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ON UNIQUENESS RESULTS FOR DIRICHLET PROBLEMS OF ELLIPTIC SYSTEMS WITHOUT DE GIORGI–NASH–MOSER REGULARITY

PASCAL AUSCHER AND MORITZ EGERT

We study uniqueness of Dirichlet problems of second-order divergence-form elliptic systems with transversally independent coefficients on the upper half-space in the absence of regularity of solutions. To this end, we develop a substitute for the fundamental solution used to invert elliptic operators on the whole space by means of a representation via abstract single-layer potentials. We also show that such layer potentials are uniquely determined.

1. Introduction

Consider the elliptic system of m equations in $n+1$ dimensions, $n \geq 1$, given by

$$-\sum_{i,j=0}^{n} \sum_{\beta=1}^{m} \partial_i (A_{i,j}^{\alpha,\beta}(x)\, \partial_j u^\beta(t,x)) = 0, \quad \alpha = 1, \dots, m,\ t > 0,\ x \in \mathbb{R}^n, \tag{1-1}$$

where $\partial_0 := \partial/\partial t$ and $\partial_i := \partial/\partial x_i$ if $i = 1, \dots, n$, with measurable coefficients A that do not depend on the variable t transversal to the boundary. Ellipticity will be described below but when $m = 1$, the uniformly elliptic equations will be included in our considerations. For short, we shall write $\mathcal{L}u = -\operatorname{div} A\nabla u = 0$ instead of (1-1).

Given $f \in L^p(\mathbb{R}^n; \mathbb{C}^m)$, following [Dahlberg 1977], the L^p Dirichlet problem on the upper half-space can be posed in the sense that one asks for a weak solution u with a certain nontangential maximal function controlled in L^p and which converges to the boundary data f almost everywhere in a nontangential sense. When $f \in \dot{W}^{1,p}(\mathbb{R}^n; \mathbb{C}^m)$, following [Kenig and Pipher 1993], the Dirichlet problem with data f, also known as the regularity problem, can be posed by asking for a maximal nontangential control on ∇u and convergence of u to f at the boundary as before. Existence and uniqueness to these problems are usually obtained by different arguments. For an overview on the topic the reader can refer to [Kenig 1994].

Our first goal is to prove duality results of the following type under minimal assumptions: existence in one of the boundary value problems for $(\mathcal{L}^*, p')$ implies uniqueness in the other problem for $(\mathcal{L}, p)$ in some range of p, which depends on $\mathcal{L}$, where p' is the conjugate exponent to p. We shall also consider the case $p \leq 1$ for the regularity problem, in which case the adjoint Dirichlet problem must be posed with data in BMO or in a Hölder space.

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Similar uniqueness results, requiring "dual" information, appear in [Kenig and Pipher 1993; Alfonseca et al. 2011; Hofmann et al. 2015c], to cite just the most relevant to our situation. The arguments are culminations of many earlier results on Laplace's equation and real symmetric equations in Lipschitz domains [Dahlberg 1977; Dahlberg and Kenig 1987; Jerison and Kenig 1981; Verchota 1984]. In those works, t-independence of the coefficients is not always assumed, but when restricted to this hypothesis, the so-called de Giorgi–Nash–Moser regularity properties of solutions (DGNM) are also used in a strong way to bring into play either harmonic measure techniques for real equations or representations and estimates with fundamental solutions for complex equations enjoying (DGNM). It seems that [Hofmann et al. 2015c] contains the most advanced results in this direction up to now.

Here we want to dispense with the assumption (DGNM) and, of course, harmonic measure is not available. In a similar direction, [Auscher and Mourgoglou 2019] establishes existence-uniqueness relations between the L^p regularity problem and a dual $L^{p'}$ Dirichlet problem which for $1 < p < \infty$ is posed with a different, less classical interior control, namely the square function. Uniqueness in this situation, however, does not suffice to conclude for uniqueness of the Dirichlet problem when posed with a nontangential maximal control. On the contrary, when $p \leq 1$, the results in [Auscher and Mourgoglou 2019] do apply and for clarity we shall put them into context in Section 6.

Our general strategy is to develop a substitute for the fundamental solution used to invert the elliptic operator $\mathcal{L}$ on $\mathbb{R}^{1+n}$. This is interesting in its own right. Surprisingly, not using the fundamental solution and its kernel estimates will make the arguments for uniqueness conceptually and technically simpler. It also allows us to reach minimal assumptions, even when assuming further (DGNM). Let us explain in formal terms the substitution idea.

In the case of transversally independent coefficients, the fundamental solution $\Gamma(t, x, s, y)$ of $\mathcal{L}$, constructed in [Hofmann and Kim 2007] under (DGNM) and more recently without this assumption in [Barton 2016], has time translation invariance; that is, it depends on $t - s$. Its restriction to fixed times $(t, 0)$, $t \neq 0$, is called the single-layer potential $\mathcal{S}_t(x, y)$ at time t. Formally writing

$$(\mathcal{L}^{-1} f)(t, x) = \iint_{\mathbb{R}^{1+n}} \Gamma(t, x, s, y) f(s, y) \, ds \, dy = \iint_{\mathbb{R}^{1+n}} \mathcal{S}_{t-s}(x, y) f(s, y) \, dy \, ds$$

allows one to recover the fundamental solution by a convolution in time with $\mathcal{S}_t(x, y)$. A difficulty is to give a meaning to the last term as a converging integral in order to obtain further estimates on $\mathcal{L}^{-1} f$. However, one can interpret this formula at the level of operators by writing

$$(\mathcal{L}^{-1} f)(t, x) = \int_{\mathbb{R}} (\mathcal{S}_{t-s} f(s, \cdot))(x) \, ds, \tag{1-2}$$

provided the operator $\mathcal{S}_t$ with kernel $\mathcal{S}_t(x, y)$ has the expected boundedness properties. Indeed, [Rosén 2013] shows the remarkable fact that $\mathcal{S}_t$ is bounded from $L^2(\mathbb{R}^n; \mathbb{C}^m)$ into $\dot{W}^{1,2}(\mathbb{R}^n; \mathbb{C}^m)$ whether or not (DGNM) holds and that, when (DGNM) is assumed, its kernel agrees with (or can be used to define) $\Gamma(t, x, 0, y)$.

This suggests that knowledge on the operator $\mathcal{S}_t$ alone is sufficient to recover $\mathcal{L}^{-1}$. This is what we shall prove and use, thereby giving a precise meaning to the representation (1-2). We shall also prove that knowledge of $\mathcal{L}^{-1}$ alone uniquely determines the operator $\mathcal{S}_t$, which we decide to call the single-layer operator (associated with $\mathcal{L}$).

Having (1-2) at hand, more operator bounds of $\mathcal{S}_t$ can be plugged in this formula to give further estimates on $\mathcal{L}^{-1} f$. Under (DGNM), [Hofmann et al. 2015d] proves some bounds by Calderón–Zygmund theory. But, following [Rosén 2013], we may also compute $\mathcal{S}_t$ (recall it is unique) using the connection between $\mathcal{L}$ and a first-order Dirac operator DB proposed in [Auscher et al. 2010]. Thus, the operator bounds proved in [Auscher and Stahlhut 2016] become available. Such bounds, including the ones of [Hofmann et al. 2015d], hold for a range of spaces determined by the coincidence of abstract Hardy spaces associated with DB and the corresponding concrete Hardy spaces associated with D. At the heart of this treatment lies the H^∞ functional calculus of DB proved in [Axelsson et al. 2006] by a remarkable elaboration on the solution of the Kato problem for elliptic systems.

The organisation of the article is as follows. First, we present our main results and the strategy to prove uniqueness (Section 2). We next present proofs of our main results in the case $p = 2$ because the arguments there do not require any use of the single-layer operators and still contain the main ideas (Section 3). Then, we state in what sense (1-2) holds (Section 4) and move to $p \neq 2$ (Section 5). In Section 6 we discuss the regularity problem with Hardy–Sobolev data versus the Dirichlet problem with BMO or Hölder continuous data. We prove (1-2) in various ways (Section 7). Some technical lemmas are presented in the final Section 8.

2. Setup, results and strategy of proofs

2A. ***Notation and general assumptions.*** We shall use the following notation for spaces. We denote by $C_0^\infty(\mathbb{R}^d)$ the space of compactly supported smooth complex-valued functions on $\mathbb{R}^d$. For $1 < p < \infty$, the inhomogeneous Sobolev space on $\mathbb{R}^d$ consists of those $f \in L^p(\mathbb{R}^d; \mathbb{C})$ for which ∇f is p-integrable. It contains $C_0^\infty(\mathbb{R}^d)$ as a dense subspace. The homogeneous Sobolev space $\dot{W}^{1,p}(\mathbb{R}^d)$ consists of all distributions on $\mathbb{R}^d$ for which ∇f is in $L^p(\mathbb{R}^d; \mathbb{C}^d)$. It is a Banach space when modding out the constants and it can be realised as the closure of $W^{1,p}(\mathbb{R}^d)$ modulo constants for the seminorm $\|\nabla f\|_p$. Its dual $\dot{W}^{-1,p'}(\mathbb{R}^d)$ is identified to the space of distributions div F with $F \in L^{p'}(\mathbb{R}^d; \mathbb{C}^d)$. The space of continuous complex-valued functions on $\mathbb{R}$ that vanish at $\pm\infty$ is denoted by $C_0(\mathbb{R})$ and $C_0([0, \infty))$ denotes the space of continuous functions on $[0, \infty)$ that vanish at $+\infty$. All these spaces have an E-valued extension (denoted by $L^p(\mathbb{R}^d; E)$ and so on) when E is a complex Banach space. Occasionally, we use the subscript "loc" to indicate that certain conditions hold only on compact subsets.

We denote points in $\mathbb{R}^{1+n} = \mathbb{R} \times \mathbb{R}^n$ by (t, x) etc. We set $\mathbb{R}^{1+n}_+ := (0, \infty) \times \mathbb{R}^n$. For short, we write $\mathcal{L}u = -\operatorname{div} A\nabla u = 0$ to mean (1-1), where we assume that the matrix

$$A(x) = (A^{\alpha,\beta}_{i,j}(x))^{\alpha,\beta=1,\dots,m}_{i,j=0,\dots,n} \in L^\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^{m(1+n)})) \tag{2-1}$$

is bounded and measurable, independent of t (transversal independence), and satisfies the following strict accretivity condition on the subspace $\mathcal{H}$ of $L^2(\mathbb{R}^n; \mathbb{C}^{m(1+n)})$ defined by $(f^\alpha_i)_{i=1,\dots,n}$ being curl-free in $\mathbb{R}^n$

for all α: for some $\lambda > 0$ and all $f \in \mathcal{H}$,

$$\int_{\mathbb{R}^n} \operatorname{Re}(A(x)f(x) \cdot \overline{f(x)})\, dx \geq \lambda \sum_{i=0}^{n} \sum_{\alpha=1}^{m} \int_{\mathbb{R}^n} |f_i^\alpha(x)|^2 \, dx. \tag{2-2}$$

In particular situations, we may weaken this condition to the well-known Gårding inequality; see Remark 2.11 below. The system (1-1) is considered in the sense of distributions with weak solutions in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^{1+n}_+; \mathbb{C}^m)$.

As weak solutions to elliptic systems might not be regular, we use the Whitney average variants of the usual nontangential maximal functions. But when we get back to systems where solutions have meaningful pointwise values, these variants turn out to be equivalent to the usual pointwise control. Consider, for $0 < q < \infty$, the q-adapted nontangential maximal function

$$\widetilde{N}_{*,q} F(x) := \sup_{t>0} \biggl(⨌_{(c_0^{-1}t, c_0 t) \times B(x, c_1 t)} |F(s,y)|^q \, ds\, dy \biggr)^{1/q}, \quad x \in \mathbb{R}^n, \tag{2-3}$$

for some fixed parameters $c_0 > 1$, $c_1 > 0$. We use $B(x, r)$ for the Euclidean ball centred at x with radius r and denote averages by dashed integrals. When $q = 2$, we simply write $\widetilde{N}_*$. For fixed $p > 0$ and $q > 0$, a covering argument reveals that changing the parameters yields equivalent $\|\widetilde{N}_{*,q} F\|_p$ norms. In the following, we shall use

$$W(t, x) := \Bigl(\frac{t}{2}, 2t\Bigr) \times B(x, t) \tag{2-4}$$

for simplicity.

2B. *Main results and consequences.* For $1 < p < \infty$, the L^p Dirichlet problem with nontangential maximal control can be formulated as follows: given $f \in L^p(\mathbb{R}^n; \mathbb{C}^m)$, uniquely solve

$$\begin{cases} \mathcal{L}u = 0 & \text{on } \mathbb{R}^{1+n}_+, \\ \widetilde{N}_* u \in L^p(\mathbb{R}^n), \\ \lim_{t\to 0} ⨌_{W(t,x)} |u(s,y) - f(x)| \, ds\, dy = 0 & \text{for a.e. } x \in \mathbb{R}^n. \end{cases} \tag*{$(D)^{\mathcal{L}}_p$}$$

The L^p regularity problem consists in solving uniquely (modulo constants), given $f \in \dot{W}^{1,p}(\mathbb{R}^n; \mathbb{C}^m)$,

$$\begin{cases} \mathcal{L}u = 0 & \text{on } \mathbb{R}^{1+n}_+, \\ \widetilde{N}_* (\nabla u) \in L^p(\mathbb{R}^n), \\ \lim_{t\to 0} ⨌_{W(t,x)} |u(s,y) - f(x)| \, ds\, dy = 0 & \text{for a.e. } x \in \mathbb{R}^n. \end{cases} \tag*{$(R)^{\mathcal{L}}_p$}$$

We have fixed the parameters for $W(t, x)$ but from Lebesgue's differentiation theorem applied to f and a covering argument we can again see that the convergence of Whitney averages of $|u - f|$ is independent of their particular choice.

To formulate our main results we implicitly use a certain perturbed first-order operator DB associated with $\mathcal{L}$ and the associated abstract Hardy spaces H^p_{DB} defined and studied in [Auscher and Stahlhut 2016]. At this stage, the reader need not be aware of their definitions as we are only going to use the conclusions drawn in that paper.

There is an exponent $p_+(DB) > 2$ that is related to resolvent estimates for DB in L^p and a coercivity property of B and B^* in L^p and $L^{p'}$, respectively; the precise definition can be found in [Auscher and Stahlhut 2016, Section 3.2]. It is also shown there that the set of those $p \in \left(\frac{n}{n+1}, p_+(DB)\right)$ such that we have the coincidence $H^p_{DB} = H^p_D$ of abstract and concrete Hardy spaces is an interval and that $p_+(DB)$ is also its upper endpoint. This interval is called I_L in [Auscher and Mourgoglou 2019]. We call it $\mathcal{H}_{\mathcal{L}}$ in this article. This is an open interval containing 2 and there is a corresponding interval $\mathcal{H}_{\mathcal{L}^*}$.

Theorem 2.1. *Let $1 < p < \infty$ with $p' \in \mathcal{H}_{\mathcal{L}^*}$. Existence for $(R)^{\mathcal{L}^*}_{p'}$ implies uniqueness for $(D)^{\mathcal{L}}_p$.*

Theorem 2.2. *Let $1 < p < \infty$ with $p \in \mathcal{H}_{\mathcal{L}}$. Existence for $(D)^{\mathcal{L}^*}_{p'}$ implies uniqueness for $(R)^{\mathcal{L}}_p$ (modulo constants).*

The interval $\mathcal{H}_{\mathcal{L}}$ equals $(1-\varepsilon'(\mathcal{L}), 2+\varepsilon(\mathcal{L}))$ in case of (DGNM) for $\mathcal{L}^*$ for example (with $0 < \varepsilon(\mathcal{L}) \le \infty$) and even some conditions weaker than (DGNM) suffice; see [Auscher and Stahlhut 2016, Section 13] for details. We note that [Hofmann et al. 2015c] uses a similar exponent $2+\varepsilon$ but we do not know whether it agrees with our $2+\varepsilon(\mathcal{L})$. More specifically, we have the following corollaries; compare with [Hofmann et al. 2015c, Proposition 8.19(i)–(ii)].

Corollary 2.3. *Assume (DGNM) for $\mathcal{L}$ and $(2+\varepsilon(\mathcal{L}^*))' < p < \infty$. Existence for $(R)^{\mathcal{L}^*}_{p'}$ implies uniqueness for $(D)^{\mathcal{L}}_p$.*

Corollary 2.4. *Assume (DGNM) for $\mathcal{L}^*$ and $1 < p < 2+\varepsilon(\mathcal{L})$. Existence for $(D)^{\mathcal{L}^*}_{p'}$ implies uniqueness for $(R)^{\mathcal{L}}_p$ (modulo constants).*

Well-posedness of a boundary value problem is the conjunction of both existence of a solution for all data and uniqueness. A stronger notion, appearing implicitly in many earlier works, is that of compatible well-posedness: it means well-posedness and that the unique solution agrees with the energy solution obtained from the Lax–Milgram lemma, whenever the boundary data is admissible for energy solutions. Theorem 2.1 then has the following interesting consequence we shall discuss in detail in Section 5C. We define the square function SF of a measurable function F by

$$SF(x) := \left(\iint_{\Gamma_a(x)} |F(t,y)|^2 \frac{dt\,dy}{t^{n+1}} \right)^{1/2}, \quad x \in \mathbb{R}^n, \tag{2-5}$$

where $a > 0$ is a fixed number called the aperture of the cone $\Gamma_a(x) := \{(t,y) : t > 0, |x-y| < at\}$.

Corollary 2.5. *Let $1 < p < \infty$ with $p' \in \mathcal{H}_{\mathcal{L}^*}$. Assume $(R)^{\mathcal{L}^*}_{p'}$ is well-posed (resp. compatible well-posed) modulo constants. Then so is $(D)^{\mathcal{L}}_p$. Moreover, given $f \in L^p(\mathbb{R}^n;\mathbb{C}^m)$, the weak solution u with data f has further regularity $u \in C_0([0,\infty); L^p(\mathbb{R}^n;\mathbb{C}^m))$, satisfies the square function estimate $\|S(t\nabla u)\|_p < \infty$ and there is comparability*

$$\|\widetilde{N}_*(u)\|_p \sim \|S(t\nabla u)\|_p \sim \sup_{t\ge 0} \|u(t,\cdot)\|_p \sim \|f\|_p. \tag{2-6}$$

In addition, the nontangential convergence improves to L^2 averages; that is, for a.e. $x \in \mathbb{R}^n$,

$$\lim_{t\to 0} \fint\!\!\fint_{W(t,x)} |u(s,y) - f(x)|^2 \, ds\,dy = 0. \tag{2-7}$$

2C. ***Comparison to earlier results.*** We comment here on the formulations of the problems and statements in relation to existing literature.

Remark 2.6 (on convergence at the boundary for the Dirichlet problem). There is no trace theorem for the space of measurable functions u with $\|\widetilde{N}_* u\|_p < \infty$. Hence, existence of boundary values is part of the Dirichlet problem and does not follow from the interior control. If we look for nontangential approach almost everywhere, this the weakest possible condition. But we may also choose a different convergence to the boundary data, such as strong L^p convergence $u(t, \cdot) \to f$ as $t \to 0$ on compact subsets of $\mathbb{R}^n$ as considered in [Hofmann et al. 2015c] under (DGNM). In that case, $\widetilde{N}_* u$ can be replaced by the usual pointwise supremum on cones denoted by $N_* u$. As $\|u(t, \cdot)\|_p \le \|N_* u\|_p$ for all p and $t > 0$, the L^p convergence on compact sets, is also natural. We shall see that a minor modification of our arguments will cover this formulation of the Dirichlet problem and even a weaker form of L^p_{loc} convergence. Note carefully that $\|\widetilde{N}_* u\|_p < \infty$ does not imply L^p boundedness for solutions when $p > 2$ (see below).

Remark 2.7 (on the formulation of the Dirichlet problem). We use $\widetilde{N}_* = \widetilde{N}_{*,2}$ in the Dirichlet problem. For complex equations, it makes a difference to consider $\widetilde{N}_*$ or $\widetilde{N}_{*,q}$ with $q = p$, see [Mayboroda 2010], as solutions may not be locally p-integrable. The choice $q = 2$ is most natural to overcome this difficulty and we could even use $\widetilde{N}_{*,1}$ by invoking reverse Hölder estimates.

Remark 2.8 (on convergence at the boundary for the regularity problem). For the regularity problem, there is a trace theorem for the space of L^2_{loc} functions satisfying $\|\widetilde{N}_*(\nabla u)\|_p < \infty$, as is implicit in [Kenig and Pipher 1993]. The Whitney averages converge almost everywhere in approaching the boundary, the limit belongs to the homogeneous Sobolev space $\dot{W}^{1,p}(\mathbb{R}^n)$ and Cesàro means $⨍_t^{2t} u \, ds$ converge in the sense of distributions modulo constants to the same limit; see Lemma 8.3 . Hence, the boundary condition in the regularity problem is implied by the interior control. Actually, [Auscher and Mourgoglou 2019, Theorem 1.1] shows that all solutions in this class for the range of p in the statement enjoy convergence $\nabla_x u(t, \cdot) \to \nabla_x u(0, \cdot)$ strongly in L^p as $t \to 0$. So, this could be taken as definition for the convergence to the boundary data as well.

Remark 2.9 (on comparability of S and $\widetilde{N}_*$). For p as in Corollary 2.5, [Auscher and Mourgoglou 2019, Theorems 1.6 and 1.9] show that (compatible) well-posedness for $(R)^{\mathcal{L}^*}_{p'}$ is equivalent to (compatible) well-posedness for a variant $(\widetilde{D})^{\mathcal{L}}_p$ of the Dirichlet problem with the nontangential maximal function being replaced by the square function $S(t\nabla u)$ in the L^p-control. Owing to [Auscher and Mourgoglou 2019, Corollary 1.4], we have $\|\widetilde{N}_*(u)\|_p \lesssim \|S(t\nabla u)\|_p$ a priori for any weak solution in this range of p and $u(t, \cdot)$ converges to its boundary data strongly in L^p if the right-hand side is finite. Thus, Corollary 2.5 can rephrased as saying that the (compatible) well-posedness of $(\widetilde{D})^{\mathcal{L}}_p$ implies that of $(D)^{\mathcal{L}}_p$.

It would be interesting to prove the converse, at least for the range of p above. For equations, that is $m = 1$, with real-valued t-independent coefficients, the real-variable argument in [Hofmann et al. 2015b] shows $\|\widetilde{N}_*(u)\|_p \sim \|S(t\nabla u)\|_p$ for any weak solution. Hence, both Dirichlet problems are a priori the same and the converse holds. Using the equivalence between $(\widetilde{D})^{\mathcal{L}}_p$ and $(R)^{\mathcal{L}^*}_{p'}$ mentioned above, this also provides a direct way for deducing the main result on well-posedness of the regularity problem $(R)^{\mathcal{L}^*}_{p'}$ in

[Hofmann et al. 2015a] for real coefficients from [Hofmann et al. 2015b]. For equations with complex coefficients and systems though, similar conclusions remain unknown.

Remark 2.10 (on representation by layer potentials). Another aspect of the theory is whether u in Corollary 2.5 can be represented as $u = \mathcal{D}_t(\mathcal{D}_{0^+})^{-1} f$, where $\mathcal{D}_t$ is the double-layer operator, also defined abstractly and proved to be bounded on L^2 in [Rosén 2013] and on L^p in this range of p in [Auscher and Stahlhut 2016]. There is no reason to believe that $\mathcal{D}_{0^+}$ is invertible under the assumptions in Corollary 2.5. Even well-posedness of the regularity problem on both half-spaces is not enough to conclude this: however, it gives the different representation $u = \mathcal{S}_t(\mathcal{S}_0)^{-1} f$ using the single-layer operator. As of today, the only available method to prove invertibility is via the so-called Rellich estimates. This was done first in [Verchota 1984] when $p \geq 2$ for Laplace's equation in Lipschitz domains and has been extended to a larger class of equations (perturbations of real symmetric coefficients) in [Alfonseca et al. 2011] by developing the layer potential approach and using the Rellich estimates of [Jerison and Kenig 1981] for invertibility. Note that the Rellich estimates give access to solvability of Neumann problems as well, which is strong additional information.

Remark 2.11 (on the ellipticity condition). Given $u \in \dot{W}^{1,2}(\mathbb{R}^{1+n})$, we can take $f(x) = \nabla u(t, x)$ for each $t \in \mathbb{R}$ in (2-2) and integrate in t to obtain Gårding's inequality

$$\iint_{\mathbb{R}^{1+n}} \operatorname{Re}(A(x)\nabla u(t, x) \cdot \nabla u(t, x))\, dx\, dt \geq \lambda \iint_{\mathbb{R}^{1+n}} |\nabla u(t, x)|^2\, dx\, dt. \tag{2-8}$$

We shall observe that our proofs of Theorem 2.1 and 2.2 in the case $p = 2$ — and even p nearby — only require (2-8). In particular, this gives access to uniqueness of boundary value problems for Lamé-type systems [Martell et al. 2017], which typically satisfy Gårding's inequality but not the strict accretivity condition (2-2).

2D. ***Strategy to the proofs.*** The formal strategy is the same for both theorems and is adopted from earlier references, in particular [Alfonseca et al. 2011; Hofmann et al. 2015c]. Let u be a solution of $\mathcal{L}u = 0$ on $\mathbb{R}^{1+n}_+$ with zero boundary condition. We take $G \in C_0^\infty(\mathbb{R}^{1+n}; \mathbb{C}^m)$ with support contained in some region $[a, b] \times B(0, c)$ contained in $\mathbb{R}^{1+n}_+$. We want to show that $\langle u, G\rangle = 0$. We then pick a second function θ supported in $\mathbb{R}^{1+n}_+$, real-valued, Lipschitz continuous and equal to 1 on the support of G. Finally, we let H be a weak solution of $\mathcal{L}^* H = G$ on $\mathbb{R}^{1+n}$. As $u\theta$ is a test function for this equation, we have

$$\langle u, G\rangle = \langle u\theta, G\rangle = \langle A\nabla(u\theta), \nabla H\rangle.$$

Next,

$$\begin{aligned}\langle A\nabla(u\theta), \nabla H\rangle &= \langle Au\nabla\theta, \nabla H\rangle + \langle A\theta\nabla u, \nabla H\rangle \\ &= \langle Au\nabla\theta, \nabla H\rangle - \langle A\nabla u, H\nabla\theta\rangle + \langle A\nabla u, \nabla(\theta H)\rangle,\end{aligned}$$

and the last term vanishes because θH is a test function for $\mathcal{L}u = 0$. All brackets here can be expressed by L^2 complex inner products, and in accordance with our shorthand notation $-\operatorname{div} A\nabla u = 0$ for (1-1) we abbreviated $Au\nabla\theta = A_{i,j}^{\alpha,\beta} u^\beta \partial_j \theta$ and $H\nabla\theta = H^\alpha \partial_i \theta$, where sums are taken over repeated indices.

Now the existence hypothesis comes into play. We let $h := H(0, \cdot\,)$ (provided it makes sense) and let H_1 be a solution to the adjoint problem $\mathcal{L}^* H_1 = 0$ on $\mathbb{R}^{1+n}_+$ with boundary condition h. We may apply the same decomposition to $\langle A\nabla(u\theta), \nabla H_1\rangle$ and remark that this term vanishes since $u\theta$ is a test function for $\mathcal{L}^* H_1 = 0$. Hence, we obtain

$$\langle u, G\rangle = \langle Au\nabla\theta, \nabla(H - H_1)\rangle - \langle A\nabla u, (H - H_1)\nabla\theta\rangle. \tag{2-9}$$

We remark that u and $H - H_1$ both vanish at the boundary. In fact, the reason to use H_1 is to help convergence near the boundary. The symmetry in u and $H - H_1$ also indicates why the results can go both ways.

The goal is then to show that these two terms tend to 0 if we let $\theta \to 1$ everywhere on $\mathbb{R}^{1+n}_+$. The heart of the matter is to prove estimates on H and h using our assumption instead of relying on estimates for the fundamental solutions to represent H in [Alfonseca et al. 2011; Hofmann et al. 2015c] under (DGNM). For us, the assumption implies boundedness properties of single-layer operators for a certain range of spaces and we shall use this as a black box, once we have shown the representation (1-2).

Some particular choice of θ will facilitate the proofs. We are going to pick θ as follows. We fix $\chi \in C_0^\infty(\mathbb{R}^n)$ to be 1 on $B(0, 1)$ and with support in $B(0, 2)$. We let η be the continuous, piecewise linear function, which is 0 on $\left[0, \frac{2}{3}\right]$ and 1 on $\left[\frac{3}{2}, \infty\right)$ and linear in between. We pick $M > 2c,\ 0 < \varepsilon < \frac{a}{4}$ and $2b < R < \infty$ to finally set

$$\theta(t, x) := \chi\left(\frac{x}{M}\right)\eta\left(\frac{t}{\varepsilon}\right)\left(1 - \eta\left(\frac{t}{R}\right)\right).$$

2E. ***Standard estimates on weak solutions.*** Here are some standard properties on weak solutions to $\mathcal{L}u = 0$ in a domain $\Omega \subset \mathbb{R}^{1+n}$ we shall freely use throughout: The reader can refer to [Giaquinta 1984] for the elliptic equations or to [Barton 2016] for systems. With regard to these references, we remark that reverse Hölder inequalities share the general feature that Lebesgue exponents on both sides can be lowered as one pleases; see [Iwaniec and Nolder 1985, Theorem 2] or [Bernicot et al. 2016, Theorem B.1] for a particularly simple proof:

$$\text{Caccioppoli's inequality:} \qquad ⨌_{W(t,x)} |\nabla u|^2 \lesssim \frac{1}{t^2} ⨌_{\widetilde{W}(t,x)} |u|^2.$$

$$\text{Reverse Hölder inequality on } \nabla u\text{:} \qquad \left(⨌_{W(t,x)} |\nabla u|^2\right)^{1/2} \lesssim ⨌_{\widetilde{W}(t,x)} |\nabla u|.$$

$$\text{Reverse Hölder inequality on } u\text{:} \qquad \left(⨌_{W(t,x)} |u|^2\right)^{1/2} \lesssim ⨌_{\widetilde{W}(t,x)} |u|.$$

Here $\widetilde{W}(t, x)$ is another Whitney region, with strictly larger parameters than $W(t, x)$ and we assume that the former is compactly included in Ω. The implied constants depend only on ellipticity of A, Whitney parameters and the distance of $\widetilde{W}(t, x)$ to $\partial\Omega$. The reverse Hölder inequalities can also come with L^p-averages for some $p > 2$ on the left but we shall not need such improvements.

3. The case $p = 2$

To illustrate the simplicity of our argument, we present here the proofs of our main results in the case $p = 2$. For the purpose of this section only, it will be sufficient to assume the weaker ellipticity condition (2-8).

3A. *Estimate for $\mathcal{L}^{-1}$.* First, $\mathcal{L} : \dot{W}^{1,2}(\mathbb{R}^{1+n}; \mathbb{C}^m) \to \dot{W}^{-1,2}(\mathbb{R}^{1+n}; \mathbb{C}^m)$ is invertible as a consequence of the Lax–Milgram lemma and (2-8). This is how we understand $\mathcal{L}^{-1}$. The adjoint of $\mathcal{L}$ is associated with the matrix A^*.

Lemma 3.1. *Let $G \in C_0^\infty(\mathbb{R}^{1+n}; \mathbb{C}^m) \cap \dot{W}^{-1,2}(\mathbb{R}^{1+n}; \mathbb{C}^m)$ and let $H \in \dot{W}^{1,2}(\mathbb{R}^{1+n}; \mathbb{C}^m)$ solve $\mathcal{L}H = G$ in $\mathbb{R}^{1+n}$. Then*

(i) *For all integers $k \geq 1$, $\partial_t^k H$ exists in $W^{1,2}(\mathbb{R}^{1+n}; \mathbb{C}^m)$ and $\mathcal{L}(\partial_t^k H) = \partial_t^k G$.*

(ii) *For all integers $k \geq 1$, $\partial_t^k H \in C_0(\mathbb{R}; W^{1,2}(\mathbb{R}^n; \mathbb{C}^m))$ with $t \in \mathbb{R}$ the distinguished variable.*

Proof. Note that for $n \geq 3$ we have $C_0^\infty(\mathbb{R}^{1+n}; \mathbb{C}^m) \subset \dot{W}^{-1,2}(\mathbb{R}^{1+n}; \mathbb{C}^m)$ by Sobolev embeddings. For $n = 1, 2$, the necessary and sufficient condition on G is $\iint_{\mathbb{R}^{1+n}} G = 0$. The solution $H \in \dot{W}^{1,2}(\mathbb{R}^n; \mathbb{C}^m)$ is defined by requiring for all $\varphi \in \dot{W}^{1,2}(\mathbb{R}^{1+n}; \mathbb{C}^m)$ that

$$\langle A\nabla H, \nabla\varphi\rangle = \langle G, \varphi\rangle, \tag{3-1}$$

where the second bracket is the (complex) duality between $\dot{W}^{-1,2}(\mathbb{R}^{1+n}; \mathbb{C}^m)$ and $\dot{W}^{1,2}(\mathbb{R}^{1+n}; \mathbb{C}^m)$. As A is t-independent, and since $\partial_t^k G \in \dot{W}^{-1,2}(\mathbb{R}^{1+n}; \mathbb{C}^m)$ for all $k \geq 1$, the method of difference quotients and induction on k allows us to differentiate (3-1) and to obtain $\partial_t^k H \in \dot{W}^{1,2}(\mathbb{R}^{1+n}; \mathbb{C}^m)$ with

$$\langle A\nabla\partial_t^k H, \nabla\varphi\rangle = \langle \partial_t^k G, \varphi\rangle$$

for all $\varphi \in \dot{W}^{1,2}(\mathbb{R}^{1+n}; \mathbb{C}^m)$. This means $\mathcal{L}(\partial_t^k H) = \partial_t^k G$. Moreover, $\nabla(\partial_t^k H) \in L^2(\mathbb{R}^{1+n}, \mathbb{C}^{m(1+n)})$ for all integers $k \geq 0$, showing in particular $\partial_t^{k+1} H \in L^2(\mathbb{R}^{1+n}; \mathbb{C}^m)$ for all $k \geq 0$. This completes the proof of (i).

For (ii), we use the vector-valued embedding $W^{1,2}(\mathbb{R}; L^2(\mathbb{R}^n; \mathbb{C}^m)) \subset C_0(\mathbb{R}; L^2(\mathbb{R}^n; \mathbb{C}^m))$. Since $\partial_t^k H, \partial_t^{k+1} H$ are in $L^2(\mathbb{R}^{1+n}; \mathbb{C}^m)$, which we identify with $L^2(\mathbb{R}; L^2(\mathbb{R}^n; \mathbb{C}^m))$ via Fubini's theorem, we obtain $\partial_t^k H \in C_0(\mathbb{R}; L^2(\mathbb{R}^n; \mathbb{C}^m))$. Similarly we have $\nabla_x\partial_t^k H, \nabla_x\partial_t^{k+1} H \in L^2(\mathbb{R}^{1+n}; \mathbb{C}^{mn})$ and hence $\nabla_x\partial_t^k H \in C_0(\mathbb{R}; L^2(\mathbb{R}^n; \mathbb{C}^{mn}))$ as well. The conclusion follows. □

Lemma 3.2. *Let $\widetilde{G} \in C_0^\infty(\mathbb{R}^{1+n}; \mathbb{C}^m) \cap \dot{W}^{-1,2}(\mathbb{R}^{1+n}; \mathbb{C}^m)$ and $\widetilde{H} := \mathcal{L}^{-1}(\widetilde{G})$. Set $G = \partial_t\widetilde{G}$ and $H = \partial_t\widetilde{H}$. Then $G \in C_0^\infty(\mathbb{R}^{1+n}; \mathbb{C}^m) \cap \dot{W}^{-1,2}(\mathbb{R}^{1+n}; \mathbb{C}^m)$ and $H \in W^{1,2}(\mathbb{R}^{1+n}; \mathbb{C}^m)$ solves $\mathcal{L}H = G$ in $\mathbb{R}^{1+n}$ with estimates*

$$\|\widetilde{N}_{*,1} H\|_2 + \|\widetilde{N}_{*,1}(\nabla H)\|_2 < \infty.$$

Remark 3.3. With a little more work the reader may check $\|\widetilde{N}_* H\|_2 < \infty$ and $\|\widetilde{N}_*(\nabla H)\|_2 < \infty$. We do not need this improvement.

Proof. As a derivative of an $L^2(\mathbb{R}^{1+n}; \mathbb{C}^m)$-function, $\partial_t\widetilde{G}$ is in $\dot{W}^{-1,2}(\mathbb{R}^{1+n}; \mathbb{C}^m)$. Observe that by Lemma 3.1 we have

$$H = \partial_t\widetilde{H} = \mathcal{L}^{-1}(\partial_t\widetilde{G}) = \mathcal{L}^{-1}(G) \in W^{1,2}(\mathbb{R}^n; \mathbb{C}^m).$$

In particular, H is square-integrable. Let $a, b \in \mathbb{R}$ such that $\operatorname{supp} G \subset [a, b] \times \mathbb{R}^n$. We may assume for simplicity $b \geq 2$.

To establish $\|\widetilde{N}_{*,1}H\|_2 < \infty$, we split the supremum defining $\widetilde{N}_{*,1}H$ in two parts according to $t < 4b$ and $4b \le t$. In the first case we note that $h := H(0,\cdot)$ is defined in $W^{1,2}(\mathbb{R}^n;\mathbb{C}^m)$ due to Lemma 3.1 to give

$$\fint\!\!\fint_{W(t,x)} |H(s,y)|\,ds\,dy \le \fint\!\!\fint_{W(t,x)} |H(s,y)-h(y)|\,ds\,dy + \fint\!\!\fint_{W(t,x)} |h(y)|\,ds\,dy.$$

Since $s \mapsto H(s,\cdot)$ is smooth with values in $L^2(\mathbb{R}^n;\mathbb{C}^m)$ again by Lemma 3.1, and as $t < 4b$, we can write

$$\begin{aligned}\fint\!\!\fint_{W(t,x)} |H(s,y)|\,ds\,dy &\le \fint_{B(x,t)} \int_0^{8b} |\partial_s H(s,y)|\,ds\,dy + \fint_{B(x,t)} |h(y)|\,ds\,dy \\ &\le \mathcal{M}(F)(x) + \mathcal{M}(h)(x),\end{aligned} \tag{3-2}$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal operator on $\mathbb{R}^n$ and $F(x) := \int_0^{8b} |\partial_s H(s,x)|\,ds$. We know that $h \in L^2(\mathbb{R}^n;\mathbb{C}^m)$, and also $F \in L^2(\mathbb{R}^n)$ since $\partial_s H \in L^2(\mathbb{R}^{1+n};\mathbb{C}^m)$ and

$$\int_{\mathbb{R}^n} |F(x)|^2\,dx \le 8b \int_{\mathbb{R}^n}\int_0^{8b} |\partial_s H(s,x)|^2\,ds\,dx.$$

Taking the supremum over $t < 4b$ in (3-2), we obtain the L^2 bound from the maximal theorem. Assume now that $t \ge 4b$. Then for $T \ge 2t$,

$$\fint\!\!\fint_{W(t,x)} |H(s,y)|\,ds\,dy \le \fint_{B(x,t)} \int_{2b}^{T} |\partial_s H(s,y)|\,ds\,dy + \fint_{B(x,t)} |H(T,y)|\,dy.$$

Applying Lemma 3.1(ii) to $\partial_t \widetilde{H} = H$, we see that the second term on the right-hand side tends to 0 as $T \to \infty$. Thus,

$$\sup_{t \ge 4b} \fint\!\!\fint_{W(t,x)} |H(s,y)|\,ds\,dy \le M(F_1)(x), \tag{3-3}$$

with $F_1(x) := \int_{2b}^{\infty} |\partial_s H(s,x)|\,ds$. Now,

$$\int_{\mathbb{R}^n} |F_1(x)|^2\,dx \le \frac{1}{2b} \int_{\mathbb{R}^n}\int_{2b}^{\infty} s^2 |\partial_s H(s,x)|^2\,ds\,dx$$

and in the domain of integration, H is a weak solution to $\mathcal{L}H = 0$. Thus, covering this region by Whitney cubes for $\mathbb{R}^{1+n}_+$, that is, cubes having side-length half their distance to the boundary, we may apply Caccioppoli's inequality on each cube and sum up, using bounded overlap, to get

$$\int_{\mathbb{R}^n}\int_{2b}^{\infty} s^2 |\partial_s H(s,x)|^2\,ds\,dx \lesssim \int_{\mathbb{R}^n}\int_{b}^{\infty} |H(s,x)|^2\,ds\,dx < \infty.$$

Going back to (3-3), the claim follows again from the maximal theorem.

We next turn to establishing $\|\widetilde{N}_{*,1}(\nabla H)\|_2 < \infty$. The control for the t-derivative $\partial_t H$ is the same upon replacing H by $\partial_t H$ and $\widetilde{G}$ by $\partial_t \widetilde{G}$ in the argument above, which satisfy the same hypotheses. Let us turn to $\nabla_y H$. Again we split the supremum into two parts $t < 4b$ and $4b \le t$. As for the first one, we argue as in (3-2), using that $s \mapsto \nabla_y H(s,\cdot)$ is smooth with values in $L^2(\mathbb{R}^n;\mathbb{C}^{mn})$ by Lemma 3.1, to give

$$\fint\!\!\fint_{W(t,x)} |\nabla_y H(s,y)|\,ds\,dy \le \mathcal{M}(\widetilde{F})(x) + \mathcal{M}(\nabla h)(x), \tag{3-4}$$

where $\widetilde{F}(x) := \int_0^{8b} |\partial_s \nabla_x H(s,x)|\, ds$. We know that $\nabla h \in L^2(\mathbb{R}^n; \mathbb{C}^{mn})$, and also $\widetilde{F} \in L^2(\mathbb{R}^n)$ since by Lemma 3.1,

$$\int_{\mathbb{R}^n} |\widetilde{F}(x)|^2\, dx \le 8b \int_{\mathbb{R}^n} \int_0^{8b} |\partial_s \nabla_x H(s,x)|^2\, ds\, dx < \infty.$$

Taking the supremum over $t < 4b$ in (3-4), we obtain again an L^2 bound from the maximal theorem. If $t \ge 4b$, we argue as in (3-3) to find

$$\sup_{t\ge 4b} \iint_{W(t,x)} |\nabla_y H(s,y)|\, ds\, dy \le \mathcal{M}(\widetilde{F}_1)(x), \tag{3-5}$$

with $\widetilde{F}_1(x) := \int_{2b}^{\infty} |\partial_s \nabla_x H(s,x)|\, ds$. Next, we can use the same covering argument as before to bring into play Caccioppoli's inequality and deduce

$$\int_{\mathbb{R}^n} |\widetilde{F}_1(x)|^2\, dx \le \frac{1}{2b} \int_{\mathbb{R}^n} \int_{2b}^{\infty} s^2 |\partial_s \nabla_x H(s,x)|^2\, ds\, dx \lesssim \int_{\mathbb{R}^n} \int_b^{\infty} |\partial_s H(s,x)|^2\, ds\, dx.$$

Lemma 3.1 guarantees that the rightmost term is finite and a final application of the maximal theorem yields the L^2 bound in (3-5). □

3B. ***Proof of Theorem 2.1 when $p = 2$.*** We assume $\mathcal{L}u = 0$ on $\mathbb{R}^{1+n}_+$, the control $\widetilde{N}_* u \in L^2(\mathbb{R}^n)$ and we have the convergence

$$\lim_{t\to 0} \iint_{W(t,x)} |u(s,y)|\, ds\, dy = 0 \tag{3-6}$$

for almost every $x \in \mathbb{R}^n$. We have to show $u = 0$ almost everywhere. To this end, we apply the strategy presented in Section 2.

For a reason that will appear later in the proof, we pick G of the form $G = \partial_t \widetilde{G}$, with $\widetilde{G} \in C_0^\infty(\mathbb{R}^{1+n}; \mathbb{C}^m) \cap \dot{W}^{-1,2}(\mathbb{R}^{1+n}; \mathbb{C}^m)$. Assume we have already proved $\langle u, G\rangle = 0$. This means $\langle \partial_t u, \widetilde{G}\rangle = 0$. When $n \ge 2$, Sobolev embeddings show that $\widetilde{G}$ can be any test function and so this implies $u(t,x) = f(x)$. When $n = 1$, we can take any test function with zero average and obtain $u(t,x) = ct + f(x)$, with c constant. The equations hold a.e. and we have $f \in L^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$ since $u \in L^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$. Due to the limit of Whitney averages at $t = 0$, we obtain $f = 0$ a.e. in both cases by Lebesgue's differentiation theorem. When $n \ge 2$, we are done. When $n = 1$, this yields $u(t,x) = ct$; hence $\iint_{W(t,x)} |u(s,y)|^2\, ds\, dy = \frac{5}{4} c^2 t^2$. As the supremum in $t > 0$ is finite a.e., we must have $c = 0$.

To show $\langle u, G\rangle = 0$, we have to make sense of H_1 and control both terms on the right-hand side of (2-9). We let $\widetilde{H} := (\mathcal{L}^*)^{-1}(\widetilde{G})$ and have $H = \partial_t \widetilde{H}$ due to Lemma 3.1(i). As $\nabla h \in L^2(\mathbb{R}^n; \mathbb{C}^{mn})$ by Lemma 3.1(ii), existence in the regularity problem for $\mathcal{L}^*$ yields a solution H_1 to $\mathcal{L}^* H_1 = 0$ in $\mathbb{R}^{1+n}_+$ with $\widetilde{N}_*(\nabla H_1) \in L^2(\mathbb{R}^n)$ and boundary trace h. Due to the explicit form of θ, we easily obtain for the first integral in (2-9),

$$|\langle A u \nabla\theta, \nabla(H - H_1)\rangle| \lesssim I_M + J_\varepsilon + J_R,$$

with

$$I_M := \frac{1}{M} \int_{|x|\ge M} \int_{2\varepsilon/3}^{3R/2} |u|\, |\nabla(H - H_1)|\, ds\, dy,$$

$$J_\alpha := \int_{\mathbb{R}^n} \fint_{2\alpha/3}^{3\alpha/2} |u|\, |\nabla(H - H_1)|\, ds\, dy.$$

First, I_M tends to 0 as $M \to \infty$. Indeed, let $\Omega := \big[\frac{2\varepsilon}{3}, \frac{3R}{2}\big] \times \{|x| \geq M\}$. By Lemma 8.2,

$$I_M \lesssim \frac{1}{M} \|\widetilde{N}_* u\|_2 \, \|\widetilde{N}_*(1_\Omega |\nabla(H - H_1)|)\|_2.$$

As $H - H_1$ is a solution to $\mathcal{L}^*(H - H_1) = 0$ on a neighbourhood of Ω, we can use reverse Hölder inequalities and a change of Whitney parameters to obtain

$$\|\widetilde{N}_*(1_\Omega |\nabla(H - H_1)|)\|_2 \lesssim \|\widetilde{N}_{*,1}(\nabla(H - H_1))\|_2,$$

which is finite by Lemma 3.2 and the construction of H_1.

Next, set $w(\varepsilon, x) := \big(\frac{2\varepsilon}{3}, \frac{3\varepsilon}{2}\big) \times B\big(x, \frac{\varepsilon}{2}\big)$, which is a Whitney region compactly contained in $W(\varepsilon, x)$. Using the averaging trick with balls of $\mathbb{R}^n$ having radii $\frac{\varepsilon}{2}$ and Tonelli's theorem, we obtain

$$\begin{aligned} J_\varepsilon &\leq \int_{\mathbb{R}^n} \bigg(⨌_{w(\varepsilon,x)} |u| \, |\nabla(H - H_1)| \bigg) \, dx \\ &\lesssim \int_{\mathbb{R}^n} \bigg(⨌_{w(\varepsilon,x)} |u|^2 \bigg)^{1/2} \bigg(⨌_{w(\varepsilon,x)} |\nabla(H - H_1)|^2 \bigg)^{1/2} dx \\ &\lesssim \int_{\mathbb{R}^n} \bigg(⨌_{W(\varepsilon,x)} |u| \bigg) \bigg(⨌_{W(\varepsilon,x)} |\nabla(H - H_1)| \bigg) \, dx, \end{aligned}$$

where we have used the reverse Hölder inequality for u and $\nabla(H - H_1)$. Observe that the integrand is controlled by $\widetilde{N}_* u \cdot \widetilde{N}_{*,1}(\nabla(H - H_1))$, which as a product of two L^2-functions is integrable. Moreover, $⨌_{W(\varepsilon,x)} |u|$ tends to 0 as $\varepsilon \to 0$ by assumption. Thus, we conclude $J_\varepsilon \to 0$ as $\varepsilon \to 0$ by dominated convergence. Finally, we have similarly

$$J_R \lesssim \int_{\mathbb{R}^n} \bigg(⨌_{W(R,x)} |u| \bigg) \bigg(⨌_{W(R,x)} |\nabla(H - H_1)| \bigg) \, dx,$$

so that we get the same L^1-control, while $⨌_{W(R,x)} |u| \to 0$ as $R \to \infty$ follows from Lemma 8.1 and $\widetilde{N}_*(u) \in L^2(\mathbb{R}^n)$.

Finally, we treat $\langle A\nabla u, (H - H_1)\nabla\theta \rangle$, which is the other term on the right-hand side of (2-9), exactly as above upon replacing $|u| \, |\nabla(H - H_1)|$ by $|\nabla u| \, |H - H_1| = (s|\nabla u|)(|H - H_1|/s)$. We observe that Caccioppoli's inequality and the reverse Hölder inequality reveal

$$\bigg(⨌_{w(t,x)} s^2 |\nabla u|^2 \bigg)^{1/2} \lesssim ⨌_{W(t,x)} |u|.$$

Hence we can apply the same argument as before, using $\|\widetilde{N}_{*,1}((H - H_1)/t)\|_2 \in L^2(\mathbb{R}^n)$ from Lemma 8.3 and $H = H_1$ on the boundary (write $H - H_1 = H - h + h - H_1$), which is used for the first time here.

Remark 3.4. With regard to Remark 2.6, we give the modification of the proof when instead of the nontangential convergence (3-6) we assume $⨍_{t/2}^{2t} |u(s, \cdot)| \, ds \to 0$ as $t \to 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$. The only difference is in the treatment of the limit of J_ε. To this end, pick any $\delta > 0$ and choose $r > 0$ such that

$$\int_{^cB(0,r)} \widetilde{N}_*(u) \widetilde{N}_{*,1}(\nabla(H - H_1)) \, dx < \delta.$$

Then for $\varepsilon < 1$ we obtain from Tonelli's theorem

$$
\begin{aligned}
J_\varepsilon &\lesssim \delta + \int_{B(0,r)} \biggl(\fint\!\!\fint_{W(\varepsilon,x)} |u(s,y)| \, ds \, dy \biggr) \widetilde{N}_{*,1}(\nabla(H - H_1))(x) \, dx \\
&\lesssim \delta + \int_{B(0,r+1)} \biggl(\fint_{\varepsilon/2}^{2\varepsilon} |u(s,y)| \, ds \biggr) \biggl(\fint_{B(y,\varepsilon)} \widetilde{N}_{*,1}(\nabla(H - H_1))(x) \, dx \biggr) dy \\
&\le \delta + \biggl\| \fint_{\varepsilon/2}^{2\varepsilon} |u(s,\cdot)| \, ds \biggr\|_{L^2(B(0,r+1))} \Bigl\| \mathcal{M}(\widetilde{N}_{*,1}(\nabla(H - H_1))) \Bigr\|_{L^2(\mathbb{R}^n)},
\end{aligned}
$$

which in the limit superior $\varepsilon \to 0$ is bounded by δ using the hypothesis and the maximal theorem. The modifications for the integral involving $(s|\nabla u|)(|H - H_1|/s)$ are similar, incorporating Caccioppoli's inequality to get back to averages of u.

3C. *Proof of Theorem 2.2 when p = 2.* We assume $\mathcal{L}u = 0$ on $\mathbb{R}^{1+n}_+$ with $\widetilde{N}_*(\nabla u) \in L^2(\mathbb{R}^n)$ and convergence $\lim_{t\to 0} \fint\!\!\fint_{W(t,x)} |u(s,y)| \, ds \, dy = 0$ for a.e. $x \in \mathbb{R}^n$. We have to show $u = 0$ almost everywhere. We first remark that by Lemma 8.3 we also have $\widetilde{N}_{*,1}(u/t) \in L^2(\mathbb{R}^n)$.

As in the proof of Theorem 2.1, we pick G of the form $G = \partial_t \widetilde{G}$ with $\widetilde{G} \in C_0^\infty(\mathbb{R}^{1+n}; \mathbb{C}^m) \cap \dot{W}^{-1,2}(\mathbb{R}^{1+n}; \mathbb{C}^m)$. We claim that it is again enough to show $\langle u, G \rangle = 0$: indeed, when $n \ge 2$ we may conclude as before, and when $n = 1$ we reach the point where $u(t,x) = ct$ a.e. but as $\widetilde{N}_*(u/t) \in L^2(\mathbb{R}^n)$ we must have $c = 0$.

To actually show $\langle u, G \rangle = 0$, we have again to control both terms on the right-hand side of (2-9). We let $\widetilde{H} := (\mathcal{L}^*)^{-1}(\widetilde{G})$, noting that by Lemma 3.1 we have $H = (\mathcal{L}^*)^{-1}(G) = \partial_t \widetilde{H}$. As $h \in L^2(\mathbb{R}^n; \mathbb{C}^m)$ by Lemma 3.1(ii), existence for the Dirichlet problem yields a solution H_1 to $\mathcal{L}^* H_1 = 0$ in $\mathbb{R}^{1+n}_+$, with $\widetilde{N}_*(H_1) \in L^2(\mathbb{R}^n)$ and boundary trace h in the sense that $\fint\!\!\fint_{W(\varepsilon,x)} |H_1(s,y) - h(x)| \to 0$ for a.e. $x \in \mathbb{R}^n$ as $\varepsilon \to 0$. We are now ready to estimate the integrals in (2-9). We still have

$$
|\langle Au\nabla\theta, \nabla(H - H_1)\rangle| \lesssim I_M + J_\varepsilon + J_R,
$$

with

$$
\begin{aligned}
I_M &= \frac{1}{M} \int_{|x|\ge M} \int_{2\varepsilon/3}^{3R/2} \frac{|u|}{s} \cdot |s\nabla(H - H_1)| \, ds \, dy, \\
J_\alpha &= \int_{\mathbb{R}^n} \fint_{2\alpha/3}^{3\alpha/2} \frac{|u|}{s} \cdot |s\nabla(H - H_1)| \, ds \, dy.
\end{aligned}
$$

First, I_M tends to 0 as $M \to \infty$. Indeed, let $\Omega := \bigl[\frac{2\varepsilon}{3}, \frac{3R}{2}\bigr] \times \{|x| \ge M\}$. By Lemma 8.2,

$$
I_M \lesssim \frac{1}{M} \Bigl\| \widetilde{N}_*\Bigl(\frac{u}{s}\Bigr) \Bigr\|_2 \|\widetilde{N}_*(1_\Omega |s\nabla(H - H_1)|)\|_2.
$$

As $\mathcal{L}u = 0$ and $s \sim t$ on Whitney regions $W(t,x)$, we have

$$
\Bigl\| \widetilde{N}_*\Bigl(\frac{u}{s}\Bigr) \Bigr\|_2 \lesssim \Bigl\| \widetilde{N}_{*,1}\Bigl(\frac{u}{s}\Bigr) \Bigr\|_2 < \infty
$$

by reverse Hölder estimates and change of parameters (which will be implicit in all of the following steps). Also $\mathcal{L}^*(H - H_1) = 0$ holds on a neighbourhood of Ω. Thus, we can use Caccioppoli inequalities to obtain

$$\|\widetilde{N}_*(1_\Omega s|\nabla(H-H_1)|)\|_2 \lesssim \|\widetilde{N}_*(1_{\widetilde{\Omega}}(H-H_1))\|_2,$$

where $\widetilde{\Omega}$ is a slightly bigger region, still at some large distance from the support of $\widetilde{G}$, so that we may use reverse Hölder inequalities to conclude

$$\|\widetilde{N}_*(1_{\widetilde{\Omega}}(H-H_1))\|_2 \lesssim \|\widetilde{N}_{*,1}H\|_2 + \|\widetilde{N}_*H_1\|_2.$$

The latter are finite by Lemma 3.2 and the construction of H_1.

Next, using the averaging trick with balls of radii $\frac{\varepsilon}{2}$ and the Whitney regions $w(\varepsilon,x) := \bigl(\frac{2\varepsilon}{3},\frac{3\varepsilon}{2}\bigr)\times B\bigl(x,\frac{\varepsilon}{2}\bigr)$ and $\tilde{w}(\varepsilon,x) = \bigl(\frac{4\varepsilon}{7},\frac{7\varepsilon}{4}\bigr)\times B\bigl(x,\frac{2\varepsilon}{3}\bigr)$, both compactly contained in $W(\varepsilon,x)$, we obtain

$$\begin{aligned}
J_\varepsilon &\lesssim \int_{\mathbb{R}^n}\biggl(\fint\!\!\!\fint_{w(\varepsilon,x)} \frac{|u|}{s}\cdot|s\nabla(H-H_1)|\biggr)\,dx \\
&\lesssim \int_{\mathbb{R}^n}\biggl(\fint\!\!\!\fint_{w(\varepsilon,x)} \frac{|u|^2}{s^2}\biggr)^{1/2}\biggl(\fint\!\!\!\fint_{w(\varepsilon,x)} |s\nabla(H-H_1)|^2\biggr)^{1/2}\,dx \\
&\lesssim \int_{\mathbb{R}^n}\biggl(\fint\!\!\!\fint_{W(\varepsilon,x)} \frac{|u|}{s}\biggr)\biggl(\fint\!\!\!\fint_{\tilde{w}(\varepsilon,x)} |H-H_1|^2\biggr)\,dx \\
&\lesssim \int_{\mathbb{R}^n}\biggl(\fint\!\!\!\fint_{W(\varepsilon,x)} \frac{|u|}{s}\biggr)\biggl(\fint\!\!\!\fint_{W(\varepsilon,x)} |H-H_1|\biggr)\,dx,
\end{aligned}$$

where we have used the reverse Hölder inequality for u and $H-H_1$, and Caccioppoli inequalities for $H-H_1$, observing that $\mathcal{L}^*(H-H_1)=0$ holds on a neighbourhood of the domain of integration. We note that the integrand is controlled by $\widetilde{N}_*(u/s)\widetilde{N}_{*,1}(H-H_1)$, which is integrable by assumption on u and Lemma 8.3, plus Lemma 3.2 for H and the construction of H_1. As for the pointwise convergence, we use $H-H_1$ and write

$$\fint\!\!\!\fint_{W(\varepsilon,x)} |H-H_1| \le \fint\!\!\!\fint_{W(\varepsilon,x)} |H(s,y)-h(x)|\,ds\,dy + \fint\!\!\!\fint_{W(\varepsilon,x)} |H_1(s,y)-h(x)|\,ds\,dy.$$

Letting $\varepsilon\to 0$, the first term goes to 0 by Lemma 3.2 combined with Lemma 8.3. By construction of H_1, so does the second one. Thus, $J_\varepsilon\to 0$ as $\varepsilon\to 0$ by dominated convergence. Finally, we have similarly,

$$J_R \lesssim \int_{\mathbb{R}^n}\biggl(\fint\!\!\!\fint_{W(R,x)} \frac{|u|}{s}\biggr)\biggl(\fint\!\!\!\fint_{W(R,x)} |H-H_1|\biggr)\,dx,$$

so that we get the same L^1-control, and to obtain convergence to 0 we can use $\fint\!\!\fint_{W(R,x)}|H-H_1|\to 0$ as $R\to\infty$, which follows from Lemma 8.1 since $\widetilde{N}_{*,1}(H-H_1)\in L^2(\mathbb{R}^n)$.

Finally, we can treat $\langle A\nabla u,(H-H_1)\nabla\theta\rangle$, which is the second term on the right-hand side of (2-9), as above upon replacing $(|u|/s)(s|\nabla(H-H_1)|)$ by $|\nabla u||H-H_1|$. The arguments for convergences when $\varepsilon\to 0$ and $R\to\infty$ are almost identical and we leave the details to the reader.

4. Representation by single-layer operators

In order not to disrupt the flow of the proofs of uniqueness, we only summarise here the needed results for the single-layer operators of [Rosén 2013], which we denote by $\mathcal{S}_t^{\mathcal{L}}$, postponing proofs until Section 7.

Throughout, we use the notation $f_s : x \mapsto f(s,x)$. We begin with the result summarising their boundedness properties.

Lemma 4.1. *Let $1 < p < \infty$. Let $k \geq 0$ be an integer. Set $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.*

(1) *If $p \in \mathcal{H}_{\mathcal{L}}$, we have the mapping properties* (*of extensions by density of*)

$$(t\partial_t)^k \mathcal{S}_t^{\mathcal{L}} : L^p(\mathbb{R}^n;\mathbb{C}^m) \to \dot{W}^{1,p}(\mathbb{R}^n;\mathbb{C}^m), \quad t \in \mathbb{R}^*,$$
$$(t\partial_t)^k \partial_t \mathcal{S}_t^{\mathcal{L}} : L^p(\mathbb{R}^n;\mathbb{C}^m) \to L^p(\mathbb{R}^n;\mathbb{C}^m), \qquad t \in \mathbb{R}^*.$$

Moreover, *the operator norms are uniform with respect to t*, *and as functions of t*, *these operators are strongly continuous on $\mathbb{R}^*$*, *have strong limits at $0^\pm$ and vanish strongly at infinity. Moreover*, *the limits at $0^\pm$ are the same except for $\partial_t \mathcal{S}_t^{\mathcal{L}}$ on L^p.*

(2) *If $p' \in \mathcal{H}_{\mathcal{L}^*}$, we have the mapping properties* (*of extensions by density of*)

$$(t\partial_t)^k \mathcal{S}_t^{\mathcal{L}} : \dot{W}^{-1,p}(\mathbb{R}^n;\mathbb{C}^m) \to L^p(\mathbb{R}^n;\mathbb{C}^m), \quad t \in \mathbb{R}^*.$$

Moreover, *the operator norms are uniform with respect to t*, *and as functions of t*, *these operators are strongly continuous on $\mathbb{R}^*$*, *have strongly continuous extensions at 0 and vanish strongly at infinity.*

Remark 4.2. Recall that the intervals $\mathcal{H}_{\mathcal{L}}, \mathcal{H}_{\mathcal{L}^*}$ were defined in Section 2B. They are open intervals and contain 2, so $\mathcal{H}_{\mathcal{L}} \cap (\mathcal{H}_{\mathcal{L}^*})'$, where $I' = \{p' : p \in I\}$, is an open interval around 2 and for p in this interval both sets of estimates hold.

Remark 4.3. In the first case, it follows from the stated properties that $(t\partial_t)^k \nabla_x \mathcal{S}_t^{\mathcal{L}}$ are L^p bounded uniformly in $t \neq 0$. The same is true for the conormal derivatives $(t\partial_t)^k \partial_{\nu_A} \mathcal{S}_t^{\mathcal{L}}$ and these operators are not continuous at 0 when $k = 0$ (jump relations). In the second case, these operators are $\dot{W}^{-1,p}$ bounded uniformly in t.

Remark 4.4. More mapping properties on fractional Sobolev and Besov spaces can be drawn by interpolating these two sets of inequalities. We do not need those here. Note that for $1 < p < \infty$, the condition $p' \in \mathcal{H}_{\mathcal{L}^*}$ is closely related to the identification of certain negative-order Sobolev spaces $\dot{W}_{DB}^{-1,p} = \dot{W}_D^{-1,p}$ as $\dot{W}_D^{-1,p}$ is a natural space for $(\partial_{\nu_A} u|_{t=0}, \nabla_x f)$ if u is a solution to $\mathcal{L}u = 0$ with Dirichlet data $f \in L^p$. The reader can refer to [Amenta and Auscher 2018] for more on this issue.

Next, we state the representation formula by the above single-layer operators for $\mathcal{L}^{-1}$. The proof will only use the properties stated above for $p = 2$.

Proposition 4.5 (representation by single-layer operators). *Assume $f \in C_0^\infty(\mathbb{R}^{1+n};\mathbb{C}^m)$. If $n = 1, 2$ assume furthermore that $f = \operatorname{div}_x F$ for some $F \in C_0^\infty(\mathbb{R}^{1+n};\mathbb{C}^{mn})$. Then*

$$(\mathcal{L}^{-1} f)(t,x) = \text{p.v.}\int_{\mathbb{R}} \mathcal{S}_{t-s}^{\mathcal{L}} f_s(x)\,ds \quad \text{in } \dot{W}^{1,2}(\mathbb{R}^{1+n};\mathbb{C}^m), \tag{4-1}$$

where

$$\text{p.v.}\int_{\mathbb{R}} \mathcal{S}_{t-s}^{\mathcal{L}} f_s(x)\,ds := \lim_{\varepsilon\to 0,\, R\to\infty} \int_{\varepsilon<|t-s|<R} \mathcal{S}_{t-s}^{\mathcal{L}} f_s(x)\,ds,$$

and

$$\mathcal{L}^{-1}(\partial_t f)(t,x) = (\partial_t \mathcal{L}^{-1} f)(t,x) = \text{p.v.}\int_{\mathbb{R}} \partial_t \mathcal{S}^{\mathcal{L}}_{t-s} f_s(x)\,ds$$
$$= \text{p.v.}\int_{\mathbb{R}} \mathcal{S}^{\mathcal{L}}_{t-s}(\partial_s f_s)(x)\,ds, \quad \textit{in } L^2(\mathbb{R}^{1+n};\mathbb{C}^m). \tag{4-2}$$

Furthermore, with $W^{1,2}(\mathbb{R}^n;\mathbb{C}^m)$-convergence uniformly for $t \in \mathbb{R}$, we have norm-convergent (Bochner) integrals in $W^{1,2}(\mathbb{R}^n;\mathbb{C}^m)$,

$$(\mathcal{L}^{-1}(\partial_t f))_t = (\partial_t \mathcal{L}^{-1} f)_t = \int_{\mathbb{R}} \partial_t \mathcal{S}^{\mathcal{L}}_{t-s} f_s\,ds = \int_{\mathbb{R}} \mathcal{S}^{\mathcal{L}}_{t-s}(\partial_s f_s)\,ds. \tag{4-3}$$

Remark 4.6. It is classical that $f_t(x) = \operatorname{div}_x F_t(x)$ is equivalent to $\int_{\mathbb{R}^n} f_t(x)\,dx = 0$. We note that the replacement of f by $\partial_t f$, which has all the other properties required for f, yields a better convergence of the principal value towards $\mathcal{L}^{-1}(\partial_t f)$. Also for fixed t, the integrals in (4-3) have good behaviour: it is only to identify the limit in $L^2(\mathbb{R}^{1+n};\mathbb{C}^m)$ that we have to take principal values.

We also provide a result implying that the operators $\mathcal{S}^{\mathcal{L}}_t$ are the unique bounded operators $L^2(\mathbb{R}^n;\mathbb{C}^m) \to \dot{W}^{1,2}(\mathbb{R}^n;\mathbb{C}^m)$ depending strongly continuously on $t \in \mathbb{R}$ for which the representation (4-1) holds true.

Proposition 4.7. *Let $f \in C_0^\infty(\mathbb{R}^n;\mathbb{C}^m)$ with $\int_{\mathbb{R}^n} f = 0$ if $n = 1, 2$. Let $\chi_\varepsilon(s) = \frac{1}{\varepsilon}\chi\left(\frac{s}{\varepsilon}\right)$ with $\varepsilon > 0$ and $\chi \in C_0^\infty(\mathbb{R})$ satisfying $\int_{\mathbb{R}} \chi(s)\,ds = 1$. Set $f_\varepsilon(s,x) := \chi_\varepsilon(s) g(x)$. Then $\mathcal{L}^{-1} f_\varepsilon$ belongs to $C_0(\mathbb{R};\dot{W}^{1,2}(\mathbb{R}^n;\mathbb{C}^m))$ and for all $t \in \mathbb{R}$, $(\mathcal{L}^{-1} f_\varepsilon)_t$ converges in $\dot{W}^{1,2}(\mathbb{R}^n;\mathbb{C}^m)$ to $\mathcal{S}^{\mathcal{L}}_t f$.*

5. The case $p \neq 2$

Let us mention that the case when $p-2$ is small could be treated similarly to the case $p = 2$ without the representation by layer potentials and assuming only Gårding's inequality (2-8). This would use a basic extension of Lemmas 3.1 and 3.2, taking into account that $\mathcal{L} : \dot{W}^{1,p}(\mathbb{R}^{1+n}) \to \dot{W}^{-1,p}(\mathbb{R}^{1+n})$ remains invertible for such p due to Shneiberg's lemma [1974]. But this does not apply when p gets "far" from 2.

Henceforth, we assume (2-2) and begin with a lemma analogous to Lemma 3.2 in our range of p.

Lemma 5.1. *Let $\widetilde{G} \in C_0^\infty(\mathbb{R}^{1+n};\mathbb{C}^m)$ be such that $\widetilde{G} = \operatorname{div}_x G^\sharp$ for some $G^\sharp \in C_0^\infty(\mathbb{R}^{1+n};\mathbb{C}^{nm})$. Set $\widetilde{H} := (\mathcal{L}^*)^{-1}(\widetilde{G})$, $G := \partial_t \widetilde{G}$ and $H := \partial_t \widetilde{H}$. Let $1 < p < \infty$:*

(i) *If $p \in \mathcal{H}_{\mathcal{L}}$, then $\partial_t^k H \in C_0(\mathbb{R}; L^{p'}(\mathbb{R}^n;\mathbb{C}^m))$ for all integers $k \geq 0$ and $\|\widetilde{N}_{*,1} H\|_{p'} < \infty$.*

(ii) *If $p' \in \mathcal{H}_{\mathcal{L}^*}$, then $\partial_t^k H \in C_0(\mathbb{R}; W^{1,p'}(\mathbb{R}^n;\mathbb{C}^m))$ for all integers $k \geq 0$ and $\|\widetilde{N}_{*,1} H\|_{p'} < \infty$, as well as $\|\widetilde{N}_{*,1}(\nabla H)\|_{p'} < \infty$.*

In both statements, the distinguished variable in the regularity estimates is $t \in \mathbb{R}$.

Proof. Let $a, b \in \mathbb{R}$ such that $\operatorname{supp} \widetilde{G} \subset [a,b] \times \mathbb{R}^n$. We may assume $b \geq 2$ for simplicity. Note that we have the assumptions of Proposition 4.5 for $\widetilde{G}$, G and so the representations apply.

We look at (i) first. By (4-3) we have

$$H_t = \partial_t \widetilde{H}_t = \int_{\mathbb{R}} \mathcal{S}^{\mathcal{L}^*}_{t-s} \partial_s \widetilde{G}_s\,ds = \int_{\mathbb{R}} \mathcal{S}^{\mathcal{L}^*}_{t-s} G_s\,ds, \tag{5-1}$$

with $G_s = \operatorname{div}_x(\partial_s G_s^\sharp) \in \dot{W}^{-1,p'}(\mathbb{R}^n;\mathbb{C}^m)$ by assumption. Due to Lemma 4.1(ii) — but replacing $(p,\mathcal{L})$ by $(p',\mathcal{L}^*)$ therein — we can bound the norm of $\mathcal{S}_t^{\mathcal{L}^*}$ as a bounded operator from $\dot{W}^{-1,p'}(\mathbb{R}^n;\mathbb{C}^m)$ to $L^{p'}(\mathbb{R}^n;\mathbb{C}^m)$ uniformly in $t \in \mathbb{R}\setminus\{0\}$. From Minkowski's inequality and the fact that G is smooth with compact support, we can infer

$$\|H_t\|_{p'} \lesssim \int_{\mathbb{R}} \|\mathcal{S}_{t-s}^{\mathcal{L}^*} G_s\|_{p'}\, ds \lesssim \int_a^b \|G_s\|_{\dot{W}^{-1,p'}}\, ds < \infty$$

uniformly for all $t \in \mathbb{R}$. Owing to Lemma 3.1, an analogous formula applies to $\partial_t^k H$ with $\partial_s^k G$ in the integral for k an integer and so we also have $\sup_{t\in\mathbb{R}} \|\partial_t^k H_t\|_{p'} < \infty$. Continuity of $t \mapsto \partial_t^k H_t$ and the limits at $\pm\infty$ both in $L^{p'}(\mathbb{R}^n;\mathbb{C}^m)$ follow by applying the dominated convergence theorem to (5-1): indeed, for $s \in \mathbb{R}$ fixed, Lemma 4.1(ii) shows that $t \mapsto \mathcal{S}_{t-s}^{L^*}\partial_s^k G_s$ is continuous on $\mathbb{R}\setminus\{s\}$ and bounded with values in $L^{p'}(\mathbb{R}^n;\mathbb{C}^m)$.

Finally, we prove to the maximal estimate $\|\widetilde{N}_{*,1}H\|_{p'} < \infty$. Proceeding as when $p=2$ in the proof of Lemma 3.2 — and with the same notation — it suffices to show that $F(x) = \int_0^{8b} |\partial_t H(t,x)|\, dt$ and $F_1(x) = \int_{2b}^\infty |\partial_t H(t,x)|\, dt$ belong to $L^{p'}(\mathbb{R}^n)$. First,

$$\int_{\mathbb{R}^n} |F(x)|^{p'}\, dx \le (8b)^{p'/p} \int_{\mathbb{R}^n}\int_0^{8b} |\partial_s H(s,x)|^{p'}\, ds\, dx \le (8b)^{p'} \sup_{t\in\mathbb{R}} \|\partial_t H_t\|_{p'}^{p'} < \infty.$$

For F_1 we take a different approach. By Minkowski's inequality

$$\|F_1\|_{p'} \le \int_{2b}^\infty \|\partial_t H_t\|_{p'}\, dt$$

and we can use integration by parts twice along with $G_s = \partial_s \widetilde{G}_s$ to obtain from the representation (4-2)

$$\partial_t H_t = \int_a^b \mathcal{S}_{t-s}^{\mathcal{L}^*}\partial_s G_s\, ds = \int_a^b \partial_s^2 \mathcal{S}_{t-s}^{\mathcal{L}^*}\widetilde{G}_s\, ds.$$

Note that because of $t-b > b$ we stay away from $t-s=0$ and can use the decay of the single layer. More precisely, the norm of $\partial_s^2\mathcal{S}_{t-s}^{\mathcal{L}^*} : \dot{W}^{-1,p'} \to L^{p'}$ is bounded by $|t-s|^{-2}$ due to Lemma 4.1 and we obtain $\|\partial_t H_t\|_{p'} \lesssim t^{-2}$, which in turn warrants $\|F_1\|_{p'} < \infty$. This completes the proof of (i).

We turn to (ii). Using (4-1) and (4-3), we have

$$\nabla_x \partial_t^k H(t,x) = \text{p.v.}\int_{\mathbb{R}} \nabla_x \mathcal{S}_{t-s}^{\mathcal{L}^*}\partial_s^k G(s,x)\, ds$$

in $L^2(\mathbb{R}^{1+n};\mathbb{C}^{mn})$ and for fixed t the integrals are bona fide Bochner integrals in $L^2(\mathbb{R}^n;\mathbb{C}^{nm})$. Again by (4-3), we have $\partial_t\partial_t^k H_t = \int_{\mathbb{R}} \partial_t \mathcal{S}_{t-s}^{\mathcal{L}^*}\partial_s^k G_s\, ds$ with the same meaning. From here, the proof of t-regularity is entirely analogous to (i), relying instead on Lemma 4.1(i) but with $(p',\mathcal{L}^*)$ replacing $(p,\mathcal{L})$ as our assumption is $p' \in \mathcal{H}_{\mathcal{L}^*}$. In particular, we obtain $\|\widetilde{N}_{*,1}H\|_{p'} < \infty$ by the same argument.

As for the nontangential maximal estimate $\|\widetilde{N}_{*,1}(\nabla H)\|_{p'} < \infty$, we follow again the proof for the $p=2$ case. We also have to estimate $\widetilde{F}(x) = \int_0^{8b} |\partial_t \nabla_x H(t,x)|\, dt$ and $\widetilde{F}_1(x) = \int_{2b}^\infty |\partial_t\nabla_x H(t,x)|\, dt$ in

$L^{p'}(\mathbb{R}^n)$. First,

$$\int_{\mathbb{R}^n} |\widetilde{F}(x)|^{p'}\, dx \le (8b)^{p'/p} \int_{\mathbb{R}^n} \int_0^{8b} |\partial_t \nabla_x H(t,x)|^{p'}\, dt\, dx \le (8b)^{p'} \sup_{t\in\mathbb{R}} \|\nabla_x(\partial_t H)\|_{p'}^{p'} < \infty.$$

Next,

$$\|\widetilde{F}_1\|_{p'} \le \int_{2b}^{\infty} \|\partial_t \nabla_x H_t\|_{p'}\, dt.$$

For t in this range we have $t - s > b$ in the integral that represents $\partial_t \nabla_x H_t$ and integrating by parts twice we obtain

$$\partial_t \nabla_x H_t = \int_a^b \nabla_x \mathcal{S}_{t-s}^{\mathcal{L}^*} \partial_s G_s\, ds = \int_a^b \partial_s^2 \nabla_x \mathcal{S}_{t-s}^{\mathcal{L}^*} \widetilde{G}_s\, ds.$$

Since the $L^{p'}$-operator norm of $\partial_s^2 \nabla_x \mathcal{S}_{t-s}^{\mathcal{L}^*}$ is controlled by $|t-s|^{-2}$, see Lemma 4.1, we have

$$\|\partial_t \nabla_x H_t\|_{p'} \lesssim t^{-2},$$

which warrants $\|\widetilde{F}_1\|_{p'} < \infty$. $\square$

Remark 5.2. We have not tried to get optimal hypotheses on G for obtaining the desired estimates. In (ii) we did not use $\widetilde{G} = \operatorname{div}_x G^\sharp$. For (i), it can also be lifted provided we have $C_0^\infty(\mathbb{R}^n) \subset \dot{W}^{-1,p'}(\mathbb{R}^n)$, which holds for $p' > \frac{n}{n-1}$ by Sobolev embeddings. It is simpler and enough for us, however, to make this assumption throughout.

5A. ***Proof of Theorem 2.1 when p ≠ 2.*** We assume $p' \in \mathcal{H}_{\mathcal{L}^*}$. We consider a weak solution to $\mathcal{L}u = 0$ on $\mathbb{R}^{1+n}_+$ such that $\widetilde{N}_* u \in L^p(\mathbb{R}^n)$ and

$$\lim_{t\to 0} \iint_{W(t,x)} |u(s,y)|\, ds\, dy = 0$$

for a.e. $x \in \mathbb{R}^n$. Our task is to show $u = 0$ almost everywhere.

This time, we pick G of the form $G = \partial_t \widetilde{G}$ with $\widetilde{G} \in C_0^\infty(\mathbb{R}^{1+n}; \mathbb{C}^m)$ and $\widetilde{G} = \operatorname{div}_x G^\sharp$. Assume, we had managed to prove $\langle u, G\rangle = 0$, that is, $\langle \partial_t u, \widetilde{G}\rangle = 0$. Then $\langle \nabla_x \partial_t u, G^\sharp\rangle = 0$, where $G^\sharp$ is an arbitrary test function in $\mathbb{R}^{1+n}_+$. Hence, $\partial_t u \in L^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+; \mathbb{C}^m)$ is independent of x and we obtain $u(t,x) = g(t) + f(x)$ with $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{C}^m)$ and $g : (0,\infty) \to \mathbb{C}^m$ continuous. Let

$$v(t,x) := \iint_{W(t,x)} u(s,y)\, ds\, dy = \fint_{t/2}^{2t} g(s)\, ds + \fint_{B(x,t)} f(y)\, dy.$$

We know $v(t,x) \to 0$ as $t \to 0$ for a.e. $x \in \mathbb{R}^n$. Applying Lebesgue's differentiation theorem to f, it follows that $\fint_{t/2}^{2t} g(s)\, ds$ has a limit when $t \to 0$. Call it $\alpha \in \mathbb{C}^m$. Then $\alpha + f(x) = 0$ almost everywhere, which in turn implies that u is independent of x. But due to $\widetilde{N}_* u \in L^p(\mathbb{R}^n)$ we must have $\widetilde{N}_* u = 0$. This yields $u = 0$ as desired.

Next, the proof of $\langle u, G\rangle = 0$ is line by line the same as the one for $p = 2$ in Section 3B. Indeed, thanks to Lemma 5.1, $h := H(0,\cdot) \in \dot{W}^{1,p'}(\mathbb{R}^n; \mathbb{C}^m)$, where $H := (\mathcal{L}^*)^{-1} G$. Thus, existence for $(R)_{p'}^{\mathcal{L}^*}$ yields a solution to $\mathcal{L}^* H_1 = 0$ in $\mathbb{R}^{1+n}_+$ with $\widetilde{N}_*(\nabla H_1) \in L^{p'}(\mathbb{R}^n)$ and boundary trace h. The nontangential estimates needed to run the argument are $\widetilde{N}_* u \in L^p(\mathbb{R}^n)$ by assumption, $\widetilde{N}_{*,1}(\nabla(H - H_1)) \in L^{p'}(\mathbb{R}^n)$ by Lemma 5.1(ii) and the construction of H_1, and $\widetilde{N}_{*,1}((H - H_1)/t) \in L^{p'}(\mathbb{R}^n)$ by Lemma 8.3.

Remark 5.3. We can modify the proof of convergence to 0 of J_ε as in Remark 3.4 if we only assume $\fint_{t/2}^{2t} |u(s,\cdot)|\,ds \to 0$ in $L^p_{\mathrm{loc}}(\mathbb{R}^n)$ as $t \to 0$.

5B. *Proof of Theorem 2.2 when p ≠ 2.* Let $p \in \mathcal{H}_{\mathcal{L}}$. We assume that $\mathcal{L}u = 0$ on $\mathbb{R}^{1+n}_+$, that $\widetilde{N}_*(\nabla u) \in L^p(\mathbb{R}^n)$ and that

$$\lim_{t\to 0} \fint\!\!\fint_{W(t,x)} |u(s,y)|\,ds\,dy = 0$$

for a.e. $x \in \mathbb{R}^n$. We have to show $u = 0$ almost everywhere. To this end, we pick G as in the previous proof. The same argument then shows that it suffices to check $\langle u, G\rangle = 0$, noting that we can use $\widetilde{N}_{*,1}(u/t) \in L^p(\mathbb{R}^n)$ from Lemma 8.3 to deduce $u = 0$ once it has been seen to depend on t only.

Next, the argument to show $\langle u, G\rangle = 0$ is identical to the one for $p = 2$ presented in Section 3C: Existence for $(D)^{\mathcal{L}^*}_{p'}$ yields a solution to $\mathcal{L}^* H_1 = 0$ in $\mathbb{R}^{1+n}_+$ with $\widetilde{N}_*(H_1) \in L^{p'}(\mathbb{R}^n)$ and boundary trace $h := H(0,\cdot) \in L^{p'}(\mathbb{R}^n;\mathbb{C}^m)$, where $H := (\mathcal{L}^*)^{-1}G$. The required nontangential estimates are $\widetilde{N}_{*,1}(u/t) \in L^p(\mathbb{R}^n)$, as seen above, as well as $\widetilde{N}_{*,1}(H - H_1) \in L^{p'}(\mathbb{R}^n)$ by Lemma 5.1(i) and the construction of H_1.

5C. *Proof of Corollary 2.5.* Assume $1 < p < \infty$ with $p' \in \mathcal{H}_{\mathcal{L}^*}$. Theorem 1.6 of [Auscher and Mourgoglou 2019] shows that well-posedness of $(R)^{\mathcal{L}^*}_{p'}$ is equivalent to well-posedness of a modified Dirichlet problem $(\widetilde{D})^{\mathcal{L}}_p$. Combining (1) and (2) of Theorem 1.9 in [Auscher and Mourgoglou 2019], we have that the compatible well-posedness of $(R)^{\mathcal{L}^*}_{p'}$ is equivalent to the compatible well-posedness of this modified Dirichlet problem $(\widetilde{D})^{\mathcal{L}}_p$,

$$\begin{cases} \mathcal{L}u = 0 & \text{on } \mathbb{R}^{1+n}_+, \\ S(t\nabla u) \in L^p(\mathbb{R}^n), \\ \lim_{t\to 0} u(t,\cdot) = f & \text{in } L^p(\mathbb{R}^n;\mathbb{C}^m). \end{cases} \tag{$(\widetilde{D})^{\mathcal{L}}_p$}$$

The following was shown in [Auscher and Mourgoglou 2019, Corollary 1.4]: Any solution to $\mathcal{L}u = 0$ in $\mathbb{R}^{1+n}_+$ such that $S(t\nabla u) \in L^p(\mathbb{R}^n)$ for p in this range, is, up to a constant, in $C_0([0,\infty); L^p(\mathbb{R}^n;\mathbb{C}^m))$, yielding the existence of $u(0,\cdot)$ and so the limit makes sense. Moreover, there are estimates

$$\sup_{t\geq 0} \|u(t,\cdot)\|_p \lesssim \|S(t\nabla u)\|_p,$$
$$\|\widetilde{N}_*(u)\|_p \lesssim \|S(t\nabla u)\|_p,$$

as well as an almost everywhere limit

$$\lim_{t\to 0} \fint\!\!\fint_{W(t,x)} |u(s,y) - u(0,x)|^2\,ds\,dy = 0.$$

(In fact, a weaker form is stated in [Auscher and Mourgoglou 2019] but this stronger form holds and is a consequence of [Auscher and Stahlhut 2016, Theorem 9.9].)

Thus, given $f \in L^p(\mathbb{R}^n;\mathbb{C}^m)$, the unique solution (the constant is eliminated) of $(\widetilde{D})^{\mathcal{L}}_p$ with data f satisfies $\|S(t\nabla u)\|_p \lesssim \|f\|_p$. Note then that $\|f\|_p = \|u(0,\cdot)\|_p \leq \sup_{t\geq 0}\|u(t,\cdot)\|_p$ and also $\|u(0,\cdot)\|_p \lesssim \|\widetilde{N}_* u\|_p$ by Fatou's lemma. In particular, u is a solution to $(D)^{\mathcal{L}}_p$ and (2-6), (2-7) hold. Uniqueness of $(D)^{\mathcal{L}}_p$ on the other hand follows from Theorem 2.1.

6. Regularity with Hardy Sobolev data vs Dirichlet with BMO or Hölder continuous data

This section is mostly contained in [Auscher and Mourgoglou 2019] but we feel it is informative to review it in light of what we just proved. When $\frac{n}{n+1} < p \le 1$, the regularity problem becomes the following: given $f \in \dot{H}^{1,p}(\mathbb{R}^n;\mathbb{C}^m)$, solve uniquely (modulo constants)

$$\begin{cases} \mathcal{L}u = 0 & \text{on } \mathbb{R}^{1+n}_+, \\ \widetilde{N}_*(\nabla u) \in L^p(\mathbb{R}^n), \\ \lim_{t\to 0} ⨌_{W(t,x)} |u(s,y) - f(x)|\, ds\, dy = 0 & \text{for a.e. } x \in \mathbb{R}^n. \end{cases} \qquad (R)^{\mathcal{L}}_p$$

Here, $\dot{H}^{1,p}(\mathbb{R}^n;\mathbb{C}^m)$ is the Hardy–Sobolev space of order 1 above the real Hardy space H^p. The dual problem is the following Dirichlet problem: given $f \in \dot{\Lambda}^\alpha(\mathbb{R}^n;\mathbb{C}^m)$, $0 \le \alpha < 1$, solve uniquely (modulo constants)

$$\begin{cases} \mathcal{L}^*u = 0 & \text{on } \mathbb{R}^{1+n}_+, \\ C_\alpha(t\nabla u) \in L^\infty(\mathbb{R}^n), \\ \lim_{t\to 0} u(t,\cdot) = f & \text{in } \mathcal{D}'(\mathbb{R}^n;\mathbb{C}^m)/\mathbb{C}^m, \\ \lim_{t\to \infty} u(t,\cdot) = 0 & \text{in } \mathcal{D}'(\mathbb{R}^n;\mathbb{C}^m)/\mathbb{C}^m, \end{cases} \qquad (D)^{\mathcal{L}^*}_\alpha$$

where

$$C_\alpha F(x) := \sup\biggl(\frac{1}{r^{2\alpha}|B(y,r)|}\iint_{T_{y,r}} |F(t,z)|^2 \frac{dt\, dz}{t}\biggr)^{1/2}$$

taken over all open balls $B(y,r)$ containing x, with $T_{y,r} = (0,r)\times B(y,r)$. Here, $\dot{\Lambda}^\alpha(\mathbb{R}^n;\mathbb{C}^m)$ designates $\mathrm{BMO}(\mathbb{R}^n;\mathbb{C}^m)$ for $\alpha = 0$ and for $\alpha > 0$ it designates the homogeneous Hölder space of exponent α. Note the condition at infinity that does not follow from the interior control in general. As the interior condition is on the gradient, the problem is posed modulo constants.

The situation of interest is when the interval of Hardy spaces coincidence contains exponents below $p = 1$, as is the case under (DGNM).

Theorem 6.1. *Let $\frac{n}{n+1} < p \le 1$ with $p \in \mathcal{H}_{\mathcal{L}}$ and let $\alpha = n\bigl(\frac{1}{p} - 1\bigr)$:*

(i) *Existence for $(R)^{\mathcal{L}}_p$ implies uniqueness for $(D)^{\mathcal{L}^*}_\alpha$.*

(ii) *Existence for $(D)^{\mathcal{L}^*}_\alpha$ implies uniqueness for $(R)^{\mathcal{L}}_p$.*

(iii) *(Compatible) well-posedness for $(R)^{\mathcal{L}}_p$ implies (compatible) well-posedness for $(D)^{\mathcal{L}^*}_\alpha$.*

Proof. For item (iii), the implication concerning well-posedness is in Theorem 1.6 of [Auscher and Mourgoglou 2019]; concerning compatible well-posedness, the implication is contained in Theorem 1.9(1) of the same work. For (i) and (ii) we recall that Theorems 1.1, 1.3 and 1.7 of that same work yield the following properties:

(A) For any weak solution u to $\mathcal{L}u = 0$ on $\mathbb{R}^{1+n}_+$ with $\widetilde{N}_*(\nabla u) \in L^p(\mathbb{R}^n)$, the conormal derivative $\partial_{\nu_A} u|_{t=0}$ exists in $H^p(\mathbb{R}^n;\mathbb{C}^m)$, $u|_{t=0}$ exists in $\dot{H}^{1,p}(\mathbb{R}^n;\mathbb{C}^m)$, and u is constant if and only if $u|_{t=0} = 0$ and $\partial_{\nu_A} u|_{t=0} = 0$ in the respective spaces.

(B) For any weak solution w to $\mathcal{L}^* w = 0$ on $\mathbb{R}^{1+n}_+$ with $C_\alpha(t\nabla w) \in L^\infty(\mathbb{R}^n)$ and $w(t,\cdot)$ converging to 0 in $\mathcal{D}'(\mathbb{R}^n;\mathbb{C}^m)$ modulo constants as $t \to \infty$, $w|_{t=0}$ exists in $\dot{\Lambda}^\alpha(\mathbb{R}^n;\mathbb{C}^m)$ and $\partial_{\nu_{A^*}} w|_{t=0}$ exists in

$\dot{\Lambda}^{\alpha-1}(\mathbb{R}^n;\mathbb{C}^m)$, and w is constant if and only if $w|_{t=0}=0$ and $\partial_{\nu_{A^*}}w|_{t=0}=0$ in the respective spaces. Here, a distribution is in $\dot{\Lambda}^{\alpha-1}(\mathbb{R}^n;\mathbb{C}^m)$ if it is the divergence of an element in $\dot{\Lambda}^{\alpha}(\mathbb{R}^n;\mathbb{C}^{mn})$.

(C) With u and w as above, there is a Green's formula

$$\langle \partial_{\nu_A}u|_{t=0}, w|_{t=0}\rangle = \langle u|_{t=0}, \partial_{\nu_{A^*}}w|_{t=0}\rangle.$$

Here, the first pairing is the $\langle H^p(\mathbb{R}^n;\mathbb{C}^m), \dot{\Lambda}^{\alpha}(\mathbb{R}^n;\mathbb{C}^m)\rangle$ sesquilinear duality, while the second one is the $\langle \dot{H}^{1,p}(\mathbb{R}^n;\mathbb{C}^m), \dot{\Lambda}^{\alpha-1}(\mathbb{R}^n;\mathbb{C}^m)\rangle$ sesquilinear duality.

With this at hand the arguments are very simple. For (i), assume that w is given with $w|_{t=0}=0$. Let $\varphi\in C_0^\infty(\mathbb{R}^n;\mathbb{C}^m)$ and solve $(R)_p^{\mathcal{L}}$ with $u|_{t=0}=\varphi$ (modulo constants). It follows from (C) that $\langle\varphi, \partial_{\nu_{A^*}}w|_{t=0}\rangle=0$. This means that also $\partial_{\nu_{A^*}}w|_{t=0}=0$ as a distribution; hence w is constant using (B). For (ii), assume that u is given with $u|_{t=0}=0$. Let $\varphi\in C_0^\infty(\mathbb{R}^n;\mathbb{C}^m)$ and solve $(D)_\alpha^{\mathcal{L}^*}$ with $w|_{t=0}=\varphi$ (modulo constants). By (C) we have $\langle \partial_{\nu_A}u|_{t=0},\varphi\rangle=0$. This means $\partial_{\nu_A}u|_{t=0}=0$ as a distribution; hence u is constant using (A). □

7. Proof of the layer potential representation

Now, we come to justify (1-2) of the Introduction in proving Proposition 4.5. For this we set, for $0<\varepsilon<R<\infty$,

$$(\Gamma_{\varepsilon,R}f)_t = \int_{\varepsilon<|t-s|<R} \mathcal{S}_{t-s}^{\mathcal{L}} f_s\,ds, \qquad t\in\mathbb{R}.$$

We give the main properties of these approximants and then show how they converge to $\mathcal{L}^{-1}$. The proofs depend only on the case $p=2$ of Lemma 4.1.

Lemma 7.1. *Assume $f\in C_0^\infty(\mathbb{R}^{1+n};\mathbb{C}^m)$. If $n=1,2$, assume furthermore that $f=\operatorname{div}_x F$ coordinatewise for some $F\in C_0^\infty(\mathbb{R}^n;\mathbb{C}^{mn})$. Then for fixed $0<\varepsilon<R<\infty$, we have $\Gamma_{\varepsilon,R}f\in W^{1,2}(\mathbb{R}^{1+n};\mathbb{C}^m)$ and $t\mapsto(\Gamma_{\varepsilon,R}f)_t\in C^\infty(\mathbb{R};W^{1,2}(\mathbb{R}^n;\mathbb{C}^m))$ with all derivatives bounded.*

Proof. Sobolev embeddings yield $f\in W^{-1,2}(\mathbb{R}^{1+n};\mathbb{C}^m)$, except when $n=1,2$, where the condition $f=\operatorname{div}_x F$ is also required. This is the case under our assumption. Thus $u=\mathcal{L}^{-1}f$ is well-defined in $\dot{W}^{1,2}(\mathbb{R}^{1+n};\mathbb{C}^m)$. Also, $t\mapsto f_t$ is in $C_0^\infty(\mathbb{R};L^2(\mathbb{R}^n;\mathbb{C}^m)\cap\dot{W}^{-1,2}(\mathbb{R}^n;\mathbb{C}^m))$, using again that for $n=1,2$ we have $\int_{\mathbb{R}^n} f_t=0$ for all $t\in\mathbb{R}$ by assumption. Lemma 4.1 yields that $t\mapsto\mathcal{S}_t^{\mathcal{L}}$ is uniformly bounded and in particular belongs to $L^1_{\mathrm{loc}}(\mathbb{R};\mathcal{L}(X,Y))$, where

$$(X,Y)=(\dot{W}^{-1,2}(\mathbb{R}^n;\mathbb{C}^m), L^2(\mathbb{R}^n;\mathbb{C}^m)) \quad\text{or}\quad (X,Y)=(L^2(\mathbb{R}^n;\mathbb{C}^m), \dot{W}^{1,2}(\mathbb{R}^n;\mathbb{C}^m)).$$

By (operator-valued) convolution $t\mapsto(\Gamma_{\varepsilon,R}f)_t$ is in $L^\infty(\mathbb{R};W^{1,2}(\mathbb{R}^n;\mathbb{C}^m))\cap L^2(\mathbb{R};W^{1,2}(\mathbb{R}^n;\mathbb{C}^m))$. In particular, $\Gamma_{\varepsilon,R}f\in L^2(\mathbb{R}^{1+n};\mathbb{C}^m)$ and $\nabla_x\Gamma_{\varepsilon,R}f\in L^2(\mathbb{R}^{1+n};\mathbb{C}^{nm})$. We next compute $\partial_t\Gamma_{\varepsilon,R}f$ by differentiating the integral and integrating by parts. Setting $w_{t,\varepsilon} := \mathcal{S}^{\mathcal{L}}_{\varepsilon} f_{t-\varepsilon} - \mathcal{S}^{\mathcal{L}}_{-\varepsilon} f_{t+\varepsilon}$, we obtain

$$\begin{aligned}(\partial_t \Gamma_{\varepsilon,R} f)_t &= \int_{\varepsilon<|t-s|<R} \partial_t \mathcal{S}^{\mathcal{L}}_{t-s} f_s \, ds + w_{t,\varepsilon} - w_{t,R} \\ &= -\int_{\varepsilon<|t-s|<R} \partial_s \mathcal{S}^{\mathcal{L}}_{t-s} f_s \, ds + w_{t,\varepsilon} - w_{t,R} \\ &= \int_{\varepsilon<|t-s|<R} \mathcal{S}^{\mathcal{L}}_{t-s} \partial_s f_s \, ds = \Gamma_{\varepsilon,R}(\partial_s f)_t. \end{aligned} \tag{7-1}$$

Since $\partial_s f$ has the same properties as f, we can apply the above to $\Gamma_{\varepsilon,R}(\partial_s f)$ and conclude that $\partial_t \Gamma_{\varepsilon,R} f$ is in $L^\infty(\mathbb{R}; W^{1,2}(\mathbb{R}^n; \mathbb{C}^m)) \cap L^2(\mathbb{R}; W^{1,2}(\mathbb{R}^n; \mathbb{C}^m))$. Altogether, we have seen $\Gamma_{\varepsilon,R} f \in W^{1,2}(\mathbb{R}^{1+n}; \mathbb{C}^m)$ and that $t \mapsto (\Gamma_{\varepsilon,R} f)_t$ is bounded and Lipschitz continuous into $W^{1,2}(\mathbb{R}^n; \mathbb{C}^m)$. Iterating t-derivatives from $\partial_t(\Gamma_{\varepsilon,R} f) = \Gamma_{\varepsilon,R}(\partial_s f)$ and using Lemma 4.1 yields the claim. □

The following lemma will be useful in the proof of convergence.

Lemma 7.2. *Let X, Y be Banach spaces, $T : [0,\infty) \to \mathcal{L}(X, Y)$ uniformly bounded and strongly continuous and let $f \in C_0(\mathbb{R}; X)$. Then for any $s_0 \in [0,\infty]$ there is convergence in Y of*

$$\lim_{s\to s_0} T_s f_{t+s} = T_{s_0} f_{t+s_0},$$

uniformly in $t \in \mathbb{R}$. If in addition $f \in L^p(\mathbb{R}; X)$ for some $p \in [1,\infty)$, then convergence also holds in $L^p(\mathbb{R}, dt; Y)$.

Proof. We set $f(\pm\infty) := 0$, so that f becomes (uniformly) continuous on $\mathbb{R} \cup \{\pm\infty\}$, viewed as a compact topological space. Since T is uniformly bounded,

$$\|T_s f_{t+s} - T_{s_0} f_{t+s_0}\|_Y \lesssim \|f_{t+s} - f_{t+s_0}\|_Y + \|(T_s - T_{s_0}) f_{t+s_0}\|_Y. \tag{7-2}$$

In the limit $s \to s_0$ the first term on the right tends to 0 uniformly in $t \in \mathbb{R}$ since f is uniformly continuous. For the second term we first note that $K := \{f_{t+s_0} : t \in \mathbb{R} \cup \{\pm\infty\}\}$ is the continuous image of a compact set, and hence is compact in X. Thus, the strong convergence $T_s - T_{s_0} \to 0$ as $s \to s_0$ improves to uniform strong convergence on K. Therefore we also get convergence to 0 uniformly in $t \in \mathbb{R}$ for the second term above and the proof of the first claim is complete.

Now suppose in addition $f \in L^p(\mathbb{R}; X)$. Taking (7-2) to the p-th power and integrating in t gives

$$\begin{aligned}&\int_{\mathbb{R}} \|T_s f_{t+s} - T_{s_0} f_{t+s_0}\|_Y^p \, dt \\ &\qquad \lesssim \int_{(-R,R)} \|f_{t+s} - f_{t+s_0}\|_X^p \, dt + \int_{\mathbb{R}\setminus(-R,R)} \|f_{t+s} - f_{t+s_0}\|_X^p \, dt + \int_{\mathbb{R}} \|(T_s - T_{s_0}) f_{t+s_0}\|_Y^p \, dt,\end{aligned}$$

where $R > 0$ is a degree of freedom. Given $\varepsilon > 0$, we can first choose R large enough to guarantee that for all $s \in (s_0 - 1, s_0 + 1)$ the middle term is bounded by ε. Then the first integral vanishes in the limit $s \to s_0$ by uniform continuity of f and the third one vanishes by dominated convergence taking into account boundedness and strong continuity of T. □

In order to proceed, we eventually have to give the abstract definition of the single-layer $\mathcal{S}^{\mathcal{L}}_t$. All of the following material and further background can be found in [Auscher and Stahlhut 2016] and we report here only on the essentials required to follow the line of reasoning. We identify $\mathbb{C}^{(1+n)m}$ with $\mathbb{C}^m \times \mathbb{C}^{nm}$ and represent vectors as

$$F = \begin{bmatrix} F_\perp \\ F_\parallel \end{bmatrix}$$

accordingly.

Following the calculation on p. 68 of [Auscher and Axelsson 2011], there are a constant-coefficient first-order differential operator D acting on $\mathbb{C}^{(1+n)m}$-valued functions, a bounded multiplication operator $B = B(x)$ on $L^2(\mathbb{R}^n; \mathbb{C}^{(1+n)m})$ and purely algebraic way of rewriting the elliptic system $\mathcal{L}u = f$ as

$$\partial_t F + DB\, F = \begin{bmatrix} \mathcal{L}u \\ 0 \end{bmatrix}, \quad \text{where } F = \nabla_A u := \begin{bmatrix} (A\nabla u)_\perp \\ \nabla_x u \end{bmatrix}.$$

Here, $u \in \mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbb{R}^{1+n}; \mathbb{C}^m)$ and the autonomous first-order equation for its conormal gradient $F = \nabla_A u$ is understood in the sense of distributions. The multiplication operator B is designed in such a way that

$$B \begin{bmatrix} (AF)_\perp \\ F_\parallel \end{bmatrix} = \begin{bmatrix} F_\perp \\ (AF)_\parallel \end{bmatrix}, \quad F \in \mathbb{C}^{(1+n)m}. \tag{7-3}$$

The operator DB has an H^∞-functional calculus on $\mathcal{H} = \overline{\mathsf{R}(D)}$, the closure of the range of D in $L^2(\mathbb{R}^n; \mathbb{C}^{(1+n)m})$, allowing us to define a bounded operator $\varphi(DB)$ on $\mathcal{H}$ for any bounded and holomorphic function φ on a suitably large double sector around the real axis. Recall that $\mathcal{H}$ has been defined in connection with the ellipticity condition (2-2) and that it contains $L^2(\mathbb{R}^n; \mathbb{C}^m) \times \{0\}$. With $\chi^\pm$ the indicator functions of $\mathbb{C}^\pm$, functional calculus provides a means of defining $e^{\mp t\, DB}\chi^\pm(DB)$ for $t > 0$ as a bounded operator on $\mathcal{H}$. These are the bounded holomorphic C_0-semigroups generated by $\mp DB$ on $\chi^\pm(DB)\mathcal{H}$.

From [Auscher and Stahlhut 2016, pp. 100–101] with correction of an unfortunate typo in (82) and (84), we then have

$$\mathcal{S}_t^{\mathcal{L}} f := \begin{cases} -\big(D^{-1} e^{-t\, DB}\chi^+(DB)\big[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\big]\big)_\perp & \text{if } t > 0, \\ \big(D^{-1} e^{-t\, DB}\chi^-(DB)\big[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\big]\big)_\perp & \text{if } t < 0, \end{cases} \tag{7-4}$$

and

$$\nabla_A \mathcal{S}_t^{\mathcal{L}} f = \begin{cases} +e^{-t\, DB}\chi^+(DB)\big[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\big] & \text{if } t > 0, \\ -e^{-t\, DB}\chi^-(DB)\big[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\big] & \text{if } t < 0. \end{cases} \tag{7-5}$$

Here, D^{-1} is the closed extension of $D^{-1} : \mathsf{R}(D) \to \mathsf{D}(D)$ to $\overline{\mathsf{R}(D)} = \mathcal{H}$ with values in the abstract D-adapted Sobolev space $\dot{\mathcal{H}}^1_D$.

Proof of Proposition 4.5. We begin with (4-1). As $f \in \dot{W}^{-1,2}(\mathbb{R}^{1+n})$, we can define $u := \mathcal{L}^{-1} f$ in $\dot{W}^{1,2}(\mathbb{R}^{1+n})$. Defining the conormal gradient $F := \nabla_A u$, we have $F_t \in \mathcal{H}$ for every $t > 0$ by construction and $F \in L^2(\mathbb{R}^{1+n}; \mathbb{C}^{(1+n)m})$. We have seen above that in the distributional sense $\partial_t F + DB\, F = \big[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\big]$ but as both F and f are smooth functions of t valued in $L^2(\mathbb{R}^n; \mathbb{C}^{(1+n)m})$, see Lemma 3.1, this first-order equation also holds in the strong sense. Replacing $\big[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\big]$ by this differential equation in (7-5) and integrating by parts, we obtain, for all $t \in \mathbb{R},\ 0 < \varepsilon < R < \infty$,

$$\int_{\varepsilon<|t-s|<R} \nabla_A \mathcal{S}^{\mathcal{L}}_{t-s} f_s \, ds \\ = \mathrm{e}^{-\varepsilon DB}\chi^+(DB)F_{t-\varepsilon} + \mathrm{e}^{\varepsilon DB}\chi^-(DB)F_{t+\varepsilon} - \mathrm{e}^{-R DB}\chi^+(DB)F_{t-R} - \mathrm{e}^{R DB}\chi^+(DB)F_{t+R}.$$

Limits in ε, R for the terms on the right all fall under the scope of Lemma 7.2 applied with $X = Y = \mathcal{H} \subset L^2(\mathbb{R}^n; \mathbb{C}^{(1+n)m})$. Thus,

$$\lim_{\varepsilon\to 0,\, R\to\infty} \int_{\varepsilon<|t-s|<R} \nabla_A \mathcal{S}^{\mathcal{L}}_{t-s} f_s \, ds = \chi^+(DB)F_t + \chi^-(DB)F_t = F_t \tag{7-6}$$

in $L^2(\mathbb{R}^n; \mathbb{C}^{(1+n)m})$ uniformly in $t \in \mathbb{R}$ and in $L^2(\mathbb{R}; L^2(\mathbb{R}^n; \mathbb{C}^{(1+n)m})) \simeq L^2(\mathbb{R}^{1+n}; \mathbb{C}^{(1+n)m})$. Taking the $\|$-component, we obtain in particular convergence of $\nabla_x \Gamma_{\varepsilon,R} f$ to $F_\| = \nabla_x u = \nabla_x \mathcal{L}^{-1} f$ in $L^2(\mathbb{R}^{1+n}; \mathbb{C}^{nm})$. As for convergence of $\partial_t \Gamma_{\varepsilon,R} f$, we keep (7-3) in mind, multiply the previous equation by the bounded operator B on $L^2(\mathbb{R}^n; \mathbb{C}^{(1+n)m})$ and take the $\perp$-component, to give

$$\lim_{\varepsilon\to 0,\, R\to\infty} \int_{\varepsilon<|t-s|<R} (B\nabla_A \mathcal{S}^{\mathcal{L}}_{t-s} f_s)_\perp \, ds = \lim_{\varepsilon\to 0,\, R\to\infty} \int_{\varepsilon<|t-s|<R} \partial_t \mathcal{S}^{\mathcal{L}}_{t-s} f_s \, ds = (BF_t)_\perp = \partial_t u$$

in $L^2(\mathbb{R}^n; \mathbb{C}^m)$ uniformly in $t \in \mathbb{R}$ and in $L^2(\mathbb{R}^{1+n}; \mathbb{C}^m)$. Now, let us have a look at (7-1). We have seen that the integral in the first line enjoys the desired convergence. The convergence of $w_{t,\varepsilon} - w_{t,R}$ will once again be a direct application of Lemma 7.2. Indeed, $\mathcal{S}^{\mathcal{L}}_t$ viewed as a bounded operator $(L^2 \cap \dot{W}^{-1,2})(\mathbb{R}^n; \mathbb{C}^m) \to W^{1,2}(\mathbb{R}^n; \mathbb{C}^m))$ is uniformly bounded and strongly continuous with respect to $t \in \mathbb{R}$ from Lemma 4.1 and $t \mapsto f_t$ valued in $(L^2 \cap \dot{W}^{-1,2})(\mathbb{R}^n; \mathbb{C}^m)$ is continuous with compact support. Thus,

$$\lim_{\varepsilon\to 0,\, R\to\infty} w_{t,\varepsilon} - w_{t,R} = (\mathcal{S}^{\mathcal{L}}_0 f_t - \mathcal{S}^{\mathcal{L}}_0 f_t) - 0 = 0 \tag{7-7}$$

even in $W^{1,2}(\mathbb{R}^n; \mathbb{C}^m)$ uniformly in $t \in \mathbb{R}$ and in $L^2(\mathbb{R}; W^{1,2}(\mathbb{R}^n; \mathbb{C}^m))$. In particular, all four lines in (7-1) share convergence in $L^2(\mathbb{R}^{1+n}; \mathbb{C}^m)$ to the limit $\partial_t u = \partial_t \mathcal{L}^{-1} f t$. By looking just at $\partial_t \Gamma_{\varepsilon,R} f$, we complete the proof of (4-1).

However, the other lines of (7-1) — together with the equality $\mathcal{L}^{-1}(\partial_t f) = \partial_t \mathcal{L}^{-1} f$ in $L^2(\mathbb{R}^{1+n}; \mathbb{C}^m)$ noted in Lemma 3.1 — also give all the limits stated in (4-2) and those stated in (4-3) in the sense of $L^2(\mathbb{R}^n; \mathbb{C}^m)$-convergence, uniformly in $t \in \mathbb{R}$. The missing uniform $L^2(\mathbb{R}^n; \mathbb{C}^m)$-convergence for $\int_{\mathbb{R}} \nabla_x \mathcal{S}^{\mathcal{L}}_{t-s}(\partial_s f_s)\, ds$ follows from (7-6) with f replaced by $\partial_s f$ and that of $\int_{\mathbb{R}} \nabla_x \partial_t \mathcal{S}^{\mathcal{L}}_{t-s} f_s \, ds$ is a consequence of (7-1) since we have uniform $W^{1,2}(\mathbb{R}^n; \mathbb{C}^m)$-convergence of the error terms in (7-7).

Finally, the integrals in (4-3) are norm-convergent in $W^{1,2}(\mathbb{R}^n; \mathbb{C}^m)$ for fixed t since for all integers $k \geq 0$, Lemma 4.1 guarantees that $\partial_s^k S_s : (L^2 \cap \dot{W}^{-1,2})(\mathbb{R}^n; \mathbb{C}^m) \to W^{1,2}(\mathbb{R}^n; \mathbb{C}^m)$ is uniformly bounded in $s \in \mathbb{R}$ and by assumption $s \mapsto \partial_s^k f_s$ is continuous with compact support valued in $(L^2 \cap \dot{W}^{-1,2})(\mathbb{R}^n; \mathbb{C}^m)$. $\square$

Next, we shall explain how the properties stated in Lemma 4.1 follow from [Auscher and Stahlhut 2016].

Proof of Lemma 4.1. This is Theorem 12.6, items (1), (3) and (5), of [Auscher and Stahlhut 2016], the case $p = 2$ being mostly from [Rosén 2013], except for the global strong continuity and the limits at $\pm\infty$. Only the strong limits at $0^{\pm}$ are explained there. But continuity at other points is even easier with the arguments there, using the boundedness properties of the *DB*-semigroups in [Auscher and Stahlhut 2016, Corollary 8.3]. This being said, the limits at $\pm\infty$ come similarly from the fact that a holomorphic C_0-semigroup converges strongly to 0 at ∞ on the closure of the range of its generator. $\square$

We conclude with:

Proof of Proposition 4.7. For any $\varepsilon > 0$, we have that f_ε belongs to $\dot{W}^{-1,2}(\mathbb{R}^{n+1}; \mathbb{C}^m)$; hence $\mathcal{L}^{-1} f_\varepsilon$ exists in $\dot{W}^{1,2}(\mathbb{R}^{n+1}; \mathbb{C}^m)$ and Lemma 3.1 shows that in fact it belongs to $C_0(\mathbb{R}; \dot{W}^{1,2}(\mathbb{R}^n; \mathbb{C}^m))$. It remains to prove the convergence. By (4-1), we have

$$\nabla_x(\mathcal{L}^{-1} f_\varepsilon)(t,x) = \text{p.v.}\int_{\mathbb{R}} \nabla_x \mathcal{S}^{\mathcal{L}}_{t-s} f(x)\,\chi_\varepsilon(s)\,ds \quad \text{in } L^2(\mathbb{R}^{n+1}; \mathbb{C}^{mn}).$$

Boundedness and strong continuity of $\mathcal{S}^{\mathcal{L}}_t$ yield that the integral on the right converges in norm and the equality holds in $C_0(\mathbb{R}; L^2(\mathbb{R}^n; \mathbb{C}^{mn}))$ for fixed $\varepsilon > 0$. Now we fix $t \in \mathbb{R}$. Changing variables

$$\nabla_x(\mathcal{L}^{-1} f_\varepsilon)(t,x) = \int_{\mathbb{R}} \nabla_x \mathcal{S}^{\mathcal{L}}_{s} f(x)\,\chi_\varepsilon(t-s)\,ds$$

and using the continuity of $s \mapsto \nabla_x \mathcal{S}^{\mathcal{L}}_s f$ valued in $L^2(\mathbb{R}^n; \mathbb{C}^{nm})$, the integral converges to $\nabla_x \mathcal{S}^{\mathcal{L}}_t f$ in $L^2(\mathbb{R}^n; \mathbb{C}^{mn})$ as ε tends to 0. $\square$

8. Generic technical lemmas

For the convenience of the reader, here are some technical lemmas involving nontangential maximal functions used throughout the paper.

Lemma 8.1. *Let $0 < q, p < \infty$. Let $F : \mathbb{R}^{1+n}_+ \to \mathbb{R}$ be a measurable function with $\|\widetilde{N}_{*,q} F\|_p < \infty$. Then, for all $x \in \mathbb{R}^n$,*

$$\lim_{t\to\infty} ⨌_{W(t,x)} |F(s,y)|^q\,ds\,dy = 0.$$

Proof. Let G be the q-adapted nontangential function of F with parameters $c_0 = 2,\ c_1 = 2$. Thus $\|G\|_p \sim \|\widetilde{N}_{*,q} F\|_p < \infty$. Next, for all $z \in B(x,t)$,

$$\left(⨌_{W(t,x)} |F(s,y)|^q\,ds\,dy \right)^{p/q} \le 2^{np/q} G(z)^p,$$

and hence

$$\left(⨌_{W(t,x)} |F(s,y)|^q\,ds\,dy \right)^{p/q} \lesssim ⨍_{B(x,t)} G(z)^p\,dz \lesssim t^{-n} \|G\|_p^p.$$

The conclusion follows. $\square$

Lemma 8.2. *Let $1 < p < \infty$. Let $F, H : \mathbb{R}^{1+n}_+ \to \mathbb{R}$ be measurable functions with $\|\widetilde{N}_* F\|_p < \infty$ and $\|\widetilde{N}_* H\|_{p'} < \infty$. Then, for any fixed $0 < \varepsilon < R < \infty$,*

$$\iint_{(\varepsilon,R)\times\mathbb{R}^n} |FH|\,dx\,dt \lesssim \|\widetilde{N}_* F\|_p \|\widetilde{N}_* H\|_{p'}$$

and

$$\lim_{M\to\infty}\iint_{(\varepsilon,R)\times\{|x|>M\}} |FH|\,dx\,dt = 0.$$

Proof. By covering the interval (ε, R) with a finite number of intervals of the form $(c_0^{-1}a, c_0a)$, we may reduce to a single such interval. In that case, the averaging trick in the x-variable and Hölder's inequality show that

$$\int_{c_0^{-1}a}^{c_0a}\int_{\mathbb{R}^n} |FH|\,dx\,dt = \int_{\mathbb{R}^n} a(c_0-c_0^{-1})\biggl(\fint\!\!\fint_{W(a,y)} |FH|\,dx\,dt\biggr)\,dy \le a(c_0-c_0^{-1})\|\widetilde{N}_*F\|_p\|\widetilde{N}_*H\|_{p'} < \infty.$$

The limit follows by dominated convergence. □

Lemma 8.3. *Let $1 \le q < \infty$, $1 < p < \infty$ and let $H \in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$ be such that $\|\widetilde{N}_{*,q}(\nabla H)\|_p < \infty$. Then there exists a measurable function $h : \mathbb{R}^n \to \mathbb{R}$ such that, for a.e. $x \in \mathbb{R}^n$,*

$$\lim_{t\to 0}\fint\!\!\fint_{W(t,x)} |H(s,y) - h(x)|\,ds\,dy = 0,$$

as well as

$$\lim_{t\to 0}\fint_{t/2}^{2t} H(s,\cdot)\,ds = h$$

in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Moreover, $h \in \dot{W}^{1,p}(\mathbb{R}^n)$ with $\|\nabla_x h\|_p \lesssim \|\widetilde{N}_{*,q}(\nabla H)\|_p$ and

$$\left\|\widetilde{N}_{*,1}\Bigl(\frac{H-h}{t}\Bigr)\right\|_p \lesssim \|\widetilde{N}_{*,q}(\nabla H)\|_p.$$

Proof. It is enough to assume $q = 1$ throughout as $\widetilde{N}_{*,1}(\nabla H) \le \widetilde{N}_{*,q}(\nabla H)$. This is then essentially in [Kenig and Pipher 1993, pp. 461–462] up to minor modifications of the proof (working directly with averages) and is a simple consequence of Poincaré inequalities and change of parameters c_0, c_1. Details of this modification are written out for example in [Amenta and Auscher 2018, Section 6.6]. □

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EIGENVALUE BOUNDS FOR NON-SELF-ADJOINT SCHRÖDINGER OPERATORS WITH NONTRAPPING METRICS

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We study eigenvalues of non-self-adjoint Schrödinger operators on nontrapping asymptotically conic manifolds of dimension $n \geq 3$. Specifically, we are concerned with the following two types of estimates. The first one deals with Keller-type bounds on individual eigenvalues of the Schrödinger operator with a complex potential in terms of the L^p-norm of the potential, while the second one is a Lieb–Thirring-type bound controlling sums of powers of eigenvalues in terms of the L^p-norm of the potential. We extend the results of Frank (2011), Frank and Sabin (2017), and Frank and Simon (2017) on the Keller- and Lieb–Thirring-type bounds from the case of Euclidean spaces to that of nontrapping asymptotically conic manifolds. In particular, our results are valid for the operator $\Delta_g + V$ on $\mathbb{R}^n$ with g being a nontrapping compactly supported (or suitably short-range) perturbation of the Euclidean metric and $V \in L^p$ complex-valued.

1. Introduction and statement of results

The purpose of this paper is to establish bounds of Keller- and Lieb–Thirring-type for eigenvalues of non-self-adjoint Schrödinger operators on nontrapping asymptotically conic manifolds. Before stating our results, let us proceed to describe these two types of bounds in the more familiar Euclidean setting, motivating the significance of extending them to the case of asymptotically conic manifolds.

1A. ***Keller- and Lieb–Thirring-type bounds in the Euclidean case.*** Recently there have been numerous works devoted to the study of eigenvalues of the Schrödinger operator $\mathcal{P} = \Delta + V$ in $L^2(\mathbb{R}^n)$, with Δ being the nonnegative Laplace operator and V being a complex-valued potential. Of particular interest here is the problem of obtaining quantitative information concerning the localization and distribution of the eigenvalues of $\mathcal{P}$ under the sole assumption that $V \in L^p(\mathbb{R}^n)$ for some $1 \leq p < \infty$. Here we may note that the spectrum of $\mathcal{P}$ in $\mathbb{C} \setminus [0, \infty)$ consists then of isolated eigenvalues of finite algebraic multiplicity; see [Frank 2018, Proposition B.2].

The following two types of results are of particular interest for this problem. The first one deals with Keller-type bounds [1961] on the individual eigenvalues of $\mathcal{P}$ in terms of the L^p-norm of the potential. If V is real-valued, so that $\mathcal{P}$ admits a natural self-adjoint realization, then the eigenvalues of $\mathcal{P}$ in $\mathbb{C} \setminus [0, \infty)$

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are negative and by the variational principle and Sobolev's inequalities, for any eigenvalue $\lambda < 0$ of $\mathcal{P}$, we have the scale-invariant bounds

$$|\lambda|^{\gamma} \leq C_{\gamma,n} \int_{\mathbb{R}^n} |V(x)|^{\gamma+\frac{n}{2}}\, dx \tag{1-1}$$

for every $\gamma \geq \frac{1}{2}$ if $n = 1$ and every $\gamma > 0$ if $n \geq 2$. Here the constant $C_{\gamma,n} > 0$ depends on γ and n only; see [Frank and Simon 2017; Keller 1961; Lieb and Thirring 1976].

If the potential V is complex-valued, the problem is more involved due to the lack of variational techniques and the absence of a spectral resolution theorem. In dimension $n = 1$ the bound (1-1) with $\gamma = \frac{1}{2}$ was proved by Abramov, Aslanyan, and Davies [Abramov et al. 2001]. In dimensions $n \geq 2$, Frank [2011] established the bound (1-1) for all eigenvalues $\lambda \in \mathbb{C} \setminus [0, \infty)$ and for all $0 < \gamma \leq \frac{1}{2}$; see also [Frank and Simon 2017]. The work [Frank 2018] gives a replacement of the bound (1-1) for all $\gamma > \frac{1}{2}$. We refer to [Cuenin 2017; Cuenin and Kenig 2017; Enblom 2016; Laptev and Safronov 2009; Mizutani 2016] for some other recent works on bounds on the individual eigenvalues for non-self-adjoint operators of Schrödinger type.

The second type of result is concerned with bounds on sums of powers of absolute values of eigenvalues of $\mathcal{P}$, generalizing the classical Lieb–Thirring bounds [1976] to the non-self-adjoint case. If V is real-valued then the Lieb–Thirring inequality has the form

$$\sum |\lambda|^{\gamma} \leq C_{\gamma,n} \int_{\mathbb{R}^n} V_-(x)^{\gamma+\frac{n}{2}}\, dx, \tag{1-2}$$

where $V_- = \max(-V, 0)$, $\gamma \geq \frac{1}{2}$ if $n = 1$, $\gamma > 0$ if $n = 2$, and $\gamma \geq 0$ if $n \geq 3$. The summation in the left-hand side in (1-2) extends over all negative eigenvalues of $\mathcal{P}$, counted with their multiplicities. The situation in the non-self-adjoint case is less clear. In particular, Bögli [2017] established that for any $p > n$, there exists a nonreal potential $V \in L^p(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$ such that the Schrödinger operator $\mathcal{P}$ has infinitely many nonreal eigenvalues accumulating at every point of the essential spectrum $[0, \infty)$, thus showing that inequalities like (1-2) cannot hold in the non-self-adjoint case for $p > n$. A possible modification of the Lieb–Thirring inequality (1-2) to the non-self-adjoint case was suggested in [Demuth et al. 2013b]:

$$\sum \frac{d(\lambda)^{\gamma+\frac{n}{2}}}{|\lambda|^{\frac{n}{2}}} \leq C_{\gamma,n} \int_{\mathbb{R}^n} |V(x)|^{\gamma+\frac{n}{2}}\, dx, \tag{1-3}$$

where

$$d(\lambda) = \operatorname{dist}(\lambda, [0, \infty)). \tag{1-4}$$

We refer to [Demuth et al. 2009; 2013a; Frank et al. 2006; Frank and Sabin 2017; Sambou 2014] for some of the important contributions to generalizations of the Lieb–Thirring inequality (1-2) to the setting of complex potentials.

A crucial idea of Frank [2011] in establishing bounds (1-1) on the individual eigenvalues of the Schrödinger operator $\mathcal{P}$ with a complex-valued potential was to make use of the uniform L^p resolvent estimates for Δ of Kenig, Ruiz, and Sogge [Kenig et al. 1987]. Recently, this approach was extended to the case of non-self-adjoint Schrödinger operators with inverse-square potentials in [Mizutani 2019], to the case of magnetic Schrödinger and Pauli operators with complex electromagnetic potentials in [Cuenin and Kenig 2017], and to the case of the Dirac and fractional Schrödinger operators with complex potentials in [Cuenin 2017].

Developing the idea of [Frank 2011] further, Frank and Sabin [2017] obtained some very interesting uniform weighted bounds for the resolvent of Δ in suitable Schatten classes, and applied these bounds to derive uniform estimates on the sums of eigenvalues of non-self-adjoint Schrödinger operators, thus obtaining some results towards proving the conjectured Lieb–Thirring inequality (1-3) in the case of complex potentials. Recently, this approach was extended in [Cuenin 2017] to the case of the Dirac and fractional Schrödinger operators with complex potentials.

1B. ***Asymptotically conic manifolds.*** Notice that in all the works described above the principal part of the operators considered has constant coefficients. It is nevertheless of significant interest to extend both types of results to the case of complex potential perturbations of the Laplace–Beltrami operator Δ_g considered on $\mathbb{R}^n$ or more generally, on a class of complete noncompact Riemannian manifolds.

The class we consider here is the class of *asymptotically conic manifolds*, whose Riemannian metric outside a compact set is asymptotic to the end of a metric cone. Metric cones are Riemannian manifolds of the form $N \times (0,\infty)_r$ with metric $dr^2 + r^2 G$ for some metric G on N. They were studied in [Cheeger 1983; Cheeger and Taylor 1982] but have a long history going back to [Sommerfeld 1896]. As defined by Melrose [1994] (who used the term "scattering metric"), (M, g) is *asymptotically conic* if M is the interior of a smooth compact manifold with boundary $\overline{M}$ and g is a smooth metric on M satisfying the following property: there exists a smooth boundary-defining function[1] x on $\overline{M}$ such that (M, g) is isometric outside a compact set to a collar $(0,\varepsilon)_x \times \partial\overline{M}$ equipped with the metric of the form

$$\frac{dx^2}{x^4} + \frac{h(x)}{x^2} = \frac{dx^2}{x^4} + \frac{\sum_{j,k} h_{jk}(x, y)\, dy_j\, dy_k}{x^2} \tag{1-5}$$

for some smooth one-parameter family of metrics h on the boundary $\partial\overline{M}$. If $y = (y_1, \dots, y_{n-1})$ stands for local coordinates on $\partial\overline{M}$ and (x, y) are the corresponding local coordinates on M near $\partial\overline{M}$, the function $r = \frac{1}{x}$ near $x = 0$ can be thought of as a "radial" variable near infinity and $y = (y_1, \dots, y_{n-1})$ can be regarded as $n-1$ "angular" variables. Rewriting (1-5) in the (r, y)-coordinates, we have

$$g = dr^2 + r^2 h(r^{-1}) = dr^2 + r^2 \sum h_{jk}(r^{-1}, y)\, dy^j\, dy^k, \tag{1-6}$$

and we observe that the metric g is asymptotic to an exact conic metric $dr^2 + r^2 h(0)$ on $(r_0, \infty)_r \times \partial\overline{M}$ as $r \to \infty$. The most important example of an asymptotically conic manifold is Euclidean space $M = \mathbb{R}^n$ equipped with a short-range perturbation of the Euclidean metric (δ_{ij}), which is of the form

$$g_{ij} = \delta_{ij} + |z|^{-2} k_{ij}\left(\frac{z}{|z|}, \frac{1}{|z|}\right), \quad |z| \to \infty, \tag{1-7}$$

where k_{ij} are smooth on $\mathbb{S}^{n-1} \times [0, 1)$; see [Melrose and Zworski 1996].

[1] A boundary-defining function is a nonnegative smooth function x such that $\partial\overline{M} = x^{-1}(\{0\})$ and $dx|_{\partial M}$ does not vanish on $\partial\overline{M}$.

Let $z = (z_1, \dots, z_n)$ be local coordinates away from $\partial\overline{M}$. We say that M is *nontrapping* if every geodesic $z(s)$ in M reaches $\partial\overline{M}$ as $s \to \pm\infty$. This places restrictions on the compactification $\overline{M}$. For example, a compact perturbation of the Euclidean metric is nontrapping provided that it is sufficiently small in C^2; see [Hassell et al. 2006]. However, a nontrapping asymptotically conic metric g may be far from asymptotically Euclidean. Indeed, there is such a nontrapping metric g on $\mathbb{R}^n$ for every limiting metric $h(0)$ on the sphere $\mathbb{S}^{n-1}$, identified with $\partial\overline{M}$ in this case.

In terms of the Weyl calculus, the symbol of the Laplacian for an asymptotically conic metric on $\mathbb{R}^n$ is in the calculus corresponding to the metric on $T^*\mathbb{R}^n$

$$\frac{dz^2}{\langle z\rangle^2} + \frac{d\xi^2}{\langle \xi\rangle^2}.$$

This class of symbols was studied by Parenti [1972], Cordes [1976], Schrohe [1987], Hörmander [1985, Equation (19.3.11) and Theorem 19.3.1′] and others. Melrose [1994] adopted a different point of view, working from the outset on the compactification $\overline{M}$ (which can be any manifold with boundary) and introducing the *scattering calculus* as the natural class of pseudodifferential operators associated with the *scattering Lie algebra* of vector fields on $\overline{M}$. He seems to have been the first to exploit the fact that in this calculus one has propagation of singularities at spatial infinity *at all finite frequencies*. Using the scattering calculus, the second author in collaboration with Vasy, Wunsch, the first author, and Sikora, worked out detailed properties of the spectral measure; see [Guillarmou et al. 2013a; Hassell and Vasy 2001; Hassell and Wunsch 2008].

Let us remark on why we elect to work with the class of nontrapping asymptotically conic manifolds. On the one hand, it is a sufficiently *general* class which includes compactly supported or suitable short-range perturbations of Euclidean space as well as geometrically interesting examples such as metrics with strictly negative curvature, which are not present in the class of asymptotically Euclidean manifolds. On the other hand, it is sufficiently *restricted* to allow us to obtain detailed results on the resolvent and spectral measure, analogous in some sense to that for flat Euclidean space.

1C. ***Main results.*** Throughout the paper, we let (M, g) be an asymptotically conic nontrapping manifold of dimension $n \geq 3$. Since g is complete, the Laplacian Δ_g associated with the metric g is nonnegative self-adjoint on $L^2(M)$ with domain $H^2(M)$. The spectrum of Δ_g is purely absolutely continuous and is given by $\mathrm{Spec}(\Delta_g) = [0, \infty)$: the absence of singular continuous spectrum follows for example from [Froese and Hislop 1989] using a Mourre estimate, and the absence of embedded L^2-eigenvalues follows from adapting [Hörmander 1985, Theorem 17.2.8] as in [Melrose 1994, Section 10].

Our starting point is the following uniform L^p resolvent estimate of Kenig–Ruiz–Sogge-type for the Laplace operator Δ_g on an asymptotically conic nontrapping manifold, established in [Guillarmou and Hassell 2014].

Theorem 1. *Let (M, g) be an asymptotically conic nontrapping manifold of dimension $n \geq 3$. Then for all $p \in \left[\frac{2n}{n+2}, \frac{2(n+1)}{n+3}\right]$, there is a constant $C > 0$ such that for all $z \in \mathbb{C}$ and for all $f \in L^p(M)$, we have*

$$\|(\Delta_g - z)^{-1} f\|_{L^{p'}(M)} \leq C |z|^{n(\frac{1}{p}-\frac{1}{2})-1} \|f\|_{L^p(M)}. \tag{1-8}$$

Here $\frac{1}{p} + \frac{1}{p'} = 1$.

As explained in [Guillarmou and Hassell 2014], when $z \in (0, +\infty)$, the operator in (1-8) may be taken to be either the outgoing or incoming resolvent $(\Delta_g - (z \pm i0))^{-1}$, defined by

$$(\Delta_g - (z \pm i0))^{-1} = \lim_{\delta \to 0^+} (\Delta_g - (z \pm i\delta))^{-1}$$

as a map $x^{1/2+\varepsilon} L^2(M) \to x^{-1/2-\varepsilon} L^2(M)$ for all $\varepsilon > 0$, where x is the boundary-defining function, thanks to the limiting absorption principle; see [Hassell and Vasy 2001; Melrose 1994] for details.

The main technical contribution of the present paper is the following weighted uniform Schatten class estimate for the resolvent of Δ_g, generalizing [Frank and Sabin 2017, Theorem 12], obtained in the Euclidean setting. This result is the key ingredient which allows us to extend the Lieb–Thirring-type bounds of [Frank and Sabin 2017; Frank 2018] to our setting. Below, $\mathcal{C}_q(L^2(M))$ denotes the Schatten space of order q (see Section 2A for definition).

Theorem 2. *Let (M, g) be an asymptotically conic nontrapping manifold of dimension $n \geq 3$. Let $p \in \left[\frac{n}{2}, \frac{n+1}{2}\right]$. Then there exists $C > 0$ such that for all $z \in \mathbb{C} \setminus \{0\}$ and all $W_1, W_2 \in L^{2p}(M)$, we have $W_1(\Delta_g - z)^{-1} W_2 \in \mathcal{C}_q(L^2(M))$, $q = \frac{p(n-1)}{n-p} \in [n-1, n+1]$, and*

$$\|W_1(\Delta_g - z)^{-1} W_2\|_{\mathcal{C}_q(L^2(M))} \leq C|z|^{-1+\frac{n}{2p}} \|W_1\|_{L^{2p}(M)} \|W_2\|_{L^{2p}(M)}. \tag{1-9}$$

Remark 1.1. When $z \in (0, +\infty)$, the operator in (1-9) may be taken to be either the outgoing or incoming resolvent $(\Delta_g - (z \pm i0))^{-1}$.

In what follows we shall write $E_{\sqrt{\Delta_g}}(\lambda) = 1_{(-\infty,\lambda)}(\sqrt{\Delta_g})$, $\lambda > 0$, for the spectral projection of $\sqrt{\Delta_g}$, and remark that the spectral measures $d(E_{\sqrt{\Delta_g}}(\lambda)u, u)_{L^2(M)}$ are absolutely continuous with respect to the Lebesgue measure for any $u \in L^2(M)$. Let us write

$$dE_{\sqrt{\Delta_g}}(\lambda) := \frac{d}{d\lambda} E_{\sqrt{\Delta_g}}(\lambda).$$

The proof of Theorem 2 is based on the following weighted Schatten norm estimates on the spectral measure $dE_{\sqrt{\Delta_g}}(\lambda)$ of $\sqrt{\Delta_g}$, which extend the corresponding estimates of [Frank and Sabin 2017, Theorem 2], obtained in the Euclidean setting. We believe that these estimates may be of some independent interest.

Theorem 3. *Let (M, g) be an asymptotically conic nontrapping manifold of dimension $n \geq 3$. Let $p \in \left[1, \frac{n+1}{2}\right]$. Then there exists $C > 0$ such that for all $\lambda > 0$ and all $W_1, W_2 \in L^{2p}(M)$, we have $W_1 dE_{\sqrt{\Delta_g}}(\lambda) W_2 \in \mathcal{C}_q(L^2(M))$, $q = \frac{p(n-1)}{n-p} \in [1, n+1]$, and*

$$\|W_1 dE_{\sqrt{\Delta_g}}(\lambda) W_2\|_{\mathcal{C}_q(L^2(M))} \leq C\lambda^{-1+\frac{n}{p}} \|W_1\|_{L^{2p}(M)} \|W_2\|_{L^{2p}(M)}. \tag{1-10}$$

Remark 1.2. If the nontrapping assumption is dropped, the estimates in Theorem 3, and therefore also Theorem 2, may fail. Instead, the estimates will hold for all $\lambda \leq \lambda_0$ for a constant C which depends on λ_0. A "metric bottle" example illustrating this, for which the best $C(\lambda_0)$ grows exponentially in λ_0, is given in [Guillarmou et al. 2013b, Remark 8.8].

Let us now consider the Schrödinger operator $\Delta_g + V$ with a complex-valued potential $V \in L^p(M)$, $\frac{n}{2} \leq p < \infty$. As explained in Section 6, this operator has a natural m-sectorial realization on $L^2(M)$, and the spectrum of $\Delta_g + V$ in $\mathbb{C} \setminus [0, \infty)$ consists of isolated eigenvalues of finite algebraic multiplicity.

As an application of Theorem 1, we have the following generalization of the results of [Frank 2011; 2018; Frank and Simon 2017] concerning Keller-type bounds on the individual eigenvalues of non-self-adjoint Schrödinger operators in the Euclidean setting to that of an asymptotically conic nontrapping manifold; see also [Fanelli et al. 2018].

Theorem 4. *Let (M, g) be an asymptotically conic nontrapping manifold of dimension $n \geq 3$:*

(i) *Let $V \in L^{\gamma+n/2}(M)$ for some $0 < \gamma \leq \frac{1}{2}$. Then any eigenvalue $\lambda \in \mathbb{C}$ of the operator $\Delta_g + V$ satisfies*

$$|\lambda|^{\gamma} \leq C_{\gamma,n} \|V\|^{\gamma+\frac{n}{2}}_{L^{\gamma+n/2}(M)}, \tag{1-11}$$

where the constant $C_{\gamma,n} > 0$ depends on γ and n only.

(ii) *If $V \in L^{n/2}(M)$ is such that $\|V\|_{L^{n/2}(M)}$ is sufficiently small, then the operator $\Delta_g + V$ has no eigenvalues.*

(iii) *Let $V \in L^{\gamma+n/2}(M)$ for some $\gamma > \frac{1}{2}$. Then any eigenvalue $\lambda \in \mathbb{C}$ of the operator $\Delta_g + V$ satisfies*

$$d(\lambda)^{\gamma-\frac{1}{2}} |\lambda|^{\frac{1}{2}} \leq C_{\gamma,n} \|V\|^{\gamma+\frac{n}{2}}_{L^{\gamma+n/2}(M)}, \tag{1-12}$$

where $d(\lambda)$ is given by (1-4) *and the constant $C_{\gamma,n} > 0$ depends on γ and n only.*

Remark 1.3. Parts (i) and (ii) of Theorem 4 have been established in [Guillarmou and Hassell 2014, Proposition 7.2] without specifying the radius of the disk containing the eigenvalues of $\Delta_g + V$ in part (i).

As a consequence of Theorem 2, we obtain the following analogue of [Frank and Sabin 2017, Theorem 16], concerning Lieb–Thirring-type inequalities for the sums of eigenvalues of $\Delta_g + V$ in the case of a short-range potential $V \in L^p(M)$, $p = \frac{n}{2} + \gamma$, where $0 \leq \gamma \leq \frac{1}{2}$.

Theorem 5. *Let (M, g) be an asymptotically conic nontrapping manifold of dimension $n \geq 3$, and let $V \in L^p(M)$ with p such that*

$$\tfrac{n}{2} \leq p \leq \tfrac{n+1}{2}.$$

Let us denote by λ_j the eigenvalues of $\Delta_g + V$ in $\mathbb{C} \setminus [0, \infty)$, repeated according to their algebraic multiplicities. The following estimates then hold:

(i) *If $p = \frac{n}{2}$, we have*

$$\sum_j \frac{\operatorname{Im} \sqrt{\lambda_j}}{1 + |\lambda_j|} < \infty, \tag{1-13}$$

where the branch of the square root is chosen to have positive imaginary part.

(ii) *If $\frac{n}{2} < p \leq \frac{n+1}{2}$, then*

$$\sum_j \frac{d(\lambda_j)}{|\lambda_j|^{\frac{1-\varepsilon}{2}}} \leq C_{\varepsilon,p,n} \|V\|^{\frac{(1+\varepsilon)p}{2p-n}}_{L^p(M)} \tag{1-14}$$

for all ε satisfying

$$\begin{cases} \varepsilon \geq 0, & \frac{n}{2} < p < \frac{n^2}{2n-1}, \\ \varepsilon > \frac{p(2n-1)-n^2}{n-p} \geq 0, & \frac{n^2}{2n-1} \leq p \leq \frac{n+1}{2}. \end{cases}$$

Remark 1.4. If $\frac{n}{2} < p \le \frac{n+1}{2}$, then by Theorem 4 we know that the eigenvalues of $\Delta_g + V$ are confined to an open disk centered at the origin. Furthermore, it follows from (1-14) that if a sequence of eigenvalues $\lambda_{j_k} \in \mathbb{C} \setminus [0,\infty)$ converges to $E > 0$ then $\operatorname{Im}\lambda_{j_k} \in \ell^1$. In the case $p = \frac{n}{2}$ the bound (1-13) controls a possible accumulation rate of eigenvalues in $\mathbb{C} \setminus [0,\infty)$ at infinity, and it implies in particular, with the help of

$$\operatorname{Im}(\sqrt{\lambda}) = \frac{|\operatorname{Im}\lambda|}{\sqrt{2(|\lambda| + \operatorname{Re}\lambda)}},$$

that if a sequence of eigenvalues $\lambda_{j_k} \in \mathbb{C} \setminus [0,\infty)$ converges to $E > 0$ then $\operatorname{Im}\lambda_{j_k} \in \ell^1$.

As another application of the Schatten class estimates for the resolvent of Δ_g given in Theorem 2, we get the following generalization of [Frank 2018, Theorem 1.2], concerning Lieb–Thirring-type inequalities for the sums of eigenvalues $\Delta_g + V$ in the case of a long-range potential $V \in L^p(M)$, $p = \gamma + \frac{n}{2}$, $\gamma > \frac{1}{2}$.

Theorem 6. *Let (M, g) be an asymptotically conic nontrapping manifold of dimension $n \ge 3$, and let $V \in L^p(M)$ with $p = \gamma + \frac{n}{2}$, $\gamma > \frac{1}{2}$. Then the eigenvalues $\lambda_j \in \mathbb{C} \setminus [0,\infty)$ of $\Delta_g + V$, repeated according to their algebraic multiplicities, satisfy the following bounds: for any $\varepsilon > 0$,*

$$\Biggl(\sum_{|\lambda_j|^{\gamma} \le C_{\gamma,n} \int_M |V|^{\gamma+n/2}\,dx} d(\lambda_j)^{2\gamma+\varepsilon}\Biggr)^{\frac{\gamma}{2\gamma+\varepsilon}} \le L_{\varepsilon,\gamma,n} \int_M |V|^{\gamma+\frac{n}{2}}\,dx,$$

and for any $\varepsilon > 0$, $0 < \varepsilon' < \frac{\gamma}{\gamma+n/2}$, and $\mu \ge 1$,

$$\Biggl(\sum_{|\lambda_j|^{\gamma} \ge \mu C_{\gamma,n} \int_M |V|^{\gamma+n/2}\,dx} \frac{d(\lambda_j)^{2\gamma+\varepsilon}}{|\lambda_j|^{2\gamma - \frac{\gamma}{\gamma+n/2} + \varepsilon + \varepsilon'}}\Biggr)^{\frac{\gamma(\gamma+n/2)}{\gamma - \varepsilon'(\gamma+n/2)}} \le L_{\varepsilon,\varepsilon',\gamma,n}\, \mu^{-\frac{\varepsilon'(\gamma+n/2)}{\gamma-\varepsilon'(\gamma+n/2)}} \int_M |V|^{\gamma+\frac{n}{2}}\,dx.$$

Remark 1.5. As observed in [Frank 2018], Theorem 6 has the following consequence: let $\gamma > \frac{1}{2}$ and $V \in L^{\gamma+n/2}(M)$. If $(\lambda_j)_{j=1}^{\infty}$ is a sequence of eigenvalues of $\Delta_g + V$ with $\lambda_j \to \lambda_0 \in [0,\infty)$ then $\operatorname{Im}\lambda_j \in l^p$ for any $p > 2\gamma$.

Remark 1.6. Let us emphasize once more that all our results, Theorems 2–6, are valid for the metric Schrödinger operator in the Euclidean space $\mathbb{R}^n$, with a metric that is a nontrapping short-range perturbation of the Euclidean one, in the sense of (1-7). In particular, the results hold true for the metric Schrödinger operator in the Euclidean space $\mathbb{R}^n$, with a metric that is a sufficiently small compactly supported perturbation of the Euclidean one.

1D. ***Outline of the paper.*** The plan of the paper is as follows. In Section 2 we present our strategy for proving Theorem 2, which is the main result of the paper. Section 3 is devoted to the proof of Theorem 3, giving Schatten norm estimates on the spectral measure. In Section 4 we derive some Schatten norm estimates on the resolvent of the Laplacian, as a direct consequence of the Schatten norm estimates on the spectral measure, and give their analogues at the endpoint case $p = \frac{n}{2}$, needed in the proof of Theorem 2. The principal step in the proof of Theorem 2, corresponding to the estimates on the spectrum, is carried out in Section 5. Section 6 contains the proof of Theorem 4, which follows the arguments of [Frank 2018; Frank and Simon 2017] closely, relying on Theorem 1, with some small adjustments due to the fact that we are no longer in the Euclidean setting. Finally, we observe in Section 7 that Theorems 5 and 6 are direct consequences of Theorem 2 combined with the arguments of [Frank and Sabin 2017, Theorem 16] and [Frank 2018, Theorem 1.2]. Appendix A contains the proof of Lemma 5.5, needed in the main text. Appendix B is concerned with the analysis of the microlocal structure of the spectrally localized outgoing and incoming resolvent, used in the proof of Theorem 2.

2. Strategy of the proof of Theorem 2

2A. ***Schatten norm estimates.*** We first recall the definition of the Schatten spaces of operators on $L^2(M)$; see [Simon 1979]. Let A be a compact operator on $L^2(M)$, and let $\mu_j(A)$ be the singular values of A, given by $\mu_j(A) = \lambda_j((A^*A)^{1/2})$. Here $\lambda_j(B)$ denotes the eigenvalues of a positive self-adjoint compact operator B, arranged in decreasing order. The Schatten norm of A of order $1 \le q < \infty$ is defined as

$$\|A\|^q_{\mathcal{C}_q(L^2(M))} = \sum_{j=1}^{\infty} \mu_j(A)^q = \operatorname{tr}((A^*A)^{\frac{q}{2}}).$$

The basic mechanism for proving the Schatten norm estimates of Theorems 2 and 3 comes from the fact that the Schatten spaces are complex interpolation spaces, see [Simon 1979, Theorem 2.9; 2015, p. 154], and from [Frank and Sabin 2017, Proposition 1].

Proposition 2.1. *Let T_s be an analytic family of operators, defined on the strip $\{s \in \mathbb{C} \mid -\lambda_0 \le \operatorname{Re} s \le 0\}$ for some $\lambda_0 > 1$, acting on functions on M. Assume that we have operator norm bounds*

$$\|T_{ir}\|_{L^2(M)\to L^2(M)} \le M_0 e^{a|r|}, \quad \|T_{-\lambda_0+ir}\|_{L^1(M)\to L^\infty(M)} \le M_1 e^{a|r|} \quad \textit{for all } r \in \mathbb{R},$$

for some $a \ge 0$ and $M_0, M_1 > 0$. Then for any $W_1, W_2 \in L^{2\lambda_0}(M)$, the operator $W_1 T_{-1} W_2$ belongs to the Schatten class $\mathcal{C}_{2\lambda_0}(L^2(M))$ and we have the estimate

$$\|W_1 T_{-1} W_2\|_{\mathcal{C}_{2\lambda_0}} \le M_0^{1-\frac{1}{\lambda_0}} M_1^{\frac{1}{\lambda_0}} \|W_1\|_{L^{2\lambda_0}(M)} \|W_2\|_{L^{2\lambda_0}(M)}.$$

Let us recall briefly the proof of Proposition 2.1. The result is established by considering the analytic family of operators $S_s = |W_1|^{-1-s} W_1 T_s W_2 |W_2|^{-1-s}$. This family has the property that $S_{-1} = W_1 T_{-1} W_2$ and it satisfies the following estimates on the boundary of the strip. For $s = ir$, r real, we have

$$\|S_{ir}\|_{L^2(M)\to L^2(M)} \le \|T_{ir}\|_{L^2(M)\to L^2(M)} \le M_0 e^{a|r|},$$

and for $s = -\lambda_0 + ir$, we note that T_s has its Schwartz kernel bounded pointwise by $M_1 e^{a|r|}$ (due to the $L^1 \to L^\infty$ bound on T_s) and $|W_1|^{-s}$, $|W_2|^{-s}$ are L^2 functions; hence S_s is a Hilbert–Schmidt operator with the Hilbert–Schmidt norm bounded by $M_1 e^{a|r|} \|W_1\|^{\lambda_0}_{L^{2\lambda_0}(M)} \|W_2\|^{\lambda_0}_{L^{2\lambda_0}(M)}$. Interpolating between the operator norm and the Hilbert–Schmidt norm gives us a bound on the Schatten norms, in particular at $s = -1$, where we obtain the Schatten norm at exponent $2\lambda_0$.

2B. ***Strategy.*** The principal idea of the proof of the Euclidean analogue of Theorem 2, which is due to Frank and Sabin [2017, Theorem 12], is to establish the following pointwise bound for the Schwartz kernel of the powers of the resolvent $(\Delta - z)^{-\alpha}$:

$$|(\Delta - z)^{-\alpha}(x, y)| \le C e^{C(\mathrm{Im}(\alpha))^2} |z|^{\frac{n-1}{4} - \frac{\mathrm{Re}(\alpha)}{2}} |x - y|^{\mathrm{Re}(\alpha) - \frac{n+1}{2}}, \quad x, y \in \mathbb{R}^n. \tag{2-1}$$

Here $z \in \mathbb{C} \setminus [0, \infty)$, $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) \in [\frac{n-1}{2}, \frac{n+1}{2}]$. The desired Schatten bound (1-9) in the Euclidean case is therefore a consequence of (2-1) combined with the Hölder and Hardy–Littlewood–Sobolev inequalities as well as an interpolation argument.

Unfortunately, the natural analogue of the pointwise bound (2-1) does not hold in general, for z close to the spectrum of Δ_g, for asymptotically conic manifolds, essentially because there can be conjugate points for the geodesic flow, and to prove the bound (1-9) we have to proceed differently.

Our strategy of the proof of Theorem 2 is to establish the Schatten norm estimate (1-9) for $W_1(\Delta_g - z)^{-1} W_2$ for z on the negative real axis, and for z just above and below the spectrum, that is, for $W_1(\Delta_g - (z \pm i0))^{-1} W_2$ for $z > 0$. We then use the Phragmén–Lindelöf theorem to obtain the result on the whole of the complex plane, excluding the origin.

Let us give the proof of Theorem 2, assuming that it has been established for $z < 0$ and for $z \pm i0$, $z > 0$. Let $W_1, W_2 \in L^{2p}(M)$ with $p \in \left[\frac{n}{2}, \frac{n+1}{2}\right]$, and let us consider the following bilinear form for $z \in \mathbb{C} \setminus [0, \infty)$:

$$B_z(W_1, W_2) := W_1(\Delta_g - z)^{-1} W_2. \tag{2-2}$$

When $z \in (0, \infty)$, we extend the definition of B_z by taking the outgoing resolvent $(\Delta_g - (z + i0))^{-1}$ in (2-2). Thus, we know that for $z \in \mathbb{R} \setminus \{0\}$, B_z is a bounded bilinear form

$$B_z : L^{2p}(M) \times L^{2p}(M) \to \mathcal{C}_q(L^2(M)), \quad p \in \left[\tfrac{n}{2}, \tfrac{n+1}{2}\right], \quad q = \tfrac{p(n-1)}{n-p},$$

such that

$$\|B_z(W_1, W_2)\|_{\mathcal{C}_q} \le C |z|^{-1 + \frac{n}{2p}} \|W_1\|_{L^{2p}(M)} \|W_2\|_{L^{2p}(M)}. \tag{2-3}$$

We now complete the proof of Theorem 2 by a Phragmén–Lindelöf argument. In doing so, let $W_1, W_2 \in C_0^\infty(M)$. We claim that the function $H(z) := B_z(W_1, W_2)$ is holomorphic in $\mathrm{Im}\, z > 0$ with values in $\mathcal{C}_q(L^2(M))$ such that

$$\|H(z)\|_{\mathcal{C}_q} \le C(|z|^{-\frac{1}{2}} + |z|^{\frac{1}{2}}).$$

Indeed, for $\mathrm{Im}\, z > 0$, the operator $W_1(\Delta_g - z)^{-1} W_2 : L^2(M) \to H^2(M) \cap \mathcal{E}'(K)$ is bounded, where K is a compact set containing the support of W_1. Furthermore, it depends holomorphically on z with $\mathrm{Im}\, z > 0$ and satisfies the bound

$$\|W_1(\Delta_g - z)^{-1} W_2\|_{\mathcal{L}(L^2(M), H^2(M))} \le C(|z|^{-\frac{1}{2}} + |z|^{\frac{1}{2}}), \quad \mathrm{Im}\, z \ge 0,\ z \ne 0;$$

see [Melrose 1994] for intermediate values of z, [Vasy and Zworski 2000] for $|z| \to \infty$ and [Rodnianski and Tao 2015, Proposition 1.26] for $|z| \to 0$. Now the embedding $H^2(M) \cap \mathcal{E}'(K) \to L^2(M)$ is an operator in $\mathcal{C}_{n/2+\varepsilon}$ for all $\varepsilon > 0$ in view of the Weyl law for the Laplacian on a compact manifold. Since $q > \frac{n}{2}$, we deduce the claim.

The function $H(z)$ is continuous for $\operatorname{Im} z \geq 0,\ z \neq 0$, with values in $\mathcal{C}_q(L^2(M))$, and to avoid the problem at $z = 0$, we consider the map

$$F(z) := \langle H(e^z), T\rangle e^{(1-\frac{n}{2p})z}$$

for a fixed $T \in \mathcal{C}_{q'}(L^2(M))$ with norm $\|T\|_{\mathcal{C}_{q'}} = 1$. Here $\frac{1}{q'} + \frac{1}{q} = 1$ and the product is the duality pairing between the Banach space $\mathcal{C}_q$ and its dual $\mathcal{C}_{q'}$. Then $F(z)$ is holomorphic in $\operatorname{Im} z \in (0, \pi)$, continuous on the closure, and enjoys the bounds

$$|F(z)| \leq Ce^{C|z|} \quad \text{for } 0 \leq \operatorname{Im} z \leq \pi,$$
$$|F(z)| \leq C\|W_1\|_{L^{2p}(M)}\|W_2\|_{L^{2p}(M)} \quad \text{for } \operatorname{Im} z \in \{0, \pi\}$$

in view of (2-3). Applying the Phragmén–Lindelöf principle, we deduce that

$$|F(z)| \leq C\|W_1\|_{L^{2p}(M)}\|W_2\|_{L^{2p}(M)}$$

for all $z \in \mathbb{C}$ such that $0 \leq \operatorname{Im} z \leq \pi$, and therefore

$$\|H(z)\|_{\mathcal{C}_q} \leq C|z|^{-1+\frac{n}{2p}}\|W_1\|_{L^{2p}(M)}\|W_2\|_{L^{2p}(M)}, \quad \operatorname{Im} z \geq 0,\ z \neq 0.$$

By a density argument, we obtain the bound (1-9) for $\operatorname{Im} z \geq 0,\ z \neq 0$. By considering the adjoint of the operator B_z, we complete the proof of Theorem 2.

This argument reduces the problem to proving estimate (1-9) for $z \in \mathbb{R} \setminus \{0\}$. We find it convenient to first prove the corresponding estimate for the spectral measure given in Theorem 3. The proof of Theorem 3 relies crucially on the TT^* structure of the spectral measure.

When $z \in (-\infty, 0)$ and $p \in \left(\frac{n}{2}, \frac{n+1}{2}\right]$, the Schatten norm estimate (1-9) is a direct consequence of Theorem 3, and at the endpoint case $p = \frac{n}{2}$, the Schatten norm estimate (1-9) follows from the heat kernel estimates due to [Grigoryan 1997; Varopoulos 1985].

Establishing the Schatten norm estimate (1-9) for $W_1(\Delta_g - (z \pm i0))^{-1}W_2$ with $z > 0$ represents the main difficulty in the proof of Theorem 2. When doing so, following [Guillarmou and Hassell 2014; Guillarmou et al. 2013b; Hassell and Zhang 2016], we use a microlocal partition of the identity

$$\sum_{i=1}^{N} Q_i(\eta) = \mathrm{Id},$$

where $Q_i(\eta)$are pseudodifferential operators depending on the energy parameter $0 < \eta \sim |z|^{1/2}$, constructed in [Guillarmou et al. 2013b]. Splitting up the operator $W_1(\Delta_g - (z \pm i0))^{-1}W_2$ by means of the partition of the identity, we are led to estimate the individual terms $W_1 Q_i(\eta)^*(\Delta_g - (z \pm i0))^{-1}Q_j(\eta)W_2$, and here the most interesting contributions arise when $i = j$. When handling those, we proceed by establishing pointwise bounds for the Schwartz kernel of the operator

$$Q_i(\eta)^*\phi\left(\frac{\Delta_g}{z}\right)(\Delta_g - (z \pm i0))^{-s}Q_j(\eta), \quad \operatorname{Re} s \in \left[\tfrac{n-1}{2}, \tfrac{n+1}{2}\right],$$

analogous to the Euclidean estimates (2-1). Here ϕ is a cut-off near 1.

3. Schatten norm estimates on the spectral measure: proof of Theorem 3

Our starting point for the proof is the operator partition of unity, $\mathrm{Id} = \sum_{i=1}^{N} Q_i(\eta)$, depending on $\eta > 0$, constructed in [Guillarmou et al. 2013b]. This partition of unity enjoys the following estimates in particular: there exists $\delta > 0$ sufficiently small but fixed such that, for all $k = 0, 1, 2, \dots$, there is $C_k > 0$ such that, for all $m, m' \in M$, we have

$$\left|\partial_\lambda^k \big(Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(\lambda) Q_i(\eta)\big)(m, m')\right| \\ \le C_k \lambda^{n-1-k}(1 + \lambda d(m, m'))^{-\frac{(n-1)}{2}+k}, \quad \lambda \in [(1-\delta)\eta, (1+\delta)\eta], \tag{3-1}$$

with $d(\cdot\,,\cdot)$ being the Riemannian distance on M. We say more about this partition of the identity in Section 5A below; here, we can use results of [Chen 2018; Guillarmou et al. 2013b] as a "black box". Then for all $\lambda \in \big[\big(1 - \frac{\delta}{2}\big)\eta, \big(1 + \frac{\delta}{2}\big)\eta\big]$, we use the partition of unity to decompose the spectral measure sandwiched between two L^{2p} functions:

$$W_1 dE_{\sqrt{\Delta_g}}(\lambda) W_2 = \sum_{i,j=1}^{N} W_1 Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(\lambda) Q_j(\eta) W_2. \tag{3-2}$$

Let $p \in \big[1, \frac{n+1}{2}\big]$ and $q = \frac{p(n-1)}{n-p} \in [1, n+1]$. In the first step, we shall prove microlocalized estimates of the form

$$\|W_1 Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(\lambda) Q_i(\eta) W_2\|_{\mathcal{C}_q} \le C \lambda^{-1+\frac{n}{p}} \|W_1\|_{L^{2p}(M)} \|W_2\|_{L^{2p}(M)} \tag{3-3}$$

for the diagonal $(i = j)$ terms of the decomposition (3-2). In doing so, we shall follow [Frank and Sabin 2017, proof of Theorem 2] and start by showing (3-3) at the endpoints $p = \frac{n+1}{2}$ and $p = 1$; i.e.,

$$\|W_1 Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(\lambda) Q_i(\eta) W_2\|_{\mathcal{C}_{n+1}} \le C \lambda^{\frac{n-1}{n+1}} \|W_1\|_{L^{n+1}(M)} \|W_2\|_{L^{n+1}(M)}, \tag{3-4}$$

$$\|W_1 Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(\lambda) Q_i(\eta) W_2\|_{\mathcal{C}_1} \le C \lambda^{n-1} \|W_1\|_{L^2(M)} \|W_2\|_{L^2(M)}, \tag{3-5}$$

respectively. Once the estimates (3-4) and (3-5) have been established, the bound (3-3) follows by a complex interpolation argument applied to the analytic family of operators

$$\zeta \mapsto W_1^{\frac{2}{n+1}+\zeta\frac{n-1}{n+1}} Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(\lambda) Q_i(\eta) W_2^{\frac{2}{n+1}+\zeta\frac{n-1}{n+1}}$$

in the strip $0 \le \operatorname{Re}\zeta \le 1$, with $W_j \ge 0$ being simple functions such that $\|W_j\|_{L^2(M)} = 1$, $j = 1, 2$; see [Simon 1979, Theorem 2.9].

Now to prove the estimate (3-4), we shall consider the family of operators

$$T_s := Q_i(\eta)^* \phi\left(\frac{\sqrt{\Delta_g}}{\lambda}\right) \chi_+^s(\lambda - \sqrt{\Delta_g}) Q_i(\eta), \quad -\frac{(n+1)}{2} \le \operatorname{Re} s \le 0,$$

introduced in [Chen 2018; Guillarmou et al. 2013b, Definition 3.2]. Here $\phi \in C_0^\infty\big(\big(1 - \frac{\delta}{4}, 1 + \frac{\delta}{4}\big)\big)$ is such that $\phi(t) = 1$ in a neighborhood of $t = 1$, and χ_+^s is the family of distributions on $\mathbb{R}$, entire analytic in $s \in \mathbb{C}$ and such that

$$\chi_+^s(\lambda) = \frac{\lambda_+^s}{\Gamma(s+1)}, \quad \operatorname{Re} s > -1,$$

where $\lambda_+ = \max(\lambda, 0)$; see [Hörmander 1990, Section 3.2]. Note that, at least formally, we have

$$\chi_+^0(\lambda - \sqrt{\Delta_g}) = E_{\sqrt{\Delta_g}}(\lambda), \quad \chi_+^{-k}(\lambda - \sqrt{\Delta_g}) = \Big(\frac{d}{d\lambda}\Big)^{k-1} dE_{\sqrt{\Delta_g}}(\lambda), \quad k = 1, 2, \dots.$$

Recall from [Guillarmou et al. 2013b, Definition 3.2] that T_s is the operator whose Schwartz kernel is given by

$$\begin{aligned}\bigg(Q_i(\eta)^*\phi\bigg(\frac{\sqrt{\Delta_g}}{\lambda}\bigg)\chi_+^s(\lambda - \sqrt{\Delta_g})Q_i(\eta)\bigg)(m, m') \\ = \int \chi_+^{k+s}(\lambda-\mu)\partial_\mu^k\bigg(Q_i(\eta)^*\phi\Big(\frac{\mu}{\lambda}\Big)dE_{\sqrt{\Delta_g}}(\mu)Q_i(\eta)\bigg)(m, m')\, d\mu, \end{aligned} \tag{3-6}$$

where $k \in \mathbb{N}$ is such that $\operatorname{Re} s + k > -1$. As $\mu \in [\eta(1-\delta), \eta(1+\delta)]$ for $\lambda \in \big[\big(1-\frac{\delta}{2}\big)\eta, \big(1+\frac{\delta}{2}\big)\eta\big]$ and $\frac{\mu}{\lambda} \in \operatorname{supp}(\phi)$, thanks to the estimates (3-1) the integral in (3-6) is well-defined.

As explained in [Guillarmou et al. 2013b], the family of operators T_s is analytic in the sense of Stein in the strip $-\frac{(n+1)}{2} \le \operatorname{Re} s \le 0$. When $\operatorname{Re} s = 0$ we have

$$\|T_s\|_{L^2(M)\to L^2(M)} \le Ce^{\frac{\pi|s|}{2}},$$

and relying on the estimates (3-1) it was shown in [Chen 2018; Guillarmou et al. 2013b] that when $\operatorname{Re} s = -\frac{(n+1)}{2}$ we have

$$\|T_s\|_{L^1(M)\to L^\infty(M)} \le C(1+|r|)e^{\frac{\pi|r|}{2}}\lambda^{\frac{n-1}{2}}, \quad s = -\frac{(n+1)}{2} + ir, \quad r \in \mathbb{R}.$$

Applying Proposition 2.1, we get, for any two complex-valued functions $W_1, W_2 \in L^{n+1}(M)$,

$$\begin{aligned} W_1 T_{-1} W_2 &= W_1 Q_i(\eta)^*\phi\bigg(\frac{\sqrt{\Delta_g}}{\lambda}\bigg)\chi_+^{-1}(\lambda - \sqrt{\Delta_g})Q_i(\eta)W_2 \\ &= W_1 Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(\lambda)Q_i(\eta)W_2 \end{aligned}$$

is in the Schatten $\mathcal{C}_{n+1}$ class and (3-4) holds.

To show (3-5), we recall from [Guillarmou et al. 2013b] that we have a pointwise kernel bound on the (microlocalized) spectral measure,

$$\|Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(\lambda)Q_i(\eta)\|_{L^1(M)\to L^\infty(M)} \le C\lambda^{n-1}. \tag{3-7}$$

Also, we have

$$dE_{\sqrt{\Delta_g}}(\lambda) = (2\pi)^{-1}P(\lambda)P^*(\lambda), \tag{3-8}$$

where $P(\lambda) : L^2(\partial M) \to L^r(M)$, $r \in \big[\frac{2(n+1)}{n-1}, \infty\big]$, is the Poisson operator; see [Guillarmou et al. 2013b]. Using the T^*T trick, it follows from (3-7) and (3-8) that

$$\|Q_i(\eta)^* P(\lambda)\|_{L^2(\partial M)\to L^\infty(M)} \le C\lambda^{\frac{n-1}{2}}.$$

The Schwartz kernel $Q_i(\eta)^*P(\lambda)(m, m')$ of the operator $Q_i(\eta)^*P(\lambda)$ satisfies therefore

$$\|Q_i(\eta)^*P(\lambda)(m, \cdot)\|_{L^2(\partial M)} \le C\lambda^{\frac{n-1}{2}}$$

for almost all $m \in M$. Thus, for any $W_1 \in L^2(M)$, the operator $W_1 Q_i(\eta)^* P(\lambda) : L^2(\partial M) \to L^2(M)$ is Hilbert–Schmidt with the norm bounded by $C\lambda^{(n-1)/2}\|W_1\|_{L^2(M)}$. Taking adjoints, we find that $P(\lambda)^* Q_i(\eta) W_2$ is a Hilbert–Schmidt operator with norm bounded by $C\lambda^{(n-1)/2}\|W_2\|_{L^2(M)}$. Therefore, $(2\pi)^{-1}$ times the composition of these two operators, which is precisely $W_1 Q_i(\eta)^* dE_{\sqrt{\Delta}}(\lambda) Q_i(\eta) W_2$, is of trace class and (3-5) follows.

In the second step, we shall bound the Schatten norm of the off-diagonal ($i \neq j$) terms in the decomposition (3-2); i.e., we shall prove the estimate

$$\|W_1 Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(\lambda) Q_j(\eta) W_2\|_{\mathcal{C}_q} \le C\lambda^{-1+\frac{n}{p}}\|W_1\|_{L^{2p}(M)}\|W_2\|_{L^{2p}(M)}. \tag{3-9}$$

As above, we shall exploit the T^*T structure of the spectral measure.

Let $T : L^2(M) \to L^2(\partial M)$ be a compact operator and $q \ge 1$. Then $T^*T \in \mathcal{C}_q(L^2(M))$ if and only if $T \in \mathcal{C}_{2q}(L^2(M), L^2(\partial M))$, and moreover, $\|T^*T\|_{\mathcal{C}_q} = \|T\|^2_{\mathcal{C}_{2q}}$. This is a consequence of the following equality for the singular values:

$$\mu_k(T^*T) = \mu_k(T)^2. \tag{3-10}$$

Moreover, if T_1, T_2 are in $\mathcal{C}_{2q}(L^2(M), L^2(\partial M))$, then $T_1^* T_2$ is in $\mathcal{C}_q(L^2(M))$, and

$$\|T_1^* T_2\|^q_{\mathcal{C}_q} \le \|T_1^* T_1\|^q_{\mathcal{C}_q} + \|T_2^* T_2\|^q_{\mathcal{C}_q}; \tag{3-11}$$

see for example [McCarthy 1967]. Using (3-8), we write

$$W_1 Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(\lambda) Q_j(\eta) W_2 = (2\pi)^{-1} T_1^* T_2, \tag{3-12}$$

where $T_1 = P(\lambda)^* Q_i(\eta) \overline{W}_1$, and $T_2 = P(\lambda)^* Q_j(\eta) W_2$. Now it follows from (3-3) that $T_1^* T_1 \in \mathcal{C}_q(L^2(M))$, $T_2^* T_2 \in \mathcal{C}_q(L^2(M))$, and we have

$$\|T_1^* T_1\|_{\mathcal{C}_q} \le C\lambda^{-1+\frac{n}{p}}\|W_1\|^2_{L^{2p}(M)}, \quad \|T_2^* T_2\|_{\mathcal{C}_q} \le C\lambda^{-1+\frac{n}{p}}\|W_2\|^2_{L^{2p}(M)}.$$

By the discussion above, this is equivalent to the fact that $T_1 \in C_{2q}(L^2(M), L^2(\partial M))$ and $T_2 \in C_{2q}(L^2(M), L^2(\partial M))$. It follows from (3-12) and discussion above that $W_1 Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(\lambda) Q_j(\eta) W_2 \in \mathcal{C}_q(L^2(M))$, and using (3-11), we get that

$$\|W_1 Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(\lambda) Q_j(\eta) W_2\|_{\mathcal{C}_q} \le C\lambda^{-1+\frac{n}{p}}(\|W_1\|^2_{L^{2p}(M)} + \|W_2\|^2_{L^{2p}(M)}).$$

Thus, (3-9) follows by bilinearity in W_1, W_2. This completes the proof of Theorem 3.

4. Consequences of the spectral measure estimates for $p \in \left(\frac{n}{2}, \frac{n+1}{2}\right]$ and their analogues at the endpoint $p = \frac{n}{2}$

4A. ***Consequences of the spectral measure Schatten norm estimate.*** Using Theorem 3 and Minkowski's integral inequality, we can deduce some Schatten estimates on the resolvent. In this subsection, we only treat the case $p > \frac{n}{2}$.

The first result applies for z in any sector excluding the positive real axis.

Proposition 4.1. *Let $p \in (\frac{n}{2}, \frac{n+1}{2}]$, and suppose $W_1, W_2 \in L^{2p}(M)$. Let $\varepsilon > 0$ be arbitrary. Then for $z \in \mathbb{C}$ such that $z \neq 0$, $\arg z \in [\varepsilon, 2\pi - \varepsilon]$, we have the sandwiched resolvent $W_1(\Delta_g - z)^{-1} W_2$ is in the Schatten class $\mathcal{C}_q(L^2(M))$ with $q = \frac{p(n-1)}{n-p} \in (n-1, n+1]$ and*

$$\|W_1(\Delta_g - z)^{-1} W_2\|_{\mathcal{C}_q} \le C|z|^{-1+\frac{n}{2p}} \|W_1\|_{L^{2p}(M)} \|W_2\|_{L^{2p}(M)},$$

where C depends on p, ε and (M, g), but not z.

Proof. We express the operator $W_1(\Delta_g - z)^{-1} W_2$ as

$$W_1(\Delta_g - z)^{-1} W_2 = \int_0^\infty (\lambda^2 - z)^{-1} W_1 \, dE_{\sqrt{\Delta_g}}(\lambda) W_2 \, d\lambda.$$

The result follows by estimating the Schatten norm of $W_1 \, dE_{\sqrt{\Delta_g}}(\lambda) W_2$ using Theorem 3 and noting that, provided $p > \frac{n}{2}$, we have

$$\int_0^\infty |\lambda^2 - z|^{-1} \lambda^{-1+\frac{n}{p}} \, d\lambda \le C|z|^{-1+\frac{n}{2p}},$$

where C depends on p and ε but does not depend on z in the given sector. □

In a similar manner we obtain "elliptic" estimates on the resolvent, where we remove the singularity in the spectral multiplier. In this way we can obtain estimates on the positive real axis. To state these, we fix a function $\phi : [0, \infty) \to [0, 1]$ such that $\phi(t) = 1$ for t in a neighborhood of $t = 1$ and has support in a slightly bigger neighborhood of $t = 1$.

Proposition 4.2. *Let $p \in \left(\frac{n}{2}, \frac{n+1}{2}\right]$, and suppose $W_1, W_2 \in L^{2p}(M)$. Then for $z \in \mathbb{C} \setminus \{0\}$, the operator $W_1(1-\phi)(\Delta_g/|z|)(\Delta_g - z)^{-1} W_2$ is in the Schatten class $\mathcal{C}_q(L^2(M))$ with $q = \frac{p(n-1)}{n-p} \in (n-1, n+1]$, and we have*

$$\left\| W_1(1-\phi)\left(\frac{\Delta_g}{|z|}\right)(\Delta_g - z)^{-1} W_2 \right\|_{\mathcal{C}_q} \le C|z|^{-1+\frac{n}{2p}} \|W_1\|_{L^{2p}(M)} \|W_2\|_{L^{2p}(M)},$$

where C depends on p and on (M, g), but not z.

Proof. Again we express the operator using an integral over the spectral measure, and estimate the Schatten norm of the spectral measure using Theorem 3 and Minkowski's integral inequality. This time we obtain the integral

$$\int_0^\infty |\lambda^2 - z|^{-1} (1-\phi)\left(\frac{\lambda^2}{|z|}\right) \lambda^{-1+\frac{n}{p}} \, d\lambda$$

and it is straightforward to check that this is bounded by $C|z|^{-1+n/(2p)}$ uniformly in z. □

4B. ***Analogues at the endpoint $p = \frac{n}{2}$.*** In the case $p = \frac{n}{2}$, the arguments used in the proofs of Propositions 4.1 and 4.2 are no longer valid and need to be replaced. In view of the Phragmén–Lindelöf argument, explained in Section 2B, we only need to do this for z negative in the case of Proposition 4.1 and z positive in the case of Proposition 4.2. To this end we prove the following two results.

Proposition 4.3. *Let $p = \frac{n}{2}$. There is $C > 0$ such that for all $z < 0$ and for all $W_1, W_2 \in L^n(M)$, the operator $W_1(\Delta_g - z)^{-1}W_2$ is in $\mathcal{C}_{n-1}(L^2(M))$ and we have*

$$\|W_1(\Delta_g - z)^{-1}W_2\|_{\mathcal{C}_{n-1}} \leq C\|W_1\|_{L^n(M)}\|W_2\|_{L^n(M)}. \tag{4-1}$$

Proof. Here we use a slight variation of Proposition 2.1. Let W_1, W_2 be nonnegative simple functions and consider the analytic family of operators

$$S_s = W_1^{-s}(\Delta_g - z)^s W_2^{-s}, \quad -\tfrac{(n-1)}{2} \leq \operatorname{Re} s \leq 0.$$

Clearly, when $\operatorname{Re} s = 0$, we have

$$\|S_s\|_{L^2(M)\to L^2(M)} \leq C. \tag{4-2}$$

Next, we will show that, when $\operatorname{Re} s = -\frac{(n-1)}{2}$, then S_s is Hilbert–Schmidt and we have

$$\|S_s\|_{\mathcal{C}_2} \leq Ce^{C|\operatorname{Im} s|}\|W_1\|_{L^n(M)}^{\frac{n-1}{2}}\|W_2\|_{L^n(M)}^{\frac{n-1}{2}}. \tag{4-3}$$

This allows us to run the interpolation argument in the proof of Proposition 2.1.

To prove (4-3), on the line $\operatorname{Re} s = -\frac{(n-1)}{2}$, we express $(\Delta_g - z)^s$ in terms of the heat kernel:

$$\Gamma(-s)(\Delta_g - z)^s(m, m') = \int_0^\infty t^{-s-1}e^{tz}e^{-t\Delta_g}(m, m')\,dt. \tag{4-4}$$

We now use heat kernel estimates. Due to [Varopoulos 1985], we have the estimate $\|e^{-t\Delta_g}\|_{L^1\to L^\infty} \leq Ct^{-n/2}$ and by a result of [Grigoryan 1997], this implies a pointwise upper Gaussian estimate on the heat kernel

$$|e^{-t\Delta_g}(m, m')| \leq Ct^{-\frac{n}{2}}e^{-\frac{cd(m,m')^2}{t}}, \quad t > 0, \tag{4-5}$$

for some $c > 0$. The integral in (4-4) is convergent for all $m \neq m'$ due to (4-5). We thus get for all $m \neq m'$ and $z \in (-\infty, 0)$, and uniformly for all s such that $\operatorname{Re} s = -\frac{(n-1)}{2}$

$$\begin{aligned}
|\Gamma(-s)(\Delta_g - z)^s(m, m')| &\leq C\int_0^\infty t^{-\frac{3}{2}}e^{-\frac{cd(m,m')^2}{t}+zt}\,dt \\
&\leq Cd(m, m')^{-1}\int_0^\infty t^{-\frac{3}{2}}e^{-\frac{c}{t}+zd(m,m')^2t}\,dt \\
&\leq Cd(m, m')^{-1}.
\end{aligned} \tag{4-6}$$

Using Hölder's inequality, the generalized Hardy–Littlewood–Sobolev inequality of [García-Cuerva and Gatto 2004] and (4-6), we obtain for $\operatorname{Re} s = -\frac{(n-1)}{2}$,

$$\begin{aligned}
\|W_1^{-s}(\Delta_g - z)^s W_2^{-s}\|_{\mathcal{C}_2(M)}^2 &\leq C|\Gamma(-s)|^{-1}\int_{M\times M} W_1(m)^{n-1}d(m, m')^{-2}W_2(m')^{n-1}\,dV_g(m)\,dV_g(m') \\
&\leq C|\Gamma(-s)|^{-1}\|W_1^{n-1}\|_{L^{n/(n-1)}(M)}\|W_2^{n-1}\|_{L^{n/(n-1)}(M)} \\
&\leq Ce^{C|\operatorname{Im} s|}\|W_1\|_{L^n(M)}^{n-1}\|W_2\|_{L^n(M)}^{n-1},
\end{aligned}$$

where the factor $e^{C|\operatorname{Im} s|}$ is contributed by the Gamma function. This shows (4-3).

We now interpolate using the family S_s between (4-2) and (4-3), as in the proof of Proposition 2.1, and we obtain at $s = -1$

$$\|W_1(\Delta_g - z)^{-1} W_2\|_{\mathcal{C}_{n-1}} \le C \|W_1\|_{L^n(M)} \|W_2\|_{L^n(M)}, \tag{4-7}$$

which completes the proof for W_1 and W_2 nonnegative and simple. The extension to general $W_1, W_2 \in L^n(M)$ is standard. $\square$

We now prove an analogue of Proposition 4.2.

Proposition 4.4. *Let $p = \frac{n}{2}$ and suppose $W_1, W_2 \in L^n(M)$, and let ϕ be as in Proposition 4.2. Then for $z > 0$, the operator $W_1(1-\phi)(\Delta_g/z)(\Delta_g - z)^{-1} W_2$ is in the Schatten class $\mathcal{C}_{n-1}(L^2(M))$ and*

$$\left\| W_1(1-\phi)\left(\frac{\Delta_g}{z}\right)(\Delta_g - z)^{-1} W_2 \right\|_{\mathcal{C}_{n-1}} \le C \|W_1\|_{L^n(M)} \|W_2\|_{L^n(M)}$$

uniformly in z.

Proof. We first note that for $z > 0$, the operator

$$W_1 \phi\left(\frac{\Delta_g}{z}\right)(\Delta_g + z)^{-1} W_2$$

is in the Schatten class $\mathcal{C}_{n-1}(L^2(M))$, and

$$\left\| W_1 \phi\left(\frac{\Delta_g}{z}\right)(\Delta_g + z)^{-1} W_2 \right\|_{\mathcal{C}_{n-1}} \le C \|W_1\|_{L^n(M)} \|W_2\|_{L^n(M)}$$

uniformly in z. This follows from the spectral measure estimate (1-10), since

$$\int_0^\infty \lambda \phi\left(\frac{\lambda^2}{z}\right)(\lambda^2 + z)^{-1} \, d\lambda$$

is bounded uniformly in z. Combining this with Proposition 4.3, we see that $W_1(1-\phi)(\Delta_g/z)(\Delta_g+z)^{-1} W_2$ is in $\mathcal{C}_{n-1}(L^2(M))$ and we have

$$\left\| W_1(1-\phi)\left(\frac{\Delta_g}{z}\right)(\Delta_g + z)^{-1} W_2 \right\|_{\mathcal{C}_{n-1}} \le C \|W_1\|_{L^n(M)} \|W_2\|_{L^n(M)} \tag{4-8}$$

uniformly in z.

Now we write

$$\begin{aligned} W_1(1-\phi)\left(\frac{\Delta_g}{z}\right)(\Delta_g - z)^{-1} W_2 &= W_1(1-\phi)\left(\frac{\Delta_g}{z}\right)(\Delta_g + z)^{-1} W_2 \\ &\quad + 2z W_1(1-\phi)\left(\frac{\Delta_g}{z}\right)(\Delta_g + z)^{-1}(\Delta_g - z)^{-1} W_2. \end{aligned} \tag{4-9}$$

The first term in the right-hand side of (4-9) has already been shown to lie in $\mathcal{C}_{n-1}$ with the bound (4-8). We write the second term on the right-hand side of (4-9) in terms of the spectral measure and apply

Minkowski's integral inequality together with the spectral measure estimate (1-10), and find that the norm in $\mathcal{C}_{n-1}$ is bounded by

$$C\left(z\int_0^\infty (1-\phi)\left(\frac{\lambda^2}{z}\right)(\lambda^2+z)^{-1}(\lambda^2-z)^{-1}\lambda\,d\lambda\right)\|W_1\|_{L^n(M)}\|W_2\|_{L^n(M)}$$

and a change of variable shows that this integral is convergent and independent of z, completing the proof. □

5. Resolvent estimates on the spectrum: completion of the proof of Theorem 2

The key difficulty in proving Theorem 2 is to obtain estimates on the limiting resolvent at the spectrum $(\Delta_g-(z+i0))^{-1}$ for $z>0$. Given Propositions 4.2 and 4.4, we only need to do this localized near the singularity at z of the spectral multiplier $(\lambda^2-z)^{-1}$. In doing so, following [Guillarmou and Hassell 2014; Guillarmou et al. 2013b; Hassell and Zhang 2016], we shall use a microlocal partition of unity.

5A. ***Operator partition of unity.*** We begin by recalling some results of [Guillarmou and Hassell 2014; Hassell and Zhang 2016] on high- and low-frequency microlocal estimates on the spectral measure and resolvents of Δ_g.

Proposition 5.1. ***High-frequency microlocal estimates.*** *For all high energies $\eta\geq\frac{1}{2}$, there exists a family of bounded operators $Q_i(\eta):L^2(M)\to L^2(M)$, $i=1,\dots,N_h$, with N_h independent of η and with the norm satisfying*

$$\|Q_i(\eta)\|_{L^2(M)\to L^2(M)}\leq C\quad\text{for some } C \text{ independent of } \eta, \tag{5-1}$$

so that the following properties hold:

(1) *The operators $Q_i(\eta)$ form an operator partition of unity*:

$$\sum_{i=1}^{N_h} Q_i(\eta)=\mathrm{Id}. \tag{5-2}$$

(2) *Let $\eta\geq\frac{1}{2}$ and $(i,j)\in\{1,\dots,N_h\}^2$. There exists $\delta>0$ small such that for all $z>0$ such that $\sqrt{z}\in[(1-\delta)\eta,(1+\delta)\eta]$, one of the following three alternatives holds*:

(2.i) *One has for the **outgoing** resolvent*

$$\big(Q_i(\eta)^*(\Delta_g-(z+i0))^{-1}Q_j(\eta)\big)(m,m')\in x(m)^\infty x(m')^\infty z^{-\infty}C^\infty(\overline{M}\times\overline{M}) \tag{5-3}$$

for all $m,m'\in M$, where the $C^\infty(\overline{M}\times\overline{M})$-part depends also on z and is uniformly bounded in z in the smooth topology.

(2.ii) *One has for the **incoming** resolvent*

$$\big(Q_i(\eta)^*(\Delta_g-(z-i0))^{-1}Q_j(\eta)\big)(m,m')\in x(m)^\infty x(m')^\infty z^{-\infty}C^\infty(\overline{M}\times\overline{M}) \tag{5-4}$$

for all $m,m'\in M$.

(2.iii) *The spectral measure satisfies, for* $\lambda = \sqrt{z} \in [(1-\delta)\eta, (1+\delta)\eta]$, *the following bounds: for all* $k = 0, 1, 2, \dots$, *there is* $C_k > 0$ *such that for all* $m, m' \in M$

$$\left|\partial_\lambda^k \big(Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(\lambda) Q_j(\eta)\big)(m,m')\right| \le C_k \lambda^{n-1-k}(1+\lambda d(m,m'))^{-\frac{(n-1)}{2}+k}, \tag{5-5}$$

$$\big(Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(\lambda) Q_j(\eta)\big)(m,m') = \lambda^{n-1}\Big(\sum_{\pm} e^{\pm i\lambda d(m,m')} a_\pm(\lambda,m,m') + b(\lambda,m,m')\Big), \tag{5-6}$$

with $a_\pm, b$ *satisfying the estimates, for all* $k = 0, 1, 2, \dots$,

$$|\partial_\lambda^k a_\pm(\lambda,m,m')| \le C_k \lambda^{-k}(1+\lambda d(m,m'))^{-\frac{(n-1)}{2}}, \tag{5-7}$$

$$|\partial_\lambda^k b(\lambda,m,m')| \le C_k \lambda^{-k}(1+\lambda d(m,m'))^{-K} \quad \text{for all } K > 1. \tag{5-8}$$

Moreover the alternative (2.iii) *always holds if* $i = j$.

Low-frequency microlocal estimates. *Similarly, for all low energies* $\eta \le 2$, *there exists a family of bounded operators* $Q_i(\eta) : L^2(M) \to L^2(M)$, $i = 0, *, 1, \dots, N_l$, *with* N_l *independent of* η *satisfying* (5-1) *and* (5-2) (*with the sum in this case ranging over* $i = 0, *, 1, \dots, N_l$), *satisfying the following*:

(3) *Let* $0 < \eta \le 2$ *and* i, j *range independently in* $\{0, *, 1, \dots, N_l\}$. *There exists* $\delta > 0$ *small such that, for all* $z > 0$ *satisfying* $\lambda := \sqrt{z} \in [(1-\delta)\eta, (1+\delta)\eta]$, *one of the following three alternatives holds*:

(3.i) *One has the pointwise kernel bound for the* ***outgoing*** *resolvent* (*for all* $N \in \mathbb{N}$)

$$\left|\big(Q_i(\eta)^*(\Delta_g-(z+i0))^{-1} Q_j(\eta)\big)(m,m')\right| \le C_N \left(\frac{x}{x+\lambda}\right)^N \left(\frac{x'}{x'+\lambda}\right)^N \frac{(xx')^{\frac{n-1}{2}}\big(\chi\big(\frac{x}{\lambda}\big)+\chi\big(\frac{x'}{\lambda}\big)\big)}{x+x'+\lambda}, \tag{5-9}$$

where $x = x(m)$, $x' = x(m')$, *and* $\chi \in C_0^\infty((-\varepsilon,\varepsilon),[0,\infty))$ *is such that* $\chi = 1$ *in* $\left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right]$. *Here* $\varepsilon > 0$ *is small enough.*

(3.ii) *One has the pointwise kernel bound for the* ***incoming*** *resolvent* (*for all* $N \in \mathbb{N}$)

$$\left|\big(Q_i(\eta)^*(\Delta_g-(z-i0))^{-1} Q_j(\eta)\big)(m,m')\right| \le C_N \left(\frac{x}{x+\lambda}\right)^N \left(\frac{x'}{x'+\lambda}\right)^N \frac{(xx')^{\frac{n-1}{2}}\big(\chi\big(\frac{x}{\lambda}\big)+\chi\big(\frac{x'}{\lambda}\big)\big)}{x+x'+\lambda}. \tag{5-10}$$

(3.iii) *For all* $k = 0, 1, 2, \dots$, *there is* $C_k > 0$ *such that* (5-5), (5-6), (5-7) *and* (5-8) *hold.*

Moreover if $i = j$, *the alternative* (3.iii) *holds.*

Remark 5.2. The two partitions of the identity do not quite match up in the intermediate energy regime, $\frac{1}{2} \le \eta \le 2$. Because of this, it would be more notationally accurate to label the partitions Q_i^{high} and Q_j^{low}; to avoid cumbersome notation, we do not do this. We emphasize that in this intermediate regime either partition can be used.

Remark 5.3. In the low-energy case, $\eta \le 2$, let us first point out the meaning of the right-hand side of (5-9) and (5-10). In [Guillarmou et al. 2013a] it was shown that the Schwartz kernel of the resolvent $(\Delta_g - (\lambda^2 \pm i0))^{-1}$ for $\lambda \in [0, \lambda_0]$ has some polyhomogeneous structure on the "low-energy space", which is a blowup of $\overline{M} \times \overline{M} \times [0, \lambda_0]$. Ignoring the artificial boundary at $\lambda = \lambda_0$, this blown-up space has seven boundary hypersurfaces corresponding to seven different types of asymptotics displayed by the resolvent kernel. These are the left boundary lb, the right boundary rb, which arise from $\partial\overline{M}\times\overline{M}\times[0,\lambda_0]$ and $\overline{M}\times\partial\overline{M}\times[0,\lambda_0]$; the b-face bf, which arises from blowing up $\partial\overline{M}\times\partial\overline{M}\times[0,\lambda_0]$; the "zero face" zf, arising from $\overline{M}\times\overline{M}\times\{0\}$; and three faces at $\lambda=0$ produced by blowing up. These are bf_0, arising from blowing up $\partial\overline{M}\times\partial\overline{M}\times\{0\}$; the face lb_0, arising from blowing up $\partial\overline{M}\times\overline{M}\times\{0\}$; and lastly rb_0, arising from blowing up $\overline{M}\times\partial\overline{M}\times\{0\}$. See Figure 1 of [Guillarmou et al. 2013a].

The resolvent (microlocally away from the conormal bundle of the diagonal) was shown in [Guillarmou et al. 2013a] to be polyhomogeneous and vanish to order $n-2$ at the boundary hypersurfaces lb_0, rb_0, bf_0, and to vanish to order $\frac{n-1}{2}$ at lb and rb. Cases (3.i) and (3.ii) apply when the microlocalizing operators Q_i and Q_j remove the wavefront set at lb, rb and bf, meaning there is infinite-order vanishing there. Moreover, the cutoff factor $\chi\big(\frac{x}{\lambda}\big)+\chi\big(\frac{x'}{\lambda}\big)$ vanishes in a neighborhood of zf. Now notice that x vanishes to first order at lb, lb_0 and bf_0, while x' vanishes to first order at rb, rb_0 and bf_0 and $x+x'+\lambda$ vanishes to first order at bf_0. So the product on the right-hand side of (5-9) and (5-10) precisely encodes the order of vanishing at these remaining boundary hypersurfaces.

Proof. This is a combination of several results from [Guillarmou and Hassell 2014; Guillarmou et al. 2013b]. In the high-energy case, $\eta\ge\frac{1}{2}$, Lemma 5.3 of [Guillarmou and Hassell 2014] tells us that the pairs (i,j) split into four cases. In the first two cases, $Q_i(\eta)^*$ is either not-incoming or not-outgoing related to $Q_j(\eta)$, and then Proposition 6.7 of [Guillarmou and Hassell 2014] applies; note that the estimates in (2.i) and (2.ii) above appear in the proof, rather than the statement, of Proposition 6.7. In the third and fourth cases, Theorem 1.12 of [Guillarmou et al. 2013b] applies and shows that estimates (5-5) hold; see also Proposition 6.4 of [Guillarmou and Hassell 2014]. Also in the third and fourth cases, Proposition 1.5 of [Hassell and Zhang 2016] holds and gives the estimates (5-6), (5-7) and (5-8). Note that [Hassell and Zhang 2016, Proposition 1.5] is written in the case when $i=j$ but the proof of that proposition shows that it remains valid more generally when $i\neq j$ but the microsupports are close enough.

In the low-energy case, as shown in Section 6 of [Guillarmou and Hassell 2014], case (3.iii) applies to the pairs $(0,0)$, $(*,*)$, and (i,j) where $i,j\ge1$ and $|i-j|\le1$. Moreover, case (3.iii) also applies to any pair where either $i=*$ or $j=*$. That is because in these cases, the operator $Q_*(\eta)$ annihilates the wavefront set of the spectral measure at bf, with the consequence that the spectral measure estimates

$$\big|\partial_\lambda^k\big(Q_i(\eta)^*dE_{\sqrt{\Delta_g}}(\lambda)Q_j(\eta)\big)(m,m')\big|\le C_k\lambda^{n-1-k}(1+\lambda d(m,m'))^{-\frac{(n-1)}{2}+k} \tag{5-11}$$

hold if either $i=*$ or $j=*$, and this leads to estimates (5-5) as in the high-energy case. For (3.iii) with $i,j\ge1$, the estimates (5-6), (5-7) and (5-8) are proven in [Hassell and Zhang 2016, Proposition 1.5] in the case when $i=j$ but the proof shows that it remains valid more generally when $i\neq j$ but the microsupports are close enough. The case $i,j\in\{0,*\}$ in (3.iii) is also shown in [Hassell and Zhang 2016, Proposition 1.5].

The cases $i=0$ and $j\ge1$, and $i\ge1$ and $j=0$, fit any one of the cases (3.i), (3.ii), (3.iii) above. This is because here the wavefront set at bf is wiped out by $Q_0(\eta)$, while the wavefront set at fiber-infinity is wiped out by $Q_j(\eta)$ for $j\ge1$.

The final case remaining, where $i, j \geq 1$ and $|i-j| \geq 2$, fits into cases (3.i) or (3.ii) according to whether $Q_i(\eta)^*$ is not incoming-related or not outgoing-related to $Q_j(\eta)$, as shown in Proposition 6.9 of [Guillarmou and Hassell 2014]. □

Cases (3.i) and (3.ii) will be treated using the following lemma.

Lemma 5.4. *Let (M, g) be an asymptotically conic manifold of dimension $n \geq 3$. Then if an integral operator K has kernel $K(m, m')$ bounded pointwise by*

$$C \frac{(xx')^{\frac{n-1}{2}}\left(\chi\left(\frac{x}{\lambda}\right)+\chi\left(\frac{x'}{\lambda}\right)\right)}{x+x'+\lambda}, \quad 0<\lambda \leq 3,$$

then for $W_1, W_2 \in L^{2p}(M)$, $p \in \left[\frac{n}{2}, \frac{n+1}{2}\right]$, the operator $W_1 K W_2$ is Hilbert–Schmidt and we have

$$\|W_1 K W_2\|_{\mathcal{C}_2} \leq C\lambda^{-2+\frac{n}{p}} \|W_1\|_{L^{2p}(M)} \|W_2\|_{L^{2p}(M)}. \tag{5-12}$$

Proof. Using Hölder's inequality with $\frac{1}{p'}+\frac{1}{p}=1$ and $p' \in \left[\frac{n+1}{n-1}, \frac{n}{n-2}\right]$, we get

$$\begin{aligned} &\|W_1 K W_2\|_{\mathcal{C}_2} \\ &\quad \leq \|W_1\|_{L^{2p}} \|W_2\|_{L^{2p}} \left(\int_{M\times M} \frac{(x(m)x(m'))^{(n-1)p'}\left(\chi\left(\frac{x(m)}{\lambda}\right)+\chi\left(\frac{x(m')}{\lambda}\right)\right)^{2p'}}{(x(m)+x(m')+\lambda)^{2p'}}\, dV_g(m)\, dV_g(m')\right)^{\frac{1}{2p'}}. \end{aligned}$$

We use the coordinates $m=(x, y)$, $m'=(x', y')$ near the boundary, where the measure $dV_g(m)$ is comparable to $dx\, dy/x^{n+1}$. Let us introduce the polar coordinates $(x, x')=(R\sin(\theta), R\cos(\theta))$ with $\theta \in \left[0, \frac{\pi}{2}\right]$, near $x=x'=0$. Using that $(n-1)p'-(n+1) \geq 0$ and $x+x' \sim R$, we get

$$\begin{aligned} &\left(\int_{M\times M} \frac{(xx')^{(n-1)p'}\chi\left(\frac{x}{\lambda}\right)}{(x+x'+\lambda)^{2p'}}\, dV_g\, dV_{g'}\right)^{\frac{1}{2p'}} \\ &\qquad \leq C\left(\int_{0<x<2\lambda} \frac{(xx')^{(n-1)p'-(n+1)}}{(x+x'+\lambda)^{2p'}}\, dx\, dx'\right)^{\frac{1}{2p'}} \\ &\qquad \leq C\left(\int_0^\infty \int_{0<\sin\theta<\frac{2\lambda}{R}} \frac{R^{2(n-1)p'-2n-1}}{(R+\lambda)^{2p'}}\, dR\, d\theta\right)^{\frac{1}{2p'}} \\ &\qquad \leq C\frac{1}{\lambda}\left(\int_0^{2\lambda} R^{2(n-1)p'-2n-1}\, dR\right)^{\frac{1}{2p'}} + C\left(\int_{2\lambda}^\infty \int_{0<\theta\leq\frac{\widetilde{C}\lambda}{R}} R^{2(n-1)p'-2p'-2n-1}\, dR\, d\theta\right)^{\frac{1}{2p'}} \\ &\qquad \leq C\lambda^{\frac{n}{p}-2} + C\lambda^{\frac{1}{2p'}}\left(\int_{2\lambda}^\infty R^{2(n-2)p'-2n-2}\, dR\right)^{\frac{1}{2p'}} \leq C\lambda^{\frac{n}{p}-2}. \end{aligned}$$

Here we used that $(n-1)p' > n$ and $2(n-2)p'-2n-1<0$. The same argument works with the term involving $\chi\left(\frac{x'}{\lambda}\right)$ and the estimate (5-12) follows. □

5B. ***Analytic family of operators.*** In this section we closely follow Section 4 of [Guillarmou and Hassell 2014], especially Remark 4.2 (which is essentially due to Adam Sikora). Let $\phi \in C_0^\infty\left(\left(\left(1-\frac{\delta}{4}\right)^2, \left(1+\frac{\delta}{4}\right)^2\right)\right)$ be such that $\phi(t)=1$ in a neighborhood of $t=1$, where $\delta>0$ is small, and consider the analytic family of operators in $\mathrm{Re}(s) \le 0$

$$H_{s,z,\varepsilon}(\Delta_g) = \phi\Big(\frac{\Delta_g}{z}\Big)(\Delta_g - (z + i\varepsilon))^s, \quad z > 0,\ \varepsilon > 0.$$

By the spectral theorem, we have

$$H_{s,z,\varepsilon}(\Delta_g) = z^{s+\frac{1}{2}} \int_0^\infty \Big(\lambda - \Big(1 + i\frac{\varepsilon}{z}\Big)\Big)^s \frac{\phi(\lambda)}{2\sqrt{\lambda}} dE_{\sqrt{\Delta_g}}(z^{\frac{1}{2}}\lambda^{\frac{1}{2}})\, d\lambda. \tag{5-13}$$

Let $\eta > 0$ be such that $z^{1/2} \in \big[\big(1 - \frac{\delta}{2}\big)\eta, \big(1 + \frac{\delta}{2}\big)\eta\big]$ and let $Q_i(\eta)$ and $Q_j(\eta)$ be such that the condition (2.iii) or (3.iii) of Proposition 5.1 holds, in the high-energy, respectively, low-energy case. Then using (5-13), we have on the level of Schwartz kernels, for $m, m' \in M$,

$$\big(Q_i(\eta)^* H_{s,z,\varepsilon}(\Delta_g) Q_j(\eta)\big)(m, m') = z^{s+\frac{1}{2}} \int_0^\infty \Big(\lambda - \Big(1 + i\frac{\varepsilon}{z}\Big)\Big)^s \psi(\lambda)\, d\lambda, \tag{5-14}$$

where

$$\psi(\lambda) = \frac{\phi(\lambda)}{2\sqrt{\lambda}} Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(z^{\frac{1}{2}}\lambda^{\frac{1}{2}}) Q_j(\eta)(m, m').$$

Here, as $\delta > 0$ is small, we have $z^{1/2}\lambda^{1/2} \in [(1-\delta)\eta, (1+\delta)\eta]$ when $z^{1/2} \in \big[\big(1 - \frac{\delta}{2}\big)\eta, \big(1 + \frac{\delta}{2}\big)\eta\big]$ and $\lambda \in \mathrm{supp}(\phi)$, and therefore, in view of (5-5), we have $\psi(\lambda) \in C_0^\infty(\mathbb{R})$.

Letting $\varepsilon \to 0$ in (5-14), we define $Q_i(\eta)^* H_{s,z,0}(\Delta_g) Q_j(\eta)$ when $z^{1/2} \in \big[\big(1-\frac{\delta}{2}\big)\eta, \big(1+\frac{\delta}{2}\big)\eta\big]$ as operators whose Schwartz kernels are given by

$$\begin{aligned}\big(Q_i(\eta)^* H_{s,z,0}(\Delta_g) Q_j(\eta)\big)(m, m') &= z^{s+\frac{1}{2}} \int_0^\infty (\lambda - (1 + i0))^s \psi(\lambda)\, d\lambda \\ &= z^{s+\frac{1}{2}}((\lambda - i0)^s * \psi(\lambda))(1).\end{aligned} \tag{5-15}$$

We are interested in pointwise estimates for the kernel of $Q_i(\eta)^* H_{s,z,0}(\Delta_g) Q_j(\eta)$ and to this end we shall need the following result of [Guillarmou and Hassell 2014, Remark 4.2]. Even though the proof is almost the same as that of [Guillarmou et al. 2013b, Lemma 3.3], for completeness we provide a proof in Appendix A.

Lemma 5.5. *Let $a < b < c \le 0$ and let us write $b = \theta a + (1-\theta)c$, $0 < \theta < 1$. Then there is $C > 0$ such that, for all $f \in C_0^\infty(\mathbb{R})$, all $t \in \mathbb{R}$, and all $0 < \varepsilon \ll 1$, we have*

$$\|(\lambda \pm i\varepsilon)^{b+it} * f\|_{L^\infty_\lambda} \le C(1 + |t|)e^{\frac{3\pi|t|}{2}} \|\chi_+^a * f\|^\theta_{L^\infty_\lambda} \|\chi_+^c * f\|^{1-\theta}_{L^\infty_\lambda}. \tag{5-16}$$

We have the following result.

Proposition 5.6. *Suppose that (i, j) are such that the condition (2.iii) or (3.iii) holds in the high-energy, respectively, low-energy case. Then there is $C > 0$ such that the kernel of the operator $Q_i(\eta)^* H_{s,z,0}(\Delta_g) Q_j(\eta)$ with $z > 0$ and $z^{1/2} \in \big[\big(1-\frac{\delta}{2}\big)\eta, \big(1+\frac{\delta}{2}\big)\eta\big]$ has the following pointwise estimates,*

(i) *For $\mathrm{Re}(s) = -\frac{(n+1)}{2}$, we have*

$$|Q_i(\eta)^* H_{s,z,0}(\Delta_g) Q_j(\eta)(m, m')| \le Ce^{C|\mathrm{Im}(s)|} z^{-\frac{1}{2}} \tag{5-17}$$

for all $m, m' \in M$, uniformly in z and η.

(ii) *For* $\mathrm{Re}(s) = -\frac{(n-1)}{2}$, *we have*

$$|Q_i(\eta)^* H_{s,z,0}(\Delta_g) Q_j(\eta)(m,m')| \le C e^{C|\mathrm{Im}(s)|} d(m,m')^{-1} \tag{5-18}$$

for all $m, m' \in M$, *uniformly in* z *and* η.

Proof. Estimate (5-17) is proved in [Guillarmou and Hassell 2014, Remark 4.2]. Estimate (5-18) is proved in the same way, except for the case $n = 3$, relying on the estimates (5-5) only. Indeed, in the case $n \ge 5$ is odd, we take $a = -\frac{(n+1)}{2}$ and $c = -\frac{(n-3)}{2}$ in Lemma 5.5 and using that

$$\chi_+^{-k} = \delta_0^{(k-1)}, \quad k = 1, 2, \dots,$$

we get

$$\begin{aligned} |Q_i(\eta)^* H_{s,z,0}(\Delta_g) Q_j(\eta)(m,m')| &\le C z^{\frac{2-n}{2}} (1 + |\mathrm{Im}(s)|) e^{\frac{3\pi|\mathrm{Im}(s)|}{2}} \\ &\quad \times \left\| \partial_\lambda^{\frac{n-1}{2}} \left(\frac{\phi(\lambda)}{2\sqrt{\lambda}} Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(z^{\frac12}\lambda^{\frac12}) Q_j(\eta)(m,m') \right) \right\|_{L^\infty}^{\frac12} \\ &\quad \times \left\| \partial_\lambda^{\frac{n-5}{2}} \left(\frac{\phi(\lambda)}{2\sqrt{\lambda}} Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(z^{\frac12}\lambda^{\frac12}) Q_j(\eta)(m,m') \right) \right\|_{L^\infty}^{\frac12}, \end{aligned}$$

and therefore, using (5-5), we obtain

$$\begin{aligned} |Q_i(\eta)^* H_{s,z,0}(\Delta_g) Q_j(\eta)(m,m')| &\le C e^{C|\mathrm{Im}(s)|} z^{\frac12} (1 + z^{\frac12} d(m,m'))^{-1} \\ &\le C e^{C|\mathrm{Im}(s)|} d(m,m')^{-1}. \end{aligned} \tag{5-19}$$

For $n \ge 4$ even, taking $a = -\frac{n}{2}$, $c = -\frac{(n-2)}{2}$ in Lemma 5.5 and using (5-5), we also get (5-19). We have therefore established (5-18) for all $n \ge 4$.

When $n = 3$, using Lemma 5.5 with $a = -2$ and $c = 0$, and the fact that $\chi_+^0(\lambda) = H(\lambda)$ is the Heaviside function, we obtain

$$\begin{aligned} |Q_i(\eta)^* H_{s,z,0}(\Delta_g) Q_j(\eta)(m,m')| &\le C z^{-\frac12} (1+|\mathrm{Im}(s)|) e^{\frac{3\pi|\mathrm{Im}(s)|}{2}} \\ &\quad \times \left\| \partial_\lambda \left(\frac{\phi(\lambda)}{2\sqrt{\lambda}} Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(z^{\frac12}\lambda^{\frac12}) Q_j(\eta)(m,m') \right) \right\|_{L^\infty}^{\frac12} \\ &\quad \times \left\| H * \left(\frac{\phi(\lambda)}{2\sqrt{\lambda}} Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(z^{\frac12}\lambda^{\frac12}) Q_j(\eta)(m,m') \right) \right\|_{L^\infty}^{\frac12}. \end{aligned} \tag{5-20}$$

By (5-5), we get

$$\left\| \partial_\lambda \left(\frac{\phi(\lambda)}{2\sqrt{\lambda}} Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(z^{\frac12}\lambda^{\frac12}) Q_j(\eta)(m,m') \right) \right\|_{L^\infty} \le C z. \tag{5-21}$$

Now if we show that

$$\left\| H * \left(\frac{\phi(\lambda)}{2\sqrt{\lambda}} Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(z^{\frac12}\lambda^{\frac12}) Q_j(\eta)(m,m') \right) \right\|_{L^\infty} \le C d(m,m')^{-2}, \tag{5-22}$$

then the estimate (5-18) will follow from (5-20), (5-21) and (5-22). To prove (5-22), using (5-6), we write

$$\begin{aligned}
&H*\Big(\frac{\phi(\lambda)}{2\sqrt{\lambda}}Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(z^{\frac12}\lambda^{\frac12})Q_j(\eta)(m,m')\Big)(\lambda)\\
&\qquad=\int_0^{\lambda^{1/2}}\phi(\mu^2)Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(z^{\frac12}\mu)Q_j(\eta)(m,m')\,d\mu\\
&\qquad=\int_0^{\lambda^{1/2}}\phi(\mu^2)z\mu^2\Big[\sum_{\pm}e^{\pm iz^{1/2}\mu d(m,m')}a_\pm(z^{\frac12}\mu,m,m')+b(z^{\frac12}\mu,m,m')\Big]\,d\mu.
\end{aligned}\tag{5-23}$$

The terms involving $a_\pm$ in (5-23) can be treated similarly and in what follows we shall only consider the term involving a_+ and drop the sign $+$. To estimate this term, we integrate by parts and get

$$\begin{aligned}
&\int_0^{\lambda^{1/2}}\phi(\mu^2)z\mu^2 e^{iz^{1/2}\mu d(m,m')}a(z^{\frac12}\mu,m,m')\,d\mu\\
&\quad=\frac{1}{iz^{\frac12}d(m,m')}\Big[\phi(\mu^2)z\mu^2e^{iz^{1/2}\mu d(m,m')}a(z^{\frac12}\mu,m,m')|_{\mu=0}^{\mu=\lambda^{1/2}}\\
&\qquad\qquad-\int_0^{\lambda^{1/2}}\partial_\mu\big(\phi(\mu^2)z\mu^2a(z^{\frac12}\mu,m,m')\big)e^{iz^{1/2}\mu d(m,m')}\,d\mu\Big].
\end{aligned}\tag{5-24}$$

Estimating the terms in the left-hand side of (5-24) with the help of (5-7), we obtain that

$$\Big|\int_0^{\lambda^{1/2}}\phi(\mu^2)z\mu^2e^{iz^{\frac12}\mu d(m,m')}a(z^{\frac12}\mu,m,m')\,d\mu\Big|\le C\lambda^{\frac12}d(m,m')^{-2},\tag{5-25}$$

uniformly in z. To estimate the term involving the remainder b in (5-23), we use (5-8) with $K=2$ and get

$$\begin{aligned}
\int_0^{\lambda^{1/2}}\phi(\mu^2)z\mu^2|b(z^{\frac12}\mu,m,m')|\,d\mu&\le C\int_0^{\lambda^{1/2}}\phi(\mu^2)z\mu^2(1+z^{\frac12}\mu d(m,m'))^{-2}\,d\mu\\
&\le Cd(m,m')^{-2}.
\end{aligned}\tag{5-26}$$

Now (5-22) follows from (5-23), (5-25) and (5-26). This completes the proof of estimate (5-18). □

When proving the Schatten bound on the resolvent on the spectrum in Section 5C below, the cases (2.iii) and (3.iii) of Proposition 5.1 will be treated using the following result.

Proposition 5.7. *Suppose that (i,j) are such that the condition (2.iii) or (3.iii) holds in the high-energy, respectively low-energy case. Let $p\in[\frac{n}{2},\frac{n+1}{2}]$. Then there is $C>0$ such that, for all $z\in(0,\infty)$, $z^{1/2}\in\big[\big(1-\frac{\delta}{2}\big)\eta,\big(1+\frac{\delta}{2}\big)\eta\big]$ and all $W_1,W_2\in L^{2p}(M)$, we have $W_1Q_i(\eta)^*H_{-1,z,0}(\Delta_g)Q_j(\eta)W_2\in\mathcal{C}_q(L^2(M))$, $q=\frac{p(n-1)}{n-p}$, and*

$$\|W_1Q_i(\eta)^*H_{-1,z,0}(\Delta_g)Q_j(\eta)W_2\|_{\mathcal{C}_q}\le Cz^{-1+\frac{n}{2p}}\|W_1\|_{L^{2p}(M)}\|W_2\|_{L^{2p}(M)}.\tag{5-27}$$

Proof. First thanks to Proposition 5.6, case (i), we know that for $\operatorname{Re}s=-\frac{(n+1)}{2}$,

$$\|Q_i(\eta)^*H_{s,z,0}(\Delta_g)Q_j(\eta)\|_{L^1(M)\to L^\infty(M)}\le Ce^{C|\operatorname{Im}(s)|}z^{-\frac12}.$$

By the spectral theorem, we also know that for $\operatorname{Re} s = 0$

$$\|Q_i(\eta)^* H_{s,z,0}(\Delta_g) Q_j(\eta)\|_{L^2(M)\to L^2(M)} \le C e^{\pi|\operatorname{Im}(s)|}.$$

Hence, Proposition 2.1 implies that $W_1 Q_i(\eta)^* H_{-1,z,0}(\Delta_g) Q_j(\eta) W_2 \in \mathcal{C}_{n+1}(L^2(M))$ and, moreover,

$$\|W_1 Q_i(\eta)^* H_{-1,z,0}(\Delta_g) Q_j(\eta) W_2\|_{\mathcal{C}_{n+1}} \le C z^{-\frac{1}{n+1}} \|W_1\|_{L^{n+1}(M)} \|W_2\|_{L^{n+1}(M)}. \tag{5-28}$$

Now when $\operatorname{Re} s = -\frac{(n-1)}{2}$, thanks to Proposition 5.6(ii), the kernel of the operator $Q_i(\eta)^* H_{s,z,0}(\Delta_g) Q_j(\eta)$ has the bound (5-18), which is the same as the bound (4-6) in the proof of Proposition 4.3. Proceeding exactly as in the proof of Proposition 4.3, we get

$$\|W_1 Q_i(\eta)^* H_{-1,z,0}(\Delta_g) Q_j(\eta) W_2\|_{\mathcal{C}_{n-1}} \le C \|W_1\|_{L^n(M)} \|W_2\|_{L^n(M)}. \tag{5-29}$$

In view of (5-28) and (5-29), the bound (5-27) follows by a complex interpolation argument applied to the analytic family of operators

$$\zeta \mapsto W_1^{\frac{2}{n+1}+\zeta\frac{2}{n(n+1)}} Q_i(\eta)^* H_{-1,z,0}(\Delta_g) Q_j(\eta) W_2^{\frac{2}{n+1}+\zeta\frac{2}{n(n+1)}}$$

in the strip $0 \le \operatorname{Re}\zeta \le 1$, with $W_j \ge 0$ being simple functions such that $\|W_j\|_{L^2(M)} = 1$, $j = 1, 2$; see [Simon 2015, p. 154]. □

5C. *Resolvent estimates on the spectrum.* The final ingredient in the proof of Theorem 2 is the following result.

Proposition 5.8. *Let $\phi \in C_0^\infty\big(\big((1-\frac{\delta}{4})^2, (1+\frac{\delta}{4})^2\big)\big)$ be such that $\phi(t) = 1$ in a neighborhood of $t = 1$, where $\delta > 0$ is small, and let $p \in \big[\frac{n}{2}, \frac{n+1}{2}\big]$. Then there is $C > 0$ such that for all $z \in (0,\infty)$ and all $W_1, W_2 \in L^{2p}(M)$, for $q = \frac{p(n-1)}{n-p}$ we have $W_1\phi(\Delta_g/z)(\Delta_g - (z+i0))^{-1} W_2 \in \mathcal{C}_q(L^2(M))$ and*

$$\left\| W_1 \phi\left(\frac{\Delta_g}{z}\right)(\Delta_g - (z+i0))^{-1} W_2 \right\|_{\mathcal{C}_q} \le C z^{-1+\frac{n}{2p}} \|W_1\|_{L^{2p}(M)} \|W_2\|_{L^{2p}(M)}. \tag{5-30}$$

Proof. Let us first take the high-energy case $z \ge 1$ and let $\eta \ge 1$ be such that $\sqrt{z} \in \big[(1-\frac{\delta}{2})\eta, (1+\frac{\delta}{2})\eta\big]$. We decompose the spectrally localized outgoing resolvent $\phi(\Delta_g/z)(\Delta_g - (z+i0))^{-1}$ into microlocalized pieces

$$W_1 \phi\left(\frac{\Delta_g}{z}\right)(\Delta_g - (z+i0))^{-1} W_2 = \sum_{i,j=1}^{N_h} W_1 Q_i(\eta)^* \phi\left(\frac{\Delta_g}{z}\right)(\Delta_g - (z+i0))^{-1} Q_j(\eta) W_2.$$

The bound (5-30) will follow if we show that for all (i, j) we have

$$\left\| W_1 Q_i(\eta)^* \phi\left(\frac{\Delta_g}{z}\right)(\Delta_g - (z+i0))^{-1} Q_j(\eta) W_2 \right\|_{\mathcal{C}_q} \le C z^{-1+\frac{n}{2p}} \|W_1\|_{L^{2p}(M)} \|W_2\|_{L^{2p}(M)}. \tag{5-31}$$

To that end, the pairs (i, j) will be divided into three cases as in Proposition 5.1.

In the first case, (2.i), in view of (5-3) and Corollary B.5, we know that the Schwartz kernel of the operator $Q_i(\eta)^*\phi(\Delta_g/z)(\Delta_g - z - i0)^{-1} Q_j(\eta)$ is $\mathcal{O}(z^{-N})$ in $L^{2p'}(M \times M)$ with $\frac{1}{p'} + \frac{1}{p} = 1$. Using this together with the fact that $q \geq 2$ and Hölder's inequality, we get

$$\begin{aligned}\left\| W_1 Q_i(\eta)^* \phi\left(\frac{\Delta_g}{z}\right)(\Delta_g - (z+i0))^{-1} Q_j(\eta) W_2 \right\|_{\mathcal{C}_q} &\leq \left\| W_1 Q_i(\eta)^* \phi\left(\frac{\Delta_g}{z}\right)(\Delta_g - (z+i0))^{-1} Q_j(\eta) W_2 \right\|_{\mathcal{C}_2} \\ &\leq \mathcal{O}(z^{-N}) \|W_1\|_{L^{2p}(M)} \|W_2\|_{L^{2p}(M)}\end{aligned}$$

for any $N \in \mathbb{N}$, showing (5-31).

In the second case, (2.ii), using Stone's formula, we write

$$\begin{aligned}W_1 Q_i(\eta)^* \phi\left(\frac{\Delta_g}{z}\right)&(\Delta_g - (z+i0))^{-1} Q_j(\eta) W_2 \\ &= W_1 Q_i(\eta)^* \phi\left(\frac{\Delta_g}{z}\right)(\Delta_g - (z-i0))^{-1} Q_j(\eta) W_2 \\ &\qquad + \frac{\pi i}{\lambda} W_1 Q_i(\eta)^* dE_{\sqrt{\Delta_g}}(\lambda) Q_j(\eta) W_2, \quad \lambda = \sqrt{z}. \qquad (5\text{-}32)\end{aligned}$$

Then the estimate for the term involving the incoming resolvent in (5-32) follows exactly as in case (2.i). On the other hand, we have already proved the corresponding estimate (3-9) for the spectral measure, which leads to the estimate (5-31) in this case.

In the third case, (2.iii), we get

$$W_1 Q_i(\eta)^* \phi\left(\frac{\Delta_g}{z}\right)(\Delta_g - (z+i0))^{-1} Q_j(\eta) W_2 = W_1 Q_i(\eta)^* H_{-1,z,0}(\Delta_g) Q_j(\eta) W_2, \qquad (5\text{-}33)$$

where the operator $Q_i(\eta)^* H_{-1,z,0}(\Delta_g) Q_j(\eta)$ is defined in (5-15). The required estimate for this term therefore is a consequence of Proposition 5.7.

In the low-energy case, $0 < z \leq 1$, the argument is similar. In cases (3.i) and (3.ii) we use Corollary B.5 together with Lemma 5.4 and the bound (3-9) for the spectral measure to deduce the Schatten norm estimate. In case (3.iii), the argument is the same as for case (2.iii). This concludes the proof of the proposition. □

6. Bounds on individual eigenvalues: proof of Theorem 4

In this section we shall follow some of the arguments of [Frank 2018; Frank and Simon 2017], making some necessary changes due to the fact that we are no longer in the Euclidean setting.

Let us recall that $n = \dim(M) \geq 3$. We have the following result which is a generalization of [Frank 2018, Lemma 4.2] to the case of the Laplace operator on asymptotically conic manifolds.

Proposition 6.1. *Let $V \in L^p(M)$ with $\frac{n}{2} \leq p < \infty$. The operator $\sqrt{|V|}(\Delta_g + 1)^{-1/2}$ is compact on $L^2(M)$.*

Proof. We follow [Frank 2018, Lemma 4.2]. First we shall show that

$$\|W(\Delta_g + 1)^{-\frac{1}{2}}\|_{\mathcal{L}(L^2(M), L^2(M))} \leq C \|W\|_{L^{2p}(M)}, \quad W \in L^{2p}(M). \qquad (6\text{-}1)$$

Indeed, we have

$$(\Delta_g + 1)^{-\frac{1}{2}} : L^2(M) \to H^1(M) \qquad (6\text{-}2)$$

is bounded, and therefore, by Sobolev's embedding $H^1(M) \subset L^{2n/(n-2)}(M)$, which is valid on an asymptotically conic manifold of dimension $n \geq 3$, see [Guillarmou and Hassell 2014, Proposition 2.1], we get

$$(\Delta_g + 1)^{-\frac{1}{2}} : L^2(M) \to L^{\frac{2n}{n-2}}(M) \tag{6-3}$$

is also bounded. Using Hölder's inequality, the logarithmic convexity of L^p norms, and (6-2), (6-3), we obtain

$$\begin{aligned} \|W(\Delta_g+1)^{-\frac{1}{2}} f\|_{L^2(M)} &\leq \|W\|_{L^{2p}(M)} \|(\Delta_g+1)^{-\frac{1}{2}} f\|_{L^{2p/(p-1)}(M)} \\ &\leq \|W\|_{L^{2p}(M)} \|(\Delta_g+1)^{-\frac{1}{2}} f\|_{L^2(M)}^{1-\frac{n}{2p}} \|(\Delta_g+1)^{-\frac{1}{2}} f\|_{L^{2n/(n-2)}(M)}^{\frac{n}{2p}} \\ &\leq C\|W\|_{L^{2p}(M)} \|f\|_{L^2(M)}, \end{aligned}$$

showing (6-1).

Let $W_j \in C_0^\infty(M)$ be such that $W_j \to \sqrt{|V|}$ in $L^{2p}(M)$. By Rellich's compactness theorem, the operator $W_j(\Delta_g+1)^{-1/2}$ is compact on $L^2(M)$, and it follows from (6-1) that $W_j(\Delta_g+1)^{-1/2} \to \sqrt{|V|}(\Delta_g+1)^{-1/2}$ in $\mathcal{L}(L^2(M), L^2(M))$. □

Setting

$$\sqrt{V(x)} = \begin{cases} V(x)/\sqrt{|V(x)|}, & V(x) \neq 0, \\ 0, & V(x) = 0, \end{cases}$$

and combining Proposition 6.1 with [Frank 2018, Lemma B.1], we get that the quadratic form

$$\|(\Delta_g)^{\frac{1}{2}} u\|_{L^2(M)}^2 + (\sqrt{V} u, \sqrt{|V|} u)_{L^2(M)},$$

equipped with the domain $H^1(M)$, is closed and sectorial. Associated to the quadratic form is an m–sectorial operator with domain $\subset H^1(M)$, which we shall denote by $\Delta_g + V$. The spectrum of $\Delta_g + V$ in $\mathbb{C} \setminus [0, \infty)$ consists of isolated eigenvalues of finite algebraic multiplicity; see [Frank 2018, Proposition B.2].

Now interpolating between the estimate, valid for $z \in \mathbb{C} \setminus [0, \infty)$,

$$\|(\Delta_g - z)^{-1}\|_{L^2(M) \to L^2(M)} = \frac{1}{d(z)},$$

and the uniform estimate (1-8), with $p = \frac{2(n+1)}{n+3}$, we obtain the following result.

Corollary 6.2. *Let (M, g) be an asymptotically conic nontrapping manifold of dimension $n \geq 3$. Then for all $p \in \left[\frac{2(n+1)}{n+3}, 2\right]$ there is a constant $C > 0$ such that for all $z \in \mathbb{C} \setminus [0, \infty)$,*

$$\|(\Delta_g - z)^{-1}\|_{L^p(M) \to L^{p'}(M)} \leq C d(z)^{(n+1)(\frac{1}{p} - \frac{1}{2}) - 1} |z|^{\frac{1}{2} - \frac{1}{p}}. \tag{6-4}$$

We shall now proceed to prove Theorem 4. In doing so we shall follow [Frank and Simon 2017, Theorem 3.2]. Let $\lambda \in \mathbb{C}$ be an eigenvalue and $\psi \in H^1(M)$ be the corresponding eigenfunction of $\Delta_g + V$,

$$(\Delta_g + V)\psi = \lambda\psi.$$

(i) Let $0 < \gamma \le \frac{1}{2}$. Assume first that $\lambda \in \mathbb{C} \setminus [0, \infty)$. Let us choose $p > 1$ such that

$$\gamma + \tfrac{n}{2} = \tfrac{p}{2-p}, \tag{6-5}$$

and notice that then $\frac{2n}{n+2} < p \le \frac{2(n+1)}{n+3}$ and $\frac{2(n+1)}{n-1} \le p' < \frac{2n}{n-2}$.

By Sobolev's embedding, we have $\psi \in L^{2n/(n-2)}(M)$, and thus, $\psi \in L^r(M)$ for $r \in \left[2, \frac{2n}{n-2}\right]$, by interpolation. In particular, $\psi \in L^{p'}(M)$, and by Hölder's inequality, we get

$$\|V\psi\|_{L^p(M)} \le \|V\|_{L^{p/(2-p)}(M)} \|\psi\|_{L^{p'}(M)} = \|V\|_{L^{\gamma+n/2}(M)} \|\psi\|_{L^{p'}(M)}.$$

We have

$$\psi = (\Delta_g - \lambda)^{-1}(\Delta_g - \lambda)\psi = -(\Delta_g - \lambda)^{-1}(V\psi).$$

Hence, using (1-8), we get

$$\begin{aligned} \|\psi\|_{L^{p'}(M)} &\le \|(\Delta_g - \lambda)^{-1}\|_{L^p(M)\to L^{p'}(M)} \|V\psi\|_{L^p(\mathbb{R}^n)} \\ &\le C|\lambda|^{\frac{n}{2}(\frac{2}{p}-1)-1} \|V\|_{L^{\gamma+n/2}(M)} \|\psi\|_{L^{p'}(M)}, \end{aligned} \tag{6-6}$$

which implies (1-11) in view of

$$\tfrac{n}{2}\left(\tfrac{2}{p} - 1\right) - 1 = -\frac{\gamma}{\gamma + \frac{n}{2}}.$$

Assume now that $\lambda \in (0, \infty)$. Then for $\varepsilon > 0$, we set

$$\psi_\varepsilon = (\Delta_g - \lambda - i\varepsilon)^{-1}(\Delta_g - \lambda)\psi = f_\varepsilon(\Delta_g)\psi,$$

where

$$f_\varepsilon(t) = \frac{t-\lambda}{t-\lambda-i\varepsilon}, \quad t \in \mathbb{R}.$$

By the spectral theorem, we have

$$\|\psi_\varepsilon - \psi\|^2_{L^2(M)} = \|f_\varepsilon(\Delta_g)\psi - \psi\|^2_{L^2(M)} = \int |f_\varepsilon(t) - 1|^2 \, d(E_{\Delta_g}(t)\psi, \psi)_{L^2(M)},$$

where $dE_{\Delta_g}(t)$ is the spectral measure of Δ_g. Using the dominated convergence theorem together with the fact that $f_\varepsilon(t) \to 1$ as $\varepsilon \to 0$ for all $t \ne \lambda$, and that $E_\lambda = 0$ as λ is not an eigenvalue of Δ_g, we conclude that $\psi_\varepsilon \to \psi$ in $L^2(M)$.

On the other hand, we have

$$\psi_\varepsilon = -(\Delta_g - \lambda - i\varepsilon)^{-1}(V\psi).$$

Choosing $p > 1$ satisfying (6-5) and using (1-8), we obtain

$$\|\psi_\varepsilon\|_{L^{p'}(M)} \le C|\lambda|^{\frac{n}{2}(\frac{2}{p}-1)-1} \|V\|_{L^{\gamma+n/2}(M)} \|\psi\|_{L^{p'}(M)}, \tag{6-7}$$

i.e., ψ_ε is uniformly bounded in $L^{p'}(M)$. Passing to a subsequence, we may assume that there exists $\tilde\psi \in L^{p'}(M)$ such that $\psi_\varepsilon \to \tilde\psi$ in the weak-$*$ topology of $L^{p'}(M)$. It follows that $\psi = \tilde\psi \in L^{p'}(M)$. By the lower semicontinuity of the norm and (6-7), we get

$$\|\psi\|_{L^{p'}(M)} \le \liminf_{\varepsilon\to 0} \|\psi_\varepsilon\|_{L^{p'}(M)} \le C|\lambda|^{\frac{n}{2}(\frac{2}{p}-1)-1} \|V\|_{L^{\gamma+n/2}(M)} \|\psi\|_{L^{p'}(M)}, \tag{6-8}$$

which shows (1-11) when $\lambda \in (0, \infty)$.

(ii) Let $V \in L^{n/2}(M)$. Setting $p = \frac{2n}{n+2}$, and arguing as in the case (i) above, for $\lambda \in \mathbb{C} \setminus \{0\}$, we obtain

$$\|\psi\|_{L^{p'}(M)} \le C\|V\|_{L^{n/2}(M)}\|\psi\|_{L^{p'}(M)}.$$

The case $\lambda = 0$ is handled similarly using that

$$\|(\Delta_g - i\varepsilon)^{-1}\|_{L^p(M)\to L^{p'}(M)} \le \mathcal{O}(1),$$

in view of (1-8). The claim (ii) follows.

(iii) Let $\gamma > \frac{1}{2}$, and let $\lambda \in \mathbb{C}\setminus[0,\infty)$ be an eigenvalue of $\Delta_g + V$, and $\psi \in H^1(M)$ be the corresponding eigenfunction. Choosing $p > 1$ satisfying (6-5), we have $\frac{2(n+1)}{n+3} < p < 2$ and $2 < p' < \frac{2(n+1)}{n-1}$. Using that $\psi \in L^{p'}(M)$ and (6-4), similarly to above, we obtain

$$\begin{aligned}\|\psi\|_{L^{p'}(M)} &\le \|(\Delta_g - \lambda)^{-1}\|_{L^p(M)\to L^{p'}(M)}\|V\psi\|_{L^p(M)} \\ &\le C\delta(\lambda)^{(n+1)(\frac{1}{p}-\frac{1}{2})-1}|\lambda|^{\frac{1}{2}-\frac{1}{p}}\|V\|_{L^{\gamma+n/2}(M)}\|\psi\|_{L^{p'}(M)},\end{aligned}$$

which implies (1-12) in view of the fact that

$$\frac{1}{p} = \frac{1+\gamma+\frac{n}{2}}{2\big(\gamma+\frac{n}{2}\big)}.$$

This completes the proof of Theorem 4.

7. Bounds on sums of eigenvalues for Schrödinger operators with complex potentials

7A. ***Short-range potentials*: *proof of Theorem 5.*** Let $V \in L^p(M)$, $\frac{n}{2} \le p \le \frac{n+1}{2}$, and let $q = \frac{p(n-1)}{n-p}$. Then Theorem 2 implies that for $z \in \mathbb{C}\setminus[0,\infty)$, we have $\sqrt{V}(\Delta_g - z)^{-1}\sqrt{|V|} \in \mathcal{C}_q(L^2(M))$ and

$$\|\sqrt{V}(\Delta_g - z)^{-1}\sqrt{|V|}\|_{\mathcal{C}_q(L^2(M))} \le C|z|^{-1+\frac{n}{2p}}\|V\|_{L^p(M)}. \tag{7-1}$$

We claim that the map

$$\mathbb{C}\setminus[0,\infty) \ni z \mapsto \sqrt{V}(\Delta_g - z)^{-1}\sqrt{|V|} \tag{7-2}$$

is holomorphic with values in $\mathcal{C}_q(L^2(M))$. First let us check that (7-2) is holomorphic with values in $\mathcal{L}(L^2(M), L^2(M))$. Indeed, letting $z_0 \in \mathbb{C}\setminus[0,\infty)$, we write

$$\sqrt{V}(\Delta_g - z)^{-1}\sqrt{|V|} = \sqrt{V}\sum_{j=0}^{\infty}(z-z_0)^j(\Delta_g - z_0)^{-j-1}\sqrt{|V|} \tag{7-3}$$

and notice that

$$\begin{aligned}\|\sqrt{V}(\Delta_g - z_0)^{-j-1}\sqrt{|V|}\|_{\mathcal{L}(L^2(M),L^2(M))} &\le \|\sqrt{V}(\Delta_g - z_0)^{-1}\|_{\mathcal{L}(L^2(M),L^2(M))}, \\ \|(-\Delta - z_0)^{-1}\sqrt{|V|}\|_{\mathcal{L}(L^2(M),L^2(M))}&\|(\Delta_g - z_0)^{-1}\|^{j-1}_{\mathcal{L}(L^2(M),L^2(M))} \le C^{j+1}\end{aligned}$$

for some $C > 0$. Here we have used that the operators $\sqrt{V}(-\Delta - z_0)^{-1}$, $(\Delta_g - z_0)^{-1}\sqrt{|V|}$ are bounded on $L^2(M)$, as seen by arguing as in the proof of (6-1). This shows that the series (7-3) converges in $\mathcal{L}(L^2(M), L^2(M))$ for $|z-z_0|$ small, and therefore, the map (7-2) is holomorphic with values in $\mathcal{L}(L^2(M), L^2(M))$. In particular, if $T \in \mathcal{C}_1(L^2(M))$, i.e., of trace class, the map

$$\mathbb{C} \setminus [0,\infty) \ni z \mapsto \langle \sqrt{V}(\Delta_g - z)^{-1}\sqrt{|V|}, T\rangle \tag{7-4}$$

is holomorphic. Using the density of $\mathcal{C}_1(L^2(M))$ in $\mathcal{C}_{q'}(L^2(M))$, the bound (7-1), and Hölder's inequality in Schatten classes, we conclude that the map (7-4) is holomorphic for all $T \in \mathcal{C}_{q'}(L^2(M))$, establishing the claim.

Consider the holomorphic function

$$h(z) := \det_{\lceil q\rceil}(1 + \sqrt{V}(\Delta_g - z)^{-1}\sqrt{|V|}), \quad z \in \mathbb{C} \setminus [0,\infty),$$

where $\lceil q\rceil$ is the smallest integer $\geq q$, and $\det_{\lceil q\rceil}$ is the regularized determinant; see [Simon 1979, Chapter 9]. As explained in [Frank and Sabin 2017, proof of Theorem 16], using (7-1), we get

$$\log|h(z)| \leq C\|\sqrt{V}(\Delta_g - z)^{-1}\sqrt{|V|}\|^q_{\mathcal{C}_q} \leq C|z|^{(-1+\frac{n}{2p})q}\|V\|^q_{L^p(M)}, \tag{7-5}$$

uniformly in $z \in \mathbb{C} \setminus [0,\infty)$.

Combining Proposition 6.1 and Lemma B.1 of [Frank 2018], we conclude that the following version of the Birman–Schwinger principle holds: $z \in \mathbb{C} \setminus [0,\infty)$ is an eigenvalue of $\Delta_g + V$ if and only if

$$\operatorname{Ker}(1 + \sqrt{V}(\Delta_g - z)^{-1}\sqrt{|V|}) \neq \{0\}. \tag{7-6}$$

An application of Lemma 3.2 of [Frank 2018] gives that (7-6) is equivalent to the fact that $h(z) = 0$ and that the order of vanishing of h at z agrees with the algebraic multiplicity of z as an eigenvalue of $\Delta_g + V$.

At this point we are exactly in the same situation as in [Frank and Sabin 2017, Theorem 16]. Here we may remark that the proof of that result is based on a result of Borichev, Golinskii and Kupin [Borichev et al. 2009] concerning the distribution of zeros of a holomorphic function in the unit disc growing rapidly at a boundary point. The proof of Theorem 5 is therefore complete.

7B. *Long-range potentials*: *proof of Theorem 6*. First we have the following result: Let $\gamma \geq \frac{1}{2}$. Then there exists a constant $C > 0$ such that for all $W \in L^{2(\gamma+n/2)}(M)$ and all $z \in \mathbb{C} \setminus [0,\infty)$,

$$\|W(\Delta_g - z)^{-1}W\|_{\mathcal{C}_{2(\gamma+n/2)}} \leq C d(z)^{-1+\frac{n+1}{2(\gamma+n/2)}}|z|^{-\frac{1}{2(\gamma+n/2)}}\|W\|^2_{L^{2(\gamma+n/2)}(M)}. \tag{7-7}$$

Indeed this follows as in [Frank 2018, Proposition 2.1] by interpolation between (1-9) with $p = \frac{n+1}{2}$ and the standard bound

$$\|W(\Delta_g - z)^{-1}W\|_{L^2(M)\to L^2(M)} \leq d(z)^{-1}\|W\|^2_{L^\infty(M)}.$$

Now an application of [Frank 2018, Theorem 3.1] to the holomorphic family $K(z) = \sqrt{V}(\Delta_g - z)^{-1}\sqrt{|V|}$ completes the proof of Theorem 6 exactly in the same way as in [Frank 2018, Theorem 1.2].

Appendix A: Proof of Lemma 5.5

We shall follow the proof of Lemma 3.3 in [Guillarmou et al. 2013b] closely. Let $a < b < c \le 0$ and let $\alpha := a - c - 1 < -1$ and $\beta := b - c - 1 < -1$. We shall show the estimate (5-16) for $\|(\lambda - i\varepsilon)^{b+it} * f\|_{L^\infty_\lambda}$, as the bound (5-16) for $\|(\lambda + i\varepsilon)^{b+it} * f\|_{L^\infty_\lambda}$ can be proved similarly.

To that end, let χ^z_- be the family of distributions on $\mathbb{R}$ holomorphic in $z \in \mathbb{C}$ given by

$$\chi^z_-(\lambda) = \frac{\lambda^z_-}{\Gamma(z+1)}, \quad \operatorname{Re} z > -1,$$

where

$$\lambda^z_- = \begin{cases} 0 & \text{if } \lambda > 0, \\ |\lambda|^z & \text{if } \lambda < 0. \end{cases}$$

We have $\chi^z_-(-\lambda) = \chi^z_+(\lambda)$. Recall from [Hörmander 1990, Section 3.2] that when $\operatorname{Re} z > -1$, we have

$$(\lambda - i0)^z = \lambda^z_+ + e^{-i\pi z}\lambda^z_- \tag{A-1}$$

and from [Hörmander 1990, Example 7.1.17] that for $\varepsilon > 0$ and $z \in \mathbb{C}$, we have

$$\mathcal{F}((\lambda - i\varepsilon)^{-z})(\xi) = 2\pi e^{\frac{iz\pi}{2}} e^{\varepsilon\xi} \chi^{z-1}_-(\xi), \tag{A-2}$$

and

$$\mathcal{F}(\chi^z_+)(\xi) = e^{-i(z+1)\frac{\pi}{2}}(\xi - i0)^{-z-1}. \tag{A-3}$$

Consider the family of operators A_t for $t \in \mathbb{R}$ given by

$$A_t : C^\infty_0(\mathbb{R}) \to \mathcal{D}'(\mathbb{R}), \quad A_t f := \eta_t * f, \tag{A-4}$$

where

$$\hat{\eta}_t(\xi) = \frac{2\pi e^{i(-\beta-it)\frac{\pi}{2} - i\pi(c+1)} e^{\varepsilon\xi} \xi_-^{-\beta-1-it}}{\Gamma(-b-it)(\sigma + e^{-i(\alpha+1)\frac{\pi}{2}}(\xi - i0)^{-\alpha-1})} \tag{A-5}$$

when $c < 0$, and

$$\hat{\eta}_t(\xi) = \frac{2\pi e^{-i(b-1+it)\frac{\pi}{2}} e^{\varepsilon\xi} \xi_-^{-b-it}}{\Gamma(-b-it)(\sigma - e^{-\frac{i\pi a}{2}}(\xi - i0)^{-a})} \tag{A-6}$$

when $c = 0$, and $\sigma \in \mathbb{C}$, $|\sigma| = 1$ and $\sigma \notin \{ie^{-i\alpha\pi/2}, -ie^{i\alpha\pi/2}, e^{ia\pi/2}\}$. In view of (A-1), we see that $\hat{\eta}_t \in \mathcal{S}'(\mathbb{R})$.

We notice that for all $t \in \mathbb{R}$, $\hat{\eta}_t \in L^1_{\text{loc}}(\mathbb{R})$. Furthermore, using that

$$\left| \frac{1}{\Gamma(-b-it)} \right| \le Ce^{\pi|t|},$$

we have, for $|\xi| \ge 1$,

$$|\partial_\xi \hat{\eta}_t(\xi)| \le Ce^{\frac{3\pi|t|}{2}}(1+|t|)|\xi|^{-\beta+\alpha-1}, \tag{A-7}$$

and for $|\xi| \le 1$ we get

$$|\partial_\xi \hat{\eta}_t(\xi)| \le Ce^{\frac{3\pi|t|}{2}}(1+|t|)|\xi|^{-\beta-2}, \tag{A-8}$$

and therefore,

$$\partial_\xi \hat{\eta}_t \in L^p(\mathbb{R}) \cap L^1(\mathbb{R}, \langle\xi\rangle^\delta d\xi) \quad \text{for some } p \in (1,2),\ \delta > 0.$$

By the Hausdorff–Young inequality, we see that $u(\lambda) := \lambda\eta_t(\lambda) \in L^{p'}(\mathbb{R})$ with $p' \in (2,\infty)$ being the dual exponent to p. We also have

$$\begin{aligned} |u(\lambda)-u(\lambda')| &\le (2\pi)^{-1}\int |e^{i\xi\lambda}-e^{i\xi\lambda'}||\hat{u}(\xi)|\,d\xi \le C\int |\xi|^\delta|\lambda-\lambda'|^\delta|\hat{u}(\xi)|\,d\xi \\ &\le C|\lambda-\lambda'|^\delta\|\hat{u}\|_{L^1(\mathbb{R},\langle\xi\rangle^\delta d\xi)}, \end{aligned} \tag{A-9}$$

showing that $u = \lambda\eta_t \in C^\delta(\mathbb{R})$. Thus, by the Hölder inequality, we get

$$\int_{\mathbb{R}} |\eta_t(\lambda)|\,d\lambda \le C\left(\int_{|\lambda|>1} |\lambda\eta_t|^{p'}\,d\lambda\right)^{\frac{1}{p'}} + \|\lambda\eta_t\|_{C^\delta}\int_{|\lambda|<1}|\lambda|^{-1+\delta}\,d\lambda < \infty. \tag{A-10}$$

It follows from (A-10) combined with the Hausdorff–Young inequality, (A-7), (A-8) and (A-9) that

$$\|\eta_t\|_{L^1(\mathbb{R})} \le C(1+|t|)e^{\frac{3\pi|t|}{2}},$$

and therefore, A_t extends as a bounded operator on L^∞ with norm

$$\|A_t\|_{L^\infty(\mathbb{R})\to L^\infty(\mathbb{R})} \le C(1+|t|)e^{\frac{3\pi|t|}{2}},$$

where the constant $C > 0$ is independent of ε and t.

Next let B be the operator

$$B : C_0^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), \quad Bf := (\sigma\chi_+^c + \chi_+^a) * f,$$

which is also equal to

$$B = \mathcal{F}^{-1}\mu\mathcal{F}, \tag{A-11}$$

with

$$\mu(\xi) := \sigma e^{-i(c+1)\frac{\pi}{2}}(\xi - i0)^{-c-1} + e^{-i(a+1)\frac{\pi}{2}}(\xi - i0)^{-a-1}, \tag{A-12}$$

in view of (A-3).

If $c < 0$ then $\mu \in L^1_{\text{loc}}(\mathbb{R}) \cap C^\infty(\mathbb{R}\setminus\{0\})$. Using also the fact that the distribution $(\xi - i0)^z$ is of polynomial growth when $\operatorname{Re} z > -1$, we have $\mu\hat{f} \in L^1(\mathbb{R})$ for any $f \in C_0^\infty(\mathbb{R})$. Thus, the operator $B : C_0^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ is bounded.

Now if $c = 0$ then $Bf := \sigma H * f + \chi_+^a * f$, where H is the Heaviside function. The fact that the convolution with the Heaviside function maps C_0^∞ functions into L^∞ functions implies that the operator $B : C_0^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ is bounded also in the case $c = 0$.

Thus, the composition $A_tB : C_0^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ is bounded in all cases $c \le 0$. We claim that

$$A_tBf = (\lambda - i\varepsilon)^{b+it} * f, \quad f \in C_0^\infty(\mathbb{R}). \tag{A-13}$$

Indeed, (A-13) follows from (A-4), (A-11), and the equality

$$\hat{\eta}_t\mu = \mathcal{F}((\lambda - i\varepsilon)^{b+it})$$

obtained from (A-5), (A-6) (A-12), and (A-2). In the case $c = 0$, we also use that

$$\xi_-^{-b-it}(\xi - i0)^{-1} = \xi^{-b-1-it}, \quad b < 0.$$

We thus get for all $\varepsilon > 0$ and $t \in \mathbb{R}$

$$\|(\lambda - i\varepsilon)^{b+it} * f\|_{L^\infty} \le C(1+|t|)e^{\frac{3\pi|t|}{2}}(\|\chi_+^c * f\|_{L^\infty} + \|\chi^a * f\|_{L^\infty}). \tag{A-14}$$

Now a scaling argument as in the proof of Lemma 3.3 of [Guillarmou et al. 2013b] finishes the proof. Indeed, letting $f_\tau(\lambda) = f(\tau\lambda)$, we have

$$\chi_+^z * f_\tau(\lambda) = \tau^{-z-1}(\chi_+^z * f)(\tau\lambda), \quad (\lambda - i\varepsilon)^z * f_\tau(\lambda) = \tau^{-z-1}((\lambda - i\tau\varepsilon)^z * f)(\tau\lambda) \tag{A-15}$$

for all $\tau > 0$ and $z \in \mathbb{C}$. It follows from (A-14) and (A-14) that for each $\tau > 0$

$$\tau^{-b}\|(\lambda - i\tau\varepsilon)^{b+it} * f\|_{L^\infty} \le C(1+|t|)e^{\frac{3\pi|t|}{2}}(\tau^{-c}\|\chi_+^c * f\|_{L^\infty} + \tau^{-a}\|\chi_+^a * f\|_{L^\infty})$$

and choosing $\tau := \|\chi_+^a * f\|_{L^\infty}^{1/(a-c)}\|\chi_+^c * f\|_{L^\infty}^{-1/(a-c)}$, we obtain the desired estimate (5-16). The proof of Lemma 5.5 is complete.

Appendix B: Microlocal structure of the spectrally localized resolvent

We now analyze the microlocal structure of the spectrally localized resolvent $\phi(\Delta_g/z)(\Delta_g - (z \pm i0))^{-1}$, where $z > 0$ and $\phi \in C_0^\infty\big(\big((1-\frac{\delta}{4})^2, (1+\frac{\delta}{4})^2\big)\big)$ is such that $\phi(t) = 1$ for $t \in \big((1-\frac{\delta}{8})^2, (1+\frac{\delta}{8})^2\big)$, for $\delta > 0$ small. In doing so, we use the notation and results established in [Guillarmou et al. 2013a; 2013b; Hassell and Wunsch 2008].

Proposition B.1. *Let ϕ be as above. For all $\mu > 0$, the operator $\phi(\Delta_g/\mu^2)$ is a pseudodifferential operator in the following senses*:

(i) ***High-energy case***. *For $h = \mu^{-1} \le 2$, the operator $\phi(h^2\Delta_g)$ is a semiclassical scattering pseudodifferential operator with microsupport in $\big\{(z,\zeta) \,\big|\, |\zeta|_g \in \big((1-\frac{\delta}{4})^2, (1+\frac{\delta}{4})^2\big)\big\}$, where ζ is the semiclassically rescaled cotangent variable*; *i.e., ζ_i is the symbol of $-ih\partial_{z_i}$.*

(ii) ***Low-energy case***. *For $\mu \in (0,2)$, the operator $\phi(\Delta_g/\mu^2)$ is a pseudodifferential operator in the class $\Psi_k^0(M, \Omega_{k,b}^{1/2}) + \mathcal{A}^{\mathcal{E}}(M_{k,b}^2, \Omega_{k,b}^{1/2})$ where $\mathcal{E}$ is an index family for the boundary hypersurfaces of $M_{k,b}^2$, satisfying $\mathcal{E}_{\mathrm{bf}_0} = 0$, $\mathcal{E}_{\mathrm{zf}} = n$, $\mathcal{E}_{\mathrm{lb}_0} = \mathcal{E}_{\mathrm{rb}_0} = \frac{n}{2}$, $\mathcal{E}_{\mathrm{lb}} = \mathcal{E}_{\mathrm{rb}} = \mathcal{E}_{\mathrm{bf}} = \infty$. That is, it is the sum of a pseudodifferential operator in the class defined in [Guillarmou et al. 2013a, Section 5] and a conormal function which is smooth across the diagonal, but has nontrivial behavior at the boundary hypersurfaces* lb_0 *and* rb_0.

Proof. (i) This follows by expressing the operator $\phi(h^2\Delta_g)$ using the Helffer–Sjöstrand formula for the self-adjoint functional calculus,

$$\phi(h^2\Delta_g) = \frac{1}{2\pi i}\int_{\mathbb{C}} \bar\partial\tilde\phi(z)(h^2\Delta_g - z)^{-1}\, d\bar z \wedge dz,$$

where $\tilde\phi$ is an almost holomorphic extension of ϕ; see [Dimassi and Sjöstrand 1999, Theorem 8.1]. In terms of the notation for the spaces of semiclassical scattering pseudodifferential operators used in [Vasy and Zworski 2000], we have $\phi(h^2\Delta_g) \in \Psi_{\mathrm{sc},h}^{-\infty,0,0}(M)$.

(ii) The same argument applies to show that the operator $\phi(\Delta_g/\mu^2)$ is pseudodifferential in a neighborhood of the diagonal on the space $M^2_{k,\mathrm{sc}}$. We also need to understand the behavior of the kernel of this operator away from the diagonal. Here, we recall from [Guillarmou et al. 2013a] that the spectral measure is conormal and vanishes to order $n-1$ at zf, order $\frac{n}{2}-1$ at lb_0 and rb_0 and order -1 at bf_0 as a b-half-density on $M^2_{k,b}$, while it is Legendrian (oscillatory) at lb, rb and bf. As a result, the integral

$$\phi\Big(\frac{\Delta_g}{\mu^2}\Big) = \int \phi\Big(\frac{\lambda^2}{\mu^2}\Big) dE_{\sqrt{\Delta_g}}(\lambda)\, d\lambda \tag{B-1}$$

is conormal on $M^2_{k,b}$ and vanishes to order n at zf, order $\frac{n}{2}$ at lb_0 and rb_0, order 0 at bf_0 and order ∞ at lb, rb and bf. □

Remark B.2. The pseudodifferential nature of $\phi(h^2\Delta_g)$ can also be proved via the spectral measure using the results of [Guillarmou et al. 2013a]. Recall from this article that the spectral measure $dE_{\sqrt{\Delta_g}}(\lambda)$ for $\lambda \geq 1$ is a Legendre distribution associated to a pair of Legendre submanifolds $(L, L_2^\#)$, where L is the flowout by (left) bicharacteristic flow starting from $N^*\mathrm{Diag}_b \cap \Sigma_l$, where $N^*\mathrm{Diag}_b$ is the conormal bundle to the diagonal in M_b^2. Here Σ_l denotes the "left" characteristic variety of the operator $h^2\Delta_g - 1$, that is, the set $\{(z,\zeta,z',\zeta') \mid |\zeta|_g = 1\}$ where the semiclassical symbol of $h^2\Delta_g - 1$, acting in the left variable z, vanishes. Being a Legendre distribution, the spectral measure may be expressed (up to a trivial kernel, that is, one that is smooth and rapidly vanishing both as $h \to 0$ and as one approaches the boundary of M_b^2) as a finite sum of oscillatory integrals associated to neighborhoods of the submanifold L. The phase function for this oscillatory integral takes the form $\lambda\Phi$, where Φ is independent of λ. If we then integrate in the λ-variable as in (B-1) (with $h = \mu^{-1}$ in the high-energy case), then it is straightforward to check that the phase function $\lambda\Phi$ parametrizes the conormal bundle to the diagonal, and the result is a semiclassical scattering pseudodifferential operator of order 0.

Remark B.3. It is not hard to see that the operator $\phi(\Delta_g/\mu^2)$ is microlocally equal to the identity for $|\zeta|_g \in \big((1-\frac{\delta}{8})^2, (1+\frac{\delta}{8})^2\big)$, where ζ is the rescaled cotangent variable. First, the operator $\phi(\Delta_g/\mu^2)$ is elliptic in this region. Next, choose a function ϕ_1 supported in the interior of the region where $\phi = 1$. Then by functional calculus, $\phi_1(\Delta_g/\mu^2) = \phi(\Delta_g/\mu^2)\phi_1(\Delta/\mu^2)$, from which it follows that $\phi(\Delta_g/\mu^2)$ is microlocally equal to the identity on the elliptic set of $\phi_1(\Delta_g/\mu^2)$, which is an arbitrary subset of $\big\{(z,\zeta) \bigm| |\zeta|_g \in \big((1-\frac{\delta}{8})^2, (1+\frac{\delta}{8})^2\big)\big\}$.

We next consider the microlocal structure of the spectrally localized resolvent.

Proposition B.4. *The microlocal structure of the operator* $\phi(\Delta_g/z)(\Delta_g - (z \pm i0))^{-1}$, $z > 0$, *is as follows*:

(i) ***High-energy case.*** *Here we use semiclassical notation and we write* $z = h^{-2}$. *The operator* $\phi(h^2\Delta_g)(h^2\Delta_g - (1 \pm i0))^{-1}$, *acting on half-densities, lies in the same microlocal space as the semiclassical resolvent* (as detailed in [Hassell and Wunsch 2008, Theorem 1.1]), *indeed in a "better" space as the differential order is* $-\infty$ *rather than* -2. *That is, the spectrally localized resolvent is a sum of three terms* $S_1 + S_2 + S_3$, *where*

- S_1 *is a semiclassical pseudodifferential operator of differential order* $-\infty$ *and semiclassical order* 0,

- *S_2 is an intersecting Legendre distribution associated to the conormal bundle $N^*\mathrm{Diag}_b$ and to the propagating Legendrian L, and*
- *S_3 is a conic Legendre pair associated to L and to the outgoing Legendrian $L_2^\#$.*

Moreover, $S_2 + S_3$ are microlocally identical to the full resolvent in a neighborhood of the characteristic variety Σ_l of $h^2\Delta_g - 1$.

(ii) ***Low-energy case.*** *Let $z \in (0, 2)$. The operator $\phi(\Delta_g/z)(\Delta_g - (z \pm i0))^{-1}$, acting on half-densities, lies in the same microlocal space as the resolvent* (as detailed in [Guillarmou et al. 2013a, Theorem 3.9]), *indeed in a better space as the differential order is $-\infty$ rather than -2. In detail, the operator $\phi(\Delta_g/z)(\Delta_g - (z \pm i0))^{-1}$ can be decomposed as $S_1 + S_2 + S_3 + S_4$ (with $\sqrt{z}$ playing the role of the spectral parameter on $M^2_{k,b}$), where*

- *$S_1 \in \Psi^{-\infty}(M, \Omega^{1/2}_{k,b})$ is a pseudodifferential operator of order $-\infty$ in the calculus of operators defined in [Guillarmou et al. 2013b],*
- *$S_2 \in I^{-1/2,\mathcal{B}}(M^2_{k,b}, ({}^{\mathrm{sc}}N^*_{\mathrm{Diag}_b}, L^{\mathrm{bf}}_+); \Omega^{1/2}_{k,b})$ is an intersecting Legendre distribution on $M^2_{k,b}$, microsupported close to ${}^{\mathrm{sc}}N^*_{\mathrm{Diag}_b}$,*
- *$S_3 \in I^{-1/2,(n-2)/2;(n-1)/2,(n-1)/2;\mathcal{B}}(M^2_{k,b}, (L^{\mathrm{bf}}_+, L^\#_+); \Omega^{1/2}_{k,b})$ is a Legendre distribution on $M^2_{k,b}$ associated to the intersecting pair of Legendre submanifolds with conic points $(L^{\mathrm{bf}}_+, L^\#_+)$, microsupported away from ${}^{\mathrm{sc}}N^*_{\mathrm{Diag}_b}$,*
- *S_4 is supported away from* bf *and is such that $e^{\pm i\lambda r}e^{\pm i\lambda r'}R_4$ is polyhomogeneous conormal on $M^2_{k,b}$.*

Here $\mathcal{B} = (\mathcal{B}_{\mathrm{bf}_0}, \mathcal{B}_{\mathrm{lb}_0}, \mathcal{B}_{\mathrm{rb}_0}, \mathcal{B}_{\mathrm{zf}})$ is an index family with minimal exponents (i.e., order of vanishing) $\min \mathcal{B}_{\mathrm{bf}_0} = -2,\ \min \mathcal{B}_{\mathrm{lb}_0} = \min \mathcal{B}_{\mathrm{rb}_0} = \frac{n}{2} - 2,\ \min \mathcal{B}_{\mathrm{zf}} = 0$. *In addition S_4 vanishes to order ∞ at* lb *and* bf *and to order $\frac{n-1}{2}$ at* rb.

Corollary B.5. *The estimates* (5-3), (5-4), (5-9) *and* (5-10) *hold if the resolvent $(\Delta_g - (z \pm i0))^{-1}$ is replaced by the spectrally localized resolvent $\phi(\Delta_g/z)(\Delta_g - (z \pm i0))^{-1}$.*

Proof of Corollary B.5. The proofs of these estimates only used the location of the wavefront set of the resolvent kernel, together with the vanishing orders of the resolvent on the boundary hypersurfaces of $M^2_{k,b}$ at $z = 0$. In view of Proposition B.4, the same proof applies verbatim to the spectrally localized resolvent. □

Proof of Proposition B.4. (i) We study the composition of the operator $\phi(h^2\Delta_g)$ with the incoming or outgoing resolvent, $(h^2\Delta_g - (1 \pm i0))^{-1}$. We know from [Hassell and Wunsch 2008, Theorem 1.1] that the actual resolvent can be decomposed into a sum of three terms $R_1 + R_2 + R_3$ as in the proposition (except that R_1 will have differential order -2). We may assume that R_2 and R_3 are microsupported in the region where $|\zeta|_g \in \big((1 - \frac{\delta}{8})^2, (1 + \frac{\delta}{8})^2\big)$, and R_1 is microsupported in the region where $|\zeta|_g \notin \big((1 - \frac{\delta}{16})^2, (1 + \frac{\delta}{16})^2\big)$. The composition $S_1 := \phi(h^2\Delta_g)R_1$ is another semiclassical pseudodifferential operator, of semiclassical order 0 and differential order $-\infty$. On the other hand, the operator $\phi(h^2\Delta_g)$ is microlocally equal to the identity on the microsupport of R_2 and R_3, so using [Guillarmou et al. 2013b, Section 7], we find that the composition of $\phi(h^2\Delta_g)$ with $R_2 + R_3$ is equal to $R_2 + R_3$ up to an operator that is residual in all senses, that is, a smooth kernel that vanishes rapidly as $h \to 0$ or upon approach to the boundary of M_b^2. So we can take $S_2 = R_2$ and $S_3 = R_3$ up to a residual kernel.

(ii) Similarly, in the low-energy case the actual resolvent has a decomposition into $R_1 + R_2 + R_3 + R_4$ having properties as in the proposition (with R_1 of differential order -2). We also need to decompose the operator $\phi(\Delta_g/z) = B_1 + B_2$ into two parts, where B_1 is supported close to the diagonal on the space $M^2_{k,b}$, and B_2 has empty wavefront set. This second piece B_2 can be taken to vanish to infinite order at bf, lb and rb, and to be polyhomogeneous conormal to $\mathrm{bf}_0, \mathrm{lb}_0, \mathrm{rb}_0$ and zf vanishing to order 0 at bf_0, order $\frac{n}{2}$ at lb_0 and rb_0 and order n at zf. When we apply B_1 to the resolvent, the argument is just as in the high-energy case, using [Guillarmou et al. 2013b, Section 5] instead of Section 7 of that work.

To understand what happens when we apply B_2 to the resolvent, we view the composition of operators as the pushforward of the product of the Schwartz kernels on a "triple space" $M^3_{k,b}$ down to $M^2_{k,b}$, as was done in the appendix of [Guillarmou and Hassell 2008]. As a multiple of a nonvanishing b-half-density on $M^2_{k,b}$ we find that B_2 (multiplied by $\left|\frac{dk}{k}\right|^{1/2}$, $k = \sqrt{z}$, which is a purely formal factor) is polyhomogeneous conormal, with no log terms at leading order, and vanishes to order n at zf, 0 at bf_0 and $\frac{n}{2}$ at lb_0 and rb_0. On the other hand, we can decompose the resolvent kernel as the sum of $R_1 + R_2$, supported near the diagonal, and $R_3 + R_4$, which is microsupported in the set where $|\zeta|_g \in \left(\left(1-\frac{\delta}{8}\right)^2, \left(1+\frac{\delta}{8}\right)^2\right)$, where ζ is the cotangent variable rescaled by a factor $\sqrt{z}$.

The composition of B_2 with R_1+R_2 can be treated by lifting both kernels to the space $M^3_{k,b}$ and pushing forward. Since B_2 has no wavefront set, the composition has no wavefront set, so it is polyhomogeneous conormal, and the order of vanishing can be read off as n at zf, $\frac{n}{2}$ at lb_0, $\frac{n}{2}-2$ at rb_0, -2 at bf_0, and ∞ at lb, rb and bf. This lies in a better space than claimed in the proposition.

The composition of B_2 with $R_3 + R_4$ can also be analyzed by lifting both kernels to $M^3_{k,b}$ and then pushing forward. Although $R_3 + R_4$ is not polyhomogeneous conormal at the boundary hypersurfaces bf, lb and rb, when lifted to $M^3_{k,b}$ and multiplied by the lift of B_2, the rapid vanishing of B_2 at bf and rb means that the product of the two kernels is rapidly decreasing as the "middle variable" (the right variable of B_2 and the left variable of $R_3 + R_4$) tends to the boundary. As for the right variable of $R_3 + R_4$, after multiplying the kernel of $R_3 + R_4$ by $e^{\mp i\lambda r'}$ (where $r' = \frac{1}{x'}$ is the right radial variable) it becomes polyhomogeneous conormal also at rb. So the product of the kernels B_2 (in the left and middle variables) and $(R_3 + R_4)e^{\mp i\lambda r'}$ (in the middle and right variables) on $M^3_{k,b}$ *is* polyhomogeneous conormal. After pushing forward to $M^2_{k,b}$ a calculation similar to that done in [Guillarmou and Hassell 2008, Appendix] shows that the result is $e^{\mp i\lambda r'}$ times a polyhomogeneous kernel which vanishes to order $n-2$ at zf, -2 at bf_0, $\min\left(\frac{n}{2}, n-2\right)$ at lb_0, $\frac{n}{2}-2$ at rb_0, $\frac{n-1}{2}$ at rb and ∞ at lb and bf, with no log terms to leading order except possibly at lb_0 in the case $n = 4$. Again this is in a better space than is claimed in the proposition. □

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A PROOF OF THE INSTABILITY OF ADS FOR THE EINSTEIN-NULL DUST SYSTEM WITH AN INNER MIRROR

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In 2006, Dafermos and Holzegel formulated the so-called AdS instability conjecture, stating that there exist *arbitrarily small* perturbations to AdS initial data which, under evolution by the Einstein vacuum equations for $\Lambda < 0$ with reflecting boundary conditions on conformal infinity $\mathcal{I}$, lead to the formation of black holes. The numerical study of this conjecture in the simpler setting of the spherically symmetric Einstein-scalar field system was initiated by Bizón and Rostworowski (*Phys. Rev. Lett.* **107**:3 (2011), art. id. 031102), followed by a vast number of numerical and heuristic works by several authors.

In this paper, we provide the first rigorous proof of the AdS instability conjecture in the simplest possible setting, namely for the spherically symmetric Einstein-massless Vlasov system, in the case when the Vlasov field is moreover supported only on radial geodesics. This system is equivalent to the Einstein-null dust system, allowing for both ingoing and outgoing dust. In order to overcome the breakdown of this system occurring once the null dust reaches the center $r = 0$, we place an inner mirror at $r = r_0 > 0$ and study the evolution of this system on the exterior domain $\{r \geq r_0\}$. The structure of the maximal development and the Cauchy stability properties of general initial data in this setting are studied in our companion paper (2017, arXiv: 1704.08685).

The statement of the main theorem is as follows: We construct a family of mirror radii $r_{0\varepsilon} > 0$ and initial data $\mathcal{S}_\varepsilon$, $\varepsilon \in (0, 1]$, converging, as $\varepsilon \to 0$, to the AdS initial data $\mathcal{S}_0$ in a suitable norm, such that, for any $\varepsilon \in (0, 1]$, the maximal development $(\mathcal{M}_\varepsilon, g_\varepsilon)$ of $\mathcal{S}_\varepsilon$ contains a black hole region. Our proof is based on purely physical space arguments and involves the arrangement of the null dust into a large number of beams which are successively reflected off $\{r = r_{0\varepsilon}\}$ and $\mathcal{I}$, in a configuration that forces the energy of a certain beam to increase after each successive pair of reflections. As $\varepsilon \to 0$, the number of reflections before a black hole is formed necessarily goes to $+\infty$. We expect that this instability mechanism can be applied to the case of more general matter fields.

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1. Introduction

Anti-de Sitter spacetime $(\mathcal{M}^{n+1}_{\mathrm{AdS}}, g_{\mathrm{AdS}})$, $n \geq 3$, is the simplest solution of the *Einstein vacuum equations*

$$\mathrm{Ric}_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{1-1}$$

with a negative cosmological constant Λ. In the standard polar coordinate chart on $\mathcal{M}_{\mathrm{AdS}}$, the AdS metric takes the form

$$g_{\mathrm{AdS}} = -\Big(1 - \frac{2}{n(n-1)}\Lambda r^2\Big) dt^2 + \Big(1 - \frac{2}{n(n-1)}\Lambda r^2\Big)^{-1} dr^2 + r^2 g_{\mathbb{S}^{n-1}}, \tag{1-2}$$

where $g_{\mathbb{S}^{n-1}}$ is the round metric on the $(n-1)$-dimensional sphere; see [Hawking and Ellis 1973].

Despite being geodesically complete, $(\mathcal{M}_{\mathrm{AdS}}, g_{\mathrm{AdS}})$ fails to be globally hyperbolic. In particular, it can be conformally identified with the interior of $(\mathbb{R} \times \mathbb{S}^n_+, g_E)$, where $\mathbb{S}^n_+$ is the closed upper hemisphere of $\mathbb{S}^n$ and g_E is the metric

$$g_E = -d\bar{t}^2 + \Big(-\frac{n(n-1)}{2\Lambda}\Big) g_{\mathbb{S}^n}. \tag{1-3}$$

Through this identification, the time-like boundary

$$\mathcal{I}^n = \mathbb{R} \times \partial\mathbb{S}^n_+ \simeq \mathbb{R} \times \mathbb{S}^{n-1} \tag{1-4}$$

of $(\mathbb{R}\times\mathbb{S}^n_+, g_E)$ is naturally attached to $(\mathcal{M}_{\mathrm{AdS}}, g_{\mathrm{AdS}})$ as a "conformal boundary at infinity"; see [Hawking and Ellis 1973].

Maldacena [1998], Gubser, Klebanov, and Polyakov [Gubser et al. 1998] and Witten [1998] proposed the *AdS/CFT conjecture*, suggesting a correspondence between certain conformal field theories defined on $\mathcal{I}^n$ (in the strongly coupled regime) and supergravity on spacetimes asymptotically of the form $(\mathcal{M}^{n+1}_{\mathrm{AdS}} \times S^k, g_{\mathrm{AdS}} + g_{S^k})$, where (S^k, g_{S^k}) is a suitable compact Riemannian manifold of dimension k. Following the introduction of this conjecture, asymptotically AdS spacetimes (i.e., spacetimes $(\mathcal{M}, g)$ with an asymptotic region with geometry resembling that of $(\mathcal{M}_{\mathrm{AdS}}, g_{\mathrm{AdS}})$ in the vicinity of $\mathcal{I}$) became a subject of intense study in the high-energy physics literature; see, e.g., [Aharony et al. 2000; Hartnoll 2009; Ammon and Erdmenger 2015].

The correct setting for the study of the dynamics of asymptotically AdS solutions $(\mathcal{M}, g)$ to (1-1) is that of an *initial value problem* with appropriate *boundary conditions* prescribed asymptotically on $\mathcal{I}$. The issue of the right boundary conditions on $\mathcal{I}$ leading to well-posedness for the resulting initial-boundary value problem for (1-1) was first addressed by Friedrich [1995]. Well-posedness for more general boundary conditions and matter fields in the spherically symmetric case was obtained in [Holzegel and Smulevici 2012; Holzegel and Warnick 2015]; see also [Holzegel et al. 2020; Friedrich 2014]. In general, most physically interesting boundary conditions on $\mathcal{I}$ leading to a well-posed initial-boundary value problem can be classified as either *reflecting* (for which an appropriate "energy flux" for g through $\mathcal{I}$ vanishes) or *dissipative* (allowing for a nonvanishing outgoing "energy flux" for g through $\mathcal{I}$), with substantially different global dynamics associated to each case; see the discussion in [Holzegel et al. 2020].

Dafermos and Holzegel [2006a], see also [Dafermos 2006], suggested the following conjecture:

AdS instability conjecture. *There exist arbitrarily small perturbations to the initial data of* $(\mathcal{M}_{\mathrm{AdS}}, g_{\mathrm{AdS}})$ *for the vacuum Einstein equations* (1-1) *with a reflecting boundary condition on* $\mathcal{I}$ *which lead to the development of trapped surfaces and, thus, black hole regions. In particular,* $(\mathcal{M}_{\mathrm{AdS}}, g_{\mathrm{AdS}})$ *is nonlinearly unstable.*

This conjecture was motivated in [Dafermos and Holzegel 2006a] by the study of asymptotically AdS solutions to (1-1) with biaxial Bianchi IX symmetry in $4+1$ dimensions, a symmetry class in which the vacuum Einstein equations (1-1) reduce to a $1+1$ hyperbolic system with nontrivial dynamics. This model was introduced in [Bizoń et al. 2005]. In this setting, it was observed in [Dafermos and Holzegel 2006a] that perturbations of the initial data of $(\mathcal{M}_{\mathrm{AdS}}, g_{\mathrm{AdS}})$ (which, if not trivial, necessarily have strictly positive ADM mass M_{ADM}, in view of [Gibbons et al. 1983]) cannot settle down to a horizonless static spacetime, since M_{ADM} is conserved along $\mathcal{I}$ under reflecting boundary conditions and no static asymptotically AdS solution of (1-1) with $M_{\mathrm{ADM}} > 0$ exists (according to [Boucher et al. 1984]). This picture was supported by results of [Anderson 2006].

The following remarks should be made regarding the statement of the AdS instability conjecture:

- The perturbations referred to in the conjecture are assumed to be small with respect to a norm for which (1-1) is well-posed and $(\mathcal{M}_{\mathrm{AdS}}, g_{\mathrm{AdS}})$ is *Cauchy stable* as a solution to (1-1) (otherwise, the conjecture is trivial).[1] For such perturbations, Cauchy stability implies that the "time" elapsed before the formation of a trapped surface tends to $+\infty$ as the size of the initial perturbation shrinks to 0.

- The AdS instability conjecture stands in contrast to the nonlinear stability of Minkowski space $(\mathbb{R}^{3+1}, \eta)$, in the case $\Lambda = 0$ (see [Christodoulou and Klainerman 1993]), or de Sitter space $(\mathcal{M}_{\mathrm{dS}}, g_{\mathrm{dS}})$, in the case $\Lambda > 0$ (see [Friedrich 1986]). The proof of the nonlinear stability of $(\mathbb{R}^{3+1}, \eta)$ and $(\mathcal{M}_{\mathrm{dS}}, g_{\mathrm{dS}})$ is based on a stability mechanism related to the fact that linear fields on those spacetimes satisfy sufficiently strong decay rates. The decay rates are, however, borderline in the case $\Lambda = 0$, and thus the stability of $(\mathbb{R}^{3+1}, \eta)$ is a deep fact depending on the precise nonlinear structure of the system (1-1), whereas, in the case $\Lambda > 0$, the decay is exponential and stability can be inferred relatively easily. In contrast, on $(\mathcal{M}_{\mathrm{AdS}}, g_{\mathrm{AdS}})$, it can be shown that linear fields satisfying a reflecting boundary condition on $\mathcal{I}$ remain bounded, but do *not* decay in time. It is precisely the lack of a sufficiently fast decay rate at the linear level which is associated to the possibility of nonlinear instability.

- The prescription of a reflecting boundary condition on $\mathcal{I}$ is essential for the conjecture: for maximally dissipative boundary conditions, it is expected that $(\mathcal{M}_{\mathrm{AdS}}, g_{\mathrm{AdS}})$ is nonlinearly *stable*, in view of the quantitative decay rates obtained for the linearized vacuum Einstein equations (and other linear fields) around $(\mathcal{M}_{\mathrm{AdS}}, g_{\mathrm{AdS}})$ by Holzegel, Luk, Smulevici, and Warnick [Holzegel et al. 2020].

- In the biaxial Bianchi IX symmetry class, all perturbations of $(\mathcal{M}_{\mathrm{AdS}}, g_{\mathrm{AdS}})$ leading to the formation of a trapped surface can be shown to possess a complete conformal infinity $\mathcal{I}$ and are expected to settle down to a member of the Schwarzschild-AdS family; see [Dafermos and Holzegel 2006a; 2006b]. However,

[1] Here, Cauchy stability of $(\mathcal{M}_{\mathrm{AdS}}, g_{\mathrm{AdS}})$ refers to Cauchy stability of the conformal compactification of $(\mathcal{M}_{\mathrm{AdS}}, g_{\mathrm{AdS}})$ (including, therefore, the time-like boundary $\mathcal{I}$); see the discussion in the next section.

in the absence of any symmetry, the picture regarding the end state of the evolution of general vacuum perturbations of $(\mathcal{M}_{\mathrm{AdS}}, g_{\mathrm{AdS}})$ is complicated; see the discussion in the next section.

Starting from the pioneering work [Bizoń and Rostworowski 2011], a plethora of numerical and heuristic results have been obtained in the direction of establishing the AdS instability conjecture, mainly in the context of the spherically symmetric Einstein-scalar field system. See the discussion in Section 1A.

In this paper, we will prove the AdS instability conjecture in the simplest possible setting, namely for the Einstein-massless Vlasov system in spherical symmetry, further reduced to the case when the Vlasov field f is supported only on radial geodesics. We will call this system the *spherically symmetric Einstein-radial massless Vlasov system*. In fact, this is a singular reduction; the resulting system is equivalent to the *spherically symmetric Einstein-null dust system*, allowing for both ingoing and outgoing dust. This system has been studied in the $\Lambda = 0$ case in [Poisson and Israel 1990].

A serious problem with the spherically symmetric Einstein-null dust system is that it suffers from a severe breakdown when the null dust reaches the center $r = 0$. In particular, in any reasonable initial-data topology, the spherically symmetric Einstein-null dust system is *not* well-posed and $(\mathcal{M}_{\mathrm{AdS}}, g_{\mathrm{AdS}})$ is *not* a Cauchy stable solution of it. One way to restore the well-posedness of this system (a necessary step for the study of the AdS instability conjecture in this setting) is to place an *inner mirror* at some radius sphere $\{r = r_0\}$ with $r_0 > 0$ and study the evolution of the system in the exterior region $\{r \geq r_0\}$. However, fixing the mirror radius r_0 results in a trivial global stability statement for $(\mathcal{M}_{\mathrm{AdS}}, g_{\mathrm{AdS}})$, as initial data perturbations with total ADM mass $\tilde{m}_{\mathrm{ADM}} < \frac{1}{2} r_0$ cannot form a black hole. Thus, it is necessary to allow the radius r_0 to shrink to 0 as the total ADM mass of the initial data shrinks to 0, in order to address the AdS instability conjecture in this setting. See the discussion in Section 1B.

A nontechnical statement of our result is the following:

Theorem 1 (rough version). *The AdS spacetime $(\mathcal{M}^{3+1}_{\mathrm{AdS}}, g_{\mathrm{AdS}})$ is nonlinearly unstable under evolution by the spherically symmetric Einstein-radial massless Vlasov system with a reflecting boundary condition on $\mathcal{I}$ and an inner mirror*, *in the following sense.*

There exists a one-parameter family of spherically symmetric initial data $\mathcal{S}_\varepsilon$, $\varepsilon \in (0, 1]$, and a family of inner mirror radii $r = r_{0\varepsilon}$ (with $r_{0\varepsilon} \xrightarrow{\varepsilon \to 0} 0$) satisfying the following properties:

(1) *As $\varepsilon \to 0$, $\|\mathcal{S}_\varepsilon\|_{\mathcal{CS}} \to 0$; i.e.*, *the $\mathcal{S}_\varepsilon$ converge to the initial data $\mathcal{S}_0$ of $(\mathcal{M}_{\mathrm{AdS}}, g_{\mathrm{AdS}})$.*

(2) *For any $\varepsilon > 0$, the maximal future development $(\mathcal{M}_\varepsilon, g_\varepsilon)$ of $\mathcal{S}_\varepsilon$ contains a trapped surface and*, *thus*, *a black hole region. Moreover*, *$(\mathcal{M}_\varepsilon, g_\varepsilon)$ possesses a complete conformal infinity $\mathcal{I}$.*

The norm $\|\cdot\|_{\mathcal{CS}}$ in 1 *measures the concentration of the energy of $\mathcal{S}_\varepsilon$ in annuli of width $\sim r_{0\varepsilon}$ and has the property that the radial Einstein-massless Vlasov system is well-posed and $(\mathcal{M}_{\mathrm{AdS}}, g_{\mathrm{AdS}})$ is Cauchy stable with respect to $\|\cdot\|_{\mathcal{CS}}$ independently of the precise value of $r_{0\varepsilon}$. See Figure 1.*

For progressively more detailed statements of Theorem 1, see Sections 1C and 4. For further discussion on the need of an inner mirror at $r \sim r_{0\varepsilon}$ and its relation to natural dispersive mechanisms appearing in other matter models, see Section 1B.

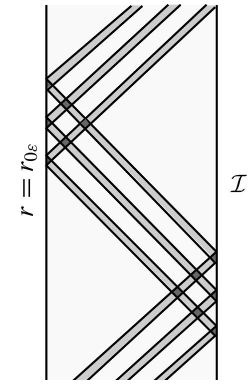

Figure 1. The family of initial data $\mathcal{S}_\varepsilon$ that we construct for the proof of Theorem 1 gives rise to a large number of Vlasov beams, which are successively reflected off $\mathcal{I}$ and the inner mirror at $r = r_{0\varepsilon}$. The Cauchy stability statement for $\|\cdot\|_{\mathcal{CS}}$ implies that the number of reflections necessarily goes to $+\infty$ as $\varepsilon \to 0$ (see the remark after the statement of the AdS instability conjecture).

We should also note the following:

- Except for the condition $r_{0\varepsilon} < 2(M_{\mathrm{ADM}})_\varepsilon$ referred to earlier, where $(M_{\mathrm{ADM}})_\varepsilon$ is the ADM mass of $\mathcal{S}_\varepsilon$, there is considerable flexibility in the choice of the mirror radii $r_{0\varepsilon}$ in the statement of Theorem 1 and this can be exploited to one's advantage. For simplicity, we choose $r_{0\varepsilon}$ to satisfy $r_{0\varepsilon} \sim (M_{\mathrm{ADM}})_\varepsilon$ (see also the discussion in Section 1C).

- While we do not address the issue of the end state of the evolution of $\mathcal{S}_\varepsilon$, it can be easily inferred from our proof of Theorem 1 that the spacetimes $(\mathcal{M}_\varepsilon, g_\varepsilon)$ settle down to a member of the Schwarzschild-AdS family; see also [Moschidis 2017].

The trivial instability at $r = 0$ occurring for the spherically symmetric Einstein-null dust system is absent in the case of smooth solutions to the general spherically symmetric Einstein-massless Vlasov system (not reduced to the radial case). In particular, the smooth initial value problem for the spherically symmetric Einstein-massless Vlasov system is well-posed, and placing an inner mirror at $r = r_0 > 0$ is not necessary.[2] For a proof of the AdS instability in this setting, see our forthcoming [Moschidis 2018].

1A. ***Earlier numerical and heuristic works.*** Restricted under spherical symmetry, all solutions to the Einstein vacuum equations (1-1) are locally isometric to a member of the Schwarzschild-AdS family; see [Eiesland 1925]. Thus, any attempt to search for unstable vacuum perturbations of $(\mathcal{M}_{\mathrm{AdS}}, g_{\mathrm{AdS}})$ for (1-1) in $3+1$ dimensions cannot be reduced to a problem for a $1+1$ hyperbolic system (where the wide variety of available tools would make the problem more tractable).[3] For this reason, instead of (1-1), numerical

[2] In fact, well-posedness for the smooth initial value problem for the Einstein–Vlasov system also holds outside spherical symmetry; see [Choquet-Bruhat 1971]. In the case $\Lambda = 0$, the stability of Minkowski spacetime for the Einstein-massless Vlasov system without any symmetry assumptions was recently established in [Taylor 2017].

[3] This problem is circumvented in $4+1$ dimensions by the biaxial Bianchi IX symmetry class referred to earlier; see [Bizoń et al. 2005].

and heuristic works on the AdS instability have so far mainly focused on the *Einstein-scalar field system*

$$\begin{cases} \mathrm{Ric}_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}[\varphi], \\ \Box_g \varphi = 0, \\ T_{\mu\nu}[\varphi] \doteq \partial_\mu \varphi \, \partial_\nu \varphi - \frac{1}{2} g_{\mu\nu} \partial^\alpha \varphi \, \partial_\alpha \varphi. \end{cases} \tag{1-5}$$

The system (1-5), whose mathematical study in the case $\Lambda = 0$ was pioneered in [Christodoulou 1999], admits nontrivial dynamics in spherical symmetry, and spherically symmetric solutions to (1-5) share many qualitative properties with general solutions of (1-1). Reduced under spherical symmetry in a double null gauge (u, v) in $3+1$ dimensions, i.e., a gauge where

$$g = -\Omega^2 \, du \, dv + r^2 g_{\mathbb{S}^2}, \tag{1-6}$$

the system (1-5) takes the form

$$\begin{cases} \partial_u \partial_v (r^2) = -\frac{1}{2}(1 - \Lambda r^2)\Omega^2, \\ \partial_u \partial_v \log(\Omega^2) = \dfrac{\Omega^2}{2r^2}(1 + 4\Omega^{-2} \, \partial_u r \, \partial_v r) - 8\pi \, \partial_u \varphi \, \partial_v \varphi, \\ \partial_v (\Omega^{-2} \, \partial_v r) = -4\pi r \Omega^{-2} (\partial_v \varphi)^2, \\ \partial_u (\Omega^{-2} \, \partial_u r) = -4\pi r \Omega^{-2} (\partial_u \varphi)^2, \\ \partial_u \partial_v (r\varphi) = -\dfrac{\Omega^2 - 4 \, \partial_u r \, \partial_v r}{4r^2} \cdot r\varphi. \end{cases} \tag{1-7}$$

The well-posedness of the asymptotically AdS initial-boundary value problem for the system (1-7) with reflecting boundary conditions on $\mathcal{I}$ was established by [Holzegel and Smulevici 2012].

Numerical results in the direction of establishing the AdS instability conjecture were first obtained by Bizoń and Rostworowski [2011], who studied the evolution of spherically symmetric perturbations of $(\mathcal{M}_{\mathrm{AdS}}, g_{\mathrm{AdS}})$ for (1-5) in Schwarzschild-type coordinates. More precisely, [Bizoń and Rostworowski 2011] numerically simulated the evolution of initial data for (1-5) with φ initially arranged into small amplitude wave packets. It was found that, for certain families of initial arrangements of this form (of "size" ε), after a finite number of reflections on $\mathcal{I}$ (proportional to ε^{-2}), the energy of the wave packets becomes substantially concentrated, leading to a breakdown of the coordinate system associated with the threshold of trapped surface formation.

Following [Bizoń and Rostworowski 2011], a vast amount of numerical and heuristic works have been dedicated to the understanding of the global dynamics of perturbations of $(\mathcal{M}_{\mathrm{AdS}}, g_{\mathrm{AdS}})$ for the system (1-5); see, e.g., [Dias et al. 2012a; 2012b; Buchel et al. 2012; Maliborski and Rostworowski 2013; Balasubramanian et al. 2014; Craps et al. 2014; 2015; Bizoń et al. 2015; Dimitrakopoulos et al. 2015; 2016; 2018; Green et al. 2015; Horowitz and Santos 2015; Dimitrakopoulos and Yang 2015]. In these works, the picture that arises regarding the long-time dynamics of *generic* spherically symmetric perturbations is rather complicated: Apart from perturbations that lead to instability and trapped surface formation [Dias et al. 2012b; Buchel et al. 2012], it appears that there exist certain types of perturbations (dubbed "islands of stability") which remain close to $(\mathcal{M}_{\mathrm{AdS}}, g_{\mathrm{AdS}})$ for long times; see [Dias et al. 2012a; Maliborski and Rostworowski 2013; Balasubramanian et al. 2014; Dimitrakopoulos and Yang 2015]. Perturbations of the latter type might in fact occupy an open set in the moduli space of spherically symmetric initial data for (1-5); see [Balasubramanian et al. 2014; Dimitrakopoulos and Yang 2015]. The question of existence of open "corners" of initial data around $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$ leading to trapped surface formation has also been studied; see, e.g., [Dimitrakopoulos et al. 2015].

Another interesting problem in this context is the characterization of the possible end states of the evolution of unstable perturbations of $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$. Holzegel and Smulevici [2013b] established that the Schwarzschild-AdS spacetime $(\mathcal{M}_{\text{Sch}}, g_{\text{Sch}})$ is an asymptotically stable solution of the system (1-5) in spherical symmetry, with perturbations decaying at an exponential rate.[4] This result supports the expectation that all spherically symmetric perturbations of $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$ for the system (1-5) leading to the formation of a trapped surface eventually settle down to a member of the Schwarzschild-AdS family; see [Dafermos and Holzegel 2006a; 2006b]. However, beyond spherical symmetry, Holzegel and Smulevici [2013a; 2014] showed that solutions to the linear scalar wave equation

$$\Box_{g_{\text{Sch}}}\varphi = 0 \tag{1-8}$$

on $(\mathcal{M}_{\text{Sch}}, g_{\text{Sch}})$ (and, more generally, on Kerr-AdS) decay at a slow (logarithmic) rate, which is insufficient in itself to yield the nonlinear stability of $(\mathcal{M}_{\text{Sch}}, g_{\text{Sch}})$ (see our remark below the statement of the AdS instability conjecture). Thus, [Holzegel and Smulevici 2014] conjectured that $(\mathcal{M}_{\text{Sch}}, g_{\text{Sch}})$ is nonlinearly unstable. On the other hand, based on a detailed analysis of quasinormal modes on $(\mathcal{M}_{\text{Sch}}, g_{\text{Sch}})$, Dias, Horowitz, Marolf, and Santos [Dias et al. 2012a] suggested that sufficiently regular, nonlinear perturbations of $(\mathcal{M}_{\text{Sch}}, g_{\text{Sch}})$ still remain small, at least for long times. As a result, the picture regarding the end state of the evolution of generic perturbations of $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$ outside spherical symmetry remains unclear; see also [Horowitz and Santos 2015; Dias and Santos 2016; Rostworowski 2017].

Following [Bizoń and Rostworowski 2011], the bulk of heuristic works have implemented a frequency space analysis in the study of the AdS instability conjecture. A notable exception is the work of Dimitrakopoulos, Freivogel, Lippert, and Yang [Dimitrakopoulos et al. 2015], where a physical space mechanism possibly leading to instability for the system (1-7) is suggested. We will revisit the mechanism of [Dimitrakopoulos et al. 2015] and compare it with the results of this paper at the end of Section 1D.

1B. ***The Einstein-null dust system in spherical symmetry.*** A spherically symmetric model for (1-1) which is even simpler than (1-5) is the *Einstein-massless Vlasov* system; see [Andréasson 2011; Rein 1995]. The case where the Vlasov field is supported only on radial geodesics is a singular reduction of this system which is equivalent to the *Einstein-null dust* system, allowing for both ingoing and outgoing dust; see [Rendall 1997]. This system was studied in the seminal work [Poisson and Israel 1990] on mass

[4] A similar result can presumably also be deduced for the vacuum Einstein equations (1-1) reduced under the biaxial Bianchi IX symmetry in $4 + 1$ dimensions, following by an amalgamation of the proofs of [Holzegel 2010] and [Holzegel and Smulevici 2013b].

inflation. In $3+1$ dimensions, it takes the form (in double null coordinates (u, v))

$$\begin{cases} \partial_u \partial_v (r^2) = -\frac{1}{2}(1-\Lambda r^2)\Omega^2, \\ \partial_u \partial_v \log(\Omega^2) = \dfrac{\Omega^2}{2r^2}(1+4\Omega^{-2}\,\partial_u r\,\partial_v r), \\ \partial_v(\Omega^{-2}\,\partial_v r) = -4\pi r^{-1}\Omega^{-2}\bar{\tau}, \\ \partial_u(\Omega^{-2}\,\partial_u r) = -4\pi r^{-1}\Omega^{-2}\tau, \\ \partial_u \bar{\tau} = 0, \\ \partial_v \tau = 0. \end{cases} \tag{1-9}$$

In certain cases, the Einstein-null dust system (1-9) can be formally viewed as a high-frequency limit of the Einstein-scalar field system (1-7) (as was already discussed in [Poisson and Israel 1990]): Setting

$$\tau \doteq r^2(\partial_u\varphi)^2, \quad \bar{\tau} \doteq r^2(\partial_v\varphi)^2$$

in (1-5) and dropping all lower-order terms from the wave equation for φ, one formally obtains (1-9) in the region where $\partial_u\varphi\,\partial_v\varphi$ is negligible, i.e., outside the intersection of the supports of $\tau, \bar{\tau}$. While this formal limiting procedure can be rigorously justified away from $r=0$, the dynamical similarities between (1-7) and (1-9) break down close to $r=0$. A fundamental difference between these systems is the fact that, while small-data asymptotically AdS solutions to (1-7) satisfying a reflecting boundary condition at $\mathcal{I}$ remain regular (and "small") for large times, all nontrivial solutions to the system (1-9) break down once the support of $\bar{\tau}$ reaches the axis γ (i.e., the time-like portion of $\{r=0\}$), independently of the boundary conditions imposed at $\mathcal{I}$. This is an ill-posedness statement for (1-9), which needs to be addressed before any attempt to study the AdS instability conjecture in the setting of (1-9).

We will now proceed to discuss this difference of (1-7) and (1-9) in more detail.

Cauchy stability for the Einstein-scalar field system. The following Cauchy stability result holds for the system (1-7):

Proposition 1 (Cauchy stability for (1-7); see [Holzegel and Smulevici 2012]). *For a suitable initial-data norm $\|\cdot\|_{\text{initial}}$, $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$ is Cauchy stable as a solution of the system* (1-7) *with reflecting boundary conditions on $\mathcal{I}$. That is to say, for all fixed times $T_* > 0$, any perturbation of the initial data of $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$ which is small enough (when measured in terms of $\|\cdot\|_{\text{initial}}$) with respect to T_* gives rise to a solution of* (1-7) *which is regular and close to $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$ for times up to T_*.*

Remark. In the statement of Proposition 1, Cauchy stability of $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$ refers to stability over fixed compact subsets of the *conformal compactification* of $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$, such as subsets of the form $\{0 \le t \le T_*\}$ in the $(t, r, \vartheta, \varphi)$ coordinate chart. Any such subset contains, in particular, a compact subset of the time-like boundary $\mathcal{I}$.

The initial-data norm $\|\cdot\|_{\text{initial}}$, for which the Cauchy stability of $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$ follows from [Holzegel and Smulevici 2012], is a higher-order, suitably weighted C^k norm. However, this is not the only norm for which $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$ can be shown to be Cauchy stable: An additional, highly nontrivial example of such a norm is the bounded variation norm of [Christodoulou 1993] (modified with suitable r-weights near $r=\infty$). Similar low-regularity norms will also play an important role in this paper (see Section 1B).

Assuming, for simplicity, that initial data are prescribed on the outgoing null hypersurface corresponding to $u=0$, for $0 \le v \le v_*$, a necessary condition for Cauchy stability of $(\mathcal{M}_{\mathrm{AdS}}, g_{\mathrm{AdS}})$ for the system (1-7) with respect to an initial-data norm $\|\cdot\|_{\mathrm{initial}}$ is that, for any given $R_0 > 0$, $\|\cdot\|_{\mathrm{initial}}$ controls the quantity

$$\mathscr{M} \doteq \sup_{\substack{0\le v_1<v_2\le v_* \\ r(0,v_2)/r(0,v_1)<3/2 \\ r(0,v_2)\le R_0}} \frac{\tilde{m}(0,v_2)-\tilde{m}(0,v_1)}{(r(0,v_2)-r(0,v_1))|\log(r(0,v_2)/r(0,v_1)-1)|}, \tag{1-10}$$

where $\tilde{m}$ is the *renormalized Hawking mass*, defined in terms of the Hawking mass m,

$$m \doteq \tfrac{1}{2} r(1-4\Omega^{-2}\,\partial_u r\,\partial_v r), \tag{1-11}$$

by the relation

$$\tilde{m} \doteq m - \tfrac{1}{6}\Lambda r^3. \tag{1-12}$$

This is a consequence of the fact that, when $\mathscr{M}$ exceeds a certain threshold (depending on R_0), there exists a point $p = (u_\dagger, v_\dagger)$ in the development of the initial data, with $0 < u_\dagger < v_*$, such that

$$\frac{2m}{r}(u_\dagger, v_\dagger) > 1, \tag{1-13}$$

a result proven in [Christodoulou 1991].[5] The bound (1-13) implies that

$$\partial_u r(u_\dagger, v_\dagger) < 0, \quad \partial_v r(u_\dagger, v_\dagger) < 0, \tag{1-14}$$

i.e., that the symmetry sphere associated to $(u_\dagger, v_\dagger)$ is a *trapped surface*. In particular, $(u_\dagger, v_\dagger)$ is contained in a black hole.[6]

As a corollary, it follows that the total ADM mass of the initial data, though expressible as a coercive functional on the space of initial data of (1-7), does not yield a norm for which $(\mathcal{M}_{\mathrm{AdS}}, g_{\mathrm{AdS}})$ is Cauchy stable for (1-7), since the ADM mass manifestly fails to control (1-10).

Break down at $r=0$ and "trivial" Cauchy instability for the Einstein-null dust system. The following instability result holds for the system (1-9) (see [Moschidis 2017]):

Proposition 2 (Cauchy instability for (1-9)). *Any globally hyperbolic spherically symmetric solution $(\mathcal{M}, g; \tau, \bar{\tau})$ of* (1-9) *with nonempty axis γ "breaks down" at the first point when a radial geodesic in the support of $\bar{\tau}$ reaches γ: Beyond that point, $(\mathcal{M}, g; \tau, \bar{\tau})$ is C^0 inextendible as a spherically symmetric solution to* (1-9)*. As a result, $(\mathcal{M}_{\mathrm{AdS}}, g_{\mathrm{AdS}})$ is not a Cauchy stable solution of* (1-9) *for any "reasonable" initial-data topology.*

For the precise definition of the notion of C^0 inextendibility as a spherically symmetric solution to (1-9), see [Moschidis 2017]. Note that this is a *stronger* statement than $(\mathcal{M}, g; \tau, \bar{\tau})$ breaking down as a smooth solution of (1-9). We should also note the following regarding Proposition 2:

[5] The result of [Christodoulou 1991] was restricted to the case $\Lambda = 0$, but the proof can be readily modified to include the case $\Lambda < 0$.

[6] We should remark that (1-14) follows from (1-13) under the assumption that $\partial_u r < 0$ (which always holds provided, initially, $\partial_u r|_{u=0} < 0$, see [Christodoulou 1993]).

• Proposition 2 holds independently of the value of the cosmological constant Λ. In particular, Minkowski spacetime $(\mathbb{R}^{3+1}, \eta)$ is not Cauchy stable for (1-9) with $\Lambda = 0$ for any "reasonable" initial-data topology.

• Proposition 2 yields a uniform upper bound on the time of existence of solutions $(\mathcal{M}, g)$ to (1-9) for any initial data set for which $\bar{\tau}$ is not identically equal to 0, depending only on the distance of the initial support of $\bar{\tau}$ from the axis and, thus, independent of the proximity of the initial data to the trivial data (in any reasonable initial-data norm). We should also highlight that the instability of Proposition 2 has nothing to do with trapped surface formation: Up to the first retarded time when a radial geodesic in the support of $\bar{\tau}$ reaches γ, any solution $(\mathcal{M}, g)$ to (1-9) arising from smooth initial data close to $(\mathcal{M}_{\mathrm{AdS}}, g_{\mathrm{AdS}})$ remains smooth and close to $(\mathcal{M}_{\mathrm{AdS}}, g_{\mathrm{AdS}})$, and $(\mathcal{M}, g)$ *contains no trapped surface*. In fact, in this case, despite being C^0 inextendible as a globally hyperbolic spherically symmetric solution to (1-9), $(\mathcal{M}, g)$ is globally C^∞-extendible as a spherically symmetric Lorentzian manifold; see [Moschidis 2017].

• The Cauchy stability statement for $(\mathcal{M}_{\mathrm{AdS}}, g_{\mathrm{AdS}})$ for the system (1-7) stated in Proposition 1 can be informally interpreted as the result of a natural dispersive mechanism close to the axis γ displayed by the system (1-7), which does not allow the energy of φ to concentrate on scales smaller than $\tilde{m}$ in $O(1)$ time, provided a suitable initial norm of φ (controlling at least (1-10)) is small enough. No such mechanism is present for the system (1-9), as is illustrated by Proposition 2.

Resolution of the "trivial" instability of (1-9) *through an inner mirror.* In order to turn the spherically symmetric Einstein-null dust system (1-9) into a well-posed, Cauchy-stable system (a necessary step for converting (1-9) into an effective model of the vacuum Einstein equations (1-1)), it is necessary to explicitly add to (1-9) a mechanism that prevents the breakdown at $r = 0$ described by Proposition 2, so that, moreover, an analogue of Proposition 1 holds for (1-9). This can be achieved by placing an *inner mirror* at $r = r_0 > 0$, i.e., by restricting (1-9) on $\{r \ge r_0\}$, for some $r_0 > 0$, and imposing a reflecting boundary condition on the portion γ_0 of the set $\{r = r_0\}$ which is time-like.

Remark. The reflecting boundary condition on γ_0 can be motivated by the fact that, for smooth spherically symmetric solutions $(\mathcal{M}, g; \varphi)$ to (1-5), the function φ, viewed as a function on the quotient of $(\mathcal{M}, g)$ by the spheres of symmetry, satisfies a reflecting boundary condition on the axis.

The well-posedness and the properties of the maximal development for the system (1-9) with reflecting boundary conditions on $\mathcal{I}$ and γ_0 are addressed in the companion paper [Moschidis 2017]. The following result is established in that paper:

Theorem 2 (well-posedness for (1-9) with an inner mirror). *For any $r_0 > 0$ and any smooth asymptotically AdS initial data set $(r, \Omega^2, \tau, \bar{\tau})|_{u=0}$ on $u = 0$, there exists a unique smooth maximal future development $(r, \Omega^2, \tau, \bar{\tau})$ on $\{r \ge r_0\}$, solving* (1-9) *with reflecting boundary conditions on $\mathcal{I}$ and γ_0, where $r|_{\gamma_0} = r_0$ and γ_0 coincides with the portion of the curve $\{r = r_0\}$ which is time-like* (*fixing the gauge freedom by imposing a reflecting gauge condition on both $\mathcal{I}$ and γ_0*). *For this development, $\mathcal{I}$ is complete and $\{r = r_0\}$ is time-like in the past of $\mathcal{I}$* (*see Figure 2*).

In the case when the future event horizon $\mathcal{H}^+$ is nonempty, it is smooth and future complete. A necessary condition for $\mathcal{H}^+$ to be nonempty is the existence of a point $(u_\dagger, v_\dagger)$ where (1-13) *holds. If the total mass*

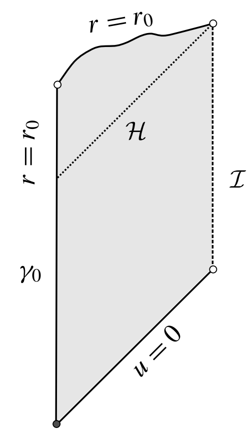

Figure 2. Schematic depiction of the domain on which the maximal future development $(r, \Omega^2, \tau, \bar{\tau})$ of a smooth initial data set on $u = 0$ (with reflecting boundary conditions on $\mathcal{I}$ and γ_0) is defined. A gauge condition ensures that $\mathcal{I}$ and γ_0 are straight vertical lines. Conformal infinity $\mathcal{I}$ is always complete in this setting. In the case when the future event horizon $\mathcal{H}^+$ is nonempty, it is smooth and has infinite affine length. In this case, apart from the mirror γ_0, the boundary of the domain has a space-like portion on which $\{r = r_0\}$.

$\tilde{m}|_{\mathcal{I}}$ and the mirror radius r_0 satisfy

$$\frac{2\tilde{m}|_{\mathcal{I}}}{r_0} \le 1 - \tfrac{1}{3}\Lambda r_0^2, \tag{1-15}$$

then necessarily $\mathcal{H}^+ = \varnothing$.

For a more detailed statement of Theorem 2, see Section 3 and [Moschidis 2017].

In view of the fact that $\mathcal{H}^+ = \varnothing$ in the case when the total mass $\tilde{m}|_{\mathcal{I}}$ and the mirror radius r_0 satisfy (1-15), in order to address the AdS instability conjecture for the system (1-9) with reflecting boundary conditions on $\mathcal{I}$ and γ_0, it is necessary to allow r_0 to shrink to 0 with the size of the data. Thus, addressing the AdS instability conjecture in this setting requires establishing a Cauchy stability statement for $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$ which is *independent of the precise value of the mirror radius r_0*. This is the statement of the following result, proved in our companion paper [Moschidis 2017]:

Theorem 3 (Cauchy stability for (1-9) uniformly in r_0). *Given $\varepsilon > 0$, $u_* > 0$, there exists a $\delta > 0$ such that the following statement holds: for any $r_0 > 0$ and any initial data set $(r, \Omega^2, \tau, \bar{\tau})|_{u=0}$ satisfying*

$$\begin{aligned} &\|(r, \Omega^2, \tau, \bar{\tau})\|_{u=0} \\ &\quad \doteq \sup_{\bar{v}} \int_{u=0} \frac{\bar{\tau}(0, v)}{|\rho(0, v) - \rho(0, \bar{v})| + \tan^{-1}(\sqrt{-\Lambda} r_0)} \frac{\sqrt{-\Lambda}\, dv}{\partial_v \rho(0, v)} + \sup_{u=0}\left(\left|\left(1 - \frac{2\tilde{m}}{r}\right)^{-1} - 1\right| + \sqrt{-\Lambda}\tilde{m}\right) \\ &\quad \le \varepsilon, \end{aligned} \tag{1-16}$$

where

$$\rho(0, v) \doteq \tan^{-1}(\sqrt{-\Lambda} r)(0, v), \tag{1-17}$$

the corresponding solution $(r, \Omega^2, \tau, \bar{\tau})$ *to* (1-9) *with reflecting boundary conditions on* $\mathcal{I}$ *and* γ_0 *will satisfy*

$$\sup_{0 \le \bar{u} \le u_*} \|(r, \Omega^2, \tau, \bar{\tau})\|_{u=\bar{u}} \le \delta. \tag{1-18}$$

For a more detailed statement of Theorem 3, see Section 3 and [Moschidis 2017].

Notice that the norm (1-16) vanishes only for the trivial initial data $(r, \Omega^2, 0, 0)$. Informally, Theorem 3 implies that, if the energy of the initial data concentrated on scales proportional to the mirror radius r_0 is small enough, then the energy of the solution to (1-9) (with reflecting boundary conditions on $\mathcal{I}$ and γ_0) will remain similarly dispersed for times less than any given constant. In particular, no trapped surface can form in this timescale if ε is chosen sufficiently small.

In Section 3, we will also present a Cauchy stability statement for general solutions of (1-9) with reflecting boundary conditions on $\mathcal{I}$ and γ_0, which will be used in the proof of Theorem 1 (see Theorem 3.6).

1C. *Statement of Theorem 1: the nonlinear instability of AdS.* According to Theorem 3, a Cauchy stability statement holds for $(\mathcal{M}_{\mathrm{AdS}}, g_{\mathrm{AdS}})$ for time intervals which are independent of the precise value of the mirror radius r_0, depending only on the smallness of the initial-data norm (1-16). As a result, it is possible to study the AdS instability conjecture for the system (1-9) with reflecting boundary conditions on $\mathcal{I}$ and γ_0, for perturbations which are small with respect to (1-16), *allowing the mirror radius r_0 to shrink with the size of the data*. In this paper, we will prove the following result:

Theorem 1 (more precise version). *There exists a family of positive numbers $r_{0\varepsilon}$ (satisfying $r_{0\varepsilon} \xrightarrow{\varepsilon \to 0} 0$) and smooth initial data $(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}|_{u=0}$ for the system* (1-9) *satisfying the following properties*:

(1) *In the norm $\|\cdot\|_{u=0}$ defined by* (1-16),

$$\|(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}\|_{u=0} \xrightarrow{\varepsilon \to 0} 0. \tag{1-19}$$

(2) *For any $\varepsilon > 0$, the maximal development $(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}$ of $(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}|_{u=0}$ for the system* (1-9) *with reflecting boundary conditions on $\mathcal{I}$ and γ_0, $r|_{\gamma_0} = r_{0\varepsilon}$ contains a trapped sphere*; *i.e., there exists a point $(u_\varepsilon, v_\varepsilon)$ such that*

$$\frac{2m^{(\varepsilon)}}{r^{(\varepsilon)}}(u_\varepsilon, v_\varepsilon) > 1. \tag{1-20}$$

Thus, in view of Theorem 2, $(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}$ contains a nonempty, smooth and future complete event horizon $\mathcal{H}^+$ and a complete conformal infinity $\mathcal{I}$.

For the definitive statement of Theorem 1, see Section 4. The following remarks should be made concerning Theorem 1:

- In view of the Cauchy stability of $(\mathcal{M}_{\mathrm{AdS}}, g_{\mathrm{AdS}})$ with respect to (1-16) (see Theorem 3), the time[7] required to elapse before $(1 - 2m^{(\varepsilon)}/r^{(\varepsilon)})$ becomes negative necessarily tends to $+\infty$ as $\varepsilon \to 0$. However, our proof yields a quantitative upper bound (increasing to $+\infty$ as $\varepsilon \to 0$) on the time of trapped sphere formation for the development of $(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}|_{u=0}$ (see (4-6)).

[7] Where time is measured with respect to the (dimensionless) coordinate function $\bar{t} = \sqrt{-\Lambda}(u + v)$.

• In view of the fact that $\mathcal{H}^+ = \varnothing$ when (1-15) holds, in order for $(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}$ to satisfy both (1-19) and (1-20), it is necessary that $r_{0\varepsilon} \to 0$ as $\varepsilon \to 0$, at a rate which is at least as fast as that of $2\tilde{m}^{(\varepsilon)}|_{\mathcal{I}}$,[8] i.e.,

$$r_{0\varepsilon} \leq 2\tilde{m}^{(\varepsilon)}|_{\mathcal{I}}. \tag{1-21}$$

In fact, we choose the family $r_{0\varepsilon}$, $(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}|_{u=0}$ of Theorem 1 to saturate the bound (1-21) in the limit $\varepsilon \to 0$; i.e.,

$$\lim_{\varepsilon\to 0} \frac{r_{0\varepsilon}}{2\tilde{m}^{(\varepsilon)}|_{\mathcal{I}}} = 1. \tag{1-22}$$

For the proof of Theorem 1, (1-22) is not essential and can be relaxed; however, it is fundamental for our proof that $r_{0\varepsilon}$ is bounded from below by some small multiple of $\tilde{m}^{(\varepsilon)}|_{\mathcal{I}}$.

• It follows from the proof of Theorem 2 that, in the case $\Lambda = 0$, Minkowski spacetime $(\mathbb{R}^{3+1}, \eta)$ is *globally stable* (for the system (1-9) with reflecting boundary conditions on the inner mirror $\{r = r_0\}$) to initial data perturbations which are small with respect to the norm (1-16), independently of the precise choice of r_0. This fact further justifies the choice of the matter model and the norm (1-16) as a setting for establishing the AdS instability conjecture.

• The proof of Theorem 1 also applies in the case $\Lambda = 0$ when placing an *outer mirror* at $r = R_0 \gg r_0$ (in addition to the inner mirror at $r = r_0$), i.e., restricting the solutions of (1-9) in the region $\{r_0 \leq r \leq R_0\}$ and imposing reflecting boundary conditions on both $\{r = r_0\}$ and $\{r = R_0\}$. This is in accordance with the numerical results of [Buchel et al. 2012] for the system (1-5).

• It can be readily inferred by Cauchy stability (see Theorem 3.6 in Section 3D) that, for any $y_\varepsilon \doteq (r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}|_{u=0}$ in the family of initial data of Theorem 1, there exists an open neighborhood $\mathcal{W}_\varepsilon$ of initial data around y_ε such that, for all $y \in \mathcal{W}_\varepsilon$, the maximal future development of y also contains a trapped surface. In particular, the set of initial data leading to trapped surface formation is open. An even stronger genericity statement would be the existence of an *open instability corner* in the space of initial data around $(\mathcal{M}_{\mathrm{AdS}}, g_{\mathrm{AdS}})$ (see [Dimitrakopoulos et al. 2015]), i.e., the existence of a $c_1 > 0$ such that $\{y : \mathrm{dist}(y, y_\varepsilon) \leq c_1 \|y_\varepsilon\|_{u=0}\} \subset \mathcal{W}_\varepsilon$ for all $\varepsilon > 0$ (with $\mathrm{dist}(\,\cdot\,,\,\cdot\,)$ being the distance function associated to (1-16) for $r_0 = r_{0\varepsilon}$). While we have not addressed the issue of genericity of the unstable initial data in this paper, we expect that the proof of Theorem 1 can be adapted to yield the existence of an instability corner around $(\mathcal{M}_{\mathrm{AdS}}, g_{\mathrm{AdS}})$.

• A plethora of numerical works, see, e.g., [Bizoń and Rostworowski 2011; Buchel et al. 2012; Bizoń et al. 2015; Dimitrakopoulos et al. 2015], suggest that, in the case of the Einstein-scalar field system (1-5), for families of initial data $(\varphi_\varepsilon^{(0)}, \varphi_\varepsilon^{(1)})$ for the scalar field φ of the form

$$(\varphi_\varepsilon^{(0)}, \varphi_\varepsilon^{(1)}) = (\varepsilon\varphi^{(0)}, \varepsilon\varphi^{(1)}) \tag{1-23}$$

(where $(\varphi^{(0)}, \varphi^{(1)})$ is a fixed initial profile), trapped surface formation occurs at time $\sim \varepsilon^{-2}$. However, any rigorous formulation of this statement for general families of initial data, not necessarily of the form (1-23), requires fixing an initial-data norm for which the initial data size is measured, with different

[8]Note that the renormalized Hawking mass $\tilde{m}^{(\varepsilon)}$ is constant on $\mathcal{I}$ when imposing a reflecting boundary condition.

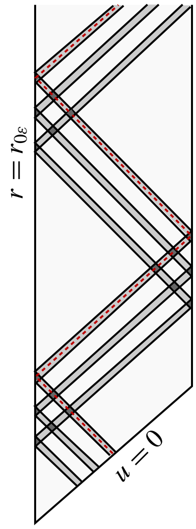

Figure 3. The initial data $(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}|_{u=0}$ give rise to a bundle of ingoing beams which are successively reflected off $\{r = r_{0\varepsilon}\}$ and $\mathcal{I} = \{r = +\infty\}$. While the number of beams goes to infinity as $\varepsilon \to 0$, for simplicity, we only depict here a bundle of three beams. As long as the total width of the bundle of beams remains small, the interaction set naturally splits into a part which lies close to $\{r = r_{0\varepsilon}\}$ and a part near $\mathcal{I}$. We have also marked with a red dashed line the beam lying (initially) to the future of the rest.

choices of (scale-invariant) norms possibly leading to different time scales of trapped surface formation for initial data of size $\sim \varepsilon$. For this reason, given that the initial data $(\tau, \bar{\tau})^{(\varepsilon)}|_{u=0}$ in Theorem 1 cannot be viewed as a rescaling of a fixed profile of the form (1-23), we have not tried to optimize the time required for trapped surface formation in Theorem 1 in terms of the initial norm (1-16).

1D. ***Sketch of the proof and remarks on Theorem 1.*** We will now proceed to sketch the main arguments involved in the proof of Theorem 1.

Construction of the initial data. The family of initial data $(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}|_{u=0}$ in Theorem 1 is chosen so that its total ADM mass $\tilde{m}^{(\varepsilon)}|_{\mathcal{I}}$ and the mirror radius $r_{0\varepsilon}$ satisfy (for $\varepsilon \ll 1$)

$$r_{0\varepsilon}, \tilde{m}^{(\varepsilon)}|_{\mathcal{I}} \sim \varepsilon(-\Lambda)^{-1/2}. \tag{1-24}$$

In particular, fixing a function $h(\varepsilon)$ in terms of ε such that

$$\varepsilon \ll h(\varepsilon) \ll 1,$$

the initial data $(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}|_{u=0}$ are constructed so that the null dust initially forms a bundle of narrow ingoing beams emanating from the region $r \sim 1$; see Figure 3. The number of the beams is chosen to be large, i.e., of order $\sim (h(\varepsilon))^{-1}$, and the beams are initially separated by gaps of r-width $\sim (h(\varepsilon))^{-1}\varepsilon(-\Lambda)^{-1/2}$. The large number of beams and their initial separation are chosen so that

$$\|(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}\|_{u=0} \sim h(\varepsilon) \xrightarrow{\varepsilon \to 0} 0. \tag{1-25}$$

Remarks on the configuration of the null dust beams. As the solution $(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}$ arising from the initial data set $(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}|_{u=0}$ evolves according to (1-9), the null dust beams are reflected successively off $\gamma_0 = \{r = r_{0\varepsilon}\}$ and $\mathcal{I}$, as depicted in Figure 3. The beams separate the spacetime into vacuum regions (the larger rectangular regions between the beams in Figure 3), where the renormalized Hawking mass $\tilde{m}^{(\varepsilon)}$ is constant (recall the definitions of the Hawking mass and the renormalized Hawking mass by (1-11) and (1-12), respectively). The *interaction set* of the beams consists of all the points in the spacetime where two different beams intersect (depicted in Figure 3 as the union of all the smaller dark rectangles, lying in the intersection of any two beams). As long as the total width of the bundle of beams remains small, the interaction set can be split into two sets, one consisting of the intersections occurring close to the mirror γ_0 and one consisting of the intersections near $\mathcal{I}$ (see Figure 3).

Every beam is separated by the interaction set into several components. To each such component, we can associate the *mass difference* $\mathfrak{D}\tilde{m}$ between the two vacuum regions which are themselves separated by that beam component. The mass difference $\mathfrak{D}\tilde{m}$ measures the energy content of each beam component and, in view of the nonlinearity of the system (1-9), it is not necessarily conserved along the beam after an intersection with another beam. Precisely determining the resulting change in the mass difference after the interaction of two beams will be the crux of the proof of Theorem 1.

Beam interactions and change in mass difference. In Figure 4, the region around the intersection of an incoming null dust beam ζ_{in} and an outgoing null dust beam ζ_{out} is depicted. This region is separated by the beams into four vacuum subregions $\mathcal{R}_1, \dots, \mathcal{R}_4$ with associated renormalized Hawking masses $\tilde{m}_1, \dots, \tilde{m}_4$ (see Figure 4). Before the intersection of the two beams, the mass difference of the incoming beam ζ_{in} is

$$\bar{\mathfrak{D}}_{-}\tilde{m} = \tilde{m}_3 - \tilde{m}_4, \tag{1-26}$$

while the mass difference of the outgoing beam ζ_{out} is

$$\mathfrak{D}_{-}\tilde{m} = \tilde{m}_4 - \tilde{m}_2. \tag{1-27}$$

After the intersection of the beams, the mass differences associated to ζ_{in} and ζ_{out} become

$$\bar{\mathfrak{D}}_{+}\tilde{m} = \tilde{m}_1 - \tilde{m}_2, \tag{1-28}$$

$$\mathfrak{D}_{+}\tilde{m} = \tilde{m}_3 - \tilde{m}_1, \tag{1-29}$$

respectively.

Assuming that

$$\frac{2m}{r} < 1, \tag{1-30}$$

$$\partial_u r < 0 < \partial_v r, \tag{1-31}$$

we can readily obtain the following differential relations for r and $\tilde{m}$ from (1-9):

$$\begin{aligned} \partial_u \log\left(\frac{\partial_v r}{1 - 2m/r}\right) &= -\frac{4\pi}{r}\frac{\tau}{-\partial_u r}, \\ \partial_v \log\left(\frac{-\partial_u r}{1 - 2m/r}\right) &= \frac{4\pi}{r}\frac{\bar{\tau}}{\partial_v r} \end{aligned} \tag{1-32}$$

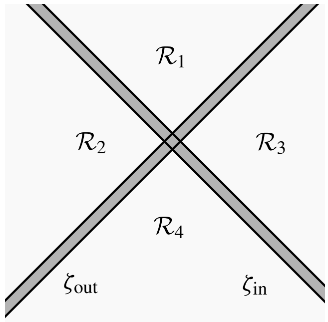

Figure 4. The region in the (u, v)-plane around the intersection of an incoming beam ζ_{in} and an outgoing beam ζ_{out}. The regions $\mathcal{R}_i$, $i = 1, \dots, 4$, are vacuum and the renormalized Hawking mass $\tilde{m}$ is constant (and equal to $\tilde{m}_i$) on each of the $\mathcal{R}_i$'s.

and

$$\begin{aligned}\partial_u \tilde{m} &= -2\pi \left(\frac{1-2m/r}{-\partial_u r}\right)\tau,\\ \partial_v \tilde{m} &= 2\pi \left(\frac{1-2m/r}{\partial_v r}\right)\bar{\tau}.\end{aligned} \tag{1-33}$$

We will also assume that:

- The null dust beams ζ_{in} and ζ_{out} are sufficiently narrow so that, on their intersection $\zeta_{\text{in}} \cap \zeta_{\text{out}}$, r can be considered nearly constant:[9]

$$\sup_{\zeta_{\text{in}} \cap \zeta_{\text{out}}} r - \inf_{\zeta_{\text{in}} \cap \zeta_{\text{out}}} r \ll \varepsilon(-\Lambda)^{-1/2}. \tag{1-34}$$

- $\overline{\mathfrak{D}}_+ \tilde{m} - \overline{\mathfrak{D}}_- \tilde{m}$ and $\mathfrak{D}_+ \tilde{m} - \mathfrak{D}_- \tilde{m}$ are relatively small.[10]

Then, (1-32)–(1-33), combined with the conservation laws

$$\partial_u \bar{\tau} = 0, \quad \partial_v \tau = 0,$$

yield the following relations for the change in the mass difference associated to ζ_{in} and ζ_{out} after their intersection:

$$\overline{\mathfrak{D}}_+ \tilde{m} = \overline{\mathfrak{D}}_- \tilde{m} \cdot \exp\left(\frac{2}{r}\frac{\mathfrak{D}_- \tilde{m}}{1-2m_2/r} + \mathfrak{Err}_{\text{in}}\right), \tag{1-35}$$

$$\mathfrak{D}_+ \tilde{m} = \mathfrak{D}_- \tilde{m} \cdot \exp\left(-\frac{2}{r}\frac{\overline{\mathfrak{D}}_- \tilde{m}}{1-2m_2/r} + \mathfrak{Err}_{\text{out}}\right), \tag{1-36}$$

where the error terms $\mathfrak{Err}_{\text{in}}$, $\mathfrak{Err}_{\text{out}}$ are negligible compared to the other terms in (1-35), (1-36) (see also the relations (6-51) and (6-52) in Section 6A2). In particular, whenever an ingoing and an outgoing null dust beam intersect, *the mass difference of the ingoing beam increases, while that of the outgoing beam decreases*.

[9]This is possible in view of the fact that, for solutions $(r, \Omega^2, \tau, \bar{\tau})$ to (1-9), r remains uniformly continuous in the limit when τ, $\bar{\tau}$ tend to δ-functions in the u-, v-variables, respectively.

[10]Note that, necessarily, $\overline{\mathfrak{D}}_+ \tilde{m} - \overline{\mathfrak{D}}_- \tilde{m} = -(\mathfrak{D}_+ \tilde{m} - \mathfrak{D}_- \tilde{m})$.

Remark. Notice that, according to (1-35) and (1-36), the change in the mass difference of each of the beams ζ_{in}, ζ_{out} after their intersection can be estimated in terms of the mass difference of the other beam and the value of r and $\inf(1-2m/r)$ in the region of intersection. A relation for the change of the mass difference of two infinitely thin, intersecting null dust beams was also obtained in [Poisson and Israel 1990].

The instability mechanism. Let us now consider, among the null dust beams arising from the initial data $(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}|_{u=0}$, the beam ζ_0 which initially lies to the future of the rest (this is the beam marked with a dashed line in Figure 3). Defining

$$\mathcal{E}_{\zeta_0}[t_*] \doteq \text{mass difference associated to } \zeta_0 \text{ at } \zeta_0 \cap \{u+v=t_*\}, \tag{1-37}$$

we will examine how $\mathcal{E}_{\zeta_0}$ changes along ζ_0, after each successive intersection of ζ_0 with the rest of the beams:

(1) Starting from $u=0$ up to the first reflection of ζ_0 off the inner mirror γ_0, the beam ζ_0 is ingoing and intersects all the other beams *after* they are reflected off γ_0. Thus, applying (1-35) successively at each intersection of ζ_0 with an outgoing beam, we infer that $\mathcal{E}_{\zeta_0}$ *increases* at this step by a multiplicative factor

$$A_{\mathrm{in}} \geq \exp\biggl(\frac{2(\tilde{m}^{(\varepsilon)}|_{\mathcal{I}} - \mathcal{E}_{\zeta_0}|_{u=0})}{r_{\gamma_0}}(1-\varepsilon)\biggr), \tag{1-38}$$

where r_{γ_0} is the value of r at the region of intersection of ζ_0 with the first beam which is reflected off $\{r = r_{0\varepsilon}\}$ (note that r_{γ_0} is also the r-width of the bundle of beams when ζ_0 first reaches the mirror γ_0). In obtaining (1-38), we have assumed that

$$r_{0\varepsilon} \ll r_{\gamma_0} \ll (-\Lambda)^{-1/2}, \quad \tilde{m}^{(\varepsilon)}|_{\mathcal{I}} \sim r_{0\varepsilon}, \quad \mathcal{E}_{\zeta_0}|_{u=0} \ll \tilde{m}^{(\varepsilon)}|_{\mathcal{I}}$$

(where the latter holds in view of the way the initial data where chosen).

(2) The mass difference $\mathcal{E}_{\zeta_0}$ right before and right after the reflection of ζ_0 off γ_0 is the same, in view of the reflecting boundary conditions on γ_0.

(3) From its first reflection off γ_0 up to its first reflection off $\mathcal{I}$, the beam ζ_0 is outgoing and intersects (again) the rest of the beams in the region close to $\mathcal{I}$ (*after* these beams are reflected off $\mathcal{I}$). Applying (1-36) successively at each intersection, we infer that $\mathcal{E}_{\zeta_0}$ *decreases* at this step, being multiplied by a factor

$$1 > A_{\mathrm{out}} \geq \exp\biggl(-\frac{2(\tilde{m}^{(\varepsilon)}|_{\mathcal{I}} - \mathcal{E}_{\zeta_0}|_{u=0})}{r_{\mathcal{I}}}\biggl(\frac{1}{\bigl(1-\varepsilon-\frac{1}{3}\Lambda r_{\mathcal{I}}^2\bigr)}+\varepsilon\biggr)\biggr), \tag{1-39}$$

where $r_{\mathcal{I}}$ is the value of r at the region of intersection of ζ_0 with the first beam which is reflected off $\mathcal{I}$. In obtaining (1-39), we have assumed that $r_{\mathcal{I}} \gg (-\Lambda)^{-1/2}$ (which holds in view of the way the initial data where chosen).

(4) The mass difference $\mathcal{E}_{\zeta_0}$ right before and right after the reflection of ζ_0 off $\mathcal{I}$ is the same, in view of the reflecting boundary conditions on $\mathcal{I}$.

Therefore, provided $r_{\gamma_0} \ll (-\Lambda)^{-1/2} \ll r_{\mathcal{I}}$, we infer that, after the first reflection of ζ_0 off γ_0 and $\mathcal{I}$, the mass difference $\mathcal{E}_{\zeta_0}$ increases by a factor

$$\begin{aligned} A_{\mathrm{tot}} = A_{\mathrm{in}} \cdot A_{\mathrm{out}} &\geq \exp\biggl(\frac{2(\tilde{m}^{(\varepsilon)}|_{\mathcal{I}} - \mathcal{E}_{\zeta_0}|_{u=0})}{r_{\gamma_0}} (1-\varepsilon) - \frac{2(\tilde{m}^{(\varepsilon)}|_{\mathcal{I}} - \mathcal{E}_{\zeta_0}|_{u=0})}{r_{\mathcal{I}}} \biggl(\frac{1}{\bigl(1-\varepsilon-\frac{1}{3}\Lambda r_{\mathcal{I}}^2\bigr)} + \varepsilon \biggr) \biggr) \\ &\geq \exp\biggl(\frac{\tilde{m}^{(\varepsilon)}|_{\mathcal{I}} - \mathcal{E}_{\zeta_0}|_{u=0}}{r_{\gamma_0}} \biggr). \end{aligned} \tag{1-40}$$

The steps (1)–(4) in the above procedure can then be repeated for each successive reflection of ζ_0 off γ_0 and $\mathcal{I}$, as long as

$$r_{0\varepsilon} \ll r_{\gamma_0;n} \ll (-\Lambda)^{-1/2} \ll r_{\mathcal{I};n}, \tag{1-41}$$

where $r_{\gamma_0;n}, r_{\mathcal{I};n}$ are the values of $r_{\gamma_0}, r_{\mathcal{I};n}$ after the n-th reflection of ζ_0 on γ_0 and $\mathcal{I}$ (note that $r_{\gamma_0;n}$ is also the r-width of the bundle of beams at the n-th reflection of ζ_0 off γ_0). Thus, as long as (1-41) holds, denoting with $\mathcal{E}_{\zeta_0;n}$ the value of $\mathcal{E}_{\zeta_0}$ at the n-th reflection of ζ_0 off γ_0, the following inductive bound holds:

$$\mathcal{E}_{\zeta_0;n} \geq A_{\mathrm{tot};n} \cdot \mathcal{E}_{\zeta_0;n-1}, \tag{1-42}$$

where the multiplicative factor

$$A_{\mathrm{tot};n} \doteq \exp\biggl(\frac{\tilde{m}^{(\varepsilon)}|_{\mathcal{I}} - \mathcal{E}_{\zeta_0;n}}{r_{\gamma_0;n}} \biggr) \tag{1-43}$$

is always greater than 1, since $\mathcal{E}_{\zeta_0;n} < \tilde{m}^{(\varepsilon)}|_{\mathcal{I}}$ (see also the relation (6-13) in Section 6A). *This is the main mechanism driving the instability,* and the proof of Theorem 1 is aimed at showing that, for some large enough $n(\varepsilon)$ depending on ε,

$$\prod_{n=0}^{n(\varepsilon)} A_{\mathrm{tot};n} > \frac{r_{0\varepsilon}}{2\mathcal{E}_{\zeta_0}|_{u=0}}. \tag{1-44}$$

Inequality (1-44) implies (in view of (1-42)) that

$$\frac{2\mathcal{E}_{\zeta_0;n(\varepsilon)}}{r_{0\varepsilon}} > 1, \tag{1-45}$$

i.e., that, after the $n(\varepsilon)$-th successive reflection of ζ_0 on γ_0 and $\mathcal{I}$, the mass difference $\mathcal{E}_{\zeta_0}$ has become so large that a trapped surface (in particular, a point where $2m/r > 1$) necessarily forms before ζ_0 reaches the mirror $\gamma_0 = \{r = r_{0\varepsilon}\}$ for the $(n(\varepsilon)+1)$-th time (provided ζ_0 was initially chosen sufficiently "narrow").[11]

Control of $r_{\gamma_0;n}$ and the final step before trapped surface formation. The main obstacle to establishing (1-44) (and, thus, Theorem 1) is the following: Once $\mathcal{E}_{\zeta_0}$ exceeds $c \cdot r_{0\varepsilon}$ for some fixed (small) $c > 0$, the total r-width of the bundle of beams close to γ_0, i.e., $r_{\gamma_0;n}$ in (1-43), increases after each successive

[11] We should remark that, once a trapped surface $\mathcal{S}$ has formed, $\{r = r_{0\varepsilon}\} \cap J^+(\mathcal{S})$ (where $J^+(\mathcal{S})$ is the future of $\mathcal{S}$) will be space-like and we will not study the evolution of the spacetime beyond $\{r = r_{0\varepsilon}\} \cap J^+(\mathcal{S})$. In particular, no more reflections of the beams will occur in the future of $\mathcal{S}$. See Theorem 2.

reflection off γ_0 and $\mathcal{I}$. Thus, the multiplicative factor (1-43) decreases as n grows. The increase in $r_{\gamma_0;n}$ is more dramatic when the spacetime is close to having a trapped surface, i.e., when $2m/r$ is close to 1.[12]

Controlling the growth of $r_{\gamma_0;n}$ is achieved by establishing an inductive bound of the form

$$r_{\gamma_0;n} \le r_{\gamma_0;n-1} \cdot \left(1 + C_0 \frac{r_{0\varepsilon}}{r_{\gamma_0;n-1}} \left(\left| \log\left(1 - \frac{2\mathcal{E}_{\zeta_0;n-1}}{r_{0\varepsilon}}\right) \right| + 1 \right)\right) \tag{1-46}$$

(see also the relation (6-14) in Section 6A). Obtaining the bound (1-46) is one of the most demanding parts in the proof of Theorem 1 and requires controlling the r-distance $r^{(1)}_{\gamma_0;n}$ of ζ_0 from the second-to-top beam ζ_1 at the n-th reflection off γ_0 for all $n \le n(\varepsilon)$, i.e., establishing a bound of the form

$$\frac{r^{(1)}_{\gamma_0;n}}{r_{0\varepsilon}} \ge 1 + c_0 \left(\frac{\mathcal{E}_{\zeta_0;0}}{r_{0\varepsilon}} \right). \tag{1-47}$$

(see (6-12) in Section 6A). The bound (1-47) is in turn obtained by establishing an inductive bound of the form

$$\log\left(\frac{r^{(1)}_{\gamma_0;n-1}}{r^{(1)}_{\gamma_0;n}} \right) \le C_0 \log\left(\frac{\mathcal{E}_{\zeta_0;n}}{\mathcal{E}_{\zeta_0;n-1}} \right), \tag{1-48}$$

estimating the decrease of $r^{(1)}_{\gamma_0;n}$ by the increase of $\mathcal{E}_{\zeta_0;n}$ at each reflection (see (6-129) in Section 6A2). The bound (1-47) is inferred from (1-48), in view of the fact that $\mathcal{E}_{\zeta_0;n} \ge \mathcal{E}_{\zeta_0;n-1}$ and

$$\sum_{n=1}^{n(\varepsilon)} \log\left(\frac{\mathcal{E}_{\zeta_0;n}}{\mathcal{E}_{\zeta_0;n-1}} \right) = \log\left(\frac{\mathcal{E}_{\zeta_0;n(\varepsilon)}}{\mathcal{E}_{\zeta_0;0}} \right) \le \log\left(\frac{r_{0\varepsilon}}{2\mathcal{E}_{\zeta_0;0}} \right). \tag{1-49}$$

At the level of the initial data, obtaining (1-46) and (1-48) requires introducing a certain hierarchy for the scales of the r-distances and mass differences associated to the beams initially (see (5-2) and (5-3) in Section 5A).

Combining (1-42) and (1-46), we can show that there exists a large $n(\varepsilon)$ such that, after $n(\varepsilon)$ reflections of ζ_0 off γ_0 (but not earlier!), we have

$$\frac{2\mathcal{E}_{\zeta_0;n(\varepsilon)}}{r_{0\varepsilon}} > 1 - c(\varepsilon), \tag{1-50}$$

where $c(\varepsilon) \ll h(\varepsilon)$ is a fixed function of ε. Note that, compared to (1-45), (1-50) is a slightly weaker bound, which just stops short of implying that a trapped surface is formed. In order to complete the proof of Theorem 1, we therefore have to consider two different scenarios for $\mathcal{E}_{\zeta_0;n(\varepsilon)}$:

Case 1: In the case when (1-45) holds, the proof of Theorem 1 follows readily, since (1-45) implies that, before ζ_0 reaches $\{r = r_{0\varepsilon}\}$ for the $(n(\varepsilon)+1)$-th time, a point arises where $2m/r > 1$.

[12] The example of two outgoing null rays in the exterior of Schwarzschild-AdS, with mass $M \ll (-\Lambda)^{-1/2}$, serves to illustrate this phenomenon: The r-separation of two rays emanating from the region close to the future event horizon $\mathcal{H}^+$, where $2m/r \sim 1$, increases dramatically by the time they reach the region $r \sim (-\Lambda)^{-1/2}$. This is, of course, nothing other than the celebrated *red-shift* effect.

Case 2: In the case when (1-50) holds but (1-45) is violated, we have the bound

$$1-c(\varepsilon)<\frac{2\mathcal{E}_{\zeta_0;n(\varepsilon)}}{r_{0\varepsilon}}\leq 1. \tag{1-51}$$

In this case, ζ_0 reaches $\{r=r_{0\varepsilon}\}$ for the $(n(\varepsilon)+1)$-th time before a trapped surface has formed. One would be tempted to repeat the above procedure for one more reflection, in an attempt to establish that a trapped surface has formed before the $(n(\varepsilon)+2)$-th reflection of ζ_0 off γ_0. However, the bound (1-51) implies that most of the bootstrap assumptions needed for the proof of Theorem 1 (which we have suppressed in this sketch for the sake of simplicity) are violated beyond the $(n(\varepsilon)+1)$-th reflection and, thus, the above procedure cannot be repeated. For this reason, we choose a different path: Applying a Cauchy stability statement backwards in time (see Theorem 3.6), we show that there exists a small perturbation $(r',(\Omega')^2,\tau',\bar{\tau}')^{(\varepsilon)}|_{u=0}$ of the initial data $(r,\Omega^2,\tau,\bar{\tau})^{(\varepsilon)}|_{u=0}$ (satisfying (1-25)), such that the perturbed solution $(r',(\Omega')^2,\tau',\bar{\tau}')^{(\varepsilon)}$ to (1-9) satisfies (1-45) and, furthermore,

$$\frac{2\mathcal{E}'_{\zeta_0;n(\varepsilon)}}{r_{0\varepsilon}}>1 \tag{1-52}$$

(where $\mathcal{E}'_{\zeta_0}$ is similarly defined by the relation (1-37) for $(r',(\Omega')^2,\tau',\bar{\tau}')^{(\varepsilon)}$ in place of $(r,\Omega^2,\tau,\bar{\tau})^{(\varepsilon)}$). Thus, we end up in the scenario of Case 1, and the proof of Theorem 1 follows readily.

Further remarks on the proof of Theorem 1. The proof of Theorem 1 involves many technical issues related to the final step of the evolution before a trapped surface is formed. Most of these technical issues simplify considerably in the case when one restricts to showing a weaker instability statement for $(\mathcal{M}_{\text{AdS}},g_{\text{AdS}})$, e.g., by replacing (1-20) with

$$\Big(1-\frac{2m}{r}\Big)^{(\varepsilon)}\Big|_{(u_\varepsilon,v_\varepsilon)}<\frac{1}{2}. \tag{1-53}$$

See Sections 4 and 6A1 for more details.

The mechanism leading to trapped surface formation in the proof of Theorem 1 only made use of the fact that we chose the initial data $(r,\Omega^2,\tau,\bar{\tau})^{(\varepsilon)}|_{u=0}$ so that the matter was supported in narrow null beams, successively reflected off γ_0 and $\mathcal{I}$, while the matter model satisfied the condition

$$T_{uv}=\Omega^2 g^{AB}T_{AB}=0. \tag{1-54}$$

Thus, we expect that the same mechanism can be adapted to the case of more general matter fields, which allow for matter to be arranged into narrow and sufficiently localised null beams, satisfying (in a region around the set of intersection of the beams)

$$T_{uv},\ |\Omega^2 g^{AB}T_{AB}|\ll T_{uu}+T_{vv}, \tag{1-55}$$

with such a configuration arising moreover from initial data which are small in a norm for which $(\mathcal{M}_{\text{AdS}},g_{\text{AdS}})$ is Cauchy stable. For an application of this mechanism in the case of the spherically symmetric Einstein-massless Vlasov system (without reducing to the radial case and without an inner mirror), see our forthcoming [Moschidis 2018].

Finally, let us remark that the general mechanism of instability suggested by the proof of Theorem 1 can be summarized as follows: In a configuration consisting of a relatively narrow bundle of nearly null beams of matter that are successively reflected on $\mathcal{I}$ and $r = 0$ (on an approximately $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$ background), the energy content of the "top" beam will increase after each pair of reflections. A similar physical space mechanism was described for the Einstein-scalar field system (1-7) in [Dimitrakopoulos et al. 2015], where it was suggested that, on a nearly null scalar field beam successively reflected off $\mathcal{I}$ and the center $r = 0$, the energy density on the top part of the beam tends to increase.

1E. ***Outline of the paper.*** This paper is organized as follows:

In Section 2, we will introduce the spherically symmetric Einstein-radial massless Vlasov system in double null coordinates. We will also formulate the notion of reflecting boundary conditions for this system on $\mathcal{I}$ and on time-like hypersurfaces of the form $\{r = r_0\}$.

In Section 3, we will formulate the asymptotically AdS characteristic initial-boundary value problem for the spherically symmetric Einstein-radial massless Vlasov system. We will then recall the main results established in [Moschidis 2017] regarding the structure of the maximal development and the Cauchy stability properties for this system.

In Section 4, we will provide a technical statement of the main result of this paper, namely the instability of AdS for the Einstein-radial massless Vlasov system with reflecting boundary conditions on $\{r = r_0\}$ and $\mathcal{I}$. The proof of this result will occupy Sections 5 and 6.

2. The Einstein-massless Vlasov system in spherical symmetry

In this section, we will review the basic properties of the spherically symmetric Einstein-massless Vlasov system in $3+1$ dimensions, expressed in double null coordinates, following the conventions introduced in [Dafermos and Rendall 2016]. We will also introduce the notion of the reflecting boundary condition on time-like hypersurfaces for the radial massless Vlasov equation. To this end, we will follow the conventions adopted in our companion paper [Moschidis 2017].

2A. ***Spherically symmetric spacetimes in double null coordinates.*** Let $(\mathcal{M}^{3+1}, g)$ be a smooth Lorentzian manifold such that $\mathcal{M}$ is of the form

$$\mathcal{M} \simeq \mathcal{U} \times \mathbb{S}^2, \tag{2-1}$$

where $\mathcal{U}$ is an open domain of $\mathbb{R}^2$ with piecewise Lipschitz boundary $\partial\mathcal{U}$ and, in the standard (u, v)-coordinates on $\mathcal{U}$, g takes the form

$$g = -\Omega^2(u, v)\, du\, dv + r^2(u, v) g_{\mathbb{S}^2}, \tag{2-2}$$

where $g_{\mathbb{S}^2}$ is the standard round metric on $\mathbb{S}^2$ and $\Omega, r : \mathcal{U} \to (0, +\infty)$ are smooth functions. In addition, we will assume that

$$\inf_{\mathcal{U}} r > 0. \tag{2-3}$$

We will also fix a time orientation on $\mathcal{M}$ by requiring that the time-like vector field $N = \partial_u + \partial_v$ is future-directed.

Remark. Notice that the action of SO(3) on $(\mathcal{M}, g)$ through rotations of the $\mathbb{S}^2$ factor of (2-1) is an isometric action.

We will also define the *Hawking mass* $m : \mathcal{M} \to \mathbb{R}$ by the expression

$$m = \tfrac{1}{2} r(1 - g(\nabla r, \nabla r)). \tag{2-4}$$

Viewed as a function on $\mathcal{U}$, m takes the form

$$m = \tfrac{1}{2} r(1 + 4\Omega^{-2}\, \partial_u r\, \partial_v r). \tag{2-5}$$

Equivalently, we have

$$\Omega^2 = 4\frac{(-\partial_u r)\, \partial_v r}{1 - 2m/r}. \tag{2-6}$$

In any local coordinate chart (y^1, y^2) on $\mathbb{S}^2$, the nonzero Christoffel symbols of (2-2) in the (u, v, y^1, y^2) local coordinate chart on $\mathcal{M}$ are computed as

$$\begin{gathered}
\Gamma^u_{uu} = \partial_u \log(\Omega^2), \quad \Gamma^v_{vv} = \partial_u \log(\Omega^2), \\
\Gamma^u_{AB} = \Omega^{-2}\, \partial_v(r^2)(g_{\mathbb{S}^2})_{AB}, \quad \Gamma^v_{AB} = \Omega^{-2}\, \partial_u(r^2)(g_{\mathbb{S}^2})_{AB}, \\
\Gamma^A_{uB} = r^{-1}\, \partial_u r \delta^A_B, \quad \Gamma^A_{vB} = r^{-1}\, \partial_v r\, \delta^A_B, \\
\Gamma^A_{BC} = (\Gamma_{\mathbb{S}^2})^A_{BC},
\end{gathered} \tag{2-7}$$

where the indices A, B, C are associated to the spherical coordinates y^1, y^2, δ^A_B is the Kronecker delta and $\Gamma_{\mathbb{S}^2}$ are the Christoffel symbols of the round sphere in the (y^1, y^2) coordinate chart.

For any pair of smooth functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ with $f_1', f_2' \neq 0$, the coordinate transformation

$$(\bar{u}, \bar{v}) = (f_1(u), f_2(v)), \tag{2-8}$$

mapping $\mathcal{U}$ to $\bar{\mathcal{U}} \subset \mathbb{R}^2$, can be used to diffeomorphically identify $\mathcal{M}$ with $\bar{\mathcal{U}} \times \mathbb{S}^2$. In these new coordinates, the metric g takes the form

$$g = -\bar{\Omega}^2(\bar{u}, \bar{v})\, d\bar{u}\, d\bar{v} + r^2(\bar{u}, \bar{v}) g_{\mathbb{S}^2}, \tag{2-9}$$

where

$$\bar{\Omega}^2(\bar{u}, \bar{v}) = \frac{1}{f_1' f_2'} \Omega^2(f_1^{-1}(\bar{u}), f_2^{-1}(\bar{v})), \tag{2-10}$$

$$r(\bar{u}, \bar{v}) = r(f_1^{-1}(\bar{u}), f_2^{-1}(\bar{v})). \tag{2-11}$$

We will frequently make use of such coordinate transformations, without renaming the coordinates each time.

Note that m is invariant under coordinate transformations of the form $(u, v) \to (f_1(u), f_2(v))$; i.e.,

$$m(\bar{u}, \bar{v}) = m(f_1^{-1}(\bar{u}), f_2^{-1}(\bar{v})). \tag{2-12}$$

2B. ***The radial massless Vlasov equation.*** Let $(\mathcal{M}, g)$ be as in Section 2A. Let $f \geq 0$ be a measure on $T\mathcal{M}$ which is constant along the geodesic flow, that is to say, in any local coordinate chart (x^0, x^1, x^2, x^3) on $\mathcal{M}$ with associated momentum coordinates (p^0, p^1, p^2, p^3) on the fibers of $T\mathcal{M}$, f satisfies (as a distribution) the first-order equation

$$p^\alpha \partial_{x^\alpha} f - \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \partial_{p^\alpha} f = 0, \tag{2-13}$$

where $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols of g in the chart (x^0, x^1, x^2, x^3). We will call f a *massless Vlasov field* if it is supported on the set $P \subset T\mathcal{M}$ of future-directed null vectors; any such vector $(x; p) \in T\mathcal{M}$ will satisfy

$$g_{\alpha\beta}(x) p^\alpha p^\beta = 0. \tag{2-14}$$

Associated to f is a symmetric (0, 2)-form on $\mathcal{M}$ (possibly defined only in the sense of distributions), the *energy momentum* tensor of f, given by the expression

$$T_{\alpha\beta}(x) = \int_{\pi^{-1}(x)} p_\alpha p_\beta f, \tag{2-15}$$

where $\pi^{-1}(x)$ denotes the fiber of $T\mathcal{M}$ over $x \in \mathcal{M}$ and the indices of the momentum coordinates are lowered with the use of the metric g, i.e.,

$$p_\gamma = g_{\gamma\delta}(x) p^\delta. \tag{2-16}$$

Remark. In this paper, we will only consider distributions f for which the expression (2-15) is finite for all $x \in \mathcal{M}$ and depends smoothly on $x \in \mathcal{M}$.

We will consider only distributions f which are spherically symmetric, i.e., invariant under the action of SO(3) on $\mathcal{M}$. In that case, in any (u, v, y^1, y^2) local coordinate chart as in Section 2A, the energy-momentum tensor T is of the form

$$T = T_{uu}(u, v)\, du^2 + 2T_{uv}(u, v)\, du\, dv + T_{vv}(u, v)\, dv^2 + T_{AB}(u, v)\, dy^A\, dy^B. \tag{2-17}$$

Furthermore, we will restrict to *radial* Vlasov fields f, i.e., fields supported only on radial null vectors which are normal to the orbits of the action of SO(3) on $\mathcal{M}$. In any (u, v, y^1, y^2) local coordinate chart as in Section 2A (with associated momentum coordinates (p^u, p^v, p^1, p^2)), a spherically symmetric, radial massless Vlasov field f has the form

$$f(u, v, y^1, y^2; p^u, p^v, p^1, p^2) = (\bar{f}_{\text{in}}(u, v; p^u) + \bar{f}_{\text{out}}(u, v; p^v))\, \delta\big(\sqrt{(g_{\mathbb{S}^2})_{AB} p^A p^B}\big)\, \delta(\Omega^2 p^u p^v), \tag{2-18}$$

where $\bar{f}_{\text{in}}, \bar{f}_{\text{out}} \geq 0$ and δ is the Dirac delta function on $\mathbb{R}$. In this case, the only nonzero components of the energy momentum tensor (2-15) are the T_{uu} and T_{vv} components. In particular, in terms of $\bar{f}_{\text{in}}, \bar{f}_{\text{out}}$, we (formally) compute

$$T_{uu}(u, v) = \int_0^{+\infty} \Omega^4 (p^v)^2 \bar{f}_{\text{out}}(u, v; p^v)\, r^2 \frac{dp^v}{p^v}, \tag{2-19}$$

$$T_{vv}(u, v) = \int_0^{+\infty} \Omega^4 (p^u)^2 \bar{f}_{\text{in}}(u, v; p^u)\, r^2 \frac{dp^u}{p^u}. \tag{2-20}$$

Remark. In this paper, we will only consider the case when $\bar{f}_{\text{in}}$, $\bar{f}_{\text{out}}$ are smooth and compactly supported in the p^u-, p^v-variables, respectively.

In the case when f is of the form (2-18), equation (2-13) is equivalent to the following system for $\bar{f}_{\text{in}}$ and $\bar{f}_{\text{out}}$:

$$\partial_u(\Omega^4 r^4 p^u \bar{f}_{\text{in}}) + \partial_{p^u}(\Omega^2 \partial_u(\Omega^2) r^4 (p^u)^2 \bar{f}_{\text{in}}) = 0, \tag{2-21}$$

$$\partial_v(\Omega^4 r^4 p^v \bar{f}_{\text{out}}) + \partial_{p^v}(\Omega^2 \partial_v(\Omega^2) r^4 (p^v)^2 \bar{f}_{\text{out}}) = 0. \tag{2-22}$$

The equations (2-21)–(2-22) readily yield the following transport equations for T_{uu}, T_{vv}:

$$\partial_v(r^2 T_{uu}) = 0, \tag{2-23}$$

$$\partial_u(r^2 T_{vv}) = 0. \tag{2-24}$$

Remark. Under a coordinate transformation of the form (2-8), $\bar{f}_{\text{in}}$, $\bar{f}_{\text{out}}$ transform as

$$\bar{f}_{\text{in}}^{(\text{new})}(f_1(u), f_2(v); f_1'(u)p) = \bar{f}_{\text{in}}(u, v; p), \tag{2-25}$$

$$\bar{f}_{\text{out}}^{(\text{new})}(f_1(u), f_2(v); f_2'(v)p) = \bar{f}_{\text{out}}(u, v; p). \tag{2-26}$$

2C. ***The spherically symmetric Einstein-radial massless Vlasov system.*** Let $(\mathcal{M}, g)$ be a smooth Lorentzian manifold and let $\Lambda < 0$. Let also f be a nonnegative measure on $T\mathcal{M}$. The *Einstein–Vlasov* system for $(\mathcal{M}, g; f)$ with cosmological constant Λ is

$$\begin{cases} \mathrm{Ric}_{\mu\nu}(g) - \frac{1}{2}R(g)g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}, \\ p^\alpha \partial_{x^\alpha} f - \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \partial_{p^\alpha} f = 0, \end{cases} \tag{2-27}$$

where $T_{\mu\nu}$ is expressed in terms of f by (2-15).

Restricting to the case where $(\mathcal{M}, g)$ is a spherically symmetric spacetime as in Section 2A and f is a radial massless Vlasov field (i.e., has the form (2-18)), the system (2-27) is equivalent to the following system for $(r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})$:

$$\partial_u\partial_v(r^2) = -\tfrac{1}{2}(1 - \Lambda r^2)\Omega^2, \tag{2-28}$$

$$\partial_u\partial_v \log(\Omega^2) = \frac{\Omega^2}{2r^2}(1 + 4\Omega^{-2}\partial_u r\, \partial_v r), \tag{2-29}$$

$$\partial_v(\Omega^{-2}\partial_v r) = -4\pi r T_{vv}\Omega^{-2}, \tag{2-30}$$

$$\partial_u(\Omega^{-2}\partial_u r) = -4\pi r T_{uu}\Omega^{-2}, \tag{2-31}$$

$$\partial_u(\Omega^4 r^4 p^u \bar{f}_{\text{in}}) = -\partial_{p^u}(\Omega^2 \partial_u(\Omega^2) r^4 (p^u)^2 \bar{f}_{\text{in}}), \tag{2-32}$$

$$\partial_v(\Omega^4 r^4 p^v \bar{f}_{\text{out}}) = -\partial_{p^v}(\Omega^2 \partial_v(\Omega^2) r^4 (p^v)^2 \bar{f}_{\text{out}}), \tag{2-33}$$

where T_{uu}, T_{vv} are expressed in terms of $\bar{f}_{\text{out}}$, $\bar{f}_{\text{in}}$ by (2-19), (2-20), respectively. Notice that the system (2-28)–(2-33) reduces to the following system for $(r, \Omega^2, T_{uu}, T_{vv})$:

$$\partial_u\partial_v(r^2) = -\tfrac{1}{2}(1 - \Lambda r^2)\Omega^2, \tag{2-34}$$

$$\partial_u\partial_v \log(\Omega^2) = \frac{\Omega^2}{2r^2}(1 + 4\Omega^{-2}\partial_u r\, \partial_v r), \tag{2-35}$$

$$\partial_v(\Omega^{-2}\,\partial_v r) = -4\pi r T_{vv}\Omega^{-2}, \tag{2-36}$$
$$\partial_u(\Omega^{-2}\,\partial_u r) = -4\pi r T_{uu}\Omega^{-2}, \tag{2-37}$$
$$\partial_u(r^2 T_{vv}) = 0, \tag{2-38}$$
$$\partial_v(r^2 T_{uu}) = 0. \tag{2-39}$$

Remark. The system (2-34)–(2-39) is the Einstein-null dust system with both ingoing and outgoing dust (used as a model for self-gravitating radiation already in [Poisson and Israel 1990]). In the notation of Section 1B of the Introduction,

$$r^2 T_{vv} = \bar{\tau}, \tag{2-40}$$
$$r^2 T_{uu} = \tau. \tag{2-41}$$

Defining the renormalized Hawking mass as

$$\tilde{m} \doteq m - \tfrac{1}{6}\Lambda r^3, \tag{2-42}$$

and using the relation (2-5), equations (2-34)–(2-39) formally give rise to the following system for $(r, \tilde{m}, T_{uu}, T_{vv})$ (valid in the region of $\mathcal{U}$ where $\partial_v r > 0$, $\partial_u r < 0$ and $1 - 2m/r > 0$):

$$\partial_u \log\left(\frac{\partial_v r}{1-2m/r}\right) = -4\pi r^{-1}\frac{r^2 T_{uu}}{-\partial_u r}, \tag{2-43}$$
$$\partial_v \log\left(\frac{-\partial_u r}{1-2m/r}\right) = 4\pi r^{-1}\frac{r^2 T_{vv}}{\partial_v r}, \tag{2-44}$$
$$\partial_u\partial_v r = -\frac{2\tilde{m} - \frac{2}{3}\Lambda r^3}{r^2}\frac{(-\partial_u r)\,\partial_v r}{1-2m/r}, \tag{2-45}$$
$$\partial_u \tilde{m} = -2\pi\frac{(1-2m/r)}{-\partial_u r} r^2 T_{uu}, \tag{2-46}$$
$$\partial_v \tilde{m} = 2\pi\frac{(1-2m/r)}{\partial_v r} r^2 T_{vv} \tag{2-47}$$
$$\partial_u(r^2 T_{vv}) = 0, \tag{2-48}$$
$$\partial_v(r^2 T_{uu}) = 0. \tag{2-49}$$

2D. ***The reflecting boundary condition for the radial Vlasov equation.*** Let $(\mathcal{M}, g)$ be as in Section 2A. Recall that $\mathcal{M}$ splits topologically as the product

$$\mathcal{M} \simeq \mathcal{U} \times \mathbb{S}^2.$$

Let $\partial_{\mathrm{tim}}\mathcal{U}$ be the subset of the boundary $\partial\mathcal{U}$ of $\mathcal{U} \subset \mathbb{R}^2$ consisting of a union of connected, time-like Lipschitz curves with respect to the comparison metric

$$g_{\mathrm{comp}} = -du\,dv \tag{2-50}$$

on $\mathbb{R}^2$. Recall that a connected Lipschitz curve γ in $\mathbb{R}^2$ is said to be time-like with respect to (2-50) if, for every point $p = (u_*, v_*) \in \gamma$, we have

$$\gamma\backslash p \subset I^+(p) \cup I^-(p) \doteq (\{u > u_*\} \cap \{v > v_*\}) \cup (\{u < u_*\} \cap \{v < v_*\}). \tag{2-51}$$

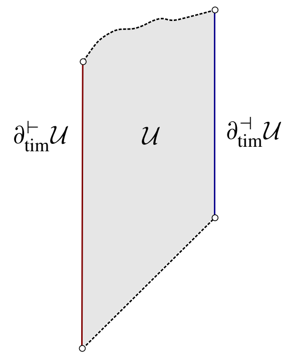

Figure 5. For a domain $\mathcal{U} \subset \mathbb{R}^2$ as depicted above, the time-like portion $\partial_{\mathrm{tim}}\mathcal{U}$ of the boundary $\partial\mathcal{U}$ splits as the union of a "left" component $\partial^{\vdash}_{\mathrm{tim}}\mathcal{U}$ and a "right" component $\partial^{\dashv}_{\mathrm{tim}}\mathcal{U}$. In general, $\partial^{\vdash}_{\mathrm{tim}}\mathcal{U}$ and $\partial^{\dashv}_{\mathrm{tim}}\mathcal{U}$ need not necessarily be straight line segments as depicted above. However, in the following sections, we will impose a gauge condition on the domains under consideration that will indeed fix $\partial^{\vdash}_{\mathrm{tim}}\mathcal{U}$ and $\partial^{\dashv}_{\mathrm{tim}}\mathcal{U}$ to be vertical line segments (see Definitions 3.2 and 3.3).

Let us fix $w : \mathcal{U} \cup \partial_{\mathrm{tim}}\mathcal{U} \to \mathbb{R}$ to be a smooth boundary-defining function of $\partial_{\mathrm{tim}}\mathcal{U}$; i.e.,

$$w|_{\partial_{\mathrm{tim}}\mathcal{U}} = 0, \quad dw|_{\partial_{\mathrm{tim}}\mathcal{U}} \neq 0, \quad w|_{\mathcal{U}} > 0.$$

We can split $\partial_{\mathrm{tim}}\mathcal{U}$ into its "left" and "right" components as

$$\partial_{\mathrm{tim}}\mathcal{U} = \partial^{\vdash}_{\mathrm{tim}}\mathcal{U} \cup \partial^{\dashv}_{\mathrm{tim}}\mathcal{U} \tag{2-52}$$

(see Figure 5), where

$$\partial^{\vdash}_{\mathrm{tim}}\mathcal{U} = \{(u_0, v_0) \in \partial_{\mathrm{tim}}\mathcal{U} : \partial_v w(u_0, v_0) > 0\},$$
$$\partial^{\dashv}_{\mathrm{tim}}\mathcal{U} = \{(u_0, v_0) \in \partial_{\mathrm{tim}}\mathcal{U} : \partial_v w(u_0, v_0) < 0\}.$$

Remark. Notice that any future-directed radial null geodesic of $\mathcal{M} = \mathcal{U} \times \mathbb{S}^2$ with a future-limiting point on $\partial^{\vdash}_{\mathrm{tim}}\mathcal{U} \times \mathbb{S}^2$ (in the ambient $\mathbb{R}^2 \times \mathbb{S}^2$ topology of $\bar{\mathcal{U}} \times \mathbb{S}^2$) is necessarily ingoing. Similarly, future-directed radial null geodesics "terminating" at $\partial^{\dashv}_{\mathrm{tim}}\mathcal{U} \times \mathbb{S}^2$ are necessarily outgoing.

In the next sections, we will only consider the reflection of radial null geodesics on parts of $\partial_{\mathrm{tim}}\mathcal{U}$ for which either $r - r_0$ (for some constant $r_0 > 0$) or $1/r$ is a boundary-defining function.

Following [Moschidis 2017], we will define the reflecting boundary condition on $\partial_{\mathrm{tim}}\mathcal{U}$ for the radial massless Vlasov equation as follows:

Definition. A radial massless Vlasov field f on $T\mathcal{M}$ will be said to satisfy the *reflecting boundary condition* on $\partial_{\mathrm{tim}}\mathcal{U} \times \mathbb{S}^2$ if and only if

- for any $(u_0, v_0) \in \partial^{\vdash}_{\mathrm{tim}}\mathcal{U}$ and any $p > 0$

$$\lim_{h \to 0^+} \left(\frac{\bar{f}_{\mathrm{out}}\big(u_0, v_0 + h; (-\partial_u w / \partial_v w)(u_0, v_0) \cdot \Omega^{-2}(u_0, v_0 + h) \cdot p\big)}{\bar{f}_{\mathrm{in}}(u_0 - h, v_0;\ \Omega^{-2}(u_0 - h, v_0) \cdot p)} \right) = 1, \tag{2-53}$$

- for any $(u_1, v_1) \in \partial^{\dashv}_{\text{tim}}\mathcal{U}$ and any $p > 0$

$$\lim_{h\to 0^+}\left(\frac{\bar{f}_{\text{in}}\big(u_1+h, v_1; (-\partial_v w/\partial_u w)(u_1, v_1)\cdot \Omega^{-2}(u_1+h, v_1)\cdot p\big)}{\bar{f}_{\text{out}}(u_1, v_1-h;\ \Omega^{-2}(u_1, v_1-h)\cdot p)}\right) = 1. \tag{2-54}$$

Remark. In the above definition, the relations (2-53)–(2-54) are equivalent to the condition that the radial Vlasov field f is constant along the *reflection* of radial null geodesics off $\partial_{\text{tim}}\mathcal{U}$; for a precise definition of the notion of a reflected radial null geodesic, see [Moschidis 2017].

Note that the relations (2-53) and (2-54) for $\bar{f}_{\text{in}}$, $\bar{f}_{\text{out}}$ imply the following boundary relations for the components (2-19)–(2-20) of the energy momentum tensor T:

- For any $(u_0, v_0) \in \partial^{\vdash}_{\text{tim}}\mathcal{U}$,

$$\lim_{h\to 0^+}\frac{r^2 T_{uu}(u_0, v_0+h)}{r^2 T_{vv}(u_0-h, v_0)} = \left(\frac{-\partial_u w}{\partial_v w}(u_0, v_0)\right)^2. \tag{2-55}$$

- For any $(u_1, v_1) \in \partial^{\dashv}_{\text{tim}}\mathcal{U}$,

$$\lim_{h\to 0^+}\frac{r^2 T_{vv}(u_1+h, v_1)}{r^2 T_{uu}(u_1, v_1-h)} = \left(\frac{-\partial_v w}{\partial_u w}(u_0, v_0)\right)^2. \tag{2-56}$$

3. The boundary-characteristic initial value problem: well-posedness and Cauchy stability

In this section, we will formulate the asymptotically AdS initial value problem for the system (2-28)–(2-33) with reflecting boundary conditions on $\{r = r_0\}$ and $\mathcal{I}$ for some $r_0 > 0$. We will then recall the main results established in [Moschidis 2017] regarding the well-posedness and the structure of the maximal development for this system.

3A. ***Asymptotically AdS characteristic initial data.*** The following was introduced in [Moschidis 2017]:

Definition 3.1 [Moschidis 2017, Definition 3.1]. For any $v_1 < v_2$ and any $r_0 > 0$, let $r_/ : [v_1, v_2) \to [r_0, +\infty)$, $\Omega_/ : [v_1, v_2) \to (0, +\infty)$ and $\bar{f}_{\text{in}/}, \bar{f}_{\text{out}/} : [v_1, v_2) \times (0, +\infty) \to [0, +\infty)$ be C^∞ functions, such that

$$r_/(v_1) = r_0, \tag{3-1}$$

$$\lim_{v\to v_2} r_/(v) = +\infty. \tag{3-2}$$

Let us define $(\partial_u r)_/ : [v_1, v_2) \to (-\infty, 0)$ by the relation

$$(\partial_u r)_/(v) = \frac{1}{r_/(v)}\left(-r_/\,\partial_v r_/(v_1) - \frac{1}{4}\int_{v1}^{v}(1-\Lambda r_/^2(\bar{v}))\Omega_/^2(\bar{v})\,d\bar{v}\right). \tag{3-3}$$

We will call $(r_/, \Omega_/^2, \bar{f}_{\text{in}/}, \bar{f}_{\text{out}/})$ an *asymptotically AdS boundary-characteristic initial data set* on $[v_1, v_2)$ for the system (2-28)–(2-33) satisfying the reflecting gauge condition at $r = r_0, +\infty$ if:

- $(r_/, \Omega_/)$ satisfies the constraint equation

$$\partial_v(\Omega_/^{-2}\,\partial_v r_/) = -4\pi r_/ (T_{vv})_/ \Omega_/^{-2}, \tag{3-4}$$

where

$$(T_{vv})_/(v) \doteq \int_0^{+\infty}\Omega_/^4(v)(p^u)^2\,\bar{f}_{\text{in}/}(v;\, p^u)\, r_/^2(v)\frac{dp^u}{p^u}. \tag{3-5}$$

- $\bar{f}_{\text{out}/}$ solves the massless radial Vlasov equation

$$\partial_v\big(\Omega^4_/(v)r^4_/(v)p^v\bar{f}_{\text{out}/}(v,p^v)\big)+\partial_{p^v}\big(\Omega^2_/\,\partial_v(\Omega^2_/)r^4_/(v)(p^v)^2\bar{f}_{\text{out}/}(v,p^v)\big)=0. \tag{3-6}$$

- $(\partial_u r)_/$ satisfies

$$\lim_{v\to v_2^-}\frac{(\partial_u r)_/}{\partial_v r_/}=1. \tag{3-7}$$

- $\bar{f}_{\text{out}/}$, $\bar{f}_{\text{in}/}$ satisfy the following compatibility conditions at $v=v_1, v_2$ for any $p>0$:

$$\frac{\bar{f}_{\text{out}/}\big(v_1;(-(\partial_u r)_//\partial_v r_/)(v_1)\cdot\Omega_/^{-2}(v_1)\cdot p\big)}{\bar{f}_{\text{in}/}(v_1;\Omega_/^{-2}(v_1)\cdot p)}=1, \tag{3-8}$$

and

$$\lim_{h\to 0^+}\left(\frac{\bar{f}_{\text{in}/}\big(v_2-h;(\partial_v r_//-(\partial_u r)_/)(v_2-h)\cdot\Omega_/^{-2}(v_2-h)\cdot p\big)}{\bar{f}_{\text{out}/}(v_2-h;\Omega_/^{-2}(v_2-h)\cdot p)}\right)=1. \tag{3-9}$$

Remark. Notice that the constraint equation (3-4) implies

$$\partial_v(\Omega_/^{-2}\,\partial_v r_/)\le 0. \tag{3-10}$$

Thus, (3-2) yields

$$\partial_v r_/>0 \tag{3-11}$$

everywhere on $[v_1,v_2)$.

Given any asymptotically AdS boundary-characteristic initial data set $(r_/,\Omega^2_/,\bar{f}_{\text{in}/},\bar{f}_{\text{out}/})$ on $[v_1,v_2)$ with reflecting gauge conditions at $r=r_0,+\infty$, we will also define the initial Hawking mass $m_/$ and initial renormalized Hawking mass $\tilde{m}_/$ on $[v_1.v_2)$ by the relations

$$m_/\doteq\tfrac{1}{2}r_/(1-4\Omega_/^{-2}(\partial_u r)_/\,\partial_v r_/), \tag{3-12}$$

$$\tilde{m}_/\doteq m_/-\tfrac{1}{6}\Lambda r_/^3, \tag{3-13}$$

in accordance with (2-6), (2-42).

3B. ***Developments with reflecting boundary conditions on $r=r_0,+\infty$.*** We will only consider solutions $(r,\Omega^2,\bar{f}_{\text{in}},\bar{f}_{\text{out}})$ to (2-28)–(2-33) satisfying a reflecting gauge condition on $\partial_{\text{tim}}\mathcal{U}$, which fixes $\partial_{\text{tim}}\mathcal{U}$ to be a union of vertical straight lines in the (u,v)-plane. This motivates defining the following class of domains $\mathcal{U}$ in the plane (see [Moschidis 2017]):

Definition 3.2 [Moschidis 2017, Definition 3.3]. For any $v_0>0$, let $\mathscr{U}_{v_0}$ be the set of all connected open domains $\mathcal{U}$ of the (u,v)-plane with piecewise Lipschitz boundary $\partial\mathcal{U}$, with the property that $\partial\mathcal{U}$ splits as the union

$$\partial\mathcal{U}=\gamma_0\cup\mathcal{I}\cup\mathcal{S}_{v_0}\cup\text{clos}(\gamma), \tag{3-14}$$

where, for some $0<u_{\gamma_0},u_{\mathcal{I}}\le+\infty$,

$$\gamma_0=\{u=v\}\cap\{0\le u<u_{\gamma_0}\}, \tag{3-15}$$

$$\mathcal{I}=\{u=v-v_0\}\cap\{0\le u<u_{\mathcal{I}}\}, \tag{3-16}$$

$$\mathcal{S}_{v_0}=\{0\}\times[0,v_0], \tag{3-17}$$

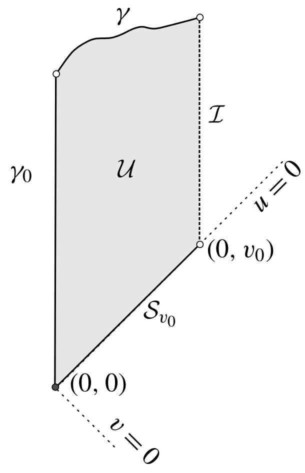

Figure 6. A typical domain $\mathcal{U} \in \mathscr{U}_{v_0}$ would be as depicted above. In the case when the boundary set γ is empty, it is necessary that both γ_0 and $\mathcal{I}$ are unbounded (i.e., extend all the way to $u+v=\infty$).

and $\gamma : (x_1, x_2) \to \mathbb{R}^2$ is a Lipschitz, achronal (with respect to the reference Lorentzian metric (2-50)) curve, which is allowed to be empty (the closure $\mathrm{clos}(\gamma)$ of γ in (3-14) is considered with respect to the standard topology of $\mathbb{R}^2$). See Figure 6.

Remark. It follows readily from Definition 3.2 that $\mathcal{U}$ is necessarily contained in the future domain of dependence of $\mathcal{S}_{v_0} \cup \gamma_0 \cup \mathcal{I}$ (with respect to the comparison metric (2-50)). In the case when $\gamma = \varnothing$ in (3-14), it is necessary that both γ_0 and $\mathcal{I}$ extend all the way to $u+v=+\infty$.

A development of an asymptotically AdS boundary-characteristic initial data set for the system (2-28)–(2-33) with reflecting boundary conditions on $r = r_0, +\infty$ is defined as follows (see [Moschidis 2017]):

Definition 3.3 [Moschidis 2017, Definition 3.4]. For any $v_0 > 0$ and $r_0 > 0$, let $(r_/, \Omega^2_/, \bar{f}_{\mathrm{in}/}, \bar{f}_{\mathrm{out}/})$ be a smooth asymptotically AdS boundary-characteristic initial data set on $[0, v_0)$ for the system (2-28)–(2-33) satisfying the reflecting gauge condition at $r = r_0, +\infty$, according to Definition 3.1. A *future development* of $(r_/, \Omega^2_/, \bar{f}_{\mathrm{in}/}, \bar{f}_{\mathrm{out}/})$ will consist of an open set $\mathcal{U} \in \mathscr{U}_{v_0}$ (see Definition 3.2) and smooth functions $r : \mathcal{U} \to (r_0, +\infty)$, $\Omega^2 : \mathcal{U} \to (0, +\infty)$ and $\bar{f}_{\mathrm{in}}, \bar{f}_{\mathrm{out}} : \mathcal{U} \times (0, +\infty) \to [0, +\infty)$ satisfying the following properties:

(1) The functions $r, \Omega^2, \bar{f}_{\mathrm{in}}, \bar{f}_{\mathrm{out}}$ solve the system (2-28)–(2-33) on $\mathcal{U}$.

(2) The functions $r, \Omega^2, \bar{f}_{\mathrm{in}}, \bar{f}_{\mathrm{out}}$ satisfy the given initial conditions on $\mathcal{S}_{v_0} = \{0\} \times [0, v_0)$, i.e.,

$$(r, \Omega^2, \bar{f}_{\mathrm{in}}, \bar{f}_{\mathrm{out}})|_{\mathcal{S}_{v_0}} = (r_/, \Omega^2_/, \bar{f}_{\mathrm{in}/}, \bar{f}_{\mathrm{out}/}). \tag{3-18}$$

(3) The functions $(r, \bar{f}_{\mathrm{in}}, \bar{f}_{\mathrm{out}})$ satisfy on γ_0 the boundary conditions

$$r|_{\gamma_0} = r_0, \tag{3-19}$$

$$\bar{f}_{\mathrm{out}}(u_*, v_*;\ p) = \bar{f}_{\mathrm{in}}(u_*, v_*;\ p), \tag{3-20}$$

for all $(u_*, v_*) \in \gamma_0$ and $p > 0$, and on $\mathcal{I}$ the boundary conditions

$$(1/r)|_{\mathcal{I}} = 0, \tag{3-21}$$

$$\lim_{h\to 0^+}\left(\frac{\bar{f}_{\text{in}}(u_*+h, v_*;\ \Omega^{-2}(u_*+h, v_*)\cdot p)}{\bar{f}_{\text{out}}(u_*, v_*-h;\ \Omega^{-2}(u_*, v_*-h)\cdot p)}\right) = 1, \tag{3-22}$$

for all $(u_*, v_*) \in \mathcal{I}$ and $p > 0$.

(4) The following are satisfied on γ_0 and $\mathcal{I}$:

$$\partial_u r|_{\gamma_0} = -\partial_v r|_{\gamma_0}, \tag{3-23}$$

$$\partial_u(1/r)|_{\mathcal{I}} = -\partial_v(1/r)|_{\mathcal{I}}. \tag{3-24}$$

Remark. Notice that the boundary conditions (3-19) and (3-21), combined with the form (3-15) and (3-16) of γ_0 and $\mathcal{I}$, respectively, imply the relations (3-23) and (3-24). However, the relations (3-23) and (3-24) should be viewed as *gauge conditions* fixing, in conjunction with (3-19) and (3-21), the form (3-15) and (3-16) of γ_0 and $\mathcal{I}$.

If $\mathscr{D} = (\mathcal{U}; r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})$ and $\mathscr{D}' = (\mathcal{U}'; r', (\Omega')^2, \bar{f}'_{\text{in}}, \bar{f}'_{\text{out}})$ are two future developments of the same initial data $(r_/, \Omega^2_/, \bar{f}_{\text{in}/}, \bar{f}_{\text{out}/})$, we will say that $\mathscr{D}'$ is an extension of $\mathscr{D}$, writing $\mathscr{D} \subseteq \mathscr{D}'$, if $\mathcal{U} \subseteq \mathcal{U}'$ and the restriction of $(r', (\Omega')^2, \bar{f}'_{\text{in}}, \bar{f}'_{\text{out}})$ on $\mathcal{U}$ coincides with $(r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})$.

Remark. If $\mathscr{D} = (\mathcal{U}; r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})$ and $\mathscr{D}' = (\mathcal{U}'; r', (\Omega')^2, \bar{f}'_{\text{in}}, \bar{f}'_{\text{out}})$ are two future developments of the same initial data $(r_/, \Omega^2_/, \bar{f}_{\text{in}/}, \bar{f}_{\text{out}/})$, then

$$(r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})|_{\mathcal{U}\cap\mathcal{U}'} = (r', (\Omega')^2, \bar{f}'_{\text{in}}, \bar{f}'_{\text{out}})|_{\mathcal{U}\cap\mathcal{U}'}; \tag{3-25}$$

see [Moschidis 2017].

3C. *The maximal development.*

The following result was established in [Moschidis 2017]:

Theorem 3.4 [Moschidis 2017, Theorem 1]. *For any $v_0 > 0$ and $r_0 > 0$, let $(r_/, \Omega^2_/, \bar{f}_{\text{in}/}, \bar{f}_{\text{out}/})$ be a smooth asymptotically AdS boundary-characteristic initial data set on* $[0, v_0)$ *for the system* (2-28)–(2-33) *satisfying the reflecting gauge condition at $r = r_0, +\infty$, according to Definition 3.1, such that the quantities $\Omega^2_/ / \left(1 - \frac{1}{3}\Lambda r^2_/\right)$, $r^2_/(T_{vv})_/$, and $\tan^{-1} r_/$ extend smoothly on $v = v_0$. Then, there exists a unique, smooth future development $(\mathcal{U}; r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})$ of $(r_/, \Omega^2_/, \bar{f}_{\text{in}/}, \bar{f}_{\text{out}/})$ which is maximal*; *i.e., any other future development $(\mathcal{U}'; r', (\Omega')^2, \bar{f}'_{\text{in}}, \bar{f}'_{\text{out}})$ of $(r_/, \Omega^2_/, \bar{f}_{\text{in}/}, \bar{f}_{\text{out}/})$ with $r' \geq r_0$ everywhere on $\mathcal{U}'$ satisfies $\mathcal{U}' \subseteq \mathcal{U}$ and $r', (\Omega')^2, \bar{f}'_{\text{in}}, \bar{f}'_{\text{out}}$ are the restrictions of $r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}}$ on $\mathcal{U}'$.*

The maximal future development $(\mathcal{U}; r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})$ satisfies the following properties (*for the definition of the curves $\gamma_0, \mathcal{I}, \gamma$, see Definition 3.2*):

(1) *The renormalized Hawking mass $\tilde{m}$ is conserved on γ_0 and $\mathcal{I}$, i.e.*,

$$\tilde{m}|_{\gamma_0} = \tilde{m}|_{\gamma_0\cap\{u=0\}}, \tag{3-26}$$

$$\tilde{m}|_{\mathcal{I}} = \tilde{m}|_{\mathcal{I}\cap\{u=0\}}. \tag{3-27}$$

(2) *The curve $\mathcal{I}$ is conformally complete*; *i.e.*, $\Omega^2/\big(1-\frac{1}{3}\Lambda r^2\big)$ *has a finite limit on $\mathcal{I}$ and*

$$\int_{\mathcal{I}} \sqrt{\frac{\Omega^2}{1-\frac{1}{3}\Lambda r^2}}\bigg|_{\mathcal{I}}\, du = +\infty. \tag{3-28}$$

(3) *We have*

$$\partial_u r < 0, \tag{3-29}$$

$$\Big(1-\frac{2m}{r}\Big)\Big|_{J^-(\mathcal{I})\cup J^-(\gamma_0)} > 0 \tag{3-30}$$

$$\partial_v r|_{J^-(\mathcal{I})\cup J^-(\gamma_0)} > 0, \tag{3-31}$$

where

$$J^-(\mathcal{I}) = \big\{0 \le u < \sup_{\mathcal{I}} u\big\} \cap \mathcal{U} \tag{3-32}$$

is the causal past of $\mathcal{I}$ and

$$J^-(\gamma_0) = \big\{0 \le v < \sup_{\gamma_0} v\big\} \cap \mathcal{U} \tag{3-33}$$

is the causal past of γ_0 (*with respect to the reference Lorentzian metric* (2-50)).

(4) *In the case $\mathcal{U}\backslash J^-(\mathcal{I}) \neq \varnothing$, the future event horizon*

$$\mathcal{H}^+ \doteq \mathcal{U} \cap \partial J^-(\mathcal{I}) = \big\{u = \sup_{\mathcal{I}} u\big\} \cap \mathcal{U} \tag{3-34}$$

has the following properties:

(a) *$\mathcal{H}^+$ has infinite affine length*, *i.e.*,

$$\int_{\mathcal{H}^+} \Omega^2\, dv = +\infty. \tag{3-35}$$

(b) *We have*

$$\sup_{\mathcal{H}^+} r = r_S \tag{3-36}$$

and

$$\inf_{\mathcal{H}^+}\Big(1-\frac{2m}{r}\Big) = 0, \tag{3-37}$$

where r_S defined by the relation

$$1-2\frac{\lim_{v\to v_0^-} \tilde{m}_/(v)}{r_S} - \tfrac{1}{3}\Lambda r_S^2 = 0. \tag{3-38}$$

(5) *In the case $\mathcal{H}^+ \neq \varnothing$, the curve γ_0 is bounded and satisfies*

$$\gamma_0 \nsubseteq J^-(\mathcal{I}); \tag{3-39}$$

i.e., *γ_0 contains points lying to the future of $\mathcal{H}^+$.*

(6) *In the case $\mathcal{H}^+ \neq \varnothing$, the curve γ is nonempty, piecewise smooth and r extends continuously on γ with $r|_{\gamma_0} = r_0$. Furthermore, for any point $(u_1, v_1) \in \gamma$, the line $\{v = v_1\}$ intersects $\mathcal{I}$.*[13]

[13] In other words, there is no point in γ which lies on the curve $\{v = v_{\mathcal{I}}\}$, where $(u_{\mathcal{I}}, v_{\mathcal{I}})$ is the future limit point of $\mathcal{I}$.

Remark. In the case when $\mathcal{U}\backslash J^{-}(\mathcal{I}) \neq \varnothing$ (and thus $\mathcal{H}^{+} \neq \varnothing$), in view of (3-36), (3-38) and the fact that $r > r_0$ on $\mathcal{U}$, it is necessary that

$$2\frac{\lim_{v\to v_0^-} \tilde{m}_{/}(v)}{r_0} > 1 - \tfrac{1}{3}\Lambda r_0^2. \tag{3-40}$$

In a similar way, we can uniquely define the *maximal past development* $(\mathcal{U}; r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})$ of $(r_{/}, \Omega^2_{/}, \bar{f}_{\text{in}/}, \bar{f}_{\text{out}/})$, satisfying the properties outlined by Theorem 3.4 after performing a "time reversal" transformation $(u, v) \to (-v, -u)$. Notice that such a coordinate transformation turns an asymptotically AdS boundary-characteristic initial data set on $u = 0$ into an asymptotically AdS boundary-characteristic initial data set on $v = 0$. However, Theorem 3.4 also holds (with exactly the same proof) for such initial data sets.

3D. ***Cauchy stability in a rough norm, uniformly in r_0.*** In [Moschidis 2017], the following "norm" was introduced for smooth asymptotically AdS boundary-characteristic initial data sets $(r_{/}, \Omega^2_{/}, \bar{f}_{\text{in}/}, \bar{f}_{\text{out}/})$ on $[0, v_0)$ for the system (2-28)–(2-33):

$$\begin{aligned}\|(r_{/}, \Omega^2_{/}, \bar{f}_{\text{in}/}, \bar{f}_{\text{out}/})\|_{\mathcal{CS}} &\\ \doteq \sqrt{-\Lambda} \sup_{0\le v<v_0} |\tilde{m}_{/}(v)| + (-\Lambda) &\sup_{0\le v<v_0} \int_0^{v_0} \frac{1}{\rho_{/}(v) - \rho_{/}(\bar{v}) + \rho_{/}(0)} \left(\frac{r^2_{/}(T_{vv})_{/}}{\partial_v \rho_{/}}\right)(\bar{v})\, d\bar{v} \\ &+ \sup_{0\le v<v_0} \max\left\{\frac{2m_{/}}{r_{/}}, 0\right\},\end{aligned} \tag{3-41}$$

where

$$\rho_{/}(v) \doteq \tan^{-1}(\sqrt{-\Lambda}\, r_{/}(v)). \tag{3-42}$$

Remark. Note that, in (3-41),

$$\rho_{/}(0) = \tan^{-1}(\sqrt{-\Lambda}\, r_0).$$

The expression (3-41) is invariant under gauge transformations, as well as scale transformations of the form $(u, v) \to (\lambda u, \lambda v)$, $(r, \tilde{m}, \Lambda) \to (\lambda r, \lambda\tilde{m}, \lambda^{-2}\Lambda)$, $r_0 \to \lambda r_0$, $(\bar{f}_{\text{in}}, \bar{f}_{\text{out}}) \to (\lambda^{-4}\bar{f}_{\text{in}}, \lambda^{-4}\bar{f}_{\text{out}})$. Moreover, $\|(r_{/}, \Omega^2_{/}, \bar{f}_{\text{in}/}, \bar{f}_{\text{out}/})\|_{\mathcal{CS}} = 0$ if and only if $(r_{/}, \Omega^2_{/}, \bar{f}_{\text{in}/}, \bar{f}_{\text{out}/})$ are the initial data for pure AdS spacetime on $\{r \ge r_0\}$, i.e., if $\bar{f}_{\text{in}/} = \bar{f}_{\text{out}/} = 0$ and $\tilde{m} = 0$.

The following Cauchy stability result for the trivial initial data was established in [Moschidis 2017]:

Proposition 3.5 [Moschidis 2017, Corollary 1]. *For any* (*possibly large*) $l_* > 0$, *there exists a* (*small*) $\varepsilon_0 > 0$ *and a constant* $C_{l_*} > 0$ *depending only on* l_*, *so that the following statement holds*: *for any* $v_0 > 0$ *and* $0 < r_0 < (-\Lambda)^{-1/2}$, *if* $(r_{/}, \Omega^2_{/}, \bar{f}_{\text{in}/}, \bar{f}_{\text{out}/})$ *is a smooth asymptotically AdS boundary-characteristic initial data set on* $[0, v_0)$ *for the system* (2-28)–(2-33) *satisfying the reflecting gauge condition at* $r = r_0, +\infty$, *according to Definition 3.1, such that the quantities* $\Omega^2_{/}/\big(1 - \frac{1}{3}\Lambda r^2_{/}\big)$, $r^2_{/}(T_{vv})_{/}$, *and* $\tan^{-1} r_{/}$ *extend smoothly on* $v = v_0$ *and moreover*

$$\|(r_{/}, \Omega^2_{/}, \bar{f}_{\text{in}/}, \bar{f}_{\text{out}/})\|_{\mathcal{CS}} < \varepsilon \tag{3-43}$$

for some $0 < \varepsilon \le \varepsilon_0$, *then the maximal development* $(\mathcal{U}; r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})$ *satisfies*

$$\mathcal{W}_{l_*} \doteq \{0 < u \le l_* v_0\} \cap \{u < v < u + v_0\} \subset \mathcal{U} \tag{3-44}$$

and

$$\sqrt{-\Lambda}\sup_{\mathcal{W}_{l_*}}|\tilde{m}|+\sup_{\mathcal{W}_{l_*}}\log\Big(\frac{1-\frac{1}{3}\Lambda r^2}{1-\max\{2m/r,0\}}\Big) \\ +\sup_{\bar{u}}\int_{\{u=\bar{u}\}\cap\mathcal{W}_{l_*}}\frac{rT_{vv}}{\partial_v r}\,dv+\sup_{\bar{v}}\int_{\{v=\bar{v}\}\cap\mathcal{W}_{l_*}}\frac{rT_{uu}}{(-\partial_u r)}\,du<C_{l_*}\varepsilon. \quad (3\text{-}45)$$

Remark. Proposition 3.5 should be interpreted as a Cauchy stability statement for the pure AdS initial data set with respect to the topology defined by (3-41) which is independent of the radius r_0 of the reflecting boundary.

Considering the spherically symmetric Einstein-scalar field system (1-7) with an inner mirror placed at $\{r=r_0\}$, the analogue of the initial-data norm (3-41) (obtained using the substitution $(T_{vv})_/\to(\partial_v\varphi)|_{u=0}$) is rougher compared to the bounded variation norm of Christodoulou [1993]. It is not known whether (1-7), restricted to the exterior of an inner mirror at $\{r=r_0\}$, satisfies a Cauchy stability estimate with respect to the analogue of the initial-data norm (3-41) which is independent of r_0 (although local existence and uniqueness follow trivially in this case for fixed r_0).

In fact, Proposition 3.5 is a special case of the following Cauchy stability estimate established in [Moschidis 2017]:

Theorem 3.6 [Moschidis 2017, Theorem 2]. *For any $v_1<v_2$ and $0<r_0<(-\Lambda)^{-1/2}$, let $(r_{/i},\Omega^2_{/i},\bar{f}_{\text{in}/i},\bar{f}_{\text{out}/i})$, $i=1,2$, be two smooth asymptotically AdS boundary-characteristic initial data sets on $[v_1,v_2)$ for the system (2-28)–(2-33) satisfying the reflective gauge condition at $r=r_0,+\infty$, according to Definition 3.1, such that the quantities $\Omega^2_{/i}/\big(1-\frac{1}{3}\Lambda r^2_{/i}\big)$, $r^2_{/i}(T_{vv})_{/i}$, and $\tan^{-1}r_{/i}$ extend smoothly on $v=v_2$. Assume, also, the following conditions:*

(1) *For some $u_0>0$, the maximal future development $(\mathcal{U}_1;r_1,\Omega^2_1,\bar{f}_{\text{in}1},\bar{f}_{\text{out}1})$ of $(r_{/1},\Omega^2_{/1},\bar{f}_{\text{in}/1},\bar{f}_{\text{out}/1})$ satisfies*

$$\mathcal{W}_{u_0}\doteq\{0<u<u_0\}\cap\{u+v_1<v<u+v_2\}\subset\mathcal{U}_1 \quad (3\text{-}46)$$

and

$$\sup_{\mathcal{W}_{u_0}}\Big\{\Big|\log\Big(\frac{\Omega_1^2}{1-\frac{1}{3}\Lambda r_1^2}\Big)\Big|+\Big|\log\Big(\frac{2\partial_v r_1}{1-2m_1/r_1}\Big)\Big|+\Big|\log\Big(\frac{1-2m_1/r_1}{1-\frac{1}{3}\Lambda r_1^2}\Big)\Big|+\sqrt{-\Lambda}|\tilde{m}_1|\Big\} \\ +\sup_{\bar{u}}\int_{\{u=\bar{u}\}\cap\mathcal{W}_{u_0}}r_1\frac{(T_{vv})_1}{\partial_v r_1}\,dv+\sup_{\bar{v}}\int_{\{v=\bar{v}\}\cap\mathcal{W}_{u_0}}r_1\frac{(T_{uu})_1}{-\partial_u r_1}\,du=C_0<+\infty. \quad (3\text{-}47)$$

(2) *The $(r_{/i},\Omega^2_{/i},\bar{f}_{\text{in}/i},\bar{f}_{\text{out}/i})$, $i=1,2$, are δ-close in the following sense:*

$$\sup_{v\in[v_1,v_2)}\Big\{\Big|\log\Big(\frac{\Omega^2_{/1}}{1-\frac{1}{3}\Lambda r^2_{/1}}\Big)-\log\Big(\frac{\Omega^2_{/2}}{1-\frac{1}{3}\Lambda r^2_{/2}}\Big)\Big| \\ +\Big|\log\Big(\frac{2\partial_v r_{/1}}{1-2m_{/1}/r_{/1}}\Big)-\log\Big(\frac{2\partial_v r_{/2}}{1-2m_{/2}/r_{/2}}\Big)\Big|+ \\ +\Big|\log\Big(\frac{1-2m_{/1}/r_{/1}}{1-\frac{1}{3}\Lambda r^2_{/1}}\Big)-\log\Big(\frac{1-2m_{/2}/r_{/2}}{1-\frac{1}{3}\Lambda r^2_{/2}}\Big)\Big|+\sqrt{-\Lambda}|\tilde{m}_{/1}-\tilde{m}_{/2}|\Big\}(v)\le\delta \quad (3\text{-}48)$$

and

$$\sup_{v\in[v_1,v_2]}(-\Lambda)\int_{v_1}^{v_2}\left|r_{/1}^2\frac{(T_{vv})_{/1}}{\partial_v\rho_1}(\bar{v})-r_{/2}^2\frac{(T_{vv})_{/2}}{\partial_v\rho_2}(\bar{v})\right|\big(|\rho_/(v)-\rho_/(\bar{v})|+\rho_/(v_1)\big)^{-1}\,d\bar{v}\le\delta, \tag{3-49}$$

where C_1 is a large fixed absolute constant, δ satisfies

$$0\le\delta\le\delta_0\doteq\exp\Big(-\exp\Big(C_1(1+C_0)\frac{u_0}{v_2-v_1}\Big)\Big) \tag{3-50}$$

and $\rho_/$ is defined by the relation

$$\rho_/(v)\doteq\tan^{-1}\Big(\sqrt{-\frac{\Lambda}{3}}r_/(v)\Big). \tag{3-51}$$

Then, the maximal development $(\mathcal{U}_2;\, r_2,\, \Omega_2^2,\, \bar{f}_{\text{in}2},\, \bar{f}_{\text{out}2})$ of $(r_{/2},\, \Omega_{/2}^2,\, \bar{f}_{\text{in}/2},\, \bar{f}_{\text{out}/2})$ satisfies

$$\mathcal{W}_{u_0}\subset\mathcal{U}_2 \tag{3-52}$$

and

$$\begin{aligned}\sup_{\mathcal{W}_{u_0}}\bigg\{&\bigg|\log\Big(\frac{\Omega_1^2}{1-\frac{1}{3}\Lambda r_1^2}\Big)-\log\Big(\frac{\Omega_2^2}{1-\frac{1}{3}\Lambda r_2^2}\Big)\bigg|+\bigg|\log\Big(\frac{2\partial_v r_1}{1-2m_1/r_1}\Big)-\log\Big(\frac{2\partial_v r_2}{1-2m_2/r_2}\Big)\bigg|\\&+\bigg|\log\Big(\frac{1-2m_1/r_1}{1-\frac{1}{3}\Lambda r_1^2}\Big)-\log\Big(\frac{1-2m_2/r_2}{1-\frac{1}{3}\Lambda r_2^2}\Big)\bigg|+\sqrt{-\Lambda}|\tilde{m}_1-\tilde{m}_2|\bigg\}\\&+\sup_{\bar{u}}\int_{\{u=\bar{u}\}\cap\mathcal{W}_{u_0}}|r_1(T_{vv})_1-r_2(T_{vv})_2|\,dv+\sup_{\bar{v}}\int_{\{v=\bar{v}\}\cap\mathcal{W}_{u_0}}|r_1(T_{uu})_1-r_2(T_{uu})_2|\,du\\&\le\exp\Big(\exp(C_1(1+C_0))\frac{u_0}{v_2-v_1}\Big)\delta.\end{aligned} \tag{3-53}$$

Remark. By repeating the proof of Theorem 3.6, the Cauchy stability estimate (3-53) also holds in the case when $(\mathcal{U}_i;\, r_i,\, \Omega_i^2,\, \bar{f}_{\text{in};i},\, \bar{f}_{\text{out};i})$, $i=1,2$, are the maximal *past* developments of $(r_{/i},\, \Omega_{/i}^2,\, \bar{f}_{\text{in}/i},\, \bar{f}_{\text{out}/i})$, i.e., when $\mathcal{W}_{u_0}$ is replaced by

$$\mathcal{W}_{u_0}^{(-)}\doteq\{-u_0\le u<0\}\cap\{u+v_1<v<u+v_2\} \tag{3-54}$$

and (3-47) holds on $\mathcal{W}_{u_0}^{(-)}$ in place of $\mathcal{W}_{u_0}$.

4. Final statement of Theorem 1: the nonlinear instability of AdS

The main result of this paper is the following:

Theorem 1 (final version). *For any $\varepsilon\in(0,1]$, there exist $r_{0\varepsilon}$, $v_{0\varepsilon}$ depending smoothly on ε such that*

$$r_{0\varepsilon}\xrightarrow{\varepsilon\to0}0, \tag{4-1}$$

$$\sqrt{-\Lambda}v_{0\varepsilon}\xrightarrow{\varepsilon\to0}\sqrt{3}\pi, \tag{4-2}$$

as well as a family $(r_/^{(\varepsilon)},\, (\Omega_/^{(\varepsilon)})^2,\, \bar{f}_{\text{in}/}^{(\varepsilon)},\, \bar{f}_{\text{out}/}^{(\varepsilon)})$ of smooth asymptotically AdS boundary-characteristic initial data sets for the system (2-28)–(2-33) *satisfying the reflecting gauge condition at $r=r_0,+\infty$, such that the following hold*:

(1) *The family* $(r_{/}^{(\varepsilon)}, (\Omega_{/}^{(\varepsilon)})^2, \bar{f}_{\mathrm{in}/}^{(\varepsilon)}, \bar{f}_{\mathrm{out}/}^{(\varepsilon)})$ *satisfies*

$$\|(r_{/}^{(\varepsilon)}, (\Omega_{/}^{(\varepsilon)})^2, \bar{f}_{\mathrm{in}/}^{(\varepsilon)}, \bar{f}_{\mathrm{out}/}^{(\varepsilon)})\|_{\mathcal{CS}} \xrightarrow{\varepsilon\to 0} 0, \tag{4-3}$$

where $\|\cdot\|_{\mathcal{CS}}$ *is the norm defined by* (3-41).

(2) *There exists a trapped sphere, i.e., a point* $(u_\dagger, v_\dagger)$ *in the maximal future development* $(\mathcal{U}_\varepsilon; r_\varepsilon, \Omega_\varepsilon^2, \bar{f}_{\mathrm{in}\varepsilon}, \bar{f}_{\mathrm{out}\varepsilon})$ *of* $(r_{/}^{(\varepsilon)}, (\Omega_{/}^{(\varepsilon)})^2, \bar{f}_{\mathrm{in}/}^{(\varepsilon)}, \bar{f}_{\mathrm{out}/}^{(\varepsilon)})$ *such that*

$$\frac{2m}{r}(u_\dagger, v_\dagger) > 1. \tag{4-4}$$

In particular, in view of Theorem 3.4, $(\mathcal{U}_\varepsilon; r_\varepsilon, \Omega_\varepsilon^2, \bar{f}_{\mathrm{in}\varepsilon}, \bar{f}_{\mathrm{out}\varepsilon})$ *has a nonempty future event horizon* $\mathcal{H}^+$ *(defined by* (5-30)), *satisfying the properties* (4*a*) *and* (4*b*) *of Theorem 3.4, and a complete conformal infinity* $\mathcal{I}$ *(satisfying* (3-28)).

Remark. If

$$\delta(\varepsilon) \doteq \|(r_{/}^{(\varepsilon)}, (\Omega_{/}^{(\varepsilon)})^2, \bar{f}_{\mathrm{in}/}^{(\varepsilon)}, \bar{f}_{\mathrm{out}/}^{(\varepsilon)})\|_{\mathcal{CS}}, \tag{4-5}$$

the point $(u_\dagger, v_\dagger)$ satisfies the upper bound

$$u_\dagger \le \exp(\exp(\delta^6(\varepsilon)))v_0. \tag{4-6}$$

On the other hand, in view of Proposition 3.5, we necessarily have

$$u_\dagger \xrightarrow{\varepsilon\to 0} +\infty. \tag{4-7}$$

In the simpler case when one is interested in a weaker instability statement, such as the existence of a point $(u_\dagger, v_\dagger)$ where

$$\frac{2m}{r}\bigg|_{(u_\dagger, v_\dagger)} > \frac{1}{2} \tag{4-8}$$

(instead of the stronger bound (4-4)), the proof of Theorem 1 can be substantially simplified. In the case of (4-8), the upper bound (4-6) can be improved into a polynomial bound

$$u_\dagger \le (\delta(\varepsilon))^{-C_1} v_0 \tag{4-9}$$

for some fixed $C_1 > 0$.

5. Construction of the initial data and notation

As described already in Section 1D of the Introduction, the initial data family in Theorem 1 will be such that their development consists of a large number of initially ingoing Vlasov beams. In this section, we will construct such a family $(r_{/\varepsilon}, \Omega_{/\varepsilon}^2, \bar{f}_{\mathrm{in}/\varepsilon}, \bar{f}_{\mathrm{out}/\varepsilon})$ of asymptotically AdS boundary-characteristic initial data for (2-28)–(2-33). The family $(r_{/}^{(\varepsilon)}, (\Omega_{/}^{(\varepsilon)})^2, \bar{f}_{\mathrm{in}/}^{(\varepsilon)}, \bar{f}_{\mathrm{out}/}^{(\varepsilon)})$ in the statement of Theorem 1 will be eventually obtained from $(r_{/\varepsilon}, \Omega_{/\varepsilon}^2, \bar{f}_{\mathrm{in}/\varepsilon}, \bar{f}_{\mathrm{out}/\varepsilon})$ after possibly adding a suitable perturbation (see Section 6).

This section is organized as follows: In Section 5A, we will introduce a certain hierarchy of parameters that will be necessary for the construction of $(r_{/\varepsilon}, \Omega^2_{/\varepsilon}, \bar{f}_{\mathrm{in}/\varepsilon}, \bar{f}_{\mathrm{out}/\varepsilon})$ in Section 5B. In Section 5C, we will introduce some basic notation related to the maximal future development $(\mathcal{U}_\varepsilon; r_\varepsilon, \Omega^2_\varepsilon, \bar{f}_{\mathrm{in}\varepsilon}, \bar{f}_{\mathrm{out}\varepsilon})$ of $(r_{/\varepsilon}, \Omega^2_{/\varepsilon}, \bar{f}_{\mathrm{in}/\varepsilon}, \bar{f}_{\mathrm{out}/\varepsilon})$. Finally, in Section 5D, we will perform some basic geometric constructions on $(\mathcal{U}_\varepsilon; r_\varepsilon, \Omega^2_\varepsilon, \bar{f}_{\mathrm{in}\varepsilon}, \bar{f}_{\mathrm{out}\varepsilon})$, related to the separation of $\mathcal{U}_\varepsilon$ into various subregions by the Vlasov beams arising from the initial data.

5A. ***Parameters and auxiliary functions.*** Let us fix some smooth and strictly increasing functions $h_0, h_1, h_2 : (0, 1) \to (0, 1)$, so that

$$\lim_{\varepsilon\to 0^+} h_0(\varepsilon) = \lim_{\varepsilon\to 0^+} h_1(\varepsilon) = \lim_{\varepsilon\to 0^+} h_2(\varepsilon) = 0, \tag{5-1}$$

$$\lim_{\varepsilon\to 0^+} \varepsilon \cdot \exp\Big(\frac{1}{(h_1(\varepsilon))^6}\Big) = \lim_{\varepsilon\to 0^+} h_1(\varepsilon) \cdot \exp\Big(\exp\Big(\frac{1}{(h_0(\varepsilon))^6}\Big)\Big) = 0, \tag{5-2}$$

$$\lim_{\varepsilon\to 0^+} h_2(\varepsilon) \cdot \exp(\varepsilon^{-2}) = 0. \tag{5-3}$$

In particular, the following relations hold for $\varepsilon \ll 1$:

$$h_2(\varepsilon) \ll \varepsilon \ll h_1(\varepsilon) \ll h_0(\varepsilon) \ll 1. \tag{5-4}$$

Let $\chi : \mathbb{R} \to [0, 1]$ be a smooth cut-off function, satisfying $\chi|_{[-1,1]} = 1$, $\chi|_{\mathbb{R}\setminus[-2,2]} = 0$ and

$$\chi|_{(-2,2)} > 0, \tag{5-5}$$

and let $\varepsilon_0 \ll 1$ be a small enough absolute constant. For any $0 < \varepsilon < \varepsilon_0$ and any $r_0 > 0$ satisfying

$$1 - \exp(-2(h_0(\varepsilon))^{-4}) < \frac{r_0}{(2/\sqrt{-\Lambda})\varepsilon - \frac{1}{3}\Lambda r_0^3} < 1 - \tfrac{1}{2}\exp(-2(h_0(\varepsilon))^{-4}) \tag{5-6}$$

(note that (5-6) implies that $r_0\sqrt{-\Lambda}/(2\varepsilon) = 1 + O(\exp(-2(h_0(\varepsilon))^{-4}))$ as $\varepsilon \to 0$), we will define the following function on $[0, +\infty) \times (0, +\infty)$:

$$\bar{f}_\varepsilon(v, p^u) \doteq C_{\varepsilon r_0} \sum_{j=0}^{\lceil 1/h_1(\varepsilon)\rceil} \chi(p^u - 3) \cdot \frac{1}{h_2(\varepsilon)} \chi\Big(\frac{(v - v^{(j)})\sqrt{-\Lambda} - 2h_2(\varepsilon)}{h_2(\varepsilon)}\Big) \cdot h_{(j)}(\varepsilon) \cdot \varepsilon \tag{5-7}$$

for some constant $C_{\varepsilon r_0}$ to be specified in terms of ε, r_0 later, where $\lceil 1/h_1(\varepsilon)\rceil$ denotes the least integer greater than or equal to $1/h_1(\varepsilon)$

$$v^{(j)} = \frac{\pi}{\sqrt{-\Lambda}} - j\frac{\varepsilon}{h_1(\varepsilon)\sqrt{-\Lambda}} \tag{5-8}$$

for any $0 \le j \le \lceil 1/h_1(\varepsilon)\rceil$,

$$h_{(0)} = h_0, \tag{5-9}$$

$$h_{(j)} = h_1 \tag{5-10}$$

for all $1 \le j \le \lceil 1/h_1(\varepsilon)\rceil$.

5B. ***Construction of the initial data family*.** For any $0<\varepsilon<\varepsilon_0$ and any r_0 satisfying (5-6), we will define the following asymptotically AdS boundary-characteristic initial data set according to Definition 3.1:

Definition 5.1. For any $0<\varepsilon<\varepsilon_0$ and any r_0 satisfying (5-6), we define $v_0=v_0(r_0,\varepsilon)>0$ and the set of smooth functions $r_{/\varepsilon}:[0,v_0)\to[r_0,+\infty)$, $\Omega^2_{/\varepsilon}:[0,v_0)\to(0,+\infty)$, $\bar{f}_{\text{in}/\varepsilon}:[0,v_0)\times(0,+\infty)\to[0,+\infty)$, and $\bar{f}_{\text{out}/\varepsilon}:[0,v_0)\times(0,+\infty)\to[0,+\infty)$ by the requirement that $(r_{/\varepsilon},\Omega^2_{/\varepsilon},\bar{f}_{\text{in}/\varepsilon},\bar{f}_{\text{out}/\varepsilon})$ is an asymptotically AdS boundary-characteristic initial data set on $[0,v_0)$ for the system (2-28)–(2-33) satisfying the reflecting gauge condition at $r=r_0,+\infty$, so that

$$\begin{cases}\dfrac{\partial_v r_{/\varepsilon}}{1-2m_{/\varepsilon}/r_{/\varepsilon}}=\dfrac{\Omega^2_{/\varepsilon}}{4\partial_v r_{/\varepsilon}},\\[2ex] \left.\dfrac{\partial_v r_{/\varepsilon}}{1-\frac{1}{3}\Lambda r_{/\varepsilon}}\right|_{v=0}=\dfrac{1}{2},\end{cases}\tag{5-11}$$

$$\bar{f}_{\text{out}/\varepsilon}=0,\tag{5-12}$$

$$\bar{f}_{\text{in}/\varepsilon}(v,p^u)=\bar{f}_\varepsilon(v,p^u)\tag{5-13}$$

for all $0\le v\le v_0$ and $p^u>0$. The constant $C_{\varepsilon r_0}$ in (5-7) is fixed in terms of ε, r_0 by the requirement that

$$\lim_{v\to v_0}\tilde{m}_{/\varepsilon}=\frac{\varepsilon}{\sqrt{-\Lambda}}\tag{5-14}$$

(in particular, there exists some fixed (large) $C_0>1$, independent of ε, r_0, so that $C_{\varepsilon r_0}\in[C_0^{-1},C_0]$ for any $0<\varepsilon<\varepsilon_0$, any r_0 satisfying (5-6).[14] See Figure 7.

Remark. The conditions (5-11)–(5-13) determine (implicitly) v_0 and $r_{/\varepsilon}$, $\Omega^2_{/\varepsilon}$ uniquely in terms of ε, r_0, provided ε_0 has been fixed small enough: in view of the fact that (5-11) is equivalent to

$$\begin{cases}\dfrac{\partial_v r_{/\varepsilon}}{1-2m_{/\varepsilon}/r_{/\varepsilon}}=\dfrac{-\partial_u r_{/\varepsilon}}{1-2m_{/\varepsilon}/r_{/\varepsilon}},\\[2ex] \left.\dfrac{\partial_v r_{/\varepsilon}}{1-\frac{1}{3}\Lambda r_{/\varepsilon}}\right|_{v=0}=\dfrac{1}{2}\end{cases}\tag{5-15}$$

(where $m_{/\varepsilon}$ is defined in terms of $r_{/\varepsilon}$, $\Omega^2_{/\varepsilon}$ by (3-12)), from (5-14) and (2-44) (which is equivalent to the constraint equation (3-4)) we infer that

$$\frac{\partial_v r_{/\varepsilon}}{1-\frac{1}{3}\Lambda r_{/\varepsilon}}=\frac{1}{2}+O(\varepsilon)\tag{5-16}$$

and, therefore,

$$v_0=\sqrt{-\frac{3}{\Lambda}}\pi+O(\varepsilon).\tag{5-17}$$

While $(r_{/\varepsilon},\Omega^2_{/\varepsilon},\bar{f}_{\text{in}/\varepsilon},\bar{f}_{\text{out}/\varepsilon})$ depend on both ε and r_0, we will only use the subscript ε in their notation, since most of the estimates that we will later establish for their maximal development will depend only on ε.

[14] In fact, it suffices to choose $C_0=50$.

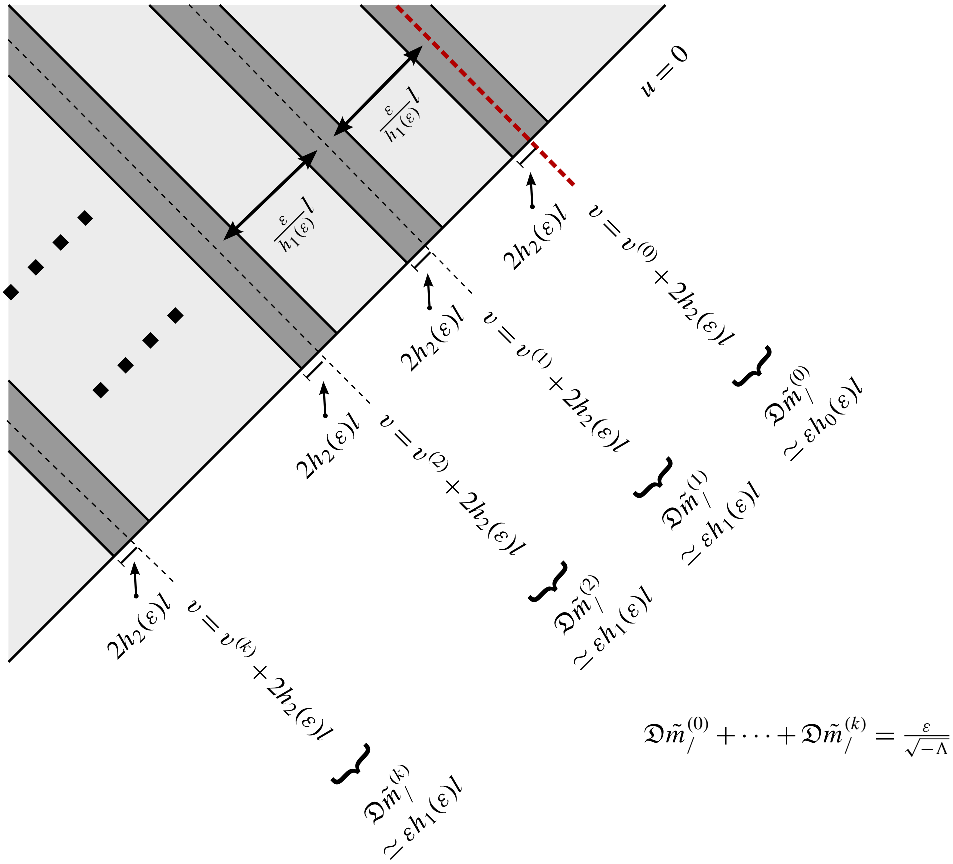

Figure 7. The initial data $(r_{\varepsilon/}, \Omega^2_{\varepsilon/}, \bar{f}_{\text{in}\varepsilon/}, \bar{f}_{\text{out}\varepsilon/})$ give rise to $k+1$ Vlasov beams, initially arranged as depicted above (using, for convenience, the abbreviation $k = \lceil 1/h_1(\varepsilon) \rceil$ and $l = (-\Lambda)^{-1/2}$). For any $0 \le j \le k$, we can associate to the beam centered around $v = v^{(j)} + 2h_2(\varepsilon)l$ the initial mass difference $\mathfrak{D}\tilde{m}^{(j)}_{/}$, defined as the renormalized Hawking mass difference between the two vacuum regions surrounding the beam. The top beam (centered initially around $v = v^0 + 2h_2(\varepsilon)l$) has a greater initial mass difference than the rest of the beams.

The initial data $(r^{(\varepsilon)}_{/}, (\Omega^{(\varepsilon)}_{/})^2, \bar{f}^{(\varepsilon)}_{\text{in}/}, \bar{f}^{(\varepsilon)}_{\text{out}/})$ in the statement of Theorem 1 will eventually be chosen to be small perturbations of $(r_{/\varepsilon}, \Omega^2_{/\varepsilon}, \bar{f}_{\text{in}/\varepsilon}, \bar{f}_{\text{out}/\varepsilon})$ (see Section 6).

5C. ***Notational conventions and basic computations.*** For any $0 < \varepsilon < \varepsilon_0$ and any r_0 satisfying (5-6), let $(r_{/\varepsilon}, \Omega^2_{/\varepsilon}, \bar{f}_{\text{in}/\varepsilon}, \bar{f}_{\text{out}/\varepsilon})$ be the initial data set defined by Definition 5.1. Assuming that ε_0 is fixed small enough, for any $0 < \varepsilon < \varepsilon_0$ and any r_0 satisfying (5-6), the initial data set $(r_{/\varepsilon}, \Omega^2_{/\varepsilon}, \bar{f}_{\text{in}/\varepsilon}, \bar{f}_{\text{out}/\varepsilon})$ satisfies the following estimate depending only on ε:

$$\|(r_{/\varepsilon}, \Omega^2_{/\varepsilon}, \bar{f}_{\text{in}/\varepsilon}, \bar{f}_{\text{out}/\varepsilon})\|_{\mathcal{CS}} \le C h_0(\varepsilon), \tag{5-18}$$

where $\|\cdot\|_{\mathcal{CS}}$ is defined by (3-41) and $C > 0$ is a fixed constant.

Let $(\mathcal{U}_\varepsilon; r_\varepsilon, \Omega_\varepsilon^2, \bar{f}_{\text{in}\varepsilon}, \bar{f}_{\text{out}\varepsilon})$ be the maximal future development of $(r_{/\varepsilon}, \Omega^2_{/\varepsilon}, \bar{f}_{\text{in}/\varepsilon}, \bar{f}_{\text{out}/\varepsilon})$ (see Theorem 3.4). In view of Proposition 3.5, the bound (5-18) implies that, for any fixed $u_* > 0$ and any $\delta > 0$, there exists an $\varepsilon_{\delta u_*} > 0$ sufficiently small depending only on δ and u_* so that, for any $0 \le \varepsilon < \varepsilon_{\delta u_*}$,

$$\mathcal{W}_{u_*} \doteq \{0 < u < u_*\} \cap \{u < v < u + v_0\} \subset \mathcal{U} \tag{5-19}$$

and

$$\sqrt{-\Lambda} \sup_{\mathcal{W}_{u_*}} |\tilde{m}_\varepsilon| + \sup_{\mathcal{W}_{u_*}} \log\left(\frac{1 - \frac{1}{3}\Lambda r_\varepsilon^2}{1 - \max\{2m_\varepsilon / r_\varepsilon, 0\}}\right) + \sup_{\bar{u}} \int_{\{u=\bar{u}\}\cap\mathcal{W}_{u_*}} \frac{r_\varepsilon (T_{vv})_\varepsilon}{\partial_v r_\varepsilon}\, dv + \sup_{\bar{v}} \int_{\{v=\bar{v}\}\cap\mathcal{W}_{u_*}} \frac{r_\varepsilon (T_{uu})_\varepsilon}{(-\partial_u r_\varepsilon)}\, du < \delta, \tag{5-20}$$

where $m_\varepsilon, \tilde{m}_\varepsilon, (T_{uu})_\varepsilon, (T_{vv})_\varepsilon$ are defined in terms of $r_\varepsilon, \Omega_\varepsilon^2 \bar{f}_{\text{in}\varepsilon}, \bar{f}_{\text{out}\varepsilon}$ by (2-5), (2-42), (2-19) and (2-20). In particular, if

$$g_{\text{AdS}} = -\Omega^2_{\text{AdS},r_0,v_0}\, du\, dv + r_{\text{AdS},r_0,v_0} g_{\mathbb{S}^2} \tag{5-21}$$

is the pure AdS metric in a spherically symmetric coordinate chart (u, v) such that $r_{\text{AdS},r_0,v_0} = r_0$ on $\{u = v\}$ and $r_{\text{AdS},r_0,v_0} = +\infty$ on $\{u = v - v_0\}$,[15] then $(\mathcal{U}_\varepsilon; r_\varepsilon, \Omega_\varepsilon^2, \bar{f}_{\text{in}\varepsilon}, \bar{f}_{\text{out}\varepsilon})$, when restricted on $\mathcal{W}_{u_*}$, is δ-close to $(\mathcal{W}_{u_*}; r_{\text{AdS},r_0,v_0}, \Omega^2_{\text{AdS},r_0,v_0}, 0, 0)$ with respect to the (gauge-invariant) distance defined by (3-41). Notice also that (5-20) implies that, provided δ is small enough, the spacetime $(\mathcal{W}_{u_*} \times \mathbb{S}^2, g_\varepsilon)$ does not contain any trapped surface, where

$$g_\varepsilon = -\Omega_\varepsilon^2\, du\, dv + r_\varepsilon^2 g_{\mathbb{S}^2}. \tag{5-22}$$

Notice that, in view of the conservation of $\tilde{m}$ on γ_0 and $\mathcal{I}$ (see (3-26) and (3-27)), we have

$$\tilde{m}_\varepsilon|_{\gamma_0} = 0, \tag{5-23}$$

$$\tilde{m}_\varepsilon|_{\mathcal{I}} = \lim_{v\to v_0} \tilde{m}_{/\varepsilon}(v) = \frac{\varepsilon}{\sqrt{-\Lambda}}. \tag{5-24}$$

For each $0 \le j \le \lceil 1/h_1(\varepsilon) \rceil$, we can associate to the beam centered at

$$v = v^{(j)} + \frac{2}{\sqrt{-\Lambda}} h_2(\varepsilon)$$

the mass difference

$$\mathfrak{D}\tilde{m}_{/}^{(j)} \doteq \tilde{m}_{/\varepsilon}\Big(v^{(j)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)\Big) - \tilde{m}_{/\varepsilon}(v^{(j)}). \tag{5-25}$$

Notice that

$$\mathcal{D}\tilde{m}_{/}^{(0)} \simeq \frac{h_0(\varepsilon)\varepsilon}{\sqrt{-\Lambda}} \tag{5-26}$$

and, for all $1 \le j \le \lceil 1/h_1(\varepsilon) \rceil$,

$$\mathcal{D}\tilde{m}_{/}^{(j)} \simeq \frac{h_1(\varepsilon)\varepsilon}{\sqrt{-\Lambda}}. \tag{5-27}$$

[15] Note that such a coordinate chart is not unique.

Furthermore

$$\sum_{j=0}^{\lceil 1/h_1(\varepsilon)\rceil} \mathcal{D}\tilde{m}_{/}^{(j)} = \lim_{v\to v_0} \tilde{m}_{/\varepsilon}(v) = \frac{\varepsilon}{\sqrt{-\Lambda}}. \tag{5-28}$$

5D. ***Some geometric constructions on $\mathcal{U}_\varepsilon$.*** For any $0<\varepsilon<\varepsilon_0$ and any r_0 satisfying (5-6), we will define some special subsets of the domain $\mathcal{U}_\varepsilon$ of the maximal future development $(\mathcal{U}_\varepsilon; r_\varepsilon, \Omega_\varepsilon^2, \bar{f}_{\text{in}\varepsilon}, \bar{f}_{\text{out}\varepsilon})$ of the initial data set $(r_{/\varepsilon}, \Omega_{/\varepsilon}^2, \bar{f}_{\text{in}/\varepsilon}, \bar{f}_{\text{out}/\varepsilon})$.

Remark. In the rest of this section, we will adopt the convention that the boundary ∂A of a subset $A\subseteq\mathcal{U}_\varepsilon$ is the boundary of A as a subset of $\mathbb{R}^2$ (with respect to the ambient topology of $\mathbb{R}^2$).

Let us define the domain of outer communications $\mathcal{D}_\varepsilon$ of $\mathcal{U}_\varepsilon$ as

$$\mathcal{D}_\varepsilon \doteq J^-(\mathcal{I})\cap\mathcal{U}_\varepsilon, \tag{5-29}$$

where $J^-(\mathcal{I})$ is the causal past of $\mathcal{I}$ with respect to the reference metric (2-50) (see (3-32)). In accordance with Theorem 3.4, we will also define the future event horizon $\mathcal{H}_\varepsilon^+$ of $\mathcal{U}_\varepsilon$ as

$$\mathcal{H}_\varepsilon^+ \doteq \partial\mathcal{D}_\varepsilon\cap\mathcal{U}_\varepsilon. \tag{5-30}$$

Note that we allow $\mathcal{H}_\varepsilon^+$ to be empty. In view of Theorem 3.4, in the case when $\mathcal{H}_\varepsilon^+$ is nonempty, it is necessarily of the form

$$\mathcal{H}_\varepsilon^+ = \{u = u_{\mathcal{H}_\varepsilon^+}\}\cap\mathcal{U}_\varepsilon \tag{5-31}$$

and has infinite affine length.

We will also define

$$\mathcal{J}_\varepsilon \doteq J^-(\gamma_0)\cap\mathcal{U}_\varepsilon. \tag{5-32}$$

Notice that, as a consequence of Theorem 3.4, on $\mathcal{J}_\varepsilon\cup\mathcal{D}_\varepsilon$ we have

$$1-\frac{2m}{r} > 0; \tag{5-33}$$

i.e., trapped spheres can only appear in the region $\mathcal{U}_\varepsilon\backslash(\mathcal{J}_\varepsilon\cup\mathcal{D}_\varepsilon)$. In the case $\mathcal{H}_\varepsilon^+\neq\varnothing$, Theorem 3.4 also implies that $\mathcal{J}_\varepsilon\backslash\mathcal{D}_\varepsilon\neq\varnothing$.

For any $v_*\in[0, v_0]$ and any integer $n\geq 1$, we will define

$$U_n(v_*) \doteq v_* + (n-1)v_0, \tag{5-34}$$

$$V_n(v_*) = v_* + nv_0. \tag{5-35}$$

We will also set

$$V_0(v_*) \doteq v_*. \tag{5-36}$$

Notice that the segment $\{u=U_n(v_*)\}\cap\mathcal{U}_\varepsilon$ is the image of the ingoing null geodesic of $\mathcal{U}_\varepsilon$ emanating from the point $(0, v_*)$ after n reflections off γ_0 and $n-1$ reflections off $\mathcal{I}$, while the segment $\{v=V_n(v_*)\}\cap\mathcal{U}_\varepsilon$ is the image of the same null geodesic after n reflections off γ_0 and n reflections off $\mathcal{I}$; see Figure 8.

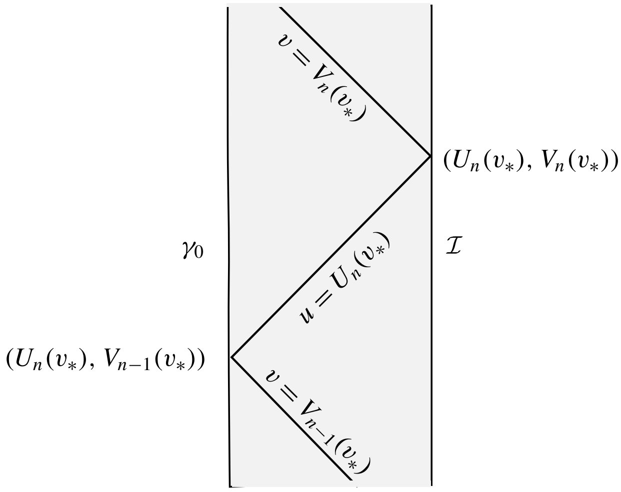

Figure 8. Schematic depiction of the lines $v = V_{n-1}(v_*)$, $u = U_n(v_*)$, and $v = V_n(v_*)$.

Let us define the domains $\mathcal{R}^{(i,j)}_{\varepsilon n} \subset \mathcal{U}_\varepsilon$ for any $n \in \mathbb{N}$, $0 \le i \le \lceil 1/h_1(\varepsilon)\rceil$, and $i \le j \le \lceil 1/h_1(\varepsilon)\rceil + i + 1$ by the relation

$$\mathcal{R}^{(i,j)}_{\varepsilon n} = \Big\{ U_n\Big(v^{(i)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)\Big) < u < U_n(v^{(i-1)}) \Big\} \cap \Big\{ V_n\Big(v^{(j)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)\Big) < v < V_n(v^{(j-1)}) \Big\} \cap \mathcal{U}_\varepsilon, \quad (5\text{-}37)$$

where we have used the following conventions in the expression (5-37):

(1) $U_n(v^{(-1)}) \doteq U_{n+1}(v^{(\lceil 1/h_1(\varepsilon)\rceil)})$.

(2) $V_n(v^{(\lceil 1/h_1(\varepsilon)\rceil + l)} + c) \doteq V_{n-1}(v^{(l-1)} + c)$ for any integer $1 \le l \le \lceil 1/h_1(\varepsilon)\rceil$ and any $c \ge 0$.

Remark. The boundary of the domains $\mathcal{R}^{(i,i)}_{\varepsilon n}$, $0 \le i \le \lceil 1/h_1(\varepsilon)\rceil$, contains a segment $\mathcal{I}$, while the boundary of the domains $\mathcal{R}^{(i,\lceil 1/h_1(\varepsilon)\rceil+1+i)}_{\varepsilon n}$, $0 \le i \le \lceil 1/h_1(\varepsilon)\rceil$, contains a segment in γ_0. See Figure 9.

Notice that $T_{uu} = T_{vv} = 0$ in $\mathcal{R}^{(i,j)}_{\varepsilon n}$. In particular, all the domains $(\mathcal{R}^{(i,j)}_{\varepsilon n} \times \mathbb{S}^2, g_\varepsilon)$ are isometric to a region of a member of the Schwarzschild-AdS family (or to a region of pure AdS spacetime), and the renormalized mass function $\tilde{m}_\varepsilon$ is constant on them. We will define for any $0 \le i \le \lceil 1/h_1(\varepsilon)\rceil$, $i \le j \le \lceil 1/h_1(\varepsilon)\rceil + i + 1$ and $n \in \mathbb{N}$ such that $\mathcal{R}^{(i,j)}_{\varepsilon n} \ne \varnothing$,

$$\tilde{m}^{(i,j)}_{\varepsilon n} \doteq \tilde{m}_\varepsilon|_{\mathcal{R}^{(i,j)}_{\varepsilon n}}. \quad (5\text{-}38)$$

In view of (5-23) and (5-24), we immediately calculate that for all $0 \le i \le \lceil 1/h_1(\varepsilon)\rceil$ and all $n \in \mathbb{N}$ such that $\mathcal{R}^{(i,i)}_{\varepsilon n}, \mathcal{R}^{(i,\lceil 1/h_1(\varepsilon)\rceil+1+i)}_{\varepsilon n} \ne \varnothing$,

$$\tilde{m}^{(i,\lceil 1/h_1(\varepsilon)\rceil+1+i)}_{\varepsilon n} = 0, \quad (5\text{-}39)$$

$$\tilde{m}^{(i,i)}_{\varepsilon n} = \tilde{m}^{(i,i)}_{\varepsilon 0} = \frac{\varepsilon}{\sqrt{-\Lambda}}. \quad (5\text{-}40)$$

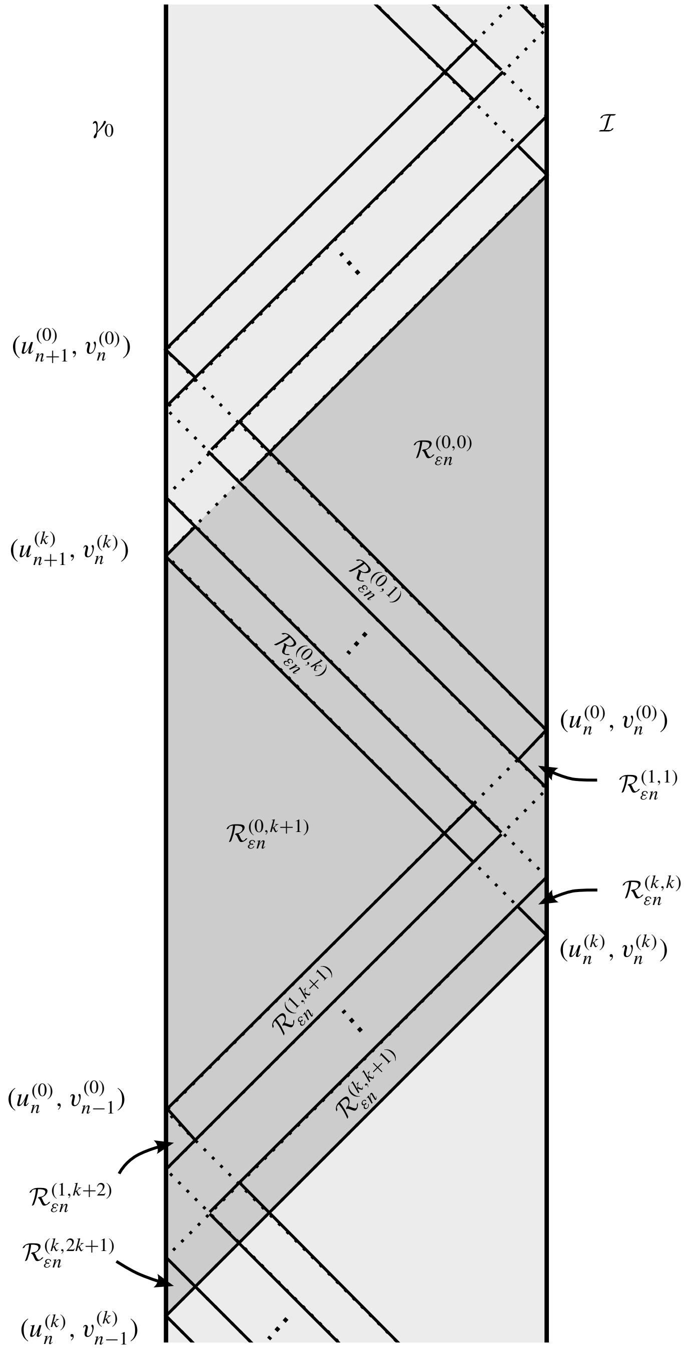

Figure 9. Schematic depiction of the domains $\mathcal{R}^{(i,j)}_{\varepsilon n}$ for $0 \le i \le k$ and $i \le j \le k+i+1$ (where $k = \lceil 1/h_1(\varepsilon) \rceil$). We have used the shorthand notation $u^{(j)}_n = U_n(v^{(j)})$ and $v^{(j)}_n = V_n(v^{(j)})$. Having assumed that $h_2(\varepsilon) \ll \varepsilon$), all the beams of the form $\{U_n(v^{(i)}) \le u \le U_n(v^{(i)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))\}$ and $\{V_n(v^{(j)}) \le v \le V_n(v^{(j)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))\}$, which separate the domains $\mathcal{R}^{(i,j)}_{\varepsilon n}$, are depicted as straight line segments.

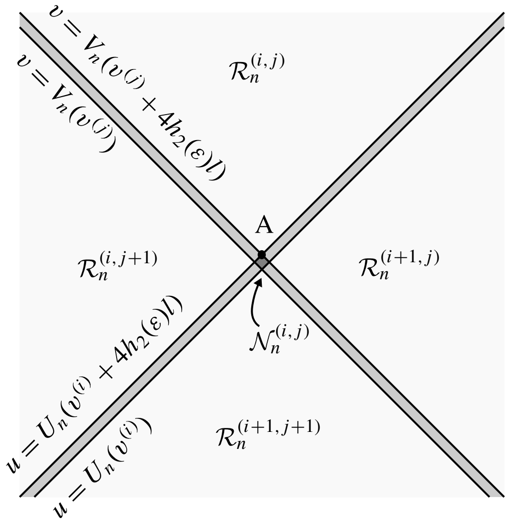

Figure 10. Typical arrangement of neighboring vacuum domains not intersecting γ_0 or $\mathcal{I}$. The point A at the lower corner of $\mathcal{R}_n^{(i,j)}$ satisfies $r(A) = r_n^{(i,j)}$. For simplicity, we have used the shorthand notation $l = (-\Lambda)^{-1/2}$.

For any $n \in \mathbb{N}$, $0 \le i \le \lceil 1/h_1(\varepsilon)\rceil$ and $i+1 \le j \le \lceil 1/h_1(\varepsilon)\rceil + i$, we will define the interaction regions

$$\mathcal{N}_{\varepsilon n}^{(i,j)} \doteq \Big\{U_n(v^{(i)}) \le u \le U_n\Big(v^{(i)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)\Big)\Big\} \cap \Big\{V_n(v^{(j)}) \le u \le V_n\Big(v^{(j)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)\Big)\Big\} \cap \mathcal{U}_\varepsilon, \tag{5-41}$$

where the conventions stated below (5-37) hold regarding indices smaller than 0 or larger than $\lceil 1/h_1(\varepsilon)\rceil$. See Figure 10.

Let us define for any $0 \le i \le \lceil 1/h_1(\varepsilon)\rceil$, $i \le j \le \lceil 1/h_1(\varepsilon)\rceil + i + 1$, and $n \in \mathbb{N}$ such that $\mathcal{R}_{\varepsilon n}^{(i,j)} \neq \varnothing$

$$r_{\varepsilon n}^{(i,j)} \doteq r_\varepsilon\Big(U_n\Big(v^{(i)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)\Big), V_n\Big(v^{(j)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)\Big)\Big). \tag{5-42}$$

Note that $r_{\varepsilon n}^{(i,i)} = +\infty$ and $r_{\varepsilon n}^{(i,\lceil 1/h_1(\varepsilon)\rceil + i + 1)} = r_{0\varepsilon}$.

Finally, let us remark that, in view of property (6-204) of the cut-off used in the construction of the initial data and (2-48)–(2-49), for any $1 \le n \le n_f$, $0 \le i \le \lceil 1/h_1(\varepsilon)\rceil$, $i \le j \le \lceil 1/h_1(\varepsilon)\rceil + i + 1$, we have

$$T_{uu} > 0 \quad \text{on } \Big\{U_n(v^{(i)}) < u < U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))\Big\}, \tag{5-43}$$

$$T_{vv} > 0 \quad \text{on } \Big\{V_n(v^{(j)}) < u < V_n(v^{(j)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))\Big\}. \tag{5-44}$$

6. Proof of Theorem 1

In this section, we will prove Theorem 1. In order to simplify our notation, from now on, we will often drop the subscripts ε in notation related to the maximal future development $(\mathcal{U}_\varepsilon; r_\varepsilon, \Omega_\varepsilon^2, \bar{f}_{\text{in}\varepsilon}, \bar{f}_{\text{out}\varepsilon})$ of the initial data $(r_{/\varepsilon}, \Omega_{/\varepsilon}^2, \bar{f}_{\text{in}/\varepsilon}, \bar{f}_{\text{out}/\varepsilon})$ (see Definition 5.1).

For any $0 < \varepsilon < \varepsilon_0$ (provided ε_0 is fixed sufficiently small) and any $r_0 > 0$ satisfying (5-6), let $(\mathcal{U}_\varepsilon; r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})$ be the maximal future development of $(r_{/\varepsilon}, \Omega^2_{/\varepsilon}, \bar{f}_{\text{in}/\varepsilon}, \bar{f}_{\text{out}/\varepsilon})$, and let us define

$$u_+ \doteq \sup\Big\{u_* > 0 : 1 - \frac{2m}{r} > h_3(\varepsilon) \text{ on } \mathcal{U}_\varepsilon \cap \{u < u_*\}\Big\} \tag{6-1}$$

and

$$\mathcal{U}_\varepsilon^+ = \mathcal{U}_\varepsilon \cap \{u < \min\{u_+, (h_1(\varepsilon))^{-2} v_{0\varepsilon}\}\}, \tag{6-2}$$

where

$$h_3(\varepsilon) = \exp\{-\exp((h_1(\varepsilon))^{-5} \exp(-2(h_0(\varepsilon))^{-4}))\}. \tag{6-3}$$

Let us also set

$$k \doteq \lceil 1/h_1(\varepsilon) \rceil, \tag{6-4}$$

$$n_f \doteq \lfloor (u_+ - v^{(0)})/v_0 \rfloor, \tag{6-5}$$

where $\lceil x \rceil$ denotes the least integer greater than or equal to x, while $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

The proof of Theorem 1 will follow in two steps: First, in Section 6B, we will show that

$$\sup_{\mathcal{U}_\varepsilon^+} \Big(1 - \frac{2m}{r}\Big) = h_3(\varepsilon), \tag{6-6}$$

i.e., that $\mathcal{U}_\varepsilon^+$ contains a nearly trapped sphere. Then, in Section 6C, we will show that, at the final step of the evolution, either a trapped sphere is formed, or there exists a small perturbation of the initial data $(r_{/\varepsilon}, \Omega^2_{/\varepsilon}, \bar{f}_{\text{in}/\varepsilon}, \bar{f}_{\text{out}/\varepsilon})$ giving rise to a trapped sphere.

Before proving (6-6), we need to establish some necessary bounds for the evolution of $(r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})$ in the region $\mathcal{U}_\varepsilon^+$. These bounds, which will be obtained in Section 6A, will be used both in Section 6B and in Section 6C.

6A. *Inductive bounds for the evolution in the region $\mathcal{U}_\varepsilon^+$.* In this section, we will establish a number of useful bounds for $(\mathcal{U}_\varepsilon^+; r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})$. These bounds will include a number of inductive bounds for the quantities $\tilde{m}_n^{(1,k+1)}$, $r_n^{(k,k+1)}$, and $r_n^{(1,k+1)}$ (with k defined by (6-4)) that will be of fundamental significance in the proof of Theorem 1.

In particular, we will prove the following result:

Proposition 6.1. *For any $0 < \varepsilon < \varepsilon_0$, the following bounds hold for $(\mathcal{U}_\varepsilon^+; r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})$:*

(1) *On $\mathcal{U}_\varepsilon^+$, we can estimate*

$$\bigg|\log\bigg(\frac{-\partial_u r}{1 - \frac{1}{3}\Lambda r^2}\bigg)\bigg| + \bigg|\log\bigg(\frac{\partial_v r}{1 - 2m/r}\bigg)\bigg| \le (h_1(\varepsilon))^{-4} \log((h_3(\varepsilon))^{-1}). \tag{6-7}$$

(2) *For any $1 \le n \le n_f$,*

$$r_n^{(0,k)} \ge \frac{\varepsilon^{-1/2}}{\sqrt{-\Lambda}}, \tag{6-8}$$

$$r_n^{(k,k+1)} \le \frac{\varepsilon^{1/2}}{\sqrt{-\Lambda}}, \tag{6-9}$$

$$\frac{2(\tilde{m}|_{\mathcal{I}} - \tilde{m}_n^{(1,k+1)})}{r_0} \geq \exp(-2(h_0(\varepsilon))^{-4}), \tag{6-10}$$

$$\frac{2(\tilde{m}|_{\mathcal{I}} - \tilde{m}_n^{(1,k+1)})}{r_0} \leq 1 - \frac{1}{C_0} h_0(\varepsilon), \tag{6-11}$$

$$\frac{r_n^{(1,k+1)}}{r_0} - 1 \geq \exp(-(h_0(\varepsilon))^{-4}), \tag{6-12}$$

where $C_0 > 1$ *is a large fixed constant* (*independent of all the parameters*).

(3) *For any* $2 \leq n \leq n_f$,

$$\frac{\tilde{m}_n^{(1,k+1)}}{\tilde{m}_{n-1}^{(1,k+1)}} \geq 1 + \tfrac{1}{4} \exp(-2(h_0(\varepsilon))^{-4}) \frac{r_0}{r_n^{(k,k+1)}}, \tag{6-13}$$

$$\frac{r_n^{(k,k+1)} - r_0}{r_{n-1}^{(k,k+1)} - r_0} \leq 1 + 2C_0 \frac{r_0}{r_{n-1}^{(k,k+1)}} \left(\left| \log\left(1 - \frac{2\tilde{m}_{n-1}^{(1,k+1)}}{r_0}\right) \right| + (h_0(\varepsilon))^{-4} \right). \tag{6-14}$$

Before presenting the proof of Proposition 6.1 (in Section 6A2), we will briefly comment on the nature of the bounds (6-7)–(6-14) and their relation with the specific choice of the parameters (5-2)–(5-3).

6A1. *Remarks on Proposition 6.1.* The bounds (6-7)–(6-14) in Proposition 6.1 lie at the heart of the proof of Theorem 1. The precise form of the initial data (5-7), the range (5-6) for the mirror radius r_0 and the asymptotic bounds (5-2)–(5-3) on the parameters h_0, h_1, h_2 were carefully chosen so that (6-7)–(6-14) can be obtained. We will now proceed to briefly comment on the role of the bounds (6-7)–(6-14) in the proof of Theorem 1. The reader is advised to review first the sketch of the proof in Section 1D of the Introduction. Let us remark that, in the notation of Section 1D,

$$\mathcal{E}_{\zeta_0;n} = \tilde{m}_n^{(1,k+1)}, \tag{6-15}$$

$$r_{\gamma_0;n} = r_n^{(k,k+1)}, \tag{6-16}$$

$$r_{\gamma_0;n}^{(1)} = r_n^{(1,k+1)}. \tag{6-17}$$

The bound (6-7) is a "trivial" bound controlling quantities related to the chosen gauge. The right-hand side of (6-7), upon integration across any specific beam (in a direction transversal to the beam), will yield a small quantity, in view of the fact that the width of the null beams emanating from $u = 0$, $v \sim v^{(j)}$ was chosen to be $\sim h_2(\varepsilon)$ and, moreover, $h_2(\varepsilon)$ was chosen in (5-3) to be small compared to the right-hand side of (6-7). This fact will prove convenient for the proof of Proposition 6.1 and Theorem 1, as it will enable us to "ignore" the variation of certain quantities across the width of any specific beam. That is to say, the bound (6-7) will enable us to frequently treat the null beams as line segments having negligible width.

The bounds (6-8)–(6-9) are quantitative expressions of the fact that the set of interactions of the beams splits into two portions, one close to $r = r_0$ and one close to $\mathcal{I}$.

The lower bound (6-10) is necessary in order to establish (6-13). In order to obtain (6-10), it is necessary that r_0 satisfies the upper bound of (5-6).

The upper bound (6-11) implies that a trapped sphere (i.e., a sphere where $2m/r > 1$) cannot be formed at $\mathcal{R}^{(i,j)}_{\varepsilon n}$ for any $j > k+1$, since one can also show that $\tilde{m} \le \tilde{m}|_{\mathcal{I}} - \tilde{m}^{(1,k+1)}_n$ in those regions. In order to obtain (6-11), it is necessary that the mirror radius r_0 satisfy the lower bound of (5-6).

In the language of Section 1D of the Introduction, the bound (6-12) states that, when ζ_0 reaches $\{r = r_0\}$ for the n-th time, the r-distance of the top beam ζ_0 from the second-to-top beam ζ_1, i.e., $r^{(1,k+1)}_n - r_0$, can be bounded from below by a small multiple of r_0 which is large compared to $h_1(\varepsilon) r_0$. As a consequence of (6-12) and the bound (5-6) for r_0, for any $i \ne 1$, $\mathcal{R}^{(i,k+1)}_{\varepsilon n}$ does not contain a trapped sphere. As a result, combining (6-11), (6-8) and (6-12), we infer that, among all regions $\mathcal{R}^{(i,j)}_{\varepsilon n}$, a trapped sphere can only appear for $i = 1$, $j = k+1$. This fact serves to simplify the proof of Theorem 1, by avoiding considering multiple scenarios of trapped surface formation. Furthermore, it is crucial in obtaining (6-14).

Establishing (6-12) is the most demanding part of the proof of Proposition 6.1. It requires obtaining a lower bound in the rate of decrease of $r^{(1,k+1)}_n$ in terms of the rate of increase of $\tilde{m}^{(1,k+1)}_n$, using also the fact that $\tilde{m}^{(1,k+1)}_n \lesssim r_0$ before a trapped sphere is formed (see the relations (6-129) and (6-154) in the next section).

The bound (6-13) is a technical version of the bound (1-42), and its proof follows from the ideas outlined in Section 1D. In obtaining (6-13), the lower bound of (6-10) is necessary.

Finally, the bound (6-14) is a technical version of the bound (1-46) in Section 1D and provides an estimate for the decrease of the multiplicative factor in the right-hand side of (6-13). In obtaining (6-14) when $2m/r \simeq 1$, the fact that $2m/r$ is bounded away from 1 everywhere but on $\mathcal{R}^{(i,j)}_{\varepsilon n}$ is crucially used (in particular, the bound (6-12) is necessary for (6-14)).

Remark. As is evident from the above discussion, most of the technical difficulties in the proof of Proposition 6.1 are associated to issues related with the near-trapped regime $2m/r \simeq 1$. In the case when, instead of the stronger bound (4-4), one is merely interested in establishing the weaker instability estimate (4-8), the proof of Proposition 6.1 simplifies substantially: In that case, it is not necessary to demand that the worst instability scenario take place in $\mathcal{R}^{(1,k+1)}_{\varepsilon n}$. In particular, the bounds (6-11) and (6-12) can be omitted from the proof. Moreover, the lower bound for r_0 in (5-6) can be relaxed, and the exponentials in the relations (5-2) between $h_0(\varepsilon)$, $h_1(\varepsilon)$ can be replaced by polynomial functions.

6A2. *Proof of Proposition 6.1.* In this section, we will make use of the $O(\cdot)$ convention: for any pair of functions $\mathcal{F}, \mathcal{G}$ defined on the same domain, with $\mathcal{G} \ge 0$, the notation

$$\mathcal{F} = O(\mathcal{G})$$

will imply that

$$|\mathcal{F}| \le C \cdot \mathcal{G}$$

for some universal constant $C > 0$ which is independent of all the parameters in the statement of Theorem 1. We should also remark that, throughout this proof, we will adopt the convention on the indices stated under (5-37); i.e.,

(1) $U_n(v^{(-1)}) \doteq U_{n+1}(v^{(k)})$,

(2) $V_n(v^{(k+l)} + c) \doteq V_{n-1}(v^{(l-1)} + c)$ for any integer $1 \le l \le k$ and any $c \ge 0$.

In view of (6-1), on $\mathcal{U}_\varepsilon^+$ we have

$$\partial_u r < 0 < \partial_v r, \tag{6-18}$$

$$\partial_u \tilde{m} \le 0 \le \partial_v \tilde{m}. \tag{6-19}$$

We will split the proof of Theorem 1 into two parts: In the first (and shortest) part, we will establish the bound (6-7) through a standard continuity argument. The proof of (6-7) will also yield (6-8) and (6-9). In the second (and more extended) part, we will establish the bounds (6-10)–(6-14) by induction on n.

<u>Part I</u>: proof of (6-7)–(6-9). Let $u_* > 0$ be such that on

$$\mathcal{U}_\varepsilon^* \doteq \mathcal{U}_\varepsilon^+ \cap \{u < u_*\}, \tag{6-20}$$

we can bound

$$\left|\log\left(\frac{-\partial_u r}{1-\frac{1}{3}\Lambda r^2}\right)\right| + \left|\log\left(\frac{\partial_v r}{1-2m/r}\right)\right| \le 2(h_1(\varepsilon))^{-4}\log((h_3(\varepsilon))^{-1}). \tag{6-21}$$

By showing that (6-7) holds on $\mathcal{U}_\varepsilon^*$, it will follow (by applying a standard continuity argument) that (6-7) holds on the whole of $\mathcal{U}_\varepsilon^+$.

Inductive formulas for $\partial_u r$ and $\partial_v r$ and proof of (6-7). From (2-45), we can readily derive the following renormalized equation:

$$\partial_v\partial_u\left\{\sqrt{-\frac{3}{\Lambda}}\tan^{-1}\left(\sqrt{-\frac{\Lambda}{3}}r\right)\right\} = -2\frac{\tilde{m}}{r^2}\frac{(1-\Lambda r^2)}{\left(1-\frac{1}{3}\Lambda r^2\right)}\left(\frac{-\partial_u r}{1-\frac{1}{3}\Lambda r^2}\right)\left(\frac{\partial_v r}{1-2m/r}\right). \tag{6-22}$$

Let $n\ge 1,\ 0\le i\le k,\ i\le j\le k+i,\ \bar{u}<u_*$ and v_b be such that

$$U_n\left(v^{(i)}+\frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)\right)\le \bar{u}\le U_n(v^{(i-1)})$$

and

$$V_n(v^{(j)})\le v_b\le V_n\left(v^{(j)}+\frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)\right).$$

Integrating (6-22) for $u=\bar{u}$ from $v=V_n(v^{(j)})$ up to $v=v_b$, using also the fact that

$$\partial_u\left\{\sqrt{-\frac{3}{\Lambda}}\tan^{-1}\left(\sqrt{-\frac{\Lambda}{3}}r\right)\right\} = \frac{\partial_u r}{1-\frac{1}{3}\Lambda r^2},$$

we obtain

$$\frac{-\partial_u r}{1-\frac{1}{3}\Lambda r^2}\bigg|_{(\bar{u},V_n(v^{(j)}))} = \frac{-\partial_u r}{1-\frac{1}{3}\Lambda r^2}\bigg|_{(\bar{u},v_b)} + O\left(\sup_{\{u=\bar{u}\}\cap\{V_n(v^{(j)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))\le v\le V_n(v^{(j-1)})\}}\left|\frac{\tilde{m}}{r^2}\left(\frac{-\partial_u r}{1-\frac{1}{3}\Lambda r^2}\right)\left(\frac{\partial_v r}{1-2m/r}\right)\right|h_2(\varepsilon)\right). \tag{6-23}$$

Using the bootstrap bound (6-21), combined with the trivial bounds

$$\tilde{m}\le \tilde{m}|_{\mathcal{I}} = \frac{\varepsilon}{\sqrt{-\Lambda}}, \tag{6-24}$$

$$r\ge r_0 \tag{6-25}$$

(following from (2-46), (2-47) and (6-18)), as well as the relation (5-3) for $h_2(\varepsilon)$, the relation (6-23) yields

$$\frac{-\partial_u r}{1-\frac{1}{3}\Lambda r^2}\Big|_{(\bar{u},V_n(v^{(j)}))} = \frac{-\partial_u r}{1-\frac{1}{3}\Lambda r^2}\Big|_{(\bar{u},v_b)} + O((h_2(\varepsilon))^{1/2}). \tag{6-26}$$

Similarly, integrating (6-22) for $v=\bar{v}$ from $u=U_n(v^{(i)})$ up to $u=u_b$ for any

$$V_n\Big(v^{(j)}+\frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)\Big) \le \bar{v} \le V_n(v^{(j-1)})$$

and any

$$U_n(v^{(i)}) \le u_b \le U_n\Big(v^{(i)}+\frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)\Big)$$

(assuming that $u_b<u_*$), we infer

$$\frac{\partial_v r}{1-\frac{1}{3}\Lambda r^2}\Big|_{(U_n(v^{(i)}),\bar{v})} = \frac{\partial_v r}{1-\frac{1}{3}\Lambda r^2}\Big|_{(u_b,\bar{v})} + O((h_2(\varepsilon))^{1/2}). \tag{6-27}$$

By multiplying and dividing each factor by

$$1-\frac{2m}{r} = 1-\frac{2\tilde{m}}{r}-\frac{1}{3}\Lambda r^2,$$

the relations (6-26) and (6-27) are equivalent to

$$\frac{-\partial_u r}{1-2m/r}\Big|_{(\bar{u},V_n(v^{(j)}))} = \frac{-\partial_u r}{1-2m/r}\Big|_{(\bar{u},v_b)} \cdot \left(\frac{1-2\tilde{m}/\big(r\big(1-\frac{1}{3}\Lambda r^2\big)\big)\big|_{(\bar{u},v_b)}}{1-2\tilde{m}/\big(r\big(1-\frac{1}{3}\Lambda r^2\big)\big)\big|_{(\bar{u},V_n(v^{(j)}))}} + O((h_2(\varepsilon))^{1/2})\right) \tag{6-28}$$

and

$$\frac{\partial_v r}{1-2m/r}\Big|_{(U_n(v^{(i)}),\bar{v})} = \frac{\partial_v r}{1-2m/r}\Big|_{(u_b,\bar{v})} \cdot \left(\frac{1-2\tilde{m}/\big(r\big(1-\frac{1}{3}\Lambda r^2\big)\big)\big|_{(u_b,\bar{v})}}{1-2\tilde{m}/\big(r\big(1-\frac{1}{3}\Lambda r^2\big)\big)\big|_{(U_n(v^{(i)}),\bar{v})}} + O((h_2(\varepsilon))^{1/2})\right). \tag{6-29}$$

Remark. In the vacuum case, where $\tilde{m}$ is constant, the factors in the right-hand side of (6-28) and (6-29) become identically 1. In our case, however, where matter is present, by relaxing our definition of h_2 and considering the limit $h_2 \to 0$ for fixed ε, the dominant terms in the factors in the right-hand side of (6-28) and (6-29), i.e., the first summands, do *not* converge to 1. This is because, in this limit, while the function r remains C^1, the renormalized Hawking mass $\tilde{m}$ has a jump discontinuity across the beam.

Since $T_{uu}=T_{vv}=0$ on $\mathcal{R}_{\varepsilon n}^{(i,j)}$ for any $n\ge 1$, any $0\le i\le k$ and any $i\le j\le k+i+1$, the relations (2-43)–(2-44) imply

$$\partial_v\left(\frac{-\partial_u r}{1-2m/r}\right)\Big|_{\mathcal{R}_{\varepsilon n}^{(i,j)}} = \partial_u\left(\frac{\partial_v r}{1-2m/r}\right)\Big|_{\mathcal{R}_{\varepsilon n}^{(i,j)}} = 0. \tag{6-30}$$

In particular, along lines of the form $\{u=\bar{u}\}$, the quantity $-\partial_u r/(1-2m/r)$ remains constant on $\{u=\bar{u}\}\cap\mathcal{R}_{\varepsilon n}^{(i,j)}$ for any $\bar{u}<u_*$ such that $\{u=\bar{u}\}\cap\mathcal{R}_{\varepsilon n}^{(i,j)}$ is nontrivial. In view of (6-28), the quantities

$-\partial_u r/(1-2m/r)|_{\{u=\bar{u}\}\cap\mathcal{R}_{\varepsilon n}^{(i,j+1)}}$ and $-\partial_u r/(1-2m/r)|_{\{u=\bar{u}\}\cap\mathcal{R}_{\varepsilon n}^{(i,j)}}$ (for any $\bar{u}<u_*$ such that $\{u=\bar{u}\}\cap\mathcal{R}_{\varepsilon n}^{(i,j)}$ is nontrivial) are related by

$$\begin{aligned}\frac{-\partial_u r}{1-2m/r}\Big|_{\{u=\bar{u}\}\cap\mathcal{R}_{\varepsilon n}^{(i,j)}} &\\ = \frac{-\partial_u r}{1-2m/r}\Big|_{\{u=\bar{u}\}\cap\mathcal{R}_{\varepsilon n}^{(i,j+1)}} &\cdot\left(\frac{1-\big(2\tilde{m}/r\big(1-\frac{1}{3}\Lambda r^2\big)\big)\big|_{(\bar{u},V_n(v^{(j)}))}}{1-2\tilde{m}/\big(r\big(1-\frac{1}{3}\Lambda r^2\big)\big)\big|_{(\bar{u},V_n(v^{(j)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)))}}+O((h_2(\varepsilon))^{1/2})\right).\end{aligned} \tag{6-31}$$

Similarly, the quantity $\partial_v r/(1-2m/r)$ remains constant along segments of the form $\{v=\bar{v}\}\cap\mathcal{R}_{\varepsilon n}^{(i,j)}$, and $\partial_v r/(1-2m/r)|_{\{v=\bar{v}\}\cap\mathcal{R}_{\varepsilon n}^{(i+1,j)}}$ and $\partial_v r/(1-2m/r)|_{\{v=\bar{v}\}\cap\mathcal{R}_{\varepsilon n}^{(i,j)}}$ are related (in view of (6-29)) by

$$\begin{aligned}\frac{\partial_v r}{1-2m/r}\Big|_{\{v=\bar{v}\}\cap\mathcal{R}_{\varepsilon n}^{(i,j)}} = \frac{\partial_v r}{1-2m/r}\Big|_{\{v=\bar{v}\}\cap\mathcal{R}_{\varepsilon n}^{(i+1,j)}}&\\ \cdot\left(\frac{1-2\tilde{m}/\big(r\big(1-\frac{1}{3}\Lambda r^2\big)\big)\big|_{(U_n(v^{(i)}),\bar{v})}}{1-2\tilde{m}/\big(r\big(1-\frac{1}{3}\Lambda r^2\big)\big)\big|_{(U_n(v^{(i)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),\bar{v})}}+O((h_2(\varepsilon))^{1/2})\right)&.\end{aligned} \tag{6-32}$$

We infer, therefore, that for any point $(\bar{u},\bar{v})\in\mathcal{R}_{\varepsilon,n}^{(i,j)}$ for some $n\geq 2$, $0\leq i\leq k$ and $i\leq j\leq k+i+1$ such that $\bar{u}<u_*$, the following relations hold between $(\bar{u},\bar{v})$ and $(\bar{u}-v_0,\bar{v}-v_0)\in\mathcal{R}_{\varepsilon,n-1}^{(i,j)}$:

$$\begin{aligned}\frac{-\partial_u r}{1-2m/r}\Big|_{(\bar{u},\bar{v})} = \frac{-\partial_u r}{1-2m/r}\Big|_{(\bar{u}-v_0,\bar{v}-v_0)}&\\ \times\prod_{\bar{j}=j}^{k+i}\left(\frac{1-2\tilde{m}/\big(r\big(1-\frac{1}{3}\Lambda r^2\big)\big)\big|_{(\bar{u},V_n(v^{(\bar{j})}))}}{1-2\tilde{m}/\big(r\big(1-\frac{1}{3}\Lambda r^2\big)\big)\big|_{(\bar{u},V_n(v^{(\bar{j})}+(4/\sqrt{-\Lambda})h_2(\varepsilon)))}}+O((h_2(\varepsilon))^{1/2})\right)&\\ \times\prod_{\bar{i}=i}^{k+i}\left(\frac{1-2\tilde{m}/\big(r\big(1-\frac{1}{3}\Lambda r^2\big)\big)\big|_{(U_n(v^{(\bar{i})}),\bar{u})}}{1-2\tilde{m}/\big(r\big(1-\frac{1}{3}\Lambda r^2\big)\big)\big|_{(U_n(v^{(\bar{i})}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),\bar{u})}}+O((h_2(\varepsilon))^{1/2})\right)&\\ \times\prod_{\bar{j}=k+i+1}^{k+j}\left(\frac{1-2\tilde{m}/\big(r\big(1-\frac{1}{3}\Lambda r^2\big)\big)\big|_{(\bar{u}-v_0,V_n(v^{(\bar{j})}))}}{1-2\tilde{m}/\big(r\big(1-\frac{1}{3}\Lambda r^2\big)\big)\big|_{(\bar{u}-v_0,V_n(v^{(\bar{j})}+(4/\sqrt{-\Lambda})h_2(\varepsilon)))}}+O((h_2(\varepsilon))^{1/2})\right)&\end{aligned} \tag{6-33}$$

and

$$\begin{aligned}\frac{\partial_v r}{1-2m/r}\Big|_{(\bar{u},\bar{v})} = \frac{\partial_v r}{1-2m/r}\Big|_{(\bar{u}-v_0,\bar{v}-v_0)}&\\ \times\prod_{\bar{i}=i}^{j-1}\left(\frac{1-2\tilde{m}/\big(r\big(1-\frac{1}{3}\Lambda r^2\big)\big)\big|_{(U_n(v^{(\bar{i})}),\bar{v})}}{1-2\tilde{m}/\big(r\big(1-\frac{1}{3}\Lambda r^2\big)\big)\big|_{(U_n(v^{(\bar{i})}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),\bar{v})}}+O((h_2(\varepsilon))^{1/2})\right)&\\ \times\prod_{\bar{j}=j}^{k+j}\left(\frac{1-2\tilde{m}/\big(r\big(1-\frac{1}{3}\Lambda r^2\big)\big)\big|_{(\bar{u}-v_0,V_n(v^{(\bar{j})}))}}{1-2\tilde{m}/\big(r\big(1-\frac{1}{3}\Lambda r^2\big)\big)\big|_{(\bar{u}-v_0,V_n(v^{(\bar{j})}+(4/\sqrt{-\Lambda})h_2(\varepsilon)))}}+O((h_2(\varepsilon))^{1/2})\right)&\\ \times\prod_{\bar{i}=j}^{k+i}\left(\frac{1-2\tilde{m}/\big(r\big(1-\frac{1}{3}\Lambda r^2\big)\big)\big|_{(U_n(v^{(\bar{i})}),\bar{v}-v_0)}}{1-2\tilde{m}/\big(r\big(1-\frac{1}{3}\Lambda r^2\big)\big)\big|_{(U_n(v^{(\bar{i})}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),\bar{v}-v_0)}}+O((h_2(\varepsilon))^{1/2})\right)&.\end{aligned} \tag{6-34}$$

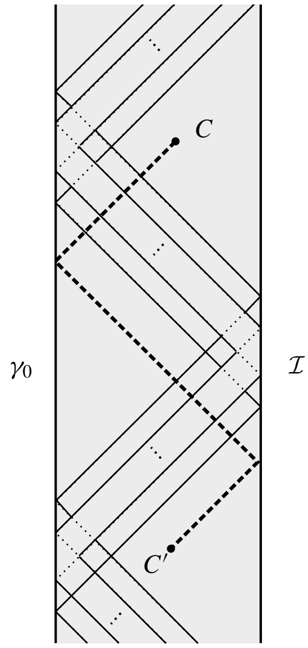

Figure 11. In order to obtain the formula (6-33) relating $-\partial_u r/(1-2m/r)$ at the point $C=(\bar{u},\bar{v})$ with the same quantity at the point $C'=(\bar{u}-v_0,\bar{v}-v_0)$, we apply the relations (6-31) and (6-32) along the dashed path depicted above, using also the reflecting gauge condition on γ_0 and $\mathcal{I}$.

The relation (6-33) is obtained as follows (see also Figure 11): First, (6-31) determines the evolution of $-\partial_u r/(1-2m/r)$ (according to (6-31)) along the line $\{u=\bar{u}\}$ in the past direction, from $(\bar{u},\bar{v})$ up to γ_0. Then, using the boundary relation

$$\frac{-\partial_u r}{1-2m/r}\bigg|_{\gamma_0}=\frac{\partial_v r}{1-2m/r}\bigg|_{\gamma_0}, \tag{6-35}$$

one repeats the same procedure for $\partial_v r/(1-2m/r)$ along $\{v=\bar{u}\}$ from γ_0 up to $\mathcal{I}$. Finally, using

$$\frac{-\partial_u r}{1-2m/r}\bigg|_{\mathcal{I}}=\frac{\partial_v r}{1-2m/r}\bigg|_{\mathcal{I}}, \tag{6-36}$$

and following the evolution of $-\partial_u r/(1-2m/r)$ along $\{u=\bar{u}-v_0\}$ from $\mathcal{I}$ up to $(\bar{u}-v_0,\bar{v}-v_0)$, one arrives at (6-33). The relation (6-34) is similarly obtained by following the same procedure along the lines $\{v=\bar{v}\}$ (up to $\mathcal{I}$), $\{u=\bar{v}-v_0\}$ (from $\mathcal{I}$ up to γ_0) and $\{v=\bar{v}-v_0\}$ (from $\mathcal{I}$ up to $(\bar{u}-v_0,\bar{v}-v_0)$).

In view of the bound

$$1-\frac{2m}{r}\geq h_3(\varepsilon) \tag{6-37}$$

on $\mathcal{U}_\varepsilon^+$ (see (6-1)), we can estimate in the region $\{r\leq\varepsilon^{1/2}(-\Lambda)^{1/2}\}\cap\mathcal{U}_\varepsilon^+$

$$1-\frac{2\tilde{m}}{r\bigl(1-\frac{1}{3}\Lambda r^2\bigr)}=\frac{1-2m/r}{1-\frac{1}{3}\Lambda r^2}\geq\tfrac{1}{2}h_1(\varepsilon). \tag{6-38}$$

On the other hand, in the region $\{r \geq \varepsilon^{1/2}(-\Lambda)^{1/2}\} \cap \mathcal{U}_{\varepsilon}^{+}$, using (5-24) to bound $\tilde{m}$ we can trivially estimate (in view also of (5-2))

$$1 - \frac{2\tilde{m}}{r\left(1 - \frac{1}{3}\Lambda r^2\right)} \geq 1 - \frac{2\varepsilon}{\varepsilon^{1/2}} \geq h_1(\varepsilon). \tag{6-39}$$

Combining (6-38) and (6-39), using also that $\tilde{m} \geq \tilde{m}|_{\gamma_0} = 0$ on $\mathcal{U}_{\varepsilon}^{+}$, we can bound $1 - 2\tilde{m}/\left(r\left(1 - \frac{1}{3}\Lambda r^2\right)\right)$ from above and below everywhere on $\mathcal{U}_{\varepsilon}^{+}$ as

$$\tfrac{1}{2}h_3(\varepsilon) \leq 1 - \frac{2\tilde{m}}{r\left(1 - \frac{1}{3}\Lambda r^2\right)} \leq 1. \tag{6-40}$$

Thus, by considering the logarithm of the relations (6-33)–(6-34) and noting that the resulting right-hand side contains $\sim k = \lceil 1/h_1(\varepsilon) \rceil$ summands, each controlled with the help of (6-40), we readily obtain for any $n \geq 2,\ 0 \leq i \leq k$, and $i \leq j \leq k+i+1$ and any point $(\bar{u}, \bar{v}) \in \mathcal{R}_{\varepsilon n}^{(i,j)}$ with $\bar{u} < u_*$

$$\left| \log\left(\frac{-\partial_u r}{1 - 2m/r}\right)\Big|_{(\bar{u},\bar{v})} - \log\left(\frac{-\partial_u r}{1 - 2m/r}\right)\Big|_{(\bar{u}-v_0,\bar{v}-v_0)} \right| \leq \frac{C}{h_1(\varepsilon)} \log((h_3(\varepsilon))^{-1}) \tag{6-41}$$

and

$$\left| \log\left(\frac{\partial_v r}{1 - 2m/r}\right)\Big|_{(\bar{u},\bar{v})} - \log\left(\frac{\partial_v r}{1 - 2m/r}\right)\Big|_{(\bar{u}-v_0,\bar{v}-v_0)} \right| \leq \frac{C}{h_1(\varepsilon)} \log((h_3(\varepsilon))^{-1}). \tag{6-42}$$

In view of (6-28)–(6-29), the bounds (6-41) and (6-42) (stated in the case when $(\bar{u}, \bar{v})$ belongs to a vacuum region $\mathcal{R}_{\varepsilon n}^{(i,j)}$) also hold when $(\bar{u}, \bar{v})$ belongs to a beam, i.e., when

$$U_n(v^{(i)}) \leq \bar{u} \leq U_n\left(v^{(i-1)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)\right)$$

or

$$V_n(v^{(j)}) \leq \bar{v} \leq V_n\left(v^{(i-1)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)\right)$$

for some $n \geq 2,\ 0 \leq i \leq k$, and $i \leq j \leq k+i+1$. Therefore, for any $n \geq 2$, the bounds (6-28)–(6-29) hold on the whole of

$$\mathcal{U}_{\varepsilon;n}^{*} \doteq \{U_n(v^{(k)}) \leq u \leq U_{n+1}(v^{(k)})\} \cap \mathcal{U}_{\varepsilon}^{*}. \tag{6-43}$$

From (6-2) and the definition (6-5), it follows that

$$n_f \leq (h_1(\varepsilon))^{-2}. \tag{6-44}$$

Since $n \leq n_f$ (because $\mathcal{U}_{\varepsilon}^{*} \subset \mathcal{U}_{\varepsilon}^{+}$), by substituting $(\bar{u}, \bar{v}) \to (\bar{u} - v_0, \bar{v} - v_0)$ in (6-41)–(6-42) $n-2$ times and using (6-44), (5-16), (6-40) as well as the Cauchy stability estimate of Proposition 3.5 for the region $\{0 \leq u \leq 2v_0\}$, we readily obtain

$$\sup_{\mathcal{U}_{\varepsilon}^{*}} \left\{ \left| \log\left(\frac{-\partial_u r}{1 - \frac{1}{3}\Lambda r^2}\right) \right| + \left| \log\left(\frac{\partial_v r}{1 - 2m/r}\right) \right| \right\} \leq \frac{C}{(h_1(\varepsilon))^3} \log((h_3(\varepsilon))^{-1}). \tag{6-45}$$

Thus, (6-7) holds on $\mathcal{U}_{\varepsilon}^{*}$ in view of the relation (5-2) for the parameter $h_1(\varepsilon)$ (provided ε_0 is small enough). Therefore (as explained in the beginning of the proof), a standard continuity argument yields that (6-7) actually holds on the whole of $\mathcal{U}_{\varepsilon}^{+}$.

Proof of (6-8) *and* (6-9). For any $1 \le n \le n_f$, we can bound, in view of the definition (5-8) of $v^{(j)}$ and the bound (6-7),

$$\begin{aligned}
&\left| \tan^{-1}\Big(\sqrt{-\frac{\Lambda}{3}}r\Big)\Big|_{(U_n(v^{(0)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),V_n(v^{(k)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)))} - \tan^{-1}\Big(\sqrt{-\frac{\Lambda}{3}}r\Big)\Big|_{\mathcal{I}\cap\{v=V_n(v^{(k)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))\}} \right| \\
&\qquad = \sqrt{-\frac{\Lambda}{3}} \int_{U_n(v^{(k)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))}^{U_n(v^{(0)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))} \frac{-\partial_u r}{1-\frac{1}{3}\Lambda r^2}\Big|_{(u,V_n(v^{(k)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)))} du \\
&\qquad \le \frac{C\sqrt{-\Lambda}}{(h_1(\varepsilon))^4} \log((h_3(\varepsilon))^{-1}) |v^{(k)} - v^{(0)}| \\
&\qquad \le \frac{C\varepsilon}{(h_1(\varepsilon))^6} \log((h_3(\varepsilon))^{-1})
\end{aligned} \tag{6-46}$$

and

$$\begin{aligned}
&\left| \tan^{-1}\Big(\sqrt{-\frac{\Lambda}{3}}r\Big)\Big|_{(U_n(v^{(k)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),V_n(v^{(k+1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)))} - \tan^{-1}\Big(\sqrt{-\frac{\Lambda}{3}}r\Big)\Big|_{\gamma_0\cap\{v=V_n(v^{(k+1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))\}} \right| \\
&\qquad = \sqrt{-\frac{\Lambda}{3}} \int_{U_n(v^{(k)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))}^{U_n(v^{(0)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))} \frac{-\partial_u r}{1-\frac{1}{3}\Lambda r^2}\Big|_{(u,V_n(v^{(k+1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)))} du \\
&\qquad \le \frac{C\sqrt{-\Lambda}}{(h_1(\varepsilon))^4} \log((h_3(\varepsilon))^{-1}) |v^{(k)} - v^{(0)}| \\
&\qquad \le \frac{C\varepsilon}{(h_1(\varepsilon))^6} \log((h_3(\varepsilon))^{-1}).
\end{aligned} \tag{6-47}$$

From (6-46) and (6-47) we readily obtain (6-8) and (6-9), respectively, in view of the relations (5-2) and (6-3) for h_1, h_3, respectively, and the fact that $r|_{\gamma_0} = r_0$, $r|_{\mathcal{I}} = +\infty$.

Part II: proof of (6-10)–(6-14). We will now proceed to establish the bounds (6-10)–(6-14). To this end, we will first derive some useful estimates for the differences of the renormalized masses $\tilde{m}_n^{(i,j)}$ associated to the vacuum regions around each interaction region $\mathcal{N}_{\varepsilon n}^{(i,j)}$.[16]

Relations for the change in the mass difference of the beams. Let us introduce the notion of the mass difference for the beams $\{U_n(v^{(i)}) \le u \le U_n(v^{(i)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))\}$ and $\{V_n(v^{(j)}) \le v \le V_n(v^{(j)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))\}$ around their interaction region $\mathcal{N}_{\varepsilon n}^{(i,j)}$: For any $1 \le n \le n_f$, $0 \le i \le k$, and $i+1 \le j \le k+i$, we define the initial mass differences

$$\begin{aligned}
(\mathfrak{D}_-\tilde{m})_n^{(i,j)} &\doteq \tilde{m}_n^{(i+1,j+1)} - \tilde{m}_n^{(i,j+1)}, \\
(\overline{\mathfrak{D}}_-\tilde{m})_n^{(i,j)} &\doteq \tilde{m}_n^{(i+1,j)} - \tilde{m}_n^{(i+1,j+1)}
\end{aligned} \tag{6-48}$$

and the final mass differences

$$\begin{aligned}
(\mathfrak{D}_+\tilde{m})_n^{(i,j)} &\doteq \tilde{m}_n^{(i+1,j)} - \tilde{m}_n^{(i,j)}, \\
(\overline{\mathfrak{D}}_+\tilde{m})_n^{(i,j)} &\doteq \tilde{m}_n^{(i,j)} - \tilde{m}_n^{(i,j+1)}.
\end{aligned} \tag{6-49}$$

[16]A relation for the change the mass differences of two intersecting, infinitely thin null dust beams was also obtained in [Poisson and Israel 1990].

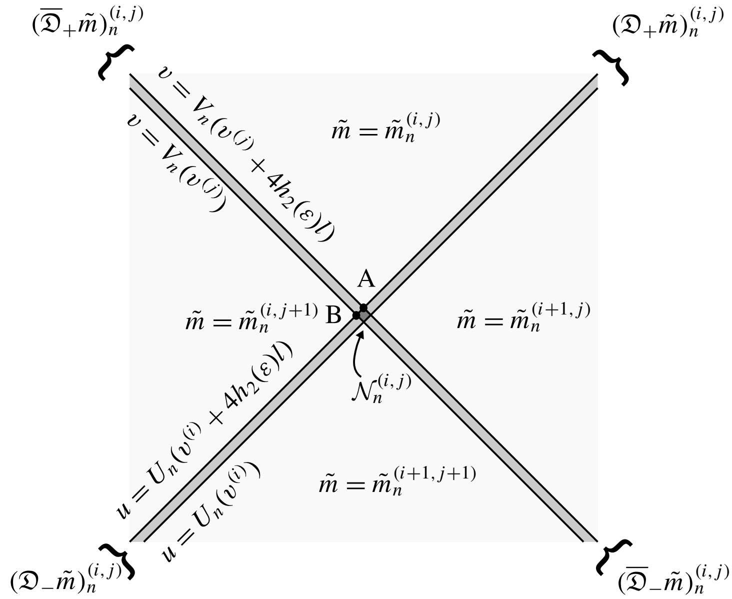

Figure 12. Schematic depiction of two intersecting beams, with associated incoming and outcoming mass differences $(\mathfrak{D}_{-}\tilde{m})_n^{(i,j)}$, $(\overline{\mathfrak{D}}_{-}\tilde{m})_n^{(i,j)}$ and $(\mathfrak{D}_{+}\tilde{m})_n^{(i,j)}$, $(\overline{\mathfrak{D}}_{+}\tilde{m})_n^{(i,j)}$, respectively. The point A satisfies $r(A) = r_n^{(i,j)}$, while the point B satisfies $r(B) = \bar{r}_n^{(i,j)}$. For simplicity, we have used the shorthand notation $l = (-\Lambda)^{-1/2}$.

Note that $(\mathfrak{D}_{-}\tilde{m})_n^{(i,j)}$ and $(\mathfrak{D}_{+}\tilde{m})_n^{(i,j)}$ are the mass differences around the outgoing beam $\{U_n(v^{(i)}) \le u \le U_n(v^{(i)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))\}$ before and after crossing the region $\mathcal{N}_{\varepsilon n}^{(i,j)}$, respectively, while $(\overline{\mathfrak{D}}_{-}\tilde{m})_n^{(i,j)}$ and $(\overline{\mathfrak{D}}_{+}\tilde{m})_n^{(i,j)}$ are the mass differences around the ingoing beam $\{V_n(v^{(j)}) \le v \le U_n(v^{(j)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))\}$ before and after crossing the region $\mathcal{N}_{\varepsilon n}^{(i,j)}$. See Figure 12. Note the trivial identity

$$(\mathfrak{D}_{-}\tilde{m})_n^{(i,j)} + (\overline{\mathfrak{D}}_{-}\tilde{m})_n^{(i,j)} = (\mathfrak{D}_{+}\tilde{m})_n^{(i,j)} + (\overline{\mathfrak{D}}_{+}\tilde{m})_n^{(i,j)}. \tag{6-50}$$

We will establish the following bounds for any $1 \le n \le n_f$, $1 \le i \le k$ and $i+1 \le j \le k+i$:

$$(\overline{\mathfrak{D}}_{+}\tilde{m})_n^{(i,j)} = (\overline{\mathfrak{D}}_{-}\tilde{m})_n^{(i,j)} \exp\left(\frac{2}{\bar{r}_n^{(i,j)}} \frac{(\mathfrak{D}_{-}\tilde{m})_n^{(i,j)}}{1 - 2\tilde{m}_n^{(i+1,j)}/\bar{r}_n^{(i,j)} - \frac{1}{3}\Lambda(\bar{r}_n^{(i,j)})^2} (1 - \mathfrak{Err}_{1,n}^{(i,j)})(1 - \mathfrak{Err}_{\backslash n}^{(i,j)}) \right), \tag{6-51}$$

$$(\mathfrak{D}_{+}\tilde{m})_n^{(i,j)} = (\mathfrak{D}_{-}\tilde{m})_n^{(i,j)} \exp\left(-\frac{2}{\bar{r}_n^{(i,j)}} \frac{(\overline{\mathfrak{D}}_{+}\tilde{m})_n^{(i,j)}}{1 - 2\tilde{m}_n^{(i+1,j)}/\bar{r}_n^{(i,j)} - \frac{1}{3}\Lambda(\bar{r}_n^{(i,j)})^2} (1 - \mathfrak{Err}_{1,n}^{(i,j)})(1 - \mathfrak{Err}_{/n}^{(i,j)}) \right), \tag{6-52}$$

where the terms $\mathfrak{Err}_{1,n}^{(i,j)}$ in (6-51) and (6-52) are allowed to be different from each other, but they both satisfy the bound

$$0 \le \mathfrak{Err}_{1,n}^{(i,j)} \le 1 - \frac{\bar{r}_n^{(i,j)} - 2\tilde{m}_n^{(i+1,j)} - \frac{1}{3}\Lambda(\bar{r}_n^{(i,j)})^3}{r|_{(U_n(v^{(i)}), V_n(v^{(j)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)))} - 2\tilde{m}_n^{(i,j+1)} - \frac{1}{3}\Lambda r^3|_{(U_n(v^{(i)}), V_n(v^{(j)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)))}} \tag{6-53}$$

and $\mathfrak{Err}^{(i,j)}_{\backslash n}$, $\mathfrak{Err}^{(i,j)}_{/n}$ satisfy the bounds

$$0 \le \mathfrak{Err}^{(i,j)}_{\backslash n} \le 1 - \frac{(\mathfrak{D}_+\tilde{m})^{(i,j)}_n}{(\mathfrak{D}_-\tilde{m})^{(i,j)}_n}, \tag{6-54}$$

$$0 \le \mathfrak{Err}^{(i,j)}_{/n} \le 1 - \frac{(\overline{\mathfrak{D}}_-\tilde{m})^{(i,j)}_n}{(\overline{\mathfrak{D}}_+\tilde{m})^{(i,j)}_n}. \tag{6-55}$$

Moreover, the following estimate will be useful in the proof of (6-12): for any $1 \le n \le n_f$, $1 \le i \le k$, and $k+1 \le j \le k+i$,

$$(\overline{\mathfrak{D}}_+\tilde{m})^{(i,j)}_n \ge (\overline{\mathfrak{D}}_-\tilde{m})^{(i,j)}_n \cdot \exp\biggl(\frac{1}{5C_0}\frac{(\mathfrak{D}_-\tilde{m})^{(i,j)}_n}{\bar{r}^{(i,j)}_n}\biggr). \tag{6-56}$$

Remark. Notice that, as a consequence of (6-51) and (6-52), during the interaction of the two beams at $\mathcal{N}^{(i,j)}_{\varepsilon n}$, the mass difference $\overline{\mathfrak{D}}\tilde{m}$ of the ingoing beam increases, while the mass difference $\mathfrak{D}\tilde{m}$ of the outgoing beam decreases.

Proof of (6-51) *and* (6-52). By differentiating (2-47) in u and using (2-43) and (2-48), we readily obtain the wave-type equation for $\tilde{m}$

$$\partial_u\partial_v\tilde{m} = -F(r,\tilde{m})\,\partial_u\tilde{m}\partial_v\tilde{m}, \tag{6-57}$$

where

$$F(r,\tilde{m}) \doteq \frac{2}{r-2\tilde{m}-\frac{1}{3}\Lambda r^3}. \tag{6-58}$$

Note that, formally, (6-57) can be rewritten as

$$\partial_v \log(-\partial_u\tilde{m}) = -F(r,\tilde{m})\,\partial_v\tilde{m} \tag{6-59}$$

or

$$\partial_u \log(\partial_v\tilde{m}) = F(r,\tilde{m})(-\partial_u\tilde{m}) \tag{6-60}$$

(note, however, that $\log(-\partial_u\tilde{m})$, $\log(\partial_v\tilde{m})$ will not be well-defined when $\partial_u\tilde{m}=0$ or $\partial_v\tilde{m}=0$).

For any $1 \le n \le n_f$, $1 \le i \le k$, and $i+1 \le j \le k+i$, integrating (6-57) first in u, for $U_n(v^{(i)})) \le u \le U_n(v^{(i)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))$, and then in v, for $V_n(v^{(j)}) \le v \le V_n(v^{(j)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))$, we obtain

$$\tilde{m}^{(i,j)}_n - \tilde{m}^{(i,j+1)}_n = \int_{V_n(v^{(j)})}^{V_n(v^{(j)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))} \partial_v\tilde{m}|_{(U_n(v^{(i)})),v)} \cdot \exp\biggl(2\int_{U_n(v^{(i)})}^{U_n(v^{(i)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))} \frac{-\partial_u\tilde{m}}{r-2m}\bigg|_{(u,v)}\,du\biggr)\,dv. \tag{6-61}$$

Remark. Note that, at the formal level, the derivation of (6-61) is easiest seen by integrating (6-60) first in u, then exponentiating, and then integrating in v. This procedure can actually be done rigorously, since $\partial_u\tilde{m} < 0 < \partial_v\tilde{m}$ in the interior of $\mathcal{N}^{(i,j)}_{\varepsilon n}$, in view of (2-46)–(2-47) and (5-43)–(5-44).

In view of (6-18)–(6-19), we can bound for any $U_n(v^{(i)}) \le u \le U_n(v^{(i)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))$ and any $V_n(v^{(j)}) \le v \le V_n(v^{(j)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))$

$$\begin{aligned}\bar{r}^{(i,j)}_n - 2\tilde{m}^{(i+1,j)}_n - \tfrac{1}{3}\Lambda(\bar{r}^{(i,j)}_n)^3 &\le (r-2m)|_{(u,v)} \\ &\le r|_{(U_n(v^{(i)}),V_n(v^{(j)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)))} - 2\tilde{m}^{(i,j+1)}_n - \tfrac{1}{3}\Lambda r^3|_{(U_n(v^{(i)}),V_n(v^{(j)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)))},\end{aligned} \tag{6-62}$$

where

$$\bar{r}_n^{(i,j)} \doteq r|_{(U_n(v^{(i)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),V_n(v^{(j)}))}.$$

Therefore, using (6-62) to estimate $1/(r-2m)$, from (6-61) we readily infer that

$$\tilde{m}_n^{(i,j)} - \tilde{m}_n^{(i,j+1)} = \int_{V_n(v^{(j)})}^{V_n(v^{(j)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))} \partial_v \tilde{m}|_{(U_n(v^{(i)})),v)} \cdot \exp\biggl(\frac{2}{\bar{r}_n^{(i,j)}} \frac{\tilde{m}|_{(U_n(v^{(i)}),v)} - \tilde{m}|_{(U_n(v^{(i)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),v)}}{1-2\tilde{m}_n^{(i+1,j)}/\bar{r}_n^{(i,j)} - \frac{1}{3}\Lambda(\bar{r}_n^{(i,j)})^2}(1-\mathfrak{Err}_{1,n}^{(i,j)}(v))\biggr)\,dv, \tag{6-63}$$

where, for any $V_n(v^{(j)}) \le v \le V_n(v^{(j)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))$, $\mathfrak{Err}_{1,n}^{(i,j)}(v)$ satisfies the bound (6-53).

Equations (2-46), (2-44) and (2-23) imply that, for any $V_n(v^{(j)}) \le v \le V_n(v^{(j)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))$,

$$\partial_v(\tilde{m}|_{(U_n(v^{(i)}),v)} - \tilde{m}|_{(U_n(v^{(i)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),v)}) \le 0 \tag{6-64}$$

and, therefore, for any $V_n(v^{(j)}) \le v \le V_n(v^{(j)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))$,

$$(\mathfrak{D}_+\tilde{m})_n^{(i,j)} \le \tilde{m}|_{(U_n(v^{(i)}),v)} - \tilde{m}|_{(U_n(v^{(i)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),v)} \le (\mathfrak{D}_-\tilde{m})_n^{(i,j)}. \tag{6-65}$$

The bound (6-65) implies that (6-63) can be expressed as

$$\tilde{m}_n^{(i,j)} - \tilde{m}_n^{(i,j+1)} = (\tilde{m}_n^{(i+1,j)} - \tilde{m}_n^{(i+1,j+1)}) \cdot \exp\biggl(\frac{2}{\bar{r}_n^{(i,j)}} \frac{(\mathfrak{D}_-\tilde{m})_n^{(i,j)}}{1-2\tilde{m}_n^{(i+1,j)}/\bar{r}_n^{(i,j)} - \frac{1}{3}\Lambda(\bar{r}_n^{(i,j)})^2}(1-\mathfrak{Err}_{1,n}^{(i,j)})(1-\mathfrak{Err}_{\backslash n}^{(i,j)})\biggr), \tag{6-66}$$

where $\mathfrak{Err}_{1,n}^{(i,j)}$ satisfies the bound (6-53) and $\mathfrak{Err}_{\backslash n}^{(i,j)}$ satisfies the bound (6-54). In view of (6-48) and (6-49), (6-66) is equivalent to (6-51).

Similarly, integrating (6-57) first in v, for $V_n(v^{(j)}) \le v \le V_n(v^{(j)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))$, and then in u, for $U_n(v^{(i)})) \le u \le U_n(v^{(i)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))$ (see also (6-59)), we obtain (6-52). □

Proof of (6-56). Recall F defined by (6-58) and let us define the function $\bar{F} : \mathcal{D}_{\bar{F}} \to (0,+\infty)$, where

$$\mathcal{D}_{\bar{F}} = \bigl\{(x,y) \in \mathbb{R}^2 x > 0 \text{ and } x - y - \tfrac{2}{3}\Lambda x^2 > 0\bigr\}, \tag{6-67}$$

by the relation

$$\bar{F}(x,y) \doteq \frac{2}{x-y-\frac{2}{3}\Lambda x^2}. \tag{6-68}$$

Note that, in view of (6-7), (6-9), (5-6) and (5-3), for any $\mu \ge 0$ for which

$$\inf_{(u,v)\in\mathcal{N}_{\varepsilon n}^{(i,j)}} \bigl\{r(u,v) - 2\mu - \tfrac{1}{3}\Lambda r^2(u,v)\bigr\} > h_3(\varepsilon), \tag{6-69}$$

we can readily bound

$$\max_{(u,v)\in\mathcal{N}_{\varepsilon n}^{(i,j)}} \bar{F}(r(u,v),\mu) < \min_{(u,v)\in\mathcal{N}_{\varepsilon n}^{(i,j)}} F(r(u,v),\mu) \tag{6-70}$$

and

$$\partial_\mu \bar{F}(r(u,v),\mu),\ \partial_\mu F(r(u,v),\mu) > 0 \tag{6-71}$$

(note that $F(r|_{\mathcal{N}_{\varepsilon n}^{(i,j)}},\mu)$ and $\bar{F}(r|_{\mathcal{N}_{\varepsilon n}^{(i,j)}},\mu)$ are well-defined and positive under the condition (6-69)).

For any $1 \le n \le n_f$, $1 \le i \le k$, and $k+1 \le j \le k+i$, let us consider the following characteristic initial value problem on $\mathcal{N}_{\varepsilon n}^{(i,j)}$:

$$\begin{cases} \partial_u \partial_v \bar{m} = -\bar{F}(r, \bar{m}) \partial_u \bar{m}\, \partial_v \bar{m} & \text{on } \mathcal{N}_{\varepsilon n}^{(i,j)}, \\ \qquad \bar{m} = \tilde{m} & \text{on } \Big[U_n(v^{(i)}), U_n\Big(v^{(i)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)\Big)\Big] \times \{V_n(v^{(j)})\} \cup \{U_n(v^{(i)})\} \\ & \qquad \times \Big[V_n(v^{(j)}), V_n\Big(v^{(j)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)\Big)\Big]. \end{cases} \tag{6-72}$$

Note that $\tilde{m}$ satisfies the same characteristic initial value problem with $F(r, \tilde{m})$ in place of $\bar{F}(r, \bar{m})$. Notice also that, in view of (2-46)–(2-47) and (5-43)–(5-44), the initial data for $\tilde{m}$ and $\bar{m}$ satisfy

$$\partial_u \tilde{m} < 0 \quad \text{on } \Big\{U_n(v^{(i)}) < u < U_n\big(v^{(i)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)\big)\Big\}, \tag{6-73}$$

$$\partial_v \tilde{m} > 0 \quad \text{on } \Big\{V_n(v^{(j)}) < u < V_n\big(v^{(j)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)\big)\Big\}. \tag{6-74}$$

Therefore, in view of (6-70)–(6-71) and (6-73)–(6-74), an application of Lemma A.1 (see Section A1 of the Appendix) with $\tilde{m}, \bar{m}$ in place of z_2, z_1, respectively, yields the following a priori bounds for a solution $\bar{m}$ of (6-72):

$$\bar{m} \le \tilde{m} \qquad \text{on } \mathcal{N}_{\varepsilon n}^{(i,j)}, \tag{6-75}$$

$$\partial_u \bar{m} < 0 < \partial_v \bar{m} \quad \text{in the interior of } \mathcal{N}_{\varepsilon n}^{(i,j)}. \tag{6-76}$$

Notice that the a priori bound (6-75) and the initial data in (6-72) imply that $\bar{m} \ge 0$ and that (6-69) holds for $\mu = \tilde{m}$ and $\mu = \bar{m}$; in particular, $\bar{F}(r, \bar{m})$ is well-defined and positive on $\mathcal{N}_{\varepsilon n}^{(i,j)}$. Thus, it readily follows (using standard arguments) that (6-72) indeed has a unique smooth solution $\bar{m}$ satisfying (6-75).

With $\bar{m}$ defined on $\mathcal{N}_{\varepsilon n}^{(i,j)}$ as above for any $1 \le n \le n_f$, $1 \le i \le k$, and $k+1 \le j \le k+i$, we will define the following modified versions of (6-48) and (6-49):

$$\begin{aligned} (\mathfrak{D}_- \bar{m})_n^{(i,j)} &\doteq \bar{m}|_{(U_n(v^{(i)}), V_n(v^{(j)}))} - \bar{m}|_{(U_n(v^{(i)} + (4/\sqrt{-\Lambda}) h_2(\varepsilon)), V_n(v^{(j)}))}, \\ (\bar{\mathfrak{D}}_- \bar{m})_n^{(i,j)} &\doteq \bar{m}|_{(U_n(v^{(i)}), V_n(v^{(j)} + (4/\sqrt{-\Lambda}) h_2(\varepsilon)))} - \bar{m}|_{(U_n(v^{(i)}), V_n(v^{(j)}))} \end{aligned} \tag{6-77}$$

and

$$\begin{aligned} (\mathfrak{D}_+ \bar{m})_n^{(i,j)} &\doteq \bar{m}|_{(U_n(v^{(i)}), V_n(v^{(j)} + (4/\sqrt{-\Lambda}) h_2(\varepsilon)))} - \bar{m}|_{(U_n(v^{(i)} + (4/\sqrt{-\Lambda}) h_2(\varepsilon)), V_n(v^{(j)} + (4/\sqrt{-\Lambda}) h_2(\varepsilon)))}, \\ (\bar{\mathfrak{D}}_+ \bar{m})_n^{(i,j)} &\doteq \bar{m}|_{(U_n(v^{(i)} + (4/\sqrt{-\Lambda}) h_2(\varepsilon)), V_n(v^{(j)} + (4/\sqrt{-\Lambda}) h_2(\varepsilon)))} - \bar{m}|_{(U_n(v^{(i)} + (4/\sqrt{-\Lambda}) h_2(\varepsilon)), V_n(v^{(j)}))}. \end{aligned} \tag{6-78}$$

Note that, in view of the initial data for (6-72),

$$\begin{aligned} (\mathfrak{D}_- \bar{m})_n^{(i,j)} &= (\mathfrak{D}_- \tilde{m})_n^{(i,j)}, \\ (\bar{\mathfrak{D}}_- \bar{m})_n^{(i,j)} &= (\bar{\mathfrak{D}}_- \tilde{m})_n^{(i,j)}, \end{aligned} \tag{6-79}$$

while, in view of the bound (6-75) (and the initial data for (6-72)),

$$\begin{aligned} (\mathfrak{D}_+ \bar{m})_n^{(i,j)} &\ge (\mathfrak{D}_+ \tilde{m})_n^{(i,j)}, \\ (\bar{\mathfrak{D}}_+ \bar{m})_n^{(i,j)} &\le (\bar{\mathfrak{D}}_+ \tilde{m})_n^{(i,j)}. \end{aligned} \tag{6-80}$$

By repeating exactly the same steps that led to (6-51) and (6-52) but using (6-72) instead of (6-57), we obtain for any $1 \le n \le n_f$, $1 \le i \le k$, and $k+1 \le j \le k+i$,

$$(\bar{\mathfrak{D}}_+\bar{m})_n^{(i,j)} = (\bar{\mathfrak{D}}_-\bar{m})_n^{(i,j)} \cdot \exp\Bigg(\frac{2}{\bar{r}_n^{(i,j)}} \frac{(\mathfrak{D}_-\bar{m})_n^{(i,j)}}{1-\tilde{m}_n^{(i+1,j)}/\bar{r}_n^{(i,j)} - \frac{2}{3}\Lambda(\bar{r}_n^{(i,j)})^2}(1-\overline{\mathfrak{Err}}_{1,n}^{(i,j)})(1-\overline{\mathfrak{Err}}_{\backslash n}^{(i,j)})\Bigg), \quad (6\text{-}81)$$

$$(\mathfrak{D}_+\bar{m})_n^{(i,j)} = (\mathfrak{D}_-\bar{m})_n^{(i,j)} \cdot \exp\Bigg(-\frac{2}{\bar{r}_n^{(i,j)}} \frac{(\bar{\mathfrak{D}}_+\bar{m})_n^{(i,j)}}{1-\tilde{m}_n^{(i+1,j)}/\bar{r}_n^{(i,j)} - \frac{2}{3}\Lambda(\bar{r}_n^{(i,j)})^2}(1-\overline{\mathfrak{Err}}_{1,n}^{(i,j)})(1-\overline{\mathfrak{Err}}_{/n}^{(i,j)})\Bigg), \quad (6\text{-}82)$$

where

$$0 \le \overline{\mathfrak{Err}}_{1,n}^{(i,j)} \le 1 - \frac{\bar{r}_n^{(i,j)} - \tilde{m}_n^{(i+1,j)} - \frac{2}{3}\Lambda(\bar{r}_n^{(i,j)})^3}{r|_{(U_n(v^{(i)}),V_n(v^{(j)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)))} - \tilde{m}_n^{(i,j+1)} - \frac{2}{3}\Lambda r^3|_{(U_n(v^{(i)}),V_n(v^{(j)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)))}}, \quad (6\text{-}83)$$

and

$$0 \le \overline{\mathfrak{Err}}_{\backslash n}^{(i,j)} \le 1 - \frac{(\mathfrak{D}_+\bar{m})_n^{(i,j)}}{(\mathfrak{D}_-\bar{m})_n^{(i,j)}}, \quad (6\text{-}84)$$

$$0 \le \overline{\mathfrak{Err}}_{/n}^{(i,j)} \le 1 - \frac{(\bar{\mathfrak{D}}_-\bar{m})_n^{(i,j)}}{(\bar{\mathfrak{D}}_+\bar{m})_n^{(i,j)}} \quad (6\text{-}85)$$

(and, as before, we allow the terms $\overline{\mathfrak{Err}}_{1,n}^{(i,j)}$ in (6-81) and (6-82) to be different).

Because

$$\tilde{m}_n^{(i+1,j)} \le \tilde{m}|_{\mathcal{I}} \le \tfrac{2}{3} r_0 \le \tfrac{2}{3}\bar{r}_n^{(i,j)} \quad (6\text{-}86)$$

(in view of (5-24), (5-6), (6-18) and (6-19)), from (6-82) (using also (6-9) and that $\overline{\mathfrak{Err}}_{1,n}^{(i,j)}, \overline{\mathfrak{Err}}_{/n}^{(i,j)} \ge 0$) we can estimate for any $1 \le n \le n_f$, $1 \le i \le k$, and $k+1 \le j \le k+i$

$$\begin{aligned}(\mathfrak{D}_+\bar{m})_n^{(i,j)} &= (\mathfrak{D}_-\bar{m})_n^{(i,j)} \cdot \exp\Bigg(-\frac{2}{\bar{r}_n^{(i,j)}} \frac{(\bar{\mathfrak{D}}_+\bar{m})_n^{(i,j)}}{1-\tilde{m}_n^{(i+1,j)}/\bar{r}_n^{(i,j)} - \frac{2}{3}\Lambda(\bar{r}_n^{(i,j)})^2}(1-\overline{\mathfrak{Err}}_{1,n}^{(i,j)})(1-\overline{\mathfrak{Err}}_{/n}^{(i,j)})\Bigg) \\ &\ge (\mathfrak{D}_-\bar{m})_n^{(i,j)} \cdot \exp\Bigg(-\frac{8(\bar{\mathfrak{D}}_+\bar{m})_n^{(i,j)}}{\bar{r}_n^{(i,j)}}\Bigg). \end{aligned} \quad (6\text{-}87)$$

In view of the fact that

$$(\bar{\mathfrak{D}}_+\bar{m})_n^{(i,j)} \le (\bar{\mathfrak{D}}_+\tilde{m})_n^{(i,j)} = \tilde{m}_n^{(i,j)} - \tilde{m}_n^{(i,j+1)} \le \tilde{m}|_{\mathcal{I}} - 0 \le \tfrac{2}{3} r_0 \le \tfrac{2}{3}\bar{r}_n^{(i,j)}$$

(following from (6-80)), (6-87) yields

$$(\mathfrak{D}_+\bar{m})_n^{(i,j)} \ge e^{-16/3}(\mathfrak{D}_-\bar{m})_n^{(i,j)}. \quad (6\text{-}88)$$

In view of (6-84), (6-88) implies

$$1 - \overline{\mathfrak{Err}}_{\backslash n}^{(i,j)} \ge \frac{1}{C_0}. \quad (6\text{-}89)$$

Using (6-89) in (6-81), we obtain

$$(\bar{\mathfrak{D}}_+\bar{m})_n^{(i,j)} \geq (\bar{\mathfrak{D}}_-\bar{m})_n^{(i,j)} \cdot \exp\left(\frac{2}{C_0\bar{r}_n^{(i,j)}} \frac{(\mathfrak{D}_-\bar{m})_n^{(i,j)}}{1-\tilde{m}_n^{(i+1,j)}/\bar{r}_n^{(i,j)} - \frac{2}{3}\Lambda(\bar{r}_n^{(i,j)})^2}(1-\overline{\mathfrak{Err}}_{1,n}^{(i,j)})\right). \tag{6-90}$$

In view of (6-86) and (6-9), we can also estimate

$$\frac{1-\overline{\mathfrak{Err}}_{1,n}^{(i,j)}}{1-\tilde{m}_n^{(i+1,j)}/\bar{r}_n^{(i,j)} - \frac{2}{3}\Lambda(\bar{r}_n^{(i,j)})^2} \geq \frac{1}{10} \tag{6-91}$$

and, thus, (6-90) yields

$$(\bar{\mathfrak{D}}_+\bar{m})_n^{(i,j)} \geq (\bar{\mathfrak{D}}_-\bar{m})_n^{(i,j)} \cdot \exp\left(\frac{1}{5C_0}\frac{(\mathfrak{D}_-\bar{m})_n^{(i,j)}}{\bar{r}_n^{(i,j)}}\right). \tag{6-92}$$

From (6-92) and the relations (6-79) and (6-80), we readily obtain (6-56). □

Proof of (6-11). For any $1 \leq n \leq n_f$, from (6-51) we readily obtain that, for any $1 \leq i \leq k$,

$$(\bar{\mathfrak{D}}_+\tilde{m})_n^{(i,k+1)} \geq (\bar{\mathfrak{D}}_-\tilde{m})_n^{(i,k+1)}. \tag{6-93}$$

Applying (6-93) successively for $i = 1, 2, \dots k$, using also the identity

$$(\bar{\mathfrak{D}}_-\tilde{m})_n^{(i,j)} = (\bar{\mathfrak{D}}_+\tilde{m})_n^{(i+1,j)} \tag{6-94}$$

(which follows from the fact that $\tilde{m}$ is constant over each $\mathcal{R}_{\varepsilon n}^{(i,j)}$), we thus infer that, for any $1 \leq i \leq k$,

$$(\bar{\mathfrak{D}}_+\tilde{m})_n^{(i,k+1)} \geq (\bar{\mathfrak{D}}_-\tilde{m})_n^{(k,k+1)} = (\tilde{m}_{n-1}^{(0,0)} - \tilde{m}_{n-1}^{(0,1)}). \tag{6-95}$$

Since

$$\tilde{m}_{n-1}^{(0,0)} = \tilde{m}_{n-1}^{(1,1)} = \tilde{m}|_{\mathcal{I}} \tag{6-96}$$

and

$$(\mathfrak{D}_+\tilde{m})_{n-1}^{(0,1)} = \tilde{m}_{n-1}^{(1,1)} - \tilde{m}_{n-1}^{(0,1)} = (\bar{\mathfrak{D}}_-\tilde{m})_n^{(k,k+1)}, \tag{6-97}$$

from (6-95) we infer that, for any $1 \leq i \leq k$,

$$(\bar{\mathfrak{D}}_+\tilde{m})_n^{(i,k+1)} \geq (\mathfrak{D}_+\tilde{m})_{n-1}^{(0,1)}. \tag{6-98}$$

Similarly as for the derivation of (6-93), applying the relation (6-52) successively for $i = 0$ and $j = 1, 2, \dots, k$ (with $n-1$ in place of n), we infer

$$\begin{aligned}(\mathfrak{D}_+\tilde{m})_{n-1}^{(0,1)} \\ = (\mathfrak{D}_-\tilde{m})_{n-1}^{(0,k)} \cdot \exp\left(-\sum_{j=1}^{k}\frac{2}{\bar{r}_{n-1}^{(0,j)}}\frac{(\bar{\mathfrak{D}}_-\tilde{m})_{n-1}^{(0,j)}}{1-2\tilde{m}_{n-1}^{(0,j)}/\bar{r}_{n-1}^{(0,j)} - \frac{1}{3}\Lambda(\bar{r}_{n-1}^{(0,j)})^2}(1-\mathfrak{Err}_{1,n-1}^{(0,j)})(1-\mathfrak{Err}_{/n-1}^{(0,j)})\right).\end{aligned} \tag{6-99}$$

In view of the bound (5-24) for the total mass $\tilde{m}|_{\mathcal{I}}$, the lower bound (6-8) for $r_n^{(0,k)}$ and the fact that

$$\bar{r}_{n-1}^{(0,j)} \leq r_{n-1}^{(0,k)} \leq \bar{r}_{n-1}^{(0,j)}(1+(h_2(\varepsilon))^{1/2})$$

for $1 \le j \le k$ (following from (6-7), and (5-3)), we can estimate

$$\sum_{j=1}^{k} \frac{2}{\bar{r}_{n-1}^{(0,j)}} \frac{(\overline{\mathfrak{D}}_{-}\tilde{m})_{n-1}^{(0,j)}}{1-2\tilde{m}_{n-1}^{(0,j)}/\bar{r}_{n-1}^{(0,j)}-\frac{1}{3}\Lambda(\bar{r}_{n-1}^{(0,j)})^2}(1-\mathfrak{Err}_{1,n-1}^{(0,j)})(1-\mathfrak{Err}_{/n-1}^{(0,j)}) \le \varepsilon^{3/2}. \tag{6-100}$$

Therefore, (6-99) yields

$$\log \frac{(\mathfrak{D}_{+}\tilde{m})_{n-1}^{(0,1)}}{(\mathfrak{D}_{-}\tilde{m})_{n-1}^{(0,k)}} \ge -\varepsilon^{3/2}. \tag{6-101}$$

From (6-98) and (6-101) we thus infer that, for any $1 \le i \le k$ and any $2 \le n \le n_f$,

$$\log \frac{(\overline{\mathfrak{D}}_{+}\tilde{m})_{n}^{(i,k+1)}}{\tilde{m}_{n-1}^{(1,k+1)}} \ge -\varepsilon^{3/2}. \tag{6-102}$$

From (6-102) for $i=1$ and the fact that, for any $1 \le n \le n_f$,

$$(\overline{\mathfrak{D}}_{+}\tilde{m})_{n}^{(1,k+1)} = (\mathfrak{D}_{-}\tilde{m})_{n}^{(0,k)} = \tilde{m}_{n}^{(1,k+1)}, \tag{6-103}$$

we thus infer that, for all $2 \le n \le n_f$,

$$\log \frac{\tilde{m}_{n}^{(1,k+1)}}{\tilde{m}_{n-1}^{(1,k+1)}} \ge -\varepsilon^{3/2}. \tag{6-104}$$

Applying (6-104) successively $n-1$ times, we thus infer, for any $2 \le n \le n_f$,

$$\log \frac{\tilde{m}_{n}^{(1,k+1)}}{\tilde{m}_{1}^{(1,k+1)}} \ge -\varepsilon^{3/2}(n-1). \tag{6-105}$$

The bound (5-6) for r_0 and the form (5-7) of the initial data imply

$$\frac{2(\tilde{m}_{/}(v^{(0)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))-\tilde{m}_{/}(v^{(0)}))}{r_0} \ge \frac{4}{C_0}h_0(\varepsilon). \tag{6-106}$$

Therefore, from (6-95) and (6-106) we infer that, for all $1 \le i \le 1$,

$$\frac{2(\overline{\mathfrak{D}}_{+}\tilde{m})_{1}^{(i,k+1)}}{r_0} \ge \frac{4}{C_0}h_0(\varepsilon). \tag{6-107}$$

From (6-105) and (6-107) for $i=1$ (when $(\overline{\mathfrak{D}}_{+}\tilde{m})_{1}^{(1,k+1)} = \tilde{m}_{1}^{(1,k+1)}$), using also the fact that $n_f \le (h_1(\varepsilon))^{-2}$, we thus deduce that, for all $1 \le n \le n_f$,

$$\frac{2\tilde{m}_{n}^{(1,k+1)}}{r_0} \ge \frac{2}{C_0}h_0(\varepsilon). \tag{6-108}$$

The relations (5-24) and (6-108) readily yield (6-11). □

Proof of (6-10). In view of the bound (6-37), we infer that, for any $1 \le n \le n_f$,

$$1-\frac{2\tilde{m}_{n}^{(1,k+1)}}{r|_{(U_n(v^{(1)}),V_n(v^{(k+1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)))}}-\frac{1}{3}\Lambda r^2|_{(U_n(v^{(1)}),V_n(v^{(k+1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)))} \ge h_3(\varepsilon). \tag{6-109}$$

Using the bounds

$$\frac{r|_{(U_n(v^{(1)}),\,V_n(v^{(k+1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)))}}{r_0} \le 1+(h_2(\varepsilon))^{1/2} \tag{6-110}$$

(derived from (6-7), (6-9) and (5-3)) and

$$\frac{r_0}{(2/\sqrt{-\Lambda})\varepsilon-\frac{1}{3}\Lambda r_0^3} < 1-\tfrac{1}{2}\exp(-2(h_0(\varepsilon))^{-4}) \tag{6-111}$$

(from (5-6)), as well as the relation (5-24) for $\tilde{m}|_{\mathcal{I}}$, we can readily derive from (6-109) that

$$\frac{2(\tilde{m}|_{\mathcal{I}}-\tilde{m}_n^{(1,k+1)})}{r_0} \ge 2(1+(h_2(\varepsilon))^{1/2})^{-1}\exp(-2(h_0(\varepsilon))^{-4})+\tfrac{1}{3}\Lambda r_0^2(1+(h_2(\varepsilon))^{1/2}). \tag{6-112}$$

The bound (6-10) follows readily from (6-111) and (6-112). □

Proof of (6-13). For any $2\le n\le n_f$, applying the relation (6-51) successively for $j=k+1$ and $i=1,2,\dots,k$, using also the identity (6-94) and the trivial bound

$$(\mathfrak{D}_-\tilde{m})_n^{(i,j)}(1-\mathfrak{Err}_{\backslash n}^{(i,k+1)}) \ge (\mathfrak{D}_+\tilde{m})_n^{(i,k+1)} \tag{6-113}$$

(following directly from (6-54)), we obtain

$$\begin{aligned}(\overline{\mathfrak{D}}_+\tilde{m})_n^{(1,k+1)} &= (\tilde{m}_{n-1}^{(0,0)}-\tilde{m}_{n-1}^{(0,1)})\cdot\exp\Bigg(\sum_{i=1}^{k}\frac{2}{\bar{r}_n^{(i,k+1)}}\frac{(\mathfrak{D}_-\tilde{m})_n^{(i,k+1)}}{1-2\tilde{m}_n^{(i+1,k+1)}/\bar{r}_n^{(i,j)}-\frac{1}{3}\Lambda(\bar{r}_n^{(i,k+1)})^2}(1-\mathfrak{Err}_{1,n}^{(i,k+1)})(1-\mathfrak{Err}_{\backslash n}^{(i,k+1)})\Bigg)\\ &\ge (\tilde{m}_{n-1}^{(0,0)}-\tilde{m}_{n-1}^{(0,1)})\cdot\exp\Bigg(\sum_{i=1}^{k}\frac{2}{\bar{r}_n^{(i,k+1)}}\frac{(\mathfrak{D}_+\tilde{m})_n^{(i,k+1)}}{1-2\tilde{m}_n^{(i+1,k+1)}/\bar{r}_n^{(i,j)}-\frac{1}{3}\Lambda(\bar{r}_n^{(i,k+1)})^2}(1-\mathfrak{Err}_{1,n}^{(i,k+1)})\Bigg).\end{aligned} \tag{6-114}$$

In view of (6-7), (5-3), (6-53), (6-9) and the fact that

$$r_0\le \bar{r}_n^{(i,k+1)}\le \bar{r}_n^{(k,k+1)},$$

for $1\le i\le k$ (following from (6-18)), we can bound, for any $1\le i\le k$,

$$\begin{aligned}&\frac{2}{\bar{r}_n^{(i,k+1)}}\frac{1}{1-2\tilde{m}_n^{(i+1,k+1)}/\bar{r}_n^{(i,j)}-\frac{1}{3}\Lambda(\bar{r}_n^{(i,k+1)})^2}(1-\mathfrak{Err}_{1,n}^{(i,k+1)})\\ &\qquad\ge 2\min\Bigg\{\frac{1}{r|_{(U_n(v^{(i)}),\,V_n(v^{(j)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)))}-2\tilde{m}_n^{(i,j+1)}-\frac{1}{3}\Lambda r^3|_{(U_n(v^{(i)}),\,V_n(v^{(j)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)))}},\\ &\qquad\qquad\frac{1}{\bar{r}_n^{(i,j)}-2\tilde{m}_n^{(i+1,k+1)}-\frac{1}{3}\Lambda(\bar{r}_n^{(i,k+1)})^3}\Bigg\}\\ &\qquad\ge\frac{2-O(\varepsilon)}{r_n^{(k,k+1)}}.\end{aligned} \tag{6-115}$$

Furthermore,

$$\sum_{i=1}^{k}(\mathfrak{D}_{+}\tilde{m})_n^{(i,k+1)} = \sum_{i=1}^{k}(\tilde{m}_n^{(i+1,k+1)} - \tilde{m}_n^{(i,k+1)}) = \tilde{m}|_{\mathcal{I}} - \tilde{m}_n^{(1,k+1)}. \tag{6-116}$$

Therefore, in view of (6-10), (6-115), (6-116) and the fact that

$$(\overline{\mathfrak{D}}_{+}\tilde{m})_n^{(1,k+1)} = \tilde{m}_n^{(1,k+1)} - \tilde{m}_n^{(1,k+2)} = \tilde{m}_n^{(1,k+1)}, \tag{6-117}$$

the bound (6-114) yields

$$\begin{aligned}\tilde{m}_n^{(1,k+1)} &\geq (\tilde{m}_{n-1}^{(0,0)} - \tilde{m}_{n-1}^{(0,1)}) \cdot \exp\left(\frac{2-O(\varepsilon)}{r_n^{(k,k+1)}}(\tilde{m}|_{\mathcal{I}} - \tilde{m}_n^{(1,k+1)})\right) \\ &\geq (\tilde{m}_{n-1}^{(0,0)} - \tilde{m}_{n-1}^{(0,1)}) \cdot \exp\left(\frac{r_0}{2r_n^{(k,k+1)}}\exp(-2(h_0(\varepsilon))^{-4})\right).\end{aligned} \tag{6-118}$$

Using the bound (6-101) and the fact that

$$\begin{aligned}(\mathfrak{D}_{+}\tilde{m})_{n-1}^{(0,1)} &= \tilde{m}_{n-1}^{(1,1)} - \tilde{m}_{n-1}^{(0,1)} = \tilde{m}_{n-1}^{(0,0)} - \tilde{m}_{n-1}^{(0,1)}, \\ (\mathfrak{D}_{-}\tilde{m})_{n-1}^{(0,k)} &= \tilde{m}_{n-1}^{(1,k+1)},\end{aligned}$$

we can estimate

$$(\tilde{m}_{n-1}^{(0,0)} - \tilde{m}_{n-1}^{(0,1)}) \geq e^{-\varepsilon^{3/2}}\tilde{m}_{n-1}^{(1,k+1)}. \tag{6-119}$$

From (6-118) and (6-119) we thus obtain (in view also of (6-9) and the properties (5-2) of $h_0(\varepsilon)$)

$$\tilde{m}_n^{(1,k+1)} \geq \tilde{m}_{n-1}^{(1,k+1)} \exp\left(\frac{r_0}{4r_n^{(k,k+1)}}\exp(-2(h_0(\varepsilon))^{-4})\right). \tag{6-120}$$

In particular, (6-13) holds for all $2 \leq n \leq n_f$. □

Proof of (6-12). Combining (6-102) and (6-11) (using also (6-107) in the case $n = 1$, as well as (5-24) and (5-6) for $\tilde{m}|_{\mathcal{I}}$, r_0), we can readily estimate for any $1 \leq n \leq n_f$ and any $1 \leq i \leq k$

$$\frac{2(\overline{\mathfrak{D}}_{+}\tilde{m})_n^{(i,k+1)}}{r_0} \geq \frac{1}{C_0}h_0(\varepsilon). \tag{6-121}$$

Similarly, in view of (6-94), (6-97) and (6-101), we can bound for any $1 \leq n \leq n_f$ and any $1 \leq i \leq k$

$$\frac{2(\overline{\mathfrak{D}}_{-}\tilde{m})_n^{(i,k+1)}}{r_0} \geq \frac{1}{C_0}h_0(\varepsilon). \tag{6-122}$$

Using the relation

$$(\overline{\mathfrak{D}}_{+}\tilde{m})_n^{(i,k+1)} = \tilde{m}_n^{(i,k+1)} - \tilde{m}_n^{(i,k+2)} \tag{6-123}$$

and the trivial bounds

$$\tilde{m}_n^{(i,k+1)} \leq \tilde{m}|_{\mathcal{I}} \tag{6-124}$$

and

$$\max_{k+2 \leq j \leq k+i+1} \tilde{m}_n^{(i,j)} = \tilde{m}_n^{(i,k+2)} \tag{6-125}$$

(following from the monotonicity properties (6-19) of $\tilde{m}$), from (6-121) we obtain for any $1 \le n \le n_f$

$$\min_{\substack{1\le i\le k\\ k+2\le j\le k+i+1}} \frac{2(\tilde{m}|_{\mathcal{I}} - \tilde{m}_n^{(i,j)})}{r_0} \ge \frac{1}{C_0} h_0(\varepsilon). \tag{6-126}$$

In view of (5-24) and (5-6) and the properties (5-2) of $h_0(\varepsilon)$, from (6-126) we infer that, for any $1 \le n \le n_f$,

$$\max_{\substack{1\le i\le k\\ k+2\le j\le k+i+1}} \frac{2\tilde{m}_n^{(i,j)}}{r_0} \le 1 - \frac{1}{3C_0} h_0(\varepsilon). \tag{6-127}$$

In particular, for any $1 \le n \le n_f$, (6-127) implies

$$\sup_{\{u\le v\le V_n(v^{(k+1)})\}\cap\{u\ge U_n(v^{(k)})\}} \frac{1}{1-2m/r} \le 4C_0 (h_0(\varepsilon))^{-1}. \tag{6-128}$$

The main estimate that will be used in the proof of (6-12) is the following bound: for any $1 \le n_1 < n_2 \le n_f$ and any $V_{n_2}(v^{(k+2)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)) \le v \le V_{n_2}(v^{(k+1)})$

$$\begin{aligned}\partial_v r|_{(U_{n_2}(v^{(1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),v)} &\ge \partial_v r|_{(U_{n_1-1}(v^{(1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),v-(n_2-n_1-1)v_0)} \\ &\quad\times \exp\biggl(-C_0^5 (h_0(\varepsilon))^{-3} \max_{n_1\le n\le n_2}\biggl\{\frac{r_n^{(1,k+1)}}{r_0}\biggr\} \log\biggl(\frac{\tilde{m}_{n_2}^{(1,k+1)}}{\tilde{m}_{n_1}^{(1,k+1)}}\biggr) - 2\varepsilon^{1/2}\biggr).\end{aligned} \tag{6-129}$$

Proof of (6-129). For any $1 \le n \le n_f$, $1 \le i \le k$, and $k+1 \le j \le k+i$, integrating (2-43) from $u = U_n(v^{(i)})$ up to $U_n(v^{(i)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))$ (and using (2-46)), we infer that, for all $V_n(v^{(j+1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)) \le \bar{v} \le V_n(v^{(j)})$,

$$\begin{aligned}\log\biggl(\frac{\partial_v r}{1-2m/r}\biggr)\biggr|_{(U_n(v^{(i)}),\bar{v})} &- \log\biggl(\frac{\partial_v r}{1-2m/r}\biggr)\biggr|_{(U_n(v^{(i)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),\bar{v})} \\ &= 4\pi \int_{U_n(v^{(i)})}^{U_n(v^{(i)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))} \frac{rT_{uu}}{-\partial_u r}\biggr|_{(u,\bar{v})} du \\ &= 2\int_{U_n(v^{(i)})}^{U_n(v^{(i)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))} \frac{-\partial_u \tilde{m}}{r(1-2m/r)}\biggr|_{(u,\bar{v})} du.\end{aligned} \tag{6-130}$$

In view of the monotonicity properties (6-18), (6-19) of r and $\tilde{m}$, we can estimate

$$\begin{aligned}&2\int_{U_n(v^{(i)})}^{U_n(v^{(i)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))} \frac{-\partial_u \tilde{m}}{r(1-2m/r)}\biggr|_{(u,\bar{v})} du \\ &\le \sup_{U_n(v^{(i)})\le\bar{u}\le U_n(v^{(i)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))} \biggl(\frac{2}{r(1-2m/r)}\biggr)\biggr|_{(u,\bar{v})} \int_{U_n(v^{(i)})}^{U_n(v^{(i)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))} (-\partial_u\tilde{m})\bigr|_{(u,\bar{v})} du \\ &\le \frac{2}{r(U_n(v^{(i)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),\bar{v})} \sup_{U_n(v^{(i)})\le\bar{u}\le U_n(v^{(i)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))} \biggl(\frac{1}{1-2m/r|_{(\bar{u},\bar{v})}}\biggr) (\mathfrak{D}_-\tilde{m})_n^{(i,j)}.\end{aligned} \tag{6-131}$$

From (6-130), (6-131) and the bound (6-128) for $1-2m/r$, we infer that

$$\log\left(\frac{\partial_v r}{1-2m/r}\right)\Big|_{(U_n(v^{(i)}),\bar{v})} - \log\left(\frac{\partial_v r}{1-2m/r}\right)\Big|_{(U_n(v^{(i)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),\bar{v})} \le 8C_0(h_0(\varepsilon))^{-1}\frac{\bar{r}_n^{(i,j)}}{r(U_n(v^{(i)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),\bar{v})}\cdot\frac{(\mathfrak{D}_-\tilde{m})_n^{(i,j)}}{\bar{r}_n^{(i,j)}}. \tag{6-132}$$

Notice that, in view of the bound (6-56), we can estimate

$$\frac{(\mathfrak{D}_-\tilde{m})_n^{(i,j)}}{\bar{r}_n^{(i,j)}} \le 5C_0\log\left(\frac{(\bar{\mathfrak{D}}_+\tilde{m})_n^{(i,j)}}{(\bar{\mathfrak{D}}_-\tilde{m})_n^{(i,j)}}\right). \tag{6-133}$$

Thus, from (6-132) and (6-133), we deduce that, for any $1\le n\le n_f$, $1\le i\le k$, and $k+1\le j\le k+i$ and any $V_n(v^{(j+1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))\le\bar{v}\le V_n(v^{(j)})$,

$$\log\left(\frac{\partial_v r}{1-2m/r}\right)\Big|_{(U_n(v^{(i)}),\bar{v})} - \log\left(\frac{\partial_v r}{1-2m/r}\right)\Big|_{(U_n(v^{(i)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),\bar{v})} \le 40C_0^2(h_0(\varepsilon))^{-1}\frac{\bar{r}_n^{(i,j)}}{r(U_n(v^{(i)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),\bar{v})}\log\left(\frac{(\bar{\mathfrak{D}}_+\tilde{m})_n^{(i,j)}}{(\bar{\mathfrak{D}}_-\tilde{m})_n^{(i,j)}}\right). \tag{6-134}$$

Applying the relation (6-134) successively for $i=1,\dots,k$ and $\bar{v}=v$, using also the fact that

$$\partial_u\left(\frac{\partial_v r}{1-2m/r}\right)=0$$

on each $\mathcal{R}_{\varepsilon n}^{(i,j)}$, we obtain

$$\frac{\partial_v r}{1-2m/r}\Big|_{(U_n(v^{(1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),\bar{v})} \ge \frac{\partial_v r}{1-2m/r}\Big|_{(U_{n-1}(v^{(0)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),\bar{v})} \times\exp\left(-40C_0^2(h_0(\varepsilon))^{-1}\sum_{i=1}^{k}\frac{\bar{r}_n^{(i,k+1)}}{r(U_n(v^{(i)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),\bar{v})}\log\left(\frac{(\bar{\mathfrak{D}}_+\tilde{m})_n^{(i,k+1)}}{(\bar{\mathfrak{D}}_-\tilde{m})_n^{(i,k+1)}}\right)\right). \tag{6-135}$$

For any $i=1,\dots,k$, integrating (2-43) in u from $u=U_n(v^{(i)})$ up to $U_n(v^{(1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))$ for $v=v_*\in[\bar{v},V_n(v^{(j)})]$ and using (2-46) and the fact that $\partial_u\tilde{m}=0$ on $\mathcal{R}_{\varepsilon n}^{(i,j)}$, we infer

$$\frac{\partial_v r}{1-2m/r}\Big|_{(U_n(v^{(1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),v_*)} = \frac{\partial_v r}{1-2m/r}\Big|_{(U_n(v^{(i)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),v_*)}\exp\left(-2\int_{U_n(v^{(i)})}^{U_n(v^{(1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))}\frac{-\partial_u\tilde{m}}{r-2\tilde{m}-\frac{1}{3}\Lambda r^3}\Big|_{(u,v_*)}du\right). \tag{6-136}$$

Using the fact that $r\ge r_0$, from (6-136) we infer (in view of the monotonicity property (6-19) for $\tilde{m}$) that

$$\begin{aligned}
&\frac{\partial_v r}{1-2m/r}\Big|_{(U_n(v^{(1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),v_*)}\\
&\quad\geq \frac{\partial_v r}{1-2m/r}\Big|_{(U_n(v^{(i)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),v_*)}\exp\biggl(-2\int_{U_n(v^{(i)})}^{U_n(v^{(1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))}\frac{-\partial_u\tilde{m}|_{(u,v_*)}}{r_0-2\tilde{m}|_{(u,v_*)}-\frac{1}{3}\Lambda r_0^3}\,du\biggr)\\
&\quad=\frac{\partial_v r}{1-2m/r}\Big|_{(U_n(v^{(i)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),v_*)}\exp\biggl(-\int_{U_n(v^{(i)})}^{U_n(v^{(1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))}\partial_u\Bigl(\log\Bigl(1-\frac{2\tilde{m}|_{(u,v_*)}}{r_0}-\frac{1}{3}\Lambda r_0^2\Bigr)\Bigr)\,du\biggr)\\
&\quad=\frac{\partial_v r}{1-2m/r}\Big|_{(U_n(v^{(i)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),v_*)}\cdot\biggl(\frac{1-2\tilde{m}|_{(U_n(v^{(i)}),v_*)}/r_0-\frac{1}{3}\Lambda r_0^2}{1-2\tilde{m}|_{(U_n(v^{(1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),v_*)}/r_0-\frac{1}{3}\Lambda r_0^2}\biggr).
\end{aligned}\tag{6-137}$$

In view of the bounds (6-9) and (6-127), (6-137) yields

$$\frac{\partial_v r}{1-2m/r}\Big|_{(U_n(v^{(1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),v_*)}\geq\frac{\partial_v r}{1-2m/r}\Big|_{(U_n(v^{(i)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),v_*)}\frac{h_0(\varepsilon)}{4C_0}.\tag{6-138}$$

Integrating (6-138) in $v_*\in[\bar{v},V_n(v^{(j)})]$ and using (6-128) (and (6-9)) for the $1/(1-2m/r)$ factors, we thus obtain

$$\bar{r}_n^{(1,k+1)}-r_0\geq\Bigl(\bar{r}_n^{(i,k+1)}-r\bigl(U_n(v^{(i)}+\frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)),\bar{v}\bigr)\Bigr)\frac{(h_0(\varepsilon))^2}{16C_0^2}\tag{6-139}$$

and, thus (in view of (6-7), (5-3) and the fact that $r_0\leq\min\{r_n^{(1,k+1)},r(U_n(v^{(i)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),\bar{v})\}$)

$$\frac{\bar{r}_n^{(i,k+1)}}{r(U_n(v^{(i)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),\bar{v})}\leq16C_0^2(h_0(\varepsilon))^{-2}\frac{r_n^{(1,k+1)}}{r_0}.\tag{6-140}$$

From (6-140) and (6-94), it follows that

$$\begin{aligned}
&\sum_{i=1}^{k}\frac{\bar{r}_n^{(i,k+1)}}{r(U_n(v^{(i)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),v)}\log\biggl(\frac{(\bar{\mathfrak{D}}_+\tilde{m})_n^{(i,k+1)}}{(\bar{\mathfrak{D}}_-\tilde{m})_n^{(i,k+1)}}\biggr)\\
&\qquad\leq16C_0^2(h_0(\varepsilon))^{-2}\frac{r_n^{(1,k+1)}}{r_0}\sum_{i=1}^{k}\log\biggl(\frac{(\bar{\mathfrak{D}}_+\tilde{m})_n^{(i,k+1)}}{(\bar{\mathfrak{D}}_-\tilde{m})_n^{(i,k+1)}}\biggr)\\
&\qquad=16C_0^2(h_0(\varepsilon))^{-2}\frac{r_n^{(1,k+1)}}{r_0}\biggl\{\sum_{i=1}^{k-1}\log\biggl(\frac{(\bar{\mathfrak{D}}_+\tilde{m})_n^{(i,k+1)}}{(\bar{\mathfrak{D}}_+\tilde{m})_n^{(i+1,k+1)}}\biggr)+\log\biggl(\frac{(\bar{\mathfrak{D}}_+\tilde{m})_n^{(k,k+1)}}{\tilde{m}|_{\mathcal{I}}-\tilde{m}_{n-1}^{(0,1)}}\biggr)\biggr\}\\
&\qquad=16C_0^2(h_0(\varepsilon))^{-2}\frac{r_n^{(1,k+1)}}{r_0}\log\biggl(\frac{(\bar{\mathfrak{D}}_+\tilde{m})_n^{(1,k+1)}}{\tilde{m}|_{\mathcal{I}}-\tilde{m}_{n-1}^{(0,1)}}\biggr)\\
&\qquad=16C_0^2(h_0(\varepsilon))^{-2}\frac{r_n^{(1,k+1)}}{r_0}\log\biggl(\frac{(\bar{\mathfrak{D}}_+\tilde{m})_n^{(1,k+1)}}{(\mathfrak{D}_+\tilde{m})_{n-1}^{(0,1)}}\biggr)\\
&\qquad\leq16C_0^2(h_0(\varepsilon))^{-2}\frac{r_n^{(1,k+1)}}{r_0}\biggl\{\log\biggl(\frac{(\bar{\mathfrak{D}}_+\tilde{m})_n^{(1,k+1)}}{(\mathfrak{D}_-\tilde{m})_{n-1}^{(0,k)}}\biggr)+\varepsilon^{3/2}\biggr\}\\
&\qquad=16C_0^2(h_0(\varepsilon))^{-2}\frac{r_n^{(1,k+1)}}{r_0}\biggl\{\log\biggl(\frac{\tilde{m}_n^{(1,k+1)}}{\tilde{m}_{n-1}^{(1,k+1)}}\biggr)+\varepsilon^{3/2}\biggr\}
\end{aligned}\tag{6-141}$$

(where the inequality at the sixth line of (6-141) follows from (6-101)). Therefore, (6-135) and (6-141) yield, for any $1 \le n \le n_f$ and any $V_n(v^{(k+2)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)) \le v \le V_n(v^{(k+1)})$,

$$\frac{\partial_v r}{1-2m/r}\Big|_{(U_n(v^{(1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),v)} \\ \ge \frac{\partial_v r}{1-2m/r}\Big|_{(U_{n-1}(v^{(0)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),v)} \exp\biggl(-C_0^5(h_0(\varepsilon))^{-3}\frac{r_n^{(1,k+1)}}{r_0}\biggl\{\log\biggl(\frac{\tilde{m}_n^{(1,k+1)}}{\tilde{m}_{n-1}^{(1,k+1)}}\biggr)+\varepsilon^{3/2}\biggr\}\biggr). \tag{6-142}$$

In view of (2-44), we can bound for any $U_{n-1}(v^{(1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)) \le u \le U_{n-1}(v^{(0)})$

$$\frac{-\partial_u r}{1-2m/r}\Big|_{(u,V_{n-1}(v^{(1)}))} \ge \frac{-\partial_u r}{1-2m/r}\Big|_{(u,V_{n-1}(v^{(k+1)})}. \tag{6-143}$$

Hence, using that

- from (5-39), (6-9),

$$\frac{\partial_v r}{1-2m/r}\Big|_{(U_n(v^{(1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),v)} = \frac{\partial_v r}{1-\frac{1}{3}\Lambda r^2}\Big|_{(U_n(v^{(1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),v)} \\ = (1+O(\varepsilon))\,\partial_v r|_{(U_n(v^{(1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),v)}, \tag{6-144}$$

- from (2-45), (5-24), (6-8) and (6-7)

$$\frac{\partial_v r}{1-2m/r}\Big|_{(U_{n-1}(v^{(0)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),v)} = \frac{\partial_v r}{1-2m/r}\Big|_{(U_{n-1}(v^{(0)}),v)}(1+O(\varepsilon)), \tag{6-145}$$

- from (2-44), (2-43) and the gauge condition (3-24),

$$\frac{\partial_v r}{1-2m/r}\Big|_{\mathcal{R}_{n-1}^{(1,1)}\cap\{v=\bar{v}\}} = \frac{-\partial_u r}{1-2m/r}\Big|_{\mathcal{R}_{n-1}^{(1,1)}\cap\{u=\bar{v}-v_0\}}, \tag{6-146}$$

the bounds (6-142) and (6-143) yield for any $V_n(v^{(k+2)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)) \le v \le V_n(v^{(k+1)})$

$$\partial_v r|_{(U_n(v^{(1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),v)} \\ \ge \frac{-\partial_u r}{1-2m/r}\Big|_{(v-v_0,V_{n-1}(v^{(k+1)})}\exp\biggl(-C_0^5(h_0(\varepsilon))^{-3}\frac{r_n^{(1,k+1)}}{r_0}\log\biggl(\frac{\tilde{m}_n^{(1,k+1)}}{\tilde{m}_{n-1}^{(1,k+1)}}\biggr)-\varepsilon^{1/2}\biggr). \tag{6-147}$$

In view of (5-39), (6-9) and the fact that

$$\frac{-\partial_u r}{1-2m/r}\Big|_{\mathcal{R}_{n-1}^{(1,k+2)}\cap\{u=\bar{u}\}} = \frac{\partial_v r}{1-2m/r}\Big|_{\mathcal{R}_{n-1}^{(1,k+2)}\cap\{v=\bar{u}\}} \tag{6-148}$$

(following from the gauge condition (3-23)), from (6-147) we infer for any $V_n(v^{(k+2)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)) \le v \le V_n(v^{(k+1)})$

$$\partial_v r|_{(U_n(v^{(1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),v)} \\ \ge \partial_v r|_{(U_{n-1}(v^{(1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),v-v_0)}\exp\biggl(-C_0^5(h_0(\varepsilon))^{-3}\frac{r_n^{(1,k+1)}}{r_0}\log\biggl(\frac{\tilde{m}_n^{(1,k+1)}}{\tilde{m}_{n-1}^{(1,k+1)}}\biggr)-2\varepsilon^{1/2}\biggr). \tag{6-149}$$

Iterating (6-149) for $n_1 < n \le n_2$, we thus obtain (6-129). □

Let $2 \leq n_1 \leq n_f$ be such that

$$r^{(1,k+1)}_{n_1} \leq 2r_0, \tag{6-150}$$

$$r^{(1,k+1)}_{n_1-1} > 2r_0. \tag{6-151}$$

Note that if no such n_1 exists, then (6-12) is automatically true ((6-151) holds for $r^{(1,k+1)}_1$ as a corollary of the Cauchy stability estimates of Proposition 3.5 and the choice of the initial data).

Let us also define

$$n_2 = \max\{n_1 \leq n \leq n_f : r^{(1,k+1)}_l \leq 2r_0 \text{ for all } n_1 \leq l \leq n\}. \tag{6-152}$$

In order to establish (6-12), it suffices to establish that, for all $n_1 \leq n \leq n_2$,

$$\frac{r^{(1,k+1)}_n}{r_0} - 1 \geq \exp\bigl(-C_0^7 (h_0(\varepsilon))^{-3} \log((h_0(\varepsilon))^{-1})\bigr). \tag{6-153}$$

In view of the fact that

$$\sup_{1 \leq l_1 < l_2 \leq n_*} \frac{\tilde{m}^{(1,k+1)}_{l_2}}{\tilde{m}^{(1,k+1)}_{l_1}} \leq C_0 (h_0(\varepsilon))^{-1} \tag{6-154}$$

(following from (5-7), (6-86) and the fact that the sequence $\tilde{m}^{(1,k+1)}_n$ is increasing in n as a consequence of (6-120)), from (6-129) we infer that, for any $n_1 \leq n \leq n_2$ and any $V_{n_2}(v^{(k+2)} + (4/\sqrt{-\Lambda}) h_2(\varepsilon)) \leq v \leq V_{n_2}(v^{(k+1)})$,

$$\begin{aligned} &\partial_v r|_{(U_n(v^{(1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),v)} \\ &\qquad \geq \partial_v r|_{(U_{n_1-1}(v^{(1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)),v-(n-n_1)v_0)} \exp\bigl(-C_0^6 (h_0(\varepsilon))^{-3} \log((h_0(\varepsilon))^{-1})\bigr). \end{aligned} \tag{6-155}$$

Thus, integrating (6-155) from $v = V_n(v^{(k+2)} + (4/\sqrt{-\Lambda}) h_2(\varepsilon))$ up to $v = V_n(v^{(k+1)})$ and using (6-151), we immediately infer (6-153). □

Proof of (6-14). In view of (5-3), (6-32), (6-31), as well as the boundary condition (3-24) and the bounds (6-8) and (6-9), the following one-sided bound holds for all $2 \leq n \leq n_f$:

$$\begin{aligned} r^{(k,k+1)}_n - r_0 &= \int_{V_n(v^{(2k)})}^{V_n(v^{(k+1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))} \partial_v r|_{(U_n(v^{(k)}),v)}\, dv \\ &\leq \int_{V_n(v^{(2k)})}^{V_n(v^{(k+1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))} \frac{\partial_v r}{1-2m/r}\bigg|_{(U_n(v^{(k)}),v)} (1+O(\varepsilon))\, dv \\ &= \int_{U_{n-1}(v^{(k)})}^{U_{n-1}(v^{(0)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))} \frac{-\partial_u r}{1-2m/r}\bigg|_{(u,V_{n-1}(v^{(k+1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)))} (1+O(\varepsilon))\, du. \end{aligned} \tag{6-156}$$

We can readily compute (using also (5-3), (6-9) and (6-45))

$$\begin{aligned}
&\int_{U_{n-1}(v^{(1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))}^{U_{n-1}(v^{(0)})} \frac{-\partial_u r}{1-2m/r}\Big|_{(u,V_{n-1}(v^{(k+1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)))} du \\
&\quad = \int_{U_{n-1}(v^{(1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))}^{U_{n-1}(v^{(0)})} \left(1 - \frac{2\tilde{m}_{n-1}^{(1,k+1)}}{r|_{(u,V_{n-1}(v^{(k+1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)))}} - \tfrac{1}{3}\Lambda r^2|_{(u,V_{n-1}(v^{(k+1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)))}\right)^{-1} \\
&\qquad\qquad \times (-\partial_u r)|_{(u,V_{n-1}(v^{(k+1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)))}\, du \\
&\quad = \int_{r_0+O((h_2(\varepsilon))^{1/2})}^{r_{n-1}^{(1,k+1)}} \left(1 - \frac{2\tilde{m}_{n-1}^{(1,k+1)}}{r} - \tfrac{1}{3}\Lambda r^2\right)^{-1} dr \\
&\quad \le r_{n-1}^{(1,k+1)} - r_0 + C_0 \tilde{m}_{n-1}^{(1,k+1)} \left|\log\left(1 - \frac{2\tilde{m}_{n-1}^{(1,k+1)}}{r_0}\right)\right|. \qquad (6\text{-}157)
\end{aligned}$$

From (5-6) and (6-12) we can similarly estimate

$$\begin{aligned}
\int_{U_{n-1}(v^{(k)})}^{U_{n-1}(v^{(1)})} \frac{-\partial_u r}{1-2m/r}\Big|_{(u,V_{n-1}(v^{(k+1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)))} du &\le \int_{r_{n-1}^{(1,k+1)}+O((h_2(\varepsilon))^{1/2})}^{r_{n-1}^{(k,k+1)}} \left(1 - \frac{2\tilde{m}|_{\mathcal{I}}}{r} + O(\varepsilon)\right)^{-1} dr \\
&\le r_{n-1}^{(k,k+1)} + C_0 \tilde{m}|_{\mathcal{I}} \left|\log(\exp(h_0(\varepsilon))^{-4}) + 1\right| \\
&\le r_{n-1}^{(k,k+1)} - r_{n-1}^{(1,k+1)} + C_0 \tilde{m}|_{\mathcal{I}} (h_0(\varepsilon))^{-4}. \qquad (6\text{-}158)
\end{aligned}$$

From (5-3), (6-45), (6-156), (6-157) and (6-158) one readily obtains the bound (6-14). □

6B. ***Formation of a nearly trapped sphere.*** In this section, we will establish (6-6), using the bounds (6-7)–(6-14) of Proposition 6.1.

Let us set

$$n_{\max} \doteq \max\{n_* \in \mathbb{N} : \mathcal{R}_n^{(1,k+1)} \subset \mathcal{U}_\varepsilon^+ \text{ for all } n \le n_*\}. \qquad (6\text{-}159)$$

Note that, in view of (6-2) and (6-5), $n_{\max}$ satisfies

$$n_f \le n_{\max} \le n_f + 1. \qquad (6\text{-}160)$$

Thus, (6-1) implies

$$n_{\max} \le (h_1(\varepsilon))^{-2}. \qquad (6\text{-}161)$$

Notice that (6-160) and the definition (6-2) imply that, if

$$n_{\max} < \tfrac{1}{2}(h_1(\varepsilon))^{-2}, \qquad (6\text{-}162)$$

then, necessarily, (6-6) holds. Thus, in order to establish (6-6), it suffices to show (6-162).

We will show (6-162) by applying Lemma A.2 (see Section A1.1 of the Appendix). In particular, setting for any $1 \le n \le n_{\max}+1$

$$\mu_n \doteq \frac{2\tilde{m}_{n-1}^{(1,k+1)}}{r_0}, \qquad (6\text{-}163)$$

$$\rho_n \doteq \frac{r_{n-1}^{(k,k+1)}}{r_0}, \qquad (6\text{-}164)$$

the inductive bounds (6-13) and (6-14) imply that μ_n, ρ_n satisfy

$$\begin{aligned} \rho_{n+1} &\le \rho_n + C_1 \log((1-\mu_n)^{-1}+1), \\ \mu_{n+1} &\ge \mu_n \exp\left(\frac{c_1}{\rho_{n+1}}\right) \end{aligned} \tag{6-165}$$

for any $1 \le n \le n_{\max}+1$, with

$$C_1 = (h_0(\varepsilon))^{-4}, \tag{6-166}$$

$$c_1 = \tfrac{1}{16}\exp(-2(h_0(\varepsilon))^{-4}). \tag{6-167}$$

Furthermore, setting

$$\delta = h_3(\varepsilon) \tag{6-168}$$

(where $h_3(\,\cdot\,)$ is defined by (6-3)), the definition (6-159) immediately implies

$$\max_{0\le n\le n_{\max}} \mu_n < 1-\delta. \tag{6-169}$$

Note that, in view of Definition 5.1 and Proposition 3.5, we have

$$\mu_0 \sim h_0(\varepsilon), \tag{6-170}$$

$$\rho_0 \sim (h_1(\varepsilon))^{-1}. \tag{6-171}$$

Hence, as a consequence of (5-2) and (6-3),

$$\left(\frac{C_1}{c_1}\right)^8 \ll \frac{\rho_0}{\mu_0} \tag{6-172}$$

and

$$\delta < \left(\frac{\mu_0}{\rho_0}\right)^{(C_1/c_1)^4}. \tag{6-173}$$

The relations (6-165)–(6-173) allow us to apply Lemma A.2 (see Section A1.1 of the Appendix) with $n_* = n_{\max}+1$ for the sequence ρ_n, μ_n. Thus, in view of Lemma A.2, we obtain the following upper bound for $n_{\max}$:

$$n_{\max}+1 \le \exp(\exp(2(h_0(\varepsilon))^{-4}))(h_1(\varepsilon))^{-1}. \tag{6-174}$$

In particular, (6-162) (and, thus, (6-6)) holds.

6C. ***The final step of the evolution.*** In this section, we will complete the proof of Theorem 1, using the near-trapping bound (6-6), the bounds (6-7)–(6-14) of Proposition 6.1, as well a backwards-in-time Cauchy stability estimate (see Lemma A.3 in Section A1.2).

The bound (6-6), combined with the estimates (6-11) and (6-12) of Proposition 6.1, imply that, necessarily (in view also of (5-2), (5-3), (6-7), (6-13) and (6-159))

$$\frac{2\tilde{m}^{(1,k+1)}_{n_{\max}+1}}{r_0} \ge 1-2h_3(\varepsilon). \tag{6-175}$$

Therefore, applying again Lemma A.2 for μ_n, ρ_n (defined by (6-163), (6-164)) and $n_* = n_{\max}+1$ yields, in view of (6-175), that, either

$$\mu_{n_{\max}+1} > 1 + h_3(\varepsilon), \tag{6-176}$$

or

$$1 - 2h_3(\varepsilon) \le \mu_{n_{\max}+1} \le 1 + h_3(\varepsilon) \tag{6-177}$$

and

$$\mu_{n_{\max}} \le 1 - \exp(-\exp(2(h_0(\varepsilon))^{-4}))(h_1(\varepsilon))^2 \tag{6-178}$$

$$\max\{\rho_{n_{\max}+1}, \rho_{n_{\max}}\} \le \exp(\exp(2(h_0(\varepsilon))^{-4}))(h_1(\varepsilon))^{-1}\log((h_1(\varepsilon))^{-1}). \tag{6-179}$$

Let us set

$$\bar{v}_* \doteq V_{n_{\max}+1}\Big(v^{(k+1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)\Big) \tag{6-180}$$

(recall that (6-180) equals $V_{n_{\max}}(v^{(0)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))$, in view of our conventions on the indices). The proof of Theorem 1 will follow by showing that

- either

$$\inf_{\mathcal{U}_\varepsilon \cap \{v=\bar{v}_*\}} \Big(1 - \frac{2m}{r}\Big) < 0 \tag{6-181}$$

(in which case $(r_/^{(\varepsilon)}, (\Omega_/^{(\varepsilon)})^2, \bar{f}_{\text{in}/}^{(\varepsilon)}, \bar{f}_{\text{out}/}^{(\varepsilon)}) = (r_{/\varepsilon}, \Omega_{/\varepsilon}^2, \bar{f}_{\text{in}/\varepsilon}, \bar{f}_{\text{out}/\varepsilon})$ in the statement of Theorem 1),

- or

$$\inf_{\mathcal{U}_\varepsilon \cap \{v=\bar{v}_*\}} \Big(1 - \frac{2m'}{r'}\Big) < 0, \tag{6-182}$$

where $(r', (\Omega')^2, \bar{f}'_{\text{in}}, \bar{f}'_{\text{out}})$ is a (possibly different) smooth solution to the system (2-28)–(2-33) arising as a future development of an asymptotically AdS boundary-characteristic initial data set $(r'_{/\varepsilon}, (\Omega'_{/\varepsilon})^2, \bar{f}'_{\text{in}/\varepsilon}, \bar{f}'_{\text{out}/\varepsilon})$ on $\{u=0\}\cap\{0 \le v \le v_{0\varepsilon}\}$ (satisfying the reflecting gauge condition at $r = r_0, +\infty$) which is $(h_1(\varepsilon))^2$ close to $(r_{/\varepsilon}, \Omega_{/\varepsilon}^2, \bar{f}_{\text{in}/\varepsilon}, \bar{f}_{\text{out}/\varepsilon})$ with respect to the norm (3-41), i.e., satisfies, in particular, (A-68) and (A-69) (in which case $(r_/^{(\varepsilon)}, (\Omega_/^{(\varepsilon)})^2, \bar{f}_{\text{in}/}^{(\varepsilon)}, \bar{f}_{\text{out}/}^{(\varepsilon)}) = (r'_{/\varepsilon}, (\Omega'_{/\varepsilon})^2, \bar{f}'_{\text{in}/\varepsilon}, \bar{f}'_{\text{out}/\varepsilon})$ in the statement of Theorem 1).

Notice that, in both cases, (4-3) follows readily from (5-18) and (5-2). To this end, we will proceed to treat the cases (6-176) and (6-177) separately.

Case I: Assume that (6-176) holds. Then, we will show that (6-181) also holds. We will argue by contradiction, assuming that

$$\inf_{\mathcal{U}_\varepsilon \cap \{v=\bar{v}_*\}} \Big(1 - \frac{2m}{r}\Big) \ge 0. \tag{6-183}$$

Let us set

$$\mathcal{C}_* \doteq \Big\{U_{n_{\max}+1}\Big(v^{(1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)\Big) \le u < U_{n_{\max}+1}(v^{(0)})\Big\} \cap \{v = \bar{v}_*\} \cap \mathcal{U}_\varepsilon. \tag{6-184}$$

The renormalized mass $\tilde{m}$ is constant on $\mathcal{C}_*$, satisfying in particular

$$\tilde{m}|_{\mathcal{C}_*} = \tilde{m}_{n_{\max}+1}^{(1,k+1)}. \tag{6-185}$$

Since $\partial_u r < 0$ on $\mathcal{U}_\varepsilon$ (see (3-29)), from (6-183) and the fact that $\mathcal{C}_*$ does not contain its future endpoint, we infer the following stronger bound:

$$1 - \frac{2m}{r}\Big|_{\mathcal{C}_*} > 0. \tag{6-186}$$

Thus, we also have

$$\partial_v r|_{\mathcal{C}_*} > 0. \tag{6-187}$$

We will now show that the future endpoint of $\mathcal{C}_*$ is exactly $(U_{n_{\max}+1}(v^{(0)}), \bar{v}_*)$. If there existed some $(u_b, \bar{v}_*) \in (\partial\mathcal{U}_\varepsilon \backslash \mathcal{I})$ such that $U_{n_{\max}+1}(v^{(1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)) \le u_b < U_{n_{\max}+1}(v^{(0)})$, then Theorem 3.4 on the structure of the maximal future development would imply that r extends continuously on $(u_b, \bar{v}_*)$ with

$$r(u_b, \bar{v}_*) = r_{0\varepsilon}. \tag{6-188}$$

However, in that case, (6-176), (6-185) and (6-188) would imply that, for some u_{b*} close enough to u_b,

$$1 - \frac{2m}{r}\Big|_{(u_{b*}, \bar{v}_*)} < 0, \tag{6-189}$$

which is a contradiction in view of (6-183). Therefore,

$$\Big\{U_{n_{\max}+1}\Big(v^{(1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)\Big) \le u < U_{n_{\max}+1}(v^{(0)})\Big\} \cap \{v = \bar{v}_*\} \cap (\partial\mathcal{U}_\varepsilon \backslash \mathcal{I}) = \varnothing,$$

and, thus

$$\mathcal{C}_* = \Big\{U_{n_{\max}+1}(v^{(1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)) \le u < U_{n_{\max}+1}(v^{(0)})\Big\} \cap \{v = \bar{v}_*\}. \tag{6-190}$$

In order to complete the proof in the case when (6-176) holds, it suffices to establish that

$$\limsup_{\bar{u} \to U_{n_{\max}+1}(v^{(0)})} \frac{r|_{(\bar{u}, \bar{v}_*)}}{r_0} \le 1 + O((h_2(\varepsilon))^{1/2}). \tag{6-191}$$

Assuming that (6-191) holds, from (6-176), (6-185) and (6-191) (in view also of (5-3), (6-3)) we readily obtain

$$\liminf_{\bar{u} \to U_{n_{\max}+1}(v^{(0)})} \Big(1 - \frac{2m}{r}\Big)\Big|_{(\bar{u}, \bar{v}_*)} < -\tfrac{1}{2}h_3(\varepsilon) < 0, \tag{6-192}$$

which is a contradiction in view of (6-183).

Let us set

$$\mathcal{B}_* \doteq \Big\{U_{n_{\max}+1}\Big(v^{(1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)\Big) \le u < U_{n_{\max}+1}(v^{(0)})\Big\} \cap \{V_{n_{\max}+1}(v^{(k)}) \le v \le \bar{v}_*\}. \tag{6-193}$$

From (6-190) and the structure of the maximal future development of general initial data sets for (2-28)–(2-33) (see Theorem 3.4), we infer that

$$\mathcal{B}_* \subset \mathcal{U}_\varepsilon.$$

Furthermore, in view of (2-30) and (6-187), we infer that

$$\partial_v r|_{\mathcal{B}_*} > 0 \tag{6-194}$$

and, thus (in view of (3-29))

$$1-\frac{2m}{r}|_{\mathcal{B}_*}>0. \tag{6-195}$$

In view of (6-159) and the bounds (6-11) and (6-12), we have

$$\{u\leq U_{n_{\max}+1}(v^{(1)})\}\cap\mathcal{U}_\varepsilon\subset\mathcal{U}_\varepsilon^+. \tag{6-196}$$

Therefore, as a consequence of (6-7), we can estimate

$$\log\Big(\frac{\partial_v r}{1-2m/r}\Big)\Big|_{\{u=U_{n_{\max}+1}(v^{(1)})\}\cap\mathcal{U}_\varepsilon}\leq(h_1(\varepsilon))^{-4}\log((h_3(\varepsilon))^{-1}). \tag{6-197}$$

Since (2-43) implies

$$\partial_u\log\Big(\frac{\partial_v r}{1-2m/r}\Big)\leq 0, \tag{6-198}$$

from (6-197) (and (6-9)) we infer the one sided bound

$$\partial_v r|_{\mathcal{B}_*}\leq 2(h_1(\varepsilon))^{-4}\log((h_3(\varepsilon))^{-1}). \tag{6-199}$$

Integrating (6-199) from $v=V_{n_{\max}+1}(v^{(k)})$ up to $V_{n_{\max}+1}(v^{(k+1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))$ using (5-3), we finally obtain (6-191). Thus, the proof in the case when (6-176) holds is complete.

Case II: Assume that (6-177) holds. Then, (6-178) and (6-179) also hold.

As a consequence of (6-11), (6-12) and (6-13), the bound (6-178) implies

$$\inf_{\{u\leq U_{n_{\max}}(v^{(0)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))\}\cap\mathcal{U}_\varepsilon}\Big(1-\frac{2\tilde{m}}{r}\Big)\geq\tfrac{1}{2}\exp(-\exp(2(h_0(\varepsilon))^{-4}))(h_1(\varepsilon))^2. \tag{6-200}$$

Therefore, using (6-11), (6-12) and (6-13) to estimate $1-2m/r$ in the region

$$\Big\{U_{n_{\max}}\Big(v^{(0)}+\frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)\Big)\leq u\leq U_{n_{\max}+1}(v^{(0)})\Big\}\backslash\mathcal{R}^{(1,k+1)}_{n_{\max}+1},$$

we infer that

$$\inf_{\{u\leq U_{n_{\max}+1}(v^{(1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))\}\cap\mathcal{U}_\varepsilon}\Big(1-\frac{2\tilde{m}}{r}\Big)\geq\tfrac{1}{2}\exp(-\exp(2(h_0(\varepsilon))^{-4}))(h_1(\varepsilon))^2. \tag{6-201}$$

Remark. Notice that, while $1-2\tilde{m}/r$ becomes $\sim h_3(\varepsilon)$ in $\{u\leq U_{n_{\max}+1}(v^{(0)})\}\cap\mathcal{U}_\varepsilon$ (in view of (6-177)), when restricting to the subregion $\{u\leq U_{n_{\max}+1}(v^{(1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))\}\cap\mathcal{U}_\varepsilon$, the improved bound (6-201) holds.

Let us set

$$u_*\doteq U_{n_{\max}+1}\Big(v^{(1)}+\frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)\Big), \tag{6-202}$$

noticing that

$$\operatorname{supp}(r^2T_{vv})\cap\{u=u_*\}\subset\{r\leq\varepsilon^{1/2}\} \tag{6-203}$$

as a consequence of (6-8). Let us also fix a smooth cut-off function $\chi_\varepsilon:[u_*,u_*+v_{0\varepsilon})\to[0,1]$ such that

$$\chi_\varepsilon(v)=1\quad\text{for } v\in\Big[V_{n_{\max}+1}(v^{(k+1)}),\,V_{n_{\max}+1}\Big(v^{(k+1)}+\frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)\Big)\Big] \tag{6-204}$$

and

$$\chi_\varepsilon(v)=0 \quad \text{for } v\in[u_*,u_*+v_{0\varepsilon})\backslash\Big[V_{n_{\max}+1}\Big(v^{(k+1)}-\frac{1}{\sqrt{-\Lambda}}h_2(\varepsilon)\Big),\,V_{n_{\max}+1}\Big(v^{(k+1)}+\frac{5}{\sqrt{-\Lambda}}h_2(\varepsilon)\Big)\Big].$$

We will then define the function $\widetilde{T}_{vv}:[u_*,u_*+v_{0\varepsilon})\to\mathbb{R}$ by the relation

$$\widetilde{T}_{vv}(v)\doteq\exp\Big(-2C_\varepsilon^2\frac{u_*}{v_0}\Big)(h_1(\varepsilon))^2\chi_\varepsilon(v)T_{vv}(u_*,v), \tag{6-205}$$

where C_ε is defined by (A-62). Notice that, since

$$2\pi\int_{V_{n_{\max}+1}(v^{(k+1)})}^{V_{n_{\max}+1}(v^{(k+1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))}\frac{(1-2m/r)}{\partial_v r}r^2T_{vv}\Big|_{(u_*,v)}\,dv=\tilde{m}^{(1,k+1)}_{n_{\max}+1}-\tilde{m}^{(1,k+2)}_{n_{\max}+1}=\tilde{m}^{(1,k+1)}_{n_{\max}+1}, \tag{6-206}$$

we can readily bound in view of (6-205), (6-12), (6-177) and (6-206)

$$\sup_{u_*\le\bar{v}\le u_*+v_{0\varepsilon}}(-\Lambda)\int_{u_*}^{u_*+v_{0\varepsilon}}\frac{r^2(u_*,v)|\tilde{T}_{vv}(v)|/\partial_v\rho(u_*,v)}{|\rho(u_*,v)-\rho(u_*,\bar{v})|+\rho(u_*,u_*)}\,dv\le\exp\Big(-C_\varepsilon^2\frac{u_*}{v_{0\varepsilon}}\Big)(h_1(\varepsilon))^2, \tag{6-207}$$

where ρ is defined in terms of r by the relation

$$\rho\doteq\tan^{-1}\Big(\sqrt{-\frac{\Lambda}{3}}r\Big). \tag{6-208}$$

Applying the backwards-in-time Cauchy stability lemma, Lemma A.3, (see Section A1.2 of the Appendix), for u_* given by (6-202) and $\widetilde{T}_{vv}$ given by (6-205) (in view of (6-201), (6-203) and (6-207)), we infer that there exists a smooth asymptotically AdS boundary-characteristic initial data set $(r'_{/\varepsilon},(\Omega'_{/\varepsilon})^2,\bar{f}'_{\text{in}/\varepsilon},\bar{f}'_{\text{out}/\varepsilon})$ on $\{u=0\}$ for the system (2-28)–(2-33) satisfying the reflecting gauge condition at $r=r_{0\varepsilon},+\infty$ with the following properties:

(1) The initial data sets $(r_{/\varepsilon},\Omega^2_{/\varepsilon},\bar{f}_{\text{in}/\varepsilon},\bar{f}_{\text{out}/\varepsilon})$ and $(r'_{/\varepsilon},(\Omega'_{/\varepsilon})^2,\bar{f}'_{\text{in}/\varepsilon},\bar{f}'_{\text{out}/\varepsilon})$ satisfy (A-68) and (A-69).

(2) The maximal development $(\mathcal{U}'_\varepsilon;r',(\Omega')^2,\bar{f}'_{\text{in}},\bar{f}'_{\text{out}})$ of $(r_{/\varepsilon},\Omega^2_{/\varepsilon},\bar{f}_{\text{in}/\varepsilon},\bar{f}_{\text{out}/\varepsilon})$ satisfies (A-70), (A-71) and (A-72).

Using primes to denote quantities associated to $(r',(\Omega')^2,\bar{f}'_{\text{in}},\bar{f}'_{\text{out}})$, we can readily estimate in view of (A-70), (A-71), (A-72) and (6-205)

$$\begin{aligned}\tilde{m}'|_{(u_*,\bar{v}_*)}&=\int_{V_{n_{\max}+1}(v^{(k+1)})}^{V_{n_{\max}+1}(v^{(k+1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))}\frac{(1-2m'/r')}{\partial_v r'}(r')^2T'_{vv}\Big|_{(u_*,v)}\,dv\\&=\int_{V_{n_{\max}+1}(v^{(k+1)})}^{V_{n_{\max}+1}(v^{(k+1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))}\frac{(1-2m'/r)}{\partial_v r}r^2(T_{vv}+\widetilde{T}_{vv})\Big|_{(u_*,v)}\,dv\\&\ge\Big(1+\exp\Big(-2C_\varepsilon^2\frac{u_*}{v_0}\Big)(h_1(\varepsilon))^2\Big)\tilde{m}\Big|_{(u_*,\bar{v}_*)}.\end{aligned} \tag{6-209}$$

Therefore, since $\tilde{m}|_{(u_*,\bar{v}_*)}=\tilde{m}^{(1,k+1)}_{n_{\max}+1}$, the bound (6-177) (in view also of (6-3) and (A-62)) implies

$$\frac{2\tilde{m}'|_{(u_*,\bar{v}_*)}}{r_0}\ge\Big(1+\exp\Big(-2C_\varepsilon^2\frac{u_*}{v_0}\Big)(h_1(\varepsilon))^2\Big)(1-2h_3(\varepsilon))\ge 1+h_3(\varepsilon). \tag{6-210}$$

Since $\tilde{m}|_{(u_*,\bar{v}_*)}$ is constant on

$$\mathcal{C}'_* \doteq \Big\{U_{n_{\max}+1}\Big(v^{(1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)\Big) \le u < U_{n_{\max}+1}(v^{(0)})\Big\} \cap \{v = \bar{v}_*\} \cap \mathcal{U}'_\varepsilon$$

and satisfies (6-210), we can now repeat the same arguments as in Case I (i.e., the case when (6-176) holds) in order to infer that (6-182) holds.

Thus, the proof of Theorem 1 is complete. □

Appendix: A maximum principle and Cauchy stability backwards in time

In this section, we will prove some lemmas which are necessary for the proof of Proposition 6.1 and Theorem 1.

A1. ***A maximum principle for $1+1$ wave-type equations.*** The following lemma provides a comparison inequality for certain $1+1$ equations of wave type, and is used in the proof of Proposition 6.1.

Lemma A.1. *For any $u_0 < u_1$, $v_0 < v_1$ and $a \in \mathbb{R}$, let $F_1, F_2 : [u_0, u_1] \times [v_0, v_1] \times (-\infty, a] \to (0, +\infty)$ be smooth functions so that*

$$\max_{(u,v)\in[u_0 u_1]\times[v_0,v_1]} F_1(u, v, z) < \min_{(u,v)\in[u_0 u_1]\times[v_0,v_1]} F_2(u, v, z) \tag{A-1}$$

for any $z \in (-\infty, a]$ and

$$\partial_z F_1(u, v, z),\ \partial_z F_2(u, v, z) \ge 0 \tag{A-2}$$

for any $(u, v, z) \in [u_0, u_1] \times [v_0, v_1] \times (-\infty, a]$. Suppose also $z_1, z_2 : [u_0, u_1] \times [v_0, v_1] \to (-\infty, a]$ are smooth solutions to the equations

$$\partial_v \partial_u z_1 = -F_1(u, v, z_1)\, \partial_u z_1\, \partial_v z_1, \tag{A-3}$$

$$\partial_v \partial_u z_2 = -F_2(u, v, z_2)\, \partial_u z_2\, \partial_v z_2, \tag{A-4}$$

satisfying the same characteristic initial data

$$z_1(u, v_0) = z_2(u, v_0) = z_\backslash(u), \tag{A-5}$$

$$z_1(u_0, v) = z_2(u_0, v) = z_/(v), \tag{A-6}$$

where $z_/ : [v_0, v_1] \to (-\infty, a)$ and $z_\backslash : [u_0, u_1] \to (-\infty, a)$ are smooth functions so that

$$z_/(v_0) = z_\backslash(v_1), \tag{A-7}$$

$$\partial_v z_/|_{(v_0,v_1)} > 0, \tag{A-8}$$

$$\partial_u z_\backslash|_{(u_0,u_1)} < 0. \tag{A-9}$$

Then, the functions z_1, z_2 satisfy

$$\partial_u z_i < 0 < \partial_v z_i, \quad i = 1, 2, \tag{A-10}$$

in $(u_0, u_1) \times (v_0, v_1)$ and

$$z_1 \le z_2 \tag{A-11}$$

everywhere on $[u_0, u_1] \times [v_0, v_1]$.

Proof. We will first establish (A-10). By applying a standard continuity argument, rewriting (A-3) as

$$\partial_v \log(-\partial_u z_1) = -\partial_v z_1 F_1(u, v, z_1) \tag{A-12}$$

and integrating in v, using also the property (A-9) of the initial data, we obtain that

$$\partial_u z_1 < 0 \tag{A-13}$$

everywhere on $(u_0, u_1) \times (v_0, v_1)$. Similarly, rewriting (A-3) as

$$\partial_u \log(\partial_v z_1) = -\partial_u z_1 \, F_1(u, v, z_1) \tag{A-14}$$

and integrating in u, using (A-9), and then repeating the same procedure for z_2, we finally obtain (A-10).

In order to establish (A-11), we will argue by continuity: Let $u_* \in [u_0, u_1)$ be such that (A-11),

$$\partial_v z_1 \le \partial_v z_2, \tag{A-15}$$

$$\partial_u z_1 \le \partial_u z_2 \tag{A-16}$$

hold on $[u_0, u_*] \times [v_0, v_1]$. Note that $u_* = u_0$ satisfies this condition: in this case, (A-11) and (A-15) follow directly from (A-6), while (A-16) follows by integrating (A-12) (and its analogue for z_2) and using (A-1). We will show that there exists a $\delta > 0$, such that (A-11), (A-15) and (A-16) hold on $[u_0, u_* + \delta) \times [v_0, v_1]$.

For any $\bar{v} \in (v_0, v_1]$, integrating (A-3) and (A-4) in v along $\{u_*\} \times [v_0, \bar{v}]$, we obtain

$$\log(-\partial_u z_1)(u_*, \bar{v}) = -\int_{v_0}^{\bar{v}} F_1(u_*, v, z_1)\, \partial_v z_1 \, dv + \log(-\partial_u z_\backslash)(u_*), \tag{A-17}$$

$$\log(-\partial_u z_2)(u_*, \bar{v}) = -\int_{v_0}^{\bar{v}} F_2(u_*, v, z_2)\, \partial_v z_2 \, dv + \log(-\partial_u z_\backslash)(u_*). \tag{A-18}$$

Let us define the auxiliary functions $F_{1;u_*\bar{v}}, F_{2;u_*\bar{v}} : (-\infty, a] \to (0, +\infty)$ by the relations

$$F_{1;u_*\bar{v}}(z) = \max_{v \in [v_0, \bar{v}]} F_1(u_*, v, z), \tag{A-19}$$

$$F_{2;u_*\bar{v}}(z) = \min_{v \in [v_0, \bar{v}]} F_2(u_*, v, z). \tag{A-20}$$

In view of (A-1), (A-2) and the fact that (A-11) holds on $\{u_*\} \times [v_0, \bar{v}]$,[17] we can bound for any $v \in [v_0, \bar{v}]$

$$F_{1;u_*\bar{v}}(z_1(u_*, v)) < F_{2;u_*\bar{v}}(z_1(u_*, v)) \le F_{2;u_*\bar{v}}(z_2(u_*, v)). \tag{A-21}$$

Thus, subtracting (A-17) and (A-18) and using (A-21) and (A-15) (and the fact that $\partial_v z_2 > 0$, $\bar{v} > v_0$), we readily infer that

$$\begin{aligned} &\log(-\partial_u z_1)(u_*, \bar{v}) - \log(-\partial_u z_2)(u_*, \bar{v}) \\ &\qquad \ge \int_{v_0}^{\bar{v}} F_{2;u_*\bar{v}}(z_2(u_*, v))\, \partial_v z_2(u_*, v)\, dv - \int_{v_0}^{\bar{v}} F_{1;u_*\bar{v}}(z_1(u_*, v))\, \partial_v z_1(u_*, v)\, dv \\ &\qquad > 0. \end{aligned}$$

[17] Note that we can immediately restrict from $[u_0, u_1] \times [v_0, v_1]$ to $\{u_*\} \times [v_0, \bar{v}]$ in (A-1).

From (A-22) we thus infer that, for any $v_0 < \bar{v} \le v_1$,

$$\partial_u z_1(u_*, \bar{v}) < \partial_u z_2(u_*, \bar{v}). \tag{A-22}$$

Therefore, since z_1, z_2 are smooth, there exists a continuous function $\delta_u : [v_0, v_1] \to [0, 1)$, with $\delta_u|_{(v_0,v_1]} > 0$, such that

$$\partial_u z_1(u, v) \le \partial_u z_2(u, v) \quad \text{for } \{v_0 \le v \le v_1\} \cap \{u_* \le u \le u_* + \delta_u(v))\}. \tag{A-23}$$

Similarly, by integrating (A-3) and (A-4) in u along $[u_0, u_1] \times \{v_0\}$ and repeating a similar procedure (using (A-5)), we also obtain that there exists a continuous function $\delta_v : [u_0, u_1] \to [0, 1)$, with $\delta_v|_{(u_0,u_1]} > 0$, such that

$$\partial_v z_1(\bar{u}, v_0) \le \partial_v z_2(\bar{u}, v_0) \quad \text{for } \{u_0 \le u \le u_1\} \cap \{v_0 \le v \le v_0 + \delta_v(u))\}. \tag{A-24}$$

From (A-22) and (A-24), we infer that there exists some $\delta > 0$, such that

$$z_1 \le z_2 \quad \text{on } (u_*, u_* + \delta) \times [v_0, v_1]. \tag{A-25}$$

In particular, (A-11) holds on $[u_0, u_* + \delta) \times [v_0, v_1]$. Furthermore, for any $\bar{u} \in (u_*, u_* + \delta)$ and any $\bar{v} \in (v_0, v_0 + \delta_v(\bar{u}))$, repeating the procedure leading to (A-22) with $\bar{u}$ in place of u_* and using (A-24) and (A-25) in place of (A-15) and (A-11), respectively, we infer that

$$\partial_u z_1(\bar{u}, \bar{v}) \le \partial_u z_2(\bar{u}, \bar{v}). \tag{A-26}$$

Thus, combining (A-23) and (A-26), we infer that (A-16) holds on $[u_0, u_* + \delta') \times [v_0, v_1]$ for some $0 < \delta' \le \delta$. Finally, the bound (A-15) on $[u_0, u_* + \delta') \times [v_0, v_1]$ follows in a way similar to the proof of (A-22), by integrating (A-3) and (A-4) in $u \in [u_0, u_* + \delta')$ for any $\bar{v} \in (v_0, v_1)$ and using (A-1), (A-11) and (A-16) (which we have shown hold on $[u_0, u_* + \delta') \times [v_0, v_1]$). We will omit the details. □

A1.1. *A lemma for a system of inductive inequalities.* The following lemma is used to show that the inductive bounds (6-13) and (6-14) for $\tilde{m}_n^{(1,k+1)}$ and $r_n^{(k,k+1)}$ indeed lead to the formation of an almost-trapped surface.

Lemma A.2. *Let* $0 < c_1 \ll 1 \ll C_1$, *and* $0 < \mu_0 \ll 1 \ll \rho_0$, $0 < \delta \ll 1$ *be given variables, such that*

$$\left(\frac{C_1}{c_1}\right)^8 \ll \frac{\rho_0}{\mu_0} \tag{A-27}$$

and

$$\delta < \left(\frac{\mu_0}{\rho_0}\right)^{(C_1/c_1)^4}. \tag{A-28}$$

Let also $\mu_n, \rho_n > 0$ *be sequences of positive numbers, with* μ_n *increasing in* n, *such that for* $0 \le n \le n_*$ *they satisfy*

$$\rho_{n+1} \le \rho_n + C_1 \log((1 - \mu_n)^{-1} + 1), \tag{A-29}$$

$$\mu_{n+1} \ge \mu_n \exp\left(\frac{c_1}{\rho_{n+1}}\right), \tag{A-30}$$

and

$$\max_{0\le n\le n_*-1}\mu_n < 1-\delta. \tag{A-31}$$

Then,

$$n_* \le \left(\frac{C_1}{c_1}\right)^3 \rho_0 \mu_0^{-(C_1/c_1)^2}. \tag{A-32}$$

Furthermore, *if* $1-\delta \le \mu_{n_*} \le 1+\delta$, *we can bound*

$$\mu_{n_*-1} \le 1-\left(\frac{c_1}{C_1}\right)^3 \rho_0^{-2}\mu_0^{2(C_1/c_1)^2} \tag{A-33}$$

and

$$\max\{\rho_{n_*},\rho_{n_*-1}\} \le \left(\frac{C_1}{c_1}\right)^4 \frac{\rho_0}{\mu_0^{(C_1/c_1)^2}}\log\left(\frac{\rho_0}{\mu_0}\right). \tag{A-34}$$

Remark. Notice that the right-hand side of (A-32) is independent of δ.

Proof. Let us define for any integer $k\ge 1$

$$n_k = \max\left\{0\le n\le n_* : \mu_l \le 1-\frac{1}{2^k} \quad \text{for all } 0\le l\le n\right\}, \tag{A-35}$$

using the convention

$$n_0 = 0. \tag{A-36}$$

Notice that, in view of the fact that the sequence μ_n is increasing, for all $k\ge 1$ and all $n_{k-1} < n \le n_k$ we can estimate

$$1-\frac{1}{2^{k-1}} \le \mu_n \le 1-\frac{1}{2^k} \tag{A-37}$$

(note that, in the case $n_{k-1}=n_k$, there is no n satisfying $n_{k-1}<n\le n_k$ and (A-37)).

Using (A-37), from (A-29) we can bound for any $k\ge 1$ such that $n_{k-1}<n_k$ and any $n_{k-1}<n\le n_k$

$$\rho_n \le \rho_{n_{k-1}} + 2C_1(\log 2)k(n-n_{k-1}) \tag{A-38}$$

and, therefore, for any $0\le n\le n_k$ we have

$$\rho_n \le 2C_1(\log 2)\left(\sum_{l=1}^{k-1} l(n_l-n_{l-1}) + k(n-n_{k-1})\right) + \rho_0 \tag{A-39}$$

(note that (A-39) holds for all $0\le n\le n_k$, while the bounds (A-37) and (A-38) are nontrivial only for those values of k for which $n_k > n_{k-1}$).

Let us set

$$k_1 \doteq 32\left\lceil \log\frac{C_1}{c_1}\right\rceil. \tag{A-40}$$

Then, (A-38) implies that, for all $0\le n\le n_{k_1}$,

$$\rho_n \le \rho_0 + 2C_1(\log 2)k_1 n \tag{A-41}$$

and, thus, (A-30) implies

$$\log\left(\frac{\mu_{n_{k_1}}}{\mu_0}\right) \geq c_1 \sum_{n=1}^{n_{k_1}} \rho_n^{-1} \geq c_1 \sum_{n=1}^{n_{k_1}} \frac{1}{\rho_0 + 2C_1(\log 2)k_1 n}$$
$$\geq c_1 \log\left(\frac{\rho_0 + 2C_1(\log 2)k_1 n_{k_1}}{\rho_0 + 2C_1(\log 2)k_1}\right)(4C_1(\log 2)k_1)^{-1}. \tag{A-42}$$

From (A-35) and (A-42) we readily infer that

$$c_1 \log\left(\frac{\rho_0 + 2C_1(\log 2)k_1 n_{k_1}}{\rho_0 + 2C_1(\log 2)k_1}\right)(4C_1(\log 2)k_1)^{-1} \leq -\log(\mu_0) \tag{A-43}$$

and, therefore (using also (A-27))

$$n_{k_1} \leq \frac{\rho_0}{\mu_0^{(C_1/c_1)^2}}. \tag{A-44}$$

For any $k \geq 2$ such that $n_k > n_{k-1}+1$, from (A-30), (A-35), (A-37) and (A-39) we readily infer

$$\frac{1}{2^{k-2}} \geq \log\frac{\mu_{n_k}}{\mu_{n_{k-1}+1}} \geq c_1 \sum_{n=n_{k-1}+2}^{n_k} \rho_n^{-1}$$
$$\geq \frac{c_1}{4C_1(\log 2)} \frac{1}{(k-1)+\sum_{l=2}^{k-1}(l-1)(n_l-n_{l-1})/(n_k-n_{k-1}-1)} \tag{A-45}$$

and, hence

$$n_k - n_{k-1} - 1 \leq \frac{\sum_{l=2}^{k-1}(l-1)(n_l-n_{l-1})}{2^{k-2}c_1/(4C_1(\log 2))-(k-1)}$$
$$\leq \frac{k(k-1)}{2}\frac{\max_{2\leq l\leq k-1}(n_l-n_{l-1})}{2^{k-2}c_1/(4C_1(\log 2))-(k-1)}$$
$$\leq \frac{k(k-1)}{2}\frac{1}{2^{k-2}c_1/(4C_1(\log 2))-(k-1)}n_{k-1}. \tag{A-46}$$

The relation (A-46) also holds (trivially) when $n_k \leq n_{k-1}+1$. Thus, for any $k \geq k_1$, the bound (A-46) yields

$$n_k \leq \left(1 + \frac{C_1}{c_1}2^{-(k-2)/4}\right)n_{k-1} + 1 \tag{A-47}$$

and, therefore, for any $k \geq 2$,

$$n_k \leq 16\frac{C_1}{c_1}(n_{k_1} + \max\{k-k_1, 0\}). \tag{A-48}$$

In view of (A-44), we thus obtain for any $k \geq 2$

$$n_k \leq 16\frac{C_1}{c_1}\left(\frac{\rho_0}{\mu_0^{(C_1/c_1)^2}} + \max\{k-k_1, 0\}\right). \tag{A-49}$$

Let us set

$$k_2 \doteq 4k_1 + 2\frac{\log(\rho_0/\mu_0^{(C_1/c_1)^2})}{\log 2}. \tag{A-50}$$

Note that (A-28), (A-31) and (A-35) imply

$$n_{k_2} \le n_* - 1. \tag{A-51}$$

In view of (A-49), we have

$$n_{k_2} \le \left(\frac{C_1}{c_1}\right)^3 \frac{\rho_0}{\mu_0^{(C_1/c_1)^2}} \tag{A-52}$$

and, for all $k \ge k_2$ (in view of (A-46) and (A-49))

$$n_k - n_{k-1} \le 1. \tag{A-53}$$

Furthermore, (A-39) implies (in view of (A-50) and (A-52)) that

$$\max\{\rho_{n_{k_2}+1}, \rho_{n_{k_2}}\} \le \left(\frac{C_1}{c_1}\right)^3 \frac{\rho_0}{\mu_0^{(C_1/c_1)^2}} \log\left(\frac{\rho_0}{\mu_0}\right). \tag{A-54}$$

In view of (A-35), we have

$$\mu_{n_{k_2}} \le 1 - 2^{-k_2} < \mu_{n_{k_2}+1}. \tag{A-55}$$

We will consider two cases, depending on whether $\mu_{n_{k_2}+1}$ is larger than $1-\delta$ or not.

(1) In the case $\mu_{n_{k_2}+1} \ge 1-\delta$, (A-31) implies that $n_{k_2}+1 = n_*$. Thus, (A-32) follows from (A-52). Furthermore, (A-34) follows from (A-54), while (A-33) follows from (A-55).

(2) In the case $\mu_{n_{k_2}+1} < 1-\delta$, we can assume without loss of generality that $n_{k_2} \le n_* - 2$ (otherwise, (A-32) follows from (A-52)). From (A-29), (A-54) and (A-55), we thus infer that

$$\rho_{n_{k_2}+2} \le \left(\frac{C_1}{c_1}\right)^3 \frac{\rho_0}{\mu_0^{(C_1/c_1)^2}} \log\left(\frac{\rho_0}{\mu_0}\right) + C_1 \log((1-\mu_{n_{k_2}+1})^{-1}) \tag{A-56}$$

Hence, setting

$$M \doteq \left(\frac{C_1}{c_1}\right)^3 \frac{\rho_0}{\mu_0^{(C_1/c_1)^2}} \log\left(\frac{\rho_0}{\mu_0}\right), \tag{A-57}$$

from (A-30) and (A-55) we calculate

$$\begin{aligned} \mu_{n_{k_2}+2} &\ge \mu_{n_{k_2}+1} \exp\left(\frac{c_1}{\rho_{n_{k_2}+2}}\right) \\ &\ge \begin{cases} (1-2^{-k_2}) e^{c_1/(2M)} & \text{if } \log((1-\mu_{n_{k_2}+1})^{-1}) \le M/C_1, \\ \mu_{n_{k_2}+1}\left(1 + \dfrac{c_1}{C_1 \log((1-\mu_{n_{k_2}+1})^{-1}+1)}\right) & \text{if } \log((1-\mu_{n_{k_2}+1})^{-1}) > M/C_1. \end{cases} \end{aligned} \tag{A-58}$$

If $\log((1-\mu_{n_{k_2}+1})^{-1}) \le M/C_1$, in view of (A-50) and (A-57) we can bound (using also (A-27))

$$(1-2^{-k_2}) e^{c_1/(2M)} \ge \left(1 - \frac{\mu_0^{2(C_1/c_1)^2}}{\rho_0^2}\right)\left(1 + \frac{c_1}{2}\left(\frac{c_1}{C_1}\right)^3 \frac{\mu_0^{(C_1/c_1)^2}}{\rho_0 \log(\rho_0/\mu_0)}\right) > 1+\delta. \tag{A-59}$$

If $\log((1-\mu_{n_{k_2}}+1)^{-1}) > M/C_1 \gg (C_1/c_1)^3$, we can also estimate

$$\mu_{n_{k_2}+1}\left(1+\frac{c_1}{C_1\log((1-\mu_{n_{k_2}+1})^{-1}+1)}\right) \geq (1-e^{-\log((1-\mu_{n_{k_2}+1})^{-1})})\left(1+\frac{c_1}{C_1\log((1-\mu_{n_{k_2}+1})^{-1}+1)}\right) > 1+\delta. \tag{A-60}$$

Therefore, (A-58) implies

$$\mu_{n_{k_2}+2} > 1+\delta \tag{A-61}$$

and, hence, $n_{k_2+2}=n_*$. Thus, (A-32) follows again from (A-52). □

A1.2. *Cauchy stability backwards in time.* The following lemma, which is essentially a backwards-in-time Cauchy stability estimate for late time perturbations of $(\mathcal{U}_\varepsilon; r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})$, is an easy corollary of Theorem 3.6.

Lemma A.3. *For any $0<\varepsilon<\varepsilon_0$ (provided ε_0 is sufficiently small) and any $r_0>0$ satisfying* (5-6), *let $(\mathcal{U}_\varepsilon; r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})$ be the maximal future development of $(r_{/\varepsilon}, \Omega^2_{/\varepsilon}, \bar{f}_{\text{in}/\varepsilon}, \bar{f}_{\text{out}/\varepsilon})$, and let us set*

$$C_\varepsilon \doteq \exp\big(\exp(-2(h_0(\varepsilon))^{-4})(h_1(\varepsilon))^{-4}\big). \tag{A-62}$$

Then, for any $0\leq u_*\leq (h_1(\varepsilon))^{-2}v_{0\varepsilon}$ such that

$$\mathcal{W}_{u_*} \doteq \{0<u\leq u_*\}\cap\{u<v<u+v_{0\varepsilon}\}\subset\mathcal{U}_\varepsilon, \tag{A-63}$$

$$\sup_{\mathcal{W}_{u_*}}\left(1-\frac{2\tilde{m}}{r}\right)^{-1}\leq C_\varepsilon, \tag{A-64}$$

$$u_*+v_{0\varepsilon}\notin \operatorname{supp}(r^2T_{vv}(u_*,\cdot)), \tag{A-65}$$

and for any $\tilde{T}_{vv}:(u_*, u_*+v_{0\varepsilon})\to\mathbb{R}$ smooth and compactly supported satisfying $\tilde{T}_{vv}(\cdot)\geq -T_{vv}(u_*,\cdot)$ and

$$\sup_{u_*\leq\bar{v}\leq u_*+v_{0\varepsilon}}(-\Lambda)\int_{u_*}^{u_*+v_{0\varepsilon}}\frac{r^2(u_*,v)|\tilde{T}_{vv}(v)|/\partial_v\rho(u_*,v)}{|\rho(u_*,v)-\rho(u_*,\bar{v})|+\rho(u_*,u_*)}\,dv\leq \exp\left(-C_\varepsilon^2\frac{u_*}{v_{0\varepsilon}}\right)(h_1(\varepsilon))^2, \tag{A-66}$$

with

$$\rho(u,v)\doteq\tan^{-1}\left(\sqrt{-\frac{\Lambda}{3}}r\right), \tag{A-67}$$

the following statement holds: There exists a smooth asymptotically AdS boundary-characteristic initial data set $(r'_{/\varepsilon}, (\Omega'_{/\varepsilon})^2, \bar{f}'_{\text{in}/\varepsilon}, \bar{f}'_{\text{out}/\varepsilon})$ on $\{u=0\}$ for the system (2-28)–(2-33) *satisfying the reflecting gauge condition at $r=r_{0\varepsilon}, +\infty$ with the following properties*:

(1) *The initial data sets $(r_{/\varepsilon}, \Omega^2_{/\varepsilon}, \bar{f}_{\text{in}/\varepsilon}, \bar{f}_{\text{out}/\varepsilon})$ and $(r'_{/\varepsilon}, (\Omega'_{/\varepsilon})^2, \bar{f}'_{\text{in}/\varepsilon}, \bar{f}'_{\text{out}/\varepsilon})$ are $(h_1(\varepsilon))^2$ close in the* (3-41) *norm, and in particular*

$$\sup_{v\in[0,v_{0\varepsilon})}\left\{\left|\log\left(\frac{\Omega^2_{/\varepsilon}}{1-\frac{1}{3}\Lambda r^2_{/\varepsilon}}\right)-\log\left(\frac{(\Omega'_{/\varepsilon})^2}{1-\frac{1}{3}\Lambda(r'_{/\varepsilon})^2}\right)\right| + \left|\log\left(\frac{2\partial_v r_{/\varepsilon}}{1-2m_{/\varepsilon}/r_{/\varepsilon}}\right)-\log\left(\frac{2\partial_v r'_{/\varepsilon}}{1-2m'_{/\varepsilon}/r'_{/\varepsilon}}\right)\right| + \left|\log\left(\frac{1-2m_{/\varepsilon}/r_{/\varepsilon}}{1-\frac{1}{3}\Lambda r^2_{/\varepsilon}}\right)-\log\left(\frac{1-2m'_{/\varepsilon}/r'_{/\varepsilon}}{1-\frac{1}{3}\Lambda(r'_{/\varepsilon})}\right)\right| + \sqrt{-\Lambda}|\tilde{m}_{/\varepsilon}-\tilde{m}'_{/\varepsilon}|\right\}(v)\leq(h_1(\varepsilon))^2 \tag{A-68}$$

and

$$\sup_{\bar{v}\in[0,v_{0\varepsilon}]}\int_0^{v_{0\varepsilon}}\left|r_{/\varepsilon}^2\frac{(T_{vv})_{/\varepsilon}}{\partial_v\rho_{/\varepsilon}}(v)-(r'_{/\varepsilon})^2\frac{(T_{vv})'_{/\varepsilon}}{\partial_v\rho'_{/\varepsilon}}(v)\right|\big(|\rho_{/\varepsilon}(\bar{v})-\rho_{/\varepsilon}(v)|+\rho_{/\varepsilon}(0)\big)^{-1}\,dv\le(h_1(\varepsilon))^2. \quad \text{(A-69)}$$

(2) *The maximal future development* $(\mathcal{U}'_\varepsilon; r', (\Omega')^2, \bar{f}'_{\text{in}}, \bar{f}'_{\text{out}})$ *of* $(r'_{/\varepsilon}, (\Omega'_{/\varepsilon})^2, \bar{f}'_{\text{in}/\varepsilon}, \bar{f}'_{\text{out}/\varepsilon})$ *satisfies*

$$\mathcal{W}_{u_*}\subset\mathcal{U}'_\varepsilon, \quad \text{(A-70)}$$

$$r'|_{\{u=u_*\}\cap\operatorname{supp}(T_{vv})}=r|_{\{u=u_*\}\cap\operatorname{supp}(T_{vv})}, \quad \text{(A-71)}$$

$$T'_{vv}|_{\{u=u_*\}}=T_{vv}|_{\{u=u_*\}}+\tilde{T}_{vv}. \quad \text{(A-72)}$$

Proof. In view of (5-6), (5-24), (6-11) and (6-12), we can readily estimate

$$\sup_{\mathcal{W}_{u_*}\setminus\bigcup_n\mathcal{R}_n^{(1,k+1)}}\Big(1-\frac{2\tilde{m}}{r}\Big)^{-1}\le 2\exp((h_0(\varepsilon))^{-4}). \quad \text{(A-73)}$$

Therefore, using (A-64) for $\bigcup_n\mathcal{R}_n^{(1,k+1)}$ and (A-73) for $\mathcal{W}_{u_*}\setminus\bigcup_n\mathcal{R}_n^{(1,k+1)}$, the relations (6-33) and (6-34) imply (in view of (5-3), (6-8) and the fact that $u_*\le(h_1(\varepsilon))^{-2}v_{0\varepsilon}$) that

$$\sup_{\mathcal{W}_{u_*}}\left(\left|\log\Big(\frac{-\partial_u r}{1-2m/r}\Big)\right|+\left|\log\Big(\frac{\partial_v r}{1-2m/r}\Big)\right|\right)\le(h_1(\varepsilon))^{-3}\exp((h_0(\varepsilon))^{-4}). \quad \text{(A-74)}$$

Similarly, (2-43) and (2-44), in view of the relations (6-33), (6-34) (using again the bounds (A-64) (A-73)) imply

$$\sup_{\bar{u}}\int_{\{u=\bar{u}\}\cap\mathcal{W}_{u_*}}rT_{vv}\,dv+\sup_{\bar{v}}\int_{\{v=\bar{v}\}\cap\mathcal{W}_{u_*}}rT_{uu}\,du\le(h_1(\varepsilon))^{-1}\exp((h_0(\varepsilon))^{-4}). \quad \text{(A-75)}$$

Let us fix a set of smooth functions $r_/^*$, $(\Omega_/^*)^2:[u_*,u_*+v_{0\varepsilon})\to(0,+\infty)$ and

$$\bar{f}_{\text{in}/}^*,\ \bar{f}_{\text{out}/}^*:[u_*,u_*+v_{0\varepsilon})\times(0,+\infty)\to[0,+\infty)$$

satisfying the following requirements:

(1) $(r_/^*, (\Omega_/^*)^2, \bar{f}_{\text{in}/}^*, \bar{f}_{\text{out}/}^*)$ is a smooth asymptotically AdS boundary-characteristic initial data set for the system (2-28)–(2-33) on $\{u_*\}\times[u_*,u_*+v_{0\varepsilon})$ satisfying the reflecting gauge condition at $r^*=r_{0\varepsilon},+\infty$.

(2) The function $r_/^*$ satisfies for any v such that $(u_*,v)\in\operatorname{supp}(T_{vv})$

$$r_/^*(v)=r|_{\operatorname{supp}(T_{vv})\cap\{u=u_*\}}. \quad \text{(A-76)}$$

(3) The function $\bar{f}_{\text{in}/}^*$ satisfies, for all $v\in[u_*,u_*+v_{0\varepsilon})$,

$$\int_0^{+\infty}((\Omega_/^*)^2(v)p^v)^2\bar{f}_{\text{in}/}^*(v;p^v)\,(r_/^*)^2(v)\frac{dp^v}{p^v}=T_{vv}(u_*,v)+\tilde{T}_{vv}(v). \quad \text{(A-77)}$$

(4) The function $\bar{f}^*_{\mathrm{out}/}$ satisfies, for all $(v, p^v) \in [u_*, u_* + v_{0\varepsilon}) \times (0, +\infty)$,

$$\bar{f}^*_{\mathrm{out}/}(v; p^v) = \bar{f}_{\mathrm{out}}(u_*, v; p^v). \tag{A-78}$$

(5) The initial data sets $(r, \Omega^2, \bar{f}_{\mathrm{in}}, \bar{f}_{\mathrm{out}})|_{u=u_*}$ and $(r^*_/, (\Omega^*_/)^2, \bar{f}^*_{\mathrm{in}/}, \bar{f}^*_{\mathrm{out}/})$ satisfy

$$\begin{aligned}\sup_{v\in[u_*,u_*+v_{0\varepsilon})}\Bigg\{&\bigg|\log\Big(\frac{\Omega^2}{1-\frac{1}{3}\Lambda r^2}\Big)\Big|_{u=u_*}-\log\Big(\frac{(\Omega^*_/)^2}{1-\frac{1}{3}\Lambda(r^*_/)^2}\Big)\bigg| \\ &+\bigg|\log\Big(\frac{2\partial_v r}{1-2m/r}\Big)\Big|_{u=u_*}-\log\Big(\frac{2\partial_v r^*_/}{1-2m^*_/ /r^*_/}\Big)\bigg| \\ &\quad+\bigg|\log\Big(\frac{1-2m/r}{1-\frac{1}{3}\Lambda r^2}\Big)\Big|_{u=u_*}-\log\Big(\frac{1-2m^*_/ /r^*_/}{1-\frac{1}{3}\Lambda(r^*_/)}\Big)\bigg|+\sqrt{-\Lambda}|\tilde{m}|_{u=u_*}-\tilde{m}^*_/|\Bigg\}(v) \\ &\le \exp\Big(-\frac{1}{2}C_\varepsilon^2\frac{u_*}{v_0}\Big)(h_1(\varepsilon))^2\end{aligned} \tag{A-79}$$

and

$$\sup_{v\in[v_1,v_2]}\int_{v_1}^{v_2}\frac{|r^2T_{vv}(u_*,\bar{v})-(r^*_/)^2(T_{vv})^*_/(\bar{v})|}{|\rho_/(v)-\rho_/(\bar{v})|+\rho_/(v_1)}\,d\bar{v} \le \exp\Big(-\frac{1}{2}C_\varepsilon^2\frac{u_*}{v_0}\Big)(h_1(\varepsilon))^2. \tag{A-80}$$

Remark. As a consequence of (A-65), by suitably deforming $r^*_/$ near $v = u_* + v_{0\varepsilon}$, we can always arrange that (3-7) and (A-76) are satisfied simultaneously. Furthermore, since $\tilde{T}_{vv}$ is compactly supported in $(u_*, u_* + v_{0\varepsilon})$, we can always choose $\bar{f}^*_{\mathrm{in}/} = f_{\mathrm{in}}|_{\{u=u_*\}}$ in a neighborhood of $v = u_*, u_* + v_{0\varepsilon}$, so that (3-8) and (3-9) are satisfied. Finally, $(r^*_/, (\Omega^*_/)^2, \bar{f}^*_{\mathrm{in}/}, \bar{f}^*_{\mathrm{out}/})$ can be chosen so that (A-79) and (A-80) are satisfied because of (A-66) and the relations (2-6), (2-44) and (2-47).

Let us now consider the two sets of initial data $(r, \Omega^2, \bar{f}_{\mathrm{in}}, \bar{f}_{\mathrm{out}})|_{u=u_*}$ and $(r^*_/, (\Omega^*_/)^2, \bar{f}^*_{\mathrm{in}/}, \bar{f}^*_{\mathrm{out}/})$ on $\{u = u_*\} \cap \{u_* \le v < u_* + v_{0\varepsilon}\}$. The maximal *past* development of $(r, \Omega^2, \bar{f}_{\mathrm{in}}, \bar{f}_{\mathrm{out}})|_{u=u_*}$ (see the remark below Theorem 3.4) coincides with $(\mathcal{W}_{u_*}; r, \Omega^2, \bar{f}_{\mathrm{in}}, \bar{f}_{\mathrm{out}})$ when restricted on $\{u \ge 0\}$ and, in view of (A-74) and (A-75), satisfies

$$\begin{aligned}\sup_{\mathcal{W}_{u_*}}\bigg\{\Big|\log\Big(\frac{\Omega^2}{1-\frac{1}{3}\Lambda r^2}\Big)\Big|+\Big|\log\Big(\frac{2\partial_v r}{1-2m/r}\Big)\Big|+\Big|\log\Big(\frac{1-2m/r}{1-\frac{1}{3}\Lambda r^2}\Big)\Big|+\sqrt{-\Lambda}|\tilde{m}|\bigg\}& \\ +\sup_{\bar{u}}\int_{\{u=\bar{u}\}\cap\mathcal{W}_{u_*}} rT_{vv}\,dv+\sup_{\bar{v}}\int_{\{v=\bar{v}\}\cap\mathcal{W}_{u_*}} rT_{uu}\,du& \\ \le 4(h_1(\varepsilon))^{-3}\exp(-(h_0(\varepsilon))^{-4}).&\end{aligned} \tag{A-81}$$

Therefore, in view of (A-81), (A-79) and (A-80), Theorem 3.6 applied for the past development of $(r, \Omega^2, \bar{f}_{\mathrm{in}}, \bar{f}_{\mathrm{out}})|_{u=u_*}$ on $\mathcal{W}_{u_*}$ (see the remark below Theorem 3.6) implies that the maximal past development $(\mathcal{U}^*; r^*, (\Omega^*)^2, \bar{f}^*_{\mathrm{in}}, \bar{f}^*_{\mathrm{out}})$ of $(r^*_/, (\Omega^*_/)^2, \bar{f}^*_{\mathrm{in}/}, \bar{f}^*_{\mathrm{out}/})$ satisfies

$$\mathcal{W}_{u_*} \subset \mathcal{U}^* \tag{A-82}$$

and

$$\sup_{\mathcal{W}_{u_*}}\Bigg\{\bigg|\log\bigg(\frac{\Omega^2}{1-\frac{1}{3}\Lambda r^2}\bigg)-\log\bigg(\frac{(\Omega^*)^2}{1-\frac{1}{3}\Lambda (r^*)^2}\bigg)\bigg|+\bigg|\log\bigg(\frac{2\partial_v r}{1-2m/r}\bigg)-\log\bigg(\frac{2\partial_v r^*}{1-2m^*/r^*}\bigg)\bigg|$$
$$+\bigg|\log\bigg(\frac{1-2m/r}{1-\frac{1}{3}\Lambda r^2}\bigg)-\log\bigg(\frac{1-2m^*/r^*}{1-\frac{1}{3}\Lambda (r^*)^2}\bigg)\bigg|+\sqrt{-\Lambda}|\tilde{m}-\tilde{m}^*|\Bigg\}$$
$$+\sup_{\bar{u}}\int_{\{u=\bar{u}\}\cap\mathcal{W}_{u_*}}|rT_{vv}-r^*(T_{vv})^*|\,dv+\sup_{\bar{v}}\int_{\{v=\bar{v}\}\cap\mathcal{W}_{u_*}}|rT_{uu}-r^*(T_{uu})^*|\,du$$
$$\leq (h_1(\varepsilon))^3. \tag{A-83}$$

Thus, the proof of the lemma concludes by setting

$$(r'_/, (\Omega'_/)^2, \bar{f}'_{\text{in}/}, \bar{f}'_{\text{out}/}) \doteq (r^*, (\Omega^*)^2, \bar{f}^*_{\text{in}}, \bar{f}^*_{\text{out}})|_{u=0}. \qquad \square$$

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WEAK SOLUTIONS TO THE QUATERNIONIC MONGE–AMPÈRE EQUATION

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We solve the Dirichlet problem for the quaternionic Monge–Ampère equation with a continuous boundary data and the right-hand side in L^p for $p > 2$. This is the optimal bound on p. We prove also that the local integrability exponent of quaternionic plurisubharmonic functions is 2, which turns out to be less than an integrability exponent of the fundamental solution.

1. Introduction

Pluripotential theory, initiated in the seminal papers [Bedford and Taylor 1976; 1982], has become a powerful tool for solving problems in complex analysis and geometry. It has been generalized in many directions in the last decade. The most general setting is that of calibrated geometries, which were extensively studied in a long series of papers by Harvey and Lawson [2009b]. Even before that the basics of pluripotential theory in $\mathbb{H}^n$ were recreated in [Alesker 2003b], and more generally on hypercomplex manifolds in [Alesker and Verbitsky 2006]. In this paper we wish to concentrate on the flat space $\mathbb{H}^n$.

The short historical overview is as follows. Quaternionic plurisubharmonic functions in $\mathbb{H}^n$ and their basic properties were investigated in [Alesker 2003b]. Inspired by [Bedford and Taylor 1976], Alesker developed there the foundations of pluripotential theory in the quaternionic setting showing among other things that a quaternionic Monge–Ampère operator defined for smooth functions as the Moore determinant [1922] of a quaternionic Hessian can be extended to the class of continuous functions. In [Alesker 2003a] he solved the Dirichlet problem in a quaternionic strictly pseudoconvex domain $\Omega \subset \mathbb{H}^n$ with a continuous boundary data and the Monge–Ampère mass continuous up to the boundary. Only recently Wan [2020] obtained another result in this direction. Following the approach of [Kołodziej 1995; 2005] she proved that the Dirichlet problem admits a bounded solution provided the right-hand side is a finite Borel measure and a subsolution to the problem exists. Motivated by reasoning presented in [Cegrell and Persson 1992] and using comparison of real and quaternionic Monge–Ampère operators she showed existence of continuous solutions to the Dirichlet problem for densities in L^q, $q \geq 4$. To sum up, the strongest known result concerning existence of a continuous solution to the Dirichlet problem with a degenerate right-hand side is as follows.

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Keywords: Monge–Ampère equation, pluripotential theory, quaternionic plurisubharmonic functions.

Theorem. *Suppose $\Omega \subset \mathbb{H}^n$ is a quaternionic strictly pseudoconvex domain and $f \in L^q(\Omega)$ for $q \geq 4$ is a nonnegative function. Then the Dirichlet problem*

$$\begin{cases} u \in \mathcal{QPSH}(\Omega) \cap C(\bar{\Omega}), \\ (\partial\partial_J u)^n = f\Omega_n, \\ u_{|\partial\Omega} = \phi \in C(\partial\Omega) \end{cases}$$

has a unique solution.

The regularity of solutions (except for a ball, which was discussed earlier in [Alesker 2003a]) was proven in [Zhu 2016]. More precisely using the ideas presented in [Caffarelli et al. 1985], he proved the following result.

Theorem. *For a quaternionic strictly pseudoconvex domain $\Omega \subset \mathbb{H}^n$, $f \in C^\infty(\bar{\Omega} \times \mathbb{R})$ a positive function such that f_x is nonnegative on $\bar{\Omega} \times \mathbb{R}$ and $\phi \in C^\infty(\partial\Omega)$, the Dirichlet problem*

$$\begin{cases} u \in \mathcal{QPSH}(\Omega) \cap C^\infty(\bar{\Omega}), \\ \det\left(\dfrac{\partial^2 u(q)}{\partial\bar{q}_\alpha \partial q_\beta}\right)_{\alpha,\beta\in\{1,\dots,n\}} = f(q, u(q)) \quad \textit{in } \Omega, \\ u_{|\partial\Omega} = \phi \end{cases}$$

has a unique smooth solution.

In the meantime quaternionic pluripotential theory was further developed in [Wan and Zhang 2015; Wan and Kang 2017; Wan and Wang 2017], of which we will make an extensive use. Contents of those papers will be discussed below in more detail. For results concerning Dirichlet problems in this more general approach, one can consult [Harvey and Lawson 2009a] for the flat case, [Harvey and Lawson 2011] for manifolds and [Harvey and Lawson 2019] for a degenerate case.

In this note we are interested in finding weak solutions to the Dirichlet problem for the quaternionic Monge–Ampère operator in $\mathbb{H}^n$ with a more degenerate right-hand side and a continuous boundary data. It turns out to be possible whenever densities are in L^p for $p > 2$ and the exponent is optimal as we show. To do that we follow the approach of [Kołodziej 1996; 1998]. Probably the most interesting results are these which actually allow us to apply his method of proof. Among them is comparison of a quaternionic capacity and volume (Lebesgue measure). We prove it in the quaternionic setting, coupling two things. The first is the trick of Dinew and Kołodziej [2014] which allowed them to show similar comparison for the capacity related to a complex Hesssian equation in $\mathbb{C}^n$. It reduces to noting that although plurisubharmonic functions are rare among m-subharmonic ones still they realize this m-Hessian capacity. The second is a fact that is interesting in its own right, namely the comparison of complex and quaternionic Monge–Ampère operators. To our knowledge it was not known or used before and relies on the observation that the Moore determinant of a hyperhermitian matrix is in fact the Pfaffian of an associated complex matrix. Afterwards we obtain an L^∞ estimate for the solutions. The last step before proving the main theorem is stability of solutions in terms of their densities and boundary data, but here the proofs are more standard. All of this is done in Section 4. In Section 3 we discuss the problem of finding the local integrability exponent for quaternionic plurisubharmonic functions. The proof of the main theorem there is inspired by the one presented in [Hörmander 1994] for plurisubharmonic functions in $\mathbb{C}^n$. It turns out that the class of quaternionic plurisubharmonic functions exhibits an unusual property in this context; namely, the integrability exponent of a general function is 2, which is smaller than $2n$ occurring for a fundamental solution. This phenomenon can be excluded assuming boundedness of the function near the boundary of a domain, which is proven in Section 4.

2. Preliminaries

General references for quaternionic linear algebra and basic properties of quaternionic plurisubharmonic functions are [Alesker 2003a; 2003b; Alesker and Verbitsky 2006], while for quaternionic pluripotential theory see [Wan and Zhang 2015; Wan and Kang 2017; Wan and Wang 2017]. Let us fix the notation for an algebra of quaternions

$$\mathbb{H} = \{x_0 + x_1\mathfrak{i} + x_2\mathfrak{j} + x_3\mathfrak{k} \mid x_0, x_1, x_2, x_3 \in \mathbb{R}\},$$

where $\mathfrak{i}, \mathfrak{j}, \mathfrak{k}$ satisfy quaternionic relations and we consider $\mathbb{H}^n$ as a *right* quaternionic module. With such a choice we denote by I, J, K the complex structures induced by $\mathfrak{i}, \mathfrak{j}, \mathfrak{k}$ when treating $\mathbb{H}^n$ as a flat hypercomplex manifold. We introduce two coordinate systems,

$$\mathbb{H}^n \ni (q_i)_{i=0}^{n-1} \mapsto (z_j)_{j=0}^{2n-1} \in \mathbb{C}^{2n}$$

in such a way that $q_i = z_{2i} + \mathfrak{j} z_{2i+1}$ (this is a holomorphic chart for the complex structure I), and

$$\mathbb{H}^n \ni (q_i)_{i=0}^{n-1} \mapsto (x_j)_{j=0}^{4n-1} \in \mathbb{R}^{4n}$$

in such a way that $q_i = x_{4i} + x_{4i+1}\mathfrak{i} + x_{4i+2}\mathfrak{j} + x_{4i+3}\mathfrak{k}$ (this is just a real chart). It is easy to see that

$$z_j = x_{2j} + (-1)^j x_{2j+1}\mathfrak{i} \quad \text{for } j = 0, \dots, 2n-1.$$

As always ∂ and $\bar\partial$ are the canonical differential operators induced by the complex structure I and $d = \partial + \bar\partial$, $d^c = \mathfrak{i}(\bar\partial - \partial)$. We also introduce the twisted differential

$$\partial_J := J^{-1} \circ \bar\partial \circ J,$$

considered in [Alesker and Verbitsky 2010; Verbitsky 2002], which plays the role of $\bar\partial$ in the hypercomplex setting (e.g., the quaternionic Dolbeault or Salamon complex). For its properties we refer to the mentioned papers. Most notably we will only use

$$\partial_J : \Lambda_I^{k,0}(\mathbb{H}^n) \to \Lambda_I^{k+1,0}(\mathbb{H}^n), \quad \text{since } J : \Lambda_I^{p,q}(\mathbb{H}^n) \to \Lambda_I^{q,p}(\mathbb{H}^n),$$
$$\partial\partial_J + \partial_J\partial = 0,$$
$$\partial_J^2 = 0.$$

Later on it may happen frequently that we skip the subscript I and understand that $\Lambda^{k,0}(\mathbb{H}^n)$ come from considering degrees with respect to I. One can check that for a smooth function $u : \mathbb{H}^n \to \mathbb{R}$ the following formulas hold:

$$\partial u=\sum_{i=0}^{2n-1}(\partial_{z_i}u)\,dz_i,\qquad \bar\partial u=\sum_{i=0}^{2n-1}(\partial_{\bar z_i}u)\,d\bar z_i,$$

$$\partial_J u=\sum_{i=0}^{2n-1}(-1)^{i+1}(\partial_{\bar z_{i+(-1)^i}}u)\,dz_i,\quad \partial\bar\partial u=\sum_{i,j}(\partial_{z_i}\partial_{\bar z_j}u)\,dz_i\wedge d\bar z_j,$$

$$\begin{aligned}\partial\partial_J u&=\sum_{i,j}((-1)^{j+1}\partial_{z_i}\partial_{\bar z_{j+(-1)^j}}u)\,dz_i\wedge dz_j\\&=\sum_{i<j}((-1)^{j+1}\partial_{z_i}\partial_{\bar z_{j+(-1)^j}}u-(-1)^{i+1}\partial_{z_j}\partial_{\bar z_{i+(-1)^i}})\,dz_i\wedge dz_j.\end{aligned}$$

Suppose that $f:\mathbb{H}^n\to\mathbb{H}$ is a $\mathcal{C}^2$ function; we define the formal quaternionic derivatives

$$\frac{\partial f}{\partial\bar q_\alpha}=\frac{\partial f}{\partial x_{4\alpha}}+\mathfrak{i}\frac{\partial f}{\partial x_{4\alpha+1}}+\mathfrak{j}\frac{\partial f}{\partial x_{4\alpha+2}}+\mathfrak{k}\frac{\partial f}{\partial x_{4\alpha+3}},$$

$$\frac{\partial f}{\partial q_\alpha}=\overline{\frac{\partial\bar f}{\partial\bar q_\alpha}}=\frac{\partial f}{\partial x_{4\alpha}}-\frac{\partial f}{\partial x_{4\alpha+1}}\mathfrak{i}-\frac{\partial f}{\partial x_{4\alpha+2}}\mathfrak{j}-\frac{\partial f}{\partial x_{4\alpha+3}}\mathfrak{k}.$$

Let us observe that for any $f:\mathbb{H}^n\to\mathbb{H}$ of class $\mathcal{C}^2$

$$\frac{\partial}{\partial\bar q_\alpha}\frac{\partial}{\partial q_\beta}=\frac{\partial}{\partial q_\beta}\frac{\partial}{\partial\bar q_\alpha}.$$

Furthermore for a *real*-valued f one has

$$\frac{\partial}{\partial\bar q_\alpha}\frac{\partial}{\partial q_\alpha}f=\frac{\partial^2 f}{\partial x_{4\alpha}^2}+\frac{\partial^2 f}{\partial x_{4\alpha+1}^2}+\frac{\partial^2 f}{\partial x_{4\alpha+2}^2}+\frac{\partial^2 f}{\partial x_{4\alpha+3}^2}\quad\text{and}\quad\frac{\partial}{\partial\bar q_\alpha}\left(\frac{\partial}{\partial q_\beta}f\right)=\overline{\frac{\partial}{\partial\bar q_\beta}\frac{\partial}{\partial q_\alpha}f}.$$

As a consequence the matrix

$$\mathrm{Hess}(f,\mathbb{H})=\left(\frac{\partial^2 f}{\partial\bar q_\alpha\partial q_\beta}\right)_{\alpha,\beta\in\{1,\dots,n\}}$$

is a hyperhermitian matrix for any real-valued f. The following relations are known to hold for a smooth real-valued function u:

$$(dd^c u)^{2n}=2^{2n}(\mathrm{i}\,\partial\bar\partial u)^{2n}=4^{2n}(2n)!\,\det\left(\frac{\partial^2 u}{\partial_{z_i}\partial_{\bar z_j}}\right)\left(\frac{\mathrm{i}}{2}\,dz_0\wedge d\bar z_0\right)\wedge\cdots\wedge\left(\frac{\mathrm{i}}{2}\,dz_{2n-1}\wedge d\bar z_{2n-1}\right),$$

$$(\partial\partial_J u)^n=\frac{n!}{4^n}\det\left(\frac{\partial^2 u}{\partial_{\bar q_l}\partial_{q_k}}\right)(dz_0\wedge dz_1\wedge\cdots\wedge dz_{2n-2}\wedge dz_{2n-1}),$$

where in the last expression det is the Moore determinant, see [Moore 1922] for the original definition, of a hyperhermitian matrix. The last formula was computed in [Alesker and Verbitsky 2006] and, in a different setting, in [Wan and Wang 2017]. For further simplifications we introduce some canonical differential forms:

$$\omega_{2n} = \sum_{i=0}^{2n-1} \frac{\mathrm{i}}{2}\, dz_i \wedge d\bar{z}_i,$$

$$\beta_n = \sum_{i=0}^{n-1} dz_{2i} \wedge dz_{2i+1},$$

$$\Omega_n = \frac{\beta_n^n}{n!} = dz_0 \wedge dz_1 \wedge \cdots \wedge dz_{2n-2} \wedge dz_{2n-1}.$$

Since we will extensively use facts from pluripotential theory reproved in the quaternionic setting in [Wan and Zhang 2015; Wan and Wang 2017], it is desirable to compare differential operators ∂, ∂_J, which we use with their formally defined operators d_0, d_1. Those were introduced by D. Wan and W. Wang [2017]; we refer to their work for more details. They considered the "coordinates"

$$z^{j0} = x_{2j} + (-1)^{j+1} x_{2j+1}\mathrm{i} = \bar{z}_j,$$
$$z^{j1} = (-1)^{j+1} x_{2(j+(-1)^j)} + x_{2(j+(-1)^j)+1}\mathrm{i} = (-1)^{j+1} z_{j+(-1)^j}$$

for $j = 0, \ldots, 2n-1$ and the associated formal derivatives

$$\nabla_{j0} = \partial_{x_{2j}} + (-1)^j \partial x_{2j+1}\mathrm{i} = 2\partial_{\bar{z}_j},$$
$$\nabla_{j1} = (-1)^{j+1} \partial x_{2(j+(-1)^j)} - \partial x_{2(j+(-1)^j)+1}\mathrm{i} = (-1)^{j+1} 2\partial_{z_{j+(-1)^j}}.$$

Afterwards they fixed a complex basis $\omega^0, \ldots, \omega^{2n-1}$ of $\mathbb{C}^{2n} \approx \mathbb{C}^{2n*}$ and an associated one $\omega^I = \omega_{i_1} \wedge \cdots \wedge \omega_{i_k}$, for $I = (i_1, \ldots, i_k)$ such that $i_1 < \cdots < i_k$ belong to $\{0, \ldots, 2n-1\}$, of a complex exterior product $\Lambda^k(\mathbb{C}^{2n}) \approx \Lambda^k(\mathbb{C}^{2n*})$. Finally they defined operators

$$d_i : \Lambda^{k,0}(\mathbb{H}^n) \approx C^\infty(\mathbb{H}^n, \Lambda^k\mathbb{C}^{2n}) \to C^\infty(\mathbb{H}^n, \Lambda^{k+1}\mathbb{C}^{2n}) \approx \Lambda^{k+1,0}(\mathbb{H}^n)$$

for $i = 0, 1$ in the following way. Suppose that $F = \sum_I f_I \omega^I$; then

$$d_i F = \sum_{I,k \in \{0,\ldots,2n-1\}} (\nabla_{ki} f_I)\omega^k \wedge \omega^I.$$

From formulas for ∇_{ki} we obtain

$$d_0 F = \sum_{I,k \in \{0,\ldots,2n-1\}} (\nabla_{k0} f_I)\omega^k \wedge \omega^I = \sum_{I,k \in \{0,\ldots,2n-1\}} 2(\partial_{\bar{z}_k} f_I)\omega^k \wedge \omega^I,$$
$$d_1 F = \sum_{I,k \in \{0,\ldots,2n-1\}} (\nabla_{k1} f_I)\omega^k \wedge \omega^I = \sum_{I,k \in \{0,\ldots,2n-1\}} 2(-1)^{k+1} (\partial_{z_{k+(-1)^k}} f_I)\omega^k \wedge \omega^I.$$

Proposition 1. *For the basis $\omega^k = (-1)^k\, dz_{k+(-1)^k}$*

$$d_0 = 2\partial_J, \quad d_1 = -2\partial \quad \textit{and} \quad \Delta = d_0 d_1 = 4\partial\partial_J.$$

Proof. Let us recall that $\partial_J = J^{-1} \circ \bar{\partial} \circ J$ and one can check that J acts as

$$J(dz_{2i+1}) = d\bar{z}_{2i}, \quad J(dz_{2i}) = -d\bar{z}_{2i+1}, \quad \text{i.e.,} \quad J(dz_k) = (-1)^{k+1}\, d\bar{z}_{k+(-1)^k}.$$

As before, for $F = \sum_I f_I \omega^I$, we obtain

$$\partial F = \sum_{I,k\in\{0,\dots,2n-1\}} (\partial_{z_k} f_I)\, dz_k \wedge \omega^I,$$

$$\begin{aligned}\partial_J F &= J^{-1}\circ\bar\partial\Big(\sum_I f_I J(\omega^I)\Big) = J^{-1}\Big(\sum_{I,k\in\{0,\dots,2n-1\}} (\partial_{\bar z_k} f_I)\, d\bar z_k \wedge J(\omega^I)\Big)\\ &= J^{-1}\Big(\sum_{I,k\in\{0,\dots,2n-1\}} (\partial_{\bar z_{k+(-1)^k}} f_I)\, d\bar z_{k+(-1)^k} \wedge J(\omega^I)\Big)\\ &= \sum_{I,k\in\{0,\dots,2n-1\}} (\partial_{\bar z_{k+(-1)^k}} f_I) J^{-1}(d\bar z_{k+(-1)^k}) \wedge \omega^I\\ &= \sum_{I,k\in\{0,\dots,2n-1\}} (\partial_{\bar z_{k+(-1)^k}} f_I)(-1)^{k+1}\, dz_k \wedge \omega^I.\end{aligned}$$

This results in

$$\begin{aligned} d_0 F &= 2 \sum_{I,k\in\{0,\dots,2n-1\}} (\partial_{\bar z_k} f_I)\omega^k \wedge \omega^I\\ &= 2 \sum_{I,k\in\{0,\dots,2n-1\}} (-1)^k (\partial_{\bar z_k} f_I)\, dz_{k+(-1)^k} \wedge \omega^I\\ &= 2 \sum_{I,k\in\{0,\dots,2n-1\}} (-1)^{k+1} (\partial_{\bar z_{k+(-1)^k}} f_I)\, dz_k \wedge \omega^I = \mathbf{2}\partial_J F,\end{aligned}$$

$$\begin{aligned} d_1 F &= 2 \sum_{I,k\in\{0,\dots,2n-1\}} (-1)^{k+1} (\partial_{z_{k+(-1)^k}} f_I)\omega^k \wedge \omega^I\\ &= 2 \sum_{I,k\in\{0,\dots,2n-1\}} (-1) (\partial_{z_{k+(-1)^k}} f_I)\, dz_{k+(-1)^k} \wedge \omega^I = -\mathbf{2}\partial F. \qquad \square\end{aligned}$$

Remark 2. Let us just emphasize that the choosing of ∂, ∂_J over d_0, d_1 has a deeper meaning than just the conventional one. These are the natural intrinsic operators not only in $\mathbb{H}^n$ but on any hypercomplex manifold. In fact on an abstract hypercomplex manifold, quaternionic plurisubharmonic functions are defined only with their aid, see [Alesker and Verbitsky 2006], since the local chart definition is not possible due to nonintegrability of a generic hypercomplex structure, i.e., nonexistence of quaternionic charts.

From Proposition 1 it follows that we are able to use all results from [Wan and Zhang 2015; Wan and Kang 2017; Wan and Wang 2017] as well as those from [Alesker 2003a; 2003b; Alesker and Verbitsky 2006]. We just give here the necessary details and refer to the mentioned papers for more details. The quaternionic plurisubharmonic functions were defined in [Alesker 2003b].

Definition. Let Ω be a domain in $\mathbb{H}^n$. We call an upper semicontinuous function $f : \Omega \to \mathbb{R}$ (strictly) quaternionic plurisubharmonic, qpsh for short, if f restricted to any affine right quaternionic line intersected with Ω is (strictly) subharmonic as a function on a domain in $\mathbb{R}^4$. The set of all qpsh functions on Ω is denoted by $\mathcal{QPSH}(\Omega)$.

Remark 3. If we fix $t \in \{a\mathfrak{i}+b\mathfrak{j}+c\mathfrak{k} \mid a^2+b^2+c^2=1\}$ an imaginary unit and consider $\mathbb{H}^n$ as a complex vector space where multiplication by $\mathfrak{i}$ is given by a right multiplication by t then psh functions with respect to this complex structure are qpsh since quaternionic lines are complex 2-planes. We will use that remark only for $t=\mathfrak{i}$, i.e., only for $\mathbb{H}^n$ treated as $\mathbb{C}^{2n}$ via the chart introduced in the beginning of the section.

For a smooth function being qpsh is equivalent to $\partial\partial_J u \geq 0$ in a quaternionic sense. Let us elaborate. The cones of strongly positive $SP^{2k}(\Omega) \subset \Lambda_{\mathbb{R}}^{2k,0}(\Omega)$ and positive $\Lambda_{\mathbb{R},\geq 0}^{2k,0}(\Omega) \subset \Lambda_{\mathbb{R}}^{2k,0}(\Omega)$ forms were introduced in [Alesker and Verbitsky 2006]; see also [Verbitsky 2010] for a careful and extended treatment. Here $\Lambda_{\mathbb{R}}^{2k,0}(\Omega) \subset \Lambda^{2k,0}(\Omega)$ is the space of forms α such that $\overline{J(\alpha)}=\alpha$. To introduce them we firstly argue for a point, an element $\Omega_n \in \Lambda_{\mathbb{R}}^{2n,0}(\mathbb{H}^n \approx T_0\mathbb{H}^n)$ is chosen to be strongly positive and a convex combination of elements of the form $G^*(\Omega_k)$ for $G:\mathbb{H}^n \to \mathbb{H}^k$ a quaternionic linear map is strongly positive. When the reasoning is applied pointwise we obtain the notion of strong positivity for differential forms in Ω. As always the cone of positive elements is the dual one. We have mentioned above that $(\partial\partial_J u)^n$ agrees with Moore's determinant of a quaternionic Hessian $\mathrm{Hess}(u,\mathbb{H})$ for a smooth function, Alesker [2003a], motivated by [Bedford and Taylor 1976], showed that $(\partial\partial_J u)^n$ can be interpreted as a measure for continuous u and proved certain convergence for this operator. It is a cornerstone for having proper pluripotential theory. Later in [Wan and Wang 2017] the authors proved that $\partial\partial_J u$ is a positive current (where positivity is defined using the cone of strongly positive forms) for any qpsh function. More importantly they showed that like in the complex case, see [Bedford and Taylor 1982], one can define $(\partial\partial_J u)^n$ for any locally bounded u and treat it as a measure. From there one can recreate most of theorems which hold for psh functions. Among other things they showed weak convergence of this operator on decreasing sequences of qpsh functions and Chern–Levine–Nirenberg inequalities; see [Wan and Wang 2017]. In [Wan and Zhang 2015] the quaternionic relative capacity is introduced in the spirit of Bedford and Taylor. Let $K \subset \Omega$ be a compact set; then

$$\mathrm{cap}(K,\Omega)=\sup\left\{\int_K (\partial\partial_J u)^n \,\middle|\, u \in \mathcal{QPSH}(\Omega),\ 0 \leq u \leq 1\right\},$$

and this can be extended to Borel subsets as well. What is more, the authors proved quasicontinuity of qpsh functions and most notably the comparison principle, which is probably the most powerful tool in pluripotential theory. The statement is exactly as we know it in the complex case, but we recall it for the reader's convenience.

Theorem [Wan and Zhang 2015]. *Let $u, v \in \mathcal{QPSH}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$. If, for any $\xi \in \partial\Omega$,*

$$\liminf_{\xi \leftarrow q \in \partial\Omega}(u(q)-v(q)) \geq 0$$

then

$$\int_{\{u<v\}}(\partial\partial_J v)^n \leq \int_{\{u<v\}}(\partial\partial_J u)^n.$$

In particular if $(\partial\partial_J v)^n \geq (\partial\partial_J u)^n$ as measures then $u \geq v$ in Ω.

Finally they characterize maximality of a bounded qpsh function in terms of the vanishing of its Monge–Ampère mass. Here we mean that $u \in \mathcal{QPSH}(\Omega)$ is maximal if it is above any other qpsh function on compact sets $K \subset \Omega$ provided the values of both functions are the same on ∂K.

3. Local integrability of qpsh functions

In this section we address the question of local integrability of qpsh functions in a domain $\Omega \subset \mathbb{H}^n$. For psh functions it is well known that they are locally integrable with any exponent. The proof of the proposition below is inspired by the presentation in [Hörmander 1994].

Proposition 4. *Suppose $u \in \mathcal{QPSH}(\Omega)$ is such that $u \not\equiv -\infty$. Then $u \in L^p_{\text{loc}}(\Omega)$ for any $p < 2$ and the bound on p is optimal. Additionally, if $u_j \not\equiv -\infty$ is a sequence of qpsh functions converging in $L^1_{\text{loc}}(\Omega)$ to some u, necessarily belonging to $\mathcal{QPSH}(\Omega)$, then convergence holds in $L^p_{\text{loc}}(\Omega)$ for any $p < 2$.*

Proof. Suppose without loss of generality that $u \le 0$ in a neighborhood of a quaternionic polyball $P(0, 1)$ of radius 1 centered at 0 contained in Ω, that $u(0) > -\infty$ and fix $p < 2$. Let us deal firstly with the case $n = 1$. From the Riesz representation theorem, see Theorem 3.3.6 in [Hörmander 1994],

$$u(q) = h(g) + \int_{\|\xi\|<1} G(q, \xi)\, d\mu(\xi)$$

for some nonpositive harmonic function h in $B(0, 1) := B_1$, nonnegative Borel measure μ and Green's function

$$G(q, \xi) = -\frac{1}{\|q - \xi\|^2} + \frac{1}{\|(q - \xi/|\xi|^2)|\xi|\|^2}.$$

By Harnack's inequality, see Theorem 3.1.7 in [Hörmander 1994], for any $\|q\| < \frac{1}{2}$ we have

$$0 \le -h(q) \le \frac{1 + \|q\|}{(1 - \|q\|)^3}(-h(0)) \le 12(-h(0)).$$

This shows that

$$\|h\|_{L^p(B(0,\frac{1}{2}))} \le C_p |h(0)|$$

for a constant C_p, depending only on $p < 2$, which we may still need to increase (see below).

For estimating the second component of the decomposition of u, let us introduce the notation

$$H(q, \xi) = -G(q, \xi) \ge 0;$$

for $\xi = 0$ we have

$$H(q, 0) = \frac{1}{\|q\|^2} - 1.$$

We consider two cases depending on whether ξ is close to the center or to the boundary of B_1.

In the first case, say when $\|\xi\| \le \frac{3}{4}$, we use the estimate

$$0 \le H(q, \xi) \le \frac{1}{\|q - \xi\|^2}$$

for any q and ξ; consequently

$$\begin{aligned}\left(\int_{\|q\|<\frac{1}{2}} (H(q, \xi))^p\, d\mathcal{L}^4(q)\right)^{\frac{1}{p}} &\le \left(\int_{\|q\|<\frac{1}{2}} \frac{1}{\|q - \xi\|^{2p}}\, d\mathcal{L}^4(q)\right)^{\frac{1}{p}} \\ &\le \left(\int_{\|q\|<\frac{5}{4}} \frac{1}{\|q\|^{2p}}\, d\mathcal{L}^4(q)\right)^{\frac{1}{p}} \le C'_p\left(\frac{1}{\|\xi\|^2} - 1\right)\end{aligned}$$

for a constant C_p' independent of ξ and depending only on $p<2$, since the expression $1/\|\xi\|^2-1$ is bounded from below for $\|\xi\|\le\frac{3}{4}$.

In the second case, say when $\|\xi\|\ge\frac{3}{4}$, we note that for any fixed ξ the function $H(\,\cdot\,,\xi)$ is nonnegative and harmonic in $B_{\frac{3}{4}}$. Applying Harnack's inequality for each fixed ξ we obtain that for all $\|\xi\|\ge\frac{3}{4}$ and for all $\|q\|<\frac{3}{4}$

$$0\le H(q,\xi)\le\Big(\frac{3}{4}\Big)^2\frac{\frac{3}{4}+\|q\|}{\big(\frac{3}{4}-\|q\|\big)^3}H(0,\xi);$$

hence for all $\|\xi\|\ge\frac{3}{4}$ and $\|q\|<\frac{1}{2}$

$$0\le H(q,\xi)\le 45\bigg(\frac{1}{\|\xi\|^2}-1\bigg).$$

To sum up we have proven that there exists a constant $C_p=\max\{C_p',45\}$, independent of ξ, such that for $\|\xi\|<1$

$$\|H(\,\cdot\,,\xi)\|_{L^p(B(0,\frac{1}{2}))}\le C_p\bigg(\frac{1}{\|\xi\|^2}-1\bigg).$$

From Minkowski's inequality and Minkowski's integral inequality we obtain

$$\begin{aligned}\|u\|_{L^p(B(0,\frac{1}{2}))}&\le\|h\|_{L^p(B(0,\frac{1}{2}))}+\bigg(\int_{\|q\|<\frac{1}{2}}\bigg|\int_{\|\xi\|<1}H(q,\xi)\,d\mu(\xi)\bigg|^p\,d\mathcal{L}^4(q)\bigg)^{\frac{1}{p}}\\&\le C_p|h(0)|+\int_{\|\xi\|<1}\bigg(\int_{\|q\|<\frac{1}{2}}H(q,\xi)^p\,d\mathcal{L}^4(q)\bigg)^{\frac{1}{p}}\,d\mu(\xi)\\&\le C_p\bigg(|h(0)|+\int_{\|\xi\|<1}\bigg(\frac{1}{\|\xi\|^2}-1\bigg)\,d\mu(\xi)\bigg)=C_p|u(0)|.\end{aligned}$$

Using Fubini's theorem and the estimate above, one obtains that in the case of $n\ge1$ we have

$$\|u\|_{L^p(P(0,\frac{1}{2}))}\le C_p^n|u(0)|.$$

To the end observe that Ω', the set of points in Ω in a neighborhood in which u is integrable with exponent p, is an open set by definition. It is closed by what we have just shown. This is so because if $q\in\overline{\Omega}'$ and $r>0$ is such that $P(q,3r)\Subset\Omega$ then we can find an element q' of Ω' within a distance of $\frac{r}{2}$ from q and a point $q''\in\Omega$ within a distance of $\frac{r}{2}$ from q' such that $u(q'')$ is finite. We note that $P(q'',2r)\Subset\Omega$; consequently u is integrable with the exponent p on $P(q'',r)$, and $q\in P(q'',r)$. What is more, Ω' is nonempty by the assumption $u\not\equiv-\infty$. The bound on p is optimal as the example of $-1/\|q_0\|^2$ in $\mathbb{H}^n$ for $n\ge1$ shows.

For the proof of the second assertion we note that the sequence u_j-u is bounded in $L^p_{\mathrm{loc}}(\Omega)$ for any $1\le p<2$. To prove this it is enough to show that the sequence u_j is bounded in $L^p_{\mathrm{loc}}(\Omega)$. Fix any point $q\in\Omega$ and let $r>0$ be such that $P(q,3r)\Subset\Omega$. We claim that L^p norms of u_j in $P\big(q,\frac{r}{2}\big)$ are bounded. Suppose to the contrary that they are not. Let us choose a subsequence j_k such that $\|u_{j_k}\|_{L^p(P(q,\frac{r}{2}))}\to\infty$. We know that the u_{j_k} are locally uniformly bounded from above, see Theorem 3.2.13 in [Hörmander

1994], so we may assume that u and the u_{j_k} are nonpositive in $P(q, 3r)$. It is possible to find a point q' within a distance of $\frac{r}{2}$ from q such that $u(q') > -\infty$ and $\limsup_{k\to\infty} u_{j_k}(q') = u(q')$ since both of these properties hold almost everywhere in Ω; see Theorem 3.2.13 in [Hörmander 1994]. We assume without loss of generality that $\lim_{k\to\infty} u_{j_k}(q') = u(q')$, for if not we take a subsequence again. In particular there exists $C > 0$ such that $u_{j_k}(q') > -C$ for any k. This together with the estimate we have proven shows that the sequence $\|u_{j_k}\|_{L^p(P(q',r))}$ is bounded. Because we also know that $P\big(q, \frac{r}{2}\big) \subset P(q', r)$, contradiction with $\|u_{j_k}\|_{L^p(P(q,\frac{r}{2}))} \to \infty$ is obtained. Fix $1 \le p < 2$ and observe that for any compact $K \subset \Omega$ we have

$$\begin{aligned}\int_K |u_j - u|^p \, d\mathcal{L}^{4n} &= \int_K |u_j - u|^{\frac{2-p}{2}} |u_j - u|^{\frac{3p-2}{2}} \, d\mathcal{L}^{4n} \\ &\le \left(\int_K |u_j - u|^{\frac{2-p}{2}\frac{2}{2-p}} \, d\mathcal{L}^{4n}\right)^{\frac{2-p}{2}} \left(\int_K |u_j - u|^{\frac{3p-2}{2}\frac{2}{p}} \, d\mathcal{L}^{4n}\right)^{\frac{p}{2}} \\ &= \left(\int_K |u_j - u| \, d\mathcal{L}^{4n}\right)^{\frac{2-p}{2}} \left(\int_K |u_j - u|^{3-\frac{2}{p}} \, d\mathcal{L}^{4n}\right)^{\frac{p}{2}}\end{aligned}$$

by Hölder's inequality. By the assumption the first term tends to zero, while second one is bounded since $1 \le 3 - \frac{2}{p} < 2$. This proves that the u_j tend to u in $L^p_{\mathrm{loc}}(\Omega)$ for any $1 \le p < 2$. □

Proposition 5 [Wan and Wang 2017]. *The function $f(q) = -1/\|q\|^2$ is a fundamental solution for the quaternionic Monge–Ampère operator in $\mathbb{H}^n$. More exactly*

$$(\partial\partial_J f)^n = \frac{2^n \pi^{2n} n!}{(2n)!} \delta_0.$$

We see that the fundamental solution to the quaternionic Monge–Ampère equation is in $L^p_{\mathrm{loc}}(\mathbb{H}^n)$ for any $p < 2n$, while a generic qpsh function is only for $p < 2$, which is in contrast with the case of psh functions.

4. Dirichlet problem for quaternionic Monge–Ampère equation

In this section we aim to solve the Dirichlet problem

$$\begin{cases} u \in \mathcal{QPSH}(\Omega) \cap C(\bar{\Omega}), \\ (\partial\partial_J u)^n = f\Omega_n, \\ u_{|\partial\Omega} = \phi \in C(\partial\Omega), \end{cases}$$

where $f \in L^q(\Omega)$ for $q > 2$ and $\Omega \Subset \mathbb{H}^n$ is a smoothly bounded, strictly quaternionic pseudoconvex domain, which is a global assumption for Ω in this section. Let us recall:

Definition. $\Omega \Subset \mathbb{H}^n$, a smoothly bounded domain, is strictly quaternionic pseudoconvex if there exists v, a smooth strictly qpsh function in a neighborhood of $\bar{\Omega}$, such that $v < 0$ in Ω, $v = 0$, but $\nabla v \neq 0$ on $\partial\Omega$.

Let us just mention that the Dirichlet problem for the complex Monge–Ampère equation with densities in L^p for $p > 1$ was solved in [Kołodziej 1996]. In fact he proved it for densities in appropriate Orlicz spaces being subspaces of L^1 and in particular cases reducing to L^p. For the real Monge–Ampère equation one can always solve the above problem for any density in L^1; see [Rauch and Taylor 1977].

The first goal is to compare complex and quaternionic Monge–Ampère operators. We start with smooth functions, in which case we have to compare complex and quaternionic Hessians, or rather their determinants to be precise.

Lemma 6. *For a smooth function $u : \Omega \to \mathbb{R}$ and any $l, k \in \{0, \dots, n-1\}$*

$$\begin{aligned}\partial_{\bar{q}_l}\partial_{q_k} u &= \partial_{\bar{q}_l}(\partial_{x_{4k}} u - \mathfrak{i}\partial_{x_{4k+1}} u - \mathfrak{j}\partial_{x_{4k+2}} u - \mathfrak{k}\partial_{x_{4k+3}} u)\\ &= \partial_{\bar{q}_k}(2\partial_{z_{2k}} u - 2\mathfrak{j}\partial_{\bar{z}_{2k+1}} u) = (2\partial_{\bar{z}_{2l}} + 2\mathfrak{j}\partial_{z_{2l+1}})(2\partial_{z_{2k}} u - 2\mathfrak{j}\partial_{\bar{z}_{2k+1}} u)\\ &= 4(\partial_{\bar{z}_{2l}}\partial_{z_{2k}} u + \partial_{z_{2l+1}}\partial_{\bar{z}_{2k+1}} u) + 4\mathfrak{j}(\partial_{\bar{z}_{2l+1}}\partial_{z_{2k}} u - \partial_{z_{2l}}\partial_{\bar{z}_{2k+1}} u).\end{aligned}$$

Let us recall that we distinguish the set $\mathcal{PSH}(\Omega)$ of plurisubharmonic functions in Ω by identifying $\mathbb{H}^n$ with $\mathbb{C}^{2n}$ via a chart introduced in Section 2.

Lemma 7. *For a function $u \in \mathcal{PSH}(\Omega) \cap C^2(\Omega) \subset \mathcal{QPSH}(\Omega)$ the following holds*:

$$\left(\det\left(\frac{\partial^2 u}{\partial_{\bar{q}_l}\partial_{q_k}}\right)\right)^2 \geq 4^{2n} \det\left(\frac{\partial^2 u}{\partial_{z_i}\partial_{\bar{z}_j}}\right).$$

Proof. Let us define

$$\begin{aligned}\operatorname{Hess}(u, \mathbb{C}) &= \left(\frac{\partial^2 u}{\partial_{z_i}\partial_{\bar{z}_j}}\right)_{i,j=0,\dots,2n-1},\\ \operatorname{Hess}(u, \mathbb{H}) &= \left(\frac{\partial^2 u}{\partial_{\bar{q}_l}\partial_{q_k}}\right)_{l,k=0,\dots,n-1}.\end{aligned}$$

Note that

$$\det\left(\frac{\partial^2 u}{\partial_{z_i}\partial_{\bar{z}_j}}\right)_{i,j=0,\dots,2n-1} = \det \operatorname{Hess}(u, \mathbb{C}) = \det \overline{\operatorname{Hess}(u, \mathbb{C})} = \det\left(\frac{\partial^2 u}{\partial_{\bar{z}_i}\partial_{z_j}}\right)_{i,j=0,\dots,2n-1}.$$

The last matrix is Hermitian positive since it is just $\operatorname{Hess}(u, \mathbb{C})^T$. If $\operatorname{Hess}(u, \mathbb{H}) = G + \mathfrak{j}H$ then we define

$$\psi(\operatorname{Hess}(u, \mathbb{H})) = \begin{pmatrix} G & -\overline{H} \\ H & \overline{G} \end{pmatrix}.$$

By Lemma 6 we obtain

$$\begin{aligned}\psi(\operatorname{Hess}(u, \mathbb{H})) &= 4\begin{pmatrix} [\partial_{\bar{z}_{2l}}\partial_{z_{2k}} u + \partial_{z_{2l+1}}\partial_{\bar{z}_{2k+1}} u]_{l,k} & [-\partial_{z_{2l+1}}\partial_{\bar{z}_{2k}} u + \partial_{\bar{z}_{2l}}\partial_{z_{2k+1}} u]_{l,k} \\ [\partial_{\bar{z}_{2l+1}}\partial_{z_{2k}} u - \partial_{z_{2l}}\partial_{\bar{z}_{2k+1}} u]_{l,k} & [\partial_{z_{2l}}\partial_{\bar{z}_{2k}} u + \partial_{\bar{z}_{2l+1}}\partial_{z_{2k+1}} u]_{l,k} \end{pmatrix}\\ &= 4\begin{pmatrix} [\partial_{\bar{z}_{2l}}\partial_{z_{2k}} u]_{l,k} & [\partial_{\bar{z}_{2l}}\partial_{z_{2k+1}} u]_{l,k} \\ [\partial_{\bar{z}_{2l+1}}\partial_{z_{2k}} u]_{l,k} & [\partial_{\bar{z}_{2l+1}}\partial_{z_{2k+1}} u]_{l,k} \end{pmatrix} + 4\begin{pmatrix} [\partial_{z_{2l+1}}\partial_{\bar{z}_{2k+1}} u]_{l,k} & [-\partial_{z_{2l+1}}\partial_{\bar{z}_{2k}} u]_{l,k} \\ [-\partial_{z_{2l}}\partial_{\bar{z}_{2k+1}} u]_{l,k} & [\partial_{z_{2l}}\partial_{\bar{z}_{2k}} u]_{l,k} \end{pmatrix}.\end{aligned}$$

Following [Cegrell and Persson 1992] we introduce three matrices

$$A = [\partial_{\bar{z}_{2l}}\partial_{z_{2k}} u]_{l,k}, \quad B = [\partial_{\bar{z}_{2l+1}}\partial_{z_{2k}} u]_{l,k}, \quad C = [\partial_{\bar{z}_{2l+1}}\partial_{z_{2k+1}} u]_{l,k}.$$

Under this notation

$$\psi(\operatorname{Hess}(u, \mathbb{H})) = 4\begin{pmatrix} A & \overline{B}^T \\ B & C \end{pmatrix} + 4\begin{pmatrix} \overline{C} & -\overline{B} \\ -B^T & \overline{A} \end{pmatrix}.$$

Note that

$$\begin{pmatrix} A & \bar{B}^T \\ B & C \end{pmatrix}$$

is the conjugate of a Hessian of u with respect to the coordinates $z_0, \dots, z_{2n-2}, z_1, \dots, z_{2n-1}$, so it is Hermitian positive as well. Moreover

$$\det\left(\frac{\partial^2 u}{\partial_{z_i}\partial_{\bar{z}_j}}\right) = \det\begin{pmatrix} A & \bar{B}^T \\ B & C \end{pmatrix}.$$

Consider the matrix

$$I = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

with the inverse

$$I^{-1} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

and determinant equal to 1. Note that

$$I\begin{pmatrix} \bar{C} & -\bar{B} \\ -B^T & \bar{A} \end{pmatrix} I^{-1} = \begin{pmatrix} \bar{A} & B^T \\ \bar{B} & \bar{C} \end{pmatrix},$$

and the last matrix is the conjugate of the one just shown to be Hermitian positive so as such is also Hermitian positive. Consequently

$$\begin{pmatrix} \bar{C} & -\bar{B} \\ -B^T & \bar{A} \end{pmatrix}$$

is positive, being similar to one of that kind. Now we use the equality between Moore's determinant of a matrix M and the Pfaffian of an associated complex matrix $\psi(M)$, as proved in [Dyson 1970], which results in

$$\begin{aligned}\left(\det\left(\frac{\partial^2 u}{\partial_{\bar{q}_l}\partial_{q_k}}\right)\right)^2 = \det\psi(\mathrm{Hess}(u, \mathbb{H})) &= 4^{2n}\det\left(\begin{pmatrix} A & \bar{B}^T \\ B & C \end{pmatrix} + \begin{pmatrix} \bar{C} & -\bar{B} \\ -B^T & \bar{A} \end{pmatrix}\right) \\ &\geq 4^{2n}\det\begin{pmatrix} A & \bar{B}^T \\ B & C \end{pmatrix} = 4^{2n}\det\left(\frac{\partial^2 u}{\partial_{z_i}\partial_{\bar{z}_j}}\right),\end{aligned}$$

as we desired to prove. □

Having this, the announced comparison of quaternionic and complex Monge–Ampère operators for nonsmooth functions follows from the standard approximation procedure as presented in the proof below. Real and quaternionic Monge–Ampère operators were compared in [Wan 2020].

Theorem 8. *Let $u \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$ satisfy the equation*

$$(dd^c u)^{2n} = f^2 4^{2n} \omega_{2n}^{2n}$$

for some nonnegative $f \in L^p(\Omega)$, $p > 2$. Then

$$(\partial\partial_J u)^n \geq f\Omega_n^n.$$

Proof. Since the property is local we may assume that Ω is strictly pseudoconvex, otherwise we argue as below but for some ball contained in Ω. Approximate f by a sequence of smooth positive functions f_i in L^p norm and u uniformly by a sequence of smooth functions ϕ_i on $\partial\Omega$. Let us solve the family of Dirichlet problems

$$\begin{cases} u_i \in \mathcal{PSH}(\Omega) \cap C^\infty(\bar{\Omega}), \\ (dd^c u_i)^{2n} = f_i^2 4^{2n} \omega_{2n}^{2n}, \\ u_i = \phi_i \quad \text{on } \partial\Omega, \end{cases}$$

which is possible due to [Caffarelli et al. 1985]. Observe that the u_i converge uniformly to u due to the stability of solutions in L^q, $q > 1$, for the complex Monge–Ampère equation; see [Kołodziej 1996; Dinew and Kołodziej 2014]. From Lemma 7

$$(\partial\partial_J u_i)^n = \frac{1}{4^n} \det\left(\frac{\partial^2 u_i}{\partial_{\bar{q}_l}\partial_{q_k}}\right)\Omega_n^n \geq \sqrt{\det\left(\frac{\partial^2 u_i}{\partial_{z_m}\partial_{\bar{z}_n}}\right)} = f_i \Omega_n^n$$

as measures. The right-hand sides converge as measures to $f\Omega_n^n$ and the left ones converge to $(\partial\partial_J u)^n$ since convergence of u_i is uniform, see [Wan and Wang 2017], completing the proof. □

We are going to prove an inequality between the volume and quaternionic capacity which was an essential component of Kołodziej's proof of solvability of the complex Monge–Ampère equation for densities in appropriate Orlicz spaces; see [Kołodziej 1996; 2005]. Similar inequality for the capacity associated to a complex m-Hessian equation was proven in [Dinew and Kołodziej 2014] with the use of an observation that psh functions, although an extremal example of m-subharmonic ones, still realize the m-Hessian capacity. Here we couple that trick with a comparison of quaternionic and complex Monge–Ampère operators proved in Theorem 8.

Lemma 9. *For a fixed $p \in (1, 2)$ there exists a constant $C(p, R)$ such that for any $\Omega \subset B(0, R)$ and $K \Subset \Omega$*

$$\mathcal{L}^{4n}(K) \leq C(p, R)\, \mathrm{cap}^p(K, \Omega).$$

Proof. Suppose that $\mathcal{L}(K) \neq 0$; otherwise there is nothing to prove. Take any $\epsilon \in \left(0, \frac{1}{2}\right)$ and consider $f = \mathcal{L}(K)^{2\epsilon-1}\chi_K$. Let us solve the Dirichlet problem

$$\begin{cases} u \in \mathcal{PSH}(B) \cap C(\bar{B}), \\ (dd^c u)^{2n} = f 4^{2n} \omega_{2n}^{2n}, \\ u = 0 \quad \text{on } \partial B, \end{cases}$$

which is possible due to [Cegrell 1984]. By Theorem 8 the quaternionic Monge–Ampère operator of the solution u satisfies

$$(\partial\partial_J u)^n \geq \sqrt{f}\,\Omega_n^n.$$

Taking $q = 1 + \epsilon$, one checks that

$$\int_B f^q (4^{2n}(2n)!)^q \, d\mathcal{L}^{4n} = (4^{2n}(2n)!)^q \mathcal{L}(K)^{(2\epsilon-1)(1+\epsilon)+1} = (4^{2n}(2n)!)^q \mathcal{L}(K)^{2\epsilon^2+\epsilon} \leq (4^{2n}(2n)!)^2 R^{4n};$$

i.e., the L^q norm of f is bounded by a quantity depending only on R. By Kołodziej's L^∞ estimate [1996; 1998], there exists a constant $c(\epsilon, R)$ such that

$$\|u\|_{L^\infty(B)} \le \frac{1}{c(\epsilon, R)}.$$

Put $v = c(\epsilon, R)u$; then since v is a qpsh function such that $-1 \le v \le 0$

$$\mathrm{cap}(K, \Omega) \ge \int_K (\partial\partial_J v)^n \ge n!\, c(\epsilon, R)^n (\mathcal{L}^{4n}(K))^{\frac{2\epsilon+1}{2}}$$

and consequently

$$\left(\frac{1}{n!\, c(\epsilon, R)^n}\right)^{\frac{2}{2\epsilon+1}} \mathrm{cap}^{\frac{2}{2\epsilon+1}}(K, \Omega) \ge \mathcal{L}^{4n}(K).$$

This gives the claim since when ϵ varies in $\left(0, \frac{1}{2}\right)$ the exponent $\frac{2}{2\epsilon+1}$ varies in $(1, 2)$. □

In the previous section we have proven that any qpsh function belongs to L^p for $p < 2$ locally and that this is the optimal exponent. The lemma below gives the estimates on capacity and volume for sublevel sets of certain qpsh functions. In particular it shows that in the case of $u \in \mathcal{QPSH}(\Omega)$ bounded near the boundary of Ω the local integrability of $|u|^p$ is ensured for $p < 2n$. Again this bound is optimal as the example of $-1/\|q\|^2$ shows.

Lemma 10. *Fix $p \in (1, 2)$. Let $u \in \mathcal{QPSH}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ be such that*

$$\liminf_{q \to q_0} (u(q) - v(q)) \ge 0$$

for any $q_0 \in \partial\Omega$ and some fixed $v \in \mathcal{QPSH}(\Omega) \cap C(\overline{\Omega})$. Then there exists a constant $C(p, \mathrm{diam}(\Omega))$ depending only on p and the diameter of Ω such that for $U(s) = \{u < v - s\} \Subset \Omega$

$$\mathrm{cap}(U(s), \Omega) \le \frac{\int_\Omega (\partial\partial_J u)^n}{s^n},$$

$$\mathcal{L}^{4n}(U(s)) \le C(p, \mathrm{diam}(\Omega)) \frac{\int_\Omega (\partial\partial_J u)^n}{s^{pn}}.$$

Proof. Take $\epsilon > 0$ and a compact set $K \subset U(s)$. By definition one can find $w \in \mathcal{QPSH}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ such that $-1 \le w \le 0$ and

$$\int_K (\partial\partial_J w)^n \ge \mathrm{cap}(K, \Omega) - \epsilon.$$

Due to the way we have chosen K and the comparison principle

$$\begin{aligned} \mathrm{cap}(K, \Omega) - \epsilon &\le \int_K (\partial\partial_J w)^n \le \int_{\{\frac{u}{s} < \frac{v}{s} - 1\}} (\partial\partial_J w)^n \le \int_{\{\frac{u}{s} < \frac{v}{s} + w\}} (\partial\partial_J w)^n \\ &\le \int_{\{\frac{u}{s} < \frac{v}{s} + w\}} \left(\partial\partial_J \left(\frac{v}{s} + w\right)\right)^n \le \frac{1}{s^n} \int_{\{\frac{u}{s} < \frac{v}{s} + w\}} (\partial\partial_J u)^n \le \frac{\int_\Omega (\partial\partial_J u)^n}{s^n}. \end{aligned}$$

Letting ϵ tend to 0 and taking the supremum over all compact sets K we obtain the first claim. The second one follows from Lemma 9. □

The next goal is to prove the a priori L^∞ estimate for continuous solutions of the Dirichlet problem. Firstly note that by Alesker's result [2003a] on the Dirichlet problem with continuous density and boundary value, and characterization of maximality of qpsh functions as in [Wan and Zhang 2015], we can find $v \in C(\overline{\Omega})$ solving

$$\begin{cases}(\partial\partial_J v)^n = 0,\\ v_{|\partial\Omega} = \phi \in C(\partial\Omega);\end{cases}$$

i.e., is the maximal qpsh function matching our boundary condition. For such a fixed v we set

$$U(s) = \{u < v - s\} \subset \Omega$$

and introduce the function

$$b(s) = (\operatorname{cap}(U(s), \Omega))^{\frac{1}{n}}.$$

Theorem 11. *There exists a constant* $C(q, \|f\|_{L^q(\Omega)}, \|\phi\|_{L^\infty(\partial\Omega)}, \operatorname{diam}(\Omega))$ *depending on* q, $\|f\|_{L^q(\Omega)}$, $\|\phi\|_{L^\infty(\partial\Omega)}$ *and* $\operatorname{diam}(\Omega)$ *such that any solution u of the Dirichlet problem*

$$\begin{cases}u \in \mathcal{QPSH}(\Omega) \cap C(\overline{\Omega}),\\ (\partial\partial_J u)^n = f\Omega_n,\\ u_{|\partial\Omega} = \phi \in C(\partial\Omega),\end{cases}$$

for $f \in L^q(\Omega)$ *and* $q > 2$, *satisfies* $\|u\|_{L^\infty(\Omega)} \le C$.

Proof. Take any $s > 0$, $t \in [0, 1]$ and $w \in \mathcal{QPSH}(\Omega)$ such that $0 \le w \le 1$. Then

$$\begin{aligned} t^n \int_{U(s+t)} (\partial\partial_J w)^n &= \int_{U(s+t)} (\partial\partial_J (tw - t - s))^n = \int_{\{u<v-s-t\}} (\partial\partial_J (tw - t - s))^n \\ &\le \int_{\{u<v-s+tw-t\}} (\partial\partial_J (tw - t - s))^n \\ &\le \int_{\{u<v-s+tw-t\}} (\partial\partial_J (v + tw - t - s))^n \\ &\le \int_{\{u<v-s+tw-t\}} (\partial\partial_J u)^n \le \int_{\{u<v-s\}} (\partial\partial_J u)^n = \int_{U(s)} (\partial\partial_J u)^n, \end{aligned}$$

due to inclusions of appropriate sets, superadditivity and the comparison principle. To conclude

$$t^n (b(s+t))^n \le \int_{U(s)} (\partial\partial_J u)^n.$$

Estimating the right-hand side gives

$$\begin{aligned} \int_{U(s)} (\partial\partial_J u)^n = \int_{U(s)} f\Omega_n &\le \|f\|_{L^q(\Omega)} \left(\int_{U(s)} 1\, d\mathcal{L}^{4n}\right)^{\frac{1}{q'}} \\ &\le \|f\|_{L^q(\Omega)} C(p, \operatorname{diam}(\Omega)) (\operatorname{cap}(U(s), \Omega))^{\frac{p}{q'}} = \|f\|_{L^q(\Omega)} C(p, \operatorname{diam}(\Omega)) (b(s))^{n(1+\alpha(q))}, \end{aligned}$$

where we used Hölder's inequality and Lemma 9 and p depends only on q', which is the conjugate of q and we choose it so that $\frac{p}{q'} > 1$. This reassembles to

$$tb(s+t) \le A(q, \|f\|_{L^q(\Omega)}, \operatorname{diam}(\Omega)) (b(s))^{1+\alpha(q)}$$

for any $s > 0$ and $t \in [0, 1]$.

We would like to apply the De Giorgi lemma, stated below, for the function b. Let us just note that Lemma 12(a) is satisfied since for $s_n \searrow s$ we have $U(s_n) \nearrow U(s)$ and under such an assumption $\operatorname{cap}(U(s_n), \Omega) \to \operatorname{cap}(U(s), \Omega)$; see [Wan and Kang 2017]. The condition (b) follows from the first assertion of Lemma 10 as well as the dependence of s_0 only on q, $\|f\|_{L^q(\Omega)}$ and $\operatorname{diam}(\Omega)$. Indeed, it was proven there that

$$b^{\alpha(q)}(s) = \operatorname{cap}(U(s), \Omega)^{\frac{\alpha(q)}{n}} \le \frac{\left(\int_\Omega (\partial\partial_J u)^n\right)^{\frac{\alpha(q)}{n}}}{s^{\alpha(q)}} = \frac{\|f\|_{L^1(\Omega)}^{\frac{\alpha(q)}{n}}}{s^{\alpha(q)}} \le \frac{c(q, \|f\|_{L^q(\Omega)}, \operatorname{diam}(\Omega))}{s^{\alpha(q)}},$$

so surely $s_0 \le (2Ac)^{\frac{1}{\alpha(q)}}$ and this estimate depends only on q, $\|f\|_{L^q(\Omega)}$ and $\operatorname{diam}(\Omega)$. By the De Giorgi lemma there exists $S(q, \|f\|_{L^q(\Omega)}, \operatorname{diam}(\Omega))$ such that $b(s) = 0$ for any $s > S(q, \|f\|_{L^q(\Omega)}, \operatorname{diam}(\Omega))$. This together with Lemma 9 gives our claim since then

$$\|u\|_{L^\infty} \le \sup|\phi| + S(q, \|f\|_{L^q(\Omega)}, \operatorname{diam}(\Omega)) = C(q, \|f\|_{L^q(\Omega)}, \|\phi\|_{L^\infty(\partial\Omega)}, \operatorname{diam}(\Omega)). \qquad \square$$

Lemma 12 (De Giorgi lemma [Phong et al. 2012]). *Let $f : R_+ \to R_+$ satisfy the following conditions*:

(a) *f is right-continuous.*

(b) *f decreases to* 0.

(c) *There exist positive constants α, A_α so that, for all $s \ge 0$ and all $0 \le r \le 1$, we have*

$$r f(s + r) \le A_\alpha f(s)^{1+\alpha}.$$

Then there exists s_∞, depending only on α, A_α and the smallest value s_0 for which we have $f(s_0)^\alpha \le (2A_\alpha)^{-1}$, so that $f(s) = 0$ for $s > s_\infty$. In fact, we can take

$$s_\infty = s_0 + 2A_\alpha(1 - 2^{-\alpha})^{-1} f(s_0)^\alpha.$$

The L^∞ estimate allows us to prove the stability of solutions to the Dirichlet problem in terms of densities and boundary values. This will be needed for the proof of solvability of the Dirichlet problem but is of course a result that is interesting in its own right. As we were told by S. Dinew the idea of proving stability presented in Proposition 14 is due to N. C. Nguyen.

Lemma 13. *There exists a constant $C(q, \operatorname{diam}(\Omega))$ depending on q and* $\operatorname{diam}(\Omega)$ *such that any solution u of the Dirichlet problem*

$$\begin{cases} u \in \mathcal{QPSH}(\Omega) \cap C(\bar{\Omega}), \\ (\partial\partial_J u)^n = f\Omega_n, \\ u_{|\partial\Omega} = 0, \end{cases}$$

for $f \in L^q(\Omega)$ and $q > 2$, satisfies $\|u\|_{L^\infty(\Omega)} \le C(q, \operatorname{diam}(\Omega))\|f\|_{L^q(\Omega)}^{\frac{1}{n}}$.

Proof. Suppose that $\|f\|_{L^q(\Omega)} \ne 0$; otherwise there is nothing to prove. The function

$$v := \frac{u}{\|f\|_{L^q(\Omega)}^{\frac{1}{n}}}$$

solves the Dirichlet problem

$$\begin{cases} v \in \mathcal{QPSH}(\Omega) \cap C(\bar{\Omega}), \\ (\partial\partial_J v)^n = \dfrac{f}{\|f\|_{L^q(\Omega)}}\Omega_n, \\ v_{|\partial\Omega} = 0. \end{cases}$$

By Theorem 11 there exists a constant $C(q, \operatorname{diam}(\Omega)) := C(q, 1, 0, \operatorname{diam}(\Omega))$ such that $\|v\|_{L^\infty(\Omega)} \leq C(q, \operatorname{diam}(\Omega))$; this gives the claim. □

Proposition 14. *There exists a constant $C(q, \operatorname{diam}(\Omega))$ such that if u and v satisfy*

$$\begin{cases} u \in \mathcal{QPSH}(\Omega) \cap C(\bar{\Omega}), \\ (\partial\partial_J u)^n = f\Omega_n, \\ u_{|\partial\Omega} = \phi \in C(\partial\Omega) \end{cases}$$

and

$$\begin{cases} v \in \mathcal{QPSH}(\Omega) \cap C(\bar{\Omega}), \\ (\partial\partial_J v)^n = g\Omega_n, \\ v_{|\partial\Omega} = \psi \in C(\partial\Omega) \end{cases}$$

for $f, g \in L^q(\Omega)$, $q > 2$ then

$$\|u - v\|_{L^\infty(\Omega)} \leq \sup_{\partial\Omega} |\phi - \psi| + C(q, \operatorname{diam}(\Omega))\|f - g\|_{L^q(\Omega)}^{\frac{1}{n}}.$$

Proof. Consider a function w that is the solution of

$$\begin{cases} w \in \mathcal{QPSH}(\Omega) \cap C(\bar{\Omega}), \\ (\partial\partial_J w)^n = (f - g)_+\Omega_n, \\ w_{|\partial\Omega} = 0. \end{cases}$$

Note that on $\partial\Omega$ we have $w + v + \inf(\phi - \psi) \leq u$, while

$$(\partial\partial_J(w + v + \inf(\phi - \psi)))^n \geq (f - g)_+ + g \geq f = (\partial\partial_J u)^n.$$

The comparison principle gives $w + v + \inf(\phi - \psi) \leq u$ in $\bar{\Omega}$, which by Lemma 13 results in

$$\begin{aligned} u - v \geq w + \inf(\phi - \psi) &\geq -C(q, \operatorname{diam}(\Omega))\|(f - g)_+\|_{L^q(\Omega)}^{\frac{1}{n}} - \sup|\phi - \psi| \\ &\geq -C(q, \operatorname{diam}(\Omega))\|f - g\|_{L^q(\Omega)}^{\frac{1}{n}} - \sup|\phi - \psi|. \end{aligned}$$

The same reasoning gives

$$v - u \geq -C(q, \operatorname{diam}(\Omega))\|f - g\|_{L^q(\Omega)}^{\frac{1}{n}} - \sup|\phi - \psi|.$$

This reassembles to our claim. □

Remark 15. Equicontinuity of a family of functions

$$\mathcal{P}(q, c_0, \phi) = \left\{ u \in \mathcal{QPSH}(\Omega) \cap C(\bar{\Omega}) \;\middle|\; (\partial\partial_J u)^n \in L^q(\Omega),\ \int_\Omega (\partial\partial_J u)^n \leq c_0,\ u_{|\partial\Omega} = \phi \right\}$$

for a quaternionic strictly pseudoconvex domain Ω, $q > 2$, $c_0 > 0$ and $\phi \in C(\partial\Omega)$ follows easily from Proposition 14. In the complex case it was proven in [Kołodziej 2002].

Theorem 16. *The Dirichlet problem*

$$\begin{cases} u \in \mathcal{QPSH}(\Omega) \cap C(\bar{\Omega}), \\ (\partial\partial_J u)^n = f\,\Omega_n, \\ u_{|\partial\Omega} = \phi \in C(\partial\Omega) \end{cases}$$

in a smoothly bounded, quaternionic strictly pseudoconvex domain Ω for $f \in L^q(\Omega)$, $q > 2$, has a unique solution.

Proof. Uniqueness follows from the comparison principle. For solvability we take a sequence of continuous nonnegative functions f_i converging to f in $L^q(\Omega)$. Solving Dirichlet problems for them with our boundary condition, which is possible due to [Alesker 2003a], gives a sequence of continuous solutions u_i. Since, by Proposition 14, these solutions constitute a Cauchy sequence it follows that the u_i converge uniformly to some u. This is the solution we were looking for because of convergence of Monge–Ampère masses; see [Wan and Wang 2017]. □

The example below shows that the exponent 2 is optimal in the sense that for densities in $L^p(\Omega)$ with $p < 2$ solutions may not even be bounded.

Proposition 17. *Let $f(q) = \log(\|q\|)$. Then it belongs to $\mathcal{QPSH}(\mathbb{H}^n)$ and*

$$(\partial\partial_J f)^n = \frac{n!}{2\|q\|^{2n}}\Omega_n.$$

Proof. We compute for $f_\epsilon(q) = \frac{1}{2}\log(\|q\|^2 + \epsilon)$

$$\begin{aligned}
\partial\partial_J f_\epsilon &= \partial\Bigl(\sum_{i=0}^{2n-1} (-1)^{i+1}(\partial_{\bar{z}_{i+(-1)^i}} f_\epsilon)\, dz_i\Bigr) \\
&= \frac{1}{2}\partial\Bigl(\sum_{i=0}^{2n-1} (-1)^{i+1} \frac{z_{i+(-1)^i}}{\|q\|^2+\epsilon}\, dz_i\Bigr) \\
&= \frac{1}{2}\Bigl(\sum_{i=0,\, j=0}^{2n-1} (-1)^{i+1} \frac{\delta^j_{i+(-1)^i}(\|q\|^2+\epsilon) - z_{i+(-1)^i}\bar{z}_j}{(\|q\|^2+\epsilon)^2}\, dz_j \wedge dz_i\Bigr) \\
&= \frac{1}{2}\Bigl(\sum_{i>j}^{2n-1} \Bigl((-1)^{i+1} \frac{\delta^j_{i+(-1)^i}(\|q\|^2+\epsilon) - z_{i+(-1)^i}\bar{z}_j}{(\|q\|^2+\epsilon)^2} \\
&\qquad\qquad - (-1)^{j+1} \frac{\delta^i_{j+(-1)^j}(\|q\|^2+\epsilon) - z_{j+(-1)^j}\bar{z}_i}{(\|q\|^2+\epsilon)^2}\Bigr)\, dz_j \wedge dz_i\Bigr) \\
&= \frac{1}{2}\Bigl(\sum_{i>j}^{2n-1} \Bigl(\frac{2\delta^j_{i+(-1)^i}(\|q\|^2+\epsilon) + (-1)^i z_{i+(-1)^i}\bar{z}_j + (-1)^{j+1} z_{j+(-1)^j}\bar{z}_i}{(\|q\|^2+\epsilon)^2}\Bigr)\, dz_j \wedge dz_i\Bigr).
\end{aligned}$$

Let us define

$$M_{ij} = (-1)^i z_{i+(-1)^i}\bar{z}_j + (-1)^{j+1} z_{j+(-1)^j}\bar{z}_i$$

as in [Wan and Wang 2017] and let $\delta^{j_1 i_1,\dots,j_n i_n}_{0,\dots,2n-1}$ be the sign of the permutation

$$(j_1, i_1, \dots, j_n, i_n) \to (0, 1, \dots, 2n-1).$$

With this notation we see that

$$\begin{aligned}
&2^n(\|q\|^2+\epsilon)^{2n}(\partial\partial_J f_\epsilon)^n \\
&\quad=\Biggl(\sum_{\substack{j_1,i_1,\dots,j_n,i_n:\\ \{j_1,i_1,\dots,j_n,i_n\}=\{0,\dots,2n-1\}\\ i_l>j_l,\ l\in\{1,\dots,n\}}}\Biggl(\delta^{j_1 i_1,\dots,j_n i_n}_{0,\dots,2n-1}\prod_{l\in\{1,\dots,n\}}(2\delta^{j_l}_{i_l+(-1)^{i_l}}(\|q\|^2+\epsilon)+M_{i_l j_l})\Biggr)\Biggr)\Omega_n \\
&\quad=\binom{n}{0}\sum_{\{k_1,\dots,k_n\}=\{0,\dots,n-1\}}\delta^{(2k_1)(2k_1+1),\dots,(2k_n)(2k_n+1)}_{0,\dots,2n-1}2^n(\|q\|^2+\epsilon)^n \\
&\qquad+\binom{n}{1}\sum_{\substack{\{j_1,i_1,2k_2,2k_2+1,\dots,2k_n,2k_n+1\}=\{0,\dots,2n-1\}\\ i_1>j_1\\ k_l\in\{0,\dots,n-1\}}}\delta^{j_1 i_1,\dots,(2k_n)(2k_n+1)}_{0,\dots,2n-1}2^{n-1}(\|q\|^2+\epsilon)^{(n-1)}M_{i_1 j_1} \\
&\qquad\quad+\binom{n}{2}\sum_{\substack{\{j_1,i_1,j_2,i_2,\dots,2k_n,2k_n+1\}=\{0,\dots,2n-1\}\\ i_1>j_1,\ i_2>j_2\\ k_l\in\{0,\dots,n-1\}}}\delta^{j_1 i_1,j_2,i_2,\dots,(2k_n)(2k_n+1)}_{0,\dots,2n-1}2^{n-2}(\|q\|^2+\epsilon)^{(n-2)}M_{i_1 j_1}M_{i_2 j_2} \\
&\qquad\qquad+\dots+\binom{n}{n}\sum_{\substack{\{j_1,i_1,\dots,j_n,i_n\}=\{0,\dots,2n-1\}\\ i_l>j_l,\ l\in\{1,\dots,n\}}}\delta^{j_1 i_1,\dots,j_n i_n}_{0,\dots,2n-1}\prod_{l\in\{1,\dots,n\}}M_{i_l j_l}.
\end{aligned}$$

Note that for fixed indices $j_3, i_3, \dots, j_n, i_n$ the expression

$$M'_{j_3,i_3,\dots,j_n,i_n}=\sum_{\substack{j_1,i_1,j_2,i_2:\\ \{j_1,i_1,j_2,i_2,\dots,2k_n,2k_n+1\}=\{0,\dots,2n-1\}\\ i_1>j_1,i_2>j_2}}\delta^{j_1 i_1,\dots,j_n i_n}_{0,\dots,2n-1}M_{i_1 j_1}M_{i_2 j_2}$$

vanishes; this was already noticed in [Wan and Wang 2017]. To see this let

$$\{0,\dots,2n-1\}\setminus\{j_3,i_3,\dots,j_n,i_n\}=\{k,l,m,n\}$$

and $k>l>m>n$. Then

$$\begin{aligned}
\tfrac{1}{2}M'_{j_3,i_3,\dots,j_n,i_n}&=\delta^{lk,nm,j_3i_3,\dots,j_ni_n}_{0,\dots,2n-1}M_{kl}M_{mn}+\delta^{mk,nl,j_3i_3,\dots,j_ni_n}_{0,\dots,2n-1}M_{km}M_{ln}+\delta^{nk,ml,j_3i_3,\dots,j_ni_n}_{0,\dots,2n-1}M_{kn}M_{lm} \\
&=\delta^{lk,nm,j_3i_3,\dots,j_ni_n}_{0,\dots,2n-1}(M_{kl}M_{mn}-M_{km}M_{ln}+M_{kn}M_{lm}) \\
&=\pm\Bigl(\bigl((-1)^k z_{k+(-1)^k}\bar z_l+(-1)^{l+1}z_{l+(-1)^l}\bar z_k\bigr)\bigl((-1)^m z_{m+(-1)^m}\bar z_n+(-1)^{n+1}z_{n+(-1)^n}\bar z_m\bigr) \\
&\qquad-\bigl((-1)^k z_{k+(-1)^k}\bar z_m+(-1)^{m+1}z_{m+(-1)^m}\bar z_k\bigr)\bigl((-1)^l z_{l+(-1)^l}\bar z_n+(-1)^{n+1}z_{n+(-1)^n}\bar z_l\bigr) \\
&\qquad\quad+\bigl((-1)^k z_{k+(-1)^k}\bar z_n+(-1)^{n+1}z_{n+(-1)^n}\bar z_k\bigr)\bigl((-1)^l z_{l+(-1)^l}\bar z_m+(-1)^{m+1}z_{m+(-1)^m}\bar z_l\bigr)\Bigr)
\end{aligned}$$

$$
\begin{aligned}
&= \pm\Big((-1)^{k+m+1} z_{k+(-1)^k\bar{z}_n} z_{m+(-1)^m\bar{z}_l} + (-1)^{k+m} z_{k+(-1)^k\bar{z}_l} z_{m+(-1)^m\bar{z}_n}\\
&\quad + (-1)^{l+m+1} z_{l+(-1)^l\bar{z}_k} z_{m+(-1)^m\bar{z}_n} + (-1)^{m+l} z_{m+(-1)^m\bar{z}_k} z_{l+(-1)^l\bar{z}_n}\\
&\quad + (-1)^{k+l+1} z_{k+(-1)^k\bar{z}_m} z_{l+(-1)^l\bar{z}_n} + (-1)^{k+l} z_{k+(-1)^k\bar{z}_n} z_{l+(-1)^l\bar{z}_m}\\
&\quad + (-1)^{k+n+1} z_{k+(-1)^k\bar{z}_l} z_{n+(-1)^n\bar{z}_m} + (-1)^{k+n} z_{k+(-1)^k\bar{z}_m} z_{n+(-1)^n\bar{z}_l}\\
&\quad + (-1)^{m+n+1} z_{m+(-1)^m\bar{z}_k} z_{n+(-1)^n\bar{z}_l} + (-1)^{n+m} z_{n+(-1)^n\bar{z}_k} z_{m+(-1)^m\bar{z}_l}\\
&\quad + (-1)^{n+l+1} z_{n+(-1)^n\bar{z}_k} z_{l+(-1)^l\bar{z}_m} + (-1)^{l+n} z_{l+(-1)^l\bar{z}_k} z_{n+(-1)^n\bar{z}_m}\Big)\\
&= 0.
\end{aligned}
$$

Because of that only the first two summands of the expression for $(\partial\partial_J f)^n$ do not vanish. We are left with

$$
(\partial\partial_J f_\epsilon)^n = \frac{1}{2^n(\|q\|^2+\epsilon)^{2n}}\big(n!\,2^n(\|q\|^2+\epsilon)^n - n!\,2^{n-1}(\|q\|^2+\epsilon)^{(n-1)}\|q\|^2\big)\Omega_n = \frac{n!\,(\|q\|^2+2\epsilon)}{2(\|q\|^2+\epsilon)^{n+1}}\Omega_n.
$$

Finally since measures $(\partial\partial_J f_\epsilon)^n$ converge weakly to $(\partial\partial_J f)^n$, see [Wan and Wang 2017], it is enough to find the weak limit of

$$
\frac{n!\,(\|q\|^2+2\epsilon)}{2(\|q\|^2+\epsilon)^{n+1}},
$$

which by

$$
\frac{n!\,(\|q\|^2+2\epsilon)}{2(\|q\|^2+\epsilon)^{n+1}} \le \frac{n!}{2\|q\|^{2n}}
$$

and Lebesgue's dominated convergence theorem is

$$
\frac{n!}{2\|q\|^{2n}},
$$

exactly as we wanted. □

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SPECTRAL STABILITY OF INVISCID COLUMNAR VORTICES

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Dedicated to the memory of Louis N. Howard

Columnar vortices are stationary solutions of the three-dimensional Euler equations with axial symmetry, where the velocity field only depends on the distance to the axis and has no component in the axial direction. Stability of such flows was first investigated by Lord Kelvin in 1880, but despite a long history the only analytical results available so far provide necessary conditions for instability under either planar or axisymmetric perturbations. The purpose of this paper is to show that columnar vortices are spectrally stable with respect to three-dimensional perturbations with no particular symmetry. Our result applies to a large family of velocity profiles, including the most common models in atmospheric flows and engineering applications. The proof is based on a homotopy argument which allows us, when analyzing the spectrum of the linearized operator, to concentrate on a small neighborhood of the imaginary axis, where unstable eigenvalues can be excluded using integral identities and a careful study of the so-called critical layers.

1. Introduction

An interesting open question in hydrodynamic stability theory is whether the balance between the centrifugal force and the pressure gradient in axisymmetric vortex flows may lead to an instability even if the vorticity profile is monotone and the velocity field has no axial component. For incompressible perfect fluids, partial answers have been obtained under additional symmetry assumptions. For instance, in the restricted framework of two-dimensional flows, radially symmetric vortices are known to be stable if the vorticity distribution is a monotone function of the distance to the vortex center [Rayleigh 1879; Marchioro and Pulvirenti 1994], but even in that idealized situation no sharp stability criterion seems to be available. In the three-dimensional case, the simplest vortex-like equilibria are *columnar vortices*, namely axisymmetric flows with no vertical velocity and no dependence upon the vertical coordinate.

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In such flows, all streamlines are horizontal circles centered on the vertical symmetry axis. According to a celebrated result of [Rayleigh 1917], columnar vortices are stable with respect to axisymmetric perturbations if the square of the velocity circulation along the streamlines is a nondecreasing function of the distance to the symmetry axis, and that condition is actually sharp [Synge 1933].

A natural question arises from these centennial results: *When the vorticity profile is monotone and Rayleigh's condition is satisfied, are columnar vortices stable against three-dimensional perturbations with no particular symmetry ?* Although instabilities have never been observed experimentally or numerically for such vortices in the absence of axial flow, we could not find in the literature even a plausible formal argument supporting the affirmative answer; see Section 1C below for a short historical discussion. In the present paper, we give a rigorous proof of spectral stability for a large family of inviscid columnar vortices without imposing any symmetry assumption on the class of allowed perturbations. We thus provide an answer to an important question that dates back to the pioneering work of Lord Kelvin [1880], who was the first to investigate the three-dimensional stability of vortex columns.

Before stating our results, we first describe the precise framework. We start from the incompressible Euler equation in the whole space $\mathbb{R}^3$:

$$\partial_t u + (u \cdot \nabla) u = -\nabla p, \quad \operatorname{div} u = 0, \tag{1-1}$$

where $u = u(x,t) \in \mathbb{R}^3$ denotes the fluid velocity and $p = p(x,t) \in \mathbb{R}$ the internal pressure. We mainly consider the vorticity $\omega(x,t) = \operatorname{curl} u(x,t)$, which describes the local rotation of the fluid particles. Since we are interested in the stability of axially symmetric flows, it is convenient to use cylindrical coordinates (r,θ,z) defined by $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, and $x_3 = z$. The velocity and vorticity fields are then decomposed as

$$\begin{aligned} u &= u_r(r,\theta,z,t)e_r + u_\theta(r,\theta,z,t)e_\theta + u_z(r,\theta,z,t)e_z, \\ \omega &= \omega_r(r,\theta,z,t)e_r + \omega_\theta(r,\theta,z,t)e_\theta + \omega_z(r,\theta,z,t)e_z, \end{aligned}$$

where e_r, e_θ, e_z are unit vectors in the radial, azimuthal, and vertical directions, respectively. In these coordinates, the vorticity equation $\partial_t\omega + (u\cdot\nabla)\omega - (\omega\cdot\nabla)u = 0$ becomes

$$\begin{aligned} &\partial_t\omega_r + (u\cdot\nabla)\omega_r - (\omega\cdot\nabla)u_r = 0, \\ &\partial_t\omega_\theta + (u\cdot\nabla)\omega_\theta - (\omega\cdot\nabla)u_\theta = \frac{1}{r}(u_r\omega_\theta - u_\theta\omega_r), \\ &\partial_t\omega_z + (u\cdot\nabla)\omega_z - (\omega\cdot\nabla)u_z = 0, \end{aligned} \tag{1-2}$$

where $u\cdot\nabla = u_r\partial_r + (1/r)u_\theta\partial_\theta + u_z\partial_z$ and $\omega\cdot\nabla = \omega_r\partial_r + (1/r)\omega_\theta\partial_\theta + \omega_z\partial_z$. The velocity field satisfies the incompressibility condition

$$\frac{1}{r}\partial_r(ru_r) + \frac{1}{r}\partial_\theta u_\theta + \partial_z u_z = 0, \tag{1-3}$$

and can be expressed in terms of the vorticity by solving the linear elliptic system

$$\frac{1}{r}\partial_\theta u_z - \partial_z u_\theta = \omega_r, \quad \partial_z u_r - \partial_r u_z = \omega_\theta, \quad \frac{1}{r}\partial_r(ru_\theta) - \frac{1}{r}\partial_\theta u_r = \omega_z. \tag{1-4}$$

1A. ***Columnar vortices.*** Columnar vortices are stationary solutions of (1-1)–(1-4) of the particular form

$$u = V(r)e_\theta, \quad \omega = W(r)e_z, \quad p = P(r), \tag{1-5}$$

where V is the velocity profile and W the vorticity distribution. The pressure P inside the vortex is determined, up to an irrelevant additive constant, by the centrifugal balance $rP'(r) = V(r)^2$. Instead of V, we prefer using the angular velocity $\Omega(r) = V(r)/r$, which has the same physical dimension as the vorticity W. As a consequence of (1-4), we have

$$W(r) = \frac{1}{r}\partial_r(rV(r)) = r\Omega'(r) + 2\Omega(r). \tag{1-6}$$

Here are typical examples that are often considered in the literature:

(1) *The Rankine vortex*:

$$\Omega(r) = \begin{cases} 1 & \text{if } r \le 1, \\ r^{-2} & \text{if } r \ge 1, \end{cases} \qquad W(r) = \begin{cases} 2 & \text{if } r < 1, \\ 0 & \text{if } r > 1. \end{cases} \tag{1-7}$$

As is clear from (1-7), the flow of Rankine's vortex corresponds to a rigid rotation for $r < 1$ and an irrotational motion for $r > 1$. Although nonphysical because of the singularity at $r = 1$, this flow is relatively easy to analyze mathematically due to the very simple form of the vorticity distribution W, which is a piecewise constant function. The dynamical stability of Rankine's vortex was first investigated in [Kelvin 1880].

(2) *The Kaufmann–Scully vortex*:

$$\Omega(r) = \frac{1}{1+r^2}, \quad W(r) = \frac{2}{(1+r^2)^2}, \quad r > 0. \tag{1-8}$$

This smooth vortex is characterized by a relatively slow decay of the vorticity distribution as $r \to \infty$. It has also a very simple analytical form, and is often used as a model for vortices that appear in atmospheric flows or in laboratory experiments; see, e.g., [Alekseenko et al. 2007, Section 3.3.4].

(3) *The Lamb–Oseen vortex*:

$$\Omega(r) = \frac{1}{r^2}(1 - e^{-r^2}), \quad W(r) = 2e^{-r^2}, \quad r > 0. \tag{1-9}$$

Among all solutions of the form (1-5), the Lamb–Oseen vortex plays a distinguished role in connection with the long-time asymptotics of viscous planar flows. Indeed, if viscosity is taken into account, it is known that all localized distributions of vorticity evolve toward a Gaussian vorticity profile as $t \to +\infty$; see [Gallay and Wayne 2005]. In particular, the Lamb–Oseen vortex is the only one in the above family which corresponds to a self-similar solution of the Navier–Stokes equations.

Remark 1.1. Throughout this paper, it is understood that all independent and dependent variables in the Euler equations (1-1) are *dimensionless*. Examples (1-7)–(1-9) are normalized so that the vortex core has a diameter of size $\mathcal{O}(1)$, but that choice can be modified by a simple rescaling. Also, we assume without loss of generality that all vortices are normalized so that $\Omega(0) = 1$, which implies $W(0) = 2$.

To study the dynamical stability of the columnar vortex (1-5), we look for solutions of (1-2), (1-3) of the form

$$u(r,\theta,z,t) = r\Omega(r)e_\theta + \tilde{u}(r,\theta,z,t), \qquad \omega(r,\theta,z,t) = W(r)e_z + \tilde{\omega}(r,\theta,z,t),$$

where $\Omega = V/r$ is the angular velocity of the vortex and W the vorticity distribution given by (1-6). Inserting this ansatz into (1-2), neglecting the quadratic terms in $\tilde{u}$ and $\tilde{\omega}$, and finally dropping the tildes for notational simplicity, we arrive at the linearized evolution equations

$$\begin{aligned}
\partial_t\omega_r + \Omega(r)\,\partial_\theta\omega_r &= W(r)\,\partial_z u_r, \\
\partial_t\omega_\theta + \Omega(r)\,\partial_\theta\omega_\theta &= W(r)\,\partial_z u_\theta + r\Omega'(r)\omega_r, \\
\partial_t\omega_z + \Omega(r)\,\partial_\theta\omega_z &= W(r)\,\partial_z u_z - W'(r)u_r,
\end{aligned} \tag{1-10}$$

which are the starting point of our analysis. Of course, the linear relations (1-3), (1-4) still hold for the perturbed velocity and vorticity.

It is a classical observation that (1-10) can be considered as a self-contained evolution system for the vorticity ω, provided the velocity u is expressed in terms of ω by solving the linear elliptic system (1-3), (1-4). Once this is done, we can rewrite (1-10) in the compact form

$$\partial_t\omega = L\omega, \tag{1-11}$$

where L is a vector-valued, nonlocal, first-order differential operator. Our purpose is to study the spectral properties of that operator, and to show that L has no spectrum outside the imaginary axis under general assumptions on the angular velocity Ω or the vorticity distribution W.

Another fundamental remark is that system (1-2)–(1-4) is invariant under rotations about the vertical axis, and under translations along that axis. Using a Fourier series expansion with respect to the angular variable θ and a Fourier transform in the vertical variable z, we are led to consider velocities and vorticities of the particular form

$$u(r,\theta,z,t) = u_{m,k}(r,t)e^{im\theta}e^{ikz}, \qquad \omega(r,\theta,z,t) = \omega_{m,k}(r,t)e^{im\theta}e^{ikz}, \tag{1-12}$$

where $m \in \mathbb{Z}$ is the angular Fourier mode and $k \in \mathbb{R}$ is the vertical wave number. Here u, ω are complex-valued functions, but we impose that $\bar{u}_{m,k} = u_{-m,-k}$ and $\bar{\omega}_{m,k} = \omega_{-m,-k}$ so as to obtain real functions after summing over all possible values of m, k. Dropping the subscripts m, k for notational simplicity, we see that the perturbation equations (1-10) translate into

$$\begin{aligned}
(\partial_t + im\Omega(r))\omega_r &= W(r)iku_r, \\
(\partial_t + im\Omega(r))\omega_\theta &= W(r)iku_\theta + r\Omega'(r)\omega_r, \\
(\partial_t + im\Omega(r))\omega_z &= W(r)iku_z - W'(r)u_r.
\end{aligned} \tag{1-13}$$

In addition, the following relations hold:

$$\begin{gathered}
\omega_r = \frac{im}{r}u_z - iku_\theta, \qquad \omega_\theta = iku_r - \partial_r u_z, \qquad \omega_z = \frac{1}{r}\partial_r(ru_\theta) - \frac{im}{r}u_r, \\
\frac{1}{r}\partial_r(ru_r) + \frac{im}{r}u_\theta + iku_z = 0.
\end{gathered} \tag{1-14}$$

As before, we can rewrite (1-13) in the compact form

$$\partial_t \omega = L_{m,k}\omega, \tag{1-15}$$

assuming that the velocity $u = u_{m,k}$ in (1-13) is expressed in terms of the vorticity $\omega = \omega_{m,k}$ by solving the linear relations (1-14) with appropriate boundary conditions. The main properties of the Biot–Savart map $\omega_{m,k} \mapsto u_{m,k}$ obtained in this way will be recalled in Section A1. Being an integral operator acting on functions of the sole variable r, the generator $L_{m,k}$ in (1-15) is of course easier to study than the original three-dimensional differential operator L in (1-11).

1B. ***Statement of the results.*** To state our results in a precise way, we first specify our hypotheses on the unperturbed columnar vortex. We find it convenient to formulate these assumptions at the level of the *vorticity* profile W. Note that, in view of (1-6), the angular velocity Ω can be expressed in terms of W by the formula

$$\Omega(r) = \frac{1}{r^2}\int_0^r W(s)s\,\mathrm{d}s, \qquad r > 0, \tag{1-16}$$

and the derivative of Ω is in turn given by

$$\Omega'(r) = \frac{W(r) - 2\Omega(r)}{r} = \frac{1}{r^3}\int_0^r W'(s)s^2\,\mathrm{d}s, \qquad r > 0. \tag{1-17}$$

In what follows, we set $\mathbb{R}_+ = (0,\infty)$ and $\overline{\mathbb{R}}_+ = [0,\infty)$.

Assumption H1. *The vorticity profile* $W : \overline{\mathbb{R}}_+ \to \mathbb{R}_+$ *is a* $\mathcal{C}^1$ *function satisfying* $W'(0) = 0,\ W'(r) < 0$ *for all* $r > 0$, *and the total circulation*

$$2\pi\Gamma = 2\pi\int_0^\infty W(r)r\,\mathrm{d}r \tag{1-18}$$

of the columnar vortex is finite.

Under Assumption H1 the angular velocity profile $\Omega \in \mathcal{C}^1(\overline{\mathbb{R}}_+) \cap \mathcal{C}^2(\mathbb{R}_+)$ given by (1-16) is positive and satisfies $\Omega(0) = W(0)/2,\ \Omega'(0) = 0,\ \Omega'(r) < 0$ for all $r > 0$, and $\Omega(r) \sim \Gamma/r^2$ as $r \to \infty$. In particular, the *Rayleigh function* $\Phi : [0,\infty) \to \mathbb{R}$ defined by

$$\Phi(r) = 2\Omega(r)W(r), \qquad r \ge 0, \tag{1-19}$$

is positive everywhere. As a matter of fact, in our framework Assumption H1 corresponds exactly to the combination of Rayleigh's condition [1917] and of the two-dimensional stability criterion [Rayleigh 1879; Marchioro and Pulvirenti 1994]. We supplement it with the following:

Assumption H2. *The* $\mathcal{C}^1$ *function* $J : \mathbb{R}_+ \to \mathbb{R}_+$ *defined by*

$$J(r) = \frac{\Phi(r)}{\Omega'(r)^2}, \qquad r > 0, \tag{1-20}$$

satisfies $J'(r) < 0$ *for all* $r > 0$ *and* $rJ'(r) \to 0$ *as* $r \to \infty$.

This second assumption is more technical in nature, and certainly more difficult to justify. We first observe that it is satisfied for the Kaufmann–Scully vortex (1-8), because $J(r) = 1 + 1/r^2$ in that case, and a direct calculation that can be found in Section A7 below reveals that Assumption H2 also holds for the Lamb–Oseen vortex (1-9). A quantity corresponding to (1-20) appears in the work of G. I. Taylor [1931] on the stability of stratified shear flows; in that context it is called the *local Richardson number*; see, e.g., [Drazin and Reid 1981, Chapter 6]. Its relevance for stability was confirmed in [Miles 1961; Howard 1961]. The ideas of Howard were translated into the columnar vortex framework in [Howard and Gupta 1962], where the quantity (1-20) is also shown to play an important role in the stability analysis for perturbations with nonzero angular Fourier mode m and nonzero vertical wave number k. Indeed, it is proved in [Howard and Gupta 1962] that the linear operator $L_{m,k}$ in (1-15) has no unstable eigenvalue if

$$\frac{k^2}{m^2} J(r) \geq \frac{1}{4} \quad \text{for all } r > 0; \tag{1-21}$$

see also Proposition 3.4 below. Note that, in the case of the Lamb–Oseen vortex, inequality (1-21) is always violated for large $r > 0$ because $J(r) \to 0$ as $r \to \infty$, whereas (1-21) holds for the Kaufmann–Scully vortex if and only if $m^2 \leq 4k^2$. Although Howard and Gupta's result alone is not sufficient, it plays a crucial role in our stability analysis in Section 4, where we have to distinguish two spatial regions according to whether the local Richardson number $(k^2/m^2)J(r)$ is greater or smaller than $\frac{1}{4}$. It turns out to be important for our approach that inequality (1-21) either holds for all $r \geq 0$, or is satisfied if and only if $r \leq r_*$ for some $r_* > 0$. The only way to enforce that property for all possible values of m and k is to assume that the function J in (1-20) is decreasing. However, there is no evidence that Assumption H2 is more than a technical limitation, and we hope that this question will be clarified in the future.

Remark 1.2. Although this is not immediately obvious, Assumption H2 implies the existence of a nonnegative number $\ell_\infty \geq 0$ such that

$$\lim_{r\to\infty} r^4 W(r) = \ell_\infty, \qquad \lim_{r\to\infty} r^5 W'(r) = -4\ell_\infty; \tag{1-22}$$

see Section A4 below.

Next, we specify the function space in which we study the linearized operator $L_{m,k}$ defined in (1-13), (1-15). Since we used a Fourier decomposition to reduce our analysis to functions of the form (1-12), it is natural to work in L^2-based function spaces. Given $m \in \mathbb{Z}$ and $k \in \mathbb{R}$, we thus define the enstrophy space

$$X_{m,k} = \Big\{\omega \in L^2(\mathbb{R}_+, r\,\mathrm{d}r)^3 \;\Big|\; \frac{1}{r}\partial_r(r\omega_r) + \frac{im}{r}\omega_\theta + ik\omega_z = 0\Big\}, \tag{1-23}$$

equipped with the norm

$$\|\omega\|_{L^2}^2 = \int_0^\infty |\omega(r)|^2 r\,\mathrm{d}r, \quad \text{ where } |\omega|^2 = |\omega_r|^2 + |\omega_\theta|^2 + |\omega_z|^2.$$

It is not difficult to verify that the generator $L_{m,k}$ of the linearized evolution equation (1-15) defines a bounded linear operator in the space $X_{m,k}$ if $k \neq 0$; see Proposition 2.1 below. With this observation in mind, we can formulate our first main result:

Theorem 1.3. *Consider a columnar vortex whose vorticity profile W satisfies Assumptions H1, H2 above. Given $m \in \mathbb{Z}$ and $k \neq 0$, let $L_{m,k}$ be the generator of the linearized evolution* (1-15). *Then the spectrum of $L_{m,k}$ in the enstrophy space $X_{m,k}$ satisfies*

$$\sigma(L_{m,k}) \subset i\mathbb{R}. \tag{1-24}$$

Remark 1.4. The proof actually shows that, under the normalization condition $W(0) = 2$, $\sigma(L_{m,k})$ consists of an essential spectrum filling the closed interval $\{-imb \mid 0 \le b \le 1\} \subset i\mathbb{R}$, and of a countable family of simple, purely imaginary eigenvalues that accumulate only at $-im \in i\mathbb{R}$. These eigenvalues are well studied in the physical literature (a brief account is given in Section 1C below), and the corresponding eigenfunctions are referred to as *Kelvin vibration modes*. The main contribution of the present paper is to show that the operator $L_{m,k}$ has no eigenvalue outside the imaginary axis if the vorticity profile W satisfies Assumptions H1, H2. It is interesting to note that this result remains valid for the Rankine vortex (1-7), which does not satisfy our hypotheses; see Section A2 below.

Remark 1.5. The particular case $k = 0$, which corresponds to two-dimensional perturbations, is excluded in Theorem 1.3 because the function space $X_{m,k}$ is not appropriate in that situation. This is essentially due to the fact that the two-dimensional Biot–Savart law is ill-defined for vorticities in the enstrophy space. The problem can be eliminated by introducing a radial weight that ensures a faster decay of $\omega(r)$ as $r \to \infty$, or alternatively by working in the energy space as mentioned in Remark 1.7 below. However, since the two-dimensional stability of radially symmetric vortices is already well documented, we chose to ignore these technical issues and to concentrate here on the genuinely three-dimensional case $k \neq 0$, which was essentially unexplored until now.

According to Theorem 1.3, for any $s \in \mathbb{C}$ with $\mathrm{Re}(s) \neq 0$, the resolvent operator $(s - L_{m,k})^{-1}$ is well-defined and bounded in the space $X_{m,k}$ if $m \in \mathbb{Z}$ and $k \neq 0$. Actually, one can prove that the resolvent is uniformly bounded for all $m \in \mathbb{Z}$ and for all nonzero k in the one-dimensional lattice $\mathbb{Z}k_0$, where $k_0 > 0$ is arbitrary. Returning to the full linearized evolution (1-11), this proves spectral stability of the generator L in the space

$$\dot{L}^2_{\sigma,\mathrm{per},h} = \left\{ \omega \in L^2(\mathbb{R}^2 \times \mathbb{T}_h)^3 \;\middle|\; \mathrm{div}\,\omega = 0,\ \int_0^h \omega(x_1, x_2, x_3)\,\mathrm{d}x_3 = 0 \right\}, \tag{1-25}$$

where $\mathbb{T}_h = \mathbb{R}/(\mathbb{Z}h)$ and $h = 2\pi/k_0$ is the vertical period. We can thus state our second main result:

Theorem 1.6. *Under the assumptions of Theorem 1.3, let L denote the full linearized operator in* (1-11). *Then, for any $h > 0$, the spectrum of L in the space $\dot{L}^2_{\sigma,\mathrm{per},h}$ satisfies*

$$\sigma(L) = i\mathbb{R}. \tag{1-26}$$

Remark 1.7. The reason for restricting ourselves to functions with zero average in the vertical direction was explained in Remark 1.5. The same technical limitation prevents us from considering perturbations in the enstrophy space $L^2_\sigma(\mathbb{R}^3)$, without assuming periodicity in the vertical direction, because in that case all values of the vertical wave number $k \in \mathbb{R}$ have to be taken into account. In a subsequent work [Gallay and Smets 2019], we use Theorem 1.3 to obtain an analogue of Theorem 1.6 for the Euler equation in velocity formulation. There we consider perturbations in the energy space, and we also obtain semigroup estimates for the linearized operator at a columnar vortex.

In the proof of Theorems 1.3 and 1.6, we find it convenient to normalize our velocity and vorticity profiles so that $\Omega(0) = 1$ and $W(0) = 2$. This leads to the following definition:

Definition 1.8. We denote by $\mathscr{W}$ the class of all vorticity profiles $W : \overline{\mathbb{R}}_+ \to \mathbb{R}_+$ satisfying the Assumptions H1, H2 above, as well as the normalizing condition $W(0) = 2$.

It is worth emphasizing here that Assumption H2 involves the function J defined in (1-20), which depends nonlinearly on the vorticity profile W. As a consequence, our family of admissible profiles is not a vector space, and the class $\mathscr{W}$ introduced in Definition 1.8 is not even a convex set. However, we shall prove in Section A4 that any profile $W \in \mathscr{W}$ is entirely determined by the auxiliary function

$$Q(r) = \frac{1}{\sqrt{1 + J(r)}}, \qquad r > 0, \tag{1-27}$$

and that the class $\mathscr{W}$ can be described by simple linear constraints at the level of the function Q. This makes it possible to perform continuous interpolation and approximation within the class $\mathscr{W}$, and such tools will play a crucial role in the proof of Theorem 1.3.

Remark 1.9. If we equip the class $\mathscr{W}$ with the topology of $\mathcal{C}^1_b(\overline{\mathbb{R}}_+)$, the Banach space of all bounded continuously differentiable functions on $\overline{\mathbb{R}}_+$ with bounded derivative, it is easily verified that the linear operator $L_{m,k} \in \mathcal{L}(X_{m,k})$ depends continuously on the vorticity profile $W \in \mathscr{W}$; see Lemma 4.1 below. In particular, isolated eigenvalues of $L_{m,k}$ outside the imaginary axis (if there are any) vary continuously when W is perturbed in that topology. This implies that the conclusion (1-24) of Theorem 1.3 remains valid for any vorticity profile that belongs to the closure of the class $\mathscr{W}$ in $\mathcal{C}^1_b(\overline{\mathbb{R}}_+)$. This larger class contains vorticities W that are not strictly decreasing functions of the radius r, and may even be compactly supported.

1C. *Previous results and perspectives.* The first historical contribution regarding the stability of columnar vortices in incompressible fluids is of course the seminal work [Kelvin 1880]. In that study, the focus is put on neutral modes, namely eigenmodes of the linearized Euler equation that correspond to purely imaginary eigenvalues; these were later termed "Kelvin vibration modes". As Kelvin expresses it: "*The problem thus solved is the finding of the periodic disturbance in the motion of rotating liquid* [...]". The computations in [Kelvin 1880] are performed in situations where the underlying axisymmetric flow has piecewise constant vorticity; this exactly corresponds to what was called the Rankine vortex in Section 1A above. However, Kelvin waves are observed to play an important role in the dynamics of the Euler equation for a much wider variety of profiles, and were actively studied in the literature since then (in most cases numerically, or using asymptotic expansions combined with physical arguments). In the case of the Lamb–Oseen vortex, important contributions were made in particular by Le Dizès and Lacaze [2005] and Fabre, Sipp and Jacquin [Fabre et al. 2006], both in the inviscid case and in the vanishing viscosity limit. Unlike Kelvin (who had no computer account!), the authors of [Le Dizès and Lacaze 2005; Fabre et al. 2006] also consider the possibility of eigenvalues off the imaginary axis. One of the conclusions of [Fabre et al. 2006] based on their numerical findings is that "[...]*no amplified modes were found, a result which demonstrates the stability of the Lamb–Oseen vortex.*"

In a different direction, Rayleigh [1879; 1917] initiated the study of necessary conditions for columnar vortex instability.[1] Although it may certainly be found physically convincing, the original argument [Rayleigh 1917] leading to Rayleigh's criterion cannot be easily transposed into rigorous mathematical terms. Instead, the approach followed in [Howard and Gupta 1962], which we consider one of the most interesting and important contributions so far, is both rigorous and elementary. This remarkable work contains most importantly an inconclusive but enlightening section called "Remarks on the nonaxisymmetric case", in which the partial stability criterion (1-21) can be found. The authors write: "*The overall conclusion of this consideration of the nonaxisymmetric case is thus essentially negative*: *the methods used to derive the Richardson number and semicircle results in the axisymmetric case reproduce the known results of Rayleigh for two-dimensional perturbations and pure axial flow, but seem to give very little more. In fact the present situation with regard to nonaxisymmetric perturbations seems to be very unsatisfactory from a theoretical point of view.*"

Attempts have been made to derive necessary conditions for instability extending Rayleigh's criterion to nonaxisymmetric perturbations. One such criterion was proposed in [Billant and Gallaire 2005], following [Leibovich and Stewartson 1983], and applies in a given Fourier sector. It is relatively simple to state but requires a number of a posteriori checks which could be more difficult to perform. As the authors mention, in all the situations they tested the most unstable modes were always the axisymmetric ones (this is reminiscent of Squire's theorem in the context of viscous shear flows), and therefore, in practice, Rayleigh's criterion appears to be sufficient to detect potential instabilities. Yet, a priori estimates on the possible growth in a given Fourier sector are certainly interesting per se.

Spectral stability of course does not imply stability of the flow for a Hamiltonian system such as (1-1). In celebrated works, Arnold [1965; 1966] derived a nonlinear stability criterion for stationary solutions of the Euler equations, which are viewed as critical points of the kinetic energy functional over the manifold of isovortical vector fields, and he treated in detail the case of two-dimensional flows. His approach was subsequently extended in [Szeri and Holmes 1988] and applied to axisymmetric perturbations of columnar vortices. A few years later, Rouchon [1991] proved that the conditions in Arnold's criterion are never satisfied if one considers genuinely three-dimensional perturbations of nontrivial stationary flows. An intermediate step between spectral and nonlinear stability is linear stability, which consists in controlling the growth of the semigroup generated by the linearized operator in Theorem 1.6. Preliminary results in that direction can be found in the subsequent work [Gallay and Smets 2019].

We close this section by mentioning that a number of interesting phenomena are known to arise, as far as instabilities are concerned, when the base flow possesses an additional axial component. Some of the works already quoted, and many others, do consider that situation as well. Since we did not investigate it at all in this work, we keep that discussion for another occasion.

[1] Or equivalently sufficient conditions for their stability; in the present work stability is only understood in the spectral sense, meaning the absence of eigenvalues with positive real part.

1D. ***Organization of the paper.*** Our strategy to prove Theorems 1.3 and 1.6 can be explained as follows. In a first step, we show in Section 2 that the essential spectrum of the operator $L_{m,k}$ is purely imaginary. The rest of the spectrum consists of isolated eigenvalues with finite multiplicity, and the corresponding eigenfunctions are solutions of a second-order differential equation involving a complex potential that depends on m, k, and the spectral parameter s. The eigenvalue equation is difficult to study in general, but using techniques that date back to [Rayleigh 1879; 1917] it is easy to verify that it has no nontrivial solution with $\mathrm{Re}(s) \neq 0$ when the perturbations are either axisymmetric ($m = 0$) or two-dimensional ($k = 0$). In Section 3, we establish a few preliminary results in the case where $m \neq 0$ and $k \neq 0$. In particular, we derive useful identities satisfied by any nontrivial eigenfunction, and we recover the stability criterion (1-21) of Howard and Gupta. The core of the proof of Theorem 1.3 is Section 4. We construct a suitable homotopy between the vorticity profile $W \in \mathscr{W}$ and a reference profile for which stability in the corresponding Fourier sector $X_{m,k}$ is known by Howard and Gupta's criterion. By a continuity argument, this strategy allows us to reduce the problem to proving the absence of unstable eigenvalues *arbitrarily close to the imaginary axis* for a one-parameter family of profiles in the class $\mathscr{W}$. A delicate combination of integral identities and comparison arguments relying on Assumption H2 are then used to perform such a "critical layer analysis" and hence to preclude the existence of unstable eigenvalues. Finally, in Section 5, we prove uniform resolvent estimates for the linear operator $L_{m,k}$ outside the imaginary axis, which imply that the full linearization L has indeed no spectrum in that region when acting on the space $\dot{L}^2_{\sigma,\mathrm{per},h}$ for any $h > 0$. This is precisely the conclusion of Theorem 1.6. The last section is an Appendix where several auxiliary results are established. In particular, we give useful estimates for the Biot–Savart law in the Fourier sector indexed by m, k, we prove the stability of Rankine's vortex (1-7), which is not covered by Theorem 1.3, and we explain how to perform continuous interpolation and approximation in the nonlinear class $\mathscr{W}$.

2. Formulation of the spectral problem

Let W be a vorticity profile in the class $\mathscr{W}$, and let Ω be the corresponding angular velocity defined by (1-16). For a fixed value of the angular Fourier mode $m \in \mathbb{Z}$ and of the vertical wave number $k \in \mathbb{R}$, we consider the linear operator $L_{m,k}$ introduced in (1-15). In view of (1-13), we have the natural decomposition

$$L_{m,k} = A_m + B_{m,k}, \tag{2-1}$$

where A_m is the multiplication operator defined by

$$A_m\omega = -im\Omega(r)\omega + r\Omega'(r)\omega_r e_\theta, \tag{2-2}$$

and $B_{m,k}$ is the nonlocal perturbation

$$B_{m,k}\omega = ikW(r)u - W'(r)u_r e_z. \tag{2-3}$$

Here $u = (u_r, u_\theta, u_z)$ denotes the velocity obtained from the vorticity $\omega = (\omega_r, \omega_\theta, \omega_z)$ by solving the linear PDE system (1-14) with appropriate boundary conditions. We refer the reader to Section A1 below for a discussion of the map $\omega \mapsto u$, which we call the Biot–Savart law in the Fourier subspace indexed by m and k. Our main goal in this paper is to study the spectral properties of the operator $L_{m,k}$ acting on the enstrophy space $X_{m,k}$ defined by (1-23).

The following simple result is the starting point of our analysis.

Proposition 2.1. *Fix $m \in \mathbb{Z}$ and $k \in \mathbb{R} \setminus \{0\}$:*

(1) *The linear operator A_m defined by* (2-2) *is bounded in $X_{m,k}$ with spectrum given by*

$$\sigma(A_m) = \{z \in \mathbb{C} \mid z = -imb \text{ for some } b \in [0, 1]\}. \tag{2-4}$$

This spectrum is purely continuous if $m \neq 0$, and reduces to a single eigenvalue if $m = 0$.

(2) *The linear operator $B_{m,k}$ defined by* (2-3) *is compact in $X_{m,k}$.*

Proof. Given $s \in \mathbb{C}$ and $f = (f_r, f_\theta, f_z) \in X_{m,k}$, the resolvent equation $(s - A_m)\omega = f$ is equivalent to the linear system

$$(s + im\Omega(r))\omega_r = f_r, \quad (s + im\Omega(r))\omega_\theta = f_\theta + r\Omega'(r)\omega_r, \quad (s + im\Omega(r))\omega_z = f_z. \tag{2-5}$$

As $W \in \mathscr{W}$, we know that $\Omega : [0, \infty) \to \mathbb{R}_+$ is strictly decreasing with $\Omega(0) = 1$ and $\Omega(r) \to 0$ as $r \to \infty$. Thus, if $s \neq -imb$ for all $b \in [0, 1]$, the quantity $|s + im\Omega(r)|$ is bounded away from zero, and it follows that system (2-5) has a unique solution $\omega \in X_{m,k}$ satisfying $\|\omega\|_{L^2} \le C(s)\|f\|_{L^2}$. On the other hand, if $m \neq 0$ and $s = -imb$ for some $b \in [0, 1]$, it is easy to verify that the operator $s - A_m$ is one-to-one but not onto (its range is dense but strictly contained in $X_{m,k}$), so that s belongs to the continuous spectrum of A_m. Finally, if $m = 0$, it is clear that $s = 0$ is an eigenvalue of A_m with infinite multiplicity. This proves the first part.

We next consider the operator $B_{m,k}$. If $\omega \in X_{m,k}$ and $\|\omega\|_{L^2} \le 1$, Proposition A.1 shows that the associated velocity field u satisfies $\|\partial_r u\|_{L^2} + \|ku\|_{L^2} \le C$ for some universal constant $C > 0$. This gives a uniform bound on u in $H^1(\mathbb{R}_+, r\,\mathrm{d}r)$ since we assume that $k \neq 0$. By the Fréchet–Kolmogorov theorem, we deduce that the map $\omega \mapsto B_{m,k}\omega = ikW(r)u - W'(r)u_r e_z$ is compact in $X_{m,k}$, because the functions W and W' are bounded and converge to zero as $r \to \infty$. ☐

Proposition 2.1 shows in particular that, for any $m \in \mathbb{Z}$ and any $k \in \mathbb{R} \setminus \{0\}$, the linearization $L_{m,k} = A_m + B_{m,k}$ defines a bounded operator in the space $X_{m,k}$. Moreover, as $B_{m,k}$ is compact, the *essential spectrum* of $L_{m,k}$ is the same as the (essential) spectrum of A_m, namely the closed interval $I_m = \{-imb \mid 0 \le b \le 1\} \subset i\mathbb{R}$; see [Edmunds and Evans 2018, Theorem I.4.1]. Note that, in the present case, the various definitions of the essential spectrum listed in [Edmunds and Evans 2018, Section I.4] all coincide. This implies that the spectrum of $L_{m,k}$ outside the interval I_m entirely consists of isolated eigenvalues with finite multiplicities, which can accumulate only on the essential spectrum. The proof of Theorem 1.3 is thus reduced to showing that all isolated eigenvalues of $L_{m,k}$ actually lie on the imaginary axis.

Remark 2.2. As the functions Ω, W are real-valued, it is not difficult to verify, using the definitions (2-2), (2-3) and the relations (1-14) between u and ω, that the spectrum of $L_{m,k}$ in $X_{m,k}$ has the symmetries

$$\sigma(L_{m,k}) = \sigma(L_{m,-k}) = -\sigma(L_{-m,k}) \quad \text{and} \quad \sigma(L_{m,k}) = -\overline{\sigma(L_{m,k})}. \tag{2-6}$$

The corresponding mappings between eigenspaces are also easy to establish. In particular, the last relation in (2-6) means that the spectrum of $\sigma(L_{m,k})$ is symmetric with respect to the imaginary axis, a property that will be used later on.

As a first step in the proof of Theorem 1.3, we derive an equation for the eigenfunctions of the operator $L_{m,k}$ corresponding to eigenvalues outside the essential spectrum. In what follows, we thus assume that $s \in \mathbb{C}$ is an isolated eigenvalue of $L_{m,k}$ with eigenfunction $\omega = (\omega_r, \omega_\theta, \omega_z) \in X_{m,k}$, and we denote by $u = (u_r, u_\theta, u_z)$ the velocity field associated with ω via the Biot–Savart law; see Section A1. As in [Drazin and Reid 1981], we define

$$\gamma(r) = s + im\Omega(r), \qquad r > 0. \tag{2-7}$$

Since s does not belong to the essential spectrum of $L_{m,k}$ by assumption, it follows from Proposition 2.1 that $\gamma(r) \neq 0$ for all $r > 0$.

In view of (1-13), the eigenvalue equation is given by

$$\begin{aligned} \gamma(r)\omega_r &= ikW(r)u_r, \\ \gamma(r)\omega_\theta &= ikW(r)u_\theta + r\Omega'(r)\omega_r, \\ \gamma(r)\omega_z &= ikW(r)u_z - W'(r)u_r, \end{aligned} \tag{2-8}$$

where $r\Omega'(r) = W(r) - 2\Omega(r)$ by (1-6). If we express the vorticity ω in terms of u using the relations (1-14), we obtain the equivalent system

$$ikW(r)u_r + ik\gamma(r)u_\theta - \frac{im\gamma(r)}{r}u_z = 0, \tag{2-9}$$

$$ik\gamma(r)u_r - 2ik\Omega(r)u_\theta - \partial_r(\gamma(r)u_z) = 0, \tag{2-10}$$

$$\Big(W'(r) - \frac{im\gamma(r)}{r}\Big)u_r + \gamma(r)\frac{1}{r}\partial_r(ru_\theta) - ikW(r)u_z = 0. \tag{2-11}$$

Assuming for the moment that $k \neq 0$, it is straightforward to verify that the relations (2-9)–(2-11) together imply the incompressibility condition

$$\frac{1}{r}\partial_r(ru_r) + \frac{im}{r}u_\theta + iku_z = 0. \tag{2-12}$$

To reduce system (2-9)–(2-12) to a single equation, we first express the azimuthal velocity u_θ in terms of u_r, u_z using (2-9), and replace it into (2-10), (2-12) to obtain the 2×2 system

$$\Big(\partial_r^* - \frac{imW(r)}{r\gamma(r)}\Big)u_r + ik\Big(1 + \frac{m^2}{k^2r^2}\Big)u_z = 0, \tag{2-13}$$

$$\Big(\partial_r + \frac{imW(r)}{r\gamma(r)}\Big)u_z - ik\Big(1 + \frac{\Phi(r)}{\gamma(r)^2}\Big)u_r = 0, \tag{2-14}$$

where $\Phi = 2\Omega W$ is the Rayleigh function and $\partial_r^* = \partial_r + 1/r$. Next, observing that the coefficient of u_z in (2-13) does not vanish, we can divide (2-13) by that coefficient and apply the differential operator $\partial_r + imW/(r\gamma)$ to obtain, with the help of (2-14), the following second-order differential equation for the radial velocity:

$$\left(\partial_r + \frac{imW(r)}{r\gamma(r)}\right)\frac{r^2}{m^2+k^2r^2}\left(\partial_r^* - \frac{imW(r)}{r\gamma(r)}\right)u_r = \left(1+\frac{\Phi(r)}{\gamma(r)^2}\right)u_r. \tag{2-15}$$

If we expand the product in the left-hand side, we find after straightforward calculations

$$-\partial_r\left(\frac{r^2\partial_r^* u_r}{m^2+k^2r^2}\right) + \left\{1+\frac{1}{\gamma(r)^2}\frac{k^2r^2\Phi(r)}{m^2+k^2r^2} + \frac{imr}{\gamma(r)}\partial_r\left(\frac{W(r)}{m^2+k^2r^2}\right)\right\}u_r = 0; \tag{2-16}$$

see also [Drazin and Reid 1981, equation (15.26)]. This is the desired eigenvalue equation, which will be our main concern in the rest of this paper. It is formulated in terms of the radial velocity u_r, which satisfies $u_r \in H^1(\mathbb{R}_+, r\,\mathrm{d}r)$ according to Proposition A.1. In fact, we also have $u_r \in H^2_{\mathrm{loc}}(\mathbb{R}_+)$ in view of the divergence-free condition (2-12).

Remark 2.3. In the case where $k = 0$, a much simpler calculation shows that the eigenvalue equation is still given by (2-16) if $m \neq 0$, although the derivation above is not correct. If $k = m = 0$, equation (2-16) is of course meaningless, but in that case it is obvious that system (2-8) has no nontrivial solution for $s \neq 0$.

Summarizing the arguments developed so far, the proof of Theorem 1.3 can be reduced to showing that, for all $m \in \mathbb{Z}$ and all $k \in \mathbb{R}\setminus\{0\}$, the eigenvalue equation (2-16) has no nontrivial solution $u_r \in H^1(\mathbb{R}_+, r\,\mathrm{d}r)$ if the spectral parameter $s \in \mathbb{C}$ satisfies $\mathrm{Re}(s) \neq 0$. This is a difficult task in general, which we postpone to Sections 3 and 4. For the time being, we just mention two important particular cases which are relatively easy to handle.

2A. *The axisymmetric case.* In the axisymmetric case $m = 0$, Proposition 2.1 asserts that the essential spectrum of $L_{0,k}$ is reduced to zero, and therefore away from the origin there may only exist eigenvalues with finite multiplicity. The spectral function (2-7) is constant in that case, and the stability equation (2-16) reduces to

$$-\partial_r\partial_r^* u_r + k^2\left(1+\frac{\Phi(r)}{s^2}\right)u_r = 0. \tag{2-17}$$

The following classical result dates back to [Rayleigh 1917], and is reproduced here for the reader's convenience.

Proposition 2.4. *Assume that the Rayleigh function Φ is nonnegative. Then the eigenvalue equation* (2-17) *has no nontrivial solution $u_r \in H^1(\mathbb{R}_+, r\,\mathrm{d}r)$ if* $\mathrm{Re}(s) \neq 0$.

Proof. According to Remark 2.3, we can suppose that $k \neq 0$. Assume that $u_r \in H^1(\mathbb{R}_+, r\,\mathrm{d}r)$ is a nontrivial solution of (2-17) for some $s \in \mathbb{C}\setminus\{0\}$. Multiplying both sides of (2-17) by $r\bar{u}_r$ and integrating the resulting expression over $\mathbb{R}_+$, we obtain the useful relation

$$\int_0^\infty \left\{|\partial_r^* u_r|^2 + k^2\left(1+\frac{\Phi(r)}{s^2}\right)|u_r|^2\right\} r\,\mathrm{d}r = 0. \tag{2-18}$$

By assumption we have $\int_0^\infty \Phi|u_r|^2 r\,\mathrm{d}r > 0$, because u_r is a nontrivial solution of (2-17) and Φ is a nonnegative function with $\Phi(0) > 0$. Thus taking the imaginary part of (2-18) we deduce that $\mathrm{Im}(s^2) = 0$;

hence $s \in \mathbb{R}$ or $s \in i\mathbb{R}$. The first possibility is excluded by taking the real part of (2-18); hence we conclude that $s \in i\mathbb{R}$. $\square$

Remark 2.5. Actually it was observed in [Synge 1933] that the Rayleigh stability criterion $\Phi \geq 0$ is not only sufficient, but also necessary in the axisymmetric case. Indeed, we know that $\Phi(0) = W(0)^2 > 0$, and for localized vortices we always have $\Phi(r) \to 0$ as $r \to \infty$. Now, assume that $\Phi(\bar{r}) < 0$ for some $\bar{r} > 0$, and consider the Schrödinger equation

$$-s^2 \partial_r \partial_r^* u_r + k^2 (s^2 + \Phi(r)) u_r = E u_r, \quad r > 0, \tag{2-19}$$

in the semiclassical limit where $0 < s \ll 1$. As the potential term $s^2 + \Phi(r)$ takes negative values near $r = \bar{r}$, it is well known that the operator in (2-19) has negative eigenvalues E if $s > 0$ is sufficiently small; see, e.g., [Simon 1983; Helffer and Sjöstrand 1984]. In fact, the number of negative eigenvalues increases unboundedly as $s \to 0$, and this implies by continuity that (2-19) with $E = 0$, or equivalently (2-17), has a nontrivial solution $u_r \in H^1(\mathbb{R}_+, r\,\mathrm{d}r)$ for a sequence of values of $s > 0$ that converges to zero.

We also note that the analogue of Synge's observation used for $s \in i\mathbb{R}$ instead of $s \in \mathbb{R}$ implies in contrast that, when the Rayleigh function is nonnegative, the linearized operator $L_{0,k}$ *does* possess nonzero eigenvalues on the imaginary axis, which correspond to Kelvin modes.

2B. *The two-dimensional case.* Although it is not included in Theorem 1.3, the two-dimensional case $k = 0$ is worth mentioning too. When $m \neq 0$, the eigenvalue equation (2-16) reduces to

$$-\partial_r (r^2 \partial_r^* u_r) + \left(m^2 + \frac{imr W'(r)}{\gamma(r)}\right) u_r = 0. \tag{2-20}$$

A well-known sufficient condition for stability is that the vorticity profile W be a monotone function; see, e.g., [Marchioro and Pulvirenti 1994], but unlike in the axisymmetric case no sharp criterion has been established so far. Again, for the reader's convenience, we reproduce here the easy argument showing spectral stability if W' has a constant sign.

Proposition 2.6. *Assume that the vorticity profile W is monotone. Then the eigenvalue equation* (2-20) *has no nontrivial solution $u_r \in H^1(\mathbb{R}_+, r\,\mathrm{d}r)$ if* $\mathrm{Re}(s) \neq 0$.

Proof. Assume that $u_r \in H^1(\mathbb{R}_+, r\,\mathrm{d}r)$ is a nontrivial solution of (2-20) for some $s \in \mathbb{C}$ with $\mathrm{Re}(s) \neq 0$. Multiplying both members of (2-20) by $r\bar{u}_r$ and integrating over $\mathbb{R}_+$, we obtain the relation

$$\int_0^\infty \left\{ |\partial_r (r u_r)|^2 + \left(m^2 + \frac{imr W'(r)}{\gamma(r)}\right) |u_r|^2 \right\} r\,\mathrm{d}r = 0. \tag{2-21}$$

In particular, taking the imaginary part and using (2-7), we find

$$m \,\mathrm{Re}(s) \int_0^\infty \frac{W'(r)}{|\gamma(r)|^2} |u_r|^2 r^2 \,\mathrm{d}r = 0,$$

and since W is monotone we conclude that u_r is supported in the set where W' vanishes. This is clearly impossible if W is not identically constant, because u_r is a nontrivial solution of the second-order ODE (2-20). But if W is a constant, (2-21) immediately gives the desired contradiction. $\square$

3. The eigenvalue equation for $m \neq 0$ and $k \neq 0$

In this section we begin our study of the eigenvalue equation (2-16) in the general case where $m \neq 0$ and $k \neq 0$. In view of the symmetries (2-6), we can assume without loss of generality that $m \geq 1$ and $k > 0$. We write the spectral parameter as $s = m(a - ib)$, where $a, b \in \mathbb{R}$, and we take the decomposition

$$\gamma(r) = s + im\Omega(r) = im\gamma_\star(r), \quad \text{where } \gamma_\star(r) = \Omega(r) - b - ia. \tag{3-1}$$

According to Proposition 2.1, the essential spectrum of the operator $L_{m,k}$ is the set of all $s = m(a - ib)$ such that $a = 0$ and $b \in [0, 1]$. Outside that set, the function $\gamma_\star$ is bounded away from zero for all $r > 0$ and the eigenvalue equation (2-16) becomes

$$-\partial_r(\mathcal{A}(r)\partial_r^* u_r) + \mathcal{B}(r)u_r = 0, \tag{3-2}$$

where $\partial_r^* = \partial_r + 1/r$ and

$$\mathcal{A}(r) = \frac{r^2}{m^2 + k^2 r^2}, \quad \mathcal{B}(r) = 1 - \frac{k^2}{m^2}\frac{\mathcal{A}(r)\Phi(r)}{\gamma_\star(r)^2} + \frac{r}{\gamma_\star(r)}\partial_r\left(\frac{W(r)}{m^2 + k^2 r^2}\right). \tag{3-3}$$

3A. ***Asymptotic behavior at the origin and at infinity.*** Our first goal is to determine the asymptotic behavior of the solutions of the complex ODE (3-2) as $r \to 0$ and $r \to \infty$, assuming that $a \neq 0$ or $b \notin [0, 1]$. We start with the behavior at the origin. If u_r is a solution of (3-2), we set

$$u_r(r) = \frac{1}{r}v\left(\log\frac{1}{r}\right), \quad r > 0,$$

or equivalently $v(x) = e^{-x}u_r(e^{-x})$ for $x = \log(1/r) \in \mathbb{R}$. The new function $v : \mathbb{R} \to \mathbb{R}$ satisfies the equation

$$v''(x) + 2k^2\mathcal{A}(e^{-x})v'(x) - \mathcal{C}(x)v(x) = 0, \quad \text{where } \mathcal{C}(x) = e^{-2x}\frac{\mathcal{B}(e^{-x})}{\mathcal{A}(e^{-x})}. \tag{3-4}$$

In view of (3-3) we have $\mathcal{A}(e^{-x}) = \mathcal{O}(e^{-2x})$ and $\mathcal{C}(x) = m^2 + \mathcal{O}(e^{-2x}) + \mathcal{O}(e^{-x}|W'(e^{-x})|)$ as $x \to +\infty$. Thus applying, e.g., [Coddington and Levinson 1955, Theorem 3.8.1], we deduce that (3-4) has a unique solution v such that $e^{mx}v(x) \to 1$ as $x \to +\infty$. Returning to the original variables, we conclude that (3-2) has a unique solution u_r such that $r^{1-m}u_r(r) \to 1$ as $r \to 0$. This solution u_r and its first derivative u_r' depend continuously on the various parameters in (3-2), including the vorticity profile $W \in C_b^1(\overline{\mathbb{R}}_+)$ and the spectral parameter $s = m(a - ib) \in \mathbb{C}$, uniformly in r on any bounded interval of the form $(0, R)$. Any linearly independent solution of (3-2) blows up like r^{-1-m} as $r \to 0$, and is therefore not square integrable near the origin.

We next study the behavior at infinity. If u_r is a solution of (3-2), we define $w(r) = r^{1/2}u_r(r)$ and obtain for w the equation

$$w''(r) + \frac{\mathcal{A}'(r)}{\mathcal{A}(r)}w'(r) - \mathcal{D}(r)w(r) = 0, \quad \text{where } \mathcal{D}(r) = \frac{\mathcal{B}(r)}{\mathcal{A}(r)} + \frac{3}{4r^2} - \frac{1}{2r}\frac{\mathcal{A}'(r)}{\mathcal{A}(r)}. \tag{3-5}$$

We have $\mathcal{A}'(r)/\mathcal{A}(r) = \mathcal{O}(r^{-3})$ and $\mathcal{D}(r) = k^2 + \mathcal{O}(r^{-2})$ as $r \to \infty$, because Remark 1.2 implies $W(r) = \mathcal{O}(r^{-4})$, $W'(r) = \mathcal{O}(r^{-5})$, and $\Phi(r) = \mathcal{O}(r^{-6})$ in that limit. Invoking again [Coddington and

Levinson 1955, Theorem 3.8.1], we deduce that (3-5) has a unique solution w such that $e^{kr}w(r) \to 1$ as $r \to \infty$; hence (3-2) has a unique solution satisfying $r^{1/2}e^{kr}u_r(r) \to 1$ as $r \to \infty$. This solution and its first derivative depend continuously on the parameters in (3-2), uniformly on the interval (R, ∞) for any $R > 0$. Any linearly independent solution of (3-2) grows like $r^{-1/2}e^{kr}$ as $r \to \infty$, and is therefore not square integrable.

Summarizing, we have shown:

Lemma 3.1. *If $m \neq 0$ and $k \neq 0$, any eigenvalue of the linear operator $L_{m,k} \in \mathcal{L}(X_{m,k})$ outside the essential spectrum* (2-4) *is necessarily simple. Moreover, if u_r is the radial velocity profile of the corresponding eigenfunction, there exist $\alpha, \beta \in \mathbb{C}$ such that*

$$\lim_{r\to 0} r^{1-|m|}u_r(r) = \alpha \quad \text{and} \quad \lim_{r\to\infty} r^{1/2}e^{|k|r}u_r(r) = \beta.$$

3B. ***Eigenvalues on the imaginary axis: Kelvin waves.*** In a second step, we consider the eigenvalues of the linearized operator $L_{m,k}$ on the imaginary axis. The corresponding eigenfunctions describe "vibration modes" of the columnar vortex and were first studied by Kelvin [1880] in the particular case of Rankine's vortex. Strictly speaking, this subsection is not part of the proof of Theorem 1.3, but in view of the physical relevance of the Kelvin waves it is worth mentioning a few results that can be rigorously established.

In what follows, we thus assume that $a = 0$ and $b \notin (0, 1)$, so that $\gamma_\star(r) \neq 0$ for all $r > 0$. In that case (3-2) has real coefficients, and its solutions can be studied using standard ODE techniques. For simplicity we suppose here that the vorticity profile $W \in \mathscr{W}$ is the restriction to $\mathbb{R}_+$ of a smooth even function on $\mathbb{R}$ satisfying $W''(0) < 0$, as is the case for the Kaufmann–Scully vortex (1-8) or the Lamb–Oseen vortex (1-9). We consider separately the regimes where $b \geq 1$ and $b \leq 0$.

Lemma 3.2. *For any $m \neq 0$ and $k \neq 0$, the set of all $b > 1$ such that* (3-2) *with $a = 0$ has a nontrivial solution in $H^1(\mathbb{R}_+, r\,\mathrm{d}r)$ is a countable family which accumulates only at* 1. *Moreover,* (3-2) *has no nontrivial solution in $H^1(\mathbb{R}_+, r\,\mathrm{d}r)$ if $a = 0$ and $b = 1$.*

Proof. When $b > 1$, we apply to (3-2) the change of variables $u_r = r^m \mathcal{A}(r)^{-1/2}v$, where $\mathcal{A}(r)$ is as in (3-3). A direct calculation shows that the new function v satisfies

$$-\partial_r^2 v - \frac{2m+1}{r}\partial_r v + (k^2 + \mathcal{F}(r) + \mathcal{G}(r))v = 0, \quad r > 0, \tag{3-6}$$

where

$$\mathcal{F}(r) = \frac{k^2\mathcal{A}(r)}{r^2}(-2 + 3k^2\mathcal{A}(r)), \quad \mathcal{G}(r) = -\frac{k^2}{m^2}\frac{\Phi(r)}{\gamma_\star(r)^2} + \frac{r}{\mathcal{A}(r)\gamma_\star(r)}\partial_r\left(\frac{W(r)}{m^2 + k^2r^2}\right).$$

We assume that $b = 1 + h^2$ for some small $h > 0$, and we expand

$$-\gamma_\star(r) = 1 + h^2 - \Omega(r) = h^2 + \rho r^2 + \mathcal{O}(r^4) \quad \text{as } r \to 0,$$

where $\rho = -\Omega''(0)/2 = -W''(0)/8 > 0$. If $r = hs$, it is straightforward to verify that

$$h^4(k^2 + \mathcal{F}(hs) + \mathcal{G}(hs)) = -\frac{4k^2}{m^2}\frac{1}{(1 + \rho s^2)^2} + \mathcal{O}(h^2) \quad \text{as } h \to 0,$$

uniformly for all $s>0$. Thus the new function w defined by setting $w(s)=v(hs)$ satisfies the semiclassical Schrödinger equation

$$\mathscr{L}_h w := -h^2\left(\partial_s^2 w + \frac{2m+1}{s}\partial_s w\right) - \frac{4k^2}{m^2}\frac{w}{(1+\rho s^2)^2} + U(s,h)w = 0, \tag{3-7}$$

for all $s>0$, where $U(s,h)=\mathcal{O}(h^2)$ as $h\to 0$, uniformly in s. Since the principal part of the potential term in (3-7) is negative, standard results in semiclassical analysis [Simon 1983; Helffer and Sjöstrand 1984] show that the operator $\mathscr{L}_h$ has negative eigenvalues if h is sufficiently small, and that the number of these bound states is $\mathcal{O}(h^{-1})$ as $h\to 0$. Moreover, since $\mathcal{F}(r)+\mathcal{G}(r)\to 0$ as $r\to\infty$, the bottom of the essential spectrum of $\mathscr{L}_h$ is $k^2h^4>0$ for any $h>0$. These two observations together imply that $\mathscr{L}_h$ has a zero eigenvalue for a countable sequence $h_n\to 0$, and returning to the original variables we conclude that (3-2) with $a=0$ has a nontrivial solution in $H^1(\mathbb{R}_+, r\,\mathrm{d}r)$ for a sequence $b_n = 1+h_n^2\to 1$.

When $b=1$, namely $h=0$, the leading term in the function $\mathcal{B}(r)/\mathcal{A}(r)$ satisfies

$$\frac{k^2}{m^2}\frac{\Phi(r)}{\gamma_\star(r)^2} = \frac{\Theta^2}{r^4}(1+\mathcal{O}(r^2)) \quad \text{as } r\to 0, \quad \text{where } \Theta^2 = \frac{4k^2}{m^2\rho^2}.$$

To investigate the behavior of the solutions of (3-2) near $r=0$ in that case, it is useful make the change of variables $u_r(r)=r^{-1/2}U(1/r)$. Setting $x=1/r$, this leads to an equation of the form

$$U''(x)+\widetilde{\mathcal{C}}\left(\frac{1}{x}\right)U'(x)+\widetilde{\mathcal{D}}\left(\frac{1}{x}\right)U(x)=0, \quad x>0, \tag{3-8}$$

where $\widetilde{\mathcal{C}}(r)=\mathcal{O}(r^3)$ and $\widetilde{\mathcal{D}}(r)=\Theta^2+\mathcal{O}(r^2)$ as $r\to 0$. Using [Coddington and Levinson 1955, Theorem 3.8.1], we deduce that (3-8) has two linearly independent solutions satisfying $U_\pm(x) = e^{\pm i\Theta x}(1+\mathcal{O}(1/x))$ as $x\to+\infty$. If we now return to the original variables, we conclude that (3-2) has two linearly independent solutions $\phi_\pm$ such that

$$\phi_\pm(r) = \frac{1}{\sqrt{r}}e^{\pm i\Theta/r}(1+\mathcal{O}(r)) \quad \text{as } r\to 0. \tag{3-9}$$

As is easily verified, no nontrivial linear combination of ϕ_+ and ϕ_- can belong to $H^1(\mathbb{R}_+, r\,\mathrm{d}r)$, which means that (3-2) has no nontrivial solution if $a=0$ and $b=1$. □

The situation is completely different when $b\le 0$.

Lemma 3.3. *For any $m\neq 0$ and $k\neq 0$, the set of all $b\le 0$ such that* (3-2) *with $a=0$ has a nontrivial solution in $H^1(\mathbb{R}_+, r\,\mathrm{d}r)$ is finite. Moreover*:

(1) *This set is nonempty for a finite number of values of m only.*

(2) *For both the Kaufmann–Scully vortex* (1-8) *and the Lamb–Oseen vortex* (1-9), *equation* (3-2) *has no nontrivial solution when $a=0$ and $b\le 0$ if $|m|\ge 2$.*

Proof. If $a=0$ and $b\le 0$, then $\gamma_\star(r)=\Omega(r)-b=\Omega(r)+|b|>0$. In this region, it is easy to verify that the coefficient $\mathcal{B}(r)<1$ defined in (3-3) is an increasing function of both parameters $|m|$ and $|b|$.

Moreover, using the bounds on Ω, W, and Φ which follow from Assumptions H1, H2, see Remark 1.2, we obtain the estimate

$$\sup_{r>0}(1-\mathcal{B}(r)) \le \frac{C}{m^2}\sup_{r>0}\left(\frac{\Phi(r)}{(\Omega(r)+|b|)^2}+\frac{W(r)+r|W'(r)|}{\Omega(r)+|b|}\right) \le \frac{C}{m^2}\frac{1}{1+|b|},$$

where the constant C depends only on the vorticity profile. As a consequence, we see that $\mathcal{B}(r) \ge 0$ when $|m|$ or $|b|$ is large enough, and this implies that (3-2) has no nontrivial solution; see (3-10) below. It follows that the linearized operator $L_{m,k}$ can have eigenvalues $s = m(a-ib)$ with $a = 0$ and $b \le 0$ only for a finite number of values of $m \in \mathbb{Z}$, and using Sturm–Liouville theory we also conclude that, for any $m \in \mathbb{Z}$, there exist only finitely many eigenvalues with $a = 0$ and $b \le 0$. Interestingly enough, for both the Kaufmann–Scully vortex (1-8) and the Lamb–Oseen vortex (1-9), an explicit calculation, which is reproduced in Section A7, shows that $\mathcal{B}(r) \ge 1-4/m^2$, so that there are no eigenvalues in this region when $|m| \ge 2$. □

As a final comment, we mention that when $m = \pm 1$ there are always eigenvalues with $a = 0$ and $b \le 0$. Indeed, due to translation invariance, the operator $L_{m,0}$ has a zero eigenvalue with eigenfunction

$$u = -im\Omega e_r + (W-\Omega)e_\theta, \quad \omega = W'e_z.$$

That eigenvalue bifurcates out of the essential spectrum as the parameter k varies, so that $L_{m,k}$ has at least one eigenvalue $s = -imb$ with $b < 0$ if $|m| = 1$ and $|k|$ is small enough.

3C. *Eigenvalues outside the imaginary axis*: *Howard identities.* For our next step in the study of the eigenvalue equation (3-2), we use a classical method originally due to Rayleigh [1879] to show that the linearized operator $L_{m,k}$ has no spectrum in large regions of the complex plane, which are depicted in Figure 1. The idea is to derive integral identities satisfied by the hypothetical eigenfunctions, which eventually lead to a contradiction.

Assume thus that the eigenvalue equation (3-2) has a nontrivial solution $u_r \in H^1(\mathbb{R}_+, r\,\mathrm{d}r)$ for some $s = m(a-ib) \in \mathbb{C}$, where $a \ne 0$. Multiplying both sides of (3-2) by $r\bar{u}_r$ and integrating over $\mathbb{R}_+$, we easily obtain, using the results of Section 3A

$$\int_0^\infty (\mathcal{A}(r)|\partial_r^* u_r|^2 + \mathcal{B}(r)|u_r|^2) r\,\mathrm{d}r = 0. \tag{3-10}$$

Note that the function $\mathcal{B}$ is complex-valued if $a \ne 0$, so that (3-10) gives two integral relations for the radial velocity u_r. For instance, taking the imaginary part of (3-10) and using the expression (3-3) of $\mathcal{B}$, we obtain the identity

$$a\int_0^\infty \left\{\frac{2(b-\Omega(r))}{(a^2+(\Omega-b)^2)^2}\frac{k^2}{m^2}\mathcal{A}(r)\Phi(r) + \frac{r}{a^2+(\Omega-b)^2}\partial_r\left(\frac{W(r)}{m^2+k^2r^2}\right)\right\}|u_r|^2 r\,\mathrm{d}r = 0. \tag{3-11}$$

This relation is identically satisfied if $a = 0$, but gives useful information if $a \ne 0$. For instance, if $b \le 0$, then $b - \Omega(r) < 0$ for all $r > 0$, and Assumption H1 implies

$$\Phi(r) > 0 \quad \text{and} \quad \partial_r\left(\frac{W(r)}{m^2+k^2r^2}\right) < 0 \quad \text{for all } r > 0.$$

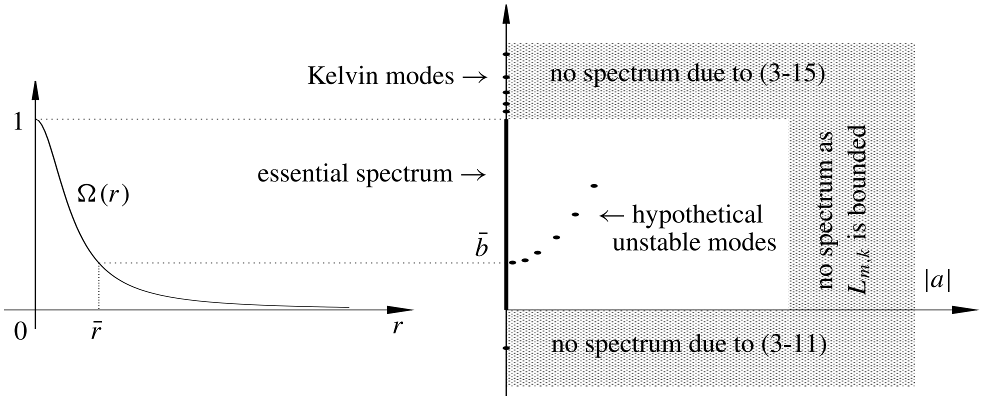

Figure 1. Right: The information obtained so far on the spectrum of the linearized operator (2-1), using the parametrization $s = m(a - ib)$. Kelvin modes are located on the imaginary axis $a = 0$, and accumulate only at the upper edge of the essential spectrum, which fills the segment $a = 0$, $b \in [0, 1]$. The rest of the spectrum, if any, consists of isolated eigenvalues which can accumulate only on the essential spectrum, and are contained in a region of the form $|a| \leq M$, $b \in [0, 1]$ according to Proposition 3.4. Left: The angular velocity profile Ω and the critical radius $\bar{r}$ associated with a spectral point $a = 0$, $b = \bar{b}$ inside the essential spectrum.

Thus the integrand in (3-11) is nonpositive and not identically zero; hence equality (3-11) cannot hold. We conclude that the operator $L_{m,k}$ has no eigenvalue $s = m(a - ib)$ with $a \neq 0$ and $b \leq 0$; see Figure 1. Unfortunately, we do not know how to use the relation (3-10) to preclude the existence of eigenvalues of $L_{m,k}$ in other regions of the complex plane.

The following approach, due to [Howard 1961; Howard and Gupta 1962], provides other identities similar to (3-10), which give further information on the possible eigenvalues. Define $u_r = q(r)v_r$, where q is a (real- or complex-valued) weight function satisfying $q(r) \neq 0$ for all $r > 0$. Then v_r is a solution to

$$-\partial_r(q(r)^2\mathcal{A}(r)\partial_r^* v_r) + \mathcal{E}(r)v_r = 0, \tag{3-12}$$

where

$$\mathcal{E}(r) = q(r)^2\mathcal{B}(r) - q(r)q'(r)\left(\mathcal{A}'(r) - \frac{\mathcal{A}(r)}{r}\right) - q(r)q''(r)\mathcal{A}(r).$$

Multiplying both sides of (3-12) by $r\bar{v}_r$ and integrating over $\mathbb{R}_+$, we deduce

$$\int_0^\infty (q(r)^2\mathcal{A}(r)|\partial_r^* v_r|^2 + \mathcal{E}(r)|v_r|^2)r\,\mathrm{d}r = 0. \tag{3-13}$$

If q is real-valued, then $q^2|v_r|^2 = |u_r|^2$ and taking the imaginary part of (3-13) we recover (3-11), but the real part gives new information. If q is complex, both the real and the imaginary parts of (3-13) provide new information.

Following [Howard and Gupta 1962], we now consider in more detail some interesting particular cases of (3-13).

Case 1: $q(r)=\gamma_\star(r)$. We then have

$$\mathcal{E}(r)=\gamma_\star(r)^2-\frac{k^2}{m^2}\mathcal{A}(r)\Phi(r)+r\gamma_\star(r)\partial_r\Bigl(\frac{W(r)}{m^2+k^2r^2}\Bigr)-\gamma_\star(r)\gamma_\star'(r)\Bigl(\mathcal{A}'(r)-\frac{\mathcal{A}(r)}{r}\Bigr)-\gamma_\star(r)\gamma_\star''(r)\mathcal{A}(r).$$

Since $r\gamma_\star'(r)=r\Omega'(r)=W(r)-2\Omega(r)$, we observe that

$$\gamma_\star'(r)\Bigl(\mathcal{A}'(r)-\frac{\mathcal{A}(r)}{r}\Bigr)+\gamma_\star''(r)\mathcal{A}(r)=r\,\partial_r\Bigl(\frac{\gamma_\star'(r)\mathcal{A}(r)}{r}\Bigr)=r\,\partial_r\Bigl(\frac{W(r)-2\Omega(r)}{m^2+k^2r^2}\Bigr), \tag{3-14}$$

and we deduce the following simpler expression of $\mathcal{E}(r)$:

$$\mathcal{E}(r)=\gamma_\star(r)^2-\frac{k^2}{m^2}\mathcal{A}(r)\Phi(r)+2r\gamma_\star(r)\partial_r\Bigl(\frac{\Omega(r)}{m^2+k^2r^2}\Bigr).$$

In particular, taking the imaginary part of (3-13), we obtain the identity

$$2a\int_0^\infty\Bigl\{(b-\Omega(r))(\mathcal{A}(r)|\partial_r^*v_r|^2+|v_r|^2)-r\,\partial_r\Bigl(\frac{\Omega(r)}{m^2+k^2r^2}\Bigr)|v_r|^2\Bigr\}r\,\mathrm{d}r=0. \tag{3-15}$$

If we now assume that $b\ge 1$, so that $b-\Omega(r)>0$ for all $r>0$, we see that all terms in the integrand of (3-15) are nonnegative, which leads to a contradiction if $a\neq 0$. We conclude that the linear operator $L_{m,k}$ has no eigenvalue $s=m(a-ib)$ if $a\neq 0$ and $b\ge 1$; see Figure 1.

Case 2: $q(r)=\gamma_\star(r)^{1/2}$. Proceeding as above, we find

$$\mathcal{E}(r)=\gamma_\star(r)-\frac{k^2}{m^2}\frac{\mathcal{A}(r)\Phi(r)}{\gamma_\star(r)}+r\,\partial_r\Bigl(\frac{W(r)}{m^2+k^2r^2}\Bigr)-\frac{1}{2}\gamma_\star'(r)\Bigl(\mathcal{A}'(r)-\frac{\mathcal{A}(r)}{r}\Bigr)-\frac{1}{2}\gamma_\star''(r)\mathcal{A}(r)+\frac{1}{4}\frac{\gamma_\star'(r)^2}{\gamma_\star(r)}\mathcal{A}(r).$$

Using again (3-14), we deduce that

$$\mathcal{E}(r)=\gamma_\star(r)-\frac{k^2}{m^2}\frac{\mathcal{A}(r)\Phi(r)}{\gamma_\star(r)}+\frac{r}{2}\partial_r\Bigl(\frac{W(r)+2\Omega(r)}{m^2+k^2r^2}\Bigr)+\frac{1}{4}\frac{\Omega'(r)^2}{\gamma_\star(r)}\mathcal{A}(r).$$

In particular, taking the imaginary part of (3-13), we obtain the identity

$$-a\int_0^\infty\Bigl\{\mathcal{A}(r)|\partial_r^*v_r|^2+|v_r|^2+\frac{\mathcal{A}(r)}{a^2+(\Omega-b)^2}\Bigl(\frac{k^2\Phi(r)}{m^2}-\frac{\Omega'(r)^2}{4}\Bigr)|v_r|^2\Bigr\}r\,\mathrm{d}r=0. \tag{3-16}$$

As a consequence, if we assume that

$$J(r)\equiv\frac{\Phi(r)}{\Omega'(r)^2}\ge\frac{m^2}{4k^2}\quad\text{for all } r>0, \tag{3-17}$$

we see that all terms in the integrand of (3-16) are nonnegative, which leads to a contradiction if $a\neq 0$. We conclude that (3-17) is a *sufficient condition for spectral stability*. Unfortunately, condition (3-17) is never met for the Lamb–Oseen vortex, because $J(r)\to 0$ as $r\to+\infty$ in that case. In the case of the Kaufmann–Scully vortex, it is satisfied only if $m^2\le 4k^2$.

The results obtained by Howard's approach can thus be summarized as follows.

Proposition 3.4. *Assume that the vorticity profile W satisfies Assumption H1 in Section 1B. Then for any $m\neq 0$ and $k\neq 0$ the following hold*:

(i) *The linearized operator* $L_{m,k}$ *has no eigenvalue* $s = m(a - ib)$ *with* $a \neq 0$ *and* $b(1-b) \leq 0$.

(ii) *If condition* (3-17) *is satisfied, then* $L_{m,k}$ *has no eigenvalue outside the imaginary axis.*

3D. ***Critical layers and embedded eigenvalues.*** We assume here that $a = 0$ and $0 < b < 1$, which means that the spectral parameter $s = m(a - ib)$ is contained in the essential spectrum of the linearized operator $L_{m,k}$ and does not coincide with one of its endpoints. The natural extension of the eigenvalue equation (3-2) to this situation[2] is

$$-\partial_r(\mathcal{A}(r)\partial_r^* u_r) + \left[1 - \frac{k^2}{m^2}\frac{\mathcal{A}(r)\Phi(r)}{(\Omega(r)-b)^2} + \frac{r}{(\Omega(r)-b)}\partial_r\left(\frac{W(r)}{m^2+k^2r^2}\right)\right]u_r = 0. \tag{3-18}$$

Since the value of b belongs to the range of the angular velocity Ω, both denominators in (3-18) vanish at exactly one point $\bar{r} > 0$, characterized by $\Omega(\bar{r}) = b$, so that (3-18) becomes singular at that point. In the physical literature, singularities of the eigenvalue equation are usually avoided by allowing the variable r to take slightly complex values, a procedure that is referred to as "critical layer analysis" in this context [Drazin and Reid 1981].

To perform such an analysis, we restrict our attention in the rest of this section to vorticity profiles W which satisfy Assumption H1 and, in addition, are real-analytic on $(0,\infty)$, so that the angular velocity Ω and the Rayleigh function Φ are analytic too. According to the usual terminology, the point $\bar{r}$ is then a *regular singular point* of (3-18); see, e.g., [Coddington and Levinson 1955, Chapter 4] or Section A3 below. Extending the range of the variable r to a neighborhood of $(0,\infty)$ in $\mathbb{C}$ allows us to make a connection between solutions of (3-18) defined on the interval $(0,\bar{r})$ and others defined on $(\bar{r},\infty)$. In a neighborhood of $\bar{r}$, the behavior of the solutions of (3-18) is determined by the roots d of the *indicial equation*

$$d(d-1) + \frac{k^2}{m^2}J(\bar{r}) = 0. \tag{3-19}$$

We distinguish three cases.

Case 1: $0 < J(\bar{r}) < m^2/(4k^2)$. The roots of (3-19) are real and simple:

$$d_\pm := \frac{1}{2} \pm \left(\frac{1}{4} - \frac{k^2}{m^2}J(\bar{r})\right)^{1/2}.$$

In particular, we have $0 < d_- < \frac{1}{2} < d_+ < 1$. The Frobenius method [Coddington and Levinson 1955, Section 4.8] can be used to construct two real-valued analytic functions $V_\pm$ on $(0,\infty)$ such that $V_\pm(\bar{r}) = 1$ and such that the functions $\phi_\pm$ defined by

$$\phi_\pm(r) = |b - \Omega(r)|^{d_\pm} e^{(i\pi/2)(1-\mathrm{sgn}(b-\Omega(r)))d_\pm} V_\pm(r), \quad r > 0, \tag{3-20}$$

are independent solutions of (3-18) on both intervals $(0,\bar{r})$ and $(\bar{r},\infty)$. Note that $\phi_\pm$ are real-valued on $(\bar{r},\infty)$, but complex-valued (although with a constant phase) on the interval $(0,\bar{r})$.

[2] We emphasize that the derivation of (3-2) from the spectral problem was performed in Section 2 under the assumption that $s = m(a - ib)$ does not belong to the essential spectrum.

Case 2: $J(\bar{r}) > m^2/(4k^2)$. The roots of (3-19) are complex conjugate:

$$d_\pm := \frac{1}{2} \pm i\delta, \quad \text{where } \delta = \Big(\frac{k^2}{m^2} J(\bar{r}) - \frac{1}{4}\Big)^{1/2}.$$

Similarly, the Frobenius method yields the existence of two independent solutions $\phi_\pm$, which we write in the form

$$\phi_\pm(r) = |b - \Omega(r)|^{1/2} e^{\pm i\delta \log|b-\Omega(r)|} e^{(i\pi/2)(1-\operatorname{sgn}(b-\Omega(r)))d_\pm} V_\pm(r). \tag{3-21}$$

Case 3: $J(\bar{r}) = m^2/(4k^2)$. Equation (3-19) possesses the unique root $\frac{1}{2}$ with multiplicity 2, and two independent solutions of (3-18) can be constructed such that

$$\begin{aligned}
\phi_+(r) &= |b - \Omega(r)|^{1/2} e^{(i\pi/4)(1-\operatorname{sgn}(b-\Omega(r)))} V_+(r),\\
\phi_-(r) &= |b - \Omega(r)|^{1/2} e^{(i\pi/4)(1-\operatorname{sgn}(b-\Omega(r)))}\\
&\qquad \times \Big[\Big(\log|b-\Omega(r)| + \tfrac{i\pi}{2}(1-\operatorname{sgn}(b-\Omega(r)))\Big) V_+(r) + V_-(r)\Big].
\end{aligned} \tag{3-22}$$

The following technical lemma emphasizes the relevance of the singular functions $\phi_\pm$ for the approximation of solutions of (3-18) by nonsingular solutions of (3-2). In the statement, the vorticity profile W, the spectral parameter $s = -imb$ and the corresponding singular radius $\bar{r}$ are defined as above. However, we consider a sequence $(u_n)_{n\in\mathbb{N}}$ of smooth solutions of the eigenvalue equation (3-2) where the spectral parameter s is replaced by some complex number s_n with nonzero real part (so that s_n does not belong to the essential spectrum), and where also the vorticity profile W is replaced by some function W_n that is allowed to depend on n.[3] We assume that $s_n \to s$ and $W_n \to W$ as $n \to \infty$. In what follows, for $\rho > 0$ we denote by $\mathbb{D}(\bar{r}, \rho) \subset \mathbb{C}$ the open disc of radius ρ centered at $\bar{r} + 0i$. When no confusion is possible, we also use the same symbols for functions of the real variable $r \in (0, \infty)$ and their analytic extensions into (part of) the complex plane.

Lemma 3.5. *Let* $(u_n)_{n\in\mathbb{N}}$ *be a sequence of solutions of* (3-2) *corresponding to a sequence of spectral parameters* $s_n = m(a_n - ib_n)$ *and of real-valued analytic profiles* W_n. *Suppose that*

(i) $a_n > 0$ *for all* n, *and* $a_n \to 0$, $b_n \to b \in (0, 1)$ *as* $n \to \infty$;

(ii) $W_n \to W$ *in* $C^1_b(\overline{\mathbb{R}}_+)$ *as* $n \to \infty$;

(iii) *there exists* $\rho > 0$ *such that, for all* $n \in \mathbb{N}$, *the radius of analyticity of* W_n *at* $\bar{r}$ *is at least equal to* ρ, *and* $W_n \to W$ *uniformly in* $\mathbb{D}(\bar{r}, \rho)$.

If $u_n(r)$ *and* $u_n'(r)$ *have a limit as* $n \to \infty$ *for some* $r \in (0, \infty) \setminus \{\bar{r}\}$, *then there exist* $\alpha_\pm \in \mathbb{C}$ *such that* $u_n \to \alpha_+\phi_+ + \alpha_-\phi_-$ *in the* C^1 *topology on compact subsets of* $(0, \infty) \setminus \{\bar{r}\}$, *where* $\phi_\pm$ *are given by* (3-20), (3-21) *or* (3-22) *depending on the roots of the indicial equation.*

The proof of Lemma 3.5 is postponed to Section A3 below where we also establish the main properties of $\phi_\pm$, in particular the analyticity of $V_\pm$ across the singularity $\bar{r}$ and the fact these functions are real-valued. For the moment, we observe that the implicit determination of logarithms we opted for in constructing the solutions $\phi_\pm$ is directly related to the assumption that $a_n > 0$ in Lemma 3.5. An approximation procedure

[3] The reason for the latter will become clear in Section 4.

valid for negative values of a_n would involve the complex conjugates of the functions $\phi_\pm$ defined in (3-20)–(3-22).

Remark 3.6. The expressions (3-20)–(3-22) show in particular that no nontrivial solution of (3-18) lies in $H^2(\bar{r}-\epsilon, \bar{r}+\epsilon)$ if $\epsilon > 0$. As we know that the radial velocity u_r associated with any vorticity vector $\omega \in X_{m,k}$ belongs to $H^1(\mathbb{R}_+, r\,\mathrm{d}r) \cap H^2_{\mathrm{loc}}(\mathbb{R}_+)$, we deduce from the observation above that the linear operator $L_{m,k}$ acting on $X_{m,k}$ has no nonzero eigenvalue s embedded in the continuous spectrum (2-4).

Finally, repeating the proof of Lemma 3.1 for (3-18), we easily obtain:

Lemma 3.7. *If* $m \neq 0$, $k \neq 0$, *and* $0 < b < 1$, *there exist a unique solution* ψ_0 *of* (3-18) *on* $(0, \bar{r})$ *and a unique solution* ψ_∞ *of* (3-18) *on* $(\bar{r}, \infty)$ *such that*

$$\lim_{r\to 0} r^{1-|m|}\psi_0(r) = 1 \quad \textit{and} \quad \lim_{r\to\infty} r^{1/2}e^{|k|r}\psi_\infty(r) = 1.$$

Moreover, both ψ_0 *and* ψ_∞ *are real-valued.*

Since (3-18) is a linear equation, we infer the existence of constants $\alpha_\pm^0, \alpha_\pm^\infty \in \mathbb{C}$ such that

$$\psi_0 = \alpha_-^0\phi_- + \alpha_+^0\phi_+ \quad \text{on } (0, \bar{r}), \qquad \psi_\infty = \alpha_-^\infty\phi_- + \alpha_+^\infty\phi_+ \quad \text{on } (\bar{r}, \infty), \tag{3-23}$$

where $\phi_\pm$ are defined in (3-20)–(3-22).

4. The homotopy argument

This section is the core of the proof of Theorem 1.3. We concentrate on the situation where the angular Fourier mode m and the vertical wave number k are both nonzero, because the cases $m = 0$ and $k = 0$ have already been treated in Sections 2A and 2B, respectively. In view of the symmetry properties (2-6), we can assume without loss of generality that $m \geq 1$ and $k > 0$.

The argument is by contradiction: given a vorticity profile W satisfying Assumptions H1, H2 in Section 1B, we assume that there exist an integer $m \geq 1$ and a real number $k > 0$ such that the linearized operator $L_{m,k}$ has at least one eigenvalue outside the imaginary axis. The strategy is then to perform a homotopy between the vorticity profile $W_0 := W$ and a reference profile W_1 for which we know a priori that the corresponding linearized operator has no eigenvalue with nonzero real part. Since eigenvalues outside the imaginary axis depend continuously on the vorticity profile, in an appropriate topology, this implies in our situation that all eigenvalues necessarily merge into the essential spectrum as the homotopy parameter varies from 0 to 1. We eventually reach a contradiction by showing that such a merger is impossible. This is achieved by a careful asymptotic analysis of the solutions of the complex ODE (3-2) in the limit where the real part of the eigenvalue $s = m(a - ib)$ vanishes. Our approach combines the results of Section 3D on critical layers, the integral identities obtained by Howard's method in Section 3C, and new ingredients which rely on the monotonicity Assumption H2.

Since we have to consider various vorticity profiles in the course the proof, the linearized operator (2-1) will sometimes be denoted by $L_{m,k}^W$ instead of $L_{m,k}$, to avoid any ambiguity. The following continuity property plays an essential role in our argument.

Lemma 4.1. *The* (*linear*) *mapping* $W \mapsto L^W_{m,k}$ *is continuous from* $\mathcal{C}^1_b(\overline{\mathbb{R}}_+)$ *into* $\mathcal{L}(X_{m,k})$.

Proof. As can be seen from definitions (2-1)–(2-3), the linearized operator $L^W_{m,k}$ has variable coefficients depending (linearly) on the functions Ω, $r\Omega'$, W, and W'. Now, we have the estimate

$$\|\Omega\|_{L^\infty(\mathbb{R}_+)} + \|r\Omega'\|_{L^\infty(\mathbb{R}_+)} \le C\|W\|_{L^\infty(\mathbb{R}_+)},$$

which follows from the representation formula (1-16) and the identity $r\Omega' = W - 2\Omega$. Thus all coefficients of $L^W_{m,k}$ are L^∞ functions that depend continuously on W in the topology of $\mathcal{C}^1_b(\overline{\mathbb{R}}_+)$, and since the Biot–Savart map $\omega \mapsto u$ is bounded in $X_{m,k}$ by Proposition A.1 below, we obtain the desired result. □

4A. *Reduction to a real analytic vorticity profile.* We now present the contradiction argument in detail. We fix $m \ge 1$, $k > 0$, and we assume that there exists a vorticity profile $W_0 \in \mathscr{W}$ such that the associated linear operator $L^{W_0}_{m,k} \in \mathcal{L}(X_{m,k})$ has at least one (isolated) eigenvalue outside the imaginary axis. Our goal is to prove that this is impossible, which is exactly the conclusion of Theorem 1.3.

In a first step, we show that one can assume without loss of generality that the profile $W_0 \in \mathscr{W}$ is *real analytic* on $\overline{\mathbb{R}}_+$. By this we mean more precisely that W_0 is the restriction to $\overline{\mathbb{R}}_+$ of a real analytic even function defined on the whole real line. Indeed, we know from Proposition 2.1 that, for any $W \in \mathscr{W}$, the spectrum of $L^W_{m,k}$ outside the imaginary axis consists of isolated eigenvalues with finite multiplicity, which are in fact simple as asserted by Lemma 3.1. Invoking Lemma 4.1 and classical perturbation theory [Kato 1966, Chapter 4, §3.5], we observe that these (hypothetical) eigenvalues depend continuously on the vorticity profile W in the topology of $\mathcal{C}^1_b(\overline{\mathbb{R}}_+)$. In particular, if W is close enough to W_0 in that topology, we are sure that the operator $L^W_{m,k}$ has at least one eigenvalue with nonzero real part.

We next invoke a density result that will be established in Section A4 below.

Lemma 4.2. *The subset* $\mathscr{W}^\omega$ *of* $\mathscr{W}$ *consisting of vorticity profiles which are also real analytic on* $\overline{\mathbb{R}}_+$ *is dense in* $\mathscr{W}$ *for the topology of* $\mathcal{C}^1_b(\overline{\mathbb{R}}_+)$.

The proof of Lemma 4.2 is not straightforward because the definition of the class $\mathscr{W}$ involves the quantity J, introduced in (1-20), which depends in a nonlinear way on the vorticity profile W. Thus, given $W \in \mathscr{W}$, we cannot construct an approximation $W_\epsilon \in \mathscr{W}^\omega$ just by taking the convolution of W with a real analytic mollifier. To avoid this difficulty, we prove in Section A4 that all quantities Ω, W, Φ are entirely determined by the auxiliary function J, and we even provide explicit reconstruction formulas. Then, at the level of J, we use a nonlinear approximation scheme of the form

$$\frac{1}{\sqrt{1+J_\epsilon}} = G_\epsilon * \frac{1}{\sqrt{1+J}}, \quad \epsilon > 0,$$

where G_ϵ denotes the heat kernel on the half-line $\mathbb{R}_+$ with Dirichlet boundary condition at $r = 0$. This provides an approximation procedure within the class $\mathscr{W}$ which allows us to prove Lemma 4.2; see Section A4 for details.

Taking advantage of Lemma 4.2 we assume from now on that the initial vorticity profile W_0 in our contradiction argument is real analytic, namely $W_0 \in \mathscr{W}^\omega$.

4B. ***Construction of the homotopy.*** In the particular example of the Kaufmann–Scully vortex (1-8), the function (1-20) reduces to $J(r) = 1 + 1/r^2 \geq 1$. By a simple rescaling we deduce that, for the vorticity profile $W_1 \in \mathscr{W}^\omega$ defined by

$$W_1(r) = \frac{2}{(1 + 4k^2r^2/m^2)^2} = \frac{2m^4}{(m^2 + 4k^2r^2)^2}, \quad r \geq 0, \tag{4-1}$$

the stability condition (3-17) is satisfied, so that the linear operator $L_{m,k}^{W_1}$ has no eigenvalue outside the imaginary axis as a consequence of Proposition 3.4. To interpolate in the class $\mathscr{W}$ between the initial profile $W_0 \in \mathscr{W}^\omega$ and the reference profile (4-1), we use the following result, whose proof is also postponed to Section A4.

Lemma 4.3. *If $W_0, W_1 \in \mathscr{W}$, there exists a Lipschitz function $\mathcal{H} : [0, 1] \to \mathcal{C}_b^1(\overline{\mathbb{R}}_+)$ such that $\mathcal{H}(0) = W_0$, $\mathcal{H}(1) = W_1$, and $W_t := \mathcal{H}(t) \in \mathscr{W}$ for any $t \in [0, 1]$. Moreover, if $W_0, W_1 \in \mathscr{W}^\omega$, then $W_t \in \mathscr{W}^\omega$ for all $t \in [0, 1]$. In that case, if $W_1''(0) < 0$, then $W_t''(0) < 0$ for all $t \in (0, 1]$.*

Since the class $\mathscr{W}$ is not convex, the linear interpolation $H(t) = (1 - t)W_0 + tW_1$ is not appropriate here. Instead, we use again the fact that a vorticity profile $W \in \mathscr{W}$ is entirely determined by the auxiliary function (1-20), and at the level of J we define the homotopy $\mathcal{H}$ by the nonlinear interpolation procedure

$$\frac{1}{\sqrt{1 + J_t}} = \frac{1 - t}{\sqrt{1 + J_0}} + \frac{t}{\sqrt{1 + J_1}} \quad \text{for all } t \in [0, 1]. \tag{4-2}$$

If J_0 and J_1 are real analytic, so is J_t for all $t \in [0, 1]$, and it follows that $W_t \in \mathscr{W}^\omega$ for all $t \in [0, 1]$. We refer to Section A4 for details.

4C. ***The bifurcation point.*** For any $t \in [0, 1]$, we denote by $W_t \in \mathscr{W}^\omega$ the vorticity profile obtained from Lemma 4.3, where $W_0 \in \mathscr{W}^\omega$ is the initial vorticity defined in Section 4A and W_1 is given by (4-1). We also introduce the associated angular velocity

$$\Omega_t(r) = \frac{1}{r^2} \int_0^r W_t(s)s \, \mathrm{d}s, \quad r > 0,$$

and we define $\Phi_t = 2\Omega_t W_t$ and $J_t = \Phi_t/(\Omega_t')^2$ as in (1-19), (1-20). We consider the family of linear operators $L_{m,k}^{W_t}$, indexed by the homotopy parameter $t \in [0, 1]$, which is uniformly bounded in $\mathcal{L}(X_{m,k})$ by Lemma 4.1. For each $t \in [0, 1]$, it follows from Proposition 2.1 and Lemma 3.1 that the spectrum of $L_{m,k}^{W_t}$ outside the imaginary axis consists of simple isolated eigenvalues. If $s = m(a - ib)$ is such an eigenvalue, we know from Proposition 3.4 that $0 < b < 1$, and by uniform boundedness there exists a constant $M > 0$ (independent of t) such that $0 < |a| \leq M$.

As the homotopy parameter t varies, the isolated eigenvalues of $L_{m,k}^{W_t}$ move continuously in the complex plane, as described, e.g., in [Kato 1966, Chapter 4, §3.5], and we chose our reference profile W_1 so that the associated linearized operator has no eigenvalue with nonzero real part. This implies that, when t increases from 0 to 1, all isolated eigenvalues of $L_{m,k}^{W_t}$ eventually merge into the essential spectrum on the imaginary axis. In particular, we can define the bifurcation point

$$t_* = \inf\{t \in (0, 1] \mid \sigma(L_{m,k}^{W_\tau}) \subset i\mathbb{R} \text{ for all } \tau \in [t, 1]\}.$$

Our assumption on W_0 and the continuity of the eigenvalues imply that $t_* > 0$ and $\sigma(L^{W_{t_*}}_{m,k}) \subset i\mathbb{R}$. Moreover, there exist an increasing sequence $t_n \to t_*$ and a sequence of isolated eigenvalues $s_n = m(a_n - ib_n)$ of $L^{W_{t_n}}_{m,k}$ such that $a_n \neq 0,\ 0 < b_n < 1$, and

$$a_n + ib_n \to i\bar{b} \quad \text{as } n \to \infty \tag{4-3}$$

for some $\bar{b} \in [0, 1]$. In view of the second identity in (2-6), we can assume without loss of generality that $a_n > 0$ for all $n \in \mathbb{N}$. Associated with $\bar{b}$, we also introduce the critical radius

$$\bar{r} = \begin{cases} \Omega_{t_*}^{-1}(\bar{b}) & \text{if } \bar{b} > 0, \\ +\infty & \text{if } \bar{b} = 0. \end{cases} \tag{4-4}$$

As $W_{t_*} \in \mathscr{W}^\omega$ by construction, we recall that $\Omega_{t_*} : \overline{\mathbb{R}}_+ \to \mathbb{R}_+$ is real analytic, strictly decreasing on $\mathbb{R}_+$, and satisfies $\Omega_{t_*}(0) = 1$ and $\Omega_{t_*}(r) \to 0$ as $r \to \infty$, so that $\bar{r} \in [0, \infty]$ is well-defined; see Figure 1.

In the sequel, for notational simplicity, we write W instead of W_{t_*} and W_n instead of W_{t_n}. Note in particular that, after this redefinition, the symbol W no longer refers to the vorticity profile that appears in the statement of Theorem 1.3! Similarly, we write

$$\Omega = \Omega_{t_*}, \quad \Phi = \Phi_{t_*}, \quad J = J_{t_*} \quad \text{and} \quad \Omega_n = \Omega_{t_n}, \quad \Phi_n = \Phi_{t_n}, \quad J_n = J_{t_n}.$$

Finally, we also set $L_{m,k} = L^W_{m,k}$ and $L^n_{m,k} = L^{W_n}_{m,k}$. We observe that $W_n \to W$ in $\mathcal{C}^1_b(\overline{\mathbb{R}}_+)$ as $n \to \infty$, due to the continuity properties of the homotopy defined in Lemma 4.3.

As is recalled at the beginning of Section 3, for each $n \in \mathbb{N}$ we may associate to the eigenvalue $s_n = m(a_n - ib_n)$ of $L^n_{m,k}$ a nontrivial solution $u_n \in H^1(\mathbb{R}_+, r\,\mathrm{d}r) \cap H^2_{\mathrm{loc}}(\mathbb{R}_+)$ of the complex differential equation

$$-\partial_r(\mathcal{A}(r)\partial_r^* u_n) + \left[1 - \frac{k^2}{m^2}\frac{\mathcal{A}(r)\Phi_n(r)}{\gamma_n(r)^2} + \frac{r}{\gamma_n(r)}\partial_r\left(\frac{W_n(r)}{m^2 + k^2r^2}\right)\right] u_n = 0, \tag{4-5}$$

where $\gamma_n(r) = \Omega_n(r) - b_n - ia_n$. As $W_n \in \mathscr{W}^\omega$, it is clear that u_n is in fact real analytic for all $n \in \mathbb{N}$. According to Lemma 3.1 there exist nonzero complex numbers α_n, β_n such that

$$\alpha_n = \lim_{r \to 0_+} r^{-m+1} u_n(r) \quad \text{and} \quad \beta_n = \lim_{r \to \infty} r^{1/2} \exp(kr) u_n(r). \tag{4-6}$$

In what follows, we often normalize u_n so that $\beta_n = 1$ for all values of n.

As $n \to \infty$, the ODE (4-5) becomes singular at the point $r = \bar{r}$, because $\gamma_n(\bar{r}) \to \Omega(\bar{r}) - \bar{b} = 0$ in view of (4-3) and (4-4). As is explained in Section 3B, the nature of the critical layer near $r = \bar{r}$ depends upon whether the quantity $J(\bar{r})$ is larger or smaller than $m^2/(4k^2)$. This motivates the following definition:

$$r_* = \begin{cases} J^{-1}\left(\dfrac{m^2}{4k^2}\right) & \text{if } \inf J < \dfrac{m^2}{4k^2}, \\ +\infty & \text{otherwise.} \end{cases} \tag{4-7}$$

Note that $J : (0, \infty) \to \mathbb{R}_+$ is strictly decreasing by Assumption H2, so that r_* is uniquely defined. Moreover,

$$r_* > 0, \quad J(r) > \frac{m^2}{4k^2} \quad \text{for } r < r_* \quad \text{and} \quad J(r) < \frac{m^2}{4k^2} \quad \text{for } r > r_*. \tag{4-8}$$

Also, since $t_* > 0$ we deduce from (4-2) and from our choice (4-1) of W_1 that

$$J(\infty) > 0. \tag{4-9}$$

In the rest of the proof of Theorem 1.3, to reach the desired contradiction, we consider various cases according to whether the critical radius $\bar{r}$ is larger, smaller or equal to 0, r_*, or $+\infty$.

4D. ***The situation $0 < \bar{r} < r_*$ is excluded.*** In this case, a contradiction is obtained from identity (3-16), or rather from its analogue for the solutions of (4-5) where the vorticity profile W_n and the spectral parameter $s_n = m(a_n - ib_n)$ depend on n. In terms of the weighted function

$$v_n(r) = \frac{u_n(r)}{(b_n + ia_n - \Omega_n(r))^{1/2}} = \frac{-iu_n(r)}{\gamma_n(r)^{1/2}}, \quad r > 0, \tag{4-10}$$

the identity becomes, after dividing by $a_n \neq 0$,

$$\int_0^\infty \left\{ \mathcal{A}(r)|\partial_r^* v_n|^2 + |v_n|^2 + \frac{\mathcal{A}(r)\Omega_n'(r)^2}{a_n^2 + (\Omega_n(r) - b_n)^2}\left(\frac{k^2}{m^2}J_n(r) - \frac{1}{4}\right)|v_n|^2 \right\} r\,\mathrm{d}r = 0. \tag{4-11}$$

We choose here to normalize the solutions u_n of (4-5) so that $\beta_n = 1$ in (4-6) for all $n \in \mathbb{N}$. This implies, in view of the analysis in Sections 3A and 3D and of the definitions in Section 4C, that $u_n(r) \to \psi_\infty(r)$ and $u_n'(r) \to \psi_\infty'(r)$ locally uniformly on $(\bar{r}, \infty]$ as $n \to \infty$, where ψ_∞ is the solution of the limiting equation (3-18) introduced in Lemma 3.7. Moreover, for any $\epsilon > 0$, the sequence (u_n) is uniformly bounded in $H^1([\bar{r} + \epsilon, \infty), r\,\mathrm{d}r)$, and so is the sequence (v_n) since $|\gamma_n(r)|$ is bounded away from zero when $r \geq \bar{r} + \epsilon$. This uniform H^1 bound means that the restriction of the integral in (4-11) to the interval $[\bar{r} + \epsilon, \infty)$ is uniformly bounded for all $n \in \mathbb{N}$. As the integral over $(0, +\infty)$ is equal to zero, we deduce that the integral over $(0, \bar{r} + \epsilon)$ is also uniformly bounded, namely

$$\sup_{n\in\mathbb{N}} \int_0^{\bar{r}+\epsilon} \left\{ \mathcal{A}(r)|\partial_r^* v_n|^2 + |v_n|^2 + \frac{\mathcal{A}(r)\Omega_n'(r)^2}{a_n^2 + (\Omega_n(r) - b_n)^2}\left(\frac{k^2}{m^2}J_n(r) - \frac{1}{4}\right)|v_n|^2 \right\} r\,\mathrm{d}r < \infty. \tag{4-12}$$

Now comes into play the assumption that $\bar{r} < r_*$. If we choose $\epsilon > 0$ small enough so that $\bar{r} + \epsilon < r_*$, we observe that, due to the definition of r_* in (4-7), the integrand in (4-12) is nonnegative when n is sufficiently large. Moreover, for all $r > \bar{r}$, we know from (4-10) that $v_n(r) \to \psi_\infty(r)(\bar{b} - \Omega(r))^{-1/2}$ as $n \to \infty$. So restricting the integral to the interval $(\bar{r}, \bar{r} + \epsilon)$ and invoking Fatou's lemma, we deduce from (4-12) that

$$\int_{\bar{r}}^{r_*} \frac{\mathcal{A}(r)\Omega'(r)^2}{(\bar{b} - \Omega(r))^3}\left(\frac{k^2}{m^2}J(r) - \frac{1}{4}\right)|\psi_\infty(r)|^2 r\,\mathrm{d}r < \infty. \tag{4-13}$$

The inequality $\bar{r} < r_*$ also means that the roots of the indicial equation (3-19) are complex conjugate so that, according to what we called Case 2 in Section 3D, we have the decomposition $\psi_\infty(r) = \alpha_-^\infty\phi_-(r) + \alpha_+^\infty\phi_+(r)$ for $r > \bar{r}$, where $\phi_\pm$ are given by (3-21). From these expressions, it is easy to deduce that (4-13) cannot hold if ψ_∞ is replaced by either ϕ_+ or ϕ_-, because the integrand is positive and behaves like $(r - \bar{r})^{-2}$ in a neighborhood of $\bar{r}$. In the general case where both coefficients $\alpha_\pm^\infty$ are nonzero, there may be cancellations between the contributions of ϕ_+ and ϕ_-, but due to the logarithmic phases in the expressions (3-21) of $\phi_\pm$ the function ψ_∞ cannot vanish on large sets. More precisely, given $\alpha_\pm^\infty \in \mathbb{C}$, there exist $\epsilon > 0$ and $0 < \rho_0 < r_* - \bar r$ such that, for any $\rho \in (0, \rho_0)$,

$$\frac{1}{\rho}\,\mathrm{meas}\big(\{r \in (\bar r, \bar r + \rho) \text{ such that } |\psi_\infty(r)|^2 \geq \epsilon |r - \bar r|\}\big) \geq \frac{1}{2},$$

and the same argument as above shows that (4-13) is impossible.

4E. ***The situation $\bar r = 0$ is excluded.*** This case is treated by the same argument as in the previous section up to and including inequality (4-13). The only difference at that level is the asymptotic behavior of the functions $\phi_\pm(r)$ as $r \to \bar r$, because $\bar r = 0$ is now an irregular singular point of the ODE (3-18). According to (3-9), we have the expansion

$$\phi_\pm(r) = \frac{1}{\sqrt{r}} \exp\left(\frac{\pm 16ik}{mW''(0)r}\right)(1 + \mathcal{O}(r)) \quad \text{as } r \to 0_+. \tag{4-14}$$

The contradiction then follows exactly as in Section 4D, the integrand of (4-13) being even more singular here since it behaves like r^{-4} in a neighborhood of 0.

4F. ***The situation $r_* < \bar r < \infty$ is excluded.*** In that case, we cannot get a contradiction from identity (4-11), because the various terms in the integrand now have different signs in a neighborhood of the singular point $\bar r$. Instead, our argument relies on a detailed analysis of the solutions of (4-5) near $\bar r$, and on monotonicity properties that follow from Assumption H2

As in the previous section, we normalize the solutions u_n of (4-5) so that $\beta_n = 1$ in (4-6) for all $n \in \mathbb{N}$. In particular, for any $r > \bar r$, we know that $u_n(r) \to \psi_\infty(r)$ and $u_n'(r) \to \psi_\infty'(r)$ as $n \to \infty$, where ψ_∞ is as in Lemma 3.7. Applying Lemma 3.5, whose assumptions are satisfied by construction of the homotopy argument, we deduce that

$$u_n(r) \to \alpha_-^\infty \phi_-(r) + \alpha_+^\infty \phi_+(r) \quad \text{and} \quad u_n'(r) \to \alpha_-^\infty \phi_-'(r) + \alpha_+^\infty \phi_+'(r) \tag{4-15}$$

for all $r \in (0, \infty) \setminus \bar r$, where $\alpha_\pm^\infty \in \mathbb{C}$ and $\phi_\pm$ are the solutions of (3-18) given by (3-20). Note that the roots $d_\pm$ of the indicial equation (3-19) are now real and distinct, so that we are in the situation referred to as Case 1 in Section 3D. The convergence (4-15) for some $r < \bar r$ implies, in view of the results in Section 3A concerning the solutions of (3-2) near the origin, that the normalizing constants α_n in (4-6) converge to some limit $\alpha_* \in \mathbb{C}$ as $n \to \infty$. We deduce that

$$\alpha_-^\infty \phi_-(r) + \alpha_+^\infty \phi_+(r) = \begin{cases} \alpha_* \psi_0(r) & \text{if } r \in (0, \bar r), \\ \psi_\infty(r) & \text{if } r \in (\bar r, \infty). \end{cases} \tag{4-16}$$

Now, the functions ψ_∞, ϕ_-, ϕ_+ are all real-valued on $(\bar r, \infty)$, and we know from (3-20) that $\phi_\pm(r) \approx (r - \bar r)^{d_\pm}$ as $r \to \bar r_+$, where $0 < d_- < \frac{1}{2} < d_+ < 1$. These observations imply that both coefficients α_-^∞ and α_+^∞ are necessarily real. On the other hand, we deduce from (4-16) that the complex function $\alpha_-^\infty \phi_- + \alpha_+^\infty \phi_+$ must have a constant phase (modulo π) on the interval $(0, \bar r)$, as it is equal to the product of the real function ψ_0 by the complex constant α_*. This, however, is impossible if both coefficients $\alpha_\pm^\infty$ are nonzero, because by (3-20) the complex functions $\phi_\pm$ have different phases when $r < \bar r$ and vanish at different rates as $r \to \bar r_-$. More precisely, since $\phi_+(r)/\phi_-(r) \to 0$ as $r \to \bar r_-$, it follows from (4-16), (3-20) that

$$\alpha_-^\infty = \alpha_* e^{-i\pi d_-}\lambda, \quad \text{where } \lambda = \lim_{r\to\bar r_-} \frac{\psi_0(r)}{|\bar b - \Omega(r)|^{d_-}} \in \mathbb{R}.$$

If $\alpha_-^\infty \neq 0$, then $\lambda \neq 0$ and $\alpha_* = \alpha_-^\infty \lambda^{-1} e^{i\pi d_-}$, so using (4-16) and (3-20) again we obtain

$$0 = \operatorname{Im}(\alpha_+^\infty e^{-i\pi d_-}\phi_+(r)) = \alpha_+^\infty \sin(\pi(d_+ - d_-))V_+(r), \quad 0 < r < \bar r.$$

As $V_+(r) \neq 0$ for r sufficiently close to $\bar r$, we conclude that $\alpha_+^\infty \sin(\pi(d_+ - d_-)) = 0$, and this implies $\alpha_+^\infty = 0$ since $0 < d_+ - d_- < 1$. Therefore, we must have $\alpha_-^\infty \alpha_+^\infty = 0$.

In the rest of this section, using totally different arguments which rely on Assumption H2, we show that necessarily $\alpha_-^\infty \alpha_+^\infty \neq 0$ in (4-16), and this will give the desired contradiction. To that purpose, we introduce the auxiliary functions

$$U_\pm(r) = (\bar b - \Omega(r))^{d_\pm}, \quad r > \bar r, \tag{4-17}$$

and we denote by $\mathscr{L} = -\partial_r \mathcal{A}(r)\partial_r^* + \mathcal{B}(r)$ the linear operator in (3-18). We claim that:

Lemma 4.4 (upper solutions). *There exists $\gamma > 0$ such that $\mathscr{L}(U_\pm) \geq \gamma U_\pm' > 0$ on $(\bar r, \infty)$.*

Proof. For notational simplicity we write U instead of $U_\pm$, d instead of $d_\pm$, and b instead of $\bar b$. Computing $\mathscr{L}(U)$ when $U = (b - \Omega)^d$, we obtain after elementary rearrangements of terms

$$\mathscr{L}(U) = (b - \Omega)^{d-2}(T_1 + T_2 + T_3),$$

where

$$\begin{aligned}
T_1 &= -\mathcal{A}\Omega'^2\Big(d(d-1) + \frac{k^2}{m^2}J\Big),\\
T_2 &= \Big(1 - \Big(\frac{\mathcal{A}}{r}\Big)'\Big)(b - \Omega)^2,\\
T_3 &= \mathcal{A}(b - \Omega)\Big(d\Omega'' + d\Big(\frac{\mathcal{A}'}{\mathcal{A}} + \frac{1}{r}\Big)\Omega' - \frac{r}{\mathcal{A}}\partial_r\Big(\frac{W}{m^2 + k^2r^2}\Big)\Big).
\end{aligned}$$

Since d is a solution of the indicial equation (3-19), we may rewrite

$$T_1(r) = \mathcal{A}(r)\Omega'^2(r)\frac{k^2}{m^2}(J(\bar r) - J(r)), \tag{4-18}$$

and our Assumption H2 on J implies $T_1 > 0$ on $(\bar r, \infty)$. Next, using the definition of $\mathcal{A}$ and the fact that $m^2 \geq 1$, we observe that

$$1 - \Big(\frac{\mathcal{A}}{r}\Big)' = 1 - \frac{m^2 - k^2r^2}{(m^2 + k^2r^2)^2} \geq 0,$$

so that $T_2 \geq 0$ on $(0, \infty)$. As for T_3, we expand

$$\frac{\mathcal{A}'}{\mathcal{A}} + \frac{1}{r} = \frac{3}{r} - \frac{2k^2r}{m^2 + k^2r^2}, \quad \frac{r}{\mathcal{A}}\partial_r\Big(\frac{W}{m^2 + k^2r^2}\Big) = \frac{W'}{r} - \frac{2k^2W}{m^2 + k^2r^2},$$

and we use the identities $W = r\Omega' + 2\Omega$ and $W' = r\Omega'' + 3\Omega'$ to derive the alternative expression

$$T_3 = \mathcal{A}(b-\Omega)\left((d-1)\frac{W'}{r} + \frac{2k^2}{m^2+k^2r^2}(W - dr\Omega')\right).$$

Since $0 < d < 1$, and since $W > 0$, $W' < 0$ and $\Omega' < 0$ by Assumption H1, we deduce that T_3 as well is positive on $(\bar{r}, \infty)$. Altogether, we have shown that $\mathscr{L}(U) > 0$ on $(\bar{r}, \infty)$.

To conclude the proof, we fix $r_0 > \bar{r}$ large enough so that $|(\mathcal{A}/r)'| \le \frac{1}{2}$ for $r \ge r_0$. In that region, we have $\mathscr{L}(U) \ge (b-\Omega)^{d-2}T_2 \ge (b-\Omega)^d/2 \ge \gamma_0 U'$ if $\gamma_0 > 0$ is small enough. On the other hand, using (4-18) and the fact that $J'(\bar{r}) < 0$, $\Omega(\bar{r}) = b$, and $\Omega'(\bar{r}) < 0$, we can find $\gamma_1 > 0$ small enough so that

$$T_1(r) = \frac{J(\bar{r}) - J(r)}{b - \Omega(r)}\mathcal{A}(r){\Omega'}^2(r)\frac{k^2}{m^2}(b-\Omega(r)) \ge \gamma_1 d(b-\Omega(r))|\Omega'(r)|$$

for all r in the compact interval $[\bar{r}, r_0]$. This implies $\mathscr{L}(U) \ge \gamma_1 U'$ on $[\bar{r}, r_0]$, and taking $\gamma = \min(\gamma_0, \gamma_1)$ we obtained the desired conclusion. □

Corollary 4.5. *The solutions $\phi_\pm(r)$ of* (3-18) *given by* (3-20) *are unbounded as $r \to \infty$.*

Proof. Assume that ϕ is a solution of $\mathscr{L}(\phi) = 0$ on $(\bar{r}, \infty)$, which is decomposed as $\phi = UV$, where U is one of the functions $U_\pm$ defined in (4-17). The equation satisfied by V is

$$0 = \mathscr{L}(UV) = -\mathcal{A}UV'' - \left(2\mathcal{A}U' + \left(\mathcal{A}' + \frac{\mathcal{A}}{r}\right)U\right)V' + \mathscr{L}(U)V, \quad r \in (\bar{r}, \infty). \tag{4-19}$$

We interpret the right-hand side of (4-19) as the action on the function V of a second-order differential operator $\mathscr{L}_U$ whose coefficients depend on U. Since $\mathcal{A}U$ is positive on $(\bar{r}, \infty)$ by construction, and $\mathscr{L}(U)$ is positive on $(\bar{r}, \infty)$ by Lemma 4.4, we observe that the maximum principle holds for the operator $\mathscr{L}_U$. As a consequence, the function V which satisfies $\mathscr{L}_U(V) = 0$ cannot have a positive maximum nor a negative minimum on the interval $(\bar{r}, \infty)$.

We first choose $\phi = \phi_+$, $U = U_-$, and we claim that ϕ_+ is unbounded on $(\bar{r}, \infty)$. Indeed, in the opposite case, the function $V(r) = \phi_+(r)/U_-(r)$ would tend to zero both as $r \to \bar{r}_+$ and as $r \to \infty$, so that $V \equiv 0$ by the maximum principle, which is clearly absurd. As a second application, we take $\phi = \phi_-$, $U = U_-$, and we claim again that ϕ_- is unbounded on $(\bar{r}, \infty)$. If not, by the maximum principle the function $V(r) = \phi_-(r)/U_-(r)$ would be nonincreasing on $(\bar{r}, \infty)$ with $V(r) \to 1$ as $r \to \bar{r}_+$ and $V(r) \to 0$ as $r \to \infty$. Note that V coincides with the function V_- in (3-20) and is therefore analytic up to the singular point $\bar{r}$. Thus, using (4-19) and Lemma 4.4, we can compute

$$V'(\bar{r}) = \lim_{r\to\bar{r}_+} V'(r) = \lim_{r\to\bar{r}_+} \frac{\mathscr{L}(U)V - \mathcal{A}UV''}{2\mathcal{A}U' + (\mathcal{A}' + \mathcal{A}/r)U} = \lim_{r\to\bar{r}_+} \frac{\mathscr{L}(U)}{2\mathcal{A}U'} \ge \frac{\gamma}{2\mathcal{A}(\bar{r})} > 0,$$

and this contradicts the claim that V is nonincreasing. □

As ψ_∞ is bounded on the interval $(\bar{r}, \infty)$, whereas both ϕ_+, ϕ_- are unbounded by Corollary 4.5, the relation $\psi_\infty = \alpha_-^\infty\phi_- + \alpha_+^\infty\phi_+$ can hold only if both coefficients α_-^∞, α_+^∞ are nonzero. This gives the desired contradiction in the case where $r_* < \bar{r} < \infty$.

4G. ***The situation $\bar r = r_* < \infty$ is excluded.*** We proceed here as in Section 4F, the only essential difference being that the exponents $d_- = d_+ = \frac{1}{2}$ are no longer distinct. We are thus in the situation referred to as Case 3 in Section 3D, where the solutions $\phi_\pm$ are given by the expressions (3-22). Applying Corollary 4.5, we obtain as above that ϕ_+ is unbounded on the interval $(\bar r, \infty)$, but the argument does not apply to the second solution ϕ_- which contains a logarithmic correction; see (3-22). Nevertheless, we deduce that the coefficient α_-^∞ in the representation (4-16) is necessarily nonzero, and this turns out to be enough to obtain the desired contradiction. Indeed, due to the logarithmic term, it is easy to verify that, if $\theta(r) = \arg(\alpha_-^\infty \phi_-(r) + \alpha_+^\infty \phi_+(r))$, then $\tan(\theta(r)) \sim -\pi^{-1}\log(\bar r - r)$ as $r \to \bar r$, which shows that the left-hand side of (4-16) cannot have a constant phase for $r \in (0, \bar r)$.

4H. ***The situation $r_* < \bar r = \infty$ is excluded.*** We next consider the case where $\bar b = 0$ in (4-3), so that $\bar r = +\infty$ according to (4-4). In that situation, the "critical layer" occurs at very large values of r, in a region where the eigenvalue equation (4-5) is already in some asymptotic regime. Here we cannot use the same arguments as in Section 4F to obtain a contradiction, because the location of the critical layer changes as n is increased. However, it is possible to obtain an accurate representation of the solution of (4-5) that decays to zero as $r \to \infty$ by comparing it with the explicit solution of a model problem, (4-25) below, which can be expressed in terms of modified Bessel functions. This approximation turns out to be sufficient to derive a contradiction when combined with the identity (3-11).

Our starting point is the equation (3-5) for $w_n(r) = r^{1/2}u_n(r)$, which reads

$$w_n''(r) + \frac{\mathcal{A}'(r)}{\mathcal{A}(r)} w_n'(r) - \mathcal{D}_n(r) w_n(r) = 0, \quad r > 0, \tag{4-20}$$

where, in view of (3-3),

$$\mathcal{D}_n(r) = k^2 + \frac{m^2 + \frac{3}{4}}{r^2} - \frac{1}{2r}\frac{\mathcal{A}'(r)}{\mathcal{A}(r)} - \frac{k^2}{m^2}\frac{\Phi_n(r)}{\gamma_n(r)^2} + \frac{r}{\mathcal{A}(r)\gamma_n(r)}\partial_r\left(\frac{W_n(r)}{m^2 + k^2 r^2}\right). \tag{4-21}$$

We recall that $\gamma_n(r) = \Omega_n(r) - b_n - ia_n$, and we observe that the function $r \mapsto |\gamma_n(r)|$ reaches its minimum at $r = r_n$, where $r_n = \Omega_n^{-1}(b_n)$. As $b_n \to 0$, it is clear that $r_n \to \infty$ as $n \to \infty$, and since $r^2\Omega_n(r)$ converges uniformly on $\mathbb{R}_+$ to $r^2\Omega(r)$ (by the results of Section A4), we even have $\lim_{n\to\infty} r_n^2 b_n = \lim_{r\to\infty} r^2\Omega(r) = \Gamma > 0$; hence

$$r_n^2 = \frac{\Gamma_n}{b_n} \quad \text{for all } n \in \mathbb{N}, \text{ where } \Gamma_n \xrightarrow[n\to\infty]{} \Gamma. \tag{4-22}$$

Similarly we have $\Omega_n'(r_n) = -d_n b_n^{3/2}$ for all $n \in \mathbb{N}$, where $d_n \to 2\Gamma^{-1/2}$ as $n \to \infty$.

Equation (4-20) has asymptotically constant coefficients, in the sense that $\mathcal{A}'(r)/\mathcal{A}(r) = \mathcal{O}(r^{-3})$ and $\mathcal{D}_n(r) \to k^2$ as $r \to \infty$. However, in general, the convergence of $\mathcal{D}_n(r)$ toward its limit k^2 is not uniform with respect to $n \in \mathbb{N}$, because of the "critical layer" that may occur at $r = r_n$. Indeed, if we expand the expression $\gamma_n(r)$ around that point, we obtain to leading order

$$\gamma_n(r) = \Omega_n(r) - \Omega_n(r_n) - ia_n \approx \Omega_n'(r_n)(r - r_n + ic_n), \tag{4-23}$$

where

$$c_n = -\frac{a_n}{\Omega_n'(r_n)} = \frac{1}{d_n}\frac{a_n}{b_n^{3/2}}. \tag{4-24}$$

It follows that, for r close to r_n,

$$\frac{\Phi_n(r)}{\gamma_n(r)^2} \approx \frac{J_n(r_n)}{(r-r_n+ic_n)^2}, \quad \text{where } J_n(r_n) = \frac{\Phi_n(r_n)}{\Omega_n'(r_n)^2}.$$

We know that $J_n(r_n) \to J(\infty)$ as $n \to \infty$, where $0 < J(\infty) < m^2/(4k^2)$ (by (4-9) and because $r_* < \infty$). Thus, the term involving $\gamma_n(r)^{-2}$ in (4-21) converges to zero as $r \to \infty$ uniformly in $n \in \mathbb{N}$ only if $c_n \to \infty$ as $n \to \infty$, which is the case if $a_n \gg b_n^{3/2}$. Otherwise, that term plays an important role and has to be taken into account.

Our strategy is thus to compare for large r the solutions of (4-20) with those of the simplified equation

$$w_n''(r) - \left(k^2 - \frac{k^2}{m^2}\frac{J_n(r_n)}{(r-r_n+ic_n)^2}\right)w_n(r) = 0, \tag{4-25}$$

which can be solved explicitly in terms of modified Bessel functions. In particular, the unique solution of (4-25) such that $w_n(r) \sim e^{-k(r-r_n)}$ as $r \to \infty$ is given by

$$w_n(r) = \chi_n(r) := \sqrt{\frac{2}{\pi}}(r-r_n+ic_n)^{1/2}K_{\nu_n}(k(r-r_n+ic_n)), \quad r > 0, \tag{4-26}$$

where K_ν is the modified Bessel function of the second kind [Abramowitz and Stegun 1964, Section 9.6], and the parameter $\nu_n \in \left(0, \frac{1}{2}\right)$ is determined by the relation

$$\nu_n^2 = \frac{1}{4} - \frac{k^2}{m^2}J_n(r_n). \tag{4-27}$$

To perform a rigorous analysis, we rewrite (4-20) in the equivalent form

$$w_n''(r) + \frac{\mathcal{A}'(r)}{\mathcal{A}(r)}w_n'(r) - \left(k^2 - \frac{k^2}{m^2}\frac{J_n(r_n)}{(r-r_n+ic_n)^2} + \mathcal{R}_n(r)\right)w_n(r) = 0, \tag{4-28}$$

where the remainder $\mathcal{R}_n$ is defined by

$$\mathcal{R}_n(r) = \mathcal{D}_n(r) - k^2 + \frac{k^2}{m^2}\frac{J_n(r_n)}{(r-r_n+ic_n)^2}, \quad r > 0. \tag{4-29}$$

The idea is now to look for a solution of (4-28) in the form

$$w_n(r) = f_n(r)\chi_n(r), \quad r > 0,$$

where χ_n is as in (4-26), and $f_n(r) \to 1$ as $r \to \infty$. The equation satisfied by f_n is easily found to be

$$\frac{\mathrm{d}}{\mathrm{d}r}\big(\mathcal{A}(r)\chi_n(r)^2 f_n'(r)\big) = \big(\mathcal{A}(r)\chi_n(r)^2\mathcal{R}_n(r) - \mathcal{A}'(r)\chi_n(r)\chi_n'(r)\big)f_n(r), \quad r > 0.$$

Integrating both sides over (r, ∞), we first obtain

$$f_n'(r) = -\frac{1}{\mathcal{A}(r)\chi_n(r)^2}\int_r^\infty \big(\mathcal{A}(s)\chi_n(s)^2\mathcal{R}_n(s) - \mathcal{A}'(s)\chi_n(s)\chi_n'(s)\big)f_n(s)\,\mathrm{d}s,$$

and a second integration, combined with an application of Fubini's theorem, gives the representation formula

$$f_n(r) = 1 + \int_r^\infty \mathcal{K}_n(r,s) f_n(s)\,\mathrm{d}s, \tag{4-30}$$

where the integral kernel $\mathcal{K}_n(r,s)$ has the expression

$$\mathcal{K}_n(r,s) = \big(\mathcal{A}(s)\chi_n(s)^2\mathcal{R}_n(s) - \mathcal{A}'(s)\chi_n(s)\chi_n'(s)\big)\int_r^s \frac{1}{\mathcal{A}(t)\chi_n(t)^2}\,\mathrm{d}t. \tag{4-31}$$

We now use the following estimate, whose proof is postponed to Section A5:

Lemma 4.6. *For any $\delta \in (0,1)$, there exists a constant $C > 0$ such that*

$$\sup_{r\ge\delta r_n}\int_r^\infty |\mathcal{K}_n(r,s)|\,\mathrm{d}s \le C b_n^{1/2}\Big(1+\log_+\frac{1}{b_n}\Big) \tag{4-32}$$

for all sufficiently large $n \in \mathbb{N}$, where $\log_+(x) = \max(\log(x), 0)$.

Assuming (4-32) for the moment, we easily deduce that the solution of (4-30) satisfies

$$\sup_{r\ge\delta r_n} |f_n(r) - 1| \le C b_n^{1/2}\Big(1+\log_+\frac{1}{b_n}\Big) \xrightarrow[n\to\infty]{} 0. \tag{4-33}$$

Also, differentiating (4-30) and using similar estimates as in the proof of Lemma 4.6, we obtain

$$\sup_{r\ge\delta r_n} |f_n'(r)| \le C b_n^{1/2}\Big(1+\log_+\frac{b_n}{a_n}\Big). \tag{4-34}$$

These estimates imply that the solution $w_n = f_n\chi_n$ of (4-28) is very close for large $n \in \mathbb{N}$ to its approximation χ_n defined by (4-26), uniformly on the interval $[\delta r_n, \infty)$ for any (small) $\delta > 0$. In particular, in view of (A-44), there exist positive constants C_1, C_2 such that

$$\begin{aligned} |w_n(r)| &\le C_1 e^{-k(r-r_n)} \quad \text{for } r \ge r_n - 1,\\ |w_n(r)| &\ge C_2 e^{-k(r-r_n)} \quad \text{for } \delta r_n \le r \le (1-\delta) r_n. \end{aligned} \tag{4-35}$$

To reach the desired contradiction, we now show that these bounds are *incompatible* with identity (3-11), which has to be satisfied for all $n \in \mathbb{N}$ by the function $u_n(r) = r^{-1/2} w_n(r)$. In terms of w_n, identity (3-11) becomes

$$\int_0^\infty \Big\{\frac{2\mathcal{A}(r)(b_n-\Omega_n)}{(a_n^2+(\Omega_n-b_n)^2)^2}\frac{k^2}{m^2}\Phi_n(r) + \frac{r}{a_n^2+(\Omega_n-b_n)^2}\partial_r\Big(\frac{W_n(r)}{m^2+k^2r^2}\Big)\Big\}|w_n(r)|^2\,\mathrm{d}r = 0.$$

The second term in the integrand is obviously negative, because $W_n(r)$ is a decreasing function of r. It follows that $\mathcal{I}_{n,1} + \mathcal{I}_{n,2} \le \mathcal{I}_{n,3}$, where

$$\mathcal{I}_{n,1} = \int_0^{r_n-1} \frac{\mathcal{A}(r)(\Omega_n(r)-b_n)}{(a_n^2+(\Omega_n(r)-b_n)^2)^2}\Phi_n(r)|w_n(r)|^2\,\mathrm{d}r, \tag{4-36}$$

$$\mathcal{I}_{n,2} = \int_{r_n-1}^{r_n+1} \frac{\mathcal{A}(r)(\Omega_n(r)-b_n)}{(a_n^2+(\Omega_n(r)-b_n)^2)^2}\Phi_n(r)|w_n(r)|^2\,\mathrm{d}r, \tag{4-37}$$

$$\mathcal{I}_{n,3} = \int_{r_n+1}^{\infty} \frac{\mathcal{A}(r)(b_n-\Omega_n(r))}{(a_n^2+(\Omega_n(r)-b_n)^2)^2}\Phi_n(r)|w_n(r)|^2\,\mathrm{d}r. \tag{4-38}$$

Since $\Omega_n(r)-b_n$ is positive for $r<r_n$ and negative for $r>r_n$, the quantities $\mathcal{I}_{n,1}$ and $\mathcal{I}_{n,3}$ are positive, whereas $\mathcal{I}_{n,2}$ has no obvious sign.

We first estimate $\mathcal{I}_{n,3}$. As $b_n-\Omega_n(r)=\Omega_n(r_n)-\Omega_n(r)\geq\Omega_n(r_n)-\Omega_n(r_n{+}1)\geq C|\Omega'(r_n)|$ for $r\geq r_n+1$, we have

$$\frac{(b_n-\Omega_n(r))\Phi_n(r)}{(a_n^2+(\Omega_n(r){-}b_n)^2)^2}\leq\frac{C\Phi_n(r)}{|\Omega'(r_n)|^3}\leq\frac{CJ_n(r)}{|\Omega'(r_n)|}\leq Cb_n^{-3/2},\quad r\geq r_n+1,$$

and using (4-35), (4-38) we deduce that $\mathcal{I}_{n,3}\leq Cb_n^{-3/2}$. Next, we bound $\mathcal{I}_{n,1}$ from below by restricting the integral in (4-36) to the region where $\delta r_n\leq r\leq(1{-}\delta)r_n$ for some small $\delta>0$. In that region we have $\epsilon_1 b_n\leq\Omega_n(r)-b_n\leq\epsilon_2 b_n$ and $|\Omega_n'(r)|\geq\epsilon_3 b_n^{3/2}$ for some $\epsilon_1,\epsilon_2,\epsilon_3>0$ depending on δ; hence

$$\frac{(\Omega_n(r)-b_n)\Phi_n(r)}{(a_n^2+(\Omega_n(r){-}b_n)^2)^2}\geq\frac{\epsilon_1 b_n\Omega_n'(r)^2J_n(r)}{(a_n^2+(\epsilon_2 b_n)^2)^2}\geq\frac{Cb_n^4}{(a_n^2+b_n^2)^2},\quad \delta r_n\leq r\leq(1{-}\delta)r_n,$$

and using (4-35) we deduce that

$$\mathcal{I}_{n,1}\geq\int_{\delta r_n}^{(1-\delta)r_n}\frac{\mathcal{A}(r)(b_n-\Omega_n(r))}{(a_n^2+(\Omega_n(r){-}b_n)^2)^2}\Phi_n(r)|w_n(r)|^2\,\mathrm{d}r\geq\frac{Cb_n^4}{(a_n^2+b_n^2)^2}e^{2k(1-\delta)r_n}.$$

Finally, when $|r-r_n|\leq 1$, we have $\Omega_n(r)-b_n\approx(r{-}r_n)\Omega_n'(r_n)$; hence

$$\frac{|\Omega_n(r)-b_n|\Phi_n(r)}{(a_n^2+(\Omega_n(r){-}b_n)^2)^2}\leq\frac{C|\Omega'(r_n)|^3|r-r_n|}{(a_n^2+\Omega'(r_n)^2(r{-}r_n)^2)^2}J_n(r_n)\leq\frac{C}{|\Omega'(r_n)|}\frac{|r-r_n|}{((r{-}r_n)^2+c_n^2)^2},$$

and using (4-35) we obtain the crude estimate

$$|\mathcal{I}_{n,2}|\leq C\int_{r_n-1}^{r_n+1}\frac{b_n^{-3/2}|r-r_n|}{((r{-}r_n)^2+c_n^2)^2}\,\mathrm{d}r\leq C\int_{-\infty}^{\infty}\frac{b_n^{-3/2}|x|}{(x^2+c_n^2)^2}\,\mathrm{d}x\leq\frac{Cb_n^{-3/2}}{c_n^2}\leq\frac{Cb_n^{3/2}}{a_n^2}.$$

As $\mathcal{I}_{n,1}\leq|\mathcal{I}_{n,2}|+\mathcal{I}_{n,3}$, the estimates obtained so far show that

$$\frac{C_3b_n^4}{(a_n^2+b_n^2)^2}e^{2k(1-\delta)r_n}\leq C_4b_n^{-3/2}\left(1+\frac{b_n^3}{a_n^2}\right)\tag{4-39}$$

for some positive constants C_3, C_4. If $a_n\geq b_n$ for a sequence of integers n, then multiplying both sides of (4-39) by $(a_n^2+b_n^2)^2$ we clearly obtain an inequality that cannot be satisfied for large n if $a_n\to 0$. Thus we can assume that $a_n\leq b_n$ for all $n\in\mathbb{N}$, in which case (4-39) implies

$$C_3b_n^{3/2}e^{2k(1-\delta)r_n}\leq 4C_4\left(1+\frac{b_n^3}{a_n^2}\right);\quad\text{hence}\quad a_n^2\leq Cb_n^{3/2}e^{-2k(1-\delta)r_n}.\tag{4-40}$$

Since $r_n=\mathcal{O}(b_n^{-1/2})$, this means that a_n is *exponentially small* when compared to b_n. With this information at hand, it is possible to compute explicitly the quantity $\mathcal{I}_{n,2}$ to leading order as $n\to\infty$. Indeed, in this parameter regime, the main contribution to the integral (4-37) comes from an extremely small neighborhood of the critical point r_n, so that

$$\begin{aligned}\mathcal{I}_{n,2} &\approx -\frac{\mathcal{A}(r_n)\Phi_n(r_n)}{|\Omega'(r_n)|^3}\int_{r_n-1}^{r_n+1}\frac{r-r_n}{(c_n^2+(r-r_n)^2)^2}|w_n(r)|^2\,\mathrm{d}r \\ &\approx -\frac{2\mathcal{A}(r_n)J_n(r_n)}{\pi|\Omega'(r_n)|}\int_{r_n-1}^{r_n+1}\frac{r-r_n}{(c_n^2+(r-r_n)^2)^{3/2}}|K_{\nu_n}(k(r-r_n+ic_n))|^2\,\mathrm{d}r, \qquad (4\text{-}41)\end{aligned}$$

where in the second line we used the fact that the solution w_n of (4-20) is well approximated for large n by the function χ_n in (4-26) in view of (4-33) and (4-34). Now, an explicit calculation which is reproduced in Section A6 shows that, for any $\nu \in \big(0, \frac{1}{2}\big)$ and any $\epsilon > 0$,

$$\mathcal{J}_\nu := \lim_{a\to 0_+}\int_{-\epsilon}^{\epsilon}\frac{-ax}{(a^2+x^2)^{3/2}}|K_\nu(x+ia)|^2\,\mathrm{d}x = \frac{2\pi\cos(\nu\pi)}{1-4\nu^2}. \qquad (4\text{-}42)$$

Assuming (4-42) for the moment, we deduce from (4-41) that

$$\mathcal{I}_{n,2} \approx \frac{2\mathcal{A}(r_n)J_n(r_n)}{\pi|\Omega'(r_n)|}\frac{\mathcal{J}_{\nu_n}}{c_n} \quad \text{as } n\to\infty,$$

which means in particular that $\mathcal{I}_{n,2} > 0$ when n is sufficiently large. Thus, for large n we must have

$$C_5 e^{2k(1-\delta)r_n} \le \mathcal{I}_{n,1} \le \mathcal{I}_{n,1} + \mathcal{I}_{n,2} \le \mathcal{I}_{n,3} \le C_6 r_n^3$$

for some positive constants C_5, C_6, which is clearly impossible since $r_n \to \infty$ as $n \to \infty$. So we have reached a contradiction in that case too.

4I. *The situation $r_* = \bar{r} = \infty$ is excluded.* Having exhausted all possibilities for which $r_* < \infty$, we finally consider the case where $r_* = \infty$. According to (4-7), this occurs if and only if $J(\infty) \ge m^2/(4k^2)$. Of course, we can assume that $\bar{r} = \infty$, because when $\bar{r} < \infty$ a contradiction has already been obtained in Section 4D or 4E. If $J(\infty) > m^2/(4k^2)$, then $J_n(r) > J_n(\infty) > m^2/(4k^2)$ for all $r > 0$ when n is sufficiently large, and in that situation we know from Proposition 3.4 that the operator $L^n_{m,k}$ has no eigenvalue outside the imaginary axis. However, if $J(\infty) = m^2/(4k^2)$, it is possible that $J_n(\infty) < m^2/(4k^2)$ for all $n \in \mathbb{N}$, in which case we cannot obtain a contradiction directly from Proposition 3.4. In that situation, we must have $J_n(r_n) \to m^2/(4k^2)$ as $n \to \infty$. Two possibilities can occur:

(1) If $J_n(r_n) < m^2/(4k^2)$ for a sequence of integers n, we can get a contradiction by following exactly the same lines as in Section 4H, the only difference being that the indices ν_n defined by (4-27) now converge to zero as $n \to \infty$. This is harmless because, as is observed for instance in Remark A.7 below, all estimates we need hold uniformly in the limit where $\nu \to 0$. We leave the details to the reader.

(2) If instead $J_n(r_n) \ge m^2/(4k^2)$ for all $n \in \mathbb{N}$, we shall prove that the quadratic form given by the left-hand side of (4-11) is positive definite for sufficiently large values of n, so that (4-11) gives the desired contradiction. To do that, we use the asymptotic expansions

$$\Omega_n(r) = \frac{\Gamma_n}{r^2} + \mathcal{O}\Big(\frac{1}{r^4}\Big), \quad \Omega_n'(r) = \frac{-2\Gamma_n}{r^3} + \mathcal{O}\Big(\frac{1}{r^5}\Big), \quad J_n'(r) = o\Big(\frac{1}{r}\Big), \quad r\to\infty, \qquad (4\text{-}43)$$

which are established in Section A4, and we observe that the integrand in (4-11) is nonnegative outside an interval of the form $[r_n, r_n+\delta_n]$, where $\delta_n \to 0$ as $n \to \infty$. Indeed, the integrand is clearly nonnegative for $r \le r_n$, and for $r \ge r_n$ we have the lower bound

$$\frac{\mathcal{A}(r)\Omega_n'(r)^2}{a_n^2+(b_n-\Omega_n(r))^2}\left(\frac{k^2}{m^2}J_n(r)-\frac{1}{4}\right) \ge \frac{1}{m^2}\frac{\Omega_n'(r)^2}{(b_n-\Omega_n(r))^2}(J_n(r)-J_n(r_n)). \tag{4-44}$$

In view of (4-43), the last member of (4-44) is bounded from below by -1 if $r \ge r_n + \mathcal{O}(|J'(r_n)|)$, which proves the claim. If we now restrict the integral in (4-11) to the interval $[r_n-1, r_n+1]$, make the change of variable $r = r_n + s$, and use the lower bound (4-44), we deduce (after a few obvious simplifications) that it is sufficient for us to show that the quadratic form

$$\mathfrak{Q}_n(v) := \int_{-1}^{1}\left\{|\partial_s v|^2 + \left(k^2 - \frac{\epsilon_n s}{c_n^2+s^2}\right)|v|^2\right\}\mathrm{d}s, \quad k>0, \tag{4-45}$$

is positive on $H^1([-1,1], \mathrm{d}s)$ for all $c_n \ne 0$ and all sufficiently small $\epsilon_n > 0$. This in turn is an easy consequence of the Sobolev embedding theorem. Indeed, decomposing $v = v(0)\chi + w$, where $\chi : [-1,1] \to [0,1]$ is smooth, even, and satisfies $\chi(0) = 1$, we first observe that

$$\int_{-1}^{1}\frac{\epsilon_n s}{c_n^2+s^2}|v(0)|^2\chi^2(s)\,\mathrm{d}s = 0,$$

by symmetry. Moreover, as $w(0) = 0$ by construction, we have $|w(s)|^2 \le C|s|\|v\|_{H^1}^2$. Combining these observations, we deduce that

$$\left|\int_{-1}^{1}\frac{\epsilon_n s}{c_n^2+s^2}|v|^2\,\mathrm{d}s\right| \le C\epsilon_n\|v\|_{H^1}^2,$$

where the constant $C > 0$ is independent of n. The quadratic form (4-45) is thus positive if ϵ_n is sufficiently small, and we deduce that (4-11) cannot be satisfied. This concludes the contradiction argument initiated in Section 4A, and hence also the proof of Theorem 1.3.

5. Uniform resolvent estimates

This section is devoted to the proof of Theorem 1.6. Given any $s \in \mathbb{C}$ with $\mathrm{Re}(s) \ne 0$, we already know from Theorem 1.3 that the resolvent operator $(s - L_{m,k})^{-1}$ is bounded in the space $X_{m,k}$ for all $m \in \mathbb{Z}$ and all nonzero $k \in \mathbb{R}$. It remains to show that, for any $k_0 > 0$, the resolvent norm $\|(s-L_{m,k})^{-1}\|$ is *uniformly bounded* for all $m \in \mathbb{Z}$ and all nonzero $k \in \mathbb{Z}k_0$.

Given $m \in \mathbb{Z}$, $k \ne 0$, and $\omega, f \in X_{m,k}$, the resolvent equation $(s - L_{m,k})\omega = f$ takes the form

$$\begin{aligned} \gamma(r)\omega_r &= ikW(r)u_r + f_r,\\ \gamma(r)\omega_\theta &= ikW(r)u_\theta + r\Omega'(r)\omega_r + f_\theta,\\ \gamma(r)\omega_z &= ikW(r)u_z - W'(r)u_r + f_z, \end{aligned} \tag{5-1}$$

where $\gamma(r) = s + im\Omega(r)$. Here and in what follows, we assume that $\mathrm{Re}(s) \ne 0$, which implies $|\gamma(r)| \ge |\mathrm{Re}(s)| > 0$ for all $r > 0$. In (5-1), it is understood that the velocity u is obtained from the vorticity ω by the Biot–Savart formula in the Fourier subspace indexed by m and k; see Section A1 below.

Proceeding as in the derivation of the eigenvalue equation (2-16) in Section 2, we can transform the resolvent system (5-2) into a single equation for the radial velocity u_r. After some calculations, we obtain the differential equation

$$-\partial_r(\mathcal{A}(r)\partial_r^* u_r) + \mathcal{B}(r)u_r = \mathcal{F}(r), \tag{5-2}$$

where the coefficients in the left-hand side are as in (3-3),

$$\mathcal{A}(r) = \frac{r^2}{m^2+k^2r^2}, \quad \mathcal{B}(r) = 1 + \frac{k^2}{\gamma(r)^2}\mathcal{A}(r)\Phi(r) + \frac{imr}{\gamma(r)}\partial_r\left(\frac{W(r)}{m^2+k^2r^2}\right), \tag{5-3}$$

and the right-hand side takes the form

$$\mathcal{F}(r) = -\partial_r\left(\frac{m}{kr\gamma(r)}\mathcal{A}(r)f_r\right) - \frac{im^2}{k\gamma(r)^2}\frac{W(r)}{m^2+k^2r^2}f_r - \frac{i}{k\gamma(r)}f_\theta + \frac{2i\Omega(r)}{k\gamma(r)^2}f_z. \tag{5-4}$$

Of course, if $f=0$, then $\mathcal{F}=0$ and (5-2) reduces to (2-16). The following result will be useful to estimate the solutions of (5-2) when $|k|$ is large.

Lemma 5.1. *For any $m\in\mathbb{Z}$ and any $s\in\mathbb{C}$ with* $\mathrm{Re}(s)\neq 0$, *there exists a positive constant $C=C(m,s)$ such that*, *for any $k\neq 0$ and any $f\in X_{m,k}$, the solution u_r of* (5-2) *satisfies*

$$\|\mathcal{A}^{1/2}\partial_r^* u_r\|_{L^2} + \|u_r\|_{L^2} \le \frac{C(m,s)}{|k|}(\|u_r\|_{L^2} + \|f\|_{L^2}). \tag{5-5}$$

Proof. As in Section 3C, we set $u_r(r) = \gamma(r)^{1/2}v(r)$. The new function v satisfies the equation

$$-\partial_r(\mathcal{A}(r)\gamma(r)\partial_r^* v) + \mathcal{E}(r)v = \gamma(r)^{1/2}\mathcal{F}(r), \tag{5-6}$$

where

$$\begin{aligned}\mathcal{E}(r) &= \gamma(r)\mathcal{B}(r) - \frac{\gamma'(r)}{2}\left(\mathcal{A}'(r) - \frac{\mathcal{A}(r)}{r}\right) - \frac{1}{2}\gamma''(r)\mathcal{A}(r) + \frac{\gamma'(r)^2\mathcal{A}(r)}{4\gamma(r)}\\ &= \gamma(r) + \frac{k^2}{\gamma(r)}\mathcal{A}(r)\Phi(r) + \frac{imr}{2}\partial_r\left(\frac{W(r)+2\Omega(r)}{m^2+k^2r^2}\right) - \frac{m^2\Omega'(r)^2}{4\gamma(r)}\mathcal{A}(r).\end{aligned} \tag{5-7}$$

We also observe that

$$\gamma^{1/2}\mathcal{F} = -\partial_r\left(\frac{m}{kr\gamma^{1/2}}\mathcal{A}f_r\right) - \frac{im^2}{2k\gamma^{3/2}}\frac{W+2\Omega}{m^2+k^2r^2}f_r - \frac{i}{k\gamma^{1/2}}f_\theta + \frac{2i\Omega}{k\gamma^{3/2}}f_z. \tag{5-8}$$

Without loss of generality, we assume that $a := \mathrm{Re}(s) > 0$. If we multiply both sides of (5-6) by $r\bar{v}$, integrate the resulting equality over $\mathbb{R}_+$ and take the real part, we obtain the identity

$$\begin{aligned}&a\int_0^\infty\left\{\mathcal{A}|\partial_r^* v|^2 + |v|^2 + \frac{\mathcal{A}}{|\gamma|^2}\left(k^2\Phi - \frac{m^2\Omega'^2}{4}\right)|v|^2\right\}r\,\mathrm{d}r\\ &\quad= \mathrm{Re}\int_0^\infty(\partial_r^*\bar{v})\frac{m\mathcal{A}}{kr\gamma^{1/2}}f_r r\,\mathrm{d}r + \mathrm{Re}\int_0^\infty\bar{v}\left(-\frac{im^2}{2k\gamma^{3/2}}\frac{W+2\Omega}{m^2+k^2r^2}f_r - \frac{i}{k\gamma^{1/2}}f_\theta + \frac{2i\Omega}{k\gamma^{3/2}}f_z\right)r\,\mathrm{d}r.\end{aligned}$$

Keeping in mind that $\Phi(r) \ge 0$, $|\gamma(r)| \ge a$, and $0 < \mathcal{A}(r) \le \min(1/k^2, r^2/m^2)$, we can estimate the various terms in a straightforward way, and we arrive at the inequality

$$\begin{aligned} &a(\|\mathcal{A}^{1/2}\partial_r^* v\|_{L^2}^2 + \|v\|_{L^2}^2) \\ &\qquad \le \frac{Cm^2}{ak^2}\|v\|_{L^2}^2 + \frac{C}{a^{1/2}|k|}\|\mathcal{A}^{1/2}\partial_r^* v\|_{L^2}\|f_r\|_{L^2} + \frac{C}{a^{3/2}|k|}\|v\|_{L^2}(\|f_r\|_{L^2} + a\|f_\theta\|_{L^2} + \|f_z\|_{L^2}), \end{aligned}$$

where $C > 0$ is a universal constant. Applying now Young's inequality, we conclude that

$$\|\mathcal{A}^{1/2}\partial_r^* v\|_{L^2} + \|v\|_{L^2} \le \frac{C(m,a)}{|k|}(\|v\|_{L^2} + \|f\|_{L^2}), \tag{5-9}$$

where the constant depends only on m and a.

Finally, we return to the original function $u_r(r) = \gamma(r)^{1/2} v(r)$. As $|\gamma(r)| \le |s| + |m|$ and

$$\mathcal{A}^{1/2}\partial_r^* u_r = \mathcal{A}^{1/2}\gamma^{1/2}\left(\partial_r^* v + \frac{imr\Omega'}{2r\gamma} v\right),$$

we have $|\mathcal{A}^{1/2}\partial_r^* u_r| + |u_r| \le C(m,s)(|\mathcal{A}^{1/2}\partial_r^* v| + |v|)$. Thus the desired inequality (5-5) follows immediately from (5-9). $\square$

Equipped with this lemma, we now establish the main result of this section.

Proposition 5.2. *Fix any $k_0 > 0$. For any $s \in \mathbb{C}$ with $\mathrm{Re}(s) \neq 0$, there exists a constant $C = C(s, k_0)$ such that, for all $m \in \mathbb{Z}$ and all nonzero $k \in \mathbb{Z}k_0$, the following estimate holds for all $f \in X_{m,k}$:*

$$\|(s - L_{m,k})^{-1} f\|_{L^2} \le C\|f\|_{L^2}. \tag{5-10}$$

Proof. We proceed by contradiction. If (5-10) does not hold, there exist sequences (m_n) in $\mathbb{Z}$, (k_n) in $\mathbb{Z}^* k_0$, and $\omega^{(n)}$, $f^{(n)}$ in $X_{m,k}$ such that $\|\omega^{(n)}\|_{L^2} = 1$ for all $n \in \mathbb{N}$, $\|f^{(n)}\|_{L^2} \to 0$ as $n \to \infty$, and $(s - L_{m,k})\omega^{(n)} = f^{(n)}$ for all $n \in \mathbb{N}$, namely

$$\begin{aligned} (s + im_n\Omega(r))\omega_r^{(n)} &= ik_n W(r) u_r^{(n)} + f_r^{(n)}, \\ (s + im_n\Omega(r))\omega_\theta^{(n)} &= ik_n W(r) u_\theta^{(n)} + r\Omega'(r)\omega_r^{(n)} + f_\theta^{(n)}, \\ (s + im_n\Omega(r))\omega_z^{(n)} &= ik_n W(r) u_z^{(n)} - W'(r) u_r^{(n)} + f_z^{(n)}. \end{aligned} \tag{5-11}$$

Step 1: We first show that the sequence (m_n) is bounded. Indeed, if this is not the case, we can assume (after extracting a subsequence) that $|m_n| \to \infty$ as $n \to \infty$. In view of the first equation in (5-11), this implies

$$\|\omega_r^{(n)}\|_{L^2} \le \left\|\frac{ik_n W u_r^{(n)}}{s + im_n\Omega}\right\|_{L^2} + \left\|\frac{f_r^{(n)}}{s + im_n\Omega}\right\|_{L^2} \xrightarrow[n\to\infty]{} 0. \tag{5-12}$$

Indeed, we know from Proposition A.1 that $\|k_n u^{(n)}\|_{L^2} \le C\|\omega^{(n)}\|_{L^2} \le C$ for all $n \in \mathbb{N}$, so that

$$\left\|\frac{ik_n W u_r^{(n)}}{s + im_n\Omega}\right\|_{L^2} \le \left\|\frac{W}{s + im_n\Omega}\right\|_{L^\infty}\|k_n u_r^{(n)}\|_{L^2} \le C\left\|\frac{W}{s + im_n\Omega}\right\|_{L^\infty} \xrightarrow[n\to\infty]{} 0,$$

and the last term in (5-12) is bounded by $|\operatorname{Re}(s)|^{-1}\|f_r^{(n)}\|_{L^2}$, a quantity that converges to zero as $n\to\infty$ by assumption. Once (5-12) is known, the same argument applied to the second equation in (5-11) shows that $\|\omega_\theta^{(n)}\|_{L^2}\to 0$. Finally, we have

$$\|u_r^{(n)}\|_{L^2}\le k_0^{-1}\|k_n u_r^{(n)}\|_{L^2}\le Ck_0^{-1},$$

because $|k_n|\ge k_0$ for all $n\in\mathbb{N}$. Applying thus the same argument again to the third equation in (5-11), we conclude that $\|\omega_z^{(n)}\|_{L^2}\to 0$, which of course contradicts the hypothesis that $\|\omega^{(n)}\|_{L^2}=1$ for all $n\in\mathbb{N}$. This means that sequence (m_n) must be bounded, and after extracting a subsequence we can therefore assume that there exists an integer $m\in\mathbb{Z}$ such that $m_n=m$ for all $n\in\mathbb{N}$.

Step 2: We next show that the sequence (k_n) is bounded. Again, if this is not the case, we can assume after extracting a subsequence that $|k_n|\to\infty$ as $n\to\infty$. In that situation, we infer from estimate (5-5) that, for n sufficiently large,

$$|k_n|\big(\|\mathcal{A}_n^{1/2}\partial_r^* u_r^{(n)}\|_{L^2}+\|u_r^{(n)}\|_{L^2}\big)\le 2C(m,s)\|f^{(n)}\|_{L^2}\xrightarrow[n\to\infty]{}0. \tag{5-13}$$

Next, we use the relation

$$k_n^2\mathcal{A}_n\Big(\partial_r^*-\frac{imW}{r\gamma}\Big)u_r^{(n)}+ik_nu_z^{(n)}=\frac{mk_n}{r\gamma}\mathcal{A}_nf_r^{(n)},$$

which reduces to (2-13) when $k_n=k$ and $f_r^{(n)}=0$. Invoking (5-13) and using the elementary bounds $0<\mathcal{A}_n(r)\le\min(1/k_n^2,r^2/m^2)$, we deduce that

$$|k_n|\|u_z^{(n)}\|_{L^2}+|m|\Big\|\frac{u_z^{(n)}}{r}\Big\|_{L^2}\xrightarrow[n\to\infty]{}0. \tag{5-14}$$

Finally, with the help of the additional relation

$$\frac{ik_nW}{\gamma}u_r^{(n)}+ik_nu_\theta^{(n)}-\frac{im}{r}u_z^{(n)}=-\frac{f_r^{(n)}}{\gamma},$$

which reduces to (2-9) when $k_n=k$ and $f_r^{(n)}=0$, we find that $|k_n|\|u_\theta^{(n)}\|_{L^2}\to 0$ as $n\to\infty$ in view of (5-13), (5-14).

Thus, we have shown that $|k_n|\|u^{(n)}\|_{L^2}\to 0$ as $n\to\infty$, and considering successively all three lines in (5-11) we easily deduce that

$$\|\omega_r^{(n)}\|_{L^2}\xrightarrow[n\to\infty]{}0,\quad \|\omega_\theta^{(n)}\|_{L^2}\xrightarrow[n\to\infty]{}0,\quad \|\omega_z^{(n)}\|_{L^2}\xrightarrow[n\to\infty]{}0.$$

This of course contradicts the assumption that $\|\omega^{(n)}\|_{L^2}=1$ for all $n\in\mathbb{N}$. The sequence (k_n) must therefore be bounded, and after extracting a subsequence we can assume that $k_n=k$ for some fixed $k\in\mathbb{Z}^*k_0$.

Step 3: Assuming that estimate (5-10) does not hold for some $s\in\mathbb{C}$ with $\operatorname{Re}(s)\neq 0$, we have reached the conclusion that, for some $m\in\mathbb{Z}$ and some $k\neq 0$, the operator $s-L_{m,k}$ has no bounded inverse in $X_{m,k}$, in contradiction with Theorem 1.3. Thus estimate (5-10) must hold, and the proof of Proposition 5.2 is complete. ☐

Appendix

A1. ***The Biot–Savart law in cylindrical coordinates.*** The Biot–Savart law defines the velocity field $u = (u_r, u_\theta, u_z)$ in terms of the vorticity vector $\omega = (\omega_r, \omega_\theta, \omega_z)$ for a fixed value of the angular Fourier mode $m \in \mathbb{Z}$ and of the vertical wave number $k \in \mathbb{R}$. The velocity is determined by the linear relations

$$\omega_r = \frac{im}{r} u_z - iku_\theta, \quad \omega_\theta = iku_r - \partial_r u_z, \quad \omega_z = \frac{1}{r}\partial_r(ru_\theta) - \frac{im}{r} u_r, \tag{A-1}$$

together with the divergence-free condition

$$\frac{1}{r}\partial_r(ru_r) + \frac{im}{r} u_\theta + iku_z = 0. \tag{A-2}$$

These equations have to be solved on the half-line $r > 0$, and we require that the velocity field $u_r e_r + u_\theta e_\theta + u_z e_z$ be regular at the origin $r = 0$ and decay to zero as $r \to \infty$. More precisely, if the vorticity ω is (for instance) compactly supported in $\mathbb{R}_+ = (0, \infty)$, the following boundary conditions hold for the associated velocity u:

- The horizontal velocities u_r, u_θ satisfy the homogeneous Dirichlet condition at $r = 0$ if $m = 0$ or $|m| \ge 2$, and the homogeneous Neumann condition if $|m| = 1$ (or $|m| \ge 3$).
- The vertical velocity u_z satisfies the homogeneous Dirichlet condition at $r = 0$ if $|m| \ge 1$, and the homogeneous Neumann condition if $m = 0$ (or $|m| \ge 2$).

It is possible to give explicit formulas for the velocity u in terms of the vorticity ω, but the bounds we need in this paper are more conveniently obtained by standard energy estimates. We recall that $\|\cdot\|_{L^2}$ denotes the usual norm in the Lebesgue space $L^2(\mathbb{R}_+, r\,\mathrm{d}r)$.

Proposition A.1. *There exists a constant $C > 0$ such that, for any $m \in \mathbb{Z}$ and any $k \in \mathbb{R}$, the following inequality holds*:

$$\begin{aligned}\|\partial_r u_r\|_{L^2}^2 + \|\partial_r u_\theta\|_{L^2}^2 + \|\partial_r u_z\|_{L^2}^2 + k^2(\|u_r\|_{L^2}^2 + \|u_\theta\|_{L^2}^2 + \|u_z\|_{L^2}^2) \\ + |m^2 - 1|\Big(\Big\|\frac{u_r}{r}\Big\|_{L^2}^2 + \Big\|\frac{u_\theta}{r}\Big\|_{L^2}^2\Big) + m^2\Big\|\frac{u_z}{r}\Big\|_{L^2}^2 \le C(\|\omega_r\|_{L^2}^2 + \|\omega_\theta\|_{L^2}^2 + \|\omega_z\|_{L^2}^2).\end{aligned} \tag{A-3}$$

Proof. We assume here for definiteness that $k \neq 0$, but the proof is similar (and in fact simpler) when $k = 0$. Without loss of generality, we also suppose that ω is continuous and compactly supported in $\mathbb{R}_+$. We first observe that the vertical velocity u_z satisfies the linear elliptic equation

$$\Big(-\partial_r^2 - \frac{1}{r}\partial_r + \frac{m^2}{r^2} + k^2\Big) u_z = \frac{1}{r}\partial_r(r\omega_\theta) - \frac{im}{r}\omega_r. \tag{A-4}$$

We multiply both sides of (A-4) by $r\bar{u}_z$ and integrate the resulting expression over $\mathbb{R}_+$. After elementary calculations, we obtain the estimate

$$\|\partial_r u_z\|_{L^2}^2 + \Big\|\frac{m}{r} u_z\Big\|_{L^2}^2 + \|ku_z\|_{L^2}^2 \le C(\|\omega_r\|_{L^2}^2 + \|\omega_\theta\|_{L^2}^2), \tag{A-5}$$

where $C > 0$ is a universal constant. As $iku_r = \partial_r u_z + \omega_\theta$ and $iku_\theta = (im/r)u_z - \omega_r$, it follows immediately from (A-5) that

$$\|ku_r\|_{L^2}^2 + \|ku_\theta\|_{L^2}^2 \le C(\|\omega_r\|_{L^2}^2 + \|\omega_\theta\|_{L^2}^2). \tag{A-6}$$

On the other hand, we deduce from (A-2) and the last relation in (A-1) that

$$\partial_r u_r + \frac{1}{r}(u_r + imu_\theta) = -iku_z, \quad \partial_r u_\theta + \frac{1}{r}(u_\theta - imu_r) = \omega_z. \tag{A-7}$$

We multiply the first equation by $r\partial_r \bar{u}_r$ and the second one by $r\partial_r \bar{u}_\theta$. Adding the resulting expressions, taking the real parts, and integrating over $\mathbb{R}_+$, we obtain the inequality

$$\|\partial_r u_r\|_{L^2}^2 + \|\partial_r u_\theta\|_{L^2}^2 \le C(\|ku_z\|_{L^2}^2 + \|\omega_z\|_{L^2}^2) \le C(\|\omega_r\|_{L^2}^2 + \|\omega_\theta\|_{L^2}^2 + \|\omega_z\|_{L^2}^2). \tag{A-8}$$

If $m = \pm 1$, this concludes the proof of (A-3). Otherwise, we deduce from (A-7) that

$$\begin{aligned} \frac{m^2-1}{r} u_r &= \partial_r(u_r - imu_\theta) + iku_z + im\omega_z, \\ \frac{m^2-1}{r} u_\theta &= \partial_r(u_\theta + imu_r) + kmu_z - \omega_z. \end{aligned} \tag{A-9}$$

If $m = 0$ or $|m| \ge 2$, these relations allow us to estimate the L^2 norm of u_r/r and u_θ/r in terms of quantities that are already controlled by (A-5) or (A-8), and we arrive at (A-3). □

A2. ***Stability of Rankine's vortex.*** We consider here in some detail the particular case of the Rankine vortex (1-7), which is of historical relevance. We do not use the functional framework of Section 2 because, as is clear from (2-3), the linearization $L_{m,k}$ does not define a bounded linear operator on $X_{m,k}$ if the vorticity profile W has a discontinuity. Instead we look for solutions of the eigenvalue equation (2-8) where the velocity field u (and not the vorticity ω) belongs to $X_{m,k}$. We always assume that $m \neq 0$ and $k \neq 0$, the other cases being similar and in fact simpler. To avoid the essential spectrum, we also suppose that the spectral parameter $s \in \mathbb{C}$ satisfies $s \neq 0$ and $s + im \neq 0$.

Following Kelvin's original approach [1880], we eliminate the radial velocity u_r in the 2×2 system (2-13)–(2-14) to obtain a closed equation for the vertical velocity u_z. In the inner region where $0 < r < 1$, we have $\gamma(r) = \gamma := s + im$ and $\Phi(r) = W(r)^2 = 4$, so that u_z satisfies the Bessel equation

$$-\frac{1}{r}\partial_r(r\partial_r u_z) + \left(\beta^2 + \frac{m^2}{r^2}\right)u_z = 0, \quad \text{where } \beta^2 = k^2\left(1 + \frac{4}{\gamma^2}\right). \tag{A-10}$$

Since u_z is regular at the origin, it follows that $u_z(r) = AI_m(\beta r)$ for $0 < r < 1$, where $A \in \mathbb{C}$ and I_m is the modified Bessel function of order m [Abramowitz and Stegun 1964, Section 9.6]. In the outer region where $r > 1$, we have $W(r) = \Phi(r) = 0$, and system (2-13)–(2-14) reduces to the (somewhat simpler) Bessel equation

$$-\frac{1}{r}\partial_r(r\partial_r u_z) + \left(k^2 + \frac{m^2}{r^2}\right)u_z = 0. \tag{A-11}$$

As $u_z(r)$ decays to zero at infinity, we must have $u_z(r) = BK_m(kr)$ for some $B \in \mathbb{C}$, where K_m is again a modified Bessel function.

At the interface $r = 1$, both velocities u_z, u_r are continuous, as can be seen from (2-10) and (2-12). Jump conditions for the first-order derivatives can be deduced from system (2-13)–(2-14) and are found to be

$$\partial_r u_r(1_+) = \partial_r u_r(1_-) - \frac{2im}{\gamma} u_r(1), \tag{A-12}$$

$$\partial_r u_z(1_+) = \frac{\gamma^2}{\gamma^2+4}\left(\partial_r u_z(1_-) + \frac{2im}{\gamma} u_z(1)\right) = iku_r(1). \tag{A-13}$$

In particular, as $u_z(r) = AI_m(\beta r)$ for $r < 1$ and $u_z(r) = BK_m(kr)$ for $r > 1$, we must have

$$AI_m(\beta) = BK_m(k), \quad \frac{A\gamma^2}{\gamma^2+4}\left(\beta I_m'(\beta) + \frac{2im}{\gamma} I_m(\beta)\right) = BkK_m'(k). \tag{A-14}$$

This linear system has a nontrivial solution (A, B) if and only if

$$\frac{I_m'(\beta)}{\beta I_m(\beta)} + \frac{2im}{\gamma\beta^2} = \frac{K_m'(k)}{kK_m(k)}, \tag{A-15}$$

where we recall that $\gamma = s + im \neq 0$ and $\beta^2 = k^2(1 + 4/\gamma^2)$.

It was already observed by Kelvin that the dispersion relation (A-15) is satisfied for a countable set of *purely imaginary* values of the spectral parameter s. More precisely, if we define $s = -imb$, so that $\gamma = im(1-b)$, equality (A-15) holds for a decreasing sequence of values of b accumulating at 1, and also for an increasing sequence accumulating at 1, all solutions being contained in the interval $|b-1| \le 2/|m|$ [Kelvin 1880]. The linearized operator at Rankine's vortex thus has a countable family of purely imaginary eigenvalues (Kelvin modes). However, it is not easy to verify that the dispersion relation (A-15) has no solution when $s \notin i\mathbb{R}$, and there is no such claim in Kelvin's work,[4] where only purely imaginary eigenvalues are considered. Thus, contrary to what is often asserted in the literature, stability of Rankine's vortex was not established by Kelvin, and we could not find any further reference where this point is clarified.

Fortunately, it is quite easy to prove spectral stability of Rankine's vortex following the approach of Section 3C. Indeed, taking into account the particular form of the vorticity profile (1-7), it is straightforward to verify that identity (3-10) becomes

$$\int_0^\infty (\mathcal{A}(r)|\partial_r^* u_r|^2 + |u_r|^2) r \,\mathrm{d}r + \int_0^1 \left\{-\frac{4k^2\mathcal{A}(r)}{m^2\gamma_\star^2} + \frac{2r}{\gamma_\star}\partial_r\left(\frac{1}{m^2+k^2r^2}\right)\right\}|u_r|^2 r\,\mathrm{d}r \\ = \frac{2\mathcal{A}(1)}{\gamma_\star}|u_r(1)|^2; \tag{A-16}$$

see also Remark A.2 below. Here $\gamma_\star = 1 - b - ia$, so that $\gamma = im\gamma_\star$. We now multiply both sides of (A-16) by $\gamma_\star$ and take the imaginary parts. We arrive at the identity

$$a\int_0^\infty (\mathcal{A}(r)|\partial_r^* u_r|^2 + |u_r|^2) r\,\mathrm{d}r + a\int_0^1 \frac{4k^2}{m^2}\frac{\mathcal{A}(r)}{(1-b)^2+a^2}|u_r|^2 r\,\mathrm{d}r = 0. \tag{A-17}$$

If $a \neq 0$, it follows from (A-17) that $u_r \equiv 0$; hence the eigenvalue equation (2-16) has no nontrivial solution if $s = m(a - ib) \notin i\mathbb{R}$. This proves that the linearized operator at Rankine's vortex has no unstable eigenvalue.

[4] Except for an ambiguous sentence asserting, without any justification, that the eigenfunctions corresponding to purely imaginary eigenvalues should form a complete family.

Remark A.2. Alternatively, one can obtain the relation (A-16) by restricting the eigenvalue equation (2-16) to the open intervals (0, 1) and (1, ∞), where the vorticity profile is smooth. On each interval, we multiply (2-16) by $r\bar{u}_r$ and we integrate over r. If we add the resulting expressions and simplify the boundary terms (which result from partial integrations) using the matching condition (A-12), we arrive at (A-16).

A3. ***Critical layers and their continuity properties.*** In this section we present the proof of Lemma 3.5. We first rewrite (3-2) for u_n in the form

$$u_n''(r) + \mathcal{P}(r)u_n'(r) + \mathcal{Q}_n(r)u_n(r) = 0, \tag{A-18}$$

where $\mathcal{P}(r) = \mathcal{A}'(r)/\mathcal{A}(r) + 1/r$ and

$$\mathcal{Q}_n(r) = \frac{k^2}{m^2}\frac{\Phi_n(r)}{\gamma_n(r)^2} - \frac{r}{\mathcal{A}(r)\gamma_n(r)}\partial_r\left(\frac{W_n(r)}{m^2+k^2r^2}\right) + \frac{\mathcal{P}(r)}{r} - \frac{2}{r^2}.$$

Here, as in Section 4C, we set $\gamma_n(r) = \Omega_n(r) - b_n - ia_n$, where Ω_n is the angular velocity associated with W_n as in (1-16). By assumption (ii) and (1-16) we have $\Omega_n \to \Omega$ in $\mathcal{C}^2$ on compact subsets of $(0, \infty)$. In view of (iii), Ω is analytic in $\mathbb{D}(\bar{r}, \rho)$ for some $\rho > 0$, and Ω_n converges uniformly to Ω on that disc as $n \to \infty$. Since $\Omega(\bar{r}) = b$ and $\Omega'(\bar{r}) < 0$, it follows from Hurwitz's theorem that, for sufficiently large n, there exists a unique $\bar{r}_n \in \mathbb{D}(\bar{r}, \rho)$ such that $\Omega_n(\bar{r}_n) = b_n + ia_n$. Moreover $\Omega_n'(\bar{r}_n) \neq 0$, $\Omega_n'(\bar{r}) < 0$ and

$$\bar{r}_n = \bar{r} + \frac{b_n + ia_n - \Omega_n(\bar{r})}{\Omega_n'(\bar{r})} + \mathcal{O}(|b_n - \Omega_n(\bar{r})|^2 + a_n^2) \quad \text{as } n \to \infty,$$

so that $\bar{r}_n \to \bar{r}$ and $\bar{r}_n \in \{z \in \mathbb{C} \mid \mathrm{Im}(z) < 0\}$ when n is sufficiently large. By construction, we also have $\gamma_n(r) = \Omega_n'(\bar{r}_n)(r - \bar{r}_n) + \mathcal{O}(|r - \bar{r}_n|^2)$ as $r \to \bar{r}_n$.

Multiplying (A-18) by z^2 and applying the change of variables $z = r - \bar{r}_n$, $w_n(z) = u_n(\bar{r}_n + z)$, we obtain the canonical form

$$z^2 w_n''(z) + zP_n(z)w_n'(z) + Q_n(z)w_n(z) = 0, \tag{A-19}$$

where $P_n(z) = z\mathcal{P}(\bar{r}_n + z)$ and $Q_n(z) = z^2\mathcal{Q}_n(\bar{r}_n + z)$ are analytic inside the disc $\mathbb{D}(0, \rho/2)$, if n is large enough so that $|\bar{r}_n - \bar{r}| < \rho/2$. In this situation, the Frobenius method [Coddington and Levinson 1955, Section 4.8], which we briefly recall now, can be used to construct solutions of (A-19) in $\mathbb{D}(0, \rho/2) \setminus \mathbb{R}_-$ of the form

$$w_n(z) = z^{d_n} v_n(z), \quad \text{where } v_n(z) = \sum_{k=0}^{\infty} c_{n,k} z^k \text{ and } c_{n,0} = 1 \text{ for all } n. \tag{A-20}$$

The coefficients $c_{n,k}$ for $k \geq 1$ are determined by substituting (A-20) into (A-19) and collecting equal powers of z. If $P_n(z) = \sum_{k=0}^{\infty} p_{n,k} z^k$ and $Q_n(z) = \sum_{k=0}^{\infty} q_{n,k} z^k$, one obtains the recursion relations

$$c_{n,k} = \frac{-1}{f_n(d_n + k)} \sum_{j=0}^{k-1} c_{n,j}[(d_n + j)p_{n,k-j} + q_{n,k-j}], \quad k \geq 1, \tag{A-21}$$

where the *indicial function* $f_n : \mathbb{R} \to \mathbb{R}$ is defined by

$$f_n(d) = d(d-1) + dp_{n,0} + q_{n,0}. \tag{A-22}$$

Assuming that $f_n(d_n+k) \neq 0$ for all $k \geq 1$, it is straightforward to verify that the formal series $w_n(z)$ defined by (A-20), (A-21) satisfies

$$z^2 w_n''(z) + z P_n(z) w_n'(z) + Q_n(z) w_n(z) = f_n(d_n) z^{d_n}, \tag{A-23}$$

and hence is a (formal) solution of (A-19) provided d_n is a root of the quadratic polynomial f_n.

In our situation we have $p_{n,0} = 0$ and $q_{n,0} = (k^2/m^2) J_n(\bar{r}_n)$, so that the indicial equation $f_n(d_n) = 0$ reduces to (3-19) as $n \to \infty$. The roots $d_n^{\pm}$ of f_n thus converge to the explicit values $d_{\pm}$ described in Section 3D, which are such that $|d_+ - d_-| < 1$. As a consequence, if n is large enough and $d_n = d_n^+$ or d_n^-, the denominator in (A-21) never vanishes, and even satisfies $|f_n(d_n+k)| \geq c_0 k^2$ for all $k \geq 1$, where $c_0 > 0$ is independent of n. This allows us to solve the recursion relations (A-21) if $d_n = d_n^{\pm}$, and it is then straightforward to verify that the series in (A-20) converges for all $z \in \mathbb{D}(0, \rho/2)$, and that the sum $v_n^{\pm}$ is uniformly bounded on compact subsets of $\mathbb{D}(0, \rho/2)$ if n is sufficiently large. We denote henceforth by

$$w_n^{\pm}(z) = z^{d_n^{\pm}} v_n^{\pm}(z)$$

the solution of (A-19) given by (A-20) with $d_n = d_n^{\pm}$.

By assumption (iii), the quantities $p_{n,k}$, $q_{n,k}$ converge as $n \to \infty$ to the Taylor coefficients of the functions $P(z)$, $Q(z)$ associated with the limiting profile W and the limiting value of the spectral parameter. Using the recursion relation (A-21), where each coefficient $c_{n,k}^{\pm}$ is entirely determined by a finite number of coefficients $p_{n,\ell}$, $q_{n,\ell}$ (namely, those with $\ell < k$), we see that

$$c_{n,k}^{\pm} \to c_k^{\pm} \quad \text{as } n \to \infty, \tag{A-24}$$

where $c_k^{\pm}$ denote the coefficients of the Frobenius solution $w^{\pm}(z) = z^{d_{\pm}} v^{\pm}(z)$ of the limiting equation (A-19), where P_n, Q_n are replaced by P, Q. In view of the uniform bounds mentioned above, this implies that $v_n^{\pm}(z)$ converges to $v^{\pm}(z)$ uniformly on compact subsets of $\mathbb{D}(0, \rho/2)$ as $n \to \infty$. Note that since P and Q are real-valued on the real axis, the recurrence relation yields that the coefficients $c_k^{\pm}$ are real too. The functions $V_{\pm}$ which appear in the formulas (3-20)–(3-22) are the only real analytic functions on $\mathbb{R}_+$ that satisfy

$$V_{\pm}(r) = v^{\pm}(r - \bar{r}) \left(\frac{r - \bar{r}}{b - \Omega(r)} \right)^{d_{\pm}}, \quad r \in \left(\bar{r} - \frac{\rho}{2}, \bar{r} + \frac{\rho}{2} \right).$$

That $V_{\pm}$ are well-defined and real analytic on the whole half-line $\mathbb{R}_+$ follows from the representation (3-20) and the ODE (3-18).

Now, consider a sequence $(u_n)_{n \in \mathbb{N}}$ of solutions of (A-18) as in the statement of Lemma 3.5, and assume first that $d_+ \neq d_-$, so that $d_n^+ \neq d_n^-$ when n is sufficiently large. Since (A-18) is a second-order differential equation, there exist complex coefficients $\alpha_n^{\pm}$ such that[5]

$$u_n(r) = \alpha_n^+ w_n^+(r - \bar{r}_n) + \alpha_n^- w_n^-(r - \bar{r}_n) \quad \text{for } r \in \left(\bar{r} - \frac{\rho}{4}, \bar{r} + \frac{\rho}{4} \right). \tag{A-25}$$

[5]Note that $\mathbb{D}(\bar{r}_n, \rho/2)$ contains the real interval $(\bar{r} - \rho/4, \bar{r} + \rho/4)$ for large values of n.

By assumption $u_n(r)$ and $u_n'(r)$ have a limit as $n \to \infty$ for some $r \neq \bar{r}$, and using elementary continuity properties for solutions of nonsingular ODEs we deduce that convergence holds locally uniformly for all $r > \bar{r}$, or for all $r < \bar{r}$. Since the functions $w_n^{\pm}(r-\bar{r}_n)$ converge uniformly to $w^{\pm}(r-\bar{r})$ in a neighborhood of $\bar{r}$, and since the limits $w^{\pm}(r-\bar{r})$ have genuinely different behaviors as $r \to \bar{r}$, this implies that the coefficients $\alpha_n^{\pm}$ in (A-25) have finite limits $\alpha_{\pm} \in \mathbb{C}$ as $n \to \infty$. In particular we have

$$u_n(r) \to \alpha_+(r-\bar{r})^{d_+}v^+(r-\bar{r}) + \alpha_-(r-\bar{r})^{d_-}v^-(r-\bar{r}),$$

uniformly for $r \in (\bar{r}, \bar{r}+\rho/4)$, and (keeping in mind that $\mathrm{Im}(\bar{r}_n) < 0$)

$$u_n(r) \to \alpha_+ e^{i\pi d_+}(\bar{r}-r)^{d_+}v^+(r-\bar{r}) + \alpha_- e^{i\pi d_-}(\bar{r}-r)^{d_-}v^-(r-\bar{r}),$$

uniformly for $r \in (\bar{r}-\rho/4, \bar{r})$. Since outside the interval $(\bar{r}-\rho/4, \bar{r}+\rho/4)$ the ODE (A-18) is asymptotically regular, this implies the desired conclusion, namely that $u_n \to \alpha_+\phi_+ + \alpha_-\phi_-$ where $\phi_{\pm}$ are as in (3-20) or (3-21).

We next consider the exceptional situation where $d_- = d_+$. Without loss of generality we may assume that either $d_n^- = d_n^+$ for all $n \in \mathbb{N}$, or $d_n^- \neq d_n^+$ for all $n \in \mathbb{N}$. In the first case we obtain from (A-20) only one solution w_n of (A-19), but we can construct a second solution by differentiating (A-23) with respect to the exponent d_n, taking into account the fact that $f_n'(d_n) = 0$ since $d_n = d_n^{\pm}$ is a double root by assumption. The new solution has the form

$$w_n^{\sharp}(z) = \log(z)w_n(z) + z^{d_n}\sum_{k=0}^{\infty}\left(\frac{\partial c_{n,k}}{\partial d_n}\right)z^k, \qquad z \in \mathbb{D}(0,\rho/2) \setminus \mathbb{R}_-,$$

and its asymptotic behavior as $z \to 0$ is clearly different from that of $w_n(z)$. This allows us to conclude the proof using the same argument as above, and we obtain that $u_n \to \alpha_+\phi_+ + \alpha_-\phi_-$, where $\phi_{\pm}$ are as in (3-22). On the other hand, when $d_n^- \neq d_n^+$ for all $n \in \mathbb{N}$ (but $d_n^- - d_n^+ \to 0$), the decomposition (A-25) is not appropriate, because in that case we cannot prove that the coefficients α_n^+ and α_n^- are bounded, yet alone have limits as $n \to \infty$. Instead, we write

$$u_n(r) = \alpha_n(w_n^+(r-\bar{r}_n) + w_n^-(r-\bar{r}_n)) + \alpha_n^{\sharp}\left(\frac{w_n^+(r-\bar{r}_n) - w_n^-(r-\bar{r}_n)}{d_n^+ - d_n^-}\right) \tag{A-26}$$

for $r \in (\bar{r}-\rho/4, \bar{r}+\rho/4)$, and this new decomposition has the property that both coefficients α_n and $\alpha_n^{\sharp}$ necessarily have limits as $n \to \infty$. We then finish the proof along the same lines as above. □

A4. ***Approximation and interpolation in the class $\mathscr{W}$.*** This section is devoted to the proofs of Lemmas 4.2 and 4.3. If W is a vorticity profile that belongs to the class $\mathscr{W}$, in the sense of Definition 1.8, we denote by Ω the corresponding angular velocity given by (1-16), and by J the function defined in (1-20). The first observation is that both Ω and W can be expressed in terms of the auxiliary function J.

Indeed, let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be defined by $\phi(r) = \Omega(r)/\Omega'(r)$ for all $r > 0$. According to (1-6), (1-19), (1-20) we have

$$J(r) = \frac{\Phi(r)}{\Omega'(r)^2} = \frac{2r\Omega(r)}{\Omega'(r)} + \frac{4\Omega(r)^2}{\Omega'(r)^2} = 2r\phi(r) + 4\phi(r)^2 \tag{A-27}$$

for all $r > 0$. Since $J(r) > 0$ and $\phi(r) < 0$ by assumption, we deduce that

$$\phi(r) = \frac{\Omega(r)}{\Omega'(r)} = -\frac{1}{4}\big(r + \sqrt{r^2 + 4J(r)}\big), \quad r > 0. \tag{A-28}$$

Integrating this differential equation and using the normalization condition $\Omega(0) = 1$, which follows from (1-16) since $W(0) = 2$, we obtain the representation formula

$$\Omega(r) = \exp\biggl(-\int_0^r \frac{4}{s + \sqrt{s^2 + 4J(s)}}\,\mathrm{d}s\biggr), \quad r > 0. \tag{A-29}$$

As $W(r) = r\Omega'(r) + 2\Omega(r)$, we also have

$$W(r) = \Omega(r)\Bigl(2 + \frac{r}{\phi(r)}\Bigr) = \Omega(r)\Bigl(2 - \frac{4r}{r + \sqrt{r^2 + 4J(r)}}\Bigr), \quad r > 0. \tag{A-30}$$

Furthermore, if we differentiate (A-30) with respect to r and observe that $\Omega' = \Omega/\phi$, we obtain

$$W' = \frac{\Omega}{\phi}\Bigl(2 + \frac{r}{\phi}\Bigr) + \Omega\Bigl(\frac{1}{\phi} - \frac{r\phi'}{\phi^2}\Bigr) = \frac{\Omega}{\phi^2}(3\phi + r - r\phi').$$

Thus, using the expression (A-28) of ϕ, we find after elementary calculations

$$W'(r) = \frac{8\Omega(r)}{\big(r + \sqrt{r^2 + 4J(r)}\big)^2}\Bigl(r - \frac{r^2 + 6J(r)}{\sqrt{r^2 + 4J(r)}} + \frac{rJ'(r)}{\sqrt{r^2 + 4J(r)}}\Bigr). \tag{A-31}$$

As $r\sqrt{r^2 + 4J} < r^2 + 6J$ when $J > 0$, this formula shows that the vorticity W is strictly decreasing as soon as the auxiliary function J satisfies $J(r) > 0$ and $J'(r) < 0$ for all $r > 0$. This observation will be used later on.

Since $W \in \mathscr{W}$ by assumption, the angular velocity satisfies $\Omega'(r) \to 0$ as $r \to 0$, and in view of (1-20) or (A-28) this implies $J(r) \to \infty$ as $r \to 0$. It then follows from (A-31) that

$$W'(r) \sim \frac{rJ'(r)}{J(r)^{3/2}} - \frac{6}{J(r)^{1/2}} \quad \text{as } r \to 0,$$

and since $W'(r)$ vanishes at the origin we deduce that $rJ'(r)J(r)^{-3/2} \to 0$ as $r \to 0$. Concerning the behavior at infinity, we observe that $\phi(r) = -2/r + \mathcal{O}(1/r^3)$ as $r \to \infty$, and in view of (A-29) this implies $\Omega(r) = \Gamma/r^2 + \mathcal{O}(1/r^4)$ as $r \to \infty$, for some $\Gamma > 0$. The expression (A-30) also shows that $r^4 W(r) \to 2\Gamma J(\infty)$ as $r \to \infty$. Finally, one infers from (A-31) that

$$W'(r) \sim \Bigl\{\frac{2\Gamma}{r^4} + \mathcal{O}\Bigl(\frac{1}{r^6}\Bigr)\Bigr\}\Bigl\{\frac{rJ'(r)}{\sqrt{r^2 + 4J(r)}} - \frac{4J(r)}{r} + \mathcal{O}\Bigl(\frac{1}{r^3}\Bigr)\Bigr\} \quad \text{as } r \to \infty,$$

and since $rJ'(r) \to 0$ as $r \to \infty$ we also obtain $r^5 W'(r) \to -8\Gamma J(\infty)$ as $r \to \infty$.

The properties of J are more conveniently expressed in terms of the new function

$$Q(r) = \frac{1}{\sqrt{1 + J(r)}}, \quad r > 0. \tag{A-32}$$

Definition A.3. We say that a $\mathcal{C}^1$ function $Q : \mathbb{R}_+ \to (0, 1]$ belongs to the class $\mathscr{Q}$ if

(i) $Q'(r) > 0$ for all $r > 0$;

(ii) $Q(r) \to 0$ and $rQ'(r) \to 0$ as $r \to 0$;

(iii) $rQ'(r) \to 0$ as $r \to \infty$.

In particular $\mathscr{Q}$ is convex, and if $Q \in \mathscr{Q}$ then $\mathcal{N}(Q) := \sup_{r>0} r|Q'(r)| < \infty$.

Lemma A.4. *A vorticity profile W belongs to the class $\mathscr{W}$ in the sense of Definition 1.8 if and only if the function Q defined by* (A-32), *with J as in* (1-20), *belongs to the class $\mathscr{Q}$.*

Proof. If $W \in \mathscr{W}$, we have just shown that the $\mathcal{C}^1$ map $J : \mathbb{R}_+ \to \mathbb{R}$ defined by (1-20) satisfies $J(r) > 0$ and $J'(r) < 0$ for all $r > 0$, $J(r) \to \infty$ and $rJ'(r)J(r)^{-3/2} \to 0$ as $r \to 0$, and $rJ'(r) \to 0$ as $r \to \infty$. These properties precisely mean that the function Q defined by (A-32) belongs to the class $\mathscr{Q}$. Conversely, if $Q \in \mathscr{Q}$, we define $J = Q^{-2} - 1$ so that (A-32) holds, and we reconstruct the angular velocity Ω and the vorticity W by the formulas (A-29), (A-30). The calculations above then show that $W \in \mathscr{W}$. In particular, since $J'(r) < 0$, formula (A-31) shows that $W'(r) < 0$ for all $r > 0$. □

The following result expresses the fact that the vorticity profile $W \in \mathscr{W}$ depends continuously on the auxiliary function $Q \in \mathscr{Q}$, in appropriate topologies.

Lemma A.5. *Assume that $Q_1, Q_2 \in \mathscr{Q}$, and take $\delta > 0$ small enough so that*

$$\delta \le \min\{Q_1(1), Q_2(1)\} \le \max\{Q_1(1), Q_2(1)\} \le \sqrt{1-\delta^2}. \tag{A-33}$$

Then there exists a constant $C > 0$, depending only on δ, such that, if W_1, W_2 denote the vorticity profiles associated with Q_1, Q_2 as in Lemma A.4, the following estimates hold:

$$\sup_{r>0}(1+r^4)|W_1(r) - W_2(r)| \le C\|Q_1 - Q_2\|_{L^\infty(\mathbb{R}_+)}, \tag{A-34}$$

$$\sup_{r>0}(1+r^5)|W_1'(r) - W_2'(r)| \le C\mathcal{N}(Q_1 - Q_2) + C(1+\mathcal{N}(Q_2))\|Q_1 - Q_2\|_{L^\infty(\mathbb{R}_+)}. \tag{A-35}$$

Proof. Let $J_i(r) = Q_i(r)^{-2} - 1$ for $i = 1, 2$. We first consider the quantity

$$\Theta(r) = \frac{1}{r+\sqrt{r^2+4J_1}} - \frac{1}{r+\sqrt{r^2+4J_2}} = \frac{4(Q_1^2 - Q_2^2)}{(Q_1\Delta_2 + Q_2\Delta_1)(rQ_1 + \Delta_1)(rQ_2 + \Delta_2)},$$

where we use the shorthand notation $\Delta_i(r) = Q_i(r)\sqrt{r^2 + 4J_i(r)} = \sqrt{4 + (r^2-4)Q_i(r)^2}$ for $i = 1, 2$. We claim that

$$\sup_{r>0}(1+r^3)|\Theta(r)| \le C\|Q_1 - Q_2\|_{L^\infty(\mathbb{R}_+)} \tag{A-36}$$

for some constant $C > 0$ depending only on δ. To prove that, we distinguish two cases:

(i) In the region where $r \le 1$, we know from (A-33) that $Q_i(r)^2 \le Q_i(1)^2 \le 1 - \delta^2$, and this implies $\Delta_i(r) \ge 2\sqrt{1 - Q_i(r)^2} \ge 2\delta$. It follows that

$$|\Theta(r)| \le \frac{4(Q_1 + Q_2)|Q_1 - Q_2|}{8\delta^3(Q_1 + Q_2)} \le \frac{|Q_1 - Q_2|}{2\delta^3}, \quad 0 < r \le 1.$$

(ii) When $r \ge 1$ we observe that $\Delta_i(r) \ge rQ_i \ge r\delta$ by (A-33), so that

$$|\Theta(r)| \le \frac{4(Q_1+Q_2)|Q_1-Q_2|}{8r^3Q_1^2Q_2^2} \le \frac{|Q_1-Q_2|}{\delta^3 r^3}, \quad r \ge 1.$$

Altogether, this proves (A-36).

As an immediate consequence, we see that the angular velocities defined by (A-29) satisfy the estimate $\|\Omega_1-\Omega_2\|_{L^\infty(\mathbb{R}_+)} \le C\|Q_1-Q_2\|_{L^\infty(\mathbb{R}_+)}$. In fact, we have a stronger result:

$$\sup_{r>0}(1+r^2)|\Omega_1(r)-\Omega_2(r)| \le C\|Q_1-Q_2\|_{L^\infty(\mathbb{R}_+)}. \tag{A-37}$$

Indeed, it follows from (A-29) that

$$r^2\Omega_i(r) = \Omega_i(1)\exp\int_1^r \left(\frac{2}{s} - \frac{4}{s+\sqrt{s^2+4J_i(s)}}\right)\mathrm{d}s, \quad r>0,\ i=1,2. \tag{A-38}$$

If we define

$$M_i(r) = \frac{1}{r} - \frac{2}{r+\sqrt{r^2+4J_i}} = \frac{4J_i}{r(r+\sqrt{r^2+4J_i})^2} = \frac{4(1-Q_i^2)}{r(rQ_i+\Delta_i)^2} > 0, \quad r>0,$$

the same estimates as above show that

$$\sup_{0<r\le 1} rM_i(r) \le \frac{1}{\delta^2} \quad \text{and} \quad \sup_{r\ge 1} r^3M_i(r) \le \frac{1}{\delta^2}. \tag{A-39}$$

This implies in particular that $r^2\Omega_i(r) \le e^{1/\delta^2}$ for $r \ge 1$; hence $(1+r^2)\Omega_i(r) \le C$ for some constant $C>0$ depending only on δ. In addition, using (A-36) and (A-38), we obtain

$$r^2|\Omega_1(r)-\Omega_2(r)| \le e^{1/\delta^2}\left(|\Omega_1(1)-\Omega_2(1)| + 4\int_1^r |\Theta(s)|\,\mathrm{d}s\right) \le C\|Q_1-Q_2\|_{L^\infty(\mathbb{R}_+)}$$

for all $r \ge 1$, and this concludes the proof of (A-37).

On the other hand, in view of (A-30), we have $W_i(r) = 2r\Omega_i(r)M_i(r)$ for $i=1,2$; hence

$$W_1(r)-W_2(r) = 2r(\Omega_1(r)-\Omega_2(r))M_1(r) - 4r\Omega_2(r)\Theta(r), \quad r>0.$$

Thus using estimates (A-36), (A-37), (A-39) we arrive at (A-34).

Finally, we deduce from (A-31) that $W_1'(r)-W_2'(r) = \Xi(r)-16\Lambda(r)$, where $\Xi(r)$ collects all terms that do not involve the derivatives J_1', J_2', and

$$\Lambda(r) = \frac{\Omega_1 rQ_1'}{(rQ_1+\Delta_1)^2\Delta_1} - \frac{\Omega_2 rQ_2'}{(rQ_2+\Delta_2)^2\Delta_2} = \Lambda_1(r)+\Lambda_2(r),$$

where

$$\Lambda_1(r) = \frac{\Omega_1 r(Q_1'-Q_2')}{(rQ_1+\Delta_1)^2\Delta_1}, \quad \Lambda_2(r) = \frac{\Omega_1 rQ_2'}{(rQ_1+\Delta_1)^2\Delta_1} - \frac{\Omega_2 rQ_2'}{(rQ_2+\Delta_2)^2\Delta_2}.$$

Proceeding exactly as above it is straightforward to verify that

$$\sup_{r>0}(1+r^5)|\Xi(r)| \le C\|Q_1-Q_2\|_{L^\infty(\mathbb{R}_+)},$$

where $C > 0$ depends only on δ. We thus concentrate on the new term Λ, which contains the derivatives Q_1', Q_2'. Again, considering separately the regions where $r \le 1$ and $r \ge 1$, and using the appropriate lower bound on $\Delta_1(r)$ in each region, we obtain

$$|\Lambda_1(r)| \le C\Omega_1(r)\frac{r|Q_1'(r) - Q_2'(r)|}{1+r^3} \le C(1+r^2)\Omega_1(r)\frac{\mathcal{N}(Q_1 - Q_2)}{1+r^5},$$

where we recall that $(1+r)^2\Omega_1(r) \le C$ for some $C > 0$ depending only on δ. The quantity Λ_2 is estimated along the same lines:

$$|\Lambda_2(r)| \le C(1+r^2)|\Omega_1(r) - \Omega_2(r)|\frac{r|Q_2'(r)|}{1+r^5} + C(1+r^2)\Omega_2(r)\frac{r|Q_2'(r)|}{1+r^5}|Q_1(r) - Q_2(r)|,$$

and using (A-37) we find $(1+r^5)|\Lambda_2(r)| \le C(1+\mathcal{N}(Q_2))\|Q_1 - Q_2\|_{L^\infty(\mathbb{R}_+)}$. Combining these estimates we arrive at (A-35). □

Proof of Lemma 4.2. Let $W \in \mathscr{W}$, and denote by $Q \in \mathscr{Q}$ the function defined by (A-32) with J as in (1-20). For any $\epsilon > 0$ we define

$$Q^{(\epsilon)}(r) = \frac{1}{\sqrt{\pi\epsilon}}\int_0^\infty (e^{-(r-s)^2/\epsilon} - e^{-(r+s)^2/\epsilon})Q(s)\,\mathrm{d}s, \quad r > 0. \tag{A-40}$$

In other words, $Q^{(\epsilon)}$ is the restriction to $\mathbb{R}_+$ of the real-analytic odd function obtained by extending Q to an odd function $\overline{Q} : \mathbb{R} \to \mathbb{R}$ and applying to $\overline{Q}$ the heat semigroup on $\mathbb{R}$ at time $t = \epsilon/4$. In particular, $Q^{(\epsilon)}$ is real-analytic on $\overline{\mathbb{R}}_+$ for any $\epsilon > 0$. Moreover, as the function $s \mapsto \overline{Q}(\mathrm{s})$ is continuous on $\mathbb{R}$ and converges to finite limits as $s \to \pm\infty$, it is clear that $Q^{(\epsilon)}$ converges uniformly to Q on $\mathbb{R}_+$ as $\epsilon \to 0$. On the other hand, differentiating (A-40) with respect to r, we obtain the relation

$$rQ^{(\epsilon)\prime}(r) = \frac{1}{\sqrt{\pi\epsilon}}\int_0^\infty (e^{-(r-s)^2/\epsilon} - e^{-(r+s)^2/\epsilon})sQ'(s)\,\mathrm{d}s + R_\epsilon(r), \quad r > 0, \tag{A-41}$$

where

$$R_\epsilon(r) = \frac{1}{\sqrt{\pi\epsilon}}\int_0^\infty ((r-s)e^{-(r-s)^2/\epsilon} + (r+s)e^{-(r+s)^2/\epsilon})Q'(s)\,\mathrm{d}s, \quad r > 0.$$

As $Q' \in L^1(\mathbb{R}_+)$ and $\mathcal{N}(Q) < \infty$, it is straightforward to verify that R_ϵ converges uniformly to zero as $\epsilon \to 0$. Moreover, since $sQ'(s) \to 0$ as $s \to 0$ and $s \to \infty$, it is clear that the integral term in (A-41) converges uniformly on $\mathbb{R}_+$ towards $rQ'(r)$ as $\epsilon \to 0$. Altogether, this shows that

$$\lim_{\epsilon \to 0}(\|Q^{(\epsilon)} - Q\|_{L^\infty(\mathbb{R}_+)} + |\mathcal{N}(Q^{(\epsilon)} - Q)|) = 0. \tag{A-42}$$

Now, let $W^{(\epsilon)} \in \mathscr{W}$ be the vorticity profile associated with $Q^{(\epsilon)}$ as in Lemma A.4. By construction $W^{(\epsilon)}$ is real-analytic on $\overline{\mathbb{R}}_+$ for any $\epsilon > 0$, and it follows from (A-42) and Lemma A.5 that $W^{(\epsilon)} \to W$ in the topology of $\mathcal{C}_b^1(\overline{\mathbb{R}}_+)$ as $\epsilon \to 0$. This is the desired density result. □

Proof of Lemma 4.3. Assume that $W_0, W_1 \in \mathscr{W}$, and let $Q_1, Q_2 \in \mathscr{Q}$ be the corresponding functions defined as in Lemma A.4. For any $t \in [0, 1]$, we define $Q_t \in \mathscr{Q}$ by the linear interpolation formula

$$Q_t(r) = (1-t)Q_0(r) + tQ_1(r), \quad r > 0, \tag{A-43}$$

and we denote by $W_t \in \mathscr{W}$ the vorticity profile associated with Q_t. As

$$\|Q_{t_1} - Q_{t_2}\|_{L^\infty(\mathbb{R}_+)} + \mathcal{N}(Q_{t_1} - Q_{t_2}) = |t_1 - t_2|(\|Q_0 - Q_1\|_{L^\infty(\mathbb{R}_+)} + \mathcal{N}(Q_0 - Q_1)),$$

it follows from Lemma A.5 that W_t is a Lipschitz function of $t \in [0, 1]$ in the topology of $\mathcal{C}^1_b(\overline{\mathbb{R}}_+)$. By construction, if W_0, W_1 are real analytic on $\overline{\mathbb{R}}_+$, so is W_t for any $t \in [0, 1]$. Moreover, in that case, if $W_1''(0) \equiv -8Q_1'(0) < 0$, then $W_t''(0) \equiv -8Q_t'(0) < 0$ for all $t \in (0, 1]$, because $W_0''(0) \equiv -8Q_0'(0) \le 0$. □

A5. *Proof of Lemma 4.6.* The proof of estimate (4-32) is lengthy but rather straightforward, and we just indicate here the main steps. Using classical estimates for the modified Bessel functions K_ν [Abramowitz and Stegun 1964, Section 9.6], we first observe that the approximate solution χ_n defined in (4-26) satisfies

$$\begin{aligned} |\chi_n(r)| &\approx \min(1, |r - r_n + ic_n|)^{1/2-\nu_n} e^{-k(r-r_n)}, \\ |\chi_n'(r)| &\approx \min(1, |r - r_n + ic_n|)^{-1/2-\nu_n} e^{-k(r-r_n)} \end{aligned} \tag{A-44}$$

for all $r > 0$. Here $A \approx B$ means that the ratio A/B is bounded from above and from below by some positive constants that are independent of n. A direct consequence of (A-44) is:

Lemma A.6. *There exists a constant* $C > 0$ *such that*, *for all* $n \in \mathbb{N}$ *and all* $s > r > 0$, *the following estimates hold*:

$$|\chi_n(s)|^2 \int_r^s \frac{1}{|\chi_n(t)|^2}\,\mathrm{d}t \le C \min(1, |s - r_n + ic_n|)^{1-2\nu_n}, \tag{A-45}$$

$$|\chi_n(s)\chi_n'(s)| \int_r^s \frac{1}{|\chi_n(t)|^2}\,\mathrm{d}t \le C \min(1, |s - r_n + ic_n|)^{-2\nu_n}. \tag{A-46}$$

Proof. We only prove (A-45), the proof of (A-46) being similar. If $c_n \ge 1$, the first estimate in (A-44) simply means that $|\chi_n(r)| \approx e^{-k(r-r_n)}$, and (A-45) follows immediately. Thus we assume henceforth that $0 < c_n \le 1$, and for simplicity we set $s = r_n + \tau$, so that the proof of (A-45) reduces to showing that

$$\mathcal{I}_n(\tau) := |\chi_n(r_n + \tau)|^2 \int_{-\infty}^{\tau} \frac{1}{|\chi_n(r_n + t)|^2}\,\mathrm{d}t \le C \min(1, |\tau + ic_n|)^{1-2\nu_n}, \quad \tau \in \mathbb{R}. \tag{A-47}$$

If $\tau \le -1$, we know from (A-44) that $|\chi_n(r_n + \tau)| \le Ce^{-k\tau}$ and $|\chi_n(r_n + t)| \ge Ce^{-kt}$ for $t \le \tau$; hence $\mathcal{I}_n(\tau) \le C$. If $|\tau| \le 1$, then $|\chi_n(r_n + \tau)| \le C|\tau + ic_n|)^{1/2-\nu_n}$, and

$$\mathcal{I}_n(\tau) \le C|\tau + ic_n|^{1-2\nu_n}\biggl(\int_{-\infty}^{-1} e^{2kt}\,\mathrm{d}t + \int_{-1}^{\tau} |t + ic_n|^{2\nu_n - 1}\,\mathrm{d}t\biggr) \le C|\tau + ic_n|^{1-2\nu_n}.$$

Finally, if $\tau \ge 1$, then

$$\mathcal{I}_n(\tau) \le Ce^{-2k\tau}\biggl(\int_{-\infty}^{-1} e^{2kt}\,\mathrm{d}t + \int_{-1}^{1} |t + ic_n|^{2\nu_n - 1}\,\mathrm{d}t + \int_1^{\tau} e^{2kt}\,\mathrm{d}t\biggr) \le C,$$

and this completes the proof of (A-45). We observe that the constant C in the right-hand side is independent of n because, as can be seen from (4-27), the exponent ν_n is bounded away from zero as $n \to \infty$. □

Remark A.7. In Section 4I we use the fact that if $\nu_n \to 0$ as $n \to \infty$, the conclusion of Lemma A.6 remains valid up to a logarithmic correction. Estimates (A-44) are not appropriate in that case, but one can use the fact that $K_{\nu_n}(z)$ is close to $K_0(z)$ for n large, where $K_0(z) \approx -\log(z)$ as $z \to 0$. In particular, the integral

$$\int_{r_n-1}^{r_n+1} \frac{1}{\chi_n(r)^2}\,\mathrm{d}r = \frac{\pi}{2}\int_{-1}^{1} \frac{1}{(t+ic_n)K_{\nu_n}(k(t+ic_n))^2}\,\mathrm{d}t$$

is uniformly bounded even if $\nu_n \to 0$ and $c_n \to 0$, because the function $t \mapsto |t|^{-1}|K_0(kt)|^{-2}$ is integrable over $[-1, 1]$. We leave the details to the reader.

Returning to the proof of Lemma 4.6, we note that $\mathcal{K}_n(r,s) = \mathcal{K}_n^{(1)}(r,s) - \mathcal{K}_n^{(2)}(r,s)$, where

$$\mathcal{K}_n^{(1)}(r,s) = \mathcal{A}(s)\chi_n(s)^2\mathcal{R}_n(s)\int_r^s \frac{1}{\mathcal{A}(t)\chi_n(t)^2}\,\mathrm{d}t,$$
$$\mathcal{K}_n^{(2)}(r,s) = \mathcal{A}'(s)\chi_n(s)\chi_n'(s)\int_r^s \frac{1}{\mathcal{A}(t)\chi_n(t)^2}\,\mathrm{d}t.$$

In what follows, we assume that $s \ge r \ge \delta r_n$ for some $\delta > 0$, where $r_n = (\Gamma_n/b_n)^{1/2} \to \infty$ as $n \to \infty$. Since $\mathcal{A}(s) \to 1/k^2$ and $\mathcal{A}'(s) = \mathcal{O}(s^{-3})$ as $s \to \infty$, it follows immediately from (A-46) that

$$\sup_{r \ge \delta r_n}\int_r^\infty |\mathcal{K}_n^{(2)}(r,s)|\,\mathrm{d}s \le C\int_{\delta r_n}^\infty s^{-3}\min(1, |s-r_n+ic_n|)^{-2\nu_n}\,\mathrm{d}s \le \frac{C}{r_n^2} \le Cb_n. \tag{A-48}$$

It thus remains to estimate the kernel $\mathcal{K}_n^{(1)}(r,s)$, which involves the remainder $\mathcal{R}_n$ defined in (4-29). We can further decompose $\mathcal{R}_n$ as $\mathcal{R}_n = \mathcal{R}_n^{(1)} + \mathcal{R}_n^{(2)} + \mathcal{R}_n^{(3)}$, where

$$\mathcal{R}_n^{(1)}(r) = \frac{m^2+\frac{3}{4}}{r^2} - \frac{1}{2r}\frac{\mathcal{A}'(r)}{\mathcal{A}(r)},$$
$$\mathcal{R}_n^{(2)}(r) = -\frac{k^2}{m^2}\left(\frac{\Phi_n(r)}{\gamma_n(r)^2} - \frac{J_n(r_n)}{(r-r_n+ic_n)^2}\right),$$
$$\mathcal{R}_n^{(3)}(r) = \frac{r}{\mathcal{A}(r)\gamma_n(r)}\partial_r\left(\frac{W_n(r)}{m^2+k^2r^2}\right).$$

We concentrate here on the term $\mathcal{R}_n^{(2)}$, which gives the main contribution to the integral (4-32).

Lemma A.8. *For any $\delta > 0$, there exists a constant $C > 0$ such that*

$$|\mathcal{R}_n^{(2)}(r)| \le \frac{C}{r|r-r_n+ic_n|} \quad \textit{for all } r \ge \delta r_n. \tag{A-49}$$

Proof. We first assume that $|r - r_n| \le r_n/2$. Using the same notation as in (4-23), we have

$$\gamma_n(r) = \Omega_n(r) - \Omega_n(r_n) - ia_n = \Omega_n'(r_n)(r-r_n+ic_n) + (\Omega'(\xi) - \Omega'(r_n))(r-r_n)$$

for some $\xi \in [r_n/2, 3r_n/2]$. As $\Omega_n'(s) \approx s^{-3}$ and $\Omega_n''(s) \approx s^{-4}$ as $s \to \infty$ (see (4-9) and the first part of Section A4), we have

$$|\Omega_n'(\xi) - \Omega_n'(r_n)| \le \frac{C}{r_n}|\Omega_n'(r_n)|\,|\xi - r_n| \le \frac{C}{r_n}|\Omega_n'(r_n)|\,|r-r_n|,$$

so that $\gamma_n(r) = \Omega_n'(r_n)(r - r_n + ic_n)[1 + \mathcal{O}(\epsilon(r))]$, where $\epsilon(r) = (r - r_n)/r_n$. Recalling that $J_n(r_n) = \Phi_n(r_n)/\Omega_n'(r_n)^2$, we obtain the expression

$$\mathcal{R}_n^{(2)}(r) = -\frac{k^2}{m^2}\frac{\Phi_n(r)[1 + \mathcal{O}(\epsilon(r))] - \Phi_n(r_n)}{\Omega'(r_n)^2(r - r_n + ic_n)^2}.$$

Now, it also follows from (4-9) and the first part of Section A4 that $\Phi_n(s) \approx s^{-6}$ and $\Phi_n'(s) \approx s^{-7}$ as $s \to \infty$. This implies $|\Phi_n(r) - \Phi_n(r_n)| \le C\Phi_n(r_n)\epsilon(r)$, and we deduce

$$|\mathcal{R}_n^{(2)}(r)| \le C J_n(r_n)\frac{\epsilon(r)}{|r-r_n+ic_n|^2} \le \frac{C}{r_n|r-r_n+ic_n|} \quad \text{if } \frac{|r - r_n|}{r_n} \le \frac{1}{2}.$$

The argument is simpler if $|r_n - r| \ge r_n/2$, because we can estimate both terms in $\mathcal{R}_n^{(2)}$ separately. Straightforward calculations lead to the lower bound

$$|\gamma_n(r)| = |\Omega_n(r) - \Omega_n(r_n) - ia_n| \ge C|\Omega_n'(r)|\,|r - r_n + ic_n|, \tag{A-50}$$

whenever $r \ge \delta r_n$ (here C depends on δ), and this implies

$$|\mathcal{R}_n^{(2)}(r)| \le C\frac{\Phi_n(r)}{|\gamma_n(r)|^2} + C\frac{J_n(r_n)}{|r-r_n+ic_n|^2} \le \frac{C}{|r-r_n+ic_n|^2} \quad \text{if } \frac{|r - r_n|}{r_n} \ge \frac{1}{2}.$$

The proof of (A-49) is thus complete. □

It is now easy to conclude the proof of Lemma 4.6. The term $\mathcal{R}_n^{(3)}$ in the remainder is estimated using the lower bound (A-50), which leads to

$$|\mathcal{R}_n^{(3)}(r)| \le \frac{C}{r^3|r-r_n+ic_n|}, \quad r \ge \delta r_n, \tag{A-51}$$

whereas $|\mathcal{R}_n^{(1)}(r)| \le C/r^2$ for all $r > 0$. In view of (A-45), we thus have

$$|\mathcal{K}_n^{(1)}(r, s)| \le C\min(1, |s-r_n+ic_n|)^{1-2\nu_n}\left(\frac{1}{s^2} + \frac{1}{s|s-r_n+ic_n|}\right), \quad s \ge r \ge \delta r_n.$$

Integrating this bound, we arrive at

$$\sup_{r \ge \delta r_n}\int_r^\infty |\mathcal{K}_n^{(1)}(r, s)|\,\mathrm{d}s \le \frac{C}{r_n}(1 + \log_+(r_n)) \le C b_n^{1/2}\left(1 + \log_+\frac{1}{b_n}\right), \tag{A-52}$$

and estimate (4-32) follows immediately from (A-48), (A-52). □

A6. *Proof of equality* (4-42). Assume that $0 < \nu < \frac{1}{2}$. Given any $\epsilon > 0$ and $a > 0$, we define

$$\mathcal{J}_\nu(a) = \int_{-\epsilon}^{\epsilon}\frac{-ax}{(a^2 + x^2)^{3/2}}|K_\nu(x + ia)|^2\,\mathrm{d}x, \tag{A-53}$$

where K_ν is the modified Bessel function. Our goal is to compute the limit of $\mathcal{J}_\nu(a)$ as $a \to 0$. We recall from [Abramowitz and Stegun 1964, Section 9.6] that

$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}, \quad \text{where } I_\nu(z) = \frac{1}{\Gamma(1+\nu)}\frac{z^\nu}{2^\nu}(1 + \mathcal{O}(z^2)) \quad \text{as } z \to 0.$$

We thus have

$$K_\nu(z) = \frac{c_\nu}{z^\nu}(1 - d_\nu z^{2\nu} + \mathcal{O}(z^2)) \quad \text{as } z \to 0,$$

where

$$c_\nu = \frac{\pi}{\sin(\nu\pi)} \frac{2^{\nu-1}}{\Gamma(1-\nu)}, \quad d_\nu = \frac{1}{2^{2\nu}} \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}.$$

It follows that

$$\begin{aligned} |K_\nu(x+ia)|^2 &= \frac{c_\nu^2}{(a^2+x^2)^\nu}(|1 - d_\nu(x+ia)^{2\nu}|^2 + \mathcal{O}(x^2+a^2)) \\ &= \frac{c_\nu^2}{(a^2+x^2)^\nu}\big(1 - 2d_\nu(a^2+x^2)^\nu \cos(2\nu \arg(x+ia)) + \mathcal{O}((x^2+a^2)^{2\nu})\big) \end{aligned} \tag{A-54}$$

as $z = x + ia \to 0$. The leading term in (A-54) is even and therefore does not contribute to the integral (A-53), where it is multiplied by an odd function. The main contribution comes from the next term, so that

$$\begin{aligned} \lim_{a\to 0} \mathcal{J}_\nu(a) &= 2d_\nu c_\nu^2 \lim_{a\to 0} \int_{-\epsilon}^{\epsilon} \frac{ax}{(a^2+x^2)^{3/2}} \cos(2\nu \arg(x+ia))\,\mathrm{d}x \\ &= 2d_\nu c_\nu^2 \int_{\mathbb{R}} \frac{y}{(1+y^2)^{3/2}} \cos(2\nu \arg(y+i))\,\mathrm{d}y, \end{aligned} \tag{A-55}$$

where the second equality is obtained by setting $x = ay$.

It remains to perform the integration in (A-55). As $\arg(y+i) = \frac{\pi}{2} - \arctan(y)$, we have

$$\cos(2\nu \arg(y+i)) = \cos(\nu\pi)\cos(2\nu \arctan(y)) + \sin(\nu\pi)\sin(2\nu \arctan(y)).$$

The first term does not contribute to the integral in (A-55), whereas setting $y = \tan(t)$ we find

$$\int_{\mathbb{R}} \frac{y}{(1+y^2)^{3/2}} \sin(2\nu \arctan(y))\,\mathrm{d}y = 2\int_0^{\pi/2} \sin(t)\sin(2\nu t)\,\mathrm{d}t = \frac{4\nu\cos(\nu\pi)}{1-4\nu^2}.$$

Summarizing, we have shown that

$$\lim_{a\to 0} \mathcal{J}_\nu(a) = 2d_\nu c_\nu^2 \sin(\nu\pi)\frac{4\nu\cos(\nu\pi)}{1-4\nu^2} = \frac{2\pi\cos(\nu\pi)}{1-4\nu^2}.$$

A7. ***Explicit calculations in some particular cases.*** We collect in this section a few results for the Kaufmann–Scully vortex (1-8) and the Lamb–Oseen vortex (1-9) which can be established by a direct calculation.

(1) We first show that the vorticity profile W of the Lamb–Oseen vortex satisfies Assumption H2 in Section 1B, hence belongs to the class $\mathscr{W}$. Indeed, in that case, the function J defined by (1-20) is given by

$$J(r) = \frac{r^4 e^{-r^2}(1-e^{-r^2})}{(1-(1+r^2)e^{-r^2})^2}, \quad r > 0,$$

so that

$$J'(r) = \frac{2r^3 e^{-r^2}}{(1-(1+r^2)e^{-r^2})^3}(2 - r^2 - (4 - r^2 + r^4)e^{-r^2} + 2e^{-2r^2}), \quad r > 0.$$

We want to show that $J'(r) < 0$ for all $r > 0$. Setting $s = r^2$, we have to verify that

$$s - 2 + (4 - s + s^2)e^{-s} - 2e^{-2s} > 0, \quad s > 0,$$

or equivalently

$$\frac{s}{1-e^{-s}} + \left(\frac{s}{1-e^{-s}}\right)^2 e^{-s} > 2, \quad s > 0.$$

Since $s > 1 - e^{-s}$, it is sufficient to show that

$$\frac{s}{\tanh(s/2)} \equiv \frac{s}{1-e^{-s}} + \frac{se^{-s}}{1-e^{-s}} > 2,$$

which is indeed true because $\tanh(x) < x$ for all $x > 0$.

(2) Next, for the Lamb–Oseen vortex, we establish the inequality $\mathcal{B}(r) \geq 1 - 4/m^2$ when $a = 0$ and $b \leq 0$, where $\mathcal{B}$ is defined in (3-3). Indeed, as $\gamma_\star(r) = \Omega(r) - b \geq \Omega(r)$, we have

$$\mathcal{B}(r) \geq 1 - \frac{1}{m^2}\frac{k^2r^2}{m^2+k^2r^2}\frac{\Phi}{\Omega^2} + \frac{r}{\Omega}\frac{W'}{m^2+k^2r^2} - \frac{2k^2r^2}{\Omega}\frac{W}{(m^2+k^2r^2)^2}. \tag{A-56}$$

As $\Phi = 2\Omega W$ and $W'(r) = -2rW(r)$ for the Lamb–Oseen vortex, inequality (A-56) can be written in the equivalent form

$$\mathcal{B}(r) \geq 1 - \frac{2W(r)}{m^2\Omega(r)}\left\{1 + \frac{m^2r^2}{m^2+k^2r^2} - \frac{m^4}{(m^2+k^2r^2)^2}\right\}. \tag{A-57}$$

Define $s = r^2 > 0$ and $\alpha = m^2/(m^2+k^2r^2) \in (0, 1)$. As $W = 2e^{-s}$ and $\Omega = s^{-1}(1-e^{-s})$, it is straightforward to verify that (A-57) implies the desired inequality $\mathcal{B}(r) \geq 1 - 4/m^2$ provided

$$\frac{se^{-s}}{1-e^{-s}}(1 + \alpha s - \alpha^2) \leq 1, \quad \text{or equivalently} \quad e^s \geq 1 + s + \alpha s(s - \alpha). \tag{A-58}$$

But

$$\sup_{\alpha\in(0,1)} (1 + s + \alpha s(s-\alpha)) = \begin{cases} 1 + s + s^3/4 & \text{if } s \leq 2, \\ 1 + s^2 & \text{if } s \geq 2, \end{cases}$$

and we conclude that the last inequality in (A-58) holds in all cases.

(3) Finally, we establish the same inequality $\mathcal{B}(r) \geq 1 - 4/m^2$ for the Kaufmann–Scully vortex. In that case $W'(r) = -4rW(r)\Omega(r)$ by (1-8); hence inequality (A-56) takes the form

$$\mathcal{B}(r) \geq 1 - \frac{2W(r)}{m^2\Omega(r)}\left\{1 + \frac{2r^2}{1+r^2}\frac{m^2}{m^2+k^2r^2} - \frac{m^4}{(m^2+k^2r^2)^2}\right\}. \tag{A-59}$$

Setting again $\alpha = m^2/(m^2+k^2r^2)$ and using the fact that $W = 2\Omega^2 = 2/(1+r^2)$ in the present case, we see that the desired inequality $\mathcal{B}(r) \geq 1 - 4/m^2$ holds provided

$$1 + \frac{2\alpha r^2}{1+r^2} - \alpha^2 \leq 1 + r^2, \quad r > 0. \tag{A-60}$$

The maximum of the left-hand side, considered as a function of $\alpha \in (0, 1)$, is reached at the point $\alpha = r^2/(1+r^2)$, and the resulting inequality becomes $(1+r^2)^2 + r^4 \leq (1+r^2)^3$, which is of course true. This concludes the proof.

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EVANESCENT ERGOSURFACE INSTABILITY

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Some exotic compact objects, including supersymmetric microstate geometries and certain boson stars, possess *evanescent ergosurfaces*: time-like submanifolds on which a Killing vector field, which is time-like everywhere else, becomes null. We show that any manifold possessing an evanescent ergosurface but no event horizon exhibits a linear instability of a peculiar kind: *either* there are solutions to the linear wave equation which concentrate a finite amount of energy into an arbitrarily small spatial region, *or* the energy of waves measured by a stationary family of observers can be amplified by an arbitrarily large amount. In certain circumstances we can rule out the first type of instability. We also provide a generalisation to asymptotically Kaluza–Klein manifolds. This instability bears some similarity with the "ergoregion instability" of Friedman (*Comm. Math. Phys.* **63**:3 (1978), 243–255), and we use many of the results from the recent proof of this instability by Moschidis (*Comm. Math. Phys.* **358**:2 (2018), 437–520).

1. Introduction

With the recent experimental detection of gravitational waves [Abbott et al. 2017] there has been a great deal of interest in *exotic compact objects* and their properties. These objects, which are often solutions to various speculative theories, are supposed to "mimic" certain aspects of black holes: they are extremely compact, with a strong localised gravitational field, while having a similar asymptotic structure to black holes. On the other hand, many of these objects are supposed to avoid some of the "pathologies" of black

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holes: in particular, they are often nonsingular. From an observational point of view, many of these exotic objects have a compact region in which null geodesics can be "trapped" (as at the "photon sphere" in Schwarzschild), and this can lead to similar gravitational wave signals to those emitted by black holes [Cardoso et al. 2016a; 2016b], at least on short time scales. Hence the recent interest in this subject: it is clearly of great importance to be able to distinguish these objects from genuine black holes, both from a theoretical and an observational point of view.

Despite mimicking black holes to some extent, in other ways these geometries can differ drastically from black holes, and this might provide a way to distinguish between them. For example, the gravitational wave signal from many of the exotic objects is expected to exhibit "echoes" in a way which black holes do not; see [Cardoso and Pani 2017] for an overview. In addition (and in some ways related to the "echoes"), many of these exotic objects are classically unstable (see, e.g., [Cardoso et al. 2006; Keir 2016; 2020; Eperon et al. 2016]), whereas black holes are expected to be classically nonlinearly stable. The question of stability will be the focus of this work.

An unusual geometric feature that is present in some of these exotic geometries is an "evanescent ergosurface" — for example, this is present in *supersymmetric microstate geometries* (see [Maldacena and Maoz 2002; Balasubramanian et al. 2001; Lunin et al. 2002; Giusto et al. 2004; 2005; Bena and Warner 2006; Berglund et al. 2006; Gibbons and Warner 2014]), as well as in boson stars[1] which are sufficiently compact and rotating at a particular rate [Grandclément 2017]. An evanescent ergosurface is a time-like submanifold where an asymptotically time-like Killing vector field, which is time-like everywhere else, becomes null.[2] Thus, these submanifolds are similar to the boundary of an ergoregion; however, unlike an ergoregion, there is no "interior" where the asymptotically time-like Killing vector field becomes space-like. Evanescent ergosurfaces are intimately related to questions of stability: for example, nonsupersymmetric microstates have an ergoregion but no horizon, and so are susceptible to the "ergoregion instability" of [Friedman 1978] (recently proved rigorously in [Moschidis 2018]). However, *supersymmetric* microstates do not have an ergoregion[3] but only an evanescent ergosurface, so it might be hoped that they avoid an instability, at least on the linear level. Similar comments hold in the boson star case: compact stars which rotate more rapidly than some critical rate admit an ergoregion and so are susceptible to the ergoregion instability, but stars rotating at precisely the critical rate only admit an evanescent ergoregion.

In [Eperon et al. 2016; Keir 2020], particular geometries with evanescent ergosurfaces were studied, and various properties of waves propagating on these geometries were discussed. In particular, it was shown that a "stable trapping" phenomena occurs, causing waves to decay extremely slowly, and it is

[1] In this context, the evanescent ergoregion has been called a "light point". This terminology comes about because the null geodesic has constant spatial coordinates in any coordinate system in which the spatial coordinates are Lie transported with respect to the asymptotically time-like Killing vector field T.

[2] For asymptotically Kaluza–Klein manifolds, a different (but functionally similar) definition can be given.

[3] There is some subtlety here: some supersymmetric microstate geometries also have an ergoregion, but this ergoregion only allows for negative energy waves if those waves also have some nonzero momentum in the Kaluza–Klein directions. In this work, when dealing with asymptotically Kaluza–Klein spacetimes, we will restrict our attention to waves which are invariant in the Kaluza–Klein directions.

conjectured that this might lead to a nonlinear instability; see also [Holzegel and Smulevici 2014; Cunha et al. 2017]. In fact, waves on these geometries decay even slower than waves on other geometries exhibiting stable trapping, a feature which is related directly to the presence of the evanescent ergosurface [Keir 2020].

Here, we will take a much more general approach. Rather than studying a particular geometry, we will study a *general* manifold with an evanescent ergosurface. As far as possible, we avoid placing other restrictions on the manifold: we require a suitable asymptotic structure and smoothness properties, and we also require *either* a certain kind of discrete isometry (satisfied, for example, by $(t-\phi)$-symmetric spacetimes) *or* an additional Killing field with suitable properties. Note that we are able to deal with both asymptotically flat and asymptotically Kaluza–Klein manifolds. Under these very general conditions, we are able to show that a kind of instability is present, which is (in a sense) stronger than the "slow decay" results of [Eperon et al. 2016; Keir 2020] but weaker than the ergoregion instability. The geometries that we study can also be expected to exhibit very slow decay of linear waves, and this alone might lead to the expectation of a nonlinear instability. However, the new instability which we find is of a different nature, and already appears at the *linear* level.

The instability that we exhibit has a lot in common with the "ergosphere instability" originally discovered by Friedman [1978] and recently proved rigorously by Moschidis [2018]. This instability occurs in all asymptotically flat spacetimes with ergoregions *but no event horizon*. Indeed, we can view the "ergosurface instability" as what is left over of the ergosphere instability, when the ergoregion degenerates into an evanescent ergosurface.

Let us now make some comments on the nature of the instability we show in this paper. First, we are focussing on *scalar* perturbations; that is, we are examining solutions to the linear wave equation. This can either be viewed as a model for the Einstein(-matter) equations, which typically involve a set of nonlinear wave equations, or it can be viewed as a model for scalar fields or scalar modes of the geometry.

Next, note that our instability is *not* associated with an exponentially growing mode solution to the wave equation. Indeed, under the geometric conditions we assume, such a solution can in fact be ruled out. Nevertheless, we believe that the kind of behaviour we demonstrate can justifiably be called an instability, as we aim to show below.

Specifically, we are able to show that at least one of the following two cases occur:

(A) Given a stationary family[4] of observers moving along time-like curves, and given any constant $C > 0$, there exist waves, arising from smooth, compactly supported initial data (depending on the constant C), such that initially the *total* energy measured by the entire family of observers is arbitrarily small, but, after some time has passed, the total energy measured by these observers is at least C. Moreover, the energy density measured by the observers in a neighbourhood of the ergosurface is $\mathcal{O}(C)$.

[4]That is, a family of observers moving along integral curves of some vector field N, where the Lie derivative of N along the asymptotically time-like Killing vector field vanishes. Note that this does *not* mean that each member of the family moves parallel to the asymptotically time-like Killing vector field!

(B) The spacetime exhibits an *Aretakis-type* instability [2011; 2015], where there are waves arising from smooth, compactly supported initial data, whose local energy[5] *fails to decay* in a neighbourhood of the ergosurface, although it decays everywhere else. In fact, a nonzero amount of energy is concentrated in a smaller and smaller region, leading to pointwise blow-up.

Note that the "energy" measured by the family of observers referred to above is *not* the energy measured with respect to the asymptotically time-like Killing vector field, which (since it is a Killing field) is conserved. In fact, the family of observers referred to above *cannot* move parallel to the asymptotically time-like Killing vector field, since this is vector field is null (rather than time-like) on the evanescent ergosurface.

Note also that, although we cannot rule it out in general, we do not know of a particular case where behaviour of type (B) is exhibited. This is in contrast to the behaviour of type (A), which (as we will show) is exhibited by the supersymmetric microstate geometries studied in [Maldacena and Maoz 2002; Balasubramanian et al. 2001; Lunin et al. 2002; Giusto et al. 2004; 2005; Bena and Warner 2006; Berglund et al. 2006; Gibbons and Warner 2014; Eperon et al. 2016; Keir 2020].

Of the two possible instability scenarios outlined above, we can guarantee that we have an instability of type (A) if there exists *another* Killing vector field (in addition to the asymptotically time-like one) such that the span of these two Killing vector fields is time-like in a neighbourhood of the evanescent ergosurface.[6] In fact, the presence of an extra Killing field of this kind allows us to show a number of other details of the instability. In particular, we can show that:

- Despite the behaviour outlined in point (A) above, the local energy *is* bounded, but it is bounded in terms of a *higher-order* initial energy, and not in terms of the initial energy.
- However, if we know the initial higher-order energy, then at later times this same higher-order energy can become arbitrarily large.
- If we want the energy measured by our family of observers to be amplified by a factor of C, then this can be achieved in some time which is bounded by $\exp(C^{1+\epsilon})$ for some $\epsilon > 0$.
- There exist (possibly nonsmooth and noncompactly supported) initial data leading to a (weak) solution of the wave equation with *unbounded* local energy.
- Also, in the presence of this additional symmetry, we do not require the discrete isometry.

1A. ***Brief overview of the instability.*** We will very briefly sketch the construction, made rigorous later in this paper, which underlies our instability result.

First, we need to invoke two notions of the energy of a solution to the wave equation: the *nondegenerate energy*, which we call $\mathcal{E}^{(N)}$, which is the energy measured by a family of time-like-moving observers, and the *conserved energy* associated with the Killing field, which we call $\mathcal{E}^{(T)}$. We also define the *local, nondegenerate energy* $\mathcal{E}^{(N)}_{\mathcal{U}}$, which is the same as the nondegenerate energy except that it is only evaluated

[5]That is, the energy measured by a subset of the family of observers mentioned above, which is such that the worldlines of these observers intersect a space-like hypersurface in a compact set.

[6]Note that this is the case for the supersymmetric microstate geometries investigated in [Eperon et al. 2016; Keir 2020].

on some subset of the space at each moment of time.[7] We choose this subset to be a neighbourhood of the ergosurface at each point in time.

Because of the presence of the ergosurface, we find that it is possible to construct initial data for the wave equation for which $\mathcal{E}^{(N)}_{\mathcal{U}}$ (and hence $\mathcal{E}^{(N)}$) is very large, and yet $\mathcal{E}^{(T)}$ is very small. Let us begin with such data, at some time far in the future, say at time τ. We will then evolve this data *backwards* in time.

There are then two options: either the wave disperses in the past, and $\mathcal{E}^{(N)}_{\mathcal{U}} \to 0$ as we go backwards in time, or else the local energy does not disperse. Let us discuss each of these cases in turn.

In the first case, the local energy $\mathcal{E}^{(N)}_{\mathcal{U}}$ decays as we evolve backwards in time. Since we began with data in the future for which the conserved energy $\mathcal{E}^{(T)}$ is very small, it will remain the case that $\mathcal{E}^{(T)}$ is small as we evolve backwards in time. At the same time, by assumption, the local energy $\mathcal{E}^{(N)}_{\mathcal{U}}$ also becomes small, say at time $t = 0$. This means that, at time 0, both $\mathcal{E}^{(N)}_{\mathcal{U}}$ and $\mathcal{E}^{(T)}$ are very small. But we also find that the *total* nondegenerate energy, $\mathcal{E}^{(N)}$, can be expressed as

$$\mathcal{E}^{(N)} \sim \mathcal{E}^{(N)}_{\mathcal{U}} + \mathcal{E}^{(T)}.$$

Hence, at time 0, the total, nondegenerate energy $\mathcal{E}^{(N)}$ is also very small.

Thus we have arrived at data at time 0 with a very small total, nondegenerate energy $\mathcal{E}^{(N)}$. In fact, all three energies, $\mathcal{E}^{(N)}$, $\mathcal{E}^{(N)}_{\mathcal{U}}$ and $\mathcal{E}^{(T)}$ are very small at time 0. And yet, if we evolve this data forward in time, we know that at time τ, the local energy $\mathcal{E}^{(N)}_{\mathcal{U}}$ becomes very large. If we now interpret this solution as a solution to the forward-in-time problem, then we find that both the nondegenerate energy $\mathcal{E}^{(N)}$ and the local nondegenerate energy $\mathcal{E}^{(N)}_{\mathcal{U}}$ have been "amplified" by a very large factor. See Figure 1 for a sketch explaining this case (case (A)).

Alternatively, it might be the case that there are some solutions to the wave equation which do not disperse as we evolve them to the past, and so $\mathcal{E}^{(N)}_{\mathcal{U}}$ does not approach zero as we evolve backwards in time. In this case, we can invoke the discrete isometry to obtain a solution to the wave equation for which the local energy does not approach zero as we evolve *to the future*. However, it is possible to show that the local energy does, in fact, tend to zero in the future, at least in every compact set which is positioned away from the ergosurface. Hence, in this case, a finite amount of energy must eventually be contained within an arbitrarily small region of space near the ergosurface. We call this an Aretakis-type instability.

1B. ***Comparison with the ergosphere instability.*** The "evanescent ergosurface instability" is somewhat weaker than the "ergosphere instability" of [Friedman 1978; Moschidis 2018]. In particular, [Moschidis 2018] showed that, when an ergoregion is present but no event horizon exists, then there are solutions to the wave equation, arising from smooth, compactly supported initial data, whose local energy is unbounded. This is not the case for the evanescent ergosurface instability — indeed, in the case where an extra symmetry is present, we can actually rule out this kind of behaviour. On the other hand, if we allow for noncompactly supported data and we do not require that "higher-order" energies are finite, then we

[7] Specifically, the local energy $\mathcal{E}^{(N)}_{\mathcal{U}}$ is evaluated on the subset $\mathcal{U} \cap \Sigma$, where Σ is the space-like hypersurface defining "space at a given time", and $\mathcal{U}$ is some subset of the manifold which is invariant under the flow generated by the stationary Killing field. We require that $\mathcal{U} \cap \Sigma$ is (pre-)compact.

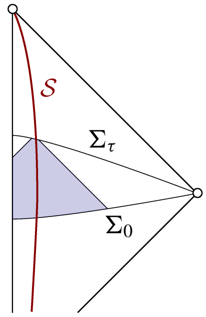

Figure 1. A sketch of a Penrose diagram illustrating the construction of the instability in case (A), that is, when the local energy of waves decays to the past. We begin by constructing initial data on the hypersurface Σ_τ such that the local nondegenerate energy $\mathcal{E}_{\mathcal{U}}^{(N)}$ at time τ is very large, say ϵ^{-1}, for some arbitrarily small ϵ. At the same time, the conserved energy $\mathcal{E}^{(T)}$ at time τ is very small, say ϵ. This data can also be chosen to be supported only in a neighbourhood of the ergosurface $\mathcal{S}$.
We then evolve this data backwards in time to the hypersurface Σ_0. If Σ_τ is sufficiently far in the future, then we know that the local energy $\mathcal{E}_{\mathcal{U}}^{(N)}$ on the hypersurface Σ_0 will be very small — say ϵ. The conserved energy $\mathcal{E}^{(T)}$ on the hypersurface Σ_0 is also ϵ. It follows that the total, nondegenerate energy $\mathcal{E}^{(N)}$ measured on the hypersurface Σ_0 is $\mathcal{O}(\epsilon)$. Hence, between the hypersurfaces Σ_0 and Σ_τ, the nondegenerate energy $\mathcal{E}^{(N)}$ has been "amplified" by a factor of ϵ^{-2}.
Note that a solution to the wave equation constructed in this way is supported only in the shaded (blue) region. In particular, the initial data on the hypersurface Σ_0 is compactly supported.

can recover similar behaviour, although the rate of growth of the energy will generally be much slower in the evanescent ergosurface case.

The key idea behind the ergosphere instability of [Friedman 1978; Moschidis 2018] is to use the ergoregion to construct initial data for the wave equation with *negative conserved energy*. Then, under the assumption that the nondegenerate energy $\mathcal{E}^{(N)}$ remains uniformly bounded over time, it is possible to show (see [Moschidis 2018]) that the local nondegenerate energy $\mathcal{E}_{\mathcal{U}}^{(N)}$ must decay, at least away from the ergoregion. It is then possible to derive a contradiction with the conservation of the (negative) conserved energy, which ensures that some part of the wave always remains trapped within the ergoregion.

Our approach is similar in many ways, and for this reason we shall make use of many of the results of [Moschidis 2018]. However, since we only have an evanescent ergosurface rather than a full ergoregion, it is not possible to produce waves with negative conserved energy. Instead, we can make use of the evanescent ergosurface to construct data for the wave equation such that its conserved energy is *much smaller* than its nondegenerate energy. This is the key fact which, as we show, leads to some kind of instability.

1C. ***Comparison with extremal black holes.*** One might wonder whether the ideas in this paper can be applied to extremal black holes. After all, the event horizon of an extremal black hole bears many similarities with an evanescent ergosurface. However, there are several technical reasons why our construction fails in this case: for example, the horizon is a null hypersurface rather than a time-like hypersurface.

Heuristically, we can understand the failure of our construction in the case of black holes in the following way. Our result relies crucially on being able to evolve data *backwards in time*: it must be the case that the time-reversed manifold "looks similar" to the original manifold in a suitable sense. This is guaranteed if the manifold admits a discrete isometry of the required kind, or if it has an additional Killing field with certain properties. However, in the case of a black hole, it fails spectacularly. If we begin with a hypersurface which intersects the future horizon and then evolve a solution to the wave equation *backwards* in time, then it behaves very differently from solutions which are evolved forwards in time. For example, at least in subextremal black holes, the red-shift effect means that the energy of a wave near the event horizon decays when evolved to the future; when evolved to the past, the energy will instead be blue-shifted. If, instead, we begin with a hypersurface that intersects the bifurcation sphere, then we do not expect the energy to decay, since we are not really "evolving" the data in this region when we flow along curves of the stationary Killing field.

Nevertheless, one kind of instability (case (B)) that might be exhibited by spacetimes with an evanescent ergosurface has a lot in common with the *Aretakis instability* of extremal horizons [2011; 2015]. In both cases, there is some nondecaying quantity on a specific hypersurface (either the event horizon or the evanescent ergosurface) which decays everywhere else, and this is responsible for a certain kind of blow-up. Note, however, that while we cannot rule out this kind of behaviour in general, we can rule it out on manifolds which have some extra symmetry. In many of the explicit examples of spacetimes with evanescent ergosurfaces, this extra symmetry is present, and so we can actually rule out this kind of instability. Instead, on these manifolds we have a different kind of instability, wherein the local energy of waves can be amplified by an arbitrarily large amount.

2. Notation

In this section, we will often refer to "the asymptotically time-like Killing vector field T". In all these cases, when considering the asymptotically Kaluza–Klein case, the vector field T should be replaced by the vector field V.

We use the following notation for inequalities: we write $A \lesssim B$ if there is some constant $C > 0$, *independent* of all of the parameters which we are varying, such that

$$A \leq CB.$$

Similarly, we write $A \gtrsim B$ if there is a constant $C > 0$, *independent* of all of the parameters which we are varying, such that

$$B \leq CA.$$

Also, we write $A \sim B$ if there are constants $c > 0$, $C > 0$, again independent of the parameters which are varying, such that

$$cB \leq A \leq CB.$$

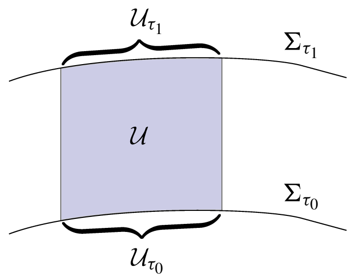

Figure 2. A sketch showing two space-like hypersurfaces Σ_{τ_0} and Σ_{τ_1}, the regions $\mathcal{U}_{\tau_0}$ and $\mathcal{U}_{\tau_1}$ and the shaded part of the region, $\mathcal{U}$ (in blue), which lies between these two regions. In this sketch, the isometry generated by the vector field T is represented by translation up the page.

When integrating over a space-like hypersurface Σ_t, we will use the notation $\mathrm{d}\,\mathrm{vol}$ for the volume form induced on Σ_t by the spacetime metric g. Also, given a set $\mathcal{U}_t \subset \Sigma_t$ we define

$$\mathrm{Volume}(\mathcal{U}_t) := \int_{\mathcal{U}_t} \mathrm{d}\,\mathrm{vol}.$$

Given some subset $U_t \subset \Sigma_t \subset \mathcal{M}$ of a hypersurface Σ_t, we define the corresponding set U as the set of T-translations of U_t, where T is the asymptotically time-like Killing vector field. In other words, we define

$$U := \{x \in \mathcal{M} \mid \text{there exists an integral curve of } T \text{ through } x \text{ and } U_t\}.$$

Similarly, if we have a foliation of $\mathcal{M}$ by hypersurfaces Σ_t, and if $U_t \subset \Sigma_t$, then we define U_τ by

$$U_\tau := \{x \in \Sigma_\tau \mid \text{there exists an integral curve of } T \text{ through } x \text{ and } U_t\}.$$

See Figure 2 for a sketch of these regions.

Also, we define the "δ-thickening" of the set $\mathcal{U}$ as follows: for $\delta > 0$, we define

$$(\mathcal{U}_{(\delta)})_0 := \{x \in \Sigma_0 \mid \mathrm{dist}(x, \mathcal{U}_0) < \delta\},$$

where $\mathrm{dist}(x, \mathcal{U})$ is the distance from x to the set $\mathcal{U}$ defined using the Riemannian metric induced on Σ_0. Then, we define $\mathcal{U}_{(\delta)}$ to be the set consisting of all the T-translates of the set $(\mathcal{U}_{(\delta)})_0$.

We use ∇ to denote the covariant derivative induced by the metric g. We also write the geometric wave operator as

$$\Box_g \phi := (g^{-1})^{\mu\nu} \nabla_\mu \nabla_\nu \phi$$

and will use the notation $\mathcal{L}_X$ to denote the Lie derivative with respect to a vector field X.

The notation $\partial\phi$, for some scalar field ϕ, will be used to denote (schematically) the collection of first derivatives of ϕ. To be more precise: assuming that the asymptotically time-like Killing field T is transverse to the space-like hypersurface Σ, we define

$$\partial\phi := \{(T\phi), (e_a\phi)\},$$

where the e_a are an orthonormal frame for the tangent space of Σ. Similarly, we define norms

$$|\partial\phi| := \sqrt{|T\phi|^2 + \sum_a |e_a\phi|^2}.$$

Given a vector field X, we will say that X is *uniformly time-like* if there are positive constants c and C such that

$$c \le -g(X, X) \le C.$$

Also, given a vector X, we define the covector $X^\flat$ by defining its action on a arbitrary vector field Y:

$$X^\flat(Y) := g(X, Y).$$

Similarly, given a covector ω, we define the vector $\omega^\sharp$ by the prescription

$$g(\omega^\sharp, Y) := \omega(Y)$$

for all vectors Y.

Finally, given a set $\mathcal{U}_0 \subset \Sigma_0$ and the associated set $\mathcal{U} \subset \mathcal{M}$, we define the notation

$$\|f\|_{L^p[\mathcal{U}]}(\tau) := \left(\int_{\mathcal{U}\cap\Sigma_\tau} |f|^p \,\mathrm{dvol}\right)^{\frac{1}{p}}$$

and similarly

$$\|f\|_{L^\infty[\mathcal{U}]}(\tau) := \operatorname*{ess\,sup}_{\mathcal{U}\cap\Sigma_\tau} |f|.$$

3. Asymptotic structure

We are interested in smooth,[8] stationary metrics that are either asymptotically flat or asymptotically Kaluza–Klein. We shall restrict attention to manifolds without an event horizon or a black hole region, so we require the manifold $\mathcal{M}$ to be identical to the causal past of future null infinity. In the Kaluza–Klein case, we also restrict to those asymptotically Kaluza–Klein metrics that are "homogeneous" in the compact directions, a notion that we will make precise below.

3A. ***Asymptotically flat, stationary spacetimes.*** Since we make use of the results of [Moschidis 2018], we shall use the same definition of an "asymptotically flat" stationary manifold given in that paper. That is, we shall consider a stationary $(d+1)$-dimensional manifold to be *asymptotically flat* if there is an open region $\mathcal{U}^{\mathrm{as}}$ with compact complement, called the "asymptotic region", which is diffeomorphic to $\mathbb{R} \times (\mathbb{R}^d \setminus B)$, for some integer $d \ge 3$, where B denotes the unit open ball in $\mathbb{R}^d$. Moreover, we require that there exist coordinates for $\mathcal{U}^{\mathrm{as}}$ such that the metric on $\mathcal{M}$ takes the form

$$g = -\Big(1 - \frac{2M}{r} + h_1(r,\sigma)\Big)\,\mathrm{d}t^2 + \Big(1 + \frac{2M}{r} + h_2(r,\sigma)\Big)\,\mathrm{d}r^2 + r^2(g_{\mathbb{S}^{d-1}} + h_3(r,\sigma)) + h_4(r,\sigma)\,\mathrm{d}t. \quad (1)$$

In the expression above r is the pull-back of the standard Euclidean radial function on $(\mathbb{R}^d \setminus B)$ by the diffeomorphism mentioned above. Moreover, there is a (related) diffeomorphism from the asymptotic

[8]But not necessarily analytic!

region to $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{S}^{d-1}$, which can be understood as a "polar coordinate chart" for $\mathcal{U}^{\text{as}}$. The function σ is then the projection $\sigma : \mathcal{U}^{\text{as}} \to \mathbb{S}^{d-1}$ naturally defined by this diffeomorphism.

Note that the constant M in the above formula corresponds to the mass of the spacetime *if* the spacetime dimension is 4, i.e., if $d = 3$.

In equation (1) above, h_1 and h_2 are smooth functions from $\mathcal{U}^{\text{as}}$ to $\mathbb{R}$. By $g_{\mathbb{S}^{d-1}}$ we denote the standard round metric on the *unit* $(d-1)$-sphere. For each value of r, $h_3(r, \cdot)$ is a smooth, symmetric rank-(0, 2) tensor field on the $(d-1)$-sphere. Finally, for each value of r, h_4 is a smooth one-form on the $(d-1)$-sphere.

In addition, the functions $h_1, \dots, h_4$ are required to decay at suitable rates as $r \to \infty$. Using the round metric $g_{\mathbb{S}^{d-1}}$ we can measure the norm of tensor fields on the sphere $\mathbb{S}^{d-1}$ in the obvious way, and the decay rates required are as follows: for some $\alpha \in (0, 1]$,

$$h_1, h_2, h_3 = \mathcal{O}_4(r^{-1-\alpha}), \qquad h_5 = \mathcal{O}_4(r^{-\alpha}),$$

where a function h is said to be $\mathcal{O}_n(r^k)$ if there is a constant C such that

$$\sum_{j=0}^{n} \sum_{j_1+j_2+j_3=j} r^{j_1+j_2} |\partial_r^{j_1} \partial_t^{j_2} \partial_\sigma^{j_3} h| \leq C r^k,$$

where ∂_σ is schematic notation for the action of some vector field on $\mathbb{S}^{d-1}$ with unit norm (measured, as usual, with respect to the round metric $g_{\mathbb{S}^{d-1}}$).

Finally, the diffeomorphism mapping $\mathcal{U}^{\text{as}} \to \mathbb{R} \times (\mathbb{R}^d \setminus B)$ will be labelled as φ_{as}.

3B. ***Asymptotically Kaluza–Klein, stationary manifolds.*** As mentioned above, the asymptotically Kaluza–Klein manifolds we shall study will be "homogeneous" in the compact directions. We now make this notion precise:

Definition 3.1 (Kaluza–Klein manifolds). Let $\mathcal{M}$ be a smooth manifold. Let $\mathcal{G}$ be a compact Lie algebra $\mathcal{G}$. Let

$$\varphi_{\mathcal{G}} : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$$

be a smooth action of $\mathcal{G}$ on the manifold $\mathcal{M}$ *by isometries*.

We require[9] that there exists a foliation of $\mathcal{M}$ by space-like hypersurfaces which are invariant under the action of $\mathcal{G}$. That is, $\mathcal{M}$ is foliated by hypersurfaces Σ_t such that Σ_t are the level sets of a function t, where t is invariant under the action of $\mathcal{G}$. Hence the smooth action of $\mathcal{G}$ on $\mathcal{M}$ descends to a smooth action on Σ_t.

Let X be a smooth compact manifold that is a homogeneous space for $\mathcal{G}$, that is, there is a transitive action of G on X. Then X is diffeomorphic to $\mathcal{G}/\mathcal{H}$ for some subgroup $\mathcal{H}$ of $\mathcal{G}$ (specifically, $\mathcal{H}$ is the isotropy subgroup of a point in X). So the points in X can be represented as left cosets of $\mathcal{H}$.

As before, we require the existence of an "asymptotic region": an open set $\mathcal{U}^{\text{as}} \subset \mathcal{M}$, with a compact complement, such that $\mathcal{U}^{\text{as}}$ is diffeomorphic to $\mathbb{R} \times (\mathbb{R}^d \setminus B) \times X$, for some $d \geq 3$. Note that $\mathcal{M}$ is not

[9] It would be possible to relax this requirement and instead require only that the leaves of the foliation are $\mathcal{G}$-invariant in a neighbourhood of the evanescent ergosurface.

required to be *globally* a product space. The action of $\mathcal{G}$ on the asymptotic region is given explicitly by

$$\mathcal{G} : \mathbb{R} \times (\mathbb{R}^d \setminus B) \times X \to \mathbb{R} \times (\mathbb{R}^d \setminus B) \times X,$$
$$g_1 : (t, x, g_2 H) \mapsto (t, x, g_1 g_2 H);$$

i.e., $\mathcal{G}$ acts on the left on X viewed as a coset space, and does not affect the t- or x-coordinates. Moreover, since $\mathcal{G}$ acts by isometries, we require X to be equipped with a $\mathcal{G}$-invariant, Riemannian metric g_X. We can then use g_X to define the norm of tensor fields on X, and for tensor fields on $(\mathbb{S}^{d-1} \times X)$ we can measure the norms using a combination of both $g_{\mathbb{S}^{d-1}}$ and g_X.

Now, in $\mathcal{U}^{\text{as}}$ we require there to exist coordinates such that the metric on $\mathcal{M}$ takes the form

$$\begin{aligned} g = -\Big(1 - \frac{2M}{r} + h_1(r,\sigma)\Big)\,\mathrm{d}t^2 + \Big(1 + \frac{2M}{r} + h_2(r,\sigma)\Big)\,\mathrm{d}r^2 + r^2(g_{\mathbb{S}^{d-1}} + h_3(r,\sigma)) + h_4(r,\sigma)\,\mathrm{d}t \\ + (g_X + h_5(r,\sigma)) + h_6(r,\sigma)\,\mathrm{d}t + h_7(r,\sigma)\,\mathrm{d}r + h_8(r,\sigma), \end{aligned} \tag{2}$$

where the functions $h_1, \dots, h_4$ are of the same type, and satisfy the same bounds, as in the asymptotically flat case, while the other functions are defined as follows:

- For each r, σ, the function h_5 is a symmetric, $\mathcal{G}$-invariant rank-$(0, 2)$ tensor on X satisfying the bound $h_5 = \mathcal{O}_4(r^{-1-\alpha})$.
- For each r, σ, the functions h_6 and h_7 are $\mathcal{G}$-invariant one-forms on X, all of which are $\mathcal{O}_4(r^{-1-\alpha})$.
- For each r, the function $h_8(r, \cdot)$ is a smooth, symmetric rank-$(0, 2)$ tensor on $(\mathbb{S}^{d-1} \times X)$ that is $\mathcal{O}_4(r^{-\alpha})$ and which is invariant under the natural action of $\mathcal{G}$ on such tensor fields.

The asymptotically time-like Killing vector field, T, is given in the coordinates above by ∂_t. Note that the above definition ensures that this vector field is invariant under the action of $\mathcal{G}$ in the asymptotic region; we require that it is, in fact, globally invariant under the action of the group $\mathcal{G}$. In other words, for any $g \in \mathcal{G}$, we require that

$$(\varphi_g)_*(T) = T.$$

To put this in yet another way, we require the action of the real line $\mathbb{R}$ generated by T to commute with the action of the group $\mathcal{G}$.

In the definition above, we view the tensor g_X as the "limiting" metric on the Kaluza–Klein fibres: asymptotically, the metric approaches $g_\infty = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 g_{\mathbb{S}^{d-1}} + g_X$.

Note that we can view the continuous action of $\mathcal{G}$ on $\mathcal{M}$ as being generated by a set of vector fields $L_{\mathcal{G}}$, which, since $\mathcal{G}$ acts by isometries, are Killing fields. Note that, since $\mathcal{G}$ acts transitively on X, the pushforwards of the vector fields $L_{\mathcal{G}}$ by the diffeomorphism φ_{as} span the tangent space of X at each point in the asymptotic region. In fact, if $\mathfrak{g}$ is the Lie algebra associated with $\mathcal{G}$, then to each element $A \in \mathfrak{g}$ we have a smooth one-parameter family of maps

$$\phi_s(A) : \mathcal{M} \to \mathcal{M}, \quad x \mapsto \varphi(\exp(sA), x),$$

for $s \in \mathbb{R}$. The vector field generating the map $\phi_s(A)$ is denoted by L_A. Then we have

$$L_{\mathcal{G}} = \{L_A \mid A \in \mathfrak{g}\}.$$

Although the vector fields L_A do not necessarily commute with one another, they do all commute with T; i.e., $[L_{\mathcal{G}}, T] = 0$.

Note that the condition that the tensor fields $h_1, \dots, h_8$ are invariant under the action of $\mathcal{G}$ means that the Lie derivatives of these tensor fields with respect to the vector fields L_A, $A \in \mathfrak{g}$, vanish, and so this gives the required bounds on the derivatives in the "Kaluza–Klein directions".

3C. ***An example: 3-charge microstate geometries.*** Our definition of an "asymptotically Kaluza–Klein" manifold is somewhat technical: we will illustrate it with the example of 3-charge microstate geometries of [Maldacena and Maoz 2002; Balasubramanian et al. 2001; Lunin et al. 2002; Giusto et al. 2004; 2005; Bena and Warner 2006; Berglund et al. 2006; Gibbons and Warner 2014]. We will also refer back to this example when discussing evanescent ergosurfaces.

The metric is given by

$$\begin{aligned} g = &-\frac{1}{h}(\mathrm{d}t^2 - \mathrm{d}z^2) + \frac{Q_p}{hf}(\mathrm{d}t - \mathrm{d}z)^2 + hf\left(\frac{\mathrm{d}r^2}{r^2 + (\tilde{\gamma}_1 + \tilde{\gamma}_2)^2\eta} + \mathrm{d}\theta^2\right) \\ &+ h\left(r^2 + \tilde{\gamma}_1(\tilde{\gamma}_1 + \tilde{\gamma}_2)\eta - \frac{(\tilde{\gamma}_1^2 - \tilde{\gamma}_2^2)\eta Q_1 Q_2 \cos^2\theta}{h^2 f^2}\right)\cos^2\theta\, \mathrm{d}\psi^2 \\ &+ h\left(r^2 + \tilde{\gamma}_2(\tilde{\gamma}_1 + \tilde{\gamma}_2)\eta - \frac{(\tilde{\gamma}_1^2 - \tilde{\gamma}_2^2)\eta Q_1 Q_2 \sin^2\theta}{h^2 f^2}\right)\sin^2\theta\, \mathrm{d}\phi^2 \\ &+ \frac{Q_p(\tilde{\gamma}_1 + \tilde{\gamma}_2)^2\eta^2}{hf}(\cos^2\theta\, \mathrm{d}\psi + \sin^2\theta\, \mathrm{d}\phi)^2 \\ &- 2\frac{\sqrt{Q_1 Q_2}}{hf}(\tilde{\gamma}_1\cos^2\theta\, \mathrm{d}\psi + \tilde{\gamma}_2 \sin^2\theta\, \mathrm{d}\phi)(\mathrm{d}t - \mathrm{d}z) \\ &- 2\frac{(\tilde{\gamma}_1 + \tilde{\gamma}_2)\eta\sqrt{Q_1 Q_2}}{hf}(\cos^2\theta\, \mathrm{d}\psi + \sin^2\theta\, \mathrm{d}\phi)\mathrm{d}z + \sqrt{\frac{H_1}{H_2}}\sum_{i=1}^{4}\mathrm{d}x_i^2, \end{aligned} \tag{3}$$

where

$$\eta = \frac{Q_1 Q_2}{Q_1 Q_2 + Q_1 Q_p + Q_2 Q_p},$$

$$\tilde{\gamma}_1 = -an, \quad \tilde{\gamma}_2 = a(n+1), \quad f = r^2 + (\tilde{\gamma}_1 + \tilde{\gamma}_2)\eta(\tilde{\gamma}_1 \sin^2\theta + \tilde{\gamma}_2\cos^2\theta),$$
$$H_1 = 1 + \frac{Q_1}{f}, \quad H_2 = 1 + \frac{Q_2}{f} \quad h = \sqrt{H_1 H_2}.$$

Here, Q_1, Q_2 and a are independent constants, while $Q_p = a^2 n(n+1)$.

The coordinate $z \in [0, 2\pi R_z]$ parametrises a circle of radius R_z, where $R_z = \sqrt{Q_1}Q_2/a$. The coordinates x^i, $i = 1, \dots, 4$, are also periodic: they parametrise a torus T^4, and their precise ranges are unimportant. The time coordinate is given by $t \in \mathbb{R}$, the radial coordinate is $r > 0$, and the coordinates θ, ϕ, ψ parametrise a 3-sphere, with $\theta \in \left[0, \frac{\pi}{2}\right]$, and ϕ, $\psi \in [0, 2\pi]$.

This metric is asymptotically Kaluza–Klein. We have

$$g = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 g_{\mathbb{S}^3} + g_X + \mathcal{O}(r^{-2}),$$

where g_X is given by

$$g_X = \mathrm{d}z^2 + \sum_{i=1}^{4} \mathrm{d}x_i^2.$$

Note that $M = 0$ in this case.

For this metric, the group $\mathcal{G}$ is given by $(U(1))^5$, and it acts by rotating the z- and x_i-coordinates. The space X can be taken to be simply $(U(1))^5$, parametrised by the coordinates (z, x_i): we do not need to quotient out by some isotropy subgroup, since the isotropy subgroup is trivial in this case. In other words, the only group element which fixes a point on X is the identity.

The vector fields L_A can be written in terms of the coordinates defined above as

$$L_z := \frac{\partial}{\partial_z}, \quad L_i := \frac{\partial}{\partial x_i}.$$

Note that these are Killing fields.

Note also that, in addition to the asymptotically time-like Killing vector field $T = \partial/\partial t$, the metric also possesses a *globally null* Killing vector field:

$$V = \frac{\partial}{\partial t} + \frac{\partial}{\partial z}.$$

4. Evanescent ergosurfaces

We shall say that an asymptotically flat (or asymptotically Kaluza–Klein) manifold possesses an evanescent ergosurface[10] if either of the following conditions holds:

(ES1) $\mathcal{M}$ is asymptotically flat and stationary, and the asymptotically time-like vector field T (given by ∂_t in the asymptotic region) is *globally causal* and nonvanishing. Moreover, there is a submanifold $\mathcal{S}$ such that:

(1) $\mathcal{S}$ is spatially compact (i.e., $\mathcal{S} \cap \Sigma_0$ is compact for some space-like Cauchy surface Σ_0) and has codimension at least 1.

(2) T is null on $\mathcal{S}$ and time-like on $\mathcal{M} \setminus \mathcal{S}$.

(3) For every open set $\mathcal{U}$ such that $\mathcal{S} \subset \mathcal{U}$, there exists some constant $c_{(T,\mathcal{U})} > 0$ such that

$$\inf_{x \in \mathcal{M} \setminus \mathcal{U}} |g(T, T)| \geq c_{(T,\mathcal{U})}.$$

(4) Either $\mathcal{M} \setminus \mathcal{S}$ consists of a single connected component, *or* $\mathcal{M} \setminus \mathcal{S}$ consists of at least two components, one[11] of which (which we call $\mathcal{M}_{(\mathrm{ext})}$) includes the asymptotic region, and where $\mathcal{M} \setminus \mathcal{M}_{(\mathrm{ext})}$ is precompact, and either

- there is some other Killing field Φ such that $[T, \Phi] = 0$ and the span of T and Φ is time-like on $\mathcal{S}$, *or*

[10] Note that the conditions we impose also exclude manifolds with black hole regions or event horizons.

[11] We can also deal with the case where $\mathcal{M} \setminus \mathcal{S}$ consists of a multiple components, if each one of the components includes an "asymptotic region", but for simplicity we stick with a single asymptotic region.

- the manifold is real analytic in a neighbourhood of $\mathcal{S}$, *or*
- $\mathcal{M} \setminus \mathcal{S}$ consists of two[12] connected components: one which includes the asymptotic region, which we label $\mathcal{U}_{\text{asy}}$, and an "enclosed" region, which we label $\mathcal{U}_{\text{enc}}$. Then, we need the following unique continuation criteria: for every solution ϕ to the wave equation $\Box_g \phi = 0$, if $\phi \equiv 0$ on $\mathcal{U}_{\text{asy}}$, then $\phi = 0$ on all of $\mathcal{M}$.

(ES2) $\mathcal{M}$ is asymptotically Kaluza–Klein in the sense of Section 3, and there is a *globally causal* Killing vector field V (which is not necessarily identical to the asymptotically time-like Killing vector field T). In addition, there is submanifold $\mathcal{S}$ such that:

(1) $\mathcal{S}$ is spatially compact (i.e., $\mathcal{S} \cap \Sigma_0$ is compact for some space-like Cauchy surface Σ_0) and has codimension at least 1.

(2) $g(V, V)$ vanishes to at least second order on $\mathcal{S}$.

(3) For all $L_A \in L_{\mathcal{G}}$, we have

$$g(V, L_A)|_{\mathcal{S}} = 0,$$

and for all $x \in \mathcal{M} \setminus \mathcal{S}$ there exists some $L_A \in L_{\mathcal{G}}$ such that $g(V, L_A) \neq 0$.

(4) For every open set $\mathcal{U}$ such that $\mathcal{S} \subset \mathcal{U}$, there exists some constant $c_{(V,\mathcal{U})} > 0$ such that

$$\inf_{x \in \mathcal{M} \setminus \mathcal{U}} \sup_{L_A \in L_{\mathcal{G}}} \frac{|g(V, L_A)|}{\sqrt{g(L_A, L_A)}} \geq c_{(V,\mathcal{U})}.$$

(5) Either $\mathcal{M} \setminus \mathcal{S}$ consists of a single connected component, *or* $\mathcal{M} \setminus \mathcal{S}$ consists of at least two components, one of which (which we call $\mathcal{M}_{(\text{ext})}$) includes the asymptotic region, and where $\mathcal{M} \setminus \mathcal{M}_{(\text{ext})}$ is precompact, and either

- there is some other Killing field Φ such that $[T, \Phi] = 0$ and the span of T and Φ is time-like on $\mathcal{S}$, *or*
- the manifold is real analytic in a neighbourhood of $\mathcal{S}$, *or*
- $\mathcal{M} \setminus \mathcal{S}$ consists of two[13] connected components: one which includes the asymptotic region, which we label $\mathcal{U}_{\text{asy}}$, and an "enclosed" region, which we label $\mathcal{U}_{\text{enc}}$. Then, we need the following unique continuation criteria: for every solution ϕ to the wave equation $\Box_g \phi = 0$, if $\phi \equiv 0$ on $\mathcal{U}_{\text{asy}}$, then $\phi = 0$ on all of $\mathcal{M}$.

Remark 4.1. In each case, the first few points give a fairly straightforward definition of an evanescent ergosurface. However, the final point in each definition (that is, point (4) in the asymptotically flat case and point (5) in the asymptotically Kaluza–Klein case) requires some additional discussion. We include this point in the definition in analogy to assumption (A1) of [Moschidis 2018]: it is made so that, in the case of an evanescent ergosurface which divides the manifold into an "inside" and an "outside", *if* a wave decays on the outside, then it is also guaranteed to decay on the inside. If we do not make this assumption, then there is a third type of behaviour which is possible for waves on manifolds with

[12] Again, if there are multiple asymptotic regions, then this condition can be modified to allow for more components.

[13] Again, if there are multiple asymptotic regions, then this condition can be modified to allow for more components.

evanescent ergosurfaces: the energy of the wave is neither amplified by an arbitrary amount, nor do we have an Aretakis-like instability, but instead the wave approaches a solution to the wave equation which vanishes outside the ergosurface but is nonzero inside of it. This is only a viable alternative if there is a solution of the wave equation which is compactly supported within the ergosurface for all time, and it is ruled out by any of the conditions in the fourth point of our definitions of an evanescent ergosurface.

Note that in many of the explicit cases of spacetimes with evanescent ergosurfaces — such as the supersymmetric microstates — the evanescent ergosurface is at least codimension-2, and so there is no issue here.

4A. ***The evanescent ergosurface on the example geometry.*** Recall the metric (3), which we are using as an example to illustrate our definitions. In this case, we have already seen that the metric is asymptotically Kaluza–Klein. We have also seen that the vector field V (given in terms of coordinate derivatives by $\partial_t + \partial_z$) is globally null.

If we define $L_i = \partial/\partial x_i$, then we find that $g(V, L_i) \equiv 0$. On the other hand, with $L_z = \partial_z$, we can compute

$$g(V, L_z) = \frac{1}{h}.$$

Thus, this spacetime has an evanescent ergosurface if h diverges somewhere, i.e., when $f = 0$. Such a region does in fact exist and it has (spatial) topology $\mathbb{S} \times \mathbb{S}^3 \times T^4$ [Gibbons and Warner 2014]. In particular, it is a smooth, compact, codimension-2 submanifold.

We can also compute

$$\frac{|g(V, L_z)|}{\sqrt{g(L_z, L_z)}} = \frac{1}{\sqrt{h}\sqrt{1 + Q_p/f}}$$

so, if $f > 0$ (and hence $h < \infty$) this quantity is positive, as required.

Finally, we note that since the ergosurface is codimension-2, $\mathcal{M} \setminus \mathcal{S}$ consists of a single connected component, so the final required property holds. In fact, the manifold is also analytic, so this gives another reason that the required "unique continuation" result holds.

4B. ***Null geodesics on the ergosurface.*** In both cases, we have the following (see also [Eperon et al. 2016]):

Proposition 4.2. *There exists a null geodesic γ lying entirely within the evanescent ergosurface $\mathcal{S}$. In the asymptotically flat case, T is tangent to an affinely parametrised null geodesic on $\mathcal{S}$, while in the Kaluza–Klein case, V is tangent to an affinely parametrised null geodesic on $\mathcal{S}$.*

Proof. Using the fact that T is a Killing vector field, we have

$$(\nabla_T T)^\mu = -T^\nu \nabla^\mu T_\nu = -\tfrac{1}{2}\nabla^\mu (T^\nu T_\nu).$$

Now, $g(T, T) = 0$ on $\mathcal{S}$ so any derivatives of $g(T, T)$ tangent to $\mathcal{S}$ vanish. Additionally, since T is a smooth vector field and the function $g(T, T)$ attains its maximum on $\mathcal{S}$, transverse derivatives of $g(T, T)$ also vanish on $\mathcal{S}$. Hence $\nabla_T T = 0$ on $\mathcal{S}$.

In the second case, since V is a Killing vector field, we also have

$$(\nabla_V V)^\mu = \partial^\mu(g(V, V))$$

since $g(V, V)$ vanishes to at least second order on $\mathcal{S}$, the right-hand side vanishes on $\mathcal{S}$, so V is tangent to affinely parametrised null geodesic. Now, let γ be such a geodesic, with tangent V, passing through $\mathcal{S}$ at some point. For any $L_A \in L_{\mathcal{G}}$ we compute

$$\begin{aligned} V(g(V, L_A))|_{\mathcal{S}} &= g(\mathcal{L}_V V, L_A)|_{\mathcal{S}} + g(V, \mathcal{L}_V L_A)|_{\mathcal{S}} \\ &= -g(V, \mathcal{L}_{L_A} V)|_{\mathcal{S}} \\ &= -\tfrac{1}{2} L_A(g(V, V)) = 0, \end{aligned}$$

where in the first and last lines we have used the fact that V and L_A are Killing fields for the metric g, respectively. Hence, if $g(V, L_A) = 0$ at some point of γ, then $g(V, L_A) = 0$ everywhere on γ; i.e., γ remains within $\mathcal{S}$. □

5. The discrete isometry

We require our manifolds to possess a discrete isometry $\mathscr{T}$ with the following properties:

- $\mathscr{T}$ is an isometry; i.e.,
$$\mathscr{T} : \mathcal{M} \to \mathcal{M},$$
$$\mathscr{T}^*(g) = g.$$
- $\mathscr{T}$ reverses the direction of Killing vector field T; i.e.,
$$\mathscr{T}_*(T) = -T.$$
- There exists some *space-like* hypersurface Σ_0 which is a Cauchy surface for $\mathcal{M}$ and which is fixed by the discrete isometry $\mathscr{T}$.

Note that we do not require that the *points* of the hypersurface Σ_0 are invariant under the discrete isometry, which would lead to the requirement that the spacetime is static. For example, "$(t-\phi)$-symmetric" spacetimes are acceptable, in which there is a discrete isometry which maps T to $-T$ and also Φ to $-\Phi$, where Φ is an axisymmetric Killing vector field. In this case, in terms of standard coordinates, the hypersurface $\{t = 0\}$ can be chosen as Σ_0: it is invariant under the discrete isometry, although a point with coordinates $\phi = \phi_1$ is mapped to a point with coordinates $\phi = -\phi_1$.

Note also that, in the case we have an additional symmetry of the right kind (see Section 11) we actually do not need this additional symmetry.

5A. ***The discrete isometry in the example geometry.*** The example geometry with metric (3) possesses a discrete isometry of the required kind. The map $\mathscr{T}$ can be specified using the coordinates: we have

$$\mathscr{T}(t, r, \theta, \phi, \psi, z, x_i) = (-t, r, \theta, \phi, \psi, 2\pi R_z - z, x_i).$$

In other words, we replace $t \mapsto -t$ and $z \mapsto 2\pi R_z - z$. This metric could be said to be "$t - z$ symmetric". We clearly have $\mathscr{T}_* V = -V$, since V is given in coordinates by $\partial_t + \partial_z$.

Examining the form of the metric (3) reveals that this is indeed an isometry. Moreover, this isometry clearly fixes the hypersurface $\Sigma_0 = \{t = 0\}$. This hypersurface is space-like: we can compute

$$g^{-1}(\mathrm{d}t, \mathrm{d}t) = -\frac{1}{hf}\left(f + Q_1 + Q_2 + Q_p + \frac{Q_1 Q_2 + Q_1 Q_p + Q_2 Q_p}{r^2 + (\tilde{\gamma}_1 + \tilde{\gamma}_2)^2 \eta}\right),$$

which turns out to be uniformly bounded and negative. Hence the isometry $\mathscr{T}$ given above fixes a space-like hypersurface as required. We also find that the hypersurfaces Σ_t are given by the level sets of t.

The discrete isometry descends to a discrete isometry on the hypersurface $t = 0$, which can also be given in coordinates:

$$\overline{\mathscr{T}}(r, \theta, \phi, \psi, z, x_i) = (r, \theta, \phi, \psi, 2\pi R_z - z, x_i).$$

It is easy to see that this is an isometry of the submanifold $t = 0$ equipped with the metric induced by g. This isometry can be viewed as a reflection in the hypersurface $z = 0$.

6. The energy momentum tensor and energy currents

For any smooth, compactly supported function ϕ on a manifold $\mathcal{M}$ with metric g, we define the associated *energy momentum tensor* Q:

$$Q_{\mu\nu}[\phi] := (\partial_\mu \phi)(\partial_\nu \phi) - \tfrac{1}{2} g_{\mu\nu} g^{\alpha\beta} (\partial_\alpha \phi)(\partial_\beta \phi).$$

Given a smooth vector field X we define the associated deformation tensor

$$^{(X)}\pi_{\mu\nu} := (\mathcal{L}_X g)_{\mu\nu} = \nabla_\mu X_\nu + \nabla_\nu X_\mu.$$

Using these we define the two energy currents

$$\begin{aligned} (^{(X)}J[\phi])_\mu &:= X^\nu Q_{\mu\nu}[\phi], \\ ^{(X)}K[\phi] &:= \tfrac{1}{2}{}^{(X)}\pi^{\mu\nu} Q_{\mu\nu}[\phi]. \end{aligned}$$

Now, for some time function t, we define the hypersurface

$$\Sigma_t := \{x \in \mathcal{M} \mid t(x) = t\}$$

as well as the (open) spacetime region

$$\mathcal{M}_{t_1}^{t_2} := \{x \in \mathcal{M} \mid t_1 < t(x) < t_2\}.$$

We choose the time function t to agree with the coordinate function t in the asymptotic region.

In general, given an initial surface Σ_0, we will choose Σ_t to be the T-translate of Σ_0, where t is a parameter such that $T(t) = 1$. Moreover, we choose the initial hypersurface Σ_0 to be a hypersurface which is fixed by the discrete isometry, as detailed in Section 5.

Then, we have the energy identity: for $t_2 \geq t_1$, using ι to denote the interior product, we have

$$\int_{\Sigma_{t_2}} \iota_{(^{(X)}J[\phi])} \,\mathrm{dvol}_g = \int_{\Sigma_{t_1}} \iota_{(^{(X)}J[\phi])} \,\mathrm{dvol}_g + \int_{\mathcal{M}_{t_1}^{t_2}} {}^{(X)}K[\phi] \,\mathrm{dvol}_g. \tag{4}$$

In particular, if X is a Killing vector field, then $^{(X)}K[\phi]$ vanishes identically, and the associated energy is conserved.

Now, suppose that X is *uniformly* time-like and future-directed. Then the energy current ${}^{(X)}J[\phi]$ is "nondegenerate" in a sense we shall make precise below. As above, let t be a time function for $\mathcal{M}$, so the hypersurfaces Σ_t are uniformly time-like (i.e., the one-form $\mathrm{d}t$ is uniformly time-like). The unit, future-directed normal to Σ_t is defined by

$$n := -\frac{(\mathrm{d}t)^\sharp}{\sqrt{-g^{-1}(\mathrm{d}t, \mathrm{d}t)}}.$$

Now, for any sufficiently small open set $\mathcal{U} \subset \mathcal{M}$, we can find an orthonormal basis for the tangent space $\mathcal{T}(\mathcal{U})$ consisting of the vector fields $\{e_0, \dots, e_d\}$, satisfying $g(e_a, e_b) = \eta_{ab}$, where $\eta_{00} = -1$, $\eta_{aa} = 1$ for $a \in \{1, 2, \dots, d\}$ and $\eta_{ab} = 0$ for other values of a and b. Furthermore, we can choose $e_0 = n$, and we can choose e_1 so that

$$X = X^0 n + X^1 e_1$$

for some smooth functions X^0 and X^1. Finally, we set

$$|\mathrm{d}t| := \sqrt{-g^{-1}(\mathrm{d}t, \mathrm{d}t)}.$$

Then, a short computation shows that in $\mathcal{U}$

$$(\iota_{({}^{(X)}J[\phi])}\mathrm{d}\mathrm{vol}_g)|_{\Sigma_t} = \frac{1}{2}\left(\frac{|\mathrm{d}t|}{X(t)}(X(\phi))^2 + \frac{|\mathrm{d}t|^2 g(X,X)}{(X(t))^2}(e_1(\phi))^2 + \sum_{a=2}^{d}\frac{(X(t))}{|\mathrm{d}t|}(e_a(\phi))^2\right)\iota_n \mathrm{d}\mathrm{vol}_g\bigg|_{\Sigma_t}. \quad (5)$$

The condition that X is uniformly time-like and future-directed ensures that

$$c|\mathrm{d}t| \le X(t) \le C|\mathrm{d}t|, \quad (6)$$

and so we find that

$$\frac{|\mathrm{d}t|}{X(t)}(X(\phi))^2 + \frac{|\mathrm{d}t|^2 g(X,X)}{(X(t))^2}(e_1(\phi))^2 + \sum_{a=2}^{d}\frac{(X(t))}{|\mathrm{d}t|}(e_a(\phi))^2 \sim \max\left\{c, \frac{1}{C}, \frac{c}{C^2}\right\}\left((X(\phi))^2 + \sum_{a=1}^{d}(e_a(\phi))^2\right),$$

so the energy associated with the vector field X is equivalent to the L^2 norm of the derivatives; i.e.,

$$\int_{\Sigma_t}(\iota_{({}^{(X)}J[\phi])}\,\mathrm{d}\mathrm{vol}_g) \sim \|\partial\phi\|^2_{L^2(\Sigma_t)},$$

where we recall that $\partial\phi$ means the collection of first derivatives of ϕ, defined using an orthonormal frame.

Now, suppose that X is future-directed and "uniformly transverse" to the leaves of the foliation Σ_t in such a way that the bounds in equation (6) hold, but this time we do not require X to be uniformly time-like. Then we can see from equation (5) that the energy current ${}^{(X)}J$ is *not* necessarily comparable to the L^2 norm of the derivatives — instead, the energy current will be "missing" a derivative wherever X is null, and will not be positive definite where X is space-like.

7. Energy currents in a spacetime with an evanescent ergosurface

Let $\mathcal{M}$ be a stationary spacetime with an evanescent ergosurface, as defined in Section 4. In the asymptotically flat case (ES1) we define the *conserved energy* at time t,

$$\mathcal{E}^{(T)}[\phi](t) := \int_{\Sigma_t}\iota_{{}^{(T)}J[\phi]}\,\mathrm{d}\mathrm{vol}_g, \quad (7)$$

while in the Kaluza–Klein case (ES2) we define

$$\mathcal{E}^{(V)}[\phi](t) := \int_{\Sigma_t} \iota_{(V)} J[\phi] \, \mathrm{d}\mathrm{vol}_g. \tag{8}$$

Note that these "energies" are conserved in the sense that, for any $t \in \mathbb{R}$,

$$\mathcal{E}^{(A)}[\phi](t) = \mathcal{E}^{(A)}[\phi](0),$$

where $A = T$ or V, depending on whether we are in the asymptotically flat case or the asymptotically Kaluza–Klein case. This follows from the fact that both T and V are Killing vector fields for their respective geometries. In addition, since both T and V are globally causal, the energy currents $\mathcal{E}^{(A)}[\phi]$ are nonnegative.

We also define the "nondegenerate energy", which is the energy associated with some *uniformly time-like* vector field N, which is also T-invariant (or V-invariant, in the Kaluza–Klein case), that is, $[T, N] = 0$ (or $[V, N] = 0$):

$$\mathcal{E}^{(N)}[\phi](t) := \int_{\Sigma_t} \iota_{(N)} J[\phi] \, \mathrm{d}\mathrm{vol}_g.$$

Note that, since N is *not* required to be a Killing vector field, this energy is not in general conserved. It represents the total energy measured by a family of observers moving along the time-like curves given by integral curves of N. Since N is T-invariant, the *family* of observers (though not each individual observer!) is stationary; that is, it is invariant under time translation.

Both the energy current $\mathcal{E}^{(T)}[\phi]$ and the current $\mathcal{E}^{(V)}[\phi]$ are "degenerate": the associated energies are *not* equivalent to the L^2 norm of the derivatives of ϕ. In case (ES1), the vector field T becomes null on $\mathcal{S}$, so the energy current $\mathcal{E}^{(T)}[\phi]$ is "missing" a derivative on the evanescent ergosurface. On the other hand, in case (ES2) the vector field V is only *globally causal*, and so the energy current $\mathcal{E}^{(V)}[\phi]$ might (in the case that V is null everywhere) be "missing" a derivative everywhere.

However, we shall see that, when the function ϕ is invariant under the action of $\mathcal{G}$, then the energy current $\mathcal{E}^{(V)}[\phi]$ is, in fact, nondegenerate away from the submanifold $\mathcal{S}$. Indeed, this is the motivation for labelling the surface $\mathcal{S}$ an "evanescent ergosurface" in case (ES2).

Proposition 7.1. *Let ϕ be a function on $\mathcal{M}$ invariant under the action of the group $\mathcal{G}$; i.e., $L_A(\phi) = 0$ for all $L_A \in L_{\mathcal{G}}$. Suppose also that t is invariant under the action of $\mathcal{G}$. Then, for any open set $\mathcal{U}$ such that $\mathcal{S} \subset \mathcal{U}$, there exists some $c = c(\mathcal{U}) > 0$ such that*

$$\int_{\Sigma_t \setminus \mathcal{U}} \iota_{(V)} J[\phi] \, \mathrm{d}\mathrm{vol} \geq c \|\partial\phi\|^2_{L^2(\Sigma_t \setminus \mathcal{U})}. \tag{9}$$

Proof. The calculations in Section 6 show that the energy current $\mathcal{E}^{(V)}[\phi]$ is almost equivalent to the L^2 norm of the derivatives, except that it is "missing" a derivative in the direction of the orthogonal projection of V onto the surface Σ_t (that is, in the e_1-direction). Note that $V^1 e_1 = V + g(V, n)n$. In order to prove the proposition, we need to show that, away from $\mathcal{S}$, we can express this derivative in terms of the vector fields L_A as well as the other vector fields on Σ_t that are orthogonal to the projection of V onto Σ_t.

At all points away from $\mathcal{S}$, there exists some $L_{\hat{A}} \in L_{\mathcal{G}}$ such that $g(L_{\hat{A}}, V) \neq 0$. We claim that

$$V + g(V, n)n = \frac{(g(V, n))^2}{g(L_{\hat{A}}, V)} \left(L_{\hat{A}} + g(L_{\hat{A}}, n)n - \sum_{A=2}^{d} g(L_{\hat{A}}, e_A)e_A \right).$$

To prove this we simply take contractions with the orthonormal base $\{n, e_1, \dots, e_d\}$. Additionally, using the conditions in the definition of the evanescent ergosurface, as well as the condition that $c|\mathrm{d}t| \leq V(t) \leq C|\mathrm{d}t|$, we also find that for any function ϕ, in the set $\mathcal{M} \setminus \mathcal{U}$ there exists some $c > 0$ such that

$$|V(u) + g(V, n)n(\phi)| \leq c \left(\frac{|L_{\hat{A}}(u)|}{\sqrt{g(L_{\hat{A}}, L_{\hat{A}})}} + |n(\phi)| + \sum_{A=2}^{d} |e_A \phi| \right).$$

In particular, if ϕ is invariant under the action of $\mathcal{G}$, then the "missing" derivative in the energy can be *uniformly* bounded in terms of the derivatives which *do* appear in the energy, proving the proposition. □

8. Estimates in asymptotically Kaluza–Klein spaces

In the following two sections we will need to make use of some of the results proved in [Moschidis 2016a; 2018]. In Section 10 we appeal to the results of [Moschidis 2018], which apply to asymptotically flat manifolds with an ergoregion but with no horizon; we can therefore apply all estimates which do not rely on the existence of a nonempty ergoregion to our manifolds in the asymptotically flat case (in the sense of Section 3). However, for manifolds with Kaluza–Klein asymptotics of the appropriate form (also defined in Section 3), some modification is needed.

In Section 11 we will also make some use of the results of [Moschidis 2016a], which establishes logarithmic decay of the local energy on a very general class of geometries. These results are proved for asymptotically flat manifolds with a *globally* time-like Killing vector T, or else on manifolds with an ergoregion which overlaps with a "red shift region". Neither of these conditions holds in the cases we are considering, and the additional symmetry assumed in Section 11 plays a crucial role in recovering some of the results. These issues concern the local geometry near the evanescent ergosurface, and will be addressed in Section 11. In this section, we shall sketch an adaptation of some of the methods of [Moschidis 2016a; 2018] to the asymptotically Kaluza–Klein case.

The key inequalities which we need to extend to the asymptotically Kaluza–Klein case are the "Carleman inequalities" (see Section 6 of [Moschidis 2018]), as well as the "p-weighted energy estimate" (see Section 5 of [Moschidis 2016a][14]). The Carleman inequality is used to establish decay in [Moschidis 2018], while [Moschidis 2016a] used the p-weighted estimates for this purpose, so there is actually some redundancy here (this redundancy is noted explicitly in [Moschidis 2018]); however, since we will quote the results of these two papers rather than rederive them, it is important that both types of estimates can be applied in our setting, i.e., on asymptotically Kaluza–Klein spacetimes.

[14]Note that this powerful technique, first developed in [Dafermos and Rodnianski 2010], was also extensively developed for use on the linear wave equation in [Moschidis 2016b] (among many others), and was also extended for use with nonlinear equations in [Yang 2013; 2015; 2016; Keir 2018].

Some of the estimates in [Moschidis 2016a; 2018] will actually *not* hold for general waves on asymptotically Kaluza–Klein spacetimes. However, they will still hold for $\mathcal{G}$-invariant waves on these spacetimes. Before discussing this further, we need the following standard result:

Proposition 8.1. *Let ϕ be a solution to the wave equation $\Box_g \phi = 0$ on an asymptotically Kaluza–Klein manifold, and suppose that the initial data for ϕ, that is, $\phi|_{\Sigma_0}$ and $n(\phi)|_{\Sigma_0}$, is invariant under the action of $\mathcal{G}$; i.e., $L_A \phi|_{\Sigma_0} = n L_A \phi|_{\Sigma_0} = 0$ for all $L_A \in L_{\mathcal{G}}$. Then $L_A \phi = 0$ everywhere in $\mathcal{M}$.*

Proof. Since the vector fields L_A are Killing vector fields for the metric g, they commute with the geometric wave operator $\Box_g$, so $L_A \phi$ satisfies

$$\Box_g (L_A \phi) = 0.$$

However, the initial data for $L_A \phi$ vanishes identically, and so by standard uniqueness results $L_A \phi = 0$ everywhere. □

Now, suppose ϕ is a solution to the wave equation which is invariant under the action of $\mathcal{G}$. Then we claim that ϕ satisfies a wave equation (with a right-hand side) on a related manifold in the "asymptotic region". In fact, away from the fixed points of the action of $\mathcal{G}$ we can use the action of $\mathcal{G}$ to define a smooth projection. We have

$$\pi_{\mathcal{G}} : \mathcal{U}^{\mathrm{as}} \to \mathcal{U}^{\mathrm{as}}/\mathcal{G}.$$

In the asymptotic region we have a diffeomorphism $\varphi : \mathcal{U}^{\mathrm{as}} \to \mathbb{R} \times (\mathbb{R}^d \setminus B) \times X$ and the group action of $\mathcal{G}$ descends to a transitive action on the space X, so the projection $\pi_{\mathcal{G}}(\mathcal{U}^{\mathrm{as}})$ of the asymptotic region may be identified with $\mathbb{R} \times (\mathbb{R}^d \setminus B)$. That is we can define the projection

$$\begin{aligned} \bar{\pi}_{\mathcal{G}} : \mathbb{R} \times (\mathbb{R}^d \setminus B) \times X &\to \mathbb{R} \times (\mathbb{R}^d \setminus B), \\ (t, x, gH) &\mapsto (t, x). \end{aligned}$$

This projection allows us to define *vertical vector fields* in $T\mathcal{M}$. To be precise, we say that a vector field X is *vertical* if its pushforward by the projection map $\pi_{\mathcal{G}}$ (or, equivalently in the asymptotic region, its pushforward by the map $\varphi \circ \bar{\pi}_{\mathcal{G}}$) vanishes.

Similarly, *in the asymptotic region* (although not necessarily elsewhere) we can make sense of the notion of *horizontal* vector fields as follows: We can define the map

$$\begin{aligned} \bar{\pi}_{\mathrm{hor}} : \mathbb{R} \times (\mathbb{R}^d \setminus B) \times X &\to X, \\ (t, x, gH) &\mapsto (gH). \end{aligned}$$

Then a vector field $X \in T\mathcal{U}^{\mathrm{as}}$ is called *horizontal* if its pushforward under the projection $\varphi_{\mathrm{as}} \circ \bar{\pi}_{\mathrm{hor}}$ vanishes. Using this, we can split the tangent space in the asymptotic region as

$$T_p(\mathcal{M}) = (T_p(\mathcal{M}))^{\mathrm{ver}} \oplus (T_p(\mathcal{M}))^{\mathrm{hor}}$$

for $p \in \mathcal{U}^{\mathrm{as}}$. Note that the spaces $(T_p(\mathcal{M}))^{\mathrm{ver}}$ and $(T_p(\mathcal{M}))^{\mathrm{hor}}$ are then linear subspaces of the tangent space at the point p, but they are not, in general, orthogonal subspaces.

We now take a nonvanishing section of the frame bundle of some sufficiently small set $\mathcal{U} \subset \mathcal{U}^{\mathrm{as}}$, such that the vectors in the frame each lie within either the vertical or horizontal subspace at each point. We shall label the frame vector fields

$$\begin{aligned}(e_a)|_p \in (T_p(\mathcal{M}))^{\mathrm{hor}} \quad &\text{for all } p \in \mathcal{U}^{\mathrm{as}}, \text{ for } a \in \{0, \dots, d\}, \\ (e_A)|_p \in (T_p(\mathcal{M}))^{\mathrm{ver}} \quad &\text{for all } p \in \mathcal{U}^{\mathrm{as}}, \text{ for } A \in \{d+1, \dots, d+n\}.\end{aligned}$$

We may moreover take the frame vectors e_A to satisfy $e_A \in L_{\mathcal{G}}$, since every vector $L_g \in L_{\mathcal{G}}$ is evidently in the vertical subspace. Hence the vectors e_A are Killing vectors for the metric g. Note, however, that we do not assume that the vector fields in the frame commute with one another, nor are they necessarily orthogonal. Note also that we will use lower case Latin letters to label vectors in the horizontal subspace, and upper case Latin letters to label vectors in the vertical subspace. When we wish to sum over the entire (co-)tangent space we shall use (lower case) Greek letters. Finally, note that, for vector fields $L_A, L_B \in L_{\mathcal{G}}$, a standard computation reveals that their commutator is given by

$$[L_A, L_B] = L_{[B,A]},$$

where on the right-hand side we use the Lie bracket associated with the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$. In particular, we have that $[L_A, L_B] \in L_{\mathcal{G}}$.

Given local coordinates $\theta^1, \dots, \theta^{d-1}$ for a coordinate patch on the sphere $\mathbb{S}^{d-1}$, we may use the coordinate vector fields $\partial_t, \partial_r, \partial_{\theta^1}, \dots, \partial_{\theta^{d-1}}$ to give a basis for the horizontal subspace. Since these vector fields are coordinate-induced, their commutators vanish. Additionally, the fact that the coordinates $t, r, \theta^1, \dots$ are invariant under the action of $\mathcal{G}$ (e.g., we have $L_A(r) = 0$ for all $A \in \mathfrak{g}$) ensures that $[e_A, e_b] \in L_{\mathcal{G}}$.

Given a vector field X, we can write

$$X = X^a e_a + X^A e_A,$$

where we use the usual summation convention to sum over both the indices labelling the horizontal subspace and those labelling the vertical subspace. Similarly, given a covector field ω, we write

$$\omega_a := \omega(e_a), \quad \omega_A := \omega(e_A).$$

Now, let ϕ be a function which is invariant under the action of $\mathcal{G}$, so $e_A(\phi) = 0$. Then we can compute

$$\Box_g \phi = g^{ab} e_a(e_b(\phi)) - (g^{ab}\Gamma^c_{ab} - g^{AB}\Gamma^c_{AB} - 2g^{aB}\Gamma^c_{aB}) e_c(\phi), \tag{10}$$

where

$$\Gamma^\alpha_{\beta\gamma} := \tfrac{1}{2}(g^{-1})^{\alpha\delta}\big(e_\beta g_{\delta\gamma} + e_\gamma g_{\delta\beta} - e_\delta g_{\gamma\beta} + g([e_\delta, e_\gamma], e_\beta) + g([e_\delta, e_\beta], e_\gamma) - g([e_\gamma, e_\beta], e_\delta)\big).$$

In the asymptotic region, we can use the map $\varphi_{\mathcal{G}} : \mathcal{M} \to \mathcal{M}/\mathcal{G}$ to define the pushforward of the inverse metric

$$(\varphi_{\mathcal{G}})_*(g^{-1}) := \bar{g}^{-1}.$$

Moreover, in the asymptotic region we can see from the form of the metric that, at least for sufficiently large r, $\bar{g}^{-1}$ is nondegenerate. Thus, we can define a symmetric, nondegenerate inverse tensor $\bar{g} \in$

$\Gamma(T_*(\mathcal{M}) \otimes T_*(\mathcal{M}))$. We can equip the orbit space $\mathcal{M}/\mathcal{G}$ with the metric $\bar{g}$, and for any $\mathcal{G}$-invariant function ϕ on $\mathcal{M}$ we can identify ϕ with a function $\bar{\phi}$ on $\mathcal{M}/\mathcal{G}$. Then, for sufficiently large R, we find that if ϕ satisfies (10), then $\bar{\phi}$ satisfies a wave equation on the Lorentzian manifold $(\mathcal{M}/\mathcal{G} \cap \{r > R\}, \bar{g})$

$$\Box_{\bar{g}}\bar{\phi} + F^a(\varphi_{\mathcal{G}})^* e_a(\bar{\phi}) = 0,$$

where

$$\begin{aligned} F^a := \tfrac{1}{2}(\bar{g}^{-1})^{ab}(\bar{g}^{-1})^{cd}\big(&e_c(\bar{g}_{be}g^{eA}g_{Ad}) + e_d(\bar{g}_{be}g^{eA}g_{Ac}) - e_b(\bar{g}_{ce}g^{eA}g_{Ad})\big) \\ &- (g^{-1})^{bc}(g^{-1})^{aD}e_b(g_{cD}) - \tfrac{1}{2}(g^{-1})^{AB}(g^{-1})^{a\mu}e_\mu(g_{AB}) \\ &- (g^{-1})^{Ab}(g^{-1})^{ac}e_b(g_{cA}) + (g^{-1})^{Ab}(g^{-1})^{ac}e_c(g_{Ab}) \\ &- (g^{-1})^{Ab}(g^{-1})^{aD}(e_b(g_{AD}) + g([e_A, e_D], e_b)). \end{aligned}$$

Note that the terms defining the functions F^a, $a \in \{0, \dots, d\}$, are $\mathcal{G}$-invariant, which follows from the form of the metric in the asymptotic region, and so their pushforward to the orbit space is well-defined. Furthermore, note that the derivatives of the metric in the directions e_a are $\mathcal{O}(r^{-2-\alpha})$, whereas the derivatives in the directions e_A generally have worse decay. For example, since the components of $[e_A, e_B]$ will generally be $\mathcal{O}(1)$, we have

$$(g^{-1})^{a\mu}e_\mu(g_{AB}) = \mathcal{O}(r^{-1-\alpha}).$$

Importantly, we find that, overall,

$$|F^a| = \mathcal{O}(r^{-1-\alpha}).$$

And so, although ϕ does not satisfy a wave equation on $\mathcal{M}/\mathcal{G}$ (even in the asymptotic region), we can consider the terms arising from F as error terms, and moreover, error terms of this kind were dealt with previously in [Yang 2016; Moschidis 2016b; Keir 2018]. Note, for example, that [Yang 2016] dealt[15] with nonlinear equations using a similar method to that used in [Dafermos and Rodnianski 2010; Moschidis 2016b], and in the process established estimates for equations of the form $\Box_g u = F$ in the asymptotic region, for $F = \mathcal{O}(r^{-2-\alpha})$. Note also that we can view the derivatives in the compact directions e_A as "bad" derivatives: taking a derivative of the metric in these directions does not in general improve the decay rate of the metric component, in contrast to the e_a derivatives. Hence we can expect $(e_A\phi) \sim r^{-1}$.

As mentioned above, we intend to make direct use of the results of [Moschidis 2016a; 2018]. Note, however, that a function ϕ satisfying

$$\Box_{\bar{g}}\phi = F(\partial\phi)$$

for $F = \mathcal{O}(r^{-1-\alpha})$ does not quite satisfy all of the estimates in [Moschidis 2016a], which are concerned with homogeneous equations of the form $\Box_{\bar{g}}\phi = 0$. One can check that an inhomogeneous term of this sort does not pose any problems for the Carleman estimates of [Moschidis 2018]. On the other hand, certain estimates in [Moschidis 2016a] need a little bit of additional consideration.

[15] In fact, using the results of [Keir 2018], error terms of the form $F \cdot \partial\phi$ with $F \sim r^{-1}$ can be handled, but this requires some subtle modification of the estimates.

To be precise, the p-weighted energy estimates of [Dafermos and Rodnianski 2010] cannot be established for the entire range $0 < p < 2$ as in the homogeneous case $F = 0$ studied in [Moschidis 2016a]. To see why, we can sketch the case where $g = m$, the Minkowski metric on $\mathbb{R}^{d+1}$, and

$$F^a e_a(\phi) = F^{\underline{L}}\underline{L}(\phi),$$

where $\underline{L} = \partial_t - \partial_r$in the standard Minkowski coordinates, and where we assume $F^{\underline{L}} = \mathcal{O}(r^{-1-\alpha})$.

First, set t^* to be the function

$$t^* := \begin{cases} t & \text{if } r \leq R, \\ t - r + R & \text{if } r \geq R, \end{cases}$$

and we set

$$\Sigma_{t^*} := \{x \in \mathcal{M} | t^*(x) = t^*\}.$$

For the purposes of this example, we will assume that an *integrated local energy decay* statement of the following type holds:

$$\int_{\tau=\tau_1}^{\tau_2} \left(\int_{\Sigma_\tau} \alpha(1+r)^{-1-\frac{\alpha}{2}} \left(\sum_{a=0}^{d} |e_a \phi|^2 \right) r^{d-1} \, \mathrm{d}r \, \mathrm{d}\mathrm{vol}_{\mathbb{S}^{d-1}} \right) \mathrm{d}\tau \lesssim \mathcal{E}^{(N)}[\phi](\tau_1),$$

where $\mathrm{d}\mathrm{vol}_{\mathbb{S}^{d-1}}$ is the standard volume form on the *unit* $(d-1)$-sphere and $\alpha > 0$ is some positive constant. Note that such an estimate can indeed be proven using the multiplier $(1-(1+r)^{-\alpha/2})\partial_r$ and an appropriately modified energy current in the asymptotic region. However, our purpose here is to sketch the argument so we will skip this computation.

Setting $L = \partial_t + \partial_r$, and defining

$$\psi := r^{\frac{d-1}{2}} \phi,$$

we find that ψ satisfies

$$-L\underline{L}\psi + \not{\Delta}\psi + r^{\frac{d-1}{2}} F^{\underline{L}}(\underline{L}\phi) - \frac{(d-1)(d-3)}{4} r^{-2}\psi = 0.$$

Multiplying by $-r^p L\psi$ and integrating by parts in the region $r \geq R,\ \tau_0 \leq \tau \leq \tau_1$ we find that

$$\begin{aligned}
&\int_{\Sigma_{t_1^*} \cap \{r \geq R\}} r^p (L\psi)^2 \mathrm{d}r \, \mathrm{d}\mathrm{vol}_{\mathbb{S}^2} \\
&\quad + \frac{1}{2} \int_{t_0^*}^{t_1^*} \left(\int_{\Sigma_\tau \cap \{r \geq R\}} \left(p r^{p-1} (L\psi)^2 + (2-p) r^{p-1} |\not{\nabla}\psi|^2 + (2-p) \frac{(d-1)(d-3)}{4} r^{p-3} |\psi|^2 \right) \mathrm{d}r \, \mathrm{d}\mathrm{vol}_{\mathbb{S}^{d-1}} \right) \mathrm{d}\tau \\
&\qquad \leq \int_{\Sigma_{\tau_0} \cap \{r \geq R\}} r^p (L\psi)^2 \, \mathrm{d}r \, \mathrm{d}\mathrm{vol}_{\mathbb{S}^{d-1}} + \int_{t=R}^{R+\tau} \left(\int_{\mathbb{S}^{d-1}} r^p ((L\psi)^2 + |\not{\nabla}\psi|^2) \, \mathrm{d}\mathrm{vol}_{\mathbb{S}^{d-1}} \right) \mathrm{d}t \\
&\qquad\qquad + \int_{\tau_0}^{\tau_1} \left(\int_{\Sigma_\tau} |F^{\underline{L}}| r^{\frac{d-1}{2}+p} |\underline{L}\phi| |L\psi| \, \mathrm{d}r \, \mathrm{d}\mathrm{vol}\mathbb{S}^{d-1} \right) \mathrm{d}\tau. \qquad (11)
\end{aligned}$$

The terms on the right-hand side which are evaluated at $\tau = \tau_0$ are to be considered part of the initial data, while the term evaluated on the surface $r = R$ can be dealt with using the integrated local energy decay estimate in a standard way. The new term which must be estimated arises from the presence of the inhomogeneity $F^{\underline{L}}$.

Since $F^{\underline{L}}$ satisfies $|F^{\underline{L}}| = \mathcal{O}(r^{-1-\alpha})$, we can estimate this term as follows: for $r \geq R$, and for any $\delta > 0$ we have

$$|F|r^{\frac{d-1}{2}+p}|\underline{L}\phi||L\psi| \lesssim \delta r^{p-1}|L\psi|^2 + \frac{1}{\delta}C(R)(1+r)^{-1+p-2\alpha}|\underline{L}\phi|^2 r^{d-1}.$$

If we now take $p = \frac{3}{2}\alpha$ and substitute back into (11), then we can estimate

$$\begin{aligned}
&\int_{\Sigma_{\tau_1}\cap\{r\geq R\}} r^{\frac{3}{2}\alpha}(L\psi)^2\,\mathrm{d}r\,\mathrm{d}\mathrm{vol}_{\mathbb{S}^2}\\
&+\int_{\tau_0}^{\tau_1}\left(\int_{\Sigma_\tau\cap\{r\geq R\}}\left(\alpha r^{\frac{3}{2}\alpha-1}(L\psi)^2+(4-3\alpha)r^{\frac{3}{2}\alpha-1}|\nabla\!\!\!/\psi|^2+(d-1)(d-3)(4-3\alpha)r^{\frac{3}{2}\alpha-3}|\psi|^2\right)\mathrm{d}r\,\mathrm{d}\mathrm{vol}_{\mathbb{S}^{d-1}}\right)\mathrm{d}\tau\\
&\qquad\lesssim \mathcal{E}^{(N)}[\phi](\tau_0)+\int_{\Sigma_{\tau_0}\cap\{r\geq R\}} r^{\frac{3}{2}\alpha}(L\psi)^2\,\mathrm{d}r\,\mathrm{d}\mathrm{vol}_{\mathbb{S}^{d-1}}\\
&\qquad\qquad+\int_{\tau_0}^{\tau_1}\left(\int_{\Sigma_\tau}\left(\delta r^{\frac{3}{2}\alpha-1}|L\psi|^2+C(R)\frac{1}{\delta}(1+r)^{-1-\frac{1}{2}\alpha}|\underline{L}\phi|^2 r^{d-1}\right)\mathrm{d}r\,\mathrm{d}\mathrm{vol}_{\mathbb{S}^{d-1}}\right)\mathrm{d}\tau. \qquad (12)
\end{aligned}$$

If we fix δ as some sufficiently small constant, then the first term in the second integral on the right-hand side can be absorbed by the corresponding term on the left-hand side. Moreover, the second term in the second integral on the right-hand side can be estimated in terms of the initial energy by using the integrated local energy decay estimate, resulting in the bound

$$\begin{aligned}
&\int_{\Sigma_{\tau_1}\cap\{r\geq R\}} r^{\frac{3}{2}\alpha}(L\psi)^2\,\mathrm{d}r\,\mathrm{d}\mathrm{vol}_{\mathbb{S}^2}\\
&+\int_{\tau_0}^{\tau_1}\left(\int_{\Sigma_\tau\cap\{r\geq R\}}\left(\alpha r^{\frac{3}{2}\alpha-1}(L\psi)^2+(4-3\alpha)r^{\frac{3}{2}\alpha-1}|\nabla\!\!\!/\psi|^2+(d-1)(d-3)(4-3\alpha)r^{\frac{3}{2}\alpha-3}|\psi|^2\right)\mathrm{d}r\,\mathrm{d}\mathrm{vol}_{\mathbb{S}^{d-1}}\right)\mathrm{d}\tau\\
&\qquad\lesssim (1+\alpha^{-1}C(R))\mathcal{E}^{(N)}[\phi](\tau_0)+\int_{\Sigma_{\tau_0}\cap\{r\geq R\}} r^{\frac{3}{2}\alpha}(L\psi)^2\,\mathrm{d}r\,\mathrm{d}\mathrm{vol}_{\mathbb{S}^{d-1}}. \qquad (13)
\end{aligned}$$

Next, we return to (11), and this time we estimate the term involving the inhomogeneity $F^{\underline{L}}$ as follows:

$$|F|r^{\frac{d-1}{2}+p}|\underline{L}\phi||L\psi| \lesssim (1+\tau)^{-1-\beta}r^p|L\psi|^2 + C(R)(1+\tau)^{1+\beta}(1+r)^{-2+p-2\alpha}|\underline{L}\phi|^2 r^{d-1},$$

where $\beta > 0$ is some constant. Substituting this into (11) and setting $p = 1+\frac{3}{2}\alpha$ we can obtain the bound

$$\begin{aligned}
&\int_{\Sigma_{\tau_1}\cap\{r\geq R\}} r^{1+\frac{3}{2}\alpha}(L\psi)^2\,\mathrm{d}r\,\mathrm{d}\mathrm{vol}_{\mathbb{S}^2}\\
&+\int_{\tau_0}^{\tau_1}\left(\int_{\Sigma_\tau\cap\{r\geq R\}}\left((1+\alpha)r^{\frac{3}{2}\alpha}(L\psi)^2+(2-3\alpha)r^{\frac{3}{2}\alpha}|\nabla\!\!\!/\psi|^2+(d-1)(d-3)(2-3\alpha)r^{\frac{3}{2}\alpha-2}|\psi|^2\right)\mathrm{d}r\,\mathrm{d}\mathrm{vol}_{\mathbb{S}^{d-1}}\right)\mathrm{d}\tau\\
&\qquad\lesssim \mathcal{E}^{(N)}[\phi](\tau_0)+\int_{\Sigma_{\tau_0}\cap\{r\geq R\}} r^{1+\frac{3}{2}\alpha}(L\psi)^2\,\mathrm{d}r\,\mathrm{d}\mathrm{vol}_{\mathbb{S}^{d-1}}\\
&\qquad\qquad+\int_{\tau_0}^{\tau_1}\left(\int_{\Sigma_\tau}\left((1+\tau)^{-1-\beta}r^{1+\frac{3}{2}\alpha}|L\psi|^2+C(R)(1+\tau)^{1+\beta}(1+r)^{-1-\frac{1}{2}\alpha}|\underline{L}\phi|^2 r^{d-1}\right)\mathrm{d}r\,\mathrm{d}\mathrm{vol}_{\mathbb{S}^{d-1}}\right)\mathrm{d}\tau.
\end{aligned}$$

The third term on the right-hand side can be estimated by making use of the integrated local energy decay statement, while the second can be controlled using Gronwall's inequality. Note, however, that this third term grows as $(1+\tau_1)^{1+\beta}$. We are led to the inequality

$$\begin{aligned}\int_{\Sigma_{\tau_1}\cap\{r\geq R\}} & r^{1+\frac{3}{2}\alpha}(L\psi)^2\,\mathrm{d}r\,\mathrm{d}\mathrm{vol}_{\mathbb{S}^2}\\ &+\int_{\tau_0}^{\tau_1}\biggl(\int_{\Sigma_\tau\cap\{r\geq R\}}\bigl((1+\alpha)r^{\frac{3}{2}\alpha}(L\psi)^2+(2-3\alpha)r^{\frac{3}{2}\alpha}|\not\nabla\psi|^2+(d-1)(d-3)(2-3\alpha)r^{\frac{3}{2}\alpha-2}|\psi|^2\bigr)\,\mathrm{d}r\,\mathrm{d}\mu_{\mathbb{S}^{d-1}}\biggr)\mathrm{d}\tau\\ &\lesssim(1+\alpha^{-1}C(R))(1+\tau_1)^{1+\beta}\mathcal{E}^{(N)}(\tau_0)+\int_{\Sigma_{\tau_0}\cap\{r\geq R\}}r^{1+\frac{3}{2}\alpha}(L\psi)^2\,\mathrm{d}r\,\mathrm{d}\mathrm{vol}_{\mathbb{S}^{d-1}}.\end{aligned}\tag{14}$$

Interpolating between (14) and (13), and making use of Hölder's inequality we can show

$$\begin{aligned}\int_{\Sigma_{\tau_1}\cap\{r\geq R\}}r(L\psi)^2\,\mathrm{d}r\,\mathrm{d}\mathrm{vol}_{\mathbb{S}^2}+\int_{\tau_0}^{\tau_1}\biggl(\int_{\Sigma_\tau\cap\{r\geq R\}}\bigl((L\psi)^2+|\not\nabla\psi|^2+(d-1)(d-3)r^{-2}|\psi|^2\bigr)\,\mathrm{d}r\,\mathrm{d}\mathrm{vol}_{\mathbb{S}^{d-1}}\biggr)\mathrm{d}\tau\\ \lesssim(1+\alpha^{-1}C(R))(1+\tau_1)^{(1+\beta)(1-\frac{3}{2}\alpha)}\widetilde{\mathcal{E}}^{(N)}_{(1+\frac{3}{2}\alpha)}[\phi](\tau_0),\end{aligned}\tag{15}$$

where we have defined the weighted energy

$$\widetilde{\mathcal{E}}^{(N)}_{(p)}[\phi](\tau):=\widetilde{\mathcal{E}}^{(N)}[\phi](\tau)+\int_{\Sigma_\tau\cap\{r\geq R\}}r^p(L\psi)^2\,\mathrm{d}r\,\mathrm{d}\mathrm{vol}_{\mathbb{S}^2},$$

where, as usual, $\psi=r^{(d-1)/2}\phi$.

Now, if $\widetilde{\mathcal{E}}^{(N)}_{(1+(3/2)\alpha)}[\phi](\tau_0)<\infty$ and if we define

$$\sigma:=1-(1+\beta)\bigl(1-\tfrac{3}{2}\alpha\bigr)$$

then, if $\sigma>0$ (e.g., if $\beta>\frac{3}{2}\alpha$) then we can use (15) to find a dyadic sequence of times τ_i such that the energy satisfies

$$\widetilde{\mathcal{E}}^{(N)}[\phi](\tau_i)\lesssim(1+\tau_i)^{-\sigma}\widetilde{\mathcal{E}}^{(N)}_{(1+\frac{3}{2}\alpha)}[\phi](\tau_0).$$

The energy boundedness assumption (A1) then allows us to drop the restriction to the dyadic sequence, and we find that for all times $\tau\geq\tau_0$ we have polynomial decay of the energy:

$$\mathcal{E}^{(N)}(\tau)[\phi]\lesssim(1+\tau)^{-\sigma}\mathcal{E}^{(N)}_{(1+\frac{3}{2}\alpha)}[\phi](\tau_0).$$

Note that we also have the important corollary: for $\tau\geq\tau_0$,

$$\widetilde{\mathcal{E}}^{(N)}_{(1+\frac{3}{2}\alpha)}(\tau)\lesssim\widetilde{\mathcal{E}}^{(N)}_{(1+\frac{3}{2}\alpha)}[\phi](\tau_0);$$

i.e., the weighted energy at future times is bounded by the weighted energy initially.

In the above sketch, we have shown that the additional, inhomogeneous term arising in the wave equation satisfied by $\bar{\phi}$ on the orbit space $\mathcal{M}/\mathcal{G}$ does not prevent polynomial decay, assuming that an appropriately weighted initial energy is bounded, and also that an integrated local energy decay estimate holds. In the cases we will consider in this paper (as in the general cases studied in [Moschidis 2016a]) this integrated local energy decay estimate will only hold for *bounded* frequencies; nevertheless, by combining this with an energy boundedness statement such as assumption (A1), this is sufficient to conclude decay at a logarithmic rate. In other words, for this argument it is only important to establish polynomial decay for bounded frequency waves: the precise exponent (which depends on the maximum value of p) is not important.

9. Adapted coordinates near an evanescent ergosurface

In this section we will construct coordinates near a point on the evanescent ergosurface. These local coordinates will be used later in order to construct initial data for the wave. We will first do this in the case of an asymptotically flat manifold admitting an evanescent ergosurface of the first kind (ES1), and then show how to adapt this construction to an asymptotically Kaluza–Klein manifold and to an ergosurface of the second kind (ES2). The coordinates we will construct are very similar to "null Fermi coordinates" [Manasse and Misner 1963; Fermi 1922].

As above, let Σ_t be a space-like hypersurface and let $p \in \mathcal{S} \cap \Sigma_t$ be a point on the intersection of the evanescent ergosurface with the hypersurface Σ_t. Let Y be the orthogonal projection of T onto the hypersurface Σ_t. That is, if n is the unit, future-directed normal to Σ_t then we define

$$Y := T + g(T, n)n.$$

Note that, at p, Y is nonvanishing, since T is null here and Σ_t is space-like. We can define a normalised version of Y:

$$\widehat{Y} := \frac{1}{\sqrt{g(Y, Y)}} Y.$$

We complete $\{T, \widehat{Y}\}$ to a form a basis for the tangent space of $\mathcal{M}$ at the point p. We can choose these other vectors to be mutually orthogonal, normalised, and also orthogonal to both Y and T. In other words, we have some set of vectors $e_a \in \mathcal{T}_p(\mathcal{M})$ such that

$$g(e_a, e_b) = \delta_{ab}, \quad g(e_a, \widehat{Y}) = 0, \quad g(e_a, T) = 0.$$

Note that these vectors are orthogonal to both T and Y, and the normal to the hypersurface Σ_t at the point p is proportional to $T - Y$. Hence these vectors are tangent to the hypersurface Σ_t at the point p.

Now, we consider an affinely parametrised geodesic $\beta_{(y,x^a)}(s)$ originating at the point p and with initial tangent vector $y\widehat{Y} + x^a e_a$. That is, the geodesic $\beta_{(y,x^a)}(s)$ satisfies

$$\begin{aligned} \beta_{(y,x^a)}(0) &= p, \\ \frac{\partial}{\partial s}\beta_{(y,x^a)}(s)\Big|_{s=0} &= y\widehat{Y} + x^a e_a. \end{aligned}$$

We can use the coordinates (sy, sx^a) to label the point reached along this geodesic $\beta_{(y,x^a)}(s)$ after an affine distance s. Note that, since $\{Y, e_a\}$ do *not* span the tangent space of $\mathcal{M}$ at the point p, we cannot yet use these coordinates in a neighbourhood of p. However, we can define the set

$$S_p := \{q \in \mathcal{M} \mid \text{there exists } s, y, x^a \in \mathbb{R} \text{ such that } q = \beta_{(y,x^a)}(s)\}.$$

This set is locally a smooth hypersurface near the point p. In other words, for any sufficiently small neighbourhood of p, the restriction of S_p to this neighbourhood defines a smooth hypersurface in the

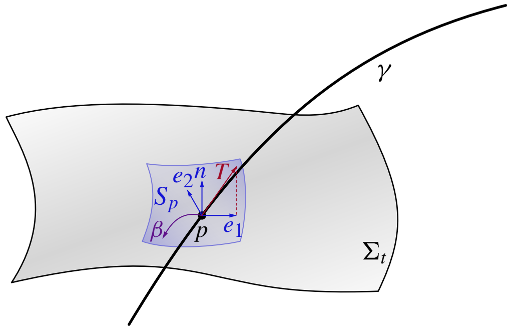

Figure 3. A sketch showing our construction of coordinates adapted to the ergosurface. Here, p is a point on the ergosurface and on the hypersurface Σ_t. The geodesic γ is an integral curve of the Killing vector field T (which should be replaced by V in the asymptotically Kaluza–Klein case). The vector n is the future-directed normal to Σ_t, and e_1 is a unit space-like vector at p in the direction of the orthogonal projection of T onto Σ_t. The vector e_2 is another unit vector at p, tangent to Σ_t and orthogonal to e_1. The purple curve β is a geodesic, with tangent at p which is in the span of $\{e_1, e_2\}$. The blue surface S_p is the surface consisting of the union of all such curves in a neighbourhood of p. Note that the two hypersurfaces S_p and Σ_t are tangent at p but do not, in general, agree away from p.

neighbourhood. Moreover, if this neighbourhood is sufficiently small, then the vector field T is transverse to S_p.

We recall that, since $p \in \mathcal{S}$, there is an affinely parametrised geodesic γ through p with tangent T. We will now define coordinates in a neighbourhood of γ. First, we define a function $\tilde{t}$ by the conditions

$$\tilde{t}|_{S_p} = 0, \quad T(\tilde{t}) = 1.$$

Next, we extend the coordinates (y, x^a) off the hypersurface S_p by imposing the condition

$$T(y) = T(x^a) = 0.$$

Note that the isometry generated by T preserves distances, so by this process we are able to obtain coordinates for a local neighbourhood of the entire geodesic γ. See Figure 3 for a sketch of this construction.

We note that, in these coordinates, we have

$$T = \partial_{\tilde{t}}.$$

In addition, *at the point* p we have

$$\partial_y = \widehat{Y}, \quad \partial_a = e_a.$$

Hence, if we define

$$A := g(T, \widehat{Y})$$

then *at the point* p the metric and its inverse are given by

$$\begin{aligned} g|_p &= 2A\, \mathrm{d}\tilde{t}\, \mathrm{d}y + \mathrm{d}y^2 + \delta_{ab}\, \mathrm{d}x^a\, \mathrm{d}x^b, \\ g^{-1}|_p &= -A^{-2}\partial_{\tilde{t}}^2 + 2A^{-1}\partial_{\tilde{t}}\partial_y + \delta^{ab}\partial_a\partial_b. \end{aligned} \tag{16}$$

Now, by construction, the curve with coordinates $(\tilde{t}, y, x^a) = (0, sy_0, sx_0^a)$ for *arbitrary constants* y_0 and x_0^a is a geodesic with affine parameter s. Since T is a Killing vector field, the curve with coordinates (t_0, sy_0, sx_0^a) for constants t_0, sy_0, sx_0^a is also a geodesic. Hence, from the geodesic equation we obtain

$$\begin{aligned}
\Gamma^{\tilde{t}}_{yy}|_\gamma &= \Gamma^{\tilde{t}}_{ya}|_\gamma = \Gamma^{\tilde{t}}_{ab}|_\gamma = 0,\\
\Gamma^{y}_{yy}|_\gamma &= \Gamma^{y}_{ya}|_\gamma = \Gamma^{y}_{ab}|_\gamma = 0,\\
\Gamma^{a}_{yy}|_\gamma &= \Gamma^{a}_{yb}|_\gamma = \Gamma^{a}_{bc}|_\gamma = 0.
\end{aligned}$$

Similarly, since the curve with coordinates $(s, 0, 0)$ is an affinely parametrised geodesic we find

$$\Gamma^{\tilde{t}}_{\tilde{t}\tilde{t}}|_\gamma = \Gamma^{y}_{\tilde{t}\tilde{t}}|_\gamma = \Gamma^{a}_{\tilde{t}\tilde{t}}|_\gamma = 0.$$

If it were the case that all the Christoffel symbols of g vanish along γ, then we would be working in "null Fermi coordinates" adapted to the geodesic γ [Manasse and Misner 1963]. However, this is not the case for the coordinates defined above: specifically, we cannot guarantee that all Christoffel symbols with mixed spatial and time indices vanish along γ. However note that at the point p we have

$$\nabla_y(\partial_{\tilde{t}})|_p = \nabla_y T|_p = \Gamma^{\tilde{t}}_{y\tilde{t}}|_p T + \Gamma^{y}_{y\tilde{t}}|_p Y + \Gamma^{a}_{y\tilde{t}}|_p e_a,$$

where we note that, although the vectors Y and e_a are only defined at the point p, the vector field T is defined globally.

Taking an inner product with T at the point p and using the expression for the metric (16) we find

$$g(\nabla_y T, T)|_p = A\Gamma^{y}_{y\tilde{t}}|_p.$$

However, the left-hand side is given by

$$g(\nabla_y T, T) = \tfrac{1}{2}\partial_y(g(T, T))$$

and $g(T, T)$ vanishes at least quadratically on the evanescent ergosurface (and so, in particular, at p). We conclude that

$$\Gamma^{y}_{y\tilde{t}}|_p = 0.$$

We can also compute

$$g(\nabla_T \partial_y, \partial_y)|_p = A\Gamma^{\tilde{t}}_{y\tilde{t}}|_p + \Gamma^{y}_{y\tilde{t}}|_p.$$

The second term on the right-hand side vanishes, as we have already seen. On the other hand, the left-hand side is given by

$$g(\nabla_T \partial_y, \partial_y) = \tfrac{1}{2} T g(\partial_y, \partial_y) = g([T, \partial_y], \partial_y) = 0,$$

where we have used the fact that $\mathcal{L}_T g = 0$ since T is a Killing vector field, as well as the fact that $[T, \partial_y] = [\partial_{\tilde{t}}, \partial_y] = 0$. Hence we find that

$$\Gamma^{\tilde{t}}_{yt}|_p = 0.$$

Similarly, by using the vector fields ∂_a in place of ∂_y we can also find that

$$\Gamma^{y}_{a\tilde{t}}|_p = 0, \quad \Gamma^{a}_{a\tilde{t}}|_p = 0,$$

where in the second line there is *no summation* over the index a.

To summarise the above calculations, we find that, with our choice of coordinates, the only Christoffel symbols which can be nonzero at p (and hence along γ) are

$$\Gamma^{\tilde{t}}_{\tilde{t}a}, \quad \Gamma^{a}_{\tilde{t}y}, \quad \Gamma^{a}_{by},$$

where $a \neq b$.

We now define

$$|\tilde{x}| := \sqrt{y^2 + \sum_a (x^a)^2}$$

and we also give labels to certain metric components: we define

$$\begin{aligned} g(T, T) = g_{\tilde{t}\tilde{t}} &:= a, \\ g(T, \partial_a) = g_{\tilde{t}a} &:= b_a, \end{aligned}$$

and we note that

$$\begin{aligned} |a| &= \mathcal{O}(|\tilde{x}|^2), \\ |b| &= \mathcal{O}(|\tilde{x}|), \end{aligned}$$

where we have defined

$$|b| := \sqrt{\sum_a (b_a)^2}.$$

Finally, we note that, since T is time-like away from the evanescent ergosurface, we have

$$a \leq 0.$$

Putting together the calculations above, we conclude that the metric near the point p is given by

$$\begin{aligned} g = 2A \, \mathrm{d}\tilde{t} \, \mathrm{d}y + \mathrm{d}y^2 + \delta_{ab} \, \mathrm{d}x^a \, \mathrm{d}x^b + a \, \mathrm{d}\tilde{t}^2 + b_a \, \mathrm{d}\tilde{t} \, \mathrm{d}x^a + \mathcal{O}(|\tilde{x}|) \, \mathrm{d}\tilde{t} \, \mathrm{d}y + \mathcal{O}(|\tilde{x}|^2) \, \mathrm{d}y^2 \\ + \mathcal{O}(|\tilde{x}|^2) \, \mathrm{d}y \, \mathrm{d}x^a + \mathcal{O}(|\tilde{x}|^2) \, \mathrm{d}x^a \, \mathrm{d}x^b. \end{aligned}$$

Consequently, the inverse metric is given by

$$\begin{aligned} g^{-1} = A^{-2}\partial_{\tilde{t}}^2 + 2A^{-1}\partial_{\tilde{t}}\partial_y + \delta^{ab}\partial_a\partial_b + \mathcal{O}(|\tilde{x}|)\partial_{\tilde{t}}^2 + \mathcal{O}(|\tilde{x}|)\partial_{\tilde{t}}\partial_y + \mathcal{O}(|\tilde{x}|)\partial_{\tilde{t}}\partial_a + \mathcal{O}(|\tilde{x}|)\partial_y\partial_a \\ + (A^{-2}|b|^2 - A^{-2}a + \mathcal{O}(|\tilde{x}|^3))\partial_y^2 + \mathcal{O}(|\tilde{x}|^2)\partial_a\partial_b. \end{aligned}$$

Finally, again making use of the expressions for the Christoffel symbols at the point p, we find that the wave operator can be expressed as

$$\begin{aligned} \Box_g u = -A^{-2}\partial_{\tilde{t}}^2 u + 2A^{-1}\partial_{\tilde{t}}\partial_y u + \delta^{ab}\partial_a\partial_b u + A^{-2}(|b|^2 - a)\partial_y^2 u + \mathcal{O}(|\tilde{x}||\partial u|) + \mathcal{O}(|\tilde{x}||\partial T u|) \\ + \mathcal{O}(|\tilde{x}|^2|\bar{\partial}\partial u|) + \mathcal{O}(|\tilde{x}|^3|\partial^2 u|), \quad (17) \end{aligned}$$

where we have defined

$$\begin{aligned} |\partial u| &:= \sqrt{|\partial_{\tilde{t}} u|^2 + |\partial_y u|^2 + \sum_a |\partial_a u|^2}, \\ |\bar{\partial} u| &:= \sqrt{|\partial_{\tilde{t}} u|^2 + \sum_a |\partial_a u|^2}. \end{aligned}$$

We now explain how to adapt this construction to the asymptotically Kaluza–Klein case, with an evanescent ergosurface of the second kind (ES2). In this case, we once again take a point $p \in \mathcal{S} \cap \Sigma_t$. We now define a linear subspace of the tangent space of p

$$G_p \subset \mathcal{T}_p(\mathcal{M}) := \{X \in \mathcal{T}_p(\mathcal{M}) \mid X = L_A \text{ for some } A \in \mathfrak{g}\}.$$

Note that, since the group $\mathcal{G}$ does not necessarily act freely, this subspace is not necessarily isomorphic to $\mathfrak{g}$. Indeed, if p is a fixed point of $\mathcal{G}$, then G_p is trivial.

Note that V is orthogonal to G_p in the sense that

$$g(V, X) = 0 \quad \text{for all } X \in G_p,$$

which follows from the definition of an evanescent ergosurface of the second kind. Consequently, the vectors X must either be space-like or proportional to V. However, if a vector $X \in G_p$ were proportional to V then, since V generates an action of $\mathbb{R}$ on $\mathcal{M}$ by isometries, $\mathcal{G}$ would have a subgroup isomorphic to $\mathbb{R}$. But this is impossible, since $\mathcal{G}$ is a compact Lie group. Hence the vectors X must all be space-like.

Analogously to the previous case, we now define

$$Y := V + g(n, V)n$$

as the orthogonal projection of V onto the hypersurface Σ_t at the point p. Note that Y is nonvanishing, since V is null at p and Σ_t is space-like. Again, we can define the normalised version of Y as

$$\widehat{Y} := \frac{1}{\sqrt{g(Y, Y)}} Y.$$

Note that $Y \notin G_p$ (and hence $\widehat{Y} \notin G_p$) because Y is *not* orthogonal to V. Again, this follows from the fact that V is null at p and Σ_t is space-like.

We now take an orthonormal basis for G_p, which we shall label as e_A. Note that this is possible since G_p is space-like. We now complete the set $\{V, \widehat{Y}, e_A\}$ to form a basis for $\mathcal{T}_p(\mathcal{M})$ by adding some vectors e_a, which are chosen to satisfy

$$g(e_a, e_b) = \delta_{ab}, \quad g(e_a, V) = 0, \quad g(e_a, \widehat{Y}) = 0, \quad g(e_a, e_A) = 0.$$

We note here that we do not necessarily have $g(e_A, \widehat{Y})|_p = 0$.

Now we repeat the previous construction to define coordinates. We first consider an affinely parametrised geodesic $\beta_{(y,x^a,z^A)}(s)$ originating at the point p and with initial tangent vector $y\widehat{Y} + x^a e_a + z^A e_A$. So, the geodesic $\beta_{(y,x^a,z^A)}(s)$ satisfies

$$\beta_{(y,x^a,z^A)}(0) = p,$$
$$\frac{\partial}{\partial s}\beta_{(y,x^a,z^A)}(s)\Big|_{s=0} = y\widehat{Y} + x^a e_a + z^A e_A.$$

We can use the coordinates (sy, sx^a, sz^A) to label the point reached along this geodesic $\beta_{(y,x^a,z^A)}(s)$ after an affine distance s. We define the set

$$S_p := \{q \in \mathcal{M} \mid \text{there exists } s, y, x^a, z^A \in \mathbb{R} \text{ such that } q = \beta_{(y,x^a,z^A)}(s)\}.$$

As before, this set is locally a smooth hypersurface near the point p. We now extend these coordinates off this hypersurface by defining

$$\tilde{t}|_{S_p} = 0, \quad V(\tilde{t}) = 1, \quad V(y) = V(x^a) = V(z^A) = 0.$$

This gives us local coordinates near the geodesic γ through p with tangent V.

Note that, although the vector fields ∂_A are tangent to the generators of the group action $\mathcal{G}$ *at* p, they do not necessarily remain so. Note also that, in these coordinates, we have

$$V = \partial_{\tilde{t}}.$$

In analogy to the previous case we define

$$A := g(V, \widehat{Y})|_p.$$

We also define

$$B_A := g(\widehat{Y}, e_A)|_p.$$

Now, the metric at the point p can be expressed as

$$g|_p = 2A\, \mathrm{d}\tilde{t}\, \mathrm{d}y + 2B_A\, \mathrm{d}\tilde{t}\, \mathrm{d}z^A + \mathrm{d}y^2 + \delta_{ab}\, \mathrm{d}x^a\, \mathrm{d}x^b + \delta_{AB}\, \mathrm{d}z^A\, \mathrm{d}z^B$$

and the inverse metric is

$$g^{-1}|_p = -A^{-2}(1 - |B|^2)\partial_{\tilde{t}}^2 + 2A^{-1}\partial_{\tilde{t}}\partial_y - 2A^{-1}B^A\partial_{\tilde{t}}\partial_A + \delta^{AB}\partial_A\partial_B + \delta^{ab}\partial_a\partial_b,$$

where we lower and raise the indices $A, B, \ldots$ using the Euclidean metric δ_{AB} and its inverse, and we have defined $|B|^2 = B^A B_A$. Note that, since $\widehat{Y}$ and e_A are space-like, unit vectors, we have $|B|^2 \leq 1$. Moreover, we cannot have $|B|^2 = 1$ because this would imply that $Y \in G_p$, which, as we have seen above, is impossible. Hence $|B|^2 < 1$, and so the coefficient of $\partial_{\tilde{t}}^2$ in the inverse metric is strictly negative.

As before, using the fact that the curve with coordinates

$$(\tilde{t}, y, x^a, z^A) = (\tilde{t}_0, sy_0, sx_0^a, sz_0^A)$$

is a geodesic with affine parameter s, we have that many of the Christoffel symbols vanish. Specifically,

$$\begin{aligned}
\Gamma^{\tilde{t}}_{yy}|_\gamma = \Gamma^{\tilde{t}}_{ya}|_\gamma = \Gamma^{\tilde{t}}_{yA}|_\gamma = \Gamma^{\tilde{t}}_{ab}|_\gamma = \Gamma^{\tilde{t}}_{aA}|_\gamma = \Gamma^{\tilde{t}}_{AB}|_\gamma = 0,\\
\Gamma^{y}_{yy}|_\gamma = \Gamma^{y}_{ya}|_\gamma = \Gamma^{y}_{yA}|_\gamma = \Gamma^{y}_{ab}|_\gamma = \Gamma^{y}_{aA}|_\gamma = \Gamma^{y}_{AB}|_\gamma = 0,\\
\Gamma^{a}_{yy}|_\gamma = \Gamma^{a}_{yb}|_\gamma = \Gamma^{a}_{yA}|_\gamma = \Gamma^{a}_{bc}|_\gamma = \Gamma^{a}_{bA}|_\gamma = \Gamma^{a}_{AB}|_\gamma = 0,\\
\Gamma^{A}_{yy}|_\gamma = \Gamma^{A}_{ya}|_\gamma = \Gamma^{A}_{yB}|_\gamma = \Gamma^{A}_{ab}|_\gamma = \Gamma^{A}_{aB}|_\gamma = \Gamma^{A}_{BC}|_\gamma = 0.
\end{aligned}$$

In addition, the curve with coordinates $(s, 0, 0, 0)$ is an affinely parametrised geodesic with affine parameter s. Hence

$$\Gamma^{\tilde{t}}_{\tilde{t}\tilde{t}} = \Gamma^{y}_{\tilde{t}\tilde{t}} = \Gamma^{a}_{\tilde{t}\tilde{t}} = \Gamma^{A}_{\tilde{t}\tilde{t}} = 0.$$

Finally, we note that we have the expression

$$\nabla_{\partial_y}\partial_{\tilde{t}} = \Gamma^{\tilde{t}}_{y\tilde{t}}\partial_{\tilde{t}} + \Gamma^{y}_{y\tilde{t}}\partial_y + \Gamma^{a}_{y\tilde{t}}\partial_a + \Gamma^{A}_{y\tilde{t}}\partial_A.$$

Noting that, at the point p we have $\partial_t = V,\ \partial_y = \widehat{Y},\ \partial_a = e_a$ and $\partial_A = e_A$, we can evaluate the expression above at the point p and then take the inner product with V. We find

$$g(\nabla_{\partial_y} V, V) = \tfrac{1}{2}\partial_y g(V, V) = A\Gamma^{y}_{y\tilde{t}}.$$

Now, since $A \neq 0$ and $g(V, V)$ vanishes to (at least) second order on the evanescent ergosurface, we also conclude that

$$\Gamma^{y}_{y\tilde{t}}|_p = 0.$$

Similarly, by considering $g(\nabla_a V, V)$ and $g(\nabla_A V, V)$ we conclude that

$$\Gamma^{y}_{a\tilde{t}}|_p = \Gamma^{y}_{A\tilde{t}}|_p = 0.$$

Defining now

$$\begin{aligned} |\tilde{x}| &= \sqrt{y^2 + \sum_a (x^a)^2 + \sum_A (z^A)^2}, \\ a &:= g(V, V) = g_{\tilde{t}\tilde{t}} = \mathcal{O}(|\tilde{x}|^2), \\ b_a &:= g(V, \partial_a) = g_{\tilde{t}a} = \mathcal{O}(|\tilde{x}|), \\ c_A &:= g(V, \partial_A) = g_{\tilde{t}A} = \mathcal{O}(|\tilde{x}|), \end{aligned}$$

where, since V is globally causal, we have $a \leq 0$.

We find that the metric near the point p can be expressed as

$$\begin{aligned} g = {} & 2A\,\mathrm{d}\tilde{t}\,\mathrm{d}y + 2B_A\,\mathrm{d}\tilde{t}\,\mathrm{d}z^A + \mathrm{d}y^2 + \delta_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b + \delta_{AB}\,\mathrm{d}z^A\,\mathrm{d}z^B + a\,\mathrm{d}\tilde{t}^2 + b_a\,\mathrm{d}\tilde{t}\,\mathrm{d}x^a + c_A\,\mathrm{d}\tilde{t}\,\mathrm{d}z^A \\ & + \mathcal{O}(|\tilde{x}|)\,\mathrm{d}\tilde{t}\,\mathrm{d}y + \mathcal{O}(|\tilde{x}|^2)\,\mathrm{d}y^2 + \mathcal{O}(|\tilde{x}|^2)\,\mathrm{d}y\,\mathrm{d}x^a + \mathcal{O}(|\tilde{x}|^2)\,\mathrm{d}y\,\mathrm{d}z^A + \mathcal{O}(|\tilde{x}|^2)\,\mathrm{d}x^a\,\mathrm{d}x^b \\ & + \mathcal{O}(|\tilde{x}|^2)\,\mathrm{d}x^a\,\mathrm{d}z^A + \mathcal{O}(|\tilde{x}|^2)\,\mathrm{d}z^A\,\mathrm{d}z^B, \end{aligned}$$

and the inverse metric can be expressed as

$$\begin{aligned} g^{-1} = {} & -A^{-2}(1 - |B|^2)\partial_{\tilde{t}}^2 + 2A^{-1}\partial_{\tilde{t}}\partial_y - 2A^{-1}B^A\partial_{\tilde{t}}\partial_A + \delta^{AB}\partial_A\partial_B + \delta^{ab}\partial_a\partial_b \\ & + A^{-2}(|b|^2 + |c|^2 - a)\partial_y^2 + \mathcal{O}(|\tilde{x}|)\partial_{\tilde{t}}^2 + \mathcal{O}(|\tilde{x}|)\partial_{\tilde{t}}\partial_y + \mathcal{O}(|\tilde{x}|)\partial_{\tilde{t}}\partial_a + \mathcal{O}(|\tilde{x}|)\partial_{\tilde{t}}\partial_A + \mathcal{O}(|\tilde{x}|^3)\partial_y^2 \\ & + \mathcal{O}(|\tilde{x}|)\partial_y\partial_a + \mathcal{O}(|\tilde{x}|)\partial_y\partial_A + \mathcal{O}(|\tilde{x}|^2)\partial_a\partial_b + \mathcal{O}(|\tilde{x}|)\partial_a\partial_A + \mathcal{O}(|\tilde{x}|)\partial_A\partial_B. \end{aligned} \tag{18}$$

Finally, we note that the wave equation can be expressed as

$$\begin{aligned} \Box_g u = {} & -A^{-2}(1 - |B|^2)\partial_{\tilde{t}}^2 u + 2A^{-1}\partial_{\tilde{t}}\partial_y u - 2A^{-1}B^A\partial_{\tilde{t}}\partial_A u + \delta^{AB}\partial_A\partial_B u + \delta^{ab}\partial_a\partial_b u \\ & + A^{-2}(|b|^2 + |c|^2 - a)\partial_y^2 u + \mathcal{O}(|\tilde{x}||\partial V u|) + \mathcal{O}(|\tilde{x}^2||\bar{\partial}\partial u|) + \mathcal{O}(|\tilde{x}||\partial u|) + \mathcal{O}(|\tilde{x}|^3|\partial^2 u|). \end{aligned} \tag{19}$$

10. Instability in the general case, without additional symmetry assumptions

The purpose of this section is to establish the existence of some kind of linear “instability” in the general case of a spacetime with an evanescent ergosurface. As above, we shall only consider spacetimes which are asymptotically flat or asymptotically Kaluza–Klein, and which do not have event horizons.

As explained in the Introduction, we allow for two different types of instability, which we refer to as case (A) and case (B). So far, we have only given a “rough” version of the statement referring to these instabilities. Here, we state the primary theorem of this paper, and in doing so we make precise the two kinds of instabilities which may be present.

Theorem 10.1 (evanescent ergosurface instability, general case). *Let $(\mathcal{M}, g)$ be a smooth, Lorentzian manifold which is stationary and either asymptotically flat in the sense of Section 3A, or asymptotically Kaluza–Klein in the sense of Section 3B. Suppose that the manifold possesses an evanescent ergosurface in the sense of* (ES1) (*in the asymptotically flat case*) *or* (ES2) (*in the asymptotically Kaluza–Klein case*). *Finally, suppose that the manifold possesses a discrete symmetry $\mathcal{I}$ as in Section 5.*

Then at least one of the following applies:

(A) *For any $C > 0$, and any open set $\mathcal{U}_0 \subset \Sigma_0$ such that $(\mathcal{S} \cap \Sigma_0) \subset \mathcal{U}_0$, there exists a solution ϕ to the wave equation $\Box_g \phi = 0$ arising from smooth, compactly supported* (*and $\mathcal{G}$-invariant, in the Kaluza–Klein case*) *data, and a time τ such that*
$$\frac{\mathcal{E}^{(N)}_{\mathcal{U}}[\phi_C](\tau)}{\mathcal{E}^{(N)}[\phi_C](0)} \geq C,$$
where we recall that $\mathcal{E}^{(N)}_{\mathcal{U}}[\phi_C](\tau)$ measures the nondegenerate energy of the wave ϕ_C in the set $\mathcal{U}_\tau$, which is the time translate of the set $\mathcal{U}_0$ onto the surface Σ_τ.

(B) *For any open set $\mathcal{U}_0 \subset \Sigma_0$ with $(\mathcal{S} \cap \Sigma_0) \subset \mathcal{U}_0$, there exists some constant $\mathring{C} > 0$ and a solution ϕ to the wave equation $\Box_g \phi = 0$, arising from smooth, compactly supported data, such that for **all** times $\tau > 0$, we have*
$$\frac{\mathcal{E}^{(N)}_{\mathcal{U}}[\phi_C](\tau)}{\mathcal{E}^{(N)}[\phi_C](0)} \geq \mathring{C}.$$
Furthermore we have the following pointwise blowup behaviour: *there exists a constant $c > 0$ such that, given any set $\mathcal{U}_\epsilon \subset \Sigma_0$ with $\mathcal{S} \cap \mathcal{U}_\epsilon \neq \varnothing$ and* $\mathrm{Volume}(\mathcal{U}_\epsilon) = \epsilon$, *there exists a solution ϕ to the wave equation $\Box_g \phi = 0$ and a time τ such that*
$$\|Y\phi\|_{L^\infty[\mathcal{U}_\epsilon](\tau)} \geq \frac{c}{\epsilon}.$$

In this general setting we cannot establish many details of the instability. Note that the two possible behaviours are not mutually exclusive. Note also that, since we will argue by contradiction, we will not obtain any details of the initial data which gives rise to these instabilities, other than smoothness and compact support: in particular, our proof is not constructive. Finally, we remark that although we have several examples of manifolds giving rise to the behaviour of case (A) (for example, supersymmetric microstate geometries), we do not have an example of a manifold giving rise to the behaviour of case (B).

This is in marked contrast to the situation in which an additional symmetry is present, discussed in Section 11 below. In that case, we can rule out case (B), and explicitly construct data giving rise to the behaviour in case (A). Moreover, we can also establish some bounds on the time at which the local energy becomes large, and the required support of the initial data. Finally, when this additional symmetry is present we can also construct (possibly nonsmooth, and noncompactly supported) initial data such that the local energy of the resulting solution is actually *unbounded* in time.

The structure of the proof of Theorem 10.1 is a little convoluted, so for clarity we outline it below:

(1) We begin by assuming that case (A) does *not* hold, that is, we assume that the local energy is bounded by some multiple of its initial value. This will be referred to as a *nondegenerate energy boundedness* statement.

(2) There are now two possibilities: either the local energy of all suitable waves *decays* over time, or it does not.

   (a) If the local energy does not decay, then we are led to an Aretakis-type instability and case (B).

   (b) On the other hand, if the local energy *does* decay, then we can construct initial data for a wave whose local energy is amplified by an arbitrarily large factor, i.e., case (A). This contradicts the boundedness assumption made in step (1) above. See Figure 1 for an overview of the construction of the instability in this case.

(3) From the above argument we see that if (A) does not hold, then we must have (B).

(4) Hence we must have at least one of (A) or (B).

10A. ***The nondegenerate energy boundedness assumption.*** In order to make progress we will need to assume a suitable *nondegenerate energy boundedness* inequality holds, which is essentially[16] the negation of the statement of case (A). Note, however, that this is *not* an assumption which limits the scope of the theorem: if this assumption does not hold, then we can show that we have an instability of type (A). On the other hand, after making this assumption, we will be able to show that a consequence of this assumption is an Aretakis-type instability in the general case. Note that, in Section 11 we will show that the nondegenerate energy boundedness inequality assumption leads directly to a contradiction in the case where an additional symmetry is present.

To make the statement precise, we make the following assumption:

Assumption (A1). *There exists some constant* $C^{(N)} > 0$ *such that, for all* ($\mathcal{G}$*-invariant*) *solutions* ϕ *to the linear wave equation* $\Box_g \phi = 0$ *and all* $t \in \mathbb{R}$,

$$\mathcal{E}^{(N)}[\phi](t) \le C^{(N)} \mathcal{E}^{(N)}[\phi](0).$$

Note that, if N were a Killing field, then it would be easy to verify (A1) using the energy estimate (4) associated with N. Likewise, if there were to exist a uniformly time-like Killing vector field, then it would be easy to verify assumption (A1), even if N is not chosen to be this Killing vector field.[17] However, the geometries we are studying only possess a globally causal (and not globally time-like!) Killing field. Thus we cannot straightforwardly verify assumption (A1), and indeed, in some cases it can lead to a contradiction. For now, we shall proceed, making the assumption (A1).

[16]Technically, the negation of the statement of case (A) only entails that there exists some open set $\mathcal{U}_0$, which includes the ergosurface, such that a nondegenerate energy boundedness statement holds in that region. However, away from the ergoregion, the conservation of the T energy already gives the required bound.

[17]In this case, the energy associated with N will not generally be conserved; however, it will still remain bounded: the energy associated with N and the energy associated with the time-like Killing vector field provide equivalent norms.

10B. ***Local energy decay away from the evanescent ergosurface.*** The results of [Moschidis 2018] imply the decay of the local energy of waves away from the evanescent ergosurface, assuming a boundedness statement of the form (A1) holds. To be precise, the following proposition is a very slight adaptation of Proposition 4.1 of [Moschidis 2018], making use of the comments above regarding asymptotically Kaluza–Klein manifolds.

We first need to define (in analogy to the "extended ergoregions" of [Moschidis 2018]) the *extended ergosurface*:

Definition 10.2 (the extended ergosurface). Suppose that $(\mathcal{M} \setminus \mathcal{S})$ consists of a number of connected components, which we can separate into two types: those that include an asymptotic region, and those that do not. For simplicity, suppose that there is only one component which includes an asymptotic region. We label this region $\mathcal{M}_{(\mathrm{outer})}$. Then we define the extended ergosurface

$$\mathcal{S}_{(\mathrm{ext})} := \mathcal{M} \setminus \mathcal{M}_{(\mathrm{outer})}.$$

Proposition 10.3. *Let $\mathcal{M}$ be an asymptotically flat or asymptotically Kaluza–Klein manifold with an evanescent ergosurface in the sense of* (ES1), *or an asymptotically Kaluza–Klein manifold with an evanescent ergosurface in the sense of* (ES2). *Let ϕ be a smooth* ($\mathcal{G}$*-invariant*) *solution to $\Box_g \phi = 0$ arising from compactly supported initial data. Suppose in addition that the initial energy of ϕ and its first three T derivatives is finite*; *that is*,

$$\sum_{j=0}^{3} \mathcal{E}^{(N)}[T^j \phi](0) < \infty.$$

Finally, *suppose that the boundedness statement* (A1) *holds.*

Then, *for any $\delta > 0$ let $\mathcal{U}_0 \subset \Sigma_0$ be any compact set such that the distance*[18] *from $\mathcal{U}_0$ to $\mathcal{S}_{(\mathrm{ext})} \cap \Sigma_0$ is at least δ. Then*, *for any $\epsilon > 0$ there is a time $\tau > 0$ such that*

$$\mathcal{E}^{(N)}_{\mathcal{U}}[T\phi](\tau) + \mathcal{E}^{(N)}_{\mathcal{U}}[T^2\phi](\tau) < \epsilon. \tag{20}$$

Note that this proposition actually follows from an application of the mean value theorem to the proposition given in [Moschidis 2018], which establishes a very similar inequality for an *integrated* energy quantity. However, we have chosen to present the proposition in the form which will be most useful for our purposes.

Note also that the distance to the extended evanescent ergosurface, δ, can be chosen to be as small as we like, although the time τ taken to decay will depend on δ. Hence, this proposition establishes decay of the local energy everywhere *away from the ergosurface*. This will play an important role in our argument for instability. Indeed, if this decay can be extended to cover the ergoregion as well, then we will find a contradiction with the boundedness assumption (A1). On the other hand, if this decay cannot be extended to the ergosurface, then we are faced with a situation in which the energy decays everywhere except for on the ergosurface. In this case, an instability of a very similar kind to that encountered in extremal black holes is present.

[18] The distance can be measured using the induced Riemannian metric on Σ_0.

10C. ***Local energy decay "inside" the evanescent ergosurface.*** In situations where the evanescent ergosurface divides the manifold $\mathcal{M}$ into an "inside" and an "outside", we also need to establish energy decay "inside" the ergosurface. In other words, we need to establish decay in the set $\mathcal{S}_{(\mathrm{ext})} \setminus \mathcal{S}$, if this is nonempty.

Here, we can use Lemma 4.2 of [Moschidis 2018], which we quote[19] here for convenience:

Lemma 10.4 [Moschidis 2018, Lemma 4.2]. *Let ϕ be a smooth solution to $\Box_g \phi = 0$ arising from compactly supported initial data, such that*

$$\sum_{j=0}^{3} \mathcal{E}^{(N)}[T^j \phi](0) < \infty.$$

Define the function $\psi_\tau : \mathcal{M} \to \mathbb{R}$ as

$$\psi_\tau(t, x) := \begin{cases} T\phi(t+\tau, x), & t \geq -\tau, \\ 0, & t < \tau. \end{cases}$$

Then, there exists an increasing sequence $\{\tau_n\}_{n\in\mathbb{N}}$ of nonnegative integers and a function $\tilde{\psi}$, with $\tilde{\psi}, T\tilde{\psi} \in H^1_{\mathrm{loc}}(\mathcal{M})$ such that $\Box_g \tilde{\psi} = 0$ and

$$\int_{-\tau_*}^{\tau_*} (\mathcal{E}^{(N)}[\tilde{\psi}](\tau) + \mathcal{E}^{(N)}[T\tilde{\psi}])\,\mathrm{d}\tau < \infty \quad \textit{for any } \tau_* > 0, \tag{21}$$

and also

$$\tilde{\psi} \equiv 0 \quad \textit{on } \mathcal{M} \setminus \mathcal{S}_{(\mathrm{ext})}. \tag{22}$$

Moreover, $(T\phi_{\tau_n}, T^2\phi_{\tau_n}) \to (\tilde{\psi}, T\tilde{\psi})$ weakly in $H^1_{(\mathrm{loc})}(\mathcal{M}) \times H^1_{(\mathrm{loc})}(\mathcal{M})$, strongly in $H^1_{(\mathrm{loc})}(\mathcal{M} \setminus \mathcal{S}_{\mathrm{ext}}) \times H^1_{(\mathrm{loc})}(\mathcal{M} \setminus \mathcal{S}_{\mathrm{ext}})$ and strongly in $L^2_{(\mathrm{loc})}(\mathcal{M}) \times L^2_{(\mathrm{loc})}(\mathcal{M})$ in the following sense:

- *For any compactly supported test functions $\{\zeta_j\}_{j=0,1} \in L^2(\mathcal{M})$ and compactly supported vector fields $\{X_j\}_{j=0,1}$ on $\mathcal{M}$ such that $|X_j|_{g_{(\mathrm{ref})}} \in L^2(\mathcal{M})$*
$$\lim_{n\to\infty} \sum_{j=0}^{1} \int_{\mathcal{M}} \big(g_{(\mathrm{ref})}(\nabla(T^j\psi_{\tau_n} - T^j\tilde{\psi}, X_j)) + (T^j\psi_{\tau_n} - T^j\tilde{\psi})\zeta_j \big)\,\mathrm{d}\mathrm{vol} = 0.$$
- *For any compact subset $\mathcal{K} \subset \mathcal{M}$ and any $\delta > 0$*
$$\lim_{n\to\infty} \left(\sum_{j=0}^{1} \int_{\mathcal{K}} |T^j\psi_{\tau_n} - T^j\tilde{\psi}|^2\,\mathrm{d}\mathrm{vol} + \sum_{j=0}^{1} \int_{\mathcal{K}\setminus(\mathcal{S}_{(\mathrm{ext})})_{(\delta)}} |\nabla(T^j\psi_{\tau_n}) - \nabla(T^j\tilde{\psi})|^2_{g_{(\mathrm{ref})}}\,\mathrm{d}\mathrm{vol} \right) = 0,$$

where $g_{(\mathrm{ref})}$ is an arbitrarily chosen smooth, T-invariant Riemannian metric on $\mathcal{M}$.

Now, by our assumption on the spacetime, since $\tilde{\psi}$ solves the wave equation and vanishes outside the ergoregion, we actually have $\tilde{\psi} \equiv 0$ everywhere on $\mathcal{M}$ (see Remark 4.1)

If we apply this lemma also to the field $T\phi$ (i.e., we take one more T derivative), then we can obtain, in particular, that there is a sequence of times τ_n such that

$$\lim_{n\to\infty} \sum_{j=0}^{2} \int_{\tau_n}^{\tau_n+1} \left(\int_{\mathcal{S}_{(\mathrm{ext})}\cap\Sigma_\tau} |T^j\psi_{\tau_n}|^2\,\mathrm{d}\mathrm{vol} \right) \mathrm{d}\tau = 0.$$

[19] With a very slight modification to account for the lack of a horizon.

Now, in $\mathcal{S}_{(\text{ext})} \setminus \mathcal{S}$, we know T is a time-like vector field. Therefore, by using elliptic estimates and the mean value theorem, we have that, for any $\delta > 0$, there is a time $\tilde{\tau}_n$ with $\tau_n \leq \tilde{\tau}_n \leq \tau_n + 1$ such that

$$\sum_{|j|\leq 2} \mathcal{E}^{(N)}_{(\mathcal{S}_{(\text{ext})}\setminus\mathcal{S}_{(\delta)})}[\partial^{(j)}\psi_{\tau_n}](\tilde{\tau}_n) \to 0. \tag{23}$$

10D. ***Local energy decay on the evanescent ergosurface and energy amplification.*** The purpose of this section is to show that, if the local energy decay of Section 10B can be extended to cover the evanescent ergosurface, then this leads to a contradiction with the boundedness assumption (A1). This lies at the heart of our argument for instability. Put another way, we show that under assumption (A1) we *cannot* extend the local energy decay to cover the evanescent ergosurface. This is the form taken by the following proposition.

Proposition 10.5. *Let $\mathcal{M}$ be an asymptotically flat or an asymptotically Kaluza–Klein manifold with an evanescent ergosurface in the sense of* (ES1), *or an asymptotically Kaluza–Klein manifold with an evanescent ergosurface in the sense of* (ES2). *Suppose also that the manifold admits a discrete isometry as in Section 5. Additionally, suppose that the boundedness statement* (A1) *holds.*

Let $\mathcal{U}_0 \subset \Sigma_0$ be any compact set such that $(\mathcal{S} \cap \Sigma_0) \subset \mathcal{U}_0$. Then there exists some positive constant $\mathring{C}$ and a solution to the wave equation $\Box_g \phi = 0$ such that, for all times $t > 0$ the local energy of its T derivatives in the set $\mathcal{U}$ is at least $\mathring{C}$; i.e.,

$$\mathcal{E}^{(N)}_{\mathcal{U}}[T\phi](t) + \mathcal{E}^{(N)}_{\mathcal{U}}[T^2\phi](t) \geq \mathring{C}. \tag{24}$$

Moreover, the solution ϕ can be chosen to be smooth and to arise from compactly supported initial data satisfying

$$\sum_{j=0}^{3} \mathcal{E}^{(N)}[T^j u](0) < \infty.$$

We can see that the conclusion of Proposition 10.5 runs counter to the conclusion of Proposition 10.3. We are showing that, if the set $\mathcal{U}_0$ is allowed to contain the ergosurface $\mathcal{S}$, then assumption (A1) leads to the exact opposite behaviour to the case where the set $\mathcal{U}_0$ is disjoint from the ergosurface. We shall see further consequences of this conclusion in the next subsection.

Proof. The proof of Proposition 10.5 will proceed by contradiction. That is, we shall suppose that, for all solutions of the wave equation arising from suitable initial data, the local energy in the set $\mathcal{U}_0$ eventually becomes arbitrarily small. We shall then derive a contradiction with assumption (A1).

To be precise, suppose the following: for all smooth solutions ϕ to $\Box_g \phi = 0$ such that ϕ arises from compactly supported initial data satisfying

$$\sum_{j=0}^{3} \mathcal{E}^{(N)}[T^j \phi](0) < \infty,$$

for all sets $\mathcal{U}_t$ as defined in Proposition 10.5 and for all $\epsilon > 0$ there is some time τ such that

$$\mathcal{E}^{(N)}_{\mathcal{U}}[T\phi](\tau) + \mathcal{E}^{(N)}_{\mathcal{U}}[T^2\phi](\tau) < \epsilon.$$

Suppose in addition that assumption (A1) holds.

Using the discrete isometry $\mathscr{T}$ we can see that the same results would hold in the *time-reversed* manifold, which is the manifold with the other choice of time orientation. To see this, we note the following: we can combine the discrete isometry $\mathscr{T}$ with the one-parameter family of isometries associated to the Killing vector field T (or V), which we label F_t, to form the isometry

$$\mathscr{T}_t := F_t \circ \mathscr{T} \circ F_{-t}.$$

Since $\mathscr{T}$ fixes the hypersurface Σ_0, we find that $\mathscr{T}_t$ is a discrete isometry fixing the hypersurface Σ_t. In particular, $\mathscr{T}_t$ descends to a discrete isometry of Σ_t together with its induced metric, which we denote by $\overline{\mathscr{T}}_t$.

Now, suppose that ϕ is a solution to the wave equation, inducing the following data on the hypersurface Σ_t:

$$\phi|_{\Sigma_t} = \phi_0, \quad T\phi|_{\Sigma_t} = \phi_1.$$

Then $(\mathscr{T}_t^{-1})^*(\phi)$ will be a solution to the wave equation on the *time-reversed* manifold, that is, on the manifold $(\mathscr{T}_t^{-1}(\mathcal{M}), (\mathscr{T}_t^{-1})^*(g)) = (\mathcal{M}, g)$. Moreover, this solution will induce data on the hypersurface Σ_t given by

$$\begin{aligned}(\mathscr{T}_t^{-1})^*(\phi)|_{\Sigma_t} &= (\overline{\mathscr{T}}_t^{-1})^*\phi_0,\\ T((\mathscr{T}_t^{-1})^*(\phi))|_{\Sigma_t} &= -(\overline{\mathscr{T}}_t^{-1})^*\phi_1.\end{aligned}$$

Since ϕ_0 and ϕ_1 are smooth and compactly supported, this initial data is also smooth and compactly supported. Moreover, $(\mathscr{T}_t^{-1})^*(\phi)$ solves $\Box_g(\mathscr{T}_t^{-1})^*(\phi) = 0$. Hence this solution will disperse in the future: for any ϵ and for any compact set $\mathcal{U}_0$ there is some time $\tau > 0$ such that

$$\mathcal{E}_{\mathcal{U}}^{(N)}[T(\mathscr{T}_t^{-1})^*(\phi)](\tau) + \mathcal{E}_{\mathcal{U}}^{(N)}[T^2(\mathscr{T}_t^{-1})^*(\phi)](\tau) < \epsilon.$$

If we now apply the discrete isometry $\mathscr{T}_t$ to this solution, we find[20] that

$$\mathcal{E}_{\mathcal{U}}^{(N)}[T\phi](t-\tau) + \mathcal{E}_{\mathcal{U}}^{(N)}[T^2\phi](t-\tau) < \epsilon.$$

Note that we have made use of the fact that the dispersion result holds for *all* initial data. It is also important that the discrete isometry fixes a Cauchy hypersurface, since this allows us to pick "time-reversed" initial data.

We now make the following claim:

Claim 10.6. *In the asymptotically flat case, for all $\delta > 0$ and for any τ_0 there exists data (on Σ_{τ_0}) for the wave equation such that*

$$\mathcal{E}^{(N)}[Tu](\tau_0) \geq \delta^{-1}, \quad \mathcal{E}^{(T)}[Tu](\tau_0) = 1, \quad \sum_{j=0}^{3} \mathcal{E}^{(N)}[T^j u](\tau_0) < \infty. \tag{25}$$

Likewise, in the asymptotically Kaluza–Klein case, for all $\delta > 0$ and for any τ_0 there exists $\mathcal{G}$-invariant data such that

$$\mathcal{E}^{(N)}[Tu](\tau_0) \geq \delta^{-1}, \quad \mathcal{E}^{(V)}[Tu](\tau_0) = 1, \quad \sum_{j=0}^{3} \mathcal{E}^{(N)}[T^j u](\tau_0) < \infty. \tag{26}$$

[20] Note that the energy, on a surface which is fixed by the discrete isometry, is invariant under this isometry.

We postpone the proof of this claim, and first show that, combined with the assumptions above, this leads to a contradiction. To exhibit this contradiction, we begin with initial data as in Claim 10.6 at some time τ_0, and evolve it *backwards* in time (equivalently, we evolve it forward in time on the time-reversed manifold). We then find that, using the dispersion property derived above, this data will then disperse in the following sense: given any set $\mathcal{U}_0$, given any $\epsilon > 0$ we can find some time τ_1 such that

$$\mathcal{E}^{(N)}_{\mathcal{U}}[T\phi](\tau_0 - \tau_1) < \epsilon.$$

In particular, we can pick $\epsilon = 1$ and also use the isometry generated by the Killing vector T to translate the solution in time, so that $\tau_0 = \tau_1$. Additionally, we can pick the set $\mathcal{U}_0$ to include the evanescent ergosurface $\mathcal{S}$. Moreover, we can apply the boundedness assumption (A1) to the waves ϕ, $T\phi$, $T^2\phi$ and $T^3\phi$ (using the fact that, since T is a Killing vector field, $T^j\phi$ also solves the wave equation). Finally, we can use the fact that the T-energy is conserved to deduce that the T-energy at the initial time is $\mathcal{E}^{(T)}[T\phi](0) = 1$.

We arrive at initial data on $t = 0$ such that, in the case of the first kind of evanescent ergosurface (ES1),

$$\mathcal{E}^{(N)}_{\mathcal{U}}[T\phi](0) < 1, \quad \mathcal{E}^{(T)}(0) = 1, \quad \sum_{j=0}^{3} \mathcal{E}^{(N)}[T^j\phi](0) < \infty.$$

Furthermore, this data is such that, at the time τ_0, we have

$$\mathcal{E}^{(N)}[T\phi](\tau_0) \geq \delta^{-1}.$$

Similarly, given a manifold with the second kind of evanescent ergosurface (ES2) we arrive at $\mathcal{G}$-invariant initial data at $t = 0$ such that

$$\mathcal{E}^{(N)}_{\mathcal{U}}[T\phi](0) < 1, \quad \mathcal{E}^{(V)}(0) = 1, \quad \sum_{j=0}^{3} \mathcal{E}^{(N)}[T^j\phi](0) < \infty.$$

Now we only need to show that the *global* N-energy is of order 1 initially, and not just the local N-energy, as stated above. Since N is *uniformly* time-like, we have

$$\mathcal{E}^{(N)}[T\phi](t) \sim \|\partial T\phi\|^2_{L^2[\Sigma_t]}.$$

Combining this estimate with the estimates in Section 6 and the definition of the evanescent ergosurface of the first kind (ES1) we see that

$$\mathcal{E}^{(T)}_{\mathcal{M}\setminus\mathcal{U}}[T\phi](t) \sim \|\partial T\phi\|^2_{L^2[\Sigma_t\setminus\mathcal{U}_t]} \sim \mathcal{E}^{(N)}_{\mathcal{M}\setminus\mathcal{U}}[T\phi](t).$$

In other words, outside of the set $\mathcal{U}_t$, the T-energy $\mathcal{E}^{(T)}$ *is* comparable to the N-energy $\mathcal{E}^{(N)}$. Similarly, we saw in Proposition 7.1 that, for $\mathcal{G}$-invariant waves, the V-energy $\mathcal{E}^{(V)}$ is comparable to the N-energy outside of the set $\mathcal{U}_t$. However, in the asymptotically flat case, the *global* T-energy is bounded by 1 (at all times, since this energy is conserved). Similarly, in the asymptotically Kaluza–Klein case, the global V-energy is bounded by 1 at all times.

Hence, we find that, for the data defined above, in the asymptotically flat case the data is such that

$$\mathcal{E}^{(N)}[T\phi](0) \lesssim 1, \quad \mathcal{E}^{(T)}[T\phi](0) = 1, \quad \sum_{j=0}^{3} \mathcal{E}^{(N)}[T^j\phi](0) < \infty.$$

Note, importantly, that the bound on the initial N-energy is *independent* of δ. Similarly, in the asymptotically Kaluza–Klein case we find initial data such that

$$\mathcal{E}^{(N)}[T\phi](0) \lesssim 1, \quad \mathcal{E}^{(V)}[T\phi](0) = 1, \quad \sum_{j=0}^{3} \mathcal{E}^{(N)}[T^j\phi](0) < \infty,$$

while, in both cases, at time τ we have

$$\mathcal{E}^{(N)}[T\phi](\tau) \geq \delta^{-1}.$$

Since we can pick δ arbitrarily small, and since $T\phi$ obeys the wave equation, we arrive at a contradiction with assumption (A1). In fact, we have shown that the claim that the local energy decay statement of Proposition 10.3 can be extended to cover also the evanescent ergosurface leads to a contradiction with assumption (A1).

Subject to proving Claim 10.6, we have finished the proof. □

10E. ***Constructing the initial data.*** To finish the proof of Proposition 10.5, we need only to construct initial data to prove Claim 10.6. This turns out to be fairly difficult, and the construction will be the subject of this subsection. We shall need to make detailed use of the adapted coordinate system we considered in Section 9.

In the case of an asymptotically flat manifold with an evanescent ergosurface of the first kind (ES1), suppose the support of the wave $(T\phi)|_{\Sigma_t}$ is contained in a region $\mathcal{U}_t$, which is sufficiently small that the coordinates in Section 9 are defined in this region. Then the N-energy of the wave $T\phi$ is given by

$$\mathcal{E}^{(N)}[T\phi](\tau) \sim \int_{\Sigma_\tau} \left((TT\phi)^2 + (\partial_y T\phi)^2 + \sum_a (\partial_{x^a} T\phi)^2 \right) \mathrm{d}y\, \mathrm{d}x^1 \cdots \mathrm{d}x^{D-1}, \tag{27}$$

whereas the conserved T-energy of the wave is

$$\mathcal{E}^{(T)}[T\phi](\tau) \sim \int_{\Sigma_\tau} \left((TT\phi)^2 + \sum_a (\partial_{x^a} T\phi)^2 + \mathcal{O}(|\tilde{x}|^2)(\partial T\phi) \right) \mathrm{d}y\, \mathrm{d}x^1 \cdots \mathrm{d}x^{D-1}. \tag{28}$$

Recall that in these coordinates we have $T = \partial_{\tilde{t}}$. Note, however, that the hypersurface Σ_t is not necessarily locally a surface of constant $\tilde{t}$, and so we must bear in mind that, on Σ_t, the coordinate $\tilde{t}$ should be considered a function of the other coordinates (y, x^a).

If we could freely prescribe both $T\phi$ and $TT\phi$ on Σ_t then it would be very easy to prescribe initial data satisfying Claim 10.6. However, we can only prescribe ϕ and $T\phi$ on Σ_t. Higher-order *spatial* derivatives of these quantities can then be obtained by taking the spatial derivatives of this data; however, the quantity $TT\phi$ is constrained by the wave equation to take on values which depend on the other derivatives.

Specifically, we have that, if ϕ solves the wave equation $\Box_g\phi = 0$ then in the adapted coordinates the expression (17) gives

$$\begin{aligned} TT\phi = 2A\partial_y T\phi + A^2\delta^{ab}\partial_a\partial_b\phi + (|b|^2 - a)\partial_y^2\phi + \mathcal{O}(|\tilde{x}|\,|\partial\phi|) + \mathcal{O}(|\tilde{x}|\,|\partial T\phi|) \\ + \mathcal{O}(|\tilde{x}^2|\,|\bar{\partial}\partial\phi|) + \mathcal{O}(|\tilde{x}|^3|\partial^2\phi|). \end{aligned} \tag{29}$$

Let χ be a smooth cut-off function such that

$$\chi(x) = 0 \quad \text{for } x \geq 1,$$
$$\chi(x) = 1 \quad \text{for } x \leq \tfrac{1}{2}.$$

As mentioned above, we are free to prescribe both $u|_{\Sigma_t}$ and $T\phi|_{\Sigma_t}$. As a preliminary step, we first make the following choices for the same quantities associated to a function u_0:

$$u_0|_{\Sigma_t} = \mathfrak{Re}(e^{i\delta_0^{-1}y} F_1(x))\chi(\delta_0^{-\frac{3}{4}}|y|),$$
$$T\phi_0|_{\Sigma_t} = \mathfrak{Re}(-i\omega e^{i\delta_0^{-1}y} F_1(x))\chi(\delta_0^{-\frac{3}{4}}|y|),$$

where δ_0 and ω are constants (to be fixed below), and $F_1(x)$ is a function (to be defined below) which depends only on the coordinates x^a (and not on y or $\tilde{t}$).

Recall that we have

$$0 \leq (|b|^2 - a) = \mathcal{O}(|\tilde{x}|^2).$$

Evaluating this at $y = 0$, in terms of the coordinates x^a we can write

$$(|b|^2 - a)|_{y=0} = M_{ab}x^a x^b + \mathcal{O}(|x|^3)$$

for some symmetric matrix M_{ab}. Since M is symmetric, we can diagonalise it by making some orthogonal transformation on the coordinates x^a; that is, we can define new variables

$$x'^a := R_b{}^a x^b,$$

where R is an orthogonal matrix, and where the matrix M is diagonal in this basis. Note that the coordinate derivatives $\partial_{x'^a}$ are still orthonormal at p and satisfy the same conditions as the original coordinate derivatives, and so the form of the metric (and hence the wave equation) is unchanged by this change of variables. From now on, we will assume that this change of basis has been made, and drop the prime on the coordinates x^a.

Since M is a positive matrix, its eigenvalues are nonnegative. Associated to each eigenvalue is a coordinate x^a. We now split the coordinates x^a into two sets: those associated with a nonzero eigenvalue for M, and those associated with a zero eigenvalue of M. That is, we define

$$X_1 = \{a \in \{1, 2, \ldots\} \mid M_a{}^b x^a = 0\},$$
$$X_2 = \{a \in \{1, 2, \ldots\} \mid M_a{}^b x^a \neq 0\}.$$

Note that either one of these sets might be empty. We also define the notation

$$|x| := \sqrt{\sum_{a\in X_1\cup X_2} (x^a)^2}, \quad |x|_1 := \sqrt{\sum_{a\in X_1} (x^a)^2}, \quad |x|_2 := \sqrt{\sum_{a\in X_2} (x^a)^2}.$$

We now define

$$F_1(x) := \chi(\delta_0^{-\frac{2}{5}}|x|_1)F_2(x),$$

where F_2 is a function *only* of the coordinates x^a for $a \in X_2$.

We now construct the function $F_2(x)$ by requiring it to satisfy

$$-\Delta F_2 + A^{-2}\delta_0^{-2}(M_{ab}x^a x^b)F_2 = 2A^{-1}\omega\delta_0^{-1}F_2,$$
$$F_2(x) = 0 \quad \text{when } |x|_2 = \delta_0^{\frac{1}{2}-\delta_1}, \tag{30}$$

where Δ is the Laplacian-type operator $\sum_{a\in X_2}\partial_a^2$ and δ_1 is some small constant that we fix below.

We wish to view (30) as an elliptic eigenvalue problem for the function F_2 and its associated eigenvalue ω. We first rescale the coordinates by a factor of $\delta_0^{-1/2}$:

$$\bar{x}^a := \delta_0^{-\frac{1}{2}}x^a \quad \text{for } a \in X_2.$$

Then, using an overbar to refer to quantities defined by replacing the coordinate x^a with $\bar{x}^a$ throughout, we arrive at the eigenvalue problem

$$-\bar{\Delta}F_2 + A^{-2}(M_{ab}\bar{x}^a\bar{x}^b)F_2 = 2A^{-1}\omega F_2,$$
$$F_2(\bar{x}) = 0 \quad \text{when } |\bar{x}|_2 = \delta_0^{-\delta_1}, \tag{31}$$

where we consider ω as the eigenvalue and $M_{ab}\bar{x}^a\bar{x}^b$ as the potential for this eigenvalue problem.

By considering the variational formulation of this problem, we can place a lower bound on the number of eigenvalues ω below some threshold $\omega_{\max}$. Specifically, let $N(\omega_{\max};\delta_0)$ be the number of positive eigenvalues ω for the problem (31) satisfying $\omega \le \omega_{\max}$. Let $N_+(\mathcal{U};\delta_0)$ be the number of positive eigenvalues ω_+ satisfying the same bound, where ω_+ is an eigenvalue for the related problem

$$-\bar{\Delta}F_2 + A^{-2}\sup_{\bar{x}\in\mathcal{U}}(M_{ab}\bar{x}^a\bar{x}^b)F_2 = 2A^{-1}\omega_+F_2,$$
$$F_2(\bar{x})|_{\partial\mathcal{U}} = 0,$$

where $\mathcal{U} \subset \{|\bar{x}|_2 \le \delta_0^{-\delta_1}\}$. Then we have

$$N(\omega_{\max};\delta_0) \ge N_+(\mathcal{U};\delta_0).$$

In particular, we can take the $\mathcal{U}$ to be the cubic region with unit volume

$$\mathcal{U} := \{|\bar{x}^a| < 1 \text{ for } a \in X_2\}.$$

Note that this set is indeed a subset of $\{|\bar{x}|_2 \le \delta_0^{-\delta_1}\}$ for all sufficiently small δ_1.

Then we can explicitly calculate the positive eigenvalues for (31): they are given by

$$\omega_+ = \frac{1}{2}\left(A^{-1}\lambda_{\max} + A\pi^2\sum_{i=1}^{|X_2|}n_i^2\right), \quad n_i \in \mathbb{N},$$

where $\lambda_{\max}$ is the largest eigenvalue of the matrix M, and where at least one of the n_i is nonzero.

In particular, this proves the following proposition:

Proposition 10.7. *For all sufficiently small δ_0, there exists a function F_2 and an eigenvalue ω solving the problem* (31). *Moreover, there exists some $\omega_{\max} > 0$, which is* independent *of δ_0, such that for all* *sufficiently small δ_0 we can find an eigenvalue ω satisfying*

$$\omega \leq \omega_{\max}. \tag{32}$$

We can also establish some basic properties of the eigenfunction F_2. By linearity, we can rescale F_2 and so we can assume that

$$\int_{|x|_2<\delta_0^{1/2-\delta_1}} (F_2)^2 \prod_{a\in X_2} \mathrm{d}x^a = 1.$$

Now, multiplying the eigenvalue equation (31) by F_2 and integrating by parts (using the boundary conditions satisfied by F_2) we find that

$$\int_{|\bar{x}|_2<\delta_0^{-\delta_1}} \left(\sum_{a\in X_2} (\partial_{\bar{x}^a} F_2)^2 + A^{-2}(M_{ab}\bar{x}^a\bar{x}^b)(F_2)^2 \right) \prod_{a\in X_2} \mathrm{d}x^a = 2A^{-1}\omega.$$

In particular, transforming back to the nonrescaled coordinates, we have

$$\int_{|x|_2<\delta_0^{\frac{1}{2}-\delta_1}} \left(\sum_{a\in X_2} (\partial_a F_2)^2 \right) \prod_{a\in X_2} \mathrm{d}x^a \leq \delta_0^{-1} 2A^{-1}\omega.$$

Recall that we are choosing ω to be bounded independently of δ_0. Hence, the L_2 norm of the derivatives of F_2 scales as δ_0^{-1}.

10E1. *Agmon estimates.* We can obtain more detailed information regarding the behaviour of the eigenfunction through the use of "Agmon estimates"; see, for example, [Holzegel and Smulevici 2014]. These quantify the size of the solution in the so-called "forbidden region". Specifically, we can define a kind of forbidden region:

$$\mathcal{U}_{\text{forbidden}}(\delta_1) := \{\bar{x} \mid |\bar{x}|_2 \leq \delta_0^{-\delta_1},\ A^{-2}M_{ab}\bar{x}^a\bar{x}^b - 2A^{-1}\omega > \delta_2\}$$

for some positive constant δ_2.

We also define the "classical region":

$$\mathcal{U}_{\text{classical}} := \{\bar{x} \mid |\bar{x}|_2 \leq C_0,\ A^{-2}M_{ab}\bar{x}^a\bar{x}^b - 2A^{-1}\omega < 0\}.$$

We define the "Agmon distance" between points x and y with coordinates $\bar{x} = (\bar{x}^a),\ a \in X_2$, and $\bar{y} = (\bar{y}^a),\ a \in X_2$. This distance is defined as

$$d_\omega(\bar{x}, \bar{y}) := \inf_{\substack{\gamma:[0,1]\to(\bar{x}^a),\ a\in X_2 \\ \gamma \text{ smooth} \\ \gamma(0)=\bar{x},\ \gamma(1)=\bar{y}}} \int_0^1 \sqrt{\left(\sum_{a\in X_2} \left(\frac{\mathrm{d}\gamma}{\mathrm{d}s}(\bar{x}^a) \right)^2 \sup(A^{-2}M_{bc}\bar{x}^b\bar{x}^c - 2A^{-1}\omega, 0) \right)}\, \mathrm{d}s.$$

In other words, d_ω is the distance function defined with respect to the metric

$$(g_{(\text{Agmon})})_{ab} = \sup(A^{-2}M_{bc}\bar{x}^b\bar{x}^c - 2A^{-1}\omega, 0)\delta_{ab}.$$

If we define, for some function $u(\bar{x}^a \mid a \in X_2)$,

$$|\bar{\nabla} u| = \sqrt{\sum_a (\partial_{\bar{x}^a} u)^2}$$

then we find that the distance function with some fixed point $\bar{x}_0$ satisfies

$$|\bar{\nabla} d_\omega(\bar{x}, \bar{x}_0)| \leq \sqrt{\sup(A^{-2} M_{bc} \bar{x}^b \bar{x}^c - 2A^{-1}\omega, 0)}.$$

We can also define the distance to the classical region:

$$d_\omega^{\text{classic}}(\bar{x}) := \inf_{\substack{|\bar{y}|_2 \leq \delta_0^{-\delta_1} \\ y \in \mathcal{U}_{\text{classical}}}} d_\omega(\bar{x}, \bar{y}).$$

Now, we can prove the following proposition, which is a kind of exponentially weighted energy estimate, and which follows from integrating by parts.

Proposition 10.8 (an exponentially weighted energy estimate). *Let ϕ, W and D be smooth, real-valued functions on $\{|\bar{x}|_2 \leq C_0\}$ such that $\phi(\bar{x}) = 0$ when $|\bar{x}|_2 = \delta_0^{-\delta_1}$. Then*

$$\int_{|\bar{x}|_2 \leq \delta_0^{-\delta_1}} \big(|\bar{\nabla}(e^D \phi)|^2 + (W - |\bar{\nabla} D|^2) e^{2D} |\phi|^2\big) \prod_{a \in X_2} \mathrm{d}\bar{x}^a = \int_{|x|_2 \leq \delta_0^{-\delta_1}} (-\bar{\Delta}\phi + Wu) u e^{2D} \prod_{a \in X_2} \mathrm{d}\bar{x}^a. \quad (33)$$

Now, we apply this proposition with the choices

$$\phi = F_2, \quad W = A^{-2}(M_{ab} x^a x^b) - 2A^{-1}\omega, \quad D = (1 - \delta_2) d_\omega^{\text{classic}}.$$

This leads to the estimate

$$\begin{aligned}
&\int_{\mathcal{U}_{\text{forbidden}}} |\bar{\nabla}(e^{(1-\delta_2) d_\omega^{\text{classic}}} F_2)|^2 \prod_{a \in X_2} \mathrm{d}\bar{x}^a \\
&\qquad + \int_{\mathcal{U}_{\text{forbidden}}} \big(A^{-2}(M_{bc} \bar{x}^b \bar{x}^c) - 2A^{-1}\omega - (1-\delta_2)^2 |\bar{\nabla} d_\omega^{\text{classic}}|^2\big) e^{2(1-\delta_2) d_\omega^{\text{classic}}} |F_2|^2 \prod_{a \in X_2} \mathrm{d}\bar{x}^a \\
&= -\int_{\{|\bar{x}|_2 \leq \delta_0^{-\delta^1}\} \setminus \mathcal{U}_{\text{forbidden}}} |\bar{\nabla}(e^{(1-\delta_2) d_\omega^{\text{classic}}} F_2)|^2 \prod_{a \in X_2} \mathrm{d}\bar{x}^a \\
&\quad + \int_{\{|\bar{x}|_2 \leq \delta_0^{-\delta_1}\} \setminus \mathcal{U}_{\text{forbidden}}} \big(2A^{-1}\omega - A^{-2}(M_{bc} \bar{x}^b \bar{x}^c) + (1-\delta_2)^2 |\bar{\nabla} d_\omega^{\text{classic}}|^2\big) e^{2(1-\delta_2) d_\omega^{\text{classic}}} |F_2|^2 \prod_{a \in X_2} \mathrm{d}\bar{x}^a.
\end{aligned}$$

Now, in the forbidden region we can calculate

$$|\bar{\nabla} d_\omega^{\text{classic}}|^2 = A^{-2}(M_{bc} \bar{x}^b \bar{x}^c) - 2A^{-1}\omega > \delta_2.$$

Some straightforward calculations now lead to the estimate

$$\begin{aligned}
&\int_{\mathcal{U}_{\text{forbidden}}} |\bar{\nabla}(e^{(1-\delta_2) d_\omega^{\text{classic}}} F_2)|^2 \prod_{a \in X_2} \mathrm{d}\bar{x}^a + \delta_2^2 \int_{\mathcal{U}_{\text{forbidden}}} |F_2|^2 e^{2(1-\delta_2) d_\omega^{\text{classic}}} \prod_{a \in X_2} \mathrm{d}\bar{x}^a \\
&\qquad \lesssim A^{-1}\omega e^{2(1-\delta_2) a_\omega(\delta_2)} \int_{\{|\bar{x}|_2 \leq \delta_0^{-\delta_1}\} \setminus \mathcal{U}_{\text{forbidden}}} |F_2|^2 \prod_{a \in X_2} \mathrm{d}\bar{x}^a
\end{aligned}$$

for sufficiently small δ_2, and where $a_\omega(\delta_2)$ is defined as

$$a_\omega(\delta_2) := \sup_{\bar{x} \in \{|\bar{x}|_2 \leq \delta_0^{-\delta_1}\} \setminus \mathcal{U}_{\text{forbidden}}} d_\omega^{\text{classic}}(\bar{x});$$

i.e., $a_\omega(\delta_2)$ is the largest Agmon distance from the complement of the forbidden region to the classical region.

As $\delta_2 \to 0$, we have $a_\omega(\delta_2) \to 0$. More precisely, if we pick any $\delta_3 > 0$, then there is some choice of $\delta_2 > 0$ such that $a_\omega(\delta_2) \le \frac{1}{2}\delta_3$. Moreover, this bound holds *independently* of the choice of δ_0, at least for all sufficiently small δ_0. We now fix this choice of δ_2.

Recall that we can choose the eigenfunction to satisfy the bound $\omega \le \omega_{\max}$ independently of δ_0. From now on, we make this choice for the eigenvalue ω. For any subset of the forbidden region, $\mathcal{U}_{\mathrm{f}} \subset \mathcal{U}_{\text{forbidden}}$, we have

$$\int_{\mathcal{U}_{\mathrm{f}}} |\bar{\nabla}(e^{(1-\delta_2)d_\omega^{\text{classic}}} F_2)|^2 \prod_{a\in X_2} \mathrm{d}\bar{x}^a + \int_{\mathcal{U}_{\mathrm{f}}} |F_2|^2 e^{2(1-\delta_2)d_\omega^{\text{classic}}} \prod_{a\in X_2} \mathrm{d}\bar{x}^a \\ \lesssim e^{\delta_3} \int_{\{|\bar{x}|_2 \le \delta_0^{-\delta_1}\}\setminus \mathcal{U}_{\text{forbidden}}} |F_2|^2 \prod_{a\in X_2} \mathrm{d}\bar{x}^a. \quad (34)$$

Note that the constants depend on δ_3, but by picking δ_2 suitably small we are able to make $\delta_3 > 0$ as small as we like. Moreover, we can make such a choice for the constant δ_2 *independent* of the value of δ_0.

We now choose the subset of the forbidden region to be defined by

$$\mathcal{U}_{\mathrm{f}} := \bigl\{\bar{x} \bigm| \tfrac{1}{2}\delta_0^{-\delta_1} < |\bar{x}|_2 \le \delta_0^{-\delta_1}\bigr\}.$$

Note that this is indeed a subset of the forbidden region for all sufficiently small δ_0. In addition, within this subset we have the lower bound

$$d_\omega^{\text{classic}} \gtrsim \delta_0^{-2\delta_1},$$

which follows from the fact that the potential grows quadratically in the forbidden region.

With this choice for the region $\mathcal{U}_{\mathrm{f}}$, we return to (34). Dropping the first term on the left-hand side, and making use of the lower bound on $d_\omega^{\text{classic}}$ in the region $\mathcal{U}_{\mathrm{f}}$, we find that there is some $c_1 > 0$ such that

$$\int_{U_{\mathrm{f}}} |F_2|^2 \prod_{a\in X_2} \mathrm{d}\bar{x}^a \lesssim e^{-c_1\delta_0^{-2\delta_1}} \int_{|\bar{x}|_2 \le \delta_0^{-\delta_1}} |F_2|^2 \prod_{a\in X_2} \mathrm{d}\bar{x}^a,$$

or, returning to the nonrescaled coordinates x^a and remembering our normalisation condition on F_2, we have

$$\int_{\frac{1}{2}\delta_0^{-\delta_1} \le |x|_2 \le \delta_0^{\delta_1}} |F_2|^2 \prod_{a\in X_2} \mathrm{d}x^a \lesssim e^{-c_1\delta_0^{-2\delta_1}}.$$

In other words, the L^2 norm of F_2 is exponentially small in the region $\mathcal{U}_{\mathrm{f}}$ for small δ_0.

Now we return again to (34), and this time we drop the second term on the left-hand side. We can expand the first term as

$$\int_{\mathcal{U}_{\mathrm{f}}} e^{2(1-\delta_2)d_\omega^{\text{classic}}} \bigl(|\bar{\nabla}F_2|^2 + 2F_2(1-\delta_2)(\bar{\nabla}d_\omega^{\text{classic}})\cdot(\bar{\nabla}F_2) + (1-\delta_2)^2|\bar{\nabla}d_\omega^{\text{classic}}|^2\bigr) \prod_{a\in X_2} \mathrm{d}\bar{x}^a.$$

The third term is positive and we can immediately drop it. For the second term, the Cauchy–Schwarz inequality, allow us to write

$$|2F_2(1-\delta_2)(\bar{\nabla} d_\omega^{\text{classic}}) \cdot (\bar{\nabla} F_2)| \leq \tfrac{1}{2}|\bar{\nabla} F_2|^2 + 4(1-\delta_2)^2 |\bar{\nabla} d_\omega^{\text{classic}}|^2 |F_2|^2.$$

Now, in the region $\mathcal{U}_{\mathrm{f}}$ we have the bound

$$|\bar{\nabla} d_\omega^{\text{classic}}|^2 \lesssim \delta_0^{-2\delta_1}.$$

Combining this bound with the bound already obtained for the L^2 norm of F_2 in the region $\mathcal{U}_{\mathrm{f}}$ we find that we can bound

$$\int_{U_{\mathrm{f}}} |\bar{\nabla} F_2|^2 \prod_{a \in X_2} \mathrm{d}\bar{x}^a \lesssim \delta_0^{-2\delta_1} e^{-c_1 \delta_0^{-2\delta_1}} \int_{|\bar{x}|_2 \leq \delta_0^{-\delta_1}} |F_2|^2 \prod_{a \in X_2} \mathrm{d}\bar{x}^a$$

for all sufficiently small δ_0. Again, we can return to our nonrescaled coordinates, and also redefine c_1 to be some slightly smaller constant, and we find that the L^2 norm of ∇F_2 is also exponentially suppressed for small δ. Putting together the results above, we have proved the following proposition:

Proposition 10.9. *For all sufficiently small δ_0, there exists a solution (F_2, ω) to the eigenvalue problem* (30). *Moreover, the eigenvalue ω can be chosen to satisfy the bound*

$$\omega \leq \omega_{\max}, \tag{35}$$

where $\omega_{\max}$ is some sufficiently large constant, but which is independent *of δ_0. With this choice of eigenvalue, the associated eigenfunction F_2 satisfies*

$$\int_{\frac{1}{2}\delta_0^{\delta_1} \leq |x|_2 \leq \delta_0^{\delta_1}} (|F_2|^2 + |\nabla F_2|^2) \prod_{a \in X_2} \mathrm{d}x^a \lesssim e^{-c_1 \delta_0^{-2\delta_1}} \int_{|x|_2 \leq \delta_0^{\delta_1}} |F_2|^2 \prod_{a \in X_2} \mathrm{d}x^a. \tag{36}$$

10E2. *Estimating the error terms.* We now wish to plug our choice of initial data into the equation for $TT\phi$ (29) and obtain bounds on the size of this term, which appears in both the conserved energy and the nondegenerate energy of the wave $T\phi$. Specifically, our choice of the initial data is the following:

$$\begin{aligned}
\phi_0|_{\Sigma_t} &= \mathfrak{Re}(e^{i\delta_0^{-1} y}) F_2(x) \chi(\delta_0^{-\frac{3}{4}}|y|) \chi(\delta_0^{-\frac{2}{5}}|x|_1) \chi(\delta_0^{-\frac{1}{2}+\delta_1}|x|_2), \\
T\phi_0|_{\Sigma_t} &= \mathfrak{Re}(-i\omega e^{i\delta_0^{-1} y}) F_2(x) \chi(\delta_0^{-\frac{3}{4}}|y|) \chi(\delta_0^{-\frac{2}{5}}|x|_1) \chi(\delta_0^{-\frac{1}{2}+\delta_1}|x|_2),
\end{aligned}$$

where F_2, ω and δ_1 are as above.

For the sake of brevity we define

$$\chi_0 := \chi(\delta_0^{-\frac{3}{4}}|y|), \quad \chi_1 := \chi(\delta_0^{-\frac{2}{5}}|x|_1), \quad \chi_2 := \chi(\delta_0^{-\frac{1}{2}+\delta_1}|x|_2).$$

Similarly, in order to easily keep track of the scaling of each quantity with respect to δ_0, we define

$$\chi_0' := \chi'(\delta_0^{-\frac{3}{4}}|y|), \quad \chi_0'' := \chi''(\delta_0^{-\frac{3}{4}}|y|),$$

and similarly for χ_1 and χ_2. Note that we have, for example,

$$\partial_y \chi(\delta_0^{-\frac{3}{4}}|y|) = \delta_0^{-\frac{3}{4}} \frac{y}{|y|} \chi_0'.$$

Also note that we have

$$\begin{aligned}(|b|^2 - a) &= M_{ab}x^a x^b + \mathcal{O}(|x|^3) + \mathcal{O}(|y|\,|\tilde{x}|)\\ &= \mathcal{O}(|x|_2^2) + \mathcal{O}(|x|^3) + \mathcal{O}(|y|\,|\tilde{x}|).\end{aligned}$$

Using (29) we now calculate

$$TT\phi = \mathfrak{Re}(e^{i\delta_0^{-1}y})(2A\omega\delta_0^{-1}F_2 + A^2\Delta F_2 - M_{ab}x^a x^b\delta_0^{-2}F_2)\chi_0\chi_1\chi_2 + \mathrm{Err},$$

where the error term is given by

$$\begin{aligned}\mathrm{Err} &= \mathfrak{Re}(\mathrm{Err}_1),\\ \mathrm{Err}_1 &= -\delta_0^{-\frac{3}{4}}2Ai\omega e^{i\delta_0^{-1}y}\frac{y}{|y|}F_2\chi_0'\chi_1\chi_2 + \delta_0^{-\frac{4}{5}}e^{i\delta_0^{-1}y}|X_1|F_2\chi_0\chi_1''\chi_2\\ &\quad + A^2 e^{i\delta_0^{-1}y}\chi_0\chi_1\left(-2C_0\delta_0^{-\frac{1}{2}+\delta_1}\chi_2'\frac{x^a}{|x|}\nabla_a F_2 + C_0^2\delta_0^{-1+2\delta_1}F_2\chi_2''\right)\\ &\quad - ((|b|^2 - a) - M_{ab}x^a x^b)\delta_0^{-2}e^{i\delta_0^{-1}y}F_2\chi_0\chi_1\chi_2\\ &\quad + (|b|^2 - a)\left(i\delta_0^{-\frac{7}{4}}e^{i\delta_0^{-1}y}\frac{y}{|y|}F_2\chi_0'\chi_1\chi_2 + \delta_0^{-\frac{3}{2}}e^{i\delta_0^{-1}y}F_2\chi_0''\chi_1\chi_2\right) + \mathcal{O}(|\tilde{x}|\,|\partial u|)\\ &\quad + \mathcal{O}(|\tilde{x}|\,|\partial T\phi|) + \mathcal{O}(|\tilde{x}|^2|\bar{\partial}\partial\phi|) + \mathcal{O}(|\tilde{x}|^3|\partial^2\phi|).\end{aligned}$$

These first two error terms are easy to estimate: we have

$$\begin{aligned}\left|-\delta_0^{-\frac{3}{4}}2Ai\omega e^{i\delta_0^{-1}y}\frac{y}{|y|}F_2\chi_0'\chi_1\chi_2\right| &\lesssim \delta_0^{-\frac{3}{4}},\\ |\delta_0^{-\frac{4}{5}}e^{i\delta_0^{-1}y}|X_1|F_2\chi_0\chi_1''\chi_2| &\lesssim \delta_0^{-\frac{4}{5}}.\end{aligned}$$

Next, due to the support of $\chi_1\chi_2\chi_3$ we have

$$|((|b|^2 - a) - M_{ab}x^a x^b)\delta_0^{-2}e^{i\delta_0^{-1}y}F_2\chi_0\chi_1\chi_2| \lesssim \delta_0^{-\frac{17}{20}}.$$

Similarly, we have

$$\left|(|b|^2 - a)\left(i\delta_0^{-\frac{7}{4}}e^{i\delta_0^{-1}y}\frac{y}{|y|}F_2\chi_0'\chi_1\chi_2 + \delta_0^{-\frac{3}{2}}e^{i\delta_0^{-1}y}F_2\chi_0''\chi_1\chi_2\right)\right| \lesssim \delta_0^{-\frac{3}{4}+2\delta_1}$$

and also

$$\left|\mathcal{O}(|\tilde{x}|\,|\partial\phi|) + \mathcal{O}(|\tilde{x}|\,|\partial T\phi|) + \mathcal{O}(|\tilde{x}|^2|\bar{\partial}\partial\phi|) + \mathcal{O}(|\tilde{x}|^3|\partial^2\phi|)\right| \lesssim \delta_0^{-\frac{4}{5}}.$$

The only terms remaining are those involving χ_2' and χ_2''. Specifically, we must bound the terms

$$A^2 e^{i\delta_0^{-1}y}\chi_0\chi_1\left(-2C_0\delta_0^{-\frac{1}{2}+\delta_1}\chi_2'\frac{x^a}{|x|}\nabla_a F_2 + C_0^2\delta_0^{-1+2\delta_1}F_2\chi_2''\right).$$

A naïve approach to bounding these terms (in a similar manner to the bounds above) suggests that the first term behaves like $\delta^{-1+\delta_1}$ and the second term like $\delta^{-1+2\delta_1}$, but these bounds are insufficient for our purposes. Instead, we use the Agmon estimates of the previous section. Since χ_2' and χ_2'' are both supported only in the region $\frac{1}{2}\delta_0^{-1/2+\delta_0}$, Proposition 10.9 gives

$$\left\| A^2 e^{i\delta_0^{-1}y}\chi_0\chi_1\left(-2C_0\delta_0^{-\frac{1}{2}+\delta_1}\chi_2'\frac{x^a}{|x|}\nabla_a F_2 + C_0^2\delta_0^{-1+2\delta_1}F_2\chi_2''\right)\right\|_{L^2} \lesssim \delta_0^{-1+2\delta_1}e^{-c_1\delta_0^{-2\delta_1}},$$

so this actually *decays* as $\delta_0 \to 0$.

Putting together all of these calculations, we find that

$$\|\mathrm{Err}\|_{L^2} \lesssim \delta_0^{-\frac{17}{20}}.$$

Finally, we rescale the data, so that it satisfies $\mathcal{E}^{(T)}[T\phi] = 1$. This means multiplying by a constant that scales as $\delta_0^{17/20}$ By the calculations above, along with the expressions for the N-energy (27) and the T-energy (28) we have found data such that

$$\mathcal{E}^{(T)}[T\phi] = 1, \quad \mathcal{E}^{(N)}[T\phi] \gtrsim \delta_0^{-\frac{3}{20}}.$$

Now by choosing δ_0 sufficiently small we can prove Claim 10.6.

The construction in the asymptotically Kaluza–Klein case follows along almost identical lines, beginning from (19) instead of (17) and constructing data which is symmetric in the Kaluza–Klein directions.

10F. ***Nondecay of energy and an Aretakis-type instability.*** Now that we have proved Proposition 10.5, we know that *if* assumption (A1) holds, then there is some constant $\mathring{C} > 0$ and some solution ϕ to the wave equation such that

$$\mathcal{E}^{(N)}_{\mathcal{U}}[T\phi](t) + \mathcal{E}^{(N)}_{\mathcal{U}}[T^2\phi](t) \geq \mathring{C}$$

for all times t, and for any T-invariant open set $\mathcal{U}$ such that $\mathcal{S} \subset \mathcal{U}$. On the other hand, Proposition 10.3, together with (23), shows that, if $\delta > 0$ and $\mathcal{U}$ is such that $S \cap \mathcal{U}_{(\delta)} = \varnothing$, then for *all* $\epsilon > 0$ there is some time τ_ϵ such that

$$\mathcal{E}^{(N)}_{\mathcal{U}}[T\phi](\tau_\epsilon) + \mathcal{E}^{(N)}_{\mathcal{U}}[T^2\phi](\tau_\epsilon) \leq \epsilon.$$

From this, it follows that, if we take any precompact, open set $\mathcal{U}_0 \subset \Sigma_0$ such that $S \subset \mathcal{U}_0$, and if $\mathcal{S}_{(\epsilon)}$ is the δ-thickening of $\mathcal{S}$ defined such that

$$\mathrm{Volume}(\mathcal{S}_{\epsilon,0}) = \epsilon$$

for some constant $\epsilon > 0$, and where the volume is defined with respect to the induced Riemannian metric on Σ_0, then, there is some time τ_ϵ such that

$$\mathcal{E}^{(N)}_{\mathcal{U}_0}[T\phi](\tau_\epsilon) + \mathcal{E}^{(N)}_{\mathcal{U}_0}[T^2\phi](\tau_\epsilon) \geq \mathring{C},$$
$$\mathcal{E}^{(N)}_{\mathcal{U}\setminus\mathcal{S}_\epsilon}[T\phi](\tau_\epsilon) + \mathcal{E}^{(N)}_{\mathcal{U}\setminus\mathcal{S}_\epsilon}[T^2\phi](\tau_\epsilon) \leq \epsilon,$$

from which it follows that, if we choose ϵ sufficiently small

$$\mathcal{E}^{(N)}_{\mathcal{S}_\epsilon}[T\phi](\tau_\epsilon) + \mathcal{E}^{(N)}_{\mathcal{S}_\epsilon}[T^2\phi](\tau_\epsilon) \geq \tfrac{1}{2}\mathring{C}.$$

This is a kind of "energy concentration" phenomenon. It is easy to see from this result that the wave ϕ must blow up pointwise: indeed, we must have

$$\sup_{\mathcal{S}_\epsilon \cap \Sigma_{\tau_\epsilon}} (|\partial T\phi|^2 + |\partial T^2\phi|^2) \geq \frac{\mathring{C}}{2\epsilon};$$

since ϵ can be taken arbitrarily small, this establishes pointwise blow-up (without a rate).

We refer to this as an "Aretakis-type" instability because of its similarity to the instability found in [Aretakis 2011; 2015]. In particular, in both cases there is some quantity which is "conserved" on one hypersurface, but which decays everywhere else, and this is the cause of some kind of pointwise blowup.

11. Spacetimes with additional symmetry

Now we turn to spacetimes with an additional symmetry. Along with the symmetry generated by the Killing vector field T, in this section we will assume the existence of another Killing vector field Φ such that the span of $\{T, \Phi\}$ is *time-like* in a neighbourhood of the ergosurface S. For simplicity, we shall assume that the Killing vector field Φ is an axial Killing field, i.e., that its integral curves are closed and space-like. Moreover, we will assume that T and Φ commute. Hence, there is a Killing vector field $\widetilde{T}$ such that

- $\widetilde{T} = \alpha T + \beta\Phi$ for some constants α and β,
- $\widetilde{T}$ is time-like and future-directed in a neighbourhood of S.

It turns out that, given the presence of an additional symmetry of this sort, we can give many additional details regarding the instability discussed above. First, we find that we do not require the discrete isometry of Section 5. In addition, we can show that the energy of waves *is* bounded, not in terms of the initial energy, but in terms of a "higher-order" energy quantity. Furthermore, we can rule out case (B), showing that the Aretakis-type instability cannot occur, and instead we will always encounter the unbounded local energy amplification of case (A). Additionally, we can provide an upper bound on the time for this energy amplification to occur, which may be important for physical applications. We can also provide an example of unit-energy initial data which is *not* compactly supported, but which gives rise to a solution of the wave equation with unbounded local energy. In other words, rather than a family of solutions, each of which exhibits energy amplification by a larger and larger factor, we can provide a *single* solution of the wave equation, for which the local energy tends to infinity along a certain sequence of times. Finally, the additional symmetry allows us to deal with the issue of *higher derivatives*. Consider the situation in which we know that the initial *higher-order* energy is small: then, in the case where this extra symmetry is present, we can prove that this same higher-order energy can be amplified by an arbitrarily large constant.

Finally, we can use our results to rule out the existence of a manifold with an evanescent ergosurface, an additional symmetry of the kind described above, and a *globally time-like* Killing vector field. Note that this is a result in pure differential geometry — a priori this has nothing to do with the wave equation. However, we can use the properties of solutions of the wave equation to prove that such a manifold cannot exist.

11A. ***"Time reversal" without a discrete isometry.*** First, we describe how to deal with the issue of "time reversal" when we lack the discrete isometry of Section 5.

Let Σ_0 be some space-like Cauchy surface for $\mathcal{M}$. Then, using the fact that T (or V) is causal and transverse to Σ_0 we can construct a foliation of $\mathcal{M}$ by leaves Σ_t, where these leaves are the level sets of the function t, defined by

$$t|_{\Sigma_0} = 0, \quad T(t) = 1.$$

As usual, we replace T with V in the asymptotically Kaluza–Klein case.

Now, we construct the "time reversal" operator as follows:

$$\mathscr{T} : \mathcal{M} \to \mathcal{M},$$
$$p \in \Sigma_t \mapsto q \in \Sigma_{-t}, \quad \text{where } q \text{ is such that an integral curve of } T \text{ passes through } p \text{ and } q.$$

Note that $\mathscr{T}$ is *not* necessarily an isometry. Nevertheless, the "time-reversed" manifold $(\mathcal{M}, \mathscr{T}^*(g))$ will possess the same important properties as the manifold $(\mathcal{M}, g)$: it will have an evanescent ergosurface and an additional symmetry of the correct kind.

11B. ***Boundedness with a loss of derivatives.*** Now, we can show that this additional symmetry leads to *energy boundedness with a loss of derivatives*. Specifically, we can prove a statement of the form (A1), but where, on the right-hand side, instead of the N-energy we see a "higher-order" N-energy. Note that this will also hold for the "time-reversed" manifold constructed in the subsection above.

Recall that, in a neighbourhood of $\mathcal{S}$, the vector field $\widetilde{T}$ is time-like. We can express $\widetilde{T}$ in terms of the frame constructed in Section 9 as

$$\widetilde{T} = \widetilde{T}^T T + \widetilde{T}^Y \widehat{Y} + \widetilde{T}^a e_a + \widetilde{T}^A e_A,$$

where the last term is absent in the case of an asymptotically flat manifold. Then, since $\widetilde{T}$ is time-like, on the ergosurface $\mathcal{S}$ we have

$$2\widetilde{T}^T \widetilde{T}^Y \frac{g(T,n)}{\sqrt{g(Y,Y)}} + (\widetilde{T}^Y)^2 + \sum_a (\widetilde{T}^a)^2 + \sum_A (\widetilde{T}^A)^2 < 0.$$

Since $\widetilde{T}$ is future-directed, it follows that $\widetilde{T}^T > 0$. In turn, this implies that $\widetilde{T}^Y > 0$ (note that $g(T,n) < 0$) everywhere on S. Since S is a compact submanifold, it follows that there is some constant $C_{\widetilde{T}} > 0$ such that $\widetilde{T}^Y \geq C_{\widetilde{T}}$ everywhere on S.

Now, if we apply the T-energy estimate to the field $\widetilde{T}\phi$, we find that

$$\mathcal{E}^{(T)}[\widetilde{T}\phi](\tau) = \mathcal{E}^{(T)}[\widetilde{T}\phi](\tau_0)$$

for all $\tau \geq \tau_0$. In particular, from (28), we see that, near S, we have

$$\mathcal{E}^{(T)}[\widetilde{T}\phi](\tau) \sim \int_{\Sigma_\tau} \left((T\widetilde{T}\phi)^2 + \sum_a (\partial_{x^a}\widetilde{T}\phi)^2 + \mathcal{O}(|\tilde{x}|^2)(\partial\widetilde{T}u) \right) \mathrm{d}y\, \mathrm{d}x^1 \cdots \mathrm{d}x^{D-1}.$$

Now, using a Hardy inequality (see, for example, [Moschidis 2018]) we find that we have a bound of the form

$$\int_{\Sigma_\tau} h(x)(\widetilde{T}\phi)^2 \,\mathrm{d}y\, \mathrm{d}x^1 \cdots \mathrm{d}x^{D-1} \lesssim \mathcal{E}^{(T)}[\widetilde{T}\phi](\tau),$$

where $h(x)$ is some positive, T-invariant (and $\mathcal{G}$-invariant, in the Kaluza–Klein case) function which tends to zero in the asymptotic region as $r \to \infty$, but which is otherwise bounded away from zero.[21] In particular, $h(x)$ is bounded away from zero in a neighbourhood of $\mathcal{S}$.

Combining this bound with the calculation above, we see that, if $\mathcal{U}$ is a sufficiently small, T-invariant neighbourhood of $\mathcal{S}$, then we have

$$\int_{\Sigma_\tau} \left((\widehat{Y}\phi)^2 + (T\phi)^2 + \sum_a (\partial_{x^a}\phi)^2 + \sum_A (\partial_{x^A}\phi)^2 \right) \mathrm{d}y\, \mathrm{d}x^1 \cdots \mathrm{d}x^{D-1} \lesssim \mathcal{E}^{(T)}[\phi](\tau) + \frac{1}{C_{\tilde{\tilde{T}}}^2} \mathcal{E}^{(T)}[\widetilde{T}\phi](\tau),$$

which in turn yields the global bound

$$\begin{aligned} \mathcal{E}^{(N)}[\phi](\tau) &\lesssim \mathcal{E}^{(T)}[\phi](\tau) + \frac{1}{C_{\tilde{\tilde{T}}}^2} \mathcal{E}^{(T)}[\widetilde{T}\phi](\tau) \\ &\lesssim \mathcal{E}^{(T)}[\phi](0) + \frac{1}{C_{\tilde{\tilde{T}}}^2} \mathcal{E}^{(T)}[\widetilde{T}\phi](0). \end{aligned} \tag{37}$$

We refer to the bound in (37) as a *uniform boundedness estimate with a loss of derivatives*. It provides "uniform boundedness" of the *nondegenerate* N-energy in terms of conserved quantities, which can therefore be evaluated in terms of the initial data. On the other hand, the quantity that we bound (namely the N-energy) involves only *first* derivatives of the solution, and yet in order to control it we find that we need information regarding the *second* derivatives of the initial data. We refer to this as a *loss of derivatives*. We will show, below, that it is actually necessary to lose derivatives in this way: there is no such uniform boundedness estimate in terms of the first derivatives alone.

11C. *Ruling out case* (B). In the general case considered in Section 10, we were led to a dichotomy: we either had amplification of the local energy by an arbitrarily large factor (case (A)), or else we had an Aretakis-type instability (case (B)). We can now show that, in the situation of enhanced symmetry now under consideration, this latter case cannot occur, and so we must have the kind of behaviour considered in case (A).

Recall that the behaviour of case (B) can occur *only if* the local energy *does not decay* towards the past, that is, if there is some smooth solution ϕ to $\Box_g \phi = 0$ such that ϕ arises from compactly supported initial data satisfying

$$\sum_{j=0}^{3} \mathcal{E}^{(N)}[T^j \phi](0) < \infty$$

and some positive constant $\mathring{C}$ such that, for all times $\tau < 0$, we have

$$\mathcal{E}_{\mathcal{U}}^{(N)}[T\phi](\tau) + \mathcal{E}_{\mathcal{U}}^{(N)}[T^2\phi](\tau) \geq \mathring{C}. \tag{38}$$

We now show that, in the case of enhanced symmetry, such a solution cannot exist.

Recall that the argument of [Moschidis 2016a] establishes logarithmic decay of the N-energy on manifolds with the same asymptotic structure as those we are considering,[22] under the assumptions that:

[21] In fact, we can choose $h(x) \sim r^{-2}$.

[22] With the required modifications for $\mathcal{G}$-invariant data on asymptotically Kaluza–Klein manifolds as discussed in Section 8.

(1) A *uniform boundedness* statement holds. Note that we now have such a uniform boundedness statement, albeit in terms of a higher-order energy of the initial data.

(2) Either the spacetime has a horizon, in which case it is allowed to have a suitably "small" ergoregion (in a way made precise in [Moschidis 2016a]), or the asymptotically time-like Killing field T is globally and uniformly time-like.

Note that the spacetimes we are considering do not obey this second condition, since the asymptotically time-like Killing field is null on the evanescent ergosurface. Nevertheless, we can modify the arguments of [loc. cit.] to show that we still have some decay of solutions to the wave equation.

Most of the estimates of [loc. cit.] still apply to the kinds of manifolds we are considering — in particular, all of the estimates in the "asymptotic region" still apply. However, the low-frequency estimates in Section 6 and the Carleman estimates in Section 7 of [loc. cit.], which are used to establish an "integrated local energy decay" estimate in the interior region, need to be significantly modified if they are to apply to the kinds of spacetimes we are considering, since it is in the proof of these estimates that the failure of the vector field T to be globally time-like causes an issue.

Fortunately, for the spacetimes under consideration here, such a modification is possible, and we will sketch the details below. The main idea is that, when we have an additional axial Killing field of the kind we are assuming, we can simultaneously decompose solutions to the wave equation into (time) frequency-localised and angular frequency-localised components. Then, the Carleman estimates performed in [loc. cit.] can be shown to apply to each of the angular frequency components separately, but with an additional degenerate factor that degenerates at high angular frequency. Finally, we can perform a double interpolation argument, first showing that the individual angular frequency components decay logarithmically, and then showing that the entire solution decays sublogarithmically.

11C1. *The angular frequency decomposition.* We first note that, since Φ is a Killing vector field with closed orbits, and since $[T, \Phi] = 0$, we can define some coordinate φ with period 2π such that (rescaling Φ if necessary) $\Phi(\varphi) = 1$, Moreover, the level sets of τ can be chosen such that the integral curves of Φ lie within the level sets of τ. Then, we can decompose a solution to the wave equation ϕ in terms of "axial modes"

$$\phi = \sum_{n=-\infty}^{\infty} \phi_{(n)}, \qquad \Phi(\phi_{(n)}) = in\phi_{(n)}.$$

In other words, each of the $\phi_{(n)}$ is of the form

$$\phi_{(n)} = e^{in\varphi}\tilde{\psi}_{(n)},$$

where $\Phi(\tilde{\psi}_{(n)}) = 0$. Note that, by orthogonality, for each n we have

$$\Box_g(\phi_{(n)}) = 0.$$

The idea is that we will apply the logarithmic decay result of [Moschidis 2016a] *separately* for each axial mode. When doing so, we will have to keep track of the dependence of various constants on the mode number n. The key estimates are the low-frequency estimates of Section 6 and the Carleman estimates of Section 7 of [Moschidis 2016a]; it is easy to see that the estimates in the asymptotic region apply to the solutions $\phi_{(n)}$. Note also that, since $[T, \Phi] = 0$, we can simultaneously perform the angular frequency decomposition and also decompose the solutions with respect to time frequencies.

Below we will sketch the required modifications to the arguments given in [loc. cit.]. Many of the arguments are almost identical, so we will only go into depth in those places which differ significantly from the argument presented in [loc. cit.].

11C2. *Integrated local energy decay for very low frequencies.* The approach of [Moschidis 2016a] involves dividing up the (time) frequency range into a low-frequency part $\omega \lesssim \omega_0$, intermediate-frequency parts $\omega \sim \omega_k$ and a high-frequency part $\omega \gtrsim \omega_+$. In our case, this division will itself depend on the value of n, the angular quantum number.

First, we define some ω_0, which is sufficiently small compared with various geometric quantities (as in [loc. cit.]) and which also satisfies

$$\omega_0 \lesssim \frac{1}{n}.$$

The corresponding frequency-localised wave will be denoted by $\psi_{0,n}$.

Then, we can repeat the calculations of Section 6 of [loc. cit.], using the current

$$J_\mu := h_R \psi_{0,n} \partial_\mu \psi_{0,n} - \tfrac{1}{2} \partial_\mu h_R \psi_{0,n}^2.$$

The construction of the function h_R is identical to that given in [loc. cit.]. We can see (using the expression for the inverse metric given in (18) together with the fact that $\{T, \Phi\}$ span a time-like direction, so that Φ must have a component in the ∂_y-direction) that

$$h_R \partial^\mu \psi_{0,n} \partial_\mu \psi_{0,n} \geq c \cdot h_R |\nabla_{\Sigma,\text{degen}} \psi_{0,n}|^2 - C \cdot h_R |T\psi_{0,n}| (|T\psi_{0,n}| + |\Phi\psi_{0,n}|),$$

where c and C are some numerical constants, and $|\nabla_{\Sigma,\text{degen}} \psi_{0,n}|^2$ includes all of the spatial derivatives of $\psi_{0,n}$ *except* for the Φ derivative on the surface $\mathcal{S}$. More precisely, the coefficient of $|\Phi\psi_{0,n}|^2$ degenerates quadratically at $\mathcal{S}$.

The remainder of the calculations proceed in the same way as those in Section 6 of [loc. cit.], except that there is no black hole horizon, and so many of the calculations are easier (see the footnotes in [loc. cit.]). Following these calculations we can show that, in the notation of [loc. cit.],

$$\int_{\{r \leq R\} \cap \mathcal{R}(0,t^*)} (|\nabla_{\Sigma,\text{degen}} \psi_{0,n}|^2 + |\psi_{0,n}|^2) \lesssim (1+\omega_0^2+n^2) \mathcal{E}^{(T)}[\psi_{0,n}](0) + \int_{\{r \leq R\} \cap \mathcal{R}(0,t^*)} ((\omega_0)(\omega_0+n)|\psi_{0,n}|^2).$$

Hence, if ω_0 is sufficiently small relative to the geometry *and* to $1/n$, then we can absorb the second term on the right-hand side by the left-hand side.

This proves a *degenerate* integrated local energy decay statement, since we still do not control all of the derivatives of $\psi_{0,n}$ — we are missing the Φ derivatives on the surface $\mathcal{S}$. We can fix this by commuting once with Φ. We obtain, in the end,

$$\int_{\{r \leq R\} \cap \mathcal{R}(0,t^*)} (|\partial\psi_{0,n}|^2 + |\psi_{0,n}|^2) \lesssim (1+\omega_0^2+n^2)(1+n^2) \mathcal{E}^{(T)}[\psi_{0,n}](0).$$

11C3. *Integrated local energy decay for intermediate frequencies.* For the intermediate frequencies ω_k we can also follow the calculations of Section 7 of [Moschidis 2016a]. The calculations in this section make use of the fact that the frequencies are bounded away from zero. Hence, when repeating these calculations, we must keep in mind the fact that, in our case, $\omega_0 \sim n^{-1}$, and so we must track the dependence of various constants on the value of ω_0.

As in [loc. cit.], we extend the function r from the asymptotic region (where it is the pullback of the spherical polar radial coordinate on $\mathbb{R}^d$ or $\mathbb{R}^d \times X$) to the entire hypersurface Σ_t by requiring that r is a Morse function. Moreover, we can arrange that, say, $r = r_0$ on $\mathcal{S}$, and $\mathrm{d}r \neq 0$ on $\mathcal{S}$. Furthermore, we can set $\Phi(r) = 0$ in a neighbourhood of $\mathcal{S}$. Finally, we can also arrange that in the region $r \leq 2r_0$ the vector field $\widetilde{T}$ is uniformly time-like.

The construction of the two Morse functions ω, ω' required for the energy currents in Section 7 of [loc. cit.] proceeds exactly as in [loc. cit.] — note that we only construct these functions away from $\mathcal{S}$, in the region where T is uniformly time-like. Note that, for $r_0 \leq r \leq 2r_0$, we have $\omega = \omega' = r$.

Note that, as in [loc. cit.], $\partial^\mu \omega\, \partial_\mu \omega \neq 0$ away from the critical points of ω. In the region $r \geq 2r_0$ this follows directly from the arguments of [loc. cit.], using the fact that T is time-like in this region. On the other hand, for $r_0 \leq r \leq 2r_0$, both the vector fields T and Φ are tangent to the level sets of ω, and so in particular the time-like vector field $\widetilde{T}$ is tangent to the level sets of ω.

The remainder of the calculations in Section 7 of [loc. cit.] proceed in almost identical fashion, with the important exception of inequality (7.22). Here, we instead have

$$|\partial \psi_{k,n}|_h^2 \leq C_1 \partial_\mu \psi_{k,n} \partial^\mu \bar{\psi}_{k,n} + C_2 |T\psi_{k,n}|^2 + C_3 |\Phi \psi_{k,n}|^2,$$

where the third term is new. Effectively, this means that the constants ω_k^2 that appear in the inequalities in Section 7 of [loc. cit.] need to be replaced with $\omega_k^2 + n^2$ in our calculation. In particular, the parameter s needs to be chosen sufficiently large compared to $\omega_k + |n|$ rather than just ω_k.

With this in mind, and noting that the various constants that depend on ω_0 are, at worst, of order $(\omega_0)^{-2}$ (see the comments in Lemma 4.6 of [loc. cit.]), we see that the key integrated local energy decay estimate, Proposition 7.2 in [loc. cit.], is replaced by:

Proposition 11.1 (ILED for bounded frequencies [Moschidis 2016a, Proposition 7.2]). *For any $R \geq r_0$ and for ω_0 sufficiently small compared with both* 1 *and n^{-1}, there exists some positive constant $C(R)$ such that, for any smooth axial mode ψ_k with compactly supported initial data, and any $\omega_+ > 1$ and $1 \leq |k| \leq n$, we have*

$$\int_{\{r \leq R\} \cap \mathcal{R}(0,t^*)} (|\partial \psi_{\leq \omega_+,n}|^2 + |\psi_{\leq \omega_+,n}|^2) \lesssim C(R) e^{C(R) n^2 (\omega_+ + |n|)} \mathcal{E}^{(T)}[\psi_{0,n}](0).$$

In other words, we can repeat all of the calculations of Section 7 of [Moschidis 2016a] for the axial mode ψ_n, at the expense of making our estimates degenerate exponentially in n.

11C4. *A double interpolation argument and sublogarithmic decay.* Performing the interpolation argument as in [Moschidis 2016a], and remembering that we must "lose derivatives" in the boundedness statement, we find that the local energy of the n-th axial mode decays logarithmically as

$$\mathcal{E}_{\mathcal{U}}^{(N)}[\phi_{(n)}](\tau) \leq C_{(\mathcal{U},m)}\left(\frac{n^{4m}}{(\log(2+\tau))^{2m}}+\frac{e^{C_{(\mathcal{U},m)}|n|^3}}{(1+\tau)^{\delta_5}}\right)\left(\int_{\Sigma_0}(r^{\delta_4})\iota_{(N)}J[\phi]\,\mathrm{d}\mathrm{vol}+\sum_{j+k\leq m+1}\mathcal{E}^{(N)}[T^j\Phi^k\phi](0)\right)$$

for any positive integer m and any real number $\delta_4 > 0$, and for some small $\delta_5 > 0$, and where the $C_{(\mathcal{U},m)}$ are some (possibly very large) constants depending only on m and the set $\mathcal{U}$. This is a slightly modified version of Corollary 2.2 of [Moschidis 2016a]. Note that the first term defines a kind of "weighted" energy; see the discussion of p-weighted energy estimates in Section 8.

Note that, although this estimate shows that the n-th axial mode decays logarithmically, we cannot simply "add up" all such estimates to show that the solution as a whole decays logarithmically. In fact, these estimates degenerate (exponentially) in n at large values of n (Note that the polynomial degeneration in n of the bounds given for time frequencies is strictly better than this). To obtain decay for the solution ϕ, rather than just the axial modes $\phi_{(n)}$, we can use the interpolation argument again, using the decay statement for axial modes with $|n| \leq n_+$, and simply using the boundedness statement (and commuting with Φ) for modes with $|n| > n_+$. Choosing $n_+ \sim (\log(2+\tau))^{1/3}$, we find

$$\mathcal{E}_{\mathcal{U}}^{(N)}[\phi](\tau) \leq \frac{C_{(\mathcal{U},m)}}{(\log(2+\tau))^{\frac{2}{3}m}}\left(\int_{\Sigma_0}(r^{\delta_4})\iota_{(N)}J[\phi]\,\mathrm{d}\mathrm{vol}+\sum_{j+k\leq m+2}\mathcal{E}^{(N)}[T^j\Phi^k\phi](0)\right), \tag{39}$$

where $\mathcal{U}_0 \subset \Sigma_0$ is *any* precompact set (including those which intersect the ergosurface $\mathcal{S}$). Note that this decay result holds in both the original manifold and the time-reversed manifold.

In short, the local N-energy of ϕ decays *sublogarithmically*. In particular, if ϕ arises from smooth, compactly supported initial data then

$$\begin{aligned}\mathcal{E}_{\mathcal{U}}^{(N)}&[T\phi](\tau)+\mathcal{E}_{\mathcal{U}}^{(N)}[T^2\phi](\tau)\\ &\leq \frac{C_{\mathcal{U}}}{(\log(2+\tau))^{\frac{2}{3}}}\left(\int_{\Sigma_0}(r^{\delta_4})\iota_{(N)}J[T\phi]\,\mathrm{d}\mathrm{vol}+\int_{\Sigma_0}(r^{\delta_4})\iota_{(N)}J[T^2\phi]\,\mathrm{d}\mathrm{vol}+\sum_{j+k\leq 4}\mathcal{E}^{(N)}[T^j\Phi^k\phi](0)\right)\\ &\leq \frac{C_{(\mathcal{U},\phi)}}{(\log(2+\tau))^{\frac{4}{3}}},\end{aligned}$$

where the first line follows from (39) and the second line follows from the fact that the data is smooth and compactly supported, so that the energy quantities (including the weighted energies) are finite initially. Note that the numerical constants $C_{\mathcal{U}}$ and $C_{(\mathcal{U},\phi)}$ are generally different.

We call this decay "sublogarithmic" because, in terms of pointwise decay rates, this would lead to decay for the fields ϕ at a rate $\phi \sim (\log(2+\tau))^{-1/3}$. Note that this is a kind of converse to the lower bound proved on microstate geometries in [Keir 2016], albeit this result shows decay at a slower rate. It is likely, therefore, that this does not represent a sharp decay rate for linear waves on these geometries.

In any case, this rules out the existence of a constant $\mathring{C} > 0$ such that (38) holds, since, if τ is sufficiently large, then we will always have

$$\frac{C_{(\mathcal{U},\phi)}}{(\log(2+\tau))^{\frac{2}{3}}} \leq \mathring{C}$$

even if the numerical constant $C_{(\mathcal{U},\phi)}$ is very large. In turn, this rules out case (B), so that, when an extra symmetry of the right kind is present, then we must have case (A), i.e., energy amplification by an arbitrarily large factor. Note also that, since the T-energy remains bounded, the energy amplification must occur near the evanescent ergosurface $\mathcal{S}$. In other words, we have proved the following:

Lemma 11.2 (local energy amplification in the presence of an additional symmetry). *Let $(\mathcal{M}, g)$ be a Lorentzian manifold which is either asymptotically flat and has an evanescent ergosurface of the first kind* (*see condition* (ES1)), *or which is asymptotically Kaluza–Klein and has an evanescent ergosurface of the second kind* (*see condition* (ES2)). *Furthermore*, *suppose there is an additional Killing vector field Φ, such that the span of T and Φ includes a Killing vector field that is time-like in a neighbourhood of $\mathcal{S}$.*

Then, *for any constant $C > 0$ and any open set $\mathcal{U}_0 \subset \Sigma_0$ such that $\mathcal{S} \cap \mathcal{U}_0 \neq \varnothing$, there is a solution $\phi_{(C,\mathcal{U})}$ to the wave equation $\Box_g \phi_{(C,\mathcal{U})} = 0$ arising from smooth*, *compactly supported initial data*, *and a time $\tau_{(C,\mathcal{U})}$ such that*

$$\mathcal{E}^{(N)}_{\mathcal{U}}[\phi_{(C,\mathcal{U})}](\tau_{(C,\mathcal{U})}) \geq C\mathcal{E}^{(N)}[\phi_{(C,\mathcal{U})}](0). \tag{40}$$

In other words, *there is a solution to the wave equation whose local energy* (*in the set $\mathcal{U}$*) *is amplified*, *relative to its* ***total*** *initial energy*, *by a factor of at least C.*

Note that, by construction, the solution $\phi_{(C,\,\mathcal{U})}$ in Lemma 11.2 can be chosen to be the T-derivative of a solution to the wave equation.

11D. ***Bounds for the amplification time and the support of the data.*** Physically, it is very important to be able to estimate the *time-scale* of any proposed instability. For example, if the time-scale of an instability of some object is very small compared to the time-scale on which those objects form, then we would not expect to find such objects in nature, whereas in the opposite case we might still expect to find these objects, despite the presence of an instability.

Now that we have shown that, in the presence of an additional symmetry, the local (nondegenerate) energy of a solution to the wave equation can grow arbitrarily large relative to its initial (nondegenerate) energy; it turns out that the extra symmetry also allows us to prove an upper bound for the time taken for the local energy to grow. At the same time, we can prove a bound on the size of the support of the initial data which leads to this growing solution, which might also have some physical relevance.

Recall that, in the case of additional symmetry, we can show both that the N-energy is bounded (by a higher-order initial energy) and that the local N-energy decays at least logarithmically (again, this bound necessarily involves higher-order initial energies). Thus, we can construct data for the wave equation as in Section 10E. This leads to initial data for ϕ at a time τ_1 such that

- $\mathcal{E}^{(T)}[T\phi](\tau_1) = 1$,
- $\mathcal{E}^{(N)}[T^2\phi](\tau_1) = \mathcal{O}(\delta_0^{-3/20})$,
- $\mathcal{E}^{(N)}[\partial T\phi](\tau_1) = \mathcal{O}(\delta_0^{-2-3/20})$.

Moreover, it is not very difficult to see that the initial data constructed in this way satisfies

$$\sum_{j+k\leq m} \mathcal{E}^{(N)}[T^j \Phi^k \phi](\tau_1) = \mathcal{O}(\delta_0^{-2m-\frac{3}{20}}).$$

Thus, solving the wave equation *backwards* in time, the decay estimate (39) tells us that the local nondegenerate energy of the wave $T\phi$ at time 0 is bounded by

$$\mathcal{E}^{(N)}_{\mathcal{U}}[T\phi](0) \leq \frac{C_{(\mathcal{U},m)}\delta_0^{-2m-6-\frac{3}{20}}}{(\log(2+\tau_1))^{\frac{2}{3}m}}$$

for any choice of $m \in \mathbb{N}$. Thus, we can guarantee that the local energy at time 0 is bounded above by 1 by choosing

$$\tau_1 = \exp((C_{(\mathcal{U},m)})^{\frac{3}{2m}}\delta_0^{-3-\frac{82}{5m}}).$$

Moreover, at this time the *total* N-energy is bounded by some constant which is *independent of* δ_0, since the N-energy away from the ergosurface is bounded by the T-energy, which is conserved and takes the value 1.

Putting this together,[23] we have shown the following:

Corollary 11.3 (bounds for the amplification time and the support of the data). *Let* $(\mathcal{M}, g)$ *be as in Lemma 11.2.*

Then, for any constant $C > 0$ *and any open set* $\mathcal{U}_0 \subset \Sigma_0$ *such that* $\mathcal{S} \subset \mathcal{U}_0$, *there is a solution* $\phi_{(C,\mathcal{U})}$ *to the wave equation* $\Box_g \phi_{(C,\mathcal{U})} = 0$ *arising from smooth, compactly supported initial data such that*

$$\sup_{\tau\in[0,\tau^*]} \frac{\mathcal{E}^{(N)}_{\mathcal{U}}[\phi_{(C,\mathcal{U})}](\tau)}{\mathcal{E}^{(N)}[\phi_{(C,\mathcal{U})}](0)} \geq C, \tag{41}$$

where τ^* *is given by*

$$\tau^* = C_{(\mathcal{U},\,m)} \exp(C^{20+\frac{328}{3m}}) \tag{42}$$

for some constant $C_{(\mathcal{U},\,m)}$ *depending only on the set* $\mathcal{U}$ *and the positive integer* m.

Moreover, the initial data for this solution is supported only in the intersection of the causal past of the set $\mathcal{U} \cap \Sigma_{\tau^*}$ *with the initial hypersurface* Σ_0.

Note that the solution $\phi_{(C,\mathcal{U})}$ appearing in the corollary is the function $T\phi$ from the calculations above. Recall that, since T is a Killing vector field, if ϕ is a solution to the wave equation then so is $T\phi$.

11E. ***Solutions with unbounded local energy.*** We have seen (in Lemma 11.2) that, in the case where an extra symmetry is present, the local nondegenerate energy can be amplified by an arbitrarily large amount compared with the initial, total, nondegenerate energy. A natural question now arises: does there exist finite-energy initial data leading to a *single* solution ϕ to the wave equation (as opposed to a sequence of solutions), such that the local energy of ϕ becomes arbitrarily large?

[23] Choosing a larger value of m appears to give an improved lower bound on the amplification bound; i.e., it leads to a bound whose functional dependence on δ_0 is better. However, the numerical constant $C_{\mathcal{U},m}$ also depends on the value of m in some uncontrolled way, so we cannot simply pass to the limit $m \to \infty$.

Using the extra symmetry, we give an affirmative answer to this question. Note, however, that the initial data we construct is not necessarily smooth, and indeed, in view of the nondegenerate energy bound with a loss of derivatives, it cannot have finite "higher-order" energies. Moreover, it is not necessarily compactly supported either.

Corollary 11.4 (a solution with unbounded local energy). *Let* $(\mathcal{M}, g)$ *be as in Lemma 11.2.*

Then there exists a (*weak*) *solution* ϕ *to* $\Box_g \phi = 0$ *such that*:

- *The initial* N*-energy of* ϕ *is* 2; *that is,* $\mathcal{E}^{(N)}[\phi](0) = 1$.
- *The local* N*-energy of* ϕ *is unbounded*; *that is,*

$$\limsup_{\tau \to \infty} \mathcal{E}^{(N)}_{\mathcal{U}}[\phi](\tau) = \infty, \tag{43}$$

where $\mathcal{U}$ *is any* T*-invariant open set such that* $\mathcal{S} \subset \mathcal{U}$.

For ease of notation, let us fix an open set $\mathcal{U}$ intersecting the ergosurface. Let $(T\phi)_n$ be a solution to the wave equation constructed as in Lemma 11.2, except that we continue evolving the solution to the past until the N-energy is bounded above[24] by $1/(C_n)^3$. Then we have the following lower bound on the local energy:

$$\mathcal{E}^{(N)}_{\mathcal{U}}[(T\phi)_n](\tau_{C_n}) \geq C_n,$$

where τ_{C_n} is some time satisfying

$$\tau_{C_n} \sim e^{(C_n)^{21}}.$$

At the same time, for the solution $(T\phi)_n$ we have the decay estimate (see (39) with the choice $m = 1$)

$$\mathcal{E}^{(N)}_{\mathcal{U}}[(T\phi)_n](\tau) \lesssim \frac{(C_n)^{41}}{(\log(2+\tau))^{\frac{2}{3}}},$$

which holds for *all* τ.

Finally note that, by applying the decay estimate to the solution as we evolve backwards in time, we find that, for all times τ such that

$$\tau \lesssim e^{C_n^{129/2}},$$

we also have the bound

$$\mathcal{E}^{(N)}_{\mathcal{U}}[(T\phi)_n](\tau) \lesssim \frac{1}{(C_n)^2}.$$

Now, we can use these bounds to construct a solution with the desired properties. First, we choose the constants $C_n = 2^{2^n}$. We see that, at some time τ_n satisfying

$$\tau_n \sim e^{2^{21 \cdot 2^n}}$$

we have

$$\mathcal{E}^{(N)}_{\mathcal{U}}[(T\phi)_n](\tau_n) \geq 2^{2^n}.$$

[24] Recall that, when we solve the wave equation backwards and use the decay estimate (39), we only obtain an *upper bound* on the initial N-energy. We want to avoid scaling the solution up by a (potentially large) factor.

Now, if $m \geq n$ and n is large enough, then we have

$$e^{2^{\frac{129}{2}\cdot 2^m}} > e^{2^{21\cdot 2^n}}$$

and so we have

$$\mathcal{E}^{(N)}_{\mathcal{U}}[(T\phi)_m](\tau_n) \lesssim \frac{1}{2^{2\cdot 2^m}}.$$

On the other hand, if $m \leq n$ then we can use the uniform decay estimate to show

$$\mathcal{E}^{(N)}_{\mathcal{U}}[(T\phi)_m](\tau_n) \lesssim \frac{2^{41\cdot 2^m}}{2^{21\cdot 2^n}}.$$

Again, if n is sufficiently large, then this term is bounded, say by $1/2^n$.

Now, we set

$$\phi_\infty := \sum_{n=1}^{\infty} (T\phi)_n.$$

By the triangle inequality, and the calculations above, we see that there is a sequence of times τ_n such that

$$\mathcal{E}^{(N)}_{\mathcal{U}}[\phi_\infty](\tau_n) \to \infty.$$

At the same time, the initial energy of this function ϕ_∞ is bounded by

$$\sum_{n=0}^{\infty} \frac{1}{2^{3\cdot 2^n}},$$

which is clearly finite. Hence the series has a limit which is a weak solution of the wave equation. Note, however, that from the bound (37), the initial energy of $\widetilde{T}\phi_\infty$ must be infinite.

11F. ***Higher derivatives.*** Another natural question in the context of the work above is whether the instability we have discussed can be "cured" by looking at higher derivatives. We could compare the situation to the case of wave equations on a Schwarzschild black hole, where the well-known "trapping" phenomena means that an *integrated local energy decay* statement cannot hold. In the language of this paper, this means that no statement of the form

$$\int_0^{\tau_1} \left(\int_{\Sigma_\tau \cap \mathcal{U}} \iota_{(N)} J[\phi] \, \mathrm{d}\mathrm{vol} \right) \mathrm{d}\tau \lesssim \mathcal{E}^{(N)}[\phi](0)$$

can hold; see [Sbierski 2015] for a very general proof that this kind of statement cannot hold on spacetimes involving trapping. However, in the case of Schwarzschild black holes, it is possible to "fix" this problem by including higher derivatives on the right-hand side, and indeed a statement of the following form can be shown to hold:

$$\int_0^{\tau_1} \left(\int_{\Sigma_\tau \cap \mathcal{U}} \iota_{(N)} J[\phi] \, \mathrm{d}\mathrm{vol} \right) \mathrm{d}\tau \lesssim \mathcal{E}^{(N)}[\phi](0) + \mathcal{E}^{(N)}[T\phi](0).$$

One might wonder whether a similar approach could be used to "cure" the instability discussed in this paper. In fact, we have already seen that, when an additional symmetry is present, the local energy *can* be bounded in terms of a higher-order energy (see (37)). However, this is not a very satisfactory result, since it leaves open the possibility that the higher-order energy is itself unbounded, and this, in turn, could be interpreted as a kind of instability (albeit of a "weaker" type).

Again, the additional symmetry enables us to resolve this issue. Let $\tilde{\psi} = T\phi$, where ϕ is the solution to the wave equation arising from the initial data we have constructed in Section 10E, and where we pose the initial data at time τ_1. Then it is not hard to see that $\tilde{\psi}$ satisfies

$$\sum_{j+k\leq m} \mathcal{E}^{(N)}[T^j \Phi^k \tilde{\psi}](\tau_1) = \mathcal{O}((\delta_0)^{-2m-\frac{3}{20}}),$$
$$\mathcal{E}^{(T)}[T^j \Phi^k \tilde{\psi}](\tau_1) = \mathcal{O}((\delta_0)^{-2m}),$$

and so, following exactly the same arguments as before, we can show that there exists a solution to the wave equation u_{C_1} and a time τ_{C_1} such that, for any $C_1 > 0$, we have

$$\begin{aligned} \sum_{j+k\leq m} \mathcal{E}^{(N)}[T^j \Phi^k u_{C_1}](0) &= 1, \\ \sum_{j+k\leq m} \mathcal{E}^{(N)}[T^j \Phi^k u_C](\tau_{C_1}) &\geq C_1. \end{aligned} \tag{44}$$

Now, since at each point on the manifold $\mathcal{M}$, there is a time-like vector in the span of T and Φ, we can use standard elliptic estimates to show that there is some numerical constant C_2 such that

$$\sum_{j\leq m} \mathcal{E}^{(N)}[\partial^j u_{C_1}](0) \leq C_2,$$
$$C_1 \leq \sum_{j\leq m} \mathcal{E}^{(N)}[\partial^j u_{C_1}](\tau_{C_1}) \leq C_2 C_1,$$

where in the second line the first inequality follows from (44). Hence, rescaling the solution by a factor of $(C_2)^{-1}$ and setting $C_3 = C_1(C_2)^{-1}$ we have found a solution to the wave equation such that

$$\sum_{j\leq m} \mathcal{E}^{(N)}[\partial^j u_{C_1}](0) \leq 1,$$
$$\sum_{j\leq m} \mathcal{E}^{(N)}[\partial^j u_{C_1}](\tau_{C_1}) \geq C_3.$$

In other words, we have proved the following corollary:

Corollary 11.5 (higher-order energies). *Suppose that the same conditions holds as for Lemma 11.2. Then, for any positive integer m, for any $C > 0$, there is a solution to the wave equation $u_{(C,m)}$ and a time $\tau_{(C,m)}$ such that*

$$\frac{\sum_{j\leq m} \mathcal{E}^{(N)}[\partial^j u_{(C,m)}](\tau_{(C,m)})}{\sum_{j\leq m} \mathcal{E}^{(N)}[\partial^j u_{(C,m)}](0)} \geq C. \tag{45}$$

This shows that we cannot fully "escape" the instability by looking at higher-order energies, at least in the case where the extra symmetry is present. To be explicit: suppose that we know, initially, that the "n-th order" energy of some wave ϕ is small. In other words, we know that $\partial\partial^n\phi$ is small in L^2. Then, although it may be the case that $\partial\partial^{n-1}\phi$ is small in L^2 at all points in the future, but we can never guarantee that $\partial\partial^n\phi$ is also small (in the same sense) at all points in the future.

Appendix: Nonexistence of manifolds with evanescent ergosurfaces and a globally time-like Killing vector field

A curious corollary of the results we have proved above allows us to rule out a certain kind of smooth Lorentzian manifold, possessing certain symmetries and a particular asymptotic structure. Note that the statement of this corollary makes no reference to the wave equation: it is a result purely in Lorentzian differential geometry. Nevertheless, our proof makes use of the wave equation!

Corollary A.1 (nonexistence of Lorentzian manifolds with an evanescent ergosurface and a globally time-like Killing vector field). *There does not exist a smooth*, *Lorentzian manifold which is either*

- *asymptotically flat and possesses an evanescent ergosurface of the first kind* (*condition* (ES1)) *or*
- *asymptotically Kaluza–Klein and possesses an evanescent ergosurface of the second kind* (*condition* (ES2))

and which also possesses a uniformly time-like *Killing vector field* $\widehat{T}$.

Proof. The proof is by contradiction. Suppose that such a manifold did exist. Let ϕ_C be the solution to the wave equation constructed in Lemma 11.2. Then, since $\widehat{T}$ is a Killing vector field, the associated energy is conserved; i.e., for all τ,

$$\mathcal{E}^{(\widehat{T})}[\phi_C](\tau) = \mathcal{E}^{(\widehat{T})}[\phi_C](0).$$

At the same time, by construction we have

$$\mathcal{E}^{(N)}[\phi_C](\tau_C) = C\mathcal{E}^{(N)}[\phi_C](0).$$

But, since both N and $\widehat{T}$ are uniformly time-like, at all times τ and for any function ϕ we have

$$\hat{c}\mathcal{E}^{(N)}[\phi](\tau) \leq \mathcal{E}^{(\widehat{T})}[\phi](\tau) \leq \widehat{C}\mathcal{E}^{(N)}[\phi_C](\tau)$$

for some constants $\hat{c}$ and $\widehat{C}$.

Combining these, we have

$$\mathcal{E}^{(\widehat{T})}[\phi_C](0) = \mathcal{E}^{(\widehat{T})}[\phi_C](\tau_C) \geq \hat{c}\mathcal{E}^{(N)}[\phi_C](\tau_C) \geq \hat{c}C\mathcal{E}^{(N)}[\phi_C](0) \geq \hat{c}\widehat{C}^{-1}C\mathcal{E}^{(\widehat{T})}[\phi_C](0).$$

The constants $\widehat{C}$ and $\hat{c}$ are independent of the solution u_C. Hence, we can choose $C > (\hat{c})^{-1}\widehat{C}$, giving the required contradiction. □

We note here that an analogous proposition holds, with "evanescent ergosurface" replaced by "ergoregion". This follows immediately from the result of [Moschidis 2016b].

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BOUNDARY VALUE PROBLEMS FOR SECOND-ORDER ELLIPTIC OPERATORS WITH COMPLEX COEFFICIENTS

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The theory of second-order complex-coefficient operators of the form $\mathcal{L} = \operatorname{div} A(x)\nabla$ has recently been developed under the assumption of p-ellipticity. In particular, if the matrix A is p-elliptic, the solutions u to $\mathcal{L}u = 0$ will satisfy a higher integrability, even though they may not be continuous in the interior. Moreover, these solutions have the property that $|u|^{p/2-1}u \in W^{1,2}_{\mathrm{loc}}$. These properties of solutions were used by Dindoš and Pipher to solve the L^p Dirichlet problem for p-elliptic operators whose coefficients satisfy a further regularity condition, a Carleson measure condition that has often appeared in the literature in the study of real, elliptic divergence form operators. This paper contains two main results. First, we establish solvability of the regularity boundary value problem for this class of operators, in the same range as that of the Dirichlet problem. The regularity problem, even in the real elliptic setting, is more delicate than the Dirichlet problem because it requires estimates on derivatives of solutions. Second, the regularity results allow us to extend the previously established range of L^p solvability of the Dirichlet problem using a theorem due to Z. Shen for general bounded sublinear operators.

1. Introduction

The theory of elliptic boundary value problems under minimal smoothness assumptions on the boundary or the coefficients has been well-studied in the real-valued setting and there is a rich literature of results and applications. By contrast, the literature in the complex-valued setting is much more limited. Some important milestones in the study of complex-coefficient operators exist: notable is the resolution of the Kato problem, which can be formulated as a "regularity" boundary value problem for operators that satisfy very specific constraints in structure [Auscher et al. 2002; Hofmann et al. 2015; Hofmann and Martell 2003]. Other milestones in this theory include perturbation results such as those of [Auscher et al. 2008]. The challenge in this theory is that solutions to complex-coefficient elliptic operators are not necessarily continuous, nor do they satisfy even a weak maximum principle, which is typically the starting point for the study of boundary value problems. Some of the results for complex-coefficient operators have been proven under the assumption of interior Hölder regularity (the De Giorgi–Nash–Moser theory), yet it is not clear how this assumption can be correlated with quantitative verifiable assumptions on the operators.

In this paper we continue the investigation of solvability of boundary value problems for complex-valued second-order divergence form elliptic operators under a structural algebraic assumption on the

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matrix known as *p-ellipticity*. This structural assumption was introduced independently in [Dindoš and Pipher 2019] and [Carbonaro and Dragičević 2020] and is a quantitative strengthening of a condition related to L^p-contractivity of elliptic operators that was discovered by Cialdea and Mazya [2006]. When the coefficients of the operator are real, or when $p = 2$, the p-ellipticity condition is equivalent to the familiar uniform ellipticity condition.

In [Dindoš and Pipher 2019] we used the p-ellipticity condition to establish a limited interior regularity for solutions to these complex-coefficient second-order divergence form operators. We think of this as a weak substitute for the De Giorgi–Nash–Moser regularity of real-valued operators and, in fact, we used a variant of Moser's iteration argument to prove it. Specifically, we considered there operators of the form

$$\mathcal{L} = \operatorname{div} A(x)\nabla + B(x)\cdot\nabla,$$

where the matrix A is p-elliptic and B satisfies a natural minimal scaling condition. This limited regularity theory allowed us to address the solvability of the L^p Dirichlet problem for a collection of operators with complex coefficients whose matrices are in canonical form, as defined below. (The paper [Dindoš and Pipher 2019] contains a discussion of how to put an operator with lower-order terms in canonical form.)

This results of this paper concern the aforementioned regularity problem, in which the boundary data is prescribed in the Sobolev space of functions whose tangential derivatives belong to some L^p space. In analogy with the Dirichlet problem, where one expects to show classical convergence of a solution nontangentially to its boundary data in L^p through the control of a nontangential maximal function, in this problem one expects to prove nontangential estimates for the gradients of the solution in terms of the derivatives of the data on the boundary. The formulation of these estimates must take into account the fact these solutions and their derivatives do not have pointwise values, but are merely measurable functions in certain Lebesgue spaces.

We now discuss the class of elliptic operators for which Dirichlet and regularity problems are considered. In [Kenig and Pipher 2001], a class of real-valued second-order operators (with drift terms like those defined below) was introduced, and the elliptic measure associated to such operators was shown to belong to the A_∞ class with respect to surface measure on the boundary. This implies that the Dirichlet problem for these operators is solvable with data in L^p for some possibly large value of p. The study of this class of operators was motivated by a question of Dahlberg, which in turn was inspired by the fact that these operators arose naturally from a change of variables mapping from Lipschitz into flat domains. Specifically, the coefficients of the matrix A was assumed to satisfy a Carleson measure. Examples showed that A_∞ was the optimal result in this regime. Later, a slight strengthening of the Carleson measure condition was shown in [Dindoš et al. 2007] to imply solvability of the Dirichlet problem for the full range $1 < p < \infty$. We refer to this condition as the "small" Carleson condition, defined in Section 2.

In [Dindoš et al. 2017b], this regularity problem was solved for equations of the form $\mathcal{L} = \operatorname{div} A(x)\nabla$, with *A real and elliptic*, satisfying this small Carleson condition. There are open questions even for operators with real coefficients that satisfy the Carleson condition of [Kenig and Pipher 2001], such as solvability of the regularity problem in L^p for p near 1.

The first main result of this paper is the solvability of the regularity problem for boundary data $\nabla_T f \in L^p$ under the assumption that the matrix A is p-elliptic and satisfies small Carleson condition.

Theorem 1.1. *Let $1 < p < \infty$, and let Ω be the upper half-space $\mathbb{R}^n_+ = \{(x_0, x') : x_0 > 0$ and $x' \in \mathbb{R}^{n-1}\}$. Consider the operator*

$$\mathcal{L}u = \partial_i (A^0_{ij}(x)\, \partial_j u)$$

and assume that $\mathcal{L}$ can be rewritten as

$$\mathcal{L}u = \partial_i (A_{ij}(x)\, \partial_j u) + B_i\, \partial_i u, \tag{1-1}$$

where the matrix A is p-elliptic with constants λ_p, Λ, $A_{00} = 1$ and $\mathcal{I}m\, A_{0j} = 0$ for all $1 \le j \le n-1$. Assume also that

$$d\mu(x) = \sup_{B_{\delta(x)/2}(x)} (|\nabla A|^2 + |B|^2)\, \delta(x)\, dx \tag{1-2}$$

is a Carleson measure in Ω.

Then there exist $K = K(\lambda_p, \Lambda, n, p) > 0$ and $C(\lambda_p, \Lambda, n, p) > 0$ such that if

$$\|\mu\|_{\mathcal{C}} < K \tag{1-3}$$

then the L^p regularity problem

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \Omega, \\ u = f & \text{for } \sigma\text{-a.e. } x \in \partial\Omega, \\ \widetilde{N}_{p,a}(\nabla u) \in L^p(\partial\Omega) \end{cases} \tag{1-4}$$

is solvable and the estimate

$$\|\widetilde{N}_{p,a}(\nabla u)\|_{L^p(\partial\Omega)} \le C\, \|\nabla_T f\|_{L^p(\partial\Omega;\mathbb{C})} \tag{1-5}$$

holds for all energy solutions u with datum f.

The second main theorem of the paper extends the range of solvability of $\mathcal{L}u = 0$ with L^p Dirichlet boundary data for variable-coefficient complex-coefficient operators satisfying these Carleson conditions on coefficients. In [Dindoš and Pipher 2019] we have considered the solvability in the range $p \in (p_0, p_0')$, where

$$p_0 = \inf\{p > 1 : \text{the matrix } A \text{ is } p\text{-elliptic}\}. \tag{1-6}$$

Thanks to the solvability of the regularity problem (Theorem 1.1) we are now able to use the technique of Z. Shen [2005; 2006] and extend the previously established range of solvability of the Dirichlet problem to a larger interval

$$p \in \left(p_0, p_0' \frac{n-1}{n-1-p_0'}\right).$$

In particular, when $n = 2, 3$ or when $p_0' > n-1$, the range of solvability is extended to all $p \in (p_0, \infty)$.

Theorem 1.2. *Consider the operator*

$$\mathcal{L}u = \partial_i (A^0_{ij}(x)\, \partial_j u)$$

in the domain $\Omega = \mathbb{R}^n_+ = \{(x_0, x') : x_0 > 0 \text{ and } x' \in \mathbb{R}^{n-1}\}$. *Assume again that* $\mathcal{L}$ *can be rewritten as* (1-1) *and let* p_0 *be defined as in* (1-6) *and let* $p_{\max} = \infty$ *when* $p_0' \geq n-1$,

$$p_{\max} = \frac{p_0'(n-1)}{n-1-p_0'}$$

otherwise. Finally consider any $p_0 < p < p_{\max}$.

Assume further that the matrix A *satisfies* $A_{00} = 1$, $\mathscr{I}m\, A_{0j} = 0$ *for all* $1 \leq j \leq n-1$ *and let*

$$d\mu(x) = \sup_{B_{\delta(x)/2}(x)} (|\nabla A|^2 + |B|^2)\delta(x)\, dx \tag{1-7}$$

be a Carleson measure in Ω.

Then there exist $K = K(\lambda_p, \Lambda, n, p) > 0$ *and* $C(\lambda_p, \Lambda, n, p) > 0$ *such that if*

$$\|\mu\|_{\mathcal{C}} < K \tag{1-8}$$

then the L^p*-Dirichlet problem*

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \Omega, \\ u = f & \text{for } \sigma\text{-a.e. } x \in \partial\Omega, \\ \widetilde{N}_{p,a}(u) \in L^p(\partial\Omega) \end{cases} \tag{1-9}$$

is solvable and the estimate

$$\|\widetilde{N}_{p,a}(u)\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega;\mathbb{C})} \tag{1-10}$$

holds for all energy solutions u *with datum* f.

In particular observe that $p_{\max} = \infty$ in dimensions 2 and 3 and that when $n \geq 4$

$$p_{\max} > \frac{2(n-1)}{n-3}.$$

Remark 1.3. We address at the end of Section 2 how we can rewrite any operator $\mathcal{L}$ as (1-1) with coefficients A_{0j} real and $A_{00} = 1$. We require this particular form of our operator in the main section, Section 4, of this paper. We note that the transformation we introduce preserves the Carleson condition and hence for this condition we might assume that A_{0j} real and $A_{00} = 1$ without loss of generality. In order to preserve p-ellipticity, smallness of the imaginary part of A^0_{0j} is needed.

In the statement of these two theorems, we've used some notation that will be defined in subsequent sections. We will also recall there the concept of Carleson measure, discuss the notions of L^p solvability and energy solutions and define $\widetilde{N}_p$ which is a variant of the nontangential maximal function defined using L^p averages of the solution u.

Remark 1.4. Lemma 2.6 of [Dindoš and Pipher 2019] shows that L^q averages of solutions on interior balls are controlled bp L^q averages for q in the range $(p_0, p_0'n/(n-2))$, extending beyond the range of p-ellipticity. Thus one can use the N_q nontangential maximal function for such q in the estimate (1-10). The arguments for Theorem 1.1 show that, similarly, the gradient ∇u of solutions to the regularity problem will be locally L^q integrable for q in the range $(p_0, p_0'n/(n-2))$. In particular, by Sobolev embedding, solvability of the regularity problem in the regime $p_0' > n-2$ implies that solutions are Hölder continuous.

The paper is organized as follows. In Section 2, we define the concept of p-ellipticity, the nontangential maximal function, the p-adapted square function, Carleson measures and the notions of solvability of these various boundary value problems. In Section 3, we establish bounds for the nontangential maximal function by the square function. The estimates for the p-adapted square functions are established in Section 4. In light of (4-9), square functions that involve tangential derivatives are easier to handle and we begin by bounding these. We then show that, essentially, the square function with the full gradient can be controlled by the square functions of tangential derivatives. In Sections 5 and 6, we present the proofs of the two main theorems.

2. Basic notions and definitions

p-ellipticity. A concept related to p-ellipticity was introduced in [Cialdea and Mazya 2005], where the authors investigated the L^p-dissipativity of second-order divergence complex-coefficient operators. Later, and independently, we [Dindoš and Pipher 2019] and Carbonaro and Dragičević [2020] gave equivalent definitions of this property — the term "p-ellipticity" was coined in [Carbonaro and Dragičević 2020] and their definition is the one we introduce below. To introduce this, we define, for $p > 1$, the $\mathbb{R}$-linear map $\mathcal{J}_p : \mathbb{C}^n \to \mathbb{C}^n$ by

$$\mathcal{J}_p(\alpha + i\beta) = \frac{\alpha}{p} + i\,\frac{\beta}{p'},$$

where $p' = p/(p-1)$ and $\alpha, \beta \in \mathbb{R}^n$.

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$. Let $A : \Omega \to M_n(\mathbb{C})$, where $M_n(\mathbb{C})$ is the space of $n \times n$ complex-valued matrices. We say that A is p-elliptic if, for a.e. $x \in \Omega$,

$$\mathscr{Re}\langle A(x)\xi, \mathcal{J}_p\xi\rangle \geq \lambda_p|\xi|^2 \quad \text{for all } \xi \in \mathbb{C}^n, \tag{2-1}$$

for some $\lambda_p > 0$ and there exists $\Lambda > 0$ such that

$$|\langle A(x)\xi, \eta\rangle| \leq \Lambda|\xi||\eta| \quad \text{for all } \xi,\ \eta \in \mathbb{C}^n. \tag{2-2}$$

It is now easy to observe that the notion of 2-ellipticity coincides with the usual ellipticity condition for complex matrices. As shown in [Carbonaro and Dragičević 2020] if A is elliptic, then there exists $\mu(A) > 0$ such that A is p-elliptic if and only if $|1 - 2/p| < \mu(A)$. Also $\mu(A) = 1$ if and only if A is real-valued, implying p-ellipticity in the full range $p \in (1, \infty)$.

2.1. *Nontangential maximal and square functions.*

On a domain of the form

$$\Omega = \{(x_0, x') \in \mathbb{R} \times \mathbb{R}^{n-1} : x_0 > \phi(x')\}, \tag{2-3}$$

where $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function with Lipschitz constant given by $L := \|\nabla\phi\|_{L^\infty(\mathbb{R}^{n-1})}$, define for each point $x = (x_0, x') \in \Omega$

$$\delta(x) := x_0 - \phi(x') \approx \operatorname{dist}(x, \partial\Omega). \tag{2-4}$$

In other words, $\delta(x)$ is comparable to the distance of the point x from the boundary of Ω.

Definition 2.2. A cone of aperture $a > 0$ is a nontangential approach region to the point $Q = (x_0, x') \in \partial\Omega$ defined as

$$\Gamma_a(Q) = \{(y_0, y') \in \Omega : a|x_0 - y_0| > |x' - y'|\}. \tag{2-5}$$

We require $1/a > L$, otherwise the aperture of the cone is too large and might not lie inside Ω. When $\Omega = \mathbb{R}^n_+$ all parameters $a > 0$ may be considered. Sometimes it is necessary to truncate $\Gamma(Q)$ at height h, in which case we write

$$\Gamma_a^h(Q) := \Gamma_a(Q) \cap \{x \in \Omega : \delta(x) \le h\}, \tag{2-6}$$

$$\|S_a(w)\|^2_{L^2(\partial\Omega)} \approx \int_\Omega |\nabla w(x)|^2 \delta(x)\, dx. \tag{2-7}$$

In [Dindoš et al. 2007], a "p-adapted" square function was introduced. The usual square function is the p-adapted square function when $p = 2$. In the following definition, when $p < 2$ we use the convention that the expression $|\nabla w(x)|^2 |w(x)|^{p-2}$ is zero whenever $\nabla w(x)$ vanishes.

Definition 2.3. For $\Omega \subset \mathbb{R}^n$, the p-adapted square function of $w : \Omega \to \mathbb{C}$ such that $w|w|^{p/2-1} \in W^{1,2}_{\mathrm{loc}}(\Omega; \mathbb{C})$ at $Q \in \partial\Omega$ relative to the cone $\Gamma_a(Q)$ is defined by

$$S_{p,a}(w)(Q) := \left(\int_{\Gamma_a(Q)} |\nabla w(x)|^2 |w(x)|^{p-2} \delta(x)^{2-n}\, dx \right)^{1/2} \tag{2-8}$$

and, for each $h > 0$, its truncated version is given by

$$S^h_{p,a}(w)(Q) := \left(\int_{\Gamma^h_a(Q)} |\nabla w(x)|^2 |w(x)|^{p-2} \delta(x)^{2-n}\, dx \right)^{1/2}. \tag{2-9}$$

We further introduce the following convention. When $w : \Omega \to \mathbb{C}^k$ with component functions $(w_i)_{1 \le i \le k}$, we denote by $S_{p,a}(w)(Q)$ the sum

$$S_{p,a}(w)(Q) := \sum_{i=1}^k S_{p,a}(w_i)(Q); \tag{2-10}$$

hence for example if $w = \nabla_T u$ then $S_{p,a}(\nabla_T u)(Q)$ denotes

$$\sum_{i=1}^{n-1} S_{p,a}(\partial_i u)(Q).$$

It is not immediately clear that the integrals appearing in (2-8) are well-defined. However, in [Dindoš and Pipher 2019], it was shown that the expressions of the form $|\nabla w(x)|^2 |w(x)|^{p-2}$, when w is a solution of $\mathcal{L}w = 0$, are locally integrable and hence the definition of $S_p(w)$ makes sense for such p whenever p-ellipticity holds. This in particular applies with some modifications to $w = \nabla_T u$ on $\mathbb{R}^n_+$. Each component of w solves a PDE $\mathcal{L}(w_k) = \partial_i((\partial_k A_{ij})w_j) - \partial_k(B_i)w_i$. The right-hand side of this PDE is good enough for the regularity theory developed in [Dindoš and Pipher 2019] to apply to this more complicated system of equations as well.

A simple application of Fubini's theorem gives

$$\|S_{p,a}(w)\|^p_{L^p(\partial\Omega)} \approx \int_\Omega |\nabla w(x)|^2 |w(x)|^{p-2} \delta(x)\, dx. \tag{2-11}$$

Definition 2.4. For $\Omega \subset \mathbb{R}^n$ as above, and for a continuous $w : \Omega \to \mathbb{C}$, the nontangential maximal function (h-truncated nontangential maximal function) of u at $Q \in \partial\Omega$ relative to the cone $\Gamma_a(Q)$ is defined by

$$N_a(w)(Q) := \sup_{x \in \Gamma_a(Q)} |w(x)| \quad \text{and} \quad N_a^h(w)(Q) := \sup_{x \in \Gamma_a^h(Q)} |w(x)|. \tag{2-12}$$

Moreover, we shall also consider a related version of the above nontangential maximal function. This is denoted by $\widetilde{N}_{p,a}$ and is defined using L^p averages over balls in the domain Ω. Specifically, given $w \in L^p_{\text{loc}}(\Omega; \mathbb{C})$ we set

$$\widetilde{N}_{p,a}(w)(Q) := \sup_{x \in \Gamma_a(Q)} w_p(x) \quad \text{and} \quad \widetilde{N}^h_{p,a}(w)(Q) := \sup_{x \in \Gamma_a^h(Q)} w_p(x) \tag{2-13}$$

for each $Q \in \partial\Omega$ and $h > 0$ where, at each $x \in \Omega$,

$$w_p(x) := \left(\fint_{B_{\delta(x)/2}(x)} |w(z)|^p\, dz \right)^{1/p}. \tag{2-14}$$

Above and elsewhere, a barred integral indicates an averaging operation. Observe that, given $w \in L^p_{\text{loc}}(\Omega; \mathbb{C})$, the function w_p associated with w as in (2-14) is continuous and $\widetilde{N}_{p,a}(w) = N_a(w_p)$ everywhere on $\partial\Omega$.

The L^2-averaged nontangential maximal function was introduced in [Kenig and Pipher 1993] in connection with the Neuman and regularity problem value problems. In the context of p-ellipticity, Proposition 3.5 of [Dindoš and Pipher 2019] shows that there is no difference between L^2 averages and L^p averages when $w = u$ solves $\mathcal{L}u = 0$ and that $\widetilde{N}_{p,a}(u)$ and $\widetilde{N}_{2,a'}(u)$ are comparable in L^r norms for all $r > 0$ and all allowable apertures a, a'.

In this paper we shall consider $w = \nabla u$. However, as it turns out a modification of the argument following (2.20) of [Dindoš and Pipher 2019] applies in our case: each component $w^k = \partial_k u$ of w solves an equation similar to one considered in that paper, namely

$$\mathcal{L}w_k = \partial_i(A_{ij}\, \partial_j w_k) = \partial_i((\partial_k A_{ij})w_j). \tag{2-15}$$

Observe that the condition $|\nabla A(x)| \le K(\delta(x))^{-1}$ implies that the right-hand side of (2-15) is the divergence of a vector in L^2 and thus the solutions w_k will belong $W^{1,2}_{\text{loc}}$. We record the regularity results in the following proposition.

Proposition 2.5. *Suppose that $u \in W^{1,2}_{\mathrm{loc}}(\Omega;\mathbb{C})$ is the weak solution of $\mathcal{L}u = \operatorname{div} A(x)\nabla u = 0$ in Ω. Let $p_0 = \inf\{p > 1 : A \text{ is } p\text{-elliptic}\}$, and suppose that A has bounded measurable coefficients satisfying*

$$|\nabla A(x)| \le K(\delta(x))^{-1} \quad \textit{for all } x \in \Omega, \tag{2-16}$$

where the constant K is uniform, and $\delta(x)$ denotes the distance of x to the boundary of Ω. Then we have the following improvement in the regularity of ∇u. For any $B_{4r}(x) \subset \Omega$ and $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\left(\fint_{B_r(x)} |\nabla u|^p \, dy\right)^{1/p} \le C_\varepsilon \left(\fint_{B_{2r}(x)} |\nabla u|^q \, dy\right)^{1/q} + \varepsilon \left(\fint_{B_{2r}(x)} |\nabla u|^2 \, dy\right)^{1/2} \tag{2-17}$$

for all $p, q \in (p_0, p_0' n/(n-2))$. (Here $p_0' = p_0/(p_0 - 1)$ and when $n = 2$ one can take $p, q \in (p_0, \infty)$.) The constant in the estimate depends on the dimension, the p-ellipticity constants, Λ, K and $\varepsilon > 0$ but not on $x \in \Omega$, $r > 0$ or u.

It follows that for any boundary ball $\Delta = \Delta_d \subset \partial\Omega$, for any $p, q \in (p_0, p_0' n/(n-2))$ and for any allowed aperture parameters $a, a' > 0$ there exists $m = m(a, a') > 1$ such that

$$\|\widetilde{N}^d_{p,a}(\nabla u)\|_{L^r(\Delta_d)} \lesssim \|\widetilde{N}^{2d}_{q,a'}(\nabla u)\|_{L^r(m\Delta_d)} \tag{2-18}$$

for all $r > 0$. We also have for the same range of p's the estimate

$$\left(r^2 \fint_{B_r(x)} |\nabla \partial_k u|^2 |\partial_k u|^{p-2} \, dy\right)^{1/p} \le C_p \left(\fint_{B_{2r}(x)} |\nabla u|^2 \, dy\right)^{1/2} \tag{2-19}$$

for all $k = 0, 1, 2, \dots, n-1$.

2.2. ***Carleson measures.*** We begin by recalling the definition of a Carleson measure in a domain Ω as in (2-3). For $P \in \mathbb{R}^n$, define the ball centered at P with the radius $r > 0$ as

$$B_r(P) := \{x \in \mathbb{R}^n : |x - P| < r\}. \tag{2-20}$$

Next, given $Q \in \partial\Omega$, by $\Delta = \Delta_r(Q)$ we denote the surface ball $\partial\Omega \cap B_r(Q)$. The Carleson region $T(\Delta_r)$ is then defined by

$$T(\Delta_r) := \Omega \cap B_r(Q). \tag{2-21}$$

Definition 2.6. A Borel measure μ in Ω is said to be Carleson if there exists a constant $C \in (0, \infty)$ such that for all $Q \in \partial\Omega$ and $r > 0$

$$\mu(T(\Delta_r)) \le C\sigma(\Delta_r), \tag{2-22}$$

where σ is the surface measure on $\partial\Omega$. The best possible constant C in the above estimate is called the Carleson norm and is denoted by $\|\mu\|_{\mathcal{C}}$.

In all that follows we now assume that the coefficients of the matrices A and B of the elliptic operator $\mathcal{L} = \operatorname{div} A(x)\nabla + B(x)\cdot\nabla$ satisfy the following natural conditions. First, we assume that the entries A_{ij} of A are in $\mathrm{Lip}_{\mathrm{loc}}(\Omega)$ and the entries of B are $L^\infty_{\mathrm{loc}}(\Omega)$. Second, we assume that

$$d\mu(x) = \sup_{B_{\delta(x)/2}(x)} [|\nabla A|^2 + |B|^2]\delta(x)\,dx \tag{2-23}$$

is a Carleson measure in Ω. Sometimes, and for certain coefficients of A, we will assume that their Carleson norm $\|\mu\|_{\mathcal{C}}$ is sufficiently small. The fact that μ is a Carleson allows one to relate integrals in Ω with respect to μ to boundary integrals involving the nontangential maximal function. We have the following result for our averaged nontangential maximal function; see [Dindoš and Pipher 2019].

Theorem 2.7. *Suppose that $d\nu = f\,dx$ and $d\mu(x) = [\sup_{B_{\delta(x)/2}(x)} |f|]\,dx$. Assume that μ is a Carleson measure. Then there exists a finite constant $C = C(L, a) > 0$ such that for every $u \in L^p_{\mathrm{loc}}(\Omega;\mathbb{C})$ one has*

$$\int_\Omega |u(x)|^p\,d\nu(x) \le C\|\mu\|_{\mathcal{C}} \int_{\partial\Omega} (\widetilde{N}_{p,a}(u))^p\,d\sigma. \tag{2-24}$$

Furthermore, consider $\Omega = \mathbb{R}^n_+$, where μ and ν are measures as above supported in Ω and $\delta(x_0, x') = x_0$. Let $h : \mathbb{R}^{n-1} \to \mathbb{R}^+$ be a Lipschitz function with Lipschitz norm L and

$$\Omega_h = \{(x_0, x') : x_0 > h(x')\}.$$

Then for any $\Delta \subset \mathbb{R}^{n-1}$ with $\sup_\Delta h \le \operatorname{diam}(\Delta)/2$ we have

$$\int_{\Omega_h\cap T(\Delta)} |u(x)|^p\,d\nu(x) \le C\|\mu\|_{\mathcal{C}} \int_{\partial\Omega_h\cap T(\Delta)} (\widetilde{N}_{p,a,h}(u))^p\,d\sigma. \tag{2-25}$$

Here for a point $Q = (h(x'), x') \in \partial\Omega_h$ we define

$$\widetilde{N}_{p,a,h}(u)(Q) = \sup_{\Gamma_a(Q)} w, \tag{2-26}$$

where

$$\Gamma_a(Q) = \Gamma_a((h(x'), x')) = \{y = (y_0, y') \in \Omega : a|h(x') - y_0| > |x' - y'|\} \tag{2-27}$$

and the L^p averages w are defined by (2-14) *where the distance δ is taken with respect to the domain $\Omega = \mathbb{R}^n_+$.*

2.3. *The L^p-Dirichlet problem.* We recall the definition of L^p solvability of the Dirichlet problem. When an operator $\mathcal{L}$ is as in Theorem 1.2 is uniformly elliptic (i.e., 2-elliptic) the Lax-Milgram lemma can be applied and guarantees the existence of weak solutions. That is, given any $f \in \dot{B}^{2,2}_{1/2}(\partial\Omega;\mathbb{C})$, the homogenous space of traces of functions in $\dot{W}^{1,2}(\Omega;\mathbb{C})$, there exists a unique (up to a constant) $u \in \dot{W}^{1,2}(\Omega;\mathbb{C})$ such that $\mathcal{L}u = 0$ in Ω and $\operatorname{Tr} u = f$ on $\partial\Omega$. We call these solutions "energy solutions" and use them to define the notion of solvability of the L^p Dirichlet problem.

Definition 2.8. Let Ω be the Lipschitz domain introduced in (2-3) and fix an integrability exponent $p \in (1,\infty)$. Also, fix an aperture parameter $a > 0$. Consider the following Dirichlet problem for a complex-valued function $u : \Omega \to \mathbb{C}$:

$$\begin{cases} 0 = \partial_i (A_{ij}(x)\,\partial_j u) + B_i(x)\,\partial_i u & \text{in } \Omega, \\ u(x) = f(x) & \text{for } \sigma\text{-a.e. } x \in \partial\Omega, \\ \widetilde{N}_{2,a}(u) \in L^p(\partial\Omega), \end{cases} \tag{2-28}$$

where the usual Einstein summation convention over repeated indices (i, j in this case) is employed.

We say the Dirichlet problem (2-28) is solvable for a given $p \in (1,\infty)$ if there exists a $C = C(p,\Omega) > 0$ such that for all boundary data $f \in L^p(\partial\Omega;\mathbb{C}) \cap \dot{B}^{2,2}_{1/2}(\partial\Omega;\mathbb{C})$ the unique energy solution satisfies the estimate

$$\|\widetilde{N}_{2,a}(u)\|_{L^p(\partial\Omega)} \le C\|f\|_{L^p(\partial\Omega;\mathbb{C})}. \tag{2-29}$$

Similarly, we say the regularity problem for the same PDE is solvable for a given $p \in (1,\infty)$ if there exists a $C = C(p,\Omega) > 0$ such that for all boundary data $f \in \dot{W}^{1,p}(\partial\Omega;\mathbb{C}) \cap \dot{B}^{2,2}_{1/2}(\partial\Omega;\mathbb{C})$ the unique (modulo constants) energy solution satisfies the estimate

$$\|\widetilde{N}_{2,a}(\nabla u)\|_{L^p(\partial\Omega)} \le C\|\nabla_T f\|_{L^p(\partial\Omega;\mathbb{C})}. \tag{2-30}$$

Remark. Given $f \in \dot{B}^{2,2}_{1/2}(\partial\Omega;\mathbb{C}) \cap L^p(\partial\Omega;\mathbb{C})$ the corresponding energy solution constructed above is unique (since the decay implied by the L^p estimates eliminates constant solutions). As the space $\dot{B}^{2,2}_{1/2}(\partial\Omega;\mathbb{C}) \cap L^p(\partial\Omega;\mathbb{C})$ is dense in $L^p(\partial\Omega;\mathbb{C})$ for each $p \in (1,\infty)$, it follows that there exists a unique continuous extension of the solution operator $f \mapsto u$ to the whole space $L^p(\partial\Omega;\mathbb{C})$, with u such that $\widetilde{N}_{2,a}(u) \in L^p(\partial\Omega)$ and the accompanying estimate $\|\widetilde{N}_{2,a}(u)\|_{L^p(\partial\Omega)} \le C\|f\|_{L^p(\partial\Omega;\mathbb{C})}$ being valid. Furthermore, as shown in the Appendix of [Dindoš and Pipher 2019] for any $f \in L^p(\partial\Omega;\mathbb{C})$ the corresponding solution u constructed by the continuous extension attains the datum f as its boundary values in the following sense. Consider the average $\tilde{u} : \Omega \to \mathbb{C}$ defined by

$$\tilde{u}(x) = \fint_{B_{\delta(x)/2}(x)} u(y)\,dy \quad \text{for all } x \in \Omega.$$

Then

$$f(Q) = \lim_{x \to Q,\, x \in \Gamma(Q)} \tilde{u}(x) \quad \text{for a.e. } Q \in \partial\Omega, \tag{2-31}$$

where the a.e. convergence is taken with respect to the $\mathcal{H}^{n-1}$ Hausdorff measure on $\partial\Omega$.

We can make a similar statement regarding nontangential convergence of gradients for solutions to the regularity problem. That is, defining

$$\widetilde{\nabla} u(x) = \fint_{B_{\delta(x)/2}(x)} \nabla u(y)\,dy \quad \text{for all } x \in \Omega,$$

the same proof in [Dindoš and Pipher 2019] yields that

$$\nabla u(Q) = \lim_{x \to Q,\, x \in \Gamma(Q)} \widetilde{\nabla} u(x) \quad \text{for a.e. } Q \in \partial\Omega, \tag{2-32}$$

and when $\Omega = \mathbb{R}^n_+$,

$$\nabla_T f(Q) = \lim_{x \to Q,\, x \in \Gamma(Q)} \widetilde{\nabla}_T u(x) \quad \text{for a.e. } Q \in \partial\Omega. \tag{2-33}$$

Let us make some observations that explain the structural assumptions we have made in Theorems 1.1 and 1.2. As we have already stated it suffices to formulate the result in the case $\Omega = \mathbb{R}^n_+$ by using the pull-back map introduced above. Since Theorem 1.2 requires that the coefficients have *small* Carleson norm, this puts a restriction on the size of the Lipschitz constant $L = \|\nabla\phi\|_{L^\infty}$ of the map ϕ that defines the domain Ω in (2-3). The constant L will have also to be small (depending on λ_p, Λ, n and p).

For technical reasons in the proof we also need that all coefficients A_{0j}, $j = 0, 1, \dots, n-1$, are real. This can be ensured as follows. When $j > 0$ observe that we have

$$\partial_0([\mathscr{I}m\, A_{0j}]\, \partial_j u) = \partial_j([\mathscr{I}m\, A_{0j}]\, \partial_0 u) + (\partial_0[\mathscr{I}m\, A_{0j}])\, \partial_j u - ([\partial_j\, \mathscr{I}m\, A_{0j}])\, \partial_0 u, \tag{2-34}$$

which allows us to move the imaginary part of the coefficient A_{0j} onto the coefficient A_{j0} at the expense of two (harmless) first-order terms. This does not work for the coefficient A_{00}. Instead we make the following observation.

Suppose that the measure (2-23) associated to an operator $\mathcal{L} = \partial_i(A_{ij}(x)\,\partial_j) + B_i(x)\,\partial_i$ is Carleson. Consider a related operator $\widetilde{\mathcal{L}} = \partial_i(\widetilde{A}_{ij}(x)\,\partial_j) + \widetilde{B}_i(x)\,\partial_i$, where $\widetilde{A} = \alpha A$ and $\widetilde{B} = \alpha B - (\partial_i\alpha)A_{ij}\,\partial_j$, and $\alpha \in L^\infty(\Omega)$ is a complex-valued function such that $|\alpha(x)| \ge \alpha_0 > 0$ and $|\nabla\alpha|^2 x_0$ is a Carleson measure.

Observe that a weak solution u to $\widetilde{\mathcal{L}}u = 0$ is also a weak solution to $\mathcal{L}u = 0$ and that the new coefficients of $\widetilde{A}$ and $\widetilde{B}$ also satisfy a Carleson measure condition as in (2-23), from the assumption on α. We will only require that the coefficient $\widetilde{A}_{00}$ is real but we may as well ensure for simplicity that it is equal to 1. Clearly, if we choose $\alpha = A_{00}^{-1}$, then the new operator $\widetilde{L}$ will have this property. If A_{00} (and hence α) is real, then $\widetilde{A}$ is p-elliptic if and only if A is p-elliptic. Similarly, if A is p-elliptic and $\mathscr{I}m\, A_{00}$ is sufficiently small (depending on the ellipticity constants), then $\widetilde{A}$ will also be p-elliptic. However, if $\mathscr{I}m\,\alpha$ is not small, the p-ellipticity, after multiplication of A by α may not be preserved. Thus, we assume in our main results (Theorems 1.1 and 1.2) the p-ellipticity of the new matrix $\widetilde{A}$, which has all coefficients $\widetilde{A}_{0j}$, $j = 0, 1, \dots, n-1$, real, as this is not implied in the general case from the p-ellipticity of the original matrix A.

3. Bounds for the nontangential maximal function by the square function

We work on $\Omega = \mathbb{R}^n_+$ and we assume that the matrix A is p-elliptic. Our aim in this section is to establish bounds for the nontangential maximal function by the square function. The approach necessarily differs from the usual argument in the real scalar elliptic case due to the fact that certain estimates, such as interior Hölder regularity of a weak solution, are unavailable for the complex-coefficient case. Here we deviate from the approach taken in [Dindoš and Pipher 2019], where we worked with the p-adapted square function, and instead focus on the estimates for the usual square function. Our approach is similar to [Dindoš et al. 2017a] for elliptic systems and when possible we refer to results from there.

The major innovation from [Dindoš et al. 2017a] is the use of an entire family of Lipschitz graphs on which the nontangential maximal function is large in lieu of a single graph constructed via a stopping time argument. This is necessary as we are using L^2 averages of solutions to define the nontangential maximal function, and hence the knowledge of certain bounds for a solution on a single graph provides no information about the L^2 averages over interior balls.

Let u be an energy solution to

$$\mathcal{L}u = \partial_i(A_{ij}\,\partial_j u) = 0 \quad \text{in } \Omega = \mathbb{R}^n_+.$$

Let $v = \nabla u$, that is, $v_k = \partial_k u,\ k = 0, 1, \dots, n-1$. Let $w = w_2$ be the L^2 averages of v, that is,

$$w(x) := \left(\fint_{B_{\delta(x)/2}(x)} |v(z)|^2\, dz \right)^{1/2}. \tag{3-1}$$

Set

$$E_{\nu,a} := \{x' \in \partial\Omega : N_a(w)(x') > \nu\} \tag{3-2}$$

(where, as usual, $a > 0$ is a fixed background parameter), and consider the map $h : \partial\Omega \to \mathbb{R}$ given at each $x' \in \partial\Omega$ by

$$h_{\nu,a}(w)(x') := \inf\{x_0 > 0 : \sup_{z\in\Gamma_a(x_0,x')} w(z) < \nu\}, \tag{3-3}$$

with the convention that $\inf \varnothing = \infty$. We remark that h differs somewhat from the function that has been used in the argument for scalar equations; see [Kenig and Pipher 2001, pp. 212] and [Kenig et al. 2000].

At this point we note that $h_{\nu,a}(w, x') < \infty$ for all points $x' \in \partial\Omega$. Since $u \in \dot W^{1,2}(\mathbb{R}^n_+; \mathbb{C})$, it follows that $v \in L^2(\mathbb{R}^n_+; \mathbb{C}^n)$. Thus w as an L^2 average of v is continuous on the upper half-space and decays to zero as $x_0 \to \infty$. Thus $h_{\nu,a}$ is finite everywhere.

We look at some further properties of this function. As in [Dindoš et al. 2017a] we have the following (with identical proof).

Lemma 3.1. *Let w be as above* (3-1). *Also, fix two positive numbers ν, a. Then the following properties hold*:

(i) *The function $h_{\nu,a}(w)$ is Lipschitz, with a Lipschitz constant $1/a$. That is,*

$$|h_{\nu,a}(w)(x') - h_{\nu,a}(w)(y')| \le a^{-1}|x' - y'| \tag{3-4}$$

for all $x', y' \in \partial\Omega$.

(ii) *Given an arbitrary $x' \in E_{\nu,a}$, let $x_0 := h_{\nu,a}(w)(x')$. Then there exists a point $y = (y_0, y') \in \partial\Gamma_a(x_0, x')$ such that $w(y) = \nu$ and $h_{\nu,a}(w)(y') = y_0$.*

We also have (as in [Dindoš et al. 2017a]) by an identical argument:

Lemma 3.2. *Let v, w be as above. For any $a > 0$ there exists $b = b(a) > a$ and $\gamma = \gamma(a) > 0$ such that the following holds. Having fixed an arbitrary $\nu > 0$, for each point x' from the set*

$$\{x' : N_a(w)(x') > \nu \text{ and } S_b(v)(x') \le \gamma\nu\} \tag{3-5}$$

there exists a boundary ball R with $x' \in 2R$ and such that

$$|w(h_{\nu,a}(w)(z'), z')| > \nu/2 \quad \text{for all } z' \in R. \tag{3-6}$$

Here $S_b = S_{2,b}$ is the usual square function of $v = \nabla u$ associated with nontangential cones $\Gamma_b(\cdot)$.

Given a Lipschitz function $h : \mathbb{R}^{n-1} \to \mathbb{R}$, denote by M_h the Hardy–Littlewood maximal function considered on the graph of h. That is, given any locally integrable function f on the Lipschitz surface $\Lambda_h = \{(h(z'), z') : z' \in \mathbb{R}^{n-1}\}$, define

$$(M_h f)(x) := \sup_{r>0} ⨍_{\Lambda_h \cap B_r(x)} |f|\, d\sigma$$

for each $x \in \Lambda_h$.

Corollary 3.3. *Let v, w be defined as above and let $a > 0$ be fixed. Associated with these, let b, γ be as in Lemma 3.2. Then there exists a finite constant $C = C(n, p) > 0$ with the property that for any $\nu > 0$ and any point $x' \in E_{\nu,a}$ such that $S_b(v)(x') \le \gamma\nu$ one has*

$$(M_{h_{\nu,a}} w)(h_{\nu,a}(x'), x') \ge C\nu. \tag{3-7}$$

The following lemma requires a modified proof, which we include below.

Lemma 3.4. *Consider the equation $\mathcal{L}u = 0$ with coefficients satisfying the assumptions of Theorem 1.2, let $v = \nabla u$ and let w be defined by* (3-1). *Then there exists $a > 0$ with the following significance. Select $\theta \in \left[\frac{1}{6}, 6\right]$ and, having picked $\nu > 0$ arbitrary, let $h_{\nu,a}(w)$ be as in* (3-3). *Also, consider the domain $\mathcal{O} = \{(x_0, x') \in \Omega : x_0 > \theta h_{\nu,a}(x')\}$ with boundary $\partial\mathcal{O} = \{(x_0, x') \in \Omega : x_0 = \theta h_{\nu,a}(x')\}$. In this context, for any surface ball $\Delta_r = B_r(Q) \cap \partial\Omega$, with $Q \in \partial\Omega$ and $r > 0$ chosen such that $h_{\nu,a}(w) \le 2r$ pointwise on Δ_{2r}, one has*

$$\begin{aligned}\int_{\Delta_r} |v(\theta h_{\nu,a}(w)(\cdot), \cdot)|^2\, dx' \le C(1+\|\mu\|_{\mathcal{C}}^{1/2}) \|S_b(v)\|_{L^p(\Delta_{2r})} \|N_{2,a}(w)\|_{L^p(\Delta_{2r})} \\ + C\|\mu\|_{\mathcal{C}}^{1/2} \|N_{2,a}(w)\|^2_{L^p(\Delta_{2r})} + C\|S_b(v)\|^2_{L^p(\Delta_{2r})} + \frac{c}{r} \iint_{\mathcal{K}} |v|^2\, dX.\end{aligned} \tag{3-8}$$

Here $C = C(\Lambda, p, n) \in (0, \infty)$ and $\mathcal{K}$ is a region inside $\mathcal{O}$ with diameter, distance to the boundary $\partial\mathcal{O}$, and distance to Q all comparable to r. Also, the parameter $b > a$ is as in Lemma 3.2, and the cones used to define the square and nontangential maximal functions in this lemma have vertices on $\partial\Omega$.

Moreover, the term $\iint_{\mathcal{K}} |v|^2\, dX$ appearing in (3-8) *may be replaced by the quantity*

$$C r^{n-1} |\tilde{v}(A_r)|^2 + C \int_{\Delta_{2r}} S_b^2(v)\, d\sigma, \tag{3-9}$$

where A_r is any point inside $\mathcal{K}$ (usually called a corkscrew point of Δ_r) and

$$\tilde{v}(X) := ⨍_{B_{\delta(X)/2}(X)} v(Z)\, dZ. \tag{3-10}$$

Finally, (3-8) *and* (3-9) *remain true even if v is replaced by $v - v_0$ for any fixed $v_0 \in \mathbb{C}^n$.*

Proof. Fix $\theta \in \left[\frac{1}{6}, 6\right]$. Consider the well-known pullback transformation $\rho : \mathbb{R}^n_+ \to \mathcal{O}$ appearing in works of Dahlberg, Nečas, Kenig and Stein and others, defined by

$$\rho(x_0, x') := (x_0 + P_{\gamma x_0} * \phi(x'), x') \quad \text{for all } (x_0, x') \in \mathbb{R}^n_+, \tag{3-11}$$

for some positive constant γ. Here ϕ is a Lipschitz function describing the boundary on $\partial\mathcal{O}$, P is a nonnegative function $P \in C_0^\infty(\mathbb{R}^{n-1})$ and, for each $\lambda > 0$,

$$P_\lambda(x') := \lambda^{-n+1} P(x'/\lambda) \quad \text{for all } x' \in \mathbb{R}^{n-1}. \tag{3-12}$$

Finally, $P_\lambda * \phi(x')$ is the convolution

$$P_\lambda * \phi(x') := \int_{\mathbb{R}^{n-1}} P_\lambda(x'-y')\phi(y')\,dy'. \tag{3-13}$$

Observe that ρ extends up to the boundary of $\mathbb{R}^n_+$ and maps one-to-one from $\partial\mathbb{R}^n_+$ onto $\partial\mathcal{O}$. Also for sufficiently small $\gamma \lesssim L$ the map ρ is a bijection from $\overline{\mathbb{R}}^n_+$ onto $\overline{\mathcal{O}}$ and, hence, is invertible.

For a solution $u \in W^{1,2}_{\mathrm{loc}}(\mathcal{O};\mathbb{C})$ to $\mathcal{L}u = 0$ in $\mathcal{O}$ with Dirichlet datum f, consider $\tilde{u} := u \circ \rho$ and $\tilde{f} := f \circ \rho$. The change of variables via the map ρ just described implies that $\tilde{u} \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n_+;\mathbb{C})$ solves a new elliptic PDE of the form

$$\partial_i(\widetilde{A}_{ij}(x)\,\partial_j\tilde{u}) = 0, \tag{3-14}$$

with boundary datum $\tilde{f}$ on $\partial\mathbb{R}^n_+$. Hence, solving a boundary value problem for u in Ω is equivalent to solving a related boundary value problem for $\tilde{u}$ in $\mathbb{R}^n_+$. Crucially, if the coefficients of the original system are such that (2-23) is a Carleson measure, then the coefficients of $\widetilde{A}$ satisfy an analogous Carleson condition in the upper-half space. If, in addition, the Carleson norm of (2-23) is small and L (the Lipschitz constant for the domain Ω) is also small, then the Carleson norm for the new coefficients $\widetilde{A}$

$$d\tilde{\mu}(x) = \Big(\sup_{B_{\delta(x)/2}(x)} |\nabla\widetilde{A}|\Big)^2 \delta(x)\,dx \tag{3-15}$$

will be correspondingly small and will only depend on the Carleson norm of the original coefficients and the Lipschitz norm of the function $h_{\nu,a}$. When the Lipschitz norm of this function goes to zero we have

$$\limsup \|\tilde{\mu}\|_{\mathcal{C}} \le \|\mu\|_{\mathcal{C}},$$

and hence the parameter $a > 0$ may be chosen large enough so that the Lipschitz norm of the function $\theta h_{\nu,a}$ is sufficiently small (at most $6/a$) such that $\|\tilde{\mu}\|_{\mathcal{C}} \le 2\|\mu\|_{\mathcal{C}}$. Moreover, this transformation also preserves ellipticity.

Having fixed a scale $r > 0$, we localize to a ball $B_r(y')$ in $\mathbb{R}^{n-1}$. Let ζ be a smooth cutoff function of the form $\zeta(x_0, x') = \zeta_0(x_0)\zeta_1(x')$, where

$$\zeta_0 = \begin{cases} 1 & \text{in } [0,r], \\ 0 & \text{in } [2r,\infty), \end{cases} \qquad \zeta_1 = \begin{cases} 1 & \text{in } B_r(y'), \\ 0 & \text{in } \mathbb{R}^n \setminus B_{2r}(y') \end{cases} \tag{3-16}$$

and

$$r|\partial_0\zeta_0| + r|\nabla_{x'}\zeta_1| \le c \tag{3-17}$$

for some constant $c \in (0,\infty)$ independent of r. Our goal is to control the L^2 norm of $\nabla u(\theta h_{\nu,a}(w)(\cdot),\cdot)$. Since after the pullback under the mapping ρ the latter is comparable with the L^2 norm of $\nabla\tilde{u}(0,\cdot)$, we proceed to estimate this quantity.

Clearly, if we establish estimate (3-8) for $\nabla\tilde{u}$ on $\Delta_r \subset \partial\mathbb{R}^n_+$ it will imply the original estimate for ∇u on the graph of $\theta h_{v,a}$.

Hence, let $\tilde{v} = \nabla\tilde{u}$. For $\tilde{v}_k$, where $k = 1, 2, \dots, n-1$, we have

$$\begin{aligned}\int_{B_{2r}(y')} |\tilde{v}_k|^2(0,x')\zeta(0,x')\,dx' &= -\iint_{[0,2r]\times B_{2r}(y')} \partial_0[|\tilde{v}_k|^2\zeta](x_0,x')\,dx_0\,dx' \\ &= -2\iint_{[0,2r]\times B_{2r}(y')} \mathscr{R}e\langle\tilde{v}_k, \partial_0\tilde{v}_k\rangle\zeta\,dx_0\,dx' \\ &\qquad -\iint_{[0,2r]\times B_{2r}(y')} |\tilde{v}_k|^2(x_0,x')\,\partial_0\zeta\,dx_0\,dx' \\ &=: \mathcal{A} + IV. \end{aligned} \tag{3-18}$$

We further expand the term $\mathcal{A}$ as a sum of three terms obtained via integration by parts with respect to x_0 as follows:

$$\begin{aligned}\mathcal{A} &= -2\iint_{[0,2r]\times B_{2r}(y')} \mathscr{R}e\langle\tilde{v}_k, \partial_0\tilde{v}_k\rangle\zeta(\partial_0 x_0)\,dx_0\,dx' \\ &= 2\iint_{[0,2r]\times B_{2r}(y')} |\partial_0\tilde{v}_k|^2 x_0\zeta\,dx_0\,dx' \\ &\quad + 2\iint_{[0,2r]\times B_{2r}(y')} \mathscr{R}e\langle\tilde{v}_k, \partial^2_{00}\tilde{v}_k\rangle x_0\zeta\,dx_0\,dx' \\ &\quad + 2\iint_{[0,2r]\times B_{2r}(y')} \mathscr{R}e\langle\tilde{v}_k, \partial_0\tilde{v}_k\rangle x_0\,\partial_0\zeta\,dx_0\,dx' \\ &=: I + II + III. \end{aligned} \tag{3-19}$$

We start by analyzing the term II. We write

$$\partial_0^2\tilde{v}_k = \partial_k\partial_0\tilde{v}_0$$

and integrate by parts moving the ∂_k derivative. This gives

$$\begin{aligned} II &= 2\iint_{[0,2r]\times B_{2r}(y')} \mathscr{R}e\langle\tilde{v}_k, \partial_k\partial_0\tilde{v}_0\rangle x_0\zeta\,dx_0\,dx' \\ &= -2\iint_{[0,2r]\times B_{2r}(y')} \mathscr{R}e\langle\partial_k\tilde{v}_k, \partial_0\tilde{v}_0\rangle x_0\zeta\,dx_0\,dx' - 2\iint_{[0,2r]\times B_{2r}(y')} \mathscr{R}e\langle\tilde{v}_0, \partial_k\tilde{v}_k\rangle x_0\,\partial_0\zeta\,dx_0\,dx' \\ &= II_1 + II_2. \end{aligned} \tag{3-20}$$

We now group together terms that are of the same type. Firstly, we have

$$I + II_1 \le C(\Lambda, n)\|S_b(v)\|^2_{L^2(B_{2r})}. \tag{3-21}$$

Here, the estimate would be true even with truncated square function $\|S_b^{2r}(\tilde{v})\|^2_{L^2(B_{2r})}$, which is at every point dominated by $\|S_b(v)\|^2_{L^2(B_{2r})}$.

Next, corresponding to the case when the derivative falls on the cutoff function ζ we have

$$
\begin{aligned}
II_2 + III &\le C(\Lambda, n) \iint_{[0,2r]\times B_{2r}} |\nabla \tilde{v}||\tilde{v}| \frac{x_0}{r}\, dx_0\, dx' \\
&\le C(\Lambda, n)\left(\iint_{[0,2r]\times B_{2r}} |\tilde{v}|^2 \frac{x_0}{r^2}\, dx_0\, dx' \right)^{1/2} \|S_b^{2r}(\tilde{v})\|_{L^2(B_{2r})} \\
&\le C(\Lambda, n) \|S_b(v)\|^{p/2}_{L^p(B_{2r})} \|N_{p,a}(w)\|_{L^2(B_{2r})}. \qquad (3\text{-}22)
\end{aligned}
$$

Finally, the interior term V, which arises from the fact that $\partial_0 \zeta$ vanishes on the set $(0, r) \cup (2r, \infty)$ may be estimated as follows:

$$
IV \le \frac{c}{r} \iint_{[r,2r]\times B_{2r}} |v|^2\, dx_0\, dx'. \qquad (3\text{-}23)
$$

Summing up all terms, the above analysis ultimately yields

$$
\begin{aligned}
\int_{B_r(y')} |\nabla_T \tilde{u}(0, x')|^2\, dx' \le C(\Lambda, n)(1 + \|\mu\|_C^{1/2}) \|S_b(v)\|_{L^p(B_{2r})} \|N_a(w)\|_{L^p(B_{2r})} \\
+ C(\Lambda, n)\|S_b(v)\|^2_{L^p(B_{2r})} + \frac{c}{r} \iint_{[r,2r]\times B_{2r}} |v|^2\, dx_0\, dx'. \qquad (3\text{-}24)
\end{aligned}
$$

Observe also we could have done the whole calculation with a constant subtracted from $\tilde{v}_k$ without any substantial modifications. It remains to consider the derivative in a transversal direction to the boundary. Instead of $\tilde{v}_0 = \partial_0 \tilde{u}$ it is more convenient to work with

$$
H = \sum_{j=0}^{n-1} \widetilde{A}_{0j} \tilde{v}_j,
$$

which will give us the desired bound since

$$
\begin{aligned}
&\int_{B_{2r}(y')} |\tilde{v}_0|^2(0, x')\zeta(0, x')\, dx' \\
&\qquad \approx \int_{B_{2r}(y')} |\widetilde{A}_{00} \tilde{v}_0(0, x')|^2 \zeta(0, x')\, dx' \\
&\qquad \le n \left(\int_{B_{2r}(y')} |H(0, x')|^2 \zeta(0, x')\, dx' + \sum_{j>0} \int_{B_{2r}(y')} |\widetilde{A}_{0j} \tilde{v}_j(0, x')|^2 \zeta(0, x')\, dx' \right) \\
&\qquad \le n \int_{B_{2r}(y')} |H(0, x')|^2 \zeta(0, x')\, dx' + C(n, \Lambda) \int_{B_r(y')} |\nabla_T \tilde{u}(0, x')|^2\, dx'. \qquad (3\text{-}25)
\end{aligned}
$$

The second term is okay as we have (3-24). We deal with the first term now. A calculation similar to (3-18)–(3-19) gives us

$$
\begin{aligned}
&\int_{B_{2r}(y')} |H|^2(0, x')\zeta(0, x')\, dx' \\
&\quad = -2 \iint_{[0,2r]\times B_{2r}(y')} \mathscr{Re}\langle H, \partial_0 H \rangle \zeta\, dx_0\, dx' - \iint_{[0,2r]\times B_{2r}(y')} |H|^2(x_0, x')\, \partial_0 \zeta\, dx_0\, dx'. \qquad (3\text{-}26)
\end{aligned}
$$

The second term has an estimate similar to (3-23). For the first term we use the fact that $\widetilde{\mathcal{L}}\tilde{u} = 0$, which implies

$$\partial_0 H = -\sum_{i>0} \partial_i(\widetilde{A}_{ij}\tilde{v}_j).$$

It follows that

$$\begin{aligned}
-2\iint_{[0,2r]\times B_{2r}(y')} &\mathcal{R}e\langle H, \partial_0 H\rangle \zeta \, dx_0\, dx' \\
&= 2\sum_{i>0}\iint_{[0,2r]\times B_{2r}(y')} \mathcal{R}e\langle H, \partial_i(\widetilde{A}_{ij}\tilde{v}_j)\rangle \zeta(\partial_0 x_0)\, dx_0\, dx' \\
&= -2\sum_{i>0}\iint_{[0,2r]\times B_{2r}(y')} \mathcal{R}e\langle \partial_0 H, \partial_i(\widetilde{A}_{ij}\tilde{v}_j)\rangle \zeta x_0\, dx_0\, dx' \\
&\quad + 2\sum_{i>0}\iint_{[0,2r]\times B_{2r}(y')} \mathcal{R}e\langle \partial_i H, \partial_0(\widetilde{A}_{ij}\tilde{v}_j)\rangle \zeta x_0\, dx_0\, dx' \\
&\quad - 2\sum_{i>0}\iint_{[0,2r]\times B_{2r}(y')} \mathcal{R}e\langle H, \partial_i(\widetilde{A}_{ij}\tilde{v}_j)\rangle (\partial_0\zeta) x_0\, dx_0\, dx' \\
&\quad + 2\sum_{i>0}\iint_{[0,2r]\times B_{2r}(y')} \mathcal{R}e\langle H, \partial_0(\widetilde{A}_{ij}\tilde{v}_j)\rangle (\partial_i\zeta) x_0\, dx_0\, dx'. \qquad (3\text{-}27)
\end{aligned}$$

We analyze this term by term. In the last two terms, if the derivative falls on $\tilde{v}_j$, these terms are of the same nature as (3-22) and are handled identically. When the derivative falls on the coefficients, these are bounded by

$$\iint_{[0,2r]\times B_{2r}(y')} |\tilde{v}|^2 |\nabla\widetilde{A}| \frac{x_0}{r}\, dx_0\, dx' \lesssim \|\mu\|_{\mathcal{C}}^{1/2} \|N_a(w)\|_{L^2}^2,$$

where we have used the Cauchy–Schwarz inequality and the Carleson condition.

The first two terms on the right-hand side of (3-27) will give us the square function of $\tilde{v}$ when both derivatives fall on $\tilde{v}$ or a mixed term like (3-22) or finally when both derivatives hit the coefficients terms bounded from above by

$$\iint_{[0,2r]\times B_{2r}(y')} |\tilde{v}|^2 |\nabla\widetilde{A}|^2 x_0\, dx_0\, dx' \lesssim \|\mu\|_{\mathcal{C}} \|N_a(w)\|_{L^2}^2.$$

With this in hand, the estimate in (3-8) follows (by passing from $\tilde{v}$ back to $v = \nabla u$ via the map ρ).

Finally, the claim that the term (3-9) can be used in the statement of the lemma follows from the Poincaré inequality. See [Dindoš et al. 2017a] for the details. □

Combining all lemmas above, we can establish the following local good-λ inequality. We omit the proof as the argument is the same as in [Dindoš et al. 2017a].

Lemma 3.5. *Consider the equation $\mathcal{L}u = 0$ with coefficients satisfying the assumptions of Theorem 1.2. Consider any boundary ball $\Delta_d = \Delta_d(Q) \subset \mathbb{R}^{n-1}$, let $A_d = (d/2, Q)$ be its corkscrew point and let*

$$\nu_0 = \left(\fint_{B_{d/4}(A_d)} |\nabla u(z)|^2 \, dz \right)^{1/2}. \tag{3-28}$$

Then for each $\gamma \in (0,1)$ there exists a constant $C(\gamma) > 0$ such that $C(\gamma) \to 0$ as $\gamma \to 0$ and with the property that for each $\nu > 2\nu_0$ and each energy solution u of $\mathcal{L}u = 0$ there holds

$$\begin{aligned} \Big| \big\{ x' \in \mathbb{R}^{n-1} : \widetilde{N}_a(\nabla u \chi_{T(\Delta_d)}) > \nu,\ (M(S_b^2(\nabla u)))^{1/2} \le \gamma\nu, \\ \big(M(S_b^2(\nabla u)) M(\widetilde{N}_a^2(\nabla u \chi_{T(\Delta_d)}))\big)^{1/4} \le \gamma\nu,\ \big(M(\|\mu\|_{\mathcal{C}}^{1/2} \widetilde{N}_a^2(\nabla u \chi_{T(\Delta_d)}))\big)^{1/2} \le \gamma\nu \big\} \Big| \\ \le C(\gamma) |\{ x' \in \mathbb{R}^{n-1} : \widetilde{N}_a(\nabla u \chi_{T(\Delta_d)})(x') > \nu/32 \}|. \end{aligned} \tag{3-29}$$

Here $\chi_{T(\Delta_d)}$ is the indicator function of the Carleson region $T(\Delta_d)$ and the square function S_b in (3-29) is truncated at the height $2d$. Similarly, the Hardy–Littlewood maximal operator M is only considered over all balls $\Delta' \subset \Delta_{md}$ for some enlargement constant $m = m(a) \ge 2$.

Finally we have the following.

Proposition 3.6. *Consider the equation $\mathcal{L}u = 0$ in $\Omega = \mathbb{R}^n_+$ with coefficients satisfying the assumptions of Theorem 1.2. Then for any $p > 0$ and $a > 0$ there exists an integer $m = m(a) \ge 2$ and finite constants $K = K(n, \lambda, \Lambda, p, a) > 0$ and $C = C(n, \lambda, \Lambda, p, a) > 0$ such that if*

$$\|\mu\|_{\mathcal{C}} < K,$$

then for all balls $\Delta_d \subset \mathbb{R}^{n-1}$ we have

$$\|\widetilde{N}_a^r(\nabla u)\|_{L^p(\Delta_d)} \le C \|S_a^{2r}(\nabla u)\|_{L^p(\Delta_{md})} + C d^{(n-1)/p} |\widetilde{\nabla u}(A_d)|, \tag{3-30}$$

where A_d denotes the corkscrew point of the ball Δ_d and $\widetilde{\nabla u}$ is as in (3-10).

We also have a global estimate for any $p > 0$ and $a > 0$. Under the same assumptions as above (*and extra a priori assumption $\|\widetilde{N}_a(\nabla u)\|_{L^p(\mathbb{R}^{n-1})} < \infty$ when $p < 2$*) *there exists a finite constant $C = C(n, \lambda, \Lambda, p, a) > 0$ such that*

$$\|\widetilde{N}_a(\nabla u)\|_{L^p(\mathbb{R}^{n-1})} \le C \|S_a(\nabla u)\|_{L^p(\mathbb{R}^{n-1})}. \tag{3-31}$$

Proof. When $p > 2$, (3-30) follows immediately by a standard argument (multiplying the good-λ inequality (3-29) by ν^{p-1} and integrating in ν over the interval $(2\nu_0, \infty)$). Note that the fact that the square function S_a^{2r} is only integrated over some enlargement of Δ_d instead of the whole $\mathbb{R}^{n-1}$ follows from the fact that the set $\{x' \in \mathbb{R}^{n-1} : \widetilde{N}_a(\nabla u \chi_{T(\Delta_d)})(x') > \nu/32\}$ on the right-hand side of (3-29) vanishes outside a ball of diameter comparable to Δ_d. For this reason the maximal operators M in (3-29) can be restricted to such an enlarged ball Δ_{md}.

The condition $\|\mu\|_{\mathcal{C}} < K$ comes from the presence of the term $(M(\|\mu\|_{\mathcal{C}}^{1/2} \widetilde{N}_a^2(\nabla u \chi_{T(\Delta_d)})))^{1/2} \le \gamma\nu$ in the good-λ inequality. The argument that shows (3-30) for all $p > 0$ can be found in [Fefferman and Stein 1972]. The local estimate (3-30) for $p > 2$ is the necessary ingredient for what is otherwise a purely real-variable argument. Further details can be found in [Fefferman and Stein 1972].

Finally taking the limit $d \to \infty$ yields (3-31). The additional assumption $\|\widetilde{N}_a(\nabla u)\|_{L^p(\mathbb{R}^{n-1})} < \infty$ when $p < 2$ comes into play it order to guarantee that the term $d^{(n-1)/p}|\widetilde{\nabla u}(A_d)|$ in (3-30) converges to zero as $d \to \infty$. □

4. Estimates for the p-adapted square function

Let $\Omega = \mathbb{R}^n_+$ and assume u is a weak solution $\mathcal{L}u = 0$, where

$$\mathcal{L}u = \partial_i(A_{ij}(x)\,\partial_j u) + B_i(x)\,\partial_i u, \tag{4-1}$$

with the Dirichlet boundary datum $f \in \dot{B}^{2,2}_{1/2}(\partial\Omega;\mathbb{C}) \cap \dot{W}^{1,p}(\partial\Omega;\mathbb{C})$. Assume that A is p-elliptic and smooth in $\mathbb{R}^n_+$ with $A_{00} = 1$ and A_{0j} real and that the measure μ defined as in (1-2) is Carleson.

Fix an arbitrary $y' \in \partial\Omega \equiv \mathbb{R}^{n-1}$ and consider $\Delta = \Delta_r(y)$, a ball of radius r in $\mathbb{R}^{n-1}$ centered at y'. Pick a smooth cutoff function ζ which is x_0-independent and satisfies

$$\zeta = \begin{cases} 1 & \text{in } \Delta, \\ 0 & \text{outside } 2\Delta, \end{cases} \tag{4-2}$$

where 2Δ is a ball of radius $2r$ centered at y'. Moreover, assume that $r|\nabla\zeta| \le c$ for some positive constant c independent of y'. We note that since

$$\partial_0(A_{0j}\,\partial_j u) = \partial_j(A_{0j}\,\partial_0 u) - (\partial_j A_{0j})\,\partial_0 u + (\partial_0 A_{0j})\,\partial_j u,$$

we may as well assume that $A_{0j} = 0,\ j > 0$, by changing coefficients A_{0j} and A_{j0} of the matrix A and modifying B. We note that this does not affect the ellipticity of A as all A_{0j} are assumed to be real. It follows that we can assume $\partial_k A_{0j} = 0$ for all $j, k = 0, 1, \dots, n-1$.

We begin by considering the integral quantity for some function w (to be specified later) such that $w|w|^{p/2-1} \in W^{1,2}_{\mathrm{loc}}(\Omega)$

$$\mathcal{I} := \mathscr{R}e \iint_{[0,s]\times 2\Delta} A_{ij}\,\partial_j w\,\partial_i(|w|^{p-2}\bar{w})x_0\zeta\,dx'\,dx_0, \tag{4-3}$$

with the usual summation convention understood. Here $s \in [0, r]$. With $\chi = x_0\zeta$ we have by Theorem 2.4 of [Dindoš and Pipher 2019] for all p for which A is p-elliptic for some $\lambda_p > 0$

$$\mathcal{I} \ge \lambda_p \iint_{[0,s]\times 2\Delta} |w|^{p-2}|\nabla w|^2 x_0\zeta\,dx'\,dx_0. \tag{4-4}$$

The objective is to ultimately apply (4-4) to $w = \partial_i u,\ i = 1, \dots, n-1$, and obtain a quantity that can be bounded from above by expressions that involve L^p norms of $|\nabla u|$, and nontangential maximal functions of $|\nabla u|$, on the boundary. To see this, we continue the calculation using the fact that we can bound the right-hand side of (4-4) by the expression $\mathcal{I}$, which brings in the equation. For the moment, we ignore the issue of finiteness of this expression, even though we use this fact in the calculations that follow. We'll return to this point after some of the basic calculations, for the sake of clarity of exposition.

The idea now is to integrate by parts the formula for $\mathcal{I}$ in order to relocate the ∂_i derivative. This gives

$$\begin{aligned}\mathcal{I} = \mathscr{R}e \int_{\partial[(0,s)\times 2\Delta]} & A_{ij}\, \partial_j w|w|^{p-2}\bar{w} x_0 \zeta \nu_{x_i}\, d\sigma \\ - \mathscr{R}e \iint_{[0,s]\times 2\Delta} & \partial_i (A_{ij}\, \partial_j w)|w|^{p-2}\bar{w} x_0 \zeta\, dx'\, dx_0 \\ - \mathscr{R}e \iint_{[0,s]\times 2\Delta} & A_{ij}\, \partial_j w|w|^{p-2}\bar{w}\, \partial_i x_0 \zeta\, dx'\, dx_0 \\ - \mathscr{R}e \iint_{[0,s]\times 2\Delta} & A_{ij}\, \partial_j w|w|^{p-2}\bar{w} x_0\, \partial_i \zeta\, dx'\, dx_0 \\ =: I + II + III + IV, &\end{aligned} \tag{4-5}$$

where ν is the outer unit normal vector to $(0,s)\times 2\Delta$. The boundary term I is nonvanishing only on the set $\{s\}\times 2\Delta$ and only when $i=0$. This gives

$$I = \mathscr{R}e \int_{\{s\}\times 2\Delta} A_{0j}\, \partial_j w|w|^{p-2}\bar{w} x_0 \zeta\, d\sigma. \tag{4-6}$$

As $\partial_i x_0 = 0$ for $i > 0$ the term III is nonvanishing only for $i = 0$. Since $A_{0j} = 0$ for $j > 0$ and $A_{00} = 1$, term III simplifies to

$$\begin{aligned} III &= -\mathscr{R}e \iint_{[0,s]\times 2\Delta} \partial_0 w|w|^{p-2}\bar{w}\zeta\, dx'\, dx_0 \\ &= -\frac{1}{p} \iint_{[0,s]\times 2\Delta} \partial_0(|w|^p)\zeta\, dx'\, dx_0 \\ &= -\frac{1}{p}\int_{2\Delta} |w|^p(s,x')\zeta\, dx' + \frac{1}{p}\int_{2\Delta} |w|^p(0,x')\zeta\, dx'. \end{aligned} \tag{4-7}$$

We add up all terms we have so far to obtain

$$\begin{aligned}\mathcal{I} \le p^{-1}\int_{2\Delta} \partial_0(|w|^p)(s,x') s\zeta\, dx' - \mathscr{R}e \iint_{[0,s]\times 2\Delta} \partial_i (A_{ij}\, \partial_j w)|w|^{p-2}\bar{w} x_0 \zeta\, dx'\, dx_0 \\ - p^{-1}\int_{2\Delta} |w|^p(s,x')\zeta\, dx' + p^{-1}\int_{2\Delta} |w|^p(0,x')\zeta\, dx' + IV.\end{aligned} \tag{4-8}$$

So far w was an arbitrary function. We now apply (4-8) to $w_k = \partial_k u,\ k = 1,2,\dots,n-1$, where u solves $\mathcal{L}u = 0$. It follows that each w_k solves

$$\begin{aligned}\mathcal{L} w_k &= \partial_i (A_{ij}\, \partial_j w_k) + B_i w_k \\ &= \partial_i ((\partial_k A_{ij}) w_j) - \partial_k (B_i) w_i.\end{aligned} \tag{4-9}$$

It follows that

$$
\begin{aligned}
II &= -\mathscr{R}e \iint_{[0,s]\times 2\Delta} \partial_i (A_{ij}\, \partial_j w_k)|w_k|^{p-2}\bar{w}_k x_0 \zeta \, dx'\, dx_0 \\
&= \mathscr{R}e \iint_{[0,s]\times 2\Delta} B_i w_k |w_k|^{p-2}\bar{w}_k x_0 \zeta \, dx'\, dx_0 \\
&\quad + \mathscr{R}e \iint_{[0,s]\times 2\Delta} (\partial_i A_{ij}) w_j |w_k|^{p-2}\bar{w}_k x_0\, \partial_k \zeta \, dx'\, dx_0 \\
&\quad - \mathscr{R}e \iint_{[0,s]\times 2\Delta} B_i w_i |w_k|^{p-2}\bar{w}_k x_0\, \partial_k \zeta \, dx'\, dx_0 \\
&\quad + \mathscr{R}e \iint_{[0,s]\times 2\Delta} (\partial_i A_{ij}) w_j\, \partial_k (|w_k|^{p-2}\bar{w}_k) x_0 \zeta \, dx'\, dx_0 \\
&\quad - \mathscr{R}e \iint_{[0,s]\times 2\Delta} (\partial_k A_{ij})(\partial_i w_j) |w_k|^{p-2}\bar{w}_k x_0 \zeta \, dx'\, dx_0 \\
&\quad - \mathscr{R}e \iint_{[0,s]\times 2\Delta} B_i\, \partial_k (w_i |w_k|^{p-2}\bar{w}_k) x_0 \zeta \, dx'\, dx_0 \\
&=: II_1 + II_2 + II_3 + II_4 + II_5 + II_6.
\end{aligned}
\tag{4-10}
$$

Here we integrated by parts terms containing two derivatives of A or one derivative of B by moving the ∂_k derivative. It is important here that $k \neq 0$ and hence $\partial_k x_0 = 0$. The first term on the right-hand side can be estimated directly using Theorem 2.7, while the last three terms we estimate using Cauchy–Schwarz inequality, the Carleson conditions for A and B and Theorem 2.7:

$$
\begin{aligned}
&|II_4| + |II_5| + |II_6| \\
&\leq \left(\iint_{[0,s]\times 2\Delta} (|\nabla A| + |B|)^2 |w|^p x_0 \zeta \, dx'\, dx_0 \right)^{1/2} \left(\iint_{[0,s]\times 2\Delta} |w|^{p-2} |\nabla w|^2 x_0 \zeta \, dx'\, dx_0 \right)^{1/2} \\
&\leq C(\lambda_p, \Lambda, p, n) \left(\|\mu\|_{\mathcal{C}} \int_{2\Delta} [\widetilde{N}^r_{p,a}(w)]^p \, dx' \right)^{1/2} \mathcal{I}^{1/2}.
\end{aligned}
\tag{4-11}
$$

In summary we get (after using the inequality between the arithmetic and geometric means (AG))

$$
|II| \leq C(\lambda_p, \Lambda, p, n) \|\mu\|_{\mathcal{C}} \int_{2\Delta} [\widetilde{N}^r_{p,a}(\nabla u)]^p \, dx' + \tfrac{1}{2}\mathcal{I} + II_2 + II_3.
$$

It follows that (4-8) simplifies to (after summing over $k = 1, 2, \dots n-1$)

$$
\begin{aligned}
\sum_{k=1}^{n-1} \mathcal{I}_k \leq p^{-1} &\int_{2\Delta} \partial_0 (|\nabla_T u|^p)(s, x') r \zeta \, dx' \\
&- p^{-1} \int_{2\Delta} |\nabla_T u|^p (s, x') \zeta \, dx' + p^{-1} \int_{2\Delta} |\nabla_T u|^p (0, x') \zeta \, dx' \\
&+ C(\lambda_p, \Lambda, p, n) \|\mu\|_{\mathcal{C}} \int_{2\Delta} [\widetilde{N}^r_{p,a}(\nabla u)]^p \, dx' + II_2 + II_3 + IV.
\end{aligned}
\tag{4-12}
$$

We estimate the terms IV. It can be bounded (up to a constant) by

$$\iint_{[0,s]\times 2\Delta} |\nabla w||w|^{p-1} x_0 |\partial_T \zeta|\, dx'\, dx_0, \tag{4-13}$$

where $\partial_T \zeta$ denotes any of the derivatives in the direction parallel to the boundary. Recall that ζ is a smooth cutoff function equal to 1 on Δ and 0 outside 2Δ. In particular, we may assume ζ to be of the form $\zeta = \eta^2$ for another smooth function η such that $|\nabla_T \eta| \le C/r$. By Cauchy–Schwarz (4-13) can be further estimated by

$$\begin{aligned}
\left(\iint_{[0,s]\times 2\Delta} |\nabla w|^2 |w|^{p-2} x_0 (\eta)^2\, dx'\, dx_0\right)^{1/2} &\left(\iint_{[0,s]\times 2\Delta} |w|^p x_0 |\nabla_T \eta|^2\, dx'\, dx_0\right)^{1/2} \\
&\lesssim \mathcal{I}^{1/2} \left(\frac{1}{r} \iint_{[0,s]\times(2\Delta\setminus\Delta)} |w|^p\, dx'\, dx_0\right)^{1/2} \\
&\le \varepsilon \mathcal{I} + C_\varepsilon \int_{2\Delta\setminus\Delta} [\widetilde{N}^r_{p,a}(\nabla u)]^p\, dx'.
\end{aligned} \tag{4-14}$$

In the last step we have used the AG-inequality and a trivial estimate of the solid integral $|u|^p$ by the p-averaged nontangential maximal function.

The terms II_2 and II_3 are also similar. We use $|\nabla A|, |B| \le \|\mu\|_C^{1/2}/x_0$ and what remains has a trivial estimate by $\int_{2\Delta\setminus\Delta} [\widetilde{N}^r_{p,a}(\nabla u)]^p\, dx$. Substituting this and (4-14) into (4-12) and by integrating in s over $[0,r]$ and dividing by r we finally obtain

$$\begin{aligned}
\iint_\Delta [S_p^{r/2}(\nabla_T u)]^p\, dx' &\le 2 \sum_{k=1}^{n-1} \iint_{[0,r]\times\Delta} |\nabla(\partial_k u)|^2 |\partial_k u|^{p-2}\, x_0 \left(1 - \frac{x_0}{r}\right) dx'\, dx_0 \\
&\lesssim p^{-1} \int_{2\Delta} |\nabla_T u|^p (0,x')\, dx' + p^{-1} \int_{2\Delta} |\nabla_T u|^p (r,x')\, dx' \\
&\quad + C\|\mu\|_C \int_{2\Delta} [\widetilde{N}^r_{p,a}(\nabla u)]^p\, dx' + C \int_{2\Delta\setminus\Delta} [\widetilde{N}^r_{p,a}(\nabla u)]^p\, dx'.
\end{aligned} \tag{4-15}$$

We return now to the issue of finiteness of the quantities in (4-4). We fix an $\varepsilon > 0$ and consider a bound for the expression

$$\iint_{[\varepsilon,s]\times 2\Delta} |w|^{p-2} |\nabla w|^2 (x_0 - \varepsilon) \zeta\, dx'\, dx_0 \tag{4-16}$$

instead of $\iint_{[0,s]\times 2\Delta} |w|^{p-2} |\nabla w|^2 x_0 \zeta\, dx'\, dx_0$. The quantity (4-16) is finite by the interior estimates (2-19). By Theorem 2.7, all of the previous calculations for the term (4-16), after averaging in s will give the estimate

$$\begin{aligned}
\sum_{k=1}^{n-1} &\iint_{[\varepsilon,r/2]\times\Delta} |\nabla(\partial_k u)|^2 |\partial_k u|^{p-2}\, (x_0 - \varepsilon))\, dx'\, dx_0 \\
&\lesssim p^{-1} \int_{2\Delta} |\nabla_T u|^p (\varepsilon,x')\, dx' + p^{-1} \int_{2\Delta} |\nabla_T u|^p (r,x')\, dx' \\
&\quad + C\|\mu\|_C \int_{2\Delta} [\widetilde{N}^r_{p,a,\varepsilon}(\nabla u)]^p\, dx' + C \int_{2\Delta\setminus\Delta} [\widetilde{N}^r_{p,a,\varepsilon}(\nabla u)]^p\, dx',
\end{aligned} \tag{4-17}$$

where $\widetilde{N}^r_{p,a,\varepsilon}(\nabla u)$ denotes the nontangential maximal function relative to the domain $\{x_0 > \varepsilon\}$ as defined in (2-27).

To deal with the quantity $\int_{2\Delta} |\nabla_T u|^p(\varepsilon, x')\, dx'$, which is not itself a priori finite, we average the inequalities above over $\varepsilon \in [\varepsilon_0/2, \varepsilon_0]$. Averaging in r as we have done earlier bounds the boundary integral $\int_{2\Delta} |\nabla_T u|^p(r, x')\, dx'$ by a solid integral and we obtain, for each $k = 1, \dots, n-1$,

$$\iint_{[\varepsilon_0, r/4]\times\Delta} |\nabla(\partial_k u)|^2 |\partial_k u|^{p-2} (x_0 - \varepsilon_0)\, dx'\, dx_0 \lesssim C \int_{2\Delta} [\widetilde{N}^r_{p,a,\varepsilon_0/2}(\nabla u)]^p\, dx'. \tag{4-18}$$

By Fatou's lemma, letting $\varepsilon_0 \to 0$, the expressions in (4-4) have an upper bound in terms of $\int_{2\Delta} [\widetilde{N}^r_{p,a}(\nabla u)]^p\, dx$. Whenever this nontangential maximal function expression is finite, the calculations leading to (4-15) that depend on the finiteness of (4-4) are justified.

To obtain a global version of (4-15), consider a sequence of disjoint boundary balls $(\Delta_r(y'_j))_{k\in\mathbb{N}}$ such that $\bigcup_j \Delta_{2r}(y'_j)$ covers $\partial\Omega = \mathbb{R}^{n-1}$ and consider a partition of unity $(\zeta_j)_{k\in\mathbb{N}}$ subordinate to this cover. That is, assume $\sum_j \zeta_j = 1$ on $\mathbb{R}^{n-1}$ and each ζ_j is supported in $\Delta_{2r}(y'_j)$. Given that $\sum_j \partial_i \zeta_j = 0$ for each i, it follows by summing over all k that

$$\sum_j II_2 + II_3 + IV = 0.$$

It follows from (4-12) (after averaging in s over $[0, r]$) that

$$\begin{aligned}\iint_{\mathbb{R}^{n-1}} [S_p^{r/2}(\nabla_T u)]^p\, dx' &\le 2\sum_{k=1}^{n-1} \iint_{[0,r]\times\mathbb{R}^{n-1}} |\nabla(\partial_k u)|^2 |\partial_k u|^{p-2}\, x_0\Big(1 - \frac{x_0}{r}\Big)\, dx'\, dx_0 \\ &\lesssim p^{-1}\int_{\mathbb{R}^{n-1}} |\nabla_T u|^p(0, x')\, dx' + p^{-1}\int_{\mathbb{R}^{n-1}} |\nabla_T u|^p(r, x')\, dx' \\ &\qquad + C\|\mu\|_{\mathcal{C}} \int_{\mathbb{R}^{n-1}} [\widetilde{N}^r_{p,a}(\nabla u)]^p\, dx'.\end{aligned} \tag{4-19}$$

We now modify our calculation above by considering a Lipschitz function $g : \mathbb{R}^{n-1} \to [0, \infty)$ such that $\sup_{2\Delta} g \le r/4$ (we only assume this to avoid integration over an empty set). We perform the same calculation starting from (4-3) but this time we integrate over the set

$$([0, s] \times 2\Delta) \cap \Omega_g,$$

where

$$\Omega_g := \{(x_0, x') \in \mathbb{R} \times \mathbb{R}^{n-1} : x_0 > g(x')\}.$$

Rather that repeating the whole calculation again we focus on the differences. We note that we will only consider $s \in [r/2, 2r]$ to avoid complications that might arise from integration over empty sets.

The first difference will be that the term I of (4-5) will contain an additional boundary and hence

$$I = \mathscr{Re}\int_{\{s\}\times 2\Delta} (\partial_0 w)|w|^{p-2}\bar{w} x_0 \zeta\, d\sigma + \mathscr{Re}\int_{([0,s]\times 2\Delta)\cap\partial\Omega_g} A_{ij}\, \partial_j w |w|^{p-2}\bar{w} x_0 \zeta \nu_i\, d\sigma, \tag{4-20}$$

where ν_i is the i-component of the outer normal of $\partial\Omega_g$. The term (4-7) becomes

$$III = -\frac{1}{p}\int_{2\Delta} |w|^p(s,x')\zeta\,dx' + \frac{1}{p}\int_{2\Delta} |w|^p(g(x'),x')\zeta\,dx'. \tag{4-21}$$

We look at the term II. As we integrate by parts to obtain (4-10) we pick up two extra boundary terms.

$$II_{\mathrm{bd}} = -\mathscr{R}e\int_{([0,s]\times 2\Delta)\cap\partial\Omega_g} (\partial_i A_{ij})w_j|w_k|^{p-2}\bar{w}_k x_0\nu_k\zeta\,d\sigma \\ + \mathscr{R}e\int_{([0,s]\times 2\Delta)\cap\partial\Omega_g} B_i w_i|w_k|^{p-2}\bar{w}_k x_0\nu_k\zeta\,d\sigma. \tag{4-22}$$

We also modify some estimates. Terms II_5, II_6 and II_7 of (4-10) are now integrated over the set $([0,s]\times 2\Delta)\cap\Omega_g$, which allows us to use the estimate (2-25) of Theorem 2.7. This gives us

$$|II_5|+|II_6|+|II_7| \lesssim \left(\|\mu\|_{\mathcal{C}}\int_{([0,s]\times 2\Delta)\cap\partial\Omega_g} [\widetilde{N}^{2r}_{p,a,g}(\nabla u)]^p\,dx'\right)^{1/2}\mathcal{I}^{1/2}. \tag{4-23}$$

A similar observation applies to terms II_2, II_3 and IV. It follows that what we have so far implies the estimate for some $c_p > 0$:

$$\begin{aligned} c_p\sum_{k=1}^{n-1}\iint_{([0,s]\times 2\Delta)\cap\Omega_g} &|\nabla(\partial_k u)|^2|\partial_k u|^{p-2}x_0\zeta\,dx'\,dx_0 \\ &\le p^{-1}\int_{2\Delta}\partial_0(|\nabla_T u|^p)(s,x')r\zeta\,dx' - p^{-1}\int_{2\Delta}|\nabla_T u|^p(s,x')\zeta\,dx' \\ &\quad + p^{-1}\int_{2\Delta}|\nabla_T u|^p(g(x'),x')\zeta\,dx' + C(\lambda_p,\Lambda,\|\mu\|_{\mathcal{C}},p,n)\int_{T(2\Delta)\times\partial\Omega_g}[\widetilde{N}^{2r}_{p,a,g}(\nabla u)]^p\,dx' \\ &\quad + \sum_{k=1}^{n-1}\mathscr{R}e\int_{([0,s]\times 2\Delta)\cap\partial\Omega_g} A_{ij}\,\partial_j(\partial_k u)|\partial_k u|^{p-2}\,\overline{\partial_k u}\,x_0\zeta\nu_i\,d\sigma + II_{\mathrm{bd}}. \end{aligned}\tag{4-24}$$

Our goal is to estimate the first two terms on the right-hand side of (4-24) by $\widetilde{N}^{2r}_{p,a,g}$. To do that we average in s twice. We first integrate (4-24) over an interval $s\in[r/2(1+\theta), r(1+\theta)]$ and then integrate the resulting inequality again in $\theta\in[0,1]$. This turns both mentioned terms into solid integrals of $|\nabla_T u|^p$ over a Whitney-type box inside Ω_g. This simplifies (4-24) to

$$\begin{aligned} c_p\sum_{k=1}^{n-1}\iint_{([0,r/2]\times\Delta)\cap\Omega_g} &|\nabla(\partial_k u)|^2|\partial_k u|^{p-2}x_0\,dx'\,dx_0 \\ &\le p^{-1}\int_{2\Delta}|\nabla_T u|^p(g(x'),x')\,dx' + C(\lambda_p,\Lambda,\|\mu\|_{\mathcal{C}},p,n)\int_{T(2\Delta)\times\partial\Omega_g}[\widetilde{N}^{2r}_{p,a,g}(\nabla u)]^p\,dx' \\ &\quad + \sum_{k=1}^{n-1}\mathscr{R}e\int_{([0,s]\times 2\Delta)\cap\partial\Omega_g} A_{ij}\,\partial_j(\partial_k u)|\partial_k u|^{p-2}\,\overline{\partial_k u}\,x_0\zeta\nu_i\,d\sigma + II_{\mathrm{bd}}. \end{aligned}\tag{4-25}$$

We shall use (4-25) in the following lemma.

Lemma 4.1. *Let $\Omega = \mathbb{R}^n_+$ and assume u is the energy solution of* (4-1). *Assume that A is p-elliptic and smooth in $\mathbb{R}^n_+$ with $A_{00} = 1$ and A_{0j} real and that the measure μ defined as in* (1-2) *is Carleson.*

Consider any $b > a > 0$. Then for each $\gamma \in (0,1)$ there exists a constant $C(\gamma) > 0$ such that $C(\gamma, a, b) \to 0$ as $\gamma \to 0$ and with the property that for each $\nu > 0$ we have

$$\left|\left\{x' \in \mathbb{R}^{n-1} : S_{p,a}(\nabla_T u)(x') > \nu,\ \widetilde{N}_b(\nabla u)(x') \le \gamma\nu\right\}\right| \le C(\gamma)\left|\left\{x' \in \mathbb{R}^{n-1} : S_{p,b}(\nabla_T u)(x') > \nu/2\right\}\right|. \tag{4-26}$$

Here $\widetilde{N}_b$ denotes the L^2 version of the nontangential maximal function defined over cones of aperture b.

Proof. We observe that $\widetilde{N}_b(\nabla u) \le \gamma\nu$ also implies $\widetilde{N}_{p,b}(\nabla u) \lesssim \gamma\nu$ thanks to Proposition 2.5. Also clearly, $\{x' \in \mathbb{R}^{n-1} : S_{p,b}(\nabla_T u)(x') > \nu/2\}$ is an open subset of $\mathbb{R}^{n-1}$. When this set is empty, or is all of $\mathbb{R}^{n-1}$, estimate (4-26) is trivial, so we focus on the case when the set in question is both nonempty and proper. Granted this, we may consider a Whitney decomposition $(\Delta_i)_{i\in I}$ of it, consisting of open cubes in $\mathbb{R}^{n-1}$. Let F^i_ν be the set appearing on the left-hand side of (4-26) intersected with Δ_i. Let r_i be the diameter of Δ_i. Due to the nature of the Whitney decomposition there exists a point $p' \in 2\Delta_i$ such that $S_{p,b}(\nabla_T u)(p') < \nu/2$. From this and the fact that $b > a$ it follows that for all $x' \in F^i_\nu$ we have

$$S^d_{p,a}(\nabla_T u)(x') > \nu/2,$$

where $S^d_{p,a}$ is the truncated version of the square function at some height $d \approx r_i$, where the precise nature of the relation between d and r_i depends on the apertures a and b.

For some $a < c < b$ consider the domain

$$\Omega_c = \bigcup_{x' \in F^i_\nu} \Gamma_c(x');$$

this is a Lipschitz domain with Lipschitz constant $1/c$. Observe that $F^i_\nu \subset \partial\Omega_c$. It follows that

$$|F^i_\nu| \le \frac{2^p}{\nu^p}\int_{F^i_\nu}[S^d_{p,a}(\nabla_T u)(x')]^p\,dx' \lesssim \nu^{-p}\sum_{k=1}^{n-1}\iint_{\Omega_c\cap T(\Delta_i)} |\nabla(\partial_k u)|^2|\partial_k u|^{p-2}x_0\,dx.$$

We apply (4-25). It follows that

$$\begin{aligned}|F^i_\nu| \lesssim \nu^{-p}\biggl\{&\int_{\partial\Omega_c\cap T(2\Delta_i)}(|\nabla_T u|_{\partial\Omega_c}|^p + [\widetilde{N}^{2d}_{p,a,c}(\nabla u)]^p)\,d\sigma \\ &+ \sum_{k=1}^{n-1}\biggl[\mathscr{Re}\int_{T((2\Delta_i)\cap\partial\Omega_c}A_{ij}(\partial^2_{jk}u)|\partial_k u|^{p-2}\,\overline{\partial_k u}\,x_0\zeta\nu_i\,d\sigma \\ &\qquad + \mathscr{Re}\int_{([0,s]\times 2\Delta)\cap\partial\Omega_g}(\partial_i A_{ij})\,\partial_j u|\partial_k u|^{p-2}\,\overline{\partial_k u}\,x_0\nu_k\zeta\,d\sigma \\ &\qquad + \mathscr{Re}\int_{([0,s]\times 2\Delta)\cap\partial\Omega_g}B_i\,\partial_i u|\partial_k u|^{p-2}\,\overline{\partial_k u}\,x_0\nu_k\zeta\,d\sigma\biggr]\biggr\},\end{aligned} \tag{4-27}$$

where $\widetilde{N}^{2d}_{p,a,c}$ is defined using nontangential cones with aperture a with vertices on $\partial\Omega_c$. Due to the fact that each of these cones is contained in one of the cones $\Gamma_b(x')$ for some $x' \in F^i_\nu$ (as $c < b$) and on F^i_ν $\widetilde{N}_b(\nabla_T u)(x') \le \gamma\nu$, we also have $\widetilde{N}^{2d}_{p,a,c}(\nabla_T u) \lesssim \gamma\nu$ everywhere on $\partial\Omega_c$. This takes care of the second term.

We still need to deal with the four other terms on the right-hand side. We do it by converting these terms into a solid integrals by averaging c over the interval $[a, (a+b)/2]$. Let us define

$$\mathcal{O} = \Omega_{(a+b)/2} \setminus \overline{\Omega}_a.$$

$\mathcal{O}$ is the set over which the four terms we want to bound will integrate after the averaging. The sets Ω_c also share F^i_ν as a common boundary; however, there we have a trivial estimate

$$\int_{F^i_\nu} |\nabla_T u|_{\partial\Omega_c}|^p \, d\sigma \le (\gamma\nu)^p |\Delta_i|$$

from the fact that $\widetilde{N}_b(\nabla_T u)(x') \le \gamma\nu$ on F^i_ν, while the last three terms of (4-27) vanish there (as $x_0 = 0$).

Given the way the set $\mathcal{O}$ is defined, geometric considerations imply that it can be covered by a nonoverlapping collection of Whitney cubes $\{Q_j\}$ in $\mathbb{R}^n_+$ with the following properties:

$$\mathcal{O} \subset \bigcup_j Q_j, \quad r_j = \operatorname{diam}(Q_j) \approx \operatorname{dist}(Q_i, \partial\mathbb{R}^n_+), \quad 2Q_j \subset \Omega_b. \tag{4-28}$$

Furthermore the projections of Q_j onto the boundary $\mathbb{R}^{n-1}$ are "almost disjoint"; that is, each such projection overlaps with at most $K = K(a,b)$ other projections. From this $\sum_j \operatorname{diam}(Q_j)^{n-1} \approx |2\Delta_i|$.

Consider the contribution of the first term on the right-hand side of (4-27) after the averaging in c on each Q_j. Such term can be bounded by

$$(\operatorname{diam}(Q_j))^{-1} \iint_{Q_j} |\nabla u|^p \, dx \lesssim (\gamma\nu)^p \operatorname{diam}(Q_j)^{n-1},$$

where the bound $\lesssim (\gamma\nu)^p$ comes from the fact that $Q_j \subset \Omega_b$ and hence the L^p average of ∇u on Q_j has this bound from our assumptions. Summing over all Q_j gives us the bound

$$\sum_j (\operatorname{diam}(Q_j))^{-1} \iint_{Q_j} |\nabla u|^p \, dx \lesssim (\gamma\nu)^p |2\Delta_i|.$$

In fact, we have this bound also for the fourth and fifth terms on the right-hand side of (4-27) since $|\nu_k| \le 1$ and $|\nabla A| x_0, |B| x_0 \lesssim \|\mu\|_{\mathcal{C}}^{1/2}$ and hence we are again dealing with a solid integral of $|\nabla u|^p$ over each Q_j. Finally, the third term on the right-hand side of (4-27) is somewhat different and on Q_j is bounded by

$$(\operatorname{diam}(Q_j))^{-1} \iint_{Q_j} |\partial_k u|^{p-1} |\nabla \partial_k u| x_0 \, dx,$$

which, since $x_0 \approx \operatorname{diam}(Q_j)$, by Cauchy–Schwarz is further bounded by

$$(\operatorname{diam}(Q_j))^{-1}\left(\iint_{Q_j} |\partial_k u|^p \, dx\right)^{1/2}\left((\operatorname{diam}(Q_j))^2 \iint_{Q_j} |\nabla \partial_k u|^2 |\partial_k u|^{p-2} \, dx\right)^{1/2},$$

where for the second term we can use (2-19) to again bound the whole expression by $C \operatorname{diam}(Q_j)^{n-1}$. It follows that we have after the averaging procedure for every term of (4-27) the same bound (up to a constant) and that

$$|F_\nu^i| \le C(a, b, \|\mu\|_{\mathcal{C}})\gamma^p |\Delta_i|.$$

Summing over all i yields (4-26) as desired. □

We will require a localized version of Lemma 4.1 as well.

Lemma 4.2. *Let u be as in Lemma 4.1. Fix $R \ge h$ and consider a boundary ball $\Delta_R \subset \mathbb{R}^{n-1}$. Let $p \ge q > 1$ for any q such that A is q-elliptic. Let*

$$\nu_0^p = C \fint_{\Delta_{2R}} [N_b^{2R}(\nabla u)]^p \, dx',$$

where C is a constant depending only on dimension (calculated in the proof below). Then for each $\gamma \in (0, 1)$ there exists a constant $C(\gamma) > 0$ such that $C(\gamma, a, b) \to 0$ as $\gamma \to 0$ and with the property that for each $\nu > \nu_0$

$$\left|\left\{x' \in \Delta_R : S_{q,a}^R(\nabla_T u)(x') > \nu, \, \tilde{N}_b^{2R}(\nabla u)(x') \le \gamma\nu\right\}\right| \le C(\gamma)\left|\left\{x' \in \Delta_R : S_{q,b}(\nabla_T u)(x') > \nu/2\right\}\right|. \tag{4-29}$$

Proof. It follows from (4-15) (by well-known averaging) that

$$\|S_{q,b}^R(\nabla_T u)\|_{L^q(\Delta_R)} \lesssim \|N_b^{2R}(\nabla u)\|_{L^q(\Delta_{2R})}. \tag{4-30}$$

Therefore,

$$\begin{aligned} |\Delta_R \cap \{S_q^R > \nu/2\}| &\lesssim \nu^{-q} \|N_b^{2R}(\nabla u)\|^q_{L^q(\Delta_{2R})} \\ &\lesssim \nu^{-q} \|N_b^{2R}(\nabla u)\|^{q/p}_{L^p(\Delta_{2R})} |\Delta_{2R}|^{1-q/p} \\ &\lesssim C_\varepsilon \nu^{-p} \int_{\Delta_{2R}} (N_b^{2R}(\nabla u))^p + \varepsilon |\Delta_R|. \end{aligned} \tag{4-31}$$

Choose $\varepsilon = \frac{1}{4}$, which determines C_ε, and we now fix $C = 4C_\varepsilon$ in the definition of ν_0. This implies that for any $\nu > \nu_0$, we have

$$|\Delta_R \cap \{S_{q,b}^R > \nu/2\}| < \tfrac{1}{2}|\Delta_R|. \tag{4-32}$$

Thus, there exists a Whitney decomposition of $\Delta_R \cap \{S_{q,b}^R > \nu/2\}$ into open cubes Δ_i with the property that $2\Delta_i \cap \Delta_R$ contains a point for which $S_{q,b}^R(\nabla_T u) < \nu/2$. From this point on, the proof proceeds as in Lemma 4.1. □

Corollary 4.3. *Under the assumption of Lemma 4.1, for any $q \ge p > 1$ and $a > 0$ there exists a finite constant $C = C(\lambda_p, \Lambda, p, q, a, \|\mu\|_{\mathcal{C}}, n) > 0$ such that*

$$\|S_{p,a}^R(\nabla_T u)\|_{L^q(\Delta_R)} \le C \|\tilde{N}_{p,a}^{2R}(\nabla u)\|_{L^q(\Delta_{2R})}, \tag{4-33}$$

$$\|S_{p,a}(\nabla_T u)\|_{L^q(\mathbb{R}^{n-1})} \le C\|\widetilde{N}_{p,a}(\nabla u)\|_{L^q(\mathbb{R}^{n-1})}. \tag{4-34}$$

The inequality (4-34) *also holds for any* $q > 0$, *provided we know a priori that* $\|S_{p,a}(\nabla_T u)\|_{L^q(\mathbb{R}^{n-1})} < \infty$.

Proof. The estimate (4-33) is a consequence of the local good-λ inequality established above and the equivalence [Coifman et al. 1985] of p-adapted square functions with different aperture in any L^q norm.

When $q \ge p$, and M is large,

$$\int_0^M \nu^{q-1}|\Delta_R \cap \{S^R_{p,a}(\nabla_T u) > \nu\}|\,d\nu \le C(M)\int_0^M \nu^{p-1}|\Delta_R \cap \{S^R_{p,a}(\nabla_T u) > \nu\}|\,d\nu.$$

By (4-30) and the fact that the coefficients are smooth, the right-hand side is finite. Therefore, the left hand side is also bounded, with a constant that may depend on M.

Now we multiply the good-λ inequality of Lemma 4.2 by ν^{p-1} and integrate separately over $(0, \nu_0)$ and (ν_0, M). This gives

$$\|S^R_{p,a}(u)\|_{L^q(\Delta_R)} \le C\|\widetilde{N}^{2R}_{p,a}(u)\|_{L^q(\Delta_{2R})},$$

after taking the limit as $M \to \infty$.

The estimate (4-34) follows by taking the limit $R \to \infty$.

When $q < p$, the local good-λ inequality is not available, which is why we need the additional a priori assumption $\|S_{p,a}(\nabla_T u)\|_{L^q(\mathbb{R}^{n-1})} < \infty$. The proof proceeds otherwise as above but using Lemma 4.1. $\square$

So far we have avoided considering the square function of $\partial_0 u$. We remedy it now. Observe that since

$$|\nabla(\partial_0 u)| \le |\partial^2_{00} u| + |\nabla(\nabla_T u)|,$$

we can use previous calculations for the square function of $\nabla_T u$ and focus on $\partial^2_{00} u$.

Since u solves $\mathcal{L}u = 0$ and $A_{00} = 1$ we have

$$\partial^2_{00} u = -\sum_{(i,j)\ne(0,0)} \partial_i(A_{ij}\,\partial_j u) - \sum_i B_i\,\partial_i u.$$

It follows that we have the estimate

$$S^R_{2,a}(\partial_0 u)(x') \le S^R_{2,a}(\nabla_T u)(x') + C\,\mathcal{T}^R_a(\nabla u)(x'), \tag{4-35}$$

where we define

$$\mathcal{T}^R_a(\nabla u)(Q) = \left(\int_{\Gamma^R_a(Q)} (|\nabla A|^2 + |B|^2)|\nabla u|^2 \delta(x)^{2-n}\,dx\right)^{1/2}. \tag{4-36}$$

Considering the same Ω_g as above we have an analogue of (4-25):

$$\iint_{([0,r]\times\Delta)\cap\Omega_g} (|\nabla A|^2 + |B|^2)|\nabla u|^2 x_0\,dx'\,dx_0 \le C\|\mu\|_{\mathcal{C}} \int_{T(2\Delta)\times\partial\Omega_g} [\widetilde{N}^{2r}_{p,a,g}(\nabla u)]^2\,dx. \tag{4-37}$$

It follows that we can establish a good-λ inequality analogous to Lemma 4.1.

Lemma 4.4. *Let $\Omega = \mathbb{R}^n_+$ and assume u is the energy solution of* (4-1). *Assume that A is 2-elliptic and smooth in $\mathbb{R}^n_+$ with $A_{00} = 1$ and A_{0j} real and that the measure μ defined as in* (1-2) *is Carleson.*

Consider any $b > a > 0$. Then for each $\gamma \in (0,1)$ there exists a constant $C(\gamma) > 0$ such that $C(\gamma, a, b) \to 0$ as $\gamma \to 0$ and with the property that for each $\nu > 0$ we have

$$\big|\{x' \in \mathbb{R}^{n-1} : \mathcal{T}_a(\nabla u)(x') > \nu,\ \|\mu\|_{\mathcal{C}}^{1/2} \widetilde{N}_b(\nabla u)(x') \le \gamma\nu\}\big| \le C(\gamma)\big|\{x' \in \mathbb{R}^{n-1} : \mathcal{T}_b(\nabla u)(x') > \nu/2\}\big|. \tag{4-38}$$

We omit the proof as it follows the same idea as the proof of Lemma 4.1 using (4-37) in place of (4-25). Also averaging in c is not needed. We also have an analogue of Lemma 4.2 by the same argument. We record the consequences of these two results.

Corollary 4.5. *Under the assumption of Lemma 4.1, for any $q \ge 2$ and $a > 0$ there exists a finite constant $C = C(\lambda_2, \Lambda, q, a, \|\mu\|_{\mathcal{C}}, n) > 0$ such that*

$$\|S^R_{2,a}(\partial_0 u)\|_{L^q(\Delta_R)} \le C[\|S^R_{2,a}(\nabla_T u)\|_{L^q(\Delta_{2R})} + \|\mu\|_{\mathcal{C}}^{1/2}\|\widetilde{N}^{2R}_{2,a}(\nabla u)\|_{L^q(\Delta_{2R})}], \tag{4-39}$$

$$\|S_{2,a}(\partial_0 u)\|_{L^q(\mathbb{R}^{n-1})} \le C[\|S_{2,a}(\nabla_T u)\|_{L^q(\mathbb{R}^{n-1})} + \|\mu\|_{\mathcal{C}}^{1/2}\|\widetilde{N}_{2,a}(\nabla u)\|_{L^q(\mathbb{R}^{n-1})}]. \tag{4-40}$$

The inequality (4-40) *also holds for any $q > 0$, provided we know a priori that $\|\mathcal{T}_a(\nabla u)\|_{L^q(\mathbb{R}^{n-1})} < \infty$.*

We are now ready to establish a local solvability result. Let us consider domains of the following the form. Let $\Delta_d \subset \mathbb{R}^{n-1}$ be a boundary ball or a cube of diameter d. We define

$$\mathcal{O}_{\Delta_d,a} = \bigcup_{x' \in \Delta_d} \Gamma_a(x'). \tag{4-41}$$

Here as before $\Gamma_a(x')$ denotes the nontangential region with aperture a at a point x' (see Definition 2.2).

Clearly, a domain such as (4-41) is a domain with Lipschitz constant $1/a$. It follows that if $\mathcal{L}$ satisfies assumptions of Theorem 1.1 on $\mathbb{R}^n_+$ it also satisfies it on any domain $\mathcal{O}_{\Delta_d,a}$, provided $1/a$ is sufficiently small. This can be seen via the pullback transformation (3-11) which transforms the problem from $\mathcal{O}_{\Delta_d,a}$ back to $\mathbb{R}^n_+$. This modifies the coefficients of our PDE to say

$$\operatorname{div}(\bar{A}\nabla v) = 0. \tag{4-42}$$

In particular, if the original PDE on $\mathcal{O}_{\Delta_d,a}$ satisfies $A_{00} = 1$ and A_{0j} are real, the modified coefficients $\bar{A}$ will fail to do so. However, we could fix that via the change of coefficients discussed in (2-34) together with the observations noted below. It follows that (4-42) can be rewritten as

$$\operatorname{div}(\widetilde{A}\nabla v) + \widetilde{B} \cdot \nabla v = 0. \tag{4-43}$$

Because $1/a$ is small the coefficient $\bar{A}_{00}$ is close to 1 and $\bar{A}_{0j}$ are almost real. It follows that rewriting (4-42) as (4-43) will not destroy the ellipticity and p-ellipticity of the matrix $\widetilde{A}$. Hence our previous results of this section apply as they were developed for operators $\mathcal{L}$ with first-order (drift) terms.

We note that in Section 3, drift terms are not allowed, but the results of this section can be applied to the PDE (4-42) because the special assumptions on $\bar{A}_{0j}$ are not used there.

To discuss solvability on domain $\mathcal{O}_{\Delta_d,a}$ we need to consider the nontangential maximal function $\widetilde{N}$ that is taken with respect to nontangential approach regions that are contained inside $\mathcal{O}_{\Delta_d,a}$; that is, we need to take regions $\Gamma_b(\cdot)$ for any $b < a$. Without loss of generality we choose $b = a/2$ and fix it for the remaining part of this section. Finally, $\nabla_T u$ at the boundary of $\mathcal{O}_{\Delta_d,a}$ is understood to be the tangential component of the gradient with respect to the boundary of this domain.

For ease of notation we drop the dependence of the domain $\mathcal{O}_{\Delta_d,a}$ on Δ_d and a and use $\mathcal{O} = \mathcal{O}_{\Delta_d,a}$. We have the following result.

Lemma 4.6. *Let $\mathcal{L}$ be as in Theorem 1.1 on the domain $\mathbb{R}^n_+$ and let A be q-elliptic for some $q \geq 2$. Let $\mathcal{O}$ be a Lipschitz domain as above and assume u is an arbitrary energy solution of $\mathcal{L}u = 0$ in $\mathbb{R}^n_+$ with the Dirichlet boundary datum $\nabla_T f \in L^q(\partial\mathcal{O}; \mathbb{R}^N)$. Then there exists $m = m(a) > 1$ and $K = K(\lambda_p, \Lambda, n, p) > 0$ such that if*

$$\|\mu\|_{\mathcal{C}} + a^{-1} < K,$$

the following estimate holds:

$$\|\widetilde{N}_{a/2}(\nabla u)\|_{L^q(\Delta_d)} \leq C_q \|\nabla_T f\|_{L^q(\partial\mathcal{O}\cap\overline{T(\Delta_{md})})} + C_q d^{(n-1)/q} \sup_{x\in\mathcal{O}\cap\{\delta(x)>d\}} W_2(x), \tag{4-44}$$

where $\delta(x) = \operatorname{dist}(x, \partial\mathbb{R}^n_+)$ and

$$W_2(x) = \left(\fint_{B_{\delta(x)/4}(x)} |\nabla u(y)|^2 \, dy\right)^{1/2}.$$

Proof. In last term of (4-44) because of the way $\mathcal{O}$ is defined we clearly have

$$\{(x_0, x') \in \mathcal{O} : x' \notin \Delta_{(1+a)d}\} \subset \mathcal{O} \cap \{\delta(x) > d\}. \tag{4-45}$$

It follows that by considering the pull-back map $\rho : \mathbb{R}^n_+ \to \mathcal{O}$ defined in (3-11), proving (4-44) is equivalent to establishing

$$\|\widetilde{N}(\nabla u)\|_{L^q(\Delta_d)} \leq C \|\nabla_T f\|_{L^q(\Delta_{md};\mathbb{R}^N)} + C d^{(n-1)/q} \sup_{x\in\mathbb{R}^n_+\setminus T(\Delta_{(1+a)d})} W_2(x), \tag{4-46}$$

where we now work on the domain $\mathbb{R}^n_+$ with u solving $\mathcal{L}u = 0$ in $\mathbb{R}^n_+$ for $\mathcal{L}$ as in Theorem 1.1. We start with the term on the left-hand side of (4-46). If follows from (3-30) that for some $m_1 > 1 + a$

$$\|\widetilde{N}_a^{(1+a)d}(\nabla u)\|^q_{L^q(\Delta_d)} \leq C \|S_a^{m_1 d}(\nabla u)\|^q_{L^q(\Delta_{m_1 d})} + C d^{n-1} |\widetilde{\nabla u}(A_d)|^q. \tag{4-47}$$

The last term above has a trivial bound by $C d^{n-1} \sup_{x\in\mathbb{R}^n_+\setminus T(\Delta_{(1+a)d})} [W_2(x)]^q$.

By (4-39) we have for $m_2 = 2m_1$

$$\|S_a^{m_1 d}(\nabla u)\|_{L^q(\Delta_{m_1 d})} \leq C [\|S_a^{m_2 d}(\nabla_T u)\|_{L^q(\Delta_{m_2 d})} + \|\mu\|_{\mathcal{C}}^{1/2} \|\widetilde{N}_a^{m_2 d}(\nabla u)\|_{L^q(\Delta_{m_2 d})}].$$

Using the Hölder inequality we have for any $x' \in \mathbb{R}^{n-1}$

$$\begin{aligned}
&[S_{2,a}(\nabla_T u)(x')]^2 \\
&\quad= \sum_{k>0} \iint_{\Gamma_a(x')} |\nabla \partial_k u|^{2/q} |\partial_k u|^{1-2/q} |\nabla \partial_k u|^{2/q'} |\partial_k u|^{1-2/q'} x_0 \, dx' \, dx_0 \\
&\quad\le \sum_{k>0} \biggl(\iint_{\Gamma_a(x')} |\nabla \partial_k u|^2 |\partial_k u|^{q-2} x_0 \, dx' \, dx_0 \biggr)^{1/q} \biggl(\iint_{\Gamma_a(x')} |\nabla \partial_k u|^2 |\partial_k u|^{q'-2} x_0 \, dx' \, dx_0 \biggr)^{1/q'} \\
&\quad\le S_{2,q}(\nabla_T u)(x') S_{2,q'}(\nabla_T u)(x'). \qquad (4\text{-}48)
\end{aligned}$$

Hence the previous line implies that for any $\varepsilon > 0$ we have

$$S_{2,a}(\nabla_T u)(x') \le C_\varepsilon S_{q,a}(\nabla_T u)(x') + \varepsilon S_{q',a}(\nabla_T u)(x'), \qquad (4\text{-}49)$$

and the same inequality holds for the truncated square functions. Observe that $q \ge q'$ and hence we can use (4-33) to estimate the second term. This gives us

$$\|S_a^{m_2 d}(\nabla_T u)\|^q_{L^q(\Delta_{m_2 d})} \le C_\varepsilon \|S_{q,a}^{m_2 d}(\nabla_T u)\|^q_{L^q(\Delta_{m_2 d})} + \varepsilon^q \|\widetilde{N}^{m_3 d}(\nabla u)\|^q_{L^q(\Delta_{m_3 d})}$$

for some $m_3 > m_2$. We choose ε so that $\varepsilon^q = \|\mu\|_{\mathcal{C}}^{q/2}$. The estimates we have so far can be combined to the following estimate:

$$\begin{aligned}
\|\widetilde{N}_a^{(1+a)d}(\nabla u)\|^q_{L^q(\Delta_d)} &\le C \|S_{q,a}^{m_2 d}(\nabla_T u)\|^q_{L^q(\Delta_{m_2 d})} \\
&\quad + C \|\mu\|_{\mathcal{C}}^{q/2} \|\widetilde{N}_a^{m_3 d}(\nabla u)\|^q_{L^q(\Delta_{m_3 d})} + C d^{n-1} \sup_{x \in \mathbb{R}^n_+ \setminus T(\Delta_{(1+a)d})} [W_2(x)]^q. \qquad (4\text{-}50)
\end{aligned}$$

To estimate the first term on the right-hand side we use (4-15). This gives

$$\begin{aligned}
\|S_{q,a}^{m_2 d}(\nabla_T u)\|^q_{L^q(\Delta_{m_2 d})} &\lesssim \int_{\Delta_{m_3 d}} |\nabla_T u(0,x')|^q \, dx' + \int_{\Delta_{m_3 d}} |\nabla_T u(m_3 d, x')|^q \, dx' \\
&\quad + \|\mu\|_{\mathcal{C}} \int_{\Delta_{m_3 d}} [\widetilde{N}^{m_3 d}(\nabla u)]^q \, dx' + C \int_{\Delta_{m_3 d} \setminus \Delta_{m_2 d}} [\widetilde{N}^{m_3 d}(\nabla u)]^p \, dx'. \qquad (4\text{-}51)
\end{aligned}$$

Observe that if the estimate above holds for certain $m_3 > 1$ it will certainly holds for any larger value, say $2m_3$. Hence we can average the estimate on the right-hand side of (4-15) between m_3 and $2m_3$. This turns the second term on the right-hand side of (4-15) into a solid integral over a set that is contained in $\mathbb{R}^n_+ \setminus T(\Delta_{(1+a)d})$ and therefore bounded by $C d^{n-1} \sup_{x \in \mathbb{R}^n_+ \setminus T(\Delta_{(1+a)d})} [W_2(x)]^q$. Hence we have for $m_4 = 2m_3$, thanks to (4-50),

$$\begin{aligned}
\|\widetilde{N}^{(1+a)d}(\nabla u)\|^q_{L^q(\Delta_d)} &\lesssim \int_{\Delta_{m_4 d}} |\nabla_T f(x')|^q \, dx' + \max\{\|\mu\|_{\mathcal{C}}, \|\mu\|_{\mathcal{C}}^{q/2}\} \int_{\Delta_{m_4 d}} [\widetilde{N}^{(1+a)d}(\nabla u)]^q \, dx' \\
&\quad + C \int_{\Delta_{m_4 d} \setminus \Delta_{m_2 d}} [\widetilde{N}^{(1+a)d}(\nabla u)]^p \, dx' + d^{n-1} \sup_{x \in \mathbb{R}^n_+ \setminus T(\Delta_{(1+a)d})} [W_2(x)]^q. \qquad (4\text{-}52)
\end{aligned}$$

Here we truncated $\widetilde{N}$ on the right-hand side at the height $(1+a)d$ instead of $m_4 d$ since everything above this height can be incorporated into the last term.

Clearly, for sufficiently small $\|\mu\|_{\mathcal{C}}$ we can hide part of the second term in the last line on the right-hand side of (4-54). Hence

$$\|\widetilde{N}^{(1+a)d}(\nabla u)\|^q_{L^q(\Delta_d)} \lesssim \int_{\Delta_{m_4 d}} |\nabla_T f(x')|^q\,dx' + C\|\widetilde{N}^{(1+a)d}(\nabla u)\|^q_{L^q(\Delta_{m_4 d}\setminus\Delta_d)} + d^{n-1}\sup_{x\in\mathbb{R}^n_+\setminus T(\Delta_{(1+a)d})}[W_2(x)]^q. \tag{4-53}$$

Clearly, the last estimate is scale-invariant and so we write it instead for an enlarged ball $\Delta_{(1+d)d}$. We do this to have in the second term $\Delta_{m_5 d}\setminus\Delta_{(1+a)d}$, where $m_5=(1+a)m_4$. Since

$$\|\widetilde{N}^{(1+a)d}(\nabla u)\|_{L^q(\Delta_d)} \le \|\widetilde{N}^{(1+a)d}(\nabla u)\|_{L^q(\Delta_{(1+a)d})}$$

this gives us

$$\|\widetilde{N}^{(1+a)d}(\nabla u)\|^q_{L^q(\Delta_d)} \lesssim \int_{\Delta_{m_5 d}} |\nabla_T f(x')|^q\,dx' + C\|\widetilde{N}^{(1+a)d}(\nabla u)\|^q_{L^q(\Delta_{m_5 d}\setminus\Delta_{(1+a)d})} + d^{n-1}\sup_{x\in\mathbb{R}^n_+\setminus T(\Delta_{(1+a)d})}[W_2(x)]^q. \tag{4-54}$$

We now push-forward (4-54) back to the original domain $\mathcal{O}$. We have

$$\|\widetilde{N}^{(1+a)d}_{a/2}(\nabla u)\|^q_{L^q(\Delta_d)} \le C\|\nabla_T f\|^q_{L^q(\partial\mathcal{O}\cap\overline{T(\Delta_{m_5 d})})} + Cd^{n-1}\sup_{x\in\mathcal{O}\cap\{\delta(x)>d\}} W_2(x)^q + C\|\widetilde{N}^{(1+a)d}(\nabla u)\|^q_{L^q(\partial\mathcal{O}\cap[T(\Delta_{m_5 d})\setminus T(\Delta_{(1+a)d})])}. \tag{4-55}$$

We would like to hide the last term. Observe that all points of $\partial O\cap[T(\Delta_{m_5 d})\setminus T(\Delta_{(1+a)d})]$ are in the interior of the original domain $\mathbb{R}^n_+$ of distance at least d away from the boundary of $\mathbb{R}^n_+$. Hence whenever we were applying Theorem 2.7 we could have in fact used (2-25) there with h being the function describing the boundary of $\mathcal{O}$. Since pointwise for $Q\in\partial\mathcal{O}\cap[T(\Delta_{m_5 d})\setminus T(\Delta_{(1+a)d})]$

$$\widetilde{N}_{a,h}(\nabla u)(Q) \le \sup_{x\in\mathcal{O}\cap\{\delta(x)>d\}} W_2(x),$$

the last term can be estimated by $Cd^{n-1}\sup_{x\in\mathcal{O}\cap\{\delta(x)>d\}} W_2(x)^q$ as well.

Finally, we can remove the truncation of $\widetilde{N}$ at height $(1+a)d$ on the left-hand side of (4-55) as for points above this height again the term $Cd^{n-1}\sup_{x\in\mathcal{O}\cap\{\delta(x)>d\}} W_2(x)^q$ controls the nontangential maximal function. This establishes our claim. □

5. Proof of Theorem 1.1

We will establish the solvability of the regularity problem assuming that the coefficients of A and B are smooth, applying the results of the previous two sections. The constants in the estimates will not depend on the degree of smoothness. Then, considering smooth approximations of $\mathcal{L}$, a limiting argument gives Theorem 1.1 in the general case.

We start with $p=2$. Assume the matrix A is 2-elliptic. It follows that Lemma 4.6 applies. For any K as in the lemma for any $\|\mu\|_{\mathcal{C}}<K$ we pick a such that $\|\mu\|_{\mathcal{C}}+a^{-1}<K$.

Consider any $f \in L^2(\partial\mathbb{R}^n_+) \cap \dot{B}^{2,2}_{1/2}(\partial\mathbb{R}^n_+)$ and let $u \in \dot{W}^{1,2}(\mathbb{R}^n_+)$ be the unique energy solution of $\mathcal{L}u = 0$ with boundary datum f. We shall additionally assume f is a smooth compactly supported function, it suffices to establish our estimates for those as such functions form a dense subset of $L^2(\partial\mathbb{R}^n_+) \cap \dot{B}^{2,2}_{1/2}(\partial\mathbb{R}^n_+)$.

Fix $d > 0$ and consider $\Delta = \Delta_d(0)$. We apply Lemma 4.6 to the domains $\mathcal{O}_\tau = \mathcal{O}_{\tau\Delta,a}$ for $\tau \in [1, 2]$. This gives us

$$\|\widetilde{N}_{a/2}(\nabla u)\|^2_{L^2(\Delta)} \leq C\|\nabla_T f\|^2_{L^2(\partial\mathcal{O}_\tau \cap \overline{T(\tau m\Delta)})} + Cd^{n-1} \sup_{x \in \mathcal{O}_\tau \cap \{\delta(x) > d\}} W_2(x)^2. \tag{5-1}$$

Note that each of the sets $\partial\mathcal{O}_\tau \cap \overline{T(\tau m\Delta)}$ consists of the "flat piece" that is just $\tau\Delta = \Delta_{\tau d}(0)$ and the remaining curve that lies inside $\mathbb{R}^n_+$. If we average the above inequality over all values of $\tau \in [1, 2]$, the latter turns into a solid integral over a set that is contained in

$$\mathcal{S}_d := (0, 2md) \times (\Delta_{2md} \setminus \Delta_d).$$

It follows that

$$\|\widetilde{N}_{a/2}(\nabla u)\|^2_{L^2(\Delta)} \leq C\|\nabla_T f\|^2_{L^2(2\Delta)} + Cd^{n-1} \sup_{\{x:\delta(x)>d\}} W_2(x)^2 + Cd^{-1} \iint_{\mathcal{S}_d} |\nabla u|^2\, dx. \tag{5-2}$$

Consider what happens as we take $d \to \infty$ in the estimate above. Recall that we know that $\nabla u \in L^2(\mathbb{R}^n_+)$ from the fact that u is an energy solution. This information implies that both

$$\iint_{B(x,\delta(x)/2)} |\nabla u|^2\, dx \to 0, \qquad \iint_{\mathcal{S}_d} |\nabla u|^2\, dx \to 0$$

for all $x \in \{x : \delta(x) > d\}$ uniformly as $d \to \infty$. From this however we see that the last two terms of (5-2) go to zero as $d \to \infty$, and hence in the limit we have

$$\|\widetilde{N}_{a/2}(\nabla u)\|^2_{L^2(\partial\mathbb{R}^n_+)} \leq C\|\nabla_T f\|^2_{L^2(\partial\mathbb{R}^n_+)},$$

which is L^2 solvability of the regularity problem. Also observe that constant C in the estimate above only depends on λ_2, Λ and n, precisely as stated in Theorem 1.1.

We now extrapolate. It has been established in [Dindoš et al. 2017a] that, from Lemma 4.6, a purely real-variable argument can be used to establish the estimate

$$\int_{E_\nu \cap \{g \leq \nu\}} [\widetilde{N}(\nabla u)(x')]^2\, dx' \leq C_\alpha \nu^2 |E_\nu| + C\alpha^{-1} \int_{E_\nu} [\widetilde{N}(\nabla u)(x')]^2\, dx', \tag{5-3}$$

where $E_\nu = \{x' \in \mathbb{R}^{n-1}_+ : \widetilde{N}_\alpha(\nabla u)(x') > \nu\}$ and

$$g(x') = \sup_{B \ni x'} \left(\fint_B |\nabla_T f(y')|^2\, dy' \right)^{1/2}.$$

See in particular Lemma 6.1 and (6.17) of [Dindoš et al. 2017a], which are completely analogous to our Lemma 4.6 and (5-3). A consequence of (5-3) is the existence of $\delta > 0$ which only depends on the constant in the estimate (4-44) such that

$$\|\widetilde{N}_{a/2}(\nabla u)\|_{L^{2+\delta}(\partial\mathbb{R}^n_+)} \le C\|\nabla_T f\|_{L^{2+\delta}(\partial\mathbb{R}^n_+)}, \tag{5-4}$$

which is the solvability of the regularity problem for $p_0 = 2+\delta$. If p_0 is such that the matrix A is p_0-elliptic we can repeat the process we did for $p=2$ above. We now apply Lemma 4.6 for the value p_0 and again take the limit $d\to\infty$. This time the solid integrals we get are

$$d^{-1}\iint_{B(x,\delta(x)/2)} |\nabla u|^{p_0}\,dx, \qquad d^{-1}\iint_{\mathcal{S}_d} |\nabla u|^{p_0}\,dx,$$

which we know go to zero uniformly for all $x\in\{x:\delta(x)>d\}$ as $d\to\infty$ thanks to the fact that (5-4) implies $\|\widetilde{N}_{p_0,a/2}(\nabla u)\|_{L^{p_0}}<\infty$. Hence taking the limit $d\to\infty$ in the analogue of (5-2) for p_0 yields

$$\|\widetilde{N}_{a/2}(\nabla u)\|_{L^{p_0}(\partial\mathbb{R}^n_+)} \le C_{p_0}\|\nabla_T f\|_{L^{p_0}(\partial\mathbb{R}^n_+)}. \tag{5-5}$$

This seemingly is just a restatement of (5-4). The difference however is that now the constant C_{p_0} in (5-5) only depends on the constant in Lemma 4.6 for the value p_0. This allows us to extrapolate again and obtain solvability of the regularity problem for some value $p_0+\delta'$. There is no difference in the structure of the argument. We can continue this bootstrapping as long as we stay in the range of p-ellipticity and as long as we can be sure that we are moving by an amount δ' which is not getting smaller at each step. This last point is assured by the fact that the constants C_{p_0} in the L^{p_0} norm inequalities (5-5) only depend on the p_0-ellipticity and the Carleson measure norm of the coefficients. If we fix $p>2$ such that the operator is p-elliptic the constants C_q for $2\le q\le p$ in Lemma 4.6 are uniformly bounded which ensures that our bootstrapping argument will reach the desired value p in finitely many steps, giving us solvability of the regularity problem and the estimate (1-5) of Theorem 1.1.

We now deal with $p<2$ such that A is p-elliptic. Assume first that we a priori know that

$$\|\widetilde{N}_{2,a}(\nabla u)\|_{L^p(\mathbb{R}^{n-1})}<\infty$$

for an energy solution u in $\mathbb{R}^n_+$ with boundary datum f. Then by (3-31) of Proposition 3.6 and by (4-40) of Corollary 4.5 we have

$$\begin{aligned}\|\widetilde{N}_{2,a}(\nabla u)\|_{L^p(\mathbb{R}^{n-1})} &\le C\|S_{2,a}(\nabla u)\|_{L^p(\mathbb{R}^{n-1})}\\ &\le C\|S_{2,a}(\nabla_T u)\|_{L^p(\mathbb{R}^{n-1})} + C\|\mu\|^{1/2}\|\widetilde{N}_{2,a}(\nabla u)\|_{L^p(\mathbb{R}^{n-1})}.\end{aligned} \tag{5-6}$$

Here in order to use Corollary 4.5, we must verify that $\|\mathcal{T}_a(\nabla u)\|_{L^p(\mathbb{R}^{n-1})}<\infty$. However under the assumption that the coefficients are smooth we have a pointwise bound $\mathcal{T}_a(\nabla u)(Q)\le S_{2,a}(u)(Q)$. We have established solvability of the L^p Dirichlet problem in [Dindoš and Pipher 2019] in the range where p-ellipticity holds and in particular we have shown the bound $\|S_{2,a}(u)\|_{L^p(\mathbb{R}^{n-1})}\lesssim\|f\|_{L^p(\mathbb{R}^{n-1})}<\infty$ (using that $f\in C_0^\infty\subset L^p$).

Hence taking sufficiently small K in Theorem 1.1 it follows that

$$\|\widetilde{N}_{2,a}(\nabla u)\|_{L^p(\mathbb{R}^{n-1})} \le C\|S_{2,a}(\nabla_T u)\|_{L^p(\mathbb{R}^{n-1})}. \tag{5-7}$$

Thus by (4-49) in conjunction with (4-34) of Corollary 4.3 and (2-18) we have

$$\|S_{2,a}(\nabla_T u)\|_{L^p(\mathbb{R}^{n-1})} \le C\|S_{p,a}(\nabla_T u)\|_{L^p(\mathbb{R}^{n-1})} + \varepsilon\|\widetilde{N}_{2,a}(\nabla u)\|_{L^p(\mathbb{R}^{n-1})}. \tag{5-8}$$

When applying (4-34) to estimate $\|S_{p',a}(\nabla_T u)\|_{L^p(\mathbb{R}^{n-1})}$ we need to know a priori that this quantity is finite. Here we use our assumption that for now the coefficients are smooth. This gives is a pointwise bound

$$S_{p',a}(\nabla_T u) \le C S_{2,a}(\nabla_T u) + C N(\nabla u),$$

where N is the pointwise maximal function. The classical L^∞ bounds of Agmon, Douglis and Nirenberg [Agmon et al. 1964] for smooth PDE systems imply $N \lesssim N_2$. We also have $\|S_{2,a}(\nabla_T u)\|_{L^p} < \infty$ from a similar estimate

$$S_{2,a}(\nabla_T u) \le C S_{p,a}(\nabla_T u) + C N(\nabla u),$$

and finally we know that $\|S_{p,a}(\nabla_T u)\|_{L^p(\mathbb{R}^{n-1})} < \infty$ by (4-19) (taking $r \to \infty$). The one "bad" term in (4-19), which is $\int_{\mathbb{R}^{n-1}} |\nabla_T u(r, x')|^p \, dx'$, can be dealt with by averaging in r first, which turns it into a solid integral. Such term can be estimated by

$$\|\widetilde{N}_{p,a}(\nabla u)\|_{L^p(\mathbb{R}^{n-1})} \lesssim \|\widetilde{N}_{2,a}(\nabla u)\|_{L^p(\mathbb{R}^{n-1})} < \infty$$

and furthermore it follows this term converges to zero as $r \to \infty$.

Hence all quantities appearing in (5-8) are finite under the assumption our coefficients are smooth, but the constants in this estimate only depend on the parameters n, p, λ_p, Λ. We choose $\varepsilon > 0$ in this inequality small enough so that we can hide this term on the left-hand side of (5-7). This gives is

$$\|\widetilde{N}_{2,a}(\nabla u)\|_{L^p(\mathbb{R}^{n-1})} \le C\|S_{p,a}(\nabla_T u)\|_{L^p(\mathbb{R}^{n-1})}. \tag{5-9}$$

We can now use again (4-19) for $S_{p,a}(\nabla_T u)$ taking $r \to \infty$. As above, the term $\int_{\mathbb{R}^{n-1}} |\nabla_T u(r, x')|^p \, dx'$ gets eliminated. It follows that (4-19) gives us

$$\|S_{p,a}(\nabla_T u)\|_{L^p(\mathbb{R}^{n-1})} \le C\|\nabla_T f\|_{L^p(\mathbb{R}^{n-1})} + C\|\mu\|_{\mathcal{C}}^{1/p}\|\widetilde{N}_{p,a}(\nabla u)\|_{L^p(\mathbb{R}^{n-1})}. \tag{5-10}$$

Hence for all $\|\mu\|_{\mathcal{C}} < K$ sufficiently small, combining (5-9), (5-10) and (2-18) yields

$$\|\widetilde{N}_{2,a}(\nabla u)\|_{L^p(\mathbb{R}^{n-1})} \le C\|\nabla_T f\|_{L^p(\mathbb{R}^{n-1})}, \tag{5-11}$$

from which solvability of the L^p regularity problem follows.

It remains to remove the a priori assumption $\|\widetilde{N}_{2,a}(\nabla u)\|_{L^p(\mathbb{R}^{n-1})} < \infty$ we have made earlier.

We again argue by extrapolation starting with $p = 2$, where we know this since we have already established solvability of the regularity problem for this value of p.

This time we shall use an extrapolation argument based on an method in [Dahlberg et al. 1986] of obtaining $L^{2-\varepsilon}$ estimates of nontangential maximal functions from L^2 estimates on sawtooth domains. See also [Dindoš et al. 2017a], where this technique was used to get solvability of the L^p Dirichlet problem for elliptic systems for $2 - \varepsilon < p < 2$. In particular, the argument of [Dahlberg et al. 1986], reproduced in Section 6 of [Dindoš et al. 2017a] for systems and hence valid in our setting, gives that

$\|\widetilde{N}_{2,a}(\nabla u)\|_{L^{p_0}(\mathbb{R}^{n-1})} < \infty$ for $p_0 = 2-\varepsilon$ and hence the same is true for $\|\widetilde{N}_{p_0,a}(\nabla u)\|_{L^{p_0}(\mathbb{R}^{n-1})}$. The quantity ε depends on the constant C_2 in the L^2 norm inequality between the nontangential maximal function and the square function S_2.

Once we know these quantities are finite, the calculation we did above holds for p_0 giving us (5-11), and hence the same estimate for ∇u for $p_0 = 2-\varepsilon$ and a constant $C_{2-\varepsilon}$.

The very same extrapolation argument, now invoking the L^{p_0} estimate gives an $L^{p_0-\varepsilon'}$ estimate where ε' now depends on $C_{2-\varepsilon}$. In other words, we apply the same argument as [Dahlberg et al. 1986] but starting from known estimates for the nontangential maximal function in L^{p_0} instead of L^2. We can continue this bootstrapping as long as we stay in the range of p-ellipticity and as long as we can be sure that we are moving by an amount ε which is not getting smaller at each step. The same argument as given previously implies that we can reach any value $p < 2$ in the p-ellipticity range of the matrix A in a finite number of steps. From this Theorem 1.1 follows.

Finally, we remove the temporary assumption that the coefficients are smooth. The key is that the constants in the estimates above depend only on $n, p, \lambda_p, \Lambda, \|\mu\|_C$ and not on any further degree of smoothness of the coefficients of $\mathcal{L}$. Hence the classical argument where we approximate our coefficients by smooth functions, and then pass from the smooth coefficient case by taking the limit can be applied. See for example Section 4 of [Dindoš and Pipher 2019], where this is discussed in more detail.

6. Proof of Theorem 1.2

The proof is based on the following abstract result [Shen 2005]; see also [Wei and Zhang 2015, Theorem 3.1] for a version on an arbitrary bounded domain.

Theorem 6.1. *Let T be a bounded sublinear operator on $L^2(\mathbb{R}^{n-1};\mathbb{C}^m)$. Suppose that for some $p > 2$, T satisfies the following L^p localization property. For any ball $\Delta = \Delta_d \subset \mathbb{R}^{n-1}$ and C^∞ function f with* $\operatorname{supp}(f) \subset \mathbb{R}^{n-1} \setminus 3\Delta$ *the following estimate holds*:

$$\left(|\Delta|^{-1}\int_\Delta |Tf|^p\,dx'\right)^{1/p} \le C\left\{\left(|2\Delta|^{-1}\int_{2\Delta}|Tf|^2\,dx'\right)^{1/2} + \sup_{\Delta'\supset\Delta}\left(|\Delta'|^{-1}\int_{\Delta'}|f|^2\,dx'\right)^{1/2}\right\} \tag{6-1}$$

for some $C > 0$ independent of f. Then T is bounded on $L^q(\mathbb{R}^{n-1};\mathbb{C}^m)$ for any $2 \le q < p$.

In our case the role of T is played by the sublinear operator $f \mapsto \widetilde{N}_{2,a}(u)$, where u is the solution of the Dirichlet problem $\mathcal{L}u = 0$ with boundary data f. Clearly, in the theorem above the factors 2Δ, 3Δ do not play significant role. Hence if we establish estimate (6-1) with 2Δ replaced by $m\Delta$ with f vanishing on $(m+1)\Delta$ for some $m > 1$ the claim of the theorem will still hold.

Clearly, our operator $T : f \mapsto \widetilde{N}_{2,a}(u)$ is sublinear and bounded on L^2 by [Dindoš and Pipher 2019] for coefficients with small Carleson norm μ. To prove (6-1) we shall establish the following reverse Hölder inequality, following ideas of [Shen 2006]:

$$\left(\frac{1}{|\Delta|}\int_\Delta |\widetilde{N}_{2,a}(u)|^p\,dx'\right)^{1/p} \le C\left(\frac{1}{|3\beta m\Delta|}\int_{3\beta m\Delta}|\widetilde{N}_{2,a}(u)|^2\,dx'\right)^{1/2}, \tag{6-2}$$

where $f = u|_{\partial\mathbb{R}^n_+}$ vanishes on $4\beta m\Delta$. Here m is determined by Lemma 4.6 and $\beta > 1$ is determined by a bootstrap argument explained later. Having this by Theorem 6.1 we have for any $q \in [2, p)$ the estimate

$$\|\widetilde{N}_{2,a}(u)\|_{L^q(\mathbb{R}^{n-1})} \leq C\|f\|_{L^q(\mathbb{R}^{n-1})}, \tag{6-3}$$

which implies L^q solvability of the Dirichlet problem for the operator $\mathcal{L}$.

It remains to establish (6-2). Let us define

$$\begin{aligned}\mathcal{M}_1(u)(x') &= \sup_{y\in\Gamma_a(x')} \{w_2(y) : \delta(y) \leq cd\},\\ \mathcal{M}_2(u)(x') &= \sup_{y\in\Gamma_a(x')} \{w_2(y) : \delta(y) > cd\},\end{aligned} \tag{6-4}$$

where $c = c(a) > 0$ is chosen such that for all $x' \in \Delta$ if $y = (y_0, y') \in \Gamma_a(x')$ and $y_0 = \delta(y) \leq cd$ then $y' \in 2\Delta$. Here $d = \operatorname{diam}(\Delta)$ and w_2 is the L^2 average of u

$$w_2(y) = \left(⨍_{B_{\delta(y)/2}(y)} |u(z)|^2\, dz\right)^{1/2}.$$

It follows that

$$\widetilde{N}_{2,a}(u) = \max\{\mathcal{M}_1(u), \mathcal{M}_2(u)\}.$$

We first estimate $\mathcal{M}_2(u)$. Pick any $x' \in \Delta$. For any $y \in \Gamma(x')$ with $\delta(y) > cd$ it follows that for a large subset A of 2Δ (of size comparable to 2Δ) we have

$$z' \in A \quad \Longrightarrow \quad y \in \Gamma_a(z') \quad \Longrightarrow \quad w_2(y) \leq \widetilde{N}_{2,a}(u)(z').$$

Hence for any $x' \in \Delta$

$$\mathcal{M}_2(u)(x') \leq C\left(\frac{1}{|2\Delta|}\int_{2\Delta} [\widetilde{N}_{2,a}(u)(z')]^2\, dz'\right)^{1/2}.$$

It remains to estimate $\mathcal{M}_1(u)$ on Δ.

We write

$$u(x_0, x') - u(0, y') = \int_0^1 \frac{\partial u}{\partial s}(sx_0, (1-s)y' + sx')\, ds.$$

Let $K = \{(y_0, y') : y' \in \Delta \text{ and } cd < y_0 < 2cd\}$. Using the previous line and the fact that u vanishes on $3\Delta \subset 4\beta m\Delta$ we have for any $x' \in \Delta$

$$\mathcal{M}_1(u)(x') \leq \sup_K w_2 + C\int_{2\Delta} \frac{\widetilde{N}_{2,a/2}(\nabla u)(y')}{|x'-y'|^{n-2}}\, dy'. \tag{6-5}$$

By the fractional integral estimate, this implies

$$\left(\frac{1}{|\Delta|}\int_\Delta [\mathcal{M}_1(u)(x')]^p\, dx'\right)^{1/p} \leq \sup_K w_2 + Cd\left(\frac{1}{|2\Delta|}\int_{2\Delta} [\widetilde{N}_{2,a/2}(\nabla u)(x')]^q\, dx'\right)^{1/q}, \tag{6-6}$$

where

$$\frac{1}{p} = \frac{1}{q} - \frac{1}{n-1} \quad \text{and} \quad 1 < q < n-1.$$

To further estimate (6-6) we use Lemma 4.6. We claim the reverse Hölder inequality

$$\left(\frac{1}{|\Delta|}\int_{\Delta}[\widetilde{N}_{2,a/2}(\nabla u)(x')]^q\,dx'\right)^{1/q} \lesssim \left(\frac{1}{|\beta\Delta|}\int_{\beta\Delta}[\widetilde{N}_{2,a/2}(\nabla u)(x')]^2\,dx'\right)^{1/2}$$

holds whenever the solution $\mathcal{L}u=0$ vanishes on at least $2\beta\Delta$.

Let d be the diameter of Δ. We apply Lemma 4.6 to the domains (4-41) $\mathcal{O}_\tau=\mathcal{O}_{\tau\Delta,a}$ for $\tau\in[1,2]$. This gives us

$$\|\widetilde{N}_{a/2}(\nabla u)\|_{L^q(\Delta)} \le C\|\nabla_T f\|_{L^q(\partial\mathcal{O}_\tau\cap\overline{T(\tau m\Delta)})} + Cd^{(n-1)/q}\sup_{x\in\mathcal{O}_\tau\cap\{\delta(x)>d\}} W_2(x). \tag{6-7}$$

Observe that for any $x\in\mathcal{O}_\tau\cap\{\delta(x)>d\}$ we have

$$|A| = |\{y'\in 2\Delta : x\in\Gamma_{a/2}(y')\}| \approx d^{n-1},$$

and clearly for each $y'\in A$ we have

$$W_2(x)\lesssim \widetilde{N}_{a/2}(\nabla u)(y'),$$

from which

$$W_2(x)\lesssim |A|^{-1}\left(\int_A[\widetilde{N}_{a/2}(\nabla u)(y')]^2\,dy'\right)^{1/2} \lesssim |2\Delta|^{-1}\left(\int_{2\Delta}[\widetilde{N}_{a/2}(\nabla u)(y')]^2\,dy'\right)^{1/2}.$$

It follows that

$$\sup_{x\in\mathcal{O}_\tau\cap\{\delta(x)>d\}} W_2(x) \lesssim |2\Delta|^{-1}\left(\int_{2\Delta}[\widetilde{N}_{a/2}(\nabla u)(y')]^2\,dy'\right)^{1/2}. \tag{6-8}$$

We use this in (6-7), integrate (6-7) in τ over the interval $[1,2]$ and divide by $d^{(n-1)/q}$. This gives, after using the fact that $u=0$ vanishes on at least $4m\Delta$,

$$\begin{aligned}&\left(\frac{1}{|\Delta|}\int_{\Delta}[\widetilde{N}_{2,a/2}(\nabla u)(x')]^q\,dx'\right)^{1/q}\\ &\qquad\lesssim \left(\frac{1}{T(2m\Delta)}\iint_{T(2m\Delta)}|\nabla u(x)|^q\,dx\right)^{1/q} + \left(\frac{1}{|2\Delta|}\int_{2\Delta}[\widetilde{N}_{2,a}(\nabla u)(x')]^2\,dx'\right)^{1/2}.\end{aligned} \tag{6-9}$$

We have also used the trivial estimate $|\nabla_T u|\le|\nabla u|$ on $\partial\mathcal{O}_\tau\cap T(2m\Delta)$. For the first term we have

$$\iint_{T(2m\Delta)}|\nabla u(x)|^q\,dx = \iint_{T(2m\Delta)\cap\{x_0<\varepsilon md\}}|\nabla u(x)|^q\,dx + \iint_{T(2m\Delta)\cap\{x_0>\varepsilon md\}}|\nabla u(x)|^q\,dx. \tag{6-10}$$

The set $T(2m\Delta)\cap\{x_0>\varepsilon md\}$ is in the interior of $\mathbb{R}^n_+$ and has diameter and distance to the boundary that are comparable to d. It follows that the interior estimate (2-17) can be used (we only enlarge this set by a small factor $\alpha > 1$ so that $\alpha[T(2m\Delta) \cap \{x_0 > \varepsilon md\}]$ fully lies in the interior of $\mathbb{R}^n_+$). It follows that

$$\begin{aligned}\frac{1}{|T(2m\Delta)|}\iint_{T(2m\Delta)\cap\{x_0>\varepsilon md\}} |\nabla u(x)|^q\,dx &\lesssim \left(\frac{1}{|T(2m\Delta)|}\iint_{\alpha[T(2m\Delta)\cap\{x_0>\varepsilon md\}]} |\nabla u(x)|^2\,dx\right)^{q/2}\\ &\lesssim \left(\frac{1}{|T(3m\Delta)|}\iint_{T(3m\Delta)} |\nabla u(x)|^2\,dx\right)^{q/2}\\ &\lesssim \left(\frac{1}{|3m\Delta|}\int_{3m\Delta} [\widetilde{N}_{a/2}(\nabla u)(x')]^2\,dx'\right)^{q/2}. \end{aligned} \tag{6-11}$$

For the term $\iint_{T(2m\Delta)\cap\{x_0<\varepsilon md\}} |\nabla u(x)|^q\,dx$ we use the trivial estimate

$$\iint_{T(2m\Delta)\cap\{x_0<\varepsilon md\}} |\nabla u(x)|^q\,dx \le \varepsilon md \int_{3m\Delta} [\widetilde{N}_{a/2}(\nabla u)(x')]^q\,dx'. \tag{6-12}$$

Combining (6-10)–(6-12) finally yields

$$\begin{aligned}&\frac{1}{|T(2m\Delta)|}\iint_{T(2m\Delta)} |\nabla u(x)|^q\,dx\\ &\qquad\le \left(\frac{C_\varepsilon}{|3m\Delta|}\int_{3m\Delta} [\widetilde{N}_{a/2}(\nabla u)(x')]^2\,dx'\right)^{q/2} + \frac{\varepsilon}{|3m\Delta|}\int_{3m\Delta} [\widetilde{N}_{a/2}(\nabla u)(x')]^q\,dx'. \end{aligned} \tag{6-13}$$

This combined with (6-9) yields

$$\begin{aligned}&\frac{1}{|\Delta|}\int_{\Delta} [\widetilde{N}_{2,a/2}(\nabla u)(x')]^q\,dx'\\ &\qquad\lesssim \left(\frac{C_\varepsilon}{|3m\Delta|}\int_{3m\Delta} [\widetilde{N}_{a/2}(\nabla u)(x')]^2\,dx'\right)^{q/2} + \frac{\varepsilon}{|3m\Delta|}\int_{3m\Delta} [\widetilde{N}_{a/2}(\nabla u)(x')]^q\,dx'. \end{aligned} \tag{6-14}$$

We now recall an abstract result from [Giaquinta 1983, Chapter 5, Proposition 1.1].

Theorem 6.2. *Let B_R be a ball in $\mathbb{R}^N$. Suppose that $g \ge 0$, $g \in L^q(B_R)$ for some $q > 1$ and for all $x \in B_{R/2}$ and $0 < r < R/16$ we have*

$$\fint_{B_r} g^q\,dx \le C\left(\fint_{B_{2r}} g\,dx\right)^q + \theta\fint_{B_{2r}} g^q\,dx$$

for some constants $C > 1$, $\theta < 1$.

Then there exists $\delta = \delta(C, \theta, N, q) > 0$ and $K = K(C, \theta, N, q) > 0$ such that for all B_r concentric with B_R of radius $0 < r < R/4$ we have

$$\left(\fint_{B_{r/2}} g^{q+\delta}\,dx\right)^{1/(q+\delta)} \le K\left(\fint_{B_r} g^q\,dx\right)^{1/q}.$$

Applying this to (6-14) with $g(x') = [\widetilde{N}_{a/2}(\nabla u)(x')]^2$ yields that for some $\alpha > 1$ we have

$$\left(\frac{1}{|\Delta|}\int_{\Delta} [\widetilde{N}_{2,a/2}(\nabla u)(x')]^{q+\delta}\,dx'\right)^{1/(q+\delta)} \lesssim \left(\frac{1}{|\alpha\Delta|}\int_{\alpha\Delta} [\widetilde{N}_{a/2}(\nabla u)(x')]^q\,dx'\right)^{1/q}. \tag{6-15}$$

Here clearly, $\delta = \delta(q)$ depends on q but as long as the constant C_ε in the estimate (6-13) stays uniform (which is for $q \in [p_0+\eta, p_0'-\eta]$ for any $\eta > 0$ where (p_0, p_0') is the interval in which we have p-ellipticity) we have

$$\inf_{q\in[2+\eta,p_0'-\eta]} \delta(q) > 0 \quad \text{for all } \eta > 0.$$

Here we are avoiding q near 2 as well since then (6-13) provides no information. However, to get us started in the bootstrap argument we may use the inequality

$$\left(\frac{1}{T(2m\Delta)}\iint_{T(2m\Delta)} |\nabla u|^{2+\delta_0}\,dx\right)^{1/(2+\delta_0)} \lesssim \left(\frac{1}{T(3m\Delta)}\iint_{T(3m\Delta)} |\nabla u|^2\,dx\right)^{1/2}$$

for some $\delta_0 > 0$ small, which is a well known consequence of Caccioppoli's inequality and Theorem 6.2. It follows using (6-9) that

$$\left(\frac{1}{|\Delta|}\int_\Delta [\widetilde{N}_{2,a/2}(\nabla u)(x')]^{2+\delta_0}\,dx'\right)^{1/(2+\delta_0)} \lesssim \left(\frac{1}{|3m\Delta|}\int_{3m\Delta} [\widetilde{N}_{2,a}(\nabla u)(x')]^2\,dx'\right)^{1/2}. \tag{6-16}$$

This is the initial inequality in the bootstrap argument after which we iteratively use (6-15), where $\delta > 0$ stays bounded away from zero as long as we take $q \le p_0' - \eta$ for some small fixed $\eta > 0$. This finally implies that for all $q < p_0'$ we have

$$\left(\frac{1}{|\Delta|}\int_\Delta [\widetilde{N}_{2,a/2}(\nabla u)(x')]^q\,dx'\right)^{1/q} \lesssim \left(\frac{1}{|\beta\Delta|}\int_{\beta\Delta} [\widetilde{N}_{2,a}(\nabla u)(x')]^2\,dx'\right)^{1/2} \tag{6-17}$$

for some $\beta > 1$ with u vanishing on $2\beta\Delta$. The implied constant in the estimate (6-17) gets progressively worse as $q \to p_0'-$. Next, we use again (6-9), this time for $q = 2$,

$$\begin{aligned}\left(\frac{1}{|\beta\Delta|}\int_{\beta\Delta} [\widetilde{N}_{2,a/2}(\nabla u)(x')]^2\,dx'\right)^{1/2}&\\ \lesssim \left(\frac{1}{T(2\beta m\Delta)}\iint_{T(2\beta m\Delta)} |\nabla u|^2\,dx\right)^{1/2} &+ \sup_{x\in\mathcal{O}_{2\beta}\cap\{\delta(x)>d\}} W_2(x),\end{aligned} \tag{6-18}$$

where we put back W_2 instead of our initial estimate (6-8). For the first term we use the boundary Caccioppoli inequality

$$\begin{aligned}\left(\frac{1}{T(2\beta m\Delta)}\iint_{T(2\beta m\Delta)} |\nabla u|^2\,dx\right)^{1/2} &\lesssim d^{-1}\left(\frac{1}{T(3\beta m\Delta)}\iint_{T(3\beta m\Delta)} |u|^2\,dx\right)^{1/2}\\ &\lesssim d^{-1}\left(\frac{1}{|3\beta m\Delta|}\int_{3\beta m\Delta} [\widetilde{N}_{2,a}(u)(z')]^2\,dz'\right)^{1/2},\end{aligned} \tag{6-19}$$

while for the second term by the interior Caccioppoli inequality we have for all $x \in \mathbb{R}^n_+$ with $\delta(x) > d$

$$W_2(x) \le Cd^{-1}w_2(x),$$

where w_2 denotes the L^2 averages of u (defined earlier). We have intentionally shrunk the size of the ball in the definition of W_2 so that this pointwise estimate holds. Since the x we consider in the supremum is in $\mathcal{O}_{2\beta}$ it then follows that

$$\sup_{x\in\mathcal{O}_{2\beta}\cap\{\delta(x)>d\}} W_2(x) \lesssim d^{-1}\left(\frac{1}{|2\beta\Delta|}\int_{2\beta\Delta}[\widetilde{N}_{2,a}(u)(z')]^2\,dz'\right)^{1/2}. \tag{6-20}$$

Using this and the previous estimates (6-17)–(6-18) we then have for all $q < p_0'$

$$\left(\frac{1}{|\Delta|}\int_{\Delta}[\widetilde{N}_{2,a/2}(\nabla u)(x')]^q\,dx'\right)^{1/q} \lesssim d^{-1}\left(\frac{1}{|3\beta m\Delta|}\int_{3\beta m\Delta}[\widetilde{N}_{2,a}(u)(z')]^2\,dz'\right)^{1/2}. \tag{6-21}$$

Finally, inserting this estimate into (6-6) yields

$$\left(\frac{1}{|\Delta|}\int_{\Delta}[\mathcal{M}_1(u)(x')]^p\,dx'\right)^{1/p} \le C\left(\frac{1}{|3\beta m\Delta|}\int_{3\beta m\Delta}[\widetilde{N}_{2,a}(u)(z')]^2\,dz'\right)^{1/2}, \tag{6-22}$$

where

$$\frac{1}{p}=\frac{1}{q}-\frac{1}{n-1} \quad\text{and}\quad 1<q<n-1$$

such that A is q-elliptic and Carleson norm of μ is small. Since we have assumed A is q-elliptic for $q\in(p_0,p_0')$ and $p_0'>2$ this implies in dimensions 2 and 3 that we can consider any $2<p<\infty$, while in dimensions $n\ge 4$ we can have $2<p<p_{\max}=p_0'(n-1)/(n-1-p_0')$ when $p_0'<n-1$, $p_{\max}=\infty$ otherwise. Observe that always $p_{\max}>2(n-1)/(n-3)$. From this the claim of Theorem 1.2 follows as we have established (6-2) for such values of p.

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ON THE SHARP UPPER BOUND RELATED TO THE WEAK MUCKENHOUPT–WHEEDEN CONJECTURE

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We construct an example showing that the upper bound $[w]_{A_1}\log(\mathrm{e}+[w]_{A_1})$ for the $L^1(w)\to L^{1,\infty}(w)$ norm of the Hilbert transform cannot be improved in general.

1. Introduction

Define the Hardy–Littlewood maximal operator on $\mathbb{R}$ by

$$Mf(x)=\sup_{I\ni x}\frac{1}{|I|}\int_I |f(y)|\,dy,$$

where the supremum is taken over all intervals $I\subset\mathbb{R}$ containing the point x.

C. Fefferman and E. Stein [1971] established the following weighted weak-type inequality for M: there exists an absolute constant $C>0$ such that, for every weight w,

$$\sup_{\alpha>0}\alpha w\{x\in\mathbb{R}:Mf(x)>\alpha\}\le C\int_{\mathbb{R}}|f(x)|Mw(x)\,dx \tag{1-1}$$

(here by a weight we mean any nonnegative locally integrable function on $\mathbb{R}$, and we use the standard notation $w(E)=\int_E w$ for a measurable set $E\subset\mathbb{R}$).

Inequality (1-1) is important for several reasons. First, it is the key ingredient in extending the Hardy–Littlewood maximal theorem to a vector-valued case. Second, this result was a precursor of the weighted theory, which had started to develop rapidly in the beginning of the 70's. In particular, if we define the $[w]_{A_1}$ constant of the weight w by $[w]_{A_1}=\|Mw/w\|_{L^\infty}$, then, assuming $[w]_{A_1}<\infty$, (1-1) implies immediately that

$$\|Mf\|_{L^{1,\infty}(w)}\le C[w]_{A_1}\|f\|_{L^1(w)}. \tag{1-2}$$

Consider now the Hilbert transform,

$$Hf(x)=\text{P.V.}\int_{\mathbb{R}}\frac{f(y)}{x-y}\,dy.$$

Inequality (1-1) with the maximal operator on the left-hand side replaced by the Hilbert transform has become known as the Muckenhoupt–Wheeden conjecture. Only recently this conjecture has been disproved by M. Reguera and C. Thiele [2012] (see also [Caldarelli et al. 2017; Criado and Soria 2016;

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Reguera 2011] for some complements and extensions). Their result, however, left open the question of whether a weaker form of the Muckenhoupt–Wheeden conjecture holds, with M replaced by H on the left-hand side of (1-2).

In [Lerner et al. 2009], it was proved that

$$\|Hf\|_{L^{1,\infty}(w)} \le C[w]_{A_1}\log(\mathrm{e}+[w]_{A_1})\|f\|_{L^1(w)}. \tag{1-3}$$

This improved a previous result in [Lerner et al. 2008], where the right-hand side contained an additional factor double logarithmic in $[w]_{A_1}$. Notice also that actually (1-3) in [Lerner et al. 2009] was proved for every Calderón–Zygmund operator on $\mathbb{R}^n$ with sufficiently smooth kernel.

On the other hand, in [Nazarov et al. 2015], it was shown for the martingale transform (and explained how to transfer the result to the Hilbert transform case) that the dependence of $[w]_{A_1}$ in the weighted weak-type-(1, 1) inequality cannot in general be made better than $[w]_{A_1}\log^{1/5}(\mathrm{e}+[w]_{A_1})$, thus disproving the weak Muckenhoupt–Wheeden conjecture. Later, in [Nazarov et al. 2018], the power of the logarithm was improved to $\frac{1}{3}$ (this was again done for the martingale transform).

Summarizing the results in [Lerner et al. 2009; Nazarov et al. 2015; 2018], if we denote by α_H the best possible exponent for which the inequality

$$\|Hf\|_{L^{1,\infty}(w)} \le C[w]_{A_1}\log^{\alpha_H}(\mathrm{e}+[w]_{A_1})\|f\|_{L^1(w)}$$

holds, then we have that $\frac{1}{3} \le \alpha_H \le 1$.

The main result of this paper shows, in particular, that $\alpha_H = 1$. For $t \ge 1$, define

$$\varphi_H(t) = \sup_{[w]_{A_1}\le t} \|H\|_{L^1(w)\to L^{1,\infty}(w)}.$$

Then (1-3) implies $\varphi_H(t) \le Ct\log(\mathrm{e}+t)$. We will show that actually $\varphi_H(t) \eqsim t\log(\mathrm{e}+t)$. Our main result is the following.

Theorem 1.1. *There exists $c' > 0$ such that, for all $t \ge 1$,*

$$\varphi_H(t) \ge c' t\log(\mathrm{e}+t).$$

As a trivial corollary we obtain that the inequality

$$\|Hf\|_{L^{1,\infty}(w)} \le \psi([w]_{A_1})\|f\|_{L^1(w)}$$

fails in general for every function ψ increasing on $[1, \infty)$ satisfying

$$\lim_{t\to\infty} \frac{\psi(t)}{t\log(\mathrm{e}+t)} = 0.$$

2. Proof of Theorem 1.1

An overview of the proof. At the first step we show that the definition of φ_H along with the standard extrapolation and dualization arguments yields

$$\|H(w\chi_{[0,1)})\|_{L^2(\sigma)} \le 4\varphi_H(2\|M\|_{L^2(\sigma)\to L^2(\sigma)})\left(\int_0^1 w\right)^{1/2}, \tag{2-1}$$

where $\sigma = w^{-1}$. Notice that $\|M\|_{L^2(\sigma)\to L^2(\sigma)} < \infty$ if and only if $w \in A_2$, that is, if

$$\sup_{I\subset\mathbb{R}} \frac{w(I)\sigma(I)}{|I|^2} < \infty.$$

Therefore, we assume here that $w \in A_2$.

The key step is to show that there exist $C_1, C_2, C_3 > 0$ such that for every $t > C_3$, there is an A_2-weight w satisfying

$$\int_0^1 w = 1, \quad \|M\|_{L^2(\sigma)\to L^2(\sigma)} \le C_1 t, \quad \|H(w\chi_{[0,1)})\|_{L^2(\sigma)} \ge C_2 t \log t. \tag{2-2}$$

Plugging these estimates into (2-1), we finish the proof.

Extrapolation and dualization. First, we apply the standard Rubio de Francia extrapolation trick. Given $g \ge 0$ with $\|g\|_{L^2(\sigma)} = 1$, define

$$\mathcal{R}g(x) = \sum_{k=0}^{\infty} \frac{M^k g(x)}{(2\|M\|_{L^2(\sigma)\to L^2(\sigma)})^k}.$$

Then $g \le \mathcal{R}g$, $\|\mathcal{R}g\|_{L^2(\sigma)} \le 2$, and $[\mathcal{R}g]_{A_1} \le 2\|M\|_{L^2(\sigma)\to L^2(\sigma)}$. These estimates along with the definition of φ_H and Hölder's inequality imply

$$\begin{aligned}
\alpha \int_{\{x:|Hf(x)|>\alpha\}} g &\le \alpha \int_{\{x:|Hf(x)|>\alpha\}} \mathcal{R}g \le \varphi_H(2\|M\|_{L^2(\sigma)\to L^2(\sigma)})\|f\|_{L^1(\mathcal{R}g)} \\
&\le \varphi_H(2\|M\|_{L^2(\sigma)\to L^2(\sigma)})\|f\|_{L^2(w)}\|\mathcal{R}g\|_{L^2(\sigma)} \\
&\le 2\varphi_H(2\|M\|_{L^2(\sigma)\to L^2(\sigma)})\|f\|_{L^2(w)}.
\end{aligned}$$

Taking here the supremum over all $g \ge 0$ with $\|g\|_{L^2(\sigma)} = 1$ yields

$$\|Hf\|_{L^{2,\infty}(w)} \le 2\varphi_H(2\|M\|_{L^2(\sigma)\to L^2(\sigma)})\|f\|_{L^2(w)}. \tag{2-3}$$

We now use the elementary estimate

$$\int_0^1 |Hf| w \le 2\|Hf\|_{L^{2,\infty}(w)} \left(\int_0^1 w\right)^{1/2}, \tag{2-4}$$

which along with (2-3) implies

$$\left|\int_{\mathbb{R}} (H(w\chi_{[0,1)}))f\right| = \left|\int_0^1 (Hf)w\right| \le 4\varphi_H(2\|M\|_{L^2(\sigma)\to L^2(\sigma)})\left(\int_0^1 w\right)^{1/2} \|f\|_{L^2(w)}.$$

Taking here the supremum over all f with $\|f\|_{L^2(w)} = 1$ proves (2-1).

To show (2-4), notice that, for every $\lambda > 0$,

$$\int_0^1 |Hf| w \le \int_\lambda^\infty w\{x : |Hf(x)| > \alpha\}\, d\alpha + \lambda \int_0^1 w \le \frac{1}{\lambda}\|Hf\|^2_{L^{2,\infty}(w)} + \lambda \int_0^1 w.$$

Optimizing this estimate with respect to λ, we obtain (2-4).

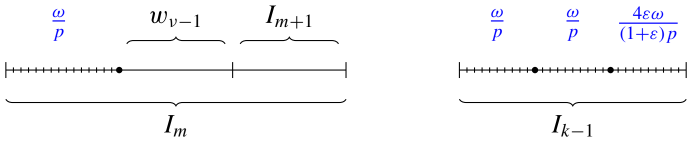

Figure 1. $w_\nu(\omega,\sigma,I)$ on intervals $I_m^{(i)}$ for $i=1,2$ and $0\le m\le k-2$ and on $I_{k-1}^{(i)}$ for $i=1,2,3$.

Construction of the weight. Fix $t\gg 1$. Take $k\in\mathbb{N}$ such that $t\le 3^k\le 3t$. Let $\varepsilon=3^{-k}$ and

$$p=\frac{1}{3\varepsilon}\left(\frac{1+\varepsilon}{2}+\frac{4\varepsilon^2}{1+\varepsilon}\right).$$

The reason for this definition of p will be clarified a bit later. Note that we will frequently use the obvious estimate $1/(6\varepsilon)\le p\le 2/\varepsilon$.

For every two positive numbers ω and σ such that $\omega\sigma=p$ and any interval $I\subset\mathbb{R}$, we define inductively the sequence of weights $w_\nu=w_\nu(\omega,\sigma,I)$ $(\nu=0,1,2,\dots)$ supported on I as follows.

Let $u=\sqrt{p}+\sqrt{p-1}$ be the larger root of $u+1/u=2\sqrt{p}$. Define

$$w_0(\omega,\sigma,I)=\frac{\omega}{\sqrt{p}}\left(u\chi_{I_-}+\frac{1}{u}\chi_{I_+}\right),$$

where I_- and I_+ are the left and the right halves of I respectively.

Suppose that $w_{\nu-1}(\omega,\sigma,I)$ is already defined for all ω,σ with $\omega\sigma=p$ and all $I\subset\mathbb{R}$. To construct $w_\nu(\omega,\sigma,I)$, first denote by I_m, $m=0,1,\dots,k-1$, the interval with the same right endpoint as I of length $3^{-m}|I|$, so

$$I_{k-1}\subset I_{k-2}\subset\cdots\subset I_0=I$$

and $|I_{k-1}|=3\varepsilon|I|$.

Given an interval J, denote by $J^{(i)}$, $i=1,2,3$, the i-th from the left subinterval of J in the partition of J into three equal intervals.

Define $w_\nu(\omega,\sigma,I)$ by

$$w_\nu(\omega,\sigma,I)=\frac{\omega}{p}\left(\sum_{m=0}^{k-2}\chi_{I_m^{(1)}}+\chi_{I_{k-1}^{(1)}\cup I_{k-1}^{(2)}}+\frac{4\varepsilon}{1+\varepsilon}\chi_{I_{k-1}^{(3)}}\right)+\sum_{m=0}^{k-2}w_{\nu-1}\left(2\omega,\frac{\sigma}{2},I_m^{(2)}\right). \tag{2-5}$$

See Figure 1.

Note that the interval $I_{k-1}^{(3)}$ plays a rather special role in the final step of this recursive construction. We shall call any interval of this type arising at any step in the construction of the weight $w_\nu(\omega,\sigma,I)$ a “tail interval”, so within I we shall have one tail interval $I_{k-1}^{(3)}$ arising at the final stage of the construction, $k-1$ tail intervals $(I_m^{(2)})_{k-1}^{(3)}$ arising in the construction of the weights $w_{\nu-1}(2\omega,\sigma/2,I_m^{(2)})$, and so on.

Finally, we define w as the 1-periodization of $w_n(1,p,[0,1))$ with $n=3^{k-1}$.

For $l=0,1,\dots,n$, we shall say that an interval I “carries w_{n-l}” if $w=w_{n-l}(2^l,2^{-l}p,I)$ on I. Denote by $\operatorname{supp}w_{n-l}$ the union of all intervals carrying w_{n-l}. For example, $\operatorname{supp}w_n=\bigcup_{k\in\mathbb{Z}}[k,k+1)=\mathbb{R}$ as $[k,k+1)$ carries w_n for every $k\in\mathbb{Z}$.

Let us now establish several useful properties of $w_\nu(\omega,\sigma,I)$.

Proposition 2.1. *For every* $\nu \geq 0$,

$$\frac{1}{|I|}\int_I w_\nu(\omega,\sigma,I)\,dx=\omega \quad \textit{and} \quad \frac{1}{|I|}\int_I w_\nu^{-1}(\omega,\sigma,I)\,dx=\sigma. \tag{2-6}$$

Proof. The proof is by induction on ν. For $\nu=0$,

$$\frac{1}{|I|}\int_I w_0(\omega,\sigma,I)\,dx=\frac{\omega}{\sqrt{p}}\frac{1}{2}\left(u+\frac{1}{u}\right)=\omega,$$
$$\frac{1}{|I|}\int_I w_0^{-1}(\omega,\sigma,I)\,dx=\frac{\sqrt{p}}{\omega}\frac{1}{2}\left(\frac{1}{u}+u\right)=\frac{p}{\omega}=\sigma.$$

Assume that the statement holds for $\nu-1$ and let us prove it for ν. Observe that $w_\nu(\omega,\sigma,I)$ equals ω/p on a subset of I of total measure

$$\frac{1-3\varepsilon}{2}|I|+2\varepsilon|I|=\frac{1+\varepsilon}{2}|I|,$$

it equals

$$\frac{4\varepsilon}{1+\varepsilon}\frac{\omega}{p}$$

on a set of measure $\varepsilon|I|$, and the average of $w_{\nu-1}(2\omega,\cdot,\cdot)$ over the remaining set of measure $\frac{1}{2}(1-3\varepsilon)|I|$ is 2ω by the inductive assumption. Thus

$$\begin{aligned}\frac{1}{|I|}\int_I w_\nu(\omega,\sigma,I)\,dx&=\frac{\omega}{p}\left(\frac{1+\varepsilon}{2}+\frac{4\varepsilon^2}{1+\varepsilon}\right)+\omega(1-3\varepsilon)\\&=\omega+\left(\frac{1}{p}\left(\frac{1+\varepsilon}{2}+\frac{4\varepsilon^2}{1+\varepsilon}\right)-3\varepsilon\right)\omega=\omega\end{aligned}$$

(it is this equation that was used to determine p).

On the other hand, $w_\nu^{-1}(\omega,\sigma,I)$ equals $p/\omega=\sigma$ on a subset of I of measure $\frac{1}{2}(1+\varepsilon)|I|$ (the same set on which w_ν is defined as ω/p), it equals

$$\frac{1+\varepsilon}{4\varepsilon}\sigma$$

on a set of measure $\varepsilon|I|$, and its average over the remaining set of measure $\frac{1}{2}(1-3\varepsilon)|I|$ equals $\frac{1}{2}\sigma$. Thus

$$\frac{1}{|I|}\int_I w_\nu^{-1}(\omega,\sigma,I)\,dx=\frac{1+\varepsilon}{2}\sigma+\frac{1+\varepsilon}{4}\sigma+\frac{(1-3\varepsilon)}{2}\frac{\sigma}{2}=\sigma,$$

which completes the proof. □

In particular, it follows from Proposition 2.1 that for the constructed weight w,

$$\int_0^1 w\,dx=\int_0^1 w_n(1,p,[0,1))\,dx=1.$$

Proposition 2.2. *Let* $I=[a,a+h)$. *Then, for every* $\nu\geq 0$ *and for all* $0<\tau<h$,

$$\frac{1}{\tau}\int_a^{a+\tau} w_\nu(\omega,\sigma,I)\leq 3\omega \quad \textit{and} \quad \frac{1}{\tau}\int_{a+h-\tau}^{a+h} w_\nu(\omega,\sigma,I)\leq \frac{9}{2}\omega. \tag{2-7}$$

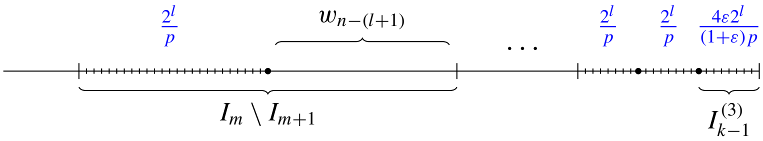

Figure 2. w on some interval I carrying w_{n-l} for $l < n$.

Proof. For $\nu = 0$ the statement is obvious since $w_0(\omega, \sigma, I) \le 2\omega$ on I. Assume that $\nu \ge 1$.

Since $w_\nu(\omega, \sigma, I) = \omega/p$ on $I^{(1)}$, we have

$$\frac{1}{\tau}\int_a^{a+\tau} w_\nu(\omega, \sigma, I) = \frac{\omega}{p}$$

for $0 < \tau < \frac{1}{3}h$. But if $\tau \ge \frac{1}{3}h$, then, by Proposition 2.1,

$$\frac{1}{\tau}\int_a^{a+\tau} w_\nu(\omega, \sigma, I)\,dx \le \frac{3}{|I|}\int_I w_\nu(\omega, \sigma, I)\,dx = 3\omega.$$

We now turn to the proof of the second estimate in (2-7). Let I_m, $m = 0, 1, \dots, k-1$, be the intervals appearing in the definition of $w_\nu(\omega, \sigma, I)$. Since $w_\nu(\omega, \sigma, I) \le \omega/p$ on I_{k-1}, the estimate is trivial if $a + h - \tau \in I_{k-1}$. Assume that $a + h - \tau \in I_m \setminus I_{m+1}$, $m = 0, \dots, k-2$. Then

$$\frac{1}{\tau}\int_{a+h-\tau}^{a+h} w_\nu(\omega, \sigma, I)\,dx \le \frac{1}{|I_{m+1}|}\int_{I_m} w_\nu(\omega, \sigma, I)\,dx. \tag{2-8}$$

Next, by Proposition 2.1,

$$\begin{aligned}\int_{I_m} w_\nu(\omega, \sigma, I)\,dx &= \sum_{j=m}^{k-2}\left(\frac{\omega}{p}|I_j^{(1)}| + \int_{I_j^{(2)}} w_{\nu-1}\left(2\omega, \frac{\sigma}{2}, I_j^{(2)}\right)dx\right) + \int_{I_{k-1}} w_\nu(\omega, \sigma, I)\,dx \\ &\le \omega\sum_{j=m}^{k-1}|I_j| \le \frac{3\omega}{2}\frac{|I|}{3^m} = \frac{9\omega}{2}|I_{m+1}|, \end{aligned} \tag{2-9}$$

which along with (2-8) completes the proof. □

Assume that I carries w_{n-l}; see Figure 2. Consider the corresponding tail intervals contained in I, that is, the intervals on which

$$w = \frac{4\varepsilon}{1+\varepsilon}\frac{2^j}{p}, \qquad j = l, \dots, n-1.$$

These intervals will play the central role in the estimate of the Hilbert transform of $w\chi_{[0,1)}$. There is only one tail interval in $I \setminus \operatorname{supp} w_{n-(l+1)}$, and its measure equals $(1/3^k)|I|$. Next, there are $k-1$ tail intervals in

$$I \cap (\operatorname{supp} w_{n-(l+1)} \setminus \operatorname{supp} w_{n-(l+2)})$$

of total measure

$$\frac{1}{2}\left(1 - \frac{1}{3^{k-1}}\right)\frac{1}{3^k}|I|.$$

Similarly, the measure of the union of tail intervals in

$$I \cap (\operatorname{supp} w_{n-(l+j)} \setminus \operatorname{supp} w_{n-(l+j+1)}), \quad j = 0, \dots, n-l-1,$$

is

$$\left(\frac{1}{2}\left(1 - \frac{1}{3^{k-1}}\right)\right)^j \frac{1}{3^k}|I|.$$

In particular, if we denote by A_l the union of tail intervals in

$$[0, 1) \cap (\operatorname{supp} w_{n-l} \setminus \operatorname{supp} w_{n-(l+1)}),$$

then

$$|A_l| = \left(\frac{1}{2}\left(1 - \frac{1}{3^{k-1}}\right)\right)^l \frac{1}{3^k}, \quad l = 0, \dots, n-1. \tag{2-10}$$

Estimate of the maximal operator. In this section, we will prove the first inequality in (2-2). We start with the reduction of this estimate to its triadic version.

Let $\mathcal{T}$ be the standard triadic lattice; that is,

$$\mathcal{T} = \{[3^j n, 3^j(n+1)) : j, n \in \mathbb{Z}\}.$$

Denote by $\mathcal{J}$ the family of all unions of two adjacent triadic intervals of equal length.

Our key tool will be the estimate

$$\|M\|_{L^2(\sigma)\to L^2(\sigma)} \le 24 \sup_{J\in\mathcal{J}} \left(\frac{1}{w(J)} \int_J (M(w\chi_J))^2 \sigma\right)^{1/2}. \tag{2-11}$$

This estimate is fairly standard and well known. For the reader's convenience, we supply the proof in the Appendix.

Combining (2-11) with the inequality $p \le 2/\varepsilon \le 6t$, we see that in order to prove the first estimate in (2-2), it suffices to show that there exists $C > 0$ such that, for every interval $J \in \mathcal{J}$,

$$\int_J (M(w\chi_J))^2 \sigma \le Cp^2 w(J). \tag{2-12}$$

Define an auxiliary 1-periodic function $\tilde{w}$ by

$$\tilde{w}(x) = \sum_{l=1}^{n} 2^l \chi_{\operatorname{supp} w_{n-(l-1)} \setminus \operatorname{supp} w_{n-l}}(x) + 2^{n+1} \chi_{\operatorname{supp} w_0}(x).$$

The role of this function is clarified in the following two propositions.

Proposition 2.3. *For all $x \in \mathbb{R}$,*

$$Mw(x) \le \tfrac{9}{2}\tilde{w}(x). \tag{2-13}$$

Proof. First, notice that for $x \in \operatorname{supp} w_0$ the statement is trivial. Indeed, $w \le 2^{n-1}/p$ on the complement of $\operatorname{supp} w_0$, and if I carries w_0, then on I

$$w_0 = \frac{2^n}{\sqrt{p}}\left((\sqrt{p} + \sqrt{p-1})\chi_{I_-} + \frac{1}{\sqrt{p} + \sqrt{p-1}}\chi_{I_+}\right).$$

Hence,

$$\|w\|_{L^\infty} \le \frac{2^n(\sqrt{p}+\sqrt{p-1})}{\sqrt{p}} \le 2^{n+1},$$

and therefore $\|Mw\|_{L^\infty} \le 2^{n+1}$.

On the other hand, for $x \in \operatorname{supp} w_{n-(l-1)} \setminus \operatorname{supp} w_{n-l}$, the estimate (2-13) follows immediately from the facts that $w \le 2^{l-1}/p$ on the complement of $\operatorname{supp} w_{n-l}$ and, by Proposition 2.2, the average of w over the intersection of any interval J carrying w_{n-l} with an interval not contained in J is at most $\frac{9}{2}\cdot 2^l$. $\square$

Recall that, given an interval I, we denoted by I_m, $m=0,\dots,k-1$, the interval with the same right endpoint as I of length $|I_m| = (1/3^m)|I|$. These intervals have already appeared in the definition of $w_\nu(\omega,\sigma,I)$.

Proposition 2.4. *Assume that I carries w_{n-l}. Then*

$$\int_{I_m} (\tilde{w})^2\sigma \le 30p^2 w(I_m), \quad l=0,\dots,n,\ m=0,\dots,k-2.$$

Proof. First, notice that the case when $l=n$ is trivial, since $2^{n+1} \le 4pw(x)$ on any interval I carrying w_0, and hence,

$$\int_J (\tilde{w})^2\sigma \le 16p^2 w(J) \quad \text{for every } J \subset \operatorname{supp} w_0. \tag{2-14}$$

Suppose now that $l \le n-1$ and consider first the case $m=0$. Assume that I carries w_{n-l}. For $j=0,\dots,n-l-1$, define

$$F_j = I \cap (\operatorname{supp} w_{n-(l+j)} \setminus \operatorname{supp} w_{n-(l+j+1)})$$

and let E_j be the union of the tail intervals contained in F_j. Observe that $w = 2^{l+j}/p$ on $F_j \setminus E_j$, and hence, $\tilde{w}(x) = 2pw(x)$ for $x \in F_j \setminus E_j$, which implies

$$\int_{\bigcup_j (F_j\setminus E_j)} (\tilde{w})^2\sigma \le 4p^2 w(I). \tag{2-15}$$

On the other hand,

$$w = \frac{4\varepsilon}{1+\varepsilon}\frac{2^{l+j}}{p}$$

on E_j and, as we have seen in the previous section,

$$|E_j| = \left(\frac{1}{2}\left(1-\frac{1}{3^{k-1}}\right)\right)^j \frac{1}{3^k}|I| \le \frac{1}{2^j}\frac{1}{3^k}|I|.$$

Combining this with Proposition 2.1 yields

$$\begin{aligned}\int_{\bigcup_j E_j} (\tilde{w})^2\sigma &\le 4\sum_{j=0}^{n-l-1} 2^{2(l+j)}\frac{(1+\varepsilon)p}{4\varepsilon 2^{l+j}}\frac{1}{2^j}\frac{1}{3^k}|I| \\ &\le \frac{2p}{\varepsilon}\frac{n}{3^k}2^l|I| \le 4p^2 2^l|I| = 4p^2 w(I).\end{aligned} \tag{2-16}$$

Further, by (2-14),

$$\int_{I\cap \operatorname{supp} w_0} (\tilde{w})^2\sigma \le 16p^2 w(I\cap \operatorname{supp} w_0) \le 16p^2 w(I).$$

Combining this estimate with (2-15) and (2-16), we obtain

$$\int_I (\tilde{w})^2\sigma = \int_{\bigcup_j (F_j\setminus E_j)} (\tilde{w})^2\sigma + \int_{\bigcup_j E_j} (\tilde{w})^2\sigma + \int_{I\cap \operatorname{supp} w_0} (\tilde{w})^2\sigma \le 24p^2 w(I), \tag{2-17}$$

and this completes the proof in the case $m=0$.

Assume now that $1\le m\le k-2$. Notice that $I_m\setminus I_{m+1} = I_m^{(1)}\cup I_m^{(2)}$, where $I_m^{(2)}$ carries $w_{n-(l+1)}$, and $\tilde{w}(x)=2pw(x)$ on $I_m^{(1)}$. Thus, by (2-17),

$$\begin{aligned}\int_{I_m\setminus I_{m+1}} (\tilde{w})^2\sigma &\le 4p^2 w(I_m^{(1)}) + 24p^2 w(I_m^{(2)})\\ &\le 24p^2 w(I_m\setminus I_{m+1}).\end{aligned} \tag{2-18}$$

Further,

$$\int_{I_{k-1}} (\tilde{w})^2\sigma \le 4(2^l)^2 \frac{(1+\varepsilon)p}{4\varepsilon 2^l}|I_{k-1}| \le 6p2^l|I|.$$

On the other hand, by Proposition 2.1,

$$w(I_m) \ge w(I_m^{(2)}) = 2^{l+1}|I_m^{(2)}| = \frac{2^{l+1}}{3^{m+1}}|I|,$$

which, combined with the previous estimate, implies

$$\int_{I_{k-1}} (\tilde{w})^2\sigma \le p3^{m+2}w(I_m) \le \frac{p}{\varepsilon}w(I_m) \le 6p^2 w(I_m).$$

Therefore, using (2-18), we obtain

$$\int_{I_m} (\tilde{w})^2\sigma = \sum_{j=m}^{k-2}\int_{I_j\setminus I_{j+1}} (\tilde{w})^2\sigma + \int_{I_{k-1}} (\tilde{w})^2\sigma \le 30p^2 w(I_m),$$

which completes the proof. □

We now turn to the proof of (2-12). Let $J\in\mathcal{J}$. First consider the simple case when $|J|\ge 1$. In this case, $|J|=k$ for some $k\in\mathbb{N}$. Using that w and $\tilde{w}$ are 1-periodic along with the fact that $\int_0^1 w=1$, and combining Propositions 2.3 and 2.4, we obtain

$$\frac{1}{w(J)}\int_J (M(w\chi_J))^2\sigma \le 25\int_0^1 (\tilde{w})^2\sigma \le 25\cdot 30p^2.$$

Suppose that $|J|<1$. We can represent J as the union of two triadic intervals $J=J_-\cup J_+$, where $J_-, J_+\in\mathcal{T}$ are the left and the right halves of J respectively. Since J_- is triadic, we must have $|J_-|\le\frac{1}{3}$. Also, by the 1-periodicity of w, one can assume that $J_-\subset[0,1)$.

Consider the case when J contains an interval carrying $\operatorname{supp} w_{n-(l+1)}$ for some l. Out of all such intervals choose the longest one. Note that since $|J|\le\frac{2}{3}$, we must have $l\ge 0$ in this case. Thus, the interval in question must be of the kind $R_m^{(2)}$, where R is an interval carrying w_{n-l}. Since neither $R = R_0$ nor $R_{m-1}^{(2)}$ (if $m \geq 1$) is contained in J, there are only three possible options:

- $J_- = R_m^{(2)},\ 0 \leq m \leq k-2$;
- $J_+ = R_m^{(2)},\ 0 \leq m \leq k-2$;
- $J_- = R_m,\ 1 \leq m \leq k-2$.

Suppose first that $J_- = R_m^{(2)}$ or $J_+ = R_m^{(2)}$. Then $J \subset R_m$. By (2-9), $w(R_m) \leq 3 \cdot 2^{l-1}|R_m|$. On the other hand, since $R_m^{(2)}$ carries $w_{n-(l+1)}$, by Proposition 2.1,

$$w(J) \geq w(R_m^{(2)}) = 2^{l+1}|R_m^{(2)}| = 2^{l+1}\frac{|R_m|}{3}, \tag{2-19}$$

which implies $w(R_m) \leq \frac{9}{4}w(J)$. Therefore, by Propositions 2.3 and 2.4,

$$\int_J (M(w\chi_J))^2\sigma \leq 25\int_{R_m}(\tilde{w})^2\sigma \leq 25\cdot 30p^2 w(R_m) \leq 25\cdot 75p^2 w(J).$$

Assume now that $J_- = R_m,\ 1 \leq m \leq k-2$. Then $w \equiv 2^{l-1}/p$ on J_+ if $l > 0$ and $w \equiv 2^l/p$ on J_+ if $l = 0$. In either case, $\tilde{w} = 2pw$ on J_+, so

$$\int_{J_+}(\tilde{w})^2\sigma = 4p^2 w(J_+),$$

and thus, by Propositions 2.3 and 2.4,

$$\begin{aligned}\int_J (M(w\chi_J))^2\sigma &\leq 25\biggl(\int_{R_m}(\tilde{w})^2\sigma + \int_{J_+}(\tilde{w})^2\sigma\biggr)\\ &\leq 25(30p^2 w(R_m) + 4p^2 w(J_+)) \leq 25\cdot 30p^2 w(J).\end{aligned}$$

It remains to consider the case when J does not contain an interval carrying $w_{n-(l+1)}$ for any $0 \leq l \leq n-1$. Denote by E the union of all tail intervals appearing in the definition of w. Notice that if $x \notin E$, then

$$\sum_{l=1}^{n} 2^l \chi_{\operatorname{supp} w_{n-(l-1)} \setminus \operatorname{supp} w_{n-l}}(x) = 2pw(x)\chi_{\mathbb{R}\setminus \operatorname{supp} w_0}.$$

Also, $2^{n+1} \leq 4pw(x)\chi_{\operatorname{supp} w_0}$. From this and from Proposition 2.3,

$$Mw(x) \leq 18pw(x) \quad (x \notin E).$$

Therefore, if $J \cap E = \varnothing$,

$$\frac{1}{w(J)}\int_J (M(w\chi_J))^2\sigma \leq 18^2 p^2. \tag{2-20}$$

Suppose that $J \cap E \neq \varnothing$. Then there exists R carrying w_{n-l} for some $0 \leq l \leq n-1$ such that $J \cap R_{k-1}^{(3)} \neq \varnothing$. Since J_-, J_+ and $R_{k-1}^{(3)}$ are triadic, we have that either one half of J is contained in $R_{k-1}^{(3)}$

or $R^{(3)}_{k-1} \subset J$. Since J cannot contain any interval carrying $\operatorname{supp} w_{n-(l+1)}$, in both cases we obtain that w can take only three possible values

$$\frac{2^l}{p}, \quad \frac{4\varepsilon 2^l}{(1+\varepsilon)p}, \quad \frac{2^{l-1}}{p}$$

on J and therefore,

$$\frac{1}{w(J)}\int_J (M(w\chi_J))^2\sigma \le \left(\frac{\sup_J w}{\inf_J w}\right)^2 \le \left(\frac{1+\varepsilon}{4\varepsilon}\right)^2 \le 9p^2,$$

which along with (2-20) implies

$$\frac{1}{w(J)}\int_J (M(w\chi_J))^2\sigma \le 18^2 p^2.$$

This completes the proof of (2-12), and therefore the first estimate in (2-2) is proved.

Estimate of the Hilbert transform. The goal of this section is to prove the second estimate in (2-2).

Denote by A^*_l, $l = 0, \dots, n-1$, the union of all intervals $\frac{1}{2}I$, where I is a tail interval contained in

$$[0, 1) \cap (\operatorname{supp} w_{n-l} \setminus \operatorname{supp} w_{n-(l+1)}).$$

In other words, A^*_l is the union of all intervals $\frac{1}{2}J^{(3)}_{k-1}$, where $J \subset [0, 1)$ carries w_{n-l}. Then, by (2-10),

$$|A^*_l| = \frac{1}{2}|A_l| = \frac{1}{2}\left(\frac{1}{2}\left(1 - \frac{1}{3^{k-1}}\right)\right)^l \frac{1}{3^k}, \quad l = 0, \dots, n-1. \tag{2-21}$$

The sets A^*_l plays the central role in establishing the lower bound for $H(w\chi_{[0,1)})$, as the following proposition shows.

Proposition 2.5. *There exists an absolute $C > 0$ such that for all $l = 0, \dots, n-1$ and for every $x \in A^*_l$*

$$|H(w\chi_{[0,1)})(x)| \ge Ck2^l. \tag{2-22}$$

Let us first show how to derive the second estimate in (2-2) from here. By (2-21) and (2-22),

$$\int_{A^*_l} |H(w\chi_{[0,1)})|^2\sigma \ge C^2k^2 2^{2l}\frac{1+\varepsilon}{4\varepsilon}\frac{p}{2^l}\frac{1}{2}\left(\frac{1}{2}\left(1 - \frac{1}{3^{k-1}}\right)\right)^l \frac{1}{3^k}.$$

Therefore,

$$\begin{aligned}\|H(w\chi_{[0,1)})\|^2_{L^2(\sigma)} &\ge \sum_{l=0}^{n-1}\int_{A^*_l}|H(w\chi_{[0,1)})|^2\sigma \\ &\ge \frac{C^2}{8}k^2 p\sum_{l=0}^{n-1}\left(1 - \frac{1}{3^{k-1}}\right)^l = \frac{C^2}{24}k^2 3^k p\left(1 - \left(1 - \frac{1}{3^{k-1}}\right)^n\right).\end{aligned}$$

Since $n = 3^{k-1}$ and $(1 - 1/n)^n < 1/\mathrm{e}$, we obtain

$$\|H(w\chi_{[0,1)})\|^2_{L^2(\sigma)} \ge \frac{C^2(1 - 1/\mathrm{e})}{24}k^2 3^k p \ge \frac{C^2(1 - 1/\mathrm{e})}{144(\log 3)^2}t^2\log^2 t.$$

Let us now turn to the proof of Proposition 2.5. Let $J=[a,b)\subset[0,1)$ be an interval carrying w_{n-l}. Assume that $x\in\frac{1}{2}J^{(3)}_{k-1}$. Write

$$\begin{aligned}H(w\chi_{[0,1)})(x)&\\&=H(w\chi_{[0,a)})(x)+\sum_{m=0}^{k-2}H(w\chi_{J_m\setminus J_{m+1}})(x)+H(w\chi_{J_{k-1}\setminus J^{(3)}_{k-1}})(x)+H(w\chi_{J^{(3)}_{k-1}})(x)+H(w\chi_{[b,1)})(x)\\&\equiv A(x)+B(x)+C(x)+D(x)+E(x).\end{aligned}$$

We will show that there are absolute constants C_1 and C_2 such that, for all $x\in\frac{1}{2}J^{(3)}_{k-1}$,

$$|B(x)|\ge C_1k2^l\quad\text{and}\quad\max\{|D(x)|,|E(x)|\}\le C_22^l.\tag{2-23}$$

Since $A(x)$, $B(x)$ and $C(x)$ are positive for all $x\in\frac{1}{2}J^{(3)}_{k-1}$, we obtain from (2-23) that

$$\begin{aligned}|H(w\chi_{[0,1)})(x)|&\ge|A(x)+B(x)+C(x)|-|D(x)|-|E(x)|\\&\ge|B(x)|-|D(x)|-|E(x)|\ge\frac{C_1}{2}k2^l\end{aligned}$$

for $k>4C_2/C_1$.

Now let us prove the first estimate in (2-23). If $y\in J_m\setminus J_{m+1}$ and $x\in\frac{1}{2}J^{(3)}_{k-1}$, then $0\le x-y\le|J_m|$. Using also that $J^{(2)}_m\subset J_m\setminus J_{m+1}$, by Proposition 2.1 we obtain

$$H(w\chi_{J_m\setminus J_{m+1}})(x)=\int_{J_m\setminus J_{m+1}}\frac{w(y)}{x-y}\,dy\ge\frac{w(J_m\setminus J_{m+1})}{|J_m|}\ge\frac{w(J^{(2)}_m)}{|J_m|}=\frac{2}{3}2^l.$$

Therefore,

$$B(x)>\tfrac{2}{3}(k-1)2^l.$$

Turn to the second part of (2-23). Let $J^{(3)}_{k-1}=[\alpha,b)$. Then, for all $x\in\frac{1}{2}J^{(3)}_{k-1}$,

$$\left|\int_{J^{(3)}_{k-1}}\frac{w(y)}{x-y}\,dy\right|=\frac{4\varepsilon}{(1+\varepsilon)}\frac{2^l}{p}\left|\log\left|\frac{x-\alpha}{x-b}\right|\right|\le4(\log3)\varepsilon\frac{2^l}{p}\le4(\log3)2^l.$$

It remains to estimate $|E(x)|$. Take the intervals $J^i=[a_i,b_i)$, $i=0,\dots,l$, such that J^i carries w_{n-l+i} and

$$J=J^0\subset J^1\subset\cdots\subset J^l=[0,1).$$

We claim that, for every $i=1,\dots,l$ and for all x such that $0<x\le b_{i-1}-|J^{i-1}|/(4\cdot3^k)$,

$$|H(w\chi_{[b_{i-1},b_i)})(x)|\le13\cdot2^{l-i}.\tag{2-24}$$

Notice first that this claim immediately implies the desired estimate for $E(x)$. Indeed, let $x\in\frac{1}{2}J^{(3)}_{k-1}$. Then $0<x\le b-|J|/(4\cdot3^k)$, and hence (2-24) holds for $i=1$. But since $x\notin(J^i)_{k-1}$ for all $i=1,\dots,l$, we obviously obtain that $0<x\le b_{i-1}-|J^{i-1}|/(4\cdot3^k)$ for all $i\le l$. Therefore, by (2-24),

$$|H(w\chi_{[b,1)})(x)|\le\sum_{i=1}^{l}|H(w\chi_{[b_{i-1},b_i)})(x)|\le13\sum_{i=1}^{l}2^{l-i}\le13\cdot2^l.$$

It remains to prove the claim. Set $x_i = b_{i-1} - |J^{i-1}|/(4\cdot 3^k)$. Observe that $|H(w\chi_{[b_{i-1},b_i)})(x)|$ is an increasing function for $x < b_{i-1}$. Therefore, it suffices to prove that

$$|H(w\chi_{[b_{i-1},b_i)})(x_i)| \le 13\cdot 2^{l-i}. \tag{2-25}$$

There exists $0 \le m \le k-2$ such that $J^{i-1} = (J^i_m)^{(2)}$. Then $[b_{i-1}, b_i) = J^i_{m+1}$. Let $h = |J^i_{m+1}|$. Split the integral in (2-25) as follows:

$$\int_{b_{i-1}}^{b_i} \frac{w(y)}{y-x_i}\,dy = \int_{b_{i-1}}^{b_{i-1}+h/3} \frac{w(y)}{y-x_i}\,dy + \int_{b_{i-1}+h/3}^{b_i} \frac{w(y)}{y-x_i}\,dy.$$

Using that $w \equiv 2^{l-i}/p$ on $[b_{i-1}, b_{i-1}+h/3)$, we obtain

$$\int_{b_{i-1}}^{b_{i-1}+h/3} \frac{w(y)}{y-x_i}\,dy \le \frac{2^{l-i}}{p}\frac{h}{3}\frac{4\cdot 3^k}{h} \le 8\cdot 2^{l-i}.$$

Next, applying (2-9) yields

$$\int_{b_{i-1}+h/3}^{b_i} \frac{w(y)}{y-x_i}\,dy \le \frac{3}{h}\frac{3}{2}2^{l-i}|J^i_{m+1}| = \frac{9}{2}\cdot 2^{l-i},$$

which along with the previous estimate proves (2-25).

This proves the claim and so Proposition 2.5. Thus, Theorem 1.1 is completely proved.

Appendix

In this section, we will show how to prove (2-11). Let us show first that, for every interval $I \subset \mathbb{R}$, there exists an interval $J \in \mathcal{J}$ containing I and such that $|J| \le 6|I|$. Indeed, let $I = [a, a+h)$. Fix $j \in \mathbb{Z}$ such that $3^{j-1} \le h < 3^j$ and take $n \in \mathbb{Z}$ such that

$$3^j n \le a < 3^j(n+1).$$

Then $I \subset J = [3^j n, 3^j(n+2))$, and

$$\frac{|J|}{|I|} \le \frac{2\cdot 3^j}{3^{j-1}} = 6.$$

It follows immediately from this property that

$$Mf(x) \le 6M^{\mathcal{J}} f(x), \tag{A-1}$$

where

$$M^{\mathcal{J}} f(x) = \sup_{J\ni x,\, J\in\mathcal{J}} \frac{1}{|J|}\int_J |f|\,dy.$$

Next, it is easy to see that the intervals from $\mathcal{J}$ can be split into two disjoint triadic lattices, $\mathcal{J} = \mathcal{T}^1 \cup \mathcal{T}^2$ (see Figure 3 for a geometric illustration of this fact).

Therefore, by (A-1),

$$\|M\|_{L^2(\sigma)\to L^2(\sigma)} \le 6(\|M^{\mathcal{T}^1}\|_{L^2(\sigma)\to L^2(\sigma)} + \|M^{\mathcal{T}^2}\|_{L^2(\sigma)\to L^2(\sigma)}). \tag{A-2}$$

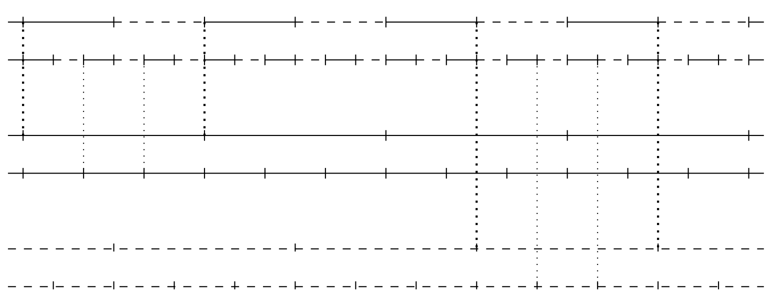

Figure 3. The lattices $\mathcal{T}$, $\mathcal{T}^1$ and $\mathcal{T}^2$ shown at two consecutive generations. The unions of solid and adjacent (from the right) dashed intervals from $\mathcal{T}_j$ form $\mathcal{T}_j^1$, and the unions of dashed and adjacent (from the right) solid intervals from $\mathcal{T}_j$ form $\mathcal{T}_j^2$. In turn, the unions of solid and dashed children from $\mathcal{T}_{j+1}$ form $\mathcal{T}_{j+1}^1$, and the unions of dashed and solid children from $\mathcal{T}_{j+1}$ form $\mathcal{T}_{j+1}^2$.

In order to estimate the right-hand side of (A-2), we invoke the following proposition.

Proposition A.1. *Let $\mathfrak{T}$ be a triadic lattice. Then*

$$\|M^{\mathfrak{T}}\|_{L^2(\sigma)\to L^2(\sigma)} \le 2 \sup_{R\in\mathfrak{T}} \left(\frac{1}{w(R)}\int_R (M^{\mathfrak{T}}(w\chi_R))^2\sigma\right)^{1/2}.$$

Remark A.2. For dyadic lattices this result can be found in [Moen 2009]. The proof there is closely related to the approach by E. Sawyer [1982] in his two-weighted characterization for the maximal operator. For triadic lattices the proof is essentially the same, and we give it for the sake of completeness.

Proof of Proposition A.1. Let $a > 1$. For $k \in \mathbb{Z}$ write the set $\Omega_k = \{M^{\mathfrak{T}} f > a^k\}$ as the union of pairwise disjoint triadic intervals I_j^k such that

$$\frac{1}{|I_j^k|}\int_{I_j^k} |f| > a^k.$$

Define $E_j^k = I_j^k \setminus \Omega_{k+1}$, and set $\alpha_{j,k} = (w(I_j^k)/|I_j^k|)^2\sigma(E_j^k)$. We have

$$\begin{aligned}\|M^{\mathfrak{T}} f\|_{L^2(\sigma)}^2 &= \sum_{k\in\mathbb{Z}} \int_{\Omega_k\setminus\Omega_{k+1}} (M^{\mathfrak{T}} f)^2\sigma \le a^2 \sum_{k,j}\left(\frac{1}{|I_j^k|}\int_{I_j^k}|f|\right)^2 \sigma(E_j^k)\\ &= a^2\sum_{k,j}\left(\frac{1}{w(I_j^k)}\int_{I_j^k}|f\sigma|w\right)^2 \alpha_{j,k}. \end{aligned} \tag{A-3}$$

Notice that for every $R \in \mathfrak{T}$,

$$\sum_{j,k:I_j^k\subset R} \alpha_{j,k} \le \int_R (M^{\mathfrak{T}}(w\chi_R))^2\sigma \le N^2 w(R), \tag{A-4}$$

where

$$N = \sup_{R\in\mathfrak{T}}\left(\frac{1}{w(R)}\int_J (M^{\mathfrak{T}}(w\chi_R))^2\sigma\right)^{1/2}.$$

For $\lambda > 0$ set

$$E_\lambda = \left\{(j,k) : \left(\frac{1}{w(I_j^k)}\int_{I_j^k}|f\sigma|w\right)^2 > \lambda\right\}.$$

Define the weighted maximal operator $M_w^{\mathfrak{T}}$ by

$$M_w^{\mathfrak{T}} f(x) = \sup_{J\ni x,\, J\in\mathfrak{T}}\frac{1}{w(J)}\int_J |f|w\,dy.$$

Writing the set $\{x : M_w^{\mathfrak{T}}(f\sigma)^2(x) > \lambda\}$ as the union of the maximal pairwise disjoint triadic intervals $\bigcup_i R_i$ and applying (A-4), we obtain

$$\sum_{(j,k)\in E_\lambda}\alpha_{j,k} \le \sum_i \sum_{j,k:I_j^k\subset R_i}\alpha_{j,k} \le N^2 w\{x : M_w^{\mathfrak{T}}(f\sigma)^2(x) > \lambda\}.$$

Therefore,

$$\begin{aligned}\sum_{k,j}\left(\frac{1}{w(I_j^k)}\int_{I_j^k}|f\sigma|w\right)^2\alpha_{j,k} &= \int_0^\infty\left(\sum_{(j,k)\in E_\lambda}\alpha_{j,k}\right)d\lambda \\ &\le N^2\|M_w^{\mathfrak{T}}(f\sigma)\|_{L^2(w)}^2 \le 4N^2\|f\sigma\|_{L^2(w)}^2 = 4N^2\|f\|_{L^2(\sigma)}^2,\end{aligned}$$

which, along with (A-3), completes the proof. □

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