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We prove global existence of instantaneously complete Yamabe flows on hyperbolic space of arbitrary dimension $m \geq 3$ starting from any smooth, conformally hyperbolic initial metric. We do not require initial completeness or curvature bounds. With the same methods, we show rigidity of hyperbolic space under the Yamabe flow.

Let (M, g_M) be a Riemannian manifold. Let $g_0 = u_0 g_M$ be a conformal metric on M defined by a smooth function $u_0 : M \rightarrow]0, \infty[$. A family $(g(t))_{t \in [0, T[}$ of Riemannian metrics on M is called *Yamabe flow* with initial metric g_0 if for all $t \in [0, T[$

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -R_{g(t)} g(t), \\ g(0) = g_0, \end{cases} \quad (1)$$

where R_g denotes the scalar curvature of the Riemannian manifold (M, g) . Richard Hamilton [1989] introduced this flow as alternative approach to the Yamabe problem and showed that solutions to (1) exist on any compact manifold (M, g_0) without boundary. Since then, a full theory on compact manifolds was developed with major contributions by Chow [1992], Ye [1994], Schwetlick and Struwe [2003] and Brendle [2005; 2007].

In dimension $\dim(M) = 2$, where the Yamabe flow coincides with the Ricci flow, Gregor Giesen and Peter Topping [Giesen and Topping 2011; Topping 2010; 2015] obtained existence and uniqueness of instantaneously complete solutions to (1) on an arbitrary surface (M, g_0) . Instantaneous completeness means that the Riemannian manifold $(M, g(t))$ is geodesically complete for all $t > 0$ even if the initial surface (M, g_0) is incomplete. In [Schulz 2019a], the author studied the question of whether Giesen and Topping's results generalise to noncompact manifolds of higher dimension and obtained affirmative results for manifolds conformally equivalent to hyperbolic space provided that the conformal factor in the initial metric and the initial scalar curvature are both uniformly bounded from above. In the present paper, we now are able to show existence of instantaneously complete Yamabe flows for *any* conformal initial metric on hyperbolic space $(\mathbb{H}, g_{\mathbb{H}})$ of dimension $m \geq 3$.

Theorem 1 (existence). *Let $g_0 = u_0 g_{\mathbb{H}}$ be any conformal Riemannian metric on hyperbolic space $(\mathbb{H}, g_{\mathbb{H}})$ of dimension $m \geq 3$. Then, there exists an instantaneously complete Yamabe flow $(g(t))_{t \in [0, \infty[}$ on \mathbb{H} satisfying*

- (1) $g(0) = g_0$,
- (2) $g(t) \geq m(m-1)t g_{\mathbb{H}}$ for all $t > 0$.

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Remark. Recall that hyperbolic space is a noncompact, simply connected Riemannian manifold of constant sectional curvature -1 and scalar curvature $R_{g_{\mathbb{H}}} = -m(m-1)$. Theorem 1 improves the existence result obtained in [Schulz 2019a], which depended on the uniform upper bounds $g_0 \leq C_0 g_{\mathbb{H}}$ and $R_{g_0} \leq K$ on the initial metric and its scalar curvature. Here, we are able to drop these assumptions entirely, so the existence of instantaneously complete Yamabe flows on hyperbolic space of any dimension is true with the same level of generality as in dimension 2. It is likely that the proof of Theorem 1 can be generalised to apply on any noncompact manifold (M, g_M) with strictly negative scalar curvature $-\kappa_2 \leq R_{g_M} \leq -\kappa_1 < 0$ in place of hyperbolic space.

However, there exist (geodesically incomplete) initial manifolds (M, g_0) which do *not* allow any instantaneously complete solution to the Yamabe flow. Conformally flat examples (M, g_0) such as the punctured sphere in dimension $m \geq 3$ are given in [Schulz 2019a, Theorem 3]. The incompleteness of Yamabe flows on arbitrary punctured manifolds will be analysed in a forthcoming article.

For the more general class of complete, noncompact background manifolds (M, g_M) with nonpositive, bounded scalar curvature and positive Yamabe invariant, the author proves global existence of complete Yamabe flows in [Schulz 2019b], provided the initial metric $g_0 = u_0 g_M$ already is complete with $c_1 \leq u_0 \leq c_2$ for some constants $c_1, c_2 > 0$, so that cases like the punctured sphere are excluded. However, no assumptions on the curvature of g_0 are required.

If g_0 is any Riemannian metric on some noncompact manifold M , then existence of a global Yamabe flow on M with initial metric g_0 was shown by Yinglian An and Li Ma [1999] provided that (M, g_0) is complete, with Ricci curvature bounded from below and with bounded, nonpositive scalar curvature, and also by Li Ma [2016] under the assumption that (M, g_0) is complete with nonnegative scalar curvature R_{g_0} which allows a positive solution $w > 0$ of the equation

$$-\Delta_{g_0} w = \frac{m-2}{4(m-1)} R_{g_0}$$

in M .

Bahuaud and Vertman [2014; 2019] constructed Yamabe flows starting from spaces with incomplete edge singularities which are preserved along the flow. Recently, Choi, Daskalopoulos, and King [Choi et al. 2018] constructed solutions to the Yamabe flow on \mathbb{R}^m which develop a type II singularity in finite time.

In general, solutions to problem (1) on noncompact manifolds M are not unique. An example on $M = \mathbb{H}$ is the flat metric $g_0 = g_{\mathbb{E}}$, which is conformally equivalent to $g_{\mathbb{H}}$ according to the Poincaré ball model. By Theorem 1, there exists an instantaneously complete Yamabe flow $(g(t))_{t \in [0, \infty[}$ on \mathbb{H} with $g(0) = g_{\mathbb{E}}$. However, the constant flow given by $\bar{g}(t) = g_{\mathbb{E}}$ for all t is also a solution to (1) because $R_{g_{\mathbb{E}}} = 0$. In [Schulz 2019a] we conjectured that uniqueness holds in the class of instantaneously complete Yamabe flows and obtained a partial result in the class of rotationally symmetric, instantaneously complete flows. While the conjecture is still open in general, the methods used in the proof of Theorem 1 yield the following result without the assumption of symmetry or completeness.

Theorem 2 (rigidity). *Let $(\mathbb{H}, g_{\mathbb{H}})$ be hyperbolic space of dimension $m \geq 3$. Let $(g(t))_{t \in [0, T[}$ be a Yamabe flow on \mathbb{H} with $g(0) = g_{\mathbb{H}}$. Then, the flow is uniquely given by $g(t) = (m(m-1)t + 1)g_{\mathbb{H}}$ for all $t \in [0, T[$.*

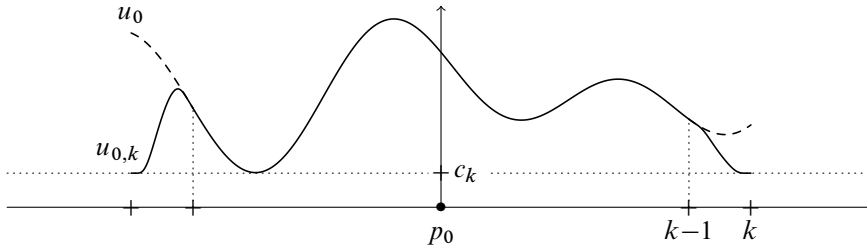


Figure 1. The construction of $u_{0,k}$ given u_0 .

Proofs of the main results

Let $u_0 : \mathbb{H} \rightarrow]0, \infty[$ be the conformal factor of the given metric $g_0 = u_0 g_{\mathbb{H}}$ on \mathbb{H} . We assume that the restriction of u_0 to any smooth, bounded domain $\Omega \subset \mathbb{H}$ is in the Hölder space $C^{2,\alpha}(\Omega)$ for some $0 < \alpha < 1$. Since the Yamabe flow preserves the conformal class, any Yamabe flow $(g(t))_{t \in [0, T[}$ on \mathbb{H} with $g(0) = g_0$ is of the form $g(t) = u(\cdot, t)g_{\mathbb{H}}$ with a conformal factor $u : \mathbb{H} \times [0, T[\rightarrow]0, \infty[$ satisfying

$$\begin{cases} \frac{\partial}{\partial t} u = -R_g u, \\ u(\cdot, 0) = u_0. \end{cases} \tag{2}$$

In dimension $m = \dim(\mathbb{H}) \geq 3$ we may introduce the exponent

$$\eta = \frac{m-2}{4}$$

and the function $U = u^\eta$ in order to express the scalar curvature of the conformal metric $g = u g_{\mathbb{H}}$ by

$$R_g = U^{-\frac{m+2}{m-2}} \left(R_{g_{\mathbb{H}}} U - 4 \frac{m-1}{m-2} \Delta_{g_{\mathbb{H}}} U \right)$$

and to formulate the following equivalent evolution equations for U and u respectively:

$$\frac{1}{m-1} \frac{\partial U}{\partial t} = (m\eta U + \Delta_{g_{\mathbb{H}}} U) U^{-\frac{1}{\eta}}, \tag{3}$$

$$\frac{1}{m-1} \frac{\partial u}{\partial t} = m + \frac{\Delta_{g_{\mathbb{H}}} u}{u} + \frac{(m-6)}{4} \frac{|\nabla u|_{g_{\mathbb{H}}}^2}{u^2}. \tag{4}$$

Let $B_k = B_k(p_0) \subset \mathbb{H}$ be the open metric ball of radius k around some origin $p_0 \in \mathbb{H}$. Let $\chi_k : \mathbb{H} \rightarrow [0, 1]$ be smooth with compact support in B_k satisfying $\chi_k(x) = 1$ for all $x \in B_{k-1}$. For any $k > 2$ we define

$$c_k := \inf_{B_k} u_0 > 0, \tag{5}$$

$$u_{0,k} := (1 - \chi_k) c_k + \chi_k u_0, \tag{6}$$

$$\phi_k(t) := c_k + m(m-1)t. \tag{7}$$

Then, $u_{0,k} \in C^{2,\alpha}(B_k)$ coincides with u_0 in B_{k-1} and takes the constant value c_k in some neighbourhood of ∂B_k as shown in Figure 1. Moreover, $u_{0,k}$ and ϕ_k satisfy the first-order compatibility conditions

for (4). As shown in [Schulz 2019a, Lemma 1.1] there exists some $T_k > 0$ and a solution $u > 0$ of

$$\begin{cases} \frac{1}{m-1} \frac{\partial u}{\partial t} = m + \frac{\Delta_{g_{\mathbb{H}}} u}{u} + \frac{(m-6)}{4} \frac{|\nabla u|_{g_{\mathbb{H}}}^2}{u^2} & \text{in } B_k \times [0, T_k[, \\ u = \phi_k & \text{on } \partial B_k \times [0, T_k[, \\ u = u_{0,k} & \text{on } B_k \times \{0\}. \end{cases} \tag{8}$$

The following pointwise estimate is analogous to [Schulz 2019a, Lemma 1.3].

Lemma 1. *Let u be a positive solution to problem (8) with initial and boundary data as given in (6) and (7). Then, for every $0 \leq t < T_k$*

$$\inf_{B_k} u_0 \leq u(\cdot, t) - m(m-1)t \leq \sup_{B_k} u_0.$$

Proof. Given any constant $c \in \mathbb{R}$ the function $w(\cdot, t) = u(\cdot, t) - m(m-1)t - c$ satisfies

$$\frac{1}{m-1} \frac{\partial w}{\partial t} - \frac{\Delta_{g_{\mathbb{H}}} w}{u} - \frac{(m-6)\langle \nabla u, \nabla w \rangle_{g_{\mathbb{H}}}}{4u^2} = 0 \quad \text{in } B_k \times [0, T_k[. \tag{9}$$

Since $u > 0$, equation (9) is uniformly parabolic. For $c = \inf_{B_k} u_0$ (respectively $c = \sup_{B_k} u_0$) we have $w \geq 0$ (respectively $w \leq 0$) on $(\partial B_k \times [0, T_k[) \cup (B_k \times \{0\})$ by (7) and the parabolic maximum principle (see [Schulz 2019a, Proposition A.2]) implies $w \geq 0$ (respectively $w \leq 0$) in $B_k \times [0, T_k[$. \square

Lemma 2 (global existence on bounded domains). *For every $k > 2$, there exists a unique global solution $0 < u_k \in C^{2;1}(B_k \times [0, \infty[)$ to problem (8) with $T_k = \infty$, boundary data (7) and initial data (6).*

Proof. Since any solution $u \in C^{2;1}(B_k \times [0, T_k[)$ to problem (8) with $T_k < \infty$ satisfies

$$0 < \inf_{B_k} u_0 \leq u(\cdot, t) \leq \sup_{B_k} u_0 + m(m-1)T_k \quad \text{in } B_k$$

for every $t \in [0, T_k[$ according to Lemma 1, the same approach as in [Schulz 2019b, Lemma 1.2] using parabolic De Giorgi–Nash–Moser theory applies. In fact, a similar argument is used in the proof of Theorem 1. \square

The estimate obtained in Lemma 1 is not uniform in k because we do not assume any uniform bounds on u_0 . To pass to the limit $k \rightarrow \infty$ we require local bounds which do not depend on k . The nonlinearity of the equation is helpful for upper bounds. Lower bounds however are delicate. We will make use of the following estimate.

Lemma 3. *For any real numbers $a, c > 0$ there exists $\lambda > 0$ such that the function $f :]0, 1[\rightarrow \mathbb{R}$ given by $f(r) = (1 - r^2)^2$ satisfies*

$$f''(r) + \frac{c}{r} f'(r) \geq -\lambda (f(r))^{1+a}$$

for all $r \in]0, 1[$.

Proof. Since $f'(r) = -4r(1 - r^2)$ and $f''(r) = -4 + 12r^2$, we have

$$f''(r) + \frac{c}{r} f'(r) = (-4 + 12r^2) - 4c(1 - r^2) = 8 - 4(3 + c)(1 - r^2) = y \left(\frac{1}{1 - r^2} \right) (f(r))^{1+a},$$

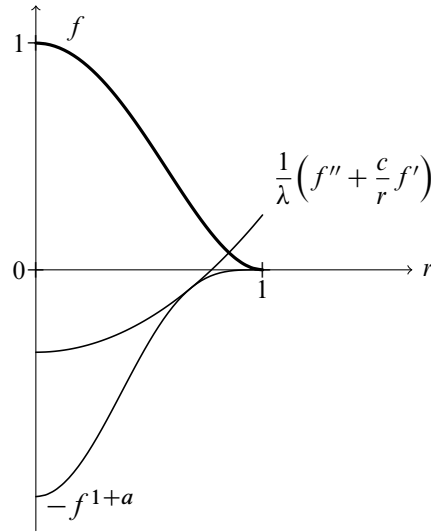


Figure 2. Visualisation of Lemma 3 for $a = 1$, $c = 2$ and $\lambda = 33$.

where we introduced the function $y : [1, \infty[\rightarrow \mathbb{R}$ given by

$$y(x) = 8x^{2+2a} - 4(3 + c)x^{1+2a}.$$

Since the leading term $8x^{2+2p}$ has a positive coefficient, the function $y : [1, \infty[\rightarrow \mathbb{R}$ is bounded from below by some constant $-\lambda < 0$ depending only on the parameters a and c . □

Lemma 4 (radial subsolution). *Let $B_1(x_0) \subset \mathbb{H}$ be the open metric unit ball in $(\mathbb{H}, g_{\mathbb{H}})$ around $x_0 \in \mathbb{H}$ and let $a, b, h_0 > 0$. Then there exists a constant $C > 0$ depending only on a, b, h_0 and $m = \dim \mathbb{H}$ such that the map $V : B_1 \times [0, t_0[\rightarrow]0, h_0]$ given by*

$$V(\cdot, t) = (h_0^a - Ct)^{\frac{1}{a}}(1 - r^2)^2,$$

where $t_0 = \frac{1}{C}h_0^a$ and where $r : B_1 \rightarrow [0, \infty[$ is the Riemannian distance function from x_0 in $(\mathbb{H}, g_{\mathbb{H}})$, satisfies

$$\frac{\partial}{\partial t} V^{1+a} \leq b \Delta_{g_{\mathbb{H}}} V. \tag{10}$$

Proof. A function of the form $V(\cdot, t) = h(t)(f \circ r)$ satisfies (10) if

$$\frac{d}{dt} h^{1+a} = -b\lambda h, \quad -\lambda f^{1+a}(r) \leq \Delta_{g_{\mathbb{H}}}(f(r)),$$

for some $\lambda > 0$. Let $f :]0, 1[\rightarrow \mathbb{R}$ and $\lambda > 0$ be as in Lemma 3 with

$$c = \frac{m - 1}{\tanh(1)}.$$

Then

$$\Delta_{g_{\mathbb{H}}}(f(r)) = f''(r) + \frac{m - 1}{\tanh(r)} f'(r) \geq f''(r) + \frac{c}{r} f'(r) \geq -\lambda f^{1+a}(r).$$

The equation for h implies

$$\frac{d}{dt}(h^a) = -\frac{ab\lambda}{a+1},$$

which integrates to

$$h^a(t) = h^a(0) - \frac{ab\lambda t}{a+1}.$$

Choosing $h(0) = h_0$ we arrive at

$$V(\cdot, t) = \left(h_0^a - \frac{ab\lambda t}{a+1} \right)^{\frac{1}{a}} (1-r^2)^2, \tag{11}$$

which completes the proof with constant

$$C = \frac{ab\lambda}{a+1}. \quad \square$$

Remark. The equation corresponding to (10) is called *fast diffusion equation* (see [Vázquez 2007]), which is well-studied even for domains in Riemannian manifolds of negative curvature: Bonforte, Grillo and Vázquez [Bonforte et al. 2008] proved existence of (weak) solutions to the fast diffusion equation in a more general setting and provided more refined estimates of the extinction time. Grillo and Muratori [2014] studied radial solutions of $\frac{\partial}{\partial t} V^{1+a} = \Delta_{g_{\mathbb{H}}} V$ on hyperbolic space of dimension $m \geq 3$ in the subcritical range $1+a < \frac{m+2}{m-2}$ and analysed their fine asymptotics near the extinction time. In the following we will specialise to the critical exponent $1+a = \frac{m+2}{m-2}$ which corresponds to the Yamabe flow.

It is surprising that the simple profile $f(r) = (1-r^2)^2$ allows the construction of a compactly supported subsolution to the Yamabe flow on hyperbolic space of any dimension $m \geq 3$. The same approach works on \mathbb{R}^m for $m \geq 3$ if we choose $c = (m-1)$. On manifolds of dimension 2, however, the Yamabe flow behaves differently: according to [Giesen and Topping 2013, Theorem A.3] there exist Yamabe flows starting from the flat 2-dimensional unit disc with arbitrarily small extinction time.

Lemma 5 (local lower bound). *Let $\Omega \subseteq \mathbb{H}$ be any open subset of hyperbolic space of dimension $m \geq 3$ containing the metric ball $B_{r_0} \subset \mathbb{H}$ of radius $r_0 > 1$. Let $(g(t))_{t \in [0, T[}$ be any Yamabe flow on Ω given by $g(t) = u(\cdot, t)g_{\mathbb{H}}|_{\Omega}$. Then, there exists a constant $C_m > 0$ depending only on m and not on Ω such that for all $t \in [0, T[$*

$$u(\cdot, t) \geq \inf_{B_{r_0}} u(\cdot, 0) - C_m t \quad \text{in } B_{r_0-1}.$$

Proof. Let $\eta = \frac{m-2}{4}$ as before. According to (3) the function $U = u^\eta$ satisfies

$$\frac{\eta}{(m-1)(\eta+1)} \frac{\partial}{\partial t} U^{1+\frac{1}{\eta}} = m\eta U + \Delta_{g_{\mathbb{H}}} U \geq \Delta_{g_{\mathbb{H}}} U$$

in $B_{r_0} \subset \Omega$. Let $x_0 \in B_{r_0-1}$ be arbitrary and let $V : B_1(x_0) \times [0, t_0[\rightarrow \mathbb{R}$ be as in Lemma 4 with parameters

$$a = \frac{1}{\eta}, \quad b = \frac{(m-1)(\eta+1)}{\eta}, \quad h_0 = \inf_{B_{r_0}} U(\cdot, 0). \tag{12}$$

According to Lemma 2 we may assume $T > t_0$. We consider the difference

$$w := V^{1+\frac{1}{\eta}} - U^{1+\frac{1}{\eta}},$$

define the function $w_+ : B_1(x_0) \times [0, t_0[\rightarrow [0, \infty[$ by $w_+(x, t) = \max\{w(x, t), 0\}$ and study the evolution of the quantity

$$J(t) = \int_{B_1(x_0)} w_+(\cdot, t) d\mu_{g_{\mathbb{H}}}.$$

For any $0 < \tau \leq t < t_0$ we have

$$\begin{aligned} J(t) - J(t - \tau) &= \int_{B_1(x_0)} w_+(\cdot, t) d\mu_{g_{\mathbb{H}}} - \int_{B_1(x_0)} w_+(\cdot, t - \tau) d\mu_{g_{\mathbb{H}}} \\ &\leq \int_{\{w(\cdot, t) > 0\}} (w_+(\cdot, t) - w_+(\cdot, t - \tau)) d\mu_{g_{\mathbb{H}}} \\ &\leq \int_{\{w(\cdot, t) > 0\}} (w(\cdot, t) - w(\cdot, t - \tau)) d\mu_{g_{\mathbb{H}}}. \end{aligned}$$

Hence,

$$\begin{aligned} \limsup_{\tau \searrow 0} \frac{J(t) - J(t - \tau)}{\tau} &\leq \int_{\{w(\cdot, t) > 0\}} \frac{\partial w}{\partial t}(\cdot, t) d\mu_{g_{\mathbb{H}}} \\ &\leq b \int_{\{(V-U)(\cdot, t) > 0\}} \Delta_{g_{\mathbb{H}}}(V - U)(\cdot, t) d\mu_{g_{\mathbb{H}}} \leq 0, \end{aligned} \tag{13}$$

where we use the following Lemma 6 to obtain the last inequality. We proceed similarly to an argument by Richard Hamilton [1986, Lemma 3.1] (see also [Schulz 2019a, Lemma A.5]). Let $\varepsilon > 0$ be arbitrary. Estimate (13) implies that there exists $\delta > 0$ such that

$$\text{for all } \tau \in [0, \delta[, \quad J(t) - J(t - \tau) \leq \varepsilon\tau. \tag{14}$$

We may assume that $\delta \in]0, t]$ is maximal with this property. By continuity of $t \mapsto J(t)$, estimate (14) extends to

$$J(t) - J(t - \delta) \leq \varepsilon\delta. \tag{15}$$

If $t - \delta > 0$, we repeat the argument to find $\delta' > 0$ such that,

$$\text{for all } \tau \in [0, \delta'[, \quad J(t - \delta) - J(t - \delta - \tau) \leq \varepsilon\tau. \tag{16}$$

In particular, (15) and (16) can be combined to

$$J(t) - J(t - \delta - \tau) \leq \varepsilon(\delta + \tau)$$

for all $\tau \in [0, \delta'[,$ in contradiction to the maximality of δ . Hence, $\delta = t$ and we obtain

$$J(t) - J(0) \leq \varepsilon t.$$

By the choice of h_0 we have $J(0) = 0$. Since $\varepsilon > 0$ is arbitrary, $J(t) \leq 0$ follows and implies $U(\cdot, t) \geq V(\cdot, t)$ in B_1 . In particular, using formula (11) with parameters (12) for V , we have

$$U(x_0, t) \geq V(x_0, t) = \left(\inf_{B_{r_0}} u(\cdot, 0) - \frac{(m-1)\lambda t}{\eta} \right)^\eta.$$

Since $x_0 \in B_{r_0-1}$ and $t \in]0, t_0[$ are arbitrary and $U = u^\eta$, the claim follows with constant $C_m = \frac{m-1}{\eta}\lambda$. \square

Lemma 6. *Let $\Omega \subset \mathbb{H}$ be a smooth, bounded domain and let $f \in C^2(\Omega)$ satisfy $f \leq 0$ on $\partial\Omega$. Let $\{f > 0\} := \{x \in \Omega \mid f(x) > 0\}$. Then,*

$$\int_{\{f>0\}} \Delta_{g_{\mathbb{H}}} f \, d\mu_{g_{\mathbb{H}}} \leq 0.$$

Proof. For any regular value $y \geq 0$ of f , the set $\{f > y\} \subset \Omega$ is regular, open and bounded with outer unit normal ν in the direction of $-\nabla f$. Therefore, we may integrate by parts to obtain

$$\int_{\{f>y\}} \Delta_{g_{\mathbb{H}}} f \, d\mu_{g_{\mathbb{H}}} = \int_{\partial\{f>y\}} \langle \nabla f, \nu \rangle_{g_{\mathbb{H}}} \, d\mu_{g_{\mathbb{H}}} \leq 0. \tag{17}$$

If $y = 0$ is not a regular value for f , we choose a sequence $(y_k)_{k \in \mathbb{N}}$ of regular values for f with $y_k \rightarrow 0$ as $k \rightarrow \infty$ and pass to the limit in (17). □

The following lemma about upper bounds is a local version of [Schulz 2019a, Proposition 2.1] and complements the local lower bound obtained in Lemma 5. In [Schulz 2019a], the estimate is derived from (3) for $U = u^n$. Here, we give a slightly different proof using (4) instead.

Lemma 7 (local upper bound). *Let $\Omega \subseteq \mathbb{H}$ be any open subset of hyperbolic space of dimension $m \geq 3$ containing the metric ball $B_{r_0} \subset \mathbb{H}$ of radius $r_0 > 1$. Let $(g(t))_{t \in [0, T[}$ be any Yamabe flow on Ω given by $g(t) = u(\cdot, t)g_{\mathbb{H}}|_{\Omega}$. Then, there exists a constant $c_m > 0$ depending only on m and not on Ω such that for all $t \in [0, T[$*

$$u(\cdot, t) \leq \sup_{B_{r_0}} u(\cdot, 0) + (m - 1)(m + c_m)t \quad \text{in } B_{r_0-1}.$$

Proof. Let $\psi : \Omega \rightarrow [0, 1]$ be a smooth cutoff function with support in $B_{r_0} \subset \Omega$ such that $\psi(x) = 1$ for all $x \in B_{r_0-1}$ and such that

$$\left(\frac{(m + 2)}{4\psi} |\nabla \psi|_{g_{\mathbb{H}}}^2 - \Delta_{g_{\mathbb{H}}} \psi \right) \leq c_m \tag{18}$$

in B_{r_0} with some constant $c_m > 0$ depending only on the dimension m . Such cutoff functions exist as shown in [Schulz 2019a, Lemma A.3–4]. Consider the spatially constant function

$$w(t) = \sup_{B_{r_0}} u(\cdot, 0) + (m - 1)(m + c_m)t.$$

Recalling (4), but suppressing the index $g_{\mathbb{H}}$ to ease notation of derivatives and inner products, we have

$$\begin{aligned} & \frac{1}{m-1} \frac{\partial}{\partial t} (u\psi - w) \\ &= m\psi - (m + c_m) + \frac{\Delta u}{u} \psi + \frac{(m-6)}{4} \frac{|\nabla u|^2}{u^2} \psi \\ &= m\psi - (m + c_m) + \frac{\Delta(u\psi)}{u} - \frac{m+2}{4u} \langle \nabla u, \nabla \psi \rangle - \Delta \psi + \frac{m-6}{4u^2} \langle \nabla(\psi u), \nabla u \rangle \\ &= m\psi - (m + c_m) + \frac{\Delta(u\psi)}{u} - \frac{m+2}{4u\psi} \langle \nabla(u\psi), \nabla \psi \rangle + \frac{m-6}{4u^2} \langle \nabla(\psi u), \nabla u \rangle + \frac{m+2}{4\psi} |\nabla \psi|^2 - \Delta \psi \\ &\leq \frac{\Delta(u\psi - w)}{u} - \frac{m+2}{4u\psi} \langle \nabla(u\psi - w), \nabla \psi \rangle + \frac{m-6}{4u^2} \langle \nabla(\psi u - w), \nabla u \rangle. \end{aligned} \tag{19}$$

Since $B_{r_0} \subset \Omega$ is a bounded domain, $u|_{B_{r_0} \times [0, T]}$ is strictly bounded away from zero and from above. Moreover, $u\psi - w \leq 0$ on $(B_{r_0} \times \{0\}) \cup (\partial B_{r_0} \times [0, T])$. Hence, the parabolic maximum principle [Schulz 2019a, Proposition A.2] applies to inequality (19) and yields $u\psi - w \leq 0$ in $B_{r_0} \times [0, T]$, which by the choice of ψ implies $u(\cdot, t) \leq w(t)$ in B_{r_0-1} for all $t \in [0, T]$ as claimed. \square

Proof of Theorem 1. Let $r > 1$ and $T > 1$ be arbitrary but fixed. For every $k \in \mathbb{N}$ let $u_k : B_k \times [0, \infty[\rightarrow]0, \infty[$ be the solution to problem (8) with boundary data (7) and initial data (6) as given in Lemma 2. Combining the lower bounds from Lemmas 1 and 5, for every $k \geq r + 3$ we obtain

$$u_k|_{\bar{B}_{r+2} \times [0, T]} \geq \frac{m(m-1)}{m(m-1) + C_m} \inf_{B_{r+3}} u_0 > 0, \tag{20}$$

where the constant $C_m > 0$ is the same as in Lemma 5. Here we use that we have

$$\max\{at, b - ct\} \geq \frac{ab}{a + c}$$

for any $a, b, t > 0$. In fact, $\max\{at, b - ct\}$ is minimal when $at = b - ct$, that is, when $t = b/(a + c)$. By Lemma 7, we also have

$$u_k|_{\bar{B}_{r+2} \times [0, T]} \leq \sup_{B_{r+3}} u_0 + (m-1)(m + c_m)T. \tag{21}$$

Recalling $\eta = \frac{m-2}{4}$, we write (8) in divergence form

$$\frac{1}{m-1} \frac{\partial u^{\eta+1}}{\partial t} = \frac{m(\eta+1)u^{\eta+1}}{u} + \operatorname{div}_{g_{\mathbb{H}}} \left(\frac{1}{u} \nabla u^{\eta+1} \right) \tag{22}$$

and interpret it as linear parabolic equation for $u_k^{\eta+1}$ in $B_{r+2} \times [0, T]$ with coefficients which are uniformly bounded due to (20) and (21). Since $u_k|_{B_{r+2} \times \{0\}} = u_0|_{B_{r+2}}$ is Hölder continuous by assumption, we may apply parabolic De Giorgi–Nash–Moser theory [Ladyzhenskaya et al. 1968, p. 204, Theorem III.10.1] (see also [Trudinger 1968, §4]) to (22) in order to obtain the interior Hölder bound

$$\|u_k^{\eta+1}\|_{C^{0,\alpha;0,\alpha/2}(\bar{B}_{r+1} \times [0, T])} \leq C(m, T, u_0|_{B_{r+3}})$$

for some $0 < \alpha < 1$ with a constant C depending only on m, T , the upper and lower bounds (20) and (21) and the Hölder bound on $u_0|_{B_{r+2}}$, but not on k . Together with (20) and (21), it follows that the coefficient $1/u_k$ in (22) is Hölder continuous in $\bar{B}_{r+1} \times [0, T]$ satisfying a similar estimate. Since we assume $u_0 \in C^{2,\alpha}(B_{r+1})$, linear parabolic theory [Ladyzhenskaya et al. 1968, p. 351, Theorem IV.10.1] yields

$$\|u_k^{\eta+1}\|_{C^{2,\alpha;1,\alpha/2}(\bar{B}_r \times [0, T])} \leq C'(m, T, u_0|_{B_{r+3}}).$$

By compactness of the embedding $C^{2,\alpha;1,\alpha/2}(\bar{B}_r \times [0, T]) \hookrightarrow C^{2;1}(\bar{B}_r \times [0, T])$ a subsequence of $\{u_k|_{B_r \times [0, T]}\}_{r+2 \leq k \in \mathbb{N}}$ converges to a solution of (4) in $\bar{B}_r \times [0, T]$. We repeat this argument to obtain a further subsequence which converges to a solution of (4) in $\bar{B}_{2r} \times [0, 2T]$.

A diagonal argument allows us to find a subsequence of $\{u_k\}_{k \in \mathbb{N}}$ which converges everywhere to a limit $u \in C^{2;1}(\mathbb{H} \times [0, \infty[)$ satisfying the Yamabe flow (4). Since the uniform lower bound $u(\cdot, t) \geq m(m-1)t$

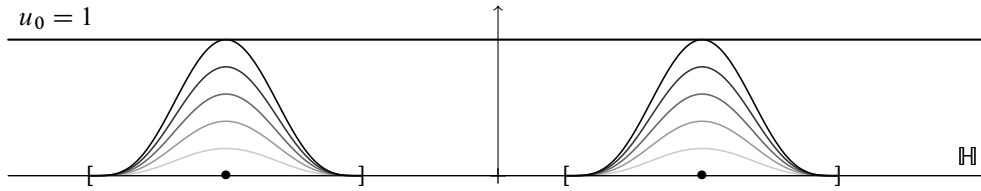


Figure 3. Applying Lemma 5 uniformly.

from Lemma 1 is preserved in the limit, the Yamabe flow given by $g(t) = u(\cdot, t)g_{\mathbb{H}}$ is instantaneously complete. \square

Proof of Theorem 2. Let $(g(t))_{t \in [0, T[}$ be any Yamabe flow on \mathbb{H} with $g(0) = g_{\mathbb{H}}$. Let $u : \mathbb{H} \times]0, T[\rightarrow]0, \infty[$ be such that $g(t) = u(\cdot, t)g_{\mathbb{H}}$ for all $t \in [0, T[$. In order to obtain a sharp bound on u from below it is convenient to aim for an upper bound on the so-called pressure $v = \frac{1}{u}$ which evolves by the equation (see [Schulz 2019a])

$$\frac{1}{m-1} \frac{\partial}{\partial t} v = -mv^2 + v\Delta_{g_{\mathbb{H}}} v - \frac{m+2}{4} |\nabla v|_{g_{\mathbb{H}}}^2. \tag{23}$$

Let $r : \mathbb{H} \rightarrow [0, \infty[$ denote the Riemannian distance function in $(\mathbb{H}, g_{\mathbb{H}})$ with respect to some origin in \mathbb{H} . Given $0 < \varepsilon < \frac{1}{2}$ let $\psi : \mathbb{H} \rightarrow]0, \infty[$ be defined by

$$\psi = \frac{1}{\cosh(\varepsilon r)}.$$

Then we have $|\nabla \psi|_{g_{\mathbb{H}}}^2 \leq \varepsilon^2 \psi^2$ and

$$-\Delta_{g_{\mathbb{H}}} \psi = -\frac{\partial^2 \psi}{\partial r^2} - \frac{(m-1)}{\tanh(r)} \frac{\partial \psi}{\partial r} = \varepsilon^2 \frac{1 - \sinh^2(\varepsilon r)}{\cosh^3(\varepsilon r)} + \frac{(m-1)\varepsilon \tanh(\varepsilon r)}{\tanh(r) \cosh(\varepsilon r)} \leq \varepsilon^2 \psi + (m-1)\varepsilon \psi$$

in \mathbb{H} . This implies

$$\begin{aligned} \frac{1}{m-1} \frac{\partial}{\partial t} (\psi v) &= -m\psi v^2 + v\Delta_{g_{\mathbb{H}}}(\psi v) - v^2 \Delta_{g_{\mathbb{H}}} \psi - 2v \langle \nabla \psi, \nabla v \rangle_{g_{\mathbb{H}}} - \frac{m+2}{4} \psi |\nabla v|_{g_{\mathbb{H}}}^2 \\ &\leq -m\psi v^2 + v\Delta_{g_{\mathbb{H}}}(\psi v) + \left(-\Delta_{g_{\mathbb{H}}} \psi + \frac{4|\nabla \psi|_{g_{\mathbb{H}}}^2}{(m+2)\psi} \right) v^2 \\ &\leq -m\psi v^2 + v\Delta_{g_{\mathbb{H}}}(\psi v) + m\varepsilon \psi v^2 \\ &\leq -m(1-\varepsilon)(\psi v)^2 + v\Delta_{g_{\mathbb{H}}}(\psi v), \end{aligned} \tag{24}$$

where we used $\psi \leq 1$ in the last step. Since $u(\cdot, 0) = 1$ we may apply Lemma 5 uniformly in \mathbb{H} and obtain a constant $C_m > 0$ depending only on the dimension m such that $u(\cdot, t) \geq 1 - C_m t$ in \mathbb{H} for all $t \in [0, T[$ as illustrated in Figure 3. This implies

$$v(\cdot, t) \leq \frac{1}{1 - C_m t}$$

in \mathbb{H} for all $t \in [0, T_0[$, where $T_0 := \min\{T, \frac{1}{C_m}\}$. Hence, the function $(\psi v)(\cdot, t)$ attains a global maximum in \mathbb{H} and the map $w : [0, T_0[\rightarrow]0, \infty[$ given by

$$w(t) = \max_{\mathbb{H}}(\psi v)(\cdot, t)$$

is well-defined. Let $t_0 \in]0, T_0[$ be arbitrary but fixed. Let $q_0 \in \mathbb{H}$ such that $w(t_0) = (\psi v)(q_0, t_0)$. By (24), we have

$$\begin{aligned} \liminf_{\tau \searrow 0} \frac{1}{\tau} \left(\frac{1}{w(t_0)} - \frac{1}{w(t_0 - \tau)} \right) &\geq \liminf_{\tau \searrow 0} \frac{1}{\tau} \left(\frac{1}{(\psi v)(q_0, t_0)} - \frac{1}{(\psi v)(q_0, t_0 - \tau)} \right) \\ &= \frac{\partial}{\partial t} \Big|_{t=t_0} \frac{1}{(\psi v)(q_0, t)} = \frac{-\frac{\partial}{\partial t}(\psi v)}{(\psi v)^2}(q_0, t_0) \\ &\geq \frac{m-1}{(\psi v)^2} (m(1-\varepsilon)(\psi v)^2 - v\Delta_{g_{\mathbb{H}}}(\psi v))(q_0, t_0) \\ &\geq m(m-1)(1-\varepsilon), \end{aligned} \tag{25}$$

where we used that $-\Delta_{g_{\mathbb{H}}}(\psi v)(q_0, t_0) \geq 0$ since q_0 is a maximum. As shown in [Schulz 2019a, Lemma A.5], estimate (25) implies

$$\frac{1}{w(t)} - \frac{1}{w(0)} \geq m(m-1)(1-\varepsilon)t$$

for every $t \in]0, T_0[$, which yields

$$(\psi v)(\cdot, t) \leq w(t) \leq \frac{1}{m(m-1)(1-\varepsilon)t + 1}$$

since $w(0) = 1$. Letting $\varepsilon \rightarrow 0$ and recalling $v = \frac{1}{u}$ we conclude

$$u(\cdot, t) \geq m(m-1)t + 1 \tag{26}$$

for all $t \in [0, T_0[$. By repeating the argument with initial time T_0 if necessary, we obtain that estimate (26) holds in fact for all $t \in [0, T[$.

The reverse inequality $u(\cdot, t) \leq m(m-1)t + 1$ is similar to the statement of Lemma 7. In fact, if we choose $\Omega = \mathbb{H}$ and replace the cutoff function ψ by $\psi(\varepsilon r)$ in the proof of Lemma 7, then the constant c_m in estimate (18) can be replaced by εc_m and we may conclude by letting $\varepsilon \rightarrow 0$. \square

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