times d_\lambda}. \tag{5-15}$$

6. The Stein–Tomas argument

Throughout this section, $\mathbb{S}^1$ stands for the standard circle of unit length, and $\|\cdot\|$ stands for the distance from 0 on $\mathbb{S}^1$. Define

$$\mathcal{M}_{a,q} := \left\{ t\in\mathbb{S}^1 \;\middle|\; \left\| t-\frac{a}{q}\right\| < \frac{1}{qN}\right\},$$

where

$$a\in\mathbb{Z}_{\geq 0},\quad q\in\mathbb{N},\quad a<q,\quad (a,q)=1,\quad q<N.$$

We call such $\mathcal{M}_{a,q}$'s as *major arcs*, which are reminiscent of the Hardy–Littlewood circle method. We will prove the following key dispersive estimate.

Theorem 6.1. *Let $\boldsymbol{K}_N$ be the Schrödinger kernel* (5-5) *and T be the period* (5-8). *Then*

$$|\boldsymbol{K}_N(t,x)| \lesssim \frac{N^d}{\left(\sqrt{q}\left(1+N\left\|\frac{t}{2\pi D}-\frac{a}{q}\right\|^{1/2}\right)\right)^r}$$

for $\frac{t}{2\pi D}\in\mathcal{M}_{a,q}$, uniformly in $x\in G$.

Noting the product structure (5-5) of $\boldsymbol{K}_N$, the above theorem reduces to the cases on irreducible components of G.

Theorem 6.2. (i) *Given $G = \mathbb{T}^d = \mathbb{R}^d/2\pi\Gamma$ such that there exists $D \in \mathbb{R}$ for which $\langle \lambda, \mu\rangle \in D^{-1}\mathbb{Z}$ for all $\lambda, \mu \in \Gamma$. Then the Schrödinger kernel*

$$K_N(t, H) = \sum_{\lambda\in\Lambda} \varphi\left(\frac{|\lambda|^2}{N^2}\right) e^{-it|\lambda|^2+i\langle\lambda,H\rangle}$$

satisfies

$$|K_N(t, H)| \lesssim \left(\frac{N}{\sqrt{q}\big(1+N\big\|\frac{t}{2\pi D}-\frac{a}{q}\big\|^{1/2}\big)}\right)^d$$

for $\frac{t}{2\pi D} \in \mathcal{M}_{a,q}$, uniformly in $H \in \mathbb{T}^n$.

(ii) *Let G be a compact simply connected semisimple Lie group. Let Λ be the weight lattice for which $\langle \lambda, \mu\rangle \in D^{-1}\mathbb{Z}$ for all $\lambda, \mu \in \Lambda$ for some $D \in \mathbb{R}$. Let K_N be the Schrödinger kernel as defined in* (5-2). *Then*

$$|K_N(t, x)| \lesssim \frac{N^d}{\big(\sqrt{q}\big(1+N\big\|\frac{t}{2\pi D}-\frac{a}{q}\big\|^{1/2}\big)\big)^r} \tag{6-1}$$

for $\frac{t}{2\pi D} \in \mathcal{M}_{a,q}$, uniformly in $x \in G$.

We will prove this theorem in the next section. Now we show how this theorem implies Strichartz estimates.

Theorem 6.3. *Let $G = \mathbb{T}^n \times K_1 \times \cdots \times K_m$ be equipped with a rational metric $\tilde{g}$ and T be a period of the Schrödinger flow as in* (5-8). *Let d, r be the dimension and rank of G respectively. Let $f \in L^2(G)$, $\lambda > 0$ and define*

$$m_\lambda = \mu\{(t, x) \in \mathbb{T} \times G \mid |P_N e^{it\Delta} f(x)| > \lambda|\},$$

where $\mu = dt\cdot d\mu_G$, with dt being the standard measure on $\mathbb{T} = \mathbb{R}/T\mathbb{Z}$ and $d\mu_G$ being the Haar measure on G. Let

$$p_0 = \frac{2(r+2)}{r}.$$

Then the following statements hold true:

(I) $$m_\lambda \lesssim_\varepsilon N^{\frac{dp_0}{2}-(d+2)+\varepsilon}\lambda^{-p_0}\|f\|^{p_0}_{L^2(G)} \quad \text{for all } \lambda \gtrsim N^{\frac{d}{2}-\frac{r}{4}},\ \varepsilon > 0.$$

(II) $$m_\lambda \lesssim N^{\frac{dp}{2}-(d+2)}\lambda^{-p}\|f\|^{p}_{L^2(G)} \quad \text{for all } \lambda \gtrsim N^{\frac{d}{2}-\frac{r}{4}},\ p > p_0.$$

(III) $$\|\boldsymbol{P}_N e^{it\Delta} f\|_{L^p(\mathbb{T}\times G)} \lesssim N^{\frac{d}{2}-\frac{d+2}{p}}\|f\|_{L^2(G)} \tag{6-2}$$

holds for all $p \geq 2 + \frac{8}{r}$.

(IV) *Assume it holds that*

$$\|\boldsymbol{P}_N e^{it\Delta} f\|_{L^p(\mathbb{T}\times G)} \lesssim_\varepsilon N^{\frac{d}{2}-\frac{d+2}{p}+\varepsilon}\|f\|_{L^2(G)} \tag{6-3}$$

for some $p > p_0$; then (6-2) *holds for all $q > p$.*

The proof strategy of this theorem is a Stein–Tomas-type argument, similar to the proofs of Propositions 3.82, 3.110, 3.113 in [Bourgain 1993]. The new ingredient is the nonabelian Fourier transform. We detail the proof in the following.

Let $\omega \in C_c^\infty(\mathbb{R})$ such that $\omega \geq 0$, $\omega(x) = 1$ for all $|x| \leq 1$ and $\omega(x) = 0$ for all $|x| \geq 2$. Let N be a dyadic natural number. Define

$$\omega_{\frac{1}{N^2}} := \omega(N^2 \cdot),$$
$$\omega_{\frac{1}{NM}} := \omega(NM \cdot) - \omega(2NM \cdot),$$

where

$$1 \leq M < N, \quad M \text{ dyadic.}$$

Let

$$N_1 = \frac{N}{2^{10}}, \quad 1 \leq Q < N_1,\ Q \text{ dyadic.}$$

Then

$$\sum_{Q \leq M \leq N} \omega_{\frac{1}{NM}} = 1 \quad \text{on } \Big[-\frac{1}{NQ}, \frac{1}{NQ}\Big], \tag{6-4}$$

$$\sum_{Q \leq M \leq N} \omega_{\frac{1}{NM}} = 0 \quad \text{outside } \Big[-\frac{2}{NQ}, \frac{2}{NQ}\Big]. \tag{6-5}$$

Write

$$1 = \sum_{1 \leq Q \leq N_1} \sum_{Q \leq M \leq N} \Bigg[\Bigg(\sum_{\substack{(a,q)=1 \\ Q \leq q < 2Q}} \delta_{\frac{a}{q}}\Bigg) * \omega_{\frac{1}{NM}}\Bigg]\Big(\frac{t}{T}\Big) + \rho(t). \tag{6-6}$$

Note the major arc disjointness property

$$\Big(\frac{a_1}{q_1} + \Big[-\frac{2}{NQ_1}, \frac{2}{NQ_1}\Big]\Big) \cap \Big(\frac{a_2}{q_2} + \Big[-\frac{2}{NQ_2}, \frac{2}{NQ_2}\Big]\Big) = \varnothing$$

for $(a_i, q_i) = 1$, $Q_i \leq q_i < 2Q_i$, $i = 1, 2$, $Q_1 \leq Q_2 \leq N_1$. This in particular implies

$$0 \leq \rho(t) \leq 1 \quad \text{for all } t \in \mathbb{R}/T\mathbb{Z}, \tag{6-7}$$

$$\Bigg[\Bigg(\sum_{\substack{(a,q)=1 \\ Q \leq q < 2Q}} \delta_{\frac{a}{q}}\Bigg) * \omega_{\frac{1}{NM}}\Big(\frac{\cdot}{T}\Big)\Bigg]^\wedge(0) = \frac{1}{T}\int_0^T \Bigg(\sum_{\substack{(a,q)=1 \\ Q \leq q < 2Q}} \delta_{\frac{a}{q}}\Bigg) * \omega_{\frac{1}{NM}}\Big(\frac{t}{T}\Big)\, dt \leq \frac{2Q^2}{NM}, \tag{6-8}$$

which implies

$$1 \geq |\hat{\rho}(0)| \geq 1 - \sum_{1 \leq Q \leq N_1} \sum_{Q \leq M \leq N} \Bigg|\Bigg[\Bigg(\sum_{\substack{(a,q)=1 \\ Q \leq q < 2Q}} \delta_{\frac{a}{q}}\Bigg) * \omega_{\frac{1}{NM}}\Big(\frac{\cdot}{T}\Big)\Bigg]^\wedge(0)\Bigg| \geq 1 - \frac{8N_1}{N} \geq \frac{1}{2}. \tag{6-9}$$

By Dirichlet's lemma on rational approximations, for any $\frac{t}{T} \in \mathbb{S}^1$, there exists a, q, with $a \in \mathbb{Z}_{\geq 0}$, $q \in \mathbb{N}$, $(a, q) = 1$, $q \leq N$, such that $\left|\frac{t}{T} - \frac{a}{q}\right| < \frac{1}{qN}$. If $\rho\big(\frac{t}{T}\big) \neq 0$, then (6-4) implies $q > N_1 = N/2^{10}$. This implies by (6-1) and (6-7) that

$$\|\rho(t)K_N(t,x)\|_{L^\infty(\mathbb{T}\times G)} \lesssim N^{d-\frac{r}{2}}. \tag{6-10}$$

Now define coefficients $\alpha_{Q,M}$ such that

$$\Bigg[\Bigg(\sum_{\substack{(a,q)=1\\ Q\le q<2Q}} \delta_{\frac{a}{q}}\Bigg) * \omega_{\frac{1}{NM}}\Big(\frac{\cdot}{T}\Big)\Bigg]^{\wedge}(0) = \alpha_{Q,M}\,\hat{\rho}(0). \tag{6-11}$$

Then (6-8) and (6-9) imply

$$\alpha_{Q,M} \lesssim \frac{Q^2}{NM}. \tag{6-12}$$

Write

$$\boldsymbol{K}_N(t,x) = \sum_{Q\le N_1}\sum_{Q\le M\le N} \boldsymbol{K}_N(t,x)\Bigg[\Bigg(\Bigg(\sum_{\substack{(a,q)=1\\ Q\le q<2Q}} \delta_{\frac{a}{q}}\Bigg) * \omega_{\frac{1}{NM}}\Big(\frac{\cdot}{T}\Big)\Bigg) - \alpha_{Q,M}\,\rho\Bigg](t) + \Bigg(1+\sum_{Q,M}\alpha_{Q,M}\Bigg)\boldsymbol{K}_N(t,x)\rho(t), \tag{6-13}$$

and define

$$\Lambda_{Q,M}(t,x) := \boldsymbol{K}_N(t,x)\Bigg[\Bigg(\Bigg(\sum_{\substack{(a,q)=1\\ Q\le q<2Q}} \delta_{\frac{a}{q}}\Bigg) * \omega_{\frac{1}{NM}}\Big(\frac{\cdot}{T}\Big)\Bigg) - \alpha_{Q,M}\,\rho\Bigg](t). \tag{6-14}$$

Then from (6-1), (6-10), (6-12), we have

$$\|\Lambda_{Q,M}\|_{L^\infty(\mathbb{T}\times G)} \lesssim N^{d-\frac{r}{2}}\Big(\frac{M}{Q}\Big)^{\frac{r}{2}}. \tag{6-15}$$

Next, we estimate $\widehat{\Lambda}_{Q,M}$. From (5-15), for

$$n \in \frac{2\pi}{T}\mathbb{Z} \cong \widehat{\mathbb{T}}, \quad \lambda \in \widehat{G},$$

we have

$$\widehat{\Lambda}_{Q,M}(n,\lambda) = \lambda_{Q,M}(n,\lambda)\cdot \mathrm{Id}_{d_\lambda\times d_\lambda}, \tag{6-16}$$

where

$$\lambda_{Q,M}(n,\lambda) = \varphi(\lambda,N)\Bigg[\Bigg(\sum_{\substack{(a,q)=1\\ Q\le q<2Q}} \delta_{\frac{a}{q}}\Bigg)^{\wedge}\cdot \hat{\omega}_{\frac{1}{NM}}(T\,\cdot) - \alpha_{Q,M}\,\hat{\rho}\Bigg](n+k_\lambda). \tag{6-17}$$

Note that (6-11) immediately implies

$$\lambda_{Q,M}(n,\lambda) = 0 \quad \text{for } n+k_\lambda = 0. \tag{6-18}$$

Let $d(m,Q)$ denote the number of divisors of m less than Q; using Lemma 3.33 in [Bourgain 1993],

$$\Bigg|\Bigg(\sum_{\substack{(a,q)=1\\ Q\le q<2Q}} \delta_{\frac{a}{q}}\Bigg)^{\wedge}(Tn)\Bigg| \lesssim_\varepsilon d\Big(\frac{Tn}{2\pi},Q\Big)Q^{1+\varepsilon}, \quad n\neq 0,\ \varepsilon>0, \tag{6-19}$$

we get

$$|\lambda_{Q,M}(n,\lambda)| \lesssim_\varepsilon \varphi(\lambda, N)\frac{Q^{1+\varepsilon}}{NM} d\left(\frac{T(n+k_\lambda)}{2\pi}, Q\right) + \frac{Q^2}{NM}|\hat{\rho}(n+k_\lambda)|. \tag{6-20}$$

Using

$$d(m, Q) \lesssim_\varepsilon m^\varepsilon,$$

(6-19) and (6-6), we have

$$|\hat{\rho}(n)| \le \sum_{1\le Q\le N_1} \sum_{Q\le M\le N} \frac{d\left(\frac{Tn}{2\pi}, Q\right)Q^{1+\varepsilon}}{NM} \lesssim \frac{N^\varepsilon}{N} \quad \text{for } n \neq 0,\ |n| \lesssim N^2; \tag{6-21}$$

thus

$$\begin{aligned} |\lambda_{Q,M}(n,\lambda)| &\lesssim_\varepsilon \varphi(\lambda, N)\frac{Q}{NM}\left[Q^\varepsilon d\left(\frac{T(n+k_\lambda)}{2\pi}, Q\right) + \frac{Q}{N^{1-\varepsilon}}\right] \\ &\lesssim_\varepsilon \varphi(\lambda, N)\frac{QN^\varepsilon}{NM} \quad \text{for } |n| \lesssim N^2. \end{aligned} \tag{6-22}$$

Proposition 6.4. (i) *Assume that $f \in L^1(\mathbb{T}\times G)$. Then*

$$\|f * \Lambda_{Q,M}\|_{L^\infty(\mathbb{T}\times G)} \lesssim N^{d-\frac{r}{2}}\left(\frac{M}{Q}\right)^{\frac{r}{2}} \|f\|_{L^1(\mathbb{T}\times G)}. \tag{6-23}$$

(ii) *Assume that $f \in L^2(\mathbb{T}\times G)$. Assume also*

$$\hat{f}(n,\lambda) = 0 \quad \textit{for } |n| \gtrsim N^2. \tag{6-24}$$

Then

$$\|f * \Lambda_{Q,M}\|_{L^2(\mathbb{T}\times G)} \lesssim_\varepsilon \frac{QN^\varepsilon}{NM}\|f\|_{L^2(\mathbb{T}\times G)}, \tag{6-25}$$

$$\|f * \Lambda_{Q,M}\|_{L^2(\mathbb{T}\times G)} \lesssim_{\tau,B} \frac{Q^{1+2\tau}L}{NM}\|f\|_{L^2(\mathbb{T}\times G)} + M^{-1}L^{-\frac{B}{2}}N^{\frac{d}{2}}\|f\|_{L^1(\mathbb{T}\times G)} \tag{6-26}$$

for all

$$L > 1, \quad 0 < \tau < 1, \quad B > \frac{6}{\tau}, \quad N > (LQ)^B. \tag{6-27}$$

Proof. Using (6-15), we have

$$\|f * \Lambda_{Q,M}\|_{L^\infty(\mathbb{T}\times G)} \le \|f\|_{L^1(\mathbb{T}\times G)}\|\Lambda_{Q,M}\|_{L^\infty(\mathbb{T}\times G)} \lesssim N^{d-\frac{r}{2}}\left(\frac{M}{Q}\right)^{\frac{r}{2}}\|f\|_{L^1(\mathbb{T}\times G)}.$$

This proves (i). (6-25) is a consequence of (4-4), (6-16), and (6-22). To prove (6-26), we use (4-1), (4-3) and (6-16) to get

$$\|f * \Lambda_{Q,M}\|_{L^2(\mathbb{T}\times G)} = \left(\sum_{n,\lambda} d_\lambda \|\hat{f}(n,\lambda)\|_{\mathrm{HS}}^2 \cdot |\lambda_{Q,M}(n,\lambda)|^2\right)^{\frac{1}{2}},$$

which combined with (6-18), (6-20), and (6-21) yields

$$\begin{aligned} &\|f * \Lambda_{Q,M}\|_{L^2(\mathbb{T}\times G)} \\ &\lesssim_\varepsilon \frac{Q^{1+\varepsilon}}{NM}\left(\sum_{n,\lambda}\varphi(\lambda,N)^2 d_\lambda \|\hat{f}(n,\lambda)\|_{\mathrm{HS}}^2\, d\left(\frac{T(n+k_\lambda)}{2\pi}, Q\right)^2\right)^{\frac{1}{2}} + \frac{Q^2}{MN^{2-\varepsilon}}\|f\|_{L^2(\mathbb{T}\times G)}. \end{aligned} \tag{6-28}$$

Using Lemma 3.47 in [Bourgain 1993] and Lemma 5.2, we have

$$\begin{aligned}\left|\left\{(n,\lambda)\;\middle|\;|n|,k_\lambda\lesssim N^2, d\left(\frac{T(n+k_\lambda)}{2\pi},Q\right)>D\right\}\right|\\
&\lesssim_{\tau,B}(D^{-B}Q^\tau N^2+Q^B)\cdot\max_{|m|\lesssim N^2}|\{(n,\lambda)\mid n+k_\lambda=m\}|\\
&\lesssim_{\tau,B}(D^{-B}Q^\tau N^2+Q^B)\cdot|\{\lambda\in\widehat{G}\mid k_\lambda\lesssim N^2\}|\\
&\lesssim_{\tau,B}(D^{-B}Q^\tau N^2+Q^B)\cdot N^r.\end{aligned}\tag{6-29}$$

Now (4-5) gives

$$\|\hat{f}(n,\lambda)\|_{\mathrm{HS}}^2\lesssim d_\lambda\|f\|_{L^1(\mathbb{T}\times G)}^2,$$

and Lemma 5.2 gives

$$|\varphi(\lambda,N)d_\lambda^2|\lesssim N^{d-r},$$

which together with (6-29) imply

$$\begin{aligned}&\|f*\Lambda_{Q,M}\|_{L^2(\mathbb{T}\times G)}\\
&\quad\lesssim_{\tau,B}\left(\frac{Q^{1+\varepsilon}D}{NM}+\frac{Q^2}{MN^{2-\varepsilon}}\right)\|f\|_{L^2(\mathbb{T}\times G)}+\frac{Q^{1+\varepsilon}}{NM}\cdot Q\cdot(D^{-\frac{B}{2}}Q^\tau N+Q^{\frac{B}{2}})N^{\frac{d}{2}}\|f\|_{L^1(\mathbb{T}\times G)}.\end{aligned}\tag{6-30}$$

This implies (6-26) assuming the conditions in (6-27). □

Now interpolating (6-23) and (6-25), we get

$$\|f*\Lambda_{Q,M}\|_{L^p(\mathbb{T}\times G)}\lesssim_\varepsilon N^{d-\frac{r}{2}-\frac{2d-r+2}{p}+\varepsilon}M^{\frac{r}{2}-\frac{r+2}{p}}Q^{-\frac{r}{2}+\frac{r+2}{p}}\|f\|_{L^{p'}(\mathbb{T}\times G)}.\tag{6-31}$$

Interpolating (6-23) and (6-26) for

$$p>\frac{2(r+2)}{r}+10\tau,\quad\text{which implies }\sigma=\frac{r}{2}-\frac{r+2+4\tau}{p}>0,\tag{6-32}$$

we get

$$\begin{aligned}\|f*\Lambda_{Q,M}\|_{L^p(\mathbb{T}\times G)}\lesssim_{\tau,B}&\,N^{d-\frac{r}{2}-\frac{2d-r+2}{p}}M^{\frac{r}{2}-\frac{r+2}{p}}Q^{-\sigma}L^{\frac{2}{p}}\|f\|_{L^{p'}(\mathbb{T}\times G)}\\
&+Q^{-\frac{2}{r}(1-\frac{2}{p})}M^{\frac{r}{2}-\frac{r+2}{p}}L^{-\frac{B}{p}}N^{d-\frac{r}{2}-\frac{d-r}{p}}\|f\|_{L^1(\mathbb{T}\times G)}.\end{aligned}\tag{6-33}$$

Now we are ready to prove Theorem 6.3.

Proof of Theorem 6.3. Without loss of generality, we assume that $\|f\|_{L^2(G)}=1$. Then for $F=\boldsymbol{P}_N e^{it\Delta}f$, (3-2) implies

$$\|F\|_{L_x^2}\lesssim 1,\tag{6-34}$$

$$\|F\|_{L_x^\infty}\lesssim N^{\frac{d}{2}}.\tag{6-35}$$

Let

$$H=\chi_{|F|>\lambda}\cdot\frac{F}{|F|}.\tag{6-36}$$

Let $\tilde{\tilde{\boldsymbol{P}}}_N$ be a Littlewood–Paley projection of product type such that $\tilde{\tilde{\boldsymbol{P}}}_N \circ \boldsymbol{P}_N = \boldsymbol{P}_N$. Let $\tilde{\tilde{\boldsymbol{K}}}_N$ be the Schrödinger kernel associated to $\tilde{\tilde{\boldsymbol{P}}}_N e^{it\Delta}$. Then by (4-3), (5-13), and (5-14), we have

$$F * \tilde{\tilde{\boldsymbol{K}}}_N = F.$$

Let Q_{N^2} be the Littlewood–Paley projection operator on $L^2(\mathbb{T} \times G)$ defined by

$$(Q_{N^2} H)^\wedge := \varphi\left(\frac{-k_\lambda - n^2}{N^4}\right) \widehat{H}(n, \lambda)$$

for some bump function φ such that $Q_{N^2} \circ \boldsymbol{P}_N = \boldsymbol{P}_N$. Then by (4-2) and (5-14), we have

$$\langle F, H \rangle_{L^2_{t,x}} = \langle Q_{N^2} F, H \rangle_{L^2_{t,x}} = \langle F, Q_{N^2} H \rangle_{L^2_{t,x}}.$$

Then we can write

$$\lambda m_\lambda \le \langle F, H \rangle_{L^2_{t,x}} = \langle F * \tilde{\tilde{\boldsymbol{K}}}_N, Q_{N^2} H \rangle_{L^2_{t,x}}.$$

Using (4-1) and (4-3) again, we get

$$\begin{aligned} \lambda m_\lambda &\le \langle F, Q_{N^2} H * \tilde{\tilde{\boldsymbol{K}}}_N \rangle_{L^2_{t,x}} \le \|F\|_{L^2_{t,x}} \|Q_{N^2} H * \tilde{\tilde{\boldsymbol{K}}}_N\|_{L^2_{t,x}} \\ &\lesssim \|Q_{N^2} H * \tilde{\tilde{\boldsymbol{K}}}_N\|_{L^2_{t,x}} = \langle Q_{N^2} H * \tilde{\tilde{\boldsymbol{K}}}_N, Q_{N^2} H * \tilde{\tilde{\boldsymbol{K}}}_N \rangle_{L^2_{t,x}} \\ &= \langle Q_{N^2} H, Q_{N^2} H * (\tilde{\tilde{\boldsymbol{K}}}_N * \tilde{\tilde{\boldsymbol{K}}}_N) \rangle_{L^2_{t,x}}. \end{aligned} \tag{6-37}$$

Let

$$H' = Q_{N^2} H, \quad \tilde{\boldsymbol{K}}_N = \tilde{\tilde{\boldsymbol{K}}}_N * \tilde{\tilde{\boldsymbol{K}}}_N.$$

Note that H' by definition satisfies the assumption in (6-24) and we can apply Proposition 6.4. Also note that $\tilde{\boldsymbol{K}}_N$ is still a Schrödinger kernel associated to a Littlewood–Paley projection operator of product type. Finally note that the Bernstein-type inequalities (3-2) and the definition (6-36) of H give

$$\|H'\|_{L^p_{t,x}} \lesssim \|H\|_{L^p_{t,x}} \lesssim m_\lambda^{\frac{1}{p}}. \tag{6-38}$$

Write

$$\Lambda = \sum_{1 \le Q \le N_1} \sum_{Q \le M \le N} \Lambda_{Q,M}, \quad \tilde{\boldsymbol{K}}_N = \Lambda + (\tilde{\boldsymbol{K}}_N - \Lambda),$$

where $\Lambda_{Q,M}$ is defined as in (6-14) except that $\boldsymbol{K}_N$ is replaced by $\tilde{\boldsymbol{K}}_N$. We have by (6-37)

$$\begin{aligned} \lambda^2 m_\lambda^2 &\lesssim \langle H', H' * \Lambda \rangle_{L^2_{t,x}} + \langle H', H' * (\tilde{\boldsymbol{K}}_N - \Lambda) \rangle_{L^2_{t,x}} \\ &\lesssim \|H'\|_{L^{p'}_{t,x}} \|H' * \Lambda\|_{L^p_{t,x}} + \|H'\|^2_{L^1_{t,x}} \|\tilde{\boldsymbol{K}}_N - \Lambda\|_{L^\infty_{t,x}}. \end{aligned} \tag{6-39}$$

Using (6-31) for $p = p_0 := \frac{2(r+2)}{r}$, then summing over Q, M, and noting (6-38), we have

$$\|H'\|_{L^{p'}_{t,x}} \|H' * \Lambda\|_{L^p_{t,x}} \lesssim N^{d - \frac{2d+4}{p_0} + \varepsilon} \|H'\|^2_{L^{p_0'}_{t,x}} \lesssim N^{d - \frac{2d+4}{p_0} + \varepsilon} m_\lambda^{\frac{2}{p_0'}}.$$

From (6-10) and (6-12) we get

$$\|\tilde{\boldsymbol{K}}_N - \Lambda\|_{L^\infty_{t,x}} \lesssim N^{d - \frac{r}{2}}, \tag{6-40}$$

which implies

$$\|H'\|^2_{L^1_{t,x}} \|\widetilde{\boldsymbol{K}}_N - \Lambda\|_{L^\infty_{t,x}} \lesssim N^{d-\frac{r}{2}} \|H'\|^2_{L^1_{t,x}} \lesssim N^{d-\frac{r}{2}} m_\lambda^2. \tag{6-41}$$

Then we have

$$\lambda^2 m_\lambda^2 \lesssim N^{d-\frac{2d+4}{p_0}+\varepsilon} m_\lambda^{\frac{2}{p_0'}} + N^{d-\frac{r}{2}} m_\lambda^2,$$

which implies for $\lambda \gtrsim N^{d/2-r/4}$

$$m_\lambda \lesssim_\varepsilon N^{p_0(\frac{d}{2}-\frac{d+2}{p_0})+\varepsilon} \lambda^{-p_0}.$$

Thus part (I) is proved. To prove part (II) for some fixed p, using part (I) and (6-35), it suffices to prove it for $\lambda \gtrsim N^{d/2-\varepsilon}$. Summing (6-33) over Q, M in the range indicated by (6-27), we get

$$\|H' * \Lambda_1\|_{L^p_{t,x}} \lesssim L N^{d-\frac{2d+4}{p}} \|H'\|_{L^{p'}_{t,x}} + L^{-\frac{B}{p}} N^{d-\frac{d+2}{p}} \|H'\|_{L^1_{t,x}}, \tag{6-42}$$

where

$$\Lambda_1 := \sum_{\substack{Q<Q_1 \\ Q \le M \le N}} \Lambda_{Q,M}$$

and Q_1 is the largest Q-value satisfying (6-27). For values $Q \ge Q_1$, use (6-31) to get

$$\|H' * (\Lambda - \Lambda_1)\|_{L^p_{t,x}} \lesssim_\varepsilon N^{d-\frac{2d+4}{p}+\varepsilon} Q_1^{-(\frac{r}{2}-\frac{r+2}{p})} \|H'\|_{L^{p'}_{t,x}}. \tag{6-43}$$

Using (6-39), (6-41), (6-42) and (6-43), we get

$$\lambda^2 m_\lambda^2 \lesssim N^{d-\frac{2(d+2)}{p}} \left(L + \frac{N^\varepsilon}{Q_1^{\frac{r}{2}-\frac{r+2}{p}}} \right) m_\lambda^{\frac{2}{p}'} + L^{-\frac{B}{p}} N^{d-\frac{d+2}{p}} m_\lambda^{1+\frac{1}{p'}} + N^{d-\frac{r}{2}} m_\lambda^2.$$

For $\lambda \gtrsim N^{d/2-r/4}$, the last term of the above inequality can be dropped. Let $Q_1 = N^\delta$ such that $\delta > 0$ and

$$(LN^\delta)^B < N \tag{6-44}$$

such that (6-27) holds. Note that

$$L > 1 > \frac{N^\varepsilon}{Q_1^{\frac{r}{2}-\frac{r+2}{p}}}$$

for $p > p_0 + 10\tau$ and ε sufficiently small; thus

$$\lambda^2 m_\lambda^2 \lesssim N^{d-\frac{2(d+2)}{p}} L m_\lambda^{\frac{2}{p}'} + L^{-\frac{B}{p}} N^{d-\frac{d+2}{p}} m_\lambda^{1+\frac{1}{p'}}.$$

This implies

$$\begin{aligned} m_\lambda &\lesssim N^{p(\frac{d}{2}-\frac{d+2}{p})} L^{\frac{p}{2}} \lambda^{-p} + N^{p(d-\frac{d+2}{p})} L^{-B} \lambda^{-2p} \\ &\lesssim N^{-d-2} \left(\frac{N^{\frac{d}{2}}}{\lambda} \right)^p L^{\frac{p}{2}} + N^{-d-2} \left(\frac{N^{\frac{d}{2}}}{\lambda} \right)^{2p} L^{-B}. \end{aligned}$$

Let

$$L=\left(\frac{N^{\frac{d}{2}}}{\lambda}\right)^{\tau},\quad B>\frac{p}{\tau}$$

and δ be sufficiently small so that (6-44) holds; then

$$m_\lambda\lesssim N^{-d-2}\left(\frac{N^{\frac{d}{2}}}{\lambda}\right)^{p+\frac{p\tau}{2}}.$$

Note that conditions for p,τ indicated in (6-32) imply that $p+\frac{p\tau}{2}$ can take any exponent $>p_0=\frac{2(r+2)}{r}$. This completes the proof of part (II).

The proofs of parts (III) and (IV) are then identical to the proofs of Propositions 3.110 and 3.113 respectively in [Bourgain 1993]. □

Proof of Theorem 2.2. Part (i) is a direct consequence of Theorem 6.3(III). Part (ii) is a direct consequence of Theorem 6.3(IV) and the result from [Bourgain and Demeter 2015] that full Strichartz estimates hold on any torus with an ε-loss. □

7. Dispersive estimates on major arcs

In this section, we prove Theorem 6.2.

7A. *Weyl-type sums on rational lattices.*

Definition 7.1. Let $L=\mathbb{Z}w_1+\cdots+\mathbb{Z}w_r$ be a lattice on an inner product space $(V,\langle\cdot,\cdot\rangle)$. We say L is a *rational lattice* provided that there exists some $D\in\mathbb{R}$ such that $\langle w_i,w_j\rangle\in D^{-1}\mathbb{Z}$. We call the number D a *period* of L.

By Lemma 5.3, any weight lattice Λ is a rational lattice with respect to the Cartan–Killing form. As a sublattice of Λ, the root lattice Γ is also rational.

Let f be a function on $\mathbb{Z}^r$ and define the *difference operator* D_i by

$$D_if(n_1,\dots,n_r):=f(n_1,\dots,n_{i-1},n_i+1,n_{i+1},\dots,n_r)-f(n_1,\dots,n_r)\tag{7-1}$$

for $i=1,\dots,r$. The Leibniz rule for D_i reads

$$D_i\left(\prod_{j=1}^n f_j\right)=\sum_{l=1}^n\sum_{1\le k_1<\cdots<k_l\le n}D_if_{k_1}\cdots D_if_{k_l}\cdot\prod_{\substack{j\neq k_1,\dots,k_l\\1\le j\le n}}f_j.\tag{7-2}$$

Note that there are 2^n-1 terms in the right side of the above formula.

Definition 7.2. Let $L\cong\mathbb{Z}^r$ be a lattice of rank r. Given $A\in\mathbb{R}$, we say a function f on L is a pseudopolynomial of degree A provided for each $n\in\mathbb{Z}_{\ge0}$

$$|D_{i_1}\cdots D_{i_n}f(n_1,\dots,n_r)|\lesssim N^{A-n}\tag{7-3}$$

holds uniformly in $|n_i|\lesssim N,\ i=1,\dots,r$, for all $i_j=1,\dots,r,\ j=1,\dots,n$, and $N\ge1$.

A direct application of the Leibniz rule (7-2) gives the following lemma.

Lemma 7.3. *Let L be a lattice and f, g two functions on L. Assume f, g are pseudopolynomials of degrees A, B respectively. Then $f \cdot g$ is a pseudopolynomial of degree $A+B$.*

Now we have the following estimate on Weyl-type sums, which generalizes the classical Weyl inequality in one dimension, as in Lemma 3.18 of [Bourgain 1993].

Lemma 7.4. *Let $L = \mathbb{Z}w_1 + \cdots + \mathbb{Z}w_r$ be a rational lattice in the inner product space $(V, \langle\cdot,\cdot\rangle)$ with a period $D > 0$. Let φ be a bump function on $\mathbb{R}$ and $N \geq 1$, $A \in \mathbb{R}$. Suppose $f : L \to \mathbb{C}$ a pseudopolynomial of degree A. Let*

$$F(t, H) = \sum_{\lambda\in L} e^{-it|\lambda|^2+i\langle\lambda,H\rangle}\varphi\left(\frac{|\lambda|^2}{N^2}\right)\cdot f \tag{7-4}$$

for $t \in \mathbb{R}$ and $H \in V$. Then for $\frac{t}{2\pi D} \in \mathcal{M}_{a,q}$, we have

$$|F(t, H)| \lesssim \frac{N^{A+r}}{\left(\sqrt{q}\left(1 + N\left\|\frac{t}{2\pi D} - \frac{a}{q}\right\|^{1/2}\right)\right)^r} \tag{7-5}$$

uniformly in $H \in V$.

Note that part (i) of Theorem 6.2 is a direct consequence of this lemma.

Proof. By the Weyl differencing trick, write

$$\begin{aligned}
|F|^2 &= \sum_{\lambda_1,\lambda_2\in L} e^{-it(|\lambda_1|^2-|\lambda_2|^2)+i\langle\lambda_1-\lambda_2,H\rangle}\varphi\left(\frac{|\lambda_1|^2}{N^2}\right)\varphi\left(\frac{|\lambda_2|^2}{N^2}\right)f(\lambda_1)\overline{f(\lambda_2)} \\
&= \sum_{\mu=\lambda_1-\lambda_2} e^{-it|\mu|^2+i\langle\mu,H\rangle}\sum_{\lambda=\lambda_2} e^{-i2t\langle\mu,\lambda\rangle}\varphi\left(\frac{|\mu+\lambda|^2}{N^2}\right)\varphi\left(\frac{|\lambda|^2}{N^2}\right)f(\mu+\lambda)\overline{f(\lambda)} \\
&\leq \sum_{|\mu|\lesssim N}\left|\sum_{\lambda} e^{-i2t\langle\mu,\lambda\rangle}\varphi\left(\frac{|\mu+\lambda|^2}{N^2}\right)\varphi\left(\frac{|\lambda|^2}{N^2}\right)f(\mu+\lambda)\overline{f(\lambda)}\right|.
\end{aligned}$$

Now let $L = \mathbb{Z}w_1 + \cdots + \mathbb{Z}w_r$. Write

$$\lambda = \sum_{i=1}^{r} n_i w_i$$

and

$$g(\lambda) = \varphi\left(\frac{|\mu+\lambda|^2}{N^2}\right)\varphi\left(\frac{|\lambda|^2}{N^2}\right)f(\mu+\lambda)\overline{f(\lambda)}.$$

Note that as functions in $\lambda \in L$, both $\varphi(|\mu+\lambda|^2/N^2)|$ and $\varphi(|\lambda|^2/N^2)$ are pseudopolynomials of degree 0, and both $f(\mu+\lambda)$ and $f(\lambda)$ are pseudopolynomials of degree A, which implies by Lemma 7.3 that $g(\lambda)$ is a pseudopolynomial of degree $2A$. That is, $g(\lambda)$ satisfies

$$|D_{i_1}\cdots D_{i_n}g(\lambda)| \lesssim N^{2A-n} \tag{7-6}$$

uniformly for $|\lambda| \lesssim N$ and $N \geq 1$, for all $i_1, \dots, i_n \in \{1, \dots, r\}$. Write

$$\sum_{\lambda \in L} e^{-i2t\langle \mu,\lambda\rangle} g(\lambda) = \sum_{n_1,\dots,n_r \in \mathbb{Z}} \biggl(\prod_{i=1}^{r} e^{-itn_i\langle \mu, 2w_i\rangle}\biggr) g(\lambda). \tag{7-7}$$

By summation by parts twice, we have

$$\sum_{n_1 \in \mathbb{Z}} e^{-itn_1\langle \mu,2w_1\rangle} g = \biggl(\frac{e^{-it\langle \mu,2w_1\rangle}}{1-e^{-it\langle \mu,2w_1\rangle}}\biggr)^2 \sum_{n_1 \in \mathbb{Z}} e^{-itn_1\langle \mu,2w_1\rangle} D_1^2 g(n_1, \dots, n_r); \tag{7-8}$$

then (7-7) becomes

$$\sum_{\lambda \in L} e^{-i2t\langle \mu,\lambda\rangle} g = \biggl(\frac{e^{-it\langle \mu,2w_1\rangle}}{1-e^{-it\langle \mu,2w_1\rangle}}\biggr)^2 \sum_{n_1,\dots,n_r \in \mathbb{Z}} \biggl(\prod_{i=1}^{r} e^{-itn_i\langle \mu,2w_i\rangle}\biggr) D_1^2 g(n_1, \dots, n_r).$$

Then we can carry out the procedure of summation by parts twice with respect to other variables $n_2, \dots, n_r$. But we require that only when

$$|1 - e^{-it\langle \mu,2w_i\rangle}| \geq \tfrac{1}{N}$$

do we carry out the procedure to the variable n_i. Using (7-6), we obtain

$$\begin{aligned}\biggl|\sum_{\lambda} e^{-i2t\langle \mu,\lambda\rangle} \varphi\biggl(\frac{|\mu+\lambda|^2}{N^2}\biggr)\varphi\biggl(\frac{|\lambda|^2}{N^2}\biggr) f(\mu+\lambda)\,\overline{f(\lambda)}\biggr| &\lesssim N^{2A-r} \prod_{i=1}^{r} \frac{1}{\bigl(\max\{1-e^{-it\langle \mu,2w_i\rangle}, \frac{1}{N}\}\bigr)^2} \\ &\lesssim N^{2A-r} \prod_{i=1}^{r} \frac{1}{\bigl(\max\{\bigl\|\frac{1}{2\pi} t\langle \mu, 2w_i\rangle\bigr\|, \frac{1}{N}\}\bigr)^2}.\end{aligned}$$

Writing $\mu = \sum_{j=1}^{r} m_j w_j,\ m_j \in \mathbb{Z}$, we have

$$|F|^2 \lesssim N^{2A-r} \sum_{\substack{|m_j| \lesssim N \\ j=1,\dots,r}} \prod_{i=1}^{r} \frac{1}{\bigl(\max\{\bigl\|\frac{1}{2\pi} t \sum_{j=1}^{r} m_j\langle w_j, 2w_i\rangle\bigr\|, \frac{1}{N}\}\bigr)^2}.$$

Let

$$n_i = \sum_{j=1}^{r} m_j \langle w_j, 2w_i\rangle \cdot D, \quad i = 1, \dots, r, \tag{7-9}$$

where $D > 0$ is the period of L so that $\langle w_j, w_i\rangle \in D^{-1}\mathbb{Z}$. Then $n_i \in \mathbb{Z}$. Note that the matrix $(\langle w_j, 2w_i\rangle D)_{i,j}$ is nondegenerate, which implies that for each vector $(n_1, \dots, n_r) \in \mathbb{Z}^r$ there exists at most one vector $(m_1, \dots, m_r) \in \mathbb{Z}^r$ so that (7-9) holds; thus

$$\begin{aligned}|F|^2 &\lesssim N^{2A-r} \sum_{\substack{|n_i| \lesssim N \\ i=1,\dots,r}} \prod_{i=1}^{r} \frac{1}{\bigl(\max\{\bigl\|\frac{t}{2\pi D} n_i\bigr\|, \frac{1}{N}\}\bigr)^2} \\ &\lesssim N^{2A-r} \prod_{i=1}^{r} \biggl(\sum_{|n_i| \lesssim N} \frac{1}{\bigl(\max\{\bigl\|\frac{t}{2\pi D} n_i\bigr\|, \frac{1}{N}\}\bigr)^2}\biggr).\end{aligned}$$

Then by a standard estimate as in the proof of the classical Weyl inequality in one dimension, we have

$$\sum_{|n_i|\lesssim N} \frac{1}{\left(\max\{\left\|\frac{t}{2\pi D}n_i\right\|, \frac{1}{N}\}\right)^2} \lesssim \frac{N^3}{\left(\sqrt{q}\left(1+N\left\|\frac{t}{2\pi D}-\frac{a}{q}\right\|^{1/2}\right)\right)^2},$$

which implies the desired result

$$|F|^2 \lesssim \frac{N^{2A+2r}}{\left(\sqrt{q}\left(1+N\left\|\frac{t}{2\pi D}-\frac{a}{q}\right\|^{1/2}\right)\right)^{2r}}. \qquad \square$$

Remark 7.5. Let λ_0 be a constant vector in $\mathbb{R}^r$ and C a constant real number. Then we can slightly generalize the form of the function $F(t, H)$ in the above lemma into

$$F(t,H)=\sum_{\lambda\in L} e^{-it|\lambda+\lambda_0|^2+i\langle\lambda,H\rangle}\varphi\left(\frac{|\lambda+\lambda_0|^2+C}{N^2}\right)\cdot f$$

such that the conclusion of the lemma still holds.

7B. ***From a chamber to the whole weight lattice.*** To prove part (ii) of Theorem 6.2, we first rewrite the Schrödinger kernel as an exponential sum over the whole weight lattice Λ instead of just a chamber of it, in order to apply Lemma 7.4.

Lemma 7.6. *Recall that $D_P(H)=\sum_{s\in W}(\det s)e^{i\langle\rho,H\rangle}$ is the Weyl denominator. We have*

$$K_N(t,x)=\frac{e^{it|\rho|^2}}{\left(\prod_{\alpha\in P}\langle\alpha,\rho\rangle\right)D_P(H)}\sum_{\lambda\in\Lambda} e^{-it|\lambda|^2+i\langle\lambda,H\rangle}\varphi\left(\frac{|\lambda|^2-|\rho|^2}{N^2}\right)\prod_{\alpha\in P}\langle\alpha,\lambda\rangle \tag{7-10}$$

$$=\frac{e^{it|\rho|^2}}{\left(\prod_{\alpha\in P}\langle\alpha,\rho\rangle\right)|W|}\sum_{\lambda\in\Lambda} e^{-it|\lambda|^2}\varphi\left(\frac{|\lambda|^2-|\rho|^2}{N^2}\right)\prod_{\alpha\in P}\langle\alpha,\lambda\rangle\frac{\sum_{s\in W}(\det s)e^{i\langle s(\lambda),H\rangle}}{\sum_{s\in W}(\det s)e^{i\langle s(\rho),H\rangle}}. \tag{7-11}$$

Proof. To prove (7-11), first note that from Proposition 7.13 below, $\prod_{\alpha\in P}\langle\alpha,\cdot\rangle$ is an *anti-invariant polynomial*; that is,

$$\prod_{\alpha\in P}\langle\alpha,s(\lambda)\rangle=(\det s)\prod_{\alpha\in P}\langle\alpha,\lambda\rangle \tag{7-12}$$

for all $\lambda\in i\mathfrak{b}^*$. Recall that the Weyl group W acts on $i\mathfrak{b}^*$ isometrically; that is,

$$|s(\lambda)|=|\lambda| \quad \text{for all } s\in W,\ \lambda\in i\mathfrak{b}^*. \tag{7-13}$$

Also recalling the definition (4-12) of ρ and the definition (4-10) of the fundamental chamber C, we may rewrite K_N as in (5-2) into

$$K_N(t,x)=\frac{e^{it|\rho|^2}}{\left(\prod_{\alpha\in P}\langle\alpha,\rho\rangle\right)D_P}\sum_{\lambda\in\Lambda\cap C} e^{-it|\lambda|^2}\varphi\left(\frac{|\lambda|^2-|\rho|^2}{N^2}\right)\prod_{\alpha\in P}\langle\alpha,\lambda\rangle\sum_{s\in W}(\det s)e^{i\langle s(\lambda),H\rangle}.$$

Using the (7-12) and (7-13), we write

$$\begin{aligned}K_N(t,x)&=\frac{e^{it|\rho|^2}}{\left(\prod_{\alpha\in P}\langle\alpha,\rho\rangle\right)D_P}\sum_{s\in W}\sum_{\lambda\in\Lambda\cap C}e^{-it|\lambda|^2}\varphi\left(\frac{|\lambda|^2-|\rho|^2}{N^2}\right)\prod_{\alpha\in P}\langle\alpha,s(\lambda)\rangle e^{i\langle s(\lambda),H\rangle}\\&=\frac{e^{it|\rho|^2}}{\left(\prod_{\alpha\in P}\langle\alpha,\rho\rangle\right)D_P}\sum_{s\in W}\sum_{\lambda\in\Lambda\cap C}e^{-it|s(\lambda)|^2}\varphi\left(\frac{|s(\lambda)|^2-|\rho|^2}{N^2}\right)\prod_{\alpha\in P}\langle\alpha,s(\lambda)\rangle e^{i\langle s(\lambda),H\rangle}\\&=\frac{e^{it|\rho|^2}}{\left(\prod_{\alpha\in P}\langle\alpha,\rho\rangle\right)D_P}\sum_{\lambda\in\bigsqcup_{s\in W}s(\Lambda\cap C)}e^{-it|\lambda|^2}\varphi\left(\frac{|\lambda|^2-|\rho|^2}{N^2}\right)\prod_{\alpha\in P}\langle\alpha,\lambda\rangle e^{i\langle\lambda,H\rangle},\end{aligned}\tag{7-14}$$

which then implies by (4-11) that

$$K_N(t,x)=\frac{e^{it|\rho|^2}}{\left(\prod_{\alpha\in P}\langle\alpha,\rho\rangle\right)D_P}\sum_{\lambda\in\Lambda}e^{-it|\lambda|^2}\varphi\left(\frac{|\lambda|^2-|\rho|^2}{N^2}\right)\prod_{\alpha\in P}\langle\alpha,\lambda\rangle e^{i\langle\lambda,H\rangle}.$$

This proves (7-10). To prove (7-11), write

$$\begin{aligned}&\sum_{\lambda\in\Lambda}e^{-it|\lambda|^2+i\langle\lambda,H\rangle}\varphi\left(\frac{|\lambda|^2-|\rho|^2}{N^2}\right)\prod_{\alpha\in P}\langle\alpha,\lambda\rangle\\&\qquad=\sum_{\lambda\in\Lambda}e^{-it|s(\lambda)|^2+i\langle s(\lambda),H\rangle}\varphi\left(\frac{|s(\lambda)|^2-|\rho|^2}{N^2}\right)\prod_{\alpha\in P}\langle\alpha,s(\lambda)\rangle,\end{aligned}\tag{7-15}$$

which implies using (7-12) and (7-13) that

$$\sum_{\lambda\in\Lambda}e^{-it|\lambda|^2+i\langle\lambda,H\rangle}\varphi\left(\frac{|\lambda|^2-|\rho|^2}{N^2}\right)\prod_{\alpha\in P}\langle\alpha,\lambda\rangle=(\det s)\sum_{\lambda\in\Lambda}e^{-it|\lambda|^2+i\langle s(\lambda),H\rangle}\varphi\left(\frac{|\lambda|^2-|\rho|^2}{N^2}\right)\prod_{\alpha\in P}\langle\alpha,\lambda\rangle,$$

which further implies

$$\begin{aligned}&\sum_{\lambda\in\Lambda}e^{-it|\lambda|^2+i\langle\lambda,H\rangle}\varphi\left(\frac{|\lambda|^2-|\rho|^2}{N^2}\right)\prod_{\alpha\in P}\langle\alpha,\lambda\rangle\\&\qquad=\frac{1}{|W|}\sum_{\lambda\in\Lambda}e^{-it|\lambda|^2}\varphi\left(\frac{|\lambda|^2-|\rho|^2}{N^2}\right)\prod_{\alpha\in P}\langle\alpha,\lambda\rangle\sum_{s\in W}(\det s)e^{i\langle s(\lambda),H\rangle}.\end{aligned}$$

This combined with (7-10) yields (7-11). □

Example 7.7. Specializing (7-10) and (7-11) to the Schrödinger kernel (5-3) for $G=\mathrm{SU}(2)$, we get

$$K_N(t,\theta)=\frac{e^{it}}{e^{i\theta}-e^{-i\theta}}\sum_{m\in\mathbb{Z}}e^{-itm^2+im\theta}\varphi\left(\frac{m^2-1}{N^2}\right)m\tag{7-16}$$

$$=\frac{e^{it}}{2}\sum_{m\in\mathbb{Z}}e^{-itm^2}\varphi\left(\frac{m^2-1}{N^2}\right)m\cdot\frac{e^{im\theta}-e^{-im\theta}}{e^{i\theta}-e^{-i\theta}},\quad\theta\in\mathbb{R}/2\pi\mathbb{Z}.\tag{7-17}$$

Corollary 7.8. (6-1) *holds for the following two scenarios*:

Scenario 1: $x = 1_G$, *where* 1_G *is the identity element of* G.

Scenario 2: $\left\| \frac{1}{2\pi}\langle \alpha, H\rangle \right\| \gtrsim \frac{1}{N}$ *for any* x *conjugate to* $\exp H$. *This is to say that the variable* H *is away from all the cell walls* $\left\{H \mid \left\| \frac{1}{2\pi}\langle \alpha, H\rangle \right\| = 0 \text{ for some } \alpha \in P\right\}$ *by a distance of* $\gtrsim \frac{1}{N}$.

Proof. Scenario 1: When $x = 1_G$, the character equals $\chi_\lambda(1_G) = d_\lambda = \prod_{\alpha\in P}\langle \alpha, \lambda\rangle / \prod_{\alpha\in P}\langle \alpha, \rho\rangle$. Then by (7-11), the Schrödinger kernel at $x = 1_G$ equals

$$K_N(t, 1_G) = \frac{e^{it|\rho|^2}}{\left(\prod_{\alpha\in P}\langle \alpha, \rho\rangle\right)^2 |W|} \sum_{\lambda\in\Lambda} e^{-it|\lambda|^2} \varphi\left(\frac{|\lambda|^2 - |\rho|^2}{N^2}\right)\left(\prod_{\alpha\in P}\langle \alpha, \lambda\rangle\right)^2. \tag{7-18}$$

Note that $f(\lambda) = \left(\prod_{\alpha\in P}\langle \alpha, \lambda\rangle\right)^2$ is a polynomial in the variable $\lambda = n_1 w_1 + \cdots + n_r w_r \in \Lambda$ of degree $2|P|$, which equals $d - r$ by (4-8). Thus f is also a pseudopolynomial of degree $d - r$. Then the desired estimate is a direct consequence of Lemma 7.4.

Scenario 2: By Lemma 4.13.4 of Chapter 4 in [Varadarajan 1974], the Weyl denominator $D_P = \sum_{s\in W}(\det s) e^{i\langle s(\rho), H\rangle}$ can be rewritten as

$$D_P = e^{-i\langle \rho, H\rangle} \prod_{\alpha\in P}\left(e^{i\langle \alpha, H\rangle} - 1\right). \tag{7-19}$$

Note that

$$1 \lesssim \frac{|e^{i\langle \alpha, H\rangle} - 1|}{\left\| \frac{1}{2\pi}\langle \alpha, H\rangle \right\|} \lesssim 1.$$

Then by assumption the Weyl denominator satisfies

$$|D_P(H)| \gtrsim \prod_{\alpha\in P} \left\| \tfrac{1}{2\pi}\langle \alpha, H\rangle \right\| \gtrsim N^{-|P|}. \tag{7-20}$$

Let

$$F = \sum_{\lambda\in\Lambda} e^{-it|\lambda|^2 + i\langle \lambda, H\rangle} \varphi\left(\frac{|\lambda|^2 - |\rho|^2}{N^2}\right) \cdot f,$$

where $f = \prod_{\alpha\in P}\langle \alpha, \lambda\rangle$. Note that f is a polynomial and thus also a pseudopolynomial of degree $|P|$ in λ. Applying Lemma 7.4 to F we get

$$|K_N(t, x)| = \left| \frac{e^{it|\rho|^2}}{\left(\prod_{\alpha\in P}\langle \alpha, \rho\rangle\right) D_P(H)} \right| \cdot |F| \lesssim \left| \frac{1}{D_P(H)} \right| \cdot |F| \lesssim N^{|P|} \cdot \frac{N^{r+|P|}}{\left(\sqrt{q}\left(1 + N\left\| \frac{t}{2\pi D} - \frac{a}{q}\right\|^{1/2}\right)\right)^r}.$$

Recalling $|P| = \frac{d-r}{2}$, we establish (6-1) for Scenario 2. □

Example 7.9. We specialize the Schrödinger kernel (7-16) and (7-17) to the case of $G = \mathrm{SU}(2)$. Scenario 1 in the above corollary corresponds to when $\theta \in 2\pi\mathbb{Z}$ and

$$K_N(t, \theta) = \frac{e^{it}}{2} \sum_{m\in\mathbb{Z}} e^{-itm^2} \varphi\left(\frac{m^2 - 1}{N^2}\right) m^2, \quad |K_N(t, \theta)| \lesssim \left| \sum_{m\in\mathbb{Z}} e^{-itm^2} \varphi\left(\frac{m^2 - 1}{N^2}\right) m^2 \right|. \tag{7-21}$$

Scenario 2 corresponds to when $|e^{i\theta}-e^{-i\theta}|\gtrsim\frac{1}{N}$, equivalently, when θ is away from the cell walls $\{0,\pi\}$ by a distance $\gtrsim\frac{1}{N}$. In this case,

$$|K_N(t,\theta)|\lesssim\left|\frac{1}{e^{i\theta}-e^{-i\theta}}\right|\cdot\left|\sum_{m\in\mathbb{Z}}e^{-itm^2+im\theta}\varphi\left(\frac{m^2-1}{N^2}\right)m\right|. \tag{7-22}$$

Then we get the desired estimates for (7-21) and (7-22) using Lemma 7.4.

7C. *Pseudopolynomial behavior of characters.* We have established the key estimates (6-1) for when the variable $\exp H$ in the maximal torus is either the identity or away from all the cell walls by a distance of $\gtrsim\frac{1}{N}$. To establish (6-1) fully, we need to look at the scenarios when the variable $\exp H$ is close to the some of the cell walls within a distance of $\lesssim\frac{1}{N}$. In this section, we first deal with the scenario when the variable $\exp H$ is close to all the cell walls within a distance of $\lesssim\frac{1}{N}$. To achieve this end, we first prove the following crucial lemma on the pseudopolynomial behavior of characters.

Lemma 7.10. *Let $\mu\in i\mathfrak{b}^*$. For $\lambda\in i\mathfrak{b}^*$, define*

$$\chi^\mu(\lambda,H)=\frac{\sum_{s\in W}(\det s)e^{i\langle s(\lambda+\mu),H\rangle}}{\sum_{s\in W}(\det s)e^{i\langle s(\rho),H\rangle}}.$$

Let $L\cong\mathbb{Z}^r$ be the weight lattice or the root lattice (or any sublattice of full rank of the weight lattice), and viewing $\chi^\mu(\lambda,H)$ as a function in $\lambda\in L$, we have

$$|D_{i_1}\cdots D_{i_k}\chi^\mu(\lambda,H)|\lesssim N^{\frac{d-r}{2}-k} \tag{7-23}$$

holds uniformly in $|\lambda|\lesssim N$, $|H|\lesssim\frac{1}{N}$, and $N\geq 1$, for all $k\in\mathbb{Z}_{\geq 0}$. In other words, $\chi^\mu(\lambda,H)$ is a pseudopolynomial of degree $\frac{d-r}{2}$ in λ uniformly in $|H|\lesssim\frac{1}{N}$.

Using this lemma, applying Lemma 7.4 to the Schrödinger kernel K_N in the form of (7-11), we immediately get the following corollary.

Corollary 7.11. *Inequality* (6-1) *holds uniformly when $x\in G$ is conjugate to* $\exp H$ *such that $|H|\lesssim\frac{1}{N}$. In other words, when x is within $\lesssim\frac{1}{N}$ a distance from the identity 1_G.*

We now prove Lemma 7.10 for $L\cong\mathbb{Z}w_1+\cdots+\mathbb{Z}w_r$ being the weight lattice (the case for the root lattice or any other sublattice can be proved similarly). First note that as $|H|\lesssim\frac{1}{N}$ for N large enough, by (7-19), we have

$$\left|\frac{\prod_{\alpha\in P}\langle\alpha,H\rangle}{D_P}\right|\approx 1.$$

Thus it suffices to show (7-23) replacing $\chi^\mu(\lambda,H)$ by

$$\chi_1^\mu(\lambda,H)=\frac{\sum_{s\in W}(\det s)e^{i\langle s(\lambda+\mu),H\rangle}}{\prod_{\alpha\in P}\langle\alpha,H\rangle}. \tag{7-24}$$

7C.1. *Approach 1: via BGG-Demazure operators.* The idea is to expand the numerator of $\chi_1^{\mu}(\lambda, H)$ into a power series of polynomials in $H \in i\mathfrak{b}^*$ which are *anti-invariant* with respect to the Weyl group W, and then to estimate the quotients of these polynomial over the denominator $\prod_{\alpha\in P}\langle\alpha, H\rangle$. We will see that these quotients are in fact polynomials in $H \in i\mathfrak{b}^*$, and can be more or less explicitly computed by the *BGG-Demazure operators*. We now review the basic definitions and facts of the BGG-Demazure operators and the related invariant theory. A good reference is Chapter IV in [Hiller 1982].

From now on, we fix an inner product space $(\mathfrak{a}, \langle\cdot,\cdot\rangle)$ and let Φ be an integral root system in the dual space $(\mathfrak{a}^*, \langle\cdot,\cdot\rangle)$. Let $P(\mathfrak{a})$ be the space of polynomial functions on $\mathfrak{a}$. The orthogonal group $O(\mathfrak{a})$ with respect to the inner product on $\mathfrak{a}$, in particular the Weyl group, acts on $P(\mathfrak{a})$ by

$$(sf)(H) := f(s^{-1}H), \quad s \in O(\mathfrak{a}), \quad f \in P(\mathfrak{a}), \quad H \in \mathfrak{a}.$$

Definition 7.12. For $\alpha \in \mathfrak{a}^*$, let $s_\alpha : \mathfrak{a} \to \mathfrak{a}$ denote the reflection about the hyperplane

$$\{H \in \mathfrak{a} \mid \alpha(H) = 0\},$$

that is,

$$s_\alpha(H) := H - 2\frac{\alpha(H)}{\langle\alpha,\alpha\rangle}H_\alpha,$$

where $H \in \mathfrak{a}$. Here H_α corresponds to α through the identification $\mathfrak{a} \xrightarrow{\sim} \mathfrak{a}^*$. Define the *BGG-Demazure operator* $\Delta_\alpha : P(\mathfrak{a}) \to P(\mathfrak{a})$ associated to $\alpha \in \mathfrak{a}^*$ by

$$\Delta_\alpha(f) = \frac{f - s_\alpha(f)}{\alpha}.$$

As an example, we compute $\Delta_\alpha(\lambda^m)$ for $\lambda \in \mathfrak{a}^*$:

$$\begin{aligned}\Delta_\alpha(\lambda^m) &= \frac{\lambda^m - \lambda\big(\cdot - 2\frac{\alpha}{\langle\alpha,\alpha\rangle}H_\alpha\big)^m}{\alpha} = \frac{\lambda^m - \big(\lambda - 2\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha\big)^m}{\alpha}\\ &= \sum_{i=1}^m (-1)^{i-1}\binom{m}{i}\frac{2^i}{\langle\alpha,\alpha\rangle^i}\langle\lambda,\alpha\rangle^i\alpha^{i-1}\lambda^{m-i}.\end{aligned} \tag{7-25}$$

This computation in particular implies that for any $f \in P(\mathfrak{a})$, the operator $\Delta_\alpha(f)$ lowers the degree of f by at least 1.

Let $P(\mathfrak{a})^W$ denote the subspace of $P(\mathfrak{a})$ that is invariant under the action of the Weyl group W, that is,

$$P(\mathfrak{a})^W := \{f \in P(\mathfrak{a}) \mid sf = f \text{ for all } s \in W\}.$$

We call $P(\mathfrak{a})^W$ the space of *invariant polynomials*. We also define

$$P(\mathfrak{a})^W_{\det} := \{f \in P(\mathfrak{a}) \mid sf = (\det s)f \text{ for all } s \in W\}.$$

We call $P(\mathfrak{a})^W_{\det}$ the space of *anti-invariant polynomials*. We have the following proposition which states that $P(\mathfrak{a})^W_{\det}$ is a free $P(\mathfrak{a})^W$-module of rank 1.

Proposition 7.13 [Hiller 1982, Chapter II, Proposition 4.4]. *Define* $d_{\det} \in P(\mathfrak{a})$ *by*

$$d_{\det} = \prod_{\alpha \in P} \alpha.$$

Then $d_{\det} \in P(\mathfrak{a})^W_{\det}$ *and*

$$P(\mathfrak{a})^W_{\det} = d_{\det} \cdot P(\mathfrak{a})^W.$$

By the above proposition, given any anti-invariant polynomial f, we have $f = d \cdot g$, where g is invariant. We call g the *invariant part* of f. The BGG-Demazure operators provide a procedure that computes the invariant part of any anti-invariant polynomial. We describe this procedure as follows. The Weyl group W is generated by the reflections $s_{\alpha_1}, \ldots, s_{\alpha_r}$, where $S = \{\alpha_1, \ldots, \alpha_r\}$ is the set of simple roots. Define the *length* of $s \in W$ to be the smallest number k such that s can be written as $s = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_k}}$. The longest element s in W is of length $|P| = \frac{d-r}{2}$, and such s is unique; see Section 1.8 in [Humphreys 1990]. Write $s = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_L}}$. Set

$$\delta = \Delta_{\alpha_{i_1}} \cdots \Delta_{\alpha_{i_L}}$$

and note that it is well-defined in the sense it does not depend on the particular choice of the decomposition $s = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_L}}$; see Chapter IV, Proposition 1.7 in [Hiller 1982].

Proposition 7.14 [Hiller 1982, Chapter IV, Proposition 1.6]. *We have*

$$\delta f = \frac{|W|}{d_{\det}} \cdot f$$

for all $f \in P(\mathfrak{a})^W_{\det}$.

That is, the operator δ produces the invariant part of any anti-invariant polynomial (modulo a multiplicative constant). As an example, we compute $\delta = \Delta_{\alpha_{i_1}} \cdots \Delta_{\alpha_{i_L}}$ on λ^m. Proceed inductively using (7-25), we arrive at the following proposition.

Proposition 7.15. *Let* $m \geq L$. *Then*

$$\delta(\lambda^m) = \sum_{\theta, a(\alpha,\beta), b(\gamma), c(\zeta), \eta \in \mathbb{Z}} (-1)^\theta \prod_{\alpha \leq \beta} \langle \alpha_{i_\alpha}, \alpha_{i_\beta} \rangle^{a(\alpha,\beta)} \prod_{\gamma} \langle \lambda, \alpha_{i_\gamma} \rangle^{b(\gamma)} \prod_{\zeta} \alpha_{i_\zeta}^{c(\zeta)} \lambda^\eta$$

such that the following statements are true:

1. *In each term of the sum,* $\sum_\gamma b(\gamma) + \eta = m$.
2. *In each term of the sum,* $\sum_\zeta c(\zeta) + \eta = m - L$.
3. *In each term of the sum,* $\sum_\gamma b(\gamma) - \sum_\zeta c(\zeta) = L$.
4. *In each term of the sum,* $|a(\alpha, \beta)| \leq mL$ *and* $b(\gamma), c(\zeta), \eta = 0, 1, \ldots, m$.
5. *There are in total less than* 3^{mL} *terms in the sum.*

Note that since each BGG-Demazure operator $\Delta_{\alpha_{i_j}}$ in $\delta = \Delta_{\alpha_{i_1}} \cdots \Delta_{\alpha_{i_L}}$ lowers the degree of polynomials by at least 1, δ lowers the degree by at least L. Thus

$$\delta(\lambda^m) = 0 \quad \text{for } m < L. \tag{7-26}$$

Example 7.16. We specialize the discussion to the case $M = \mathrm{SU}(2)$. Recall that $\mathfrak{a}^* = \mathbb{R}w$, where w is the fundamental weight, and $\Phi = \{\pm\alpha\}$ with $\alpha = 2w$. $P(\mathfrak{a})$ consists of polynomials in the variable $\lambda \in \underset{1 \mapsto w}{\mathbb{R} \cong \mathbb{R}w}$. For $\lambda \in \underset{1 \mapsto w}{\mathbb{R} \cong \mathbb{R}w}$, and $f \in P(\mathfrak{a})$, we have

$$\begin{aligned}(\delta f)(\lambda) &= \frac{f(\lambda) - f(-\lambda)}{2\lambda},\\ \delta(\lambda^m) &= \begin{cases}\lambda^{m-1}, & m \text{ odd},\\ 0, & m \text{ even},\end{cases}\\ d_{\det}(\lambda) &= 2\lambda.\end{aligned} \tag{7-27}$$

We can now finish the proof of (7-23).

Proof of Lemma 7.10. Recall that it suffices to prove (7-23) replacing $\chi^{\mu}(\lambda, H)$ by $\chi^{\mu}_1(\lambda, H)$ in (7-24). Using power series expansions, write

$$\begin{aligned}\sum_{s\in W} (\det s) e^{i\langle\lambda+\mu, H\rangle} &= \sum_{s\in W} (\det s) \sum_{m=0}^{\infty} \frac{1}{m!} (i\langle s(\lambda+\mu), H\rangle)^m\\ &= \sum_{m=0}^{\infty} \frac{i^m}{m!} \sum_{s\in W} (\det s) \langle s(\lambda+\mu), H\rangle^m.\end{aligned} \tag{7-28}$$

Note that

$$f_m(H) = f_m(\lambda) = f_m(\lambda, H) := \sum_{s\in W} (\det s)\langle s(\lambda+\mu), H\rangle^m \tag{7-29}$$

is an anti-invariant polynomial in H with respect to the Weyl group W; thus by Proposition 7.14,

$$f_m(H) = \frac{d_{\det}(H)}{|W|} \cdot \delta f_m(H) = \frac{\prod_{\alpha\in P} \langle\alpha, H\rangle}{|W|} \cdot \delta f_m(H).$$

This implies that we can rewrite (7-24) as

$$\chi^{\mu_1}(\lambda, H) = \frac{1}{|W|} \sum_{m=0}^{\infty} \frac{i^m}{m!} \delta f_m(H).$$

Thus to prove (7-23), it suffices to prove that

$$\sum_{m=0}^{\infty} \frac{1}{m!} |D_{i_1} \cdots D_{i_k} (\delta f_m(\lambda))| \lesssim N^{L-k}$$

for all $k \in \mathbb{Z}_{\geq 0}$, uniformly in $|n_i| \lesssim N$, where $\lambda = n_1 w_1 + \cdots + n_r w_r$. Then by (7-29), it suffices to prove that

$$\sum_{m=0}^{\infty} \frac{1}{m!} |D_{i_1} \cdots D_{i_k} (\delta[(s(\lambda+\mu))^m])| \lesssim N^{L-k} \quad \text{for all } s \in W.$$

Without loss of generality, it suffices to show

$$\sum_{m=0}^{\infty} \frac{1}{m!} |D_{i_1} \cdots D_{i_k} (\delta[(\lambda+\mu)^m])| \lesssim N^{L-k}. \tag{7-30}$$

Noting (7-26), it suffices to consider cases when $m \geq L$. We apply Proposition 7.15 to write

$$\delta((\lambda+\mu)^m)(H) = \sum_{\theta,a(\alpha,\beta),b(\gamma),c(\zeta),\eta} (-1)^\theta \prod_{\alpha\leq\beta} \langle \alpha_{i_\alpha}, \alpha_{i_\beta}\rangle^{a(\alpha,\beta)} \prod_\gamma \langle \lambda+\mu, \alpha_{i_\gamma}\rangle^{b(\gamma)} \prod_\zeta \langle \alpha_{i_\zeta}, H\rangle^{c(\zeta)} \langle \lambda+\mu, H\rangle^\eta. \quad (7\text{-}31)$$

First note that for $\lambda = n_1 w_1 + \cdots + n_r w_r$, $|n_i| \lesssim N$, $i = 1, \dots, r$, we have

$$1 \lesssim |\langle \alpha_i, \alpha_j\rangle| \lesssim 1, \quad |\langle \lambda+\mu, \alpha_i\rangle| \lesssim N, \quad (7\text{-}32)$$

and by the assumption $|H| \lesssim \frac{1}{N}$,

$$|\langle \alpha_i, H\rangle| \lesssim \tfrac{1}{N}, \quad |\langle \lambda+\mu, H\rangle| = \left| \left(\sum_{i=1}^r n_i \langle w_i, H\rangle \right) + \langle \mu, H\rangle \right| \lesssim 1. \quad (7\text{-}33)$$

These imply

$$|\delta((\lambda+\mu)^m)(H)| \leq \sum_{\theta,a(\alpha,\beta),b(\gamma),c(\zeta),\eta} C^{\sum_{\alpha,\beta}|a(\alpha,\beta)| + \sum_\gamma b(\gamma) + \sum_\zeta c(\zeta) + \eta} N^{\sum_\gamma c(\gamma) - \sum_\zeta c(\zeta)} \quad (7\text{-}34)$$

for some constant C independent of m. Now we derive a similar estimate for $D_i(\delta[(\lambda+\mu)^m])(H)$. By (7-31),

$$D_i(\delta[(\lambda+\mu)^m])(H) = \sum_{\theta,a(\alpha,\beta),b(\gamma),c(\zeta),\eta} (-1)^\theta \prod_{\alpha\leq\beta} \langle \alpha_{i_\alpha}, \alpha_{i_\beta}\rangle^{a(\alpha,\beta)} \prod_\zeta \langle \alpha_{i_\zeta}, H\rangle^{c(\zeta)} \cdot D_i\left(\prod_\gamma \langle \lambda+\mu, \alpha_{i_\gamma}\rangle^{b(\gamma)} \langle \lambda+\mu, H\rangle^\eta \right). \quad (7\text{-}35)$$

For $\lambda = n_1 w_1 + \cdots + n_r w_r$, we compute

$$D_i(\langle \lambda+\mu, \alpha_{i_\gamma}\rangle) = \langle \alpha_i, \alpha_{i_\gamma}\rangle,$$
$$D_i(\langle \lambda+\mu, H\rangle) = \langle \alpha_i, H\rangle.$$

The above two formulas combined with (7-32), (7-33), and the Leibniz rule (7-2) for D_i imply

$$\left| D_i\left(\prod_\gamma \langle \lambda+\mu, \alpha_{i_\gamma}\rangle^{b(\gamma)} \langle \lambda+\mu, H\rangle^\eta \right) \right| \leq C^{\sum_\gamma b(\gamma)+\eta} N^{\sum_\gamma b(\gamma)-1}.$$

This combined with (7-32), (7-33) and (7-35) implies

$$|D_i(\delta[(\lambda+\mu)^m])(H)| \lesssim \sum_{\theta,a(\alpha,\beta),b(\gamma),c(\zeta),\eta} C^{\sum_{\alpha,\beta}|a(\alpha,\beta)| + \sum_\gamma b(\gamma) + \sum_\zeta c(\zeta) + \eta} N^{\sum_\gamma b(\gamma) - \sum_\zeta c(\zeta) - 1}.$$

Inductively, we have

$$|D_{i_1} \cdots D_{i_k}(\delta[(\lambda+\mu)^m])(H)| \lesssim \sum_{\theta,a(\alpha,\beta),b(\gamma),c(\zeta),\eta} C^{\sum_{\alpha,\beta}|a(\alpha,\beta)| + \sum_\gamma b(\gamma) + \sum_\zeta c(\zeta) + \eta} N^{\sum_\gamma b(\gamma) - \sum_\zeta c(\zeta) - k}$$

for some constant C independent of m. This by Proposition 7.15 then implies

$$|D_{i_1} \cdots D_{i_k}(\delta[(\lambda+\mu)^m])(H)| \leq 3^{mL} C^{CmL} N^{L-k} \leq C^m N^{L-k}$$

for some positive constant C independent of m. This estimate implies (7-30), noting that

$$\sum_{m=0}^{\infty} \frac{C^m}{m!} \lesssim 1. \tag{7-36}$$

This finishes the proof. □

7C.2. *Approach 2: via Harish-Chandra's integral formula.* This very short approach expresses $\chi_1^{\mu}(\lambda, H)$ as an integral over the group G. We apply the Harish-Chandra's integral formula [1957], which reads

$$\sum_{s\in W} (\det s) e^{\langle s\lambda, \mu\rangle} = \frac{\prod_{\alpha\in P}\langle \alpha, \lambda\rangle \cdot \prod_{\alpha\in P}\langle \alpha, \mu\rangle}{\prod_{\alpha\in P}\langle \alpha, \rho\rangle} \int_G e^{\langle \mathrm{Ad}_g(\lambda), \mu\rangle}\, dg,$$

where $\lambda, \mu \in \mathfrak{b}_{\mathbb{C}}$, and dg is the normalized Haar measure on G. Then we can rewrite $\chi_1^{\mu}(\lambda, H)$ as

$$\chi_1^{\mu}(\lambda, H) = \frac{i^{|P|}\prod_{\alpha\in P}\langle \alpha, \lambda+\rho\rangle}{\prod_{\alpha\in P}\langle \alpha, \rho\rangle} \int_G e^{i\langle \lambda+\rho, \mathrm{Ad}_g(H)\rangle}\, dg.$$

Note that

$$\frac{i^{|P|}\prod_{\alpha\in P}\langle \alpha, \lambda+\rho\rangle}{\prod_{\alpha\in P}\langle \alpha, \rho\rangle}$$

is a polynomial in $\lambda \in \Lambda$ of degree $|P| = \frac{d-r}{2}$. Also, as $|H| \lesssim \frac{1}{N}$, we have $|\mathrm{Ad}_g(H)| \lesssim \frac{1}{N}$ uniformly in $g \in G$, which implies that the integral

$$f(\lambda) = \int_G e^{i\langle \lambda+\rho, \mathrm{Ad}_g(H)\rangle}\, dg$$

as a function in λ is a pseudopolynomial of degree 0, uniformly in $|H| \lesssim \frac{1}{N}$. Then by the Leibniz rule, $\chi'(\lambda, H)$ as a function of λ is a pseudopolynomial of degree $\frac{d-r}{2}$, uniformly in $|H| \lesssim \frac{1}{N}$. This finishes the proof of Lemma 7.10.

Remark 7.17. Note that Lemma 7.10 can be stated purely in terms of an integral root system without mentioning the ambient compact Lie group, and it still holds true this way. It can be seen either by the approach via BGG-Demazure operators, which is purely a root-system-theoretic argument, or by the fact that, for any integral root system Φ, there associates to it a unique compact simply connected semisimple Lie group equipped with this root system; thus the approach via Harish-Chandra's integral formula still works, even though the argument explicitly involves the group.

7D. ***From the weight lattice to the root lattice.*** We say $\exp H$ is a *corner* in the maximal torus provided

$$\left\| \tfrac{1}{2\pi}\langle \alpha, H\rangle \right\| = 0 \quad \text{for all } \alpha \in P.$$

In this section, we extend Corollary 7.11 to the scenarios when $\exp H$ is within a distance of $\lesssim \frac{1}{N}$ from some corner. That is, when

$$\left\| \tfrac{1}{2\pi}\langle \alpha, H\rangle \right\| \lesssim \tfrac{1}{N} \quad \text{for all } \alpha \in P. \tag{7-37}$$

To this end, we rewrite the Schrödinger kernel $K_N(t,x)$ as a finite sum of exponential sums over the root lattice:

$$\begin{aligned}K_N(t,x) &= C\sum_{\mu\in\Lambda/\Gamma}\sum_{\lambda\in\mu+\Gamma} e^{-it(|\lambda|^2-|\rho|^2)}\varphi\Big(\frac{|\lambda|^2-|\rho|^2}{N^2}\Big)\frac{\prod_{\alpha\in P}\langle\alpha,\lambda\rangle}{\prod_{\alpha\in P}\langle\alpha,\rho\rangle}\frac{\sum_{s\in W}(\det s)e^{i\langle s(\lambda),H\rangle}}{\sum_{s\in W}(\det s)e^{i\langle s(\rho),H\rangle}}\\ &= C\sum_{\mu\in\Lambda/\Gamma}\sum_{\lambda\in\Gamma} e^{-it(|\lambda+\mu|^2-|\rho|^2)}\varphi\Big(\frac{|\lambda+\mu|^2-|\rho|^2}{N^2}\Big)\frac{\prod_{\alpha\in P}\langle\alpha,\lambda+\mu\rangle}{\prod_{\alpha\in P}\langle\alpha,\rho\rangle}\frac{\sum_{s\in W}(\det s)e^{i\langle s(\lambda+\mu),H\rangle}}{\sum_{s\in W}(\det s)e^{i\langle s(\rho),H\rangle}},\end{aligned} \tag{7-38}$$

where $C=e^{it|\rho|^2}/|W|$.

Proposition 7.18. *Let μ be an element in the weight lattice Λ and let*

$$K_N^\mu(t,x) = \sum_{\lambda\in\Gamma} e^{-it(|\lambda+\mu|^2-|\rho|^2)}\varphi\Big(\frac{|\lambda+\mu|^2-|\rho|^2}{N^2}\Big)\frac{\prod_{\alpha\in P}\langle\alpha,\lambda+\mu\rangle}{\prod_{\alpha\in P}\langle\alpha,\rho\rangle}\frac{\sum_{s\in W}(\det s)e^{i\langle s(\lambda+\mu),H\rangle}}{\sum_{s\in W}(\det s)e^{i\langle s(\rho),H\rangle}}, \tag{7-39}$$

where x is conjugate to $\exp H$. *Then*

$$|K_N^\mu(t,x)| \lesssim \frac{N^d}{\big(\sqrt{q}\big(1+N\big\|\frac{t}{2\pi D}-\frac{a}{q}\big\|^{1/2}\big)\big)^r} \tag{7-40}$$

for $\frac{t}{2\pi D}\in\mathcal{M}_{a,q}$, uniformly for $\big\|\frac{1}{2\pi}\langle\alpha,H\rangle\big\|\lesssim\frac{1}{N}$ for all $\alpha\in P$.

Using (7-38) and the finiteness of Λ/Γ, we have the following corollary.

Corollary 7.19. *Inequality* (6-1) *holds for the case when $\big\|\frac{1}{2\pi}\langle\alpha,H\rangle\big\|\lesssim\frac{1}{N}$ for all $\alpha\in P$.*

To prove Proposition 7.18, we first prove a variant of Lemma 7.10.

Lemma 7.20. *Let*

$$\chi^\mu(\lambda,H)=\frac{\sum_{s\in W}(\det s)e^{i\langle s(\mu+\lambda),H\rangle}}{e^{-i\langle\rho,H\rangle}\prod_{\alpha\in P}(e^{i\langle\alpha,H\rangle}-1)} \tag{7-41}$$

be defined as in Lemma 7.10. Assume in addition that $\mu\in\Lambda$. Then $\chi^\mu(\lambda,H)$ as a function in $\lambda\in\Gamma$ is a pseudopolynomial of degree $\frac{d-r}{2}$, uniformly in H such that $\big\|\frac{1}{2\pi}\alpha(H)\big\|\lesssim\frac{1}{N}$ for all $\alpha\in P$.

Proof. For all $H\in i\mathfrak{b}^*$ such that $\big\|\frac{1}{2\pi}\langle\alpha,H\rangle\big\|\lesssim\frac{1}{N}$ for all $\alpha\in P$, by considering the dual basis of the simple roots $\{\alpha_1,\dots,\alpha_r\}$, we can write

$$H=H_1+H_2 \tag{7-42}$$

such that

$$\big|\tfrac{1}{2\pi}\langle\alpha_i,H_1\rangle\big|=\big\|\tfrac{1}{2\pi}\langle\alpha_i,H\rangle\big\|\lesssim\tfrac{1}{N},\quad i=1,\dots,r, \tag{7-43}$$

and

$$\langle\alpha_i,H_2\rangle\in 2\pi\mathbb{Z},\quad i=1,\dots,r. \tag{7-44}$$

This implies that $\exp H_2$ is a corner and

$$|H_1|\lesssim\tfrac{1}{N}. \tag{7-45}$$

Then for $\lambda \in \Gamma = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_r$,

$$\chi^\mu(\lambda, H) = \chi^\mu(\lambda, H_1 + H_2) = \frac{\sum_{s\in W}(\det s) e^{i\langle s(\lambda+\mu), H_1\rangle} e^{i\langle s(\mu), H_2\rangle}}{e^{-i\langle \rho, H_1+H_2\rangle}\prod_{\alpha\in P}(e^{i\langle\alpha, H_1\rangle}-1)}. \tag{7-46}$$

Note that, see Corollary 4.13.3 in [Varadarajan 1974], $s(\mu)-\mu \in \Gamma$ for all $\mu \in \Lambda$ and $s \in W$, which combined with (7-44) implies

$$e^{i\langle s(\mu), H_2\rangle} = e^{i\langle \mu, H_2\rangle} \quad \text{for all } \mu \in \Lambda,\ s \in W.$$

Then (7-46) becomes

$$\chi^\mu(\lambda, H) = \frac{e^{i\langle\mu, H_2\rangle}}{e^{-i\langle\rho, H_2\rangle}} \cdot \frac{\sum_{s\in W}(\det s) e^{i\langle s(\lambda+\mu), H_1\rangle}}{e^{-i\langle\rho, H_1\rangle}\prod_{\alpha\in P}(e^{i\langle\alpha, H_1\rangle}-1)} = e^{i\langle\mu+\rho, H_2\rangle}\cdot\chi^\mu(\lambda, H_1), \tag{7-47}$$

which is a pseudopolynomial in $\lambda \in \Gamma$ of degree $\frac{d-r}{2}$ uniformly in $|H_1| \lesssim \frac{1}{N}$ by Lemma 7.10. □

Proof of Proposition 7.18. Since $\prod_{\alpha\in P}\langle\alpha, \lambda+\mu\rangle$ is a polynomial, and thus also a pseudopolynomial in λ of degree $|P| = \frac{d-r}{2}$, and $\chi^\mu(\lambda, H)$ is a pseudopolynomial of degree $\frac{d-r}{2}$ uniformly in $\left\|\frac{1}{2\pi}\langle\alpha, H\rangle\right\| \lesssim \frac{1}{N}$ for all $\alpha \in P$ by the previous lemma,

$$f(\lambda) = \frac{\prod_{\alpha\in P}\langle\alpha, \lambda+\mu\rangle}{\prod_{\alpha\in P}\langle\alpha, \rho\rangle}\cdot\chi^\mu(\lambda, H)$$

is then a pseudopolynomial of degree $d-r$ uniformly in $\left\|\frac{1}{2\pi}\langle\alpha, H\rangle\right\| \lesssim \frac{1}{N}$ for all $\alpha \in P$. Then the desired result comes from a direct application of Lemma 7.4. □

Example 7.21. We specialize the discussion in this section to the case $G = \mathrm{SU}(2)$. Recall that $\Lambda = \mathbb{Z}w$, $\Gamma = \mathbb{Z}\alpha$ with $\alpha = 2w$; thus $\Lambda/\Gamma \cong \{0, 1\}\cdot w$. (7-38) specializes to

$$K_N(t, \theta) = \tfrac{1}{2}e^{it}(K_N^0(t, \theta) + K_N^1(t, \theta)),$$

where

$$K_N^0 = \sum_{\substack{m=2k\\ k\in\mathbb{Z}}} e^{-itm^2}\varphi\left(\frac{m^2-1}{N^2}\right)m\cdot\frac{e^{im\theta}-e^{-im\theta}}{e^{i\theta}-e^{-i\theta}},$$

$$K_N^1 = \sum_{\substack{m=2k+1\\ k\in\mathbb{Z}}} e^{-itm^2}\varphi\left(\frac{m^2-1}{N^2}\right)m\cdot\frac{e^{im\theta}-e^{-im\theta}}{e^{i\theta}-e^{-i\theta}}$$

for $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Condition (7-37) specializes to $\left\|\frac{\theta}{\pi}\right\| \lesssim \frac{1}{N}$. Write $\theta = \theta_1 + \theta_2$, where $|\theta_1| \lesssim \frac{1}{N}$, and $\theta_2 = 0, \pi$. Then for $m = 2k,\ k \in \mathbb{Z}$,

$$\begin{aligned}\chi_m(\theta) &= \frac{1}{e^{-i\theta}(e^{i2\theta_1}-1)}\cdot(e^{im\theta_1}-e^{-im\theta_1})\\ &= \frac{1}{e^{-i\theta}(e^{i2\theta_1}-1)}\cdot\sum_{n=0}^{\infty}\frac{i^n}{n!}((m\theta_1)^n-(-m\theta_1)^n)\\ &= \frac{\theta_1}{e^{-i\theta}(e^{i2\theta_1}-1)}\cdot\sum_{n \text{ odd}}\frac{i^n}{n!}(2\theta_1^{n-1}m^n),\end{aligned} \tag{7-48}$$

and similarly for $m = 2k+1,\ k \in \mathbb{Z}$,

$$\chi_m(\theta) = \frac{e^{i\theta_2}\theta_1}{e^{-i\theta}(e^{i2\theta_1}-1)} \cdot \sum_{n \text{ odd}} \frac{i^n}{n!}(2\theta_1^{n-1}m^n).$$

Note that we are implicitly applying Proposition 7.14 so that

$$f_n(\theta_1) := (m\theta_1)^n - (-m\theta_1)^n = \theta_1 \cdot \delta f_n = \begin{cases} \theta_1 \cdot 2\theta_1^{n-1}m^n, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

If $|k| \lesssim N$, then it is clear that

$$|D^L\chi_{2k}| \lesssim N^{1-L}, \quad |D^L\chi_{2k+1}| \lesssim N^{1-L}, \quad L \in \mathbb{Z}_{\geq 0},$$

where D is the difference operator with respect to the variable k. These two inequalities will give the desired estimates for K_N^0 and K_N^1 respectively using the Weyl sum estimate Lemma 7.4 in one dimension.

7E. ***Root subsystems.*** To finish the proof of part (ii) of Theorem 6.2, considering Corollaries 7.8 and 7.19, it suffices to prove (6-1) in the scenarios when $\exp H$ is away from all the corners by a distance of $\gtrsim \frac{1}{N}$ but stays close to some cell walls within a distance of $\lesssim \frac{1}{N}$. We will identify these other walls as belonging to a *root subsystem* of the original root system Φ, and then we will decompose the character, the weight lattice and thus the Schrödinger kernel according to this root subsystem.

7E.1. *Identifying root subsystems and rewriting the character.* Given any $H \in i\mathfrak{b}^*$, let Q_H be the subset of the set Φ of roots defined by

$$Q_H := \left\{\alpha \in \Phi \;\middle|\; \left\|\tfrac{1}{2\pi}\langle\alpha, H\rangle\right\| \leq \tfrac{1}{N}\right\}.$$

Thus

$$\Phi \setminus Q_H = \left\{\alpha \in \Phi \;\middle|\; \left\|\tfrac{1}{2\pi}\langle\alpha, H\rangle\right\| > \tfrac{1}{N}\right\}.$$

Define

$$\Phi_H := \{\alpha \in \Phi \mid \alpha \text{ lies in the } \mathbb{Z}\text{-linear span of } Q_H\}. \tag{7-49}$$

Then $\Phi_H \supset Q_H$, and

$$\left\|\tfrac{1}{2\pi}\langle\alpha, H\rangle\right\| \lesssim \tfrac{1}{N} \quad \text{for all } \alpha \in \Phi_H, \tag{7-50}$$

with the implicit constant independent of H, and

$$\left\|\tfrac{1}{2\pi}\langle\alpha, H\rangle\right\| > \tfrac{1}{N} \quad \text{for all } \alpha \in \Phi \setminus \Phi_H. \tag{7-51}$$

Note that Φ_H is $\mathbb{Z}$*-closed* in Φ; that is, no element in $\Phi \setminus \Phi_H$ lies in the $\mathbb{Z}$-linear span of Φ_H.

Proposition 7.22. *Φ_H is an integral root system.*

Proof. We check the requirements for an integral root system listed on page 1182. Parts (ii) and (iv) are automatic from the fact that Φ_H is a subset of Φ. Part (i) comes from the fact that Φ_H is a $\mathbb{Z}$-linear space. Part (iii) follows from the fact that $s_\alpha\beta$ is a $\mathbb{Z}$-linear combination of α and β for all $\alpha, \beta \in \Phi_H$, and the fact that Φ_H is a $\mathbb{Z}$-linear space. □

Then we say that Φ_H is a *root subsystem* of Φ.

Let W_H be the Weyl group of Φ_H. W_H is generated by reflections s_α for $\alpha \in \Phi_H$ and W_H is a subgroup of the Weyl group W of Φ. Let P be a positive system of roots of Φ and $P_H = P \cap \Phi_H$. Then P_H is a positive system of roots of Φ_H. We rewrite the Weyl character as

$$\begin{aligned}\chi_\lambda &= \frac{\sum_{s\in W}(\det s)e^{i\langle s(\lambda),H\rangle}}{e^{-i\langle\rho,H\rangle}\prod_{\alpha\in P}(e^{i\langle\alpha,H\rangle}-1)}\\ &= \frac{(1/|W_H|)\sum_{s_H\in W_H}\sum_{s\in W}(\det(s_Hs))e^{i\langle(s_Hs)(\lambda),H\rangle}}{e^{-i\langle\rho,H\rangle}\big(\prod_{\alpha\in P\setminus P_H}(e^{i\langle\alpha,H\rangle}-1)\big)\big(\prod_{\alpha\in P_H}(e^{i\langle\alpha,H\rangle}-1)\big)}\\ &= \frac{1}{|W_H|e^{-i\langle\rho,H\rangle}\prod_{\alpha\in P\setminus P_H}(e^{i\langle\alpha,H\rangle}-1)}\sum_{s\in W}(\det s)\cdot\frac{\sum_{s_H\in W_H}(\det s_H)e^{i\langle s_H(s(\lambda)),H\rangle}}{\prod_{\alpha\in P_H}(e^{i\langle\alpha,H\rangle}-1)}\\ &= C(H)\sum_{s\in W}(\det s)\cdot\frac{\sum_{s_H\in W_H}(\det s_H)e^{i\langle s_H(s(\lambda)),H\rangle}}{\prod_{\alpha\in P_H}(e^{i\langle\alpha,H\rangle}-1)},\end{aligned}\tag{7-52}$$

where

$$C(H) := \frac{1}{|W_H|e^{-i\langle\rho,H\rangle}\prod_{\alpha\in P\setminus P_H}(e^{i\langle\alpha,H\rangle}-1)}.\tag{7-53}$$

Then by (7-51),

$$|C(H)| \lesssim N^{|P\setminus P_H|}.\tag{7-54}$$

Let V_H be the $\mathbb{R}$-linear span of Φ_H in $V = i\mathfrak{b}^*$ and let $H^\parallel$ be the orthogonal projection of H on V_H. Let $H^\perp = H - H^\parallel$. Then $H^\perp$ is orthogonal to V_H and we have

$$\begin{aligned}\chi_\lambda &= C(H)\sum_{s\in W}(\det s)\cdot\frac{\sum_{s_H\in W_H}(\det s_H)\,e^{i\langle s_H(s(\lambda)),H^\perp+H^\parallel\rangle}}{\prod_{\alpha\in P_H}(e^{i\langle\alpha,H^\perp+H^\parallel\rangle}-1)}\\ &= C(H)\sum_{s\in W}(\det s)\cdot\frac{\sum_{s_H\in W_H}(\det s_H)e^{i\langle s(\lambda),s_H(H^\perp)\rangle}e^{i\langle s_H(s(\lambda)),H^\parallel\rangle}}{\prod_{\alpha\in P_H}(e^{i\langle\alpha,H^\parallel\rangle}-1)}.\end{aligned}\tag{7-55}$$

Note that since $H^\perp$ is orthogonal to every root in Φ_H, $H^\perp$ is fixed by s_α for any $\alpha \in \Phi_H$, which in turn implies that $H^\perp$ is fixed by any $s_H \in W_H$; that is, $s_H(H^\perp) = H^\perp$. Then

$$\begin{aligned}\chi_\lambda &= C(H)\sum_{s\in W}(\det s)\cdot\frac{\sum_{s_H\in W_H}(\det s_H)e^{i\langle s(\lambda),H^\perp\rangle}e^{i\langle s_H(s(\lambda)),H^\parallel\rangle}}{\prod_{\alpha\in P_H}(e^{i\langle\alpha,H^\parallel\rangle}-1)}\\ &= C(H)\sum_{s\in W}(\det s)\cdot e^{i\langle s(\lambda),H^\perp\rangle}\cdot\frac{\sum_{s_H\in W_H}(\det s_H)e^{i\langle s_H(s(\lambda)),H^\parallel\rangle}}{\prod_{\alpha\in P_H}(e^{i\langle\alpha,H^\parallel\rangle}-1)}.\end{aligned}\tag{7-56}$$

Note that by the definition of $H^\parallel$, we have

$$\left\|\tfrac{1}{2\pi}\langle\alpha, H^\parallel\rangle\right\| \lesssim \tfrac{1}{N} \quad \text{for all } \alpha \in \Phi_H.\tag{7-57}$$

This means that $\exp H^{\parallel}$ is a corner in the maximal torus of the compact semisimple Lie group associated to the integral root system Φ_H.

Using the above formula, we rewrite the Schrödinger kernel (7-11) as

$$K_N(t,x) = \frac{C(H)e^{it|\rho|^2}}{\left(\prod_{\alpha\in P}\langle\alpha,\rho\rangle\right)|W|}\sum_{s\in W}(\det s)\cdot K_{N,s}(t,x), \tag{7-58}$$

where

$$K_{N,s}(t,x) = \sum_{\lambda\in\Lambda} e^{i\langle s(\lambda),H^{\perp}\rangle - it|\lambda|^2}\varphi\left(\frac{|\lambda|^2-|\rho|^2}{N^2}\right)\left(\prod_{\alpha\in P}\langle\alpha,\lambda\rangle\right)\frac{\sum_{s_H\in W_H}(\det s_H)e^{i\langle s_H(s(\lambda)),H^{\parallel}\rangle}}{\prod_{\alpha\in P_H}(e^{i\langle\alpha,H^{\parallel}\rangle}-1)}.$$

Noting that for any $s\in W$, $|s(\lambda)|=|\lambda|$, $\prod_{\alpha\in P}\langle\alpha,s(\lambda)\rangle = (\det s)\prod_{\alpha\in P}\langle\alpha,\lambda\rangle$ by Proposition 7.13, and $s(\Lambda)=\Lambda$, we have

$$K_{N,s}(t,x) = (\det s)K_{N,\mathbb{1}}(t,x),$$

where $\mathbb{1}$ is the identity element in W. Then (7-58) becomes

$$K_N(t,x) = \frac{C(H)e^{it|\rho|^2}}{\left(\prod_{\alpha\in P}\langle\alpha,\rho\rangle\right)}K_{N,\mathbb{1}}(t,x). \tag{7-59}$$

Proposition 7.23. *Recall that*

$$K_{N,\mathbb{1}}(t,x) = \sum_{\lambda\in\Lambda} e^{i\langle \lambda,H^{\perp}\rangle - it|\lambda|^2}\varphi\left(\frac{|\lambda|^2-|\rho|^2}{N^2}\right)\left(\prod_{\alpha\in P}\langle\alpha,\lambda\rangle\right)\frac{\sum_{s_H\in W_H}(\det s_H)e^{i\langle s_H(\lambda),H^{\parallel}\rangle}}{\prod_{\alpha\in P_H}(e^{i\langle\alpha,H^{\parallel}\rangle}-1)}. \tag{7-60}$$

Then

$$|K_{N,\mathbb{1}}(t,x)| \lesssim \frac{N^{d-|P\setminus P_H|}}{\left(\sqrt{q}\left(1+N\left|\frac{t}{2\pi D}-\frac{a}{q}\right|^{1/2}\right)\right)^r} \tag{7-61}$$

for $\frac{t}{2\pi D}\in\mathcal{M}_{a,q}$*, uniformly in* $x\in G$.

Noting (7-54) and (7-59), the above proposition implies part (ii) of Theorem 6.2.

Example 7.24. Figure 1 is an illustration of the decomposition of the maximal torus of SU(3) according to the values of $\left\|\frac{1}{2\pi}\langle\alpha,H\rangle\right\|$, $\alpha\in\Phi$. Here $P=\{\alpha_1,\alpha_2,\alpha_3=\alpha_1+\alpha_2\}$. The three proper root subsystems of Φ are $\{\pm\alpha_i\}$, $i=1,2,3$. The association of Φ_H to H is as follows:

$$H \in \text{regions of color} \quad \Longleftrightarrow \quad \Phi_H = \Phi,$$

$$H \in \text{regions of color} \quad \Longleftrightarrow \quad \Phi_H = \{\pm\alpha_1\},$$

$$H \in \text{regions of color} \quad \Longleftrightarrow \quad \Phi_H = \{\pm\alpha_2\},$$

$$H \in \text{regions of color} \quad \Longleftrightarrow \quad \Phi_H = \{\pm\alpha_3\},$$

$$H \in \text{regions of color} \quad \Longleftrightarrow \quad \Phi_H = \varnothing.$$

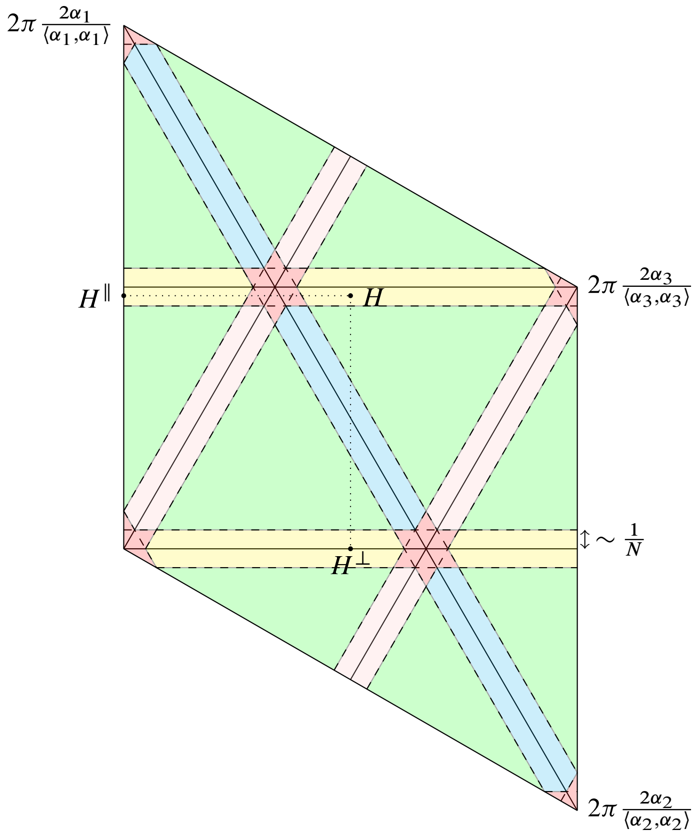

Figure 1. Decomposition of the maximal torus of SU(3) according to the values of $\left\| \frac{1}{2\pi}\langle \alpha, H\rangle \right\|$, $\alpha \in \Delta$.

7E.2. *Decomposition of the weight lattice.* To prove Proposition 7.23, we will make a decomposition of the weight lattice Λ according to the root subsystem Φ_H. First, we have the following lemma about root subsystems. Let Proj_U denote the orthogonal projection map from the ambient inner product space onto the subspace U.

Lemma 7.25. *Let Φ be an integral root system in the space V with the associated weight lattice Λ_Φ. Let Ψ be a root subsystem of Φ. Then let Γ_Ψ and Λ_Φ be the root lattice and weight lattice associated to Ψ respectively. Let V_Ψ be the $\mathbb{R}$-linear span of Ψ in V. Let Υ_Ψ be the image of the orthogonal projection of Λ_Φ onto V_Ψ. Then the following statements hold true*:

(1) *Υ_Ψ is a lattice and $\Gamma_\Psi \subset \Upsilon_\Psi \subset \Lambda_\Psi$. In particular*, *the rank of Υ_Ψ equals the rank of Γ_Ψ as well as Λ_Ψ.*

(2) *Let the ranks of Υ_Ψ and Λ_Φ be r and R respectively. Let $\{w_1, \dots, w_r\}$ be a basis of Υ_Ψ. Pick any $\{u_1, \dots, u_r\} \subset \Lambda_\Phi$ such that $\mathrm{Proj}_{V_\Psi}(u_i) = w_i$, $i = 1, \dots, r$. Then we can extend $\{u_1, \dots, u_r\}$ into a basis $\{u_1, \dots, u_r, u_{r+1}, \dots, u_R\}$ of Λ_Φ. Furthermore*, *we can pick $\{u_{r+1}, \dots, u_R\}$ such that $\mathrm{Proj}_{V_\Psi}(u_i) = 0$ for $i = r + 1, \dots, R$.*

Proof. (1) It's clear that Υ_Ψ is a lattice. Let Γ_Φ be the root lattice associated to Φ. Then $\Gamma_\Psi \subset \Gamma_\Phi$. Thus

$$\Gamma_\Psi = \mathrm{Proj}_{V_\Psi}(\Gamma_\Psi) \subset \mathrm{Proj}_{V_\Psi}(\Gamma_\Phi) \subset \mathrm{Proj}_{V_\Psi}(\Lambda_\Phi) = \Upsilon_\Psi.$$

On the other hand, for any $\mu \in \Lambda_\Phi,\ \alpha \in \Gamma_\Psi$, we have $\langle \mathrm{Proj}_{V_\Psi}(\mu), \alpha\rangle = \langle \mu, \alpha\rangle$. This in particular implies

$$2\frac{\langle \mathrm{Proj}_{V_\Psi}(\mu), \alpha\rangle}{\langle \alpha, \alpha\rangle} = 2\frac{\langle \mu, \alpha\rangle}{\langle \alpha, \alpha\rangle} \in \mathbb{Z} \quad \text{for all } \mu \in \Lambda_\Phi,\ \alpha \in \Gamma_\Psi.$$

This implies $\mathrm{Proj}_{V_\Psi}(\mu) \in \Lambda_\Psi$ for all $\mu \in \Lambda_\Phi$; that is, $\Upsilon_\Psi = \mathrm{Proj}_{V_\Psi}(\Lambda_\Phi) \subset \Lambda_\Psi$.

(2) Let $S_\Phi := \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_r$; then S_Φ is a sublattice of Λ_Φ of rank r. By the theory of modules over a principal ideal domain, there exists a basis $\{u'_1, \ldots, u'_R\}$ of Λ_Φ and positive integers $d_1|d_2|\cdots|d_r$ such that $\{d_1u'_1, \ldots, d_ru'_r\}$ is a basis of S_Φ. Then we must have $d_1 = d_2 = \cdots = d_r = 1$, since

$$\begin{aligned}\mathbb{Z}d_1 \,\mathrm{Proj}_{V_\Psi}(u'_1) + \cdots + \mathbb{Z}d_r \,\mathrm{Proj}_{V_\Psi}(u'_r) &= \mathrm{Proj}_{V_\Psi}(S_\Phi)\\ &= \mathrm{Proj}_{V_\Psi}(\Lambda_\Phi) \supset \mathbb{Z}\,\mathrm{Proj}_{V_\Psi}(u'_1) + \cdots + \mathbb{Z}\,\mathrm{Proj}_{V_\Psi}(u'_r) \end{aligned} \tag{7-62}$$

and $u'_1, \ldots, u'_r$ are $\mathbb{R}$-linear independent. Thus we have

$$S_\Phi = \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_r = \mathbb{Z}u'_1 + \cdots + \mathbb{Z}u'_r$$

and then $\{u_1, \ldots, u_r, u'_{r+1}, \ldots, u'_R\}$ is also a basis of Λ_Φ. Furthermore, by adding a $\mathbb{Z}$-linear combination of $u_1, \ldots, u_r$ to each of $u'_{r+1}, \ldots, u'_R$, we can assume that $\mathrm{Proj}_{V_\Psi}(u'_i) = 0$ for $i = r+1, \ldots, R$. □

We apply the above lemma to the root subsystem Φ_H of Φ. Let $V = i\mathfrak{b}^*$, V_H be the $\mathbb{R}$-linear span of Φ_H in V, Γ_H be the root lattice for Φ_H, and let

$$\Upsilon_H := \mathrm{Proj}_{V_H}(\Lambda). \tag{7-63}$$

Then by the above lemma, we have

$$\Upsilon_H \supset \Gamma_H. \tag{7-64}$$

Let r_H be the rank of Φ_H as well as of Γ_H and Υ_H, and let $\{w_1, \ldots, w_{r_H}\} \subset \Upsilon_H$ such that

$$\Upsilon_H = \mathbb{Z}w_1 + \cdots + \mathbb{Z}w_{r_H}.$$

Pick $\{u_1, \ldots, u_{r_H}\} \subset \Lambda$ such that

$$\mathrm{Proj}_{V_H}(u_i) = w_i, \quad i = 1, \ldots, r_H.$$

Then by the above lemma, we can extend $\{u_1, \ldots, u_{r_H}\}$ into a basis $\{u_1, \ldots, u_r\}$ of Λ such that

$$\mathrm{Proj}_{V_H}(u_i) = 0, \quad i = r_H + 1, \ldots, r, \tag{7-65}$$

with

$$\Lambda = \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_r.$$

Set

$$\Upsilon'_H = \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_{r_H} \subset \Lambda.$$

Then

$$\mathrm{Proj}_{V_H} : \Upsilon'_H \xrightarrow{\sim} \Upsilon_H .$$

Recalling (7-64), let Γ'_H be the sublattice of Υ'_H corresponding to $\Gamma_H \subset \Upsilon_H$ under this isomorphism. More precisely, let $\{\alpha_1, \dots, \alpha_{r_H}\}$ be a simple system of roots for Γ_H; then

$$\mathrm{Proj}_{V_H} : \Gamma'_H = \mathbb{Z}\alpha'_1 + \cdots + \mathbb{Z}\alpha'_{r_H} \xrightarrow{\sim} \Gamma_H = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_{r_H}, \quad \alpha'_i \mapsto \alpha_i,\ i = 1, \dots, r_H, \tag{7-66}$$

and we have

$$\Upsilon'_H / \Gamma'_H \cong \Upsilon_H / \Gamma_H, \quad |\Upsilon'_H / \Gamma'_H| = |\Upsilon_H / \Gamma_H| < \infty. \tag{7-67}$$

Decomposing the weight lattice as

$$\Lambda = \bigsqcup_{\mu \in \Upsilon'_H / \Gamma'_H} (\mu + \Gamma'_H + \mathbb{Z}u_{r_H+1} + \cdots + \mathbb{Z}u_r),$$

we have

$$K_{N,\mathbb{1}}(t,x) = \sum_{\substack{\mu \in \Upsilon'_H / \Gamma'_H \\ \lambda'_1 = n_1\alpha'_1 + \cdots + n_{r_H}\alpha'_{r_H} \\ \lambda_2 = n_{r_H+1}u_{r_H+1} + \cdots + n_r u_r}} e^{i\langle \mu + \lambda'_1 + \lambda_2, H^\perp\rangle - it|\mu + \lambda'_1 + \lambda_2|^2} \varphi\left(\frac{|\mu + \lambda'_1 + \lambda_2|^2 - |\rho|^2}{N^2}\right) \cdot \left(\prod_{\alpha \in P} \langle \alpha, \mu + \lambda'_1 + \lambda_2\rangle\right) \frac{\sum_{s_H \in W_H} (\det s_H) e^{i\langle s_H(\mu + \lambda'_1 + \lambda_2), H^\parallel\rangle}}{\prod_{\alpha \in P_H} (e^{i\langle \alpha, H^\parallel\rangle} - 1)}. \tag{7-68}$$

Note that (7-65) implies for $\lambda_2 = n_{r_H+1}u_{r_H+1} + \cdots + n_r u_r$ that

$$\langle s_H(\lambda_2), H^\parallel\rangle = \langle \lambda_2, s_H(H^\parallel)\rangle = 0,$$

and (7-66) implies for $\lambda'_1 = n_1\alpha'_1 + \cdots + n_{r_H}\alpha'_{r_H}$ that

$$\langle s_H(\lambda'_1), H^\parallel\rangle = \langle \lambda'_1, s_H(H^\parallel)\rangle = \langle \lambda_1, s_H(H^\parallel)\rangle = \langle s_H(\lambda_1), H^\parallel\rangle,$$

where $\lambda_1 = n_1\alpha_1 + \cdots + n_{r_H}\alpha_{r_H} \in V_H$. Similarly, also note that

$$\langle s_H(\mu), H^\parallel\rangle = \langle s_H(\mu^\parallel), H^\parallel\rangle, \quad \text{where } \mu^\parallel := \mathrm{Proj}_{V_H}(\mu).$$

Thus we write

$$K_{N,\mathbb{1}}(t,x) = \sum_{\mu \in \Upsilon'_H / \Gamma'_H} \sum_{\substack{\lambda'_1 = n_1\alpha'_1 + \cdots + n_{r_H}\alpha'_{r_H} \\ \lambda_1 = n_1\alpha_1 + \cdots + n_{r_H}\alpha_{r_H} \\ \lambda_2 = n_{r_H+1}u_{r_H+1} + \cdots + n_r u_r}} e^{i\langle \mu + \lambda'_1 + \lambda_2, H^\perp\rangle - it|\mu + \lambda'_1 + \lambda_2|^2} \varphi\left(\frac{|\mu + \lambda'_1 + \lambda_2|^2 - |\rho|^2}{N^2}\right) \cdot \left(\prod_{\alpha \in P} \langle \alpha, \mu + \lambda'_1 + \lambda_2\rangle\right) \frac{\sum_{s_H \in W_H} (\det s_H) e^{i\langle s_H(\mu^\parallel + \lambda_1), H^\parallel\rangle}}{\prod_{\alpha \in P_H} (e^{i\langle \alpha, H^\parallel\rangle} - 1)}. \tag{7-69}$$

Remark 7.26. We have that in the above formula

$$\chi^{\mu^\parallel}(\lambda_1, H^\parallel) := \frac{\sum_{s_H \in W_H} (\det s_H) e^{i\langle s_H(\mu^\parallel + \lambda_1), H^\parallel\rangle}}{\prod_{\alpha \in P_H} (e^{i\langle \alpha, H^\parallel\rangle} - 1)}$$

is a character of the form (7-41). Also note that $\mu^{\parallel} \in \mathrm{Proj}_{V_H}(\Lambda)$ lies in the weight lattice Λ_H of Φ_H by Lemma 7.25.

Noting (7-67), Proposition 7.23 reduces to the following.

Proposition 7.27. *For $\mu \in \Upsilon'_H/\Gamma'_H$, let*

$$K^{\mu}_{N,\mathbb{1}}(t,x) := \sum_{\substack{\lambda'_1 = n_1\alpha'_1+\cdots+n_{r_H}\alpha'_{r_H}\\ \lambda_1=n_1\alpha_1+\cdots+n_{r_H}\alpha_{r_H}\\ \lambda_2=n_{r_H+1}u_{r_H+1}+\cdots+n_r u_r\\ n_1,\dots,n_r\in\mathbb{Z}}} e^{i\langle \mu+\lambda'_1+\lambda_2, H^{\perp}\rangle - it|\mu+\lambda'_1+\lambda_2|^2}\varphi\left(\frac{|\mu+\lambda'_1+\lambda_2|^2-|\rho|^2}{N^2}\right) \cdot\left(\prod_{\alpha\in P}\langle \alpha, \mu+\lambda'_1+\lambda_2\rangle\right)\frac{\sum_{s_H\in W_H}(\det s_H)e^{i\langle s_H(\mu^{\parallel}+\lambda_1),H^{\parallel}\rangle}}{\prod_{\alpha\in P_H}(e^{i\langle\alpha,H^{\parallel}\rangle}-1)}. \tag{7-70}$$

Then

$$|K^{\mu}_{N,\mathbb{1}}(t,x)| \lesssim \frac{N^{d-|P\setminus P_H|}}{\left(\sqrt{q}\left(1+N\left|\frac{t}{2\pi D}-\frac{a}{q}\right|^{1/2}\right)\right)^r} \tag{7-71}$$

for $\frac{t}{2\pi D} \in \mathcal{M}_{a,q}$, uniformly in $x \in G$.

Proof. We apply Lemma 7.4 to the lattice $\mathbb{Z}\alpha'_1 + \cdots + \mathbb{Z}\alpha'_{r_H} + \mathbb{Z}u_{r_H+1} + \cdots + \mathbb{Z}u_r$. Write

$$\chi^{\mu^{\parallel}}(\lambda_1, H^{\parallel}) = \frac{\sum_{s_H\in W_H}(\det s_H)e^{i\langle s_H(\mu^{\parallel}+\lambda_1),H^{\parallel}\rangle}}{\prod_{\alpha\in P_H}(e^{i\langle\alpha,H^{\parallel}\rangle}-1)}.$$

Then it suffices to show that

$$\left|D_{i_1}\cdots D_{i_k}\left(\prod_{\alpha\in P}\langle\alpha,\mu+\lambda'_1+\lambda_2\rangle\chi^{\mu^{\parallel}}(\lambda_1,H^{\parallel})\right)\right| \lesssim N^{d-|P\setminus P_H|-r-k}$$

for $1 \le i_1, \dots, i_k \le r$,

$$\begin{aligned}\lambda'_1 &= n_1\alpha'_1 + \cdots + n_{r_H}\alpha'_{r_H},\\ \lambda_1 &= n_1\alpha_1 + \cdots + n_{r_H}\alpha_{r_H},\\ \lambda_2 &= n_{r_H+1}u_{r_H+1} + \cdots + n_r u_r,\end{aligned}$$

uniformly in $|n_i| \lesssim N$, $i = 1, \dots, r$. Since $\prod_{\alpha\in P}\langle\alpha, \mu+\lambda'_1+\lambda_2\rangle$ is a polynomial and thus a pseudopolynomial of degree $|P|$, it suffices to show that

$$|D_{i_1}\cdots D_{i_k}(\chi^{\mu^{\parallel}}(\lambda_1,H^{\parallel}))| \lesssim N^{d-|P\setminus P_H|-r-|P|-k} = N^{|P_H|-k}. \tag{7-72}$$

Since $\chi(\lambda_1)$ does not involve the variables $n_{r_H+1}, \dots, n_r$, it suffices to prove (7-72) for $1 \le i_1, \dots, i_k \le r_H$ uniformly in $|\lambda_1| \lesssim N$. But this follows by applying Lemma 7.20 to the root system Φ_H, noting Remark 7.26. □

7F. L^p *estimates.* We prove in this section $L^p(G)$ estimates of the Schrödinger kernel for $p<\infty$. Though we do not apply them to the proof of the main theorem, they encapsulate the essential ingredients in the proof of the $L^\infty(G)$-estimate and are of independent interest.

Proposition 7.28. *Let $K_N(t,x)$ be the Schrödinger kernel as in Theorem 6.1. Then for any $p>3$, we have*

$$\|K_N(t,\cdot)\|_{L^p(G)} \lesssim \frac{N^{d-\frac{d}{p}}}{\left(\sqrt{q}\left(1+N\left\|\frac{t}{2\pi D}-\frac{a}{q}\right\|^{\frac{1}{2}}\right)\right)^r} \tag{7-73}$$

for $\frac{t}{2\pi D}\in\mathcal{M}_{a,q}$.

Proof. As a linear combination of characters, the Schrödinger kernel $K_N(t,\cdot)$ is a central function. Then we can apply to it the Weyl integration formula (4-16)

$$\|K_N(t,\cdot)\|_{L^p(G)}^p = \frac{1}{|W|}\int_B |K_N(t,b)|^p |D_P(b)|^2\,db, \tag{7-74}$$

where B is the maximal torus with normalized Haar measure db. Recall that we can parametrize $B=\exp\mathfrak{b}$ by $H\in i\mathfrak{b}^*\cong\mathfrak{b}$, and write

$$B\cong i\mathfrak{b}^*/(2\pi\mathbb{Z}\alpha_1^\vee+\cdots+2\pi\mathbb{Z}\alpha_r^\vee)=[0,2\pi)\alpha_1^\vee+\cdots+[0,2\pi)\alpha_r^\vee, \tag{7-75}$$

where $\{\alpha_i^\vee=2\alpha_i/\langle\alpha_i,\alpha_i\rangle \mid i=1,\dots,r\}$ is the set of simple coroots associated to a system of simple roots $\{\alpha_i\mid i=1,\dots,r\}$.

We have shown in Section 7E that each $H\in i\mathfrak{b}^*$ is associated to a root subsystem Φ_H such that (7-50) and (7-51) hold. Note that there are finitely many root subsystems of a given root system; thus B is covered by finitely many subsets R of the form

$$R=\left\{H\in B \;\middle|\; \left\|\tfrac{1}{2\pi}\langle\alpha,H\rangle\right\|\lesssim\tfrac{1}{N}\text{ for all }\alpha\in\Psi,\ \left\|\tfrac{1}{2\pi}\langle\alpha,H\rangle\right\|>\tfrac{1}{N}\text{ for all }\alpha\in\Phi\setminus\Psi\right\}, \tag{7-76}$$

where Ψ is a root subsystem of Φ. Thus to prove (7-73), using (7-74), it suffices to show

$$\int_R |K_N(t,\exp H)|^p|D_p(\exp H)|^2\,dH \lesssim \left(\frac{N^d}{\left(\sqrt{q}\left(1+N\left\|\frac{t}{2\pi D}-\frac{a}{q}\right\|^{1/2}\right)\right)^r}\right)^p N^{-d}. \tag{7-77}$$

By (7-54), (7-59) and (7-61), we have

$$K_N(t,\exp H)\lesssim \frac{1}{\prod_{\alpha\in P\setminus Q}(e^{i\langle\alpha,H\rangle}-1)}\cdot\frac{N^{d-|P\setminus Q|}}{\left(\sqrt{q}\left(1+N\left|\frac{t}{2\pi D}-\frac{a}{q}\right|^{1/2}\right)\right)^r},$$

where P, Q are respectively the sets of positive roots of Φ and Ψ with $P\supset Q$. Recalling $D_P(\exp H)=\prod_{\alpha\in P}(e^{i\langle\alpha,H\rangle}-1)$, (7-77) is then reduced to

$$\int_R\left|\frac{1}{\prod_{\alpha\in P\setminus Q}(e^{i\langle\alpha,H\rangle}-1)}\right|^{p-2}\left|\prod_{\alpha\in Q}(e^{i\langle\alpha,H\rangle}-1)\right|^2 dH\lesssim N^{p|P\setminus Q|-d}.$$

Using

$$|e^{i\langle\alpha,H\rangle}-1|\lesssim\left\|\tfrac{1}{2\pi}\langle\alpha,H\rangle\right\|\lesssim|e^{i\langle\alpha,H\rangle}-1|,$$

it suffices to show

$$\int_R \left| \frac{1}{\prod_{\alpha\in P\setminus Q} \left\| \frac{1}{2\pi}\langle \alpha, H\rangle \right\|} \right|^{p-2} \left| \prod_{\alpha\in Q} \left\| \frac{1}{2\pi}\langle \alpha, H\rangle \right\| \right|^2 dH \lesssim N^{p|P\setminus Q|-d}. \tag{7-78}$$

For each $H \in B$, we write

$$H = H' + H_0$$

such that

$$\left\| \tfrac{1}{2\pi}\langle \alpha, H\rangle \right\| = \left| \tfrac{1}{2\pi}\langle \alpha, H'\rangle \right|, \quad \langle \alpha, H_0\rangle \in 2\pi\mathbb{Z} \quad \text{for all } \alpha \in P.$$

We write

$$R \subset \bigcup_{\substack{H_0\in B \\ \langle \alpha, H_0\rangle \in 2\pi\mathbb{Z}, \forall \alpha\in P}} R' + H_0, \tag{7-79}$$

where

$$R' = \left\{ H \in B \;\middle|\; \left| \tfrac{1}{2\pi}\langle \alpha, H\rangle \right| \lesssim \tfrac{1}{N} \text{ for all } \alpha \in Q,\ \left| \tfrac{1}{2\pi}\langle \alpha, H\rangle \right| > \tfrac{1}{N} \text{ for all } \alpha \in P\setminus Q \right\}. \tag{7-80}$$

Note that $\langle \alpha, \alpha_i^\vee\rangle \in \mathbb{Z}$ for all $\alpha \in P$ and $i = 1, \dots, r$ due to the integrality of the root system; using (7-75), we have that there are only finitely many $H_0 \in B$ such that $\langle \alpha, H_0\rangle \in 2\pi\mathbb{Z}$ for all $\alpha \in P$. Thus using (7-79), (7-78) is further reduced to

$$\int_{R'} \left| \frac{1}{\prod_{\alpha\in P\setminus Q} \left| \frac{1}{2\pi}\langle \alpha, H\rangle \right|} \right|^{p-2} \left| \prod_{\alpha\in Q} \left| \frac{1}{2\pi}\langle \alpha, H\rangle \right| \right|^2 dH \lesssim N^{p|P\setminus Q|-d}. \tag{7-81}$$

Now we reparametrize $B \cong [0, 2\pi)\alpha_1^\vee + \cdots + [0, 2\pi)\alpha_r^\vee$ by

$$H = \sum_{i=1}^{r} t_i w_i, \quad (t_1, \dots, t_r) \in D,$$

where $\{w_i \mid i = 1, \dots, r\}$ are the fundamental weights such that $\langle \alpha_i, w_j\rangle = \delta_{ij}|\alpha_i|^2/2,\ i, j = 1, \dots, r$, and D is a bounded domain in $\mathbb{R}^r$. Then the normalized Haar measure dH equals

$$dH = C\,dt_1 \cdots dt_r$$

for some constant C. Let $s \le r$ such that

$$\{\alpha_1, \dots, \alpha_s\} \subset P \setminus Q,$$
$$\{\alpha_{s+1}, \dots, \alpha_r\} \subset Q.$$

Using (7-80), we estimate

$$\begin{aligned} \int_{R'} \left| \frac{1}{\prod_{\alpha\in P\setminus Q} \left| t\frac{1}{2\pi}\langle \alpha, H\rangle \right|} \right|^{p-2} & \left| \prod_{\alpha\in Q} \left| \frac{1}{2\pi}\langle \alpha, H\rangle \right| \right|^2 dH \\ &\lesssim \int_{R'} \frac{1}{|t_1\cdots t_s|^{p-2}} N^{(p-2)(|P\setminus Q|-s)} N^{-2|Q|}\, dt_1\cdots dt_r \\ &\lesssim N^{(p-2)(|P\setminus Q|-s)} N^{-2|Q|} \int_{\substack{|t_1|,\dots,|t_s| \gtrsim \frac{1}{N} \\ |t_{s+1}|,\dots,|t_r| \lesssim \frac{1}{N}}} \frac{1}{|t_1\cdots t_s|^{p-2}}\, dt_1\cdots dt_r. \end{aligned} \tag{7-82}$$

If $p > 3$, the above is bounded by

$$\lesssim N^{(p-2)(|P\setminus Q|-s)} N^{-2|Q|} N^{s(p-3)-(r-s)} = N^{p|P\setminus Q|-d},$$

noting that $2|P \setminus Q| + 2|Q| + r = 2|P| + r = d$. □

Remark 7.29. The requirement $p > 3$ is by no means optimal. The estimate in (7-82) may be improved to lower the exponent p. We conjecture that (7-73) holds for all $p > p_r$ such that $\lim_{r\to\infty} p_r = 2$, where r is the rank of G.

Acknowledgments

I would like to thank my Ph.D. thesis advisors Prof. Monica Visan and Prof. Rowan Killip for their constant guidance and support. I especially thank them for giving me all the freedom to choose problems that suit my own interest. I would like to thank Prof. Raphaël Rouquier and Prof. Terence Tao for sharing their expertise on the BGG-Demazure operators and the Weyl-type sums respectively. I would also like to thank Jiayin Guo for pointing out several false conjectures of mine on root systems. My thanks also goes to Prof. Guozhen Lu who encouraged me to publish the paper. I am thankful to the editors and referees for their carefully reading the paper and their helpful suggestions on revising it. Last but not least, I am thankful to Mengmeng Zhang for her company and encouragement.

References

[Anker and Pierfelice 2009] J.-P. Anker and V. Pierfelice, "Nonlinear Schrödinger equation on real hyperbolic spaces", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26**:5 (2009), 1853–1869. MR Zbl

[Anker et al. 2011] J.-P. Anker, V. Pierfelice, and M. Vallarino, "Schrödinger equations on Damek–Ricci spaces", *Comm. Partial Differential Equations* **36**:6 (2011), 976–997. MR Zbl

[Banica 2007] V. Banica, "The nonlinear Schrödinger equation on hyperbolic space", *Comm. Partial Differential Equations* **32**:10 (2007), 1643–1677. MR Zbl

[Banica and Duyckaerts 2007] V. Banica and T. Duyckaerts, "Weighted Strichartz estimates for radial Schrödinger equation on noncompact manifolds", *Dyn. Partial Differ. Equ.* **4**:4 (2007), 335–359. MR Zbl

[Bouclet 2011] J.-M. Bouclet, "Strichartz estimates on asymptotically hyperbolic manifolds", *Anal. PDE* **4**:1 (2011), 1–84. MR Zbl

[Bouclet and Tzvetkov 2008] J.-M. Bouclet and N. Tzvetkov, "On global Strichartz estimates for non-trapping metrics", *J. Funct. Anal.* **254**:6 (2008), 1661–1682. MR Zbl

[Bourgain 1989] J. Bourgain, "On $\Lambda(p)$-subsets of squares", *Israel J. Math.* **67**:3 (1989), 291–311. MR Zbl

[Bourgain 1993] J. Bourgain, "Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, I: Schrödinger equations", *Geom. Funct. Anal.* **3**:2 (1993), 107–156. MR Zbl

[Bourgain 2013] J. Bourgain, "Moment inequalities for trigonometric polynomials with spectrum in curved hypersurfaces", *Israel J. Math.* **193**:1 (2013), 441–458. MR Zbl

[Bourgain and Demeter 2015] J. Bourgain and C. Demeter, "The proof of the l^2 decoupling conjecture", *Ann. of Math.* (2) **182**:1 (2015), 351–389. MR Zbl

[Burq et al. 2004] N. Burq, P. Gérard, and N. Tzvetkov, "Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds", *Amer. J. Math.* **126**:3 (2004), 569–605. MR Zbl

[Burq et al. 2007] N. Burq, P. Gérard, and N. Tzvetkov, "Global solutions for the nonlinear Schrödinger equation on three-dimensional compact manifolds", pp. 111–129 in *Mathematical aspects of nonlinear dispersive equations* (Princeton, 2004), edited by J. Bourgain et al., Ann. of Math. Stud. **163**, Princeton Univ. Press, 2007. MR Zbl

[Fotiadis et al. 2018] A. Fotiadis, N. Mandouvalos, and M. Marias, "Schrödinger equations on locally symmetric spaces", *Math. Ann.* **371**:3-4 (2018), 1351–1374. MR Zbl

[Ginibre and Velo 1995] J. Ginibre and G. Velo, "Generalized Strichartz inequalities for the wave equation", *J. Funct. Anal.* **133**:1 (1995), 50–68. MR Zbl

[Guo et al. 2014] Z. Guo, T. Oh, and Y. Wang, "Strichartz estimates for Schrödinger equations on irrational tori", *Proc. Lond. Math. Soc.* (3) **109**:4 (2014), 975–1013. MR Zbl

[Harish-Chandra 1957] Harish-Chandra, "Differential operators on a semisimple Lie algebra", *Amer. J. Math.* **79** (1957), 87–120. MR Zbl

[Hassell et al. 2006] A. Hassell, T. Tao, and J. Wunsch, "Sharp Strichartz estimates on nontrapping asymptotically conic manifolds", *Amer. J. Math.* **128**:4 (2006), 963–1024. MR Zbl

[Herr 2013] S. Herr, "The quintic nonlinear Schrödinger equation on three-dimensional Zoll manifolds", *Amer. J. Math.* **135**:5 (2013), 1271–1290. MR Zbl

[Herr et al. 2011] S. Herr, D. Tataru, and N. Tzvetkov, "Global well-posedness of the energy-critical nonlinear Schrödinger equation with small initial data in $H^1(\mathbb{T}^3)$", *Duke Math. J.* **159**:2 (2011), 329–349. MR Zbl

[Hiller 1982] H. Hiller, *Geometry of Coxeter groups*, Research Notes in Math. **54**, Pitman, Boston, 1982. MR Zbl

[Humphreys 1990] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Adv. Math. **29**, Cambridge Univ. Press, 1990. MR Zbl

[Ionescu and Staffilani 2009] A. D. Ionescu and G. Staffilani, "Semilinear Schrödinger flows on hyperbolic spaces: scattering H^1", *Math. Ann.* **345**:1 (2009), 133–158. MR Zbl

[Keel and Tao 1998] M. Keel and T. Tao, "Endpoint Strichartz estimates", *Amer. J. Math.* **120**:5 (1998), 955–980. MR Zbl

[Killip and Vişan 2016] R. Killip and M. Vişan, "Scale invariant Strichartz estimates on tori and applications", *Math. Res. Lett.* **23**:2 (2016), 445–472. MR Zbl

[Pierfelice 2006] V. Pierfelice, "Weighted Strichartz estimates for the radial perturbed Schrödinger equation on the hyperbolic space", *Manuscripta Math.* **120**:4 (2006), 377–389. MR Zbl

[Pierfelice 2008] V. Pierfelice, "Weighted Strichartz estimates for the Schrödinger and wave equations on Damek–Ricci spaces", *Math. Z.* **260**:2 (2008), 377–392. MR Zbl

[Procesi 2007] C. Procesi, *Lie groups: an approach through invariants and representations*, Springer, 2007. MR Zbl

[Sogge 1988] C. D. Sogge, "Concerning the L^p norm of spectral clusters for second-order elliptic operators on compact manifolds", *J. Funct. Anal.* **77**:1 (1988), 123–138. MR Zbl

[Staffilani and Tataru 2002] G. Staffilani and D. Tataru, "Strichartz estimates for a Schrödinger operator with nonsmooth coefficients", *Comm. Partial Differential Equations* **27**:7-8 (2002), 1337–1372. MR Zbl

[Strichartz 1977] R. S. Strichartz, "Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations", *Duke Math. J.* **44**:3 (1977), 705–714. MR Zbl

[Strichartz 1983] R. S. Strichartz, "Analysis of the Laplacian on the complete Riemannian manifold", *J. Funct. Anal.* **52**:1 (1983), 48–79. MR Zbl

[Varadarajan 1974] V. S. Varadarajan, *Lie groups, Lie algebras, and their representations*, Prentice-Hall, Englewood Cliffs, NJ, 1974. MR Zbl

Received 27 Aug 2018. Revised 19 Feb 2019. Accepted 18 Apr 2019.

YUNFENG ZHANG: yunfeng.zhang@uconn.edu
Department of Mathematics, University of Connecticut, Storrs, CT, United States

ANALYSIS AND PDE
Vol. 13, No. 4, 2020
dx.doi.org/10.2140/apde.2020.13.1221

PARABOLIC L^p DIRICHLET BOUNDARY VALUE PROBLEM AND VMO-TYPE TIME-VARYING DOMAINS

MARTIN DINDOŠ, LUKE DYER AND SUKJUNG HWANG

We prove the solvability of the parabolic L^p Dirichlet boundary value problem for $1 < p \leq \infty$ for a PDE of the form $u_t = \operatorname{div}(A\nabla u) + B \cdot \nabla u$ on time-varying domains where the coefficients $A = [a_{ij}(X, t)]$ and $B = [b_i]$ satisfy a certain natural small Carleson condition. This result brings the state of affairs in the parabolic setting up to the elliptic standard.

Furthermore, we establish that if the coefficients A, B of the operator satisfy a vanishing Carleson condition and the time-varying domain is of VMO type then the parabolic L^p Dirichlet boundary value problem is solvable for all $1 < p \leq \infty$. This result is related to results in papers by Maz'ya, Mitrea and Shaposhnikova, and Hofmann, Mitrea and Taylor, where the fact that the boundary of the domain has a normal in VMO or near VMO implies invertibility of certain boundary operators in L^p for all $1 < p \leq \infty$, which then (using the method of layer potentials) implies solvability of the L^p boundary value problem in the same range for certain elliptic PDEs.

Our result does not use the method of layer potentials since the coefficients we consider are too rough to use this technique, but remarkably we recover L^p solvability in the full range of p's as in the two papers mentioned above.

1. Introduction

Let us consider a parabolic differential equation on a time-varying domain Ω of the form

$$\begin{cases} u_t = \operatorname{div}(A\nabla u) + B \cdot \nabla u & \text{in } \Omega \subset \mathbb{R}^{n+1}, \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{1-1}$$

where $A = [a_{ij}(X, t)]$ is an $n \times n$ matrix satisfying the uniform ellipticity condition with $X \in \mathbb{R}^n$, $t \in \mathbb{R}$. That is, there exist positive constants λ and Λ such that

$$\lambda|\xi|^2 \leq \sum_{i,j} a_{ij}(X, t)\xi_i\xi_j \leq \Lambda|\xi|^2 \tag{1-2}$$

for almost every $(X, t) \in \Omega$ and all $\xi \in \mathbb{R}^n$. In addition, we assume that the coefficients of A and B satisfy a natural, minimal smoothness condition (1-6) and we do not assume any symmetry on A.

Dyer was supported by The Maxwell Institute Graduate School in Analysis and its Applications, a Centre for Doctoral Training funded by the UK Engineering and Physical Sciences Research Council (grant EP/L016508/01), the Scottish Funding Council, the University of Edinburgh, and Heriot-Watt University. Hwang was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2017R1D1A1B03035152).
MSC2010: primary 35K10, 35K20; secondary 35R05.
Keywords: parabolic boundary value problems, L^p solvability, VMO-type domains.

It has been observed via the method of layer potentials that when the domain on which we consider certain boundary value problems for elliptic or parabolic PDEs is sufficiently smooth, the question of L^p invertibility of a certain boundary operator can be resolved using Fredholm theory since this operator is just a compact perturbation of the identity. This observation then implies invertibility of this boundary operator for all $1 < p \le \infty$ and hence solvability of the corresponding L^p boundary value problem in this range.

The notion of how smooth the domain has to be for the above observation to hold has evolved. Initial results for constant-coefficient elliptic PDEs required domains of at least $C^{1,\alpha}$ type. This was reduced to C^1 domains by Fabes, Jodeit, and Rivière [Fabes et al. 1978]. Later the method of layer potentials was adapted to variable coefficient settings, and the results were extended to elliptic PDEs with variable coefficients [Dindoš 2008] on C^1 domains.

Further progress was made after advancements in singular integrals theory on sets that are not necessary of graph type [Semmes 1991; Hofmann et al. 2010]. It turns out that compactness of the mentioned boundary operator only requires that the normal (which must be well-defined at almost every boundary point) belongs to VMO.

This observation for the Stokes system was made in [Mazya et al. 2009], where boundary value problems for domains whose normal belongs to VMO (or is near to VMO in the BMO norm) were considered. In [Hofmann et al. 2015] symbol calculus for operators of layer potential type on surfaces with VMO normals was developed and applied to various elliptic PDEs including elliptic systems.

So far we have only mentioned elliptic results. One of the first results for the heat equation in Lipschitz cylinders is by Brown [1989]. Here the domain considered is time-independent and Fourier methods in the time variable are used. Domains of a time-varying type for the heat operator were first considered in [Lewis and Murray 1995; Hofmann and Lewis 1996] and again the method of layer potentials was used to establish L^2 solvability. The question of solvability of various boundary value problems for parabolic PDEs on time-varying domains has a long history. Recall that in the elliptic setting [Dahlberg 1977] has shown in a Lipschitz domain that the harmonic measure and surface measure are mutually absolutely continuous and that the elliptic Dirichlet problem is solvable with data in L^2 with respect to the surface measure. R. Hunt then asked whether Dalhberg's result holds for the heat equation in domains whose boundaries are given locally as functions $\phi(x, t)$, Lipschitz in the spatial variable. It was conjectured (due to the natural parabolic scaling) that the correct regularity of $\phi(x, t)$ should be a Hölder condition of order $\frac{1}{2}$ in the time variable t and Lipschitz in x. It turns out that under this assumption the parabolic measure associated with (1-1) is doubling [Nyström 1997].

However, to answer R. Hunt's question positively, one has to consider more regular classes of domains than the one just described above. This follows from the counterexample of [Kaufman and Wu 1988], where it was shown that under just the $\mathrm{Lip}\left(1, \frac{1}{2}\right)$ condition on the domain Ω the associated caloric measure (that is, the measure associated with the operator $\partial_t - \Delta$) might not be mutually absolutely continuous with the natural surface measure. The issue was resolved in [Lewis and Murray 1995], where it was established that mutual absolute continuity of caloric measure and a certain parabolic analogue of the surface measure holds when ϕ has $\frac{1}{2}$ of a time derivative in the parabolic $\mathrm{BMO}(\mathbb{R}^n)$ space, which is a slightly stronger condition than $\mathrm{Lip}\big(1, \frac{1}{2}\big)$. We shall say such domains are of Lewis–Murray type. Hofmann and Lewis [1996] subsequently showed that this condition is sharp. We thoroughly discuss these domains in Section 2A.

Further work was done in [Hofmann and Lewis 2001; Rivera-Noriega 2003; 2014] in graph domains and time-varying cylinders satisfying the Lewis–Murray condition, where they proved the L^p Dirichlet problem was solvable for all $p > p'$ for some potentially very large p' (due to the technique used, there is no control on the size of p'). Finally, [Dindoš and Hwang 2018] established L^p solvability $2 \le p \le \infty$ in domains that are local of Lewis–Murray type under a small Carleson condition.

While researching literature on domains of Lewis–Murray type and ways this concept can be localized (in the time variable the half-derivative is a nonlocal operator, and hence any condition imposed on it is difficult to localize), we have realized that important results we have planned to rely on have issues (either in their proofs or even worse are simply false; see in particular Remark 2.7 in the next section). This has prompted us to write Section 2A on parabolic domains in substantially more detail than we originally intended to. This sets the literature record straight and more importantly in detail explains the concept of localized domains of Lewis–Murray type. For readability of the paper and this section, we have moved long proofs into an Appendix.

We establish L^p solvability results for parabolic PDEs on time-varying cylinders satisfying locally the Lewis–Murray condition in the full range $1 < p \le \infty$, improving the solvability range from [Dindoš and Hwang 2018] as well as older results such as [Hofmann and Lewis 1996], where only $p = 2$ was considered. The coefficients we consider are very rough and, in particular, the method of layer potentials cannot be used. Despite this, we recover (in the parabolic setting) an analogue of [Mazya et al. 2009; Hofmann et al. 2015]. When the domain Ω, on which the parabolic PDE is considered, is of VMO type (that is, certain derivatives both in temporal and spatial variables will be in VMO) and the coefficients of the operator satisfy a vanishing Carleson condition the L^p solvability can be established for all $1 < p \le \infty$. Remarkably this is the full range of solvability that holds for smooth coefficients (via the layer potential method).

Our proof is however completely different from the layer potential method; for example at no point is compactness used. The proof is also substantially different than the case $2 \le p \le \infty$ of [Dindoš and Hwang 2018] in the following way. We were inspired by [Dindoš et al. 2007] and have used a so-called p-adapted square function to prove L^p solvability. However, due to the presence of the parabolic term, a second-square-function-type object will arise, namely

$$\int_{\Omega} |u_t(X, t)|^2 |u(X, t)|^{p-2} \delta(X, t)^3 \, dX \, dt, \tag{1-3}$$

where $\delta(X, t)$ is the parabolic distance to the boundary defined as

$$\delta(X, t) = \inf_{(Y,\tau) \in \partial\Omega} (|X - Y|^2 + |t - \tau|)^{1/2}.$$

When $p = 2$ such an object was called the "area function" and in [Dindoš and Hwang 2018] it was shown that it the usual square function can dominate it. It turns out however that the case $1 < p < 2$ is substantially more complicated and we were only able to establish required bounds for (1-3) for nonnegative u after a substantial effort.

There is also an issue of whether the p-adapted square function is actually well-defined and locally finite (as the exponent on $|u|$ is negative). We prove that when u is a solution of a parabolic PDE the p-adapted square function is indeed well-defined by adapting a recent regularity result [Dindoš and Pipher 2019]. That paper deals with complex-coefficient elliptic PDEs but the method used there can be adapted to the parabolic setting; see Theorem 4.1 for details.

Many results in the parabolic setting are motivated by previous results in the elliptic setting and ours is not different. Let us, therefore, give an overview of the major elliptic results related to our main theorem.

The papers [Kenig et al. 2000] and [Kenig and Pipher 2001] started the study of nonsymmetric divergence elliptic operators with bounded and measurable coefficients. Kenig and Pipher [2001] used [Kenig et al. 2000] to show that the elliptic measure of operators satisfying a type of Carleson measure condition is in A_∞ and hence the L^p Dirichlet problem is solvable for some, potentially large, p. In [Dindoš et al. 2007], the authors improved the result of [Kenig and Pipher 2001] in the following way. They showed that if

$$\delta(X)^{-1}\Big(\operatorname*{osc}_{B_{\delta(X)/2}(X)} a_{ij}\Big)^2 \quad \text{and} \quad \delta(X)\Big(\sup_{B_{\delta(X)/2}(X)} b_i\Big)^2 \tag{1-4}$$

are densities of Carleson measures with vanishing Carleson norms then the L^p Dirichlet problem is solvable for all $1 < p \leq \infty$. A similar result for the elliptic Neumann and regularity boundary value problem was established in [Dindoš et al. 2017].

The parabolic analogue of the elliptic Carleson condition (1-4) is that

$$\delta(X,t)^{-1}\sup_{i,j}\Big(\operatorname*{osc}_{B_{\delta(X,t)/2}(X,t)} a_{ij}\Big)^2 + \delta(X,t)\Big(\sup_{B_{\delta(X,t)/2}(X,t)} b_i\Big)^2 \tag{1-5}$$

is the density of a Carleson measure on Ω with a small Carleson norm and $\delta(X,t)$ is the parabolic distance of a point (X,t) to the boundary $\partial\Omega$.

The condition (1-5) arises naturally as follows. Let $\Omega = \{(x_0, x, t) : x_0 > \phi(x,t)\}$ for a function ϕ which satisfies the Lewis–Murray condition above. Let $\rho : U \to \Omega$ be a mapping from the upper half-space U to Ω. Consider $v = u \circ \rho$. It will follow that if u solves (1-1) in Ω then v will be a solution to a parabolic PDE similar to (1-1) in U. In particular, if ρ is chosen to be the mapping in (2-26) then the coefficients of the new PDE for v will satisfy a Carleson condition like (1-5), see Lemma 2.18, provided the original coefficients (for u) were either smooth or constant.

Furthermore, if we do not insist on control over the size of the Carleson norm, then we can still infer solvability of the L^p Dirichlet problem for large p, as in [Hofmann and Lewis 2001; Rivera-Noriega 2003; 2014].

Finally, we ready to state our main result; some notions used here are defined in detail in Section 2.

Theorem 1.1. *Let Ω be a domain as in Definition 2.10 with character (ℓ, η, N, d) and let A be bounded and elliptic as in* (1-2), *and B be measurable. Consider any $1 < p \leq \infty$ and assume that either*

(1)
$$d\mu_1 = \Big[\delta\big(X,t\big)^{-1}\sup_{i,j}\Big(\operatorname*{osc}_{B_{\delta(X,t)/2}(X,t)} a_{ij}\Big)^2 + \delta(X,t)\sup_{B_{\delta(X,t)/2}(X,t)} |B|^2\Big]\, dX\, dt \tag{1-6}$$

is a Carleson measure on Ω with Carleson norm $\|\mu_1\|_C$,

(2) *or assume in addition that* ∇A, $\partial_t A$ *are well-defined at a.e. point in* Ω *and*

$$\mathrm{d}\mu_2 = \big(\delta(X,t)|\nabla A|^2 + \delta(X,t)^3|\partial_t A|^2 + \delta(X,t)|B|^2\big)\,\mathrm{d}X\,\mathrm{d}t \tag{1-7}$$

is a Carleson measure on Ω *with Carleson norm* $\|\mu_2\|_C$ *and*

$$\delta(X,t)|\nabla A| + \delta(X,t)^2|\partial_t A| + \delta(X,t)|B| \le \|\mu_2\|_C^{1/2}. \tag{1-8}$$

Then there exists $K = K(\lambda, \Lambda, \ell, n, p) > 0$ *such that if for some* $r_0 > 0$

$$\max\{\eta, \|\mu_1\|_{C,r_0}\} < K \quad \textit{or} \quad \max\{\eta, \|\mu_2\|_{C,r_0}\} < K$$

the L^p *Dirichlet boundary value problem* (1-1) *is solvable* (*see Definition 2.26*) *and the following estimate holds for all continuous boundary data* $f \in C_0(\partial\Omega)$:

$$\|N(u)\|_{L^p(\partial\Omega,\,\mathrm{d}\sigma)} \lesssim \|f\|_{L^p(\partial\Omega,\,\mathrm{d}\sigma)},$$

where the implied constant depends only on the operator, n, p *and character* (ℓ, η, N, d), *and* $N(u)$ *is the nontangential maximal function of* u.

Corollary 1.2. *In particular*, *if* Ω *is of VMO type* (η *in the character* (ℓ, η, N, d) *can be taken arbitrary small*), *and the Carleson measure* μ *from Theorem 1.1 is a vanishing Carleson measure then the* L^p *Dirichlet boundary value problem* (1-1) *is solvable for all* $1 < p \le \infty$.

2. Preliminaries

Here and throughout we consistently use ∇u to denote the gradient in the spatial variables and u_t or $\partial_t u$ to denote the gradient in the time variable.

2A. ***Parabolic domains.*** In this subsection, we define a class of time-varying domains whose boundaries are given locally as functions $\phi(x,t)$, Lipschitz in the spatial variable and satisfying the Lewis–Murray condition in the time variable. At each time $\tau \in \mathbb{R}$ the set of points in Ω with fixed time $t = \tau$, that is, $\Omega_\tau = \Omega \cap \{t = \tau\}$, is a nonempty bounded Lipschitz domain in $\mathbb{R}^n$. We start with a discussion of the Lewis–Murray condition, give a summary and clarification of the results in the literature, and introduce some new equivalent definitions.

We define a *parabolic cube* in $\mathbb{R}^{n-1} \times \mathbb{R}$, for a constant $r > 0$, as

$$Q_r(x,t) = \{(y,s) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x_i - y_i| < r \text{ for all } 1 \le i \le n-1,\ |t-s|^{1/2} < r\}.$$

Let $J_r \subset \mathbb{R}^{n-1}$ be a *spatial cube* of radius r. For a given $f : \mathbb{R}^n \to \mathbb{R}$ let

$$f_{Q_r} = \frac{1}{|Q_r|}\int_{Q_r} f(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

When we write $f \in \mathrm{BMO}(\mathbb{R}^n)$ we mean that f belongs to the parabolic version of the usual BMO space with the norm $\|f\|_*$, where

$$\|f\|_* = \sup_{Q_r} \frac{1}{|Q_r|}\int_{Q_r} |f - f_{Q_r}|\,\mathrm{d}x\,\mathrm{d}t < \infty. \tag{2-1}$$

Recall that the Lewis–Murray condition imposed that a half-derivative in time of $\phi(x,t)$ belongs to parabolic BMO. There are a few different ways one can define half-derivatives and BMO-Sobolev spaces, and there are also some erroneous results in the literature which we correct here. To bring clarity, we start by discussing the various definitions in the global setting of a graph domain $\Omega = \{(x_0, x, t) : x_0 > \phi(x,t)\}$, where $\phi : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$. We follow the standard notation of [Hofmann and Lewis 1996].

If $g \in C_0^\infty(\mathbb{R})$ and $0 < \alpha < 2$ then the one-dimensional fractional differentiation operators D_α are defined on the Fourier side by

$$\widehat{D_\alpha g}(\tau) = |\tau|^\alpha \hat{g}(\tau).$$

If $0 < \alpha < 1$ then by standard results

$$D_\alpha g(t) = c \int_{\mathbb{R}} \frac{g(t) - g(s)}{|t-s|^{1+\alpha}} \, \mathrm{d}s.$$

Therefore, we define the *pointwise half-derivative in time* of $\phi : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ to be

$$D^t_{1/2} \phi(x,t) = c_n \int_{\mathbb{R}} \frac{\phi(x,s) - \phi(x,t)}{|s-t|^{3/2}} \, \mathrm{d}s \tag{2-2}$$

for a properly chosen constant c_n; see [Hofmann and Lewis 1996]. In order for the Fourier transform to be well-defined, ϕ should be a tempered distribution modulo first-degree polynomials in x; see [Hofmann 1995; Strichartz 1980].

However, this definition ignores the spatial coordinates. Instead by following [Fabes and Rivière 1967] we may define the *parabolic half-derivative in time* of $\phi : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ to be

$$\widehat{\mathbb{D}_n \phi}(\xi, \tau) = \frac{\tau}{\|(\xi,\tau)\|} \hat{\phi}(\xi,\tau), \tag{2-3}$$

where ξ and τ denote the spatial and temporal variables on the Fourier side respectively, and $\|(x,t)\| = |x| + |t|^{1/2}$ denotes the parabolic norm. In addition we define the *parabolic derivative* (in space and time) of $\phi : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ to be

$$\widehat{\mathbb{D}\phi}(\xi,\tau) = \|(\xi,\tau)\| \hat{\phi}(\xi,\tau). \tag{2-4}$$

$\mathbb{D}^{-1}$ is the parabolic Riesz potential. Again, we assume here that ϕ is a tempered distribution modulo first-degree polynomials in x. One can also represent $\mathbb{D}$ as

$$\mathbb{D} = \sum_{j=1}^{n} R_j \mathbb{D}_j, \tag{2-5}$$

where $\mathbb{D}_j = \partial_j$ for $1 \le j \le n-1$, $\mathbb{D}_n$ is defined above and R_j are the parabolic Riesz transforms defined on the Fourier side as

$$\begin{aligned} \widehat{R}_j(\xi,\tau) &= \frac{i\xi_j}{\|(\xi,\tau)\|} \quad \text{for } 1 \le j \le n-1, \\ \widehat{R}_n(\xi,\tau) &= \frac{\tau}{\|(\xi,\tau)\|^2}. \end{aligned} \tag{2-6}$$

Furthermore the kernels of R_j have average zero on (parabolically weighted) spheres around the origin, obey the standard Calderón–Zygmund kernel and therefore by standard Calderón–Zygmund theory each

R_j defines a bounded operator on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ and is bounded on $\mathrm{BMO}(\mathbb{R}^n)$ [Peetre 1966; Fabes and Rivière 1966; 1967; Hofmann and Lewis 1996].

We say that $\phi : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ is $\mathrm{Lip}\big(1, \frac{1}{2}\big)$ with Lipschitz constant ℓ if ϕ is Lipschitz in the spatial variables and Hölder continuous of order $\frac{1}{2}$ in the temporal variable. That is,

$$|\phi_j(x,t) - \phi_j(y,t)| \le \ell(|x-y| + |t-s|^{1/2}). \tag{2-7}$$

The *Lewis–Murray condition* on the domain Ω, for which they proved the mutual absolute continuity of the caloric measure and the natural surface measure, is $\phi \in \mathrm{Lip}\big(1, \frac{1}{2}\big)$ and $\|D^t_{1/2}\phi\|_* \le \eta$; note this BMO norm is taken over $\mathbb{R}^n$.

It is worth remarking that none of the operators $D^t_{1/2}$, $\mathbb{D}_n$ or $\mathbb{D}$ easily lend themselves to being localized to a function $\phi : Q_d \to \mathbb{R}$ due to their nonlocal natures. However, our goal is to provide a theory where the domain is locally given by graphs satisfying the Lewis–Murray condition. The parabolic nature of the PDE (especially time irreversibility and exponential decay of solutions with vanishing boundary data) suggests we should expect to need only local conditions on the functions describing the boundary.

To this end, we state the following theorems, where we show some statements equivalent to the Lewis–Murray condition for a global function $\phi : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$. Furthermore, the final conditions admit themselves to being localized easily as well as amiable to an extension; see Theorem 2.8 for details on an extension.

The equivalence of (1) and (2) below is shown in [Hofmann and Lewis 1996] with an equivalence of norms in the small and large sense; see (2.10) and Theorem 7.4 in that work for precise details, and see (2-5) and (2-6).

Theorem 2.1. *Let $\phi : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ and $\phi \in \mathrm{Lip}\big(1, \frac{1}{2}\big)$. Then the following conditions are equivalent*:

(1) $D^t_{1/2}\phi \in \mathrm{BMO}(\mathbb{R}^n)$.

(2) $\mathbb{D}_n\phi \in \mathrm{BMO}(\mathbb{R}^n)$.

(3) $\mathbb{D}\phi \in \mathrm{BMO}(\mathbb{R}^n)$.

We also note that since $\mathbb{D}_n\phi = R_n\mathbb{D}\phi$ we have $\|\mathbb{D}_n\phi\|_* \lesssim \|\mathbb{D}\phi\|_*$ by the boundedness of R_n on $\mathrm{BMO}(\mathbb{R}^n)$.

We now extend this theorem by adding three more equivalent statements. To motivate (6) of Theorem 2.3 below we first recall a characterisation of BMO from [Strichartz 1980, p. 546]. Let $M(f, Q) = (1/|Q|)\int_Q f$ denote the average of f over a cube Q, and let $\widetilde{Q}_\rho(x)$ be the cube of radius ρ with x in the upper-right corner.

Lemma 2.2 [Strichartz 1980]. *We have $f \in \mathrm{BMO}(\mathbb{R}^n)$ is equivalent to*

$$\sup_{Q_r} \sum_{k=1}^{n} \frac{1}{|Q_r|} \int_{Q_r} \int_0^r |M(f, \widetilde{Q}_\rho(x)) - M(f, \widetilde{Q}_\rho(x - \rho e_k))|^2 \frac{\mathrm{d}\rho}{\rho}\, \mathrm{d}x = B < \infty, \tag{2-8}$$

where e_k are the usual unit vectors in $\mathbb{R}^n$, and $\|f\|_*^2 \sim B$.

The equivalence of (3) and (4) in the theorem below is a generalisation of [Strichartz 1980] to the parabolic setting that is stated in [Rivera-Noriega 2003]; see also [Fefferman and Stein 1972; Calderón and Torchinsky 1975; 1977]. We have some questions about the proof given in [Rivera-Noriega 2003]; however, the argument we give for (5) also works for (4) and hence the claim in that paper is correct.

Theorem 2.3. *Let $\phi : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ and $\phi \in \mathrm{Lip}\big(1, \frac{1}{2}\big)$. Then the following conditions are equivalent*:

(3) $\mathbb{D}\phi \in \mathrm{BMO}(\mathbb{R}^n)$.

(4)
$$\sup_{Q_r} \frac{1}{|Q_r|} \int_{Q_r} \int_{\|(y,s)\| \le r} \frac{|\phi(x+y,t+s)-2\phi(x,t)+\phi(x-y,t-s)|^2}{\|(y,s)\|^{n+3}} \,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t = B_{(4)} < \infty. \tag{2-9}$$

(5) (a)
$$\sup_{Q_r} \frac{1}{|Q_r|} \int_{Q_r} \int_{|y|<r} \frac{|\phi(x+y,t)-2\phi(x,t)+\phi(x-y,t)|^2}{|y|^{n+1}} \,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t = B_{(5.\mathrm{a})} < \infty. \tag{2-10}$$

(b)
$$\sup_{Q_r = J_r \times I_r} \frac{1}{|Q_r|} \int_{Q_r} \int_{I_r} \frac{|\phi(x,t)-\phi(x,s)|^2}{|t-s|^2} \,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x = B_{(5.\mathrm{b})} < \infty. \tag{2-11}$$

(6) *Let $u = (u', u_n) \in \mathbb{S}^{n-1}$ and let e_n be the unit vector in the time direction. For $k = 1, \dots, n-1$ let*
$$A_k = \int_0^1 \rho u' \cdot \big(M(\nabla\phi, \widetilde{Q}_\rho(x+\lambda\rho u', t)) - M(\nabla\phi, \widetilde{Q}_\rho(x+\lambda\rho u' - \rho e_k, t))\big)\,\mathrm{d}\lambda,$$
$$A_n = \int_0^1 \rho u' \cdot \big(M(\nabla\phi, \widetilde{Q}_\rho(x+\lambda\rho u', t)) - M(\nabla\phi, \widetilde{Q}_\rho(x+\lambda\rho u', t-\rho^2))\big)\,\mathrm{d}\lambda.$$

Then

(a)
$$\sup_{Q_r} \sum_{k=1}^{n} \frac{1}{|Q_r|} \int_{Q_r} \int_{u \in \mathbb{S}^{n-1}} \int_0^r \frac{|A_k|^2}{\rho^3} \,\mathrm{d}\rho\,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}t = B_{(6.\mathrm{a})} < \infty, \tag{2-12}$$

(b)
$$\sup_{Q_r = J_r \times I_r} \frac{1}{|Q_r|} \int_{Q_r} \int_{I_r} \frac{|\phi(x,t)-\phi(x,s)|^2}{|t-s|^2} \,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x = B_{(6.\mathrm{b})} < \infty. \tag{2-11}$$

Furthermore we have equivalence of the norms
$$\|\mathbb{D}\phi\|_*^2 \sim B_{(4)} \sim B_{(5.\mathrm{a})} + B_{(5.\mathrm{b})} \sim B_{(6.\mathrm{a})} + B_{(6.\mathrm{b})}. \tag{2-13}$$

We give a proof of this result in the Appendix at the end of the paper.

Remark 2.4. Condition (6.a) does not immediately look too similar to its supposed motivation, (2-8) in Lemma 2.2. However, if we move back into Cartesian coordinates and undo the mean value theorem, then we obtain something very similar to a combination of (2-8) and an endpoint version of [Strichartz 1980, (3.1)]. The reason why we can obtain the endpoint, whereas [Strichartz 1980, (3.1)] can only be used for a fractional derivative smaller than 1, is due to additional integrability and cancellation coming from (A-1). Consider
$$\begin{aligned} A_k' = {} & M(\phi, \widetilde{Q}_{\|(y,s)\|}(x+y,t)) - M(\phi, \widetilde{Q}_{\|(y,s)\|}(x,t)) \\ & - M(\phi, \widetilde{Q}_{\|(y,s)\|}(x+y-\|(y,s)\|e_k, t)) + M(\phi, \widetilde{Q}_{\|(y,s)\|}(x-\|(y,s)\|e_k, t)), \\ A_n' = {} & M(\phi, \widetilde{Q}_{\|(y,s)\|}(x+y,t)) - M(\phi, \widetilde{Q}_{\|(y,s)\|}(x,t)) \\ & - M(\phi, \widetilde{Q}_{\|(y,s)\|}(x+y, t-\|(y,s)\|^2)) + M(\phi, \widetilde{Q}_{\|(y,s)\|}(x, t-\|(y,s)\|^2)). \end{aligned}$$

Then (6.a) is equivalent to

$$\sup_{Q_r} \sum_{k=1}^{n} \frac{1}{|Q_r|} \int_{Q_r} \int_{\|(y,s)\|<r} \frac{|A_k'|^2}{\|(y,s)\|^{n+3}} \,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t = \widetilde{B}_{(6.\mathrm{a})} < \infty. \tag{2-14}$$

Proposition 2.5. *Let* $\phi : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ *and* $\phi \in \mathrm{Lip}\big(1, \frac{1}{2}\big)$. *Let* $B_{(5.\mathrm{a})}$ *and* $B_{(6.\mathrm{a})}$ *be as in Theorem 2.3. Then we have*

$$B_{(6.\mathrm{a})} \lesssim \|\nabla\phi\|_*^2, \qquad B_{(5.\mathrm{a})} \lesssim \sup_t \|\nabla\phi(\,\cdot\,, t)\|^2_{\mathrm{BMO}(\mathbb{R}^{n-1})},$$

where $\mathrm{BMO}(\mathbb{R}^{n-1})$ *denotes the* BMO *norm in the spatial variables only.*

Proof. The statement $\nabla\phi \in \mathrm{BMO}(\mathbb{R}^{n-1})$ implies (5.a) follows from [Strichartz 1980, Theorem 3.3]. In order to establish the second claim, for the ease of notation let us fix Q_r and k in $1 \le k \le n-1$. Then since $|u'| \le 1$ after changing the order of integration (and the substitution $y = x + \lambda\rho u' \in Q_{2r}$) we get that $B_{(6.\mathrm{a})}$ defined by (2-12) is bounded by

$$\int_0^1 \int_{\mathbb{S}^{n-1}} \int_0^r \frac{1}{|Q_r|} \int_{Q_{2r}} \big|\big(M(\nabla u, \widetilde{Q}_\rho(y,t)) - M(\nabla u, \widetilde{Q}_\rho(y - \rho e_k, t))\big)\big|^2 \mathrm{d}y\,\mathrm{d}t\,\frac{\mathrm{d}\rho}{\rho}\,\mathrm{d}u\,\mathrm{d}\lambda.$$

Then by Lemma 2.2 the two interior integrals are bounded by $C\|\nabla\phi\|_*^2$. Therefore $B_{(6.\mathrm{a})}$ is controlled by $C\|\nabla\phi\|_*^2$. □

It is not immediately obvious whether the opposite implication is true or false due to the highly singular nature of Riesz potentials; see (2-5) and (2-6).

Corollary 2.6. *Let* $\phi : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ *and* $\phi \in \mathrm{Lip}\big(1, \frac{1}{2}\big)$. *If* $\|\nabla\phi\|_* \lesssim \eta$ *and* $B_{(5.\mathrm{b})} \lesssim \eta^2$ *then* $\|\mathbb{D}\phi\|_* \lesssim \eta$.

Here we have replaced conditions (5.a) and (6.a) by the slightly stronger but easier to verify condition $\|\nabla\phi\|_* \lesssim \eta$. We believe that, without too much extra work, one could formulate our main theorem and associated lemmas with a local version of (5.a) in place of $\|\nabla\phi\|_*$.

Remark 2.7. In [Rivera-Noriega 2003, Lemma 2.1], it is stated that another condition is equivalent to those given in Theorems 2.1 and 2.3; however this claim is not correct and only one of the stated implications holds.

Theorem 3.3 in [Strichartz 1980] states that in the one-dimensional setting $D^t_{1/2}\phi(t) \in \mathrm{BMO}(\mathbb{R})$ is equivalent to the one-dimensional version of (5.b) and (6.b)

$$\sup_{I' \subset \mathbb{R}} \left(\frac{1}{|I'|} \int_{I'} \int_{I'} \frac{|\phi(t) - \phi(s)|^2}{|t-s|^2} \,\mathrm{d}t\,\mathrm{d}s \right)^{1/2} \le B, \tag{2-15}$$

with $B \sim \|D^t_{1/2}\phi(\,\cdot\,)\|_{\mathrm{BMO}(\mathbb{R})}$.

In [Rivera-Noriega 2003, Lemma 2.1] it is claimed that given $\phi : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ and $\phi \in \mathrm{Lip}\big(1, \frac{1}{2}\big)$ the pointwise n-dimensional analogue of (2-15)

$$\sup_{x \in \mathbb{R}^{n-1}} \sup_{I' \subset \mathbb{R}} \left(\frac{1}{|I'|} \int_{I'} \int_{I'} \frac{|\phi(x,t) - \phi(x,s)|^2}{|t-s|^2} \,\mathrm{d}t\,\mathrm{d}s \right)^{1/2} \le B \tag{2-16}$$

is equivalent to $\mathbb{D}\phi \in \mathrm{BMO}(\mathbb{R}^n)$ with $B \sim \|\mathbb{D}\phi\|_{\mathrm{BMO}(\mathbb{R}^n)}$. This does not appear to be correct. The paper [Rivera-Noriega 2003] does not give a proof and provides instead a reference to [Strichartz 1980] that is irrelevant for the claim. By [Strichartz 1980] (2-16) is equivalent to $D^t_{1/2}\phi(x,\cdot) \in \mathrm{BMO}(\mathbb{R})$ pointwise for a.e. x. After some tedious and technical calculations we were able to show $\sup_x D^t_{1/2}\phi(x,\cdot) \in \mathrm{BMO}(\mathbb{R})$ implies $D^t_{1/2}\phi \in \mathrm{BMO}(\mathbb{R}^n)$ and hence $\mathbb{D}\phi \in \mathrm{BMO}(\mathbb{R}^n)$ via (4) of Theorem 2.3. However, whether the converse holds is not clear even if we assume more structure for the function $\phi(x,t)$. This is due to the fact that there is "no reasonable Fubini theorem relating $\mathrm{BMO}(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R})$" [Strichartz 1980, p. 558].

Fortunately, the lack of a converse implication does not cast doubt over the subsequent results of [Rivera-Noriega 2003] since the author only uses the claimed equivalence in the correct direction — that (2-16) implies $\mathbb{D}\phi \in \mathrm{BMO}(\mathbb{R}^n)$.

2B. *Localisation.* After the comprehensive review of the Lewis–Murray condition for a graph domain Ω we continue in our aim to construct a time-varying domain which is locally described by local graphs ϕ_j.

For a vector $x \in \mathbb{R}^{n-1}$ we consider the norm $|x|_\infty = \sup_i |x_i|$.

Consider $\phi : Q_{8d} \to \mathbb{R}^{n-1} \times \mathbb{R}$. The localised version of (2-11) from Theorem 2.3 is simply

$$\sup_{\substack{Q_r = J_r \times I_r \\ Q_r \subset Q_{8d}}} \frac{1}{|Q_r|} \int_{Q_r} \int_{I_r} \frac{|\phi(x,t) - \phi(x,s)|^2}{|t-s|^2} \,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x < \infty. \tag{2-17}$$

We denote by $\|f\|_{*,d}$ the BMO norm of f where the supremum in the BMO norm, see (2-1), is taken over all cubes Q_r with $r \le d$. For a function $f : J \times I \to \mathbb{R}$, where $J \subset \mathbb{R}^{n-1}$ and $I \subset \mathbb{R}$ are closed bounded cubes, we consider the norm $\|f\|_{*,J\times I}$ defined as above where the supremum is taken over all parabolic cubes Q_r contained in $J \times I$. The norm $\|f\|_{*,J\times I,d}$ is where the supremum is taken over all parabolic cubes Q_r with $r \le d$ contained in $J \times I$. If the context is clear we suppress the $J \times I$ and write $\|f\|_*$ or $\|f\|_{*,d}$.

Recall that $\mathrm{VMO}(\mathbb{R}^n)$ is defined as the closure of all bounded uniformly continuous functions (which we denote by $C_{b,u}(\mathbb{R}^n)$) in the BMO norm or equivalently BMO functions f such that $\|f\|_{*,d} \to 0$ as $d \to 0$. Alternatively, if we define

$$d(f, \mathrm{VMO}) := \inf_{h \in C_{b,u}(\mathbb{R}^n)} \|f - h\|_*$$

then $f \in \mathrm{VMO}$ if and only if $d(f,\mathrm{VMO}) = 0$; for $f \in \mathrm{BMO}$ this measures the distance of f to VMO. In our case, the boundary of the parabolic domains we consider can be locally described as a graph of a continuous function. However, as our domain is unbounded in time, we may potentially require an infinite family of local graphs $\{\phi_j\}$. Therefore we need to measure the distance to VMO uniformly across this infinite family.

Let $\delta : \mathbb{R}_+ \to \mathbb{R}_+$, $\delta(0) = 0$ and δ be continuous at 0. Then we define C_δ to be the set of bounded continuous functions with the same modulus of continuity δ. That is,

$$C_\delta = \{g : \mathbb{R}^n \to \mathbb{R} : |g(x) - g(y)| \le \delta(|x-y|) \text{ for all } x, y, \text{ and } g \text{ is bounded}\}. \tag{2-18}$$

Note that every family of bounded equicontinuous functions is a subset of C_δ for some modulus of continuity δ. Also $C_{b,u} = \bigcup_\delta C_\delta$. For $f : Q_{8d} \to \mathbb{R}$ we define $d(f, C_\delta)$ as

$$d(f, C_\delta) = \inf_{h \in C_\delta} \|f - h\|_{*, Q_{8d}}.$$

We are now ready to state and prove the result on the extensibility of $\phi : Q_{8d} \to \mathbb{R}$ to a global function.

Theorem 2.8. *Let $\phi : Q_{8d} \subset \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ be* $\mathrm{Lip}\big(1, \frac{1}{2}\big)$ *with Lipschitz constant ℓ. If there exist a scale r_1, a constant $\eta > 0$ and a modulus of continuity δ such that*

$$\sup_{\substack{Q_s = J_s \times I_s \\ Q_s \subset Q_{8d},\, s \le r_1}} \frac{1}{|Q_s|} \int_{Q_s} \int_{I_s} \frac{|\phi(x,t) - \phi(x,\tau)|^2}{|t - \tau|^2} \,\mathrm{d}\tau \,\mathrm{d}t \,\mathrm{d}x \le \eta^2 \tag{2-19}$$

and

$$d(\nabla \phi, C_\delta) \le \eta \tag{2-20}$$

then there exists a scale $d' \le d$ that only depends on d, δ, η, and r_1 and not ϕ such that for all $Q_r \subset Q_{4d}$ with $r \le d'$ there exists a global $\mathrm{Lip}\big(1, \frac{1}{2}\big)$ *function $\Phi : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ with the following properties for all $0 < \varepsilon < 1$:*

(i) $\Phi|_{Q_r} = \phi|_{Q_r}$.

(ii) *The* $\mathrm{Lip}\big(1, \frac{1}{2}\big)$ *constant of Φ is ℓ.*

(iii) $\|\nabla \Phi\|_* \lesssim_\varepsilon \eta^{1-\varepsilon} + \eta\ell$.

(iv) $\displaystyle \sup_{Q_s = J_s \times I_s} \frac{1}{|Q_s|} \int_{Q_s} \int_{I_s} \frac{|\Phi(x,t) - \Phi(x,\tau)|^2}{|t - \tau|^2} \,\mathrm{d}\tau \,\mathrm{d}t \,\mathrm{d}x \lesssim \eta^2.$

Therefore by Corollary 2.6, $\|\mathbb{D}\Phi\|_* \lesssim_\varepsilon \eta^{1-\varepsilon} + \eta\ell$.

We again give the proof of this result in the Appendix. We are now ready to define the class of parabolic domains on which we will work. Motivated by the usual definition of a Lipschitz domain we have:

Definition 2.9. $\mathbb{Z} \subset \mathbb{R}^n \times \mathbb{R}$ is an *ℓ-cylinder* of diameter d if there exists a coordinate system $(x_0, x, t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}$ obtained from the original coordinate system by translation in spatial and time variables and rotation only in the spatial variables such that

$$\mathbb{Z} = \{(x_0, x, t) : |x| \le d, |t|^{1/2} \le d,\ |x_0| \le (\ell + 1)d\}$$

and for $s > 0$

$$s\mathbb{Z} := \{(x_0, x, t) : |x| < sd, |t|^{1/2} \le sd,\ |x_0| \le (\ell + 1)sd\}.$$

Definition 2.10. $\Omega \subset \mathbb{R}^n \times \mathbb{R}$ is an *admissible parabolic domain* with character (ℓ, η, N, d) if there exists a positive scale r_1, and a modulus of continuity δ such that for any time $\tau \in \mathbb{R}$ there are at most N ℓ-cylinders $\{\mathbb{Z}_j\}_{j=1}^N$ of diameter d satisfying the following conditions:

(1) $\partial\Omega \cap \{|t - \tau| \le d^2\} = \bigcup_j (\mathbb{Z}_j \cap \partial\Omega)$.

(2) In the coordinate system (x_0, x, t) of the ℓ-cylinder $\mathbb{Z}_j$

$$\mathbb{Z}_j \cap \Omega \supset \{(x_0, x, t) \in \Omega : |x| < d,\ |t| < d^2,\ \delta(x_0, x, t) \le d/2\}.$$

(3) $8\mathbb{Z}_j \cap \partial\Omega$ is the graph $\{x_0 = \phi_j(x,t)\}$ of a function $\phi_j : Q_{8d} \to \mathbb{R}$, with $Q_{8d} \subset \mathbb{R}^{n-1} \times \mathbb{R}$, such that

$$|\phi_j(x,t) - \phi_j(y,s)| \le \ell(|x-y| + |t-s|^{1/2}) \quad \text{and} \quad \phi_j(0,0) = 0. \tag{2-21}$$

(4)
$$d(\nabla\phi_j, C_\delta) \le \eta \tag{2-22}$$

and

$$\sup_{\substack{Q_s = J_s \times I_s \\ Q_s \subset Q_{8d},\, s \le r_1}} \frac{1}{|Q_s|} \int_{Q_s} \int_{I_s} \frac{|\phi_j(x,t) - \phi_j(x,\tau)|^2}{|t-\tau|^2} \, d\tau \, dt \, dx \le \eta^2. \tag{2-23}$$

Here and throughout $\delta(x_0,x,t) := \operatorname{dist}((x_0,x,t),\partial\Omega)$, and by dist we denote the parabolic distance $\operatorname{dist}[(X,t),(Y,s)] = |X-Y| + |t-s|^{1/2}$.

We say that Ω is of VMO type if η in the character (ℓ,η,N,d) can be taken arbitrarily small (at the expense of a potentially smaller d and r_1, and larger N).

Remark 2.11. When (2-22) holds for small or vanishing η it follows that for a fixed time τ the normal ν to the fixed-time spatial domain $\Omega_\tau = \Omega \cap \{t = \tau\}$ can be written in local coordinates as

$$\nu = \frac{1}{|(-1,\nabla\phi_j)|}(-1,\nabla\phi_j)$$

and hence $d(\nu,\mathrm{VMO}) \lesssim \eta$. Therefore Ω_τ is similar to the domains considered in [Mazya et al. 2009; Hofmann et al. 2015], which dealt with the elliptic problems on domains with normal in or near VMO.

Remark 2.12. It follows from this definition that for each $\tau \in \mathbb{R}$ the time-slice Ω_τ of an admissible parabolic domain $\Omega \subset \mathbb{R}^n \times \mathbb{R}$ is a bounded Lipschitz domain in $\mathbb{R}^n$ and they all have a uniformly bounded diameter. That is,

$$\inf_{\tau\in\mathbb{R}} \operatorname{diam}(\Omega_\tau) \sim d \sim \sup_{\tau\in\mathbb{R}} \operatorname{diam}(\Omega_\tau),$$

where d is the scale from Definition 2.10 and the implied constants only depend on N. In particular, if $\mathcal{O} \subset \mathbb{R}^n$ is a bounded Lipschitz domain then the parabolic cylinder $\Omega = \mathcal{O} \times \mathbb{R}$ is an example of a domain satisfying Definition 2.10.

Definition 2.13. Let $\Omega \subset \mathbb{R}^n \times \mathbb{R}$ be an admissible parabolic domain with character (ℓ,η,N,d). The *measure* σ defined on sets $A \subset \partial\Omega$ is

$$\sigma(A) = \int_{-\infty}^{\infty} \mathscr{H}^{n-1}(A \cap \{(X,t) \in \partial\Omega\}) \, dt, \tag{2-24}$$

where $\mathscr{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure on the Lipschitz boundary $\partial\Omega_\tau$.

We consider solvability of the L^p Dirichlet boundary value problem with respect to this measure σ. The measure σ may not be comparable to the usual surface measure on $\partial\Omega$: in the t-direction the functions ϕ_j from Definition 2.10 are only $\frac{1}{2}$-Lipschitz and hence the standard surface measure might not be locally finite. Our definition assures that for any $A \subset 8\mathbb{Z}_j$, where $\mathbb{Z}_j$ is an ℓ-cylinder, we have

$$\mathscr{H}^n(A) \sim \sigma(\{(\phi_j(x,t),x,t) : (x,t) \in A\}), \tag{2-25}$$

where the constants in (2-25), by which these measures are comparable, only depend on ℓ of the character (ℓ, η, N, d) of the domain Ω. If Ω has a smoother boundary, such as Lipschitz (in all variables) or better, then the measure σ is comparable to the usual n-dimensional Hausdorff measure $\mathscr{H}^n$. In particular, this holds for a parabolic cylinder $\Omega = \mathcal{O} \times \mathbb{R}$.

Corollary 2.14. *Let Ω be defined as in Definition 2.10 by a family of functions $\{\phi_j\}$, $\phi_j : Q_{8d} \to \mathbb{R}$. Then there exists an extended family $\{\Phi_j\}$, $\Phi_j : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$, such that*

(i) *$\{\Phi_j|_{Q_{8r}}\}$ still describes Ω, as in Definition 2.10, but with character $(\ell, \eta, \widetilde{N}, r)$ instead of (ℓ, η, N, d), where $\widetilde{N} \geq N$ and $r \leq r_1 \leq d$ an by Theorem 2.8,*

(ii) *$\|\nabla \Phi_j\|_* \lesssim_\varepsilon \eta^{1-\varepsilon} + \eta\ell$, and*

(iii) *$\|\mathbb{D}\Phi_j\|_* \lesssim_\varepsilon \eta^{1-\varepsilon} + \eta\ell$.*

Here, $\widetilde{N}$, r depend on the original character variables ℓ, η, N, d, the modulus of continuity δ and the dimension n.

Proof. This follows from Theorem 2.8 and by tiling the support of each ϕ_j into parabolic cubes of size $8r$ with enough overlap. □

Corollary 2.15. *If Ω is a* VMO*-type domain then we may take η arbitrarily small in Corollary 2.14, or in* (2-22) *and* (2-23) *of Definition 2.10, by reducing r.*

2C. ***Pullback transformation and Carleson condition.*** We now briefly recall the pullback mapping of Dahlberg, Kenig, Nečas and Stein on the upper half-space U $\rho : U \to \Omega$, see [Hofmann and Lewis 1996; 2001], in the setting of parabolic equations defined by

$$\rho(x_0, x, t) = (x_0 + P_{\gamma x_0}\phi(x, t), x, t). \tag{2-26}$$

For simplicity, assume

$$\Omega = \{(x_0, x, t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} : x_0 > \phi(x, t)\}, \tag{2-27}$$

where $\phi(x, t) : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ and satisfies (4) and (3) of Definition 2.10. This transformation maps the upper half-space

$$U = \{(x_0, x, t) : x_0 > 0,\ x \in \mathbb{R}^{n-1},\ t \in \mathbb{R}\} \tag{2-28}$$

into Ω and allows us to consider the L^p solvability of the PDE (1-1) in the upper half-space instead of in the original domain Ω.

To complete the definition of the mapping ρ we define a parabolic approximation to the identity P to be an even nonnegative function $P(x, t) \in C_0^\infty(Q_1(0, 0))$ for $(x, t) \in \mathbb{R}^{n-1} \times \mathbb{R}$, with $\int P(x, t)\, \mathrm{d}x\, \mathrm{d}t = 1$ and set

$$P_\lambda(x, t) := \lambda^{-(n+1)} P\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right).$$

Let $P_\lambda \phi$ be the convolution operator

$$P_\lambda \phi(x, t) := \int_{\mathbb{R}^{n-1} \times \mathbb{R}} P_\lambda(x - y, t - s)\phi(y, s)\, \mathrm{d}y\, \mathrm{d}s.$$

Then P satisfies for constants γ

$$\lim_{(y_0,y,s)\to(0,x,t)} P_{\gamma y_0}\phi(y,s)=\phi(x,t)$$

and ρ defined in (2-26) extends continuously to $\rho:\overline{U}\to\overline{\Omega}$. The usual surface measure on ∂U is comparable with the measure σ defined by (2-24) on $\partial\Omega$.

Suppose that $v=u\circ\rho$ and $f^v=f\circ\rho$. Then (1-1) transforms to a new PDE for the variable v

$$\begin{cases} v_t=\operatorname{div}(A^v\nabla v)+B^v\cdot\nabla v & \text{in } U,\\ v=f^v & \text{on } \partial U,\end{cases} \tag{2-29}$$

where $A^v=[a^v_{ij}(X,t)]$, $B^v=[b^v_i(X,t)]$ are $(n\times n)$ and $(1\times n)$ matrices.

The precise relations between the original coefficients A and B and the new coefficients A^v and B^v are detailed in [Rivera-Noriega 2014, p. 448]. We note that if the constant $\gamma>0$ is chosen small enough then the coefficients $a^v_{ij}, b^v_i : U\to\mathbb{R}$ are Lebesgue measurable and A^v satisfies the standard uniform ellipticity condition with constants λ^v and Λ^v, since the original matrix A did.

Definition 2.16. Let Ω be a parabolic domain from Definition 2.10. For $(Y,s)\in\partial\Omega$, $(X,t),(Z,\tau)\in\Omega$ and $r>0$ we write

$$\begin{aligned} B_r(X,t)&=\{(Z,\tau)\in\mathbb{R}^n\times\mathbb{R}:\operatorname{dist}[(X,t),(Z,\tau)]<r\},\\ Q_r(X,t)&=\{(Z,\tau)\in\mathbb{R}^n\times\mathbb{R}:|x_i-z_i|<r \text{ for all } 0\le i\le n-1,\ |t-\tau|^{1/2}<r\},\\ \Delta_r(Y,s)&=\partial\Omega\cap B_r(Y,s),\\ T(\Delta_r)&=\Omega\cap B_r(Y,s),\\ \delta(X,t)&=\inf_{(Y,s)\in\partial\Omega}\operatorname{dist}[(X,t),(Y,s)].\end{aligned}$$

Definition 2.17 (Carleson measure). A measure $\mu:\Omega\to\mathbb{R}^+$ is a *Carleson measure* if there exists a constant $C=C(d)$ such that for all $r\le d$ and all surface balls Δ_r

$$\mu(T(\Delta_r))\le C\sigma(\Delta_r). \tag{2-30}$$

The best possible constant C is called the *Carleson norm* and is denoted by $\|\mu\|_{C,d}$. Occasionally, for brevity, we drop the d and just write $\|\mu\|_C$ if the context is clear. We say that μ is a vanishing Carleson measure if $\|\mu\|_{C,d}\to 0$ as $d\to 0+$.

When $\partial\Omega$ is locally given as a graph of a function $x_0=\phi(x,t)$ in the coordinate system (x_0,x,t) and μ is a measure supported on $\{x_0>\phi(x,t)\}$, we can reformulate the Carleson condition locally using the parabolic boundary cubes Q_r and corresponding Carleson regions $T(Q_r)$. The Carleson condition (2-30) then becomes

$$\mu(T(Q_r))\le C|Q_r|=Cr^{n+1}. \tag{2-31}$$

Note that the Carleson norms induced from (2-30) and (2-31) are not equal but are comparable.

We now return to the pullback transformation and investigate the Carleson condition on the coefficients of A and B. The following result comes directly from a careful reading of the proofs of Lemma 2.8 and Theorem 7.4 in [Hofmann and Lewis 1996] combined with Theorems 2.1 and 2.3.

Lemma 2.18. *Let σ and θ be nonnegative integers, $\alpha = (\alpha_1, \dots, \alpha_{n-1})$ a multi-index with $l = \sigma + |\alpha| + \theta$, d a scale and fix γ. If $\phi : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ satisfies for all $x, y \in \mathbb{R}^{n-1}$, $t, s \in \mathbb{R}$, and for some positive constants ℓ and η*

$$|\phi(x,t) - \phi(y,s)| \le \ell(|x-y| + |t-s|^{1/2}),\qquad \|\mathbb{D}\phi\|_* \le \eta \tag{2-32}$$

then the measure ν defined at (x_0, x, t) by

$$\mathrm{d}\nu = \left(\frac{\partial^l P_{\gamma x_0}\phi}{\partial x_0^\sigma \partial x^\alpha \partial t^\theta}\right)^2 x_0^{2l+2\theta-3}\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0$$

is a Carleson measure on cubes of diameter $\le d/4$ whenever either $\sigma + \theta \ge 1$ or $|\alpha| \ge 2$, with

$$\nu[(0,r) \times Q_r(x,t)] \lesssim \eta |Q_r(x,t)|,$$

where $r \le d/4$. Moreover, if $l \ge 1$ then at (x_0, x, t), with $x_0 \le d/4$,

$$\left|\frac{\partial^l P_{\gamma x_0}\phi}{\partial x_0^\sigma \partial x^\alpha \partial t^\theta}\right| \lesssim \eta(1+\ell) x_0^{1-l-\theta}, \tag{2-33}$$

where the implicit constants depend on d, l, n.

The drift term B^v from the pullback transformation in (2-29) includes the term

$$\frac{\partial}{\partial t} P_{\gamma x_0}\phi u_{x_0}.$$

From Lemma 2.18 with $\sigma = |\alpha| = 0$ and $\theta = 1$, we see that

$$x_0\left[\frac{\partial}{\partial t} P_{\gamma x_0}\phi(x,t)\right]^2 \mathrm{d}X\,\mathrm{d}t$$

is a Carleson measure in U. Thus it is natural to expect that

$$\mathrm{d}\mu_1(X,t) = x_0 |B^v|^2(X,t)\,\mathrm{d}X\,\mathrm{d}t \tag{2-34}$$

is a Carleson measure in U and B^v satisfies

$$x_0|B^v|(X,t) \le \Lambda_B < \|\mu_1\|_C^{1/2}. \tag{2-35}$$

Indeed, this is the case provided the original vector B satisfies the assumption that

$$\mathrm{d}\mu(X,t) = \delta(X,t)\Big[\sup_{B_{\delta(X,t)/2}(X,t)} |B|\Big]^2 \mathrm{d}X\,\mathrm{d}t \tag{2-36}$$

is a Carleson measure in Ω. Here $\|\mu_1\|_C$ depends on η and the Carleson norm of (2-36).

Similarly, for the matrix A^v if we apply Lemma 2.18 and use the calculations in [Rivera-Noriega 2014, §6] then

$$\mathrm{d}\mu_2(X,t) = (x_0|\nabla A^v|^2 + x_0^3|A_t^v|^2)(X,t)\,\mathrm{d}X\,\mathrm{d}t \tag{2-37}$$

is a Carleson measure in U and A^v satisfies

$$(x_0|\nabla A^v| + x_0^2|A_t^v|)(X,t) \le \|\mu_2\|_C^{1/2} \tag{2-38}$$

for almost every $(X,t)\in U$ provided the original matrix A satisfies that

$$d\mu(X,t)=\Big(\delta(X,t)\Big[\sup_{B_{\delta(X,t)/2}(X,t)}|\nabla A|\Big]^2+\delta(X,t)^3\Big[\sup_{B_{\delta(X,t)/2}(X,t)}|\partial_t A|\Big]^2\Big)\,dX\,dt \tag{2-39}$$

is a Carleson measure in Ω.

We note that if both $\|\mu\|_{C,r}$ and η are small then so too are the Carleson norms $\|\mu_1\|_{C,r}$ and $\|\mu_2\|_{C,r}$ of the matrix A^v and vector B^v, at least if we restrict ourselves to small Carleson regions $r\le d$; this comes from Theorem 2.8, Corollary 2.14 and Corollary 2.15. Then by Lemma 2.18 we see that $\|\mu_1\|_{C,r}$ and $\|\mu_2\|_{C,r}$ only depend on η and $\|\mu\|_{C,r}$ on Carleson regions of size $r\le d$. In particular, they are small if both η and $\|\mu\|_{C,r}$ are small. It further follows by Corollary 2.15 that we can make $\|\mu_1\|_{C,r}$ and $\|\mu_2\|_{C,r}$ as small as we like if μ is a vanishing Carleson norm and the domain Ω is of VMO type.

Observe that condition (2-39) is slightly stronger than (1-6), which we claimed to assume in Theorem 1.1. We replace condition (2-39) by the weaker condition (1-6) later via perturbation results of [Sweezy 1998].

Definition 2.19. We define $\rho_j : U\to 8\mathbb{Z}_j$ to be the local pullback mapping in $8\mathbb{Z}_j$ associated to the function Φ_j in Theorem 2.8, the extension of ϕ_j from Definition 2.10.

Remark 2.20. By [Ball and Zarnescu 2017] and its adaptation to the setting of admissible domains in [Dindoš and Hwang 2018, §2.3], one may construct a "proper generalised distance" globally when η in the character of the domain is small. The smallness of η in the character of the domain is used to guarantee that overlapping coordinate charts, generated by a local construction, are almost parallel. We may then use the result of [Ball and Zarnescu 2017, Theorem 5.1] to show there exists a domain Ω^ε of class C^∞, a homeomorphism $f^\varepsilon:\overline{\Omega}\to\overline{\Omega}^\varepsilon$ such that $f^\varepsilon(\partial\Omega)=\partial\Omega^\varepsilon$ and $f^\varepsilon:\Omega\to\Omega^\varepsilon$ is a C^∞ diffeomorphism.

2D. ***Parabolic nontangential cones, maximal functions and p-adapted square and area functions.*** We proceed with the definition of parabolic nontangential cones and define the cones in a (local) coordinate system where $\Omega=\{(x_0,x,t):x_0>\phi(x,t)\}$, which also applies to the upper half-space U.

Definition 2.21. For a constant $a>0$, we define the *parabolic nontangential cone* at a point $(x_0,x,t)\in\partial\Omega$ by

$$\Gamma_a(x_0,x,t)=\{(y_0,y,s)\in\Omega:|y-x|+|s-t|^{1/2}<a(y_0-x_0),\ x_0<y_0\}.$$

We occasionally truncate the cone Γ at the height r:

$$\Gamma_a^r(x_0,x,t)=\{(y_0,y,s)\in\Omega:|y-x|+|s-t|^{1/2}<a(y_0-x_0),\ x_0<y_0<x_0+r\}.$$

Definition 2.22 (nontangential maximal function). For a function $u:\Omega\to\mathbb{R}$, the *nontangential maximal function* $N_a(u):\partial\Omega\to\mathbb{R}$ and its truncated version at a height r are defined as

$$\begin{aligned} N_a(u)(x_0,x,t)&=\sup_{(y_0,y,s)\in\Gamma_a(x_0,x,t)}|u(y_0,y,s)|,\\ N_a^r(u)(x_0,x,t)&=\sup_{(y_0,y,s)\in\Gamma_a^r(x_0,x,t)}|u(y_0,y,s)|\quad\text{for }(x_0,x,t)\in\partial\Omega. \end{aligned} \tag{2-40}$$

The following p-adapted square function was introduced in [Dindoš et al. 2007] and has been modified appropriately for the parabolic setting. It is used to control the spatial derivatives of the solution. When

$p = 2$ it is equivalent to the usual square function and when $p < 2$ we use the convention that the expression $|\nabla u|^2|u|^{p-2}$ is zero whenever ∇u vanishes.

Definition 2.23 (p-adapted square function). For a function $u : \Omega \to \mathbb{R}$, the *$p$-adapted square function* $S_{p,a}(u) : \partial\Omega \to \mathbb{R}$ and its truncated version at a height r are defined as

$$\begin{aligned} S_{p,a}(u)(Y,s) &= \left(\int_{\Gamma_a(Y,s)} |\nabla u(X,t)|^2 |u(X,t)|^{p-2}\delta(X,t)^{-n}\,\mathrm{d}X\,\mathrm{d}t\right)^{1/p}, \\ S^r_{p,a}(u)(Y,s) &= \left(\int_{\Gamma^r_a(Y,s)} |\nabla u(X,t)|^2 |u(X,t)|^{p-2}\delta(X,t)^{-n}\,\mathrm{d}X\,\mathrm{d}t\right)^{1/p}. \end{aligned} \tag{2-41}$$

By applying Fubini we also have

$$\|S_{p,a}(u)\|^p_{L^p(\partial U)} \sim \int_U |\nabla u|^2|u|^{p-2}x_0\,\mathrm{d}x_0\,\mathrm{d}x\,\mathrm{d}t. \tag{2-42}$$

It is not known a priori if these integrals are locally integrable even for $p > 2$. However, Theorem 4.1 shows that these expressions make sense and are finite for solutions to (1-1).

We also need a p-adapted version of an object called the area function, which was introduced in [Dindoš and Hwang 2018] and is used to control the solution in the time variable. Again when $p = 2$ this is just the area function of that work.

Definition 2.24 (p-adapted area function). For a function $u : \Omega \to \mathbb{R}$, the *$p$-adapted area function* $A_{p,a}(u) : \partial\Omega \to \mathbb{R}$ and its truncated version at a height r are defined as

$$\begin{aligned} A_{p,a}(u)(Y,s) &= \left(\int_{\Gamma_a(Y,s)} |u_t|^2 |u(X,t)|^{p-2}\delta(X,t)^{2-n}\,\mathrm{d}X\,\mathrm{d}t\right)^{1/p}, \\ A^r_{p,a}(u)(Y,s) &= \left(\int_{\Gamma^r_a(Y,s)} |u_t|^2 |u(X,t)|^{p-2}\delta(X,t)^{2-n}\,\mathrm{d}X\,\mathrm{d}t\right)^{1/p}. \end{aligned} \tag{2-43}$$

Also by Fubini

$$\|A_{p,a}(u)\|^p_{L^p(\partial U)} \sim \int_U |u_t|^2|u|^{p-2}x_0^3\,\mathrm{d}x_0\,\mathrm{d}x\,\mathrm{d}t. \tag{2-44}$$

As before, it is not known a priori if these expressions are finite for solutions to (1-1) but in Lemma 4.5 we establish control of $A_{p,a}$ by $S_{p,2a}$ and use the finiteness of $S_{p,a}$ from Theorem 4.1.

2E. ***The L^p solvability of the Dirichlet boundary value problem.*** We are now in the position to define the L^p Dirichlet boundary value problem and our main results.

Definition 2.25 [Aronson 1968]. We say that u is a *weak solution* to a parabolic operator of the form (1-1) in Ω if $u, \nabla u \in L^2_{\mathrm{loc}}(\Omega)$, $\sup_t \|u(\cdot,t)\|_{L^2_{\mathrm{loc}}(\Omega_t)} < \infty$ and

$$\int_\Omega (-u\phi_t + A\nabla u\cdot\nabla\phi - \phi B\cdot\nabla u)\,\mathrm{d}X\,\mathrm{d}t = 0$$

for all $\phi \in C_0^\infty(\Omega)$.

Definition 2.26. We say that the L^p *Dirichlet problem* with boundary data in $L^p(\partial\Omega,\, d\sigma)$ is solvable if the unique solution u to (1-1) for any continuous boundary data f decaying to 0 as $t \to \pm\infty$ satisfies the nontangential maximum function estimate

$$\|N(u)\|_{L^p(\partial\Omega,\, d\sigma)} \lesssim \|f\|_{L^p(\partial\Omega,\, d\sigma)}, \tag{2-45}$$

with the implied constant depending only on the operator, n, p and Ω.

Remark 2.27. Since the space $C_0(\Omega)$ is dense in $L^p(\Omega)$ for $p < \infty$ it follows that the solution operator $f \mapsto u$ has a unique extension onto the whole space $L^p(\Omega)$ with the bound (2-45) being satisfied for such u. Hence we can assign to every boundary datum $f \in L^p(\Omega)$ a unique solution u such that (2-45) holds.

3. Basic results and interior estimates

In this section, we now recall some foundational estimates that will be used later. First, we state interior estimates of a weak solution of the parabolic operator

$$u_t = \operatorname{div}(A\nabla u) + B \cdot \nabla u. \tag{3-1}$$

Definition 3.1 [Aronson 1968]. We say that u is a *weak solution* to a parabolic operator of the form (3-1) in Ω if $u, \nabla u \in L^2_{\mathrm{loc}}(\Omega)$, $\sup_t \|u(\,\cdot\,, t)\|_{L^2_{\mathrm{loc}}(\Omega_t)} < \infty$ and

$$\int_\Omega (-u\phi_t + A\nabla u \cdot \nabla\phi - \phi B \cdot \nabla u)\, dX\, dt = 0$$

for all $\phi \in C_0^\infty(\Omega)$.

Lemma 3.2 (a Caccioppoli inequality, see [Aronson 1968]). *Let A and B satisfy* (1-2) *and* (2-35) *and suppose that u is a weak solution of* (3-1) *in $Q_{4r}(X, t)$ with $0 < r < \delta(X, t)/8$. Then there exists a constant $C = C(\lambda, \Lambda, n)$ such that*

$$\begin{aligned} r^n\Big(\sup_{Q_{r/2}(X,t)} u\Big)^2 &\le C \sup_{t-r^2\le s\le t+r^2} \int_{Q_r(X,t)\cap\{t=s\}} u^2(Y, s)\, dY + C\int_{Q_r(X,t)} |\nabla u|^2\, dY\, ds \\ &\le \frac{C^2}{r^2}\int_{Q_{2r}(X,t)} u^2(Y, s)\, dY\, ds. \end{aligned}$$

Lemmas 3.4 and 3.5 in [Hofmann and Lewis 2001] give the following estimates for weak solutions of (3-1).

Lemma 3.3 (interior Hölder continuity). *Let A and B satisfy* (1-2) *and* (2-35) *and suppose that u is a weak solution of* (3-1) *in $Q_{4r}(X, t)$ with $0 < r < \delta(X, t)/8$. Then for any $(Y, s), (Z, \tau) \in Q_{2r}(X, t)$*

$$|u(Y, s) - u(Z, \tau)| \le C\left(\frac{|Y - Z| + |s - \tau|^{1/2}}{r}\right)^{\alpha} \sup_{Q_{4r}(X,t)} |u|,$$

where $C = C(\lambda, \Lambda, n)$, $\alpha = \alpha(\lambda, \Lambda, n)$, and $0 < \alpha < 1$.

Lemma 3.4 (Harnack inequality). *Let A and B satisfy* (1-2) *and* (2-35) *and suppose that u is a weak nonnegative solution of* (3-1) *in $Q_{4r}(X,t)$, with $0<r<\delta(X,t)/8$. Suppose that $(Y,s),(Z,\tau)\in Q_{2r}(X,t)$. Then there exists $C=C(\lambda,\Lambda,n)$ such that, for $\tau<s$,*

$$u(Z,\tau)\le u(Y,s)\exp\left[C\left(\frac{|Y-Z|^2}{|s-\tau|}+1\right)\right].$$

We state a version of the maximum principle from [Dindoš and Hwang 2018] that is a modification of [Hofmann and Lewis 2001, Lemma 3.38].

Lemma 3.5 (maximum principle). *Let A and B satisfy* (1-2) *and* (2-35), *and let u and v be bounded continuous weak solutions to* (3-1) *in Ω. If $|u|,|v|\to 0$ uniformly as $t\to-\infty$ and*

$$\limsup_{(Y,s)\to(X,t)}(u-v)(Y,s)\le 0$$

for all $(X,t)\in\partial\Omega$, then $u\le v$ in Ω.

Remark 3.6 [Dindoš and Hwang 2018]. The proof of Lemma 3.38 from [Hofmann and Lewis 2001] works given the assumption that $|u|,|v|\to 0$ uniformly as $t\to-\infty$. Even with this additional assumption, the lemma as stated is sufficient for our purposes. We shall mostly use it when $u\le v$ on the boundary of $\Omega\cap\{t\ge\tau\}$ for a given time τ. Obviously then the assumption that $|u|,|v|\to 0$ uniformly as $t\to-\infty$ is not necessary. Another case when the lemma as stated here applies is when $u|_{\partial\Omega},v|_{\partial\Omega}\in C_0(\partial\Omega)$, where $C_0(\partial\Omega)$ denotes the class of continuous functions decaying to zero as $t\to\pm\infty$. This class is dense in any $L^p(\partial\Omega,\,d\sigma)$, $1<p<\infty$, allowing us to consider an extension of the solution operator from $C_0(\partial\Omega)$ to L^p.

The following result is from [Dindoš and Hwang 2018], which was adapted from the elliptic result in [Dindoš 2002].

Lemma 3.7. *Let $r>0$ and $0<a<b$. Consider the nontangential maximal functions defined using two set of cones Γ_a^r and Γ_b^r. Then for any $p>0$ there exists a constant $C_p>0$ such that for all $u:U\to\mathbb{R}$*

$$N_a^r(u)\le N_b^r(u)\quad\text{and}\quad \|N_b^r(u)\|_{L^p(\partial U)}\le C_p\|N_a^r(u)\|_{L^p(\partial U)}.$$

4. Improved regularity for p-adapted square function

Here we extend recent work of [Dindoš and Pipher 2019] for complex-coefficient elliptic equations to the real parabolic setting. The goal is to obtain an improved regularity result for weak solutions of (1-1) implying that $|\nabla u|^2|u|^{p-2}$ belongs to $L^1_{\mathrm{loc}}(\Omega)$ when $1<p<2$. Having this it follows that the p-adapted square function $S_{p,a}$ is well-defined at almost every boundary point.

Theorem 4.1 (see [Dindoš and Pipher 2019, Theorem 1.1]). *Suppose $u\in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a weak solution to $\mathcal{L}u=u_t$, where $\mathcal{L}u=\operatorname{div}(A\nabla u)+B\nabla u$, A is bounded and elliptic and B is locally bounded and satisfies*

$$\delta(X,t)|B(X,t)|\le K \tag{4-1}$$

for some uniform constant $K > 0$. Then for any parabolic ball $B_{4r}(X, t) \subset \Omega$ and $p, q \in (1, \infty)$ we have the following improvement in regularity:

$$\left(\fint_{B_r(X,t)} |u|^p\right)^{1/p} \le C_\varepsilon \left(\fint_{B_{2r}(X,t)} |u|^q\right)^{1/q} + \varepsilon \left(\fint_{B_{2r}(X,t)} |u|^2\right)^{1/2}. \tag{4-2}$$

Here the constant C_ε only depends on $p, q, \varepsilon, n, \lambda, \Lambda$, and K but not on u, (X, t) or r. In addition, for all $1 < p < \infty$

$$r^2 \fint_{B_r(X,t)} |\nabla u|^2 |u|^{p-2} \le C_\varepsilon \fint_{B_{2r}(X,t)} |u|^p + \varepsilon \left(\fint_{B_{2r}(X,t)} |u|^2\right)^{p/2}, \tag{4-3}$$

where again the constant only depends on ε, p, n, the ellipticity constants of A, and K. This also shows that $|u|^{(p-2)/2}\nabla u \in L^2_{\mathrm{loc}}(\Omega)$.

Remark 4.2. If $q \ge 2$ in (4-2) or if $p \ge 2$ in (4-3) then one can take $\varepsilon = 0$ because the L^2 averages can be controlled by the first term on the right-hand side of these inequalities.

We focus only on the case $1 < p < 2$ as the $p \ge 2$ result above follows from the Caccioppoli inequality, Lemma 3.2. We shall establish the following lemma for the $1 < p < 2$ case, which concludes the proof of Theorem 4.1.

Lemma 4.3 (see [Dindoš and Pipher 2019, Lemma 2.7]). *Let u be a weak solution to $\mathcal{L}u = u_t$ in Ω for A elliptic and bounded, and B bounded satisfying* (4-1). *Then for any $p < 2$, any ball $B_r(X, t)$ with $r < \delta(X, t)/4$, and any $\varepsilon > 0$*

$$r^2 \int_{B_r(X,t)} |\nabla u|^2 |u|^{p-2} \le C_\varepsilon \fint_{B_{2r}(X,t)} |u|^p + \varepsilon \left(\fint_{B_{2r}(X,t)} |u|^2\right)^{p/2}, \tag{4-4}$$

$$\left(\fint_{B_r(X,t)} |u|^2\right)^{1/2} \le C_\varepsilon \left(\fint_{B_{2r}(X,t)} |u|^p\right)^{1/p} + \varepsilon \left(\fint_{B_{2r}(X,t)} |u|^2\right)^{1/2}, \tag{4-5}$$

where the constants only depend on n, ε, λ, Λ and K. In particular, $|u|^{(p-2)/2}\nabla u \in L^2_{\mathrm{loc}}(\Omega)$.

Proof. We start by assuming that A and B are smooth. Then the solution u to $\mathcal{L}u = u_t$ is smooth. We prove the above inequalities have constants that do not depend on the smoothness of A or B. It follows then that the smoothness assumption can be removed by a limiting argument; that is, A and B are approximated by sequences of smooth functions for which (4-4) and (4-5) hold uniformly. This is done in detail in the elliptic setting in [Dindoš and Pipher 2019, Lemma 2.7] and a similar argument in the parabolic case is shown in [Hofmann and Lewis 2001]. We skip further details as the argument is fairly standard.

To simplify notation, we suppress the argument of the ball $B_r(X, t)$. Let

$$\rho_\delta(s) = \begin{cases} \delta^{(p-2)/2} & 0 \le s \le \delta, \\ s^{(p-2)/2} & s > \delta. \end{cases} \tag{4-6}$$

The choice of cut-off function ρ_δ in this proof is inspired by [Langer 1999, p. 311; Cialdea and Mazya 2005, p. 1088]. We multiply $\mathcal{L}u = u_t$ by $\rho_\delta^2(|u|)u$ and integrate by parts to obtain

$$\int_{B_r} \nabla(\rho_\delta^2(|u|)u) A \nabla u = \int_{B_r} \rho_\delta^2(|u|) u u_t + \int_{B_r} \rho_\delta^2(|u|) B \cdot \nabla u + \int_{\partial B_r} (\rho_\delta^2(|u|)) \nu \cdot A \nabla u \, d\sigma(y, s), \tag{4-7}$$

where ν is the outer unit normal to B_r. Consider $E_\delta = \{u > \delta\}$. Then the left-hand side of (4-7) is

$$\int_{B_r} \nabla(\rho_\delta^2(|u|)u) A\nabla u = \delta^{p-2} \int_{B_r \setminus E_\delta} \nabla u \cdot A\nabla u + \int_{B_r \cap E_\delta} A\nabla u \cdot \nabla(|u|^{p-2}u) \tag{4-8}$$

and by the ellipticity of A on the open set $B_r \cap E_\delta$ we have for some $\lambda' > 0$

$$\lambda' \int_{B_r \cap E_\delta} |u|^{p-2} |\nabla u|^2 \le \int_{B_r \cap E_\delta} A\nabla u \cdot \nabla(|u|^{p-2}u). \tag{4-9}$$

Our strategy is to let $\delta \to 0$ and show all the integrals involving $B_r \setminus E_\delta$ tend to 0.

First, we use the following result from [Langer 1999]. They proved if $u \in C^2(\bar{B}_r)$ and $u = 0$ on ∂B_r then for $q > -1$

$$\lim_{\delta \to 0} \delta^q \int_{B_r \setminus E_\delta} |\nabla u|^2 = 0. \tag{4-10}$$

To deal with the boundary integral in (4-7) we note that (4-7) to (4-9) remain valid for any enlarged ball $B_{\alpha r}$ for $1 \le \alpha \le \frac{5}{4}$. We write (4-7) for every $B_{\alpha r}$ and then average in α over the interval $\left[1, \frac{5}{4}\right]$. The last term in (4-7) then turns into a solid integral over $B_{5r/4} \setminus B_r$. Therefore,

$$\begin{aligned}
&\lambda' \int_{B_r \cap E_\delta} |u|^{p-2} |\nabla u|^2 \\
&\quad \le \sup_{\alpha \in [1,5/4]} \left| \int_{B_{\alpha r}} \rho_\delta^2(|u|) u u_t \right| + \sup_{\alpha \in [1,5/4]} \left| \int_{B_{\alpha r}} \rho_\delta^2(|u|) u B \cdot \nabla u \right| + \left| r^{-1} \int_{B_{5\alpha r/4} \setminus B_r} \rho_\delta^2(|u|) u \nu \cdot A\nabla u \right| + o(1) \\
&\quad = I + II + III + o(1),
\end{aligned}$$

where $o(1)$ contains the integral over $B_{\alpha r} \setminus E_\delta$, which tends to 0 as $\delta \to 0$. We bound II and III as [Dindoš and Pipher 2019]

$$II + III \le C_\varepsilon r^{-2} \int_{B_{5r/4}} |u|^p + \varepsilon r^{p-2} \int_{B_{5r/4}} |\nabla u|^p + o(1).$$

Now we turn to I and use the same idea as the proof of (4-10) in [Langer 1999, (3.3)] to show I converges as expected. By splitting the integral with the set E_δ, using the fact $\delta^{p-2} \le |u|^{p-2}$ on $B_{\alpha r} \setminus E_\delta$ (since $p < 2$), and the smoothness of u, which implies $|u|^{p-2} u u_t \in L^1(B_{\alpha r})$, we obtain

$$\begin{aligned}
\int_{B_{\alpha r}} \rho_\delta^2(|u|) u u_t &= \int_{B_{\alpha r} \cap E_\delta} |u|^{p-2} u u_t + \delta^{p-2} \int_{B_{\alpha r} \setminus E_\delta} u u_t \\
&\le \int_{B_{\alpha r} \cap E_\delta} |u|^{p-2} u u_t + \int_{B_{\alpha r} \setminus E_\delta} |u|^{p-2} u u_t \\
&\le \int_{B_{\alpha r}} |u|^{p-1} |u_t| < \infty.
\end{aligned}$$

Therefore by the dominated convergence theorem

$$\int_{B_{\alpha r}} \rho_\delta^2(|u|) u u_t \to \int_{B_{\alpha r}} |u|^{p-2} u u_t. \tag{4-11}$$

We change from working with balls to integrating over parabolic cubes $Q_{\alpha r}$ and denote by $Q_{\alpha r}|_s$ the cube $Q_{\alpha r}$ restricted to the hypersurface $\{t = s\}$. Using the fundamental theorem of calculus, we obtain in the limit that

$$\begin{aligned}\int_{B_{\alpha r}} |u|^{p-2} u u_t &\sim \int_{B_{\alpha r}} \frac{\partial}{\partial t}(|u|^p)\,\mathrm{d}t\,\mathrm{d}X \\ &\le \int_{Q_{\alpha r}} \frac{\partial}{\partial t}(|u|^p)\,\mathrm{d}t\,\mathrm{d}X = \int_{t_0-(\alpha r)^2}^{t_0+(\alpha r)^2} \frac{\mathrm{d}}{\mathrm{d}t}\int_{Q_{\alpha r}|_s} |u|^p\,\mathrm{d}X\,\mathrm{d}s \\ &\le \|u\|^p_{L^p_X(Q_{\alpha r}|_{t_0+(\alpha r)^2})} + \|u\|^p_{L^p_X(Q_{\alpha r}|_{t_0-(\alpha r)^2})}. \end{aligned} \tag{4-12}$$

Observe that (4-12) holds for all time-restricted cubes $Q_{\alpha r}|_{t_0 \pm (\alpha r)^2}$ with $\alpha \in [1, 1.1]$. Once again we average over these cubes to show

$$\int_{B_{\alpha r}} |u|^{p-2} u u_t \lesssim \frac{1}{r^2}\int_{Q_{1.1\alpha r}} |u|^p\,\mathrm{d}X\,\mathrm{d}t.$$

Since $Q_{1.1\alpha r} \subset B_{2r}$, in the limit as $\delta \to 0$

$$I \lesssim \frac{1}{r^2}\int_{B_{2r}} |u|^p\,\mathrm{d}X\,\mathrm{d}t.$$

Therefore grouping the estimates we have the bound

$$\lambda' \int_{B_r \cap E_\delta} |u|^{p-2}|\nabla u|^2 \lesssim C_\varepsilon r^{-2}\int_{B_{2r}} |u|^p + \varepsilon r^{p-2}\int_{B_{5r/4}} |\nabla u|^p + o(1). \tag{4-13}$$

We let $\delta \to 0$ and proceed as [Dindoš and Pipher 2019] to obtain (4-4) and (4-5) for smooth A and B. Finally, since no constants depend on the smoothness of A or B, we can remove the smoothness assumption by the same argument as in [Hofmann and Lewis 2001]. We suppose A is just elliptic and bounded, and B satisfies (4-1). Then we approximate A and B by smooth matrices and vectors respectively. For each smooth approximation, we have (4-4) and (4-5) and then passing to the limit we obtain analogous estimates for $W^{1,2}_{\mathrm{loc}}$ solutions u of $\mathcal{L}u = u_t$, with the constants having the same dependence as before. □

It follows that the p-adapted square function $S_{p,a}$ is well-defined. The paper [Dindoš and Hwang 2018] also considered an area function and established in its Lemma 5.2 that the usual square function can control this area function. The case $1 < p < 2$ is significantly more complicated so for this reason we focus only on nonnegative solutions u.

We fix a boundary point $(Y, s) \in \partial\Omega$ and consider $A_{p,a}(Y, s)$. Clearly, the nontangential cone $\Gamma_a(Y, s)$ can be covered by a nonoverlapping collection of Whitney cubes $\{Q_i\}$ with the properties:

$$\Gamma_a(Y,s) \subset \bigcup_i Q_i \subset \Gamma_{2a}(Y,s), \quad r_i := \mathrm{diam}(Q_i) \sim \mathrm{dist}(Q_i, \partial\Omega), \quad 4Q_i \subset \Omega, \tag{4-14}$$

and the cubes $\{2Q_i\}$ having only finite overlap. It follows that

$$\begin{aligned}[A_{p,a}(Y,s)]^p &\lesssim \sum_i (r_i)^{2-n}\int_{Q_i} |u_t|^2 u^{p-2}\,\mathrm{d}X\,\mathrm{d}t \\ &\lesssim \sum_i (r_i)^{2-n}\int_{Q_i} |\nabla^2 u|^2 u^{p-2} + (|\nabla A|^2 + |B|^2)|\nabla u|^2 u^{p-2}\,\mathrm{d}X\,\mathrm{d}t.\end{aligned} \tag{4-15}$$

We need the following estimate on each Q_i.

Lemma 4.4. *Assume the ellipticity condition* (1-2) *and that the coefficients A and B of* (1-1) *satisfy the conditions*

$$|\nabla A(X,t)| \le K/\delta(X,t) \quad \text{and} \quad |B(X,t)| \le K/\delta(X,t),$$

for some uniform constant $K > 0$. Then for all nonnegative solutions u of (1-1) *and any parabolic cube Q such that $4Q \subset \Omega$ we have the estimate*

$$\int_Q |\nabla^2 u|^2 u^{p-2}\,\mathrm{d}X\,\mathrm{d}t \lesssim r^{-2} \int_{2Q} |\nabla u|^2 u^{p-2}\,\mathrm{d}X\,\mathrm{d}t, \tag{4-16}$$

where $r = \operatorname{diam}(Q)$.

Proof. Since we assume differentiability of the matrix A in the spatial variables, we may also assume that A is symmetric. Let us set $W = (w_k)$, where $w_k = \partial_k u$ for $k = 0, 1, \dots, n-1$. Differentiating (1-1) we obtain the following PDE for each w_k:

$$(w_k)_t - \operatorname{div}(A\nabla w_k) = \operatorname{div}((\partial_k A)W) + \partial_k(B\cdot W). \tag{4-17}$$

We multiply (4-17) by $w_k u^{p-2}\zeta^2$, integrate over $2Q$ and integrate by parts. Here $0 \le \zeta \le 1$ is a smooth cut-off function equal to 1 on Q, vanishing outside $2Q$ and satisfying $r|\nabla\zeta| + r^2|\zeta_t| \le C$ for some $C > 0$ independent of Q. This gives

$$\begin{aligned}\int_{2Q} (w_k)_t w_k u^{p-2}\zeta^2\,\mathrm{d}X\,\mathrm{d}t &+ \int_{2Q} a_{ij}(\partial_j w_k)\partial_i(w_k u^{p-2}\zeta^2)\,\mathrm{d}X\,\mathrm{d}t \\ &= -\int_{2Q} (\partial_k a_{ij}) w_j \partial_i(w_k u^{p-2}\zeta^2)\,\mathrm{d}X\,\mathrm{d}t - \int_{2Q} b_i w_i \partial_k(w_k u^{p-2}\zeta^2)\,\mathrm{d}X\,\mathrm{d}t. \end{aligned}\tag{4-18}$$

We rearrange and group similar terms together:

$$\begin{aligned}
&\frac{1}{2}\int_{2Q} [(w_k u^{p/2-1}\zeta)^2]_t\,\mathrm{d}X\,\mathrm{d}t - \frac{p-2}{2}\int_{2Q} w_k^2 u^{p-3} u_t \zeta^2\,\mathrm{d}X\,\mathrm{d}t \\
&\quad + \int_{2Q} A(\nabla(w_k\zeta)u^{p/2-1})\cdot(\nabla(w_k\zeta)u^{p/2-1})\,\mathrm{d}X\,\mathrm{d}t \\
&\quad + (p-2)\int_{2Q} A(\nabla(w_k\zeta)u^{p/2-1})\cdot((\nabla u)w_k u^{p/2-2}\zeta)\,\mathrm{d}X\,\mathrm{d}t \\
&= \int_{2Q} |w_k|^2 u^{p-2}\zeta\zeta_t\,\mathrm{d}X\,\mathrm{d}t + \int_{2Q} |w_k|^2 u^{p-2} A\nabla\zeta\cdot\nabla\zeta\,\mathrm{d}X\,\mathrm{d}t - \int_{2Q} b_i w_i \partial_k(w_k\zeta)u^{p-2}\zeta\,\mathrm{d}X\,\mathrm{d}t \\
&\quad - (p-2)\int_{2Q} b_i w_i((\partial_k u)w_k u^{p/2-2}\zeta)u^{p/2-1}\zeta\,\mathrm{d}X\,\mathrm{d}t \\
&\quad - \int_{2Q} b_i w_i w_k u^{p-2}\zeta\zeta_k\,\mathrm{d}X\,\mathrm{d}t - \int_{2Q} (\partial_k a_{ij}) w_j w_k u^{p-2}\zeta\zeta_i\,\mathrm{d}X\,\mathrm{d}t \\
&\quad - \int_{2Q} (\partial_k a_{ij}) w_j (\partial_i w_k\zeta) u^{p-2}\zeta\,\mathrm{d}X\,\mathrm{d}t \\
&\quad - (p-2)\int_{2Q} (\partial_k a_{ij}) w_j((\partial_i u)w_k u^{p/2-2}\zeta)u^{p/2-1}\zeta\,\mathrm{d}X\,\mathrm{d}t.
\end{aligned}\tag{4-19}$$

All the terms after the equal sign are "error" terms since they either contain a derivative of ζ, or coefficients ∇A or B. These will be handled using the Cauchy–Schwarz inequality and the estimates for $|\nabla A|, |B| \le K/r$. The four main terms are on the left-hand side of (4-19). The term that needs further work is the second term, and we use the PDE (1-1) for u_t. This gives

$$-\frac{p-2}{2}\int_{2Q} w_k^2 u^{p-3} u_t \zeta^2 \,\mathrm{d}X\,\mathrm{d}t = -\frac{p-2}{2}\int_{2Q} w_k^2 u^{p-3}\operatorname{div}(A\nabla u)\zeta^2 \,\mathrm{d}X\,\mathrm{d}t - \frac{p-2}{2}\int_{2Q} w_k^2 u^{p-3} B\cdot W\zeta^2 \,\mathrm{d}X\,\mathrm{d}t. \tag{4-20}$$

Again the second term will be an "error" term. For the first term, we observe the equality

$$u^{p-3}\operatorname{div}(A\nabla u) = \operatorname{div}(A(\nabla u)u^{p-3}) - (p-3)A((\nabla u)u^{p/2-2})\cdot((\nabla u)u^{p/2-2}).$$

It follows (by integrating by parts) that

$$\begin{aligned}-\frac{p-2}{2}\int_{2Q} w_k^2 u^{p-3}\operatorname{div}(A\nabla u)\zeta^2 \,\mathrm{d}X\,\mathrm{d}t &= (p-2)\int_{2Q} A(\nabla(w_k\zeta)u^{p/2-1})\cdot((\nabla u)w_k u^{p/2-2}\zeta)\,\mathrm{d}X\,\mathrm{d}t \\ &\quad + \frac{(2-p)(3-p)}{2}\int_{2Q} A((\nabla u)w_k u^{p/2-2}\zeta)\cdot((\nabla u)w_k u^{p/2-2}\zeta)\,\mathrm{d}X\,\mathrm{d}t. \end{aligned}\tag{4-21}$$

We now group all main terms together; these are the first, second and fourth terms on the left-hand side of (4-19) and the terms of (4-21). This gives

$$\begin{aligned}\text{LHS of (4-19)} &= \frac{1}{2}\int_{2Q}[(w_k u^{p/2-1}\zeta)^2]_t \,\mathrm{d}X\,\mathrm{d}t \\ &\quad + \int_{2Q} A(\nabla(w_k\zeta)u^{p/2-1})\cdot(\nabla(w_k\zeta)u^{p/2-1})\,\mathrm{d}X\,\mathrm{d}t \\ &\quad + 2(p-2)\int_{2Q} A(\nabla(w_k\zeta)u^{p/2-1})\cdot((\nabla u)w_k u^{p/2-2}\zeta)\,\mathrm{d}X\,\mathrm{d}t \\ &\quad + \frac{(2-p)(3-p)}{2}\int_{2Q} A((\nabla u)w_k u^{p/2-2}\zeta)\cdot((\nabla u)w_k u^{p/2-2}\zeta)\,\mathrm{d}X\,\mathrm{d}t \\ &= \frac{1}{2}\int_{2Q}[(w_k u^{p/2-1}\zeta)^2]_t \,\mathrm{d}X\,\mathrm{d}t \\ &\quad + \left(1 - \frac{2(2-p)}{3-p}\right)\int_{2Q} A(\nabla(w_k\zeta)u^{p/2-1})\cdot(\nabla(w_k\zeta)u^{p/2-1})\,\mathrm{d}X\,\mathrm{d}t \\ &\quad + \int_{2Q} A\left(\sqrt{\frac{2(2-p)}{3-p}}[\nabla(w_k\zeta)u^{p/2-1}] - \sqrt{\frac{(2-p)(3-p)}{2}}[(\nabla u)w_k u^{p/2-2}\zeta]\right) \\ &\qquad \cdot\left(\sqrt{\frac{2(2-p)}{3-p}}[\nabla(w_k\zeta)u^{p/2-1}] - \sqrt{\frac{(2-p)(3-p)}{2}}[(\nabla u)w_k u^{p/2-2}\zeta]\right)\mathrm{d}X\,\mathrm{d}t \\ &\ge \frac{1}{2}\int_{2Q}[(w_k u^{p/2-1}\zeta)^2]_t \,\mathrm{d}X\,\mathrm{d}t + \frac{(p-1)\lambda}{3-p}\int_{2Q}|\nabla(w_k\zeta)u^{p/2-1}|^2\,\mathrm{d}X\,\mathrm{d}t. \end{aligned}\tag{4-22}$$

Here we have first completed the square (using the symmetry of A), and then used the ellipticity of the matrix A. The important point is that for all $1 < p < 2$ the coefficient $(p-1)\lambda/(3-p)$ is positive.

We also we could have completed the square differently and, instead of (4-22), obtained the estimate

$$\text{LHS of (4-19)} \geq \frac{1}{2}\int_{2Q} [(w_k u^{p/2-1}\zeta)^2]_t \,\mathrm{d}X\,\mathrm{d}t + \frac{(p-1)(2-p)\lambda}{2}\int_{2Q} |(\nabla u) w_k u^{p/2-2}\zeta|^2 \,\mathrm{d}X\,\mathrm{d}t. \tag{4-23}$$

It follows that we could average (4-22) and (4-23) and have both

$$\int_{2Q} |\nabla(w_k\zeta)u^{p/2-1}|^2 \,\mathrm{d}X\,\mathrm{d}t \quad \text{and} \quad \int_{2Q} |(\nabla u) w_k u^{p/2-2}\zeta|^2 \,\mathrm{d}X\,\mathrm{d}t$$

in the estimate with small positive constants.

Now we briefly mention how all the error terms of (4-19), (4-20) and (4-22) can be handled. Some can be immediately estimated from above by

$$r^{-2}\int_{2Q} |W|^2 u^{p-2} \,\mathrm{d}X\,\mathrm{d}t,$$

where the scaling factor r^{-2} comes from the estimates on $\nabla\zeta$, ζ_t, $|\nabla A|$ and $|B|$. For other terms (for example the third term of fourth line of (4-19) or the term on the fifth line) we use Cauchy–Schwarz. One of the terms in the product will be

$$\left(r^{-2}\int_{2Q} |W|^2 u^{p-2} \,\mathrm{d}X\,\mathrm{d}t\right)^{1/2},$$

while the other term is one of

$$\left(\int_{2Q} |\nabla(w_k\zeta)u^{p/2-1}|^2 \,\mathrm{d}X\,\mathrm{d}t\right)^{1/2} \quad \text{or} \quad \left(\int_{2Q} |(\nabla u) w_k u^{p/2-2}\zeta|^2 \,\mathrm{d}X\,\mathrm{d}t\right)^{1/2}.$$

It follows using the ε-Cauchy–Schwarz inequality that we can hide these on the left-hand side of (4-19). Finally, we put everything together by summing over all k and recalling that $W = \nabla u$. This gives for some constant $\varepsilon = \varepsilon(p, \lambda, n) > 0$ with $\varepsilon \to 0$ as $p \to 1$,

$$\sup_\tau \int_{Q\cap\{t=\tau\}} |\nabla u|^2 u^{p-2} \,\mathrm{d}X + \varepsilon\int_Q |\nabla^2 u|^2 u^{p-2} \,\mathrm{d}X\,\mathrm{d}t + \varepsilon\int_Q |\nabla u|^4 u^{p-4} \,\mathrm{d}X\,\mathrm{d}t \leq Cr^{-2}\int_{2Q} |\nabla u|^2 u^{p-2} \,\mathrm{d}X\,\mathrm{d}t. \tag{4-24}$$

In particular (4-16) holds. □

After using (4-16) in (4-15) we can conclude the following.

Lemma 4.5. *Let u be a nonnegative solution of* (1-1) *with matrix A satisfying the ellipticity hypothesis and the coefficients satisfying the bound $|\nabla A|, |B| \leq K/\delta$. Then given $a > 0$ there exists a constant $C = (\Lambda, \lambda, a, K, p, n)$ such that*

$$A_{p,a}(u)(X, t) \leq CS_{p,2a}(u)(X, t). \tag{4-25}$$

From this we have the global estimate

$$\|A_{p,a}(u)\|^p_{L^p(\partial\Omega)} \leq C_2\|S_{p,a}(u)\|^p_{L^p(\partial\Omega)}. \tag{4-26}$$

As far as the proof goes, the calculations above clearly work for solutions u with a uniform bound $u \geq \varepsilon > 0$. Hence considering $v_\varepsilon = u + \varepsilon$ and then taking the limit $\varepsilon \to 0+$, using Fatou's lemma yields (4-25) for all nonnegative u, where we have used the convention that $|\nabla u|^2 u^{p-2} = 0$ whenever $u = 0$ and $\nabla u = 0$ with a similar convention for the second gradient in $A_{p,a}$.

5. Bounding the p-adapted square function by the nontangential maximum function

We slightly abuse notation and only work on a Carleson region $T(\Delta_r)$ in the upper half-space U even though we formulate the following lemmas on any admissible domain Ω. The equivalence of these formulations via the pullback map ρ is discussed in Section 2C and [Dindoš and Hwang 2018], and hence we omit the details. We start with a local bound of the p-adapted square function by the nontangential maximal function.

Lemma 5.1. *Let Ω be an admissible domain from Definition 2.10 with character (ℓ, η, N, d). Let $1 < p < 2$ and u be a nonnegative solution of* (1-1), *with the Carleson conditions* (1-7) *and* (1-8) *on the coefficients A and B. Then there exists a constant $C = C(\lambda, \Lambda, N, C_0)$ such that for any solution u with boundary data f on any ball $\Delta_r \subset \partial\Omega$ with $r \leq \min\{d/4, d/(4C_0)\}$ we have*

$$\int_{T(\Delta_r)} |\nabla u|^2 |u|^{p-2} x_0 \, \mathrm{d}x_0 \, \mathrm{d}x \, \mathrm{d}t \leq C(1 + \|\mu\|)(1 + \ell^2) \int_{\Delta_{2r}} (N^{2r}(u))^2 \, \mathrm{d}x \, \mathrm{d}t. \tag{5-1}$$

In addition, we have the following global result.

Lemma 5.2. *Let Ω be an admissible domain with smooth boundary $\partial\Omega$. Let $1 < p < 2$ and u be a weak nonnegative solution of* (1-1) *satisfying* (2-34), (2-35), (2-37) *and* (2-38) *with Dirichlet boundary data $f \in L^p(\partial\Omega)$. Then there exist positive constants C_1 and C_2 independent of u such that for small $r_0 > 0$ we have*

$$\begin{aligned} &\frac{C_1}{2} \int_0^{r_0/2} \int_{\partial\Omega} |\nabla u|^2 |u|^{p-2} x_0 \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}x_0 + \frac{2}{r_0} \int_0^{r_0} \int_{\partial\Omega} u^p(x_0, x, t) \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}x_0 \\ &\quad \leq \int_{\partial\Omega} u^p(r_0, x, t) \, \mathrm{d}x \, \mathrm{d}t + \int_{\partial\Omega} u^p(0, x, t) \, \mathrm{d}x \, \mathrm{d}t \\ &\qquad + C_2(\|\mu_1\|_{C,2r} + \|\mu_2\|_{C,2r} + \|\mu_2\|_{C,2r}^{1/2}) \int_{\partial\Omega} (N^{2r}(u))^p \, \mathrm{d}x \, \mathrm{d}t. \end{aligned} \tag{5-2}$$

Proof of Lemmas 5.1 and 5.2. Let $Q_r(y, s)$ be a parabolic cube on the boundary with $r < d$ and let ζ be a smooth cut-off function independent of the x_0-variable. As long as there is no ambiguity we suppress the argument of Q_r and extensively use the Einstein summation convention. Let ζ be supported in Q_{2r}, equal 1 in Q_r and satisfy the estimate $r|\nabla\zeta| + r^2|\zeta_t| \leq C$ for some constant C.

We start by estimating

$$\int_0^r \int_{Q_{2r}} |u|^{p-2} \frac{a_{ij}}{a_{00}} (\partial_i u)(\partial_j u) \zeta^2 x_0 \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}x_0, \tag{5-3}$$

where by ellipticity we have

$$\frac{\lambda}{\Lambda} \int_0^r \int_{Q_r} |\nabla u|^2 |u|^{p-2} x_0 \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}x_0 \leq \int_0^r \int_{Q_{2r}} |u|^{p-2} \frac{a_{ij}}{a_{00}} (\partial_i u)(\partial_j u) \zeta^2 x_0 \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}x_0.$$

Now we integrate by parts whilst noting that $\nu = (1, 0, 0, \dots, 0)$ since the domain is $\{x_0 > 0\}$:

$$\begin{aligned}
&\int_0^r \int_{Q_{2r}} |u|^{p-2} \frac{a_{ij}}{a_{00}} (\partial_i u)(\partial_j u)\zeta^2 x_0 \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}x_0 \\
&\quad = \frac{1}{p} \int_{Q_{2r}} \frac{a_{0j}}{a_{00}} \partial_j (|u(r,x,t)|^p) r \zeta^2 \, \mathrm{d}x \, \mathrm{d}t - \int_0^r \int_{Q_{2r}} \frac{1}{a_{00}} |u|^{p-2} u \partial_i (a_{ij} \partial_j u) \zeta^2 x_0 \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}x_0 \\
&\qquad - \int_0^r \int_{Q_{2r}} \partial_i \left(\frac{1}{a_{00}}\right) |u|^{p-2} u a_{ij} \partial_j u \zeta^2 x_0 \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}x_0 - 2 \int_0^r \int_{Q_{2r}} \frac{a_{ij}}{a_{00}} |u|^{p-2} u (\partial_j u) \zeta \partial_i \zeta x_0 \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}x_0 \\
&\qquad - \int_0^r \int_{Q_{2r}} \frac{a_{0j}}{a_{00}} |u|^{p-2} u (\partial_j u) \zeta^2 \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}x_0 - \int_0^r \int_{Q_{2r}} \frac{a_{ij}}{a_{00}} \partial_i (|u|^{p-2}) u (\partial_j u) \zeta^2 \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}x_0 \\
&\quad = I + II + III + IV + V + VI.
\end{aligned} \tag{5-4}$$

Our strategy is to further estimate all these terms and then group similar terms together. First consider II; we use that u is a solution to (1-1):

$$II = -\int_0^r \int_{Q_{2r}} \frac{1}{a_{00}} |u|^{p-2} u u_t \zeta^2 x_0 \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}x_0 + \int_0^r \int_{Q_{2r}} \frac{1}{a_{00}} |u|^{p-2} u b_i \partial_i u \zeta^2 x_0 \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}x_0 = II_1 + II_2.$$

Using the identity $2x_0 = \partial_0 x_0^2$ we integrate by parts in x_0 to obtain

$$\begin{aligned}
II_1 &= -\frac{1}{2} \int_0^r \int_{Q_{2r}} \frac{1}{a_{00}} |u|^{p-2} u u_t \zeta^2 \partial_0 x_0^2 \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}x_0 \\
&= -\frac{1}{2} \int_{Q_{2r}} \frac{1}{a_{00}} |u(r,x,t)|^{p-2} u(r,x,t) u_t(r,x,t) \zeta^2 r^2 \, \mathrm{d}x \, \mathrm{d}t + \frac{1}{2} \int_0^r \int_{Q_{2r}} \partial_0 \left(\frac{1}{a_{00}}\right) |u|^{p-2} u u_t \zeta^2 x_0^2 \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}x_0 \\
&\quad + \frac{p-1}{2} \int_0^r \int_{Q_{2r}} \frac{1}{a_{00}} |u|^{p-2} \partial_0 u u_t \zeta^2 x_0^2 \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}x_0 + \frac{1}{2} \int_0^r \int_{Q_{2r}} \frac{1}{a_{00}} |u|^{p-2} u \partial_0 \partial_t u \zeta^2 x_0^2 \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}x_0 \\
&= II_{11} + II_{12} + II_{13} + II_{14}.
\end{aligned}$$

Consider the boundary term II_{11} and we integrate by parts in t:

$$\begin{aligned}
II_{11} &= -\frac{1}{4} \int_{Q_{2r}} \frac{1}{a_{00}} |u(r,x,t)|^{p-2} \partial_t (u^2(r,x,t)) \zeta^2 r^2 \, \mathrm{d}x \, \mathrm{d}t \\
&= \frac{1}{4} \int_{Q_{2r}} \partial_t \left(\frac{1}{a_{00}}\right) |u(r,x,t)|^{p-2} u^2(r,x,t) \zeta^2 r^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + \frac{1}{2} \int_{Q_{2r}} \frac{1}{a_{00}} |u(r,x,t)|^{p-2} u^2(r,x,t) \zeta \zeta_t r^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\qquad + \frac{p-2}{4} \int_{Q_{2r}} \frac{1}{a_{00}} |u(r,x,t)|^{p-2} u(r,x,t) u_t(r,x,t) \zeta^2 r^2 \, \mathrm{d}x \, \mathrm{d}t \\
&= II_{111} + II_{112} + II_{113}.
\end{aligned}$$

Since $p < 2$, so $p-2 < 0$, we can absorb II_{113} into II_{11} and save II_{12} to bound later on.

Considering II_{14}, we swap the order of differentiation on $\partial_0\partial_t u$ and integrate by parts in t to show

$$\begin{aligned} II_{14} &= \frac{1}{2}\int_0^r\int_{Q_{2r}} \frac{1}{a_{00}}|u|^{p-2}u\,\partial_t\partial_0 u\,\zeta^2 x_0^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 \\ &= -\frac{1}{2}\int_0^r\int_{Q_{2r}} \partial_t\Big(\frac{1}{a_{00}}\Big)|u|^{p-2}u\,\partial_0 u\,\zeta^2 x_0^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 - \frac{p-1}{2}\int_0^r\int_{Q_{2r}} \frac{1}{a_{00}}|u|^{p-2}u_t\,\partial_0 u\,\zeta^2 x_0^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 \\ &\qquad - \int_0^r\int_{Q_{2r}} \frac{1}{a_{00}}|u|^{p-2}u\,\partial_0 u\,\zeta\,\zeta_t x_0^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 \\ &= II_{141}+II_{142}+II_{143}. \end{aligned}$$

Observe that $II_{142} = -II_{13}$ so these terms cancel. We bound II_{141} by

$$\begin{aligned} II_{141} &= \frac{1}{2}\int_0^r\int_{Q_{2r}} \frac{\partial_t a_{00}}{a_{00}^2}|u|^{p-2}u\,\partial_0 u\,\zeta^2 x_0^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 \\ &\lesssim \Big(\int_0^r\int_{Q_{2r}} |A_t|^2|u|^p x_0^3\zeta^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\Big)^{1/2}\Big(\int_0^r\int_{Q_{2r}} |\nabla u|^2|u|^{p-2}x_0\zeta^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\Big)^{1/2}. \end{aligned}$$

Two parts of II_1 we have left to bound are II_{112} and II_{143}. Both of these integrals involve $\zeta\zeta_t$ and therefore if ζ is a partition of unity, when we sum over that partition these terms sum to 0.

The terms II_2 and III are simply dealt with by

$$\begin{aligned} II_2 &\lesssim \Big(\int_0^r\int_{Q_{2r}} |B|^2|u|^p x_0\zeta^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\Big)^{1/2}\Big(\int_0^r\int_{Q_{2r}} |\nabla u|^2|u|^{p-2}x_0\zeta^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\Big)^{1/2}, \\ III &\lesssim \Big(\int_0^r\int_{Q_{2r}} |\nabla A|^2|u|^p x_0\zeta^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\Big)^{1/2}\Big(\int_0^r\int_{Q_{2r}} |\nabla u|^2|u|^{p-2}x_0\zeta^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\Big)^{1/2}. \end{aligned}$$

The integral in the term IV contains the terms $\zeta\,\partial_i\zeta$ and as before if ζ is a partition of unity then after summing this term cancels out. Therefore the terms that we have yet to estimate are I, V, and VI.

We consider V in the two cases $j=0$ and $j\neq 0$ separately. Since ζ is independent of x_0 by the fundamental theorem of calculus

$$\begin{aligned} V_{\{j=0\}} &= -\int_0^r\int_{Q_{2r}} |u|^{p-2}u(\partial_0 u)\zeta^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 = -\frac{1}{p}\int_0^r\int_{Q_{2r}} \partial_0(|u|^p\zeta^2)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 \\ &= \frac{1}{p}\int_{Q_{2r}} |u(0,x,t)|^p\zeta^2\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{p}\int_{Q_{2r}} |u(r,x,t)|^p\zeta^2\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

For the $j\neq 0$ case we use that $\partial_0 x_0 = 1$ and integrate this case by parts in x_0:

$$\begin{aligned} V_{\{j\neq 0\}} &= -\frac{1}{p}\int_0^r\int_{Q_{2r}} \frac{a_{0j}}{a_{00}}\partial_j(|u|^p)\zeta^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 \\ &= -\frac{1}{p}\int_0^r\int_{Q_{2r}} \frac{a_{0j}}{a_{00}}\partial_j(|u|^p)\zeta^2\partial_0 x_0\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 \\ &= -\frac{1}{p}\int_{Q_{2r}} \frac{a_{0j}}{a_{00}}\partial_j(|u(r,x,t)|^p)\zeta^2 r\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{p}\int_0^r\int_{Q_{2r}} \frac{a_{0j}}{a_{00}}\partial_j\partial_0(|u|^p)\zeta^2 x_0\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 \\ &\qquad + \frac{1}{p}\int_0^r\int_{Q_{2r}} \partial_0\Big(\frac{a_{0j}}{a_{00}}\Big)\partial_j(|u|^p)\zeta^2 x_0\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 \\ &= V_1+V_2+V_3. \end{aligned}$$

Since $V_1 = -I_{\{j\neq 0\}}$, they cancel out. For V_2 we integrate by parts in x_j:

$$\begin{aligned} V_2 = &-\sum_{j\neq 0}\frac{1}{p}\int_{Q_{2r}}\frac{a_{0j}}{a_{00}}\partial_0(|u(r,x,t)|^p)\zeta^2 r\,\mathrm{d}x\,\mathrm{d}t \\ &\quad -\frac{1}{p}\int_0^r\int_{Q_{2r}}\partial_j\left(\frac{a_{0j}}{a_{00}}\right)\partial_0(|u|^p)\zeta^2 x_0\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 - \frac{2}{p}\int_0^r\int_{Q_{2r}}\frac{a_{0j}}{a_{00}}\partial_0(|u|^p)\zeta\,\partial_j\zeta\, x_0\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 \\ &= V_{21}+V_{22}+V_{23}. \end{aligned}$$

The terms V_{22} and V_3 are of the same type and can be estimated as III by

$$\begin{aligned} &\left|\int_0^r\int_{Q_{2r}}\nabla\left(\frac{a_{0j}}{a_{00}}\right)\nabla(|u|^p)\zeta^2 x_0\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\right| \\ &\qquad\lesssim \int_0^r\int_{Q_{2r}}|u|^{p-1}|\nabla u|\,|\nabla A|\zeta^2 x_0\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 \\ &\qquad\lesssim \left(\int_0^r\int_{Q_{2r}}|\nabla A|^2|u|^p\zeta^2 x_0\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\right)^{1/2}\left(\int_0^r\int_{Q_{2r}}|\nabla u|^2|u|^{p-2}\zeta^2 x_0\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\right)^{1/2}. \end{aligned}$$

The final term from (5-4) to estimate is VI:

$$VI = -\int_0^r\int_{Q_{2r}}\frac{a_{ij}}{a_{00}}\partial_i(|u|^{p-2})u(\partial_j u)\zeta^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 = (2-p)\int_0^r\int_{Q_{2r}}\frac{a_{ij}}{a_{00}}|u|^{p-2}(\partial_i u)(\partial_j u)\zeta^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0$$

and since $2-p<1$ we can hide VI in the left-hand side of (5-4).

We are now at the stage where we can group all the similar terms and estimate them. There are four different types of terms:

$$\begin{aligned} J_1 &= I_{\{j=0\}} + II_{111} + V_{\{j=0\}} + V_{21}, & J_2 &= II_{12}, \\ J_3 &= II_{141} + II_2 + III + \sum_{j\neq 0} V_{22} + \sum_{j\neq 0} V_3, & J_4 &= II_{112} + II_{143} + IV + \sum_{j\neq 0} V_{23}. \end{aligned}$$

We shall use the following standard result multiple times to deal with terms containing $|\nabla A|^2$, $|A_t|$ or $|B|$; a reference for this is [Stein 1993, p. 59]. Let μ be a Carleson measure and U the upper half-space. Then for any function u we have

$$\int_U |u|^p\,\mathrm{d}\mu \le \|\mu\|_C\|N(u)\|^p_{L^p(\mathbb{R}^n)}, \tag{5-5}$$

with a local version holding on Carleson boxes as well.

First we consider J_1, which consists of boundary terms at $(0,x,t)$ and (r,x,t):

$$\begin{aligned} J_1 = &\frac{1}{p}\int_{Q_{2r}}\partial_0(|u(r,x,t)|^p)\zeta^2 r\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{4}\int_{Q_{2r}}\frac{\partial_t a_{00}}{a_{00}^2}|u(r,x,t)|^{p-2}u^2(r,x,t)\zeta^2 r^2\,\mathrm{d}x\,\mathrm{d}t \\ &+\frac{1}{p}\int_{Q_{2r}}|u(0,x,t)|^p\zeta^2\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{p}\int_{Q_{2r}}|u(r,x,t)|^p\zeta^2\,\mathrm{d}x\,\mathrm{d}t \\ &-\sum_{j\neq 0}\frac{1}{p}\int_{Q_{2r}}\frac{a_{0j}}{a_{00}}\partial_0(|u(r,x,t)|^p)\zeta^2 r\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

The second term in J_1, originating from II_{111}, has the bound

$$II_{111} = -\frac{1}{4}\int_{Q_{2r}} \frac{\partial_t a_{00}}{a_{00}^2}|u(r,x,t)|^{p-2}u^2(r,x,t)\zeta^2 r^2\,\mathrm{d}x\,\mathrm{d}t \le \frac{1}{4\lambda^2}\int_{Q_{2r}} |A_t|\,|u(r,x,t)|^p\zeta^2 r^2\,\mathrm{d}x\,\mathrm{d}t \le \frac{\|\mu_2\|_{C,2r}^{1/2}}{\lambda^2}\|N^r(u)\|_{L^p(Q_{2r})}^p.$$

For the term J_2, we have

$$\begin{aligned} J_2 &= \frac{1}{2}\int_0^r\int_{Q_{2r}} \partial_0\Big(\frac{1}{a_{00}}\Big)|u|^{p-2}uu_t\zeta^2 x_0^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 \\ &\le \frac{1}{2\lambda^2}\Big(\int_0^r\int_{Q_{2r}} |\nabla A|^2|u|^p x_0\zeta^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\Big)^{1/2}\Big(\int_0^r\int_{Q_{2r}} |u_t|^2|u|^{p-2}x_0^3\zeta^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\Big)^{1/2} \\ &\le \frac{1}{\lambda^2}(\|\mu_2\|_{C,2r}\|N^r(u)\|_{L^p(Q_{2r})}^p)^{1/2}\Big(\int_0^r\int_{Q_{2r}} |u_t|^2|u|^{p-2}x_0^3\zeta^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\Big)^{1/2}. \end{aligned}$$

With a constant $C_3 = C_3(\lambda,\Lambda,n)$ we can bound J_3 by

$$\begin{aligned} J_3 &\le C_3\Big(\int_0^r\int_{Q_{2r}}(x_0|\nabla A|^2+x_0|B|^2+x_0^3|A_t|^2)|u|^p\zeta^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\Big)^{1/2}\Big(\int_0^r\int_{Q_{2r}}|\nabla u|^2|u|^{p-2}x_0\zeta^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\Big)^{1/2} \\ &\le C_3((\|\mu_1\|_{C,2r}+\|\mu_2\|_{C,2r})\|N^r(u)\|_{L^p(Q_{2r})}^p)^{1/2}\Big(\int_0^r\int_{Q_{2r}}|\nabla u|^2|u|^{p-2}x_0\zeta^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\Big)^{1/2}. \end{aligned}$$

Finally, J_4 consists of terms of the types $\zeta\partial_t\zeta$ and $\zeta\partial_i\zeta$. Later we take ζ to be a partition of unity and so when we sum up over the partition, all the terms in J_4 sum to 0.

Therefore after all these calculations

$$\begin{aligned} &\int_0^r\int_{Q_{2r}}|u|^{p-2}\frac{a_{ij}}{a_{00}}(\partial_i u)(\partial_j u)\zeta^2 x_0\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 \\ &= J_1+J_2+J_3+J_4 \\ &\le \frac{n\Lambda}{\lambda}\int_{Q_{2r}}\partial_0(|u(r,x,t)|^p)\zeta^2 r\,\mathrm{d}x\,\mathrm{d}t+\int_{Q_{2r}}|u(0,x,t)|^p\zeta^2\,\mathrm{d}x\,\mathrm{d}t \\ &\quad -\int_{Q_{2r}}|u(r,x,t)|^p\zeta^2\,\mathrm{d}x\,\mathrm{d}t+\frac{\|\mu_2\|_{C,2r}^{1/2}}{\lambda^2}\|N^r(u)\|_{L^p(Q_{2r})}^p \\ &\quad +\frac{1}{\lambda^2}(\|\mu_2\|_{C,2r}\|N^r(u)\|_{L^p(Q_{2r})}^p)^{1/2}\Big(\int_0^r\int_{Q_{2r}}|u_t|^2|u|^{p-2}x_0^3\zeta^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\Big)^{1/2} \\ &\quad +C_3((\|\mu_1\|_{C,2r}+\|\mu_2\|_{C,2r})\|N^r(u)\|_{L^p(Q_{2r})}^p)^{1/2}\Big(\int_0^r\int_{Q_{2r}}|\nabla u|^2|u|^{p-2}x_0\zeta^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0\Big)^{1/2}+J_4. \end{aligned} \tag{5-6}$$

By assuming that Ω is smooth as well as an admissible domain (see Definition 2.10) there exists a collar neighbourhood V of $\partial\Omega$ in $\mathbb{R}^{n+1}$ such that $\Omega\cap V$ can be globally parametrised by $(0,r)\times\partial\Omega$ for some small $r>0$; see Remark 2.20 and [Dindoš and Hwang 2018] for details. Using Definition 2.10, there is a collection of charts covering $\partial\Omega$ with bounded overlap, say by M. We consider a partition of unity of these charts ζ_j, with ζ_j having the same definition, support and estimates as ζ before, and $\sum_j \zeta_j = 1$ everywhere. Therefore, when we sum (5-6) over this partition of unity the term on the left-hand side is bounded below by

$$\frac{1}{\Lambda}\int_0^r\int_{\partial\Omega} |u|^{p-2}(A\nabla u\cdot\nabla u)x_0\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0,$$

which is comparable to the truncated p-adapted square function $\|S_p^r(u)\|^p_{L^p(\partial\Omega)}$. Therefore, remembering that after summing $J_4 = 0$, for any $\varepsilon > 0$ we have

$$\begin{aligned}
\frac{\lambda}{\Lambda}\|S_p^r(u)\|^p_{L^p(\partial\Omega)} &\sim \frac{\lambda}{\Lambda}\int_0^r\int_{\partial\Omega} |u|^{p-2}|\nabla u|^2 x_0\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 \\
&\le \frac{n\Lambda}{\lambda}\int_{\partial\Omega} \partial_0(|u(r,x,t)|^p)r\,\mathrm{d}x\,\mathrm{d}t + \int_{\partial\Omega}|u(0,x,t)|^p\,\mathrm{d}x\,\mathrm{d}t - \int_{\partial\Omega}|u(r,x,t)|^p\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{M\|\mu_2\|^{1/2}_{C,2r}}{\lambda^2}\|N^r(u)\|^p_{L^p(\partial\Omega)} + \frac{\|\mu_2\|_{C,2r}}{4\varepsilon\lambda^2}\|N^r(u)\|^p_{L^p(\partial\Omega)} \\
&\quad + \varepsilon\int_0^r\int_{\partial\Omega}|u_t|^2|u|^{p-2}x_0^3\zeta^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 + C_3\frac{\|\mu_1\|_{C,2r}+\|\mu_2\|_{C,2r}}{4\varepsilon}\|N^r(u)\|^p_{L^p(\partial\Omega)} \\
&\quad + \varepsilon\int_0^r\int_{\partial\Omega}|\nabla u|^2|u|^{p-2}x_0\zeta^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0.
\end{aligned} \tag{5-7}$$

By applying Lemma 4.5 to the p-adapted area function in (5-7) we see that the p-adapted square function on the right-hand side of (5-7) is always multiplied by ε. By choosing ε small enough we can absorb this p-adapted square function into the left-hand side yielding

$$\begin{aligned}
C_1\|S_p^r(u)\|^p_{L^p(\partial\Omega)} \le \int_{\partial\Omega}\partial_0(|u(r,x,t)|^p)r\,\mathrm{d}x\,\mathrm{d}t + \int_{\partial\Omega}|u(0,x,t)|^p\,\mathrm{d}x\,\mathrm{d}t - \int_{\partial\Omega}|u(r,x,t)|^p\,\mathrm{d}x\,\mathrm{d}t \\
+ C_2(\|\mu_1\|_{C,2r}+\|\mu_2\|_{C,2r}+\|\mu_2\|^{1/2}_{C,2r})\|N^r(u)\|^p_{L^p(\partial\Omega)}.
\end{aligned} \tag{5-8}$$

We integrate (5-8) in the r-variable, average over $[0, r_0]$ and use the identity $(\partial_0|u|^p)x_0 = \partial_0(|u|^p x_0) - |u|^p$ to give

$$\begin{aligned}
C_1\int_0^{r_0}\int_{\partial\Omega}\left(x_0 - \frac{x_0^2}{r_0}\right)|\nabla u|^2|u|^{p-2}\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 + \frac{2}{r_0}\int_0^{r_0}\int_{\partial\Omega}|u(x_0,x,t)|^p\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 \\
\le \int_{\partial\Omega}|u(r_0,x,t)|^p\,\mathrm{d}x\,\mathrm{d}t + \int_{\partial\Omega}|u(0,x,t)|^p\,\mathrm{d}x\,\mathrm{d}t \\
+ C_2(\|\mu_1\|_{C,2r}+\|\mu_2\|_{C,2r}+\|\mu_2\|^{1/2}_{C,2r})\|N^r(u)\|^p_{L^p(\partial\Omega)}.
\end{aligned} \tag{5-9}$$

Finally truncating the first integral on the left-hand side to $[0, r_0/2]$ gives

$$\begin{aligned}
\frac{C_1}{2}\int_0^{r_0/2}\int_{\partial\Omega}|\nabla u|^2|u|^{p-2}x_0\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 + \frac{2}{r_0}\int_0^{r_0}\int_{\partial\Omega}|u(x_0,x,t)|^p\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}x_0 \\
\le \int_{\partial\Omega}|u(r_0,x,t)|^p\,\mathrm{d}x\,\mathrm{d}t + \int_{\partial\Omega}|u(0,x,t)|^p\,\mathrm{d}x\,\mathrm{d}t \\
+ C_2(\|\mu_1\|_{C,2r}+\|\mu_2\|_{C,2r}+\|\mu_2\|^{1/2}_{C,2r})\|N^r(u)\|^p_{L^p(\partial\Omega)}.
\end{aligned} \tag{5-10}$$

The local estimate for Lemma 5.1 is obtained (exactly as in [Dindoš and Hwang 2018]) if we do not sum over all the coordinate patches but instead use the estimates derived for a single boundary cube Q_r in (5-6). □

We need to control the first integral on the right-hand side of (5-2) to achieve our goal of controlling the p-adapted square function. Thankfully this has already been done for us in the proof of [Dindoš and Hwang 2018, Corollary 5.3], which we encapsulate below.

Lemma 5.3. *Let Ω be as in Lemma 5.2 and u be a nonnegative solution to* (1-1). *For a small $r_0 > 0$ depending on the geometry of the domain Ω, there exists a constant C such that for $\varepsilon = \|\mu_1\|_{C,2r} + \|\mu_2\|_{C,2r} + \|\mu_2\|_{C,2r}^{1/2}$*

$$\int_{\partial\Omega} u(r_0, x, t)^p \, dx \, dt \le \frac{2}{r_0}\int_0^{r_0}\int_{\partial\Omega} u(x_0, x, t)^p \, dx \, dt \, dx_0 + C\varepsilon \|N^{r_0}(u)\|^p_{L^p(\partial\Omega)}.$$

Combining Lemmas 5.2 and 5.3 gives us the desired result.

Corollary 5.4. *Let Ω be as in Lemma 5.2 and u be a nonnegative solution to* (1-1). *For a small $r_0 > 0$ depending on the geometry of the domain Ω, there exist constants $C_1, C_2 > 0$ such that for $\varepsilon = \|\mu_1\|_{C,2r} + \|\mu_2\|_{C,2r} + \|\mu_2\|_{C,2r}^{1/2}$*

$$\begin{aligned} \|S_p^{r_0/2}(u)\|^p_{L^p(\partial\Omega)} &\sim \int_0^{r_0/2}\int_{\partial\Omega} |\nabla u|^2 |u|^{p-2} x_0 \, dx \, dt \, dx_0 \\ &\le C_1 \int_{\partial\Omega} |u(0, x, t)|^p \, dx \, dt + C_2\varepsilon \|N^{r_0}(u)\|^p_{L^p(\partial\Omega)}. \end{aligned} \tag{5-11}$$

6. Bounding the nontangential maximum function by the p-adapted square function

Our goal in this section has been vastly simplified due to [Rivera-Noriega 2003] proving a local good-λ inequality. We use this to bound the nontangential maximum function by the p-adapted square function. We first bound the nontangential maximum function by the usual L^2-based square function $S_2(u)$ but a simple argument from [Dindoš et al. 2007, (3.41)] shows that for $1 < p < 2$ and any $\varepsilon > 0$ we have

$$\|S_2^r(u)\|_{L^p(\partial\Omega)} \le C_\varepsilon \|S_p^r(u)\|_{L^p(\partial\Omega)} + \varepsilon \|N^r(u)\|_{L^p(\partial\Omega)}, \tag{6-1}$$

with a local version of this statement holding as well.

The good-λ inequality from [Rivera-Noriega 2003, p. 508] is expressed in the following lemma.

Lemma 6.1. *Let v be a solution to* (2-29) *and $v(X, t) = 0$ for some point $(X, t) \in Q_r$. Let $E = \{(0, x, t) \in Q_r : S_{2,a}(v) \le \lambda\}$ and $q > 2$. Then*

$$|\{(0, x, t) \in Q_r : N_a(v) > \lambda\}| \lesssim |\{(0, x, t) \in Q_r : S_{2,a}(v) > \lambda\}| + \frac{1}{\lambda^q}\int_E S_{2,a}(v)^q \, dx \, dt. \tag{6-2}$$

If $p \ge 2$ then the following lemma is immediate from [Dindoš and Hwang 2018, Lemma 6.1], which is an adaptation of [Rivera-Noriega 2003, Theorem 1.3 and Proposition 5.3].

Lemma 6.2. *Let v be a solution to* (2-29) *in U and the coefficients of* (2-29) *satisfy the Carleson estimates* (2-34), (2-35), (2-37) *and* (2-38) *on all parabolic balls of size $\le r_0$. Then there exists a constant C such that for any $r \in (0, r_0/8)$*

$$\int_{Q_r} N_{a/12}(v)^p \,\mathrm{d}x\,\mathrm{d}t \le C\left(\int_{Q_{2r}} A_{2,a}(v)^p \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_{2r}} S_{2,a}(v)^p \,\mathrm{d}x\,\mathrm{d}t\right) + r^{n+1}|v(A_{\Delta_r})|^p, \tag{6-3}$$

where A_{Δ_r} is a corkscrew point of the boundary ball Δ_r. That is, a point $2r^2$ later in time than the centre of Δ_r and at a distance comparable to r from the boundary and r from the centre of the ball Δ_r.

Proof. We first assume that $v(X,t) = 0$ for some $(X,t) \in Q_r$ and then we have the good-λ inequality (6-2). The passage from this good-λ inequality to a local L^p estimate is standard in the spirit of [Fefferman and Stein 1972]. We remove the assumption $v(X,t) = 0$ for the cost of adding the $r^{n+1}|v(A_{\Delta_r})|^p$ term in the same way as [Rivera-Noriega 2003; Dindoš and Hwang 2018]. □

From this local estimate, we can obtain the following global L^p estimate by the same proof as the global L^2 estimate from [Dindoš and Hwang 2018, Theorem 6.3].

Theorem 6.3. *Let u be a solution to* (1-1) *and the coefficients of* (1-1) *satisfy the Carleson estimates* (2-36) *and* (2-39) *then*

$$\|N^r(u)\|_{L^p(\partial\Omega)} \lesssim \|S_2^r(u)\|_{L^p(\partial\Omega)} + \|u\|_{L^p(\partial\Omega)} \tag{6-4}$$

and by (6-1)

$$\|N^r(u)\|_{L^p(\partial\Omega)} \lesssim \|S_p^r(u)\|_{L^p(\partial\Omega)} + \|u\|_{L^p(\partial\Omega)}. \tag{6-5}$$

7. Proof of Theorem 1.1

We only consider the case $1 < p < 2$ and use interpolation to obtain solvability for $p \ge 2$. First assume either the stronger Carleson condition of (2-39), or (1-7) and (1-8) hold. Therefore the Carleson conditions on the pullback coefficients (2-34), (2-35), (2-37) and (2-38) hold.

Without loss of generality, by Remark 2.20, we may assume that our domain is smooth. Consider $f^+ = \max\{0, f\}$ and $f^- = \max\{0, -f\}$, where $f \in C_0(\partial\Omega)$, and denote the corresponding solutions with these boundary data by u^+ and u^- respectively. Hence we may apply Corollary 5.4 separately to u^+ and u^-. By the maximum principle, these two solutions are nonnegative. It follows that for any such nonnegative u we have

$$\|S_p^r(u)\|^p_{L^p(\partial\Omega)} \le C\|f\|^p_{L^p(\partial\Omega)} + C(\|\mu\|_C^{1/2} + \|\mu\|_C)\|N^{2r}(u)\|^p_{L^p(\partial\Omega)}$$

and Theorem 6.3 gives

$$\|N^r(u)\|^p_{L^p(\partial\Omega)} \le C\|f\|^p_{L^p(\partial\Omega)} + C\|S_p^{2r}(u)\|^p_{L^p(\partial\Omega)},$$

where $\|\mu\|_C$ is the Carleson norm of (1-7) on Carleson regions of size $\le r_0$. As noted earlier, if, for example, Ω is of VMO type then the size of μ appearing in this estimate will only depend on the Carleson norm of coefficients on Ω, provided we only consider small Carleson regions. Hence we can choose r_0

small enough (depending on the domain Ω) such that the Carleson norm after the pullback is only twice the original Carleson norm of the coefficients over all balls of size $\le r_0$.

Since we are assuming $\|\mu\|_C$ is small, clearly we also have $\|\mu\|_C \le C\|\mu\|_C^{1/2}$. By rearranging these two inequalities and combining estimates for u^+ and u^-, we obtain, for $0 < r \le r_0/8$,

$$\|N^r(u)\|^p_{L^p(\partial\Omega)} \le C\|f\|^p_{L^p(\partial\Omega)} + C\|\mu\|_C^{1/2}\|N^{4r}(u)\|^p_{L^p(\partial\Omega)}.$$

By a simple geometric argument in [Dindoš and Hwang 2018] involving cones of different apertures, Lemmas 3.4 and 3.7 show there exists a constant M such that

$$\|N^{4r}(u)\|^p_{L^p(\partial\Omega)} \le M\|N^r(u)\|^p_{L^p(\partial\Omega)}. \tag{7-1}$$

It follows that if $CM\|\mu\|_C^{1/2} < \frac{1}{2}$ by combining the last two inequalities we obtain

$$\|N^r(u)\|^p_{L^p(\partial\Omega)} \le 2C\|f\|^p_{L^p(\partial\Omega)},$$

which is the desired estimate (for the truncated version of nontangential maximum function). The result with the nontruncated version of the nontangential maximum function $N(u)$ follows as our domain is bounded in space and hence (7-1) can be iterated finitely many times until the nontangential cones have sufficient height to cover the whole domain.

Finally, we comment on how the Carleson condition (2-39) can be relaxed to the weaker condition (1-6). The idea is the same as [Dindoš and Hwang 2018, Theorem 3.1]. As shown there, if the operator $\mathcal{L}$ satisfies the weaker condition (1-6), then it is possible (via mollification of coefficients) to find another operator $\mathcal{L}_1$ which is a small perturbation of the operator $\mathcal{L}$ and $\mathcal{L}_1$ satisfies (2-39). The solvability of the L^p Dirichlet problem in the range $1 < p < 2$ for $\mathcal{L}_1$ follows by our previous arguments. However, as $\mathcal{L}$ is a small perturbation of the operator $\mathcal{L}_1$ we have by the perturbation argument of [Sweezy 1998] L^p solvability of $\mathcal{L}$ as well.

Finally, for larger values of p we use the maximum principle and interpolation to obtain solvability results in the full range $1 < p < \infty$. □

Appendix: proofs of results from Section 2

Proof of Theorem 2.3. We begin by proving the equivalence of (3) and (6) using ideas from [Strichartz 1980] and write $F = \mathbb{D}\phi$, where F is a tempered distribution. Let

$$\varphi^k = \chi_{\widetilde{Q}_1(0,0)} - \chi_{\widetilde{Q}_1(e_k)}.$$

Then for $1 \le k \le n-1$

$$\begin{aligned}\widehat{\varphi}^k(\xi,\tau) &= \frac{2\sin^2(\xi_k/2)}{\xi_k}\,\frac{1-e^{-i\tau}}{i\tau}\prod_{j\ne k}^{n-1}\frac{1-e^{-i\xi_j}}{i\xi_j},\\ \widehat{\varphi}^n(\xi,\tau) &= \frac{2\sin^2(\tau/2)}{\tau}\prod_{j=1}^{n-1}\frac{1-e^{-i\xi_j}}{i\xi_j},\end{aligned} \tag{A-1}$$

with $\widehat{\varphi}^k(\xi,\tau) \sim \xi_k$ for small ξ_k and $1 \le k \le n-1$. We let

$$\widehat{\psi}^u = \frac{e^{i(\xi,0)\cdot u} - 1}{\|(\xi,\tau)\|}$$

and denote by $\psi^u_\rho(x,t)$ the usual parabolic dilation by ρ, that is,

$$\psi^u_\rho(x,t) = \rho^{-(n+1)}\psi^u(x/\rho, t/\rho^2).$$

It is worth noting that $(\varphi^k * \psi^u)_\rho = \varphi^k_\rho * \psi^u_\rho$. Therefore we may rewrite (6.a), by Remark 2.4, as

$$\sup_{Q_r} \sum_{k=1}^{n-1} \frac{1}{|Q_r|} \int_{Q_r} \int_{u\in\mathbb{S}^{n-1}} \int_0^r (\psi^u_\rho * \varphi^k_\rho * F)^2 \frac{\mathrm{d}\rho}{\rho} \,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}t \sim B_{(6.\mathrm{a})}. \tag{A-2}$$

Similarly if we let

$$\widehat{\psi}^u_n = \frac{e^{i(0,\tau)\cdot u} - 1}{\|(\xi,\tau)\|} \tag{A-3}$$

then we may rewrite (6.b) as

$$\sup_{Q_r} \frac{1}{|Q_r|} \int_{Q_r} \int_{u\in\mathbb{S}^{n-1}} \int_0^r (\psi^u_{n,\rho} * F)^2 \frac{\mathrm{d}\rho}{\rho} \,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}t \sim B_{(6.\mathrm{b})}. \tag{A-4}$$

The functions $\varphi^k * \psi^u$ and ψ^u_n all satisfy the following conditions for some $\varepsilon_i > 0$:

$$\begin{aligned} &\int \psi \,\mathrm{d}x\,\mathrm{d}t = 0, \\ &|\psi(x,t)| \lesssim \|(x,t)\|^{-n-1-\varepsilon_1} \quad \text{for } \|(x,t)\| \ge a > 0, \\ &|\widehat{\psi}(\xi,\tau)| \lesssim \|(\xi,\tau)\|^{\varepsilon_2} \quad \text{for } \|(\xi,\tau)\| \le 1, \\ &|\widehat{\psi}(\xi,\tau)| \lesssim \|(\xi,\tau)\|^{-\varepsilon_3} \quad \text{for } \|(\xi,\tau)\| \ge 1. \end{aligned} \tag{A-5}$$

Therefore if $\mathbb{D}\phi = F \in \mathrm{BMO}(\mathbb{R}^n)$ then $B_{(6.\mathrm{a})} \lesssim \|\mathbb{D}\phi\|^2_*$ and $B_{(6.\mathrm{b})} \lesssim \|\mathbb{D}\phi\|^2_*$ by [Strichartz 1980, Theorem 2.1]; this shows (3) implies (6).

For the converse, we proceed via an analogue of the proof of [Strichartz 1980, Theorem 2.6]. Consider

$$\hat{\theta}(\xi,\tau) = \|(\xi,\tau)\|\hat{\zeta}(\xi,\tau),$$

where $\zeta \in C^\infty_0(\mathbb{R})$. Let H^1_{00} be the dense subclass of continuous H^1 functions g such that g and all its derivatives decay rapidly; see [Stein 1970, p. 225]. Via an analogue of [Fefferman and Stein 1972, Theorem 3; Strichartz 1980, Lemma 2.7] by assuming (6.a) and (6.b) if $g \in H^1_{00}(\mathbb{R}^n)$ then for each $1 \le k \le n-1$

$$\left| \int_{\mathbb{S}^{n-1}} \int_0^\infty \iint_{\mathbb{R}^{n-1}\times\mathbb{R}} \psi^u_\rho * \varphi^k_\rho * F(x,t)\theta_\rho * g(x,t)\,\mathrm{d}x\,\mathrm{d}t \frac{\mathrm{d}\rho}{\rho}\,\mathrm{d}u \right| \lesssim B^{1/2}_{(6.\mathrm{a})}\|g\|_{H^1}, \tag{A-6}$$

$$\left| \int_{\mathbb{S}^{n-1}} \int_0^\infty \iint_{\mathbb{R}^{n-1}\times\mathbb{R}} \psi^u_{n,\rho} * F(x,t)\theta_\rho * g(x,t)\,\mathrm{d}x\,\mathrm{d}t \frac{\mathrm{d}\rho}{\rho}\,\mathrm{d}u \right| \lesssim B^{1/2}_{(6.\mathrm{b})}\|g\|_{H^1}. \tag{A-7}$$

For $1 \le k \le n-1$ let

$$\begin{aligned} m_k(\xi,\tau) &= \int_{\mathbb{S}^{n-1}} \int_0^\infty \overline{\widehat{\psi}^u(-\rho\xi,-\rho^2\tau)\widehat{\varphi}^k(-\rho\xi,-\rho^2\tau)}\,\|(\xi,\tau)\|\zeta(\rho\|(\xi,\tau)\|)\,\mathrm{d}\rho\,\mathrm{d}u, \\ m_n(\xi,\tau) &= \int_{\mathbb{S}^{n-1}} \int_0^\infty \overline{\widehat{\psi}^u_n(-\rho\xi,-\rho^2\tau)}\,\|(\xi,\tau)\|\zeta(\rho\|(\xi,\tau)\|)\,\mathrm{d}\rho\,\mathrm{d}u. \end{aligned} \tag{A-8}$$

All of these functions m_i are homogeneous of degree zero, smooth away from the origin and the associated Fourier multipliers M_k, for $1 \le k \le n$, are Calderón–Zygmund operators that preserve the class H^1_{00} and are bounded on H^1.

The nondegeneracy condition from [Calderón and Torchinsky 1975] on the family of functions $\{m_k\}_{k=1}^n$ holds — that is, the property that $\sum_k |m_k(r\xi, r^2\tau)|^2$ does not vanish identically in r for $(\xi,\tau) \neq (0,0)$. Therefore by [Calderón and Torchinsky 1975; 1977] we can find smooth homogeneous functions $u_{k,j}(\xi,\tau)$ of degree zero and positive numbers r_j such that for all $(\xi,\tau) \neq (0,0)$

$$\sum_{k=1}^{n} \sum_{j=1}^{j_0} m_{k,r_j}(\xi,\tau) u_{k,j}(\xi,\tau) = 1, \tag{A-9}$$

where m_{k,r_j} are as m_k but with $r_j\rho$ replacing ρ in the arguments of $\widehat{\psi}^u$, $\widehat{\varphi}^k$ and $\widehat{\psi}^u_2$ in (A-8) (but not ζ).

Let $M_{k,j}$ and $U_{k,j}$ be the Fourier multiplier operators associated to their respective multipliers m_{k,r_j} and $u_{k,j}$. Then $\sum\sum M_{k,j}U_{k,j}g = g$ for all $g \in H^1_{00}$. By [Fefferman and Stein 1972, Theorem 3; Strichartz 1980, Lemma 2.7] there exists $h_{k,j} \in \mathrm{BMO}(\mathbb{R}^n)$ such that $\|h_{k,j}\|_*^2 \lesssim B_{(6.\mathrm{a})}$ or $B_{(6.\mathrm{b})}$, and $(h_{k,j}, g) = (F, M_{k,j}g)$ for all $g \in H^1_{00}$. If we replace g by $U_{j,k}g \in H^1_{00}$ in the previous identity and sum over j and k we obtain $(h, g) = (F, g)$ for all $g \in H^1_{00}$, where $h = \sum_{k,j} U^*_{k,j} h_{k,j}$; furthermore by the BMO condition on $h_{k,j}$, we have $\|h\|_*^2 \lesssim B_{(6.\mathrm{a})} + B_{(6.\mathrm{b})}$. The identity (A-9) does not need to hold at the origin; therefore $\hat{h} - \widehat{F}$ may be supported at the origin and hence $F = h + p$, where p is a polynomial. Due to the assumption $\phi \in \mathrm{Lip}\big(1, \tfrac{1}{2}\big)$, clearly F must be a tempered distribution. Hence as in [Strichartz 1980] we may conclude $F = h \in \mathrm{BMO}(\mathbb{R}^n)$. This implies equivalence of (3) and (6).

Similarly we may prove the equivalence of (4) and (5) to (3). The changes needed are outlined below.

We first look at (5) $\iff$ (3). In this instance we replace the convolutions $\varphi^k * \psi^u$ by

$$\widehat{\psi}^u_1(\xi,\tau) = \frac{e^{i(\xi,0)\cdot u} - 2 - e^{-i(\xi,0)\cdot u}}{\|(\xi,\tau)\|},$$

which corresponds to (5.a), and we keep the convolution ψ^u_n as it is in (A-3). The same proof then goes through to give that (5) holds if and only if (3) holds with equivalent norms, as in (2-13).

We now consider (4) $\iff$ (3). This case is stated in [Rivera-Noriega 2003, Proposition 3.2]. Again the proof proceeds as above with one convolution

$$\widehat{\psi}^u(\xi,\tau) = \frac{e^{i(\xi,\tau)\cdot u} - 2 - e^{-i(\xi,\tau)\cdot u}}{\|(\xi,\tau)\|}. \qquad \square$$

Proof of Theorem 2.8. Without loss of generality, we only consider the case $\eta < 1$. When $\eta \ge 1$ the existence of an extension with $\|\mathbb{D}\Phi\|_* \lesssim \eta + \ell$ requires a much simpler argument.

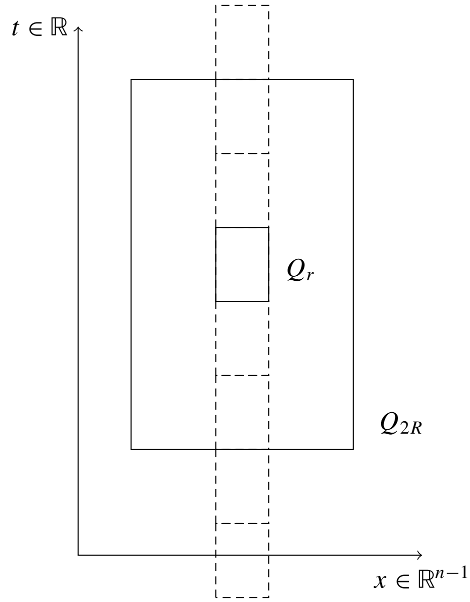

Figure 1. The reflection and tiling of the cube $Q_r \subset Q_{2R}$ defined in (A-10).

By (2-20) there exists $f \in C_\delta$ such that $\|\nabla\phi - f\|_{*,Q_{8d}} \le 2\eta$ and a scale $0 < r_0 = r_0(\delta) \le d$ such that

$$\|f\|_{*,Q_{8d},r_0} \le 2\eta.$$

Let $d' = \eta \min(r_0, r_1)/2$ and consider some $r \le d'$ and $Q_r \subset Q_{4d}$. Find a natural number k such that $R = 2^k r$ and $R\eta/2 < r \le R\eta$. By our choice of d' the cube Q_{2R}, which is an enlargement of Q_r by a factor 2^{k+1}, is still contained in the original cube Q_{8d}.

It follows that

$$\|\nabla\phi\|_{*,Q_{2R}} \lesssim \eta,$$
$$\sup_{\substack{Q_s = J_s \times I_s \\ Q_s \subset Q_{2R}}} \frac{1}{|Q_s|} \int_{Q_s}\int_{I_s} \frac{|\phi(x,t) - \phi(x,\tau)|^2}{|t-\tau|^2} \,\mathrm{d}\tau\,\mathrm{d}t\,\mathrm{d}x \le \eta^2.$$

Without loss of generality, we may now assume that the cube Q_{2R} is centred at the origin $(0,0)$ and that $\phi(0,0) = 0$, since the BMO norm is invariant under translation and ignores constants. We first define $\tilde{\phi}$ as an extension in time via reflection and tiling of the cube Q_r:

$$\tilde{\phi}(x,t) = \begin{cases} \phi(x,t), & t \in [-r^2, r^2] + 4kr^2, \\ \phi(x, 2r^2 - t), & t \in [r^2, 3r^2] + 4kr^2,\ k \in \mathbb{Z}. \end{cases} \tag{A-10}$$

See Figure 1 for an illustration of this. Clearly $\tilde{\phi}$ coincides with ϕ on Q_r.

It follows that $\tilde{\phi}$ is a function $\tilde{\phi} : \{|x|_\infty < 2R\} \times \mathbb{R} \to \mathbb{R}$ and $(\nabla\tilde{\phi})_{Q_r} = (\nabla\phi)_{Q_r}$. Consider a cut-off function ρ such that

$$\rho(x) = \begin{cases} 1 & \text{if } |x|_\infty < r, \\ 0 & \text{if } |x|_\infty > 2R, \end{cases}$$

and $|\nabla\rho| \lesssim 1/R \lesssim \eta/r$. Finally define

$$\Phi = \tilde{\phi}\rho + (1-\rho)(x \cdot (\nabla\tilde{\phi})_{Q_r}). \tag{A-11}$$

Clearly Φ is well-defined on $\mathbb{R}^{n-1} \times \mathbb{R}$ as $\rho = 0$ outside the support of $\tilde{\phi}$. We claim that Φ satisfies (i)–(iv) of Theorem 2.8, which we establish in a sequence of lemmas below. Observe also that from our definition of Φ we have

$$\nabla\Phi = (\nabla\tilde{\phi} - (\nabla\tilde{\phi})_{Q_r})\rho + \nabla\rho(\tilde{\phi} - x \cdot (\nabla\tilde{\phi})_{Q_r}) + (\nabla\tilde{\phi})_{Q_r}, \tag{A-12}$$

completing the proof. □

We start with a couple of lemmas that allow us to reduce our claim to the dyadic case; this is to make the geometry easier to handle.

Lemma A.1 ([Jones 1980, Lemma 2.3], see also [Strichartz 1980, Theorem 2.8]). *Let f be defined on $\mathbb{R}^n$ and*

$$\sup_Q \frac{1}{|Q|}\int_Q |f - f_Q| \le c(\eta), \tag{A-13}$$

where the supremum is taken over all dyadic cubes $Q \subset \mathbb{R}^n$. Further, assume that

$$\sup_{Q_1,Q_2} |f_{Q_1} - f_{Q_2}| \le c(\eta), \tag{A-14}$$

where the supremum is taken over all dyadic cubes Q_1, Q_2 of equal edge length with a touching edge. Then

$$\|f\|_* \lesssim c(\eta).$$

Below $l(Q_s) = s$ denotes the radius of a parabolic cube.

Lemma A.2 [Jones 1980, Lemma 2.1 and pp. 44-45]. *Let $f \in \mathrm{BMO}(Q)$ and $Q_0 \subset Q_1 \subset Q$. Then*

$$|f_{Q_0} - f_{Q_1}| \lesssim \log\left(2 + \frac{l(Q_1)}{l(Q_0)}\right)\|f\|_{*,Q}. \tag{A-15}$$

Furthermore, the same proof in [Jones 1980] gives the following slightly stronger result:

$$\frac{1}{|Q_0|}\int_{Q_0} |f - f_{Q_1}| \lesssim \log\left(2 + \frac{l(Q_1)}{l(Q_0)}\right)\|f\|_{*,Q}. \tag{A-16}$$

If Q_0, $Q_1 \subset Q$ and $l(Q_0) \le l(Q_1)$ but they are not necessarily nested then

$$|f_{Q_0} - f_{Q_1}| \lesssim \left(\log\left(2 + \frac{l(Q_1)}{l(Q_0)}\right) + \log\left[2 + \frac{\mathrm{dist}(Q_0, Q_1)}{l(Q_1)}\right]\right)\|f\|_{*,Q}. \tag{A-17}$$

If the cubes Q_0, Q_1 and Q are dyadic then we may replace BMO *by dyadic* BMO.

There is a typo at the top of [Jones 1980, p. 45]. It should read $l(Q_k) \le l(Q_j)$ (it currently reads the converse).

Claim A.3. *Let $\tilde{\phi}$ be defined as in* (A-10), $\|\nabla\phi\|_{*,Q_{2R}} \lesssim \eta$, *and let Q be dyadic with $r \le l(Q) \le 2R$. Then*

$$\frac{1}{|Q|}\int_Q |\nabla\tilde{\phi} - \nabla\tilde{\phi}_{Q_r}| \lesssim_\varepsilon \eta^{1-\varepsilon}. \tag{A-18}$$

Proof of claim. Let $N \in \mathbb{N}$ be such that $l(Q) = 2^N l(Q_r)$. Let $\{Q^i\}$ be the $2^{N(n-1)}$ dyadic cubes that are translations of Q_r and partition $Q \cap \{|t| \le r^2\}$. Then by Lemma A.2

$$\begin{aligned}\frac{1}{|Q|}\int_Q |\nabla\tilde{\phi} - \nabla\tilde{\phi}_{Q_r}| &= \sum_i \frac{2^{2N}|Q^i|}{|Q|}\frac{1}{|Q^i|}\int_{Q^i} |\nabla\tilde{\phi} - \nabla\tilde{\phi}_{Q_r}| \\ &\le \sum_i \frac{2^{2N}|Q^i|}{|Q|}\left(\frac{1}{|Q^i|}\int_{Q^i} |\nabla\phi - \nabla\phi_{Q^i}| + |\nabla\phi_{Q^i} - \nabla\phi_{Q_r}|\right) \\ &\lesssim (\eta + \eta\log(2+R/r)) \lesssim \eta + \eta\log(1+1/\eta) \lesssim_\varepsilon \eta^{1-\varepsilon}. \qquad \square\end{aligned}$$

Lemma A.4 [Stegenga 1976]. *Let $g, h \in L^1_{\text{loc}}$. Then*

$$\frac{1}{|Q|}\int_Q |gh - (gh)_Q| \le \frac{2}{|Q|}\int_Q |g(h-h_Q)| + \frac{|h_Q|}{|Q|}\int_Q |g - g_Q|. \tag{A-19}$$

Proof. This small reduction is from [Stegenga 1976, p. 582]. First observe

$$gh - (gh)_Q = g(h-h_Q) + h_Q(g-g_Q) + g_Q h_Q - (gh)_Q$$

and

$$|g_Q h_Q - (gh)_Q| = \left|\frac{1}{|Q|}\int_Q g h_Q - \frac{1}{|Q|}\int_Q gh\right| \le \frac{1}{|Q|}\int_Q |g(h-h_Q)|.$$

Hence

$$\left|\frac{1}{|Q|}\int_Q |gh - (gh)_Q| - \frac{|h_Q|}{|Q|}\int_Q |g - g_Q|\right| \le 2\frac{1}{|Q|}\int_Q |g(h-h_Q)|, \tag{A-20}$$

completing the proof. $\square$

We can now prove (iii) of Theorem 2.8.

Lemma A.5. *Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be defined as in* (A-11) *with $\|\nabla\phi\|_{*,Q_{2R}} \lesssim \eta$. Then $\nabla\Phi \in \mathrm{BMO}(\mathbb{R}^n)$ and for all $0 < \varepsilon < 1$*

$$\|\nabla\Phi\|_* \lesssim_\varepsilon \eta^{1-\varepsilon} + \eta\ell. \tag{A-21}$$

Proof. Recall $\nabla\Phi = (\nabla\tilde{\phi} - (\nabla\tilde{\phi})_{Q_r})\rho + \nabla\rho(\tilde{\phi} - x\cdot(\nabla\tilde{\phi})_{Q_r}) + (\nabla\tilde{\phi})_{Q_r}$; we can ignore the constant term as the BMO norm doesn't see it. Let $\psi = \nabla\tilde{\phi} - (\nabla\tilde{\phi})_{Q_r}$ and $\theta = \tilde{\phi} - x\cdot(\nabla\tilde{\phi})_{Q_r}$. We want to bound $\|\rho\psi\|_*$ and $\|\nabla\rho\theta\|_*$. We first tackle the term $\|\rho\psi\|_*$.

Step 1: (A-14) holds; that is, $\sup_{Q_1,Q_2} |(\rho\psi)_{Q_1} - (\rho\psi)_{Q_2}| \le c(\eta)$ for Q_1, Q_2 dyadic cubes of equal side length and with a touching edge.

Since $\tilde{\phi}$ is the extension in the time direction by reflection and tiling (see (A-10)), and Q_1, Q_2 and Q_r are all dyadic cubes, we may assume that if $l(Q_1) \le r$ then $Q_1, Q_2 \subset \{|t| < r^2\}$, and if $l(Q_1) > r$ then $\{|t| < r^2\} \subset Q_1$.

If $Q_1, Q_2 \subset Q_{2R}$ then $|(\rho\psi)_{Q_1} - (\rho\psi)_{Q_2}| \lesssim \|\rho\psi\|_{*,\,\text{dyadic},\,Q_{2R}}$. Therefore, if we show (A-13) for $f = \rho\psi$ then by Lemmas A.2 and A.4 clearly

$$|(\rho\psi)_{Q_1} - (\rho\psi)_{Q_2}| \lesssim \|\rho\psi\|_{*,\,\text{dyadic},\,Q_{2R}} \le \|\psi\|_{*,\,\text{dyadic},\,Q_{2R}} \le \|\nabla\tilde{\phi}\|_{*,\,\text{dyadic},\,Q_{2R}} \lesssim \eta.$$

Now look at the other cases: $Q_1 \subset Q_{2R}$ and $Q_2 \cap Q_{2R} = \varnothing$, or $Q_{2R} \subset Q_1$ and $Q_2 \cap Q_{2R} = \varnothing$. In both cases, we wish to control $|(\rho\psi)_{Q_1}|$.

Step 1.a: Case $Q_1 \subset Q_{2R},\ Q_2 \cap Q_{2R} = \varnothing$ and $l(Q_1) \lesssim R\eta/\ell$.

Q_1 is small here and touches the boundary of Q_{2R}. This means that $\|\rho\|_{L^\infty(Q_1)} \lesssim l(Q_1)/R$ since ρ is 0 outside Q_{2R}. Therefore we apply the trivial bound

$$|(\rho\psi)_{Q_1}| \le \|\rho\|_{L^\infty(Q_1)} \|\psi\|_{L^\infty(Q_1)} \lesssim \frac{l(Q_1)}{R}\ell \lesssim \eta.$$

Step 1.b: Case $Q_1 \subset Q_{2R},\ Q_2 \cap Q_{2R} = \varnothing$ and $R\eta/\ell \lesssim l(Q_1) \le 2R$.

Since $Q_1 \subset Q_{2R}$ we have $R\eta/\ell \lesssim l(Q_1) \le 2R$. Q_1 is dyadic so there exists $N \in \mathbb{Z}$ such that $l(Q_1) = 2^N l(Q_r)$.

Step 1.b.i: $N \le 0$.

This means that $l(Q_1) \le l(Q_r)$ and so by the reflection and tiling in time, (A-10), we may assume $Q_1 \subset \{|t| \le r^2\}$ and by Lemma A.2

$$\begin{aligned}|(\rho\psi)_{Q_1}| \le |\psi|_{Q_1} = \frac{1}{|Q_1|}\int_{Q_1} |\nabla\phi - \nabla\phi_{Q_r}| &\le \frac{1}{|Q_1|}\int_{Q_1} |\nabla\phi - \nabla\phi_{Q_1}| + |\nabla\phi_{Q_1} - \nabla\phi_{Q_r}| \\ &\lesssim \eta + \eta\log(1+\ell) + \eta\log(1+1/\eta) \lesssim_\varepsilon \eta^{1-\varepsilon} + \eta\log(1+\ell).\end{aligned}$$

Step 1.b.ii: $N > 0$.

By Claim A.3 we obtain

$$|(\rho\psi)_{Q_1}| \le |\psi|_{Q_1} = \frac{1}{|Q_1|}\int_{Q_1} |\nabla\tilde{\phi} - \nabla\tilde{\phi}_{Q_r}| \lesssim_\varepsilon \eta^{1-\varepsilon}.$$

Step 1.c: Case $Q_{2R} \subset Q_1,\ Q_2 \cap Q_{2R} = \varnothing$ so $l(Q_1) \ge 2R$.

Let N satisfy $l(Q_1) = 2^N l(Q_{2R})$, the number of dyadic generations separating Q_1 and Q_{2R}. Then Q_1 overlaps Q_{2R} (and its dyadic translates in time) exactly 2^{2N} times. Therefore by Claim A.3,

$$|(\rho\psi)_{Q_1}| \le |\psi|_{Q_1} \le \frac{2^{2N}}{|Q_1|}\int_{Q_{2R}} |\nabla\tilde{\phi} - \nabla\tilde{\phi}_{Q_r}| \le \frac{2^{2N}}{2^{N(n+1)}}\frac{1}{|Q_{2R}|}\int_{Q_{2R}} |\nabla\tilde{\phi} - \nabla\tilde{\phi}_{Q_r}| \lesssim_\varepsilon \eta^{1-\varepsilon}.$$

Hence, modulo the unproved statement $\|\rho\psi\|_{*,\,\text{dyadic},\,Q_{2R}} \lesssim \eta$ we have shown

$$|(\rho\psi)_{Q_1} - (\rho\psi)_{Q_2}| \lesssim_\varepsilon \eta^{1-\varepsilon} + \eta\log(1+\ell).$$

Step 2: (A-13) holds; that is, $\|\rho\psi\|_{*,\,\text{dyadic}} \lesssim c(\eta)$.

To apply Lemma A.4 we need to control two terms,

$$\sup_{Q\ \text{dyadic}} \|\rho\|_{L^\infty(Q)} \frac{1}{|Q|}\int_Q |\psi - \psi_Q|$$

and

$$\sup_{Q \text{ dyadic}} \frac{|\psi_Q|}{|Q|} \int_Q |\rho - \rho_Q|.$$

Step 2.a: Estimating

$$\sup_{Q \text{ dyadic}} \|\rho\|_{L^\infty(Q)} \frac{1}{|Q|} \int_Q |\psi - \psi_Q|.$$

In all the following cases we bound $\|\rho\|_{L^\infty(Q)} \le 1$.

Step 2.a.i: Case $l(Q) \le r$.

As before, by the reflection and tiling in time, we may assume $Q \subset \{|t| \le r^2\}$ and so $\nabla\tilde{\phi} = \nabla\phi$ on Q. Hence

$$\frac{1}{|Q|} \int_Q |\psi - \psi_Q| = \frac{1}{|Q|} \int_Q |\nabla\tilde{\phi} - (\nabla\tilde{\phi})_Q| = \frac{1}{|Q|} \int_Q |\nabla\phi - (\nabla\phi)_Q| \lesssim \eta.$$

Step 2.a.ii: Case $r < l(Q) \le 2R$.

Applying Claim A.3 gives

$$\frac{1}{|Q|} \int_Q |\psi - \psi_Q| \le |\psi|_Q \lesssim_\varepsilon \eta^{1-\varepsilon}.$$

Step 2.a.iii: Case $2R < l(Q)$.

From Step 1.c it follows that

$$\frac{1}{|Q|} \int_Q |\psi - \psi_Q| \le |\psi|_Q \lesssim_\varepsilon \eta^{1-\varepsilon}.$$

Step 2.b: Estimating

$$\sup_{Q \text{ dyadic}} \frac{|\psi_Q|}{|Q|} \int_Q |\rho - \rho_Q|.$$

We have the following three cases to consider.

Step 2.b.i: Case $Q \subset Q_{2R}$, $l(Q) \le r$ and $Q \subset \{|t| \le r^2\}$.

Because the cube Q might not be touching the boundary we can't follow Section 7 and bound

$$\frac{1}{|Q|} \int_Q |\rho - \rho_Q|$$

by $\|\rho\|_{L^\infty(Q)}$, which here is likely be 1. However, we can use the mean value theorem and get a better bound. By the intermediate value theorem there exists $(z, \tau) \in Q$ such that $\rho(z) = \rho_Q$ and using that ρ is independent of time and $|\nabla\rho| \lesssim 1/R$ we have

$$|\rho(x) - \rho_Q| = |\rho(x) - \rho(z)| \le |\nabla\rho| l(Q) \lesssim \frac{l(Q)}{R} \le \frac{l(Q)}{r}.$$

Then applying Lemma A.2 gives

$$\begin{aligned} \frac{|\psi_Q|}{|Q|} \int_Q |\rho - \rho_Q| &\lesssim \frac{l(Q)}{r} \left| \frac{1}{|Q|} \int_Q \nabla\tilde{\phi} - \nabla\tilde{\phi}_{Q_r} \right| \le \frac{l(Q)}{r} \frac{1}{|Q|} \int_Q |\nabla\phi - \nabla\phi_{Q_r}| \\ &\lesssim \frac{l(Q)}{r} \log\left(2 + \frac{r}{l(Q)}\right) \eta \lesssim \eta. \end{aligned}$$

Step 2.b.ii: Case $Q \subset Q_{2R}$ and $r < l(Q) \le 2R$.

This case is a straightforward application of Claim A.3:

$$\frac{|\psi_Q|}{|Q|}\int_Q |\rho - \rho_Q| \le |\psi_Q| \lesssim_\varepsilon \eta^{1-\varepsilon}.$$

Step 2.b.iii: Case $Q_{2R} \subset Q$ so $l(Q) > 2R$.

This follows similarly to Step 1.c; let N be defined as there and

$$\frac{|\psi_Q|}{|Q|}\int_Q |\rho - \rho_Q| \le \frac{1}{|Q|}\left|\int_Q \nabla\phi - \nabla\phi_{Q_{2R}}\right| \le \frac{2^{2N}}{2^{N(n+1)}}\|\nabla\phi\|_{*,Q_{2R}} \le \eta.$$

Therefore by Lemma A.1, $\|\rho\psi\|_* \lesssim_\varepsilon \eta^{1-\varepsilon} + \eta\log(1+\ell)$.

It remains to tackle the harder piece $\nabla\rho\theta = \nabla\rho(\tilde{\phi} - x\cdot\nabla\tilde{\phi}_{Q_r})$. Recall that $\operatorname{supp}(\nabla\rho) = \{r \le |x|_\infty \le 2R\}$.

Step 3: (A-14) holds; that is, $\sup_{Q_1,Q_2}|(\nabla\rho\theta)_{Q_1} - (\nabla\rho\theta)_{Q_2}| \le c(\eta)$, where Q_1, Q_2 are dyadic with a touching edge and $l(Q_1) = l(Q_2)$.

There are two different cases to consider:

(1) $Q_1 \cap \operatorname{supp}(\nabla\rho) \ne \varnothing$ and $Q_2 \cap \operatorname{supp}(\nabla\rho) \ne \varnothing$.

(2) $Q_1 \cap \operatorname{supp}(\nabla\rho) \ne \varnothing$ and $Q_2 \cap \operatorname{supp}(\nabla\rho) = \varnothing$.

Again (1) is controlled by $\|\nabla\rho\theta\|_{*,\text{dyadic},Q_{2R}}$ by Lemma A.2 so we only have to deal with (2) and bound $\sup_{Q_1 \text{ dyadic}}|(\nabla\rho\theta)_{Q_1}|$.

Step 3.a: Case $Q_1 \subset Q_{2R}$ and $l(Q_1) \lesssim R\eta/\ell$.

In this case Q_1 touches the boundary of the support of $\nabla\rho$ so we have the estimate $\|\nabla\rho\|_{L^\infty(Q_1)} \lesssim l(Q_1)/R^2$ since $|\nabla^2\rho| \lesssim 1/R^2$. Also $\phi(0,0) = 0$ and $\phi \in \operatorname{Lip}\big(1, \frac{1}{2}\big)$ so

$$\|\tilde{\phi}(x,t)\|_{L^\infty(Q_1)} \le \|\phi(x,t)\|_{L^\infty(Q_{2R})} \lesssim \ell R.$$

Finally $\|x\cdot\nabla\tilde{\phi}_{Q_r}\|_{L^\infty(Q_{2R})} \lesssim \ell R$. Therefore

$$|(\nabla\rho\theta)_{Q_1}| \le \|\nabla\rho\|_{L^\infty(Q_1)}|\theta|_{Q_1} \lesssim \frac{l(Q_1)}{R^2}\frac{1}{|Q_1|}\int_{Q_1}|\tilde{\phi}(x,t) - x\cdot\nabla\tilde{\phi}_{Q_r}|\,\mathrm{d}x\,\mathrm{d}t \lesssim \frac{l(Q_1)}{R^2}\ell R \lesssim \eta.$$

Step 3.b: Case $Q_1 \subset Q_{2R}$ and $R\eta/\ell \lesssim l(Q_1) \le 2R$.

By the fundamental theorem of calculus, we may write

$$\tilde{\phi}(x,t) - \tilde{\phi}\left(r\frac{x}{|x|}, t\right) = x\cdot\int_{r/|x|}^1 \nabla\tilde{\phi}(\lambda x, t)\,\mathrm{d}\lambda.$$

Therefore, we have

$$\begin{aligned}
|(\nabla\rho\theta)_{Q_1}| &\le |\nabla\rho|\,|\theta|_{Q_1} \\
&= |\nabla\rho|\left|\tilde{\phi}\left(r\frac{x}{|x|}, t\right) + x\cdot\int_{r/|x|}^1(\nabla\tilde{\phi}(\lambda x,t) - \nabla\tilde{\phi}_{Q_r})\,\mathrm{d}\lambda + x\cdot\frac{r}{|x|}\nabla\tilde{\phi}_{Q_r}\right|_{Q_1} \\
&\lesssim \frac{1}{R}\left\|\tilde{\phi}\left(r\frac{x}{|x|}, t\right)\right\|_{L^\infty(Q_1)} + \frac{R}{R}\frac{1}{|Q_1|}\int_{Q_1}\left(\int_{r/|x|}^1|\nabla\tilde{\phi}(\lambda x,t) - \nabla\tilde{\phi}_{Q_r}|\,\mathrm{d}\lambda\right)\mathrm{d}x\,\mathrm{d}t + \frac{\eta R\ell}{R}.
\end{aligned}$$

Since $\tilde{\phi}$ defined by (A-10) is tiled and reflected in time on cubes of scale r, and $(rx/|x|, 0) \in Q_r$ we control the first term above by

$$\frac{1}{R}\left\|\tilde{\phi}\left(r\frac{x}{|x|}, t\right) - 0\right\|_{L^\infty(Q_1)} \le \frac{1}{R}\|\phi - \phi(0,0)\|_{L^\infty(Q_r)} \lesssim \frac{\ell r}{R} \lesssim \ell\eta.$$

Recall that $r \sim \eta R$, $R\eta/\ell \lesssim l(Q_1) \le 2R$ and $r \le |x|_\infty \le 2R$ so $\eta/2 \le \lambda \le 1$. We apply Fubini to the second term:

$$\frac{1}{|Q_1|}\int_{Q_1}\left(\int_{r/|x|}^1 |\nabla\tilde{\phi}(\lambda x, t) - \nabla\tilde{\phi}_{Q_r}|\,\mathrm{d}\lambda\right)\mathrm{d}x\,\mathrm{d}t \le \frac{1}{|Q_1|}\int_{\eta/2}^1\int_{Q_1} |\nabla\tilde{\phi}(\lambda x, t) - \nabla\tilde{\phi}_{Q_r}|\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\lambda.$$

Let $\widetilde{Q}_1$ be the set formed by Q_1 under the transformation $(x,t) \mapsto (\lambda x, t)$. We may further cover $\widetilde{Q}_1$ by $\sim \lambda^{-2}$ translations of λQ_1 with $|\lambda Q_1|/|\widetilde{Q}_1| \lesssim \lambda^2$. Therefore a similar proof to Claim A.3, using Lemma A.2, gives

$$\begin{aligned}\frac{1}{|Q_1|}\int_{Q_1} |\nabla\tilde{\phi}(\lambda x, t) - \nabla\tilde{\phi}_{Q_r}|\,\mathrm{d}x\,\mathrm{d}t &= \frac{1}{|\widetilde{Q}_1|}\int_{\widetilde{Q}_1} |\nabla\tilde{\phi} - \nabla\tilde{\phi}_{Q_r}| \\ &\lesssim \lambda^{-2}\frac{\lambda^2}{|sQ_1|}\int_{sQ_1} |\nabla\tilde{\phi} - \nabla\tilde{\phi}_{Q_r}| \lesssim \eta\log\left(2 + \frac{r}{sl(Q_1)}\right) \lesssim \eta\log\left(1 + \frac{\ell}{\eta^2}\right) \\ &\lesssim_\varepsilon \eta^{1-\varepsilon} + \eta\log(1+\ell)\end{aligned}$$

and hence after harmlessly integrating in λ we can control the second term by

$$\int_{\eta/2}^1 \eta\log\left(1 + \frac{\ell}{\eta^2}\right)\mathrm{d}\lambda \lesssim_\varepsilon \eta^{1-\varepsilon} + \eta\log(1+\ell).$$

Step 3.c: Case $l(Q_1) \ge 2R$.

As before in Step 1.c, $|(\nabla\rho\theta)_{Q_1}| \le |(\nabla\rho\theta)_{Q_{2R}}|$, which can be further controlled by cubes that tile $\mathrm{supp}(\nabla\rho)$. Therefore, this case is bounded as in Section 7.

Step 4: (A-13) holds; that is, $\|\nabla\rho\theta\|_{*,\,\text{dyadic}} \lesssim c(\eta)$.

Here we have three cases to consider:

(1) $Q \subset Q_{2R}$.

(2) $Q \subset \mathbb{R}^n \setminus \mathrm{supp}(\nabla\rho)$.

(3) $Q_{2R} \subset Q$.

Case (2) is obvious. Case (3) reduces down to (1) by Step 1.c, the reflection and tiling of $\tilde{\phi}$, and $\mathrm{supp}(\nabla\rho)$.

Case (1): Using Lemma A.4 this reduces down to showing that

(a) $\displaystyle\frac{|\theta_Q|}{|Q|}\int_Q |\nabla\rho - (\nabla\rho)_Q| \lesssim c(\eta)$,

(b) $\displaystyle\frac{1}{|Q|}\int_Q |\nabla\rho(\theta - \theta_Q)| \lesssim c(\eta)$

for Q dyadic and $Q \subset Q_{2R}$.

Step 4.a: (a) holds for Q dyadic and $Q \subset Q_{2R}$.

Step 4.a.i: Case $Q \subset Q_{2R}$ and $l(Q) \lesssim R\eta/\ell$.

By the naive bounds in Step 3.a, $|\theta|_Q \lesssim \ell R$. If we use the mean value theorem for $\nabla\rho$ similar to Section 7 then

$$\frac{1}{|Q|}\int_Q |\nabla\rho - (\nabla\rho)_Q| \lesssim |\nabla^2\rho| l(Q) \lesssim \frac{l(Q)}{R^2}.$$

Therefore

$$\frac{|\theta_Q|}{|Q|}\int_Q |\nabla\rho - (\nabla\rho)_Q| \lesssim \ell R\frac{l(Q)}{R^2} \lesssim \eta.$$

Step 4.a.ii: Case $Q \subset Q_{2R}$ and $R\eta/\ell \lesssim l(Q) \le 2R$.

Here we apply the same technique as Section 7:

$$\frac{|\theta_Q|}{|Q|}\int_Q |\nabla\rho - (\nabla\rho)_Q| \le |\theta|_Q|\nabla\rho| \lesssim_\varepsilon \eta^{1-\varepsilon} + \eta\log(1+\ell).$$

Step 4.b: (b) holds for Q dyadic and $Q \subset Q_{2R}$.

$$\frac{1}{|Q|}\int_Q |\nabla\rho(\theta-\theta_Q)| \lesssim \frac{1}{R}\frac{1}{|Q|}\int_Q |\theta-\theta_Q|.$$

We split this into the now-usual cases.

Step 4.b.i: Case $l(Q) \lesssim R\eta/\ell$.

By the intermediate and mean value theorems $|\tilde\phi - \tilde\phi_Q| \lesssim l(Q)\ell$ and $|x - x_Q| \lesssim l(Q)$ so

$$\frac{1}{R}\frac{1}{|Q|}\int_Q |\theta-\theta_Q| = \frac{1}{R}\frac{1}{|Q|}\int_Q |\tilde\phi - \tilde\phi_Q - x\cdot\nabla\tilde\phi_{Q_r} + (x\cdot\nabla\tilde\phi_{Q_r})_Q| \lesssim \frac{1}{R} l(Q)\ell \lesssim \eta.$$

Step 4.b.ii: Case $R\eta/\ell \lesssim l(Q) < 2R$.

$$\frac{1}{R}\frac{1}{|Q|}\int_Q |\theta-\theta_Q| \lesssim \frac{1}{R}|\theta|_Q,$$

and then applying the result from Section 7 gives

$$\frac{1}{|Q|}\int_Q |\nabla\rho(\theta-\theta_Q)| \lesssim_\varepsilon \eta^{1-\varepsilon} + \eta\log(1+\ell).$$

Therefore by Lemma A.1 we have shown $\nabla\Phi \in \mathrm{BMO}(\mathbb{R}^n)$ and the bound (A-21) holds. □

To finish proving Theorem 2.8 we need to establish (iv).

Lemma A.6. *Let* $\Phi : \mathbb{R}^{n-1}\times\mathbb{R} \to \mathbb{R}$ *be defined in* (A-11) *with*

$$\sup_{\substack{Q_s = J_s\times I_s \\ Q_s\subset Q_{8d},\, s\le r_1}} \frac{1}{|Q_s|}\int_{Q_s}\int_{I_s} \frac{|\phi(x,t)-\phi(x,\tau)|^2}{|t-\tau|^2}\,\mathrm{d}\tau\,\mathrm{d}t\,\mathrm{d}x \le \eta^2 \tag{A-22}$$

then Φ *satisfies*

$$\sup_{Q_s = J_s\times I_s} \frac{1}{|Q_s|}\int_{Q_s}\int_{I_s} \frac{|\Phi(x,t)-\Phi(x,\tau)|^2}{|t-\tau|^2}\,\mathrm{d}\tau\,\mathrm{d}t\,\mathrm{d}x \lesssim \eta^2. \tag{A-23}$$

Proof. Trivially since Φ is defined globally

$$\sup_{Q_s=J_s\times I_s}\frac{1}{|Q_s|}\int_{Q_s}\int_{I_s}\frac{|\Phi(x,t)-\Phi(x,\tau)|^2}{|t-\tau|^2}\,d\tau\,dt\,dx\le\sup_{Q_s=J_s\times I_s}\frac{1}{|Q_s|}\int_{Q_s}\int_{I_s}\frac{|\tilde\phi(x,t)-\tilde\phi(x,\tau)|^2}{|t-\tau|^2}\,d\tau\,dt\,dx,$$

where we interpret the value of $\tilde\phi$ where it is undefined as 0, i.e., $\tilde\phi(x,t)=0$ when $(x,t)\notin\operatorname{supp}(\tilde\phi)$. It remains to establish

$$\sup_{I_s}\frac{1}{|I_s|}\int_{I_s}\int_{I_s}\frac{|\tilde\phi(x,t)-\tilde\phi(x,\tau)|^2}{|t-\tau|^2}\,d\tau\,dt\,dx\lesssim\sup_{I_s\subset I_r}\frac{1}{|I_s|}\int_{I_s}\int_{I_s}\frac{|\phi(x,t)-\phi(x,\tau)|^2}{|t-\tau|^2}\,d\tau\,dt\,dx \tag{A-24}$$

pointwise in x, where $Q_r=J_r\times I_r$ and is used to define Φ in (A-11). To simplify our notation, we drop the dependence on the spatial variables in $\tilde\phi$ and ϕ. We also set $A:=I_s$. Recall from (A-10) that

$$\tilde\phi(t)=\begin{cases}\phi(t), & t\in[-r^2,r^2]+4kr^2,\\ \phi(2r^2-t), & t\in[r^2,3r^2]+4kr^2,\end{cases}$$

for $k\in\mathbb{Z}$. Let $I_k=[-r^2,r^2]+4kr^2$ and $J_k=[r^2,3r^2]+4kr^2$ be intervals in time for $k\in\mathbb{Z}$. We partition A into disjoint pieces $A=\bigcup_i I_i\bigcup_j J_j\cup A_1\cup A_2$, where A_1 and A_2 are pieces that don't contain either I_i or J_j.

If $A=A_1\cup A_2$, we may as well assume (by translation and reflection) that $A_1=[a,r^2]$, $A_2=[r^2,b]$. Let τ', b' and A_2' be the images of τ, b and A_2 respectively under the map $\tau\mapsto 2r^2-\tau$. Without loss of generality we only consider the case $|A_1|>|A_2|$. Since $|t-\tau|=|t-r^2|+|\tau'-r^2|\ge|t-\tau'|$ we have for $t\in A_1,\ \tau\in A_2$

$$\begin{aligned}\int_{A_1}\int_{A_2}\frac{|\tilde\phi(t)-\tilde\phi(\tau)|^2}{|t-\tau|^2}\,d\tau\,dt&=\int_a^{r^2}\int_{b'}^{r^2}\frac{|\phi(t)-\phi(\tau')|^2}{|t-(2t^2-\tau')|^2}\,d\tau'\,dt\\&\le\int_a^{r^2}\int_{b'}^{r^2}\frac{|\phi(t)-\phi(\tau')|^2}{|t-\tau'|^2}\,d\tau'\,dt\le\int_{A_1}\int_{A_1}\frac{|\phi(t)-\phi(\tau')|^2}{|t-\tau'|^2}\,d\tau'\,dt.\end{aligned}$$

Therefore

$$\frac{1}{|A|}\int_A\int_A\frac{|\tilde\phi(t)-\tilde\phi(\tau)|^2}{|t-\tau|^2}\,d\tau\,dt=\frac{1}{|A|}\left(\int_{A_1}\int_{A_1}+2\int_{A_1}\int_{A_2}+\int_{A_2}\int_{A_2}\right)\frac{|\phi(t)-\phi(\tau')|^2}{|t-\tau'|^2}\,d\tau'\,dt\lesssim\eta^2.$$

In the general case when $A=\bigcup_{i\in\mathcal{I}}I_i\bigcup_{j\in\mathcal{J}}J_j\cup A_1\cup A_2$ we write the double integral over A in terms of integrals

$$\sum_{i,k\in\mathcal{I}}\int_{I_i}\int_{I_k}\frac{|\tilde\phi(t)-\tilde\phi(\tau)|^2}{|t-\tau|^2}\,d\tau\,dt,\qquad\sum_{i\in\mathcal{I},j\in\mathcal{J}}\int_{I_i}\int_{J_j}\frac{|\tilde\phi(t)-\tilde\phi(\tau)|^2}{|t-\tau|^2}\,d\tau\,dt$$

and integrals that involve sets A_1 or A_2 or both (those are handled similarly to the earlier calculation).

Dealing with the first case, if $i \neq k,\ t \in I_i$ and $\tau \in I_k$ then $|t-\tau| \sim r^2|i-k|$; if $i=k$ then $|t-\tau| = |t'-\tau'|$. Therefore

$$\begin{aligned}\sum_{i,k\in\mathcal{I}}\int_{I_i}\int_{I_k}\frac{|\tilde{\phi}(t)-\tilde{\phi}(\tau)|^2}{|t-\tau|^2}\,\mathrm{d}\tau\,\mathrm{d}t &\sim \sum_{i\in\mathcal{I}}\int_{I_0}\int_{I_0}\frac{|\phi(t)-\phi(\tau)|^2}{|t-\tau|^2}\,\mathrm{d}\tau\,\mathrm{d}t+\sum_{\substack{i,k\in\mathcal{I}\\ i\neq k}}\frac{1}{r^4|i-k|^2}\int_{I_0}\int_{I_0}|\phi(t)-\phi(\tau)|^2\,\mathrm{d}\tau\,\mathrm{d}t\\ &\le \sum_{i\in\mathcal{I}}\int_{I_0}\int_{I_0}\frac{|\phi(t)-\phi(\tau)|^2}{|t-\tau|^2}\,\mathrm{d}\tau\,\mathrm{d}t+\sum_{\substack{i,k\in\mathcal{I}\\ i\neq k}}\frac{1}{|i-k|^2}\int_{I_0}\int_{I_0}\frac{|\phi(t)-\phi(\tau)|^2}{|t-\tau|^2}\,\mathrm{d}\tau\,\mathrm{d}t\\ &\lesssim |\mathcal{I}|\int_{I_0}\int_{I_0}\frac{|\phi(t)-\phi(\tau)|^2}{|t-\tau|^2}\,\mathrm{d}\tau\,\mathrm{d}t.\end{aligned}$$

In the second case

$$\begin{aligned}&\sum_{i\in\mathcal{I},\,j\in\mathcal{J}}\int_{I_i}\int_{J_j}\frac{|\tilde{\phi}(t)-\tilde{\phi}(\tau)|^2}{|t-\tau|^2}\,\mathrm{d}\tau\,\mathrm{d}t\\ &\qquad\lesssim \sum_{\substack{i\in\mathcal{I},\,j\in\mathcal{J}\\ |i-j|\le 1}}\int_{I_0}\int_{I_0}\frac{|\phi(t)-\phi(\tau)|^2}{|t-\tau|^2}+\sum_{\substack{i\in\mathcal{I},\,j\in\mathcal{J}\\ |i-j|\ge 2}}\frac{1}{r^4(|i-j|-1)^2}\int_{I_0}\int_{I_0}|\phi(t)-\phi(\tau)|^2\,\mathrm{d}\tau\,\mathrm{d}t\\ &\qquad\lesssim (|\mathcal{I}|+|\mathcal{J}|)\int_{I_0}\int_{I_0}\frac{|\phi(t)-\phi(\tau)|^2}{|t-\tau|^2}\,\mathrm{d}\tau\,\mathrm{d}t.\end{aligned}$$

Since $|A| \sim (|\mathcal{I}|+|\mathcal{J}|)|I_0|$ and I_0 is one of the time intervals considered in the supremum of (A-24),

$$\frac{1}{|A|}\int_A\int_A\frac{|\phi(t)-\phi(\tau)|^2}{|t-\tau|^2}\,\mathrm{d}\tau\,\mathrm{d}t \sim \frac{1}{|I_0|}\int_{I_0}\int_{I_0}\frac{|\phi(t)-\phi(\tau)|^2}{|t-\tau|^2}\,\mathrm{d}\tau\,\mathrm{d}t \lesssim \eta^2. \qquad \square$$

References

[Aronson 1968] D. G. Aronson, "Non-negative solutions of linear parabolic equations", *Ann. Scuola Norm. Sup. Pisa* (3) **22**:4 (1968), 607–694. Addendum in **25**:2 (1971), 221–228. MR Zbl

[Ball and Zarnescu 2017] J. M. Ball and A. Zarnescu, "Partial regularity and smooth topology-preserving approximations of rough domains", *Calc. Var. Partial Differential Equations* **56**:1 (2017), art. id. 13. MR Zbl

[Brown 1989] R. M. Brown, "The method of layer potentials for the heat equation in Lipschitz cylinders", *Amer. J. Math.* **111**:2 (1989), 339–379. MR Zbl

[Calderón and Torchinsky 1975] A.-P. Calderón and A. Torchinsky, "Parabolic maximal functions associated with a distribution", *Adv. Math.* **16**:1 (1975), 1–64. MR Zbl

[Calderón and Torchinsky 1977] A.-P. Calderón and A. Torchinsky, "Parabolic maximal functions associated with a distribution, II", *Adv. Math.* **24**:1 (1977), 101–171. MR Zbl

[Cialdea and Mazya 2005] A. Cialdea and V. Maz'ya, "Criterion for the L^p-dissipativity of second order differential operators with complex coefficients", *J. Math. Pures Appl.* (9) **84**:8 (2005), 1067–1100. MR Zbl

[Dahlberg 1977] B. E. J. Dahlberg, "Estimates of harmonic measure", *Arch. Ration. Mech. Anal.* **65**:3 (1977), 275–288. MR Zbl

[Dindoš 2002] M. Dindoš, "Existence and uniqueness for a semilinear elliptic problem on Lipschitz domains in Riemannian manifolds", *Comm. Partial Differential Equations* **27**:1-2 (2002), 219–281. MR Zbl

[Dindoš 2008] M. Dindoš, *Hardy spaces and potential theory on C^1 domains in Riemannian manifolds*, Mem. Amer. Math. Soc. **894**, Amer. Math. Soc., Providence, RI, 2008. MR Zbl

[Dindoš and Hwang 2018] M. Dindoš and S. Hwang, "The Dirichlet boundary problem for second order parabolic operators satisfying a Carleson condition", *Rev. Mat. Iberoam.* **34**:2 (2018), 767–810. MR Zbl

[Dindoš and Pipher 2019] M. Dindoš and J. Pipher, "Regularity theory for solutions to second order elliptic operators with complex coefficients and the L^p Dirichlet problem", *Adv. Math.* **341** (2019), 255–298. MR Zbl

[Dindoš et al. 2007] M. Dindos, S. Petermichl, and J. Pipher, "The L^p Dirichlet problem for second order elliptic operators and a p-adapted square function", *J. Funct. Anal.* **249**:2 (2007), 372–392. MR Zbl

[Dindoš et al. 2017] M. Dindoš, J. Pipher, and D. Rule, "Boundary value problems for second-order elliptic operators satisfying a Carleson condition", *Comm. Pure Appl. Math.* **70**:7 (2017), 1316–1365. MR Zbl

[Fabes and Rivière 1966] E. B. Fabes and N. M. Rivière, "Singular integrals with mixed homogeneity", *Studia Math.* **27** (1966), 19–38. MR Zbl

[Fabes and Rivière 1967] E. B. Fabes and N. M. Rivière, "Symbolic calculus of kernels with mixed homogeneity", pp. 106–127 in *Singular integrals* (Chicago, 1966), edited by A. P. Calderón, Proc. Sympos. Pure Math. **10**, Amer. Math. Soc., Providence, RI, 1967. MR Zbl

[Fabes et al. 1978] E. B. Fabes, M. Jodeit, Jr., and N. M. Rivière, "Potential techniques for boundary value problems on C^1-domains", *Acta Math.* **141**:3-4 (1978), 165–186. MR Zbl

[Fefferman and Stein 1972] C. Fefferman and E. M. Stein, "H^p spaces of several variables", *Acta Math.* **129** (1972), 137–193. MR Zbl

[Hofmann 1995] S. Hofmann, "A characterization of commutators of parabolic singular integrals", pp. 195–210 in *Fourier analysis and partial differential equations* (Miraflores de la Sierra, Spain, 1992), edited by J. García-Cuerva et al., CRC, Boca Raton, FL, 1995. MR Zbl

[Hofmann and Lewis 1996] S. Hofmann and J. L. Lewis, "L^2 solvability and representation by caloric layer potentials in time-varying domains", *Ann. of Math.* (2) **144**:2 (1996), 349–420. MR Zbl

[Hofmann and Lewis 2001] S. Hofmann and J. L. Lewis, *The Dirichlet problem for parabolic operators with singular drift terms*, Mem. Amer. Math. Soc. **719**, Amer. Math. Soc., Providence, RI, 2001. MR Zbl

[Hofmann et al. 2010] S. Hofmann, M. Mitrea, and M. Taylor, "Singular integrals and elliptic boundary problems on regular Semmes–Kenig–Toro domains", *Int. Math. Res. Not.* **2010**:14 (2010), 2567–2865. MR Zbl

[Hofmann et al. 2015] S. Hofmann, M. Mitrea, and M. E. Taylor, "Symbol calculus for operators of layer potential type on Lipschitz surfaces with VMO normals, and related pseudodifferential operator calculus", *Anal. PDE* **8**:1 (2015), 115–181. MR Zbl

[Jones 1980] P. W. Jones, "Extension theorems for BMO", *Indiana Univ. Math. J.* **29**:1 (1980), 41–66. MR Zbl

[Kaufman and Wu 1988] R. Kaufman and J.-M. Wu, "Parabolic measure on domains of class Lip$\frac{1}{2}$", *Compositio Math.* **65**:2 (1988), 201–207. MR Zbl

[Kenig and Pipher 2001] C. E. Kenig and J. Pipher, "The Dirichlet problem for elliptic equations with drift terms", *Publ. Mat.* **45**:1 (2001), 199–217. MR Zbl

[Kenig et al. 2000] C. Kenig, H. Koch, J. Pipher, and T. Toro, "A new approach to absolute continuity of elliptic measure, with applications to non-symmetric equations", *Adv. Math.* **153**:2 (2000), 231–298. MR Zbl

[Langer 1999] M. Langer, "L^p-contractivity of semigroups generated by parabolic matrix differential operators", pp. 307–330 in *The Maz'ya anniversary collection, I* (Rostock, Germany, 1998), edited by J. Rossmann et al., Oper. Theory Adv. Appl. **109**, Birkhäuser, Basel, 1999. MR Zbl

[Lewis and Murray 1995] J. L. Lewis and M. A. M. Murray, *The method of layer potentials for the heat equation in time-varying domains*, Mem. Amer. Math. Soc. **545**, Amer. Math. Soc., Providence, RI, 1995. MR Zbl

[Mazya et al. 2009] V. Mazya, M. Mitrea, and T. Shaposhnikova, "The inhomogeneous Dirichlet problem for the Stokes system in Lipschitz domains with unit normal close to VMO", *Funktsional. Anal. i Prilozhen.* **43**:3 (2009), 65–88. In Russian; translated in *Funct. Anal. Appl.* **43**:3 (2009), 217–235. MR Zbl

[Nyström 1997] K. Nyström, "The Dirichlet problem for second order parabolic operators", *Indiana Univ. Math. J.* **46**:1 (1997), 183–245. MR Zbl

[Peetre 1966] J. Peetre, "On convolution operators leaving $L^{p,\lambda}$ spaces invariant", *Ann. Mat. Pura Appl.* (4) **72**:1 (1966), 295–304. MR Zbl

[Rivera-Noriega 2003] J. Rivera-Noriega, "Absolute continuity of parabolic measure and area integral estimates in non-cylindrical domains", *Indiana Univ. Math. J.* **52**:2 (2003), 477–525. MR Zbl

[Rivera-Noriega 2014] J. Rivera-Noriega, "Perturbation and solvability of initial L^p Dirichlet problems for parabolic equations over non-cylindrical domains", *Canad. J. Math.* **66**:2 (2014), 429–452. MR Zbl

[Semmes 1991] S. Semmes, "Chord-arc surfaces with small constant, I", *Adv. Math.* **85**:2 (1991), 198–223. MR Zbl

[Stegenga 1976] D. A. Stegenga, "Bounded Toeplitz operators on H^1 and applications of the duality between H^1 and the functions of bounded mean oscillation", *Amer. J. Math.* **98**:3 (1976), 573–589. MR Zbl

[Stein 1970] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Math. Series **30**, Princeton Univ. Press, 1970. MR Zbl

[Stein 1993] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Math. Series **43**, Princeton Univ. Press, 1993. MR Zbl

[Strichartz 1980] R. S. Strichartz, "Bounded mean oscillation and Sobolev spaces", *Indiana Univ. Math. J.* **29**:4 (1980), 539–558. MR Zbl

[Sweezy 1998] C. Sweezy, "B^q for parabolic measures", *Studia Math.* **131**:2 (1998), 115–135. MR Zbl

Received 28 Oct 2018. Revised 12 Mar 2019. Accepted 18 Apr 2019.

MARTIN DINDOŠ: m.dindos@ed.ac.uk
School of Mathematics, The University of Edinburgh and Maxwell Institute of Mathematical Sciences, Edinburgh, United Kingdom

LUKE DYER: luke.dyer@peak.ai
School of Mathematics, The University of Edinburgh and Maxwell Institute of Mathematical Sciences, Edinburgh, United Kingdom

SUKJUNG HWANG: sukjung_hwang@yonsei.ac.kr
Department of Mathematics, Yonsei University, Seoul, South Korea

Guidelines for Authors

Authors may submit manuscripts in PDF format on-line at the Submission page at msp.org/apde.

Originality. Submission of a manuscript acknowledges that the manuscript is original and and is not, in whole or in part, published or under consideration for publication elsewhere. It is understood also that the manuscript will not be submitted elsewhere while under consideration for publication in this journal.

Language. Articles in APDE are usually in English, but articles written in other languages are welcome.

Required items. A brief abstract of about 150 words or less must be included. It should be self-contained and not make any reference to the bibliography. If the article is not in English, two versions of the abstract must be included, one in the language of the article and one in English. Also required are keywords and subject classifications for the article, and, for each author, postal address, affiliation (if appropriate), and email address.

Format. Authors are encouraged to use LaTeX but submissions in other varieties of TeX, and exceptionally in other formats, are acceptable. Initial uploads should be in PDF format; after the refereeing process we will ask you to submit all source material.

References. Bibliographical references should be complete, including article titles and page ranges. All references in the bibliography should be cited in the text. The use of BibTeX is preferred but not required. Tags will be converted to the house format, however, for submission you may use the format of your choice. Links will be provided to all literature with known web locations and authors are encouraged to provide their own links in addition to those supplied in the editorial process.

Figures. Figures must be of publication quality. After acceptance, you will need to submit the original source files in vector graphics format for all diagrams in your manuscript: vector EPS or vector PDF files are the most useful.

Most drawing and graphing packages (Mathematica, Adobe Illustrator, Corel Draw, MATLAB, etc.) allow the user to save files in one of these formats. Make sure that what you are saving is vector graphics and not a bitmap. If you need help, please write to graphics@msp.org with details about how your graphics were generated.

White space. Forced line breaks or page breaks should not be inserted in the document. There is no point in your trying to optimize line and page breaks in the original manuscript. The manuscript will be reformatted to use the journal's preferred fonts and layout.

Proofs. Page proofs will be made available to authors (or to the designated corresponding author) at a Web site in PDF format. Failure to acknowledge the receipt of proofs or to return corrections within the requested deadline may cause publication to be postponed.

Analysis & PDE

Volume 13 No. 4 2020

2157-5045(2020)13:4;1-A