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WEAK SOLUTIONS TO THE QUATERNIONIC MONGE–AMPÈRE EQUATION

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We solve the Dirichlet problem for the quaternionic Monge–Ampère equation with a continuous boundary data and the right-hand side in L^p for $p > 2$. This is the optimal bound on p. We prove also that the local integrability exponent of quaternionic plurisubharmonic functions is 2, which turns out to be less than an integrability exponent of the fundamental solution.

1. Introduction

Pluripotential theory, initiated in the seminal papers [Bedford and Taylor 1976; 1982], has become a powerful tool for solving problems in complex analysis and geometry. It has been generalized in many directions in the last decade. The most general setting is that of calibrated geometries, which were extensively studied in a long series of papers by Harvey and Lawson [2009b]. Even before that the basics of pluripotential theory in $\mathbb{H}^n$ were recreated in [Alesker 2003b], and more generally on hypercomplex manifolds in [Alesker and Verbitsky 2006]. In this paper we wish to concentrate on the flat space $\mathbb{H}^n$.

The short historical overview is as follows. Quaternionic plurisubharmonic functions in $\mathbb{H}^n$ and their basic properties were investigated in [Alesker 2003b]. Inspired by [Bedford and Taylor 1976], Alesker developed there the foundations of pluripotential theory in the quaternionic setting showing among other things that a quaternionic Monge–Ampère operator defined for smooth functions as the Moore determinant [1922] of a quaternionic Hessian can be extended to the class of continuous functions. In [Alesker 2003a] he solved the Dirichlet problem in a quaternionic strictly pseudoconvex domain $\Omega \subset \mathbb{H}^n$ with a continuous boundary data and the Monge–Ampère mass continuous up to the boundary. Only recently Wan [2020] obtained another result in this direction. Following the approach of [Kołodziej 1995; 2005] she proved that the Dirichlet problem admits a bounded solution provided the right-hand side is a finite Borel measure and a subsolution to the problem exists. Motivated by reasoning presented in [Cegrell and Persson 1992] and using comparison of real and quaternionic Monge–Ampère operators she showed existence of continuous solutions to the Dirichlet problem for densities in L^q, $q \geq 4$. To sum up, the strongest known result concerning existence of a continuous solution to the Dirichlet problem with a degenerate right-hand side is as follows.

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Theorem. *Suppose $\Omega \subset \mathbb{H}^n$ is a quaternionic strictly pseudoconvex domain and $f \in L^q(\Omega)$ for $q \geq 4$ is a nonnegative function. Then the Dirichlet problem*

$$\begin{cases} u \in \mathcal{QPSH}(\Omega) \cap C(\bar{\Omega}), \\ (\partial \partial_J u)^n = f \Omega_n, \\ u_{|\partial\Omega} = \phi \in C(\partial\Omega) \end{cases}$$

has a unique solution.

The regularity of solutions (except for a ball, which was discussed earlier in [Alesker 2003a]) was proven in [Zhu 2016]. More precisely using the ideas presented in [Caffarelli et al. 1985], he proved the following result.

Theorem. *For a quaternionic strictly pseudoconvex domain $\Omega \subset \mathbb{H}^n$, $f \in C^\infty(\bar{\Omega} \times \mathbb{R})$ a positive function such that f_x is nonnegative on $\bar{\Omega} \times \mathbb{R}$ and $\phi \in C^\infty(\partial\Omega)$, the Dirichlet problem*

$$\begin{cases} u \in \mathcal{QPSH}(\Omega) \cap C^\infty(\bar{\Omega}), \\ \det\left(\dfrac{\partial^2 u(q)}{\partial \bar{q}_\alpha \partial q_\beta}\right)_{\alpha,\beta \in \{1,\dots,n\}} = f(q, u(q)) \quad \text{in } \Omega, \\ u_{|\partial\Omega} = \phi \end{cases}$$

has a unique smooth solution.

In the meantime quaternionic pluripotential theory was further developed in [Wan and Zhang 2015; Wan and Kang 2017; Wan and Wang 2017], of which we will make an extensive use. Contents of those papers will be discussed below in more detail. For results concerning Dirichlet problems in this more general approach, one can consult [Harvey and Lawson 2009a] for the flat case, [Harvey and Lawson 2011] for manifolds and [Harvey and Lawson 2019] for a degenerate case.

In this note we are interested in finding weak solutions to the Dirichlet problem for the quaternionic Monge–Ampère operator in $\mathbb{H}^n$ with a more degenerate right-hand side and a continuous boundary data. It turns out to be possible whenever densities are in L^p for $p > 2$ and the exponent is optimal as we show. To do that we follow the approach of [Kołodziej 1996; 1998]. Probably the most interesting results are these which actually allow us to apply his method of proof. Among them is comparison of a quaternionic capacity and volume (Lebesgue measure). We prove it in the quaternionic setting, coupling two things. The first is the trick of Dinew and Kołodziej [2014] which allowed them to show similar comparison for the capacity related to a complex Hesssian equation in $\mathbb{C}^n$. It reduces to noting that although plurisubharmonic functions are rare among m-subharmonic ones still they realize this m-Hessian capacity. The second is a fact that is interesting in its own right, namely the comparison of complex and quaternionic Monge–Ampère operators. To our knowledge it was not known or used before and relies on the observation that the Moore determinant of a hyperhermitian matrix is in fact the Pfaffian of an associated complex matrix. Afterwards we obtain an L^∞ estimate for the solutions. The last step before proving the main theorem is stability of solutions in terms of their densities and boundary data, but here the proofs are more standard. All of this is done in Section 4. In Section 3 we discuss the problem of finding the local integrability exponent for quaternionic plurisubharmonic functions. The proof of the main theorem there is inspired by the one presented in [Hörmander 1994] for plurisubharmonic functions in $\mathbb{C}^n$. It turns out that the class of quaternionic plurisubharmonic functions exhibits an unusual property in this context; namely, the integrability exponent of a general function is 2, which is smaller than $2n$ occurring for a fundamental solution. This phenomenon can be excluded assuming boundedness of the function near the boundary of a domain, which is proven in Section 4.

2. Preliminaries

General references for quaternionic linear algebra and basic properties of quaternionic plurisubharmonic functions are [Alesker 2003a; 2003b; Alesker and Verbitsky 2006], while for quaternionic pluripotential theory see [Wan and Zhang 2015; Wan and Kang 2017; Wan and Wang 2017]. Let us fix the notation for an algebra of quaternions

$$\mathbb{H} = \{x_0 + x_1\mathfrak{i} + x_2\mathfrak{j} + x_3\mathfrak{k} \mid x_0, x_1, x_2, x_3 \in \mathbb{R}\},$$

where $\mathfrak{i}, \mathfrak{j}, \mathfrak{k}$ satisfy quaternionic relations and we consider $\mathbb{H}^n$ as a *right* quaternionic module. With such a choice we denote by I, J, K the complex structures induced by $\mathfrak{i}, \mathfrak{j}, \mathfrak{k}$ when treating $\mathbb{H}^n$ as a flat hypercomplex manifold. We introduce two coordinate systems,

$$\mathbb{H}^n \ni (q_i)_{i=0}^{n-1} \mapsto (z_j)_{j=0}^{2n-1} \in \mathbb{C}^{2n}$$

in such a way that $q_i = z_{2i} + \mathfrak{j} z_{2i+1}$ (this is a holomorphic chart for the complex structure I), and

$$\mathbb{H}^n \ni (q_i)_{i=0}^{n-1} \mapsto (x_j)_{j=0}^{4n-1} \in \mathbb{R}^{4n}$$

in such a way that $q_i = x_{4i} + x_{4i+1}\mathfrak{i} + x_{4i+2}\mathfrak{j} + x_{4i+3}\mathfrak{k}$ (this is just a real chart). It is easy to see that

$$z_j = x_{2j} + (-1)^j x_{2j+1}\mathfrak{i} \quad \text{for } j = 0, \dots, 2n-1.$$

As always ∂ and $\bar{\partial}$ are the canonical differential operators induced by the complex structure I and $d = \partial + \bar{\partial}$, $d^c = \mathfrak{i}(\bar{\partial} - \partial)$. We also introduce the twisted differential

$$\partial_J := J^{-1} \circ \bar{\partial} \circ J,$$

considered in [Alesker and Verbitsky 2010; Verbitsky 2002], which plays the role of $\bar{\partial}$ in the hypercomplex setting (e.g., the quaternionic Dolbeault or Salamon complex). For its properties we refer to the mentioned papers. Most notably we will only use

$$\begin{gathered}\partial_J : \Lambda_I^{k,0}(\mathbb{H}^n) \to \Lambda_I^{k+1,0}(\mathbb{H}^n), \quad \text{since } J : \Lambda_I^{p,q}(\mathbb{H}^n) \to \Lambda_I^{q,p}(\mathbb{H}^n), \\ \partial\partial_J + \partial_J\partial = 0, \\ \partial_J^2 = 0.\end{gathered}$$

Later on it may happen frequently that we skip the subscript I and understand that $\Lambda^{k,0}(\mathbb{H}^n)$ come from considering degrees with respect to I. One can check that for a smooth function $u : \mathbb{H}^n \to \mathbb{R}$ the following formulas hold:

$$\partial u = \sum_{i=0}^{2n-1} (\partial_{z_i} u)\, dz_i, \qquad \bar\partial u = \sum_{i=0}^{2n-1} (\partial_{\bar z_i} u)\, d\bar z_i,$$

$$\partial_J u = \sum_{i=0}^{2n-1} (-1)^{i+1} (\partial_{\bar z_{i+(-1)^i}} u)\, dz_i, \quad \partial\bar\partial u = \sum_{i,j} (\partial_{z_i}\partial_{\bar z_j} u)\, dz_i \wedge d\bar z_j,$$

$$\begin{aligned}\partial\partial_J u &= \sum_{i,j} ((-1)^{j+1} \partial_{z_i}\partial_{\bar z_{j+(-1)^j}} u)\, dz_i \wedge dz_j \\ &= \sum_{i<j} ((-1)^{j+1} \partial_{z_i}\partial_{\bar z_{j+(-1)^j}} u - (-1)^{i+1} \partial_{z_j}\partial_{\bar z_{i+(-1)^i}})\, dz_i \wedge dz_j.\end{aligned}$$

Suppose that $f : \mathbb{H}^n \to \mathbb{H}$ is a $\mathcal{C}^2$ function; we define the formal quaternionic derivatives

$$\frac{\partial f}{\partial \bar q_\alpha} = \frac{\partial f}{\partial x_{4\alpha}} + \mathfrak{i}\frac{\partial f}{\partial x_{4\alpha+1}} + \mathfrak{j}\frac{\partial f}{\partial x_{4\alpha+2}} + \mathfrak{k}\frac{\partial f}{\partial x_{4\alpha+3}},$$

$$\frac{\partial f}{\partial q_\alpha} = \overline{\frac{\partial \bar f}{\partial \bar q_\alpha}} = \frac{\partial f}{\partial x_{4\alpha}} - \frac{\partial f}{\partial x_{4\alpha+1}}\mathfrak{i} - \frac{\partial f}{\partial x_{4\alpha+2}}\mathfrak{j} - \frac{\partial f}{\partial x_{4\alpha+3}}\mathfrak{k}.$$

Let us observe that for any $f : \mathbb{H}^n \to \mathbb{H}$ of class $\mathcal{C}^2$

$$\frac{\partial}{\partial \bar q_\alpha}\frac{\partial}{\partial q_\beta} = \frac{\partial}{\partial q_\beta}\frac{\partial}{\partial \bar q_\alpha}.$$

Furthermore for a *real*-valued f one has

$$\frac{\partial}{\partial \bar q_\alpha}\frac{\partial}{\partial q_\alpha} f = \frac{\partial^2 f}{\partial x_{4\alpha}^2} + \frac{\partial^2 f}{\partial x_{4\alpha+1}^2} + \frac{\partial^2 f}{\partial x_{4\alpha+2}^2} + \frac{\partial^2 f}{\partial x_{4\alpha+3}^2} \quad \text{and} \quad \frac{\partial}{\partial \bar q_\alpha}\left(\frac{\partial}{\partial q_\beta} f\right) = \overline{\frac{\partial}{\partial \bar q_\beta}\frac{\partial}{\partial q_\alpha} f}.$$

As a consequence the matrix

$$\mathrm{Hess}(f, \mathbb{H}) = \left(\frac{\partial^2 f}{\partial \bar q_\alpha \partial q_\beta}\right)_{\alpha,\beta\in\{1,\dots,n\}}$$

is a hyperhermitian matrix for any real-valued f. The following relations are known to hold for a smooth real-valued function u:

$$(dd^c u)^{2n} = 2^{2n} (\mathfrak{i}\,\partial\bar\partial u)^{2n} = 4^{2n} (2n)!\, \det\left(\frac{\partial^2 u}{\partial_{z_i}\partial_{\bar z_j}}\right)\left(\frac{\mathfrak{i}}{2}\, dz_0 \wedge d\bar z_0\right) \wedge \cdots \wedge \left(\frac{\mathfrak{i}}{2}\, dz_{2n-1} \wedge d\bar z_{2n-1}\right),$$

$$(\partial\partial_J u)^n = \frac{n!}{4^n} \det\left(\frac{\partial^2 u}{\partial_{\bar q_l}\partial_{q_k}}\right)(dz_0 \wedge dz_1 \wedge \cdots \wedge dz_{2n-2} \wedge dz_{2n-1}),$$

where in the last expression det is the Moore determinant, see [Moore 1922] for the original definition, of a hyperhermitian matrix. The last formula was computed in [Alesker and Verbitsky 2006] and, in a different setting, in [Wan and Wang 2017]. For further simplifications we introduce some canonical differential forms:

$$\omega_{2n} = \sum_{i=0}^{2n-1} \frac{\mathrm{i}}{2}\, dz_i \wedge d\bar{z}_i,$$

$$\beta_n = \sum_{i=0}^{n-1} dz_{2i} \wedge dz_{2i+1},$$

$$\Omega_n = \frac{\beta_n^n}{n!} = dz_0 \wedge dz_1 \wedge \cdots \wedge dz_{2n-2} \wedge dz_{2n-1}.$$

Since we will extensively use facts from pluripotential theory reproved in the quaternionic setting in [Wan and Zhang 2015; Wan and Wang 2017], it is desirable to compare differential operators ∂, ∂_J, which we use with their formally defined operators d_0, d_1. Those were introduced by D. Wan and W. Wang [2017]; we refer to their work for more details. They considered the "coordinates"

$$z^{j0} = x_{2j} + (-1)^{j+1} x_{2j+1}\mathrm{i} = \bar{z}_j,$$
$$z^{j1} = (-1)^{j+1} x_{2(j+(-1)^j)} + x_{2(j+(-1)^j)+1}\mathrm{i} = (-1)^{j+1} z_{j+(-1)^j}$$

for $j = 0, \ldots, 2n-1$ and the associated formal derivatives

$$\nabla_{j0} = \partial_{x_{2j}} + (-1)^j \partial x_{2j+1}\mathrm{i} = 2\partial_{\bar{z}_j},$$
$$\nabla_{j1} = (-1)^{j+1} \partial x_{2(j+(-1)^j)} - \partial x_{2(j+(-1)^j)+1}\mathrm{i} = (-1)^{j+1} 2\partial_{z_{j+(-1)^j}}.$$

Afterwards they fixed a complex basis $\omega^0, \ldots, \omega^{2n-1}$ of $\mathbb{C}^{2n} \approx \mathbb{C}^{2n*}$ and an associated one $\omega^I = \omega_{i_1} \wedge \cdots \wedge \omega_{i_k}$, for $I = (i_1, \ldots, i_k)$ such that $i_1 < \cdots < i_k$ belong to $\{0, \ldots, 2n-1\}$, of a complex exterior product $\Lambda^k(\mathbb{C}^{2n}) \approx \Lambda^k(\mathbb{C}^{2n*})$. Finally they defined operators

$$d_i : \Lambda^{k,0}(\mathbb{H}^n) \approx C^\infty(\mathbb{H}^n, \Lambda^k \mathbb{C}^{2n}) \to C^\infty(\mathbb{H}^n, \Lambda^{k+1}\mathbb{C}^{2n}) \approx \Lambda^{k+1,0}(\mathbb{H}^n)$$

for $i = 0, 1$ in the following way. Suppose that $F = \sum_I f_I \omega^I$; then

$$d_i F = \sum_{I,k \in \{0,\ldots,2n-1\}} (\nabla_{ki} f_I)\omega^k \wedge \omega^I.$$

From formulas for ∇_{ki} we obtain

$$d_0 F = \sum_{I,k \in \{0,\ldots,2n-1\}} (\nabla_{k0} f_I)\omega^k \wedge \omega^I = \sum_{I,k \in \{0,\ldots,2n-1\}} 2(\partial_{\bar{z}_k} f_I)\omega^k \wedge \omega^I,$$

$$d_1 F = \sum_{I,k \in \{0,\ldots,2n-1\}} (\nabla_{k1} f_I)\omega^k \wedge \omega^I = \sum_{I,k \in \{0,\ldots,2n-1\}} 2(-1)^{k+1}(\partial_{z_{k+(-1)^k}} f_I)\omega^k \wedge \omega^I.$$

Proposition 1. *For the basis* $\omega^k = (-1)^k\, dz_{k+(-1)^k}$

$$d_0 = 2\partial_J, \quad d_1 = -2\partial \quad \text{and} \quad \Delta = d_0 d_1 = 4\partial\partial_J.$$

Proof. Let us recall that $\partial_J = J^{-1} \circ \bar{\partial} \circ J$ and one can check that J acts as

$$J(dz_{2i+1}) = d\bar{z}_{2i}, \quad J(dz_{2i}) = -d\bar{z}_{2i+1}, \quad \text{i.e.,} \quad J(dz_k) = (-1)^{k+1}\, d\bar{z}_{k+(-1)^k}.$$

As before, for $F=\sum_I f_I\omega^I$, we obtain

$$
\begin{aligned}
\partial F &= \sum_{I,k\in\{0,\dots,2n-1\}} (\partial_{z_k} f_I)\, dz_k \wedge \omega^I,\\
\partial_J F &= J^{-1}\circ\bar\partial\Big(\sum_I f_I J(\omega^I)\Big) = J^{-1}\Big(\sum_{I,k\in\{0,\dots,2n-1\}} (\partial_{\bar z_k} f_I)\, d\bar z_k \wedge J(\omega^I)\Big)\\
&= J^{-1}\Big(\sum_{I,k\in\{0,\dots,2n-1\}} (\partial_{\bar z_{k+(-1)^k}} f_I)\, d\bar z_{k+(-1)^k} \wedge J(\omega^I)\Big)\\
&= \sum_{I,k\in\{0,\dots,2n-1\}} (\partial_{\bar z_{k+(-1)^k}} f_I) J^{-1}(d\bar z_{k+(-1)^k}) \wedge \omega^I\\
&= \sum_{I,k\in\{0,\dots,2n-1\}} (\partial_{\bar z_{k+(-1)^k}} f_I)(-1)^{k+1}\, dz_k \wedge \omega^I.
\end{aligned}
$$

This results in

$$
\begin{aligned}
d_0 F &= 2\sum_{I,k\in\{0,\dots,2n-1\}} (\partial_{\bar z_k} f_I)\omega^k \wedge \omega^I\\
&= 2\sum_{I,k\in\{0,\dots,2n-1\}} (-1)^k(\partial_{\bar z_k} f_I)\, dz_{k+(-1)^k} \wedge \omega^I\\
&= 2\sum_{I,k\in\{0,\dots,2n-1\}} (-1)^{k+1}(\partial_{\bar z_{k+(-1)^k}} f_I)\, dz_k \wedge \omega^I = \mathbf{2}\partial_J F,
\end{aligned}
$$

$$
\begin{aligned}
d_1 F &= 2\sum_{I,k\in\{0,\dots,2n-1\}} (-1)^{k+1}(\partial_{z_{k+(-1)^k}} f_I)\omega^k \wedge \omega^I\\
&= 2\sum_{I,k\in\{0,\dots,2n-1\}} (-1)(\partial_{z_{k+(-1)^k}} f_I)\, dz_{k+(-1)^k} \wedge \omega^I = -\mathbf{2}\partial F. \qquad \square
\end{aligned}
$$

Remark 2. Let us just emphasize that the choosing of ∂, ∂_J over d_0, d_1 has a deeper meaning than just the conventional one. These are the natural intrinsic operators not only in $\mathbb{H}^n$ but on any hypercomplex manifold. In fact on an abstract hypercomplex manifold, quaternionic plurisubharmonic functions are defined only with their aid, see [Alesker and Verbitsky 2006], since the local chart definition is not possible due to nonintegrability of a generic hypercomplex structure, i.e., nonexistence of quaternionic charts.

From Proposition 1 it follows that we are able to use all results from [Wan and Zhang 2015; Wan and Kang 2017; Wan and Wang 2017] as well as those from [Alesker 2003a; 2003b; Alesker and Verbitsky 2006]. We just give here the necessary details and refer to the mentioned papers for more details. The quaternionic plurisubharmonic functions were defined in [Alesker 2003b].

Definition. Let Ω be a domain in $\mathbb{H}^n$. We call an upper semicontinuous function $f:\Omega\to\mathbb{R}$ (strictly) quaternionic plurisubharmonic, qpsh for short, if f restricted to any affine right quaternionic line intersected with Ω is (strictly) subharmonic as a function on a domain in $\mathbb{R}^4$. The set of all qpsh functions on Ω is denoted by $\mathcal{QPSH}(\Omega)$.

Remark 3. If we fix $t \in \{a\mathfrak{i} + b\mathfrak{j} + c\mathfrak{k} \mid a^2 + b^2 + c^2 = 1\}$ an imaginary unit and consider $\mathbb{H}^n$ as a complex vector space where multiplication by $\mathfrak{i}$ is given by a right multiplication by t then psh functions with respect to this complex structure are qpsh since quaternionic lines are complex 2-planes. We will use that remark only for $t = \mathfrak{i}$, i.e., only for $\mathbb{H}^n$ treated as $\mathbb{C}^{2n}$ via the chart introduced in the beginning of the section.

For a smooth function being qpsh is equivalent to $\partial\partial_J u \geq 0$ in a quaternionic sense. Let us elaborate. The cones of strongly positive $SP^{2k}(\Omega) \subset \Lambda^{2k,0}_{\mathbb{R}}(\Omega)$ and positive $\Lambda^{2k,0}_{\mathbb{R},\geq 0}(\Omega) \subset \Lambda^{2k,0}_{\mathbb{R}}(\Omega)$ forms were introduced in [Alesker and Verbitsky 2006]; see also [Verbitsky 2010] for a careful and extended treatment. Here $\Lambda^{2k,0}_{\mathbb{R}}(\Omega) \subset \Lambda^{2k,0}(\Omega)$ is the space of forms α such that $\overline{J(\alpha)} = \alpha$. To introduce them we firstly argue for a point, an element $\Omega_n \in \Lambda^{2n,0}_{\mathbb{R}}(\mathbb{H}^n \approx T_0\mathbb{H}^n)$ is chosen to be strongly positive and a convex combination of elements of the form $G^*(\Omega_k)$ for $G : \mathbb{H}^n \to \mathbb{H}^k$ a quaternionic linear map is strongly positive. When the reasoning is applied pointwise we obtain the notion of strong positivity for differential forms in Ω. As always the cone of positive elements is the dual one. We have mentioned above that $(\partial\partial_J u)^n$ agrees with Moore's determinant of a quaternionic Hessian $\mathrm{Hess}(u, \mathbb{H})$ for a smooth function, Alesker [2003a], motivated by [Bedford and Taylor 1976], showed that $(\partial\partial_J u)^n$ can be interpreted as a measure for continuous u and proved certain convergence for this operator. It is a cornerstone for having proper pluripotential theory. Later in [Wan and Wang 2017] the authors proved that $\partial\partial_J u$ is a positive current (where positivity is defined using the cone of strongly positive forms) for any qpsh function. More importantly they showed that like in the complex case, see [Bedford and Taylor 1982], one can define $(\partial\partial_J u)^n$ for any locally bounded u and treat it as a measure. From there one can recreate most of theorems which hold for psh functions. Among other things they showed weak convergence of this operator on decreasing sequences of qpsh functions and Chern–Levine–Nirenberg inequalities; see [Wan and Wang 2017]. In [Wan and Zhang 2015] the quaternionic relative capacity is introduced in the spirit of Bedford and Taylor. Let $K \subset \Omega$ be a compact set; then

$$\mathrm{cap}(K, \Omega) = \sup\left\{ \int_K (\partial\partial_J u)^n \;\middle|\; u \in \mathcal{QPSH}(\Omega),\ 0 \leq u \leq 1 \right\},$$

and this can be extended to Borel subsets as well. What is more, the authors proved quasicontinuity of qpsh functions and most notably the comparison principle, which is probably the most powerful tool in pluripotential theory. The statement is exactly as we know it in the complex case, but we recall it for the reader's convenience.

Theorem [Wan and Zhang 2015]. *Let $u, v \in \mathcal{QPSH}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$. If, for any $\xi \in \partial\Omega$,*

$$\liminf_{\xi \leftarrow q \in \partial\Omega} (u(q) - v(q)) \geq 0$$

then

$$\int_{\{u<v\}} (\partial\partial_J v)^n \leq \int_{\{u<v\}} (\partial\partial_J u)^n.$$

In particular if $(\partial\partial_J v)^n \geq (\partial\partial_J u)^n$ as measures then $u \geq v$ in Ω.

Finally they characterize maximality of a bounded qpsh function in terms of the vanishing of its Monge–Ampère mass. Here we mean that $u \in \mathcal{QPSH}(\Omega)$ is maximal if it is above any other qpsh function on compact sets $K \subset \Omega$ provided the values of both functions are the same on ∂K.

3. Local integrability of qpsh functions

In this section we address the question of local integrability of qpsh functions in a domain $\Omega \subset \mathbb{H}^n$. For psh functions it is well known that they are locally integrable with any exponent. The proof of the proposition below is inspired by the presentation in [Hörmander 1994].

Proposition 4. *Suppose $u \in \mathcal{QPSH}(\Omega)$ is such that $u \not\equiv -\infty$. Then $u \in L^p_{\mathrm{loc}}(\Omega)$ for any $p < 2$ and the bound on p is optimal. Additionally, if $u_j \not\equiv -\infty$ is a sequence of qpsh functions converging in $L^1_{\mathrm{loc}}(\Omega)$ to some u, necessarily belonging to $\mathcal{QPSH}(\Omega)$, then convergence holds in $L^p_{\mathrm{loc}}(\Omega)$ for any $p < 2$.*

Proof. Suppose without loss of generality that $u \le 0$ in a neighborhood of a quaternionic polyball $P(0,1)$ of radius 1 centered at 0 contained in Ω, that $u(0) > -\infty$ and fix $p < 2$. Let us deal firstly with the case $n = 1$. From the Riesz representation theorem, see Theorem 3.3.6 in [Hörmander 1994],

$$u(q) = h(g) + \int_{\|\xi\|<1} G(q,\xi)\, d\mu(\xi)$$

for some nonpositive harmonic function h in $B(0,1) := B_1$, nonnegative Borel measure μ and Green's function

$$G(q,\xi) = -\frac{1}{\|q-\xi\|^2} + \frac{1}{\|(q-\xi/|\xi|^2)|\xi|\|^2}.$$

By Harnack's inequality, see Theorem 3.1.7 in [Hörmander 1994], for any $\|q\| < \frac{1}{2}$ we have

$$0 \le -h(q) \le \frac{1+\|q\|}{(1-\|q\|)^3}(-h(0)) \le 12(-h(0)).$$

This shows that

$$\|h\|_{L^p(B(0,\frac{1}{2}))} \le C_p |h(0)|$$

for a constant C_p, depending only on $p < 2$, which we may still need to increase (see below).

For estimating the second component of the decomposition of u, let us introduce the notation

$$H(q,\xi) = -G(q,\xi) \ge 0;$$

for $\xi = 0$ we have

$$H(q,0) = \frac{1}{\|q\|^2} - 1.$$

We consider two cases depending on whether ξ is close to the center or to the boundary of B_1.

In the first case, say when $\|\xi\| \le \frac{3}{4}$, we use the estimate

$$0 \le H(q,\xi) \le \frac{1}{\|q-\xi\|^2}$$

for any q and ξ; consequently

$$\begin{aligned}\left(\int_{\|q\|<\frac{1}{2}} (H(q,\xi))^p \, d\mathcal{L}^4(q)\right)^{\frac{1}{p}} &\le \left(\int_{\|q\|<\frac{1}{2}} \frac{1}{\|q-\xi\|^{2p}} \, d\mathcal{L}^4(q)\right)^{\frac{1}{p}} \\ &\le \left(\int_{\|q\|<\frac{5}{4}} \frac{1}{\|q\|^{2p}} \, d\mathcal{L}^4(q)\right)^{\frac{1}{p}} \le C'_p\left(\frac{1}{\|\xi\|^2} - 1\right)\end{aligned}$$

for a constant C'_p independent of ξ and depending only on $p<2$, since the expression $1/\|\xi\|^2-1$ is bounded from below for $\|\xi\|\le\frac{3}{4}$.

In the second case, say when $\|\xi\|\ge\frac{3}{4}$, we note that for any fixed ξ the function $H(\,\cdot\,,\xi)$ is nonnegative and harmonic in $B_{\frac{3}{4}}$. Applying Harnack's inequality for each fixed ξ we obtain that for all $\|\xi\|\ge\frac{3}{4}$ and for all $\|q\|<\frac{3}{4}$

$$0\le H(q,\xi)\le\Big(\frac{3}{4}\Big)^2\frac{\frac{3}{4}+\|q\|}{\big(\frac{3}{4}-\|q\|\big)^3}H(0,\xi);$$

hence for all $\|\xi\|\ge\frac{3}{4}$ and $\|q\|<\frac{1}{2}$

$$0\le H(q,\xi)\le 45\bigg(\frac{1}{\|\xi\|^2}-1\bigg).$$

To sum up we have proven that there exists a constant $C_p=\max\{C'_p,45\}$, independent of ξ, such that for $\|\xi\|<1$

$$\|H(\,\cdot\,,\xi)\|_{L^p(B(0,\frac{1}{2}))}\le C_p\bigg(\frac{1}{\|\xi\|^2}-1\bigg).$$

From Minkowski's inequality and Minkowski's integral inequality we obtain

$$\begin{aligned}\|u\|_{L^p(B(0,\frac{1}{2}))}&\le\|h\|_{L^p(B(0,\frac{1}{2}))}+\bigg(\int_{\|q\|<\frac{1}{2}}\bigg|\int_{\|\xi\|<1}H(q,\xi)\,d\mu(\xi)\bigg|^p\,d\mathcal{L}^4(q)\bigg)^{\frac{1}{p}}\\&\le C_p|h(0)|+\int_{\|\xi\|<1}\bigg(\int_{\|q\|<\frac{1}{2}}H(q,\xi)^p\,d\mathcal{L}^4(q)\bigg)^{\frac{1}{p}}\,d\mu(\xi)\\&\le C_p\bigg(|h(0)|+\int_{\|\xi\|<1}\bigg(\frac{1}{\|\xi\|^2}-1\bigg)\,d\mu(\xi)\bigg)=C_p|u(0)|.\end{aligned}$$

Using Fubini's theorem and the estimate above, one obtains that in the case of $n\ge1$ we have

$$\|u\|_{L^p(P(0,\frac{1}{2}))}\le C_p^n|u(0)|.$$

To the end observe that Ω', the set of points in Ω in a neighborhood in which u is integrable with exponent p, is an open set by definition. It is closed by what we have just shown. This is so because if $q\in\overline{\Omega}'$ and $r>0$ is such that $P(q,3r)\Subset\Omega$ then we can find an element q' of Ω' within a distance of $\frac{r}{2}$ from q and a point $q''\in\Omega$ within a distance of $\frac{r}{2}$ from q' such that $u(q'')$ is finite. We note that $P(q'',2r)\Subset\Omega$; consequently u is integrable with the exponent p on $P(q'',r)$, and $q\in P(q'',r)$. What is more, Ω' is nonempty by the assumption $u\not\equiv-\infty$. The bound on p is optimal as the example of $-1/\|q_0\|^2$ in $\mathbb{H}^n$ for $n\ge1$ shows.

For the proof of the second assertion we note that the sequence u_j-u is bounded in $L^p_{\mathrm{loc}}(\Omega)$ for any $1\le p<2$. To prove this it is enough to show that the sequence u_j is bounded in $L^p_{\mathrm{loc}}(\Omega)$. Fix any point $q\in\Omega$ and let $r>0$ be such that $P(q,3r)\Subset\Omega$. We claim that L^p norms of u_j in $P\big(q,\frac{r}{2}\big)$ are bounded. Suppose to the contrary that they are not. Let us choose a subsequence j_k such that $\|u_{j_k}\|_{L^p(P(q,\frac{r}{2}))}\to\infty$. We know that the u_{j_k} are locally uniformly bounded from above, see Theorem 3.2.13 in [Hörmander

1994], so we may assume that u and the u_{j_k} are nonpositive in $P(q,3r)$. It is possible to find a point q' within a distance of $\frac{r}{2}$ from q such that $u(q') > -\infty$ and $\limsup_{k\to\infty} u_{j_k}(q') = u(q')$ since both of these properties hold almost everywhere in Ω; see Theorem 3.2.13 in [Hörmander 1994]. We assume without loss of generality that $\lim_{k\to\infty} u_{j_k}(q') = u(q')$, for if not we take a subsequence again. In particular there exists $C > 0$ such that $u_{j_k}(q') > -C$ for any k. This together with the estimate we have proven shows that the sequence $\|u_{j_k}\|_{L^p(P(q',r))}$ is bounded. Because we also know that $P\big(q,\frac{r}{2}\big) \subset P(q',r)$, contradiction with $\|u_{j_k}\|_{L^p(P(q,\frac{r}{2}))} \to \infty$ is obtained. Fix $1 \le p < 2$ and observe that for any compact $K \subset \Omega$ we have

$$\begin{aligned}\int_K |u_j - u|^p \, d\mathcal{L}^{4n} &= \int_K |u_j - u|^{\frac{2-p}{2}} |u_j - u|^{\frac{3p-2}{2}} \, d\mathcal{L}^{4n} \\ &\le \left(\int_K |u_j - u|^{\frac{2-p}{2}\frac{2}{2-p}} \, d\mathcal{L}^{4n}\right)^{\frac{2-p}{2}} \left(\int_K |u_j - u|^{\frac{3p-2}{2}\frac{2}{p}} \, d\mathcal{L}^{4n}\right)^{\frac{p}{2}} \\ &= \left(\int_K |u_j - u| \, d\mathcal{L}^{4n}\right)^{\frac{2-p}{2}} \left(\int_K |u_j - u|^{3-\frac{2}{p}} \, d\mathcal{L}^{4n}\right)^{\frac{p}{2}}\end{aligned}$$

by Hölder's inequality. By the assumption the first term tends to zero, while second one is bounded since $1 \le 3 - \frac{2}{p} < 2$. This proves that the u_j tend to u in $L^p_{\mathrm{loc}}(\Omega)$ for any $1 \le p < 2$. □

Proposition 5 [Wan and Wang 2017]. *The function $f(q) = -1/\|q\|^2$ is a fundamental solution for the quaternionic Monge–Ampère operator in $\mathbb{H}^n$. More exactly*

$$(\partial\partial_J f)^n = \frac{2^n \pi^{2n} n!}{(2n)!} \delta_0.$$

We see that the fundamental solution to the quaternionic Monge–Ampère equation is in $L^p_{\mathrm{loc}}(\mathbb{H}^n)$ for any $p < 2n$, while a generic qpsh function is only for $p < 2$, which is in contrast with the case of psh functions.

4. Dirichlet problem for quaternionic Monge–Ampère equation

In this section we aim to solve the Dirichlet problem

$$\begin{cases} u \in \mathcal{QPSH}(\Omega) \cap C(\bar{\Omega}), \\ (\partial\partial_J u)^n = f\Omega_n, \\ u_{|\partial\Omega} = \phi \in C(\partial\Omega), \end{cases}$$

where $f \in L^q(\Omega)$ for $q > 2$ and $\Omega \Subset \mathbb{H}^n$ is a smoothly bounded, strictly quaternionic pseudoconvex domain, which is a global assumption for Ω in this section. Let us recall:

Definition. $\Omega \Subset \mathbb{H}^n$, a smoothly bounded domain, is strictly quaternionic pseudoconvex if there exists v, a smooth strictly qpsh function in a neighborhood of $\bar{\Omega}$, such that $v < 0$ in Ω, $v = 0$, but $\nabla v \neq 0$ on $\partial\Omega$.

Let us just mention that the Dirichlet problem for the complex Monge–Ampère equation with densities in L^p for $p > 1$ was solved in [Kołodziej 1996]. In fact he proved it for densities in appropriate Orlicz spaces being subspaces of L^1 and in particular cases reducing to L^p. For the real Monge–Ampère equation one can always solve the above problem for any density in L^1; see [Rauch and Taylor 1977].

The first goal is to compare complex and quaternionic Monge–Ampère operators. We start with smooth functions, in which case we have to compare complex and quaternionic Hessians, or rather their determinants to be precise.

Lemma 6. *For a smooth function $u:\Omega\to\mathbb{R}$ and any $l,k\in\{0,\dots,n-1\}$*

$$\begin{aligned}\partial_{\bar q_l}\partial_{q_k}u&=\partial_{\bar q_l}(\partial_{x_{4k}}u-\mathrm{i}\partial_{x_{4k+1}}u-\mathrm{j}\partial_{x_{4k+2}}u-\mathfrak{k}\partial_{x_{4k+3}}u)\\&=\partial_{\bar q_k}(2\partial_{z_{2k}}u-2\mathrm{j}\partial_{\bar z_{2k+1}}u)=(2\partial_{\bar z_{2l}}+2\mathrm{j}\partial_{z_{2l+1}})(2\partial_{z_{2k}}u-2\mathrm{j}\partial_{\bar z_{2k+1}}u)\\&=4(\partial_{\bar z_{2l}}\partial_{z_{2k}}u+\partial_{z_{2l+1}}\partial_{\bar z_{2k+1}}u)+4\mathrm{j}(\partial_{\bar z_{2l+1}}\partial_{z_{2k}}u-\partial_{z_{2l}}\partial_{\bar z_{2k+1}}u).\end{aligned}$$

Let us recall that we distinguish the set $\mathcal{PSH}(\Omega)$ of plurisubharmonic functions in Ω by identifying $\mathbb{H}^n$ with $\mathbb{C}^{2n}$ via a chart introduced in Section 2.

Lemma 7. *For a function $u\in\mathcal{PSH}(\Omega)\cap C^2(\Omega)\subset\mathcal{QPSH}(\Omega)$ the following holds*:

$$\left(\det\left(\frac{\partial^2u}{\partial_{\bar q_l}\partial_{q_k}}\right)\right)^2\ge 4^{2n}\det\left(\frac{\partial^2u}{\partial_{z_i}\partial_{\bar z_j}}\right).$$

Proof. Let us define

$$\mathrm{Hess}(u,\mathbb{C})=\left(\frac{\partial^2u}{\partial_{z_i}\partial_{\bar z_j}}\right)_{i,j=0,\dots,2n-1},$$
$$\mathrm{Hess}(u,\mathbb{H})=\left(\frac{\partial^2u}{\partial_{\bar q_l}\partial_{q_k}}\right)_{l,k=0,\dots,n-1}.$$

Note that

$$\det\left(\frac{\partial^2u}{\partial_{z_i}\partial_{\bar z_j}}\right)_{i,j=0,\dots,2n-1}=\det\mathrm{Hess}(u,\mathbb{C})=\det\overline{\mathrm{Hess}(u,\mathbb{C})}=\det\left(\frac{\partial^2u}{\partial_{\bar z_i}\partial_{z_j}}\right)_{i,j=0,\dots,2n-1}.$$

The last matrix is Hermitian positive since it is just $\mathrm{Hess}(u,\mathbb{C})^T$. If $\mathrm{Hess}(u,\mathbb{H})=G+\mathrm{j}H$ then we define

$$\psi(\mathrm{Hess}(u,\mathbb{H}))=\begin{pmatrix}G&-\overline{H}\\H&\overline{G}\end{pmatrix}.$$

By Lemma 6 we obtain

$$\begin{aligned}\psi(\mathrm{Hess}(u,\mathbb{H}))&=4\begin{pmatrix}[\partial_{\bar z_{2l}}\partial_{z_{2k}}u+\partial_{z_{2l+1}}\partial_{\bar z_{2k+1}}u]_{l,k}&[-\partial_{z_{2l+1}}\partial_{\bar z_{2k}}u+\partial_{\bar z_{2l}}\partial_{z_{2k+1}}u]_{l,k}\\ [\partial_{\bar z_{2l+1}}\partial_{z_{2k}}u-\partial_{z_{2l}}\partial_{\bar z_{2k+1}}u]_{l,k}&[\partial_{z_{2l}}\partial_{\bar z_{2k}}u+\partial_{\bar z_{2l+1}}\partial_{z_{2k+1}}u]_{l,k}\end{pmatrix}\\&=4\begin{pmatrix}[\partial_{\bar z_{2l}}\partial_{z_{2k}}u]_{l,k}&[\partial_{\bar z_{2l}}\partial_{z_{2k+1}}u]_{l,k}\\ [\partial_{\bar z_{2l+1}}\partial_{z_{2k}}u]_{l,k}&[\partial_{\bar z_{2l+1}}\partial_{z_{2k+1}}u]_{l,k}\end{pmatrix}+4\begin{pmatrix}[\partial_{z_{2l+1}}\partial_{\bar z_{2k+1}}u]_{l,k}&[-\partial_{z_{2l+1}}\partial_{\bar z_{2k}}u]_{l,k}\\ [-\partial_{z_{2l}}\partial_{\bar z_{2k+1}}u]_{l,k}&[\partial_{z_{2l}}\partial_{\bar z_{2k}}u]_{l,k}\end{pmatrix}.\end{aligned}$$

Following [Cegrell and Persson 1992] we introduce three matrices

$$A=[\partial_{\bar z_{2l}}\partial_{z_{2k}}u]_{l,k},\quad B=[\partial_{\bar z_{2l+1}}\partial_{z_{2k}}u]_{l,k},\quad C=[\partial_{\bar z_{2l+1}}\partial_{z_{2k+1}}u]_{l,k}.$$

Under this notation

$$\psi(\mathrm{Hess}(u,\mathbb{H}))=4\begin{pmatrix}A&\overline{B}^T\\B&C\end{pmatrix}+4\begin{pmatrix}\overline{C}&-\overline{B}\\-B^T&\overline{A}\end{pmatrix}.$$

Note that

$$\begin{pmatrix} A & \bar{B}^T \\ B & C \end{pmatrix}$$

is the conjugate of a Hessian of u with respect to the coordinates $z_0, \dots, z_{2n-2}, z_1, \dots, z_{2n-1}$, so it is Hermitian positive as well. Moreover

$$\det\left(\frac{\partial^2 u}{\partial_{z_i}\partial_{\bar{z}_j}}\right) = \det\begin{pmatrix} A & \bar{B}^T \\ B & C \end{pmatrix}.$$

Consider the matrix

$$I = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

with the inverse

$$I^{-1} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

and determinant equal to 1. Note that

$$I\begin{pmatrix} \bar{C} & -\bar{B} \\ -B^T & \bar{A} \end{pmatrix} I^{-1} = \begin{pmatrix} \bar{A} & B^T \\ \bar{B} & \bar{C} \end{pmatrix},$$

and the last matrix is the conjugate of the one just shown to be Hermitian positive so as such is also Hermitian positive. Consequently

$$\begin{pmatrix} \bar{C} & -\bar{B} \\ -B^T & \bar{A} \end{pmatrix}$$

is positive, being similar to one of that kind. Now we use the equality between Moore's determinant of a matrix M and the Pfaffian of an associated complex matrix $\psi(M)$, as proved in [Dyson 1970], which results in

$$\begin{aligned}\left(\det\left(\frac{\partial^2 u}{\partial_{\bar{q}_l}\partial_{q_k}}\right)\right)^2 = \det\psi(\mathrm{Hess}(u,\mathbb{H})) &= 4^{2n}\det\left(\begin{pmatrix} A & \bar{B}^T \\ B & C \end{pmatrix} + \begin{pmatrix} \bar{C} & -\bar{B} \\ -B^T & \bar{A} \end{pmatrix}\right) \\ &\geq 4^{2n}\det\begin{pmatrix} A & \bar{B}^T \\ B & C \end{pmatrix} = 4^{2n}\det\left(\frac{\partial^2 u}{\partial_{z_i}\partial_{\bar{z}_j}}\right),\end{aligned}$$

as we desired to prove. □

Having this, the announced comparison of quaternionic and complex Monge–Ampère operators for nonsmooth functions follows from the standard approximation procedure as presented in the proof below. Real and quaternionic Monge–Ampère operators were compared in [Wan 2020].

Theorem 8. *Let $u \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$ satisfy the equation*

$$(dd^c u)^{2n} = f^2 4^{2n} \omega_{2n}^{2n}$$

for some nonnegative $f \in L^p(\Omega)$, $p > 2$. Then

$$(\partial\partial_J u)^n \geq f\Omega_n^n.$$

Proof. Since the property is local we may assume that Ω is strictly pseudoconvex, otherwise we argue as below but for some ball contained in Ω. Approximate f by a sequence of smooth positive functions f_i in L^p norm and u uniformly by a sequence of smooth functions ϕ_i on $\partial\Omega$. Let us solve the family of Dirichlet problems

$$\begin{cases} u_i \in \mathcal{PSH}(\Omega) \cap C^\infty(\bar{\Omega}), \\ (dd^c u_i)^{2n} = f_i^2 4^{2n} \omega_{2n}^{2n}, \\ u_i = \phi_i \quad \text{on } \partial\Omega, \end{cases}$$

which is possible due to [Caffarelli et al. 1985]. Observe that the u_i converge uniformly to u due to the stability of solutions in L^q, $q > 1$, for the complex Monge–Ampère equation; see [Kołodziej 1996; Dinew and Kołodziej 2014]. From Lemma 7

$$(\partial\partial_J u_i)^n = \frac{1}{4^n} \det\left(\frac{\partial^2 u_i}{\partial_{\bar{q}_l} \partial_{q_k}}\right) \Omega_n^n \geq \sqrt{\det\left(\frac{\partial^2 u_i}{\partial_{z_m} \partial_{\bar{z}_n}}\right)} = f_i \Omega_n^n$$

as measures. The right-hand sides converge as measures to $f\Omega_n^n$ and the left ones converge to $(\partial\partial_J u)^n$ since convergence of u_i is uniform, see [Wan and Wang 2017], completing the proof. $\square$

We are going to prove an inequality between the volume and quaternionic capacity which was an essential component of Kołodziej's proof of solvability of the complex Monge–Ampère equation for densities in appropriate Orlicz spaces; see [Kołodziej 1996; 2005]. Similar inequality for the capacity associated to a complex m-Hessian equation was proven in [Dinew and Kołodziej 2014] with the use of an observation that psh functions, although an extremal example of m-subharmonic ones, still realize the m-Hessian capacity. Here we couple that trick with a comparison of quaternionic and complex Monge–Ampère operators proved in Theorem 8.

Lemma 9. *For a fixed $p \in (1, 2)$ there exists a constant $C(p, R)$ such that for any $\Omega \subset B(0, R)$ and $K \Subset \Omega$*

$$\mathcal{L}^{4n}(K) \leq C(p, R) \operatorname{cap}^p(K, \Omega).$$

Proof. Suppose that $\mathcal{L}(K) \neq 0$; otherwise there is nothing to prove. Take any $\epsilon \in \left(0, \frac{1}{2}\right)$ and consider $f = \mathcal{L}(K)^{2\epsilon-1}\chi_K$. Let us solve the Dirichlet problem

$$\begin{cases} u \in \mathcal{PSH}(B) \cap C(\bar{B}), \\ (dd^c u)^{2n} = f 4^{2n} \omega_{2n}^{2n}, \\ u = 0 \quad \text{on } \partial B, \end{cases}$$

which is possible due to [Cegrell 1984]. By Theorem 8 the quaternionic Monge–Ampère operator of the solution u satisfies

$$(\partial\partial_J u)^n \geq \sqrt{f}\,\Omega_n^n.$$

Taking $q = 1 + \epsilon$, one checks that

$$\int_B f^q (4^{2n}(2n)!)^q \, d\mathcal{L}^{4n} = (4^{2n}(2n)!)^q \mathcal{L}(K)^{(2\epsilon-1)(1+\epsilon)+1} = (4^{2n}(2n)!)^q \mathcal{L}(K)^{2\epsilon^2+\epsilon} \leq (4^{2n}(2n)!)^2 R^{4n};$$

i.e., the L^q norm of f is bounded by a quantity depending only on R. By Kołodziej's L^∞ estimate [1996; 1998], there exists a constant $c(\epsilon, R)$ such that

$$\|u\|_{L^\infty(B)} \le \frac{1}{c(\epsilon, R)}.$$

Put $v = c(\epsilon, R)u$; then since v is a qpsh function such that $-1 \le v \le 0$

$$\operatorname{cap}(K, \Omega) \ge \int_K (\partial\partial_J v)^n \ge n!\, c(\epsilon, R)^n (\mathcal{L}^{4n}(K))^{\frac{2\epsilon+1}{2}}$$

and consequently

$$\left(\frac{1}{n!\, c(\epsilon, R)^n}\right)^{\frac{2}{2\epsilon+1}} \operatorname{cap}^{\frac{2}{2\epsilon+1}}(K, \Omega) \ge \mathcal{L}^{4n}(K).$$

This gives the claim since when ϵ varies in $\left(0, \frac{1}{2}\right)$ the exponent $\frac{2}{2\epsilon+1}$ varies in $(1, 2)$. $\square$

In the previous section we have proven that any qpsh function belongs to L^p for $p < 2$ locally and that this is the optimal exponent. The lemma below gives the estimates on capacity and volume for sublevel sets of certain qpsh functions. In particular it shows that in the case of $u \in \mathcal{QPSH}(\Omega)$ bounded near the boundary of Ω the local integrability of $|u|^p$ is ensured for $p < 2n$. Again this bound is optimal as the example of $-1/\|q\|^2$ shows.

Lemma 10. *Fix $p \in (1, 2)$. Let $u \in \mathcal{QPSH}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ be such that*

$$\liminf_{q \to q_0} (u(q) - v(q)) \ge 0$$

for any $q_0 \in \partial\Omega$ and some fixed $v \in \mathcal{QPSH}(\Omega) \cap C(\bar{\Omega})$. Then there exists a constant $C(p, \operatorname{diam}(\Omega))$ depending only on p and the diameter of Ω such that for $U(s) = \{u < v - s\} \Subset \Omega$

$$\operatorname{cap}(U(s), \Omega) \le \frac{\int_\Omega (\partial\partial_J u)^n}{s^n},$$

$$\mathcal{L}^{4n}(U(s)) \le C(p, \operatorname{diam}(\Omega)) \frac{\int_\Omega (\partial\partial_J u)^n}{s^{pn}}.$$

Proof. Take $\epsilon > 0$ and a compact set $K \subset U(s)$. By definition one can find $w \in \mathcal{QPSH}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ such that $-1 \le w \le 0$ and

$$\int_K (\partial\partial_J w)^n \ge \operatorname{cap}(K, \Omega) - \epsilon.$$

Due to the way we have chosen K and the comparison principle

$$\begin{aligned} \operatorname{cap}(K, \Omega) - \epsilon &\le \int_K (\partial\partial_J w)^n \le \int_{\{\frac{u}{s} < \frac{v}{s} - 1\}} (\partial\partial_J w)^n \le \int_{\{\frac{u}{s} < \frac{v}{s} + w\}} (\partial\partial_J w)^n \\ &\le \int_{\{\frac{u}{s} < \frac{v}{s} + w\}} \left(\partial\partial_J \left(\frac{v}{s} + w\right)\right)^n \le \frac{1}{s^n} \int_{\{\frac{u}{s} < \frac{v}{s} + w\}} (\partial\partial_J u)^n \le \frac{\int_\Omega (\partial\partial_J u)^n}{s^n}. \end{aligned}$$

Letting ϵ tend to 0 and taking the supremum over all compact sets K we obtain the first claim. The second one follows from Lemma 9. $\square$

The next goal is to prove the a priori L^∞ estimate for continuous solutions of the Dirichlet problem. Firstly note that by Alesker's result [2003a] on the Dirichlet problem with continuous density and boundary value, and characterization of maximality of qpsh functions as in [Wan and Zhang 2015], we can find $v \in C(\bar{\Omega})$ solving

$$\begin{cases} (\partial\partial_J v)^n = 0, \\ v_{|\partial\Omega} = \phi \in C(\partial\Omega); \end{cases}$$

i.e., is the maximal qpsh function matching our boundary condition. For such a fixed v we set

$$U(s) = \{u < v - s\} \subset \Omega$$

and introduce the function

$$b(s) = (\operatorname{cap}(U(s), \Omega))^{\frac{1}{n}}.$$

Theorem 11. *There exists a constant $C(q, \|f\|_{L^q(\Omega)}, \|\phi\|_{L^\infty(\partial\Omega)}, \operatorname{diam}(\Omega))$ depending on q, $\|f\|_{L^q(\Omega)}$, $\|\phi\|_{L^\infty(\partial\Omega)}$ and $\operatorname{diam}(\Omega)$ such that any solution u of the Dirichlet problem*

$$\begin{cases} u \in \mathcal{QPSH}(\Omega) \cap C(\bar{\Omega}), \\ (\partial\partial_J u)^n = f\Omega_n, \\ u_{|\partial\Omega} = \phi \in C(\partial\Omega), \end{cases}$$

for $f \in L^q(\Omega)$ and $q > 2$, satisfies $\|u\|_{L^\infty(\Omega)} \le C$.

Proof. Take any $s > 0$, $t \in [0, 1]$ and $w \in \mathcal{QPSH}(\Omega)$ such that $0 \le w \le 1$. Then

$$\begin{aligned} t^n \int_{U(s+t)} (\partial\partial_J w)^n &= \int_{U(s+t)} (\partial\partial_J(tw - t - s))^n = \int_{\{u<v-s-t\}} (\partial\partial_J(tw - t - s))^n \\ &\le \int_{\{u<v-s+tw-t\}} (\partial\partial_J(tw - t - s))^n \\ &\le \int_{\{u<v-s+tw-t\}} (\partial\partial_J(v + tw - t - s))^n \\ &\le \int_{\{u<v-s+tw-t\}} (\partial\partial_J u)^n \le \int_{\{u<v-s\}} (\partial\partial_J u)^n = \int_{U(s)} (\partial\partial_J u)^n, \end{aligned}$$

due to inclusions of appropriate sets, superadditivity and the comparison principle. To conclude

$$t^n (b(s+t))^n \le \int_{U(s)} (\partial\partial_J u)^n.$$

Estimating the right-hand side gives

$$\begin{aligned} \int_{U(s)} (\partial\partial_J u)^n &= \int_{U(s)} f\Omega_n \le \|f\|_{L^q(\Omega)} \Big(\int_{U(s)} 1\, d\mathcal{L}^{4n}\Big)^{\frac{1}{q'}} \\ &\le \|f\|_{L^q(\Omega)} C(p, \operatorname{diam}(\Omega)) (\operatorname{cap}(U(s), \Omega))^{\frac{p}{q'}} = \|f\|_{L^q(\Omega)} C(p, \operatorname{diam}(\Omega)) (b(s))^{n(1+\alpha(q))}, \end{aligned}$$

where we used Hölder's inequality and Lemma 9 and p depends only on q', which is the conjugate of q and we choose it so that $\frac{p}{q'} > 1$. This reassembles to

$$tb(s+t) \le A(q, \|f\|_{L^q(\Omega)}, \operatorname{diam}(\Omega)) (b(s))^{1+\alpha(q)}$$

for any $s > 0$ and $t \in [0, 1]$.

We would like to apply the De Giorgi lemma, stated below, for the function b. Let us just note that Lemma 12(a) is satisfied since for $s_n \searrow s$ we have $U(s_n) \nearrow U(s)$ and under such an assumption $\operatorname{cap}(U(s_n), \Omega) \to \operatorname{cap}(U(s), \Omega)$; see [Wan and Kang 2017]. The condition (b) follows from the first assertion of Lemma 10 as well as the dependence of s_0 only on q, $\|f\|_{L^q(\Omega)}$ and $\operatorname{diam}(\Omega)$. Indeed, it was proven there that

$$b^{\alpha(q)}(s) = \operatorname{cap}(U(s), \Omega)^{\frac{\alpha(q)}{n}} \leq \frac{\left(\int_\Omega (\partial\partial_J u)^n\right)^{\frac{\alpha(q)}{n}}}{s^{\alpha(q)}} = \frac{\|f\|_{L^1(\Omega)}^{\frac{\alpha(q)}{n}}}{s^{\alpha(q)}} \leq \frac{c(q, \|f\|_{L^q(\Omega)}, \operatorname{diam}(\Omega))}{s^{\alpha(q)}},$$

so surely $s_0 \leq (2Ac)^{\frac{1}{\alpha(q)}}$ and this estimate depends only on q, $\|f\|_{L^q(\Omega)}$ and $\operatorname{diam}(\Omega)$. By the De Giorgi lemma there exists $S(q, \|f\|_{L^q(\Omega)}, \operatorname{diam}(\Omega))$ such that $b(s) = 0$ for any $s > S(q, \|f\|_{L^q(\Omega)}, \operatorname{diam}(\Omega))$. This together with Lemma 9 gives our claim since then

$$\|u\|_{L^\infty} \leq \sup|\phi| + S(q, \|f\|_{L^q(\Omega)}, \operatorname{diam}(\Omega)) = C(q, \|f\|_{L^q(\Omega)}, \|\phi\|_{L^\infty(\partial\Omega)}, \operatorname{diam}(\Omega)). \qquad \square$$

Lemma 12 (De Giorgi lemma [Phong et al. 2012]). *Let $f : R_+ \to R_+$ satisfy the following conditions*:

(a) *f is right-continuous.*

(b) *f decreases to* 0*.*

(c) *There exist positive constants α, A_α so that, for all $s \geq 0$ and all $0 \leq r \leq 1$, we have*

$$r f(s+r) \leq A_\alpha f(s)^{1+\alpha}.$$

Then there exists s_∞, depending only on α, A_α and the smallest value s_0 for which we have $f(s_0)^\alpha \leq (2A_\alpha)^{-1}$, so that $f(s) = 0$ for $s > s_\infty$. In fact, we can take

$$s_\infty = s_0 + 2A_\alpha(1 - 2^{-\alpha})^{-1} f(s_0)^\alpha.$$

The L^∞ estimate allows us to prove the stability of solutions to the Dirichlet problem in terms of densities and boundary values. This will be needed for the proof of solvability of the Dirichlet problem but is of course a result that is interesting in its own right. As we were told by S. Dinew the idea of proving stability presented in Proposition 14 is due to N. C. Nguyen.

Lemma 13. *There exists a constant $C(q, \operatorname{diam}(\Omega))$ depending on q and $\operatorname{diam}(\Omega)$ such that any solution u of the Dirichlet problem*

$$\begin{cases} u \in \mathcal{QPSH}(\Omega) \cap C(\bar{\Omega}), \\ (\partial\partial_J u)^n = f\Omega_n, \\ u_{|\partial\Omega} = 0, \end{cases}$$

for $f \in L^q(\Omega)$ and $q > 2$, satisfies $\|u\|_{L^\infty(\Omega)} \leq C(q, \operatorname{diam}(\Omega))\|f\|_{L^q(\Omega)}^{\frac{1}{n}}$.

Proof. Suppose that $\|f\|_{L^q(\Omega)} \neq 0$; otherwise there is nothing to prove. The function

$$v := \frac{u}{\|f\|_{L^q(\Omega)}^{\frac{1}{n}}}$$

solves the Dirichlet problem

$$\begin{cases} v \in \mathcal{QPSH}(\Omega) \cap C(\bar{\Omega}), \\ (\partial\partial_J v)^n = \dfrac{f}{\|f\|_{L^q(\Omega)}}\Omega_n, \\ v_{|\partial\Omega} = 0. \end{cases}$$

By Theorem 11 there exists a constant $C(q, \operatorname{diam}(\Omega)) := C(q, 1, 0, \operatorname{diam}(\Omega))$ such that $\|v\|_{L^\infty(\Omega)} \leq C(q, \operatorname{diam}(\Omega))$; this gives the claim. ☐

Proposition 14. *There exists a constant $C(q, \operatorname{diam}(\Omega))$ such that if u and v satisfy*

$$\begin{cases} u \in \mathcal{QPSH}(\Omega) \cap C(\bar{\Omega}), \\ (\partial\partial_J u)^n = f\Omega_n, \\ u_{|\partial\Omega} = \phi \in C(\partial\Omega) \end{cases}$$

and

$$\begin{cases} v \in \mathcal{QPSH}(\Omega) \cap C(\bar{\Omega}), \\ (\partial\partial_J v)^n = g\Omega_n, \\ v_{|\partial\Omega} = \psi \in C(\partial\Omega) \end{cases}$$

for $f, g \in L^q(\Omega)$, $q > 2$ then

$$\|u - v\|_{L^\infty(\Omega)} \leq \sup_{\partial\Omega} |\phi - \psi| + C(q, \operatorname{diam}(\Omega))\|f - g\|_{L^q(\Omega)}^{\frac{1}{n}}.$$

Proof. Consider a function w that is the solution of

$$\begin{cases} w \in \mathcal{QPSH}(\Omega) \cap C(\bar{\Omega}), \\ (\partial\partial_J w)^n = (f - g)_+\Omega_n, \\ w_{|\partial\Omega} = 0. \end{cases}$$

Note that on $\partial\Omega$ we have $w + v + \inf(\phi - \psi) \leq u$, while

$$(\partial\partial_J(w + v + \inf(\phi - \psi)))^n \geq (f - g)_+ + g \geq f = (\partial\partial_J u)^n.$$

The comparison principle gives $w + v + \inf(\phi - \psi) \leq u$ in $\bar{\Omega}$, which by Lemma 13 results in

$$\begin{aligned} u - v \geq w + \inf(\phi - \psi) &\geq -C(q, \operatorname{diam}(\Omega))\|(f - g)_+\|_{L^q(\Omega)}^{\frac{1}{n}} - \sup|\phi - \psi| \\ &\geq -C(q, \operatorname{diam}(\Omega))\|f - g\|_{L^q(\Omega)}^{\frac{1}{n}} - \sup|\phi - \psi|. \end{aligned}$$

The same reasoning gives

$$v - u \geq -C(q, \operatorname{diam}(\Omega))\|f - g\|_{L^q(\Omega)}^{\frac{1}{n}} - \sup|\phi - \psi|.$$

This reassembles to our claim. ☐

Remark 15. Equicontinuity of a family of functions

$$\mathcal{P}(q, c_0, \phi) = \left\{ u \in \mathcal{QPSH}(\Omega) \cap C(\bar{\Omega}) \;\middle|\; (\partial\partial_J u)^n \in L^q(\Omega),\ \int_\Omega (\partial\partial_J u)^n \leq c_0,\ u_{|\partial\Omega} = \phi \right\}$$

for a quaternionic strictly pseudoconvex domain Ω, $q > 2$, $c_0 > 0$ and $\phi \in C(\partial\Omega)$ follows easily from Proposition 14. In the complex case it was proven in [Kołodziej 2002].

Theorem 16. *The Dirichlet problem*

$$\begin{cases} u \in \mathcal{QPSH}(\Omega) \cap C(\bar{\Omega}), \\ (\partial\partial_J u)^n = f\Omega_n, \\ u_{|\partial\Omega} = \phi \in C(\partial\Omega) \end{cases}$$

in a smoothly bounded, quaternionic strictly pseudoconvex domain Ω for $f \in L^q(\Omega)$, $q > 2$, has a unique solution.

Proof. Uniqueness follows from the comparison principle. For solvability we take a sequence of continuous nonnegative functions f_i converging to f in $L^q(\Omega)$. Solving Dirichlet problems for them with our boundary condition, which is possible due to [Alesker 2003a], gives a sequence of continuous solutions u_i. Since, by Proposition 14, these solutions constitute a Cauchy sequence it follows that the u_i converge uniformly to some u. This is the solution we were looking for because of convergence of Monge–Ampère masses; see [Wan and Wang 2017]. □

The example below shows that the exponent 2 is optimal in the sense that for densities in $L^p(\Omega)$ with $p < 2$ solutions may not even be bounded.

Proposition 17. *Let $f(q) = \log(\|q\|)$. Then it belongs to $\mathcal{QPSH}(\mathbb{H}^n)$ and*

$$(\partial\partial_J f)^n = \frac{n!}{2\|q\|^{2n}}\Omega_n.$$

Proof. We compute for $f_\epsilon(q) = \frac{1}{2}\log(\|q\|^2 + \epsilon)$

$$\begin{aligned}
\partial\partial_J f_\epsilon &= \partial\Bigg(\sum_{i=0}^{2n-1}(-1)^{i+1}(\partial_{\bar{z}_{i+(-1)^i}} f_\epsilon)\, dz_i\Bigg) \\
&= \frac{1}{2}\partial\Bigg(\sum_{i=0}^{2n-1}(-1)^{i+1}\frac{z_{i+(-1)^i}}{\|q\|^2+\epsilon}\, dz_i\Bigg) \\
&= \frac{1}{2}\Bigg(\sum_{i=0,\, j=0}^{2n-1}(-1)^{i+1}\frac{\delta^j_{i+(-1)^i}(\|q\|^2+\epsilon) - z_{i+(-1)^i}\bar{z}_j}{(\|q\|^2+\epsilon)^2}\, dz_j \wedge dz_i\Bigg) \\
&= \frac{1}{2}\Bigg(\sum_{i>j}^{2n-1}\bigg((-1)^{i+1}\frac{\delta^j_{i+(-1)^i}(\|q\|^2+\epsilon) - z_{i+(-1)^i}\bar{z}_j}{(\|q\|^2+\epsilon)^2} \\
&\qquad\qquad - (-1)^{j+1}\frac{\delta^i_{j+(-1)^j}(\|q\|^2+\epsilon) - z_{j+(-1)^j}\bar{z}_i}{(\|q\|^2+\epsilon)^2}\bigg)\, dz_j \wedge dz_i\Bigg) \\
&= \frac{1}{2}\Bigg(\sum_{i>j}^{2n-1}\bigg(\frac{2\delta^j_{i+(-1)^i}(\|q\|^2+\epsilon) + (-1)^i z_{i+(-1)^i}\bar{z}_j + (-1)^{j+1} z_{j+(-1)^j}\bar{z}_i}{(\|q\|^2+\epsilon)^2}\bigg)\, dz_j \wedge dz_i\Bigg).
\end{aligned}$$

Let us define

$$M_{ij} = (-1)^i z_{i+(-1)^i}\bar{z}_j + (-1)^{j+1} z_{j+(-1)^j}\bar{z}_i$$

as in [Wan and Wang 2017] and let $\delta^{j_1 i_1,\dots,j_n i_n}_{0,\dots,2n-1}$ be the sign of the permutation

$$(j_1, i_1, \dots, j_n, i_n) \to (0, 1, \dots, 2n-1).$$

With this notation we see that

$$\begin{aligned}
&2^n(\|q\|^2+\epsilon)^{2n}(\partial\partial_J f_\epsilon)^n \\
&\quad= \Biggl(\sum_{\substack{j_1,i_1,\dots,j_n,i_n:\\ \{j_1,i_1,\dots,j_n,i_n\}=\{0,\dots,2n-1\}\\ i_l>j_l,\ l\in\{1,\dots,n\}}} \Bigl(\delta^{j_1 i_1,\dots,j_n i_n}_{0,\dots,2n-1} \prod_{l\in\{1,\dots,n\}} (2\delta^{j_l}_{i_l+(-1)^{i_l}}(\|q\|^2+\epsilon)+M_{i_l j_l})\Bigr)\Biggr)\Omega_n \\
&\quad= \binom{n}{0} \sum_{\{k_1,\dots,k_n\}=\{0,\dots,n-1\}} \delta^{(2k_1)(2k_1+1),\dots,(2k_n)(2k_n+1)}_{0,\dots,2n-1} 2^n(\|q\|^2+\epsilon)^n \\
&\qquad+ \binom{n}{1} \sum_{\substack{\{j_1,i_1,2k_2,2k_2+1,\dots,2k_n,2k_n+1\}=\{0,\dots,2n-1\}\\ i_1>j_1\\ k_l\in\{0,\dots,n-1\}}} \delta^{j_1 i_1,\dots,(2k_n)(2k_n+1)}_{0,\dots,2n-1} 2^{n-1}(\|q\|^2+\epsilon)^{(n-1)} M_{i_1 j_1} \\
&\qquad+ \binom{n}{2} \sum_{\substack{\{j_1,i_1,j_2,i_2,\dots,2k_n,2k_n+1\}=\{0,\dots,2n-1\}\\ i_1>j_1,\ i_2>j_2\\ k_l\in\{0,\dots,n-1\}}} \delta^{j_1 i_1,j_2,i_2,\dots,(2k_n)(2k_n+1)}_{0,\dots,2n-1} 2^{n-2}(\|q\|^2+\epsilon)^{(n-2)} M_{i_1 j_1} M_{i_2 j_2} \\
&\qquad+\dots+ \binom{n}{n} \sum_{\substack{\{j_1,i_1,\dots,j_n,i_n\}=\{0,\dots,2n-1\}\\ i_l>j_l,\ l\in\{1,\dots,n\}}} \delta^{j_1 i_1,\dots,j_n i_n}_{0,\dots,2n-1} \prod_{l\in\{1,\dots,n\}} M_{i_l j_l}.
\end{aligned}$$

Note that for fixed indices $j_3, i_3, \dots, j_n, i_n$ the expression

$$M'_{j_3,i_3,\dots,j_n,i_n} = \sum_{\substack{j_1,i_1,j_2,i_2:\\ \{j_1,i_1,j_2,i_2,\dots,2k_n,2k_n+1\}=\{0,\dots,2n-1\}\\ i_1>j_1,i_2>j_2}} \delta^{j_1 i_1,\dots,j_n i_n}_{0,\dots,2n-1} M_{i_1 j_1} M_{i_2 j_2}$$

vanishes; this was already noticed in [Wan and Wang 2017]. To see this let

$$\{0, \dots, 2n-1\} \setminus \{j_3, i_3, \dots, j_n, i_n\} = \{k, l, m, n\}$$

and $k > l > m > n$. Then

$$\begin{aligned}
\tfrac{1}{2} M'_{j_3,i_3,\dots,j_n,i_n} &= \delta^{lk,nm,j_3 i_3,\dots,j_n i_n}_{0,\dots,2n-1} M_{kl} M_{mn} + \delta^{mk,nl,j_3 i_3,\dots,j_n i_n}_{0,\dots,2n-1} M_{km} M_{ln} + \delta^{nk,ml,j_3 i_3,\dots,j_n i_n}_{0,\dots,2n-1} M_{kn} M_{lm} \\
&= \delta^{lk,nm,j_3 i_3,\dots,j_n i_n}_{0,\dots,2n-1} (M_{kl} M_{mn} - M_{km} M_{ln} + M_{kn} M_{lm}) \\
&= \pm\Bigl(\bigl((-1)^k z_{k+(-1)^k}\bar z_l + (-1)^{l+1} z_{l+(-1)^l}\bar z_k\bigr)\bigl((-1)^m z_{m+(-1)^m}\bar z_n + (-1)^{n+1} z_{n+(-1)^n}\bar z_m\bigr) \\
&\qquad - \bigl((-1)^k z_{k+(-1)^k}\bar z_m + (-1)^{m+1} z_{m+(-1)^m}\bar z_k\bigr)\bigl((-1)^l z_{l+(-1)^l}\bar z_n + (-1)^{n+1} z_{n+(-1)^n}\bar z_l\bigr) \\
&\qquad + \bigl((-1)^k z_{k+(-1)^k}\bar z_n + (-1)^{n+1} z_{n+(-1)^n}\bar z_k\bigr)\bigl((-1)^l z_{l+(-1)^l}\bar z_m + (-1)^{m+1} z_{m+(-1)^m}\bar z_l\bigr)\Bigr)
\end{aligned}$$

$$
\begin{aligned}
=\pm\Big(&(-1)^{k+m+1}z_{k+(-1)^k\bar{z}_n}z_{m+(-1)^m\bar{z}_l}+(-1)^{k+m}z_{k+(-1)^k\bar{z}_l}z_{m+(-1)^m\bar{z}_n}\\
&+(-1)^{l+m+1}z_{l+(-1)^l\bar{z}_k}z_{m+(-1)^m\bar{z}_n}+(-1)^{m+l}z_{m+(-1)^m\bar{z}_k}z_{l+(-1)^l\bar{z}_n}\\
&+(-1)^{k+l+1}z_{k+(-1)^k\bar{z}_m}z_{l+(-1)^l\bar{z}_n}+(-1)^{k+l}z_{k+(-1)^k\bar{z}_n}z_{l+(-1)^l\bar{z}_m}\\
&+(-1)^{k+n+1}z_{k+(-1)^k\bar{z}_l}z_{n+(-1)^n\bar{z}_m}+(-1)^{k+n}z_{k+(-1)^k\bar{z}_m}z_{n+(-1)^n\bar{z}_l}\\
&+(-1)^{m+n+1}z_{m+(-1)^m\bar{z}_k}z_{n+(-1)^n\bar{z}_l}+(-1)^{n+m}z_{n+(-1)^n\bar{z}_k}z_{m+(-1)^m\bar{z}_l}\\
&+(-1)^{n+l+1}z_{n+(-1)^n\bar{z}_k}z_{l+(-1)^l\bar{z}_m}+(-1)^{l+n}z_{l+(-1)^l\bar{z}_k}z_{n+(-1)^n\bar{z}_m}\Big)\\
=0.&
\end{aligned}
$$

Because of that only the first two summands of the expression for $(\partial\partial_J f)^n$ do not vanish. We are left with

$$(\partial\partial_J f_\epsilon)^n=\frac{1}{2^n(\|q\|^2+\epsilon)^{2n}}\big(n!\,2^n(\|q\|^2+\epsilon)^n-n!\,2^{n-1}(\|q\|^2+\epsilon)^{(n-1)}\|q\|^2\big)\Omega_n=\frac{n!\,(\|q\|^2+2\epsilon)}{2(\|q\|^2+\epsilon)^{n+1}}\Omega_n.$$

Finally since measures $(\partial\partial_J f_\epsilon)^n$ converge weakly to $(\partial\partial_J f)^n$, see [Wan and Wang 2017], it is enough to find the weak limit of

$$\frac{n!\,(\|q\|^2+2\epsilon)}{2(\|q\|^2+\epsilon)^{n+1}},$$

which by

$$\frac{n!\,(\|q\|^2+2\epsilon)}{2(\|q\|^2+\epsilon)^{n+1}}\le\frac{n!}{2\|q\|^{2n}}$$

and Lebesgue's dominated convergence theorem is

$$\frac{n!}{2\|q\|^{2n}},$$

exactly as we wanted. □

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References

[Alesker 2003a] S. Alesker, "Non-commutative linear algebra and plurisubharmonic functions of quaternionic variables", *Bull. Sci. Math.* **127**:1 (2003), 1–35. MR Zbl

[Alesker 2003b] S. Alesker, "Quaternionic Monge–Ampère equations", *J. Geom. Anal.* **13**:2 (2003), 205–238. MR Zbl

[Alesker and Verbitsky 2006] S. Alesker and M. Verbitsky, "Plurisubharmonic functions on hypercomplex manifolds and HKT-geometry", *J. Geom. Anal.* **16**:3 (2006), 375–399. MR Zbl

[Alesker and Verbitsky 2010] S. Alesker and M. Verbitsky, "Quaternionic Monge–Ampère equation and Calabi problem for HKT-manifolds", *Israel J. Math.* **176** (2010), 109–138. MR Zbl

[Bedford and Taylor 1976] E. Bedford and B. A. Taylor, "The Dirichlet problem for a complex Monge–Ampère equation", *Invent. Math.* **37**:1 (1976), 1–44. MR Zbl

[Bedford and Taylor 1982] E. Bedford and B. A. Taylor, "A new capacity for plurisubharmonic functions", *Acta Math.* **149**:1-2 (1982), 1–40. MR Zbl

[Caffarelli et al. 1985] L. Caffarelli, J. J. Kohn, L. Nirenberg, and J. Spruck, "The Dirichlet problem for nonlinear second-order elliptic equations, II: Complex Monge–Ampère, and uniformly elliptic, equations", *Comm. Pure Appl. Math.* **38**:2 (1985), 209–252. MR Zbl

[Cegrell 1984] U. Cegrell, "On the Dirichlet problem for the complex Monge–Ampère operator", *Math. Z.* **185**:2 (1984), 247–251. MR Zbl

[Cegrell and Persson 1992] U. Cegrell and L. Persson, "The Dirichlet problem for the complex Monge–Ampère operator: stability in L^2", *Michigan Math. J.* **39**:1 (1992), 145–151. MR Zbl

[Dinew and Kołodziej 2014] S. Dinew and S. Kołodziej, "A priori estimates for complex Hessian equations", *Anal. PDE* **7**:1 (2014), 227–244. MR Zbl

[Dyson 1970] F. J. Dyson, "Correlations between eigenvalues of a random matrix", *Comm. Math. Phys.* **19** (1970), 235–250. MR Zbl

[Harvey and Lawson 2009a] F. R. Harvey and H. B. Lawson, Jr., "Dirichlet duality and the nonlinear Dirichlet problem", *Comm. Pure Appl. Math.* **62**:3 (2009), 396–443. MR Zbl

[Harvey and Lawson 2009b] F. R. Harvey and H. B. Lawson, Jr., "An introduction to potential theory in calibrated geometry", *Amer. J. Math.* **131**:4 (2009), 893–944. MR Zbl

[Harvey and Lawson 2011] F. R. Harvey and H. B. Lawson, Jr., "Dirichlet duality and the nonlinear Dirichlet problem on Riemannian manifolds", *J. Differential Geom.* **88**:3 (2011), 395–482. MR Zbl

[Harvey and Lawson 2019] F. R. Harvey and H. B. Lawson, Jr., "The inhomogeneous Dirichlet problem for natural operators on manifolds", *Ann. Inst. Fourier* (*Grenoble*) **69**:7 (2019), 3017–3064.

[Hörmander 1994] L. Hörmander, *Notions of convexity*, Progr. Math. **127**, Birkhäuser, Boston, 1994. MR Zbl

[Kołodziej 1995] S. Kołodziej, "The range of the complex Monge–Ampère operator, II", *Indiana Univ. Math. J.* **44**:3 (1995), 765–782. MR Zbl

[Kołodziej 1996] S. Kołodziej, "Some sufficient conditions for solvability of the Dirichlet problem for the complex Monge–Ampère operator", *Ann. Polon. Math.* **65**:1 (1996), 11–21. MR Zbl

[Kołodziej 1998] S. Kołodziej, "The complex Monge–Ampère equation", *Acta Math.* **180**:1 (1998), 69–117. MR Zbl

[Kołodziej 2002] S. Kołodziej, "Equicontinuity of families of plurisubharmonic functions with bounds on their Monge–Ampère masses", *Math. Z.* **240**:4 (2002), 835–847. MR Zbl

[Kołodziej 2005] S. Kołodziej, *The complex Monge–Ampère equation and pluripotential theory*, Mem. Amer. Math. Soc. **840**, 2005. MR Zbl

[Moore 1922] E. H. Moore, "On the determinant of an hermitian matrix of quaternionic elements", *Bull. Amer. Math. Soc.* **28** (1922), 161–162. Zbl

[Phong et al. 2012] D. H. Phong, J. Song, and J. Sturm, "Complex Monge–Ampère equations", pp. 327–410 in *Surveys in differential geometry, XVII* (Bethlehem, PA, 2010), edited by H.-D. Cao and S.-T. Yau, Int. Press, Boston, 2012. MR Zbl

[Rauch and Taylor 1977] J. Rauch and B. A. Taylor, "The Dirichlet problem for the multidimensional Monge–Ampère equation", *Rocky Mountain J. Math.* **7**:2 (1977), 345–364. MR Zbl

[Verbitsky 2002] M. Verbitsky, "Hyperkähler manifolds with torsion, supersymmetry and Hodge theory", *Asian J. Math.* **6**:4 (2002), 679–712. MR Zbl

[Verbitsky 2010] M. Verbitsky, "Positive forms on hyperkähler manifolds", *Osaka J. Math.* **47**:2 (2010), 353–384. MR Zbl

[Wan 2020] D. Wan, "Subsolution theorem and the Dirichlet problem for the quaternionic Monge–Ampère equation", *Math. Z* (online publication Febrary 2020).

[Wan and Kang 2017] D. Wan and Q. Kang, "Potential theory for quaternionic plurisubharmonic functions", *Michigan Math. J.* **66**:1 (2017), 3–20. MR Zbl

[Wan and Wang 2017] D. Wan and W. Wang, "On the quaternionic Monge–Ampère operator, closed positive currents and Lelong–Jensen type formula on the quaternionic space", *Bull. Sci. Math.* **141**:4 (2017), 267–311. MR Zbl

[Wan and Zhang 2015] D. Wan and W. Zhang, "Quasicontinuity and maximality of quaternionic plurisubharmonic functions", *J. Math. Anal. Appl.* **424**:1 (2015), 86–103. MR Zbl

[Zhu 2016] J. Zhu, "Dirichlet problem of quaternionic Monge–Ampère equations", *Israel J. Math.* **214**:2 (2016), 597–619. MR Zbl

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