ANALYSIS & PDE

Volume 13 No. 4 2020

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ANALYSIS AND PDE
Vol. 13, No. 4, 2020
dx.doi.org/10.2140/apde.2020.13.1173

STRICHARTZ ESTIMATES FOR THE SCHRÖDINGER FLOW ON COMPACT LIE GROUPS

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We establish scale-invariant Strichartz estimates for the Schrödinger flow on any compact Lie group equipped with canonical rational metrics. In particular, full Strichartz estimates without loss for some non-rectangular tori are given. The highlights of this paper include estimates for some Weyl-type sums defined on rational lattices, different decompositions of the Schrödinger kernel that accommodate different positions of the variable inside the maximal torus relative to the cell walls, and an application of the BGG-Demazure operators or Harish-Chandra's integral formula to the estimate of the difference between characters.

1. Introduction

We start with a complete Riemannian manifold (M, g) of dimension d, associated to which are the Laplace–Beltrami operator Δ_g and the volume-form measure μ_g. Then it is well known that Δ_g is essentially self-adjoint on $L^2(M) := L^2(M, d\mu_g)$; see [Strichartz 1983] for a proof. This gives the functional calculus of Δ_g, and in particular gives the one-parameter unitary operator $e^{it\Delta_g}$, which provides the solution to the linear Schrödinger equation on (M, g). We refer to $e^{it\Delta_g}$ as the *Schrödinger flow*. The functional calculus of Δ_g also gives the definition of the Bessel potentials, and thus the definition of the Sobolev space

$$H^s(M) := \{u \in L^2(M) \mid \|u\|_{H^s(M)} := \|(I-\Delta)^{\frac{s}{2}}u\|_{L^2(M)} < \infty\}.$$

We are interested in obtaining estimates of the form

$$\|e^{it\Delta_g} f\|_{L^pL^r(I\times M)} \le C\|f\|_{H^s(M)}, \tag{1-1}$$

MSC2010: primary 42B37; secondary 22E30.
Keywords: compact Lie groups, Schrödinger equation, circle method, Strichartz estimates, BGG-Demazure operators, Harish-Chandra's integral formula.

where $I \subset \mathbb{R}$ is a fixed time interval, and $L^pL^q(I \times M)$ is the space of L^p functions on I with values in $L^q(M)$. Such estimates are often called Strichartz estimates (for the Schrödinger flow), in honor of Robert Strichartz [1977] who first derived such estimates for the wave equation on Euclidean spaces.

The significance of Strichartz estimates is evident in many ways. Strichartz estimates have important applications in the field of nonlinear Schrödinger equations, in the sense that many perturbative results often require good control on the linear solution, which is exactly provided by Strichartz estimates. Strichartz estimates can also be interpreted as Fourier restriction estimates, which play a fundamental rule in the field of classical harmonic analysis. Furthermore, the relevance of the distribution of eigenvalues and the norm of eigenfunctions of Δ_g in deriving the estimates makes Strichartz estimates also a subject in the field of spectral geometry.

Many cases of Strichartz estimates for the Schrödinger flow are known in the literature. For noncompact manifolds, first we have the sharp Strichartz estimates on the Euclidean spaces obtained in [Ginibre and Velo 1995; Keel and Tao 1998]:

$$\|e^{it\Delta} f\|_{L^pL^q(\mathbb{R}\times\mathbb{R}^d)} \le C\|f\|_{L^2(\mathbb{R}^d)}, \tag{1-2}$$

where $\frac{2}{p}+\frac{d}{q}=\frac{d}{2}$, $p,q\ge 2$, $(p,q,d)\ne(2,\infty,2)$. Such pairs (p,q) are called *admissible*. This implies by Sobolev embedding that

$$\|e^{it\Delta} f\|_{L^pL^r(\mathbb{R}\times\mathbb{R}^d)} \le C\|f\|_{H^s(\mathbb{R}^d)}, \tag{1-3}$$

where

$$s=\frac{d}{2}-\frac{2}{p}-\frac{d}{r}\ge 0, \tag{1-4}$$

$p,q\ge 2$, $(p,r,d)\ne(2,\infty,2)$. Note that the equality in (1-4) can be derived from a standard scaling argument, and we call exponent triples (p,r,s) that satisfy (1-4) as well as the corresponding Strichartz estimates *scale-invariant*. Similar Strichartz estimates hold on many noncompact manifolds. For example, see [Anker and Pierfelice 2009; Banica 2007; Ionescu and Staffilani 2009; Pierfelice 2006] for Strichartz estimates on the real hyperbolic spaces, [Anker et al. 2011; Pierfelice 2008; Banica and Duyckaerts 2007] for Damek–Ricci spaces which include all rank-1 symmetric spaces of noncompact type, [Bouclet 2011] for asymptotically hyperbolic manifolds, [Hassell et al. 2006] for asymptotically conic manifolds, [Bouclet and Tzvetkov 2008; Staffilani and Tataru 2002] for some perturbed Schrödinger equations on Euclidean spaces, and [Fotiadis et al. 2018] for symmetric spaces G/K, where G is complex.

For compact manifolds, however, Strichartz estimates such as (1-2) are expected to fail. The Sobolev exponent s in (1-1) is expected to be positive for (1-1) to possibly hold. And we also expect sharp Strichartz estimates that are *non-scale-invariant*, in the sense that the exponents (p,r,s) in (1-1) satisfy

$$s>\frac{d}{2}-\frac{2}{p}-\frac{d}{r}.$$

For example, from the results in [Staffilani and Tataru 2002; Burq et al. 2004], we know that on a general compact Riemannian manifold (M,g) it holds that, for any finite interval I,

$$\|e^{it\Delta_g} f\|_{L^pL^r(I\times M)} \le C\|f\|_{H^{1/p}(M)} \tag{1-5}$$

for all admissible pairs (p, r). These estimates are non-scale-invariant, and the special case of which when $(p, r, s) = \left(2, \frac{2d}{d-2}, \frac{1}{2}\right)$ can be shown to be sharp on spheres of dimension $d \geq 3$ equipped with canonical Riemannian metrics. On the other hand, scale-invariant estimates are out of reach of the local methods employed in [Staffilani and Tataru 2002; Burq et al. 2004], and they are not well explored yet in the literature. To my best knowledge, the only known results in the literature in this direction are on Zoll manifolds, which include all compact symmetric spaces of rank 1, the standard sphere being a typical example, and on rectangular tori. We summarize the results here. Consider the scale-invariant estimates

$$\|e^{it\Delta_g} f\|_{L^p(I\times M)} \leq C\|f\|_{H^{d/2-(d+2)/p}(M)}. \tag{1-6}$$

In the direction of Zoll manifolds, (1-6) is first proved in [Burq et al. 2007] for the standard three-sphere for $p = 6$. Then in [Herr 2013], (1-6) is proved for all $p > 4$ for any three-dimensional Zoll manifold, but the methods employed in that paper in fact prove (1-6) for $p > 4$ for any Zoll manifold with dimension $d \geq 3$ and for $p \geq 6$ for any Zoll surface ($d = 2$). The paper crucially uses the property of Zoll manifolds that the spectrum of the Laplace–Beltrami operator is clustered around a sequence of squares, and the spectral cluster estimates [Sogge 1988] which are optimal on spheres. In the direction of tori, (1-6) was first proved in [Bourgain 1993] for $p \geq \frac{2(d+4)}{d}$ on square tori, by interpolating the distributional Strichartz estimate

$$\begin{aligned}\lambda\cdot\mu\{(t,x)\in I\times\mathbb{T}^d \mid |e^{it\Delta_g}\varphi(N^{-2}\Delta_g)f(x)| > \lambda\}^{\frac{1}{p}} &\leq CN^{\frac{d}{2}-\frac{d+2}{p}}\|f\|_{L^2(\mathbb{T}^d)}\\ &\leq C\|f\|_{H^{d/2-(d+2)/p}(\mathbb{T}^d)}\end{aligned} \tag{1-7}$$

for $\lambda > N^{d/4}$, $p > \frac{2(d+2)}{d}$, $N \geq 1$, with the trivial subcritical Strichartz estimate

$$\|e^{it\Delta_g} f\|_{L^2(I\times\mathbb{T}^d)} \leq C\|f\|_{L^2(\mathbb{T}^d)}. \tag{1-8}$$

The estimate (1-7) is a consequence of an arithmetic version of dispersive estimates:

$$\|e^{it\Delta_g}\varphi(N^{-2}\Delta_g)\|_{L^\infty(\mathbb{T}^d)} \leq C\left(\frac{N}{\sqrt{q}\left(1+N\left\|\frac{t}{T}-\frac{a}{q}\right\|^{1/2}\right)}\right)^d\|f\|_{L^1(\mathbb{T}^d)}, \tag{1-9}$$

where $\|\cdot\|$ stands for the distance from 0 on the standard circle with length 1, $\left\|\frac{t}{T}-\frac{a}{q}\right\| < \frac{1}{qN}$, a, q are nonnegative integers with $a < q$ and $(a, q) = 1$, and $q < N$. Here T is the period for the Schrödinger flow $e^{it\Delta_g}$. Then in [Bourgain 2013], the author improved (1-8) into a stronger subcritical Strichartz estimate

$$\|e^{it\Delta_g} f\|_{L^{2(d+1)/d}(I\times\mathbb{T}^d)} \leq C\|f\|_{L^2(\mathbb{T}^d)}, \tag{1-10}$$

which yields (1-6) for $p \geq \frac{2(d+3)}{d}$. Eventually, (1-6) with an ε-loss is proved for the full range $p > \frac{2(d+2)}{d}$ in [Bourgain and Demeter 2015], and (1-7) can be used to remove this ε-loss. Then authors in [Guo et al. 2014; Killip and Vişan 2016] extended the results to all rectangular tori. We will see in this paper that by a slight adaptation of the methods in [Bourgain 1993], we may generalize (1-7) to all rational (not necessarily rectangular) tori $\mathbb{T}^d = \mathbb{R}^d/\Gamma$, where $\Gamma \cong \mathbb{Z}^d$ is a lattice such that there exists some $D \neq 0$

for which $\langle \lambda, \mu \rangle \in D^{-1}\mathbb{Z}$ for all $\lambda, \mu \in \Gamma$, which can also be used for the removal of the ε-loss of the results in [Bourgain and Demeter 2015] to yield (1-6) for the full range $p > \frac{2(d+2)}{d}$ on such rational tori.

The understanding of Strichartz estimates on compact manifolds is far from complete. It is not known in general how the exponents (p, r, s) in the sharp Strichartz estimates are related to the geometry and topology of the underlying manifold. Also, there still are important classes of compact manifolds on which Strichartz estimates have not been explored yet. Note that both standard tori and spheres on which Strichartz estimates are known are special cases of compact globally symmetric spaces, and since all compact globally symmetric spaces share the same behavior of geodesic dynamics as tori, from a semiclassical point of view, it's natural to conjecture that similar Strichartz estimates should hold on general compact globally symmetric spaces. An important class of such spaces is the class of compact Lie groups. The goal of this paper is to prove scale-invariant Strichartz estimates of the form (1-6) for $M = G$ being any connected compact Lie group equipped with a canonical *rational metric* in the sense that is described below, for all $p \geq \frac{2(r+4)}{r}$, r being the *rank* of G. In particular, full Strichartz estimates without loss for some nonrectangular tori will be given.

2. Statement of the main theorem

2A. *Rational metric.* Let G be a connected compact Lie group and $\mathfrak{g}$ be its Lie algebra. By the classification theorem of connected compact Lie groups, see [Procesi 2007, Chapter 10, Section 7.2, Theorem 4], there exists an exact sequence of Lie group homomorphisms

$$1 \to A \to \widetilde{G} \cong \mathbb{T}^n \times K \to G \to 1,$$

where $\mathbb{T}^n$ is the n-dimensional torus, K is a compact simply connected semisimple Lie group, and A is a finite and central subgroup of the *covering group* $\widetilde{G}$. As a compact simply connected semisimple Lie group, K is a direct product $K_1 \times K_2 \times \cdots \times K_m$ of compact simply connected simple Lie groups.

Now each K_i is equipped with the canonical bi-invariant Riemannian metric g_i that is induced from the negative of the Cartan–Killing form. We use $\langle \cdot, \cdot \rangle$ to denote the Cartan–Killing form. Then we equip the torus factor $\mathbb{T}^n$ with a flat metric g_0 inherited from its representation as the quotient $\mathbb{R}^n / 2\pi\Gamma$ and require that there exists some $D \in \mathbb{N}$ such that $\langle \lambda, \mu \rangle \in D^{-1}\mathbb{Z}$ for all $\lambda, \mu \in \Gamma$. Then we equip $\widetilde{G} \cong \mathbb{T}^n \times K_1 \times \cdots \times K_m$ with the bi-invariant metric

$$\tilde{g} = \bigotimes_{j=0}^{m} \beta_j g_j, \tag{2-1}$$

$\beta_j > 0$, $j = 0, \ldots, m$. Then $\tilde{g}$ induces a bi-invariant metric g on G.

Definition 2.1. Let g be the bi-invariant metric induced from $\tilde{g}$ in (2-1) as described above. We call g a *rational metric* provided the numbers $\beta_0, \ldots, \beta_m$ are rational multiples of each other. If not, we call it an *irrational metric*.

Provided the numbers $\beta_0, \ldots, \beta_m$ are rational multiples of each other, the periods of the Schrödinger flow $e^{it\Delta_{\tilde{g}}}$ on each factor of $\widetilde{G}$ are rational multiples of each other, which implies that the Schrödinger flow on $\widetilde{G}$, as well as on G, is also periodic (see Section 5).

2B. ***Main theorem.*** We define the *rank* of G to be the dimension of any of its maximal torus. This paper mainly proves the following theorem.

Theorem 2.2. *Let G be a connected compact Lie group equipped with a rational metric g. Let d be the dimension of G and r the rank of G. Let $I\subset\mathbb{R}$ be a finite time interval. Consider the scale-invariant Strichartz estimate*

$$\|e^{it\Delta_g}f\|_{L^p(I\times G)}\le C\|f\|_{H^{d/2-(d+2)/p}(G)}. \tag{2-2}$$

Then the following statements hold true:

(i) (2-2) *holds for all* $p\ge 2+\frac{8}{r}$.

(ii) *Let $G=\mathbb{T}^d$ be a flat torus equipped with a rational metric; that is, we can write $\mathbb{T}^d=\mathbb{R}^d/2\pi\Gamma$ such that there exists some $D\in\mathbb{R}$ for which $\langle\lambda,\mu\rangle\in D^{-1}\mathbb{Z}$ for all $\lambda,\mu\in\Gamma$. Then* (2-2) *holds for all* $p>2+\frac{4}{d}$.

The framework for the proof of this theorem will be based on [Bourgain 1993], in which the author proves some Strichartz estimates for the case of square tori, based on the Hardy–Littlewood circle method. We also refer to [Bourgain 1989] for applications of the circle method to Fourier restriction problems on tori. Note that part (ii) of the above theorem provides full expected Strichartz estimates without loss for some nonrectangular tori. We then have the following immediate corollary.

Corollary 2.3. *Let $d=3,4$ and let $\mathbb{T}^d$ be the flat torus equipped with a rational metric* (*not necessarily rectangular*). *Then the nonlinear Schrödinger equation $i\partial u_t=-\Delta u\pm|u|^{4/(d-2)}u$ is locally well-posed for initial data in $H^1(\mathbb{T}^d)$. Furthermore, for $d=3$, we have $i\partial u_t=-\Delta u\pm|u|^2u$ is locally well-posed for initial data in $H^{1/2}(\mathbb{T}^d)$.*

We refer to [Herr et al. 2011; Killip and Vișan 2016] for the definition of local well-posedness and a proof of this corollary.

Remark 2.4. To the best of my knowledge, the only known optimal range of p for (2-2) to hold is on square tori $\mathbb{T}^d$, with $p>2+\frac{4}{d}$ [Bourgain 1993], and on spheres $\mathbb{S}^d$ ($d\ge 3$), with $p>4$ [Burq et al. 2004; Herr 2013]. For a general compact Lie group, we do not yet have a conjecture about the optimal range. We will prove (Theorem 6.2) the following distributional estimate: for any $p>2+\frac{4}{r}$,

$$\lambda\cdot\mu\{(t,x)\in I\times G\mid |e^{it\Delta_g}\varphi(N^{-2}\Delta_g)f(x)|>\lambda\}^{\frac{1}{p}}\le CN^{\frac{d}{2}-\frac{d+2}{p}}\|f\|_{L^2(G)} \tag{2-3}$$

for all $\lambda\gtrsim N^{d/2-r/4}$. It seems reasonable to conjecture that the above distributional estimate could be upgraded to the estimate (2-2) for all $p>2+\frac{4}{r}$ (which is the case for the tori). But this still will not be the optimal range for a general compact Lie group, by looking at the example of the three-sphere $\mathbb{S}^3$, which is isomorphic to the group SU(2). The optimal range for $\mathbb{S}^3$ is $p>4$, while Theorem 2.2 proves the range $p\ge 10$, and the above conjecture indicates the range $p>6$. Estimate (2-2) for $\mathbb{S}^3$ on the optimal range $p>4$ is proved in [Herr 2013] by crucially using the L^p-estimates of the spectral clusters for the Laplace–Beltrami operator [Sogge 1988], which are optimal on spheres. On tori and more generally compact Lie groups with rank higher than 1, such spectral cluster estimates fail to be optimal and do not help provide the desired Strichartz estimates. On the other hand, the Stein–Tomas argument in our proof of Theorem 2.2 seems only sensitive to the L^∞-estimate of the Schrödinger kernel (Theorem 6.1) but not to the L^p-estimate (as in Proposition 7.28). This failure of incorporating L^p-estimates for either the spectral clusters or the Schrödinger kernel may be one of the reasons why Theorem 2.2 is still a step away from the optimal range.

2C. ***Organization of the paper.*** The organization of the paper is as follows. In Section 3, we will first reduce the Strichartz estimates on $G \cong \widetilde{G}/A$ to the spectrally localized Strichartz estimates with respect Littlewood–Paley projections of product type on the covering group $\widetilde{G}$. In Section 4, we will review the basic facts of structures and harmonic analysis on compact Lie groups, including the Fourier transform, root systems, structure of maximal tori, Weyl's character and dimension formulas, and the functional calculus of the Laplace–Beltrami operator. In Section 5 we will explicitly write down the Schrödinger kernel and interpret the Strichartz estimates as Fourier restriction estimates on the space-time, which then makes applicable the argument of Stein–Tomas type in Section 6. Then comes the core of the paper, Section 7, in which we will derive dispersive estimates for the Schrödinger kernel as the time variable lies in major arcs. In Section 7A, we will estimate some Weyl-type exponential sums over the so-called rational lattices, which in particular will imply the desired bound on the Schrödinger kernel for the nonrectangular rational tori. In Section 7B, we will rewrite the Schrödinger kernel for compact Lie groups into an exponential sum over the whole weight lattice instead of just one chamber of the lattice, and will prove the desired bound on the kernel for the case when the variable in the maximal torus stays away from all the cell walls by an application of the Weyl-type sum estimate established in Section 7A. In Section 7C, we will record two approaches to the pseudopolynomial behavior of characters, which will be applied to proving the desired bound on the Schrödinger kernel when the variable in the maximal torus stays close to the identity. In Section 7D, we further extend the result to the case when the variable in the maximal torus stays close to some corner. Section 7E will finally deal with the case when the variable in the maximal torus stays away from all the corners but close to some cell walls. These cell walls will be identified as those of a root subsystem, and we will then decompose the Schrödinger kernel into exponential sums over the root lattice of this root subsystem, thus reducing the problem into one similar to those already discussed in previous sections. This will finish the proof of the main theorem. In Section 7F, we will derive $L^p(G)$ estimates on the Schrödinger kernel as an upgrade of the $L^\infty(G)$-estimate.

Throughout the paper:

- $A \lesssim B$ means $A \le CB$ for some constant C.
- $A \lesssim_{a,b,\ldots} B$ means $A \le CB$ for some constant C that depends on $a, b, \ldots$.
- Δ, μ are short for the Laplace–Beltrami operator Δ_g and the associated volume-form measure μ_g respectively when the underlying Riemannian metric g is clear from context.
- L^p_x, H^s_x, L^p_t, $L^p_t L^q_x$, $L^p_{t,x}$ are short for $L^p(M)$, $H^s(M)$, $L^p(I)$, $L^p L^q(I \times M)$, $L^p(I \times M)$ respectively when the underlying manifold M and time interval I are clear from context.
- p' denotes the number such that $\frac{1}{p} + \frac{1}{p'} = 1$.

3. First reductions

3A. *Littlewood–Paley theory.* Let (M,g) be a compact Riemannian manifold and Δ be the Laplace–Beltrami operator. Let φ be a bump function on $\mathbb{R}$. Then for $N \geq 1$, $P_N := \varphi(N^{-2}\Delta)$ defines a bounded operator on $L^2(M)$ through the functional calculus of Δ. These operators P_N are often called the *Littlewood–Paley projections*. We reduce the problem of obtaining Strichartz estimates for $e^{it\Delta}$ to those for $P_N e^{it\Delta}$.

Proposition 3.1. *Fix $p, q \geq 2$, $s \geq 0$. Then the Strichartz estimate (1-1) is equivalent to the following statement: given any bump function φ,*

$$\|P_N e^{it\Delta} f\|_{L^p L^q(I\times M)} \lesssim N^s \|f\|_{L^2(M)}$$

holds for all dyadic natural numbers N (that is, for $N = 2^m$, $m \in \mathbb{Z}_{\geq 0}$). In particular, (2-2) reduces to

$$\|P_N e^{it\Delta} f\|_{L^p(I\times G)} \leq N^{\frac{d}{2}-\frac{d+2}{p}} \|f\|_{L^2(G)}. \tag{3-1}$$

This reduction is classical. We refer to [Burq et al. 2004] for a proof.

We also record here the Bernstein-type inequalities that will be useful in the sequel.

Proposition 3.2 [Burq et al. 2004, Corollary 2.2]. *Let d be the dimension of M. Then for all $1 \leq p \leq r \leq \infty$,*

$$\|P_N f\|_{L^r(M)} \lesssim N^{d(\frac{1}{p}-\frac{1}{r})} \|f\|_{L^p(M)}. \tag{3-2}$$

Note that the above proposition in particular implies that (3-1) holds for $N \lesssim 1$ or $p = \infty$.

3B. *Reduction to a finite cover.*

Proposition 3.3. *Let $\pi : (\widetilde{M}, \tilde{g}) \to (M, g)$ be a Riemannian covering map between compact Riemannian manifolds (then automatically with finite fibers). Let $\Delta_{\tilde{g}}$, Δ_g be the Laplace–Beltrami operators on $(\widetilde{M}, \tilde{g})$ and (M, g) respectively and let $\tilde{\mu}$ and μ be the normalized volume-form measures respectively, which define the L^p spaces. Let π^* be the pull-back map. Define*

$$C^\infty_\pi(\widetilde{M}) := \pi^*(C^\infty(M)),$$

and similarly define $C_\pi(\widetilde{M})$, $L^p_\pi(\widetilde{M})$ and $H^s_\pi(\widetilde{M})$. Then the following statements hold:

(i) *$\pi^* : C(M) \to C_\pi(\widetilde{M})$ and $\pi^* : C^\infty(M) \to C^\infty_\pi(\widetilde{M})$ are well-defined and are linear isomorphisms.*

(ii) *$\pi^* : L^p(M) \to L^p_\pi(\widetilde{M})$ is well-defined and is an isometry.*

(iii) *$\Delta_{\tilde{g}}$ maps $C^\infty_\pi(\widetilde{M})$ into $C^\infty_\pi(\widetilde{M})$ and the diagram*

$$\begin{array}{ccc} C^\infty(M)_{\Delta_g} & \xrightarrow{\pi^*} & C^\infty_\pi(\widetilde{M}) \\ \big\downarrow & & \big\downarrow \Delta_{\tilde{g}} \\ C^\infty(M) & \xrightarrow{\pi^*} & C^\infty_\pi(\widetilde{M}) \end{array}$$

commutes.

(iv) $e^{it\Delta_{\tilde{g}}}$ *maps* $L^2_\pi(\widetilde{M})$ *into* $L^2_\pi(\widetilde{M})$ *and is an isometry, and the diagrams*

$$\begin{array}{ccc} L^2(M)_{e^{it\Delta_g}} & \xrightarrow{\pi^*} & L^2_\pi(\widetilde{M}) \\ \downarrow & & \downarrow e^{it\Delta_{\tilde{g}}} \\ L^2(M) & \xrightarrow{\pi^*} & L^2_\pi(\widetilde{M}) \end{array} \qquad \begin{array}{ccc} L^2(M)_{P_N} & \xrightarrow{\pi^*} & L^2_\pi(\widetilde{M}) \\ \downarrow & & \downarrow P_N \\ L^2(M) & \xrightarrow{\pi^*} & L^2_\pi(\widetilde{M}) \end{array} \tag{3-3}$$

commute, where P_N *stands for both* $\varphi(N^{-2}\Delta_g)$ *and* $\varphi(N^{-2}\Delta_{\tilde{g}})$.

(v) $\pi^* : H^s(M) \to H^s_\pi(\widetilde{M})$ *is well-defined and is an isometry.*

Proof. Parts (i), (ii) and (iii) are direct consequences of the definition of a Riemannian covering map. For part (iv), note that (i), (ii) and (iii) together imply that the triples $(L^2(M), C^\infty(M), \Delta_g)$ and $(L^2_\pi(\widetilde{M}), C^\infty_\pi(\widetilde{M}), \Delta_{\tilde{g}})$ are isometric as systems of essentially self-adjoint operators on Hilbert spaces, and thus have isometric functional calculus. This implies (iv). Note that the $H^s(M)$ and $H^s_\pi(\widetilde{M})$ norms are also defined in terms of the isometric functional calculus of $(L^2(M), C^\infty(M), \Delta_g)$ and $(L^2_\pi(\widetilde{M}), C^\infty_\pi(\widetilde{M}), \Delta_{\tilde{g}})$ respectively, which implies (v). □

Combining Proposition 3.1 and 3.3, Theorem 2.2 is reduced to the following.

Theorem 3.4. *Let K_i's be simply connected simple Lie groups and let $G = \mathbb{T}^n \times K_1 \times \cdots \times K_m$ be equipped with a rational metric as in Definition 2.1. Then*

$$\|P_N e^{it\Delta} f\|_{L^p(I\times G)} \lesssim N^{\frac{d}{2}-\frac{d+2}{p}} \|f\|_{L^2(G)} \tag{3-4}$$

holds for $p \geq 2 + \frac{8}{r}$ and $N \gtrsim 1$.

3C. *Littlewood–Paley projections of product type.* Let (M, g) be the Riemannian product of the compact Riemannian manifolds (M_j, g_j), $j = 0, \ldots, m$. Any eigenfunction of the Laplace–Beltrami operator Δ on M with the eigenvalue $\lambda \leq 0$ is of the form $\prod_{j=0}^m \psi_{\lambda_j}$, where each ψ_{λ_j} is an eigenfunction of Δ_j on M_i with eigenvalue $\lambda_j \leq 0$, $j = 0, \ldots, m$, such that $\lambda = \lambda_0 + \cdots + \lambda_n$.

Given any bump function φ on $\mathbb{R}$, there always exist bump functions φ_j, $j = 0, \ldots, m$, such that for all $(x_0, \ldots, x_m) \in \mathbb{R}^{m+1}_{\leq 0}$ with $\varphi(x_0 + \cdots + x_m) \neq 0$, we have $\prod_{j=0}^m \varphi_j(x_j) = 1$. In particular,

$$\varphi \cdot \prod_{j=0}^m \varphi_j(x_j) = \varphi.$$

For $N \geq 1$, define

$$P_N := \varphi(N^{-2}\Delta),$$
$$\boldsymbol{P}_N := \varphi_0(N^{-2}\Delta_0) \otimes \cdots \otimes \varphi_m(N^{-2}\Delta_m)$$

as bounded operators on $L^2(M)$. We call $\boldsymbol{P}_N$ a *Littlewood–Paley projection of product type*. We have

$$\boldsymbol{P}_N \circ P_N = P_N.$$

This implies that we can further reduce Theorem 3.4 into the following.

Theorem 3.5. *Let $G = \mathbb{T}^n \times K_1 \times \cdots \times K_m$ be equipped with a rational metric. Let $\Delta_0, \Delta_1, \ldots, \Delta_m$ be respectively the Laplace–Beltrami operators on $\mathbb{T}^n, K_1, \ldots, K_m$. Let φ_j be any bump function for each $j = 0, \ldots, m$. For $N \geq 1$, let $\boldsymbol{P}_N = \bigotimes_{j=0}^{m} \varphi_j(N^{-2}\Delta_j)$. Then*

$$\|\boldsymbol{P}_N e^{it\Delta} f\|_{L^p(I\times G)} \lesssim N^{\frac{d}{2}-\frac{d+2}{p}} \|f\|_{L^2(G)} \tag{3-5}$$

holds for $p \geq 2 + \frac{8}{r}$ and $N \gtrsim 1$.

On the other hand, similarly, for each Littlewood–Paley projection $\boldsymbol{P}_N$ of product type, there exists a bump function φ such that $P_N = \varphi(N^{-2}\Delta)$ satisfies $P_N \circ \boldsymbol{P}_N = \boldsymbol{P}_N$. Noting that $\|\boldsymbol{P}_N f\|_{L^2} \lesssim \|f\|_{L^2}$, (3-2) then implies

$$\|\boldsymbol{P}_N f\|_{L^r(M)} \lesssim N^{d(\frac{1}{2}-\frac{1}{r})} \|f\|_{L^2(M)} \tag{3-6}$$

for all $2 \leq r \leq \infty$.

4. Preliminaries on harmonic analysis on compact Lie groups

4A. *Fourier transform.* Let G be a compact group and $\widehat{G}$ be its Fourier dual, i.e., the set of equivalent classes of irreducible unitary representations of G. For $\lambda \in \widehat{G}$, let $\pi_\lambda : V_\lambda \to V_\lambda$ be the irreducible unitary representation in the class λ, and let $d_\lambda = \dim(V_\lambda)$. Let μ be the normalized Haar measure on G. Then for $f \in L^2(G)$, define the Fourier transform

$$\hat{f}(\lambda) = \int_G f(x)\pi_\lambda(x^{-1})\, d\mu.$$

Then the inverse Fourier transform

$$f(x) = \sum_{\lambda\in\widehat{G}} d_\lambda \operatorname{tr}(\hat{f}(\lambda)\pi_\lambda(x))$$

converges in $L^2(G)$. We have the Plancherel identities

$$\|f\|_{L^2(G)} = \Big(\sum_{\lambda\in\widehat{G}} d_\lambda \|\hat{f}(\lambda)\|_{\mathrm{HS}}^2\Big)^{\frac{1}{2}}, \tag{4-1}$$

$$\langle f, g\rangle_{L^2(G)} = \sum_{\lambda\in\widehat{G}} d_\lambda \operatorname{tr}(\hat{f}(\lambda)\hat{g}(\lambda)^*). \tag{4-2}$$

Here $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert–Schmidt norm of endomorphisms.

For the convolution

$$(f * g)(x) = \int_G f(xy^{-1})g(y)\, d\mu(y),$$

we have

$$(f * g)^\wedge(j) = \hat{f}(j)\hat{g}(j). \tag{4-3}$$

If $\hat{g}(\lambda) = c_\lambda \cdot \mathrm{Id}_{d_\lambda\times d_\lambda}$, where c_λ is a scalar, then

$$\|f * g\|_{L^2(G)} \leq \sup_\lambda |c_\lambda| \cdot \|f\|_{L^2(G)}. \tag{4-4}$$

We also have the Hausdorff–Young inequality

$$\|\hat{f}(\lambda)\|_{\mathrm{HS}} \leq d_\lambda^{\frac{1}{2}} \|f\|_{L^1(G)} \quad \text{for all } \lambda \in \widehat{G}. \tag{4-5}$$

4B. ***Root system and the Laplace–Beltrami operator.*** Let G be a compact simply connected semisimple Lie group of dimension d and $\mathfrak{g}$ be its Lie algebra, and let $\mathfrak{g}_{\mathbb{C}}$ denote the complexification of $\mathfrak{g}$. Choose a *maximal torus* $B \subset G$ and let r be the dimension of B. Let $\mathfrak{b}$ be the Lie algebra of B, which is a *Cartan subalgebra* of $\mathfrak{g}$, and let $\mathfrak{b}_{\mathbb{C}}$ denote its complexification. The Fourier dual $\widehat{B}$ of B is isomorphic to a lattice $\Lambda \subset i\,\mathfrak{b}^*$, which is the *weight lattice*, under the isomorphism

$$\Lambda \xrightarrow{\sim} \widehat{B}, \quad \lambda \mapsto e^\lambda. \tag{4-6}$$

We have the *root space decomposition* $\mathfrak{g}_{\mathbb{C}} = \mathfrak{b}_{\mathbb{C}} \oplus \big(\bigoplus_{\alpha\in\Phi} \mathfrak{g}_{\mathbb{C}}^\alpha\big)$. Here $\Phi \subset i\,\mathfrak{b}^*$,

$$\mathfrak{g}_{\mathbb{C}}^\alpha = \{X \in \mathfrak{g}_{\mathbb{C}} \mid \mathrm{Ad}_b(X) = e^\alpha(b)X \text{ for all } b \in B\},$$

and $\dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}}^\alpha = 1$. This implies

$$|\Phi| + r = d. \tag{4-7}$$

The Cartan–Killing form $\langle \cdot\,, \cdot \rangle$ on $i\,\mathfrak{b}^*$ becomes a real inner product, and $(\Psi, \langle \cdot\,, \cdot \rangle)$ becomes an *integral root system*, that is, a finite set Φ in a finite-dimensional real inner product space with the following requirements:

(i) $\Phi = -\Phi$.

(ii) $\alpha \in \Phi,\ k \in \mathbb{R},\ k\alpha \in \Phi \Rightarrow k = \pm 1$.

(iii) $s_\alpha \Phi = \Phi$ for all $\alpha \in \Phi$.

(iv) $2\langle \alpha, \beta\rangle / \langle \alpha, \alpha\rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.

Here s_α is the reflection about the hyperplane $\alpha^\perp$ orthogonal to α; that is,

$$s_\alpha(x) := x - 2\frac{\langle x, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha.$$

Let P be a system of positive roots such that $\Phi = P \sqcup -P$. Then by (4-7), we have

$$|P| = \frac{d-r}{2}. \tag{4-8}$$

We can describe the weight lattice Λ purely in terms of the root system

$$\Lambda = \left\{ \lambda \in i\,\mathfrak{b}^* \;\middle|\; \frac{2\langle \lambda, \alpha\rangle}{\langle \alpha, \alpha\rangle} \in \mathbb{Z} \text{ for all } \alpha \in \Phi \right\}. \tag{4-9}$$

The set Φ of roots generate the *root lattice* Γ and we have $\Gamma \subset \Lambda$ and Λ/Γ is finite.

Let

$$\Lambda^+ := \left\{ \lambda \in i\,\mathfrak{b}^* \;\middle|\; \frac{2\langle \lambda, \alpha\rangle}{\langle \alpha, \alpha\rangle} \in \mathbb{Z}_{\geq 0} \text{ for all } \alpha \in P \right\}$$

be the set of dominant weights. We describe Λ, Λ^+ in terms of a basis. Let $\{\alpha_1, \dots, \alpha_r\}$ be the set of simple roots in P. Let $\{w_1, \dots, w_r\}$ be the corresponding fundamental weights, i.e., the dual basis to the coroot basis $\{2\alpha_1/\langle\alpha_1, \alpha_1\rangle, \dots, 2\alpha_r/\langle\alpha_r, \alpha_r\rangle\}$. Then

$$\Lambda = \mathbb{Z}w_1 + \cdots + \mathbb{Z}w_r,$$
$$\Lambda^+ = \mathbb{Z}_{\geq 0}w_1 + \cdots + \mathbb{Z}_{\geq 0}w_r.$$

Let

$$C = \mathbb{R}_{>0}w_1 + \cdots + \mathbb{R}_{>0}w_r \tag{4-10}$$

be the *fundamental Weyl chamber*, and we have the decomposition

$$i\mathfrak{b}^* = \Big(\bigsqcup_{s\in W} sC\Big) \sqcup \Big(\bigcup_{\alpha\in\Phi} \{\lambda \in i\mathfrak{b}^* \mid \langle\lambda, \alpha\rangle = 0\}\Big), \tag{4-11}$$

where W is the Weyl group. Here $\sqcup$ stands for disjoint union.

Define

$$\rho := \frac{1}{2}\sum_{\alpha\in P}\alpha = \sum_{i=1}^{r} w_i. \tag{4-12}$$

Then we have

$$\widehat{G} \cong \Lambda^+$$

such that the irreducible representation π_λ corresponding to $\lambda \in \Lambda^+$ has the character χ_λ and dimension d_λ given by Weyl's formulas

$$\chi_\lambda|_B = \frac{\sum_{s\in W}(\det s)e^{s(\lambda+\rho)}}{\sum_{s\in W}(\det s)e^{s\rho}}, \tag{4-13}$$

$$d_\lambda = \frac{\prod_{\alpha\in P}\langle\alpha, \lambda+\rho\rangle}{\prod_{\alpha\in P}\langle\alpha, \rho\rangle}. \tag{4-14}$$

Let $H \in \mathfrak{b}$. We can think of $-iH$ as a real linear functional on $i\mathfrak{b}^*$, and by the Cartan–Killing inner product on $i\mathfrak{b}^*$, we thus get a correspondence between $H \in \mathfrak{b}$ and an element in $i\mathfrak{b}^*$, still denoted as H. Under this correspondence, $e^{\lambda(H)} = e^{i\langle\lambda, H\rangle}$ and we rewrite Weyl's character formula as

$$\chi_\lambda(\exp H) = \frac{\sum_{s\in W}(\det s)e^{i\langle s(\lambda+\rho), H\rangle}}{\sum_{s\in W}(\det s)e^{i\langle\rho, H\rangle}}. \tag{4-15}$$

Also under this correspondence between $\mathfrak{b}$ and $i\mathfrak{b}^*$, we have

$$B \cong i\mathfrak{b}^*/2\pi\Gamma^\vee,$$

where

$$\Gamma^\vee = \mathbb{Z}\frac{2\alpha_1}{\langle\alpha_1, \alpha_1\rangle} + \cdots + \mathbb{Z}\frac{2\alpha_r}{\langle\alpha_r, \alpha_r\rangle}$$

is the *coroot lattice*.

We define the *cells* to be the connected components of $\{H \in i\mathfrak{b}^*/2\pi\Gamma^\vee \mid \langle\alpha, H\rangle \notin 2\pi\mathbb{Z}\}$ and call $\{H \in i\mathfrak{b}^*/2\pi\Gamma^\vee \mid \langle\alpha, H\rangle \in 2\pi\mathbb{Z}\}$ the *cell walls*.

We also record here Weyl's integral formula. Let $f \in L^1(G)$ be invariant under the adjoint action of G. Then

$$\int_G f\,d\mu = \frac{1}{|W|}\int_B f(b)|D_P(b)|^2\,db. \tag{4-16}$$

Here $d\mu, db$ are respectively the normalized Haar measures of G and B, and

$$D_P(H) = \sum_{s\in W}(\det s)e^{i\langle\rho,H\rangle}$$

is the *Weyl denominator*.

Finally we describe the functional calculus of the Laplace–Beltrami operator Δ. Given any irreducible unitary representation (π_λ, V_λ) of G in the class $\lambda \in \widehat{G} \cong \Lambda^+$, the operator Δ acts on the space $\mathcal{M}_\lambda = \{\operatorname{tr}(\pi_\lambda T) \mid T \in \operatorname{End}(V_\lambda)\}$ of matrix coefficients by

$$\Delta f = -k_\lambda f \quad \text{for all } f \in \mathcal{M}_\lambda,\ \lambda \in \widehat{G},$$

where

$$k_\lambda = |\lambda+\rho|^2 - |\rho|^2. \tag{4-17}$$

Let $f \in L^2(G)$ and consider the inverse Fourier transform $f(x) = \sum_{\lambda\in\Lambda^+} d_\lambda \operatorname{tr}(\pi_\lambda(x)\hat{f}(\lambda))$; then for any bounded Borel function $F : \mathbb{R} \to \mathbb{C}$, we have

$$F(\Delta)f = \sum_{\lambda\in\Lambda^+} F(-k_\lambda)d_\lambda \operatorname{tr}(\pi_\lambda(x)\hat{f}(\lambda)).$$

In particular, we have

$$e^{it\Delta}f = \sum_{\lambda\in\Lambda^+} e^{-itk_\lambda}d_\lambda \operatorname{tr}(\pi_\lambda(x)\hat{f}(\lambda)), \tag{4-18}$$

$$P_N e^{it\Delta}f = \sum_{\lambda\in\Lambda^+} \varphi\left(-\frac{k_\lambda}{N^2}\right)e^{-itk_\lambda}d_\lambda \operatorname{tr}(\pi_\lambda(x)\hat{f}(\lambda)). \tag{4-19}$$

Example 4.1. Let $M = \mathrm{SU}(2)$, which is of dimension 3 and rank 1. Let $\mathfrak{a} \cong \mathbb{R}$ be the Cartan subalgebra and $A \cong \mathbb{R}/2\pi\mathbb{Z}$ be the maximal torus. The root system is $\{\pm\alpha\}$, where α acts on $\mathfrak{a}$ by $\alpha(\theta) = 2\theta$. The fundamental weight is $w = \frac{1}{2}\alpha$. We normalize the Cartan–Killing form so that $|w| = 1$. The Weyl group W is of order 2, and acts on $\mathfrak{a}$ as well as $\mathfrak{a}^*$ through multiplication by ± 1. For $m \in \mathbb{Z}_{\geq 0} \cong \mathbb{Z}_{\geq 0}w = \Lambda^+$, we have

$$d_m = m+1, \tag{4-20}$$

$$\chi_m(\theta) = \frac{e^{i(m+1)\theta} - e^{-i(m+1)\theta}}{e^{i\theta} - e^{-i\theta}} = \frac{\sin(m+1)\theta}{\sin\theta}, \quad \theta \in \mathbb{R}/2\pi\mathbb{Z}, \tag{4-21}$$

$$k_m = (m+1)^2 - 1. \tag{4-22}$$

5. The Schrödinger kernel

Let $f \in L^2(G)$. Then (4-19) implies

$$(P_N e^{it\Delta}f)^\wedge(\lambda) = \varphi\left(\frac{k_\lambda}{N^2}\right)e^{-itk_\lambda}\hat{f}(\lambda).$$

Define

$$(K_N(t,\cdot))^\wedge(\lambda)=\varphi\Big(\frac{k_\lambda}{N^2}\Big)e^{-itk_\lambda}\mathrm{Id}_{d_\lambda\times d_\lambda},$$

which implies

$$K_N(t,x)=\sum_{\lambda\in\Lambda^+}\varphi\Big(\frac{k_\lambda}{N^2}\Big)e^{-itk_\lambda}d_\lambda\chi_\lambda(x). \tag{5-1}$$

Then we can write

$$P_Ne^{it\Delta}f=K_N(t,\cdot)*f=f*K_N(t,\cdot),$$

and we call $K_N(t,x)$ the *Schrödinger kernel*. Incorporating (4-14), (4-15) and (4-17) into (5-1), we get

$$K_N(t,x)=\sum_{\lambda\in\Lambda^+}e^{-it(|\lambda+\rho|^2-|\rho|^2)}\varphi\Big(\frac{|\lambda+\rho|^2-|\rho|^2}{N^2}\Big)\frac{\prod_{\alpha\in P}\langle\alpha,\lambda+\rho\rangle}{\prod_{\alpha\in P}\langle\alpha,\rho\rangle}\frac{\sum_{s\in W}(\det s)e^{i\langle s(\lambda+\rho),H\rangle}}{\sum_{s\in W}(\det s)e^{i\langle s(\rho),H\rangle}}. \tag{5-2}$$

Example 5.1. Specializing the Schrödinger kernel (5-2) to $G=\mathrm{SU}(2)$, using (4-20), (4-21), and (4-22), we have

$$K_N(t,\theta)=\sum_{m=0}^{\infty}\varphi\Big(\frac{(m+1)^2-1}{N^2}\Big)(m+1)e^{-i((m+1)^2-1)t}\frac{e^{i(m+1)\theta}-e^{-i(m+1)\theta}}{e^{i\theta}-e^{-i\theta}},\quad \theta\in\mathbb{R}/2\pi\mathbb{Z}. \tag{5-3}$$

More generally, let $G=\mathbb{R}^n/2\pi\Gamma_0\times K_1\times\cdots\times K_m$ be equipped with a rational metric g as in Definition 2.1. Let Λ_0 be the dual lattice of Γ_0 and Λ_j be the weight lattice for K_j, $j=1,\dots,m$. Let $\boldsymbol{P}_N=\bigotimes_{j=0}^m\varphi_j(N^{-2}\Delta_j)$ be a Littlewood–Paley projection of product type as described in Section 3C. Define the *Schrödinger kernel* $\boldsymbol{K}_N$ on G by

$$\boldsymbol{P}_Ne^{it\Delta}f=f*\boldsymbol{K}_N(t,\cdot)=\boldsymbol{K}_N(t,\cdot)*f. \tag{5-4}$$

Then

$$\boldsymbol{K}_N=\prod_{j=0}^m K_{N,j}, \tag{5-5}$$

where the $K_{N,j}$'s are respectively the Schrödinger kernels on each component of G

$$K_{N,0}=\sum_{\lambda_0\in\Lambda_0}\varphi_0\Big(\frac{-|\lambda_0|^2}{\beta_0N^2}\Big)e^{-it\beta_0^{-1}|\lambda_0|^2}e^{i\langle\lambda_0,H_0\rangle},$$

$$K_{N,j}=\sum_{\lambda_j\in\Lambda_j^+}\varphi_j\Big(\frac{-|\lambda_j+\rho_j|^2+|\rho_j|^2}{\beta_jN^2}\Big)e^{it\beta_j^{-1}(-|\lambda_j+\rho_j|^2+|\rho_j|^2)}d_{\lambda_j}\chi_{\lambda_j},$$

$j=1,\dots,m$. Here the ρ_j's are defined in terms of (4-12). We also write

$$\boldsymbol{K}_N=\sum_{\lambda\in\widehat{G}}\varphi(\lambda,N)e^{-itk_\lambda}d_\lambda\chi_\lambda,$$

where

$$\lambda=(\lambda_0,\dots,\lambda_m)\in\widehat{G}=\Lambda_0\times\Lambda_1^+\times\cdots\times\Lambda_m^+,$$

$$-k_\lambda=-\beta_0^{-1}|\lambda_0|^2+\sum_{j=1}^m\beta_j^{-1}(-|\lambda_j+\rho_j|^2+|\rho_j|^2), \tag{5-6}$$

$$\varphi(\lambda, N) = \varphi_0\left(\frac{-|\lambda_0|^2}{\beta_0 N^2}\right) \cdot \prod_{j=1}^{n} \varphi_j\left(\frac{-|\lambda_j+\rho_j|^2+|\rho_j|^2}{\beta_j N^2}\right), \tag{5-7}$$

$$d_\lambda = \prod_{j=1}^{m} d_{\lambda_j}, \ \ \chi_\lambda = e^{i\langle\lambda_0, H_0\rangle} \prod_{j=1}^{m} \chi_{\lambda_j}.$$

Tracking all the definitions, we get the following lemma.

Lemma 5.2. *Let d, r be respectively the dimension and rank of G*:

(i) $|\{\lambda \in \widehat{G} \mid k_\lambda \lesssim N^2\}| \lesssim N^r$.

(ii) $d_\lambda \lesssim N^{(d-r)/2}$ *uniformly for all* $\lambda \in \widehat{G}$ *such that* $k_\lambda \lesssim N^2$.

Now we interpret the Strichartz estimates on G as *Fourier restriction estimates*.

Lemma 5.3. *For a compact simply connected semisimple Lie group G and its weight lattice Λ, there exists $D \in \mathbb{N}$ such that $\langle\lambda_1, \lambda_2\rangle \in D^{-1}\mathbb{Z}$ for all $\lambda_1, \lambda_2 \in \Lambda$.*

Proof. Let Φ be the set of roots for G. Then by Lemma 4.3.5 in [Varadarajan 1974], $\langle\alpha, \beta\rangle$ are rational numbers for all $\alpha, \beta \in \Phi$. Let $S = \{\alpha_1, \dots, \alpha_r\} \subset \Phi$ be a system of simple roots. Since the set of fundamental weights $\{w_1, \dots, w_n\}$ forms a dual basis to $\{2\alpha_1/\langle\alpha_1, \alpha_1\rangle, \dots, 2\alpha_r/\langle\alpha_r, \alpha_r\rangle\}$ with respect to the Cartan–Killing form $\langle\cdot,\cdot\rangle$, and $\langle\alpha_i, \alpha_j\rangle$ are rational numbers for all $i, j = 1, \dots, r$, we have that the w_j's can be expressed as linear combinations of the α_j's with rational coefficients. This implies that $\langle w_i, w_j\rangle$ are rational numbers for all $i, j = 1, \dots, r$. Since there are only finitely many such numbers as $\langle w_i, w_j\rangle$, there exists $D \in \mathbb{N}$ so that $\langle w_i, w_j\rangle \in D^{-1}\mathbb{Z}$ for all $i, j = 1, \dots, r$. Thus $\langle\lambda_1, \lambda_2\rangle \in D^{-1}\mathbb{Z}$ for all $\lambda_1, \lambda_2 \in \Lambda$, since $\Lambda = \mathbb{Z}w_1 + \dots + \mathbb{Z}w_n$. ☐

For $G = \mathbb{R}^n/2\pi\Gamma_0 \times K_1 \times \dots \times K_m$, by the previous lemma, there exists for each $j = 1, \dots, m$ some $D_j \in \mathbb{N}$ such that $\langle\lambda, \mu\rangle \in D_j^{-1}\mathbb{Z}$ for all $\lambda, \mu \in \Lambda_j^+$, which implies by (4-12) that

$$-|\lambda_j+\rho_j|^2+|\rho_j|^2 = -|\lambda_j|^2 - \langle\lambda_j, 2\rho_j\rangle \in D_j^{-1}\mathbb{Z}$$

for all $\lambda_j \in \Lambda_j$. Also recall that we require that there exists some $D \in \mathbb{N}$ such that $\langle u, v\rangle \in D^{-1}\mathbb{Z}$ for all $u, v \in \Gamma_0$. This implies that there also exists some $D_0 \in \mathbb{N}$ such that $\langle\lambda, \mu\rangle \in D_0^{-1}\mathbb{Z}$ for all $\lambda, \mu \in \Lambda_0$. By Definition 2.1 of a rational metric, there exists some $D_* > 0$ such that

$$\beta_0^{-1}, \dots, \beta_m^{-1} \in D_*^{-1}\mathbb{N}.$$

Define

$$T = 2\pi D_* \cdot \prod_{j=0}^{m} D_j. \tag{5-8}$$

Then (5-6) implies that $Tk_\lambda \in 2\pi\mathbb{Z}$, which then implies that the Schrödinger kernel as in (5-5) is periodic in t with a period of T. Thus we may view the time variable t as living on the circle $\mathbb{T} = \mathbb{R}/T\mathbb{Z}$. Now the formal dual to the operator

$$\boldsymbol{T} : L^2(G) \to L^p(\mathbb{T} \times G), \quad f \mapsto \boldsymbol{P}_N e^{it\Delta}, \tag{5-9}$$

is computed to be

$$\boldsymbol{T}^*: L^{p'}(\mathbb{T}\times G)\to L^2(G),\quad F\mapsto \int_{\mathbb{T}} \boldsymbol{P}_N e^{-is\Delta}F(s,\cdot)\,\frac{ds}{T}, \tag{5-10}$$

and thus

$$\boldsymbol{T}\boldsymbol{T}^*: L^{p'}(\mathbb{T}\times G)\to L^p(\mathbb{T}\times G),\quad F\mapsto \int_{\mathbb{T}} \boldsymbol{P}_N^2 e^{i(t-s)\Delta}F(s,\cdot)\,\frac{ds}{T} = \widetilde{\boldsymbol{K}}_N * F, \tag{5-11}$$

where

$$\widetilde{\boldsymbol{K}}_N = \sum_{\lambda\in\widehat{G}} \varphi^2(\lambda,N)e^{-itk_\lambda}d_\lambda\chi_\lambda = \boldsymbol{K}_N * \boldsymbol{K}_N.$$

Note that the cutoff function $\varphi^2(\lambda,N)$ still defines a Littlewood–Paley projection of product type and $\widetilde{\boldsymbol{K}}_N$ is the associated Schrödinger kernel. Now the argument of $\boldsymbol{T}\boldsymbol{T}^*$ says that the boundedness of the operators (5-9), (5-10) and (5-11) are all equivalent; thus the Strichartz estimate in (3-1) is equivalent to the *space-time Strichartz estimate*

$$\|\widetilde{\boldsymbol{K}}_N * F\|_{L^p(\mathbb{T}\times G)} \lesssim N^{d-\frac{2(d+2)}{p}}\|F\|_{L^{p'}(\mathbb{T}\times G)}. \tag{5-12}$$

We have the *space-time Fourier transform* on $\mathbb{T}\times G$ as follows. For $(n,\lambda)\in \frac{2\pi}{T}\mathbb{Z}\times\widehat{G}$, we have

$$\widehat{\boldsymbol{K}}_N(n,\lambda) = \begin{cases} \varphi(\lambda,N)\cdot \mathrm{Id}_{d_\lambda\times d_\lambda} & \text{if } n=-k_\lambda,\\ 0 & \text{otherwise.}\end{cases} \tag{5-13}$$

Similarly, for $f\in L^2(G)$, we have

$$(\boldsymbol{P}_N e^{it\Delta}f(x))^\wedge(n,\lambda) = \begin{cases} \varphi(\lambda,N)\cdot \hat{f}(\lambda) & \text{if } n=-k_\lambda,\\ 0 & \text{otherwise.}\end{cases} \tag{5-14}$$

For $m(t)=\sum_{n\in(2\pi/T)\mathbb{Z}}\hat{m}(n)e^{itn}$, we compute

$$(m\boldsymbol{K}_N)^\wedge(n,\lambda) = \hat{m}(n+k_\lambda)\varphi(\lambda,N)\mathrm{Id}_{d_\lambda\times d_\lambda}. \tag{5-15}$$

6. The Stein–Tomas argument

Throughout this section, $\mathbb{S}^1$ stands for the standard circle of unit length, and $\|\cdot\|$ stands for the distance from 0 on $\mathbb{S}^1$. Define

$$\mathcal{M}_{a,q} := \left\{t\in\mathbb{S}^1 \;\middle|\; \left\|t-\frac{a}{q}\right\| < \frac{1}{qN}\right\},$$

where

$$a\in\mathbb{Z}_{\geq 0},\quad q\in\mathbb{N},\quad a<q,\quad (a,q)=1,\quad q<N.$$

We call such $\mathcal{M}_{a,q}$'s as *major arcs*, which are reminiscent of the Hardy–Littlewood circle method. We will prove the following key dispersive estimate.

Theorem 6.1. *Let $\boldsymbol{K}_N$ be the Schrödinger kernel (5-5) and T be the period (5-8). Then*

$$|\boldsymbol{K}_N(t,x)| \lesssim \frac{N^d}{\left(\sqrt{q}\left(1+N\left\|\frac{t}{2\pi D}-\frac{a}{q}\right\|^{1/2}\right)\right)^r}$$

for $\frac{t}{2\pi D}\in\mathcal{M}_{a,q}$, uniformly in $x\in G$.

Noting the product structure (5-5) of $\boldsymbol{K}_N$, the above theorem reduces to the cases on irreducible components of G.

Theorem 6.2. (i) *Given $G = \mathbb{T}^d = \mathbb{R}^d/2\pi\Gamma$ such that there exists $D \in \mathbb{R}$ for which $\langle\lambda,\mu\rangle \in D^{-1}\mathbb{Z}$ for all $\lambda,\mu \in \Gamma$. Then the Schrödinger kernel*

$$K_N(t,H) = \sum_{\lambda\in\Lambda} \varphi\left(\frac{|\lambda|^2}{N^2}\right) e^{-it|\lambda|^2 + i\langle\lambda,H\rangle}$$

satisfies

$$|K_N(t,H)| \lesssim \left(\frac{N}{\sqrt{q}\left(1+N\left\|\frac{t}{2\pi D}-\frac{a}{q}\right\|^{1/2}\right)}\right)^d$$

for $\frac{t}{2\pi D} \in \mathcal{M}_{a,q}$, uniformly in $H \in \mathbb{T}^n$.

(ii) *Let G be a compact simply connected semisimple Lie group. Let Λ be the weight lattice for which $\langle\lambda,\mu\rangle \in D^{-1}\mathbb{Z}$ for all $\lambda,\mu\in\Lambda$ for some $D \in \mathbb{R}$. Let K_N be the Schrödinger kernel as defined in (5-2). Then*

$$|K_N(t,x)| \lesssim \frac{N^d}{\left(\sqrt{q}\left(1+N\left\|\frac{t}{2\pi D}-\frac{a}{q}\right\|^{1/2}\right)\right)^r} \tag{6-1}$$

for $\frac{t}{2\pi D} \in \mathcal{M}_{a,q}$, uniformly in $x \in G$.

We will prove this theorem in the next section. Now we show how this theorem implies Strichartz estimates.

Theorem 6.3. *Let $G = \mathbb{T}^n \times K_1 \times \cdots \times K_m$ be equipped with a rational metric $\tilde{g}$ and T be a period of the Schrödinger flow as in (5-8). Let d, r be the dimension and rank of G respectively. Let $f \in L^2(G)$, $\lambda > 0$ and define*

$$m_\lambda = \mu\{(t,x) \in \mathbb{T}\times G \mid |P_N e^{it\Delta} f(x)| > \lambda|\},$$

where $\mu = dt\cdot d\mu_G$, with dt being the standard measure on $\mathbb{T} = \mathbb{R}/T\mathbb{Z}$ and $d\mu_G$ being the Haar measure on G. Let

$$p_0 = \frac{2(r+2)}{r}.$$

Then the following statements hold true:

(I) $$m_\lambda \lesssim_\varepsilon N^{\frac{dp_0}{2}-(d+2)+\varepsilon}\lambda^{-p_0}\|f\|^{p_0}_{L^2(G)} \quad \textit{for all } \lambda \gtrsim N^{\frac{d}{2}-\frac{r}{4}},\ \varepsilon > 0.$$

(II) $$m_\lambda \lesssim N^{\frac{dp}{2}-(d+2)}\lambda^{-p}\|f\|^{p}_{L^2(G)} \quad \textit{for all } \lambda \gtrsim N^{\frac{d}{2}-\frac{r}{4}},\ p > p_0.$$

(III) $$\|\boldsymbol{P}_N e^{it\Delta} f\|_{L^p(\mathbb{T}\times G)} \lesssim N^{\frac{d}{2}-\frac{d+2}{p}}\|f\|_{L^2(G)} \tag{6-2}$$

holds for all $p \ge 2 + \frac{8}{r}$.

(IV) *Assume it holds that*

$$\|\boldsymbol{P}_N e^{it\Delta} f\|_{L^p(\mathbb{T}\times G)} \lesssim_\varepsilon N^{\frac{d}{2}-\frac{d+2}{p}+\varepsilon}\|f\|_{L^2(G)} \tag{6-3}$$

for some $p > p_0$; then (6-2) holds for all $q > p$.

The proof strategy of this theorem is a Stein–Tomas-type argument, similar to the proofs of Propositions 3.82, 3.110, 3.113 in [Bourgain 1993]. The new ingredient is the nonabelian Fourier transform. We detail the proof in the following.

Let $\omega \in C_c^\infty(\mathbb{R})$ such that $\omega \geq 0$, $\omega(x) = 1$ for all $|x| \leq 1$ and $\omega(x) = 0$ for all $|x| \geq 2$. Let N be a dyadic natural number. Define

$$\omega_{\frac{1}{N^2}} := \omega(N^2 \cdot),$$
$$\omega_{\frac{1}{NM}} := \omega(NM \cdot) - \omega(2NM \cdot),$$

where

$$1 \leq M < N, \quad M \text{ dyadic}.$$

Let

$$N_1 = \frac{N}{2^{10}}, \quad 1 \leq Q < N_1, \ Q \text{ dyadic}.$$

Then

$$\sum_{Q \leq M \leq N} \omega_{\frac{1}{NM}} = 1 \quad \text{on } \Big[-\frac{1}{NQ}, \frac{1}{NQ}\Big], \tag{6-4}$$

$$\sum_{Q \leq M \leq N} \omega_{\frac{1}{NM}} = 0 \quad \text{outside } \Big[-\frac{2}{NQ}, \frac{2}{NQ}\Big]. \tag{6-5}$$

Write

$$1 = \sum_{1 \leq Q \leq N_1} \sum_{Q \leq M \leq N} \Bigg[\Bigg(\sum_{\substack{(a,q)=1 \\ Q \leq q < 2Q}} \delta_{\frac{a}{q}}\Bigg) * \omega_{\frac{1}{NM}}\Bigg]\Big(\frac{t}{T}\Big) + \rho(t). \tag{6-6}$$

Note the major arc disjointness property

$$\Big(\frac{a_1}{q_1} + \Big[-\frac{2}{NQ_1}, \frac{2}{NQ_1}\Big]\Big) \cap \Big(\frac{a_2}{q_2} + \Big[-\frac{2}{NQ_2}, \frac{2}{NQ_2}\Big]\Big) = \varnothing$$

for $(a_i, q_i) = 1$, $Q_i \leq q_i < 2Q_i$, $i = 1, 2$, $Q_1 \leq Q_2 \leq N_1$. This in particular implies

$$0 \leq \rho(t) \leq 1 \quad \text{for all } t \in \mathbb{R}/T\mathbb{Z}, \tag{6-7}$$

$$\Bigg[\Bigg(\sum_{\substack{(a,q)=1 \\ Q \leq q < 2Q}} \delta_{\frac{a}{q}}\Bigg) * \omega_{\frac{1}{NM}}\Big(\frac{\cdot}{T}\Big)\Bigg]^{\wedge}(0) = \frac{1}{T}\int_0^T \Bigg(\sum_{\substack{(a,q)=1 \\ Q \leq q < 2Q}} \delta_{\frac{a}{q}}\Bigg) * \omega_{\frac{1}{NM}}\Big(\frac{t}{T}\Big)\, dt \leq \frac{2Q^2}{NM}, \tag{6-8}$$

which implies

$$1 \geq |\hat{\rho}(0)| \geq 1 - \sum_{1 \leq Q \leq N_1} \sum_{Q \leq M \leq N} \Bigg|\Bigg[\Bigg(\sum_{\substack{(a,q)=1 \\ Q \leq q < 2Q}} \delta_{\frac{a}{q}}\Bigg) * \omega_{\frac{1}{NM}}\Big(\frac{\cdot}{T}\Big)\Bigg]^{\wedge}(0)\Bigg| \geq 1 - \frac{8N_1}{N} \geq \frac{1}{2}. \tag{6-9}$$

By Dirichlet's lemma on rational approximations, for any $\frac{t}{T} \in \mathbb{S}^1$, there exists a, q, with $a \in \mathbb{Z}_{\geq 0}$, $q \in \mathbb{N}$, $(a,q) = 1$, $q \leq N$, such that $\big|\frac{t}{T} - \frac{a}{q}\big| < \frac{1}{qN}$. If $\rho\big(\frac{t}{T}\big) \neq 0$, then (6-4) implies $q > N_1 = N/2^{10}$. This implies by (6-1) and (6-7) that

$$\|\rho(t)K_N(t,x)\|_{L^\infty(\mathbb{T}\times G)} \lesssim N^{d-\frac{r}{2}}. \tag{6-10}$$

Now define coefficients $\alpha_{Q,M}$ such that

$$\left[\left(\sum_{\substack{(a,q)=1\\ Q\le q<2Q}} \delta_{\frac{a}{q}}\right) * \omega_{\frac{1}{NM}}\left(\frac{\cdot}{T}\right)\right]^{\wedge}(0) = \alpha_{Q,M}\hat{\rho}(0). \tag{6-11}$$

Then (6-8) and (6-9) imply

$$\alpha_{Q,M} \lesssim \frac{Q^2}{NM}. \tag{6-12}$$

Write

$$\boldsymbol{K}_N(t,x) = \sum_{Q\le N_1}\sum_{Q\le M\le N} \boldsymbol{K}_N(t,x)\left[\left(\left(\sum_{\substack{(a,q)=1\\ Q\le q<2Q}} \delta_{\frac{a}{q}}\right) * \omega_{\frac{1}{NM}}\left(\frac{\cdot}{T}\right)\right) - \alpha_{Q,M}\rho\right](t) + \left(1+\sum_{Q,M}\alpha_{Q,M}\right)\boldsymbol{K}_N(t,x)\rho(t), \tag{6-13}$$

and define

$$\Lambda_{Q,M}(t,x) := \boldsymbol{K}_N(t,x)\left[\left(\left(\sum_{\substack{(a,q)=1\\ Q\le q<2Q}} \delta_{\frac{a}{q}}\right) * \omega_{\frac{1}{NM}}\left(\frac{\cdot}{T}\right)\right) - \alpha_{Q,M}\rho\right](t). \tag{6-14}$$

Then from (6-1), (6-10), (6-12), we have

$$\|\Lambda_{Q,M}\|_{L^\infty(\mathbb{T}\times G)} \lesssim N^{d-\frac{r}{2}}\left(\frac{M}{Q}\right)^{\frac{r}{2}}. \tag{6-15}$$

Next, we estimate $\widehat{\Lambda}_{Q,M}$. From (5-15), for

$$n \in \frac{2\pi}{T}\mathbb{Z} \cong \widehat{\mathbb{T}}, \quad \lambda \in \widehat{G},$$

we have

$$\widehat{\Lambda}_{Q,M}(n,\lambda) = \lambda_{Q,M}(n,\lambda)\cdot \mathrm{Id}_{d_\lambda\times d_\lambda}, \tag{6-16}$$

where

$$\lambda_{Q,M}(n,\lambda) = \varphi(\lambda,N)\left[\left(\sum_{\substack{(a,q)=1\\ Q\le q<2Q}} \delta_{\frac{a}{q}}\right)^{\wedge}\cdot \hat{\omega}_{\frac{1}{NM}}(T\,\cdot) - \alpha_{Q,M}\hat{\rho}\right](n+k_\lambda). \tag{6-17}$$

Note that (6-11) immediately implies

$$\lambda_{Q,M}(n,\lambda) = 0 \quad \text{for } n+k_\lambda = 0. \tag{6-18}$$

Let $d(m,Q)$ denote the number of divisors of m less than Q; using Lemma 3.33 in [Bourgain 1993],

$$\left|\left(\sum_{\substack{(a,q)=1\\ Q\le q<2Q}} \delta_{\frac{a}{q}}\right)^{\wedge}(Tn)\right| \lesssim_\varepsilon d\left(\frac{Tn}{2\pi}, Q\right) Q^{1+\varepsilon}, \quad n\ne 0,\ \varepsilon>0, \tag{6-19}$$

we get

$$|\lambda_{Q,M}(n,\lambda)| \lesssim_\varepsilon \varphi(\lambda,N)\frac{Q^{1+\varepsilon}}{NM} d\left(\frac{T(n+k_\lambda)}{2\pi}, Q\right) + \frac{Q^2}{NM}|\hat{\rho}(n+k_\lambda)|. \tag{6-20}$$

Using

$$d(m,Q) \lesssim_\varepsilon m^\varepsilon,$$

(6-19) and (6-6), we have

$$|\hat{\rho}(n)| \le \sum_{1\le Q\le N_1}\sum_{Q\le M\le N} \frac{d\left(\frac{Tn}{2\pi},Q\right)Q^{1+\varepsilon}}{NM} \lesssim \frac{N^\varepsilon}{N} \quad \text{for } n\ne 0,\ |n|\lesssim N^2; \tag{6-21}$$

thus

$$\begin{aligned} |\lambda_{Q,M}(n,\lambda)| &\lesssim_\varepsilon \varphi(\lambda,N)\frac{Q}{NM}\left[Q^\varepsilon d\left(\frac{T(n+k_\lambda)}{2\pi},Q\right) + \frac{Q}{N^{1-\varepsilon}}\right] \\ &\lesssim_\varepsilon \varphi(\lambda,N)\frac{QN^\varepsilon}{NM} \quad \text{for } |n|\lesssim N^2. \end{aligned} \tag{6-22}$$

Proposition 6.4. (i) *Assume that $f\in L^1(\mathbb{T}\times G)$. Then*

$$\|f*\Lambda_{Q,M}\|_{L^\infty(\mathbb{T}\times G)} \lesssim N^{d-\frac{r}{2}}\left(\frac{M}{Q}\right)^{\frac{r}{2}}\|f\|_{L^1(\mathbb{T}\times G)}. \tag{6-23}$$

(ii) *Assume that $f\in L^2(\mathbb{T}\times G)$. Assume also*

$$\hat{f}(n,\lambda) = 0 \quad \textit{for } |n|\gtrsim N^2. \tag{6-24}$$

Then

$$\|f*\Lambda_{Q,M}\|_{L^2(\mathbb{T}\times G)} \lesssim_\varepsilon \frac{QN^\varepsilon}{NM}\|f\|_{L^2(\mathbb{T}\times G)}, \tag{6-25}$$

$$\|f*\Lambda_{Q,M}\|_{L^2(\mathbb{T}\times G)} \lesssim_{\tau,B} \frac{Q^{1+2\tau}L}{NM}\|f\|_{L^2(\mathbb{T}\times G)} + M^{-1}L^{-\frac{B}{2}}N^{\frac{d}{2}}\|f\|_{L^1(\mathbb{T}\times G)} \tag{6-26}$$

for all

$$L>1,\quad 0<\tau<1,\quad B>\frac{6}{\tau},\quad N>(LQ)^B. \tag{6-27}$$

Proof. Using (6-15), we have

$$\|f*\Lambda_{Q,M}\|_{L^\infty(\mathbb{T}\times G)} \le \|f\|_{L^1(\mathbb{T}\times G)}\|\Lambda_{Q,M}\|_{L^\infty(\mathbb{T}\times G)} \lesssim N^{d-\frac{r}{2}}\left(\frac{M}{Q}\right)^{\frac{r}{2}}\|f\|_{L^1(\mathbb{T}\times G)}.$$

This proves (i). (6-25) is a consequence of (4-4), (6-16), and (6-22). To prove (6-26), we use (4-1), (4-3) and (6-16) to get

$$\|f*\Lambda_{Q,M}\|_{L^2(\mathbb{T}\times G)} = \left(\sum_{n,\lambda} d_\lambda\|\hat{f}(n,\lambda)\|_{\mathrm{HS}}^2\cdot|\lambda_{Q,M}(n,\lambda)|^2\right)^{\frac{1}{2}},$$

which combined with (6-18), (6-20), and (6-21) yields

$$\|f*\Lambda_{Q,M}\|_{L^2(\mathbb{T}\times G)} \lesssim_\varepsilon \frac{Q^{1+\varepsilon}}{NM}\left(\sum_{n,\lambda}\varphi(\lambda,N)^2 d_\lambda\|\hat{f}(n,\lambda)\|_{\mathrm{HS}}^2\, d\left(\frac{T(n+k_\lambda)}{2\pi},Q\right)^2\right)^{\frac{1}{2}} + \frac{Q^2}{MN^{2-\varepsilon}}\|f\|_{L^2(\mathbb{T}\times G)}. \tag{6-28}$$

Using Lemma 3.47 in [Bourgain 1993] and Lemma 5.2, we have

$$\begin{aligned}\left|\left\{(n,\lambda)\;\middle|\;|n|,k_\lambda\lesssim N^2, d\left(\frac{T(n+k_\lambda)}{2\pi},Q\right)>D\right\}\right| \\ &\lesssim_{\tau,B}(D^{-B}Q^\tau N^2+Q^B)\cdot\max_{|m|\lesssim N^2}|\{(n,\lambda)\mid n+k_\lambda=m\}| \\ &\lesssim_{\tau,B}(D^{-B}Q^\tau N^2+Q^B)\cdot|\{\lambda\in\widehat{G}\mid k_\lambda\lesssim N^2\}| \\ &\lesssim_{\tau,B}(D^{-B}Q^\tau N^2+Q^B)\cdot N^r.\end{aligned} \tag{6-29}$$

Now (4-5) gives

$$\|\hat{f}(n,\lambda)\|_{\mathrm{HS}}^2\lesssim d_\lambda\|f\|_{L^1(\mathbb{T}\times G)}^2,$$

and Lemma 5.2 gives

$$|\varphi(\lambda,N)d_\lambda^2|\lesssim N^{d-r},$$

which together with (6-29) imply

$$\begin{aligned}&\|f*\Lambda_{Q,M}\|_{L^2(\mathbb{T}\times G)} \\ &\quad\lesssim_{\tau,B}\left(\frac{Q^{1+\varepsilon}D}{NM}+\frac{Q^2}{MN^{2-\varepsilon}}\right)\|f\|_{L^2(\mathbb{T}\times G)}+\frac{Q^{1+\varepsilon}}{NM}\cdot Q\cdot(D^{-\frac{B}{2}}Q^\tau N+Q^{\frac{B}{2}})N^{\frac{d}{2}}\|f\|_{L^1(\mathbb{T}\times G)}.\end{aligned} \tag{6-30}$$

This implies (6-26) assuming the conditions in (6-27). □

Now interpolating (6-23) and (6-25), we get

$$\|f*\Lambda_{Q,M}\|_{L^p(\mathbb{T}\times G)}\lesssim_\varepsilon N^{d-\frac{r}{2}-\frac{2d-r+2}{p}+\varepsilon}M^{\frac{r}{2}-\frac{r+2}{p}}Q^{-\frac{r}{2}+\frac{r+2}{p}}\|f\|_{L^{p'}(\mathbb{T}\times G)}. \tag{6-31}$$

Interpolating (6-23) and (6-26) for

$$p>\frac{2(r+2)}{r}+10\tau,\quad\text{which implies }\sigma=\frac{r}{2}-\frac{r+2+4\tau}{p}>0, \tag{6-32}$$

we get

$$\begin{aligned}\|f*\Lambda_{Q,M}\|_{L^p(\mathbb{T}\times G)}\lesssim_{\tau,B}&\,N^{d-\frac{r}{2}-\frac{2d-r+2}{p}}M^{\frac{r}{2}-\frac{r+2}{p}}Q^{-\sigma}L^{\frac{2}{p}}\|f\|_{L^{p'}(\mathbb{T}\times G)} \\ &+Q^{-\frac{2}{r}(1-\frac{2}{p})}M^{\frac{r}{2}-\frac{r+2}{p}}L^{-\frac{B}{p}}N^{d-\frac{r}{2}-\frac{d-r}{p}}\|f\|_{L^1(\mathbb{T}\times G)}.\end{aligned} \tag{6-33}$$

Now we are ready to prove Theorem 6.3.

Proof of Theorem 6.3. Without loss of generality, we assume that $\|f\|_{L^2(G)}=1$. Then for $F=\boldsymbol{P}_N e^{it\Delta}f$, (3-2) implies

$$\|F\|_{L_x^2}\lesssim 1, \tag{6-34}$$

$$\|F\|_{L_x^\infty}\lesssim N^{\frac{d}{2}}. \tag{6-35}$$

Let

$$H=\chi_{|F|>\lambda}\cdot\frac{F}{|F|}. \tag{6-36}$$

Let $\widetilde{\widetilde{\boldsymbol{P}}}_N$ be a Littlewood–Paley projection of product type such that $\widetilde{\widetilde{\boldsymbol{P}}}_N \circ \boldsymbol{P}_N = \boldsymbol{P}_N$. Let $\widetilde{\widetilde{\boldsymbol{K}}}_N$ be the Schrödinger kernel associated to $\widetilde{\widetilde{\boldsymbol{P}}}_N e^{it\Delta}$. Then by (4-3), (5-13), and (5-14), we have

$$F * \widetilde{\widetilde{\boldsymbol{K}}}_N = F.$$

Let Q_{N^2} be the Littlewood–Paley projection operator on $L^2(\mathbb{T} \times G)$ defined by

$$(Q_{N^2}H)^\wedge := \varphi\left(\frac{-k_\lambda - n^2}{N^4}\right)\widehat{H}(n,\lambda)$$

for some bump function φ such that $Q_{N^2} \circ \boldsymbol{P}_N = \boldsymbol{P}_N$. Then by (4-2) and (5-14), we have

$$\langle F, H\rangle_{L^2_{t,x}} = \langle Q_{N^2}F, H\rangle_{L^2_{t,x}} = \langle F, Q_{N^2}H\rangle_{L^2_{t,x}}.$$

Then we can write

$$\lambda m_\lambda \le \langle F, H\rangle_{L^2_{t,x}} = \langle F * \widetilde{\widetilde{\boldsymbol{K}}}_N, Q_{N^2}H\rangle_{L^2_{t,x}}.$$

Using (4-1) and (4-3) again, we get

$$\begin{aligned} \lambda m_\lambda &\le \langle F, Q_{N^2}H * \widetilde{\widetilde{\boldsymbol{K}}}_N\rangle_{L^2_{t,x}} \le \|F\|_{L^2_{t,x}} \|Q_{N^2}H * \widetilde{\widetilde{\boldsymbol{K}}}_N\|_{L^2_{t,x}} \\ &\lesssim \|Q_{N^2}H * \widetilde{\widetilde{\boldsymbol{K}}}_N\|_{L^2_{t,x}} = \langle Q_{N^2}H * \widetilde{\widetilde{\boldsymbol{K}}}_N, Q_{N^2}H * \widetilde{\widetilde{\boldsymbol{K}}}_N\rangle_{L^2_{t,x}} \\ &= \langle Q_{N^2}H, Q_{N^2}H * (\widetilde{\widetilde{\boldsymbol{K}}}_N * \widetilde{\widetilde{\boldsymbol{K}}}_N)\rangle_{L^2_{t,x}}. \end{aligned} \tag{6-37}$$

Let

$$H' = Q_{N^2}H, \quad \widetilde{\boldsymbol{K}}_N = \widetilde{\widetilde{\boldsymbol{K}}}_N * \widetilde{\widetilde{\boldsymbol{K}}}_N.$$

Note that H' by definition satisfies the assumption in (6-24) and we can apply Proposition 6.4. Also note that $\widetilde{\boldsymbol{K}}_N$ is still a Schrödinger kernel associated to a Littlewood–Paley projection operator of product type. Finally note that the Bernstein-type inequalities (3-2) and the definition (6-36) of H give

$$\|H'\|_{L^p_{t,x}} \lesssim \|H\|_{L^p_{t,x}} \lesssim m_\lambda^{\frac{1}{p}}. \tag{6-38}$$

Write

$$\Lambda = \sum_{1 \le Q \le N_1} \sum_{Q \le M \le N} \Lambda_{Q,M}, \quad \widetilde{\boldsymbol{K}}_N = \Lambda + (\widetilde{\boldsymbol{K}}_N - \Lambda),$$

where $\Lambda_{Q,M}$ is defined as in (6-14) except that $\boldsymbol{K}_N$ is replaced by $\widetilde{\boldsymbol{K}}_N$. We have by (6-37)

$$\begin{aligned} \lambda^2 m_\lambda^2 &\lesssim \langle H', H' * \Lambda\rangle_{L^2_{t,x}} + \langle H', H' * (\widetilde{\boldsymbol{K}}_N - \Lambda)\rangle_{L^2_{t,x}} \\ &\lesssim \|H'\|_{L^{p'}_{t,x}} \|H' * \Lambda\|_{L^p_{t,x}} + \|H'\|^2_{L^1_{t,x}} \|\widetilde{\boldsymbol{K}}_N - \Lambda\|_{L^\infty_{t,x}}. \end{aligned} \tag{6-39}$$

Using (6-31) for $p = p_0 := \frac{2(r+2)}{r}$, then summing over Q, M, and noting (6-38), we have

$$\|H'\|_{L^{p'}_{t,x}} \|H' * \Lambda\|_{L^p_{t,x}} \lesssim N^{d - \frac{2d+4}{p_0} + \varepsilon} \|H'\|^2_{L^{p_0'}_{t,x}} \lesssim N^{d - \frac{2d+4}{p_0} + \varepsilon} m_\lambda^{\frac{2}{p_0'}}.$$

From (6-10) and (6-12) we get

$$\|\widetilde{\boldsymbol{K}}_N - \Lambda\|_{L^\infty_{t,x}} \lesssim N^{d - \frac{r}{2}}, \tag{6-40}$$

which implies

$$\|H'\|_{L^1_{t,x}}^2 \|\widetilde{\boldsymbol{K}}_N - \Lambda\|_{L^\infty_{t,x}} \lesssim N^{d-\frac{r}{2}} \|H'\|_{L^1_{t,x}}^2 \lesssim N^{d-\frac{r}{2}} m_\lambda^2. \tag{6-41}$$

Then we have

$$\lambda^2 m_\lambda^2 \lesssim N^{d-\frac{2d+4}{p_0}+\varepsilon} m_\lambda^{\frac{2}{p_0'}} + N^{d-\frac{r}{2}} m_\lambda^2,$$

which implies for $\lambda \gtrsim N^{d/2-r/4}$

$$m_\lambda \lesssim_\varepsilon N^{p_0(\frac{d}{2}-\frac{d+2}{p_0})+\varepsilon} \lambda^{-p_0}.$$

Thus part (I) is proved. To prove part (II) for some fixed p, using part (I) and (6-35), it suffices to prove it for $\lambda \gtrsim N^{d/2-\varepsilon}$. Summing (6-33) over Q, M in the range indicated by (6-27), we get

$$\|H' * \Lambda_1\|_{L^p_{t,x}} \lesssim L N^{d-\frac{2d+4}{p}} \|H'\|_{L^{p'}_{t,x}} + L^{-\frac{B}{p}} N^{d-\frac{d+2}{p}} \|H'\|_{L^1_{t,x}}, \tag{6-42}$$

where

$$\Lambda_1 := \sum_{\substack{Q<Q_1 \\ Q \le M \le N}} \Lambda_{Q,M}$$

and Q_1 is the largest Q-value satisfying (6-27). For values $Q \ge Q_1$, use (6-31) to get

$$\|H' * (\Lambda - \Lambda_1)\|_{L^p_{t,x}} \lesssim_\varepsilon N^{d-\frac{2d+4}{p}+\varepsilon} Q_1^{-(\frac{r}{2}-\frac{r+2}{p})} \|H'\|_{L^{p'}_{t,x}}. \tag{6-43}$$

Using (6-39), (6-41), (6-42) and (6-43), we get

$$\lambda^2 m_\lambda^2 \lesssim N^{d-\frac{2(d+2)}{p}} \left(L + \frac{N^\varepsilon}{Q_1^{\frac{r}{2}-\frac{r+2}{p}}} \right) m_\lambda^{\frac{2}{p}'} + L^{-\frac{B}{p}} N^{d-\frac{d+2}{p}} m_\lambda^{1+\frac{1}{p'}} + N^{d-\frac{r}{2}} m_\lambda^2.$$

For $\lambda \gtrsim N^{d/2-r/4}$, the last term of the above inequality can be dropped. Let $Q_1 = N^\delta$ such that $\delta > 0$ and

$$(L N^\delta)^B < N \tag{6-44}$$

such that (6-27) holds. Note that

$$L > 1 > \frac{N^\varepsilon}{Q_1^{\frac{r}{2}-\frac{r+2}{p}}}$$

for $p > p_0 + 10\tau$ and ε sufficiently small; thus

$$\lambda^2 m_\lambda^2 \lesssim N^{d-\frac{2(d+2)}{p}} L m_\lambda^{\frac{2}{p}'} + L^{-\frac{B}{p}} N^{d-\frac{d+2}{p}} m_\lambda^{1+\frac{1}{p'}}.$$

This implies

$$\begin{aligned} m_\lambda &\lesssim N^{p(\frac{d}{2}-\frac{d+2}{p})} L^{\frac{p}{2}} \lambda^{-p} + N^{p(d-\frac{d+2}{p})} L^{-B} \lambda^{-2p} \\ &\lesssim N^{-d-2} \left(\frac{N^{\frac{d}{2}}}{\lambda} \right)^p L^{\frac{p}{2}} + N^{-d-2} \left(\frac{N^{\frac{d}{2}}}{\lambda} \right)^{2p} L^{-B}. \end{aligned}$$

Let

$$L = \left(\frac{N^{\frac{d}{2}}}{\lambda}\right)^{\tau}, \quad B > \frac{p}{\tau}$$

and δ be sufficiently small so that (6-44) holds; then

$$m_\lambda \lesssim N^{-d-2}\left(\frac{N^{\frac{d}{2}}}{\lambda}\right)^{p+\frac{p\tau}{2}}.$$

Note that conditions for p, τ indicated in (6-32) imply that $p + \frac{p\tau}{2}$ can take any exponent $> p_0 = \frac{2(r+2)}{r}$. This completes the proof of part (II).

The proofs of parts (III) and (IV) are then identical to the proofs of Propositions 3.110 and 3.113 respectively in [Bourgain 1993]. □

Proof of Theorem 2.2. Part (i) is a direct consequence of Theorem 6.3(III). Part (ii) is a direct consequence of Theorem 6.3(IV) and the result from [Bourgain and Demeter 2015] that full Strichartz estimates hold on any torus with an ε-loss. □

7. Dispersive estimates on major arcs

In this section, we prove Theorem 6.2.

7A. *Weyl-type sums on rational lattices.*

Definition 7.1. Let $L = \mathbb{Z}w_1 + \cdots + \mathbb{Z}w_r$ be a lattice on an inner product space $(V, \langle\cdot,\cdot\rangle)$. We say L is a *rational lattice* provided that there exists some $D \in \mathbb{R}$ such that $\langle w_i, w_j\rangle \in D^{-1}\mathbb{Z}$. We call the number D a *period* of L.

By Lemma 5.3, any weight lattice Λ is a rational lattice with respect to the Cartan–Killing form. As a sublattice of Λ, the root lattice Γ is also rational.

Let f be a function on $\mathbb{Z}^r$ and define the *difference operator* D_i by

$$D_i f(n_1,\dots,n_r) := f(n_1,\dots,n_{i-1},n_i+1,n_{i+1},\dots,n_r) - f(n_1,\dots,n_r) \tag{7-1}$$

for $i = 1,\dots,r$. The Leibniz rule for D_i reads

$$D_i\left(\prod_{j=1}^{n} f_j\right) = \sum_{l=1}^{n} \sum_{1\le k_1<\cdots<k_l\le n} D_i f_{k_1}\cdots D_i f_{k_l} \cdot \prod_{\substack{j\ne k_1,\dots,k_l\\ 1\le j\le n}} f_j. \tag{7-2}$$

Note that there are $2^n - 1$ terms in the right side of the above formula.

Definition 7.2. Let $L \cong \mathbb{Z}^r$ be a lattice of rank r. Given $A \in \mathbb{R}$, we say a function f on L is a pseudopolynomial of degree A provided for each $n \in \mathbb{Z}_{\ge 0}$

$$|D_{i_1}\cdots D_{i_n} f(n_1,\dots,n_r)| \lesssim N^{A-n} \tag{7-3}$$

holds uniformly in $|n_i| \lesssim N$, $i = 1,\dots,r$, for all $i_j = 1,\dots,r$, $j = 1,\dots,n$, and $N \ge 1$.

A direct application of the Leibniz rule (7-2) gives the following lemma.

Lemma 7.3. *Let L be a lattice and f, g two functions on L. Assume f, g are pseudopolynomials of degrees A, B respectively. Then $f \cdot g$ is a pseudopolynomial of degree $A + B$.*

Now we have the following estimate on Weyl-type sums, which generalizes the classical Weyl inequality in one dimension, as in Lemma 3.18 of [Bourgain 1993].

Lemma 7.4. *Let $L = \mathbb{Z}w_1 + \cdots + \mathbb{Z}w_r$ be a rational lattice in the inner product space $(V, \langle\cdot,\cdot\rangle)$ with a period $D > 0$. Let φ be a bump function on $\mathbb{R}$ and $N \geq 1$, $A \in \mathbb{R}$. Suppose $f : L \to \mathbb{C}$ a pseudopolynomial of degree A. Let*

$$F(t, H) = \sum_{\lambda \in L} e^{-it|\lambda|^2 + i\langle\lambda, H\rangle} \varphi\left(\frac{|\lambda|^2}{N^2}\right) \cdot f \tag{7-4}$$

for $t \in \mathbb{R}$ and $H \in V$. Then for $\frac{t}{2\pi D} \in \mathcal{M}_{a,q}$, we have

$$|F(t, H)| \lesssim \frac{N^{A+r}}{\left(\sqrt{q}\left(1 + N\left\|\frac{t}{2\pi D} - \frac{a}{q}\right\|^{1/2}\right)\right)^r} \tag{7-5}$$

uniformly in $H \in V$.

Note that part (i) of Theorem 6.2 is a direct consequence of this lemma.

Proof. By the Weyl differencing trick, write

$$\begin{aligned} |F|^2 &= \sum_{\lambda_1, \lambda_2 \in L} e^{-it(|\lambda_1|^2 - |\lambda_2|^2) + i\langle\lambda_1 - \lambda_2, H\rangle} \varphi\left(\frac{|\lambda_1|^2}{N^2}\right) \varphi\left(\frac{|\lambda_2|^2}{N^2}\right) f(\lambda_1)\overline{f(\lambda_2)} \\ &= \sum_{\mu = \lambda_1 - \lambda_2} e^{-it|\mu|^2 + i\langle\mu, H\rangle} \sum_{\lambda = \lambda_2} e^{-i2t\langle\mu,\lambda\rangle} \varphi\left(\frac{|\mu+\lambda|^2}{N^2}\right) \varphi\left(\frac{|\lambda|^2}{N^2}\right) f(\mu+\lambda)\overline{f(\lambda)} \\ &\leq \sum_{|\mu| \lesssim N} \left| \sum_{\lambda} e^{-i2t\langle\mu,\lambda\rangle} \varphi\left(\frac{|\mu+\lambda|^2}{N^2}\right) \varphi\left(\frac{|\lambda|^2}{N^2}\right) f(\mu+\lambda)\overline{f(\lambda)} \right|. \end{aligned}$$

Now let $L = \mathbb{Z}w_1 + \cdots + \mathbb{Z}w_r$. Write

$$\lambda = \sum_{i=1}^r n_i w_i$$

and

$$g(\lambda) = \varphi\left(\frac{|\mu+\lambda|^2}{N^2}\right) \varphi\left(\frac{|\lambda|^2}{N^2}\right) f(\mu+\lambda)\overline{f(\lambda)}.$$

Note that as functions in $\lambda \in L$, both $\varphi(|\mu+\lambda|^2/N^2)|$ and $\varphi(|\lambda|^2/N^2)$ are pseudopolynomials of degree 0, and both $f(\mu+\lambda)$ and $f(\lambda)$ are pseudopolynomials of degree A, which implies by Lemma 7.3 that $g(\lambda)$ is a pseudopolynomial of degree $2A$. That is, $g(\lambda)$ satisfies

$$|D_{i_1} \cdots D_{i_n} g(\lambda)| \lesssim N^{2A-n} \tag{7-6}$$

uniformly for $|\lambda| \lesssim N$ and $N \geq 1$, for all $i_1, \dots, i_n \in \{1, \dots, r\}$. Write

$$\sum_{\lambda \in L} e^{-i2t\langle\mu,\lambda\rangle} g(\lambda) = \sum_{n_1,\dots,n_r \in \mathbb{Z}} \biggl(\prod_{i=1}^{r} e^{-itn_i\langle\mu,2w_i\rangle} \biggr) g(\lambda). \tag{7-7}$$

By summation by parts twice, we have

$$\sum_{n_1 \in \mathbb{Z}} e^{-itn_1\langle\mu,2w_1\rangle} g = \biggl(\frac{e^{-it\langle\mu,2w_1\rangle}}{1-e^{-it\langle\mu,2w_1\rangle}} \biggr)^2 \sum_{n_1 \in \mathbb{Z}} e^{-itn_1\langle\mu,2w_1\rangle} D_1^2 g(n_1,\dots,n_r); \tag{7-8}$$

then (7-7) becomes

$$\sum_{\lambda \in L} e^{-i2t\langle\mu,\lambda\rangle} g = \biggl(\frac{e^{-it\langle\mu,2w_1\rangle}}{1-e^{-it\langle\mu,2w_1\rangle}} \biggr)^2 \sum_{n_1,\dots,n_r \in \mathbb{Z}} \biggl(\prod_{i=1}^{r} e^{-itn_i\langle\mu,2w_i\rangle} \biggr) D_1^2 g(n_1,\dots,n_r).$$

Then we can carry out the procedure of summation by parts twice with respect to other variables $n_2, \dots, n_r$. But we require that only when

$$|1-e^{-it\langle\mu,2w_i\rangle}| \geq \tfrac{1}{N}$$

do we carry out the procedure to the variable n_i. Using (7-6), we obtain

$$\begin{aligned} \biggl| \sum_{\lambda} e^{-i2t\langle\mu,\lambda\rangle} \varphi\Bigl(\frac{|\mu+\lambda|^2}{N^2}\Bigr) \varphi\Bigl(\frac{|\lambda|^2}{N^2}\Bigr) f(\mu+\lambda)\overline{f(\lambda)} \biggr| &\lesssim N^{2A-r} \prod_{i=1}^{r} \frac{1}{\bigl(\max\{1-e^{-it\langle\mu,2w_i\rangle}, \frac{1}{N}\}\bigr)^2} \\ &\lesssim N^{2A-r} \prod_{i=1}^{r} \frac{1}{\bigl(\max\{\|\frac{1}{2\pi}t\langle\mu,2w_i\rangle\|, \frac{1}{N}\}\bigr)^2}. \end{aligned}$$

Writing $\mu = \sum_{j=1}^{r} m_j w_j$, $m_j \in \mathbb{Z}$, we have

$$|F|^2 \lesssim N^{2A-r} \sum_{\substack{|m_j| \lesssim N \\ j=1,\dots,r}} \prod_{i=1}^{r} \frac{1}{\bigl(\max\{\|\frac{1}{2\pi}t\sum_{j=1}^{r} m_j\langle w_j,2w_i\rangle\|, \frac{1}{N}\}\bigr)^2}.$$

Let

$$n_i = \sum_{j=1}^{r} m_j\langle w_j,2w_i\rangle \cdot D, \quad i = 1,\dots,r, \tag{7-9}$$

where $D > 0$ is the period of L so that $\langle w_j, w_i\rangle \in D^{-1}\mathbb{Z}$. Then $n_i \in \mathbb{Z}$. Note that the matrix $(\langle w_j, 2w_i\rangle D)_{i,j}$ is nondegenerate, which implies that for each vector $(n_1, \dots, n_r) \in \mathbb{Z}^r$ there exists at most one vector $(m_1, \dots, m_r) \in \mathbb{Z}^r$ so that (7-9) holds; thus

$$\begin{aligned} |F|^2 &\lesssim N^{2A-r} \sum_{\substack{|n_i| \lesssim N \\ i=1,\dots,r}} \prod_{i=1}^{r} \frac{1}{\bigl(\max\{\|\frac{t}{2\pi D}n_i\|, \frac{1}{N}\}\bigr)^2} \\ &\lesssim N^{2A-r} \prod_{i=1}^{r} \biggl(\sum_{|n_i| \lesssim N} \frac{1}{\bigl(\max\{\|\frac{t}{2\pi D}n_i\|, \frac{1}{N}\}\bigr)^2} \biggr). \end{aligned}$$

Then by a standard estimate as in the proof of the classical Weyl inequality in one dimension, we have

$$\sum_{|n_i|\lesssim N}\frac{1}{\left(\max\{\|\frac{t}{2\pi D}n_i\|,\frac{1}{N}\}\right)^2}\lesssim\frac{N^3}{\left(\sqrt{q}\left(1+N\|\frac{t}{2\pi D}-\frac{a}{q}\|^{1/2}\right)\right)^2},$$

which implies the desired result

$$|F|^2\lesssim\frac{N^{2A+2r}}{\left(\sqrt{q}\left(1+N\|\frac{t}{2\pi D}-\frac{a}{q}\|^{1/2}\right)\right)^{2r}}.\qquad\square$$

Remark 7.5. Let λ_0 be a constant vector in $\mathbb{R}^r$ and C a constant real number. Then we can slightly generalize the form of the function $F(t,H)$ in the above lemma into

$$F(t,H)=\sum_{\lambda\in L}e^{-it|\lambda+\lambda_0|^2+i\langle\lambda,H\rangle}\varphi\left(\frac{|\lambda+\lambda_0|^2+C}{N^2}\right)\cdot f$$

such that the conclusion of the lemma still holds.

7B. ***From a chamber to the whole weight lattice.*** To prove part (ii) of Theorem 6.2, we first rewrite the Schrödinger kernel as an exponential sum over the whole weight lattice Λ instead of just a chamber of it, in order to apply Lemma 7.4.

Lemma 7.6. *Recall that $D_P(H)=\sum_{s\in W}(\det s)e^{i\langle\rho,H\rangle}$ is the Weyl denominator. We have*

$$K_N(t,x)=\frac{e^{it|\rho|^2}}{\left(\prod_{\alpha\in P}\langle\alpha,\rho\rangle\right)D_P(H)}\sum_{\lambda\in\Lambda}e^{-it|\lambda|^2+i\langle\lambda,H\rangle}\varphi\left(\frac{|\lambda|^2-|\rho|^2}{N^2}\right)\prod_{\alpha\in P}\langle\alpha,\lambda\rangle \tag{7-10}$$

$$=\frac{e^{it|\rho|^2}}{\left(\prod_{\alpha\in P}\langle\alpha,\rho\rangle\right)|W|}\sum_{\lambda\in\Lambda}e^{-it|\lambda|^2}\varphi\left(\frac{|\lambda|^2-|\rho|^2}{N^2}\right)\prod_{\alpha\in P}\langle\alpha,\lambda\rangle\frac{\sum_{s\in W}(\det s)e^{i\langle s(\lambda),H\rangle}}{\sum_{s\in W}(\det s)e^{i\langle s(\rho),H\rangle}}. \tag{7-11}$$

Proof. To prove (7-11), first note that from Proposition 7.13 below, $\prod_{\alpha\in P}\langle\alpha,\cdot\rangle$ is an *anti-invariant polynomial*; that is,

$$\prod_{\alpha\in P}\langle\alpha,s(\lambda)\rangle=(\det s)\prod_{\alpha\in P}\langle\alpha,\lambda\rangle \tag{7-12}$$

for all $\lambda\in i\mathfrak{b}^*$. Recall that the Weyl group W acts on $i\mathfrak{b}^*$ isometrically; that is,

$$|s(\lambda)|=|\lambda|\quad\text{for all } s\in W,\ \lambda\in i\mathfrak{b}^*. \tag{7-13}$$

Also recalling the definition (4-12) of ρ and the definition (4-10) of the fundamental chamber C, we may rewrite K_N as in (5-2) into

$$K_N(t,x)=\frac{e^{it|\rho|^2}}{\left(\prod_{\alpha\in P}\langle\alpha,\rho\rangle\right)D_P}\sum_{\lambda\in\Lambda\cap C}e^{-it|\lambda|^2}\varphi\left(\frac{|\lambda|^2-|\rho|^2}{N^2}\right)\prod_{\alpha\in P}\langle\alpha,\lambda\rangle\sum_{s\in W}(\det s)e^{i\langle s(\lambda),H\rangle}.$$

Using the (7-12) and (7-13), we write

$$\begin{aligned}K_N(t,x)&=\frac{e^{it|\rho|^2}}{\left(\prod_{\alpha\in P}\langle\alpha,\rho\rangle\right)D_P}\sum_{s\in W}\sum_{\lambda\in\Lambda\cap C}e^{-it|\lambda|^2}\varphi\left(\frac{|\lambda|^2-|\rho|^2}{N^2}\right)\prod_{\alpha\in P}\langle\alpha,s(\lambda)\rangle e^{i\langle s(\lambda),H\rangle}\\&=\frac{e^{it|\rho|^2}}{\left(\prod_{\alpha\in P}\langle\alpha,\rho\rangle\right)D_P}\sum_{s\in W}\sum_{\lambda\in\Lambda\cap C}e^{-it|s(\lambda)|^2}\varphi\left(\frac{|s(\lambda)|^2-|\rho|^2}{N^2}\right)\prod_{\alpha\in P}\langle\alpha,s(\lambda)\rangle e^{i\langle s(\lambda),H\rangle}\\&=\frac{e^{it|\rho|^2}}{\left(\prod_{\alpha\in P}\langle\alpha,\rho\rangle\right)D_P}\sum_{\lambda\in\bigsqcup_{s\in W}s(\Lambda\cap C)}e^{-it|\lambda|^2}\varphi\left(\frac{|\lambda|^2-|\rho|^2}{N^2}\right)\prod_{\alpha\in P}\langle\alpha,\lambda\rangle e^{i\langle\lambda,H\rangle},\end{aligned}\tag{7-14}$$

which then implies by (4-11) that

$$K_N(t,x)=\frac{e^{it|\rho|^2}}{\left(\prod_{\alpha\in P}\langle\alpha,\rho\rangle\right)D_P}\sum_{\lambda\in\Lambda}e^{-it|\lambda|^2}\varphi\left(\frac{|\lambda|^2-|\rho|^2}{N^2}\right)\prod_{\alpha\in P}\langle\alpha,\lambda\rangle e^{i\langle\lambda,H\rangle}.$$

This proves (7-10). To prove (7-11), write

$$\begin{aligned}\sum_{\lambda\in\Lambda}e^{-it|\lambda|^2+i\langle\lambda,H\rangle}\varphi\left(\frac{|\lambda|^2-|\rho|^2}{N^2}\right)\prod_{\alpha\in P}\langle\alpha,\lambda\rangle&\\=\sum_{\lambda\in\Lambda}e^{-it|s(\lambda)|^2+i\langle s(\lambda),H\rangle}\varphi\left(\frac{|s(\lambda)|^2-|\rho|^2}{N^2}\right)\prod_{\alpha\in P}\langle\alpha,s(\lambda)\rangle,&\end{aligned}\tag{7-15}$$

which implies using (7-12) and (7-13) that

$$\sum_{\lambda\in\Lambda}e^{-it|\lambda|^2+i\langle\lambda,H\rangle}\varphi\left(\frac{|\lambda|^2-|\rho|^2}{N^2}\right)\prod_{\alpha\in P}\langle\alpha,\lambda\rangle=(\det s)\sum_{\lambda\in\Lambda}e^{-it|\lambda|^2+i\langle s(\lambda),H\rangle}\varphi\left(\frac{|\lambda|^2-|\rho|^2}{N^2}\right)\prod_{\alpha\in P}\langle\alpha,\lambda\rangle,$$

which further implies

$$\begin{aligned}\sum_{\lambda\in\Lambda}e^{-it|\lambda|^2+i\langle\lambda,H\rangle}\varphi\left(\frac{|\lambda|^2-|\rho|^2}{N^2}\right)\prod_{\alpha\in P}\langle\alpha,\lambda\rangle&\\=\frac{1}{|W|}\sum_{\lambda\in\Lambda}e^{-it|\lambda|^2}\varphi\left(\frac{|\lambda|^2-|\rho|^2}{N^2}\right)\prod_{\alpha\in P}\langle\alpha,\lambda\rangle\sum_{s\in W}(\det s)e^{i\langle s(\lambda),H\rangle}.&\end{aligned}$$

This combined with (7-10) yields (7-11). ☐

Example 7.7. Specializing (7-10) and (7-11) to the Schrödinger kernel (5-3) for $G=\mathrm{SU}(2)$, we get

$$K_N(t,\theta)=\frac{e^{it}}{e^{i\theta}-e^{-i\theta}}\sum_{m\in\mathbb{Z}}e^{-itm^2+im\theta}\varphi\left(\frac{m^2-1}{N^2}\right)m\tag{7-16}$$

$$=\frac{e^{it}}{2}\sum_{m\in\mathbb{Z}}e^{-itm^2}\varphi\left(\frac{m^2-1}{N^2}\right)m\cdot\frac{e^{im\theta}-e^{-im\theta}}{e^{i\theta}-e^{-i\theta}},\qquad\theta\in\mathbb{R}/2\pi\mathbb{Z}.\tag{7-17}$$

Corollary 7.8. (6-1) *holds for the following two scenarios*:

Scenario 1: $x = 1_G$, *where* 1_G *is the identity element of* G.

Scenario 2: $\left\| \frac{1}{2\pi}\langle \alpha, H\rangle \right\| \gtrsim \frac{1}{N}$ *for any* x *conjugate to* $\exp H$. *This is to say that the variable* H *is away from all the cell walls* $\left\{H \,\middle|\, \left\|\frac{1}{2\pi}\langle\alpha, H\rangle\right\| = 0 \text{ for some } \alpha \in P\right\}$ *by a distance of* $\gtrsim \frac{1}{N}$.

Proof. Scenario 1: When $x = 1_G$, the character equals $\chi_\lambda(1_G) = d_\lambda = \prod_{\alpha\in P}\langle\alpha,\lambda\rangle / \prod_{\alpha\in P}\langle\alpha,\rho\rangle$. Then by (7-11), the Schrödinger kernel at $x = 1_G$ equals

$$K_N(t, 1_G) = \frac{e^{it|\rho|^2}}{\left(\prod_{\alpha\in P}\langle\alpha,\rho\rangle\right)^2 |W|} \sum_{\lambda\in\Lambda} e^{-it|\lambda|^2} \varphi\left(\frac{|\lambda|^2 - |\rho|^2}{N^2}\right)\left(\prod_{\alpha\in P}\langle\alpha,\lambda\rangle\right)^2. \tag{7-18}$$

Note that $f(\lambda) = \left(\prod_{\alpha\in P}\langle\alpha,\lambda\rangle\right)^2$ is a polynomial in the variable $\lambda = n_1 w_1 + \cdots + n_r w_r \in \Lambda$ of degree $2|P|$, which equals $d - r$ by (4-8). Thus f is also a pseudopolynomial of degree $d - r$. Then the desired estimate is a direct consequence of Lemma 7.4.

Scenario 2: By Lemma 4.13.4 of Chapter 4 in [Varadarajan 1974], the Weyl denominator $D_P = \sum_{s\in W}(\det s) e^{i\langle s(\rho), H\rangle}$ can be rewritten as

$$D_P = e^{-i\langle\rho, H\rangle} \prod_{\alpha\in P}\left(e^{i\langle\alpha, H\rangle} - 1\right). \tag{7-19}$$

Note that

$$1 \lesssim \frac{|e^{i\langle\alpha,H\rangle} - 1|}{\left\|\frac{1}{2\pi}\langle\alpha, H\rangle\right\|} \lesssim 1.$$

Then by assumption the Weyl denominator satisfies

$$|D_P(H)| \gtrsim \prod_{\alpha\in P}\left\|\tfrac{1}{2\pi}\langle\alpha, H\rangle\right\| \gtrsim N^{-|P|}. \tag{7-20}$$

Let

$$F = \sum_{\lambda\in\Lambda} e^{-it|\lambda|^2 + i\langle\lambda, H\rangle}\varphi\left(\frac{|\lambda|^2 - |\rho|^2}{N^2}\right)\cdot f,$$

where $f = \prod_{\alpha\in P}\langle\alpha,\lambda\rangle$. Note that f is a polynomial and thus also a pseudopolynomial of degree $|P|$ in λ. Applying Lemma 7.4 to F we get

$$|K_N(t,x)| = \left|\frac{e^{it|\rho|^2}}{\left(\prod_{\alpha\in P}\langle\alpha,\rho\rangle\right) D_P(H)}\right| \cdot |F| \lesssim \left|\frac{1}{D_P(H)}\right|\cdot|F| \lesssim N^{|P|}\cdot \frac{N^{r+|P|}}{\left(\sqrt{q}\left(1 + N\left\|\frac{t}{2\pi D} - \frac{a}{q}\right\|^{1/2}\right)\right)^r}.$$

Recalling $|P| = \frac{d-r}{2}$, we establish (6-1) for Scenario 2. ☐

Example 7.9. We specialize the Schrödinger kernel (7-16) and (7-17) to the case of $G = \mathrm{SU}(2)$. Scenario 1 in the above corollary corresponds to when $\theta \in 2\pi\mathbb{Z}$ and

$$K_N(t,\theta) = \frac{e^{it}}{2}\sum_{m\in\mathbb{Z}} e^{-itm^2}\varphi\left(\frac{m^2-1}{N^2}\right)m^2, \quad |K_N(t,\theta)| \lesssim \left|\sum_{m\in\mathbb{Z}} e^{-itm^2}\varphi\left(\frac{m^2-1}{N^2}\right)m^2\right|. \tag{7-21}$$

Scenario 2 corresponds to when $|e^{i\theta}-e^{-i\theta}|\gtrsim \frac{1}{N}$, equivalently, when θ is away from the cell walls $\{0,\pi\}$ by a distance $\gtrsim \frac{1}{N}$. In this case,

$$|K_N(t,\theta)|\lesssim \left|\frac{1}{e^{i\theta}-e^{-i\theta}}\right|\cdot\left|\sum_{m\in\mathbb{Z}} e^{-itm^2+im\theta}\varphi\left(\frac{m^2-1}{N^2}\right)m\right|. \tag{7-22}$$

Then we get the desired estimates for (7-21) and (7-22) using Lemma 7.4.

7C. *Pseudopolynomial behavior of characters.* We have established the key estimates (6-1) for when the variable $\exp H$ in the maximal torus is either the identity or away from all the cell walls by a distance of $\gtrsim \frac{1}{N}$. To establish (6-1) fully, we need to look at the scenarios when the variable $\exp H$ is close to the some of the cell walls within a distance of $\lesssim \frac{1}{N}$. In this section, we first deal with the scenario when the variable $\exp H$ is close to all the cell walls within a distance of $\lesssim \frac{1}{N}$. To achieve this end, we first prove the following crucial lemma on the pseudopolynomial behavior of characters.

Lemma 7.10. *Let $\mu\in i\mathfrak{b}^*$. For $\lambda\in i\mathfrak{b}^*$, define*

$$\chi^\mu(\lambda,H)=\frac{\sum_{s\in W}(\det s)e^{i\langle s(\lambda+\mu),H\rangle}}{\sum_{s\in W}(\det s)e^{i\langle s(\rho),H\rangle}}.$$

Let $L\cong\mathbb{Z}^r$ be the weight lattice or the root lattice (or any sublattice of full rank of the weight lattice), and viewing $\chi^\mu(\lambda,H)$ as a function in $\lambda\in L$, we have

$$|D_{i_1}\cdots D_{i_k}\chi^\mu(\lambda,H)|\lesssim N^{\frac{d-r}{2}-k} \tag{7-23}$$

holds uniformly in $|\lambda|\lesssim N$, $|H|\lesssim\frac{1}{N}$, and $N\geq 1$, for all $k\in\mathbb{Z}_{\geq 0}$. In other words, $\chi^\mu(\lambda,H)$ is a pseudopolynomial of degree $\frac{d-r}{2}$ in λ uniformly in $|H|\lesssim\frac{1}{N}$.

Using this lemma, applying Lemma 7.4 to the Schrödinger kernel K_N in the form of (7-11), we immediately get the following corollary.

Corollary 7.11. *Inequality (6-1) holds uniformly when $x\in G$ is conjugate to $\exp H$ such that $|H|\lesssim\frac{1}{N}$. In other words, when x is within $\lesssim\frac{1}{N}$ a distance from the identity 1_G.*

We now prove Lemma 7.10 for $L\cong\mathbb{Z}\varpi_1+\cdots+\mathbb{Z}\varpi_r$ being the weight lattice (the case for the root lattice or any other sublattice can be proved similarly). First note that as $|H|\lesssim\frac{1}{N}$ for N large enough, by (7-19), we have

$$\left|\frac{\prod_{\alpha\in P}\langle\alpha,H\rangle}{D_P}\right|\approx 1.$$

Thus it suffices to show (7-23) replacing $\chi^\mu(\lambda,H)$ by

$$\chi_1^\mu(\lambda,H)=\frac{\sum_{s\in W}(\det s)e^{i\langle s(\lambda+\mu),H\rangle}}{\prod_{\alpha\in P}\langle\alpha,H\rangle}. \tag{7-24}$$

7C.1. *Approach 1: via BGG-Demazure operators.* The idea is to expand the numerator of $\chi_1^{\mu}(\lambda, H)$ into a power series of polynomials in $H \in i\mathfrak{b}^*$ which are *anti-invariant* with respect to the Weyl group W, and then to estimate the quotients of these polynomial over the denominator $\prod_{\alpha \in P} \langle \alpha, H \rangle$. We will see that these quotients are in fact polynomials in $H \in i\mathfrak{b}^*$, and can be more or less explicitly computed by the *BGG-Demazure operators*. We now review the basic definitions and facts of the BGG-Demazure operators and the related invariant theory. A good reference is Chapter IV in [Hiller 1982].

From now on, we fix an inner product space $(\mathfrak{a}, \langle \cdot, \cdot \rangle)$ and let Φ be an integral root system in the dual space $(\mathfrak{a}^*, \langle \cdot, \cdot \rangle)$. Let $P(\mathfrak{a})$ be the space of polynomial functions on $\mathfrak{a}$. The orthogonal group $O(\mathfrak{a})$ with respect to the inner product on $\mathfrak{a}$, in particular the Weyl group, acts on $P(\mathfrak{a})$ by

$$(sf)(H) := f(s^{-1}H), \quad s \in O(\mathfrak{a}), \quad f \in P(\mathfrak{a}), \quad H \in \mathfrak{a}.$$

Definition 7.12. For $\alpha \in \mathfrak{a}^*$, let $s_\alpha : \mathfrak{a} \to \mathfrak{a}$ denote the reflection about the hyperplane

$$\{H \in \mathfrak{a} \mid \alpha(H) = 0\},$$

that is,

$$s_\alpha(H) := H - 2\frac{\alpha(H)}{\langle \alpha, \alpha \rangle} H_\alpha,$$

where $H \in \mathfrak{a}$. Here H_α corresponds to α through the identification $\mathfrak{a} \xrightarrow{\sim} \mathfrak{a}^*$. Define the *BGG-Demazure operator* $\Delta_\alpha : P(\mathfrak{a}) \to P(\mathfrak{a})$ associated to $\alpha \in \mathfrak{a}^*$ by

$$\Delta_\alpha(f) = \frac{f - s_\alpha(f)}{\alpha}.$$

As an example, we compute $\Delta_\alpha(\lambda^m)$ for $\lambda \in \mathfrak{a}^*$:

$$\begin{aligned} \Delta_\alpha(\lambda^m) &= \frac{\lambda^m - \lambda\big(\cdot - 2\frac{\alpha}{\langle \alpha,\alpha \rangle} H_\alpha\big)^m}{\alpha} = \frac{\lambda^m - \big(\lambda - 2\frac{\langle \lambda,\alpha \rangle}{\langle \alpha,\alpha \rangle}\alpha\big)^m}{\alpha} \\ &= \sum_{i=1}^{m} (-1)^{i-1} \binom{m}{i} \frac{2^i}{\langle \alpha, \alpha \rangle^i} \langle \lambda, \alpha \rangle^i \alpha^{i-1} \lambda^{m-i}. \end{aligned} \tag{7-25}$$

This computation in particular implies that for any $f \in P(\mathfrak{a})$, the operator $\Delta_\alpha(f)$ lowers the degree of f by at least 1.

Let $P(\mathfrak{a})^W$ denote the subspace of $P(\mathfrak{a})$ that is invariant under the action of the Weyl group W, that is,

$$P(\mathfrak{a})^W := \{f \in P(\mathfrak{a}) \mid sf = f \text{ for all } s \in W\}.$$

We call $P(\mathfrak{a})^W$ the space of *invariant polynomials*. We also define

$$P(\mathfrak{a})^W_{\det} := \{f \in P(\mathfrak{a}) \mid sf = (\det s) f \text{ for all } s \in W\}.$$

We call $P(\mathfrak{a})^W_{\det}$ the space of *anti-invariant polynomials*. We have the following proposition which states that $P(\mathfrak{a})^W_{\det}$ is a free $P(\mathfrak{a})^W$-module of rank 1.

Proposition 7.13 [Hiller 1982, Chapter II, Proposition 4.4]. *Define $d_{\text{det}} \in P(\mathfrak{a})$ by*

$$d_{\text{det}} = \prod_{\alpha \in P} \alpha.$$

Then $d_{\text{det}} \in P(\mathfrak{a})^W_{\text{det}}$ and

$$P(\mathfrak{a})^W_{\text{det}} = d_{\text{det}} \cdot P(\mathfrak{a})^W.$$

By the above proposition, given any anti-invariant polynomial f, we have $f = d \cdot g$, where g is invariant. We call g the *invariant part* of f. The BGG-Demazure operators provide a procedure that computes the invariant part of any anti-invariant polynomial. We describe this procedure as follows. The Weyl group W is generated by the reflections $s_{\alpha_1}, \ldots, s_{\alpha_r}$, where $S = \{\alpha_1, \ldots, \alpha_r\}$ is the set of simple roots. Define the *length* of $s \in W$ to be the smallest number k such that s can be written as $s = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_k}}$. The longest element s in W is of length $|P| = \frac{d-r}{2}$, and such s is unique; see Section 1.8 in [Humphreys 1990]. Write $s = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_L}}$. Set

$$\delta = \Delta_{\alpha_{i_1}} \cdots \Delta_{\alpha_{i_L}}$$

and note that it is well-defined in the sense it does not depend on the particular choice of the decomposition $s = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_L}}$; see Chapter IV, Proposition 1.7 in [Hiller 1982].

Proposition 7.14 [Hiller 1982, Chapter IV, Proposition 1.6]. *We have*

$$\delta f = \frac{|W|}{d_{\text{det}}} \cdot f$$

for all $f \in P(\mathfrak{a})^W_{\text{det}}$.

That is, the operator δ produces the invariant part of any anti-invariant polynomial (modulo a multiplicative constant). As an example, we compute $\delta = \Delta_{\alpha_{i_1}} \cdots \Delta_{\alpha_{i_L}}$ on λ^m. Proceed inductively using (7-25), we arrive at the following proposition.

Proposition 7.15. *Let $m \geq L$. Then*

$$\delta(\lambda^m) = \sum_{\theta, a(\alpha,\beta), b(\gamma), c(\zeta), \eta \in \mathbb{Z}} (-1)^\theta \prod_{\alpha \leq \beta} \langle \alpha_{i_\alpha}, \alpha_{i_\beta} \rangle^{a(\alpha,\beta)} \prod_\gamma \langle \lambda, \alpha_{i_\gamma} \rangle^{b(\gamma)} \prod_\zeta \alpha_{i_\zeta}^{c(\zeta)} \lambda^\eta$$

such that the following statements are true:

1. *In each term of the sum, $\sum_\gamma b(\gamma) + \eta = m$.*
2. *In each term of the sum, $\sum_\zeta c(\zeta) + \eta = m - L$.*
3. *In each term of the sum, $\sum_\gamma b(\gamma) - \sum_\zeta c(\zeta) = L$.*
4. *In each term of the sum, $|a(\alpha, \beta)| \leq mL$ and $b(\gamma), c(\zeta), \eta = 0, 1, \ldots, m$.*
5. *There are in total less than 3^{mL} terms in the sum.*

Note that since each BGG-Demazure operator $\Delta_{\alpha_{i_j}}$ in $\delta = \Delta_{\alpha_{i_1}} \cdots \Delta_{\alpha_{i_L}}$ lowers the degree of polynomials by at least 1, δ lowers the degree by at least L. Thus

$$\delta(\lambda^m) = 0 \quad \text{for } m < L. \tag{7-26}$$

Example 7.16. We specialize the discussion to the case $M = \mathrm{SU}(2)$. Recall that $\mathfrak{a}^* = \mathbb{R}w$, where w is the fundamental weight, and $\Phi = \{\pm\alpha\}$ with $\alpha = 2w$. $P(\mathfrak{a})$ consists of polynomials in the variable $\lambda \in \underset{1 \mapsto w}{\mathbb{R} \cong \mathbb{R}w}$. For $\lambda \in \underset{1 \mapsto w}{\mathbb{R} \cong \mathbb{R}w}$, and $f \in P(\mathfrak{a})$, we have

$$\begin{aligned} (\delta f)(\lambda) &= \frac{f(\lambda) - f(-\lambda)}{2\lambda}, \\ \delta(\lambda^m) &= \begin{cases} \lambda^{m-1}, & m \text{ odd}, \\ 0, & m \text{ even}, \end{cases} \\ d_{\det}(\lambda) &= 2\lambda. \end{aligned} \tag{7-27}$$

We can now finish the proof of (7-23).

Proof of Lemma 7.10. Recall that it suffices to prove (7-23) replacing $\chi^{\mu}(\lambda, H)$ by $\chi_1^{\mu}(\lambda, H)$ in (7-24). Using power series expansions, write

$$\begin{aligned} \sum_{s \in W} (\det s) e^{i\langle \lambda+\mu, H\rangle} &= \sum_{s \in W} (\det s) \sum_{m=0}^{\infty} \frac{1}{m!} (i\langle s(\lambda+\mu), H\rangle)^m \\ &= \sum_{m=0}^{\infty} \frac{i^m}{m!} \sum_{s \in W} (\det s) \langle s(\lambda+\mu), H\rangle^m. \end{aligned} \tag{7-28}$$

Note that

$$f_m(H) = f_m(\lambda) = f_m(\lambda, H) := \sum_{s \in W} (\det s) \langle s(\lambda+\mu), H\rangle^m \tag{7-29}$$

is an anti-invariant polynomial in H with respect to the Weyl group W; thus by Proposition 7.14,

$$f_m(H) = \frac{d_{\det}(H)}{|W|} \cdot \delta f_m(H) = \frac{\prod_{\alpha \in P} \langle \alpha, H\rangle}{|W|} \cdot \delta f_m(H).$$

This implies that we can rewrite (7-24) as

$$\chi^{\mu_1}(\lambda, H) = \frac{1}{|W|} \sum_{m=0}^{\infty} \frac{i^m}{m!} \delta f_m(H).$$

Thus to prove (7-23), it suffices to prove that

$$\sum_{m=0}^{\infty} \frac{1}{m!} |D_{i_1} \cdots D_{i_k} (\delta f_m(\lambda))| \lesssim N^{L-k}$$

for all $k \in \mathbb{Z}_{\geq 0}$, uniformly in $|n_i| \lesssim N$, where $\lambda = n_1 w_1 + \cdots + n_r w_r$. Then by (7-29), it suffices to prove that

$$\sum_{m=0}^{\infty} \frac{1}{m!} |D_{i_1} \cdots D_{i_k} (\delta[(s(\lambda+\mu))^m])| \lesssim N^{L-k} \quad \text{for all } s \in W.$$

Without loss of generality, it suffices to show

$$\sum_{m=0}^{\infty} \frac{1}{m!} |D_{i_1} \cdots D_{i_k} (\delta[(\lambda+\mu)^m])| \lesssim N^{L-k}. \tag{7-30}$$

Noting (7-26), it suffices to consider cases when $m \geq L$. We apply Proposition 7.15 to write

$$\delta((\lambda+\mu)^m)(H) = \sum_{\theta, a(\alpha,\beta), b(\gamma), c(\zeta), \eta} (-1)^\theta \prod_{\alpha \leq \beta} \langle \alpha_{i_\alpha}, \alpha_{i_\beta} \rangle^{a(\alpha,\beta)} \prod_\gamma \langle \lambda+\mu, \alpha_{i_\gamma} \rangle^{b(\gamma)} \prod_\zeta \langle \alpha_{i_\zeta}, H \rangle^{c(\zeta)} \langle \lambda+\mu, H \rangle^\eta. \tag{7-31}$$

First note that for $\lambda = n_1 w_1 + \cdots + n_r w_r$, $|n_i| \lesssim N$, $i = 1, \ldots, r$, we have

$$1 \lesssim |\langle \alpha_i, \alpha_j \rangle| \lesssim 1, \quad |\langle \lambda+\mu, \alpha_i \rangle| \lesssim N, \tag{7-32}$$

and by the assumption $|H| \lesssim \frac{1}{N}$,

$$|\langle \alpha_i, H \rangle| \lesssim \tfrac{1}{N}, \quad |\langle \lambda+\mu, H \rangle| = \left| \left(\sum_{i=1}^r n_i \langle w_i, H \rangle \right) + \langle \mu, H \rangle \right| \lesssim 1. \tag{7-33}$$

These imply

$$|\delta((\lambda+\mu)^m)(H)| \leq \sum_{\theta, a(\alpha,\beta), b(\gamma), c(\zeta), \eta} C^{\sum_{\alpha,\beta} |a(\alpha,\beta)| + \sum_\gamma b(\gamma) + \sum_\zeta c(\zeta) + \eta} N^{\sum_\gamma c(\gamma) - \sum_\zeta c(\zeta)} \tag{7-34}$$

for some constant C independent of m. Now we derive a similar estimate for $D_i(\delta[(\lambda+\mu)^m])(H)$. By (7-31),

$$D_i(\delta[(\lambda+\mu)^m])(H) = \sum_{\theta, a(\alpha,\beta), b(\gamma), c(\zeta), \eta} (-1)^\theta \prod_{\alpha \leq \beta} \langle \alpha_{i_\alpha}, \alpha_{i_\beta} \rangle^{a(\alpha,\beta)} \prod_\zeta \langle \alpha_{i_\zeta}, H \rangle^{c(\zeta)} \cdot D_i \left(\prod_\gamma \langle \lambda+\mu, \alpha_{i_\gamma} \rangle^{b(\gamma)} \langle \lambda+\mu, H \rangle^\eta \right). \tag{7-35}$$

For $\lambda = n_1 w_1 + \cdots + n_r w_r$, we compute

$$D_i(\langle \lambda+\mu, \alpha_{i_\gamma} \rangle) = \langle \alpha_i, \alpha_{i_\gamma} \rangle,$$
$$D_i(\langle \lambda+\mu, H \rangle) = \langle \alpha_i, H \rangle.$$

The above two formulas combined with (7-32), (7-33), and the Leibniz rule (7-2) for D_i imply

$$\left| D_i \left(\prod_\gamma \langle \lambda+\mu, \alpha_{i_\gamma} \rangle^{b(\gamma)} \langle \lambda+\mu, H \rangle^\eta \right) \right| \leq C^{\sum_\gamma b(\gamma) + \eta} N^{\sum_\gamma b(\gamma) - 1}.$$

This combined with (7-32), (7-33) and (7-35) implies

$$|D_i(\delta[(\lambda+\mu)^m])(H)| \lesssim \sum_{\theta, a(\alpha,\beta), b(\gamma), c(\zeta), \eta} C^{\sum_{\alpha,\beta} |a(\alpha,\beta)| + \sum_\gamma b(\gamma) + \sum_\zeta c(\zeta) + \eta} N^{\sum_\gamma b(\gamma) - \sum_\zeta c(\zeta) - 1}.$$

Inductively, we have

$$|D_{i_1} \cdots D_{i_k}(\delta[(\lambda+\mu)^m])(H)| \lesssim \sum_{\theta, a(\alpha,\beta), b(\gamma), c(\zeta), \eta} C^{\sum_{\alpha,\beta} |a(\alpha,\beta)| + \sum_\gamma b(\gamma) + \sum_\zeta c(\zeta) + \eta} N^{\sum_\gamma b(\gamma) - \sum_\zeta c(\zeta) - k}$$

for some constant C independent of m. This by Proposition 7.15 then implies

$$|D_{i_1} \cdots D_{i_k}(\delta[(\lambda+\mu)^m])(H)| \leq 3^{mL} C^{CmL} N^{L-k} \leq C^m N^{L-k}$$

for some positive constant C independent of m. This estimate implies (7-30), noting that

$$\sum_{m=0}^{\infty} \frac{C^m}{m!} \lesssim 1. \tag{7-36}$$

This finishes the proof. □

7C.2. *Approach 2: via Harish-Chandra's integral formula.* This very short approach expresses $\chi_1^{\mu}(\lambda, H)$ as an integral over the group G. We apply the Harish-Chandra's integral formula [1957], which reads

$$\sum_{s\in W} (\det s) e^{\langle s\lambda, \mu\rangle} = \frac{\prod_{\alpha\in P}\langle\alpha,\lambda\rangle \cdot \prod_{\alpha\in P}\langle\alpha,\mu\rangle}{\prod_{\alpha\in P}\langle\alpha,\rho\rangle} \int_G e^{\langle \mathrm{Ad}_g(\lambda),\mu\rangle}\, dg,$$

where $\lambda, \mu \in \mathfrak{h}_{\mathbb{C}}$, and dg is the normalized Haar measure on G. Then we can rewrite $\chi_1^{\mu}(\lambda, H)$ as

$$\chi_1^{\mu}(\lambda, H) = \frac{i^{|P|}\prod_{\alpha\in P}\langle\alpha,\lambda+\rho\rangle}{\prod_{\alpha\in P}\langle\alpha,\rho\rangle} \int_G e^{i\langle\lambda+\rho, \mathrm{Ad}_g(H)\rangle}\, dg.$$

Note that

$$\frac{i^{|P|}\prod_{\alpha\in P}\langle\alpha,\lambda+\rho\rangle}{\prod_{\alpha\in P}\langle\alpha,\rho\rangle}$$

is a polynomial in $\lambda \in \Lambda$ of degree $|P| = \frac{d-r}{2}$. Also, as $|H| \lesssim \frac{1}{N}$, we have $|\mathrm{Ad}_g(H)| \lesssim \frac{1}{N}$ uniformly in $g \in G$, which implies that the integral

$$f(\lambda) = \int_G e^{i\langle\lambda+\rho, \mathrm{Ad}_g(H)\rangle}\, dg$$

as a function in λ is a pseudopolynomial of degree 0, uniformly in $|H| \lesssim \frac{1}{N}$. Then by the Leibniz rule, $\chi'(\lambda, H)$ as a function of λ is a pseudopolynomial of degree $\frac{d-r}{2}$, uniformly in $|H| \lesssim \frac{1}{N}$. This finishes the proof of Lemma 7.10.

Remark 7.17. Note that Lemma 7.10 can be stated purely in terms of an integral root system without mentioning the ambient compact Lie group, and it still holds true this way. It can be seen either by the approach via BGG-Demazure operators, which is purely a root-system-theoretic argument, or by the fact that, for any integral root system Φ, there associates to it a unique compact simply connected semisimple Lie group equipped with this root system; thus the approach via Harish-Chandra's integral formula still works, even though the argument explicitly involves the group.

7D. ***From the weight lattice to the root lattice.*** We say $\exp H$ is a *corner* in the maximal torus provided

$$\left\| \tfrac{1}{2\pi}\langle\alpha, H\rangle \right\| = 0 \quad \text{for all } \alpha \in P.$$

In this section, we extend Corollary 7.11 to the scenarios when $\exp H$ is within a distance of $\lesssim \frac{1}{N}$ from some corner. That is, when

$$\left\| \tfrac{1}{2\pi}\langle\alpha, H\rangle \right\| \lesssim \tfrac{1}{N} \quad \text{for all } \alpha \in P. \tag{7-37}$$

To this end, we rewrite the Schrödinger kernel $K_N(t,x)$ as a finite sum of exponential sums over the root lattice:

$$\begin{aligned} K_N(t,x) &= C\sum_{\mu\in\Lambda/\Gamma}\sum_{\lambda\in\mu+\Gamma} e^{-it(|\lambda|^2-|\rho|^2)}\varphi\Big(\frac{|\lambda|^2-|\rho|^2}{N^2}\Big)\frac{\prod_{\alpha\in P}\langle\alpha,\lambda\rangle}{\prod_{\alpha\in P}\langle\alpha,\rho\rangle}\frac{\sum_{s\in W}(\det s)e^{i\langle s(\lambda),H\rangle}}{\sum_{s\in W}(\det s)e^{i\langle s(\rho),H\rangle}} \\ &= C\sum_{\mu\in\Lambda/\Gamma}\sum_{\lambda\in\Gamma} e^{-it(|\lambda+\mu|^2-|\rho|^2)}\varphi\Big(\frac{|\lambda+\mu|^2-|\rho|^2}{N^2}\Big)\frac{\prod_{\alpha\in P}\langle\alpha,\lambda+\mu\rangle}{\prod_{\alpha\in P}\langle\alpha,\rho\rangle}\frac{\sum_{s\in W}(\det s)e^{i\langle s(\lambda+\mu),H\rangle}}{\sum_{s\in W}(\det s)e^{i\langle s(\rho),H\rangle}}, \end{aligned} \tag{7-38}$$

where $C = e^{it|\rho|^2}/|W|$.

Proposition 7.18. *Let μ be an element in the weight lattice Λ and let*

$$K_N^{\mu}(t,x) = \sum_{\lambda\in\Gamma} e^{-it(|\lambda+\mu|^2-|\rho|^2)}\varphi\Big(\frac{|\lambda+\mu|^2-|\rho|^2}{N^2}\Big)\frac{\prod_{\alpha\in P}\langle\alpha,\lambda+\mu\rangle}{\prod_{\alpha\in P}\langle\alpha,\rho\rangle}\frac{\sum_{s\in W}(\det s)e^{i\langle s(\lambda+\mu),H\rangle}}{\sum_{s\in W}(\det s)e^{i\langle s(\rho),H\rangle}}, \tag{7-39}$$

where x is conjugate to $\exp H$. *Then*

$$|K_N^{\mu}(t,x)| \lesssim \frac{N^d}{\big(\sqrt{q}\big(1+N\big\|\frac{t}{2\pi D}-\frac{a}{q}\big\|^{1/2}\big)\big)^r} \tag{7-40}$$

for $\frac{t}{2\pi D}\in\mathcal{M}_{a,q}$, uniformly for $\big\|\frac{1}{2\pi}\langle\alpha,H\rangle\big\|\lesssim\frac{1}{N}$ for all $\alpha\in P$.

Using (7-38) and the finiteness of Λ/Γ, we have the following corollary.

Corollary 7.19. *Inequality (6-1) holds for the case when $\big\|\frac{1}{2\pi}\langle\alpha,H\rangle\big\|\lesssim\frac{1}{N}$ for all $\alpha\in P$.*

To prove Proposition 7.18, we first prove a variant of Lemma 7.10.

Lemma 7.20. *Let*

$$\chi^{\mu}(\lambda,H) = \frac{\sum_{s\in W}(\det s)e^{i\langle s(\mu+\lambda),H\rangle}}{e^{-i\langle\rho,H\rangle}\prod_{\alpha\in P}(e^{i\langle\alpha,H\rangle}-1)} \tag{7-41}$$

be defined as in Lemma 7.10. Assume in addition that $\mu\in\Lambda$. Then $\chi^{\mu}(\lambda,H)$ as a function in $\lambda\in\Gamma$ is a pseudopolynomial of degree $\frac{d-r}{2}$, uniformly in H such that $\big\|\frac{1}{2\pi}\alpha(H)\big\|\lesssim\frac{1}{N}$ for all $\alpha\in P$.

Proof. For all $H\in i\mathfrak{b}^*$ such that $\big\|\frac{1}{2\pi}\langle\alpha,H\rangle\big\|\lesssim\frac{1}{N}$ for all $\alpha\in P$, by considering the dual basis of the simple roots $\{\alpha_1,\dots,\alpha_r\}$, we can write

$$H = H_1 + H_2 \tag{7-42}$$

such that

$$\big|\tfrac{1}{2\pi}\langle\alpha_i,H_1\rangle\big| = \big\|\tfrac{1}{2\pi}\langle\alpha_i,H\rangle\big\| \lesssim \tfrac{1}{N},\quad i=1,\dots,r, \tag{7-43}$$

and

$$\langle\alpha_i,H_2\rangle\in 2\pi\mathbb{Z},\quad i=1,\dots,r. \tag{7-44}$$

This implies that $\exp H_2$ is a corner and

$$|H_1|\lesssim\tfrac{1}{N}. \tag{7-45}$$

Then for $\lambda \in \Gamma = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_r$,

$$\chi^\mu(\lambda, H) = \chi^\mu(\lambda, H_1 + H_2) = \frac{\sum_{s\in W}(\det s)e^{i\langle s(\lambda+\mu),H_1\rangle}e^{i\langle s(\mu),H_2\rangle}}{e^{-i\langle\rho,H_1+H_2\rangle}\prod_{\alpha\in P}(e^{i\langle\alpha,H_1\rangle}-1)}. \tag{7-46}$$

Note that, see Corollary 4.13.3 in [Varadarajan 1974], $s(\mu)-\mu \in \Gamma$ for all $\mu \in \Lambda$ and $s \in W$, which combined with (7-44) implies

$$e^{i\langle s(\mu),H_2\rangle} = e^{i\langle\mu,H_2\rangle} \quad \text{for all } \mu \in \Lambda,\ s \in W.$$

Then (7-46) becomes

$$\chi^\mu(\lambda, H) = \frac{e^{i\langle\mu,H_2\rangle}}{e^{-i\langle\rho,H_2\rangle}} \cdot \frac{\sum_{s\in W}(\det s)e^{i\langle s(\lambda+\mu),H_1\rangle}}{e^{-i\langle\rho,H_1\rangle}\prod_{\alpha\in P}(e^{i\langle\alpha,H_1\rangle}-1)} = e^{i\langle\mu+\rho,H_2\rangle}\cdot\chi^\mu(\lambda, H_1), \tag{7-47}$$

which is a pseudopolynomial in $\lambda \in \Gamma$ of degree $\frac{d-r}{2}$ uniformly in $|H_1| \lesssim \frac{1}{N}$ by Lemma 7.10. □

Proof of Proposition 7.18. Since $\prod_{\alpha\in P}\langle\alpha, \lambda+\mu\rangle$ is a polynomial, and thus also a pseudopolynomial in λ of degree $|P| = \frac{d-r}{2}$, and $\chi^\mu(\lambda, H)$ is a pseudopolynomial of degree $\frac{d-r}{2}$ uniformly in $\left\|\frac{1}{2\pi}\langle\alpha, H\rangle\right\| \lesssim \frac{1}{N}$ for all $\alpha \in P$ by the previous lemma,

$$f(\lambda) = \frac{\prod_{\alpha\in P}\langle\alpha,\lambda+\mu\rangle}{\prod_{\alpha\in P}\langle\alpha,\rho\rangle}\cdot\chi^\mu(\lambda, H)$$

is then a pseudopolynomial of degree $d-r$ uniformly in $\left\|\frac{1}{2\pi}\langle\alpha, H\rangle\right\| \lesssim \frac{1}{N}$ for all $\alpha \in P$. Then the desired result comes from a direct application of Lemma 7.4. □

Example 7.21. We specialize the discussion in this section to the case $G = \mathrm{SU}(2)$. Recall that $\Lambda = \mathbb{Z}w$, $\Gamma = \mathbb{Z}\alpha$ with $\alpha = 2w$; thus $\Lambda/\Gamma \cong \{0, 1\}\cdot w$. (7-38) specializes to

$$K_N(t,\theta) = \tfrac{1}{2}e^{it}(K_N^0(t,\theta) + K_N^1(t,\theta)),$$

where

$$K_N^0 = \sum_{\substack{m=2k\\ k\in\mathbb{Z}}} e^{-itm^2}\varphi\left(\frac{m^2-1}{N^2}\right)m\cdot\frac{e^{im\theta}-e^{-im\theta}}{e^{i\theta}-e^{-i\theta}},$$

$$K_N^1 = \sum_{\substack{m=2k+1\\ k\in\mathbb{Z}}} e^{-itm^2}\varphi\left(\frac{m^2-1}{N^2}\right)m\cdot\frac{e^{im\theta}-e^{-im\theta}}{e^{i\theta}-e^{-i\theta}}$$

for $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Condition (7-37) specializes to $\left\|\frac{\theta}{\pi}\right\| \lesssim \frac{1}{N}$. Write $\theta = \theta_1 + \theta_2$, where $|\theta_1| \lesssim \frac{1}{N}$, and $\theta_2 = 0, \pi$. Then for $m = 2k,\ k \in \mathbb{Z}$,

$$\begin{aligned}
\chi_m(\theta) &= \frac{1}{e^{-i\theta}(e^{i2\theta_1}-1)}\cdot(e^{im\theta_1}-e^{-im\theta_1}) \\
&= \frac{1}{e^{-i\theta}(e^{i2\theta_1}-1)}\cdot\sum_{n=0}^{\infty}\frac{i^n}{n!}((m\theta_1)^n-(-m\theta_1)^n) \\
&= \frac{\theta_1}{e^{-i\theta}(e^{i2\theta_1}-1)}\cdot\sum_{n\text{ odd}}\frac{i^n}{n!}(2\theta_1^{n-1}m^n),
\end{aligned} \tag{7-48}$$

and similarly for $m = 2k+1,\ k \in \mathbb{Z}$,

$$\chi_m(\theta) = \frac{e^{i\theta_2}\theta_1}{e^{-i\theta}(e^{i2\theta_1}-1)} \cdot \sum_{n \text{ odd}} \frac{i^n}{n!}(2\theta_1^{n-1}m^n).$$

Note that we are implicitly applying Proposition 7.14 so that

$$f_n(\theta_1) := (m\theta_1)^n - (-m\theta_1)^n = \theta_1 \cdot \delta f_n = \begin{cases} \theta_1 \cdot 2\theta_1^{n-1}m^n, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

If $|k| \lesssim N$, then it is clear that

$$|D^L \chi_{2k}| \lesssim N^{1-L}, \quad |D^L \chi_{2k+1}| \lesssim N^{1-L}, \quad L \in \mathbb{Z}_{\geq 0},$$

where D is the difference operator with respect to the variable k. These two inequalities will give the desired estimates for K_N^0 and K_N^1 respectively using the Weyl sum estimate Lemma 7.4 in one dimension.

7E. *Root subsystems.* To finish the proof of part (ii) of Theorem 6.2, considering Corollaries 7.8 and 7.19, it suffices to prove (6-1) in the scenarios when $\exp H$ is away from all the corners by a distance of $\gtrsim \frac{1}{N}$ but stays close to some cell walls within a distance of $\lesssim \frac{1}{N}$. We will identify these other walls as belonging to a *root subsystem* of the original root system Φ, and then we will decompose the character, the weight lattice and thus the Schrödinger kernel according to this root subsystem.

7E.1. *Identifying root subsystems and rewriting the character.* Given any $H \in i\mathfrak{b}^*$, let Q_H be the subset of the set Φ of roots defined by

$$Q_H := \left\{\alpha \in \Phi \;\middle|\; \left\|\tfrac{1}{2\pi}\langle \alpha, H\rangle\right\| \leq \tfrac{1}{N}\right\}.$$

Thus

$$\Phi \setminus Q_H = \left\{\alpha \in \Phi \;\middle|\; \left\|\tfrac{1}{2\pi}\langle \alpha, H\rangle\right\| > \tfrac{1}{N}\right\}.$$

Define

$$\Phi_H := \{\alpha \in \Phi \mid \alpha \text{ lies in the } \mathbb{Z}\text{-linear span of } Q_H\}. \tag{7-49}$$

Then $\Phi_H \supset Q_H$, and

$$\left\|\tfrac{1}{2\pi}\langle \alpha, H\rangle\right\| \lesssim \tfrac{1}{N} \quad \text{for all } \alpha \in \Phi_H, \tag{7-50}$$

with the implicit constant independent of H, and

$$\left\|\tfrac{1}{2\pi}\langle \alpha, H\rangle\right\| > \tfrac{1}{N} \quad \text{for all } \alpha \in \Phi \setminus \Phi_H. \tag{7-51}$$

Note that Φ_H is $\mathbb{Z}$-*closed* in Φ; that is, no element in $\Phi \setminus \Phi_H$ lies in the $\mathbb{Z}$-linear span of Φ_H.

Proposition 7.22. *Φ_H is an integral root system.*

Proof. We check the requirements for an integral root system listed on page 1182. Parts (ii) and (iv) are automatic from the fact that Φ_H is a subset of Φ. Part (i) comes from the fact that Φ_H is a $\mathbb{Z}$-linear space. Part (iii) follows from the fact that $s_\alpha \beta$ is a $\mathbb{Z}$-linear combination of α and β for all $\alpha, \beta \in \Phi_H$, and the fact that Φ_H is a $\mathbb{Z}$-linear space. □

Then we say that Φ_H is a *root subsystem* of Φ.

Let W_H be the Weyl group of Φ_H. W_H is generated by reflections s_α for $\alpha \in \Phi_H$ and W_H is a subgroup of the Weyl group W of Φ. Let P be a positive system of roots of Φ and $P_H = P \cap \Phi_H$. Then P_H is a positive system of roots of Φ_H. We rewrite the Weyl character as

$$\begin{aligned}
\chi_\lambda &= \frac{\sum_{s\in W}(\det s)e^{i\langle s(\lambda),H\rangle}}{e^{-i\langle\rho,H\rangle}\prod_{\alpha\in P}(e^{i\langle\alpha,H\rangle}-1)} \\
&= \frac{(1/|W_H|)\sum_{s_H\in W_H}\sum_{s\in W}(\det(s_H s))e^{i\langle(s_H s)(\lambda),H\rangle}}{e^{-i\langle\rho,H\rangle}\big(\prod_{\alpha\in P\setminus P_H}(e^{i\langle\alpha,H\rangle}-1)\big)\big(\prod_{\alpha\in P_H}(e^{i\langle\alpha,H\rangle}-1)\big)} \\
&= \frac{1}{|W_H|e^{-i\langle\rho,H\rangle}\prod_{\alpha\in P\setminus P_H}(e^{i\langle\alpha,H\rangle}-1)}\sum_{s\in W}(\det s)\cdot\frac{\sum_{s_H\in W_H}(\det s_H)e^{i\langle s_H(s(\lambda)),H\rangle}}{\prod_{\alpha\in P_H}(e^{i\langle\alpha,H\rangle}-1)} \\
&= C(H)\sum_{s\in W}(\det s)\cdot\frac{\sum_{s_H\in W_H}(\det s_H)e^{i\langle s_H(s(\lambda)),H\rangle}}{\prod_{\alpha\in P_H}(e^{i\langle\alpha,H\rangle}-1)},
\end{aligned} \tag{7-52}$$

where

$$C(H) := \frac{1}{|W_H|e^{-i\langle\rho,H\rangle}\prod_{\alpha\in P\setminus P_H}(e^{i\langle\alpha,H\rangle}-1)}. \tag{7-53}$$

Then by (7-51),

$$|C(H)| \lesssim N^{|P\setminus P_H|}. \tag{7-54}$$

Let V_H be the $\mathbb{R}$-linear span of Φ_H in $V = i\mathfrak{b}^*$ and let $H^\parallel$ be the orthogonal projection of H on V_H. Let $H^\perp = H - H^\parallel$. Then $H^\perp$ is orthogonal to V_H and we have

$$\begin{aligned}
\chi_\lambda &= C(H)\sum_{s\in W}(\det s)\cdot\frac{\sum_{s_H\in W_H}(\det s_H)\,e^{i\langle s_H(s(\lambda)),H^\perp+H^\parallel\rangle}}{\prod_{\alpha\in P_H}(e^{i\langle\alpha,H^\perp+H^\parallel\rangle}-1)} \\
&= C(H)\sum_{s\in W}(\det s)\cdot\frac{\sum_{s_H\in W_H}(\det s_H)e^{i\langle s(\lambda),s_H(H^\perp)\rangle}e^{i\langle s_H(s(\lambda)),H^\parallel\rangle}}{\prod_{\alpha\in P_H}(e^{i\langle\alpha,H^\parallel\rangle}-1)}.
\end{aligned} \tag{7-55}$$

Note that since $H^\perp$ is orthogonal to every root in Φ_H, $H^\perp$ is fixed by s_α for any $\alpha \in \Phi_H$, which in turn implies that $H^\perp$ is fixed by any $s_H \in W_H$; that is, $s_H(H^\perp) = H^\perp$. Then

$$\begin{aligned}
\chi_\lambda &= C(H)\sum_{s\in W}(\det s)\cdot\frac{\sum_{s_H\in W_H}(\det s_H)e^{i\langle s(\lambda),H^\perp\rangle}e^{i\langle s_H(s(\lambda)),H^\parallel\rangle}}{\prod_{\alpha\in P_H}(e^{i\langle\alpha,H^\parallel\rangle}-1)} \\
&= C(H)\sum_{s\in W}(\det s)\cdot e^{i\langle s(\lambda),H^\perp\rangle}\cdot\frac{\sum_{s_H\in W_H}(\det s_H)e^{i\langle s_H(s(\lambda)),H^\parallel\rangle}}{\prod_{\alpha\in P_H}(e^{i\langle\alpha,H^\parallel\rangle}-1)}.
\end{aligned} \tag{7-56}$$

Note that by the definition of $H^\parallel$, we have

$$\left\|\tfrac{1}{2\pi}\langle\alpha,H^\parallel\rangle\right\| \lesssim \tfrac{1}{N} \quad \text{for all } \alpha\in\Phi_H. \tag{7-57}$$

This means that $\exp H^{\parallel}$ is a corner in the maximal torus of the compact semisimple Lie group associated to the integral root system Φ_H.

Using the above formula, we rewrite the Schrödinger kernel (7-11) as

$$K_N(t,x) = \frac{C(H)e^{it|\rho|^2}}{\left(\prod_{\alpha\in P}\langle\alpha,\rho\rangle\right)|W|}\sum_{s\in W}(\det s)\cdot K_{N,s}(t,x), \tag{7-58}$$

where

$$K_{N,s}(t,x) = \sum_{\lambda\in\Lambda} e^{i\langle s(\lambda),H^{\perp}\rangle - it|\lambda|^2}\varphi\left(\frac{|\lambda|^2-|\rho|^2}{N^2}\right)\left(\prod_{\alpha\in P}\langle\alpha,\lambda\rangle\right)\frac{\sum_{s_H\in W_H}(\det s_H)e^{i\langle s_H(s(\lambda)),H^{\parallel}\rangle}}{\prod_{\alpha\in P_H}(e^{i\langle\alpha,H^{\parallel}\rangle}-1)}.$$

Noting that for any $s\in W$, $|s(\lambda)|=|\lambda|$, $\prod_{\alpha\in P}\langle\alpha,s(\lambda)\rangle = (\det s)\prod_{\alpha\in P}\langle\alpha,\lambda\rangle$ by Proposition 7.13, and $s(\Lambda)=\Lambda$, we have

$$K_{N,s}(t,x) = (\det s)K_{N,\mathbb{1}}(t,x),$$

where $\mathbb{1}$ is the identity element in W. Then (7-58) becomes

$$K_N(t,x) = \frac{C(H)e^{it|\rho|^2}}{\left(\prod_{\alpha\in P}\langle\alpha,\rho\rangle\right)}K_{N,\mathbb{1}}(t,x). \tag{7-59}$$

Proposition 7.23. *Recall that*

$$K_{N,\mathbb{1}}(t,x) = \sum_{\lambda\in\Lambda} e^{i\langle\lambda,H^{\perp}\rangle - it|\lambda|^2}\varphi\left(\frac{|\lambda|^2-|\rho|^2}{N^2}\right)\left(\prod_{\alpha\in P}\langle\alpha,\lambda\rangle\right)\frac{\sum_{s_H\in W_H}(\det s_H)e^{i\langle s_H(\lambda),H^{\parallel}\rangle}}{\prod_{\alpha\in P_H}(e^{i\langle\alpha,H^{\parallel}\rangle}-1)}. \tag{7-60}$$

Then

$$|K_{N,\mathbb{1}}(t,x)| \lesssim \frac{N^{d-|P\setminus P_H|}}{\left(\sqrt{q}\left(1+N\left|\frac{t}{2\pi D}-\frac{a}{q}\right|^{1/2}\right)\right)^r} \tag{7-61}$$

for $\frac{t}{2\pi D}\in\mathcal{M}_{a,q}$, *uniformly in* $x\in G$.

Noting (7-54) and (7-59), the above proposition implies part (ii) of Theorem 6.2.

Example 7.24. Figure 1 is an illustration of the decomposition of the maximal torus of SU(3) according to the values of $\left\|\frac{1}{2\pi}\langle\alpha,H\rangle\right\|$, $\alpha\in\Phi$. Here $P=\{\alpha_1,\alpha_2,\alpha_3=\alpha_1+\alpha_2\}$. The three proper root subsystems of Φ are $\{\pm\alpha_i\}$, $i=1,2,3$. The association of Φ_H to H is as follows:

$H\in$ regions of color (red) $\iff$ $\Phi_H=\Phi$,

$H\in$ regions of color (yellow) $\iff$ $\Phi_H=\{\pm\alpha_1\}$,

$H\in$ regions of color (pink) $\iff$ $\Phi_H=\{\pm\alpha_2\}$,

$H\in$ regions of color (blue) $\iff$ $\Phi_H=\{\pm\alpha_3\}$,

$H\in$ regions of color (green) $\iff$ $\Phi_H=\varnothing$.

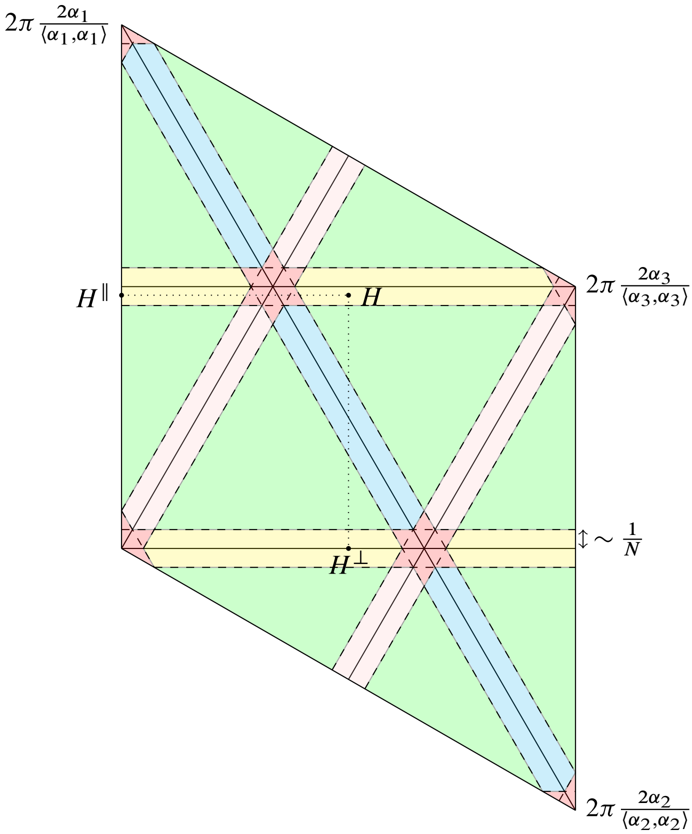

Figure 1. Decomposition of the maximal torus of SU(3) according to the values of $\left\|\frac{1}{2\pi}\langle\alpha, H\rangle\right\|$, $\alpha \in \Delta$.

7E.2. *Decomposition of the weight lattice.* To prove Proposition 7.23, we will make a decomposition of the weight lattice Λ according to the root subsystem Φ_H. First, we have the following lemma about root subsystems. Let Proj_U denote the orthogonal projection map from the ambient inner product space onto the subspace U.

Lemma 7.25. *Let Φ be an integral root system in the space V with the associated weight lattice Λ_Φ. Let Ψ be a root subsystem of Φ. Then let Γ_Ψ and Λ_Φ be the root lattice and weight lattice associated to Ψ respectively. Let V_Ψ be the $\mathbb{R}$-linear span of Ψ in V. Let Υ_Ψ be the image of the orthogonal projection of Λ_Φ onto V_Ψ. Then the following statements hold true*:

(1) *Υ_Ψ is a lattice and $\Gamma_\Psi \subset \Upsilon_\Psi \subset \Lambda_\Psi$. In particular, the rank of Υ_Ψ equals the rank of Γ_Ψ as well as Λ_Ψ.*

(2) *Let the ranks of Υ_Ψ and Λ_Φ be r and R respectively. Let $\{w_1, \dots, w_r\}$ be a basis of Υ_Ψ. Pick any $\{u_1, \dots, u_r\} \subset \Lambda_\Phi$ such that $\mathrm{Proj}_{V_\Psi}(u_i) = w_i$, $i = 1, \dots, r$. Then we can extend $\{u_1, \dots, u_r\}$ into a basis $\{u_1, \dots, u_r, u_{r+1}, \dots, u_R\}$ of Λ_Φ. Furthermore, we can pick $\{u_{r+1}, \dots, u_R\}$ such that $\mathrm{Proj}_{V_\Psi}(u_i) = 0$ for $i = r+1, \dots, R$.*

Proof. (1) It's clear that Υ_Ψ is a lattice. Let Γ_Φ be the root lattice associated to Φ. Then $\Gamma_\Psi \subset \Gamma_\Phi$. Thus

$$\Gamma_\Psi = \mathrm{Proj}_{V_\Psi}(\Gamma_\Psi) \subset \mathrm{Proj}_{V_\Psi}(\Gamma_\Phi) \subset \mathrm{Proj}_{V_\Psi}(\Lambda_\Phi) = \Upsilon_\Psi.$$

On the other hand, for any $\mu \in \Lambda_\Phi,\ \alpha \in \Gamma_\Psi$, we have $\langle \mathrm{Proj}_{V_\Psi}(\mu), \alpha\rangle = \langle \mu, \alpha\rangle$. This in particular implies

$$2\frac{\langle \mathrm{Proj}_{V_\Psi}(\mu), \alpha\rangle}{\langle \alpha, \alpha\rangle} = 2\frac{\langle \mu, \alpha\rangle}{\langle \alpha, \alpha\rangle} \in \mathbb{Z} \quad \text{for all } \mu \in \Lambda_\Phi,\ \alpha \in \Gamma_\Psi.$$

This implies $\mathrm{Proj}_{V_\Psi}(\mu) \in \Lambda_\Psi$ for all $\mu \in \Lambda_\Phi$; that is, $\Upsilon_\Psi = \mathrm{Proj}_{V_\Psi}(\Lambda_\Phi) \subset \Lambda_\Psi$.

(2) Let $S_\Phi := \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_r$; then S_Φ is a sublattice of Λ_Φ of rank r. By the theory of modules over a principal ideal domain, there exists a basis $\{u'_1, \ldots, u'_R\}$ of Λ_Φ and positive integers $d_1 | d_2 | \cdots | d_r$ such that $\{d_1 u'_1, \ldots, d_r u'_r\}$ is a basis of S_Φ. Then we must have $d_1 = d_2 = \cdots = d_r = 1$, since

$$\begin{aligned}\mathbb{Z}d_1 \mathrm{Proj}_{V_\Psi}(u'_1) + \cdots + \mathbb{Z}d_r \mathrm{Proj}_{V_\Psi}(u'_r) &= \mathrm{Proj}_{V_\Psi}(S_\Phi) \\ &= \mathrm{Proj}_{V_\Psi}(\Lambda_\Phi) \supset \mathbb{Z}\,\mathrm{Proj}_{V_\Psi}(u'_1) + \cdots + \mathbb{Z}\,\mathrm{Proj}_{V_\Psi}(u'_r)\end{aligned} \tag{7-62}$$

and $u'_1, \ldots, u'_r$ are $\mathbb{R}$-linear independent. Thus we have

$$S_\Phi = \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_r = \mathbb{Z}u'_1 + \cdots + \mathbb{Z}u'_r$$

and then $\{u_1, \ldots, u_r, u'_{r+1}, \ldots, u'_R\}$ is also a basis of Λ_Φ. Furthermore, by adding a $\mathbb{Z}$-linear combination of $u_1, \ldots, u_r$ to each of $u'_{r+1}, \ldots, u'_R$, we can assume that $\mathrm{Proj}_{V_\Psi}(u'_i) = 0$ for $i = r+1, \ldots, R$. □

We apply the above lemma to the root subsystem Φ_H of Φ. Let $V = i\mathfrak{b}^*$, V_H be the $\mathbb{R}$-linear span of Φ_H in V, Γ_H be the root lattice for Φ_H, and let

$$\Upsilon_H := \mathrm{Proj}_{V_H}(\Lambda). \tag{7-63}$$

Then by the above lemma, we have

$$\Upsilon_H \supset \Gamma_H. \tag{7-64}$$

Let r_H be the rank of Φ_H as well as of Γ_H and Υ_H, and let $\{w_1, \ldots, w_{r_H}\} \subset \Upsilon_H$ such that

$$\Upsilon_H = \mathbb{Z}w_1 + \cdots + \mathbb{Z}w_{r_H}.$$

Pick $\{u_1, \ldots, u_{r_H}\} \subset \Lambda$ such that

$$\mathrm{Proj}_{V_H}(u_i) = w_i, \quad i = 1, \ldots, r_H.$$

Then by the above lemma, we can extend $\{u_1, \ldots, u_{r_H}\}$ into a basis $\{u_1, \ldots, u_r\}$ of Λ such that

$$\mathrm{Proj}_{V_H}(u_i) = 0, \quad i = r_H + 1, \ldots, r, \tag{7-65}$$

with

$$\Lambda = \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_r.$$

Set

$$\Upsilon'_H = \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_{r_H} \subset \Lambda.$$

Then

$$\mathrm{Proj}_{V_H} : \Upsilon'_H \xrightarrow{\sim} \Upsilon_H.$$

Recalling (7-64), let Γ'_H be the sublattice of Υ'_H corresponding to $\Gamma_H \subset \Upsilon_H$ under this isomorphism. More precisely, let $\{\alpha_1, \dots, \alpha_{r_H}\}$ be a simple system of roots for Γ_H; then

$$\mathrm{Proj}_{V_H} : \Gamma'_H = \mathbb{Z}\alpha'_1 + \cdots + \mathbb{Z}\alpha'_{r_H} \xrightarrow{\sim} \Gamma_H = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_{r_H}, \quad \alpha'_i \mapsto \alpha_i,\ i = 1, \dots, r_H, \tag{7-66}$$

and we have

$$\Upsilon'_H/\Gamma'_H \cong \Upsilon_H/\Gamma_H, \quad |\Upsilon'_H/\Gamma'_H| = |\Upsilon_H/\Gamma_H| < \infty. \tag{7-67}$$

Decomposing the weight lattice as

$$\Lambda = \bigsqcup_{\mu \in \Upsilon'_H/\Gamma'_H} (\mu + \Gamma'_H + \mathbb{Z}u_{r_H+1} + \cdots + \mathbb{Z}u_r),$$

we have

$$K_{N,\mathbb{1}}(t,x) = \sum_{\substack{\mu \in \Upsilon'_H/\Gamma'_H \\ \lambda'_1 = n_1\alpha'_1 + \cdots + n_{r_H}\alpha'_{r_H} \\ \lambda_2 = n_{r_H+1}u_{r_H+1} + \cdots + n_r u_r}} e^{i\langle \mu+\lambda'_1+\lambda_2, H^\perp\rangle - it|\mu+\lambda'_1+\lambda_2|^2} \varphi\left(\frac{|\mu+\lambda'_1+\lambda_2|^2 - |\rho|^2}{N^2}\right) \cdot \left(\prod_{\alpha\in P} \langle \alpha, \mu+\lambda'_1+\lambda_2\rangle\right) \frac{\sum_{s_H \in W_H} (\det s_H) e^{i\langle s_H(\mu+\lambda'_1+\lambda_2), H^\parallel\rangle}}{\prod_{\alpha\in P_H} (e^{i\langle \alpha, H^\parallel\rangle} - 1)}. \tag{7-68}$$

Note that (7-65) implies for $\lambda_2 = n_{r_H+1}u_{r_H+1} + \cdots + n_r u_r$ that

$$\langle s_H(\lambda_2), H^\parallel\rangle = \langle \lambda_2, s_H(H^\parallel)\rangle = 0,$$

and (7-66) implies for $\lambda'_1 = n_1\alpha'_1 + \cdots + n_{r_H}\alpha'_{r_H}$ that

$$\langle s_H(\lambda'_1), H^\parallel\rangle = \langle \lambda'_1, s_H(H^\parallel)\rangle = \langle \lambda_1, s_H(H^\parallel)\rangle = \langle s_H(\lambda_1), H^\parallel\rangle,$$

where $\lambda_1 = n_1\alpha_1 + \cdots + n_{r_H}\alpha_{r_H} \in V_H$. Similarly, also note that

$$\langle s_H(\mu), H^\parallel\rangle = \langle s_H(\mu^\parallel), H^\parallel\rangle, \quad \text{where } \mu^\parallel := \mathrm{Proj}_{V_H}(\mu).$$

Thus we write

$$K_{N,\mathbb{1}}(t,x) = \sum_{\mu \in \Upsilon'_H/\Gamma'_H} \sum_{\substack{\lambda'_1 = n_1\alpha'_1 + \cdots + n_{r_H}\alpha'_{r_H} \\ \lambda_1 = n_1\alpha_1 + \cdots + n_{r_H}\alpha_{r_H} \\ \lambda_2 = n_{r_H+1}u_{r_H+1} + \cdots + n_r u_r}} e^{i\langle \mu+\lambda'_1+\lambda_2, H^\perp\rangle - it|\mu+\lambda'_1+\lambda_2|^2} \varphi\left(\frac{|\mu+\lambda'_1+\lambda_2|^2 - |\rho|^2}{N^2}\right) \cdot \left(\prod_{\alpha\in P} \langle \alpha, \mu+\lambda'_1+\lambda_2\rangle\right) \frac{\sum_{s_H \in W_H} (\det s_H) e^{i\langle s_H(\mu^\parallel+\lambda_1), H^\parallel\rangle}}{\prod_{\alpha\in P_H} (e^{i\langle \alpha, H^\parallel\rangle} - 1)}. \tag{7-69}$$

Remark 7.26. We have that in the above formula

$$\chi^{\mu^\parallel}(\lambda_1, H^\parallel) := \frac{\sum_{s_H \in W_H} (\det s_H) e^{i\langle s_H(\mu^\parallel+\lambda_1), H^\parallel\rangle}}{\prod_{\alpha\in P_H} (e^{i\langle \alpha, H^\parallel\rangle} - 1)}$$

is a character of the form (7-41). Also note that $\mu^{\parallel} \in \mathrm{Proj}_{V_H}(\Lambda)$ lies in the weight lattice Λ_H of Φ_H by Lemma 7.25.

Noting (7-67), Proposition 7.23 reduces to the following.

Proposition 7.27. *For $\mu \in \Upsilon'_H/\Gamma'_H$, let*

$$K^{\mu}_{N,\mathbb{1}}(t,x) := \sum_{\substack{\lambda'_1 = n_1\alpha'_1+\cdots+n_{r_H}\alpha'_{r_H} \\ \lambda_1 = n_1\alpha_1+\cdots+n_{r_H}\alpha_{r_H} \\ \lambda_2 = n_{r_H+1}u_{r_H+1}+\cdots+n_r u_r \\ n_1,\ldots,n_r\in\mathbb{Z}}} e^{i\langle \mu+\lambda'_1+\lambda_2, H^{\perp}\rangle - it|\mu+\lambda'_1+\lambda_2|^2} \varphi\left(\frac{|\mu+\lambda'_1+\lambda_2|^2-|\rho|^2}{N^2}\right) \cdot\left(\prod_{\alpha\in P}\langle \alpha, \mu+\lambda'_1+\lambda_2\rangle\right)\frac{\sum_{s_H\in W_H}(\det s_H)e^{i\langle s_H(\mu^{\parallel}+\lambda_1),H^{\parallel}\rangle}}{\prod_{\alpha\in P_H}(e^{i\langle\alpha,H^{\parallel}\rangle}-1)}. \tag{7-70}$$

Then

$$|K^{\mu}_{N,\mathbb{1}}(t,x)| \lesssim \frac{N^{d-|P\setminus P_H|}}{\left(\sqrt{q}\left(1+N\left|\frac{t}{2\pi D}-\frac{a}{q}\right|^{1/2}\right)\right)^r} \tag{7-71}$$

for $\frac{t}{2\pi D} \in \mathcal{M}_{a,q}$, uniformly in $x \in G$.

Proof. We apply Lemma 7.4 to the lattice $\mathbb{Z}\alpha'_1+\cdots+\mathbb{Z}\alpha'_{r_H}+\mathbb{Z}u_{r_H+1}+\cdots+\mathbb{Z}u_r$. Write

$$\chi^{\mu^{\parallel}}(\lambda_1, H^{\parallel}) = \frac{\sum_{s_H\in W_H}(\det s_H)e^{i\langle s_H(\mu^{\parallel}+\lambda_1),H^{\parallel}\rangle}}{\prod_{\alpha\in P_H}(e^{i\langle\alpha,H^{\parallel}\rangle}-1)}.$$

Then it suffices to show that

$$\left|D_{i_1}\cdots D_{i_k}\left(\prod_{\alpha\in P}\langle\alpha,\mu+\lambda'_1+\lambda_2\rangle\chi^{\mu^{\parallel}}(\lambda_1,H^{\parallel})\right)\right| \lesssim N^{d-|P\setminus P_H|-r-k}$$

for $1 \le i_1,\ldots,i_k \le r$,

$$\begin{aligned} \lambda'_1 &= n_1\alpha'_1+\cdots+n_{r_H}\alpha'_{r_H}, \\ \lambda_1 &= n_1\alpha_1+\cdots+n_{r_H}\alpha_{r_H}, \\ \lambda_2 &= n_{r_H+1}u_{r_H+1}+\cdots+n_r u_r, \end{aligned}$$

uniformly in $|n_i| \lesssim N$, $i = 1,\ldots,r$. Since $\prod_{\alpha\in P}\langle\alpha,\mu+\lambda'_1+\lambda_2\rangle$ is a polynomial and thus a pseudopolynomial of degree $|P|$, it suffices to show that

$$|D_{i_1}\cdots D_{i_k}(\chi^{\mu^{\parallel}}(\lambda_1,H^{\parallel}))| \lesssim N^{d-|P\setminus P_H|-r-|P|-k} = N^{|P_H|-k}. \tag{7-72}$$

Since $\chi(\lambda_1)$ does not involve the variables $n_{r_H+1},\ldots,n_r$, it suffices to prove (7-72) for $1 \le i_1,\ldots,i_k \le r_H$ uniformly in $|\lambda_1| \lesssim N$. But this follows by applying Lemma 7.20 to the root system Φ_H, noting Remark 7.26. □

7F. L^p estimates. We prove in this section $L^p(G)$ estimates of the Schrödinger kernel for $p<\infty$. Though we do not apply them to the proof of the main theorem, they encapsulate the essential ingredients in the proof of the $L^\infty(G)$-estimate and are of independent interest.

Proposition 7.28. *Let $K_N(t,x)$ be the Schrödinger kernel as in Theorem 6.1. Then for any $p>3$, we have*

$$\|K_N(t,\cdot)\|_{L^p(G)} \lesssim \frac{N^{d-\frac{d}{p}}}{\left(\sqrt{q}\left(1+N\left\|\frac{t}{2\pi D}-\frac{a}{q}\right\|^{\frac{1}{2}}\right)\right)^r} \tag{7-73}$$

for $\frac{t}{2\pi D}\in\mathcal{M}_{a,q}$.

Proof. As a linear combination of characters, the Schrödinger kernel $K_N(t,\cdot)$ is a central function. Then we can apply to it the Weyl integration formula (4-16)

$$\|K_N(t,\cdot)\|^p_{L^p(G)} = \frac{1}{|W|}\int_B |K_N(t,b)|^p|D_P(b)|^2\,db, \tag{7-74}$$

where B is the maximal torus with normalized Haar measure db. Recall that we can parametrize $B=\exp\mathfrak{b}$ by $H\in i\mathfrak{b}^*\cong\mathfrak{b}$, and write

$$B\cong i\mathfrak{b}^*/(2\pi\mathbb{Z}\alpha_1^\vee+\cdots+2\pi\mathbb{Z}\alpha_r^\vee)=[0,2\pi)\alpha_1^\vee+\cdots+[0,2\pi)\alpha_r^\vee, \tag{7-75}$$

where $\{\alpha_i^\vee=2\alpha_i/\langle\alpha_i,\alpha_i\rangle \mid i=1,\dots,r\}$ is the set of simple coroots associated to a system of simple roots $\{\alpha_i\mid i=1,\dots,r\}$.

We have shown in Section 7E that each $H\in i\mathfrak{b}^*$ is associated to a root subsystem Φ_H such that (7-50) and (7-51) hold. Note that there are finitely many root subsystems of a given root system; thus B is covered by finitely many subsets R of the form

$$R=\left\{H\in B \;\middle|\; \left\|\tfrac{1}{2\pi}\langle\alpha,H\rangle\right\|\lesssim\tfrac{1}{N}\text{ for all }\alpha\in\Psi,\ \left\|\tfrac{1}{2\pi}\langle\alpha,H\rangle\right\|>\tfrac{1}{N}\text{ for all }\alpha\in\Phi\setminus\Psi\right\}, \tag{7-76}$$

where Ψ is a root subsystem of Φ. Thus to prove (7-73), using (7-74), it suffices to show

$$\int_R |K_N(t,\exp H)|^p|D_p(\exp H)|^2\,dH \lesssim \left(\frac{N^d}{\left(\sqrt{q}\left(1+N\left\|\frac{t}{2\pi D}-\frac{a}{q}\right\|^{1/2}\right)\right)^r}\right)^p N^{-d}. \tag{7-77}$$

By (7-54), (7-59) and (7-61), we have

$$K_N(t,\exp H)\lesssim \frac{1}{\prod_{\alpha\in P\setminus Q}(e^{i\langle\alpha,H\rangle}-1)}\cdot\frac{N^{d-|P\setminus Q|}}{\left(\sqrt{q}\left(1+N\left|\frac{t}{2\pi D}-\frac{a}{q}\right|^{1/2}\right)\right)^r},$$

where P,Q are respectively the sets of positive roots of Φ and Ψ with $P\supset Q$. Recalling $D_P(\exp H)=\prod_{\alpha\in P}(e^{i\langle\alpha,H\rangle}-1)$, (7-77) is then reduced to

$$\int_R\left|\frac{1}{\prod_{\alpha\in P\setminus Q}(e^{i\langle\alpha,H\rangle}-1)}\right|^{p-2}\left|\prod_{\alpha\in Q}(e^{i\langle\alpha,H\rangle}-1)\right|^2 dH\lesssim N^{p|P\setminus Q|-d}.$$

Using

$$|e^{i\langle\alpha,H\rangle}-1|\lesssim\left\|\tfrac{1}{2\pi}\langle\alpha,H\rangle\right\|\lesssim|e^{i\langle\alpha,H\rangle}-1|,$$

it suffices to show

$$\int_R \left| \frac{1}{\prod_{\alpha\in P\setminus Q} \left\| \frac{1}{2\pi}\langle \alpha, H\rangle \right\|} \right|^{p-2} \left| \prod_{\alpha\in Q} \left\| \frac{1}{2\pi}\langle \alpha, H\rangle \right\| \right|^2 dH \lesssim N^{p|P\setminus Q|-d}. \tag{7-78}$$

For each $H \in B$, we write

$$H = H' + H_0$$

such that

$$\left\| \tfrac{1}{2\pi}\langle \alpha, H\rangle \right\| = \left| \tfrac{1}{2\pi}\langle \alpha, H'\rangle \right|, \quad \langle \alpha, H_0\rangle \in 2\pi\mathbb{Z} \quad \text{for all } \alpha \in P.$$

We write

$$R \subset \bigcup_{\substack{H_0\in B \\ \langle \alpha, H_0\rangle \in 2\pi\mathbb{Z}, \forall \alpha\in P}} R' + H_0, \tag{7-79}$$

where

$$R' = \left\{ H \in B \;\middle|\; \left|\tfrac{1}{2\pi}\langle \alpha, H\rangle\right| \lesssim \tfrac{1}{N} \text{ for all } \alpha \in Q,\ \left|\tfrac{1}{2\pi}\langle \alpha, H\rangle\right| > \tfrac{1}{N} \text{ for all } \alpha \in P \setminus Q \right\}. \tag{7-80}$$

Note that $\langle \alpha, \alpha_i^\vee\rangle \in \mathbb{Z}$ for all $\alpha \in P$ and $i = 1, \dots, r$ due to the integrality of the root system; using (7-75), we have that there are only finitely many $H_0 \in B$ such that $\langle \alpha, H_0\rangle \in 2\pi\mathbb{Z}$ for all $\alpha \in P$. Thus using (7-79), (7-78) is further reduced to

$$\int_{R'} \left| \frac{1}{\prod_{\alpha\in P\setminus Q} \left| \frac{1}{2\pi}\langle \alpha, H\rangle \right|} \right|^{p-2} \left| \prod_{\alpha\in Q} \left| \frac{1}{2\pi}\langle \alpha, H\rangle \right| \right|^2 dH \lesssim N^{p|P\setminus Q|-d}. \tag{7-81}$$

Now we reparametrize $B \cong [0, 2\pi)\alpha_1^\vee + \cdots + [0, 2\pi)\alpha_r^\vee$ by

$$H = \sum_{i=1}^r t_i w_i, \quad (t_1, \dots, t_r) \in D,$$

where $\{w_i \mid i = 1, \dots, r\}$ are the fundamental weights such that $\langle \alpha_i, w_j\rangle = \delta_{ij}|\alpha_i|^2/2,\ i, j = 1, \dots, r$, and D is a bounded domain in $\mathbb{R}^r$. Then the normalized Haar measure dH equals

$$dH = C dt_1 \cdots dt_r$$

for some constant C. Let $s \le r$ such that

$$\{\alpha_1, \dots, \alpha_s\} \subset P \setminus Q,$$
$$\{\alpha_{s+1}, \dots, \alpha_r\} \subset Q.$$

Using (7-80), we estimate

$$\begin{aligned} \int_{R'} \left| \frac{1}{\prod_{\alpha\in P\setminus Q} \left| t\frac{1}{2\pi}\langle \alpha, H\rangle \right|} \right|^{p-2} & \left| \prod_{\alpha\in Q} \left| \frac{1}{2\pi}\langle \alpha, H\rangle \right| \right|^2 dH \\ &\lesssim \int_{R'} \frac{1}{|t_1 \cdots t_s|^{p-2}} N^{(p-2)(|P\setminus Q|-s)} N^{-2|Q|}\, dt_1 \cdots dt_r \\ &\lesssim N^{(p-2)(|P\setminus Q|-s)} N^{-2|Q|} \int_{\substack{|t_1|,\dots,|t_s|\gtrsim \frac{1}{N} \\ |t_{s+1}|,\dots,|t_r|\lesssim \frac{1}{N}}} \frac{1}{|t_1 \cdots t_s|^{p-2}}\, dt_1 \cdots dt_r. \end{aligned} \tag{7-82}$$

If $p > 3$, the above is bounded by

$$\lesssim N^{(p-2)(|P\setminus Q|-s)} N^{-2|Q|} N^{s(p-3)-(r-s)} = N^{p|P\setminus Q|-d},$$

noting that $2|P \setminus Q| + 2|Q| + r = 2|P| + r = d$. □

Remark 7.29. The requirement $p > 3$ is by no means optimal. The estimate in (7-82) may be improved to lower the exponent p. We conjecture that (7-73) holds for all $p > p_r$ such that $\lim_{r\to\infty} p_r = 2$, where r is the rank of G.

Acknowledgments

I would like to thank my Ph.D. thesis advisors Prof. Monica Visan and Prof. Rowan Killip for their constant guidance and support. I especially thank them for giving me all the freedom to choose problems that suit my own interest. I would like to thank Prof. Raphaël Rouquier and Prof. Terence Tao for sharing their expertise on the BGG-Demazure operators and the Weyl-type sums respectively. I would also like to thank Jiayin Guo for pointing out several false conjectures of mine on root systems. My thanks also goes to Prof. Guozhen Lu who encouraged me to publish the paper. I am thankful to the editors and referees for their carefully reading the paper and their helpful suggestions on revising it. Last but not least, I am thankful to Mengmeng Zhang for her company and encouragement.

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Received 27 Aug 2018. Revised 19 Feb 2019. Accepted 18 Apr 2019.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

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Analysis & PDE

Volume 13 No. 4 2020