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OPERATORS

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# UNIFORM SOBOLEV ESTIMATES FOR SCHRÖDINGER OPERATORS WITH SCALING-CRITICAL POTENTIALS AND APPLICATIONS

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We prove uniform Sobolev estimates for the resolvent of Schrödinger operators with large scaling-critical potentials without any repulsive condition. As applications, global-in-time Strichartz estimates including some nonadmissible retarded estimates, a Hörmander-type spectral multiplier theorem, and Keller-type eigenvalue bounds with complex-valued potentials are also obtained.

## 1. Introduction and main results

This paper is a continuation of [Bouclet and Mizutani 2018; Mizutani 2019], where uniform estimates for the resolvent  $(H - z)^{-1}$  of the Schrödinger operator  $H = -\Delta + V(x)$  on  $\mathbb{R}^n$  with a real-valued potential  $V(x)$  exhibiting one critical singularity were investigated under some *repulsive* conditions so that  $H$  is nonnegative and its spectrum  $\sigma(H)$  is purely absolutely continuous. In the present paper we improve upon and extend those previous results to a class of scaling-critical potentials without any repulsive condition such that  $H$  may have (finitely many) negative eigenvalues and multiple scaling-critical singularities. Applications to Strichartz estimates, a Hörmander-type multiplier theorem for  $H$  and eigenvalue bounds for  $H + W$  with complex potential  $W$  are also established.

We first recall some known results in the free case,  $H = -\Delta$ , describing the motivation of this paper. The classical Hardy–Littlewood–Sobolev (HLS for short) inequality states that

$$\|(-\Delta)^{-s/2} f\|_{L^q} \leq C \|f\|_{L^p}$$

for  $f \in \mathcal{S}(\mathbb{R}^n)$ ,  $0 < s < n$ ,  $1 < p < q < \infty$  and  $1/p - 1/q = s/n$ , where  $\mathcal{S}(\mathbb{R}^n)$  denotes the space of Schwarz functions,  $(-\Delta)^{-s/2} = \mathcal{F}^{-1} |\xi|^{-s} \mathcal{F}$  is the Riesz potential of order  $s$  and  $\mathcal{F}$  stands for the Fourier transform in  $\mathbb{R}^n$ . An equivalent form is Sobolev's inequality

$$\|f\|_{L^q} \leq C \|(-\Delta)^{s/2} f\|_{L^p}.$$

When  $s = 2$ , the HLS inequality can be regarded as the  $L^p$ - $L^q$  boundedness of the free resolvent  $(-\Delta - z)^{-1}$  at  $z = 0$ . In this context, the HLS inequality was extended to nonzero energies  $z \neq 0$  in [Kenig, Ruiz, and Sogge 1987; Kato and Yajima 1989; Gutiérrez 2004] as follows:

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**Proposition 1.1** (uniform Sobolev estimates). *Let  $n \geq 3$ ,  $1 \leq r \leq \infty$  and  $(p, q)$  satisfy*

$$\frac{2}{n+1} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{2}{n}, \quad \frac{2n}{n+3} < p < \frac{2n}{n+1}, \quad \frac{2n}{n-1} < q < \frac{2n}{n-3}. \quad (1-1)$$

*Then the free resolvent  $R_0(z) = (-\Delta - z)^{-1}$  satisfies*

$$\|R_0(z)f\|_{L^{q,r}} \leq C|z|^{(n/2)(1/p-1/q)-1}\|f\|_{L^{p,r}} \quad (1-2)$$

*uniformly in  $f \in L^{p,r}(\mathbb{R}^n)$ ,  $z \in \mathbb{C} \setminus [0, \infty)$  and  $r$ , where  $L^{p,r}(\mathbb{R}^n)$  denotes the Lorentz space.*

*Sketch of proof.* By virtue of real interpolation (see [Theorem A.1](#) in the [Appendix](#)), we may replace without loss of generality  $L^{p,r}$  and  $L^{q,r}$  by  $L^p$  and  $L^q$ , respectively. Then the case  $1/p + 1/q = 1$  was proved independently by [\[Kenig, Ruiz, and Sogge 1987, Theorem 2.3\]](#) and [\[Kato and Yajima 1989, \(3.29\), p. 493\]](#); the case  $1/p - 1/q = 2/n$  is due to [\[Kenig, Ruiz, and Sogge 1987, Theorem 2.2\]](#); otherwise, we refer to [\[Gutiérrez 2004, Theorem 6\]](#).  $\square$

Note that, when  $1/p - 1/q = 2/n$ , the estimate is uniform in  $z$ , as its name suggests.

Uniform Sobolev estimates can be used in the study of broad areas including the spectral and scattering theory for Schrödinger operators. In [\[Kenig, Ruiz, and Sogge 1987\]](#), the authors applied [\(1-2\)](#) to study unique continuation properties of  $-\Delta + V$  with  $V \in L^{n/2}$ . In [\[Kato and Yajima 1989; Goldberg and Schlag 2004; Ionescu and Schlag 2006\]](#), [\(1-2\)](#) was used to show the limiting absorption principle and asymptotic completeness of wave operators for  $-\Delta + L$  with a large class of singular perturbations  $L$ . In [\[Frank 2011\]](#), [\(1-2\)](#) was used to prove the Keller-type inequality for  $-\Delta + W(x)$  with a complex potential  $W \in L^p$  with some  $p \geq n/2$ , which is a quantitative estimate of the spectral radius of  $\sigma_p(-\Delta + W)$ . In [\[Gutiérrez 2004\]](#), [\(1-2\)](#) was applied to show the existence of  $L^q$ -solutions for the stationary Ginzburg–Landau equation under some radiation condition.

In a more abstract setting, the following observations are satisfied for not only  $\Delta$  but also a general nonnegative self-adjoint operator  $L$  on  $L^2(X, \mu)$ :

- The uniform Sobolev estimate with  $p = 2n/(n+2)$  and  $q = 2n/(n-2)$  implies that, for any  $w \in L^n$ , the weighted resolvent  $w(L-z)^{-1}w$  is bounded on  $L^2$  uniformly in  $z \in \mathbb{C} \setminus [0, \infty)$ . As observed by [\[Kato 1966; Kato and Yajima 1989; Rodnianski and Schlag 2004\]](#), such a weighted estimate is closely connected with dispersive properties of the solution to [\(1-4\)](#) such as Kato-smoothing effects, time-decay and Strichartz estimates, which are fundamental tools in the study of nonlinear Schrödinger equations; see [\[Tao 2006\]](#).
- Uniform Sobolev estimates imply that the spectral measure  $dE_L(\lambda)$  associated with  $L$  is bounded from  $L^p$  to  $L^{p'}$  for

$$\frac{2n}{n+2} \leq p \leq \frac{2(n+1)}{n+3}.$$

This is an important input to prove the Hörmander-type theorem on the  $L^p$  boundedness of the spectral multiplier  $f(L)$ ; see [\[Chen, Ouhabaz, Sikora, and Yan 2016\]](#).

Motivated by those observations, we are interested in extending (1-2) to the Schrödinger operator  $H = -\Delta + V(x)$ . If  $V$  is of very short range type in the sense that, with some  $\varepsilon > 0$ ,

$$|V(x)| \leq C(1 + |x|)^{-2-\varepsilon}, \quad x \in \mathbb{R}^n, \quad (1-3)$$

then there is a vast literature on uniform weighted  $L^2$ -estimates for  $(H - z)^{-1}$  without any additional repulsive condition such as suitable smallness of the negative part of  $V$ ; see, e.g., [Jensen and Kato 1979; Rodnianski and Tao 2015]. Weighted  $L^2$ -estimates were also obtained for a class of potentials satisfying  $|x|^2 V \in L^\infty$  under some additional repulsive conditions [Burq, Planchon, Stalker, and Tahvildar-Zadeh 2004; Barceló, Vega, and Zubeldia 2013]. In our previous works [Bouclet and Mizutani 2018; Mizutani 2019], we proved uniform Sobolev estimates for  $H$  with a class of critical potentials  $V \in L^{n/2, \infty}$  under some repulsive conditions so that  $H$  has purely absolutely continuous spectrum. However, in these works, the range of  $(p, q)$  has been restricted on the line  $1/p + 1/q = 1$ . Furthermore, the situation for (large) critical potentials without any repulsive condition is less understood.

The main goal of this paper is to prove the full set of uniform Sobolev estimates for  $H = -\Delta + V(x)$  with a large scaling-critical potential  $V \in L_0^{n/2, \infty}$  without any repulsive condition. The following three types of applications are also established in the paper:

- (i) We prove global-in-time Strichartz estimates for the Schrödinger equation

$$i\partial_t u(t, x) = Hu(t, x) + F(t, x), \quad (t, x) \in \mathbb{R}^{1+n}, \quad u(0, x) = \psi, \quad x \in \mathbb{R}^n, \quad (1-4)$$

for all admissible cases and several nonadmissible cases.

- (ii) A Hörmander-type spectral multiplier theorem for  $f(H)$  is obtained provided that  $H$  is nonnegative.
- (iii) We obtain Keller-type estimates for the eigenvalues (including possible embedded eigenvalues) of the operator  $H + W$  with complex potentials  $W \in L^p$ ,  $n/2 < p \leq (n+1)/2$ .

Finally, we mention that the results in this paper could be used to study spectral and scattering theory for both linear and nonlinear Schrödinger equations with potentials  $V \in L_0^{n/2, \infty}$ .

**Notation.**  $A \lesssim B$  (resp.  $A \gtrsim B$ ) means  $A \leq cB$  (resp.  $A \geq cB$ ) with some universal constant  $c > 0$ . By  $\langle x \rangle$  we denote  $\sqrt{1 + |x|^2}$  and we set  $\mathbb{C}^\pm := \{z \in \mathbb{C} \mid \pm \operatorname{Im} z > 0\}$ . Given two Banach spaces  $X$  and  $Y$ ,  $\mathbb{B}(X, Y)$  is the Banach space of bounded linear operators from  $X$  to  $Y$  and  $\mathbb{B}(X) = \mathbb{B}(X, X)$ , and  $\mathbb{B}_\infty(X, Y)$  and  $\mathbb{B}_\infty(X)$  are families of compact operators. By  $\langle f, g \rangle = \int f \bar{g} dx$  we denote the inner product in  $L^2$ . We also use the same notation  $\langle \cdot, \cdot \rangle$  for the dual coupling between  $L^p$  and  $L^{p'}$ , where  $p' = p/(p-1)$  denotes the Hölder conjugate of  $p$ .  $L_T^p \mathcal{X}_x = L^p(\mathbb{R}; \mathcal{X})$  is the Bochner–Lebesgue space with norm  $\|F\|_{L_T^p \mathcal{X}} = \|\|F(t, x)\|_{\mathcal{X}_x}\|_{L_T^p}$ .  $L_T^p L_x^q := L^p([-T, T]; L^q(\mathbb{R}^n))$ . Let  $\langle \cdot, \cdot \rangle_T$  be the inner product in  $L_T^2 L_x^2$  defined by

$$\langle F, G \rangle_T = \int_{-T}^T \langle F(\cdot, t), G(\cdot, t) \rangle dt.$$

$\mathcal{H}^s(\mathbb{R}^n)$  and  $\mathcal{J}^s(\mathbb{R}^n)$  are inhomogeneous and homogeneous  $L^2$ -Sobolev spaces, respectively.  $\mathcal{W}^{s,p}(\mathbb{R}^n)$  is the  $L^p$ -Sobolev space.  $L^{p,q}(\mathbb{R}^n)$  denotes the Lorentz space (see the Appendix).

**1A. Main results.** Throughout the paper we assume that  $n \geq 3$  and that  $V \in L_0^{n/2,\infty}(\mathbb{R}^n)$  is a real-valued function, where  $L_0^{p,\infty}(\mathbb{R}^n)$  is the completion of  $C_0^\infty(\mathbb{R}^n)$  with respect to the norm  $\|\cdot\|_{L^{p,\infty}}$ . It follows from Hölder's and Sobolev's inequalities for Lorentz norms (see the [Appendix](#)) that  $V$  is  $\Delta$ -form compact. Then the KLMN theorem [Reed and Simon 1975, Theorem X.17] yields that there exists a unique lower semibounded self-adjoint operator  $H$  on  $L^2(\mathbb{R}^n)$  with form domain  $\mathcal{H}^1(\mathbb{R}^n)$  such that

$$\langle Hu, v \rangle = \langle (-\Delta + V)u, v \rangle, \quad u \in D(H), \quad v \in \mathcal{H}^1(\mathbb{R}^n),$$

and that its domain  $D(H) = \{u \in \mathcal{H}^1(\mathbb{R}^n) \mid Hu \in L^2(\mathbb{R}^n)\}$  is dense in  $\mathcal{H}^1(\mathbb{R}^n)$ . In other words,  $H$  is defined as the Friedrichs extension of the sesquilinear form  $\langle (-\Delta + V)u, v \rangle$ .

**Remark 1.2.** Note that  $L^{n/2,q} \hookrightarrow L_0^{n/2,\infty}$  for all  $1 \leq q < \infty$ . Also note that the class  $L_0^{n/2,\infty}$  is scaling-critical in the sense that the norm  $\|V\|_{L^{n/2,\infty}}$  is invariant under the scaling  $V \mapsto V_\lambda$ , where  $V_\lambda(x) = \lambda^2 V(\lambda x)$ . In particular, if  $V$  itself is invariant under this scaling, the potential energy  $\langle Vu, u \rangle$  has the same scale-invariant structure as that for the kinetic energy  $\langle -\Delta u, u \rangle$ .

Let  $\mathcal{E} \subset \sigma(H)$  be the exceptional set of  $H$ , the set of all eigenvalues and resonances of  $H$  (see [Definition 2.6](#)). Note that  $\mathcal{E} \cap (-\infty, 0)$  is equal to  $\sigma_d(H)$ , the discrete spectrum of  $H$ , and that  $\mathcal{E}$  is bounded in  $\mathbb{R}$  (see [Remark 3.4](#)). For the absence of embedded eigenvalues and resonances, we have the following simple criterion (see also [Remark 1.18](#)):

**Lemma 1.3.** *Let  $V$  be as above. Then the following statements are satisfied:*

- (1) *If  $V \in L^{n/2}$  then there are no positive eigenvalues and resonances; that is,  $\mathcal{E} \cap (0, \infty) = \emptyset$ .*
- (2) *If  $-\Delta + V \geq -\delta \Delta$  with some  $\delta > 0$  in the sense of forms on  $C_0^\infty$  then  $0 \notin \mathcal{E}$ .*

*Proof.* The proof will be given in [Section 2B](#). □

Define  $\mathcal{E}_\delta := \{z \in \mathbb{C} \mid \text{dist}(z, \mathcal{E}) < \delta\}$  if  $\mathcal{E} \neq \emptyset$  and  $\mathcal{E}_\delta := \emptyset$  if  $\mathcal{E} = \emptyset$ . For  $z \in \mathbb{C} \setminus \sigma(H)$ , we denote the resolvent of  $H$  by  $R(z) = (H - z)^{-1}$ .

Then the main result in this paper is as follows.

**Theorem 1.4.** *Suppose that  $(p, q)$  satisfies (1-1). Then  $R(z)$  extends to a bounded operator from  $L^{p,2}$  to  $L^{q,2}$  for all  $z \in \mathbb{C} \setminus \sigma(H)$ . Moreover, for any  $\delta > 0$  there exists  $C_\delta > 0$  such that*

$$\|R(z)f\|_{L^{q,2}} \leq C_\delta |z|^{(n/2)(1/p-1/q)-1} \|f\|_{L^{p,2}} \quad (1-5)$$

for all  $z \in \mathbb{C} \setminus ([0, \infty) \cup \mathcal{E}_\delta)$  and  $f \in L^{p,2}$ . In particular, if  $\mathcal{E} = \emptyset$ , then (1-5) holds uniformly with respect to  $z \in \mathbb{C} \setminus [0, \infty)$  and  $f \in L^{p,2}$ .

As a corollary, the limiting absorption principle in the same topology is derived.

**Corollary 1.5.** *Let  $(p, q)$  satisfy (1-1). Then the following statements are satisfied:*

- (1) *The boundary values  $R(\lambda \pm i0) = \lim_{\varepsilon \searrow 0} R(\lambda \pm i\varepsilon) \in \mathbb{B}(L^{p,2}, L^{q,2})$  exist for all  $\lambda \in (0, \infty) \setminus \mathcal{E}$ . Moreover, for any  $\delta > 0$  there exists  $C_\delta > 0$  such that*

$$\|R(\lambda \pm i0)f\|_{L^{q,2}} \leq C_\delta \lambda^{(n/2)(1/p-1/q)-1} \|f\|_{L^{p,2}}, \quad f \in L^{p,2}(\mathbb{R}^n), \quad \lambda \in (0, \infty) \setminus \mathcal{E}_\delta. \quad (1-6)$$

*In particular, if  $\mathcal{E} \cap [0, \infty) = \emptyset$ , then (1-6) holds uniformly in  $\lambda > 0$ .*

(2) Assume in addition that  $1/p - 1/q = 2/n$  and  $0 \notin \mathcal{E}$ . Then  $R(0 \pm i0) \in \mathbb{B}(L^{p,2}, L^{q,2})$  exist and  $R(0 + i0) = R(0 - i0)$ . Moreover,  $HR(0 + i0)f = f$  and  $R(0 + i0)Hg = g$  for all  $f, g \in \mathcal{S}$  in the sense of distributions. In particular, one has the HLS-type inequality

$$\|H^{-1}f\|_{L^{q,2}} \leq C\|f\|_{L^{p,2}}, \quad f \in L^{p,2}(\mathbb{R}^n). \quad (1-7)$$

As a byproduct of [Theorem 1.4](#), we also obtain the  $L^p$ - $L^q$  boundedness of  $R(z)$  for fixed  $z$  with a wider range than [\(1-1\)](#).

**Corollary 1.6.** *For any  $z \in \mathbb{C} \setminus \sigma(H)$ , the resolvent  $R(z)$  is bounded from  $L^{p,2}$  to  $L^{q,2}$  whenever*

$$0 \leq \frac{1}{p} - \frac{1}{q} \leq \frac{2}{n}, \quad \frac{2n}{n+3} < p, q < \frac{2n}{n-3}. \quad (1-8)$$

In particular,  $D(H) \subset D(w)$  for any  $w \in L^{n/s, \infty}$  with  $0 \leq s < \frac{3}{2}$ . Here  $D(w)$  denotes the domain of the multiplication operator by  $w(x)$ .

**Remark 1.7.** Since  $L^p \hookrightarrow L^{p,2}$  and  $L^{q,2} \hookrightarrow L^q$  if  $p \leq 2 \leq q$ , one has  $\mathbb{B}(L^{p,2}, L^{q,2}) \subset \mathbb{B}(L^p, L^q)$ . Moreover, by virtue of real interpolation (see [Theorem A.1](#)), [Theorem 1.4](#) and [Corollaries 1.5](#) and [1.6](#) also hold with  $L^{p,2}$  and  $L^{q,2}$  replaced respectively by  $L^{p,r}$  and  $L^{q,r}$  for any  $1 \leq r \leq \infty$ .

As explained in the [Introduction](#), the resolvent  $R(z)$  has a close relation with the spectral measure  $E_H$  associated with  $H$  through Stone's formula

$$E'_H(\lambda) = \frac{1}{2\pi i} \lim_{\varepsilon \searrow 0} (R(\lambda + i\varepsilon) - R(\lambda - i\varepsilon)), \quad \lambda \in (0, \infty) \setminus \sigma_p(H), \quad (1-9)$$

where  $E'_H(\lambda) = (dE_H/d\lambda)(\lambda)$  is the density of  $E_H$ . Using this formula and above theorems, we also obtain the following restriction-type estimates.

**Theorem 1.8.** *Assume that  $\mathcal{E} \cap [0, \infty) = \emptyset$ . Then, for any*

$$\frac{2n}{n+3} < p \leq \frac{2(n+1)}{n+3},$$

we have

$$\|E'_H(\lambda)\|_{\mathbb{B}(L^p, L^{p'})} \leq C\lambda^{(n/2)(1/p-1/p')-1}, \quad \lambda > 0. \quad (1-10)$$

**Remark 1.9.** The existence of  $R(\lambda \pm i0)$  in  $\mathbb{B}(L^{2(n+1)/(n+3)}, L^{2(n+1)/(n-1)})$  for each  $\lambda > 0$  was proved in [\[Ionescu and Schlag 2006\]](#) for the case when  $V \in L^p$  with  $n/2 \leq p \leq (n+1)/2$ . The uniform estimate [\(1-6\)](#) in the high energy regime  $\lambda \geq \lambda_0 > 0$  was obtained in [\[Goldberg and Schlag 2004\]](#) for the case when  $n = 3$ ,  $V \in L^{3/2} \cap L^r$  with  $r > \frac{3}{2}$  and  $(p, q) = (\frac{4}{3}, 4)$ . Recently, [\(1-6\)](#) for  $\lambda > 0$  and

$$(p, q) = \left( \frac{2(n+1)}{n-1}, \frac{2(n+1)}{n+3} \right)$$

was proved in [\[Huang, Yao, and Zheng 2018\]](#) provided that  $V \in L^{n/2} \cap L^{n/2+\varepsilon}$  and  $0 \notin \mathcal{E}$  (note that, in this case,  $\mathcal{E} \cap (0, \infty) = \emptyset$  as in [Lemma 1.3](#)). Compared with those previous works, the main new contributions of [Theorem 1.4](#) and [Corollary 1.5](#) are threefold. At first, we obtain the uniform estimates [\(1-5\)](#) and [\(1-6\)](#) with respect to  $z$  or  $\lambda$  in both high- and low-energy regimes, under the condition  $\mathcal{E} \cap [0, \infty) = \emptyset$ . This is an important input to prove global-in-time Strichartz estimates without any low- or high-energy cut-off. Next,

the full set of uniform Sobolev estimates is obtained, while the above previous references considered the case  $1/p + 1/q = 1$  only. In particular, (1-5) and (1-6) for  $(p, q)$  away from the line  $1/p + 1/q = 1$  seems to be new even under the condition (1-3). Such “off-diagonal” estimates play an important role in the proof of Strichartz estimates for nonadmissible pairs and  $L^p$ -boundedness of the spectral multiplier  $f(H)$  for a wider range of  $p$  than that obtained by the “diagonal” estimate on the line  $1/p + 1/q = 1$  (see Sections 4 and 5, respectively). Finally, we obtain the above results for large critical potentials  $V \in L_0^{n/2, \infty}$  without any additional regularity or repulsive condition. Concerning  $L^p$ - $L^q$  boundedness of  $R(z)$  for each  $z \in \mathbb{C} \setminus [0, \infty)$ , a result similar to Corollary 1.6 was previously obtained in [Simon 1982] for Kato class potentials. However, to our best knowledge, this corollary seems to be new for the present class of potentials.

In this paper we also study several applications of the above resolvent estimates to the time-dependent problem, harmonic analysis and spectral theory associated with  $H$ .

We first consider global-in-time estimates for the Schrödinger equation (1-4). Let  $e^{-itH}$  be the unitary group generated by  $H$  via Stone’s theorem. For  $F \in L_{\text{loc}}^1(\mathbb{R}; L^2(\mathbb{R}^n))$ , we define

$$\Gamma_H F(t) = \int_0^t e^{-i(t-s)H} F(s) ds.$$

For  $\psi \in L^2(\mathbb{R}^n)$  and  $F \in L_{\text{loc}}^1(\mathbb{R}; L^2(\mathbb{R}^n))$ , a unique (mild) solution to (1-4) is then given by

$$u = e^{-itH} \psi - i \Gamma_H F. \quad (1-11)$$

The next theorem generalizes a result in [Ben-Artzi and Klainerman 1992], where the case when  $|V(x)| \lesssim \langle x \rangle^{-2-\varepsilon}$  was considered.

**Theorem 1.10.** *Assume that  $\mathcal{E} \cap [0, \infty) = \emptyset$ . Then, for any  $\rho > \frac{1}{2}$ ,*

$$\|\langle x \rangle^{-\rho} |D|^{1/2} e^{-itH} P_{\text{ac}}(H) \psi\|_{L_t^2 L_x^2} \leq C_\rho \|\psi\|_{L_x^2},$$

where  $P_{\text{ac}}(H)$  is the projection onto the absolutely continuous subspace associated with  $H$ .

To state the result on Strichartz estimates, we recall some standard notation.

**Definition 1.11.** When  $n \geq 3$ , a pair  $(p, q) \in \mathbb{R}^2$  is said to be admissible if

$$p, q \geq 2, \quad \frac{2}{p} = n \left( \frac{1}{2} - \frac{1}{q} \right). \quad (1-12)$$

**Theorem 1.12.** *Suppose that  $\mathcal{E} \cap [0, \infty) = \emptyset$ . Then, for any admissible pairs  $(p_1, q_1)$  and  $(p_2, q_2)$ , the solution  $u$  to (1-4) satisfies*

$$\|P_{\text{ac}}(H)u\|_{L_t^{p_1} L_x^{q_1}} \lesssim \|\psi\|_{L^2} + \|F\|_{L_t^{p_2'} L_x^{q_2'}}, \quad \psi \in L^2, \quad F \in L_t^{p_2'} L_x^{q_2'}. \quad (1-13)$$

For any

$$\frac{n}{2(n-1)} \leq s \leq \frac{3n-4}{2(n-1)},$$

we also obtain nonadmissible inhomogeneous Strichartz estimates:

$$\|\Gamma_H P_{\text{ac}}(H)F\|_{L_t^2 L_x^{2n/(n-2s)}} \lesssim \|F\|_{L_t^2 L_x^{2n/(n+2(2-s))}}, \quad F \in L_t^2 L_x^{2n/(n+2(2-s))}. \quad (1-14)$$

**Remark 1.13.** For the admissible case or the case when

$$\frac{n}{2(n-1)} < s < \frac{3n-4}{2(n-1)},$$

we can actually obtain stronger estimates than (1-13) and (1-14):

$$\begin{aligned} \|P_{\text{ac}}(H)u\|_{L_t^{p_1} L_x^{q_1,2}} &\lesssim \|\psi\|_{L^2} + \|F\|_{L_t^{p'_2} L_x^{q'_2,2}}, \\ \|\Gamma_H P_{\text{ac}}(H)F\|_{L_t^2 L_x^{2n/(n-2s),2}} &\lesssim \|F\|_{L_t^2 L_x^{2n/(n+2(2-s)),2}}, \quad \frac{n}{2(n-1)} < s < \frac{3n-4}{2(n-1)}, \end{aligned}$$

Inhomogeneous estimates for some other nonadmissible pairs may be also deduced from (1-14) and usual inhomogeneous estimates. For instance, if we interpolate between (1-14) and the trivial estimate  $\|\Gamma_H P_{\text{ac}}(H)F\|_{L_t^\infty L_x^2} \leq \|F\|_{L_t^1 L_x^2}$  then

$$\|\Gamma_H P_{\text{ac}}F\|_{L_t^p L_x^q} \lesssim \|F\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}},$$

where

$$\frac{n}{2(n-1)} \leq s \leq \frac{3n-4}{2(n-1)} \quad \text{and} \quad \frac{n}{s} \left( \frac{1}{2} - \frac{1}{q} \right) = \frac{2}{p} = \frac{2}{\tilde{p}} = \frac{n}{2-s} \left( \frac{1}{2} - \frac{1}{\tilde{q}} \right).$$

Inhomogeneous Strichartz estimates with nonadmissible pairs for the free Schrödinger equation have been studied by several authors [Kato 1994; Keel and Tao 1998; Foschi 2005; Vilela 2007; Koh and Seo 2016] under suitable conditions on  $(p, q)$ ; see [Foschi 2005; Koh and Seo 2016]. The estimates (1-14) correspond to the endpoint cases for such conditions. It is also worth noting that, as well as the estimates for admissible pairs, nonadmissible estimates can be used in the study of nonlinear Schrödinger equations; see [Kato 1994].

**Remark 1.14.** There is a vast literature on Strichartz estimates for Schrödinger equations with potentials. We refer to [Rodnianski and Schlag 2004; Goldberg 2009; Beceanu 2011; Bouclet and Mizutani 2018]. We also note that the dispersive  $L^1$ - $L^\infty$  estimate for  $e^{-itH} P_{\text{ac}}(H)$  and  $L^p$ -boundedness of wave operators  $W_\pm$ , which imply Strichartz estimates, have been also extensively studied; see [Rodnianski and Schlag 2004; Beceanu and Goldberg 2012; Yajima 1995; Beceanu 2014]. In particular, Goldberg [2009] proved the endpoint Strichartz estimates for  $e^{-itH} P_{\text{ac}}$  under the conditions  $V \in L^{n/2}$ ,  $0 \notin \mathcal{E}$  and  $n \geq 3$ . When  $n = 3$ , Strichartz estimates for all admissible cases and some nonadmissible cases (which are different from (1-14)) for  $V \in L_0^{3/2, \infty}$  were obtained in [Beceanu 2011]. Compared with those previous works, a new contribution of this theorem is that we obtain the full set of admissible Strichartz estimates (1-13), including the inhomogeneous double endpoint case for all  $n \geq 3$ . Moreover, nonadmissible estimates (1-14) are new even for  $V \in L^{n/2}$ .

The next application of resolvent estimates in this paper is the  $L^p$ -boundedness of the spectral multiplier  $F(H)$ , which is defined by the spectral decomposition theorem, namely

$$F(H) = \int_{\sigma(H)} F(\lambda) dE_H(\lambda).$$

For the free case  $H = -\Delta$ , Hörmander's multiplier theorem [1960] implies that if  $F \in L^\infty$  satisfies

$$\sup_{t>0} \|\psi(\cdot)F(t\cdot)\|_{\mathcal{H}^\beta} < \infty, \quad (1-15)$$

with some nontrivial  $\psi \in C_0^\infty(\mathbb{R})$  supported in  $(0, \infty)$  and  $\beta > n/2$ , then  $F(-\Delta)$  is bounded on  $L^p$  for all  $1 < p < \infty$ . The following theorem is a generalization of this result to nonnegative Schrödinger operators with scaling-critical potentials.

**Theorem 1.15.** *Suppose that  $\mathcal{E} \cap [0, \infty) = \emptyset$  and  $H \geq 0$ . Then, for any  $F \in L^\infty(\mathbb{R})$  satisfying (1-15) with some nontrivial  $\psi \in C_0^\infty(\mathbb{R})$  supported in  $(0, \infty)$  and  $\beta > \frac{3}{2}$ ,  $F(\sqrt{H})$  is bounded on  $L^p$  for all*

$$\frac{2n}{n+3} < p < \frac{2n}{n-3}$$

and satisfies

$$\|F(\sqrt{H})\|_{\mathbb{B}(L^p)} \leq C \left( \sup_{t>0} \|\psi(\cdot)F(t\cdot)\|_{\mathcal{H}^\beta} + |F(0)| \right). \quad (1-16)$$

It is easy to check that  $F$  satisfies (1-15) if and only if  $G(\lambda) = F(\lambda^2)$  does. Therefore, (1-16) also holds with  $F(\sqrt{H})$  replaced by  $F(H)$ . Also note that, in the proof of this theorem, the restriction estimates (1-10) will play an essential role and the restriction for the range of  $p$  when  $n \geq 4$  is due to the condition  $p > 2n/(n+3)$  for (1-10).

**Remark 1.16.** Some applications of Theorem 1.15 will be also established (see Section 5). First we obtain the equivalence between the Sobolev norms  $\|(-\Delta)^{s/2}u\|_{L^2}$  and  $\|H^{s/2}u\|_{L^2}$  for  $0 \leq s < \frac{3}{2}$ . Then we shall prove square function estimates for the Littlewood–Paley decomposition via the spectral multiplier associated with  $H$ . These are known to play an important role in the study of nonlinear Schrödinger equations with potentials; see, e.g., [Killip, Miao, Visan, Zhang, and Zheng 2018].

**Remark 1.17.** If the Schrödinger semigroup  $e^{-tH}$  satisfies the Gaussian estimate or some generalized Gaussian-type estimates, then Hörmander’s multiplier theorem for  $F(H)$  has been extensively studied; see [Chen, Ouhabaz, Sikora, and Yan 2016]. Compared with such cases, the interest of Theorem 1.15 is that we obtain Hörmander’s multiplier theorem under a scaling-critical condition  $V \in L_0^{n/2,\infty}$ , while it is not known for such a class of potentials whether  $H$  satisfies (generalized) Gaussian estimates or not, even if  $H$  is assumed to be nonnegative.

**Remark 1.18.** To ensure the nonnegativity of  $H$ , it suffices to assume  $\|V_-\|_{L^{n/2,\infty}} \leq S_n^{-1}$ , where  $V_- = \max\{0, -V\}$  is the negative part of  $V$  and

$$S_n := \frac{n(n-2)}{4} 2^{2/n} \pi^{1+1/n} \Gamma\left(\frac{n+1}{2}\right)^{-2/n}$$

is the best constant in Sobolev’s inequality.  $\|f\|_{L^{2n/(n-2)}} \leq S_n \|\nabla f\|_{L^2}$ . Moreover, if  $\|V_-\|_{L^{n/2}} < S_n^{-1}$  then  $0 \notin \mathcal{E}$  by Lemma 1.3.

The last application of Theorem 1.4 in the paper is the Keller-type inequality for individual eigenvalues of a non-self-adjoint Schrödinger operator. Let  $0 < \gamma < \infty$  and  $W \in L^{n/2+\gamma}(\mathbb{R}^n; \mathbb{C})$  be a possibly complex-valued potential. Then  $W$  is  $H$ -form compact and we define the operator  $H_W = H + W$  as a form sum. Under this setting, it is known that  $\sigma(H_W)$  is contained in a sector  $\{z \in \mathbb{C} \mid |\arg(z-z_0)| \leq \theta\}$  for some  $z_0 \in \mathbb{R}$  and  $\theta \in [0, \frac{\pi}{2}]$  (see [Kato 1966]), but the point spectrum  $\sigma_p(H_W)$  could be unbounded in  $\mathbb{C}$  in general even if  $V \equiv 0$  and  $W$  is smooth. The following theorem, however, shows that this is not the case if  $0 < \gamma \leq \frac{1}{2}$ .

**Theorem 1.19.** *Let  $\delta > 0$ . If  $0 < \gamma \leq \frac{1}{2}$ , any eigenvalue  $E \in \mathbb{C} \setminus \mathcal{E}_\delta$  of  $H_W$  satisfies*

$$|E|^\gamma \leq C_{\gamma, \delta} \|W\|_{L^{n/2+\gamma}}^{n/2+\gamma}. \quad (1-17)$$

*Moreover, if  $\gamma > \frac{1}{2}$ , any eigenvalue  $E \in \mathbb{C} \setminus \mathcal{E}_\delta$  of  $H_W$  satisfies*

$$|E|^{1/2} \operatorname{dist}(E, [0, \infty))^{\gamma-1/2} \leq C_{\gamma, \delta} \|W\|_{L^{n/2+\gamma}}^{n/2+\gamma}. \quad (1-18)$$

*Here the constant  $C_{\gamma, \delta} = C(\gamma, \delta, n, V) > 0$  may be taken uniformly in  $W$ .*

**Remark 1.20.** *Theorem 1.19 implies the following spectral consequence. If  $0 < \gamma \leq \frac{1}{2}$  then*

$$\sigma_p(H_W) \subset \mathcal{E}_\delta \cup \{z \in \mathbb{C} \mid |z|^\gamma \leq C_{\gamma, \delta} \|W\|_{L^{n/2+\gamma}}^{n/2+\gamma}\}.$$

In particular, since  $\mathcal{E}$  is bounded in  $\mathbb{R}$  (see Remark 3.4),  $\sigma_p(H_W)$  is bounded in  $\mathbb{C}$ . On the other hand, if  $\gamma > \frac{1}{2}$  and  $\operatorname{Re} E > 0$ , then  $E$  satisfies

$$|\operatorname{Im} E| \leq C_{\gamma, \delta} |E|^{-1/(2(\gamma-1/2))} \|W\|_{L^{n/2+\gamma}}^{(n+2\gamma)/(2\gamma-1)}.$$

This implies that, for any sequence  $\{E_j\} \subset \sigma_p(H_W) \setminus [0, \infty)$  satisfying  $\operatorname{Re} E_j \rightarrow +\infty$  as  $j \rightarrow \infty$ , we have  $|\operatorname{Im} E_j| \rightarrow 0$  as  $j \rightarrow \infty$ .

**Remark 1.21.** For a complex potential  $W(x)$ , the estimates (1-17) and (1-18) were first proved by Frank [2011; 2018] for the case when  $-\Delta + W(x)$  and then extended to the operator  $-\Delta - a|x|^{-2} + W(x)$  with  $a \leq (n-2) - \frac{2}{4}$  by [Mizutani 2019]. In both cases, the free Hamiltonians  $-\Delta$  and  $-\Delta - a|x|^{-2}$  are nonnegative and purely absolutely continuous. Theorem 1.19 shows that the same result still holds even if the free Hamiltonian has (embedded) eigenvalues or resonances.

The rest of the paper is devoted to the proof of above results. We here outline the plan of the paper, describing rough idea of the proofs. Following the classical scheme, the proof of the uniform Sobolev estimates is based on the resolvent identity  $R(z) = (I + R_0(z)V)^{-1}R_0(z)$ .

In Section 2 we collect several properties on the free resolvent  $R_0(z)$  used throughout the paper and, then, study basic properties of the exceptional set  $\mathcal{E}$ . In particular, we show that  $R_0(z)V$  extends to a  $\mathbb{B}_\infty(L^q)$ -valued continuous function on  $\overline{\mathbb{C}^+}$ . This fact plays an important role to justify the above resolvent identity. The proof of Lemma 1.3 is also given in Section 2.

Using materials prepared in Section 2 and the Fredholm alternative theorem, we prove Theorem 1.4, Corollaries 1.5 and 1.6 and Theorem 1.8 in Section 3.

Section 4 is devoted to proving Theorems 1.10 and 1.12. The proof follows an abstract scheme by [Rodnianski and Schlag 2004] (see also [Burq, Planchon, Stalker, and Tahvildar-Zadeh 2004; Bouclet and Mizutani 2018]), which is based on Duhamel's formulas

$$e^{-itH} = e^{it\Delta} - i\Gamma_0 V \Gamma_H, \quad \Gamma_H = \Gamma_0 - i\Gamma_0 V \Gamma_H,$$

where  $\Gamma_0 = \Gamma_{-\Delta}$ . Using these identities, the proof can be reduced to that of corresponding estimates for the free propagators  $e^{it\Delta}$  and  $\Gamma_0$ , which are well known, and  $L_t^2 L_x^2$  estimates for  $V_1 e^{-itH} P_{\operatorname{ac}}(H)$  and

$V_1 \Gamma_H P_{\text{ac}}(H) V_2$  with a suitable decomposition  $V = V_1 V_2$ . Kato's smooth perturbation theory [1966] allows us to deduce such  $L_t^2 L_x^2$ -estimates from the resolvent estimate

$$\sup_{z \in \mathbb{C} \setminus \mathbb{R}} \|V_1 R(z) P_{\text{ac}}(H) V_2\|_{\mathbb{B}(L^2)} < \infty,$$

which follows from uniform Sobolev estimates for  $P_{\text{ac}}(H) R(z)$  (which are also proved as a corollary of [Theorem 1.4](#) in the end of [Section 3](#)) and Hölder's inequality. A rigorous justification of the above Duhamel's formulas in the sense of forms are also given in [Section 4](#).

Proofs of the spectral multiplier theorem and its applications are given in [Section 5](#). The proof of [Theorem 1.15](#) employs an abstract method by [Chen, Ouhabaz, Sikora, and Yan 2016], which allows us to deduce [Theorem 1.15](#) from the restriction estimates (1-10) and the so-called Davies–Gaffney estimate for the Schrödinger semigroup  $e^{-tH}$ . In the proof of the Davies–Gaffney estimate, we use the condition that  $H$  is nonnegative.

[Section 6](#) is devoted to the proof of [Theorem 1.19](#), which follows basically the same line as in [Frank 2011; 2018] and is based on the estimates (1-5), (1-6) and the Birman–Schwinger principle.

The [Appendix](#) is devoted to a brief introduction of real interpolation and Lorentz spaces.

## 2. Preliminaries

In this section we first study several properties of the free resolvent, which will often appear in the sequel. The second part is devoted to a detailed study of the exceptional set of  $H$ .

**2A. The free resolvent.** For  $z \notin \mathbb{C} \setminus [0, \infty)$ ,  $R_0(z) = (-\Delta - z)^{-1}$  denotes the free resolvent, which is defined as a Fourier multiplier with symbol  $(|\xi|^2 - z)^{-1}$ . The integral kernel of  $R_0(z)$  is given by

$$R_0(z, x, y) = \frac{i}{4} \left( \frac{z^{1/2}}{2\pi|x-y|} \right)^{n/2-1} H_{n/2-1}^{(1)}(z^{1/2}|x-y|), \quad \text{Im } z^{1/2} > 0,$$

where  $H_{n/2-1}^{(1)}$  is the Hankel function of the first kind. The pointwise estimate

$$|H_{n/2-1}^{(1)}(w)| \leq C_n \begin{cases} |w|^{-n/2+1} & \text{for } |w| \leq 1, \\ |w|^{-1/2} & \text{for } |w| > 1, \end{cases}$$

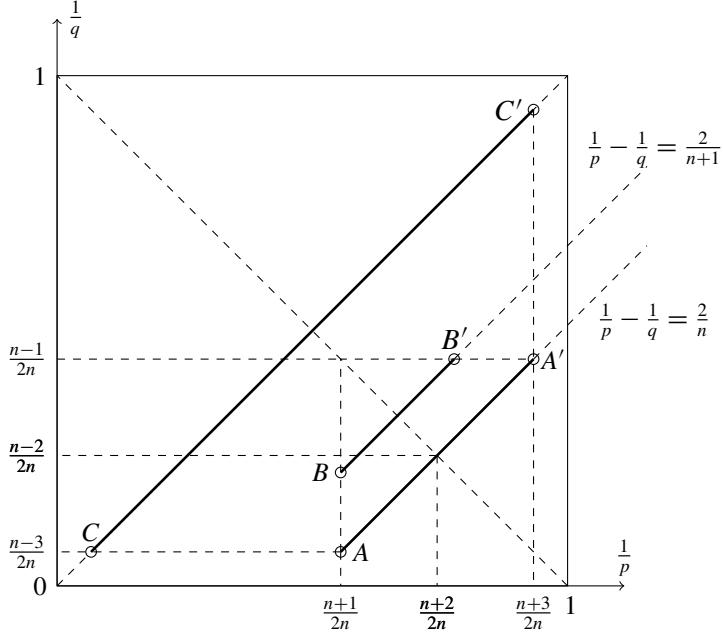
then implies that there exists  $C_n > 0$  depending only on  $n$  such that

$$|R_0(z, x, y)| \leq C_n (|x-y|^{-n+2} + |x-y|^{-(n-1)/2}) \langle z \rangle^{(n-3)/4}, \quad (2-1)$$

see [Jensen 1980]. For  $s \in \mathbb{R}$ , we let  $L_s^2 = L^2(\mathbb{R}^n, \langle x \rangle^{2s} dx)$  and  $\mathcal{H}_s^2 = \{u \mid \partial^\alpha u \in L_s^2, |\alpha| \leq 2\}$ . Then the following limiting absorption principle in weighted  $L^2$ -spaces is well known; see [Agmon 1975; Jensen and Kato 1979; Jensen 1980; 1984].

**Lemma 2.1.** *Let  $s > (n+1)/2$ . Then  $R_0(z)$  is bounded from  $L_s^2$  to  $L_{-s}^2$  uniformly in  $z \in \mathbb{C} \setminus [0, \infty)$ . Moreover, the following statements are satisfied:*

- *Boundary values  $R_0(\lambda \pm i0) = \lim_{\varepsilon \rightarrow 0} R_0(\lambda \pm i\varepsilon) \in \mathbb{B}_\infty(L_s^2, L_{-s}^2)$  exist on  $[0, \infty)$  such that  $R_0(0 \pm i0) = (-\Delta)^{-1}$ . Moreover,  $R_0(\lambda \pm i0) \in \mathbb{B}_\infty(L_s^2, \mathcal{H}_{-s}^2)$  if  $\lambda > 0$ .*



**Figure 1.** The set of  $(1/p, 1/q)$  satisfying (1-1) is the trapezium  $ABB'A'$  with two closed line segments  $\overline{AB}, \overline{B'A'}$  removed. The set of  $(1/p, 1/q)$  satisfying (1-8) is the trapezium  $ACC'A'$  with two closed line segments  $\overline{AC}, \overline{C'A'}$  removed.

- Define the extended free resolvent  $R_0^\pm(z)$  by  $R_0^\pm(z) = R_0(z)$  if  $z \in \mathbb{C} \setminus [0, \infty)$  and  $R_0^\pm(z) = R_0(z \pm i0)$  if  $z \geq 0$ . Then  $R_0^\pm(z)$  are  $\mathbb{B}_\infty(L_s^2, L_{-s}^2)$ -valued continuous functions on  $\overline{\mathbb{C}^\pm}$ .
- For any  $z \in \overline{\mathbb{C}^\pm}$  and  $f \in L_s^2$ , we have  $(-\Delta - z)R_0^\pm(z)f = f$  in the sense of distributions.

The following corollaries are immediate consequences of [Lemma 2.1](#) and [Proposition 1.1](#).

**Corollary 2.2.** Let  $(p, q)$  satisfy (1-1) and

$$\frac{2n}{n+3} < r < \frac{2n}{n+1};$$

see [Figure 1](#). Then:

(1)  $R_0^\pm(z)$  extend to elements in  $\mathbb{B}(L^{p,2}, L^{q,2})$  and satisfy

$$\|R_0^\pm(z)\|_{\mathbb{B}(L^{p,2}, L^{q,2})} \leq C|z|^{(n/2)(1/p-1/q)-1}, \quad z \in \overline{\mathbb{C}^\pm} \setminus \{0\}. \quad (2-2)$$

(2) For any  $f \in L^{p,2}$  and  $g \in L^{q',2}$ , we have  $\langle R_0^\pm(z)f, g \rangle$  are continuous on  $\overline{\mathbb{C}^\pm} \setminus \{0\}$ .  
(3) For any  $z \in \overline{\mathbb{C}^\pm}$  and  $f \in L^{r,2}$ , we have  $(-\Delta - z)R_0^\pm(z)f = f$  in the sense of distributions.

Assuming in addition that  $1/p - 1/q = 2/n$ , the statements (1) and (2) hold for all  $z \in \overline{\mathbb{C}^\pm}$ .

Throughout the paper, we frequently use the notation

$$p_s = \frac{2n}{n+2(2-s)}, \quad q_s = \frac{2n}{n-2s}. \quad (2-3)$$

Note that

$$\left\{ (p_s, q_s) \mid \frac{1}{2} < s < \frac{3}{2} \right\} = \left\{ (p, q) \mid (p, q) \text{ satisfies (1-1) and } \frac{1}{p} - \frac{1}{q} = \frac{2}{n} \right\}.$$

**Corollary 2.3.** *Let  $\frac{1}{2} < s < \frac{3}{2}$ ,  $V_1 \in L_0^{n/s, \infty}$  and  $V_2 \in L_0^{n/(2-s), \infty}(\mathbb{R}^n)$ . Then  $V_1 R_0^\pm(z) V_2$  are  $\mathbb{B}_\infty(L^2)$ -valued continuous functions of  $z \in \overline{\mathbb{C}^\pm}$ .*

*Proof.* Corollary 2.2(1) with  $(p, q) = (p_s, q_s)$  and Hölder's inequality (A-1) imply

$$\sup_{z \in \overline{\mathbb{C}^\pm}} \|V_1 R_0^\pm(z) V_2\|_{\mathbb{B}(L^2)} \lesssim \|V_1\|_{L^{n/s, \infty}} \|V_2\|_{L^{n/(2-s), \infty}}.$$

Since  $C_0^\infty$  is dense in  $L_0^{p, \infty}$  for all  $1 < p < \infty$  and an operator norm limit of compact operators is compact, we observe from this uniform bound and a standard  $\varepsilon/3$  argument that it suffices to show the corollary for  $V_1, V_2 \in C_0^\infty$ . In this case, the corollary follows from Lemma 2.1.  $\square$

The following proposition plays an essential role throughout the paper.

**Proposition 2.4.** *Let  $w \in L_0^{n/2, \infty}(\mathbb{R}^n)$ ,  $\frac{1}{2} < s < \frac{3}{2}$  and  $q_s$  as above. Then  $R_0(z)w \in \mathbb{B}_\infty(\mathcal{H}^1)$  for all  $z \in \mathbb{C} \setminus [0, \infty)$ . Moreover,  $R_0^\pm(z)w$  are  $\mathbb{B}_\infty(L^{q_s, 2})$ -valued continuous functions on  $\overline{\mathbb{C}^\pm}$ .*

**Remark 2.5.**  $R_0^\pm(z)w$  are also  $\mathbb{B}_\infty(L^{q_s})$ -valued continuous functions on  $\overline{\mathbb{C}^\pm}$ . The proof is completely the same.

*Proof.* The facts  $R_0(z)w \in \mathbb{B}(\mathcal{H}^1) \cap \mathbb{B}(L^{q_s, 2})$  and  $R_0^\pm(z)w \in \mathbb{B}(L^{q_s, 2})$  follow from the continuity of  $R_0(z) : \mathcal{H}^{-1} \rightarrow \mathcal{H}^1$ , uniform Sobolev estimates (1-2) and Hölder's inequality for Lorentz norms.

To prove the compactness and the continuity (in  $z$ ), by virtue of these estimates and the same argument as above, we may assume without loss of generality that  $w \in C_0^\infty$  and  $w(x) = 0$  for  $|x| \geq c_0$  with some  $c_0 > 0$ . Then it was proved by [Ionescu and Schlag 2006, Lemma 4.2] that there is a Banach space  $X$  satisfying the continuous embedding  $X \hookrightarrow \mathcal{H}^{-1}$  such that  $w : X^* \rightarrow X$  is compact as a multiplication operator.  $R_0(z)w$  is therefore compact on  $\mathcal{H}^1$  for  $z \in \mathbb{C} \setminus [0, \infty)$ .

Next we shall prove that  $R_0^\pm(z)w$  are compact on  $L^{q_s, 2}$  for  $z \in \overline{\mathbb{C}^\pm}$ . As before, we only consider  $R_0^+(z)$ . By virtue of real interpolation (Theorem A.1), it suffices to show that  $R_0^+(z)w$  is compact on  $L^{q_s}$  for all  $\frac{1}{2} < s < \frac{3}{2}$ . Assume that  $f_j \in L^{q_s}$  and  $\|f_j\|_{L^{q_s}} \leq 1$ . Extracting a subsequence if necessary we may assume  $f_j \rightarrow 0$  weakly in  $L^{q_s}$ . Then it remains to show that there exists a subsequence  $\{\tilde{f}_j\} \subset \{f_j\}$  such that  $R_0^+(z)w\tilde{f}_j \rightarrow 0$  strongly in  $L^{q_s}$ . To this end, we decompose  $R_0^+(z)w$  into two regions  $B_r^c$  and  $B_r$ , where  $B_r = \{x \in \mathbb{R}^n \mid |x| \leq r\}$ . For the former case, the pointwise estimate (2-1) yields

$$|R_0^+(z)wf_j(x)| \leq C_n \langle z \rangle^{(n-3)/4} |x|^{-(n-1)/2} \|wf_j\|_{L^1} \leq C_{n,z} |x|^{-(n-1)/2} \|w\|_{L^{2n/(n+2s)}}$$

uniformly in  $|x| \geq r$ ,  $r \geq 2c_0$  and  $j \geq 0$ . Let us fix  $\varepsilon > 0$  arbitrarily. Since

$$\| |x|^{-(n-1)/2} \|_{L^{q_s}(B_r^c)} \leq C r^{-(s-1/2)},$$

we can find  $r_0 = r_0(n, \varepsilon, z, w) > 0$  such that

$$\|R_0^+(z)wf_j\|_{L^{q_s}(B_{r_0}^c)} < \varepsilon. \tag{2-4}$$

For the latter case, we observe that  $R_0^+(z)w : L^{q_s}(\mathbb{R}^n) \rightarrow \mathcal{W}^{2,q_s}(\mathbb{R}^n)$  is bounded since

$$(-\Delta + 1)R_0^+(z)wf = (-\Delta - z)R_0^+(z)wf + (z + 1)R_0^+(z)wf = wf + (z + 1)R_0^+(z)wf \quad (2-5)$$

for all  $f \in L^{q_s}$  by [Corollary 2.2\(3\)](#). In particular,  $\{R_0^+(z)wf_j\}_j$  is bounded in  $\mathcal{W}^{2,q_s}(B_{r_0})$ . Since  $\mathcal{W}^{2,q_s}(B_{r_0})$  embeds compactly into  $L^{q_s}(B_{r_0})$  by the Rellich–Kondrachov compactness theorem, one can find a subsequence  $\{\tilde{f}_j\} \subset \{f_j\}$  such that

$$\lim_{j \rightarrow \infty} \|R_0^+(z)w\tilde{f}_j\|_{L^{q_s}(B_{r_0})} = 0. \quad (2-6)$$

It follows from [\(2-4\)](#) and [\(2-6\)](#) that

$$\limsup_{j \rightarrow \infty} \|R_0^+(z)w\tilde{f}_j\|_{L^{q_s}(\mathbb{R}^n)} \leq \varepsilon.$$

By extracting further a subsequence, we conclude that  $R_0^+(z)w\tilde{f}_j \rightarrow 0$  strongly in  $L^{q_s}$ .

To prove the continuity, let us fix a bounded set  $\Lambda \subset \overline{\mathbb{C}^+}$  arbitrarily. We first show that, for any  $z, z_j \in \Lambda$  and  $g, g_j \in L^{q_s,2}$  satisfying  $z_j \rightarrow z$  and  $g_j \rightarrow g$  weakly in  $L^{q_s,2}$  as  $j \rightarrow \infty$ ,

$$R_0^+(z_j)wg_j \rightarrow R_0^+(z)wg \quad \text{strongly in } L^{q_s,2} \text{ as } j \rightarrow \infty. \quad (2-7)$$

To this end, we write

$$R_0^+(z_j)wg_j - R_0^+(z)wg = (R_0^+(z_j)w - R_0^+(z)w)g_j + R_0^+(z)w(g_j - g).$$

The second term  $R_0^+(z)w(g_j - g)$  converges to 0 strongly in  $L^{q_s,2}$  since  $R_0^+(z)w$  is compact on  $L^{q_s,2}$  and  $g_j \rightarrow g$  weakly. For the first part, we set  $h_j = (R_0^+(z_j)w - R_0^+(z)w)g_j$  and shall show that  $h_j \rightarrow 0$  strongly in  $L^{q_s,2}$ . Since  $\{g_j\} \subset L^{q_s,2}$  is bounded, say  $\|g_j\|_{L^{q_s,2}} \leq M$  with  $M > 0$  being independent of  $j$ , we have by the same argument as above that, with some  $\gamma_j = \gamma_j(s, n) > 0$ ,

$$\|R_0^+(\zeta)wg_j\|_{L^{q_s,2}(B_r^c)} \leq C_{n,M,w} \langle \zeta \rangle^{\gamma_1} r^{-\gamma_2}$$

for all  $\zeta \in \overline{\mathbb{C}^+}$ ,  $j \geq 1$  and  $r \geq 2c_0$ , where  $C_{n,M,w}$  may be taken uniformly in  $j$  and  $r$ . This estimate yields that, for any  $\varepsilon > 0$ , there exists  $0 < r_\varepsilon = r(n, M, w, \Lambda, \varepsilon) \sim \varepsilon^{-1/\gamma_2}$  such that

$$\sup_{j \geq 1} \|h_j\|_{L^{q_s,2}(B_{r_\varepsilon}^c)} \leq \sup_{j \geq 1} (\|R_0^+(z_j)wg_j\|_{L^{q_s,2}(B_{r_\varepsilon}^c)} + \|R_0^+(z)wg_j\|_{L^{q_s,2}(B_{r_\varepsilon}^c)}) < \varepsilon. \quad (2-8)$$

On the other hand, it follows from Sobolev's embedding on  $\mathbb{R}^n$  that

$$\|h_j\|_{L^{q_s,2}(B_{r_\varepsilon})} \leq C_{\varepsilon,N} \|(-\Delta + 1)\langle x \rangle^{-N} h_j\|_{L^2(\mathbb{R}^n)} \leq C_{\varepsilon,N} \|\langle x \rangle^{-N} (-\Delta + 1)h_j\|_{L^2(\mathbb{R}^n)}$$

for all  $N \geq 0$ , where we have used the fact that  $(-\Delta + 1)\langle x \rangle^{-N}(-\Delta + 1)^{-1}\langle x \rangle^N$  is a pseudodifferential operator of order 0 and thus bounded on  $L^p$  for all  $1 < p < \infty$ . Equation [\(2-5\)](#) then yields

$$\begin{aligned} \|\langle x \rangle^{-N} (-\Delta + 1)h_j\|_{L^2} &\leq |z - z_j| \|\langle x \rangle^{-N} R_0^+(z_j) \langle x \rangle^{-N}\|_{\mathbb{B}(L^2)} \|\langle x \rangle^N w g_j\|_{L^2} \\ &\quad + (|z| + 1) \|\langle x \rangle^{-N} (R_0^+(z_j) - R_0^+(z)) \langle x \rangle^{-N}\|_{\mathbb{B}(L^2)} \|\langle x \rangle^N w g_j\|_{L^2}. \end{aligned}$$

Let  $N \geq (n+1)/2$ . Since  $\langle x \rangle^{-N} R_0^+(z) \langle x \rangle^{-N}$  is bounded on  $L^2$  uniformly in  $z \in \overline{\mathbb{C}^+}$  and continuous on  $\overline{\mathbb{C}^+}$  in the operator norm topology of  $\mathbb{B}(L^2)$  by [Lemma 2.1](#) and

$$\|\langle x \rangle^N w g_j\|_{L^2} \leq CM \|\langle x \rangle^N w\|_{L^{2n/(n+2s),2}} \leq C_{N,M,\omega}$$

uniformly in  $j$ , we see that  $\lim_{j \rightarrow \infty} \|\langle x \rangle^{-N} (-\Delta + 1) h_j\|_{L^2} = 0$ , which, together with [\(2-8\)](#), shows that there exists  $j_\varepsilon \in \mathbb{N}$  such that, for all  $j \geq j_\varepsilon$ , we have  $\|h_j\|_{L^{q_s,2}(\mathbb{R}^n)} < \varepsilon$ . Since  $\varepsilon > 0$  is arbitrarily small, this shows that  $h_j \rightarrow 0$  strongly in  $L^{q_s,2}$  and [\(2-7\)](#) follows.

Finally, we shall show  $R_0^+(z)w$  is continuous on  $\overline{\mathbb{C}^+}$  in the operator norm topology of  $\mathbb{B}(L^{q_s,2})$ . Assume for contradiction that this is not the case. Then there exist  $z_j, z \in \overline{\mathbb{C}^+}$  with  $z_j \rightarrow z$  and  $g_j \in L^{q_s,2}$  with  $\|g_j\|_{L^{q_s,2}} \leq 1$  such that  $\liminf_{j \rightarrow \infty} \|(R_0^+(z_j)w - R_0^+(z)w)g_j\|_{L^{q_s,2}} > 0$ . Extracting a subsequence if necessary we may assume  $g_j \rightarrow g$  with some  $g \in L^{q_s}$  weakly in  $L^{q_s}$ . Then, by the argument as above and the compactness of  $R_0^+(z)w$ , we have  $\lim_{j \rightarrow \infty} R_0^+(z_j)w g_j = R_0^+(z)w g = \lim_{j \rightarrow \infty} R_0^+(z)w g_j$ , which gives a contradiction, proving the desired assertion.  $\square$

**2B. The exceptional set.** Having [Proposition 2.4](#) in mind, we define the exceptional set of  $H$  as follows.

**Definition 2.6.** We say that  $\lambda \in \mathcal{E}$  if there exist  $\frac{1}{2} < s < \frac{3}{2}$  and  $f \in L^{q_s,2}(\mathbb{R}^n) \setminus \{0\}$  such that  $f = -R_0(\lambda)Vf$ , where  $q_s = 2n/(n-2s)$  and  $R_0(\lambda)$  is replaced by  $R_0(\lambda+i0)$  if  $\lambda \geq 0$ .  $\mathcal{E}$  is said to be the *exceptional set* of  $H$ , and  $z \in \mathcal{E} \setminus \sigma_p(H)$  is called a *resonance* of  $H$ . For  $\lambda \in \mathcal{E}$ , we denote the family of corresponding solutions by  $\mathcal{N}_s(\lambda)$ :

$$\mathcal{N}_s(\lambda) := \{f \in L^{q_s,2}(\mathbb{R}^n) \setminus \{0\} \mid f = -R_0(\lambda)Vf\},$$

where  $R_0(\lambda)$  is replaced by  $R_0^+(\lambda)$  if  $\lambda \geq 0$ .

Note that, since  $R_0(\lambda-i0)f = \overline{R_0(\lambda+i0)\bar{f}}$ , one has

$$\mathcal{N}_s(\lambda) = \{f \in L^{q_s,2}(\mathbb{R}^n) \setminus \{0\} \mid f = -R_0^-(\lambda)Vf\}, \quad \lambda \geq 0. \quad (2-9)$$

The next lemma collects some basic properties of  $\mathcal{E}$ .

**Proposition 2.7.** (1)  $\mathcal{E} \subset \sigma(H)$ ,  $\sigma_p(H) \subset \mathcal{E}$  and  $\mathcal{E} \cap (-\infty, 0) = \sigma_d(H)$ . Moreover,  $\mathcal{N}_s(\lambda)$  is finite-dimensional.

(2)  $\mathcal{N}_s(\lambda)$  is independent of  $\frac{1}{2} < s < \frac{3}{2}$ ; that is,  $\mathcal{N}_s(\lambda) = \mathcal{N}_{s'}(\lambda)$  for any  $\frac{1}{2} < s, s' < \frac{3}{2}$ .

*Proof of Proposition 2.7(1).* To prove  $\mathcal{E} \subset \sigma(H)$ , we first claim that

$$\mathcal{N}_s(\lambda) = \{f \in \dot{\mathcal{H}}^s \mid f = -R_0(\lambda)Vf\}, \quad \lambda \in \mathbb{C} \setminus (0, \infty). \quad (2-10)$$

Indeed, if we set  $\tilde{\mathcal{N}}_s(\lambda) := \{f \in \dot{\mathcal{H}}^s \mid f = -R_0(\lambda)Vf\}$  then the inclusion  $\tilde{\mathcal{N}}_s(\lambda) \subset \mathcal{N}_s(\lambda)$  is obvious since  $\dot{\mathcal{H}}^s \subset L^{q_s,2}$  by the HLS inequality [\(A-2\)](#). On the other hand, the HLS inequality [\(A-2\)](#) shows that  $R_0(\lambda)V \in \mathbb{B}(L^{q_s,2}, \dot{\mathcal{H}}^s)$  for  $\lambda \in \mathbb{C} \setminus (0, \infty)$  and the opposite inclusion  $\tilde{\mathcal{N}}_s(\lambda) \supset \mathcal{N}_s(\lambda)$  thus holds. Next, we let  $f \in \mathcal{N}_s(\lambda)$  with some  $\lambda \in \mathbb{C} \setminus \sigma(H)$ . Then  $Vf \in \dot{\mathcal{H}}^{2-s} \cap L^{p_s,2}$  by the HLS and Hölder's inequalities for Lorentz norms. Therefore, by [Corollary 2.2\(3\)](#),  $(-\Delta - \lambda)f = -Vf$  holds in the distribution sense. In particular,  $\lambda f = (-\Delta + V)f \in \dot{\mathcal{H}}^{2-s} \cap \dot{\mathcal{H}}^s \subset L^2$  and thus  $f \in D(H)$ . Since  $\sigma(H) \subset \mathbb{R}$ , this shows  $f \equiv 0$ . Therefore, we obtain  $\mathcal{E} \subset \sigma(H)$ .

The inclusion  $\sigma_p(H) \subset \mathcal{E}$  is obvious since  $D(H) \subset \mathcal{H}^1 \subset \dot{\mathcal{H}}^1$ . This inclusion, together with the fact  $\sigma(H) \cap (-\infty, 0) = \sigma_d(H)$ , implies  $\mathcal{E} \cap (-\infty, 0) = \sigma_d(H)$ . Finally, since  $R_0^\pm(z)V$  are compact operators on  $L^{q_s,2}$ , one has  $\dim \mathcal{N}_s(\lambda) < \infty$ .  $\square$

To prove the second part of [Proposition 2.7](#), we need the following:

**Lemma 2.8.** *For  $\frac{1}{2} < s < \frac{3}{2}$  and real-valued functions  $V_1 \in L_0^{n/s,\infty}$ ,  $V_2 \in L_0^{n/(2-s),\infty}$  with  $V = V_1 V_2$ , we set  $K_s^+(\lambda) := V_1 R^+(\lambda) V_2$ . Then, for  $\lambda \in \mathbb{R}$ ,*

$$\dim \mathcal{N}_s(\lambda) = \dim \text{Ker}(I + K_s^+(\lambda)) = \dim \text{Ker}(I + K_s^+(\lambda)^*) = \dim \mathcal{N}_{2-s}(\lambda).$$

**Remark 2.9.** Such  $V_1, V_2$  always exist. Indeed, one can take  $V_1 = |V|^{s/2}$  and  $V_2 = \text{sgn } V |V|^{(2-s)/2}$ .

*Proof.* Hölder's inequality [\(A-1\)](#) and [\(2-2\)](#) yield that

$$\|V_1 f\|_{L^2} \leq C \|V\|_{L^{n/s,\infty}} \|f\|_{L^{q_s,2}}, \quad \|R_0^\pm(\lambda) V_2 u\|_{L^{q_s,2}} \lesssim \|V_2\|_{L^{n/(2-s),\infty}} \|u\|_{L^2},$$

from which one has two continuous maps

$$\mathcal{N}_s(\lambda) \ni f \mapsto V_1 f \in \text{Ker}(I + K_s^+(\lambda)), \quad \text{Ker}(I + K_s^+(\lambda)) \ni u \mapsto -R_0^+(\lambda) V_2 u \in \mathcal{N}_s(\lambda).$$

Furthermore, one also has, for  $f \in \mathcal{N}_s(\lambda)$  and  $u \in \text{Ker}(I + K_s^+(\lambda))$ ,

$$-R_0^+(\lambda) V_2 V_1 f = -R_0^+(\lambda) V f = f, \quad -V_1 R_0^+(\lambda) V_2 u = u.$$

Therefore, the multiplication by  $V_1$  is a bijection between  $\mathcal{N}_s(\lambda)$  and  $\text{Ker}(I + K_s^+(\lambda))$  and its inverse is given by  $-R_0^+(\lambda) V_2$ . In particular,  $\dim \text{Ker}(I + K_s^+(\lambda)) = \dim \mathcal{N}_s(\lambda)$ .

Taking the facts  $R_0^\pm(z)^* = R_0^\mp(\bar{z})$  and [\(2-9\)](#) into account, it can be seen from the same argument that the multiplication by  $V_2$  is a bijection between  $\mathcal{N}_{2-s}(\lambda)$  and  $\text{Ker}(I + K_s^+(\lambda)^*)$ , and its inverse is given by  $-R_0^-(\lambda) V_1$ . In particular,  $\dim \mathcal{N}_{2-s}(\lambda) = \dim \text{Ker}(I + K_s^+(\lambda)^*)$ .

For the part  $\dim \text{Ker}(I + K_s^+(\lambda)) = \dim \text{Ker}(I + K_s^+(\lambda)^*)$ , since  $K_s^+(\lambda)$  is compact on  $L^2$  (see [Corollary 2.3](#)),  $I + K_s^+(\lambda)$  is Fredholm and its index satisfies

$$\dim \text{Ker}(I + K_s^+(\lambda)) - \text{codim } \text{Ran}(I + K_s^+(\lambda)) = \text{ind}(I + K_s^+(\lambda)) = \text{ind } I = 0.$$

Therefore, taking the fact  $L^2 / \text{Ran}(I + K_s^+(\lambda)) \cong [\text{Ran}(I + K_s^+(\lambda))]^\perp$  into account, one has

$$\dim \text{Ker}(I + K_s^+(\lambda)) = \dim [\text{Ran}(I + K_s^+(\lambda))]^\perp = \dim \text{Ker}(I + K_s^+(\lambda)^*),$$

which completes the proof.  $\square$

*Proof of [Proposition 2.7](#)(2).* Let  $f \in \mathcal{N}_s(\lambda)$  and  $\frac{1}{2} < s \leq s' < \frac{3}{2}$ . Let  $V = v_1 + v_2$  be such that  $v_1 \in C_0^\infty$  and  $\|v_2\|_{L^{n/2,\infty}} \leq \varepsilon$ . Then  $f = -R_0^+(\lambda) v_1 f - R_0^+(\lambda) v_2 f$ . By [Proposition 2.4](#), the map  $I + R_0^+(\lambda) v_2 : L^{2n/(n-2r),2} \rightarrow L^{2n/(n-2r),2}$  is bounded and invertible for  $r = s, s'$  and small  $\varepsilon > 0$ . If  $E_r$  denotes the inverse of  $I + R_0^+(\lambda) v_2 : L^{2n/(n-2r),2} \rightarrow L^{2n/(n-2r),2}$ , then  $E_s = E_{s'}$  on  $L^{2n/(n-2s),2} \cap L^{2n/(n-2s'),2}$ . Taking the inequality  $s - s' > -1$  into account, the HLS inequality [\(A-2\)](#) implies

$$\|R_0^+(\lambda) v_1 f\|_{L^{2n/(n-2s'),2}} \lesssim \|v_1 f\|_{L^{2n/(n+2(2-s'))}} \lesssim \|v_1\|_{L^{n/(2+2(s-s'))}} \|f\|_{L^{2n/(n-2s)}}.$$

Thus  $R_0^+(\lambda)v_1f \in L^{2n/(n-2s),2} \cap L^{2n/(n-2s'),2}$  and  $f = E_s R_0^+(\lambda)v_1f = E_{s'} R_0^+(\lambda)v_1f \in L^{2n/(n-2s'),2}$ , which implies  $f \in \mathcal{N}_{s'}(\lambda)$ . Therefore  $\mathcal{N}_s(\lambda)$  is monotonically increasing in  $s$ . Combined with the fact  $\dim \mathcal{N}_s(\lambda) = \dim \mathcal{N}_{2-s}(\lambda) < \infty$  (see [Lemma 2.8](#)), this monotonicity implies  $\mathcal{N}_s(\lambda) = \mathcal{N}_{s'}(\lambda)$ .  $\square$

We conclude this subsection to prove [Lemma 1.3](#). For the first part, we employ the following results of [\[Ionescu and Jerison 2003; Ionescu and Schlag 2006\]](#).

**Proposition 2.10** [\[Ionescu and Jerison 2003, Theorem 2.1\]](#). *Let  $n \geq 3$  and  $V \in L^{n/2}$ . Suppose that  $f \in \mathcal{H}_{\text{loc}}^1$  and  $\langle x \rangle^{-1/2+\delta} f \in L^2$  with some  $\delta > 0$ . If  $-\Delta f + Vf = \lambda f$  for some  $\lambda > 0$ , then  $f \equiv 0$ .*

Let us set  $X = \mathcal{W}^{-1/(n+1), 2(n+1)/(n+3)} + S_1(B)$ , where  $B$  is the Agmon–Hörmander space and  $S_1(B)$  is the image of  $B$  under  $S_1 = (1 - \Delta)^{1/2}$ ; see [\[Ionescu and Schlag 2006\]](#). Then

$$X^* = \mathcal{W}^{1/(n+1), 2(n+1)/(n-1)} \cap S_{-1}(B^*)$$

and we have the continuous embeddings  $L^{2n/(n+2)} \subset X$  and  $X^* \subset L^{2n/(n-2)}$ . Moreover, it was proved in [\[Ionescu and Schlag 2006, Lemma 4.1\(b\)\]](#) that  $R_0^\pm(\lambda) \in \mathbb{B}(X, X^*)$  for all  $\lambda \in \mathbb{R} \setminus \{0\}$ .

**Proposition 2.11** [\[Ionescu and Schlag 2006, Lemma 4.4\]](#). *Let  $n \geq 3$  and  $V \in L^{n/2}$ . Assume that  $f$  belongs to  $X^*$  and satisfies  $f + R_0^\pm(\lambda)Vf = 0$  for some  $\lambda \in \mathbb{R} \setminus \{0\}$ . Then, for any  $N \geq 0$ ,*

$$\|\langle x \rangle^N f\|_{X^*} \leq C_{N,\lambda} \|f\|_{X^*}.$$

*Proof of Lemma 1.3.* For the proof of the part (1), we let  $f \in \mathcal{N}_1(\lambda)$  with  $\lambda > 0$ . As observed in the proof of [Proposition 2.4](#),  $R_0^+(\lambda)V$  maps from  $L^{2n/(n-2)}(\mathbb{R}^n)$  into  $\mathcal{W}^{2, 2n/(n-2)}(\mathbb{R}^n)$  (see [\(2-5\)](#)) and thus  $f = -R_0^+(\lambda)Vf \in \mathcal{H}_{\text{loc}}^1$ . Moreover, since  $Vf \in L^{2n/(n+2)} \subset X$  and  $R_0^\pm(\lambda) \in \mathbb{B}(X, X^*)$ , we have  $f \in X^*$ . [Proposition 2.11](#) then implies that  $f \in L^2$ . Using [Proposition 2.10](#), we conclude that  $f \equiv 0$ . For part (2), we let  $f \in \mathcal{N}_1(0)$ . Since  $-\Delta f + Vf \in \mathcal{H}^{-1}$ , the form  $\langle -\Delta f + Vf, f \rangle$  is well-defined. By assumption, we have  $0 = \langle -\Delta f + Vf, f \rangle \geq \delta \|f\|_{\mathcal{H}^1}$ , which implies  $f \equiv 0$ .  $\square$

### 3. Uniform Sobolev estimates

This section is devoted to the proof of [Theorem 1.4](#), [Corollaries 1.5](#) and [1.6](#) and [Theorem 1.8](#). We begin with the following proposition which plays an important role in the proof.

**Proposition 3.1.** *Assume  $\frac{1}{2} < s < \frac{3}{2}$  and let  $(p_s, q_s)$  be as in [\(2-3\)](#). Then  $(I + R_0^\pm(z)V)^{-1}$  are  $\mathbb{B}(L^{q_s,2})$ -valued continuous functions on  $\overline{\mathbb{C}^\pm} \setminus \mathcal{E}$ , respectively. Furthermore, for any  $\delta > 0$ ,*

$$\sup_{z \in \overline{\mathbb{C}^\pm} \setminus \mathcal{E}_\delta} \|(I + R_0^\pm(z)V)^{-1}\|_{\mathbb{B}(L^{q_s,2})} < \infty. \quad (3-1)$$

*In particular, if  $\mathcal{E} \cap [0, \infty) = \emptyset$ , then  $\sup_{z \in \mathbb{C} \setminus \mathbb{R}} \|(I + R_0(z)V)^{-1}\|_{\mathbb{B}(L^{q_s,2})} < \infty$ .*

The proof of [Proposition 3.1](#) is divided into a series of lemmas. Let us prove the proposition for  $z \in \overline{\mathbb{C}^+} \setminus \mathcal{E}$  only, as the proof for the case  $z \in \overline{\mathbb{C}^-} \setminus \mathcal{E}$  is analogous.

**Lemma 3.2.**  *$(I + R_0^+(z)V)^{-1}$  is a  $\mathbb{B}(L^{q_s,2})$ -valued continuous function on  $\overline{\mathbb{C}^+} \setminus \mathcal{E}$ .*

*Proof.* By [Proposition 2.4](#),  $R_0^+(z)V$  is compact. Since  $\mathcal{N}_s(z) = \{0\}$  for  $z \in \overline{\mathbb{C}^+} \setminus \mathcal{E}$  by definition, the Fredholm alternative ensures the existence of  $(I + R_0^+(z)V)^{-1} \in \mathbb{B}(L^{q_s,2})$ . Moreover, since  $R_0^+(z)V$  is continuous on  $\overline{\mathbb{C}^+}$  in the operator norm topology of  $\mathbb{B}(L^{q_s,2})$  by [Proposition 2.4](#),  $(I + R_0^+(z)V)^{-1}$  is also continuous on  $\overline{\mathbb{C}^+} \setminus \mathcal{E}$  in the same topology.  $\square$

The proof of the uniform bound (3-1) is divided into high-, intermediate- and low-energy parts.

**Lemma 3.3** (the high-energy estimate). *There exists  $L \geq 1$  such that  $(I + R_0^+(z)V)^{-1}$  is bounded on  $L^{2n/(n-2s),2}$  uniformly in  $z \in \overline{\mathbb{C}^+} \cap \{|z| \geq L\}$ .*

*Proof.* Let  $V_k \in C_0^\infty(\mathbb{R}^n)$  be such that  $\lim_{k \rightarrow \infty} \|V - V_k\|_{L^{n/2,\infty}} = 0$  and set  $Q_k^+(z) := R_0^+(z)(V - V_k)$ . By [Corollary 2.2](#) with  $(p_s, q_s)$ , one can find  $k_0 \geq 1$  such that

$$\sup_{z \in \overline{\mathbb{C}^+}} \|Q_{k_0}^+(z)\|_{\mathbb{B}(L^{q_s,2})} \leq \frac{1}{2}.$$

Hence  $(I + Q_{k_0}(z))^{-1}$  is defined by the Neumann series  $\sum_{n=0}^{\infty} (-Q_{k_0}^+(z))^n$  and satisfies

$$M_1 := \sup_{z \in \overline{\mathbb{C}^+}} \|(I + Q_{k_0}^+(z))^{-1}\|_{\mathbb{B}(L^{q_s,2})} \leq 2.$$

Next if we take  $p_\delta$  and small  $\delta > 0$  such that  $1/p_\delta = 1/p_s - \delta$  and  $(p_\delta, q_s)$  satisfies (1-1), [Corollary 2.2](#) implies

$$\|R_0^+(z)V_{k_0}f\|_{L^{q_s,2}} \lesssim |z|^{-\delta} \|V_{k_0}f\|_{L^{p_\delta,2}} \lesssim |z|^{-\delta} \|V_{k_0}\|_{L^r} \|f\|_{L^{q_s,2}}$$

uniformly in  $|z| \geq 1$  and  $f \in L^{q_s,2}$ , where  $1/r = 1/p_\delta - 1/q_s = 2/n - \delta$ . Hence one can find  $L = L_{k_0}$  so large that  $M_2 := \|R_0^+(z)V_{k_0}\|_{\mathbb{B}(L^{q_s,2})} \leq \frac{1}{4}$  for  $|z| \geq L$ . Then, writing

$$I + R_0^+(z)V = I + Q_{k_0}^+(z) + R_0^+(z)V_{k_0} = (I + Q_{k_0}^+(z))(I + (I + Q_{k_0}^+(z))^{-1}R_0^+(z)V_{k_0}),$$

we see that  $(I + R_0^+(z)V)^{-1} = (I + (I + Q_{k_0}^+(z))^{-1}R_0^+(z)V_{k_0})^{-1}(I + Q_{k_0}^+(z))^{-1}$  and

$$\sup_{z \in \overline{\mathbb{C}^+} \cap \{|z| \geq L\}} \|(I + R_0^+(z)V)^{-1}\|_{\mathbb{B}(L^{q_s,2})} \leq M_1 \sum_{n=1}^{\infty} (M_1 M_2)^n \leq 4. \quad \square$$

**Remark 3.4.** This lemma particularly implies  $\mathcal{E} \cap [L, \infty) = \emptyset$  and thus  $\mathcal{E}$  is bounded in  $\mathbb{R}$ .

**Lemma 3.5** (the intermediate-energy estimate). *For any  $\delta, L > 0$ , the function  $(I + R_0^+(z)V)^{-1}$  is bounded on  $L^{q_s,2}$  uniformly in  $z \in (\overline{\mathbb{C}^+} \setminus \mathcal{E}_\delta) \cap \{\delta < |z| < L\}$ .*

*Proof.* We follow the argument in [\[Ionescu and Schlag 2006, Lemma 4.6\]](#) closely. Let

$$\Lambda_{\delta,L} = (\overline{\mathbb{C}^+} \setminus \mathcal{E}_\delta) \cap \{\delta < |z| < L\}.$$

Note that  $\overline{\Lambda_{\delta,L}} \cap \mathcal{E} = \emptyset$ . Assume for contradiction that

$$\sup_{z \in \Lambda_{\delta,L}} \|(I + R_0^+(z)V)^{-1}\|_{\mathbb{B}(L^{q_s,2})} = \infty.$$

Then one can find  $f_j \in L^{q_s,2}$  with  $\|f_j\|_{L^{q_s,2}} = 1$  and  $z_j \in \Lambda_{\delta,L}$  such that

$$\|(I + R_0^+(z_j)V)f_j\|_{\mathbb{B}(L^{q_s,2})} \rightarrow 0, \quad j \rightarrow \infty. \quad (3-2)$$

By passing to a subsequence, we may assume  $z_j \rightarrow z_\infty \in \overline{\Lambda_{\delta,L}}$  as  $j \rightarrow \infty$ . Since  $R_0^+(z_\infty)V$  is compact on  $L^{q_s,2}$ , by passing to a subsequence, we may assume without loss of generality that there exists  $g \in L^{q_s,2}$  such that  $R_0^+(z_\infty)Vf_j \rightarrow g$  strongly in  $L^{q_s,2}$ . By virtue of (3-2) and the condition  $\|f_j\|_{L^{q_s,2}} = 1$ , we have  $g \not\equiv 0$ . Now we claim that  $g$  belongs to  $\mathcal{N}_s(z_\infty)$ , which implies  $z_\infty \in \mathcal{E}$ . This contradicts  $z_\infty \in \overline{\Lambda_{\delta,L}}$ .

In order to prove the claim, we write  $f_j$  as

$$f_j = (I + R_0^+(z_j)V)f_j - (R_0^+(z_j) - R_0^+(z_\infty))Vf_j - R_0^+(z_\infty)Vf_j.$$

By virtue of (3-2) and the continuity of  $R_0^+(z)V$  (see [Proposition 2.4](#)) and the fact  $\|f_j\|_{L^{q_s,2}} = 1$ , the right-hand side converges to  $-g$  strongly in  $L^{q_s,2}$  as  $j \rightarrow \infty$ . Therefore, we have  $g = -R_0^+(z_\infty)Vg$ . Moreover, since  $\|f_j\| = 1$ , we have  $g \not\equiv 0$  and hence  $g \in \mathcal{N}_s(z_\infty)$  follows.  $\square$

[Lemmas 3.3 and 3.5](#) give the desired bound (3-1) for the case when  $0 \in \mathcal{E}$ . When  $0 \notin \mathcal{E}$ , we need the following lemma to complete the proof of [Proposition 3.1](#).

**Lemma 3.6** (the low-energy estimate). *Suppose that  $0 \notin \mathcal{E}$ . Then there exists  $\delta > 0$  such that the function  $(I + R_0^+(z)V)^{-1}$  is bounded on  $L^{q_s,2}$  uniformly in  $z \in \overline{\mathbb{C}^+} \cap \{|z| \leq \delta\}$ .*

*Proof.* Since  $I + R_0^+(0)V$  is invertible if  $0 \notin \mathcal{E}$  by [Lemma 3.2](#), one can write

$$I + R_0^+(z)V = (I + R_0^+(0)V)(I + (I + R_0^+(0)V)^{-1}(R_0^+(z) - R_0^+(0))V).$$

Since  $\overline{\mathbb{C}^+} \ni z \mapsto R_0^+(z)V \in \mathbb{B}(L^{q_s,2})$  is continuous by [Proposition 2.4](#), one has

$$\sup_{z \in \overline{\mathbb{C}^+} \cap \{|z| \leq \delta\}} \|(R_0^+(z) - R_0^+(0))V\|_{\mathbb{B}(L^{q_s,2})} \leq \frac{1}{2\|(I + R_0^+(0)V)^{-1}\|}$$

for  $\delta > 0$  small enough. Therefore,  $I + R_0^+(z)V$  is invertible on  $L^{q_s,2}$  and

$$\sup_{z \in \overline{\mathbb{C}^+} \cap \{|z| \leq \delta\}} \|(I + R_0^+(z)V)^{-1}\|_{\mathbb{B}(L^{q_s,2})} \leq 2 \sup_{z \in \overline{\mathbb{C}^+} \cap \{|z| \leq \delta\}} \|(I + R_0^+(0)V)^{-1}\|_{\mathbb{B}(L^{q_s,2})} < \infty,$$

which completes the proof.  $\square$

By [Lemmas 3.2–3.5](#), we have completed the proof of [Proposition 3.1](#).

We next give a rigorous justification of the second resolvent equation.

**Lemma 3.7.** *Let  $z \in \mathbb{C} \setminus \sigma(H)$ . Then, as a bounded operator from  $L^2$  to  $D(H)$ ,*

$$R(z) = (I + R_0(z)V)^{-1}R_0(z) = R_0(z) - R_0(z)VR(z). \quad (3-3)$$

Moreover, we also obtain for  $z, z' \in \mathbb{C} \setminus \sigma(H)$ ,

$$R(z) - R(z') = (I + R_0(z')V)^{-1}(R_0(z) - R_0(z'))(I - VR(z)). \quad (3-4)$$

*Proof.* It follows from [Proposition 2.7](#)(1) and the fact  $\mathcal{H}^1 \subset L^{2n/(n-2),2}$  that  $\text{Ker}_{\mathcal{H}^1}(I + R_0(z)V)$  is trivial. Since  $R_0(z)V \in \mathbb{B}_\infty(\mathcal{H}^1)$  by [Proposition 2.4](#),  $I + R_0(z)V$  is invertible on  $\mathcal{H}^1$  by the Fredholm alternative theorem. Thus  $(I + R_0(z)V)^{-1}R_0(z)$  is a bounded operator from  $L^2$  to  $\mathcal{H}^1$ . Let  $f \in L^2$  and set  $g = (I + R_0(z)V)^{-1}R_0(z)f \in \mathcal{H}^1$ . Since

$$(I + R_0(z)V)(I + R_0(z)V)^{-1}R_0(z) = R_0(z)$$

as a bounded operator from  $L^2$  to  $\mathcal{H}^1$ , we see that

$$g = R_0(z)f - R_0(z)Vg. \quad (3-5)$$

Then, for any  $\varphi \in \mathcal{H}^1$ ,

$$\langle(-\Delta - z)g, \varphi\rangle = \langle f, \varphi\rangle - \langle Vg, \varphi\rangle = \langle f, \varphi\rangle - \langle V_1g, V_2\varphi\rangle,$$

where  $V_1, V_2 \in L_0^{n/2,\infty}(\mathbb{R}^n; \mathbb{R})$  satisfies  $V = V_1V_2$ . Therefore, we obtain

$$\langle(H - z)g, \varphi\rangle = \langle(-\Delta - z)g, \varphi\rangle + \langle V_1g, V_2\varphi\rangle = \langle f, \varphi\rangle,$$

which shows  $(H - z)(I + R_0(z)V)^{-1}R_0(z) = I$  on  $L^2$ . For  $f \in D(H)$ , we similarly obtain

$$(I + R_0(z)V)^{-1}R_0(z)(H - z)f = (I + R_0(z)V)^{-1}f + (I + R_0(z)V)^{-1}R_0(z)Vf = f,$$

which gives us  $(I + R_0(z)V)^{-1}R_0(z)(H - z) = I$  on  $D(H)$  and the first identity in [\(3-3\)](#) thus follows. The second identity in [\(3-3\)](#) follows from the first identity and [\(3-5\)](#).

Now we shall show [\(3-4\)](#). It follows from [\(3-3\)](#) that

$$(I + R_0(z')V)(R(z) - R(z')) = (R_0(z) - R_0(z'))(I - VR(z))$$

on  $L^2$ . Since  $R_0(z) - R_0(z')$ ,  $R(z) - R(z') : L^2 \rightarrow \mathcal{H}^1$  are continuous and  $I + R_0(z')V$  is invertible on  $\mathcal{H}^1$ , we have the desired identity [\(3-4\)](#).  $\square$

Now we are in position to prove [Theorem 1.4](#), [Corollaries 1.5](#) and [1.6](#) and [Theorem 1.8](#).

*Proof of Theorem 1.4.* Assume that  $(p, q)$  satisfies [\(1-1\)](#). It follows from [Propositions 1.1](#) and [3.1](#) and [Lemma 3.7](#) that for any  $\delta > 0$  there exists  $C_\delta > 0$  such that

$$\|R(z)f\|_{L^{q,2}} \leq C_\delta(1 + \|(I + R_0(z)V)^{-1}\|_{\mathbb{B}(L^{q,2})})\|R_0(z)f\|_{L^{q,2}} \leq C_\delta|z|^{(n/2)(1/p-1/q)-1}\|f\|_{L^{p,2}}$$

for all  $f \in L^2 \cap L^{p,2}$  and  $z \in \mathbb{C} \setminus ([0, \infty) \cup \mathcal{E}_\delta)$ . Since  $L^2 \cap L^{p,2}$  is dense in  $L^{p,2}$ , this implies that  $R(z) \in \mathbb{B}(L^{p,2}, L^{q,2})$  and that [\(1-5\)](#) holds uniformly in  $z \in \mathbb{C} \setminus ([0, \infty) \cup \mathcal{E}_\delta)$ .  $\square$

*Proof of Corollary 1.5.* As before, we shall prove the corollary for  $R(\lambda + i0)$  only. We also consider the case  $1/p - 1/q = 2/n$  only, as the proofs for other cases are similar. At first, we claim that, for any  $\chi_1, \chi_2 \in C_0^\infty(\mathbb{R}^n)$ ,  $\chi_1 R(z) \chi_2$  defined for  $z \in \mathbb{C}^+$  extends to a  $\mathbb{B}(L^2)$ -valued continuous function  $\chi_1 R^+(z) \chi_2$  on  $\overline{\mathbb{C}^+} \setminus \mathcal{E}$ . It follows from this claim that, for any  $u, v \in C_0^\infty(\mathbb{R}^n)$ ,  $\langle R^+(z)u, v \rangle$  is a continuous function on  $\overline{\mathbb{C}^+} \setminus \mathcal{E}$ . Then, by letting  $\varepsilon \searrow 0$  in the estimate

$$|\langle R(\lambda + i\varepsilon)u, v \rangle| \lesssim \|u\|_{L^{p,2}}\|v\|_{L^{q',2}},$$

which follows from [Theorem 1.4](#), and by using the density argument, we obtain that  $R(\lambda + i0)$  extends to an element in  $\mathbb{B}(L^{p,2}, L^{q,2})$  and satisfies

$$\sup_{\lambda \in [0, \infty) \setminus \mathcal{E}} \|R(\lambda + i0)\|_{\mathbb{B}(L^{p,2}, L^{q,2})} < \infty. \quad (3-6)$$

This shows the first statement (1). For the second statement (2), it follows by setting  $z = \lambda \pm i\varepsilon$  and then letting  $\varepsilon \searrow 0$  in [\(3-3\)](#) that, for any  $f \in L^{q,2} \cap L^2$  and  $\lambda \in [0, \infty) \setminus \mathcal{E}$ ,

$$R(\lambda \pm i0)f = R_0(\lambda \pm i0)(I - VR(\lambda \pm i0))f \quad (3-7)$$

in the sense of distributions, which particularly implies that, under the condition  $0 \notin \mathcal{E}$ , we have  $R(0+i0) = R(0-i0)$  since  $R_0(0 \pm i0) = (-\Delta)^{-1}$ . Moreover, we also know by [\(3-7\)](#) that

$$\begin{aligned} (-\Delta + V - \lambda)R(\lambda + i0)u &= (I + VR_0(\lambda + i0))(I - VR(\lambda + i0))u \\ &= u + V[R_0(\lambda + i0) - R(\lambda + i0) - R_0(\lambda + i0)VR(\lambda + i0)]u = u \end{aligned}$$

for all  $u \in L^2 \cap L^{p,2}$  and that, for all  $v \in \mathcal{S}$ ,

$$\begin{aligned} R(\lambda + i0)(-\Delta + V - \lambda)v &= R_0(\lambda + i0)(I - VR(\lambda + i0))(-\Delta + V - \lambda)v \\ &= v - R_0(\lambda + i0)Vv - R_0(\lambda + i0)Vv = v \end{aligned}$$

in the sense of distributions. These two identities and [\(3-6\)](#) imply [\(1-7\)](#).

It remains to show the above claim. Let  $V_1, V_2 \in L_0^{n,\infty}(\mathbb{R}^n; \mathbb{R})$  be such that  $V = V_1 V_2$  and set  $K_1(z) = V_1 R_0(z) V_2$ . The resolvent identity [\(3-3\)](#) then yields

$$V_1 R(z) \chi_2 = V_1 R_0(z) \chi_2 - K_1(z) V_1 R(z) \chi_2$$

on  $L^2$  for all  $z \in \mathbb{C} \setminus \sigma(H)$ . Since  $K_1(z) \in \mathbb{B}_\infty(L^2)$  by [Corollary 2.3](#) and  $\text{Ker}_{L^2}(I + K_1(z)) = \emptyset$  for all  $z \in \mathbb{C} \setminus \sigma(H)$  by [Proposition 2.7](#) and [Lemma 2.8](#), we have by this identity that

$$V_1 R(z) \chi_2 = (I + K_1(z))^{-1} V_1 R_0(z) \chi_2, \quad z \in \mathbb{C} \setminus \sigma(H),$$

on  $L^2$ . It follows from again [Corollary 2.3](#) that  $V_1 R_0(z) \chi_2$  and  $K_1(z)$  extend to  $\mathbb{B}_\infty(L^2)$ -valued continuous functions  $V_1 R_0^+(z) \chi_2$  and  $K_1^+(z) = V_1 R_0^+(z) V_2$  on  $\overline{\mathbb{C}^+}$ . Since  $\text{Ker}(I + K_1^+(z)) = \emptyset$  for  $z \in \overline{\mathbb{C}^+} \setminus \mathcal{E}$ ,  $(I + K_1(z))^{-1}$  also extends to a  $\mathbb{B}(L^2)$ -valued continuous function  $(I + K_1^+(z))^{-1}$  on  $\overline{\mathbb{C}^+} \setminus \mathcal{E}$ . Thus  $V_1 R(z) \chi_2$  extends to a  $\mathbb{B}(L^2)$ -valued continuous function  $V_1 R^+(z) \chi_2$  on  $\overline{\mathbb{C}^+} \setminus \mathcal{E}$  satisfying  $V_1 R^+(z) \chi_2 = (I + K_1^+(z))^{-1} V_1 R_0^+(z)$ . Finally, the claim follows from the formula

$$\chi_1 R(z) \chi_2 = \chi_1 R_0(z) \chi_2 - \chi_1 R_0(z) V_2 V_1 R(z) \chi_2$$

and the continuity of  $\chi_1 R_0^+(z) \chi_2$ ,  $\chi_1 R_0^+(z) V_2$  and  $V_1 R_0^+(z) \chi_2$  on  $\overline{\mathbb{C}^+} \setminus \mathcal{E}$ .  $\square$

*Proof of Corollary 1.6.* Let us fix  $z \in \mathbb{C} \setminus \sigma(H)$  and take  $\delta > 0$  so small that  $z \notin \mathcal{E}_\delta$ . Recall that  $R_0(z) \in \mathbb{B}(L^p)$  for all  $1 \leq p \leq \infty$  and thus  $R_0(z) \in \mathbb{B}(L^{p,2})$  for all  $1 < p < \infty$  by [Theorem A.1](#).

The proof of the first assertion is divided into two cases:

$$\frac{2n}{n+3} < p = q < \frac{2n}{n+1}$$

and otherwise. In the first case one can find

$$\frac{2n}{n-1} < q_0 < \frac{2n}{n-3}$$

such that  $1/p - 1/q_0 = 2/n$ . Applying [Theorem 1.4](#) to the resolvent equation [\(3-3\)](#) implies that, for all  $f \in L^2 \cap L^{p,2}$ ,

$$\|R(z)f\|_{L^{p,2}} \lesssim \|R_0(z)f\|_{L^{p,2}} + \|R_0(z)\|_{\mathbb{B}(L^{p,2})} \|V\|_{L^{n/2,\infty}} \|R(z)f\|_{L^{q_0,2}} \leq C_\delta \|f\|_{L^{p,2}}.$$

Combined with a density argument, this implies  $R(z) \in \mathbb{B}(L^{p,2})$  for each  $z \in \mathbb{C} \setminus \sigma(H)$ .

Next, by taking the adjoint and using the fact  $R(z)^* = R(\bar{z})$ , we see that  $R(z) \in \mathbb{B}(L^{p,2})$  for all

$$\frac{2n}{n-1} < p < \frac{2n}{n-3}.$$

Interpolating these two cases yields that  $R(z) \in \mathbb{B}(L^{p,2})$  for all

$$\frac{2n}{n+3} < p < \frac{2n}{n-3}.$$

Then the other cases in the first assertion follow by interpolating between the estimates on the two lines  $1/p - 1/q = 0$  and  $1/p - 1/q = 2/n$  under the conditions  $2n/(n+3) < p$  and  $q < 2n/(n-3)$ .

Finally, assuming  $\frac{1}{2} < s < \frac{3}{2}$  without loss of generality, the second assertion follows from

$$\|wR(M)f\|_{L^2} \lesssim \|w\|_{L^{n/s,\infty}} \|R(M)f\|_{L^{2n/(n-2s),2}} \lesssim \|w\|_{L^{n/s,\infty}} \|f\|_{L^2}$$

for  $M < \inf \sigma(H) - 1$ , which is a particular case of the first assertion.  $\square$

*Proof of Theorem 1.8.* When

$$\frac{2n}{n+2} \leq p \leq \frac{2(n+1)}{n+3},$$

[\(1-10\)](#) follows from [\(1-6\)](#) and Stone's formula [\(1-9\)](#). When

$$\frac{2n}{n+3} < p < \frac{2n}{n+2},$$

there are two main ingredients.

At first, it is known that  $E'_{-\Delta}(\lambda) \in \mathbb{B}(L^p, L^{p'})$  for all

$$1 \leq p \leq \frac{2(n+1)}{n+3}$$

and satisfies

$$\|E'_{-\Delta}(\lambda)\|_{\mathbb{B}(L^p, L^{p'})} \lesssim \lambda^{(n/2)(1/p - 1/p') - 1}, \quad \lambda > 0. \quad (3-8)$$

Indeed,  $E'_{-\Delta}(\lambda)$  can be brought to the form  $E'_{-\Delta}(\lambda) = (2\pi)^{-n} \lambda^{(n-1)/2} R_{\sqrt{\lambda}}^* R_{\sqrt{\lambda}}$ , where

$$R_\mu u(\omega) := \int_{\mathbb{R}^n} e^{-2\pi i \mu \omega \cdot x} u(x) dx, \quad \mu > 0, \quad \omega \in \mathbb{S}^{n-1}.$$

Then the Stein–Tomas restriction theorem (see [Tomas 1975; Stein 1970]) and the  $TT^*$ -argument show that  $R_1^*R_1$  is bounded from  $L^p$  to  $L^{p'}$  for all

$$1 \leq p \leq \frac{2(n+1)}{n+3},$$

which particularly implies (3-8) by scaling.

Secondly, we claim that the following identity holds for all  $f, g \in \mathcal{S}$  and  $\lambda \in (0, \infty)$ :

$$\langle E'_H(\lambda)f, g \rangle = \langle (I + R_0(\lambda - i0)V)^{-1}E'_{-\Delta}(\lambda)(I - VR(\lambda + i0))f, g \rangle. \quad (3-9)$$

Since  $VR(\lambda + i0) \in \mathbb{B}(L^p)$  and  $(I + R_0(\lambda - i0)V)^{-1} \in \mathbb{B}(L^{p'})$  for

$$\frac{2n}{n+3} < p < \frac{2n}{n+1}$$

by Corollary 1.5 and Proposition 3.1, the desired assertion (1-10) follows from (3-8), (3-9) and a density argument.

It remains to show the identity (3-9). Let  $f, g \in \mathcal{S}$  and set

$$F(z) = \frac{1}{\pi}(I + R_0(\bar{z})V)^{-1} \operatorname{Im} R_0(z)(I - VR(z)), \quad z \in \mathbb{C}^+,$$

which is a bounded operator from  $L^2$  to  $\mathcal{H}^1$  (see the proof of Lemma 3.7), where

$$\operatorname{Im} R_0(z) = (2i)^{-1}(R_0(z) - R_0(\bar{z})).$$

By (3-4) with  $z = \lambda + i\varepsilon$ ,  $z' = \bar{z}$ , one has  $\pi^{-1} \operatorname{Im} R(z) = F(z)$ . Moreover,

$$\langle E'_H(\lambda)f, g \rangle = \pi^{-1} \lim_{\varepsilon \searrow 0} \langle \operatorname{Im} R(\lambda + i\varepsilon)f, g \rangle$$

exists by Corollary 1.5. For the operator  $F(z)$ , we write

$$F(z)f = \frac{1}{\pi}(I + R_0(\bar{z})V)^{-1}(\operatorname{Im} R_0(z)\langle x \rangle^{-3} - \operatorname{Im} R_0(z)VR(z)\langle x \rangle^{-3})\langle x \rangle^3 f.$$

By Proposition 2.4, all of  $(I + R_0(\bar{z})V)^{-1}$ ,  $\operatorname{Im} R_0(z)\langle x \rangle^{-3}$ ,  $\operatorname{Im} R_0(z)V$  and  $R(z)\langle x \rangle^{-3}$  extend to  $\mathbb{B}(L^{p'})$ -valued continuous function on  $\overline{\mathbb{C}^+} \setminus \mathcal{E}$ . Therefore,  $\langle F(\lambda + i0)f, g \rangle = \lim_{\varepsilon \searrow 0} \langle F(\lambda + i\varepsilon)f, g \rangle$  exists and coincides with the right-hand side of (3-9). Therefore (3-9) follows.  $\square$

The remaining part of the section is devoted to the following theorem, which plays a crucial role in the proof of Strichartz estimates.

**Theorem 3.8.** *Suppose that  $\mathcal{E} \cap [0, \infty) = \emptyset$ . Let  $(p, q)$  be such that*

$$\frac{1}{p} - \frac{1}{q} = \frac{2}{n} \quad \text{and} \quad \frac{2n}{(n+3)} < p < \frac{2n}{(n+1)}.$$

*Then*

$$\sup_{z \in \mathbb{C} \setminus [0, \infty)} \|P_{\operatorname{ac}}(H)R(z)\|_{\mathbb{B}(L^{p,2}, L^{q,2})} < \infty. \quad (3-10)$$

We first prove some  $L^p$ -boundedness of the projection  $P_{\text{ac}}(H)$ . At first note that, under the condition  $0 \notin \mathcal{E}$ ,  $H$  may have at most finitely many negative eigenvalues of finite multiplicities. Indeed, since  $\sigma_p(H) \cap (-\infty, 0) = \sigma_d(H)$ , each negative eigenvalue has finite multiplicity and their only possible accumulation point is  $z = 0$ . Moreover, [Lemma 3.6](#) and the Fredholm alternative show that, for sufficiently small  $\delta > 0$ ,  $(-\delta, \delta) \cap \mathcal{E} = \emptyset$  as long as  $0 \notin \mathcal{E}$ . Therefore,  $H$  may have at most finitely many negative eigenvalues. In this case  $P_{\text{ac}}(H)$  is written in the form

$$P_{\text{ac}}(H) = I - \sum_{j=1}^N P_j, \quad P_j := \langle \cdot, \psi_j \rangle \psi_j, \quad (3-11)$$

where  $\psi_j$  are eigenfunctions of  $H$  and  $N < \infty$ .

**Lemma 3.9.** *We have  $\psi_j \in L^{q,2}$  and  $P_{\text{ac}}(H) \in \mathbb{B}(L^{q,2})$  for all*

$$\frac{2n}{n+3} < q < \frac{2n}{n-3}.$$

*Proof.* Let  $\psi$  be an eigenfunction of  $H$  with an eigenvalue  $\lambda < 0$ . By virtue of (3-11) and real interpolation, it suffices to show  $\psi \in L^{q,2}$ . For a given  $\varepsilon > 0$ , we decompose  $V$  as  $V = v_1 + v_2$  with  $v_1 \in C_0^\infty(\mathbb{R}^n)$  and  $\|v_2\|_{L^{n/2,\infty}} \leq \varepsilon$ . We first let

$$\frac{2n}{n-1} < q < \frac{2n}{n-3}.$$

By Sobolev's inequality and [Proposition 1.1](#), one has

$$\begin{aligned} \|R_0(\lambda)v_1\psi\|_{L^q} &\lesssim \|R_0(\lambda)v_1\psi\|_{\mathcal{H}^{n(1/2-1/q)}} \leq C_\lambda \|v_1\psi\|_{L^2} \leq C_\lambda \|v_1\|_{L^\infty} \|\psi\|_{L^2}, \\ \|R_0(\lambda)v_2\|_{\mathbb{B}(L^q)} &\lesssim \|v_2\|_{L^{n/2,\infty}}. \end{aligned}$$

For  $\varepsilon > 0$  small enough,  $I + R_0(\lambda)v_2$  thus is invertible on  $L^q$  and

$$\psi = -R_0(\lambda)V\psi = R_0(\lambda)v_1\psi - R_0(\lambda)v_2\psi = -(I + R_0(\lambda)v_2)^{-1}R_0(\lambda)v_1\psi \in L^q.$$

Next, since  $R_0(\lambda) \in \mathbb{B}(L^p)$  for all  $1 < p < \infty$ , we have by Hölder's inequality that

$$\|\psi\|_{L^p} = \|R_0(\lambda)V\psi\|_{L^p} \leq C_\lambda \|V\psi\|_{L^p} \leq C_\lambda \|V\|_{L^{n/2,\infty}} \|\psi\|_{L^q}$$

if  $1/p - 1/q = 2/n$ . This shows  $\psi \in L^p$  for all

$$\frac{2n}{n+3} < p < \frac{2n}{n+1}.$$

Interpolating these two cases, we conclude that  $\psi \in L^q$  for all

$$\frac{2n}{n+3} < q < \frac{2n}{n-3}. \quad \square$$

*Proof of Theorem 3.8.* Assume that  $\mathcal{E} \cap [0, \infty) = \emptyset$ . Then one can find  $\delta > 0$  small enough such that  $\text{dist}(\mathcal{E}_\delta, [0, \infty)) \geq \delta/2$ . The proof is divided into two cases:  $z \in \mathbb{C} \setminus ([0, \infty) \cup \mathcal{E}_\delta)$  and  $z \in \mathcal{E}_\delta$ . For the case when  $z \in \mathbb{C} \setminus ([0, \infty) \cup \mathcal{E}_\delta)$ , since

$$\frac{2n}{n-1} < q, p' < \frac{2n}{n-3}$$

and  $P_j R(z) = (\lambda_j - z)^{-1} \langle \cdot, \psi_j \rangle \psi_j$ , [Lemma 3.9](#) implies

$$\|P_j R(z) f\|_{L^{p',2}} \leq \delta^{-1} \|\psi_j\|_{L^{q,2}} \|\psi_j\|_{L^{p',2}} \|f\|_{L^{p,2}},$$

which, together with [Theorem 1.4](#) and the formula (3-11), gives us the desired bound

$$\sup_{z \in \mathbb{C} \setminus ([0, \infty) \cup \mathcal{E}_\delta)} \|P_{\text{ac}}(H) R(z)\|_{\mathbb{B}(L^{p,2}, L^{q,2})} \lesssim \delta^{-1}. \quad (3-12)$$

When  $z \in \mathcal{E}_\delta$ , we use twice the first resolvent equation  $R(z) = R(z') - (z - z')R(z')R(z)$  to write

$$P_{\text{ac}}(H) R(z) = P_{\text{ac}}(H) R(M) + (z + M) P_{\text{ac}}(H) R(M)^2 + (z + M)^2 R(M) P_{\text{ac}}(H) R(z) R(M),$$

where we have taken  $M < \inf \sigma(H) - 1$ . Note that  $|z + M| \leq 2|M| + \delta$  for  $z \in \mathcal{E}_\delta$  since  $\mathcal{E}$  is a bounded set in  $\mathbb{R}$ . Moreover, we have by [Lemma 3.9](#) and [Corollary 1.6](#) and [Theorem A.1](#) that

$$\|P_{\text{ac}}(H) R(M)\|_{\mathbb{B}(L^{p,2}, L^{q,2})} \leq \|P_{\text{ac}}(H)\|_{\mathbb{B}(L^{q,2})} \|R(M)\|_{\mathbb{B}(L^{p,2}, L^{q,2})} \leq C_M,$$

$$\|R(M)\|_{\mathbb{B}(L^2, L^{q,2})} + \|R(M)\|_{\mathbb{B}(L^{p,2}, L^2)} \leq C_M$$

for some  $C_M$  independent of  $z$ . It follows from these two bounds and the trivial  $L^2$ -bound

$$\sup_{z \in \mathcal{E}_\delta} \|P_{\text{ac}}(H) R(z)\|_{\mathbb{B}(L^2)} \leq \text{dist}(\mathcal{E}_\delta, [0, \infty))^{-1} \leq 2\delta^{-1}$$

that there exists  $C_{M,\delta} > 0$ , independent of  $z$ , such that

$$\sup_{z \in \mathcal{E}_\delta} \|P_{\text{ac}}(H) R(z)\|_{\mathbb{B}(L^{p,2}, L^{q,2})} \leq C_{M,\delta}. \quad (3-13)$$

The assertion of the theorem then follows from (3-12) and (3-13).  $\square$

#### 4. Kato smoothing and Strichartz estimates

This section is devoted to the proof of Theorems 1.10 and 1.12. We first prepare several lemmas. Let  $e^{it\Delta}$  be the free Schrödinger unitary group and define

$$\Gamma_0 F(t) := \int_0^t e^{i(t-s)\Delta} F(s) ds, \quad F \in L^1_{\text{loc}}(\mathbb{R}; L^2(\mathbb{R}^n)).$$

The estimates for the free Schrödinger equation used in this section are summarized as follows:

**Lemma 4.1.** *Let  $(p, q)$  satisfy (1-12),  $(p_s, q_s)$  be as in (2-3) and  $\rho > \frac{1}{2}$ . Then*

$$\|e^{it\Delta} \psi\|_{L_t^p L_x^{q,2}} \lesssim \|\psi\|_{L_x^2}, \quad (4-1)$$

$$\|\Gamma_0 F\|_{L_t^2 L_x^{q_s,2}} \lesssim \|F\|_{L_t^2 L_x^{p_s,2}} \quad \text{for } \frac{n}{2(n-1)} < s < \frac{3n-4}{2(n-1)}, \quad (4-2)$$

$$\|\Gamma_0 F\|_{L_t^2 L_x^{q_s}} \lesssim \|F\|_{L_t^2 L_x^{p_s}} \quad \text{for } s = \frac{n}{2(n-1)}, \frac{3n-4}{2(n-1)}, \quad (4-3)$$

$$\|\langle x \rangle^{-\rho} |D|^{1/2} e^{it\Delta} \psi\|_{L_t^2 L_x^2} \lesssim \|\psi\|_{L_x^2}, \quad (4-4)$$

$$\|\langle x \rangle^{-\rho} |D|^{1/2} \Gamma_0 F\|_{L_t^2 L_x^2} \lesssim \|F\|_{L_t^2 L_x^{2n/(n+2),2}}. \quad (4-5)$$

*Proof.* Inequality (4-1) for  $p > 2$  is due to [Strichartz 1977; Ginibre and Velo 1985]. Inequality (4-1) with  $p = 2$  and (4-2) with  $s = 1$  were settled in [Keel and Tao 1998]. Inequality (4-2) was proved independently by [Foschi 2005] and [Vilela 2007]. Inequality (4-3) was settled recently in [Koh and Seo 2016]. Kato-smoothing (4-4) was proved in [Kenig, Ponce, and Vega 1991]. Finally, (4-5) can be found in [Mizutani 2018, Lemma 3.2].  $\square$

The following lemma, which was proved in [Kato 1966] (see also [Reed and Simon 1978; D'Ancona 2015]), shows the equivalence of the uniform weighted resolvent estimate and the Kato smoothing estimate.

**Lemma 4.2.** *Let  $L$  be a self-adjoint operator on a Hilbert space  $\mathcal{H}$ , let  $A$  be a densely defined closed operator on  $\mathcal{H}$ , and let  $a > 0$ . Then the following two estimates are equivalent to each other:*

$$\begin{aligned} |\langle \text{Im}(L - z)^{-1} A^* u, A^* u \rangle_{\mathcal{H}}| &\leq a \|u\|_{\mathcal{H}}^2, \quad u \in D(A^*), z \in \mathbb{C} \setminus \mathbb{R}, \\ \|A e^{-itL} v\|_{L_t^2 \mathcal{H}} &\leq 2\sqrt{a} \|v\|_{\mathcal{H}}, \quad v \in \mathcal{H}. \end{aligned}$$

The following concerns the equivalence of Sobolev norms generated by  $\Delta$  and  $H$ .

**Lemma 4.3.** *Assume that  $\mathcal{E} \cap [0, \infty) = \emptyset$  and  $0 \leq s < \frac{3}{2}$ . Then*

$$\|(-\Delta + M)^{s/2} (H + M)^{-s/2}\|_{\mathbb{B}(L^2)} + \|(H + M)^{s/2} (-\Delta + M)^{-s/2}\|_{\mathbb{B}(L^2)} < \infty. \quad (4-6)$$

*Proof.* The proof will be given in the next section.  $\square$

Recall that  $\langle \cdot, \cdot \rangle_T$  is the inner product in  $L_T^2 L_x^2$  defined by  $\langle F, G \rangle_T = \int_{-T}^T \langle F(t), G(t) \rangle dt$ . It is not hard to check that  $\langle \Gamma_H F, G \rangle_T = \langle F, \Gamma_H^* G \rangle_T$  with

$$\Gamma_H^* G(t) = \mathbb{1}_{[0, \infty)}(t) \int_t^T e^{-i(t-s)H} G(s) ds - \mathbb{1}_{(-\infty, 0]}(t) \int_{-T}^t e^{-i(t-s)H} G(s) ds.$$

The following lemma gives the rigorous definition of Duhamel's formula (in the sense of forms).

**Lemma 4.4.** *Let  $\frac{1}{2} < s < \frac{3}{2}$ ,  $V_1 \in L_0^{n/s, \infty}(\mathbb{R}^n; \mathbb{R})$  and  $V_2 \in L_0^{n/(2-s), \infty}(\mathbb{R}^n; \mathbb{R})$  be such that  $V = V_1 V_2$ . Then, for all  $\psi \in L^2$  and all simple functions  $F, G : \mathbb{R} \rightarrow \mathcal{S}$ ,*

$$\langle e^{-itH} P_{\text{ac}}(H) \psi, G \rangle_T = \langle e^{it\Delta} P_{\text{ac}}(H) \psi, G \rangle_T - i \langle V_1 P_{\text{ac}}(H) e^{-itH} \psi, V_2 \Gamma_0^* G \rangle_T, \quad (4-7)$$

$$\langle \Gamma_H P_{\text{ac}}(H) F, G \rangle_T = \langle \Gamma_0 P_{\text{ac}}(H) F, G \rangle_T - i \langle V_1 \Gamma_H P_{\text{ac}}(H) F, V_2 \Gamma_0^* G \rangle_T, \quad (4-8)$$

$$= \langle \Gamma_0 F, P_{\text{ac}}(H) G \rangle_T - i \langle V_2 \Gamma_0 F, V_1 \Gamma_H^* P_{\text{ac}}(H) G \rangle_T. \quad (4-9)$$

*Proof.* The proof is basically same as that in [Bouclet and Mizutani 2018, Proposition 4.4], where the case  $s = 1$  was considered. We shall show (4-8), since the other proofs are similar. We start from the formula

$$\langle e^{-itH} P_{\text{ac}}(H) u, v \rangle - \langle e^{it\Delta} P_{\text{ac}}(H) u, v \rangle = -i \int_0^t \langle V_1 e^{-i\tau H} P_{\text{ac}}(H) u, V_2 e^{i(t-\tau)\Delta} v \rangle d\tau \quad (4-10)$$

for  $u, v \in \mathcal{S}$ , which follows by computing  $\frac{d}{dt} \langle e^{-itH} P_{\text{ac}}(H) u, e^{it\Delta} v \rangle$ . Here note that the HLS inequality (A-2) and Lemma 4.3 yield

$$|\langle V_1 e^{-i\tau H} P_{\text{ac}}(H) u, V_2 e^{i(t-\tau)\Delta} v \rangle| \lesssim \|V_1\|_{L^{n/s, \infty}} \|V_2\|_{L^{n/(2-s), \infty}} \|(-\Delta + 1)^{s/2} u\|_{L^2} \|(-\Delta + 1)^{1-s/2} v\|_{L^2} < \infty$$

and, hence, the right-hand side of (4-10) makes sense. Changing  $t$  to  $t-s$ , plugging in  $u = F(s)$ ,  $v = G(t)$  and integrating in  $s$  over  $[0, t]$ , we obtain

$$\begin{aligned} \langle \Gamma_H P_{\text{ac}}(H)F(t), G(t) \rangle - \langle \Gamma_0 P_{\text{ac}}(H)F(t), G(t) \rangle \\ = -i \int_0^t \int_s^t \langle V_1 e^{-i(\tau-s)H} P_{\text{ac}}(H)F(s), V_2 e^{i(\tau-t)\Delta} G(t) \rangle d\tau dt, \end{aligned}$$

where, by the same argument as above, the integrand of the right-hand side is finite and thus integrable in  $(\tau, s) \in [0, t]^2$ . Therefore, by Fubini's theorem,

$$\begin{aligned} \langle \Gamma_H P_{\text{ac}}(H)F(t), G(t) \rangle - \langle \Gamma_0 P_{\text{ac}}(H)F(t), G(t) \rangle \\ = -i \int_0^t \langle V_1 \Gamma_H P_{\text{ac}}(H)F(\tau), V_2 e^{i(\tau-t)\Delta} G(t) \rangle d\tau. \quad (4-11) \end{aligned}$$

Finally, observing from the same argument as above that  $|\langle V_1 \Gamma_H F(\tau), V_2 e^{i(\tau-t)\Delta} G(t) \rangle|$  is finite, we integrate (4-11) in  $t$  and use Fubini's theorem to obtain the desired formula (4-8).  $\square$

**Remark 4.5.** When  $s = 1$ , the identities (4-7), (4-8) and (4-9) also hold for all  $F, G \in L_{\text{loc}}^1 L^2$ ; see [Bouclet and Mizutani 2018, Proposition 4.4].

Using these lemmas, we first prove Kato smoothing estimates.

*Proof of Theorem 1.10.* The following argument is basically same as that in [Burq, Planchon, Stalker, and Tahvildar-Zadeh 2004]. With the above remark at hand, we use (4-7) with  $G$  replaced by  $|D|^{1/2} \langle x \rangle^{-\rho} G$  to obtain

$$\begin{aligned} \langle \langle x \rangle^{-\rho} |D|^{1/2} e^{-itH} P_{\text{ac}}(H) \psi, G \rangle_T \\ = \langle \langle x \rangle^{-\rho} |D|^{1/2} e^{it\Delta} P_{\text{ac}}(H) \psi, G \rangle_T - i \langle V_1 P_{\text{ac}}(H) e^{-itH} \psi, V_2 \Gamma_0^* |D|^{1/2} \langle x \rangle^{-\rho} G \rangle_T \end{aligned}$$

for all  $\psi \in L^2$  and a simple function  $G(t) : \mathbb{R} \rightarrow \mathcal{S}$ . By (4-4), the first term obeys

$$| \langle \langle x \rangle^{-\rho} |D|^{1/2} e^{it\Delta} P_{\text{ac}}(H) \psi, G \rangle_T | \lesssim \|\psi\|_{L^2} \|G\|_{L_t^2 L_x^2} \quad (4-12)$$

uniformly in  $T > 0$ . On the other hand, we have by the dual estimate of (4-5) that

$$| \langle V_1 P_{\text{ac}}(H) e^{-itH} \psi, V_2 \Gamma_0^* G \rangle_T | \lesssim \|V_1 P_{\text{ac}}(H) e^{-itH} \psi\|_{L_t^2 L_x^2} \|G\|_{L_t^2 L_x^2} \quad (4-13)$$

uniformly in  $T > 0$ . For the term  $\|V_1 P_{\text{ac}}(H) e^{-itH} \psi\|_{L_t^2 L_x^2}$ , we use Lemma 4.2 to deduce

$$\|V_1 P_{\text{ac}}(H) e^{-itH} \psi\|_{L_t^2 L_x^2} \lesssim \|\psi\|_{L_x^2} \quad (4-14)$$

from the following uniform weighted resolvent estimate

$$\sup_{z \in \mathbb{C} \setminus \mathbb{R}} \|V_1 P_{\text{ac}}(H) R(z) P_{\text{ac}}(H) V_1\|_{\mathbb{B}(L^2)} < \infty,$$

which is a consequence of Theorem 3.8 and Hölder's inequality (A-1), where we note that  $P_{\text{ac}}(H)^2 = P_{\text{ac}}(H)$  since  $P_{\text{ac}}(H)$  is an orthogonal projection. Finally, (4-12)–(4-14) imply

$$| \langle \langle x \rangle^{-\rho} |D|^{1/2} e^{-itH} P_{\text{ac}}(H) \psi, G \rangle_T | \lesssim \|\psi\|_{L^2} \|G\|_{L_t^2 L_x^2},$$

which, together with duality and density arguments, gives us the assertion.  $\square$

In order to prove Strichartz estimates, we need one more lemma.

**Lemma 4.6.** *Assume  $\mathcal{E} \cap [0, \infty) = \emptyset$ . Then, for any  $\frac{1}{2} < s < \frac{3}{2}$  there exists  $C > 0$  such that, for all  $w \in L^{n/(2-s), \infty}$ ,  $\chi \in C_0^\infty(\mathbb{R}^n)$  and  $T > 0$ ,*

$$\|\chi \Gamma_H P_{\text{ac}}(H) w F\|_{L_T^2 L_x^2} \leq C \|\chi\|_{L^{n/s, \infty}} \|w\|_{L^{n/(2-s), \infty}} \|F\|_{L_T^2 L_x^2}. \quad (4-15)$$

*Proof.* The proof is essentially based on the argument in [D'Ancona 2015, Theorem 2.3]. At first note that it suffices to show (4-15) with  $[-T, T]$  replaced by  $\mathbb{R}$ . Indeed, since  $s \in [-T, T]$  if  $t \in [-T, T]$  and  $s \in [0, t]$  (or  $s \in [t, 0]$ ), (4-15) with  $[-T, T]$  replaced by  $\mathbb{R}$  implies

$$\|\chi \Gamma_H P_{\text{ac}}(H) w F\|_{L_T^2 L_x^2} \lesssim \|\mathbb{1}_{[-T, T]}(s) F\|_{L_T^2 L_x^2} = \|F\|_{L_T^2 L_x^2}.$$

We may assume, by a density argument, that  $F(t) : \mathbb{R} \rightarrow \mathcal{S}$  is a simple function. Set  $A_1 = \chi(x) P_{\text{ac}}(H)$  and  $A_2 = w P_{\text{ac}}(H)$ . For a function  $v(t) : \mathbb{R} \rightarrow L^2$ ,  $\tilde{v}$  denotes its Laplace transform:

$$\tilde{v}(z) = \pm \int_0^{\pm\infty} e^{izt} v(t) dz, \quad \pm \text{Im } z > 0.$$

A direct calculation yields that if  $v(t) = \Gamma_H A_2^* F(t)$  then  $\tilde{v}(z) = -i R(z) A_2^* \tilde{F}(z)$ , where the identity  $\tilde{A_2^* F} = A_2^* \tilde{F}$  follows from the estimate  $\|A_2 F\|_{L_{\text{loc}}^1 L_x^2} \lesssim \|w\|_{L^{n/(2-s), \infty}} \|F\|_{L_{\text{loc}}^1 \mathcal{H}^{2-s}} < \infty$  and Hille's theorem [Hille and Phillips 1957, Theorem 3.7.12]. Also we see that  $v(t) \in D(A_1)$  for each  $t$ . Indeed, writing  $F(t) = \sum_{j=1}^N \mathbb{1}_{E_j}(t) f_j$  with some  $f_j \in \mathcal{S}(\mathbb{R}^n)$ , we have for each  $t$

$$\|A_1 v(t)\|_{L^2} \leq \sum_{j=1}^N \int_0^{|t|} \|A_1 e^{isH} e^{-itH} P_{\text{ac}}(H) w f_j\|_{L^2} ds \lesssim |t| \|w\|_{L^{n/(2-s), \infty}} \sum_{j=1}^N \|f_j\|_{\mathcal{H}^{2-s}} < \infty.$$

Then one can use Parseval's theorem to obtain

$$\pm \int_0^{\pm\infty} e^{-2\varepsilon|t|} \langle v(t), A_1^* G(t) \rangle dt = 2\pi \int_{\mathbb{R}} \langle \tilde{v}(\lambda \pm i\varepsilon), A_1^* \tilde{G}(\lambda \pm i\varepsilon) \rangle d\lambda, \quad \varepsilon > 0,$$

for any simple function  $G : \mathbb{R} \rightarrow \mathcal{S}$ . By virtue of uniform Sobolev estimates (3-10) with

$$(p, q) = \left( \frac{2n}{n+2(2-s)}, \frac{2n}{n-2s} \right)$$

and Hölder's inequality (A-1), the integrand of the right-hand side obeys

$$|\langle \tilde{v}(\lambda \pm i\varepsilon), A_1^* \tilde{G}(\lambda \pm i\varepsilon) \rangle| \leq \|\chi\|_{L^{n/2, \infty}} \|w\|_{L^{n/(2-s), \infty}} \|\tilde{F}(\lambda \pm i\varepsilon)\|_{L_x^2} \|\tilde{G}(\lambda \pm i\varepsilon)\|_{L_x^2}.$$

Applying again Parseval's theorem, we have

$$\begin{aligned} \left| \int_0^{\pm\infty} e^{-2\varepsilon|t|} \langle v(t), A_1^* G(t) \rangle dt \right| &= \left| \int_{\mathbb{R}} \langle \tilde{v}(\lambda \pm i\varepsilon), A_1^* \tilde{G}(\lambda \pm i\varepsilon) \rangle d\lambda \right| \\ &\lesssim \|\chi\|_{L^{n/2, \infty}} \|w\|_{L^{n/(2-s), \infty}} \|\tilde{F}(\lambda \pm i\varepsilon)\|_{L_x^2} \|\tilde{G}(\lambda \pm i\varepsilon)\|_{L_x^2} \\ &\lesssim \|\chi\|_{L^{n/2, \infty}} \|w\|_{L^{n/(2-s), \infty}} \|e^{-\varepsilon|t|} F(t)\|_{L^2(\mathbb{R}_\pm; L^2(\mathbb{R}^n))} \|e^{-\varepsilon|t|} G(t)\|_{L^2(\mathbb{R}_\pm; L^2(\mathbb{R}^n))}, \end{aligned}$$

which, together with the density of simple functions with values in  $\mathcal{S}$ , shows

$$\|e^{-\varepsilon|t|} A_1 \Gamma_H A_2 F\|_{L_t^2 L_x^2} \lesssim \|\chi\|_{L^{n/2, \infty}} \|w\|_{L^{n/(2-s), \infty}} \|e^{-\varepsilon|t|} F\|_{L_t^2 L_x^2}, \quad F \in L_t^2 L_x^2.$$

The result then follows by letting  $\varepsilon \rightarrow 0$ .  $\square$

**Remark 4.7.** If  $\frac{1}{2} < s \leq 1$ , (4-15) also holds for any  $\chi \in L^{n/s, \infty}$ . The proof is completely the same. When  $1 < s < \frac{3}{2}$ , we do not, a priori, know  $\chi e^{-itH} P_{\text{ac}}(H) w F(s) \in L_x^2$  for each  $t, s$  under the condition  $\chi \in L^{n/s, \infty}$  only, even if  $F : \mathbb{R} \rightarrow \mathcal{S}$ . This is the reason why we have assumed  $\chi \in C_0^\infty$ . We however stress that Lemma 4.6 is sufficient for our purpose.

We are now ready to show our Strichartz estimates.

*Proof of Theorem 1.12.* Using (4-1) and (4-2) with  $s = 1$  instead of (4-4) and (4-5), respectively, one can see that the proof of the homogeneous endpoint Strichartz estimate of the form

$$\|e^{-itH} P_{\text{ac}}(H) \psi\|_{L_t^2 L_x^{2n/(n-2), 2}} \lesssim \|\psi\|_{L^2} \quad (4-16)$$

is similar to that of Theorem 1.10 and even easier than that of (1-14). We thus omit the proof.

We shall show (1-14). Let

$$\frac{n}{2(n-1)} < s < \frac{3n-4}{2(n-1)},$$

and  $V_1 \in L_0^{n/s, \infty}$  and  $V_2 \in L_0^{n/(2-s), \infty}$  be real-valued such that  $V = V_1 V_2$ . Take a sequence  $V_{1,j} \in C_0^\infty$  such that  $\|V_1 - V_{1,j}\|_{L^{n/s, \infty}} \rightarrow 0$ . Let  $F : \mathbb{R} \rightarrow \mathcal{S}$  be a simple function in  $t$ . As in the proof of Lemma 4.4, we see that  $\Gamma_H P_{\text{ac}}(H) F \in L_T^2 L_x^{q_s, 2}$  for each  $T > 0$  by Lemma 4.3. Then, by the duality argument, we have

$$\|\Gamma_H P_{\text{ac}}(H) F\|_{L_T^2 L_x^{q_s, 2}} \lesssim \sup\{|\langle \Gamma_H P_{\text{ac}}(H) F, G \rangle_T| \mid \|G\|_{L_T^2 L_x^{q_s', 2}} = 1\}, \quad (4-17)$$

where we may assume by a density argument that  $G : \mathbb{R} \rightarrow \mathcal{S}$  is a simple function. Then, it follows from Duhamel's formula (4-8), (4-2), Lemma 3.9 and Hölder's inequality (A-1) that

$$\begin{aligned} |\langle \Gamma_H P_{\text{ac}}(H) F, G \rangle_T| &\lesssim \|P_{\text{ac}}(H)\|_{\mathbb{B}(L^{ps, 2})} \|F\|_{L_t^2 L_x^{ps, 2}} + \|V_{1,j} \Gamma_H P_{\text{ac}}(H) F\|_{L_T^2 L_x^2} \|V_2\|_{L^{n/(2-s), \infty}} \\ &\quad + \|V_1 - V_{1,j}\|_{L^{n/s, \infty}} \|\Gamma_H P_{\text{ac}}(H) F\|_{L_T^2 L_x^{q_s, 2}} \end{aligned}$$

uniformly in  $T > 0$ . Taking  $j$  large enough (which can be taken independently of  $T$ ), the last term can be absorbed into the left-hand side of (4-17), implying

$$\|\Gamma_H P_{\text{ac}}(H) F\|_{L_T^2 L_x^{q_s, 2}} \lesssim \|F\|_{L_t^2 L_x^{ps, 2}} + \|V_{1,j} \Gamma_H P_{\text{ac}}(H) F\|_{L_T^2 L_x^2}$$

uniformly in  $T > 0$ . To deal with the term  $\|V_{1,j} \Gamma_H P_{\text{ac}}(H) F\|_{L_T^2 L_x^2}$ , we use (4-9) to write

$$\langle V_{1,j} \Gamma_H P_{\text{ac}}(H) F, \tilde{G} \rangle_T = \langle \Gamma_0 F, P_{\text{ac}}(H) V_{1,j} \tilde{G} \rangle_T - i \langle V_2 \Gamma_0 F, V_1 \Gamma_H^* P_{\text{ac}}(H) V_{1,j} \tilde{G} \rangle_T$$

for all simple functions  $\tilde{G} : \mathbb{R} \rightarrow \mathcal{S}$  satisfying  $\|\tilde{G}\|_{L_T^2 L_x^2} = 1$ . By (4-2) the first term enjoys

$$|\langle \Gamma_0 F, P_{\text{ac}}(H) V_{1,j} \tilde{G} \rangle_T| \lesssim \|V_{1,j}\|_{L^{n/s, \infty}} \|F\|_{L_t^2 L_x^{ps, 2}} \lesssim \|F\|_{L_t^2 L_x^{ps, 2}}$$

uniformly in  $T > 0$  and  $j$ . On the other hand, since  $V_2 \Gamma_H^* P_{\text{ac}}(H) V_{1,j} \tilde{G} \in L_T^2 L_x^2$  by [Lemma 4.6](#) and  $V_1 \Gamma_0 F \in L_T^2 L_x^2$  by [\(4-2\)](#), the last term can be rewritten in the form

$$\langle V_2 \Gamma_0 F, V_1 \Gamma_H^* P_{\text{ac}}(H) V_{1,j} \tilde{G} \rangle_T = \langle V_1 \Gamma_0 F, V_2 \Gamma_H^* P_{\text{ac}}(H) V_{1,j} \tilde{G} \rangle_T.$$

Using [\(4-2\)](#), [Lemma 4.6](#) and a duality argument, we then obtain

$$|\langle V_1 \Gamma_0 F, V_2 \Gamma_H^* P_{\text{ac}}(H) V_{1,j} \tilde{G} \rangle_T| \lesssim \|F\|_{L_t^2 L_x^{p_s,2}}.$$

Putting it all together, we conclude that

$$\|\Gamma_H P_{\text{ac}}(H) F\|_{L_T^2 L_x^{q_s,2}} \lesssim \|F\|_{L_t^2 L_x^{p_s,2}}$$

uniformly in  $T > 0$ , which implies the desired estimates [\(1-14\)](#) for

$$\frac{n}{2(n-1)} < s < \frac{3n-4}{2(n-1)}.$$

The cases  $s = n/(2(n-1))$  and  $(3n-4)/(2(n-1))$  can be obtained analogously by using [\(4-3\)](#) instead of [\(4-2\)](#).  $\square$

## 5. Spectral multiplier theorem

This section is devoted to the proof of [Lemma 4.3](#) and [Theorem 1.15](#). Proofs are based on an abstract method in [\[Chen, Ouhabaz, Sikora, and Yan 2016\]](#), which, in the Euclidean case, can be stated as follows.

**Proposition 5.1** [\[Chen, Ouhabaz, Sikora, and Yan 2016, Theorem A\]](#). *Let  $1 \leq p_0 < 2$  and  $1 \leq q \leq \infty$ . Let  $L$  be a nonnegative self-adjoint operator on  $L^2(\mathbb{R}^n)$  satisfying the following two conditions:*

- *Davies–Gaffney’s estimate: for any open sets  $U_j \subset \mathbb{R}^n$  and  $\psi_j \in L^2(U_j)$ ,  $j = 1, 2$ ,*

$$|\langle e^{-tL} \psi_1, \psi_2 \rangle| \leq \exp\left(-\frac{d(U_1, U_2)^2}{4t}\right) \|\psi_1\|_{L^2} \|\psi_2\|_{L^2}, \quad (5-1)$$

where  $d(U_1, U_2) := \inf_{x_1 \in U_1, x_2 \in U_2} |x_1 - x_2|$  is the distance between  $U_1$  and  $U_2$ .

- *Stein–Tomas-type restriction estimate: for any  $a > 0$  and any bounded Borel function  $F_0$  on  $\mathbb{R}$  supported in  $[0, a]$ , we have  $F_0(\sqrt{L}) \in \mathbb{B}(L^{p_0}, L^2)$  and*

$$\|F_0(\sqrt{L}) \mathbb{1}_{B(x,r)}\|_{\mathbb{B}(L^{p_0}, L^2)} \lesssim a^{n(1/p_0 - 1/2)} \|F_0(a \cdot)\|_{L^q} \quad (5-2)$$

for all  $x \in \mathbb{R}^n$  and  $r \geq a^{-1}$ , where  $B(x, r) = \{y \mid |y - x| < r\}$ .

Then, for any bounded Borel function  $F$  on  $\mathbb{R}$  satisfying

$$|F|_{\mathcal{W}(\beta,q)} := \sup_{t>0} \|\psi(\cdot) F(t \cdot)\|_{\mathcal{W}^{\beta,q}(\mathbb{R})} < \infty, \quad (5-3)$$

with some nontrivial  $\psi \in C_0^\infty$  supported in  $(0, \infty)$  and

$$\beta > \max\left\{n\left(\frac{1}{p_0} - \frac{1}{2}\right), \frac{1}{q}\right\}$$

such that  $\beta$  is an integer if  $q = \infty$ , we have  $F(\sqrt{L})$  is bounded on  $L^p$  for all  $p_0 < p < p'_0$  and satisfies

$$\|F(\sqrt{L})\|_{\mathbb{B}(L^p)} \leq C_\beta (|F|_{\mathcal{W}(\beta, q)} + |F(0)|).$$

Strictly speaking, instead of Davies–Gaffney’s estimate, it was assumed in [Chen, Ouhabaz, Sikora, and Yan 2016] that  $L$  satisfies the so-called finite-speed propagation property; see (FS) on page 229 of [loc. cit.]. However, these two conditions are known to be equivalent; see [loc. cit., Theorem 3.4]. Moreover, (5-1) is always satisfied for nonnegative Schrödinger operators  $-\Delta + V(x)$  as shown in [Coulhon and Sikora 2008].

**Lemma 5.2** [Coulhon and Sikora 2008, Theorem 3.3]. *Let  $L = -\Delta + V(x)$  with real-valued  $V \in L^1_{\text{loc}}(\mathbb{R}^n)$  such that  $L \geq 0$  as a quadratic form. Then (5-1) is satisfied.*

When  $q = \infty$ , (5-2) can be replaced by an  $L^p$ - $L^2$  estimate of the Schrödinger semigroup.

**Lemma 5.3.** *Let  $1 \leq p_0 < 2$ . Then (5-2) with  $q = \infty$  follows from*

$$\|e^{-t^2 L}\|_{\mathbb{B}(L^{p_0}, L^2)} \lesssim t^{-n(1/p_0 - 1/2)}, \quad t > 0. \quad (5-4)$$

*Proof.* By [Chen, Ouhabaz, Sikora, and Yan 2016, Proposition 1.3], (5-2) with  $q = \infty$  is equivalent to

$$\|e^{-t^2 L} \mathbb{1}_{B(x, r)}\|_{\mathbb{B}(L^{p_0}, L^2)} \lesssim |B(x, r)|^{1/p_0 - 1/2} (rt^{-1})^{n(1/p_0 - 1/2)}, \quad t > 0, \quad x \in \mathbb{R}^n, \quad r \geq t,$$

which clearly follows from (5-4) since  $|B(x, r)| \leq C_n r^n$ .  $\square$

Now we show Lemma 4.3 whose proof is classical and based on Stein’s complex interpolation theorem. Let us fix  $M > |\inf \sigma(H)| + 1$  so that  $H + M \geq I$ . A key observation is the following.

**Lemma 5.4.** *For any  $\alpha \in \mathbb{R}$  and*

$$\frac{2n}{n+3} < p < \frac{2n}{n-3},$$

*we have*

$$\|(H + M)^{i\alpha}\|_{\mathbb{B}(L^p)} \leq C_M \langle \alpha \rangle^n.$$

*Proof.* It is easy to see that  $F(x) = x^{2i\alpha}$  satisfies  $|F|_{\mathcal{W}(n, \infty)} \leq C_n \langle \alpha \rangle^n$  and  $|F(0)| = 1$ . Let us fix

$$\frac{2n}{n+3} < p_0 \leq \frac{2n}{n+2}$$

arbitrarily. By virtue of Proposition 5.1 and Lemmas 5.2 and 5.3, it suffices to show that  $L := H + M$  satisfies (5-4). Decompose  $e^{-t^2 L}$  into the absolutely continuous part  $e^{-t^2 L} P_{\text{ac}}(H)$  and the discrete part  $\sum_{j=1}^N e^{-t^2 L} P_j$ .

For the discrete part, since  $\lambda_j + M \geq 1$ , we know by Lemma 3.9 that

$$\|e^{-t^2 L} P_j f\|_{L^2} = \|e^{-t^2(\lambda_j + M)} P_j f\|_{L^2} \leq e^{-t^2} \|\varphi_j\|_{L^2} \|\varphi_j\|_{L^{p'_0}} \|f\|_{L^{p_0}} \lesssim e^{-t^2} \|f\|_{L^{p_0}}.$$

On the other hand, it follows from the spectral decomposition theorem that

$$e^{-t^2 L} P_{\text{ac}}(H) (e^{-t^2 L} P_{\text{ac}}(H))^* = e^{-2t^2 L} P_{\text{ac}}(H) = \int_0^\infty e^{-2t^2(\lambda + M)} dE_H(\lambda).$$

**Theorem 1.8** then implies

$$\|e^{-2t^2L} P_{\text{ac}}(H)\|_{\mathbb{B}(L^{p_0}, L^{p'_0})} \lesssim \int_0^\infty e^{-2t^2(\lambda+M)} \lambda^{(n/2)(1/p_0-1/p'_0)-1} d\lambda \lesssim t^{-n(1/p_0-1/p'_0)} = t^{-2n(1/p_0-1/2)}.$$

Since  $\|e^{-t^2L} P_{\text{ac}}(H)\|_{\mathbb{B}(L^{p_0}, L^2)} \leq \|e^{-2t^2L} P_{\text{ac}}(H)\|_{\mathbb{B}(L^{p_0}, L^{p'_0})}^{1/2}$  by the duality, (5-4) follows.  $\square$

*Proof of Lemma 4.3.* We may assume  $1 < s < \frac{3}{2}$  without loss of generality since the case when  $0 \leq s \leq 1$  follows from Stein's complex interpolation [1956] and the estimate

$$\|(-\Delta + M)^{1/2}(H + M)^{-1/2}\|_{\mathbb{B}(L^2)} + \|(-\Delta + M)^{-1/2}(H + M)^{1/2}\|_{\mathbb{B}(L^2)} < \infty,$$

which is a consequence of the fact that the form domain of  $H$  is  $\mathcal{H}^1$ .

For  $f, g \in \mathcal{S}$ , we consider a function  $G(z) = \langle (H + M)^{-z} f, (-\Delta + M)^z g \rangle$  which is continuous on  $0 \leq \text{Re } z \leq 1$  and analytic in  $0 < \text{Re } z < 1$ . By **Corollary 1.6** and **Lemma 5.4**, for

$$\frac{2n}{n+3} < r_1 < \frac{2n}{n-3} \quad \text{and} \quad \frac{2n}{n+3} < r_2 < \frac{2n}{n+1},$$

we have

$$|G(it)| \leq \|(H + M)^{-it} f\|_{L^{r_1}} \|(-\Delta + M)^{it} g\|_{L^{r'_1}} \lesssim \langle t \rangle^{2n} \|f\|_{L^{r_1}} \|g\|_{L^{r'_1}},$$

$$|G(1+it)| \leq \|(-\Delta + M)(H + M)^{-1-it} f\|_{L^{r_2}} \|(-\Delta + M)^{it} g\|_{L^{r'_2}} \lesssim \langle t \rangle^{2n} \|f\|_{L^{r_2}} \|g\|_{L^{r'_2}},$$

where, since  $(-\Delta + M)(H + M)^{-1} = 1 - V(H + M)^{-1}$ , the second estimate can be verified as

$$\|(-\Delta + M)(H + M)^{-1}\|_{\mathbb{B}(L^{r_2})} \leq 1 + \|V(H + M)^{-1}\|_{\mathbb{B}(L^{r_2})} \leq 1 + C_M \|V\|_{L^{n/2, \infty}}.$$

Let

$$r_1 = \frac{2n}{n-2s} \quad \text{and} \quad r_2 = \frac{2n}{n+2(2-s)}.$$

Since

$$\frac{1}{2} = \left(1 - \frac{s}{2}\right) \left(\frac{1}{r_1}\right) + \frac{s}{2} \cdot \frac{1}{r_2},$$

we apply Stein's complex interpolation theorem to  $G$ , implying  $|G(s/2)| \leq C_t \|f\|_{L^2} \|g\|_{L^2}$ . This gives us

$$\|(-\Delta + M)^{s/2}(H + M)^{-s/2}\|_{\mathbb{B}(L^2)} < \infty.$$

Applying the same argument to a function  $G(z) = \langle (-\Delta + M)^{-z} f, (H + M)^z g \rangle$ , we also have

$$\|(H + M)^{s/2}(-\Delta + M)^{-s/2}\|_{\mathbb{B}(L^2)} < \infty. \quad \square$$

Next we shall show **Theorem 1.15**.

*Proof of Theorem 1.15.* Since  $H$  is assumed to be nonnegative, the Davies–Gaffney estimate (5-1) is satisfied. It thus remains to check the Stein–Tomas-type restriction estimate (5-2) with  $q = 2$ . Let

$$\frac{2n}{n+3} < p_0 < \frac{2n}{n+2}$$

and  $F_0 \in L^\infty(\mathbb{R})$  be such that  $\text{supp } F_0 \subset [0, a]$ . By [Theorem 1.8](#),

$$\begin{aligned} \|F_0(\sqrt{H})^2\|_{\mathbb{B}(L^{p_0}, L^{p'_0})} &\lesssim \int_0^{a^2} |F_0(\sqrt{\lambda})|^2 \lambda^{(n/2)(1/p_0 - 1/p'_0) - 1} d\lambda \\ &\lesssim \|F_0\|_{L^2([0, a])}^2 a^{n(1/p_0 - 1/p'_0) - 1} \lesssim a^{n(1/p_0 - 1/p'_0)} \|F_0(a \cdot)\|_{L^2}^2. \end{aligned}$$

Finally, by the duality, we have  $\|F_0(\sqrt{H})\|_{\mathbb{B}(L^{p_0}, L^2)} \leq \|F_0(\sqrt{H})^2\|_{\mathbb{B}(L^{p_0}, L^{p'_0})}^{1/2}$ , which, combined with the above estimate for  $\|F_0(\sqrt{H})^2\|_{\mathbb{B}(L^{p_0}, L^{p'_0})}$ , implies [\(5-2\)](#) with  $q = 2$ .  $\square$

We conclude this section with two immediate consequences of [Theorem 1.15](#).

**Corollary 5.5.** *Suppose that  $\mathcal{E} \cap [0, \infty) = \emptyset$ ,  $H \geq 0$  and  $0 \leq s < \frac{3}{2}$ . Then*

$$\|(-\Delta)^{s/2} H^{-s/2}\|_{\mathbb{B}(L^2)} + \|H^{s/2}(-\Delta)^{-s/2}\|_{\mathbb{B}(L^2)} < \infty.$$

*Proof.* The proof is analogous to that of [Lemma 2.8](#).  $\square$

**Corollary 5.6.** *Suppose that  $\mathcal{E} \cap [0, \infty) = \emptyset$  and  $H \geq 0$ . Let  $\varphi \in C_0^\infty(\mathbb{R})$  be such that  $\text{supp } \varphi \subset (\frac{1}{2}, 2)$ ,  $0 \leq \varphi \leq 1$  and  $\sum_{j \in \mathbb{Z}} \varphi(2^{-j} \lambda) = 1$  for all  $\lambda > 0$ . Then, for any*

$$\frac{2n}{n+3} < p < \frac{2n}{n-3},$$

*there exists  $C_p > 0$  such that*

$$C_p^{-1} \|f\|_{L^p} \leq \left\| \left( \sum_{j \in \mathbb{Z}} |\varphi(2^{-j} H) f(x)|^2 \right)^{1/2} \right\|_{L^p} \leq C_p \|f\|_{L^p}.$$

*In particular, if  $2 \leq p < 2n/(n-3)$ , then*

$$\|f\|_{L^p} \lesssim \left( \sum_{j \in \mathbb{Z}} \|\varphi(2^{-j} H) f\|_{L^p}^2 \right)^{1/2}.$$

*Proof.* With [Theorem 1.15](#) at hand, the corollary follows from a standard method in [\[Stein 1970\]](#). The proof is completely the same as that for the usual Littlewood–Paley estimate and we omit it.  $\square$

## 6. Eigenvalue bounds

This section is devoted to the proof of [Theorem 1.19](#). The proof is based on a method of Frank [\[2011; 2018\]](#). Recall that  $W \in L^{n/2+\gamma}(\mathbb{R}^n; \mathbb{C})$  with  $0 < \gamma < \infty$ . Then  $W$  is  $H$ -form compact. Indeed, taking  $M > -\inf \sigma(H)$ , we see that  $|W|^{1/2}(1 - \Delta)^{-1/2}$  is compact and  $(1 - \Delta)^{1/2}(H + M)^{-1/2}$  is bounded. Hence  $|W|^{1/2}(H + M)^{-1/2} = |W|^{1/2}(1 - \Delta)^{-1/2}(1 - \Delta)^{1/2}(H + M)^{-1/2}$  is also compact. Then there exists a unique  $m$ -sectorial operator  $H_W$  such that  $D(H_W) \subset Q(H_W) = \mathcal{H}^1$  and  $\langle H_W u, v \rangle = \langle (H + W)u, v \rangle$  for  $u \in D(H_W)$  and  $v \in \mathcal{H}^1$ . We also have  $D(H_W)$  is dense in  $\mathcal{H}^1$ , and  $\sigma(H_W)$  is contained in a sector  $\{z \in \mathbb{C} \mid |\arg(z - z_0)| \leq \theta\}$  for some  $z_0 \in \mathbb{R}$  and  $\theta \in [0, \frac{\pi}{2}]$ ; see [\[Kato 1966, Theorems VI.3.9 and VI.2.1\]](#). We fix a factorization  $W = W_1 W_2$  with  $W_1 = |W|^{1/2} \text{sgn } W$  and  $W_2 = |W|^{1/2}$ , where  $\text{sgn } W(x) = W(x)/|W(x)|$  if  $W(x) \neq 0$  and  $\text{sgn } W(x) = 0$  if  $W(x) = 0$ . Let  $d(z) = \text{dist}(z, [\infty))$ . We begin with the following lemma.

**Lemma 6.1.** *Suppose that  $E \in \mathbb{C} \setminus \sigma(H)$  is an eigenvalue of  $H_W$ . Then  $-1$  is an eigenvalue of  $W_1 R(E) W_2$ . Moreover, if  $0 < \gamma \leq \frac{1}{2}$ , the same statement also holds for  $E \in (0, \infty) \setminus \mathcal{E}$  with  $R(E)$  replaced by  $R(E+i0)$ .*

*Proof.* We show the lemma for the case  $E \in (0, \infty) \setminus \mathcal{E}$  only, since, in the case  $E \in \mathbb{C} \setminus \sigma(H)$ , the lemma is a consequence of the well-known Birman–Schwinger principle (see, e.g., [Frank 2018, Section 4]), and the proof is easier. Let  $f \in \text{Ker}_{L^2}(H_W - E)$ . We let  $\varphi \in \mathcal{S}$  and plug  $v = R(E - i\varepsilon)W_1\varphi \in \mathcal{H}^1$  into the identity  $\langle (H - E)f, v \rangle + \langle W_1 f, W_2 v \rangle = 0$ , letting  $\varepsilon \searrow 0$  and then using [Corollary 1.5\(2\)](#) to obtain

$$\langle W_1 f, \varphi \rangle + \langle W_1 R(E + i0)W_2 W_1 f, \varphi \rangle = 0.$$

Since  $\|W_1 f\|_{L^2} \lesssim \|W_1\|_{L^{n+2\gamma}} \|f\|_{\mathcal{H}^1} < \infty$ , this shows  $W_1 f \in \text{Ker}_{L^2}(I + W_1 R(E + i0)W_2)$ .  $\square$

Since  $W_1 R(E) W_2$  is a compact operator on  $L^2$ , if  $-1$  is an eigenvalue of  $W_1 R(E) W_2$  then  $\|W_1 R(E) W_2\|_{\mathbb{B}(L^2)} \geq 1$  at least. With this remark at hand, it is easy to see that [Theorem 1.19](#) follows from the following lemma.

**Lemma 6.2.** *For any  $\delta > 0$  and  $0 \leq \gamma \leq \frac{1}{2}$ , one has*

$$\|W_1 R(z) W_2\|_{\mathbb{B}(L^2)} \leq C_\delta |z|^{-\gamma/(n/2+\gamma)} \|W\|_{L^{n/2+\gamma}}, \quad z \in \mathbb{C} \setminus \mathcal{E}_\delta, \quad (6-1)$$

where  $R(z)$  is replaced by  $R(z + i0)$  if  $z \in (0, \infty) \setminus \mathcal{E}_\delta$ . Moreover, for any  $\gamma > \frac{1}{2}$ ,

$$\|W_1 R(z) W_2\|_{\mathbb{B}(L^2)} \leq C_{\gamma, \delta} |z|^{-(1/2)/(n/2+\gamma)} d(z)^{(\gamma-1/2)/(n/2+\gamma)} \|W\|_{L^{n/2+\gamma}}, \quad z \in \mathbb{C} \setminus (\mathcal{E}_\delta \cup [0, \infty)). \quad (6-2)$$

*Proof.* Inequality (6-1) is a direct consequence of [\(1-5\)](#) and [\(1-6\)](#) with

$$\frac{1}{p} = \frac{1}{2} + \frac{1}{n+2\gamma} \quad \text{and} \quad q = p'.$$

For the proof of (6-2), we take

$$\theta = \frac{2\gamma-1}{n+2\gamma} \in (0, 1)$$

so that

$$1 - \theta = \frac{n+1}{n+2\gamma}.$$

Interpolating between [\(1-5\)](#) with

$$p = \frac{2(n+1)}{n+3} \quad \text{and} \quad q = p'$$

and the trivial bound  $\|R(z)\|_{\mathbb{B}(L^2)} = \text{dist}(z, [0, \infty))^{-1}$  and, then, using Hölder's inequality, we obtain

$$\begin{aligned} \|W_1 R(E) W_2\|_{\mathbb{B}(L^2)} &\leq C_{\gamma, \delta} |z|^{-(1-\theta)/(n+1)} d(z)^{-\theta} \|W\|_{L^{n/2+\gamma}} \\ &= C_{\gamma, \delta} |z|^{-1/2/(n/2+\gamma)} d(z)^{(\gamma-1/2)/(n/2+\gamma)} \|W\|_{L^{n/2+\gamma}}, \end{aligned}$$

which completes the proof.  $\square$

## Appendix: Real interpolation and Lorentz space

Here a brief summary of real interpolation spaces and Lorentz spaces is given without proofs. One can find a much more detailed exposition in [Bergh and Löfström 1976; Grafakos 2008].

A pair of Banach spaces  $(\mathcal{A}, \mathcal{B})$  is said to be a Banach couple if both  $\mathcal{A}, \mathcal{B}$  are algebraically and topologically embedded in a Hausdorff topological vector space  $\mathcal{C}$ . Note that one can always take  $\mathcal{C}$  to be a Banach space  $\mathcal{A}_0 + \mathcal{A}_1$ . Given a Banach couple  $(\mathcal{A}_0, \mathcal{A}_1)$  and  $0 < \theta < 1$  and  $1 \leq q \leq \infty$ , one can define a Banach space  $\mathcal{A}_{\theta, q} = (\mathcal{A}_0, \mathcal{A}_1)_{\theta, q}$  by the so-called  $K$ -method, which satisfies that  $(\mathcal{A}_0, \mathcal{A}_0)_{\theta, q} = \mathcal{A}_0$  and  $(\mathcal{A}_0, \mathcal{A}_1)_{\theta, q} = (\mathcal{A}_1, \mathcal{A}_0)_{1-\theta, q}$  with equivalent norms and that if  $1 \leq q_1 \leq q_2 \leq \infty$  then  $(\mathcal{A}_0, \mathcal{A}_1)_{\theta, 1} \hookrightarrow (\mathcal{A}_0, \mathcal{A}_1)_{\theta, q_1} \hookrightarrow (\mathcal{A}_0, \mathcal{A}_1)_{\theta, q_2} \hookrightarrow (\mathcal{A}_0, \mathcal{A}_1)_{\theta, \infty}$ . Then the following real interpolation theorem is frequently used in this paper.

**Theorem A.1** [Bergh and Löfström 1976, Theorem 3.1.2; Cobos, Edmunds, and Potter 1990]. *Let  $(\mathcal{A}_0, \mathcal{A}_1)$  and  $(\mathcal{B}_0, \mathcal{B}_1)$  be two Banach couples,  $0 < \theta < 1$  and  $1 \leq q \leq \infty$ . Suppose that  $T$  is a bounded linear operator from  $(\mathcal{A}_0, \mathcal{A}_1)$  to  $(\mathcal{B}_0, \mathcal{B}_1)$  in the sense that  $T : \mathcal{A}_j \rightarrow \mathcal{B}_j$  and  $\|T\|_{\mathcal{B}(\mathcal{A}_j, \mathcal{B}_j)} \leq M_j$ ,  $j = 0, 1$ . Then  $T$  is bounded from  $\mathcal{A}_{\theta, q}$  to  $\mathcal{B}_{\theta, q}$  and satisfies  $\|T\|_{\mathcal{B}(\mathcal{A}_{\theta, q}, \mathcal{B}_{\theta, q})} \leq M_0^{1-\theta} M_1^\theta$ . Moreover, if both  $T : \mathcal{A}_0 \rightarrow \mathcal{B}_0$  and  $T : \mathcal{A}_1 \rightarrow \mathcal{B}_1$  are compact, then  $T : \mathcal{A}_{\theta, q} \rightarrow \mathcal{B}_{\theta, q}$  is also compact.*

Next we recall the definition and basic properties of Lorentz spaces. Given a  $\mu$ -measurable function  $f$  on  $\mathbb{R}^n$ , we let  $\mu_f(\alpha) = \mu(\{x \mid |f(x)| > \alpha\})$ . If we define the decreasing rearrangement of  $f$  by  $f^*(t) = \inf\{\alpha \mid \mu_f(\alpha) \leq t\}$  then the Lorentz space  $L^{p, q}(\mathbb{R}^n)$  is the set of measurable  $f$  such that the following quasinorm is finite:

$$\|f\|_{L^{p, q}}^* := \|t^{1/p-1/q} f^*(t)\|_{L^q(\mathbb{R}_+, dt)} = p^{1/q} \|\alpha \mu_f(\alpha)^{1/p}\|_{L^q(\mathbb{R}_+, \alpha^{-1} d\alpha)} < \infty.$$

Moreover, if  $1 < p < \infty$  and  $1 \leq q \leq \infty$  (which are sufficient for our purpose), then

$$\|f\|_{L^{p, q}} := \|f^{**}\|_{L^{p, q}}^*, \quad f^{**}(t) := \frac{1}{t} \int_0^t f^*(\alpha) d\alpha,$$

becomes a norm on  $L^{p, q}$  which makes  $L^{p, q}$  a Banach space. Furthermore,  $\|\cdot\|_{L^{p, q}}$  is equivalent to  $\|\cdot\|_{L^{p, q}}^*$  in the sense that  $\|f\|_{L^{p, q}}^* \leq \|f\|_{L^{p, q}} \leq C(p, q) \|f\|_{L^{p, q}}^*$  with some constant  $C(p, q) > 0$ . Thus all continuity estimates for linear operators can be expressed in terms of  $\|\cdot\|_{L^{p, q}}^*$ .  $L^{p, q}$  is increasing in  $q$ :  $L^{p, 1} \hookrightarrow L^{p, q_1} \hookrightarrow L^{p, p} = L^p \hookrightarrow L^{p, q_2} \hookrightarrow L^{p, \infty}$  if  $1 < q_1 < p < q_2 < \infty$ . Moreover,  $L^{p, q}$  is characterized by real interpolation: for  $0 < \theta < 1$ ,  $1 < p_1 < p_2 < \infty$  with

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$$

and  $1 \leq q \leq \infty$ , one has  $(L^{p_0}, L^{p_2})_{\theta, q} = L^{p, q}$  with equivalent norms. If  $1 < p, q < \infty$  then  $L^{p, q}(X; \mathbb{C})' = L^{p', q'}(X; \mathbb{C})$ , where  $r' = r/(r-1)$  is the Hölder conjugate of  $r$ .

Finally we record two inequalities used frequently in this paper. First, for  $1 \leq p, p_j < \infty$  and  $1 \leq q, q_j \leq \infty$  with

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and} \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2},$$

one has Hölder's inequality

$$\|fg\|_{L^{p,q}} \leq C\|f\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}}, \quad \|fg\|_{L^{p,q}} \leq C\|f\|_{L^\infty}\|g\|_{L^{p,q}}. \quad (\text{A-1})$$

Secondly, for  $1 < s < n$ ,  $1 < p < q < \infty$ ,  $1/p - 1/q = 2/n$  and  $1 \leq r \leq \infty$ , we have the HLS inequality

$$\|(-\Delta)^{-s/2}f\|_{L^{q,r}} \leq C\|f\|_{L^{p,r}}. \quad (\text{A-2})$$

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