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## ON THE SHARP UPPER BOUND RELATED TO THE WEAK MUCKENHOUP-T-WHEEDEN CONJECTURE

ANDREI K. LERNER, FEDOR NAZAROV AND SHELDY OMBROSI

We construct an example showing that the upper bound  $[w]_{A_1} \log(e + [w]_{A_1})$  for the  $L^1(w) \rightarrow L^{1,\infty}(w)$  norm of the Hilbert transform cannot be improved in general.

### 1. Introduction

Define the Hardy–Littlewood maximal operator on  $\mathbb{R}$  by

$$Mf(x) = \sup_{I \ni x} \frac{1}{|I|} \int_I |f(y)| dy,$$

where the supremum is taken over all intervals  $I \subset \mathbb{R}$  containing the point  $x$ .

C. Fefferman and E. Stein [1971] established the following weighted weak-type inequality for  $M$ : there exists an absolute constant  $C > 0$  such that, for every weight  $w$ ,

$$\sup_{\alpha > 0} \alpha w \{x \in \mathbb{R} : Mf(x) > \alpha\} \leq C \int_{\mathbb{R}} |f(x)| Mw(x) dx \quad (1-1)$$

(here by a weight we mean any nonnegative locally integrable function on  $\mathbb{R}$ , and we use the standard notation  $w(E) = \int_E w$  for a measurable set  $E \subset \mathbb{R}$ ).

Inequality (1-1) is important for several reasons. First, it is the key ingredient in extending the Hardy–Littlewood maximal theorem to a vector-valued case. Second, this result was a precursor of the weighted theory, which had started to develop rapidly in the beginning of the 70’s. In particular, if we define the  $[w]_{A_1}$  constant of the weight  $w$  by  $[w]_{A_1} = \|Mw/w\|_{L^\infty}$ , then, assuming  $[w]_{A_1} < \infty$ , (1-1) implies immediately that

$$\|Mf\|_{L^{1,\infty}(w)} \leq C[w]_{A_1} \|f\|_{L^1(w)}. \quad (1-2)$$

Consider now the Hilbert transform,

$$Hf(x) = \text{P.V.} \int_{\mathbb{R}} \frac{f(y)}{x - y} dy.$$

Inequality (1-1) with the maximal operator on the left-hand side replaced by the Hilbert transform has become known as the Muckenhoupt–Wheeden conjecture. Only recently this conjecture has been disproved by M. Reguera and C. Thiele [2012] (see also [Caldarelli et al. 2017; Criado and Soria 2016;

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[Reguera 2011] for some complements and extensions). Their result, however, left open the question of whether a weaker form of the Muckenhoupt–Wheeden conjecture holds, with  $M$  replaced by  $H$  on the left-hand side of (1-2).

In [Lerner et al. 2009], it was proved that

$$\|Hf\|_{L^{1,\infty}(w)} \leq C[w]_{A_1} \log(e + [w]_{A_1}) \|f\|_{L^1(w)}. \tag{1-3}$$

This improved a previous result in [Lerner et al. 2008], where the right-hand side contained an additional factor double logarithmic in  $[w]_{A_1}$ . Notice also that actually (1-3) in [Lerner et al. 2009] was proved for every Calderón–Zygmund operator on  $\mathbb{R}^n$  with sufficiently smooth kernel.

On the other hand, in [Nazarov et al. 2015], it was shown for the martingale transform (and explained how to transfer the result to the Hilbert transform case) that the dependence of  $[w]_{A_1}$  in the weighted weak-type-(1, 1) inequality cannot in general be made better than  $[w]_{A_1} \log^{1/5}(e + [w]_{A_1})$ , thus disproving the weak Muckenhoupt–Wheeden conjecture. Later, in [Nazarov et al. 2018], the power of the logarithm was improved to  $\frac{1}{3}$  (this was again done for the martingale transform).

Summarizing the results in [Lerner et al. 2009; Nazarov et al. 2015; 2018], if we denote by  $\alpha_H$  the best possible exponent for which the inequality

$$\|Hf\|_{L^{1,\infty}(w)} \leq C[w]_{A_1} \log^{\alpha_H}(e + [w]_{A_1}) \|f\|_{L^1(w)}$$

holds, then we have that  $\frac{1}{3} \leq \alpha_H \leq 1$ .

The main result of this paper shows, in particular, that  $\alpha_H = 1$ . For  $t \geq 1$ , define

$$\varphi_H(t) = \sup_{[w]_{A_1} \leq t} \|H\|_{L^1(w) \rightarrow L^{1,\infty}(w)}.$$

Then (1-3) implies  $\varphi_H(t) \leq Ct \log(e + t)$ . We will show that actually  $\varphi_H(t) \approx t \log(e + t)$ . Our main result is the following.

**Theorem 1.1.** *There exists  $c' > 0$  such that, for all  $t \geq 1$ ,*

$$\varphi_H(t) \geq c' t \log(e + t).$$

As a trivial corollary we obtain that the inequality

$$\|Hf\|_{L^{1,\infty}(w)} \leq \psi([w]_{A_1}) \|f\|_{L^1(w)}$$

fails in general for every function  $\psi$  increasing on  $[1, \infty)$  satisfying

$$\lim_{t \rightarrow \infty} \frac{\psi(t)}{t \log(e + t)} = 0.$$

## 2. Proof of Theorem 1.1

**An overview of the proof.** At the first step we show that the definition of  $\varphi_H$  along with the standard extrapolation and dualization arguments yields

$$\|H(w\chi_{[0,1)})\|_{L^2(\sigma)} \leq 4\varphi_H(2\|M\|_{L^2(\sigma) \rightarrow L^2(\sigma)}) \left(\int_0^1 w\right)^{1/2}, \tag{2-1}$$

where  $\sigma = w^{-1}$ . Notice that  $\|M\|_{L^2(\sigma) \rightarrow L^2(\sigma)} < \infty$  if and only if  $w \in A_2$ , that is, if

$$\sup_{I \subset \mathbb{R}} \frac{w(I)\sigma(I)}{|I|^2} < \infty.$$

Therefore, we assume here that  $w \in A_2$ .

The key step is to show that there exist  $C_1, C_2, C_3 > 0$  such that for every  $t > C_3$ , there is an  $A_2$ -weight  $w$  satisfying

$$\int_0^1 w = 1, \quad \|M\|_{L^2(\sigma) \rightarrow L^2(\sigma)} \leq C_1 t, \quad \|H(w\chi_{[0,1]})\|_{L^2(\sigma)} \geq C_2 t \log t. \tag{2-2}$$

Plugging these estimates into (2-1), we finish the proof.

**Extrapolation and dualization.** First, we apply the standard Rubio de Francia extrapolation trick. Given  $g \geq 0$  with  $\|g\|_{L^2(\sigma)} = 1$ , define

$$\mathcal{R}g(x) = \sum_{k=0}^{\infty} \frac{M^k g(x)}{(2\|M\|_{L^2(\sigma) \rightarrow L^2(\sigma)})^k}.$$

Then  $g \leq \mathcal{R}g$ ,  $\|\mathcal{R}g\|_{L^2(\sigma)} \leq 2$ , and  $[\mathcal{R}g]_{A_1} \leq 2\|M\|_{L^2(\sigma) \rightarrow L^2(\sigma)}$ . These estimates along with the definition of  $\varphi_H$  and Hölder’s inequality imply

$$\begin{aligned} \alpha \int_{\{x:|Hf(x)|>\alpha\}} g &\leq \alpha \int_{\{x:|Hf(x)|>\alpha\}} \mathcal{R}g \leq \varphi_H(2\|M\|_{L^2(\sigma) \rightarrow L^2(\sigma)}) \|f\|_{L^1(\mathcal{R}g)} \\ &\leq \varphi_H(2\|M\|_{L^2(\sigma) \rightarrow L^2(\sigma)}) \|f\|_{L^2(w)} \|\mathcal{R}g\|_{L^2(\sigma)} \\ &\leq 2\varphi_H(2\|M\|_{L^2(\sigma) \rightarrow L^2(\sigma)}) \|f\|_{L^2(w)}. \end{aligned}$$

Taking here the supremum over all  $g \geq 0$  with  $\|g\|_{L^2(\sigma)} = 1$  yields

$$\|Hf\|_{L^{2,\infty}(w)} \leq 2\varphi_H(2\|M\|_{L^2(\sigma) \rightarrow L^2(\sigma)}) \|f\|_{L^2(w)}. \tag{2-3}$$

We now use the elementary estimate

$$\int_0^1 |Hf|w \leq 2\|Hf\|_{L^{2,\infty}(w)} \left(\int_0^1 w\right)^{1/2}, \tag{2-4}$$

which along with (2-3) implies

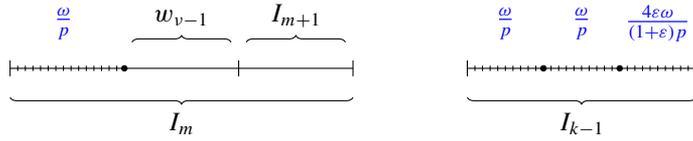
$$\left| \int_{\mathbb{R}} (H(w\chi_{[0,1]}))f \right| = \left| \int_0^1 (Hf)w \right| \leq 4\varphi_H(2\|M\|_{L^2(\sigma) \rightarrow L^2(\sigma)}) \left(\int_0^1 w\right)^{1/2} \|f\|_{L^2(w)}.$$

Taking here the supremum over all  $f$  with  $\|f\|_{L^2(w)} = 1$  proves (2-1).

To show (2-4), notice that, for every  $\lambda > 0$ ,

$$\int_0^1 |Hf|w \leq \int_{\lambda}^{\infty} w\{x : |Hf(x)| > \alpha\} d\alpha + \lambda \int_0^1 w \leq \frac{1}{\lambda} \|Hf\|_{L^{2,\infty}(w)}^2 + \lambda \int_0^1 w.$$

Optimizing this estimate with respect to  $\lambda$ , we obtain (2-4).



**Figure 1.**  $w_v(\omega, \sigma, I)$  on intervals  $I_m^{(i)}$  for  $i = 1, 2$  and  $0 \leq m \leq k - 2$  and on  $I_{k-1}^{(i)}$  for  $i = 1, 2, 3$ .

**Construction of the weight.** Fix  $t \gg 1$ . Take  $k \in \mathbb{N}$  such that  $t \leq 3^k \leq 3t$ . Let  $\varepsilon = 3^{-k}$  and

$$p = \frac{1}{3\varepsilon} \left( \frac{1 + \varepsilon}{2} + \frac{4\varepsilon^2}{1 + \varepsilon} \right).$$

The reason for this definition of  $p$  will be clarified a bit later. Note that we will frequently use the obvious estimate  $1/(6\varepsilon) \leq p \leq 2/\varepsilon$ .

For every two positive numbers  $\omega$  and  $\sigma$  such that  $\omega\sigma = p$  and any interval  $I \subset \mathbb{R}$ , we define inductively the sequence of weights  $w_v = w_v(\omega, \sigma, I)$  ( $v = 0, 1, 2, \dots$ ) supported on  $I$  as follows.

Let  $u = \sqrt{p} + \sqrt{p-1}$  be the larger root of  $u + 1/u = 2\sqrt{p}$ . Define

$$w_0(\omega, \sigma, I) = \frac{\omega}{\sqrt{p}} \left( u\chi_{I_-} + \frac{1}{u}\chi_{I_+} \right),$$

where  $I_-$  and  $I_+$  are the left and the right halves of  $I$  respectively.

Suppose that  $w_{v-1}(\omega, \sigma, I)$  is already defined for all  $\omega, \sigma$  with  $\omega\sigma = p$  and all  $I \subset \mathbb{R}$ . To construct  $w_v(\omega, \sigma, I)$ , first denote by  $I_m$ ,  $m = 0, 1, \dots, k - 1$ , the interval with the same right endpoint as  $I$  of length  $3^{-m}|I|$ , so

$$I_{k-1} \subset I_{k-2} \subset \dots \subset I_0 = I$$

and  $|I_{k-1}| = 3\varepsilon|I|$ .

Given an interval  $J$ , denote by  $J^{(i)}$ ,  $i = 1, 2, 3$ , the  $i$ -th from the left subinterval of  $J$  in the partition of  $J$  into three equal intervals.

Define  $w_v(\omega, \sigma, I)$  by

$$w_v(\omega, \sigma, I) = \frac{\omega}{p} \left( \sum_{m=0}^{k-2} \chi_{I_m^{(1)}} + \chi_{I_{k-1}^{(1)} \cup I_{k-1}^{(2)}} + \frac{4\varepsilon}{1 + \varepsilon} \chi_{I_{k-1}^{(3)}} \right) + \sum_{m=0}^{k-2} w_{v-1} \left( 2\omega, \frac{\sigma}{2}, I_m^{(2)} \right). \tag{2-5}$$

See Figure 1.

Note that the interval  $I_{k-1}^{(3)}$  plays a rather special role in the final step of this recursive construction. We shall call any interval of this type arising at any step in the construction of the weight  $w_v(\omega, \sigma, I)$  a “tail interval”, so within  $I$  we shall have one tail interval  $I_{k-1}^{(3)}$  arising at the final stage of the construction,  $k - 1$  tail intervals  $(I_m^{(2)})_{k-1}^{(3)}$  arising in the construction of the weights  $w_{v-1}(2\omega, \sigma/2, I_m^{(2)})$ , and so on.

Finally, we define  $w$  as the 1-periodization of  $w_n(1, p, [0, 1))$  with  $n = 3^{k-1}$ .

For  $l = 0, 1, \dots, n$ , we shall say that an interval  $I$  “carries  $w_{n-l}$ ” if  $w = w_{n-l}(2^l, 2^{-l}p, I)$  on  $I$ . Denote by  $\text{supp } w_{n-l}$  the union of all intervals carrying  $w_{n-l}$ . For example,  $\text{supp } w_n = \bigcup_{k \in \mathbb{Z}} [k, k + 1) = \mathbb{R}$  as  $[k, k + 1)$  carries  $w_n$  for every  $k \in \mathbb{Z}$ .

Let us now establish several useful properties of  $w_v(\omega, \sigma, I)$ .

**Proposition 2.1.** For every  $v \geq 0$ ,

$$\frac{1}{|I|} \int_I w_v(\omega, \sigma, I) dx = \omega \quad \text{and} \quad \frac{1}{|I|} \int_I w_v^{-1}(\omega, \sigma, I) dx = \sigma. \tag{2-6}$$

*Proof.* The proof is by induction on  $v$ . For  $v = 0$ ,

$$\begin{aligned} \frac{1}{|I|} \int_I w_0(\omega, \sigma, I) dx &= \frac{\omega}{\sqrt{p}} \frac{1}{2} \left( u + \frac{1}{u} \right) = \omega, \\ \frac{1}{|I|} \int_I w_0^{-1}(\omega, \sigma, I) dx &= \frac{\sqrt{p}}{\omega} \frac{1}{2} \left( \frac{1}{u} + u \right) = \frac{p}{\omega} = \sigma. \end{aligned}$$

Assume that the statement holds for  $v - 1$  and let us prove it for  $v$ . Observe that  $w_v(\omega, \sigma, I)$  equals  $\omega/p$  on a subset of  $I$  of total measure

$$\frac{1 - 3\varepsilon}{2} |I| + 2\varepsilon |I| = \frac{1 + \varepsilon}{2} |I|,$$

it equals

$$\frac{4\varepsilon}{1 + \varepsilon} \frac{\omega}{p}$$

on a set of measure  $\varepsilon |I|$ , and the average of  $w_{v-1}(2\omega, \cdot, \cdot)$  over the remaining set of measure  $\frac{1}{2}(1 - 3\varepsilon)|I|$  is  $2\omega$  by the inductive assumption. Thus

$$\begin{aligned} \frac{1}{|I|} \int_I w_v(\omega, \sigma, I) dx &= \frac{\omega}{p} \left( \frac{1 + \varepsilon}{2} + \frac{4\varepsilon^2}{1 + \varepsilon} \right) + \omega(1 - 3\varepsilon) \\ &= \omega + \left( \frac{1}{p} \left( \frac{1 + \varepsilon}{2} + \frac{4\varepsilon^2}{1 + \varepsilon} \right) - 3\varepsilon \right) \omega = \omega \end{aligned}$$

(it is this equation that was used to determine  $p$ ).

On the other hand,  $w_v^{-1}(\omega, \sigma, I)$  equals  $p/\omega = \sigma$  on a subset of  $I$  of measure  $\frac{1}{2}(1 + \varepsilon)|I|$  (the same set on which  $w_v$  is defined as  $\omega/p$ ), it equals

$$\frac{1 + \varepsilon}{4\varepsilon} \sigma$$

on a set of measure  $\varepsilon |I|$ , and its average over the remaining set of measure  $\frac{1}{2}(1 - 3\varepsilon)|I|$  equals  $\frac{1}{2}\sigma$ . Thus

$$\frac{1}{|I|} \int_I w_v^{-1}(\omega, \sigma, I) dx = \frac{1 + \varepsilon}{2} \sigma + \frac{1 + \varepsilon}{4} \sigma + \frac{(1 - 3\varepsilon)}{2} \frac{\sigma}{2} = \sigma,$$

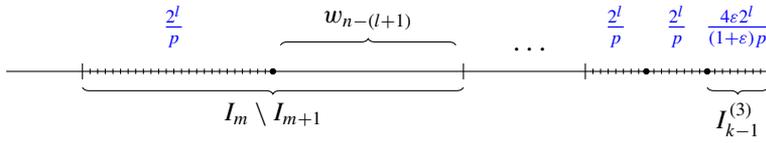
which completes the proof. □

In particular, it follows from [Proposition 2.1](#) that for the constructed weight  $w$ ,

$$\int_0^1 w dx = \int_0^1 w_n(1, p, [0, 1)) dx = 1.$$

**Proposition 2.2.** Let  $I = [a, a + h)$ . Then, for every  $v \geq 0$  and for all  $0 < \tau < h$ ,

$$\frac{1}{\tau} \int_a^{a+\tau} w_v(\omega, \sigma, I) \leq 3\omega \quad \text{and} \quad \frac{1}{\tau} \int_{a+h-\tau}^{a+h} w_v(\omega, \sigma, I) \leq \frac{9}{2}\omega. \tag{2-7}$$



**Figure 2.**  $w$  on some interval  $I$  carrying  $w_{n-l}$  for  $l < n$ .

*Proof.* For  $\nu = 0$  the statement is obvious since  $w_0(\omega, \sigma, I) \leq 2\omega$  on  $I$ . Assume that  $\nu \geq 1$ .

Since  $w_\nu(\omega, \sigma, I) = \omega/p$  on  $I^{(1)}$ , we have

$$\frac{1}{\tau} \int_a^{a+\tau} w_\nu(\omega, \sigma, I) = \frac{\omega}{p}$$

for  $0 < \tau < \frac{1}{3}h$ . But if  $\tau \geq \frac{1}{3}h$ , then, by Proposition 2.1,

$$\frac{1}{\tau} \int_a^{a+\tau} w_\nu(\omega, \sigma, I) dx \leq \frac{3}{|I|} \int_I w_\nu(\omega, \sigma, I) dx = 3\omega.$$

We now turn to the proof of the second estimate in (2-7). Let  $I_m, m = 0, 1, \dots, k - 1$ , be the intervals appearing in the definition of  $w_\nu(\omega, \sigma, I)$ . Since  $w_\nu(\omega, \sigma, I) \leq \omega/p$  on  $I_{k-1}$ , the estimate is trivial if  $a + h - \tau \in I_{k-1}$ . Assume that  $a + h - \tau \in I_m \setminus I_{m+1}, m = 0, \dots, k - 2$ . Then

$$\frac{1}{\tau} \int_{a+h-\tau}^{a+h} w_\nu(\omega, \sigma, I) dx \leq \frac{1}{|I_{m+1}|} \int_{I_m} w_\nu(\omega, \sigma, I) dx. \tag{2-8}$$

Next, by Proposition 2.1,

$$\begin{aligned} \int_{I_m} w_\nu(\omega, \sigma, I) dx &= \sum_{j=m}^{k-2} \left( \frac{\omega}{p} |I_j^{(1)}| + \int_{I_j^{(2)}} w_{\nu-1} \left( 2\omega, \frac{\sigma}{2}, I_j^{(2)} \right) dx \right) + \int_{I_{k-1}} w_\nu(\omega, \sigma, I) dx \\ &\leq \omega \sum_{j=m}^{k-1} |I_j| \leq \frac{3\omega}{2} \frac{|I|}{3^m} = \frac{9\omega}{2} |I_{m+1}|, \end{aligned} \tag{2-9}$$

which along with (2-8) completes the proof. □

Assume that  $I$  carries  $w_{n-l}$ ; see Figure 2. Consider the corresponding tail intervals contained in  $I$ , that is, the intervals on which

$$w = \frac{4\epsilon}{1 + \epsilon} \frac{2^j}{p}, \quad j = l, \dots, n - 1.$$

These intervals will play the central role in the estimate of the Hilbert transform of  $w \chi_{[0,1]}$ . There is only one tail interval in  $I \setminus \text{supp } w_{n-(l+1)}$ , and its measure equals  $(1/3^k)|I|$ . Next, there are  $k - 1$  tail intervals in

$$I \cap (\text{supp } w_{n-(l+1)} \setminus \text{supp } w_{n-(l+2)})$$

of total measure

$$\frac{1}{2} \left( 1 - \frac{1}{3^{k-1}} \right) \frac{1}{3^k} |I|.$$

Similarly, the measure of the union of tail intervals in

$$I \cap (\text{supp } w_{n-(l+j)} \setminus \text{supp } w_{n-(l+j+1)}), \quad j = 0, \dots, n-l-1,$$

is

$$\left(\frac{1}{2}\left(1 - \frac{1}{3^{k-1}}\right)\right)^j \frac{1}{3^k} |I|.$$

In particular, if we denote by  $A_l$  the union of tail intervals in

$$[0, 1) \cap (\text{supp } w_{n-l} \setminus \text{supp } w_{n-(l+1)}),$$

then

$$|A_l| = \left(\frac{1}{2}\left(1 - \frac{1}{3^{k-1}}\right)\right)^l \frac{1}{3^k}, \quad l = 0, \dots, n-1. \tag{2-10}$$

**Estimate of the maximal operator.** In this section, we will prove the first inequality in (2-2). We start with the reduction of this estimate to its triadic version.

Let  $\mathcal{T}$  be the standard triadic lattice; that is,

$$\mathcal{T} = \{[3^j n, 3^j(n+1)) : j, n \in \mathbb{Z}\}.$$

Denote by  $\mathcal{J}$  the family of all unions of two adjacent triadic intervals of equal length.

Our key tool will be the estimate

$$\|M\|_{L^2(\sigma) \rightarrow L^2(\sigma)} \leq 24 \sup_{J \in \mathcal{J}} \left(\frac{1}{w(J)} \int_J (M(w\chi_J))^2 \sigma\right)^{1/2}. \tag{2-11}$$

This estimate is fairly standard and well known. For the reader’s convenience, we supply the proof in the [Appendix](#).

Combining (2-11) with the inequality  $p \leq 2/\varepsilon \leq 6t$ , we see that in order to prove the first estimate in (2-2), it suffices to show that there exists  $C > 0$  such that, for every interval  $J \in \mathcal{J}$ ,

$$\int_J (M(w\chi_J))^2 \sigma \leq Cp^2 w(J). \tag{2-12}$$

Define an auxiliary 1-periodic function  $\tilde{w}$  by

$$\tilde{w}(x) = \sum_{l=1}^n 2^l \chi_{\text{supp } w_{n-(l-1)} \setminus \text{supp } w_{n-l}}(x) + 2^{n+1} \chi_{\text{supp } w_0}(x).$$

The role of this function is clarified in the following two propositions.

**Proposition 2.3.** For all  $x \in \mathbb{R}$ ,

$$Mw(x) \leq \frac{9}{2} \tilde{w}(x). \tag{2-13}$$

*Proof.* First, notice that for  $x \in \text{supp } w_0$  the statement is trivial. Indeed,  $w \leq 2^{n-1}/p$  on the complement of  $\text{supp } w_0$ , and if  $I$  carries  $w_0$ , then on  $I$

$$w_0 = \frac{2^n}{\sqrt{p}} \left( (\sqrt{p} + \sqrt{p-1}) \chi_{I_-} + \frac{1}{\sqrt{p} + \sqrt{p-1}} \chi_{I_+} \right).$$

Hence,

$$\|w\|_{L^\infty} \leq \frac{2^n(\sqrt{p} + \sqrt{p-1})}{\sqrt{p}} \leq 2^{n+1},$$

and therefore  $\|Mw\|_{L^\infty} \leq 2^{n+1}$ .

On the other hand, for  $x \in \text{supp } w_{n-(l-1)} \setminus \text{supp } w_{n-l}$ , the estimate (2-13) follows immediately from the facts that  $w \leq 2^{l-1}/p$  on the complement of  $\text{supp } w_{n-l}$  and, by Proposition 2.2, the average of  $w$  over the intersection of any interval  $J$  carrying  $w_{n-l}$  with an interval not contained in  $J$  is at most  $\frac{9}{2} \cdot 2^l$ .  $\square$

Recall that, given an interval  $I$ , we denoted by  $I_m$ ,  $m = 0, \dots, k-1$ , the interval with the same right endpoint as  $I$  of length  $|I_m| = (1/3^m)|I|$ . These intervals have already appeared in the definition of  $w_\nu(\omega, \sigma, I)$ .

**Proposition 2.4.** *Assume that  $I$  carries  $w_{n-l}$ . Then*

$$\int_{I_m} (\tilde{w})^2 \sigma \leq 30p^2 w(I_m), \quad l = 0, \dots, n, m = 0, \dots, k-2.$$

*Proof.* First, notice that the case when  $l = n$  is trivial, since  $2^{n+1} \leq 4pw(x)$  on any interval  $I$  carrying  $w_0$ , and hence,

$$\int_J (\tilde{w})^2 \sigma \leq 16p^2 w(J) \quad \text{for every } J \subset \text{supp } w_0. \tag{2-14}$$

Suppose now that  $l \leq n-1$  and consider first the case  $m = 0$ . Assume that  $I$  carries  $w_{n-l}$ . For  $j = 0, \dots, n-l-1$ , define

$$F_j = I \cap (\text{supp } w_{n-(l+j)} \setminus \text{supp } w_{n-(l+j+1)})$$

and let  $E_j$  be the union of the tail intervals contained in  $F_j$ . Observe that  $w = 2^{l+j}/p$  on  $F_j \setminus E_j$ , and hence,  $\tilde{w}(x) = 2pw(x)$  for  $x \in F_j \setminus E_j$ , which implies

$$\int_{\bigcup_j (F_j \setminus E_j)} (\tilde{w})^2 \sigma \leq 4p^2 w(I). \tag{2-15}$$

On the other hand,

$$w = \frac{4\varepsilon}{1+\varepsilon} \frac{2^{l+j}}{p}$$

on  $E_j$  and, as we have seen in the previous section,

$$|E_j| = \left( \frac{1}{2} \left( 1 - \frac{1}{3^{k-1}} \right) \right)^j \frac{1}{3^k} |I| \leq \frac{1}{2^j} \frac{1}{3^k} |I|.$$

Combining this with Proposition 2.1 yields

$$\begin{aligned} \int_{\bigcup_j E_j} (\tilde{w})^2 \sigma &\leq 4 \sum_{j=0}^{n-l-1} 2^{2(l+j)} \frac{(1+\varepsilon)p}{4\varepsilon 2^{l+j}} \frac{1}{2^j} \frac{1}{3^k} |I| \\ &\leq \frac{2p}{\varepsilon} \frac{n}{3^k} 2^l |I| \leq 4p^2 2^l |I| = 4p^2 w(I). \end{aligned} \tag{2-16}$$

Further, by (2-14),

$$\int_{I \cap \text{supp } w_0} (\tilde{w})^2 \sigma \leq 16p^2 w(I \cap \text{supp } w_0) \leq 16p^2 w(I).$$

Combining this estimate with (2-15) and (2-16), we obtain

$$\int_I (\tilde{w})^2 \sigma = \int_{\cup_j (F_j \setminus E_j)} (\tilde{w})^2 \sigma + \int_{\cup_j E_j} (\tilde{w})^2 \sigma + \int_{I \cap \text{supp } w_0} (\tilde{w})^2 \sigma \leq 24p^2 w(I), \tag{2-17}$$

and this completes the proof in the case  $m = 0$ .

Assume now that  $1 \leq m \leq k - 2$ . Notice that  $I_m \setminus I_{m+1} = I_m^{(1)} \cup I_m^{(2)}$ , where  $I_m^{(2)}$  carries  $w_{n-(l+1)}$ , and  $\tilde{w}(x) = 2pw(x)$  on  $I_m^{(1)}$ . Thus, by (2-17),

$$\begin{aligned} \int_{I_m \setminus I_{m+1}} (\tilde{w})^2 \sigma &\leq 4p^2 w(I_m^{(1)}) + 24p^2 w(I_m^{(2)}) \\ &\leq 24p^2 w(I_m \setminus I_{m+1}). \end{aligned} \tag{2-18}$$

Further,

$$\int_{I_{k-1}} (\tilde{w})^2 \sigma \leq 4(2^l)^2 \frac{(1 + \varepsilon)p}{4\varepsilon 2^l} |I_{k-1}| \leq 6p2^l |I|.$$

On the other hand, by Proposition 2.1,

$$w(I_m) \geq w(I_m^{(2)}) = 2^{l+1} |I_m^{(2)}| = \frac{2^{l+1}}{3^{m+1}} |I|,$$

which, combined with the previous estimate, implies

$$\int_{I_{k-1}} (\tilde{w})^2 \sigma \leq p3^{m+2} w(I_m) \leq \frac{p}{\varepsilon} w(I_m) \leq 6p^2 w(I_m).$$

Therefore, using (2-18), we obtain

$$\int_{I_m} (\tilde{w})^2 \sigma = \sum_{j=m}^{k-2} \int_{I_j \setminus I_{j+1}} (\tilde{w})^2 \sigma + \int_{I_{k-1}} (\tilde{w})^2 \sigma \leq 30p^2 w(I_m),$$

which completes the proof. □

We now turn to the proof of (2-12). Let  $J \in \mathcal{J}$ . First consider the simple case when  $|J| \geq 1$ . In this case,  $|J| = k$  for some  $k \in \mathbb{N}$ . Using that  $w$  and  $\tilde{w}$  are 1-periodic along with the fact that  $\int_0^1 w = 1$ , and combining Propositions 2.3 and 2.4, we obtain

$$\frac{1}{w(J)} \int_J (M(w\chi_J))^2 \sigma \leq 25 \int_0^1 (\tilde{w})^2 \sigma \leq 25 \cdot 30p^2.$$

Suppose that  $|J| < 1$ . We can represent  $J$  as the union of two triadic intervals  $J = J_- \cup J_+$ , where  $J_-, J_+ \in \mathcal{T}$  are the left and the right halves of  $J$  respectively. Since  $J_-$  is triadic, we must have  $|J_-| \leq \frac{1}{3}$ . Also, by the 1-periodicity of  $w$ , one can assume that  $J_- \subset [0, 1)$ .

Consider the case when  $J$  contains an interval carrying  $\text{supp } w_{n-(l+1)}$  for some  $l$ . Out of all such intervals choose the longest one. Note that since  $|J| \leq \frac{2}{3}$ , we must have  $l \geq 0$  in this case. Thus, the

interval in question must be of the kind  $R_m^{(2)}$ , where  $R$  is an interval carrying  $w_{n-l}$ . Since neither  $R = R_0$  nor  $R_{m-1}^{(2)}$  (if  $m \geq 1$ ) is contained in  $J$ , there are only three possible options:

- $J_- = R_m^{(2)}$ ,  $0 \leq m \leq k - 2$ ;
- $J_+ = R_m^{(2)}$ ,  $0 \leq m \leq k - 2$ ;
- $J_- = R_m$ ,  $1 \leq m \leq k - 2$ .

Suppose first that  $J_- = R_m^{(2)}$  or  $J_+ = R_m^{(2)}$ . Then  $J \subset R_m$ . By (2-9),  $w(R_m) \leq 3 \cdot 2^{l-1} |R_m|$ . On the other hand, since  $R_m^{(2)}$  carries  $w_{n-(l+1)}$ , by Proposition 2.1,

$$w(J) \geq w(R_m^{(2)}) = 2^{l+1} |R_m^{(2)}| = 2^{l+1} \frac{|R_m|}{3}, \tag{2-19}$$

which implies  $w(R_m) \leq \frac{9}{4} w(J)$ . Therefore, by Propositions 2.3 and 2.4,

$$\int_J (M(w\chi_J))^2 \sigma \leq 25 \int_{R_m} (\tilde{w})^2 \sigma \leq 25 \cdot 30 p^2 w(R_m) \leq 25 \cdot 75 p^2 w(J).$$

Assume now that  $J_- = R_m$ ,  $1 \leq m \leq k - 2$ . Then  $w \equiv 2^{l-1}/p$  on  $J_+$  if  $l > 0$  and  $w \equiv 2^l/p$  on  $J_+$  if  $l = 0$ . In either case,  $\tilde{w} = 2pw$  on  $J_+$ , so

$$\int_{J_+} (\tilde{w})^2 \sigma = 4p^2 w(J_+),$$

and thus, by Propositions 2.3 and 2.4,

$$\begin{aligned} \int_J (M(w\chi_J))^2 \sigma &\leq 25 \left( \int_{R_m} (\tilde{w})^2 \sigma + \int_{J_+} (\tilde{w})^2 \sigma \right) \\ &\leq 25(30 p^2 w(R_m) + 4 p^2 w(J_+)) \leq 25 \cdot 30 p^2 w(J). \end{aligned}$$

It remains to consider the case when  $J$  does not contain an interval carrying  $w_{n-(l+1)}$  for any  $0 \leq l \leq n - 1$ . Denote by  $E$  the union of all tail intervals appearing in the definition of  $w$ . Notice that if  $x \notin E$ , then

$$\sum_{l=1}^n 2^l \chi_{\text{supp } w_{n-(l-1)} \setminus \text{supp } w_{n-l}}(x) = 2pw(x) \chi_{\mathbb{R} \setminus \text{supp } w_0}.$$

Also,  $2^{n+1} \leq 4pw(x) \chi_{\text{supp } w_0}$ . From this and from Proposition 2.3,

$$Mw(x) \leq 18pw(x) \quad (x \notin E).$$

Therefore, if  $J \cap E = \emptyset$ ,

$$\frac{1}{w(J)} \int_J (M(w\chi_J))^2 \sigma \leq 18^2 p^2. \tag{2-20}$$

Suppose that  $J \cap E \neq \emptyset$ . Then there exists  $R$  carrying  $w_{n-l}$  for some  $0 \leq l \leq n - 1$  such that  $J \cap R_{k-1}^{(3)} \neq \emptyset$ . Since  $J_-$ ,  $J_+$  and  $R_{k-1}^{(3)}$  are triadic, we have that either one half of  $J$  is contained in  $R_{k-1}^{(3)}$

or  $R_{k-1}^{(3)} \subset J$ . Since  $J$  cannot contain any interval carrying  $\text{supp } w_{n-(l+1)}$ , in both cases we obtain that  $w$  can take only three possible values

$$\frac{2^l}{p}, \quad \frac{4\varepsilon 2^l}{(1+\varepsilon)p}, \quad \frac{2^{l-1}}{p}$$

on  $J$  and therefore,

$$\frac{1}{w(J)} \int_J (M(w\chi_J))^2 \sigma \leq \left( \frac{\sup_J w}{\inf_J w} \right)^2 \leq \left( \frac{1+\varepsilon}{4\varepsilon} \right)^2 \leq 9p^2,$$

which along with (2-20) implies

$$\frac{1}{w(J)} \int_J (M(w\chi_J))^2 \sigma \leq 18^2 p^2.$$

This completes the proof of (2-12), and therefore the first estimate in (2-2) is proved.

**Estimate of the Hilbert transform.** The goal of this section is to prove the second estimate in (2-2).

Denote by  $A_l^*$ ,  $l = 0, \dots, n-1$ , the union of all intervals  $\frac{1}{2}I$ , where  $I$  is a tail interval contained in

$$[0, 1) \cap (\text{supp } w_{n-l} \setminus \text{supp } w_{n-(l+1)}).$$

In other words,  $A_l^*$  is the union of all intervals  $\frac{1}{2}J_{k-1}^{(3)}$ , where  $J \subset [0, 1)$  carries  $w_{n-l}$ . Then, by (2-10),

$$|A_l^*| = \frac{1}{2}|A_l| = \frac{1}{2} \left( \frac{1}{2} \left( 1 - \frac{1}{3^{k-1}} \right) \right)^l \frac{1}{3^k}, \quad l = 0, \dots, n-1. \tag{2-21}$$

The sets  $A_l^*$  plays the central role in establishing the lower bound for  $H(w\chi_{[0,1)})$ , as the following proposition shows.

**Proposition 2.5.** *There exists an absolute  $C > 0$  such that for all  $l = 0, \dots, n-1$  and for every  $x \in A_l^*$*

$$|H(w\chi_{[0,1)})(x)| \geq Ck2^l. \tag{2-22}$$

Let us first show how to derive the second estimate in (2-2) from here. By (2-21) and (2-22),

$$\int_{A_l^*} |H(w\chi_{[0,1)})|^2 \sigma \geq C^2 k^2 2^{2l} \frac{1+\varepsilon}{4\varepsilon} \frac{p}{2^l} \frac{1}{2} \left( \frac{1}{2} \left( 1 - \frac{1}{3^{k-1}} \right) \right)^l \frac{1}{3^k}.$$

Therefore,

$$\begin{aligned} \|H(w\chi_{[0,1)})\|_{L^2(\sigma)}^2 &\geq \sum_{l=0}^{n-1} \int_{A_l^*} |H(w\chi_{[0,1)})|^2 \sigma \\ &\geq \frac{C^2}{8} k^2 p \sum_{l=0}^{n-1} \left( 1 - \frac{1}{3^{k-1}} \right)^l = \frac{C^2}{24} k^2 3^k p \left( 1 - \left( 1 - \frac{1}{3^{k-1}} \right)^n \right). \end{aligned}$$

Since  $n = 3^{k-1}$  and  $(1 - 1/n)^n < 1/e$ , we obtain

$$\|H(w\chi_{[0,1)})\|_{L^2(\sigma)}^2 \geq \frac{C^2(1-1/e)}{24} k^2 3^k p \geq \frac{C^2(1-1/e)}{144(\log 3)^2} t^2 \log^2 t.$$

Let us now turn to the proof of [Proposition 2.5](#). Let  $J = [a, b) \subset [0, 1)$  be an interval carrying  $w_{n-l}$ . Assume that  $x \in \frac{1}{2}J_{k-1}^{(3)}$ . Write

$$\begin{aligned} H(w\chi_{[0,1)})(x) &= H(w\chi_{[0,a)})(x) + \sum_{m=0}^{k-2} H(w\chi_{J_m \setminus J_{m+1}})(x) + H(w\chi_{J_{k-1} \setminus J_{k-1}^{(3)}})(x) + H(w\chi_{J_{k-1}^{(3)}})(x) + H(w\chi_{[b,1)})(x) \\ &\equiv A(x) + B(x) + C(x) + D(x) + E(x). \end{aligned}$$

We will show that there are absolute constants  $C_1$  and  $C_2$  such that, for all  $x \in \frac{1}{2}J_{k-1}^{(3)}$ ,

$$|B(x)| \geq C_1 k 2^l \quad \text{and} \quad \max\{|D(x)|, |E(x)|\} \leq C_2 2^l. \tag{2-23}$$

Since  $A(x)$ ,  $B(x)$  and  $C(x)$  are positive for all  $x \in \frac{1}{2}J_{k-1}^{(3)}$ , we obtain from [\(2-23\)](#) that

$$\begin{aligned} |H(w\chi_{[0,1)})(x)| &\geq |A(x) + B(x) + C(x)| - |D(x)| - |E(x)| \\ &\geq |B(x)| - |D(x)| - |E(x)| \geq \frac{C_1}{2} k 2^l \end{aligned}$$

for  $k > 4C_2/C_1$ .

Now let us prove the first estimate in [\(2-23\)](#). If  $y \in J_m \setminus J_{m+1}$  and  $x \in \frac{1}{2}J_{k-1}^{(3)}$ , then  $0 \leq x - y \leq |J_m|$ . Using also that  $J_m^{(2)} \subset J_m \setminus J_{m+1}$ , by [Proposition 2.1](#) we obtain

$$H(w\chi_{J_m \setminus J_{m+1}})(x) = \int_{J_m \setminus J_{m+1}} \frac{w(y)}{x - y} dy \geq \frac{w(J_m \setminus J_{m+1})}{|J_m|} \geq \frac{w(J_m^{(2)})}{|J_m|} = \frac{2}{3} 2^l.$$

Therefore,

$$B(x) > \frac{2}{3}(k - 1)2^l.$$

Turn to the second part of [\(2-23\)](#). Let  $J_{k-1}^{(3)} = [\alpha, b)$ . Then, for all  $x \in \frac{1}{2}J_{k-1}^{(3)}$ ,

$$\left| \int_{J_{k-1}^{(3)}} \frac{w(y)}{x - y} dy \right| = \frac{4\varepsilon}{(1 + \varepsilon)} \frac{2^l}{p} \left| \log \left| \frac{x - \alpha}{x - b} \right| \right| \leq 4(\log 3)\varepsilon \frac{2^l}{p} \leq 4(\log 3)2^l.$$

It remains to estimate  $|E(x)|$ . Take the intervals  $J^i = [a_i, b_i)$ ,  $i = 0, \dots, l$ , such that  $J^i$  carries  $w_{n-l+i}$  and

$$J = J^0 \subset J^1 \subset \dots \subset J^l = [0, 1).$$

We claim that, for every  $i = 1, \dots, l$  and for all  $x$  such that  $0 < x \leq b_{i-1} - |J^{i-1}|/(4 \cdot 3^k)$ ,

$$|H(w\chi_{[b_{i-1}, b_i)})(x)| \leq 13 \cdot 2^{l-i}. \tag{2-24}$$

Notice first that this claim immediately implies the desired estimate for  $E(x)$ . Indeed, let  $x \in \frac{1}{2}J_{k-1}^{(3)}$ . Then  $0 < x \leq b - |J|/(4 \cdot 3^k)$ , and hence [\(2-24\)](#) holds for  $i = 1$ . But since  $x \notin (J^i)_{k-1}$  for all  $i = 1, \dots, l$ , we obviously obtain that  $0 < x \leq b_{i-1} - |J^{i-1}|/(4 \cdot 3^k)$  for all  $i \leq l$ . Therefore, by [\(2-24\)](#),

$$|H(w\chi_{[b,1)})(x)| \leq \sum_{i=1}^l |H(w\chi_{[b_{i-1}, b_i)})(x)| \leq 13 \sum_{i=1}^l 2^{l-i} \leq 13 \cdot 2^l.$$

It remains to prove the claim. Set  $x_i = b_{i-1} - |J^{i-1}|/(4 \cdot 3^k)$ . Observe that  $|H(w\chi_{[b_{i-1}, b_i)})(x)|$  is an increasing function for  $x < b_{i-1}$ . Therefore, it suffices to prove that

$$|H(w\chi_{[b_{i-1}, b_i)})(x_i)| \leq 13 \cdot 2^{l-i}. \tag{2-25}$$

There exists  $0 \leq m \leq k - 2$  such that  $J^{i-1} = (J_m^i)^{(2)}$ . Then  $[b_{i-1}, b_i) = J_{m+1}^i$ . Let  $h = |J_{m+1}^i|$ . Split the integral in (2-25) as follows:

$$\int_{b_{i-1}}^{b_i} \frac{w(y)}{y - x_i} dy = \int_{b_{i-1}}^{b_{i-1}+h/3} \frac{w(y)}{y - x_i} dy + \int_{b_{i-1}+h/3}^{b_i} \frac{w(y)}{y - x_i} dy.$$

Using that  $w \equiv 2^{l-i}/p$  on  $[b_{i-1}, b_{i-1} + h/3)$ , we obtain

$$\int_{b_{i-1}}^{b_{i-1}+h/3} \frac{w(y)}{y - x_i} dy \leq \frac{2^{l-i}}{p} \frac{h}{3} \frac{4 \cdot 3^k}{h} \leq 8 \cdot 2^{l-i}.$$

Next, applying (2-9) yields

$$\int_{b_{i-1}+h/3}^{b_i} \frac{w(y)}{y - x_i} dy \leq \frac{3}{h} \frac{3}{2} 2^{l-i} |J_{m+1}^i| = \frac{9}{2} \cdot 2^{l-i},$$

which along with the previous estimate proves (2-25).

This proves the claim and so Proposition 2.5. Thus, Theorem 1.1 is completely proved.

### Appendix

In this section, we will show how to prove (2-11). Let us show first that, for every interval  $I \subset \mathbb{R}$ , there exists an interval  $J \in \mathcal{J}$  containing  $I$  and such that  $|J| \leq 6|I|$ . Indeed, let  $I = [a, a + h)$ . Fix  $j \in \mathbb{Z}$  such that  $3^{j-1} \leq h < 3^j$  and take  $n \in \mathbb{Z}$  such that

$$3^j n \leq a < 3^j(n + 1).$$

Then  $I \subset J = [3^j n, 3^j(n + 2))$ , and

$$\frac{|J|}{|I|} \leq \frac{2 \cdot 3^j}{3^{j-1}} = 6.$$

It follows immediately from this property that

$$Mf(x) \leq 6M^{\mathcal{J}}f(x), \tag{A-1}$$

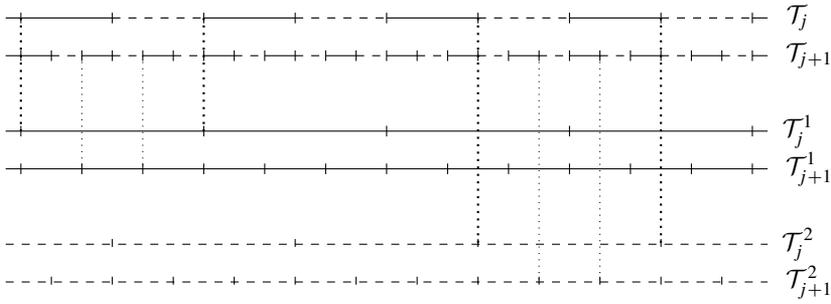
where

$$M^{\mathcal{J}}f(x) = \sup_{J \ni x, J \in \mathcal{J}} \frac{1}{|J|} \int_J |f| dy.$$

Next, it is easy to see that the intervals from  $\mathcal{J}$  can be split into two disjoint triadic lattices,  $\mathcal{J} = \mathcal{T}^1 \cup \mathcal{T}^2$  (see Figure 3 for a geometric illustration of this fact).

Therefore, by (A-1),

$$\|M\|_{L^2(\sigma) \rightarrow L^2(\sigma)} \leq 6(\|M^{\mathcal{T}^1}\|_{L^2(\sigma) \rightarrow L^2(\sigma)} + \|M^{\mathcal{T}^2}\|_{L^2(\sigma) \rightarrow L^2(\sigma)}). \tag{A-2}$$



**Figure 3.** The lattices  $\mathcal{T}$ ,  $\mathcal{T}^1$  and  $\mathcal{T}^2$  shown at two consecutive generations. The unions of solid and adjacent (from the right) dashed intervals from  $\mathcal{T}_j$  form  $\mathcal{T}_j^1$ , and the unions of dashed and adjacent (from the right) solid intervals from  $\mathcal{T}_j$  form  $\mathcal{T}_j^2$ . In turn, the unions of solid and dashed children from  $\mathcal{T}_{j+1}$  form  $\mathcal{T}_{j+1}^1$ , and the unions of dashed and solid children from  $\mathcal{T}_{j+1}$  form  $\mathcal{T}_{j+1}^2$ .

In order to estimate the right-hand side of (A-2), we invoke the following proposition.

**Proposition A.1.** *Let  $\mathfrak{T}$  be a triadic lattice. Then*

$$\|M^{\mathfrak{T}}\|_{L^2(\sigma) \rightarrow L^2(\sigma)} \leq 2 \sup_{R \in \mathfrak{T}} \left( \frac{1}{w(R)} \int_R (M^{\mathfrak{T}}(w\chi_R))^2 \sigma \right)^{1/2}.$$

**Remark A.2.** For dyadic lattices this result can be found in [Moen 2009]. The proof there is closely related to the approach by E. Sawyer [1982] in his two-weighted characterization for the maximal operator. For triadic lattices the proof is essentially the same, and we give it for the sake of completeness.

*Proof of Proposition A.1.* Let  $a > 1$ . For  $k \in \mathbb{Z}$  write the set  $\Omega_k = \{M^{\mathfrak{T}} f > a^k\}$  as the union of pairwise disjoint triadic intervals  $I_j^k$  such that

$$\frac{1}{|I_j^k|} \int_{I_j^k} |f| > a^k.$$

Define  $E_j^k = I_j^k \setminus \Omega_{k+1}$ , and set  $\alpha_{j,k} = (w(I_j^k)/|I_j^k|)^2 \sigma(E_j^k)$ . We have

$$\begin{aligned} \|M^{\mathfrak{T}} f\|_{L^2(\sigma)}^2 &= \sum_{k \in \mathbb{Z}} \int_{\Omega_k \setminus \Omega_{k+1}} (M^{\mathfrak{T}} f)^2 \sigma \leq a^2 \sum_{k,j} \left( \frac{1}{|I_j^k|} \int_{I_j^k} |f| \right)^2 \sigma(E_j^k) \\ &= a^2 \sum_{k,j} \left( \frac{1}{w(I_j^k)} \int_{I_j^k} |f \sigma| w \right)^2 \alpha_{j,k}. \end{aligned} \tag{A-3}$$

Notice that for every  $R \in \mathfrak{T}$ ,

$$\sum_{j,k: I_j^k \subset R} \alpha_{j,k} \leq \int_R (M^{\mathfrak{T}}(w\chi_R))^2 \sigma \leq N^2 w(R), \tag{A-4}$$

where

$$N = \sup_{R \in \mathfrak{T}} \left( \frac{1}{w(R)} \int_J (M^{\mathfrak{T}}(w\chi_R))^2 \sigma \right)^{1/2}.$$

For  $\lambda > 0$  set

$$E_\lambda = \left\{ (j, k) : \left( \frac{1}{w(I_j^k)} \int_{I_j^k} |f\sigma|w \right)^2 > \lambda \right\}.$$

Define the weighted maximal operator  $M_w^{\mathfrak{T}}$  by

$$M_w^{\mathfrak{T}} f(x) = \sup_{J \ni x, J \in \mathfrak{T}} \frac{1}{w(J)} \int_J |f|w \, dy.$$

Writing the set  $\{x : M_w^{\mathfrak{T}}(f\sigma)^2(x) > \lambda\}$  as the union of the maximal pairwise disjoint triadic intervals  $\bigcup_i R_i$  and applying (A-4), we obtain

$$\sum_{(j,k) \in E_\lambda} \alpha_{j,k} \leq \sum_i \sum_{j,k: I_j^k \subset R_i} \alpha_{j,k} \leq N^2 w\{x : M_w^{\mathfrak{T}}(f\sigma)^2(x) > \lambda\}.$$

Therefore,

$$\begin{aligned} \sum_{k,j} \left( \frac{1}{w(I_j^k)} \int_{I_j^k} |f\sigma|w \right)^2 \alpha_{j,k} &= \int_0^\infty \left( \sum_{(j,k) \in E_\lambda} \alpha_{j,k} \right) d\lambda \\ &\leq N^2 \|M_w^{\mathfrak{T}}(f\sigma)\|_{L^2(w)}^2 \leq 4N^2 \|f\sigma\|_{L^2(w)}^2 = 4N^2 \|f\|_{L^2(\sigma)}^2, \end{aligned}$$

which, along with (A-3), completes the proof. □

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