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REGULARITY RESULTS FOR GENERALIZED DOUBLE PHASE FUNCTIONALS

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We consider a wide class of functionals with the property of changing their growth and ellipticity properties according to the modulating coefficients in the framework of Musielak–Orlicz spaces. In particular, we provide an optimal condition on the modulating coefficient to establish the Hölder regularity and Harnack inequality for quasiminimizers of the generalized double phase functional with (G, H)-growth for two Young functions G and H.

1. Introduction

There have been systematic and extensive research activities on the variational problems with nonstandard growth. In particular, functionals whose structure exhibits a phase transition have attracted increasing attention over the last couple of decades. These functionals intervene in the homogenization of strongly anisotropic materials [Zhikov 1986; Zhikov et al. 1994] and in the Lavrentiev phenomenon [Zhikov 1993; 1995]. In this paper, we are concerned with the functionals of the type

$$v \in W^{1,1}(\Omega) \mapsto \mathcal{F}(v, \Omega) := \int_\Omega [G(|Dv|) + a(x)H(|Dv|)]\,dx, \tag{1-1}$$

where $G, H : [0, \infty) \to [0, \infty)$ are Young functions satisfying a suitable gap condition, see (2-24), $a : \Omega \to [0, \infty)$ is a continuous function, and Ω is a bounded domain in $\mathbb{R}^n$ with $n \geq 2$.

The main feature of the functional (1-1) is that the energy density changes its growth and ellipticity properties according to the modulating coefficient $a(\cdot)$. The double phase functional (1-1) is a natural generalization of the one with (p, q)-type

$$v \in W^{1,1}(\Omega) \mapsto \int_\Omega [|Dv|^p + a(x)|Dv|^q]\,dx, \quad q > p > 1, \tag{1-2}$$

and the one in a borderline case

$$v \in W^{1,1}(\Omega) \mapsto \int_\Omega [|Dv|^p + a(x)|Dv|^p \ln(1 + |Dv|)]\,dx, \quad p > 1. \tag{1-3}$$

Zhikov [1986; 1994] first introduced a family of functionals including (1-2) for the purpose of describing a feature of strongly anisotropic materials: the modulating coefficient $a(\cdot)$ presents the geometry of the mixture of two different materials. As shown in [Esposito et al. 2004; Fonseca et al. 2004; Zhikov 1995; 1997], such functionals exhibit Lavrentiev phenomenon whereby minimizers are irregular and even

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discontinuous. On the other hand, the functionals (1-2) and (1-3) belong to the class of functionals having (p, q)-growth condition. These are functionals of the type

$$v \in W^{1,1}(\Omega) \mapsto \int_\Omega F(x, Dv)\,dx, \tag{1-4}$$

where the energy density $F(x, \xi)$ satisfies

$$|\xi|^p \lesssim F(x, \xi) \lesssim |\xi|^q + 1, \quad q > p > 1. \tag{1-5}$$

This (p, q)-growth condition was first treated by Marcellini [1986; 1989; 1991] and extensively studied in recent years; see [Breit 2012; Esposito et al. 1999; 2002; 2004; Fonseca et al. 2004; Fusco and Sbordone 1990; Schmidt 2008; 2009].

In the case $p > n$, it is clear from the Sobolev embedding theorem that quasiminimizers of the functionals (1-2) and (1-3) are locally bounded and Hölder continuous. Recently, Baroni, Colombo and Mingione [Baroni et al. 2015a; Colombo and Mingione 2015a; 2015b] found that when $p \le n$, the optimal condition for Hölder continuity of quasiminimizers of the functional (1-2) is $a(\cdot) \in C^{0,\alpha}(\Omega)$, with $\alpha \in (0, 1]$ and $q \le p + \alpha$. For the functional (1-3), the log-Hölder continuity of $a(\cdot)$ is sufficient in order to obtain the Hölder continuity of quasiminimizers; see [Baroni et al. 2015a; 2015b]. These results show that the regularity of the modulating coefficient $a(\cdot)$ is closely related to how to control the size of the associated phase transition. In addition, $C^{1,\beta}$-regularity results for minimizers of the double phase functionals (1-2) and (1-3) have been obtained in [Baroni et al. 2015b; 2018; Colombo and Mingione 2015a; 2015b] and the regularity of the modulating coefficient is directly linked to the gap between two phases. For further regularity results including $C^{0,1}$-regularity for minimizers of functionals with general (p, q)-growth, we refer the reader to [Beck and Mingione 2018; Cupini et al. 2017; 2018; Esposito et al. 2006].

The main object of this paper is to investigate an optimal condition on the modulating coefficient $a(\cdot)$ in the functional (1-1) under which the Hölder regularity result holds for local quasiminimizers. We provide a reasonable condition on the modulus of continuity of $a(\cdot)$, see (4-6), and prove local boundedness, Hölder continuity via De Giorgi's method and the Harnack inequality under this condition. Harjulehto, Hästö and Toivanen [Harjulehto et al. 2017] considered a general setting and developed a set of assumptions on the energy density. Some of the assumptions in [Harjulehto et al. 2017] are the same as ours in the setting of the double phase functionals, see Remark 3.3, but we introduce refined conditions on G and H, and prove that these are sharp conditions for the absence of the Lavrentiev phenomenon, see Theorem 3.1, which also yields the regularity of local quasiminimizers for the generalized double phase functionals. The results in [Harjulehto et al. 2017] and ours complement each other. We also remark that our condition agrees with the known one in the classical case, see Remark 3.2, and serves the natural assumption for the modulating coefficient in a wide variety of double phase functionals such as

$$v \in W^{1,1}(\Omega) \mapsto \int_\Omega [|Dv|^p + a(x)|Dv|^p[\ln(1 + |Dv|)]^\gamma]\,dx, \quad p > 1,\ \gamma > 0,$$

and

$$v \in W^{1,1}(\Omega) \mapsto \int_\Omega [|Dv|^p + a(x)|Dv|^p \ln\ln(e + |Dv|)]\,dx, \quad p > 1;$$

see Remark 4.13.

The method used in this paper is influenced by [Baroni et al. 2015b; 2018; Colombo and Mingione 2015a; 2015b]. For the Hölder continuity of quasiminimizers, we first derive a Caccioppoli-type inequality which is similar to the one that holds for the functional $v \mapsto \int_\Omega G(|Dv|)\,dx$ by using the condition (4-8) on the modulus of continuity of $a(\cdot)$. We then consider a sequence of nested and shrinking balls $\{B_{4^{-i}r_0}\}_{i=0}^{\infty}$ in order to control the oscillation of quasiminimizers along the sequence of balls. Here we should verify for each ball whether the condition (4-8) holds true. If this condition holds true for every ball, then we obtain the Hölder continuity of quasiminimizers. Otherwise, we reduce the oscillation until we reach the exit time for ball $B_{4^{-j}r_0}$, and then we use the existing regularity theory, see Lemma 4.11, for the frozen functional

$$v \in W^{1,1}(B_{4^{-j}r_0}) \mapsto \int_{B_{f^{-j}r_0}} [G(|Dv|) + a_0 H(|Dv|)]\,dx, \qquad a_0 = \sup_{B_{4^{-j}r_0}} a(\cdot).$$

For the proof of the Harnack inequality, we first deduce the weak Harnack inequality and the local sup-estimates under the assumption (4-8). Then we apply the exit-time argument as above to obtain the desired inequality.

This paper is organized as follows. In the next section, we introduce some background and investigate the gap conditions. Section 3 deals with the Lavrentiev phenomenon. In Section 4, we establish the local boundedness and the Hölder continuity for (1-1). Section 5 is devoted to proving the Harnack inequality.

2. Preliminaries

Notation. We start this section with introducing notation that will be used in this paper.

Let $B_\rho(y) = \{x \in \mathbb{R}^n : |x-y| < \rho\}$ be the open ball in $\mathbb{R}^n$ centered at $y \in \mathbb{R}^n$ with radius $\rho > 0$. If the center is clear in the context, we shall denote it by $B_\rho \equiv B_\rho(y)$.

For a function v, we write $v_\pm := \max\{\pm v, 0\}$.

For $k \in \mathbb{R}$, $\rho > 0$ and a quasiminimizer u of the functional $\mathcal{F}$, we set

$$A(k,\rho) := \{x \in B_\rho : u(x) > k\} \quad \text{and} \quad A^-(k,\rho) := \{x \in B_\rho : u(x) \le k\}.$$

Hereafter, for the sake of the convenience, we employ the letter c to denote any universal constants which can be explicitly computed in terms of known quantities, and so c might vary from line to line.

Orlicz spaces and Musielak–Orlicz spaces. A Young function $\Phi : [0,\infty) \to [0,\infty)$ is an increasing convex function satisfying

$$\Phi(0) = 0, \quad \lim_{t\to\infty} \Phi(t) = \infty, \quad \lim_{t\to 0+} \frac{\Phi(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty.$$

Definition 2.1. Let Φ be a Young function:

(1) Φ is said to satisfy the Δ_2-condition, denoted by $\Phi \in \Delta_2$, if there exists a positive number $\Delta_2(\Phi)$ such that $\Phi(2t) \le \Delta_2(\Phi)\Phi(t)$ for all $t \ge 0$.

(2) Φ is said to satisfy the ∇_2-condition, denoted by $\Phi \in \nabla_2$, if there exists a positive number $\nabla_2(\Phi) > 1$ such that $\Phi(\nabla_2(\Phi)t) \ge 2\nabla_2(\Phi)\Phi(t)$ for all $t \ge 0$.

(3) We write $\Phi \in \Delta_2 \cap \nabla_2$ if $\Phi \in \Delta_2$ and $\Phi \in \nabla_2$.

We note that if $\Phi \in \Delta_2$, then $\Delta_2(\Phi) > 2$. Indeed, by the convexity of Φ, we get

$$\Phi(2t) \le \Delta_2(\Phi)\Phi(t) \le \frac{\Delta_2(\Phi)}{2}\Phi(2t) \quad \text{for all } t \ge 0, \tag{2-1}$$

and hence $\Delta_2(\Phi) \ge 2$. If $\Delta_2(\Phi) = 2$, then it follows from (2-1) that $\Phi(2t) = 2\Phi(t)$ for all $t \ge 0$, and so $\Phi(t) \equiv \Phi(1)t$ is not a Young function. Thus $\Delta_2(\Phi) > 2$.

For a given Young function Φ, we define the complementary Young function Φ^* of Φ by

$$\Phi^*(t) = \sup\{st - \Phi(s) : s \ge 0\}.$$

We remark that Φ^* satisfies all the conditions to be a Young function and that $(\Phi^*)^* = \Phi$. Moreover, $\Phi \in \nabla_2$ if and only if $\Phi^* \in \Delta_2$ with $2\nabla_2(\Phi) = \Delta_2(\Phi^*)$.

We will use the following basic properties of Young functions satisfying Δ_2 and ∇_2 conditions; see for instance [Adams and Fournier 2003; Ok 2016; Rao and Ren 1991].

Lemma 2.2. *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$:*

(1) *For any $1 \le \Lambda < \infty$ and $t \ge 0$, we have*

$$\Phi(\Lambda t) \le \Delta_2(\Phi)\Lambda^{\log_2 \Delta_2(\Phi)}\Phi(t). \tag{2-2}$$

(2) *For any $0 < \lambda \le 1$ and $t \ge 0$, we have*

$$\Phi(\lambda t) \le 2\nabla_2(\Phi)\lambda^{1+\log_{\nabla_2(\Phi)} 2}\Phi(t). \tag{2-3}$$

(3) (*Young's inequality*) *For any $\varepsilon \in (0, 1]$, there exists a positive constant c depending only on $\Delta_2(\Phi)$, $\nabla_2(\Phi)$ and ε such that*

$$st \le \varepsilon\Phi(s) + c\Phi^*(t) \quad \text{for all } s, t \ge 0. \tag{2-4}$$

(4) *If $\Phi \in C^1([0, \infty))$, then for any $t \ge 0$, we have*

$$c_1^{-1}\Phi(t) \le t\Phi'(t) \le c_1\Phi(t) \tag{2-5}$$

and

$$\Phi^*(\Phi'(t)) \le c_2\Phi(t) \tag{2-6}$$

for some constants $c_1, c_2 > 1$ depending only on $\Delta_2(\Phi)$ and $\nabla_2(\Phi)$.

(5) (*a modified form of Young's inequality*) *If $\Phi \in C^1([0, \infty))$, then for any $\varepsilon \in (0, 1]$, there exists a positive constant c depending only on $\Delta_2(\Phi)$, $\nabla_2(\Phi)$ and ε such that*

$$s\Phi'(t) \le \varepsilon\Phi(s) + c\Phi(t) \quad \text{for all } s, t \ge 0. \tag{2-7}$$

For a Young function Φ, the Orlicz class $K^\Phi(\Omega; \mathbb{R}^N)$, $N \in \mathbb{N}$, consists of all measurable functions $v : \Omega \to \mathbb{R}^N$ satisfying

$$\int_\Omega \Phi(|v(x)|)\,dx < +\infty.$$

The Orlicz space $L^{\Phi}(\Omega;\mathbb{R}^N)$ is the vector space generated by the Orlicz class $K^{\Phi}(\Omega;\mathbb{R}^N)$. If $\Phi\in\Delta_2$, then $K^{\Phi}(\Omega;\mathbb{R}^N)=L^{\Phi}(\Omega;\mathbb{R}^N)$ and this space is a Banach space under the Luxemburg norm

$$\|v\|_{L^{\Phi}(\Omega;\mathbb{R}^N)}=\inf\left\{\sigma>0:\int_\Omega\Phi\left(\frac{|v(x)|}{\sigma}\right)dx\le 1\right\}.$$

For $N=1$, we simply write $L^{\Phi}(\Omega):=L^{\Phi}(\Omega;\mathbb{R})$.

We state some relevant inequalities regarding the Luxemburg norm; see [Rao and Ren 1991].

Lemma 2.3. *Let Φ be a Young function with $\Phi\in\Delta_2$:*

(1) $\|v\|_{L^{\Phi}(\Omega;\mathbb{R}^N)}\le 1 \implies \int_\Omega\Phi(|v|)\,dx\le\|v\|_{L^{\Phi}(\Omega;\mathbb{R}^N)}$.

(2) $\|v\|_{L^{\Phi}(\Omega;\mathbb{R}^N)}\ge 1 \implies \int_\Omega\Phi(|v|)\,dx\ge\|v\|_{L^{\Phi}(\Omega;\mathbb{R}^N)}$.

(3) $\|v\|_{L^{\Phi}(\Omega;\mathbb{R}^N)}\le 1 \iff \int_\Omega\Phi(|v|)\,dx\le 1$.

(4) $0<\|v\|_{L^{\Phi}(\Omega;\mathbb{R}^N)}<\infty \implies \int_\Omega\Phi\left(\frac{|v|}{\|v\|_{L^{\Phi}(\Omega;\mathbb{R}^N)}}\right)dx=1$.

(5) (*Hölder's inequality*) *For any $v\in L^{\Phi}(\Omega)$ and $w\in L^{\Phi^*}(\Omega)$,*

$$\int_\Omega|vw|\,dx\le 2\|v\|_{L^{\Phi}(\Omega)}\|w\|_{L^{\Phi^*}(\Omega)}. \tag{2-8}$$

We now introduce a partial order relation between Young functions, see [Verde 2011], and present a series of lemmas which will be used frequently throughout the paper.

Definition 2.4. Let Φ_1, Φ_2 be Young functions. We shall write

$$\Phi_1\prec\Phi_2$$

if $\Phi_2\circ\Phi_1^{-1}$ is a Young function.

Lemma 2.5. *Let Φ_1, Φ_2 be Young functions with $\Phi_1\prec\Phi_2$. Then*

$$\Phi_1(t)\le\frac{1}{(\Phi_2\circ\Phi_1^{-1})(1)}\Phi_2(t)\quad\text{for all } t\ge\Phi_1^{-1}(1). \tag{2-9}$$

Proof. We first note that for a Young function Φ, there holds

$$\Phi(1)s\le\Phi(s)\quad\text{for all } s\ge 1.$$

Indeed, this follows from the convexity of Φ. Since $\Phi_1\prec\Phi_2$, we have

$$(\Phi_2\circ\Phi_1^{-1})(1)s\le(\Phi_2\circ\Phi_1^{-1})(s)\quad\text{for all } s\ge 1.$$

Setting $t=\Phi_1^{-1}(s)$, we obtain the desired conclusion (2-9). □

Corollary 2.6. *Let Φ_1, Φ_2 be Young functions with $\Phi_1 \prec \Phi_2$. Then*

$$\Phi_1(t) \le c(\Phi_2(t)+1) \quad \text{for all } t \ge 0, \tag{2-10}$$

where c is a positive constant depending only on Φ_1 and Φ_2.

Lemma 2.7. *Let Φ_1, Φ_2 be Young functions with $\Phi_1 \prec \Phi_2$. Then the function*

$$t \mapsto \left(\frac{\Phi_2}{\Phi_1}\right)(t) = \frac{\Phi_2(t)}{\Phi_1(t)}$$

is nondecreasing.

Proof. We first note that the function Φ_2/Φ_1 is nondecreasing if and only if the function $(\Phi_2/\Phi_1)\circ\Phi_1^{-1}$ is nondecreasing, as $t \mapsto \Phi_1(t)$ is increasing and continuous. Since $\Phi_1 \prec \Phi_2$, we see that $\Phi_2 \circ \Phi_1^{-1}$ is a Young function. Hence, it follows from the convexity of $\Phi_2 \circ \Phi_1^{-1}$ that the function

$$t \mapsto \left(\frac{\Phi_2}{\Phi_1}\circ\Phi_1^{-1}\right)(t) = \frac{(\Phi_2\circ\Phi_1^{-1})(t)}{t}$$

is nondecreasing. □

The following lemma and its proof can be found in [Lieberman 1991; Rao and Ren 1991, Chapter II].

Lemma 2.8. *Let $\Phi \in C^1([0,\infty)) \cap C^2((0,\infty))$ be a Young function satisfying*

$$\frac{1}{c_\Phi} \le \frac{t\Phi''(t)}{\Phi'(t)} \le c_\Phi \quad \text{for all } t > 0, \tag{2-11}$$

for some $c_\Phi \ge 1$. Then:

(1) *$\Phi \in \Delta_2 \cap \nabla_2$, and the constants $\Delta_2(\Phi)$, $\nabla_2(\Phi)$ depend only on c_Φ.*

(2) *For any $1 \le \Lambda < \infty$ and $t \ge 0$, we have*

$$\Phi(\Lambda t) \le \Lambda^{c_\Phi+1}\Phi(t). \tag{2-12}$$

(3) *For any $0 < \lambda \le 1$ and $t \ge 0$, we have*

$$\Phi(\lambda t) \le \lambda^{(1/c_\Phi)+1}\Phi(t). \tag{2-13}$$

Lemma 2.9. *Let Φ be a Young function with $\Phi \in C^1([0,\infty)) \cap C^2((0,\infty))$. If*

$$\frac{t\Phi''(t)}{\Phi'(t)} \le c_\Phi \quad \text{for all } t > 0,$$

for some $c_\Phi \ge 1$, then $t \mapsto \Phi(t^{1/(1+c_\Phi)})$ is a concave function.

Proof. Set $\varphi(t) := \Phi(t^{1/(1+c_\Phi)})$ for $t \ge 0$. Then we have

$$\varphi'(t) = \frac{1}{1+c_\Phi}\Phi'(t^{1/(1+c_\Phi)})\,t^{-c_\Phi/(1+c_\Phi)},$$

and hence

$$\begin{aligned}\varphi''(t) &= \frac{1}{(1+c_\Phi)^2}\Phi''(t^{1/(1+c_\Phi)})(t^{-c_\Phi/(1+c_\Phi)})^2 - \frac{c_\Phi}{(1+c_\Phi)^2}\Phi'(t^{1/(1+c_\Phi)})t^{-c_\Phi/(1+c_\Phi)-1} \\ &= \frac{1}{(1+c_\Phi)^2}t^{-c_\Phi/(1+c_\Phi)-1}[t^{1/(1+c_\Phi)}\Phi''(t^{1/(1+c_\Phi)}) - c_\Phi\Phi'(t^{1/(1+c_\Phi)})] \le 0\end{aligned}$$

for all $t > 0$. □

We now introduce the Musielak–Orlicz spaces which generalize the Orlicz spaces. Let $\Phi : \Omega\times[0, \infty) \to [0, \infty)$ be a function satisfying the following conditions:

(1) $\Phi(x, \cdot)$ is a Young function for every $x \in \Omega$.

(2) $\Phi(\cdot, t)$ is a measurable function for every $t \ge 0$.

Such a function $\Phi(x, t)$ is called a Musielak–Orlicz function. As before, we present some definitions and properties regarding Musielak–Orlicz functions.

Definition 2.10. Let Φ be a Musielak–Orlicz function:

(1) Φ is said to satisfy the Δ_2-condition, denoted by $\Phi \in \Delta_2$, if there exists a positive number $\Delta_2(\Phi)$ such that $\Phi(x, 2t) \le \Delta_2(\Phi)\Phi(x, t)$ for all $x \in \Omega$ and $t \ge 0$.

(2) Φ is said to satisfy the ∇_2-condition, denoted by $\Phi \in \nabla_2$, if there exists a positive number $\nabla_2(\Phi) > 1$ such that $\Phi(x, \nabla_2(\Phi)t) \ge 2\nabla_2(\Phi)\Phi(x, t)$ for all $x \in \Omega$ and $t \ge 0$.

(3) We write $\Phi \in \Delta_2 \cap \nabla_2$ if $\Phi \in \Delta_2$ and $\Phi \in \nabla_2$.

For a given Musielak–Orlicz function Φ, we define the complementary Φ^* of Φ by, for each $x \in \Omega$,

$$\Phi^*(x, t) = \sup\{st - \Phi(x, s) : s \ge 0\}.$$

Then Φ^* satisfies all the conditions to be a Musielak–Orlicz function. Also we note that $(\Phi^*)^* = \Phi$ and that $\Phi \in \nabla_2$ if and only if $\Phi^* \in \Delta_2$ with $2\nabla_2(\Phi) = \Delta_2(\Phi^*)$.

The following lemma can be directly obtained from the definitions of Δ_2-condition, ∇_2-condition and the complementary of Musielak–Orlicz function.

Lemma 2.11. *Let Φ be a Musielak–Orlicz function with $\Phi \in \Delta_2 \cap \nabla_2$:*

(1) *For any $1 \le \Lambda < \infty$, $t \ge 0$ and $x \in \Omega$, we have*

$$\Phi(x, \Lambda t) \le \Delta_2(\Phi)\Lambda^{\log_2 \Delta_2(\Phi)}\Phi(x, t). \tag{2-14}$$

(2) *For any $0 < \lambda \le 1$, $t \ge 0$ and $x \in \Omega$, we have*

$$\Phi(x, \lambda t) \le 2\nabla_2(\Phi)\lambda^{1+\log_{\nabla_2(\Phi)} 2}\Phi(x, t). \tag{2-15}$$

(3) (*Young's inequality*) *For any $\varepsilon \in (0, 1]$, there exists a positive constant c depending only on $\Delta_2(\Phi)$, $\nabla_2(\Phi)$ and ε such that*

$$st \le \varepsilon\Phi(x, s) + c\Phi^*(x, t) \tag{2-16}$$

for all $s, t \ge 0$ and $x \in \Omega$.

For a Musielak–Orlicz function Φ, the Musielak–Orlicz class $K^{\Phi}(\Omega;\mathbb{R}^N)$, $N\in\mathbb{N}$, consists of all measurable functions $v:\Omega\to\mathbb{R}^N$ satisfying

$$\int_\Omega \Phi(x,|v(x)|)\,dx<+\infty.$$

The Musielak–Orlicz space $L^{\Phi}(\Omega;\mathbb{R}^N)$ is the vector space generated by $K^{\Phi}(\Omega;\mathbb{R}^N)$. If $\Phi\in\Delta_2$, then $K^{\Phi}(\Omega;\mathbb{R}^N)=L^{\Phi}(\Omega;\mathbb{R}^N)$ and this space is a Banach space under the Luxemburg norm

$$\|v\|_{L^{\Phi}(\Omega;\mathbb{R}^N)}=\inf\left\{\sigma>0:\int_\Omega \Phi\left(x,\frac{|v(x)|}{\sigma}\right)dx\le 1\right\}.$$

The Musielak–Orlicz–Sobolev space $W^{1,\Phi}(\Omega;\mathbb{R}^N)$ is the function space of all measurable functions $v\in L^{\Phi}(\Omega;\mathbb{R}^N)$ such that its distributional gradient vector Dv belongs to $L^{\Phi}(\Omega;\mathbb{R}^{Nn})$. For $v\in W^{1,\Phi}(\Omega;\mathbb{R}^N)$, we define its norm to be

$$\|v\|_{W^{1,\Phi}(\Omega;\mathbb{R}^N)}=\|v\|_{L^{\Phi}(\Omega;\mathbb{R}^N)}+\|Dv\|_{L^{\Phi}(\Omega;\mathbb{R}^{Nn})}.$$

The space $W_0^{1,\Phi}(\Omega;\mathbb{R}^N)$ is defined as the closure of $C_0^\infty(\Omega;\mathbb{R}^N)$ in $W^{1,\Phi}(\Omega;\mathbb{R}^N)$. For $N=1$, we simply write $L^{\Phi}(\Omega):=L^{\Phi}(\Omega;\mathbb{R})$ and $W^{1,\Phi}(\Omega):=W^{1,\Phi}(\Omega;\mathbb{R})$. For a detailed discussion of the Musielak–Orlicz space and the associated Sobolev space, we refer the reader to [Benkirane and Sidi El Vally 2014; Diening 2005; Fan 2012; Fan and Guan 2010; Harjulehto et al. 2016; Musielak 1983; Sidi El Vally 2013].

Gap conditions. We now consider the double phase functional

$$\mathcal{F}(v,\Omega)=\int_\Omega [G(|Dv|)+a(x)H(|Dv|)]\,dx,\qquad v\in W^{1,1}(\Omega),$$

and investigate gap conditions on two Young functions G and H.

In the rest of the paper we shall use the notation

$$\Psi(x,\xi):=G(|\xi|)+a(x)H(|\xi|), \tag{2-17}$$

when $x\in\Omega$ and $\xi\in\mathbb{R}^n$. By abuse of notation, we will continue to write $\Psi(x,\xi)$ also when $x\in\Omega$ and $\xi\in\mathbb{R}$.

Proposition 2.12. *Let G, $H:[0,\infty)\to[0,\infty)$ be Young functions. Suppose that the function $a=a(\cdot):\Omega\to[0,\infty)$ has a modulus of continuity ω satisfying*

$$\limsup_{\rho\to 0+}\omega(\rho)\frac{(H\circ G^{-1})(\rho^{-n})}{\rho^{-n}}<\infty. \tag{2-18}$$

If $H\succ G^\kappa$ for some $\kappa>1+1/n$, then $a(\cdot)$ is a constant function.

Proof. It follows from the condition (2-18) that there exists a constant $L>0$ such that

$$\omega(\rho)\frac{(H\circ G^{-1})(\rho^{-n})}{\rho^{-n}}\le L$$

for all $0 < \rho \le 1$. Since $H \succ G^\kappa$, we have

$$\omega(\rho)\frac{(G^\kappa \circ G^{-1})(\rho^{-n})}{\rho^{-n}} \le c\omega(\rho)\frac{(H \circ G^{-1})(\rho^{-n})}{\rho^{-n}} \le cL \tag{2-19}$$

for all small $\rho > 0$. Here, we see that

$$\omega(\rho)\frac{(G^\kappa \circ G^{-1})(\rho^{-n})}{\rho^{-n}} = \omega(\rho)\frac{[(G \circ G^{-1})(\rho^{-n})]^\kappa}{\rho^{-n}} = \omega(\rho)\rho^{-n(\kappa-1)}. \tag{2-20}$$

Combining (2-19) with (2-20) yields

$$\omega(\rho) \le cL\rho^{n(\kappa-1)} \quad \text{for all } \rho \le \rho_0, \tag{2-21}$$

for some small $\rho_0 > 0$. Then we conclude from the definition of the modulus of continuity that

$$\frac{|a(x) - a(y)|}{|x - y|} \le cL|x - y|^{n(\kappa-1)-1} \tag{2-22}$$

for every $x, y \in \Omega$ with $0 < |x - y| \le \rho_0$. Since $n(\kappa - 1) - 1 > 0$, it follows immediately that $a(\cdot)$ is a constant function. □

Proposition 2.13. *Let $G, H : [0, \infty) \to [0, \infty)$ be Young functions. Suppose that the function $a = a(\cdot) : \Omega \to [0, \infty)$ has a modulus of continuity ω satisfying*

$$\limsup_{\rho \to 0+} \omega(\rho)\frac{H(\rho^{-1})}{G(\rho^{-1})} < \infty. \tag{2-23}$$

If $H \succ G^\kappa$ for some $\kappa > 2$, then $a(\cdot)$ is a constant function.

Proof. It follows from the condition (2-23) that there exists a constant $L > 0$ such that

$$\omega(\rho)\frac{H(\rho^{-1})}{G(\rho^{-1})} \le L$$

for all $0 < \rho \le 1$. We note from the convexity of G that

$$G(1)s \le G(s) \quad \text{for all } s \ge 1.$$

Since $H \succ G^\kappa$, we get

$$\omega(\rho)\rho^{-(\kappa-1)} \le c\omega(\rho)[G(\rho^{-1})]^{\kappa-1} = c\omega(\rho)\frac{[G(\rho^{-1})]^\kappa}{G(\rho^{-1})} \le c\omega(\rho)\frac{H(\rho^{-1})}{G(\rho^{-1})} \le cL$$

for all small $\rho > 0$. As in the previous proof, we conclude that $a(\cdot)$ is a constant function if $\kappa > 2$. □

Remark 2.14. If $G(t) \succ t^n$, then it follows from Lemmas 2.5 and 2.7 that

$$\frac{(H \circ G^{-1})(\rho^{-n})}{\rho^{-n}} = \left(\frac{H}{G}\right)(G^{-1}(\rho^{-n})) \le \left(\frac{H}{G}\right)(c\rho^{-1}) \le c\frac{H(\rho^{-1})}{G(\rho^{-1})},$$

and hence the condition (2-23) implies (2-18). On the contrary, if $G(t) \prec t^n$, then

$$\frac{H(\rho^{-1})}{G(\rho^{-1})} = \left(\frac{H}{G}\right)(\rho^{-1}) \le \left(\frac{H}{G}\right)(cG^{-1}(\rho^{-n})) \le c\frac{(H \circ G^{-1})(\rho^{-n})}{\rho^{-n}},$$

and consequently the condition (2-18) implies (2-23). These agree with the known results in the classical case; see Remark 3.2 below.

From this point of view, we shall assume that $G, H : [0,\infty) \to [0,\infty)$ are Young functions with $G, H \in \Delta_2 \cap \nabla_2$ and

$$G \prec H \prec G^{1+1/n}. \tag{2-24}$$

We remark that $\Psi \in \Delta_2 \cap \nabla_2$. To get regularity results, we shall concentrate on nice Young functions, or the N-functions. Thus we further assume that $G, H \in C^1([0,\infty)) \cap C^2((0,\infty))$ and there exist constants $c_G, c_H \geq 1$ such that

$$\frac{1}{c_G} \leq \frac{tG''(t)}{G'(t)} \leq c_G \quad \text{and} \quad \frac{1}{c_H} \leq \frac{tH''(t)}{H'(t)} \leq c_H \tag{2-25}$$

hold for all $t > 0$.

3. Lavrentiev phenomenon

When considering the functionals of the type

$$v \in W^{1,1}(\Omega) \mapsto \int_\Omega F(x, Dv)\,dx,$$

with

$$G(|\xi|) \lesssim F(x,\xi) \lesssim H(|\xi|) + 1, \quad G \prec H,$$

the Lavrentiev phenomenon

$$\inf_{v \in W^{1,G}(\Omega)} \int_\Omega F(x, Dv)\,dx < \inf_{v \in W^{1,H}(\Omega)} \int_\Omega F(x, Dv)\,dx$$

may occur. However, for the functional $\mathcal{F}$ defined in (1-1), there is no Lavrentiev phenomenon under a suitable condition on the modulating coefficient $a(\cdot)$.

Theorem 3.1. *Let $\mathcal{F}$ be the functional defined in* (1-1):

(1) *If the modulating coefficient $a(\cdot)$ has a modulus of continuity ω satisfying*

$$\limsup_{\rho \to 0+} \omega(\rho) \frac{(H \circ G^{-1})(\rho^{-n})}{\rho^{-n}} < \infty, \tag{3-1}$$

then for every function $v \in W^{1,1}_{\text{loc}}(\Omega)$ and balls $B \Subset \widetilde{B} \Subset \Omega$ with $\mathcal{F}(v, \widetilde{B}) < \infty$, there exists a sequence $\{v_k\} \subset W^{1,\infty}(B)$ such that

$$v_k \to v \quad \text{in } W^{1,G}(B) \qquad \text{and} \qquad \mathcal{F}(v_k, B) \to \mathcal{F}(v, B). \tag{3-2}$$

(2) *If the modulating coefficient $a(\cdot)$ has a modulus of continuity ω satisfying*

$$\limsup_{\rho \to 0+} \omega(\rho) \frac{H(\rho^{-1})}{G(\rho^{-1})} < \infty, \tag{3-3}$$

then for every function $v \in W^{1,1}_{\mathrm{loc}}(\Omega) \cap L^{\infty}_{\mathrm{loc}}(\Omega)$ and balls $B \Subset \widetilde{B} \Subset \Omega$ with $\mathcal{F}(v, \widetilde{B}) < \infty$, there exists a sequence $\{v_k\} \subset W^{1,\infty}(B)$ such that

$$v_k \to v \quad \textit{in } W^{1,G}(B) \qquad \textit{and} \qquad \mathcal{F}(v_k, B) \to \mathcal{F}(v, B). \tag{3-4}$$

Proof. Let $R > 0$ be the radius of the ball B. Take $\varepsilon_0 \in (0, 1)$ in such a way that $B \equiv B_R \Subset B_{R+\varepsilon_0} \Subset \widetilde{B} \Subset \Omega$. Let $\varphi \in C_0^{\infty}(B_1)$ be a mollifier with $\varphi \geq 0$, $\int_{\mathbb{R}^n} \varphi\, dx = 1$, and set

$$\varphi_\varepsilon(x) := \frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right)$$

for $x \in B_\varepsilon$ with $\varepsilon > 0$. Then it is obvious that $\varphi_\varepsilon \in C_0^{\infty}(B_\varepsilon)$, $\int_{\mathbb{R}^n} \varphi_\varepsilon\, dx = 1$, $0 \leq \varphi_\varepsilon \leq c(n)\varepsilon^{-n}$ and $|D\varphi_\varepsilon| \leq c(n)\varepsilon^{-(n+1)}$. Now we define, for $0 < \varepsilon < \varepsilon_0$,

$$v_\varepsilon(x) := (v * \varphi_\varepsilon)(x), \quad a_\varepsilon(x) := \inf_{y \in B_\varepsilon(x)} a(y), \quad \Psi_\varepsilon(x, \xi) := G(|\xi|) + a_\varepsilon(x) H(|\xi|)$$

for $x \in B_R$ and $\xi \in \mathbb{R}^n$.

(1) It follows from Jensen's inequality that

$$G(|Dv_\varepsilon(x)|) = G(|Dv * \varphi_\varepsilon(x)|) \leq \int_{\mathbb{R}^n} G(|Dv(x-y)|)\varphi_\varepsilon(y)\, dy \leq c\varepsilon^{-n}$$

for every $x \in B_R$. By the definitions of $a_\varepsilon(\,\cdot\,)$, we obtain

$$\begin{aligned}\Psi(x, Dv_\varepsilon(x)) &\leq |a(x) - a_\varepsilon(x)| H(|Dv_\varepsilon(x)|) + \Psi_\varepsilon(x, Dv_\varepsilon(x)) \\ &\leq c\omega(\varepsilon) H(|Dv_\varepsilon(x)|) + \Psi_\varepsilon(x, Dv_\varepsilon(x)).\end{aligned}$$

We now observe from Lemmas 2.2 and 2.7 that

$$\begin{aligned}H(|Dv_\varepsilon(x)|) &= \left(\frac{H}{G}\right)(|Dv_\varepsilon(x)|) G(|Dv_\varepsilon(x)|) \\ &\leq \left(\frac{H}{G}\right)(G^{-1}(c\varepsilon^{-n})) G(|Dv_\varepsilon(x)|) = \frac{(H \circ G^{-1})(c\varepsilon^{-n})}{c\varepsilon^{-n}} G(|Dv_\varepsilon(x)|) \\ &\leq c\frac{(H \circ G^{-1})(\varepsilon^{-n})}{\varepsilon^{-n}} G(|Dv_\varepsilon(x)|) \leq c\frac{(H \circ G^{-1})(\varepsilon^{-n})}{\varepsilon^{-n}} \Psi_\varepsilon(x, Dv_\varepsilon(x)).\end{aligned}$$

Therefore, we see from (3-1) that

$$\begin{aligned}\Psi(x, Dv_\varepsilon(x)) &\leq c\omega(\varepsilon)\frac{(H \circ G^{-1})(\varepsilon^{-n})}{\varepsilon^{-n}} \Psi_\varepsilon(x, Dv_\varepsilon(x)) + \Psi_\varepsilon(x, Dv_\varepsilon(x)) \\ &\leq c\Psi_\varepsilon(x, Dv_\varepsilon(x)).\end{aligned} \tag{3-5}$$

By Jensen's inequality, we have

$$\begin{aligned}\Psi_\varepsilon(x, Dv_\varepsilon(x)) &\leq \int_{B_\varepsilon(x)} \Psi_\varepsilon(x, Dv(y))\varphi_\varepsilon(x-y)\, dy \leq \int_{B_\varepsilon(x)} \Psi(y, Dv(y))\varphi_\varepsilon(x-y)\, dy \\ &= [\Psi(\,\cdot\,, Dv(\,\cdot\,)) * \varphi_\varepsilon](x) =: [\Psi(\,\cdot\,, Dv(\,\cdot\,))]_\varepsilon(x).\end{aligned} \tag{3-6}$$

Combining (3-5) and (3-6), we deduce that

$$\Psi(x, Dv_\varepsilon(x)) \le c[\Psi(\,\cdot\,, Dv(\,\cdot\,))]_\varepsilon(x). \tag{3-7}$$

Using the fact that $[\Psi(\,\cdot\,, Dv(\,\cdot\,))]_\varepsilon \to \Psi(\,\cdot\,, Dv(\,\cdot\,))$ strongly in $L^1(B_R)$, we can apply a generalized version of the Lebesgue dominated convergence theorem to obtain a sequence of functions $\{v_k\} := \{v_{\varepsilon_k}\} \subset C_0^\infty(B_R)$ satisfying (3-2) for a suitable sequence $\varepsilon_k \to 0$.

(2) Since v is locally bounded in Ω, we have

$$\begin{aligned}|Dv_\varepsilon(x)| = |v * D\varphi_\varepsilon(x)| \le \int_{\mathbb{R}^n} |v(x-y)|\,|D\varphi_\varepsilon(y)|\,dy &\le \|v\|_{L^\infty(\widetilde{B})} \int_{B_\varepsilon} |D\varphi_\varepsilon(y)|\,dy \\ &\le \|v\|_{L^\infty(\widetilde{B})} c(n)\varepsilon^{-(n+1)}|B_\varepsilon| \le c\varepsilon^{-1}\end{aligned}$$

for every $x \in B_R$. Then we obtain from Lemmas 2.2 and 2.7 that

$$\begin{aligned}H(|Dv_\varepsilon(x)|) &= \left(\frac{H}{G}\right)(|Dv_\varepsilon(x)|)G(|Dv_\varepsilon(x)|) \\ &\le \left(\frac{H}{G}\right)(c\varepsilon^{-1})G(|Dv_\varepsilon(x)|) = \frac{H(c\varepsilon^{-1})}{G(c\varepsilon^{-1})}G(|Dv_\varepsilon(x)|) \\ &\le c\frac{H(\varepsilon^{-1})}{G(\varepsilon^{-1})}G(|Dv_\varepsilon(x)|) \le c\frac{H(\varepsilon^{-1})}{G(\varepsilon^{-1})}\Psi_\varepsilon(x, Dv_\varepsilon(x)).\end{aligned}$$

As in the proof of (1), it follows from (3-3) and (3-6) that

$$\begin{aligned}\Psi(x, Dv_\varepsilon(x)) &\le c\omega(\varepsilon)H(|Dv_\varepsilon(x)|) + \Psi_\varepsilon(x, Dv_\varepsilon(x)) \\ &\le c\omega(\varepsilon)\frac{H(\varepsilon^{-1})}{G(\varepsilon^{-1})}\Psi_\varepsilon(x, Dv_\varepsilon(x)) + \Psi_\varepsilon(x, Dv_\varepsilon(x)) \\ &\le c\Psi_\varepsilon(x, Dv_\varepsilon(x)) \le c[\Psi(\,\cdot\,, Dv(\,\cdot\,))]_\varepsilon(x).\end{aligned}$$

Again, by a generalized version of the Lebesgue dominated convergence theorem, we get a sequence of functions $\{v_k\} := \{v_{\varepsilon_k}\} \subset C_0^\infty(B_R)$ satisfying (3-4) for a suitable sequence $\varepsilon_k \to 0$. □

Remark 3.2. In the special case $(G(t), H(t)) = (t^p, t^q)$ with $1 < p < q$, and $a(\,\cdot\,) \in C^{0,\alpha}(\Omega)$ with $\alpha \in (0, 1]$, a simple computation shows that

$$\text{the condition (3-1)} \quad \Longleftrightarrow \quad \frac{q}{p} \le 1 + \frac{\alpha}{n},$$

and

$$\text{the condition (3-3)} \quad \Longleftrightarrow \quad q \le p + \alpha.$$

Therefore, Theorem 3.1 generalizes [Colombo and Mingione 2015a, Proposition 3.6; 2015b, Theorem 4.1]. In addition, as in Remark 2.14 and [Colombo and Mingione 2015b], one can check that the condition (3-3) implies the condition (3-1) if $G(t) \succ t^n$, and that the condition (3-1) implies the condition (3-3) if $G(t) \prec t^n$.

Moreover, in the case $(G(t), H(t)) = (t^p, t^p \ln(1+t))$ with $p > 1$, we see that the condition (3-1) and the condition (3-3) are equivalent to

$$\limsup_{\rho\to 0+} \omega(\rho) \ln\left(\frac{1}{\rho}\right) < \infty.$$

This shows that when $a(\cdot)$ is log-Hölder continuous, the functional

$$v \in W^{1,1}(\Omega) \mapsto \int_\Omega [|Dv|^p + a(x)|Dv|^p \ln(1+|Dv|)]\,dx, \quad p > 1,$$

has no Lavrentiev phenomenon.

Remark 3.3. In the setting of the generalized double phase functionals, the conditions (A1) and (A1-n) in [Harjulehto et al. 2017] are same as the conditions (3-1) and (3-3), respectively. From this, it is to be expected that the functionals of the general type (1-4) satisfying the conditions introduced in [Harjulehto et al. 2017] have no Lavrentiev phenomenon.

Remark 3.4. The conditions in Theorem 3.1 are sharp for the absence of the Lavrentiev phenomenon. Indeed, for any ball $B \subset \Omega$, there exist Young functions G, H satisfying (2-24), a nonnegative coefficient $a(\cdot)$ which has a modulus of continuity ω satisfying

$$\lim_{\rho\to 0+} \omega(\rho) \frac{H(\rho^{-1})}{G(\rho^{-1})} = \infty \tag{3-8}$$

and a boundary datum $v_0 \in W^{1,G}(B) \cap L^\infty(B)$ such that

$$\inf_{v\in v_0+W_0^{1,G}(B)} \mathcal{F}(v, B) < \inf_{v\in v_0+W_0^{1,G}(B)\cap W_{\mathrm{loc}}^{1,H}(B)} \mathcal{F}(v, B). \tag{3-9}$$

That is, local minimizers of $\mathcal{F}$ do not belong to $W^{1,H}_{\mathrm{loc}}(B)$ in general. Moreover, they can be discontinuous.

To see this, let us consider the classical case $G(t) = t^p$, $H(t) = t^q$ and $a(\cdot) \in C^{0,\alpha}(\Omega)$ with $1 < p < q$, $\alpha \in (0, 1]$ satisfying

$$1 < p < n < n+\alpha < q. \tag{3-10}$$

Then it follows from [Colombo and Mingione 2015b, Theorem 4.1; Esposito et al. 2004, Section 3] that there exists a coefficient function $a(\cdot) \in C^{0,\alpha}(\Omega)$ and a boundary datum $v_0 \in W^{1,p}(B) \cap L^\infty(B)$ such that the Lavrentiev phenomenon (3-9) occurs. Also we deduce from Remark 3.2 and (3-10) that the coefficient function $a(\cdot)$ has a modulus of continuity ω satisfying (3-8). Furthermore, the modulus of continuity ω does not satisfy the condition (3-1).

4. Local boundedness and Hölder continuity

In the following, we deal with local quasiminimizers of $\mathcal{F}$.

Definition 4.1. We say that $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is a local quasiminimizer of $\mathcal{F}$ for $Q \geq 1$, or a local Q-minimizer of $\mathcal{F}$, if for any $v \in W^{1,1}_{\mathrm{loc}}(\Omega)$ with $K := \operatorname{supp}(u-v) \Subset \Omega$, we have $\mathcal{F}(u, K) < +\infty$ and

$$\mathcal{F}(u, K) \leq Q\mathcal{F}(v, K).$$

If $Q = 1$, we say that u is a local minimizer of $\mathcal{F}$.

We remark that if $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is a local minimizer of the functional

$$v \in W^{1,1}(\Omega) \mapsto \int_\Omega F(x, v, Dv)\, dx$$

under the assumption that

$$c_1 \Psi(x, \xi) \le F(x, z, \xi) \le c_2 \Psi(x, \xi)$$

for all $x \in \Omega,\ z \in \mathbb{R},\ \xi \in \mathbb{R}^n$ with some constants $0 < c_1 \le 1 \le c_2$, then u is also a local quasiminimizer of the functional (1-1) with $Q = c_2/c_1 \ge 1$.

To prove the local boundedness of quasiminimizers of $\mathcal{F}$, we derive the following growth condition on the energy density $\Psi(x, \xi)$ of $\mathcal{F}$.

Lemma 4.2. *Suppose that the gap condition* (2-24) *holds. If $a \in L^\infty(\Omega)$, then*

$$G(|\xi|) \le \Psi(x, \xi) \le c(1 + [G(|\xi|)]^{1+1/n}) \tag{4-1}$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$, where c is a positive constant depending only on n, G, H and $\|a\|_{L^\infty(\Omega)}$.

Proof. Since $a(\cdot) \ge 0$, it is clear that

$$G(|\xi|) \le G(|\xi|) + a(x)H(|\xi|) = \Psi(x, \xi)$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$. Moreover, it follows from Corollary 2.6 and (2-24) that

$$\begin{aligned} \Psi(x, \xi) &= G(|\xi|) + a(x)H(|\xi|) \le G(|\xi|) + \|a\|_{L^\infty(\Omega)} H(|\xi|) \\ &\le ([G(|\xi|)]^{1+1/n} + 1) + c\|a\|_{L^\infty(\Omega)}([G(|\xi|)]^{1+1/n} + 1) \\ &\le c([G(|\xi|)]^{1+1/n} + 1) \end{aligned}$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$. □

We notice that

$$1 + \frac{1}{n} < 1 + \frac{1}{n-1} = 1^*,$$

where 1^* is the Sobolev exponent of 1. The local boundedness of quasiminimizers of $\mathcal{F}$ now follows from the result of [Cupini et al. 2015, Theorem 2.1].

Theorem 4.3 (local boundedness). *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local quasiminimizer of $\mathcal{F}$ under the assumption* (2-24), *with $a \in L^\infty_{\mathrm{loc}}(\Omega)$. Then u is locally bounded in Ω.*

Once the local boundedness of quasiminimizers has been obtained, we can prove the Hölder continuity of u without the assumption (2-24). Therefore, we shall consider an a priori bounded quasiminimizer $u \in W^{1,1}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ of $\mathcal{F}$ from Lemma 4.7 on.

Let us start the proof of the Hölder continuity of locally bounded quasiminimizers of $\mathcal{F}$. First, we present some technical lemmas.

Lemma 4.4 [Ladyzhenskaya and Uraltseva 1968]. *Let $\{Y_i\}_{i=0}^{\infty}$ be a sequence of nonnegative numbers satisfying the recursive inequalities*

$$Y_{i+1} \le C b^i Y_i^{1+\delta}, \quad i = 0, 1, 2, \dots, \tag{4-2}$$

where $C, b > 1$ and $\delta > 0$ are given numbers. If

$$Y_0 \le C^{-1/\delta} b^{-1/\delta^2}, \tag{4-3}$$

then $Y_i \to 0$ as $i \to \infty$.

Lemma 4.5 [Ladyzhenskaya and Uraltseva 1968]. *Let $v \in W^{1,1}(B_\rho)$. For any $l > k$, we have*

$$(l-k)|B_\rho \cap \{v > l\}|^{1-1/n} \le \frac{c|B_\rho|}{|B_\rho \setminus \{v > k\}|} \int_{B_\rho \cap \{k < v \le l\}} |Dv|\, dx$$

for some positive constant c depending only on n.

We now state and prove the following Caccioppoli-type inequality.

Lemma 4.6 (Caccioppoli inequality). *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a Q-minimizer of $\mathcal{F}$. Then there exists a constant $c = c(Q, \Delta_2(G), \Delta_2(H)) > 0$ such that for any concentric balls $B_{\rho'} \subset B_\rho \subset \Omega$ with $0 < \rho' < \rho < \infty$, and $k \in \mathbb{R}$, we have*

$$\int_{B_{\rho'}} \Psi(x, D(u-k)_\pm)\, dx \le c \int_{B_\rho} \Psi\left(x, \frac{(u-k)_\pm}{\rho - \rho'}\right) dx. \tag{4-4}$$

Proof. We note that it suffices to prove the version with $(u-k)_+$, as $-u$ is also a Q-minimizer of $\mathcal{F}$. Let $\eta \in C_0^\infty(B_\rho)$ be a cut-off function with $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_{\rho'}$, and $|D\eta| \le 2/(\rho-\rho')$. We set $v := u - \eta(u-k)_+$, to be used as a competitor. Note that $\mathrm{supp}(u-v) \subset A(k,\rho)$. Then the Q-minimality of u gives

$$\begin{aligned}
\int_{A(k,\rho')} \Psi(x, Du)\, dx &\le Q \int_{A(k,\rho)} \Psi(x, Dv)\, dx \\
&= Q \int_{A(k,\rho)} \Psi(x, (1-\eta)Du - (u-k)_+ D\eta)\, dx \\
&\le c_* \left(\int_{A(k,\rho) \setminus A(k,\rho')} \Psi(x, Du)\, dx + \int_{A(k,\rho)} \Psi\left(x, \frac{u-k}{\rho-\rho'}\right) dx \right)
\end{aligned}$$

for some constant $c_* = c_*(Q, \Delta_2(\Psi)) = c_*(Q, \Delta_2(G), \Delta_2(H)) \ge 1$. We now use the "hole-filling" method; that is, we add to both sides the quantity

$$c_* \int_{A(k,\rho')} \Psi(x, Du)\, dx,$$

and divide by $c_* + 1$. Then we discover that

$$\int_{A(k,\rho')} \Psi(x, Du)\, dx \le \vartheta \int_{A(k,\rho)} \Psi(x, Du)\, dx + \int_{A(k,\rho)} \Psi\left(x, \frac{u-k}{\rho-\rho'}\right) dx, \tag{4-5}$$

where $\vartheta = c_*/(c_*+1) < 1$, for any $0 < \rho' < \rho < \infty$ with $B_\rho \subset \Omega$.

Now fix $\rho' < \rho$ and consider a sequence

$$\rho_0 := \rho' \quad \text{and} \quad \rho_{i+1} = (1-\lambda)\lambda^i(\rho-\rho') + \rho_i, \quad i = 0, 1, 2, \ldots,$$

where $\lambda \in (0, 1)$ is to be chosen later. Applying (4-5) inductively, we obtain from (2-14) that

$$\begin{aligned}
\int_{A(k,\rho')} \Psi(x, Du)\,dx &\le \vartheta \int_{A(k,\rho_1)} \Psi(x, Du)\,dx + \int_{A(k,\rho_1)} \Psi\left(x, \frac{u-k}{(1-\lambda)(\rho-\rho')}\right) dx \\
&\le \vartheta^2 \int_{A(k,\rho_2)} \Psi(x, Du)\,dx + \int_{A(k,\rho)} \Psi\left(x, \frac{u-k}{(1-\lambda)(\rho-\rho')}\right) dx \\
&\qquad + \vartheta \int_{A(k,\rho_2)} \Psi\left(x, \frac{u-k}{(1-\lambda)\lambda(\rho-\rho')}\right) dx \\
&\le \vartheta^2 \int_{A(k,\rho_2)} \Psi(x, Du)\,dx + \int_{A(k,\rho)} \Psi\left(x, \frac{u-k}{(1-\lambda)(\rho-\rho')}\right) dx \\
&\qquad + \Delta_2(\Psi)\vartheta\lambda^{-\log_2 \Delta_2(\Psi)} \int_{A(k,\rho)} \Psi\left(x, \frac{u-k}{(1-\lambda)(\rho-\rho')}\right) dx \\
&\le \vartheta^i \int_{A(k,\rho_i)} \Psi(x, Du)\,dx \\
&\qquad + \Delta_2(\Psi) \sum_{j=0}^{i-1} (\vartheta\lambda^{-\log_2 \Delta_2(\Psi)})^j \int_{A(k,\rho)} \Psi\left(x, \frac{u-k}{(1-\lambda)(\rho-\rho')}\right) dx \\
&\le \vartheta^i \int_{A(k,\rho_i)} \Psi(x, Du)\,dx \\
&\qquad + \frac{\Delta_2(\Psi)}{(1-\lambda)^{\log_2 \Delta_2(\Psi)}} \sum_{j=0}^{i-1} (\vartheta\lambda^{-\log_2 \Delta_2(\Psi)})^j \int_{A(k,\rho)} \Psi\left(x, \frac{u-k}{\rho-\rho'}\right) dx.
\end{aligned}$$

Finally, choosing $\lambda = \lambda(Q, \Delta_2(\Psi)) = \lambda(Q, \Delta_2(G), \Delta_2(H)) \in (0, 1)$ in such a way that $\vartheta\lambda^{-\log_2 \Delta_2(\Psi)} < 1$ and passing to the limit for $i \to \infty$, we get

$$\int_{A(k,\rho')} \Psi(x, Du)\,dx \le \frac{\Delta_2(\Psi)}{(1-\lambda)^{\log_2 \Delta_2(\Psi)}(1-\vartheta\lambda^{-\log_2 \Delta_2(\Psi)})} \int_{A(k,\rho)} \Psi\left(x, \frac{u-k}{\rho-\rho'}\right) dx,$$

which proves the lemma. □

For the Hölder continuity of local quasiminimizers of $\mathcal{F}$, we assume that the modulating coefficient $a(\cdot)$ has a modulus of continuity ω satisfying

$$\limsup_{\rho\to 0+} \omega(\rho)\frac{H(\rho^{-1})}{G(\rho^{-1})} < \infty, \tag{4-6}$$

or, in other words

$$\omega(\rho)\frac{H(\rho^{-1})}{G(\rho^{-1})} \le L \quad \text{for every } 0 < \rho \le 1, \tag{4-7}$$

for some $L > 0$.

We remark that when $(G(t), H(t)) = (t^p, t^q)$ with $1 < p < q$, and $a(\cdot) \in C^{0,\alpha}(\Omega)$ with $\alpha \in (0, 1]$, the condition (4-6) is equivalent to $q \le p + \alpha$. In addition, when $(G(t), H(t)) = (t^p, t^p \ln(1+t))$ with

$p > 1$, the condition (4-6) is equivalent to

$$\limsup_{\rho\to 0+} \omega(\rho)\ln\left(\frac{1}{\rho}\right) < \infty.$$

Therefore, the condition (4-6) agrees with the classical ones essentially used in [Baroni et al. 2015a; 2015b; Colombo and Mingione 2015a; 2015b].

In addition, the condition (4-7) ensures that quasiminimizers of $\mathcal{F}$ satisfy the following Caccioppoli-type inequality provided the modulating coefficient $a(\cdot)$ is suitably small in the right scale.

Lemma 4.7 (almost standard Caccioppoli inequality). *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a locally bounded Q-minimizer of $\mathcal{F}$ under the assumptions* (2-25) *and* (4-7), *and let $B_r \Subset \Omega$ be a ball with $r \le 1$. Suppose that*

$$\sup_{x\in B_r} a(x) \le 4\omega(r). \tag{4-8}$$

Then for every $r/2 \le r_1 < r_2 \le r$ and $k \in \mathbb{R}$ with $|k| \le \|u\|_{L^\infty(B_r)}$,

$$\fint_{B_{r_1}} G(|D(u-k)_\pm|)\,dx \le c\left(\frac{r}{r_2-r_1}\right)^{c_G+c_H+2} \fint_{B_{r_2}} G\left(\frac{(u-k)_\pm}{r}\right)dx \tag{4-9}$$

holds for some constant $c = c(Q, c_G, c_H, L, \|u\|_{L^\infty(B_r)}) > 0$.

Proof. If follows from Lemmas 2.2, 2.7 and 4.6, and (4-8) and (2-12) that

$$\begin{aligned}
\int_{B_{r_1}} G(|D(u-k)_\pm|)\,dx &\le \int_{B_{r_1}} \Psi(x, D(u-k)_\pm)\,dx \le c\int_{B_{r_2}} \Psi\left(x, \frac{(u-k)_\pm}{r_2-r_1}\right)dx \\
&= c\int_{B_{r_2}} \left(1 + a(x)\left(\frac{H}{G}\right)\left(\frac{(u-k)_\pm}{r_2-r_1}\right)\right) G\left(\frac{(u-k)_\pm}{r_2-r_1}\right)dx \\
&\le c\int_{B_{r_2}} \left(1 + \omega(r)\left(\frac{H}{G}\right)\left(\frac{2\|u\|_{L^\infty(B_r)}}{r_2-r_1}\right)\right) G\left(\frac{(u-k)_\pm}{r}\frac{r}{r_2-r_1}\right)dx \\
&\le c\left(\frac{r}{r_2-r_1}\right)^{c_G+1}\left(1 + \omega(r)\left(\frac{H}{G}\right)\left(\frac{2\|u\|_{L^\infty(B_r)}}{r_2-r_1}\right)\right)\int_{B_{r_2}} G\left(\frac{(u-k)_\pm}{r}\right)dx.
\end{aligned}$$

We observe from Lemma 2.7, (2-12) and (4-7) that

$$\begin{aligned}
\omega(r)\left(\frac{H}{G}\right)\left(\frac{2\|u\|_{L^\infty(B_r)}}{r_2-r_1}\right) &\le \omega(r)\left(\frac{H}{G}\right)\left(\frac{2(\|u\|_{L^\infty(B_r)}+1)r}{r_2-r_1}\,\frac{1}{r}\right) \\
&\le \omega(r)\left(\frac{2(\|u\|_{L^\infty(B_r)}+1)r}{r_2-r_1}\right)^{c_H+1}\left(\frac{H}{G}\right)\left(\frac{1}{r}\right) \\
&\le c\left(\frac{r}{r_2-r_1}\right)^{c_H+1}\omega(r)\frac{H(r^{-1})}{G(r^{-1})} \le c\left(\frac{r}{r_2-r_1}\right)^{c_H+1} L,
\end{aligned}$$

which completes the proof. ∎

Lemma 4.8. *Under the assumptions of Lemma 4.7, we further suppose that the density condition*

$$\left|\left\{x \in B_{r/2} : u(x) > \sup_{B_r} u - \frac{1}{2} \operatorname*{osc}_{B_r} u\right\}\right| \le \frac{1}{2}|B_{r/2}| \tag{4-10}$$

holds. Then for any $\tau \in (0,1)$, there exists a large natural number $m \ge 3$ depending on n, Q, c_G, c_H, L, $\|u\|_{L^\infty(B_r)}$ and τ such that

$$\left|\left\{x \in B_{r/2} : u(x) > \sup_{B_r} u - \frac{1}{2^m} \operatorname*{osc}_{B_r} u\right\}\right| \le \tau|B_{r/2}|.$$

Proof. Let $m \ge 3$ be a large natural number as selected below. Define for $i = 1, 2, \dots, m$,

$$k_i := \sup_{B_r} u - \frac{1}{2^i} \operatorname*{osc}_{B_r} u, \qquad D_i := A\Big(k_i, \frac{r}{2}\Big) \setminus A\Big(k_{i+1}, \frac{r}{2}\Big),$$

and

$$w_i(x) := \begin{cases} k_{i+1} - k_i & \text{if } u(x) > k_{i+1}, \\ u(x) - k_i & \text{if } k_i < u(x) \le k_{i+1}, \\ 0 & \text{if } u(x) \le k_i. \end{cases}$$

We note that $G(w_i) \in W^{1,1}(B_{r/2})$ and $G(w_i) = 0$ in $B_{r/2} \setminus A(k_1, r/2)$ for all $i = 1, 2, \dots, m$, and that $|B_{r/2} \setminus A(k_1, r/2)| \ge \frac{1}{2}|B_{r/2}|$. Using Hölder's inequality, Sobolev's inequality and a modified form of Young's inequality (2-7) with $\varepsilon \in (0,1)$, we have

$$\begin{aligned}
\left|A\Big(k_{i+1}, \frac{r}{2}\Big)\right| G\Big(\frac{k_{i+1} - k_i}{r/2}\Big) &\le \int_{A(k_i, r/2)} G\Big(\frac{w_i}{r/2}\Big)\,dx \\
&\le \left|A\Big(k_i, \frac{r}{2}\Big)\right|^{1/n} \bigg(\int_{A(k_i, r/2)} \Big[G\Big(\frac{w_i}{r/2}\Big)\Big]^{n/(n-1)} dx\bigg)^{(n-1)/n} \\
&\le cr \bigg(\int_{A(k_i, r/2)} \Big[G\Big(\frac{w_i}{r/2}\Big)\Big]^{n/(n-1)} dx\bigg)^{(n-1)/n} \\
&\le c \int_{D_i} G'\Big(\frac{u - k_i}{r/2}\Big)|Du|\,dx \\
&\le \varepsilon \int_{D_i} G(|Du|)\,dx + c(\varepsilon) \int_{D_i} G\Big(\frac{u - k_i}{r/2}\Big)\,dx.
\end{aligned} \tag{4-11}$$

It follows from Lemma 4.7 that

$$\begin{aligned}
\int_{D_i} G(|Du|)\,dx \le c \int_{A(k_i, r)} G\left(\left|\frac{u - k_i}{r}\right|\right) dx \le c \int_{A(k_i, r)} G\Big(\frac{1}{2^i r} \operatorname*{osc}_{B_r} u\Big)\,dx \\
= cG\Big(\frac{k_{i+1} - k_i}{r/2}\Big)|A(k_i, r)| \le cG\Big(\frac{k_{i+1} - k_i}{r/2}\Big) r^n.
\end{aligned} \tag{4-12}$$

Also, it is clear that

$$\int_{D_i} G\Big(\frac{u - k_i}{r/2}\Big)\,dx \le \int_{D_i} G\Big(\frac{k_{i+1} - k_i}{r/2}\Big)\,dx = G\Big(\frac{k_{i+1} - k_i}{r/2}\Big)|D_i|. \tag{4-13}$$

Combining (4-11) with (4-12) and (4-13), we see that, for $i = 1, 2, \dots, m-1$,

$$\left|A\left(k_{m-1}, \frac{r}{2}\right)\right| \le \left|A\left(k_{i+1}, \frac{r}{2}\right)\right| \le c\varepsilon r^n + c(\varepsilon)|D_i|.$$

Summing over i from 1 to $m-1$ yields that

$$\begin{aligned}(m-1)\left|A\left(k_{m-1}, \frac{r}{2}\right)\right| &\le c(m-1)\varepsilon r^n + c(\varepsilon)\left|A\left(k_1, \frac{r}{2}\right)\right| \\ &\le (c(m-1)\varepsilon + c(\varepsilon))r^n\end{aligned}$$

and hence

$$\left|A\left(k_{m-1}, \frac{r}{2}\right)\right| \le \left(c\varepsilon + \frac{c(\varepsilon)}{m-1}\right)r^n \le \tau|B_{r/2}|$$

by taking sufficiently small $\varepsilon = \varepsilon(n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_r)}, \tau) \in (0, 1)$ and sufficiently large $m = m(n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_r)}, \tau) \in \mathbb{N}$. □

Lemma 4.9. *Under the assumptions of Lemma 4.8, we further find that there exists a small* $\tau_0 = \tau_0(n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_r)}) \in (0, 2^{-(n+1)})$ *such that if*

$$0 < \nu < \frac{1}{2}\operatorname*{osc}_{B_r} u \quad \textit{and} \quad \left|A\left(k_0, \frac{r}{2}\right)\right| \le \tau_0|B_{r/2}|, \tag{4-14}$$

where $k_0 := \sup_{B_r} u - \nu$, *then*

$$\sup_{B_{r/4}} u \le k_0 + \frac{\nu}{2} = \sup_{B_r} u - \frac{\nu}{2}. \tag{4-15}$$

Proof. We first set the sequences

$$\rho_i := \frac{r}{4}\left(1 + \frac{1}{2^i}\right) \quad \text{and} \quad k_i := k_0 + \frac{\nu}{2}\left(1 - \frac{1}{2^i}\right), \quad i = 0, 1, 2, \dots,$$

and define

$$D_{i+1} := A(k_i, \rho_{i+1}) \setminus A(k_{i+1}, \rho_{i+1}) \quad \text{and} \quad Y_i := \frac{|A(k_i, \rho_i)|}{|B_{r/2}|}.$$

We note from the definitions of k_i that $(u - k_i)_+ \le \nu \le \|u\|_{L^\infty(B_r)}$. Then we discover from (4-9) and (4-14) that

$$\begin{aligned}\int_{A(k_i, \rho_{i+1})} G(|Du|)\,dx &\le c2^{(i+3)(c_G+c_H+2)} \int_{A(k_i, \rho_i)} G\left(\frac{(u-k_i)_+}{r}\right)dx \\ &\le c2^{i(c_G+c_H+2)} G\left(\frac{\nu}{r}\right)|A(k_i, \rho_i)|.\end{aligned}$$

It follows from the convexity of G that

$$\begin{aligned}G\left(\fint_{D_{i+1}} |Du|\,dx\right) &\le \fint_{D_{i+1}} G(|Du|)\,dx \le c2^{i(c_G+c_H+2)} \frac{|A(k_i, \rho_i)|}{|D_{i+1}|} G\left(\frac{\nu}{r}\right) \\ &\le G\left(c2^{i(c_G+c_H+2)} \frac{|A(k_i, \rho_i)|}{|D_{i+1}|} \frac{\nu}{r}\right).\end{aligned}$$

Therefore, we obtain

$$\fint_{D_{i+1}} |Du|\,dx \le c2^{i(c_G+c_H+2)} \frac{|A(k_i,\rho_i)|}{|D_{i+1}|}\frac{\nu}{r}.$$

On the other hand, using Lemma 4.5 and the fact that $\tau_0 \in (0, 2^{-(n+1)})$, we have

$$\begin{aligned}\int_{D_{i+1}} |Du|\,dx &\ge c(k_{i+1}-k_i)|A(k_{i+1},\rho_{i+1})|^{1-1/n}|B_{\rho_{i+1}} \setminus A(k_i,\rho_{i+1})|\rho_{i+1}^{-n}\\ &\ge c2^{-i}\nu|A(k_{i+1},\rho_{i+1})|^{1-1/n}(|B_{r/4}|-\tau_0|B_{r/2}|)r^{-n}\\ &\ge c2^{-i}\nu|A(k_{i+1},\rho_{i+1})|^{1-1/n}\\ &\ge c2^{-i}\nu r^{n-1}Y_{i+1}^{1-1/n}.\end{aligned}$$

Combining these inequalities gives

$$Y_{i+1}^{1-1/n} \le c2^{i(c_G+c_H+3)}r^{-n}|A(k_i,\rho_i)| \le c2^{i(c_G+c_H+3)}Y_i,$$

and hence

$$Y_{i+1} \le c_* 2^{n(c_G+c_H+3)/(n-1)i}\,Y_i^{1+1/(n-1)}$$

for some constant $c_* = c_*(n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_r)}) > 1$.

Consequently, Lemma 4.4 implies that $Y_i \to 0$ as $i \to \infty$, provided

$$Y_0 = \frac{|A(k_0, r/2)|}{|B_{r/2}|} \le \tau_0 \le c_*^{-(n-1)}2^{-n(n-1)(c_G+c_H+3)}.$$

Then we obtain

$$\left|A\left(k_0+\frac{\nu}{2},\frac{r}{4}\right)\right| = 0,$$

which implies (4-15). □

The following proposition follows from the above lemma in a standard way by taking $\nu = (1/2^m)\operatorname{osc}_{B_r} u$; see for instance [Baroni et al. 2015b; DiBenedetto 1995].

Proposition 4.10. *Under the assumptions of Lemma 4.8, let $m \ge 3$ be the natural number determined in Lemma 4.8 with $\tau = \tau_0 \in (0, 2^{-(n+1)})$ which is given in Lemma 4.9. Then we see that $m \in \mathbb{N}$ depends only on $n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_r)}$, and we have*

$$\operatorname*{osc}_{B_{r/4}} u \le \left(1-\frac{1}{2^{m+1}}\right)\operatorname*{osc}_{B_r} u. \tag{4-16}$$

The following lemma provides the Hölder continuity of quasiminimizers of the functional

$$v \in W^{1,1}(\Omega) \mapsto \mathcal{F}_0(v,\Omega) := \int_\Omega [G(|Dv|)+a_0H(|Dv|)]\,dx, \tag{4-17}$$

where $0 \le a_0 \le \|a\|_{L^\infty(\Omega)}$ is a fixed constant. For simplicity, we set

$$\Psi_0(t) := G(t)+a_0H(t) \tag{4-18}$$

for $t \ge 0$.

Lemma 4.11. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a Q-minimizer of $\mathcal{F}_0$ under the assumption* (2-25). *Then there exist $\beta_0 \in (0,1)$ and $c > 0$, both depending on n, Q, c_G, c_H, but independent of a_0 and u, such that for any fixed ball $B_{r_0} \Subset \Omega$*

$$\operatorname*{osc}_{B_r} u \le c\left(\frac{r}{r_0}\right)^{\beta_0} \operatorname*{osc}_{B_{r_0}} u \tag{4-19}$$

holds for every $0 < r \le r_0$.

Proof. We first observe from [Baroni et al. 2015b, Remark 3.1] that

$$\frac{1}{2\max\{c_G, c_H\}} \le \frac{t\Psi_0''(t)}{\Psi_0'(t)} \le 2\max\{c_G, c_H\} \quad \text{for all } t > 0.$$

We deduce from Theorem 4.3 that u is locally bounded in Ω. Therefore, the result (4-19) follows from [Lieberman 1991, Section 6]. □

We are now ready to prove the Hölder continuity of locally bounded quasiminimizers of $\mathcal{F}$.

Theorem 4.12 (Hölder continuity). *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a locally bounded Q-minimizer of $\mathcal{F}$ under the assumptions* (2-25) *and* (4-7). *Then for every open subset $\Omega' \Subset \Omega$ there exists $\beta \in (0,1)$, depending on n, Q, c_G, c_H, L and $\|u\|_{L^\infty(\Omega')}$, such that*

$$u \in C^{0,\beta}_{\mathrm{loc}}(\Omega').$$

Proof. Since the proof is analogous to that of [Baroni et al. 2015b, Theorem 4.1], we only sketch the proof. We shall show that for a fixed ball $B_{8r_0} \subset \Omega'$ with $8r_0 \le 1$, there holds

$$\operatorname*{osc}_{B_r} u \le c\left(\frac{r}{r_0}\right)^{\beta} \operatorname*{osc}_{B_{r_0}} u \quad \text{for all } r \in (0, r_0], \tag{4-20}$$

for some positive constant c depending only on n, Q, c_G, c_H, L and $\|u\|_{L^\infty(\Omega')}$.

Let us define

$$\mathcal{J} := \left\{ i \in \mathbb{N}_0 : \text{(4-8) does not hold for } r = \frac{r_0}{4^i} \right\},$$

and

$$j := \begin{cases} \min \mathcal{J} & \text{if } \mathcal{J} \ne \varnothing, \\ \infty & \text{if } \mathcal{J} = \varnothing. \end{cases}$$

If $j \ge 1$, then we obtain from Proposition 4.10 that for each $r = 4^{-i} r_0$ with $i = 0, \dots, j-1$,

$$\operatorname*{osc}_{B_{r/4}} u \le \left(1 - \frac{1}{2^{m+1}}\right) \operatorname*{osc}_{B_r} u,$$

which yields

$$\operatorname*{osc}_{B_r} u \le 4\left(\frac{r}{r_0}\right)^{\beta_1} \operatorname*{osc}_{B_{r_0}} u \quad \text{for all } r \in (4^{-(j+1)} r_0, r_0], \tag{4-21}$$

for some $\beta_1 \in (0,1)$. If $j = \infty$, then (4-21) holds for every $r \in (0, r_0]$, which is the desired conclusion (4-20) with $\beta = \beta_1$.

In the case $1 \le j < \infty$, one can check that u is a $(2Q)$-minimizer of the functional

$$v \mapsto \int_{B_{4^{-j}r_0}} [G(|Dv|) + a_0 H(|Dv|)]\,dx, \quad a_0 = \sup_{B_{4^{-j}r_0}} a(\cdot).$$

Now, Lemma 4.11 gives

$$\operatorname*{osc}_{B_r} u \le c\left(\frac{r}{4^{-j}r_0}\right)^{\beta_0} \operatorname*{osc}_{B_{4^{-j}r_0}} u \tag{4-22}$$

for every $r \in (0, 4^{-j}r_0]$. Here, $\beta_0 \in (0,1)$ and $c > 0$ both depend only on n, Q, c_G, c_H. Combining (4-21) and (4-22), we conclude that (4-20) holds for $\beta = \min\{\beta_0, \beta_1\}$. Finally, if $j = 0$, then u is a $(2Q)$-minimizer of the functional

$$v \mapsto \int_{B_{r_0}} [G(|Dv|) + a_0 H(|Dv|)]\,dx, \quad a_0 = \sup_{B_{r_0}} a(\cdot),$$

and hence we have the desired conclusion (4-20) with $\beta = \beta_0$. □

Remark 4.13. Our condition (4-6) provides a characterization of the modulating coefficient $a(\cdot)$. More precisely, a modulus of continuity of $a(\cdot)$ is exactly calibrated to the size of the phase transition. For example, it is evident that the natural assumption for the modulating coefficient in the functional

$$v \in W^{1,1}(\Omega) \mapsto \int_\Omega [|Dv|^p + a(x)|Dv|^p[\ln(1+|Dv|)]^\gamma]\,dx,$$

with $p > 1$ and $\gamma > 0$, is

$$\limsup_{\rho\to 0+} \omega(\rho)\left[\ln\left(\frac{1}{\rho}\right)\right]^\gamma < \infty.$$

Similarly, for the functional

$$v \in W^{1,1}(\Omega) \mapsto \int_\Omega [|Dv|^p + a(x)|Dv|^p \ln\ln(e+|Dv|)]\,dx,$$

with $p > 1$, the natural assumption for the modulating coefficient is

$$\limsup_{\rho\to 0+} \omega(\rho)\ln\ln\left(\frac{1}{\rho}\right) < \infty.$$

5. The Harnack inequality

In this section, we prove the Harnack inequality for locally bounded quasiminimizers of $\mathcal{F}$. We first present some technical tools.

Lemma 5.1 [Ladyzhenskaya and Uraltseva 1968]. *Let $v \in W^{1,1}(B_\rho)$. For any $l > k$, we have*

$$(l-k)|B_\rho \cap \{v < k\}|^{1-1/n} \le \frac{c|B_\rho|}{|B_\rho \setminus \{v < l\}|}\int_{B_\rho\cap\{k<v\le l\}} |Dv|\,dx$$

for some positive constant c depending only on n.

Lemma 5.2 [Giusti 2003]. *Let ψ be a bounded nonnegative function in the interval $[\rho, r]$ such that*

$$\psi(t) \leq \vartheta\psi(s) + \frac{A}{(s-t)^{\kappa}} \quad \textit{for every } \rho \leq t < s \leq r,$$

with $A \geq 0$, $\kappa > 0$ and $0 \leq \vartheta < 1$. Then we have

$$\psi(\rho) \leq c(\kappa, \vartheta)\frac{A}{(r-\rho)^{\kappa}}.$$

The following lemma provides the weak Harnack inequality of quasiminimizers of the functional $\mathcal{F}_0$ in (4-17); see [Lieberman 1991].

Lemma 5.3. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a Q-minimizer of $\mathcal{F}_0$ under the assumption* (2-25), *and let $B \Subset \Omega$ be a ball. Then for any exponent $q_+ > 0$ and every $0 < t < s < 1$, we have*

$$\sup_{tB} |u| \leq c^*\left(\fint_{sB} |u|^{q_+}\,dx\right)^{1/q_+} \tag{5-1}$$

for some constant $c^* = c^*(n, Q, c_G, c_H, s-t, q_+) > 1$. Moreover, if u is nonnegative, then there exists an exponent $q_- = q_-(n, Q, c_G, c_H) \in (0, 1)$ such that for every $t, s \in (0, 1)$

$$\inf_{tB} u \geq \frac{1}{c_*}\left(\fint_{sB} u^{q_-}\,dx\right)^{1/q_-} \tag{5-2}$$

holds for some constant $c_* = c_*(n, Q, c_G, c_H, t, s) > 1$.

Analysis similar to that in the proof of Lemma 4.8 gives the following lemma.

Lemma 5.4. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a nonnegative and locally bounded Q-minimizer of $\mathcal{F}$ under the assumptions* (2-25) *and* (4-7), *and let $B_{3r} \Subset \Omega$ be a ball with $3r \leq 1$. Suppose that*

$$\sup_{x \in B_{3r}} a(x) \leq 12\omega(r). \tag{5-3}$$

For any $\tau_1, \tau_2 \in (0, 1)$, there exists a large natural number m depending on $n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_{3r})}, \tau_1$ and τ_2 such that for any $0 < \lambda \leq \|u\|_{L^\infty(B_{3r})}$ if

$$|\{x \in B_r : u(x) \geq \lambda\}| \geq \tau_1 |B_r| \tag{5-4}$$

holds, then

$$|\{x \in B_{2r} : u(x) \leq 2^{-m}\lambda\}| \leq \tau_2 |B_{2r}|. \tag{5-5}$$

Now we can obtain a lower bound of u under some density condition as follows.

Proposition 5.5. *Let the assumptions in Lemma 5.4 hold. For any $\tau \in (0, 1)$, there exists a small $\delta_1 = \delta_1(n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_{3r})}, \tau) > 0$ such that for any $0 < \lambda \leq \|u\|_{L^\infty(B_{3r})}$, if*

$$|\{x \in B_r : u(x) \geq \lambda\}| \geq \tau |B_r| \tag{5-6}$$

holds, then

$$\inf_{B_r} u \geq \delta_1 \lambda. \tag{5-7}$$

Proof. We first note that it suffices to prove the proposition for $\tau \in (0, 2^{-(n+1)})$. We fix $m_0 \in \mathbb{N}$, and set the sequences

$$\rho_i := r\left(1 + \frac{1}{2^i}\right) \quad \text{and} \quad k_i := \left(\frac{1}{2} + \frac{1}{2^i}\right)2^{-m_0}\lambda, \quad i = 0, 1, 2, \ldots.$$

We also define

$$D^-_{i+1} := A^-(k_i, \rho_{i+1}) \setminus A^-(k_{i+1}, \rho_{i+1}) \quad \text{and} \quad Y_i := \frac{|A^-(k_i, \rho_i)|}{|B_{\rho_i}|}.$$

Since u is nonnegative, we have $(u - k_i)_- \le 2^{-m_0}\lambda$. By (4-9), we get

$$\begin{aligned}\int_{A^-(k_i,\rho_{i+1})} G(|Du|)\,dx &\le c2^{(i+3)(c_G+c_H+2)} \int_{A^-(k_i,\rho_i)} G\left(\frac{(u-k_i)_-}{2r}\right)dx \\ &\le c2^{i(c_G+c_H+2)} G\left(\frac{2^{-m_0}\lambda}{r}\right)|A^-(k_i, \rho_i)|.\end{aligned}$$

We deduce from the convexity of G that

$$\begin{aligned}G\left(\fint_{D^-_{i+1}} |Du|\,dx\right) \le \fint_{D^-_{i+1}} G(|Du|)\,dx &\le c2^{i(c_G+c_H+2)} \frac{|A^-(k_i, \rho_i)|}{|D^-_{i+1}|} G\left(\frac{2^{-m_0}\lambda}{r}\right) \\ &\le G\left(c2^{i(c_G+c_H+2)} \frac{|A^-(k_i, \rho_i)|}{|D^-_{i+1}|} \frac{2^{-m_0}\lambda}{r}\right).\end{aligned}$$

Therefore, we obtain

$$\fint_{D^-_{i+1}} |Du|\,dx \le c2^{i(c_G+c_H+2)} \frac{|A^-(k_i, \rho_i)|}{|D^-_{i+1}|} \frac{2^{-m_0}\lambda}{r}.$$

On the other hand, using Lemma 5.1 and the fact that $\tau \in (0, 2^{-(n+1)})$, we have

$$\begin{aligned}\int_{D^-_{i+1}} |Du|\,dx &\ge c(k_i - k_{i+1})|A^-(k_{i+1}, \rho_{i+1})|^{1-1/n} |B_{\rho_{i+1}} \setminus A^-(k_i, \rho_{i+1})|\rho_{i+1}^{-n} \\ &\ge c2^{-i} \cdot 2^{-m_0}\lambda |A^-(k_{i+1}, \rho_{i+1})|^{1-1/n} (|B_{2r}| - \tau|B_r|)r^{-n} \\ &\ge c2^{-i} \cdot 2^{-m_0}\lambda |A^-(k_{i+1}, \rho_{i+1})|^{1-1/n} \\ &\ge c2^{-i} \cdot 2^{-m_0}\lambda r^{n-1} Y_{i+1}^{1-1/n}.\end{aligned}$$

Combining these inequalities gives

$$Y_{i+1}^{1-1/n} \le c2^{i(c_G+c_H+3)} r^{-n} |A^-(k_i, \rho_i)| \le c2^{i(c_G+c_H+3)} Y_i,$$

and hence

$$Y_{i+1} \le c_0 2^{in(c_G+c_H+3)/(n-1)} Y_i^{1+1/(n-1)}$$

for some constant $c_0 = c_0(n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_{3r})}) > 1$. Here we note from Lemma 5.4 that there exists a large natural number m_0 depending only on $n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_{3r})}$ such that

$$|\{x \in B_{2r} : u(x) \le 2^{-m_0}\lambda\}| \le c_0^{-(n-1)} 2^{-n(n-1)(c_G+c_H+3)} |B_{2r}|.$$

Then it is clear that

$$Y_0 = \frac{|A^-(k_0, 2r)|}{|B_{2r}|} = \frac{|\{x \in B_{2r} : u(x) \le 2^{-m_0}\lambda\}|}{|B_{2r}|} \le c_0^{-(n-1)} 2^{-n(n-1)(c_G+c_H+3)},$$

and hence $Y_i \to 0$ as $i \to \infty$ by Lemma 4.4. Consequently, we obtain

$$|A^-(2^{-(m_0+1)}\lambda, r)| = 0,$$

which implies (5-7) with $\delta_1 = 2^{-(m_0+1)}$. □

Proposition 5.6. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a nonnegative and locally bounded Q-minimizer of $\mathcal{F}$ under the assumptions* (2-25) *and* (4-7), *and let $B_{3r} \Subset \Omega$ be a ball with $3r \le 1$. Suppose that*

$$\sup_{x \in B_{3r}} a(x) > 12\omega(r). \tag{5-8}$$

For any $\tau \in (0, 1)$, there exists a small $\delta_2 = \delta_2(n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_{3r})}, \tau) > 0$ such that if

$$|\{x \in B_r : u(x) \ge \lambda\}| \ge \tau |B_r| \tag{5-9}$$

for $0 < \lambda \le \|u\|_{L^\infty(B_{3r})}$, then

$$\inf_{B_r} u \ge \delta_2 \lambda. \tag{5-10}$$

Proof. By (5-8), there exists $x_M \in \bar{B}_{3r}$ such that $a(x_M) = a_0 > 12\omega(r)$. Then for every $x \in B_{3r}$

$$a(x_M) - a(x) \le \omega(6r) \le 6\omega(r),$$

and hence

$$a_0 \le 2a_0 - 12\omega(r) \le 2a(x) \le 2a_0.$$

Since $\Psi(x, Du) \in L^1(B_{3r})$, it follows that

$$G(|Dv|) + a_0 H(|Dv|) \in L^1(B_{3r}).$$

Furthermore, one can see that u is a $(2Q)$-minimizer of the functional

$$v \mapsto \int_{B_{3r}} [G(|Dv|) + a_0 H(|Dv|)]\, dx, \quad a_0 = \sup_{B_{3r}} a(\cdot).$$

Now, using (5-2) in Lemma 5.3 with $B \equiv B_{3r}$ and $t = s = \frac{1}{3}$, we see from (5-9) that

$$\inf_{B_r} u \ge \frac{\tau^{1/q_-}\lambda}{c_*},$$

which implies (5-10) with $\delta_2 := \tau^{1/q_-} c_*^{-1}$. □

An immediate consequence of Propositions 5.5 and 5.6 is the following.

Corollary 5.7. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a nonnegative and locally bounded Q-minimizer of $\mathcal{F}$ under the assumptions* (2-25) *and* (4-7), *and let $B_{3r} \Subset \Omega$ be a ball with $3r \le 1$. For any $\tau \in (0, 1)$, there exists a small $\delta = \delta(n, Q, c_G, c_H, L, \|u\|_{L^\infty(B_{3r})}, \tau) > 0$ such that if*

$$|\{x \in B_r : u(x) \ge \lambda\}| \ge \tau |B_r|$$

for $0 < \lambda \le \|u\|_{L^\infty(B_{3r})}$, then

$$\inf_{B_r} u \ge \delta\lambda.$$

From Corollary 5.7 and the covering arguments in [Kinnunen and Shanmugalingam 2001, Section 7], we obtain the following weak Harnack inequality for quasiminimizers of $\mathcal{F}$. For the proof we refer the reader to [Baroni et al. 2015a, Theorem 3.5; Harjulehto et al. 2008, Theorem 5.7].

Theorem 5.8 (the weak Harnack inequality). *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a nonnegative and locally bounded Q-minimizer of $\mathcal{F}$ under the assumptions* (2-25) *and* (4-7), *and let $B_{9r} \equiv B_{9r}(x_0) \Subset \Omega$ with $9r \le 1$. Then there exists an exponent $q_- > 0$ and a constant $c > 1$, depending on n, Q, c_G, c_H, L and $\|u\|_{L^\infty(B_{9r})}$, such that*

$$\inf_{B_r} u \ge \frac{1}{c}\left(\fint_{B_{2r}} u^{q_-}\,dx\right)^{1/q_-}. \tag{5-11}$$

To prove the sup-estimate for quasiminimizers of $\mathcal{F}$, we now introduce the scaled functions and the corresponding functional. Let us define, for $R \in (0, 1]$ and $r > 0$ with $B_r \Subset \Omega$,

$$u_R(x) := \frac{u(Rx)}{R}, \quad a_R(x) := a(Rx), \quad x \in B_r,$$

and

$$\mathcal{F}_R(v, K) := \int_K [G(|Dv|) + a_R(x)H(|Dv|)]\,dx, \quad K \Subset B_r.$$

Lemma 5.9. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a Q-minimizer of $\mathcal{F}$. Let $R \in (0, 1]$ and suppose that $B_r \Subset \Omega$. Then u_R is a Q-minimizer of $\mathcal{F}_R$ in B_r.*

Proof. We first observe that $Du_R(x) = Du(Rx)$. Since $B_r \Subset \Omega$, we see that $\mathcal{F}(u, B_r) < +\infty$, and hence

$$\begin{aligned}\mathcal{F}_R(u_R, B_r) &= \int_{B_r} [G(|Du(Rx)|) + a(Rx)H(|Du(Rx)|)]\,dx \\ &= \frac{1}{R^n}\int_{B_{Rr}} [G(|Du(y)|) + a(y)H(|Du(y)|)]\,dy \\ &\le \frac{1}{R^n}\int_{B_r} [G(|Du(y)|) + a(y)H(|Du(y)|)]\,dy \\ &= \frac{1}{R^n}\mathcal{F}(u, B_r) < +\infty.\end{aligned}$$

Furthermore, for any $v_R \in W^{1,1}_{\mathrm{loc}}(B_r)$ with $K := \operatorname{supp}(u_R - v_R) \Subset B_r$, we have

$$\operatorname{supp}(u - v) = \{Rx : x \in K\} =: RK,$$

and

$$\begin{aligned}\mathcal{F}_R(u_R, K) &= \int_K [G(|Du(Rx)|) + a(Rx)H(|Du(Rx)|)]\,dx \\ &= \frac{1}{R^n}\int_{RK} [G(|Du(y)|) + a(y)H(|Du(y)|)]\,dy \\ &\le \frac{Q}{R^n}\int_{RK} [G(|Dv(y)|) + a(y)H(|Dv(y)|)]\,dy \\ &= Q\int_K [G(|Dv(Rx)|) + a(Rx)H(|Dv(Rx)|)]\,dx = Q\mathcal{F}_R(v_R, K).\end{aligned}$$

Therefore, u_R is a Q-minimizer of $\mathcal{F}_R$ in B_r. □

From the definition of the scaled function $a_R(\,\cdot\,)$, one can directly obtain the following lemma.

Lemma 5.10. *Let $R \in (0, 1]$ and suppose that $B_{4r} \subset B_1 \subset \Omega$. Then the function $a_R : B_{1/R} \to [0, \infty)$ has a modulus of continuity ω_R satisfying*

$$\omega_R(\rho) = \omega(R\rho) \quad \textit{for all } 0 < \rho \le \frac{1}{R}.$$

Moreover, we have

$$\sup_{x \in B_{3r}} a(x) \le 12\omega(r) \quad \Longleftrightarrow \quad \sup_{x \in B_{3r/R}} a_R(x) \le 12\omega_R\left(\frac{r}{R}\right).$$

We now prove the sup-estimate for quasiminimizers of $\mathcal{F}$. For this, we consider two cases separately, as in the proof of the weak Harnack inequality.

Proposition 5.11. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a locally bounded Q-minimizer of $\mathcal{F}$ under the assumptions* (2-25) *and* (4-7), *and let $B_{4r} \Subset \Omega$ be a ball with $4r \le 1$. Suppose that*

$$\sup_{x \in B_{3r}} a(x) \le 12\omega(r).$$

Then for any exponent $q_+ > 0$, we have the estimate

$$\sup_{B_r} |u| \le c\left(\fint_{B_{2r}} |u|^{q_+}\,dx\right)^{1/q_+} \tag{5-12}$$

for some constant $c > 1$ depending on n, Q, c_G, c_H, L, $\|u\|_{L^\infty(B_{4r})}$ and q_+.

Proof. Let us consider the scaled functions

$$u_r(x) = \frac{u(rx)}{r}, \quad a_r(x) = a(rx), \quad x \in B_4.$$

Then by Lemmas 5.9 and 5.10, we see that the Caccioppoli inequality (4-9) holds for u_r. For $1 \le t < s \le 2$, we now set the sequences

$$\rho_i := t + \frac{s-t}{2^i} \quad \text{and} \quad k_i := 2d\left(1 - \frac{1}{2^{i+1}}\right), \quad i = 0, 1, 2, \ldots,$$

where $d > 0$ is to be chosen later. We further define

$$\tilde{\rho}_i := \frac{\rho_i + \rho_{i+1}}{2} \quad \text{and} \quad Y_i := \frac{1}{G(d)} \int_{A_r(k_i, \rho_i)} G(u_r - k_i)\,dx,$$

where

$$A_r(k, \rho) := \{x \in B_\rho : u_r > k\}.$$

Let $\eta_i \in C_0^\infty(B_{\tilde{\rho}_i})$ be a cut-off function with $0 \le \eta_i \le 1$, $\eta_i \equiv 1$ on $B_{\rho_{i+1}}$, and

$$|D\eta_i| \le \frac{4}{\rho_i - \rho_{i+1}}.$$

Using Hölder's inequality, Sobolev's inequality and a modified form of Young's inequality (2-7) with $\varepsilon = 1$, we have

$$\begin{aligned}
G(d)Y_{i+1} &\le \int_{B_{\tilde{\rho}_i}} G((u_r - k_{i+1})_+\eta_i)\,dx \\
&\le |A_r(k_{i+1},\rho_i)|^{1/n}\biggl(\int_{B_{\tilde{\rho}_i}} [G((u_r - k_{i+1})_+\eta_i)]^{n/(n-1)}\,dx\biggr)^{(n-1)/n} \\
&\le c|A_r(k_{i+1},\rho_i)|^{1/n}\int_{B_{\tilde{\rho}_i}} G'((u_r - k_{i+1})_+\eta_i)[|D(u_r - k_{i+1})_+|\eta_i + (u_r - k_{i+1})_+|D\eta_i|]\,dx \\
&\le c|A_r(k_{i+1},\rho_i)|^{1/n}\int_{B_{\tilde{\rho}_i}} G'((u_r - k_{i+1})_+)|D(u_r - k_{i+1})_+|\,dx \\
&\qquad\qquad + c|A_r(k_{i+1},\rho_i)|^{1/n}\frac{2^{i+3}}{s-t}\int_{B_{\tilde{\rho}_i}} G'((u_r - k_{i+1})_+)(u_r - k_{i+1})_+\,dx \\
&\le c|A_r(k_{i+1},\rho_i)|^{1/n}\biggl[\int_{B_{\tilde{\rho}_i}} G(|D(u_r - k_{i+1})_+|)\,dx + \int_{B_{\tilde{\rho}_i}} G((u_r - k_{i+1})_+)\,dx\biggr] \\
&\qquad\qquad + c|A_r(k_{i+1},\rho_i)|^{1/n}\frac{2^{i+3}}{s-t}\int_{B_{\tilde{\rho}_i}} G((u_r - k_{i+1})_+)\,dx \\
&\le c|A_r(k_{i+1},\rho_i)|^{1/n}\biggl[\int_{B_{\tilde{\rho}_i}} G(|D(u_r - k_{i+1})_+|)\,dx + \frac{2^{i+3}}{s-t}\int_{B_{\tilde{\rho}_i}} G((u_r - k_{i+1})_+)\,dx\biggr] \\
&\le c|A_r(k_{i+1},\rho_i)|^{1/n}\biggl(\frac{2^{i+3}}{s-t}\biggr)^{c_G+c_H+2}\int_{B_{\rho_i}} G((u_r - k_{i+1})_+)\,dx.
\end{aligned}$$

Here we observe from (2-12) that

$$\begin{aligned}
|A_r(k_{i+1},\rho_i)| &\le \frac{1}{G(k_{i+1}-k_i)}\int_{A_r(k_{i+1},\rho_i)} G(u_r - k_i)\,dx \\
&= \frac{1}{G(d/2^{i+1})}\int_{A_r(k_{i+1},\rho_i)} G(u_r - k_i)\,dx \\
&\le \frac{G(d)}{G(d/2^{i+1})}Y_i \le 2^{(i+1)(c_G+1)}Y_i \le c\biggl(\frac{2^{i+3}}{s-t}\biggr)^{c_G+c_H+2}Y_i
\end{aligned}$$

and

$$\int_{B_{\rho_i}} G((u_r - k_{i+1})_+)\,dx = \int_{A_r(k_{i+1},\rho_i)} G(u_r - k_{i+1})\,dx \le \int_{A_r(k_i,\rho_i)} G(u_r - k_i)\,dx = G(d)Y_i.$$

Combining these inequalities yields

$$Y_{i+1} \le \frac{c_0}{(s-t)^\kappa}2^{i\kappa}Y_i^{1+1/n}$$

for some constant $c_0 > 1$ depending only on n, Q, c_G, c_H, L and $\|u\|_{L^\infty(B_{4r})}$, where

$$\kappa = \Bigl(1+\frac{1}{n}\Bigr)(c_G + c_H + 2) > 1.$$

Applying Lemma 4.4, we have $Y_i \to 0$ as $i \to \infty$, provided

$$Y_0 = \frac{1}{G(d)} \int_{A_r(d,s)} G(u_r - d)\, dx \le \left[\frac{c_0}{(s-t)^\kappa} \right]^{-n} 2^{-n^2\kappa}. \tag{5-13}$$

It is clear that (5-13) is satisfied if we choose $d > 0$ such that

$$G(d) = \frac{2^{n^2\kappa} c_0^n}{(s-t)^{n\kappa}} \int_{B_s} G((u_r)_+)\, dx. \tag{5-14}$$

Then we obtain $u_r \le 2d$ in B_t, which together with (5-14) implies

$$G\big(\sup_{B_t}(u_r)_+\big) \le \frac{c}{(s-t)^{n\kappa}} \fint_{B_s} G((u_r)_+)\, dx. \tag{5-15}$$

We note from Lemma 2.9 that there exists $\gamma = \gamma(c_G) > 1$ such that $t \mapsto G(t^{1/\gamma})$ is a concave function. Then it follows from (5-15) and Jensen's inequality that

$$\begin{aligned} G\big(\sup_{B_t}(u_r)_+\big) &\le \frac{c}{(s-t)^{n\kappa}} \fint_{B_s} G((u_r)_+)\, dx = \frac{c}{(s-t)^{n\kappa}} \fint_{B_s} G(((u_r)_+^\gamma)^{1/\gamma})\, dx \\ &\le \frac{c}{(s-t)^{n\kappa}} G\bigg(\Big(\fint_{B_s} (u_r)_+^\gamma\, dx\Big)^{1/\gamma}\bigg) \le G\bigg(\frac{c}{(s-t)^{n\kappa}} \Big(\fint_{B_s} (u_r)_+^\gamma\, dx\Big)^{1/\gamma}\bigg), \end{aligned}$$

and hence

$$\sup_{B_t}(u_r)_+ \le \frac{c}{(s-t)^{n\kappa}} \Big(\fint_{B_s} (u_r)_+^\gamma\, dx\Big)^{1/\gamma}.$$

Since $-u$ is also a Q-minimizer of $\mathcal{F}$, we get

$$\sup_{B_t} |u_r| \le \frac{c}{(s-t)^{n\kappa}} \Big(\fint_{B_s} |u_r|^\gamma\, dx\Big)^{1/\gamma}.$$

Moreover, for $0 < q_+ < \gamma$, we obtain from Young's inequality that

$$\begin{aligned} \sup_{B_t} |u_r| &\le \frac{c}{(s-t)^{n\kappa}} \big[\sup_{B_s} |u_r|\big]^{1-q_+/\gamma} \Big(\fint_{B_s} |u_r|^{q_+}\, dx\Big)^{1/\gamma} \\ &\le \frac{1}{2} \sup_{B_s} |u_r| + \frac{c}{(s-t)^{n\kappa\gamma/q_+}} \Big(\fint_{B_2} |u_r|^{q_+}\, dx\Big)^{1/q_+} \end{aligned}$$

as $1 \le t < s \le 2$. Then Lemma 5.2 with $\psi(t) := \sup_{B_t} |u_r|$ yields

$$\sup_{B_1} |u_r| \le c \Big(\fint_{B_2} |u_r|^{q_+}\, dx\Big)^{1/q_+}, \tag{5-16}$$

where c is a positive constant depending on n, Q, c_G, c_H, L, $\|u\|_{L^\infty(B_{4r})}$ and q_+.

On the other hand, the inequality (5-16) also holds for $q_+ \ge \gamma$ by Hölder's inequality. Finally, from the definition of u_r, we obtain the desired conclusion (5-12). □

Proposition 5.12. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a locally bounded Q-minimizer of $\mathcal{F}$ under the assumptions* (2-25) *and* (4-7), *and let $B_{4r} \Subset \Omega$ be a ball with $4r \le 1$. Suppose that*

$$\sup_{x\in B_{3r}} a(x) > 12\omega(r).$$

Then for any exponent $q_+ > 0$, we have the estimate

$$\sup_{B_r} |u| \le c\left(\fint_{B_{2r}} |u|^{q_+}\,dx\right)^{1/q_+} \tag{5-17}$$

for some constant $c > 1$ depending on n, Q, c_G, c_H, L, $\|u\|_{L^\infty(B_{4r})}$ and q_+.

Proof. As in the proof of Proposition 5.6, we see that u is a $(2Q)$-minimizer of the functional

$$v \mapsto \int_{B_{3r}} [G(|Dv|) + a_0 H(|Dv|)]\,dx, \quad a_0 = \sup_{B_{3r}} a(\,\cdot\,) > 0.$$

Therefore, (5-1) in Lemma 5.3 with $B \equiv B_{3r}$, $t = \frac{1}{3}$ and $s = \frac{2}{3}$ directly gives (5-17). □

Combining Propositions 5.11 and 5.12 yields the following sup-estimate.

Corollary 5.13. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a locally bounded Q-minimizer of $\mathcal{F}$ under the assumptions* (2-25) *and* (4-7), *and let $B_{4r} \Subset \Omega$ be a ball with $4r \le 1$ Then for any exponent $q_+ > 0$, we have the estimate*

$$\sup_{B_r} |u| \le c\left(\fint_{B_{2r}} |u|^{q_+}\,dx\right)^{1/q_+} \tag{5-18}$$

for some constant $c > 1$ depending on n, Q, c_G, c_H, L, $\|u\|_{L^\infty(B_{4r})}$ and q_+.

Finally, from Theorem 5.8 and Corollary 5.13 with $q_+ = q_-$, we obtain the Harnack inequality of quasiminimizers of $\mathcal{F}$. We remark that the following theorem has no extra term in (5-19), so it can be regarded as a refined version of the result in [Harjulehto et al. 2017] for the generalized double phase case.

Theorem 5.14 (the Harnack inequality). *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a nonnegative and locally bounded Q-minimizer of $\mathcal{F}$ under the assumptions* (2-25) *and* (4-7), *and let $B_{9r} \Subset \Omega$ be a ball with $9r \le 1$. Then there exists a constant $c > 1$, depending on n, Q, c_G, c_H, L and $\|u\|_{L^\infty(B_{9r})}$, such that*

$$\sup_{B_r} u \le c \inf_{B_r} u. \tag{5-19}$$

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EPSILON-REGULARITY FOR p-HARMONIC MAPS AT A FREE BOUNDARY ON A SPHERE

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We prove an ϵ-regularity theorem for vector-valued p-harmonic maps, which are critical with respect to a partially free boundary condition, namely that they map the boundary into a round sphere.

This does not seem to follow from the reflection method that Scheven used for harmonic maps with free boundary (i.e., the case $p = 2$): the reflected equation can be interpreted as a p-harmonic map equation into a manifold, but the regularity theory for such equations is only known for round targets.

Instead, we follow the spirit of Schikorra's recent work on free boundary harmonic maps and choose a good frame directly at the free boundary. This leads to growth estimates, which, in the critical regime $p = n$, imply Hölder regularity of solutions. In the supercritical regime, $p < n$, we combine the growth estimate with the geometric reflection argument: the reflected equation is supercritical, but, under the assumption of growth estimates, solutions are regular.

In the case $p < n$, for stationary p-harmonic maps with free boundary, as a consequence of a monotonicity formula we obtain partial regularity up to the boundary away from a set of $(n-p)$-dimensional Hausdorff measure.

1. Introduction

Over the last few years the theory of half-harmonic maps received a lot of attention, beginning with the pioneering work of Da Lio and Rivière [2011a; 2011b]; see also [Schikorra 2012; 2015c; Da Lio 2013; Millot and Sire 2015]. Half-harmonic maps appear in nature as free boundary problems — e.g., they are connected to critical points of the energy

$$\|\nabla u\|^2_{L^2(D,\mathbb{R}^N)} \quad \text{such that } u(\partial D) \subset \mathcal{N} \text{ in the a.e. trace sense.}$$

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Here, $D \subset \mathbb{R}^n$ is an open set and $\mathcal{N} \subset \mathbb{R}^N$ is a smooth closed manifold. The Euler–Lagrange equations of the latter problem are

$$\begin{cases} \Delta u = 0 & \text{in } D, \\ \partial_\nu u \perp T_u\mathcal{N} & \text{on } \partial D, \end{cases} \tag{1-1}$$

where ν denotes the outer normal vector.

For $D = \mathbb{R}^n_+$ and $\partial D = \mathbb{R}^{n-1} \times \{0\}$ the equation (1-1) is equivalent to

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+, \\ (-\Delta)^{1/2}_{\mathbb{R}^{n-1}} u \perp T_u\mathcal{N} & \text{on } \mathbb{R}^{n-1} \times \{0\}. \end{cases} \tag{1-2}$$

Here, $(-\Delta)^{1/2}_{\mathbb{R}^{n-1}}$ denotes the half-Laplacian acting on functions defined on $\mathbb{R}^{n-1} \times \{0\}$. The equation $(-\Delta)^{1/2}_{\mathbb{R}^{n-1}} u \perp T_u\mathcal{N}$ is the half-harmonic map equation; for an overview see [Da Lio and Rivière 2011b].

The equivalence of (1-1) and (1-2) is crucially related to the fact that we are considering critical points of an L^2-energy. Several notions of fractional p-harmonic maps have been proposed. $H^{s,p}$-harmonic maps, i.e., critical points of

$$\|(-\Delta)^{s/2} u\|^p_{L^p(\mathbb{R}^{n-1},\mathbb{R}^N)} \quad \text{such that } u(x) \in \mathcal{N} \text{ for a.e. } x \in \mathbb{R}^{n-1}, \tag{1-3}$$

were considered in [Da Lio and Schikorra 2014; 2017]. In [Schikorra 2015b] energies with a gradient-type structure were studied, namely

$$\|D^s u\|^p_{L^p(\mathbb{R}^{n-1},\mathbb{R}^N)} \quad \text{such that } u(x) \in \mathcal{N} \text{ for a.e. } x \in \mathbb{R}^{n-1}, \tag{1-4}$$

where $D^s = DI^{1-s}$ is the Riesz-fractional gradient; see also [Shieh and Spector 2015; 2018]. Finally, $W^{s,p}$-harmonic maps, that is, critical points of the energy

$$\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \quad \text{such that } u(x) \in \mathcal{N} \text{ for a.e. } x \in \mathbb{R}^{n-1}, \tag{1-5}$$

were studied in [Schikorra 2015a]; see also [Mazowiecka and Schikorra 2018]. All these versions of fractional p-harmonic maps have one thing in common: they do not seem related to a free boundary equation (1-1). For (1-3) and (1-4) this is clear, since the energies are defined on the "wrong" function space $H^{s,p}$. Indeed, a map in $W^{1,p}(D)$ has a trace in $W^{1-1/p,p}(\partial D)$, but $W^{1-1/p,p}(\partial D) \neq H^{1-1/p,p}(\partial D)$ for $p \neq 2$. For the $W^{s,p}$-energy (1-5) it is an interesting open problem if it is possible to find a p-harmonic extension that interprets this problem as a free boundary problem.

In this work we concentrate on free boundary problems. We focus on smooth bounded domains, so in the sequel D is such a domain. We prove regularity at the free boundary for critical points $u : D \to \mathbb{R}^N$ of the energy

$$\|\nabla u\|^p_{L^p(D,\mathbb{R}^N)} \quad \text{such that } u(\partial D) \subset \mathcal{N} \text{ in the a.e. trace sense.} \tag{1-6}$$

It is not clear that the space $\mathcal{A} := \{u \in W^{1,p}(D, \mathbb{R}^N) : u(\partial D) \subset \mathcal{N}\}$ possesses a natural structure of a smooth Banach manifold. That is why we shall define what we mean by critical point.

Definition 1.1. We say that u is a critical point of $\int_D |\nabla u|^p$ in the space $\mathcal{A}$ if u satisfies

$$\int_D |\nabla u|^{p-2}\,\nabla u\cdot\nabla\phi = 0 \tag{1-7}$$

for all ϕ in $W^{1,p}(D,\mathbb{R}^N)$ such that its trace $\phi(x)|_{\partial D}$ is in $T_{u(x)}\mathcal{N}$ a.e. Such a critical point is called a *p-harmonic map with free boundary*.

Equation (1-7) is obtained by requiring that for every C^1-path $\gamma:(-1,1)\to\mathcal{A}$ such that $\gamma(0)=u$ we have

$$\frac{d}{dt}\Big|_{t=0}\int_D |\nabla\gamma(t)|^p = 0. \tag{1-8}$$

Remark. Although this is not relevant for our purpose, let us remark that equation (1-7) can be interpreted as u satisfying in a distributional sense

$$\begin{cases}\operatorname{div}(|\nabla u|^{p-2}\,\nabla u)=0 & \text{in } D,\\ |\nabla u|^{p-2}\partial_\nu u\perp T_u\mathcal{N} & \text{on } \partial D.\end{cases} \tag{1-9}$$

Note that, by definition, u is a solution of (1-9) in the sense of distributions if and only if

$$\int_D |\nabla u|^{p-2}\,\nabla u\cdot\nabla\phi = 0 \tag{1-10}$$

for all $\phi\in C^\infty(\overline{D},\mathbb{R}^N)$ with $\phi(x)\in T_{u(x)}\mathcal{N}$ for $\mathcal{H}^{n-1}$-a.e. $x\in\partial D$. Indeed, taking $\phi\in C_c^\infty(D,\mathbb{R}^N)$ we obtain the interior equation

$$\operatorname{div}(|\nabla u|^{p-2}\,\nabla u)=0 \quad \text{in } D.$$

As for the boundary equation, we can see that if u is smooth enough and satisfies (1-10) then after an integration by parts we find

$$\int_{\partial D} |\nabla u|^{p-2}\,\partial_\nu u\cdot\phi = 0. \tag{1-11}$$

Since any $\phi\in C^\infty(\partial D,\mathbb{R}^N)$ with $\phi(x)\in T_{u(x)}\mathcal{N}$ can be extended in a function $\phi\in C^\infty(\overline{D},\mathbb{R}^N)$, (1-11) implies

$$|\nabla u|^{p-2}\,\partial_\nu u\perp T_u\mathcal{N} \quad \text{on } \partial D.$$

The equivalence between being a solution of (1-9) in the sense of distributions and being a critical point of the p-energy in the space $\mathcal{A}$ is true if u is smooth enough; for example $u\in C^1(\overline{D},\mathbb{R}^n)$ is sufficient. Indeed, in this case we can see that we have density of $\{\phi\in C^\infty(\overline{D},\mathbb{R}^N):\phi\in T_u\mathcal{N}\}$ in $\{\phi\in W^{1,p}(D,\mathbb{R}^N):\phi|_{\partial D}\in T_u\mathcal{N}\}$.

The natural starting point, when studying equations of the form (1-9), is the regularity theory. The interior regularity is known and follows from the interior equation and results of [Uhlenbeck 1977; Tolksdorf 1984]; see also [Kuusi and Mingione 2018]. Hence, the main difficulty is the regularity up to the boundary. For an arbitrary manifold $\mathcal{N}$ a regularity theory for a solution (1-9) is out of reach: even the regularity theory for the interior problem

$$\operatorname{div}(|\nabla u|^{p-2}\,\nabla u)\perp T_u\mathcal{N}$$

is known only for homogeneous targets $\mathcal{N}$; see [Fuchs 1993; Takeuchi 1994; Toro and Wang 1995; Strzelecki 1994; 1996; Schikorra and Strzelecki 2017]. For this reason we shall restrict our attention to the sphere $\mathbb{S}^{N-1} \subset \mathbb{R}^N$. In the rest of the paper we consider the problem

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 & \text{in } D, \\ |\nabla u|^{p-2}\partial_\nu u \perp T_u\mathbb{S}^{N-1} & \text{on } \partial D, \\ u(\partial D) \subset \mathbb{S}^{N-1}. \end{cases} \tag{1-12}$$

We remark that the free boundary conditions can be viewed as boundary conditions mixed between Dirichlet and homogeneous Neumann boundary conditions. Indeed, in the sphere case we have a Dirichlet boundary condition for the norm of u, $|u| = 1$ on ∂D, and a homogeneous Neumann condition for the "phase", $\partial_\nu(u/|u|) = 0$. To see that, in the case of a general manifold we can use Fermi coordinates near some points of $\mathcal{N}$, as explained in [Fraser 2000, pp. 938–939] in the context of minimal surfaces with free boundaries (for more on minimal surfaces with free boundaries, see also [Fraser and Schoen 2013]).

Our main theorem is the following ϵ-regularity-type theorem.

Theorem 1.2 (ϵ-regularity). *Let $D \subset \mathbb{R}^n$ be a smooth, bounded domain and $p \geq 2$. Then there exist $\epsilon = \epsilon(p, n, D) > 0$ and $\alpha = \alpha(p, n, D) > 0$ such that for any $u \in W^{1,p}(D, \mathbb{R}^N)$ solution to* (1-12) *the following holds: If for some $R > 0$ and for some $x_0 \in \overline{D}$*

$$\sup_{|y_0 - x_0| < R} \sup_{\rho < R} \rho^{p-n} \int_{B(y_0,\rho)\cap D} |\nabla u|^p < \epsilon, \tag{1-13}$$

then u and ∇u are Hölder continuous in $B(x_0, R/2) \cap \overline{D}$. Moreover, we have the estimates

$$\sup_{x,y \in B(x_0,R/2)} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \precsim R^{-\alpha} \left(\sup_{|y_0 - x_0| < R} \sup_{\rho < R} \rho^{p-n} \int_{B(y_0,\rho)\cap D} |\nabla u|^p \right)^{1/p},$$

$$\sup_{x,y \in B(x_0,R/2)} \frac{|\nabla u(x) - \nabla u(y)|}{|x - y|^\alpha} \precsim R^{-\alpha-1} \left(\sup_{|y_0 - x_0| < R} \sup_{\rho < R} \rho^{p-n} \int_{B(y_0,\rho)\cap D} |\nabla u|^p \right)^{1/p}.$$

When $p = n$ this ϵ-regularity implies directly (from the absolute continuity of the Lebesgue integral) that n-harmonic maps with free boundary and their gradients are Hölder continuous.

Corollary 1.3. *Let u and α be as in Theorem 1.2 with $p = n$. Then u is in $C^{1,\alpha}(\overline{D}, \mathbb{R}^N)$.*

As usual, an ϵ-regularity result such as Theorem 1.2 implies partial regularity for *stationary* p-harmonic maps with free boundary; see (6-1) for the definition.

Theorem 1.4 (partial regularity). *Let $D \subset \mathbb{R}^n$ be a smooth, bounded domain, $p \geq 2$, and assume that $u \in W^{1,p}(D, \mathbb{R}^N)$, with trace $u \in W^{1-1/p,p}(\partial D, \mathbb{S}^{N-1})$, is a* stationary *point of the energy* (1-6) *with free boundary. Then there exists a closed set $\Sigma \subset \overline{D}$ such that $\mathcal{H}^{n-p}(\Sigma) = 0$ and $u \in C^{1,\alpha}(\overline{D} \setminus \Sigma)$, where $\alpha > 0$ is from Theorem 1.2.*

Remark. Although some of our results work for unbounded domains we note that finite energy, stationary p-harmonic maps with free boundary satisfy a Liouville-type theorem; see Proposition 6.3. This is why we focus on bounded domains.

Moreover, besides giving regularity in the case $p = n$ and partial regularity in the case $p < n$, an ϵ-regularity could be useful to describe the possible loss of compactness of sequences of n-harmonic maps with free boundaries and an energy decomposition theorem. In the case $p = n = 2$, i.e., for harmonic maps with free boundaries, such a result was proven in [Da Lio 2015; Laurain and Petrides 2017]. Our case requires completely different methods, due to the nonlinearity of the p-Laplacian for $p \neq 2$.

Let us comment on our strategy for the proof of Theorem 1.2. The natural first attempt to prove a result like Theorem 1.2 is to adapt the beautiful geometric reflection method used in [Scheven 2006] to obtain an ϵ-regularity result up to the free boundary for harmonic maps, i.e., for the case $p = 2$ (see also [Berlyand and Mironescu 2008], where the authors also devised a reflection technique to prove regularity up to the boundary of solutions of some Ginzburg–Landau equations with free boundary conditions). This way, one would hope to be able to rewrite the Neumann condition at the boundary as an interior equation. For $p = 2$ the reflected equation has again the structure of a harmonic map (with a new metric in the reflected domain). Thus, the regularity theory for harmonic maps with a free boundary follows from the interior regularity for harmonic maps developed in [Hélein 1991]; see also [Rivière 2007]. For $p > 2$ there is a major drawback to that strategy: as mentioned above, the regularity theory for the interior p-harmonic map equation is only understood for round targets. It was not clear to us how to interpret the reflected equation as a map into such a round target. The reflection, which generates a somewhat "unnatural metric" seems to destroy our boundary sphere-structure. Indeed, up to now, only the regularity theory for *minimizing* p-harmonic maps with free boundary was understood; see [Duzaar and Gastel 1998; Müller 2002], where it is shown that such a map is in $C^{1,\alpha}$, for some α, outside a singular set $\mathcal{S}$ with $\dim_{\mathcal{H}}(\mathcal{S}) = n - \lfloor p \rfloor - 1$ and $\mathcal{S}$ is discrete if $n - 1 \leq p < n$. For $p = 2$, free boundary problems for *minimizing* harmonic maps were studied in [Duzaar and Steffen 1989; Hardt and Lin 1989].

In this work we follow in spirit the recent work [Schikorra 2018], which does not use a reflection technique, but rather computes an equation along the free boundary and applies a moving frame technique to this free boundary part of the equation itself. This strategy leads to *growth* estimates, Proposition 2.1, which for the critical case $n = p$ implies directly Hölder regularity of solutions. Once the growth estimates are established we can apply the reflection. Since the reflection is explicit, it is easy to see that the growth estimates still hold for the reflected solution, which we shall call v. Now v solves a critical or supercritical equation of the form

$$|\operatorname{div}(|\nabla v|^{p-2} \nabla v)| \precsim |\nabla v|^p.$$

In principle, solutions to this equation may be singular, e.g., $x/|x|$ or $\log \log 1/|x|$. But with the growth estimates from Proposition 2.1, which transfers to v, one can employ a blow-up argument due to [Hardt et al. 1986; Hardt and Lin 1987] and then bootstrap for higher regularity.

The outline of the paper is as follows: In Section 2 we state and prove the crucial growth estimate for solutions to (1-12). In Section 3 we show how this implies Hölder continuity of solutions for the case $p = n$. For $p < n$ we show in Section 4 how a generic supercritical system implies Hölder regularity of solutions once the growth estimates from Proposition 2.1 are guaranteed. Combining this with Scheven's reflection argument, we give in Section 5 the proof of Theorem 1.2. Finally, in Section 6, we prove the partial regularity of solutions, i.e., Theorem 1.4.

Notation. We denote by $B(x, r)$ the ball of radius r centered at $x \in \mathbb{R}^n$. We write $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times (0, \infty)$, $\mathbb{R}^n_- = \mathbb{R}^n \times (-\infty, 0)$, and $B^+(x, r) = B(x, r) \cap \mathbb{R}^n_+$. By $(u)_\Omega$ we denote the mean value of a map u on a set Ω; i.e., $(u)_\Omega = (1/|\Omega|) \int_\Omega u$.

2. The growth estimates

Recall that we assume that D is a bounded set with a smooth boundary. In view of Lemma A.1 we know that $|u| \le 1$ holds for any solution to (1-12). The arguments can be also extended to unbounded domains like $\mathbb{R}^n_+$ under the assumption that $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n_+)$; see Lemma A.2. Note that in principle, the constants may depend on the L^∞-norm of u.

The main result in this section, and the crucial argument in this work, is the following growth estimate that one could interpret as a kind of Caccioppoli-type estimate. We were not able to obtain such an estimate by a geometric reflection argument, since that reflection changes the metric, and only in the case of round targets, such as the sphere, is regularity theory (and in particular the related growth estimates) known.

Proposition 2.1 (growth estimates). *Let $p \ge 2$. There exists a radius R_0 depending only on ∂D such that for any $u \in W^{1,p}(D, \mathbb{R}^N)$ satisfying* (1-12) *the following holds*:

Whenever $B(x_0, R) \subset \mathbb{R}^n$, $R \in (0, R_0)$, is such that for some $\lambda \in (0, \infty)$ it holds

$$\sup_{B(y_0,r)\subset B(x_0,R)} r^{p-n} \int_{B(y_0,r)\cap D} |\nabla u|^p < \lambda^p, \tag{2-1}$$

we have, for any $B(y_0, 4r) \subset B(x_0, R)$ and any $\mu > 0$,

$$\int_{B(y_0,r)\cap D} |\nabla u|^p \le C(\lambda + \mu^{p-1}) \int_{B(y_0,4r)\cap D} |\nabla u|^p + C\mu^{-1} \int_{(B(y_0,4r)\setminus B(y_0,r))\cap D} |\nabla u|^p. \tag{2-2}$$

Alternatively, we have the following estimates:

If $B(y_0, 2r)\setminus D = \varnothing$, then

$$\int_{B(y_0,r)} |\nabla u|^p \le C\lambda \int_{B(y_0,4r)\cap D} |\nabla u|^p + C\lambda^{1-p} r^{-p} \int_{B(y_0,4r)\cap D} |u - (u)_{B(y_0,4r)\cap D}|^p. \tag{2-3}$$

If $B(y_0, 2r)\setminus D \ne \varnothing$, then

$$\int_{B(y_0,r)\cap D} |\nabla u|^p \le C\lambda \int_{B(y_0,4r)\cap D} |\nabla u|^p + C\lambda^{1-p} r^{-p} \int_{B(y_0,4r)\cap D} |u - (u)_{B(y_0,4r)\cap D}|^p + C\lambda^{1-p} r^{-p} \int_{B(y_0,4r)\cap D} ||u|^2 - 1|^p \tag{2-4}$$

for a constant $C = C(n, p, D)$.

Our strategy, in principle, is to adapt the method for harmonic maps into spheres developed in [Hélein 1990]; see [Strzelecki 1994] for the n-harmonic case. To motivate our approach, we briefly outline their strategy for a p-harmonic map $w \in W^{1,p}(D, \mathbb{S}^{N-1})$, i.e., a solution to

$$\mathrm{div}(|\nabla w|^{p-2} \nabla w) \perp T_w \mathbb{S}^{N-1}. \tag{2-5}$$

The first step is to rewrite this equation. Since $w \in \mathbb{S}^{N-1}$ we have $w \in (T_w\mathbb{S}^{N-1})^{\perp}$. Consequently, (2-5) can be rewritten in distributional sense as

$$\int_D |\nabla w|^{p-2}\, \nabla w^i \cdot \nabla\phi = \int_D |\nabla w|^{p-2}\, \nabla w^k \cdot \nabla(w^k w^i \phi), \tag{2-6}$$

which holds for all $\phi \in C_c^\infty(D)$ and $i = 1, \dots, N$. Here and henceforth, we use the summation convention.

Next, from $|w| \equiv 1$, we get $w^k \nabla w^k \equiv \frac{1}{2}\nabla|w|^2 = 0$. Consequently, (2-6) can be written as

$$\int_D |\nabla w|^{p-2}\, \nabla w^i \cdot \nabla\phi = \int_D |\nabla w|^{p-2}\, \nabla w^k \cdot (\nabla w^k\, w^i - \nabla w^i\, w^k)\,\phi. \tag{2-7}$$

Now one observes that from (2-6) a conservation law follows, a fact that for $p = n = 2$ was discovered by Shatah [1988],

$$\operatorname{div}(|\nabla w|^{p-2}(\nabla w^k\, w^i - \nabla w^i\, w^k)) = 0 \quad \text{in } D. \tag{2-8}$$

Thus, $|\nabla w|^{p-2}\, \nabla w^k \cdot (\nabla w^k\, w^i - \nabla w^i\, w^k)$ is a div-curl term and with the help of the celebrated result of Coifman, Lions, Meyer, and Semmes [Coifman et al. 1993], one obtains a growth estimate.

The above argument heavily relied on the fact that $w^k \nabla w^k \equiv 0$. It is important to observe that this trick will not work in the situation from Theorem 1.2: if we only know that $u|_{\partial D} \subset \mathbb{S}^{N-1}$, then there is no reason that $u \cdot \nabla u = 0$ in D. Nevertheless, we will stubbornly follow the strategy outlined above, just along the boundary ∂D, keeping the extra terms that involve $u^k \nabla u^k$. First, we find:

Lemma 2.2. *For $u \in W^{1,p}(D, \mathbb{R}^N)$ satisfying* (1-12) *we have*

$$\int_D |\nabla u|^{p-2}\, \nabla u^i \cdot \nabla\phi = \int_D |\nabla u|^{p-2}\, \nabla u^k \cdot \nabla(u^k u^i \phi)$$

for any $\phi \in W^{1,p}(D)$.

Let us stress that the test function ϕ above does not need to vanish at the boundary.

Proof. Let $\Phi = (0, \dots, \phi, \dots, 0)$ (only the i-th coordinate is nonzero and equal to ϕ). Observe that

$$\Phi - u\langle u, \Phi\rangle_{\mathbb{R}^N} \in T_u\mathbb{S}^{N-1} \quad \text{a.e. on } \partial D.$$

The claim follows now from the definition of p-harmonic maps with free boundary (1-7). □

Also we have the following conservation law.

Lemma 2.3. *Let $u \in W^{1,p}(D, \mathbb{R}^N)$ satisfy* (1-12). *Then, for*

$$\Omega_{ij} := (u^i \nabla u^j - u^j \nabla u^i),$$

we have

$$\operatorname{div}(|\nabla u|^{p-2}\Omega_{ij}) = 0 \quad \textit{in } D$$

up to the boundary. That is, for any $\phi \in C^\infty(\overline{D})$ and any $i, j = 1, \dots, N$,

$$\int_D |\nabla u|^{p-2}\, \Omega_{ij} \cdot \nabla\phi = 0. \tag{2-9}$$

Additionally, (2-9) *is also satisfied for every ϕ in $W^{1,p} \cap L^\infty(D)$.*

Proof. By the product rule,

$$\int_D \nabla\phi \cdot |\nabla u|^{p-2}\,(u^i\,\nabla u^j - u^j\,\nabla u^i) = \int_D (\nabla(\phi u^i)\cdot|\nabla u|^{p-2}\,\nabla u^j - \nabla(\phi u^j)\cdot|\nabla u|^{p-2}\,\nabla u^i).$$

Therefore, by Lemma 2.2, we find

$$\int_D |\nabla u|^{p-2}\,\Omega_{ij}\cdot\nabla\phi = \int_D |\nabla u|^{p-2}\,\nabla u^k\cdot\nabla(u^k u^i u^j\phi) - \int_D |\nabla u|^{p-2}\,\nabla u^k\cdot\nabla(u^k u^j u^i\phi) = 0. \qquad \square$$

We combine Lemmas 2.3 and 2.2. In contrast to the argument for the p-harmonic map w, we find additional terms. Namely, instead of having $w^k\,\nabla w^k \equiv 0$ we merely have $u^k\,\nabla u^k = \frac{1}{2}\nabla(|u|^2-1)$. However, it is an improvement, because $|u|^2-1 \in W_0^{1,p}(D)$.

Lemma 2.4. *Let $u \in W^{1,p}(D,\mathbb{R}^N)$ satisfy* (1-12). *Then for any $\phi \in W^{1,p}(D)$ we have*

$$\int_D |\nabla u|^{p-2}\,\nabla u^i\cdot\nabla\phi$$
$$= \int_D |\nabla u|^{p-2}\,\nabla u^k\cdot\Omega_{ik}\,\phi + \int_D |\nabla u|^{p-2}\,\nabla u^i\cdot\nabla(|u|^2-1)\,\phi + \frac{1}{2}\int_D |\nabla u|^{p-2}\,\nabla\phi\cdot\nabla(|u|^2-1)u^i.$$

It is important to observe that in particular we do not obtain an equation of the form $|\operatorname{div}(|\nabla u|^{p-2}\,\nabla u)| \precsim |\nabla u|^p$, as in the case for p-harmonic maps (i.e., the interior situation). This is why for $p < n$ we are forced to combine our growth estimate with the geometric reflection argument; see Proposition 5.3.

Proof of Lemma 2.4. By Lemma 2.2 we have for any $\phi \in C^\infty(\overline{D})$

$$\int_D |\nabla u|^{p-2}\,\nabla u^i\cdot\nabla\phi = \int_D |\nabla u|^{p-2}\,\nabla u^k\cdot\nabla u^k\,u^i\phi + \int_D |\nabla u|^{p-2}\,\nabla u^k\cdot u^k\nabla(u^i\phi).$$

Using the definition of Ω_{ik} from Lemma 2.3 we write

$$\int_D |\nabla u|^{p-2}\,\nabla u^i\cdot\nabla\phi = \int_D |\nabla u|^{p-2}\,\nabla u^k\cdot\Omega_{ik}\,\phi + 2\int_D |\nabla u|^{p-2}\,\nabla u^i\cdot\nabla u^k\,u^k\,\phi + \int_D |\nabla u|^{p-2}\,\nabla u^k\,u^i\,u^k\cdot\nabla\phi.$$

Since $u^k\,\nabla u^k = \frac{1}{2}\nabla(|u|^2-1)$, we have shown that

$$\int_D |\nabla u|^{p-2}\,\nabla u^i\cdot\nabla\phi$$
$$= \int_D |\nabla u|^{p-2}\,\nabla u^k\cdot\Omega_{ik}\,\phi + \int_D |\nabla u|^{p-2}\,\nabla u^i\cdot\nabla(|u|^2-1)\,\phi + \frac{1}{2}\int_D |\nabla u|^{p-2}\,\nabla\phi\cdot\nabla(|u|^2-1)u^i\cdot \qquad \square$$

For the second and third terms on the right-hand side of the equation in Lemma 2.4 we observe that $|u|^2-1$ has zero boundary values on ∂D. In addition, and this is another crucial ingredient here, we can choose u or (its coordinates) as a test function in Lemmas 2.2, 2.3, and 2.4 since u is in $W^{1,p}\cap L^\infty(D,\mathbb{R}^N)$ from Lemma A.1.

Moreover, in view of the interior equation for u, (1-9),

$$\int_D |\nabla u|^{p-2}\,\nabla u^i\cdot\nabla(|u|^2-1) = 0.$$

Proof of Proposition 2.1. For notational simplicity we prove the growth estimates when the boundary is flat. More precisely we treat the case where $B^+(0, R) \subset D \subset \mathbb{R}^n_+$ for some $R > 0$, and $\partial D \cap B(0, R) = \partial\mathbb{R}^n_+ \cap B(0, R)$. The following argument can be easily adapted to general D — here is where one has to choose $R_0 = R_0(D)$ for flattening the boundary. We leave the details to the reader. We also recall that, since we work in a smooth bounded domain, from Lemma A.1 we have that $\|u\|_{L^\infty(D)} \le 1$.

Let $\eta \in C_c^\infty(B(0, 2))$ be the typical bump function that is constantly 1 in $B(0, 1)$. Let $y_0 \in \mathbb{R}^n$, $r > 0$, be such that $B(y_0, 4r) \subset B(0, R)$. Define

$$\eta_{B(y_0,r)}(x) := \eta\left(\frac{x - y_0}{r}\right).$$

Set

$$\begin{aligned} \tilde{u} &:= \eta_{B(y_0,r)}(u - (u)_{B^+(y_0,2r)}), \\ \hat{u} &:= (1 - \eta_{B(y_0,r)})\eta_{B(y_0,r)}(u - (u)_{B^+(y_0,2r)}). \end{aligned}$$

Since $\eta_{B(y_0,r)} \equiv 1$ on $B(y_0, r)$ we have

$$\int_{B^+(y_0,r)} |\nabla u|^p \le \int_{\mathbb{R}^n_+} |\nabla u|^{p-2}\, \nabla\tilde{u} \cdot \nabla\tilde{u}.$$

We compute

$$\nabla\tilde{u} \cdot \nabla\tilde{u} = \nabla u \cdot \nabla\tilde{u} - \nabla u \cdot \nabla\hat{u} - \nabla\eta_{B(y_0,r)} \cdot \nabla u\, \tilde{u} + \nabla\eta_{B(y_0,r)}\, (u - (u)_{B^+(y_0,2r)}) \cdot \nabla\tilde{u}. \tag{2-10}$$

Since $|\nabla\eta_{B(y_0,r)}| \precsim r^{-1}$,

$$\int_{\mathbb{R}^n_+} |\nabla u|^{p-2}(\nabla\eta_{B(y_0,r)}\tilde{u}) \cdot \nabla u \precsim r^{-1} \int_{B^+(y_0,2r)\setminus B^+(y_0,r)} |\nabla u|^{p-1}|\tilde{u}|. \tag{2-11}$$

This can be further estimated in two ways. For the estimate (2-2), by Young and Poincaré inequalities we have for any $\mu > 0$

$$\int_{\mathbb{R}^n_+} |\nabla u|^{p-2}(\nabla\eta_{B(y_0,r)}\tilde{u}) \cdot \nabla u \precsim \frac{1}{\mu} \int_{B^+(y_0,2r)\setminus B^+(y_0,r)} |\nabla u|^p + \mu^{p-1} \int_{B^+(y_0,2r)} |\nabla u|^p.$$

For the estimates (2-3) and (2-4), by Young's inequality we have for any $\lambda > 0$

$$\int_{\mathbb{R}^n_+} |\nabla u|^{p-2}\, \nabla\eta_{B(y_0,r)} \cdot \nabla u\tilde{u} \precsim \lambda \int_{B^+(y_0,2r)} |\nabla u|^p + \lambda^{1-p} r^{-p} \int_{B^+(y_0,2r)} |u - (u)_{B^+(y_0,2r)}|^p.$$

For the last term of (2-10)

$$\begin{aligned} &\int_{\mathbb{R}^n_+} |\nabla u|^{p-2}\, \nabla\eta_{B(y_0,r)}\, (u - (u)_{B^+(y_0,2r)}) \cdot \nabla\tilde{u} \\ &\precsim r^{-2} \int_{B^+(y_0,2r)\setminus B^+(y_0,r)} |\nabla u|^{p-2}\, |u - (u)_{B^+(y_0,2r)}|^2 + r^{-1} \int_{B^+(y_0,2r)\setminus B^+(y_0,r)} |\nabla u|^{p-1}\, |u - (u)_{B^+(y_0,2r)}|. \end{aligned}$$

By a similar estimate, we easily get for any $\mu > 0$

$$\int_{\mathbb{R}^n_+} |\nabla u|^{p-2}\, \nabla\eta_{B(y_0,2r)}\, (u - (u)_{B^+(y_0,2r)}) \cdot \nabla\tilde{u} \precsim \frac{1}{\mu} \int_{B^+(y_0,2r)\setminus B^+(y_0,r)} |\nabla u|^p + \mu^{p-1} \int_{B^+(y_0,2r)} |\nabla u|^p$$

and for any $\lambda > 0$

$$\int_{\mathbb{R}^n_+} |\nabla u|^{p-2}\,\nabla \eta_{B(y_0,2r)}\,(u-(u)_{B^+(y_0,2r)})\cdot\nabla\tilde{u} \precsim \lambda \int_{B^+(y_0,2r)} |\nabla u|^p + \lambda^{1-p} r^{-p} \int_{B^+(y_0,2r)} |u-(u)_{B^+(y_0,2r)}|^p.$$

Consequently, we found

$$\int_{B^+(y_0,r)} |\nabla u|^p \precsim \left|\int_{\mathbb{R}^n_+} |\nabla u|^{p-2}\,\nabla u\cdot\nabla\tilde{u}\right| + \left|\int_{\mathbb{R}^n_+} |\nabla u|^{p-2}\,\nabla u\cdot\nabla\hat{u}\right| + \frac{1}{\mu}\int_{B^+(y_0,2r)\setminus B^+(y_0,r)} |\nabla u|^p + \mu^{p-1}\int_{B^+(y_0,2r)} |\nabla u|^p \tag{2-12}$$

and

$$\int_{B^+(y_0,r)} |\nabla u|^p \precsim \left|\int_{\mathbb{R}^n_+} |\nabla u|^{p-2}\,\nabla u\cdot\nabla\tilde{u}\right| + \left|\int_{\mathbb{R}^n_+} |\nabla u|^{p-2}\,\nabla u\cdot\nabla\hat{u}\right| + \lambda\int_{B^+(y_0,2r)} |\nabla u|^p + \lambda^{1-p} r^{-p}\int_{B^+(y_0,2r)} |u-(u)_{B^+(y_0,2r)}|^p. \tag{2-13}$$

If we are in the interior case, i.e., $B(y_0, 2r) \subset B^+(0, R)$, then $\operatorname{supp}\tilde{u} \cup \operatorname{supp}\hat{u} \subset B^+(0, R)$ and thus $\operatorname{div}(|\nabla u|^{p-2}\,\nabla u) = 0$ in $B^+(0, R)$ implies

$$\left|\int_{\mathbb{R}^n_+} |\nabla u|^{p-2}\,\nabla u\cdot\nabla\tilde{u}\right| + \left|\int_{\mathbb{R}^n_+} |\nabla u|^{p-2}\,\nabla u\cdot\nabla\hat{u}\right| = 0.$$

Thus, for $B(y_0, 2r) \subset B^+(0, R)$ the claim is proven.

From now on we assume that the ball $B(y_0, r)$ is close to the boundary; i.e,

$$B(y_0, 2r) \cap \{\mathbb{R}^{n-1} \times \{0\}\} \neq \varnothing.$$

By Lemma 2.4,

$$\int_{\mathbb{R}^n_+} |\nabla u|^{p-2}\,\nabla u^i\cdot\nabla\tilde{u}^i = I + II + \tfrac{1}{2} III,$$

where

$$I := \int_{\mathbb{R}^n_+} |\nabla u|^{p-2}\,\nabla u^k\cdot\Omega_{ik}\,\tilde{u}^i,$$

$$II := \int_{\mathbb{R}^n_+} |\nabla u|^{p-2}\,\nabla u^i\cdot\nabla(|u|^2-1)\tilde{u}^i,$$

$$III := \int_{\mathbb{R}^n_+} |\nabla u|^{p-2}\,\nabla\tilde{u}^i\cdot\nabla(|u|^2-1)u^i.$$

Since u is p-harmonic and by Lemma 2.3, all three terms above contain products of divergence-free and rotation-free quantities. However, the div-curl estimate by Coifman, Lions, Meyer, Semmes [Coifman et al. 1993] is only applicable when at least one term vanishes at the boundary; otherwise there are counterexamples. See [Da Lio and Palmurella 2017; Hirsch 2019].

The term I: Let $\widetilde{B} \subset B^+(0, R)$ be a smooth, bounded, open, and convex set such that $B^+(y_0, 2r) \subset \widetilde{B} \subset B(y_0, 3r)$ and $\partial \widetilde{B} \cap \partial \mathbb{R}^n_+ = B(y_0, 2r) \cap \partial \mathbb{R}^n_+$. By Hodge decomposition[1] (see [Iwaniec and Martin 2001, (10.4)]) we find $\xi_{ik} \in W^{1,p'}(\widetilde{B})$, with $p' = p/(p-1)$, and $\zeta_{ik} \in W^{1,p'}_0(\widetilde{B}, \bigwedge^2 \mathbb{R}^n)$ such that

$$|\nabla u|^{p-2}\Omega_{ik} = \nabla \xi_{ik} + \operatorname{Curl} \zeta_{ik} \quad \text{in } \widetilde{B}. \tag{2-14}$$

Moreover, we have

$$\|\zeta_{ik}\|_{W^{1,p'}(\widetilde{B})} \precsim \||\nabla u|^{p-2}\Omega_{ik}\|_{L^{p'}(B(y_0,3r))}. \tag{2-15}$$

The boundary data of ζ and Lemma 2.3 imply

$$\int_{\widetilde{B}} \nabla \xi_{ik} \cdot \nabla \phi = \int_{\widetilde{B}} |\nabla u|^{p-2}\Omega_{ik} \cdot \nabla \phi - \int_{\widetilde{B}} \operatorname{Curl} \zeta_{ik} \cdot \nabla \phi = 0 \quad \text{for any } \phi \in C^\infty(\overline{\widetilde{B}}).$$

That is, ξ_{ik} is harmonic with trivial Neumann data, and thus ξ_{ik} is constant. In particular, (2-14) simplifies to

$$|\nabla u|^{p-2}\Omega_{ik} = \operatorname{Curl} \zeta_{ik} \quad \text{in } \widetilde{B}. \tag{2-16}$$

Consequently,

$$I = \int_{\mathbb{R}^n_+} \operatorname{Curl} \zeta_{ik} \cdot \nabla u^k \, \tilde{u}^i = \int_{\mathbb{R}^n} \operatorname{Curl} \zeta_{ik} \cdot \nabla u^k \, \tilde{u}^i.$$

The last equality is true, since ζ_{ik} vanishes on $\partial \mathbb{R}^n_+ \cap B(0, R)$ and we can extend it by zero to $\mathbb{R}^n_- \cap B(0, R)$. Now we use the div-curl structure and apply the result by Coifman, Lions, Meyer, Semmes [Coifman et al. 1993]. Recall that BMO is the space of functions f with finite seminorm $[f]_{\mathrm{BMO}} < \infty$. Here,

$$[f]_{\mathrm{BMO}} := \sup_B |B|^{-1} \int_B |f - (f)_B|,$$

where the supremum is taken over all balls B. Observe that by the Poincaré inequality,

$$[f]_{\mathrm{BMO}} \precsim \sup_{x_0 \in \mathbb{R}^n,\, \rho > 0} \left(\rho^{p-n} \int_{B(x_0,\rho)} |\nabla f|^p \right)^{1/p}. \tag{2-17}$$

Coifman, Lions, Meyer, Semmes [Coifman et al. 1993] showed that the inequality

$$\int_{\mathbb{R}^n} F \cdot G \phi \precsim \|F\|_{L^p(\mathbb{R}^n)} \|G\|_{L^{p'}(\mathbb{R}^n)} [\phi]_{\mathrm{BMO}}$$

holds whenever F and G are vector fields such that $\operatorname{div} F = 0$ and $\operatorname{curl} G = 0$. See also [Lenzmann and Schikorra 2020] for a different proof. In our situation this inequality implies[2]

$$\begin{aligned} |I| &\precsim \||\nabla u|^{p-2}\Omega_{ik}\|_{L^{p'}(B^+(y_0,4r))} \|\nabla u\|_{L^p(B^+(y_0,4r))} [\tilde{u}]_{\mathrm{BMO}} \\ &\precsim \|\nabla u\|^p_{L^p(B^+(y_0,4r))} [\tilde{u}]_{\mathrm{BMO}}. \end{aligned} \tag{2-18}$$

[1] More precisely, one argues, e.g., as in [Schikorra 2010, (3.6), (3.7)]: One solves the system $\Delta \zeta_{ik} = \operatorname{curl}(|\nabla u|^{p-2}\Omega_{ik})$ in $\widetilde{B}$, $\zeta_{ik} = 0$ on $\partial \widetilde{B}$, such that (2-15) is satisfied. Then one sets $H := |\nabla u|^{p-2}\Omega_{ik} - \operatorname{Curl} \zeta_{ik}$. By the Poincaré lemma we can write $H = \nabla \xi$.

[2] Here, $\tilde{u}$ is extended into the whole space $\mathbb{R}^n$ in such a way that $[\tilde{u}]_{\mathrm{BMO}} \precsim \lambda$. This can be done by an appropriate reflection of u outside of $B^+(y_0, 3r)$.

The last estimate follows readily from the definition of Ω in Lemma 2.3. Thus, for the λ from (2-1) we obtain

$$|I| \precsim \lambda \int_{B^+(y_0,4r)} |\nabla u|^p.$$

<u>The term II</u>: Since $\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0$ in $B^+(0, R)$, there exists $\zeta_i \in W^{1,p}\big(B^+(y_0, 2r), \bigwedge^2 \mathbb{R}^n\big)$ such that

$$|\nabla u|^{p-2} \nabla u^i = \operatorname{Curl} \zeta_i \quad \text{in } B^+(y_0, 2r).$$

We can extend ζ to all of $\mathbb{R}^n$ so that

$$\|\zeta\|_{W^{1,p'}(\mathbb{R}^n)} \precsim \|\nabla u\|^{p-1}_{L^p(B^+(y_0,2r))}.$$

Also, since u is assumed to be bounded we have $|u|^2 \in W^{1,p}(B^+(0, R))$, and in the sense of traces $|u|^2 \equiv 1$ on $B(0, R) \cap \{\mathbb{R}^{n-1} \times \{0\}\}$. This is equivalent to saying that the extension of $|u|^2 - 1$ by zero to $B(0, R) \cap \mathbb{R}^n_-$ belongs to $W^{1,p}(B(0, R))$; that is, we have, $(|u|^2 - 1)\chi_{\mathbb{R}^n_+} \in W^{1,p}(B(0, R))$ and the distributional gradient satisfies

$$\nabla((|u|^2 - 1)\chi_{\mathbb{R}^n_+}) = \chi_{\mathbb{R}^n_+} \nabla |u|^2 \quad \text{a.e. in } B(0, R).$$

In particular, since $(|u|^2 - 1)\chi_{\mathbb{R}^n_+}$ is zero on $B(y_0, 2r) \cap \mathbb{R}^n_-$ we can use the Poincaré inequality to get

$$\||u|^2 - 1\|_{L^p(B^+(y_0,2r))} \precsim r\, \|u\|_{L^\infty(B^+(y_0,4r))}\, \|\nabla u\|_{L^p(B^+(y_0,4r))}. \tag{2-19}$$

By using that $|\nabla \eta_{B(y_0,2r)}| \precsim r^{-1}$, (2-17), the triangle inequality in L^p and (2-19), for the λ from (2-1),

$$[(|u|^2 - 1)\chi_{\mathbb{R}^n_+} \eta_{B(y_0,2r)}]_{\mathrm{BMO}} \precsim \lambda.$$

We also observe that $\nabla \tilde{u} \equiv \eta_{B(y_0,2r)} \nabla \tilde{u}$. Thus, integrating by parts we obtain

$$II = -\int_{\mathbb{R}^n} \operatorname{Curl} \zeta \cdot \nabla \tilde{u}^i \;\; (|u|^2 - 1)\chi_{\mathbb{R}^n_+} \eta_{B(y_0,2r)}.$$

Hence, with the div-curl theorem from [Coifman et al. 1993], see also the localized version [Strzelecki 1994, Corollary 3], we find

$$|II| \precsim \lambda \|\nabla u\|^p_{L^p(B^+(y_0,4r))}.$$

<u>The term III</u>: Observe that

$$\begin{aligned}
&\nabla \tilde{u}^i \cdot \nabla(|u|^2 - 1)u^i \\
&= \nabla u^i \cdot \nabla(|u|^2 - 1)\eta_{B(y_0,r)} u^i + \nabla \eta_{B(y_0,r)}\, (u^i - (u^i)_{B^+(y_0,2r)}) \cdot \nabla(|u|^2 - 1)u^i \\
&= \nabla u^i \cdot \nabla(|u|^2 - 1)\tilde{u}^i + \nabla u^i \cdot \nabla(|u|^2 - 1)\eta_{B(y_0,r)}(u^i)_{B^+(y_0,2r)} + \nabla \eta_{B(y_0,r)}\, (u^i - (u^i)_{B^+(y_0,2r)}) \cdot \nabla(|u|^2 - 1)u^i.
\end{aligned}$$

By integration by parts, using that $\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0$ in $B^+(0, R)$, $|u|^2 - 1$ is zero on $\partial\mathbb{R}^n_+ \cap B(0, R)$ and then arguing as in the argument for II,

$$\int_{\mathbb{R}^n_+} |\nabla u|^{p-2} \nabla u^i \cdot \nabla(|u|^2 - 1)\tilde{u}^i = -\int_{\mathbb{R}^n_+} |\nabla u|^{p-2} \nabla u^i \cdot \nabla \tilde{u}^i (|u|^2 - 1) \precsim \lambda\, \|\nabla u\|^p_{L^p(B^+(y_0,4r))}.$$

Moreover, again since $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$ in $B^+(0,R)$ and $|u|^2-1$ is zero on $\partial\mathbb{R}^n_+ \cap B(0,R)$,

$$\left|\int_{\mathbb{R}^n_+} |\nabla u|^{p-2}\nabla u^i \cdot \nabla(|u|^2-1)\eta_{B(y_0,r)}(u^i)_{B^+(y_0,2r)}\right|$$
$$= \left|\int_{\mathbb{R}^n_+} |\nabla u|^{p-2}\nabla u^i \cdot (|u|^2-1)\nabla\eta_{B(y_0,r)}(u^i)_{B^+(y_0,2r)}\right|$$
$$\precsim r^{-1}\,\|u\|_{L^\infty(B^+(0,R))}\,\|\nabla u\|^{p-1}_{L^p(B^+(y_0,2r)\setminus B^+(y_0,r))}\|\,|u|^2-1\|_{L^p(B^+(y_0,2r))}.$$

This leads to two estimates. Firstly, if we want to find (2-4), by Young's inequality,

$$\int_{\mathbb{R}^n_+} |\nabla u|^{p-2}\nabla u^i \cdot \nabla(|u|^2-1)\eta_{B(y_0,r)}(u^i)_{B^+(y_0,r)} \precsim \lambda\|\nabla u\|^p_{L^p(B^+(y_0,2r))} + \lambda^{1-p}\,r^{-p}\|\,|u|^2-1\|^p_{L^p(B^+(y_0,2r))}.$$

Secondly, for (2-2) by (2-19) and by Young's inequality we have for any $\mu > 0$

$$\int_{\mathbb{R}^n_+} |\nabla u|^{p-2}\nabla u^i \cdot \nabla(|u|^2-1)\eta_{B(y_0,r)}(u^i)_{B^+(y_0,2r)} \precsim \mu^{-1}\|\nabla u\|^p_{L^p(B^+(y_0,2r)\setminus B^+(y_0,r))} + \mu^{p-1}\|\nabla u\|^p_{L^p(B^+(y_0,2r))}.$$

The last remaining term can be treated in a similar way and we have

$$\int_{\mathbb{R}^n_+} |\nabla u|^{p-2}\nabla\eta_{B(y_0,r)}\,(u^i-(u^i)_{B^+(y_0,2r)})\cdot\nabla(|u|^2-1)\,u^i$$
$$\precsim \mu^{-1}\|\nabla u\|^p_{L^p(B^+(y_0,2r)\setminus B^+(y_0,r))} + \mu^{p-1}\|\nabla u\|^p_{L^p(B^+(y_0,2r))}$$

and

$$\int_{\mathbb{R}^n_+} |\nabla u|^{p-2}\nabla\eta_{B(y_0,r)}\,(u^i-(u^i)_{B^+(y_0,2r)})\cdot\nabla(|u|^2-1)\,u^i$$
$$\precsim \lambda\|\nabla u\|^p_{L^p(B^+(y_0,2r))} + \lambda^{1-p}\,r^{-p}\|u-(u)_{B^+(y_0,2r)}\|^p_{L^p(B^+(y_0,2r))}.$$

Combining the estimates of *I*, *II*, and *III* and plugging them into estimates (2-12) and (2-13), we conclude. □

3. Hölder regularity for the case $p = n$

For the case $p = n$, Hölder continuity of the solution u from Theorem 1.2 follows from Proposition 2.1 by a standard iteration argument. For higher regularity, and for $p < n$, we need to combine the growth estimates from Proposition 2.1 with the reflection method.

Proposition 3.1 (ϵ-regularity for $p = n$: Hölder continuity). *Let $D \subset \mathbb{R}^n$ be a smooth, bounded domain. Then there are positive constants $\epsilon = \epsilon(n, D)$, $\alpha = \alpha(n, D)$ such that the following holds for $p = n$:*

Any solution $u \in W^{1,n}(D, \mathbb{R}^N)$ to (1-12) *that satisfies, for $R > 0$ and for $x_0 \in \overline{D}$,*

$$\int_{B(x_0,R)\cap D} |\nabla u|^n < \epsilon$$

is Hölder continuous in $B(x_0, R/2) \cap \overline{D}$. Moreover, we have the estimate

$$\sup_{x,y\in B(x_0,R/2)\cap\overline{D}} \frac{|u(x)-u(y)|}{|x-y|^\alpha} \precsim R^{-\alpha}\|\nabla u\|_{L^n(B(x_0,R)\cap D)}.$$

Proof. Let $\lambda := \epsilon^{1/n}$ and apply Proposition 2.1 to any $B(y_0, 4r) \subset B(x_0, R/2)$, for $\mu > 0$ to be chosen below. We add

$$C\mu^{-1} \int_{B(y_0,r)\cap D} |\nabla u|^n$$

to both sides of (2-2). Then we find

$$(1 + C\mu^{-1}) \int_{B(y_0,r)\cap D} |\nabla u|^n \le C\, (\epsilon^{1/n} + \mu^{n-1} + \mu^{-1}) \int_{B(y_0,4r)\cap D} |\nabla u|^n.$$

We choose $\epsilon, \mu > 0$ small enough so that $\tau < 1$, where

$$\tau := \left(\frac{C(\epsilon^{1/n} + \mu^{n-1} + \mu^{-1})}{1 + C\mu^{-1}} \right)^{1/n}.$$

We have for any $B(y_0, 4r) \subset B(x_0, R/2)$

$$\|\nabla u\|_{L^n(B(y_0,r)\cap D)} \le \tau \|\nabla u\|_{L^n(B(y_0,4r)\cap D)}.$$

Iterating this on successively smaller balls, see, e.g., [Giaquinta 1983, Chapter III, Lemma 2.1], we find that for a uniform $\alpha = \alpha(\tau) > 0$ and for any $B(y_0, 4r) \subset B(x_0, R/2)$,

$$\|\nabla u\|_{L^n(B(y_0,r)\cap D)} \precsim \left(\frac{r}{R}\right)^{\alpha} \|\nabla u\|_{L^n(B(x_0,R)\cap D)}.$$

In particular, we have by the Poincaré inequality

$$\sup_{B(y_0,4r)\subset B(x_0,R/2)} r^{-\alpha-1} \|u - (u)_{B(y_0,r)\cap D}\|_{L^n(B(y_0,r)\cap D)} \precsim R^{-\alpha}\, \|\nabla u\|_{L^n(B(x_0,R)\cap D)}.$$

By the characterization of Campanato spaces and Hölder spaces, e.g., see [Giaquinta 1983, Chapter III, p. 75], this implies

$$\sup_{x,y\in B(x_0,R/2)\cap \overline{D}} \frac{|u(x) - u(y)|}{|x-y|^{\alpha}} \precsim R^{-\alpha} \|\nabla u\|_{L^n(B(x_0,R)\cap D)}. \qquad \square$$

4. Hölder-continuity for solutions to a supercritical system

In Proposition 2.1 we showed that solutions from Theorem 1.2 satisfy certain growth estimates. For $p = n$ these growth estimates imply Hölder continuity by an iteration argument, as we have seen in Proposition 3.1.

For $p < n$ more work is needed. The following proposition shows that, under a smallness assumption, solutions to systems satisfying

$$|\operatorname{div}(|\nabla u|^{p-2}\, \nabla u)| \precsim |\nabla u|^p \tag{4-1}$$

are Hölder continuous once the growth conditions from Proposition 2.1 are satisfied, that is, when (4-5) and (4-6) below are assumed a priori. Observe that without assuming a priori the growth conditions (4-5) and (4-6) below on the solution u, there is no hope for proving *any* regularity for solutions to systems that have the structure of (4-1). Indeed, it is easy to check that $\log\log(2/|x|)$ and $\sin\log\log(2/|x|)$ satisfy (4-1) for $p = n$.

In the next section, in order to prove Theorem 1.2, we use the reflection method from [Scheven 2006] to obtain an equation of the form (4-2). Since we already obtained the necessary growth estimates in Proposition 2.1, the following proposition then leads to regularity.

Proposition 4.1. *Let $D \subset \mathbb{R}^n$ be a smooth, bounded domain and let $\mathcal{M}$ be a smooth, compact $(n-1)$-dimensional manifold. Assume that $u \in W^{1,p}(D, \mathbb{R}^N)$ is a solution to*

$$\operatorname{div}(|G(x)\nabla u(x)|^{p-2}\, G(x)\nabla u(x)) = f_u(x), \tag{4-2}$$

where $f_u \in L^1(D, \mathbb{R}^N)$ satisfies the estimate

$$|f_u(x)| \le C|\nabla u(x)|^p \tag{4-3}$$

and $G \in C^\infty(\bar{D}, \mathrm{GL}(n))$.

Moreover, assume a priori that for every $B(x_0, R) \subset D$, $\lambda > 0$, such that

$$\sup_{B(y_0,r)\subset B(x_0,R)} r^{p-n} \int_{B(y_0,r)} |\nabla u|^p < \lambda^p, \tag{4-4}$$

the solution u already satisfies the following growth condition on any $B(y_0, 4r) \subset B(x_0, R)$:

If $B(y_0, 2r) \cap \mathcal{M} = \varnothing$, then

$$\int_{B(y_0,r)} |\nabla u|^p \le C\lambda \int_{B(y_0,4r)} |\nabla u|^p + C\lambda^{1-p} r^{-p} \int_{B(y_0,4r)} |u - (u)_{B(y_0,4r)}|^p \tag{4-5}$$

and, if $B(y_0, 2r) \cap \mathcal{M} \neq \varnothing$, then

$$\begin{aligned}\int_{B(y_0,r)} |\nabla u|^p \le C\lambda \int_{B(y_0,4r)} |\nabla u|^p &+ C\lambda^{1-p} r^{-p} \int_{B(y_0,4r)} |u - (u)_{B(y_0,4r)}|^p \\ &+ C\lambda^{1-p} r^{-p} \int_{B(y_0,4r)} |u - (u)_{B(y_0,4r)\cap\mathcal{M}}|^p \\ &+ C\lambda^{1-p} r^{1-p} \int_{B(y_0,4r)\cap\mathcal{M}} |u - (u)_{B(y_0,4r)\cap\mathcal{M}}|^p.\end{aligned} \tag{4-6}$$

Then there exist constants $\alpha = \alpha(G, p, n, C, D)$, $\epsilon > 0$ such that if (4-4) *holds on some $B(x_0, R) \subset D$ for $\lambda < \epsilon$, then $u \in C^\alpha(B(x_0, R/2), \mathbb{R}^N)$. Moreover, we have the estimate*

$$\sup_{x,y\in B(x_0,R/2)} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \le C_0\, R^{-\alpha} \Big(\sup_{B(y_0,r)\subset B(x_0,R)} r^{p-n} \int_{B(y_0,r)} |\nabla u|^p \Big)^{1/p}.$$

The constant C_0 depends on $\mathcal{M}$, D, C, and G.

To prove Proposition 4.1 we follow the strategy developed in [Hardt et al. 1986; Hardt and Lin 1987, Theorem 2.4]. The crucial result is that the equation for u together with the growth assumptions (4-5) and (4-6) on u imply the following decay estimate.

Proposition 4.2. *There are uniform constants $\epsilon, \theta \in (0, 1)$ and $\bar{R} = \bar{R}(\mathcal{M}) \in (0, 1)$ so that the following holds:*

Let u and D be as in Proposition 4.1 and assume that for a ball $B(x_0, R) \subset D$ and $R \in (0, \bar{R})$ it holds

$$E(x_0, R)(u) := \sup_{B(y_0,r)\subset B(x_0,R)} r^{p-n} \int_{B(y_0,r)} |\nabla u|^p < \epsilon^p. \tag{4-7}$$

Then

$$E(x_0, \theta R)(u) \le \tfrac{1}{2} E(x_0, R)(u). \tag{4-8}$$

Proof. It suffices to prove

$$(\theta R)^{p-n} \int_{B(y_0,\theta R)} |\nabla u|^p \le \tfrac{1}{2} E(x_0, R)(u) \quad \text{for any } B(y_0, 4\theta R) \subset B(x_0, R/2). \tag{4-9}$$

Indeed, (4-8) follows from (4-9) by taking smaller θ and observing that $B(x_1, R_1) \subset B(x_2, R_2)$ implies $E(x_1, R_1)(u) \le E(x_2, R_2)(u)$.

Assume the claim (4-9) is false. Then, for any $\theta \in (0, 1)$ we have a sequence of balls with $B(y_i, 4\theta R_i) \subset B(x_i, R_i/2) \subset D$, a sequence $(\epsilon_i)_{i=1}^{\infty}$ satisfying $\lim_{i\to\infty} \epsilon_i = 0$, and a sequence $(u_i)_{i=1}^{\infty} \subset W^{1,p}(D, \mathbb{R}^N)$ of solutions to (4-2) satisfying the growth assumptions of Proposition 4.1 such that

$$\sup_{B(y,r)\subset B(x_i,R_i)} r^{p-n} \int_{B(y,r)} |\nabla u_i|^p = \epsilon_i^p, \tag{4-10}$$

but

$$(\theta R_i)^{p-n} \int_{B(y_i,\theta R_i)} |\nabla u_i|^p > \tfrac{1}{2}\epsilon_i^p. \tag{4-11}$$

For simplicity, we assume that $R_i \equiv R_0$ and $x_i \equiv x_0$ for some $R_0 > 0$ and $x_0 \in \mathbb{R}^n$.

This is no loss of generality, since we can rescale the maps u by the factor R_0/R_i. Observe that this rescales the manifold $\mathcal{M}$, but in a way that (4-6) still holds. Set

$$w_i := \frac{1}{\epsilon_i}(u_i - (u_i)_{B(x_0,R_0)}).$$

Clearly,

$$(w_i)_{B(x_0,R_0)} = 0 \quad \text{for all } i \in \mathbb{N}.$$

Thus, we can apply the Poincaré inequality and have by (4-10)

$$\sup_{i\in\mathbb{N}} \|\nabla w_i\|^p_{L^p(B(x_0,R_0))} \precsim R_0^{n-p} \quad \text{and} \quad \sup_{i\in\mathbb{N}} \|w_i\|^p_{L^p(B(x_0,R_0))} \precsim R_0^{n-p+1}.$$

Thus, up to a subsequence denoted again by w_i, we find $w \in W^{1,p}(B(x_0, R_0), \mathbb{R}^N)$ such that as $i \to \infty$

$$\begin{aligned}
&w_i \rightharpoonup w \quad \text{weakly in } W^{1,p}(B(x_0, R_0)),\\
&w_i \to w \quad \text{strongly in } L^p(B(x_0, R_0)),\\
&w_i \to w \quad \text{strongly in } L^p(B(x_0, R_0) \cap \mathcal{M}, d\mathcal{H}^{n-1}),\\
&w_i \to w \quad \mathcal{H}^n\text{-a.e. on } B(x_0, R_0) \text{ and } \mathcal{H}^{n-1}\text{-a.e. on } B(x_0, R_0) \cap \mathcal{M}.
\end{aligned}$$

In particular,

$$(w)_{B(x_0,R_0)} = 0, \tag{4-12}$$

and also

$$\|\nabla w\|^p_{L^p(B(x_0,R_0))} \precsim R_0^{n-p} \quad \text{and} \quad \|w\|^p_{L^p(B(x_0,R_0))} \precsim R_0^{n-p+1}.$$

Moreover, for any $\phi \in C_c^\infty(B(x_0, R_0))$,

$$\int_{B(x_0,R_0)} |G\nabla w_i|^{p-2}\, G\nabla w_i \cdot \nabla\phi = (\epsilon_i)^{1-p} \int_{B(x_0,R_0)} |G\nabla u_i|^{p-2}\, G\nabla u_i \cdot \nabla\phi.$$

Now, by (4-2) and (4-3),

$$\left|\int_{B(x_0,R_0)} |G\nabla w_i|^{p-2}\, G\nabla w_i \cdot \nabla\phi\right| \precsim (\epsilon_i)^{1-p} \|\phi\|_{L^\infty(B(x_0,R_0))}\, \|\nabla u_i\|^p_{L^p(B(x_0,R_0))}.$$

That is, by (4-10)

$$\left|\int_{B(x_0,R_0)} |G\nabla w_i|^{p-2}\, G\nabla w_i \cdot \nabla\phi\right| \precsim \|\phi\|_{L^\infty(B(x_0,R_0))} R_0^{n-p} \epsilon_i \le \epsilon_i \|\phi\|_{L^\infty(B(x_0,R_0))}.$$

Now as in [Dolzmann et al. 1997, Section 4]

$$\operatorname{div}(|G\nabla w|^{p-2}\, G\nabla w) = 0 \quad \text{in } B(x_0, R_0). \tag{4-13}$$

From (4-12) and the Lipschitz estimates for solutions to (4-13), see [Uhlenbeck 1977; Mingione 2011; Duzaar and Mingione 2011; Kuusi and Mingione 2012, (1.7)], we have for any $B(z, r) \subset B(x_0, R_0/2)$

$$r^{-n} \int_{B(z,r)} |w - (w)_{B(z,r)}|^p \precsim r^p,$$

and if additionally $B(z, r) \cap \mathcal{M} \neq \varnothing$ and $r < \bar{R}$ for $\bar{R} = \bar{R}(\mathcal{M})$ small enough, then

$$r^{1-n} \int_{\mathcal{M}\cap B(z,r)} |w - (w)_{\mathcal{M}\cap B(z,r)}|^p + r^{-n} \int_{B(z,r)} |w - (w)_{\mathcal{M}\cap B(z,r)}|^p \precsim r^p.$$

On the other hand, by strong L^p-convergence of w_i to w, we find $i(\theta) \in \mathbb{N}$ so that for $i \ge i(\theta)$ and for any $r \in (\theta R_0, R_0)$ such that $B(z, r) \subset B(x_0, R_0)$

$$r^{1-n} \int_{B(z,r)\cap\mathcal{M}} |w_i - w|^p + r^{-n} \int_{B(z,r)} |w_i - w|^p \le \theta^p.$$

Combining these estimates we get for any $i \ge i(\theta)$ and for any $r \in (\theta R_0, R_0)$ such that $B(z, r) \subset B(x_0, R_0/2)$

$$r^{-n} \int_{B(z,r)} |u_i - (u_i)_{B(z,r)}|^p = \epsilon_i^p r^{-n} \int_{B(z,r)} |w_i - (w_i)_{B(z,r)}|^p \precsim \epsilon_i^p (r^p + \theta^p).$$

If additionally $B(z, r) \cap \mathcal{M} \neq \varnothing$, then

$$r^{-n} \int_{B(z,r)} |u_i - (u_i)_{B(z,r)\cap\mathcal{M}}|^p = \epsilon_i^p\, r^{-n} \int_{B(z,r)} |w_i - (w_i)_{B(z,r)\cap\mathcal{M}}|^p \precsim \epsilon_i^p (r^p + \theta^p)$$

and

$$r^{1-n} \int_{B(z,r)\cap\mathcal{M}} |u_i - (u_i)_{B(z,r)\cap\mathcal{M}}|^p \precsim \epsilon_i^p (r^p + \theta^p).$$

We now apply the growth estimates (4-5) and (4-6) of the solutions u_i with $\lambda = \epsilon_0 \geq \epsilon_i$ to find

$$(\theta R_0)^{p-n} \int_{B(y_i, \theta R_0)} |\nabla u_i|^p \leq C \epsilon_i^p (\epsilon_0 + \epsilon_0^{1-p} \theta^p).$$

Choosing ϵ_0 and θ sufficiently small so that $\epsilon_0 + \epsilon_0^{1-p}\theta^p < \frac{1}{2}$, we arrive at a contradiction with (4-11). □

Proof of Proposition 4.1. We argue as in the proof of Proposition 3.1: Assume that (4-4) is satisfied on $B(x_0, R)$ for some $\lambda < \epsilon$. Iterating the estimate from Proposition 4.2 on successively smaller balls, see [Giaquinta 1983, Chapter III, Lemma 2.1], we find a small $\alpha > 0$ such that for all $r < R$ and $B(y_0, r) \subset B(x_0, R/2)$

$$r^{p-n} \int_{B(y_0, r)} |\nabla u|^p \precsim \left(\frac{r}{R}\right)^{\alpha p} E(x_0, R).$$

In particular, for all $r < R$ and $B(y_0, r) \subset B(x_0, R/2)$,

$$r^{-\alpha p - n} \int_{B(y_0, r)} |u - (u)_{B(y_0, r)}|^p \precsim r^{p - \alpha p - n} \int_{B(y_0, r)} |\nabla u|^p \precsim R^{-\alpha p} E(x_0, R).$$

We conclude by the identification of Campanato and Hölder spaces; see [Giaquinta 1983, Chapter III, p. 75]. □

5. ϵ-regularity: proof of Theorem 1.2

The proof of Theorem 1.2 is a combination of the growth estimate for solutions, Proposition 2.1, the reflection method as in [Scheven 2006], and Proposition 4.1. More precisely, we use the reflection method to find a solution to (4-2) from Proposition 4.1. The growth estimates (4-5) and (4-6) required in Proposition 4.1 come from Proposition 2.1: they hold for the unreflected solution and by an easy argument hold also for the reflection. To set up the reflection method we first gather some standard results.

Lemma 5.1. *Let D be a smooth, bounded domain in $\mathbb{R}^n$. There exists some $R_0 = R_0(D)$ such that the following holds for any $R \in (0, R_0)$. Let $u \in W^{1,p}(D, \mathbb{R}^N)$ be a solution to* (1-12) *and $\epsilon \in (0, 1)$. If*

$$\sup_{B(y_0, r) \subset B(x_0, R)} r^{p-n} \int_{B(y_0, r) \cap D} |\nabla u|^p < \epsilon^p \tag{5-1}$$

and $B(x_0, R/2) \cap \partial D \neq \varnothing$, then

$$\sup_{x \in B(x_0, R/2) \cap D} \operatorname{dist}(u(x), \mathbb{S}^{N-1}) \leq C\epsilon.$$

Here C is a constant depending on ∂D.

Proof. Fix $x \in B(x_0, R/2) \cap D$. Let $r := \frac{1}{10} \operatorname{dist}(x, \partial D)$. Then by (5-1) and the interior Lipschitz regularity for the p-Laplace equation, see [Kuusi and Mingione 2012, (1.7)],

$$|u(x) - (u)_{B(x,r)}|^p \precsim r^{p-n} \int_{B(x, 5r)} |\nabla u|^p \leq \epsilon^p.$$

Denote by $z_1 \in \partial D \cap B(x_0, R/2)$ the projection of x onto $\partial D \cap B(x_0, R/2)$. Here we assume that $R < R_0$ for $R_0 = R_0(D)$ small enough such that z_1 is well-defined.

Let $y_0, y_1, \dots, y_{10}$ be pairwise equidistant points on the line $[x, z_1]$, where $y_0 = x$ and $y_{10} = z_1$. That is, $|y_i - y_{i+1}| = r$.

By the triangle inequality, the Poincaré inequality and again by (5-1),

$$\begin{aligned}|(u)_{B(x,r)} - (u)_{B(z_1,r)\cap D}|^p &\precsim \sum_{i=0}^{10} |(u)_{B(y_i,r)\cap D} - (u)_{B(y_{i+1},r)\cap D}|^p \\ &\precsim \sum_{i=0}^{10} r^{p-n} \int_{B(y_i,4r)\cap D} |\nabla u|^p \precsim \epsilon^p.\end{aligned}$$

From the first to second line, before applying the Poincaré inequality, we also used that $|y_i - y_{i+1}| = r$, and thus (see footnote 3)

$$|(u)_{B(y_i,r)\cap D} - (u)_{B(y_{i+1},r)\cap D}|^p \precsim \fint_{B(y_i,4r)\cap D} |u - (u)_{B(y_i,4r)\cap D}|^p.$$

Now for any $z_2 \in \partial D$

$$\operatorname{dist}((u)_{B(z_1,r)\cap D}, \mathbb{S}^{N-1}) \precsim r^{-n} \int_{B(z_1,r)\cap D} |u(z_3) - u(z_2)|\, dz_3.$$

Integrating z_2 over $\partial D \cap B(z_1, r)$ we find

$$\begin{aligned}&\operatorname{dist}((u)_{B(z_1,r)\cap D}, \mathbb{S}^{N-1}) \\ &\quad\precsim r^{-n} \int_{B(z_1,r)\cap D} |u(z_3) - (u)_{B(z_1,r)\cap\partial D}|\, dz_3 + r^{1-n} \int_{B(z_1,r)\cap\partial D} |u(z_2) - (u)_{B(z_1,r)\cap\partial D}|\, dz_2.\end{aligned}$$

By the Poincaré inequality, the trace theorem, and (5-1),

$$\operatorname{dist}((u)_{B(z_1,r)\cap D}, \mathbb{S}^{N-1}) \precsim \epsilon.$$

Now the claim follows by the triangle inequality for the distance,

$$\operatorname{dist}(u(x), \mathbb{S}^{N-1}) \le |u(x) - (u)_{B(x,r)}| + |(u)_{B(x,r)} - (u)_{B(z_1,r)\cap D}| + \operatorname{dist}((u)_{B(z_1,r)\cap D}, \mathbb{S}^{N-1}). \qquad \square$$

As an immediate corollary we obtain:

Corollary 5.2. *Let u and D be as in Theorem 1.2. There exists $\epsilon_0 > 0$ such that if $B(x_0, R/2) \cap \partial D \neq \varnothing$ and* (5-1) *holds for some $\epsilon < \epsilon_0$, then $|u| > \frac{1}{2}$ in $B(x_0, R/2) \cap D$.*

As a consequence, when we reflect the maps from Theorem 1.2, we obtain a critical equation with the growth estimates such that Proposition 4.1 is applicable.

Proposition 5.3. *Let u and D be as in Theorem 1.2. There exists $\epsilon_0 = \epsilon_0(D) > 0$ such that for any $B(x_0, 4R) \subset \mathbb{R}^n$ on which u satisfies* (5-1) *for some $\epsilon < \epsilon_0$ there exists $v \in W^{1,p}(B(x_0, R), \mathbb{R}^N)$ such that*

$$\begin{aligned} v &= u &&\text{in } B(x_0, R) \cap D, \\ |\operatorname{div}(|\nabla v|^{p-2} \nabla v)| &\precsim |\nabla v|^p &&\text{in } B(x_0, R). \end{aligned} \tag{5-2}$$

Moreover, v satisfies the growth conditions from Proposition 4.1.

Proof. The main point is to prove that v satisfies the growth conditions. The estimate (5-2) follows from the geometric reflection, more precisely [Scheven 2004, Lemma 2.5]. But for the reader's convenience we state the argument in full in the case where the boundary is flat. This means that we work in a ball $B(x_0, 4R)$ such that $B^+(x_0, 4R) \subset D \subset \mathbb{R}^n_+$ and $\partial D \cap B(x_0, 4R) = \partial\mathbb{R}^n_+ \cap B(x_0, 4R)$.

If $B(x_0, R) \subset \mathbb{R}^n_+$ then we can just take $v \equiv u$. So assume that $B(x_0, R) \cap \partial\mathbb{R}^n_+ \neq \varnothing$. Then for ϵ_0 small enough we have $|u| > \frac{1}{2}$ in $B^+(x_0, R)$ by Corollary 5.2.

Denote by $\tilde{u}$ the even reflection; i.e.,

$$\tilde{u}(x', x_n) := u(x', |x_n|).$$

Moreover, set

$$\sigma(q) := \frac{q}{|q|^2}, \quad q \in \mathbb{R}^n \backslash \{0\}.$$

Now we define the geometric reflection v as

$$v(x) := \begin{cases} u(x), & x \in B^+(x_0, R), \\ \sigma(\tilde{u}(x)), & x \in B(x_0, R) \backslash \mathbb{R}^n_+. \end{cases}$$

Since $|u| > \frac{1}{2}$ and u is uniformly bounded by Lemma A.1, v is well-defined in $B(x_0, R)$.

We also set

$$\Sigma_{ij}(q) := \partial_i \sigma^j(q) = \frac{\delta_{ij} - 2q^i q^j / |q|^2}{|q|^2}.$$

That is, for $x \in B(x_0, R) \backslash \mathbb{R}^n_+$,

$$\nabla v(x) = \Sigma(\tilde{u}(x))\, \nabla \tilde{u}(x). \tag{5-3}$$

Observe that Σ is symmetric, and

$$\Sigma(q) = \frac{1}{|q|^2}\left(I - 2\frac{q}{|q|} \otimes \frac{q}{|q|}\right)$$

and that $q/|q|$ is an eigenfunction to the eigenvalue $-1/|q|^2$, and any orthonormal basis of $(q/|q|)^\perp$ is the basis of the eigenspace of the eigenvalue $1/|q|^2$. In particular,

$$|\Sigma(q) w| = \frac{1}{|q|^2} |w| \quad \text{for all } w \in \mathbb{R}^N.$$

Thus,

$$|\nabla v(x)| = \begin{cases} |\nabla \tilde{u}(x)|, & x \in B^+(x_0, R), \\ |\nabla \tilde{u}(x)| / |\tilde{u}(x)|^2, & x \in B(x_0, R) \setminus \mathbb{R}^n_+. \end{cases} \tag{5-4}$$

Also observe that for $|q| = 1$,

$$\Sigma(q) v = \Pi(q) v - \Pi^\perp(q) v \quad \text{for all } v \in \mathbb{R}^N,$$

where $\Pi(q) := I - q \otimes q$ is the orthogonal projection onto $T_q\mathbb{S}^{N-1} = q^\perp$ and $\Pi^\perp(q) := q \otimes q$ is the orthogonal projection onto $(T_q\mathbb{S}^{N-1})^\perp = \operatorname{span}\{q\}$.

Therefore, for $\phi \in C_c^\infty(B(x_0,R),\mathbb{R}^N)$, since $\partial_\nu u \perp T_u\mathbb{S}^{N-1}$,

$$\int_{B^+(x_0,R)} |\nabla u|^{p-2}\,\nabla u\cdot\nabla\phi + \int_{B(x_0,R)\setminus\mathbb{R}^n_+} |\nabla\tilde u|^{p-2}\,\nabla\tilde u\cdot\nabla(\Sigma(\tilde u)\phi) = 0.$$

In particular,

$$\int_{B(x_0,R)} |\nabla\tilde u|^{p-2}\,\nabla v\cdot\nabla\phi = -\int_{B(x_0,R)\setminus\mathbb{R}^n_+} |\nabla\tilde u|^{p-2}\,\nabla\tilde u\cdot\nabla(\Sigma(\tilde u))\,\phi.$$

Combining this with (5-4),

$$\int_{B(x_0,R)} |\nabla v|^{p-2}\,\nabla v\cdot\nabla(m\phi) = -\int_{B(x_0,R)\setminus\mathbb{R}^n_+} |\nabla\tilde u|^{p-2}\,\nabla\tilde u\cdot\nabla(\Sigma(\tilde u))\,\phi + \int_{B(x_0,R)} |\nabla v|^{p-2}\,\nabla v\cdot\nabla m\,\phi,$$

where

$$m(x) = \begin{cases} 1, & \text{in } B^+(x_0,R),\\ |\tilde u(x)|^{2(p-2)}, & \text{in } B(x_0,R)\setminus\mathbb{R}^n_+.\end{cases}$$

Observe that $m(x)$ and $m(x)^{-1} \in L^\infty\cap W^{1,p}(B(x_0,R))$. Now (5-2) follows from (5-4).

It remains to establish the growth estimates from Proposition 4.1 which follow from Proposition 2.1. Indeed, set $\mathcal{M} := B(x_0,R)\cap\partial\mathbb{R}^n_+$.

To obtain (4-5) let $B(y_0,4r)\subset B(x_0,R)$ and $B(y_0,2r)\cap\mathcal{M}=\varnothing$. Let us consider first $B(y_0,2r)\subset\mathbb{R}^n_-$. Then we observe that by (5-4) combined with the fact that $|u|>\frac{1}{2}$ on $B^+(x_0,R)$ we have $\int_{B(y_0,r)}|\nabla v|^p \precsim \int_{B(\tilde y_0,r)}|\nabla u|^p$, where $\tilde y_0$ is the point $y_0=(y_0^1,\dots,y_0^n)$ reflected along the hyperplane $\partial\mathbb{R}^n_+$, i.e., $\tilde y_0 = (y_0^1,\dots,-y_0^n)$. Now applying (2-3) to u, we obtain

$$\begin{aligned}\int_{B(y_0,r)}|\nabla v|^p &\precsim C\lambda\int_{B^+(\tilde y_0,4r)}|\nabla u|^p + C\lambda^{1-p}r^{-p}\int_{B^+(\tilde y_0,4r)}|u-(u)_{B^+(\tilde y_0,4r)}|^p\\ &\le C\lambda\int_{B(y_0,4r)}|\nabla v|^p + C\lambda^{1-p}r^{-p}\int_{B^-(y_0,4r)}|\tilde u-(\tilde u)_{B^-(y_0,4r)}|^p. \qquad (5\text{-}5)\end{aligned}$$

To estimate the remaining part we note that since $v=\tilde u/|\tilde u|^2$ we have $\tilde u = v/|v|^2$ in $\mathbb{R}^n_-$ and for any $A\subset B(x_0,R)\setminus\mathbb{R}^n_+$:

$$\begin{aligned}\fint_A\left|\frac{v}{|v|^2}-\left(\frac{v}{|v|^2}\right)_A\right|^p &\precsim \fint_A\fint_A\left|\frac{v(x)}{|v(x)|^2}-\frac{v(y)}{|v(x)|^2}\right|^p + \fint_A\fint_A\left|\frac{v(y)}{|v(x)|^2}-\frac{v(y)}{|v(y)|^2}\right|^p\\ &\precsim \|v^{-1}\|^{2p}_{L^\infty}\fint_A\fint_A|v(x)-v(y)|^p + \|v^{-1}\|^{3p}_{L^\infty}\fint_A\fint_A\big||v(x)|^2-|v(y)|^2\big|^p. \qquad (5\text{-}6)\end{aligned}$$

Now, since for any $a,\ b$,

$$|a|^2-|b|^2 = (|a|+|b|)(|a|-|b|) \le (|a|+|b|)|a-b|,$$

we have

$$\begin{aligned}\fint_A\fint_A\big||v(x)|^2-|v(y)|^2\big|^p &\precsim \|v\|^p_{L^\infty(A)}\fint_A\fint_A|v(x)-v(y)|^p\\ &\precsim \|v\|^p_{L^\infty(A)}\fint_A|v-(v)_A|^p, \qquad (5\text{-}7)\end{aligned}$$

where the last inequality was obtained by adding and subtracting $(v)_A$ and by the triangle inequality. We deduce from (5-6) and (5-7) that

$$\fint_A \left| \frac{v}{|v|^2} - \left(\frac{v}{|v|^2}\right)_A \right|^p \precsim \|v^{-1}\|^{2p}_{L^\infty(A)} (1 + \|v\|^p_{L^\infty(A)} \|v^{-1}\|^p_{L^\infty(A)}) \fint_A |v - (v)_A|^p.$$

Due to the fact that $|u| > \frac{1}{2}$ and u is uniformly bounded we get

$$\fint_A |\tilde{u} - (\tilde{u})_A|^p \precsim \fint_A |v - (v)_A|^p \quad \text{for any } A \subset B(x_0, R) \setminus \mathbb{R}^n_+. \tag{5-8}$$

To conclude, we note[3] that since $B(y_0, 2r) \subset \mathbb{R}^n_-$ we have $|B(y_0, 4r)|/|B^-(y_0, 4r)| \approx 1$; thus

$$\fint_{B^-(y_0,4r)} |v - (v)_{B^-(y_0,4r)}|^p \precsim \fint_{B(y_0,4r)} |v - (v)_{B(y_0,4r)}|^p. \tag{5-9}$$

Combining estimates (5-5), (5-8), and (5-9) we obtain (4-5). The second case $B(y_0, 2r) \subset \mathbb{R}^n_+$ is easier and we leave it to the reader.

Finally, for (4-6) we apply (2-4) and observe that $|u|^2 \equiv 1$ on $\mathcal{I} := B(y_0, 4r) \cap \partial\mathbb{R}^n_+$. Thus,

$$\int_{B^+(y_0,4r)} ||u|^2 - 1|^p \precsim (\|u\|_{L^\infty} + 1) \int_{B^+(y_0,4r)} ||u| - (|u|)_{\mathcal{I}}|^p.$$

Now

$$||u(z)| - (|u|)_{\mathcal{I}}| \le \fint_{\mathcal{I}} ||u(z)| - |u(z_2)||\, dz_2 \le \fint_{\mathcal{I}} |u(z) - u(z_2)|\, dz_2$$

and thus

$$\fint_{B^+(y_0,4r)} ||u| - (|u|)_{\mathcal{I}}|^p \precsim \fint_{B^+(y_0,4r)} |u - (u)_{\mathcal{I}}|^p + \fint_{\mathcal{I}} |u - (u)_{\mathcal{I}}|^p.$$

Proposition 5.3 is now established. □

Proof of Theorem 1.2. For $p = n$, Hölder continuity for u follows from Proposition 3.1. For $p < n$, it follows from the combination of Propositions 5.3 and 4.1. Now $C^{1,\alpha}$-regularity follows from the reflection, Proposition 5.3, and the fact that a Hölder continuous solution to the reflected system is $C^{1,\alpha}$ for some $\alpha > 0$; see [Hardt and Lin 1987, Theorem 3.1] (which is stated for minimizers but actually only uses the continuity of the solution and the equation). See also [Rivière and Strzelecki 2005, Theorem 1.2].

Note that for $p = n$ there is also a more elegant argument to pass from C^α regularity to $C^{1,\alpha}$. Testing (1-12) in x and $x + h$ with $\phi(x) := \eta(x)(v(x+h) - v(x))$ for a suitable cutoff function η one obtains from the Hölder continuity of u that for some $\sigma > 0$ we have $\nabla v \in W^{1+\sigma,n}$. In particular, by Sobolev embedding $\nabla v \in L^{(n,1)}_{\mathrm{loc}}$, and by [Duzaar and Mingione 2010] we get a Lipschitz bound for v. Now, $C^{1,\alpha}$-regularity is a consequence of the potential estimates for p-Laplace equations; see [Kuusi and Mingione 2012; 2018]. We leave the details to the reader. □

[3] Indeed, for any $\tilde{A} \subset A$ we have by enlarging the domain of integration and applying Jensen's inequality $\fint_{\tilde{A}} |w - (w)_{\tilde{A}}|^p \precsim (|A|/|\tilde{A}|) \fint_A |w - (w)_A|^p$.

6. Partial regularity: proof of Theorem 1.4

For simplicity we assume in this section that $B^+(0,R) \subset D \subset \mathbb{R}^n_+$ and $\partial D \cap B(0,R) = \partial\mathbb{R}^n_+ \cap B(0,R)$. We begin with recalling that a map $u \in W^{1,p}(B^+(0,R), \mathbb{R}^N)$ is said to be *stationary p-harmonic* with respect to the free boundary condition $u(\partial D \cap B(0,R)) \subset \mathbb{S}^{N-1}$ if in addition to (1-9) it is a critical point of the energy with respect to variations in the domain. The latter is equivalent to u satisfying

$$\int_{B^+(0,R)} |\nabla u|^{p-2}(|\nabla u|^2\delta_{ij} - p\,\partial_i u\,\partial_j u)\,\partial_i \xi^j = 0 \tag{6-1}$$

for $\xi = (\xi^1, \dots, \xi^n) \in C_c^\infty(\overline{\mathbb{R}}^n_+ \cap B(0,R), \mathbb{R}^n)$ with $\xi(\partial\mathbb{R}^n_+) \subset \partial\mathbb{R}^n_+$.

By choosing the test function as $\xi(x) := \psi(x)(x_0 - x)$ in (6-1), where $\psi \in C_c^\infty(\overline{\mathbb{R}}^n_+ \cap B(0,R), [0,1])$ is a suitable bump function, one obtains the following.

Lemma 6.1 (monotonicity formula). *Let $u \in W^{1,p}(B^+(0,R), \mathbb{R}^N)$ be a stationary p-harmonic map with respect to the free boundary condition $u(B^+(0,R) \cap \{x_n = 0\}) \subset \mathbb{S}^{N-1}$ and let $x_0 \in B^+(0,R) \cap \{x_n = 0\}$. Then, the normalized p-energy is monotone. In particular,*

$$r^{p-n} \int_{B^+(x_0,r)} |\nabla u|^p - \rho^{p-n} \int_{B^+(x_0,\rho)} |\nabla u|^p = p \int_{B^+(x_0,r)\setminus B^+(x_0,\rho)} |x - x_0|^{p-n} |\nabla u|^{p-2} \left|\frac{\partial u}{\partial \nu}\right|^2 \tag{6-2}$$

for all $0 < \rho < r < R - |x_0|$, where ν is the outward-pointing unit normal for $\partial B(x_0, r)$, $\nu(x) := (x - x_0)/|x - x_0|$. For $x_0 \in B^+(0,R) \setminus \partial\mathbb{R}^n_+$ the same holds if r is such that $B^+(x_0,r) = B(x_0,r) \subset \mathbb{R}^n_+$.

This well-known fact was proved for Yang–Mills fields and stationary harmonic maps in [Price 1983]; see [Evans 1991; Bethuel 1993; Simon 1996, Section 2.4]. Fuchs [1989] observed that (6-2) holds for stationary p-harmonic maps. As pointed out in [Scheven 2006, p. 137] the proof holds true in the case of a free boundary condition.

We will need the following lemma; see, e.g., [Ziemer 1989, Corollary 3.2.3].

Lemma 6.2 (Frostman's lemma). *If $f \in L^p(\mathbb{R}^n)$, $p \geq 1$, and $0 \leq \alpha < n$, then for*

$$E := \left\{ x \in \mathbb{R}^n : \limsup_{r\to 0} r^{-\alpha} \int_{B(x,r)} |f(y)|^p > 0 \right\},$$

we have $\mathcal{H}^\alpha(E) = 0$.

We shall show, using monotonicity formula (6-2) and Frostman's lemma (Lemma 6.2), that the set outside which the condition (1-13) is satisfied is of zero $(n-p)$-Hausdorff measure. We then obtain Theorem 1.4 from Theorem 1.2.

Proof of Theorem 1.4. Let

$$S := \left\{ x \in \overline{\mathbb{R}}^n_+ : \limsup_{r\to 0} r^{p-n} \int_{B^+(x,r)} |\nabla u|^p > 0 \right\};$$

by Lemma 6.2, we have $\mathcal{H}^{n-p}(S) = 0$.

We define for ϵ as in Theorem 1.2

$$\Sigma_\epsilon := \left\{ x \in \overline{\mathbb{R}}^n_+ : \text{for all } R > 0 \sup_{|y_0 - x| < R} \sup_{\rho < R} \rho^{p-n} \int_{B^+(y_0,\rho)} |\nabla u|^p \geq \epsilon \right\};$$

clearly Σ_ϵ is a closed set. We will prove that $\mathcal{H}^{n-p}(\Sigma_\epsilon) = 0$. Then Theorem 1.4 is a consequence of Theorem 1.2.

Let A_ϵ be the set on which the condition (1-13) is satisfied for ϵ; i.e.,

$$A_\epsilon := \overline{\mathbb{R}}^n_+ \setminus \Sigma_\epsilon = \left\{ x \in \overline{\mathbb{R}}^n_+ : \text{there exists } R > 0 \text{ such that } \sup_{|y_0 - x| < R} \sup_{\rho < R} \rho^{p-n} \int_{B^+(y_0,\rho)} |\nabla u|^p < \epsilon \right\}.$$

In order to prove the theorem it suffices to show that $(\overline{\mathbb{R}}^n_+ \setminus S) \subseteq A_\epsilon$.

Let $x_0 \in (\overline{\mathbb{R}}^n_+ \setminus S)$, i.e., be such that $\limsup_{r \to 0} r^{p-n} \int_{B^+(x_0,r)} |\nabla u|^p = 0$. There exists an $R > 0$ such that

$$R^{p-n} \int_{B^+(x_0,R)} |\nabla u|^p < 4^{p-n} \epsilon.$$

We shall show that

$$\sup_{|y_0 - x_0| < R/4} \sup_{\rho < R/4} \rho^{p-n} \int_{B^+(y_0,\rho)} |\nabla u|^p < \epsilon.$$

Choose any y_0 such that $|y_0 - x_0| < R/4$ and any radius $\rho < R/4$. First observe that we may take $y_0 \in \overline{\mathbb{R}}^n_+$. Indeed, suppose that $y_1 \in B(x_0, R/4) \cap \mathbb{R}^n_-$; then for any $\rho < R/4$ we can choose $y_0 \in B(x_0, R/4) \cap \overline{\mathbb{R}}^n_+$ such that $B(y_1, \rho) \cap \overline{\mathbb{R}}^n_+ \subset B(y_0, \rho) \cap \overline{\mathbb{R}}^n_+$. Thus

$$\sup_{|y_1 - x_0| < R/4} \sup_{\rho < R/4} \rho^{p-n} \int_{B^+(y_1,\rho)} |\nabla u|^p = \sup_{y_0 \in B(x_0, R/4) \cap \overline{\mathbb{R}}^n_+} \sup_{\rho < R/4} \rho^{p-n} \int_{B^+(y_0,\rho)} |\nabla u|^p.$$

Now assume that $y_0 \in \partial \mathbb{R}^n_+$. We have $B^+(y_0, \rho) \subset B^+(y_0, R/4) \subset B^+(x_0, R)$. Thus

$$\rho^{p-n} \int_{B^+(y_0,\rho)} |\nabla u|^p \leq \left(\frac{R}{4}\right)^{p-n} \int_{B^+(y_0,R/4)} |\nabla u|^p \leq 4^{n-p} R^{p-n} \int_{B^+(x_0,R)} |\nabla u|^p < \epsilon,$$

where the first inequality is a consequence of the monotonicity formula (6-2).

Now, let us assume that $y_0 \notin \partial \mathbb{R}^n_+$. Let $\bar{\rho} = \operatorname{dist}(y_0, \partial \mathbb{R}^n_+)$ and $\bar{y}_0$ be the projection of y_0 onto $\partial \mathbb{R}^n_+$. We can assume that $\rho < \bar{\rho}$. Indeed, if not we would have

$$\begin{aligned} \rho^{p-n} \int_{B^+(y_0,\rho)} |\nabla u|^p &\leq \rho^{p-n} \int_{B^+(\bar{y}_0,2\rho)} |\nabla u|^p = 2^{n-p} (2\rho)^{p-n} \int_{B^+(\bar{y}_0,2\rho)} |\nabla u|^p \\ &\leq 2^{n-p} \left(\frac{R}{2}\right)^{p-n} \int_{B^+(\bar{y}_0,R/2)} |\nabla u|^p \leq 4^{n-p} R^{p-n} \int_{B^+(x_0,R)} |\nabla u|^p < \epsilon. \end{aligned}$$

Next, we note that $\bar{\rho} < R/4$ and observe the inclusions

$$B(y_0, \rho) \subset B(y_0, \bar{\rho}) \subset B^+(\bar{y}_0, 2\bar{\rho}) \subset B^+(\bar{y}_0, R/2) \subset B^+(x_0, R)$$

and the following inequalities which are consequences of the monotonicity formula (6-2):

$$\rho^{p-n}\int_{B(y_0,\rho)}|\nabla u|^p \le (\bar\rho)^{p-n}\int_{B(y_0,\bar\rho)}|\nabla u|^p,$$
$$(2\bar\rho)^{p-n}\int_{B^+(\bar y_0,2\bar\rho)}|\nabla u|^p \le \left(\frac{R}{2}\right)^{p-n}\int_{B^+(\bar y_0,R/2)}|\nabla u|^p.$$

Thus

$$\begin{aligned}\rho^{p-n}\int_{B(y_0,\rho)}|\nabla u|^p &\le (\bar\rho)^{p-n}\int_{B(y_0,\bar\rho)}|\nabla u|^p \le 2^{n-p}(2\bar\rho)^{p-n}\int_{B^+(\bar y_0,2\bar\rho)}|\nabla u|^p\\ &\le 2^{n-p}\left(\frac{R}{2}\right)^{p-n}\int_{B^+(\bar y_0,R/2)}|\nabla u|^p \le 4^{n-p}R^{p-n}\int_{B^+(x_0,R)}|\nabla u|^p < \epsilon,\end{aligned}$$

which gives $x_0 \in A_\epsilon$.

We conclude $\Sigma_\epsilon \subset S$ and thus $\mathcal{H}^{n-p}(\Sigma_\epsilon) = 0$. □

A Liouville-type result. We note that the monotonicity formula in Lemma 6.1 can be used to prove partial regularity but also Liouville-type results in the spirit of [Liu 2010]. Indeed, if we work in $\mathbb{R}^n_+$, for $u \in \dot W^{1,p}(\mathbb{R}^n_+, \mathbb{R}^N)$ we can say that u is stationary p-harmonic with respect to the free boundary condition $u(\partial\mathbb{R}^n_+) \subset \mathbb{S}^{N-1}$ if u satisfies (1-9) and

$$\int_{\mathbb{R}^n_+}|\nabla u|^{p-2}(|\nabla u|^2\delta_{ij} - p\,\partial_i u\,\partial_j u)\,\partial_i\xi^j = 0 \tag{6-3}$$

for $\xi = (\xi^1,\dots,\xi^n) \in C_c^\infty(\overline{\mathbb{R}}^n_+, \mathbb{R}^n)$ with $\xi(\partial\mathbb{R}^n_+) \subset \partial\mathbb{R}^n_+$. We then have:

Proposition 6.3. *Let $2 \le p < n$ and $u \in \dot W^{1,p}(\mathbb{R}^n_+, \mathbb{R}^N)$ be such that u is a finite-energy, stationary p-harmonic map with respect to the free boundary condition $u(\partial\mathbb{R}^n_+) \subset \mathbb{S}^{N-1}$. Then u is constant.*

Proof. By contradiction, assume u is not a constant. Then there exists $R_0 > 0$ such that $\int_{B^+(0,R_0)}|\nabla u|^p \ge c > 0$. Now by the monotonicity formula (Lemma 6.1) we have that for any $R > R_0$

$$\int_{B^+(0,R)}|\nabla u|^p \ge \left(\frac{R}{R_0}\right)^{n-p}\int_{B^+(0,R_0)}|\nabla u|^p \ge \left(\frac{R}{R_0}\right)^{n-p}c. \tag{6-4}$$

We can then let R go to $+\infty$ and we obtain that the p-energy of u in $\mathbb{R}^n_+$ is infinite. This is a contradiction since we assumed that $u \in \dot W^{1,p}(\mathbb{R}^n_+, \mathbb{R}^N)$. □

Appendix: On boundedness of p-harmonic maps

The following lemma is well-known. However, we could not find it explicitly in the literature, so we state it here for the convenience of the reader.

Lemma A.1. *Let $D \subset \mathbb{R}^n$ be a smooth, bounded domain. Assume that $u \in W^{1,p}(D, \mathbb{R}^N)$ is a solution to*

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \quad \textit{in } D.$$

If $u|_{\partial D} \in L^\infty(\partial D)$, then $\|u\|_{L^\infty(D)} \le \|u\|_{L^\infty(\partial D)}$.

Proof. For scalar functions this is a consequence of the weak maximum principle for the p-Laplacian; see [Lindqvist 2006, Theorem 2.15]. However, here we work with a system. For $\epsilon \in (0, 1)$ we find smooth solutions $u_\epsilon \in W^{1,p} \cap C^\infty(D, \mathbb{R}^N)$ of the uniformly elliptic system

$$\begin{cases} \operatorname{div}((\epsilon + |\nabla u_\epsilon|^2)^{(p-2)/2} \nabla u_\epsilon) = 0 & \text{in } D, \\ u_\epsilon = u & \text{on } \partial D. \end{cases} \tag{A-1}$$

The solution is smooth in the interior, and a direct computation shows that

$$\operatorname{div}((\epsilon + |\nabla u_\epsilon|^2)^{(p-2)/2} \nabla |u_\epsilon|^2) \geq 0. \tag{A-2}$$

Thus the weak maximum principle for scalar solutions of uniformly elliptic operators in divergence form implies

$$\sup_{\epsilon \in (0,1)} \|u_\epsilon\|_{L^\infty(D)} \leq \|u\|_{L^\infty(\partial D)}. \tag{A-3}$$

Moreover, we can test (A-1) with $u_\epsilon - u$, which is trivial on ∂D, and thus

$$\int_D |\nabla u_\epsilon|^p \leq \int_D (\epsilon + |\nabla u_\epsilon|^2)^{(p-2)/2} \, |\nabla u_\epsilon|^2 = \int_D (\epsilon + |\nabla u_\epsilon|^2)^{(p-2)/2} \, \nabla u_\epsilon \cdot \nabla u;$$

consequently, by Young's inequality,

$$\int_D |\nabla u_\epsilon|^p \leq \frac{1}{2} \int_D |\nabla u_\epsilon|^p + C \int_D |\nabla u|^p + C(|D|, p).$$

Thus, u_ϵ is uniformly bounded in $W^{1,p}$,

$$\sup_{\epsilon \in (0,1)} \int_D |\nabla u_\epsilon|^p < \infty. \tag{A-4}$$

On the other hand,

$$\int_D ((\epsilon + |\nabla u_\epsilon|^2)^{(p-2)/2} \, \nabla u_\epsilon - |\nabla u|^{p-2} \, \nabla u) \cdot (\nabla u_\epsilon - \nabla u) = 0.$$

Applying then the well-known inequality

$$|a - b|^p \precsim (|a|^{p-2} a - |b|^{p-2} b)(a - b),$$

we find that as $\epsilon \to 0$

$$\int_D |\nabla u - \nabla u_\epsilon|^p \precsim o(1) \int_D (|\nabla u|^{p-1} + |\nabla u_\epsilon|^{p-1}).$$

Therefore, in view of (A-4) and the boundedness of D,

$$u_\epsilon \xrightarrow{\epsilon \to 0} u \quad \text{in } W^{1,p}(D).$$

In particular, up to a subsequence, we have pointwise almost everywhere convergence, and from (A-3) we have

$$\|u\|_{L^\infty(D)} \leq \|u\|_{L^\infty(\partial D)}.$$

□

Lemma A.2. *Let $D \subset \mathbb{R}^n$ be a possibly unbounded domain with smooth boundary ∂D. Assume that $p > n-1$ and $u \in \dot{W}^{1,p}(D, \mathbb{R}^N)$ is a solution to*

$$\operatorname{div}(|\nabla u|^{p-2}\,\nabla u) = 0 \quad \text{in } D.$$

If $u|_{\partial D} \in L^\infty(\partial D)$, then for every compact set $K \subset \overline{D}$ we have

$$\|u\|_{L^\infty(K)} < \infty.$$

Proof. For compact K we find by Fubini's theorem a smooth, bounded domain $\widetilde{D} \supset K$ such that

$$u|_{\partial \widetilde{D} \cap D} \in W^{1,p}.$$

Since $p > n-1$ we conclude that, by Morrey–Sobolev embedding, u is continuous on $\partial\widetilde{D} \cap D$, and in particular $u \in L^\infty(\partial\widetilde{D})$. Now we can apply Lemma A.1 to $\widetilde{D}$ to obtain the result. □

We now prove a maximum principle analog of Lemma A.1 but for maps defined in the half-space $\mathbb{R}^n_+$. We work with maps with finite energy; i.e., we work with

$$\dot{W}^{1,p}(\mathbb{R}^n_+, \mathbb{R}^N) := \{v \in \mathcal{D}'(\mathbb{R}^n_+, \mathbb{R}^N) : \nabla v \in L^p(\mathbb{R}^n_+, \mathbb{R}^N)\}.$$

We remark that a map in $\dot{W}^{1,p}(\mathbb{R}^n_+, \mathbb{R}^N)$ is also in $L^p_{\text{loc}}(\mathbb{R}^n_+, \mathbb{R}^N)$ and hence has a trace on $\partial\mathbb{R}^n_+ := \mathbb{R}^{n-1} \times \{0\}$ which is well-defined.

Proposition A.3. *Let $u \in \dot{W}^{1,p}(\mathbb{R}^n_+, \mathbb{R}^N)$ be a solution to*

$$\operatorname{div}(|\nabla u|^{p-2}\,\nabla u) = 0 \quad \text{in } \mathbb{R}^n_+,$$

that is,

$$\int_{\mathbb{R}^n_+} |\nabla u|^{p-2}\,\nabla u \cdot \nabla \phi = 0 \quad \text{for all } \phi \in C^\infty_c(\overline{\mathbb{R}}^n_+).$$

Assume that $u|_{\mathbb{R}^{n-1}\times\{0\}} \in L^\infty(\mathbb{R}^{n-1} \times \{0\})$. Then $u \in L^\infty(\mathbb{R}^n_+)$ and

$$\|u\|_{L^\infty(\mathbb{R}^n_+)} \le \|u\|_{L^\infty(\partial\mathbb{R}^n_+)}.$$

Proof. We define $g := u|_{\mathbb{R}^{n-1}\times\{0\}}$ and $M := \|g\|_{L^\infty(\partial\mathbb{R}^n_+)}$. From Proposition A.4 below we know that u is the unique minimizer of the energy $\int_{\mathbb{R}^n_+} |\nabla v|^p$ in

$$X := \{v \in \dot{W}^{1,p}(\mathbb{R}^n_+, \mathbb{R}^N) : v|_{\mathbb{R}^{n-1}\times 0} = g \text{ in the trace sense}\}.$$

Now we define

$$\tilde{u} := \begin{cases} u & \text{if } |u| \le M, \\ Mu/|u| & \text{if } |u| > M. \end{cases}$$

By a direct computation we can see

$$\int_{\mathbb{R}^n_+} |\nabla \tilde{u}|^p \le \int_{\mathbb{R}^n_+} |\nabla u|^p.$$

Additionally, we have $\tilde{u}|_{\partial R^n_+} = g$. Thus by uniqueness we deduce that $\tilde{u} = u$ and $|u| \le M$ in $\mathbb{R}^n_+$. □

It remains to prove:

Proposition A.4. *Let $u \in \dot{W}^{1,p}(\mathbb{R}^n_+, \mathbb{R}^N)$ be as in Proposition A.3 a solution to*

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0 \quad \text{in } \mathbb{R}^n_+.$$

Let us denote by $g = u|_{\mathbb{R}^{n-1}\times 0}$ the trace of u. Then u is the unique minimizer of the energy $\int_{\mathbb{R}^n_+} |\nabla v|^p$ in

$$X := \{v \in \dot{W}^{1,p}(\mathbb{R}^n_+, \mathbb{R}^N) : v|_{\mathbb{R}^{n-1}\times 0} = g \text{ in the trace sense}\}.$$

Proof. By the direct method of calculus of variations we can prove that there exists a minimizer u_0 of $\int_{\mathbb{R}^n_+} |\nabla u|^p$ in X. Besides, by strict convexity of the p-energy we have that this minimizer is unique and it is the unique critical point of the p-energy in X. That is, there is at most one map with a trace equal to g which satisfies

$$\int_{\mathbb{R}^n_+} |\nabla u_0|^{p-2} \nabla u_0 \cdot \nabla \phi = 0 \quad \text{for all } \phi \in \dot{W}^{1,p}(\mathbb{R}^n_+, \mathbb{R}^N), \quad \phi|_{\mathbb{R}^{n-1}\times\{0\}} = 0. \tag{A-5}$$

Observe that $C_c^\infty(\mathbb{R}^n_+, \mathbb{R}^N)$ is dense in the space

$$Y := \{\phi \in \dot{W}^{1,p}(\mathbb{R}^n_+, \mathbb{R}^N) : \phi|_{\mathbb{R}^{n-1}\times\{0\}} = 0\},$$

which can be proven as in, e.g., [Willem 2013, Proposition 6.2.5]. We conclude that there is at most one map with a trace equal to g which satisfies

$$\int_{\mathbb{R}^n_+} |\nabla u_0|^{p-2} \nabla u_0 \cdot \nabla \phi = 0 \quad \text{for all } \phi \in C_c^\infty(\mathbb{R}^n_+). \tag{A-6}$$

This implies the claim. □

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UNIFORM SOBOLEV ESTIMATES FOR SCHRÖDINGER OPERATORS WITH SCALING-CRITICAL POTENTIALS AND APPLICATIONS

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We prove uniform Sobolev estimates for the resolvent of Schrödinger operators with large scaling-critical potentials without any repulsive condition. As applications, global-in-time Strichartz estimates including some nonadmissible retarded estimates, a Hörmander-type spectral multiplier theorem, and Keller-type eigenvalue bounds with complex-valued potentials are also obtained.

1. Introduction and main results

This paper is a continuation of [Bouclet and Mizutani 2018; Mizutani 2019], where uniform estimates for the resolvent $(H-z)^{-1}$ of the Schrödinger operator $H=-\Delta+V(x)$ on $\mathbb{R}^n$ with a real-valued potential $V(x)$ exhibiting one critical singularity were investigated under some *repulsive* conditions so that H is nonnegative and its spectrum $\sigma(H)$ is purely absolutely continuous. In the present paper we improve upon and extend those previous results to a class of scaling-critical potentials without any repulsive condition such that H may have (finitely many) negative eigenvalues and multiple scaling-critical singularities. Applications to Strichartz estimates, a Hörmander-type multiplier theorem for H and eigenvalue bounds for $H+W$ with complex potential W are also established.

We first recall some known results in the free case, $H=-\Delta$, describing the motivation of this paper. The classical Hardy–Littlewood–Sobolev (HLS for short) inequality states that

$$\|(-\Delta)^{-s/2}f\|_{L^q}\le C\|f\|_{L^p}$$

for $f\in\mathcal{S}(\mathbb{R}^n)$, $0<s<n$, $1<p<q<\infty$ and $1/p-1/q=s/n$, where $\mathcal{S}(\mathbb{R}^n)$ denotes the space of Schwarz functions, $(-\Delta)^{-s/2}=\mathcal{F}^{-1}|\xi|^{-s}\mathcal{F}$ is the Riesz potential of order s and $\mathcal{F}$ stands for the Fourier transform in $\mathbb{R}^n$. An equivalent form is Sobolev's inequality

$$\|f\|_{L^q}\le C\|(-\Delta)^{s/2}f\|_{L^p}.$$

When $s=2$, the HLS inequality can be regarded as the L^p-L^q boundedness of the free resolvent $(-\Delta-z)^{-1}$ at $z=0$. In this context, the HLS inequality was extended to nonzero energies $z\neq 0$ in [Kenig, Ruiz, and Sogge 1987; Kato and Yajima 1989; Gutiérrez 2004] as follows:

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Proposition 1.1 (uniform Sobolev estimates). *Let $n \ge 3$, $1 \le r \le \infty$ and (p,q) satisfy*

$$\frac{2}{n+1} \le \frac{1}{p} - \frac{1}{q} \le \frac{2}{n}, \quad \frac{2n}{n+3} < p < \frac{2n}{n+1}, \quad \frac{2n}{n-1} < q < \frac{2n}{n-3}. \tag{1-1}$$

Then the free resolvent $R_0(z) = (-\Delta - z)^{-1}$ satisfies

$$\|R_0(z)f\|_{L^{q,r}} \le C|z|^{(n/2)(1/p-1/q)-1}\|f\|_{L^{p,r}} \tag{1-2}$$

uniformly in $f \in L^{p,r}(\mathbb{R}^n)$, $z \in \mathbb{C} \setminus [0,\infty)$ and r, where $L^{p,r}(\mathbb{R}^n)$ denotes the Lorentz space.

Sketch of proof. By virtue of real interpolation (see Theorem A.1 in the Appendix), we may replace without loss of generality $L^{p,r}$ and $L^{q,r}$ by L^p and L^q, respectively. Then the case $1/p + 1/q = 1$ was proved independently by [Kenig, Ruiz, and Sogge 1987, Theorem 2.3] and [Kato and Yajima 1989, (3.29), p. 493]; the case $1/p - 1/q = 2/n$ is due to [Kenig, Ruiz, and Sogge 1987, Theorem 2.2]; otherwise, we refer to [Gutiérrez 2004, Theorem 6]. □

Note that, when $1/p - 1/q = 2/n$, the estimate is uniform in z, as its name suggests.

Uniform Sobolev estimates can be used in the study of broad areas including the spectral and scattering theory for Schrödinger operators. In [Kenig, Ruiz, and Sogge 1987], the authors applied (1-2) to study unique continuation properties of $-\Delta + V$ with $V \in L^{n/2}$. In [Kato and Yajima 1989; Goldberg and Schlag 2004; Ionescu and Schlag 2006], (1-2) was used to show the limiting absorption principle and asymptotic completeness of wave operators for $-\Delta + L$ with a large class of singular perturbations L. In [Frank 2011], (1-2) was used to prove the Keller-type inequality for $-\Delta + W(x)$ with a complex potential $W \in L^p$ with some $p \ge n/2$, which is a quantitative estimate of the spectral radius of $\sigma_{\mathrm{p}}(-\Delta + W)$. In [Gutiérrez 2004], (1-2) was applied to show the existence of L^q-solutions for the stationary Ginzburg–Landau equation under some radiation condition.

In a more abstract setting, the following observations are satisfied for not only Δ but also a general nonnegative self-adjoint operator L on $L^2(X,\mu)$:

- The uniform Sobolev estimate with $p = 2n/(n+2)$ and $q = 2n/(n-2)$ implies that, for any $w \in L^n$, the weighted resolvent $w(L-z)^{-1}w$ is bounded on L^2 uniformly in $z \in \mathbb{C} \setminus [0,\infty)$. As observed by [Kato 1966; Kato and Yajima 1989; Rodnianski and Schlag 2004], such a weighted estimate is closely connected with dispersive properties of the solution to (1-4) such as Kato-smoothing effects, time-decay and Strichartz estimates, which are fundamental tools in the study of nonlinear Schrödinger equations; see [Tao 2006].

- Uniform Sobolev estimates imply that the spectral measure $dE_L(\lambda)$ associated with L is bounded from L^p to $L^{p'}$ for

$$\frac{2n}{n+2} \le p \le \frac{2(n+1)}{n+3}.$$

This is an important input to prove the Hörmander-type theorem on the L^p boundedness of the spectral multiplier $f(L)$; see [Chen, Ouhabaz, Sikora, and Yan 2016].

Motivated by those observations, we are interested in extending (1-2) to the Schrödinger operator $H = -\Delta + V(x)$. If V is of very short range type in the sense that, with some $\varepsilon > 0$,

$$|V(x)| \le C(1+|x|)^{-2-\varepsilon}, \quad x \in \mathbb{R}^n, \tag{1-3}$$

then there is a vast literature on uniform weighted L^2-estimates for $(H-z)^{-1}$ without any additional repulsive condition such as suitable smallness of the negative part of V; see, e.g., [Jensen and Kato 1979; Rodnianski and Tao 2015]. Weighted L^2-estimates were also obtained for a class of potentials satisfying $|x|^2 V \in L^\infty$ under some additional repulsive conditions [Burq, Planchon, Stalker, and Tahvildar-Zadeh 2004; Barceló, Vega, and Zubeldia 2013]. In our previous works [Bouclet and Mizutani 2018; Mizutani 2019], we proved uniform Sobolev estimates for H with a class of critical potentials $V \in L^{n/2,\infty}$ under some repulsive conditions so that H has purely absolutely continuous spectrum. However, in these works, the range of (p, q) has been restricted on the line $1/p + 1/q = 1$. Furthermore, the situation for (large) critical potentials without any repulsive condition is less understood.

The main goal of this paper is to prove the full set of uniform Sobolev estimates for $H = -\Delta + V(x)$ with a large scaling-critical potential $V \in L^{n/2,\infty}_0$ without any repulsive condition. The following three types of applications are also established in the paper:

(i) We prove global-in-time Strichartz estimates for the Schrödinger equation

$$i\partial_t u(t,x) = Hu(t,x) + F(t,x), \quad (t,x) \in \mathbb{R}^{1+n}, \qquad u(0,x) = \psi, \quad x \in \mathbb{R}^n, \tag{1-4}$$

for all admissible cases and several nonadmissible cases.

(ii) A Hörmander-type spectral multiplier theorem for $f(H)$ is obtained provided that H is nonnegative.

(iii) We obtain Keller-type estimates for the eigenvalues (including possible embedded eigenvalues) of the operator $H + W$ with complex potentials $W \in L^p$, $n/2 < p \le (n+1)/2$.

Finally, we mention that the results in this paper could be used to study spectral and scattering theory for both linear and nonlinear Schrödinger equations with potentials $V \in L^{n/2,\infty}_0$.

Notation. $A \lesssim B$ (resp. $A \gtrsim B$) means $A \le cB$ (resp. $A \ge cB$) with some universal constant $c > 0$. By $\langle x \rangle$ we denote $\sqrt{1+|x|^2}$ and we set $\mathbb{C}^\pm := \{z \in \mathbb{C} \mid \pm \operatorname{Im} z > 0\}$. Given two Banach spaces X and Y, $\mathbb{B}(X,Y)$ is the Banach space of bounded linear operators from X to Y and $\mathbb{B}(X) = \mathbb{B}(X,X)$, and $\mathbb{B}_\infty(X,Y)$ and $\mathbb{B}_\infty(X)$ are families of compact operators. By $\langle f, g \rangle = \int f\bar{g}\,dx$ we denote the inner product in L^2. We also use the same notation $\langle \cdot, \cdot \rangle$ for the dual coupling between L^p and $L^{p'}$, where $p' = p/(p-1)$ denotes the Hölder conjugate of p. $L^p_t \mathcal{X}_x = L^p(\mathbb{R}; \mathcal{X})$ is the Bochner–Lebesgue space with norm $\|F\|_{L^p_t \mathcal{X}} = \| \|F(t,x)\|_{\mathcal{X}_x} \|_{L^p_t}$. $L^p_T L^q_x := L^p([-T,T]; L^q(\mathbb{R}^n))$. Let $\langle \cdot, \cdot \rangle_T$ be the inner product in $L^2_T L^2_x$ defined by

$$\langle F, G \rangle_T = \int_{-T}^{T} \langle F(\cdot, t), G(\cdot, t) \rangle \, dt.$$

$\mathcal{H}^s(\mathbb{R}^n)$ and $\dot{\mathcal{H}}^s(\mathbb{R}^n)$ are inhomogeneous and homogeneous L^2-Sobolev spaces, respectively. $\mathcal{W}^{s,p}(\mathbb{R}^n)$ is the L^p-Sobolev space. $L^{p,q}(\mathbb{R}^n)$ denotes the Lorentz space (see the Appendix).

1A. ***Main results.*** Throughout the paper we assume that $n \geq 3$ and that $V \in L_0^{n/2,\infty}(\mathbb{R}^n)$ is a real-valued function, where $L_0^{p,\infty}(\mathbb{R}^n)$ is the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $\|\cdot\|_{L^{p,\infty}}$. It follows from Hölder's and Sobolev's inequalities for Lorentz norms (see the Appendix) that V is Δ-form compact. Then the KLMN theorem [Reed and Simon 1975, Theorem X.17] yields that there exists a unique lower semibounded self-adjoint operator H on $L^2(\mathbb{R}^n)$ with form domain $\mathcal{H}^1(\mathbb{R}^n)$ such that

$$\langle Hu, v\rangle = \langle(-\Delta + V)u, v\rangle, \quad u \in D(H),\ v \in \mathcal{H}^1(\mathbb{R}^n),$$

and that its domain $D(H) = \{u \in \mathcal{H}^1(\mathbb{R}^n) \mid Hu \in L^2(\mathbb{R}^n)\}$ is dense in $\mathcal{H}^1(\mathbb{R}^n)$. In other words, H is defined as the Friedrichs extension of the sesquilinear form $\langle(-\Delta + V)u, v\rangle$.

Remark 1.2. Note that $L^{n/2,q} \hookrightarrow L_0^{n/2,\infty}$ for all $1 \leq q < \infty$. Also note that the class $L_0^{n/2,\infty}$ is scaling-critical in the sense that the norm $\|V\|_{L^{n/2,\infty}}$ is invariant under the scaling $V \mapsto V_\lambda$, where $V_\lambda(x) = \lambda^2 V(\lambda x)$. In particular, if V itself is invariant under this scaling, the potential energy $\langle Vu, u\rangle$ has the same scale-invariant structure as that for the kinetic energy $\langle -\Delta u, u\rangle$.

Let $\mathcal{E} \subset \sigma(H)$ be the exceptional set of H, the set of all eigenvalues and resonances of H (see Definition 2.6). Note that $\mathcal{E} \cap (-\infty, 0)$ is equal to $\sigma_{\mathrm{d}}(H)$, the discrete spectrum of H, and that $\mathcal{E}$ is bounded in $\mathbb{R}$ (see Remark 3.4). For the absence of embedded eigenvalues and resonances, we have the following simple criterion (see also Remark 1.18):

Lemma 1.3. *Let V be as above. Then the following statements are satisfied*:

(1) *If $V \in L^{n/2}$ then there are no positive eigenvalues and resonances*; *that is*, $\mathcal{E} \cap (0, \infty) = \varnothing$.

(2) *If $-\Delta + V \geq -\delta\Delta$ with some $\delta > 0$ in the sense of forms on C_0^∞ then* $0 \notin \mathcal{E}$.

Proof. The proof will be given in Section 2B. □

Define $\mathcal{E}_\delta := \{z \in \mathbb{C} \mid \operatorname{dist}(z, \mathcal{E}) < \delta\}$ if $\mathcal{E} \neq \varnothing$ and $\mathcal{E}_\delta := \varnothing$ if $\mathcal{E} = \varnothing$. For $z \in \mathbb{C} \setminus \sigma(H)$, we denote the resolvent of H by $R(z) = (H - z)^{-1}$.

Then the main result in this paper is as follows.

Theorem 1.4. *Suppose that (p, q) satisfies* (1-1). *Then $R(z)$ extends to a bounded operator from $L^{p,2}$ to $L^{q,2}$ for all $z \in \mathbb{C} \setminus \sigma(H)$. Moreover*, *for any $\delta > 0$ there exists $C_\delta > 0$ such that*

$$\|R(z)f\|_{L^{q,2}} \leq C_\delta |z|^{(n/2)(1/p-1/q)-1} \|f\|_{L^{p,2}} \tag{1-5}$$

for all $z \in \mathbb{C} \setminus ([0, \infty) \cup \mathcal{E}_\delta)$ and $f \in L^{p,2}$. In particular, *if $\mathcal{E} = \varnothing$*, *then* (1-5) *holds uniformly with respect to $z \in \mathbb{C} \setminus [0, \infty)$ and $f \in L^{p,2}$.*

As a corollary, the limiting absorption principle in the same topology is derived.

Corollary 1.5. *Let (p, q) satisfy* (1-1). *Then the following statements are satisfied*:

(1) *The boundary values $R(\lambda \pm i0) = \lim_{\varepsilon \searrow 0} R(\lambda \pm i\varepsilon) \in \mathbb{B}(L^{p,2}, L^{q,2})$ exist for all $\lambda \in (0, \infty) \setminus \mathcal{E}$. Moreover*, *for any $\delta > 0$ there exists $C_\delta > 0$ such that*

$$\|R(\lambda \pm i0)f\|_{L^{q,2}} \leq C_\delta \lambda^{(n/2)(1/p-1/q)-1} \|f\|_{L^{p,2}}, \quad f \in L^{p,2}(\mathbb{R}^n),\ \lambda \in (0, \infty) \setminus \mathcal{E}_\delta. \tag{1-6}$$

In particular, *if $\mathcal{E} \cap [0, \infty) = \varnothing$*, *then* (1-6) *holds uniformly in $\lambda > 0$.*

(2) *Assume in addition that* $1/p - 1/q = 2/n$ *and* $0 \notin \mathcal{E}$. *Then* $R(0 \pm i0) \in \mathbb{B}(L^{p,2}, L^{q,2})$ *exist and* $R(0+i0) = R(0-i0)$. *Moreover*, $HR(0+i0)f = f$ *and* $R(0+i0)Hg = g$ *for all* $f, g \in \mathcal{S}$ *in the sense of distributions. In particular*, *one has the HLS-type inequality*

$$\|H^{-1}f\|_{L^{q,2}} \le C\|f\|_{L^{p,2}}, \quad f \in L^{p,2}(\mathbb{R}^n). \tag{1-7}$$

As a byproduct of Theorem 1.4, we also obtain the L^p-L^q boundedness of $R(z)$ for fixed z with a wider range than (1-1).

Corollary 1.6. *For any* $z \in \mathbb{C} \setminus \sigma(H)$, *the resolvent* $R(z)$ *is bounded from* $L^{p,2}$ *to* $L^{q,2}$ *whenever*

$$0 \le \frac{1}{p} - \frac{1}{q} \le \frac{2}{n}, \quad \frac{2n}{n+3} < p, q < \frac{2n}{n-3}. \tag{1-8}$$

In particular, $D(H) \subset D(w)$ *for any* $w \in L^{n/s,\infty}$ *with* $0 \le s < \frac{3}{2}$. *Here* $D(w)$ *denotes the domain of the multiplication operator by* $w(x)$.

Remark 1.7. Since $L^p \hookrightarrow L^{p,2}$ and $L^{q,2} \hookrightarrow L^q$ if $p \le 2 \le q$, one has $\mathbb{B}(L^{p,2}, L^{q,2}) \subset \mathbb{B}(L^p, L^q)$. Moreover, by virtue of real interpolation (see Theorem A.1), Theorem 1.4 and Corollaries 1.5 and 1.6 also hold with $L^{p,2}$ and $L^{q,2}$ replaced respectively by $L^{p,r}$ and $L^{q,r}$ for any $1 \le r \le \infty$.

As explained in the Introduction, the resolvent $R(z)$ has a close relation with the spectral measure E_H associated with H through Stone's formula

$$E'_H(\lambda) = \frac{1}{2\pi i} \lim_{\varepsilon \searrow 0} (R(\lambda + i\varepsilon) - R(\lambda - i\varepsilon)), \quad \lambda \in (0, \infty) \setminus \sigma_{\mathrm{p}}(H), \tag{1-9}$$

where $E'_H(\lambda) = (dE_H/d\lambda)(\lambda)$ is the density of E_H. Using this formula and above theorems, we also obtain the following restriction-type estimates.

Theorem 1.8. *Assume that* $\mathcal{E} \cap [0, \infty) = \varnothing$. *Then*, *for any*

$$\frac{2n}{n+3} < p \le \frac{2(n+1)}{n+3},$$

we have

$$\|E'_H(\lambda)\|_{\mathbb{B}(L^p, L^{p'})} \le C\lambda^{(n/2)(1/p - 1/p') - 1}, \quad \lambda > 0. \tag{1-10}$$

Remark 1.9. The existence of $R(\lambda \pm i0)$ in $\mathbb{B}(L^{2(n+1)/(n+3)}, L^{2(n+1)/(n-1)})$ for each $\lambda > 0$ was proved in [Ionescu and Schlag 2006] for the case when $V \in L^p$ with $n/2 \le p \le (n+1)/2$. The uniform estimate (1-6) in the high energy regime $\lambda \ge \lambda_0 > 0$ was obtained in [Goldberg and Schlag 2004] for the case when $n = 3$, $V \in L^{3/2} \cap L^r$ with $r > \frac{3}{2}$ and $(p, q) = \left(\frac{4}{3}, 4\right)$. Recently, (1-6) for $\lambda > 0$ and

$$(p, q) = \left(\frac{2(n+1)}{n-1}, \frac{2(n+1)}{n+3}\right)$$

was proved in [Huang, Yao, and Zheng 2018] provided that $V \in L^{n/2} \cap L^{n/2+\varepsilon}$ and $0 \notin \mathcal{E}$ (note that, in this case, $\mathcal{E} \cap (0, \infty) = \varnothing$ as in Lemma 1.3). Compared with those previous works, the main new contributions of Theorem 1.4 and Corollary 1.5 are threefold. At first, we obtain the uniform estimates (1-5) and (1-6) with respect to z or λ in both high- and low-energy regimes, under the condition $\mathcal{E} \cap [0, \infty) = \varnothing$. This is an important input to prove global-in-time Strichartz estimates without any low- or high-energy cut-off. Next, the full set of uniform Sobolev estimates is obtained, while the above previous references considered the case $1/p+1/q=1$ only. In particular, (1-5) and (1-6) for (p,q) away from the line $1/p+1/q=1$ seems to be new even under the condition (1-3). Such “off-diagonal” estimates play an important role in the proof of Strichartz estimates for nonadmissible pairs and L^p-boundedness of the spectral multiplier $f(H)$ for a wider range of p than that obtained by the “diagonal” estimate on the line $1/p+1/q=1$ (see Sections 4 and 5, respectively). Finally, we obtain the above results for large critical potentials $V\in L^{n/2,\infty}_0$ without any additional regularity or repulsive condition. Concerning L^p-L^q boundedness of $R(z)$ for each $z\in\mathbb{C}\setminus[0,\infty)$, a result similar to Corollary 1.6 was previously obtained in [Simon 1982] for Kato class potentials. However, to our best knowledge, this corollary seems to be new for the present class of potentials.

In this paper we also study several applications of the above resolvent estimates to the time-dependent problem, harmonic analysis and spectral theory associated with H.

We first consider global-in-time estimates for the Schrödinger equation (1-4). Let e^{-itH} be the unitary group generated by H via Stone's theorem. For $F\in L^1_{\mathrm{loc}}(\mathbb{R};L^2(\mathbb{R}^n))$, we define

$$\Gamma_H F(t)=\int_0^t e^{-i(t-s)H}F(s)\,ds.$$

For $\psi\in L^2(\mathbb{R}^n)$ and $F\in L^1_{\mathrm{loc}}(\mathbb{R};L^2(\mathbb{R}^n))$, a unique (mild) solution to (1-4) is then given by

$$u=e^{-itH}\psi-i\Gamma_H F. \tag{1-11}$$

The next theorem generalizes a result in [Ben-Artzi and Klainerman 1992], where the case when $|V(x)|\lesssim\langle x\rangle^{-2-\varepsilon}$ was considered.

Theorem 1.10. *Assume that $\mathcal{E}\cap[0,\infty)=\varnothing$. Then, for any $\rho>\frac{1}{2}$,*

$$\|\langle x\rangle^{-\rho}|D|^{1/2}e^{-itH}P_{\mathrm{ac}}(H)\psi\|_{L^2_tL^2_x}\le C_\rho\|\psi\|_{L^2_x},$$

where $P_{\mathrm{ac}}(H)$ is the projection onto the absolutely continuous subspace associated with H.

To state the result on Strichartz estimates, we recall some standard notation.

Definition 1.11. When $n\ge 3$, a pair $(p,q)\in\mathbb{R}^2$ is said to be admissible if

$$p,q\ge 2,\quad \frac{2}{p}=n\Big(\frac{1}{2}-\frac{1}{q}\Big). \tag{1-12}$$

Theorem 1.12. *Suppose that $\mathcal{E}\cap[0,\infty)=\varnothing$. Then, for any admissible pairs (p_1,q_1) and (p_2,q_2), the solution u to* (1-4) *satisfies*

$$\|P_{\mathrm{ac}}(H)u\|_{L^{p_1}_tL^{q_1}_x}\lesssim\|\psi\|_{L^2}+\|F\|_{L^{p_2'}_tL^{q_2'}_x},\quad \psi\in L^2,\ F\in L^{p_2'}_tL^{q_2'}_x. \tag{1-13}$$

For any

$$\frac{n}{2(n-1)}\le s\le\frac{3n-4}{2(n-1)},$$

we also obtain nonadmissible inhomogeneous Strichartz estimates:

$$\|\Gamma_H P_{\mathrm{ac}}(H)F\|_{L^2_tL^{2n/(n-2s)}_x}\lesssim\|F\|_{L^2_tL^{2n/(n+2(2-s))}_x},\quad F\in L^2_tL^{2n/(n+2(2-s))}_x. \tag{1-14}$$

Remark 1.13. For the admissible case or the case when

$$\frac{n}{2(n-1)} < s < \frac{3n-4}{2(n-1)},$$

we can actually obtain stronger estimates than (1-13) and (1-14):

$$\|P_{\mathrm{ac}}(H)u\|_{L_t^{p_1}L_x^{q_1,2}} \lesssim \|\psi\|_{L^2} + \|F\|_{L_t^{p_2'}L_x^{q_2',2}},$$

$$\|\Gamma_H P_{\mathrm{ac}}(H)F\|_{L_t^2 L_x^{2n/(n-2s),2}} \lesssim \|F\|_{L_t^2 L_x^{2n/(n+2(2-s)),2}}, \qquad \frac{n}{2(n-1)} < s < \frac{3n-4}{2(n-1)},$$

Inhomogeneous estimates for some other nonadmissible pairs may be also deduced from (1-14) and usual inhomogeneous estimates. For instance, if we interpolate between (1-14) and the trivial estimate $\|\Gamma_H P_{\mathrm{ac}}(H)F\|_{L_t^\infty L_x^2} \le \|F\|_{L_t^1 L_x^2}$ then

$$\|\Gamma_H P_{\mathrm{ac}} F\|_{L_t^p L_x^q} \lesssim \|F\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}},$$

where

$$\frac{n}{2(n-1)} \le s \le \frac{3n-4}{2(n-1)} \quad \text{and} \quad \frac{n}{s}\Big(\frac{1}{2}-\frac{1}{q}\Big) = \frac{2}{p} = \frac{2}{\tilde{p}} = \frac{n}{2-s}\Big(\frac{1}{2}-\frac{1}{\tilde{q}}\Big).$$

Inhomogeneous Strichartz estimates with nonadmissible pairs for the free Schrödinger equation have been studied by several authors [Kato 1994; Keel and Tao 1998; Foschi 2005; Vilela 2007; Koh and Seo 2016] under suitable conditions on (p, q); see [Foschi 2005; Koh and Seo 2016]. The estimates (1-14) correspond to the endpoint cases for such conditions. It is also worth noting that, as well as the estimates for admissible pairs, nonadmissible estimates can be used in the study of nonlinear Schrödinger equations; see [Kato 1994].

Remark 1.14. There is a vast literature on Strichartz estimates for Schrödinger equations with potentials. We refer to [Rodnianski and Schlag 2004; Goldberg 2009; Beceanu 2011; Bouclet and Mizutani 2018]. We also note that the dispersive L^1-L^∞ estimate for $e^{-itH}P_{\mathrm{ac}}(H)$ and L^p-boundedness of wave operators $W_\pm$, which imply Strichartz estimates, have been also extensively studied; see [Rodnianski and Schlag 2004; Beceanu and Goldberg 2012; Yajima 1995; Beceanu 2014]. In particular, Goldberg [2009] proved the endpoint Strichartz estimates for $e^{-itH}P_{\mathrm{ac}}$ under the conditions $V \in L^{n/2}$, $0 \notin \mathcal{E}$ and $n \ge 3$. When $n = 3$, Strichartz estimates for all admissible cases and some nonadmissible cases (which are different from (1-14)) for $V \in L_0^{3/2,\infty}$ were obtained in [Beceanu 2011]. Compared with those previous works, a new contribution of this theorem is that we obtain the full set of admissible Strichartz estimates (1-13), including the inhomogeneous double endpoint case for all $n \ge 3$. Moreover, nonadmissible estimates (1-14) are new even for $V \in L^{n/2}$.

The next application of resolvent estimates in this paper is the L^p-boundedness of the spectral multiplier $F(H)$, which is defined by the spectral decomposition theorem, namely

$$F(H) = \int_{\sigma(H)} F(\lambda)\, dE_H(\lambda).$$

For the free case $H = -\Delta$, Hörmander's multiplier theorem [1960] implies that if $F \in L^\infty$ satisfies

$$\sup_{t>0} \|\psi(\cdot)F(t\,\cdot)\|_{\mathcal{H}^\beta} < \infty, \tag{1-15}$$

with some nontrivial $\psi \in C_0^\infty(\mathbb{R})$ supported in $(0, \infty)$ and $\beta > n/2$, then $F(-\Delta)$ is bounded on L^p for all $1 < p < \infty$. The following theorem is a generalization of this result to nonnegative Schrödinger operators with scaling-critical potentials.

Theorem 1.15. *Suppose that $\mathcal{E} \cap [0, \infty) = \varnothing$ and $H \geq 0$. Then, for any $F \in L^\infty(\mathbb{R})$ satisfying* (1-15) *with some nontrivial $\psi \in C_0^\infty(\mathbb{R})$ supported in $(0, \infty)$ and $\beta > \frac{3}{2}$, $F(\sqrt{H})$ is bounded on L^p for all*

$$\frac{2n}{n+3} < p < \frac{2n}{n-3}$$

and satisfies

$$\|F(\sqrt{H})\|_{\mathbb{B}(L^p)} \leq C\big(\sup_{t>0} \|\psi(\,\cdot\,)F(t\,\cdot\,)\|_{\mathcal{H}^\beta} + |F(0)|\big). \tag{1-16}$$

It is easy to check that F satisfies (1-15) if and only if $G(\lambda) = F(\lambda^2)$ does. Therefore, (1-16) also holds with $F(\sqrt{H})$ replaced by $F(H)$. Also note that, in the proof of this theorem, the restriction estimates (1-10) will play an essential role and the restriction for the range of p when $n \geq 4$ is due to the condition $p > 2n/(n+3)$ for (1-10).

Remark 1.16. Some applications of Theorem 1.15 will be also established (see Section 5). First we obtain the equivalence between the Sobolev norms $\|(-\Delta)^{s/2}u\|_{L^2}$ and $\|H^{s/2}u\|_{L^2}$ for $0 \leq s < \frac{3}{2}$. Then we shall prove square function estimates for the Littlewood–Paley decomposition via the spectral multiplier associated with H. These are known to play an important role in the study of nonlinear Schrödinger equations with potentials; see, e.g., [Killip, Miao, Visan, Zhang, and Zheng 2018].

Remark 1.17. If the Schrödinger semigroup e^{-tH} satisfies the Gaussian estimate or some generalized Gaussian-type estimates, then Hörmander's multiplier theorem for $F(H)$ has been extensively studied; see [Chen, Ouhabaz, Sikora, and Yan 2016]. Compared with such cases, the interest of Theorem 1.15 is that we obtain Hörmander's multiplier theorem under a scaling-critical condition $V \in L_0^{n/2,\infty}$, while it is not known for such a class of potentials whether H satisfies (generalized) Gaussian estimates or not, even if H is assumed to be nonnegative.

Remark 1.18. To ensure the nonnegativity of H, it suffices to assume $\|V_-\|_{L^{n/2,\infty}} \leq S_n^{-1}$, where $V_- = \max\{0, -V\}$ is the negative part of V and

$$S_n := \frac{n(n-2)}{4} 2^{2/n} \pi^{1+1/n} \Gamma\Big(\frac{n+1}{2}\Big)^{-2/n}$$

is the best constant in Sobolev's inequality. $\|f\|_{L^{2n/(n-2)}} \leq S_n \|\nabla f\|_{L^2}$. Moreover, if $\|V_-\|_{L^{n/2}} < S_n^{-1}$ then $0 \notin \mathcal{E}$ by Lemma 1.3.

The last application of Theorem 1.4 in the paper is the Keller-type inequality for individual eigenvalues of a non-self-adjoint Schrödinger operator. Let $0 < \gamma < \infty$ and $W \in L^{n/2+\gamma}(\mathbb{R}^n; \mathbb{C})$ be a possibly complex-valued potential. Then W is H-form compact and we define the operator $H_W = H + W$ as a form sum. Under this setting, it is known that $\sigma(H_W)$ is contained in a sector $\{z \in \mathbb{C} \mid |\arg(z - z_0)| \leq \theta\}$ for some $z_0 \in \mathbb{R}$ and $\theta \in \big[0, \frac{\pi}{2}\big)$ (see [Kato 1966]), but the point spectrum $\sigma_{\mathrm{p}}(H_W)$ could be unbounded in $\mathbb{C}$ in general even if $V \equiv 0$ and W is smooth. The following theorem, however, shows that this is not the case if $0 < \gamma \leq \frac{1}{2}$.

Theorem 1.19. *Let $\delta > 0$. If $0 < \gamma \le \frac{1}{2}$, any eigenvalue $E \in \mathbb{C} \setminus \mathcal{E}_\delta$ of H_W satisfies*

$$|E|^\gamma \le C_{\gamma,\delta} \|W\|_{L^{n/2+\gamma}}^{n/2+\gamma}. \tag{1-17}$$

Moreover, if $\gamma > \frac{1}{2}$, any eigenvalue $E \in \mathbb{C} \setminus \mathcal{E}_\delta$ of H_W satisfies

$$|E|^{1/2} \operatorname{dist}(E, [0,\infty))^{\gamma-1/2} \le C_{\gamma,\delta} \|W\|_{L^{n/2+\gamma}}^{n/2+\gamma}. \tag{1-18}$$

Here the constant $C_{\gamma,\delta} = C(\gamma,\delta,n,V) > 0$ may be taken uniformly in W.

Remark 1.20. Theorem 1.19 implies the following spectral consequence. If $0 < \gamma \le \frac{1}{2}$ then

$$\sigma_{\mathrm{p}}(H_W) \subset \mathcal{E}_\delta \cup \{z \in \mathbb{C} \mid |z|^\gamma \le C_{\gamma,\delta} \|W\|_{L^{n/2+\gamma}}^{n/2+\gamma}\}.$$

In particular, since $\mathcal{E}$ is bounded in $\mathbb{R}$ (see Remark 3.4), $\sigma_{\mathrm{p}}(H_W)$ is bounded in $\mathbb{C}$. On the other hand, if $\gamma > \frac{1}{2}$ and $\operatorname{Re} E > 0$, then E satisfies

$$|\operatorname{Im} E| \le C_{\gamma,\delta} |E|^{-1/(2(\gamma-1/2))} \|W\|_{L^{n/2+\gamma}}^{(n+2\gamma)/(2\gamma-1)}.$$

This implies that, for any sequence $\{E_j\} \subset \sigma_{\mathrm{p}}(H_W) \setminus [0,\infty)$ satisfying $\operatorname{Re} E_j \to +\infty$ as $j \to \infty$, we have $|\operatorname{Im} E_j| \to 0$ as $j \to \infty$.

Remark 1.21. For a complex potential $W(x)$, the estimates (1-17) and (1-18) were first proved by Frank [2011; 2018] for the case when $-\Delta + W(x)$ and then extended to the operator $-\Delta - a|x|^{-2} + W(x)$ with $a \le (n-2) - \frac{2}{4}$ by [Mizutani 2019]. In both cases, the free Hamiltonians $-\Delta$ and $-\Delta - a|x|^{-2}$ are nonnegative and purely absolutely continuous. Theorem 1.19 shows that the same result still holds even if the free Hamiltonian has (embedded) eigenvalues or resonances.

The rest of the paper is devoted to the proof of above results. We here outline the plan of the paper, describing rough idea of the proofs. Following the classical scheme, the proof of the uniform Sobolev estimates is based on the resolvent identity $R(z) = (I + R_0(z)V)^{-1} R_0(z)$.

In Section 2 we collect several properties on the free resolvent $R_0(z)$ used throughout the paper and, then, study basic properties of the exceptional set $\mathcal{E}$. In particular, we show that $R_0(z)V$ extends to a $\mathbb{B}_\infty(L^q)$-valued continuous function on $\overline{\mathbb{C}^+}$. This fact plays an important role to justify the above resolvent identity. The proof of Lemma 1.3 is also given in Section 2.

Using materials prepared in Section 2 and the Fredholm alternative theorem, we prove Theorem 1.4, Corollaries 1.5 and 1.6 and Theorem 1.8 in Section 3.

Section 4 is devoted to proving Theorems 1.10 and 1.12. The proof follows an abstract scheme by [Rodnianski and Schlag 2004] (see also [Burq, Planchon, Stalker, and Tahvildar-Zadeh 2004; Bouclet and Mizutani 2018]), which is based on Duhamel's formulas

$$e^{-itH} = e^{it\Delta} - i\Gamma_0 V \Gamma_H, \quad \Gamma_H = \Gamma_0 - i\Gamma_0 V \Gamma_H,$$

where $\Gamma_0 = \Gamma_{-\Delta}$. Using these identities, the proof can be reduced to that of corresponding estimates for the free propagators $e^{it\Delta}$ and Γ_0, which are well known, and $L_t^2 L_x^2$ estimates for $V_1 e^{-itH} P_{\mathrm{ac}}(H)$ and

$V_1\Gamma_H P_{\rm ac}(H)V_2$ with a suitable decomposition $V = V_1V_2$. Kato's smooth perturbation theory [1966] allows us to deduce such $L^2_tL^2_x$-estimates from the resolvent estimate

$$\sup_{z\in\mathbb{C}\setminus\mathbb{R}} \|V_1R(z)P_{\rm ac}(H)V_2\|_{\mathbb{B}(L^2)} < \infty,$$

which follows from uniform Sobolev estimates for $P_{\rm ac}(H)R(z)$ (which are also proved as a corollary of Theorem 1.4 in the end of Section 3) and Hölder's inequality. A rigorous justification of the above Duhamel's formulas in the sense of forms are also given in Section 4.

Proofs of the spectral multiplier theorem and its applications are given in Section 5. The proof of Theorem 1.15 employs an abstract method by [Chen, Ouhabaz, Sikora, and Yan 2016], which allows us to deduce Theorem 1.15 from the restriction estimates (1-10) and the so-called Davies–Gaffney estimate for the Schrödinger semigroup e^{-tH}. In the proof of the Davies–Gaffney estimate, we use the condition that H is nonnegative.

Section 6 is devoted to the proof of Theorem 1.19, which follows basically the same line as in [Frank 2011; 2018] and is based on the estimates (1-5), (1-6) and the Birman–Schwinger principle.

The Appendix is devoted to a brief introduction of real interpolation and Lorentz spaces.

2. Preliminaries

In this section we first study several properties of the free resolvent, which will often appear in the sequel. The second part is devoted to a detailed study of the exceptional set of H.

2A. *The free resolvent.* For $z \notin \mathbb{C}\setminus[0,\infty)$, $R_0(z) = (-\Delta - z)^{-1}$ denotes the free resolvent, which is defined as a Fourier multiplier with symbol $(|\xi|^2 - z)^{-1}$. The integral kernel of $R_0(z)$ is given by

$$R_0(z,x,y) = \frac{i}{4}\left(\frac{z^{1/2}}{2\pi|x-y|}\right)^{n/2-1} H^{(1)}_{n/2-1}(z^{1/2}|x-y|), \quad \operatorname{Im} z^{1/2} > 0,$$

where $H^{(1)}_{n/2-1}$ is the Hankel function of the first kind. The pointwise estimate

$$|H^{(1)}_{n/2-1}(w)| \le C_n \begin{cases} |w|^{-n/2+1} & \text{for } |w|\le 1,\\ |w|^{-1/2} & \text{for } |w|>1,\end{cases}$$

then implies that there exists $C_n > 0$ depending only on n such that

$$|R_0(z,x,y)| \le C_n(|x-y|^{-n+2} + |x-y|^{-(n-1)/2})\langle z\rangle^{(n-3)/4}; \tag{2-1}$$

see [Jensen 1980]. For $s\in\mathbb{R}$, we let $L^2_s = L^2(\mathbb{R}^n, \langle x\rangle^{2s}dx)$ and $\mathcal{H}^2_s = \{u \mid \partial^\alpha u \in L^2_s,\ |\alpha|\le 2\}$. Then the following limiting absorption principle in weighted L^2-spaces is well known; see [Agmon 1975; Jensen and Kato 1979; Jensen 1980; 1984].

Lemma 2.1. *Let $s > (n+1)/2$. Then $R_0(z)$ is bounded from L^2_s to L^2_{-s} uniformly in $z\in\mathbb{C}\setminus[0,\infty)$. Moreover, the following statements are satisfied*:

- *Boundary values $R_0(\lambda\pm i0) = \lim_{\varepsilon\to 0} R_0(\lambda\pm i\varepsilon) \in \mathbb{B}_\infty(L^2_s, L^2_{-s})$ exist on $[0,\infty)$ such that $R_0(0\pm i0) = (-\Delta)^{-1}$. Moreover, $R_0(\lambda\pm i0)\in\mathbb{B}_\infty(L^2_s,\mathcal{H}^2_{-s})$ if $\lambda>0$.*

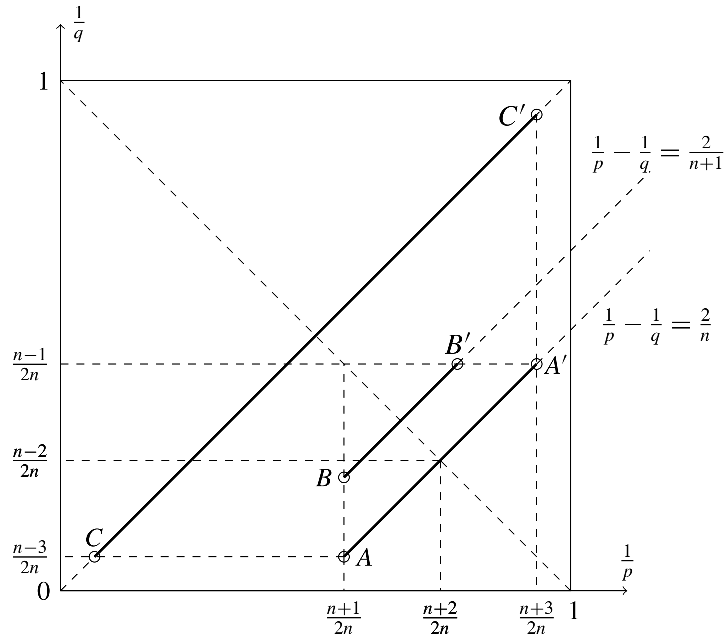

Figure 1. The set of $(1/p, 1/q)$ satisfying (1-1) is the trapezium $ABB'A'$ with two closed line segments $\overline{AB}$, $\overline{B'A'}$ removed. The set of $(1/p, 1/q)$ satisfying (1-8) is the trapezium $ACC'A'$ with two closed line segments $\overline{AC}$, $\overline{C'A'}$ removed.

- *Define the extended free resolvent* $R_0^\pm(z)$ *by* $R_0^\pm(z) = R_0(z)$ *if* $z \in \mathbb{C} \setminus [0, \infty)$ *and* $R_0^\pm(z) = R_0(z \pm i0)$ *if* $z \geq 0$. *Then* $R_0^\pm(z)$ *are* $\mathbb{B}_\infty(L^2_s, L^2_{-s})$*-valued continuous functions on* $\overline{\mathbb{C}^\pm}$.
- *For any* $z \in \overline{\mathbb{C}^+}$ *and* $f \in L^2_s$, *we have* $(-\Delta - z)R_0^\pm(z) f = f$ *in the sense of distributions.*

The following corollaries are immediate consequences of Lemma 2.1 and Proposition 1.1.

Corollary 2.2. *Let* (p, q) *satisfy* (1-1) *and*

$$\frac{2n}{n+3} < r < \frac{2n}{n+1};$$

see Figure 1. Then:

(1) $R_0^\pm(z)$ *extend to elements in* $\mathbb{B}(L^{p,2}, L^{q,2})$ *and satisfy*

$$\|R_0^\pm(z)\|_{\mathbb{B}(L^{p,2}, L^{q,2})} \leq C|z|^{(n/2)(1/p-1/q)-1}, \quad z \in \overline{\mathbb{C}^\pm} \setminus \{0\}. \tag{2-2}$$

(2) *For any* $f \in L^{p,2}$ *and* $g \in L^{q',2}$, *we have* $\langle R_0^\pm(z) f, g\rangle$ *are continuous on* $\overline{\mathbb{C}^\pm} \setminus \{0\}$.

(3) *For any* $z \in \overline{\mathbb{C}^\pm}$ *and* $f \in L^{r,2}$, *we have* $(-\Delta - z)R_0^\pm(z) f = f$ *in the sense of distributions.*

Assuming in addition that $1/p - 1/q = 2/n$, *the statements* (1) *and* (2) *hold for all* $z \in \overline{\mathbb{C}^\pm}$.

Throughout the paper, we frequently use the notation

$$p_s = \frac{2n}{n+2(2-s)}, \quad q_s = \frac{2n}{n-2s}. \tag{2-3}$$

Note that

$$\left\{(p_s, q_s) \,\middle|\, \frac{1}{2} < s < \frac{3}{2}\right\} = \left\{(p, q) \,\middle|\, (p, q) \text{ satisfies (1-1) and } \frac{1}{p} - \frac{1}{q} = \frac{2}{n}\right\}.$$

Corollary 2.3. *Let $\frac{1}{2} < s < \frac{3}{2}$, $V_1 \in L_0^{n/s,\infty}$ and $V_2 \in L_0^{n/(2-s),\infty}(\mathbb{R}^n)$. Then $V_1 R_0^{\pm}(z) V_2$ are $\mathbb{B}_\infty(L^2)$-valued continuous functions of $z \in \overline{\mathbb{C}^{\pm}}$.*

Proof. Corollary 2.2(1) with $(p, q) = (p_s, q_s)$ and Hölder's inequality (A-1) imply

$$\sup_{z \in \overline{\mathbb{C}^+}} \|V_1 R_0^{\pm}(z) V_2\|_{\mathbb{B}(L^2)} \lesssim \|V_1\|_{L^{n/s,\infty}} \|V_2\|_{L^{n/(2-s),\infty}}.$$

Since C_0^∞ is dense in $L_0^{p,\infty}$ for all $1 < p < \infty$ and an operator norm limit of compact operators is compact, we observe from this uniform bound and a standard $\varepsilon/3$ argument that it suffices to show the corollary for $V_1, V_2 \in C_0^\infty$. In this case, the corollary follows from Lemma 2.1. □

The following proposition plays an essential role throughout the paper.

Proposition 2.4. *Let $w \in L_0^{n/2,\infty}(\mathbb{R}^n)$, $\frac{1}{2} < s < \frac{3}{2}$ and q_s as above. Then $R_0(z) w \in \mathbb{B}_\infty(\mathcal{H}^1)$ for all $z \in \mathbb{C} \setminus [0, \infty)$. Moreover, $R_0^{\pm}(z) w$ are $\mathbb{B}_\infty(L^{q_s,2})$-valued continuous functions on $\overline{\mathbb{C}^{\pm}}$.*

Remark 2.5. $R_0^{\pm}(z) w$ are also $\mathbb{B}_\infty(L^{q_s})$-valued continuous functions on $\overline{\mathbb{C}^{\pm}}$. The proof is completely the same.

Proof. The facts $R_0(z) w \in \mathbb{B}(\mathcal{H}^1) \cap \mathbb{B}(L^{q_s,2})$ and $R_0^{\pm}(z) w \in \mathbb{B}(L^{q_s,2})$ follow from the continuity of $R_0(z) : \mathcal{H}^{-1} \to \mathcal{H}^1$, uniform Sobolev estimates (1-2) and Hölder's inequality for Lorentz norms.

To prove the compactness and the continuity (in z), by virtue of these estimates and the same argument as above, we may assume without loss of generality that $w \in C_0^\infty$ and $w(x) = 0$ for $|x| \ge c_0$ with some $c_0 > 0$. Then it was proved by [Ionescu and Schlag 2006, Lemma 4.2] that there is a Banach space X satisfying the continuous embedding $X \hookrightarrow \mathcal{H}^{-1}$ such that $w : X^* \to X$ is compact as a multiplication operator. $R_0(z) w$ is therefore compact on $\mathcal{H}^1$ for $z \in \mathbb{C} \setminus [0, \infty)$.

Next we shall prove that $R_0^{\pm}(z) w$ are compact on $L^{q_s,2}$ for $z \in \overline{\mathbb{C}^{\pm}}$. As before, we only consider $R_0^+(z)$. By virtue of real interpolation (Theorem A.1), it suffices to show that $R_0^+(z) w$ is compact on L^{q_s} for all $\frac{1}{2} < s < \frac{3}{2}$. Assume that $f_j \in L^{q_s}$ and $\|f\|_{L^{q_s}} \le 1$. Extracting a subsequence if necessary we may assume $f_j \to 0$ weakly in L^{q_s}. Then it remains to show that there exists a subsequence $\{\tilde{f}_j\} \subset \{f_j\}$ such that $R_0^+(z) w \tilde{f}_j \to 0$ strongly in L^{q_s}. To this end, we decompose $R_0^+(z) w$ into two regions B_r^c and B_r, where $B_r = \{x \in \mathbb{R}^n \mid |x| \le r\}$. For the former case, the pointwise estimate (2-1) yields

$$|R_0^+(z) w f_j(x)| \le C_n \langle z \rangle^{(n-3)/4} |x|^{-(n-1)/2} \|w f_j\|_{L^1} \le C_{n,z} |x|^{-(n-1)/2} \|w\|_{L^{2n/(n+2s)}}$$

uniformly in $|x| \ge r$, $r \ge 2c_0$ and $j \ge 0$. Let us fix $\varepsilon > 0$ arbitrarily. Since

$$\||x|^{-(n-1)/2}\|_{L^{q_s}(B_r^c)} \le C r^{-(s-1/2)},$$

we can find $r_0 = r_0(n, \varepsilon, z, w) > 0$ such that

$$\|R_0^+(z) w f_j\|_{L^{q_s}(B_{r_0}^c)} < \varepsilon. \tag{2-4}$$

For the latter case, we observe that $R_0^+(z)w : L^{q_s}(\mathbb{R}^n) \to \mathcal{W}^{2,q_s}(\mathbb{R}^n)$ is bounded since

$$(-\Delta+1)R_0^+(z)wf = (-\Delta-z)R_0^+(z)wf + (z+1)R_0^+(z)wf = wf + (z+1)R_0^+(z)wf \tag{2-5}$$

for all $f \in L^{q_s}$ by Corollary 2.2(3). In particular, $\{R_0^+(z)wf_j\}_j$ is bounded in $\mathcal{W}^{2,q_s}(B_{r_0})$. Since $\mathcal{W}^{2,q_s}(B_{r_0})$ embeds compactly into $L^{q_s}(B_{r_0})$ by the Rellich–Kondrachov compactness theorem, one can find a subsequence $\{\tilde{f}_j\} \subset \{f_j\}$ such that

$$\lim_{j\to\infty} \|R_0^+(z)w\tilde{f}_j\|_{L^{q_s}(B_{r_0})} = 0. \tag{2-6}$$

It follows from (2-4) and (2-6) that

$$\limsup_{j\to\infty} \|R_0^+(z)w\tilde{f}_j\|_{L^{q_s}(\mathbb{R}^n)} \le \varepsilon.$$

By extracting further a subsequence, we conclude that $R_0^+(z)w\tilde{f}_j \to 0$ strongly in L^{q_s}.

To prove the continuity, let us fix a bounded set $\Lambda \subset \overline{\mathbb{C}^+}$ arbitrarily. We first show that, for any $z, z_j \in \Lambda$ and $g, g_j \in L^{q_s,2}$ satisfying $z_j \to z$ and $g_j \to g$ weakly in $L^{q_s,2}$ as $j \to \infty$,

$$R_0^+(z_j)wg_j \to R_0^+(z)wg \quad \text{strongly in } L^{q_s,2} \text{ as } j \to \infty. \tag{2-7}$$

To this end, we write

$$R_0^+(z_j)wg_j - R_0^+(z)wg = (R_0^+(z_j)w - R_0^+(z)w)g_j + R_0^+(z)w(g_j-g).$$

The second term $R_0^+(z)w(g_j-g)$ converges to 0 strongly in $L^{q_s,2}$ since $R_0^+(z)w$ is compact on $L^{q_s,2}$ and $g_j \to g$ weakly. For the first part, we set $h_j = (R_0^+(z_j)w - R_0^+(z)w)g_j$ and shall show that $h_j \to 0$ strongly in $L^{q_s,2}$. Since $\{g_j\} \subset L^{q_s,2}$ is bounded, say $\|g_j\|_{L^{q_s,2}} \le M$ with $M > 0$ being independent of j, we have by the same argument as above that, with some $\gamma_j = \gamma_j(s,n) > 0$,

$$\|R_0^+(\zeta)wg_j\|_{L^{q_s,2}(B_r^c)} \le C_{n,M,w}\langle\zeta\rangle^{\gamma_1} r^{-\gamma_2}$$

for all $\zeta \in \overline{\mathbb{C}^+}$, $j \ge 1$ and $r \ge 2c_0$, where $C_{n,M,w}$ may be taken uniformly in j and r. This estimate yields that, for any $\varepsilon > 0$, there exists $0 < r_\varepsilon = r(n, M, w, \Lambda, \varepsilon) \sim \varepsilon^{-1/\gamma_2}$ such that

$$\sup_{j\ge1,} \|h_j\|_{L^{q_s,2}(B_{r_\varepsilon}^c)} \le \sup_{j\ge1}(\|R_0^+(z_j)wg_j\|_{L^{q_s,2}(B_{r_\varepsilon}^c)} + \|R_0^+(z)wg_j\|_{L^{q_s,2}(B_{r_\varepsilon}^c)}) < \varepsilon. \tag{2-8}$$

On the other hand, it follows from Sobolev's embedding on $\mathbb{R}^n$ that

$$\|h_j\|_{L^{q_s,2}(B_{r_\varepsilon})} \le C_{\varepsilon,N}\|(-\Delta+1)\langle x\rangle^{-N}h_j\|_{L^2(\mathbb{R}^n)} \le C_{\varepsilon,N}\|\langle x\rangle^{-N}(-\Delta+1)h_j\|_{L^2(\mathbb{R}^n)}$$

for all $N \ge 0$, where we have used the fact that $(-\Delta+1)\langle x\rangle^{-N}(-\Delta+1)^{-1}\langle x\rangle^N$ is a pseudodifferential operator of order 0 and thus bounded on L^p for all $1 < p < \infty$. Equation (2-5) then yields

$$\begin{aligned}\|\langle x\rangle^{-N}(-\Delta+1)h_j\|_{L^2} &\le |z-z_j|\|\langle x\rangle^{-N}R_0^+(z_j)\langle x\rangle^{-N}\|_{\mathbb{B}(L^2)}\|\langle x\rangle^N wg_j\|_{L^2} \\ &\quad + (|z|+1)\|\langle x\rangle^{-N}(R_0^+(z_j)-R_0^+(z))\langle x\rangle^{-N}\|_{\mathbb{B}(L^2)}\|\langle x\rangle^N wg_j\|_{L^2}.\end{aligned}$$

Let $N \ge (n+1)/2$. Since $\langle x\rangle^{-N} R_0^+(z)\langle x\rangle^{-N}$ is bounded on L^2 uniformly in $z \in \overline{\mathbb{C}^+}$ and continuous on $\overline{\mathbb{C}^+}$ in the operator norm topology of $\mathbb{B}(L^2)$ by Lemma 2.1 and

$$\|\langle x\rangle^N w g_j\|_{L^2} \le CM \|\langle x\rangle^N w\|_{L^{2n/(n+2s),2}} \le C_{N,M,\omega}$$

uniformly in j, we see that $\lim_{j\to\infty} \|\langle x\rangle^{-N}(-\Delta+1)h_j\|_{L^2} = 0$, which, together with (2-8), shows that there exists $j_\varepsilon \in \mathbb{N}$ such that, for all $j \ge j_\varepsilon$, we have $\|h_j\|_{L^{q_s,2}(\mathbb{R}^n)} < \varepsilon$. Since $\varepsilon > 0$ is arbitrarily small, this shows that $h_j \to 0$ strongly in $L^{q_s,2}$ and (2-7) follows.

Finally, we shall show $R_0^+(z)w$ is continuous on $\overline{\mathbb{C}^+}$ in the operator norm topology of $\mathbb{B}(L^{q_s,2})$. Assume for contradiction that this is not the case. Then there exist $z_j, z \in \overline{\mathbb{C}^+}$ with $z_j \to z$ and $g_j \in L^{q_s,2}$ with $\|g_j\|_{L^{q_s,2}} \le 1$ such that $\liminf_{j\to\infty} \|(R_0^+(z_j)w - R_0^+(z)w)g_j\|_{L^{q_s,2}} > 0$. Extracting a subsequence if necessary we may assume $g_j \to g$ with some $g \in L^{q_s}$ weakly in L^{q_s}. Then, by the argument as above and the compactness of $R_0^+(z)w$, we have $\lim_{j\to\infty} R_0^+(z_j)wg_j = R_0^+(z)wg = \lim_{j\to\infty} R_0^+(z)wg_j$, which gives a contradiction, proving the desired assertion. □

2B. ***The exceptional set.*** Having Proposition 2.4 in mind, we define the exceptional set of H as follows.

Definition 2.6. We say that $\lambda \in \mathcal{E}$ if there exist $\frac12 < s < \frac32$ and $f \in L^{q_s,2}(\mathbb{R}^n)\setminus\{0\}$ such that $f = -R_0(\lambda)Vf$, where $q_s = 2n/(n-2s)$ and $R_0(\lambda)$ is replaced by $R_0(\lambda+i0)$ if $\lambda \ge 0$. $\mathcal{E}$ is said to be the *exceptional set* of H, and $z \in \mathcal{E}\setminus\sigma_{\mathrm{p}}(H)$ is called a *resonance* of H. For $\lambda \in \mathcal{E}$, we denote the family of corresponding solutions by $\mathcal{N}_s(\lambda)$:

$$\mathcal{N}_s(\lambda) := \{f \in L^{q_s,2}(\mathbb{R}^n)\setminus\{0\} \mid f = -R_0(\lambda)Vf\},$$

where $R_0(\lambda)$ is replaced by $R_0^+(\lambda)$ if $\lambda \ge 0$.

Note that, since $R_0(\lambda - i0)f = \overline{R_0(\lambda+i0)\bar f}$, one has

$$\mathcal{N}_s(\lambda) = \{f \in L^{q_s,2}(\mathbb{R}^n)\setminus\{0\} \mid f = -R_0^-(\lambda)Vf\}, \quad \lambda \ge 0. \tag{2-9}$$

The next lemma collects some basic properties of $\mathcal{E}$.

Proposition 2.7. (1) $\mathcal{E} \subset \sigma(H)$, $\sigma_{\mathrm{p}}(H) \subset \mathcal{E}$ *and* $\mathcal{E} \cap (-\infty, 0) = \sigma_{\mathrm{d}}(H)$. *Moreover*, $\mathcal{N}_s(\lambda)$ *is finite-dimensional.*

(2) $\mathcal{N}_s(\lambda)$ *is independent of* $\frac12 < s < \frac32$; *that is*, $\mathcal{N}_s(\lambda) = \mathcal{N}_{s'}(\lambda)$ *for any* $\frac12 < s, s' < \frac32$.

Proof of Proposition 2.7(1). To prove $\mathcal{E} \subset \sigma(H)$, we first claim that

$$\mathcal{N}_s(\lambda) = \{f \in \dot{\mathcal{H}}^s \mid f = -R_0(\lambda)Vf\}, \quad \lambda \in \mathbb{C}\setminus(0,\infty). \tag{2-10}$$

Indeed, if we set $\widetilde{\mathcal{N}}_s(\lambda) := \{f \in \dot{\mathcal{H}}^s \mid f = -R_0(\lambda)Vf\}$ then the inclusion $\widetilde{\mathcal{N}}_s(\lambda) \subset \mathcal{N}_s(\lambda)$ is obvious since $\dot{\mathcal{H}}^s \subset L^{q_s,2}$ by the HLS inequality (A-2). On the other hand, the HLS inequality (A-2) shows that $R_0(\lambda)V \in \mathbb{B}(L^{q_s,2}, \dot{\mathcal{H}}^s)$ for $\lambda \in \mathbb{C}\setminus(0,\infty)$ and the opposite inclusion $\widetilde{\mathcal{N}}(\lambda)_s \supset \mathcal{N}_s(\lambda)$ thus holds. Next, we let $f \in \mathcal{N}_s(\lambda)$ with some $\lambda \in \mathbb{C}\setminus\sigma(H)$. Then $Vf \in \dot{\mathcal{H}}^{2-s} \cap L^{p_s,2}$ by the HLS and Hölder's inequalities for Lorentz norms. Therefore, by Corollary 2.2(3), $(-\Delta-\lambda)f = -Vf$ holds in the distribution sense. In particular, $\lambda f = (-\Delta+V)f \in \dot{\mathcal{H}}^{2-s} \cap \dot{\mathcal{H}}^s \subset L^2$ and thus $f \in D(H)$. Since $\sigma(H) \subset \mathbb{R}$, this shows $f \equiv 0$. Therefore, we obtain $\mathcal{E} \subset \sigma(H)$.

The inclusion $\sigma_{\mathrm{p}}(H) \subset \mathcal{E}$ is obvious since $D(H) \subset \mathcal{H}^1 \subset \dot{\mathcal{H}}^1$. This inclusion, together with the fact $\sigma(H) \cap (-\infty, 0) = \sigma_{\mathrm{d}}(H)$, implies $\mathcal{E} \cap (-\infty, 0) = \sigma_{\mathrm{d}}(H)$. Finally, since $R_0^{\pm}(z)V$ are compact operators on $L^{q_s,2}$, one has $\dim \mathcal{N}_s(\lambda) < \infty$. □

To prove the second part of Proposition 2.7, we need the following:

Lemma 2.8. *For $\frac{1}{2} < s < \frac{3}{2}$ and real-valued functions $V_1 \in L_0^{n/s,\infty}$, $V_2 \in L_0^{n/(2-s),\infty}$ with $V = V_1 V_2$, we set $K_s^+(\lambda) := V_1 R^+(\lambda) V_2$. Then, for $\lambda \in \mathbb{R}$,*

$$\dim \mathcal{N}_s(\lambda) = \dim \mathrm{Ker}(I + K_s^+(\lambda)) = \dim \mathrm{Ker}(I + K_s^+(\lambda)^*) = \dim \mathcal{N}_{2-s}(\lambda).$$

Remark 2.9. Such V_1, V_2 always exist. Indeed, one can take $V_1 = |V|^{s/2}$ and $V_2 = \operatorname{sgn} V |V|^{(2-s)/2}$.

Proof. Hölder's inequality (A-1) and (2-2) yield that

$$\|V_1 f\|_{L^2} \le C \|V\|_{L^{n/s,\infty}} \|f\|_{L^{q_s,2}}, \quad \|R_0^{\pm}(\lambda) V_2 u\|_{L^{q_s,2}} \lesssim \|V_2\|_{L^{n/(2-s),\infty}} \|u\|_{L^2},$$

from which one has two continuous maps

$$\mathcal{N}_s(\lambda) \ni f \mapsto V_1 f \in \mathrm{Ker}(I + K_s^+(\lambda)), \quad \mathrm{Ker}(I + K_s^+(\lambda)) \ni u \mapsto -R_0^+(\lambda) V_2 u \in \mathcal{N}_s(\lambda).$$

Furthermore, one also has, for $f \in \mathcal{N}_s(\lambda)$ and $u \in \mathrm{Ker}(I + K_s(\lambda))$,

$$-R_0^+(\lambda) V_2 V_1 f = -R_0^+(\lambda) V f = f, \quad -V_1 R_0^+(\lambda) V_2 u = u.$$

Therefore, the multiplication by V_1 is a bijection between $\mathcal{N}_s(\lambda)$ and $\mathrm{Ker}(I + K_s^+(\lambda))$ and its inverse is given by $-R_0^+(\lambda) V_2$. In particular, $\dim \mathrm{Ker}(I + K_s^+(\lambda)) = \dim \mathcal{N}_s(\lambda)$.

Taking the facts $R_0^{\pm}(z)^* = R_0^{\mp}(\bar{z})$ and (2-9) into account, it can be seen from the same argument that the multiplication by V_2 is a bijection between $\mathcal{N}_{2-s}(\lambda)$ and $\mathrm{Ker}(I + K_s^+(\lambda)^*)$, and its inverse is given by $-R_0^-(\lambda) V_1$. In particular, $\dim \mathcal{N}_{2-s}(\lambda) = \dim \mathrm{Ker}(I + K_s^+(\lambda)^*)$.

For the part $\dim \mathrm{Ker}(I + K_s^+(\lambda)) = \dim \mathrm{Ker}(I + K_s^+(\lambda)^*)$, since $K_s^+(\lambda)$ is compact on L^2 (see Corollary 2.3), $I + K_s^+(\lambda)$ is Fredholm and its index satisfies

$$\dim \mathrm{Ker}(I + K_s^+(\lambda)) - \operatorname{codim} \mathrm{Ran}(I + K_s^+(\lambda)) = \operatorname{ind}(I + K_s^+(\lambda)) = \operatorname{ind} I = 0.$$

Therefore, taking the fact $L^2 / \mathrm{Ran}(I + K_s^+(\lambda)) \cong [\mathrm{Ran}(I + K_s^+(\lambda))]^{\perp}$ into account, one has

$$\dim \mathrm{Ker}(I + K_s^+(\lambda)) = \dim [\mathrm{Ran}(I + K_s^+(\lambda))]^{\perp} = \dim \mathrm{Ker}(I + K_s^+(\lambda)^*),$$

which completes the proof. □

Proof of Proposition 2.7(2). Let $f \in \mathcal{N}_s(\lambda)$ and $\frac{1}{2} < s \le s' < \frac{3}{2}$. Let $V = v_1 + v_2$ be such that $v_1 \in C_0^{\infty}$ and $\|v_2\|_{L^{n/2,\infty}} \le \varepsilon$. Then $f = -R_0^+(\lambda) v_1 f - R_0^+(\lambda) v_2 f$. By Proposition 2.4, the map $I + R_0^+(\lambda) v_2 : L^{2n/(n-2r),2} \to L^{2n/(n-2r),2}$ is bounded and invertible for $r = s, s'$ and small $\varepsilon > 0$. If E_r denotes the inverse of $I + R_0^+(\lambda) v_2 : L^{2n/(n-2r),2} \to L^{2n/(n-2r),2}$, then $E_s = E_{s'}$ on $L^{2n/(n-2s),2} \cap L^{2n/(n-2s'),2}$. Taking the inequality $s - s' > -1$ into account, the HLS inequality (A-2) implies

$$\|R_0^+(\lambda) v_1 f\|_{L^{2n/(n-2s'),2}} \lesssim \|v_1 f\|_{L^{2n/(n+2(2-s'))}} \lesssim \|v_1\|_{L^{n/(2+2(s-s'))}} \|f\|_{L^{2n/(n-2s)}}.$$

Thus $R_0^+(\lambda)v_1 f \in L^{2n/(n-2s),2} \cap L^{2n/(n-2s'),2}$ and $f = E_s R_0^+(\lambda)v_1 f = E_{s'} R_0^+(\lambda)v_1 f \in L^{2n/(n-2s'),2}$, which implies $f \in \mathcal{N}_{s'}(\lambda)$. Therefore $\mathcal{N}_s(\lambda)$ is monotonically increasing in s. Combined with the fact $\dim \mathcal{N}_s(\lambda) = \dim \mathcal{N}_{2-s}(\lambda) < \infty$ (see Lemma 2.8), this monotonicity implies $\mathcal{N}_s(\lambda) = \mathcal{N}_{s'}(\lambda)$. □

We conclude this subsection to prove Lemma 1.3. For the first part, we employ the following results of [Ionescu and Jerison 2003; Ionescu and Schlag 2006].

Proposition 2.10 [Ionescu and Jerison 2003, Theorem 2.1]. *Let $n \ge 3$ and $V \in L^{n/2}$. Suppose that $f \in \mathcal{H}^1_{\mathrm{loc}}$ and $\langle x\rangle^{-1/2+\delta} f \in L^2$ with some $\delta > 0$. If $-\Delta f + V f = \lambda f$ for some $\lambda > 0$, then $f \equiv 0$.*

Let us set $X = \mathcal{W}^{-1/(n+1),2(n+1)/(n+3)} + S_1(B)$, where B is the Agmon–Hörmander space and $S_1(B)$ is the image of B under $S_1 = (1-\Delta)^{1/2}$; see [Ionescu and Schlag 2006]. Then

$$X^* = \mathcal{W}^{1/(n+1),2(n+1)/(n-1)} \cap S_{-1}(B^*)$$

and we have the continuous embeddings $L^{2n/(n+2)} \subset X$ and $X^* \subset L^{2n/(n-2)}$. Moreover, it was proved in [Ionescu and Schlag 2006, Lemma 4.1(b)] that $R_0^\pm(\lambda) \in \mathbb{B}(X, X^*)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$.

Proposition 2.11 [Ionescu and Schlag 2006, Lemma 4.4]. *Let $n \ge 3$ and $V \in L^{n/2}$. Assume that f belongs to X^* and satisfies $f + R_0^\pm(\lambda) V f = 0$ for some $\lambda \in \mathbb{R} \setminus \{0\}$. Then, for any $N \ge 0$,*

$$\|\langle x\rangle^N f\|_{X^*} \le C_{N,\lambda} \|f\|_{X^*}.$$

Proof of Lemma 1.3. For the proof of the part (1), we let $f \in \mathcal{N}_1(\lambda)$ with $\lambda > 0$. As observed in the proof of Proposition 2.4, $R_0^+(\lambda)V$ maps from $L^{2n/(n-2)}(\mathbb{R}^n)$ into $\mathcal{W}^{2,2n/(n-2)}(\mathbb{R}^n)$ (see (2-5)) and thus $f = -R_0^+(\lambda)Vf \in \mathcal{H}^1_{\mathrm{loc}}$. Moreover, since $Vf \in L^{2n/(n+2)} \subset X$ and $R_0^\pm(\lambda) \in \mathbb{B}(X, X^*)$, we have $f \in X^*$. Proposition 2.11 then implies that $f \in L^2$. Using Proposition 2.10, we conclude that $f \equiv 0$. For part (2), we let $f \in \mathcal{N}_1(0)$. Since $-\Delta f + V f \in \dot{\mathcal{H}}^{-1}$, the form $\langle -\Delta f + V f, f\rangle$ is well-defined. By assumption, we have $0 = \langle -\Delta f + V f, f\rangle \ge \delta \|f\|_{\dot{\mathcal{H}}^1}$, which implies $f \equiv 0$. □

3. Uniform Sobolev estimates

This section is devoted to the proof of Theorem 1.4, Corollaries 1.5 and 1.6 and Theorem 1.8. We begin with the following proposition which plays an important role in the proof.

Proposition 3.1. *Assume $\frac{1}{2} < s < \frac{3}{2}$ and let (p_s, q_s) be as in* (2-3). *Then $(I + R_0^\pm(z)V)^{-1}$ are $\mathbb{B}(L^{q_s,2})$-valued continuous functions on $\overline{\mathbb{C}^\pm} \setminus \mathcal{E}$, respectively. Furthermore, for any $\delta > 0$,*

$$\sup_{z \in \overline{\mathbb{C}^\pm} \setminus \mathcal{E}_\delta} \|(I + R_0^\pm(z)V)^{-1}\|_{\mathbb{B}(L^{q_s,2})} < \infty. \tag{3-1}$$

In particular, if $\mathcal{E} \cap [0,\infty) = \varnothing$, then $\sup_{z \in \mathbb{C}\setminus\mathbb{R}} \|(I + R_0(z)V)^{-1}\|_{\mathbb{B}(L^{q_s,2})} < \infty$.

The proof of Proposition 3.1 is divided into a series of lemmas. Let us prove the proposition for $z \in \overline{\mathbb{C}^+} \setminus \mathcal{E}$ only, as the proof for the case $z \in \overline{\mathbb{C}^-} \setminus \mathcal{E}$ is analogous.

Lemma 3.2. *$(I + R_0^+(z)V)^{-1}$ is a $\mathbb{B}(L^{q_s,2})$-valued continuous function on $\overline{\mathbb{C}^+} \setminus \mathcal{E}$.*

Proof. By Proposition 2.4, $R_0^+(z)V$ is compact. Since $\mathcal{N}_s(z)=\{0\}$ for $z\in\overline{\mathbb{C}^+}\setminus\mathcal{E}$ by definition, the Fredholm alternative ensures the existence of $(I+R_0^+(z)V)^{-1}\in\mathbb{B}(L^{q_s,2})$. Moreover, since $R_0^+(z)V$ is continuous on $\overline{\mathbb{C}^+}$ in the operator norm topology of $\mathbb{B}(L^{q_s,2})$ by Proposition 2.4, $(I+R_0^+(z)V)^{-1}$ is also continuous on $\overline{\mathbb{C}^+}\setminus\mathcal{E}$ in the same topology. □

The proof of the uniform bound (3-1) is divided into high-, intermediate- and low-energy parts.

Lemma 3.3 (the high-energy estimate). *There exists $L\ge 1$ such that $(I+R_0^+(z)V)^{-1}$ is bounded on $L^{2n/(n-2s),2}$ uniformly in $z\in\overline{\mathbb{C}^+}\cap\{|z|\ge L\}$.*

Proof. Let $V_k\in C_0^\infty(\mathbb{R}^n)$ be such that $\lim_{k\to\infty}\|V-V_k\|_{L^{n/2,\infty}}=0$ and set $Q_k^+(z):=R_0^+(z)(V-V_k)$. By Corollary 2.2 with (p_s,q_s), one can find $k_0\ge 1$ such that

$$\sup_{z\in\overline{\mathbb{C}^+}}\|Q_{k_0}^+(z)\|_{\mathbb{B}(L^{q_s,2})}\le\tfrac12.$$

Hence $(I+Q_{k_0}(z))^{-1}$ is defined by the Neumann series $\sum_{n=0}^\infty(-Q_{k_0}^+(z))^n$ and satisfies

$$M_1:=\sup_{z\in\overline{\mathbb{C}^+}}\|(I+Q_{k_0}^+(z))^{-1}\|_{\mathbb{B}(L^{q_s,2})}\le 2.$$

Next if we take p_δ and small $\delta>0$ such that $1/p_\delta=1/p_s-\delta$ and (p_δ,q_s) satisfies (1-1), Corollary 2.2 implies

$$\|R_0^+(z)V_{k_0}f\|_{L^{q_s,2}}\lesssim|z|^{-\delta}\|V_{k_0}f\|_{L^{p_\delta,2}}\lesssim|z|^{-\delta}\|V_{k_0}\|_{L^r}\|f\|_{L^{q_s,2}}$$

uniformly in $|z|\ge 1$ and $f\in L^{q_s,2}$, where $1/r=1/p_\delta-1/q_s=2/n-\delta$. Hence one can find $L=L_{k_0}$ so large that $M_2:=\|R_0^+(z)V_{k_0}\|_{\mathbb{B}(L^{q_s,2})}\le\frac14$ for $|z|\ge L$. Then, writing

$$I+R_0^+(z)V=I+Q_{k_0}^+(z)+R_0^+(z)V_{k_0}=(I+Q_{k_0}^+(z))\big(I+(I+Q_{k_0}^+(z))^{-1}R_0^+(z)V_{k_0}\big),$$

we see that $(I+R_0^+(z)V)^{-1}=\big(I+(I+Q_{k_0}^+(z))^{-1}R_0^+(z)V_{k_0}\big)^{-1}(I+Q_{k_0}^+(z))^{-1}$ and

$$\sup_{z\in\overline{\mathbb{C}^+}\cap\{|z|\ge L\}}\|(I+R_0^+(z)V)^{-1}\|_{\mathbb{B}(L^{q_s,2})}\le M_1\sum_{n=1}^\infty(M_1M_2)^n\le 4.$$ □

Remark 3.4. This lemma particularly implies $\mathcal{E}\cap[L,\infty)=\varnothing$ and thus $\mathcal{E}$ is bounded in $\mathbb{R}$.

Lemma 3.5 (the intermediate-energy estimate). *For any $\delta, L>0$, the function $(I+R_0^+(z)V)^{-1}$ is bounded on $L^{q_s,2}$ uniformly in $z\in(\overline{\mathbb{C}^+}\setminus\mathcal{E}_\delta)\cap\{\delta<|z|<L\}$.*

Proof. We follow the argument in [Ionescu and Schlag 2006, Lemma 4.6] closely. Let

$$\Lambda_{\delta,L}=(\overline{\mathbb{C}^+}\setminus\mathcal{E}_\delta)\cap\{\delta<|z|<L\}.$$

Note that $\overline{\Lambda_{\delta,L}}\cap\mathcal{E}=\varnothing$. Assume for contradiction that

$$\sup_{z\in\Lambda_{\delta,L}}\|(I+R_0^+(z)V)^{-1}\|_{\mathbb{B}(L^{q_s,2})}=\infty.$$

Then one can find $f_j \in L^{q_s,2}$ with $\|f_j\|_{L^{q_s,2}} = 1$ and $z_j \in \Lambda_{\delta,L}$ such that

$$\|(I + R_0^+(z_j)V)f_j\|_{\mathbb{B}(L^{q_s,2})} \to 0, \quad j \to \infty. \tag{3-2}$$

By passing to a subsequence, we may assume $z_j \to z_\infty \in \overline{\Lambda_{\delta,L}}$ as $j \to \infty$. Since $R_0^+(z_\infty)V$ is compact on $L^{q_s,2}$, by passing to a subsequence, we may assume without loss of generality that there exists $g \in L^{q_s,2}$ such that $R_0^+(z_\infty)Vf_j \to g$ strongly in $L^{q_s,2}$. By virtue of (3-2) and the condition $\|f_j\|_{L^{q_s,2}} = 1$, we have $g \not\equiv 0$. Now we claim that g belongs to $\mathcal{N}_s(z_\infty)$, which implies $z_\infty \in \mathcal{E}$. This contradicts $z_\infty \in \overline{\Lambda_{\delta,L}}$.

In order to prove the claim, we write f_j as

$$f_j = (I + R_0^+(z_j)V)f_j - (R_0^+(z_j) - R_0^+(z_\infty))Vf_j - R_0^+(z_\infty)Vf_j.$$

By virtue of (3-2) and the continuity of $R_0^+(z)V$ (see Proposition 2.4) and the fact $\|f_j\|_{L^{q_s,2}} = 1$, the right-hand side converges to $-g$ strongly in $L^{q_s,2}$ as $j \to \infty$. Therefore, we have $g = -R_0^+(z_\infty)Vg$. Moreover, since $\|f_j\| = 1$, we have $g \not\equiv 0$ and hence $g \in \mathcal{N}_s(z_\infty)$ follows. □

Lemmas 3.3 and 3.5 give the desired bound (3-1) for the case when $0 \in \mathcal{E}$. When $0 \notin \mathcal{E}$, we need the following lemma to complete the proof of Proposition 3.1.

Lemma 3.6 (the low-energy estimate). *Suppose that* $0 \notin \mathcal{E}$. *Then there exists* $\delta > 0$ *such that the function* $(I + R_0^+(z)V)^{-1}$ *is bounded on* $L^{q_s,2}$ *uniformly in* $z \in \overline{\mathbb{C}^+} \cap \{|z| \le \delta\}$.

Proof. Since $I + R_0^+(0)V$ is invertible if $0 \notin \mathcal{E}$ by Lemma 3.2, one can write

$$I + R_0^+(z)V = (I + R_0^+(0)V)(I + (I + R_0^+(0)V)^{-1}(R_0^+(z) - R_0^+(0))V).$$

Since $\overline{\mathbb{C}^+} \ni z \mapsto R_0^+(z)V \in \mathbb{B}(L^{q_s,2})$ is continuous by Proposition 2.4, one has

$$\sup_{z \in \overline{\mathbb{C}^+} \cap \{|z| \le \delta\}} \|(R_0^+(z) - R_0^+(0))V\|_{\mathbb{B}(L^{q_s,2})} \le \frac{1}{2\|(I + R_0^+(0)V)^{-1}\|}$$

for $\delta > 0$ small enough. Therefore, $I + R_0^+(z)V$ is invertible on $L^{q_s,2}$ and

$$\sup_{z \in \overline{\mathbb{C}^+} \cap \{|z| \le \delta\}} \|(I + R_0^+(z)V)^{-1}\|_{\mathbb{B}(L^{q_s,2})} \le 2 \sup_{z \in \overline{\mathbb{C}^+} \cap \{|z| \le \delta\}} \|(I + R_0^+(0)V)^{-1}\|_{\mathbb{B}(L^{q_s,2})} < \infty,$$

which completes the proof. □

By Lemmas 3.2–3.5, we have completed the proof of Proposition 3.1.

We next give a rigorous justification of the second resolvent equation.

Lemma 3.7. *Let* $z \in \mathbb{C} \setminus \sigma(H)$. *Then, as a bounded operator from* L^2 *to* $D(H)$,

$$R(z) = (I + R_0(z)V)^{-1}R_0(z) = R_0(z) - R_0(z)VR(z). \tag{3-3}$$

Moreover, we also obtain for $z, z' \in \mathbb{C} \setminus \sigma(H)$,

$$R(z) - R(z') = (I + R_0(z')V)^{-1}(R_0(z) - R_0(z'))(I - VR(z)). \tag{3-4}$$

Proof. It follows from Proposition 2.7(1) and the fact $\mathcal{H}^1 \subset L^{2n/(n-2),2}$ that $\mathrm{Ker}_{\mathcal{H}^1}(I + R_0(z)V)$ is trivial. Since $R_0(z)V \in \mathbb{B}_\infty(\mathcal{H}^1)$ by Proposition 2.4, $I + R_0(z)V$ is invertible on $\mathcal{H}^1$ by the Fredholm alternative theorem. Thus $(I + R_0(z)V)^{-1}R_0(z)$ is a bounded operator from L^2 to $\mathcal{H}^1$. Let $f \in L^2$ and set $g = (I + R_0(z)V)^{-1}R_0(z)f \in \mathcal{H}^1$. Since

$$(I + R_0(z)V)(I + R_0(z)V)^{-1}R_0(z) = R_0(z)$$

as a bounded operator from L^2 to $\mathcal{H}^1$, we see that

$$g = R_0(z)f - R_0(z)Vg. \tag{3-5}$$

Then, for any $\varphi \in \mathcal{H}^1$,

$$\langle(-\Delta - z)g, \varphi\rangle = \langle f, \varphi\rangle - \langle Vg, \varphi\rangle = \langle f, \varphi\rangle - \langle V_1 g, V_2\varphi\rangle,$$

where $V_1, V_2 \in L_0^{n/2,\infty}(\mathbb{R}^n; \mathbb{R})$ satisfies $V = V_1V_2$. Therefore, we obtain

$$\langle(H - z)g, \varphi\rangle = \langle(-\Delta - z)g, \varphi\rangle + \langle V_1 g, V_2\varphi\rangle = \langle f, \varphi\rangle,$$

which shows $(H - z)(I + R_0(z)V)^{-1}R_0(z) = I$ on L^2. For $f \in D(H)$, we similarly obtain

$$(I + R_0(z)V)^{-1}R_0(z)(H - z)f = (I + R_0(z)V)^{-1}f + (I + R_0(z)V)^{-1}R_0(z)Vf = f,$$

which gives us $(I + R_0(z)V)^{-1}R_0(z)(H - z) = I$ on $D(H)$ and the first identity in (3-3) thus follows. The second identity in (3-3) follows from the first identity and (3-5).

Now we shall show (3-4). It follows from (3-3) that

$$(I + R_0(z')V)(R(z) - R(z')) = (R_0(z) - R_0(z'))(I - VR(z))$$

on L^2. Since $R_0(z) - R_0(z'), R(z) - R(z') : L^2 \to \mathcal{H}^1$ are continuous and $I + R_0(z')V$ is invertible on $\mathcal{H}^1$, we have the desired identity (3-4). □

Now we are in position to prove Theorem 1.4, Corollaries 1.5 and 1.6 and Theorem 1.8.

Proof of Theorem 1.4. Assume that (p, q) satisfies (1-1). It follows from Propositions 1.1 and 3.1 and Lemma 3.7 that for any $\delta > 0$ there exists $C_\delta > 0$ such that

$$\|R(z)f\|_{L^{q,2}} \le C_\delta(1 + \|(I + R_0(z)V)^{-1}\|_{\mathbb{B}(L^{q,2})})\|R_0(z)f\|_{L^{q,2}} \le C_\delta|z|^{(n/2)(1/p-1/q)-1}\|f\|_{L^{p,2}}$$

for all $f \in L^2 \cap L^{p,2}$ and $z \in \mathbb{C} \setminus ([0,\infty) \cup \mathcal{E}_\delta)$. Since $L^2 \cap L^{p,2}$ is dense in $L^{p,2}$, this implies that $R(z) \in \mathbb{B}(L^{p,2}, L^{q,2})$ and that (1-5) holds uniformly in $z \in \mathbb{C} \setminus ([0,\infty) \cup \mathcal{E}_\delta)$. □

Proof of Corollary 1.5. As before, we shall prove the corollary for $R(\lambda + i0)$ only. We also consider the case $1/p - 1/q = 2/n$ only, as the proofs for other cases are similar. At first, we claim that, for any $\chi_1, \chi_2 \in C_0^\infty(\mathbb{R}^n)$, $\chi_1 R(z)\chi_2$ defined for $z \in \mathbb{C}^+$ extends to a $\mathbb{B}(L^2)$-valued continuous function $\chi_1 R^+(z)\chi_2$ on $\overline{\mathbb{C}^+} \setminus \mathcal{E}$. It follows from this claim that, for any $u, v \in C_0^\infty(\mathbb{R}^n)$, $\langle R^+(z)u, v\rangle$ is a continuous function on $\overline{\mathbb{C}^+} \setminus \mathcal{E}$. Then, by letting $\varepsilon \searrow 0$ in the estimate

$$|\langle R(\lambda + i\varepsilon)u, v\rangle| \lesssim \|u\|_{L^{p,2}}\|v\|_{L^{q',2}},$$

which follows from Theorem 1.4, and by using the density argument, we obtain that $R(\lambda+i0)$ extends to an element in $\mathbb{B}(L^{p,2}, L^{q,2})$ and satisfies

$$\sup_{\lambda\in[0,\infty)\setminus\mathcal{E}} \|R(\lambda+i0)\|_{\mathbb{B}(L^{p,2},L^{q,2})} < \infty. \tag{3-6}$$

This shows the first statement (1). For the second statement (2), it follows by setting $z=\lambda\pm i\varepsilon$ and then letting $\varepsilon\searrow 0$ in (3-3) that, for any $f\in L^{q,2}\cap L^2$ and $\lambda\in[0,\infty)\setminus\mathcal{E}$,

$$R(\lambda\pm i0)f = R_0(\lambda\pm i0)(I-VR(\lambda\pm i0))f \tag{3-7}$$

in the sense of distributions, which particularly implies that, under the condition $0\notin\mathcal{E}$, we have $R(0+i0)=R(0-i0)$ since $R_0(0\pm i0)=(-\Delta)^{-1}$. Moreover, we also know by (3-7) that

$$\begin{aligned}(-\Delta+V-\lambda)R(\lambda+i0)u &= (I+VR_0(\lambda+i0))(I-VR(\lambda+i0))u\\ &= u+V[R_0(\lambda+i0)-R(\lambda+i0)-R_0(\lambda+i0)VR(\lambda+i0)]u = u\end{aligned}$$

for all $u\in L^2\cap L^{p,2}$ and that, for all $v\in\mathcal{S}$,

$$\begin{aligned}R(\lambda+i0)(-\Delta+V-\lambda)v &= R_0(\lambda+i0)(I-VR(\lambda+i0))(-\Delta+V-\lambda)v\\ &= v-R_0(\lambda+i0)Vv-R_0(\lambda+i0)Vv = v\end{aligned}$$

in the sense of distributions. These two identities and (3-6) imply (1-7).

It remains to show the above claim. Let $V_1, V_2\in L^{n,\infty}_0(\mathbb{R}^n;\mathbb{R})$ be such that $V=V_1V_2$ and set $K_1(z)=V_1R_0(z)V_2$. The resolvent identity (3-3) then yields

$$V_1R(z)\chi_2 = V_1R_0(z)\chi_2 - K_1(z)V_1R(z)\chi_2$$

on L^2 for all $z\in\mathbb{C}\setminus\sigma(H)$. Since $K_1(z)\in\mathbb{B}_\infty(L^2)$ by Corollary 2.3 and $\operatorname{Ker}_{L^2}(I+K_1(z))=\varnothing$ for all $z\in\mathbb{C}\setminus\sigma(H)$ by Proposition 2.7 and Lemma 2.8, we have by this identity that

$$V_1R(z)\chi_2 = (I+K_1(z))^{-1}V_1R_0(z)\chi_2, \quad z\in\mathbb{C}\setminus\sigma(H),$$

on L^2. It follows from again Corollary 2.3 that $V_1R_0(z)\chi_2$ and $K_1(z)$ extend to $\mathbb{B}_\infty(L^2)$-valued continuous functions $V_1R_0^+(z)\chi_2$ and $K_1^+(z)=V_1R_0^+(z)V_2$ on $\overline{\mathbb{C}^+}$. Since $\operatorname{Ker}(I+K_1^+(z))=\varnothing$ for $z\in\overline{\mathbb{C}^+}\setminus\mathcal{E}$, $(I+K_1(z))^{-1}$ also extends to a $\mathbb{B}(L^2)$-valued continuous function $(I+K_1^+(z))^{-1}$ on $\overline{\mathbb{C}^+}\setminus\mathcal{E}$. Thus $V_1R(z)\chi_2$ extends to a $\mathbb{B}(L^2)$-valued continuous function $V_1R^+(z)\chi_2$ on $\overline{\mathbb{C}^+}\setminus\mathcal{E}$ satisfying $V_1R^+(z)\chi_2=(I+K_1^+(z))^{-1}V_1R_0^+(z)$. Finally, the claim follows from the formula

$$\chi_1R(z)\chi_2 = \chi_1R_0(z)\chi_2 - \chi_1R_0(z)V_2V_1R(z)\chi_2$$

and the continuity of $\chi_1R_0^+(z)\chi_2$, $\chi_1R_0^+(z)V_2$ and $V_1R_0^+(z)\chi_2$ on $\overline{\mathbb{C}^+}\setminus\mathcal{E}$. □

Proof of Corollary 1.6. Let us fix $z\in\mathbb{C}\setminus\sigma(H)$ and take $\delta>0$ so small that $z\notin\mathcal{E}_\delta$. Recall that $R_0(z)\in\mathbb{B}(L^p)$ for all $1\le p\le\infty$ and thus $R_0(z)\in\mathbb{B}(L^{p,2})$ for all $1<p<\infty$ by Theorem A.1.

The proof of the first assertion is divided into two cases:

$$\frac{2n}{n+3} < p = q < \frac{2n}{n+1}$$

and otherwise. In the first case one can find

$$\frac{2n}{n-1} < q_0 < \frac{2n}{n-3}$$

such that $1/p - 1/q_0 = 2/n$. Applying Theorem 1.4 to the resolvent equation (3-3) implies that, for all $f \in L^2 \cap L^{p,2}$,

$$\|R(z)f\|_{L^{p,2}} \lesssim \|R_0(z)f\|_{L^{p,2}} + \|R_0(z)\|_{\mathbb{B}(L^{p,2})}\|V\|_{L^{n/2,\infty}}\|R(z)f\|_{L^{q_0,2}} \le C_\delta\|f\|_{L^{p,2}}.$$

Combined with a density argument, this implies $R(z) \in \mathbb{B}(L^{p,2})$ for each $z \in \mathbb{C} \setminus \sigma(H)$.

Next, by taking the adjoint and using the fact $R(z)^* = R(\bar{z})$, we see that $R(z) \in \mathbb{B}(L^{p,2})$ for all

$$\frac{2n}{n-1} < p < \frac{2n}{n-3}.$$

Interpolating these two cases yields that $R(z) \in \mathbb{B}(L^{p,2})$ for all

$$\frac{2n}{n+3} < p < \frac{2n}{n-3}.$$

Then the other cases in the first assertion follow by interpolating between the estimates on the two lines $1/p - 1/q = 0$ and $1/p - 1/q = 2/n$ under the conditions $2n/(n+3) < p$ and $q < 2n/(n-3)$.

Finally, assuming $\frac{1}{2} < s < \frac{3}{2}$ without loss of generality, the second assertion follows from

$$\|wR(M)f\|_{L^2} \lesssim \|w\|_{L^{n/s,\infty}}\|R(M)f\|_{L^{2n/(n-2s),2}} \lesssim \|w\|_{L^{n/s,\infty}}\|f\|_{L^2}$$

for $M < \inf \sigma(H) - 1$, which is a particular case of the first assertion. □

Proof of Theorem 1.8. When

$$\frac{2n}{n+2} \le p \le \frac{2(n+1)}{n+3},$$

(1-10) follows from (1-6) and Stone's formula (1-9). When

$$\frac{2n}{n+3} < p < \frac{2n}{n+2},$$

there are two main ingredients.

At first, it is known that $E'_{-\Delta}(\lambda) \in \mathbb{B}(L^p, L^{p'})$ for all

$$1 \le p \le \frac{2(n+1)}{n+3}$$

and satisfies

$$\|E'_{-\Delta}(\lambda)\|_{\mathbb{B}(L^p,L^{p'})} \lesssim \lambda^{(n/2)(1/p-1/p')-1}, \quad \lambda > 0. \tag{3-8}$$

Indeed, $E'_{-\Delta}(\lambda)$ can be brought to the form $E'_{-\Delta}(\lambda) = (2\pi)^{-n}\lambda^{(n-1)/2}R^*_{\sqrt{\lambda}}R_{\sqrt{\lambda}}$, where

$$R_\mu u(\omega) := \int_{\mathbb{R}^n} e^{-2\pi i\mu\omega\cdot x}u(x)\,dx, \quad \mu > 0,\ \omega \in \mathbb{S}^{n-1}.$$

Then the Stein–Tomas restriction theorem (see [Tomas 1975; Stein 1970]) and the TT^*-argument show that $R_1^* R_1$ is bounded from L^p to $L^{p'}$ for all

$$1 \le p \le \frac{2(n+1)}{n+3},$$

which particularly implies (3-8) by scaling.

Secondly, we claim that the following identity holds for all $f, g \in \mathcal{S}$ and $\lambda \in (0, \infty)$:

$$\langle E'_H(\lambda) f, g\rangle = \langle (I + R_0(\lambda - i0)V)^{-1} E'_{-\Delta}(\lambda)(I - VR(\lambda + i0)) f, g\rangle. \tag{3-9}$$

Since $VR(\lambda + i0) \in \mathbb{B}(L^p)$ and $(I + R_0(\lambda - i0)V)^{-1} \in \mathbb{B}(L^{p'})$ for

$$\frac{2n}{n+3} < p < \frac{2n}{n+1}$$

by Corollary 1.5 and Proposition 3.1, the desired assertion (1-10) follows from (3-8), (3-9) and a density argument.

It remains to show the identity (3-9). Let $f, g \in \mathcal{S}$ and set

$$F(z) = \tfrac{1}{\pi}(I + R_0(\bar z)V)^{-1} \operatorname{Im} R_0(z)(I - VR(z)), \quad z \in \mathbb{C}^+,$$

which is a bounded operator from L^2 to $\mathcal{H}^1$ (see the proof of Lemma 3.7), where

$$\operatorname{Im} R_0(z) = (2i)^{-1}(R_0(z) - R_0(\bar z)).$$

By (3-4) with $z = \lambda + i\varepsilon,\ z' = \bar z$, one has $\pi^{-1} \operatorname{Im} R(z) = F(z)$. Moreover,

$$\langle E'_H(\lambda) f, g\rangle = \pi^{-1} \lim_{\varepsilon \searrow 0} \langle \operatorname{Im} R(\lambda + i\varepsilon) f, g\rangle$$

exists by Corollary 1.5. For the operator $F(z)$, we write

$$F(z) f = \tfrac{1}{\pi}(I + R_0(\bar z)V)^{-1}(\operatorname{Im} R_0(z)\langle x\rangle^{-3} - \operatorname{Im} R_0(z)VR(z)\langle x\rangle^{-3})\langle x\rangle^3 f.$$

By Proposition 2.4, all of $(I + R_0(\bar z)V)^{-1}$, $\operatorname{Im} R_0(z)\langle x\rangle^{-3}$, $\operatorname{Im} R_0(z)V$ and $R(z)\langle x\rangle^{-3}$ extend to $\mathbb{B}(L^{p'})$-valued continuous function on $\overline{\mathbb{C}^+} \setminus \mathcal{E}$. Therefore, $\langle F(\lambda + i0) f, g\rangle = \lim_{\varepsilon \searrow 0} \langle F(\lambda + i\varepsilon) f, g\rangle$ exists and coincides with the right-hand side of (3-9). Therefore (3-9) follows. □

The remaining part of the section is devoted to the following theorem, which plays a crucial role in the proof of Strichartz estimates.

Theorem 3.8. *Suppose that $\mathcal{E} \cap [0, \infty) = \varnothing$. Let (p, q) be such that*

$$\frac{1}{p} - \frac{1}{q} = \frac{2}{n} \quad \text{and} \quad \frac{2n}{(n+3)} < p < \frac{2n}{(n+1)}.$$

Then

$$\sup_{z \in \mathbb{C} \setminus [0,\infty)} \|P_{\mathrm{ac}}(H)R(z)\|_{\mathbb{B}(L^{p,2}, L^{q,2})} < \infty. \tag{3-10}$$

We first prove some L^p-boundedness of the projection $P_{\mathrm{ac}}(H)$. At first note that, under the condition $0 \notin \mathcal{E}$, H may have at most finitely many negative eigenvalues of finite multiplicities. Indeed, since $\sigma_{\mathrm{p}}(H) \cap (-\infty, 0) = \sigma_{\mathrm{d}}(H)$, each negative eigenvalue has finite multiplicity and their only possible accumulation point is $z = 0$. Moreover, Lemma 3.6 and the Fredholm alternative show that, for sufficiently small $\delta > 0$, $(-\delta, \delta) \cap \mathcal{E} = \varnothing$ as long as $0 \notin \mathcal{E}$. Therefore, H may have at most finitely many negative eigenvalues. In this case $P_{\mathrm{ac}}(H)$ is written in the form

$$P_{\mathrm{ac}}(H) = I - \sum_{j=1}^{N} P_j, \qquad P_j := \langle \cdot\,, \psi_j \rangle \psi_j, \tag{3-11}$$

where ψ_j are eigenfunctions of H and $N < \infty$.

Lemma 3.9. *We have* $\psi_j \in L^{q,2}$ *and* $P_{\mathrm{ac}}(H) \in \mathbb{B}(L^{q,2})$ *for all*

$$\frac{2n}{n+3} < q < \frac{2n}{n-3}.$$

Proof. Let ψ be an eigenfunction of H with an eigenvalue $\lambda < 0$. By virtue of (3-11) and real interpolation, it suffices to show $\psi \in L^{q,2}$. For a given $\varepsilon > 0$, we decompose V as $V = v_1 + v_2$ with $v_1 \in C_0^\infty(\mathbb{R}^n)$ and $\|v_2\|_{L^{n/2,\infty}} \le \varepsilon$. We first let

$$\frac{2n}{n-1} < q < \frac{2n}{n-3}.$$

By Sobolev's inequality and Proposition 1.1, one has

$$\begin{aligned} \|R_0(\lambda) v_1 \psi\|_{L^q} &\lesssim \|R_0(\lambda) v_1 \psi\|_{\mathcal{H}^{n(1/2-1/q)}} \le C_\lambda \|v_1 \psi\|_{L^2} \le C_\lambda \|v_1\|_{L^\infty} \|\psi\|_{L^2}, \\ \|R_0(\lambda) v_2\|_{\mathbb{B}(L^q)} &\lesssim \|v_2\|_{L^{n/2,\infty}}. \end{aligned}$$

For $\varepsilon > 0$ small enough, $I + R_0(\lambda) v_2$ thus is invertible on L^q and

$$\psi = -R_0(\lambda) V \psi = R_0(\lambda) v_1 \psi - R_0(\lambda) v_2 \psi = -(I + R_0(\lambda) v_2)^{-1} R_0(\lambda) v_1 \psi \in L^q.$$

Next, since $R_0(\lambda) \in \mathbb{B}(L^p)$ for all $1 < p < \infty$, we have by Hölder's inequality that

$$\|\psi\|_{L^p} = \|R_0(\lambda) V \psi\|_{L^p} \le C_\lambda \|V \psi\|_{L^p} \le C_\lambda \|V\|_{L^{n/2,\infty}} \|\psi\|_{L^q}$$

if $1/p - 1/q = 2/n$. This shows $\psi \in L^p$ for all

$$\frac{2n}{n+3} < p < \frac{2n}{n+1}.$$

Interpolating these two cases, we conclude that $\psi \in L^q$ for all

$$\frac{2n}{n+3} < q < \frac{2n}{n-3}. \qquad \square$$

Proof of Theorem 3.8. Assume that $\mathcal{E} \cap [0, \infty) = \varnothing$. Then one can find $\delta > 0$ small enough such that $\operatorname{dist}(\mathcal{E}_\delta, [0, \infty)) \ge \delta/2$. The proof is divided into two cases: $z \in \mathbb{C} \setminus ([0, \infty) \cup \mathcal{E}_\delta)$ and $z \in \mathcal{E}_\delta$. For the case when $z \in \mathbb{C} \setminus ([0, \infty) \cup \mathcal{E}_\delta)$, since

$$\frac{2n}{n-1} < q, p' < \frac{2n}{n-3}$$

and $P_jR(z)=(\lambda_j-z)^{-1}\langle\,\cdot\,,\psi_j\rangle\psi_j$, Lemma 3.9 implies

$$\|P_jR(z)f\|_{L^{p',2}}\le\delta^{-1}\|\psi_j\|_{L^{q,2}}\|\psi_j\|_{L^{p',2}}\|f\|_{L^{p,2}},$$

which, together with Theorem 1.4 and the formula (3-11), gives us the desired bound

$$\sup_{z\in\mathbb{C}\setminus([0,\infty)\cup\mathcal{E}_\delta)}\|P_{\mathrm{ac}}(H)R(z)\|_{\mathbb{B}(L^{p,2},L^{q,2})}\lesssim\delta^{-1}. \tag{3-12}$$

When $z\in\mathcal{E}_\delta$, we use twice the first resolvent equation $R(z)=R(z')-(z-z')R(z')R(z)$ to write

$$P_{\mathrm{ac}}(H)R(z)=P_{\mathrm{ac}}(H)R(M)+(z+M)P_{\mathrm{ac}}(H)R(M)^2+(z+M)^2R(M)P_{\mathrm{ac}}(H)R(z)R(M),$$

where we have taken $M<\inf\sigma(H)-1$. Note that $|z+M|\le 2|M|+\delta$ for $z\in\mathcal{E}_\delta$ since $\mathcal{E}$ is a bounded set in $\mathbb{R}$. Moreover, we have by Lemma 3.9 and Corollary 1.6 and Theorem A.1 that

$$\|P_{\mathrm{ac}}(H)R(M)\|_{\mathbb{B}(L^{p,2},L^{q,2})}\le\|P_{\mathrm{ac}}(H)\|_{\mathbb{B}(L^{q,2})}\|R(M)\|_{\mathbb{B}(L^{p,2},L^{q,2})}\le C_M,$$
$$\|R(M)\|_{\mathbb{B}(L^2,L^{q,2})}+\|R(M)\|_{\mathbb{B}(L^{p,2},L^2)}\le C_M$$

for some C_M independent of z. It follows from these two bounds and the trivial L^2-bound

$$\sup_{z\in\mathcal{E}_\delta}\|P_{\mathrm{ac}}(H)R(z)\|_{\mathbb{B}(L^2)}\le\operatorname{dist}(\mathcal{E}_\delta,[0,\infty))^{-1}\le 2\delta^{-1}$$

that there exists $C_{M,\delta}>0$, independent of z, such that

$$\sup_{z\in\mathcal{E}_\delta}\|P_{\mathrm{ac}}(H)R(z)\|_{\mathbb{B}(L^{p,2},L^{q,2})}\le C_{M,\delta}. \tag{3-13}$$

The assertion of the theorem then follows from (3-12) and (3-13). □

4. Kato smoothing and Strichartz estimates

This section is devoted to the proof of Theorems 1.10 and 1.12. We first prepare several lemmas. Let $e^{it\Delta}$ be the free Schrödinger unitary group and define

$$\Gamma_0F(t):=\int_0^t e^{i(t-s)\Delta}F(s)\,ds,\qquad F\in L^1_{\mathrm{loc}}(\mathbb{R};L^2(\mathbb{R}^n)).$$

The estimates for the free Schrödinger equation used in this section are summarized as follows:

Lemma 4.1. *Let* (p,q) *satisfy* (1-12), (p_s,q_s) *be as in* (2-3) *and* $\rho>\frac{1}{2}$. *Then*

$$\|e^{it\Delta}\psi\|_{L^p_tL^{q,2}_x}\lesssim\|\psi\|_{L^2_x}, \tag{4-1}$$

$$\|\Gamma_0F\|_{L^2_tL^{q_s,2}_x}\lesssim\|F\|_{L^2_tL^{p_s,2}_x}\qquad \textit{for } \frac{n}{2(n-1)}<s<\frac{3n-4}{2(n-1)}, \tag{4-2}$$

$$\|\Gamma_0F\|_{L^2_tL^{q_s}_x}\lesssim\|F\|_{L^2_tL^{p_s}_x}\qquad \textit{for } s=\frac{n}{2(n-1)},\ \frac{3n-4}{2(n-1)}, \tag{4-3}$$

$$\|\langle x\rangle^{-\rho}|D|^{1/2}e^{it\Delta}\psi\|_{L^2_tL^2_x}\lesssim\|\psi\|_{L^2_x}, \tag{4-4}$$

$$\|\langle x\rangle^{-\rho}|D|^{1/2}\Gamma_0F\|_{L^2_tL^2_x}\lesssim\|F\|_{L^2_tL^{2n/(n+2),2}_x}. \tag{4-5}$$

Proof. Inequality (4-1) for $p > 2$ is due to [Strichartz 1977; Ginibre and Velo 1985]. Inequality (4-1) with $p = 2$ and (4-2) with $s = 1$ were settled in [Keel and Tao 1998]. Inequality (4-2) was proved independently by [Foschi 2005] and [Vilela 2007]. Inequality (4-3) was settled recently in [Koh and Seo 2016]. Kato-smoothing (4-4) was proved in [Kenig, Ponce, and Vega 1991]. Finally, (4-5) can be found in [Mizutani 2018, Lemma 3.2]. □

The following lemma, which was proved in [Kato 1966] (see also [Reed and Simon 1978; D'Ancona 2015]), shows the equivalence of the uniform weighted resolvent estimate and the Kato smoothing estimate.

Lemma 4.2. *Let L be a self-adjoint operator on a Hilbert space $\mathcal{H}$, let A be a densely defined closed operator on $\mathcal{H}$, and let $a > 0$. Then the following two estimates are equivalent to each other*:

$$|\langle \operatorname{Im}(L-z)^{-1}A^*u, A^*u\rangle_{\mathcal{H}}| \le a\|u\|_{\mathcal{H}}^2, \qquad u \in D(A^*),\ z \in \mathbb{C}\setminus\mathbb{R},$$
$$\|Ae^{-itL}v\|_{L^2_t\mathcal{H}} \le 2\sqrt{a}\|v\|_{\mathcal{H}}, \quad v \in \mathcal{H}.$$

The following concerns the equivalence of Sobolev norms generated by Δ and H.

Lemma 4.3. *Assume that $\mathcal{E} \cap [0,\infty) = \varnothing$ and $0 \le s < \frac{3}{2}$. Then*

$$\|(-\Delta+M)^{s/2}(H+M)^{-s/2}\|_{\mathbb{B}(L^2)} + \|(H+M)^{s/2}(-\Delta+M)^{-s/2}\|_{\mathbb{B}(L^2)} < \infty. \tag{4-6}$$

Proof. The proof will be given in the next section. □

Recall that $\langle\cdot,\cdot\rangle_T$ is the inner product in $L^2_TL^2_x$ defined by $\langle F, G\rangle_T = \int_{-T}^T \langle F(t), G(t)\rangle\,dt$. It is not hard to check that $\langle \Gamma_H F, G\rangle_T = \langle F, \Gamma_H^* G\rangle_T$ with

$$\Gamma_H^*G(t) = \mathbb{1}_{[0,\infty)}(t)\int_t^T e^{-i(t-s)H}G(s)\,ds - \mathbb{1}_{(-\infty,0]}(t)\int_{-T}^t e^{-i(t-s)H}G(s)\,ds.$$

The following lemma gives the rigorous definition of Duhamel's formula (in the sense of forms).

Lemma 4.4. *Let $\frac{1}{2} < s < \frac{3}{2}$, $V_1 \in L_0^{n/s,\infty}(\mathbb{R}^n;\mathbb{R})$ and $V_2 \in L_0^{n/(2-s),\infty}(\mathbb{R}^n;\mathbb{R})$ be such that $V = V_1V_2$. Then, for all $\psi \in L^2$ and all simple functions $F, G : \mathbb{R} \to \mathcal{S}$,*

$$\langle e^{-itH}P_{\mathrm{ac}}(H)\psi, G\rangle_T = \langle e^{it\Delta}P_{\mathrm{ac}}(H)\psi, G\rangle_T - i\langle V_1P_{\mathrm{ac}}(H)e^{-itH}\psi, V_2\Gamma_0^*G\rangle_T, \tag{4-7}$$

$$\langle \Gamma_H P_{\mathrm{ac}}(H)F, G\rangle_T = \langle \Gamma_0 P_{\mathrm{ac}}(H)F, G\rangle_T - i\langle V_1\Gamma_H P_{\mathrm{ac}}(H)F, V_2\Gamma_0^*G\rangle_T, \tag{4-8}$$

$$= \langle \Gamma_0 F, P_{\mathrm{ac}}(H)G\rangle_T - i\langle V_2\Gamma_0F, V_1\Gamma_H^*P_{\mathrm{ac}}(H)G\rangle_T. \tag{4-9}$$

Proof. The proof is basically same as that in [Bouclet and Mizutani 2018, Proposition 4.4], where the case $s = 1$ was considered. We shall show (4-8), since the other proofs are similar. We start from the formula

$$\langle e^{-itH}P_{\mathrm{ac}}(H)u, v\rangle - \langle e^{it\Delta}P_{\mathrm{ac}}(H)u, v\rangle = -i\int_0^t \langle V_1e^{-i\tau H}P_{\mathrm{ac}}(H)u, V_2e^{i(t-\tau)\Delta}v\rangle\,d\tau \tag{4-10}$$

for $u, v \in \mathcal{S}$, which follows by computing $\frac{d}{dt}\langle e^{-itH}P_{\mathrm{ac}}(H)u, e^{it\Delta}v\rangle$. Here note that the HLS inequality (A-2) and Lemma 4.3 yield

$$|\langle V_1e^{-i\tau H}P_{\mathrm{ac}}(H)u, V_2e^{i(t-\tau)\Delta}v\rangle| \lesssim \|V_1\|_{L^{n/s,\infty}}\|V_2\|_{L^{n/(2-s),\infty}}\|(-\Delta+1)^{s/2}u\|_{L^2}\|(-\Delta+1)^{1-s/2}v\|_{L^2} < \infty$$

and, hence, the right-hand side of (4-10) makes sense. Changing t to $t-s$, plugging in $u=F(s)$, $v=G(t)$ and integrating in s over $[0,t]$, we obtain

$$\langle \Gamma_H P_{\mathrm{ac}}(H)F(t), G(t)\rangle - \langle \Gamma_0 P_{\mathrm{ac}}(H)F(t), G(t)\rangle = -i\int_0^t\int_s^t \langle V_1 e^{-i(\tau-s)H}P_{\mathrm{ac}}(H)F(s), V_2 e^{i(\tau-t)\Delta}G(t)\rangle\, d\tau\, dt,$$

where, by the same argument as above, the integrand of the right-hand side is finite and thus integrable in $(\tau,s)\in[0,t]^2$. Therefore, by Fubini's theorem,

$$\langle \Gamma_H P_{\mathrm{ac}}(H)F(t), G(t)\rangle - \langle \Gamma_0 P_{\mathrm{ac}}(H)F(t), G(t)\rangle = -i\int_0^t \langle V_1 \Gamma_H P_{\mathrm{ac}}(H)F(\tau), V_2 e^{i(\tau-t)\Delta}G(t)\rangle\, d\tau. \tag{4-11}$$

Finally, observing from the same argument as above that $|\langle V_1\Gamma_H F(\tau), V_2 e^{i(\tau-t)\Delta}G(t)\rangle|$ is finite, we integrate (4-11) in t and use Fubini's theorem to obtain the desired formula (4-8). □

Remark 4.5. When $s=1$, the identities (4-7), (4-8) and (4-9) also hold for all $F, G\in L^1_{\mathrm{loc}}L^2$; see [Bouclet and Mizutani 2018, Proposition 4.4].

Using these lemmas, we first prove Kato smoothing estimates.

Proof of Theorem 1.10. The following argument is basically same as that in [Burq, Planchon, Stalker, and Tahvildar-Zadeh 2004]. With the above remark at hand, we use (4-7) with G replaced by $|D|^{1/2}\langle x\rangle^{-\rho}G$ to obtain

$$\langle \langle x\rangle^{-\rho}|D|^{1/2}e^{-itH}P_{\mathrm{ac}}(H)\psi, G\rangle_T = \langle \langle x\rangle^{-\rho}|D|^{1/2}e^{it\Delta}P_{\mathrm{ac}}(H)\psi, G\rangle_T - i\langle V_1 P_{\mathrm{ac}}(H)e^{-itH}\psi, V_2\Gamma_0^*|D|^{1/2}\langle x\rangle^{-\rho}G\rangle_T$$

for all $\psi\in L^2$ and a simple function $G(t):\mathbb{R}\to\mathcal{S}$. By (4-4), the first term obeys

$$|\langle \langle x\rangle^{-\rho}|D|^{1/2}e^{it\Delta}P_{\mathrm{ac}}(H)\psi, G\rangle_T| \lesssim \|\psi\|_{L^2}\|G\|_{L^2_tL^2_x} \tag{4-12}$$

uniformly in $T>0$. On the other hand, we have by the dual estimate of (4-5) that

$$|\langle V_1 P_{\mathrm{ac}}(H)e^{-itH}\psi, V_2\Gamma_0^* G\rangle_T| \lesssim \|V_1 P_{\mathrm{ac}}(H)e^{-itH}\psi\|_{L^2_tL^2_x}\|G\|_{L^2_tL^2_x} \tag{4-13}$$

uniformly in $T>0$. For the term $\|V_1 P_{\mathrm{ac}}(H)e^{-itH}\psi\|_{L^2_tL^2_x}$, we use Lemma 4.2 to deduce

$$\|V_1 P_{\mathrm{ac}}(H)e^{-itH}\psi\|_{L^2_tL^2_x} \lesssim \|\psi\|_{L^2_x} \tag{4-14}$$

from the following uniform weighted resolvent estimate

$$\sup_{z\in\mathbb{C}\setminus\mathbb{R}} \|V_1 P_{\mathrm{ac}}(H)R(z)P_{\mathrm{ac}}(H)V_1\|_{\mathbb{B}(L^2)}<\infty,$$

which is a consequence of Theorem 3.8 and Hölder's inequality (A-1), where we note that $P_{\mathrm{ac}}(H)^2=P_{\mathrm{ac}}(H)$ since $P_{\mathrm{ac}}(H)$ is an orthogonal projection. Finally, (4-12)–(4-14) imply

$$|\langle \langle x\rangle^{-\rho}|D|^{1/2}e^{-itH}P_{\mathrm{ac}}(H)\psi, G\rangle_T| \lesssim \|\psi\|_{L^2}\|G\|_{L^2_tL^2_x},$$

which, together with duality and density arguments, gives us the assertion. □

In order to prove Strichartz estimates, we need one more lemma.

Lemma 4.6. *Assume* $\mathcal{E} \cap [0, \infty) = \varnothing$. *Then, for any* $\frac{1}{2} < s < \frac{3}{2}$ *there exists* $C > 0$ *such that, for all* $w \in L^{n/(2-s),\infty}$, $\chi \in C_0^\infty(\mathbb{R}^n)$ *and* $T > 0$,

$$\|\chi \Gamma_H P_{\mathrm{ac}}(H) w F\|_{L^2_T L^2_x} \le C \|\chi\|_{L^{n/s,\infty}} \|w\|_{L^{n/(2-s),\infty}} \|F\|_{L^2_T L^2_x}. \tag{4-15}$$

Proof. The proof is essentially based on the argument in [D'Ancona 2015, Theorem 2.3]. At first note that it suffices to show (4-15) with $[-T, T]$ replaced by $\mathbb{R}$. Indeed, since $s \in [-T, T]$ if $t \in [-T, T]$ and $s \in [0, t]$ (or $s \in [t, 0]$), (4-15) with $[-T, T]$ replaced by $\mathbb{R}$ implies

$$\|\chi \Gamma_H P_{\mathrm{ac}}(H) w F\|_{L^2_T L^2_x} \lesssim \|\mathbb{1}_{[-T,T]}(s) F\|_{L^2_t L^2_x} = \|F\|_{L^2_T L^2_x}.$$

We may assume, by a density argument, that $F(t) : \mathbb{R} \to \mathcal{S}$ is a simple function. Set $A_1 = \chi(x) P_{\mathrm{ac}}(H)$ and $A_2 = w P_{\mathrm{ac}}(H)$. For a function $v(t) : \mathbb{R} \to L^2$, $\tilde{v}$ denotes its Laplace transform:

$$\tilde{v}(z) = \pm \int_0^{\pm\infty} e^{izt} v(t)\, dz, \quad \pm \operatorname{Im} z > 0.$$

A direct calculation yields that if $v(t) = \Gamma_H A_2^* F(t)$ then $\tilde{v}(z) = -i R(z) A_2^* \widetilde{F}(z)$, where the identity $\widetilde{A_2^* F} = A_2^* \widetilde{F}$ follows from the estimate $\|A_2 F\|_{L^1_{\mathrm{loc}} L^2_x} \lesssim \|w\|_{L^{n/(2-s),\infty}} \|F\|_{L^1_{\mathrm{loc}} \mathcal{H}^{2-s}} < \infty$ and Hille's theorem [Hille and Phillips 1957, Theorem 3.7.12]. Also we see that $v(t) \in D(A_1)$ for each t. Indeed, writing $F(t) = \sum_{j=1}^N \mathbb{1}_{E_j}(t) f_j$ with some $f_j \in \mathcal{S}(\mathbb{R}^n)$, we have for each t

$$\|A_1 v(t)\|_{L^2} \le \sum_{j=1}^N \int_0^{|t|} \|A_1 e^{isH} e^{-itH} P_{\mathrm{ac}}(H) w f_j\|_{L^2}\, ds \lesssim |t| \|w\|_{L^{n/(2-s),\infty}} \sum_{j=1}^N \|f_j\|_{\mathcal{H}^{2-s}} < \infty.$$

Then one can use Parseval's theorem to obtain

$$\pm \int_0^{\pm\infty} e^{-2\varepsilon|t|} \langle v(t), A_1^* G(t)\rangle\, dt = 2\pi \int_{\mathbb{R}} \langle \tilde{v}(\lambda \pm i\varepsilon), A_1^* \widetilde{G}(\lambda \pm i\varepsilon)\rangle\, d\lambda, \quad \varepsilon > 0,$$

for any simple function $G : \mathbb{R} \to \mathcal{S}$. By virtue of uniform Sobolev estimates (3-10) with

$$(p, q) = \left(\frac{2n}{n+2(2-s)}, \frac{2n}{n-2s}\right)$$

and Hölder's inequality (A-1), the integrand of the right-hand side obeys

$$|\langle \tilde{v}(\lambda \pm i\varepsilon), A_1^* \widetilde{G}(\lambda \pm i\varepsilon)\rangle| \le \|\chi\|_{L^{n/2,\infty}} \|w\|_{L^{n/(2-s),\infty}} \|\widetilde{F}(\lambda \pm i\varepsilon)\|_{L^2_x} \|\widetilde{G}(\lambda \pm i\varepsilon)\|_{L^2_x}.$$

Applying again Parseval's theorem, we have

$$\begin{aligned}\left|\int_0^{\pm\infty} e^{-2\varepsilon|t|} \langle v(t), A_1^* G(t)\rangle\, dt\right| &= \left|\int_{\mathbb{R}} \langle \tilde{v}(\lambda \pm i\varepsilon), A_1^* \widetilde{G}(\lambda \pm i\varepsilon)\rangle\, d\lambda\right| \\ &\lesssim \|\chi\|_{L^{n/2,\infty}} \|w\|_{L^{n/(2-s),\infty}} \|\widetilde{F}(\lambda \pm i\varepsilon)\|_{L^2_\lambda L^2_x} \|\widetilde{G}(\lambda \pm i\varepsilon)\|_{L^2_\lambda L^2_x} \\ &\lesssim \|\chi\|_{L^{n/2,\infty}} \|w\|_{L^{n/(2-s),\infty}} \|e^{-\varepsilon|t|} F(t)\|_{L^2(\mathbb{R}_\pm; L^2(\mathbb{R}^n))} \|e^{-\varepsilon|t|} G(t)\|_{L^2(\mathbb{R}_\pm; L^2(\mathbb{R}^n))},\end{aligned}$$

which, together with the density of simple functions with values in $\mathcal{S}$, shows

$$\|e^{-\varepsilon|t|}A_1\Gamma_H A_2 F\|_{L^2_tL^2_x} \lesssim \|\chi\|_{L^{n/2,\infty}}\|w\|_{L^{n/(2-s),\infty}}\|e^{-\varepsilon|t|}F\|_{L^2_tL^2_x}, \quad F\in L^2_tL^2_x.$$

The result then follows by letting $\varepsilon\to 0$. □

Remark 4.7. If $\frac12 < s \le 1$, (4-15) also holds for any $\chi\in L^{n/s,\infty}$. The proof is completely the same. When $1 < s < \frac32$, we do not, a priori, know $\chi e^{-itH}P_{\mathrm{ac}}(H)wF(s)\in L^2_x$ for each t, s under the condition $\chi\in L^{n/s,\infty}$ only, even if $F:\mathbb{R}\to\mathcal{S}$. This is the reason why we have assumed $\chi\in C_0^\infty$. We however stress that Lemma 4.6 is sufficient for our purpose.

We are now ready to show our Strichartz estimates.

Proof of Theorem 1.12. Using (4-1) and (4-2) with $s=1$ instead of (4-4) and (4-5), respectively, one can see that the proof of the homogeneous endpoint Strichartz estimate of the form

$$\|e^{-itH}P_{\mathrm{ac}}(H)\psi\|_{L^2_tL^{2n/(n-2),2}_x} \lesssim \|\psi\|_{L^2} \tag{4-16}$$

is similar to that of Theorem 1.10 and even easier than that of (1-14). We thus omit the proof.

We shall show (1-14). Let

$$\frac{n}{2(n-1)} < s < \frac{3n-4}{2(n-1)},$$

and $V_1\in L^{n/s,\infty}_0$ and $V_2\in L^{n/(2-s),\infty}_0$ be real-valued such that $V=V_1V_2$. Take a sequence $V_{1,j}\in C_0^\infty$ such that $\|V_1-V_{1,j}\|_{L^{n/s,\infty}}\to 0$. Let $F:\mathbb{R}\to\mathcal{S}$ be a simple function in t. As in the proof of Lemma 4.4, we see that $\Gamma_H P_{\mathrm{ac}}(H)F\in L^2_TL^{q_s,2}_x$ for each $T>0$ by Lemma 4.3. Then, by the duality argument, we have

$$\|\Gamma_H P_{\mathrm{ac}}(H)F\|_{L^2_TL^{q_s,2}_x} \lesssim \sup\{|\langle \Gamma_H P_{\mathrm{ac}}(H)F, G\rangle_T| \mid \|G\|_{L^2_TL^{q_s',2}_x}=1\}, \tag{4-17}$$

where we may assume by a density argument that $G:\mathbb{R}\to\mathcal{S}$ is a simple function. Then, it follows from Duhamel's formula (4-8), (4-2), Lemma 3.9 and Hölder's inequality (A-1) that

$$\begin{aligned}|\langle \Gamma_H P_{\mathrm{ac}}(H)F, G\rangle_T| \lesssim \|P_{\mathrm{ac}}(H)\|_{\mathbb{B}(L^{p_s,2})}\|F\|_{L^2_tL^{p_s,2}_x} + \|V_{1,j}\Gamma_H P_{\mathrm{ac}}(H)F\|_{L^2_TL^2_x}\|V_2\|_{L^{n/(2-s),\infty}} \\ + \|V_1-V_{1,j}\|_{L^{n/s,\infty}}\|\Gamma_H P_{\mathrm{ac}}(H)F\|_{L^2_TL^{q_s,2}_x}\end{aligned}$$

uniformly in $T>0$. Taking j large enough (which can be taken independently of T), the last term can be absorbed into the left-hand side of (4-17), implying

$$\|\Gamma_H P_{\mathrm{ac}}(H)F\|_{L^2_TL^{q_s,2}_x} \lesssim \|F\|_{L^2_tL^{p_s,2}_x} + \|V_{1,j}\Gamma_H P_{\mathrm{ac}}(H)F\|_{L^2_TL^2_x}$$

uniformly in $T>0$. To deal with the term $\|V_{1,j}\Gamma_H P_{\mathrm{ac}}(H)F\|_{L^2_TL^2_x}$, we use (4-9) to write

$$\langle V_{1,j}\Gamma_H P_{\mathrm{ac}}(H)F, \widetilde{G}\rangle_T = \langle \Gamma_0 F, P_{\mathrm{ac}}(H)V_{1,j}\widetilde{G}\rangle_T - i\langle V_2\Gamma_0 F, V_1\Gamma_H^* P_{\mathrm{ac}}(H)V_{1,j}\widetilde{G}\rangle_T$$

for all simple functions $\widetilde{G}:\mathbb{R}\to\mathcal{S}$ satisfying $\|\widetilde{G}\|_{L^2_TL^2_x=1}=1$. By (4-2) the first term enjoys

$$|\langle \Gamma_0 F, P_{\mathrm{ac}}(H)V_{1,j}\widetilde{G}\rangle_T| \lesssim \|V_{1,j}\|_{L^{n/s,\infty}}\|F\|_{L^2_tL^{p_s,2}_x} \lesssim \|F\|_{L^2_tL^{p_s,2}_x}$$

uniformly in $T > 0$ and j. On the other hand, since $V_2\Gamma_H^* P_{\mathrm{ac}}(H)V_{1,j}\widetilde{G} \in L^2_T L^2_x$ by Lemma 4.6 and $V_1\Gamma_0 F \in L^2_T L^2_x$ by (4-2), the last term can be rewritten in the form

$$\langle V_2\Gamma_0 F, V_1\Gamma_H^* P_{\mathrm{ac}}(H)V_{1,j}\widetilde{G}\rangle_T = \langle V_1\Gamma_0 F, V_2\Gamma_H^* P_{\mathrm{ac}}(H)V_{1,j}\widetilde{G}\rangle_T.$$

Using (4-2), Lemma 4.6 and a duality argument, we then obtain

$$|\langle V_1\Gamma_0 F, V_2\Gamma_H^* P_{\mathrm{ac}}(H)V_{1,j}\widetilde{G}\rangle_T| \lesssim \|F\|_{L^2_t L^{p_s,2}_x}.$$

Putting it all together, we conclude that

$$\|\Gamma_H P_{\mathrm{ac}}(H)F\|_{L^2_T L^{q_s,2}_x} \lesssim \|F\|_{L^2_t L^{p_s,2}_x}$$

uniformly in $T > 0$, which implies the desired estimates (1-14) for

$$\frac{n}{2(n-1)} < s < \frac{3n-4}{2(n-1)}.$$

The cases $s = n/(2(n-1))$ and $(3n-4)/(2(n-1))$ can be obtained analogously by using (4-3) instead of (4-2). □

5. Spectral multiplier theorem

This section is devoted to the proof of Lemma 4.3 and Theorem 1.15. Proofs are based on an abstract method in [Chen, Ouhabaz, Sikora, and Yan 2016], which, in the Euclidean case, can be stated as follows.

Proposition 5.1 [Chen, Ouhabaz, Sikora, and Yan 2016, Theorem A]. *Let $1 \le p_0 < 2$ and $1 \le q \le \infty$. Let L be a nonnegative self-adjoint operator on $L^2(\mathbb{R}^n)$ satisfying the following two conditions*:

- *Davies–Gaffney's estimate*: *for any open sets $U_j \subset \mathbb{R}^n$ and $\psi_j \in L^2(U_j)$, $j = 1, 2$,*

$$|\langle e^{-tL}\psi_1, \psi_2\rangle| \le \exp\left(-\frac{d(U_1, U_2)^2}{4t}\right)\|\psi_1\|_{L^2}\|\psi_2\|_{L^2}, \tag{5-1}$$

where $d(U_1, U_2) := \inf_{x_1\in U_1, x_2\in U_2} |x_1 - x_2|$ is the distance between U_1 and U_2.

- *Stein–Tomas-type restriction estimate*: *for any $a > 0$ and any bounded Borel function F_0 on $\mathbb{R}$ supported in $[0, a]$, we have $F_0(\sqrt{L}) \in \mathbb{B}(L^{p_0}, L^2)$ and*

$$\|F_0(\sqrt{L})\mathbb{1}_{B(x,r)}\|_{\mathbb{B}(L^{p_0},L^2)} \lesssim a^{n(1/p_0-1/2)}\|F_0(a\,\cdot)\|_{L^q} \tag{5-2}$$

for all $x \in \mathbb{R}^n$ and $r \ge a^{-1}$, where $B(x, r) = \{y \mid |y - x| < r\}$.

Then, *for any bounded Borel function F on $\mathbb{R}$ satisfying*

$$|F|_{\mathcal{W}(\beta,q)} := \sup_{t>0} \|\psi(\,\cdot\,)F(t\,\cdot\,)\|_{\mathcal{W}^{\beta,q}(\mathbb{R})} < \infty, \tag{5-3}$$

with some nontrivial $\psi \in C_0^\infty$ supported in $(0, \infty)$ and

$$\beta > \max\left\{n\left(\frac{1}{p_0} - \frac{1}{2}\right), \frac{1}{q}\right\}$$

such that β is an integer if $q=\infty$, we have $F(\sqrt{L})$ is bounded on L^p for all $p_0<p<p_0'$ and satisfies

$$\|F(\sqrt{L})\|_{\mathbb{B}(L^p)} \le C_\beta(|F|_{\mathcal{W}(\beta,q)}+|F(0)|).$$

Strictly speaking, instead of Davies–Gaffney's estimate, it was assumed in [Chen, Ouhabaz, Sikora, and Yan 2016] that L satisfies the so-called finite-speed propagation property; see (FS) on page 229 of [loc. cit.]. However, these two conditions are known to be equivalent; see [loc. cit., Theorem 3.4]. Moreover, (5-1) is always satisfied for nonnegative Schrödinger operators $-\Delta+V(x)$ as shown in [Coulhon and Sikora 2008].

Lemma 5.2 [Coulhon and Sikora 2008, Theorem 3.3]. *Let $L=-\Delta+V(x)$ with real-valued $V\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that $L\ge 0$ as a quadratic form. Then* (5-1) *is satisfied.*

When $q=\infty$, (5-2) can be replaced by an L^p-L^2 estimate of the Schrödinger semigroup.

Lemma 5.3. *Let $1\le p_0<2$. Then* (5-2) *with $q=\infty$ follows from*

$$\|e^{-t^2L}\|_{\mathbb{B}(L^{p_0},L^2)} \lesssim t^{-n(1/p_0-1/2)}, \quad t>0. \tag{5-4}$$

Proof. By [Chen, Ouhabaz, Sikora, and Yan 2016, Proposition 1.3], (5-2) with $q=\infty$ is equivalent to

$$\|e^{-t^2L}\mathbb{1}_{B(x,r)}\|_{\mathbb{B}(L^{p_0},L^2)} \lesssim |B(x,r)|^{1/p_0-1/2}(rt^{-1})^{n(1/p_0-1/2)}, \quad t>0,\ x\in\mathbb{R}^n,\ r\ge t,$$

which clearly follows from (5-4) since $|B(x,r)|\le C_n r^n$. ☐

Now we show Lemma 4.3 whose proof is classical and based on Stein's complex interpolation theorem. Let us fix $M>|\inf\sigma(H)|+1$ so that $H+M\ge I$. A key observation is the following.

Lemma 5.4. *For any $\alpha\in\mathbb{R}$ and*

$$\frac{2n}{n+3}<p<\frac{2n}{n-3},$$

we have

$$\|(H+M)^{i\alpha}\|_{\mathbb{B}(L^p)} \le C_M\langle\alpha\rangle^n.$$

Proof. It is easy to see that $F(x)=x^{2i\alpha}$ satisfies $|F|_{\mathcal{W}(n,\infty)}\le C_n\langle\alpha\rangle^n$ and $|F(0)|=1$. Let us fix

$$\frac{2n}{n+3}<p_0\le\frac{2n}{n+2}$$

arbitrarily. By virtue of Proposition 5.1 and Lemmas 5.2 and 5.3, it suffices to show that $L:=H+M$ satisfies (5-4). Decompose e^{-t^2L} into the absolutely continuous part $e^{-t^2L}P_{\mathrm{ac}}(H)$ and the discrete part $\sum_{j=1}^N e^{-t^2L}P_j$.

For the discrete part, since $\lambda_j+M\ge 1$, we know by Lemma 3.9 that

$$\|e^{-t^2L}P_jf\|_{L^2}=\|e^{-t^2(\lambda_j+M)}P_jf\|_{L^2}\le e^{-t^2}\|\varphi_j\|_{L^2}\|\varphi_j\|_{L^{p_0'}}\|f\|_{L^{p_0}}\lesssim e^{-t^2}\|f\|_{L^{p_0}}.$$

On the other hand, it follows from the spectral decomposition theorem that

$$e^{-t^2L}P_{\mathrm{ac}}(H)(e^{-t^2L}P_{\mathrm{ac}}(H))^*=e^{-2t^2L}P_{\mathrm{ac}}(H)=\int_0^\infty e^{-2t^2(\lambda+M)}\,dE_H(\lambda).$$

Theorem 1.8 then implies

$$\|e^{-2t^2L}P_{\mathrm{ac}}(H)\|_{\mathbb{B}(L^{p_0},L^{p_0'})}\lesssim\int_0^\infty e^{-2t^2(\lambda+M)}\lambda^{(n/2)(1/p_0-1/p_0')-1}\,d\lambda\lesssim t^{-n(1/p_0-1/p_0')}=t^{-2n(1/p_0-1/2)}.$$

Since $\|e^{-t^2L}P_{\mathrm{ac}}(H)\|_{\mathbb{B}(L^{p_0},L^2)}\le\|e^{-2t^2L}P_{\mathrm{ac}}(H)\|^{1/2}_{\mathbb{B}(L^{p_0},L^{p_0'})}$ by the duality, (5-4) follows. □

Proof of Lemma 4.3. We may assume $1<s<\frac{3}{2}$ without loss of generality since the case when $0\le s\le 1$ follows from Stein's complex interpolation [1956] and the estimate

$$\|(-\Delta+M)^{1/2}(H+M)^{-1/2}\|_{\mathbb{B}(L^2)}+\|(-\Delta+M)^{-1/2}(H+M)^{1/2}\|_{\mathbb{B}(L^2)}<\infty,$$

which is a consequence of the fact that the form domain of H is $\mathcal{H}^1$.

For $f,g\in\mathcal{S}$, we consider a function $G(z)=\langle (H+M)^{-z}f,(-\Delta+M)^z g\rangle$ which is continuous on $0\le\operatorname{Re}z\le 1$ and analytic in $0<\operatorname{Re}z<1$. By Corollary 1.6 and Lemma 5.4, for

$$\frac{2n}{n+3}<r_1<\frac{2n}{n-3}\quad\text{and}\quad\frac{2n}{n+3}<r_2<\frac{2n}{n+1},$$

we have

$$|G(it)|\le\|(H+M)^{-it}f\|_{L^{r_1}}\|(-\Delta+M)^{it}g\|_{L^{r_1'}}\lesssim\langle t\rangle^{2n}\|f\|_{L^{r_1}}\|g\|_{L^{r_1'}},$$
$$|G(1+it)|\le\|(-\Delta+M)(H+M)^{-1-it}f\|_{L^{r_2}}\|(-\Delta+M)^{it}g\|_{L^{r_2'}}\lesssim\langle t\rangle^{2n}\|f\|_{L^{r_2}}\|g\|_{L^{r_2'}},$$

where, since $(-\Delta+M)(H+M)^{-1}=1-V(H+M)^{-1}$, the second estimate can be verified as

$$\|(-\Delta+M)(H+M)^{-1}\|_{\mathbb{B}(L^{r_2})}\le 1+\|V(H+M)^{-1}\|_{\mathbb{B}(L^{r_2})}\le 1+C_M\|V\|_{L^{n/2,\infty}}.$$

Let

$$r_1=\frac{2n}{n-2s}\quad\text{and}\quad r_2=\frac{2n}{n+2(2-s)}.$$

Since

$$\frac{1}{2}=\Big(1-\frac{s}{2}\Big)\Big(\frac{1}{r_1}\Big)+\frac{s}{2}\cdot\frac{1}{r_2},$$

we apply Stein's complex interpolation theorem to G, implying $|G(s/2)|\le C_t\|f\|_{L^2}\|g\|_{L^2}$. This gives us

$$\|(-\Delta+M)^{s/2}(H+M)^{-s/2}\|_{\mathbb{B}(L^2)}<\infty.$$

Applying the same argument to a function $G(z)=\langle(-\Delta+M)^{-z}f,(H+M)^z g\rangle$, we also have

$$\|(H+M)^{s/2}(-\Delta+M)^{-s/2}\|_{\mathbb{B}(L^2)}<\infty.$$ □

Next we shall show Theorem 1.15.

Proof of Theorem 1.15. Since H is assumed to be nonnegative, the Davies–Gaffney estimate (5-1) is satisfied. It thus remains to check the Stein–Tomas-type restriction estimate (5-2) with $q=2$. Let

$$\frac{2n}{n+3}<p_0<\frac{2n}{n+2}$$

and $F_0 \in L^\infty(\mathbb{R})$ be such that $\operatorname{supp} F_0 \subset [0, a]$. By Theorem 1.8,

$$\|F_0(\sqrt{H})^2\|_{\mathbb{B}(L^{p_0}, L^{p_0'})} \lesssim \int_0^{a^2} |F_0(\sqrt{\lambda})|^2 \lambda^{(n/2)(1/p_0 - 1/p_0')-1}\, d\lambda$$
$$\lesssim \|F_0\|^2_{L^2([0,a])} a^{n(1/p_0-1/p_0')-1} \lesssim a^{n(1/p_0-1/p_0')} \|F_0(a\,\cdot)\|^2_{L^2}.$$

Finally, by the duality, we have $\|F_0(\sqrt{H})\|_{\mathbb{B}(L^{p_0}, L^2)} \le \|F_0(\sqrt{H})^2\|^{1/2}_{\mathbb{B}(L^{p_0}, L^{p_0'})}$, which, combined with the above estimate for $\|F_0(\sqrt{H})^2\|_{\mathbb{B}(L^{p_0}, L^{p_0'})}$, implies (5-2) with $q = 2$. □

We conclude this section with two immediate consequences of Theorem 1.15.

Corollary 5.5. *Suppose that $\mathcal{E} \cap [0, \infty) = \varnothing$, $H \ge 0$ and $0 \le s < \frac{3}{2}$. Then*

$$\|(-\Delta)^{s/2} H^{-s/2}\|_{\mathbb{B}(L^2)} + \|H^{s/2}(-\Delta)^{-s/2}\|_{\mathbb{B}(L^2)} < \infty.$$

Proof. The proof is analogous to that of Lemma 2.8. □

Corollary 5.6. *Suppose that $\mathcal{E} \cap [0, \infty) = \varnothing$ and $H \ge 0$. Let $\varphi \in C_0^\infty(\mathbb{R})$ be such that* $\operatorname{supp} \varphi \subset \left(\frac{1}{2}, 2\right)$, *$0 \le \varphi \le 1$ and $\sum_{j \in \mathbb{Z}} \varphi(2^{-j}\lambda) = 1$ for all $\lambda > 0$. Then, for any*

$$\frac{2n}{n+3} < p < \frac{2n}{n-3},$$

there exists $C_p > 0$ such that

$$C_p^{-1} \|f\|_{L^p} \le \left\| \left(\sum_{j \in \mathbb{Z}} |\varphi(2^{-j}H) f(x)|^2 \right)^{1/2} \right\|_{L^p} \le C_p \|f\|_{L^p}.$$

In particular, if $2 \le p < 2n/(n-3)$, then

$$\|f\|_{L^p} \lesssim \left(\sum_{j \in \mathbb{Z}} \|\varphi(2^{-j}H) f\|^2_{L^p} \right)^{1/2}.$$

Proof. With Theorem 1.15 at hand, the corollary follows from a standard method in [Stein 1970]. The proof is completely the same as that for the usual Littlewood–Paley estimate and we omit it. □

6. Eigenvalue bounds

This section is devoted to the proof of Theorem 1.19. The proof is based on a method of Frank [2011; 2018]. Recall that $W \in L^{n/2+\gamma}(\mathbb{R}^n; \mathbb{C})$ with $0 < \gamma < \infty$. Then W is H-form compact. Indeed, taking $M > -\inf \sigma(H)$, we see that $|W|^{1/2}(1-\Delta)^{-1/2}$ is compact and $(1-\Delta)^{1/2}(H+M)^{-1/2}$ is bounded. Hence $|W|^{1/2}(H+M)^{-1/2} = |W|^{1/2}(1-\Delta)^{-1/2}(1-\Delta)^{1/2}(H+M)^{-1/2}$ is also compact. Then there exists a unique m-sectorial operator H_W such that $D(H_W) \subset Q(H_W) = \mathcal{H}^1$ and $\langle H_W u, v\rangle = \langle (H+W)u, v\rangle$ for $u \in D(H_W)$ and $v \in \mathcal{H}^1$. We also have $D(H_W)$ is dense in $\mathcal{H}^1$, and $\sigma(H_W)$ is contained in a sector $\{z \in \mathbb{C} \mid |\arg(z - z_0)| \le \theta\}$ for some $z_0 \in \mathbb{R}$ and $\theta \in \left[0, \frac{\pi}{2}\right)$; see [Kato 1966, Theorems VI.3.9 and VI.2.1]. We fix a factorization $W = W_1 W_2$ with $W_1 = |W|^{1/2} \operatorname{sgn} W$ and $W_2 = |W|^{1/2}$, where $\operatorname{sgn} W(x) = W(x)/|W(x)|$ if $W(x) \ne 0$ and $\operatorname{sgn} W(x) = 0$ if $W(x) = 0$. Let $d(z) = \operatorname{dist}(z, [\infty)$. We begin with the following lemma.

Lemma 6.1. *Suppose that $E \in \mathbb{C}\setminus\sigma(H)$ is an eigenvalue of H_W. Then -1 is an eigenvalue of $W_1 R(E) W_2$. Moreover, if $0 < \gamma \le \frac{1}{2}$, the same statement also holds for $E \in (0,\infty)\setminus\mathcal{E}$ with $R(E)$ replaced by $R(E+i0)$.*

Proof. We show the lemma for the case $E \in (0,\infty)\setminus\mathcal{E}$ only, since, in the case $E \in \mathbb{C}\setminus\sigma(H)$, the lemma is a consequence of the well-known Birman–Schwinger principle (see, e.g., [Frank 2018, Section 4]), and the proof is easier. Let $f \in \operatorname{Ker}_{L^2}(H_W - E)$. We let $\varphi \in \mathcal{S}$ and plug $v = R(E - i\varepsilon) W_1 \varphi \in \mathcal{H}^1$ into the identity $\langle (H-E)f, v\rangle + \langle W_1 f, W_2 v\rangle = 0$, letting $\varepsilon \searrow 0$ and then using Corollary 1.5(2) to obtain

$$\langle W_1 f, \varphi\rangle + \langle W_1 R(E+i0) W_2 W_1 f, \varphi\rangle = 0.$$

Since $\|W_1 f\|_{L^2} \lesssim \|W_1\|_{L^{n+2\gamma}} \|f\|_{\mathcal{H}^1} < \infty$, this shows $W_1 f \in \operatorname{Ker}_{L^2}(I + W_1 R(E+i0) W_2)$. ☐

Since $W_1 R(E) W_2$ is a compact operator on L^2, if -1 is an eigenvalue of $W_1 R(E) W_2$ then $\|W_1 R(E) W_2\|_{\mathbb{B}(L^2)} \ge 1$ at least. With this remark at hand, it is easy to see that Theorem 1.19 follows from the following lemma.

Lemma 6.2. *For any $\delta > 0$ and $0 \le \gamma \le \frac{1}{2}$, one has*

$$\|W_1 R(z) W_2\|_{\mathbb{B}(L^2)} \le C_\delta |z|^{-\gamma/(n/2+\gamma)} \|W\|_{L^{n/2+\gamma}}, \quad z \in \mathbb{C}\setminus\mathcal{E}_\delta, \tag{6-1}$$

where $R(z)$ is replaced by $R(z+i0)$ if $z \in (0,\infty)\setminus\mathcal{E}_\delta$. Moreover, for any $\gamma > \frac{1}{2}$,

$$\|W_1 R(z) W_2\|_{\mathbb{B}(L^2)} \le C_{\gamma,\delta} |z|^{-(1/2)/(n/2+\gamma)} d(z)^{(\gamma-1/2)/(n/2+\gamma)} \|W\|_{L^{n/2+\gamma}}, \quad z \in \mathbb{C}\setminus(\mathcal{E}_\delta \cup [0,\infty)). \tag{6-2}$$

Proof. Inequality (6-1) is a direct consequence of (1-5) and (1-6) with

$$\frac{1}{p} = \frac{1}{2} + \frac{1}{n+2\gamma} \quad \text{and} \quad q = p'.$$

For the proof of (6-2), we take

$$\theta = \frac{2\gamma - 1}{n+2\gamma} \in (0,1)$$

so that

$$1 - \theta = \frac{n+1}{n+2\gamma}.$$

Interpolating between (1-5) with

$$p = \frac{2(n+1)}{n+3} \quad \text{and} \quad q = p'$$

and the trivial bound $\|R(z)\|_{\mathbb{B}(L^2)} = \operatorname{dist}(z,[0,\infty))^{-1}$ and, then, using Hölder's inequality, we obtain

$$\begin{aligned}\|W_1 R(E) W_2\|_{\mathbb{B}(L^2)} &\le C_{\gamma,\delta} |z|^{-(1-\theta)/(n+1)} d(z)^{-\theta} \|W\|_{L^{n/2+\gamma}} \\ &= C_{\gamma,\delta} |z|^{-1/2/(n/2+\gamma)} d(z)^{(\gamma-1/2)/(n/2+\gamma)} \|W\|_{L^{n/2+\gamma}},\end{aligned}$$

which completes the proof. ☐

Appendix: Real interpolation and Lorentz space

Here a brief summery of real interpolation spaces and Lorentz spaces is given without proofs. One can find a much more detailed exposition in [Bergh and Löfström 1976; Grafakos 2008].

A pair of Banach spaces $(\mathcal{A}, \mathcal{B})$ is said to be a Banach couple if both $\mathcal{A}, \mathcal{B}$ are algebraically and topologically embedded in a Hausdorff topological vector space $\mathcal{C}$. Note that one can always take $\mathcal{C}$ to be a Banach space $\mathcal{A}_0 + \mathcal{A}_1$. Given a Banach couple $(\mathcal{A}_0, \mathcal{A}_1)$ and $0 < \theta < 1$ and $1 \le q \le \infty$, one can define a Banach space $\mathcal{A}_{\theta,q} = (\mathcal{A}_0, \mathcal{A}_1)_{\theta,q}$ by the so-called K-method, which satisfies that $(\mathcal{A}_0, \mathcal{A}_0)_{\theta,q} = \mathcal{A}_0$ and $(\mathcal{A}_0, \mathcal{A}_1)_{\theta,q} = (\mathcal{A}_1, \mathcal{A}_0)_{1-\theta,q}$ with equivalent norms and that if $1 \le q_1 \le q_2 \le \infty$ then $(\mathcal{A}_0, \mathcal{A}_1)_{\theta,1} \hookrightarrow (\mathcal{A}_0, \mathcal{A}_1)_{\theta,q_1} \hookrightarrow (\mathcal{A}_0, \mathcal{A}_1)_{\theta,q_2} \hookrightarrow (\mathcal{A}_0, \mathcal{A}_1)_{\theta,\infty}$. Then the following real interpolation theorem is frequently used in this paper.

Theorem A.1 [Bergh and Löfström 1976, Theorem 3.1.2; Cobos, Edmunds, and Potter 1990]. *Let* $(\mathcal{A}_0, \mathcal{A}_1)$ *and* $(\mathcal{B}_0, \mathcal{B}_1)$ *be two Banach couples*, $0 < \theta < 1$ *and* $1 \le q \le \infty$. *Suppose that T is a bounded linear operator from* $(\mathcal{A}_0, \mathcal{A}_1)$ *to* $(\mathcal{B}_0, \mathcal{B}_1)$ *in the sense that* $T : \mathcal{A}_j \to \mathcal{B}_j$ *and* $\|T\|_{\mathbb{B}(\mathcal{A}_j,\mathcal{B}_j)} \le M_j$, $j = 0, 1$. *Then T is bounded from* $\mathcal{A}_{\theta,q}$ *to* $\mathcal{B}_{\theta,q}$ *and satisfies* $\|T\|_{\mathbb{B}(\mathcal{A}_{\theta,q},\mathcal{B}_{\theta,q})} \le M_0^{1-\theta} M_1^{\theta}$. *Moreover, if both* $T : \mathcal{A}_0 \to \mathcal{B}_0$ *and* $T : \mathcal{A}_1 \to \mathcal{B}_1$ *are compact, then* $T : \mathcal{A}_{\theta,q} \to \mathcal{B}_{\theta,q}$ *is also compact.*

Next we recall the definition and basic properties of Lorentz spaces. Given a μ-measurable function f on $\mathbb{R}^n$, we let $\mu_f(\alpha) = \mu(\{x \mid |f(x)| > \alpha\})$. If we define the decreasing rearrangement of f by $f^*(t) = \inf\{\alpha \mid \mu_f(\alpha) \le t\}$ then the Lorentz space $L^{p,q}(\mathbb{R}^n)$ is the set of measurable f such that the following quasinorm is finite:

$$\|f\|^*_{L^{p,q}} := \|t^{1/p-1/q} f^*(t)\|_{L^q(\mathbb{R}_+,dt)} = p^{1/q} \|\alpha \mu_f(\alpha)^{1/p}\|_{L^q(\mathbb{R}_+,\alpha^{-1}d\alpha)} < \infty.$$

Moreover, if $1 < p < \infty$ and $1 \le q \le \infty$ (which are sufficient for our purpose), then

$$\|f\|_{L^{p,q}} := \|f^{**}\|^*_{L^{p,q}}, \qquad f^{**}(t) := \frac{1}{t} \int_0^t f^*(\alpha)\, d\alpha,$$

becomes a norm on $L^{p,q}$ which makes $L^{p,q}$ a Banach space. Furthermore, $\|\cdot\|_{L^{p,q}}$ is equivalent to $\|\cdot\|^*_{L^{p,q}}$ in the sense that $\|f\|^*_{L^{p,q}} \le \|f\|_{L^{p,q}} \le C(p,q)\|f\|^*_{L^{p,q}}$ with some constant $C(p,q) > 0$. Thus all continuity estimates for linear operators can be expressed in terms of $\|\cdot\|^*_{L^{p,q}}$. $L^{p,q}$ is increasing in q: $L^{p,1} \hookrightarrow L^{p,q_1} \hookrightarrow L^{p,p} = L^p \hookrightarrow L^{p,q_2} \hookrightarrow L^{p,\infty}$ if $1 < q_1 < p < q_2 < \infty$. Moreover, $L^{p,q}$ is characterized by real interpolation: for $0 < \theta < 1,\ 1 < p_1 < p_2 < \infty$ with

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$$

and $1 \le q \le \infty$, one has $(L^{p_0}, L^{p_2})_{\theta,q} = L^{p,q}$ with equivalent norms. If $1 < p, q < \infty$ then $L^{p,q}(X; \mathbb{C})' = L^{p',q'}(X; \mathbb{C})$, where $r' = r/(r-1)$ is the Hölder conjugate of r.

Finally we record two inequalities used frequently in this paper. First, for $1 \le p, p_j < \infty$ and $1 \le q, q_j \le \infty$ with

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and} \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2},$$

one has Hölder's inequality

$$\|fg\|_{L^{p,q}} \le C\|f\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}}, \quad \|fg\|_{L^{p,q}} \le C\|f\|_{L^\infty}\|g\|_{L^{p,q}}. \tag{A-1}$$

Secondly, for $1 < s < n$, $1 < p < q < \infty$, $1/p - 1/q = 2/n$ and $1 \le r \le \infty$, we have the HLS inequality

$$\|(-\Delta)^{-s/2} f\|_{L^{q,r}} \le C\|f\|_{L^{p,r}}. \tag{A-2}$$

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WHEN DOES A PERTURBED MOSER–TRUDINGER INEQUALITY ADMIT AN EXTREMAL?

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We are interested in several questions raised mainly by Mancini and Martinazzi (2017) (see also work of McLeod and Peletier (1989) and Pruss (1996)). We consider the perturbed Moser–Trudinger inequality $I_\alpha^g(\Omega)$ at the critical level $\alpha = 4\pi$, where g, satisfying $g(t) \to 0$ as $t \to +\infty$, can be seen as a perturbation with respect to the original case $g \equiv 0$. Under some additional assumptions, ensuring basically that g does not oscillate too fast as $t \to +\infty$, we identify a new condition on g for this inequality to have an extremal. This condition covers the case $g \equiv 0$ studied by Carleson and Chang (1986), Struwe (1988), and Flucher (1992). We prove also that this condition is sharp in the sense that, if it is not satisfied, $I_{4\pi}^g(\Omega)$ may have no extremal.

1. Introduction

Let Ω be a smooth, bounded domain of $\mathbb{R}^2$ and let $H_0^1 = H_0^1(\Omega)$ be the standard Sobolev space, obtained as the completion of the set of smooth functions with compact support in Ω, with respect to the norm $\|\cdot\|_{H_0^1}$ given by

$$\|u\|_{H_0^1}^2 = \int_\Omega |\nabla u(x)|^2\, dx.$$

Throughout the paper, Ω is assumed to be connected. Let g be such that

$$g \in C^1(\mathbb{R}), \qquad \lim_{s\to+\infty} g(s) = 0, \qquad g(t) > -1 \quad \text{and} \quad g(t) = g(-t) \quad \text{for all } t \tag{1-1}$$

(see also Remark 1.6). Then, we have

$$C_{g,\alpha}(\Omega) := \sup_{u\in H_0^1 : \|u\|_{H_0^1}^2 \le \alpha} \int_\Omega (1 + g(u)) \exp(u^2)\, dx \tag{$I_\alpha^g(\Omega)$}$$

is finite for $0 < \alpha \le 4\pi$ and equals $+\infty$ for $\alpha > 4\pi$. This result was first obtained in [Moser 1971] in the unperturbed case $g \equiv 0$. Still by that work, we easily extend the $g \equiv 0$ case to the case of g as in (1-1). Finally, [Moser 1971] gives also the existence of an extremal for $(I_\alpha^g(\Omega))$ if $0 < \alpha < 4\pi$ (see Lemma 3.1). If now $\alpha = 4\pi$, getting the existence of an extremal is more challenging; however, Carleson and Chang [1986], Struwe [1988] and Flucher [1992] were also able to prove that $(I_{4\pi}^0(\Omega))$ admits an extremal in the unperturbed case $g \equiv 0$. Yet, surprisingly, McLeod and Peletier [1989] conjectured that there should exist a g as in (1-1) such that $(I_{4\pi}^g(\Omega))$ does not admit any extremal function. Through a nice but very

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implicit procedure, Pruss [1996] was able prove that such a g does exist. Observe that, since $g(u) \to 0$ as $u \to +\infty$ in (1-1), $(1+g(u))\exp(u^2)$ in $(I_\alpha^g(\Omega))$ seems like a very mild perturbation of $\exp(u^2)$ as $u \to +\infty$ and then, this naturally raises the following question:

Question 1.1. To what extent does the existence of an extremal for the critical Moser–Trudinger inequality $(I_{4\pi}^0(\Omega))$ really depend on asymptotic properties of the function $t \mapsto \exp(t^2)$ as $t \to +\infty$?

To investigate this question, we may rephrase it as follows: for what g satisfying (1-1) does $(I_{4\pi}^g(\Omega))$ admit an extremal? This is Open Problem 2 in [Mancini and Martinazzi 2017], stated in this paper for $\Omega = \mathbb{D}^2$, the unit disk of $\mathbb{R}^2$. In order to state our main general result, we introduce now some notation. For a first reading, one can go directly to Corollary 1.3, which aims to give a less general but more readable statement. We let $H : (0, +\infty) \to \mathbb{R}$ be given by

$$H(t) = 1 + g(t) + \frac{g'(t)}{2t}, \tag{1-2}$$

so that we have

$$[(1+g(t))\exp(t^2)]' = 2tH(t)\exp(t^2). \tag{1-3}$$

We set $tH(t) = 0$ for $t = 0$, so that $t \mapsto tH(t)$ is continuous at 0 by (1-1). This function H comes into play, since the Euler–Lagrange associated to $(I_\alpha^g(\Omega))$ reads as

$$\begin{cases} \Delta u = \lambda u H(u)\exp(u^2) & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases} \tag{1-4}$$

where $\lambda \in \mathbb{R}$ is a Lagrange multiplier and $\Delta = -\partial_{xx} - \partial_{yy}$ (see also Lemma 3.1 below). Now, we make some further assumptions on the behavior of g at $+\infty$ and at 0. First, we assume that there exist $\delta_0 \in (0, 1)$ and a sequence of real numbers $A = (A(\gamma))_\gamma$ such that:

(1-5a) $H(\gamma - t/\gamma) = H(\gamma)(1 + A(\gamma)t + o(|A(\gamma)| + \gamma^{-4}))$ in $C^0_{\text{loc}}(\mathbb{R}_t)$ as $\gamma \to +\infty$.

(1-5b) There exists $C > 0$ such that $|H(\gamma - t/\gamma) - H(\gamma)| \le C|H(\gamma)|(|A(\gamma)| + \gamma^{-4})\exp(\delta_0 t)$ for all $\gamma \gg 1$ and all $0 \le t \le \gamma^2$.

(1-5c) $\lim_{\gamma\to+\infty} A(\gamma) = 0$.

In (1-5a) and (1-6a), γ is a parameter and the $C^0_{\text{loc}}([0, +\infty))$ convergence is in the t-variable. We also assume that there exist $\delta_0' \in (0, 1)$, $\kappa \ge 0$, $\tilde{\varepsilon}_0 \in \{-1, +1\}$, F given by $F(t) := \tilde{\varepsilon}_0 t^\kappa$, and a sequence $B = (B(\gamma))_\gamma$ of positive real numbers such that:

(1-6a) $(t/\gamma)H(t/\gamma) = B(\gamma)F(t) + o(|B(\gamma)| + \gamma^{-1})$ in $C^0_{\text{loc}}((0, +\infty)_t)$ as $\gamma \to +\infty$.

(1-6b) There exists $C > 0$ such that $|(t/\gamma)H(t/\gamma)| \le C(|B(\gamma)| + \gamma^{-1})\exp(\delta_0' t)$ for all $\gamma \gg 1$ and all $0 \le t \le \gamma^2$.

Observe that we may have $B(\gamma) = o(\gamma^{-1})$ as $\gamma \to +\infty$, in which case the precise formula for F is not really significant. Since $t \mapsto (1+g(t))\exp(t^2)$ is an even C^1 function, we have

$$\lim_{\gamma\to+\infty} B(\gamma) = 0, \tag{1-7}$$

in view of (1-3) and (1-6). Following rather standard notation, we may split the Green's function G of Δ, with zero Dirichlet boundary conditions in Ω, according to

$$G_x(y) = \frac{1}{4\pi}\left(\log \frac{1}{|x-y|^2} + \mathcal{H}_x(y)\right) \tag{1-8}$$

for all $x \neq y$ in Ω, where $\mathcal{H}_x$ is harmonic in Ω and coincides with $-\log 1/|x-\cdot|^2$ in $\partial\Omega$. Then the Robin function $x \mapsto \mathcal{H}_x(x)$ is smooth in Ω, and goes to $-\infty$ as $x \to \partial\Omega$, so that we may set

$$\begin{aligned} M &= \max_{x\in\Omega} \mathcal{H}_x(x), \\ K_\Omega &= \{y \in \Omega : \mathcal{H}_y(y) = M\}, \\ S &= \max_{z\in K_\Omega} \int_\Omega G_z(y) F(4\pi G_z(y))\, dy, \end{aligned} \tag{1-9}$$

where F is as in (1-6). For $N \geq 1$, we let g_N be given by

$$(1+g_N(t))\exp(t^2) = (1+g(t))(1+t^2) + (1+g(t))\left(\sum_{k=N+1}^{+\infty} \frac{t^{2k}}{k!}\right), \tag{1-10}$$

so that $g_N \leq g,\ g_N(0) = g(0)$ for all $N \geq 1$, while $g = g_N$ for $N = 1$. We also set

$$\Lambda_g(\Omega) := \max_{u\in H_0^1 : \|u\|^2_{H_0^1} \leq 4\pi} \int_\Omega \big((1+g(u))(1+u^2) - (1+g(0))\big)\, dx. \tag{1-11}$$

We are now in position to state our main result, giving a new, very general and basically sharp picture about the existence of an extremal for the perturbed Moser–Trudinger inequality $(I^g_{4\pi}(\Omega))$.

Theorem 1.2 (existence and nonexistence of an extremal). *Let Ω be a smooth bounded domain of $\mathbb{R}^2$. Let g be such that* (1-1) *and* (1-5)–(1-6) *hold true for H as in* (1-2), *and let A, B and F be thus given. Assume that*

$$l = \lim_{\gamma\to+\infty} \frac{\gamma^{-4} + \frac{1}{2}A(\gamma) + 4\gamma^{-3}\exp(-1-M)B(\gamma)S}{\gamma^{-4} + |A(\gamma)| + \gamma^{-3}|B(\gamma)|} \tag{1-12}$$

exists, where M and S are given by (1-9). *Then*:

(1) *If $l > 0$ or $\Lambda_g(\Omega) \geq \pi \exp(1+M)$, then $(I^g_{4\pi}(\Omega))$ admits an extremal, where $\Lambda_g(\Omega)$ is as in* (1-11).

(2) *If $l < 0$ and $\Lambda_g(\Omega) < \pi \exp(1+M)$, there exists $N_0 \geq 1$ such that $(I^{g_N}_{4\pi}(\Omega))$ admits no extremal for all $N \geq N_0$, where g_N is given by* (1-10).

Observe that, for all given $N \geq 1$, g_N satisfies (1-1) and (1-5)–(1-6), with the same A, B and F as the original g, in view of $H(\gamma) \to 1$ as $\gamma \to +\infty$; see (3-3). Moreover it is clear that $\Lambda_{g_N}(\Omega) \leq \Lambda_g(\Omega)$. Then, this second assertion in Theorem 1.2 proves that the assumptions on g in the first assertion are basically sharp to get the existence of an extremal for $(I^g_{4\pi}(\Omega))$. As a remark, Pruss [1996] concludes that the existence of an extremal for the critical Moser–Trudinger inequality is in some sense accidental and relies on nonasymptotic properties of $\exp(u^2)$. Theorem 1.2 clarifies this tricky situation: the existence or nonexistence of an extremal for $(I^g_{4\pi}(\Omega))$ may really depend on a balance of the asymptotic properties of g

both at infinity (given by $A(\gamma)$) and at zero (given by $B(\gamma)$). Yet, it may also depend on the nonasymptotic quantity $\Lambda_g(\Omega)$ (see Corollary 1.4). Observe that $\Lambda_0(\Omega) = 4\pi/\lambda_1(\Omega)$ in the unperturbed case $g \equiv 0$, where $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue of Δ in Ω.

From now on, we illustrate Theorem 1.2 by two corollaries dealing with less general but more explicit situations. Let $c, c' \in \mathbb{R}$, $(a, b), (a', b') \in \mathcal{E}$, where

$$\mathcal{E} = \{(a, b) \in [0, +\infty) \times \mathbb{R} : b > 0 \text{ if } a = 0\}. \tag{1-13}$$

Let $R' > 0$ be a large positive constant. If one picks g such that

$$g(t) = \begin{cases} g_0(t) := g(0) + ct^{a+1}[\log(1/t)]^{-b} & \text{in } (0, 1/R'], \\ g_\infty(t) := c't^{-a'}[\log t]^{-b'} & \text{in } [R', +\infty), \end{cases} \tag{1-14}$$

l in (1-12) of Theorem 1.2 can be made more explicit. Indeed, we can then set

$$\begin{aligned} B(\gamma) &= \frac{1+g(0)}{\gamma} + \frac{c(a+1)}{2\gamma^a(\log\gamma)^b}, \\ F(t) &= \begin{cases} t^{\min(a,1)} & \text{if } c \neq 0, \\ t & \text{otherwise}, \end{cases} \\ A(\gamma) &= c' \times \begin{cases} a'\gamma^{-(a'+2)}(\log\gamma)^{-b'} & \text{if } a' > 0, \\ b'\gamma^{-2}(\log\gamma)^{-(b'+1)} & \text{if } a' = 0 \end{cases} \end{aligned} \tag{1-15}$$

(see also Lemma 3.3). Theorem 1.2 is even more explicit in the particular case $\Omega = \mathbb{D}^2$. Indeed, in this case we have that $K_{\mathbb{D}^2} = \{0\}$ in (1-9) and

$$G_0(x) = \frac{1}{2\pi}\log\frac{1}{|x|}.$$

Still on the unit disk $\mathbb{D}^2$, it is known that

$$\Lambda_0(\mathbb{D}^2) = \frac{4\pi}{\lambda_1(\mathbb{D}^2)} < \pi e \tag{1-16}$$

($\lambda_1(\mathbb{D}^2) \simeq 5.78$). Property (1-16) shows in particular that the second assertion $\Lambda_0(\mathbb{D}^2) \geq \pi e$ of Theorem 1.2(1) is not satisfied. In some sense, this is an additional motivation for the nice approach of [Carleson and Chang 1986], proving the existence of an extremal for $(I^0_{4\pi}(\mathbb{D}^2))$ via asymptotic analysis. As an illustration and a very particular case of Theorem 1.2, we get the following corollary.

Corollary 1.3 (case $\Omega = \mathbb{D}^2$). *Assume that or $\Omega = \mathbb{D}^2$. Let $c' \neq 0$ and $(a', b') \in \mathcal{E}$ be given, where $\mathcal{E}$ is as in* (1-13). *Let g_∞ be as in* (1-14):

(1) *If we assume $a' > 2$ or $c' > 0$, then for any even function $g \in C^1(\mathbb{R})$ such that $g > -1$,*

$$(g - g(0))^{(i)}(t) = o(t^{2-i}) \tag{1-17}$$

as $t \to 0$ and

$$g^{(i)}(t) = g_\infty^{(i)}(t)(1 + o(1)) \tag{1-18}$$

as $t \to +\infty$ for all $i \in \{0, 1\}$, the inequality $(I^g_{4\pi}(\mathbb{D}^2))$ admits an extremal.

(2) *If we assume* $a' < 2$ *and* $c' < 0$, *there exists an even function* $g \in C^1(\mathbb{R})$ *such that* $g > -1$ *and* (1-17) *and* (1-18) *hold true, while* $(I^g_{4\pi}(\mathbb{D}^2))$ *admits no extremal.*

Our main concern in Corollary 1.3 is to write a readable statement. In this result, the existence of an extremal in the unperturbed case $g \equiv 0$ is recovered for quickly decaying g's, namely if $a' > 2$; see [Mancini and Martinazzi 2017]. But a threshold phenomenon appears (only if $c' < 0$) and there are no more extremals for less decaying g's, namely for $a' < 2$. Note that Theorem 1.2 also allows us to point out the existence of a threshold $c' < 0$ in the border case $a' = 2,\ b' = 0$ (see Remark 1.5). Indeed, proving Corollary 1.3 basically reduces to giving an explicit formula for l in (1-12), which only depends on Ω and on the asymptotics of g at $+\infty$ and at 0. On the contrary, we do not care about the precise asymptotics of g in the following corollary, thus illustrating the role of $\Lambda_g(\Omega)$ in Theorem 1.2.

Corollary 1.4 (extremal for $\Lambda_g(\Omega)$ large). *Let* Ω *be a smooth bounded domain of* $\mathbb{R}^2$. *Let* $\lambda_1(\Omega) > 0$ *be the first Dirichlet eigenvalue of* Δ *in* Ω *and* M *be given as in* (1-9). *Let* $\bar{A}$ *be such that* $4(1+\bar{A}) > \lambda_1(\Omega)\exp(1+M)$ *and let* $C > \bar{A}$ *be given. Then there exists* $R \gg 1$ *such that* $(I^g_{4\pi}(\Omega))$ *admits an extremal for all* g *satisfying* (1-1) *and*

$$g(0) = \bar{A}, \qquad g \ge g(0) \quad \text{in } [1/R, R] \qquad \text{and} \qquad |g| \le C \quad \text{in } \mathbb{R}. \tag{1-19}$$

We give now an overview of the proof of Theorem 1.2, since it is a bit intricate. First, we comment on part (1). For all $0 < \varepsilon \ll 1$ small, we start by picking an extremal function u_ε for $(I^g_{4\pi(1-\varepsilon)}(\mathbb{D}^2))$. Under the assumptions of part (1), we only need to rule out the case where (2-1) holds true, as described in the proof of Theorem 1.2(1) in Section 2. Then we assume by contradiction that (2-1) holds true. By Lemma 3.4, Case 2, we get expansions of the u_ε's, and then expansions both of the Moser–Trudinger functional (see (2-4)) and of the Dirichlet energy (see (2-5)). These results are gathered in Proposition 2.1 below, whose proof (see Section 4) amounts to showing that not only M but also S in (1-9) may have to be attained at a blow-up point of our sequence of *maximizers* $(u_\varepsilon)_\varepsilon$ (see Lemma 4.1). Observe that this two-fold maximization property is necessary to get a sharp picture in Theorem 1.2. Moreover, this is not seen when restricting to the case $\Omega = \mathbb{D}^2$, where K_Ω in (1-9) contains only the single point 0, so that expanding the Dirichlet energy of a blowing-up sequence of *critical points* $(u_\varepsilon)_\varepsilon$ is sufficient; see [Mancini and Martinazzi 2017]. Theorem 1.2(1) is eventually obtained by getting a contradiction with (2-1): either by comparing (2-4) with our assumption $\Lambda_g(\Omega) \ge \pi\exp(1+M)$, or by comparing $\|u_\varepsilon\|^2_{H^1_0} = 4\pi(1-\varepsilon)$ (see (3-8) in Lemma 3.4) and (2-5) with our assumption $l > 0$.

Now we comment on part (2). Making our assumptions of part (2) and assuming also by contradiction that there exists an extremal function u_ε for $(I^{g_{N_\varepsilon}}_{4\pi}(\Omega))$ such that $N_\varepsilon \to +\infty$ as $\varepsilon \to 0$, we get from Lemma 3.4, Case 1 that our assumption $\Lambda_g(\Omega) < \pi\exp(1+M)$ automatically implies (2-1) (see Step 2), so that we may get expansions of the u_ε's and then (2-10). This gives a contradiction by comparing

$$\|u_\varepsilon\|^2_{H^1_0} = 4\pi$$

and (2-10) with our assumption $l < 0$, as developed in the proof of Theorem 1.2(2) in Section 2. These key ingredients are gathered in Proposition 2.3. In comparison with the expansions of part (1), the key observation is that the delicate N_ε-dependence generates additional terms which may only *reduce* the Dirichlet energy, as explained in the proof of Proposition 2.3 of Section 4.

Overall, the proof of Lemma 3.4, Case 1 is the most delicate part: we need to use first that the u_ε's are *maximizers* to check that we are in a Moser–Trudinger critical regime (see Step 2 and Remark 3.5) and that the pointwise and global gradient estimate (3-52) is true. In both Cases 1 and 2, resuming the approach of [Druet and Thizy 2017], this last point is the key ingredient to be in position to use the radial model B_ε studied in the Appendix. To conclude, the case of a general domain Ω addressed by Theorem 1.2 requires sharp estimates, not only at small scales close to a blow-up point x_ε, as performed in the radial case by [Mancini and Martinazzi 2017], but also in the whole Ω (see (3-99) or (4-1)). This allows in particular to get a useful accurate expansion of the Lagrange multiplier λ_ε in (4-12), when proving Proposition 2.1. As a remark, in the process of the proof below (see Remark 2.2), we answer the very interesting Open Problem 6 of [Mancini and Martinazzi 2017].

Remark 1.5 (links between Theorem 1.2 and [Carleson and Chang 1986; Flucher 1992; Mancini and Martinazzi 2017; Struwe 1988]). For $\Omega = \mathbb{D}^2$, part (1) of Theorem 1.2 implies in general [Mancini and Martinazzi 2017, Corollary 3], which gives itself the existence of an extremal function for $(I^0_{4\pi}(\Omega))$ pioneered by [Carleson and Chang 1986] in the original case $g \equiv 0$. Even if both [Mancini and Martinazzi 2017, Corollary 3] and Theorem 1.2 are much more general, we restrict there for simplicity to g's satisfying (1-17) and coinciding with g_∞ for all $t \gg 1$; see (1-14). Then [Mancini and Martinazzi 2017, Corollary 3] covers the fast decaying case $a' > 2$ (or $c' = 0$) on the disk. By (1-15), thanks to the explicit formulas above (1-16) for $\Omega = \mathbb{D}^2$ and since $\int_{\mathbb{D}^2} (\log|x|)^2\,dx = \frac{\pi}{2}$, it is easy to check that we have in this latter case that $l > 0$ in (1-12), since we have

$$\gamma^{-4} + \tfrac{1}{2} A(\gamma) + 4\gamma^{-3} \exp(-1-M) B(\gamma) S = \gamma^{-4}\Big(1 + \frac{2}{e}(1+g(0))\Big) + o(\gamma^{-4})$$

as $\gamma \to +\infty$. Pushing further their asymptotic analysis, Mancini and Martinazzi [2017] cover also the case $a' = 2$ and then suspect (see Theorem 4–Open Problem 2 in that work) that there could be no extremal function for $(I^g_{4\pi}(\mathbb{D}^2))$, if, in addition, c' is a sufficiently large negative constant. Corollary 1.4 claims that there can actually be an extremal for such a g, whatever c' is, and even independently from the precise behavior of g close to 0 or $+\infty$. However, part (2) of Theorem 1.2 gives with (1-15) the following picture in this threshold case $a' = 2$:

$$\text{if } c' > -\Big(1 + \frac{2}{e}(1+g(0))\Big) \text{ or } \Lambda_g(\mathbb{D}^2) \ge \pi e, \text{ there is an extremal for } (I^g_{4\pi}(\mathbb{D}^2)),$$
$$\text{if } c' < -\Big(1 + \frac{2}{e}(1+g(0))\Big),\ \Lambda_g(\mathbb{D}^2) < \pi e, \text{ and } N \gg 1, \text{ there is no extremal for } (I^{g_N}_{4\pi}(\mathbb{D}^2)).$$

Observe that there are many ways of building such g's satisfying $\Lambda_g(\mathbb{D}^2) < \pi e$: one is given in the proof of Corollary 1.3 in Section 2 (see also (1-16)). As observed just below Theorem 1.2, this gives a basically sharp picture about how far we can get the existence of an extremal function for $(I^g_{4\pi}(\Omega))$, *relying only on the asymptotic properties of* g (see Question 1.1). Theorem 1.2 gives a similar picture on any domain Ω, and then gives back (for $c' = 0$) the results of [Flucher 1992; Struwe 1988]. Stronger perturbations, for instance $a' < 2$ or even $a' = 0$ and $b' > 0$, are also covered by Theorem 1.2.

We conclude this introductory section by the following remark about the relevance of the assumption (1-1) on g introduced in [Mancini and Martinazzi 2017]. We also mention the nice and early result of [de Figueiredo and Ruf 1995].

Remark 1.6 (about assumption (1-1)). Indeed, assume that g is a C^1, even function such that $1 + g > 0$ in $\mathbb{R}$. Assume also that $\bar{g} = \lim_{t\to+\infty} g(t) \in [-1, +\infty]$ exists. Firstly, if $\bar{g} = +\infty$, it is easy to check with the test functions of Step 1 that $C_{g,4\pi}(\Omega) = +\infty$. Secondly, if $\bar{g} = -1$, it follows from standard integration theory (see for instance [Mancini and Martinazzi 2017, Lemma 7]) and from Moser's result [1971] that there exists an extremal function for $(I^g_{4\pi}(\Omega))$. Thus, up to replacing $1 + g$ by $(1 + g)/(1 + \bar{g})$, we have that (1-1) holds true in the remaining more sensitive case $\bar{g} \in (-1, +\infty)$. To end this remark, we mention that [de Figueiredo and Ruf 1995] already studied (1-4) in $\mathbb{D}^2$, permitting one to recover the existence of an extremal in some subcases where $\bar{g} = -1$. First, assuming that H given by (1-3) is positive in $(0, +\infty)$, it is clear that a nonnegative extremal for $(I^g_{4\pi}(\Omega))$ turns out to be a positive solution of (1-4) (for some $\lambda > 0$). Now following [de Figueiredo and Ruf 1995], assume also that $\Omega = \mathbb{D}^2$, that $t \mapsto tH(t)$ is C^2 and that, given $a > 0$, there exist $K, C, \sigma > 0$ such that $tH(t) = Kt^{-a}$ for all $t \gg 1$ and such that $H(t) \le CKt^{\sigma}$ for all $t > 0$ close to 0. Then, [de Figueiredo and Ruf 1995, Theorem 1.1] allows us to claim that there exists no positive solution of (1-4) for all $0 < \lambda \ll 1$ small enough if $a \ge 1$, while there exists a family of positive solutions of (1-4) blowing-up as $\lambda \to 0$ if $a < 1$. From by now standard arguments, this first property directly gives back the existence of an extremal in the subcase $a \ge 1$. However, observe that $\bar{g} = -1$ for all $a > 0$, since $1 + g(t) \sim 2Ke^{-t^2} \int_1^t s^{-a}\, e^{s^2}\, ds = O(t^{a+1}) \to 0$ as $t \to +\infty$, so that an extremal also exists in the subcase $a \in (0, 1)$. Actually we assert that a more precise analysis in the spirit of [Mancini and Martinazzi 2017] allows us to exclude that the aforementioned blow-up solutions of (1-4) are maximizers and to recover the existence of an extremal also in the subcase $a \in (0, 1)$ through this approach using the Euler–Lagrange equation.

2. Proof of the main results

We begin by proving Corollary 1.3, assuming that Theorem 1.2 holds true.

Proof of Corollary 1.3. The first part of Corollary 1.3 is a direct consequence of the first part of Theorem 1.2: plugging the formulas of (1-15) in (1-12), we get that $l > 0$ for g as in case (1) of Corollary 1.3. In order to prove the second part of Corollary 1.3, we apply the second part of Theorem 1.2. Let χ be a smooth nonnegative function in $\mathbb{R}$ such that $\chi(t) = 0$ for all $t \le \frac{1}{2}$ and $\chi(t) = 1$ for all $t \ge 1$. By the Sobolev inequality and standard integration theory, we can check that $g_R := g_\infty \times \chi(\cdot/R)$ satisfies $\Lambda_{g_R}(\mathbb{D}^2) \to \Lambda_0(\mathbb{D}^2)$ as $R \to +\infty$. Then, by (1-15), (1-16), assuming $a' < 2,\ c' < 0$, the second part of Theorem 1.2 applies, starting from $g = g_R$ for $R \gg 1$ fixed sufficiently large. Observe that, for all given $N \gg 1$, $(g_R)_N$ (given by (1-10) for $g = g_R$) satisfies (1-17)–(1-18). □

Proof of Corollary 1.4. Let Ω, $\bar{A}$, $\lambda_1(\Omega)$, C be as in the statement of the corollary. It is sufficient to prove that there exists $R \gg 1$ such that for all g satisfying (1-1) and (1-19), we have $\Lambda_g(\Omega) \ge \pi \exp(1 + M)$, where $\Lambda_g(\Omega)$ is as in (1-11). Let $v > 0$ in Ω be the first eigenvalue of Δ normalized according to $\|v\|^2_{H^1_0} = 4\pi$. For all g satisfying (1-19), we have

$$\begin{aligned}\Lambda_g(\Omega) &\geq \int_\Omega ((1+g(0))v^2 + (g(v)-g(0))(1+v^2))\,dx \\ &\geq (1+\bar{A})\frac{4\pi}{\lambda_1(\Omega)} + \int_{\{v\notin[1/R,R]\}} (g(v)-g(0))(1+v^2)\,dx,\end{aligned}$$

and, since we have

$$\left|\int_{\{v\notin[1/R,R]\}} (g(v)-g(0))(1+v^2)\,dx\right| \leq (|\bar{A}|+C)(1+\|v\|^2_{L^\infty})|\{v\notin[1/R,R]\}| \to 0$$

as $R \to +\infty$, we get the result using that $4(1+\bar{A}) > \lambda_1(\Omega)\exp(1+M)$. □

The following result is the core of the argument to get the existence of an extremal in Theorem 1.2(1). Its proof is postponed until Section 4. It uses the tools developed in [Druet and Thizy 2017] that allow us to push the asymptotic analysis of a concentrating sequence of extremals $(u_\varepsilon)_\varepsilon$ further than in previous works. In the process of the proof of Proposition 2.1 (see Lemma 4.1), we show first that a concentration point $\bar{x}$ of such u_ε's realizes M in (1-9). But in the case where $|B(\gamma)|$ matters in (1-12) or, in other words, where $\gamma^3|A(\gamma)|+\gamma^{-1} \lesssim |B(\gamma)|$ as $\gamma \to +\infty$, we also show that S in (1-9) has to be attained at $\bar{x}$.

Proposition 2.1. *Let Ω be a smooth bounded domain of $\mathbb{R}^2$. Let g be such that* (1-1) *and* (1-5)–(1-6) *hold true for H as in* (1-2), *and let A, B and F be thus given. Let $(u_\varepsilon)_\varepsilon$ be a sequence of nonnegative functions such that u_ε is a maximizer for $(I^g_{4\pi(1-\varepsilon)}(\Omega))$ for all $0<\varepsilon\ll 1$. Assume that*

$$u_\varepsilon \rightharpoonup 0 \quad \text{in } H^1_0 \tag{2-1}$$

as $\varepsilon\to 0$. Then, $\|u_\varepsilon\|^2_{H^1_0} = 4\pi(1-\varepsilon)$, there exists a sequence $(\lambda_\varepsilon)_\varepsilon$ of real numbers such that u_ε solves in H^1_0

$$\begin{cases}\Delta u_\varepsilon = \lambda_\varepsilon u_\varepsilon H(u_\varepsilon)\exp(u_\varepsilon^2), & u_\varepsilon > 0 \text{ in } \Omega,\\ u_\varepsilon = 0 \text{ on } \partial\Omega,\end{cases} \tag{2-2}$$

$u_\varepsilon \in C^{1,\theta}(\bar{\Omega})$ $(0<\theta<1)$ and we have

$$\gamma_\varepsilon := \max_{y\in\Omega} u_\varepsilon \to +\infty. \tag{2-3}$$

Moreover, we have

$$\lim_{\varepsilon\to 0}\int_\Omega (1+g(u_\varepsilon))\exp(u_\varepsilon^2)\,dx = |\Omega|(1+g(0)) + \pi\exp(1+M) \tag{2-4}$$

and

$$\|u_\varepsilon\|^2_{H^1_0} = 4\pi\big(1+I(\gamma_\varepsilon)+o(\gamma_\varepsilon^{-4}+|A(\gamma_\varepsilon)|+\gamma_\varepsilon^{-3}|B(\gamma_\varepsilon)|)\big) \tag{2-5}$$

as $\varepsilon\to 0$, where

$$I(\gamma_\varepsilon) := \gamma_\varepsilon^{-4} + \tfrac{1}{2}A(\gamma_\varepsilon) + 4\gamma_\varepsilon^{-3}\exp(-1-M)B(\gamma_\varepsilon)S, \tag{2-6}$$

where $|\Omega|$ stands for the volume of the domain Ω and where M and S are as in (1-9).

Remark 2.2. Let g, H be such that (1-1), (1-2), (1-5)–(1-7) hold true. Let u_ε be a maximizer for $(I^g_{4\pi(1-\varepsilon)})$ such that (2-1) holds true, as in Proposition 2.1. Then, for such a sequence $(u_\varepsilon)_\varepsilon$ satisfying in particular (2-2) and (2-3), we get in the process of the proof (see (3-16) below) that the term $I(\gamma_\varepsilon)$ in (2-5) is necessarily smaller than $o(\gamma_\varepsilon^{-2})$ as $\varepsilon \to 0$. Moreover this threshold $o(\gamma_\varepsilon^{-2})$ is sharp, in the sense that this term may be for instance of size $\gamma_\varepsilon^{-(2+a')}$ for all given $a' \in (0, 2]$. This can be seen by picking an appropriate g such that $I^g_{4\pi}(\Omega)$ has no extremal, as in Corollary 1.3, and by using Proposition 2.1. Observe that, for such a g, assumption (2-1) is indeed automatically true. This gives an answer to Open Problem 6 in [Mancini and Martinazzi 2017].

Proof of Theorem 1.2(1): existence of an extremal for $(I^g_{4\pi}(\Omega))$. We first prove the existence of an extremal stated in part (1) of Theorem 1.2. Let g be such that (1-1) and (1-5)–(1-6) hold true for H as in (1-2), and let A, B and F be thus given. Assume either that $l > 0$ in (1-12) or that $\Lambda_g(\Omega) \geq \pi \exp(1+M)$. Using Lemma 3.1, let $(u_\varepsilon)_\varepsilon$ be a sequence of nonnegative functions such that u_ε is a maximizer for $(I^g_{4\pi(1-\varepsilon)}(\Omega))$ for all $0 < \varepsilon \ll 1$. Then, up to a subsequence, $(u_\varepsilon)_\varepsilon$ converges a.e. and weakly in H^1_0 to some u_0. Independently, we check that

$$\lim_{\varepsilon\to 0} C_{g,4\pi(1-\varepsilon)}(\Omega) = C_{g,4\pi}(\Omega), \tag{2-7}$$

where $C_{g,\alpha}(\Omega)$ is as in $(I^g_\alpha(\Omega))$. Indeed, if one assumes by contradiction that the $C_{g,4\pi(1-\varepsilon)}(\Omega)$'s increase to some $\bar{l} < C_{g,4\pi}(\Omega)$ as $\varepsilon \to 0$, then we may choose some nonnegative u such that $\|u\|^2_{H^1_0} \leq 4\pi$ and

$$\int_\Omega (1+g(u)) \exp(u^2)\, dx > \bar{l}.$$

But, picking $v_\varepsilon = u\sqrt{1-\varepsilon}$, we have $\|v_\varepsilon\|^2_{H^1_0} \leq 4\pi(1-\varepsilon)$ and

$$\lim_{\varepsilon\to 0} \int_\Omega (1+g(v_\varepsilon)) \exp(v_\varepsilon^2)\, dx = \int_\Omega (1+g(u)) \exp(u^2)\, dx$$

by the dominated convergence theorem, using (1-1), $v_\varepsilon^2 \leq u^2$ and $\exp(u^2) \in L^1(\Omega)$. But this contradicts the definition of $\bar{l}$ and concludes the proof of (2-7). Now, by (2-7) and since $\|u_0\|^2_{H^1_0} \leq 4\pi$, in order to get that u_0 is the extremal for $(I^g_{4\pi}(\Omega))$ we look for, it is sufficient to prove that

$$\lim_{\varepsilon\to 0} \int_\Omega (1+g(u_\varepsilon)) \exp(u_\varepsilon^2)\, dx = \int_\Omega (1+g(u_0)) \exp(u_0^2)\, dx. \tag{2-8}$$

If $u_0 = 0$, then Proposition 2.1 gives a contradiction: either by (2-4) and (2-7) if $\Lambda_g(\Omega) \geq \pi \exp(1+M)$, since it is clear that

$$C_{g,4\pi}(\Omega) > \Lambda_g(\Omega) + (1+g(0))|\Omega|,$$

or by (2-5) and (2-6) if $l > 0$, since $\|u_\varepsilon\|_{H^1_0} \leq 4\pi$. Thus, we necessarily have that $u_0 \neq 0$. Then, noting that

$$\|u_\varepsilon - u_0\|^2_{H^1_0} \leq 4\pi - \|u_0\|^2_{H^1_0} + o(1),$$

the standard Moser–Trudinger inequality $(I^0_{4\pi}(\Omega))$ and Vitali's theorem give that (2-8) still holds true, and part (1) of Theorem 1.2 is proved in any case. □

The following proposition is the core of the argument to get the nonexistence of an extremal in Theorem 1.2(2). Its proof is postponed until Section 4.

Proposition 2.3. *Let Ω be a smooth bounded domain of $\mathbb{R}^2$. Let g be such that* (1-1) *and* (1-5)–(1-6) *hold true for H as in* (1-2), *and let A, B and F be thus given. Assume that $\Lambda_g(\Omega) < \pi \exp(1+M)$, where M is as in* (1-9) *and $\Lambda_g(\Omega)$ as in* (1-11). *Assume that there exists a sequence of positive integers $(N_\varepsilon)_\varepsilon$ such that*

$$\lim_{\varepsilon\to 0} N_\varepsilon = +\infty \tag{2-9}$$

and such that $(I_{4\pi}^{g_{N_\varepsilon}}(\Omega))$ admits a nonnegative extremal u_ε for all $\varepsilon > 0$, where g_{N_ε} is as in (1-10). *Then we have* (2-1) *and $\|u_\varepsilon\|_{H_0^1}^2 = 4\pi$ for all $0 < \varepsilon \ll 1$. Moreover, we have $u_\varepsilon \in C^{1,\theta}(\overline{\Omega})$ $(0 < \theta < 1)$,* (2-3) *and*

$$\|u_\varepsilon\|_{H_0^1}^2 \le 4\pi\big(1 + I(\gamma_\varepsilon) + o(\gamma_\varepsilon^{-4} + |A(\gamma_\varepsilon)| + \gamma_\varepsilon^{-3}|B(\gamma_\varepsilon)|)\big) \tag{2-10}$$

as $\varepsilon \to 0$, where $I(\gamma_\varepsilon)$ is given by (2-6).

Proof of Theorem 1.2(2): nonexistence of an extremal for $(I_{4\pi}^{g_N}(\Omega))$, $N \ge N_0$. Let g be such that (1-1) and 1-5)–(1-6) hold true for H as in (1-2), and let A, B and F be thus given. Assume $l < 0$ and $\Lambda_g(\Omega) < \pi \exp(1+M)$, where l is as in (1-12), Λ_g is as in (1-11) and M is as in (1-9). In order to prove part (2) of Theorem 1.2, we assume by contradiction that there exists a sequence $(N_\varepsilon)_\varepsilon$ of positive integers satisfying (2-9) and such that $(I_{4\pi}^{g_{N_\varepsilon}}(\Omega))$ admits an extremal for g_{N_ε} as in (1-10). We let $(u_\varepsilon)_\varepsilon$ be a sequence of nonnegative functions such that u_ε is a maximizer for $(I_{4\pi}^{g_{N_\varepsilon}}(\Omega))$ for all $\varepsilon > 0$. But this is not possible by Proposition 2.3, since $\|u_\varepsilon\|_{H_0^1}^2 = 4\pi$ contradicts (2-10), since we also assume now $l < 0$. □

3. Blow-up analysis in the strongly perturbed Moser–Trudinger regime

We now aim to prove the main blow-up analysis results that we need to get both Propositions 2.1 and 2.3. The following preliminary lemma deals with the existence of an extremal for the perturbed Moser–Trudinger inequality $(I_\alpha^g(\Omega))$ in the subcritical case $0 < \alpha < 4\pi$. Its proof relies on integration theory combined with $(I_{4\pi}^0(\Omega))$ and on standard variational techniques. It is omitted here and the interested reader may find more details in the proof of Proposition 6 of [Mancini and Martinazzi 2017].

Lemma 3.1. *Let Ω be a smooth bounded domain of $\mathbb{R}^2$. Let g be such that* (1-1) *holds true. Then, $(I_\alpha^g(\Omega))$ admits a nonnegative extremal u_α for all $0 < \alpha < 4\pi$. Moreover, we have that*

(1) *either $\|u_\alpha\|_{H_0^1}^2 < \alpha$ and $u_\alpha H(u_\alpha) = 0$ a.e., or*

(2) *$\|u_\alpha\|_{H_0^1}^2 = \alpha$ and there exists $\lambda \in \mathbb{R}$ such that u_α solves in H_0^1 the Euler–Lagrange equation* (1-4).

Remark 3.2. The first alternative in Lemma 3.1 may occur in general, but does not if $t \mapsto (1+g(t))\exp(t^2)$ increases in $(0, +\infty)$.

The following lemma investigates more precisely the behavior of g and H when we assume (1-1) together with (1-5)–(1-6).

Lemma 3.3. *Let Ω be a smooth bounded domain of $\mathbb{R}^2$. Let g be such that* (1-1), (1-5) *and* (1-6) *hold true for H as in* (1-2), *and let A, B and δ_0, δ_0', F, κ be thus given. Then*:

(3-1a) *We have*

$$\left(1+g\left(\frac{t}{\gamma}\right)\right)\exp\left(\frac{t^2}{\gamma^2}\right)=(1+g(0))+\frac{2B(\gamma)F(t)t}{\gamma(\kappa+1)}+o\left(\frac{|B(\gamma)|}{\gamma}+\frac{1}{\gamma^2}\right)$$

in $C^0_{\mathrm{loc}}((0,+\infty)_t)$ as $\gamma\to+\infty$.

(3-1b) *There exists $C>0$ such that*

$$\left|\left(1+g\left(\frac{t}{\gamma}\right)\right)\exp\left(\frac{t^2}{\gamma^2}\right)-(1+g(0))\right|\le C\left(\frac{|B(\gamma)|}{\gamma}+\frac{1}{\gamma^2}\right)t\exp(\delta_0't)$$

for all $\gamma\gg 1$ and all $0\le t\le 2\gamma$.

(3-1c) $\|g\|_{L^\infty(\mathbb{R})}<+\infty$.

Additionally:

(3-2a) *We have*

$$1+g\left(\gamma-\frac{t}{\gamma}\right)=H(\gamma)\big(1+A(\gamma)\big(t+\tfrac{1}{2}\big)+o(|A(\gamma)|+\gamma^{-4})\big)$$

in $C^0_{\mathrm{loc}}(\mathbb{R}_t)$ as $\gamma\to+\infty$.

(3-2b) *There exists $C>0$ such that*

$$\left|1+g\left(\gamma-\frac{t}{\gamma}\right)-H(\gamma)\right|\le C|H(\gamma)|(|A(\gamma)|+\gamma^{-4})\exp(\delta_0 t)$$

for all $\gamma\gg 1$ and all $0\le t\le 2\gamma$.

In particular, *we have*

$$H(\gamma)\to 1\quad\text{as }\gamma\to+\infty. \tag{3-3}$$

Proof of Lemma 3.3. We first prove (3-3). Using (1-3), we write

$$(1+g(r))\exp(r^2)-(1+g(0))=2\int_0^r sH(s)\exp(s^2)\,ds \tag{3-4}$$

for all $r\ge 0$. Then, as $\gamma\to+\infty$, setting $r=\gamma$, we can write

$$\begin{aligned}1+g(\gamma)&=\exp(-\gamma^2)(1+g(0))+2\int_0^{\gamma^2}\left(1-\frac{u}{\gamma^2}\right)H\left(\gamma-\frac{u}{\gamma}\right)\exp\left(-2u+\frac{u^2}{\gamma^2}\right)du\\&=O(\exp(-\gamma^2))+2H(\gamma)\int_0^{\gamma^2}\left(1-\frac{u}{\gamma^2}\right)\exp\left(-2u+\frac{u^2}{\gamma^2}\right)du\\&\qquad+O\left(|H(\gamma)|(|A(\gamma)|+\gamma^{-4})\int_0^{\gamma^2}\exp(-(1-\delta_0)u)\exp\left(-u\left(1-\frac{u}{\gamma^2}\right)\right)du\right)\\&=O(\exp(-\gamma^2))+H(\gamma)(1-\exp(-\gamma^2))+o(H(\gamma)),\end{aligned}$$

using (1-5). This proves (3-3) since g satisfies (1-1). Observe that (3-1a) and (3-1b) follow from (1-6) and (3-4) with $r = t/\gamma$, while (3-1c) is a straightforward consequence of (1-1). We prove now (3-2b). As $\gamma \to +\infty$, we write for all $0 \le t \le \gamma$

$$\begin{aligned}
&\left(1+g\left(\gamma-\frac{t}{\gamma}\right)\right)\exp\left(\left(\gamma-\frac{t}{\gamma}\right)^2\right)-(1+g(\gamma-1))\exp((\gamma-1)^2)\\
&=2\int_{\gamma-1}^{\gamma-t/\gamma} rH(r)\exp(r^2)\,dr\\
&=2\int_t^\gamma\left(1-\frac{u}{\gamma^2}\right)H\left(\gamma-\frac{u}{\gamma}\right)\exp\left(\gamma^2-2u+\frac{u^2}{\gamma^2}\right)du\\
&=H(\gamma)\left(\exp\left(\left(\gamma-\frac{t}{\gamma}\right)^2\right)-\exp((\gamma-1)^2)\right)+O\left(|H(\gamma)|(|A(\gamma)|+\gamma^{-4})\int_t^\gamma\exp(\gamma^2-(2-\delta_0)u)\,du\right), \quad (3\text{-}5)
\end{aligned}$$

using (1-5b). Multiplying the above identity by $\exp(-(\gamma-(t/\gamma))^2)$, using $t \le \gamma$, (1-1) and (3-3), (3-2b) easily follows. Using now (1-5a) in the second-to-last line of (3-5), we also get (3-2a). ☐

In the sequel, for all integers $N \ge 1$, we let φ_N be given by (see also (3-36) below)

$$\varphi_N(t) = \sum_{k=N+1}^{+\infty} \frac{t^k}{k!}. \tag{3-6}$$

The main results of this section are stated in the following lemma.

Lemma 3.4. *Let Ω be a smooth bounded domain of $\mathbb{R}^2$. Let g be such that* (1-1) *and* (1-5)–(1-6) *hold true for H as in* (1-2), *and let A, B and F be thus given. Let $(\alpha_\varepsilon)_\varepsilon$ be a sequence of numbers in* $(0, 4\pi]$. *Let $(N_\varepsilon)_\varepsilon$ be a sequence of positive integers. Assume*

$$\lim_{\varepsilon\to 0} \alpha_\varepsilon = 4\pi \quad \textit{and} \quad u_\varepsilon \ge 0 \textit{ is an extremal for } (I_{\alpha_\varepsilon}^{g_{N_\varepsilon}}(\Omega)) \tag{3-7}$$

for all $0 < \varepsilon \ll 1$, where g_{N_ε} is as in (1-10). *Assume in addition that we are in one of the following two cases*:

Case 1: $\lim_{\varepsilon\to0} N_\varepsilon = +\infty$, $\alpha_\varepsilon = 4\pi$ *for all* ε, *and* $\Lambda_g(\Omega) < \pi\exp(1+M)$, *where $\Lambda_g(\Omega)$ is as in* (1-11) *and M is as in* (1-9).

Case 2: $N_\varepsilon = 1$ *for all ε and* (2-1) *holds true.*

Then, up to a subsequence,

$$\|u_\varepsilon\|_{H_0^1}^2 = \alpha_\varepsilon, \tag{3-8}$$

and $u_\varepsilon \in C^{1,\theta}(\bar\Omega)$ $(0 < \theta < 1)$ solves

$$\begin{cases} \Delta u_\varepsilon = \lambda_\varepsilon u_\varepsilon H_{N_\varepsilon}(u_\varepsilon)\exp(u_\varepsilon^2), & u_\varepsilon > 0 \ \textit{in } \Omega,\\ u_\varepsilon = 0 \ \textit{on } \partial\Omega, \end{cases} \tag{3-9}$$

where $H_N(t) = 1 + g_N(t) + g_N'(t)/(2t)$. Moreover, by (2-4) *we have*

$$\lambda_\varepsilon = \frac{4+o(1)}{\gamma_\varepsilon^2\exp(1+M)}, \tag{3-10}$$

$$A(\gamma_\varepsilon) - 2\xi_\varepsilon = o(\tilde{\zeta}_\varepsilon), \tag{3-11}$$

$$x_\varepsilon \to \bar{x} \quad (\bar{x} \in K_\Omega) \tag{3-12}$$

as $\varepsilon \to 0$, *where* $x_\varepsilon, \gamma_\varepsilon$ *satisfy*

$$u_\varepsilon(x_\varepsilon) = \max_\Omega u_\varepsilon = \gamma_\varepsilon \to +\infty \tag{3-13}$$

as $\varepsilon \to 0$, *where* ξ_ε *is given by*

$$\xi_\varepsilon = \frac{\gamma_\varepsilon^{2(N_\varepsilon - 1)}}{\varphi_{N_\varepsilon - 1}(\gamma_\varepsilon^2)(N_\varepsilon - 1)!}, \tag{3-14}$$

and where $\tilde{\zeta}_\varepsilon$ *is given by*

$$\tilde{\zeta}_\varepsilon = \max\left(\frac{1}{\gamma_\varepsilon^2}, |A(\gamma_\varepsilon)|, \xi_\varepsilon\right). \tag{3-15}$$

Finally, (3-97)–(3-99) *below hold true for* μ_ε *as in* (3-40) *and* t_ε *as in* (3-41).

Observe that $N_\varepsilon = 1$ in Case 2 reduces to say that $g_{N_\varepsilon} = g$. From (3-30) obtained in the process of the proof below, we get that $\xi_\varepsilon = o(1/\gamma_\varepsilon^2)$ in Case 2, so that (3-11) is then equivalent to

$$A(\gamma_\varepsilon) = o\left(\frac{1}{\gamma_\varepsilon^2}\right), \tag{3-16}$$

as discussed in Remark 2.2.

Proof of Lemma 3.4. We start by several basic steps. First, a test function computation gives the following result.

Step 1. *For all* g *such that* (1-1) *holds true*, *we have*

$$C_{g,4\pi}(\Omega) \geq |\Omega|(1 + g(0)) + \pi \exp(1 + M),$$

where $C_{g,4\pi}(\Omega)$ *is as in* $(I_\alpha^g(\Omega))$ $(\alpha = 4\pi)$ *and where* M *is as in* (1-9).

Proof of Step 1. In order to get Step 1, it is sufficient to prove that there exist functions $f_\varepsilon \in H_0^1$ such that $\|f_\varepsilon\|_{H_0^1}^2 = 4\pi$ and such that

$$\int_\Omega (1 + g(f_\varepsilon)) \exp(f_\varepsilon^2)\, dy \geq |\Omega|(1 + g(0)) + \pi \exp(1 + M) + o(1) \tag{3-17}$$

as $\varepsilon \to 0$. In order to reuse these computations later, we fix any sequence $(z_\varepsilon)_\varepsilon$ of points in Ω such that

$$\frac{\varepsilon^2}{d(z_\varepsilon, \partial\Omega)^2} = o\left(\left(\log \frac{1}{\varepsilon}\right)^{-1}\right). \tag{3-18}$$

For $0 < \varepsilon < 1$, we let v_ε be given by

$$v_\varepsilon(y) = \log \frac{1}{\varepsilon^2 + |y - z_\varepsilon|^2} + \mathcal{H}_{z_\varepsilon, \varepsilon},$$

where $\mathcal{H}_{z_\varepsilon,\varepsilon}$ is harmonic in Ω and such that v_ε is zero on $\partial\Omega$. Then, by the maximum principle and (1-8), we have

$$\mathcal{H}_{z_\varepsilon,\varepsilon}(y) = \mathcal{H}_{z_\varepsilon}(y) + O\left(\frac{\varepsilon^2}{d(z_\varepsilon,\partial\Omega)^2}\right) \quad \text{for all } y \in \Omega, \tag{3-19}$$

where $\mathcal{H}_{z_\varepsilon}$ is as in (1-8). Then, integrating by parts, we compute

$$\begin{aligned}
\|v_\varepsilon\|_{H_0^1}^2 &= \int_\Omega v_\varepsilon \Delta v_\varepsilon \, dy \\
&= \int_\Omega \frac{4}{\varepsilon^2(1+|z_\varepsilon - y|^2/\varepsilon^2)^2}\left(\log\frac{1}{\varepsilon^2} + \log\frac{1}{1+|y-z_\varepsilon|^2/\varepsilon^2} + \mathcal{H}_{z_\varepsilon,\varepsilon}(y)\right) dy \\
&= 4\pi\left(\log\frac{1}{\varepsilon^2} + o(1)\right) - 4\pi(1+o(1)) + 4\pi(\mathcal{H}_{z_\varepsilon}(z_\varepsilon) + o(1)) \\
&= 4\pi\left(\log\frac{1}{\varepsilon^2} - 1 + \mathcal{H}_{z_\varepsilon}(z_\varepsilon)\right) + o(1),
\end{aligned} \tag{3-20}$$

where the change of variable $z = (y - z_\varepsilon)/\varepsilon$, (3-18), (3-19) and

$$\mathcal{H}_{z_\varepsilon}(z_\varepsilon + \varepsilon z) = \mathcal{H}_{z_\varepsilon}(z_\varepsilon) + O\left(\frac{\varepsilon|z|}{d(z_\varepsilon,\partial\Omega)}\right), \tag{3-21}$$

(see for instance Appendix B in [Druet and Thizy 2017]) are used. From now on $\liminf_{\varepsilon\to 0} d(z_\varepsilon,\partial\Omega) > 0$ is assumed. Let f_ε be given by $4\pi v_\varepsilon^2 = f_\varepsilon^2 \|v_\varepsilon\|_{H_0^1}^2$. We can write

$$\begin{aligned}
f_\varepsilon(y)^2 = \left(\left(\log\frac{1}{|z_\varepsilon - y|^2+\varepsilon^2}\right)^2 + 2\mathcal{H}_{z_\varepsilon,\varepsilon}(y)\log\frac{1}{|z_\varepsilon - y|^2+\varepsilon^2} + \mathcal{H}_{z_\varepsilon,\varepsilon}(y)^2\right) \\
\times\left(\log\frac{1}{\varepsilon^2}\left(1+\frac{\mathcal{H}_{z_\varepsilon}(z_\varepsilon)-1}{\log 1/\varepsilon^2} + o\left(\frac{1}{\log 1/\varepsilon}\right)\right)\right)^{-1},
\end{aligned}$$

using (3-20). Then, writing

$$\log\frac{1}{|z_\varepsilon - y|^2+\varepsilon^2} = \log\frac{1}{\varepsilon^2} + \log\frac{1}{1+|z_\varepsilon - y|^2/\varepsilon^2},$$

we get

$$\begin{aligned}
&\int_{B_{z_\varepsilon}(\check r_\varepsilon)\cap\Omega} (1+g(f_\varepsilon))\exp(f_\varepsilon^2)\, dy \\
&\quad= \int_{B_{z_\varepsilon}(\check r_\varepsilon)\cap\Omega} (1+o(1))\frac{\exp(-2\check t_\varepsilon(y) + 2\mathcal{H}_{z_\varepsilon,\varepsilon}(y) - \mathcal{H}_{z_\varepsilon}(z_\varepsilon)+1)}{\varepsilon^2} \\
&\qquad\qquad \times \exp\left(\frac{\check t_\varepsilon^2}{\log 1/\varepsilon^2} + O\left(\frac{1+\check t_\varepsilon}{\log 1/\varepsilon^2} + \frac{1+\check t_\varepsilon^2}{(\log 1/\varepsilon^2)^2}\right)\right) dy \\
&\quad= \pi\exp(\mathcal{H}_{z_\varepsilon}(z_\varepsilon)+1)(1+o(1))
\end{aligned} \tag{3-22}$$

as $\varepsilon \to 0$, using (1-1), (3-19) and (3-21), where $\check t_\varepsilon(y) = \log(1+|z_\varepsilon - y|^2/\varepsilon^2)$ and where $\check r_\varepsilon$ is given by

$$\log\left(1+\frac{\check r_\varepsilon^2}{\varepsilon^2}\right) = \frac{1}{2}\log\frac{1}{\varepsilon^2}.$$

Now, we can check that

$$f_\varepsilon(y)^2 \le \left(\log\frac{1}{\varepsilon^2}+O(1)\right)^{-1}\left(\log\frac{1}{|z_\varepsilon-y|^2}+O(1)\right)^2$$
$$\le \left(\log\frac{1}{|z_\varepsilon-y|^2}+O(1)\right)\left(\tfrac12+o(1)\right) \quad \text{for all } y\in\Omega\backslash B_{z_\varepsilon}(\check r_\varepsilon),$$

using (1-8), (3-19) and our definition of $\check r_\varepsilon$, so that we also get

$$\int_{\Omega\backslash B_{z_\varepsilon}(\check r_\varepsilon)}(1+g(f_\varepsilon))\exp(f_\varepsilon^2)\,dy \to (1+g(0))|\Omega| \tag{3-23}$$

as $\varepsilon\to 0$, by the dominated convergence theorem, using (1-1). Property (3-17) and then Step 1 follow from (3-22) and (3-23), choosing $z_\varepsilon\in K_\Omega$ as in (1-9). □

From now on, we make the assumptions of Lemma 3.4. In particular, we assume that either Case 1 or Case 2 holds true. Given an integer $N\ge 1$, observe that Step 1 applies to g_N, since g_N satisfies (1-1), if g does. Then, using $\alpha_\varepsilon=4\pi$ in Case 1, or (2-7) and $g_{N_\varepsilon}=g$ in Case 2, we get

$$|\Omega|(1+g(0))+\pi\exp(1+M)\le \begin{cases} C_{g_{N_\varepsilon},4\pi} & \text{in Case 1,}\\ C_{g,\alpha_\varepsilon}+o(1) & \text{in Case 2}\end{cases} \tag{3-24}$$

as $\varepsilon\to 0^+$, where $C_{g,\alpha}(\Omega)$ is as in formula $(I_\alpha^g(\Omega))$ and where M is as in (1-9). Let us rewrite now (3-9) in a more convenient way. Let Ψ_N be given by

$$\Psi_N(t)=(1+g_N(t))\exp(t^2). \tag{3-25}$$

Observe in particular that

$$(1+g(t))(1+t^2)\le\Psi_N(t)\le(1+g(t))\exp(t^2)$$

for all t and all N, by (1-1). Using (1-2), (1-3) and (1-10), we may rewrite (3-9) as

$$\begin{cases}\Delta u_\varepsilon=\frac12\lambda_\varepsilon\Psi'_{N_\varepsilon}(u_\varepsilon), & u_\varepsilon>0 \text{ in } \Omega,\\ u_\varepsilon=0 \text{ on } \partial\Omega,\end{cases} \tag{3-26}$$

with

$$\Psi'_N(t)=2tH(t)(1+t^2+\varphi_N(t^2))+2t(1+g(t))\left(\frac{t^{2N}}{N!}-t^2\right)$$
$$=2tH(t)\varphi_N(t^2)+2t\left(1+\frac{t^{2N}}{N!}\right)(1+g(t))+g'(t)(1+t^2). \tag{3-27}$$

Indeed, in (3-9), it turns out that

$$H_N(t)=\frac{\Psi'_N(t)\exp(-t^2)}{2t}. \tag{3-28}$$

Observe that by (1-1) and (3-3), using the first line of (3-27), we clearly have that there exists $C>0$ such that

$$|\Psi'_{N_\varepsilon}(t)|\le Ct\exp(t^2) \tag{3-29}$$

for all $t\ge 0$ and all ε. In Case 2, (2-1) is assumed to be true.

Step 2. *Assume that we are in Case* 1. *Then* (2-1) *holds true. Moreover*, *if* $\gamma_\varepsilon := \operatorname{ess\,sup} u_\varepsilon < +\infty$ *for all* ε, *we have*

$$\liminf_{\varepsilon\to 0} \underbrace{\frac{\varphi_{N_\varepsilon}(\gamma_\varepsilon^2)}{\exp(\gamma_\varepsilon^2)}}_{:=\delta_\varepsilon\in(0,1)} > 0, \tag{3-30}$$

and, *in other words*,

$$\liminf_{\varepsilon\to 0} \frac{\gamma_\varepsilon^2 - N_\varepsilon}{\sqrt{N_\varepsilon}} > -\infty, \tag{3-31}$$

where φ_N *is as in* (3-6).

Proof of Step 2. By (3-7) and (3-24), we get

$$\int_\Omega \Psi_{N_\varepsilon}(u_\varepsilon)\,dy \geq (1+g(0))|\Omega| + \pi\exp(1+M). \tag{3-32}$$

Writing now

$$\Psi_N(t) = (1+g(0)) + \big((1+g(t))(1+t^2) - (1+g(0))\big) + (1+g(t))\varphi_N(t^2)$$

and using (1-1) we also get

$$\int_\Omega \Psi_{N_\varepsilon}(u_\varepsilon)\,dy \leq (1+g(0))|\Omega| + \Lambda_g(\Omega) + \int_\Omega (1+g(u_\varepsilon))\varphi_{N_\varepsilon}(u_\varepsilon^2)\,dy, \tag{3-33}$$

where Λ_g is as in (1-11). Then by (1-1) and Case 1, we get from (3-32) and (3-33) that

$$\liminf_{\varepsilon\to 0} \int_\Omega \varphi_{N_\varepsilon}(u_\varepsilon^2)\,dy > 0. \tag{3-34}$$

Up to a subsequence, $u_\varepsilon \rightharpoonup u_0$ in H_0^1 for some $u_0 \in H_0^1$ such that $\|u_0\|_{H_0^1}^2 \leq 4\pi$. Let $0<\beta\ll 1$ be given. First we have

$$u_\varepsilon^2 \leq (1+\beta)(u_\varepsilon-u_0)^2 + \Big(1+\frac{1}{\beta}\Big)u_0^2.$$

Independently, by the Moser–Trudinger inequality, we have

$$u \in H_0^1 \quad\Longrightarrow\quad \text{for all } p\in[1,+\infty),\ \ \exp(u^2)\in L^p. \tag{3-35}$$

Therefore, if $u_0 \not\equiv 0$ and $\lim_{\varepsilon\to 0}\|u_\varepsilon - u_0\|_{H_0^1}^2 < 4\pi$, then there exists $p_0 > 1$ such that $(\exp(u_\varepsilon^2))_\varepsilon$ is bounded in L^{p_0}, by Moser's and Hölder's inequalities. Then, by Vitali's theorem, since $\varphi_{N_\varepsilon} \leq \exp$ in $[0,+\infty)$ and since $N_\varepsilon \to +\infty$ in Case 1, we get

$$u_0 \not\equiv 0 \quad\Longrightarrow\quad \int_\Omega \varphi_{N_\varepsilon}(u_\varepsilon^2)\,dy = o(1)$$

as $\varepsilon\to 0$, which proves (2-1), in view of (3-34). Noting that the function $t\mapsto \varphi_N(t)\exp(-t)$ increases in $[0,+\infty)$, we can write

$$\int_\Omega \varphi_{N_\varepsilon}(u_\varepsilon^2)\,dy \leq \frac{\varphi_{N_\varepsilon}(\gamma_\varepsilon^2)}{\exp(\gamma_\varepsilon^2)} \int_\Omega \exp(u_\varepsilon^2)\,dy$$

and conclude that (3-30) holds true by (3-34) and Moser's inequality. Observe that

$$\varphi_N(\Gamma) = \exp(\Gamma) \int_0^\Gamma \exp(-s) \frac{s^N}{N!}\, ds. \tag{3-36}$$

Setting $\Gamma = \gamma_\varepsilon^2,\ N = N_\varepsilon$ and $s = N_\varepsilon + u\sqrt{N_\varepsilon}$, we get (3-31) from (3-30), using Stirling's formula and

$$\left(1 + \frac{u}{\sqrt{N}}\right)^N e^{-u\sqrt{N}} \le e^{-u^2/2}$$

for $-\sqrt{N} < u < 0$. □

The next steps applies in both Case 1 and Case 2.

Step 3. *We have that* (3-8), (3-9) *hold true, and that* u_ε *is in* $C^{1,\theta}(\overline{\Omega})$.

Proof of Step 3. Assume by contradiction that (3-8) does not hold true, or in other words that $\|u_\varepsilon\|^2_{H_0^1} < \alpha_\varepsilon$ for all $\varepsilon \ll 1$, up to a subsequence; then it follows from the fact that u_ε is an (unconstrained) critical point of our functional that $\Psi'_\varepsilon(u_\varepsilon) = 0$ a.e. in Ω. The key property is now that the Lebesgue measure of $\{t_0 < u_\varepsilon \le t_1\}$ is positive for all $0 \le t_0 < t_1 \le \gamma_\varepsilon$, as it follows by $\int_\Omega |\nabla T u_\varepsilon|^2 > 0$, where $T u_\varepsilon \in H_0^1$ is the truncation of $u_\varepsilon - t_0$ as 0 when $u_\varepsilon \le t_0$ and as $t_1 - t_0$ when $u_\varepsilon > t_1$; this shows that $\Psi'_{N_\varepsilon} = 0$ in $(0, \gamma_\varepsilon)$ and then

$$(1 + g(t)) = \frac{1 + g(0)}{1 + t^2 + \varphi_{N_\varepsilon}(t^2)} \tag{3-37}$$

for all $t \in [0, \gamma_\varepsilon)$. If $\gamma_\varepsilon = +\infty$, a contradiction arises; then $\gamma_\varepsilon < +\infty$ and one can use Step 2 to show that $\gamma_\varepsilon \to +\infty$, still reaching a contradiction. Then (3-8) is proved, so that (3-9) holds true in H_0^1. Thus for all given ε, u_ε is uniformly bounded and then in $C^{1,\theta}$ by (3-9) and elliptic theory. We also use there that g appearing in the formula (3-27) of Ψ'_N is assumed to be C^1 in (1-1). □

The previous steps give in particular that (3-13) makes sense and holds true.

Step 4. *It holds that* $\lambda_\varepsilon > 0$ *for all* $0 < \varepsilon \ll 1$. *Moreover*

$$\lambda_\varepsilon \to 0 \tag{3-38}$$

as $\varepsilon \to 0$, *where* λ_ε *is as in* (3-9).

Proof of Step 4. By (2-1), we have $u_\varepsilon \to 0$ a.e. and in L^p for all $p < +\infty$. Since $\int_{u_\varepsilon \le M_0} \Psi_{N_\varepsilon}(u_\varepsilon)\, dx \to (1 + g(0))|\Omega|$, by (3-24) one has

$$\liminf_{\varepsilon \to 0} \int_{u_\varepsilon > M_0} \Psi_{N_\varepsilon}(u_\varepsilon)\, dx \ge \pi \exp(1 + M)$$

for all given $M_0 > 0$; one can now use (3-27) with (1-1), (3-3) and some standard integration argument to get

$$\liminf_{\varepsilon \to 0} \int_\Omega [\Psi'_{N_\varepsilon}(u_\varepsilon) + 2(1 + g(u_\varepsilon))u_\varepsilon^3] u_\varepsilon\, dx = +\infty. \tag{3-39}$$

Then, multiplying (3-26) by u_ε and integrating by parts, we get $\lambda_\varepsilon > 0$ and

$$4\pi + o(1) = \int_\Omega |\nabla u_\varepsilon|^2\, dx \gg \lambda_\varepsilon,$$

which proves (3-38). □

Then, using (3-3), we may let $\mu_\varepsilon > 0$ be given by

$$\lambda_\varepsilon H(\gamma_\varepsilon)\mu_\varepsilon^2\gamma_\varepsilon^2\varphi_{N_\varepsilon-1}(\gamma_\varepsilon^2) = 4, \tag{3-40}$$

where φ_N is as in (3-6). Before starting the core of the proof, we would like to make a parenthetical remark.

Remark 3.5. Case 1 is particularly delicate to handle, since the nonlinearities $(\Psi'_{N_\varepsilon})_\varepsilon$ are not of *uniform critical growth*, even in the very general framework of [Druet 2006, Definition 1]. A more intuitive way to see this is the following: if $(\tilde\gamma_\varepsilon)_\varepsilon$ is a sequence of positive real numbers such that $\tilde\gamma_\varepsilon \to +\infty$, but not too fast, in the sense that $\tilde\gamma_\varepsilon^2 \ll N_\varepsilon$, then it can be checked with (1-1) and (3-3) that

$$\tfrac{1}{2}\lambda_\varepsilon \Psi'_{N_\varepsilon}(\tilde\gamma_\varepsilon) = \tilde\lambda_\varepsilon(1+o(1))\tilde\gamma_\varepsilon^{2N_\varepsilon+1}$$

as $\varepsilon \to 0$, where $\tilde\lambda_\varepsilon = \lambda_\varepsilon/(N_\varepsilon!)$. Then, in the regime $0 \le u_\varepsilon \le \tilde\gamma_\varepsilon$, at least formally, (3-26) looks at first order like the Lane–Emden problem, namely

$$\begin{cases} \Delta u_\varepsilon = \tilde\lambda_\varepsilon u_\varepsilon^{2N_\varepsilon+1}, \quad u_\varepsilon > 0 \text{ in } \Omega, \\ u_\varepsilon = 0 \text{ on } \partial\Omega, \\ N_\varepsilon \to +\infty, \end{cases}$$

for which very interesting, but very different concentration phenomena were pointed out; see for instance [Adimurthi and Grossi 2004; De Marchis et al. 2016; 2017; Esposito et al. 2006; Ren and Wei 1994; 1996]. A real difficulty in concluding the subsequent proofs is to extend the analysis developed in [Adimurthi and Druet 2004; Druet 2006; Druet and Thizy 2017] for the Moser–Trudinger "purely critical" regime, in order to deal also with such other intermediate regimes. As a last remark, a much simpler version of the techniques developed here permits us also to answer some open questions about the Lane–Emden problem, as performed in [Thizy 2019].

We let t_ε be given by

$$t_\varepsilon(x) = \log\left(1 + \frac{|x-x_\varepsilon|^2}{\mu_\varepsilon^2}\right). \tag{3-41}$$

Here and in the sequel, for a radially symmetric function f around of x_ε (resp. around 0), we will often write $f(r)$ instead of $f(x)$ for $|x-x_\varepsilon| = r$ (resp. $|x| = r$).

Step 5. *We have*

$$\gamma_\varepsilon(\gamma_\varepsilon - u_\varepsilon(x_\varepsilon + \mu_\varepsilon\,\cdot)) \to T_0 := \log(1+|\cdot|^2) \quad \text{in } C^{1,\theta}_{\text{loc}}(\mathbb{R}^2), \tag{3-42}$$

where γ_ε, x_ε are as in (3-13) *and μ_ε is as in* (3-40). *Moreover, we have*

$$\liminf_{\varepsilon\to 0} \lambda_\varepsilon\gamma_\varepsilon^2 > 0. \tag{3-43}$$

At this stage, by taking the log of (3-40), by estimating λ_ε with (3-38) and (3-43) we get from (3-3) and (3-30) that

$$\log\frac{1}{\mu_\varepsilon^2} = \gamma_\varepsilon^2(1+o(1)) \tag{3-44}$$

as $\varepsilon \to 0$. Observe in particular that (3-44) holds true in Case 1.

Proof of Step 5. We first sketch the proof of (3-42). In Case 2, (3-42) follows closely Step 1 of the proof of [Druet 2006, Proposition 1]. Thus, we focus now on the proof of (3-42) in Case 1. Observe that

$$\sup_{t\in\mathbb{R}} \frac{t^{2N}}{N!}\exp(-t^2) = \frac{N^N}{N!}\exp(-N) \underset{N\to+\infty}{=} \frac{1+o(1)}{\sqrt{2\pi N}} \tag{3-45}$$

by Stirling's formula. Then, by (1-1), (3-3), (3-13), (3-27) and (3-30), we have

$$\begin{aligned}\tfrac{1}{2}\Psi'_{N_\varepsilon}(u_\varepsilon) &= u_\varepsilon H(u_\varepsilon)\varphi_{N_\varepsilon}(u_\varepsilon^2) + u_\varepsilon(1+g(u_\varepsilon))\frac{u_\varepsilon^{2N_\varepsilon}}{N_\varepsilon!} + O(\gamma_\varepsilon^3)\\ &\le (1+o(1))\gamma_\varepsilon\varphi_{N_\varepsilon-1}(\gamma_\varepsilon^2).\end{aligned} \tag{3-46}$$

Observe that, by (3-13) and elliptic theory, we must have $\sup_\Omega \lambda_\varepsilon\Psi'_{N_\varepsilon}(u_\varepsilon)\to+\infty$ as $\varepsilon\to 0$. Then, (3-46) implies $\lambda_\varepsilon\gamma_\varepsilon\varphi_{N_\varepsilon-1}(\gamma_\varepsilon^2)\to+\infty$ and then $\mu_\varepsilon\to 0$ as $\varepsilon\to 0$, by (3-40). Let τ_ε be given in $(\Omega-x_\varepsilon)/\mu_\varepsilon$ by

$$u_\varepsilon(x_\varepsilon+\mu_\varepsilon\,\cdot\,) = \gamma_\varepsilon - \frac{\tau_\varepsilon}{\gamma_\varepsilon}.$$

Then, since $\Delta\tau_\varepsilon = -\mu_\varepsilon^2\gamma_\varepsilon(\Delta u_\varepsilon)(x_\varepsilon+\mu_\varepsilon\,\cdot\,)$, we get from (3-26), (3-40) and (3-46) that there exists $C>0$ such that $|\Delta\tau_\varepsilon|\le C$, while $\tau_\varepsilon\ge 0,\ \tau_\varepsilon(0)=0$. As in [Druet 2006, p. 231], we have $\mu_\varepsilon = o(d(x_\varepsilon,\partial\Omega))$. Then, by standard elliptic theory, there exists τ_0 such that

$$\tau_\varepsilon\to\tau_0 \quad \text{in } C^{1,\theta}_{\text{loc}}(\mathbb{R}^2) \tag{3-47}$$

as $\varepsilon\to 0$. Note that for all Γ, $T>0$ and all N, we have

$$\varphi_N(T) = \varphi_N(\Gamma)\exp(-(\Gamma-T)) - \exp(T)\int_T^\Gamma \exp(-s)\frac{s^N}{N!}\,ds. \tag{3-48}$$

Writing the previous identity for $N=N_\varepsilon-1,\ \Gamma=\gamma_\varepsilon^2$ and $T=u_\varepsilon^2=\gamma_\varepsilon^2-2\tau_\varepsilon+\tau_\varepsilon^2/\gamma_\varepsilon^2$, noting from (3-45) and (3-47) that

$$\int_{u_\varepsilon^2}^{\gamma_\varepsilon^2}\exp(-s)\frac{s^{N_\varepsilon-1}}{(N_\varepsilon-1)!}\,ds = O\left(\frac{1}{\sqrt{N_\varepsilon}}\right)$$

in $\mathbb{R}^2_{\text{loc}}$ and resuming the arguments to get (3-46), we get

$$\Delta(-\tau_0) = 4\exp(-2\tau_0)$$

using also (3-26), (3-30) and (3-40). Now, choosing $R\gg 1$ such that $|g(t)|<1$ and $H(t)>0$ for all $t\ge R$, we easily see that there exists $C_R>0$ such that

$$u_\varepsilon[\Psi'_{N_\varepsilon}(u_\varepsilon)]^- \le C_R|u_\varepsilon| + 4u_\varepsilon^4, \tag{3-49}$$

by (1-1), (3-3) and (3-27), where $t^- = -\min(t,0)$. Then, we have

$$\frac{\lambda_\varepsilon}{2}\int_\Omega u_\varepsilon[\Psi'_{N_\varepsilon}(u_\varepsilon)]^+\,dy = 4\pi + o(1),$$

by (3-8), (3-26), (3-38) and (3-49), where $t^+ = \max(t, 0)$. For all $A \gg 1$, we get

$$4\int_{B_0(A)} \exp(-2\tau_0)\,dy \le \liminf_{\varepsilon\to 0} \frac{\lambda_\varepsilon}{2}\int_\Omega u_\varepsilon[\Psi'_{N_\varepsilon}(u_\varepsilon)]^+\,dy,$$

by (3-47) and, since A is arbitrary, we get then that $\int_{\mathbb{R}^2}\exp(-2\tau_0)\,dy < +\infty$. Thus, by the classification result of [Chen and Li 1991], since $\tau_0 \ge 0$ and $\tau_0(0) = 0$, we get $\tau_0(y) = \log(1+|y|^2)$. Thus (3-42) is proved by (3-47). Similarly, we may also choose some A_ε's such that $A_\varepsilon \to +\infty$ and such that

$$\frac{\lambda_\varepsilon}{2}\int_{B_{x_\varepsilon}(A_\varepsilon\mu_\varepsilon)} \Psi_{N_\varepsilon}(u_\varepsilon)\,dy = \frac{2\pi + o(1)}{\gamma_\varepsilon^2}.$$

We use (3-45) to write

$$\frac{\varphi_{N_\varepsilon}(\gamma_\varepsilon^2)}{\varphi_{N_\varepsilon-1}(\gamma_\varepsilon^2)} = 1 - \frac{\gamma_\varepsilon^{2N_\varepsilon}}{N_\varepsilon!\,\varphi_{N_\varepsilon-1}(\gamma_\varepsilon^2)} = 1 + o(1)$$

as $\varepsilon \to 0$. Thus, since $0 < \Psi_{N_\varepsilon}(t) \le (1+g(t))\exp(t^2)$ for all $t \ge 0$, and since $C_{g,4\pi}(\Omega) < +\infty$, we get (3-43) from (1-1). □

By Step 5 and estimates in its proof, since we assume $\|u_\varepsilon\|^2_{H_0^1} \le 4\pi$, we get that

$$\lim_{R\to+\infty}\lim_{\varepsilon\to 0}\int_{\Omega\setminus B_{x_\varepsilon}(R\mu_\varepsilon)} (\Delta u_\varepsilon(y))^+ u_\varepsilon\,dy = 0. \tag{3-50}$$

We let Ω_ε be given by

$$\Omega_\varepsilon = \begin{cases} \{y\in\Omega : \varphi_{N_\varepsilon-1}(u_\varepsilon(y)^2) \ge u_\varepsilon(y)^2 + 1\} & \text{in Case 1,}\\ \Omega & \text{in Case 2.}\end{cases}$$

Now, despite the difficulty pointed out in Remark 3.5, we are able to get the following weak, but global pointwise estimates.

Step 6. *There exists $C > 0$ such that*

$$|\cdot - x_\varepsilon|^2 |\Delta u_\varepsilon| u_\varepsilon \le C \quad \text{in } \Omega_\varepsilon \tag{3-51}$$

and such that

$$|\cdot - x_\varepsilon|\,|\nabla u_\varepsilon| u_\varepsilon \le C \quad \text{in } \Omega_\varepsilon \tag{3-52}$$

for all ε.

In Case 2, it is not so difficult to adapt the arguments of [Druet 2006, §3,4] to get Step 6. Thus, in the proof of Step 6 just below, we assume that we are in Case 1. Then observe that $\Omega_\varepsilon \ne \varnothing$ by Step 2. Given $\eta_0 \in (0, 1)$, writing

$$\varphi_{N_\varepsilon-1}(tN_\varepsilon) = \frac{t^{N_\varepsilon}N_\varepsilon^{N_\varepsilon}}{N_\varepsilon!}\left(\sum_{k=0}^{+\infty} t^k + o(1)\right) = \frac{(et)^{N_\varepsilon}}{\sqrt{2\pi N_\varepsilon}}\left(\frac{1}{1-t} + o(1)\right),$$

by Stirling's formula, where $o(1) \to 0$ as $\varepsilon \to 0$ uniformly in $|t| \le \eta_0$, the unique positive solution Γ_ε of $\varphi_{N_\varepsilon-1}(\Gamma_\varepsilon) = \Gamma_\varepsilon + 1$ satisfies $\Gamma_\varepsilon = (1+o(1))(N_\varepsilon/e)$. Then, since $\varphi_{N_\varepsilon-1}/(1+\,\cdot\,)$ increases in $(0, +\infty)$, we clearly get

$$(1+o(1))\frac{N_\varepsilon}{e} \le \min_{\Omega_\varepsilon} u_\varepsilon^2. \tag{3-53}$$

Observe also that (3-53) almost characterizes Ω_ε in the following sense: given $\delta > 0$, for all $\varepsilon \ll 1$ so that $(1+\delta)(N_\varepsilon/e) \ge \Gamma_\varepsilon$, one has that $u_\varepsilon(y)^2 \ge (1+\delta)(N_\varepsilon/e)$ implies $y \in \Omega_\varepsilon$.

Proof of Step 6, formula (3-51). As previously mentioned, we still assume that we are in Case 1. Thus, in particular, we assume that $N_\varepsilon \to +\infty$ as $\varepsilon \to 0$. Assume now by contradiction that

$$\max_{y\in\Omega_\varepsilon} |y-x_\varepsilon|^2|\Delta u_\varepsilon(y)|u_\varepsilon(y) = |y_\varepsilon - x_\varepsilon|^2|\Delta u_\varepsilon(y_\varepsilon)|u_\varepsilon(y_\varepsilon) \to +\infty \tag{3-54}$$

as $\varepsilon \to 0$, for some y_ε's such that $y_\varepsilon \in \Omega_\varepsilon$. First for any sequence $(\check{z}_\varepsilon)_\varepsilon$ such that $\check{z}_\varepsilon \in \Omega_\varepsilon$, we have $\Delta u_\varepsilon(\check{z}_\varepsilon) > 0$, $g'(u_\varepsilon(\check{z}_\varepsilon)) = o(u_\varepsilon(\check{z}_\varepsilon))$ and

$$\Psi'_{N_\varepsilon}(u_\varepsilon(\check{z}_\varepsilon)) = (1+o(1))2u_\varepsilon(\check{z}_\varepsilon)\varphi_{N_\varepsilon-1}(u_\varepsilon(\check{z}_\varepsilon)^2) \tag{3-55}$$

as $\varepsilon \to 0$, using (1-1), (3-3), (3-27) and (3-53). Additionally, we have

$$u_\varepsilon(y_\varepsilon) \to +\infty \tag{3-56}$$

as $\varepsilon \to 0$. Let $\nu_\varepsilon > 0$ be given by

$$\nu_\varepsilon^2|\Delta u_\varepsilon(y_\varepsilon)|u_\varepsilon(y_\varepsilon) = 1.$$

Then, using also (3-54), we have

$$\lim_{\varepsilon\to 0} \frac{|y_\varepsilon - x_\varepsilon|}{\nu_\varepsilon} = +\infty, \tag{3-57}$$

and, in view of Step 5,

$$\lim_{\varepsilon\to 0} \frac{|y_\varepsilon - x_\varepsilon|}{\mu_\varepsilon} = +\infty. \tag{3-58}$$

For $R > 0$, we set $\Omega_{R,\varepsilon} = B_{y_\varepsilon}(R\nu_\varepsilon) \cap \Omega$ and $\widetilde{\Omega}_{R,\varepsilon} = (\Omega_{R,\varepsilon} - y_\varepsilon)/\nu_\varepsilon$. Up to harmless rotations and since Ω is smooth, we may assume that there exists $B \in [0, +\infty]$ such that $\widetilde{\Omega}_{R,0} \to (-\infty, B) \times \mathbb{R}$ as $R \to +\infty$, where $\widetilde{\Omega}_{R,\varepsilon} \to \widetilde{\Omega}_{R,0}$ as $\varepsilon \to 0$. In this proof, for $z \in \widetilde{\Omega}_{R,\varepsilon}$, we write $z_\varepsilon = y_\varepsilon + \nu_\varepsilon z \in \Omega_{R,\varepsilon}$. Let $\tilde{u}_\varepsilon$ be given by

$$\tilde{u}_\varepsilon(z) = u_\varepsilon(y_\varepsilon)(u_\varepsilon(z_\varepsilon) - u_\varepsilon(y_\varepsilon)), \tag{3-59}$$

so that we get

$$(\Delta\tilde{u}_\varepsilon)(z) = \frac{(\Delta u_\varepsilon)(z_\varepsilon)}{(\Delta u_\varepsilon)(y_\varepsilon)} = \frac{\Psi'_{N_\varepsilon}(u_\varepsilon(z_\varepsilon))}{\Psi'_{N_\varepsilon}(u_\varepsilon(y_\varepsilon))}. \tag{3-60}$$

First, we prove that for all $R > 0$ there exists $C_R > 0$ such that

$$|\Delta\tilde{u}_\varepsilon| \le C_R \quad \text{in } \widetilde{\Omega}_{R,\varepsilon} \tag{3-61}$$

for all $0 < \varepsilon \ll 1$. Otherwise, by (3-60), assume by contradiction that there exists $z_\varepsilon \in \Omega_{R,\varepsilon}$ such that

$$|\Psi'_{N_\varepsilon}(u_\varepsilon(z_\varepsilon))| \gg \Psi'_{N_\varepsilon}(u_\varepsilon(y_\varepsilon)) \tag{3-62}$$

as $\varepsilon \to 0$. If, still by contradiction, $z_\varepsilon \notin \Omega_\varepsilon$, we have $u_\varepsilon(z_\varepsilon) < u_\varepsilon(y_\varepsilon)$ and

$$\varphi_{N_\varepsilon-1}(u_\varepsilon(z_\varepsilon)^2) < \varphi_{N_\varepsilon-1}(u_\varepsilon(y_\varepsilon)^2),$$

by the definition of Ω_ε and since $\varphi_N/(1+\cdot)$ increases in $[0,+\infty)$, and then

$$|\Psi'_{N_\varepsilon}(u_\varepsilon(z_\varepsilon))| \lesssim u_\varepsilon(z_\varepsilon)(1+u_\varepsilon(z_\varepsilon)^2+\varphi_{N_\varepsilon-1}(u_\varepsilon(z_\varepsilon)^2)) \lesssim \Psi'_{N_\varepsilon}(u_\varepsilon(y_\varepsilon)),$$

using (1-1), (3-3), (3-27), (3-55) and $y_\varepsilon \in \Omega_\varepsilon$ again. This contradicts (3-62) and then it must be the case that $z_\varepsilon \in \Omega_\varepsilon$. Thus, since y_ε is a maximizer on Ω_ε in (3-54), we get from (3-57) and (3-62) that $u_\varepsilon(z_\varepsilon) \ll u_\varepsilon(y_\varepsilon)$. But this is not possible by (3-55) and (3-62), which proves (3-61). Now we prove that, for all $R>0$,

$$\limsup_{\varepsilon\to 0} \sup_{z\in\widetilde{\Omega}_{R,\varepsilon}} \tilde{u}_\varepsilon(z) \le 0. \tag{3-63}$$

Until the end of this proof, we set $\tilde{\gamma}_\varepsilon := u_\varepsilon(y_\varepsilon)$. If (3-63) does not hold true, since $\tilde{u}_\varepsilon(0)=0$ and by continuity, we may assume that there exist $z_\varepsilon \in \Omega_{R,\varepsilon}$ such that

$$\beta_\varepsilon := [\tilde{\gamma}_\varepsilon(u_\varepsilon(z_\varepsilon)-\tilde{\gamma}_\varepsilon)] \to \beta_0 \in (0,+\infty) \tag{3-64}$$

as $\varepsilon \to 0$. Since $u_\varepsilon(z_\varepsilon) > u_\varepsilon(y_\varepsilon)$ for $0<\varepsilon \ll 1$ by (3-64), we have $z_\varepsilon \in \Omega_\varepsilon$. Moreover, since y_ε is maximizing in (3-54), we then get from (3-55), (3-56) and (3-57) that

$$\varphi_{N_\varepsilon-1}(u_\varepsilon(z_\varepsilon)^2) \le (1+o(1))\,\varphi_{N_\varepsilon-1}(\tilde{\gamma}_\varepsilon^2).$$

Independently, since φ_N is convex, we get

$$\begin{aligned}\varphi_{N_\varepsilon-1}(u_\varepsilon(z_\varepsilon)^2) &\ge \varphi_{N_\varepsilon-1}(\tilde{\gamma}_\varepsilon^2)+\varphi'_{N_\varepsilon-1}(\tilde{\gamma}_\varepsilon^2)(u_\varepsilon(z_\varepsilon)^2-\tilde{\gamma}_\varepsilon^2)\\ &\ge (1+2\beta_0(1+o(1)))\varphi_{N_\varepsilon-1}(\tilde{\gamma}_\varepsilon^2),\end{aligned} \tag{3-65}$$

using (3-64) and $\varphi'_N(t) \ge \varphi_N(t)$ for $t \ge 0$. But (3-64)–(3-65) cannot hold true simultaneously, which proves (3-63). As in [Druet 2006, p. 231], $\tilde{u}_\varepsilon(0)=0$, $u_\varepsilon=0$ on $\partial\Omega$, (3-61) and (3-63) imply

$$\lim_{\varepsilon\to 0}\frac{d(y_\varepsilon,\partial\Omega)}{\nu_\varepsilon}=+\infty. \tag{3-66}$$

Moreover, by standard elliptic theory, $\tilde{u}_\varepsilon(0)=0$, (3-61), (3-63) and (3-66) give

$$\tilde{u}_\varepsilon \to u_0 \quad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^2) \tag{3-67}$$

as $\varepsilon \to 0$, for some $u_0 \in C^1(\mathbb{R}^2)$. Given $R>0$, we prove now

$$\liminf_{\varepsilon\to 0} \inf_{z\in\widetilde{\Omega}_{R,\varepsilon}} (\Delta\tilde{u}_\varepsilon)(z) > 0. \tag{3-68}$$

Using (3-27), (3-56) and (3-67), we have

$$\Psi'_{N_\varepsilon}(u_\varepsilon) = 2\tilde{\gamma}_\varepsilon\varphi_{N_\varepsilon-1}(u_\varepsilon^2)(1+o(1))+o(\tilde{\gamma}_\varepsilon^3),$$

uniformly in $\Omega_{R,\varepsilon}$. Then, coming back to (3-60), using (3-55) and $y_\varepsilon \in \Omega_\varepsilon$, we get

$$(\Delta \tilde{u}_\varepsilon)(z) = (1+o(1)) \frac{\varphi_{N_\varepsilon - 1}(u_\varepsilon(z_\varepsilon)^2)}{\varphi_{N_\varepsilon - 1}(\tilde{\gamma}_\varepsilon^2)} + o(1),$$

uniformly in $z \in \widetilde{\Omega}_{R,\varepsilon}$. Now, we write (3-48) with $\Gamma = \tilde{\gamma}_\varepsilon^2$ and $T = u_\varepsilon^2$, where u_ε stands for $u_\varepsilon(z_\varepsilon)$ here and below. Then, in order to conclude the proof of (3-68), using also (3-36), it is sufficient to check that there exists $\eta_R < 1$ such that

$$I_\varepsilon := \frac{\exp(u_\varepsilon^2)}{\varphi_{\widetilde{N}_\varepsilon}(\tilde{\gamma}_\varepsilon^2)\exp(-(\tilde{\gamma}_\varepsilon^2 - u_\varepsilon^2))} \int_{u_\varepsilon^2}^{\tilde{\gamma}_\varepsilon^2} \exp(-s)\frac{s^{\widetilde{N}_\varepsilon}}{\widetilde{N}_\varepsilon !}\, ds = \frac{\int_{u_\varepsilon^2}^{\tilde{\gamma}_\varepsilon^2} \exp(-s)(s^{\widetilde{N}_\varepsilon}/\widetilde{N}_\varepsilon !)\, ds}{\int_0^{\tilde{\gamma}_\varepsilon^2} \exp(-s)(s^{\widetilde{N}_\varepsilon}/\widetilde{N}_\varepsilon !)\, ds} \le \eta_R \tag{3-69}$$

for all $0 < \varepsilon \ll 1$, uniformly in $\Omega_{R,\varepsilon}$, where $\widetilde{N}_\varepsilon = N_\varepsilon - 1$. If $u_\varepsilon \ge \tilde{\gamma}_\varepsilon$, the last inequality in (3-69) is obvious. If now $u_\varepsilon < \tilde{\gamma}_\varepsilon$, we write

$$I_\varepsilon = \frac{\int_{u_\varepsilon^2 - \tilde{\gamma}_\varepsilon^2}^0 \exp(-t)(1+t/\tilde{\gamma}_\varepsilon^2)^{\widetilde{N}_\varepsilon}\, dt}{\int_{-\tilde{\gamma}_\varepsilon^2}^0 \exp(-t)(1+t/\tilde{\gamma}_\varepsilon^2)^{\widetilde{N}_\varepsilon}\, dt} \le \frac{\int_{u_\varepsilon^2 - \tilde{\gamma}_\varepsilon^2}^0 \exp(t(\widetilde{N}_\varepsilon/\tilde{\gamma}_\varepsilon^2 - 1) + O(\widetilde{N}_\varepsilon t^2/\tilde{\gamma}_\varepsilon^4))\, dt}{\int_{2(u_\varepsilon^2 - \tilde{\gamma}_\varepsilon^2)}^0 \exp(t(\widetilde{N}_\varepsilon/\tilde{\gamma}_\varepsilon^2 - 1) + O(\widetilde{N}_\varepsilon t^2/\tilde{\gamma}_\varepsilon^4))\, dt} \le \eta_R$$

using (3-67), where I_ε is as in (3-69). We get the last inequality using (3-53) and $y_\varepsilon \in \Omega_\varepsilon$: (3-69) and then (3-68) are proved in any case. Let $R > 0$ be given. By (3-57), (3-58) and (3-68), we clearly get

$$\int_{\Omega \setminus B_{x_\varepsilon}(R\mu_\varepsilon)} (\Delta u_\varepsilon(y))^+ u_\varepsilon\, dy \ge \int_{B_{y_\varepsilon}(\nu_\varepsilon)} \Delta u_\varepsilon(y)\, u_\varepsilon(y)\, dy$$

for all ε small enough. Using now (3-56) and (3-67), we write that $u_\varepsilon = \tilde{\gamma}_\varepsilon(1+o(1))$ uniformly in $B_{y_\varepsilon}(\nu_\varepsilon)$, so that

$$\liminf_{\varepsilon \to 0} \int_{B_{y_\varepsilon}(\nu_\varepsilon)} \Delta u_\varepsilon(y)\, u_\varepsilon(y)\, dy = \liminf_{\varepsilon \to 0} \int_{B_0(1)} \Delta \tilde{u}_\varepsilon(z)(1+o(1))\, dz > 0,$$

by (3-68). Since this last term is independent of $R > 0$, this contradicts (3-50), which concludes the proof of (3-51). □

Proof of Step 6, formula (3-52). Remember that we assume that Case 1 holds true. Assume then by contradiction that there exists $(y_\varepsilon)_\varepsilon$ such that $y_\varepsilon \in \Omega_\varepsilon$ and

$$\max_{y \in \Omega_\varepsilon} |y - x_\varepsilon| |\nabla u_\varepsilon(y)| u_\varepsilon(y) = |y_\varepsilon - x_\varepsilon| |\nabla u_\varepsilon(y_\varepsilon)| u_\varepsilon(y_\varepsilon) := C_\varepsilon \to +\infty \tag{3-70}$$

as $\varepsilon \to 0$. Then, by (3-53), (3-56) holds true. Let $\nu_\varepsilon > 0$ be given by

$$\nu_\varepsilon = \min(|x_\varepsilon - y_\varepsilon|, d(y_\varepsilon, \partial\Omega)). \tag{3-71}$$

For all $R > 1$ and all ε, we let $\Omega_{R,\varepsilon}$ and $\widetilde{\Omega}_{R,\varepsilon}$ be given by the formulas above (3-59). Let w_ε be given by

$$w_\varepsilon(z) = u_\varepsilon(y_\varepsilon + \nu_\varepsilon z).$$

Since $\|u_\varepsilon\|_{H_0^1}^2 \le 4\pi$, we get from Moser's inequality that $\int_\Omega \exp(u_\varepsilon^2)\, dy = O(1)$ and then that, for all given $p \ge 1$,

$$\|\nu_\varepsilon^{2/p} w_\varepsilon\|_{L^p(\widetilde{\Omega}_{R,\varepsilon})} = O(1) \tag{3-72}$$

for all ε. Set $\tilde{x}_\varepsilon = (x_\varepsilon - y_\varepsilon)/\nu_\varepsilon$. Now, for any given $R > 1$ and any sequence $(z_\varepsilon)_\varepsilon$ such that $z_\varepsilon \in \Omega_{R,\varepsilon}\backslash\{x_\varepsilon\}$ (i.e., $\tilde{z}_\varepsilon := (z_\varepsilon - y_\varepsilon)/\nu_\varepsilon \in \widetilde{\Omega}_{R,\varepsilon}\backslash\{\tilde{x}_\varepsilon\}$), we get

$$|\Delta w_\varepsilon(\tilde{z}_\varepsilon)| = \nu_\varepsilon^2 |\Delta u_\varepsilon(z_\varepsilon)| \lesssim \begin{cases} 1/(u_\varepsilon(z_\varepsilon)|\tilde{z}_\varepsilon - \tilde{x}_\varepsilon|^2) & \text{if } z_\varepsilon \in \Omega_\varepsilon, \\ \lambda_\varepsilon \nu_\varepsilon^2 |\Psi'_{N_\varepsilon}(u_\varepsilon(z_\varepsilon))| = O(\lambda_\varepsilon \nu_\varepsilon^2 (1 + u_\varepsilon(z_\varepsilon)^3)) & \text{if } z_\varepsilon \notin \Omega_\varepsilon, \end{cases}$$

using (3-51) for the first line, and (3-27) for the second one. Then, using either (3-53) or (3-38) with (3-72), we get

$$\|\Delta w_\varepsilon\|_{L^p(\widetilde{\Omega}_{R,\varepsilon}\backslash B_{\tilde{x}_\varepsilon}(1/R))} \to 0 \tag{3-73}$$

as $\varepsilon \to 0$. Independently, since $\|u_\varepsilon\|_{H_0^1} = O(1)$, we easy get

$$\int_{\widetilde{\Omega}_{R,\varepsilon}} |\nabla w_\varepsilon|^2 \, dz = O(1). \tag{3-74}$$

Observe that $|\tilde{x}_\varepsilon| \geq 1$. Now, we claim that up to a subsequence,

$$\nu_\varepsilon \to 0 \quad \text{and} \quad \frac{d(y_\varepsilon, \partial\Omega)}{|x_\varepsilon - y_\varepsilon|} \to +\infty \tag{3-75}$$

as $\varepsilon \to 0$. In particular, by (3-71), this implies $\nu_\varepsilon = |x_\varepsilon - y_\varepsilon|$. Now we prove (3-75). Indeed, if we assume by contradiction that (3-75) does not hold, for all $R \gg 1$ sufficiently large, we get that the $(w_\varepsilon/u_\varepsilon(y_\varepsilon))$'s converge locally out of $B_{\tilde{x}_\varepsilon}\left(\frac{1}{2}\right)$ to some C^1 function which is 1 at 0 and 0 on the nonempty and smooth boundary of $\lim_{R\to+\infty} \lim_{\varepsilon\to 0} \widetilde{\Omega}_{R,\varepsilon}$ (maybe after a harmless rotation). We use here the Harnack inequality and elliptic theory with (3-56), (3-73) (with $p > 2$) and (3-74), since $u_\varepsilon = 0$ in $\partial\Omega$. This clearly contradicts (3-74) and (3-75) is proved. Up to a subsequence, we may now assume

$$\tilde{x}_\varepsilon \to \tilde{x}, \quad |\tilde{x}| = 1, \tag{3-76}$$

as $\varepsilon \to 0$. By (3-56), (3-73), (3-74), and similar arguments including again Harnack's principle, we get

$$\frac{w_\varepsilon}{u_\varepsilon(y_\varepsilon)} \to 1 \quad \text{in } C^1_{\text{loc}}(\mathbb{R}^2\backslash\{\tilde{x}\}), \tag{3-77}$$

using also (3-75). By (3-72) and (3-77), we get that for all $p \geq 1$

$$\nu_\varepsilon^{2/p} u_\varepsilon(y_\varepsilon) = O(1) \tag{3-78}$$

as $\varepsilon \to 0$. Let now $\tilde{w}_\varepsilon$ be given by

$$\tilde{w}_\varepsilon = \frac{w_\varepsilon - w_\varepsilon(0)}{\nu_\varepsilon |\nabla u_\varepsilon(y_\varepsilon)|},$$

so that $|\nabla \tilde{w}_\varepsilon(0)| = 1$. For any given $R > 1$ and any sequence $(z_\varepsilon)_\varepsilon$ such that $\tilde{z}_\varepsilon := (z_\varepsilon - y_\varepsilon)/\nu_\varepsilon \in \widetilde{\Omega}_{R,\varepsilon}\backslash B_{\tilde{x}}(1/R)$, we get

$$|\Delta \tilde{w}_\varepsilon(\tilde{z}_\varepsilon)| = \frac{u_\varepsilon(y_\varepsilon)}{C_\varepsilon} |\Delta w_\varepsilon(\tilde{z}_\varepsilon)| \lesssim \begin{cases} 1/(C_\varepsilon |\tilde{z}_\varepsilon - \tilde{x}_\varepsilon|^2) & \text{if } z_\varepsilon \in \Omega_\varepsilon, \\ (\lambda_\varepsilon / C_\varepsilon) \nu_\varepsilon^2 u_\varepsilon(y_\varepsilon)^4 & \text{if } z_\varepsilon \notin \Omega_\varepsilon \end{cases}$$

for all ε, using (3-51), (3-70) and (3-77). Then, since $\lambda_\varepsilon = o(1)$, we get from (3-70), (3-75) and (3-78) (with $p \geq 4$) that

$$\Delta \tilde{w}_\varepsilon \to 0 \quad \text{in } L^\infty_{\text{loc}}(\mathbb{R}^2 \backslash \{\tilde{x}\}) \tag{3-79}$$

as $\varepsilon \to 0$. By (3-70), (3-76) and (3-77), given $R > 1$ and $\tilde{z}_\varepsilon \in \widetilde{\Omega}_{R,\varepsilon} \backslash B_{\tilde{x}}(1/R)$, we get

$$|\nabla \tilde{w}_\varepsilon(\tilde{z}_\varepsilon)| = \frac{|\nabla u_\varepsilon(z_\varepsilon)|}{|\nabla u_\varepsilon(y_\varepsilon)|} \leq \frac{u_\varepsilon(y_\varepsilon)}{u_\varepsilon(z_\varepsilon)} \frac{1}{|\tilde{x}_\varepsilon - \tilde{z}_\varepsilon|} \leq \frac{1+o(1)}{|\tilde{x}_\varepsilon - \tilde{z}_\varepsilon|} \tag{3-80}$$

for all $0 < \varepsilon \ll 1$. Then, by (3-79), (3-80) and since $\tilde{w}_\varepsilon(0) = 0$, there exists a harmonic function $\mathcal{H}$ in $\mathbb{R}^2 \backslash \{\tilde{x}\}$ such that $\lim_{\varepsilon \to 0} \tilde{w}_\varepsilon = \mathcal{H}$ in $C^1_{\text{loc}}(\mathbb{R}^2 \backslash \{\tilde{x}\})$. Now, for all given $\beta > 0$, integrating by parts, we get

$$\int_{\partial B_{x_\varepsilon}(\beta \nu_\varepsilon)} u_\varepsilon \partial_\nu u_\varepsilon \, d\sigma = C_\varepsilon \left(\int_{\partial B_{\tilde{x}}(\beta)} \partial_\nu \mathcal{H} d\sigma + o(1) \right) \leq \int_\Omega |\nabla u_\varepsilon|^2 \, dy + \int_\Omega u_\varepsilon (\Delta u_\varepsilon)^+ \, dy = O(1),$$

using (3-70) and (3-77), as $\varepsilon \to 0$. Since $C_\varepsilon \to +\infty$, this implies $\int_{\partial B_{\tilde{x}}(\beta)} \partial_\nu \mathcal{H} \, d\sigma = 0$. Then, also by (3-80), β being arbitrary, $\mathcal{H}$ is bounded around $\tilde{x}$ and then the singularity at $\tilde{x}$ is removable. By the Liouville theorem, $\mathcal{H}$ is constant in $\mathbb{R}^2$, which is not possible since $|\nabla \tilde{w}_\varepsilon(0)| = |\nabla \mathcal{H}(0)| = 1$. This concludes the proof of (3-52). ☐

Remark 3.6. We do not assume that the continuous function Ψ'_{N_ε} is positive and increasing in $[0, +\infty)$. Then, standard moving plane techniques [Adimurthi and Druet 2004; Gidas et al. 1979; Han 1991; de Figueiredo et al. 1982] do not apply. We use in the proof below the variational characterization (3-7) of the u_ε's to get that $\bar{x} \in K_\Omega$, K_Ω as in (1-9), and that, in particular, $\bar{x} \notin \partial\Omega$ in (3-12).

Let B_ε be the radial solution around x_ε of

$$\begin{cases} \Delta B_\varepsilon = \frac{1}{2} \lambda_\varepsilon \Psi'_{N_\varepsilon}(B_\varepsilon), \\ B_\varepsilon(x_\varepsilon) = \gamma_\varepsilon, \end{cases} \tag{3-81}$$

where γ_ε is still given by (3-13). Let $\bar{u}_\varepsilon$ be given by

$$\bar{u}_\varepsilon(z) = \frac{1}{2\pi |x_\varepsilon - z|} \int_{\partial B_{x_\varepsilon}(|x_\varepsilon - z|)} u_\varepsilon \, d\sigma \tag{3-82}$$

for all $z \neq x_\varepsilon$ and $\bar{u}_\varepsilon(x_\varepsilon) = u_\varepsilon(x_\varepsilon) = \gamma_\varepsilon$. Let $\varepsilon_0 \in (\sqrt{1/e}, 1)$ be given. Let $\rho_\varepsilon > 0$ be given by

$$t_\varepsilon(\rho_\varepsilon) = (1 - \varepsilon_0) \gamma_\varepsilon^2. \tag{3-83}$$

By (3-44), we have

$$\rho_\varepsilon^2 = \exp(-(\varepsilon_0 + o(1)) \gamma_\varepsilon^2). \tag{3-84}$$

Let r_ε be given by

$$r_\varepsilon = \sup\{r \in (0, \rho_\varepsilon] : |\bar{u}_\varepsilon - B_\varepsilon| \leq 1/\gamma_\varepsilon \text{ in } B_{x_\varepsilon}(r)\}. \tag{3-85}$$

Observe that $r_\varepsilon \gg \mu_\varepsilon$ by Step 5 and the Appendix. Then, we state the following key result.

Step 7. *We have*

$$\bar{u}_\varepsilon(r_\varepsilon) = B_\varepsilon(r_\varepsilon) + o\left(\frac{1}{\gamma_\varepsilon}\right) \tag{3-86}$$

and then $r_\varepsilon = \rho_\varepsilon$ *for all* $0 < \varepsilon \ll 1$. *Moreover, there exists* $C > 0$ *such that*

$$|\nabla(B_\varepsilon - u_\varepsilon)| \le \frac{C}{\rho_\varepsilon \gamma_\varepsilon} \quad \text{in } B_{x_\varepsilon}(\rho_\varepsilon) \tag{3-87}$$

for all $0 < \varepsilon \ll 1$, *where* $(x_\varepsilon)_\varepsilon$ *is as in* (3-13), B_ε *is as in* (3-81), $\bar{u}_\varepsilon$ *is as in* (3-82), ρ_ε *is as in* (3-83) *and* r_ε *is as in* (3-85).

Since $B_\varepsilon(x_\varepsilon) = u_\varepsilon(x_\varepsilon) = \gamma_\varepsilon$, (3-87) obviously implies

$$|B_\varepsilon - u_\varepsilon| \le C \frac{|\cdot - x_\varepsilon|}{\rho_\varepsilon \gamma_\varepsilon} \quad \text{in } B_{x_\varepsilon}(\rho_\varepsilon) \tag{3-88}$$

for all $0 < \varepsilon \ll 1$. Then, combined with the Appendix, Step 7 provides pointwise estimates of the u_ε's in $B_{x_\varepsilon}(\rho_\varepsilon)$.

Proof of Step 7. The proof of Step 7 follows the lines of [Druet and Thizy 2017, Section 3]. We only recall here the argument in the more delicate Case 1. Let v_ε be given by

$$u_\varepsilon = B_\varepsilon + v_\varepsilon. \tag{3-89}$$

By the Appendix, we have that B_ε is well-defined, radially decreasing in $B_{x_\varepsilon}(\rho_\varepsilon)$, and

$$B_\varepsilon = \gamma_\varepsilon - \frac{t_\varepsilon}{\gamma_\varepsilon} + o\left(\frac{t_\varepsilon}{\gamma_\varepsilon}\right) \tag{3-90}$$

uniformly in $B_{x_\varepsilon}(\rho_\varepsilon)$ as $\varepsilon \to 0$, where t_ε is given by (3-41). Then, we get first from (3-83) and (3-90) the lower bound

$$\min_{B_{x_\varepsilon}(r_\varepsilon)} B_\varepsilon \ge \gamma_\varepsilon(\varepsilon_0 + o(1)).$$

Let us introduce now an intermediate radius $\tilde{r}_\varepsilon$ given by

$$\tilde{r}_\varepsilon = \sup\left\{r \in (0, r_\varepsilon] : \tfrac{1}{2}\varepsilon_0 \gamma_\varepsilon |x_\varepsilon - \cdot||\nabla u_\varepsilon| \le C \text{ in } B_{x_\varepsilon}(r)\right\}$$

for C as in (3-52). We prove now that $\tilde{r}_\varepsilon = r_\varepsilon$ for all $\varepsilon \ll 1$. Indeed, by Wirtinger's inequality on $\partial B_0(r)$, $0 < r \le \tilde{r}_\varepsilon$, we have

$$|\bar{u}_\varepsilon - u_\varepsilon| \le \frac{2C}{\varepsilon_0 \gamma_\varepsilon} \pi,$$

so that, by (3-85),

$$|v_\varepsilon| = |B_\varepsilon - u_\varepsilon| \le \left(\frac{2\pi C}{\varepsilon_0} + 1\right) \gamma_\varepsilon^{-1}$$

in $B_{x_\varepsilon}(\tilde{r}_\varepsilon)$. Then, we get a lower bound on u_ε as well, namely

$$\min_{B_{x_\varepsilon}(\tilde{r}_\varepsilon)} u_\varepsilon \ge \gamma_\varepsilon(\varepsilon_0 + o(1)), \tag{3-91}$$

so that, by (3-52), the condition in the definition of $\tilde{r}_\varepsilon$ never saturates: $\tilde{r}_\varepsilon = r_\varepsilon$ for all $\varepsilon \ll 1$. Observe for this that (3-91) combined with (3-53) (see also the paragraph below (3-53)) and with our assumption

$e\varepsilon_0^2 > 1$ implies $B_{x_\varepsilon}(\tilde r_\varepsilon) \subset \Omega_\varepsilon$. Observe in particular that (3-31) provides $\gamma_\varepsilon^2 \geq N_\varepsilon(1+o(1))$. Summarizing what we have just obtained in $B_{x_\varepsilon}(r_\varepsilon)$, we may write

$$\| |x_\varepsilon - \cdot| |\nabla u_\varepsilon| \|_{L^\infty(B_{x_\varepsilon}(r_\varepsilon))} = O\left(\frac{1}{\gamma_\varepsilon}\right),$$

and

$$\|v_\varepsilon\|_{L^\infty(B_{x_\varepsilon}(r_\varepsilon))} = O\left(\frac{1}{\gamma_\varepsilon}\right). \tag{3-92}$$

We also have

$$B_\varepsilon \leq \gamma_\varepsilon \tag{3-93}$$

in $B_{x_\varepsilon}(r_\varepsilon)$. By combining (3-26) and (3-81), (3-92) allows us to linearize (3-81) to control v_ε. More precisely, (1-5) and Lemma 3.3 permit us to compute the variations of Ψ'_{N_ε} in (3-27), even if g is only C^1 in (1-1), so that Ψ'_{N_ε} is only continuous. Namely, we get from (1-5a) and (1-5c) and from Lemma 3.3 (for $\gamma = B_\varepsilon$) that

$$|\Delta v_\varepsilon| = |\Delta(u_\varepsilon - B_\varepsilon)| \leq C'\lambda_\varepsilon \gamma_\varepsilon^2 \varphi_{N_\varepsilon - 2}(B_\varepsilon^2)\left[|v_\varepsilon| + o\left(\frac{1}{\gamma_\varepsilon}\right)\right] \quad \text{in } B_{x_\varepsilon}(r_\varepsilon)$$

for all ε, using (3-48), (3-91)–(3-93) and some computations. Then, (3-90) gives

$$|\Delta v_\varepsilon| \leq C'' \frac{\exp(-2t_\varepsilon(1+o(1)) + t_\varepsilon^2/\gamma_\varepsilon^2)}{\mu_\varepsilon^2}\left[|v_\varepsilon| + o\left(\frac{1}{\gamma_\varepsilon}\right)\right] \quad \text{in } B_{x_\varepsilon}(r_\varepsilon) \tag{3-94}$$

using (3-30), (3-40) and (3-45). Starting now from (3-92)–(3-94), we can compute and argue as in [Druet and Thizy 2017, Section 3] in order to get (3-86)–(3-87). □

Conclusion of the proof of Lemma 3.4. Let $\varepsilon_0' \in (\varepsilon_0, 1)$ be fixed and let $\rho_\varepsilon' > 0$ be given by

$$t_\varepsilon(\rho_\varepsilon') = (1 - \varepsilon_0')\gamma_\varepsilon^2, \tag{3-95}$$

so that, by (3-44),

$$(\rho_\varepsilon')^2 = \exp(-\varepsilon_0'(1+o(1))\gamma_\varepsilon^2). \tag{3-96}$$

In order to conclude the proof of Lemma 3.4, by Steps 1–7, it remains to prove (2-4), (3-10)–(3-12), that

$$\left| u_\varepsilon(y) - \frac{4\pi G_{x_\varepsilon}(y)}{\gamma_\varepsilon} \right| = o\left(\frac{G_{x_\varepsilon}(y)}{\gamma_\varepsilon}\right) \tag{3-97}$$

uniformly in $B_{x_\varepsilon}(\rho_\varepsilon')^c$, that

$$u_\varepsilon = \gamma_\varepsilon - \frac{t_\varepsilon}{\gamma_\varepsilon} + \frac{S_{0,\varepsilon}}{\gamma_\varepsilon^3} + \frac{S_{1,\varepsilon}}{\gamma_\varepsilon^5} + (A(\gamma_\varepsilon) - 2\xi_\varepsilon)\frac{S_{2,\varepsilon}}{\gamma_\varepsilon} + o\left(\bar t_\varepsilon \frac{\zeta_\varepsilon}{\gamma_\varepsilon}\right) \tag{3-98}$$

uniformly in $B_{x_\varepsilon}(\rho_\varepsilon')$, where the $S_{i,\varepsilon}$'s are as in (A-5), and that

$$u_\varepsilon(y) = G_{x_\varepsilon}(y)\left(\frac{4\pi}{\gamma_\varepsilon} + \sum_{i=0}^{1} \frac{A_i}{\gamma_\varepsilon^{3+2i}} + \frac{A_2(A(\gamma_\varepsilon) - 2\xi_\varepsilon)}{\gamma_\varepsilon}\right) + \frac{4B(\gamma_\varepsilon)}{\gamma_\varepsilon^2 \exp(1 + \mathcal{H}_{x_\varepsilon}(x_\varepsilon))}\int_\Omega G_y(x) F(4\pi G_{x_\varepsilon}(x))\,dx + o\left(\frac{\zeta_\varepsilon}{\gamma_\varepsilon} G_{x_\varepsilon}(y) + \frac{|B(\gamma_\varepsilon)|}{\gamma_\varepsilon^2}\right), \tag{3-99}$$

uniformly in $B_{x_\varepsilon}(\rho'_\varepsilon)^c$, as $\varepsilon \to 0$, where F and $B(\gamma_\varepsilon)$ are given in (1-6), where the A_i's are as in (A-3), and where ζ_ε is given in (A-8).

(1) In this first point, we aim to get pointwise estimates of the u_ε's out of $B_{x_\varepsilon}(\rho'_\varepsilon)$. Let G be the Green's function in (1-8). It is known that (see for instance [Druet and Thizy 2017, Appendix B]) there exists $C > 0$ such that

$$|\nabla_y G_x(y)| \le \frac{C}{|x-y|} \quad \text{and} \quad 0 < G_x(y) \le \frac{1}{2\pi} \log \frac{C}{|x-y|} \tag{3-100}$$

for all $x, y \in \Omega$, $x \neq y$. By (3-87) and since $\|u_\varepsilon\|^2_{H^1_0} \le 4\pi$, it is possible to prove (see for instance the proof of [Druet and Thizy 2017, Claim 4.6]) that, given $p < 1/\varepsilon'_0$,

$$\| \exp(u_\varepsilon^2)\|_{L^p(B_{x_\varepsilon}(\rho'_\varepsilon/2)^c)} = O(1) \tag{3-101}$$

for all ε, where $B_{x_\varepsilon}(\rho'_\varepsilon/2)^c = \Omega \backslash B_{x_\varepsilon}(\rho'_\varepsilon/2)$. In the sequel, $p' > 1$ is chosen such that

$$\frac{1}{p} + \frac{1}{p'} < 1.$$

Let now $(z_\varepsilon)_\varepsilon$ be any sequence of points in $B_{x_\varepsilon}(\rho'_\varepsilon)^c$. By the Green's representation formula and (3-26), we can write

$$u_\varepsilon(z_\varepsilon) = \frac{\lambda_\varepsilon}{2} \int_\Omega G_{z_\varepsilon}(y) \Psi'_{N_\varepsilon}(u_\varepsilon(y))\, dy.$$

By (3-100), we have that there exists $C > 0$ such that

$$|G_{z_\varepsilon}(x_\varepsilon) - G_{z_\varepsilon}| \le C \frac{|x_\varepsilon - \cdot\,|}{\rho'_\varepsilon} \tag{3-102}$$

in $B_{x_\varepsilon}(\rho'_\varepsilon/2)$ for all ε. Set $\bar{t}_\varepsilon = 1 + t_\varepsilon$. By (3-44) and (3-84), we have

$$\frac{|\cdot - x_\varepsilon|}{\gamma_\varepsilon \rho_\varepsilon} = o\left(\frac{\bar{t}_\varepsilon}{\gamma_\varepsilon^5}\right) \quad \text{in } \widetilde{\Omega}_\varepsilon := \{y : t_\varepsilon(y) \le \gamma_\varepsilon\}$$

as $\varepsilon \to 0$, and then, by (3-88), (A-9) holds true for v_ε as in (3-89). Independently, using (3-29), (3-40), (3-88) and (A-3) with (A-7), we clearly get that there exists $C > 0$ such that

$$\lambda_\varepsilon |\Psi'_{N_\varepsilon}(u_\varepsilon)| \le C \frac{\exp(-2t_\varepsilon + t_\varepsilon^2/\gamma_\varepsilon^2)}{\mu_\varepsilon^2 \gamma_\varepsilon} \quad \text{in } B_{x_\varepsilon}\big(\tfrac{1}{2}\rho'_\varepsilon\big) \tag{3-103}$$

for all ε. Then, we get

$$\begin{aligned} u_\varepsilon(z_\varepsilon) &= G_{z_\varepsilon}(x_\varepsilon) \int_{B_{x_\varepsilon}(\rho'_\varepsilon/2)} \frac{\lambda_\varepsilon \Psi'_{N_\varepsilon}(u_\varepsilon)}{2}\, dy + O\left(\int_{B_{x_\varepsilon}(\rho'_\varepsilon/2)} \frac{\exp(-2t_\varepsilon + t_\varepsilon^2/\gamma_\varepsilon^2)|\cdot - x_\varepsilon|}{\mu_\varepsilon^2 \gamma_\varepsilon \rho'_\varepsilon}\, dy\right) + O(\lambda_\varepsilon \|u_\varepsilon\|_{L^{p'}}) \\ &= G_{z_\varepsilon}(x_\varepsilon) \frac{4\pi}{\gamma_\varepsilon}\left(1 + \frac{1}{\gamma_\varepsilon^2} + \frac{A(\gamma_\varepsilon) - 2\xi_\varepsilon}{2} + o(\tilde{\zeta}_\varepsilon)\right) + o\left(\frac{1}{\gamma_\varepsilon}\right) + o(\|u_\varepsilon\|_{L^{p'}}), \end{aligned} \tag{3-104}$$

where $\tilde{\zeta}_\varepsilon$ is given by (3-15). We start by focusing on the first equality of (3-104): (3-102) and (3-103) are used to get the first two terms. The last term is obtained from (3-29), (3-100), (3-101) and Hölder's inequality. We focus now on the second equality of (3-104), resuming the previous one term by term:

The first term is easily computed by integrating (A-9) in $\widetilde{\Omega}_\varepsilon$ and by plugging the values of the A_i's from (A-2)–(A-4) on the one hand, and by estimating roughly in $B_{x_\varepsilon}(\rho'_\varepsilon)\backslash\widetilde{\Omega}_\varepsilon$ with (3-103) on the other hand. The last term obviously follows from $\lambda_\varepsilon = o(1)$. As for the $o(1/\gamma_\varepsilon)$, we get first $O(\mu_\varepsilon/(\rho'_\varepsilon\gamma_\varepsilon))$ using $\varepsilon_0 > \frac{1}{2}$, which clearly concludes by (3-95). Using first that $u_\varepsilon \le \gamma_\varepsilon$ and (3-84) in $B_{x_\varepsilon}(\rho_\varepsilon)$, and then (3-104) with (3-100) in $\Omega\backslash B_{x_\varepsilon}(\rho_\varepsilon)$, we get

$$\|u_\varepsilon\|_{L^{p'}} = o\left(\frac{1}{\gamma_\varepsilon} + \|u_\varepsilon\|_{L^{p'}}\right) + O\left(\frac{1}{\gamma_\varepsilon}\right).$$

This implies with (3-104)

$$u_\varepsilon(z_\varepsilon) = \frac{4\pi G_{z_\varepsilon}(x_\varepsilon)}{\gamma_\varepsilon}\left(1 + \frac{1}{\gamma_\varepsilon^2} + \frac{A(\gamma_\varepsilon) - 2\xi_\varepsilon}{2} + o(\tilde{\zeta}_\varepsilon)\right) + o\left(\frac{1}{\gamma_\varepsilon}\right). \tag{3-105}$$

(2) In this second point, we prove

$$\lambda_\varepsilon \le \frac{4 + o(1)}{\gamma_\varepsilon^2 \exp(1+M)} \tag{3-106}$$

as $\varepsilon \to 0$, for M as in (1-9). Observe that (3-105) implies

$$u_\varepsilon = (1 + o(1))\frac{4\pi G_{x_\varepsilon} + o(1)}{\gamma_\varepsilon}$$

in $\Omega\backslash B_{x_\varepsilon}(\rho_\varepsilon)$. By (1-1) and (3-100), our definition of ρ_ε and the dominated convergence theorem, this implies

$$\lim_{\varepsilon\to 0}\int_{\Omega\backslash B_{x_\varepsilon}(\rho_\varepsilon)} \Psi_{N_\varepsilon}(u_\varepsilon)\,dy = |\Omega|(1 + g(0)). \tag{3-107}$$

Independently, (A-7) and (3-88) give

$$u_\varepsilon = \gamma_\varepsilon - \frac{(1+o(1))t_\varepsilon}{\gamma_\varepsilon} \tag{3-108}$$

in $B_{x_\varepsilon}(\rho_\varepsilon)$, since $\mu_\varepsilon \ll \rho_\varepsilon$. Then, using (3-30), (3-45), $\varepsilon_0^2 > 1/e$ and resuming the arguments to get (3-55), we have

$$\Psi_{N_\varepsilon}(u_\varepsilon) = (1+o(1))\varphi_{N_\varepsilon-1}(u_\varepsilon^2) \quad \text{and} \quad \Psi'_{N_\varepsilon}(u_\varepsilon) = 2(1+o(1))\, u_\varepsilon\varphi_{N_\varepsilon-1}(u_\varepsilon^2) \tag{3-109}$$

in $B_{x_\varepsilon}(\rho_\varepsilon)$. Independently, observe that, for all $\Gamma, \delta > 0$,

$$\varphi_N(\Gamma) = \delta\exp(\Gamma) \quad \Longrightarrow \quad \text{for all } T \in [0,\Gamma],\ \ \varphi_N(T) \le \delta\exp(T), \tag{3-110}$$

since $\varphi'_N \ge \varphi_N$ in $[0,+\infty]$. Then we get

$$\int_{B_{x_\varepsilon}(\rho_\varepsilon)} \Psi_{N_\varepsilon}(u_\varepsilon)\,dy = \frac{4\pi(1+o(1))}{\gamma_\varepsilon^2\lambda_\varepsilon} \tag{3-111}$$

as $\varepsilon \to 0$, by (3-30), (3-40), (3-108), (3-109), with (3-48) for $|y - x_\varepsilon| \lesssim \mu_\varepsilon$, or with (3-110) and the dominated convergence theorem for $|y - x_\varepsilon| \gg \mu_\varepsilon$. Then, because of (3-7), we get that (3-106) holds true, by combining (3-107), (3-111) with (3-24).

(3) In this point, we conclude the proof of (3-10), and prove (2-4) and (3-12). For $R > 1$, let $\chi_{\varepsilon,R}$ be given in $\Omega_{\varepsilon,R} := \Omega \backslash B_{x_\varepsilon}(R\mu_\varepsilon)$ by

$$\chi_{\varepsilon,R} = 4\pi \Lambda_{\varepsilon,R} G_{x_\varepsilon}$$

for $\Lambda_{\varepsilon,R} > 0$ to be chosen later such that

$$\chi_{\varepsilon,R} \le u_\varepsilon \quad \text{on } \partial B_{x_\varepsilon}(R\mu_\varepsilon). \tag{3-112}$$

Integrating by parts, we can write

$$\begin{aligned}\int_{\Omega_{\varepsilon,R}} |\nabla u_\varepsilon|^2\,dy &= \int_{\Omega_{\varepsilon,R}} |\nabla \chi_{\varepsilon,R}|^2\,dy - 2\int_{\partial B_{x_\varepsilon}(R\mu_\varepsilon)} (\partial_\nu \chi_{\varepsilon,R})(u_\varepsilon - \chi_{\varepsilon,R})\,d\sigma + \int_{\Omega_{\varepsilon,R}} |\nabla(u_\varepsilon - \chi_{\varepsilon,R})|^2\,dy \\ &\ge \int_{\Omega_{\varepsilon,R}} |\nabla \chi_{\varepsilon,R}|^2\,dy,\end{aligned} \tag{3-113}$$

where ν is the unit outward normal to the boundary of $B_{x_\varepsilon}(R\mu_\varepsilon)$, using (3-112). Indeed, by [Druet and Thizy 2017, Appendix B] for instance, since $d(x_\varepsilon, \partial\Omega) \gg \mu_\varepsilon$ by Step 5, we have

$$\partial_\nu G_{x_\varepsilon} = -\frac{1}{2\pi R\mu_\varepsilon} + O\left(\frac{1}{d(x_\varepsilon, \partial\Omega)}\right) \quad \text{on } \partial B_{x_\varepsilon}(R\mu_\varepsilon). \tag{3-114}$$

Now, by (3-3), (3-40), (3-42), (3-45), (3-84), in order to have (3-112), we can choose $\Lambda_{\varepsilon,R}$ such that

$$\Lambda_{\varepsilon,R} = \frac{1}{\gamma_\varepsilon}\left(1 - \frac{\log(1+R^2) + o(1)}{\gamma_\varepsilon^2}\right)\left(1 + \frac{\log(\delta_\varepsilon \lambda_\varepsilon \gamma_\varepsilon^2/(4R^2)) + \mathcal{H}_{x_\varepsilon}(x_\varepsilon) + o(1)}{\gamma_\varepsilon^2}\right)^{-1}, \tag{3-115}$$

with $\delta_\varepsilon \in (0, 1]$ as in (3-30). In (3-115), we use

$$|\mathcal{H}_{x_\varepsilon} - \mathcal{H}_{x_\varepsilon}(x_\varepsilon)| = O\left(\frac{\mu_\varepsilon}{d(x_\varepsilon, \partial\Omega)}\right) = o(1)$$

uniformly in $\partial B_{x_\varepsilon}(R\mu_\varepsilon)$, using Step 5 and computing as in (3-21). Now, by (1-8), (3-44), (3-84), and (3-114), we compute and get first that

$$\begin{aligned}\int_{\Omega_{\varepsilon,R}} |\nabla \chi_{\varepsilon,R}|^2\,dy &\ge -\int_{\partial B_{x_\varepsilon}(R\mu_\varepsilon)} (\partial_\nu \chi_{\varepsilon,R})\chi_{\varepsilon,R}\,d\sigma \\ &\ge 4\pi\left(1 - \frac{2\log(1+R^2) + o(1)}{\gamma_\varepsilon^2}\right)\left(1 + \frac{\log(\delta_\varepsilon \lambda_\varepsilon \gamma_\varepsilon^2/(4R^2)) + \mathcal{H}_{x_\varepsilon}(x_\varepsilon) + o(1)}{\gamma_\varepsilon^2}\right)^{-1},\end{aligned}$$

using also (3-115). Independently, we compute and get also that

$$\int_{B_{x_\varepsilon}(R\mu_\varepsilon)} |\nabla u_\varepsilon|^2\,dy = \frac{4\pi}{\gamma_\varepsilon^2}\left(\log(1+R^2) - \frac{R^2}{1+R^2} + o(1)\right),$$

by (3-42). Thus, since $\|u_\varepsilon\|_{H_0^1}^2 \le 4\pi$ and by (3-7) and (3-113), we eventually get

$$\frac{\log \delta_\varepsilon \lambda_\varepsilon + \mathcal{H}_{x_\varepsilon}(x_\varepsilon)}{\gamma_\varepsilon^2} \ge o(1).$$

Moreover, using the definition (1-9) of M, (3-106), $\delta_\varepsilon \le 1$ and that $R > 0$ may be arbitrarily large, we get

$$\delta_\varepsilon \to 1, \tag{3-116}$$

and that (3-10) and (3-12) hold true. As a remark, in Case 2 where $N_\varepsilon = 1$, (3-116) is a direct consequence of the definition (3-30) of δ_ε. Then, (2-4) follows from (3-10), (3-107) and (3-111).

(4) Now we prove (3-11). Since $\varepsilon_0' > \varepsilon_0$, we get from (3-84), (3-88), (3-96) and (A-7) that

$$u_\varepsilon = \gamma_\varepsilon - \frac{t_\varepsilon}{\gamma_\varepsilon} - \frac{t_\varepsilon}{\gamma_\varepsilon^3} - (A(\gamma_\varepsilon) - 2\xi_\varepsilon)\frac{t_\varepsilon}{2\gamma_\varepsilon} + o\left(\frac{t_\varepsilon \tilde{\zeta}_\varepsilon}{\gamma_\varepsilon}\right) \tag{3-117}$$

uniformly in $\{y \in B_{x_\varepsilon}(\rho_\varepsilon') : t_\varepsilon \geq \gamma_\varepsilon/4\}$, using also (A-3). Then, noting that the averages of (3-105) and (3-117) have to match on $\partial B_{x_\varepsilon}(\rho_\varepsilon')$, we compute and get

$$\lambda_\varepsilon = \frac{4}{\gamma_\varepsilon^2 \exp\left(1 + M + \frac{1}{2}\gamma_\varepsilon^2 (A(\gamma_\varepsilon) - 2\xi_\varepsilon) + o(\tilde{\zeta}_\varepsilon \gamma_\varepsilon^2)\right)}, \tag{3-118}$$

by (3-12), (3-116) and (3-40) with (3-3) and (3-45). Observe in particular that

$$1 \lesssim \gamma_\varepsilon^{-2} G_{x_\varepsilon} \lesssim 1, \quad 1 \lesssim \gamma_\varepsilon^{-2} t_\varepsilon \lesssim 1$$

on $\partial B_{x_\varepsilon}(\rho_\varepsilon')$, by (3-95) and (3-96) with (1-8) and (3-12). By (3-10) and (3-118), (3-11) is proved.

(5) Here, we conclude the proof of Lemma 3.4. As an immediate consequence of (3-105), we get that (3-97) holds true. Pushing now one step further the above computations with very similar arguments, we get that (3-98) holds true as well. At last, using in particular (3-10) with (1-6) to improve the estimates in point (1) of this proof, we get (3-99). □

Lemma 3.4 is proved. □

4. Proof of Proposition 2.1

Proof of Proposition 2.1. We make the assumptions of Lemma 3.4 in Case 2 with $\alpha_\varepsilon = 4\pi(1-\varepsilon)$. In particular, we assume that u_ε is a maximizer for $(I^g_{4\pi(1-\varepsilon)}(\Omega))$, for all $0 < \varepsilon \ll 1$, and that (2-1) holds true. Then, Lemma 3.4 in Case 2 will be currently applied in the sequel. In particular, we may let λ_ε, γ_ε, x_ε, μ_ε be thus given and it only remains to prove (2-5)–(2-6) to get Proposition 2.1.

Let $z \in \Omega$ be given. In view of (3-99), for $\gamma, \mu > 0$, we let now $U_{\mu,\gamma,z}$ be given by

$$\begin{aligned} U_{\mu,\gamma,z}(x) &= \frac{1}{\gamma}\left(-\log\left(1 + \frac{|x-z|^2}{\mu^2}\right) + \log\frac{1}{\mu^2} + \mathcal{H}_{-1,\mu,z}(x)\right) \\ &\quad + \sum_{i=0}^{1} \frac{1}{\gamma^{3+2i}}\left(S_i\left(\frac{x-z}{\mu}\right) + \frac{A_i}{4\pi}\left(\log\frac{1}{\mu^2} + \mathcal{H}_{i,\mu,z}(x)\right) - B_i\right) \\ &\quad + \frac{A(\gamma)}{\gamma}\left(S_2\left(\frac{x-z}{\mu}\right) + \frac{A_2}{4\pi}\left(\log\frac{1}{\mu^2} + \mathcal{H}_{2,\mu,z}(x)\right) - B_2\right) \\ &\quad + \frac{4B(\gamma)}{\gamma^2 \exp(1 + \mathcal{H}_z(z))} \int_\Omega G_x(y) F(4\pi G_z(y))\, dy, \end{aligned} \tag{4-1}$$

where the S_i are given by (A-2), where the A_i, B_i are as in (A-3), where $\mathcal{H}$ is as in (1-8), where the $\mathcal{H}_{j,\mu,z}$ are harmonic in Ω and given by

$$\mathcal{H}_{-1,\mu,z} = -\log\left(\frac{1}{\mu^2 + |z - \cdot|^2}\right) \quad \text{or} \quad \mathcal{H}_{j,\mu,z} = -\frac{4\pi}{A_j}\left(S_j\left(\frac{\cdot - z}{\mu}\right) - B_j\right) + \log \mu^2$$

on $\partial\Omega$, for $j \in \{0, 1, 2\}$. By the maximum principle and (A-3), we have that $\mathcal{H}_{j,\mu,z}(x) \to \mathcal{H}_z(x)$ and $|\partial_\mu \mathcal{H}_{j,\mu,z}(x)| \le C\mu$ uniformly in $x \in \Omega$ as $\mu \to 0$, for all j. Then, setting $f_\gamma(\mu) = \gamma^{-1} U_{\mu,\gamma,z}(z) - 1$, using that $S_i(0) = 0$ and (4-1), it may be easily checked that $f_\gamma(\mu) = -\gamma^{-2}\log\mu^2(1+o(1)) - 1$, C^1-uniformly in $\mu \in (0, \mu(\gamma))$ as $\gamma \to +\infty$, where $\mu(\gamma)$ is given by $-\log\mu(\gamma)^2 = \frac{1}{2}\gamma^2$. In particular, there exists $\tilde{\gamma} \gg 1$ such that $\lim_{\mu\to 0} f_\gamma(\mu) = +\infty$, $f_\gamma(\mu(\gamma)) < 0$ and $f'_\gamma < 0$ in $(0, \mu(\gamma))$, so that there exists a unique $\tilde{\mu}(\gamma, z) \in (0, \mu(\gamma))$ such that $f_\gamma(\tilde{\mu}(\gamma, z)) = 0$ for all $\gamma \ge \tilde{\gamma}$. Fixing K a compact subset of Ω, it is clear that $\tilde{\gamma}$ can be chosen independent of $z \in K$; in particular, we may let $\tilde{\mu}_\varepsilon := \tilde{\mu}(\gamma_\varepsilon, z)$ be the unique $\mu \in (0, \mu(\gamma_\varepsilon))$ given by

$$U_{\mu,\gamma_\varepsilon,z}(z) = \gamma_\varepsilon \tag{4-2}$$

for all ε small. We write from now on $\widetilde{\mathcal{H}}_{j,\varepsilon,z} := \mathcal{H}_{j,\tilde{\mu}_\varepsilon,z}$ and $U_{\varepsilon,z} := U_{\tilde{\mu}_\varepsilon,\gamma_\varepsilon,z}$. The following result concludes the proof of Proposition 2.1.

Lemma 4.1. *We have*

$$S = \int_\Omega G_{\bar{x}}(y) F(4\pi G_{\bar{x}}(y))\, dy \quad \text{if } \frac{\gamma_\varepsilon^{-3} B(\gamma_\varepsilon)}{\gamma_\varepsilon^{-4} + |A(\gamma_\varepsilon)|} \not\to 0 \tag{4-3}$$

as $\varepsilon \to 0$, *where* S *is as in* (1-9) *and* $\bar{x}$ *as in* (3-12). *Moreover*, (2-5) *holds true in any case.*

Proof of Lemma 4.1. Let K be a compact subset of Ω and $(z_\varepsilon)_\varepsilon$ be a given sequence of points of K. For simplicity, we let in the proof below $\check{\zeta}_\varepsilon$ be given by

$$\check{\zeta}_\varepsilon = \max\left(\frac{1}{\gamma_\varepsilon^4}, |A(\gamma_\varepsilon)|, \frac{|B(\gamma_\varepsilon)|}{\gamma_\varepsilon^3}\right). \tag{4-4}$$

(1) We first derive the following more explicit expression of the $\tilde{\mu}_\varepsilon$ from (4-2):

$$\begin{aligned}\frac{4}{\tilde{\mu}_\varepsilon^2 \exp(\gamma_\varepsilon^2)\gamma_\varepsilon^2} = \frac{4}{\gamma_\varepsilon^2 \exp(1 + \mathcal{H}_{z_\varepsilon}(z_\varepsilon))}&(1 + O(\check{\zeta}_\varepsilon + \gamma_\varepsilon^4 |A(\gamma_\varepsilon)|^2)) \\ &\times \left(1 - \frac{\gamma_\varepsilon^2 A(\gamma_\varepsilon)}{2} - \frac{4B(\gamma_\varepsilon)}{\gamma_\varepsilon \exp(1 + \mathcal{H}_{z_\varepsilon}(z_\varepsilon))}\int_\Omega G_{z_\varepsilon}(y) F(4\pi G_{z_\varepsilon}(y))\, dy\right)\end{aligned} \tag{4-5}$$

as $\varepsilon \to 0$. By the maximum principle and (A-3), we get that there exists $C_K > 0$ such that $|\widetilde{\mathcal{H}}_{j,\varepsilon,z_\varepsilon}| \le C_K$ in Ω, so that, by elliptic theory, the $\widetilde{\mathcal{H}}_{j,\varepsilon,z_\varepsilon}$'s are also bounded in $C^1_{\text{loc}}(\Omega)$ for all ε and j. We get from (4-2) that $|\log 1/\tilde{\mu}_\varepsilon^2 - \gamma_\varepsilon^2| \le C'_K$, and then that

$$|\widetilde{\mathcal{H}}_{j,\varepsilon,z_\varepsilon} - \mathcal{H}_{z_\varepsilon}| \le C''_K \gamma_\varepsilon^8 \exp(-\gamma_\varepsilon^2) \quad \text{in } \Omega, \tag{4-6}$$

for all $0<\varepsilon\ll 1$ and $j\in\{-1,\dots,2\}$, by the maximum principle, (1-8) and (A-3). Rewriting then (4-2) as

$$\gamma_\varepsilon^2=\log\frac{1}{\tilde\mu_\varepsilon^2}\left(1+\frac{A_0}{4\pi\gamma_\varepsilon^2}+\frac{A_1}{4\pi\gamma_\varepsilon^4}+\frac{A(\gamma_\varepsilon)A_2}{4\pi}\right)+\mathcal{H}_{z_\varepsilon}(z_\varepsilon)\left(1+\frac{A_0}{4\pi\gamma_\varepsilon^2}\right)\\ -\frac{B_0}{\gamma_\varepsilon^2}+\frac{4B(\gamma_\varepsilon)}{\gamma_\varepsilon\exp(1+\mathcal{H}_{z_\varepsilon}(z_\varepsilon))}\int_\Omega G_{z_\varepsilon}(y)F(4\pi G_{z_\varepsilon}(y))\,dy+O(\gamma_\varepsilon^{-4}+|A(\gamma_\varepsilon)|),$$

we easily get (4-5), using (3-16) and (A-3) with $A_1/(4\pi)-A_0^2/(16\pi^2)-B_0=0$.

(2) We prove now that

$$\int_\Omega|\nabla U_{\varepsilon,z_\varepsilon}|^2\,dx=4\pi(1+I_{z_\varepsilon}(\gamma_\varepsilon)+o(\check\zeta_\varepsilon)) \tag{4-7}$$

as $\varepsilon\to0$, where $I_{z_\varepsilon}(\gamma_\varepsilon)$ is given by

$$I_{z_\varepsilon}(\gamma_\varepsilon)=\gamma_\varepsilon^{-4}+\frac{A(\gamma_\varepsilon)}{2}+\frac{4B(\gamma_\varepsilon)}{\gamma_\varepsilon^3\exp(1+\mathcal{H}_{z_\varepsilon}(z_\varepsilon))}\int_\Omega G_{z_\varepsilon}(y)F(4\pi G_{z_\varepsilon}(y))\,dy \tag{4-8}$$

and where $U_{\varepsilon,z_\varepsilon}$ is given by (4-1)–(4-2). By (1-6) and elliptic theory,

$$\left(x\mapsto\int_\Omega G_x(y)F(4\pi G_{z_\varepsilon}(y))\,dy\right)_\varepsilon \text{ is a bounded sequence in } C^1(\bar\Omega). \tag{4-9}$$

By the construction of the $\widetilde{\mathcal{H}}_{j,\varepsilon,z_\varepsilon}$, we can write

$$\begin{aligned}\int_\Omega|\nabla U_{\varepsilon,z_\varepsilon}(y)|^2\,dy\\ &=\int_\Omega\Delta U_{\varepsilon,z_\varepsilon}(y)\,U_{\varepsilon,z_\varepsilon}(y)\,dy,\\ &=\int_{\{y:\tilde t_\varepsilon(y)\le\gamma_\varepsilon\}}\left(\frac{\Delta(-\tilde t_\varepsilon)}{\gamma_\varepsilon}+\frac{\Delta\widetilde S_{0,\varepsilon}}{\gamma_\varepsilon^3}+\frac{\Delta\widetilde S_{1,\varepsilon}}{\gamma_\varepsilon^5}+\frac{A(\gamma_\varepsilon)\Delta\widetilde S_{2,\varepsilon}}{\gamma_\varepsilon}\right)\\ &\qquad\times\left(\gamma_\varepsilon-\frac{\tilde t_\varepsilon}{\gamma_\varepsilon}+\frac{\widetilde S_{0,\varepsilon}}{\gamma_\varepsilon^3}+O\left(\left(\frac{|A(\gamma_\varepsilon)|}{\gamma_\varepsilon}+\frac{1}{\gamma_\varepsilon^5}\right)(1+\tilde t_\varepsilon)+\frac{|y-z_\varepsilon|}{\gamma_\varepsilon}\right)\right)dy+o(\gamma_\varepsilon^{-4})\\ &\quad+\int_{\{y:\tilde t_\varepsilon(y)\ge\gamma_\varepsilon(\gamma_\varepsilon-1)\}}\left(O(\tilde\mu_\varepsilon^2\gamma_\varepsilon^4)+\frac{4B(\gamma_\varepsilon)}{\gamma_\varepsilon^2\exp(1+\mathcal{H}_{z_\varepsilon}(z_\varepsilon))}F(4\pi G_{z_\varepsilon}(y))\right)\\ &\qquad\times\left(\frac{4\pi G_{z_\varepsilon}(y)}{\gamma_\varepsilon}+O\left(\frac{G_{z_\varepsilon}(y)}{\gamma_\varepsilon^3}+\frac{|B(\gamma_\varepsilon)|}{\gamma_\varepsilon^2}\right)\right)dy,\end{aligned} \tag{4-10}$$

where $\tilde t_\varepsilon(y)=\log(1+|y-z_\varepsilon|^2/\tilde\mu_\varepsilon^2)$ and $\widetilde S_{i,\varepsilon}=S_i(|y-z_\varepsilon|/\tilde\mu_\varepsilon)$. We use also here (1-8) with (3-16), and the estimates of point (1) of this proof, including (4-5)–(4-6). The integral on $\{\tilde t_\varepsilon\in(\gamma_\varepsilon,\gamma_\varepsilon(\gamma_\varepsilon-1))\}$ gives an $o(\gamma_\varepsilon^{-4})$-term. Estimate (4-7) follows from (4-10), the Appendix and some computations that we do not develop here again; see also [Mancini and Martinazzi 2017, §5].

(3) We prove now that

$$\int_\Omega|\nabla u_\varepsilon|^2\,dx=4\pi(1+I_{x_\varepsilon}(\gamma_\varepsilon)+o(\check\zeta_\varepsilon)) \tag{4-11}$$

as $\varepsilon \to 0$, where $I_{x_\varepsilon}(\gamma_\varepsilon)$ is given by (4-8), for $(x_\varepsilon)_\varepsilon$ as in (3-13). Now, we can push one step further the argument involving (3-118), writing now that both formulas (3-98) and (3-99) must also coincide on $\partial B_{x_\varepsilon}(\rho'_\varepsilon)$, where $\rho'_\varepsilon > 0$ is as in (3-95). We compute and then get for μ_ε in (3-40) the analogue of (4-5) for $\tilde{\mu}_\varepsilon$

$$\begin{aligned}\lambda_\varepsilon H(\gamma_\varepsilon) &= \frac{4}{\mu_\varepsilon^2 \exp(\gamma_\varepsilon^2)\gamma_\varepsilon^2}\left(1+o\left(\frac{1}{\gamma_\varepsilon^4}\right)\right)\\ &= \frac{4}{\gamma_\varepsilon^2 \exp(1+\mathcal{H}_{x_\varepsilon}(x_\varepsilon))}(1+o(\gamma_\varepsilon^2\check{\zeta}_\varepsilon))\\ &\qquad\times\left(1-\frac{\gamma_\varepsilon^2 A(\gamma_\varepsilon)}{2}-\frac{4B(\gamma_\varepsilon)}{\gamma_\varepsilon \exp(1+\mathcal{H}_{x_\varepsilon}(x_\varepsilon))}\int_\Omega G_{x_\varepsilon}(y)F(4\pi G_{x_\varepsilon}(y))\,dy\right), \quad (4\text{-}12)\end{aligned}$$

using (1-8), (3-16), (A-3)–(A-7). Independently, integrating by parts, resuming some computations in the Appendix and using (2-2), (3-12), (3-44), point (1), and (3-97)–(3-99) (see also (3-89) and (A-9)), we get

$$\int_\Omega |\nabla u_\varepsilon|^2\,dx = \int_\Omega u_\varepsilon(\lambda_\varepsilon H(u_\varepsilon)u_\varepsilon \exp(u_\varepsilon^2))\,dx, = \int_\Omega U_{\varepsilon,x_\varepsilon}\Delta U_{\varepsilon,x_\varepsilon}\,dx + o(\check{\zeta}_\varepsilon). \quad (4\text{-}13)$$

In order to get the second equality and to apply the dominated convergence theorem, it may be useful to split Ω as

$$\Omega = \{y : t_\varepsilon(y) \le \gamma_\varepsilon\} \cup \left\{y : t_\varepsilon(y) > \gamma_\varepsilon \text{ and } \log\frac{1}{|x_\varepsilon - y|^2} \ge \frac{1-\delta'_0}{2}\gamma_\varepsilon^2\right\} \cup \left\{y : \log\frac{1}{|x_\varepsilon - y|^2} < \frac{1-\delta'_0}{2}\gamma_\varepsilon^2\right\},$$

where δ'_0 is as in (1-6), and to use the first line of (4-12) with (1-5) (resp. with (3-29)) in the first region (resp. in the second region), or (1-6)–(1-7) in the last region. Observe that the argument here is to show that $U_{\varepsilon,x_\varepsilon}$ (resp. $\Delta U_{\varepsilon,x_\varepsilon}$) is in some sense the main part of the expansion of u_ε (resp. Δu_ε). Thus we get (4-11) from (4-7) and (4-13).

(4) We prove now that, for any fixed sequence $(\eta_\varepsilon)_\varepsilon$ of real numbers such that $\eta_\varepsilon = o(\gamma_\varepsilon^{-2})$, we have

$$\begin{aligned}\int_\Omega (1+g(V_{\varepsilon,z_\varepsilon}))\exp(V_{\varepsilon,z_\varepsilon}^2)\,dy &= |\Omega|(1+g(0)) + \pi\exp(1+\mathcal{H}_{z_\varepsilon}(z_\varepsilon))(1-\eta_\varepsilon\gamma_\varepsilon^2)\\ &\qquad\times H(\gamma_\varepsilon)\left(1+\gamma_\varepsilon^2 I_{z_\varepsilon}(\gamma_\varepsilon)+\frac{1}{\gamma_\varepsilon^2}+o(\gamma_\varepsilon^2(\check{\zeta}_\varepsilon+|\eta_\varepsilon|))\right)\\ &\qquad\times\left(1+\frac{8B(\gamma_\varepsilon)}{\gamma_\varepsilon(\kappa+1)\exp(1+\mathcal{H}_{z_\varepsilon}(z_\varepsilon))}\int_\Omega G_{z_\varepsilon}(y)F(4\pi G_{z_\varepsilon}(y))\,dy\right), \quad (4\text{-}14)\end{aligned}$$

where κ is as in (1-6) and where $V_{\varepsilon,z_\varepsilon} \ge 0$ is given by

$$V_{\varepsilon,z_\varepsilon}^2 = (1-\eta_\varepsilon)U_{\varepsilon,z_\varepsilon}^2, \quad (4\text{-}15)$$

where $U_{\varepsilon,z_\varepsilon}$ is given in (4-1). Computations in the spirit of the proof of (4-13) give

$$\int_\Omega (1+g(U_{\varepsilon,x_\varepsilon}))\exp(U_{\varepsilon,x_\varepsilon}^2)\,dy = \int_\Omega (1+g(u_\varepsilon))\exp(u_\varepsilon^2)\,dy + o(\gamma_\varepsilon^2\check{\zeta}_\varepsilon), \quad (4\text{-}16)$$

not only by combining (1-1), (1-5)–(1-6), Lemma 3.3, (3-12), (3-97)–(3-99) and the Appendix, and by splitting Ω as in (4-10), but also by using (4-5) and (4-12). In particular, once (4-14) is proved, choosing $\eta_\varepsilon = 0$ and $z_\varepsilon = x_\varepsilon$, we get from (4-16) that

$$\begin{aligned}\int_\Omega (1+g(u_\varepsilon))\exp(u_\varepsilon^2)\,dy = |\Omega|(1+g(0))+\pi\exp(1+\mathcal{H}_{x_\varepsilon}(x_\varepsilon))H(\gamma_\varepsilon)\\ \times\left(1+\gamma_\varepsilon^2 I_{x_\varepsilon}(\gamma_\varepsilon)+\frac{1}{\gamma_\varepsilon^2}+o(\gamma_\varepsilon^2\check{\zeta}_\varepsilon)\right)\\ \times\left(1+\frac{8B(\gamma_\varepsilon)}{\gamma_\varepsilon(\kappa+1)\exp(1+\mathcal{H}_{x_\varepsilon}(x_\varepsilon))}\int_\Omega G_{x_\varepsilon}(y)F(4\pi G_{x_\varepsilon}(y))\,dy\right). \quad (4\text{-}17)\end{aligned}$$

It remains to prove (4-14). We compute and get

$$U_{\varepsilon,z_\varepsilon}(y)^2 = \gamma_\varepsilon^2 - 2\tilde{t}_\varepsilon + \frac{\tilde{t}_\varepsilon^2}{\gamma_\varepsilon^2} + \frac{2\widetilde{S}_{0,\varepsilon}}{\gamma_\varepsilon^2} + O\big((|A(\gamma_\varepsilon)|+\gamma_\varepsilon^{-4})(1+\tilde{t}_\varepsilon(y)^2)+|y-z_\varepsilon|\big) \quad (4\text{-}18)$$

for all y such that $\tilde{t}_\varepsilon(y) \le \gamma_\varepsilon$, using (1-7), (4-1)–(4-2), (4-5), (4-9) and (A-3). Then we get

$$\begin{aligned}&\int_{\{\tilde{t}_\varepsilon\le\gamma_\varepsilon\}}(1+g(V_{\varepsilon,z_\varepsilon}))\exp(V_{\varepsilon,z_\varepsilon}^2)\,dy\\ &\quad=\int_{\{\tilde{t}_\varepsilon\le\gamma_\varepsilon\}} H(\gamma_\varepsilon)(1+O(|A(\gamma_\varepsilon)|\exp(\delta_0\tilde{t}_\varepsilon)))\exp(\gamma_\varepsilon^2)\exp(-2\tilde{t}_\varepsilon)\exp(-\eta_\varepsilon\gamma_\varepsilon^2)\\ &\qquad\times\exp\left(\frac{\tilde{t}_\varepsilon^2+2\widetilde{S}_{0,\varepsilon}}{\gamma_\varepsilon^2}\right)\exp\big(O((|\eta_\varepsilon|+|A(\gamma_\varepsilon)|+\gamma_\varepsilon^{-4})(1+\tilde{t}_\varepsilon^2))+|y-z_\varepsilon|\big)\,dy,\end{aligned}$$

using (3-2) and (4-15) with (4-18). Then combining $\eta_\varepsilon = o(\gamma_\varepsilon^{-2})$, (3-16), (4-5), computing explicitly $\int_{\mathbb{R}^2}\exp(-2T_0)S_0\,dy = 0$ and $\int_{\mathbb{R}^2}\exp(-2T_0)T_0^2\,dy = 2\pi$ for T_0 as in (3-42), we get

$$\begin{aligned}&\int_{\{\tilde{t}_\varepsilon\le\gamma_\varepsilon\}}(1+g(V_{\varepsilon,z_\varepsilon}))\exp(V_{\varepsilon,z_\varepsilon}^2)\,dy\\ &=\frac{(1-\eta_\varepsilon\gamma_\varepsilon^2)H(\gamma_\varepsilon)\exp(\mathcal{H}_{z_\varepsilon}(z_\varepsilon)+1)}{4}\big(1+o(\gamma_\varepsilon^2(|A(\gamma_\varepsilon)|+|\eta_\varepsilon|)+\gamma_\varepsilon^{-2})\big)\\ &\quad\times\left(1+\frac{\gamma_\varepsilon^2 A(\gamma_\varepsilon)}{2}+\frac{4B(\gamma_\varepsilon)}{\gamma_\varepsilon\exp(\mathcal{H}_{z_\varepsilon}(z_\varepsilon)+1)}\int_\Omega G_{z_\varepsilon}(x)F(4\pi G_{z_\varepsilon}(x))\,dx+o\left(\frac{B(\gamma_\varepsilon)}{\gamma_\varepsilon}\right)\right)4\pi\left(1+\frac{2}{\gamma_\varepsilon^2}\right). \quad (4\text{-}19)\end{aligned}$$

Independently, we get from (1-6), (3-1) (parts (a) and (b) in $\{y,\,4\pi G_{z_\varepsilon}(y)\le\frac{1}{2}\gamma_\varepsilon\}$, or part (c) otherwise), (4-1), (4-5) and the dominated convergence theorem that

$$\begin{aligned}&\int_{\{\tilde{t}_\varepsilon\ge\gamma_\varepsilon\}}(1+g(V_{\varepsilon,z_\varepsilon}))\exp(V_{\varepsilon,z_\varepsilon}^2)\,dy\\ &\qquad= |\Omega|(1+g(0))+\frac{8\pi B(\gamma_\varepsilon)}{\gamma_\varepsilon(\kappa+1)}\int_\Omega G_{z_\varepsilon}(y)F(4\pi G_{z_\varepsilon}(y))\,dy+o\left(\frac{|B(\gamma_\varepsilon)|}{\gamma_\varepsilon}+\frac{1}{\gamma_\varepsilon^2}\right). \quad (4\text{-}20)\end{aligned}$$

Combining (4-19) and (4-20), we conclude that (4-14) holds true, using (3-3) and (4-5).

(5) We are now in position to conclude the proof of Lemma 4.1. Let $\bar{x}_0$ be a point in the compact set $K_\Omega \Subset \Omega$ where S is attained in the third equation of (1-9). Let η_ε be given by

$$(1-\eta_\varepsilon) = \frac{4\pi(1-\varepsilon)}{\|U_{\varepsilon,\bar{x}_0}\|^2_{H^1_0}}. \tag{4-21}$$

First, we can check that

$$\eta_\varepsilon = I_{\bar{x}_0}(\gamma_\varepsilon) - I_{x_\varepsilon}(\gamma_\varepsilon) + o(\check{\zeta}_\varepsilon), \tag{4-22}$$

so that the condition $\eta_\varepsilon = o(\gamma_\varepsilon^{-2})$ above (4-14) is satisfied, using (1-7), (3-7), (3-16), (4-7) and (4-11). Additionally, we have $\|V_{\varepsilon,\bar{x}_0}\|^2_{H^1_0} = 4\pi(1-\varepsilon)$, by our choice (4-21) of η_ε, and then, by (3-7),

$$\int_\Omega (1+g(u_\varepsilon))\exp(u_\varepsilon^2)\,dy \ge \int_\Omega (1+g(V_{\varepsilon,\bar{x}_0}))\exp(V^2_{\varepsilon,\bar{x}_0})\,dy;$$

this implies, in view of (4-14), (4-17), (4-22) and of our choice of $\bar{x}_0$, that (4-3) is true and then, by (4-11) again, that (2-5)–(2-6) are true as well. This concludes the proof of Lemma 4.1. □

Proposition 2.1 is proved. □

Proof of Proposition 2.3. Let Ω be a smooth bounded domain of $\mathbb{R}^2$. Let g be such that (1-1) and (1-5)–(1-6) hold true for H as in (1-2), and let A, B and F be thus given. Assume that $\Lambda_g(\Omega) < \pi\exp(1+M)$, where M is as in (1-9) and $\Lambda_g(\Omega)$ as in (1-11). Assume that there exists a sequence of positive integers $(N_\varepsilon)_\varepsilon$ such that (2-9) holds true and such that $(I_{4\pi}^{g_{N_\varepsilon}}(\Omega))$ admits a nonnegative extremal u_ε for all $\varepsilon > 0$, where g_{N_ε} is as in (1-10). Then, by Lemma 3.4 in Case 1, we have (2-1) and that (3-8) holds true for $\alpha_\varepsilon = 4\pi$ for all $0 < \varepsilon \ll 1$. Moreover, we have $u_\varepsilon \in C^{1,\theta}(\bar{\Omega})$ $(0<\theta<1)$ and (2-3) by (3-13). In order to conclude the proof of Proposition 2.3, it remains to prove (2-10). Still by Lemma 3.4 in Case 1, (3-97)–(3-99) and (A-9) (v_ε as in (3-89)) hold true. Concerning (3-97)–(3-99) and (A-9), observe that, contrary to Case 2, the term ξ_ε cannot be neglected in Case 1, which we are facing here. Indeed, using also now (3-30), (3-40), (3-110) and (A-9), we can resume computations of (4-10), (4-13) and the Appendix (now with (3-11)) to get

$$\|u_\varepsilon\|^2_{H^1_0} = 4\pi\left(1 + \check{I}(\gamma_\varepsilon) + o(\gamma_\varepsilon^{-4} + |A(\gamma_\varepsilon)| + \gamma_\varepsilon^{-3}|B(\gamma_\varepsilon)| + \xi_\varepsilon)\right)$$

as $\varepsilon \to 0$, where

$$\check{I}(\gamma_\varepsilon) := \gamma_\varepsilon^{-4} + \tfrac{1}{2}(A(\gamma_\varepsilon) - 2\xi_\varepsilon) + 4\gamma_\varepsilon^{-3}\exp(-1-M)B(\gamma_\varepsilon)S,$$

so that (2-10) holds true, which concludes the proof. □

Appendix: Radial analysis

Let $(x_\varepsilon)_\varepsilon$ be a sequence of points in $\mathbb{R}^2$ and $(\gamma_\varepsilon)_\varepsilon$ be a sequence of positive real numbers such that $\gamma_\varepsilon \to +\infty$ as $\varepsilon \to 0$. Let g be such that (1-1) and (1-5) holds true for H as in (1-2), and let A be thus given. Let $(N_\varepsilon)_\varepsilon$ be a sequence of integers. We assume that we are in one of the following two cases:

Case 1′: $N_\varepsilon \to +\infty$ as $\varepsilon \to 0$, and (3-30)—(3-31) hold true.

Case 2′: $N_\varepsilon = 1$ for all ε.

Let B_ε be the radial solution around x_ε in $\mathbb{R}^2$ of (3-81), for Ψ_N as in (3-25), where $(\lambda_\varepsilon)_\varepsilon$ is any given sequence of positive real numbers. Let T_0 be given in $\mathbb{R}^2$ by

$$T_0(x) = \log(1+|x|^2). \tag{A-1}$$

Let $S_i,\ i = 0, 1, 2$, be the radially symmetric solutions around 0 in $\mathbb{R}^2$ of

$$\begin{aligned} &\Delta S_0 - 8\exp(-2T_0)S_0 = 4\exp(-2T_0)(T_0^2 - T_0),\\ &\Delta S_1 - 8\exp(-2T_0)S_1 = 4\exp(-2T_0)\big(S_0 + 2S_0^2 - 4T_0S_0 + 2S_0T_0^2 - T_0^3 + \tfrac{1}{2}T_0^4\big),\\ &\Delta S_2 - 8\exp(-2T_0)S_2 = 4\exp(-2T_0)T_0 \end{aligned} \tag{A-2}$$

such that $S_i(0) = 0$. In the sequel, we will use the C^1 expansions of the S_i's given by

$$\begin{aligned} S_0(r) &= \frac{A_0}{4\pi}\log\frac{1}{r^2} + B_0 + O(\log(r)^2 r^{-2}), \quad \text{where } A_0 = 4\pi,\ B_0 = \frac{\pi^2}{6} + 2,\\ S_1(r) &= \frac{A_1}{4\pi}\log\frac{1}{r^2} + B_1 + O(\log(r)^4 r^{-2}), \quad \text{where } A_1 = 4\pi\Big(3 + \frac{\pi^2}{6}\Big),\ B_1 \in \mathbb{R},\\ S_2(r) &= \frac{A_2}{4\pi}\log\frac{1}{r^2} + B_2 + O(\log(r) r^{-2}), \quad \text{where } A_2 = 2\pi,\ B_2 \in \mathbb{R}, \end{aligned} \tag{A-3}$$

as $r = |x| \to +\infty$. Note that in particular

$$A_i = \int_{\mathbb{R}^2} \Delta S_i\, dx. \tag{A-4}$$

The explicit formula for S_0

$$S_0(r) = -T_0(r) + \frac{2r^2}{1+r^2} - \tfrac{1}{2}T_0(r)^2 + \frac{1-r^2}{1+r^2}\int_1^{1+r^2}\frac{\log t}{1-t}\,dt,$$

and the expansions in (A-3) are derived in [Malchiodi and Martinazzi 2014; Mancini and Martinazzi 2017]. Let $\varepsilon_0 \in (\sqrt{1/e}, 1)$ be given. Let μ_ε be given by (3-40) and t_ε by (3-41). Let $\rho_\varepsilon > 0$ be given by (3-83) and satisfying (3-84). Let $S_{i,\varepsilon}$ be then given by

$$S_{i,\varepsilon}(x) = S_i\left(\frac{|x - x_\varepsilon|}{\mu_\varepsilon}\right) \tag{A-5}$$

for $i = 0, 1, 2$. Let $\xi_\varepsilon > 0$ be given by (3-14). In Case 1′ where $N_\varepsilon \to +\infty$ as $\varepsilon \to 0$, we get $\xi_\varepsilon = O(N_\varepsilon^{-1/2})$ by (3-30) and (3-45). Then, in any case, we clearly have

$$\xi_\varepsilon \to 0 \tag{A-6}$$

as $\varepsilon \to 0$. Then we are in position to state the main result of this section.

Proposition A.1. *We have*

$$B_\varepsilon = \gamma_\varepsilon - \frac{t_\varepsilon}{\gamma_\varepsilon} + \frac{S_{0,\varepsilon}}{\gamma_\varepsilon^3} + \frac{S_{1,\varepsilon}}{\gamma_\varepsilon^5} + (A(\gamma_\varepsilon) - 2\xi_\varepsilon)\frac{S_{2,\varepsilon}}{\gamma_\varepsilon} + o\left(t_\varepsilon\left(\frac{1}{\gamma_\varepsilon^5} + \frac{|A(\gamma_\varepsilon)| + \xi_\varepsilon}{\gamma_\varepsilon}\right)\right) \tag{A-7}$$

uniformly in $[0, \rho_\varepsilon]$ *as* $\varepsilon \to 0$.

In particular, using also (1-1) and (3-3), it can be checked that B_ε is positive and radially decreasing in $[0, \rho_\varepsilon]$. Observe also that $\xi_\varepsilon \ll \gamma_\varepsilon^{-4}$ can be seen as a remainder term in Case 2′. Let $\zeta_\varepsilon > 0$ be given by

$$\zeta_\varepsilon = \max\left(\frac{1}{\gamma_\varepsilon^4}, |A(\gamma_\varepsilon)|, \xi_\varepsilon\right). \tag{A-8}$$

Set $\bar{t}_\varepsilon = 1 + t_\varepsilon$. Resuming the computations below, we get as a byproduct of Proposition A.1 that $v_\varepsilon = o(\bar{t}_\varepsilon \gamma_\varepsilon^{-5})$ implies

$$\frac{\lambda_\varepsilon \Psi_\varepsilon'(B_\varepsilon + v_\varepsilon)}{2} = \frac{4\ \exp(-2t_\varepsilon)}{\mu_\varepsilon^2 \gamma_\varepsilon}\Bigg[1 + \frac{(\Delta S_0)((\cdot - x_\varepsilon)/\mu_\varepsilon)}{\gamma_\varepsilon^2} + \frac{(\Delta S_1)((\cdot - x_\varepsilon)/\mu_\varepsilon)}{\gamma_\varepsilon^4} + (A(\gamma_\varepsilon) - 2\xi_\varepsilon)(\Delta S_2)\left(\frac{\cdot - x_\varepsilon}{\mu_\varepsilon}\right) + o(\zeta_\varepsilon \exp(\tilde{\delta}_0 t_\varepsilon))\Bigg] \tag{A-9}$$

uniformly in $\{y : t_\varepsilon(y) \le \gamma_\varepsilon\}$, for some given $\tilde{\delta}_0 \in (\delta_0, 1)$, for δ_0 as in (1-5).

Proof of Proposition A.1. Since both arguments are very similar to prove for Case 1′ and Case 2′, for the sake of readability, we only write the proof of Proposition A.1 in the more delicate Case 1′. Then, assume that we are in Case 1′. We let τ_ε be given by

$$B_\varepsilon = \gamma_\varepsilon - \frac{\tau_\varepsilon}{\gamma_\varepsilon}. \tag{A-10}$$

Let $\bar{w}_\varepsilon$ be given by

$$B_\varepsilon = \gamma_\varepsilon - \frac{t_\varepsilon}{\gamma_\varepsilon} + \frac{S_{0,\varepsilon}}{\gamma_\varepsilon^3} + \frac{S_{1,\varepsilon}}{\gamma_\varepsilon^5} + (A(\gamma_\varepsilon) - 2\xi_\varepsilon)\frac{S_{2,\varepsilon}}{\gamma_\varepsilon} + \frac{\zeta_\varepsilon \bar{w}_\varepsilon}{\gamma_\varepsilon}. \tag{A-11}$$

Let $\bar{\delta} > 0$ be fixed and let $\bar{r}_\varepsilon \ge 0$ be given by

$$\bar{r}_\varepsilon = \sup\{r > 0 : |\bar{w}_\varepsilon| \le \bar{\delta} t_\varepsilon \text{ in } [0, r]\}. \tag{A-12}$$

Now, since $\bar{\delta} > 0$ may be arbitrarily small, in order to get Proposition A.1, it is sufficient to prove that $\bar{r}_\varepsilon = \rho_\varepsilon$ for all $0 < \varepsilon \ll 1$. Using (A-12), we perform computations in $[0, \bar{r}_\varepsilon]$ and the subsequent $o(1)$ are uniformly small in this set as $\varepsilon \to 0$. First, by (1-5), (A-3), (A-6) and (A-12), we have

$$\tau_\varepsilon = t_\varepsilon(1 + o(1)). \tag{A-13}$$

Observe that, as soon as we have $\Delta B_\varepsilon > 0$ in $[0, \bar{r}_\varepsilon]$, the solution B_ε is radially decreasing and (3-93) holds true in $[0, \bar{r}_\varepsilon]$. Let L_ε^H and L_ε^g be given by

$$H(B_\varepsilon) = H(\gamma_\varepsilon)(1 + L_\varepsilon^H), \quad \text{and then} \quad (1 + g(B_\varepsilon)) = H(\gamma_\varepsilon)(1 + L_\varepsilon^H + L_\varepsilon^g). \tag{A-14}$$

In view of (A-10) and (A-13), estimates of L_ε^H, L_ε^g are given by (1-5) and (3-2), respectively. We are now in position to expand the right-hand side of (3-81). From now on, it is convenient to write

$$\widetilde{N}_\varepsilon = N_\varepsilon - 1. \tag{A-15}$$

Going back to (3-27), we have

$$\frac{\Psi'_{N_\varepsilon}(B_\varepsilon)}{2} = B_\varepsilon H(\gamma_\varepsilon)\left[(1+L_\varepsilon^H)(1+\varphi_{\widetilde{N}_\varepsilon}(B_\varepsilon^2)) + L_\varepsilon^g\left(\frac{B_\varepsilon^{2N_\varepsilon}}{N_\varepsilon!} - B_\varepsilon^2\right)\right]. \tag{A-16}$$

By (3-83), (A-10) and (A-13) and since $\bar{r}_\varepsilon \le \rho_\varepsilon$, we have

$$\min_{[0,\bar{r}_\varepsilon]} B_\varepsilon \ge (\varepsilon_0 + o(1))\gamma_\varepsilon \to +\infty \tag{A-17}$$

as $\varepsilon \to 0$. Thus, by Stirling's formula, we get

$$\frac{B_\varepsilon^{2N_\varepsilon}}{N_\varepsilon!} \ge \exp\left(N_\varepsilon\left(\log\frac{\gamma_\varepsilon^2}{N_\varepsilon} + (\log\varepsilon_0^2 + 1) + o(1)\right)\right)$$

and then, for all given integers $k \ge 0$,

$$B_\varepsilon^k = o(1)\frac{B_\varepsilon^{2N_\varepsilon}}{N_\varepsilon!} \tag{A-18}$$

in $[0, \bar{r}_\varepsilon]$ as $\varepsilon \to 0$, using $\varepsilon_0^2 > 1/e$ with (3-31). Similarly, for all given integers $k \ge 0$, we have

$$\frac{B_\varepsilon^k}{\varphi_{N_\varepsilon}(B_\varepsilon^2)} = o(1) \tag{A-19}$$

in $[0, \bar{r}_\varepsilon]$ as $\varepsilon \to 0$. Then, by (3-40), (A-10), (A-19) and (A-18), we may rewrite (A-16) as

$$\frac{\lambda_\varepsilon\Psi'_{N_\varepsilon}(B_\varepsilon)}{2} = \frac{4}{\mu_\varepsilon^2\gamma_\varepsilon}\left(1-\frac{\tau_\varepsilon}{\gamma_\varepsilon^2}\right)\left[O(\exp(-\gamma_\varepsilon^2)) + \frac{\varphi_{\widetilde{N}_\varepsilon}(B_\varepsilon^2)}{\varphi_{\widetilde{N}_\varepsilon}(\gamma_\varepsilon^2)}\left(1 + L_\varepsilon^H + O\left(\frac{B_\varepsilon^{2N_\varepsilon}}{N_\varepsilon!\,\varphi_{\widetilde{N}_\varepsilon}(B_\varepsilon^2)}L_\varepsilon^g\right)\right)\right] \tag{A-20}$$

in $[0, \bar{r}_\varepsilon]$, as $\varepsilon \to 0$. Indeed, by (A-17), we have

$$L_\varepsilon^H = o(1) \quad \text{and} \quad L_\varepsilon^g = o(1) \tag{A-21}$$

in $[0, \bar{r}_\varepsilon]$ as $\varepsilon \to 0$, using (1-1), (3-3) and (A-14). In (A-20), the term $O(\exp(-\gamma_\varepsilon^2))$ is equal to $(1+L_\varepsilon^H)/\varphi_{\widetilde{N}_\varepsilon}(\gamma_\varepsilon^2)$ and we thus get this control by (3-30) and (A-21). In the following lines, we expand the terms of (A-20). By (3-48) with $\Gamma = \gamma_\varepsilon^2$ and $T = B_\varepsilon^2$, we get

$$\frac{\varphi_{\widetilde{N}_\varepsilon}(B_\varepsilon^2)}{\varphi_{\widetilde{N}_\varepsilon}(\gamma_\varepsilon^2)} = \exp(B_\varepsilon^2 - \gamma_\varepsilon^2) - F_\varepsilon,$$

where F_ε satisfies in $[0, \bar{r}_\varepsilon]$

$$\begin{aligned} F_\varepsilon &= \frac{B_\varepsilon^{2\widetilde{N}_\varepsilon}}{\widetilde{N}_\varepsilon!\,\varphi_{\widetilde{N}_\varepsilon}(\gamma_\varepsilon^2)}\int_0^{\gamma_\varepsilon^2 - B_\varepsilon^2}\exp(-u)\left(1+\frac{u}{B_\varepsilon^2}\right)^{\widetilde{N}_\varepsilon}du \\ &= \frac{\exp(B_\varepsilon^2)}{\varphi_{\widetilde{N}_\varepsilon}(\gamma_\varepsilon^2)}\int_{B_\varepsilon^2}^{\gamma_\varepsilon^2}\exp(-s)\frac{s^{\widetilde{N}_\varepsilon}}{\widetilde{N}_\varepsilon!}\,ds = \xi_\varepsilon\exp(B_\varepsilon^2-\gamma_\varepsilon^2)\int_{B_\varepsilon^2-\gamma_\varepsilon^2}^0\exp(-y)\left(1+\frac{y}{\gamma_\varepsilon^2}\right)^{\widetilde{N}_\varepsilon}dy. \end{aligned} \tag{A-22}$$

By (A-10) and (A-11), we may write

$$\tau_\varepsilon = t_\varepsilon - \frac{S_{0,\varepsilon}}{\gamma_\varepsilon^2} - \frac{S_{1,\varepsilon}}{\gamma_\varepsilon^4} - (A(\gamma_\varepsilon) - 2\xi_\varepsilon)S_{2,\varepsilon} - \zeta_\varepsilon\bar{w}_\varepsilon.$$

Then, keeping in mind (A-3), (A-6), (A-12), (A-13) and $t_\varepsilon \le \gamma_\varepsilon^2$, we may compute

$$\exp(B_\varepsilon^2-\gamma_\varepsilon^2)=\exp\left(-2\tau_\varepsilon+\frac{\tau_\varepsilon^2}{\gamma_\varepsilon^2}\right)=\exp\left[-2\tau_\varepsilon+\frac{1}{\gamma_\varepsilon^2}\left(t_\varepsilon^2-\frac{2t_\varepsilon S_{0,\varepsilon}}{\gamma_\varepsilon^2}+O(\zeta_\varepsilon\bar{t}_\varepsilon^2)\right)\right] \tag{A-23}$$

in $[0,\bar{r}_\varepsilon]$ as $\varepsilon\to 0$. Observe that

$$\left|\exp(y)-\sum_{j=0}^{N}\frac{y^j}{j!}\right|\le\frac{|y|^{N+1}}{(N+1)!}\exp(|y|)$$

for all $y\in\mathbb{R}$ and all integers $N\ge 0$. Then we draw from (A-23) that

$$\begin{aligned}\left(1-\frac{\tau_\varepsilon}{\gamma_\varepsilon^2}\right)&\exp(B_\varepsilon^2-\gamma_\varepsilon^2)\\ &=\exp(-2t_\varepsilon)\Bigg[1+\frac{1}{\gamma_\varepsilon^2}(2S_{0,\varepsilon}+t_\varepsilon^2-t_\varepsilon)+\frac{1}{\gamma_\varepsilon^4}\big(2S_{1,\varepsilon}+2S_{0,\varepsilon}^2+\tfrac{1}{2}t_\varepsilon^4+2S_{0,\varepsilon}t_\varepsilon^2-4S_{0,\varepsilon}t_\varepsilon-t_\varepsilon^3+S_{0,\varepsilon}\big)\\ &\qquad+2(A(\gamma_\varepsilon)-2\xi_\varepsilon)S_{2,\varepsilon}+2\zeta_\varepsilon\bar{w}_\varepsilon+O\left(\left(\frac{\bar{t}_\varepsilon^6}{\gamma_\varepsilon^6}+\frac{\zeta_\varepsilon\bar{t}_\varepsilon^3}{\gamma_\varepsilon^2}+\zeta_\varepsilon^2\bar{t}_\varepsilon^3\right)\exp\left(o(t_\varepsilon)+\frac{t_\varepsilon^2}{\gamma_\varepsilon^2}\right)\right)\Bigg]\end{aligned} \tag{A-24}$$

in $[0,\bar{r}_\varepsilon]$ as $\varepsilon\to 0$. Independently, by (3-30), (3-45), (A-10), (A-12), (A-13) and since $B_\varepsilon(x_\varepsilon)=\gamma_\varepsilon$, for all given $R>0$, we have

$$\left\|\frac{B_\varepsilon^{2\widetilde{N}_\varepsilon}}{\widetilde{N}_\varepsilon!\,\varphi_{\widetilde{N}_\varepsilon}(B_\varepsilon^2)}+\frac{B_\varepsilon^{2N_\varepsilon}}{N_\varepsilon!\,\varphi_{\widetilde{N}_\varepsilon}(B_\varepsilon^2)}\right\|_{L^\infty([0,\min(R\mu_\varepsilon,\bar{r}_\varepsilon)]}=O\left(\frac{1}{\sqrt{N_\varepsilon}}\right)\quad\text{and}\quad\frac{B_\varepsilon^{2N_\varepsilon}}{N_\varepsilon!\,\varphi_{\widetilde{N}_\varepsilon}(B_\varepsilon^2)}\le 1 \tag{A-25}$$

in $[0,\bar{r}_\varepsilon]$, the second inequality being obvious by (3-6) and (A-15). In the sequel, by (3-31), we may assume that

$$\beta_\varepsilon:=\frac{\widetilde{N}_\varepsilon}{\gamma_\varepsilon^2}\quad\text{satisfies}\quad\lim_{\varepsilon\to 0}\beta_\varepsilon=\beta_0\in[0,1], \tag{A-26}$$

up to a subsequence. Now, we give estimates for F_ε given in (A-22). Up to a subsequence, we can split our results according to the following two cases:

$$\lim_{\varepsilon\to 0}\frac{\gamma_\varepsilon^2-\widetilde{N}_\varepsilon}{\sqrt{\widetilde{N}_\varepsilon}}=+\infty, \tag{A-27a}$$

$$\frac{\gamma_\varepsilon^2-\widetilde{N}_\varepsilon}{\sqrt{\widetilde{N}_\varepsilon}}=O(1). \tag{A-27b}$$

Observe that, since we assume (3-31), all the possible situations are considered in (A-27). Let $(r_\varepsilon)_\varepsilon$ be any sequence such that

$$r_\varepsilon\in[0,\bar{r}_\varepsilon] \tag{A-28}$$

for all ε. We prove that, in the case of (A-27a),

$$F_\varepsilon(r_\varepsilon)=\begin{cases}O\big(\xi_\varepsilon\gamma_\varepsilon\exp\big(-2t_\varepsilon(r_\varepsilon)(\beta_0+o(1))\big)\big) & \text{if } B_\varepsilon(r_\varepsilon)^2\ge\widetilde{N}_\varepsilon+\sqrt{\widetilde{N}_\varepsilon},\\ O\big(\exp\big(-(1+\varepsilon_0+o(1))t_\varepsilon(r_\varepsilon)\big)\big) & \text{if } B_\varepsilon(r_\varepsilon)^2<\widetilde{N}_\varepsilon+\sqrt{\widetilde{N}_\varepsilon},\end{cases} \tag{A-29}$$

while we get in the case of (A-27b)

$$F_\varepsilon(r_\varepsilon)=\begin{cases}2t_\varepsilon(r_\varepsilon)\xi_\varepsilon\exp\bigl(-2t_\varepsilon(r_\varepsilon)(1+o(1))\bigr) & \text{if } t_\varepsilon(r_\varepsilon)=o(\gamma_\varepsilon),\\ O\bigl(t_\varepsilon(r_\varepsilon)\xi_\varepsilon\exp\bigl(-(1+\varepsilon_0+o(1))t_\varepsilon(r_\varepsilon)\bigr)\bigr) & \text{if } \gamma_\varepsilon=O(t_\varepsilon(r_\varepsilon)).\end{cases}\tag{A-30}$$

Now we prove (A-29). We start with the first estimate of (A-29). Then, we assume that $B_\varepsilon(r_\varepsilon)^2\geq\widetilde{N}_\varepsilon+\sqrt{\widetilde{N}_\varepsilon}$, and thus in particular that

$$1-\frac{\widetilde{N}_\varepsilon}{B_\varepsilon(r_\varepsilon)^2}\geq\frac{1+o(1)}{\sqrt{\widetilde{N}_\varepsilon}}.\tag{A-31}$$

Writing now F_ε according to the first formula of (A-22), using (3-93), (A-17) and

$$\log(1+t)\leq t\quad\text{for all } t>-1,\tag{A-32}$$

we get first that

$$F_\varepsilon(r_\varepsilon)\leq\xi_\varepsilon\exp(-2\tau_\varepsilon(r_\varepsilon)\beta_\varepsilon)\int_0^{\gamma_\varepsilon^2-B_\varepsilon^2}\exp\biggl(-y\biggl(1-\frac{\widetilde{N}_\varepsilon}{B_\varepsilon(r_\varepsilon)^2}\biggr)\biggr)\,dy,$$

and conclude the proof of the first estimate of (A-29), by (3-31), (A-13) and (A-31). In order to prove the second estimate of (A-29), it is sufficient to write F_ε according to the second formula of (A-22), to check that

$$\int_{\mathbb{R}}\exp(-s)\frac{s^{\widetilde{N}_\varepsilon}}{\widetilde{N}_\varepsilon!}\,ds=1,$$

that $r_\varepsilon\leq\bar{r}_\varepsilon\leq\rho_\varepsilon$ implies

$$t_\varepsilon(\bar{r}_\varepsilon)\leq(1-\varepsilon_0)\gamma_\varepsilon^2,\tag{A-33}$$

and to use (A-10), (A-13) and (3-30). Now we turn to the proof of (A-30). Then, we assume (A-27b) holds true and in particular

$$1-\beta_\varepsilon=O\biggl(\frac{1}{\gamma_\varepsilon}\biggr)\quad\text{in (A-27b)}.\tag{A-34}$$

Writing F_ε according to the third estimate of (A-22), we get

$$F_\varepsilon=\xi_\varepsilon\exp\biggl(-\tau_\varepsilon\biggl(2-\frac{\tau_\varepsilon}{\gamma_\varepsilon^2}\biggr)\biggr)(\gamma_\varepsilon^2-B_\varepsilon^2)\int_0^1\exp\biggl((\gamma_\varepsilon^2-B_\varepsilon^2)y+\widetilde{N}_\varepsilon\log\biggl(1-\frac{(\gamma_\varepsilon^2-B_\varepsilon^2)y}{\gamma_\varepsilon^2}\biggr)\biggr)\,dy\tag{A-35}$$

at r_ε. Expanding the log, we easily get the first estimate of (A-30) from (A-13), (A-34), (A-35) and the assumption $t_\varepsilon(r_\varepsilon)=o(\gamma_\varepsilon)$. The second estimate of (A-30) can also be obtained from (A-35) by (A-13), (A-32), (A-33) and (A-34). This concludes the proof of (A-30). Now, we prove that, in the case of (A-27a), we have

$$\int_0^{\bar{r}_\varepsilon}F_\varepsilon(r)r\,dr=o\biggl(\frac{\mu_\varepsilon^2}{\gamma_\varepsilon^4}\biggr).\tag{A-36}$$

Since $r_\varepsilon \le \rho_\varepsilon$, we get from (3-14), (3-30), (3-31), (A-29) and by Stirling's formula that

$$\int_{\{r\in[0,\bar r_\varepsilon],\,B_\varepsilon(r)^2\ge \widetilde N_\varepsilon+\sqrt{\widetilde N_\varepsilon}\}} F_\varepsilon(r) r\,dr \lesssim \exp(\gamma_\varepsilon^2[f(\beta_\varepsilon)+O((\log\gamma_\varepsilon)/\gamma_\varepsilon^2)]) \begin{cases} \mu_\varepsilon^2 & \text{if } \beta_0>\frac12,\\ \mu_\varepsilon^2\exp\bigl(\gamma_\varepsilon^2(1-\varepsilon_0)(1-2\beta_0+o(1))\bigr) & \text{if } \beta_0\le\frac12,\end{cases} \tag{A-37}$$

where f is the continuous function in [0, 1] given for $\beta\in(0,1]$ by

$$f(\beta)=\beta\log\frac{1}{\beta}+\beta-1.$$

Independently, since $\bar r_\varepsilon \le \rho_\varepsilon$, if

$$r_\varepsilon\in J_\varepsilon := \bigl\{r\in[0,\bar r_\varepsilon] : B_\varepsilon(r)^2<\widetilde N_\varepsilon+\sqrt{\widetilde N_\varepsilon}\bigr\},$$

then $J_\varepsilon\ne\varnothing$ and $\gamma_\varepsilon^2\lesssim \widetilde N_\varepsilon$, by (A-10), (A-13) and (A-33). Thus we have

$$\gamma_\varepsilon\lesssim\sqrt{\widetilde N_\varepsilon}\ll t_\varepsilon(r_\varepsilon),$$

using that we are in the case (A-27a) for the last estimate. Then, we get from (A-29) that

$$\int_{J_\varepsilon} F_\varepsilon(r) r\,dr \lesssim \int_{\{r\le\rho_\varepsilon,\,t_\varepsilon\ge\gamma_\varepsilon\}} \exp\bigl(-(1+\varepsilon_0+o(1))t_\varepsilon(r)\bigr) r\,dr = o\biggl(\frac{\mu_\varepsilon^2}{\gamma_\varepsilon^4}\biggr). \tag{A-38}$$

Observe that f and $\beta\mapsto f(\beta)+\frac12(1-2\beta)$ are negative in [0, 1) and $\bigl[0,\frac12\bigr]$ respectively. Moreover, because of (A-27a) and by (3-31), we can check that

$$\beta_\varepsilon=\frac{\widetilde N_\varepsilon}{\gamma_\varepsilon^2}\le\frac{1}{1+1/\sqrt{\widetilde N_\varepsilon}}\le 1-\frac{1+o(1)}{\sqrt{\widetilde N_\varepsilon}}\le 1-\frac{1+o(1)}{\gamma_\varepsilon},$$

since $\gamma_\varepsilon^2\ge\widetilde N_\varepsilon+\sqrt{\widetilde N_\varepsilon}$, and then that

$$0<-f(\beta_\varepsilon)\lesssim\frac{1}{\gamma_\varepsilon}. \tag{A-39}$$

Thus, we get (A-36) from the first estimate of (A-37) with (A-39), from the second estimate of (A-37) with $1-\varepsilon_0<1-\sqrt{1/e}<\frac12$ and from (A-38). Computing as in (A-37), we get also that

$$\xi_\varepsilon = o\biggl(\frac{1}{\gamma_\varepsilon^4}\biggr) \tag{A-40}$$

in (A-27a) (see (A-39)). By (A-13) and the second part of (A-25), using that $\bar r_\varepsilon\le\rho_\varepsilon$, we may rewrite (A-20) as

$$\begin{aligned}\frac{\lambda_\varepsilon\Psi'_{N_\varepsilon}(B_\varepsilon)}{2} &= \frac{4}{\mu_\varepsilon^2\gamma_\varepsilon}\biggl[\biggl(1-\frac{\tau_\varepsilon}{\gamma_\varepsilon^2}+L_\varepsilon^H\biggr)\exp(B_\varepsilon^2-\gamma_\varepsilon^2)-F_\varepsilon+O\biggl(\frac{t_\varepsilon}{\gamma_\varepsilon^2}|F_\varepsilon|+\exp(-\gamma_\varepsilon^2)\biggr)\\ &\quad+O\biggl(\biggl(\frac{t_\varepsilon}{\gamma_\varepsilon^2}\exp(B_\varepsilon^2-\gamma_\varepsilon^2)+|F_\varepsilon|\biggr)(|L_\varepsilon^H|+|L_\varepsilon^g|)\biggr)+O\biggl(|L_\varepsilon^g|\exp(B_\varepsilon^2-\gamma_\varepsilon^2)\frac{B_\varepsilon^{2N_\varepsilon}}{N_\varepsilon!\,\varphi_{\widetilde N_\varepsilon}(B_\varepsilon^2)}\biggr)\biggr].\end{aligned} \tag{A-41}$$

By (3-84), we clearly have

$$\int_0^{\rho_\varepsilon} \exp(-\gamma_\varepsilon^2) r\, dr = o\left(\frac{\mu_\varepsilon^2}{\gamma_\varepsilon^4}\right). \tag{A-42}$$

Integrating by parts, observe that $\bar{w}_\varepsilon$ given by (A-11) satisfies

$$\bar{w}_\varepsilon(0) = 0 \quad \text{and} \quad -r_\varepsilon \bar{w}_\varepsilon'(r_\varepsilon) = \int_0^{r_\varepsilon} (\Delta \bar{w}_\varepsilon) r\, dr, \tag{A-43}$$

where, still using radial notation,

$$\bar{w}_\varepsilon'(r) = \frac{d\bar{w}_\varepsilon}{dr}(r).$$

From now on, we estimate $\bar{w}_\varepsilon$ in $[0, \bar{r}_\varepsilon]$ with (A-43). By (1-5) and (A-14), we may expand L_ε^H in (A-41). Now, since (3-81) holds true, by taking the laplacian of B_ε, we get from (A-11) and (A-41) an estimate of $\Delta \bar{w}_\varepsilon$ and then of the right-hand side of (A-43) for r_ε still as in (A-28). The key observation is that the precise form of the ODEs in (A-2) generates a cancellation, when plugging (A-24) in (A-41). The lower-order terms when taking the laplacian of (A-11) are estimated thanks to (A-3). We are left with estimating the lower-order terms in (A-41), in both cases of (A-27). Assume first that we are in the case of (A-27a). Estimating these lower-order terms amounts to gathering the appropriate previous estimates (see (A-21), (A-25), (A-29), (A-36), (A-40), (A-42)). This gives after some slightly long, but elementary computations that

$$\int_0^{r_\varepsilon} |(\Delta \bar{w}_\varepsilon)| r\, dr = O\left(\|\bar{w}_\varepsilon'\|_{L^\infty([0,r_\varepsilon])} \int_0^{r_\varepsilon/\mu_\varepsilon} \frac{\mu_\varepsilon r^2\, dr}{(1+r^2)^{1+\varepsilon_0+o(1)}}\right) + o\left(\int_0^{r_\varepsilon/\mu_\varepsilon} \frac{r\, dr}{(1+r^2)^{1+\varepsilon_0+o(1)}}\right). \tag{A-44}$$

We also use (1-5) and (3-2) to estimate L_ε^H and L_ε^g. The first term in the right-hand side of (A-44) uses that, for all $r \in [0, r_\varepsilon]$,

$$|\bar{w}_\varepsilon(r)| \le r \|\bar{w}_\varepsilon'\|_{L^\infty([0,r_\varepsilon])}.$$

Observe now that (A-44) still holds true in the case of (A-27b), replacing (A-29), (A-36) and (A-40) by (A-30) in the above argument. Since $\varepsilon_0 > \frac{1}{2}$, we clearly get from (A-43) and (A-44) that, in both cases of (A-27),

$$r_\varepsilon |\bar{w}_\varepsilon'(r_\varepsilon)| = O\left(\|\bar{w}_\varepsilon'\|_{L^\infty([0,r_\varepsilon])} \frac{\mu_\varepsilon (r_\varepsilon/\mu_\varepsilon)^3}{1 + (r_\varepsilon/\mu_\varepsilon)^3}\right) + o\left(\frac{(r_\varepsilon/\mu_\varepsilon)^2}{1 + (r_\varepsilon/\mu_\varepsilon)^2}\right). \tag{A-45}$$

Now we prove that

$$\mu_\varepsilon \|\bar{w}_\varepsilon'\|_{L^\infty([0,\bar{r}_\varepsilon])} = o(1). \tag{A-46}$$

If (A-46) does not hold true, then, by (A-45), there exists $s_\varepsilon \in [0, \bar{r}_\varepsilon]$ such that $s_\varepsilon = O(\mu_\varepsilon)$, $\mu_\varepsilon = O(s_\varepsilon)$,

$$|\bar{w}_\varepsilon'(s_\varepsilon)| = \|\bar{w}_\varepsilon'\|_{L^\infty([0,\bar{r}_\varepsilon])} \quad \text{and} \quad \limsup_{\varepsilon \to 0} \mu_\varepsilon |\bar{w}_\varepsilon'(s_\varepsilon)| > 0. \tag{A-47}$$

In particular, up to a subsequence, we may assume that there exists $\alpha_0 \in (0, +\infty]$ such that $\bar{r}_\varepsilon/\mu_\varepsilon \to \alpha_0$ as $\varepsilon \to 0$. Let $\tilde{w}_\varepsilon$ be given by

$$\tilde{w}_\varepsilon(y) = \bar{w}_\varepsilon(\mu_\varepsilon y)/(\mu_\varepsilon \|\bar{w}_\varepsilon'\|_{L^\infty([0,\bar{r}_\varepsilon])}).$$

By (A-45) and (A-47), we get that $(\|(1+\cdot)\tilde{w}'_\varepsilon\|_{L^\infty([0,\bar{r}_\varepsilon/\mu_\varepsilon])})_\varepsilon$ is a bounded sequence. Then, computing as in (A-44) and by radial elliptic theory with (3-81), we get $\tilde{w}_\varepsilon \to \tilde{w}_0$ in $C^2([0,\alpha_0])$ if $\alpha_0 < +\infty$ or in $C^2_{\mathrm{loc}}([0,\alpha_0))$ if $\alpha_0 = +\infty$, where $\tilde{w}_0$ solves

$$\begin{cases} \Delta\tilde{w}_0 = 8\exp(-2T_0)\tilde{w}_0 \ \text{ in } B_0(\alpha_0), \\ \tilde{w}_0(0) = 0, \\ \tilde{w}_0 \text{ is radial around } 0 \in \mathbb{R}^2, \end{cases}$$

still making usual radial identifications, and where T_0 is given in (A-1). By the standard theory of radial elliptic equations, this implies $\tilde{w}_0 \equiv 0$, which contradicts (A-47) and proves (A-46). Then, since $\bar{w}_\varepsilon(0) = 0$ and by the fundamental theorem of calculus, we get from (A-45) with (A-46) that $\bar{r}_\varepsilon = \rho_\varepsilon$ in (A-12). By the discussion just above (A-13), this concludes the proof of Proposition A.1. □

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WELL-POSEDNESS OF THE HYDROSTATIC NAVIER–STOKES EQUATIONS

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We address the local well-posedness of the *hydrostatic Navier–Stokes* equations. These equations, sometimes called *reduced Navier–Stokes/Prandtl*, appear as a formal limit of the Navier–Stokes system in thin domains, under certain constraints on the aspect ratio and the Reynolds number. It is known that without any structural assumption on the initial data, real-analyticity is both necessary and sufficient for the local well-posedness of the system. In this paper we prove that for convex initial data, local well-posedness holds under simple Gevrey regularity.

1. Introduction

The present paper is devoted to the study of the two-dimensional system

$$\partial_t u + u\,\partial_x u + v\,\partial_y u + \partial_x p - \eta\,\partial_y^2 u = 0, \quad (x, y) \in \mathbb{T} \times (0, 1), \tag{1-1a}$$

$$\partial_y p = 0, \quad (x, y) \in \mathbb{T} \times (0, 1), \tag{1-1b}$$

$$\partial_x u + \partial_y v = 0, \quad (x, y) \in \mathbb{T} \times (0, 1), \tag{1-1c}$$

$$u|_{y=0,1} = v|_{y=0,1} = 0, \quad x \in \mathbb{T}, \tag{1-1d}$$

where $\eta > 0$. The unknowns of this system are $(u, v) = (u, v)(x, y, t)$ and $p = p(x, y, t)$, which model respectively the velocity field and pressure of a fluid flow. The boundary condition (1-1d) corresponds to a no-slip condition at the walls $y = 0, 1$. With respect to the tangential variable x we impose $\mathbb{T}$-periodic (lateral) boundary conditions.

Note that upon integrating in y the incompressibility equation (1-1c), using the boundary condition for v (1-1d) we obtain the compatibility condition

$$\partial_x \int_0^1 u(x, y, t)\,dy = 0 \tag{1-2}$$

for all $x \in \mathbb{T}$ and $t \geq 0$, so that the vertical mean of u is just a function of time. Condition (1-2) allows us to compute the pressure gradient, see (2-4) below, and to obtain the boundary condition for the vorticity, see (2-6b) below.

System (1-1) is formally obtained [Lagrée and Lorthois 2005; Renardy 2009] when considering the asymptotics of the two-dimensional Navier–Stokes equation in a thin domain: $\Omega = (0, L) \times (0, l)$ with

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$\delta = l/L \ll 1$. After a proper rescaling

$$t := \frac{Ut}{L}, \quad x := \frac{x}{L}, \quad y := \frac{y}{l}, \quad u := \frac{u}{U}, \quad v := \frac{v}{\delta U},$$

the Navier–Stokes equation becomes

$$\partial_t u + u\,\partial_x u + v\,\partial_y u + \partial_x p - \eta\delta^2\,\partial_x^2 - \eta\,\partial_y^2 u = 0, \quad (x, y) \in \mathbb{T} \times (0, 1), \tag{1-3a}$$

$$\delta^2(\partial_t v + u\,\partial_x v + v\,\partial_y v) + \partial_y p - \eta\delta^4\,\partial_x^2 v - \eta\delta^2\,\partial_y^2 v = 0, \quad (x, y) \in \mathbb{T} \times (0, 1), \tag{1-3b}$$

$$\partial_x u + \partial_y v = 0, \quad (x, y) \in \mathbb{T} \times (0, 1), \tag{1-3c}$$

where

$$\eta = \frac{1}{\delta^2\,\mathrm{Re}},$$

with $\mathrm{Re} = UL/\nu$ the Reynolds number. If we assume $\eta \sim 1$ and keep the leading-order terms as $\delta \to 0$, or if we assume $\eta \ll 1$ and keep both the leading-order and next-order terms in (1-3), we end up with (1-1).

Our concern here will be the local-in-time well-posedness of (1-1). Besides its mathematical relevance, this problem is meaningful from the point of view of hydrodynamic stability, notably with regards to the properties of the so-called *primitive equations*

$$\partial_t u + u\,\partial_x u + v\,\partial_y u + \partial_x p - \eta'\,\partial_x^2 - \eta\,\partial_y^2 u = 0, \quad (x, y) \in \mathbb{T} \times (0, 1), \tag{1-4a}$$

$$\partial_y p = 0, \quad (x, y) \in \mathbb{T} \times (0, 1), \tag{1-4b}$$

$$\partial_x u + \partial_y v = 0, \quad (x, y) \in \mathbb{T} \times (0, 1). \tag{1-4c}$$

This model and its three-dimensional counterpart are very important in atmospheric sciences, after accounting for gravity and many other features [Lions et al. 1992a; 1992b; Temam and Ziane 2004; Petcu et al. 2009]. For positive values of tangential and transverse viscosity coefficients, they are known to be globally well-posed in the Sobolev setting in both the two- and the three-dimensional case [Ziane 1997; Bresch et al. 2003; 2005; Temam and Ziane 2004; Cao and Titi 2007; Kobelkov 2007; Kukavica and Ziane 2007; 2008], and the vanishing viscosity limit $\eta, \eta' \to 0$ can be characterized in the real-analytic category [Kukavica et al. 2016]. Yet, in the absence of additional turbulent viscosity, the dimensional analysis of (1-3) shows that the tangential diffusion coefficient η' is expected to be very small. This allows to relate the well-/ill-posedness of (1-1) and the stability/instability properties of (1-4). For instance, assume that (1-1) is linearly ill-posed without analyticity in x: a result in this direction was shown in [Renardy 2009], and will be discussed later on. It roughly means that, at least in the early stages of the evolution, there are perturbations with wave number $k \gg 1$ in x that grow like $e^{|k|t}$. From there, if η' is small enough so that $\eta'|k|^2 \ll 1$, one can expect the tangential diffusion $-\eta'\,\partial_x^2$ to stay negligible, and the perturbation to be an approximate solution of (1-4) (with Dirichlet conditions). This can result in a growth almost as strong as $e^{t/\sqrt{\eta'}}$, showing the strong instability of (1-4). We note that if one keeps $\eta' > 0$ in (1-4) while setting $\eta = 0$, the local well-posedness can be established for Sobolev initial datum [Cao et al. 2016; 2017], confirming that the horizontal dissipation dominated equation is much more stable than the hydrostatic Navier–Stokes system (1-1) considered in this paper.

From a mathematical perspective, system (1-3) is reminiscent of the two-dimensional Prandtl system, describing boundary layer flows. The latter is set in a half-plane, say $\mathbb{T} \times \mathbb{R}_+$, and reads

$$\partial_t u + u\,\partial_x u + v\,\partial_y u + \partial_x p - \eta\,\partial_y^2 u = 0, \quad (x, y) \in \mathbb{T} \times \mathbb{R}_+, \tag{1-5a}$$

$$\partial_y p = 0, \quad (x, y) \in \mathbb{T} \times \mathbb{R}_+, \tag{1-5b}$$

$$\partial_x u + \partial_y v = 0, \quad (x, y) \in \mathbb{T} \times \mathbb{R}_+, \tag{1-5c}$$

$$u|_{y=0} = v|_{y=0} = 0, \tag{1-5d}$$

$$\lim_{y \to +\infty} u = u^\infty, \tag{1-5e}$$

$$\lim_{y \to +\infty} p = p^\infty. \tag{1-5f}$$

Hence, the only difference with (1-1) lies in the domain and in the boundary conditions. Here, u^∞ and p^∞ are given data, related to the Euler flow above the boundary layer. In particular, as p does not depend on y, it is no longer an unknown of the system. This is a major difference with (1-1), where p can be seen as a Lagrange multiplier, associated to the constraint that $v = -\int_0^y \partial_x u$ vanishes at $y = 1$ (see (2-4) below).

The well-posedness properties of (1-5) are now well-understood, and depend on the monotonicity properties of the initial data. Roughly, if the data have Sobolev regularity, and if furthermore the initial data are monotonic in y, (1-5) has local-in-time Sobolev solutions [Oleinik 1966; Masmoudi and Wong 2015]. On the other hand, without monotonicity, system (1-5) is ill-posed in Sobolev spaces [Gérard-Varet and Dormy 2010; Gérard-Varet and Nguyen 2012]. Local-in-time well-posedness can be achieved when the initial datum is real analytic [Sammartino and Caflisch 1998; Kukavica and Vicol 2013], and even under the milder condition of Gevrey regularity in x [Gérard-Varet and Masmoudi 2015]. We refer to [E and Engquist 1997; Xin and Zhang 2004; Gérard-Varet et al. 2018; Ignatova and Vicol 2016; Kukavica et al. 2017; Dalibard and Masmoudi 2018] for more results on the Prandtl system such as singularities, long-time behavior, and Gevrey-class stability. Interestingly, the instability mechanism that yields ill-posedness in the Sobolev setting involves in a crucial manner the lack of monotonicity and the diffusion term $-\eta\,\partial_y^2 u$. Indeed, the inviscid version of Prandtl, that is,

$$\partial_t u + u\,\partial_x u + v\,\partial_y u + \partial_x p = 0, \quad (x, y) \in \mathbb{T} \times \mathbb{R}_+, \tag{1-6a}$$

$$\partial_y p = 0, \quad (x, y) \in \mathbb{T} \times \mathbb{R}_+, \tag{1-6b}$$

$$\partial_x u + \partial_y v = 0, \quad (x, y) \in \mathbb{T} \times \mathbb{R}_+, \tag{1-6c}$$

$$v|_{y=0} = 0, \tag{1-6d}$$

$$\lim_{y \to +\infty} p = p^\infty \tag{1-6e}$$

has local smooth solutions for smooth data, as can be shown by the method of characteristics [Hong and Hunter 2003].

With regards to this recent understanding of the Prandtl system, it is very natural to ask about the local well-posedness of (1-1), and to start from the consideration of the inviscid case $\eta = 0$, namely

$$\partial_t u + u\,\partial_x u + v\,\partial_y u + \partial_x p = 0, \quad (x, y) \in \mathbb{T} \times (0, 1), \tag{1-7a}$$

$$\partial_y p = 0, \quad (x, y) \in \mathbb{T} \times (0, 1), \tag{1-7b}$$

$$\partial_x u + \partial_y v = 0, \quad (x, y) \in \mathbb{T} \times (0, 1), \tag{1-7c}$$

$$v|_{y=0,1} = 0. \tag{1-7d}$$

This *hydrostatic Euler system* has been the matter of many studies [Brenier 1999; 2003; Grenier 1999; Renardy 2009; Kukavica et al. 2011; 2014; Masmoudi and Wong 2012; Cao et al. 2015; Wong 2015]. Contrary to (1-6), existence of local strong solutions requires a structural assumption, namely the uniform convexity (or concavity) in the variable y of the initial data. A contrario, the presence of an inflection point may trigger high-frequency instability. This point was established in [Renardy 2009], where the author considered the linearization of (1-7) around shear flows $u = U_s(y),\ v = 0$. More precisely, he showed that if the equation $\int_0^1 (U_s(y) - c)^{-2}\,dy = 0$ has complex roots, then the linearized hydrostatic Euler system admits perturbations which have wave number k in x and grow like $e^{\delta k t},\ \delta > 0$, for all $k \gg 1$. Returning to the nonlinear problem (1-7), one can only expect to show short-time stability for data whose Fourier transform in x behaves like $e^{-\delta|k|}$ for large k. This corresponds to analytic data in x. Local well-posedness in the analytic setting was established in [Kukavica et al. 2011]. Moreover, it is mentioned in [Renardy 2009] that this high-frequency instability persists in the case of the viscous system (1-1), at least for small enough η.

Considering all these results, the remaining task is to analyze the viscous system (1-1) for convex (or concave) initial data. This is the purpose of this paper. It raises strong mathematical issues, related to the control of x derivatives of the solution. In particular, we find

$$\partial_t(\partial_x u) + (u\,\partial_x + v\,\partial_y)(\partial_x u) + (\partial_x u)^2 + (\partial_x v)\,\partial_y u + \partial_x(\partial_x p) - \eta\,\partial_y^2(\partial_x u) = 0.$$

One of the main problems in controlling $\partial_x u$ is the term $\partial_x v\,\partial_y u$. Indeed, $\partial_x v = -\int_0^y \partial_x^2 u$ is recovered from the divergence-free condition, so that it can be seen as a first-order operator in x applied to $\partial_x u$. As this first-order term has no skew-symmetry, it does not disappear from energy estimates, so that standard energy arguments can only be conclusive with the help of analyticity. In the case of the hydrostatic Euler system, the way out of this difficulty consists in considering the (approximate) vorticity $\omega = \partial_y u$. Its tangential derivative is seen to satisfy

$$\partial_t(\partial_x \omega) + (u\,\partial_x + v\,\partial_y)(\partial_x \omega) + (\partial_x u)(\partial_x \omega) + (\partial_x v)\,\partial_y \omega = 0.$$

Under a uniform convexity or concavity assumption $|\partial_y \omega| \geq \alpha$, the idea is to test the equation against $\partial_x \omega / \partial_y \omega$ rather than $\partial_x \omega$, to take advantage of the cancellation

$$\int \partial_x v\,\partial_x \omega = -\int \partial_y \partial_x v\,\partial_x u = \int \partial_x^2 u\,\partial_x u = 0.$$

This allows one to get rid of the bad term and is the starting point of the local well-posedness argument. Such an idea was used previously in [Grenier 2000; Masmoudi and Wong 2012].

Unfortunately, this manipulation, which we will call *the hydrostatic trick*, is not fully appropriate for the viscous system (1-1). The reason is that in the estimate for $\partial_x \omega$ the viscous term generates extra boundary integrals such as

$$I^\flat = \eta \int_{\mathbb{T}\times\{0\}} \partial_y \partial_x \omega \, \frac{\partial_x \omega}{\partial_y \omega}\, dx, \qquad I^\sharp = \eta \int_{\mathbb{T}\times\{1\}} \partial_y \partial_x \omega \, \frac{\partial_x \omega}{\partial_y \omega}\, dx.$$

The value of $\partial_y \partial_x \omega$ at the boundary can be obtained from the equation on $\partial_x u$, and yields for instance (the computation will be detailed later)

$$\partial_y \partial_x \omega|_{y=0} = \partial_x^2 p = -2\partial_x \int_0^1 u\, \partial_x u\, dy + \partial_x \omega|_{y=1} - \partial_x \omega|_{y=0}.$$

The issue comes from the first term on the right hand-side, which is again a first-order term in $\partial_x u$ without any skew-symmetric structure. In other words, *there is an additional loss of derivative compared to the Prandtl equation*, so that obtaining well-posedness below analytic regularity is challenging. This is our goal in what follows, and we prove in Theorem 2.1 below the local well-posedness under Gevrey regularity of class $\frac{9}{8}$ in the x-variable, under an extra convexity assumption in y.

2. Main result and strategy

For notational simplicity, from now one we will set $\eta = 1$ in (1-1). Let $\Omega = \mathbb{T} \times (0, 1)$. For $\tau > 0,\ \gamma \geq 1$, we define the Gevrey norm

$$\|f\|_{\gamma,\tau}^2 = \sum_{j=0}^{\infty} \tau^{2j} (j!)^{-2\gamma} \|\partial_x^j f\|_{L^2(\Omega)}^2.$$

Functions f satisfying $\|f\|_{\gamma,\tau} < +\infty$ are in Gevrey class γ with respect to x, measured in L^2 in the y-variable. Our main result is the following:

Theorem 2.1 (well-posedness for convex Gevrey-class initial datum). *Let* $\tau^0 > \tau_1 > 0,\ \gamma \leq \frac{9}{8}$. *Let* u_0 *be a function satisfying the regularity condition*

$$\|\partial_y u_0\|_{\gamma,\tau^0} + \|\partial_y^3 u_0\|_{\gamma,\tau^0} < +\infty, \tag{2-1}$$

the convexity condition

$$\inf_{\Omega} \partial_y^2 u_0 > 0, \tag{2-2}$$

and the compatibility conditions $\partial_x \int_0^1 u_0\, dy = 0,\ u_0|_{y=0,1} = 0$,

$$\partial_y^2 u_0|_{y=0,1} = \int_0^1 (-\partial_x u_0^2 + \partial_y^2 u_0)\, dy - \int_\Omega \partial_y^2 u_0.$$

Then there exists $T > 0$, *and a unique solution* u *of* (1-1) *with initial data* u_0 *that satisfies*

$$\sup_{t\in[0,T]} (\|\partial_y u(t)\|_{\gamma,\tau_1} + \|\partial_y^3 u(t)\|_{\gamma,\tau_1}) < +\infty.$$

and

$$\inf_{t\in[0,T]\times\Omega} \partial_y^2 u > 0. \tag{2-3}$$

A few remarks are in order:

• The main point in our result is that we prove local well-posedness without analyticity, reaching exponents $\gamma > 1$. The value $\gamma = \frac{9}{8}$ is due to technical limitations, and could certainly be improved. The optimal value that can be expected for γ, or even the possibility of well-posedness in the Sobolev setting are interesting open questions. Our conjecture — based on a formal parallel with Tollmien–Schlichting instabilities for Navier–Stokes [Grenier et al. 2016] — is that the best exponent possible should be $\gamma = \frac{3}{2}$, but such result is for the time being out of reach. If confirmed, it would emphasize the destabilizing role of viscosity.

• We lose on the radius τ of Gevrey regularity, going from τ^0 to τ_1 in positive time. This loss is very standard [Sammartino and Caflisch 1998; Kukavica et al. 2011; Kukavica and Vicol 2013; Gérard-Varet and Masmoudi 2015].

• Besides the Gevrey regularity assumption (2-1), the key assumption is $\inf_\Omega \partial_y^2 u_0 > 0$, which corresponds to a strictly convex initial data. The strict concavity condition $\sup_\Omega \partial_y^2 u_0 < 0$ would work as well. On the opposite, as discussed before, we do not expect such well-posedness to hold for data with inflection points [Renardy 2009].

• The first compatibility condition $\partial_x \int_0^1 u_0 = 0$ is here to ensure that (1-2) holds for all time. Note that we can use (1-2) to determine $\partial_x p$: applying ∂_x to (1-1a), taking the mean over $y \in (0,1)$, integrating by parts in the term $\int_0^1 v\, \partial_y u\, dy$, and using the periodic lateral boundary conditions, we find

$$\partial_x p = \tilde{\omega}|_{y=1} - \tilde{\omega}|_{y=0} - \partial_x \int_0^1 u^2\, dy, \quad x \in \mathbb{T}, \tag{2-4}$$

where $\omega = \partial_y u$ is the vorticity, and we have denoted by

$$\tilde{\omega}(x,y,t) = \omega(x,y,t) - \int_{\mathbb{T}} \omega(x,y,t)\, dx, \quad y \in \{0,1\}, \tag{2-5}$$

the zero mean (in x) boundary vorticity. We will use the notation (2-5) throughout the paper. Note that for $y \in \{0,1\}$, the functions ω and $\tilde{\omega}$ only differ by a function of time.

• The second and third compatibility conditions can be explained as follows. Most of our analysis relies on the control of the vorticity $\omega = \partial_y u$. We notably need some bound on $\sup_{t\in[0,T]} \|\omega\|_{\gamma,\tau}$ for $\tau \in [\tau_1, \tau^0)$. If we leave aside the Gevrey regularity in x, this corresponds to an $L_t^\infty H_y^1$ bound on u. As u satisfies a heat-type equation with Dirichlet condition, it is well known that such an $L_t^\infty H_y^1$ bound requires the compatibility condition $u|_{t=0}|_{y=0,1} = u|_{y=0,1}|_{t=0}$. In view of (1-1c), this amounts to the second compatibility condition of the theorem: $u_0|_{y=0,1} = 0$.

Similarly, the last compatibility condition is related to the fact that we need a bound for $\sup_{t\in[0,T]} \|\partial_t \omega\|_{\gamma,\tau}$ for $\tau \in [\tau_1, \tau^0)$. More precisely, this condition can be derived from the system obeyed by $\omega = \partial_y u$, which is

$$\partial_t \omega + u\, \partial_x \omega + v\, \partial_y \omega - \partial_y^2 \omega = 0, \quad (x,y) \in \mathbb{T} \times (0,1), \tag{2-6a}$$

$$\partial_y \omega|_{y=0,1} = \tilde{\omega}|_{y=1} - \tilde{\omega}|_{y=0} - \partial_x \int_0^1 u^2\, dy. \tag{2-6b}$$

Indeed, (2-6a) follows from differentiating (1-1a) in y, while the boundary condition (2-6b) is obtained by evaluating (1-1a) at $y = 0, 1$, using the Dirichlet boundary conditions for u and v in (1-1d), and the formula for the pressure gradient (2-4). Now, from (2-6a), it appears that an $L^\infty_t L^2_y$ control of $\partial_t \omega$ is similar to an $L^\infty_t L^2_y$ control of $\partial^2_y \omega$, meaning an $L^\infty_t H^1_y$ control of $\partial_y \omega$. By differentiating (2-6a), one sees that $\partial_y \omega$ satisfies a heat-like equation, and by (2-6a), it also satisfies a Dirichlet-type condition. Again, an $L^\infty_t H^1_y$ control requires $\partial_y \omega|_{t=0}|_{y=0,1} = \partial_y \omega|_{y=0,1}|_{t=0}$, which by (2-6b) amounts to the third compatibility condition.

General strategy of the proof. Our analysis is based on the vorticity evolution (2-6). We want to benefit from the so-called hydrostatic trick, which consists in establishing L^2 estimates for the weighted derivatives $\partial^j_x \omega / \sqrt{\partial_y \omega}$. The difficulty is that these estimates are not compatible with the diffusion $-\partial^2_y \omega$, which creates boundary terms involving ∂^j_x, $\partial_y \omega|_{y=0}$. Because of the extra x-derivative at the right-hand side of (2-6b), one cannot close an estimate at the Sobolev level.

To overcome this difficulty, our first idea is to write $\omega = \omega^{\text{in}} + \omega^{\text{bl}}$, where ω^{bl} is a boundary corrector which solves (approximately)

$$\partial_t \omega^{\text{bl}} - \partial^2_y \omega^{\text{bl}} = 0, \quad \partial_y \omega^{\text{bl}}|_{y=0,1} = -\partial_x \int_0^1 u^2 \, dy,$$

where the right side of the Neumann boundary condition is seen as a given data. With this splitting, the bad term is removed from the Neumann condition on ω^{in}, so that we may apply the hydrostatic trick to this quantity. Still, this approach is obviously not enough: the equation for ω^{in} still involves ω, either directly or through ω^{bl}, so that no closed estimate is available on ω^{in}.

This is where we shall take advantage of Gevrey regularity. To explain this point, it is simpler to consider the linearization of (2-6) around a shear flow $(u_s(y), 0)$:

$$\partial_t \omega + u_s \, \partial_x \omega + u''_s v - \partial^2_y \omega = 0, \quad \partial_x u + \partial_y v = 0, \quad \partial_y \omega|_{y=0,1} = \tilde{\omega}|_{y=1} - \tilde{\omega}|_{y=0} - 2\partial_x \int_0^1 u_s u \, dy.$$

As this system has x-independent coefficients, one can Fourier transform in x. More precisely, looking for local well-posedness in Gevrey class γ, it is natural to look for solutions in the form $\omega = e^{k^{1/\gamma} t} e^{ikx} \hat{\omega}_k(t, y)$. We end up with the following system for the boundary layer corrector:

$$(k^{1/\gamma} + \partial_t) \hat{\omega}^{\text{bl}}_k - \partial^2_y \hat{\omega}^{\text{bl}}_k = 0, \quad \partial_y \hat{\omega}^{\text{bl}}_k|_{y=0,1} = -2ik \int_0^1 u_s \hat{u}_k \, dy.$$

Note that, when taking the boundary layer corrector as a solution of this heat-type system, we implicitly assume that the other terms in the equation, notably the convection term $u_s \, \partial_x \omega \sim iky\hat{\omega}^{\text{bl}}_k$, are negligible in the boundary layer. A formal analysis shows that this should hold as long as $\gamma > \frac{3}{2}$, which is the range considered here. In the limit case $\gamma = \frac{3}{2}$, conjectured to be optimal for well-posedness (see remark above), one should probably replace the heat operator by an Airy-type one, as in [Grenier et al. 2016].

Explicit calculations on the boundary layer system reveal that Gevrey regularity in x is converted into spatial localization in y: for $k \gg 1$, $\hat{\omega}^{\text{bl}}_k$ has a boundary layer behavior, with concentration near $y = 0, 1$

at scale $k^{-1/(2\gamma)}$. Roughly, neglecting the upper boundary, one can think of

$$\hat{\omega}_k^{\mathrm{bl}} \approx k^{1-1/(2\gamma)} W(t, k^{1/(2\gamma)} y) \int_0^1 u_s \hat{u}_k \, dy,$$
$$\hat{u}_k^{\mathrm{bl}} \approx k^{1-1/\gamma} U(t, k^{1/(2\gamma)} y) \int_0^1 u_s \hat{u}_k \, dy.$$

Now, the idea is to write

$$\int_0^1 u_s \hat{u}_k \, dy = \int_0^1 u_s \hat{u}_k^{\mathrm{bl}} + \int_0^1 u_s \hat{u}_k^{\mathrm{in}} = \left(k^{1-1/\gamma} \int_0^1 u_s(y) U(t, k^{1/(2\gamma)} y) \, dy \right) \int_0^1 u_s \hat{u}_k \, dy + \int_0^1 u_s \hat{u}_k^{\mathrm{in}}.$$

In short, one can check that for $\gamma \le 2$, we have $k^{1-1/\gamma} \int_0^1 u_s(y) U(t, k^{1/(2\gamma)} y) \, dy = o(1)$ in the limit of large k, so that the first term on the right-hand side can be absorbed in the left-hand side. This leads to a control of $\int_0^1 u_s u$, and thus of ω^{bl}, in terms of ω^{in}. From there, one can get closed estimates on ω^{in}.

Of course, this strategy is made more difficult when dealing with the x-dependent and nonlinear system (2-6). In particular, the Fourier approach is no longer convenient, and we must use the characterization of Gevrey regularity in the physical space, through the family $\{\partial_x^j \omega\}_{j \in \mathbb{N}}$. In order to take advantage of the boundary layer phenomenon, we shall introduce Gevrey norms with extra weight $(j+1)^r$; see (3-1). The boundary layer phenomenon will be reflected by the fact that multiplication by y or integration in y will generate a gain in the exponent r; see Lemma 3.1. Such gain will make possible the control of boundary layer quantities by ω^{in}; see Lemma 3.4.

From there, the analysis will focus on weighted estimates for ω^{in}, using the hydrostatic trick. As usual in nonlinear problems, these estimates will be obtained conditional to certain bounds (notably a lower bound on $\partial_y \omega$, to benefit from convexity). We will show that such bounds are preserved in small time, which will require estimates on the time derivative $\partial_t \omega$, as well as maximum principle arguments for $\partial_y \omega$.

3. Preliminaries

As usual in this kind of analysis, we will focus on a priori estimates. This means that from Section 3 to Section 6, we will assume implicitly that we already have a solution of (1-1) on $[0, T]$ with all necessary smoothness, and we will collect properties and estimates about this solution. Only in Section 7 will we describe the way of constructing solutions.

Norms and notation. Let $\gamma \ge 1$, $r \in \mathbb{R}$, $\tau > 0$. We introduce a refined two-dimensional Gevrey norm

$$\|f\|_{\gamma,r,\tau}^2 = \sum_{j \ge 0} M_j^2 \|\partial_x^j f\|_{L^2_{x,y}(\mathbb{T}\times[0,1])}^2, \quad \text{where} \quad M_j = \frac{(j+1)^r \tau^{j+1}}{(j!)^\gamma}. \tag{3-1}$$

Note that the L^2 norm in space is only used on $\Omega = \mathbb{T} \times [0, 1]$, although the functions may be defined on the half-space $\mathbb{T} \times [0, \infty)$. We note that if $r' \ge r$ then $\|\cdot\|_{\gamma,r',\tau} \ge \|\cdot\|_{\gamma,r,\tau}$.

For functions which are independent of the y-variable, we use the one-dimensional counterpart

$$|f|_{\gamma,r,\tau}^2 = \sum_{j \ge 0} M_j^2 \|\partial_x^j f\|_{L^2_x(\mathbb{T})}^2,$$

where M_j is defined as before. Similarly, if $r' \ge r$ then $|\cdot|_{\gamma,r',\tau} \ge |\cdot|_{\gamma,r,\tau}$.

Let τ^0, τ_1 be as in Theorem 2.1, and let τ_0 such that $\tau^0 > \tau_0 > \tau_1$. Throughout the paper, the Gevrey-class radius τ will be defined by

$$\tau(t) = \tau_0 \exp(-\beta t), \tag{3-2}$$

where $\beta \geq 1$, $t \in [0, T]$, and T is always small enough so that $\tau(t) \geq \tau_1$. In particular $\dot{\tau}(t) = -\beta\tau(t)$.

We will use $a \lesssim b$ to denote the existence of a constant $C > 0$, which may depend only on γ, τ_0, τ_1, and r, such that $a \leq Cb$. Similarly, will use $a \ll b$ to denote the existence of a sufficiently large constant $C > 0$, which may depend only on γ, τ_0, τ_1, and r, such that $Ca \leq b$.

For any function f we use the notation

$$f_j = M_j\, \partial_x^j f, \tag{3-3}$$

where M_j is defined in (3-1) and depends on r, γ, and τ. With this notation we have

$$\|f\|^2_{\gamma,r,\tau} = \sum_{j\geq 0} \|f_j\|^2_{L^2_{x,y}} \quad \text{and} \quad |f|^2_{\gamma,r,\tau} = \sum_{j\geq 0} \|f_j\|^2_{L^2_x}.$$

A boundary layer lift. The boundary condition (2-6b) in the vorticity evolution (2-6) motivates the introduction of a boundary layer lift for the vorticity, which we describe next. Throughout the paper we appeal to Gevrey estimates for the system

$$(\partial_t - \partial_y^2)\omega^\flat = 0, \tag{3-4a}$$

$$(\partial_y\omega^\flat + 2\omega^\flat)|_{y=0} = \partial_x h|_{y=0}, \tag{3-4b}$$

$$\omega^\flat|_{t=0} = 0, \tag{3-4c}$$

posed for $t \in [0, T]$, $x \in \mathbb{T}$, and $y \in \mathbb{R}_+$. Here h is a placeholder for $-\big(\int_0^1 u^2\,dy - \int_{\mathbb{T}}\int_0^1 u^2\,dy\,dx\big)$. Since the boundary datum for $\omega^\flat$ is a pure x-derivative (and this is the only nontrivial datum), we note that (3-4) immediately implies $\int_{\mathbb{T}} \omega^\flat(x, y, t)\,dx = 0$ for any $y \geq 0$. We also define

$$u^\flat(x, y) = \int_{+\infty}^{y} \omega^\flat(x, z)\,dz, \tag{3-5}$$

$$v^\flat(x, y) = \int_{y}^{+\infty} \partial_x u^\flat(x, z)\,dz. \tag{3-6}$$

Lemma 3.1. *Let $r \in \mathbb{R}$, $\beta \geq 1$ and $T > 0$ such that $\tau(t) \geq \tau_1$ for $t \in [0, T]$. The boundary layer vorticity $\omega^\flat$ obeys*

$$\int_0^t \|\omega^\flat(s)\|^2_{\gamma,r,\tau(s)}\,ds \lesssim \frac{1}{\beta^{3/2}} \int_0^t |h(s)|^2_{\gamma,r+\gamma-3/4,\tau(s)}\,ds, \tag{3-7a}$$

$$\int_0^t \|y\,\omega^\flat(s)\|^2_{\gamma,r,\tau(s)}\,ds \lesssim \frac{1}{\beta^{5/2}} \int_0^t |h(s)|^2_{\gamma,r+\gamma-5/4,\tau(s)}\,ds, \tag{3-7b}$$

$$\int_0^t \|\partial_y\omega^\flat(s)\|^2_{\gamma,r,\tau(s)}\,ds \lesssim \frac{1}{\beta^{1/2}} \int_0^t |h(s)|^2_{\gamma,r+\gamma-1/4,\tau(s)}\,ds, \tag{3-7c}$$

$$\int_0^t \|y\,\partial_y\omega^\flat(s)\|^2_{\gamma,r,\tau(s)}\,ds \lesssim \frac{1}{\beta^{3/2}}\int_0^t |h(s)|^2_{\gamma,r+\gamma-3/4,\tau(s)}\,ds, \tag{3-7d}$$

$$\int_0^t |\omega^\flat(s)|_{y=1}|^2_{\gamma,r,\tau(s)}\,ds \lesssim \frac{1}{\beta^{20}}\int_0^t |h(s)|^2_{\gamma,r+\gamma-10,\tau(s)}\,ds, \tag{3-7e}$$

$$\int_0^t |\partial_y\omega^\flat(s)|_{y=1}|^2_{\gamma,r,\tau(s)}\,ds \lesssim \frac{1}{\beta^{20}}\int_0^t |h(s)|^2_{\gamma,r+\gamma-10,\tau(s)}\,ds, \tag{3-7f}$$

the boundary layer velocity $u^\flat$ *obeys*

$$\int_0^t \|u^\flat(s)\|^2_{\gamma,r,\tau(s)}\,ds \lesssim \frac{1}{\beta^{5/2}}\int_0^t |h(s)|^2_{\gamma,r+\gamma-5/4,\tau(s)}\,ds, \tag{3-8a}$$

$$\int_0^t \|yu^\flat(s)\|^2_{\gamma,r,\tau(s)}\,ds \lesssim \frac{1}{\beta^{7/2}}\int_0^t |h(s)|^2_{\gamma,r+\gamma-7/4,\tau(s)}\,ds, \tag{3-8b}$$

$$\int_0^t |u^\flat(s)|_{y=1/2}|^2_{\gamma,r,\tau(s)}\,ds \lesssim \frac{1}{\beta^{20}}\int_0^t |h(s)|^2_{\gamma,r+\gamma-10,\tau(s)}\,ds, \tag{3-8c}$$

and the boundary layer velocity $v^\flat$ *satisfies*

$$\int_0^t \|v^\flat(s)\|^2_{\gamma,r,\tau(s)}\,ds \lesssim \frac{1}{\beta^{7/2}}\int_0^t |h(s)|^2_{\gamma,r+2\gamma-7/4,\tau(s)}\,ds, \tag{3-9a}$$

$$\int_0^t |v^\flat|_{y=0}(s)|^2_{\gamma,r,\tau(s)}\,ds \lesssim \frac{1}{\beta^{3}}\int_0^t |h(s)|^2_{\gamma,r+2\gamma-3/2,\tau(s)}\,ds, \tag{3-9b}$$

$$\int_0^t |v^\flat|_{y=1}(s)|^2_{\gamma,r,\tau(s)}\,ds \lesssim \frac{1}{\beta^{20}}\int_0^t |h(s)|^2_{\gamma,r+\gamma-10,\tau(s)}\,ds \tag{3-9c}$$

for all $t \in [0, T]$.

Proof of Lemma 3.1. In view of (3-2), (3-3), and (3-4), the function $\omega_j^\flat = M_j\,\partial_x^j\omega^\flat$ obeys equations

$$(\partial_t + \beta(j+1) - \partial_y^2)\omega_j^\flat = 0, \tag{3-10a}$$

$$(\partial_y\omega_j^\flat + 2\omega_j^\flat)|_{y=0} = \partial_x h_j|_{y=0} = \frac{M_j}{M_{j+1}}h_{j+1}, \tag{3-10b}$$

$$\omega_j^\flat|_{t=0} = 0. \tag{3-10c}$$

For fixed $x \in \mathbb{T}$ we define $f_j(x,t) = (M_j/M_{j+1})h_{j+1}(x,t)$ for $t \in [0,T]$, and $f_j(x,t) = 0$ for $t \in \mathbb{R}\setminus[0,T]$. Pointwise in x and y we take a Fourier transform in time and solve in $L^2(\mathbb{R}_t \times \mathbb{T}_x \times \mathbb{R}_y^+)$ the system

$$(\partial_t + \beta(j+1) - \partial_y^2)\bar{\omega}_j^\flat = 0,$$
$$(\partial_y\bar{\omega}_j^\flat + 2\bar{\omega}_j^\flat)|_{y=0} = f_j.$$

The solution is obtained by taking the inverse Fourier transform in time (we let ζ denote the dual Fourier variable to t) of the function

$$\hat{\bar{\omega}}_j^\flat(\zeta,x,y) = \frac{\hat{f}_j(\zeta,x)}{2-\sqrt{\beta(j+1)+i\zeta}}e^{-y\sqrt{\beta(j+1)+i\zeta}}. \tag{3-12}$$

We implicitly assume here that $\beta > 4$ so that for all $j \in \mathbb{N}$, for all ζ with $\mathcal{I}m\,\zeta \le 0$,

$$|2-\sqrt{\beta(j+1)+i\zeta}| \ge |\sqrt{\beta(j+1)+i\zeta}|-2 \ge \sqrt{\beta(j+1)-\mathcal{I}m\,\zeta}-2 \ge \sqrt{\beta}-2 > 0. \tag{3-13}$$

We will make a crucial use of:

Lemma 3.2. *The following two properties hold*:

- $\bar{\omega}_j^\flat \equiv 0$ *for* $t < 0$.
- $\bar{\omega}_j^\flat \equiv \omega_j^\flat$ *for* $t \in [0, T]$.

The proof is postponed to the Appendix. This lemma will allow us to use the explicit formula (3-12) to obtain estimates on $\omega_j^\flat$, starting with (3-7a)–(3-7f).

Let us detail the derivation of (3-7a). A simple calculation based on (3-12) yields

$$\|\hat{\bar{\omega}}_j^\flat\|^2_{L^2_{\zeta,x,y}} \le \frac{C}{(\beta(j+1))^{3/2}}\|\hat{f}_j\|^2_{L^2_{\zeta,x}}$$

for a constant C independent of j (and obviously independent of T, which is only involved in the definition of f_j). By the Plancherel formula in time

$$\|\bar{\omega}_j^\flat\|^2_{L^2_{t,x,y}} \le \frac{C}{(\beta(j+1))^{3/2}}\|f_j\|^2_{L^2_{t,x}} = \frac{C}{\beta(j+1)^{3/2}}\left(\frac{M_j}{M_{j+1}}\right)^2\int_0^T \|h_{j+1}(s)\|^2_{L^2_x}\,ds. \tag{3-14}$$

This implies (by the second item of Lemma 3.2)

$$\int_0^T \|\omega_j^\flat(s)\|^2_{L^2_{x,y}}\,ds \le \frac{C'}{\beta^{3/2}}(j+1)^{2\gamma-3/2}\int_0^T \|h_{j+1}(s)\|^2_{L^2_x}\,ds.$$

Multiplying by $(j+1)^{2r}$ and summing over j, we obtain the inequality (3-7a) *in the special case* $t = T$. For the general case $t \in (0, T)$, the idea is to slightly modify $\bar{\omega}_j^\flat$. Namely, instead of extending $(M_j/M_{j+1})h_{j+1}$ by zero outside $(0, T)$ and then solving the heat equation with the extension f_j as a boundary data, we extend $(M_j/M_{j+1})h_{j+1}|_{(0,t)}$ by zero outside $(0, t)$. We then solve the heat equation with this modified boundary data f_j^t, which is zero outside $(0, t)$, resulting in a new $\bar{\omega}_j^{\flat,t}$. Obviously, Lemma 3.2 and the previous calculation remain true with T replaced by t and $\bar{\omega}_j^\flat$ replaced by $\bar{\omega}_j^{\flat,t}$. This yields (3-7a). Inequalities (3-7b)–(3-8b) follow very similar arguments, which we skip for brevity.

In the case of (3-9a), we need to take into account one more x-derivative. A simple calculation yields (with obvious notation)

$$\|\hat{\bar{v}}_j^\flat\|^2_{L^2_{\zeta,x,y}} \le \frac{C}{(\beta(j+1))^{7/2}}\|\partial_x\hat{f}_j\|^2_{L^2_{\zeta,x}}.$$

The extra factor of $(\beta(j+1))^2$ in the denominator compared to (3-14) comes from taking two antiderivatives in y, while $\hat{f}_j$ is replaced by $\partial_x\hat{f}_j$ due to the extra x-derivative in (3-6). It follows that

$$\int_0^T \|v_j^\flat(s)\|^2_{L^2_{x,y}}\,ds \le \frac{C}{\beta^{7/2}}(j+1)^{2\gamma-7/2}\int_0^T \|\partial_x h_{j+1}(s)\|^2_{L^2_x}\,ds$$

and using that $|\partial_x h_{j+1}| \lesssim (M_{j+1}/M_{j+2})|h_{j+2}| \lesssim (j+2)^\gamma |h_{j+2}|$, we get

$$\int_0^T \|v_j^\flat(s)\|_{L^2_{x,y}}^2\, ds \le \frac{C}{\beta^{7/2}}(j+1)^{4\gamma-7/2} \int_0^T \|h_{j+2}(s)\|_{L^2_x}^2\, ds.$$

Multiplying by $(j+1)^{2r}$ and summing over j yields (3-9a) for $t=T$, while the case of an arbitrary time t is treated with the modification explained above. The pointwise estimate (3-9b), taken at $y=0$, follows from the inequality

$$\|\hat{\bar{v}}_j^\flat|_{y=0}\|_{L^2_{\zeta,x}}^2 \le \frac{C}{(\beta(j+1))^3}\|\partial_x \hat{f}_j\|_{L^2_{\zeta,x}}^2.$$

The pointwise estimates (3-7f), (3-8c), and (3-9c), taken at $y=1$ or $y=\frac{1}{2}$ are much better: all boundary layer terms taken at $y=1$ contain an exponential factor $e^{-\sqrt{\beta(j+1)+i\xi}}$ which allows us to gain an arbitrary number of powers of βj (which explains the arbitrary factor $1/\beta^{20}$ and the index $r-\gamma-10$). ☐

Lemma 3.3. *Let $r\in\mathbb{R}$, $\beta\ge 1$ and $T>0$ such that $\tau(t)\ge\tau_1$ for $t\in[0,T]$. We have*

$$\sup_{[0,t]}\|\omega^\flat(s)\|_{\gamma,r,\tau(s)}^2 \lesssim \frac{1}{\beta^{1/2}}\int_0^t |h(s)|_{\gamma,r+\gamma-1/4,\tau(s)}^2\, ds \tag{3-15a}$$

for all $t\in[0,T]$.

Proof of Lemma 3.3. In order to establish the estimate (3-15a), we rely on the explicit formula (3-12), which gives an L^1 control of the Fourier transform:

$$\begin{aligned}
\|\hat{\bar{\omega}}_j^\flat\|_{L^1_\zeta(L^2_{x,y})} &\lesssim \int_{\mathbb{R}} \frac{1}{|\sqrt{\beta(j+1)+i\zeta}-2|}\left(\int_{\mathbb{R}_+}\int_{\mathbb{T}} |e^{-2y\sqrt{\beta(j+1)+i\zeta}}|\,|\hat{f}_j(\zeta,x)|^2\,dx\,dy\right)^{1/2} d\zeta\\
&\lesssim \int_{\mathbb{R}} \frac{1}{|\sqrt{\beta(j+1)+i\zeta}|^{3/4}}\left(\int_{\mathbb{T}} |\hat{f}_j(\zeta,x)|^2\,dx\right)^{1/2} d\zeta\\
&\lesssim \left(\int_{\mathbb{R}} \frac{1}{|\sqrt{\beta(j+1)+i\zeta}|^{3/2}}\,d\zeta\right)^{1/2}\left(\int_{\mathbb{R}}\int_{\mathbb{T}} |\hat{f}_j(\zeta,x)|^2\,dx\,d\zeta\right)^{1/2}\\
&\lesssim \frac{1}{(\beta(j+1))^{1/4}}\left(\int_{\mathbb{R}}\int_{\mathbb{T}} |\hat{f}_j(\zeta,x)|^2\,dx\,d\zeta\right)^{1/2}.
\end{aligned}$$

This implies

$$\sup_{t\in\mathbb{R}}\|\bar{\omega}_j^\flat(t)\|_{L^2_{x,y}} \lesssim \frac{1}{(\beta(j+1))^{1/4}}\left(\int_{\mathbb{R}}\int_{\mathbb{T}} |f_{j+1}(t,x)|^2\,dt\right)^{1/2}.$$

Restricting the left-hand side to the supremum over $(0,T)$, we get

$$\sup_{t\in(0,T)}\|\omega_j^\flat(t)\|_{L^2_{x,y}}^2 \lesssim \frac{1}{(\beta(j+1))^{-2\gamma+1/2}}\int_0^T\int_{\mathbb{T}} \|h_{j+1}(t,x)\|^2\,dt.$$

Multiplying by $(j+1)^{2r}$ and summing over j, we get (3-15a) for $t=T$. The general case of $t\in(0,T)$ is treated as in the proof of Lemma 3.1. ☐

The interior vorticity controls the boundary layer lift. So far, we have only focused on the lower boundary layer lift, which is very small near $y=0$. We introduce the notation

$$\omega^{\mathrm{bl}}(x,y,t)=\omega^{\flat}(x,y,t)-\omega^{\flat}(x,1-y,t), \tag{3-16a}$$

$$u^{\mathrm{bl}}(x,y,t)=u^{\flat}(x,y,t)+u^{\flat}(x,1-y,t), \tag{3-16b}$$

$$v^{\mathrm{bl}}(x,y,t)=-\int_0^y \partial_x u^{\mathrm{bl}}(x,z,t)\,dz \tag{3-16c}$$

to denote the cumulative boundary layer profile, and

$$\omega^{\mathrm{in}}(x,y,t)=\omega(x,y,t)-\omega^{\mathrm{bl}}(x,y,t), \tag{3-17a}$$

$$u^{\mathrm{in}}(x,y,t)=u(x,y,t)-u^{\mathrm{bl}}(x,y,t), \tag{3-17b}$$

$$v^{\mathrm{in}}(x,y,t)=v(x,y,t)-v^{\mathrm{bl}}(x,y,t) \tag{3-17c}$$

to denote the interior vorticity, horizontal velocity component, and vertical velocity component. In view of (3-3), (3-16) and (3-17) we also define the objects ω_j^{bl}, u_j^{bl}, v_j^{bl} in terms of the function h, and ω_j^{in}, u_j^{in}, v_j^{in} in terms of h and ω.

Lemma 3.4. *Let $\gamma\in\left[1,\frac{5}{4}\right]$, $r>2\gamma+2$, $M>0$. Assume $\omega=\partial_y u$ is such that*

$$\sup_{[0,T]}\|\omega(t)\|_{\gamma,r/4,\tau(t)}\le M \tag{3-18}$$

and define

$$h(x,t)=-\int_0^1(u(x,y,t))^2\,dy+\int_{\mathbb{T}}\int_0^1(u(x,y,t))^2\,dy\,dx.$$

With h as above, let $\omega^{\flat}$ be defined via (3-4), *and let ω^{in} be as defined in* (3-17). *Then there exists $\beta_*=\beta_*(\tau_0,\tau_1,\gamma,r,M)$ such that if $\beta\ge\beta_*$ and if T is such that $\tau(t)\ge\tau_1$ for $t\in[0,T]$, then*

$$\int_0^t|h(s)|^2_{\gamma,r,\tau(s)}ds\lesssim M^2\int_0^t\|\omega^{\mathrm{in}}(s)\|^2_{\gamma,r,\tau(s)}\,ds$$

for any $t\in[0,T]$.

Note that with h defined as above we have

$$\partial_x h=-\partial_x\int_0^1 u^2\,dy,$$

so that the additional kinetic energy term in h is not seen by ω^{bl}. Combining Lemmas 3.1, 3.3 and 3.4, we see that condition (3-18) implies a sharp control of the Gevrey norm of the boundary layer profiles ω^{bl}, u^{bl}, and v^{bl}, solely in terms of the Gevrey norm of the interior vorticity ω^{in} and of the constants M and β.

Proof of Lemma 3.4. For $j=0$ we have $h_0=M_0h=\tau h$, and since

$$\int_{\mathbb{T}}h(x,t)\,dx=0,$$

we may apply the Poincaré inequality in the x-variable:

$$\|h_0\|_{L^2_x} \lesssim \|\partial_x h_0\|_{L^2_x} \lesssim \|h_1\|_{L^2_x}. \tag{3-19}$$

Hence, it is enough to estimate h_j for $j \geq 1$. By the Leibniz rule we have

$$-h_j(x,t) = \sum_{\ell=0}^{j} \binom{j}{\ell} \frac{M_j}{M_{j-\ell} M_\ell} \int_0^1 u_\ell(x,y,t) u_{j-\ell}(x,y,t)\, dy. \tag{3-20}$$

We can without loss of generality estimate only the half-sum $\sum_{0 \leq \ell \leq j/2}$, as the other half-sum can be put in the same form through the change of index $\ell' = j - \ell$.

First let us treat the case $\ell \geq 1$. The compatibility condition (1-2) yields $\int_0^1 u_\ell(x,y)\, dy = 0$, which directly implies

$$\int_0^1 u_\ell(x,y) u^{\text{in}}_{j-\ell}(x,y)\, dy = \int_0^1 u_\ell(x,y) \left(u^{\text{in}}_{j-\ell}(x,y) - \int_0^1 u^{\text{in}}_{j-\ell}(x,z)\, dz \right) dy.$$

Using the one-dimensional Gagliardo–Nirenberg inequality, the one-dimensional Hardy inequality, the one-dimensional Poincaré inequality, and the fact that $u_\ell|_{y=0} = u_\ell|_{y=1} = 0$, we have, for $\ell \geq 1$,

$$\begin{aligned}
\left\| \int_0^1 u_\ell(x,y) u_{j-\ell}(x,y)\, dy \right\|_{L^2_x} &\leq \left\| \int_0^1 u_\ell(x,y) u^{\text{in}}_{j-\ell}(x,y)\, dy \right\|_{L^2_x} + \left\| \int_0^1 u_\ell(x,y) u^{\text{bl}}_{j-\ell}(x,y)\, dy \right\|_{L^2_x} \\
&\leq \|u_\ell\|_{L^\infty_x L^2_y} \left\| u^{\text{in}}_{j-\ell} - \int_0^1 u^{\text{in}}_{j-\ell}\, dz \right\|_{L^2_{x,y}} + \left\| \frac{u_\ell}{y(1-y)} \right\|_{L^\infty_x L^2_y} \|y(1-y) u^{\text{bl}}_{j-\ell}\|_{L^2_{x,y}} \\
&\lesssim \|u_\ell\|^{1/2}_{L^2_{x,y}} \|\partial_x u_\ell\|^{1/2}_{L^2_{x,y}} \|\omega^{\text{in}}_{j-\ell}\|_{L^2_{x,y}} + \|\omega_\ell\|^{1/2}_{L^2_{x,y}} \|\partial_x \omega_\ell\|^{1/2}_{L^2_{x,y}} \|y(1-y) u^{\text{bl}}_{j-\ell}\|_{L^2_{x,y}} \\
&\lesssim \frac{M_\ell^{1/2}}{M_{\ell+1}^{1/2}} \|\omega_\ell\|^{1/2}_{L^2_{x,y}} \|\omega_{\ell+1}\|^{1/2}_{L^2_{x,y}} (\|\omega^{\text{in}}_{j-\ell}\|_{L^2_{x,y}} + \|y(1-y) u^{\text{bl}}_{j-\ell}\|_{L^2_{x,y}}).
\end{aligned}$$

For $\ell = 0$, we estimate the L^2_x norm of $\int_0^1 u_0\, u^{\text{bl}}_j\, dy$ precisely as in the case $\ell \geq 1$. For the interior piece, since $j \geq 1$ we may use (1-2) and the Poincaré inequality in y to estimate

$$\begin{aligned}
\left\| \int_0^1 u_0(x,y) u^{\text{in}}_j(x,y)\, dy \right\|_{L^2_x} &\lesssim \|u_0\|_{L^\infty_x L^2_y} \left(\left\| u^{\text{in}}_j(x,y) - \int_0^1 u^{\text{in}}_j(x,z)\, dz \right\|_{L^2_{x,y}} + \left\| \int_0^1 u^{\text{bl}}_j(x,z)\, dz \right\|_{L^2_{x,y}} \right) \\
&\lesssim M \left(\|\omega^{\text{in}}_j\|_{L^2_{x,y}} + \left\| \int_0^1 u^{\text{bl}}_j(x,z)\, dz \right\|_{L^2_x} \right)
\end{aligned}$$

since

$$\|u_0\|_{L^\infty_x L^2_y} \lesssim \|\omega_0\|_{L^\infty_x L^2_y} \lesssim \|\omega_0\|_{L^2_{x,y}} + \|\omega_1\|_{L^2_{x,y}} \lesssim M.$$

At this point we note that

$$\int_0^1 u^{\text{bl}}_j(x,y)\, dy = -\int_0^{1/2} y \omega^{\text{bl}}_j(x,y)\, dy + u^{\text{bl}}_j\big(x, \tfrac{1}{2}\big) + \int_{1/2}^1 (1-y) \omega^{\text{bl}}_j(x,y)\, dz$$

so that

$$\left\| \int_0^1 u_j^{\mathrm{bl}}(x,y)\,dy \right\|_{L^2_x} \lesssim \|y\omega_j^\flat\|_{L^2_{x,y}} + \left\|u_j^\flat\big(x,\tfrac12\big)\right\|_{L^2_x}.$$

Returning to (3-20), and using that in this range of ℓ, namely less than $\frac12 j$, we have

$$\binom{j}{\ell}\frac{M_j}{M_{j-\ell}M_\ell^{1/2}M_{\ell+1}^{1/2}} \lesssim \frac{1}{\tau^{1/2}}\binom{j}{\ell}^{1-\gamma}\frac{1}{(\ell+1)^{r-\gamma/2}} \lesssim \frac{1}{(\ell+1)^{r-\gamma/2}},$$

for $j \geq 1$ we obtain

$$\begin{aligned}\|h_j\|_{L^2_x} &\lesssim \sum_{\ell=1}^{\lceil j/2\rceil}\binom{j}{\ell}\frac{M_j}{M_{j-\ell}M_\ell^{1/2}M_{\ell+1}^{1/2}}\|\omega_\ell\|_{L^2_{x,y}}^{1/2}\|\omega_{\ell+1}\|_{L^2_{x,y}}^{1/2}(\|\omega_{j-\ell}^{\mathrm{in}}\|_{L^2_{x,y}} + \|y(1-y)u_{j-\ell}^{\mathrm{bl}}\|_{L^2_{x,y}}) \\ &\qquad + M\big(\|\omega_j^{\mathrm{in}}\|_{L^2_{x,y}} + \|yu_j^\flat\|_{L^2_{x,y}} + \|y\omega_j^\flat\|_{L^2_{x,y}} + \left\|u_j^\flat\big(x,\tfrac12\big)\right\|_{L^2_x}\big) \\ &\lesssim \sum_{\ell=1}^{\lceil j/2\rceil}\frac{(l+1)^{-3r/4}\|\omega_\ell\|_{L^2_{x,y}}^{1/2}\|\omega_{\ell+1}\|_{L^2_{x,y}}^{1/2}}{(\ell+1)^{r/4-\gamma/2}}(\|\omega_{j-\ell}^{\mathrm{in}}\|_{L^2_{x,y}} + \|yu_{j-\ell}^\flat\|_{L^2_{x,y}}) \\ &\qquad + M\big(\|\omega_j^{\mathrm{in}}\|_{L^2_{x,y}} + \|yu_j^\flat\|_{L^2_{x,y}} + \|y\omega_j^\flat\|_{L^2_{x,y}} + \left\|u_j^\flat\big(x,\tfrac12\big)\right\|_{L^2_x}\big).\end{aligned} \tag{3-21}$$

From (3-19) and (3-21), using the discrete Hölder and Young inequalities, inequalities (3-8b), (3-8c), (3-7b) and assumption (3-18) we obtain from the above that

$$\begin{aligned}\int_0^t |h(s)|^2_{\gamma,r,\tau(s)}\,ds &= \int_0^t \sum_{j\geq 0}\|h_j(s)\|^2_{L^2_x}\,ds \\ &\lesssim \sup_{[0,t]}\bigg(\sum_{j\geq0}\frac{(j+1)^{-3r/4}(\|\omega_j\|_{L^2_{x,y}}+\|\omega_{j+1}\|_{L^2_{x,y}})}{(j+1)^{r/4-\gamma/2}}\bigg)^2\int_0^t\bigg(\sum_{j\geq0}\|\omega_j^{\mathrm{in}}\|^2_{L^2_{x,y}}+\sum_{j\geq0}\|yu_j^\flat\|^2_{L^2_{x,y}}\bigg)\,ds \\ &\quad + M^2\int_0^t\big(\|\omega^{\mathrm{in}}(s)\|^2_{\gamma,r,\tau(s)}+\|yu^\flat(s)\|^2_{\gamma,r,\tau(s)}+\|y\omega^\flat(s)\|^2_{\gamma,r,\tau(s)}+|u^\flat(s)|_{y=1/2}|^2_{\gamma,r,\tau(s)}\big)\,ds \\ &\lesssim M^2\bigg(\int_0^t\|\omega^{\mathrm{in}}(s)\|^2_{\gamma,r,\tau(s)}\,ds+\int_0^t\|yu^\flat(s)\|^2_{\gamma,r,\tau(s)}+\|y\omega^\flat\|^2_{\gamma,r,\tau(s)}+|u^\flat|_{y=1/2}|^2_{\gamma,r,\tau(s)}\,ds\bigg) \\ &\lesssim M^2\bigg(\int_0^t\|\omega^{\mathrm{in}}(s)\|^2_{\gamma,r,\tau(s)}\,ds+\frac{1}{\beta^{5/2}}\int_0^t|h(s)|^2_{\gamma,r+\gamma-5/4,\tau(s)}\,ds\bigg).\end{aligned}$$

Here we have used that $\frac14 r - \frac12\gamma > \frac12$. The proof is completed using that $M^2\beta^{-5/2} \ll 1$, which follows once β_* is taken sufficiently large, and the fact that $\gamma \leq \frac54$, which allows us to absorb the second term in the right side of the above into the left side. □

4. Estimates involving ω^{in}

From the vorticity evolution (2-6) and the definition of ω^{bl} (3-16) (which obeys $\int_{\mathbb{T}}\omega^{\mathrm{bl}}(x,y,t)\,dx = 0$ for any $y \geq 0$), we obtain that the equation obeyed by the interior vorticity is

$$\partial_t \omega^{\text{in}} - \partial_y^2 \omega^{\text{in}} + u\,\partial_x \omega^{\text{in}} + v\,\partial_y \omega^{\text{in}} = -u\,\partial_x \omega^{\text{bl}} - v\,\partial_y \omega^{\text{bl}}, \tag{4-1a}$$

$$\partial_y \omega^{\text{in}}|_{y=0,1} = \tilde{\omega}^{\text{in}}|_{y=1} - \tilde{\omega}^{\text{in}}|_{y=0} + 2\omega^{\flat}|_{y=1} - \partial_y \omega^{\flat}|_{y=1}, \tag{4-1b}$$

$$\omega^{\text{in}}(0) = \omega_0. \tag{4-1c}$$

The initial condition for ω^{in} is obtained from the fact that $\omega^{\text{bl}}(0) = 0$, which holds in view of (3-4c). The main a priori estimate for ω^{in} is provided by the following proposition.

Proposition 4.1. *Let M, δ_0, $\gamma \in \left[1, \frac{9}{8}\right]$ be given, and let β_* be as in Lemma 3.4. There exists $r_0 = r_0(\gamma)$ such that for all $r \geq r_0$, one can find $\beta_0 = \beta_0(M, \delta_0, \tau_0, \tau_1, r, \gamma) > \max(\beta_*, 4)$ satisfying: if $\beta \geq \beta_0$ and $T \leq 1$ is small enough so that $\tau(t) \geq \tau_1$ for all $t \in [0, T]$, under the assumptions*

$$\sup_{t\in[0,T]} \|\omega(t)\|_{\gamma,3r/4,\tau(t)} + \sup_{t\in[0,T]} \|\partial_y \omega(t)\|_{\gamma,r/2,\tau(t)} \leq M, \tag{4-2}$$

$$\delta_0 \leq \partial_y \omega \leq \frac{1}{\delta_0}, \tag{4-3}$$

$$\sup_{t\in[0,T]} \|\partial_y^2 \omega(t)\|_{L_x^\infty L_y^2} \leq M, \tag{4-4}$$

we have

$$\sup_{s\in[0,t]} \|\omega^{\text{in}}(s)\|_{\gamma,r,\tau(s)}^2 + \int_0^t \|\partial_y \omega^{\text{in}}(s)\|_{\gamma,r,\tau(s)}^2\,ds + \beta \int_0^t \|\omega^{\text{in}}(s)\|_{\gamma,r+1/2,\tau(s)}^2\,ds \leq \frac{1}{\delta_0^2} \|\omega(0)\|_{\gamma,r,\tau_0}^2 \tag{4-5}$$

holds for all $t \in [0, T]$. Moreover, as a consequence we obtain

$$\sup_{s\in[0,t]} \|\omega(s)\|_{\gamma,r-\gamma+3/4,\tau(s)}^2 + \int_0^t \|\partial_y \omega(s)\|_{\gamma,r-\gamma+3/4,\tau(s)}^2\,ds + \beta \int_0^t \|\omega(s)\|_{\gamma,r-\gamma+5/4,\tau(s)}^2\,ds \leq \frac{4}{\delta_0^2} \|\omega(0)\|_{\gamma,r,\tau_0}^2 \tag{4-6}$$

for all $t \in [0, T]$.

Proof of Proposition 4.1. Using the convention (3-3), from (4-1) we obtain

$$(\partial_t + \beta(j+1) - \partial_y^2)\omega_j^{\text{in}} + (u\,\partial_x + v\,\partial_y)\omega_j^{\text{in}} + v_j^{\text{in}}\,\partial_y \omega = -(u\,\partial_x + v\,\partial_y)\omega_j^{\text{bl}} - v_j^{\text{bl}}\,\partial_y \omega - M_j[\partial_x^j, u\,\partial_x + v\,\partial_y]\omega + v_j\,\partial_y \omega, \tag{4-7a}$$

$$\partial_y \omega_j^{\text{in}}|_{y=0,1} = \tilde{\omega}_j^{\text{in}}|_{y=1} - \tilde{\omega}_j^{\text{in}}|_{y=0} + 2\omega_j^{\flat}|_{y=1} - \partial_y \omega_j^{\flat}|_{y=1}. \tag{4-7b}$$

Note that as soon as $j \geq 1$, we may replace $\tilde{\omega}_j^{\text{in}}|_{y=0,1} = \omega_j^{\text{in}}|_{y=0,1}$ in (4-7b). We perform a "hydrostatic energy estimate" on (4-7), which is permissible in view of (4-3). That is, we multiply (4-7a) by $\omega_j^{\text{in}}/\partial_y \omega$ and integrate over $\Omega = \mathbb{T} \times [0, 1]$. We notably use the "hydrostatic trick", which in this case gives

$$\begin{aligned}\int_\Omega v_j^{\text{in}} \omega_j^{\text{in}}\,dx\,dy &= -\int_\Omega \left(\int_0^y \partial_x u_j^{\text{in}}\right) \partial_y u_j^{\text{in}}\,dx\,dy \\ &= \int_\Omega \partial_x u_j^{\text{in}} u_j^{\text{in}}\,dx\,dy - \int_{\mathbb{T}} \left(\int_0^1 \partial_x u_j^{\text{in}}\right) u_j^{\text{in}}|_{y=1}\,dx \\ &= -\int_{\mathbb{T}} \left(\int_0^1 \partial_x u_j^{\text{bl}}(x, y)\,dy\right) u_j^{\text{bl}}(x, 1)\,dx,\end{aligned}$$

taking into account that $\int_0^1 \partial_x u_j(x,y)\,dy = 0$ and that $u_j|_{y=1} = 0$. Thus, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\frac{\omega_j^{\mathrm{in}}}{\sqrt{\partial_y\omega}}\right\|_{L^2}^2 + \beta(j+1)\left\|\frac{\omega_j^{\mathrm{in}}}{\sqrt{\partial_y\omega}}\right\|_{L^2}^2 + \left\|\frac{\partial_y\omega_j^{\mathrm{in}}}{\sqrt{\partial_y\omega}}\right\|_{L^2}^2 \\
&\quad = \int_{\mathbb{T}}\left(\frac{\partial_y\omega_j^{\mathrm{in}}\omega_j^{\mathrm{in}}}{\partial_y\omega}\bigg|_{y=1} - \frac{\partial_y\omega_j^{\mathrm{in}}\omega_j^{\mathrm{in}}}{\partial_y\omega}\bigg|_{y=0}\right)dx + \int_{\mathbb{T}}\left(\int_0^1 \partial_x u_j^{\mathrm{bl}}(x,y)\,dy\right)u_j^{\mathrm{bl}}(x,1)\,dx \\
&\qquad + \int_\Omega \frac{\partial_y\omega_j^{\mathrm{in}}\omega_j^{\mathrm{in}}}{\partial_y\omega}\frac{\partial_y^2\omega}{\partial_y\omega}\,dx\,dy - \frac{1}{2}\int_\Omega \frac{(\omega_j^{\mathrm{in}})^2}{\partial_y\omega}\frac{(u\,\partial_x + v\,\partial_y)\,\partial_y\omega}{\partial_y\omega}\,dx\,dy \\
&\qquad - \int_\Omega u\,\partial_x\omega_j^{\mathrm{bl}}\frac{\omega_j^{\mathrm{in}}}{\partial_y\omega}\,dx\,dy - \int_\Omega v\,\partial_y\omega_j^{\mathrm{bl}}\frac{\omega_j^{\mathrm{in}}}{\partial_y\omega}dxdy - \int_\Omega v_j^{\mathrm{bl}}\omega_j^{\mathrm{in}}\,dx\,dy \\
&\qquad - \sum_{k=1}^{j}\frac{M_j}{M_k M_{j-k+1}}\binom{j}{k}\int_\Omega u_k\omega_{j-k+1}\frac{\omega_j^{\mathrm{in}}}{\partial_y\omega}\,dx\,dy - \sum_{k=1}^{j-1}\frac{M_j}{M_k M_{j-k}}\binom{j}{k}\int_\Omega v_k\,\partial_y\omega_{j-k}\frac{\omega_j^{\mathrm{in}}}{\partial_y\omega}\,dx\,dy \\
&\quad =: T_{1j} + T_{2j} + T_{3j} - T_{4j} - T_{5j} - T_{6j} - T_{7j} - T_{8j} - T_{9j}.
\end{aligned} \tag{4-8}$$

Summing over j, and integrating on $[0,t)$, with $t \le T$, we obtain

$$\begin{aligned}
&\|\omega^{\mathrm{in}}(t)\|_{\gamma,r,\tau(t)}^2 + 2\beta\int_0^t \|\omega^{\mathrm{in}}\|_{\gamma,r+1/2,\tau}^2 + \int_0^t \|\partial_y\omega^{\mathrm{in}}\|_{\gamma,r,\tau}^2 \\
&\quad \le \frac{1}{\delta_0^2}\|\omega_0^{\mathrm{in}}\|_{\gamma,r,\tau_0}^2 + \frac{1}{\delta_0}\int_0^t \sum_{j\ge 0}\left(|T_{1j}| - \frac{1}{2}\left\|\frac{\partial_y\omega_j^{\mathrm{in}}}{\sqrt{\partial_y\omega}}\right\|_{L^2}^2\right) + |T_{2j}| + \left(|T_{3j}| + |T_{4j}| - \frac{1}{2}\left\|\frac{\partial_y\omega_j^{\mathrm{in}}}{\sqrt{\partial_y\omega}}\right\|_{L^2}^2\right)ds \\
&\qquad + \frac{1}{\delta_0}\int_0^t \sum_{j\ge 0}|T_{5j}| + |T_{6j}| + |T_{7j}| + |T_{8j}| + |T_{9j}|\,ds.
\end{aligned} \tag{4-9}$$

The rest of the proof is dedicated to estimating the nine terms on the right side of (4-9).

The T_{1j} bound: From (2-6b) and (4-7b) we obtain

$$\begin{aligned}
T_{1j} &= \int_{\mathbb{T}} \frac{\partial_y\omega_j^{\mathrm{in}}|_{y=0,1}(\omega_j^{\mathrm{in}}|_{y=1} - \omega_j^{\mathrm{in}}|_{y=0})}{\partial_y\omega|_{y=0,1}}\,dx \\
&= \int_{\mathbb{T}} \frac{(\tilde\omega_j^{\mathrm{in}}|_{y=1} - \tilde\omega_j^{\mathrm{in}}|_{y=0})(\omega_j^{\mathrm{in}}|_{y=1} - \omega_j^{\mathrm{in}}|_{y=0})}{\partial_y\omega|_{y=0,1}}\,dx + \int_{\mathbb{T}} \frac{(2\omega_j^{\flat}|_{y=1} - \partial_y\omega_j^{\flat}|_{y=1})(\omega_j^{\mathrm{in}}|_{y=1} - \omega_j^{\mathrm{in}}|_{y=0})}{\partial_y\omega|_{y=0,1}}\,dx \\
&= T_{11j} + T_{12j}.
\end{aligned}$$

From the Gagliardo–Nirenberg inequality $\|f\|_{L^\infty(0,1)} \le \|f\|_{L^2(0,1)} + 2\|f\|_{L^2(0,1)}^{1/2}\|\partial_y f\|_{L^2(0,1)}^{1/2}$, we have

$$|T_{11j}| \lesssim \frac{1}{\delta_0}(\|\omega_j^{\mathrm{in}}\|_{L^2_{x,y}}^2 + \|\omega_j^{\mathrm{in}}\|_{L^2_{x,y}}\|\partial_y\omega_j^{\mathrm{in}}\|_{L^2_{x,y}}).$$

Using Cauchy–Schwarz, we similarly obtain

$$|T_{12j}| \lesssim |T_{11j}| + \frac{1}{\delta_0}(\|\omega_j^{\flat}|_{y=1}\|_{L^2_x}^2 + \|\partial_y\omega_j^{\flat}|_{y=1}\|_{L^2_x}^2).$$

Summing up the above two estimates, and summing over $j \geq 0$ we obtain

$$\sum_{j\geq 0}\left(|T_{1j}| - \frac{1}{2}\left\|\frac{\partial_y \omega_j^{\mathrm{in}}}{\sqrt{\partial_y \omega}}\right\|_{L^2}^2\right) \lesssim \frac{1}{\delta_0^2}\|\omega^{\mathrm{in}}\|_{\gamma,r,\tau}^2 + \frac{1}{\delta_0}(|\omega_j^\flat|_{y=1}|_{\gamma,r,\tau}^2 + |\partial_y \omega_j^\flat|_{y=1}|_{\gamma,r,\tau}^2).$$

Using (3-7e)–(3-7f), and combining the resulting bound with Lemma 3.4 (which may be used due to assumption (4-2)), we arrive at

$$\int_0^t \sum_{j\geq 0}\left(|T_{1j}| - \frac{1}{2}\left\|\frac{\partial_y \omega_j^{\mathrm{in}}}{\sqrt{\partial_y \omega}}\right\|_{L^2}^2\right) \lesssim \frac{1}{\delta_0^2}\int_0^t \|\omega^{\mathrm{in}}\|_{\gamma,r,\tau}^2 + \frac{1}{\delta_0 \beta^{20}}\int_0^t |h|_{\gamma,r+\gamma-10,\tau}^2 \lesssim \frac{1}{\delta_0^2}\int_0^t \|\omega^{\mathrm{in}}\|_{\gamma,r,\tau}^2, \quad (4\text{-}10)$$

where we have used that $\delta_0 M^2 \leq \beta^{20}$.

The T_{2j} bound: From (3-16) we obtain

$$\begin{aligned} T_{2j} &= 2\int_{\mathbb{T}}\left(\int_0^1 \partial_x u_j^\flat(x,y)\,dy\right)(u_j^\flat(x,0) + u_j^\flat(x,1))\,dx \\ &= 2\int_{\mathbb{T}}(v_j^\flat(x,0) - v_j^\flat(x,1))(u_j^\flat(x,0) + u_j^\flat(x,1))\,dx, \end{aligned}$$

and thus, also appealing to Gagliardo–Nirenberg, we obtain

$$\begin{aligned} |T_{2j}| &\leq 2(\|v_j^\flat|_{y=0}\|_{L_x^2} + \|v_j^\flat|_{y=1}\|_{L_x^2})(\|u_j^\flat|_{y=0}\|_{L_x^2} + \|u_j^\flat|_{y=1}\|_{L_x^2}) \\ &\lesssim \frac{\|v_j^\flat|_{y=0}\|_{L_x^2} + \|v_j^\flat|_{y=1}\|_{L_x^2}}{(j+1)^{3/2-\gamma}}\big((j+1)^{3/2-\gamma}\|u_j^\flat\|_{L_{x,y}^2} + (j+1)^{7/8-\gamma/2}\|u_j^\flat\|_{L_{x,y}^2}^{1/2}(j+1)^{5/8-\gamma/2}\|\omega_j^\flat\|_{L_{x,y}^2}^{1/2}\big), \end{aligned}$$

and summing over j we arrive at

$$\sum_{j\geq 0}|T_{2j}| \lesssim (|v^\flat|_{y=0}|_{\gamma,r+\gamma-3/2,\tau} + |v_j^\flat|_{y=1}|_{\gamma,r+\gamma-3/2,\tau})(\|u^\flat\|_{\gamma,r+3/2-\gamma,\tau} + \|u^\flat\|_{\gamma,r+7/4-\gamma,\tau}^{1/2}\|\omega^\flat\|_{\gamma,r+5/4-\gamma,\tau}^{1/2}).$$

Upon integrating on $[0,t)$, the above terms are bounded using (3-7a), (3-8a), (3-9b), and (3-9c), after which Lemma 3.4 is used to yield

$$\begin{aligned} \int_0^t \sum_{j\geq 0}|T_{2j}| &\lesssim \frac{1}{\beta^{5/2}}\left(\int_0^t |h|_{\gamma,r+3\gamma-3,\tau}^2\right)^{1/2}\left(\left(\int_0^t |h|_{\gamma,r+1/4,\tau}^2\right)^{1/2} + \left(\int_0^t |h|_{\gamma,r+1/2,\tau}^2\right)^{1/2}\right) \\ &\lesssim \frac{M^2}{\beta^{5/2}}\left(\int_0^t \|\omega^{\mathrm{in}}\|_{\gamma,r+3\gamma-3,\tau}^2\right)^{1/2}\left(\int_0^t \|\omega^{\mathrm{in}}\|_{\gamma,r+1/2,\tau}^2\right)^{1/2}. \end{aligned}$$

For the last inequality, we have applied Lemma 3.4 to both factors on the right-hand side, which is legitimate under the assumptions

$$r + \min\{3\gamma - 3, \tfrac{1}{2}\} \geq 2\gamma + 2, \qquad \sup_{[0,T]}\|\omega(t)\|_{\gamma,(1/4)(r+\max\{3\gamma-3,1/2\}),\tau(t)} \leq M.$$

Both assumptions are satisfied for $r > r(\gamma)$ large enough, the second one being deduced from (4-2). Thus we have proven

$$\int_0^t \sum_{j\geq 0} |T_{2j}| \lesssim \frac{M^2}{\beta^{5/2}} \int_0^t \|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau}. \tag{4-11}$$

The T_{3j} and T_{4j} bounds: These are the only terms for which assumption (4-4) is used. In view of (4-3)–(4-4) and the Gagliardo–Nirenberg inequality in y, we immediately obtain

$$\begin{aligned}\sum_{j\geq 0}\biggl(|T_{3j}|-\frac{1}{4}\biggl\|\frac{\partial_y\omega_j^{\mathrm{in}}}{\sqrt{\partial_y\omega}}\biggr\|^2_{L^2}\biggr) &\lesssim \sum_{j\geq 0}\biggl(\frac{M}{\delta_0^{3/2}}\biggl\|\frac{\partial_y\omega_j^{\mathrm{in}}}{\sqrt{\partial_y\omega}}\biggr\|_{L^2_xL^2_y}\|\omega_j^{\mathrm{in}}\|_{L^2_xL^\infty_y}-\frac{1}{8}\biggl\|\frac{\partial_y\omega_j^{\mathrm{in}}}{\sqrt{\partial_y\omega}}\biggr\|^2_{L^2}\biggr)\\ &\lesssim \sum_{j\geq 0}\biggl(\frac{M}{\delta_0^{3/2}}\biggl\|\frac{\partial_y\omega_j^{\mathrm{in}}}{\sqrt{\partial_y\omega}}\biggr\|_{L^2}\biggl(\|\omega_j^{\mathrm{in}}\|_{L^2}+\frac{1}{\delta_0^{1/4}}\|\omega_j^{\mathrm{in}}\|^{1/2}_{L^2}\biggl\|\frac{\partial_y\omega_j^{\mathrm{in}}}{\sqrt{\partial_y\omega}}\biggr\|^{1/2}_{L^2}\biggr)-\frac{1}{8}\biggl\|\frac{\partial_y\omega_j^{\mathrm{in}}}{\sqrt{\partial_y\omega}}\biggr\|^2_{L^2}\biggr)\\ &\lesssim \frac{M^4}{\delta_0^7}\|\omega^{\mathrm{in}}\|^2_{\gamma,r,\tau},\end{aligned}$$

and using (4-2) combined with (4-3)–(4-4) we also obtain

$$\begin{aligned}\sum_{j\geq 0}\biggl(|T_{4j}|-\frac{1}{4}\biggl\|\frac{\partial_y\omega_j^{\mathrm{in}}}{\sqrt{\partial_y\omega}}\biggr\|^2_{L^2}\biggr) &\lesssim \sum_{j\geq 0}\biggl(\frac{M^2}{\delta_0^2}\|\omega_j^{\mathrm{in}}\|_{L^2_xL^2_y}\|\omega_j^{\mathrm{in}}\|_{L^2_xL^\infty_y}-\frac{1}{4}\biggl\|\frac{\partial_y\omega_j^{\mathrm{in}}}{\sqrt{\partial_y\omega}}\biggr\|^2_{L^2}\biggr)\\ &\lesssim \sum_{j\geq 0}\biggl(\frac{M^2}{\delta_0^2}\|\omega_j^{\mathrm{in}}\|_{L^2}\biggl(\|\omega_j^{\mathrm{in}}\|_{L^2}+\frac{1}{\delta_0^{1/4}}\|\omega_j^{\mathrm{in}}\|^{1/2}_{L^2}\biggl\|\frac{\partial_y\omega_j^{\mathrm{in}}}{\sqrt{\partial_y\omega}}\biggr\|^{1/2}_{L^2}\biggr)-\frac{1}{4}\biggl\|\frac{\partial_y\omega_j^{\mathrm{in}}}{\sqrt{\partial_y\omega}}\biggr\|^2_{L^2}\biggr)\\ &\lesssim \frac{M^{8/3}}{\delta_0^3}\|\omega^{\mathrm{in}}\|^2_{\gamma,r,\tau}.\end{aligned}$$

Here we have also used the second term on the left side of (4-2), in order to estimate $\|\partial_x\partial_y\omega\|_{L^\infty_xL^2_y}$. Thus,

$$\int_0^t \sum_{j\geq 0}\biggl(|T_{3j}|+|T_{4j}|-\frac{1}{4}\biggl\|\frac{\partial_y\omega_j^{\mathrm{in}}}{\sqrt{\partial_y\omega}}\biggr\|^2_{L^2}\biggr) \lesssim \frac{M^4}{\delta_0^7}\int_0^t \|\omega^{\mathrm{in}}\|^2_{\gamma,r,\tau}. \tag{4-12}$$

The T_{5j} bound: As it turns out, this term creates the most stringent assumption on γ, namely that $\gamma \leq \frac{9}{8}$. Since $u|_{y=0,1}=0$, using (4-2) and (4-4), we have

$$\begin{aligned}|T_{5j}| &\leq \frac{1}{\delta_0}\biggl\|\frac{u}{y(1-y)}\biggr\|_{L^\infty}\|y(1-y)\partial_x\omega_j^{\mathrm{bl}}\|_{L^2}\|\omega_j^{\mathrm{in}}\|_{L^2}\\ &\lesssim \frac{\|\omega\|_{L^\infty}}{\delta_0}\frac{M_j}{M_{j+1}(j+1)^{1/2}}\|y\omega^\flat_{j+1}\|_{L^2}(j+1)^{1/2}\|\omega_j^{\mathrm{in}}\|_{L^2},\end{aligned}$$

and thus, upon summing over j and integrating on $[0,t]$ we arrive at

$$\int_0^t \sum_{j\geq 0}|T_{5j}| \lesssim \frac{M}{\delta_0}\biggl(\int_0^t \|y\omega^\flat\|^2_{\gamma,r+\gamma-1/2,\tau}\biggr)^{1/2}\biggl(\int_0^t \|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau}\biggr)^{1/2}.$$

We now appeal to (3-7b) and to Lemma 3.4, which is again legitimate for $r > r(\gamma)$ large enough. We obtain

$$\begin{aligned}\int_0^t \sum_{j\geq 0} |T_{5j}| &\lesssim \frac{M}{\delta_0 \beta^{5/4}} \biggl(\int_0^t |h|^2_{\gamma,r+2\gamma-7/4,\tau}\biggr)^{1/2} \biggl(\int_0^t \|\omega^{\text{in}}\|^2_{\gamma,r+1/2,\tau}\biggr)^{1/2} \\ &\lesssim \frac{M^2}{\delta_0 \beta^{5/4}} \biggl(\int_0^t \|\omega^{\text{in}}\|^2_{\gamma,r+2\gamma-7/4,\tau}\biggr)^{1/2} \biggl(\int_0^t \|\omega^{\text{in}}\|^2_{\gamma,r+1/2,\tau}\biggr)^{1/2} \\ &\lesssim \frac{M^2}{\delta_0 \beta^{5/4}} \int_0^t \|\omega^{\text{in}}\|^2_{\gamma,r+1/2,\tau}. \end{aligned} \tag{4-13}$$

In the last inequality we have used that $2\gamma - \frac{7}{4} \leq \frac{1}{2}$, which holds since $\gamma \leq \frac{9}{8}$.

The T_{6j} bound: Similarly, using that $v|_{y=0,1} = 0$, we obtain

$$\begin{aligned}|T_{6j}| &\leq \frac{1}{\delta_0} \left\| \frac{v}{y(1-y)} \right\|_{L^\infty} \|y(1-y)\partial_y \omega_j^{\text{bl}}\|_{L^2} \|\omega_j^{\text{in}}\|_{L^2} \\ &\lesssim \frac{\|\partial_x u\|_{L^\infty}}{\delta_0} \|y\, \partial_y \omega_j^\flat\|_{L^2} \|\omega_j^{\text{in}}\|_{L^2} \\ &\lesssim \frac{M}{\delta_0} \frac{\|y\, \partial_y \omega_j^\flat\|_{L^2}}{(j+1)^{1/2}} ((j+1)^{1/2} \|\omega_j^{\text{in}}\|_{L^2}),\end{aligned}$$

so that

$$\int_0^t \sum_{j\geq 0} |T_{6j}| \lesssim \frac{M}{\delta_0} \biggl(\int_0^t \|y\, \partial_y \omega^\flat\|^2_{\gamma,r-1/2,\tau}\biggr)^{1/2} \biggl(\int_0^t \|\omega^{\text{in}}\|^2_{\gamma,r+1/2,\tau}\biggr)^{1/2}.$$

Using (3-7d), and then Lemma 3.4 (applicable for $r > r(\gamma)$ large enough, by (4-2)), we obtain

$$\begin{aligned}\int_0^t \sum_{j\geq 0} |T_{6j}| &\lesssim \frac{M}{\delta_0 \beta^{3/4}} \biggl(\int_0^t |h|^2_{\gamma,r+\gamma-7/4,\tau}\biggr)^{1/2} \biggl(\int_0^t \|\omega^{\text{in}}\|^2_{\gamma,r+1/2,\tau}\biggr)^{1/2} \\ &\lesssim \frac{M^2}{\delta_0 \beta^{3/4}} \biggl(\int_0^t \|\omega^{\text{in}}\|^2_{\gamma,r,\tau}\biggr)^{1/2} \biggl(\int_0^t \|\omega^{\text{in}}\|^2_{\gamma,r+1/2,\tau}\biggr)^{1/2}\end{aligned} \tag{4-14}$$

since $\gamma \leq \frac{7}{4}$.

The T_{7j} bound: For T_{7j} we directly estimate

$$\sum_{j\geq 0} |T_{7j}| \leq \sum_{j\geq 0} \frac{1}{\delta_0} (j+1)^{-1/2} \|v_j^\flat\|_{L^2} (j+1)^{1/2} \|\omega_j^{\text{in}}\|_{L^2} \lesssim \frac{1}{\delta_0} \|v^\flat\|_{\gamma,r-1/2,\tau} \|\omega^{\text{in}}\|_{\gamma,r+1/2,\tau}.$$

Integrating in time, appealing to (3-9a), and still using Lemma 3.4 we obtain

$$\begin{aligned}\int_0^t \sum_{j\geq 0} |T_{7j}| &\lesssim \frac{1}{\delta_0 \beta^{7/4}} \biggl(\int_0^t |h|^2_{\gamma,r+2\gamma-9/4,\tau}\biggr)^{1/2} \biggl(\int_0^t \|\omega^{\text{in}}\|^2_{\gamma,r+1/2,\tau}\biggr)^{1/2} \\ &\lesssim \frac{M}{\delta_0 \beta^{7/4}} \biggl(\int_0^t \|\omega^{\text{in}}\|^2_{\gamma,r+2\gamma-9/4,\tau}\biggr)^{1/2} \biggl(\int_0^t \|\omega^{\text{in}}\|^2_{\gamma,r+1/2,\tau}\biggr)^{1/2} \\ &\lesssim \frac{M}{\delta_0 \beta^{7/4}} \int_0^t \|\omega^{\text{in}}\|^2_{\gamma,r+1/2,\tau}\end{aligned} \tag{4-15}$$

as $2\gamma - \frac{9}{4} \leq \frac{1}{2}$.

The T_{8j} bound: We note that

$$\frac{M_j}{M_k M_{j-k+1}}\binom{j}{k} \lesssim \binom{j}{k}^{1-\gamma}\frac{(j+1)^r}{(k+1)^r(j-k+1)^{r-\gamma}},$$

and for $1 \le k \le [j/2]$ it is convenient to use $\binom{j}{k} \ge (j-k+1)/k$. We obtain

$$\begin{aligned}|T_{8j}| &\lesssim \sum_{k=1}^{[j/2]} \frac{j^{1/2}(j-k+1)^{1/2}}{(k+1)^{r-\gamma+1}}\left|\int_\Omega u_k\omega_{j-k+1}\frac{\omega_j^{\text{in}}}{\partial_y\omega}\right| + \sum_{k=[j/2]+1}^{j}\frac{1}{(j-k+1)^{r-\gamma}}\left|\int_\Omega u_k\omega_{j-k+1}\frac{\omega_j^{\text{in}}}{\partial_y\omega}\right|\\ &=: T_{8j,\text{low}} + T_{8j,\text{high}}.\end{aligned}$$

In order to estimate $T_{8j,\text{low}}$, we split ω_{j-k+1} into $\omega_{j-k+1} = \omega_{j-k+1}^{\text{in}} + \omega_{j-k+1}^{\text{bl}}$. First, using the Gagliardo–Nirenberg inequality on Ω and the Poincaré inequality in x (since $k \ge 1$) we may bound

$$\begin{aligned}\|\omega_k\|_{L^\infty} &\lesssim \|\omega_k\|_{L^2} + \|\partial_x\omega_k\|_{L^2} + (\|\omega_k\|_{L^2}^{1/2} + \|\partial_x\omega_k\|_{L^2}^{1/2})(\|\partial_y\omega_k\|_{L^2}^{1/2} + \|\partial_x\,\partial_y\omega_k\|_{L^2}^{1/2})\\ &\lesssim \|\partial_x\omega_k\|_{L^2} + \|\partial_x\omega_k\|_{L^2}^{1/2}\|\partial_x\,\partial_y\omega_k\|_{L^2}^{1/2}\\ &\lesssim k^\gamma(\|\omega_{k+1}\|_{L^2} + \|\partial_y\omega_{k+1}\|_{L^2}),\end{aligned} \tag{4-16}$$

from which we conclude that we estimate

$$\begin{aligned}\left|\int_\Omega u_k\omega_{j-k+1}^{\text{bl}}\frac{\omega_j^{\text{in}}}{\partial_y\omega}\,dx\,dy\right| &\lesssim \frac{1}{\delta_0}\left\|\frac{u_k}{y(1-y)}\right\|_{L^\infty}\|y(1-y)\omega_{j-k+1}^{\text{bl}}\|_{L^2}\|\omega_j^{\text{in}}\|_{L^2}\\ &\lesssim \frac{k^\gamma}{\delta_0}(\|\omega_{k+1}\|_{L^2} + \|\partial_y\omega_{k+1}\|_{L^2})\|y\omega_{j-k+1}^\flat\|_{L^2}\|\omega_j^{\text{in}}\|_{L^2}\\ &\lesssim \frac{k^{\gamma+r/2}}{\delta_0}\frac{\|\omega_{k+1}\|_{L^2} + \|\partial_y\omega_{k+1}\|_{L^2}}{k^{r/2}}\|y\omega_{j-k+1}^\flat\|_{L^2}\|\omega_j^{\text{in}}\|_{L^2}.\end{aligned}$$

Similarly,

$$\left|\int_\Omega u_k\omega_{j-k+1}^{\text{in}}\frac{\omega_j^{\text{in}}}{\partial_y\omega}\,dx\,dy\right| \lesssim \frac{k^{\gamma+r/2}}{\delta_0}\frac{\|\omega_{k+1}\|_{L^2} + \|\partial_y\omega_{k+1}\|_{L^2}}{k^{r/2}}\|\omega_{j-k+1}^{\text{in}}\|_{L^2}\|\omega_j^{\text{in}}\|_{L^2}$$

so that from the discrete Young and Hölder inequalities, we obtain

$$\begin{aligned}&\sum_{j\ge0} T_{8j,\text{low}}\\ &\quad\lesssim \frac{1}{\delta_0}\left(\sum_{j\ne0}\frac{j^{\gamma+r/2}}{(j+1)^{r-\gamma+1}}\frac{\|\omega_{j+1}\|_{L^2} + \|\partial_y\omega_{j+1}\|_{L^2}}{j^{r/2}}\right)(\|y\omega^\flat\|_{\gamma,r+1/2,\tau} + \|\omega^{\text{in}}\|_{\gamma,r+1/2,\tau})\|\omega^{\text{in}}\|_{\gamma,r+1/2,\tau}\\ &\quad\lesssim \frac{1}{\delta_0}(\|\omega\|_{\gamma,r/2} + \|\partial_y\omega\|_{\gamma,r/2})(\|y\omega^\flat\|_{\gamma,r+1/2,\tau} + \|\omega^{\text{in}}\|_{\gamma,r+1/2,\tau})\|\omega^{\text{in}}\|_{\gamma,r+1/2,\tau}\\ &\quad\lesssim \frac{M}{\delta_0}(\|y\,\omega^\flat\|_{\gamma,r+1/2,\tau} + \|\omega^{\text{in}}\|_{\gamma,r+1/2,\tau})\|\omega^{\text{in}}\|_{\gamma,r+1/2,\tau}.\end{aligned} \tag{4-17}$$

For the second inequality, we have assumed that $\frac{1}{2}r - 2\gamma + 1 > \frac{1}{2}$ (so that $j^{\gamma+r/2}/(j+1)^{r-\gamma+1}$ is square summable), and for the third inequality we have appealed to (4-2).

In order to bound $T_{8j,\mathrm{high}}$, we use that $u_k|_{y=0,1} = 0$, and the one-dimensional Poincaré inequality to obtain

$$\begin{aligned}\left|\int_\Omega u_k \omega_{j-k+1} \frac{\omega_j^{\mathrm{in}}}{\partial_y \omega}\,dx\,dy\right| &\lesssim \frac{1}{\delta_0} \|u_k\|_{L^2_x L^\infty_y} \|\omega_{j-k+1}\|_{L^\infty_x L^2_y} \|\omega_j^{\mathrm{in}}\|_{L^2} \\ &\lesssim \frac{(j-k+1)^\gamma}{\delta_0} \|\omega_k\|_{L^2} \|\omega_{j-k+2}\|_{L^2} \|\omega_j^{\mathrm{in}}\|_{L^2} \\ &\lesssim \frac{(j-k+1)^\gamma}{\delta_0} \frac{\|\omega_k^{\mathrm{in}}\|_{L^2} + \|\omega_k^{\mathrm{bl}}\|_{L^2}}{(k+1)^{1/2}} \|\omega_{j-k+2}\|_{L^2} (j+1)^{1/2} \|\omega_j^{\mathrm{in}}\|_{L^2}.\end{aligned}$$

We again rely on discrete Young and Hölder inequalities, assume that $r > \frac{8}{3}\gamma + \frac{2}{3}$ (so that $(j+1)^{2\gamma-3r/4}$ is square summable), and use (4-2) to arrive at

$$\begin{aligned}\sum_{j\geq 0} T_{8j,\mathrm{high}} &\lesssim \frac{1}{\delta_0} \left(\sum_j (j+1)^{2\gamma-3r/4} \frac{\|\omega_j\|_{L^2}}{(j+1)^{r/4}} \right) \|\omega^{\mathrm{in}}\|_{\gamma,r+1/2,\tau} (\|\omega^{\mathrm{in}}\|_{\gamma,r,\tau} + \|\omega^\flat\|_{\gamma,r-1/2,\tau}) \\ &\lesssim \frac{M}{\delta_0} \|\omega^{\mathrm{in}}\|_{\gamma,r+1/2,\tau} (\|\omega^{\mathrm{in}}\|_{\gamma,r-1/2,\tau} + \|\omega^\flat\|_{\gamma,r-1/2,\tau}).\end{aligned} \tag{4-18}$$

Combining (4-17), (4-18), integrating in time, using (3-7a), (3-7b), and Lemma 3.4 (which is applicable by assumption (4-2)), we arrive at

$$\begin{aligned}\int_0^t \sum_{j\geq 0} T_{8j} &\lesssim \frac{M}{\delta_0} \left(\left(\int_0^t \|y\,\omega^\flat\|^2_{\gamma,r+1/2,\tau} \right)^{1/2} + \left(\int_0^t \|\omega^\flat\|^2_{\gamma,r-1/2,\tau} \right)^{1/2} \right) \left(\int_0^t \|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau} \right)^{1/2} \\ &\qquad + \frac{M}{\delta_0} \int_0^t \|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau} \\ &\lesssim \frac{M}{\delta_0 \beta^{3/4}} \left(\left(\int_0^t |h|^2_{\gamma,r+\gamma-3/4,\tau} \right)^{1/2} + \left(\int_0^t |h|^2_{\gamma,r+\gamma-5/4,\tau} \right)^{1/2} \right) \left(\int_0^t \|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau} \right)^{1/2} \\ &\qquad + \frac{M}{\delta_0} \int_0^t \|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau} \\ &\lesssim \frac{M^2}{\delta_0 \beta^{3/4}} \left(\int_0^t \|\omega^{\mathrm{in}}\|^2_{\gamma,r+\gamma-3/4,\tau} \right)^{1/2} \left(\int_0^t \|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau} \right)^{1/2} + \frac{M}{\delta_0} \int_0^t \|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau} \\ &\lesssim \frac{M^2}{\delta_0} \int_0^t \|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau}\end{aligned} \tag{4-19}$$

since $\gamma \leq \frac{5}{4}$.

The T_{9j} bound: In order to estimate T_{9j} we note that for $1 \leq k \leq j-1$ we have

$$\frac{M_j}{M_k M_{j-k}} \binom{j}{k} \lesssim \binom{j}{k}^{1-\gamma} \frac{(j+1)^r}{(k+1)^r (j-k+1)^r} \lesssim \left(\frac{j}{\min\{k, j-k\}} \right)^{1-\gamma} \frac{1}{(\min\{k, j-k\})^r}$$

and similarly to T_{8j} we take the decomposition

$$\begin{aligned}T_{9j} &\lesssim \sum_{k=1}^{[j/2]} \frac{1}{k^r} \left| \int_\Omega v_k\, \partial_y \omega_{j-k} \frac{\omega_j^{\mathrm{in}}}{\partial_y \omega} \right| + \sum_{k=[j/2]+1}^{j-1} \frac{1}{(j-k)^{r-\gamma+1} j^{\gamma-1}} \left| \int_\Omega v_k\, \partial_y \omega_{j-k} \frac{\omega_j^{\mathrm{in}}}{\partial_y \omega} \right| \\ &=: T_{9j,\mathrm{low}} + T_{9j,\mathrm{high}}.\end{aligned} \tag{4-20}$$

First we treat the case $k \le j/2$. Using the Poincaré inequality in y (which is allowed since $u_{k+1}|_{y=0,1} = 0$) we obtain

$$\begin{aligned}\left|\int_\Omega v_k\, \partial_y \omega_{j-k} \frac{\omega_j^{\text{in}}}{\partial_y \omega}\, dx\, dy\right| &\lesssim \frac{1}{\delta_0} \left\| \frac{v_k}{y(1-y)} \right\|_{L^\infty} \|y(1-y)\, \partial_y \omega_{j-k}\|_{L^2} \|\omega_j^{\text{in}}\|_{L^2} \\ &\lesssim \frac{1}{\delta_0} \|\partial_x u_k\|_{L^\infty} (\|\partial_y \omega_{j-k}^{\text{in}}\|_{L^2} + \|y\, \partial_y \omega_{j-k}^{\flat}\|_{L^2}) \|\omega_j^{\text{in}}\|_{L^2} \\ &\lesssim \frac{k^\gamma}{\delta_0} \|\omega_{k+1}\|_{L^\infty_x L^2_y} (\|\partial_y \omega_{j-k}^{\text{in}}\|_{L^2} + \|y\, \partial_y \omega_{j-k}^{\flat}\|_{L^2}) \|\omega_j^{\text{in}}\|_{L^2}.\end{aligned}$$

Furthermore, using the one-dimensional Gagliardo–Nirenberg and Poincaré inequalities in x, for $1 \le k \le [j/2]$ we arrive at

$$\left|\int_\Omega v_k\, \partial_y \omega_{j-k} \frac{\omega_j^{\text{in}}}{\partial_y \omega}\, dx\, dy\right| \lesssim \frac{k^{2\gamma + r/4}}{\delta_0} \frac{\|\omega_{k+2}\|_{L^2}}{k^{r/4}} (\|\partial_y \omega_{j-k}^{\text{in}}\|_{L^2} + \|y\, \partial_y \omega_{j-k}^{\flat}\|_{L^2}) \|\omega_j^{\text{in}}\|_{L^2}.$$

Summing over j, assuming that $r > \frac{8}{3}\gamma + \frac{2}{3}$, and appealing to (4-2) we obtain

$$\begin{aligned}\sum_{j \ge 0} |T_{9j,\text{low}}| &\lesssim \frac{\|\omega\|_{\gamma,3r/4,\tau}}{\delta_0} (\|\partial_y \omega^{\text{in}}\|_{\gamma,r,\tau} + \|y\, \partial_y \omega^\flat\|_{\gamma,r,\tau}) \|\omega^{\text{in}}\|_{\gamma,r,\tau} \\ &\lesssim \frac{M}{\delta_0} (\|\partial_y \omega^{\text{in}}\|_{\gamma,r,\tau} + \|y\, \partial_y \omega^\flat\|_{\gamma,r,\tau}) \|\omega^{\text{in}}\|_{\gamma,r,\tau}. \qquad (4\text{-}21)\end{aligned}$$

For the case $k \ge j/2$, we first note that the compatibility condition (1-2) allows us to write

$$\int_{\mathbb{T}} \int_0^1 u_{k+1}^2\, dy\, dx = \int_{\mathbb{T}} \int_0^1 u_{k+1} u_{k+1}^{\text{bl}}\, dy\, dx + \int_{\mathbb{T}} \int_0^1 u_{k+1} \left(u_{k+1}^{\text{in}} - \int_0^1 u_{k+1}^{\text{in}}\, dz \right) dy\, dx.$$

By Cauchy–Schwarz and the Poincaré inequality in y (for zero-mean functions) we conclude

$$\|u_{k+1}\|_{L^2}^2 \lesssim \|u_{k+1}^{\text{bl}}\|_{L^2}^2 + \|\omega_{k+1}^{\text{in}}\|_{L^2}^2.$$

Then we similarly estimate

$$\begin{aligned}&\left|\int_\Omega v_k\, \partial_y \omega_{j-k} \frac{\omega_j^{\text{in}}}{\partial_y \omega}\, dx\, dy\right| \\ &\quad \lesssim \frac{1}{\delta_0} \|v_k\|_{L^2_x L^\infty_y} \|\partial_y \omega_{j-k}\|_{L^\infty_x L^2_y} \|\omega_j^{\text{in}}\|_{L^2} \\ &\quad \lesssim \frac{1}{\delta_0} \|\partial_x u_k\|_{L^2} \|\partial_x \partial_y \omega_{j-k}\|_{L^2} \|\omega_j^{\text{in}}\|_{L^2} \\ &\quad \lesssim \frac{(j-k)^\gamma j^{\gamma-1}}{\delta_0} k^{1/2} \|u_{k+1}\|_{L^2} \|\partial_y \omega_{j-k+1}\|_{L^2} (j^{1/2} \|\omega_j^{\text{in}}\|_{L^2}) \\ &\quad \lesssim \frac{(j-k)^{\gamma + r/2} j^{\gamma-1}}{\delta_0} (k^{1/2} \|\omega_{k+1}^{\text{in}}\|_{L^2} + k^{1/2} \|u_{k+1}^\flat\|_{L^2}) \frac{\|\partial_y \omega_{j-k+1}\|_{L^2}}{(j-k)^{r/2}} (j^{1/2} \|\omega_j^{\text{in}}\|_{L^2}).\end{aligned}$$

Summing over j, noting that the powers of j precisely cancel, we find for $r > r(\gamma)$ large enough

$$\begin{aligned}\sum_{j\geq 0}|T_{9j,\mathrm{high}}| &\lesssim \frac{\|\partial_y\omega\|_{\gamma,r/2}}{\delta_0}(\|\omega^{\mathrm{in}}\|_{\gamma,r+1/2,\tau}+\|u^\flat\|_{\gamma,r+1/2,\tau})\|\omega^{\mathrm{in}}\|_{\gamma,r+1/2,\tau}\\ &\lesssim \frac{M}{\delta_0}(\|\omega^{\mathrm{in}}\|_{\gamma,r+1/2,\tau}+\|u^\flat\|_{\gamma,r+1/2,\tau})\|\omega^{\mathrm{in}}\|_{\gamma,r+1/2,\tau}. \end{aligned}\tag{4-22}$$

Integrating in time the sum of (4-21) and (4-22), appealing to (3-7a) and (3-7d), and using Lemma 3.4 (which is applicable for $r > r(\gamma)$ large enough, by assumption (4-2)), we obtain

$$\begin{aligned}\int_0^t\sum_{j\geq 0}|T_{9j}|-\frac{1}{2}\int_0^t\|\partial_y\omega^{\mathrm{in}}\|^2_{\gamma,r,\tau} &\lesssim \int_0^t(\|y\,\partial_y\omega^\flat\|^2_{\gamma,r,\tau}+\|u^\flat\|^2_{\gamma,r+1/2,\tau})+\frac{M^2}{\delta_0^2}\int_0^t\|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau}\\ &\lesssim \frac{1}{\beta^{3/2}}\int_0^t|h|^2_{\gamma,r+\gamma-3/4,\tau}+\frac{M^2}{\delta_0^2}\int_0^t\|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau}\\ &\lesssim \left(\frac{M^2}{\beta^{3/2}}+\frac{M^2}{\delta_0^2}\right)\int_0^t\|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau}\end{aligned}\tag{4-23}$$

since $\gamma-\frac{3}{4}\leq\frac{1}{2}$.

Conclusion of the proof: Inserting the bounds (4-10)–(4-15), (4-19), and (4-23) into estimate (4-9), we obtain

$$\begin{aligned}&\|\omega^{\mathrm{in}}(t)\|^2_{\gamma,r,\tau(t)}+2\beta\int_0^t\|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau}\,ds+\int_0^t\|\partial_y\omega^{\mathrm{in}}\|^2_{\gamma,r,\tau}\,ds-\frac{1}{\delta_0^2}\|\omega_0^{\mathrm{in}}\|^2_{\gamma,r,\tau_0}\\ &\quad\lesssim\left(\frac{1}{\delta_0^3}+\frac{M^4}{\delta_0^8}+\frac{M}{\delta_0\beta^{3/2}}\right)\int_0^t\|\omega^{\mathrm{in}}\|^2_{\gamma,r,\tau}\,ds\\ &\qquad+\left(\frac{M^2}{\delta_0\beta^{5/2}}+\frac{M^2}{\delta_0^2\beta^{5/4}}+\frac{M^2}{\delta_0\beta^{3/2}}+\frac{M}{\delta_0^2\beta^{7/4}}+\frac{M^2}{\delta_0^2\beta^{3/4}}+\frac{M^2}{\delta_0^3}\right)\int_0^t\|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau}\,ds.\end{aligned}\tag{4-24}$$

Note that $\|\omega^{\mathrm{in}}\|^2_{\gamma,r,\tau}\leq\|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau}$, so that we may combine the last two terms on the right side of (4-24). Choosing β_0 large enough, depending on $M\geq 1$, $\delta_0\leq 1$, and the implicit constant in (4-24), for any $\beta\geq\beta_0$ we obtain

$$\|\omega^{\mathrm{in}}(t)\|^2_{\gamma,r,\tau(t)}+\beta\int_0^t\|\omega^{\mathrm{in}}\|^2_{\gamma,r+1/2,\tau}\,ds+\int_0^t\|\partial_y\omega^{\mathrm{in}}\|^2_{\gamma,r,\tau}\,ds\leq\frac{1}{\delta_0^2}\|\omega_0^{\mathrm{in}}\|^2_{\gamma,r,\tau_0}.$$

The estimate (4-5) now follows directly from the above estimate.

Finally, in order to prove (4-6), we appeal to (3-15a), Lemma 3.4, and estimate (4-5), to obtain

$$\begin{aligned}\sup_{[0,t]}\|\omega^\flat\|^2_{\gamma,r-\gamma+3/4,\tau(s)} &\lesssim\frac{1}{\beta^{1/2}}\int_0^t|h(s)|^2_{\gamma,r+1/2,\tau(s)}\,ds\\ &\lesssim\frac{M^2}{\beta^{1/2}}\int_0^t\|\omega^{\mathrm{in}}(s)\|^2_{\gamma,r+1/2,\tau(s)}\,ds\leq\frac{1}{2\delta_0^2}\|\omega^{\mathrm{in}}(0)\|^2_{\gamma,r,\tau_0}\end{aligned}\tag{4-25}$$

upon ensuring that β is sufficiently large, depending on M, δ_0. Moreover, from (3-7c) and (3-7a) we similarly obtain

$$\int_0^t \|\partial_y \omega^\flat(s)\|^2_{\gamma,r-\gamma+3/4,\tau(s)}\,ds + \beta\int_0^t \|\omega^\flat(s)\|^2_{\gamma,r-\gamma+5/4,\tau(s)}\,ds \lesssim \frac{1}{\beta^{1/2}}\int_0^t |h(s)|^2_{\gamma,r+1/2,\tau(s)}\,ds \le \frac{1}{2\delta_0^2}\|\omega^{\mathrm{in}}(0)\|^2_{\gamma,r,\tau_0} \tag{4-26}$$

as above. Summing (4-25)–(4-26) with (4-5) (and using $(a+b)^2 \le 2a^2+2b^2$) we obtain

$$\sup_{s\in[0,t]}\|\omega(s)\|^2_{\gamma,r-\gamma+3/4,\tau(s)} + \int_0^t \|\partial_y\omega(s)\|^2_{\gamma,r-\gamma+3/4,\tau(s)}\,ds + \beta\int_0^t \|\omega(s)\|^2_{\gamma,r-\gamma+5/4,\tau(s)}\,ds \le \frac{4}{\delta_0^2}\|\omega^{\mathrm{in}}(0)\|^2_{\gamma,r,\tau_0}$$

by using $\gamma \le \frac{5}{4}$. This concludes the proof of (4-6). □

As an easy consequence of the estimate (4-6), we state:

Corollary 4.2. *Let M, δ_0 and $\gamma \in \left[1, \frac{9}{8}\right]$ be given. For $r \ge r_0(\gamma)$, $\beta \ge \beta_0$ and T such that $\tau(t) \ge \tau_1$ for all $t \in [0, T]$, if*

$$\frac{4}{\delta_0^2}\|\omega_0\|_{\gamma,r,\tau_0} \le \frac{M}{2} \tag{4-27}$$

then

$$\sup_{t\in[0,T]}\|\omega(t)\|_{\gamma,3r/4,\tau(t)} \le \frac{M}{2}.$$

5. Estimates for $\partial_t\omega$

In order to emphasize the linear nature of the estimates in this section we write $\partial_t\omega = \dot\omega$. The equation obeyed by $\dot\omega$ is

$$\partial_t\dot\omega - \partial_y^2\dot\omega + (u\,\partial_x + v\,\partial_y)\dot\omega + (\dot u\,\partial_x + \dot v\,\partial_y)\omega = 0, \tag{5-1a}$$

$$\partial_y\dot\omega|_{y=0,1} = (\tilde{\dot\omega}|_{y=1} - \tilde{\dot\omega}|_{y=0}) - \partial_x\left(2\int_0^1 u\dot u\,dy\right). \tag{5-1b}$$

Proposition 5.1. *Let M, δ_0 and $\gamma \in \left[1, \frac{9}{8}\right]$ be given. There exists $r_1 = r_1(\gamma) \ge r_0$ such that: for all r, r' satisfying $r' \ge r_1$, $\frac{3}{4}r - r' \ge r_1$, one can find $\beta_1 = \beta_1(M, \delta_0, \tau_0, \tau_1, r, r', \gamma) \ge \beta_0$ satisfying: if $\beta \ge \beta_0$, if $T \le 1$ small enough so that $\tau(t) \ge \tau_1$ for all $t \in [0, T]$, and if (4-2)–(4-4) hold, we have*

$$\sup_{s\in[0,t]}\|\dot\omega(s)\|^2_{\gamma,r'-\gamma+3/4,\tau(s)} + \int_0^t \|\partial_y\dot\omega(s)\|^2_{\gamma,r'-\gamma+3/4,\tau(s)}\,ds + \beta\int_0^t \|\dot\omega(s)\|^2_{\gamma,r'-\gamma+5/4,\tau(s)}\,ds \le \frac{4}{\delta_0^2}\|\dot\omega(0)\|^2_{\gamma,r',\tau_0}. \tag{5-2}$$

Proof of Proposition 5.1. The proof is very similar to that of Proposition 4.1, since one may view (5-1) as a linearization of (2-6) about ω itself (respectively u for the boundary condition). In order to avoid redundancy, we only emphasize the essential differences.

Estimate (5-2) follows directly from estimates for $\dot\omega^{\text{in}}$ which are analogous to (4-5). In order to define $\dot\omega^{\text{in}}$, we define $\dot\omega^\flat$ as the solution of system (3-4) with boundary datum given by

$$\partial_x \dot h = -2\partial_x \int_0^1 u\dot u \, dy,$$

which is consistent with (5-1b). The function $\dot\omega^\flat$ obeys all the estimates claimed in Lemma 3.1, except that on the right side we need to replace h with $\dot h$. As in (3-16) we define the boundary layer functions corresponding to $\dot\omega$, and according to (3-17) we define the interior functions corresponding to $\dot\omega$. Note that as before we impose $\dot\omega^{\text{bl}}(0) = 0$, and thus $\dot\omega^{\text{in}}(0) = \dot\omega_0$, where by (2-6a)

$$\dot\omega_0 = -u_0 \, \partial_x \omega_0 - v_0 \, \partial_y \omega_0 - \partial_y^2 \omega_0.$$

At this stage, we can prove an analogous statement to the one provided by Lemma 3.4, with h being replaced by

$$\dot h = 2\int_0^1 u\dot u \, dy - 2\int_{\mathbb{T}}\int_0^1 u\dot u \, dy \, dx.$$

Namely, we can show that for any r as in Proposition 4.1 and any r' such that

$$\tfrac{3}{4}r - \tfrac{1}{2}\gamma - 1 \geq r' > 2\gamma + 2,$$

we have

$$\int_0^t |\dot h(s)|^2_{\gamma,r',\tau(s)} \, ds \lesssim M^2 \int_0^t \|\dot\omega^{\text{in}}(s)\|^2_{\gamma,r',\tau(s)} \, ds. \tag{5-3}$$

Indeed, defining for all f

$$f'_j = (j+1)^{r'-r} f_j = M'_j \, \partial_x^j f, \quad \text{where } M'_j = \frac{(j+1)^{r'} \tau^{j+1}}{(j!)^\gamma},$$

similarly to (3-19) we obtain $\|\dot h_0\|_{L^2_x} \lesssim \|\dot h_1\|_{L^2_x}$, while for $j \geq 1$, as a substitute to (3-21) we obtain the inequality

$$\begin{aligned}\|\dot h'_j\|_{L^2_x} \lesssim \sum_{\ell=1}^{j} \binom{j}{\ell} \frac{M'_j}{M'_{j-\ell} M_\ell'^{1/2} M_{\ell+1}'^{1/2}} \|\omega'_\ell\|^{1/2}_{L^2_{x,y}} \|\omega'_{\ell+1}\|^{1/2}_{L^2_{x,y}} (\|\dot\omega^{\text{in}'}_{j-\ell}\|_{L^2_{x,y}} + \|y(1-y)\dot u^{\text{bl}'}_{j-\ell}\|_{L^2_{x,y}}) \\ + M\big(\|\dot\omega^{\text{in}}_j\|_{L^2_{x,y}} + \|y\dot u^\flat_j\|_{L^2_{x,y}} + \|y\dot\omega^\flat_j\|_{L^2_{x,y}} + \big\|\dot u^\flat_j\big(x, \tfrac{1}{2}\big)\big\|_{L^2_x}\big).\end{aligned}$$

The half sum $\sum_{\ell=1}^{\lceil j/2 \rceil}$ and the last term on the right-hand side can be treated as before, resulting in

$$\begin{aligned}\int_0^t \bigg(M\big(\|\dot\omega^{\text{in}}_j\|_{L^2_{x,y}} + \dots + \big\|\dot u^\flat_j\big(x, \tfrac{1}{2}\big)\big\|_{L^2_x}\big) + \sum_{\ell=1}^{j/2} \dots \bigg)^2 \\ \lesssim M^2 \bigg(\int_0^t \|\dot\omega^{\text{in}}(s)\|^2_{\gamma,r',\tau(s)} \, ds + \frac{1}{\beta^{5/2}} \int_0^t |\dot h(s)|^2_{\gamma,r'+\gamma-5/4,\tau(s)} \, ds \bigg)\end{aligned}$$

if

$$\sup_{t\in[0,T]} \|\omega(t)\|_{\gamma,r'/4,\tau(t)} \leq M,$$

which is satisfied by assumption (4-2) as soon as $r' \leq 3r$.

For the half-sum $\sum_{\ell=\lceil j/2\rceil+1}^{j}$, we cannot proceed symmetrically as in the proof of Lemma 3.4: as we want an L^2-in-time control by $\dot{\omega}$, the bound

$$\binom{j}{\ell}\frac{M'_j}{M'_{j-\ell}M'^{1/2}_{\ell}M'^{1/2}_{\ell+1}}\lesssim(l+1)^{\gamma/2}$$

yields by a discrete convolution inequality

$$\int_0^t\biggl(\sum_{\ell=\lceil j/2\rceil+1}^{j}\dots\biggr)^2\lesssim\biggl(\sup_{[0,t]}\sum_{\ell\geq 1}(\ell+1)^{\gamma/2}\|\omega'_\ell\|_{L^2}\biggr)^2\int_0^t(\|\dot{\omega^{\mathrm{in}}}(s)\|^2_{\gamma,r',\tau(s)}+\|y\dot{u}^\flat(s)\|^2_{\gamma,r',\tau(s)})\,ds.$$

Writing

$$\sum_\ell(\ell+1)^{\gamma/2}\|\omega'_\ell\|_{L^2}=\sum_\ell\frac{1}{\ell+1}((\ell+1)^{\gamma/2+1}\|\omega'_\ell\|_{L^2})$$

and using Cauchy–Schwarz, we find

$$\begin{aligned}\int_0^t\biggl(\sum_{\ell=\lceil j/2\rceil+1}^{j}\dots\biggr)^2&\lesssim\sup_{[0,t]}\|\omega(s)\|^2_{\gamma,r'+\gamma/2+1,\tau(s)}\biggl(\int_0^t\|\dot{\omega^{\mathrm{in}}}(s)\|^2_{\gamma,r',\tau(s)}\,ds+\frac{1}{\beta^{7/2}}\int_0^t|\dot{h}(s)|^2_{\gamma,r'+\gamma-7/4,\tau(s)}\,ds\biggr)\\&\lesssim M^2\biggl(\int_0^t\|\dot{\omega^{\mathrm{in}}}(s)\|^2_{\gamma,r',\tau(s)}\,ds+\frac{1}{\beta^{7/2}}\int_0^t|\dot{h}(s)|^2_{\gamma,r'+\gamma-7/4,\tau(s)}\,ds\biggr),\end{aligned}$$

where the last inequality comes from (4-2), under the assumption that $r'+\frac{1}{2}\gamma+1\leq\frac{3}{4}r$. Gathering the two previous inequalities yields (5-3) for β sufficiently large.

Now, similarly to (4-7), we have

$$\begin{aligned}&(\partial_t+\beta(j+1)-\partial_y^2)\dot{\omega}_j^{\mathrm{in}'}+(u\,\partial_x+v\,\partial_y)\dot{\omega}_j^{\mathrm{in}'}+\dot{v}_j^{\mathrm{in}'}\,\partial_y\omega\\&\quad=-(u\,\partial_x+v\,\partial_y)\dot{\omega}_j^{\mathrm{bl}'}-\dot{v}_j^{\mathrm{bl}'}\,\partial_y\omega-M'_j[\partial_x^j,u\,\partial_x+v\,\partial_y]\dot{\omega}-M'_j\,\partial_x^j(\dot{u}\,\partial_x\omega)-M'_j[\partial_x^j,\partial_y\omega]\dot{v},\end{aligned}\tag{5-4a}$$

$$\partial_y\dot{\omega}_j^{\mathrm{in}}|_{y=0,1}=\tilde{\dot{\omega}}_j^{\mathrm{in}'}|_{y=1}-\tilde{\dot{\omega}}_j^{\mathrm{in}'}|_{y=0}+2\dot{\omega}_j^{\flat'}|_{y=1}-\partial_y\dot{\omega}_j^{\flat'}|_{y=1}.\tag{5-4b}$$

Note that (5-4b) is the same as (4-7b), the left side of (5-4a) is the same as the left side of (4-7a), and the first two terms on the right side of (5-4a) are the same as the first two terms on the right side of (4-7a). The difference comes from the last three terms on the right-side of (4-7a), namely the quadratic terms. The main point is that they now lack symmetry: they involve not only $(\dot{\omega}^{\mathrm{in}'},\dot{\omega}^{\mathrm{bl}'})$ but also ω. In particular, all terms containing ω must be controlled uniformly in time, to allow for the L^2_t control of $\dot{\omega}^{\mathrm{in}'}$ on the left-hand side. This is why we take r' less than $\frac{3}{4}r$: with such a margin we can still use (4-2) to control uniformly in time the terms where most derivatives fall on ω.

More precisely, proceeding as in the proof of (5-3) to handle the linear terms (see the estimates of $T_{1j},\dots,T_{7j}$), we can show that for β large enough

$$\begin{aligned}&\|\dot{\omega}^{\mathrm{in}}(t)\|^2_{\gamma,r',\tau(t)}+2\beta\int_0^t\|\dot{\omega}^{\mathrm{in}}\|^2_{\gamma,r'+1/2,\tau}\,ds+\frac{3}{2}\int_0^t\|\partial_y\dot{\omega}^{\mathrm{in}}\|^2_{\gamma,r',\tau}\,ds-\frac{1}{\delta_0^2}\|\dot{\omega}_0\|^2_{\gamma,r',\tau_0}\\&\lesssim\frac{M^4}{\delta_0^7}\int_0^t\|\dot{\omega}^{\mathrm{in}}\|^2_{\gamma,r',\tau}\,ds+\frac{M^2}{\delta_0\beta^{3/4}}\int_0^t\|\dot{\omega}^{\mathrm{in}}\|^2_{\gamma,r'+1/2,\tau}\,ds+\sum_{j\geq 0}\int_0^t(S_{1j}+S_{2j}+S_{3j}+S_{4j})(s)\,ds,\end{aligned}\tag{5-5}$$

where

$$S_{1j} = -\int_\Omega M'_j[\partial_x^j, u\,\partial_x]\dot{\omega}\frac{\dot{\omega}_j^{\text{in}'}}{\partial_y\omega}, \quad S_{2j} = -\int_\Omega M'_j[\partial_x^j, v\,\partial_y]\dot{\omega}\frac{\dot{\omega}_j^{\text{in}'}}{\partial_y\omega},$$

$$S_{3j} = -\int_\Omega M'_j\,\partial_x^j(\dot{u}\,\partial_x\omega)\frac{\dot{\omega}_j^{\text{in}'}}{\partial_y\omega}, \quad S_{4j} = -\int_\Omega M'_j[\partial_x^j, \partial_y\omega]\dot{v}\frac{\dot{\omega}_j^{\text{in}'}}{\partial_y\omega}.$$

The first term is analogous to T_{8j}. One can write

$$S_{1j} = -\Bigg(\sum_{k=1}^{\lceil j/2\rceil} + \sum_{k=\lceil j/2\rceil+1}^{j}\Bigg)\binom{j}{k}\frac{M'_j}{M'_kM'_{j-k+1}}\int_\Omega u'_k\dot{\omega}'_{j-k+1}\frac{\dot{\omega}_j^{\text{in}'}}{\partial_y\omega} = S_{1j,\text{low}} + S_{1j,\text{high}}.$$

The treatment of $S_{1j,\text{low}}$ is exactly the same as the one of $T_{8j,\text{low}}$. Similarly to (4-17), (4-19), we get

$$\sum\int_0^t S_{1j,\text{low}}(s)\,ds \lesssim \frac{M^2}{\delta_0}\int_0^t \|\dot{\omega}^{\text{in}}(s)\|^2_{\gamma,r'+1/2,\tau(s)}\,ds.$$

To treat $S_{1j,\text{high}}$, we use the inequality

$$\binom{j}{k}\frac{M'_j}{M'_kM'_{j-k+1}} \lesssim (j-k+1)^{\gamma-r'}$$

for $k \geq \lceil j/2\rceil + 1$, so that

$$\begin{aligned} S_{1j,\text{high}} &\lesssim \sum_{k=\lceil j/2\rceil+1}^{j}\frac{1}{\delta_0}\|u'_k\|_{L^\infty}(j-k+1)^{\gamma-r'}\|\dot{\omega}'_{j-k+1}\|_{L^2}\|\dot{\omega}_j^{\text{in}'}\|_{L^2} \\ &\lesssim \sum_{k=\lceil j/2\rceil+1}^{j}\frac{k^\gamma}{\delta_0}\|\omega'_{k+1}\|_{L^2}(j-k+1)^{\gamma-r'}\|\dot{\omega}'_{j-k+1}\|_{L^2}\|\dot{\omega}_j^{\text{in}'}\|_{L^2}, \end{aligned}$$

so that by the discrete Young's inequality

$$\begin{aligned} \sum\int_0^t S_{1j,\text{high}}(s)\,ds &\lesssim \frac{1}{\delta_0}\sup_{s\in[0,t]}\sum_k k^\gamma\|\omega'_k(s)\|_{L^2}\int_0^t\|\dot{\omega}(s)\|_{\gamma,\gamma,\tau(s)}\|\dot{\omega}^{\text{in}}\|_{\gamma,r',\tau(s)} \\ &\lesssim \frac{1}{\delta_0}\sup_{s\in[0,t]}\|\omega(s)\|_{\gamma,r'+\gamma+1,\tau(s)}\int_0^t\|\dot{\omega}(s)\|_{\gamma,\gamma,\tau(s)}\|\dot{\omega}^{\text{in}}\|_{\gamma,r',\tau(s)}. \end{aligned}$$

The sup in time is controlled as usual by assumption (4-2), under the constraint $r'+\gamma+1 \leq \frac{3}{4}r$. As regards the second factor, one can split $\|\dot{\omega}(s)\|_{\gamma,\gamma,\tau(s)}$ into

$$\|\dot{\omega}(s)\|_{\gamma,\gamma,\tau(s)} \leq \|\dot{\omega}^{\text{in}}(s)\|_{\gamma,\gamma,\tau(s)} + \|\dot{\omega}^{\text{bl}}(s)\|_{\gamma,\gamma,\tau(s)}$$

and control the second term by the analogue of Lemma 3.1, followed by (5-3). For $r' \geq \gamma + \big(\gamma + \frac{3}{4}\big)$ we find that

$$\sum\int_0^t S_{1j,\text{high}}(s)\,ds \lesssim \frac{M^2}{\delta_0}\int_0^t\|\dot{\omega}^{\text{in}}(s)\|^2_{\gamma,r',\tau(s)}\,ds.$$

Estimates on S_{2j} (which is analogous to T_{9j}) and S_{3j} can be established in the same way. We find for r' and $\frac{3}{4}r - r'$ large enough (with thresholds depending on γ)

$$\sum_j \int_0^t S_{2j} \le \eta \int_0^t \|\partial_y \dot{\omega}^{\text{in}}(s)\|^2_{\gamma,\gamma+r',\tau(s)}\, ds + \frac{C}{\eta}\frac{M^4}{\delta_0^2}\int_0^t \|\dot{\omega}^{\text{in}}(s)\|^2_{\gamma,\gamma+r'+1/2,\tau(s)}\, ds,$$

with $C > 0$, η arbitrarily small, and

$$\sum_j \int_0^t S_{3j} \le \frac{M^2}{\delta_0}\int_0^t \|\dot{\omega}^{\text{in}}(s)^2\|_{\gamma,\gamma+r',\tau(s)}\, ds.$$

To handle S_{4j}, we proceed slightly differently. We start with the decomposition

$$\begin{aligned} S_{4j} &= -\Bigg(\sum_{k=0}^{\lceil j/2\rceil} + \sum_{k=\lceil j/2\rceil+1}^{j-1}\Bigg)\binom{j}{k}\frac{M'_j}{M'_k M'_{j-k}}\int_\Omega \partial_y \omega'_{j-k}\dot{v}'_k \frac{\dot{\omega}_j^{\text{in}'}}{\partial_y\omega} \\ &= S_{4j,\text{low}} + S_{4j,\text{high}}. \end{aligned}$$

$S_{4j,\text{high}}$ can be treated similarly to $T_{9j,\text{high}}$. We obtain, see (4-22),

$$\begin{aligned} \sum_j \int_0^t S_{4j,\text{high}} &\lesssim \frac{1}{\delta_0}\sup_{[0,t]}\|\partial_y\omega\|_{\gamma,r'/2}\int_0^t (\|\dot{\omega}^{\text{in}}(s)\|_{\gamma,r'+1/2,\tau(s)} + \|\dot{u}^\flat\|_{\gamma,r'+1/2,\tau(s)})\|\dot{\omega}^{\text{in}}(s)\|_{\gamma,r'+1/2,\tau(s)}\, ds \\ &\lesssim \frac{M^2}{\delta_0}\int_0^t \|\dot{\omega}^{\text{in}}(s)\|^2_{\gamma,r'+1/2,\tau(s)}\, ds. \end{aligned}$$

Here, we have used the Gevrey control of $\partial_y\omega$ given by (4-2) to bound the first factor, and the analogue of Lemma 3.1 followed by (5-3) to control the boundary layer term in the second factor. As for $S_{4j,\text{low}}$, we integrate by parts in y. As $\dot{v}$ vanishes at the boundary, no boundary term appears, and we get

$$\begin{aligned} S_{4j,\text{low}} &= \sum_{k=0}^{\lceil j/2\rceil}\binom{j}{k}\frac{M'_j}{M'_k M'_{j-k}}\int_\Omega\Bigg(\omega'_{j-k}\,\partial_y\dot{v}'_k\frac{\dot{\omega}_j^{\text{in}'}}{\partial_y\omega} - \omega'_{j-k}\dot{v}'_k\frac{\partial_y^2\omega}{(\partial_y\omega)^2}\dot{\omega}_j^{\text{in}'} + \omega'_{j-k}\dot{v}'_k\frac{\partial_y\dot{\omega}_j^{\text{in}'}}{\partial_y\omega}\Bigg) \\ &= S_{4j,\text{low},1} + S_{4j,\text{low},2} + S_{4j,\text{low},3}. \end{aligned}$$

We can bound $S_{4j,\text{low},1}$ with the same ideas as before. For r' and $\frac{3}{4}r - r'$ large enough we have

$$\int_0^t \sum_j S_{4j,\text{low},1} \lesssim \frac{M^2}{\delta_0}\int_0^t \|\dot{\omega}^{\text{in}}(s)\|^2_{\gamma,r'+1/2,\tau(s)}\, ds.$$

As for $S_{4j,\text{low},2}$, we start from the bound

$$\begin{aligned} S_{4j,\text{low},2} &\lesssim \frac{1}{\delta_0^2}\sum_{k=0}^{\lceil j/2\rceil}\|\omega'_{j-k}\|_{L^\infty_x L^2_y}(k+1)^{-r'}\|\dot{v}'_k\|_{L^\infty}\|\partial_y^2\omega\|_{L^\infty_x L^2_y}\|\dot{\omega}_j^{\text{in}'}\|_{L^2_x L^\infty_y} \\ &\lesssim \frac{M}{\delta_0^2}\sum_{k=0}^{\lceil j/2\rceil}\|\omega'_{j-k}\|_{L^\infty_x L^2_y}(k+1)^{-r'}\|\dot{v}'_k\|_{L^\infty}\|\dot{\omega}_j^{\text{in}'}\|_{L^2_x L^\infty_y}, \end{aligned}$$

where the last inequality comes from (4-4) to control $\partial_y^2\omega$. It follows that

$$S_{4j,\mathrm{low},2} \lesssim \frac{M}{\delta_0^2}\sum_{k=0}^{\lceil j/2\rceil}(j-k+1)^{\gamma}\|\omega'_{j-k+1}\|_{L^2}(k+1)^{-r'+2\gamma}\|\dot{u}'_{k+2}\|_{L^2}(\|\dot{\omega}_j^{\mathrm{in}'}\|_{L^2}+\|\partial_y\dot{\omega}_j^{\mathrm{in}'}\|_{L^2}).$$

From there, for r' and $\frac{3}{4}r-r'$ large enough (with thresholds depending on γ),

$$\int_0^t\sum_j S_{4j,\mathrm{low},2} \le \eta\int_0^t \|\partial_y\dot{\omega}^{\mathrm{in}}(s)\|^2_{\gamma,\gamma+r',\tau(s)}\,ds+\frac{C}{\eta}\frac{M^6}{\delta_0^4}\int_0^t\|\dot{\omega}^{\mathrm{in}}(s)\|^2_{\gamma,\gamma+r',\tau(s)}\,ds.$$

With similar manipulations, we get the bound

$$\int_0^t\sum_j S_{4j,\mathrm{low},3} \le \eta\int_0^t \|\partial_y\dot{\omega}^{\mathrm{in}}(s)\|^2_{\gamma,\gamma+r',\tau(s)}\,ds+\frac{C}{\eta}\frac{M^4}{\delta_0^2}\int_0^t\|\dot{\omega}^{\mathrm{in}}(s)\|^2_{\gamma,\gamma+r',\tau(s)}\,ds.$$

Injecting the previous estimates in (5-5), we get for large enough β

$$\|\dot{\omega}^{\mathrm{in}}(t)\|^2_{\gamma,r',\tau(t)}+\beta\int_0^t\|\dot{\omega}^{\mathrm{in}}\|^2_{\gamma,r'+1/2,\tau}\,ds+\int_0^t\|\partial_y\dot{\omega}^{\mathrm{in}}\|^2_{\gamma,r',\tau}\,ds \le \frac{1}{\delta_0^2}\|\dot{\omega}_0\|^2_{\gamma,r',\tau_0}.$$

Estimate (5-2) follows from this inequality, in the same way as (4-6) is deduced from (4-5). □

Corollary 5.2. *Let M, δ_0 and $\gamma\in\left[1,\frac{9}{8}\right]$ be given. There exists $r_2=r_2(\gamma)\ge r_1$ such that for $r\ge r_2(\gamma)$, one can find $\beta_2=\beta_2(M,\delta_0,\tau_0,\tau_1,\gamma,r)\ge\beta_1$ and*

$$T_0=T_0(M,\delta_0,\beta,\tau_0,\tau_1,\gamma,r,\|\dot{\omega}_0\|_{\gamma,r/2+\gamma-3/4,\tau_0})>0$$

satisfying: *if $\beta\ge\beta_0$, if $T\le T_0$, if* (4-2)–(4-4) *hold, and if*

$$\|\partial_y\omega_0\|_{\gamma,r/2,\tau_0}\le\frac{M}{4}, \tag{5-6}$$

then

$$\sup_{t\in[0,T]}\|\partial_y\omega(t)\|_{\gamma,r/2,\tau(t)}\le\frac{M}{2}. \tag{5-7}$$

Proof of Corollary 5.2. We write $\partial_y\omega(t)=\partial_y\omega_0+\int_0^t\partial_y\dot{\omega}(s)\,ds$, so that for all $t\in[0,T]$

$$\begin{aligned}\|\partial_y\omega(t)\|_{\gamma,r/2,\tau(t)} &\le \|\partial_y\omega_0\|_{\gamma,r/2,\tau(t)}+\int_0^t\|\partial_y\dot{\omega}(s)\|_{\gamma,r/2,\tau(t)}\,ds\\ &\le \|\partial_y\omega_0\|_{\gamma,r/2,\tau(0)}+\int_0^t\|\partial_y\dot{\omega}(s)\|_{\gamma,r/2,\tau(s)}\,ds\\ &\le \|\partial_y\omega_0\|_{\gamma,r/2,\tau(0)}+\sqrt{t}\left(\int_0^t\|\partial_y\dot{\omega}(s)\|^2_{\gamma,r/2,\tau(s)}\,ds\right)^{1/2}.\end{aligned}$$

Taking for instance $r_2=4r_1+4\gamma+3$, where r_1 was introduced in Proposition 5.1, and $r\ge r_2$, we ensure that $r':=\frac{1}{2}r+\gamma-\frac{3}{4}$ satisfies $r'\ge r_1$ and $\frac{3}{4}r-r'\ge r_1$. By Proposition 5.1, for $\beta\ge\beta_0$ large enough, and

T such that $\tau(t) \in [\tau_1, \tau_0]$ for all $t \in [0, T]$, we get

$$\sup_{t\in[0,T]} \|\partial_y \omega(t)\|_{\gamma,r/2,\tau(t)} \le \|\partial_y \omega_0\|_{\gamma,r/2,\tau(0)} + 2\sqrt{T}/\delta_0 \|\dot{\omega}(0)\|_{\gamma,r/2+\gamma-3/4,\tau_0}. \tag{5-8}$$

The result follows from the assumption on $\partial_y \omega_0$, once T_0 is taken small enough to ensure that

$$\frac{2\sqrt{T_0}}{\delta_0} \|\dot{\omega(0)}\|_{\gamma,r/2+\gamma-3/4,\tau_0} \le \frac{M}{4}$$

holds. □

Corollary 5.3. *Let M, δ_0 and $\gamma \in \left[1, \frac{9}{8}\right]$ be given. There exists $r_3 = r_3(\gamma) \ge r_2$ such that for $r \ge r_3(\gamma)$, one can find $\beta_3 = \beta_3(M, \delta_0, \tau_0, \tau_1, \gamma, r) \ge \beta_2$, $c_0 = c_0(\tau_0, \tau_1, \gamma, r) > 0$ and*

$$T_0 = T_0(M, \delta_0, \beta, \tau_0, \tau_1, \gamma, r, \|\omega(0)\|_{\gamma,r,\tau_0}, \|\dot{\omega}(0)\|_{\gamma,r/2+\gamma-3/4,\tau_0}) > 0 \tag{5-9}$$

satisfying: *if $\beta \ge \beta_0$, if $T \le T_0$, if* (4-2)–(4-4) *hold, and if*

$$\frac{1}{\delta_0}\|\dot{\omega}_0\|_{\gamma,r/2+\gamma-3/4,\tau_0} + \frac{1}{\delta_0^2}\|\omega_0\|^2_{\gamma,r,\tau_0} + \frac{1}{\delta_0}\|\omega_0\|_{\gamma,r,\tau_0}\|\partial_y \omega_0\|_{\gamma,r/2,\tau_0} \le \frac{c_0 M}{4}, \tag{5-10}$$

then

$$\sup_{t\in[0,T]} \|\partial_y^2 \omega(t)\|_{L^\infty_x L^2_y} \le \frac{M}{2}.$$

Proof of Corollary 5.3. We write the vorticity equation in the form

$$\partial_y^2 \omega = \dot{\omega} + u\,\partial_x \omega + v\,\partial_y \omega.$$

Hence, for all $t \in [0, T]$,

$$\|\partial_y^2 \omega(t)\|_{L^\infty_x L^2_y} \le \|\dot{\omega}(t)\|_{L^\infty_x L^2_y} + \|u(t)\|_{L^\infty_{x,y}}\|\partial_x \omega(t)\|_{L^\infty_x L^2_y} + \|v(t)\|_{L^\infty_{x,y}}\|\partial_y \omega(t)\|_{L^\infty_x L^2_y}.$$

For r large enough, we obtain

$$\|\partial_y^2 \omega(t)\|_{L^\infty_x L^2_y} \lesssim \|\dot{\omega}(t)\|_{\gamma,r/2,\tau(t)} + \|\omega(t)\|^2_{\gamma,r-\gamma+3/4,\tau(t)} + \|\omega(t)\|_{\gamma,r-\gamma+3/4,\tau(t)}\|\partial_y \omega(t)\|_{\gamma,r/2,\tau(t)}.$$

By Propositions 4.1 and 5.1 applied respectively with r and $r' = \frac{1}{2}r + \gamma - \frac{3}{4}$, and by inequality (5-8), we find

$$\sup_{t\in[0,T]} \|\partial_y^2 \omega(t)\|_{L^\infty_x L^2_y} \lesssim \frac{1}{\delta_0}\|\dot{\omega}_0\|_{\gamma,r/2+\gamma-3/4,\tau_0} + \frac{1}{\delta_0^2}\|\omega_0\|^2_{\gamma,r,\tau_0} + \frac{1}{\delta_0}\|\omega_0\|_{\gamma,r,\tau_0}\left(\|\partial_y \omega_0\|_{\gamma,r/2,\tau_0} + \frac{\sqrt{T}}{\delta_0}\|\dot{\omega}_0\|_{\gamma,r/2+\gamma-3/4,\tau_0}\right).$$

Upon taking T sufficiently small, this concludes the proof of the corollary. □

6. Minimum and maximum principle for $\partial_y \omega$

The quantity $\partial_y \omega$ obeys a (degenerate) parabolic equation with Dirichlet boundary conditions

$$\partial_t(\partial_y \omega) - \partial_y^2(\partial_y \omega) + (u\,\partial_x + v\,\partial_y)(\partial_y \omega) + (\partial_x u)(\partial_y \omega) = \omega\,\partial_x \omega, \tag{6-1a}$$

$$\partial_y \omega|_{y=0,1} = (\tilde{\omega}|_{y=1} - \tilde{\omega}|_{y=0}) - \partial_x \int_0^1 u^2\,dy. \tag{6-1b}$$

Our goal is to combine this fact with $L_t^2 L_{x,y}^\infty$ estimates on $\omega\,\partial_x \omega$ and the Dirichlet datum, to deduce that the convexity of u is conserved for small time.

Proposition 6.1. *Let $M, \delta_0 > 0$ and $\gamma \in \left[1, \frac{9}{8}\right]$ be given. There exists $r_4 = r_4(\gamma) \geq r_3$ such that for $r \geq r_4(\gamma)$, one can find $\beta_4 = \beta_4(M, \delta_0, \tau_0, \tau_1, \gamma, r) \geq \beta_3$ and T_0 as in* (5-9) *satisfying*: *if $\beta \geq \beta_0$, if $T \leq T_0$, if* (4-2)–(4-4) *hold*, *and if*

$$4\delta_0 \leq \partial_y \omega_0 \leq \frac{1}{4\delta_0}, \tag{6-2}$$

then

$$2\delta_0 \leq \partial_y \omega(t) \leq \frac{1}{2\delta_0} \quad \textit{for all } t \in [0, T]. \tag{6-3}$$

Proof of Proposition 6.1. We wish to apply a version of the parabolic minimum/maximum principle for the following degenerate parabolic problem posed in $\Omega \times (0, T)$, with Ω being the periodic-in-x strip $(x, y) \in \mathbb{T} \times (0, 1)$:

$$(\partial_t - \partial_y^2 + b(x, y, t) \cdot \nabla_{x,y} + c(x, y, t))\psi = d(x, y, t) \quad \text{in } \Omega \times (0, T), \tag{6-4a}$$

$$\psi = a(x, t) \quad \text{on } \partial\Omega \times [0, T), \tag{6-4b}$$

$$\psi|_{t=0} = \psi_0(x, y) \quad \text{in } \Omega. \tag{6-4c}$$

Here $\psi = \partial_y \omega,\ b = (u, v)$ is incompressible and vanishes on the boundary $\mathbb{T} \times \{0, 1\},\ c = \partial_x u$ vanishes at the boundary $\mathbb{T} \times \{0, 1\},\ d = \omega\,\partial_x \omega$, and the boundary data is $a = (\tilde{\omega}|_{y=1} - \tilde{\omega}|_{y=0}) - \partial_x \int_0^1 u^2\,dy$. As emphasized after Theorem 2.1, the third compatibility condition of the theorem corresponds to the relation $a(x, 0) = \psi_0(x, 0)$.

By (6-2), the initial datum ψ_0 is taken to obey $0 < 4\delta_0 \leq \psi_0(x, y) \leq 1/(4\delta_0)$, for some $\delta_0 \in \left(0, \frac{1}{4}\right)$, uniformly on Ω. Thus, by the compatibility of the initial datum and of the boundary condition, we have $0 < 4\delta_0 \leq a(x, 0) \leq 1/(4\delta_0)$, uniformly on $\mathbb{T}$. Thanks to the Gagliardo–Nirenberg inequality

$$\|f\|_{L_y^\infty} \leq C\|f\|_{L_y^2}^{1/2}(\|f\|_{L_y^2}^{1/2} + \|\partial_y f\|_{L_y^2}^{1/2})$$

and the estimate (5-2), we have

$$\begin{aligned}\|\partial_t a(x, t)\|_{L^2(0,T;L_x^\infty)} &\leq 4\|\dot{\omega}\|_{L^2(0,T;L^\infty)} + 2\left\|\partial_x \int_0^1 u\,\dot{u}\,dy\right\|_{L^2(0,T;L_x^\infty)} \\ &\lesssim \frac{1}{\delta_0^2}\left(\frac{1}{\beta^{1/4}} + \frac{M}{\beta^{1/2}}\right)\|\dot{\omega}_0\|_{\gamma,r/2+\gamma-3/4,\tau_0} \leq \|\dot{\omega}_0\|_{\gamma,r/2+\gamma-3/4,\tau_0}\end{aligned}$$

for β sufficiently large. By the fundamental theorem of calculus in time, and the Cauchy–Schwarz inequality we thus obtain

$$3\delta_0 \le 4\delta_0 - \sqrt{T}\|\dot{\omega}_0\|_{\gamma,r/2+\gamma-3/4,\tau_0} \le a(x,t) \le \frac{1}{4\delta_0} + \sqrt{T}\|\dot{\omega}_0\|_{\gamma,r/2+\gamma-3/4,\tau_0} \le \frac{1}{3\delta_0}$$

uniformly on $\mathbb{T}\times(0,T)$, upon taking T sufficiently small. Thus, on the parabolic boundary $\Omega\times\{0\}\cup\partial\Omega\times(0,T)$, we have $\psi \ge 3\delta_0$.

By the same Gagliardo–Nirenberg inequality, the Poincaré inequality in y, and estimate (4-6), we have

$$\sup_{t\in[0,T]}\|c(t)\|_{L_x^\infty L_y^\infty} = \sup_{t\in[0,T]}\|\partial_x u(t)\|_{L_x^\infty L_y^\infty} \le \frac{C_1}{\delta_0}\|\omega_0\|_{\gamma,r,\tau_0},$$

where $C_1 = C_1(\tau_0,\tau_1,\gamma,r)$. Setting

$$C_* = 1 + \frac{C_1}{\delta_0}\|\omega_0\|_{\gamma,r,\tau_0}, \tag{6-5}$$

the above estimate implies

$$c(x,y,t) + C_* \ge 1.$$

Lastly, we note that by the Gagliardo–Nirenberg inequality and (4-6) we have

$$\int_0^t \|d(s)\|_{L_x^\infty L_y^\infty}\,ds = \int_0^t \|\omega(s)\|_{L_x^\infty L_y^\infty}\|\partial_x\omega(s)\|_{L_x^\infty L_y^\infty}\,ds \lesssim \frac{\sqrt{t}}{\delta_0^2}\|\omega_0\|_{\gamma,r,\tau_0}^2$$

so that for $T \le 1$ we have that

$$\begin{aligned} e(t) &:= t + \int_0^t e^{-C_* s}\|d(s) - 3\delta_0 c(s)\|_{L_x^\infty L_y^\infty}\,ds \\ &\lesssim t + \sqrt{t}\|\omega_0\|_{\gamma,r,\tau_0}^2 + tC_1\|\omega_0\|_{\gamma,r,\tau_0} \\ &\le C_2\sqrt{t}(1 + \|\omega_0\|_{\gamma,r,\tau_0}^2 + \|\omega_0\|_{\gamma,r,\tau_0}) = \sqrt{t}D_* \end{aligned} \tag{6-6}$$

hold for all $t \in [0,T]$, where C_2 is a constant that only depends on γ, r, τ_0, and τ_1, and we have set

$$D_* = C_2(1 + \|\omega_0\|_{\gamma,r,\tau_0}^2 + \|\omega_0\|_{\gamma,r,\tau_0}).$$

With this notation, we make the following change of unknowns:

$$\bar{\psi} = e^{-C_* t}(\psi(x,y,t) - 3\delta_0) + e(t), \tag{6-7a}$$

$$\bar{a} = e^{-C_* t}(a(x,t) - 3\delta_0) + e(t), \tag{6-7b}$$

$$\bar{d} = e^{-C_* t}(d(x,y,t) - 3\delta_0 c(x,y,t)), \tag{6-7c}$$

$$\bar{c} = c(x,y,t) + C_*, \tag{6-7d}$$

$$\bar{\psi}_0 = \psi_0(x,y) - 3\delta_0. \tag{6-7e}$$

The quantity $e(t)$ was chosen so that $\dot{e}(t) = 1 + \|\bar{d}(t)\|_{L^\infty}$. One may then verify directly that

$$(\partial_t - \partial_y^2 + b \cdot \nabla_{x,y} + \bar{c})\bar{\psi} = (\bar{d} + \|\bar{d}\|_{L^\infty}) + 1 + \bar{c}e \geq 1 > 0, \tag{6-8a}$$

$$\bar{\psi}|_{y\in\{0,1\}} = \bar{a} \geq t \geq 0, \tag{6-8b}$$

$$\bar{\psi}|_{t=0} = \bar{\psi}_0 \geq \delta_0 > 0. \tag{6-8c}$$

The parabolic minimum principle then guarantees that

$$\bar{\psi}(x, y, t) \geq 0 \quad \text{on } \Omega \times [0, T]. \tag{6-9}$$

Indeed, if a strictly negative minimum were attained by $\bar{\psi}$, then this point minimum could not lie on the parabolic boundary (since $\bar{a} \geq 0$ and $\bar{\psi}_0 > 0$). If this point lay in the interior, at this point we would need to have $\nabla_{t,x,y}\bar{\psi} = 0$, whereas $(-\partial_y^2 + \bar{c})\bar{\psi} < 0$ since $\bar{c} > 0$. This contradicts $(\bar{d} + \|\bar{d}\|_{L^\infty}) + 1 + \bar{c}e > 0$, which thus proves (6-9).

Working backwards from the definition of $\bar{\psi}$, we see that (6-5), (6-6), and (6-9) imply

$$\psi(x, y, t) \geq 3\delta_0 - e^{C_* t} e(t) \geq 3\delta_0 - \sqrt{T} e^{C_* T} D_* \geq 2\delta_0$$

as long as T is chosen sufficiently small in terms of C_*, D_*, and δ_0, consistent with the dependence given in (5-9). This proves the lower bound in (6-3).

The proof of the upper bound in (6-3) follows from very similar arguments, reducing the problem to a maximum principle for a parabolic equation. To avoid redundancy, we omit these details. □

7. Proof of Theorem 2.1

The proof of the main theorem proceeds as follows. Let $\gamma \leq \frac{9}{8}$ and $r \geq r_4(\gamma)$. For any $\tau_0 < \tau^0$ assumption (2-1) implies that $\omega_0 = \partial_y u_0$ satisfies

$$\|\omega_0\|_{\gamma,r,\tau_0} + \|\partial_y^2 \omega_0\|_{\gamma,r,\tau_0} < +\infty.$$

We fix $\tau_0 \in (\tau_1, \tau^0)$. We then fix δ_0 small enough and M large enough, so that the initial constraints (4-27), (5-6), (5-10) and (6-2) hold. Let $\beta \geq \beta_4$ and $\varepsilon > 0$. We consider the approximate system

$$\partial_t u + u\,\partial_x u + v\,\partial_y u + \partial_x p - \partial_y^2 u - \varepsilon \partial_x^2 u = 0, \quad (x, y) \in \mathbb{T} \times (0, 1), \tag{7-1a}$$

$$\partial_y p = 0, \quad (x, y) \in \mathbb{T} \times (0, 1), \tag{7-1b}$$

$$\partial_x u + \partial_y v = 0, \quad (x, y) \in \mathbb{T} \times (0, 1), \tag{7-1c}$$

$$u|_{y=0,1} = v|_{y=0,1} = 0, \tag{7-1d}$$

with the same initial condition $u|_{t=0} = u_0$. System (7-1) is called the two-dimensional primitive equations, and has been widely studied, in various geometries and under various boundary conditions [Bresch et al. 2003; 2005; Temam and Ziane 2004]. In particular, Gevrey or analytic regularity results were obtained in both periodic and bounded geometries [Petcu 2004; Petcu et al. 2004; Kukavica et al. 2016]. In the context of system (7-1), the well-posedness result stated in Theorem 2.1 can be proved without much difficulty. In fact, the presence of $-\varepsilon\partial_x^2 u$ allows for a classical treatment, and the existence of solutions at fixed $\varepsilon > 0$ follows, e.g., from a Galerkin approximation procedure (which is compatible with the hydrostatic trick [Masmoudi and Wong 2012]). Moreover, the compatibility conditions are the same for (1-1) and (7-1). We find in this way a unique local solution u^ε with the regularity requirements stated in Theorem 2.1. We can then consider $T_{\varepsilon,*}$ the maximal time on which $\|\omega_\varepsilon\|_{\gamma,0,\tau_1} < +\infty$. In particular, if $T_{\varepsilon,*}$ is small enough so that $\tau(T_{\varepsilon,*}) \geq \tau_1$, one has

$$\sup_{t\in[0,T_{\varepsilon,*})} \|\omega_\varepsilon(t)\|_{\gamma,3r/4,\tau(t)} = +\infty. \tag{7-2}$$

By the initial constraint (4-27), the fact that $\tau_0 < \tau^0$, and the continuity of the solution, there exists a maximal time $0 < T_\varepsilon \leq T_{\varepsilon,*}$ on which the conditions (4-2)–(4-4) are satisfied with u replaced by u_ε and T replaced by T_ε. Note that all the estimates that we established for a solution u of (1-1) adapt straightforwardly to a solution u^ε of (7-1). The only notable change is the inclusion of the $-\varepsilon\partial_x^2$ term in (3-4) for defining the boundary layer lift $\omega^{\flat,\varepsilon}$. However, since all estimates for $\omega^{\flat,\varepsilon}$ are obtained by performing a Fourier transform in x and using Plancherel to obtain the desired L^2_x bound, this modification is routine (see also [Ignatova and Vicol 2016] for ε-independent bounds for solutions of the ε-regularization of the Prandtl system that are analytic in x and Sobolev in y). Applying Corollaries 4.2, 5.2, and 5.3, and Proposition 6.1 at positive ε, we see that there exists $T > 0$ independent of ε, such that for all $t \in [0, \min(T_\varepsilon, T)]$, the conditions (4-2)–(4-4) still hold with M replaced by $\frac{1}{2}M$, and δ_0 replaced by $2\delta_0$. If $T_\varepsilon < T$, then one has necessarily $T_\varepsilon = T_{\varepsilon,*}$, otherwise by continuity the inequalities (4-2)–(4-4) would be satisfied beyond T_ε. But then there is a contradiction between (7-2) and the first half of (4-2). Hence, $T_\varepsilon \geq T$, and so $T_{\varepsilon,*} \geq T$.

We have just shown that the approximations u_ε are all defined on a time interval independent of ε, and satisfy uniform Gevrey bounds on it. This allows us to let ε go to zero, and conclude by standard compactness arguments to the existence of a solution.

For the uniqueness of solutions, the equation obeyed by the difference is basically a linearized version of the equation, very similar to the equation obeyed by $\dot\omega$. Then an estimate similar to the one from Proposition 5.1, gives the good estimate for the difference of two solutions, implying uniqueness.

Appendix: Proof of Lemma 3.2

To prove the first item, we adapt arguments of [Fernandez et al. 2016, pages 1805–1807]. We fix $x \in \mathbb{T}$, $y > 0$, and drop them from the notation. We write

$$\hat{\hat{\omega}}{}^\flat_j(\eta) = \hat f_j(\zeta)\, g_j(\zeta), \quad g_j(\zeta) = \frac{1}{2-\sqrt{\beta(j+1)+i\zeta}} e^{-y\sqrt{\beta(j+1)+i\zeta}}.$$

Clearly, as f_j is equal to 0 for $t < 0$ and belongs to $L^1(\mathbb{R})$,

$$\hat f_j(\zeta) = \int_{\mathbb{R}_+} f_j(t) e^{-i\zeta t}\, dt$$

is holomorphic for $\mathcal{I}m\,\zeta < 0$, and continuous for $\mathcal{I}m\,\zeta \leq 0$. Moreover,

$$\lim_{\mathcal{I}m\,\zeta\to+\infty} \hat f_j(\zeta) = 0 \text{ uniformly for } \mathcal{R}e\,\zeta \in \mathbb{R}, \qquad \lim_{\mathcal{R}e\,\zeta\to\pm\infty} \hat f_j(\zeta) = 0 \text{ uniformly for } \mathcal{I}m\,\zeta \leq 0. \tag{A-1}$$

The first limit follows directly from the inequality

$$|\hat{f}_j(\zeta)| \le \int_{\mathbb{R}_+} |f_j(t)| e^{-\mathcal{I}m\,\zeta t}\, dt$$

and the dominated convergence theorem. The second limit follows from a close look at the Riemann–Lebesgue lemma: given $\varepsilon > 0$, and some $f_j^\varepsilon \in C_c^1(\mathbb{R}_+)$ with $\int_{\mathbb{R}_+} |f_j - f_j^\varepsilon| \le \varepsilon$, we get

$$|\hat{f}_j(\zeta)| \le \int_{\mathbb{R}+} |f_j - f_j^\varepsilon| + \Big|\int_{\mathbb{R}+} f_j^\varepsilon(t) e^{-i\zeta t}\, dt\Big| \le \varepsilon + \frac{M_\varepsilon}{|\mathcal{R}e\,\zeta|},$$

where the second bound follows from an integration by parts of the second integral.

Obviously, g_j is also holomorphic in $\mathcal{I}m\,\zeta < 0$, continuous over $\mathcal{I}m\,\zeta \le 0$, with bound

$$|g_j(\zeta)| \le \frac{1}{\beta - 2} e^{-\sqrt{|\zeta|}y}; \tag{A-2}$$

see (3-13). We finally apply the Cauchy formula: for any $t < 0$, for any $\mu > 0$,

$$\begin{aligned}\bar{\omega}_j^\flat(t) &= \lim_{s\to+\infty} \frac{1}{2\pi} \int_{-s}^{s} \hat{f}_j(\zeta)\, g_j(\zeta) e^{i\zeta t}\, d\zeta \\ &= -\lim_{s\to+\infty} \frac{1}{2\pi} \bigg(\int_{[-s,s]-i\mu} \hat{f}_j(\zeta)\, g_j(\zeta) e^{i\zeta t}\, d\zeta + \int_{[s,s-i\mu]} \hat{f}_j(\zeta)\, g_j(\zeta) e^{i\zeta t}\, d\zeta \\ &\qquad\qquad + \int_{[-s-i\mu,-s]} \hat{f}_j(\zeta)\, g_j(\zeta) e^{i\zeta t}\, d\zeta \bigg).\end{aligned}$$

As $t < 0$, taking into account the first limit in (A-1), the first integral at the right-hand side goes to zero when $\mu \to +\infty$, while the two other integrals over the vertical segments converge to the integrals over the vertical half-lines:

$$\begin{aligned}\bar{\omega}_j^\flat(t) &= \lim_{s\to+\infty} \frac{1}{2\pi} \bigg(\int_{[s,s-i\infty]} \hat{f}_j(\zeta)\, g_j(\zeta) e^{i\zeta t}\, d\zeta + \int_{[-s-i\infty,-s]} \hat{f}_j(\zeta)\, g_j(\zeta) e^{i\zeta t}\, d\zeta \bigg) \\ &= \lim_{s\to+\infty} \frac{1}{2\pi} \bigg(\int_{[0,-i\infty]} \hat{f}_j(s+\zeta)\, g_j(s+\zeta) e^{i(s+\zeta)t}\, d\zeta + \int_{[-i\infty,0]} \hat{f}_j(-s+\zeta)\, g_j(-s+\zeta) e^{i(-s+\zeta)t}\, d\zeta \bigg).\end{aligned}$$

Using the second limit in (A-1) and the bound (A-2), we can conclude that the limit on the right-hand side is zero thanks to the dominated convergence theorem.

To prove the second item of the lemma, we remark from formula (3-12) that

$$(1+|\zeta|)^{3/4} \hat{\omega}_j^\flat \in L^2_\zeta(\mathbb{R}, L^2_y(\mathbb{R}_+, H^k_x(\mathbb{T}))), \quad (1+|\zeta|)^{1/4} \hat{\omega}_j^\flat \in L^2_\zeta(\mathbb{R}, H^1_y(\mathbb{R}_+, H^k_x(\mathbb{T}))) \quad \text{for all } k$$

using the smoothness of $\hat{f}_j$ with respect to x. We deduce that

$$\bar{\omega}_j^\flat \in H_t^{3/4}(\mathbb{R}, L^2_y(\mathbb{R}_+, H^k_x(\mathbb{T}))), \quad \bar{\omega}_j^\flat \in H_t^{1/4}(\mathbb{R}, H^1_y(\mathbb{R}_+, H^k_x(\mathbb{T})) \quad \text{for all } k. \tag{A-3}$$

Moreover, using again (3-12) and Plancherel in time, we get that, for any $\varphi = \varphi(t, x, y)$ smooth and quickly decreasing as $t \to \pm\infty$ and $y \to +\infty$,

$$\int_{\mathbb{R}\times\mathbb{R}_+\times\mathbb{T}} \bar{\omega}_j^\flat (\beta(j+1) - \partial_t)\varphi + \int_{\mathbb{R}\times\mathbb{R}_+\times\mathbb{T}} \partial_y \bar{\omega}_j^\flat\, \partial_y \varphi - \int_{\mathbb{R}\times\mathbb{T}} (2\bar{\omega}_j^\flat|_{y=0} + f_j)\varphi|_{y=0} = 0.$$

If we take φ with support in time included in $(-\infty, T)$, taking into account that $\bar{\omega}_j^\flat$ is zero for negative times, we end up with

$$\int_{(0,T)\times\mathbb{R}_+\times\mathbb{T}} \bar{\omega}_j^\flat (\beta(j+1) - \partial_t)\varphi + \int_{(0,T)\times\mathbb{R}_+\times\mathbb{T}} \partial_y \bar{\omega}_j^\flat\, \partial_y \varphi - \int_{(0,T)\times\mathbb{T}} \left(2\bar{\omega}_j^\flat|_{y=0} + \frac{M_j}{M_{j+1}} h_{j+1}\right) \varphi|_{y=0} = 0.$$

We recognize the weak formulation of system (3-10). The identity $\bar{\omega}_j^\flat = \omega_j^\flat$ over $(0, T)$ follows from the uniqueness of solutions to this system (for example in the regularity class given by (A-3)).

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SHARP VARIATION-NORM ESTIMATES FOR OSCILLATORY INTEGRALS RELATED TO CARLESON'S THEOREM

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We prove variation-norm estimates for certain oscillatory integrals related to Carleson's theorem. Bounds for the corresponding maximal operators were first proven by Stein and Wainger. Our estimates are sharp in the range of exponents, up to endpoints. Such variation-norm estimates have applications to discrete analogues and ergodic theory. The proof relies on square function estimates for Schrödinger-like equations due to Lee, Rogers and Seeger. In dimension 1, our proof additionally relies on a local smoothing estimate. Though the known endpoint local smoothing estimate by Rogers and Seeger is more than sufficient for our purpose, we also give a proof of certain local smoothing estimates using Bourgain–Guth iteration and the Bourgain–Demeter ℓ^2 decoupling theorem. This may be of independent interest, because it improves the previously known range of exponents for spatial dimensions $n \geq 4$.

1. Introduction

Let $n \geq 1$ and $\alpha > 1$ be fixed. Given a Calderón–Zygmund kernel $K : \mathbb{R}^n \to \mathbb{R}$ we define a modulated singular integral by

$$\mathcal{H}^{(u)} f(x) := \int_{\mathbb{R}^n} f(x-t) e^{iu|t|^\alpha} K(t)\, dt, \quad u \in \mathbb{R}. \tag{1-1}$$

The maximal operator

$$\sup_{u \in \mathbb{R}} |\mathcal{H}^{(u)} f| \tag{1-2}$$

was introduced in [Stein and Wainger 2001] as a generalization of the Carleson operator studied in [Carleson 1966; Fefferman 1973; Lacey and Thiele 2000]. In this paper, we study variation-norm estimates for the family $\{\mathcal{H}^{(u)} f\}_{u \in \mathbb{R}}$. Apart from the intrinsic interest in such bounds, another strong motivation is given by the connection to certain discrete analogues of (1-2) that are the subject of recent works [Krause and Lacey 2017; Krause 2018] (see Section 1A below).

If $\mathcal{J}$ is a subset of $\mathbb{R}$ and $\{a_u : u \in \mathcal{J}\}$ is a family of complex numbers indexed by $\mathcal{J}$, then for any $1 \leq r < \infty$ the r-variational norm of $\{a_u\}_{u \in \mathcal{J}}$ is defined to be

$$V^r\{a_u : u \in \mathcal{J}\} := \sup_{J \in \mathbb{N}} \sup_{\substack{u_0, u_1, \dots, u_J \in \mathcal{J} \\ u_0 < u_1 < \dots < u_J}} \left(\sum_{j=1}^{J} |a_{u_j} - a_{u_{j-1}}|^r \right)^{1/r}.$$

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Closely related to it is the jump function of the family $\{a_u\}_{u\in\mathcal{J}}$: for $\lambda > 0$, the λ-jump function of $\{a_u\}_{u\in\mathcal{J}}$, namely $N_\lambda\{a_u : u \in \mathcal{J}\}$, is defined to be the supremum of all positive integers N for which there exists a strictly increasing sequence $s_1 < t_1 < s_2 < t_2 < \cdots < s_N < t_N$, all of which are in $\mathcal{J}$, such that

$$|a_{t_j} - a_{s_j}| > \lambda$$

for all $j = 1, \ldots, N$. For $r \in (1, \infty)$ and $p \in (1, \infty)$, we will study the L^p mapping properties of the maps

$$f \longmapsto V^r\{\mathcal{H}^{(u)} f : u \in \mathbb{R}\},$$
$$f \longmapsto \lambda[N_\lambda\{\mathcal{H}^{(u)} f : u \in \mathbb{R}\}]^{1/r}, \quad \lambda > 0.$$

Henceforth f will always be a Schwartz function on $\mathbb{R}^n$; the goal is to establish a priori bounds for all such f. If in dimension $n = 1$ we take $\alpha = 1$ and replace $|t|$ by t, then this corresponds to the variation-norm Carleson operator, which has been studied in [Oberlin, Seeger, Tao, Thiele, and Wright 2012; Uraltsev 2016]. We refer the reader to [Bourgain 1989; Pisier and Xu 1988; Campbell, Jones, Reinhold, and Wierdl 2000; 2003; Jones, Seeger, and Wright 2008] for earlier results concerning jump function and variation-norm inequalities for other operators arising in harmonic analysis.

Let us assume that K is a homogeneous Calderón–Zygmund kernel, in the sense that

$$K(x) = \text{p.v.}\, \frac{\Omega(x)}{|x|^n}$$

for some function Ω that is smooth on $\mathbb{R}^n \setminus \{0\}$, homogeneous of degree 0. The assumption that K is homogeneous is not strictly necessary. It is there to help simplify the presentation of the proof of the theorem. We also assume that $\int_{\mathbb{S}^{n-1}} \Omega(x)\, d\sigma(x) = 0$, where σ denotes the surface measure on $\mathbb{S}^{n-1}$.

Theorem 1.1. *Let $n \geq 1$, $\alpha \in (1, \infty)$ and define $\mathcal{H}^{(u)}$ as in* (1-1). *If $r \in (2, \infty)$, $p \in (1, \infty)$ and $r > p'/n$, then we have*

$$\|V^r\{\mathcal{H}^{(u)} f : u \in \mathbb{R}\}\|_p \leq C\|f\|_p. \tag{1-3}$$

In addition, if $n \geq 2$ and $p \in (2n/(2n-1), \infty)$, then

$$\big\|\lambda\sqrt{N_\lambda\{\mathcal{H}^{(u)} f : u \in \mathbb{R}\}}\big\|_p \leq C\|f\|_p.$$

Here the constant C is allowed to depend on n, α, p and r.

Moreover, up to endpoints, we show that this is the best we can expect:

Theorem 1.2. *The estimate* (1-3) *fails if $r < p'/n$.*

Thus, the range of exponents for which estimate (1-3) holds is given by the quadrilateral in Figure 1 below (up to endpoints).

It is natural to ask what happens when α is less than 1. Our methods do not seem to be able to handle this case. But if $n = 1$, an easy adaptation of our methods allows us to obtain a positive result where the phase function $|t|^\alpha$ in (1-1) is replaced by $\text{sgn}(t)|t|^\alpha$. In particular, if α is an odd positive integer, we may replace $|t|^\alpha$ in (1-1) by t^α and still obtain a positive result.

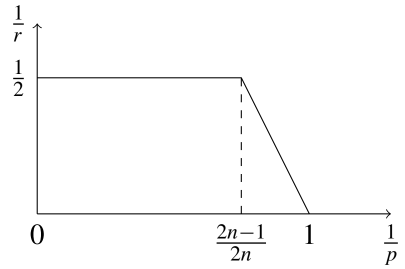

Figure 1. The range of exponents for which estimate (1-3) holds.

The inequality (1-3) can be understood as an extension of the well-known result from [Stein and Wainger 2001] (also see [Guo 2016] for the case when α is not an integer):

$$\Big\|\sup_{u\in\mathbb{R}} |\mathcal{H}^{(u)} f|\Big\|_p \lesssim \|f\|_p \quad \text{for every } p > 1. \tag{1-4}$$

1A. ***Connection with discrete analogues.*** Further motivation stems from the study of a discrete analogue of the maximal operator (1-2). Fix an integer $d \geq 2$ and let $u \in \mathbb{R}$. Consider the following operator $\mathcal{H}_{\mathbb{Z}}^{(u)}$ acting on functions $f : \mathbb{Z} \to \mathbb{C}$:

$$\mathcal{H}_{\mathbb{Z}}^{(u)} f(x) = \sum_{t\in\mathbb{Z}\setminus\{0\}} f(x-t)e^{iut^d}\frac{1}{t}, \quad x \in \mathbb{Z}.$$

This is a discrete analogue of our operator $\mathcal{H}^{(u)}$ for $n = 1$ and $\alpha = d$. Bounding the associated maximal operator $f \mapsto \sup_{u\in\mathbb{R}} |\mathcal{H}_{\mathbb{Z}}^{(u)} f|$ on $\ell^p(\mathbb{Z})$ is significantly more difficult than bounding Stein and Wainger's maximal operator and until recently, no such bounds were known. For the recent progress on this problem and further discussion of discrete analogues, we refer to [Krause 2018; Krause and Lacey 2017]. A careful analysis of the multiplier of $\mathcal{H}_{\mathbb{Z}}^{(u)}$, which is much in the spirit of the Hardy–Littlewood circle method, reveals a natural splitting of the problem into a number-theoretic and an analytic component. In the case $p = 2$, the core estimate for the analytic component is a variant of Bourgain's classical maximal multifrequency lemma [1989, Lemma 4.1]. The precise statements can be found in [Krause and Lacey 2017, Section 3; Krause 2018, Sections 5 and 10.2]; see, in particular, Theorem 3.5 of [Krause and Lacey 2017]. Using a small refinement of our Theorem 1.1 (see Theorem B.3 below), together with the argument from [Bourgain 1989], one can obtain an alternative simple proof of (a small extension of) Theorem 3.5 of [Krause and Lacey 2017]; we include some details in Appendix B.

Discrete analogues are intimately related to ergodic theorems and this connection provides a further application of our variation-norm estimates. Krause [2018, Theorem 1.2] made use of a variant of the estimate (1-3) in his recent work on a pointwise ergodic theorem.

1B. ***Outline of the proof.*** We now briefly describe an outline of the proof of Theorem 1.1. To control the left-hand side of the estimate (1-3), we split the contribution into two parts: *long variations* and *short variations*. For each $j \in \mathbb{Z}$, define the short variation on the u-interval $[2^{j\alpha}, 2^{(j+1)\alpha}]$ by

$$V_j^r \mathcal{H} f(x) := V^r\{\mathcal{H}^{(u)} f(x) : u \in [2^{j\alpha}, 2^{(j+1)\alpha}]\}.$$

Also define

$$S_r(\mathcal{H}f)(x) := \Big(\sum_{j\in\mathbb{Z}} |V_j^r \mathcal{H}f(x)|^r\Big)^{1/r},$$

$$N_\lambda^{\mathrm{dyad}}(\mathcal{H}f)(x) := N_\lambda\{\mathcal{H}^{(2^{j\alpha})}f(x) : j\in\mathbb{Z}\}.$$

We will use the following lemma (see, for example, [Jones, Seeger, and Wright 2008]):

Lemma 1.3. *For $r\in[2,\infty)$ we have*

$$\lambda[N_\lambda\{\mathcal{H}^{(u)}f : u>0\}]^{1/r} \lesssim S_r(\mathcal{H}f) + \lambda[N_{\lambda/3}^{\mathrm{dyad}}(\mathcal{H}f)]^{1/r}$$

uniformly in $\lambda>0$.

(Hereafter, $A\lesssim B$ means $A\le CB$ for some absolute constant C.)

By this lemma, and by Bourgain's argument [1989] of passing from jump norms to variation-norms (see also [Jones, Seeger, and Wright 2008, Section 2]), to prove Theorem 1.1 it suffices to prove the following two propositions.

Proposition 1.4. *For every $p\in(1,\infty)$ and $r\in[2,\infty)$ we have*

$$\|\lambda[N_{\lambda/3}^{\mathrm{dyad}}(\mathcal{H}f)]^{1/r}\|_p \lesssim \|f\|_p$$

uniformly in $\lambda>0$.

Proposition 1.5. *Let $n\ge 1$ and $p\in(1,\infty)$, $r\in(2,\infty)$ with $r>p'/n$. Then we have*

$$\|S_r(\mathcal{H}f)\|_p \lesssim \|f\|_p.$$

If $n\ge 2$, then the inequality also holds for $r=2$.

The proof of Proposition 1.4 depends on a jump function inequality of [Jones, Seeger, and Wright 2008] that is based on a Lépingle inequality for martingales.

By interpolation with the inequality (1-4) of [Stein and Wainger 2001], it suffices to consider the case $p\in(2n/(2n-1),\infty)$ to prove Proposition 1.5. The proof of Proposition 1.5 then depends on a square function estimate for Schrödinger-like equations, which is due to [Lee, Rogers, and Seeger 2012]. In one dimension, we additionally need a local smoothing estimate for these equations. The following local smoothing result is more than sufficient for our needs: indeed we will only need the following estimate for $n=1$ and some $p<\infty$. We are including the full theorem here only because it may be of independent interest.

Theorem 1.6. *Let $\gamma>1$ be a real number and let I be a compact time interval. For any dimension $n\ge 1$ and exponent $p<\infty$ satisfying*

$$\begin{cases} p>\dfrac{2(4n+7)}{4n+1} & \text{if } n\equiv -1 \pmod 3,\\ p>\dfrac{2n+3}{n} & \text{if } n\equiv 0 \pmod 3,\\ p>\dfrac{4(n+2)}{2n+1} & \text{if } n\equiv 1 \pmod 3, \end{cases} \tag{1-5}$$

we have

$$\left(\int_{\mathbb{R}^n\times I}\left|\int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{f}(\xi)e^{it|\xi|^\gamma}\,d\xi\right|^p dx\,dt\right)^{1/p} \lesssim_\epsilon \|f\|_{W^{\beta+\epsilon,p}(\mathbb{R}^n)} \tag{1-6}$$

whenever $\epsilon > 0$ *and*

$$\frac{\beta}{\gamma} = n\Big(\frac{1}{2}-\frac{1}{p}\Big)-\frac{1}{p}.$$

Here we write $W^{s,p}(\mathbb{R}^n) = (I-\Delta)^{-s/2}L^p(\mathbb{R}^n)$ *to denote the standard Bessel potential space.*

Let us take a moment to compare Theorem 1.6 with results in the existing literature. Rogers [2008] considered the case $\gamma = 2$, namely a local smoothing estimate for the Schrödinger propagator $e^{it\Delta}$. He proved that (1-6) holds whenever $\gamma = 2$, $p \in (2+4/(n+1),\infty)$ and $\epsilon > 0$ (in the rest of this section β will always be as specified in Theorem 1.6). This was improved subsequently by Rogers and Seeger [2010], who obtained the endpoint case $\epsilon = 0$ for all $\gamma > 1$: they established that (1-6) holds with $\epsilon = 0$ for all $p \in (2+4/(n+1),\infty)$ and all $\gamma > 1$. In particular, this implies Theorem 1.6 for $n = 1, 2, 3$. Theorem 1.6 gives a larger range of p in dimensions $n \ge 4$, albeit with an ϵ-loss in smoothness. We also note that in the case $\gamma = 2$ (i.e., for the Schrödinger propagator), Lee, Rogers and Seeger [2013] obtained an improvement of the aforementioned result of [Rogers and Seeger 2010]; in particular, in Proposition 5.2 of [Lee, Rogers, and Seeger 2013], they proved that if the dual Fourier restriction conjecture holds at an exponent q_0, in the sense that

$$\|Ef\|_{L^{q_0}(\mathbb{R}^{n+1})} \lesssim \|f\|_{L^{q_0}([0,1]^n)}$$

for some exponent $\gamma_0 < 2(n+3)/(n+1)$, where E is the Fourier extension operator for the paraboloid in $\mathbb{R}^{n+1}$ given by

$$Ef(x,t) = \int_{[0,1]^n} f(\xi)e^{i(x\cdot\xi+t|\xi|^2)}\,d\xi, \qquad (x,t)\in\mathbb{R}^n\times\mathbb{R}, \tag{1-7}$$

then (1-6) holds for $\gamma = 2$ with $\epsilon = 0$ whenever $p \in (q_*,\infty)$, where q_* is defined by

$$q_* := \frac{2(n+3)}{n+1}(1-\gamma(n,q_0)), \qquad \text{with } \gamma(n,q_0) := \frac{1/q_0-(n+1)/(2(n+3))}{n((n+1)/2-(n+2)/q_0)}.$$

A direct computation shows that

$$q_* = 2+\frac{4}{n}-\frac{2}{n^2-n(4-q_0)/(q_0-2)}.$$

As a result, even if one can establish (1-7) in all dimensions n with $q_0 = q_0(n)$ that decays like $q_0(n) = 2+(2+\lambda)/n+O(1/n^2)$ for some $\lambda > 0$ (the Fourier restriction conjecture shows that the best one can hope for is $\lambda = 0$), using the above result of Lee, Rogers and Seeger, one can only establish the local smoothing estimate (1-6) for $p \in (q_*(n),\infty)$, where

$$q_*(n) = 2+\frac{4}{n}+O\Big(\frac{1}{n^2}\Big).$$

On the contrary, if $p_*(n)$ is the Bourgain–Guth exponent given by the right-hand sides of (1-5), we see that

$$p_*(n) = 2 + \frac{3}{n} + O\left(\frac{1}{n^2}\right),$$

so our range of the exponent p is larger than that of Lee, Rogers and Seeger in high dimensions n, even for the Schrödinger equation case.

Contrary to [Rogers and Seeger 2010], which relied on bilinear restriction estimates, our proof of Theorem 1.6 relies on the Bourgain–Guth argument [2011] (see also the presentation in [Bourgain and Demeter 2017]), and the Bourgain–Demeter decoupling inequality [2015]; see [Wolff 2000; Łaba and Wolff 2002] for some earlier foundational work on decoupling inequalities, and their applications to local smoothing estimates. The multilinear estimates developed in [Guth 2018] might be useful in establishing (1-6) for a larger range of exponents, but we did not pursue this here.

Organization of the paper. In Section 2 we state two preliminary results, namely a consequence of the classical Lépingle inequality, and a consequence of the Plancherel–Pólya inequality. In Section 3 we control long jumps; that is, we will prove Proposition 1.4. The treatment for short jumps (that is, the proof of Proposition 1.5) will be split into two parts. In Section 4 we prove Proposition 1.5 in two special cases: $n \geq 2,\ p > 2(n+2)/n$, and $n = 1,\ p > 2$. These are the main cases to be considered. In Section 5 we indicate the modifications necessary to prove the remaining case of Proposition 1.5: namely, $n \geq 2$ and $2n/(2n-1) < p \leq 2(n+2)/n$. The proof of Theorem 1.2 is in Section 6. In Section 7 we provide the proof of a vector-valued generalization of a multiplier theorem of [Seeger 1988], which we used in the proof of the short jump estimates in Section 4. In Appendix A we prove the local smoothing estimates in Theorem 1.6. In Appendix B we refine our Theorem 1.1 by obtaining a good bound on the growth of the constant C in (1-3) as $p = r \to 2^+$ (see Theorem B.3), and use it to provide an alternative simple proof of a maximal multifrequency estimate of Krause and Lacey [2017, Theorem 3.5].

2. Prerequisites

2A. ***A jump function inequality of Jones, Seeger and Wright.*** We recall a jump function inequality for convolutions with dyadic dilations of a fixed measure from [Jones, Seeger, and Wright 2008, Theorem 1.1]. It is a consequence of the more classical Lépingle inequality for martingales.

Proposition 2.1 [Jones, Seeger, and Wright 2008]. *Let σ be a compactly supported finite nonnegative Borel measure on $\mathbb{R}^n$ whose Fourier transform satisfies*

$$|\hat{\sigma}(\xi)| \leq C|\xi|^{-a}$$

for some $a > 0$. For $k \in \mathbb{Z}$, define σ_k by

$$\int_{\mathbb{R}^n} f(x)\, d\sigma_k(x) = \int_{\mathbb{R}^n} f(2^{-k}x)\, d\sigma(x).$$

Then

$$\left\| \lambda \sqrt{N_\lambda\{f * \sigma_k : k \in \mathbb{Z}\}} \right\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}$$

for all $1 < p < \infty$, uniformly in $\lambda > 0$.

We will apply this proposition as follows. Let S be a nonnegative smooth function with compact support in $[-1, 1]^n$ and $\int_{\mathbb{R}^n} S(x)\,dx = 1$. For $k \in \mathbb{Z}$ and any Schwartz function f on $\mathbb{R}^n$, let

$$S_k f(x) = f * S_k(x),$$

where $S_k(x) = 2^{kn} S(2^k x)$. If σ is the measure on $\mathbb{R}^n$ given by

$$\int_{\mathbb{R}^n} f(x)\,d\sigma(x) = \int_{\mathbb{R}^n} f(x) S(x)\,dx,$$

then $\sigma_k(x)$ coincides with $S_k(x)dx$, and hence $f * \sigma_k = S_k f$ for all $k \in \mathbb{Z}$. Proposition 2.1 then gives

$$\big\|\lambda\sqrt{N_\lambda\{S_k f : k \in \mathbb{Z}\}}\big\|_{L^p(\mathbb{R}^n)} \le C_p \|f\|_{L^p(\mathbb{R}^n)} \tag{2-1}$$

for all $1 < p < \infty$, uniformly in $\lambda > 0$. Note that $\widehat{S}(0) = 1$ and $\widehat{S}(\xi)$ decreases rapidly to zero as $|\xi| \to \infty$. So later it helps to think of $\widehat{S}(\xi)$ as localized to $|\xi| \lesssim 1$, and interpret $S_k f$ as a localization of f to frequency $\lesssim 2^k$.

Next, let $\{c_\ell\}_{\ell=0}^\infty$ be a complex sequence with $|c_\ell| = O(2^{-\alpha\ell})$ for some $\alpha > 0$. Let $\widetilde{S}_k$ be the operator defined by

$$\widetilde{S}_k f := \sum_{\ell=0}^\infty c_\ell S_{k-\ell} f. \tag{2-2}$$

We will use (2-1) to prove that

$$\big\|\lambda\sqrt{N_\lambda\{\widetilde{S}_k f : k \in \mathbb{Z}\}}\big\|_{L^p(\mathbb{R}^n)} \le C_p \|f\|_{L^p(\mathbb{R}^n)} \tag{2-3}$$

for all $1 < p < \infty$, uniformly in $\lambda > 0$. Recall the definition of the jump norm $N_\lambda\{\widetilde{S}_k f(x) : k \in \mathbb{Z}\}$: it is the supremum of all positive integers N for which there exists a strictly increasing sequence $s_1 < t_1 < s_2 < t_2 < \cdots < s_N < t_N$, all of which are in $\mathbb{Z}$, such that

$$|\widetilde{S}_{t_j} f(x) - \widetilde{S}_{s_j} f(x)| > \lambda \tag{2-4}$$

for all $j = 1, \dots, N$. But if $s_1 < t_1 < s_2 < t_2 < \cdots < s_N < t_N$ is as such, then for all $j = 1, \dots, N$ we have

$$|S_{t_j-\ell} f(x) - S_{s_j-\ell} f(x)| \gtrsim 2^{\ell\alpha/2}\lambda$$

for at least one $\ell \ge 0$. Hence,

$$N_\lambda\{\widetilde{S}_k f(x) : k \in \mathbb{Z}\} \lesssim \sum_{\ell=0}^\infty N_{2^{\ell\alpha/2}\lambda}\{S_k f(x) : k \in \mathbb{Z}\},$$

which implies

$$\sqrt{N_\lambda\{\widetilde{S}_k f : k \in \mathbb{Z}\}} \lesssim \sum_{\ell=0}^\infty \sqrt{N_{2^{\ell\alpha/2}\lambda}\{S_k f : k \in \mathbb{Z}\}}.$$

This further implies

$$\big\|\lambda\sqrt{N_\lambda\{\widetilde{S}_k f : k \in \mathbb{Z}\}}\big\|_{L^p(\mathbb{R}^n)} \lesssim \sum_{\ell=0}^\infty 2^{-\ell\alpha/2} \big\|2^{\ell\alpha/2}\lambda\sqrt{N_{2^{\ell\alpha/2}\lambda}\{S_k f : k \in \mathbb{Z}\}}\big\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_p.$$

This finishes the proof of the estimate (2-3).

2B. ***An inequality of Plancherel and Pólya.*** Next, let $F(u)$ be an L^2 function on $\mathbb{R}$ whose Fourier transform $\widehat{F}(\xi)$ is supported on the set $|\xi| \le 1$. Such an F is sometimes said to be in a Paley–Wiener space. An inequality of Plancherel and Pólya [1936; 1937] says that for any such F and any $r \in [1, \infty)$, we have

$$\sum_{j \in \mathbb{Z}} |F(j)|^r \le C_r \int_{\mathbb{R}} |F(u)|^r \, du, \tag{2-5}$$

where C_r is a constant independent of F. This holds because if $\widehat{F}$ is supported on $|\xi| \le 1$, then, by the uncertainty principle, F is essentially constant on every interval of length 1 (see also [Young 1980] for an alternative proof based on complex analysis).

From (2-5) we can deduce the following variation-norm estimate (see also page 6729 of [Jones, Seeger, and Wright 2008]):

Proposition 2.2. *Let $F(u)$ be a function on $\mathbb{R}$ whose Fourier transform $\widehat{F}(\xi)$ is supported on the set $\{|\xi| \le \lambda\}$. Then for every $1 \le q \le r < \infty$, we have*

$$V^r\{F(u) : u \in \mathbb{R}\} \le A_{q,r} \lambda^{1/q} \|F\|_{L^q}, \tag{2-6}$$

with a constant $A_{q,r}$ depending only on q and r.

Proof. By rescaling we may assume that $\lambda = 1$. Now let $k \in \mathbb{N}$ and $u_1 < \cdots < u_k$ be a strictly increasing sequence in $\mathbb{R}$. We let $\kappa(0) = 1$, $n_1 = \lfloor u_{\kappa(0)} \rfloor$ and let $\kappa(1)$ be the largest integer in $\{1, \dots, k\}$ such that $u_{\kappa(1)} < n_1 + 1$. If $\kappa(1) < k$, we let $n_2 = \lfloor u_{\kappa(1)+1} \rfloor$ and let $\kappa(2)$ be the largest integer in $\{1, \dots, k\}$ such that $u_{\kappa(2)} < n_2 + 1$. Clearly this process will terminate in finitely many, say m, steps. In this way we collect the points $u_1, \dots, u_k$ into intervals $[n_1, n_1+1], [n_2, n_2+1], \dots, [n_m, n_m+1]$ of length at most 1. Now for $s = 1, \dots, m-1$, by the triangle inequality, we have

$$|F(u_{\kappa(s)}) - F(u_{\kappa(s)+1})|^r \lesssim |F(u_{\kappa(s)}) - F(n_s+1)|^r + |F(n_s+1)|^r + |F(n_{s+1})|^r + |F(n_{s+1}) - F(u_{\kappa(s)+1})|^r.$$

This shows

$$\begin{aligned}\sum_{i=1}^{k-1} &|F(u_i) - F(u_{i+1})|^r \\ &\lesssim \sum_{s=1}^{m} (|F(n_s)|^r + |F(n_s+1)|^r) + \sum_{s=1}^{m} \Bigg(|F(n_s) - F(u_{\kappa(s-1)})|^r \\ &\qquad + \sum_{\kappa(s-1) \le i < \kappa(s)} |F(u_i) - F(u_{i+1})|^r + |F(u_{\kappa(s)}) - F(n_s+1)|^r \Bigg).\end{aligned}$$

(Indeed, for $s = 1$, we do not need the terms $|F(n_s)|^r$ and $|F(n_s) - F(u_{\kappa(s-1)})|^r$ on the right-hand side; similarly for $s = m$, we do not need the terms $|F(n_s+1)|^r$ and $|F(u_{\kappa(s)}) - F(n_s+1)|^r$. But there is no harm putting them in, which makes the expression on the right-hand side more symmetric.) By the mean-value theorem, for $s = 1, \dots, m$, we have

$$\begin{aligned}|F(n_s) - F(u_{\kappa(s-1)})|^r + \sum_{\kappa(s-1) \le i < \kappa(s)} |F(u_i) - F(u_{i+1})|^r + |F(u_{\kappa(s)}) - F(n_s+1)|^r \\ \le \|F'\|_{L^\infty}^r \Bigg(|n_s - u_{\kappa(s-1)}|^r + \sum_{\kappa(s-1) \le i < \kappa(s)} |u_i - u_{i+1}|^r + |u_{\kappa(s)} - (n_s+1)|^r \Bigg),\end{aligned}$$

and the quantity inside the parentheses in the last line is ≤ 1 since we have the elementary inequality

$$t_1^r + \cdots + t_\sigma^r \leq (t_1 + \cdots + t_\sigma)^r$$

whenever $t_1, \ldots, t_\sigma \geq 0$ and $1 \leq r < \infty$. Now since $\widehat{F}$ is supported on $|\xi| \leq 1$, Bernstein's inequality implies

$$\|F'\|_{L^\infty} \lesssim_r \|F\|_{L^r}$$

whenever $1 \leq r < \infty$. Altogether, we see that

$$\begin{aligned}\sum_{i=1}^{k-1} |F(u_i) - F(u_{i+1})|^r &\lesssim_r \|F\|_{L^r}^r + \sum_{s=1}^{m} (|F(n_s)|^r + |F(n_s+1)|^r) \\ &\lesssim_r \|F\|_{L^r}^r + \sum_{j\in\mathbb{Z}} |F(j)|^r \lesssim_r \|F\|_{L^r}^r\end{aligned}$$

whenever $1 \leq r < \infty$, the last inequality following from (2-5). Since $\widehat{F}$ is supported on $\{|\xi| \leq 1\}$ and $1 \leq q \leq r$, Bernstein's inequality again implies $\|F\|_{L^r} \lesssim_{q,r} \|F\|_{L^q}$. This completes the proof of (2-6). □

3. Long jump estimates

Our goal in this section is to prove Proposition 1.4. Indeed, we will prove something slightly stronger, including the case $0 < \alpha < 1$.

Proposition 3.1. *Fix $\alpha > 0$, $\alpha \neq 1$. For $1 < p < \infty$, we have*

$$\|\lambda\sqrt{N_\lambda\{\mathcal{H}^{(2^{k\alpha})} f : k \in \mathbb{Z}\}}\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \tag{3-1}$$

uniformly in $\lambda > 0$. Here $\mathcal{H}^{(2^{k\alpha})}$ is defined as in (1-1).

First we decompose $\mathcal{H}^{(2^{k\alpha})}$ into

$$\begin{aligned}\mathcal{H}^{(2^{k\alpha})} f(x) &= \int_{|t|\leq 2^{-k}} f(x-t) e^{i2^{k\alpha}|t|^\alpha} K(t)\,dt + \int_{|t|>2^{-k}} f(x-t) e^{i2^{k\alpha}|t|^\alpha} K(t)\,dt \\ &=: \mathcal{H}_{k,-\infty} f(x) + \mathcal{H}_{k,\infty} f(x).\end{aligned}$$

In the term $\mathcal{H}_{k,-\infty} f$, we are integrating over small t, and the exponential $e^{i2^{k\alpha}|t|^\alpha}$ is approximately 1. This motivates us to further decompose $\mathcal{H}_{k,-\infty} f$ as

$$\begin{aligned}\mathcal{H}_{k,-\infty} f(x) &= \int_{|t|\leq 2^{-k}} f(x-t) K(t)\,dt + \int_{|t|\leq 2^{-k}} f(x-t)(e^{i2^{k\alpha}|t|^\alpha} - 1) K(t)\,dt \\ &=: \widetilde{\mathcal{H}}_{k,0} f(x) + \mathcal{H}_{k,0} f(x).\end{aligned} \tag{3-2}$$

For the other term, we take the decomposition

$$\mathcal{H}_{k,\infty} f(x) = \sum_{\ell=1}^{\infty} \mathcal{H}_{k,\ell} f(x) := \sum_{\ell=1}^{\infty} \int_{2^{-k+\ell-1}<|t|\leq 2^{-k+\ell}} f(x-t) e^{i2^{k\alpha}|t|^\alpha} K(t)\,dt.$$

The former term in (3-2) is a truncated singular integration. We have:

Lemma 3.2 [Campbell, Jones, Reinhold, and Wierdl 2003, Theorem A].

$$\big\|\lambda\sqrt{N_\lambda\{\widetilde{\mathcal{H}}_{k,0}f : k\in\mathbb{Z}\}}\big\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$$

for all $1<p<\infty$.

Hence it remains to estimate the jump norms of $\mathcal{H}_{k,0}f(x)+\sum_{\ell=1}^\infty \mathcal{H}_{k,\ell}f(x)=\sum_{\ell=0}^\infty \mathcal{H}_{k,\ell}f(x)$. To do so, we carry out a Littlewood–Paley decomposition. For each $\ell\ge 0$, apply

$$\mathcal{H}_{k,\ell}f=\mathcal{H}_{k,\ell}S_{k-\ell}f+\mathcal{H}_{k,\ell}(f-S_{k-\ell}f).$$

(see Section 2 for the precise definition of S_kf). Notice that $S_{k-\ell}f$ is approximately constant at the physical scale $2^{-k+\ell}$. Thus, $\mathcal{H}_{k,\ell}S_{k-\ell}f$ is almost just a multiple of $S_{k-\ell}f$. This motivates us to further take the decomposition

$$\mathcal{H}_{k,\ell}S_{k-\ell}f=c_\ell S_{k-\ell}f+(\mathcal{H}_{k,\ell}S_{k-\ell}f-c_\ell S_{k-\ell}f),$$

where

$$c_0:=\int_{|t|\le 1}(e^{i|t|^\alpha}-1)K(t)\,dt \quad\text{and}\quad c_\ell:=\int_{1/2<|t|\le 1} e^{i2^{\ell\alpha}|t|^\alpha}K(t)\,dt \quad \text{for } \ell\ge 1 \tag{3-3}$$

are constants. Here we choose the constants c_0 and c_ℓ as such because K is assumed to be homogeneous. Hence

$$\sum_{\ell=0}^\infty \mathcal{H}_{k,\ell}f(x)=\sum_{\ell=0}^\infty c_\ell S_{k-\ell}f+\sum_{\ell=0}^\infty(\mathcal{H}_{k,\ell}S_{k-\ell}f-c_\ell S_{k-\ell}f)+\sum_{\ell=0}^\infty \mathcal{H}_{k,\ell}(f-S_{k-\ell}f).$$

Since a simple integration-by-parts argument shows that $|c_\ell|=O(2^{-\alpha\ell})$, the contribution from the first term to the desired jump norm can be controlled using (2-3). To handle the latter two terms we use a square function. It suffices to show that

$$\sum_{\ell=0}^\infty\Big\|\Big(\sum_{k\in\mathbb{Z}}|\mathcal{H}_{k,\ell}S_{k-\ell}f-c_\ell S_{k-\ell}f|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}\lesssim\|f\|_{L^p(\mathbb{R}^n)}, \tag{3-4}$$

$$\sum_{\ell=0}^\infty\Big\|\Big(\sum_{k\in\mathbb{Z}}|\mathcal{H}_{k,\ell}(f-S_{k-\ell}f)|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}\lesssim\|f\|_{L^p(\mathbb{R}^n)} \tag{3-5}$$

since the square functions dominate the desired jump norms pointwisely. To establish these estimates we apply a finer frequency decomposition. Let

$$\Delta(x):=2^nS(2x)-S(x)\quad\text{and}\quad\Delta_k(x):=2^{kn}\Delta(2^kx)$$

and write $\Delta_kf:=f*\Delta_k$ so that

$$S_{k-\ell}f=\sum_{j=1}^\infty\Delta_{k-\ell-j}f\quad\text{and}\quad f-S_{k-\ell}f=\sum_{j=0}^\infty\Delta_{k-\ell+j}f.$$

By the triangle inequality, to prove (3-4) and (3-5), it suffices to prove the existence of some constant $\gamma > 0$ such that

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} |\mathcal{H}_{k,\ell}\Delta_{k-\ell-j} f - c_\ell \Delta_{k-\ell-j} f|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^n)} \lesssim 2^{-\gamma(j+\ell)} \|f\|_{L^p(\mathbb{R}^n)}, \tag{3-6}$$

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} |\mathcal{H}_{k,\ell}\Delta_{k-\ell+j} f|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^n)} \lesssim 2^{-\gamma(j+\ell)} \|f\|_{L^p(\mathbb{R}^n)} \tag{3-7}$$

for every $j, \ell \geq 0$ and every $1 < p < \infty$. Throughout the paper, we use γ to denote a positive real number that might vary from line to line, if not otherwise stated.

Now each of the estimates (3-6) and (3-7) holds for $1 < p < \infty$ without the small factors on the right, since $|\mathcal{H}_{k,\ell} f| \lesssim Mf$ where M is the Hardy–Littlewood maximal operator on $\mathbb{R}^n$, allowing us to invoke the Fefferman–Stein vector-valued inequality for the maximal function [Stein 1993, Chapter II.1]. Hence by real interpolation, it suffices to prove the case $p = 2$. To do so, fix $\alpha > 0$, $\alpha \neq 1$ and $\ell \in \mathbb{N}$. Let $m_\ell(\xi)$ be the multiplier defined by

$$\begin{aligned} m_0(\xi) &:= \int_{|t|\leq 1} (e^{i|t|^\alpha} - 1) e^{-it\cdot\xi} K(t)\, dt, \\ m_\ell(\xi) &:= \int_{1/2<|t|\leq 1} e^{i|2^\ell t|^\alpha} e^{-it\cdot\xi} K(t)\, dt \quad \text{for } \ell \geq 1. \end{aligned}$$

Let $\tilde{m}_\ell(\xi)$ be the multiplier defined by

$$\begin{aligned} \tilde{m}_0(\xi) &:= \int_{|t|\leq 1} (e^{i|t|^\alpha} - 1)(e^{-it\cdot\xi} - 1) K(t)\, dt, \\ \tilde{m}_\ell(\xi) &:= \int_{1/2<|t|\leq 1} e^{i|2^\ell t|^\alpha} (e^{-it\cdot\xi} - 1) K(t)\, dt \quad \text{for } \ell \geq 1. \end{aligned}$$

Since K is assumed to be homogeneous, for $\ell \geq 0$ the multiplier for $\mathcal{H}_{k,\ell}$ is $m_\ell(2^{-k+\ell}\xi)$. It follows that for $\ell \geq 0$ the multiplier for $\mathcal{H}_{k,\ell} - c_\ell$ is $\tilde{m}(2^{-k+\ell}\xi)$. Then (3-6) and (3-7) with $p = 2$ follow from the pointwise estimates for multipliers

$$\Big(\sum_{k\in\mathbb{Z}} |\hat{\Delta}(2^{-k+\ell+j}\xi) \tilde{m}_\ell(2^{-k+\ell}\xi)|^2 \Big)^{1/2} + \Big(\sum_{k\in\mathbb{Z}} |\hat{\Delta}(2^{-k+\ell-j}\xi) m_\ell(2^{-k+\ell}\xi)|^2 \Big)^{1/2} \lesssim 2^{-\gamma(\ell+j)}. \tag{3-8}$$

We need the following lemma, which is a consequence of the van der Corput lemma (details omitted):

Lemma 3.3. *We have*

$$|m_\ell(\xi)| \lesssim \min\{2^{-\gamma\ell}, 2^{\alpha\ell}|\xi|^{-\gamma}\} \quad \textit{for all } \xi \in \mathbb{R}. \tag{3-9}$$

In particular,

$$|m_\ell(\xi)| \lesssim (2^{-\gamma\ell} \cdot 2^{\alpha\ell}|\xi|^{-\gamma})^{1/2} \quad \textit{for all } \xi \in \mathbb{R}. \tag{3-10}$$

We also have

$$|\tilde{m}_\ell(\xi)| \lesssim \begin{cases} \min\{2^{-\gamma\ell}, |\xi|\} \lesssim 2^{-\gamma\ell/2}|\xi|^{1/2} & \textit{for } |\xi| \leq 1, \\ 1 & \textit{for } |\xi| \geq 1. \end{cases}$$

We are ready to prove (3-8). The estimate is invariant upon replacing ξ by 2ξ; hence we only need to prove it when $|\xi| \simeq 1$. First consider the first term on the left-hand side of (3-8). When $k \le 0$, we bound $|\tilde{m}_\ell(2^{-k+\ell}\xi)| \lesssim 1$ and $|\hat{\Delta}(2^{-k+\ell+j}\xi)| \lesssim 2^{-10(-k+\ell+j)}$. Summing over $k \le 0$, we obtain $2^{-10(\ell+j)}$.

When $k \ge 0$, we bound $|\tilde{m}_\ell(2^{-k+\ell}\xi)| \lesssim 2^{-\gamma\ell/2}2^{-k/2+\ell/2}$ and

$$|\hat{\Delta}(2^{-k+\ell+j}\xi)| \lesssim \begin{cases} 2^{-10(-k+\ell+j)} & \text{if } 0 \le k \le \ell+j, \\ 2^{-k+\ell+j} & \text{if } k \ge \ell+j. \end{cases}$$

Summing over $k \ge 0$, we obtain $2^{-\gamma(\ell+j)}$ for some $\gamma > 0$. This finishes the proof of the first half of (3-8).

Next we turn to the second term on the left-hand side of (3-8). What we need to prove can also be written as

$$\left(\sum_{k\in\mathbb{Z}} |\hat{\Delta}(2^k\xi)m_\ell(2^{k+j}\xi)|^2\right)^{1/2} \lesssim 2^{-\gamma(\ell+j)} \quad \text{for } |\xi| \simeq 1. \tag{3-11}$$

We work on two different cases. Let $C_\alpha > 0$ be a sufficiently large constant. Assume that we are in the case $j \ge C_\alpha \ell$. We bound the left-hand side of (3-11) by

$$\sum_{k\ge 0} 2^{-10k}2^{\alpha\ell}2^{-\gamma k-\gamma j} + \sum_{k<0} 2^k(2^{\alpha\ell}\cdot 2^{-\gamma\ell}2^{-\gamma k-\gamma j})^{1/2} \lesssim 2^{-\gamma(\ell+j)}.$$

Here for the case $k \ge 0$ we applied (3-9), and for the case $k < 0$ we applied (3-10).

Finally, we assume that $0 \le j \le C_\alpha \ell$. We bound the left-hand side of (3-11) by

$$\sum_{k\ge 0} 2^{-10k}2^{-\gamma\ell} + \sum_{k<0} 2^k 2^{-\gamma\ell} \lesssim 2^{-\gamma(\ell+j)}.$$

Here in both cases $k \ge 0$ and $k < 0$ we applied (3-9).

4. Short jump estimates for large *p*

We are now going to start the proof of Proposition 1.5. Recall that by interpolation, we only need to establish Proposition 1.5 when $p \in (2n/(2n-1), \infty)$ and $r \in (2, \infty)$ (see discussion following Proposition 1.5). In this section we will do so for all sufficiently large values of p. More precisely, let $\alpha > 1$, let $\mathcal{H}^{(u)}$ be as in (1-1), and let $V_j^r \mathcal{H} f(x) = V^r\{\mathcal{H}^{(u)} f(x) : u \in [2^{j\alpha}, 2^{(j+1)\alpha}]\}$. We prove

$$\left\| \left(\sum_{j\in\mathbb{Z}} |V_j^r(\mathcal{H}f)|^r\right)^{1/r} \right\|_p \lesssim \|f\|_p \tag{4-1}$$

whenever

$$p \in (2, \infty), \quad n = 1, \quad r \in (2, \infty) \tag{4-2}$$

or

$$p \in \left(2 + \frac{4}{n}, \infty\right), \quad n \ge 2, \quad r \in [2, \infty). \tag{4-3}$$

This proves Proposition 1.5 when $n = 1$. In the next section, we extend (4-1) to all $p \in (2n/(2n-1), \infty)$ when $n \ge 2$, $r \in [2, \infty)$. That would complete the proof of Proposition 1.5 when $n \ge 2$.

4A. ***Main tool: a square function estimate for the semigroup*** $e^{it(-\Delta)^{\lambda/2}}$**.** The main input to our proof of (4-1) under condition (4-2) or (4-3) is a square function estimate, due to [Lee, Rogers, and Seeger 2012]:

Proposition 4.1 [Lee, Rogers, and Seeger 2012]. (1) *Let* $n = 1$, $p \in [2, \infty)$ *and* $\lambda > 1$. *Then for any compact time interval* I,

$$\left\| \left(\int_I \left| \int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi) e^{it|\xi|^\lambda} \, d\xi \right|^2 dt \right)^{1/2} \right\|_{L^p(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}.$$

(2) *Let* $n \geq 2$, $p \in (2(n+2)/n, \infty)$ *and* $\lambda > 1$. *Then for any compact time interval* I,

$$\left\| \left(\int_I \left| \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi) e^{it|\xi|^\lambda} \, d\xi \right|^2 dt \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{W^{\beta,p}(\mathbb{R}^n)},$$

with

$$\frac{\beta}{\lambda} = n\Big(\frac{1}{2} - \frac{1}{p}\Big) - \frac{1}{2}.$$

We will apply the above estimates with $\lambda = \alpha' := \alpha/(\alpha - 1)$ (remember $\alpha > 1$). Recall that we are interested in the variation of $\mathcal{H}^{(u)} f(x)$, where u is restricted to the range $[2^{j\alpha}, 2^{(j+1)\alpha}]$ for some $j \in \mathbb{Z}$. To estimate this, we decompose the kernel $e^{iu|t|^\alpha} K(t)$ into a part where oscillation plays no role and a part where the oscillation becomes important. More precisely, for $\ell \in \mathbb{Z}$, let

$$\mathcal{H}_\ell^{(u)} f(x) := \int_{\mathbb{R}^n} f(x-t) e^{iu|t|^\alpha} \varphi_\ell(t) K(t) \, dt, \tag{4-4}$$

where $\varphi_\ell(t) = \varphi_0(2^{-\ell} t)$ and φ_0 is radial, smooth and compactly supported on an annulus $\{|t| \simeq 1\}$ so that for $t \neq 0$ we have $\sum_{\ell \in \mathbb{Z}} \varphi_\ell(t) = 1$. When $u \simeq 2^{j\alpha}$, $|t| \simeq 2^{\ell - j}$, the phase $e^{iu|t|^\alpha}$ in (4-4) is approximately 1 precisely when $\ell < 0$. Thus, it makes sense to take the decomposition

$$\mathcal{H}^{(u)} f(x) = \sum_{\ell \in \mathbb{Z}} \mathcal{H}_{\ell - j}^{(u)} f(x) \tag{4-5}$$

and expect that the terms $\ell < 0$ in the above sum are essentially nonoscillatory.

It suffices to show that

$$\sum_{\ell \in \mathbb{Z}} \left\| \left(\sum_{j \in \mathbb{Z}} |V_j^r \mathcal{H}_{\ell - j}^{(u)} f|^r \right)^{1/r} \right\|_p \lesssim \|f\|_p. \tag{4-6}$$

To do so, we introduce a Littlewood–Paley decomposition in the x-variable. Let P_k be a multiplier operator defined by $\widehat{P_k f}(\xi) = \psi(2^{-k}\xi) \hat{f}(\xi)$, where ψ is a smooth function with compact support on the annulus $\frac{1}{2} \leq |\xi| \leq 2$ so that for $\xi \neq 0$, we have $\sum_{k \in \mathbb{Z}} \psi(2^{-k}\xi) = 1$. We further take the decomposition

$$\mathcal{H}_{\ell - j}^{(u)} f(x) = \sum_{k \in \mathbb{Z}} \mathcal{H}_{\ell - j}^{(u)} P_{j+k} f(x). \tag{4-7}$$

We will estimate

$$\| \| V_j^r \mathcal{H}_{\ell - j}^{(u)} P_{j+k} f \|_{\ell_j^r} \|_p \tag{4-8}$$

for each $k, \ell \in \mathbb{Z}$, and sum the estimates at the end. (Hereafter, for compactness of notation, we write ℓ_j^r for the ℓ^r norm over all $j \in \mathbb{Z}$.)

4B. ***Estimates for $\ell \le -k/(2(\alpha+1))$: bounding the V_j^r norm by the $\dot W^{1,1}$ norm.*** First there are two simple estimates for (4-8). One way to estimate (4-8) is to bound the V_j^r norm by the V_j^1 norm, which in turn is bounded by the $\dot W^{1,1}$ norm on the u interval $[2^{j\alpha}, 2^{(j+1)\alpha}]$. We get

$$V_j^r \mathcal{H}_{\ell-j}^{(u)} P_{j+k} f(x) \lesssim \int_{|u|\simeq 2^{j\alpha}} 2^{(\ell-j)\alpha} \int_{|t|\simeq 2^{\ell-j}} |P_{j+k} f(x-t)||K(t)|\, dt\, du \lesssim 2^{\ell\alpha} M P_{j+k} f(x),$$

where M is the Hardy–Littlewood maximal function, so by the Fefferman–Stein inequality and the Littlewood–Paley inequality, we have

$$\| \| V_j^r \mathcal{H}_{\ell-j}^{(u)} P_{j+k} f(x) \|_{\ell_j^r} \|_{L_x^p} \lesssim 2^{\ell\alpha} \|f\|_{L^p}, \quad 1 < p < \infty. \tag{4-9}$$

For the second simple estimate, recall that $\int_{|t|=R} K(t)\, d\sigma(t) = 0$ for all $R \in (0,\infty)$. Since φ was chosen to be radial, we have

$$\int_{\mathbb{R}^n} e^{iu|t|^\alpha} \varphi_{\ell-j}(t) K(t)\, dt = 0.$$

Thus, in computing $V_j^r \mathcal{H}_{\ell-j}^{(u)} P_{j+k} f(x)$, we could have instead computed the V_j^r norm of

$$\mathcal{H}_{\ell-j}^{(u)} P_{j+k} f(x) - P_{j+k} f(x) \int_{\mathbb{R}^n} e^{iu|t|^\alpha} \varphi_{\ell-j}(t) K(t)\, dt.$$

This expression is equal to

$$\int_{\mathbb{R}^n} [P_{j+k} f(x-t) - P_{j+k} f(x)] e^{iu|t|^\alpha} \varphi_{\ell-j}(t) K(t)\, dt.$$

The variational norm of this expression is controlled by its $\dot W^{1,1}$ norm in the u interval $[2^{j\alpha}, 2^{(j+1)\alpha}]$, which in turn is controlled by

$$2^{j+k} 2^{\ell-j} 2^{\ell\alpha} M \widetilde{P}_{j+k} f(x),$$

where $\widetilde{P}_{j+k}$ is a variant of the Littlewood–Paley projection P_{j+k}, so arguing as before, we see that

$$\| \| V_j^r \mathcal{H}_{\ell-j}^{(u)} P_{j+k} f(x) \|_{\ell_j^r} \|_{L_x^p} \lesssim 2^{\ell+k} 2^{\ell\alpha} \|f\|_{L^p}, \quad 1 < p < \infty. \tag{4-10}$$

We can sum (4-10) over all pairs (k,ℓ) with $\ell \le -k/(2(\alpha+1))$ and $k \le 0$. We can also sum (4-9) over all (k,ℓ) with $\ell \le -k/(2(\alpha+1))$ and $k \ge 0$. Thus, it remains to bound (4-8) when

$$\ell > -\frac{k}{2(\alpha+1)} \tag{4-11}$$

and sum over all such pairs of (k,ℓ).

4C. ***Estimates for $\ell > -k/(2(\alpha+1))$: division into three cases.*** First we look at $\mathcal{H}_{\ell-j}^{(u)} P_{j+k} f(x)$ in terms of its multiplier:

$$\mathcal{H}_{\ell-j}^{(u)} P_{j+k} f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat f(\xi) \biggl(\psi(2^{-j-k}\xi) \int_{\mathbb{R}^n} e^{-it\cdot\xi} e^{iu|t|^\alpha} \varphi_{\ell-j}(t) K(t)\, dt \biggr) e^{ix\cdot\xi}\, dx.$$

The multiplier is an oscillatory integral in t with phase $\phi(t) = -t\cdot\xi + u|t|^\alpha$, which (assuming $|u| \simeq 2^{\alpha j}$ and $|\xi| \simeq 2^{j+k}$) has a critical point in the annulus $\{|t| \simeq 2^{\ell-j}\}$ if and only if $2^{k+\ell} \simeq 2^{\ell\alpha}$, that is, if and only if

$k=\ell(\alpha-1)+O(1)$. In that case, using stationary phase (see, for example, [Stein 1993, Chapter VIII.5.7] or [Sogge 1993, Theorem 1.2.1]), the multiplier can be written as

$$\psi(2^{-j-k}\xi)\big(e^{ic_\alpha(2^{-j\alpha}u)^{-1/(\alpha-1)}(2^{-j}|\xi|)^{\alpha'}}a(2^\ell 2^{-j}\xi,2^{\ell\alpha}2^{-j\alpha}u)+e(2^\ell 2^{-j}\xi,2^{\ell\alpha}2^{-j\alpha}u)\big), \tag{4-12}$$

where $\alpha'=\alpha/(\alpha-1)$, $c_\alpha=(\alpha-1)/\alpha^{\alpha'}$, $a\in S^{-n/2}(\mathbb{R}^{n+1})$ and $e\in S^{-\infty}(\mathbb{R}^{n+1})$. If there are no critical points in the annulus $\{|t|\simeq 2^{\ell-j}\}$, then the multiplier is simply

$$\psi(2^{-j-k}\xi)e(2^\ell 2^{-j}\xi,2^{\ell\alpha}2^{-j\alpha}u). \tag{4-13}$$

(In the above, by $a\in S^{-n/2}(\mathbb{R}^{n+1})$ we mean

$$|\partial_\xi^{\alpha'}\partial_u^{\alpha''}a(\xi,u)|\lesssim_\alpha (1+|\xi|+|u|)^{-n/2-|\alpha|}$$

for every multiindex $\alpha=(\alpha',\alpha'')\in\mathbb{Z}_{\geq 0}^{n+1}$, and by $e\in S^{-\infty}(\mathbb{R}^{n+1})$ we mean

$$|\partial_\xi^{\alpha'}\partial_u^{\alpha''}e(\xi,u)|\lesssim_{N,\alpha} (1+|\xi|+|u|)^{-N-|\alpha|}$$

for any positive integers N and any multiindex α.)

The above motivates us to consider three cases separately (under our earlier standing assumption (4-11)):

Case 1: $\ell\geq 0,\ k=\ell(\alpha-1)+O(1)$.

Case 2: $k>\ell(\alpha-1)+C$ for some $C>0$.

Case 3: $k<\ell(\alpha-1)-C$ for some $C>0$.

4D. *Estimates in Case 1.* Now we consider Case 1. Our goal is to bound (4-8) given k and ℓ as in Case 1. We proceed in a few steps.

4D1. *Application of Plancherel–Pólya.* First we will essentially show that if $r\in[2,\infty)$, then

$$\|\|V_j^r\mathcal{H}_{\ell-j}^{(u)}P_{j+k}f\|_{\ell_j^r}\|_{L_x^p}\lesssim 2^{\ell\alpha/q}\|\|\|\chi(u)\mathcal{H}_{\ell-j}^{(2^{j\alpha}u)}P_{j+k}f(x)\|_{L_u^q}\|_{\ell_j^q}\|_{L_x^p} \tag{4-14}$$

for any $q\in[2,r]$ and any $p\in[1,\infty]$; here $\chi(u)$ is a smooth function with compact support on $\left[\frac{1}{2},2^{\alpha+1}\right]$ that is identically equal to 1 on $[1,2^\alpha]$. Indeed, when $n\geq 2$ (and p,r are as in (4-3)), we will only need (4-14) for $q=2$. But for $n=1$ (and p,r as in (4-2)), we will need (4-14) for both $q=2$ and $q=r$. We will see that this is the case after we prove (4-14).

To prove (4-14), let us temporarily write $g=P_{j+k}f$. As a function of u, $\mathcal{H}_{\ell-j}^{(u)}g$ has frequency morally supported on the annulus of size $\simeq 2^{(\ell-j)\alpha}$ centered at the origin. Thus, we introduce Littlewood–Paley projections in the u-variable (denoted by $P^{(2)}$ so that $P_{(\ell-j)\alpha}^{(2)}$ is projection onto frequency $\simeq 2^{(\ell-j)\alpha}$) and estimate

$$|V_j^r\mathcal{H}_{\ell-j}^{(u)}g(x)| \leq |V^r(P_{\leq(\ell-j)\alpha}^{(2)}[\chi(2^{-j\alpha}u)\mathcal{H}_{\ell-j}^{(u)}g(x)])|+\sum_{k=1}^{\infty}|V^r(P_{(\ell-j+k)\alpha}^{(2)}[\chi(2^{-j\alpha}u)\mathcal{H}_{\ell-j}^{(u)}g(x)])|. \tag{4-15}$$

(Here $P_{\leq(\ell-j)\alpha}^{(2)}:=\sum_{k\leq\ell-j}P_{k\alpha}^{(2)}$.)

The first term on the right-hand side of (4-15) is the main term and can be estimated using Proposition 2.2. In particular, it is bounded by

$$2^{(\ell-j)\alpha/q}\|\chi(2^{-j\alpha}u)\mathcal{H}^{(u)}_{\ell-j}g(x)\|_{L^q_u}$$

(recall $q\in[2,r]$). By changing variable in u, this is just

$$2^{\ell\alpha/q}\|\chi(u)\mathcal{H}^{(2^{j\alpha}u)}_{\ell-j}g(x)\|_{L^q_u}.$$

Hence the contribution of the first term of (4-15) to the left-hand side of (4-14) is bounded by

$$2^{\ell\alpha/q}\|\|\|\chi(u)\mathcal{H}^{(2^{j\alpha}u)}_{\ell-j}P_{j+k}f(x)\|_{L^q_u}\|_{\ell^r_j}\|_{L^p_x}.$$

Since $r\geq q$, we have ℓ^r norm bounded by ℓ^q norm; hence the above is bounded by the right-hand side of (4-14).

On the other hand, for the second term on the right-hand side of (4-15), since $k>-C$, one can integrate by parts in u, using the fact that the multiplier for $P^{(2)}_{(\ell-j+k)\alpha}$ vanishes to infinite order at 0, and obtain

$$|P^{(2)}_{(\ell-j+k)\alpha}[\chi(2^{-j\alpha}u)\mathcal{H}^{(u)}_{\ell-j}g(x)]|\lesssim_N 2^{-k\alpha N}\widetilde{P}^{(2)}_{(\ell-j+k)\alpha}[\chi(2^{-j\alpha}u)\widetilde{\mathcal{H}}^{(u)}_{\ell-j}g(x)] \tag{4-16}$$

for any positive integer N, where $\widetilde{P}^{(2)}$ is a Littlewood–Paley projection similar to $P^{(2)}$, and $\widetilde{\mathcal{H}}^{(u)}_{\ell-j}$ is the same as $\mathcal{H}^{(u)}_{\ell-j}$ defined in (4-4), except that the cutoff φ is replaced by a smooth multiple $\tilde{\varphi}$ of φ. Hence by repeating the above argument, and summing over k using the additional convergence factors $2^{-k\alpha N}$ that we gained in (4-16), the contribution of the second term of (4-15) to the left-hand side of (4-14) is bounded by

$$2^{\ell\alpha/q}\|\|\|\chi(u)\widetilde{\mathcal{H}}^{(2^{j\alpha}u)}_{\ell-j}P_{j+k}f(x)\|_{L^q_u}\|_{\ell^q_j}\|_{L^p_x}. \tag{4-17}$$

Since $\widetilde{\mathcal{H}}$ and $\mathcal{H}$ satisfy the same estimates, we will not distinguish the two, and declare that we can also bound (4-17) once we can bound the right-hand side of (4-14).

4D2. *Application of the square function estimate.* Now fix k,ℓ as in Case 1. In other words, fix $k,\ell\geq 0$ with $k=\ell(\alpha-1)+O(1)$. We will try to bound the right-hand side of (4-14) when $q=2$. The multiplier for $\mathcal{H}^{(2^{j\alpha}u)}_{\ell-j}P_{j+k}f$ is given by (4-12) with u replaced by $2^{j\alpha}u$. For $u\in\mathbb{R}$, let $\tilde{m}_u(\xi)$ be the multiplier

$$\tilde{m}_u(\xi)=\chi(u)\psi(2^{-k}\xi)(e^{ic_\alpha u^{-1/(\alpha-1)}|\xi|^{\alpha'}}a(2^\ell\xi,2^{\ell\alpha}u)+e(2^\ell\xi,2^{\ell\alpha}u)), \tag{4-18}$$

where $a\in S^{-n/2}(\mathbb{R}^{n+1})$ and $e\in S^{-\infty}(\mathbb{R}^{n+1})$ are as in (4-12). Then the multiplier of the operator $\chi(u)\mathcal{H}^{(2^{j\alpha}u)}_{\ell-j}P_{j+k}$ is precisely $\tilde{m}_u(2^{-j}\xi)$. Now expand $\chi(u)a(2^\ell\xi,2^{\ell\alpha}u)$ in Fourier series in u: let c be a small enough constant depending on α so that the support of $\chi(u)$ is contained in $[0,c^{-1}]$. Using the smoothness in the variable u, we get

$$\chi(u)a(2^\ell\xi,2^{\ell\alpha}u)=\sum_{\kappa\in c\mathbb{Z}}(1+|\kappa|)^{-2}a_\kappa(2^\ell\xi)e^{i\kappa u}$$

for $u\in[0,c^{-1}]$, where $a_\kappa\in S^{-n/2}(\mathbb{R}^n)$ uniformly for every $\kappa\in c\mathbb{Z}$. Similarly, expand $\chi(u)e(2^\ell\xi,2^{\ell\alpha}u)$ in Fourier series in u:

$$\chi(u)e(2^\ell\xi,2^{\ell\alpha}u)=\sum_{\kappa\in c\mathbb{Z}}(1+|\kappa|)^{-2}e_\kappa(2^\ell\xi)e^{i\kappa u}$$

for $u \in [0, c^{-1}]$, where $e_\kappa \in S^{-\infty}(\mathbb{R}^n)$ uniformly for every $\kappa \in c\mathbb{Z}$. This shows

$$\tilde{m}_u(\xi) = \sum_{\kappa \in c\mathbb{Z}} (1+|\kappa|)^{-2} e^{i\kappa u} \psi(2^{-k}\xi)(a_\kappa(2^\ell \xi) e^{ic_\alpha u^{-1/(\alpha-1)}|\xi|^{\alpha'}} + e_\kappa(2^\ell \xi))$$

for $u \in [0, c^{-1}]$. Temporarily let g be the function such that $\hat{g}(\xi) = \hat{f}(\xi)\psi(2^{-k}\xi)a_\kappa(2^\ell\xi)$; note that when $k \ge 0$, we have $\|g\|_{L^p_\beta(\mathbb{R}^n)} \lesssim 2^{k\beta} 2^{-(k+\ell)n/2} \|f\|_{L^p(\mathbb{R}^n)}$ by the Hörmander–Mikhlin multiplier theorem, with an implicit constant independent of κ. This is further bounded by $2^{\ell(\alpha-1)\beta} 2^{-\ell\alpha n/2} \|f\|_{L^p(\mathbb{R}^n)}$ since we are in Case 1, where $k = \ell(\alpha-1) + O(1)$. We apply Proposition 4.1 with g in place of f and obtain

$$\left\| \left\| \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi)\psi(2^{-k}\xi) a_\kappa(2^\ell\xi) e^{ic_\alpha u^{-1/(\alpha-1)}|\xi|^{\alpha'}} \, d\xi \right\|_{L^2_u[0,c^{-1}]} \right\|_{L^p(\mathbb{R}^n)} \lesssim \begin{cases} 2^{-\ell\alpha/2} \|f\|_{L^p(\mathbb{R})} & \text{if } p \in [2,\infty) \text{ and } n = 1, \\ 2^{\ell\alpha[n(1/2-1/p)-1/2]} 2^{-\ell\alpha n/2} \|f\|_{L^p(\mathbb{R}^n)} & \text{if } p \in \left(2+\frac{4}{n}, \infty\right) \text{ and } n \ge 2. \end{cases} \tag{4-19}$$

We get a better decay if $a_\kappa(2^\ell\xi) e^{ic_\alpha u^{-1/(\alpha-1)}|\xi|^{\alpha'}}$ above is replaced by $e_\kappa(2^\ell\xi)$. Summing over κ, and simplifying the exponent in the case $n \ge 2$, we get

$$\left\| \left\| \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi) \tilde{m}_u(\xi) \, d\xi \right\|_{L^2_u} \right\|_{L^p(\mathbb{R}^n)} \lesssim \begin{cases} 2^{-\ell\alpha/2} \|f\|_{L^p(\mathbb{R})} & \text{if } p \in [2,\infty) \text{ and } n = 1, \\ 2^{-\ell\alpha/2} 2^{-\ell\alpha n/p} \|f\|_{L^p(\mathbb{R}^n)} & \text{if } p \in \left(2+\frac{4}{n}, \infty\right) \text{ and } n \ge 2. \end{cases}$$

But recall that the multiplier of the operator $\chi(u)\mathcal{H}^{(2^{j\alpha}u)}_{\ell-j} P_{j+k}$ is precisely $\tilde{m}_u(2^{-j}\xi)$. By scale invariance, we have

$$\left\| \left\| \chi(u)\mathcal{H}^{(2^{j\alpha}u)}_{\ell-j} P_{j+k} f \right\|_{L^2_u} \right\|_{L^p(\mathbb{R}^n)} \lesssim \begin{cases} 2^{-\ell\alpha/2} \|f\|_{L^p(\mathbb{R})} & \text{if } p \in [2,\infty) \text{ and } n = 1, \\ 2^{-\ell\alpha/2} 2^{-\ell\alpha n/p} \|f\|_{L^p(\mathbb{R}^n)} & \text{if } p \in \left(2+\frac{4}{n}, \infty\right) \text{ and } n \ge 2 \end{cases} \tag{4-20}$$

for all $j \in \mathbb{Z}$, where the implicit constants are independent of j. (The Fourier series expansions used to remove the dependence on u are very reminiscent of the method used to prove L^2 boundedness of multipliers in S^0; see, for example, [Stein 1993, Chapter VI.2].)

Recall that our goal now is to bound the right-hand side of (4-14) when $q = 2$. Hence we need a vector-valued version of (4-20), where we will have an additional ℓ^2 norm over $j \in \mathbb{Z}$ inside the L^p norm on the left-hand side of (4-20). To do so, we need Proposition 4.2.

4D3. *Application of Seeger's theorem for multipliers with localized bounds.* First we state a vector-valued variant of a theorem of Seeger, about multipliers with localized bounds:

Proposition 4.2 [Jones, Seeger, and Wright 2008; Seeger 1988]. *Let $I \subset \mathbb{R}$ be a compact interval. Let $\{\tilde{m}_u(\xi) : u \in I\}$ be a family of Fourier multipliers on $\mathbb{R}^n$, each of which is compactly supported on $\left\{\xi : \frac{1}{2} \le |\xi| \le 2\right\}$ and satisfies*

$$\sup_{u\in I} |\partial_\xi^\tau \tilde{m}_u(\xi)| \le B \quad \textit{for each } 0 \le |\tau| \le n+1$$

for some constant B. For $u \in I$ and $j \in \mathbb{Z}$, denote by $T_{u,j}$ the multiplier operator with multiplier $\tilde{m}_u(2^{-j}\xi)$. Fix some $p \in [2,\infty)$. Assume that there exists some constant A such that

$$\sup_{j\in\mathbb{Z}} \| \|T_{u,j} f\|_{L^2(I)} \|_{L^s(\mathbb{R}^n)} \le A \|f\|_{L^s(\mathbb{R}^n)} \tag{4-21}$$

for both $s=p$ and $s=2$. Then

$$\| \| \| T_{u,j} f \|_{L^2(I)} \|_{\ell^2(\mathbb{Z})} \|_{L^p(\mathbb{R}^n)} \lesssim A \left| \log\left(2+\frac{B}{A}\right) \right|^{1/2-1/p} \| f \|_{L^p(\mathbb{R}^n)}.$$

This proposition was stated without proof on page 6737 of [Jones, Seeger, and Wright 2008]. It is a vector-valued analogue of Theorem 1 of [Seeger 1988], and we provide a proof of this proposition in Section 7 for the convenience of the reader.

Recall that our goal is to bound the right-hand side of (4-14) when $q=2$. Also recall that if $\tilde{m}_u(\xi)$ is defined as in (4-18), and $T_{u,j}$ is the multiplier operator with multiplier $\tilde{m}_u(2^{-j}\xi)$ as in Proposition 4.2, then $T_{u,j} f$ is precisely $\chi(u)\mathcal{H}^{(2^{j\alpha}u)}_{\ell-j} P_{j+k} f$. Thus, if we could apply Proposition 4.2, we would obtain a bound about the right-hand side of (4-14) when $q=2$. To do so we verify the hypothesis of Proposition 4.2. From the explicit expression (4-18), we have

$$\sup_{u\in I} |\partial_\xi^\tau \tilde{m}_u(\xi)| \lesssim 2^{\ell N}$$

for some large positive integer N if $|\tau| \le n+1$. The hypothesis (4-21) for $s=p$ is given by (4-20), where A can be chosen to be relatively small if ℓ is large. On the other hand, by considering the L^∞ norm of the multipliers, we also get

$$\| \| \chi(u)\mathcal{H}^{(2^{j\alpha}u)}_{\ell-j} P_{j+k} f \|_{L^2_u} \|_{L^2(\mathbb{R}^n)} \lesssim 2^{-\ell\alpha n/2} \| f \|_{L^2(\mathbb{R}^n)} \quad \text{for all } n \ge 1, \tag{4-22}$$

which gives us the hypothesis (4-21) for $s=2$, where A can be chosen to be relatively small if ℓ is large.

More precisely, suppose first $n\ge 2$ and $p\in(2+4/n,\infty)$. Then we invoke (4-20) and (4-22). Since $2^{-\ell\alpha n/2} \le 2^{-\ell\alpha/2}2^{-\ell\alpha n/p}$, we may apply Proposition 4.2 with $A=2^{-\ell\alpha/2}2^{-\ell\alpha n/p}$ and $B=2^{\ell N}$ for some large positive integer N depending only on α. Thus, if $n\ge 2$ and $p\in(2+4/n,\infty)$, then we get

$$\| \| \| \chi(u)\mathcal{H}^{(2^{j\alpha}u)}_{\ell-j} P_{j+k} f(x) \|_{L^2_u} \|_{\ell^2_j} \|_{L^p_x} \lesssim_\epsilon 2^{-\ell\alpha/2}2^{-\ell\alpha n/p}2^{\ell\epsilon} \| f \|_{L^p(\mathbb{R}^n)}$$

for any $\epsilon>0$. Taking $q=2$ in (4-14), this shows that

$$\| \| V^r_j \mathcal{H}^{(u)}_{\ell-j} P_{j+k} f \|_{\ell^r_j} \|_p \lesssim_\epsilon 2^{-\ell\alpha n/p}2^{\ell\epsilon} \| f \|_{L^p(\mathbb{R}^n)} \quad \text{if } n\ge 2,\ p\in\left(2+\frac{4}{n},\infty\right) \text{ and } r\in[2,\infty).$$

Note that the power of 2 here is negative. So this estimate can be summed over all $\ell\ge 0$, and this gives the desired bound for (4-8) when $n\ge 2$, $p\in(2+4/n,\infty)$ and $r\in[2,\infty)$ for k,ℓ as in Case 1.

On the other hand, if $n=1$ and $p\in[2,\infty)$, then in light of (4-20) and (4-22), we may apply Proposition 4.2 with $A=2^{-\ell\alpha/2}$ and $B=2^{\ell N}$ for some large positive integer N depending on α. We obtain

$$\| \| \| \chi(u)\mathcal{H}^{(2^{j\alpha}u)}_{\ell-j} P_{j+k} f(x) \|_{L^2_u} \|_{\ell^2_j} \|_{L^p_x} \lesssim_\epsilon 2^{-\ell\alpha/2}2^{\ell\epsilon} \| f \|_{L^p(\mathbb{R})} \quad \text{if } n=1 \text{ and } p\in[2,\infty)$$

for any $\epsilon>0$. Taking $q=2$ in (4-14), this shows that

$$\| \| V^r_j \mathcal{H}^{(u)}_{\ell-j} P_{j+k} f \|_{\ell^r_j} \|_p \lesssim_\epsilon 2^{\ell\epsilon} \| f \|_{L^p(\mathbb{R})} \quad \text{if } n=1,\ p\in[2,\infty) \text{ and } r\in[2,\infty). \tag{4-23}$$

This is not good enough to be summed over all $\ell\ge 0$, so we need to gain a slightly better decay in ℓ. This is achieved via the local smoothing estimate in Theorem 1.6.

4D4. *Application of a local smoothing estimate in dimension $n = 1$.* The goal of this subsection is to prove that

$$\| \| V_j^r \mathcal{H}_{\ell-j}^{(u)} P_{j+k} f \|_{\ell_j^r} \|_p \lesssim 2^{-\ell\alpha/p} \|f\|_{L^p(\mathbb{R})} \quad \text{if } n = 1,\ p = r \in (4, \infty). \tag{4-24}$$

Assume for the moment that this has been established. Interpolating (4-24) against (4-23) using complex interpolation of vector-valued L^p spaces (see [Bergh and Löfström 1976, Theorem 5.1.2]), we get

$$\| \| V_j^r \mathcal{H}_{\ell-j}^{(u)} P_{j+k} f \|_{\ell_j^r} \|_p \lesssim 2^{-\gamma\ell} \|f\|_{L^p(\mathbb{R})} \quad \text{if } n = 1,\ p \in (2, \infty),\ r \in (2, \infty), \tag{4-25}$$

where $\gamma = \gamma(p, r)$ is a positive constant. This can be summed over all $\ell > 0$, and this gives the desired bound for (4-8) when $n = 1$, $p \in (2, \infty)$ and $r \in (2, \infty)$ for k, ℓ as in Case 1.

To prove (4-24) we use the local smoothing estimate in Theorem 1.6. Suppose $n = 1$, $p = r \in (4, \infty)$. We use (4-14) with $q = r = p$. Thus, the left-hand side of (4-24) is bounded up to a constant by

$$2^{\ell\alpha/p} \| \| \| \chi(u) \mathcal{H}_{\ell-j}^{(2^{j\alpha}u)} P_{j+k} f(x) \|_{L_u^p} \|_{\ell_j^p} \|_{L_x^p}. \tag{4-26}$$

Consider first

$$\begin{aligned} &\left\| \left\| \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi) \tilde{m}_u(\xi)\, d\xi \right\|_{L_u^p} \right\|_{L_x^p} \\ &\qquad = \left\| \left\| \chi(u) \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi) \psi(2^{-k}\xi) (e^{ic_\alpha u^{-1/(\alpha-1)} |\xi|^{\alpha'}} a(2^\ell \xi, 2^{\ell\alpha} u) + e(2^\ell\xi, 2^{\ell\alpha}u))\, d\xi \right\|_{L_u^p} \right\|_{L_x^p}. \end{aligned}$$

We first use Fubini's theorem to interchange the integrals in u and x, and use the Hörmander–Mikhlin multiplier theorem (for each fixed u) to get rid of the multiplier $a(2^\ell\xi, 2^{\ell\alpha}u)$. Since $k = \ell(\alpha - 1) + O(1)$, this gives

$$\begin{aligned} &\left\| \left\| \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi) \tilde{m}_u(\xi)\, d\xi \right\|_{L_u^p} \right\|_{L_x^p} \\ &\qquad \lesssim 2^{-\ell\alpha/2} \left\| \left\| \chi(u) \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi) \psi(2^{-k}\xi) e^{ic_\alpha u^{-1/(\alpha-1)} |\xi|^{\alpha'}}\, d\xi \right\|_{L_u^p} \right\|_{L_x^p} + 2^{-\ell N} \|f\|_{L^p(\mathbb{R})} \end{aligned}$$

for any positive integer N. Thus, Theorem 1.6 applies, and when $k \geq 0$ we have

$$\left\| \left\| \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi) \tilde{m}_u(\xi)\, d\xi \right\|_{L_u^p} \right\|_{L_x^p} \lesssim 2^{-\ell\alpha/2} 2^{k\alpha'[(1/2-1/p)-1/p]} \|f\|_{L^p(\mathbb{R})} \quad \text{if } n = 1,\ p \in (4, \infty).$$

But recall that the multiplier of the operator $\chi(u)\mathcal{H}_{\ell-j}^{(2^{j\alpha}u)} P_{j+k}$ is precisely $\tilde{m}_u(2^{-j}\xi)$. By scale invariance, and remembering that $k = \ell(\alpha - 1) + O(1)$, we have

$$\| \| \chi(u) \mathcal{H}_{\ell-j}^{(2^{j\alpha}u)} P_{j+k} f(x) \|_{L_u^p} \|_{L_x^p} \lesssim 2^{-2\ell\alpha/p} \|f\|_{L^p(\mathbb{R})} \quad \text{if } n = 1,\ p \in (4, \infty).$$

Replacing f by $\widetilde{P}_{j+k} f$, taking the ℓ_j^p norm on both sides, and using the Littlewood–Paley inequality (remember $p \geq 2$), we get

$$\| \| \| \chi(u) \mathcal{H}_{\ell-j}^{(2^{j\alpha}u)} P_{j+k} f(x) \|_{L_u^p} \|_{\ell_j^p} \|_{L_x^p} \lesssim 2^{-2\ell\alpha/p} \|f\|_{L^p(\mathbb{R})} \quad \text{if } n = 1,\ p \in (4, \infty).$$

Thus, (4-26) is $\lesssim 2^{-\ell\alpha/p} \|f\|_{L^p(\mathbb{R})}$. This establishes (4-24), and our treatment for Case 1 is now complete.

4E. ***Estimates in Cases 2 and 3: further gains over Case 1.*** Next we estimate (4-8) for k, ℓ as in Case 2. Fix k, ℓ such that $k > \ell(\alpha-1)+C$ for some positive constant C. If C is large enough, then the multiplier for $\mathcal{H}_{\ell-j}^{(2^{j\alpha}u)} P_{j+k} f$ is given by (4-13), since the phase function of the oscillatory integral defining the multiplier has no critical point in that case. For $u \in \mathbb{R}$, let $\tilde{m}_u(\xi)$ be the multiplier

$$\tilde{m}_u(\xi) = \chi(u)\psi(2^{-k}\xi)e(2^{\ell}\xi, 2^{\ell\alpha}u),$$

where $e \in S^{-\infty}(\mathbb{R}^{n+1})$ is as in (4-13). Then the multiplier of the operator $\chi(u)\mathcal{H}_{\ell-j}^{(2^{j\alpha}u)} P_{j+k}$ is precisely $\tilde{m}_u(2^{-j}\xi)$. For every $N \in \mathbb{N}$ we can write

$$\tilde{m}_u(\xi) = 2^{-(k+\ell)N}\chi(u)\psi(2^{-k}\xi)\tilde{e}_N(2^{\ell}\xi, 2^{\ell\alpha}u)$$

for some symbol $\tilde{e}_N \in S^{-\infty}(\mathbb{R}^{n+1})$. Thus, applying Proposition 4.2 as in the proof of (4-23), we get

$$\| \| V_j^r \mathcal{H}_{\ell-j}^{(u)} P_{j+k} f \|_{\ell_j^r} \|_p \lesssim_N 2^{-(k+\ell)N} \|f\|_{L^p(\mathbb{R}^n)}$$

whenever one of the following two conditions is fulfilled: $n = 1$, $p \in [2, \infty)$ and $r \in [2, \infty)$, or $n \geq 2$, $p \in (2+4/n, \infty)$, and $r \in [2, \infty)$. The right-hand side in the above display equation can be summed over all k, ℓ that satisfy $k > \ell(\alpha-1)+C$ and the standing assumption (4-11), and this gives the bound for (4-8) for such p, n, r for all k, ℓ as in Case 2.

Finally we estimate (4-8) for k, ℓ as in Case 3. Fix k, ℓ such that $k < \ell(\alpha-1)-C$ for some positive constant C. As in Case 2, if C is large enough, then the multiplier for $\mathcal{H}_{\ell-j}^{(2^{j\alpha}u)} P_{j+k} f$ is given by (4-13). For $u \in \mathbb{R}$, let $\tilde{m}_u(\xi)$ be the multiplier

$$\tilde{m}_u(\xi) = \chi(u)\psi(2^{-k}\xi)e(2^{\ell}\xi, 2^{\ell\alpha}u),$$

where $e \in S^{-\infty}(\mathbb{R}^{n+1})$ is as in (4-13). Then the multiplier of the operator $\chi(u)\mathcal{H}_{\ell-j}^{(2^{j\alpha}u)} P_{j+k}$ is precisely $\tilde{m}_u(2^{-j}\xi)$. For every $N \in \mathbb{N}$ we can write

$$\tilde{m}_u(\xi) = (2^{-\ell\alpha}u)^N \chi(u)\psi(2^{-k}\xi)\tilde{e}_N(2^{\ell}\xi, 2^{\ell\alpha}u)$$

for some symbol $\tilde{e}_N \in S^{-\infty}(\mathbb{R}^{n+1})$. Thus, applying Proposition 4.2 as in the proof of (4-23), we get

$$\| \| V_j^r \mathcal{H}_{\ell-j}^{(u)} P_{j+k} f \|_{\ell_j^r} \|_p \lesssim_N 2^{-\ell\alpha N} \|f\|_{L^p(\mathbb{R}^n)}$$

whenever one of the following two conditions is fulfilled: $n = 1$, $p \in [2, \infty)$ and $r \in [2, \infty)$, or $n \geq 2$, $p \in (2+4/n, \infty)$, and $r \in [2, \infty)$. The right-hand side in the above displayed equation can be summed over all k, ℓ that satisfy $k < \ell(\alpha-1)-C$ and the standing assumption (4-11). This gives the bound for (4-8) for such p, n, r for all k, ℓ as in Case 3.

We have thus completed the proof of (4-1) for all p, n, r satisfying (4-2) or (4-3).

5. Short jump estimates for $p \leq 2$

In this section, we establish

$$\Big\| \Big(\sum_{j \in \mathbb{Z}} |V_j^r(\mathcal{H}f)|^r \Big)^{1/r} \Big\|_p \lesssim \|f\|_p, \tag{5-1}$$

whenever $n \geq 2$, $2n/(2n-1) < p \leq 2$, and $r \in [2, \infty)$. By complex interpolation (see [Bergh and Löfström 1976, Theorem 5.1.2]) with (4-1), we will then have (5-1) whenever $n \geq 2$, $p \in (2n/(2n-1), \infty)$, and $r \in [2, \infty)$, which concludes the proof of Proposition 1.5.

The key here is the following square function estimate.

Proposition 5.1. *Let $n \geq 2$, $1 < p \leq 2$ and $\lambda > 1$. Then for any compact time interval I,*

$$\left\| \left(\int_I \left| \int_{\mathbb{R}^n} e^{ix\xi} \hat{f}(\xi) e^{it|\xi|^\lambda} \, d\xi \right|^2 dt \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{W^{\beta,p}(\mathbb{R}^n)},$$

with

$$\frac{\beta}{\lambda} = n\Big(\frac{1}{p} - \frac{1}{2}\Big).$$

The proof of this proposition is postponed to the end of this section.

Now let $2n/(2n-1) < p \leq 2$, $n \geq 2$, and $r \in [2, \infty)$. We proceed to establish (5-1). As in Section 4, we decompose $\mathcal{H}^{(u)} f$ as in (4-5) and (4-7), and estimate (4-8) for every $k, \ell \in \mathbb{Z}$. The inequalities (4-9) and (4-10) still hold under our current assumptions of p, n, r, and these estimates can be summed whenever $\ell \leq -k/(2(\alpha+1))$. Thus, it remains to consider pairs of (k, ℓ) for which (4-11) holds, and we still divide into Cases 1, 2, 3 as before. We will only treat Case 1 here which is the main case; an easy adaptation of this argument gives Cases 2 and 3.

So let $\ell \geq 0$ and $k = \ell(\alpha - 1) + O(1)$. By (4-14) with $q = 2$, the left-hand side of (5-1) is bounded by

$$2^{\ell\alpha/2} \| \| \| \chi(u) \mathcal{H}^{(2^{j\alpha}u)}_{\ell-j} P_{j+k} f(x) \|_{L^2_u} \|_{\ell^2_j} \|_{L^p_x}. \tag{5-2}$$

We analyze the multiplier of $\mathcal{H}^{(2^{j\alpha}u)}_{\ell-j} P_{j+k}$ as before, but in (4-19) we use Proposition 5.1 instead of Proposition 4.1 (since now $p \in (2n/(2n-1), 2)$). So instead of (4-20), we get

$$\| \| \chi(u) \mathcal{H}^{(2^{j\alpha}u)}_{\ell-j} P_{j+k} f \|_{L^2_u} \|_{L^p(\mathbb{R}^n)} \lesssim 2^{\ell\alpha n(1/p-1/2)} 2^{-\ell\alpha n/2} \|f\|_{L^p(\mathbb{R}^n)} = 2^{-\ell\alpha n/p'} \|f\|_{L^p(\mathbb{R}^n)} \tag{5-3}$$

uniformly in $j \in \mathbb{Z}$. Thus, we apply Proposition 4.2 with $A = 2^{-\ell\alpha n/p'}$ and $B = 2^{\ell N}$ for some large integer N depending only on α. This gives

$$\| \| \| \chi(u) \mathcal{H}^{(2^{j\alpha}u)}_{\ell-j} P_{j+k} f \|_{L^2_u} \|_{\ell^2_j} \|_{L^p(\mathbb{R}^n)} \lesssim_\epsilon 2^{-\ell\alpha n/p'} 2^{\ell\epsilon} \|f\|_{L^p(\mathbb{R}^n)}$$

for all $\epsilon > 0$. Continuing from (5-2), we see that the left-hand side of (5-1) is bounded by

$$2^{\ell\alpha/2} 2^{-\ell\alpha n/p'} 2^{\ell\epsilon} \|f\|_{L^p(\mathbb{R}^n)}.$$

Since $p \in (2n/(2n-1), \infty)$, the above exponent of 2 is negative if ϵ is sufficiently small. Thus, we can sum over all $\ell \geq 0$ in this case, establishing the bound for (4-8) for all k, ℓ in Case 1. A similar argument establishes a bound for (4-8) for all k, ℓ in Cases 2 and 3. This completes the proof of (5-1), modulo the proof of Proposition 5.1.

Proof of Proposition 5.1. We will prove a slightly more general result. Let us write

$$T_u f(x) = \int_{\mathbb{R}^n} e^{ix\xi} \hat{f}(\xi) m_u(\xi) \, d\xi,$$

where

$$m_u(\xi) = e^{iu|\xi|^\lambda}(1+|\xi|^2)^{-(\beta+i\gamma)/2}.$$

Theorem 5.2. *Let I be a compact interval not containing* 0. *If $\lambda > 1$, $\beta = n\lambda/2$ and $\gamma \in \mathbb{R}$, then*

$$\left\| \left(\int_I |T_u f(x)|^2 \, du \right)^{1/2} \right\|_{L^1(\mathbb{R}^n)} \lesssim \|f\|_{H^1(\mathbb{R}^n)};$$

that is, T maps the Hardy space $H^1(\mathbb{R}^n)$ boundedly into $L^1_x(\mathbb{R}^n;\, L^2_u(I))$.

All implied constants may depend on λ, β, n, I, but not on f, γ, x, u.

Theorem 5.2 implies Proposition 5.1 via complex interpolation; see, for example, [Stein 1993, Chapter IV.6.17] for a discussion of interpolation between Hardy spaces. For the scalar theory of the multiplier m_u (for fixed u) we refer to [Miyachi 1981; Fefferman and Stein 1972; Fefferman 1970].

Recall that an H^1 atom of radius r is a bounded function a on $\mathbb{R}^n$ that is supported in a ball of radius r and satisfies $\|a\|_\infty \le r^{-n}$ and $\int_{\mathbb{R}^n} a = 0$. The Hardy space $H^1(\mathbb{R}^n)$ is a Banach space consisting of functions of the form $f = \sum_j c_j a_j$, with $\sum_j |c_j| < \infty$, where the $(a_j)_j$ are H^1 atoms. Its norm is defined as

$$\|f\|_{H^1(\mathbb{R}^n)} = \inf \sum_j |c_j|,$$

where the infimum is taken over all atomic decompositions of $f = \sum_j c_j a_j$.

To prove Theorem 5.2 it suffices to show that

$$\|Ta\|_{L^1_x(L^2_u)} \lesssim 1 \tag{5-4}$$

holds for every H^1 atom a of radius r. We may assume the support of a to be centered at the origin.

For $j > 0$, let P_j denote the usual Littlewood–Paley projection with $\widehat{P_j f} = \psi_j \hat{f}$, ψ_j supported on $|\xi| \simeq 2^j$. Let $\widehat{P_0 f} = \psi_0 \hat{f}$, where ψ_0 is such that

$$1 = \psi_0 + \sum_{j>0} \psi_j.$$

(Note that P_0 here is actually $P_{\le 0}$ from the previous section.) For $j \ge 0$ we denote by $\tilde{\psi}_j$ a smooth positive function that equals 1 on the support of ψ_j and whose support is contained in a small neighborhood of the support of ψ_j. Define

$$K_u^{(j)}(x) = \hat{m}_u * \hat{\psi}_j(x) = \int_{\mathbb{R}^n} e^{ix\xi + iu|\xi|^\lambda}(1+|\xi|^2)^{-\beta/2}\psi_j(\xi)\, d\xi.$$

Before we begin, we record the following pointwise estimates for $K_u^{(j)}(x)$. From estimating the second derivative of the phase we obtain

$$|K_u^{(j)}(x)| \lesssim 2^{-jn(\lambda-1)} \quad \text{for all } x \in \mathbb{R}^n,\, u \in I. \tag{5-5}$$

Here we used that $\beta = n\lambda/2$. Estimating the first derivative of the phase we obtain

$$|K_u^{(j)}(x)| \lesssim_N 2^{-j\beta} 2^{jn} (2^j|x|)^{-N} \quad \text{for } |x| \gtrsim 2^{j(\lambda-1)},\, u \in I \tag{5-6}$$

for all $N \ge 0$. Note that these estimates are uniform in $u \in I$.

Let us prove the main estimate (5-4). The first step is to apply the triangle inequality:

$$\|Ta\|_{L^1_x(L^2_u)} \le \sum_{j\ge 0} \|K_u^{(j)} * a\|_{L^1_x(L^2_u)}.$$

We will estimate the summand in two different ways: in particular, it will be shown below that

$$\|K_u^{(j)} * a\|_{L^1_x(L^2_u)} \lesssim (2^j r)^{-n/2} + 2^{-j\beta}, \tag{5-7}$$

$$\|K_u^{(j)} * a\|_{L^1_x(L^2_u)} \lesssim 2^j r. \tag{5-8}$$

These estimates immediately imply (5-4).

To prove (5-7) we first split up[1] the integral in x:

$$\|K_u^{(j)} * a\|_{L^1_x(L^2_u)} \le \mathrm{I} + \mathrm{II},$$

where

$$\mathrm{I} = \|K_u^{(j)} * a\|_{L^1_x(\mathbb{R}^n \setminus B(C2^{j(\lambda-1)}+r); L^2_u(I))},$$

$$\mathrm{II} = \|K_u^{(j)} * a\|_{L^1_x(B(C2^{j(\lambda-1)}+r); L^2_u(I))}.$$

We claim that $\mathrm{I} \lesssim_N 2^{-jN}$. Indeed, we see from (5-6) that

$$\begin{aligned}\mathrm{I} &\le \int_{|x|\ge C2^{j(\lambda-1)}+r} \left(\int_I \left(\int_{|y|\le r} |K_u^{(j)}(x-y)a(y)|\,dy \right)^2 du \right)^{1/2} dx \\ &\lesssim_N 2^{-j\beta} 2^{jn} 2^{-jN} \int_{|x|\ge C2^{j(\lambda-1)}+r} \int_{|y|\le r} |x-y|^{-N} |a(y)|\,dy\,dx \\ &\le 2^{-j\beta} 2^{jn} 2^{-jN} \|a\|_1 \int_{|x|\gtrsim 2^{j(\lambda-1)}} |x|^{-N}\,dx \lesssim 2^{-\beta j} 2^{jn\lambda} 2^{-jN\lambda},\end{aligned}$$

which implies the claim (since N is arbitrary). To estimate the second part we use the Cauchy–Schwarz inequality:

$$\mathrm{II} \le (C2^{j(\lambda-1)} + r)^{n/2} \|K_u^{(j)} * a\|_{L^2_x(L^2_u)}.$$

Then we have by the Fubini and Plancherel theorems that

$$\|K_u^{(j)} * a\|_{L^2_x(L^2_u)} = \| \|K_u^{(j)} * a\|_2 \|_{L^2_u(I)} \lesssim 2^{-j\beta} \|a\|_2 \lesssim 2^{-j\beta} r^{-n/2},$$

which implies

$$\mathrm{II} \lesssim (2^{j(\lambda-1)n/2} + r^{n/2}) 2^{-j\beta} r^{-n/2} = (2^j r)^{-n/2} + 2^{-j\beta},$$

as desired (we used that $\beta = n\lambda/2$). This proves (5-7).

It remains to show (5-8). Clearly we have

$$\|K_u^{(j)} * a\|_{L^1_x(L^2_u)} \lesssim \| \sup_{u\in I} |K_u^{(j)} * a| \|_1. \tag{5-9}$$

We claim that

$$\| \sup_{u\in I} |K_u^{(j)} * a| \|_1 \lesssim \|P_j a\|_1. \tag{5-10}$$

[1] C is a constant that may depend on the parameters λ, β, n.

To see this replace ψ_j by $\tilde{\psi}_j$ in the definition of $K_u^{(j)}$ and call the resulting kernel $\widetilde{K}_u^{(j)}$. Then we have

$$K_u^{(j)} * a = \widetilde{K}_u^{(j)} * P_j a.$$

It is clear that $\widetilde{K}_u^{(j)}$ satisfies the same pointwise estimates (5-5), (5-6) (possibly with larger constants). Thus, there exists a positive function w_j on $\mathbb{R}^n$ such that $\|w_j\|_1 \lesssim 1$ and

$$|\widetilde{K}_u^{(j)}(x)| \le w_j(x)$$

for all $x \in \mathbb{R}^n$ and $u \in I$. As a consequence,

$$\| \sup_{u\in I} |K_u^{(j)} * a|\|_1 \le \| \sup_{u\in I} |\widetilde{K}_u^{(j)}| * |P_j a|\|_1 \le \|w_j * |P_j a|\|_1 \lesssim \|P_j a\|_1,$$

which is our claim (5-10). But by the mean zero property of a and the mean value theorem we have

$$P_j a(x) = \int_{\mathbb{R}^n} (\hat{\psi}_j(x-y) - \hat{\psi}_j(x)) a(y)\, dy = -\int_{\mathbb{R}^n} \int_0^1 y \cdot \nabla \hat{\psi}_j(x-ty)\, dt\, a(y)\, dy.$$

This implies

$$\|P_j a\|_1 \le \int_{\mathbb{R}^n} \int_{|y|\le r} \int_0^1 |y| |\nabla \hat{\psi}_j(x-ty)| |a(y)|\, dt\, dy\, dx \lesssim 2^j r.$$

In view of (5-9) and (5-10), this establishes (5-8). □

6. A counterexample: the proof of Theorem 1.2

Let ϕ be a smooth test function supported in the annulus $\frac{1}{2} \le |\xi| \le 2$ and define f_k for $k \in \mathbb{Z}$ by

$$\hat{f}_k(\xi) = \phi(2^{-k}\xi).$$

On the one hand, clearly,

$$\|f_k\|_p = 2^{nk} \left(\int_{\mathbb{R}^n} |\hat{\phi}(2^k x)|^p\, dx \right)^{1/p} \approx 2^{nk/p'}.$$

On the other hand, we claim that

$$\|V^r\{\mathcal{H}^{(u)} f_k : u \in \mathbb{R}\}\|_p \gtrsim 2^{k(-n(\alpha-1)(1/p')+\alpha/r)}. \tag{6-1}$$

If (1-3) were to hold, then plugging in f_k into the estimate (1-3) and letting $k \to \infty$, we see that

$$-n(\alpha-1)\frac{1}{p'} + \frac{\alpha}{r} \le \frac{n}{p'},$$

which is equivalent to $p' \le nr$.

For simplicity we will verify this only in the case $n = 1$, $K(t) = \text{p.v.}(1/t)$, $\alpha = 2$. The general case can be treated in the same manner. In this case (6-1) takes the form

$$\|V^r\{\mathcal{H}^{(u)} f_k : u \in \mathbb{R}\}\|_p \gtrsim 2^{k(-1+1/p+2/r)}. \tag{6-2}$$

We can choose φ such that

$$\frac{1}{t} = \sum_{j\in\mathbb{Z}} \varphi_j(t)$$

for all $t \neq 0$, where $\varphi_j(t) = 2^{-j}\varphi(2^{-j}t)$. By Fourier inversion we have

$$\mathcal{H}^{(u)} f_k(x) = \sum_{j\in\mathbb{Z}} \iint \phi(2^{-k}\xi)e^{ix\xi}e^{-it\xi+iut^2}\varphi_{j+k}(t)\,dt\,d\xi$$
$$= \sum_{j\in\mathbb{Z}} 2^k \int e^{i2^kx\xi}\phi(\xi)\int e^{-i2^{2k}t\xi+i2^{2k}ut^2}\varphi_j(t)\,dt\,d\xi = \sum_{j\in\mathbb{Z}} \mathrm{I}_j.$$

(Keep in mind that I_j also depends on k, x, u.) Let us take $u \in [1, 2]$. Then the phase of the oscillatory integral in t has no critical points if $|j| > 10$. This motivates us to set

$$\mathrm{I}^{\mathrm{main}} = \sum_{|j|\le 10} \mathrm{I}_j \quad \text{and} \quad \mathrm{I}^{\mathrm{err}} = \sum_{|j|>10} \mathrm{I}_j.$$

Write $B = [2^k, 2^{k+1}]$ and estimate

$$\|V^r\{\mathcal{H}^{(u)} f_k : u\in\mathbb{R}\}\|_p \ge \|V^r\{\mathcal{H}^{(u)} f_k : u\in[1,2]\}\|_{L^p(B)}$$
$$\ge \|V^r\{\mathrm{I}^{\mathrm{main}} : u\in[1,2]\}\|_{L^p(B)} - \|V^r\{\mathrm{I}^{\mathrm{err}} : u\in[1,2]\}\|_{L^p(B)}.$$

In order to verify (6-2) it suffices to show that

$$\|V^r\{\mathrm{I}^{\mathrm{main}} : u\in[1,2]\}\|_{L^p([2^k,2^{k+1}])} \gtrsim 2^{k(-1+1/p+2/r)}, \tag{6-3}$$
$$\|V^r\{\mathrm{I}^{\mathrm{err}} : u\in[1,2]\}\|_{L^p([2^k,2^{k+1}])} \lesssim 2^{-2k}. \tag{6-4}$$

We begin with the proof of (6-3). Write

$$\mathrm{I}^{\mathrm{main}} = 2^k \int e^{i2^kx\xi}\phi(\xi)\int e^{-i2^{2k}t\xi+i2^{2k}ut^2}\rho(t)\,dt\,d\xi,$$

where $\rho(t) = \sum_{|j|\le 10}\varphi_j(t)$. Note that the phase of the integral in t has a critical point at $t_c = \xi/(2u)$. By the principle of stationary phase [Stein 1993, Chapter VIII.5.7; Sogge 1993, Theorem 1.2.1] we have

$$\int e^{-i2^{2k}t\xi+i2^{2k}ut^2}\rho(t)\,dt = 2^{-k}c_0e^{i2^{2k}c_1\xi^2u^{-1}}u^{-1/2}\rho\left(\frac{\xi}{2u}\right) + O(2^{-2k}).$$

Here c_0, c_1 are nonzero constants. To simplify the calculation, let us take $c_0 = c_1 = 1$. Thus, the main contribution to $\mathrm{I}^{\mathrm{main}}$ is

$$\int e^{i2^{2k}(\tilde{x}\xi+\xi^2u^{-1})}a(\xi,u)\,d\xi,$$

where $x = 2^k\tilde{x} \in [2^k, 2^{k+1}]$ and $a(\xi,u) = \phi(\xi)u^{-1/2}\rho(\xi/(2u))$. Note that the u-derivative of the error term coming from the stationary phase is also $O(2^{-2k})$. Therefore that term contributes only $O(2^{-2k})$ to the variation-norm and we can ignore it. From another application of the stationary phase principle we see that the previous integral can essentially be written in the form

$$2^{-k}e^{ix^2u}b(u) + O(2^{-2k}),$$

where $b(u) = \phi(\tilde{x}u/2)\rho(\tilde{x}/4)$. Let $u_\ell = \ell\pi/x^2$ for $x^2/\pi < \ell < 2x^2/\pi$. Then

$$\left(\sum_{x^2/\pi<\ell<2x^2/\pi} |e^{ix^2u_{\ell+1}} - e^{ix^2u_\ell}|^r\right)^{1/r} \approx 2^{2k/r},$$

which implies the claim (6-3) (the contribution of $b(u)$ is negligible).

It remains to treat $\mathrm{I}^{\mathrm{err}}$. Compute

$$\begin{aligned}\partial_u \mathrm{I}_j &= i2^{3k+2j} \int \phi(\xi) e^{i2^k x\xi} \int e^{-i2^{2k+j}t\xi + i2^{2k+2j}ut^2} t^2 \varphi(t)\, dt\, d\xi \\ &= i2^{3k+2j} \int \hat{\phi}(2^{2k+j}t - 2^k x) e^{i2^{2k+2j}ut^2} t^2 \varphi(t)\, dt.\end{aligned}$$

Observe that if $x \in [2^k, 2^{k+1}]$, $|t| \in \left[\frac{1}{2}, 2\right]$ and $|j| > 10$ we have

$$|2^{2k+j}t - 2^k x| \approx 2^{2k} \max(1, 2^j).$$

Since $\hat{\phi}$ decays rapidly, we obtain

$$V^r\{\mathrm{I}_j : u \in [1,2]\} \lesssim \|\partial_u \mathrm{I}_j\|_{L^1_u([1,2])} \lesssim_N 2^{3k+2j-2Nk} \min(1, 2^{-Nj})$$

for every $N \geq 1$. Taking N large enough ($N = 3$ suffices) and summing over $|j| > 10$, we obtain (6-4).

7. Proof of Proposition 4.2

In this section we provide a proof of Proposition 4.2, which was stated in [Jones, Seeger, and Wright 2008] without proof. Indeed, Proposition 4.2 is a vector-valued analogue of Theorem 1 of [Seeger 1988]. The proof of Proposition 4.2 follows closely that of the scalar-valued case in [Seeger 1988]. On the other hand, at one point in the scalar-valued case, Seeger used a duality argument between L^p and $L^{p'}$, which is not available in the vector-valued setting. This is why we had to assume that hypothesis (4-21) holds not just for $s = p$, but also at $s = 2$.

To prove Proposition 4.2, one key tool is the Fefferman–Stein sharp function. Let B be a Banach space. For us we will only need the case $B = \ell^2(\mathbb{Z})L^2(I)$. For each measurable function $F : \mathbb{R}^n \to B$, define its Hardy–Littlewood maximal function $\mathcal{M}F$ by

$$\mathcal{M}F(x) = \sup_{x \in Q} \fint_Q |F(y)|_B\, dy$$

for each $x \in \mathbb{R}^n$, where the supremum is over all cubes Q containing x. Also define the sharp function $F^\sharp$ of F by

$$F^\sharp(x) = \sup_{x \in Q} \fint_Q |F(y) - F_Q|_B\, dy,$$

where $F_Q = \fint_Q F(y)\, dy$; again the supremum is over all cubes Q containing x. We have the following lemma about $F^\sharp$:

Lemma 7.1. *Suppose* $0 < p < \infty$. *Let* $F \in L^{p_0}(\mathbb{R}^n, B)$ *for some* $0 < p_0 \leq p$. *If* $F^\sharp \in L^p(\mathbb{R}^n)$, *then* $\mathcal{M}F \in L^p(\mathbb{R}^n)$, *and*

$$\|\mathcal{M}F\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|F^\sharp\|_{L^p(\mathbb{R}^n)}.$$

We give a proof of this lemma at the end.

Now given $f \in L^p \cap L^2(\mathbb{R}^n)$, define $Tf : \mathbb{R}^n \to B$ by

$$Tf(x) = (T_{u,j} f(x))_{u \in I, j \in \mathbb{Z}}.$$

Note $Tf \in L^2(\mathbb{R}^n, B)$. Then

$$\| \| \| T_{u,j} f \|_{L^2(I)} \|_{\ell^2(\mathbb{Z})} \|_{L^p(\mathbb{R}^n)} = \| |Tf|_B \|_{L^p(\mathbb{R}^n)} \le \| \mathcal{M}(Tf) \|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \| (Tf)^\sharp \|_{L^p(\mathbb{R}^n)},$$

where in the last inequality we invoked the lemma with $p_0 = 2$. Note that for a.e. $x \in \mathbb{R}^n$

$$(Tf)^\sharp(x) \simeq \fint_{Q_x} |Tf(y) - (Tf)_{Q_x}|_B \, dy$$

for some cube Q_x containing x; we may choose Q_x such that the side length of Q_x is $2^{r(x)}$ for some integer $r(x)$. Then we split

$$(Tf)^\sharp(x) \lesssim \sigma_1((T_{u,j} f), x) + \sigma_2((P_j f), x),$$

where N is a positive integer to be chosen; here

$$\sigma_1(G, x) = \fint_{Q_x} \Bigg(\sum_{|j + r(x)| \le N} \| G_{u,j}(y) - (G_{u,j})_{Q_x} \|_{L^2(I)}^2 \Bigg)^{1/2} dy,$$

$$\sigma_2(H, x) = \fint_{Q_x} \Bigg(\sum_{|j + r(x)| > N} \| T_{u,j} H_j(y) - (T_{u,j} H_j)_{Q_x} \|_{L^2(I)}^2 \Bigg)^{1/2} dy$$

for any functions $G = (G_{u,j}) : \mathbb{R}^n \to B$ and $H = (H_j) : \mathbb{R}^n \to \ell^2(\mathbb{Z})$. We claim that

$$\| \sigma_1(G, x) \|_{L^p(\mathbb{R}^n)} \lesssim N^{1/2 - 1/p} \| \| \| G_{u,j}(x) \|_{L^2(I)} \|_{\ell^p(\mathbb{Z})} \|_{L^p(\mathbb{R}^n)}, \tag{7-1}$$

$$\| \sigma_2(H, x) \|_{L^p(\mathbb{R}^n)} \lesssim (A + B 2^{-N}) \| \| H_j(x) \|_{\ell^2(\mathbb{Z})} \|_{L^p(\mathbb{R}^n)} \tag{7-2}$$

for any $G = (G_{u,j}) : \mathbb{R}^n \to B$ and $H = (H_j) : \mathbb{R}^n \to \ell^2(\mathbb{Z})$. But when $G_{u,j} = T_{u,j} f$, we have

$$\begin{aligned} \| \| \| G_{u,j}(x) \|_{L^2(I)} \|_{\ell^p(\mathbb{Z})} \|_{L^p(\mathbb{R}^n)} &= \| \| \| T_{u,j} f(x) \|_{L^2(I)} \|_{L^p(\mathbb{R}^n)} \|_{\ell^p(\mathbb{Z})} \\ &\lesssim A \| \| P_j f(x) \|_{L^p(\mathbb{R}^n)} \|_{\ell^p(\mathbb{Z})} \\ &\lesssim A \| \| P_j f(x) \|_{\ell^2(\mathbb{Z})} \|_{L^p(\mathbb{R}^n)} \\ &\lesssim A \| f \|_{L^p(\mathbb{R}^n)} \end{aligned}$$

(we used assumption (4-21) in the second inequality, $p \in [2, \infty)$ in the third inequality, and Littlewood–Paley inequality in the last). Also, when $H_j = P_j f$, we have

$$\| \| H_j(x) \|_{\ell^2(\mathbb{Z})} \|_{L^p(\mathbb{R}^n)} \lesssim \| f \|_{L^p(\mathbb{R}^n)}.$$

Hence

$$\| (Tf)^\sharp \|_{L^p(\mathbb{R}^n)} \lesssim (A N^{1/2 - 1/p} + A + B 2^{-N}) \| f \|_{L^p(\mathbb{R}^n)}.$$

Choosing $N \simeq \log(2 + B/A)$ gives the desired conclusion of the proposition. It remains to prove (7-1) and (7-2).

To prove (7-1), we interpolate between $p = 2$ and $p = \infty$. Indeed, we prove

$$\|\sigma_1(G, x)\|_{L^2(\mathbb{R}^n)} \lesssim \| \| \|G_{u,j}(x)\|_{L^2(I)} \|_{\ell^2(\mathbb{Z})} \|_{L^2(\mathbb{R}^n)}, \tag{7-3}$$

$$\|\sigma_1(G, x)\|_{L^\infty(\mathbb{R}^n)} \lesssim N^{1/2} \| \| \|G_{u,j}(x)\|_{L^2(I)} \|_{\ell^\infty(\mathbb{Z})} \|_{L^\infty(\mathbb{R}^n)}. \tag{7-4}$$

The desired estimate (7-1) then follows by complex interpolation and linearizing σ_1.

The estimate (7-3) follows since

$$\sigma_1(G, x) \le 2 \fint_{Q_x} \biggl(\sum_{|j+r(x)| \le N} \|G_{u,j}(y)\|^2_{L^2(I)} \biggr)^{1/2} dy, \tag{7-5}$$

so

$$\sigma_1(G, x) \lesssim \fint_{Q_x} \| \|G_{u,j}(y)\|_{L^2(I)} \|_{\ell^2(\mathbb{Z})} \, dy \lesssim M \| \|G_{u,j}\|_{L^2(I)} \|_{\ell^2(\mathbb{Z})}(x),$$

where M is the standard (scalar-valued) Hardy–Littlewood maximal function on $\mathbb{R}^n$. Hence

$$\|\sigma_1(G, x)\|_{L^2(\mathbb{R}^n)} \lesssim \| \| \|G_{u,j}(x)\|_{L^2(I)} \|_{\ell^2(\mathbb{Z})} \|_{L^2(\mathbb{R}^n)}$$

as in (7-3).

To prove (7-4), note that for each $x \in \mathbb{R}^n$, we have, from (7-5), that

$$\sigma_1(G, x) \le 2 \sup_{y \in \mathbb{R}^n} \biggl(\sum_{|j+r(x)| \le N} \|G_{u,j}(y)\|^2_{L^2(I)} \biggr)^{1/2} \lesssim N^{1/2} \sup_{y \in \mathbb{R}^n} \sup_{j \in \mathbb{Z}} \|G_{u,j}(y)\|_{L^2(I)},$$

with constants uniform in x. This gives (7-4).

Next, to prove (7-2), we will prove

$$\|\sigma_2(H, x)\|_{L^2(\mathbb{R}^n)} \lesssim A \| \|H_j(x)\|_{\ell^2(\mathbb{Z})} \|_{L^2(\mathbb{R}^n)}, \tag{7-6}$$

$$\|\sigma_2(H, x)\|_{L^\infty(\mathbb{R}^n)} \lesssim (A + B2^{-N}) \| \|H_j(x)\|_{\ell^2(\mathbb{Z})} \|_{L^\infty(\mathbb{R}^n)}. \tag{7-7}$$

The desired estimate (7-2) then follows by complex interpolation and linearizing σ_2.

To prove (7-6), note that

$$\sigma_2(H, x) \le 2 \fint_{Q_x} \biggl(\sum_{|j+r(x)| > N} \|T_{u,j} H_j\|^2_{L^2(I)} \biggr)^{1/2} dy,$$

so

$$\sigma_2(H, x) \lesssim \fint_{Q_x} \| \|T_{u,j} H_j\|_{L^2(I)} \|_{\ell^2(\mathbb{Z})} \, dy \lesssim M \| \|T_{u,j} H_j\|_{L^2(I)} \|_{\ell^2(\mathbb{Z})}(x).$$

Hence

$$\|\sigma_2(H, x)\|_{L^2(\mathbb{R}^n)} \lesssim \| \| \|T_{u,j} H_j(x)\|_{L^2(I)} \|_{\ell^2(\mathbb{Z})} \|_{L^2(\mathbb{R}^n)}.$$

We commute the $\ell^2(\mathbb{Z})$ norm outside. Since

$$\| \|T_{u,j} H_j(x)\|_{L^2(I)} \|_{L^2(\mathbb{R}^n)} \lesssim A \|H_j(x)\|_{L^2(\mathbb{R}^n)}, \tag{7-8}$$

one can conclude that

$$\|\sigma_2(H, x)\|_{L^2(\mathbb{R}^n)} \lesssim A \| \|H_j(x)\|_{L^2(\mathbb{R}^n)} \|_{\ell^2(\mathbb{Z})},$$

which gives (7-6) upon a further change in the order of the norms on the right-hand side.

Now we proceed to prove (7-7). For each $x \in \mathbb{R}^n$, we take the decomposition

$$H_j(y) = (\chi_{2Q_x} H_j)(y) + (\chi_{(2Q_x)^c} H_j)(y)$$

for all $y \in \mathbb{R}^n$. Then we plug this back into the formula for $\sigma_2(H, x)$. We find that

$$\sigma_2(H, x) \lesssim \mathrm{I}(x) + \mathrm{II}(x),$$

where

$$\mathrm{I}(x) = \fint_{Q_x} \biggl(\sum_{j \in \mathbb{Z}} \| T_{u,j}(\chi_{2Q_x} H_j)(y) - [T_{u,j}(\chi_{2Q_x} H_j)]_{Q_j} \|_{L^2(I)}^2 \biggr)^{1/2} dy,$$

$$\mathrm{II}(x) = \fint_{Q_x} \biggl(\sum_{|j + r(x)| > N} \| T_{u,j}(\chi_{(2Q_x)^c} H_j)(y) - [T_{u,j}(\chi_{(2Q_x)^c} H_j)]_{Q_j} \|_{L^2(I)}^2 \biggr)^{1/2} dy.$$

We estimate $\mathrm{I}(x)$ by

$$\begin{aligned} \mathrm{I}(x) &\le 2 \fint_{Q_x} \biggl(\sum_{j \in \mathbb{Z}} \| T_{u,j}(\chi_{2Q_x} H_j)(y) \|_{L^2(I)}^2 \biggr)^{1/2} dy \\ &\lesssim \biggl(\fint_{Q_x} \sum_{j \in \mathbb{Z}} \| T_{u,j}(\chi_{2Q_x} H_j)(y) \|_{L^2(I)}^2 \, dy \biggr)^{1/2} \\ &\lesssim \frac{1}{|Q_x|^{1/2}} \| \| \| T_{u,j}(\chi_{2Q_x} H_j)(y) \|_{L^2(I)} \|_{L^2(\mathbb{R}^n)} \|_{\ell^2(\mathbb{Z})} \\ &\lesssim \frac{A}{|Q_x|^{1/2}} \| \| (\chi_{2Q_x} H_j)(y) \|_{L^2(\mathbb{R}^n)} \|_{\ell^2(\mathbb{Z})}, \end{aligned}$$

where in the last inequality we used the estimate (7-8). Then

$$\mathrm{I}(x) \lesssim A \biggl(\fint_{2Q_x} \| H_j(y) \|_{\ell^2(\mathbb{Z})} \, dy \biggr)^{1/2} \lesssim A \sup_{y \in \mathbb{R}^n} \| H_j(y) \|_{\ell^2(\mathbb{Z})},$$

which shows that

$$\| \mathrm{I}(x) \|_{L^\infty(\mathbb{R}^n)} \lesssim A \| \| H_j(x) \|_{\ell^2(\mathbb{Z})} \|_{L^\infty(\mathbb{R}^n)}.$$

Next we estimate $\mathrm{II}(x)$. Let $K_{u,j}$ be the convolution kernel of $T_{u,j}$. Then

$$K_{u,j}(x) = 2^{jn} K_u(2^j x),$$

where

$$K_u(x) = \int_{\mathbb{R}^n} \tilde{m}_u(\xi) e^{2\pi i x \cdot \xi} \, d\xi.$$

Now by our assumption on $\partial_\xi^\tau \tilde{m}_u(\xi)$, we have

$$\sup_{u \in I} |K_u(x)| + \sup_{u \in I} |\nabla_x K_u(x)| \lesssim \frac{B}{(1 + |x|)^{n+1}}.$$

We claim now

$$\sup_{y,z \in Q_x} \biggl(\sum_{|j + r(x)| > N} \biggl(\int_{(2Q_x)^c} \sup_{u \in I} |K_{u,j}(y - w) - K_{u,j}(z - w)| \, dw \biggr)^2 \biggr)^{1/2} \lesssim B 2^{-N} \tag{7-9}$$

uniformly for $x \in \mathbb{R}^n$. Indeed, suppose $y, z \in Q_x$, and $j + r(x) > -N$. Then we have the estimate

$$\begin{aligned}\int_{(2Q_x)^c} \sup_{u \in I} |K_{u,j}(y-w) - K_{u,j}(z-w)|\, dw &\lesssim \int_{(2Q_x)^c} \sup_{u\in I} (|K_{u,j}(y-w)| + |K_{u,j}(z-w)|)\, dw \\ &\lesssim 2^{jn} \int_{|w-x| \gtrsim 2^{r(x)}} \frac{B}{(2^j |w-x|)^{n+1}}\, dw \lesssim B 2^{-j-r(x)}.\end{aligned}$$

On the other hand, if $y, z \in Q_x$, and $j + r(x) < -N$, then

$$\int_{(2Q_x)^c} \sup_{u \in I} |K_{u,j}(y-w) - K_{u,j}(z-w)|\, dw$$

is bounded by a constant times

$$\int_0^1 \int_{(2Q_x)^c} \sup_{u \in I} |y-z| |\nabla_x K_{u,j}((1-t)y + tz - w)|\, dw\, dt \lesssim 2^j 2^{r(x)} \int_{\mathbb{R}^n} 2^{jn} |(\nabla_x K_u)(2^j w)|\, dw \lesssim B 2^{j+r(x)}.$$

Summing over j such that $j + r(x) > N$ and $j + r(x) < -N$ respectively, we see that (7-9) follows.

Finally, it suffices to observe that

$$\begin{aligned}\mathrm{II}(x) &\lesssim \fint_{Q_x} \fint_{Q_x} \biggl(\sum_{|j+r(x)|>N} \|T_{u,j}(\chi_{(2Q_x)^c} H_j)(y) - T_{u,j}(\chi_{(2Q_x)^c} H_j)(z)\|^2_{L^2(I)} \biggr)^{1/2} dy\, dz \\ &= \sup_{y,z \in Q_x} \biggl(\sum_{|j+r(x)|>N} \biggl\| \int_{(2Q_x)^c} [K_{u,j}(y-w) - K_{u,j}(z-w)] H_j(w)\, dw \biggr\|^2_{L^2(I)} \biggr)^{1/2} \\ &\lesssim \sup_{y,z \in Q_x} \biggl(\sum_{|j+r(x)|>N} \biggl(\int_{(2Q_x)^c} \|K_{u,j}(y-w) - K_{u,j}(z-w)\|_{L^2(I)}\, dw \biggr)^2 \biggr)^{1/2} \| \|H_j\|_{\ell^\infty(\mathbb{Z})} \|_{L^\infty(\mathbb{R}^n)} \\ &\lesssim \sup_{y,z \in Q_x} \biggl(\sum_{|j+r(x)|>N} \biggl(\int_{(2Q_x)^c} \sup_{u\in I} |K_{u,j}(y-w) - K_{u,j}(z-w)|\, dw \biggr)^2 \biggr)^{1/2} \| \|H_j\|_{\ell^2(\mathbb{Z})} \|_{L^\infty(\mathbb{R}^n)}.\end{aligned}$$

Invoking (7-9) yields

$$\|\mathrm{II}(x)\|_{L^\infty(\mathbb{R}^n)} \lesssim B 2^{-N} \| \|H_j(x)\|_{\ell^2(\mathbb{Z})} \|_{L^\infty(\mathbb{R}^n)},$$

which together with our earlier estimate about $\|\mathrm{I}(x)\|_{L^\infty(\mathbb{R}^n)}$ gives (7-7).

Proof of Lemma 7.1. The key is a relative distribution inequality. Fix the Banach space B. For any $n \geq 1$, we claim that there exists $b_n \in (0, 1)$ such that for any $b, c > 0$ with $b \leq b_n$ we have

$$|\{x \in \mathbb{R}^n : \mathcal{M}F(x) > \alpha,\, F^\sharp(x) \leq c\alpha\}| \lesssim_n c |\{x \in \mathbb{R}^n : \mathcal{M}F(x) > b\alpha\}|.$$

If this is true, then by taking c sufficiently small, we can use Lemma 2 of Chapter IV.3.5 of [Stein 1993] (see also the remark on the bottom of page 152 there) and conclude the proof of Lemma 7.1.

To prove the above relative distributional inequality, let $b \in (0, 1)$ first. Let $F \in L^p(\mathbb{R}^n, B)$. Decompose the open set $\{x \in \mathbb{R}^n : \mathcal{M}F(x) > b\alpha\}$ into an essentially disjoint union of Whitney cubes $\{Q\}$, so that the distance of each Q from the complement of this set is bounded by 4 times the diameter of Q. Now since

$\{x \in \mathbb{R}^n : \mathcal{M}F(x) > \alpha,\ F^\sharp(x) \le c\alpha\}$ is a subset of $\{x \in \mathbb{R}^n : \mathcal{M}F(x) > b\alpha\}$, we just need to show that for each Whitney cube Q as above, we have

$$|\{x \in Q : \mathcal{M}F(x) > \alpha,\ F^\sharp(x) \le c\alpha\}| \lesssim_n c|Q|.$$

This inequality would be trivial if the set on the left-hand side were empty. So let's assume there exists a point $x_0 \in Q$ such that $F^\sharp(x_0) \le c\alpha$. Now let $\widetilde{Q}$ be any cube that intersects Q and that has diameter at least that of Q. Then $20\widetilde{Q}$ will contain a point y where $\mathcal{M}F(y) \le b\alpha$. Hence $⨍_{\widetilde{Q}} |F|_B \le 20^n b\alpha$ for all such cubes $\widetilde{Q}$. If $x \in Q$ and $\mathcal{M}F(x) > \alpha$, then by taking $b < 20^{-n}$, we see that $\mathcal{M}(F\chi_{3Q})(x) > \alpha$. We also have $⨍_{3Q} |F|_B \le 20^n b\alpha$. Thus,

$$\{x \in Q : \mathcal{M}F(x) > \alpha,\ F^\sharp(x) \le c\alpha\} \subset \left\{x \in Q : \mathcal{M}\left(F\chi_{3Q} - ⨍_{3Q} F\right)(x) > (1-20^n b)\alpha\right\},$$

where the measure of the right-hand side is bounded by

$$\frac{C_n}{(1-20^n b)\alpha}\int_Q |F\chi_{3Q}(y) - F_{3Q}|_B\, dy \le \frac{C_n}{(1-20^n b)\alpha}|3Q|F^\sharp(x_0) \le \frac{3^n C_n}{(1-20^n b)} c|Q|,$$

where C_n is the constant arising in the weak-type (1,1) bound of $\mathcal{M} : L^1(\mathbb{R}^n, B) \to L^{1,\infty}(\mathbb{R}^n)$. This proves the desired relative distributional inequality. □

Appendix A: An improved local smoothing estimate

In this section we prove Theorem 1.6. Let $\chi : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative smooth bump function supported on $1 \le |\xi| \le 2$. Define

$$E_0 f(x,t) := \int_{\mathbb{R}^n} f(\xi)\chi(\xi)e^{ix\cdot\xi + it|\xi|^\gamma}\, d\xi.$$

We will prove

$$\|E_0 f\|_{L^p(\mathbb{R}^n \times [-\lambda,\lambda])} \lesssim \lambda^{n(1/2-1/p)+\epsilon}\|\hat{f}\|_{L^p(\mathbb{R}^n)} \tag{A-1}$$

for every $\lambda \ge 1$. Once this is proved, a rescaling argument shows that

$$\left(\int_{\mathbb{R}^n \times I}\left|\int_{\mathbb{R}^n} e^{ix\xi}\hat{f}(\xi)e^{it|\xi|^\gamma}\, d\xi\right|^p dx\, dt\right)^{1/p} \lesssim 2^{k\gamma n(1/2-1/p) - k\gamma/p + k\epsilon}\|f\|_{L^p(\mathbb{R}^n)} \tag{A-2}$$

whenever $\hat{f}(\xi)$ is supported on the k-th annulus, that is, $|\xi| \approx 2^k$. As a consequence we obtain Theorem 1.6.

We will prove (A-1) for every elliptic phase. Let c_0 be a small positive real number. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a smooth function with

$$|\phi(\xi)| + |\nabla\phi(\xi)| \lesssim 1, \quad c_0 I_n \le (\nabla^2\phi)(\xi) \le \frac{1}{c_0} I_n \quad \text{for every } |\xi| < 10, \tag{A-3}$$

where I_n is the identity matrix of order $n \times n$. Let $\chi_0 : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative smooth bump function supported on $|\xi| \le 2$. Define

$$E_\phi f(x,t) := \int_{\mathbb{R}^n} f(\xi)\chi_0(\xi)e^{ix\cdot\xi + it\phi(\xi)}\, d\xi.$$

We will prove

$$\sup_{\phi:\text{(A-3)}} \|E_\phi f\|_{L^p(\mathbb{R}^n\times[-\lambda,\lambda])} \lesssim \lambda^{n(1/2-1/p)+\epsilon}\|\hat{f}\|_{L^p(\mathbb{R}^n)} \tag{A-4}$$

for every $\lambda \geq 1$.

For a ball $B_\lambda \subset \mathbb{R}^{n+1}$ of radius λ, we will let B_λ^- denote its projection in the first n variables, i.e., spatial variables. That is, B_λ^- is a ball of radius λ in $\mathbb{R}^n$. We also define the associated weight

$$w_{B_\lambda^-}(x) := \frac{1}{(1+\|x-c\|/\lambda)^{100n}} \quad \text{for } x \in \mathbb{R}^n.$$

Here c denotes the center of B_λ^-. We remark that in the argument below, various implicit constants depend on this choice of weight. However, this dependence is not important, and to avoid unnecessary technicalities, we will not make these details explicit. We refer the interested reader to [Li 2020], which contains all the necessary details that are required to run the Bourgain–Guth argument [2011].

We prove (A-4) by an inductive argument. Denote by Q_λ the smallest constant such that

$$\sup_{\phi:\text{(A-3)}} \|E_\phi f\|_{L^p(B_\lambda)} \leq Q_\lambda \cdot \lambda^{n(1/2-1/p)}\|\hat{f}\|_{L^p(w_{B_\lambda^-})}$$

for every ball $B_\lambda \subset \mathbb{R}^{n+1}$ with $B_\lambda \subset B_\lambda^- \times [-\lambda,\lambda]$. Of course our goal is to prove that

$$Q_\lambda \lesssim_\epsilon \lambda^\epsilon \tag{A-5}$$

for every $\epsilon > 0$. In the following, for the sake of simplicity, we will always abbreviate $E_\phi f$ to Ef.

First, by translation invariance, we will assume that B_λ^- is centered at 0. We normalize f such that

$$\|\hat{f}\|_{L^p(w_{B_\lambda^-})}\lambda^{n(1/2-1/p)} = 1. \tag{A-6}$$

Next, let K_n be a large integer that is to be determined, satisfying

$$K_n \ll \lambda^\epsilon.$$

For a large dyadic integer K, let Col_K denote the collection of all dyadic cubes of length $1/K$. We write

$$Ef = \sum_{\alpha_n\in\mathrm{Col}_{K_n}} Ef_{\alpha_n}, \quad \text{with } f_{\alpha_n} := f\cdot\mathbb{1}_{\alpha_n}.$$

Here $\{\mathbb{1}_{\alpha_n}\}_n$ forms a smooth partition of unity, and $\mathbb{1}_{\alpha_n}$ is supported on $2\alpha_n$. On every ball $B_{K_n} \subset \mathbb{R}^{n+1}$ of radius K_n, by the uncertainty principle, we know that $|Ef_{\alpha_n}|$ is essentially a constant for every $\alpha_n \in \mathrm{Col}_{K_n}$. We let $|Ef_{\alpha_n}|(B_{K_n})$ denote this constant. Denote by α_n^* the cube that maximizes

$$\{|Ef_{\alpha_n}|(B_{K_n})\}_{\alpha_n\in\mathrm{Col}_{K_n}}.$$

Consider the collection

$$\mathrm{Col}^*_{K_n} := \{\alpha_n \in \mathrm{Col}_{K_n} : |Ef_{\alpha_n}|(B_{K_n}) \geq K_n^{-n}|Ef_{\alpha_n^*}|(B_{K_n})\}.$$

Here the choice of the coefficient K_n^{-n} is not strict. One can also use K_n^{-2n} or something even smaller.

There are three cases.

Case 1: There exists an integer $1 \leq j \leq n$, and cubes $\alpha_n^{(1)}, \dots, \alpha_n^{(j)} \in \mathrm{Col}^*_{K_n}$ which are $(1/K_{n-1})$-separated such that every cube within $\mathrm{Col}^*_{K_n}$ is in the $1/K_{n-1}$ neighborhood of some $\alpha_n^{(j')}$.

Here $K_{n-1} \ll K_n^{\epsilon}$ is also to be determined. Next, we have:

Case 2: There exist cubes $\alpha_n^{(1)}, \dots, \alpha_n^{(n+1)} \in \mathrm{Col}^*_{K_n}$ that are $(1/K_{n-1})$-separated and do not lie in the $100/K_n$ neighborhood of any $(n-1)$-dimensional subspace.

If Case 1 and Case 2 are not satisfied, then we have:

Case 3: All cubes in $\mathrm{Col}^*_{K_n}$ lie in the $C(K_{n-1})/K_n$ neighborhood of a subspace of dimension $n-1$.

Here $C(K_{n-1})$ is a large constant depending on K_{n-1} which may change from line to line (it always suffices to take, say, $C(K_{n-1}) = K_{n-1}^{100n}$).

We deal with these three cases separately. In Case 1, we have

$$|Ef| \lesssim \max_{\alpha_n \in \mathrm{Col}_{K_n}} |Ef_{\alpha_n}| + \max_{\alpha_{n-1} \in \mathrm{Col}_{K_{n-1}}} |Ef_{\alpha_{n-1}}|. \tag{A-7}$$

In Case 2, we use

$$|Ef| \lesssim K_n^{2n} \biggl(\prod_{j=1}^{n+1} |Ef_{\alpha_n^{(j)}}| \biggr)^{1/(n+1)}.$$

In Case 3, we use

$$|Ef| \lesssim \max_{\alpha_n \in \mathrm{Col}_{K_n}} |Ef_{\alpha_n}| + \max_{\substack{L_{n-1}:\text{ subspace}\\ \text{of dimension } (n-1)}} \biggl| \sum_{\alpha_{n-1}:\mathrm{dist}(\alpha_{n-1}, L_{n-1}) \le 1/K_{n-1}} Ef_{\alpha_{n-1}} \biggr|. \tag{A-8}$$

In the last summation, we implicitly assumed that $\alpha_{n-1} \in \mathrm{Col}_{K_{n-1}}$, and that

$$\frac{C(K_{n-1})}{K_n} \le \frac{1}{K_{n-1}}.$$

Here we agree upon a convention: for $1 \le j \le n$, whenever the symbol α_j appears, we always assume that $\alpha_j \in \mathrm{Col}_{K_j}$, to keep notation simpler. Combining (A-7)–(A-8), we obtain

$$\begin{aligned} |Ef| \lesssim \max_{\alpha_n \in \mathrm{Col}_{K_n}} |Ef_{\alpha_n}| + \max_{\alpha_{n-1} \in \mathrm{Col}_{K_{n-1}}} |Ef_{\alpha_{n-1}}| + K_n^{2n} \biggl(\prod_{j=1}^{n+1} |Ef_{\alpha_n^{(j)}}| \biggr)^{1/(n+1)} \\ + \max_{\substack{L_{n-1}:\text{ subspace}\\ \text{of dimension } (n-1)}} \biggl| \sum_{\alpha_{n-1}:\mathrm{dist}(\alpha_{n-1}, L_{n-1}) \le 1/K_{n-1}} Ef_{\alpha_{n-1}} \biggr|. \end{aligned}$$

We raise both sides to the p-th power, then integrate over B_{K_n}, and in the end sum over balls B_{K_n} inside B_λ,

$$\begin{aligned} \int_{B_\lambda} |Ef|^p \lesssim \sum_{\alpha_n \in \mathrm{Col}_{K_n}} \int_{B_\lambda} |Ef_{\alpha_n}|^p + \sum_{\alpha_{n-1} \in \mathrm{Col}_{K_{n-1}}} \int_{B_\lambda} |Ef_{\alpha_{n-1}}|^p \\ + \sum_{\alpha_n^{(1)}, \dots, \alpha_n^{(n+1)} \text{ in Case 2}} \int_{B_\lambda} K_n^{2pn} \biggl(\prod_{j=1}^{n+1} |Ef_{\alpha_n^{(j)}}| \biggr)^{p/(n+1)} \\ + \sum_{B_{K_n} \subset B_\lambda} \max_{L_{n-1}} \int_{B_{K_n}} \biggl| \sum_{\alpha_{n-1}:\mathrm{dist}(\alpha_{n-1}, L_{n-1}) \le 1/K_{n-1}} Ef_{\alpha_{n-1}} \biggr|^p. \end{aligned} \tag{A-9}$$

There are four terms on the right-hand side. It is the contribution from the last term that gives us the ultimate constraint for the exponent p, as stated in (1-5).

Let us be more precise. The first and second summands on the right-hand side of (A-9) can be taken care of by parabolic rescaling. We will deal with the third summand using multilinear restriction estimates due to [Bennett, Carbery, and Tao 2006]. The last term requires further careful analysis.

For the first and second summands, we will apply rescaling. The argument is the same in both cases. Hence we will only write down the rescaling argument for the first summand.

$$\int_{B_\lambda} |Ef_{\alpha_n}|^p = \int_{B_\lambda} \left| \int f_{\alpha_n}(\xi) e^{i\xi x + it\phi(\xi)} d\xi \right|^p dx\, dt.$$

Here we apply the change of variable

$$\xi \to \frac{\xi}{K_n} + c_{\alpha_n}, \quad \text{with } c_{\alpha_n} \text{ being the center of } \alpha_n.$$

We obtain

$$K_n^{-np} \int_{B_\lambda} \left| \int f_{\alpha_n}\left(\frac{\xi}{K_n} + c_{\alpha_n}\right) e^{i(\xi/K_n + c_{\alpha_n})x + it\phi(\xi/K_n + c_{\alpha_n})} d\xi \right|^p dx\, dt$$
$$= K_n^{-np} \int_{B_\lambda} \left| \int f_{\alpha_n}\left(\frac{\xi}{K_n} + c_{\alpha_n}\right) e^{i\xi/K_n x + iK_n^2 \phi(\xi/K_n + c_{\alpha_n})(t/K_n^2)} d\xi \right|^p dx\, dt.$$

Next we apply the change of variables

$$x/K_n \to x \quad \text{and} \quad t/K_n^2 \to t$$

to obtain

$$K_n^{-np+n+2} \int_{\widetilde{B}_\lambda} \left| \int f_{\alpha_n}\left(\frac{\xi}{K_n} + c_{\alpha_n}\right) e^{i\xi x + it\cdot K_n^2 \phi(\xi/K_n + c_{\alpha_n})} d\xi \right|^p dx\, dt. \tag{A-10}$$

Here $\widetilde{B}_\lambda \subset \mathbb{R}^{n+1}$ is a rectangular box of dimensions $\lambda/K_n \times \cdots \times \lambda/K_n \times \lambda/K_n^2$ centered at 0. The reason of writing it in this form is that

$$\tilde{\phi}(\xi) := K_n^2 \phi\left(\frac{\xi}{K_n} + c_{\alpha_n}\right) - K_n \langle (\nabla\phi)(c_{\alpha_n}), \xi\rangle - K_n^2 \phi(c_{\alpha_n}) \text{ still satisfies (A-3).}$$

By a change of variable, (A-10) can be bounded by

$$K_n^{-np+n+2} \int_{2\widetilde{B}_\lambda} \left| \int f_{\alpha_n}\left(\frac{\xi}{K_n} + c_{\alpha_n}\right) e^{i\xi x + it\tilde{\phi}(\xi)} d\xi \right|^p dx\, dt.$$

Next we split the rectangular box $2\widetilde{B}_\lambda$ into a union of cubes of side-length λ/K_n^2.

$$K_n^{-np} K_n^{n+2} \sum_{B_{\lambda/K_n^2} \subset 2\widetilde{B}_\lambda} \int_{B_{\lambda/K_n^2}} \left| \int f_{\alpha_n}\left(\frac{\xi}{K_n} + c_{\alpha_n}\right) e^{i\xi x + it\tilde{\phi}(\xi)} d\xi \right|^p dx\, dt$$
$$\lesssim K_n^{-np} K_n^{n+2} \sum_{B^-_{\lambda/K_n^2} \subset B^-_{\lambda/K_n}} \left(\frac{\lambda}{K_n^2}\right)^{np(1/2-1/p)} Q^p_{\lambda/K_n^2} \int_{\mathbb{R}^n} \left| \widehat{f_{\alpha_n}\left(\frac{\cdot}{K_n} + c_{\alpha_n}\right)}(x) \right|^p w_{B^-_{\lambda/K_n^2}}(x)\, dx.$$

Here we applied the induction hypothesis. We may now sum the weights over all $B^-_{\lambda/K_n^2} \subset B^-_{\lambda/K_n}$, and bound the above by

$$\lesssim K_n^{-np} K_n^{n+2} \Big(\frac{\lambda}{K_n^2}\Big)^{np(1/2-1/p)} Q^p_{\lambda/K_n^2} \int_{\mathbb{R}^n} \Big| \widehat{f_{\alpha_n}\Big(\frac{\cdot}{K_n} + c_{\alpha_n}\Big)}(x) \Big|^p w_{B^-_{\lambda/K_n}}(x)\,dx$$
$$\lesssim K_n^{-np} K_n^{n+2} K_n^{np-n} \Big(\frac{\lambda}{K_n^2}\Big)^{np(1/2-1/p)} Q^p_{\lambda/K_n^2} \int_{\mathbb{R}^n} |\hat{f}_{\alpha_n}(\xi)|^p w_{B^-_\lambda}(x)\,dx$$

Summing over α_n, we obtain

$$K_n^{-np} K_n^{n+2} \Big(\frac{\lambda}{K_n^2}\Big)^{np(1/2-1/p)} Q^p_{\lambda/K_n^2} K_n^{-n} K_n^{np} \sum_{\alpha_n} \int_{\mathbb{R}^n} |\hat{f}_{\alpha_n}|^p w_{B^-_\lambda}$$
$$\lesssim K_n^{-np} K_n^{n+2} \Big(\frac{1}{K_n^2}\Big)^{np(1/2-1/p)} Q^p_{\lambda/K_n^2} K_n^{-n} K_n^{np}.$$

In the last step, we applied the normalization condition (A-6). Let the exponent of K_n be equal to zero and we obtain

$$2 - 2np\Big(\frac{1}{2} - \frac{1}{p}\Big) = 0 \quad \Longrightarrow \quad p = \frac{2(n+1)}{n}.$$

This is exactly the exponent in the Fourier restriction conjecture. Moreover, the last display tells us that, for the contribution from the first and second terms in (A-9), the induction can be closed whenever $p > 2(n+1)/n$.

Now we deal with the third summand on the right-hand side of (A-9). When $p \geq 2(n+1)/n$, by multilinear restriction of [Bennett, Carbery, and Tao 2006] and by Bernstein's inequality and Hölder's inequality,

$$\sum_{\alpha_n^{(1)},\dots,\alpha_n^{(n+1)} \text{ in Case 2}} \int_{B_\lambda} K_n^{2n} \Big(\prod_{j=1}^{n+1} |Ef_{\alpha_n^{(j)}}| \Big)^{p/(n+1)} \lesssim K_n^{2n} K_n^{100n!} \lambda^\epsilon.$$

Again we see that there is no problem for this term as K_n can be chosen to be much smaller compared with λ^ϵ.

In the end, we come to the last summand on the right-hand side of (A-9). Fix a ball $B_{K_n} \subset \mathbb{R}^{n+1}$. Assume that the maximum is attained at the $(n-1)$-dimensional subspace L_{n-1}. We need to consider

$$\int_{B_{K_n}} \Big| \sum_{\alpha_{n-1}:\mathrm{dist}(\alpha_{n-1}, L_{n-1}) \leq 1/K_{n-1}} Ef_{\alpha_{n-1}} \Big|^p.$$

Notice that each $|Ef_{\alpha_{n-1}}|$ is essentially a constant on $B_{K_{n-1}}$, a ball of radius K_{n-1} which is much smaller compared with K_n^ϵ. Hence tentatively we fix a ball $B_{K_{n-1}} \subset B_{K_n}$. Let α^*_{n-1} denote the cube that maximizes

$$\{|Ef_{\alpha_{n-1}}|(B_{K_{n-1}})\}_{\alpha_{n-1}:\mathrm{dist}(\alpha_{n-1}, L_{n-1}) \leq 1/K_{n-1}}.$$

Consider the collection

$$\mathrm{Col}^*_{K_{n-1}} := \Big\{ \alpha_{n-1} : \mathrm{dist}(\alpha_{n-1}, L_{n-1}) \leq \frac{1}{K_{n-1}} \text{ and } |Ef_{\alpha_{n-1}}|(B_{K_{n-1}}) \geq K_{n-1}^{-n} |Ef_{\alpha^*_{n-1}}|(B_{K_{n-1}}) \Big\}.$$

There are three further cases:

Case 3.1: There exists an integer $1 \le j \le n-1$, and cubes $\alpha_{n-1}^{(1)}, \dots, \alpha_{n-1}^{(j)} \in \mathrm{Col}^*_{K_{n-1}}$ which are $(1/K_{n-2})$-separated such that every cube within $\mathrm{Col}^*_{K_{n-1}}$ is in the $1/K_{n-2}$ neighborhood of some $\alpha_{n-1}^{(j')}$.

Here $K_{n-2} \ll K_{n-1}^{\epsilon}$ is also to be determined. Moreover, we have

Case 3.2: There exist cubes $\alpha_{n-1}^{(1)}, \dots, \alpha_{n-1}^{(n)} \in \mathrm{Col}^*_{K_{n-1}}$ that are $(1/K_{n-2})$-separated and do not lie in the $100/K_{n-1}$ neighborhood of any $(n-2)$-dimensional subspace.

If the above two cases are not satisfied, then we must have

Case 3.3: All cubes in $\mathrm{Col}^*_{K_{n-1}}$ lie in the $C(K_{n-2})/K_{n-1}$ neighborhood of a linear subspace of dimension $n-2$.

Here $C(K_{n-2})$ is a large constant depending on K_{n-2}. It suffices to take $C(K_{n-2}) = K_{n-2}^{100n}$.

Similarly to (A-7)–(A-8), we have that for every point $(x,t) \in B_{K_{n-1}}$

$$|Ef| \lesssim \max_{\alpha_n \in \mathrm{Col}_{K_n}} |Ef_{\alpha_n}| + \max_{\alpha_{n-1} \in \mathrm{Col}_{K_{n-1}}} |Ef_{\alpha_{n-1}}| + \max_{\alpha_{n-2} \in \mathrm{Col}_{K_{n-2}}} |Ef_{\alpha_{n-2}}| + K_{n-1}^{2n} \Bigl(\prod_{j=1}^{n} |Ef_{\alpha_{n-1}^{(j)}}| \Bigr)^{1/n} + \max_{L_{n-2}} \Bigl| \sum_{\alpha_{n-2} : \mathrm{dist}(\alpha_{n-2}, L_{n-2}) \le 1/K_{n-2}} Ef_{\alpha_{n-2}} \Bigr|.$$

We first raise both sides to the p-th power, integrate over $B_{K_{n-1}}$, and then sum over all balls $B_{K_{n-1}} \subset B_{K_n}$, and in the end sum over all balls $B_{K_n} \subset B_\lambda$:

$$\begin{aligned}\int_{B_\lambda} |Ef|^p \lesssim{}& \sum_{\alpha_n \in \mathrm{Col}_{K_n}} \int_{B_\lambda} |Ef_{\alpha_n}|^p + \sum_{\alpha_{n-1} \in \mathrm{Col}_{K_{n-1}}} \int_{B_\lambda} |Ef_{\alpha_{n-1}}|^p \\ &+ \sum_{\alpha_{n-2} \in \mathrm{Col}_{K_{n-2}}} \int_{B_\lambda} |Ef_{\alpha_{n-2}}|^p + \sum_{\alpha_{n-1}^{(1)}, \dots, \alpha_{n-1}^{(n)} \text{ in Case 3.2}} \int_{B_\lambda} K_{n-1}^{2n} \Bigl(\prod_{j=1}^{n} |Ef_{\alpha_{n-1}^{(j)}}| \Bigr)^{p/n} \\ &+ \sum_{B_{K_n} \subset B_\lambda} \sum_{B_{K_{n-1}} \subset B_{K_n}} \max_{L_{n-2}} \int_{B_{K_{n-1}}} \Bigl| \sum_{\alpha_{n-2} : \mathrm{dist}(\alpha_{n-2}, L_{n-2}) \le 1/K_{n-2}} Ef_{\alpha_{n-2}} \Bigr|^p.\end{aligned}$$

There are five terms on the right-hand side of the last display. By the same scaling argument as above, we can handle the first three summands. For the fourth summand, we again apply multilinear restrictions due to [Bennett, Carbery, and Tao 2006]. However, notice that we are applying an n-linear restriction estimate in $\mathbb{R}^{n+1}$. This will not give us the restriction exponent $2(n+1)/n$, but something larger. To be precise, we have

$$\sum_{\alpha_{n-1}^{(1)}, \dots, \alpha_{n-1}^{(n)} \text{ in Case 3.2}} \int_{B_\lambda} K_{n-1}^{2n} \Bigl(\prod_{j=1}^{n} |Ef_{\alpha_{n-1}^{(j)}}| \Bigr)^{p/n} \lesssim K_{n-1}^{2n} K_{n-1}^{100n!} \lambda^{\epsilon}$$

for every

$$p \ge \frac{2n}{n-1},$$

which we assume.

Hence it remains to handle the last summand

$$\sum_{B_{K_{n-1}}\subset B_\lambda} \max_{L_{n-2}} \int_{B_{K_{n-1}}} \Bigg| \sum_{\alpha_{n-2}:\mathrm{dist}(\alpha_{n-2},L_{n-2})\le 1/K_{n-2}} Ef_{\alpha_{n-2}} \Bigg|^p.$$

We repeat this iteration until we reach

$$\sum_{B_{K_{n-k}}\subset B_\lambda} \max_{L_{n-k-1}} \int_{B_{K_{n-k}}} \Bigg| \sum_{\alpha_{n-k-1}:\mathrm{dist}(\alpha_{n-k-1},L_{n-k-1})\le 1/K_{n-k-1}} Ef_{\alpha_{n-k-1}} \Bigg|^p, \tag{A-11}$$

where k is the largest positive integer such that

$$k \le \frac{n-1}{3}; \tag{A-12}$$

in other words,

$$k = \begin{cases} (n-3)/3 & \text{if } n \equiv 0 \pmod 3, \\ (n-1)/3 & \text{if } n \equiv 1 \pmod 3, \\ (n-2)/3 & \text{if } n \equiv 2 \pmod 3. \end{cases} \tag{A-13}$$

Collecting all the constraints on the exponent p from applying the multilinear restriction estimate, we obtain

$$p \ge \frac{2(n-k+1)}{n-k}. \tag{A-14}$$

Instead of running the previous argument again on (A-11), we apply the decoupling inequalities of [Bourgain and Demeter 2015]:

$$\int_{B_{K_{n-k}}} \Bigg| \sum_{\alpha_{n-k-1}:\mathrm{dist}(\alpha_{n-k-1},L_{n-k-1})\le 1/K_{n-k-1}} Ef_{\alpha_{n-k-1}} \Bigg|^p \lesssim (K_{n-k-1})^{(n-k-1)(1/2-1/p)p+\epsilon} \sum_{\alpha_{n-k-1}} \int_{B_{K_{n-k}}} |Ef_{\alpha_{n-k-1}}|^p.$$

The above inequality will hold as long as

$$p \le \frac{2(n-k+1)}{n-k-1}. \tag{A-15}$$

In the end, we sum over all balls $B_{K_{n-k}}$ inside B_λ, and obtain

$$(K_{n-k-1})^{(n-k-1)(1/2-1/p)p+\epsilon} \sum_{\alpha_{n-k-1}} \int_{B_\lambda} |Ef_{\alpha_{n-k-1}}|^p.$$

It is clear now that we should apply parabolic rescaling. This gives us

$$(K_{n-k-1})^{(n-k-1)(1/2-1/p)p+\epsilon} \times (K_{n-k-1})^{2-2np(1/2-1/p)} \times Q^p_{\lambda/K^2_{n-k-1}}.$$

By equating the exponent of K_{n-k-1} with zero we obtain the constraint

$$p > \frac{2(n+k+3)}{n+k+1}. \tag{A-16}$$

This constraint is more restrictive than (A-14), by condition (A-12). By choosing k as in (A-13), and substituting that into (A-16), we obtain the constraint on p in (1-5). To summarize, we have shown that for p satisfying (1-5) and (A-15), we can close the induction. Thus, we have established (A-5) and therefore

(1-6) for such p's. Finally, by the result of [Rogers and Seeger 2010], we already know that (1-6) holds for all $p > 2+4/(n+1)$ (even with $\epsilon = 0$), which completes the proof of Theorem 1.6 for the claimed range in p.

Appendix B: A maximal multifrequency estimate of Krause and Lacey

Fix $\ell_0 \in \mathbb{Z}$ and $d \in \mathbb{N}$ with $d \geq 2$. Take the decomposition $1 = \sum_{\ell \in \mathbb{Z}} \varphi_\ell(t)$, where $\varphi_\ell(t) := \varphi_0(2^{-\ell} t)$ for some smooth even function φ_0 supported on $|t| \simeq 1$. For $u > 0$, let

$$T^{(u)} f(x) = \sum_{\ell \geq \ell_0} T_\ell^{(u)} f(x),$$

where

$$T_\ell^{(u)} f(x) := \int_{\mathbb{R}} f(x-t) \varphi_\ell(t) e^{iut^d} \frac{dt}{t}.$$

Following the method in [Bourgain 1989], we will prove the following maximal multifrequency estimate.

Theorem B.1. *There exists a constant C such that for any $\tau > 0$, $M \in \mathbb{N}$ and any $\theta_1, \dots, \theta_M \in \mathbb{R}$ with $\min_{1 \leq i < j \leq M} |\theta_i - \theta_j| > 2\tau$, we have*

$$\left\| \sup_{u>0} \left| \sum_{m=1}^{M} \mathrm{Mod}_{\theta_m} T^{(u)} P_\tau f_m \right| \right\|_{L^2(\mathbb{R})} \leq C (\log M)^2 \left(\sum_{m=1}^{M} \|f_m\|_{L^2(\mathbb{R})}^2 \right)^{1/2} \tag{B-1}$$

for all $f_1, \dots, f_M \in L^2(\mathbb{R})$, where P_τ is the Littlewood–Paley projection onto the frequency interval $[-\tau, \tau]$.

Here $\mathrm{Mod}_\theta f(x)$ is the modulation $\mathrm{Mod}_\theta f(x) := e^{i\theta x} f(x)$.

As a corollary, we obtain Theorem 3.5 of [Krause and Lacey 2017]:

Corollary B.2 [Krause and Lacey 2017]. *There exists a constant C such that for any $\tau > 0$, $M \in \mathbb{N}$ and any $\theta_1, \dots, \theta_M \in \mathbb{R}$ with $\min_{1 \leq i < j \leq M} |\theta_i - \theta_j| > 2\tau$, we have*

$$\left\| \sup_{u>0} \left| \sum_{m=1}^{M} \mathrm{Mod}_{\theta_m} T^{(u)} (P_\tau \mathrm{Mod}_{-\theta_m} f) \right| \right\|_{L^2(\mathbb{R})} \leq C (\log M)^2 \|f\|_{L^2(\mathbb{R})}$$

for any $f \in L^2(\mathbb{R})$.

Indeed, one can obtain the corollary by applying Theorem B.1 to $f_m := P_\tau \mathrm{Mod}_{-\theta_m} f$, and noting that then $\sum_{m=1}^M \|f_m\|_{L^2(\mathbb{R})}^2 \leq \|f\|_{L^2(\mathbb{R})}^2$.

The corollary is slightly stronger than Theorem 3.5 of [Krause and Lacey 2017] because it allows one to take supremum over all $u > 0$ (not just over $u \in (0, \tau^2)$).

To prove Theorem B.1, we use the following variant of our Theorem 1.1.

Theorem B.3. *There exists a constant C such that for all $p \in (2, 3)$ we have*

$$\|V^p \{T^{(u)} f : u > 0\}\|_{L^p(\mathbb{R})} \leq C (p-2)^{-1} \|f\|_{L^p(\mathbb{R})}. \tag{B-2}$$

Stein and Wainger proved that

$$\|\sup\{T^{(u)} f : u > 0\}\|_{L^q(\mathbb{R})} \leq C_q \|f\|_{L^q(\mathbb{R})} \tag{B-3}$$

for $1 < q < \infty$. By complex interpolation, we then get the following corollary:

Corollary B.4. *There exists a constant C such that for all $r \in (2,3)$ we have*

$$\|V^r\{T^{(u)}f : u > 0\}\|_{L^2(\mathbb{R})} \le C(r-2)^{-1}\|f\|_{L^2(\mathbb{R})}. \tag{B-4}$$

Indeed, for any $r \in (2,\infty)$, one can obtain (B-4) by interpolating between (B-2) with $p = (r+6)/4$, and (B-3) with $q = \frac{3}{2}$.

Below we first prove Theorem B.3, and then use Corollary B.4 to prove Theorem B.1.

Proof of Theorem B.3. By the argument in Section 3, we have

$$\sup_{\lambda>0} \|\lambda\sqrt{N_\lambda\{T^{(2^{kd})}f : k \in \mathbb{Z}\}}\|_{L^p(\mathbb{R})} \le C_p\|f\|_{L^p(\mathbb{R})}$$

for $1 < p < \infty$. By the real interpolation argument in Lemma 3.3 of [Bourgain 1989] (see also Lemma 2.1 of [Jones, Seeger, and Wright 2008]), we have

$$\|V^p\{T^{(2^{kd})}f : k \in \mathbb{Z}\}\|_{L^p(\mathbb{R})} \le C(p-2)^{-1}\|f\|_{L^p(\mathbb{R})} \tag{B-5}$$

for all $2 < p < 3$. Furthermore, by the argument in Section 4, we have

$$\|\|V_j^p T^{(u)} f\|_{\ell^p(j\in\mathbb{Z})}\|_{L^p(\mathbb{R})} \le C(p-2)^{-1}\|f\|_{L^p(\mathbb{R})} \tag{B-6}$$

for all $2 < p < 3$, where $V_j^p T^{(u)} f(x) := V^p\{T^{(u)}f(x) : u \in [2^{jd}, 2^{(j+1)d}]\}$. Indeed, the left-hand side above is bounded by

$$\sum_{k,\ell\in\mathbb{Z}} \|\|V_j^p T_{\ell-j}^{(u)} P_{j+k} f\|_{\ell^p\{j:\ell-j\ge\ell_0\}}\|_{L^p(\mathbb{R})},$$

and the arguments of Sections 4B and 4E show that

$$\sum_{\substack{k,\ell\in\mathbb{Z}\\ \ell\le -k/(2(d+1))}} \|\|V_j^p T_{\ell-j}^{(u)} P_{j+k} f\|_{\ell_j^p}\|_{L^p(\mathbb{R})} \le C_p\|f\|_{L^p(\mathbb{R})} \tag{B-7}$$

for $1 < p < \infty$, and

$$\sum_{\substack{k,\ell\in\mathbb{Z}\\ \ell>-k/(2(d+1)),\, k>\ell(d-1)+C}} + \sum_{\substack{k,\ell\in\mathbb{Z}\\ \ell>-k/(2(d+1)),\, k<\ell(d-1)-C}} \|\|V_j^p T_{\ell-j}^{(u)} P_{j+k} f\|_{\ell_j^p}\|_{L^p(\mathbb{R})} \le C_p\|f\|_{L^p(\mathbb{R})} \tag{B-8}$$

for $2 \le p < \infty$, where the constants C_p satisfy $\sup_{2\le p\le 3} C_p < \infty$. Furthermore, the arguments in Section 4D show that there exist absolute constants C and $\delta > 0$ such that if $\ell \ge 0$ and $k = \ell(d-1) + O(1)$, then

$$\|\|V_j^p T_{\ell-j}^{(u)} P_{j+k} f\|_{\ell_j^p}\|_{L^p(\mathbb{R})} \le C2^{-\ell\delta d(p-2)}\|f\|_{L^p(\mathbb{R})}$$

for all $2 < p < 3$. Summing these up, we get

$$\sum_{\ell\ge 0}\sum_{k=\ell(d-1)+O(1)} \|\|V_j^p T_{\ell-j}^{(u)} P_{j+k} f\|_{\ell_j^p}\|_{L^p(\mathbb{R})} \le C(p-2)^{-1}\|f\|_{L^p(\mathbb{R})} \tag{B-9}$$

for all $2 < p < 3$. Inequality (B-6) then follows from (B-7), (B-8) and (B-9). Since

$$V^p\{T^{(u)}f : u > 0\} \le V^p\{T^{(2^{kd})}f : k \in \mathbb{Z}\} + \|V_j^p T^{(u)} f\|_{\ell^p(j\in\mathbb{Z})},$$

we obtain the desired conclusion (B-2) from (B-5) and (B-6). □

Proof of Theorem B.1. We will deduce Theorem B.1 from Theorem B.3, following [Bourgain 1989] closely (see Lemma 4.13 there). Suppose $\tau > 0$, and $\theta_1, \dots, \theta_M \in \mathbb{R}$ are such that $\min_{1\le i<j\le M} |\theta_i - \theta_j| > 2\tau$. First, to prove (B-1), it will suffice to show that

$$\left(\fint_{w\in[0,1/(100\tau)]} \left\| \sup_{u>0} \left| \sum_{m=1}^{M} e^{i\theta_m x} T^{(u)} P_\tau f_m(x+w) \right| \right\|_{L^2(\mathbb{R})}^2 dw \right)^{1/2} \le C(\log M)^2 \left(\sum_{m=1}^{M} \|f_m\|_{L^2(\mathbb{R})}^2 \right)^{1/2}. \quad \text{(B-10)}$$

Morally speaking, this is the uncertainty principle at work: note that for every $u > 0$, the function $x \mapsto T^{(u)} P_\tau f_m(x)$ has Fourier support contained in $[-\tau, \tau]$, and hence $|\operatorname{Mod}_{\theta_m} T^{(u)} P_\tau f_m(x)|$ can be thought of as locally constant on an interval of length $\simeq 1/\tau$. To be precise, for each $w \in [0, 1/(100\tau)]$, we have, by Plancherel's identity, that

$$\|P_\tau f_m(\cdot) - P_\tau f_m(\cdot + w)\|_{L^2} \le \tfrac{1}{2} \|f_m\|_{L^2}$$

whenever $w \in [0, 1/(100\tau)]$. Thus if B is the best constant for which (B-1) holds, then for all $w \in [0, 1/(100\tau)]$, we have

$$\left\| \sup_{u>0} \left| \sum_{m=1}^{M} \operatorname{Mod}_{\theta_m} T^{(u)} P_\tau f_m(x) \right| \right\|_{L^2(\mathbb{R})} \le \left\| \sup_{u>0} \left| \sum_{m=1}^{M} e^{i\theta_m x} T^{(u)} P_\tau f_m(x+w) \right| \right\|_{L^2(\mathbb{R})} + \frac{B}{2} \left(\sum_{m=1}^{M} \|f_m\|_{L^2(\mathbb{R})}^2 \right)^{1/2},$$

so taking L^2 average over all $w \in [0, 1/(100\tau)]$, and using (B-10), we have

$$B \le C(\log M)^2 + \frac{B}{2};$$

i.e., $B \le 2C(\log M)^2$ as desired. Thus, it remains to establish (B-10), which can be rewritten as

$$\left\| \left(\fint_{w\in[0,1/(100\tau)]} \sup_{u>0} \left| \sum_{m=1}^{M} e^{-i\theta_m w} e^{i\theta_m x} T^{(u)} P_\tau f_m(x) \right|^2 dw \right)^{1/2} \right\|_{L^2(\mathbb{R})} \le C(\log M)^2 \left(\sum_{m=1}^{M} \|f_m\|_{L^2(\mathbb{R})}^2 \right)^{1/2} \quad \text{(B-11)}$$

by first changing variable $x \mapsto x - w$, and then interchanging the integrals in x and w.

To prove (B-11), for each $x \in \mathbb{R}$, consider the (bounded) set $A_x \subset \mathbb{R}^M$, given by

$$A_x := \{(T^{(u)} P_\tau f_1(x), \dots, T^{(u)} P_\tau f_M(x)) : u > 0\}.$$

If λ is bigger than the diameter of A_x, let $E_\lambda(x) = 0$; otherwise let $E_\lambda(x)$ be the minimal number of balls in $\mathbb{R}^M$ of radius λ that is required to cover A_x. ($E_\lambda(x)$ is sometimes called the entropy number.) One then observes that for every $s \in \mathbb{Z}$, there exists a finite subset $B_s(x) \subset A_x - A_x$ of cardinality at most $E_{2^s}(x)$ such that

$$|b_s| \le 2^{s+1} \quad \text{for every } b_s \in B_s(x)$$

and such that every element a of A_x admits a decomposition

$$a = \sum_{s\in\mathbb{Z}} b_s \quad \text{with } b_s \in B_s(x) \text{ for every } s \in \mathbb{Z}.$$

Then the left-hand side of (B-11) is bounded by

$$\Bigl\| \sum_{s\in\mathbb{Z}} \max_{b_s\in B_s(x)} \Bigl(\fint_{w\in[0,1/(100\tau)]} \Bigl| \sum_{m=1}^{M} e^{-i\theta_m w} e^{i\theta_m x} b_{s,m} \Bigr|^2 dw \Bigr)^{1/2} \Bigr\|_{L^2(\mathbb{R})};$$

here $b_s = (b_{s,1}, \dots, b_{s,m})$. Using Cauchy–Schwarz for the sum over m, the above displayed equation is further bounded by

$$\Bigl\| \sum_{s\in\mathbb{Z}} \min\Bigl\{ 2^{s+1} M^{1/2}, \Bigl(\sum_{b_s\in B_s(x)} \fint_{w\in[0,1/(100\tau)]} \Bigl| \sum_{m=1}^{M} e^{-i\theta_m w} e^{i\theta_m x} b_{s,m} \Bigr|^2 dw \Bigr)^{1/2} \Bigr\} \Bigr\|_{L^2(\mathbb{R})}. \tag{B-12}$$

To estimate the integral in w above, observe that from the separation of the $\theta_1, \dots, \theta_M$, we have

$$\Bigl(\fint_{w\in[0,1/(100\tau)]} \Bigl| \sum_{m=1}^{M} e^{-i\theta_m w} c_m \Bigr|^2 dw \Bigr)^{1/2} \lesssim \Bigl(\sum_{m=1}^{M} |c_m|^2 \Bigr)^{1/2};$$

indeed, the key is that if $\mathfrak{h} : \mathbb{R}^M \to \mathbb{R}^M$ is defined by

$$(\mathfrak{h}a)_m := \sum_{m=1}^{M} \frac{\tau}{\theta_m - \theta_n} a_n,$$

then the operator norm of $\mathfrak{h}$ is bounded independent of M, which can be deduced, for instance, by comparing it to the (continuous) Hilbert transform on $\mathbb{R}$. Thus (B-12) is bounded by

$$\Bigl\| \sum_{s\in\mathbb{Z}} \min\{2^{s+1} M^{1/2}, 2^{s+1} E_{2^s}(x)^{1/2}\} \Bigr\|_{L^2(\mathbb{R})} = 2 \Bigl\| \sum_{s\in\mathbb{Z}} 2^s \min\{M^{1/2}, E_{2^s}(x)^{1/2}\} \Bigr\|_{L^2(\mathbb{R})}. \tag{B-13}$$

Now let

$$F(x) := \Bigl(\sum_{m=1}^{M} \sup_{u>0} |T^{(u)} P_\tau f_m(x)|^2 \Bigr)^{1/2}.$$

Then the diameter of A_x is at most $2F(x)$. Hence the entropy number satisfies $E_{2^s}(x) = 0$ whenever $2^s > 2F(x)$. We are then led to sum

$$\sum_{2^s \le 2F(x)} 2^s \min\{M^{1/2}, E_{2^s}(x)^{1/2}\}.$$

We split this sum into two, one where $2^s \le M^{-1/2} F(x)$, and another where $M^{-1/2} F(x) \le 2^s \le 2F(x)$. The former sum is bounded by $F(x)$, while the latter sum is bounded by $(\log M) M^{1/2-1/r} \sup_{s\in\mathbb{Z}} 2^s E_{2^s}(x)^{1/r}$ for any $r \in (2, \infty)$. Now pick $r \in (2, \infty)$ such that

$$\frac{1}{2} - \frac{1}{r} = (\log M)^{-1},$$

so that $M^{1/2-1/r} \simeq 1$. Then (B-13) is bounded by

$$\Bigl\| F(x) + (\log M) \sup_{s\in\mathbb{Z}} 2^s E_{2^s}(x)^{1/r} \Bigr\|_{L^2(\mathbb{R})}.$$

But by Stein and Wainger's inequality (B-3), we have

$$\|F\|_{L^2(\mathbb{R})} \lesssim \left(\sum_{m=1}^{M} \|f_m\|_{L^2(\mathbb{R})}^2\right)^{1/2}.$$

Furthermore, one can relate the entropy $E_{2^s}(x)^{1/r}$ with the r-th variation norm pointwise:

$$\sup_{s\in\mathbb{Z}} 2^s E_{2^s}(x)^{1/r} \le \left(\sum_{m=1}^{M} |V^r\{T^{(u)} P_\tau f_m(x) : u > 0\}|^2\right)^{1/2}.$$

Hence

$$\Big\|(\log M) \sup_{s\in\mathbb{Z}} 2^s E_{2^s}(x)^{1/r}\Big\|_{L^2(\mathbb{R})} \le (\log M)\left(\sum_{m=1}^{M} \|V^r\{T^{(u)} P_\tau f_m : u > 0\}\|_{L^2(\mathbb{R})}^2\right)^{1/2}.$$

By Corollary B.4, the latter is bounded by

$$C(\log M)(r-2)^{-1}\left(\sum_{m=1}^{M} \|f_m\|_{L^2(\mathbb{R})}^2\right)^{1/2},$$

and since $(r-2)^{-1} \simeq \log M$ by our choice of r, this completes the proof of Theorem B.1. □

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FEDERER'S CHARACTERIZATION OF SETS OF FINITE PERIMETER IN METRIC SPACES

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Federer's characterization of sets of finite perimeter states (in Euclidean spaces) that a set is of finite perimeter if and only if the measure-theoretic boundary of the set has finite Hausdorff measure of codimension 1. In complete metric spaces that are equipped with a doubling measure and support a Poincaré inequality, the "only if" direction was shown by Ambrosio (2002). By applying fine potential theory in the case $p = 1$, we prove that the "if" direction holds as well.

1. Introduction

In the past two decades, there has been great interest in studying problems of first-order analysis in the setting of general metric measure spaces; see, e.g., [Ambrosio 2002; Ambrosio et al. 2004; Björn and Björn 2011; Heinonen and Koskela 1998; Miranda 2003; Shanmugalingam 2000]. In particular, Sobolev functions (sometimes called Newton–Sobolev functions in the metric setting) and functions of bounded variation (BV functions) have been topics of central interest. In much of the literature (as well as in the current paper) one assumes that the space is complete, equipped with a doubling measure, and supports a Poincaré inequality; see Section 2 for definitions. Studying questions in such an abstract setting provides an opportunity to unify the theories developed in specific settings such as weighted Euclidean spaces, Riemannian manifolds, Carnot groups, etc. Moreover, without having the Euclidean structure available, one is forced to develop novel methods and proofs, giving new insight into various problems.

In the theory of BV functions in the Euclidean setting, a key result originally due to De Giorgi states that if E is a set of finite perimeter, then the perimeter measure $P(E,\cdot)$ coincides with the $(n-1)$-dimensional Hausdorff measure restricted to the measure-theoretic boundary ∂^*E. In particular, $P(E,\mathbb{R}^n)<\infty$ implies $\mathcal{H}^{n-1}(\partial^*E)<\infty$. By a deep result of [Federer 1969, Section 4.5.11], the converse holds as well, so in fact $P(E,\mathbb{R}^n)<\infty$ if and only if $\mathcal{H}^{n-1}(\partial^*E)<\infty$. This is known as Federer's characterization of sets of finite perimeter. In the metric setting, where it is natural to formulate this kind of result by means of the *codimension-1* Hausdorff measure $\mathcal{H}$, the "only if" direction of the characterization was shown in [Ambrosio 2002], but the "if" direction has remained an open problem. In the current paper, we show that this direction holds as well.

Theorem 1.1. *Let $\Omega\subset X$ be an open set, let $E\subset X$ be a μ-measurable set, and suppose $\mathcal{H}(\partial^*E\cap\Omega)<\infty$. Then $P(E,\Omega)<\infty$.*

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The “only if” direction of Federer’s characterization is part of a more general structure theorem for sets of finite perimeter, which in the metric setting states that the perimeter measure is comparable to the Hausdorff measure of codimension 1 restricted to the measure-theoretic boundary. This structure theorem is an indispensable tool in analysis of sets of finite perimeter, and hence more general BV functions as well. While not equally essential, the “if” direction of Federer’s characterization has a number of applications as well. For example, in [Kinnunen et al. 2012] the authors proved a characterization of Newton–Sobolev functions with zero boundary values by means of a natural Lebesgue point-type condition on the boundary. However, the proof relied on assuming that Federer’s characterization holds; now we know that this is the case under the usual assumptions on the space. We will discuss other applications in Section 5.

Previously there have been some partial results toward a proof of the “if” direction. The paper [Korte et al. 2015] showed that if the metric space is assumed to contain a “thick” bundle of curves between each pair of points, then the “if” direction can be proved by mimicking the Euclidean proof. In the current paper we take a completely different approach, which relies on *fine potential theory*. In the case $1 < p < \infty$, fine potential theory deals with superharmonic functions as understood by means of the *fine topology*; see [Adams and Hedberg 1996; Heinonen et al. 1993; Malý and Ziemer 1997] for the theory and its history in the Euclidean setting, and the recent papers [Björn and Björn 2015; Björn et al. 2015; 2016; 2018] for similar results in the metric setting. In [Lahti 2017a], the author proved some analogous results in the case $p = 1$, by relying on certain continuity properties of BV functions proved earlier in [Lahti 2017b; Lahti and Shanmugalingam 2017]. An application of these results led to the following characterization of sets of finite perimeter, which is in the same vein as Federer’s characterization. Below, $\partial^1 I_E$ denotes the fine boundary of E, or more precisely of its measure-theoretic interior; one always has $\partial^* E \subset \partial^1 I_E$.

Theorem 1.2 [Lahti 2017a, Theorem 1.1]. *For an open set $\Omega \subset X$ and a μ-measurable set $E \subset X$, we have $P(E, \Omega) < \infty$ if and only if $\mathcal{H}(\partial^1 I_E \cap \Omega) < \infty$. Furthermore, $\mathcal{H}((\partial^1 I_E \setminus \partial^* E) \cap \Omega) = 0$.*

In the current paper, our main goal is to show that if $\mathcal{H}(\partial^* E \cap \Omega) < \infty$, then $\mathcal{H}((\partial^1 I_E \setminus \partial^* E) \cap \Omega) = 0$ and thus Theorem 1.1 follows from Theorem 1.2. The proofs will be given in Section 4, and they rely mostly on properties of the 1-fine topology proved in [Lahti 2017a; 2020], as well as boxing inequality-type arguments. Our methods and the underlying theory should be of interest already in Euclidean spaces, where Federer’s original argument has remained (as far as we know) essentially the only known proof for the characterization.

2. Preliminaries

In this section we introduce the standard definitions, notation, and assumptions used in the paper.

Throughout this paper, (X, d, μ) is a complete metric space that is equipped with a metric d and a Borel regular outer measure μ satisfying a doubling property, meaning that there exists a constant $C_d \geq 1$ such that

$$0 < \mu(B(x, 2r)) \leq C_d \mu(B(x, r)) < \infty$$

for every ball $B(x,r) := \{y \in X : d(y,x) < r\}$. We assume that X consists of at least two points. Given a ball $B = B(x,r)$ and $\beta > 0$, we sometimes abbreviate $\beta B := B(x, \beta r)$. Note that in metric spaces, a ball (as a set) does not necessarily have a unique center point and radius, but we understand these to be prescribed for all balls that we consider. When we want to state that a constant C depends on the parameters $a, b, \dots$, we write $C = C(a, b, \dots)$.

All functions defined on X or its subsets will take values in $[-\infty, \infty]$. A complete metric space equipped with a doubling measure is proper; that is, closed and bounded sets are compact. For any open set $\Omega \subset X$, we define $\mathrm{Lip}_{\mathrm{loc}}(\Omega)$ as the set of functions that are in the class $\mathrm{Lip}(\Omega')$ for every open $\Omega' \Subset \Omega$; here $\Omega' \Subset \Omega$ means that $\overline{\Omega}'$ is a compact subset of Ω. Other local function spaces are defined analogously.

For any set $A \subset X$ and $0 < R < \infty$, the restricted spherical Hausdorff content of codimension 1 is defined as

$$\mathcal{H}_R(A) := \inf\left\{ \sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} : A \subset \bigcup_{i=1}^{\infty} B(x_i, r_i),\ r_i \le R \right\}.$$

The codimension-1 Hausdorff measure of $A \subset X$ is then defined as

$$\mathcal{H}(A) := \lim_{R \to 0} \mathcal{H}_R(A).$$

In the Euclidean space $\mathbb{R}^n$ (which we always equip with the Euclidean metric and the Lebesgue measure $\mathcal{L}^n$, unless otherwise specified) this is comparable to the $(n-1)$-dimensional Hausdorff measure.

By a curve we mean a nonconstant rectifiable continuous mapping from a compact interval of the real line into X. A nonnegative Borel function g on X is an upper gradient of a function u on X if for all curves γ, we have

$$|u(x) - u(y)| \le \int_\gamma g\, ds, \tag{2.1}$$

where x and y are the endpoints of γ and the curve integral is defined by using an arc-length parametrization; see [Heinonen and Koskela 1998, Section 2], where upper gradients were originally introduced. We interpret $|u(x) - u(y)| = \infty$ whenever at least one of $|u(x)|$, $|u(y)|$ is infinite. An upper gradient of a function $u \in C^1(\mathbb{R}^n)$ is given by $|\nabla u|$.

Let $1 \le p < \infty$ (we will work almost exclusively with $p = 1$). The p-modulus of a family of curves Γ is defined as

$$\mathrm{Mod}_p(\Gamma) := \inf \int_X \rho^p \, d\mu,$$

where the infimum is taken over all admissible test functions ρ, which are nonnegative Borel functions such that $\int_\gamma \rho\, ds \ge 1$ for every $\gamma \in \Gamma$. A property is said to hold for p-almost every curve if it fails only for a curve family with zero p-modulus. If g is a nonnegative μ-measurable function on X and (2.1) holds for p-almost every curve, we say that g is a p-weak upper gradient of u. By only considering curves γ in a set $A \subset X$, we can talk about a function g being a (p-weak) upper gradient of u in A.

Given an open set $\Omega \subset X$, we let

$$\|u\|_{N^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \inf \|g\|_{L^p(\Omega)},$$

where the infimum is taken over all p-weak upper gradients g of u in Ω. The Newton–Sobolev space is defined as

$$N^{1,p}(\Omega) := \{u : \|u\|_{N^{1,p}(\Omega)} < \infty\}.$$

In $\mathbb{R}^n$ this coincides, up to a choice of pointwise representatives, with the usual Sobolev space $W^{1,p}(\Omega)$; this is shown in Theorem 4.5 of [Shanmugalingam 2000], where the Newton–Sobolev space was originally introduced.

We understand a Newton–Sobolev function to be defined at every $x \in \Omega$ (even though $\|\cdot\|_{N^{1,p}(\Omega)}$ is then only a seminorm). It is known that for any $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ there exists a minimal p-weak upper gradient of u in Ω, always denoted by g_u, satisfying $g_u \le g$ almost everywhere in Ω, for any p-weak upper gradient $g \in L^p_{\mathrm{loc}}(\Omega)$ of u in Ω; see [Björn and Björn 2011, Theorem 2.25]. In $\mathbb{R}^n$, the minimal p-weak upper gradient coincides (a.e.) with $|\nabla u|$; see [Björn and Björn 2011, Corollary A.4].

The space of Newton–Sobolev functions with zero boundary values is defined as

$$N^{1,p}_0(\Omega) := \{u|_\Omega : u \in N^{1,p}(X) \text{ and } u = 0 \text{ on } X \setminus \Omega\}.$$

This class can be understood to be a subclass of $N^{1,p}(X)$ in a natural way.

The p-capacity of a set $A \subset X$ is defined as

$$\mathrm{Cap}_p(A) := \inf \|u\|_{N^{1,p}(X)},$$

where the infimum is taken over all functions $u \in N^{1,p}(X)$ such that $u \ge 1$ in A.

The variational 1-capacity of a set $A \subset \Omega$ with respect to an open set $\Omega \subset X$ is defined as

$$\mathrm{cap}_1(A, \Omega) := \inf \int_X g_u \, d\mu,$$

where the infimum is taken over functions $u \in N^{1,1}_0(\Omega)$ such that $u \ge 1$ on A, and g_u is the minimal 1-weak upper gradient of u (in X). For basic properties satisfied by capacities, such as monotonicity and countable subadditivity, see [Björn and Björn 2011].

We will assume throughout the paper that X supports a $(1,1)$-Poincaré inequality, meaning that there exist constants $C_P > 0$ and $\lambda \ge 1$ such that for every ball $B(x,r)$, every $u \in L^1_{\mathrm{loc}}(X)$, and every upper gradient g of u, we have

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \le C_P r \fint_{B(x,\lambda r)} g \, d\mu,$$

where

$$u_{B(x,r)} := \fint_{B(x,r)} u \, d\mu := \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u \, d\mu.$$

The standard example of a complete metric space equipped with a doubling measure and supporting a $(1,1)$-Poincaré inequality is the (unweighted) Euclidean space. Other examples include certain weighted Euclidean spaces (see, e.g., [Heinonen et al. 1993, Section 15]), complete Riemannian manifolds with nonnegative Ricci curvature (see [Saloff-Coste 2002, Section 3.3.5]), as well as Carnot groups which we will discuss briefly in Section 5.

Next we recall the definition and basic properties of functions of bounded variation on metric spaces, following [Miranda 2003]. See also, e.g., [Ambrosio et al. 2000; Evans and Gariepy 1992; Federer 1969; Giusti 1984; Ziemer 1989] for the classical theory in the Euclidean setting. Let $\Omega \subset X$ be an open set. Given a function $u \in L^1_{\mathrm{loc}}(\Omega)$, we define the total variation of u in Ω as

$$\|Du\|(\Omega) := \inf\left\{\liminf_{i\to\infty} \int_\Omega g_{u_i}\, d\mu : u_i \in N^{1,1}_{\mathrm{loc}}(\Omega),\ u_i \to u \text{ in } L^1_{\mathrm{loc}}(\Omega)\right\},$$

where each g_{u_i} is the minimal 1-weak upper gradient of u_i in Ω. In $\mathbb{R}^n$ this agrees with the usual Euclidean definition involving distributional derivatives; see, e.g., [Ambrosio et al. 2000, Proposition 3.6, Theorem 3.9]. (In [Miranda 2003], local Lipschitz constants were used instead of upper gradients, but the properties of the total variation can be proved similarly with either definition.) We say that a function $u \in L^1(\Omega)$ is of bounded variation, and write $u \in \mathrm{BV}(\Omega)$ if $\|Du\|(\Omega) < \infty$. For an arbitrary set $A \subset X$, we define

$$\|Du\|(A) := \inf\{\|Du\|(W) : A \subset W,\ W \subset X \text{ is open}\}.$$

If $u \in L^1_{\mathrm{loc}}(\Omega)$ and $\|Du\|(\Omega) < \infty$, then $\|Du\|(\cdot)$ is a Radon measure on Ω by [Miranda 2003, Theorem 3.4]. A μ-measurable set $E \subset X$ is said to be of finite perimeter if $\|D\chi_E\|(X) < \infty$, where χ_E is the characteristic function of E. The perimeter of E in Ω is also denoted by

$$P(E,\Omega) := \|D\chi_E\|(\Omega).$$

Sets of finite perimeter include for example bounded domains with a Lipschitz boundary in $\mathbb{R}^n$, and in this case the perimeter is simply the $(n-1)$-dimensional Hausdorff measure of the boundary. However, sets of finite perimeter can also be highly irregular, as demonstrated by Example 2.6 below.

Applying the Poincaré inequality to sequences of approximating locally Lipschitz functions in the definition of the total variation, we get the following BV version: for every ball $B(x,r)$ and every $u \in L^1_{\mathrm{loc}}(X)$, we have

$$\fint_{B(x,r)} |u - u_{B(x,r)}|\, d\mu \le C_P r\, \frac{\|Du\|(B(x,\lambda r))}{\mu(B(x,\lambda r))}.$$

For a μ-measurable set $E \subset X$, this implies the relative isoperimetric inequality

$$\min\{\mu(B(x,r)\cap E),\ \mu(B(x,r)\setminus E)\} \le 2C_P r\, P(E, B(x,\lambda r)); \tag{2.2}$$

see, e.g., [Korte and Lahti 2014, equation (3.1)].

The measure-theoretic interior of a set $E \subset X$ is defined as

$$I_E := \left\{x \in X : \lim_{r\to 0} \frac{\mu(B(x,r)\setminus E)}{\mu(B(x,r))} = 0\right\},$$

and the measure-theoretic exterior as

$$O_E := \left\{x \in X : \lim_{r\to 0} \frac{\mu(B(x,r)\cap E)}{\mu(B(x,r))} = 0\right\}.$$

The measure-theoretic boundary $\partial^* E$ is defined as the set of points $x \in X$ at which both E and its complement have strictly positive upper density; i.e.,

$$\limsup_{r\to 0} \frac{\mu(B(x,r)\cap E)}{\mu(B(x,r))} > 0 \quad \text{and} \quad \limsup_{r\to 0} \frac{\mu(B(x,r)\setminus E)}{\mu(B(x,r))} > 0. \tag{2.3}$$

Then $X = I_E \cup O_E \cup \partial^* E$.

For an open set $\Omega \subset X$ and a μ-measurable set $E \subset X$ with $P(E,\Omega) < \infty$, we know that for any Borel set $A \subset \Omega$,

$$P(E, A) = \int_{\partial^* E\cap A} \theta_E \, d\mathcal{H}, \tag{2.4}$$

where $\theta_E : \partial^* E \to [\alpha, C_d]$, with $\alpha = \alpha(C_d, C_P, \lambda) > 0$; see [Ambrosio 2002, Theorem 5.3] and [Ambrosio et al. 2004, Theorem 4.6].

If $\Omega \subset X$ is an open set and $u, v \in L^1_{\mathrm{loc}}(\Omega)$, then

$$\|D\min\{u,v\}\|(\Omega) + \|D\max\{u,v\}\|(\Omega) \le \|Du\|(\Omega) + \|Dv\|(\Omega); \tag{2.5}$$

for a proof see, e.g., [Lahti 2018, Lemma 3.1].

The measure-theoretic boundary $\partial^* E$ is always a subset of the topological boundary ∂E but the boundaries can be quite different, as shown by the following example of the so-called enlarged rationals.

Example 2.6. Let $X = \mathbb{R}^2$ (unweighted). Let $\{q_j\}_{j=1}^\infty$ be an enumeration of $\mathbb{Q}^2$. Define

$$E := \bigcup_{j=1}^{\infty} B(q_j, 2^{-j}).$$

Then by (2.5) and the fact that the perimeter is clearly lower semicontinuous with respect to convergence in $L^1(\mathbb{R}^2)$, we get

$$P(E,\mathbb{R}^2) \le \sum_{j=1}^{\infty} P(B(q_j, 2^{-j}), \mathbb{R}^2) = 2\pi \sum_{j=1}^{\infty} 2^{-j} = 2\pi.$$

Thus E is of finite perimeter, and so by (2.4) we get $\mathcal{H}^1(\partial^* E) < \infty$, where $\mathcal{H}^1$ is the 1-dimensional Hausdorff measure. On the other hand, E is dense in $\mathbb{R}^2$, so that $\partial E = \mathbb{R}^2 \setminus E$ and thus $\mathcal{L}^2(\partial E) = \infty$. This illustrates that the measure-theoretic boundary is the natural boundary to consider.

The lower and upper approximate limits of a function u on X are defined respectively by

$$u^\wedge(x) := \sup\left\{ t \in \mathbb{R} : \lim_{r\to 0} \frac{\mu(B(x,r)\cap\{u<t\})}{\mu(B(x,r))} = 0 \right\}$$

and

$$u^\vee(x) := \inf\left\{ t \in \mathbb{R} : \lim_{r\to 0} \frac{\mu(B(x,r)\cap\{u>t\})}{\mu(B(x,r))} = 0 \right\}.$$

Unlike Newton–Sobolev functions, we understand BV functions to be μ-equivalence classes. To consider fine properties, we need to consider the pointwise representatives $u^\wedge$ and $u^\vee$. We note that for $u = \chi_E$ with $E \subset X$, we have $x \in I_E$ if and only if $u^\wedge(x) = u^\vee(x) = 1$, $x \in O_E$ if and only if $u^\wedge(x) = u^\vee(x) = 0$, and $x \in \partial^* E$ if and only if $u^\wedge(x) = 0$ and $u^\vee(x) = 1$.

Throughout this paper we assume that (X, d, μ) *is a complete metric space that is equipped with the doubling measure* μ *and supports a* $(1, 1)$*-Poincaré inequality.*

3. The 1-fine topology

In this section we have gathered all the results concerning the 1-fine topology that our argument will rely on. For these, we refer to [Lahti 2017a; 2017b; 2020]. Most of the results are analogous to those that hold in the case $1 < p < \infty$, which has been studied in the metric setting in [Björn and Björn 2015; Björn et al. 2018; 2015].

Definition 3.1. We say that $A \subset X$ is 1-thin at the point $x \in X$ if

$$\lim_{r\to 0} r \frac{\operatorname{cap}_1(A \cap B(x, r), B(x, 2r))}{\mu(B(x, r))} = 0.$$

We also say that a set $U \subset X$ is 1-finely open if $X \setminus U$ is 1-thin at every $x \in U$. Then we define the 1-fine topology as the collection of 1-finely open sets on X.

We denote the 1-fine interior of a set $H \subset X$, i.e., the largest 1-finely open set contained in H, by fine-int H. We denote the 1-fine closure of a set $H \subset X$, i.e., the smallest 1-finely closed set containing H, by $\overline{H}^1$. The 1-fine boundary of H is $\partial^1 H := \overline{H}^1 \setminus \text{fine-int}\, H$. Finally, the 1-base $b_1 H$ is defined as the set of points where H is *not* 1-thin.

See [Lahti 2017b, Section 4] for discussion on this definition, and for a proof of the fact that the 1-fine topology is indeed a topology. By [Björn and Björn 2011, Proposition 6.16], a set $A \subset X$ is 1-thin at $x \in X$ if and only if

$$\lim_{r\to 0} \frac{\operatorname{cap}_1(A \cap B(x, r), B(x, 2r))}{\operatorname{cap}_1(B(x, r), B(x, 2r))} = 0,$$

and so it is clear that $W \subset b_1 W$ for any open set $W \subset X$.

Now we collect some facts concerning the 1-fine topology proved in [Lahti 2017a]. According to Corollary 3.5 of that work, the 1-fine closure of $A \subset X$ can be characterized in the following way:

$$\overline{A}^1 = A \cup b_1 A. \tag{3.2}$$

From this it easily follows that for any $A \subset X$ and any ball $B(x, r)$, we have $\overline{A}^1 \cap B(x, r) \subset \overline{A \cap B(x, r)}^1$, and then by [Lahti 2017a, Proposition 3.3] we get

$$\operatorname{cap}_1(\overline{A}^1 \cap B(x, r), B(x, 2r)) = \operatorname{cap}_1(A \cap B(x, r), B(x, 2r)). \tag{3.3}$$

By Lemma 4.6 of the same work the 1-fine boundary of a measure-theoretic interior can be characterized as follows: for any μ-measurable set $E \subset X$,

$$\partial^1 I_E = b_1 I_E \cap b_1(X \setminus I_E). \tag{3.4}$$

By [Lahti 2017a, Lemma 3.1] we know that for any μ-measurable set $E \subset X$,

$$\partial^* E \subset \partial^1 I_E. \tag{3.5}$$

Conversely, if $\Omega \subset X$ is open and $E \subset X$ is μ-measurable such that $P(E, \Omega) < \infty$, then by Theorem 1.2,

$$\mathcal{H}((\partial^1 I_E \setminus \partial^* E) \cap \Omega) = 0.$$

Combining this with (2.4) gives

$$\alpha \mathcal{H}(\partial^1 I_E \cap \Omega) \le P(E, \Omega) \le C_d \mathcal{H}(\partial^1 I_E \cap \Omega). \tag{3.6}$$

In fact this holds for every μ-measurable $E \subset X$; to see this we can assume that $\mathcal{H}(\partial^1 I_E \cap \Omega) < \infty$, and then $P(E, \Omega) < \infty$ by Theorem 1.2.

We also have the following version of the relative isoperimetric inequality: for every ball $B(x, r)$ and every μ-measurable $E \subset X$,

$$\min\{\mu(B(x, r) \cap E),\ \mu(B(x, r) \setminus E)\} \le 2C_P C_d r \mathcal{H}(\partial^1 I_E \cap B(x, \lambda r)); \tag{3.7}$$

this follows from the ordinary relative isoperimetric inequality (2.2) and (3.6).

Remark 3.8. It may seem strange to talk about $\partial^1 I_E$, as it seems that we are first taking the interior in one topology and then the boundary in another. However, if we define the measure topology more axiomatically, then I_E is actually *not* the interior of E in the measure topology, and should be seen as a measure-theoretic quantity rather than a topological one; see [Lahti 2017a, Remark 4.9]. Moreover, $\partial^* E$ is actually the boundary of I_E in the measure topology; let us denote it by $\partial^0 I_E$. Thus $\partial^1 I_E$ is a natural set to consider as well. Finally, we can note that $\partial^1 I_E = \partial^1 O_E$; see [Lahti 2017a, Lemma 4.8].

The following *weak Cartan property* in the case $p = 1$ was proved in [Lahti 2020, Theorem 5.2].

Theorem 3.9. *Let $A \subset X$ and let $x \in X \setminus A$ be such that A is 1-thin at x. Then there exist $R > 0$ and $E_0, E_1 \subset X$ such that $\chi_{E_0}, \chi_{E_1} \in \mathrm{BV}(X)$, $\max\{\chi_{E_0}^{\wedge}, \chi_{E_1}^{\wedge}\} = 1$ in $A \cap B(x, R)$, $\chi_{E_0}^{\vee}(x) = 0 = \chi_{E_1}^{\vee}(x)$, $\{\max\{\chi_{E_0}^{\vee}, \chi_{E_1}^{\vee}\} > 0\}$ is 1-thin at x, and*

$$\lim_{r\to 0} r \frac{P(E_0, B(x, r))}{\mu(B(x, r))} = 0, \quad \lim_{r\to 0} r \frac{P(E_1, B(x, r))}{\mu(B(x, r))} = 0. \tag{3.10}$$

The following simpler formulation will be sufficient for our purposes.

Corollary 3.11. *Let $A \subset X$ and let $x \in X \setminus A$ be such that A is 1-thin at x. Then there exist $R > 0$ and $F \subset X$ such that $\chi_F \in \mathrm{BV}(X)$, $A \cap B(x, R) \subset I_F$, I_F is 1-thin at x, and*

$$\lim_{r\to 0} r \frac{P(F, B(x, r))}{\mu(B(x, r))} = 0. \tag{3.12}$$

Proof. Take $E_0, E_1 \subset X$ as given by Theorem 3.9, and set $F := E_0 \cup E_1$. By (2.5) we obtain $\chi_F \in \mathrm{BV}(X)$, and (2.5) and (3.10) together give (3.12). From the fact that $\max\{\chi_{E_0}^{\wedge}, \chi_{E_1}^{\wedge}\} = 1$ in $A \cap B(x, R)$ we obtain

$$A \cap B(x, R) \subset I_{E_0} \cup I_{E_1} \subset I_F.$$

Finally, since $\{\max\{\chi_{E_0}^\vee, \chi_{E_1}^\vee\} > 0\}$ is 1-thin at x, so is

$$I_{E_0} \cup I_{E_1} \cup \partial^* E_0 \cup \partial^* E_1 \supset I_F. \qquad \square$$

In [Lahti 2020, Lemma 4.4] it was also shown that if $A \subset X$ is 1-thin at a point $x \in X \setminus A$, then there exists an open set that contains A and is also 1-thin at x; that is:

$$\text{If } x \notin A \cup b_1 A, \text{ then there exists an open } W \supset A \text{ such that } x \notin b_1 W. \tag{3.13}$$

4. Proof of the characterization

In [Korte and Lahti 2014, Theorem 3.11] it was shown that for any μ-measurable set $E \subset X$, we have $\overline{\partial^* E} = \partial I_E$; that is, the closure of the measure-theoretic boundary (in the metric topology) is the whole topological boundary of a suitable representative of E (namely the measure-theoretic interior I_E). Now we prove the analogous result with the metric topology replaced by the 1-fine topology. This will be the crux of our proof of Federer's characterization.

Theorem 4.1. *For any μ-measurable set $E \subset X$, we have $\overline{\partial^* E}^1 = \partial^1 I_E$.*

Note that by Remark 3.8, the above can be written as $\overline{\partial^0 I_E}^1 = \partial^1 I_E$, showing that the result describes the interplay between the measure topology and the 1-fine topology. It is natural to ask which other sets and topologies would satisfy an analogous property, but we will not pursue this problem here. Previously, properties of the measure topology and fine topologies have been studied in [Lukeš et al. 1986].

Proof. By (3.5) we have

$$\overline{\partial^* E}^1 \subset \overline{\partial^1 I_E}^1 = \partial^1 I_E,$$

where the last equality follows from the fact that boundaries are closed sets in every topology. Thus we only need to show that $\overline{\partial^* E}^1 \supset \partial^1 I_E$. Let $x_0 \in \partial^1 I_E$ and let $U \ni x_0$ be a 1-finely open set. We need to show that $\partial^* E \cap U \neq \varnothing$. By (3.13) there exists an open set $W \supset X \setminus U$ that is 1-thin at x_0. Since $\overline{W}^1 = W \cup b_1 W = b_1 W$ by (3.2), we have $x_0 \notin \overline{W}^1$. We will show that $\partial^* E \setminus W \neq \varnothing$; suppose that instead $\partial^* E \setminus W = \varnothing$.

Claim. *Let $x \in \partial^1 I_E \setminus \overline{W}^1$ and $s_1 > 0$. Then there exists $y \in B(x, s_1) \cap \partial^1 I_E \setminus \overline{W}^1$ and $0 < s_2 \le s_1/2$ such that*

$$\frac{1}{2C_d^{\lceil \log_2(3\lambda) \rceil + 1}} \le \frac{\mu(E \cap B(y, s_2))}{\mu(B(y, s_2))} \le 1 - \frac{1}{2C_d^{\lceil \log_2(3\lambda) \rceil}},$$

where $\lceil a \rceil$ is the smallest integer at least $a \in \mathbb{R}$.

Proof of claim. Step 1: We can assume that $x \in O_E$; the case $x \in I_E$ is handled analogously (recall that $\partial^1 I_E = \partial^1 O_E$ from Remark 3.8). Since $x \in \partial^1 I_E$, by (3.4) we have

$$\limsup_{r \to 0} r \frac{\operatorname{cap}_1(I_E \cap B(x, r), B(x, 2r))}{\mu(B(x, r))} > 0. \tag{4.2}$$

Since x belongs to the 1-finely open set $X \setminus \overline{W}^1$, we have

$$\lim_{r\to 0} r\frac{\mathrm{cap}_1(\overline{W}^1 \cap B(x,r), B(x,2r))}{\mu(B(x,r))} = 0.$$

We apply Corollary 3.11 to find $R > 0$ and $F \subset X$ such that $I_F \supset \overline{W}^1 \cap B(x, R)$ and I_F is 1-thin at x. Then by (3.3), also

$$\lim_{r\to 0} r\frac{\mathrm{cap}_1(\overline{I_F}^1 \cap B(x,r), B(x,2r))}{\mu(B(x,r))} = 0.$$

Combining this with (4.2), by subadditivity of the variational 1-capacity we get

$$\limsup_{r\to 0} r\frac{\mathrm{cap}_1((I_E \setminus \overline{I_F}^1) \cap B(x,r), B(x,2r))}{\mu(B(x,r))} > 0. \tag{4.3}$$

According to Corollary 3.11, the set F also satisfies

$$\lim_{r\to 0} r\frac{P(F, B(x,r))}{\mu(B(x,r))} = 0.$$

Thus by (3.6) and the doubling property of μ,

$$\lim_{r\to 0} r\frac{\mathcal{H}(\partial^1 I_F \cap B(x,2r))}{\mu(B(x,r))} = 0. \tag{4.4}$$

Combining the fact that $x \in O_E$ with (4.3) and (4.4), we find a number $a > 0$ and a radius

$$0 < r_f \le \frac{\min\{R, s_1\}}{2} \tag{4.5}$$

such that

$$\frac{\mu(E \cap B(x, 2r_f))}{\mu(B(x,2r_f))} \le \frac{1}{2C_d^{\lceil \log_2(60\lambda)\rceil}} \tag{4.6}$$

and

$$r_f\frac{\mathrm{cap}_1((I_E \setminus \overline{I_F}^1) \cap B(x,r_f), B(x,2r_f))}{\mu(B(x,r_f))} > a \tag{4.7}$$

and

$$r_f\frac{\mathcal{H}(\partial^1 I_F \cap B(x,2r_f))}{\mu(B(x,r_f))} < \frac{a}{16C_d^{\lceil \log_2(10\lambda)\rceil+2}C_P}. \tag{4.8}$$

<u>Step 2</u>: Let D consist of all points $z \in B(x, r_f) \setminus \overline{I_F}^1$ for which there exists a radius $0 < t \le (10\lambda)^{-1}r_f$ such that

$$\frac{\mu(F \cap B(z,t))}{\mu(B(z,t))} > \frac{1}{4C_d}.$$

Consider $z \in D$ and the corresponding radius t. Since $z \notin \overline{I_F}^1 \supset I_F \cup \partial^* F$ (recall (3.5)), we have

$$\lim_{r\to 0} \frac{\mu(F \cap B(z,r))}{\mu(B(z,r))} = 0.$$

Take the smallest $k = 0, 1, \ldots$ such that

$$\frac{\mu(F \cap B(z, 2^{-k}t))}{\mu(B(z, 2^{-k}t))} \le \frac{1}{2}. \tag{4.9}$$

If $k = 0$, let $r_z := t$ so that

$$\frac{1}{4C_d} < \frac{\mu(F \cap B(z, r_z))}{\mu(B(z, r_z))} \le \frac{1}{2}.$$

If $k \ge 1$, let $r_z := 2^{-k+1}t$, and then

$$\frac{\mu(F \cap B(z, r_z))}{\mu(B(z, r_z))} > \frac{1}{2}$$

and

$$\begin{aligned}\frac{\mu(F \cap B(z, r_z))}{\mu(B(z, r_z))} &= \frac{\mu(F \cap B(z, 2^{-k+1}t))}{\mu(B(z, 2^{-k+1}t))}\\ &\le \frac{\mu(B(z, 2^{-k+1}t)) - \mu(B(z, 2^{-k}t) \setminus F)}{\mu(B(z, 2^{-k+1}t))}\\ &\le \frac{\mu(B(z, 2^{-k+1}t)) - \mu(B(z, 2^{-k}t))/2}{\mu(B(z, 2^{-k+1}t))} \quad \text{by (4.9)}\\ &\le 1 - \frac{1}{2C_d}.\end{aligned}$$

Thus in both cases, we have $r_z \le (10\lambda)^{-1} r_f$ and

$$\frac{1}{4C_d} < \frac{\mu(F \cap B(z, r_z))}{\mu(B(z, r_z))} \le 1 - \frac{1}{2C_d}.$$

By the relative isoperimetric inequality (3.7) we now obtain

$$\mu(B(z, r_z)) \le 8C_d^2 C_P r_z \mathcal{H}(\partial^1 I_F \cap B(x, \lambda r_z)). \tag{4.10}$$

Performing the same for every $z \in D$, we obtain a covering $\{B(z, \lambda r_z)\}_{z\in D}$. By the 5-covering theorem, we can extract a countable collection $\{B_j = B(z_j, r_j)\}_{j=1}^{\infty}$ such that the balls λB_j are pairwise disjoint and $D \subset \bigcup_{j=1}^{\infty} 5\lambda B_j$. For each $j \in \mathbb{N}$, define the Lipschitz function

$$\eta_j := \max\left\{0, 1 - \frac{\operatorname{dist}(\cdot, 5\lambda B_j)}{5\lambda r_j}\right\},$$

so that $\eta_j = 1$ on $5\lambda B_j$, $\eta_j = 0$ outside $10\lambda B_j$, and the minimal 1-weak upper gradient satisfies $g_{\eta_j} \le (5\lambda r_j)^{-1} \chi_{10\lambda B_j}$; see [Björn and Björn 2011, Corollary 2.21]. Moreover, $r_j \le (10\lambda)^{-1} r_f$ and so $\eta_j \in N_0^{1,1}(B(x,2r_f))$ for all $j \in \mathbb{N}$. Now we have

$$\begin{aligned}\operatorname{cap}_1(D, B(x,2r_f)) &\le \operatorname{cap}_1\Big(\bigcup_{j=1}^{\infty} 5\lambda B_j, B(x,2r_f)\Big) \le \sum_{j=1}^{\infty} \operatorname{cap}_1(5\lambda B_j, B(x,2r_f))\\ &\le \sum_{j=1}^{\infty}\int_X g_{\eta_j}\,d\mu \le \sum_{j=1}^{\infty}\frac{\mu(10\lambda B_j)}{5\lambda r_j} \le C_d^{\lceil \log_2(10\lambda)\rceil}\sum_{j=1}^{\infty}\frac{\mu(B_j)}{r_j}\\ &\le 8C_d^{\lceil \log_2(10\lambda)\rceil+2}C_P\sum_{j=1}^{\infty}\mathcal{H}(\partial^1 I_F \cap \lambda B_j) \quad \text{by (4.10)}\\ &\le 8C_d^{\lceil \log_2(10\lambda)\rceil+2}C_P\mathcal{H}(\partial^1 I_F \cap B(x,2r_f)).\end{aligned}$$

Thus by (4.8),

$$r_f\frac{\operatorname{cap}_1(D, B(x,2r_f))}{\mu(B(x,r_f))} < \frac{a}{2}$$

and so

$$\begin{aligned} r_f & \frac{\operatorname{cap}_1((I_E \setminus (\overline{I_F}^1 \cup D)) \cap B(x,r_f), B(x,2r_f))}{\mu(B(x,r_f))}\\ &\ge r_f\frac{\operatorname{cap}_1((I_E \setminus \overline{I_F}^1) \cap B(x,r_f), B(x,2r_f)) - \operatorname{cap}_1(D, B(x,2r_f))}{\mu(B(x,r_f))} > \frac{a}{2} \end{aligned} \tag{4.11}$$

by (4.7).

<u>Step 3</u>: Now consider $z \in (I_E \setminus (\overline{I_F}^1 \cup D)) \cap B(x,r_f)$. We have

$$\lim_{r\to 0}\frac{\mu(E\cap B(z,r))}{\mu(B(z,r))} = 1,$$

and so we can choose $0 < t \le (20\lambda)^{-1}r_f$ such that

$$\frac{\mu(E\cap B(z,t))}{\mu(B(z,t))} > \frac{1}{2}.$$

Note also that for any $r \in [(20\lambda)^{-1}r_f, (10\lambda)^{-1}r_f]$, we have $B(x,2r_f) \subset B(z,60\lambda r)$ and so

$$\frac{\mu(E\cap B(z,r))}{\mu(B(z,r))} \le C_d^{\lceil \log_2(60\lambda)\rceil}\frac{\mu(E\cap B(x,2r_f))}{\mu(B(x,2r_f))} \le \frac{1}{2}$$

by (4.6). Set $r_z := 2^k t$ for the smallest $k \in \mathbb{N}$ such that

$$\frac{\mu(E\cap B(z,2^k t))}{\mu(B(z,2^k t))} \le \frac{1}{2}.$$

Then we have $0 < r_z \le (10\lambda)^{-1}r_f$ and

$$\frac{1}{2C_d} < \frac{\mu(E\cap B(z,r_z))}{\mu(B(z,r_z))} \le \frac{1}{2} \tag{4.12}$$

and since $z \notin D$,

$$\frac{1}{4C_d} \le \frac{\mu((E \setminus F) \cap B(z, r_z))}{\mu(B(z, r_z))} \le \frac{1}{2}.$$

Then by the relative isoperimetric inequality (3.7), we have

$$\frac{\mu(B(z, r_z))}{r_z} \le 8C_d^2 C_P \mathcal{H}(\partial^1 I_{E\setminus F} \cap B(z, \lambda r_z)). \tag{4.13}$$

(Note that the right-hand side could be infinity.) Let

$$A := \bigcup_{z\in(I_E\setminus(\overline{I_F}^1\cup D))\cap B(x,r_f)} B(z, \lambda r_z) \subset B(x, 2r_f). \tag{4.14}$$

Consider the covering $\{B(z, \lambda r_z)\}_{z\in(I_E\setminus(\overline{I_F}^1\cup D))\cap B(x,r_f)}$. By the 5-covering theorem, we can extract a countable collection $\{B_j = B(z_j, r_j)\}_{j=1}^{\infty}$ such that the balls λB_j are pairwise disjoint and

$$(I_E \setminus (\overline{I_F}^1 \cup D)) \cap B(x, r_f) \subset \bigcup_{j=1}^{\infty} 5\lambda B_j.$$

Just as in the previous step, for each $j \in \mathbb{N}$ define the Lipschitz function

$$\eta_j := \max\left\{0, 1 - \frac{\operatorname{dist}(\,\cdot\,, 5\lambda B_j)}{5\lambda r_j}\right\},$$

so that $\eta_j = 1$ on $5\lambda B_j$, $\eta_j = 0$ outside $10\lambda B_j$, and the minimal 1-weak upper gradient satisfies $g_{\eta_j} \le (5\lambda r_j)^{-1}\chi_{10\lambda B_j}$. Moreover, $r_j \le (10\lambda)^{-1} r_f$ and so $\eta_j \in N_0^{1,1}(B(x, 2r_f))$ for all $j \in \mathbb{N}$. Now we have

$$\begin{aligned}
\operatorname{cap}_1((I_E\setminus(\overline{I_F}^1\cup D))\cap B(x, r_f), B(x, 2r_f)) &\le \operatorname{cap}_1\left(\bigcup_{j=1}^{\infty} 5\lambda B_j, B(x, 2r_f)\right) \le \sum_{j=1}^{\infty} \operatorname{cap}_1(5\lambda B_j, B(x, 2r_f)) \\
&\le \sum_{j=1}^{\infty} \int_X g_{\eta_j}\, d\mu \le \sum_{j=1}^{\infty} \frac{\mu(10\lambda B_j)}{5\lambda r_j} \le C_d^{\lceil \log_2(10\lambda)\rceil} \sum_{j=1}^{\infty} \frac{\mu(B_j)}{r_j} \\
&\le 8C_d^{\lceil \log_2(10\lambda)\rceil+2} C_P \sum_{j=1}^{\infty} \mathcal{H}(\partial^1 I_{E\setminus F} \cap \lambda B_j) \quad \text{by (4.13)} \\
&\le 8C_d^{\lceil \log_2(10\lambda)\rceil+2} C_P \mathcal{H}(\partial^1 I_{E\setminus F} \cap A).
\end{aligned} \tag{4.15}$$

Step 4: Next we show that

$$\partial^1 I_{E\setminus F} \subset (\partial^1 I_E \setminus \overline{I_F}^1) \cup \partial^1 I_F. \tag{4.16}$$

To see this, note that $X \setminus \overline{I_F}^1 \subset O_F$ (recall (3.5)) and so $I_{E\setminus F} \setminus \overline{I_F}^1 = I_E \setminus \overline{I_F}^1$. Since $X \setminus \overline{I_F}^1$ is a 1-finely open set, it follows that $\partial^1 I_{E\setminus F} \setminus \overline{I_F}^1 = \partial^1 I_E \setminus \overline{I_F}^1$. Moreover, $I_{E\setminus F} \cap \operatorname{fine-int} I_F = \varnothing$ and $\operatorname{fine-int} I_F$ is 1-finely open, and so $\partial^1 I_{E\setminus F} \cap \operatorname{fine-int} I_F = \varnothing$. From these, (4.16) follows.

By (4.11) and (4.15),

$$r_f \frac{\mathcal{H}(\partial^1 I_{E\setminus F} \cap A)}{\mu(B(x, r_f))} \ge \frac{a}{16C_d^{\lceil \log_2(10\lambda)\rceil+2} C_P}.$$

Now by first using (4.16) and the fact that $A \subset B(x, 2r_f)$ (recall (4.14)), and then (4.8) and the above inequality, we get

$$r_f \frac{\mathcal{H}((\partial^1 I_E \setminus \overline{I_F}^1) \cap A)}{\mu(B(x, r_f))} \geq r_f \frac{\mathcal{H}(\partial^1 I_{E\setminus F} \cap A) - \mathcal{H}(\partial^1 I_F \cap B(x, 2r_f))}{\mu(B(x, r_f))} > 0.$$

In particular, there exists a point $y \in (\partial^1 I_E \setminus \overline{I_F}^1) \cap A$. Recall from (4.5) that $0 < r_f \leq \min\{R, s_1\}/2$, and so $\overline{W}^1 \cap B(x, 2r_f) \subset I_F \cap B(x, 2r_f)$. Thus

$$y \in (\partial^1 I_E \setminus \overline{W}^1) \cap B(x, 2r_f) \subset (\partial^1 I_E \setminus \overline{W}^1) \cap B(x, s_1),$$

as desired. By the definition of A (recall (4.12), (4.14)) there is a point $z \in X$ and a radius $0 < r_z \leq (10\lambda)^{-1} r_f$ such that $y \in B(z, \lambda r_z)$ and

$$\frac{1}{2C_d} \leq \frac{\mu(E \cap B(z, r_z))}{\mu(B(z, r_z))} \leq \frac{1}{2}.$$

For $s_2 := 2\lambda r_z \leq s_1/2$ we then have $B(z, r_z) \subset B(y, s_2) \subset B(z, 3\lambda r_z)$, and so

$$\frac{1}{2C_d^{\lceil \log_2(3\lambda)\rceil+1}} \leq \frac{\mu(E \cap B(y, s_2))}{\mu(B(y, s_2))} \leq \frac{\mu(B(y, s_2)) - \mu(B(z, r_z))/2}{\mu(B(y, s_2))} \leq 1 - \frac{1}{2C_d^{\lceil \log_2(3\lambda)\rceil}}.$$

This completes the proof of the claim. □

Define $r_0 = 1$. We use the claim repeatedly, first with the choice $x = x_0$ and $s_1 = r_0$, to find a sequence of points $x_j \in B(x_{j-1}, r_{j-1}) \cap \partial^1 I_E \setminus \overline{W}^1$ and a sequence of numbers $0 < r_j \leq r_{j-1}/2$ such that

$$\min\left\{\frac{\mu(B(x_j, r_j) \cap E)}{\mu(B(x_j, r_j))}, \frac{\mu(B(x_j, r_j) \setminus E)}{\mu(B(x_j, r_j))}\right\} \geq \frac{1}{2C_d^{\lceil \log_2(3\lambda)\rceil+1}}$$

for all $j \in \mathbb{N}$. By completeness of the space and the fact that W is open, we find $x \in X \setminus W$ such that $x_j \to x$. For each $j \in \mathbb{N}$ we have

$$d(x, x_j) \leq \sum_{k=j}^{\infty} d(x_k, x_{k+1}) \leq \sum_{k=j}^{\infty} r_k \leq 2r_j.$$

Thus $B(x_j, r_j) \subset B(x, 3r_j) \subset B(x_j, 5r_j)$ for all $j \in \mathbb{N}$, and so

$$\frac{\mu(B(x, 3r_j) \cap E)}{\mu(B(x, 3r_j))} \geq \frac{\mu(B(x_j, r_j) \cap E)}{\mu(B(x, 3r_j))} \geq \frac{1}{C_d^3} \frac{\mu(B(x_j, r_j) \cap E)}{\mu(B(x_j, r_j))} \geq \frac{1}{2C_d^{\lceil \log_2(3\lambda)\rceil+4}}$$

and similarly

$$\frac{\mu(B(x, 3r_j) \setminus E)}{\mu(B(x, 3r_j))} \geq \frac{\mu(B(x_j, r_j) \setminus E)}{\mu(B(x, 3r_j))} \geq \frac{1}{C_d^3} \frac{\mu(B(x_j, r_j) \setminus E)}{\mu(B(x_j, r_j))} \geq \frac{1}{2C_d^{\lceil \log_2(3\lambda)\rceil+4}}.$$

Thus

$$\limsup_{r \to 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} > 0 \quad \text{and} \quad \limsup_{r \to 0} \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} > 0,$$

and so $x \in \partial^* E \setminus W$, which proves the theorem by the discussion in the first paragraph of the proof. □

By using another argument involving Lipschitz cutoff functions, it is easy to see that, for any $A \subset X$ and any ball $B(x,r)$,

$$\operatorname{cap}_1(A \cap B(x,r), B(x,2r)) \le C_d \mathcal{H}(A \cap B(x,r)). \tag{4.17}$$

Theorem 4.18. *Let $\Omega \subset X$ be open and let $E \subset X$ be μ-measurable with $\mathcal{H}(\partial^* E \cap \Omega) < \infty$. Then $\mathcal{H}((\partial^1 I_E \setminus \partial^* E) \cap \Omega) = 0$.*

Proof. By a standard covering argument (see, e.g., the proof of [Kinnunen et al. 2014, Lemma 2.6]) we find that

$$\lim_{r\to 0} r \frac{\mathcal{H}(\partial^* E \cap B(x,r))}{\mu(B(x,r))} = 0$$

for all $x \in \Omega \setminus (\partial^* E \cup N)$, with $\mathcal{H}(N) = 0$. Then by (4.17), also

$$\begin{aligned}\limsup_{r\to 0} r \frac{\operatorname{cap}_1((\partial^* E \cup N) \cap B(x,r), B(x,2r))}{\mu(B(x,r))} &\le C_d \limsup_{r\to 0} r \frac{\mathcal{H}((\partial^* E \cup N) \cap B(x,r))}{\mu(B(x,r))} \\ &= C_d \limsup_{r\to 0} r \frac{\mathcal{H}(\partial^* E \cap B(x,r))}{\mu(B(x,r))} = 0\end{aligned}$$

for all $x \in \Omega \setminus (\partial^* E \cup N)$. Thus $\Omega \setminus (\partial^* E \cup N)$ is a 1-finely open set. Now by Theorem 4.1,

$$\partial^1 I_E \cap (\Omega \setminus (\partial^* E \cup N)) = \varnothing$$

and the result follows. □

Now we can prove our main theorem.

Proof of Theorem 1.1. By Theorem 4.18 we have $\mathcal{H}(\partial^1 I_E \cap \Omega) < \infty$. Then by Theorem 1.2 we have $P(E,\Omega) < \infty$. □

5. Some consequences and discussion

Theorem 1.1 is, in particular, new in Carnot groups, which are a class of connected and simply connected Lie groups. We point out that the metric space theory we consider is consistent with the Carnot group theory involving vector fields, since the so-called horizontal Sobolev spaces defined in Carnot groups by means of distributional derivatives coincide with the Newton–Sobolev spaces defined in metric spaces. See [Hajłasz and Koskela 2000, Theorems 11.6 and 11.7]. The Lebesgue measure on a Carnot group is doubling and supports a (1, 1)-Poincaré inequality; see [Hajłasz and Koskela 2000, Proposition 11.17] as well as [Heinonen et al. 2015, Section 14.2].

We state Federer's characterization in metric spaces as follows.

Corollary 5.1. *Let $\Omega \subset X$ be open and let $E \subset X$ be μ-measurable. Then $P(E,\Omega) < \infty$ if and only if $\mathcal{H}(\partial^* E \cap \Omega) < \infty$.*

Proof. This follows by combining Theorem 1.1 and (2.4). □

In general, the sets $\partial^* E$ and $\partial^1 I_E$ can be quite different.

Example 5.2. Let $X = \mathbb{R}$ (unweighted). Let $\{q_j\}_{j=1}^{\infty}$ be an enumeration of all rational numbers and let $E := \bigcup_{j=1}^{\infty} B(q_j, 2^{-j})$. Then $\mathcal{L}^1(I_E) \le 2$ and $\mathcal{L}^1(\partial^* E) = 0$ by Lebesgue's differentiation theorem. On the other hand, it is straightforward to check that for any $A \subset \mathbb{R}$, we have $\partial^1 A = \partial A$. Thus $\partial^1 I_E = \partial I_E \supset \mathbb{R} \setminus I_E$ and so $\mathcal{L}^1(\partial^1 I_E) = \infty$.

In the Euclidean setting, the "if" direction of Federer's characterization is proved by first showing that almost every coordinate line intersecting I_E and O_E also intersects $\partial^* E$; see [Federer 1969, Section 4.5.11] or [Evans and Gariepy 1992, pp. 222–226]. Proving this fact relies heavily on the Euclidean structure, but now we have the following generalization to metric spaces.

Proposition 5.3. *Let $E \subset X$ be μ-measurable and suppose that $\mathcal{H}(\partial^* E) < \infty$. Then* 1*-almost every curve intersecting I_E and O_E also intersects $\partial^* E$.*

Proof. By Theorem 1.1, $P(E, X) < \infty$. Then the result follows from [Lahti and Shanmugalingam 2017, Corollary 6.4]. □

If the above property were true also in the case $\mathcal{H}(\partial^* E) = \infty$, this could be useful when proving generalizations of Federer's characterization, e.g., to noncomplete spaces. One might expect that when the measure-theoretic boundary is very large, then it should be "easier" for curves to intersect it. However, the intersection property turns out to be false in this case.

Example 5.4. Let $X = \mathbb{R}^2$ equipped with the Euclidean metric and the weighted Lebesgue measure $d\mu := w\, d\mathcal{L}^2$, with $w(x) = |x|^{-3/2}$. It is straightforward to check that w is a Muckenhoupt A_1-weight, and thus μ is doubling and supports a (1, 1)-Poincaré inequality; see, e.g., [Heinonen et al. 1993, Chapter 15] for these concepts. Let

$$E := \{x = (x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 \le 2,\ -x_1^{3/2} < x_2 < x_1^{3/2}\}.$$

It is easy to check that

$$\partial^* E = \{x \in \mathbb{R}^2 : x_1 > 0,\ |x_2| = x_1^{3/2}\}$$

and then that $\mathcal{H}(\partial^* E) = \infty$. Now consider the curve family Γ consisting of the curves

$$\gamma_{r,t}(s) := (s, ts^{3/2}), \quad 0 < r \le 1,\ -1 < t < 1,\ s \in [0, r].$$

For all $0 < r \le 1$ and $-1 < t < 1$, clearly $\gamma_{r,t}(0) \in O_E$ and $\gamma_{r,t}(r) \in I_E$, but $\gamma_{r,t}$ does not intersect $\partial^* E$.

Denote the image of $\gamma_{r,t}$ in X by the same symbol. Let ρ be an admissible function for Γ. This means that $\int_{\gamma_{r,t}} \rho\, d\mathcal{H}^1 \ge 1$ for all $0 < r \le 1$ and $-1 < t < 1$, where $\mathcal{H}^1$ is the 1-dimensional Lebesgue measure. It follows that for every $-1 < t < 1$ and every $k \in \mathbb{N}$,

$$\sum_{j=k}^{\infty} \int_{\gamma_{1,t}|_{[2^{-j}, 2^{-j+1}]}} \rho\, d\mathcal{H}^1 = \int_{\gamma_{2^{-k+1},t}} \rho\, d\mathcal{H}^1 \ge 1,$$

and so

$$\sum_{j=1}^{\infty} \int_{\gamma_{1,t}|_{[2^{-j}, 2^{-j+1}]}} \rho\, d\mathcal{H}^1 = \infty. \tag{5.5}$$

Define the sets

$$A_j := \{x \in E : 2^{-j} < x_1 \le 2^{-j+1}\}, \quad j \in \mathbb{N}.$$

By the classical coarea formula we have

$$\begin{aligned}\int_{\mathbb{R}^2} \rho \, d\mu \ge \sum_{j=1}^{\infty} 2^{3j/2-3} \int_{A_j} \rho \, d\mathcal{L}^2 &\ge \sum_{j=1}^{\infty} 2^{3j/2-3} 2^{-3j/2} \int_{-1}^{1} \int_{\gamma_{1,t}|_{[2^{-j},2^{-j+1}]}} \rho \, d\mathcal{H}^1 \, dt \\ &= 2^{-3} \sum_{j=1}^{\infty} \int_{-1}^{1} \int_{\gamma_{1,t}|_{[2^{-j},2^{-j+1}]}} \rho \, d\mathcal{H}^1 \, dt = \infty\end{aligned}$$

by (5.5). It follows that $\operatorname{Mod}_1(\Gamma) = \infty$.

It is reasonable to expect Federer's characterization to find various applications especially in the metric setting, where certain tools of Euclidean BV theory, such as the Gauss–Green theorem, are not available. One likely application is in the study of images of sets of finite perimeter under quasiconformal mappings (see [Kelly 1973] for the Euclidean case), since such mappings are known to preserve the measure-theoretic boundary (see [Korte et al. 2012, Theorem 6.1]).

Now we discuss some existing applications. From the characterization it follows that the space supports the following *strong relative isoperimetric inequality* introduced in [Kinnunen et al. 2012]; compare this with (2.2) and (3.7).

Corollary 5.6. *For every ball $B(x,r)$ and every μ-measurable $E \subset X$, we have*

$$\min\{\mu(B(x,r) \cap E),\ \mu(B(x,r) \setminus E)\} \le 2C_P C_d r \mathcal{H}(\partial^* E \cap B(x, \lambda r)).$$

Proof. We can assume that the right-hand side is finite. By Theorem 1.1 we know that $P(E, B(x, \lambda r)) < \infty$, and now the result follows by combining the relative isoperimetric inequality (2.2) and (2.4). □

In [Kinnunen et al. 2012] the authors worked with the same standing assumptions as we do in the current paper, but additionally they assumed that the space supports the above strong relative isoperimetric inequality. Now we know that this does not need to be separately assumed, and the following theorem [Kinnunen et al. 2012, Theorem 1.1] holds under our standing assumptions (completeness, doubling, and Poincaré).

Theorem 5.7. *Let $\Omega \subset X$ be a bounded open set and let $u \in N^{1,p}(\Omega)$ with $1 \le p < \infty$. Then $u \in N_0^{1,p}(\Omega)$ if and only if*

$$\lim_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{\Omega \cap B(x,r)} |u| \, d\mu = 0$$

for Cap_p*-almost every $x \in \partial\Omega$.*

Theorem 6.1 in [Lahti and Shanmugalingam 2018] considered an analogous characterization of a class of BV functions with zero boundary values, also under the additional assumption of a strong relative isoperimetric inequality. Such a class and the characterization are needed in an ongoing study of new fine properties of BV functions and capacities, see [Lahti 2019; 2018], and this was in fact a key motivation for the current paper. The strong relative isoperimetric inequality was also used in proving approximation results for BV functions; see [Lahti and Shanmugalingam 2018, Corollary 6.7, Theorem 6.9], as well as [Lahti and Shanmugalingam 2017, Corollary 7.6] and the comment after it. Now we know that all of these results hold in every complete metric space equipped with a doubling measure and supporting a Poincaré inequality.

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SPECTRAL THEORY OF PSEUDODIFFERENTIAL OPERATORS OF DEGREE 0 AND AN APPLICATION TO FORCED LINEAR WAVES

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We extend the results of our paper "Attractors for two-dimensional waves with homogeneous Hamiltonians of degree 0," written with Laure Saint-Raymond, to the case of forced linear wave equations in any dimension. We prove that, in dimension 2, if the foliation on the boundary at infinity of the energy shell is Morse–Smale, we can apply Mourre's theory and hence get the asymptotics of the forced solution. We also characterize the wavefront sets of the limit Schwartz distribution using radial propagation estimates.

Introduction

This paper contains new developments of some ideas already introduced in our paper [Colin de Verdière and Saint-Raymond 2020] concerning the spectral theory of self-adjoint pseudodifferential operators of degree 0 on closed manifolds. The main motivation comes from the study of forced internal or inertial waves in physics; see [Bajars et al. 2013; Brouzet 2016; Gostiaux et al. 2006; Maas et al. 1997; Maas and Lam 1995; Ogilvie 2005; Pillet 2018; Rieutord and Valdettaro 2010; 2018] and many other works. In what follows, H is a classical self-adjoint scalar pseudodifferential operator of degree 0 on a compact manifold M of dimension n without boundary, f is a smooth function and the spectral parameter ω is a real number. The main object to study is the linear forced wave equation

$$\frac{1}{i}\frac{du}{dt} + Hu = fe^{-i\omega t}, \quad u(0) = 0. \tag{1}$$

We are interested in the behaviour of $u(t)$ as $t \to +\infty$. Thanks to the spectral theorem, we can relate this behaviour to the spectral theory of H and hence to the Hamiltonian dynamics of the principal symbol $h : T^\star M \setminus 0 \to \mathbb{R}$, which is a smooth homogeneous function of degree 0. The main tools that we use are already classical: they are, on one hand, the general theory of pseudodifferential operators, culminating in the works of Lars Hörmander, Hans Duistermaat [1973], Alan Weinstein [1971; 1975] and many others, see also [Folland 1989; Dyatlov and Zworski 2019a], and, on the other hand, the theory initiated by Eric Mourre [1981; 1983] in order to get a flexible way to have a limit absorption principle, see also [Jensen et al. 1984; Gérard 2008; Cattaneo 2005].

What is the content, beyond that of [Colin de Verdière and Saint-Raymond 2020]? The main result is Theorem 6.2 where we extend the result of that work to the generic Morse–Smale case, still in dimension 2. The other new contribution is a precise description in arbitrary dimension of the dynamical assumptions

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allowing one to apply Mourre theory thanks to the Gårding inequality (see Section 3) by constructing a *global escape function*.

After recalling general facts on the Hamiltonian dynamics of a homogeneous Hamiltonian h of degree 0 in Section 1 and on the spectral theory of H in Section 2, we give, in Section 3, a necessary and sufficient condition on the dynamics at infinity, which ensures the existence of an escape function that will be the key input in order to apply Mourre's theory thanks to the Gårding inequality. In Section 4, we recall general facts that we got in [Colin de Verdière and Saint-Raymond 2020] for the forced wave equation from Mourre's theory. In Section 5, we use radial propagation estimates (see [Dyatlov and Zworski 2019a; 2019b]), going back to works of Melrose and Vasy, in order to locate the wavefront set of the Schwartz distribution u_∞, which is the limit (modulo bounded functions in L^2) of $u(t)$ as $t \to +\infty$.

We consider then, in Section 6, the case where M is a surface ($n = 2$), extending our results of [Colin de Verdière and Saint-Raymond 2020] to the generic case where the foliation is Morse–Smale and can have singular points (foci, nodes or saddles). Finally, we consider, in Section 7, the case where M is a 3-dimensional manifold with a free S^1-action leaving H invariant, which is important for applications to physics. We end the paper with a short review of related problems in Section 8 and two Appendices.

1. Hamiltonian of degree 0: classical dynamics

In what follows, we fix the following notation: M is a smooth connected compact manifold of dimension $n \geq 2$ without boundary, q is the generic point of M and $|dq|$ a smooth density on M. The Hamiltonian h is a smooth positively homogeneous function $h : T^\star M \setminus 0 \to \mathbb{R}$. We denote by (q, p) some local canonical coordinates on $T^\star M$ and by extension a generic point of $T^\star M$. The Hamiltonian vector field of h is denoted by $\mathcal{X}_h$ and we fix the "symplectic" conventions so that

$$\mathcal{X}_h = \frac{\partial h}{\partial p}\partial_q - \frac{\partial h}{\partial q}\partial_p, \quad \mathcal{X}_h f = \{h, f\}$$

and denote by Φ_t the flow of $\mathcal{X}_h$. Because of the homogeneity of h, we have $pdq(\mathcal{X}_h) = 0$ and $\mathcal{X}_h$ is homogeneous of degree -1. Let us fix $\omega \in \mathbb{R}$ and define the energy shell $\Sigma_\omega := h^{-1}(\omega)$. We will assume in what follows that ω is *not a critical value of* h and hence Σ_ω is a smooth conic hypersurface in $T^\star M \setminus 0$. We need to introduce $Z_\omega := \Sigma_\omega/\mathbb{R}^+$, which is a smooth closed manifold of dimension $2n - 2$ and will be seen as the boundary at infinity of Σ_ω. The vector field $\mathcal{X}_h$ defines by projection a conformal class of vector fields on Z_ω, which we will call an (oriented) foliation and denote by $\mathcal{F}$. This foliation can admit singular points corresponding to the lines $\mathbb{R}^+ \cdot (q, p)$, where $\mathcal{X}_h$ is parallel to the cone direction $p\partial_p$. Note that we can and will often reduce ourselves to the case $\omega = 0$ by looking at the Hamiltonian $h - \omega$.

2. Hamiltonian of degree 0: spectral theory

Let us choose a self-adjoint pseudodifferential operator H of degree 0 acting on $L^2(M, |dq|)$ and of principal symbol h. Note that H is a bounded operator. In what follows, all pseudodifferential operators are "classical"; this means that the symbols do have full expansions in homogeneous functions with integer degrees. We are mainly interested by the spectral theory of H. As a warm up, we have:

Theorem 2.1. *The essential spectrum of H is the interval $J := [h_-, h_+]$, with $h_- := \min h$ and $h_+ := \max h$.*

Proof. If $\omega \in \mathbb{C} \setminus J$, then $H - \omega$ is elliptic and hence admits an inverse $R(\omega)$ modulo compact operators which can be chosen holomorphic in ω by taking $R(\omega) := \mathrm{Op}(h-\omega)^{-1}$, where Op is a fixed quantization on M:

$$R(\omega)(H-\omega) = \mathrm{Id} + K(\omega),$$

with K compact and holomorphic in ω. On the other hand, since H is bounded, $(H-\omega)$ is invertible for large values of ω. It follows from the Fredholm analytic theorem that the operator $H-\omega$ is invertible outside a discrete set where the kernels are finite-dimensional.

On the other hand, if $\omega \in J$, with $h(q_0, p_0) = \omega$ and $\epsilon > 0$ is fixed, choose a small neighbourhood U of q_0 so that, if $q \in U$, then $|h(q, p_0) - \omega| \leq \epsilon$. Pick then $\phi \in C_o^\infty(U)$ with $\int_M |\phi|^2(q)\,|dq| = 1$. Let us check that, for t large enough,

$$\|(H-\omega)(\phi e^{itqp_0})\|_{L^2(M)} \leq 2\epsilon. \tag{2}$$

It follows from the general properties of the principal symbols that

$$H(\phi(q)e^{itqp_0}) = h(q, p_0)\phi(q)e^{itqp_0} + O\Big(\frac{1}{t}\Big).$$

Take t so that the L^2 norm of the remainder is smaller than ϵ. We get inequality (2) by applying the triangular inequality. Hence

$$\|(H-\omega)(\phi e^{itqp_0})\|_{L^2(M)} \leq 2\epsilon \|\phi e^{itqp_0}\|_{L^2(M)},$$

which proves that $\sigma(H) \cap [\omega - 2\epsilon, \omega + 2\epsilon] \neq \varnothing$. $\square$

3. Escape functions

The key object of this paper is an escape function for h on the energy shell Σ_0:

Definition 3.1. A smooth function $k : \Sigma_0 \to \mathbb{R}$, positively homogeneous of degree 1, is called an *escape function* if there exists $\delta > 0$ so that the Poisson bracket $\{h, k\} = \mathcal{X}_h k$ is larger than δ on Σ_0.

A key observation is:

Remark 3.2. If we extend k to $T^\star M \setminus 0$ as a smooth homogeneous function $\tilde{k}$ of degree 1, then $\tilde{k}$ restricted to Σ_ω is still an escape function on Σ_ω for ω small enough.

We first give a general dynamical assumption on the oriented foliation $\mathcal{F}$ which turns out to be equivalent to the existence of a global *escape function*. We need some definitions, using the definitions of Appendix B:

Definition 3.3. We will say that the oriented 1-dimensional foliation $\mathcal{F}$ of the manifold Z_0 admits a *simple* structure (K_+, K_-) if $Z_0 = K_+ \cup K_- \cup \Omega$ as a disjoint union where:

- K_+ is an attractor of the oriented foliation $\mathcal{F}$, the *sink*.
- K_- is a repellor of the oriented foliation $\mathcal{F}$, the *source*.

- All leaves of points in Ω converge to K_+ at "$+\infty$" and to K_- at "$-\infty$"; in particular, the basin of K_+ is $\Omega \cup K_+$ and the basin of K_- for the reversed orientation of $\mathcal{F}$ is $\Omega \cup K_-$.

Definition 3.4. We say that a compact invariant set K_+ is weakly hyperbolic, denoted by (WH), if there exists, in some neighbourhood of K_+, a vector field W generating $\mathcal{F}$ and a smooth density $d\mu$ so that $\operatorname{div}_{d\mu}(W) < 0$. Similarly for K_-, we have $\operatorname{div}_{d\mu}(W) > 0$.

Our main result in this section is:

Theorem 3.5. *If the foliation $\mathcal{F}$ has a simple structure (K_+, K_-) with K_+ and K_- satisfying* (WH), *then there exists an escape function.*

The converse is true: the existence of an escape function implies that the foliation $\mathcal{F}$ has a simple structure (K_+, K_-) so that K_+ and of K_- satisfy (WH). *This simple structure is uniquely determined by $\mathcal{F}$.*

3A. ***Dynamical assumptions implying weak hyperbolicity.*** Let us choose a vector field W generating $\mathcal{F}$, whose flow is denoted by $\phi_t,\ t \in \mathbb{R}$, and equip Z_0 with a smooth density $d\mu$.

Let us describe properties of closed invariant sets of $\mathcal{F}$ from which we can deduce (WH):

(1) If some component of K_+ is an isolated point a, the assumption (WH) says that the trace of the linearized vector field of W at the point a is negative. This is independent of the choice of W. The case where the singular point is hyperbolic is studied in [Guillemin and Schaeffer 1977]. They show that, in the generic situation, there exists a pseudodifferential normal form for such points. Independently, the classical part of this normal form is also described in dimension 2 in [Davydov 1985; Arnold 1983; Davydov et al. 2008].

(2) If some component of K_+ is a closed curve γ, the assumption (WH) says that the modulus of the determinant of the linearized Poincaré map is < 1. In dimension $n = 2$, this is equivalent to our assumption (M2) in [Colin de Verdière and Saint-Raymond 2020].

(3) They are more complicated attractors which satisfy (WH). The Lorenz attractor is one of them: the vector field generating it has negative divergence.

3B. ***Construction of an escape function.*** We construct an escape function assuming that $\mathcal{F}$ has a simple structure with $K_\pm$ satisfying (WH).

3B.1. *Escape function near* Γ_+. Let $\Gamma_\pm$ be the subcones of Σ_0 generated by the sets $K_\pm$. We will construct in this section an escape function k_+ in some conic neighbourhood U_+ of Γ_+. A similar construction can be done on the basin of Γ_-.

Let us first construct "polar coordinates" (ρ, θ) on Σ_0, where $\rho \in \mathbb{R}^+ \setminus 0,\ \theta \in Z_0$ and the dilations on Σ_0 act by $\lambda \cdot (\rho, \theta) = (\lambda\rho, \theta)$:

Lemma 3.6. *If W is a given vector field on Z_0 generating $\mathcal{F}$, there exist polar coordinates $(\rho, \theta) \in (\mathbb{R}^+ \setminus 0) \times Z_0$ on Σ_0 so that*

$$\mathcal{X}_h = a(\theta)\partial_\rho + \frac{1}{\rho} W.$$

Proof. We start with arbitrary polar coordinates (ρ_1, θ): for example identify Z_0 with the cosphere bundle $S_1^\star$ for some Riemannian metric on M and define $\rho_1(q, p)$ so that $(q, p/\rho_1(q, p)) \in S_1^\star$. We get, using the homogeneity of $\mathcal{X}_h$ and the fact that W span $\mathcal{F}$,

$$\mathcal{X}_h = a_1(\theta)\partial_{\rho_1} + \frac{1}{\rho} W,$$

with $\rho = A(\theta)\rho_1$ and hence $\partial_{\rho_1} = A(\theta)\partial_\rho$. □

The Liouville measure $dL_0 := |dq\, dp/dh|$ on Σ_0, being homogeneous of degree n, with respect to dilations, can be written as $dL_0 = \rho^{n-1}|d\rho|\, d\mu$, where $d\mu$ is a smooth measure on Z_0.

The fact that

$$\operatorname{div}_{dL_0}(\mathcal{X}_h) = 0$$

can be rewritten as

$$(n-1)a + \operatorname{div}_\mu(W) = 0. \tag{3}$$

The assumption (WH) implies that we have a smooth > 0 function F defined near K_+ so that

$$\operatorname{div}_{F\mu}(W) = \frac{dF(W)}{F} + \operatorname{div}_\mu(W) \leq -c < 0.$$

Then, if $k_+ := F^{-1/(n-1)}\rho$, we get

$$dk_+(\mathcal{X}_h) = -\frac{1}{n-1} F^{-1/(n-1)}\left(\frac{dF(W)}{F} - (n-1)a\right),$$

which is equal to

$$dk_+(\mathcal{X}_h) = -\frac{1}{n-1} F^{-1/(n-1)} \operatorname{div}_{F\mu}(W),$$

and we get that the function k_+ is an escape function in some conical neighbourhood of Γ_+. □

We define similarly $k_- := -F^{-1/(n-1)}\rho$.

Note that k_+ tends to $+\infty$ as z tends to K_+ viewed as a set of points at infinity of Σ_0. We have also $k_+ \sim \langle p \rangle$ from the definition and the fact that F is positive.

Similarly, the function k_- defined near Γ_- tends to $-\infty$ as z tends to K_-.

3B.2. *Extension to* Σ_0. We choose a positive function m on Σ_0 which is smooth, homogeneous of degree 0 and equal to $m_\pm := \{h, k_\pm\}$ in some conical neighbourhoods $U_\pm$ of $\Gamma_\pm$. It follows from Proposition B.2(3) that we can choose U_+ so that $\Phi_t(U_+) \subset U_+$ for $t \geq 0$ and similarly for U_-.

Let z be in the basin of Γ_+ and define

$$l_+(z) = \lim_{t\to+\infty}\left(k_+(\Phi_t(z)) - \int_0^t m(\Phi_s(z))\, ds\right).$$

The limit exists because the expression of which we take the limit is independent of t for t large enough. Moreover the limit is smooth: if z is given and $\Phi_T(z) \in U_+$ for all $T \geq T_0$, there exists a neighbourhood V

of z so that $\Phi_{T_0}(V) \subset U_+$ and hence $\Phi_T(V) \subset U_+$ for all $T \geq T_0$. We have then, for $w \in V$,

$$l_+(w) = k_+(\Phi_{T_0}(w)) - \int_0^{T_0} m(\Phi_s(w))\, ds,$$

which is clearly smooth.

We define similarly l_-. The functions $l_\pm$ are escape functions in the basins of $\Gamma_\pm$ and satisfy in the respective basins $\{h, l_\pm\} = m$.

Let Γ_0 be the cone $\Gamma_0 := \{l_+ = 0\}$, which is smooth and transversal to $\mathcal{X}_h$ because $dl_+(\mathcal{X}_h) = m > 0$. On Γ_0 we have now the two functions $l_\pm$. The difference $\delta(z) = l_+(z) - l_-(z)$ is homogeneous of degree 1 and is constant along the flow lines. We will define k on the Hamiltonian trajectories $t \to \Phi_t(z)$ starting from $z \in \Gamma_0^1 := \{g^\star = 1\} \cap \Gamma_0$. For further use, we denote by S this hypersurface of $T^\star M$. The set Γ_0^1 is compact and hence the function $|\delta|$ is bounded by some constant $C > 0$ on it. Let us put $m_0 := \min m > 0$ and let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying

- $\psi(t) = 0$ if $t \leq 0$,
- $\psi(t) = 1$ if $t \geq 4C/m_0$,
- $|\psi'| \leq m_0/2C$.

We define now, for $z \in \Gamma_0^1$,

$$k(\phi_t(z)) = (1 - \psi(t))\, l_-(\Phi_t(z)) + \psi(t)\, l_+(\Phi_t(z)).$$

The derivative of k with respect to $\mathcal{X}_h$ is then equal to $m + \psi'(l_+ - l_-) \geq m_0/2$. We then extend k by homogeneity.

3C. ***Deriving the properties of $\mathcal{F}$ from the existence of an escape function.*** In what follows, we assume only the existence of an escape function k.

Let us give a construction of $\Gamma_\pm$ using only the dynamics of $\mathcal{X}_h$. We will see that these sets are defined independently of the choice of k: Γ_+ is the set of points $z \in \Sigma_0$ so that there exists $t_0 < 0$ with $\Phi_t(z) \to 0$ as $t \to t_0^+$; i.e., the trajectory of $\mathcal{X}_h$ is not complete as $t \to -\infty$. We similarly define Γ_- with $t_1 > 0$. We define $K_\pm$ so that they generate the cones $\Gamma_\pm$. Note that $\Gamma_+ \cap \Gamma_- = \varnothing$: if not, let $z \in \Gamma_+ \cap \Gamma_-$, and then $\Phi_t(z)$ tends to the zero section of $T^\star X$ as $t = t_0 + 0$, because the Hamiltonian flow is complete near the infinity of $T^\star X$. $\Phi_t(z)$ tends also to the zero section as $t = t_1 - 0$. This is not possible because the escape function tends to 0 at the zero section and is monotonic along the orbits.

Let us recall that we view $K_\pm$ as sets at infinity of the energy shell, namely the bases at infinity of the cones $\Gamma_\pm$.

Proposition 3.7. The picture of the dynamics is as follows:

- If $z \in \Sigma_0 \setminus (\Gamma_+ \cup \Gamma_-)$, then $\Phi_t(z)$ is defined for all $t \in \mathbb{R}$, $\Phi_t(z) \to K_+$ as $t \to +\infty$ and $\Phi_t(z) \to K_-$ as $t \to -\infty$.
- If $z \in \Gamma_+$, then $\Phi_t(z)$ is defined for all $t > t_0(z)$, $\Phi_t(z) \to K_+$ as $t \to +\infty$ and $\Phi_t(z) \to 0$ as $t \to t_0(z)$.

- If $z \in \Gamma_-$, then $\Phi_t(z)$ is defined for all $t < t_0(z)$, $\Phi_t(z) \to K_-$ as $t \to -\infty$ and $\Phi_t(z) \to 0$ as $t \to t_0(z)$.

Proof. Let us choose a metric g on M and consider the set $C_0 := k^{-1}(0) \cap (g^\star)^{-1}(1)$, where $g^\star$ is the dual metric. The set C_0 is a generating set for the cone $C := k^{-1}(0)$. If $z \in C_0$, the trajectory $t \to \Phi_t(z)$ is complete, because $t \to k(\Phi_t(z))$ is strictly monotonic and hence does not tend to the zero section where $k = 0$ at $t = \pm\infty$. Conversely, every complete trajectory cuts C_0 exactly in one point. This way we get a subset S of Σ_0 generating $\Sigma_0 \setminus (\Gamma_+ \cup \Gamma_-)$:

$$S := \{\Phi_t(z) \mid z \in C_0,\ t \in \mathbb{R}\}.$$

The orbits sitting in S have no limit points in S because the flow derivative of k is bounded below by some positive number. Let us consider the projections on Z_0 of S, Γ_+ and Γ_+, say Ω, K_+ and K_-. We have a disjoint union $Z_0 = \Omega \cup K_+ \cup K_-$. Each set is invariant by the foliation. Let us look at a leaf γ in Ω: γ has no limit points in Ω (because the foliation in Ω is diffeomorphic to the flow foliation in C). The limit points are then in $K_+ \cup K_-$. We have $\Gamma_+ \subset \{k > 0\}$ and $\Gamma_- \subset \{k < 0\}$. Hence the limit points at $+\infty$ are in K_+ and the limit points at $-\infty$ are in K_-. The set K_+ is an attractor: it is enough to consider the neighbourhoods U_N of K_+ which are the projections of the sets $(\{k \geq N\} \cap C) \cup \Gamma_+$. □

Let us show that the existence of an escape function implies that K_+ satisfies (WH): we choose polar coordinates (ρ, θ) near Γ_+ with $\rho = k$ and we have, from the equations derived in Section 3B.1, that $dk(\mathcal{X}_h) = a > 0$ and hence $\operatorname{div}_\mu W = -a/n < 0$: all components of K_+ satisfy (WH). A similar argument works for K_-.

3D. ***Radial sink and sources.*** Let us recall and introduce some notation: the radial compactification of $T^\star M$ is denoted by $\overline{T^\star M}$ and the boundary at infinity which we can identify with the sphere bundle is $S^\star M := T^\star M/\mathbb{R}^+$. The compactification of Σ_0 is $\overline{\Sigma}_0$ with the boundary at infinity $Z_0 = S\Sigma_0 \subset \overline{T^\star M}$.

Let us rephrase Definition E.52 of [Dyatlov and Zworski 2019a] in our context:

Definition 3.8. Let us introduce the symbol $r = -kh$, with k, an escape function (homogeneous of degree 1), and denote by ψ_t the flow of r extended to the boundary. The compact set $K_- \subset Z_0$ is a radial source for r if there exists a neighbourhood $U \subset \overline{T^\star M}$ of K_-, so that, uniformly for $z \in U$:

(1) For $t \leq 0$, $|k|(\psi_t(z)) \geq Ce^{\theta|t|}$ for some $C, \theta > 0$.

(2) $\psi_t(z) \to K_-$ as $t \to -\infty$.

We have:

Proposition 3.9. If k is an escape function, K_- is a radial source for $r = -kh$.

Proof. In the domain where $k < 0$, in particular near K_-, we have $\mathcal{X}_r = |k|\mathcal{X}_h - h\mathcal{X}_k$. The vector field $\mathcal{X}_r$ is homogeneous of degree 0 and hence projects onto $S^\star M$. We denote by Y_r this projection. Note that Y_r is tangent to Z_0, where it generates the foliation $\mathcal{F}$.

(1) We have $\mathcal{X}_r(|k|) = |k|\mathcal{X}_h|k| \leq -\delta|k|$. This implies that in a neighbourhood U_0 of K_-, where $k \leq -1$, for $t \leq 0$, we have $|k|(\psi_t(z)) \geq Ce^{\delta|t|}$.

(2) Let us choose V_0 a neighbourhood of K_- inside $S^\star M$ as follows: we choose first a neighbourhood V_1 of K_- in Z_0, with a smooth boundary, so that $\mathcal{Y}_r$ is outgoing and transversal to the boundary; take V_1 as the closure of the projection of the sets $\{k \leq -b\} \cap S$ for b large enough with S defined in Section 3B.2. We take for V_0 a neighbourhood of K_- in $S^\star M$ which is of the form $\{\exp(u\mathcal{Y}_r)(m) \mid m \in V_1, |u| \leq a\}$. If a is small enough, the vector field Y_r is transversal and outgoing at the boundary of V_0, because $Y_r(h) = h\{h,k\}$ and $\{h,k\} \geq \delta > 0$. Hence we get a repellor $L_- := \bigcap_{t\leq 0} \psi_t(V_1)$. The repellor L_- contains K_- and being invariant by the dynamics of $\mathcal{Y}_r$ restricted to Z_0 is equal to K_-. We then take for U_1 a small neighbourhood of V_0 in $\overline{T^\star M}$ and we get (2) by taking for U in the definition of a radial source the intersection $U_0 \cap U_1$. □

4. Applying Mourre's theory

Let us first recall some results of [Colin de Verdière and Saint-Raymond 2020]. Let us fix $\omega = 0$ for simplicity and assume that there exists an escape function k on the energy shell Σ_0. Then k can be extended to $T^\star M \setminus 0$ as an escape function in the cone $|h| \leq a$ with some $a > 0$. Let K be a self-adjoint operator of degree 1 of principal symbol k. Using the Gårding's inequality (see [Folland 1989, pp. 129–136]), one gets that K is a conjugate operator in the sense of Mourre: if J is a small enough open interval containing 0 and π_J is the spectral projector of H associated to the interval J, then

$$i\pi_J[H,K]\pi_J \geq c\pi_J + R,$$

where $c > 0$ and R is compact. Moreover the operator H is K-smooth; i.e., the map $t \to e^{itK}He^{-itK}$ is smooth with values into the bounded self-adjoint operators. Let us define the K-Sobolev spaces, denoted by $\mathcal{H}_K^s$, in the usual way using the s-powers of $(1+K^2)^{1/2}$. The usual Sobolev spaces will be denoted by $\mathcal{H}^s$. Let us give a comparison between the K-Sobolev spaces and the usual ones. There is a shift in the exponents due to the fact that the pseudodifferential calculus does not apply to nonelliptic operators like K.

Lemma 4.1. *If $f \in \mathcal{H}^1$, then $f \in \mathcal{H}_K^s$ for any $s \leq 1$. If $f \in \mathcal{H}_K^{-1}$, then $f \in \mathcal{H}^{-s}$ for any $s \leq -1$.*

Proof. If $f \in \mathcal{H}^1$, then $\langle (1+K^2)f \mid f\rangle < \infty$ because K^2 is a pseudodifferential operator of order 2 and hence $f \in \mathcal{H}_K^1$. The other inclusion follows by duality with respect to the L^2 product. □

It follows then from Mourre theory [Mourre 1981; 1983; Jensen et al. 1984; Gérard 2008] that:

Theorem 4.2 (Mourre). *The operator H has a finite number of eigenvalues in J, and they have finite multiplicity. Assuming that 0 is not an eigenvalue, the resolvent $(H-z)^{-1}$ defined for $\Im z > 0$ admits a boundary value $\omega \to (H-\omega-i0)^{-1}$ for ω real, close to 0, which, for any $\epsilon > 0$, is Hölder continuous for some positive Hölder exponent, depending on ϵ, from the Sobolev space $\mathcal{H}_K^{1/2+\epsilon}$ into $\mathcal{H}_K^{-1/2-\epsilon}$ for all $\epsilon > 0$.*

Moreover, if Π_- is the spectral projector on the negative part of the spectrum of K, then $f \in \mathcal{H}_K^{1+\epsilon}$ implies $\Pi_-((H-i0)^{-1}f) \in L^2$.

It follows then in our context:

Theorem 4.3 [Colin de Verdière and Saint-Raymond 2020]. *Assuming the existence of an escape function at $\omega = 0$ and that 0 is not an eigenvalue of H, then the solution $u(t)$ of the forced wave equation* (1) *with a smooth forcing f can uniquely be written as*

$$u(t) = u_\infty + \eta(t) + r(t),$$

where

- $u_\infty = (H - i0)^{-1}(f)$ *belongs to* $\mathcal{H}_K^{-1/2-\epsilon} \subset \mathcal{H}^{-1}$ *for all* $\epsilon > 0$,
- $\eta(t) \to 0$ *in* $\mathcal{H}_K^{-1/2-\epsilon} \subset \mathcal{H}^{-1}$ *for all* $\epsilon > 0$,
- *The function* $t \to r(t)$ *is bounded in* L^2 *has a Fourier transform vanishing near* 0,
- $\|u(t)\|_{L^2}^2 \sim ct$ *as* $t \to +\infty$ *with in general* $c > 0$.

5. Using radial source and sink propagation results

5A. *Wavefront set of u_∞.* We will now derive results on the distribution u_∞ using the radial propagation estimates of Dyatlov and Zworski, based on earlier ideas of Richard Melrose [1994] and Andras Vasy [2013].

Theorem 5.1. *The wavefront set of u_∞ is contained in the cone Γ_+.*

Proof. The result follows from the argument explained in the revised version of [Dyatlov and Zworski 2019b, Section 3.1]. This uses only the fact that K_- is a source (see Section 3D). They introduce an operator $\langle D \rangle$ which is elliptic self-adjoint invertible of degree 1. We choose it so that its principal symbol near Γ_- is $|k|$. They introduce then

$$v_\epsilon := \langle D \rangle^{-1/2}(H - i\epsilon)^{-1}\langle D \rangle^{-1/2}(g),$$

with $g = \langle D \rangle^{1/2}(f)$ and $u_\epsilon = (H - i\epsilon)^{-1}(f) = \langle D \rangle^{1/2} v_\epsilon$. Using a refined version of Theorem E.54 of [Dyatlov and Zworski 2019a], they show that there exists A, elliptic near Γ_- of degree 0, so that, for any s, the norms $\|Av_\epsilon\|_s$ are uniformly bounded in $\epsilon > 0$. We need to use here, in inequality (3.2) of [Dyatlov and Zworski 2019b], that $\|v_\epsilon\|_{-N}$ is bounded; we know it from Mourre theory for $N \geq 1$. Passing to the limit which is known to exist in $\mathcal{H}^{-1}$ by Theorem 4.3, we get that u_∞ is smooth near Γ_-. The usual propagation of singularities applied to the equation $Hu_\infty \in C^\infty$ gives the result. □

Proposition 5.2. If $Hu = 0$ and $u \in L^2(M)$, then u is smooth.

Proof. It follows from Exercise 33 in Appendix E7 of [Dyatlov and Zworski 2019a] that u is smooth near Γ_- and changing H into $-H$, u is also smooth near Γ_+. □

Remark 5.3. In the case $n = 2$, not all closed conical invariants subsets of Γ_+ can be wavefront sets of some u_∞. If the wavefront set contains the line generated by a (ws) saddle point, it contains also one of the two branches of the associated unstable manifold and hence, being closed, also an attractive invariant set. This is proved in [Guillemin and Schaeffer 1977], at least for generic cases.

5B. ***Sobolev regularity of u_∞.*** We saw in Section 4 that u_∞ belongs to $\mathcal{H}^{-1}$. Let us show that the radial sink estimates of [Dyatlov and Zworski 2019a] allow to get:

Theorem 5.4. *Under the assumption of the existence of an escape function, we have, for all $\epsilon > 0$, $u_\infty \in \mathcal{H}^{-1/2-\epsilon}$.*

Proof. We use the fact that K_+ is a sink as defined in [Dyatlov and Zworski 2019a, Definition E.52]: this is proved exactly the same way that we proved that K_- is a source in Section 3D, or just by reversing the orientations. We use then Theorem E.56 of [Dyatlov and Zworski 2019a] directly for the operator H knowing already that u_∞ is smooth away from Γ_+. Replacing $\langle \xi \rangle$ by $\langle k_+ \rangle$ we see that the threshold condition (E.5.44) is satisfied for $s < -\frac{1}{2}$. □

6. The 2-dimensional case

In this section $n = 2$.

6A. ***Morse–Smale foliation.***

Definition 6.1. A hyperbolic singular point of $\mathcal{F}$ is called *weakly stable* if the trace of the linearization of any smooth vector field generating $\mathcal{F}$ is < 0. We define similarly *weakly unstable* hyperbolic singular points. We denote these properties respectively by (ws) and (wu).

Note that if $dh \neq 0$ on Σ_0, any saddle point is either weakly stable or weakly unstable depending on whether $\mathcal{X}_h$ is pointing to infinity or not; this follows from (3) where $a \neq 0$.

Let us recall that a vector field on a surface is *Morse–Smale* if the nonwandering points are singular hyperbolic points and closed hyperbolic cycles and there is no saddle connection, i.e., there is no leave whose both limit points are saddle points. We extend this definition to oriented foliations of surfaces by choosing any vector field generating the foliation.

Theorem 6.2. *Let n be equal to 2. Let us assume that the foliation $\mathcal{F}$ is Morse–Smale. Then there exists an escape function. The set K_+ is the union of all the attracting cycles and points and all the unstable manifolds of the (ws) saddle points. The set K_- is constructed in a similar way.*

Remark 6.3. Any generic foliation of a closed surface satisfies the previous properties: Maurício Peixoto proved in the sixties that Morse–Smale vector fields on surfaces are generic; see [Palis and de Melo 1982, Chapter 4] for a detailed proof. As pointed out to me by Sylvain Courte, this genericity property extends to our context, i.e., to singular foliations of a surface embedded in a contact manifold, as it is proved in the Ph.D. thesis of Emmanuel Giroux [1991, Lemme 1.3].

Proof. Note first that K_+ and K_- are compact. They are also disjoint because there is no saddle connection.

Let us prove that K_+ is an attractor. Let K_0 be the union of the attracting component of K_+. The compact set K_0 itself is an attractor. Let us assume for simplicity that there exists an unique (ws) saddle-point b. Near b the foliation has a local normal form: the level sets of the function xy in a ball B contained in $\mathbb{R}^2_{x,y}$ with the orientation given by $x\partial_x - y\partial_y$. Let us consider a neighbourhood U_0 of K_0 satisfying the conclusion of Proposition B.2. The basin of K_0 is the complement in Z_0 of the union of all unstable cycles and all the stable manifolds of the saddle points. In particular by taking $\phi_{-T}(U_0)$ with T large enough instead of U_0 one can assume that U_0 contains $L := \{|x| \geq a, |y| \leq b\} \cap B$ with $a, b > 0$. Let us take now for the neighbourhood of K_+ the set $U := U_0 \cup L$. Clearly $\bigcap_{t \geq 0} \phi_t(U) = K_+$.

Remark 6.4. $K_0 \cup \{b\}$ is not an attractor!

Let us fix a density $d\mu$ on Z_0 and construct a vector field W generating $\mathcal{F}$ near K_+ whose divergence is nonpositive on K_+. First, we construct a vector field W_b with $\operatorname{div}(W_j) < 0$ in some neighbourhood U_b of each (ws) saddle point b. We construct also (see Appendix A.2) a vector field W_a in the basin of each attractive cycle or point a with nonpositive divergence. Let us choose a positive function l_a tending to $+\infty$ at the boundary of the basin of a. Then, for L_a large enough the set $\{l_a \geq L_a\}$ intersects the unstable manifolds Y_j of each (ws) saddle point b_j inside U_{b_j}. We choose $\chi_a \in C_o^\infty(\mathbb{R}, [0, 1])$ so that $\chi_a(s) = 1$ for $0 \leq s \leq L_a$ and $\chi_a'(s) \leq 0$ for $s \geq 0$. Then we take

$$W = \sum_a \Big((\chi_a \circ l_a) W_a + C \sum_{b_j(ws)} \psi_j W_{b_j} \Big)$$

where ψ_j satisfies

- $\psi_j \in C_0^\infty(U_{b_j}, R_+)$,
- $\psi_j = 1$ on $\{l_a \geq L_a\} \cap Y_j$,
- $d\psi_j(W) \leq 0$ on $Y_j \cap U_{b_j}$,

and $C \gg 1$. This smooth vector field is well-defined near K_+ and has negative divergence on K_+. □

6B. ***Lagrangian distributions associated to hyperbolic closed leaves.*** Let $\Gamma \subset T^\star X \setminus 0$ be a conic component of Γ_+ generated by a closed hyperbolic cycle $K_{+,0}$ of the foliation $\mathcal{F}$. The cone Γ is a conic Lagrangian submanifold of $T^\star X \setminus 0$: the Euler identity implies $\omega(\mathcal{X}_h, p\partial_p) = 0$. A theorem of Alan Weinstein [1971] implies that there is a homogeneous canonical transformation χ defined in a conic neighbourhood C of Γ whose image is a conic neighbourhood of the zero section of $T^\star\Gamma$ and so that $\chi(\Gamma)$ is the zero section of $T^\star\Gamma$. More precisely χ restricted to Γ identifies Γ to the zero section of its own cotangent bundle. Taking polar coordinates $(x, \eta) \in (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}_+ \setminus 0)$ on the cone Γ, the cotangent bundle of Γ admits coordinates $(x, \eta; \xi, y)$ with the symplectic form $d\xi \wedge dx + dy \wedge d\eta$. Note that they are not the symplectic coordinates of $T^\star X$, but those of $T^\star\Gamma$! Let X_0 be defined as $X_0 := (\mathbb{R}/2\pi\mathbb{Z})_x \times \mathbb{R}_y$. The symplectic map $(x, \eta; \xi, y) \to (x, -y; \xi, \eta)$ from $T^\star\Gamma$ onto $T^\star X_0$ identifies $T^\star\Gamma$ with $T^\star X_0$. With this identification, Γ is moved into $\Gamma_0 = \{y = 0, \ \xi = 0\}$, which is the conormal bundle of the circle of $\gamma_0 \subset X_0$ defined by $y = 0$. The Hamiltonian vector field $\mathcal{X}_0$ of $h_0 := h \circ \chi^{-1}$ preserves Γ_0. Along Γ_0, it is then given by $\mathcal{X}_0 = \partial_\xi h_0 \partial_x - \partial_y h_0 \partial_\eta$ and there $\partial_x h_0 = \partial_\eta h_0 = 0$. Because the foliation $\mathcal{F}$ is nonsingular near $K_{+,0}$, we have $\partial_\xi h_0 \neq 0$. Hence the image of the energy shell Σ_0 is given by $\xi/\eta = F(x, y)$. The projection $\pi : Z_0 \to X_0$ is a local diffeomorphism near $K_{+,0}$. Because it is a diffeomorphism on the cycle $K_{+,0}$, it is even a global diffeomorphism.

Using the tools introduced by Alan Weinstein [1975], we can build an FIO microlocally unitary $U : L^2(X) \to L^2(X_0, M)$ with M a flat bundle, called the Maslov bundle, so that $UHU^\star - K$ is smoothing in C and $\sigma_p(K) = h \circ \chi^{-1}$, $\mathrm{sub}(K) = 0$. We are then reduced to the case already studied in [Colin de Verdière and Saint-Raymond 2020] where the projection of γ onto M is a diffeomorphism.

This proves, following then [Colin de Verdière and Saint-Raymond 2020]:

Theorem 6.5. *If Γ is a component of Γ_+ generated by a closed hyperbolic stable cycle of $\mathcal{F}$, the distribution u_∞ is microlocally near Γ a Lagrangian distribution.*

7. The 3-dimensional case with S^1 invariance

Quite often in physical situations, there is an invariance of the problem by rotation or translation: internal waves in some canal [Maas and Lam 1995], inertial waves inside the earth or some stars [Rieutord and Valdettaro 2018], etc. We will study the case where $M = N_q \times S^1_\theta$ is a 3-manifold with the canonical action of S^1 by translation on the second factor. We denote by $(q, p; \theta, \tau)$ some local canonical coordinates on $T^\star M$ and assume that N is equipped with a smooth density $|dq|$ and M with $|dq\, d\theta|$. Let us give a smooth Hamiltonian $h = h(q, p, \tau)$, homogeneous of degree 0, on $T^\star M \setminus 0$ and a self-adjoint pseudodifferential operator of degree 0, H, of principal symbol h, acting on $L^2(M, |dq\, d\theta|)$. We assume that H commutes with the S^1-action. The operator H is then a direct sum of operators on M:

$$H = \bigoplus_{n\in\mathbb{Z}} H_n,$$

where H_n acts on $L^2(N, |dq|)$ as a self-adjoint pseudodifferential operator of principal symbol $h_n(q, p) := h(q, p, n)$ which is also equal to $h(q, p/n, 1)$ if $n \neq 0$.

The spectrum of H is clearly the closure of the union of the spectra of the H_n's.

7A. ***Spectra of H and the H_n's.*** Let us define $h_0(q, p) := h(q, p, 0)$ and $h_1(q, p) = h(q, p, 1)$. Note that h_1 is a smooth symbol of degree 0 on $T^\star N$ which is asymptotic to h_0 at infinity. The essential spectrum of H is the interval $I_\infty := [a_\infty, b_\infty]$, where $a_\infty = \inf h_1$ and $b_\infty = \sup h_1$. The essential spectrum of the H_n's is quite different: from the identities

$$h(q, p, n) = h\left(q, \frac{p}{|p|}, \frac{n}{|p|}\right) = h_0(q, p) + O\left(\frac{1}{|p|}\right),$$

one gets that the principal symbol of H_n is h_0. Hence the essential spectrum of any of the H_n's is $I_0 := [a_0, b_0]$, where $a_0 = \inf h_0$ and $b_0 = \sup h_0$. Note that we have $I_0 \subset I_\infty$ and they are often identical in the applications to physical problems.

We are interested in more precise properties of the spectra: we claim that, in $I_\infty \setminus I_0$, the spectrum of H is pure point dense; i.e., there is a basis of L^2 pairwise orthogonal eigenfunctions. Moreover the eigenvalues of H_n obey a Weyl rule when $n \to \infty$. One expects that the spectrum has no embedded eigenvalues in the interior of I_0. But quasimodes of the type "well in an island" are possible if the dynamics of h_1 has stable bounded invariant sets (see Section 7B).

Theorem 7.1 (Weyl law). *The spectra $\sigma(H_n)$ of the operators H_n in $I_\infty \setminus I_0$ are discrete. For any compact interval J included in $I_\infty \setminus I_0$, we have*

$$\#\{\sigma(H_n) \cap J\} \sim_{n\to\infty} \frac{n^2}{4\pi^2} \operatorname{vol}(\{q, p \mid h_1(q, p) \in J\}),$$

where the volume is defined with the Liouville measure on $T^\star N$ and the eigenvalues of H_n in J are counted with multiplicities.

Proof. The full symbol of H can be written as

$$\tilde{h} = h(q, p, \tau) + \sum_{j=1}^{\infty} k_j(q, p, \tau),$$

with k_j homogeneous of degree j. Hence H_n can be viewed as a semiclassical pseudodifferential operator on N of semiclassical symbol

$$\tilde{h}_n = h_1(q, \hbar p) + \sum_{j=1}^{\infty} \hbar^j k_j(q, \hbar p, 1),$$

with $\hbar = 1/n$. The theorem follows hence from the semiclassical Weyl asymptotics. □

7B. *Classical dynamics.* We will assume that the frequency $\omega = 0$ is fixed and the 2-dimensional Hamiltonian $h_0(q, p) := h(q, p, 0)$ admits an escape function. We will look at the dynamics of $h_1 := h(q, p, 1)$. Note that the dynamics of h reduces on each set $\tau = a$ with $a \neq 0$ to that of h_1 by some simple rescaling of the time. Moreover

$$\lim_{p\to\infty} h_1(q, p) = h_0(q, p).$$

Near infinity the dynamics still admits an escape (Lyapunov function) and hence the orbits, if they come close enough to infinity, will converge to K_+ at $+\infty$ and K_- at $-\infty$. The dynamics $t \to \phi_t$ of h_1 is hence complete. We split the phase space into three pieces: $T^\star M = \Omega \cup C_+ \cup C_-$, where

- Ω is the set of (q, p) so that $\phi_t(q, p) \to K_\pm$ as $t \to \pm\infty$,
- C_+ is the set of (q, p) so that $\phi_t(q, p)$ stays bounded for $t \geq 0$,
- C_- is the set of (q, p) so that $\phi_t(q, p)$ stays bounded for $t \leq 0$.

Finally, we define $C := C_+ \cap C_-$, the set (q, p) so that $\phi_t(q, p)$ stays bounded for $t \in \mathbb{R}$. In the literature, C is called the *trapped set*.

It could happen that C supports some quasimodes associated to the semiclassical parameter $1/n$. Generically, these quasimodes are not close to true L^2-eigenfunctions because such eigenfunctions do not exist. They are still visible in the wave dynamics for a very long time…

8. Open problems

There are still many open problems. Let us describe a few of them:

- How does the spectral picture extend outside the intervals with a.c. spectra? This problem is already not solved in the simple case where Z_0 is a 2-torus, assuming the existence of a global transversal to the foliation, and the Poincaré map loses its hyperbolicity in a generic way.

- More generally, can we study what happens at the critical values of h assuming that this function is Morse or even Morse–Bott on $S^\star M$?

- What can we do in the case of a manifold with boundary? In particular, can we say something in the case of a polygon which is studied in the experiments of the Thierry Dauxois's team [Brouzet 2016].

- Prove the generic absence of embedded eigenvalues.

- Consider the *viscous case*, namely the forced equation

$$\frac{du}{dt} + iHu - \sigma\Delta u = fe^{-i\omega t}, \quad u(0) = 0. \tag{4}$$

where σ is a positive number and Δ is the Laplacian associated to some Riemannian metric on M. Study the "small viscosity" limit $\sigma \to 0$? In particular, do the limits $\sigma \to 0^+$ and $t \to +\infty$ commute?

- There is a discrete analogue of Mourre's theory for unitary maps; see for example [Fernández et al. 2013]. What can be said from the spectral theory of the unitary action of a diffeomorphism of a closed manifold on half-densities? For example, what is the spectral theory of a diffeomorphism of the circle with irrational rotation number which is not C^1-conjugated to a rotation?

Appendix A: Divergences

A1. ***Formulae.*** Let us give a smooth vector field W whose flow is denoted by ϕ_t, $t \in \mathbb{R}$, and a smooth density $d\mu$. The divergence of W with respect to $d\mu$ is the function defined by

$$\operatorname{div}_{d\mu}(W) := \frac{\mathcal{L}_W d\mu}{d\mu},$$

where the Lie derivative $\mathcal{L}_W d\mu$ is defined by

$$\mathcal{L}_W d\mu := \frac{d}{dt}_{|t=0} \phi_t^\star(d\mu).$$

Cartan's formula gives

$$\operatorname{div}_{d\mu}(W) = \frac{d(\iota(W)d\mu)}{d\mu},$$

where $\iota(\cdot)$ is the inner product. In particular, we get the useful formulae

$$\operatorname{div}_{d\mu}(fW) = df(W) + f \operatorname{div}_{d\mu}(W),$$
$$\operatorname{div}_{gd\mu}(W) = \frac{dg(W)}{g} + \operatorname{div}_{d\mu}(W).$$

A2. ***Extending vector fields with negative divergence.***

Lemma A.1. *Let us assume that the invariant compact set K admits a smooth (Liapounov) function l defined in the basin B of K with $dl(W) < 0$ outside K and $l(K) = 0$ and $l \to +\infty$ at the boundary of B (this is the case in particular if the attractor K is hyperbolic). If the vector field W satisfies $\operatorname{div}_{d\mu}(W) < 0$ in some open neighbourhood V of K, then there exists a vector field $W_1 = FW$ in B, so that $F > 0$ and $\operatorname{div}_{d\mu}(W_1) < 0$ in B.*

Proof. Let us choose $r > 0$ so that $\{l \le r\} \subset V$. It is enough to take $F = 1$ in $\{l \le r\}$ and, for any $x \in B$ with $l(x) = r$ and any $t \ge 0$,

$$F(\phi_t(x)) := e^{\int_0^t \Phi(\phi_s(x))\,ds},$$

with Φ smooth, $\Phi = 0$ near $l(y) \le r$ and, for all y with $l(y) > r$, $\Phi(y) < -\operatorname{div}_{d\mu}(W)(y)$. □

Appendix B: Attractors and their basins

We give here some useful definitions and elementary properties of dynamical systems. We consider a smooth closed manifold X with a smooth vector field V whose flow is the 1-parameter group of diffeomorphisms of X denoted by ϕ_t, $t \in \mathbb{R}$. The definitions and statements are taken from [Hurley 1982]. We have the following:

Definition B.1. (1) If $K \subset X$ is a compact invariant set, i.e., a subset of X preserved by the flow, K is called an *attractor* if there exists an open neighbourhood U of K in X so that $K = \bigcap_{t \ge 0} \phi_t(U)$.

(2) If K is an attractor, the *basin* of K is the set of points x so that $\phi_t(x) \to K$ as $t \to +\infty$.

(3) A point $x \in X$ is wandering if there exists a neighbourhood U of x so that $\phi_t(U) \cap U = \varnothing$ for t large enough.

The set of wandering points is open. The basins are open subsets of X. We will need the following properties [Hurley 1982, Lemma 1.6]:

Proposition B.2. If K is an attractor, and V a neighbourhood of K, there exists an open set U satisfying:

(1) $K \subset U \subset V$.

(2) $\bigcap_{t \ge 0} \phi_t(\bar{U}) = K$.

(3) For all $t \ge 0$, we have $\phi_t(U) \subset U$.

The convergence of $\phi_t(m)$ to K is uniform on every compact subset of the basin of K.

The previous sets are the same for V and fV, where $f : X \to\,]0, +\infty[$ is smooth. They can therefore be defined for a 1-dimensional oriented foliation generated by a smooth vector field. In particular the open set U of the previous proposition is independent of f.

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GLOBAL EXISTENCE FOR THE DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION WITH ARBITRARY SPECTRAL SINGULARITIES

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We show that the derivative nonlinear Schrödinger (DNLS) equation is globally well-posed in the weighted Sobolev space $H^{2,2}(\mathbb{R})$. Our result exploits the complete integrability of the DNLS equation and removes certain spectral conditions on the initial data required by our previous work, thanks to Zhou's analysis (*Comm. Pure Appl. Math.* **42**:7 (1989), 895–938) on spectral singularities in the context of inverse scattering.

1. Introduction

We prove global well-posedness of the Cauchy problem for the derivative nonlinear Schrödinger (DNLS) equation

$$\begin{cases} iu_t + u_{xx} - i\varepsilon(|u|^2u)_x = 0, \quad \varepsilon = \pm 1, \\ u(x, t=0) = u_0(x), \end{cases} \tag{1-1}$$

with initial condition u_0 in the weighted Sobolev space

$$H^{2,2}(\mathbb{R}) = \{u \in L^2(\mathbb{R}) : u''(x),\ x^2u(x) \in L^2(\mathbb{R})\}.$$

In contrast to previous work using PDE methods [Fukaya et al. 2017; Hayashi and Ozawa 1992; Wu 2015], we impose no upper bound on the L^2-norm of the initial data (although we require more smoothness and decay than these authors), and in contrast to previous work using completely integrable methods [Jenkins et al. 2018a; 2018b; Lee 1989; Liu et al. 2016; Pelinovsky et al. 2017], we make no spectral restrictions to "generic initial data" that rule out singularities of the spectral data associated to the initial condition. We use the complete integrability of DNLS discovered by Kaup and Newell [1978]. As we will explain,

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an essential ingredient of our work is Zhou's approach [1989a; 1998] to inverse scattering with arbitrary spectral singularities building on the work of [Beals and Coifman 1984]; to our knowledge, the present paper constitutes the first application of these techniques to global well-posedness questions for integrable PDEs that involves *no* spectral assumptions on the initial data. This is significant in that Zhou's methods are quite general and are likely applicable to well-posedness questions for other integrable PDEs in one space dimension.

To describe our results more precisely, we recall that the invertible gauge transformation

$$\mathcal{G}(u)(x) = u(x)\exp\left(i\varepsilon\int_x^\infty |u(y)|^2\,dy\right)$$

maps solutions of (1-1) to solutions of

$$\begin{cases} iq_t + q_{xx} + i\varepsilon q^2\bar{q}_x + \frac{1}{2}|q|^4 q = 0, \quad \varepsilon = \pm 1, \\ q(x, t=0) = q_0(x). \end{cases} \tag{1-2}$$

Equation (1-2) is more directly amenable to inverse scattering. It is shown in [Colliander et al. 2002] that $\mathcal{G}$ is a continuous map from H^s to H^s, $s > \frac{1}{2}$. It is straightforward to check that it is a locally Lipschitz continuous map from $H^{2,2}(\mathbb{R})$ to itself. Indeed, setting $\psi(u) = \int_x^\infty |u(y)|^2\,dy$ and writing $\mathcal{G}(u) = u + (e^{i\varepsilon\psi} - 1)u$, it is sufficient to prove the map $u \mapsto e^{i\varepsilon\psi}$ is Lipschitz continuous from $H^{2,2}(\mathbb{R})$ into $W^{2,\infty}(\mathbb{R})$ and apply the Leibniz rule. In particular, one easily proves that

$$\|1 - e^{i\varepsilon\int_x^\infty(|u(y)|^2 - |v(y)|^2)}\,dy\|_{W^{2,\infty}} \lesssim \|u - v\|_{H^{2,2}},$$

where the implied constants may depend on $\|u\|_{H^{2,2}}$ and $\|v\|_{H^{2,2}}$. Global well-posedness in $H^{2,2}(\mathbb{R})$ for (1-1) and (1-2) are thus equivalent. In the following, we fix $\varepsilon = -1$, since solutions of (1-2) with $\varepsilon = 1$ are mapped to solutions of (1-2) with $\varepsilon = -1$ by $q(x,t) \mapsto q(-x,t)$. The main result of the paper is the following theorem:

Theorem 1.1. *Suppose $q_0 \in H^{2,2}(\mathbb{R})$. There exists a unique solution $q(x,t)$ of* (1-2) *with $q(x,t=0) = q_0$ and $t \mapsto q(\cdot\,,t) \in C([-T,T], H^{2,2}(\mathbb{R}))$ for every $T > 0$. Moreover, the map $q_0 \mapsto q$ is Lipschitz continuous from $H^{2,2}(\mathbb{R})$ to $C([-T,T], H^{2,2}(\mathbb{R}))$ for every $T > 0$.*

The Cauchy problem for (1-1) is locally well-posed in $H^1(\mathbb{R})$ as well as in weighted spaces $H^{m,0} \cap H^{0,m}$ ($m \geq 1$) and it is globally well-posed for small initial data [Tsutsumi and Fukuda 1980; Hayashi and Ozawa 1992]. More precisely, it was proved in [Hayashi and Ozawa 1992] that for any initial condition $u_0 \in H^1(\mathbb{R})$ such that $\|u_0\|_{L^2} < \sqrt{2\pi}$, global well-posedness holds in $H^1(\mathbb{R})$. The smallness condition was recently improved to $\|u_0\|_{L^2} < \sqrt{4\pi}$ (or $\|u_0\|_{L^2} = \sqrt{4\pi}$ with additional conditions on initial data) [Wu 2015; Fukaya et al. 2017].

The present paper also builds on previous work of the coauthors which proved global well-posedness of the DNLS equation for initial conditions u_0 in weighted Sobolev spaces under some additional conditions that exclude the so-called *spectral singularities* [Jenkins et al. 2018b; Liu 2017; Pelinovsky et al. 2017]. In this context, we proved global well-posedness for data in an open and dense set of $H^{2,2}(\mathbb{R})$ which allows finitely many resonances, which refer to eigenvalues away from the continuous spectrum but no spectral singularities, and also established the long-time behavior of solutions in the form of the soliton resolution [Jenkins et al. 2018a]. We will discuss precisely in Section 2 the meaning of spectral singularities. In the present paper, we remove all spectral assumptions on the initial data and obtain global well-posedness of the DNLS equation for general initial condition in $H^{2,2}(\mathbb{R})$.

Our approach is inspired by the work of Zhou [1989a; 1995; 1998], who developed new tools to construct direct and inverse scattering maps that are insensitive to singularities of the spectral data. We emphasize that spectral singularities may affect the long-time behavior of solutions in the same way that eigenvalues affect the long-time behavior of solutions through soliton resolution (see [Jenkins et al. 2018a], where the soliton resolution conjecture is proved for generic initial data). In the case of the focusing cubic nonlinear Schrödinger equation, Kamvissis [1996] studied the effect of a single spectral singularity on the large-time behavior of solutions. He showed that the latter is limited to the region of the (x, t)-plane in which the spectral singularity is close to the point of stationary phase, and there, slightly modifies the rate of decay. In a future paper, we will investigate how spectral singularities affect the long-time behavior of DNLS solutions. A new version of the inverse scattering transform has been recently introduced in [Bilman and Miller 2019] to deal with arbitrary-order poles and spectral singularities in the context of focusing NLS with nonzero boundary conditions. This method relies on the initial value problem for the Lax pair and avoids the use of a cut-off potential.

Occurrence of spectral singularities in the spectral problem is not an exceptional phenomenon. In the context of the focusing NLS equation, Zhou [1989a] constructed one example in which Schwartz class potential leads to infinitely many eigenvalues accumulating on the real line to form a spectral singularity and another example where infinitely many spectral singularities accumulate. In Appendix B of [Jenkins et al. 2018b], we analyzed a family of potentials of the form $q(x) = A\,\mathrm{sech}(x)e^{i\phi(x)}$ for which one can explicitly compute the scattering data, thus illustrating various characterizations of the spectral map. In particular, we exhibit potentials for which the associated spectral problem has either no discrete spectrum, or exactly n eigenvalues and no spectral singularities, or n eigenvalues and one spectral singularity.

To explain our methods, we will sketch the completely integrable method for (1-2) as discovered in [Kaup and Newell 1978] in two steps. First, we describe how the method works when the initial data do not support solitons or spectral singularities. Next, we describe how Zhou's method [1989a; 1998] can be extended to the DNLS equation to construct global solutions in the presence of solitons and spectral singularities.

1A. ***The inverse scattering method*: *no singularities.*** Kaup and Newell [1978] showed that the flow determined by (1-2) may be linearized by spectral data associated to the linear problem

$$\frac{d}{dx}\Psi(x,\zeta) = -i\zeta^2\sigma\Psi + \zeta Q(x)\Psi + P(x)\Psi, \quad \zeta \in \mathbb{R}\cup i\mathbb{R}, \tag{1-3}$$

where $\Psi(x,\zeta)$ is a 2×2 matrix-valued function of x and

$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q(x) = \begin{pmatrix} 0 & q(x) \\ -\overline{q(x)} & 0 \end{pmatrix}, \quad P(x) = \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix} = \frac{i}{2}\begin{pmatrix} |q(x)|^2 & 0 \\ 0 & -|q(x)|^2 \end{pmatrix}. \tag{1-4}$$

Later, it will be convenient to set $\Psi(x,\zeta)=m(x,\zeta)e^{-ix\zeta^2\sigma}$, so that m solves the equation

$$\frac{d}{dx}m(x,\zeta)=-i\zeta^2\,\mathrm{ad}(\sigma)m+\zeta Q(x)m+P(x)m, \tag{1-5}$$

where

$$\mathrm{ad}(\sigma)A=\sigma A-A\sigma.$$

Equation (1-3) admits bounded solutions provided $q\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$ and $\zeta\in\mathbb{R}\cup i\mathbb{R}$. There exist unique solutions $\Psi^{\pm}(x,\zeta)$ of (1-3) satisfying the respective boundary conditions

$$\lim_{x\to\pm\infty}\Psi^{\pm}(x,\zeta)e^{i\zeta^2x\sigma}=I,\qquad I=\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

These *Jost solutions* have determinant 1 and define action-angle variables a and b for the flow (1-2) through the relation

$$\Psi^+(x,\zeta)=\Psi^-(x,\zeta)\begin{pmatrix}a(\zeta)&b(\zeta)\\ \breve{b}(\zeta)&\breve{a}(\zeta)\end{pmatrix}.$$

That is, if $q(x,t)$ solves (1-2), and $a(\zeta,t)$ and $b(\zeta,t)$ are the corresponding scattering data for $q(\,\cdot\,,t)$, then

$$\dot{a}(\zeta,t)=0,\quad \dot{b}(\zeta,t)=-4i\zeta^4b(\zeta,t). \tag{1-6}$$

Thus, if the map $q\mapsto(a,b)$ can be inverted, one can hope to solve (1-2) via a composition of the direct scattering map $q\to(a,b)$, the flow map defined by (1-6), and the inverse map $(a,b)\mapsto q$.

The functions a and $\breve{a}$ have analytic extensions to the respective regions $\Omega^-=\{\operatorname{Im}z^2<0\}$ and $\Omega^+=\{\operatorname{Im}z^2>0\}$ (see Figure 1). Zeros of a (resp. $\breve{a}$) in Ω^- (resp. Ω^+) are associated to soliton solutions of (1-2), while zeros of a or $\breve{a}$ on $\mathbb{R}\cup i\mathbb{R}$ are called *spectral singularities*. For the moment, we assume that a and $\breve{a}$ are zero-free in their respective regions of definition. This allows us to define the reflection coefficients

$$r(\zeta)=\frac{\breve{b}(\zeta)}{a(\zeta)},\quad \breve{r}(\zeta)=\frac{b(\zeta)}{\breve{a}(\zeta)},\quad \zeta\in\mathbb{R}\cup i\mathbb{R}. \tag{1-7}$$

The map $q\mapsto r$ is the *direct scattering map*. One can recover a and b from r by solving a scalar Riemann–Hilbert problem. By symmetry one has that $\breve{r}(\zeta)=-\overline{r(\bar{\zeta})}$.

In his thesis, J.-H. Lee [1983] formulated the inverse scattering map as a Riemann–Hilbert problem (RHP) in which r and $\breve{r}$ enter as jump data for a piecewise analytic function. To describe it, denote by $\mathbb{R}\cup i\mathbb{R}$ the oriented contour, shown in Figure 1, left, that bounds $\Omega^{\pm}$ with Ω^+ to the left and Ω^- to the right. An oriented contour that divides $\mathbb{C}$ into two such regions Ω^+ and Ω^- is called a *complete* contour.

Denote by $m^{\pm}$ the *renormalized Jost solutions* $m^{\pm}=\Psi^{\pm}e^{ix\zeta^2\sigma}$. Let m_1^+ and m_2^+ denote the first and second columns of m^+, with a similar notation m_1^-,m_2^- for the columns of m^-. From the integral equations (2-2)–(2-5), it is easy to see that, for each x, $m_1^-(x,\zeta)$ and $m_2^+(x,\zeta)$ extend to analytic functions of $z\in\Omega^+$, while $m_1^+(x,\zeta)$ and $m_2^-(x,\zeta)$ extend to analytic functions of $z\in\Omega^-$. From these columns, one can construct left and right *Beals–Coifman solutions* $M(x,z)$ of (1-5) which are piecewise analytic for $z\in\mathbb{C}\setminus(\mathbb{R}\cup i\mathbb{R})$ and normalized so that $\lim_{x\to\infty}M(x,z)=I$ (right-normalized, (2-8)) or

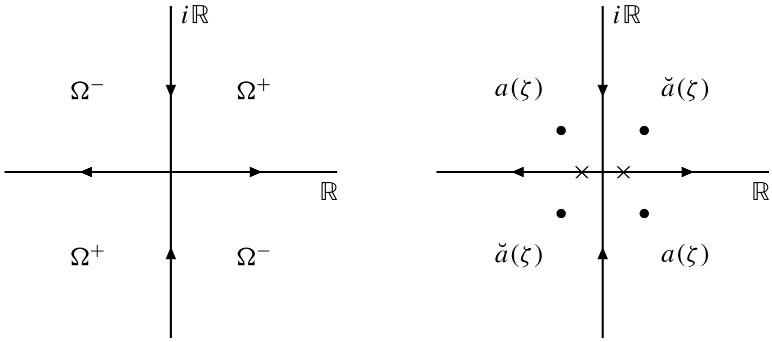

Figure 1. Left: the regions $\Omega^\pm$ and the contour $\mathbb{R} \cup i\mathbb{R}$. Right: zeros of a and $\breve{a}$.

$\lim_{x\to-\infty} M(x, z) = I$ (left-normalized, (2-9)). In what follows, we discuss the right-normalized solution. Enforcing these normalizations involves division by a and $\breve{a}$ so any zeros of a and $\breve{a}$ would create new singularities.

The Beals–Coifman solution solves a Riemann–Hilbert problem (RHP) in the z-variable. Thus x plays the role of a parameter and, for each x, the function $M(x, z)$ is piecewise analytic in z with prescribed asymptotics as $z \to \infty$ and prescribed multiplicative jumps along the contour $\mathbb{R} \cup i\mathbb{R}$.

More precisely, for each x, the piecewise analytic function $M(x, \cdot)$ solves the following Riemann–Hilbert problem.

Riemann–Hilbert Problem 1.2. *For each $x \in \mathbb{R}$, find an analytic*[1] *function $M(x, \cdot) : \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R}) \to$ $\mathrm{SL}(2, \mathbb{C})$ with*

(i) $\lim_{z\to\infty} M(x, z) = I$,

(ii) *M has continuous boundary values $M_\pm$ as $z \to \zeta \in \mathbb{R} \cup i\mathbb{R}$ from $\Omega^\pm$, and*

(iii) *$M_\pm$ obey the jump relation*

$$M_+(x, \zeta) = M_-(x, \zeta) e^{-ix\zeta^2 \operatorname{ad}\sigma} v(\zeta),$$

where

$$e^{-ix\zeta^2 \operatorname{ad}\sigma} v(\zeta) = \begin{pmatrix} 1 + |r(\zeta)|^2 & e^{-2ix\zeta^2} r(\zeta) \\ -e^{2ix\zeta^2} \breve{r}(\zeta) & 1 \end{pmatrix}.$$

The matrix $e^{-ix\zeta^2 \operatorname{ad}\sigma} v$ is called the *jump matrix* for the RHP 1.2. We recover $q(x)$ through the asymptotic formula

$$q(x) = 2i \lim_{z\to\infty} z M_{12}(x, z), \tag{1-8}$$

which may easily be deduced from the large-z-expansion for $M(x, z)$ and the fact that $M(x, z)$ satisfies (1-5).

RHP 1.2 and the reconstruction formula (1-8) define the *inverse scattering map*.

[1] If a has zeros, M is meromorphic and discrete data for each pole must be added to close the problem. For the present, we assume that a and $\breve{a}$ are zero-free.

1B. *The inverse scattering method: singularities.* So far, we have assumed that a and $\breve{a}$ are zero-free; however, zeros of a and $\breve{a}$ do occur for data of physical interest. By the symmetries

$$\breve{a}(\zeta) = \overline{a(\bar{\zeta})}, \quad a(-\zeta) = a(\zeta), \tag{1-9}$$

zeros of a and $\breve{a}$ in $\mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$ occur in "quartets", as shown in Figure 1, right. These quartets correspond to soliton solutions of (1-2). The further symmetry

$$b(-\zeta) = -b(\zeta), \quad \breve{b}(\zeta) = -\overline{b(\bar{\zeta})} \tag{1-10}$$

and the determinant condition

$$a(\zeta)\breve{a}(\zeta) - b(\zeta)\breve{b}(\zeta) = 1$$

imply

$$|a(it)|^2 - |b(it)|^2 = 1$$

for all real t, so a has no zeros on the imaginary axis. However, zeros of a on the real axis may occur and correspond to *spectral singularities*. RHP 1.2 is no longer solvable since the jump matrix v now has singularities on the contour $\mathbb{R} \cup i\mathbb{R}$; moreover, any zeros of a and $\breve{a}$ in their domains of analyticity will make the Beals–Coifman solutions meromorphic rather than analytic.

On the other hand, any zeros of a and $\breve{a}$ lie in the disc

$$B(0, R) = \{z : |z| < R\},$$

where R is determined by $\|q\|_{H^{2,2}}$ (see, for example, [Liu 2017, Proposition 3.2.5]). Moreover, for $\|q\|_{H^{2,2}}$ sufficiently small, a and $\breve{a}$ are zero-free on their respective domains. We will say that such a potential has *zero-free scattering data*.

Zhou's insight [1989a; 1998] is that RHP 1.2 can be modified in the following way. First, choose R so large that a and $\breve{a}$ have no zeros in $\mathbb{C} \setminus B(0, R)$, and denote by Σ_R the circle of radius R centered at 0.

Choose $x_0 > 0$ sufficiently large so that the potential

$$q_{x_0}(x) = \begin{cases} 0, & x \le x_0, \\ q(x), & x > x_0, \end{cases} \tag{1-11}$$

has zero-free scattering data; a sufficient condition to achieve this is that

$$\sup_{|z| \le R} \|zQ + P\|_{L^1(x > x_0)} < \tfrac{1}{2}$$

(see Section 2, (2-12) and the discussion that follows).

Note that both x_0 and R may be chosen uniformly for q in a bounded subset of $H^{2,2}(\mathbb{R})$. Next, let $M^{(0)}(x, z)$ denote the solution to RHP 1.2 for q_{x_0}. The function $M^{(0)}$ is analytic in $\mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$ with continuous boundary values $M_\pm^{(0)}$ on $\mathbb{R} \cup i\mathbb{R}$. Indeed, resonances and spectral singularities for q_{x_0} are ruled out by the small norm assumption.

Remark 1.3. Although the sharp cut-off potential q_{x_0} is not in the $H^{2,2}$ space, it is in $H^{0,2}$ and we will only need this decay property to construct $H^{2,0}$ scattering data on a bounded set.

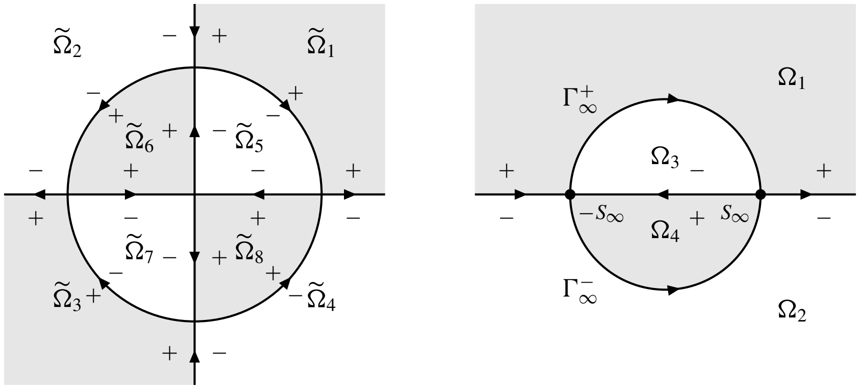

Figure 2. The augmented contour $\Gamma = \mathbb{R} \cup \Gamma_\infty$ (right) in the λ-plane for the modified Riemann–Hilbert problem RHP 3.1 and its preimage $\Sigma = \mathbb{R} \cup i\mathbb{R} \cup \Sigma_\infty$ (left) in the ζ-plane. The regions $\Omega_+ = \Omega_1 \cup \Omega_4$ (shaded) and $\Omega_- = \Omega_2 \cup \Omega_3$ lie, respectively, to the left and right of Γ.

Denote by $M^{(1)}$ the unique solution of the Volterra integral equation

$$M^{(1)}(x,\zeta) = I + \int_{x_0}^{x} e^{i(y-x)\zeta^2 \operatorname{ad}\sigma}\big(\zeta Q(y) M^{(1)}(y,\zeta) + P(y) M^{(1)}(y,\zeta)\big)\,dy \tag{1-12}$$

and define

$$M^{(2)}(x,\zeta) = M^{(1)}(x,\zeta) e^{-i(x-x_0)\zeta^2 \operatorname{ad}\sigma} M^{(0)}(x_0,\zeta). \tag{1-13}$$

Since $M^{(2)}$ and $M^{(0)}$ agree at $x = x_0$, it follows by uniqueness that $M^{(2)}(x,\zeta) = M^{(0)}(x,\zeta)$ for all $x \geq x_0$. We notice that $M^{(1)}(x_0, z)$ is entire in z; thus $M^{(2)}(x_0, z)$ and $M^{(0)}(x_0, z)$ share the same domain of analyticity. Define the contour

$$\Sigma = \mathbb{R} \cup i\mathbb{R} \cup \Sigma_\infty, \quad \Sigma_\infty = \{|z| = R\},$$

oriented as in Figure 2, left, and define

$$\boldsymbol{M}(x,z) = \begin{cases} M(x,z), & z \in \mathbb{C} \setminus (B(0,R) \cup \Sigma), \\ M^{(2)}(x,z), & z \in B(0,R) \setminus \Sigma. \end{cases} \tag{1-14}$$

The function $M(x,z)$ is piecewise analytic on $\mathbb{C} \setminus (B(0,R) \cup (\mathbb{R} \cup i\mathbb{R}))$ because a and $\breve{a}$ are zero-free for $|z| > R$. By construction, the function $M^{(2)}$ is piecewise analytic in $B(0,R) \setminus (\mathbb{R} \cup i\mathbb{R})$. The new unknown $\boldsymbol{M}(x,z)$ obeys RHP 3.7.

The jump matrix of the Riemann–Hilbert problem for $\boldsymbol{M}(x,z)$ is unchanged outside the circle Σ_∞ but is replaced inside by new jump data that may be explicitly computed from q_0 and q_{x_0}; see Section 2 for a full discussion. Since $\boldsymbol{M}(x,\zeta) = M(x,\zeta)$ in a neighborhood of infinity, we can still recover q from the reconstruction formula (1-8). To carry out the analysis, we change variables from ζ to $\lambda = \zeta^2$ and actually analyze RHP 3.1.

To analyze the direct map (from the given potential q_0 to the jump matrix for the augmented contour Σ) and the inverse map (from the jump matrix to the recovered potential) it is helpful to exploit the symmetry reduction of the spectral problem (1-3) to the spectral variable $\lambda = \zeta^2$. Under the map $\zeta \mapsto \zeta^2$, the augmented contour Σ is mapped to the contour

$$\Gamma = \mathbb{R} \cup \Gamma_\infty, \quad \Gamma_\infty = \{|z| = S_\infty\}, \tag{1-15}$$

with induced orientation as shown in Figure 2, right; the shaded and unshaded regions shown in Figure 2, left, are mapped to the shaded and unshaded regions shown in Figure 2, right. The circle Σ_∞ is mapped to Γ_∞, the circle of radius $S_\infty = R^2$; we let $\Gamma_\infty^\pm = \Gamma_\infty \cap \mathbb{C}^\pm$. The augmented contour Γ in Figure 2, right, decomposes $\mathbb{C} \setminus \Gamma$ into two sets. The notation for the sets $\Omega_\pm$, consistent with our use of subscripts for boundary values, should not be confused with the superscripted sets $\Omega^\pm = \{\pm \operatorname{Im} z^2 > 0\}$ previously introduced:

$$\Omega_+ = \Omega_1 \cup \Omega_4 \quad \text{and} \quad \Omega_- = \Omega_2 \cup \Omega_3 \tag{1-16}$$

such that Ω_+ (resp. Ω_-) lies everywhere to the left (resp. right) of Γ. The contour Γ can be viewed simultaneously as the boundary of Ω_+ or Ω_-, and we will write

$$\Gamma_+ = \partial\Omega_+ \quad \text{or} \quad \Gamma_- = \partial\Omega_- \tag{1-17}$$

when we want to emphasize either interpretation. Finally, in what follows, we will set

$$\mathbb{R}_\infty = \mathbb{R} \setminus [-S_\infty, S_\infty]; \tag{1-18}$$

that is, $\mathbb{R}_\infty$ is the part of the contour $\mathbb{R}$ outside the circle Γ_∞.

In the rest of the paper, the letter z is used as a general notation for a complex variable off contours, while ζ refers the variable on the contour Σ and $\lambda = \zeta^2$ to the variable on the contour Γ.

One can compute the jump data for the Riemann–Hilbert problem on the contour Γ explicitly in terms of scattering data for q, scattering data for q_{x_0}, and normalized Jost solutions for q (see Figure 3 and Proposition 2.2); it is then easy to show that the direct spectral map from $q \in H^{2,2}(\mathbb{R})$ to these scattering data is continuous in a natural topology on the jump data (see Theorem 2.7 for a precise statement).

It remains to show that the scattering data can be time-evolved continuously and that RHP with scattering data as described in Theorem 2.7 can be uniquely solved and used to recover the potential q. To do so, much as in [Jenkins et al. 2018b; Liu et al. 2016], we show that the Riemann–Hilbert problem in the λ-variable is equivalent to a Riemann–Hilbert problem in the ζ-variable which is uniquely solvable. We then apply Zhou's uniqueness theorem (see Proposition 2.1 and [Zhou 1989b]) to obtain unique solvability. We also need to show that the recovered potential is continuous in the scattering data; this will follow from Zhou's results [1998] and our previous results on the scattering transform in [Liu et al. 2016].

Finally, we sketch the content of the paper.

Section 2 is devoted to the direct scattering map. In Section 2A, we recall the basic properties of the scattering problem and Beals–Coifman solutions in the ζ-variable. In Sections 2B and 2C, we construct the scattering data in the ζ- and λ-variables. The goal is to choose the scattering data so the inverse scattering problem will allow a reconstruction formula for the potential. For this purpose, we implement Zhou's method to deal with spectral singularities. In this setting, the usual Beals–Coifman solutions are changed to piecewise analytic functions according to (1-14). We give explicit formulas for the corresponding jump matrices along the augmented contours.[2] We use the approach of [Zhou 1998] (see also [Trogdon and Olver 2016]) to address the matching conditions at the intersection points of the contours and give a full description of the jump matrices and their factorization. In Section 2D, we establish the time evolution of the scattering data. Finally, as shown in [Deift and Zhou 2003, Lemma 3.4] in the absence of spectral singularities, right and left RHPs are needed to obtain decay rate of the potential as $x \to \pm\infty$; there are separate left and right augmented RHPs for the same purpose in this paper. In Section 2E, we compute the auxiliary matrix that relates their corresponding jump matrices (see (2-34)). This result allows us to focus on the right RHP thereafter.

In Section 3, we show that the RHP with the augmented contour and the jump matrices, as derived in Sections 2B and 2C, has a unique solution. The proof follows the lines of the proof given in [Liu 2017]. Suppose the RHP in λ has a null vector $\boldsymbol{N}$, i.e., a solution which satisfies the jump conditions but vanishes as $z \to \infty$. This null vector corresponds to a homogeneous solution ν of the Beals–Coifman integral equation for the RHP, and induces a homogeneous solution μ to the Beals–Coifman integral equation for the RHP in the ζ-variable which is the zero solution due to Zhou's vanishing lemma [1989b, Theorem 9.3]. It follows from Fredholm theory that the Beals–Coifman equation for μ is uniquely solvable, and hence that the RHP in the λ-variable is also uniquely solvable.

As in [Jenkins et al. 2018b], we establish the existence and uniqueness of solutions to the RHP for scattering data in a larger space Y (see Definition 3.3) in order to obtain uniform resolvent estimates for scattering data in bounded sets of a smaller space.

In Section 4, we establish the mapping properties of the inverse scattering map and estimate the potential obtained from the reconstruction formula in the λ-variable. This analysis requires another technical step taken from Zhou's method [1998]. As seen in Figure 2, right, the orientation of the piece of the contour (S_∞^-, S_∞^+) goes from right to left. A second augmentation shown in Figure 5 allows the new contour to have the usual orientation thus allowing standard estimates of the Cauchy projectors on $\mathbb{R}$ to be used to obtain decay estimates on the potential. The Lipschitz continuity follows from the second resolvent identity.

To analyze Riemann–Hilbert problems with self-intersecting contours, we make use of certain Sobolev spaces of functions that obey continuity conditions at self-intersection points. For the reader's convenience, we briefly describe these Sobolev spaces in Appendix A. In Appendix B, we present the necessary abstract functional analysis tools used to prove uniform resolvent estimates needed for the Lipschitz continuity of the inverse scattering map presented in Section 4.

We end the introduction by discussing the role that factorization of the jump matrix plays in our application of the Beals–Coifman approach to inverse scattering. In Figure 2, right, the oriented contour divides the complex plane into positive and negative regions. We factorize the jump matrix as

$$J(\lambda) = J_-(\lambda)^{-1} J_+(\lambda),$$

where $W_+ = J_+ - I$ and $W_- = I - J_-$ belong to $H^1(\Gamma_\pm)$ and are continuous across the intersections between straight-line contours and circular arcs, respecting the orientations. This continuity means that the

[2] In [Zhou 1998], the nonzero off-diagonal entries are not calculated explicitly.

matrix pair (W_+, W_-) belongs to a pair of decomposing algebras $(H^1(\Gamma_+), H^1(\Gamma_-))$ in the sense of Zhou; see [Zhou 1989b, §9], where a general theory of Riemann–Hilbert problems on self-intersecting contours is presented. As shown by Zhou, this decomposition implies that the Beals–Coifman integral operator (3-2) is Fredholm. Unique solvability of (3-2) then follows from the Fredholm alternative and an appropriate vanishing lemma (the statement that the homogeneous version of (3-2) has no nonzero solutions).

In our case, we need to show that the Beals–Coifman operator

$$C_J f = C_\Gamma^+(f W_-) + C_\Gamma^-(f W_+)$$

is Fredholm. For this purpose, following Zhou, we approximate $W_\pm$ by rational functions; in this approximation, the operator $C^\pm_{W_\mp} \circ C^\mp_{W_\pm}$ is compact. We thus obtain a Fredholm regulator of the Beals–Coifman operator (see [Zhou 1989b, Proposition 4.1]). Another way to think about the compactness is that continuity across intersection points prevents singularities near these points which might otherwise occur, spoiling the compactness.

2. The direct scattering map

2A. ***The scattering problem in the ζ-variable.*** The system (1-5) can be written in the form of an integral equation for the 2×2 matrix $m(x, \zeta)$:

$$m(x,\zeta) = I + \int_\delta^x e^{i(y-x)\zeta^2 \operatorname{ad}\sigma}(\zeta Q(y) m(y,\zeta) + P(y) m(y,\zeta))\, dy, \tag{2-1}$$

where the lower limit δ can be different for various choices of normalization. We will use several solutions of (2-1). The standard AKNS method starts with the following two Volterra integral equations as special cases of (2-1) for $\operatorname{Im}\zeta^2 = 0$:

$$m^\pm(x,\zeta) = I + \int_{\pm\infty}^x e^{i(y-x)\zeta^2 \operatorname{ad}\sigma}(\zeta Q(y) m^\pm(y,\zeta) + P(y) m^\pm(y,\zeta))\, dy,$$

which are expressed in componentwise form as

$$\begin{pmatrix} m_{11}^+(x,\zeta) \\ m_{21}^+(x,\zeta) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \int_x^\infty \begin{pmatrix} \zeta q m_{21}^+ + p_1 m_{11}^+ \\ e^{2i\zeta^2(x-y)}[-\zeta \bar{q} m_{11}^+ + p_2 m_{21}^+] \end{pmatrix} dy, \tag{2-2}$$

$$\begin{pmatrix} m_{12}^+(x,\zeta) \\ m_{22}^+(x,\zeta) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \int_x^\infty \begin{pmatrix} e^{-2i\zeta^2(x-y)}[\zeta q m_{22}^+ + p_1 m_{12}^+] \\ -\zeta \bar{q} m_{12}^+ + p_2 m_{22}^+ \end{pmatrix} dy, \tag{2-3}$$

$$\begin{pmatrix} m_{11}^-(x,\zeta) \\ m_{21}^-(x,\zeta) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{-\infty}^x \begin{pmatrix} \zeta q m_{21}^- + p_1 m_{11}^- \\ e^{2i\zeta^2(x-y)}[-\zeta \bar{q} m_{11}^- + p_2 m_{21}^-] \end{pmatrix} dy, \tag{2-4}$$

$$\begin{pmatrix} m_{12}^-(x,\zeta) \\ m_{22}^-(x,\zeta) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_{-\infty}^x \begin{pmatrix} e^{-2i\zeta^2(x-y)}[\zeta q m_{22}^- + p_1 m_{12}^-] \\ -\zeta \bar{q} m_{12}^- + p_2 m_{22}^- \end{pmatrix} dy. \tag{2-5}$$

By uniqueness theory for ODEs and the normalizations of $m^{\pm}$ as $x \to \pm\infty$, $m^{\pm}(x,\zeta)$ defined by (2-2)–(2-5) are related by a matrix $A(\zeta)$ with $\det A(\zeta) = 1$ in the form

$$m^{+}(x,\zeta) = m^{-}(x,\zeta) e^{-ix\zeta^2 \operatorname{ad}\sigma} A(\zeta), \quad A(\zeta) = \begin{pmatrix} a & \breve{b} \\ b & \breve{a} \end{pmatrix}.$$

The matrix-valued function $A(\zeta)$ is expressed in terms of $m^{(\pm)}$ as

$$a(\zeta) = 1 - \int_{-\infty}^{\infty} (\zeta q m_{21}^{+} + p_1 m_{11}^{+})\, dy = 1 + \int_{-\infty}^{\infty} (-\zeta \bar{q} m_{12}^{-} + p_2 m_{22}^{-})\, dy, \tag{2-6}$$

$$\breve{a}(\zeta) = 1 - \int_{-\infty}^{\infty} (-\zeta \bar{q} m_{12}^{+} + p_2 m_{22}^{+})\, dy = 1 + \int_{-\infty}^{\infty} (\zeta q m_{21}^{-} + p_1 m_{11}^{-})\, dy, \tag{2-7}$$

and

$$b(\zeta) = \int_{-\infty}^{\infty} e^{-2i\zeta^2 y} (\zeta \bar{q} m_{11}^{+} - p_2 m_{21}^{+})\, dy = \int_{-\infty}^{\infty} e^{-2i\zeta^2 y} (\zeta \bar{q} m_{11}^{-} - p_2 m_{21}^{-})\, dy,$$

$$\breve{b}(\zeta) = -\int_{-\infty}^{\infty} e^{2i\zeta^2 y} (\zeta q m_{22}^{+} + p_1 m_{12}^{+})\, dy = -\int_{-\infty}^{\infty} e^{2i\zeta^2 y} (\zeta q m_{22}^{-} + p_1 m_{12}^{-})\, dy.$$

We now construct the Beals–Coifman solutions needed for the RHP in the form of piecewise analytic matrix functions. The left and right Beals–Coifman solutions are constructed from the normalized Jost solutions as follows:

$$M_R(x,z) = \begin{cases} \left[\dfrac{m_1^{-}(x,z)}{\breve{a}(z)},\ m_2^{+}(x,z) \right], & \operatorname{Im} z^2 > 0, \\ \left[m_1^{+}(x,z),\ \dfrac{m_2^{-}(x,z)}{a(z)} \right], & \operatorname{Im} z^2 < 0, \end{cases} \tag{2-8}$$

$$M_L(x,z) = \begin{cases} \left[m_1^{-}(x,z),\ \dfrac{m_2^{+}(x,z)}{\breve{a}(z)} \right], & \operatorname{Im} z^2 > 0, \\ \left[\dfrac{m_1^{+}(x,z)}{a(z)},\ m_2^{-}(x,z) \right], & \operatorname{Im} z^2 < 0. \end{cases} \tag{2-9}$$

The Beals–Coifman solutions are piecewise meromorphic with continuous boundary values denoted by $M_{L,\pm}$ and $M_{R,\pm}$ as $\pm \operatorname{Im} z^2 \downarrow 0$, in the absence of spectral singularities. The Beals–Coifman solutions corresponding to the potential q_{x_0} are constructed similarly.

From here onward, we will analyze the right Beals–Coifman solution (2-8) and drop the subscripts R and L. The left RHP is connected to the right RHP through multiplication by an auxiliary scattering matrix which is constructed in Section 2E.

2B. ***Construction of the scattering data in the ζ-variable.*** In this subsection, we construct the piecewise analytic function $M^{(2)}(x,z)$ introduced in (1-14) and defined inside the circle Σ_∞ from which one extracts scattering data in the form of jump matrices along the contour Σ_∞. In this subsection, M denotes the right-normalized Beals–Coifman solution M_R.

Combining (2-4) and (2-3), we obtain

$$[m_1^-, m_2^+] = I + \int_\delta^x e^{i(y-x)\zeta^2 \operatorname{ad}\sigma} \big((\zeta Q(y) + P(y))[m_1^-(y), m_2^+(y)]\big)\, dy, \tag{2-10}$$

where δ is chosen differently for the different entries of the matrix, namely $\delta = -\infty$ for the (1, 1)- and (2, 1)-entries and $\delta = +\infty$ for the (1, 2)- and (2, 2)-entries. Using (2-7), we rewrite (2-10) as

$$[m_1^-, m_2^+] = \begin{pmatrix} \breve{a} & 0 \\ 0 & 1 \end{pmatrix} + \int_\delta^x e^{i(y-x)\zeta^2 \operatorname{ad}\sigma} \big((\zeta Q(y) + P(y))[m_1^-(y), m_2^+(y)]\big)\, dy,$$

where $\delta = -\infty$ for the (2, 1)-entry and $\delta = +\infty$ for the (1, 1)-, (1, 2)- and (2, 2)-entries. If the inverse of $\left(\begin{smallmatrix} \breve{a} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ exists, we obtain a Fredholm equation for M defined in (2-8):

$$\left[\frac{m_1^-}{\breve{a}},\ m_2^+\right] = I + \int_\delta^x e^{i(y-x)\zeta^2 \operatorname{ad}\sigma} \left((\zeta Q(y) + P(y))\left[\frac{m_1^-}{\breve{a}},\ m_2^+\right]\right) dy \tag{2-11}$$

and $[m_1^-/\breve{a},\ m_2^+]$ solves (2-11) if and only if $\breve{a}(\zeta) \neq 0$.

The right-normalized Beals–Coifman solution M is analytic in the intersection of $\pm \operatorname{Im} z^2 > 0$ and $|z| > R$, where R is chosen so large that any zeros of a and $\breve{a}$ are contained inside the disc $B(0, R) = \{z : |z| \leq R\}$. We now show how to construct solutions $M^{(2)}(x, \zeta)$, analytic inside this disc, and modify the Riemann–Hilbert problem accordingly.

Recall from (1-11), for $x_0 \gg 1$, let

$$q_{x_0}(x) = \begin{cases} 0, & x \leq x_0, \\ q(x), & x > x_0, \end{cases}$$

and denote by Q_{x_0} and P_{x_0} the matrices (1-4) with q replaced by q_{x_0}. We choose x_0 so that

$$\sup_{\zeta \in B(0,R)} \|\zeta Q_{x_0} + P_{x_0}\|_{L^1} < \tfrac{1}{2}. \tag{2-12}$$

This condition guarantees that there is a bounded Beals–Coifman solution $M^{(0)}$ normalized as $x \to \infty$ associated to the potential q_{x_0} in the form of (2-8). To see this, first note that the equation

$$m(x, \zeta) = I + \int_\delta^x e^{i(y-x)\zeta^2 \operatorname{ad}\sigma} (\zeta Q_{x_0}(y) m(y, \zeta) + P_{x_0}(y) m(y, \zeta))\, dy \tag{2-13}$$

(where $\delta = -\infty$ for the (2, 1)-entry and $\delta = +\infty$ for the (1, 1)-, (1, 2)- and (2, 2)-entry) is uniquely solvable for $\zeta \in B(0, R)$ owing to the smallness condition (2-12).

Next, we claim that $\breve{a}_0(z)$ associated to q_{x_0} is nonzero for $z \in \Omega^+ \cap B(0, R)$; we prove this statement by contradiction. Suppose that there exists z_0 such that (2-12) holds and $m(x, z_0)$ solves (2-13), but $\breve{a}(z_0) = 0$. By uniqueness, there exists a nonsingular matrix $B(z_0)$ such that

$$m(x, z_0) = [m_1^-, m_2^+] e^{-ixz_0^2 \operatorname{ad}\sigma} B(z_0).$$

Here, m_1^-, m_2^+ are the Jost solutions (2-3) and (2-4) associated to the potential q_{x_0}. Letting $x \to +\infty$ and using (2-7), we obtain

$$\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ * & 1 \end{pmatrix} e^{-ixz_0^2 \operatorname{ad}\sigma} B(z_0),$$

which leads to a contradiction. Thus the cut-off potential q_{x_0} supports neither eigenvalues nor spectral singularities in $B(0, R)$, so that we can construct a bounded Beals–Coifman solution of the form (2-8) associated to the potential q_{x_0} and normalized as $x \to \infty$. We denote by $M^{(0)}$ this unique bounded solution.

Using the solutions $M(z)$ and $M^{(0)}$ corresponding to the initial data potential q and the related potential q_{x_0}, respectively, one defines a new function $\boldsymbol{M}$ using Zhou's constructions as described in (1-12)–(1-14) above. The matrix $\boldsymbol{M}$ is analytic in $\mathbb{C} \setminus \Sigma$, and we can compute the jump matrix

$$v(\zeta) = e^{ix\zeta^2 \operatorname{ad}\sigma} \boldsymbol{M}_-(x, \zeta)^{-1} \boldsymbol{M}_+(x, \zeta)$$

explicitly across the various parts of the augmented contour Σ. Along the contour $\mathbb{R} \cup i\mathbb{R}$, outside of the circle,

$$v(\zeta) = \begin{pmatrix} 1 - r(\zeta)\breve{r}(\zeta) & r(\zeta) \\ -\breve{r}(\zeta) & 1 \end{pmatrix}. \tag{2-14}$$

Along the contour $\mathbb{R} \cup i\mathbb{R}$ inside of the circle,

$$v(\zeta) = \begin{pmatrix} 1 & -r_0(\zeta) \\ \breve{r}_0(\zeta) & 1 - r_0(\zeta)\breve{r}_0(\zeta) \end{pmatrix}.$$

Here, the subscript "0" denotes the scattering data generated by q_{x_0}.

Since both M and $M^{(2)}$ are solutions of (1-5) with nonvanishing determinant,we have

$$v(\zeta) = e^{ix\zeta^2 \operatorname{ad}\sigma} (M^{(2)}(x, \zeta)^{-1} M(x, \zeta)) \tag{2-15}$$

along the circle Σ_∞. In particular, setting $x = x_0$, we obtain $v(\zeta)$ in terms of Jost functions. Across the arc in the first and third quadrant, we have

$$e^{-ix_0\zeta^2 \operatorname{ad}\sigma} v(\zeta) = M^{(2)}(x_0, \zeta)^{-1} M(x_0, \zeta) = \begin{pmatrix} 1 & 0 \\ \dfrac{m_{21}^-(x_0, \zeta)}{\breve{a}(\zeta)\breve{a}_0(\zeta)} & 1 \end{pmatrix}. \tag{2-16}$$

Across the arc in the second and fourth quadrant, we have

$$e^{-ix_0\zeta^2 \operatorname{ad}\sigma} v(\zeta) = M^{(2)}(x_0, \zeta)^{-1} M(x_0, \zeta) = \begin{pmatrix} 1 & -\dfrac{m_{12}^-(x_0, \zeta)}{a(\zeta)a_0(\zeta)} \\ 0 & 1 \end{pmatrix}. \tag{2-17}$$

Denote by $A^\dagger$ the hermitian conjugate of the matrix A. The following property of v will be used later to prove the unique solvability of the RHP (Proposition 3.9).

Proposition 2.1. *The jump matrix v along the contour Σ, defined in* (2-14)-(2-17), *satisfies*:

(i) $v(\zeta) + v(\zeta)^\dagger$ *is positive definite for* $\zeta \in \mathbb{R}$.

(ii) $v(\bar{\zeta}) = v(\zeta)^\dagger$ *for* $\zeta \in \Sigma \setminus \mathbb{R}$.

Proof. This is an immediate consequence of the definitions (2-14)–(2-17) and (1-7) as well as the symmetries (1-9)–(1-10). □

2C. ***Construction of the scattering data in the λ-variable.*** In the absence of eigenvalues and spectral singularities, we reduced the scattering problem (1-5) of $\zeta \in \mathbb{R} \cup i\mathbb{R}$ to a scattering problem for $\lambda = \zeta^2 \in \mathbb{R}$, and identified a single scattering datum $\rho(\lambda)$ defining the direct scattering map [Liu et al. 2016]; we can carry out a similar reduction here. Let $m(x, \zeta)$ be a solution to (1-5). We set

$$m^{\sharp}(x,\zeta) = \begin{pmatrix} m_{11}(x,\zeta) & \zeta^{-1} m_{12}(x,\zeta) \\ \zeta m_{21}(x,\zeta) & m_{22}(x,\zeta) \end{pmatrix}.$$

Note $m^{\sharp}$ is an even function of ζ. Defining $\lambda = \zeta^2$ and $n(x,\lambda) = m^{\sharp}(x,\zeta)$, the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & \zeta^{-1} b \\ \zeta c & d \end{pmatrix}$$

is an automorphism of 2×2 matrices and commutes with differentiation in x. It follows that the functions $n^{\pm}$ obtained from $m^{\pm}$ by this map obey

$$\frac{dn^{\pm}}{dx} = -i\lambda \operatorname{ad}\sigma(n^{\pm}) + \begin{pmatrix} 0 & q \\ -\lambda\bar{q} & 0 \end{pmatrix} n^{\pm} + P n^{\pm}, \tag{2-18a}$$

$$\lim_{x \to \pm\infty} n^{\pm}(x,\lambda) = I \tag{2-18b}$$

and satisfy

$$n^{+}(x,\lambda) = n^{-}(x,\lambda) e^{-i\lambda x \operatorname{ad}\sigma} \begin{pmatrix} \alpha(\lambda) & \beta(\lambda) \\ \lambda\breve{\beta}(\lambda) & \breve{\alpha}(\lambda) \end{pmatrix} = n^{-}(x,\lambda) e^{-i\lambda x \operatorname{ad}\sigma} \begin{pmatrix} \alpha(\lambda) & \beta(\lambda) \\ -\lambda\overline{\beta(\lambda)} & \overline{\alpha(\lambda)} \end{pmatrix}, \tag{2-19}$$

where $\alpha(\lambda) = a(\zeta)$, $\beta(\lambda) = \zeta^{-1}\breve{b}(\zeta)$ and the relation $|\alpha(\lambda)|^2 + \lambda|\beta(\lambda)|^2 = 1$ holds.

In the presence of spectral singularities, we perform the change of variable $\zeta \to \lambda$ in the same way as in [Liu et al. 2016] and obtain the corresponding row-vector-valued Beals–Coifman solutions $N^{(0)}$, $N^{(2)}$ and N:

$$N^{(0)} = \text{first row of } \begin{pmatrix} \zeta^{-1/2} & 0 \\ 0 & \zeta^{1/2} \end{pmatrix} M^{(0)} \begin{pmatrix} \zeta^{1/2} & 0 \\ 0 & \zeta^{-1/2} \end{pmatrix},$$

$$N^{(2)} = \text{first row of } \begin{pmatrix} \zeta^{-1/2} & 0 \\ 0 & \zeta^{1/2} \end{pmatrix} M^{(2)} \begin{pmatrix} \zeta^{1/2} & 0 \\ 0 & \zeta^{-1/2} \end{pmatrix},$$

$$N = \text{first row of } \begin{pmatrix} \zeta^{-1/2} & 0 \\ 0 & \zeta^{1/2} \end{pmatrix} M \begin{pmatrix} \zeta^{1/2} & 0 \\ 0 & \zeta^{-1/2} \end{pmatrix}.$$

The contour Γ for the new RHP, defined by (1-15) is the image in Figure 2, right, of the contour Σ in Figure 2, left, under the change of variable $\lambda = \zeta^2$.

Notice that the direction of the contour that consists of the part of the real axis inside the circle is from right to left. Define the piecewise analytic function $\boldsymbol{N}$ as

$$\boldsymbol{N}(x,z) = \begin{cases} N(x,z), & z \in \Omega_1 \cup \Omega_2, \\ N^{(2)}(x,z), & z \in \Omega_3 \cup \Omega_4. \end{cases} \tag{2-20}$$

By setting

$$\alpha(\lambda)=a(\zeta), \qquad \breve{\alpha}(\lambda)=\breve{a}(\zeta),$$
$$\rho(\lambda)=\zeta^{-1}r(\zeta), \qquad \rho_0(\lambda)=\zeta^{-1}r_0(\zeta),$$
$$n_{21}^-(x,\lambda)=\zeta m_{21}^-(x,\zeta), \quad n_{12}^-(x,\lambda)=\zeta^{-1}m_{12}^-(x,\zeta),$$

we obtain from (2-14)–(2-17) the jump matrices $J(\lambda)$ for the piecewise row vector $\boldsymbol{N}$.

Proposition 2.2. *The jump matrices for* $\boldsymbol{N}$ *along the various parts of the contour* Γ, *where*

$$\boldsymbol{N}_+(x,\lambda)=\boldsymbol{N}_-(x,\lambda)e^{-i\lambda\,\mathrm{ad}\,\sigma}J(\lambda),$$

are given as follows:

(i) *On* $\mathbb{R}_\infty$ *the part of the real line outside the circle*,

$$J(\lambda)=\begin{pmatrix}1+\lambda|\rho(\lambda)|^2 & \rho(\lambda)\\ \lambda\overline{\rho(\lambda)} & 1\end{pmatrix}. \tag{2-21}$$

(ii) *On* $(-S_\infty, S_\infty)$ *the part of the real line inside the circle*,

$$J(\lambda)=\begin{pmatrix}1 & -\rho_0(\lambda)\\ -\lambda\overline{\rho_0(\lambda)} & 1+\lambda|\rho_0(\lambda)|^2\end{pmatrix}. \tag{2-22}$$

(iii) *On the semicircular arc* Γ_∞^+ *in* $\mathbb{C}^+$,

$$J(\lambda)=\begin{pmatrix}1 & 0\\ e^{-2ix_0\lambda}\dfrac{n_{21}^-(x_0,\lambda)}{\breve{\alpha}(\lambda)\breve{\alpha}_0(\lambda)} & 1\end{pmatrix}. \tag{2-23}$$

(iv) *On the semicircular arc* Γ_∞^- *in* $\mathbb{C}^-$,

$$J(\lambda)=\begin{pmatrix}1 & -e^{2ix_0\lambda}\dfrac{n_{12}^-(x_0,\lambda)}{\alpha(\lambda)\alpha_0(\lambda)}\\ 0 & 1\end{pmatrix}. \tag{2-24}$$

Remark 2.3. The *scattering data* associated to the potential q are defined as the entries of the different jump matrices along Γ, as listed in Proposition 2.2 and shown in Figure 3. We show the factorizations of jump matrices exist and obtain estimates in appropriate Sobolev spaces. The choice of scattering data is motivated by the inverse problem. From these spectral data, we will, in the next section, define an inverse map and a reconstruction of the potential. Note that the scattering data depend on the choice of x_0 as well as the choice of the large circle Γ_∞. Indeed in [Zhou 1989a], scattering data are seen as an equivalence class. In the study of the inverse map, we will need the fact that the reconstruction formula does not depend on x_0 and Γ_∞. This is because the reconstruction formula involves a limit as λ tends to infinity of the entry $(1,2)$ of the solution of an RHP and will not be affected by the exact position of the cut-off point or the circle Γ_∞, although the RHP itself depends on it. For more details, we refer to [Zhou 1989a, Theorem 3.3.15].

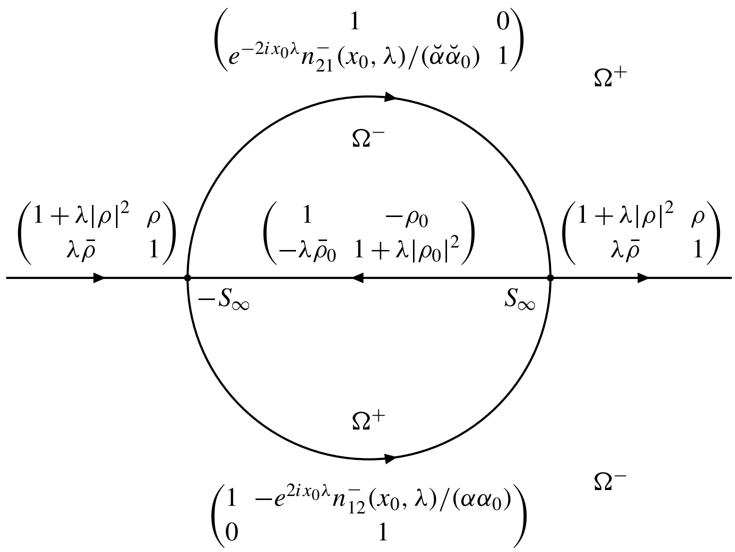

Figure 3. Scattering data for q.

To give a full characterization of the scattering data, we use the Sobolev spaces $H_z^k(\Gamma)$ and $H_\pm^k(\Gamma)$ defined on self-intersecting contours (see Appendix A) and the notion of k-regularity [Trogdon and Olver 2016, Definition 2.54] of a given jump matrix along an admissible contour. All contours under consideration here are admissible in the sense of [Trogdon and Olver 2016, Definition 2.40].

Definition 2.4. A jump matrix J defined on an admissible contour Γ is *k-regular* if Γ is complete and J has a factorization

$$J(s) = J_-^{-1}(s) J_+(s),$$

where $J_\pm(s) - I$ and $J_\pm^{-1}(s) - I \in H_\pm^k(\Gamma)$.

Definition 2.5. Assume $a \in \gamma_0$, where γ_0 is the set of self-intersections of Γ. Let $\Gamma_1, \dots, \Gamma_m$ be a counterclockwise ordering of subcomponents of Γ which contain $z = a$ as an endpoint. For $J \in H^k(\Gamma)$, we define $\hat{J}_i$ as the restriction $J \restriction_{\Gamma_i}$ if Γ_i is oriented outwards and by $(J \restriction_{\Gamma_i})^{-1}$ otherwise. We say that J *satisfies the* $(k-1)$*-th-order product condition* if, using the $(k-1)$-th-order Taylor expansion of each J_i, we have

$$\prod_{i=1}^{m} \hat{J}_i = I + \mathcal{O}(|\lambda - a|^k) \quad \text{for all } a \in \gamma_0. \tag{2-25}$$

The following theorem is due to [Zhou 1999]; see also [Trogdon and Olver 2016, Theorem 2.56].

Theorem 2.6. *The two following statements are equivalent*:

(i) *$J - I$ and $J^{-1} - I \in H^k(\Gamma)$ away from points of self-intersection and J satisfies the $(k-1)$-th-order product condition.*

(ii) *J is k-regular.*

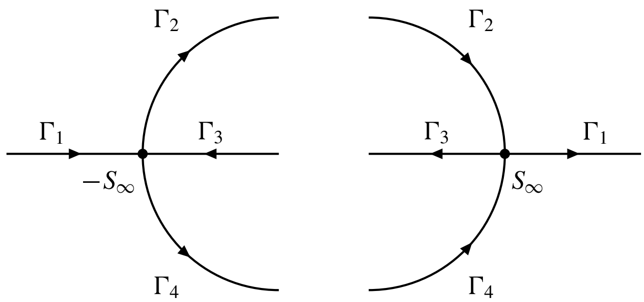

Figure 4. Zero-sum conditions $-S_\infty$ (left) and S_∞ (right).

In the next theorem, we check that the jump matrix $J(\lambda)$ satisfies condition (i) of Theorem 2.6 and characterize the large-λ decay of scattering data in weighted Sobolev spaces. Let

$$H^{2,2}(\partial\Omega_2) = \{f \in H^2(\partial\Omega_2) : f|_{\mathbb{R}_\infty} \in H^{2,2}(\mathbb{R}_\infty)\},$$
$$H^{1,1}(\partial\Omega_1) = \{f \in H^1(\partial\Omega_1) : f|_{\mathbb{R}_\infty} \in H^{1,1}(\mathbb{R}_\infty)\}.$$

Theorem 2.7 should be compared to (C2.28) of [Zhou 1995], where the scattering matrix is characterized as belonging to H^k for any $k \geq 1$ given initial data q_0 is in Schwartz class. Theorem 2.7 shows that the direct scattering transform maps a potential q in the weighted Sobolev space $H^{2,2}(\mathbb{R})$ into scattering data in appropriate *weighted* Sobolev spaces.

Theorem 2.7. *The matrix $J(\lambda)$ admits a triangular factorization*

$$J(\lambda) = J_-^{-1}(\lambda) J_+(\lambda),$$

where:

(i) $J_-(\lambda) - I \in H^{2,2}(\partial\Omega_2)$, $J_-(\lambda) - I \in H^2(\partial\Omega_3)$, $J_+(\lambda) - I \in H^2(\partial\Omega_4)$ *and* $J_+(\lambda) - I \in H^{1,1}(\partial\Omega_1)$.[3]

(ii) $J_+ \restriction_{\partial\Omega_1} - I$ *and* $J_- \restriction_{\partial\Omega_3} - I$ *are strictly lower triangular*, *while* $J_- \restriction_{\partial\Omega_2} - I$ *and* $J_+ \restriction_{\partial\Omega_4} - I$ *are strictly upper triangular.*

(iii) *The matrix $J(\lambda)$ satisfies the first-order product condition at the intersection points $\pm S_\infty$ of the real λ-axis.*

Proof. Let J_i be the restriction of J to Γ_i, $1 \leq i \leq 5$, where the contours Γ_i are shown in Figure 4, and set $\mathbb{R}_\infty = \mathbb{R} \setminus [-S_\infty, S_\infty]$. On $\mathbb{R}_\infty$, the scattering matrix J_1 admits the factorization

$$J_1(\lambda) = J_{1,-}(\lambda)^{-1} J_{1,+}(\lambda) = \begin{pmatrix} 1 & \rho(\lambda) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \lambda\overline{\rho(\lambda)} & 1 \end{pmatrix}, \tag{2-26}$$

with the same decomposition for $J_5(\lambda)$, while on $(-S_\infty, S_\infty)$, the scattering matrix J_3 admits the factorization

$$J_3(\lambda) = J_{3,-}(\lambda)^{-1} J_{3,+}(\lambda) = \begin{pmatrix} 1 & 0 \\ -\lambda\overline{\rho_0(\lambda)} & 1 \end{pmatrix} \begin{pmatrix} 1 & -\rho_0(\lambda) \\ 0 & 1 \end{pmatrix}. \tag{2-27}$$

[3]The asymmetry of the regularity properties of the terms in the $J(\lambda)$ factorization on the various parts of the contour comes from the fact that the (1, 2)- and (2, 1)-entries in the expression for J (see (2-21)) differ by a weight λ.

(i) The methods of Section 3 of [Liu et al. 2016] can be used to show that $\rho \in H^{2,2}(\mathbb{R}_\infty)$. It follows from this fact and the explicit factorization (2-26) that

$$[J_{1,-}(\lambda) - I]|_{\mathbb{R}_\infty} \in H^{2,2}(\mathbb{R}_\infty) \quad \text{and} \quad [J_{1,+}(\lambda) - I]|_{\mathbb{R}_\infty} \in H^{1,1}(\mathbb{R}_\infty).$$

Similarly, the restrictions of $J_\pm(\lambda) - I$ to $(-S_\infty, S_\infty)$ belong to $H^2(-S_\infty, S_\infty)$. The remaining Sobolev estimates all involve the bounded semicircular contours $\Gamma^\pm_\infty$. The contours $\Gamma^\pm_\infty$ are open and of finite length, so $H^2(\Gamma^\pm_\infty)$ is equivalent to $H^2(0, \pm\pi)$ after parametrization by an angle θ. The H^2-norm of a function f controls the L^∞-norm of f and f' and thus H^2 is an algebra by the Leibnitz rule. Using (2-23) and (2-24), it suffices to show that the functions $n^-_{12}(x_0, \lambda)$, $1/\breve{\alpha}(\lambda)$, and $1/\breve{\alpha}_0(\lambda)$ belong to $H^2(\Gamma^+_\infty)$ and that the functions $n^-_{21}(x, \lambda)$, $1/\alpha(\lambda)$, and $1/\alpha_0(\lambda)$ belong to $H^2(\Gamma^-_\infty)$. This is easily deduced from the Volterra integral equations corresponding to (2-18) and the integral representations for α and $\breve{\alpha}$ that can be deduced from (2-19).

(ii) The assertions about triangularity follow from the factorizations (2-26) and (2-27) together with the formulas (2-23) and (2-24).

(iii) Using the relation (2-19), the scattering matrices J_2 and J_4 are given at the self-intersection point S_∞ respectively by

$$\begin{aligned} J_2(S_\infty) &= \begin{pmatrix} 1 & 0 \\ e^{-2ix_0S_\infty}\dfrac{n^-_{21}(x_0, S_\infty)}{\breve{\alpha}(S_\infty)\breve{\alpha}_0(S_\infty)} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{-2ix_0S_\infty}\dfrac{n^+_{21}(S_\infty)}{\breve{\alpha}_0(S_\infty)} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \dfrac{S_\infty\overline{\beta(S_\infty)}n^+_{22}(S_\infty)}{\breve{\alpha}(S_\infty)\breve{\alpha}_0(S_\infty)} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ -S_\infty\overline{\rho_0(S_\infty)} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ S_\infty\overline{\rho(S_\infty)} & 1 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} J_4(S_\infty) &= \begin{pmatrix} 1 & -e^{2ix_0S_\infty}\dfrac{n^-_{12}(x_0, S_\infty)}{\alpha(S_\infty)\alpha_0(S_\infty)} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -e^{2ix_0S_\infty}\dfrac{n^+_{12}(S_\infty)}{\alpha_0(S_\infty)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \dfrac{\beta(S_\infty)n^+_{11}(S_\infty)}{\alpha(S_\infty)\alpha_0(S_\infty)} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \rho(S_\infty) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\rho_0(S_\infty) \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

The factorizations of J_2 and J_4 along the arcs are obtained by polynomial interpolation between $S_{\pm\infty}$ (see for example equation (5.19) of [Liu 2019]).

We want to establish (2-25) for $k = 2$, that is,

$$\prod_{i=1}^{4} \hat{J}_i = I + \mathcal{O}(|\lambda - S_\infty|^2). \tag{2-28}$$

Denoting by $\mathfrak{J}_i$ the first-order Taylor polynomial of J_i at S_∞, $i = 1, \dots, 4$, proving (2-28) is equivalent to proving that

$$\mathfrak{J}_1\mathfrak{J}_2^{-1} = \mathfrak{J}_4\mathfrak{J}_3^{-1} + \mathcal{O}(|\lambda - S_\infty|^2).$$

It is clear that $J_1(S_\infty)J_2(S_\infty)^{-1} = J_4(S_\infty)J_3(S_\infty)^{-1}$. We also have to check that

$$(J_1J_2^{-1})_\lambda(S_\infty) = (J_4J_3^{-1})_\lambda(S_\infty).$$

To achieve this, we need to show that

$$\frac{d}{d\lambda}\frac{e^{-2ix_0\lambda}n_{21}^{-}(x_0,\lambda)}{\breve{\alpha}(\lambda)\breve{\alpha}_0(\lambda)}\bigg|_{\lambda=S_\infty}=\rho'(S_\infty)-\rho_0'(S_\infty). \tag{2-29}$$

This is done by first letting $\lambda\to\mathbb{R}$, which leads to the factorization of J_4

$$\begin{pmatrix}1 & -e^{2ix_0\lambda}\dfrac{n_{12}^{-}(x_0,\lambda)}{\alpha(\lambda)\alpha_0(\lambda)}\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & \rho(\lambda)\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & -\rho_0(\lambda)\\ 0 & 1\end{pmatrix},$$

and taking the derivative along $\mathbb{R}$. In the same way, we can show

$$\frac{d}{d\lambda}\frac{e^{2ix_0\lambda}n_{12}^{-}(x_0,\lambda)}{\alpha(\lambda)\alpha_0(\lambda)}\bigg|_{\lambda=S_\infty}=-S_\infty\overline{\rho_0'(S_\infty)}-\overline{\rho_0(S_\infty)}+S_\infty\overline{\rho'(S_\infty)}+\overline{\rho(S_\infty)}.$$

We thus verify the $(k-1)$-th order product condition for $k=2$ and part (i) of Theorem 2.6 holds for the matrix J. We conclude that J is k-regular, which in turn implies that $J_+(\lambda)$ satisfies the matching condition (A-1) and $J_-(\lambda)$ satisfies the matching condition (A-2) at the nonsmooth point $(S_\infty,0)$. A similar proof shows that the same conclusion holds for $(-S_\infty,0)$. □

The following propositions stating Lipschitz continuity results can be obtained by the methods of [Liu et al. 2016; Liu 2017], in particular Propositions 3.2 and 3.3 of [Liu et al. 2016]. The exclusion of the disk $|\lambda|<R$ implies that $|\alpha(\lambda)|$ is strictly positive so division by α does not affect the estimates. Proposition 2.8, formulas (2-21)–(2-24), and the factorizations (2-26) and (2-27) imply also the Lipschitz continuity of the scattering data (Proposition 2.9).

Proposition 2.8. *The maps*

$$q\mapsto\rho|_{\mathbb{R}_\infty}\in H^{2,2}(\mathbb{R}_\infty),\quad q\mapsto n_{21}^{-}(x_0,\cdot)\in H^2(\Gamma_\infty^{+}),\quad q\mapsto n_{12}^{-}(x_0,\cdot)\in H^2(\Gamma_\infty^{-}),$$
$$q\mapsto\frac{1}{\breve{\alpha}}\in H^2(\Gamma_\infty^{+}),\quad q\mapsto\frac{1}{\breve{\alpha}_0}\in H^2(\Gamma_\infty^{+}),\quad q\mapsto\frac{1}{\alpha}\in H^2(\Gamma_\infty^{-}),\quad q\mapsto\frac{1}{\alpha_0}\in H^2(\Gamma_\infty^{-})$$

are locally Lipschitz continuous from $H^{2,2}(\mathbb{R})$ into the respective ranges. Moreover, the map

$$q_{x_0}\mapsto\rho_0\in H^2(\mathbb{R})$$

is locally Lipschitz continuous from $H^{0,2}(\mathbb{R})$ to $H^{2,0}(\mathbb{R})$.

Proposition 2.9. *The maps*

$$\begin{aligned}&q\mapsto J_1^{\pm}(\lambda)-I\in H^{2,2}(\partial\Omega_2),\\&q\mapsto J_2(\lambda)-I\in H^2(\partial\Omega_3),\\&q\mapsto J_3^{\pm}(\lambda)-I\in H^{1,1}(\partial\Omega_4),\\&q\mapsto J_4(\lambda)-I\in H^2(\partial\Omega_4)\end{aligned}$$

are locally Lipschitz mappings from $H^{2,2}(\mathbb{R})$ into their respective ranges.

2D. ***Time evolution of the scattering data.*** A key property of the inverse scattering method is the simple time evolution of its scattering data. In [Liu 2017], we calculated the time evolution of the scattering data where they reduce to a reflection coefficient and discrete data. We need to complement the analysis by examining the time evolution of the jump matrix on the additional section of the contour Γ_∞ (see Figure 2, right). As before, we work in the ζ-variable and carry out the change of variable $\zeta \to \lambda$. Given

$$\boldsymbol{M}^{+}(x,t;\zeta) = \boldsymbol{M}^{-}(x,t;\zeta) v_x(t;\zeta),$$

where $v_x(t;\zeta) = e^{-i\zeta^2 x \operatorname{ad}\sigma} v(t;\zeta)$, we compute the time derivative

$$\boldsymbol{M}^{+}(x,t;\zeta)_t = \boldsymbol{M}^{-}(x,t;\zeta)_t v_x(t;\zeta) + \boldsymbol{M}^{-}(x,t;\zeta) v_x(t;\zeta)_t. \tag{2-30}$$

We recall that $\boldsymbol{M}^{\pm}$ are fundamental solutions for the Lax equations

$$\frac{\partial M}{\partial x}(x,t;\zeta) = -i\zeta^2 \operatorname{ad}\sigma(M) + \zeta Q(x,t)M + P(x,t)M, \tag{2-31a}$$

$$\frac{\partial M}{\partial t}(x,t;\zeta) = -2i\zeta^4 \operatorname{ad}\sigma(M) + A(x,t;\zeta)M, \tag{2-31b}$$

where σ, P, Q are given in terms of $q = q(x,t)$ by (1-4) and

$$A(x,t;\zeta) = 2\zeta^3 \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix} + i\zeta^2 \begin{pmatrix} |q|^2 & 0 \\ 0 & -|q|^2 \end{pmatrix} + i\zeta \begin{pmatrix} 0 & q_x \\ \bar{q}_x & 0 \end{pmatrix} + \frac{i}{4} \begin{pmatrix} |q|^4 & 0 \\ 0 & -|q|^4 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -q_x\bar{q} + q\bar{q}_x & 0 \\ 0 & -q\bar{q}_x + q_x\bar{q} \end{pmatrix}. \tag{2-32}$$

Taking the limit $x \to +\infty$ in (2-30), using the normalization of $\boldsymbol{M}^{\pm}$ at $+\infty$, and using the fact that $\operatorname{ad}\sigma$ is a derivation, we obtain

$$v_x(t;\zeta)_t = -2i\zeta^4 \operatorname{ad}\sigma\, v_x(t;\zeta).$$

Integrating we obtain

$$v_x(t;\zeta) = e^{-2i\zeta^4 t \operatorname{ad}\sigma} v_x(0;\zeta) \tag{2-33}$$

or equivalently for $J_x(\lambda) = e^{-i\lambda x \operatorname{ad}\sigma} J(0;\lambda)$

$$J_x(\lambda,t) = e^{-2i\lambda^2 t \operatorname{ad}\sigma} J_x(\lambda).$$

The map $(f,t) \mapsto e^{-2i\lambda^2 t} f$ is a bounded continuous map from $X \times [-T,T]$ to X for $X = H^{2,2}(\Omega_2)$, $H^{1,1}(\Omega_1)$, $H^2(\Omega_3)$ and $H^2(\Omega_4)$. This map is also Lipschitz continuous in X uniformly for f in a bounded subset of X and $t \in [-T,T]$ for a fixed $T > 0$.

From Proposition 2.9 and these facts, we deduce the following continuity result.

Proposition 2.10. *Suppose that $q_0 \in H^{2,2}(\mathbb{R})$ and that $J(\lambda)$ is the scattering data associated to q_0. Denote by $J_\pm(\lambda,t)$ the matrices $e^{i\lambda^2 t \operatorname{ad}\sigma} J_\pm(\lambda)$, where $J_\pm(\lambda)$ are the factors given in Theorem 2.7. For* *any $T > 0$, the maps*

$$H^{2,2}(\mathbb{R}) \times [-T, T] \ni (q_0, t) \mapsto J_-(\lambda, t) - I \in H^{2,2}(\partial\Omega_2),$$
$$H^{2,2}(\mathbb{R}) \times [-T, T] \ni (q_0, t) \mapsto J_-(\lambda, t) - I \in H^{2}(\partial\Omega_3),$$
$$H^{2,2}(\mathbb{R}) \times [-T, T] \ni (q_0, t) \mapsto J_+(\lambda, t) - I \in H^{2}(\partial\Omega_4),$$
$$H^{2,2}(\mathbb{R}) \times [-T, T] \ni (q_0, t) \mapsto J_+(\lambda, t) - I \in H^{1,1}(\partial\Omega_1)$$

are all continuous, and uniformly Lipschitz in q_0 for $t \in [-T, T]$ and q_0 in a bounded subset of $H^{2,2}(\mathbb{R})$.

2E. *Auxiliary scattering matrix.* In Section 2B, we have chosen $x_0 \in \mathbb{R}$ such that the cut-off potential $q_{x_0} = q\chi_{(x_0,\infty)}$ satisfies the smallness condition (2-12). By increasing x_0 if necessary, we assume $\tilde{q}_{x_0} = q\chi_{(-\infty,-x_0)}$ also satisfies (2-12). Let $\widetilde{\boldsymbol{N}}$ be constructed in the same way as $\boldsymbol{N}$ (see (2-20)) but with potential q_0 changed to $\tilde{q}_0$ with normalization at $x \to -\infty$. We define the auxiliary matrix s:

$$s(\lambda) = e^{ix\lambda \operatorname{ad}\sigma} \widetilde{\boldsymbol{N}}^{-1}(x, \lambda) \boldsymbol{N}(x, \lambda). \tag{2-34}$$

For $\lambda \in \Omega_1 \cup \Omega_2$

$$s(\lambda) = \begin{pmatrix} \delta(\lambda)^{-1} & 0 \\ 0 & \delta(\lambda) \end{pmatrix}, \qquad \delta(\lambda) = \begin{cases} \breve{\alpha}(\lambda), & \operatorname{Im}\lambda > 0, \\ \alpha(\lambda), & \operatorname{Im}\lambda < 0. \end{cases} \tag{2-35}$$

The jump matrix $\tilde{J}$ for $\widetilde{\boldsymbol{N}}$ is obtained from J by conjugation, as $\tilde{J} = s_-^{-1} J s_+$. In analogy with Theorem 2.7, we have:

Theorem 2.11. *The matrix $\tilde{J} = s_-^{-1} J s_+$ admits a triangular factorization $\tilde{J}(\lambda) = \tilde{J}_-^{-1}(\lambda)\tilde{J}_+(\lambda)$, where:*

(i) $\tilde{J}_+(\lambda) - I \in H^{2,2}(\partial\Omega_1)$, $\tilde{J}_-(\lambda) - I \in H^{1,1}(\partial\Omega_2)$, $\tilde{J}_-(\lambda) - I \in H^{2}(\partial\Omega_3)$ *and* $\tilde{J}_+(\lambda) - I \in H^{2}(\partial\Omega_4)$.

(ii) $\tilde{J}_+ \restriction_{\partial\Omega_1} - I$ *and* $\tilde{J}_- \restriction_{\partial\Omega_3} - I$ *are strictly upper triangular, while* $\tilde{J}_- \restriction_{\partial\Omega_2} - I$ *and* $\tilde{J}_+ \restriction_{\partial\Omega_4} - I$ *are strictly lower triangular.*

Remark 2.12. The reason for working with the Beals–Coifman solutions normalized at $-\infty$ is to obtain the desired decay in x at $-\infty$. The basic idea is to guarantee that the Fourier variable satisfies $|\xi| \geq |x|$. See [Zhou 1998, Lemma 2.3] for details.

3. Unique solvability of the RHP

The goal of this section is to prove the unique solvability of the Riemann–Hilbert Problem 3.1 on the contour $\Gamma = \mathbb{R} \cup \Gamma_\infty$, shown in Figure 2, right.

Riemann–Hilbert Problem 3.1. *Fix $x \in \mathbb{R}$. Find a row-vector-valued function $\boldsymbol{N}(x, \cdot)$ on $\mathbb{C} \setminus \Gamma$ with the following properties:*

(i) (*analyticity*) *$\boldsymbol{N}(x, z)$ is an analytic function of z for $z \in \mathbb{C} \setminus \Gamma$.*

(ii) (*normalization*) *$\boldsymbol{N}(x, z) = (1, 0) + \mathcal{O}(z^{-1})$ as $z \to \infty$.*

(iii) (*jump condition*) *For each $\lambda \in \Gamma$, $\boldsymbol{N}$ has continuous boundary values $\boldsymbol{N}_\pm(\lambda)$ as $z \to \lambda$ from $\Omega_\pm$. Moreover, the jump relation*

$$\boldsymbol{N}_+(x, \lambda) = \boldsymbol{N}_-(x, \lambda) J_x(\lambda)$$

holds, *where*

$$J_x(\lambda) = e^{-i\lambda x \,\mathrm{ad}\,\sigma} \begin{cases} \begin{pmatrix} 1+\lambda|\rho(\lambda)|^2 & \rho(\lambda) \\ \lambda\overline{\rho(\lambda)} & 1 \end{pmatrix}, & \lambda \in \mathbb{R}_\infty, \\ \begin{pmatrix} 1 & -\rho_0(\lambda) \\ -\lambda\overline{\rho_0(\lambda)} & 1+\lambda|\rho_0(\lambda)|^2 \end{pmatrix}, & \lambda \in (-S_\infty, S_\infty), \\ \begin{pmatrix} 1 & 0 \\ e^{-2ix_0\lambda}\dfrac{n_{21}^-(x_0,\lambda)}{\breve{\alpha}(\lambda)\breve{\alpha}_0(\lambda)} & 1 \end{pmatrix}, & \lambda \in \Gamma_\infty^+, \\ \begin{pmatrix} 1 & -e^{2ix_0\lambda}\dfrac{n_{12}^-(x_0,\lambda)}{\alpha(\lambda)\alpha_0(\lambda)} \\ 0 & 1 \end{pmatrix}, & \lambda \in \Gamma_\infty^-. \end{cases}$$

Definition 3.2. We say that the row-vector-valued function $\boldsymbol{N}(x,z)$ is a *null vector* for RHP 3.1 if $\boldsymbol{N}(x,z)$ satisfies (i) and (iii) above but $\boldsymbol{N}(x,z) = \mathcal{O}(z^{-1})$ as $|z| \to \infty$.

The scattering data that determine the jump matrix J are the functions

$$\mathrm{SD} = (\rho, \rho_0, \alpha, \alpha_0, \breve{\alpha}_0, n_{12}^-(x_0,\cdot\,), n_{21}^-(x_0,\cdot\,)).$$

Although these functions are not independent, for the purpose of proving existence and uniqueness of solutions to RHP 3.1 we may consider them so. Recalling (1-18), we seek a Banach space Y_0, consisting of functions $\rho : \mathbb{R}_\infty \to \mathbb{C}$, with the following properties:

(a) There is an injection $i : H^{2,2}(\mathbb{R}_\infty) \to Y_0$ that maps bounded subsets of $H^{2,2}(\mathbb{R})$ to precompact subsets of Y_0.

(b) For each $\rho \in Y_0$, we have $(1+|\cdot|)\rho(\cdot) \in L^2(\mathbb{R}_\infty) \cap L^\infty(\mathbb{R}_\infty)$.

(c) Each $\rho \in Y_0$ is a continuous function with $\lim_{\lambda\to\infty} \lambda\rho(\lambda) = 0$. This will allow uniform rational approximation of $(\cdot)\rho(\cdot)$ in L^∞.

Consider the weighted Sobolev spaces

$$H^{\alpha,\beta}(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \langle\xi\rangle^\alpha \hat{f}(\xi),\ \langle x\rangle^\beta f \in L^2(\mathbb{R})\}$$

and recall that for any $\varepsilon > 0$, we have

$$H^{1/2+\varepsilon,0}(\mathbb{R}) \subset C_0(\mathbb{R}),$$

where $C_0(\mathbb{R})$ denotes the continuous functions vanishing at infinity. Also, recall that the embedding $i : H^{\alpha,\beta}(\mathbb{R}) \hookrightarrow H^{\alpha',\beta'}(\mathbb{R})$ is compact for $\alpha > \alpha'$ and $\beta > \beta'$. From the estimates

$$\|\langle\cdot\rangle\rho(\cdot)\|_{L^2} \le \|\rho\|_{H^{0,1}(\mathbb{R})}, \quad \|\langle\cdot\rangle\rho(\cdot)\|_{H^{1,0}(\mathbb{R})} \le \|\rho\|_{H^{2,2}(\mathbb{R})}$$

it follows by interpolation that for any $\varepsilon > 0$,

$$\|\langle\,\cdot\,\rangle\rho(\,\cdot\,)\|_{H^{1/2+\varepsilon,0}(\mathbb{R})} \leq \|\rho\|_{H^{1+2\varepsilon,3/2+\varepsilon}}.$$

We know $Y_0 = H^{1+2\varepsilon,3/2+\varepsilon}(\mathbb{R}_\infty)$ is the image of the fractional Sobolev space $H^{1+2\varepsilon,3/2+\varepsilon}(\mathbb{R})$ under the restriction map $f \mapsto f|_{\mathbb{R}_\infty}$. This space satisfies the required properties (a), (b), (c) above.

Definition 3.3. We denote by Y the Banach space of scattering data SD with $\rho \in Y_0$ and all other data in H^1.

Remark 3.4. Note that, for SD $\in Y$, the entries of J all belong to $L^2 \cap L^\infty$.

Let $Z_0 = H^{2,2}(\mathbb{R}_\infty)$. By Proposition 2.8, the range of the direct scattering map actually lies in the following stronger space:

Definition 3.5. We denote by Z the set of scattering data SD with $\rho \in Z_0$ and all other data in H^2.

We choose to consider SD in the larger space in order to obtain uniform resolvent estimates for scattering data in bounded subsets of Z later by a continuity-compactness argument (see Appendix B). We will exploit the fact that, under the natural continuous embedding of Z in Y, bounded subsets of Z are identified with precompact subsets of Y. We will prove:

Theorem 3.6. *Suppose that the scattering data $J(\lambda)$ are given by* (2-21)–(2-24) *with* SD $\in Y$. *Then RHP* 3.1 *has a unique solution for each $x_0 \in \mathbb{R}$.*

Following the pattern of the uniqueness result in [Jenkins et al. 2018b; Liu 2017], we will prove the existence and uniqueness of solutions in the following way. First, we show that RHP 3.1 is equivalent to a Fredholm integral equation (the Beals–Coifman integral equation, (3-2), for an unknown function $\nu(x,\cdot)$) on Γ. By the Fredholm alternative, it suffices to show that the corresponding homogeneous equation, (3-3), has only the trivial solution. In order to do so, in Section 3B, we derive similar integral equations associated to an equivalent Riemann–Hilbert Problem, RHP 3.7, on the contour Σ. These integral equations involve an unknown function μ; the inhomogeneous equation is (3-6) and the homogeneous equation is (3-7). We can use Zhou's uniqueness theorem to show that RHP 3.7 is uniquely solvable, or, equivalently, (3-7) has only the trivial solution. Finally, we show that any solution ν of the homogeneous equation (3-2) induces a solution of (3-7), It then follows from explicit formulae connecting ν and μ that $\nu = 0$, establishing the Fredholm alternative for the *original* Beals–Coifman equation (3-2).

3A. ***RHPs and singular integral equations.*** We now derive the Beals–Coifman integral equation for RHP 3.1. The unique solvability of RHP 3.1 is equivalent to the unique solvability of its associated integral equation. We define the nilpotent matrices W_x^+ and W_x^- in the various parts of the contour as

$$J_x(\lambda) = (J_{x-})^{-1} J_{x+} = (I - W_x^-)^{-1}(I + W_x^+)$$

and the Beals–Coifman solution

$$\nu = \boldsymbol{N}^+(I + W_x^+)^{-1} = \boldsymbol{N}^-(I - W_x^-)^{-1}, \tag{3-1}$$

where

$$(W_x^+, W_x^-) = \begin{cases} \left(\begin{pmatrix} 0 & 0 \\ \lambda\overline{\rho(\lambda)} e^{2i\lambda x} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \rho(\lambda) e^{-2i\lambda x} \\ 0 & 0 \end{pmatrix} \right), & \lambda \in \mathbb{R}_\infty, \\ \left(\begin{pmatrix} 0 & -\rho_0(\lambda) e^{-2i\lambda x} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -\lambda\overline{\rho_0(\lambda)} e^{2i\lambda x} & 0 \end{pmatrix} \right), & \lambda \in (-S_\infty, S_\infty), \\ \left(\begin{pmatrix} 0 & 0 \\ e^{2i\lambda x} S_1(\lambda) & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ e^{2i\lambda x} S_2(\lambda) & 0 \end{pmatrix} \right), & \lambda \in \Gamma_\infty^+, \\ \left(\begin{pmatrix} 0 & e^{-2i\lambda x} S_3(\lambda) \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & e^{-2i\lambda x} S_4(\lambda) \\ 0 & 0 \end{pmatrix} \right), & \lambda \in \Gamma_\infty^-. \end{cases}$$

The coefficients $S_i(\lambda)$, $i = 1, \dots, 4$, are not explicitly determined. Only the sums $S_1(\lambda) + S_2(\lambda)$ and $S_3(\lambda) + S_4(\lambda)$ identify to the entries $(2, 1)$ and $(1, 2)$ of the jump matrix $J_x(\lambda)$ respectively, in the corresponding part of the contour. If $\mathrm{SD} \in Y$, then $W_x^\pm$ in $L^\infty \cap L^2$, while if $\mathrm{SD} \in Z$, we have $W_x^\pm \in H^1$.

We can write the Beals–Coifman solution $\nu(x, \lambda)$ explicitly in terms of the Jost functions. From (3-1), we have two equivalent formulas:

$$\nu(x, \lambda) = \begin{cases} \begin{pmatrix} \dfrac{n_{11}^-(x, \lambda)}{\breve{a}(\lambda)} & n_{12}^+(x, \lambda) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -e^{2i\lambda x}\lambda\overline{\rho(\lambda)} & 1 \end{pmatrix}, \\ \begin{pmatrix} n_{11}^+(x, \lambda) & \dfrac{n_{12}^-(x, \lambda)}{\alpha(\lambda)} \end{pmatrix} \begin{pmatrix} 1 & \rho(\lambda) e^{-2i\lambda x} \\ 0 & 1 \end{pmatrix}, \end{cases} \qquad \lambda \in \mathbb{R}_\infty,$$

$$\nu(x, \lambda) = \begin{cases} \begin{pmatrix} \dfrac{n_{11}^-(x, \lambda)}{\breve{a}(\lambda)} & n_{12}^+(x, \lambda) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -e^{2i\lambda x} S_1(\lambda) & 1 \end{pmatrix}, \\ (N_{11+}^{(2)}(x, \lambda) \quad N_{12+}^{(2)}(x, \lambda)) \begin{pmatrix} 1 & 0 \\ e^{2i\lambda x} S_2(\lambda) & 1 \end{pmatrix}, \end{cases} \qquad \lambda \in \Gamma_\infty^+,$$

$$\nu(x, \lambda) = \begin{cases} (N_{11+}^{(2)}(x, \lambda) \quad N_{12+}^{(2)}(x, \lambda)) \begin{pmatrix} 1 & \rho_0(\lambda) e^{-2i\lambda x} \\ 0 & 1 \end{pmatrix}, \\ (N_{11-}^{(2)}(x, \lambda) \quad N_{12-}^{(2)}(x, \lambda)) \begin{pmatrix} 1 & 0 \\ -\lambda\overline{\rho_0(\lambda)} e^{2i\lambda x} & 1 \end{pmatrix} \end{cases} \qquad \lambda \in (-S_\infty, S_\infty),$$

$$\nu(x, \lambda) = \begin{cases} (N_{11-}^{(2)}(x, \lambda) \quad N_{12-}^{(2)}(x, \lambda)) \begin{pmatrix} 1 & -e^{-2i\lambda x} S_3(\lambda) \\ 0 & 1 \end{pmatrix}, \\ \begin{pmatrix} n_{11}^+(x, \lambda) & \dfrac{n_{12}^-(x, \lambda)}{\alpha(\lambda)} \end{pmatrix} \begin{pmatrix} 1 & e^{-2i\lambda x} S_4(\lambda) \\ 0 & 1 \end{pmatrix}, \end{cases} \qquad \lambda \in \Gamma_\infty^-.$$

From (3-1), we have

$$\boldsymbol{N}^+ - \boldsymbol{N}^- = \nu(W_x^+ + W_x^-).$$

The Plemelj formula and the normalization condition (ii) in RHP 3.1 provide the Beals–Coifman integral equation

$$\nu(x,\lambda) = (1,0) + (\mathcal{C}_{W_x}\nu)(\lambda), \tag{3-2}$$

where

$$\mathcal{C}_{W_x}\nu = C_\Gamma^+(\nu W_x^-) + C_\Gamma^-(\nu W_x^+).$$

RHP 3.1 is equivalent to the integral equation (3-2) [Zhou 1989b, Proposition 3.3]. Similarly, if $\boldsymbol{N}$ is a null vector for RHP 3.1 in the sense of Definition 3.2 and ν is defined in (3-1), we have

$$\nu(x,\lambda) = \mathcal{C}_{W_x}\nu(\lambda). \tag{3-3}$$

If SD $\in Y$, equation (3-2) (resp. (3-3)) is seen as an integral equation for $\nu - 1 \in L^2(\Gamma)$ (resp. $\nu \in L^2(\Gamma)$), while if SD $\in Z$, it is an integral equation for $\nu - 1 \in H^1(\Gamma)$ (resp. $\nu \in H^1(\Gamma)$).

For $\lambda \in \mathbb{R}_\infty$, (3-2) reads

$$\begin{aligned}\nu_{11}(x,\lambda) = 1 + \int_{\mathbb{R}_\infty} \frac{\nu_{12}(x,s)s\overline{\rho(s)}e^{2isx}}{s-\lambda+i0}\frac{ds}{2\pi i} - \int_{-S_\infty}^{S_\infty} \frac{\nu_{12}(x,s)s\overline{\rho_0(s)}e^{2isx}}{s-\lambda}\frac{ds}{2\pi i}\\ + \int_{\Gamma_\infty^+} \frac{\nu_{12}(x,s)(S_1(s)+S_2(s))e^{2isx}}{s-\lambda}\frac{ds}{2\pi i},\end{aligned}$$

$$\begin{aligned}\nu_{12}(x,\lambda) = \int_{\mathbb{R}_\infty} \frac{\nu_{11}(x,s)\rho(s)e^{-2isx}}{s-\lambda-i0}\frac{ds}{2\pi i} - \int_{-S_\infty}^{S_\infty} \frac{\nu_{11}(x,s)\rho_0(s)e^{-2isx}}{s-\lambda}\frac{ds}{2\pi i}\\ + \int_{\Gamma_\infty^-} \frac{\nu_{11}(x,s)(S_3(\lambda)+S_4(\lambda))e^{-2isx}}{s-\lambda}\frac{ds}{2\pi i}.\end{aligned}$$

The integral equations for $\lambda \in (-S_\infty, S_\infty)$ and $\lambda \in \Gamma_\infty^\pm$ are obtained analogously. The solution to RHP 3.1 is given, in terms of $\nu = (\nu_{11}, \nu_{12})$, by

$$\boldsymbol{N}(x,z) = (1,0) + \frac{1}{2\pi i}\int_\Gamma \frac{\nu(x,s)(W_x^+(s)+W_x^-(s))}{s-z}\,ds. \tag{3-4}$$

The goal is an existence and uniqueness result for solution to RHP 3.1. To make use of the symmetry relations of the jump conditions and Zhou's vanishing lemma, we need to consider the equivalent RHP in the ζ-variable with jump contour $\mathbb{R} \cup i\mathbb{R} \cup \Sigma_\infty$ given by Figure 2, left.

Riemann–Hilbert Problem 3.7. *Fix $x \in \mathbb{R}$. Find a matrix-valued function $\boldsymbol{M}(x,\cdot)$ with the following properties*:

(i) (*analyticity*) *$\boldsymbol{M}(x,z)$ is a 2×2 matrix-valued analytic function of z for $z \in \mathbb{C}\setminus\Sigma$ where the contour Σ is given by Figure* 2, *left.*

(ii) (*normalization*)

$$\boldsymbol{M}(x,z) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \mathcal{O}(z^{-1}) \quad \text{as } z \to \infty.$$

(iii) (*jump condition*) *For each* $\zeta \in \Sigma$, *the function* $\boldsymbol{M}$ *has continuous boundary values* $\boldsymbol{M}_\pm(\lambda)$ *as* $z \to \zeta$ *from* $\Omega_\pm$. *Moreover*, *the jump relation*

$$\boldsymbol{M}_+(x,\zeta) = \boldsymbol{M}_-(x,\zeta) e^{-ix\zeta^2 \operatorname{ad}\sigma} v(\zeta)$$

holds, *where* $v(\zeta)$ *is given by* (2-14)–(2-17).

Definition 3.8. We say that a matrix-valued function $\boldsymbol{M}(x,z)$ is a *null vector* for RHP 3.7 if $\boldsymbol{M}(x,z)$ satisfies (i) and (iii) above and

$$\boldsymbol{M}(x,z) = \mathcal{O}(z^{-1}) \quad \text{as } |z| \to \infty.$$

Observe that, given scattering data SD for RHP 3.1 in the space Y from Definition 3.3, the induced scattering data for RHP 3.7 consist of bounded continuous functions, square-integrable on the unbounded contours. Thus RHP 3.7 is well-defined with the $\mathcal{O}(z^{-1})$ condition replaced by an L^2-condition on $\boldsymbol{M}_\pm - I$ (and the condition $\boldsymbol{M}_\pm \in L^2$ for an L^2-null vector).

Proposition 3.9. *The only* L^2 *null vector for RHP* 3.7 *with scattering data induced from* SD $\in Y$ *is the zero vector.*

Proof. The proof is a direct consequence of Proposition 2.1 and [Zhou 1989b, Theorem 9.3]. □

It is useful to formulate Proposition 3.9 in terms of the homogeneous Beals–Coifman equation associated to RHP 3.7, which we now derive.

The jump matrix $v_x(\zeta)$ admits the factorization

$$v_x(\zeta) = (1 - w_x^-)^{-1}(1 + w_x^+).$$

We set

$$\mu = \boldsymbol{M}^+(1 + w_x^+)^{-1} = \boldsymbol{M}^-(1 - w_x^-)^{-1}. \tag{3-5}$$

In analogy with RHP 3.1, the Beals–Coifman integral equation for RHP 3.7 is

$$\mu = I + \mathcal{C}_{w_x}\mu = I + C_\Sigma^+(\mu w_x^-) + C_\Sigma^-(\mu w_x^+), \tag{3-6}$$

where I is the 2×2 identity matrix. If $\boldsymbol{M}$ is a null vector in the sense of Definition 3.8 and μ is defined by (3-5), then

$$\mu = \mathcal{C}_{w_x}\mu. \tag{3-7}$$

We can now reformulate Proposition 3.9 as follows:

Proposition 3.10. *Assume that* $w^\pm$ *are obtained from scattering data* SD *in* Y. *Then*, *the only solution to* (3-7) *in* $L^2(\Sigma)$ *is the zero vector.*

3B. *A mapping between null spaces.* To complete the proof of Theorem 3.6, we show that any solution ν of (3-3) induces a solution μ of (3-7) and that if $\mu = 0$, then $\nu = 0$. For notational brevity, we suppress the dependence of μ and ν on x, which remains fixed throughout the discussion.

Lemma 3.11. *Suppose that $W_x^{\pm}$ are generated from scattering data* SD $\in Y$. *For $\nu = (\nu_1, \nu_2)$ a solution of the homogeneous Beals–Coifman equation* (3-3) *in $L^2(\Gamma)$, define*

$$\mu(x,\zeta) = \begin{pmatrix} \mu_{11}(x,\zeta) & \mu_{12}(x,\zeta) \\ \mu_{21}(x,\zeta) & \mu_{22}(x,\zeta) \end{pmatrix} = \begin{pmatrix} \nu_{11}(x,\zeta^2) & \zeta\nu_{12}(x,\zeta^2) \\ -\zeta\overline{\nu_{12}(x,\bar{\zeta}^2)} & \overline{\nu_{11}(x,\bar{\zeta}^2)} \end{pmatrix}. \tag{3-8}$$

Then $\mu \in L^2(\Sigma)$ solves (3-7).

Remark 3.12. We can invert (3-8) to recover ν via the formulas

$$\nu_{11}(x,\zeta^2) = \mu_{11}(x,\zeta), \quad \nu_{12}(x,\zeta^2) = \frac{\mu_{12}(x,\zeta)}{\zeta}. \tag{3-9}$$

In particular, if $\mu = 0$, then $\nu = 0$.

Proof. Define a matrix-valued function μ by (3-9) for a given solution ν of (3-3). It is easy to see that

$$\mu_{11}(x,-\zeta) = \mu_{11}(x,\zeta), \quad \mu_{12}(x,-\zeta) = -\mu_{12}(x,\zeta).$$

In [Liu 2017, Lemma 5.2.2] we have shown that for $\nu \in L^2(\Gamma)$ and $\rho \in Y_0$,

$$\mu_{11}(x,\zeta)r(\zeta) = \nu_{11}(x,\zeta^2)\zeta\rho(\zeta^2), \quad \mu_{12}(x,\zeta)\breve{r}(\zeta) = \zeta\nu_{12}(x,\zeta^2)\zeta\overline{\rho(\zeta^2)}$$

are both square-integrable on the part of the Σ contour outside the circle Σ_∞. Thus $\mu w_x^{\pm}$ is an L^2 function on Σ. Once (3-7) is obtained from (3-3), $\mu \in L^2(\Sigma)$ follows from the boundedness of Cauchy projection on L^2-functions.

In [Liu 2017, Chapter 5], the second author established the transition from (3-3) to (3-7) when $\Gamma = \mathbb{R}$ and $\Sigma = \mathbb{R} \cup i\mathbb{R}$. Thus, we only consider the contour integrals

$$I^+ := \int_{\Gamma_\infty^+} \frac{\nu_{12}(x,s)(S_1(s)+S_2(s))e^{2isx}}{s-\lambda}\frac{ds}{2\pi i}, \tag{3-10}$$

$$I^- := \int_{\Gamma_\infty^-} \frac{\nu_{11}(x,s)(S_3(\lambda)+S_4(\lambda))e^{-2isx}}{s-\lambda}\frac{ds}{2\pi i}. \tag{3-11}$$

Let $\lambda = \zeta^2$ and fix the branch $[0, 2\pi)$. Then

$$\begin{aligned} I^+ &= \int_{\Gamma_\infty^+} \frac{\nu_{12}(x,\lambda)n_{21}^-(x_0,\lambda)e^{2i\lambda(x-x_0)}}{(\lambda-\lambda_0)\breve{a}(\lambda)\breve{a}_0(\lambda)}\frac{d\lambda}{2\pi i} \\ &= \int_C \frac{\zeta^{-1}\mu_{12}(x,\zeta)\zeta m_{21}^-(x_0,\zeta)e^{2i\zeta^2(x-x_0)}}{(\zeta^2-\zeta_0^2)\breve{a}(\zeta)\breve{a}_0(\zeta)}\frac{d\zeta^2}{2\pi i} \\ &= \int_C \left(\frac{\mu_{12}(x,\zeta)m_{21}^-(x_0,\zeta)e^{2i\zeta^2(x-x_0)}}{(\zeta-\zeta_0)\breve{a}(\zeta)\breve{a}_0(\zeta)} - \frac{\mu_{12}(x,\zeta)m_{21}^-(x_0,\zeta)e^{2i\zeta^2(x-x_0)}}{(-\zeta-\zeta_0)\breve{a}(\zeta)\breve{a}_0(\zeta)}\right)\frac{1}{2\zeta}\frac{d\zeta^2}{2\pi i} \\ &= \int_C \frac{\mu_{12}(x,\zeta)m_{21}^-(x_0,\zeta)e^{2i\zeta^2(x-x_0)}}{(\zeta-\zeta_0)\breve{a}(\zeta)\breve{a}_0(\zeta)}\frac{d\zeta}{2\pi i} - \int_C \frac{\mu_{12}(x,\zeta)m_{21}^-(x_0,\zeta)e^{2i\zeta^2(x-x_0)}}{(-\zeta-\zeta_0)\breve{a}(\zeta)\breve{a}_0(\zeta)}\frac{d\zeta}{2\pi i} = I_1^+ + I_2^+. \end{aligned}$$

Setting $\lambda = R^2 e^{i\theta}$ with $\theta \in (\pi, 0)$, we integrate I_1^+ over the arc $\zeta = Re^{i\eta}$, where η goes from $\frac{\pi}{2}$ to 0. For I_2^+, we make the change of variable $\zeta \to -\zeta$, then using of the oddness of μ_{12} and m_{21}, we obtain

$$-\int \frac{\mu_{12}(x,\zeta)m_{21}^-(x_0,\zeta)e^{2i\zeta^2(x-x_0)}}{(-\zeta-\zeta_0)\breve{a}(\zeta)\breve{a}_0(\zeta)}\,\frac{d\zeta}{2\pi i} = \int \frac{\mu_{12}(x,\zeta)m_{21}^-(x_0,\zeta)e^{2i\zeta^2(x-x_0)}}{(\zeta-\zeta_0)\breve{a}(\zeta)\breve{a}_0(\zeta)}\,\frac{d\zeta}{2\pi i}.$$

For I_2^+, we integrate over the arc $Re^{i\eta}$ with η going from $\frac{3\pi}{2}$ to π. This completes the change of variable for (3-10). Similarly, we have

$$\begin{aligned}
I^- &= -\int_{\Gamma_\infty^-} \frac{\nu_{11}(x,\lambda)n_{12}^-(x_0,\lambda)e^{2i\lambda(x_0-x)}}{(\lambda-\lambda_0)\alpha(\lambda)\alpha_0(\lambda)}\,\frac{d\lambda}{2\pi i} \\
&= -\int_C \frac{\mu_{11}(x,\zeta)m_{12}^-(x_0,\zeta)e^{2i\zeta^2(x_0-x)}}{\zeta(\zeta^2-\zeta_0^2)a(\zeta)a_0(\zeta)}\,\frac{d\zeta^2}{2\pi i} \\
&= -\int_C \frac{1}{2\zeta^2}\left(\frac{1}{\zeta-\zeta_0}+\frac{1}{\zeta+\zeta_0}\right)\frac{\mu_{11}(x,\zeta)m_{12}^-(x_0,\zeta)e^{2i\zeta^2(x_0-x)}}{a(\zeta)a_0(\zeta)}\,\frac{d\zeta^2}{2\pi i} \\
&= -\int_C \frac{\mu_{11}(x,\zeta)m_{12}^-(x_0,\zeta)e^{2i\zeta^2(x_0-x)}}{\zeta_0(\zeta-\zeta_0)a(\zeta)a_0(\zeta)}\,\frac{d\zeta}{2\pi i} + \int_C \frac{\mu_{11}(x,\zeta)m_{12}^-(x_0,\zeta)e^{2i\zeta^2(x_0-x)}}{\zeta_0(\zeta+\zeta_0)a(\zeta)a_0(\zeta)}\,\frac{d\zeta}{2\pi i} = I_1^- + I_2^-.
\end{aligned}$$

We write $\lambda = R^2 e^{i\theta}$ with $\theta \in (\pi, 2\pi)$. For I_1^-, we integrate over the arc $\zeta = Re^{i\eta}$, where η goes from $\frac{\pi}{2}$ to π. For I_2^-, we make the change of variable $\zeta \to -\zeta$, and then make use of the evenness and oddness of μ_{11} and m_{12} respectively to obtain

$$\begin{aligned}
\int \frac{\mu_{11}(x,\zeta)m_{12}^-(x_0,\zeta)e^{2i\zeta^2(x_0-x)}}{\zeta_0(\zeta+\zeta_0)\breve{a}(\zeta)\breve{a}_0(\zeta)}\,\frac{d\zeta}{2\pi i} &= -\int \frac{\mu_{11}(x,\zeta)m_{12}^-(x_0,\zeta)e^{2i\zeta^2(x_0-x)}}{\zeta_0(-\zeta-\zeta_0)\breve{a}(\zeta)\breve{a}_0(\zeta)}\,\frac{d\zeta}{2\pi i} \\
&= -\int \frac{\mu_{11}(x,\zeta)m_{12}^-(x_0,\zeta)e^{2i\zeta^2(x_0-x)}}{\zeta_0(\zeta-\zeta_0)\breve{a}(\zeta)\breve{a}_0(\zeta)}\,\frac{d\zeta}{2\pi i}.
\end{aligned}$$

For I_2^-, we integrate over the arc $Re^{i\eta}$, where η goes from $\frac{3\pi}{2}$ to 2π. Integrals involving μ_{21} and μ_{22} are derived using complex conjugations. □

Proof of Theorem 3.6. First, for scattering data SD $\in Y$, the operator $(I - \mathcal{C}_{W_x})$ is a Fredholm operator on $L^2(\Gamma)$. This follows from [Trogdon and Olver 2016, Lemma 2.60] since we allow uniform rational approximation of the function $(\cdot)\rho(\cdot)$ for $\rho \in Y_0$. Next, we claim that $\ker_{L^2(\Gamma)}(I - \mathcal{C}_{W_x})$ is trivial. If $\nu \in \ker_{L^2(\Gamma)}(I - \mathcal{C}_{W_x})$, then by Lemma 3.11, ν induces a vector $\mu \in \ker_{L^2(\Sigma)}(I - \mathcal{C}_{w_x})$, which must be the zero vector by Proposition 3.10. It follows from Remark 3.12 that $\nu = 0$. Finally, from Fredholm theory, $(I - \mathcal{C}_{W_x})$ is invertible in $L^2(\Gamma)$, which is equivalent to unique solvability of RHP 3.1. □

Corollary 3.13. *The resolvent $(I - \mathcal{C}_{W_x})^{-1}$ exists for all $x \in \mathbb{R}$ and all* SD $\in Y$.

4. Mapping properties of the inverse scattering map

Recall that the potential q is reconstructed by solving the "right" Riemann–Hilbert Problem 3.1 (for a solution normalized as $x \to +\infty$). As shown in Section 2E, the "left" Riemann–Hilbert problem (with jump matrix characterized by Theorem 2.11) can be conjugated to the "right" Riemann–Hilbert problem, so we concentrate on the mapping properties of the reconstruction from the right. We omit the (standard) proof that the left and right reconstructions agree, as well as the proof that the inverse map composed with the direct map is the identity map on $H^{2,2}(\mathbb{R})$. Thus, in the statements of Theorems 4.2 and 4.5, an assertion is made about the reconstructed potential on $\mathbb{R}$, but details of the proof are only given for the restriction of q to a half-line of the form (c, ∞).

We start with the reconstruction formula for the potential q from given scattering data $J_\pm$ as characterized in Theorem 2.7:

$$\begin{aligned} q(x) &= \left(-\frac{1}{\pi}\int_\Gamma \nu(x,\lambda)(W_x^+(\lambda)+W_x^-(\lambda))\,d\lambda\right)_{12} \\ &= \left(-\frac{1}{\pi}\int_\Gamma \nu(x,\lambda)e^{-i\lambda x\,\mathrm{ad}\,\sigma}(J_+(\lambda)-J_-(\lambda))\,d\lambda\right)_{12}, \end{aligned} \tag{4-1}$$

where the "12" subscript denotes the second entry of the row vector, and Γ is the contour shown in Figure 2, right.

Let $\boldsymbol{A}(\Omega_\pm)$ denote the space of analytic functions in the region $\Omega_\pm$ of the complex plane and $R(\partial\Omega_\pm)$ the space of functions whose restrictions on $\partial\Omega_\pm$ are rational. Following a reduction technique of [Zhou 1998], we construct functions $\omega_\pm \in \boldsymbol{A}(\Omega_\pm)$ such that, for $k=2$:

(1) $\omega_\pm \in R(\partial\Omega_\pm)$ and $\omega_\pm - I = O(z^{-2})$ as $z \to \infty$.

(2) $\omega_\pm$ has the same triangularity as $J_\pm$.

(3) $\omega_\pm(z) = J_\pm(z) + o((z-a)^{k-1})$ for $a = \pm S_\infty$.

The construction of $\omega_\pm$ is given in [Zhou 1989a, Appendix I]. For example, consider the approximation of $J_- \restriction_{\partial\Omega_2}$. Since $(J_- - I)\restriction_{\partial\Omega_2}$ is in H^2, we construct a rational function ω_- such that $(\omega_- - J_-)\restriction \partial\Omega_2$ vanishes at $\pm S_\infty$ to order 1. Explicitly

$$\omega_-(\pm S_\infty) - I = \rho(\pm S_\infty), \quad \omega_-'(\pm S_\infty) = \rho'(\pm S_\infty).$$

This is performed by the following steps:

(i) Choose $z_0 \notin \overline{\Omega}_2$ and denote by $p_\pm$ the Taylor polynomial of degree 1 of $(z-z_0)^n\rho(z)$ at $z=\pm S_\infty$.We choose $n \geq 6$.

(ii) By [Zhou 1989a, Lemma A1.2], there is a polynomial $p(z)$ of degree at most 3 such that

$$p(z) - p_\pm(z) = O(z \mp S_\infty)^2.$$

(iii) Set $\omega_-(z) = (z-z_0)^{-n}p(z)$. Clearly, $\omega_-(z) - \rho(z)$ vanishes at $\pm S_\infty$ to order 1. Since $n \geq 6$, $\omega_- \in H^{2,2}(\partial\Omega_2)$ and ω is analytic in Ω_2.

By construction,

$$J = \omega_-^{-1}(J_-\omega_-^{-1})^{-1}(J_+\omega_+^{-1})\omega_+ \equiv \omega_-^{-1}\mathcal{J}_-^{-1}\mathcal{J}_+\omega_+ \equiv \omega_-^{-1}\mathcal{J}\omega_+. \tag{4-2}$$

The advantage of working with $\mathcal{J}$ is that $\mathcal{J}_\pm - I$ vanishes at $\pm S_\infty$ to order 1:

$$\mathcal{J}_\pm(\lambda) = I + o((\lambda-a)^1), \quad a = \pm S_\infty. \tag{4-3}$$

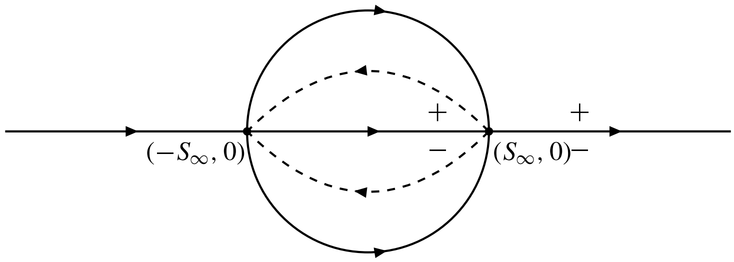

Figure 5. The newly modified contour Γ_m.

The continuity of $\mathcal{J}_\pm$ and its derivative at $\lambda = \pm S_\infty$ is a key point to performing the decay estimates for the reconstructed potential. Notice that $\mathcal{J}_\pm$ are defined like $J_\pm$ in Theorem 2.7 and they will be used when establishing estimates such as (4-9) and (4-10). For $x \geq 0$, $\mathcal{J}_x$ is the jump matrix for the RHP

$$\mathcal{N}_+(x, \lambda) = \mathcal{N}_-(x, \lambda)\mathcal{J}_x(\lambda) \quad \lambda \in \Gamma,$$

if and only if J_x is the jump matrix for the RHP 3.1. Here $\boldsymbol{N} = \mathcal{N} e^{-i\lambda x \operatorname{ad}\sigma}\omega$, where $e^{-i\lambda x \operatorname{ad}\sigma}\omega \in \boldsymbol{AL}^\infty(\mathbb{C} \setminus \Gamma) \cap \boldsymbol{AL}^2(\mathbb{C} \setminus \Gamma)$ is guaranteed by construction. Note that $\boldsymbol{N}$ and $\mathcal{N}$ give rise to the same ν, a solution of the associated Beals–Coifman equation. The potential is given by

$$q(x) = \left(-\frac{1}{\pi}\int_\Gamma \nu(x, \lambda) e^{-i\lambda x \operatorname{ad}\sigma}(\mathcal{J}_+(\lambda) - \mathcal{J}_-(\lambda))\, d\lambda\right)_{12}.$$

Due to the large z-behavior of $\omega_\pm(z)$, we have

$$\big(\lim_{z\to\infty} 2izN(x, z)\big)_{12} = \big(\lim_{z\to\infty} 2iz\mathcal{N}(x, z)\big)_{12},$$

which shows that ω gives no contribution to the reconstruction of q for $x \geq 0$. We may thus as well work with $\mathcal{J}$.

The next step consists in augmenting the contour as in Figure 5. The newly modified contour is denoted by Γ_m. The advantage of Γ_m is that it reverses the orientation of the segment (S_∞^-, S_∞^+) and thus allows to prove usual estimates of the Cauchy projections when the contour is restricted to $\mathbb{R}$. The added (dashed) contours have no effect on the RHP since the jump matrices there are chosen to be the identity.

We redefine $\mathcal{J}_\pm$ as follows:

(1) $\mathcal{J}_\pm = I$ on the added (dashed) contours.

(2) $\mathcal{J}_+(\lambda)$ and $\mathcal{J}_-(\lambda)$ are, respectively, the lower and upper triangular factors in the factorization of $\mathcal{J}(\lambda)$, $\lambda \in \mathbb{R}$, $|\lambda| > S_\infty$.

(3) On the other hand, $\mathcal{J}_+(\lambda)$ and $\mathcal{J}_-(\lambda)$ are, respectively, the lower and upper triangular factors in the factorization of $\mathcal{J}^{-1}(\lambda)$, $\lambda \in \mathbb{R}$, $|\lambda| < S_\infty$.

(4) For $\lambda \in \Gamma_\infty$, $\mathcal{J}_\pm(\lambda) = I$ for $\operatorname{Im}\lambda \lesseqgtr 0$ and $\mathcal{J}_\pm(\lambda) = \mathcal{J}(\lambda)$ for $\operatorname{Im}\lambda \gtreqless 0$.

The newly defined $\mathcal{J}_\pm$ satisfy all properties listed in Theorem 2.7. To analyze the scattering map, we will use the revised reconstruction formula

$$q(x) = \left(-\frac{1}{\pi}\int_{\Gamma_m} \nu(x, \lambda) e^{-i\lambda x \operatorname{ad}\sigma}(\mathcal{J}_+(\lambda) - \mathcal{J}_-(\lambda))\, d\lambda\right)_{12}. \tag{4-4}$$

We will at first suppress dependence of the scattering data on t (Sections 4A and 4B), but recall it again in Section 4C.

Associated to the Riemann–Hilbert problem with jump matrix $\mathcal{J}_x$ is a Beals–Coifman integral equation, where the Beals–Coifman operator is given by

$$C_{\mathcal{J}_x}\phi = C^+[\phi(\mathcal{J}_{x+} - I)] + C^-[\phi(I - \mathcal{J}_{x-})]. \tag{4-5}$$

Throughout the analysis we will use the following uniform resolvent bound.

Proposition 4.1. *Suppose that* $\mathcal{J} = \omega_- J \omega_+^{-1}$, *where* $\mathcal{J}$ *has the form of* (2-21)–(2-24), *is constructed from scattering data in a bounded subset* B *of* Z (*see Definition 3.5*), *and admits an algebraic factorization* $\mathcal{J} = \mathcal{J}_-^{-1}\mathcal{J}_+$, *where* $\mathcal{J}_\pm$ *have the same triangularities as in Theorem 2.7. Then*, *for fixed* $a \in \mathbb{R}$ *and all* $x \geq a$ *the estimate*

$$\sup_{\mathcal{J}\in B} \|(I - C_{\mathcal{J}_x})^{-1}\|_{L^2\to L^2} < \infty \tag{4-6}$$

holds. Finally, *the map*

$$\mathcal{J} \mapsto (x \mapsto (I - C_{\mathcal{J}_x})^{-1})$$

is Lipschitz continuous into the space $C([a,\infty); \mathscr{B}(L^2))$.

Proof. We check hypotheses (i)–(iii) of Proposition B.1 with $X = L^2(\mathbb{R})$, Y as given in Definition 3.3, and Z as given in Definition 3.5.

(i) The continuity of the map $(\mathcal{J}, x) \mapsto C_{\mathcal{J}_x}$ and the uniform continuity estimate follow immediately from (4-5).

(ii) The proof of Corollary 3.13 applies with no essential change to show that $(I - C_{\mathcal{J}_x})^{-1}$ exists for all $x \in \mathbb{R}$ and $\mathcal{J} \in Y$.

(iii) To prove this estimate we need to show that $(I - C_{\mathcal{J}_x})^{-1}$ is bounded as $x \to +\infty$ for each fixed $\mathcal{J}$. To do this we use a standard parametrix construction and approximation argument due to [Zhou 1989b].

Let $\check{\mathcal{J}}_\pm = (\mathcal{J}_\pm)^{-1}$. A standard computation shows that

$$I - T_{\mathcal{J}_x} = (I - C_{\mathcal{J}_x})(I - C_{\check{\mathcal{J}}_x}),$$

where

$$T_{\mathcal{J}_x}\phi = C^+(C^-\phi(\mathcal{J}_{x+} - \mathcal{J}_{x-}))(I - \check{\mathcal{J}}_{x-}) + C^-(C^+\phi(\mathcal{J}_{x+} - \mathcal{J}_{x-}))(\check{\mathcal{J}}_{x+} - I).$$

The operator $T_{\mathcal{J}_x}$ is compact so that $(I - C_{\check{\mathcal{J}}_x})$ is a Fredholm regulator for $(I - C_{\mathcal{J}_x})$. It suffices to show

$$\lim_{x\to+\infty} \|T_{\mathcal{J}_x}\|_{L^2\to L^2} = 0$$

since

$$(I - C_{\mathcal{J}_x})^{-1} = (I - C_{\check{\mathcal{J}}_x})(I - T_{\mathcal{J}_x})^{-1}$$

and $\|(I - C_{\check{\mathcal{J}}_x})\|_{L^2\to L^2}$ is bounded uniformly in x. By mimicking the proof of [Zhou 1989b, Theorem 6.1] (with some sign changes since we consider the limit $x \to +\infty$ rather than $x \to -\infty$) we can show by rational approximation that, for *fixed* $\mathcal{J}$, we have $\|T_{\mathcal{J}_x}\|_{L^2\to L^2} \to 0$ as $x \to -\infty$. Taking b so that

$\|T_{\mathcal{J}_x}\|_{L^2\to L^2} < \frac{1}{2}$ for $x \geq b$, we obtain a uniform bound on $\|(I - C_{\mathcal{J}_x})^{-1}\|_{L^2\to L^2}$ for $x \geq b$. Since $x \mapsto (I - C_{\mathcal{J}_x})^{-1}$ is continuous, this implies that $\sup_{x\geq a}\|(I - C_{\mathcal{J}_x})^{-1}\|_{L^2\to L^2}$ is bounded for any $a \in \mathbb{R}$.

We can now apply Proposition B.1 to obtain the uniform bound and the asserted Lipschitz continuity. □

4A. ***Decay property of the reconstructed potential.*** Let

$$H^{0,2}(\mathbb{R}) = \{q \in L^2(\mathbb{R}) : x^2 q(x) \in L^2(\mathbb{R})\}.$$

Theorem 4.2. *If J is given as in Theorem 2.7 and q is defined by* (4-4), *then $q \in H^{0,2}(\mathbb{R})$. Moreover, the map from data $\mathcal{J} = \mathcal{J}_-^{-1}\mathcal{J}_+$, defined in* (4-2) *and obeying the hypothesis of Theorem 3.6, to $q \in H^{0,2}(\mathbb{R})$ is Lipschitz continuous.*

Definition 4.3. Define the subsets of the contour Γ_m

$$\Gamma_\pm := \mathbb{R} \cup (\{\operatorname{Im}\lambda \gtrless 0\} \cap \Gamma_m),$$
$$\Gamma' := \Gamma_\infty^\pm = \text{either } \Gamma_\infty \cap \{\operatorname{Im}\lambda \gtrless 0\} \text{ or } \mathbb{R}.$$

Lemma 4.4 (See [Zhou 1998, Lemma 2.9]). *For $x \geq 0$,*

$$\|C^+_{\Gamma'\to\Gamma_+}(I - \mathcal{J}_{x-})\|_{L^2} \leq \frac{c}{1+x^2}\|\mathcal{J}_- - I\|_{H^2}, \tag{4-7}$$

$$\|C^-_{\Gamma'\to\Gamma_-}(I - \mathcal{J}_{x+})\|_{L^2} \leq \frac{c}{(1+x^2)^{1/2}}\|\mathcal{J}_+ - I\|_{H^1}, \tag{4-8}$$

$$\|\mathcal{J}_{x+} - I\|_{L^2(\Gamma_\infty^+)} \leq \frac{c}{(1+x^2)^{1/2}}\|\mathcal{J}_+ - I\|_{H^1}, \tag{4-9}$$

$$\|\mathcal{J}_{x-} - I\|_{L^2(\Gamma_\infty^-)} \leq \frac{c}{1+x^2}\|\mathcal{J}_- - I\|_{H^2}, \tag{4-10}$$

$$(\|(C_{\mathcal{J}_x})^2 I\|_{L^2(\Gamma)})_{11} \leq \frac{c}{1+x^2}\|\mathcal{J}_+ - I\|_{H^1}\|\mathcal{J}_- - I\|_{H^2}, \tag{4-11}$$

$$(\|(C_{\mathcal{J}_x})^2 I\|_{L^2(\Gamma)})_{22} \leq \frac{c}{(1+x^2)^{1/2}}\|\mathcal{J}_+ - I\|_{H^1}\|\mathcal{J}_- - I\|_{H^2}. \tag{4-12}$$

Proof of Theorem 4.2. Proposition 4.1 and Lemma 4.4 provide the tools for estimating the decay of the potential q, recalling that ν appearing in (4-1) is equal to $(I - C_{\mathcal{J}_x^\pm})^{-1}(1, 0)$. We decompose the following integral into the sum of four integrals

$$\int ((I - C_{\mathcal{J}_x})^{-1} I) e^{-i\lambda x \operatorname{ad}\sigma}(\mathcal{J}_+ - \mathcal{J}_-)\, d\lambda = \int_1 + \int_2 + \int_3 + \int_4, \tag{4-13}$$

where the integrals on the right-hand-side are defined in (4-14)–(4-17). We extract information on $q(x)$ from the $(1, 2)$-entry. Here and thereafter, the integral sign without subscripts refers to an integral taken on the entire contour displayed in Figure 5. We write

$$\int_1 := \int_{\mathbb{R}} (\mathcal{J}_{x+} - \mathcal{J}_{x-}) + \int_{\Gamma_\infty^+} (\mathcal{J}_{x+} - I) + \int_{\Gamma_\infty^-} (I - \mathcal{J}_{x-}). \tag{4-14}$$

Notice that $\mathcal{J}_x^- - I$ is strictly upper triangular on Γ and $\mathcal{J}^- - I$ is in H^2 so we conclude that the $(1, 2)$-entry of the integral above is in $H^{0,2}$ by mapping properties of the Fourier transform and (4-10).

The second integral

$$\int_2 := \int (C_{\mathcal{J}_x} I)(\mathcal{J}_{x+} - \mathcal{J}_{x-}) \tag{4-15}$$

is zero on the $(1,2)$-entry; it thus makes no contribution to the reconstruction of q. For the third integral,

$$\begin{aligned}\int_3 &:= \int ((C_{\mathcal{J}_x})^2 I)(\mathcal{J}_{x+} - \mathcal{J}_{x-}) \\ &= \int \big(C^+(C^-(\mathcal{J}_{x+}-I))(I-\mathcal{J}_{x-})\big)(\mathcal{J}_{x+}-I) + \int \big(C^-(C^+(I-\mathcal{J}_{x-}))(\mathcal{J}_{x+}-I)\big)(I-\mathcal{J}_{x-}).\end{aligned} \tag{4-16}$$

The $(1,2)$-entry is

$$\int_{\Gamma_\infty^-} \big(C^-_{\Gamma_+\to\Gamma}(C^+_{\Gamma\to\Gamma_+}(I-\mathcal{J}_{x-}))(\mathcal{J}_{x+}-I)\big)(I-\mathcal{J}_{x-}) + \int_{\mathbb{R}} \big(C^-_{\Gamma^+\to\Gamma}(C^+_{\Gamma\to\Gamma_+}(I-\mathcal{J}_{x-}))(\mathcal{J}_{x+}-I)\big) C^+_{\mathbb{R}}(I-\mathcal{J}_{x-}),$$

and from (4-7) and (4-10), we conclude that

$$\left|\left(\int_3\right)_{12}\right| \le \frac{c}{(1+x^2)^2}.$$

Finally we set

$$g = (1 - C_{\mathcal{J}_x})^{-1}((C_{\mathcal{J}_x})^2 I)$$

and write

$$\left|\int_4\right| := \left|\int [(C^+ g(I-\mathcal{J}_{x-}))(\mathcal{J}_{x+} - I) + (C^- g(\mathcal{J}_{x+} - I))(I - \mathcal{J}_{x-})]\right|. \tag{4-17}$$

Again, the $(1,2)$-entry is given by

$$\int_{\Gamma_\infty^-} (C^- g(\mathcal{J}_{x+} - I))(I - \mathcal{J}_{x-}) + \int_{\mathbb{R}} (C^-_{\Gamma^+\to\mathbb{R}} g(\mathcal{J}_{x+} - I)) C^+_{\mathbb{R}}(I - \mathcal{J}_{x-})$$

and from (4-7), (4-10), (4-11), and Proposition 4.1 we conclude that

$$\left|\left(\int_4\right)_{12}\right| \le \frac{c}{(1+x^2)^2}.$$

The estimate for $x \in (-\infty, a)$ is obtained by considering the RHP with jump condition described in Theorem 2.11. Lipschitz continuity of the map follows from Proposition 4.1 and (4-13). □

4B. *Smoothness property of the reconstructed potential.*

Theorem 4.5. *If the jump matrix J is given by Theorem 2.7 and q is defined by* (4-4), *then $q \in H^{2,0}(\mathbb{R})$. Moreover, the map from data $\mathcal{J}$ defined as in* (4-2) *and obeying the hypothesis of Theorem 3.6 to $q \in H^{2,0}(\mathbb{R})$ is Lipschitz continuous.*

In order to study smoothness properties of the reconstructed potential, we first show that the functions $M_\pm$ solving RHP 3.7 solve a differential equation in the x-variable. It follows that the same is true of the solution μ of (3-6) since μ is obtained from either M_+ or M_- through postmultiplication by a matrix of the form $e^{-ix\zeta^2 \operatorname{ad}\sigma} A(\zeta)$. We can then change variables to find a differential equation in x obeyed by the (matrix-valued) solution ν of RHP 3.1.

Proposition 4.6. *The functions $M_\pm$ obey the differential equation* (1-5) *where M, P and Q are constructed from the solution μ of* (3-6) *as follows*:

$$M(x,\zeta) = I + \int_\Sigma \frac{\mu(x,s)(w_x^+(s)+w_x^-(s))}{s-\zeta}\,\frac{ds}{2\pi i},$$

$$Q(x) = -\frac{1}{2\pi}\operatorname{ad}\sigma\left(\int_\Sigma \mu(x,\zeta)(w_x^+(\zeta)+w_x^-(\zeta))\,d\zeta\right),$$

$$P(x) = Q(x)i(\operatorname{ad}\sigma)^{-1}Q(x).$$

The proof of the above proposition is a slight modification of the proof of Proposition 5.3.1 in [Liu 2017]. Here we need take into account of the integration along the additional circle Σ_∞.

Proof of Theorem 4.5. We first notice that μ given by (3-5) solves the linear problem (1-5):

$$\frac{d}{dx}\mu = (-i\zeta^2\operatorname{ad}\sigma + \zeta Q(x) + P(x))\mu.$$

We now use the change of variable $\zeta \to \lambda$ to obtain

$$\frac{d}{dx}\nu = \left(-i\lambda\operatorname{ad}\sigma + \begin{pmatrix} 0 & q \\ -\lambda\bar{q} & 0\end{pmatrix} + P\right)\nu.$$

We further write

$$\frac{d}{dx}(\nu e^{-i\lambda x\operatorname{ad}\sigma}(\mathcal{J}_+ - \mathcal{J}_-)) = \left(-i\lambda\operatorname{ad}\sigma + \begin{pmatrix} 0 & q \\ -\lambda\bar{q} & 0\end{pmatrix} + P\right)(\nu e^{-i\lambda x\operatorname{ad}\sigma}(\mathcal{J}_+ - \mathcal{J}_-)). \tag{4-18}$$

Unlike RHP 3.1 in which ν appears as a row vector, here ν is a 2×2 matrix,

$$\nu = \begin{pmatrix} \nu_{11}(x,\lambda) & \nu_{12}(x,\lambda) \\ -\lambda\overline{\nu_{12}(x,\bar{\lambda})} & \overline{\nu_{11}(x,\bar{\lambda})}\end{pmatrix},$$

and its first row (ν_{11},ν_{12}) is the solution to RHP 3.1. We integrate both sides of (4-18) along the contour shown in Figure 5:

$$\frac{d}{dx}\int(\nu e^{-i\lambda x\operatorname{ad}\sigma}(\mathcal{J}_+ - \mathcal{J}_-)) = \int\left(-i\lambda\operatorname{ad}\sigma + \begin{pmatrix} 0 & q \\ -\lambda\bar{q} & 0\end{pmatrix} + P\right)(\nu e^{-i\lambda x\operatorname{ad}\sigma}(\mathcal{J}_+ - \mathcal{J}_-)).$$

The potential q is given by the $(1,2)$-entry of this matrix form integral. Using that $\mathcal{J}_- \in H^{2,2}(\Gamma)$, the $(1,2)$-entry of

$$\int -i\lambda\operatorname{ad}\sigma(\nu e^{-i\lambda x\operatorname{ad}\sigma}(\mathcal{J}_+ - \mathcal{J}_-))$$

is an L^2-function of x, following the same argument as in the proof of Theorem 4.2. To show that the $(1,2)$-entry of

$$\int\left(\begin{pmatrix} 0 & q \\ -\lambda\bar{q} & 0\end{pmatrix} + P\right)(\nu e^{-i\lambda x\operatorname{ad}\sigma}(\mathcal{J}_+ - \mathcal{J}_-))$$

is an L^2-function of x, we use that $q \in L^2\cap L^\infty$, which comes from the fact that $|q| \le c/(1+x^2)^2$, shown in Theorem 4.2. This proves that $q_x \in L^2$. To estimate q_{xx}, we differentiate (4-18) with respect to x.

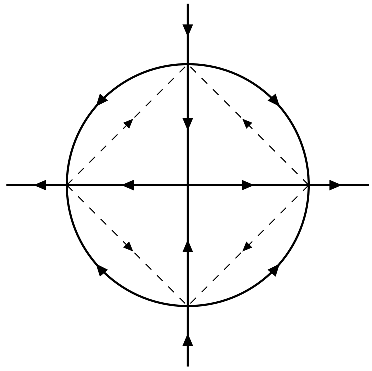

Figure 6. The modified contour Σ_m.

Explicitly, we have

$$\frac{d^2}{dx^2}\int (\nu e^{-i\lambda x\,\mathrm{ad}\,\sigma}(\mathcal{J}_+ - \mathcal{J}_-)) = \int_1 + \int_2,$$

where

$$\int_1 := \int \left(-i\lambda\,\mathrm{ad}\,\sigma + \begin{pmatrix} 0 & q \\ -\lambda\bar{q} & 0 \end{pmatrix} + P\right)^2 (\nu e^{-i\lambda x\,\mathrm{ad}\,\sigma}(\mathcal{J}_+ - \mathcal{J}_-)),$$

$$\int_2 := \int \left(\begin{pmatrix} 0 & q \\ -\lambda\bar{q} & 0 \end{pmatrix}_x + P_x\right)(\nu e^{-i\lambda x\,\mathrm{ad}\,\sigma}(\mathcal{J}_+ - \mathcal{J}_-)).$$

Again following the previous argument and using that $\mathcal{J}_- \in H^{2,2}$ and $q \in H^1$, we conclude that $q_{xx} \in L^2(-a, +\infty)$. A similar argument using scattering data given by Theorem 2.11 and solving the corresponding RHP shows that $q_{xx} \in L^2(-\infty, a)$. Lipschitz continuity of the map follows from the uniform boundedness of the resolvent operator given by (4-6) and Proposition B.1. □

4C. ***Time evolution of the reconstructed potential.*** We now recall the explicit time-dependence of $q(x,t)$ on t through the law of evolution (2-33), and write the reconstruction formula as

$$q(x,t) = \left(-\frac{1}{\pi}\int_{\Gamma_m} \nu(x,\lambda,t)e^{it\theta(x,t,\lambda)}(\mathcal{J}_+(\lambda) - \mathcal{J}_-(\lambda))\,d\lambda\right)_{12},$$

where Γ_m is the contour shown in Figure 2, right, and

$$\theta(x,t,\lambda) = -2\lambda^2 - (x/t)\lambda.$$

To study the time-evolution of $q(x,t)$, it will be convenient to work in the ζ-variable and write

$$q(x,t) = \left(-\frac{1}{\pi}\int_{\Sigma_m} \mu(x,\zeta,t)e^{it\theta(x,\zeta^2,t)}(\widetilde{\mathcal{J}}_+(\zeta) - \widetilde{\mathcal{J}}_-(\zeta))\,d\zeta\right)_{12}, \tag{4-19}$$

where Σ_m, shown in Figure 6, is the inverse image of Γ_m under the map $\zeta \mapsto \lambda = \zeta^2$, $\tilde{J}^{\pm}$ are the scattering data corresponding to $\mathcal{J}^{\pm}$ under the change of variables and μ solves the Beals–Coifman integral equation corresponding to the scattering data $\tilde{J}^{\pm}$.

The continuity of the direct and inverse maps implies that, given Cauchy data $q_0 \in H^{2,2}(\mathbb{R})$, we may approximate q_0 by a sequence $\{q_n\}$ from $\mathcal{S}(\mathbb{R})$ that converges in $H^{2,2}(\mathbb{R})$ to q_0 and obtain a sequence of approximants $q_n(x,t)$ for $q(x,t)$ which converge in $H^{2,2}(\mathbb{R})$ as $n \to \infty$ uniformly in t in any bounded interval. It follows that, in order to prove that $q(x,t)$ given by (4-19) is a weak solution of (1-2), it suffices to assume that $q_0 \in \mathcal{S}(\mathbb{R})$ and argue by approximation.

The following proposition can be proved using the same technique used to prove [Liu 2017, Proposition 7.0.4]).

Proposition 4.7. *Suppose that $M_\pm$ solve RHP* 3.7. *Let*

$$Q(x,t) = -\frac{1}{2\pi}\,\mathrm{ad}\,\sigma\left[\int_{\Sigma_m} \mu(x,\zeta)(\tilde{J}^+(\zeta) - \tilde{J}^-(\zeta))\,d\zeta\right] = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix},$$

$$P(x,t) = i\,Q(x,t)(\mathrm{ad}\,\sigma)^{-1}Q(x,t) = \frac{i}{2}\begin{pmatrix} |q|^2 & 0 \\ 0 & -|q|^2 \end{pmatrix},$$

and $A(x,t)$ be given as in (2-32). *Then $M_\pm$ are fundamental solutions of the Lax equations* (2-31).

Given a fundamental solution of the Lax equations (2-31), it now follows by a standard argument [Liu et al. 2016, Appendix B] that $q(x,t)$, defined as the $(1,2)$-entry of $Q(x,t)$, solves the integrable equation (1-2). Thus:

Proposition 4.8. *Suppose that $q_0 \in H^{2,2}(\mathbb{R})$ and let $J_\pm$ be the corresponding scattering data. Then $q(x,t)$ defined by* (4-19) *solves* (1-2).

4D. ***Proof of Theorem 1.1.*** Combining the results of Sections 4A, 4B, and 4C we can now prove the main theorem.

Proof of Theorem 1.1. Given initial data $q_0 \in H^{2,2}(\mathbb{R})$, the direct scattering map has the continuity properties asserted in Proposition 2.9, so that the time-evolved scattering data has the continuity properties asserted in Proposition 2.10. By Theorem 3.6, RHP 3.1 is uniquely solvable for each x, t, and by Theorems 4.2 and 4.5, the map from scattering data $J(\,\cdot\,,t)$ to reconstructed potential $q(\,\cdot\,,t)$ is Lipschitz continuous into $H^{2,2}(\mathbb{R})$. The map $(q_0,t) \mapsto q(x,t)$ defined by the composition of the direct scattering map, the flow map, and the inverse scattering map (i) maps $(q_0, 0)$ to q_0 (by standard arguments which we omit here), (ii) is jointly continuous in (q_0,t) (by the continuity of the direct and inverse maps), and (iii) is locally Lipschitz continuous in q_0 (by the Lipschitz continuity asserted in Theorems 4.2 and 4.5). Finally, it follows from Proposition 4.8 that $q(x,t)$ is a weak solution of (1-2). □

Appendix A: Sobolev spaces on self-intersecting contours

In this appendix, we define the Sobolev spaces $H^k_\pm(\Gamma)$ and $H^k_z(\Gamma)$ needed for the analysis of RHP 3.1. These spaces were introduced in [Zhou 1989b]; see [Trogdon and Olver 2016, §2.6–2.7] for a discussion on their role in the analysis of Beals–Coifman integral equations associated to RHPs.

If $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_n$ and the Γ_i are either half-lines, line segments, or arcs, the space $H^k(\Gamma)$ consists of functions f on Γ with the property that $f|_{\Gamma_i} \in H^k(\Gamma_i)$. The space $H^k(\Gamma_i)$ is well-defined since each Γ_i

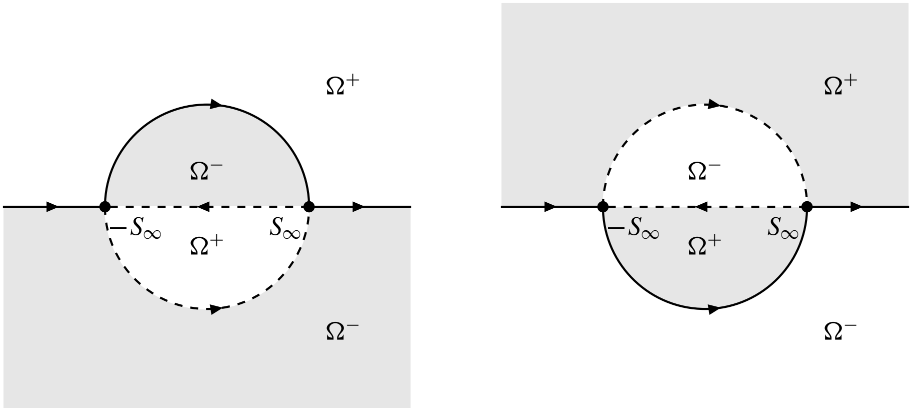

Figure 7. Boundary components of Ω^+ (left) and Ω^- (right).

can be parametrized by arc length and functions on Γ_i viewed as functions on a subset of $\mathbb{R}$. A function $f \in H^k(\Gamma_i)$ has a representative which is continuous, together with its derivatives $f^{(j)}$ up to order $k-1$. Limits of $f^{(j)}$ at the endpoints of Γ_i are well-defined for $0 \le j \le k-1$. The spaces $H^k_+(\Gamma)$ and $H^k_-(\Gamma)$ consist of the functions of $H^k(\Gamma)$ which are continuous together with their derivatives up to order $k-1$ along the solid and dashed components, respectively, shown in Figure 7.

To describe the continuity conditions, let

$$(f_i^j)_\pm = \lim_{z \to S_{\pm\infty}, z \in \Gamma_i} f^{(j)}(z),$$

where the contours Γ_i are as shown in Figure 4. A function $f \in H^k_+(\Gamma)$ obeys the conditions

$$(f_1^j)_- = (f_2^j)_-, \quad (f_3^j)_- = (f_4^j)_-, \qquad (f_2^j)_+ = (f_1^j)_+, \quad (f_3^j)_+ = (f_4^j)_+ \tag{A-1}$$

for $0 \le j \le k-1$, where in each case the first condition comes from continuity across the solid contour, and the second from continuity across the dashed contour. Similarly, a function $f \in H^k_-(\Gamma)$ obeys the conditions

$$(f_1^j)_- = (f_3^j)_-, \quad (f_2^j)_- = (f_4^j)_-, \qquad (f_3^j)_+ = (f_1^j)_+, \quad (f_2^j)_+ = (f_4^j)_+ \tag{A-2}$$

for $0 \le j \le k-1$.

The space $H^k_z(\Gamma)$ consists of those functions in $H^k(\Gamma)$ which obey the following zero-sum conditions at the two intersection points $\pm S_\infty$:

$$\begin{aligned} (f_1^j)_- + (f_4^j)_- - (f_2^j)_- - (f_3^j)_- = 0, \\ (f_2^j)_+ + (f_3^j)_+ - (f_4^j)_+ - (f_1^j)_+ = 0, \end{aligned} \tag{A-3}$$

where the $\pm$ signs are determined by the orientation of the contour as indicated in Figure 7.

It is easy to see from (A-1), (A-2), and (A-3) that $H^k_\pm(\Gamma) \subset H^k_z(\Gamma)$. In [Trogdon and Olver 2016, Lemma 2.51], it is shown that if $f \in H^k_z(\Gamma)$, then the Cauchy projectors $C^\pm_\Gamma f$ are in $H^k_\pm(\Gamma)$. This mapping property is very natural since $C_\Gamma^+ f$ (resp. $C_\Gamma^- f$) is the boundary value of a function analytic in Ω_+ (resp. Ω^-). It follows that

$$H_z^k(\Gamma) = H_+^k(\Gamma) + H_-^k(\Gamma),$$

with the decomposition given by $f = C_\Gamma^+(f) + (-C_\Gamma^-(f))$.

Appendix B: The continuity-compactness argument

In this appendix, we give the abstract functional-analytic argument needed to prove uniform resolvent estimates required in Section 4 for the Lipschitz continuity of the inverse scattering map. Proposition B.1 can also be used to simplify proofs of analogous uniform estimates in [Jenkins et al. 2018b; Liu et al. 2016]. In what follows $\mathscr{B}(X)$ denotes the Banach space of bounded operators on the Banach space X.

Proposition B.1. *Let X, Y, and Z be Banach spaces and suppose that there is a continuous embedding $i : Z \to Y$ with the property that bounded subsets of Z map to precompact subsets of Y. Suppose that $C_{J,x}$ is a family of bounded operators on a Banach space X indexed by $J \in Y$ and $x \in \mathbb{R}$. Finally, suppose that*:

(i) *The map $(J, x) \mapsto C_{J,x}$ is continuous as a map from $Y \times \mathbb{R}$ into $\mathscr{B}(X)$, and the estimate*

$$\sup_{x\in\mathbb{R}} \|C_{J,x} - C_{J',x}\|_{\mathscr{B}(X)} \lesssim \|J - J'\|_Y$$

holds.

(ii) *The resolvent $(I - C_{J,x})^{-1}$ exists for each $x \in \mathbb{R}$ and $J \in Y$.*

(iii) *For each $J \in Y$, the estimate*

$$\sup_{x\in[a,\infty)} \|(I - C_{J,x})^{-1}\|_{\mathscr{B}(X)} < \infty$$

holds.

Then for any bounded subset B of Z,

$$\sup_{J\in B}\Big(\sup_{x\in[a,\infty)} \|(I - C_{J,x})^{-1}\|_{\mathscr{B}(X)}\Big) < \infty$$

and the map

$$J \mapsto \{x \mapsto (I - C_{J,x})^{-1}\}$$

is locally Lipschitz continuous as a map from Z into $C([a, \infty); \mathscr{B}(X))$.

Remark B.2. (1) In applications, (i) is easy to prove from the explicit form of the Beals–Coifman integral operators, (ii) follows from Fredholm theory and a vanishing theorem for the RHP, and (iii) follows from the continuity of the map

$$x \mapsto (I - C_{J,x})^{-1}$$

and the fact that, in the limit $x \to \infty$, the integral kernel of the operator is highly oscillatory.

(2) In applications, the bound in hypothesis (iii) is typically only true for half-lines. One can replace $[a, \infty)$ by $(-\infty, a]$ and obtain the same result.

Proof. Denote by $C([a,\infty),\mathscr{B}(X))$ the Banach space of continuous $\mathscr{B}(X)$-valued functions of $x\in[a,\infty)$ equipped with the norm

$$\|f\|_{C([a,\infty),\mathscr{B}(X))}=\sup_{x\in\mathbb{R}}\|f(x)\|_{\mathscr{B}(X)}.$$

Consider the map

$$Y\to C([a,\infty),\mathscr{B}(X)),\quad J\mapsto (x\mapsto (I-C_{J,x})^{-1}). \tag{B-1}$$

Assumptions (i), (ii), (iii) and the second resolvent formula show that this map is well-defined and continuous. Using the injection i we can identify bounded subsets of Z with precompact subsets of Y. We can then use the continuity of the map (B-1) to conclude that the image of any bounded subset of Z has compact closure in $C([a,\infty),\mathscr{B}(X))$ and hence is bounded. The local Lipschitz continuity now follows from the identity

$$(I-C_{J,x})^{-1}-(I-C_{J',x})^{-1}=(I-C_{J,x})^{-1}(C_{J',x}-C_{J,x})(I-C_{J',x})^{-1}$$

(the "second resolvent formula") owing to the uniform bounds. □

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UNCONDITIONAL EXISTENCE OF CONFORMALLY HYPERBOLIC YAMABE FLOWS

MARIO B. SCHULZ

We prove global existence of instantaneously complete Yamabe flows on hyperbolic space of arbitrary dimension $m \geq 3$ starting from any smooth, conformally hyperbolic initial metric. We do not require initial completeness or curvature bounds. With the same methods, we show rigidity of hyperbolic space under the Yamabe flow.

Let (M, g_M) be a Riemannian manifold. Let $g_0 = u_0 g_M$ be a conformal metric on M defined by a smooth function $u_0 : M \to \,]0, \infty[$. A family $(g(t))_{t \in [0,T[}$ of Riemannian metrics on M is called *Yamabe flow* with initial metric g_0 if for all $t \in [0, T[$

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -\mathrm{R}_{g(t)} g(t), \\ \quad g(0) = g_0, \end{cases} \tag{1}$$

where R_g denotes the scalar curvature of the Riemannian manifold (M, g). Richard Hamilton [1989] introduced this flow as alternative approach to the Yamabe problem and showed that solutions to (1) exist on any compact manifold (M, g_0) without boundary. Since then, a full theory on compact manifolds was developed with major contributions by Chow [1992], Ye [1994], Schwetlick and Struwe [2003] and Brendle [2005; 2007].

In dimension $\dim(M) = 2$, where the Yamabe flow coincides with the Ricci flow, Gregor Giesen and Peter Topping [Giesen and Topping 2011; Topping 2010; 2015] obtained existence and uniqueness of instantaneously complete solutions to (1) on an arbitrary surface (M, g_0). Instantaneous completeness means that the Riemannian manifold $(M, g(t))$ is geodesically complete for all $t > 0$ even if the initial surface (M, g_0) is incomplete. In [Schulz 2019a], the author studied the question of whether Giesen and Topping's results generalise to noncompact manifolds of higher dimension and obtained affirmative results for manifolds conformally equivalent to hyperbolic space provided that the conformal factor in the initial metric and the initial scalar curvature are both uniformly bounded from above. In the present paper, we now are able to show existence of instantaneously complete Yamabe flows for *any* conformal initial metric on hyperbolic space $(\mathbb{H}, g_{\mathbb{H}})$ of dimension $m \geq 3$.

Theorem 1 (existence). *Let $g_0 = u_0 g_{\mathbb{H}}$ be any conformal Riemannian metric on hyperbolic space $(\mathbb{H}, g_{\mathbb{H}})$ of dimension $m \geq 3$. Then, there exists an instantaneously complete Yamabe flow $(g(t))_{t \in [0,\infty[}$ on $\mathbb{H}$ satisfying*

(1) $g(0) = g_0$,

(2) $g(t) \geq m(m-1)t \, g_{\mathbb{H}}$ *for all* $t > 0$.

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Keywords: Yamabe flow, instantaneously complete, unbounded curvature.

Remark. Recall that hyperbolic space is a noncompact, simply connected Riemannian manifold of constant sectional curvature -1 and scalar curvature $\mathrm{R}_{g_{\mathbb{H}}} = -m(m-1)$. Theorem 1 improves the existence result obtained in [Schulz 2019a], which depended on the uniform upper bounds $g_0 \leq C_0 g_{\mathbb{H}}$ and $\mathrm{R}_{g_0} \leq K$ on the initial metric and its scalar curvature. Here, we are able to drop these assumptions entirely, so the existence of instantaneously complete Yamabe flows on hyperbolic space of any dimension is true with the same level of generality as in dimension 2. It is likely that the proof of Theorem 1 can be generalised to apply on any noncompact manifold (M, g_M) with strictly negative scalar curvature $-\kappa_2 \leq \mathrm{R}_{g_M} \leq -\kappa_1 < 0$ in place of hyperbolic space.

However, there exist (geodesically incomplete) initial manifolds (M, g_0) which do *not* allow any instantaneously complete solution to the Yamabe flow. Conformally flat examples (M, g_0) such as the punctured sphere in dimension $m \geq 3$ are given in [Schulz 2019a, Theorem 3]. The incompleteness of Yamabe flows on arbitrary punctured manifolds will be analysed in a forthcoming article.

For the more general class of complete, noncompact background manifolds (M, g_M) with nonpositive, bounded scalar curvature and positive Yamabe invariant, the author proves global existence of complete Yamabe flows in [Schulz 2019b], provided the initial metric $g_0 = u_0 g_M$ already is complete with $c_1 \leq u_0 \leq c_2$ for some constants $c_1, c_2 > 0$, so that cases like the punctured sphere are excluded. However, no assumptions on the curvature of g_0 are required.

If g_0 is any Riemannian metric on some noncompact manifold M, then existence of a global Yamabe flow on M with initial metric g_0 was shown by Yinglian An and Li Ma [1999] provided that (M, g_0) is complete, with Ricci curvature bounded from below and with bounded, nonpositive scalar curvature, and also by Li Ma [2016] under the assumption that (M, g_0) is complete with nonnegative scalar curvature R_{g_0} which allows a positive solution $w > 0$ of the equation

$$-\Delta_{g_0} w = \frac{m-2}{4(m-1)} \mathrm{R}_{g_0}$$

in M.

Bahuaud and Vertman [2014; 2019] constructed Yamabe flows starting from spaces with incomplete edge singularities which are preserved along the flow. Recently, Choi, Daskalopoulos, and King [Choi et al. 2018] constructed solutions to the Yamabe flow on $\mathbb{R}^m$ which develop a type II singularity in finite time.

In general, solutions to problem (1) on noncompact manifolds M are not unique. An example on $M = \mathbb{H}$ is the flat metric $g_0 = g_{\mathbb{E}}$, which is conformally equivalent to $g_{\mathbb{H}}$ according to the Poincaré ball model. By Theorem 1, there exists an instantaneously complete Yamabe flow $(g(t))_{t\in[0,\infty[}$ on $\mathbb{H}$ with $g(0) = g_{\mathbb{E}}$. However, the constant flow given by $\bar{g}(t) = g_{\mathbb{E}}$ for all t is also a solution to (1) because $\mathrm{R}_{g_{\mathbb{E}}} = 0$. In [Schulz 2019a] we conjectured that uniqueness holds in the class of instantaneously complete Yamabe flows and obtained a partial result in the class of rotationally symmetric, instantaneously complete flows. While the conjecture is still open in general, the methods used in the proof of Theorem 1 yield the following result without the assumption of symmetry or completeness.

Theorem 2 (rigidity). *Let $(\mathbb{H}, g_{\mathbb{H}})$ be hyperbolic space of dimension $m \geq 3$. Let $(g(t))_{t\in[0,T[}$ be a Yamabe flow on $\mathbb{H}$ with $g(0) = g_{\mathbb{H}}$. Then, the flow is uniquely given by $g(t) = (m(m-1)t + 1)g_{\mathbb{H}}$ for all $t \in [0, T[$.*

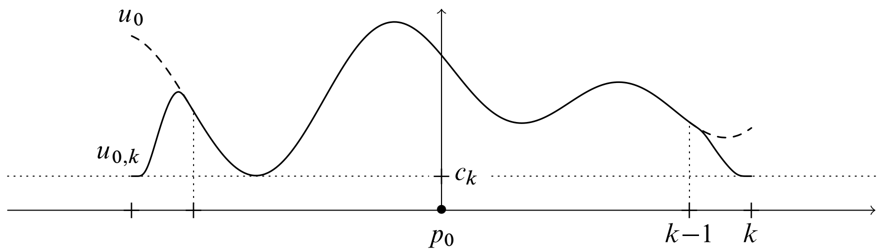

Figure 1. The construction of $u_{0,k}$ given u_0.

Proofs of the main results

Let $u_0 : \mathbb{H} \to \,]0,\infty[$ be the conformal factor of the given metric $g_0 = u_0 g_{\mathbb{H}}$ on $\mathbb{H}$. We assume that the restriction of u_0 to any smooth, bounded domain $\Omega \subset \mathbb{H}$ is in the Hölder space $\mathrm{C}^{2,\alpha}(\Omega)$ for some $0<\alpha<1$. Since the Yamabe flow preserves the conformal class, any Yamabe flow $(g(t))_{t\in[0,T[}$ on $\mathbb{H}$ with $g(0)=g_0$ is of the form $g(t)=u(\,\cdot\,,t)g_{\mathbb{H}}$ with a conformal factor $u:\mathbb{H}\times[0,T[\,\to\,]0,\infty[$ satisfying

$$\begin{cases} \frac{\partial}{\partial t}u = -\mathrm{R}_g u, \\ u(\,\cdot\,,0) = u_0. \end{cases} \tag{2}$$

In dimension $m=\dim(\mathbb{H})\geq 3$ we may introduce the exponent

$$\eta = \frac{m-2}{4}$$

and the function $U=u^\eta$ in order to express the scalar curvature of the conformal metric $g=ug_{\mathbb{H}}$ by

$$\mathrm{R}_g = U^{-\frac{m+2}{m-2}}\left(\mathrm{R}_{g_{\mathbb{H}}}U - 4\frac{m-1}{m-2}\Delta_{g_{\mathbb{H}}}U\right)$$

and to formulate the following equivalent evolution equations for U and u respectively:

$$\frac{1}{m-1}\frac{\partial U}{\partial t} = (m\eta U + \Delta_{g_{\mathbb{H}}}U)U^{-\frac{1}{\eta}}, \tag{3}$$

$$\frac{1}{m-1}\frac{\partial u}{\partial t} = m + \frac{\Delta_{g_{\mathbb{H}}}u}{u} + \frac{(m-6)}{4}\frac{|\nabla u|^2_{g_{\mathbb{H}}}}{u^2}. \tag{4}$$

Let $B_k = B_k(p_0)\subset\mathbb{H}$ be the open metric ball of radius k around some origin $p_0\in\mathbb{H}$. Let $\chi_k:\mathbb{H}\to[0,1]$ be smooth with compact support in B_k satisfying $\chi_k(x)=1$ for all $x\in B_{k-1}$. For any $k>2$ we define

$$c_k := \inf_{B_k} u_0 > 0, \tag{5}$$

$$u_{0,k} := (1-\chi_k)c_k + \chi_k\, u_0, \tag{6}$$

$$\phi_k(t) := c_k + m(m-1)t. \tag{7}$$

Then, $u_{0,k}\in\mathrm{C}^{2,\alpha}(B_k)$ coincides with u_0 in B_{k-1} and takes the constant value c_k in some neighbourhood of ∂B_k as shown in Figure 1. Moreover, $u_{0,k}$ and ϕ_k satisfy the first-order compatibility conditions for (4). As shown in [Schulz 2019a, Lemma 1.1] there exists some $T_k > 0$ and a solution $u > 0$ of

$$\begin{cases} \dfrac{1}{m-1}\dfrac{\partial u}{\partial t} = m + \dfrac{\Delta_{g_{\mathbb{H}}} u}{u} + \dfrac{(m-6)}{4}\dfrac{|\nabla u|^2_{g_{\mathbb{H}}}}{u^2} & \text{in } B_k \times [0, T_k[, \\ u = \phi_k & \text{on } \partial B_k \times [0, T_k[, \\ u = u_{0,k} & \text{on } B_k \times \{0\}. \end{cases} \tag{8}$$

The following pointwise estimate is analogous to [Schulz 2019a, Lemma 1.3].

Lemma 1. *Let u be a positive solution to problem* (8) *with initial and boundary data as given in* (6) *and* (7). *Then, for every* $0 \le t < T_k$

$$\inf_{B_k} u_0 \le u(\cdot\,, t) - m(m-1)t \le \sup_{B_k} u_0.$$

Proof. Given any constant $c \in \mathbb{R}$ the function $w(\cdot\,, t) = u(\cdot\,, t) - m(m-1)t - c$ satisfies

$$\frac{1}{m-1}\frac{\partial w}{\partial t} - \frac{\Delta_{g_{\mathbb{H}}} w}{u} - \frac{(m-6)\langle \nabla u, \nabla w\rangle_{g_{\mathbb{H}}}}{4u^2} = 0 \quad \text{in } B_k \times [0, T_k[. \tag{9}$$

Since $u > 0$, equation (9) is uniformly parabolic. For $c = \inf_{B_k} u_0$ (respectively $c = \sup_{B_k} u_0$) we have $w \ge 0$ (respectively $w \le 0$) on $(\partial B_k \times [0, T_k[) \cup (B_k \times \{0\})$ by (7) and the parabolic maximum principle (see [Schulz 2019a, Proposition A.2]) implies $w \ge 0$ (respectively $w \le 0$) in $B_k \times [0, T_k[$. ☐

Lemma 2 (global existence on bounded domains). *For every $k > 2$, there exists a unique global solution* $0 < u_k \in \mathrm{C}^{2;1}(B_k \times [0, \infty[)$ *to problem* (8) *with* $T_k = \infty$, *boundary data* (7) *and initial data* (6).

Proof. Since any solution $u \in \mathrm{C}^{2;1}(B_k \times [0, T_k[)$ to problem (8) with $T_k < \infty$ satisfies

$$0 < \inf_{B_k} u_0 \le u(\cdot\,, t) \le \sup_{B_k} u_0 + m(m-1)T_k \quad \text{in } B_k$$

for every $t \in [0, T_k[$ according to Lemma 1, the same approach as in [Schulz 2019b, Lemma 1.2] using parabolic De Giorgi–Nash–Moser theory applies. In fact, a similar argument is used in the proof of Theorem 1. ☐

The estimate obtained in Lemma 1 is not uniform in k because we do not assume any uniform bounds on u_0. To pass to the limit $k \to \infty$ we require local bounds which do not depend on k. The nonlinearity of the equation is helpful for upper bounds. Lower bounds however are delicate. We will make use of the following estimate.

Lemma 3. *For any real numbers $a, c > 0$ there exists $\lambda > 0$ such that the function $f :]0, 1[\to \mathbb{R}$ given by $f(r) = (1-r^2)^2$ satisfies*

$$f''(r) + \frac{c}{r} f'(r) \ge -\lambda (f(r))^{1+a}$$

for all $r \in]0, 1[$.

Proof. Since $f'(r) = -4r(1-r^2)$ and $f''(r) = -4 + 12r^2$, we have

$$f''(r) + \frac{c}{r} f'(r) = (-4 + 12r^2) - 4c(1-r^2) = 8 - 4(3+c)(1-r^2) = y\left(\frac{1}{1-r^2}\right)(f(r))^{1+a},$$

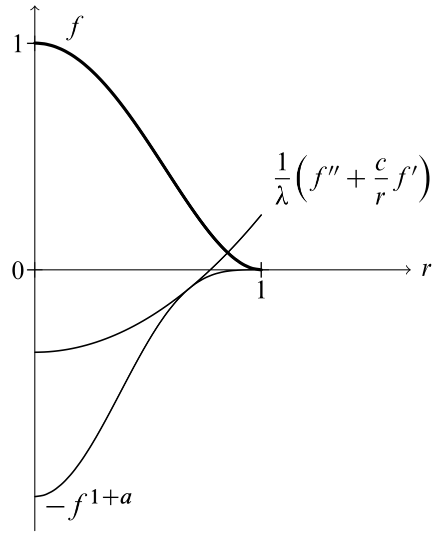

Figure 2. Visualisation of Lemma 3 for $a = 1,\ c = 2$ and $\lambda = 33$.

where we introduced the function $y : [1, \infty[\to \mathbb{R}$ given by

$$y(x) = 8x^{2+2a} - 4(3 + c)x^{1+2a}.$$

Since the leading term $8x^{2+2p}$ has a positive coefficient, the function $y : [1, \infty[\to \mathbb{R}$ is bounded from below by some constant $-\lambda < 0$ depending only on the parameters a and c. □

Lemma 4 (radial subsolution). *Let $B_1(x_0) \subset \mathbb{H}$ be the open metric unit ball in $(\mathbb{H}, g_{\mathbb{H}})$ around $x_0 \in \mathbb{H}$ and let $a, b, h_0 > 0$. Then there exists a constant $C > 0$ depending only on a, b, h_0 and $m = \dim \mathbb{H}$ such that the map $V : B_1 \times [0, t_0[\to]0, h_0]$ given by*

$$V(\,\cdot\,, t) = (h_0^a - Ct)^{\frac{1}{a}}(1 - r^2)^2,$$

where $t_0 = \frac{1}{C}h_0^a$ and where $r : B_1 \to [0, \infty[$ is the Riemannian distance function from x_0 in $(\mathbb{H}, g_{\mathbb{H}})$, satisfies

$$\frac{\partial}{\partial t}V^{1+a} \le b\Delta_{g_{\mathbb{H}}}V. \tag{10}$$

Proof. A function of the form $V(\,\cdot\,, t) = h(t)(f \circ r)$ satisfies (10) if

$$\tfrac{d}{dt}h^{1+a} = -b\lambda h, \quad -\lambda f^{1+a}(r) \le \Delta_{g_{\mathbb{H}}}(f(r)),$$

for some $\lambda > 0$. Let $f :]0, 1[\to \mathbb{R}$ and $\lambda > 0$ be as in Lemma 3 with

$$c = \frac{m-1}{\tanh(1)}.$$

Then

$$\Delta_{g_{\mathbb{H}}}(f(r)) = f''(r) + \frac{m-1}{\tanh(r)}f'(r) \ge f''(r) + \frac{c}{r}f'(r) \ge -\lambda f^{1+a}(r).$$

The equation for h implies

$$\frac{d}{dt}(h^a) = -\frac{ab\lambda}{a+1},$$

which integrates to

$$h^a(t) = h^a(0) - \frac{ab\lambda t}{a+1}.$$

Choosing $h(0) = h_0$ we arrive at

$$V(\,\cdot\,,t) = \left(h_0^a - \frac{ab\lambda t}{a+1}\right)^{\frac{1}{a}} (1-r^2)^2, \tag{11}$$

which completes the proof with constant

$$C = \frac{ab\lambda}{a+1}.$$ □

Remark. The equation corresponding to (10) is called *fast diffusion equation* (see [Vázquez 2007]), which is well-studied even for domains in Riemannian manifolds of negative curvature: Bonforte, Grillo and Vázquez [Bonforte et al. 2008] proved existence of (weak) solutions to the fast diffusion equation in a more general setting and provided more refined estimates of the extinction time. Grillo and Muratori [2014] studied radial solutions of $\frac{\partial}{\partial t}V^{1+a} = \Delta_{g_{\mathbb{H}}} V$ on hyperbolic space of dimension $m \geq 3$ in the subcritical range $1+a < \frac{m+2}{m-2}$ and analysed their fine asymptotics near the extinction time. In the following we will specialise to the critical exponent $1+a = \frac{m+2}{m-2}$ which corresponds to the Yamabe flow.

It is surprising that the simple profile $f(r) = (1-r^2)^2$ allows the construction of a compactly supported subsolution to the Yamabe flow on hyperbolic space of any dimension $m \geq 3$. The same approach works on $\mathbb{R}^m$ for $m \geq 3$ if we choose $c = (m-1)$. On manifolds of dimension 2, however, the Yamabe flow behaves differently: according to [Giesen and Topping 2013, Theorem A.3] there exist Yamabe flows starting from the flat 2-dimensional unit disc with arbitrarily small extinction time.

Lemma 5 (local lower bound). *Let $\Omega \subseteq \mathbb{H}$ be any open subset of hyperbolic space of dimension $m \geq 3$ containing the metric ball $B_{r_0} \subset \mathbb{H}$ of radius $r_0 > 1$. Let $(g(t))_{t\in[0,T[}$ be any Yamabe flow on Ω given by $g(t) = u(\,\cdot\,,t) g_{\mathbb{H}}|_{\Omega}$. Then, there exists a constant $C_m > 0$ depending only on m and not on Ω such that for all $t \in [0,T[$*

$$u(\,\cdot\,,t) \geq \inf_{B_{r_0}} u(\,\cdot\,,0) - C_m t \quad \text{in } B_{r_0 - 1}.$$

Proof. Let $\eta = \frac{m-2}{4}$ as before. According to (3) the function $U = u^\eta$ satisfies

$$\frac{\eta}{(m-1)(\eta+1)} \frac{\partial}{\partial t} U^{1+\frac{1}{\eta}} = m\eta U + \Delta_{g_{\mathbb{H}}} U \geq \Delta_{g_{\mathbb{H}}} U$$

in $B_{r_0} \subset \Omega$. Let $x_0 \in B_{r_0-1}$ be arbitrary and let $V : B_1(x_0) \times [0,t_0[\to \mathbb{R}$ be as in Lemma 4 with parameters

$$a = \frac{1}{\eta}, \quad b = \frac{(m-1)(\eta+1)}{\eta}, \quad h_0 = \inf_{B_{r_0}} U(\,\cdot\,,0). \tag{12}$$

According to Lemma 2 we may assume $T > t_0$. We consider the difference

$$w := V^{1+\frac{1}{\eta}} - U^{1+\frac{1}{\eta}},$$

define the function $w_+ : B_1(x_0)\times[0,t_0[\to[0,\infty[$ by $w_+(x,t)=\max\{w(x,t),0\}$ and study the evolution of the quantity

$$J(t)=\int_{B_1(x_0)} w_+(\,\cdot\,,t)\,d\mu_{g_{\mathbb{H}}}.$$

For any $0<\tau\le t<t_0$ we have

$$\begin{aligned} J(t)-J(t-\tau) &= \int_{B_1(x_0)} w_+(\,\cdot\,,t)\,d\mu_{g_{\mathbb{H}}} - \int_{B_1(x_0)} w_+(\,\cdot\,,t-\tau)\,d\mu_{g_{\mathbb{H}}} \\ &\le \int_{\{w(\cdot,t)>0\}} (w_+(\,\cdot\,,t)-w_+(\,\cdot\,,t-\tau))\,d\mu_{g_{\mathbb{H}}} \\ &\le \int_{\{w(\cdot,t)>0\}} (w(\,\cdot\,,t)-w(\,\cdot\,,t-\tau))\,d\mu_{g_{\mathbb{H}}}. \end{aligned}$$

Hence,

$$\begin{aligned} \limsup_{\tau\searrow 0}\frac{J(t)-J(t-\tau)}{\tau} &\le \int_{\{w(\cdot,t)>0\}} \frac{\partial w}{\partial t}(\,\cdot\,,t)\,d\mu_{g_{\mathbb{H}}} \\ &\le b\int_{\{(V-U)(\cdot,t)>0\}} \Delta_{g_{\mathbb{H}}}(V-U)(\,\cdot\,,t)\,d\mu_{g_{\mathbb{H}}}\le 0, \end{aligned} \tag{13}$$

where we use the following Lemma 6 to obtain the last inequality. We proceed similarly to an argument by Richard Hamilton [1986, Lemma 3.1] (see also [Schulz 2019a, Lemma A.5]). Let $\varepsilon>0$ be arbitrary. Estimate (13) implies that there exists $\delta>0$ such that

$$\text{for all } \tau\in[0,\delta[,\quad J(t)-J(t-\tau)\le\varepsilon\tau. \tag{14}$$

We may assume that $\delta\in\,]0,t]$ is maximal with this property. By continuity of $t\mapsto J(t)$, estimate (14) extends to

$$J(t)-J(t-\delta)\le\varepsilon\delta. \tag{15}$$

If $t-\delta>0$, we repeat the argument to find $\delta'>0$ such that,

$$\text{for all } \tau\in[0,\delta'[,\quad J(t-\delta)-J(t-\delta-\tau)\le\varepsilon\tau. \tag{16}$$

In particular, (15) and (16) can be combined to

$$J(t)-J(t-\delta-\tau)\le\varepsilon(\delta+\tau)$$

for all $\tau\in[0,\delta'[$ in contradiction to the maximality of δ. Hence, $\delta=t$ and we obtain

$$J(t)-J(0)\le\varepsilon t.$$

By the choice of h_0 we have $J(0)=0$. Since $\varepsilon>0$ is arbitrary, $J(t)\le 0$ follows and implies $U(\,\cdot\,,t)\ge V(\,\cdot\,,t)$ in B_1. In particular, using formula (11) with parameters (12) for V, we have

$$U(x_0,t)\ge V(x_0,t)=\left(\inf_{B_{r_0}} u(\,\cdot\,,0)-\frac{(m-1)\lambda t}{\eta}\right)^{\eta}.$$

Since $x_0\in B_{r_0-1}$ and $t\in\,]0,t_0[$ are arbitrary and $U=u^\eta$, the claim follows with constant $C_m=\frac{m-1}{\eta}\lambda$. □

Lemma 6. *Let $\Omega \subset \mathbb{H}$ be a smooth, bounded domain and let $f \in C^2(\Omega)$ satisfy $f \leq 0$ on $\partial\Omega$. Let $\{f > 0\} := \{x \in \Omega \mid f(x) > 0\}$. Then,*

$$\int_{\{f>0\}} \Delta_{g_{\mathbb{H}}} f \, d\mu_{g_{\mathbb{H}}} \leq 0.$$

Proof. For any regular value $y \geq 0$ of f, the set $\{f > y\} \subset \Omega$ is regular, open and bounded with outer unit normal ν in the direction of $-\nabla f$. Therefore, we may integrate by parts to obtain

$$\int_{\{f>y\}} \Delta_{g_{\mathbb{H}}} f \, d\mu_{g_{\mathbb{H}}} = \int_{\partial\{f>y\}} \langle \nabla f, \nu \rangle_{g_{\mathbb{H}}} \, d\mu_{g_{\mathbb{H}}} \leq 0. \tag{17}$$

If $y = 0$ is not a regular value for f, we choose a sequence $(y_k)_{k\in\mathbb{N}}$ of regular values for f with $y_k \to 0$ as $k \to \infty$ and pass to the limit in (17). □

The following lemma about upper bounds is a local version of [Schulz 2019a, Proposition 2.1] and complements the local lower bound obtained in Lemma 5. In [Schulz 2019a], the estimate is derived from (3) for $U = u^\eta$. Here, we give a slightly different proof using (4) instead.

Lemma 7 (local upper bound). *Let $\Omega \subseteq \mathbb{H}$ be any open subset of hyperbolic space of dimension $m \geq 3$ containing the metric ball $B_{r_0} \subset \mathbb{H}$ of radius $r_0 > 1$. Let $(g(t))_{t\in[0,T[}$ be any Yamabe flow on Ω given by $g(t) = u(\,\cdot\,, t) g_{\mathbb{H}}|_\Omega$. Then, there exists a constant $c_m > 0$ depending only on m and not on Ω such that for all $t \in [0, T[$*

$$u(\,\cdot\,, t) \leq \sup_{B_{r_0}} u(\,\cdot\,, 0) + (m-1)(m+c_m)t \quad \text{in } B_{r_0-1}.$$

Proof. Let $\psi : \Omega \to [0, 1]$ be a smooth cutoff function with support in $B_{r_0} \subset \Omega$ such that $\psi(x) = 1$ for all $x \in B_{r_0-1}$ and such that

$$\left(\frac{(m+2)}{4\psi}|\nabla\psi|^2_{g_{\mathbb{H}}} - \Delta_{g_{\mathbb{H}}}\psi\right) \leq c_m \tag{18}$$

in B_{r_0} with some constant $c_m > 0$ depending only on the dimension m. Such cutoff functions exist as shown in [Schulz 2019a, Lemma A.3–4]. Consider the spatially constant function

$$w(t) = \sup_{B_{r_0}} u(\,\cdot\,, 0) + (m-1)(m+c_m)t.$$

Recalling (4), but suppressing the index $g_{\mathbb{H}}$ to ease notation of derivatives and inner products, we have

$$\begin{aligned}
&\frac{1}{m-1}\frac{\partial}{\partial t}(u\psi - w) \\
&\quad = m\psi - (m+c_m) + \frac{\Delta u}{u}\psi + \frac{(m-6)}{4}\frac{|\nabla u|^2}{u^2}\psi \\
&\quad = m\psi - (m+c_m) + \frac{\Delta(u\psi)}{u} - \frac{m+2}{4u}\langle\nabla u, \nabla\psi\rangle - \Delta\psi + \frac{m-6}{4u^2}\langle\nabla(\psi u), \nabla u\rangle \\
&\quad = m\psi - (m+c_m) + \frac{\Delta(u\psi)}{u} - \frac{m+2}{4u\psi}\langle\nabla(u\psi), \nabla\psi\rangle + \frac{m-6}{4u^2}\langle\nabla(\psi u), \nabla u\rangle + \frac{m+2}{4\psi}|\nabla\psi|^2 - \Delta\psi \\
&\quad \leq \frac{\Delta(u\psi - w)}{u} - \frac{m+2}{4u\psi}\langle\nabla(u\psi - w), \nabla\psi\rangle + \frac{m-6}{4u^2}\langle\nabla(\psi u - w), \nabla u\rangle.
\end{aligned} \tag{19}$$

Since $B_{r_0} \subset \Omega$ is a bounded domain, $u|_{B_{r_0}\times[0,T[}$ is strictly bounded away from zero and from above. Moreover, $u\psi - w \le 0$ on $(B_{r_0}\times\{0\}) \cup (\partial B_{r_0}\times[0,T[)$. Hence, the parabolic maximum principle [Schulz 2019a, Proposition A.2] applies to inequality (19) and yields $u\psi - w \le 0$ in $B_{r_0}\times[0,T[$, which by the choice of ψ implies $u(\,\cdot\,,t) \le w(t)$ in B_{r_0-1} for all $t\in[0,T[$ as claimed. □

Proof of Theorem 1. Let $r>1$ and $T>1$ be arbitrary but fixed. For every $k\in\mathbb{N}$ let $u_k : B_k\times[0,\infty[\to]0,\infty[$ be the solution to problem (8) with boundary data (7) and initial data (6) as given in Lemma 2. Combining the lower bounds from Lemmas 1 and 5, for every $k \ge r+3$ we obtain

$$u_k|_{\overline{B}_{r+2}\times[0,T]} \ge \frac{m(m-1)}{m(m-1)+C_m} \inf_{B_{r+3}} u_0 > 0, \tag{20}$$

where the constant $C_m > 0$ is the same as in Lemma 5. Here we use that we have

$$\max\{at, b-ct\} \ge \frac{ab}{a+c}$$

for any $a,b,t>0$. In fact, $\max\{at,b-ct\}$ is minimal when $at = b-ct$, that is, when $t = b/(a+c)$. By Lemma 7, we also have

$$u_k|_{\overline{B}_{r+2}\times[0,T]} \le \sup_{B_{r+3}} u_0 + (m-1)(m+c_m)T. \tag{21}$$

Recalling $\eta = \frac{m-2}{4}$, we write (8) in divergence form

$$\frac{1}{m-1}\frac{\partial u^{\eta+1}}{\partial t} = \frac{m(\eta+1)u^{\eta+1}}{u} + \operatorname{div}_{g_{\mathbb{H}}}\Big(\frac{1}{u}\nabla u^{\eta+1}\Big) \tag{22}$$

and interpret it as linear parabolic equation for $u_k^{\eta+1}$ in $B_{r+2}\times[0,T]$ with coefficients which are uniformly bounded due to (20) and (21). Since $u_k|_{B_{r+2}\times\{0\}} = u_0|_{B_{r+2}}$ is Hölder continuous by assumption, we may apply parabolic De Giorgi–Nash–Moser theory [Ladyzhenskaya et al. 1968, p. 204, Theorem III.10.1] (see also [Trudinger 1968, §4]) to (22) in order to obtain the interior Hölder bound

$$\|u_k^{\eta+1}\|_{C^{0,\alpha;0,\alpha/2}(\overline{B}_{r+1}\times[0,T])} \le C(m,T,u_0|_{B_{r+3}})$$

for some $0<\alpha<1$ with a constant C depending only on m, T, the upper and lower bounds (20) and (21) and the Hölder bound on $u_0|_{B_{r+2}}$, but not on k. Together with (20) and (21), it follows that the coefficient $1/u_k$ in (22) is Hölder continuous in $\overline{B}_{r+1}\times[0,T]$ satisfying a similar estimate. Since we assume $u_0 \in C^{2,\alpha}(B_{r+1})$, linear parabolic theory [Ladyzhenskaya et al. 1968, p. 351, Theorem IV.10.1] yields

$$\|u_k^{\eta+1}\|_{C^{2,\alpha;1,\alpha/2}(\overline{B}_r\times[0,T])} \le C'(m,T,u_0|_{B_{r+3}}).$$

By compactness of the embedding $C^{2,\alpha;1,\alpha/2}(\overline{B}_r\times[0,T]) \hookrightarrow C^{2;1}(\overline{B}_r\times[0,T])$ a subsequence of $\{u_k|_{B_r\times[0,T]}\}_{r+2\le k\in\mathbb{N}}$ converges to a solution of (4) in $\overline{B}_r\times[0,T]$. We repeat this argument to obtain a further subsequence which converges to a solution of (4) in $\overline{B}_{2r}\times[0,2T]$.

A diagonal argument allows us to find a subsequence of $\{u_k\}_{k\in\mathbb{N}}$ which converges everywhere to a limit $u\in C^{2;1}(\mathbb{H}\times[0,\infty[)$ satisfying the Yamabe flow (4). Since the uniform lower bound $u(\,\cdot\,,t)\ge m(m-1)t$

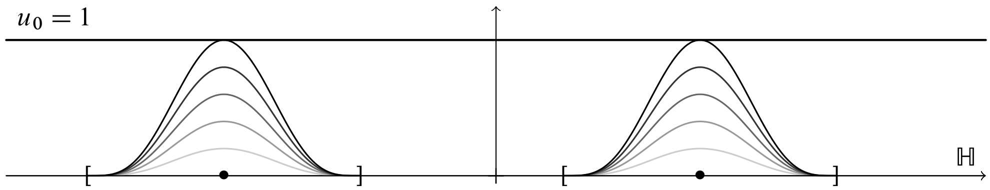

Figure 3. Applying Lemma 5 uniformly.

from Lemma 1 is preserved in the limit, the Yamabe flow given by $g(t) = u(\,\cdot\,, t)g_{\mathbb{H}}$ is instantaneously complete. ☐

Proof of Theorem 2. Let $(g(t))_{t\in[0,T[}$ be any Yamabe flow on $\mathbb{H}$ with $g(0) = g_{\mathbb{H}}$. Let $u : \mathbb{H}\times[0, T[\,\to\,]0, \infty[$ be such that $g(t) = u(\,\cdot\,, t)g_{\mathbb{H}}$ for all $t \in [0, T[$. In order to obtain a sharp bound on u from below it is convenient to aim for an upper bound on the so-called pressure $v = \frac{1}{u}$ which evolves by the equation (see [Schulz 2019a])

$$\frac{1}{m-1}\frac{\partial}{\partial t}v = -mv^2 + v\Delta_{g_{\mathbb{H}}}v - \frac{m+2}{4}|\nabla v|^2_{g_{\mathbb{H}}}. \tag{23}$$

Let $r : \mathbb{H} \to [0, \infty[$ denote the Riemannian distance function in $(\mathbb{H}, g_{\mathbb{H}})$ with respect to some origin in $\mathbb{H}$. Given $0 < \varepsilon < \frac{1}{2}$ let $\psi : \mathbb{H} \to\,]0, \infty[$ be defined by

$$\psi = \frac{1}{\cosh(\varepsilon r)}.$$

Then we have $|\nabla\psi|^2_{g_{\mathbb{H}}} \le \varepsilon^2\psi^2$ and

$$-\Delta_{g_{\mathbb{H}}}\psi = -\frac{\partial^2\psi}{\partial r^2} - \frac{(m-1)}{\tanh(r)}\frac{\partial\psi}{\partial r} = \varepsilon^2\frac{1-\sinh^2(\varepsilon r)}{\cosh^3(\varepsilon r)} + \frac{(m-1)}{\tanh(r)}\frac{\varepsilon\tanh(\varepsilon r)}{\cosh(\varepsilon r)} \le \varepsilon^2\psi + (m-1)\varepsilon\psi$$

in $\mathbb{H}$. This implies

$$\begin{aligned}\frac{1}{m-1}\frac{\partial}{\partial t}(\psi v) &= -m\psi v^2 + v\Delta_{g_{\mathbb{H}}}(\psi v) - v^2\Delta_{g_{\mathbb{H}}}\psi - 2v\langle\nabla\psi, \nabla v\rangle_{g_{\mathbb{H}}} - \frac{m+2}{4}\psi|\nabla v|^2_{g_{\mathbb{H}}}\\ &\le -m\psi v^2 + v\Delta_{g_{\mathbb{H}}}(\psi v) + \left(-\Delta_{g_{\mathbb{H}}}\psi + \frac{4|\nabla\psi|^2_{g_{\mathbb{H}}}}{(m+2)\psi}\right)v^2\\ &\le -m\psi v^2 + v\Delta_{g_{\mathbb{H}}}(\psi v) + m\varepsilon\psi v^2\\ &\le -m(1-\varepsilon)(\psi v)^2 + v\Delta_{g_{\mathbb{H}}}(\psi v),\end{aligned} \tag{24}$$

where we used $\psi \le 1$ in the last step. Since $u(\,\cdot\,, 0) = 1$ we may apply Lemma 5 uniformly in $\mathbb{H}$ and obtain a constant $C_m > 0$ depending only on the dimension m such that $u(\,\cdot\,, t) \ge 1 - C_m t$ in $\mathbb{H}$ for all $t \in [0, T[$ as illustrated in Figure 3. This implies

$$v(\,\cdot\,, t) \le \frac{1}{1 - C_m t}$$

in $\mathbb{H}$ for all $t \in [0, T_0[$, where $T_0 := \min\{T, \frac{1}{C_m}\}$. Hence, the function $(\psi v)(\cdot, t)$ attains a global maximum in $\mathbb{H}$ and the map $w : [0, T_0[\to]0, \infty[$ given by

$$w(t) = \max_{\mathbb{H}}(\psi v)(\cdot, t)$$

is well-defined. Let $t_0 \in]0, T_0[$ be arbitrary but fixed. Let $q_0 \in \mathbb{H}$ such that $w(t_0) = (\psi v)(q_0, t_0)$. By (24), we have

$$\begin{aligned}\liminf_{\tau \searrow 0} \frac{1}{\tau}\left(\frac{1}{w(t_0)} - \frac{1}{w(t_0-\tau)}\right) &\geq \liminf_{\tau \searrow 0} \frac{1}{\tau}\left(\frac{1}{(\psi v)(q_0,t_0)} - \frac{1}{(\psi v)(q_0,t_0-\tau)}\right) \\ &= \frac{\partial}{\partial t}\bigg|_{t=t_0} \frac{1}{(\psi v)(q_0,t)} = \frac{-\frac{\partial}{\partial t}(\psi v)}{(\psi v)^2}(q_0,t_0) \\ &\geq \frac{m-1}{(\psi v)^2}(m(1-\varepsilon)(\psi v)^2 - v\Delta_{g_{\mathbb{H}}}(\psi v))(q_0,t_0) \\ &\geq m(m-1)(1-\varepsilon), \end{aligned} \tag{25}$$

where we used that $-\Delta_{g_{\mathbb{H}}}(\psi v)(q_0,t_0) \geq 0$ since q_0 is a maximum. As shown in [Schulz 2019a, Lemma A.5], estimate (25) implies

$$\frac{1}{w(t)} - \frac{1}{w(0)} \geq m(m-1)(1-\varepsilon)t$$

for every $t \in]0, T_0[$, which yields

$$(\psi v)(\cdot, t) \leq w(t) \leq \frac{1}{m(m-1)(1-\varepsilon)t + 1}$$

since $w(0) = 1$. Letting $\varepsilon \to 0$ and recalling $v = \frac{1}{u}$ we conclude

$$u(\cdot, t) \geq m(m-1)t + 1 \tag{26}$$

for all $t \in [0, T_0[$. By repeating the argument with initial time T_0 if necessary, we obtain that estimate (26) holds in fact for all $t \in [0, T[$.

The reverse inequality $u(\cdot, t) \leq m(m-1)t + 1$ is similar to the statement of Lemma 7. In fact, if we choose $\Omega = \mathbb{H}$ and replace the cutoff function ψ by $\psi(\varepsilon r)$ in the proof of Lemma 7, then the constant c_m in estimate (18) can be replaced by εc_m and we may conclude by letting $\varepsilon \to 0$. □

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SHARPENING THE TRIANGLE INEQUALITY: ENVELOPES BETWEEN L^2 AND L^p SPACES

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Motivated by the inequality $\|f+g\|_2^2 \le \|f\|_2^2 + 2\|fg\|_1 + \|g\|_2^2$, Carbery (2009) raised the question of what is the "right" analogue of this estimate in L^p for $p \neq 2$. Carlen, Frank, Ivanisvili and Lieb (2018) recently obtained an L^p version of this inequality by providing upper bounds for $\|f+g\|_p^p$ in terms of the quantities $\|f\|_p^p$, $\|g\|_p^p$ and $\|fg\|_{p/2}^{p/2}$ when $p \in (0,1] \cup [2,\infty)$, and lower bounds when $p \in (-\infty,0) \cup (1,2)$, thereby proving (and improving) the suggested possible inequalities of Carbery. We continue investigation in this direction by refining the estimates of Carlen, Frank, Ivanisvili and Lieb. We obtain upper bounds for $\|f+g\|_p^p$ also when $p \in (-\infty,0) \cup (1,2)$ and lower bounds when $p \in (0,1] \cup [2,\infty)$. For $p \in [1,2]$ we extend our upper bounds to any finite number of functions. In addition, we show that all our upper and lower bounds of $\|f+g\|_p^p$ for $p \in \mathbb{R}$, $p \neq 0$, are the best possible in terms of the quantities $\|f\|_p^p$, $\|g\|_p^p$ and $\|fg\|_{p/2}^{p/2}$, and we characterize the equality cases.

1. Introduction

For any real-valued functions $f, g \in L^p$ on an arbitrary measure space, and any $p \ge 1$, one has the inequality

$$\|f+g\|_p^p \le 2^{p-1}(\|f\|_p^p + \|g\|_p^p). \tag{1}$$

The estimate (1) follows from the fact that the map $x \mapsto |x|^p$ is convex. If $f = g$ in (1) then the constant 2^{p-1} is sharp and the inequality becomes equality. On the other hand, if f and g have disjoint supports then the constant 2^{p-1} is not needed. We remark that the estimate (1) reflects the convexity of the unit ball in L^p, which is equivalent to the usual L^p triangle (Minkowski) inequality; see, e.g., [Carlen et al. 2020a].

Carbery [2009] asked under what conditions on the sequence of functions $\{f_j\} \subset L^p$ the inequality $\sum \|f_j\|_p^p < \infty$ would imply $\sum f_j \in L^p$. If we try to adapt the inequality (1) to say n functions $f_1, f_2, \dots, f_n$ instead of two, then the constant 2^{p-1} should be replaced by n^{p-1}, which grows with n. To remove dependence on n, Carbery suggested several extensions of inequality (1) which were motivated by the estimate $\|f+g\|_2^2 \le \|f\|_2^2 + 2\|fg\|_1 + \|g\|_2^2$. All of them involve the extra parameter $\|fg\|_{p/2}^{p/2}$, which measures the "overlap" between the functions, and the strongest one in the case of two functions he could prove only for indicator functions of sets. Recently a sharpened form of the triangle inequality was obtained [Carlen et al. 2020a], which implied the proposed estimates of Carbery. Namely, take any

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$p \in \mathbb{R} \setminus \{0\}$, and put

$$\|f\|_p := \left(\int_X |f|^p \, d\mu\right)^{1/p} \quad \text{and} \quad \Gamma_p := \frac{2\|fg\|_{p/2}^{p/2}}{\|f\|_p^p + \|g\|_p^p}.$$

Then

$$\|f+g\|_p^p \leq (1+\Gamma_p^{2/p})^{p-1}(\|f\|_p^p + \|g\|_p^p) \tag{2}$$

holds true if $p \in (0,1] \cup [2,\infty)$, and the inequality reverses if $p \in (-\infty, 0) \cup (1,2)$, where in the latter case we assume that f, g are positive almost everywhere. Since by Cauchy–Schwarz $\Gamma_p \in [0,1]$ for all $p \in \mathbb{R} \setminus \{0\}$, we see that (2) improves on the trivial bound (1).

In this paper we continue investigation in this direction and we address the following questions:

Question 1. Can one further sharpen the right-hand side of the estimate (2) if we are allowed to use only the quantities $\|f\|_p$, $\|g\|_p$, $\|fg\|_{p/2}$?

Question 2. What is the optimal upper bound on $\|f+g\|_p^p$ in terms of the quantities $\|f\|_p$, $\|g\|_p$, $\|fg\|_{p/2}$, also when $p \in (-\infty,0) \cup (1,2)$? Additionally we consider the same question about lower bounds on $\|f+g\|_p^p$, also when $p \in (0,1] \cup [2,\infty)$.

Question 3. Can one extend these estimates to more than two functions?

We will give complete answers to Questions 1 and 2, and we will provide an answer to Question 3 when $p > 0$. In particular we show that, for $p \in [1,2]$, if $\sum_j \|f_j\|_p^p < \infty$ and $\sum_{i<j} \|f_i f_j\|_{p/2}^{p/2} < \infty$, then $\sum_j f_j \in L^p$.

2. Main results

Let $(X, \mathcal{A}, \mu)$ be an arbitrary measure space. In what follows we consider functions f, g on X that are measurable and nonnegative. Given $p \in \mathbb{R} \setminus \{0\}$ we will always assume that $\|f\|_p^p, \|g\|_p^p < \infty$. When $p < 0$ we allow f, g to take the value $+\infty$, where we understand $f^p, g^p = 0$.

Theorem 2.1. *For any $p \in (0,1] \cup [2,\infty)$, and any nonnegative f, g on any measure space we have*

$$\|f+g\|_p^p \leq \left(\left(\frac{1+\sqrt{1-\Gamma_p^2}}{2}\right)^{1/p} + \left(\frac{1-\sqrt{1-\Gamma_p^2}}{2}\right)^{1/p}\right)^p (\|f\|_p^p + \|g\|_p^p). \tag{3}$$

The inequality reverses if $p \in (-\infty,0) \cup [1,2]$. Equality holds if $(fg)^{p/2} = k(f^p + g^p)$ for some constant $k \in \left[0, \frac{1}{2}\right]$.

Remark 2.2. The right-hand side of (3) is the best possible in the following sense: consider the measure space $([0,1], \mathcal{B}, dx)$. Pick any nonnegative numbers x, y and z such that $0 \leq z \leq \sqrt{xy}$. Then, for any $p \in (0,1] \cup [2,\infty)$ the supremum of the left-hand side of (3) over all nonnegative f, g with fixed $\|f\|_p^p = x$, $\|g\|_p^p = y$, $\|fg\|_{p/2}^{p/2} = z$ coincides with the right-hand side of (3). Similarly, for any $p \in (-\infty,0) \cup [1,2]$ the infimum of the left-hand side of (3) over all such f, g coincides with the right-hand side of (3). We justify this remark in Section 3.

Remark 2.2 implies in particular that Theorem 2.1 refines the estimate (2). As a consequence we have the following peculiar estimate:

Corollary 2.3. *For any* $p \in (0,1] \cup [2,\infty)$ *and any number* $\Gamma \in [0,1]$, *we have*

$$\left(\left(\frac{1+\sqrt{1-\Gamma^2}}{2}\right)^{1/p} + \left(\frac{1-\sqrt{1-\Gamma^2}}{2}\right)^{1/p}\right)^p \le (1+\Gamma^{2/p})^{p-1}. \tag{4}$$

The inequality reverses if $p \in (-\infty,0) \cup [1,2]$.

If we set $\Gamma := 2(ab)^{p/2}/(a^p+b^p)$ for nonnegative a, b, then after a short computation inequality (4) becomes

$$\frac{(a+b)^p}{a^p+b^p} \le \left(1+\left(2\frac{(ab)^{p/2}}{a^p+b^p}\right)^{2/p}\right)^{p-1}. \tag{5}$$

This estimate was previously obtained in [Carlen et al. 2020a] (where it was also shown to be equivalent to the inequality (2)), and the arguments are quite involved.

Remark 2.4. If we let $q := 1/p$ and $x = \sqrt{1-\Gamma^2}$, then inequality (4) can also be written as the two-point-type inequality

$$\frac{(1+x)^q + (1-x)^q}{2} \le \left(\frac{1+(1-x^2)^q}{2}\right)^{1-q} \tag{6}$$

for all $q \in \left(-\infty, \frac{1}{2}\right] \cup [1,\infty)$, $x \in [0,1]$, and the inequality reverses if $q \in \left[\frac{1}{2}, 1\right)$. This inequality is reminiscent of Bonami's two-point inequality

$$\left(\frac{|y+u\sqrt{(p-1)/(q-1)}|^q + |y-u\sqrt{(p-1)/(q-1)}|^q}{2}\right)^{1/q} \le \left(\frac{|y+u|^p+|y-u|^p}{2}\right)^{1/p}, \tag{7}$$

which holds true for all $y, u \in \mathbb{R}$ and $1 \le p \le q < \infty$; see [Bonami 1970]. Indeed, if we take $y = 1$, $p = 2$, and $u = x\sqrt{q-1}$ then we get

$$\frac{|1+x|^q + |1-x|^q}{2} \le (1+(q-1)x^2)^{q/2}. \tag{8}$$

The right sides of inequalities (6) and (8) are not comparable. For example, when $x = 1$ the estimate (6) gives better upper bounds for $q > 2$, while near $x = 0$ it gives worse upper bounds.

Next, let $p \in \mathbb{R} \setminus \{0\}$, and set[1]

$$C_p := \frac{\min\{\|f\|_p^p, \|g\|_p^p, \|fg\|_{p/2}^{p/2}\}}{\|fg\|_{p/2}^{p/2}}.$$

Theorem 2.5. *For any* $p \in (1,2)$ *and any nonnegative* f, g *on any measure space we have*

$$\|f+g\|_p^p \le \|f\|_p^p + \|g\|_p^p + ((C_p^{-1/p} + C_p^{1/p})^p - C_p^{-1} - C_p)\|fg\|_{p/2}^{p/2}. \tag{9}$$

[1] If $\|fg\|_{p/2}^{p/2} = 0$ then we set $C_p = 1$.

The inequality reverses if $p \in (0, 1] \cup [2, \infty)$. *Equality holds in* (9) *if one of the following three conditions holds*: $f = g$ *on* $\{fg > 0\}$, $g = \lambda f$ *on* $\{f > 0\}$ *for some* $\lambda \geq 1$, *or* $f = \lambda g$ *on* $\{g > 0\}$ *for some* $\lambda \geq 1$.

For $p \in (-\infty, 0)$ *we have*

$$\|f+g\|_p^p \leq (C_p^{-1/p} + C_p^{1/p})^p \|fg\|_{p/2}^{p/2}. \tag{10}$$

Equality holds in (10) *if one of the following three conditions holds*: $f = g$ *on* $\{fg < \infty\}$, $g = \lambda f$ *on* $\{f < \infty\}$ *for some* $\lambda \leq 1$, *or* $f = \lambda g$ *on* $\{g < \infty\}$ *for some* $\lambda \leq 1$.

Exactly the same remark as before applies to Theorem 2.5; that is, the right-hand sides of (9) and (10) are the best possible. Together, Theorems 2.1 and 2.5, along with the remarks about optimality, answer Questions 1 and 2.

Finally, we state a partial answer to Question 3 in the case $p > 0$.

Corollary 2.6. *For any* $p \in [1, 2]$, *and any sequence of nonnegative functions* $\{f_j\}_{j \geq 1}$ *we have*

$$\Big\|\sum_j f_j\Big\|_p^p \leq \sum_j \|f_j\|_p^p + (2^p - 2) \sum_{i<j} \|f_i f_j\|_{p/2}^{p/2}.$$

If $p \in (0, 1] \cup [2, \infty)$ *the inequality reverses. Equality holds if and only if*

$$\Big(\sum_j f_j\Big)^p = \sum_j f_j^p + (2^p - 2) \sum_{i<j} (f_j f_j)^{p/2}$$

almost everywhere.

In particular, when $p \in [1, 2]$ we have $\sum_j f_j \in L^p$ provided $\sum_j \|f_j\|_p^p < \infty$ and $\sum_{i<j} \|f_j f_j\|_{p/2}^{p/2} < \infty$.

Remark 2.7. After we finished writing this paper we received the preprint [Carlen et al. 2020b], in which the authors obtain an upper bound for the L^p norm of a sum of N functions in the case $p \geq 2$, in terms of a certain analogue for N functions of the quantity Γ_p. Their estimate complements our result Corollary 2.6, which holds for $p \in (1, 2)$, and is obtained using different techniques.

The rest of the paper is organized as follows. In Section 3 we reduce the proofs of Theorems 2.1 and 2.5, as well as the remarks about their optimality, to computing the concave and convex envelopes of a certain function defined on the boundary of a convex cone in $\mathbb{R}^3$. In Section 4 we compute these envelopes. Finally, in Section 5 we prove Corollary 2.6 using an observation about the proof of Theorem 2.5.

3. Reductions

In this section we reduce Theorems 2.1 and 2.5 to computing explicitly the convex and concave envelopes of a certain function defined on the boundary of a convex cone in $\mathbb{R}^3$. Let

$$\Omega := \{x, y \geq 0,\ 0 \leq z \leq \sqrt{xy}\}$$

be the convex cone in $\mathbb{R}^3$ whose vertical cross-sections $\Omega \cap \{x + y = c > 0\}$ are half-ellipses. For $p \in \mathbb{R} \backslash \{0\}$ define φ_p on $\partial\Omega$ by

$$\varphi_p(x, y, \sqrt{xy}) = (x^{1/p} + y^{1/p})^p, \quad x, y > 0, \qquad \varphi_p(x, y, 0) = \begin{cases} x + y, & p > 0, \\ 0, & p < 0. \end{cases}$$

Let f and g be nonnegative functions on an arbitrary measure space $(X, \mathcal{A}, \mu)$ with $\|f\|_p^p, \|g\|_p^p < \infty$. Note that the triple $(\|f\|_p^p, \|g\|_p^p, \|fg\|_{p/2}^{p/2})$ is in Ω by the Cauchy–Schwarz inequality. By the equality case, if the triple is in $\partial\Omega$ we have $\|f+g\|_p^p = \varphi_p(\|f\|_p^p, \|g\|_p^p, \|fg\|_{p/2}^{p/2})$. Our approach is based on the following lemma:

Lemma 3.1. *Let $p \in \mathbb{R}\backslash\{0\}$, and assume that $H \in C(\Omega)$ is a concave, one-homogeneous function on Ω with $H|_{\partial\Omega} = \varphi_p$. Then*

$$\|f+g\|_p^p \le H(\|f\|_p^p, \|g\|_p^p, \|fg\|_{p/2}^{p/2}).$$

If H is convex, the inequality reverses.

Proof. By the boundary conditions, we have

$$1 = H\left(\frac{f^p}{(f+g)^p}, \frac{g^p}{(f+g)^p}, \frac{(fg)^{p/2}}{(f+g)^p}\right)$$

on the set $X' = \{f+g > 0\}$ when $p > 0$, or $\{f+g < \infty\}$ when $p < 0$. Integrating this identity with respect to the probability measure $(f+g)^p \, d\mu/\|f+g\|_p^p$ on X' and applying Jensen's inequality gives

$$1 \le H\left(\frac{\|f\|_p^p}{\|f+g\|_p^p}, \frac{\|g\|_p^p}{\|f+g\|_p^p}, \frac{\|fg\|_{p/2}^{p/2}}{\|f+g\|_p^p}\right)$$

when H is concave, and the other inequality for H convex. The result follows from the one-homogeneity of H. □

Lemma 3.1 reduces our problem to computing the concave and convex envelopes of φ_p on Ω. By concave envelope we mean the infimum of linear functions on Ω that are greater than φ_p on $\partial\Omega$, and by convex envelope we mean the supremum of linear functions on Ω that are smaller than φ_p on $\partial\Omega$. Let $\overline{H}_p$ denote the concave envelope, and $\underline{H}_p$ the convex envelope. For $(x, y, z) \in \Omega$, define

$$w(x, y, z) := \frac{2z}{x+y}, \qquad v(x, y, z) := \min\left\{\frac{x}{z}, \frac{y}{z}, 1\right\},$$

where we take $w = 0$ at the origin and $v = 1$ on $\Omega \cap \{z = 0\}$. Define the one-homogeneous functions F_p, G_p on Ω by

$$F_p(x, y, z) := \frac{x+y}{2}\big((1+\sqrt{1-w^2})^{1/p} + (1-\sqrt{1-w^2})^{1/p}\big)^p, \tag{11}$$

$$G_p(x, y, z) := \begin{cases} x + y + ((v^{1/p} + v^{-1/p})^p - (v + v^{-1}))\, z, & p > 0, \\ (v^{1/p} + v^{-1/p})^p\, z, & p < 0. \end{cases} \tag{12}$$

Proposition 3.2. *The concave and convex envelopes $\overline{H}_p$, $\underline{H}_p$ of φ_p in Ω are in $C(\Omega)$ and are given explicitly by the formulae*

$$\overline{H}_p = \begin{cases} F_p, & p \in (0, 1] \cup [2, \infty), \\ G_p, & p \in (-\infty, 0) \cup (1, 2) \end{cases}$$

and

$$\underline{H}_p = \begin{cases} F_p, & p \in (-\infty, 0) \cup (1, 2), \\ G_p, & p \in (0, 1] \cup [2, \infty). \end{cases}$$

We delay the proof of Proposition 3.2 to Section 4, and immediately note that Theorems 2.1 and 2.5 follow quickly:

Proof of Theorems 2.1 and 2.5. To prove the inequalities, just apply Lemma 3.1 to the functions $\overline{H}_p$ and $\underline{H}_p$. To check the equality cases, observe that in the proof of Lemma 3.1, we have equality in Jensen provided $\{(f^p, g^p, (fg)^{p/2})\}$ lie in a set where H is linear.

Since F_p is linear when restricted to the hyperplanes $\{z = k(x+y)\} \cap \Omega$, which are nontrivial when $k \in \left[0, \frac{1}{2}\right]$, we obtain the equality case in Theorem 2.1.

We note that G_p is linear on the triangular cone $\{z \le \min\{x, y\}\} \cap \Omega$, and on the hyperplanes $\{z = \gamma x\} \cap \Omega$ and $\{z = \gamma y\} \cap \Omega$ for each $\gamma \ge 1$. The first condition gives $(fg)^{p/2} \le \min\{f^p, g^p\}$, so $f = g$ on $\{fg > 0\}$ in the case $p > 0$ and on $\{fg < \infty\}$ in the case $p < 0$. The second condition gives $(fg)^{p/2} = \gamma f^p$, and the third $(fg)^{p/2} = \gamma g^p$. When $p > 0$, the second condition gives that $g = \lambda f$ on $\{f > 0\}$ for some $\lambda \ge 1$, and the third gives that $f = \lambda g$ on $\{g > 0\}$ for some $\lambda \ge 1$; when $p < 0$ the second condition gives $g = \lambda f$ on $\{f < \infty\}$ for some $\lambda \le 1$, and the third gives that $f = \lambda g$ on $\{g < \infty\}$ for some $\lambda \le 1$. □

To conclude the section we address the optimality of Theorems 2.1 and 2.5 in the measure space $(X, \mathcal{A}, \mu) = ([0, 1], \mathcal{B}, dx)$. We define

$$\overline{B}_p(x, y, z) = \sup\{\|f+g\|_p^p : (\|f\|_p^p, \|g\|_p^p, \|fg\|_{p/2}^{p/2}) = (x, y, z)\},$$
$$\underline{B}_p(x, y, z) = \inf\{\|f+g\|_p^p : (\|f\|_p^p, \|g\|_p^p, \|fg\|_{p/2}^{p/2}) = (x, y, z)\}.$$

It is easy to see that $\overline{B}_p$, $\underline{B}_p$ are defined on a cone $\Omega_p \subset \Omega$, are locally bounded by the inequalities $(f+g)^p \le 2^{p-1}(f^p + g^p)$ for $p \in (-\infty, 0) \cup [1, \infty)$ and $(f+g)^p \le f^p + g^p$ for $p \in (0, 1]$, are one-homogeneous, and equal φ_p on $\partial\Omega$ (by the equality case of Cauchy–Schwarz). Furthermore, by Lemma 3.1 we have

$$\underline{H}_p \le \underline{B}_p \le \overline{B}_p \le \overline{H}_p$$

on the common domain of definition.

Lemma 3.3. *If $\overline{B}_p$ ($\underline{B}_p$) is defined on all of Ω and is concave* (*convex*), *then*

$$\overline{H}_p = \overline{B}_p \quad (\underline{B}_p = \underline{H}_p).$$

Proof. Local boundedness and concavity of $\overline{B}_p$ implies continuity in the interior of Ω, and since $\overline{B}_p$ is trapped between envelopes that attain the data continuously, we have $\overline{B}_p \in C(\Omega)$. Since $\overline{H}_p$ is the smallest such concave function, we conclude that $\overline{B}_p \ge \overline{H}_p$. The argument is similar for $\underline{B}_p$. □

Thus, it just remains to show that when $(X, \mathcal{A}, \mu) = ([0, 1], \mathcal{B}, dx)$, the domain of definition for $\overline{B}_p$ and $\underline{B}_p$ is all of Ω, and that $\overline{B}_p$ is concave and $\underline{B}_p$ is convex.

Lemma 3.4. *For $(X, \mathcal{A}, \mu) = ([0, 1], \mathcal{B}, dx)$ we have $\Omega_p = \Omega$ for all $p \ne 0$, that $\overline{B}_p$ is concave in Ω, and that $\underline{B}_p$ is convex in Ω.*

The optimality of the inequalities in Theorems 2.1 and 2.5 follows:

Proof of optimality statements. For either inequality, given

$$(x, y, z) = (\|f\|_p^p, \|g\|_p^p, \|fg\|_{p/2}^{p/2}),$$

the functions $\overline{B}_p(x, y, z)$ and $\underline{B}_p(x, y, z)$ are by definition the best we can do. These are equal to the envelopes $\overline{H}_p$, $\underline{H}_p$ by Lemmas 3.3 and 3.4. □

Remark 3.5. For given $(x, y, z) \in \Omega$ and $p \in \mathbb{R}\backslash\{0\}$, the supremum (infimum) in the definition of $\overline{B}_p$ ($\underline{B}_p$) is in fact attained.

For equality in (3) consider pairs of the form $(f, g) = (a, b)\chi_{[0,c]} + (b, a)\chi_{[c,1]}$ for a, b, c chosen appropriately.

For equality in (9), consider pairs of the form

$$(f, g) = (a, a)\chi_{[0,1/2]} + (b, 0)\chi_{[1/2,3/4]} + (0, c)\chi_{[3/4,1]}$$

for a, b, c appropriately chosen when $z \le \min\{x, y\}$, and $(f, g) = (a, b)\chi_{[0,1/2]} + (c, d)\chi_{[1/2,1]}$ when $z > \min\{x, y\}$ for appropriate a, b, c, d, with one of c, d equal to 0.

For equality in (10), consider pairs of the form

$$(f, g) = (a, a)\chi_{[0,1/2]} + (b, \infty)\chi_{[1/2,3/4]} + (\infty, c)\chi_{[3/4,1]}$$

for a, b, c appropriately chosen when $z \le \min\{x, y\}$, and $(f, g) = (a, b)\chi_{[0,1/2]} + (c, d)\chi_{[1/2,1]}$ when $z > \min\{x, y\}$ for appropriate a, b, c, d, with one of c, d equal to ∞.

Proof of Lemma 3.4. For the first part, if $p > 0$ take $f_s = (2x)^{1/p}\chi_{[s,s+1/2]}$ for $s \in \left[0, \frac{1}{2}\right]$ and let $g = (2y)^{1/p}\chi_{[1/2,1]}$. Then $\|f_s\|_p^p = x$ and $\|g\|_p^p = y$. Furthermore, we have $h(s) := \|f_s g\|_{p/2}^{p/2}$ is continuous, increasing, and $h(0) = 0$, $h\left(\frac{1}{2}\right) = \sqrt{xy}$. When $p < 0$, use the same example but set $f_s, g = \infty$ where they were previously zero.

For the second part, let $(x_i, y_i, z_i) \in \Omega$ with $i = 1, 2$, and for $\epsilon > 0$ choose f_i, g_i such that $(x_i, y_i, z_i) = (\|f_i\|_p^p, \|g_i\|_p^p, \|f_i g_i\|_{p/2}^{p/2})$ and

$$\|f_i + g_i\|_p^p \ge \overline{B}_p(x_i, y_i, z_i) - \epsilon, \quad i = 1, 2.$$

Extend f_i, g_i to be zero outside of $[0, 1]$, and define the rescalings

$$\tilde{f}_1(s) = 2^{1/p} f_1(2s), \quad \tilde{g}_1(s) = 2^{1/p} g_1(2s), \quad \tilde{f}_2(s) = 2^{1/p} f_2(2s-1), \quad \tilde{g}_2(s) = 2^{1/p} g_2(2s-1),$$

so that $\tilde{f}_i, \tilde{g}_i$ are supported in $\left[0, \frac{1}{2}\right]$ for $i = 1$ and in $\left[\frac{1}{2}, 1\right]$ for $i = 2$. We then have

$$\begin{aligned}\tfrac{1}{2}(\overline{B}_p(x_1, y_1, z_1) + \overline{B}_p(x_2, y_2, z_2)) - \epsilon &\le \tfrac{1}{2}(\|\tilde{f}_1 + \tilde{g}_1\|_{L^p([0,1/2])}^p + \|\tilde{f}_2 + \tilde{g}_2\|_{L^p([1/2,1])}^p) \\ &= \tfrac{1}{2}\|\tilde{f}_1 + \tilde{g}_1 + \tilde{f}_2 + \tilde{g}_2\|_p^p \\ &= \left\| \frac{\tilde{f}_1 + \tilde{f}_2}{2^{1/p}} + \frac{\tilde{g}_1 + \tilde{g}_2}{2^{1/p}} \right\|_p^p \\ &\le \overline{B}_p\left(\tfrac{1}{2}(x_1 + x_2, y_1 + y_2, z_1 + z_2)\right).\end{aligned}$$

For the last inequality, we used that for $f_0 := 2^{-1/p}(\tilde{f}_1 + \tilde{f}_2)$, $g_0 := 2^{-1/p}(\tilde{g}_1 + \tilde{g}_2)$ we have

$$\|f_0\|_p^p = \tfrac{1}{2}(x_1 + x_2), \quad \|g_0\|_p^p = \tfrac{1}{2}(y_1 + y_2), \quad \|f_0 g_0\|_{p/2}^{p/2} = \tfrac{1}{2}(z_1 + z_2).$$

Taking $\epsilon \to 0$, we conclude that $\bar{B}_p$ is concave. The convex direction is similar. □

Remark 3.6. Lemma 3.4 holds for any measure space with translation and scaling properties similar to $([0, 1], \mathcal{B}, dx)$, e.g., $(B_1 \subset \mathbb{R}^n, \mathcal{B}, dx)$.

Remark 3.7. The fact that $\bar{B}_p$ is concave also follows from Theorem 1 in [Ivanisvili 2018]. Since the argument is simple, we decided to include it for the reader's convenience.

4. Envelopes

In this section we prove Proposition 3.2. We begin with some simple observations.

First, to check concavity (convexity) in Ω and continuity up to $\partial\Omega$ of $\overline{H}_p$ ($\underline{H}_p$), by one-homogeneity it suffices to check these properties on the half-ellipse

$$D := \Omega \cap \{x + y = 2\}.$$

More generally, any one-homogeneous function B in a convex cone in $\mathbb{R}^n$ (say contained in $\{x_n > 0\}$) is concave (convex) if it is concave (convex) when restricted to a cross-section of the cone (say $\{x_n = 1\}$). Indeed, by one-homogeneity we have

$$B\left(\frac{x+y}{2}\right) = \frac{x_n + y_n}{2} B\left(\lambda \frac{x}{x_n} + (1-\lambda)\frac{y}{y_n}\right)$$

where $\lambda = x_n/(x_n + y_n)$, and the statement follows by applying concavity/convexity of B on the cross-section and then using one-homogeneity once more.

Second, to prove that $\overline{H}_p$ ($\underline{H}_p$) is the concave (convex) envelope of φ_p, it suffices to check that each point in the interior of D lies on a segment that connects boundary points of D, on which $\overline{H}_p$ ($\underline{H}_p$) is linear. Indeed, then any linear function larger (smaller) than φ_p on $\partial\Omega$ will then be larger than $\overline{H}_p$ (smaller than $\underline{H}_p$) in the interior of Ω.

Proof of Proposition 3.2. We first examine F_p, and then G_p.

The function F_p: On D we can write $F_p(1+s, 1-s, t) = u(t)$, where

$$u(t) := [(1 + \sqrt{1-t^2})^{1/p} + (1 - \sqrt{1-t^2})^{1/p}]^p, \quad t \in [0, 1].$$

It is clear that F_p is continuous up to ∂D for each $p \in \mathbb{R}\backslash\{0\}$, and $u(0) = \varphi_p$ (that is, 2 if $p > 0$ and 0 if $p < 0$) on the bottom of D and

$$F_p(1-s, 1+s, \sqrt{1-s^2}) = ((1+s)^{1/p} + (1-s)^{1/p})^p = \varphi_p$$

on the top of D. Since F_p is constant along the horizontal segments in D, it suffices to check that u is concave when $p \in (0, 1] \cup [2, \infty)$, and convex otherwise. To that end, we let $t = \sin(x)$, with $x \in \left[0, \frac{\pi}{2}\right]$.

Then

$$u(\sin(x)) = [(1+\cos(x))^{1/p} + (1-\cos(x))^{1/p}]^p.$$

Let us rewrite the last equality as

$$\tfrac{1}{2}u(\sin(2s)) = [\sin^{2/p}(s) + \cos^{2/p}(s)]^p,$$

where $s = \frac{x}{2} \in \left[0, \frac{\pi}{4}\right]$. Differentiating both sides of the equality in s, we obtain

$$\begin{aligned} u'(\sin(2s))\cos(2s) &= p[\sin^{2/p}(s)+\cos^{2/p}(s)]^{p-1}\frac{2}{p}(\sin^{2/p-1}(s)\cos(s) - \cos^{2/p-1}(s)\sin(s)) \\ &= p[\sin^{2/p}(s)+\cos^{2/p}(s)]^{p-1}\frac{2\cos^{2/p}(s)}{p}(\tan^{2/p-1}(s) - \tan(s)). \end{aligned}$$

Taking the derivative a second time we obtain

$$\begin{aligned} &2u''(\sin(2s))\cos^2(2s) - 2u'(\sin(2s))\sin(2s) \\ &\quad = p(p-1)[\sin^{2/p}(s)+\cos^{2/p}(s)]^{p-2}\left[\frac{2\cos^{2/p}(s)}{p}(\tan^{2/p-1}(s)-\tan(s))\right]^2 + p[\sin^{2/p}(s)+\cos^{2/p}(s)]^{p-1} \\ &\qquad \times\left(-\frac{4\cos^{2/p}(s)\tan(s)}{p^2}(\tan^{2/p-1}(s)-\tan(s)) + \frac{2\cos^{2/p}(s)}{p}\left(\left(\frac{2}{p}-1\right)\tan^{2/p-2}(s)-1\right)(1+\tan^2(s))\right). \end{aligned}$$

Therefore

$$\begin{aligned} &2u''(\sin(2s))\cos^2(2s) \\ &\quad = [\sin^{2/p}(s)+\cos^{2/p}(s)]^{p-2}\frac{4}{p}\cos^{4/p}(s) \\ &\qquad \times\Bigg[(p-1)[(\tan^{2/p-1}(s)-\tan(s))]^2 \\ &\qquad\qquad + [1+\tan^{2/p}(s)]\left(-\tan(s)(\tan^{2/p-1}(s)-\tan(s)) + \left(\left(1-\frac{p}{2}\right)\tan^{2/p-2}(s)-\frac{p}{2}\right)(1+\tan^2(s))\right) \\ &\qquad\qquad + p\tan(2s)[1+\tan^{2/p}(s)](\tan^{2/p-1}(s)-\tan(s))\Bigg]. \end{aligned}$$

Since $\tan(2s) = 2\tan(s)/(1-\tan^2(s))$, after setting $\tan(s) = w \in [0, 1]$ we obtain

$$\begin{aligned} \frac{2u''(\sin(2s))\cos^2(2s)}{[\sin^{2/p}(s)+\cos^{2/p}(s)]^{p-2}\cos^{4/p}(s)} &= \frac{4(p-1)}{p}(w^{2/p-1}-w)^2 \\ &\quad + \frac{4(1+w^{2/p})}{p}\left(-w^{2/p}+w^2+\left(\left(1-\frac{p}{2}\right)w^{2/p-2}-\frac{p}{2}\right)(1+w^2)\right) \\ &\quad + \frac{8w}{1-w^2}(1+w^{2/p})(w^{2/p-1}-w) \\ &= \frac{2(1+w^2)^2}{1-w^2}\left(w^{4/p-2}+\left(\frac{2}{p}-1\right)w^{2/p-2}(1-w^2)-1\right). \end{aligned}$$

(The last equality is a tedious computation, but can be checked by hand). Since

$$\frac{2(1+w^2)^2}{1-w^2} > 0,$$

we see after defining $x := w^2 \in [0, 1]$ that $\operatorname{sgn}(u'') = \operatorname{sgn}(v(x))$, where

$$v(x) = x^{2/p-1} + \left(\frac{2}{p} - 1\right) x^{1/p-1}(1-x) - 1, \quad x \in [0, 1].$$

Let us study the sign of $v(x)$. Without loss of generality assume that $p \neq 1, 2$, otherwise the claims about concavity/convexity of u are trivial. First notice that $v(1) = 0$, and

$$v'(x) = x^{1/p-2}\left(\frac{2}{p} - 1\right)\left(x^{1/p} - \left(1 + \frac{1}{p}(x-1)\right)\right).$$

Therefore, if $p \in (2, \infty)$ it follows from concavity of $x \mapsto x^{1/p}$ that $v' \geq 0$, and hence $v \leq 0$; i.e., u is concave. Similarly, if $p \in (1, 2)$, then $v \geq 0$; i.e., u is convex. Next, if $p \in (0, 1)$ then $x \mapsto x^{1/p}$ is convex, and hence $v' \geq 0$, i.e., u is concave. Finally, if $p \in (-\infty, 0)$ then $x \mapsto x^{1/p}$ is convex, and therefore $v' \leq 0$; i.e., u is convex.

The function G_p: Let $b_p(s, z) = G_p(1+s, 1-s, z)$, with (s, z) in the upper half-disc. For $p > 0$ we can write b_p explicitly as

$$b_p(s, z) = 2 + \begin{cases} w(1-|s|, z), & z \geq 1-|s|, \\ (2^p - 2)z, & z < 1-|s|, \end{cases}$$

where w is the one-homogeneous function given by

$$w(t, z) := \left(t^{1/p} + \left(\frac{z^2}{t}\right)^{1/p}\right)^p - \left(t + \frac{z^2}{t}\right),$$

with $(t, z) \in (0, 1)^2$. It is easy to check that b_p continuously takes the boundary values

$$b_p(s, 0) = 2 = \varphi_p,$$
$$b_p(s, \sqrt{1-s^2}) = ((1+s)^{1/p} + (1-s)^{1/p})^p = \varphi_p.$$

Let

$$h(t) := w(t, 1) = (t^{1/p} + t^{-1/p})^p - (t + t^{-1}), \quad t \in (0, 1).$$

By the one-homogeneity of w and the fact that b_p is linear on the triangle $\{z < 1-|s|\}$ with vertical gradient, if we show that $h'(1) = 0$ and that h is concave/convex on $[0, 1]$, then b_p is C^1 away from $(s, z) = (\pm 1, 0)$ and concave/convex. Furthermore, b_p is linear when restricted to the segments through $(s, z) = (\pm 1, 0)$ that lie outside of the triangle $\{z \leq 1-|s|\}$, so G_p is the concave/convex envelope provided the above conditions on h are confirmed. To that end we compute the first two derivatives of h. The first derivative is

$$h'(t) = (t^{1/p} + t^{-1/p})^{p-1}(t^{1/p-1} - t^{-1/p-1}) - (1 - t^{-2}).$$

This confirms that $h'(1) = 0$. The second derivative is

$$\begin{aligned}
h''(t) &= \frac{p-1}{p}(t^{1/p}+t^{-1/p})^{p-2}(t^{1/p-1}-t^{-1/p-1})^2 \\
&\qquad + \frac{1}{p}(t^{1/p}+t^{-1/p})^{p-1}((1-p)t^{1/p-2}+(1+p)t^{-1/p-2}) - 2t^{-3} \\
&= \frac{1}{p}(t^{1/p}+t^{-1/p})^{p-2}[(p-1)(t^{1/p-1}-t^{-1/p-1})^2 \\
&\qquad + (t^{1/p}+t^{-1/p})((1-p)t^{1/p-2}+(1+p)t^{-1/p-2})] - 2t^{-3} \\
&= \frac{2}{p}(t^{1/p}+t^{-1/p})^{p-2}[pt^{-2/p-2}+(2-p)t^{-2}] - 2t^{-3} \\
&= 2t^{-3}[(t^{1/p}+t^{-1/p})^{p-2}(t^{1-2/p}+(2/p-1)t) - 1] \\
&= 2t^{-3}[(1+t^{2/p})^{p-2}(1+(2/p-1)t^{2/p}) - 1].
\end{aligned}$$

Let $x := t^{2/p} \in [0, 1]$. It suffices to show that

$$g_p(x) := \left(1 + \left(\frac{2}{p} - 1\right)x\right) - (1+x)^{2-p}$$

satisfies $g_p \le 0$ on $[0, 1]$ for $p \in (1, 2)$ and $g_p \ge 0$ on $[0, 1]$ for $p \in (0, 1] \cup [2, \infty)$. Note that $g_p(0) = 0$. The desired inequality for $g_p(1)$ is equivalent to the fact that the linear function p crosses the convex function 2^{p-1} at $p = 1$ and $p = 2$. Finally, we observe that the first term in g_p is linear, and the second term is convex for $p \in (1, 2)$ and concave for $p \in (0, 1) \cup (2, \infty)$. The desired inequality for $g_p(x)$ with $x \in (0, 1)$ follows immediately from this observation and the inequalities at the endpoints $x = 0$ and $x = 1$.

When $p < 0$ we can write b_p explicitly as

$$b_p(s, z) = \begin{cases} \tilde{w}(1-|s|, z), & z \ge 1-|s|, \\ 2^p z, & z < 1-|s|, \end{cases}$$

where $\tilde{w}$ is the one-homogeneous function given by

$$\tilde{w}(t, z) := (t^{1/p} + (z^2/t)^{1/p})^p$$

with $(t, z) \in (0, 1)^2$. The same considerations as above reduce the problem to showing that

$$\tilde{h}(t) := \tilde{w}(t, 1) = (t^{1/p} + t^{-1/p})^p$$

satisfies $\tilde{h}'(1) = 0$ and $\tilde{h}$ is concave on $[0, 1]$. We have

$$\begin{aligned}
\tilde{h}' &= (t^{1/p}+t^{-1/p})^{p-1}(t^{1/p-1}-t^{-1/p-1}) \quad \Longrightarrow \quad \tilde{h}'(1) = 0, \\
\tilde{h}'' &= 2t^{-2}(t^{1/p}+t^{-1/p})^{p-2}[t^{-2/p}+(2/p-1)],
\end{aligned}$$

and the conclusion follows quickly using $p < 0$. □

Remark 4.1. It follows from the concavity/convexity properties of G_p that

$$G_p(x, y, z) \le x + y + (2^p - 2)z$$

when $p \in [1, 2]$, and the inequality reverses for $p \in (0, 1] \cup [2, \infty)$. Indeed, G_p agrees with the linear function on the right-hand side on an open set. We conclude from Theorem 2.5 that for any nonnegative numbers a, b, and any $p \in [1, 2]$, we have

$$(a+b)^p \le a^p + b^p + (2^p - 2)(ab)^{p/2},$$

and the inequality reverses if $p \in (0, 1] \cup [2, \infty)$.

5. Proof of Corollary 2.6

In this final section we prove Corollary 2.6.

Proof of Corollary 2.6. Recall from Remark 4.1 that for any nonnegative numbers a, b, and any $p \in [1, 2]$, we have

$$(a+b)^p \le a^p + b^p + (2^p - 2)(ab)^{p/2},$$

and the inequality reverses for $p \in (0, 1] \cup [2, \infty)$. Since for $p \in [0, 2]$ we have $(a+b)^{p/2} \le a^{p/2} + b^{p/2}$, and the reverse inequality holds if $p \ge 2$, it follows by induction that for any nonnegative numbers $a_j \ge 0$ we have

$$\Big(\sum a_j\Big)^p \le \sum_j a_j^p + (2^p - 2) \sum_{i<j} (a_i a_j)^{p/2} \tag{13}$$

for $p \in [1, 2]$, and the reverse inequality holds if $p \in (0, 1] \cup [2, \infty)$. Finally it remains to put $a_j = f_j(x)$ and integrate the inequality. □

Remark 5.1. When $p < 0$, inequality (13) does not hold with three or more a_j. Take, e.g., $a_j = 1$ for $j \le 3$.

6. Concluding remarks on envelopes

An important challenge in this work was to compute the envelopes (11) and (12). In this section we briefly explain how we found them.

We recall from Section 3 that for the measure space $([0, 1], \mathcal{B}, dx)$ we have $\overline{B}_p = \overline{H}_p$ is defined on Ω, one-homogeneous, and equal to φ_p on $\partial\Omega$; that is, $\overline{H}_p(x, y, \sqrt{xy}) = (x^{1/p} + y^{1/p})^p$. We also recall from the discussion at the beginning of Section 4 that by one-homogeneity, to compute $\overline{H}_p$ it is enough to restrict our attention to the cross-section $D = \Omega \cap \{x + y = 2\}$. Writing $D = \{(1+s, 1-s, z)\}$ with (s, z) in the upper half-disc, this reduces the problem to understanding how the upper boundary of the convex envelope of the space curve

$$\gamma(s) = (s, \sqrt{1-s^2}, ((1-s)^{1/p} + (1+s)^{1/p})^p), \quad s \in [-1, 1],$$

looks. One can show that the torsion τ_γ of the space curve γ changes sign only once from $-$ to $+$, at $s = 0$, when $p \in (0, 1) \cup (2, \infty)$, and from $+$ to $-$ when $p \in (-\infty, 0) \cup (1, 2)$. Consider the case $p \in (0, 1) \cup (2, \infty)$. Then it follows from Lemma 29 of Section 3.2 in [Ivanisvili 2015] that locally, say for some $\delta \in (0, 1]$, there exists a function $a(s) : [0, \delta] \to [-1, 0]$ such that $a(0) = 0$, $a(s)$ is strictly decreasing, and the function $B(u, w)$, defined parametrically by

$$B(\lambda(a(s), \sqrt{1-a(s)^2}) + (1-\lambda)(s, \sqrt{1-s^2})) = \lambda((1-a(s)^{1/p} + (1+a(s))^{1/p})^p + (1-\lambda)((1-s)^{1/p} + (1+s)^{1/p})^p$$

for $\lambda \in [0, 1]$, $s \in [0, \delta]$, is concave. In other words B has the prescribed boundary condition, i.e., $B(s, \sqrt{1-s^2}) = ((1-s)^{1/p} + (1+s)^{1/p})^p$, it is linear along the line segments

$$\ell(s) := [(a(s), \sqrt{1-a(s)^2}), (s, \sqrt{1-s^2})],$$

and B is concave. It follows that "locally" B is a concave envelope. Because of the symmetry in x and y of the boundary data φ_p, one can show that the line segments $\ell(s)$ must be horizontal; i.e., $a(s) = -s$, and in fact $\delta = 1$. This means that B is a global concave envelope

$$B(u, w) = ((1-\sqrt{1-w^2})^{1/p} + (1+\sqrt{1-w^2})^{1/p})^p$$

for all $|u| \le 1$ and $0 \le w \le \sqrt{1-u^2}$. Now it remains to change variables back to recover the envelope (11).

The case $p \in (-\infty, 0) \cup (1, 2)$ is different because τ_γ changes sign from $+$ to $-$, and in this case an "angle" arises with vertex sitting around the point $s = 0$; see Section 3 in [Ivanisvili 2015].

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