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NONEXISTENCE OF GLOBAL CHARACTERISTICS FOR VISCOSITY SOLUTIONS

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Two different types of generalized solutions, namely viscosity and variational solutions, were introduced to solve the first-order evolutionary Hamilton–Jacobi equation. They coincide if the Hamiltonian is convex in the momentum variable. We prove that there exists no other class of integrable Hamiltonians sharing this property. To do so, we build for any nonconvex, nonconcave integrable Hamiltonian a smooth initial condition such that the graph of the viscosity solution is not contained in the wavefront associated with the Cauchy problem. The construction is based on a new example for a saddle Hamiltonian and a precise analysis of the one-dimensional case, coupled with reduction and approximation arguments.

1. Introduction

Let $H : \mathbb{R} \times T^\star\mathbb{R}^d \to \mathbb{R}$ be a $\mathcal{C}^2$ Hamiltonian. We study the Cauchy problem associated with the evolutionary Hamilton–Jacobi equation

$$\partial_t u(t,q) + H(t,q,\partial_q u(t,q)) = 0, \tag{HJ}$$

where $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is the unknown function, with a Lipschitz initial datum $u(0,\cdot) = u_0$.

The method of characteristics shows that a classical solution of this equation is given by characteristics (see Section 1A). If the projections of characteristics associated with u_0 cross, the method gives rise to a multivalued solution, with a multigraph called a *wavefront* and denoted by $\mathcal{F}_{u_0}$ (see (F)). This implies in particular that for some u_0 and H, even if if smooth, the evolutionary Hamilton–Jacobi equation does not admit classical solutions in large time.

A first type of generalized solution, called a *viscosity solution* (see Section 1B), was introduced by Lions, Crandall and Evans in the early 80s for Hamilton–Jacobi equations. It possesses multiple assets: it is well-defined, unique and stable in a large range of assumptions on the Hamiltonian and the initial condition. It has a local definition avoiding the delicate question of how to choose a solution amongst the multivalued solution and its associated characteristics. This local definition can be extended effortlessly to larger classes of elliptic PDEs, which is another major asset of viscosity solutions. Also, the operator giving the viscosity solution satisfies a convenient semigroup property.

When the Hamiltonian is convex in the fiber (more precisely when it is Tonelli), this viscosity operator is given by the Lax–Oleinik semigroup, which by definition gives a section of the wavefront. The main

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result of this article addresses the converse question, in the case of integrable (i.e., depending only on the fiber variable) Hamiltonians.

Theorem 1. *If $p \mapsto H(p)$ is a neither convex nor concave, integrable Hamiltonian with bounded second derivative, there exists a smooth Lipschitz initial condition u_0 such that the graph of the viscosity solution associated with u_0 is not included in the wavefront $\mathcal{F}_{u_0}$.*

The term of *variational solution* (see Section 1C) does not appear in this statement but the idea of this other generalized solution is essential in the whole article: roughly speaking, they can be defined as continuous functions whose graph is included in the wavefront. The notion was introduced in the early 90s by Sikorav and Chaperon, who found a way to choose a continuous section of the wavefront by selecting the min-max value of the generating family for the Lagrangian geometrical solution. Joukovskaia [1994] showed that their construction coincides with the Lax–Oleinik semigroup in the fiberwise convex case. The study of the variational operator given by this Chaperon–Sikorav method gives local estimates on the variational solutions. These estimates can be used regardless of the construction of the variational solution thanks to Proposition 1.9, which gives an elementary characterization of the variational solution for semiconcave initial data. This fact makes the whole article accessible to a reader with no specific background on symplectic geometry.

To show Theorem 1, we reduce the problem to the study of two key situations in dimensions 1 and 2; see Propositions 3.1 and 2.4. The example for the dimension 1 was already well-studied. It appears in [Chenciner 1975]; see also [Izumiya and Kossioris 1996]. The creation of the example for the saddle Hamiltonian in dimension 2 is the main contribution of this article. Special care was then taken to state the reduction and approximation arguments finishing the demonstration.

Recent breakthroughs have been made in the study of the singularities of the viscosity solution of (HJ) for convex Hamiltonians; see [Cannarsa et al. 2015; 2017; Cannarsa and Cheng 2018] for a survey. A natural question following from Theorem 1 is to compare these singularities for viscosity and variational solutions when the Hamiltonian is not convex anymore. On the close topic of multitime Hamilton–Jacobi equations, let us also highlight a recent discussion about the nonexistence of viscosity solutions when convexity assumptions are dropped; see [Davini and Zavidovique 2015]. This gives another point of comparison with variational solutions, which are well-defined for this framework; see [Cardin and Viterbo 2008].

Since Proposition 1.9 holds for nonintegrable Hamiltonians, we present the different objects in the nonintegrable framework. We will underline how they simplify in the integrable case. In that purpose, we introduce a second Hypothesis on H, automatically satisfied by integrable Hamiltonians with bounded second derivative, that provides the existence of both viscosity and variational solutions in the nonintegrable case.

Hypothesis 1.1. *There is a $C > 0$ such that for each (t, q, p) in $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$,*

$$\|\partial^2_{(q,p)} H(t, q, p)\| < C, \quad \|\partial_{(q,p)} H(t, q, p)\| < C(1 + \|p\|),$$

where $\partial_{(q,p)} H$ and $\partial^2_{(q,p)} H$ denote the first- and second-order spatial derivatives of H.

1A. ***Classical solutions: the method of characteristics.*** In this section we only assume that d^2H is bounded by C. The *Hamiltonian system*

$$\begin{cases} \dot{q}(t) = \partial_p H(t, q(t), p(t)), \\ \dot{p}(t) = -\partial_q H(t, q(t), p(t)) \end{cases} \tag{HS}$$

hence admits a complete *Hamiltonian flow* ϕ_s^t, meaning that $t \mapsto \phi_s^t(q, p)$ is the unique solution of (HS) with initial conditions $(q(s), p(s)) = (q, p)$. We denote by (Q_s^t, P_s^t) the coordinates of ϕ_s^t. We call a function $t \mapsto (q(t), p(t))$ solving the Hamiltonian system (HS) a *Hamiltonian trajectory*. The *Hamiltonian action* of a $\mathcal{C}^1$ path $\gamma(t) = (q(t), p(t)) \in T^\star\mathbb{R}^d$ is denoted by

$$\mathcal{A}_s^t(\gamma) = \int_s^t p(\tau) \cdot \dot{q}(\tau) - H(\tau, q(\tau), p(\tau))\, d\tau.$$

Note that in the case of an integrable Hamiltonian (that depends only on p), the flow is given by $\phi_s^t(q, p) = (q + (t - s)\nabla H(p), p)$ and the action of a Hamiltonian path is reduced to $\mathcal{A}_s^t(\gamma) = (t - s)(p \cdot \nabla H(p) - H(p))$.

The method of characteristics states that if u_0 is a $\mathcal{C}^2$ function with second derivative bounded by $B > 0$, there exists T depending only on C and B (for example $T = 1/(BC)$ for an integrable Hamiltonian) such that the Cauchy problem (HJ) with initial condition u_0 has a unique $\mathcal{C}^2$ solution on $[0, T] \times \mathbb{R}^d \to \mathbb{R}$. Furthermore, if u is a $\mathcal{C}^2$ solution on $[0, T] \times \mathbb{R}^d$, for all (t, q) in $[0, T] \times \mathbb{R}^d$, there exists a unique q_0 in $\mathbb{R}^d$ such that $Q_0^t(q_0, du_0(q_0)) = q$ and if γ denotes the Hamiltonian trajectory issued from $(q_0, du_0(q_0))$, the $\mathcal{C}^2$ solution is given by the Hamiltonian action as

$$u(t, q) = u_0(q_0) + \mathcal{A}_0^t(\gamma),$$

and its derivative satisfies $\partial_q u(t, q) = P_0^t(q_0, du_0(q))$ at the point $q = Q_0^t(q_0, du_0(q_0))$. As a consequence, if the image $\phi_0^t(\text{graph}(du_0))$ of the graph of du_0 by the Hamiltonian flow is not a graph for some t, there is no classical solution on $[0, t] \times \mathbb{R}^d$, whence the necessity to introduce generalized solutions.

1B. ***Viscosity solutions.*** The viscosity solutions were introduced in the framework of Hamilton–Jacobi equations by Lions, Evans and Crandall in the early 80's; see [Crandall and Lions 1983]. We will use the following definition.

Definition 1.2. A continuous function u is a *subsolution* of (HJ) on the set $(0, T) \times \mathbb{R}^d$ if for each $\mathcal{C}^1$ function $\phi : (0, T) \times \mathbb{R}^d \to \mathbb{R}$ such that $u - \phi$ admits a (strict) local maximum at a point $(t, q) \in (0, T) \times \mathbb{R}^d$, we have

$$\partial_t \phi(t, q) + H(t, q, \partial_q \phi(t, q)) \leq 0.$$

A continuous function u is a *supersolution* of (HJ) on the set $(0, T) \times \mathbb{R}^d \to \mathbb{R}$ if for each $\mathcal{C}^1$ function $\phi : (0, T) \times \mathbb{R}^d$ such that $u - \phi$ admits a (strict) local minimum at a point $(t, q) \in (0, T) \times \mathbb{R}^d$, we have

$$\partial_t \phi(t, q) + H(t, q, \partial_q \phi(t, q)) \geq 0.$$

A viscosity solution is both a sub- and supersolution of (HJ).

The set of assumptions of this paper is well-adapted to the theory of viscosity solutions developed by Crandall, Lions and Ishii [Crandall et al. 1992], from which one can deduce the following well-posedness property.

Proposition 1.3. *If H satisfies Hypothesis 1.1, the Cauchy problem associated with the (HJ) equation and a Lipschitz initial condition admits a unique Lipschitz solution. This defines a viscosity operator $(V_s^t)_{s\le t}$ on the set of Lipschitz functions $\mathcal{C}^{0,1}(\mathbb{R}^d)$ which is monotonic*:

$$V_s^t u \le V_s^t v \quad \text{if } u \le v.$$

Furthermore, if u and v are Lipschitz with bounded difference,

$$\|V_s^t u - V_s^t v\|_\infty \le \|u-v\|_\infty \quad \text{for all } s \le t.$$

In dimension 1, the theory of viscosity solutions of the (HJ) equation is the counterpart of the theory of entropy solutions for conservation laws: if $p(t,q) = \partial_q u(t,q)$ and u satisfies (HJ),

$$\partial_t p(t,q) + \partial_q (H(t,q,p(t,q))) = 0.$$

The following entropy condition, first proposed by O. Oleinik [1959] for conservation laws, gives a geometric criterion to decide if a function solves the (HJ) equation in the viscosity sense at a point of shock. It is proved for example in [Kossioris 1993, Theorem 2.2] in the modern viscosity terms, as a direct application of Theorem 1.3 in [Crandall et al. 1984]. We give the statement for H integrable, i.e., depending only on p.

Definition 1.4 (Oleinik's entropy condition). Let $H:\mathbb{R}\to\mathbb{R}$ be a $\mathcal{C}^2$ Hamiltonian. If $(p_1,p_2)\in\mathbb{R}^2$, we say that *Oleinik's entropy condition* is (strictly) satisfied between p_1 and p_2 if

$$H(\mu p_1 + (1-\mu)p_2) \overset{(<)}{\le} \mu H(p_1) + (1-\mu)H(p_2) \quad \text{for all } \mu \in (0,1),$$

i.e., if and only if the graph of H lies (strictly) under the cord joining $(p_1, H(p_1))$ and $(p_2, H(p_2))$.

We say that the *Lax condition* is (strictly) satisfied if

$$H'(p_1)(p_2-p_1) \overset{(<)}{\le} H(p_2) - H(p_1) \overset{(<)}{\le} H'(p_2)(p_2-p_1),$$

which is implied by the entropy condition.

Proposition 1.5. *Let $u = \min(f_1, f_2)$ on an open neighborhood U of (t,q) in $\mathbb{R}_+\times\mathbb{R}$, with f_1 and f_2 $\mathcal{C}^1$ solutions on U of the Hamilton–Jacobi equation (HJ). Let p_1 and p_2 denote respectively $\partial_q f_1(t,q)$ and $\partial_q f_2(t,q)$. If $f_1(t,q) = f_2(t,q)$, then u is a viscosity solution at (t,q) if and only if the entropy condition is satisfied between p_1 and p_2.*

Oleinik's entropy condition is also valid in higher dimensions (for shock along a smooth hypersurface); see Theorem 3.1 in [Izumiya and Kossioris 1996], and can be generalized when u is the minimum of more than two functions: see [Bernard 2013].

1C. *Variational solutions.* If u_0 is a $\mathcal{C}^1$ initial condition, the *wavefront* associated with the Cauchy problem for u_0 is denoted by $\mathcal{F}_{u_0}$ and defined by

$$\mathcal{F}_{u_0} = \{(t, q, u_0(q_0) + \mathcal{A}_0^t(\gamma)) \mid t \geq 0,\ q \in \mathbb{R}^d,\ p_0 = du_0(q_0),\ Q_0^t(q_0, p_0) = q\}. \tag{F}$$

With this definition, the method of characteristics explained in Section 1A states that if u is a $\mathcal{C}^2$ solution on $[0, T] \times \mathbb{R}^d$, the restrictions on $[0, T] \times \mathbb{R}^d$ of the graph of u and of the wavefront coincide.

If u_0 is $\mathcal{C}^1$, we will call a *variational solution* of the Cauchy problem associated with u_0 a continuous function whose graph is included in the wavefront $\mathcal{F}_{u_0}$, i.e., a continuous function $g : [0, T] \times \mathbb{R}^d$ such that for all (t, q) in $[0, \infty) \times \mathbb{R}^d$ there exists (q_0, p_0) such that $p_0 = d_{q_0} u_0,\ Q_0^t(q_0, p_0) = q$ and

$$g(t, q) = u_0(q_0) + \mathcal{A}_0^t(\gamma),$$

where γ denotes the Hamiltonian trajectory issued from (q_0, p_0).

A family of operators $(R_s^t)_{s \leq t}$ mapping $\mathcal{C}^{0,1}(\mathbb{R}^d)$ into itself is called a *variational operator* if it satisfies the following conditions:

(1) Monotonicity: if $u \leq v$ are Lipschitz on $\mathbb{R}^d$, then $R_s^t u \leq R_s^t v$ on $\mathbb{R}^d$ for each $s \leq t$.

(2) Additivity: if u is Lipschitz on $\mathbb{R}^d$ and $c \in \mathbb{R}$, then $R_s^t(c + u) = c + R_s^t u$.

(3) Variational property: for each $\mathcal{C}^1$ Lipschitz function u_s, q in $\mathbb{R}^d$ and $s \leq t$, there exists (q_s, p_s) such that $p_s = d_{q_s} u_s,\ Q_s^t(q_s, p_s) = q$ and

$$R_s^t u_s(q) = u_s(q_s) + \mathcal{A}_s^t(\gamma),$$

where γ denotes the Hamiltonian trajectory issued from $(q(s), p(s)) = (q_s, p_s)$.

In the case of a compactly supported Hamiltonian, the existence of such a variational operator was introduced by Sikorav to his peers in 1990 and reported in [Chaperon 1991]. The author proceeded to its construction without compactness assumptions in [Roos 2019]; see Proposition 1.7.

The third property means that the variational operator maps initial data in variational solutions. There may be more than one variational solution associated with a Cauchy problem, and Proposition 1.9 states that some of them cannot be given by a variational operator. Example 1.10 presents such a situation with a nonsmooth initial value.

Remark 1.6. If a family of operators R satisfies (1) and (2), and if u and v are two Lipschitz functions on $\mathbb{R}^d$ with bounded difference, then

$$\|R_s^t u - R_s^t v\|_\infty \leq \|u - v\|_\infty.$$

As a consequence, for all $s \leq t$, R_s^t is a weak contraction, and it is continuous for the uniform norm.

Existence and local estimates. The existence of such a variational operator is given by the method of Sikorav and Chaperon; see [Viterbo 1996]. It is possible to obtain localized estimates on this family of variational operators that are also valid for the viscosity operator (in fact, they are obtained for the viscosity operator by a limit iterating process, see [Wei 2014]). They are stated explicitly for integrable Hamiltonians in [Roos 2019, Addendum 2.26].

Proposition 1.7. *There exists a family of variational operators $(R^t_{s,H})_H$ such that if $H(p)$ and $\widetilde{H}(p)$ are two integrable Hamiltonians with bounded second derivatives, then for $0 \le s \le t$ and u L-Lipschitz*

- $\|R^t_{s,\widetilde{H}}u - R^t_{s,H}u\|_\infty \le (t-s)\|\widetilde{H} - H\|_{\bar{B}(0,L)}$,
- $\|V^t_{s,\widetilde{H}}u - V^t_{s,H}u\|_\infty \le (t-s)\|\widetilde{H} - H\|_{\bar{B}(0,L)}$,

where $\bar{B}(0, L)$ denotes the closed ball of radius L centered in 0 and $\|f\|_K := \sup_K |f|$.

1D. *Extension to nonsmooth initial data.*

Lipschitz initial data. We will denote by $\partial u(q)$ the *Clarke derivative* of a function $u : \mathbb{R}^d \to \mathbb{R}$ at a point $q \in \mathbb{R}^d$. If u is Lipschitz, it is the convex envelope of the set of reachable derivatives:

$$\partial u(q) = \mathrm{co}\Big(\Big\{ \lim_{n\to\infty} du(q_n) \,\Big|\, q_n \to q \text{ as } n \to \infty,\ q_n \in \mathrm{dom}(du)\Big\}\Big).$$

It is the singleton $\{du(q)\}$ if u is $\mathcal{C}^1$ on a neighborhood of q. Variational property (3) can be extended to include a Lipschitz initial condition with the help of this generalized derivative.

Proposition 1.8. *If R^t_s is a variational operator, for each Lipschitz function u_s, q in $\mathbb{R}^d$ and $s \le t$, there exists (q_s, p_s) such that $p_s \in \partial_{q_s} u_s$, $Q^t_s(q_s, p_s) = q$ and if γ denotes the Hamiltonian trajectory issued from $(q(s), p(s)) = (q_s, p_s)$,*

$$R^t_s u_s(q) = u_s(q_s) + \mathcal{A}^t_s(\gamma).$$

The proof of this proposition can be found in [Roos 2017, Proposition 1.22].

If u_0 is a Lipschitz initial condition, the generalized wavefront associated with the Cauchy problem for u_0 is still denoted by $\mathcal{F}_{u_0}$ and defined by

$$\mathcal{F}_{u_0} = \{(t, q, u_0(q_0) + \mathcal{A}^t_0(\gamma)) \mid t \ge 0,\ q \in \mathbb{R}^d,\ p_0 \in \partial u_0(q_0),\ Q^t_0(q_0, p_0) = q\}. \tag{F$'$}$$

Proposition 1.8 implies that a variational operator applied to u_0 gives a continuous section of the wavefront $\mathcal{F}_{u_0}$. We will still call a variational solution a Lipschitz function whose graph is contained in the generalized wavefront.

Semiconcave initial data. A function $u : \mathbb{R}^d \to \mathbb{R}$ is *B-semiconcave* if $q \mapsto u(q) - \frac{1}{2}B\|q\|^2$ is concave. The function u is *semiconcave* if there exists B for which u is B-semiconcave.

The following theorem states that every variational operator maps semiconcave initial data onto the minimal section of the wavefront $\mathcal{F}_{u_0}$, at least for $[0, T]$, where T depends only on the semiconcavity constant and on the constant C given by Hypothesis 1.1.

Proposition 1.9. *If R^t_s is a variational operator and if u_0 is a Lipschitz B-semiconcave initial condition for some $B > 0$, then there exists $T > 0$ depending only on C and B such that, for all (t, q) in $[0, T] \times \mathbb{R}^d$,*

$$\begin{aligned} R^t_0 u_0(q) &= \inf\{S \mid (t, q, S) \in \mathcal{F}_{u_0}\} \\ &= \inf\{u_0(q_0) + \mathcal{A}^t_0(\gamma) \mid (q_0, p_0) \in \mathbb{R}^d \times \mathbb{R}^d,\ p_0 \in \partial u_0(q_0),\ Q^t_0(q_0, p_0) = q\}, \end{aligned} \tag{1}$$

where γ denotes the Hamiltonian trajectory issued from $(q(0), p(0)) = (q_0, p_0)$.

Moreover if H is integrable (i.e., depends only on p), we can choose $T = 1/(BC)$.

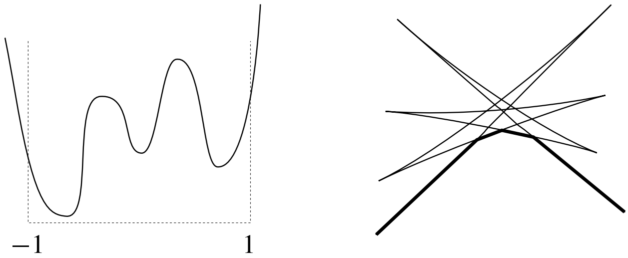

Figure 1. Left: graph of H. Right: cross-section of the wavefront $\mathcal{F}_{u_0}$ at time t.

This theorem implies on one hand that for a semiconcave initial condition, the minimal section of the wavefront is continuous for small times. On the other hand, it yields that the variational operator gives in that case a variational solution which is pointwise less than or equal to all other variational solutions on $[0, T] \times \mathbb{R}^d$.

Example 1.10. In dimension 1, if $u_0(q) = -|q|$ and if the Hamiltonian is integrable and has the shape of Figure 1, left, the wavefront at time t has the shape of Figure 1, right, and its minimal section, thickened in the figure, gives the value of $R_0^t u_0$ above each point q. In this example, there are five different variational solutions, but only the minimal one is given by a variational operator.

An analogous argument to the one proving Proposition 1.9 gives a first element of comparison between viscosity and variational solutions in the semiconcave framework. It is originally due to P. Bernard [2013].

Proposition 1.11. *Let H be a Hamiltonian satisfying Hypothesis 1.1 with constant C. If R_s^t is a variational operator and u_0 is a Lipschitz B-semiconcave initial condition for some $B > 0$, then there exists $T > 0$ depending only on C and B such that*

$$V_0^t u_0 \leq R_0^t u_0$$

for all $0 \leq t \leq T$. Consequently, the viscosity solution is smaller than any variational solution on $[0, T] \times \mathbb{R}^d$.

Moreover if H is integrable, we can choose $T = 1/(BC)$.

The article is organized as follows: Section 6 is independent from the rest; in it we prove Propositions 1.9 and 1.11 for any Hamiltonian satisfying Hypothesis 1.1. The rest of the article deals with integrable Hamiltonians: In Section 2 we prove Corollary 2.2, which is a Lipschitz version of Theorem 1. It is a corollary of Proposition 2.1, stated in terms of semiconcave initial conditions, which is proved by reduction to one- or two-dimensional considerations, contained in Propositions 2.3 and 2.4. In Section 3 we study the case of dimension 1 and prove Proposition 2.3. In Section 4 we study an example for the saddle Hamiltonian in dimension 2 in order to prove Proposition 2.4. In Section 5 we deduce Theorem 1 from its Lipschitz counterpart Corollary 2.2 by approximation.

2. Nonsmooth version of Theorem 1

Nonsmooth refers here to the initial condition. In this section we prove the following proposition, from which we deduce Corollary 2.2, which is the counterpart of Theorem 1 for a nonsmooth initial condition.

Proposition 2.1. *If $p \mapsto H(p)$ is a neither convex nor concave, integrable Hamiltonian with second derivative bounded by C, there exist $B > 0$ and a Lipschitz B-semiconcave initial condition u_0 such that the variational solution given by the minimal section of the wavefront does not solve (HJ) in the viscosity sense at some point (t, q) of $[0, 1/(BC)] \times \mathbb{R}^d$.*

Corollary 2.2. *If $p \mapsto H(p)$ is a neither convex nor concave, integrable Hamiltonian with bounded second derivative, there exists a Lipschitz initial condition u_0 such that the graph of the viscosity solution associated with u_0 is not included in the wavefront $\mathcal{F}_{u_0}$.*

To be more precise, the initial condition can be chosen so that the graph of the viscosity solution is below the minimal section of the wavefront for small times:

Proof of Corollary 2.2. Take a B-semiconcave initial condition u_0 as in Proposition 2.1. If C is a bound on d^2H, Proposition 1.9 states on one hand that the minimal section of the wavefront coincides with a variational solution on $[0, 1/(BC)] \times \mathbb{R}^d$, and on the other hand Proposition 1.11 gives that on the same set, the viscosity solution associated with u_0 is pointwise less than or equal to any variational solution. As a consequence the graph of the viscosity solution lies below the wavefront, and cannot coincide with the minimal section by Proposition 2.1. Hence there is a point of $[0, 1/(BC)] \times \mathbb{R}^d$ above which the graph of the viscosity solution lies strictly below the wavefront. □

The outline of the proof of Proposition 2.1 is the following: we give the statements in dimension 1 (Proposition 2.3) and for $H(p_1, p_2) = p_1 p_2$ (Proposition 2.4), and then reduce the situation to the first case or to an approximation of the second case. Proposition 2.5 gives for that purpose a characterization of neither convex nor concave functions, and Proposition 2.7 deals with the effect on the variational and viscosity operators of an affine transformation or dimensional reduction of the Hamiltonian.

Proposition 2.3 (one-dimensional case). *If $H : \mathbb{R} \to \mathbb{R}$ is a neither convex nor concave, integrable Hamiltonian with bounded second derivative, there exists $\delta > 0$ and a semiconcave Lipschitz initial condition u_0 such that*

$$R^t_{0,H} u_0 \neq V^t_{0,H} u_0 \quad \textit{for all } t < \delta.$$

Note that δ will be small enough so that $R^t_{0,H} u_0$ is uniquely defined, by Proposition 1.9. This proposition is proved in Section 3A, and is really based on the example in dimension 1 known at least since [Chenciner 1975]. In contrast, the following two-dimensional example is the main novelty of this work.

Proposition 2.4 (saddle Hamiltonian). *If $H(p_1, p_2) = p_1 p_2$, for all $L > 0$ there exists an L-Lipschitz, L-semiconcave initial condition u_0 such that*

$$R^t_{0,H} u_0 \neq V^t_{0,H} u_0 \quad \textit{for all } t < \frac{1}{2L}.$$

Note that $R^t_{0,H} u_0$ is uniquely defined when $t < 1/(2L)$ by Proposition 1.9. This proposition is proved in Section 4, where we explicitly state a suitable initial condition for which the wavefront has a single continuous section with a shock denying the entropy condition.

The following proposition makes precise the idea that a nonconvex, nonconcave function is either a wave or a saddle. We will proceed further with the reduction of a one-dimensional nonconvex, nonconcave function in Lemma 3.4.

Proposition 2.5. *A $\mathcal{C}^2$ function $f : \mathbb{R}^n \to \mathbb{R}$ is neither convex nor concave if and only if it is neither convex nor concave along a straight line, or there exists x in $\mathbb{R}^n$ such that the Hessian $\mathcal{H}f(x)$ admits both positive and negative eigenvalues.*

Proof. We denote by $S_n^+(\mathbb{R})$ and $S_n^-(\mathbb{R})$ respectively the sets of nonnegative and nonpositive symmetric matrices.

Since a $\mathcal{C}^2$ function is convex (resp. concave) if and only if its Hessian admits only nonnegative (resp. nonpositive) eigenvalues, it is enough to prove the following statement: if f is a nonconvex and nonconcave $\mathcal{C}^2$ function with $\mathcal{H}f(x) \in S_n^+(\mathbb{R}) \cup S_n^-(\mathbb{R})$ for all x, there exists a straight line along which f is neither concave nor convex.

Under the assumptions of this statement, the sets $U_1 = \{x \in \mathbb{R}^n \mid \mathcal{H}f(x) \in S_n^-(\mathbb{R}) \setminus \{0\}\}$ and $U_2 = \{x \in \mathbb{R}^n \mid \mathcal{H}f(x) \in S_n^+(\mathbb{R}) \setminus \{0\}\}$ are open and nonempty: if U_1 is empty, f is necessarily convex. If x_1 is in U_1, then $\mathcal{H}f(x_1)$ admits a negative eigenvalue. Hence for x close enough to x_1, $\mathcal{H}f(x)$ admits a negative eigenvalue and since $\mathcal{H}f(x) \in S_n^+(\mathbb{R}) \cup S_n^-(\mathbb{R})$ by hypothesis, necessarily $\mathcal{H}f(x)$ is in U_1. We are going to apply the following lemma to the continuous function $A = \mathcal{H}f$ and the sets U_1 and U_2.

Lemma 2.6. *If $A : \mathbb{R}^n \to M_n(\mathbb{R})$ is a continuous function and U_1 and U_2 are two disjoint open sets on which A does not vanish, there exists $(x_1, x_2) \in U_1 \times U_2$ such that*

$$x_1 - x_2 \notin \operatorname{Ker} A(x_1) \cup \operatorname{Ker} A(x_2).$$

Now, let us take (x_1, x_2) in $U_1 \times U_2$ such that $x_1 - x_2 \notin \operatorname{Ker} \mathcal{H}f(x_1) \cup \operatorname{Ker} \mathcal{H}f(x_2)$ and define $g(t) = f(tx_1 + (1-t)x_2)$. To show that the $\mathcal{C}^2$ function g is neither concave nor convex, we evaluate its second derivative

$$g''(t) = \mathcal{H}f(tx_1 + (1-t)x_2)(x_1 - x_2) \cdot (x_1 - x_2).$$

If A is in $S_n^+(\mathbb{R}) \cup S_n^-(\mathbb{R})$, $Ax \cdot x = 0$ if and only if $Ax = 0$. Since $\mathcal{H}f(x_1)$ is in $S_n^-(\mathbb{R})$ and $\mathcal{H}f(x_2)$ is in $S_n^+(\mathbb{R})$, and $x_1 - x_2 \notin \operatorname{Ker} \mathcal{H}f(x_1) \cup \operatorname{Ker} \mathcal{H}f(x_2)$, we obtain $g''(1) = \mathcal{H}f(x_1)(x_1 - x_2) \cdot (x_1 - x_2) < 0$ since $x_1 - x_2$ is not in $\operatorname{Ker} \mathcal{H}f(x_1)$, and $g''(0) = \mathcal{H}f(x_2)(x_1 - x_2) \cdot (x_1 - x_2) > 0$ since $x_1 - x_2$ is not in $\operatorname{Ker} \mathcal{H}f(x_2)$. Thus, g is neither concave nor convex. □

Proof of Lemma 2.6. For each $x_1^\circ \in U_1$, since $A(x_1^\circ)$ is a nonzero matrix, there exists x_2° in the open set U_2 such that $A(x_1^\circ)(x_1^\circ - x_2^\circ) \neq 0$. Since $(x_1, x_2) \mapsto A(x_1)(x_1 - x_2)$ is continuous, we may assume up to a reduction of U_1 and U_2 that $A(x_1)(x_1 - x_2) \neq 0$ for all $(x_1, x_2) \in U_1 \times U_2$.

Now let us fix x_2° in U_2. Again, since $A(x_2^\circ)$ is nonzero, there exists x_1° in the open set U_1 such that $A(x_2^\circ)(x_1^\circ - x_2^\circ) \neq 0$, and the previous argument gives that $A(x_1^\circ)(x_1^\circ - x_2^\circ) \neq 0$, hence the conclusion. □

The next proposition deals with the behavior of the variational and viscosity operators when reducing or transforming the Hamiltonian. Let us first describe formally the effect of such transformations on the classical solutions.

Affine transformations. Let H be a Hamiltonian on $\mathbb{R}^d$. Let A be an invertible matrix, b and n be vectors of $\mathbb{R}^d$, α a real value and λ a nonzero real value, and define $\overline{H}(p) = \frac{1}{\lambda} H(Ap+b) + p \cdot n + \alpha$. If $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is $\mathcal{C}^1$ and $v(t,q) = u(\lambda t, {}^tAq + \lambda tn) + b \cdot q + \alpha\lambda t$, then for all (t,q)

$$\partial_t u(\tilde{t}, \tilde{q}) + \overline{H}(\partial_q u(\tilde{t}, \tilde{q})) = 0 \quad \Longleftrightarrow \quad \partial_t v(t,q) + H(\partial_q v(t,q)) = 0,$$

with $(\tilde{t}, \tilde{q}) = (\lambda t, {}^tAq + \lambda tn)$.

Reduction. Assume that H is defined on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Let us fix p_2 in $\mathbb{R}^{d_2}$ and define $\overline{H}(p_1) = H(p_1, p_2)$. If $u : \mathbb{R} \times \mathbb{R}^{d_1} \to \mathbb{R}$ is $\mathcal{C}^1$ and $v(t, q_1, q_2) = u(t, q_1) + p_2 \cdot q_2$, then for all (t, q_1, q_2)

$$\partial_t u(t, q_1) + \overline{H}(\partial_{q_1} u(t, q_1)) = 0 \quad \Longleftrightarrow \quad \partial_t v(t, q_1, q_2) + H(\partial_{q_1} v(t, q_1, q_2), \partial_{q_2} v(t, q_1, q_2)) = 0.$$

Let us translate this in terms of operators.

Proposition 2.7. *Let H be a $\mathcal{C}^2$ Hamiltonian with second derivative bounded by C:*

(1) *Affine transformations: Let u_0 be a Lipschitz B-semiconcave initial condition. If $\overline{H}(p) = \frac{1}{\lambda} H(Ap+b) + p \cdot n + \alpha$ and $v_0(q) = u_0({}^tAq) + b \cdot q$, then*

$$V^t_{0,H} v_0(q) = V^{\lambda t}_{0,\overline{H}} u_0({}^tAq + \lambda tn) + b \cdot q + \alpha\lambda t$$

for all (t,q) and

$$R^t_{0,H} v_0(q) = R^{\lambda t}_{0,\overline{H}} u_0({}^tAq + \lambda tn) + b \cdot q + \alpha\lambda t$$

as long as $t < 1/(\|A\|^2 BC)$.

(2) *Reduction: Assume that H is defined on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, fix p_2 in $\mathbb{R}^{d_2}$ and define $\overline{H}(p_1) = H(p_1, p_2)$. If u_0 is a Lipschitz B-semiconcave function on $\mathbb{R}^{d_1}$, and $v_0(q_1, q_2) = u_0(q_1) + p_2 \cdot q_2$, then*

$$V^t_{0,H} v_0(q_1, q_2) = V^t_{0,\overline{H}} u_0(q_1) + p_2 \cdot q_2$$

for all (t, q_1, q_2) and

$$R^t_{0,H} v_0(q_1, q_2) = R^t_{0,\overline{H}} u_0(q_1) + p_2 \cdot q_2,$$

as long as $t < 1/(BC)$.

Proof. The viscosity equality is obtained by applying the formal transformation or reduction on the test functions (see Definition 1.2), and the variational equality is obtained for small times by applying Proposition 1.9 with the domain of validity given for integrable Hamiltonians, which is the same for $(\overline{H}, u_0)$ and (H, v_0) in both cases:

Affine transformations: Since v_0 is $B\|A\|^2$-semiconcave, the domain of validity for (H, v_0) is at least $[0, 1/(\|A\|^2 BC))$. But $\|d^2\overline{H}\| \leq C\|A\|^2/\lambda$, and hence the domain of validity for $(\overline{H}, u_0)$ is at least $[0, \lambda/(\|A\|^2 BC))$ and λt is in this domain if $t < 1/(\|A\|^2 BC)$.

Reduction: Since $\|d^2\overline{H}\| \leq C$ and v_0 is B-semiconcave, the domain of validity for both $(\overline{H}, u_0)$ and (H, v_0) is at least $[0, 1/(BC)]$. □

Proof of Proposition 2.1. If H is a neither convex nor concave, integrable Hamiltonian, Proposition 2.5 states that there is either a straight line along which H is neither convex nor concave, or a point p_0 such that the Hessian matrix $\mathcal{H}H(p_0)$ has both a positive and a negative eigenvalue.

In the first case, applying an affine transformation on the vector space we may assume without loss of generality (see Proposition 2.7(1)) that $p \in \mathbb{R} \mapsto H(p, 0, \dots, 0)$ is neither convex nor concave, and we denote by $\overline{H}(p) = H(p, 0, \dots, 0)$ the reduced Hamiltonian. Applied to $\overline{H}$, Proposition 2.3 gives a semiconcave initial condition u_0 such that $R^t_{0,\overline{H}} u_0 \neq V^t_{0,\overline{H}} u_0$ for all $t < T$. With Proposition 2.7(2), we get from u_0 a semiconcave Lipschitz initial condition $v_0 : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ for which $R^t_{0,H} v_0 \neq V^t_{0,H} v_0$ for all $t < T$.

In the second case, we may assume that the point of interest is a (strict) saddle point at 0: if p_0 denotes the point for which $\mathcal{H}H(p_0)$ has both a positive and a negative eigenvalue, take $\widetilde{H}(p) = H(p_0 - p) + p \cdot \nabla H(p_0) - H(p_0)$ and apply Proposition 2.7(1).

Then, up to another linear transformation on the vector space, the Hamiltonian may even be taken as

$$H(p_1, p_2, \dots, p_d) = p_1 p_2 + K(p_1, p_2, \dots, p_d),$$

where K is a $\mathcal{C}^2$ Hamiltonian with partial derivatives with respect to p_1 and p_2 vanishing at the second order:

$$K(0, \dots, 0) = 0, \quad \partial_{p_1} K(0, \dots, 0) = 0, \quad \partial_{p_2} K(0, \dots, 0) = 0, \quad \partial^2_{(p_1, p_2)} K(0, \dots, 0) = 0.$$

We denote by $\overline{H}$ and $\overline{K}$ the reduced Hamiltonians such that

$$\overline{H}(p_1, p_2) = H(p_1, p_2, 0, \dots, 0) = p_1 p_2 + \overline{K}(p_1, p_2).$$

We still denote by C a bound on the second derivatives of H and $\overline{H}$.

Now, we define

$$\overline{H}_\varepsilon(p_1, p_2) = \frac{1}{\varepsilon^2} \overline{H}(\varepsilon p_1, \varepsilon p_2) = p_1 p_2 + \frac{1}{\varepsilon^2} \overline{K}(\varepsilon p_1, \varepsilon p_2)$$

and

$$\overline{H}_0(p_1, p_2) = p_1 p_2.$$

We fix $L > 0$ and take u_0 as in Proposition 2.4: for all $0 < t < 1/(2L)$, there exists a point q_t such that $R^t_{0,\overline{H}_0} u_0(q_t) \neq V^t_{0,\overline{H}_0} u_0(q_t)$. Let us now fix t in $(0, 1/(2L))$.

Proposition 1.7 gives

$$\| R^t_{0,\overline{H}_\varepsilon} u_0(q_t) - R^t_{0,\overline{H}_0} u_0(q_t) \| \leq t \sup_{\|p\| \leq L} \frac{1}{\varepsilon^2} \overline{K}(\varepsilon p),$$

$$\| V^t_{0,\overline{H}_\varepsilon} u_0(q_t) - V^t_{0,\overline{H}_0} u_0(q_t) \| \leq t \sup_{\|p\| \leq L} \frac{1}{\varepsilon^2} \overline{K}(\varepsilon p).$$

Since $\overline{K}$ is zero until second order at 0, we know that $(1/\varepsilon^2)\overline{K}(\varepsilon p) = \circ(\|p\|^2)$ and $\sup_{\|p\| \leq L} (1/\varepsilon^2) \overline{K}(\varepsilon p)$ tends to 0 when ε tends to 0. Thus, there exists $\varepsilon > 0$ (depending on t) such that

$$\sup_{\|p\| \leq L} \frac{1}{\varepsilon^2} \overline{K}(\varepsilon p) < \frac{1}{3t} | R^t_{0,\overline{H}_0} u_0(q_t) - V^t_{0,\overline{H}_0} u_0(q_t) |,$$

and for such an ε, we then have $R^t_{0,\overline{H}_\varepsilon} u_0(q_t) \neq V^t_{0,\overline{H}_\varepsilon} u_0(q_t)$.

Let us go back to $\overline{H}$, using Proposition 2.7(1) with $\lambda = \varepsilon^2$, $A = \varepsilon\mathrm{id}$ and n, b and α equal to zero. Defining $v_0(q) = u_0(\varepsilon q)$, we get

$$V^{t/\varepsilon^2}_{0,\overline{H}} v_0\left(\frac{q_t}{\varepsilon}\right) = V^t_{0,\overline{H}_\varepsilon} u_0(q_t),$$

$$R^{t/\varepsilon^2}_{0,\overline{H}} v_0\left(\frac{q_t}{\varepsilon}\right) = R^t_{0,\overline{H}_\varepsilon} u_0(q_t),$$

as long as

$$\frac{t}{\varepsilon^2} < \frac{1}{\varepsilon^2 LC}$$

(which is the case since $C > 2$and $t < 1/(2L)$), and as a consequence

$$V^{t/\varepsilon^2}_{0,\overline{H}} v_0\left(\frac{q_t}{\varepsilon}\right) \neq R^{t/\varepsilon^2}_{0,\overline{H}} v_0\left(\frac{q_t}{\varepsilon}\right).$$

Note that since v_0 is $\varepsilon^2 L$-semiconcave, t/ε^2 belongs to the domain of validity of Proposition 1.9, which is here $(0, 1/(\varepsilon^2 LC))$. As in the previous case we get the semiconcave initial condition suiting the nonreduced Hamiltonian H via Proposition 2.7(2). □

3. One-dimensional integrable Hamiltonian

With the help of Lemma 3.4, stated and proved at the end of this section, we reduce Proposition 2.3, the one-dimensional counterpart of Proposition 2.1 (see Section 2), to the following statement, giving a situation where there is only one variational solution, which does not match with the viscosity solution.

Proposition 3.1. *Let H be a $\mathcal{C}^2$ Hamiltonian with bounded second derivative such that $H(-1) = H(1) = H'(1) = 0$, $H'(-1) < 0$, $H''(1) < 0$, and $H < 0$ on $(-1, 1)$.*

Then if f is a $\mathcal{C}^2$ Lipschitz function with $f(0) = f'(0) = 0$, with bounded second derivative and strictly convex on $\mathbb{R}_+$, and $u_0(q) = -|q| + f(q)$, the unique variational solution $(t, q) \mapsto R^t_0 u_0(q)$ does not solve the Hamilton–Jacobi equation (HJ) in the viscosity sense for all t small enough.

With the vocabulary of Definition 1.4, we work here on a specific case where the entropy condition is strictly satisfied between the derivatives at 0 of the initial condition, and the Lax condition is strictly satisfied on one side with equality on the other side; see Figure 2, left.

The proof consists in showing that under the assumptions of Proposition 3.1, when t is small enough, the wavefront at time t presents a unique continuous section, with a shock that does not satisfy Oleinik's entropy condition (see Proposition 1.5).

3A. ***Proof of Proposition 3.1.*** Let us fix the notation for the parametrization that follows directly from the wavefront definition (see (F′)). Since u_0 is differentiable on $\mathbb{R} \setminus \{0\}$, its Clarke derivative is reduced to a point outside zero and is the segment $[-1, 1]$ at zero. The wavefront is hence the union of three pieces $\mathcal{F}^\ell_t$, $\mathcal{F}^r_t$ and $\mathcal{F}^0_t$ respectively issued from the left part, the right part, and the singularity of the

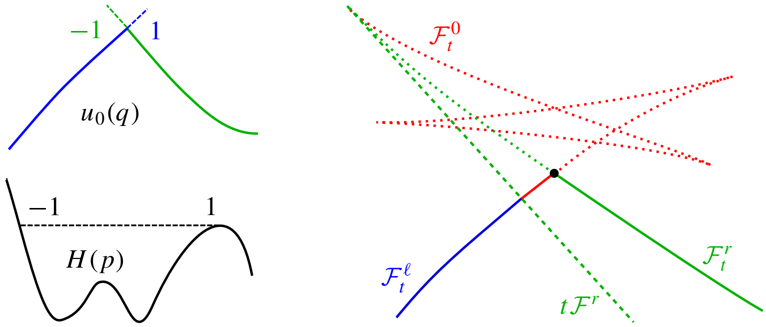

Figure 2. The variational solution, given by the unique continuous section of the wavefront, does not solve the (HJ) equation in the viscosity sense at the dot.

initial condition, with the following parametrizations:

$$\mathcal{F}_t^\ell : \begin{cases} q + tH'(u_0'(q)), \\ u_0(q) + tu_0'(q)H'(u_0'(q)) - tH(u_0'(q)), \end{cases} \qquad q < 0,$$

$$\mathcal{F}_t^r : \begin{cases} q + tH'(u_0'(q)), \\ u_0(q) + tu_0'(q)H'(u_0'(q)) - tH(u_0'(q)), \end{cases} \qquad q > 0,$$

$$\mathcal{F}_t^0 : \begin{cases} tH'(p), \\ t(pH'(p) - H(p)), \end{cases} \qquad p \in [-1, 1].$$

The structure of the wavefront for small times is addressed by Lemma 3.2. Figure 2 presents an example of the situation.

Lemma 3.2. *Under the assumptions of Proposition 3.1, there exists $\delta > 0$ such that for all $0 < t < \delta$, the wavefront $\mathcal{F}_t$ has a unique continuous section, presenting a shock between $\mathcal{F}_t^0$ and $\mathcal{F}_t^r$.*

With the previous parametrization, we may easily compute the slopes and convexity of the wavefront. We still denote by C and B the bounds on the second derivatives of H and u_0.

Proposition 3.3. (1) *Slopes on the wavefront: If $H''(p) \neq 0$ and $t > 0$, the slope of $\mathcal{F}_t^0$ at the point of parameter p is p. If $t < 1/(BC)$, the slope of $\mathcal{F}_t^r$ at the point of parameter q is $u_0'(q)$.*

(2) *Convexity of the right arm: If u_0 is convex (resp. concave) on $[0, \delta]$, then for $t < 1/(BC)$, the portion of $\mathcal{F}_t^r$ parametrized by $q \in (0, \delta]$ is convex (resp. concave).*

Proof. (1) If $(x(u), y(u))$ is the parametrization of a curve, the slope at the point of parameter u is given by $y'(u)/x'(u)$ when $x'(u)$ is nonzero. For $\mathcal{F}_t^0$, we have $x'(p) = tH''(p)$ and $y'(p) = px'(p)$, which proves the statement. For $\mathcal{F}_t^r$, if $t < 1/(BC)$, then $x'(q) = 1 + tu_0''(q)H''(u_0'(q)) > 0$ since u_0'' and H'' are respectively bounded by B and C, and the statement results from $y'(q) = u_0'(q)x'(q)$.

(2) The convexity of $\mathcal{F}_t^r$ at a point of parameter q is given by the sign of the ratio

$$\frac{x'(q)y''(q) - x''(q)y'(q)}{x'(q)^3}.$$

For $t < 1/(BC)$, we have $x'(q) > 0$ and as $y'(q) = u_0'(q)x'(q)$,

$$\frac{x'(q)y''(q) - x''(q)y'(q)}{x'(q)^3} = \frac{x'(u_0''x' + u_0'x'') - x''u_0'x'}{x'^3} = \frac{u_0''(q)}{x'(q)},$$

which proves the statement. □

The fact that $\mathcal{F}_t^0$ depends homothetically on t suggests to look for each $t > 0$ at the homothetic reduction of the wavefront at time t, where both coordinates are divided by t. We call it *reduced wavefront* and denote it by $\widetilde{\mathcal{F}}_t$. It admits the following parametrizations, where $q = tx$:

$$\widetilde{\mathcal{F}}_t^\ell : \begin{cases} x + H'(u_0'(tx)), \\ \dfrac{u_0(tx)}{t} + u_0'(tx)H'(u_0'(tx)) - H(u_0'(tx)), \end{cases} \qquad x < 0,$$

$$\widetilde{\mathcal{F}}_t^r : \begin{cases} x + H'(u_0'(tx)), \\ \dfrac{u_0(tx)}{t} + u_0'(tx)H'(u_0'(tx)) - H(u_0'(tx)), \end{cases} \qquad x > 0,$$

$$\widetilde{\mathcal{F}}_t^0 : \begin{cases} H'(p), \\ pH'(p) - H(p), \end{cases} \qquad p \in [-1, 1].$$

The asset of the reduced wavefront is that it admits a nontrivial limit when t tends to 0. The piece issued from the singularity $\mathcal{F}^0 = \widetilde{\mathcal{F}}_t^0$ does not depend on t, while $\widetilde{\mathcal{F}}_t^r$ and $\widetilde{\mathcal{F}}_t^\ell$ converge to straight half-lines denoted by $\mathcal{F}^r$ and $\mathcal{F}^\ell$. These half-lines coincide respectively with $\widetilde{\mathcal{F}}_t^r$ and $\widetilde{\mathcal{F}}_t^\ell$ at their fixed endpoints; see $t\mathcal{F}^r$ and $\mathcal{F}_t^r$ in Figure 2. A consequence of Proposition 3.3 is that $\widetilde{\mathcal{F}}_t^\ell$ is a graph as long as $t < 1/(BC)$, and the same applies to $\widetilde{\mathcal{F}}_t^r$.

Proof of Lemma 3.2. It is enough to prove the result for the reduced wavefront $\widetilde{\mathcal{F}}_t$, where both coordinates are divided by t. Using the left and right derivatives of u_0 and the fact that $H(1) = H(-1) = H'(1) = 0$, we write the parametrization of the limit of the reduced wavefront:

$$\mathcal{F}^\ell : \begin{cases} x, \\ x, \end{cases} \qquad x < 0,$$

$$\mathcal{F}^r : \begin{cases} x + H'(-1), \\ -x - H'(-1), \end{cases} \qquad x > 0,$$

$$\mathcal{F}^0 : \begin{cases} H'(p), \\ pH'(p) - H(p), \end{cases} \qquad p \in [-1, 1].$$

The left and right arms of the limit front are respectively the graph of $-\mathrm{id}$ and id on $(-\infty, 0)$ and on $(H'(-1), \infty)$, where $H'(-1) < 0$. The assumption $H < 0$ on $(-1, 1)$ implies that for all p in $(-1, 1)$,

$$pH'(p) - H(p) > -|H'(p)|, \tag{2}$$

and this inequality is also satisfied for $p = -1$ since $H(-1) = 0$ and $H'(-1) < 0$. The unique continuous section of the limit front is hence the graph of $x \mapsto -|x|$. It presents a shock at $(0, 0)$, which belongs to $\mathcal{F}^r$ and $\mathcal{F}^0$ respectively with parameters $x = -H'(-1) > 0$ and $p = 1$. Furthermore, (2) implies that this shock is not a double point of $\mathcal{F}^0$.

Proposition 3.3 states that since f is strictly convex on $\mathbb{R}_+$, $\mathcal{F}_t^r$ and hence $\widetilde{\mathcal{F}}_t^r$ are strictly convex curves for all $t > 0$. Looking at the slope for a parameter $x \to 0$ shows that $\widetilde{\mathcal{F}}_t^r$ admits the right arm of the limit front, $\mathcal{F}^r$, as a tangent at its endpoint. Since $\widetilde{\mathcal{F}}_t^r$ is convex, it is hence positioned strictly above $\mathcal{F}^r$. Since $\widetilde{\mathcal{F}}_t^\ell$ is for all $t < 1/(BC)$ a graph with fixed endpoint at $(0, 0)$, we may focus on what happens on the half-plane situated over the second diagonal.

As $H''(1) < 0$, there exists $\eta > 0$ such that $H'' < 0$ on $(1-\eta, 1]$, and the piece of $\mathcal{F}^0$ parametrized by $p \in (1-\eta, 1]$, denoted by $\mathcal{F}^0_{(1-\eta,1]}$, is immersed. Since $\mathcal{F}^0$ is compact, we may assume up to taking a smaller η that $\mathcal{F}^0_{(1-\eta,1]}$ does not contain any double point either. With this choice of η, the intersection $\mathcal{F}^r \cap \mathcal{F}^0_{(1-\eta,1]}$ is exactly the point $(0, 0)$ and is transverse, since the slopes at the shock are -1 and 1.

Let us denote the family of parametrizations of $\widetilde{\mathcal{F}}_r^t \cup \mathcal{F}^r$ by

$$g^r(t, x) = \begin{cases} (x + H'(u_0'(tx)), u_0(tx)/t + u_0'(tx)H'(u_0'(tx)) - H(u_0'(tx))) & \text{if } t \neq 0, \\ (x + H'(-1), -x - H'(-1)) & \text{if } t = 0. \end{cases}$$

The function $t \mapsto g^r(t, \cdot)$ is continuous on $[0, \infty)$ in the $\mathcal{C}^1$-topology since the function

$$(t, x) \mapsto \begin{cases} u_0(tx)/t & \text{if } t > 0, \\ -x & \text{if } t = 0 \end{cases}$$

is $\mathcal{C}^1$ on $[0, \infty) \times [0, \infty)$. The transverse intersection hence persists by the implicit function theorem in an intersection between $\widetilde{\mathcal{F}}_t^r$ and $\mathcal{F}^0_{(1-\eta,1]}$, since $\widetilde{\mathcal{F}}_t^r$ is contained in the half-plane situated over the second diagonal.

There is no other continuous section in $\widetilde{\mathcal{F}}_t$: for small times t, $\widetilde{\mathcal{F}}_t^r$ and $\widetilde{\mathcal{F}}_t^\ell$ do not cross and do not present double points; the existence of a second continuous section would then imply the existence of an intersection between $\mathcal{F}^0$ and the part of $\widetilde{\mathcal{F}}_t^r$ at the right of the shock, or an intersection between $\mathcal{F}^0$ and $\widetilde{\mathcal{F}}_t^\ell$, and neither can exist, by continuity. □

It is now enough to prove that the obtained shock denies the Lax condition.

Proof of Proposition 3.1. For all t, the graph of a variational solution is included in the wavefront $\mathcal{F}_t$. Lemma 3.2 gives a $\delta > 0$ for which every $\mathcal{F}_t$ has a unique continuous section if $t \leq \delta$, which implies that the variational solution is given by this section for small t. Lemma 3.2 states also that this section presents a shock between $\mathcal{F}_t^0$ and $\mathcal{F}_t^r$.

Let us prove that Lax condition is violated at this shock. A fortiori, Oleinik's entropy condition is violated, which by Proposition 1.5 will imply that the variational solution is not a viscosity solution. For all t in $(0, \delta)$, the shock is given by parameters (q_t, p_t) such that $q_t > 0$, $p_t \in [-1, 1]$ and

$$\begin{cases} q_t + tH'(u_0'(q_t)) = tH'(p_t), \\ u_0(q_t) + tu_0'(q_t)H'(u_0'(q_t)) - tH(u_0'(q_t)) = tp_tH'(p_t) - tH(p_t). \end{cases}$$

Substituting the first equation multiplied by $u_0'(q_t)$ into the second gives, after reorganization,

$$t\big(H(p_t) - H(u_0'(q_t)) - (p_t - u_0'(q_t))H'(p_t)\big) = q_t u_0'(q_t) - u_0(q_t).$$

The linear part of u_0 cancels in the right-hand side, which equals $q_t f'(q_t) - f(q_t)$. The strict convexity of f implies that $f'(h) > f(h)/h$ for all $h > 0$; hence the right-hand side is positive for $t > 0$. As a

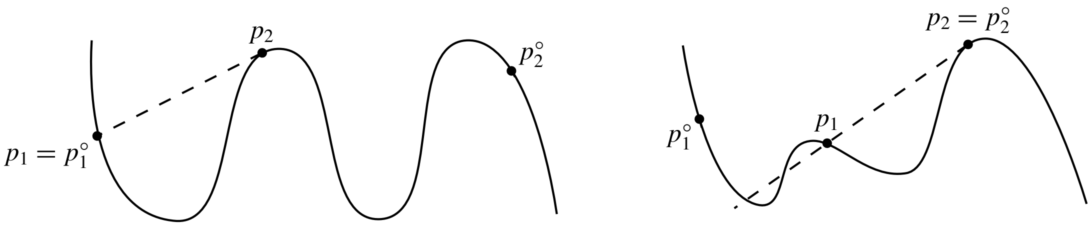

Figure 3. Both figures present a graph of H with a dashed tangent at p_2. Left: entropy condition not satisfied between p_1° and p_2°. Right: entropy condition satisfied between p_1° and p_2°.

consequence, for t in $(0, \delta)$,

$$H(p_t) - H(u_0'(q_t)) > (p_t - u_0'(q_t))H'(p_t).$$

By Proposition 3.3, the slopes at the shock are $u_0'(q_t)$ and p_t. Comparing with Definition 1.4, this inequality reads as the opposite of the Lax condition; hence Oleinik's entropy condition is violated for the shock presented by the variational solution for $t < \delta$, and the conclusion holds. □

3B. *Proof of Proposition 2.3.* The idea behind Lemma 3.4 is that for any nonconvex nonconcave Hamiltonian in dimension 1, there is a frame of variables over which the Hamiltonian looks like Figure 2, left.

Lemma 3.4. *If $H : \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^2$, neither convex nor concave Hamiltonian, up to a change of function $p \mapsto H(-p)$, there exist $p_1 < p_2$ such that $H''(p_2) < 0$, and,*

$$\text{for all } p \in (p_1, p_2), \quad \frac{H(p) - H(p_1)}{p - p_1} < \frac{H(p_2) - H(p_1)}{p_2 - p_1}, \tag{3}$$

$$H'(p_1) < \frac{H(p_2) - H(p_1)}{p_2 - p_1} = H'(p_2). \tag{4}$$

In terms of Definition 1.4, (3) means that the entropy condition is strictly satisfied between p_1 and p_2, and (4) that the Lax condition is an equality at p_2 and an inequality at p_1. We are now just one affine step away from the hypotheses of Proposition 3.1.

Proof. If $H : \mathbb{R} \to \mathbb{R}$ is neither convex nor concave, there exist in particular p_1° and p_2° such that $H''(p_1^\circ) > 0$ and $H''(p_2^\circ) < 0$, and we may assume up to the change of Hamiltonian $p \mapsto H(-p)$ that $p_1^\circ < p_2^\circ$.

Sketch of proof. The proof consists in choosing adequate p_1 and p_2, which will be done differently depending on the entropy condition between p_1° and p_2° being satisfied or not. An impatient reader could be satisfied by the choice of p_1 and p_2 suggested in Figure 3. If the entropy condition is not satisfied, we take $p_1 = p_1^\circ$ and p_2 such that the slope of the cord joining p_1 and p_2 is maximal. We then need to slightly perturb p_1 in order to get the condition $H''(p_2) < 0$. If the entropy condition is satisfied, we take $p_2 = p_2^\circ$ and p_1 is given by the last (before p_2) intersection between the tangent at p_2 and the graph of H. Again, a perturbation will be done to ensure that $H'(p_1) < H'(p_2)$.

- If the entropy condition is not satisfied between p_1° and p_2°, we define $p_1 = p_1^\circ$ and

$$p_2 = \inf\left\{ p \in (p_1, p_2^\circ) \;\middle|\; \frac{H(p) - H(p_1)}{p - p_1} = \sup_{\tilde{p} \in (p_1, p_2^\circ]} \frac{H(\tilde{p}) - H(p_1)}{\tilde{p} - p_1} \right\}.$$

Let us show that these quantities are well-defined, and prove (3) and (4).

The function

$$f : p \mapsto \frac{H(p) - H(p_1)}{p - p_1}$$

may be extended continuously at p_1 by $H'(p_1)$; hence it reaches a maximum M on $[p_1, p_2^\circ]$. It cannot be attained at p_1, or else the Taylor expansion of

$$\frac{H(p) - H(p_1)}{p - p_1} \le H'(p_1)$$

gives $H''(p_1) \le 0$, which is excluded. As a consequence $M > H'(p_1)$. It cannot be attained at p_2° because

$$\frac{H(p) - H(p_1)}{p - p_1} \le \frac{H(p_2^\circ) - H(p_1)}{p_2^\circ - p_1}$$

for all p in $[p_1, p_2^\circ)$ if and only if the entropy condition is satisfied between p_1 and p_2°, which is excluded. We hence proved that the supremum is attained on (p_1, p_2°). The infimum thus exists and belongs to $[p_1, p_2^\circ)$. By the continuity of f, we have $f(p_2) = M$. This implies that $p_2 > p_1$ since $f(p_1) = H'(p_1) < M$; hence the infimum is a minimum. The equality (3) follows directly from the definition of p_2.

Since p_2 is in (p_1, p_2°) and maximizes f, it is a critical point of f, which gives

$$H'(p_2) = \frac{H(p_2) - H(p_1)}{p_2 - p_1} = M.$$

Since $H'(p_1) < M$, (4) is proved.

Since p_2 maximizes f, we have $f''(p_2) \le 0$ and as a consequence $H''(p_2) \le 0$. In order to get $H''(p_2) < 0$, let us prove that if p_2° is fixed, $p_1 \mapsto H'(p_2)$ is increasing in a neighborhood of p_1.

For $\varepsilon > 0$ small enough, $p_1 + \varepsilon < p_2$, $H''(p_1 + \varepsilon) > 0$ and the entropy condition is not satisfied between $p_1 + \varepsilon$ and p_2°. We denote by $p_{2,\varepsilon}$ the quantity associated with $p_1 + \varepsilon$ and p_2° as before.

On one hand, by the definition of p_2, the entropy condition is strictly satisfied between p_1 and p_2, and in particular since $p_1 + \varepsilon$ is in (p_1, p_2),

$$\frac{H(p_2) - H(p_1 + \varepsilon)}{p_2 - (p_1 + \varepsilon)} > \frac{H(p_2) - H(p_1)}{p_2 - p_1} = H'(p_2).$$

On the other hand, the previous work applied to $p_{2,\varepsilon}$ gives

$$H'(p_{2,\varepsilon}) = \max_{p \in (p_1 + \varepsilon, p_2^\circ]} \frac{H(p) - H(p_1 + \varepsilon)}{p - (p_1 + \varepsilon)} \ge \frac{H(p_2) - H(p_1 + \varepsilon)}{p_2 - (p_1 + \varepsilon)},$$

and the two inequalities combined give $H'(p_{2,\varepsilon}) > H'(p_2)$.

Since $p_1 \mapsto H'(p_2)$ is increasing in a neighborhood of p_1, using Sard's theorem, we may assume without loss of generality that $H'(p_2)$ is a regular value of H', up to a perturbation of p_1 within the open set $\{H'' > 0\}$. As a consequence, $H''(p_2) < 0$, and the pair (p_1, p_2) satisfies Lemma 3.4.

- If the entropy condition is satisfied between p_1° and p_2°, we define $p_2 = p_2^\circ$ and

$$p_1 = \sup\left\{ p_1^\circ \le p \le p_2 \;\middle|\; \frac{H(p_2) - H(p)}{p_2 - p} = H'(p_2) \right\}. \tag{5}$$

As $H''(p_2)$ is negative, the graph of H is situated strictly under its tangent at p_2 over a neighborhood of p_2; hence

$$\frac{H(p_2) - H(p)}{p_2 - p} > H'(p_2)$$

on this neighborhood. The entropy condition satisfied between p_1° and p_2 implies the Lax condition

$$\frac{H(p_2) - H(p_1^\circ)}{p_2 - p_1^\circ} \ge H'(p_2).$$

By the mean value theorem, the considered set is nonempty and its supremum belongs to $[p_1^\circ, p_2)$, and by continuity of

$$p \mapsto \frac{H(p_2) - H(p)}{p_2 - p},$$

we have

$$\frac{H(p_2) - H(p_1)}{p_2 - p_1} = H'(p_2).$$

The entropy condition is strictly satisfied between p_1 and p_2 by the maximality of p_1. The mean value theorem and the maximality of p_1 make it clear that $H'(p_1) \le H'(p_2)$ and that if $H'(p_1) = H'(p_2)$, $H''(p_1) \le 0$. Let us prove that up to a perturbation we can assume $H'(p_1) < H'(p_2)$.

Let us hence assume that $H'(p_1) = H'(p_2)$. First, by Sard's theorem, up to a perturbation of p_2°, we may assume that $H'(p_2^\circ)$ is not a critical value of H', which ensures, since $H'(p_1) = H'(p_2^\circ)$, that $H''(p_1)$ is nonzero, and hence negative (note that the sign of $H''(p_1^\circ)$ had no influence in the previous paragraph). We set $p_1^\circ = p_1$ and look at the previous construction for this p_1° fixed and for a new p_2 close to p_2°. Without loss of generality we suppose that $H'(p_2^\circ) = H'(p_1^\circ) = H(p_2^\circ) = H(p_1^\circ) = 0$. Since $H''(p_1^\circ)$ and $H''(p_2^\circ)$ are negative, there exists δ such that H'' is negative on $[p_1^\circ, p_1^\circ + \delta] \cup [p_2^\circ - \delta, p_2^\circ]$. By compactness, H admits a maximum on $[p_1^\circ + \delta, p_2^\circ - \delta]$ which is negative, since the entropy condition is strictly satisfied between p_1° and p_2°.

Since H' is decreasing on $[p_2^\circ - \delta, p_2^\circ]$, there exists $p_2 \in [p_2^\circ - \delta, p_2^\circ]$ such that

$$0 < H'(p_2) < -\frac{\frac{1}{2}m}{p_2^\circ - p_1^\circ}.$$

For such a p_2, the tangent of the graph of H at p_2 lies strictly below the graph of H over $[p_1^\circ + \delta, p_2^\circ - \delta]$ by definition of m, and also over $[p_2^\circ - \delta, p_2^\circ]$ by the concavity of H. Equation (5) then defines a p_1 which is necessarily in $(p_1^\circ, p_1^\circ + \delta]$: as $H(p_1^\circ) = 0$ and $H(p_2) < 0$, the point $(p_1^\circ, H(p_1^\circ))$ is situated over the tangent of the graph of H at p_2 which has a positive slope $H'(p_2)$. By concavity of H on $[p_1^\circ, p_1^\circ+\delta]$, $H'(p_1) < H'(p_1^\circ) = 0$, and as a consequence $H'(p_1) < H'(p_2)$. The previous work proves that all the conditions of the proposition are then gathered for p_1 and p_2. □

We may now prove Proposition 2.3, joining Lemma 3.4 and Proposition 3.1.

Proof of Proposition 2.3. Let H be a nonconvex, nonconcave Hamiltonian with bounded second derivative. Using Proposition 2.7(1) with $A = -\mathrm{id}$, we may apply Lemma 3.4 up to the change of function $p \mapsto H(-p)$. It gives $p_1 < p_2$ such that $H''(p_2) < 0$, and

$$H'(p_1) < \frac{H(p) - H(p_1)}{p - p_1} < H'(p_2) = \frac{H(p_2) - H(p_1)}{p_2 - p_1}$$

for all p in (p_1, p_2). We define

$$\widetilde{H}(p) = H(p) - H(p_2) - (p - p_2)H'(p_2),$$

so that $\widetilde{H}(p_2) = \widetilde{H}'(p_2) = \widetilde{H}(p_1) = 0$. Note also that $\widetilde{H}'(p_1) = H'(p_1) - H'(p_2) < 0$. The second-order derivatives as well as the entropy condition are preserved by this transformation: $\widetilde{H}''(p_2) = H''(p_2) < 0$, and $\widetilde{H} < 0$ on (p_1, p_2).

At last, we take the affine transformation $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi(-1) = p_1$ and $\phi(1) = p_2$ and define

$$\overline{H}(p) = \widetilde{H}(\phi(p)),$$

so that $\overline{H}$ satisfies the assumptions of Proposition 3.1: $\overline{H}(-1) = \overline{H}(1) = \overline{H}'(1) = 0$ and $\overline{H}'(-1) < 0$ since $\phi' > 0$, $\overline{H}''(1) < 0$ and $\overline{H} < 0$ on $(-1, 1)$. Proposition 3.1 then gives a Lipschitz semiconcave initial condition $\bar{u}_0$ such that the variational solution denies the (HJ) equation associated with $\overline{H}$ for all t small enough. Proposition 2.7(1) applied to the two successive transformations gives then a Lipschitz semiconcave initial condition u_0, with right and left derivatives at 0 respectively equal to p_1 and p_2, such that the variational solution denies the (HJ) equation associated with H for all t small enough. □

4. Example for the saddle Hamiltonian: proof of Proposition 2.4

In this section we assume that $H(p_1, p_2) = p_1 p_2$, with $(p_1, p_2) \in \mathbb{R}^2$, and prove Proposition 2.4 by presenting a suitable initial condition. For a convex-concave Hamiltonian, [Bernardi and Cardin 2011; Wei 2013] proved that the variational solution coincides with the viscosity solution for initial conditions with separated variables; hence the wanted initial condition cannot be elementary.

We choose an initial condition that coincides with the piecewise quadratic function $u(q_1, q_2) = \min(a(q_1^2 - q_2), b(q_1^2 - q_2))$ on a large enough subset while being Lipschitz and semiconcave. We make explicit the value of the variational solution for this initial condition on a large enough subset.

Proposition 4.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a compactly supported C^2 function coinciding with $x \mapsto x^2$ on $[-1, 1]$.*

Let $u(q_1, q_2) = \min(a(f(q_1) - q_2), b(f(q_1) - q_2))$ with $b > a > 0$.

Then if $-1 \le q_1 \le -\frac{3b}{2}t$,

$$R_0^t u(q_1, q_2) = \min(a((q_1 + at)^2 - q_2), b((q_1 + bt)^2 - q_2)).$$

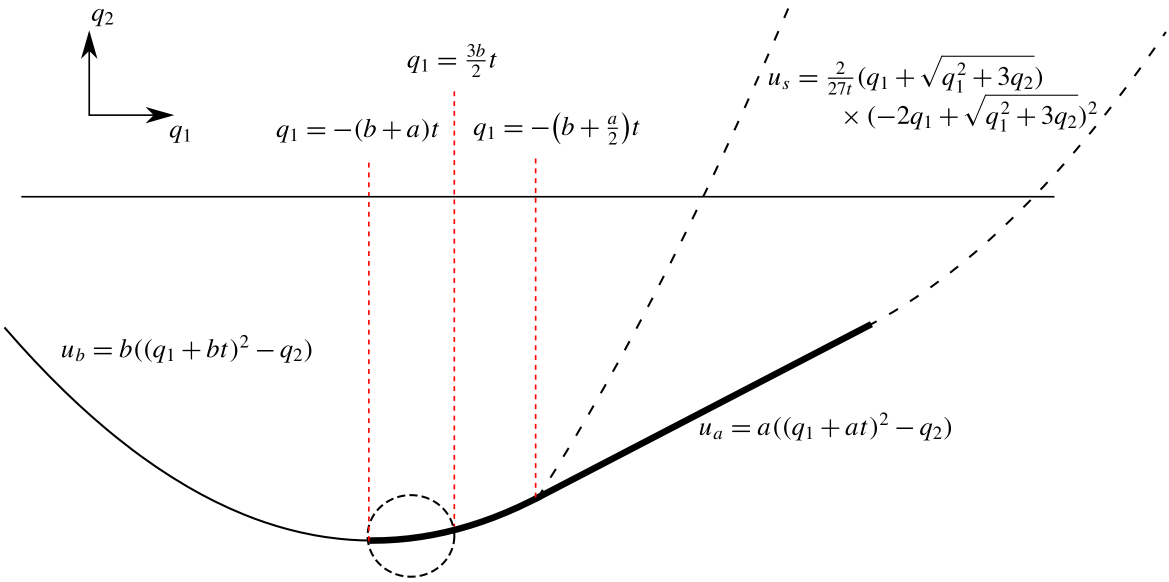

Figure 4. Value of the variational solution associated with u at time t, here for $a = 1$, $b = 2$ and $t = \frac{1}{10}$.

Figure 4 shows the explicit value of the variational solution for small times t, which is given by the unique continuous section in the wavefront. The plain curve represents a shock of the variational solution, whereas the different expressions coincide $\mathcal{C}^1$-continuously along the dotted curves. One can show that the variational solution does not satisfy the Hamilton–Jacobi equation in the viscosity sense along the thick portion of the shock, and also that it does satisfy the Hamilton–Jacobi equation in the viscosity sense everywhere except on this portion. For the purpose of this article, it is enough to show that the variational solution does not satisfy the Hamilton–Jacobi equation in the viscosity sense along the parabola circled in Figure 4. This is included in the domain concerned by Proposition 4.1, which can be proved by using an efficient convexity argument that spares us many computations.

Proof of Proposition 4.1. Using general arguments stated in Section 6, we are first going to prove that

$$R_0^t u(q_1, q_2) = \min_{c\in[a,b]} u_c(t, q_1, q_2) \quad \text{for all } t \geq 0,\ (q_1, q_2) \in \mathbb{R}^2,$$

where $u_c(t, q_1, q_2) = c(f(q_1 + ct) - q_2)$ is the unique $\mathcal{C}^2$ solution of the Cauchy problem associated with $H(p_1, p_2) = p_1 p_2$ and the initial condition $u_c^0 : (q_1, q_2) \mapsto c(f(q_1) - q_2)$.

We want to apply Proposition 6.2, observing that $u = \min_{c\in[a,b]} u_c^0$. To do so, we only need to check that the family $\{u_c^0, c \in [a, b]\}$ satisfies the conditions of Lemma 6.1, i.e., that for all (q, p) in the graph of the Clarke derivative ∂u, there exists $c \in [a, b]$ such that $u_c^0(q) = u(q)$ and $du_c^0(q) = p$.

Let us compute the Clarke derivative of u. If $f(q_1) > q_2$, then $u(q_1, q_2) = a(f(q_1) - q_2)$ on a neighborhood of (q_1, q_2); hence $\partial u(q_1, q_2)$ is reduced to the point $a\binom{f'(q_1)}{-1}$, which is also the derivative of u_a^0 at (q_1, q_2). If $f(q_1) < q_2$, then $\partial u(q_1, q_2)$ is reduced to the point $b\binom{f'(q_1)}{-1}$ which is also the derivative of u_b^0 at (q_1, q_2). If $f(q_1) = q_2$, then $\partial u(q_1, q_2)$ is the segment $\left\{c\binom{f'(q_1)}{-1} \,\middle|\, c \in [a, b]\right\}$. For all $c \in [a, b]$, $c\binom{f'(q_1)}{-1}$ is the derivative of u_c^0 at the point $(q_1, q_2 = f(q_1))$.

We hence proved that the family $\{u_c^0, c \in [a, b]\}$ satisfies the condition of Lemma 6.1; thus by Proposition 6.2

$$R_0^t u(q_1, q_2) = \min_{c\in[a,b]} u_c(t, q_1, q_2) \quad \text{for all } t \geq 0,\ (q_1, q_2) \in \mathbb{R}^2.$$

Now, for all $-1 < q_1 < -bt$, we have $f(q_1 + ct) = (q_1 + ct)^2$ since $f(x) = x^2$ for x in $[-1, 1]$, and $c \in (0, b]$. Hence if $-1 < q_1 < -bt$,

$$R_0^t u(q_1, q_2) = \min_{c\in[a,b]} c((q_1 + ct)^2 - q_2).$$

The second derivative of $g : c \mapsto c((q_1 + ct)^2 - q_2)$ is $g''(c) = 2t(2q_1 + 3ct)$. Hence if $q_1 < -\frac{3b}{2}t$, then g is concave on $[a, b]$ and the minimum defining $R_0^t u(q_1, q_2)$ is attained at an endpoint of $[a, b]$.

Thus, we proved that for $-1 < q_1 < -\frac{3b}{2}t$,

$$R_0^t u(q_1, q_2) = \min(a((q_1 + at)^2 - q_2), b((q_1 + bt)^2 - q_2)).$$

□

Proof of Proposition 2.4. Let $b > 0$, $a \in \left(\frac{b}{2}, b\right)$ and u be defined as in Proposition 4.1: f is a compactly supported $\mathcal{C}^2$ function coinciding with $x \mapsto x^2$ on $[-1, 1]$ and

$$u(q_1, q_2) = \min(a(f(q_1) - q_2), b(f(q_1) - q_2)).$$

We define $u_a : (t, q_1, q_2) \mapsto a((q_1 + at)^2 - q_2)$ and $u_b : (t, q_1, q_2) \mapsto b((q_1 + bt)^2 - q_2)$ (note that the notations slightly differ from the previous proof), so that Proposition 4.1 gives that for $-1 < q_1 < -\frac{3b}{2}t$

$$R_0^t u(q_1, q_2) = \min(u_a(t, q_1, q_2), u_b(t, q_1, q_2)).$$

Let us prove that this variational solution does not satisfy the Hamilton–Jacobi equation at the point (t, q_1, q_2) if

$$\begin{cases} q_2 = q_1^2 + 2(a + b)tq_1 + t^2(a^2 + ab + b^2), \\ -1 < q_1 < -\frac{3b}{2}t, \\ -(a + b)t < q_1. \end{cases}$$

This corresponds to the piece of parabola circled in Figure 4, which exists only if $a > \frac{b}{2}$ and $t < \frac{2}{3b}$. Note that the first line is just an equation of this parabola, which is obtained by solving $u_a = u_b$.

Let us exhibit a test function denying the viscosity equation: we define the mean function $\phi = \frac{1}{2}(u_a + u_b)$, which is $\mathcal{C}^1$, larger than $\min(u_a, u_b)$ on a neighborhood of (t, q_1, q_2) and equal to it at (t, q_1, q_2) since $u_a(t, q_1, q_2) = u_b(t, q_1, q_2)$, so that $R_0^t u - \phi$ attains a local maximum at (t, q_1, q_2). The derivatives of ϕ are given by

$$\partial_t \phi(t, q_1, q_2) = a^2(q_1 + at) + b^2(q_1 + bt),$$
$$\partial_{q_1} \phi(t, q_1, q_2) = a(q_1 + at) + b(q_1 + bt),$$
$$\partial_{q_2} \phi(t, q_1, q_2) = -\tfrac{1}{2}(a + b).$$

We compute

$$\begin{aligned}\partial_t\phi(t,q_1,q_2)+H(\partial_q\phi(t,q_1,q_2)) &= a^2(q_1+at)+b^2(q_1+bt)-\tfrac{1}{2}(a+b)(a(q_1+at)+b(q_1+bt))\\ &= \tfrac{1}{2}(a-b)^2(at+bt+q_1)>0\end{aligned}$$

when $q_1>-(a+b)t$, and as a consequence the variational solution is not a viscosity subsolution at the point (t,q_1,q_2).

Note that b can be chosen as small as needed, and hence for all L we are able to take the initial condition u L-Lipschitz and L-semiconcave, with $b\le L$. The previous work shows that for all $t<\frac{2}{3b}$, the variational solution does not satisfy the Hamilton–Jacobi solution in the viscosity sense at some point (t,q). But since $L\ge b$, we have $\frac{L}{2}\le\frac{2}{3b}$ and we hence have proved Proposition 2.4. □

5. Proof of Theorem 1

In this section we will deduce Theorem 1 from Corollary 2.2. To do so, we approach the Lipschitz initial condition of Corollary 2.2 by a smooth initial condition, keeping the Hausdorff distance between the (Clarke) derivatives small. We will use elementary properties of the Hausdorff distance, stated in Lemmas 5.1 and 5.2 and proved for completeness.

The Hausdorff distance d_{Haus} is defined (though not necessarily finite) by

$$d_{\text{Haus}}(X,Y)=\sup\bigl(\sup_{x\in X}d(x,Y),\ \sup_{y\in Y}d(y,X)\bigr)$$

for X and Y closed subsets of a metric space (E,d) (d being the euclidean distance on $\mathbb{R}^d$ in our context). The following approximation result is proved in [Czarnecki and Rifford 2006, Theorem 2.2] and its Corollary 2.1:

Theorem 2. *If $u:\mathbb{R}^d\to\mathbb{R}$ is locally Lipschitz, there exists a sequence of smooth functions u_n such that*

$$\lim_{n\to\infty}\|u_n-u\|_\infty=0,$$
$$\lim_{n\to\infty}d_{\text{Haus}}(\text{graph}(du_n),\text{graph}(\partial u))=0,$$

where ∂ denotes the Clarke derivative.

Here is a sketch of the proof: for H an integrable nonconvex, nonconcave Hamiltonian with bounded second derivative, Corollary 2.2 gives a Lipschitz initial condition u_L such that the graph of the viscosity solution is not included in the wavefront $\mathcal{F}_{u_L}$ for some time $t>0$. We are going to approach u_L by a Lipschitz smooth function u such that both the viscosity solutions at time t are close, and the Hausdorff distance between the wavefronts at time t is small. The following enhanced triangle inequality will conclude that the graph of the viscosity solution associated with u is not included in the wavefront $\mathcal{F}_u$.

Lemma 5.1 (enhanced triangle inequality). *If (E,d) is a metric space and X and Y are subsets of E, then for all x and y in E*

$$d(x,X)\le d(x,y)+d(y,Y)+d_{\text{Haus}}(X,Y).$$

Proof. The triangle inequality for d gives that, for all x, $\tilde{x}$ and y, we have $d(x,\tilde{x}) \le d(x,y)+d(y,\tilde{x})$, and taking the infimum for $\tilde{x}$ on X gives

$$d(x,X) \le d(x,y)+d(y,X) \quad \text{for all } x, y \in E. \tag{6}$$

We change the variables in (6): for all y and $\tilde{y}$,

$$d(y,X) \le d(y,\tilde{y})+d(\tilde{y},X).$$

If $\tilde{y}$ is in Y, by definition of the Hausdorff distance we get

$$d(y,X) \le d(y,\tilde{y})+d_{\text{Haus}}(X,Y)$$

and taking the infimum for $\tilde{y}$ on Y gives

$$d(y,X) \le d(y,Y)+d_{\text{Haus}}(X,Y).$$

We conclude by injecting this last inequality into (6). □

To bound the Hausdorff distance between the wavefronts, we will describe the wavefront at time t as the image of the (Clarke) derivative of the initial condition by a suitable function ψ depending on the initial condition, which will allow us to apply the following elementary continuity result for the Hausdorff distance.

Lemma 5.2 (continuity for the Hausdorff distance). *Let $f, g : (F,\tilde{d}) \mapsto (E,d)$ be two functions between two topological spaces, and X and Y be two subsets of F:*

1. *If $d(f(x),g(x)) \le a$ for all x in X, then $d_{\text{Haus}}(f(X),g(X)) \le a$.*
2. *If f is uniformly continuous on X, i.e., for all $\alpha > 0$, there exists $\varepsilon > 0$ such that for all $(x,y) \in X$ $\tilde{d}(x,y) < \varepsilon$ implies $d(f(x),f(y)) < \alpha$, then*

$$\tilde{d}_{\text{Haus}}(X,Y) < \varepsilon \quad \Longrightarrow \quad d_{\text{Haus}}(f(X),f(Y)) < \alpha.$$

Proof of Lemma 5.2. (1) By the definition of the Hausdorff distance, it is enough to observe that $d(f(x),g(X)) \le a$ for all x in X, since this quantity is smaller than $d(f(x),g(x))$.

(2) Using the symmetry of the definition of d_{Haus}, it is enough to prove that if $\tilde{d}_{\text{Haus}}(X,Y) < \varepsilon$, $d(f(x),f(Y)) < \alpha$ for all x in X. For all x in X, there exists a sequence y_n in Y such that $\tilde{d}(x,y_n) \to \tilde{d}(x,Y)$ as $n \to \infty$. Since $\tilde{d}(x,Y) \le \tilde{d}_{\text{Haus}}(X,Y)$, this implies that $\tilde{d}(x,y_n) < \varepsilon$ for n large enough, and the uniform continuity of f gives that $d(f(x),f(y_n)) < \alpha$ for n large enough; hence $d(f(x),f(Y)) < \alpha$. □

Proof of Theorem 1. Let H be an integrable nonconvex, nonconcave Hamiltonian with bounded second derivative. Corollary 2.2 gives a Lipschitz initial condition u_L for which there exist $t > 0$ and q such that

$$d((q, V_0^t u_L(q)), \mathcal{F}^t_{u_L}) > 0,$$

where $\mathcal{F}^t_{u_L}$ denotes the section of $\mathcal{F}_{u_L}$ at time t. We denote by α this positive quantity.

Let us denote by L the Lipschitz constant of u_L.

We propose another description of the wavefront at time t: if v is a Lipschitz function, we define

$$\psi_v^t : T^\star\mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R},$$
$$(q, p) \mapsto (q + t\nabla H(p), v(q) + t(p\cdot\nabla H(p) - H(p))),$$

in such a way that $\mathcal{F}_v^t = \psi_v^t(\mathrm{graph}(\partial v))$ (see (F′) for a comparison).

Note that ψ_v^t is Lipschitz, and hence uniformly continuous on every $\mathbb{R}^d \times \{\|p\| \le R\}$ for $R > 0$: it is Lipschitz with respect to q because v is, and its derivative with respect to p, $(td^2H(p), tp\cdot d^2H(p))$, is bounded on this set since d^2H is bounded.

The uniform continuity of $\psi_{u_L}^t$ on $\mathbb{R}^d \times \{p \le L+1\}$ gives an $\varepsilon \in (0, 1)$ such that

$$\begin{cases} \|(q, p) - (\tilde{q}, \tilde{p})\| < \varepsilon, \\ \|p\|, \|\tilde{p}\| \le L+1, \end{cases} \qquad \Longrightarrow \qquad \|\psi_{u_L}^t(q, p) - \psi_{u_L}^t(\tilde{q}, \tilde{p})\| < \tfrac{1}{4}\alpha.$$

By Theorem 2, there exists a smooth function u such that

$$\|u - u_L\|_\infty < \tfrac{1}{4}\alpha, \tag{7}$$
$$d_{\mathrm{Haus}}(\mathrm{graph}(du), \mathrm{graph}(\partial u_L)) < \varepsilon. \tag{8}$$

Note that since $\varepsilon \in (0, 1)$, u is $(L+1)$-Lipschitz.

On the one hand, Proposition 1.3 gives the comparison between the viscosity solutions:

$$\|V_0^t u - V_0^t u_L\|_\infty \le \|u - u_L\|_\infty \le \tfrac{1}{4}\alpha.$$

On the other hand, we estimate the Hausdorff distance between the wavefronts, using the definition of ψ:

$$\begin{aligned} d_{\mathrm{Haus}}(\mathcal{F}_u^t, \mathcal{F}_{u_L}^t) &= d_{\mathrm{Haus}}(\psi_u^t(\mathrm{graph}(du)), \psi_{u_L}^t(\mathrm{graph}(\partial u_L))) \\ &\le d_{\mathrm{Haus}}(\psi_u^t(\mathrm{graph}(du)), \psi_{u_L}^t(\mathrm{graph}(du))) + d_{\mathrm{Haus}}(\psi_{u_L}^t(\mathrm{graph}(du)), \psi_{u_L}^t(\mathrm{graph}(\partial u_L))). \end{aligned}$$

The first part of Lemma 5.2 applied with $f = \psi_{u_L}^t$, $g = \psi_u^t$, $X = \mathrm{graph}(du)$ gives that the first term of the right-hand side is bounded by $\|\psi_u^t - \psi_{u_L}^t\|_\infty = \|u - u_L\|_\infty \le \frac{1}{4}\alpha$.

The second part of Lemma 5.2 applied with $f = \psi_{u_L}^t$, $X = \mathrm{graph}(\partial u_L)$ and $Y = \mathrm{graph}(du)$ gives that the second term of the right-hand side is smaller than $\frac{1}{4}\alpha$, by uniform continuity of $\psi_{u_L}^t$, since $\mathrm{graph}(du)$ and $\mathrm{graph}(\partial u_L)$ are both contained in $\mathbb{R}^d \times \{p \le L+1\}$ and are ε-close for the Hausdorff distance; see (8). We hence proved that

$$d_{\mathrm{Haus}}(\mathcal{F}_u^t, \mathcal{F}_{u_L}^t) \le \tfrac{1}{2}\alpha.$$

Let us now apply Lemma 5.1 with $x = (q, V_0^t u_L(q))$, $y = (q, V_0^t u_L(q))$, $X = \mathcal{F}_{u_L}^t$ and $Y = \mathcal{F}_u^t$:

$$\begin{aligned} \alpha &= d((q, V_0^t u_L(q)), \mathcal{F}_{u_L}^t) \\ &\le \underbrace{d((q, V_0^t u_L(q)), (q, V_0^t u(q)))}_{\le \|V_0^t u_L - V_0^t u\|_\infty \le \frac{1}{4}\alpha} + d((q, V_0^t u(q)), \mathcal{F}_u^t) + \underbrace{d_{\mathrm{Haus}}(\mathcal{F}_{u_L}^t, \mathcal{F}_u^t)}_{\le \frac{1}{2}\alpha}. \end{aligned}$$

As a consequence, $d((q, V_0^t u(q)), \mathcal{F}_u^t) \ge \frac{1}{4}\alpha > 0$ and the graph of the viscosity solution associated with the smooth initial condition u is not contained in the wavefront $\mathcal{F}_u$. □

6. Semiconcavity arguments

This section contains the proofs of Propositions 1.9 and 1.11, as well as an additional Proposition 6.2 used in the proof of the two-dimensional case (see Section 4). The three proofs rely on the following lemma, proved in [Bernard 2013, Lemma 6]:

Lemma 6.1. *If u is a Lipschitz and B-semiconcave function on $\mathbb{R}^d$, there exists a family F of $\mathcal{C}^2$ equi-Lipschitz functions with second derivatives bounded by B such that*

- $u(q) = \min_{f\in F} f(q)$ *for any* q,
- *for each q in $\mathbb{R}^d$ and p in $\partial u(q)$, there exists f in F such that*

$$\begin{cases} f(q) = u(q), \\ df(q) = p. \end{cases}$$

Proof of Proposition 1.9. Proposition 1.8 states that the variational solution gives a section of the generalized wavefront. As a consequence

$$R_0^t u_0(q) \geq \inf\{u_0(q_0) + \mathcal{A}_0^t(\gamma) \mid (q_0, p_0) \in \mathbb{R}^d \times \mathbb{R}^d,\ p_0 \in \partial u_0(q_0),\ Q_0^t(q_0, p_0) = q\}.$$

If u_0 is L-Lipschitz and B-semiconcave, take T such that the method of characteristics is valid ($T = 1/(BC)$ if H is integrable). Let us fix definitively q, q_0, $p_0 \in \partial u_0(q_0)$ and $0 \leq t \leq T$ such that $Q_0^t(q_0, p_0) = q$ and show that $R_0^t u_0(q) \leq u_0(q_0) + \mathcal{A}_0^t(\gamma)$, where γ is the Hamiltonian trajectory issued from (q_0, p_0).

Lemma 6.1 gives a $\mathcal{C}^2$ function f_0 of F such that $f_0(q_0) = u_0(q_0)$ and $df_0(q_0) = p_0$. Since this function is $\mathcal{C}^2$ with second derivative bounded by B, the method of characteristics gives that q_0 is the only point such that $Q_0^t(q_0, df_0(q_0)) = q$, and the variational operator applied to the initial condition f_0 gives necessarily the $\mathcal{C}^2$ solution:

$$R_0^t f_0(t, q) = f_0(q_0) + \mathcal{A}_0^t(\gamma).$$

But by the definition of F, f_0 is larger than u_0 on $\mathbb{R}^d$, and the monotonicity of the variational operator brings the conclusion

$$R_0^t u_0(q) \leq R_0^t f_0(q) = f_0(q_0) + \mathcal{A}_0^t(\gamma) = u_0(q_0) + \mathcal{A}_0^t(\gamma). \qquad \square$$

Proof of Proposition 1.11. Take T such that the method of characteristics is valid (for example $T = 1/(BC)$ if H is integrable).

If t and q are fixed, Proposition 1.8 gives the existence of (q_0, p_0) in $gr(\partial u_0)$ such that $Q_0^t(q_0, p_0) = q$ and that $R_0^t u_0(q) = u_0(q_0) + \mathcal{A}_0^t(\gamma)$, where γ is the Hamiltonian trajectory issued from (q_0, p_0).

Lemma 6.1 gives a $\mathcal{C}^2$ function f_0 of F such that $f_0(q_0) = u_0(q_0)$ and $df_0(q_0) = p_0$. The method of characteristics states that there exists on $[0, T] \times \mathbb{R}^d$ a unique $\mathcal{C}^2$ solution of the (HJ) equation with initial condition f_0, which satisfies in particular

$$f(t, q) = f_0(q_0) + \mathcal{A}_0^t(\gamma).$$

Since a $\mathcal{C}^1$ solution is a viscosity solution, the uniqueness of viscosity solutions hence gives that $V_0^t f = f(t, \cdot)$ for all t in $(0, T)$, and in particular

$$V_0^t f_0(q) = f(t, q) = f_0(q_0) + \mathcal{A}_0^t(\gamma).$$

But by the definition of F, f_0 is larger than u_0 on $\mathbb{R}^d$, and the monotonicity of the viscosity operator V_0^t brings the conclusion

$$V_0^t u_0(q) \leq V_0^t f_0(t, q) = f_0(q_0) + \mathcal{A}_0^t(\gamma) = R_0^t u_0(q).$$

Since $(t, q) \mapsto R_0^t u_0(q)$ is pointwise less than or equal to any variational solution as long as $t < T$ (Proposition 1.9), this implies that for all variational solutions g, $V_0^t u_0(q) \leq g(t, q)$ on $[0, T] \times \mathbb{R}^d$. □

We end this section with another result of the same flavor, used in the proof of Proposition 2.4.

Proposition 6.2. *Let F be as in Lemma 6.1 and $u = \min_{f \in F} f$. If $T > 0$ denotes a time of shared existence of $\mathcal{C}^2$ solutions for initial conditions in F, and u_f denotes the $\mathcal{C}^2$ solution of the Hamilton–Jacobi equation associated with the $\mathcal{C}^2$ initial condition f, then for all $0 \leq t \leq T$*

$$R_0^t u(q) = \min_{f \in F} u_f(t, q).$$

Proof. Since $u \leq f$ for all f in F, the monotonicity of the variational operator guarantees that $R_0^t u(q) \leq \min_{f \in F} R_0^t f(q)$. The method of characteristics implies that the variational operator is given by the classical solution if it exists; hence $R_0^t f(q) = u_f(t, q)$ for all t in $[0, T]$ and thus

$$R_0^t u(q) \leq \min_{f \in F} u_f(t, q). \tag{9}$$

Now, for all (t, q), the variational property gives the existence of a (q_0, p_0) in the graph of ∂u such that

$$R_0^t u(q) = u(q_0) + \mathcal{A}_0^t(\gamma),$$

where γ denotes the Hamiltonian trajectory issued from (q_0, p_0). Since F is as in Lemma 6.1, there exists f in F such that $f(q_0) = u(q_0)$ and $df(q_0) = p_0$. The method of characteristics implies furthermore that $u_f(t, q) = f(q_0) + \mathcal{A}_0^t(\gamma)$. Summing all this up, we get

$$R_0^t u(q) = u(q_0) + \mathcal{A}_0^t(\gamma) = f(q_0) + \mathcal{A}_0^t(\gamma) = u_f(t, q)$$

and the inequality (9) is an equality. □

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