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YVES COLIN DE VERDIÈRE

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SPECTRAL THEORY OF PSEUDODIFFERENTIAL OPERATORS OF DEGREE 0 AND AN APPLICATION TO FORCED LINEAR WAVES

YVES COLIN DE VERDIÈRE

We extend the results of our paper "Attractors for two-dimensional waves with homogeneous Hamiltonians of degree 0," written with Laure Saint-Raymond, to the case of forced linear wave equations in any dimension. We prove that, in dimension 2, if the foliation on the boundary at infinity of the energy shell is Morse–Smale, we can apply Mourre's theory and hence get the asymptotics of the forced solution. We also characterize the wavefront sets of the limit Schwartz distribution using radial propagation estimates.

Introduction

This paper contains new developments of some ideas already introduced in our paper [Colin de Verdière and Saint-Raymond 2020] concerning the spectral theory of self-adjoint pseudodifferential operators of degree 0 on closed manifolds. The main motivation comes from the study of forced internal or inertial waves in physics; see [Bajars et al. 2013; Brouzet 2016; Gostiaux et al. 2006; Maas et al. 1997; Maas and Lam 1995; Ogilvie 2005; Pillet 2018; Rieutord and Valdettaro 2010; 2018] and many other works. In what follows, H is a classical self-adjoint scalar pseudodifferential operator of degree 0 on a compact manifold M of dimension n without boundary, f is a smooth function and the spectral parameter ω is a real number. The main object to study is the linear forced wave equation

$$\frac{1}{i}\frac{du}{dt} + Hu = fe^{-i\omega t}, \quad u(0) = 0. \tag{1}$$

We are interested in the behaviour of $u(t)$ as $t \to +\infty$. Thanks to the spectral theorem, we can relate this behaviour to the spectral theory of H and hence to the Hamiltonian dynamics of the principal symbol $h : T^\star M \setminus 0 \to \mathbb{R}$, which is a smooth homogeneous function of degree 0. The main tools that we use are already classical: they are, on one hand, the general theory of pseudodifferential operators, culminating in the works of Lars Hörmander, Hans Duistermaat [1973], Alan Weinstein [1971; 1975] and many others, see also [Folland 1989; Dyatlov and Zworski 2019a], and, on the other hand, the theory initiated by Eric Mourre [1981; 1983] in order to get a flexible way to have a limit absorption principle, see also [Jensen et al. 1984; Gérard 2008; Cattaneo 2005].

What is the content, beyond that of [Colin de Verdière and Saint-Raymond 2020]? The main result is Theorem 6.2 where we extend the result of that work to the generic Morse–Smale case, still in dimension 2. The other new contribution is a precise description in arbitrary dimension of the dynamical assumptions

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allowing one to apply Mourre theory thanks to the Gårding inequality (see Section 3) by constructing a *global escape function*.

After recalling general facts on the Hamiltonian dynamics of a homogeneous Hamiltonian h of degree 0 in Section 1 and on the spectral theory of H in Section 2, we give, in Section 3, a necessary and sufficient condition on the dynamics at infinity, which ensures the existence of an escape function that will be the key input in order to apply Mourre's theory thanks to the Gårding inequality. In Section 4, we recall general facts that we got in [Colin de Verdière and Saint-Raymond 2020] for the forced wave equation from Mourre's theory. In Section 5, we use radial propagation estimates (see [Dyatlov and Zworski 2019a; 2019b]), going back to works of Melrose and Vasy, in order to locate the wavefront set of the Schwartz distribution u_∞, which is the limit (modulo bounded functions in L^2) of $u(t)$ as $t \to +\infty$.

We consider then, in Section 6, the case where M is a surface ($n = 2$), extending our results of [Colin de Verdière and Saint-Raymond 2020] to the generic case where the foliation is Morse–Smale and can have singular points (foci, nodes or saddles). Finally, we consider, in Section 7, the case where M is a 3-dimensional manifold with a free S^1-action leaving H invariant, which is important for applications to physics. We end the paper with a short review of related problems in Section 8 and two Appendices.

1. Hamiltonian of degree 0: classical dynamics

In what follows, we fix the following notation: M is a smooth connected compact manifold of dimension $n \geq 2$ without boundary, q is the generic point of M and $|dq|$ a smooth density on M. The Hamiltonian h is a smooth positively homogeneous function $h : T^\star M \setminus 0 \to \mathbb{R}$. We denote by (q, p) some local canonical coordinates on $T^\star M$ and by extension a generic point of $T^\star M$. The Hamiltonian vector field of h is denoted by $\mathcal{X}_h$ and we fix the "symplectic" conventions so that

$$\mathcal{X}_h = \frac{\partial h}{\partial p}\partial_q - \frac{\partial h}{\partial q}\partial_p, \qquad \mathcal{X}_h f = \{h, f\}$$

and denote by Φ_t the flow of $\mathcal{X}_h$. Because of the homogeneity of h, we have $pdq(\mathcal{X}_h) = 0$ and $\mathcal{X}_h$ is homogeneous of degree -1. Let us fix $\omega \in \mathbb{R}$ and define the energy shell $\Sigma_\omega := h^{-1}(\omega)$. We will assume in what follows that ω is *not a critical value of* h and hence Σ_ω is a smooth conic hypersurface in $T^\star M \setminus 0$. We need to introduce $Z_\omega := \Sigma_\omega/\mathbb{R}^+$, which is a smooth closed manifold of dimension $2n - 2$ and will be seen as the boundary at infinity of Σ_ω. The vector field $\mathcal{X}_h$ defines by projection a conformal class of vector fields on Z_ω, which we will call an (oriented) foliation and denote by $\mathcal{F}$. This foliation can admit singular points corresponding to the lines $\mathbb{R}^+ \cdot (q, p)$, where $\mathcal{X}_h$ is parallel to the cone direction $p\partial_p$. Note that we can and will often reduce ourselves to the case $\omega = 0$ by looking at the Hamiltonian $h - \omega$.

2. Hamiltonian of degree 0: spectral theory

Let us choose a self-adjoint pseudodifferential operator H of degree 0 acting on $L^2(M, |dq|)$ and of principal symbol h. Note that H is a bounded operator. In what follows, all pseudodifferential operators are "classical"; this means that the symbols do have full expansions in homogeneous functions with integer degrees. We are mainly interested by the spectral theory of H. As a warm up, we have:

Theorem 2.1. *The essential spectrum of H is the interval $J := [h_-, h_+]$, with $h_- := \min h$ and $h_+ := \max h$.*

Proof. If $\omega \in \mathbb{C} \setminus J$, then $H - \omega$ is elliptic and hence admits an inverse $R(\omega)$ modulo compact operators which can be chosen holomorphic in ω by taking $R(\omega) := \mathrm{Op}(h-\omega)^{-1}$, where Op is a fixed quantization on M:

$$R(\omega)(H-\omega) = \mathrm{Id} + K(\omega),$$

with K compact and holomorphic in ω. On the other hand, since H is bounded, $(H-\omega)$ is invertible for large values of ω. It follows from the Fredholm analytic theorem that the operator $H-\omega$ is invertible outside a discrete set where the kernels are finite-dimensional.

On the other hand, if $\omega \in J$, with $h(q_0, p_0) = \omega$ and $\epsilon > 0$ is fixed, choose a small neighbourhood U of q_0 so that, if $q \in U$, then $|h(q, p_0) - \omega| \le \epsilon$. Pick then $\phi \in C_o^\infty(U)$ with $\int_M |\phi|^2(q)\,|dq| = 1$. Let us check that, for t large enough,

$$\|(H-\omega)(\phi e^{itqp_0})\|_{L^2(M)} \le 2\epsilon. \tag{2}$$

It follows from the general properties of the principal symbols that

$$H(\phi(q)e^{itqp_0}) = h(q, p_0)\phi(q)e^{itqp_0} + O\Big(\frac{1}{t}\Big).$$

Take t so that the L^2 norm of the remainder is smaller than ϵ. We get inequality (2) by applying the triangular inequality. Hence

$$\|(H-\omega)(\phi e^{itqp_0})\|_{L^2(M)} \le 2\epsilon\|\phi e^{itqp_0}\|_{L^2(M)},$$

which proves that $\sigma(H) \cap [\omega - 2\epsilon, \omega + 2\epsilon] \ne \varnothing$. □

3. Escape functions

The key object of this paper is an escape function for h on the energy shell Σ_0:

Definition 3.1. A smooth function $k : \Sigma_0 \to \mathbb{R}$, positively homogeneous of degree 1, is called an *escape function* if there exists $\delta > 0$ so that the Poisson bracket $\{h, k\} = \mathcal{X}_h k$ is larger than δ on Σ_0.

A key observation is:

Remark 3.2. If we extend k to $T^\star M \setminus 0$ as a smooth homogeneous function $\tilde{k}$ of degree 1, then $\tilde{k}$ restricted to Σ_ω is still an escape function on Σ_ω for ω small enough.

We first give a general dynamical assumption on the oriented foliation $\mathcal{F}$ which turns out to be equivalent to the existence of a global *escape function*. We need some definitions, using the definitions of Appendix B:

Definition 3.3. We will say that the oriented 1-dimensional foliation $\mathcal{F}$ of the manifold Z_0 admits a *simple* structure (K_+, K_-) if $Z_0 = K_+ \cup K_- \cup \Omega$ as a disjoint union where:

- K_+ is an attractor of the oriented foliation $\mathcal{F}$, the *sink*.
- K_- is a repellor of the oriented foliation $\mathcal{F}$, the *source*.

- All leaves of points in Ω converge to K_+ at "$+\infty$" and to K_- at "$-\infty$"; in particular, the basin of K_+ is $\Omega \cup K_+$ and the basin of K_- for the reversed orientation of $\mathcal{F}$ is $\Omega \cup K_-$.

Definition 3.4. We say that a compact invariant set K_+ is weakly hyperbolic, denoted by (WH), if there exists, in some neighbourhood of K_+, a vector field W generating $\mathcal{F}$ and a smooth density $d\mu$ so that $\mathrm{div}_{d\mu}(W) < 0$. Similarly for K_-, we have $\mathrm{div}_{d\mu}(W) > 0$.

Our main result in this section is:

Theorem 3.5. *If the foliation $\mathcal{F}$ has a simple structure (K_+, K_-) with K_+ and K_- satisfying* (WH), *then there exists an escape function.*

The converse is true: the existence of an escape function implies that the foliation $\mathcal{F}$ has a simple structure (K_+, K_-) so that K_+ and of K_- satisfy (WH). *This simple structure is uniquely determined by $\mathcal{F}$.*

3A. *Dynamical assumptions implying weak hyperbolicity.* Let us choose a vector field W generating $\mathcal{F}$, whose flow is denoted by ϕ_t, $t \in \mathbb{R}$, and equip Z_0 with a smooth density $d\mu$.

Let us describe properties of closed invariant sets of $\mathcal{F}$ from which we can deduce (WH):

(1) If some component of K_+ is an isolated point a, the assumption (WH) says that the trace of the linearized vector field of W at the point a is negative. This is independent of the choice of W. The case where the singular point is hyperbolic is studied in [Guillemin and Schaeffer 1977]. They show that, in the generic situation, there exists a pseudodifferential normal form for such points. Independently, the classical part of this normal form is also described in dimension 2 in [Davydov 1985; Arnold 1983; Davydov et al. 2008].

(2) If some component of K_+ is a closed curve γ, the assumption (WH) says that the modulus of the determinant of the linearized Poincaré map is < 1. In dimension $n = 2$, this is equivalent to our assumption (M2) in [Colin de Verdière and Saint-Raymond 2020].

(3) They are more complicated attractors which satisfy (WH). The Lorenz attractor is one of them: the vector field generating it has negative divergence.

3B. *Construction of an escape function.* We construct an escape function assuming that $\mathcal{F}$ has a simple structure with $K_\pm$ satisfying (WH).

3B.1. *Escape function near Γ_+.* Let $\Gamma_\pm$ be the subcones of Σ_0 generated by the sets $K_\pm$. We will construct in this section an escape function k_+ in some conic neighbourhood U_+ of Γ_+. A similar construction can be done on the basin of Γ_-.

Let us first construct "polar coordinates" (ρ, θ) on Σ_0, where $\rho \in \mathbb{R}^+ \setminus 0$, $\theta \in Z_0$ and the dilations on Σ_0 act by $\lambda \cdot (\rho, \theta) = (\lambda\rho, \theta)$:

Lemma 3.6. *If W is a given vector field on Z_0 generating $\mathcal{F}$, there exist polar coordinates $(\rho, \theta) \in (\mathbb{R}^+ \setminus 0) \times Z_0$ on Σ_0 so that*

$$\mathcal{X}_h = a(\theta)\partial_\rho + \frac{1}{\rho} W.$$

Proof. We start with arbitrary polar coordinates (ρ_1, θ): for example identify Z_0 with the cosphere bundle $S_1^\star$ for some Riemannian metric on M and define $\rho_1(q, p)$ so that $(q, p/\rho_1(q, p)) \in S_1^\star$. We get, using the homogeneity of $\mathcal{X}_h$ and the fact that W span $\mathcal{F}$,

$$\mathcal{X}_h = a_1(\theta)\partial_{\rho_1} + \frac{1}{\rho} W,$$

with $\rho = A(\theta)\rho_1$ and hence $\partial_{\rho_1} = A(\theta)\partial_\rho$. □

The Liouville measure $dL_0 := |dq\, dp/dh|$ on Σ_0, being homogeneous of degree n, with respect to dilations, can be written as $dL_0 = \rho^{n-1}|d\rho|\, d\mu$, where $d\mu$ is a smooth measure on Z_0.

The fact that

$$\operatorname{div}_{dL_0}(\mathcal{X}_h) = 0$$

can be rewritten as

$$(n-1)a + \operatorname{div}_\mu(W) = 0. \tag{3}$$

The assumption (WH) implies that we have a smooth > 0 function F defined near K_+ so that

$$\operatorname{div}_{F\mu}(W) = \frac{dF(W)}{F} + \operatorname{div}_\mu(W) \le -c < 0.$$

Then, if $k_+ := F^{-1/(n-1)}\rho$, we get

$$dk_+(\mathcal{X}_h) = -\frac{1}{n-1} F^{-1/(n-1)} \left(\frac{dF(W)}{F} - (n-1)a \right),$$

which is equal to

$$dk_+(\mathcal{X}_h) = -\frac{1}{n-1} F^{-1/(n-1)} \operatorname{div}_{F\mu}(W),$$

and we get that the function k_+ is an escape function in some conical neighbourhood of Γ_+. □

We define similarly $k_- := -F^{-1/(n-1)}\rho$.

Note that k_+ tends to $+\infty$ as z tends to K_+ viewed as a set of points at infinity of Σ_0. We have also $k_+ \sim \langle p \rangle$ from the definition and the fact that F is positive.

Similarly, the function k_- defined near Γ_- tends to $-\infty$ as z tends to K_-.

3B.2. *Extension to* Σ_0. We choose a positive function m on Σ_0 which is smooth, homogeneous of degree 0 and equal to $m_\pm := \{h, k_\pm\}$ in some conical neighbourhoods $U_\pm$ of $\Gamma_\pm$. It follows from Proposition B.2(3) that we can choose U_+ so that $\Phi_t(U_+) \subset U_+$ for $t \ge 0$ and similarly for U_-.

Let z be in the basin of Γ_+ and define

$$l_+(z) = \lim_{t \to +\infty} \left(k_+(\Phi_t(z)) - \int_0^t m(\Phi_s(z))\, ds \right).$$

The limit exists because the expression of which we take the limit is independent of t for t large enough. Moreover the limit is smooth: if z is given and $\Phi_T(z) \in U_+$ for all $T \ge T_0$, there exists a neighbourhood V

of z so that $\Phi_{T_0}(V) \subset U_+$ and hence $\Phi_T(V) \subset U_+$ for all $T \geq T_0$. We have then, for $w \in V$,

$$l_+(w) = k_+(\Phi_{T_0}(w)) - \int_0^{T_0} m(\Phi_s(w))\, ds,$$

which is clearly smooth.

We define similarly l_-. The functions $l_\pm$ are escape functions in the basins of $\Gamma_\pm$ and satisfy in the respective basins $\{h, l_\pm\} = m$.

Let Γ_0 be the cone $\Gamma_0 := \{l_+ = 0\}$, which is smooth and transversal to $\mathcal{X}_h$ because $dl_+(\mathcal{X}_h) = m > 0$. On Γ_0 we have now the two functions $l_\pm$. The difference $\delta(z) = l_+(z) - l_-(z)$ is homogeneous of degree 1 and is constant along the flow lines. We will define k on the Hamiltonian trajectories $t \to \Phi_t(z)$ starting from $z \in \Gamma_0^1 := \{g^\star = 1\} \cap \Gamma_0$. For further use, we denote by S this hypersurface of $T^\star M$. The set Γ_0^1 is compact and hence the function $|\delta|$ is bounded by some constant $C > 0$ on it. Let us put $m_0 := \min m > 0$ and let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying

- $\psi(t) = 0$ if $t \leq 0$,
- $\psi(t) = 1$ if $t \geq 4C/m_0$,
- $|\psi'| \leq m_0/2C$.

We define now, for $z \in \Gamma_0^1$,

$$k(\phi_t(z)) = (1 - \psi(t))l_-(\Phi_t(z)) + \psi(t)l_+(\Phi_t(z)).$$

The derivative of k with respect to $\mathcal{X}_h$ is then equal to $m + \psi'(l_+ - l_-) \geq m_0/2$. We then extend k by homogeneity.

3C. ***Deriving the properties of $\mathcal{F}$ from the existence of an escape function.*** In what follows, we assume only the existence of an escape function k.

Let us give a construction of $\Gamma_\pm$ using only the dynamics of $\mathcal{X}_h$. We will see that these sets are defined independently of the choice of k: Γ_+ is the set of points $z \in \Sigma_0$ so that there exists $t_0 < 0$ with $\Phi_t(z) \to 0$ as $t \to t_0^+$; i.e., the trajectory of $\mathcal{X}_h$ is not complete as $t \to -\infty$. We similarly define Γ_- with $t_1 > 0$. We define $K_\pm$ so that they generate the cones $\Gamma_\pm$. Note that $\Gamma_+ \cap \Gamma_- = \varnothing$: if not, let $z \in \Gamma_+ \cap \Gamma_-$, and then $\Phi_t(z)$ tends to the zero section of $T^\star X$ as $t = t_0 + 0$, because the Hamiltonian flow is complete near the infinity of $T^\star X$. $\Phi_t(z)$ tends also to the zero section as $t = t_1 - 0$. This is not possible because the escape function tends to 0 at the zero section and is monotonic along the orbits.

Let us recall that we view $K_\pm$ as sets at infinity of the energy shell, namely the bases at infinity of the cones $\Gamma_\pm$.

Proposition 3.7. The picture of the dynamics is as follows:

- If $z \in \Sigma_0 \setminus (\Gamma_+ \cup \Gamma_-)$, then $\Phi_t(z)$ is defined for all $t \in \mathbb{R}$, $\Phi_t(z) \to K_+$ as $t \to +\infty$ and $\Phi_t(z) \to K_-$ as $t \to -\infty$.
- If $z \in \Gamma_+$, then $\Phi_t(z)$ is defined for all $t > t_0(z)$, $\Phi_t(z) \to K_+$ as $t \to +\infty$ and $\Phi_t(z) \to 0$ as $t \to t_0(z)$.

- If $z \in \Gamma_-$, then $\Phi_t(z)$ is defined for all $t < t_0(z)$, $\Phi_t(z) \to K_-$ as $t \to -\infty$ and $\Phi_t(z) \to 0$ as $t \to t_0(z)$.

Proof. Let us choose a metric g on M and consider the set $C_0 := k^{-1}(0) \cap (g^\star)^{-1}(1)$, where $g^\star$ is the dual metric. The set C_0 is a generating set for the cone $C := k^{-1}(0)$. If $z \in C_0$, the trajectory $t \to \Phi_t(z)$ is complete, because $t \to k(\Phi_t(z))$ is strictly monotonic and hence does not tend to the zero section where $k = 0$ at $t = \pm\infty$. Conversely, every complete trajectory cuts C_0 exactly in one point. This way we get a subset S of Σ_0 generating $\Sigma_0 \setminus (\Gamma_+ \cup \Gamma_-)$:

$$S := \{\Phi_t(z) \mid z \in C_0,\ t \in \mathbb{R}\}.$$

The orbits sitting in S have no limit points in S because the flow derivative of k is bounded below by some positive number. Let us consider the projections on Z_0 of S, Γ_+ and Γ_+, say Ω, K_+ and K_-. We have a disjoint union $Z_0 = \Omega \cup K_+ \cup K_-$. Each set is invariant by the foliation. Let us look at a leaf γ in Ω: γ has no limit points in Ω (because the foliation in Ω is diffeomorphic to the flow foliation in C). The limit points are then in $K_+ \cup K_-$. We have $\Gamma_+ \subset \{k > 0\}$ and $\Gamma_- \subset \{k < 0\}$. Hence the limit points at $+\infty$ are in K_+ and the limit points at $-\infty$ are in K_-. The set K_+ is an attractor: it is enough to consider the neighbourhoods U_N of K_+ which are the projections of the sets $(\{k \geq N\} \cap C) \cup \Gamma_+$. □

Let us show that the existence of an escape function implies that K_+ satisfies (WH): we choose polar coordinates (ρ, θ) near Γ_+ with $\rho = k$ and we have, from the equations derived in Section 3B.1, that $dk(\mathcal{X}_h) = a > 0$ and hence $\operatorname{div}_\mu W = -a/n < 0$: all components of K_+ satisfy (WH). A similar argument works for K_-.

3D. *Radial sink and sources.* Let us recall and introduce some notation: the radial compactification of $T^\star M$ is denoted by $\overline{T^\star M}$ and the boundary at infinity which we can identify with the sphere bundle is $S^\star M := T^\star M/\mathbb{R}^+$. The compactification of Σ_0 is $\overline{\Sigma}_0$ with the boundary at infinity $Z_0 = S\Sigma_0 \subset \overline{T^\star M}$.

Let us rephrase Definition E.52 of [Dyatlov and Zworski 2019a] in our context:

Definition 3.8. Let us introduce the symbol $r = -kh$, with k, an escape function (homogeneous of degree 1), and denote by ψ_t the flow of r extended to the boundary. The compact set $K_- \subset Z_0$ is a radial source for r if there exists a neighbourhood $U \subset \overline{T^\star M}$ of K_-, so that, uniformly for $z \in U$:

(1) For $t \leq 0$, $|k|(\psi_t(z)) \geq Ce^{\theta|t|}$ for some $C, \theta > 0$.

(2) $\psi_t(z) \to K_-$ as $t \to -\infty$.

We have:

Proposition 3.9. If k is an escape function, K_- is a radial source for $r = -kh$.

Proof. In the domain where $k < 0$, in particular near K_-, we have $\mathcal{X}_r = |k|\mathcal{X}_h - h\mathcal{X}_k$. The vector field $\mathcal{X}_r$ is homogeneous of degree 0 and hence projects onto $S^\star M$. We denote by Y_r this projection. Note that Y_r is tangent to Z_0, where it generates the foliation $\mathcal{F}$.

(1) We have $\mathcal{X}_r(|k|) = |k|\mathcal{X}_h|k| \leq -\delta|k|$. This implies that in a neighbourhood U_0 of K_-, where $k \leq -1$, for $t \leq 0$, we have $|k|(\psi_t(z)) \geq Ce^{\delta|t|}$.

(2) Let us choose V_0 a neighbourhood of K_- inside $S^\star M$ as follows: we choose first a neighbourhood V_1 of K_- in Z_0, with a smooth boundary, so that $\mathcal{Y}_r$ is outgoing and transversal to the boundary; take V_1 as the closure of the projection of the sets $\{k \leq -b\} \cap S$ for b large enough with S defined in Section 3B.2. We take for V_0 a neighbourhood of K_- in $S^\star M$ which is of the form $\{\exp(u\mathcal{Y}_r)(m) \mid m \in V_1, |u| \leq a\}$. If a is small enough, the vector field Y_r is transversal and outgoing at the boundary of V_0, because $Y_r(h) = h\{h,k\}$ and $\{h,k\} \geq \delta > 0$. Hence we get a repellor $L_- := \bigcap_{t\leq 0} \psi_t(V_1)$. The repellor L_- contains K_- and being invariant by the dynamics of $\mathcal{Y}_r$ restricted to Z_0 is equal to K_-. We then take for U_1 a small neighbourhood of V_0 in $\overline{T^\star M}$ and we get (2) by taking for U in the definition of a radial source the intersection $U_0 \cap U_1$. □

4. Applying Mourre's theory

Let us first recall some results of [Colin de Verdière and Saint-Raymond 2020]. Let us fix $\omega = 0$ for simplicity and assume that there exists an escape function k on the energy shell Σ_0. Then k can be extended to $T^\star M \setminus 0$ as an escape function in the cone $|h| \leq a$ with some $a > 0$. Let K be a self-adjoint operator of degree 1 of principal symbol k. Using the Gårding's inequality (see [Folland 1989, pp. 129–136]), one gets that K is a conjugate operator in the sense of Mourre: if J is a small enough open interval containing 0 and π_J is the spectral projector of H associated to the interval J, then

$$i\pi_J[H, K]\pi_J \geq c\pi_J + R,$$

where $c > 0$ and R is compact. Moreover the operator H is K-smooth; i.e., the map $t \to e^{itK}He^{-itK}$ is smooth with values into the bounded self-adjoint operators. Let us define the K-Sobolev spaces, denoted by $\mathcal{H}^s_K$, in the usual way using the s-powers of $(1+K^2)^{1/2}$. The usual Sobolev spaces will be denoted by $\mathcal{H}^s$. Let us give a comparison between the K-Sobolev spaces and the usual ones. There is a shift in the exponents due to the fact that the pseudodifferential calculus does not apply to nonelliptic operators like K.

Lemma 4.1. *If $f \in \mathcal{H}^1$, then $f \in \mathcal{H}^s_K$ for any $s \leq 1$. If $f \in \mathcal{H}^{-1}_K$, then $f \in \mathcal{H}^{-s}$ for any $s \leq -1$.*

Proof. If $f \in \mathcal{H}^1$, then $\langle (1+K^2)f \mid f\rangle < \infty$ because K^2 is a pseudodifferential operator of order 2 and hence $f \in \mathcal{H}^1_K$. The other inclusion follows by duality with respect to the L^2 product. □

It follows then from Mourre theory [Mourre 1981; 1983; Jensen et al. 1984; Gérard 2008] that:

Theorem 4.2 (Mourre). *The operator H has a finite number of eigenvalues in J, and they have finite multiplicity. Assuming that 0 is not an eigenvalue, the resolvent $(H-z)^{-1}$ defined for $\Im z > 0$ admits a boundary value $\omega \to (H - \omega - i0)^{-1}$ for ω real, close to 0, which, for any $\epsilon > 0$, is Hölder continuous for some positive Hölder exponent, depending on ϵ, from the Sobolev space $\mathcal{H}^{1/2+\epsilon}_K$ into $\mathcal{H}^{-1/2-\epsilon}_K$ for all $\epsilon > 0$.*

Moreover, if Π_- is the spectral projector on the negative part of the spectrum of K, then $f \in \mathcal{H}^{1+\epsilon}_K$ implies $\Pi_-((H-i0)^{-1}f) \in L^2$.

It follows then in our context:

Theorem 4.3 [Colin de Verdière and Saint-Raymond 2020]. *Assuming the existence of an escape function at $\omega = 0$ and that 0 is not an eigenvalue of H, then the solution $u(t)$ of the forced wave equation (1) with a smooth forcing f can uniquely be written as*

$$u(t) = u_\infty + \eta(t) + r(t),$$

where

- $u_\infty = (H - i0)^{-1}(f)$ *belongs to* $\mathcal{H}_K^{-1/2-\epsilon} \subset \mathcal{H}^{-1}$ *for all* $\epsilon > 0$,
- $\eta(t) \to 0$ *in* $\mathcal{H}_K^{-1/2-\epsilon} \subset \mathcal{H}^{-1}$ *for all* $\epsilon > 0$,
- *The function* $t \to r(t)$ *is bounded in* L^2 *has a Fourier transform vanishing near* 0,
- $\|u(t)\|_{L^2}^2 \sim ct$ *as* $t \to +\infty$ *with in general* $c > 0$.

5. Using radial source and sink propagation results

5A. *Wavefront set of u_∞.* We will now derive results on the distribution u_∞ using the radial propagation estimates of Dyatlov and Zworski, based on earlier ideas of Richard Melrose [1994] and Andras Vasy [2013].

Theorem 5.1. *The wavefront set of u_∞ is contained in the cone Γ_+.*

Proof. The result follows from the argument explained in the revised version of [Dyatlov and Zworski 2019b, Section 3.1]. This uses only the fact that K_- is a source (see Section 3D). They introduce an operator $\langle D\rangle$ which is elliptic self-adjoint invertible of degree 1. We choose it so that its principal symbol near Γ_- is $|k|$. They introduce then

$$v_\epsilon := \langle D\rangle^{-1/2}(H - i\epsilon)^{-1}\langle D\rangle^{-1/2}(g),$$

with $g = \langle D\rangle^{1/2}(f)$ and $u_\epsilon = (H - i\epsilon)^{-1}(f) = \langle D\rangle^{1/2} v_\epsilon$. Using a refined version of Theorem E.54 of [Dyatlov and Zworski 2019a], they show that there exists A, elliptic near Γ_- of degree 0, so that, for any s, the norms $\|Av_\epsilon\|_s$ are uniformly bounded in $\epsilon > 0$. We need to use here, in inequality (3.2) of [Dyatlov and Zworski 2019b], that $\|v_\epsilon\|_{-N}$ is bounded; we know it from Mourre theory for $N \geq 1$. Passing to the limit which is known to exist in $\mathcal{H}^{-1}$ by Theorem 4.3, we get that u_∞ is smooth near Γ_-. The usual propagation of singularities applied to the equation $Hu_\infty \in C^\infty$ gives the result. □

Proposition 5.2. If $Hu = 0$ and $u \in L^2(M)$, then u is smooth.

Proof. It follows from Exercise 33 in Appendix E7 of [Dyatlov and Zworski 2019a] that u is smooth near Γ_- and changing H into $-H$, u is also smooth near Γ_+. □

Remark 5.3. In the case $n = 2$, not all closed conical invariants subsets of Γ_+ can be wavefront sets of some u_∞. If the wavefront set contains the line generated by a (ws) saddle point, it contains also one of the two branches of the associated unstable manifold and hence, being closed, also an attractive invariant set. This is proved in [Guillemin and Schaeffer 1977], at least for generic cases.

5B. ***Sobolev regularity of u_∞.*** We saw in Section 4 that u_∞ belongs to $\mathcal{H}^{-1}$. Let us show that the radial sink estimates of [Dyatlov and Zworski 2019a] allow to get:

Theorem 5.4. *Under the assumption of the existence of an escape function, we have, for all $\epsilon > 0$, $u_\infty \in \mathcal{H}^{-1/2-\epsilon}$.*

Proof. We use the fact that K_+ is a sink as defined in [Dyatlov and Zworski 2019a, Definition E.52]: this is proved exactly the same way that we proved that K_- is a source in Section 3D, or just by reversing the orientations. We use then Theorem E.56 of [Dyatlov and Zworski 2019a] directly for the operator H knowing already that u_∞ is smooth away from Γ_+. Replacing $\langle \xi \rangle$ by $\langle k_+ \rangle$ we see that the threshold condition (E.5.44) is satisfied for $s < -\frac{1}{2}$. □

6. The 2-dimensional case

In this section $n = 2$.

6A. *Morse–Smale foliation.*

Definition 6.1. A hyperbolic singular point of $\mathcal{F}$ is called *weakly stable* if the trace of the linearization of any smooth vector field generating $\mathcal{F}$ is < 0. We define similarly *weakly unstable* hyperbolic singular points. We denote these properties respectively by (ws) and (wu).

Note that if $dh \neq 0$ on Σ_0, any saddle point is either weakly stable or weakly unstable depending on whether $\mathcal{X}_h$ is pointing to infinity or not; this follows from (3) where $a \neq 0$.

Let us recall that a vector field on a surface is *Morse–Smale* if the nonwandering points are singular hyperbolic points and closed hyperbolic cycles and there is no saddle connection, i.e., there is no leave whose both limit points are saddle points. We extend this definition to oriented foliations of surfaces by choosing any vector field generating the foliation.

Theorem 6.2. *Let n be equal to 2. Let us assume that the foliation $\mathcal{F}$ is Morse–Smale. Then there exists an escape function. The set K_+ is the union of all the attracting cycles and points and all the unstable manifolds of the* (ws) *saddle points. The set K_- is constructed in a similar way.*

Remark 6.3. Any generic foliation of a closed surface satisfies the previous properties: Maurício Peixoto proved in the sixties that Morse–Smale vector fields on surfaces are generic; see [Palis and de Melo 1982, Chapter 4] for a detailed proof. As pointed out to me by Sylvain Courte, this genericity property extends to our context, i.e., to singular foliations of a surface embedded in a contact manifold, as it is proved in the Ph.D. thesis of Emmanuel Giroux [1991, Lemme 1.3].

Proof. Note first that K_+ and K_- are compact. They are also disjoint because there is no saddle connection.

Let us prove that K_+ is an attractor. Let K_0 be the union of the attracting component of K_+. The compact set K_0 itself is an attractor. Let us assume for simplicity that there exists an unique (ws) saddle-point b. Near b the foliation has a local normal form: the level sets of the function xy in a ball B contained in $\mathbb{R}^2_{x,y}$ with the orientation given by $x\partial_x - y\partial_y$. Let us consider a neighbourhood U_0 of K_0 satisfying the conclusion of Proposition B.2. The basin of K_0 is the complement in Z_0 of the union of all unstable cycles and all the stable manifolds of the saddle points. In particular by taking $\phi_{-T}(U_0)$ with T large enough instead of U_0 one can assume that U_0 contains $L := \{|x| \geq a, |y| \leq b\} \cap B$ with $a, b > 0$. Let us take now for the neighbourhood of K_+ the set $U := U_0 \cup L$. Clearly $\bigcap_{t \geq 0} \phi_t(U) = K_+$.

Remark 6.4. $K_0 \cup \{b\}$ is not an attractor!

Let us fix a density $d\mu$ on Z_0 and construct a vector field W generating $\mathcal{F}$ near K_+ whose divergence is nonpositive on K_+. First, we construct a vector field W_b with $\operatorname{div}(W_j) < 0$ in some neighbourhood U_b of each (ws) saddle point b. We construct also (see Appendix A.2) a vector field W_a in the basin of each attractive cycle or point a with nonpositive divergence. Let us choose a positive function l_a tending to $+\infty$ at the boundary of the basin of a. Then, for L_a large enough the set $\{l_a \geq L_a\}$ intersects the unstable manifolds Y_j of each (ws) saddle point b_j inside U_{b_j}. We choose $\chi_a \in C_o^\infty(\mathbb{R}, [0, 1])$ so that $\chi_a(s) = 1$ for $0 \leq s \leq L_a$ and $\chi_a'(s) \leq 0$ for $s \geq 0$. Then we take

$$W = \sum_a \Big((\chi_a \circ l_a) W_a + C \sum_{b_j(ws)} \psi_j W_{b_j} \Big)$$

where ψ_j satisfies

- $\psi_j \in C_0^\infty(U_{b_j}, R_+)$,
- $\psi_j = 1$ on $\{l_a \geq L_a\} \cap Y_j$,
- $d\psi_j(W) \leq 0$ on $Y_j \cap U_{b_j}$,

and $C \gg 1$. This smooth vector field is well-defined near K_+ and has negative divergence on K_+. □

6B. *Lagrangian distributions associated to hyperbolic closed leaves.* Let $\Gamma \subset T^\star X \setminus 0$ be a conic component of Γ_+ generated by a closed hyperbolic cycle $K_{+,0}$ of the foliation $\mathcal{F}$. The cone Γ is a conic Lagrangian submanifold of $T^\star X \setminus 0$: the Euler identity implies $\omega(\mathcal{X}_h, p\partial_p) = 0$. A theorem of Alan Weinstein [1971] implies that there is a homogeneous canonical transformation χ defined in a conic neighbourhood C of Γ whose image is a conic neighbourhood of the zero section of $T^\star\Gamma$ and so that $\chi(\Gamma)$ is the zero section of $T^\star\Gamma$. More precisely χ restricted to Γ identifies Γ to the zero section of its own cotangent bundle. Taking polar coordinates $(x, \eta) \in (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}_+ \setminus 0)$ on the cone Γ, the cotangent bundle of Γ admits coordinates $(x, \eta; \xi, y)$ with the symplectic form $d\xi \wedge dx + dy \wedge d\eta$. Note that they are not the symplectic coordinates of $T^\star X$, but those of $T^\star\Gamma$! Let X_0 be defined as $X_0 := (\mathbb{R}/2\pi\mathbb{Z})_x \times \mathbb{R}_y$. The symplectic map $(x, \eta; \xi, y) \to (x, -y; \xi, \eta)$ from $T^\star\Gamma$ onto $T^\star X_0$ identifies $T^\star\Gamma$ with $T^\star X_0$. With this identification, Γ is moved into $\Gamma_0 = \{y = 0, \xi = 0\}$, which is the conormal bundle of the circle of $\gamma_0 \subset X_0$ defined by $y = 0$. The Hamiltonian vector field $\mathcal{X}_0$ of $h_0 := h \circ \chi^{-1}$ preserves Γ_0. Along Γ_0, it is then given by $\mathcal{X}_0 = \partial_\xi h_0 \partial_x - \partial_y h_0 \partial_\eta$ and there $\partial_x h_0 = \partial_\eta h_0 = 0$. Because the foliation $\mathcal{F}$ is nonsingular near $K_{+,0}$, we have $\partial_\xi h_0 \neq 0$. Hence the image of the energy shell Σ_0 is given by $\xi/\eta = F(x, y)$. The projection $\pi : Z_0 \to X_0$ is a local diffeomorphism near $K_{+,0}$. Because it is a diffeomorphism on the cycle $K_{+,0}$, it is even a global diffeomorphism.

Using the tools introduced by Alan Weinstein [1975], we can build an FIO microlocally unitary $U : L^2(X) \to L^2(X_0, M)$ with M a flat bundle, called the Maslov bundle, so that $UHU^\star - K$ is smoothing in C and $\sigma_p(K) = h \circ \chi^{-1}$, $\mathrm{sub}(K) = 0$. We are then reduced to the case already studied in [Colin de Verdière and Saint-Raymond 2020] where the projection of γ onto M is a diffeomorphism.

This proves, following then [Colin de Verdière and Saint-Raymond 2020]:

Theorem 6.5. *If Γ is a component of Γ_+ generated by a closed hyperbolic stable cycle of $\mathcal{F}$, the distribution u_∞ is microlocally near Γ a Lagrangian distribution.*

7. The 3-dimensional case with S^1 invariance

Quite often in physical situations, there is an invariance of the problem by rotation or translation: internal waves in some canal [Maas and Lam 1995], inertial waves inside the earth or some stars [Rieutord and Valdettaro 2018], etc. We will study the case where $M = N_q \times S^1_\theta$ is a 3-manifold with the canonical action of S^1 by translation on the second factor. We denote by $(q, p; \theta, \tau)$ some local canonical coordinates on $T^\star M$ and assume that N is equipped with a smooth density $|dq|$ and M with $|dq\, d\theta|$. Let us give a smooth Hamiltonian $h = h(q, p, \tau)$, homogeneous of degree 0, on $T^\star M \setminus 0$ and a self-adjoint pseudodifferential operator of degree 0, H, of principal symbol h, acting on $L^2(M, |dq\, d\theta|)$. We assume that H commutes with the S^1-action. The operator H is then a direct sum of operators on M:

$$H = \bigoplus_{n\in\mathbb{Z}} H_n,$$

where H_n acts on $L^2(N, |dq|)$ as a self-adjoint pseudodifferential operator of principal symbol $h_n(q, p) := h(q, p, n)$ which is also equal to $h(q, p/n, 1)$ if $n \neq 0$.

The spectrum of H is clearly the closure of the union of the spectra of the H_n's.

7A. *Spectra of H and the H_n's.*

Let us define $h_0(q, p) := h(q, p, 0)$ and $h_1(q, p) = h(q, p, 1)$. Note that h_1 is a smooth symbol of degree 0 on $T^\star N$ which is asymptotic to h_0 at infinity. The essential spectrum of H is the interval $I_\infty := [a_\infty, b_\infty]$, where $a_\infty = \inf h_1$ and $b_\infty = \sup h_1$. The essential spectrum of the H_n's is quite different: from the identities

$$h(q, p, n) = h\left(q, \frac{p}{|p|}, \frac{n}{|p|}\right) = h_0(q, p) + O\left(\frac{1}{|p|}\right),$$

one gets that the principal symbol of H_n is h_0. Hence the essential spectrum of any of the H_n's is $I_0 := [a_0, b_0]$, where $a_0 = \inf h_0$ and $b_0 = \sup h_0$. Note that we have $I_0 \subset I_\infty$ and they are often identical in the applications to physical problems.

We are interested in more precise properties of the spectra: we claim that, in $I_\infty \setminus I_0$, the spectrum of H is pure point dense; i.e., there is a basis of L^2 pairwise orthogonal eigenfunctions. Moreover the eigenvalues of H_n obey a Weyl rule when $n \to \infty$. One expects that the spectrum has no embedded eigenvalues in the interior of I_0. But quasimodes of the type "well in an island" are possible if the dynamics of h_1 has stable bounded invariant sets (see Section 7B).

Theorem 7.1 (Weyl law). *The spectra $\sigma(H_n)$ of the operators H_n in $I_\infty \setminus I_0$ are discrete. For any compact interval J included in $I_\infty \setminus I_0$, we have*

$$\#\{\sigma(H_n) \cap J\} \sim_{n\to\infty} \frac{n^2}{4\pi^2} \operatorname{vol}(\{q, p \mid h_1(q, p) \in J\}),$$

where the volume is defined with the Liouville measure on $T^\star N$ and the eigenvalues of H_n in J are counted with multiplicities.

Proof. The full symbol of H can be written as

$$\tilde{h} = h(q, p, \tau) + \sum_{j=1}^{\infty} k_j(q, p, \tau),$$

with k_j homogeneous of degree j. Hence H_n can be viewed as a semiclassical pseudodifferential operator on N of semiclassical symbol

$$\tilde{h}_n = h_1(q, \hbar p) + \sum_{j=1}^{\infty} \hbar^j k_j(q, \hbar p, 1),$$

with $\hbar = 1/n$. The theorem follows hence from the semiclassical Weyl asymptotics. □

7B. *Classical dynamics.* We will assume that the frequency $\omega = 0$ is fixed and the 2-dimensional Hamiltonian $h_0(q, p) := h(q, p, 0)$ admits an escape function. We will look at the dynamics of $h_1 := h(q, p, 1)$. Note that the dynamics of h reduces on each set $\tau = a$ with $a \neq 0$ to that of h_1 by some simple rescaling of the time. Moreover

$$\lim_{p\to\infty} h_1(q, p) = h_0(q, p).$$

Near infinity the dynamics still admits an escape (Lyapunov function) and hence the orbits, if they come close enough to infinity, will converge to K_+ at $+\infty$ and K_- at $-\infty$. The dynamics $t \to \phi_t$ of h_1 is hence complete. We split the phase space into three pieces: $T^\star M = \Omega \cup C_+ \cup C_-$, where

- Ω is the set of (q, p) so that $\phi_t(q, p) \to K_\pm$ as $t \to \pm\infty$,
- C_+ is the set of (q, p) so that $\phi_t(q, p)$ stays bounded for $t \geq 0$,
- C_- is the set of (q, p) so that $\phi_t(q, p)$ stays bounded for $t \leq 0$.

Finally, we define $C := C_+ \cap C_-$, the set (q, p) so that $\phi_t(q, p)$ stays bounded for $t \in \mathbb{R}$. In the literature, C is called the *trapped set*.

It could happen that C supports some quasimodes associated to the semiclassical parameter $1/n$. Generically, these quasimodes are not close to true L^2-eigenfunctions because such eigenfunctions do not exist. They are still visible in the wave dynamics for a very long time…

8. Open problems

There are still many open problems. Let us describe a few of them:

• How does the spectral picture extend outside the intervals with a.c. spectra? This problem is already not solved in the simple case where Z_0 is a 2-torus, assuming the existence of a global transversal to the foliation, and the Poincaré map loses its hyperbolicity in a generic way.

• More generally, can we study what happens at the critical values of h assuming that this function is Morse or even Morse–Bott on $S^\star M$?

• What can we do in the case of a manifold with boundary? In particular, can we say something in the case of a polygon which is studied in the experiments of the Thierry Dauxois's team [Brouzet 2016].

• Prove the generic absence of embedded eigenvalues.

• Consider the *viscous case*, namely the forced equation

$$\frac{du}{dt} + iHu - \sigma\Delta u = fe^{-i\omega t}, \quad u(0) = 0. \tag{4}$$

where σ is a positive number and Δ is the Laplacian associated to some Riemannian metric on M. Study the "small viscosity" limit $\sigma \to 0$? In particular, do the limits $\sigma \to 0^+$ and $t \to +\infty$ commute?

• There is a discrete analogue of Mourre's theory for unitary maps; see for example [Fernández et al. 2013]. What can be said from the spectral theory of the unitary action of a diffeomorphism of a closed manifold on half-densities? For example, what is the spectral theory of a diffeomorphism of the circle with irrational rotation number which is not C^1-conjugated to a rotation?

Appendix A: Divergences

A1. *Formulae.* Let us give a smooth vector field W whose flow is denoted by ϕ_t, $t \in \mathbb{R}$, and a smooth density $d\mu$. The divergence of W with respect to $d\mu$ is the function defined by

$$\operatorname{div}_{d\mu}(W) := \frac{\mathcal{L}_W d\mu}{d\mu},$$

where the Lie derivative $\mathcal{L}_W d\mu$ is defined by

$$\mathcal{L}_W d\mu := \frac{d}{dt}_{|t=0} \phi_t^\star(d\mu).$$

Cartan's formula gives

$$\operatorname{div}_{d\mu}(W) = \frac{d(\iota(W)d\mu)}{d\mu},$$

where $\iota(\,\cdot\,)$ is the inner product. In particular, we get the useful formulae

$$\operatorname{div}_{d\mu}(fW) = df(W) + f\operatorname{div}_{d\mu}(W),$$
$$\operatorname{div}_{g\,d\mu}(W) = \frac{dg(W)}{g} + \operatorname{div}_{d\mu}(W).$$

A2. ***Extending vector fields with negative divergence.***

Lemma A.1. *Let us assume that the invariant compact set K admits a smooth (Liapounov) function l defined in the basin B of K with $dl(W) < 0$ outside K and $l(K) = 0$ and $l \to +\infty$ at the boundary of B (this is the case in particular if the attractor K is hyperbolic). If the vector field W satisfies $\operatorname{div}_{d\mu}(W) < 0$ in some open neighbourhood V of K, then there exists a vector field $W_1 = FW$ in B, so that $F > 0$ and $\operatorname{div}_{d\mu}(W_1) < 0$ in B.*

Proof. Let us choose $r > 0$ so that $\{l \le r\} \subset V$. It is enough to take $F = 1$ in $\{l \le r\}$ and, for any $x \in B$ with $l(x) = r$ and any $t \ge 0$,

$$F(\phi_t(x)) := e^{\int_0^t \Phi(\phi_s(x))\, ds},$$

with Φ smooth, $\Phi = 0$ near $l(y) \le r$ and, for all y with $l(y) > r$, $\Phi(y) < -\operatorname{div}_{d\mu}(W)(y)$. □

Appendix B: Attractors and their basins

We give here some useful definitions and elementary properties of dynamical systems. We consider a smooth closed manifold X with a smooth vector field V whose flow is the 1-parameter group of diffeomorphisms of X denoted by ϕ_t, $t \in \mathbb{R}$. The definitions and statements are taken from [Hurley 1982]. We have the following:

Definition B.1. (1) If $K \subset X$ is a compact invariant set, i.e., a subset of X preserved by the flow, K is called an *attractor* if there exists an open neighbourhood U of K in X so that $K = \bigcap_{t \ge 0} \phi_t(U)$.

(2) If K is an attractor, the *basin* of K is the set of points x so that $\phi_t(x) \to K$ as $t \to +\infty$.

(3) A point $x \in X$ is wandering if there exists a neighbourhood U of x so that $\phi_t(U) \cap U = \varnothing$ for t large enough.

The set of wandering points is open. The basins are open subsets of X. We will need the following properties [Hurley 1982, Lemma 1.6]:

Proposition B.2. If K is an attractor, and V a neighbourhood of K, there exists an open set U satisfying:

(1) $K \subset U \subset V$.

(2) $\bigcap_{t \ge 0} \phi_t(\bar{U}) = K$.

(3) For all $t \ge 0$, we have $\phi_t(U) \subset U$.

The convergence of $\phi_t(m)$ to K is uniform on every compact subset of the basin of K.

The previous sets are the same for V and fV, where $f : X \to]0, +\infty[$ is smooth. They can therefore be defined for a 1-dimensional oriented foliation generated by a smooth vector field. In particular the open set U of the previous proposition is independent of f.

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YVES COLIN DE VERDIÈRE: yves.colin-de-verdiere@univ-grenoble-alpes.fr
Université Grenoble Alpes, Institut Fourier, Unité mixte de recherche CNRS-UGA 5582, Saint Martin d'Hères, France

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