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FOR SCHRÖDINGER OPERATORS**

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We develop refined Strichartz estimates at L^2 regularity for a class of time-dependent Schrödinger operators. Such refinements quantify near-optimizers of the Strichartz estimate and play a pivotal part in the global theory of mass-critical NLS. On one hand, the harmonic analysis is quite subtle in the L^2 -critical setting due to an enormous group of symmetries, while on the other hand, the space-time Fourier analysis employed by the existing approaches to the constant-coefficient equation are not adapted to nontranslation-invariant situations, especially with potentials as large as those considered in this article.

Using phase-space techniques, we reduce to proving certain analogues of (adjoint) bilinear Fourier restriction estimates. Then we extend Tao's bilinear restriction estimate for paraboloids to more general Schrödinger operators. As a particular application, the resulting inverse Strichartz theorem and profile decompositions constitute a key harmonic analysis input for studying large-data solutions to the L^2 -critical NLS with a harmonic oscillator potential in dimensions ≥ 2 . This article builds on recent work of Killip, Visan, and the author in one space dimension.

1. Introduction

We prove sharpened forms of the Strichartz inequality for nontranslation-invariant linear Schrödinger equations with L^2 initial data. Recall that solutions to the linear constant-coefficient Schrödinger equation

$$i \partial_t u = -\frac{1}{2} \Delta u, \quad u(0, \cdot) = u_0 \in L^2(\mathbb{R}^d), \quad (1)$$

satisfy the Strichartz inequality [1977]

$$\|u\|_{L_{t,x}^{2(d+2)/d}(\mathbb{R} \times \mathbb{R}^d)} \leq C \|u(0, \cdot)\|_{L^2(\mathbb{R}^d)}. \quad (2)$$

On the other hand, it is also known if u a solution that comes close to saturating this inequality, then it must exhibit some “concentration”; see [Carles and Keraani 2007; Merle and Vega 1998; Moyua et al. 1999; Bégout and Vargas 2007]. Such inverse theorems may be equivalently formulated as a refined estimate

$$\|u\|_{L_{t,x}^{2(d+2)/d}} \lesssim \|u\|_X^\theta \|u(0, \cdot)\|_{L^2(\mathbb{R}^d)}^{1-\theta}, \quad (3)$$

where the norm X is weaker than the right side of (2) but measures the “microlocal concentration” of the solution. We pursue analogues of such refinements when the right side of (1) is replaced by a more general Schrödinger operator $-\frac{1}{2} \Delta + V(t, x)$.

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Inverse theorems for the Strichartz inequality have provided a key input to the study of the L^2 -critical NLS

$$i\partial_t u = -\frac{1}{2}\Delta u \pm |u|^{\frac{4}{d}}u, \quad u(0, \cdot) \in L^2(\mathbb{R}^d), \quad (4)$$

so termed because the rescaling $u \mapsto u_\lambda(t, x) := \lambda^{d/2}u(\lambda^2t, \lambda x)$ preserves both (1) and the L^2 -norm $M[u] := \|u(t)\|_{L^2(\mathbb{R}^d)} = \|u(0)\|_{L^2(\mathbb{R}^d)}$. Indeed, they are used to construct the profile decompositions underpinning the Bourgain–Kenig–Merle concentration compactness and rigidity method by identifying potential blowup scenarios for nonlinear solutions with large data. Using this method, the large-data global regularity problem for (4) was recently settled by Dodson [2012; 2015; 2016a; 2016b], building on earlier work of Killip, Visan, Tao, and Zhang [Killip et al. 2008; 2009; Tao et al. 2007]. For further discussion of this equation we refer the interested reader to the lecture notes [Killip and Visan 2013].

The large group of symmetries for the inequality (2) is a significant obstruction to characterizing its near-optimizers. Besides translation and scaling symmetry, both sides are also invariant under *Galilei* transformations

$$u \mapsto u_{\xi_0}(t, x) := e^{i[(x, \xi_0) - \frac{1}{2}t|\xi_0|^2]}u(t, x - t\xi_0), \quad \xi_0 \in \mathbb{R}^d.$$

This last symmetry emerges only at L^2 regularity and creates an additional layer of complexity. In particular, while the Littlewood–Paley decomposition is extremely well-adapted to higher Sobolev regularity variants of (2), such as the \dot{H}^1 -critical estimate

$$\|u\|_{L_{t,x}^{2(d+2)/(d-2)}} \lesssim \|\nabla u(0)\|_{L^2(\mathbb{R}^d)},$$

it is useless for inverting the L^2 -critical estimate because one has no a priori knowledge of where the solution is concentrated in frequency. Instead, the mass-critical refinements cited above combine space-time Fourier-analytic arguments with restriction theory for the paraboloid.

In physical applications, one is naturally led to consider variants of the mass-critical equation (4) with external potentials, such as the harmonic oscillator

$$i\partial_t u = \left(-\frac{1}{2}\Delta + \sum_j \omega_j^2 x_j^2\right)u \pm |u|^{\frac{4}{d}}u, \quad u(0, \cdot) \in L^2(\mathbb{R}^d). \quad (5)$$

For instance, the cubic equation (with a $|u|^2u$ nonlinearity) has been proposed as a model for Bose–Einstein condensates in a laboratory trap [Zhang 2000], where $\|u(t)\|_{L^2}$ represents the total number of particles, and in two space dimensions the critical Sobolev norm for this equation is precisely L^2 .

While introducing the potential breaks scaling symmetry, one nonetheless expects solutions with highly concentrated initial data to be approximated, for short times, by solutions to the scale-invariant equation (4). Less obviously, the equation is invariant under “generalized” Galilei boosts, detailed in [Lemma 1.1](#) below, where the spatial and frequency parameters act together on the solutions; in the constant-coefficient setting, this reduces to the usual independent space translation and Galilei boost symmetries.

This article develops refined Strichartz estimates for the linear equation

$$i\partial_t u = \left(-\frac{1}{2}\Delta + V\right)u, \quad u(0, \cdot) \in L^2(\mathbb{R}^d),$$

for a class \mathcal{V} of real-valued potentials $V(t, x)$ that merely satisfy similar bounds as the harmonic oscillator and possibly also depend on time. Specifically, define

$$\mathcal{V} := \{V : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R} : \|\partial_x^\alpha V\|_{L_{t,x}^\infty} \leq M_{|\alpha|} \text{ for } 2 \leq |\alpha| \leq N = N(d)\} \quad (6)$$

for fixed constants $0 < M_1, M_2, \dots, M_N$. These estimates play a key role in the large-data theory for nontranslation-invariant L^2 -critical Cauchy problems typified by (5). We briefly discuss the nonlinear problem in the last section of the introduction.

The case of one space dimension was treated in a previous joint work with Killip and Visan [Jao et al. 2019]. This paper extends the methods introduced there to higher dimensions.

1A. The setup. To clarify the structure of our arguments we begin with a slightly more general setup. Hence we consider time-dependent, real-valued symbols $a(t, x, \xi)$ which are measurable in t and satisfy

$$|\partial_x^\alpha \partial_\xi^\beta a| \leq c_{\alpha\beta} \quad \text{for all } |\alpha| + |\beta| \geq 2. \quad (7)$$

Further, we assume the characteristic curvature condition

$$||\det a_{\xi\xi}| - 1| + ||\|a_{\xi\xi}\| - 1| \leq \varepsilon \quad (8)$$

for some small $0 < \varepsilon < 1$. For concreteness, all matrix norms in this article denote the Hilbert–Schmidt norm, but the exact choice of norm is inessential.

These hypotheses encompass several interesting situations:

- Schrödinger Hamiltonians with time-dependent scalar potentials $a = \frac{1}{2}|\xi|^2 + V(t, x)$, where $V \in \mathcal{V}$.
- Electromagnetic-type symbols $a = \frac{1}{2}|\xi|^2 + b(x, \xi) + V(t, x)$, where the first-order symbol $b(x, \xi)$ is real and satisfies $|\partial_x^\alpha \partial_\xi^\beta b| \leq c_{\alpha\beta}$ for all $|\alpha| + |\beta| \geq 1$, and $V \in \mathcal{V}$ is a scalar potential as before.
- The frequency-1 portion of the Laplacian on a curved background.

For a symbol as defined above, write $a^w(t, x, D)$ for its Weyl quantization. Let $U(t, s)$ denote its unitary propagator on $L^2(\mathbb{R}^d)$, so that $u := U(t, s)u_s$ is the solution to the equation

$$(D_t + a^w(t, x, D))u = 0, \quad u(s, \cdot) = u_s \in L^2(\mathbb{R}^d). \quad (9)$$

Evolution equations of this type were studied in [Koch and Tataru 2005]. While translations and modulations do not preserve (9), they do preserve the class of equations defined by our assumptions. For an element (x_0, ξ_0) of classical phase space, define the “phase-space translation” operator $\pi(x_0, \xi_0)$ by

$$\pi(z_0) f(x) = e^{i\langle x - x_0, \xi_0 \rangle} f(x - x_0).$$

Then a direct computation, as in the proof of [Koch and Tataru 2005, Proposition 4.3], yields:

Lemma 1.1. *If $U(t, s)$ is the propagator for the symbol a and $\sigma \mapsto z^\sigma = (x^\sigma, \xi^\sigma)$ is a bicharacteristic of a , then*

$$U(t, s)\pi(z_0^s)f = e^{i(\phi(t, z_0) - \phi(s, z_0))}\pi(z_0^t)U^{z_0}(t, s),$$

where U^{z_0} is the propagator for the equation

$$\begin{aligned} [D_t + (a^{z_0})^w(t, x, D)]u &= 0, \\ a^{z_0}(t, z) &= a(t, z_0^t + z) - \langle x, a_x(t, z_0^t) \rangle - \langle \xi, a_\xi(t, z_0^t) \rangle - a(z_0^t), \end{aligned}$$

and the phase is defined by

$$\phi(t, z_0) = \int_0^t \langle a_\xi(\tau, z_0^\tau), \xi_0^\tau \rangle - a(\tau, z_0^\tau) d\tau.$$

Observe that the transformed symbol a^{z_0} satisfies the same estimates assumed of a . As a special case, symbols of the form $a = \frac{1}{2}|\xi|^2 + \langle A(t, x), \xi \rangle + \omega_{jk}(t)x^j x^k$ are themselves preserved by the mapping $a \mapsto a^{z_0}$ if $A = A_j dx^j$ is a 1-form whose components are linear functions of the space variables with time-dependent coefficients. In two and three space dimensions, such A are potentials for uniform magnetic fields.

The preceding hypotheses imply that (9) satisfies a local-in-time dispersive estimate:

Lemma 1.2. *If the symbol a satisfies the conditions (7) and (8), there exists $T_0 > 0$ such that the propagator $U(t, s)$ for the evolution equation (9) satisfies the estimate*

$$\|U(t, s)\|_{L_x^1 \rightarrow L_x^\infty} \lesssim |t - s|^{-\frac{d}{2}} \quad \text{for all } |t - s| \leq T_0. \quad (10)$$

Hence, the solutions to (9) satisfy local-in-time Strichartz estimates

$$\|u\|_{L_t^q L_x^r(I \times \mathbb{R}^d)} \lesssim_{|I|} \|u_s\|_{L^2(\mathbb{R}^d)}$$

for any compact time interval I , and for all Strichartz exponents (q, r) satisfying $2 \leq q, r \leq \infty$, $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$, and $(q, r, d) \neq (2, \infty, 2)$.

Proof sketch. The dispersive estimate is shown in [Koch and Tataru 2005, Proposition 4.7] using wavepacket parametrices. Standard arguments (see [Ginibre and Velo 1995; Keel and Tao 1998]) then yield the Strichartz estimates. \square

It suffices to choose the time increment T_0 so that

$$T_0 \leq 1, \quad T_0 \|a_{x\xi}\| + T_0^2 \|a_{xx}\| \leq \eta, \quad (11)$$

where $\eta = \eta(d)$ is a small parameter depending only on the dimension.

Remark. The concrete cases of scalar potentials and magnetic potentials were studied much earlier by Fujiwara [1979] and Yajima [1991], respectively, who proved the dispersive bound using Fourier integral parametrices.

We seek refinements of the Strichartz inequality analogous to those for the constant-coefficient equation. The earlier arguments for the constant-coefficient equation relied crucially on subtle bilinear estimates from Fourier restriction theory. We isolate and reformulate the technical lynchpin in the present context.

Hypothesis 1. *There exist $T_0 > 0$ and $1 < p < \frac{d+2}{d}$ such that the following holds: if $f, g \in L^2(\mathbb{R}^d)$ have frequency supports in sets of diameter $\lesssim N$ which are separated by distance $\sim N$, then*

$$\|U_\lambda^s(t) f U_\lambda^s(t) g\|_{L_{t,x}^p([-T_0, T_0] \times \mathbb{R}^d)} \lesssim N^{-\delta} \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)} \quad (12)$$

for all $s \in [-1, 1]$ and all $0 < \lambda \leq 1$, where $U_\lambda^s(t) = U_\lambda^s(t, 0)$ are the propagators for the time-translated and rescaled symbols $a_\lambda^s := \lambda^2 a(s + \lambda^2 t, \lambda x, \lambda^{-1} \xi)$.

When $a = \frac{1}{2}|\xi|^2$, the scaling and translation parameters λ, s are extraneous, and inequalities of the form (12) are called (adjoint) bilinear Fourier restriction estimates. They were utilized in [Béginout and Vargas 2007] to obtain mass-critical Strichartz refinements in dimension 3 and higher (the results in dimensions 1 and 2, due to Carles and Keraani [2007], Merle and Vega [1998], and Moyua, Vargas and Vega [Moyua et al. 1999] utilized linear restriction estimates). For further discussion of such estimates, see for instance [Tao 2003].

In the first part of this paper, we connect (12) to Strichartz refinements. To measure concentration in the solution we test it against scaled, modulated, and translated wavepackets. Set

$$\psi(x) = c_d e^{-\frac{|x|^2}{2}}, \quad \psi_{x_0, \xi_0} = \pi(x_0, \xi_0) \psi, \quad c_d = 2^{-\frac{d}{2}} \pi^{-\frac{3d}{4}}, \quad (13)$$

where S_λ is the unitary rescaling $S_\lambda f(x) := \lambda^{-d/2} f(\lambda^{-1} x)$.

Theorem 1.3. *If Hypothesis 1 holds, then there exists $0 < \theta < 1$ such that for all initial data $u_0 \in L^2(\mathbb{R}^d)$ the solution u to (9) satisfies*

$$\|u\|_{L^{2(d+2)/d}([-1, 1] \times \mathbb{R}^d)} \lesssim \left(\sup_{0 < \lambda \leq 1, |t| \leq 1, (x_0, \xi_0) \in T^* \mathbb{R}^d} |\langle S_\lambda \psi_{x_0, \xi_0}, u(t) \rangle_{L^2(\mathbb{R}^d)}| \right)^\theta \|u_0\|_{L^2(\mathbb{R}^d)}^{1-\theta}. \quad (14)$$

The generality of our hypotheses requires us to formulate the estimates locally in time. Indeed, for most potentials the left side of the Strichartz estimate (14) is infinite if one integrates over $\mathbb{R} \times \mathbb{R}^d$; for instance, the harmonic oscillator potential $V = |x|^2$ admits periodic-in-time solutions. Nonetheless, our methods do yield (a new proof of) a global-in-time refined Strichartz estimate

$$\|u\|_{L_{t,x}^{2(d+2)/d}(\mathbb{R} \times \mathbb{R}^d)} \lesssim \left(\sup_{\lambda > 0, t \in \mathbb{R}, (x_0, \xi_0) \in T^* \mathbb{R}^d} |\langle S_\lambda \psi_{x_0, \xi_0}, u(t) \rangle_{L^2(\mathbb{R}^d)}| \right)^\theta \|u_0\|_{L^2(\mathbb{R}^d)}^{1-\theta}$$

for solutions to the constant-coefficient equation (1).

In applications to PDEs, such a refined estimate is nowadays interpreted in the framework of concentration compactness and yields profile decompositions via repeated application of the following:

Lemma 1.4. *Assume the estimate (14) holds. Let $u_n := U(t) f_n$ be a sequence of linear solutions with initial data $u_n(0) = f_n \in L^2(\mathbb{R}^d)$ such that $\|f_n\|_{L^2(\mathbb{R}^d)} \leq A < \infty$ and $\|u_n\|_{L_{t,x}^{2(d+2)/d}} \geq \varepsilon > 0$. Then, after passing to a subsequence, there exist parameters*

$$\{(\lambda_n, t_n, x_n, \xi_n)\}_n \subset (0, 1] \times [-1, 1] \times \mathbb{R}_x^d \times \mathbb{R}_\xi^d$$

and a function $0 \neq \phi \in L^2(\mathbb{R}^d)$ such that

$$\begin{aligned} \pi(x_n, \xi_n)^{-1} S_{\lambda_n}^{-1} u_n &\rightharpoonup \phi \quad \text{in } L^2, \\ \|\phi\|_{L^2} &\gtrsim \varepsilon \left(\frac{\varepsilon}{A} \right)^{\frac{1-\theta}{\theta}}. \end{aligned}$$

Further,

$$\|f_n\|_{L^2}^2 - \|f_n - U(t_n)^{-1} S_{\lambda_n} \pi(x_n, \xi_n) S_{\lambda_n} \phi\|_{L^2}^2 - \|U(t_n)^{-1} S_{\lambda_n} \pi(x_n, \xi_n) S_{\lambda_n} \phi\|_{L^2}^2 \rightarrow 0.$$

Proof. By the estimate (14), there exist $\lambda_n, t_n, x_n, \xi_n$ such that

$$|\langle S_{\lambda_n} \psi_{x_n, \xi_n}, U(t_n) f_n \rangle| = |\langle \psi, \pi(x_n, \xi_n)^{-1} S_{\lambda_n}^{-1} U(t_n) f_n \rangle| \gtrsim \varepsilon \left(\frac{\varepsilon}{A} \right)^{\frac{1-\theta}{\theta}}.$$

The sequence $\pi(x_n, \xi_n)^{-1} S_{\lambda_n}^{-1} U(t_n) f_n$ is bounded in L^2 and therefore converges weakly in L^2 to some ϕ after passing to a subsequence. The lower bound on $\|\phi\|_{L^2}$ is immediate, while

$$\begin{aligned} \|f_n\|_{L^2}^2 - \|f_n - U(t_n)^{-1} S_{\lambda_n} \pi(x_n, \xi_n) \phi\|_{L^2}^2 - \|U(t_n)^{-1} S_{\lambda_n} \pi(x_n, \xi_n) \phi\|_{L^2}^2 \\ = 2 \operatorname{Re} \langle f_n - U(t_n)^{-1} S_{\lambda_n} \pi(x_n, \xi_n) \phi, U(t_n)^{-1} S_{\lambda_n} \pi(x_n, \xi_n) \phi \rangle \\ = 2 \operatorname{Re} \langle \pi(x_n, \xi_n)^{-1} S_{\lambda_n}^{-1} U(t_n) f_n - \phi, \phi \rangle \rightarrow 0. \end{aligned} \quad \square$$

Further discussion of profile decompositions and inverse Strichartz theorems may be found in the lecture notes [Killip and Visan 2013].

In the second part of this paper, we verify Hypothesis 1 for scalar potentials.

Theorem 1.5. *Consider a Schrödinger operator of the form $H(t) = -\frac{1}{2}\Delta + V(t, x)$, where $V \in \mathcal{V}$. Suppose $S_1, S_2 \subset \mathbb{R}_\xi^d$ are subsets of Fourier space with $\operatorname{diam}(S_j) \leq N$ and $c^{-1}N \geq \operatorname{dist}(S_1, S_2) \geq cN$ for some $0 < c < 1$. There exists a constant $\eta = \eta(c) > 0$ such that if $\tau_0 > 0$ satisfies*

$$(\tau_0 + \tau_0^2) \|\partial_x^2 V\|_{L^\infty} < \eta,$$

then, for any $f, g \in L^2(\mathbb{R}^d)$ with $\operatorname{supp}(\hat{f}) \subset S_1$ and $\operatorname{supp}(\hat{g}) \subset S_2$, the corresponding linear solutions $u = U(t, 0)f$ and $v = U(t, 0)g$ satisfy the estimate

$$\|uv\|_{L^q([-T_0, T_0] \times \mathbb{R}^d)} \lesssim_\varepsilon N^{d - \frac{d+2}{q} + \varepsilon} \|f\|_{L^2} \|g\|_{L^2} \quad \text{for all } \frac{d+3}{d+1} \leq q < \frac{d+2}{d}, \quad (15)$$

for any $\varepsilon > 0$, $N \geq 1$, and $V \in \mathcal{V}$.

For $V = 0$, the above estimate was conjectured by Klainerman and Machedon without the epsilon loss, and first proved in [Wolff 2001] for the wave equation and subsequently in [Tao 2003] for the Schrödinger equation (both with the epsilon loss). Strictly speaking, the time truncation is not present in the original formulations of those estimates, but may be easily removed by a rescaling and limiting argument.

Finally, while we make no attempt to address general magnetic potentials, a simple case with some physical relevance does essentially follow from the proof for scalar potentials. The necessary modifications for the following theorem are sketched in the last section.

Theorem 1.6. *The conclusion of the previous theorem holds for Schrödinger operators of the form $H(t) = -\frac{1}{2}(\nabla - iA)^2 + V(t, x)$, where $A = A_j dx^j$ is a 1-form whose components are linear in the space variables (i.e., the vector potential for a uniform magnetic field), and the condition on the time increment τ_0 is replaced by*

$$\tau_0 \|a_{x\xi}\| + (\tau_0 + \tau_0^2) \|a_{xx}\| < \eta.$$

We remark that the restriction estimate (12) does *not* hold for all symbols satisfying the conditions (7) and (8). For instance, it was observed by in [Vargas 2005] that when $U(t) = e^{it\partial_x \partial_y}$ is the “nonelliptic” Schrödinger propagator in two space dimensions (thus $a = \xi_x \xi_y$), the bilinear restriction estimate (7) can fail unless the frequency supports of the two inputs are not only disjoint but also separated in both Fourier coordinates. In fact, the refinement (14) as stated is *false* for the nonelliptic equation; for a correct formulation, one should enlarge the symmetry group on the right side to include the hyperbolic rescalings $u(x, y) \mapsto u(\mu x, \mu^{-1} y)$; see [Rogers and Vargas 2006].

While the classical bicharacteristics of elliptic and nonelliptic propagators seemingly have no qualitative difference—and indeed the dispersive estimates hold equally well for both—the quantum propagators have radically different behavior in terms of oscillations in time. If one compares the traveling wave solutions

$$e^{i[x\xi_x + y\xi_y - \frac{t}{2}(\xi_x^2 + \xi_y^2)]}, \quad e^{i[x\xi_x + y\xi_y - t\xi_x \xi_y]},$$

it is evident that unlike in the elliptic case two solutions to the nonelliptic equation which are well-separated in spatial frequency need not decouple in time.

The lesson of this counterexample is that while the dispersive and Strichartz estimates follow directly from properties of the classical Hamiltonian flow, an *inverse* Strichartz estimate depends more subtly on the temporal oscillations of the quantum evolution, which is connected to the bilinear decoupling estimates.

1B. The main ideas. Suppose one has initial data $u_0 \in L^2$ such that the corresponding solution u has nontrivial Strichartz norm. Then, we need to identify a bubble of concentration in u , characterized by several parameters that reflect the underlying symmetries in the problem. In the L^2 -critical setting, the relevant features consist of a significant length scale λ_0 as well as the position x_0 , frequency ξ_0 , and time t_0 when concentration occurs.

The existing proofs of Strichartz refinements for the constant-coefficient equation first use space-time Fourier analysis (including restriction estimates) to identify a cube Q in Fourier space accounting for a significant portion of the space-time norm of u , which reveals the frequency center ξ_0 and scale λ_0 of the concentration. For example, [Bégout and Vargas 2007] first establishes an estimate of the form

$$\|e^{\frac{it\Delta}{2}} f\|_{L^{2(d+2)/d}} \lesssim \left(\sup_{Q \text{ dyadic cubes}} |Q|^{1-\frac{p}{2}} \int_Q |\hat{f}(\xi)|^p d\xi \right)^\mu \|f\|_{L^2(\mathbb{R}^d)}^{1-\mu p}.$$

Then, the time t_0 and position x_0 are recovered via a separate physical-space argument. These arguments ultimately rely on the fact that when $V = 0$, the equation is diagonalized by the Fourier transform.

For equations with variable coefficients, it is more natural to consider position x_0 and frequency ξ_0 together as a point in phase space, which propagates along the bicharacteristics for the equation. Following the approach in [Jao et al. 2019] for the one-dimensional equation, we work in the physical space and first isolate a significant time interval $[t_0 - \lambda_0^2, t_0 + \lambda_0^2]$, which also suggests a characteristic scale λ_0 . Then x_0 and ξ_0 are recovered by phase-space techniques.

The first part of the argument in [Jao et al. 2019] carries over essentially unchanged; however, the ensuing phase-space analysis in higher dimensions is more involved and occupies the bulk of this article.

1C. An application to mass-critical NLS. This article was originally motivated by the problem of proving global well-posedness for the mass-critical quantum harmonic oscillator

$$i\partial_t u = \left(-\frac{1}{2}\Delta + \sum_j \omega_j^2 x_j^2 \right) u \pm |u|^{\frac{4}{d}} u. \quad (16)$$

By spectral theory, the Cauchy problem for (16) is naturally posed in the “harmonic” Sobolev spaces

$$u_0 \in \mathcal{H}^s := \left\{ u_0 \in L^2 : \left(-\Delta + \sum_j \omega_j^2 |x|^2 \right)^{\frac{s}{2}}, u_0 \in L^2 \right\}.$$

Global existence for data in the “energy” space \mathcal{H}^1 was studied in [Zhang 2005]. More recently, Poiret, Robert, and Thomann [Poiret et al. 2014] established probabilistic well-posedness in two space dimensions for all subcritical cases $0 < s < 1$, as well as for other supercritical problems. Another recent contribution by Burq, Thomann, and Tzvetkov [Burq et al. 2013] constructs Gibbs measures and proves probabilistic global well-posedness for the critical case in one dimension.

It is well-known that the *isotropic* harmonic oscillator $\omega_j \equiv \frac{1}{2}$ may be “trivially” solved; to construct solutions on unit-length time intervals for arbitrary L^2 data, it suffices to observe that u is a solution of (4) on $\mathbb{R}_t \times \mathbb{R}_x^d$ if and only if its *Lens transform*

$$\mathcal{L}u(t, x) := \frac{1}{(\cos t)^{\frac{d}{2}}} u\left(\tan t, \frac{x}{\cos t}\right) e^{-\frac{i|x|^2 \tan t}{2}}$$

solves (16) on $(-\frac{\pi}{2} \times \frac{\pi}{2})_t \times \mathbb{R}_x^d$ with the same initial data. However, this trick relies on algebraic cancellations that no longer hold for more general harmonic oscillators. For further discussion of the nonlinear harmonic oscillator as well as its connection with the Lens transform, consult [Carles 2011].

To solve (16) for large data in the critical space L^2 , the concentration compactness and rigidity approach is much more promising. Experience has shown that constructing suitable profile decompositions is a core difficulty in implementing this strategy for dispersive equations with broken symmetries (e.g., loss of translation-invariance). For instance, see [Jao 2016] for the energy-critical variant of the quantum harmonic oscillator, as well as [Ionescu et al. 2012; Killip et al. 2016] for other energy-critical NLS on non-Euclidean domains. Thus this article supplies the main harmonic analysis input for the deterministic large-data theory of (16) at the critical regularity.

2. Preliminaries

2A. Notation. We use the Japanese bracket notation $\langle x \rangle := (1 + |x|^2)^{1/2}$.

2B. Classical flow estimates. We collect some elementary properties of the classical Hamiltonian flow

$$\begin{cases} \dot{x} = a_\xi, & x(0) = y, \\ \dot{\xi} = -a_x, & \xi(0) = \eta. \end{cases} \quad (17)$$

Solutions to this system are *bicharacteristics*. For a point $z = (x, \xi)$ in phase space, let $\sigma \mapsto z^\sigma = (x^\sigma, \xi^\sigma)$ denote the bicharacteristic initialized at (x, ξ) . Write $(y, \eta) \mapsto (x^t(y, \eta), \xi^t(y, \eta))$ for the flow map.

The linearization of (17) satisfies the following Gronwall estimates:

Lemma 2.1. *Suppose $|t| \|\partial_{x,\xi}^2 a\|_{L^\infty} \leq 1$. Then*

$$\begin{aligned} \frac{\partial x^t}{\partial \eta} &= \int_0^t a_{\xi\xi}(\tau, x^\tau, \xi^\tau) d\tau + O(t^2 \|a_{x\xi}\| \|a_{\xi\xi}\|) + O(t^3 \|a_{xx}\| \|a_{\xi\xi}\|^2), \\ \frac{\partial \xi^t}{\partial \eta} &= I + O(t \|a_{\xi x}\|) + O(t^2 \|a_{xx}\| \|a_{\xi\xi}\|), \\ \frac{\partial x^t}{\partial y} &= I + O(t \|a_{x\xi}\|) + O(t^2 \|a_{xx}\| \|a_{\xi\xi}\|), \\ \frac{\partial \xi^t}{\partial y} &= \int_0^t -a_{xx}(\tau, x^\tau, \xi^\tau) d\tau + O(t^2 \|a_{xx}\| \|a_{x\xi}\|) + O(t^3 \|a_{xx}\|^2 \|a_{\xi\xi}\|). \end{aligned} \quad (18)$$

Proof. The linearized system takes the form

$$\begin{aligned} \dot{y} &= a_{\xi x} y + a_{\xi\xi} \eta, \\ \dot{\eta} &= -a_{xx} y - a_{x\xi} \eta. \end{aligned}$$

A preliminary application of Gronwall implies $|y(t)| + |\eta(t)| \lesssim |y(0)| + |\eta(0)|$.

Consider initial data $y(0) = I$, $\eta(0) = 0$. Then

$$|\eta(t)| \leq \int_0^t |a_{xx} y| d\tau + \int_0^t |a_{x\xi} \eta(\tau)| d\tau,$$

so $|\eta(t)| \lesssim t \|a_{xx}\|$. Substituting this into the equation for y , we deduce

$$|y - I| \leq \int_0^t |a_{\xi x} y| d\tau + \int_0^t |a_{\xi\xi} \eta| d\tau \lesssim t \|a_{\xi x}\| + t^2 \|a_{\xi\xi}\| \|a_{xx}\|.$$

This in turn yields the refinement

$$\left| \eta(t) + \int_0^t a_{xx} d\tau \right| \lesssim t^2 \|a_{xx}\| \|a_{\xi x}\| + t^3 \|a_{xx}\|^2 \|a_{\xi\xi}\|.$$

The case $y(0) = 0$, $\eta(0) = I$ is similar. We have

$$|y(t)| \leq \int_0^t |a_{\xi\xi} \eta| d\tau + \int_0^t |a_{\xi x} y| d\tau \implies |y(t)| \lesssim t \|a_{\xi\xi}\|,$$

which yields

$$\begin{aligned} |\eta(t) - I| &\lesssim \int_0^t \|a_{xx}\| \|a_{\xi\xi}\| \tau d\tau + \int_0^t |a_{x\xi}\eta| d\tau \lesssim t \|a_{x\xi}\| + t^2 \|a_{xx}\| \|a_{\xi\xi}\|, \\ \left| y(t) - \int_0^t a_{\xi\xi} d\tau \right| &\lesssim t^2 \|a_{\xi x}\| \|a_{\xi\xi}\| + t^3 \|a_{xx}\| \|a_{\xi\xi}\|^2. \end{aligned}$$

□

These imply, in view of the normalizations (8), the integrated estimates

$$\begin{aligned} x_1^t - x_2^t &= x_1^s - x_2^s + [I' + O(\varepsilon)](t-s)(\xi_1^s - \xi_2^s) \\ &\quad + O(|t-s| \|a_{x\xi}\|)(|x_1^s - x_2^s| + |t-s| |\xi_1^s - \xi_2^s|) \\ &\quad + O(|t-s|^2 \|a_{xx}\|)(|x_1^s - x_2^s| + |t-s| |\xi_1^s - \xi_2^s|), \\ \xi_1^t - \xi_2^t &= \xi_1^s - \xi_2^s + O(|t-s| \|a_{xx}\|)|x_1^s - x_2^s| \\ &\quad + O(|t-s|^2 \|a_{xx}\| \|a_{x\xi}\|)|x_1^s - x_2^s| + O(|t-s| \|a_{x\xi}\|)|\xi_1^s - \xi_2^s| \\ &\quad + O(|t-s|^3 \|a_{xx}\|^2)|x_1^s - x_2^s| + O(|t-s|^2 \|a_{xx}\|)|\xi_1^s - \xi_2^s|, \end{aligned} \tag{19}$$

where I' is an orthogonal matrix which equals the identity if $a_{\xi\xi}$ is positive-definite. In particular, we have:

Corollary 2.2. *If $|x_1^s - x_2^s| \leq r$, then $|x_1^t - x_2^t| \geq Cr$ whenever $2Cr/|\xi_1^s - \xi_2^s| \leq |t-s| \leq T_0$.*

Physically, this means that two particles colliding with sufficiently large relative velocity will only interact once in the time window of interest.

Next, we record a technical lemma first proved in the 1-dimensional case [Jao et al. 2019, Lemma 2.2]. This is used in the proof of Lemma 4.3 below but the computations use the preceding estimates.

Lemma 2.3. *There exists a constant $C = C(\|\partial^2 a\|) > 0$ so that if $Q_\eta = (0, \eta) + [-1, 1]^{2d} \subset T^*\mathbb{R}^d$ and $r \geq 1$, then*

$$\bigcup_{|t-t_0| \leq \min(|\eta|^{-1}, 1)} \Phi(t)^{-1}(z_0^t + rQ_\eta) \subset \Phi(t_0)^{-1}(z_0^{t_0} + CrQ_\eta).$$

In other words, if the bicharacteristic z^t starting at $z \in T^*\mathbb{R}^d$ passes through the cube $z_0^t + rQ_\eta$ in phase space during some time window $|t-t_0| \leq \min(|\eta|^{-1}, 1)$, then it must lie in the dilate $z_0^{t_0} + CrQ_\eta$ at time t_0 .

Proof. If $z \in \Phi(t)^{-1}(z_0^t + rQ_\eta)$, by definition we have $|x^t - x_0^t| \leq r$ and $|\xi^t - \xi_0^t - \eta| \leq r$. Assuming that $|\eta| \geq 1$, the estimates (19) imply

$$\begin{aligned} |x^{t_0} - x_0^{t_0}| &\leq r + |\eta|^{-1}(|\eta| + r) + O(|\eta|^{-1} \|\partial^2 a\|)(r + |\eta|^{-1}(|\eta| + r)) + O(|\eta|^{-2} \|\partial^2 a\|)(r + |\eta|^{-1}(|\eta| + r)) \\ &\leq Cr, \end{aligned}$$

$$\begin{aligned} |\xi^{t_0} - \xi_0^{t_0} - \eta| &\leq r + O(|\eta|^{-1} \|a_{xx}\|)r + (|\eta|^{-2} \|a_{xx}\| \|a_{x\xi}\|)r + O(|\eta|^{-1} \|a_{x\xi}\|)(|\eta| + r) \\ &\quad + (|\eta|^{-3} \|a_{xx}\|^2)r + O(|\eta|^{-2} \|a_{xx}\|)(|\eta| + r) \\ &\leq Cr. \end{aligned}$$

The case $|\eta| < 1$ is similar. □

2C. Wavepackets. Let $R \geq 1$ be a scale and $z_0 = (x_0, \xi_0)$ be a point in phase space. A scale- R *wavepacket* at z_0 is a Schwartz function ϕ_{z_0} such that ϕ_{z_0} and its Fourier transform $\hat{\phi}_{z_0}$ concentrate in the regions $|x - x_0| \leq R^{1/2}$ and $|\xi - \xi_0| \leq R^{-1/2}$, respectively:

$$|(R^{\frac{1}{2}} \partial_x)^k \phi_{z_0}(x)| \lesssim_{k,N} \left(\frac{x - x_0}{R^{\frac{1}{2}}} \right)^{-N}, \quad |(R^{-\frac{1}{2}} \partial_\xi)^k \hat{\phi}_{z_0}(\xi)| \lesssim_{k,N} \left(\frac{\xi - \xi_0}{R^{-\frac{1}{2}}} \right)^{-N} \quad \text{for all } k, N \geq 0.$$

There are many ways to decompose L^2 functions into linear combinations of wavepackets. For the first part of this article, it is technically more convenient to use a continuous decomposition. Later on in [Section 6C](#), we switch to a discrete version which is more common in the restriction theory literature.

In this section we recall a standard continuous wavepacket transform. To keep things simple we work at unit scale since that is all we shall need. For a function $f \in L^2(\mathbb{R}^d)$, its *Bargmann transform* or *FBI transform* is the function $Tf \in L^2(T^*\mathbb{R}^d)$ defined by

$$Tf(z) = \langle f, \psi_z \rangle_{L^2(\mathbb{R}^d)}, \quad \psi_z = \pi(z)\psi \quad \text{as in (13).}$$

The transform satisfies a Plancherel identity $\|Tf\|_{L^2(T^*\mathbb{R}^d)} = \|f\|_{L^2(\mathbb{R}^d)}$; dually, for any wavepacket coefficients $F \in L^2(T^*\mathbb{R}_z^d)$, one has

$$\|T^*F\|_{L_x^2} = \left\| \int_{T^*\mathbb{R}^d} F(z) \psi_z \, dz \right\|_{L_x^2} \leq \|F\|_{L_z^2}.$$

Indeed, TT^* is the orthogonal projection onto $TL^2(\mathbb{R}^d)$. Then as $T^*T = I$, any $f \in L^2(\mathbb{R}^d)$ can be resolved (nonuniquely) into a continuous superposition of wavepackets

$$f(x) = \int_{T^*\mathbb{R}^d} f_z \psi_z(x) \, dz.$$

Applying the propagator $U(t)$ to both sides and using linearity and the next lemma, one obtains a wavepacket decomposition

$$u(t, x) = \int u_z(t, x) \, dz, \quad u_z(t, x) = f_z[U(t)\psi_z](x),$$

of Schrödinger solutions. For brevity we sometimes omit the arguments and write $f = \int f_z \, dz$, $u = \int u_z \, dz$.

Lemma 2.4 (evolution of a packet). *If ψ_{z_0} is a scale-1 wavepacket, $U(t)$ is the propagator for (9), and $z_0 \mapsto z_0^t$ is the bicharacteristic starting at z_0 , then $U(t)\psi_{z_0}$ is a scale-1 wavepacket concentrated at z_0^t for all $|t| = O(1)$.*

Proof sketch. Using [Lemma 1.1](#) we reduce to the case $z_0 = 0$ and also ensure that the symbol $a(t, x, \xi)$ vanishes to second order at $(x, \xi) = (0, 0)$ in addition to satisfying the bounds (7). Then it suffices to show that propagator $U(t)$ for such symbols maps Schwartz functions to Schwartz functions on unit time scales. This is done using weighted Sobolev estimates as in [\[Koch and Tataru 2005, Section 4\]](#). \square

The term *wavepacket* shall also refer to space-time functions of the form $U(t)\psi_z$, not just the fixed time slices. Later it will be essential to exploit not just the space-time localization of wavepackets but also their phase as described in [Lemma 1.1](#).

3. Choosing a length scale

We begin with the following lemma from [\[Jao et al. 2019, Proposition 3.1\]](#), obtained by a variant of the usual TT^* derivation of the Strichartz estimates. While that article concerned just Schrödinger operators with scalar potentials, the proof works equally well in the current more general setting.

Proposition 3.1. *Suppose $U(t, s)$ satisfies a local-in-time dispersive estimate as in [Lemma 1.2](#). Let (q, r) be Strichartz exponents (i.e., satisfying the conditions in that lemma) with $2 < q < \infty$. Assume that $f \in L^2(\mathbb{R}^d)$ satisfies $\|f\|_{L^2(\mathbb{R}^d)} = 1$ and*

$$\|U(t)f\|_{L_t^q L_x^r([-1, 1] \times \mathbb{R}^d)} \geq \varepsilon.$$

Then there is a time interval $J \subset [-1, 1]$ such that

$$\|U(t, s)f\|_{L_t^{q-1} L_x^r(J \times \mathbb{R}^d)} \gtrsim |J|^{\frac{1}{q(q-1)}} \varepsilon^{\frac{q}{q-2}}.$$

Equivalently,

$$\|U(t, s)f\|_{L^q L^r} \lesssim \left(\sup_{J \in [-1, 1]} |J|^{-\frac{1}{q(q-1)}} \|U(t, s)f\|_{L_t^{q-1} L_x^r(J \times \mathbb{R}^d)} \right)^{1-\frac{2}{q}} \|f\|_{L^2(\mathbb{R}^d)}^{\frac{2}{q}}.$$

Note that by pigeonholing we may always assume that $|J| \leq T_0$, where T_0 is the time increment selected in [\(11\)](#).

Now let (q, r) be the Strichartz exponents determined by the conditions $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$ and $q - 1 = r$. It is easy to see that $2 < r < \frac{2(d+2)}{d} < q < \infty$.

For each $J = [s - \mu, s + \mu] \subset [-1, 1]$, we write

$$U(t, s)f = \left(\frac{T_0}{\mu} \right)^{\frac{d}{4}} \tilde{U} \left(\frac{T_0}{\mu}(t - s), 0 \right) \tilde{f} \left(\sqrt{\frac{T_0}{\mu}} x \right), \quad \tilde{f} = \left(\frac{\mu}{T_0} \right)^{\frac{d}{4}} f \left(\sqrt{\frac{\mu}{T_0}} x \right),$$

where $\tilde{U}(t, s)$ is the propagator for the rescaled equation $(D_t + \tilde{a}^w)\tilde{u} = 0$, and

$$\tilde{a}(t, x, \xi) := \frac{\mu}{T_0} a \left(s + \frac{\mu}{T_0} t, \sqrt{\frac{\mu}{T_0}} x, \sqrt{\frac{T_0}{\mu}} \xi \right).$$

Changing variables, we obtain

$$|J|^{-\frac{1}{q(q-1)}} \|U(t, s)f\|_{L_t^{q-1} L_x^r(J \times \mathbb{R}^d)} = \|\tilde{U}(t)\tilde{f}\|_{L_t^{q-1} L_x^r([-T_0, T_0] \times \mathbb{R}^d)}.$$

By interpolating with $L_{t,x}^2([-T_0, T_0] \times \mathbb{R}^d)$, which is bounded by unitarity, we see that [Theorem 1.3](#) would follow if we prove that for some $2 < q_0 < \frac{2(d+2)}{d}$ and $0 < \theta < 1$, the scale-1 refined estimate

$$\|U_\lambda^s(t)f\|_{L^{q_0}([-T_0, T_0] \times \mathbb{R}^d)} \lesssim \left(\sup_z |\langle \psi_z, f \rangle| \right)^\theta \|f\|_{L^2}^{1-\theta} \quad (20)$$

holds for all $s \in [-1, 1]$, $0 < \lambda \leq 1$, where the notation $U_\lambda^s(t)$ is as in [Hypothesis 1](#).

Over the next two sections we establish:

Proposition 3.2. *If Hypothesis 1 holds, then so does the estimate (20).*

4. A refined bilinear L^2 estimate

In previous work [Jao et al. 2019], we proved (20) when $d = 1$ with $q_0 = 4$ by viewing the inequality as a bilinear L^2 estimate and exploit orthogonality. Such a direct approach fails in $d \geq 2$ dimensions; since $2 < \frac{2(d+2)}{d} \leq 4$, the left side of (20) could well be infinite when $q_0 = 4$. To obtain a refined linear L^{q_0} estimate for $q_0 < \frac{2(d+2)}{d}$, we also begin by interpreting it as a refined bilinear $L^{q_0/2}$ estimate, but use dyadic decomposition and interpolate between two microlocalized estimates:

- A refined bilinear L^2 estimate (“refined” in the sense of exhibiting a sup over wavepacket coefficients) with some loss in the frequency separation of the inputs.
- A bilinear L^p estimate for some $p < \frac{d+2}{d}$ which yields gains in the frequency separation, essentially the content of Hypothesis 1.

This section discusses the former. In the next section we put together the two estimates, and the L^p estimate is established in the remainder of the paper.

Proposition 4.1. *Suppose $f = \int f_z \psi_z dz$ and $g = \int g_z \psi_z dz$ are $L^2(\mathbb{R}^d)$ initial data with corresponding Schrödinger evolutions $u = \int u_z dz$ and $v = \int v_z dz$, where $u_z(t, x) = f_z[U(t)\psi_z](x)$, $v_z(t, x) = g_z[U(t)\psi_z](x)$. Then*

$$\left\| \int_{|\xi_1 - \xi_2| \sim N} u_{z_1} v_{z_2} dz_1 dz_2 \right\|_{L^2([-T_0, T_0] \times \mathbb{R}^d)} \lesssim N^\alpha \left(\sup_z |f_z|^{\frac{1}{p'}} \|f_z\|_{L_z^2}^{\frac{1}{p}} \right) \left(\sup_z |g_z|^{\frac{1}{p'}} \|g_z\|_{L_z^2}^{\frac{1}{p}} \right) \quad (21)$$

for some $\alpha = \alpha(d)$ and $1 < p < 2$.

Proof. Square the left side and expand

$$\int f_{z_1} g_{z_2} \bar{f}_{z_3} \bar{g}_{z_4} K_N(z_1, z_2, z_3, z_4) dz_1 dz_2 dz_3 dz_4,$$

where $K_N := K \chi_{|\xi_1 - \xi_2| \sim N, |\xi_3 - \xi_4| \sim N}$, and

$$K(z_1, z_2, z_3, z_4) = \langle U(t)\psi_{z_1} U(t)\psi_{z_2}, U(t)\psi_{z_3} U(t)\psi_{z_4} \rangle_{L_{t,x}^2([-T_0, T_0] \times \mathbb{R}^d)}.$$

The estimate would follow if we could show that

$$N^{-\alpha} \langle z_1 - z_2 \rangle^\theta \langle z_3 - z_4 \rangle^\theta |K_N(\vec{z})| \text{ is a bounded operator on } L_{z_1, z_2}^2 \text{ for some } \theta > 0, \quad (22)$$

as Young’s inequality would then imply

$$\begin{aligned} \left\| \int u_z dz \right\|_{L^4}^2 &\lesssim \left(\int |f_{z_1} g_{z_2}|^2 \langle z_1 - z_2 \rangle^{-2\theta} dz_1 dz_2 \right)^{\frac{1}{2}} \left(\int |f_{z_3} g_{z_4}|^2 \langle z_3 - z_4 \rangle^{-2\theta} dz_3 dz_4 \right)^{\frac{1}{2}} \\ &\lesssim \sup_z |f_z|^{\frac{2}{p'}} \sup_z |g_z|^{\frac{2}{p'}} \|f\|_{L^2}^{\frac{2}{p}} \|g\|_{L^2}^{\frac{2}{p}} \quad \text{for some } 1 < p < 2. \end{aligned}$$

In view of the crude bound $|K(\vec{z})| \lesssim \min_{j,k} \langle z_j - z_k \rangle^{-1}$, which follows simply from the space-time supports of the wavepackets, (22) would follow from:

Lemma 4.2. *The localized kernel K_N satisfies*

$$\| |K_N|^{1-\delta} \|_{L_{z_1 z_2}^2 \rightarrow L_{z_3 z_4}^2} \lesssim N^\alpha,$$

where α is a constant depending only on the dimension.

Proof of Lemma 4.2. In view of the unit scale spatial localization of the wavepackets and the propagation estimates (19), we may further truncate the kernel to the phase-space region

$$R = \{ |x_1 - x_2| \leq 4|\xi_1 - \xi_2|, |x_3 - x_4| \leq 4|\xi_3 - \xi_4| \}.$$

For instance, if $|x_1^s - x_2^s| \geq 4|\xi_1^s - \xi_2^s|$ and $|t - s| \leq T_0$ with the parameter η in (11) chosen sufficiently small,

$$\begin{aligned} |x_1^t - x_2^t| &\geq (1 - |t - s|^2 \|\partial_x^2 V\|_{L^\infty} e^{|t-s|^2 \|\partial_x^2 V\|_{L^\infty}}) |x_1^s - x_2^s| \\ &\quad - (|t - s| + |t - s|^3 \|\partial_x^2 V\|_{L^\infty} e^{|t-s|^2 \|\partial_x^2 V\|_{L^\infty}}) |\xi_1^s - \xi_2^s| \\ &\geq \frac{1}{2} |x_1^s - x_2^s| - \frac{3}{2} |t - s| |\xi_1^s - \xi_2^s| \\ &\geq \frac{1}{8} |x_1^s - x_2^s|. \end{aligned}$$

Therefore $|K_N(1 - \chi_R)| \lesssim_M \langle x_1 - x_2 \rangle^{-M} \langle x_3 - x_4 \rangle^{-M} N^{-M}$ for any $M > 0$. Thus it suffices to prove that

$$\|K_N \chi_R\|_{L^2 \rightarrow L^2} \lesssim N^\alpha.$$

An estimate of this flavor was proved in the 1-dimensional case [Jao et al. 2019]. We shall argue similarly, but the proof is somewhat simpler since we aim for a cruder bound at this stage, completely ignoring temporal oscillations, and defer the more delicate analysis to the bilinear L^p estimate.

Partition the 4-particle phase space $(T^* \mathbb{R}^d)^4$ according to the degree of physical interaction between the particles. Let

$$\begin{aligned} E_0 &= \{ \vec{z} \in (T^* \mathbb{R}^d)^4 : \min_{|t| \leq T_0} \max_{j,k} |x_j^t - x_k^t| \leq 1 \}, \\ E_k &= \{ \vec{z} \in (T^* \mathbb{R}^d)^4 : 2^{k-1} < \min_{|t| \leq T_0} \max_{j,k} |x_j^t - x_k^t| \leq 2^k \}, \end{aligned}$$

and decompose the kernel into $K_N = \sum_{k \geq 0} K_N \chi_{E_k}$. Then we have the pointwise bound

$$|K(\vec{z})| \lesssim_M 2^{-kM} \frac{\langle \xi_1^{t(\vec{z})} + \xi_2^{t(\vec{z})} - \xi_3^{t(\vec{z})} - \xi_4^{t(\vec{z})} \rangle^{-M}}{\langle |\xi_1^{t(\vec{z})} - \xi_2^{t(\vec{z})}| + |\xi_3^{t(\vec{z})} - \xi_4^{t(\vec{z})}| \rangle}, \quad \vec{z} \in E_k, \quad (23)$$

where $t(\vec{z})$ is a time minimizing the “mutual distance” $\max_{i,j} |x_i^t - x_j^t|$. Further, the additional localization to R implies, by the estimates (19), that

$$\begin{aligned} |\xi_1^t - \xi_2^t - (\xi_1 - \xi_2)| &\lesssim \frac{1}{10} |\xi_1 - \xi_2|, \\ |\xi_3^t - \xi_4^t - (\xi_3 - \xi_4)| &\lesssim \frac{1}{10} |\xi_3 - \xi_4| \end{aligned}$$

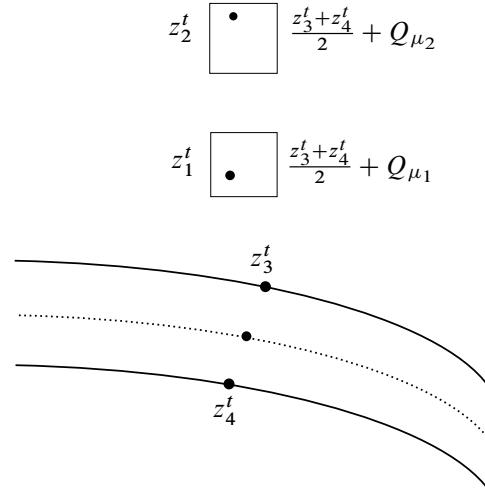


Figure 1. Z_{μ_1, μ_2} comprises all (z_1, z_2) such that z_1^t and z_2^t belong to the depicted phase-space boxes for t in the interval I .

for all $|t| \leq T_0$. In particular $|\xi_1^t(\vec{z}) - \xi_2^t(\vec{z})| \sim |\xi_3^t(\vec{z}) - \xi_4^t(\vec{z})| \sim N$; thus, while the ξ_j^t may vary rapidly with time if x_j^t are extremely far from the origin, the *relative frequencies* retain the same order of magnitude.

Assuming the bound (23) for the moment, we apply Schur's test to complete the proof of Lemma 4.2. Fix (z_3, z_4) belonging to the projection $E_k \rightarrow T^* \mathbb{R}_{z_3}^d \times T^* \mathbb{R}_{z_4}^d$, define

$$E_k(z_3, z_4) = \{(z_1, z_2) : (z_1, z_2, z_3, z_4) \in E_k\},$$

and let t_1 be the time minimizing $|x_3^{t_1} - x_4^{t_1}| \leq 2^k$. For any $(z_1, z_2) \in E_k(z_3, z_4)$, the mutual distance $\max_{j,k} |x_j^t - x_k^t|$ between $x_1^t, x_2^t, x_3^t, x_4^t$ is minimized in the time window

$$I = \left\{ t : |t - t_1| \lesssim \min\left(1, \frac{2^k}{|\xi_3 - \xi_4|}\right) \right\},$$

as for all other times we have $|x_3^t - x_4^t| \gg 2^k$ (Corollary 2.2).

We estimate the size of the level sets of $|K|$. For a momentum $\xi \in \mathbb{R}^d$, denote by $Q_\xi = (0, \xi) + [-1, 1]^d \times [-1, 1]^d \subset T^* \mathbb{R}^d$ the unit phase-space box centered at $(0, \xi)$, and write $\Phi^t = \Phi(t, 0)$ for the propagator on classical phase space relative to time 0 for the Hamiltonian $h(x, \xi) = \frac{1}{2}|\xi|^2 + V(t, x)$. For $\mu_1, \mu_2 \in \mathbb{R}^d$, define

$$Z_{\mu_1, \mu_2} = \bigcup_{t \in I} (\Phi^t \otimes \Phi^t)^{-1} \left(\frac{z_3^t + z_4^t}{2} + 2^k Q_{\mu_1} \right) \times \left(\frac{z_3^t + z_4^t}{2} + 2^k Q_{\mu_2} \right).$$

This set is depicted schematically in Figure 1 when $k = 0$, and corresponds to the pairs of wave packets $(z_1, z_2) \in E_m(z_3, z_4)$ with momenta (μ_1, μ_2) relative to the wavepackets (z_3, z_4) at the “collision time” $t(\vec{z})$.

We note that $E_k(z_3, z_4) \subset \bigcup_{\mu_1, \mu_2 \in \mathbb{Z}^d} Z_{\mu_1, \mu_2}$, and recall the following estimate from the 1-dimensional paper, whose proof we reproduce below for convenience:

Lemma 4.3.

$$|Z_{\mu_1, \mu_2}| \lesssim 2^{4dk} \max(1, |\mu_1|, |\mu_2|) |I|. \quad (24)$$

Proof. Without loss assume $|\mu_1| \geq |\mu_2|$. Partition the interval I into subintervals of length $|\mu_1|^{-1}$ if $\mu_1 \neq 0$ and into subintervals of length 1 if $\mu_1 = 0$. For each t' in the partition, [Lemma 2.3](#) implies that for some constant $C > 0$ we have

$$\begin{aligned} \bigcup_{|t-t'| \leq \min(1, |\mu_1|^{-1})} \Phi(t)^{-1} \left(\frac{z_3^t + z_4^t}{2} + 2^k Q_{\mu_1} \right) &\subset \Phi(t')^{-1} \left(\frac{z_3^{t'} + z_4^{t'}}{2} + C 2^k Q_{\mu_1} \right), \\ \bigcup_{|t-t'| \leq \min(1, |\mu_1|^{-1})} \Phi(t)^{-1} \left(\frac{z_3^t + z_4^t}{2} + 2^k Q_{\mu_2} \right) &\subset \Phi(t')^{-1} \left(\frac{z_3^{t'} + z_4^{t'}}{2} + C 2^k Q_{\mu_2} \right), \end{aligned}$$

and so

$$\begin{aligned} \bigcup_{|t-t'| \leq \min(1, |\mu_1|^{-1})} (\Phi(t) \otimes \Phi(t))^{-1} \left(\frac{z_3^t + z_4^t}{2} + 2^k Q_{\mu_1} \right) \times \left(\frac{z_3^t + z_4^t}{2} + 2^k Q_{\mu_2} \right) \\ \subset (\Phi(t') \otimes \Phi(t'))^{-1} \left(\frac{z_3^{t'} + z_4^{t'}}{2} + C 2^k Q_{\mu_1} \right) \times \left(\frac{z_3^{t'} + z_4^{t'}}{2} + C 2^k Q_{\mu_2} \right). \end{aligned}$$

By Liouville's theorem, the right side has measure $O(2^{4dk})$ in $(T^* \mathbb{R}^d)^2$. The claim follows by summing over the partition. \square

For each $(z_1, z_2) \in E_k(z_3, z_4) \cap Z_{\mu_1, \mu_2}$, we have by definition

$$z_j^{t(\vec{z})} \in \frac{z_3^{t(\vec{z})} + z_4^{t(\vec{z})}}{2} + 2^k Q_{\mu_j}.$$

Thus

$$\begin{aligned} \xi_1^{t(\vec{z})} + \xi_2^{t(\vec{z})} - \xi_3^{t(\vec{z})} - \xi_4^{t(\vec{z})} &= \mu_1 + \mu_2 + O(2^k), \\ \xi_1^{t(\vec{z})} - \xi_2^{t(\vec{z})} &= \mu_1 - \mu_2 + O(2^k). \end{aligned}$$

Hence when $(z_1, z_2) \in Z_{\mu_1, \mu_2}$, for any M we have

$$|K(\vec{z})| \lesssim_M 2^{-Mk} \frac{\langle \mu_1 + \mu_2 \rangle^{-M}}{\langle |\mu_1 - \mu_2| + |\xi_3^{t(\vec{z})} - \xi_4^{t(\vec{z})}| \rangle}. \quad (25)$$

To apply Schur's test, we combine the estimates [\(24\)](#), [\(25\)](#), and evaluate

$$\begin{aligned} \int |K_N(z_1, z_2, z_3, z_4)|^{1-\delta} \chi_{E_k}(\vec{z}) dz_1 dz_2 &\leq \sum_{\mu_1, \mu_2 \in \mathbb{Z}^d} \int_{Z_{\mu_1, \mu_2}} |K_N^{1-\delta} \chi_{E_k}| dz_2 \\ &\lesssim_M 2^{-Mk} \sum_{|\mu_1 - \mu_2| \lesssim N+2^k} 2^{-Mk} \langle \mu_1 + \mu_2 \rangle^{-M} \\ &\lesssim N^d 2^{-(M-d)k}. \end{aligned}$$

For fixed z_1, z_2 , the integral over z_3 and z_4 is estimated the same way. This concludes the proof of [Lemma 4.2](#), modulo some remarks on the crucial pointwise bound [\(23\)](#).

To obtain that estimate, we use [Lemma 1.1](#) to write

$$K(\vec{z}) = \int e^{i\Phi} \prod_{j=1}^4 U^{z_j}(t) \psi(x - x_j^t) dx dt,$$

$$\Phi(t, x; \vec{z}) = \sum_j \sigma_j [(\langle x - x_j^t, \xi_j^t \rangle + \phi(t, x_0, \xi_0)],$$

where $\sigma = (+, +, -, -)$, and we set $\prod_j c_j := c_1 c_2 \bar{c}_3 \bar{c}_4$.

It is convenient to partition the integral further, writing

$$U^{\vec{z}_j}(t) \psi(x - x_j^t) = \sum_{\ell_j \geq 0} U^{\vec{z}_j}(t) \psi(x - x_j^t) \theta_{\ell_j}(x - x_j^t),$$

where $\sum_{\ell \geq 0} \theta_\ell$ is a partition of unity with θ_ℓ supported on the dyadic annulus of radius $\sim 2^\ell$. For $\vec{z} \in E_k$, only the terms

$$K_{\vec{\ell}}(\vec{z}) := \int e^{i\Phi} \prod_{j=1}^4 U^{z_j}(t) \psi(x - x_j^t) \theta_{\ell_j}(x - x_j^t) dx dt,$$

with $\ell^* := \max_j \ell_j \gtrsim k$, will be nonzero.

By [Lemma 2.1](#), the integral is supported on the space-time region

$$\left\{ (t, x) : |t - t(\vec{z})| \lesssim \min \left(1, \frac{2^{\ell^*}}{\max_{i,j} |\xi_i^{t(\vec{z})} - \xi_k^{t(\vec{z})}|} \right) \text{ and } |x - x_j^t| \lesssim 2^{\ell_j} \right\},$$

and for all such t we have

$$|x_j^t - x_k^t| \lesssim 2^{\ell^*}, \quad |\xi_j^t - \xi_k^t - (\xi_1^{t(\vec{z})} - \xi_k^{t(\vec{z})})| \lesssim 2^{\ell^*}.$$

Integrating by parts in x , we may produce as many factors of $|\xi_1^t + \xi_2^t - \xi_3^t - \xi_4^t|^{-1}$ as desired and freeze $t = t(\vec{z})$ to obtain

$$|K_{\vec{\ell}}(\vec{z})| \lesssim_M 2^{-\ell^* M} \frac{\langle \xi_1^{t(\vec{z})} + \xi_2^{t(\vec{z})} - \xi_3^{t(\vec{z})} - \xi_4^{t(\vec{z})} \rangle^{-M}}{\langle |\xi_1^{t(\vec{z})} - \xi_2^{t(\vec{z})}| + |\xi_3^{t(\vec{z})} - \xi_4^{t(\vec{z})}| \rangle} \quad \text{for any } M \geq 0,$$

and the bound [\(23\)](#) follows upon summing over $\vec{\ell}$. □

This completes the proof of [Proposition 4.1](#). □

5. Proof of Theorem 1.3

We prove [Proposition 3.2](#) and hence [Theorem 1.3](#). Begin with a Whitney decomposition of

$$(\mathbb{R}^d \times \mathbb{R}^d) \setminus \{(\xi, \xi) : \xi \in \mathbb{R}^d\} = \bigcup_{N \in 2^{\mathbb{Z}}} \bigcup_{Q \in \mathcal{Q}_N} Q,$$

where \mathcal{Q}_N is the set of dyadic cubes in $\mathbb{R}^d \times \mathbb{R}^d$ with diameter $\sim N$ and distance $\sim N$ to the diagonal. For each $Q \in \mathcal{Q}_N$, its characteristic function factors into

$$\chi_N^Q(\xi_1, \xi_2) = \chi_N^{Q,1}(\xi_1) \chi_N^{Q,2}(\xi_2),$$

where $\chi_N^{Q,j}$ are characteristic functions of d -dimensional cubes of width N . Then we can take the decomposition

$$1(\xi_1, \xi_2) = \chi_0(\xi_1, \xi_2) + \sum_{N \geq 1} \sum_{Q \in \mathcal{Q}_N} \chi_N^{Q,1}(\xi_1) \chi_N^{Q,2}(\xi_2),$$

where $\chi_0(\xi_1, \xi_2)$ is supported on the set $|\xi_1 - \xi_2| \lesssim 1$.

Now suppose u and v are linear solutions with initial data $f = \int f_z \psi_z dz$ and $g = \int g_z \psi_z dz$, respectively, where $f_z = \langle f, \psi_z \rangle$ and $g_z = \langle g, \psi_z \rangle$. Writing $u_z = f_z U(t) \psi_z$, $v_z = g_z U(t) \psi_z$, we deduce as a consequence of [Hypothesis 1](#) that

$$\left\| \sum_{Q \in \mathcal{Q}_N} \int_Q u_{z_1} v_{z_2} dz_1 dz_2 \right\|_{L^q([-T_0, T_0] \times \mathbb{R}^d)} \lesssim N^{-\delta} \|f_z\|_{L_z^2} \|g_z\|_{L_z^2} \quad (26)$$

for each $N \geq 1$. Indeed, for each cube Q the integral has a product structure

$$\begin{aligned} \int_Q u_{z_1} v_{z_2} dz_1 dz_2 &= \left(\int u_{z_1} \chi_N^{Q,1}(\xi_1) dx_1 d\xi_1 \right) \left(\int v_{z_2} \chi_N^{Q,2}(\xi_2) dx_2 d\xi_2 \right) \\ &= U(t) \left[\int f_{z_1} \chi_N^{Q,1}(\xi_1) \psi_{z_1} dx_1 d\xi_1 \right] U(t) \left[\int g_{z_2} \chi_N^{Q,2}(\xi_2) \psi_{z_2} dx_2 d\xi_2 \right]. \end{aligned}$$

By the rapid decay of the wavepackets, we may harmlessly insert frequency cutoffs $\tilde{\chi}_N^{Q,j}(D)$, where $\tilde{\chi}_N^{Q,j}$ are slightly fattened versions of $\chi_N^{Q,j}$ and still have supports separated by distance $\sim N$, and apply [Hypothesis 1](#) to estimate

$$\begin{aligned} \left\| \int_Q u_{z_1} v_{z_2} dz_1 dz_2 \right\|_{L^q} &\lesssim N^{-\delta} \left\| \int f_{z_1} \chi_N^{Q,1}(\xi_1) dx_1 d\xi_1 \right\|_{L^2(\mathbb{R}^d)} \left\| \int g_{z_2} \chi_N^{Q,2}(\xi_2) dx_2 d\xi_2 \right\|_{L^2(\mathbb{R}^d)} \\ &\lesssim N^{-\delta} \|f_z \chi_N^{Q,1}(\xi)\|_{L_z^2} \|g_z \chi_N^{Q,2}(\xi)\|_{L_z^2}. \end{aligned}$$

The left side of (26) is therefore bounded by

$$\begin{aligned} \sum_{Q \in \mathcal{Q}_N} N^{-\delta} \|f_z \chi_N^{Q,1}(\xi)\|_{L_z^2} \|g_z \chi_N^{Q,2}(\xi)\|_{L_z^2} &\leq N^{-\delta} \left(\sum_{Q \in \mathcal{Q}_N} \|f_z \chi_N^{Q,1}(\xi)\|_{L_z^2}^2 \right)^{\frac{1}{2}} \left(\sum_{Q \in \mathcal{Q}_N} \|g_z \chi_N^{Q,2}(\xi)\|_{L_z^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim N^{-\delta} \|f_z\|_{L_z^2} \|g_z\|_{L_z^2}, \end{aligned}$$

as claimed.

Now decompose the product uv into

$$uv = \int u_{z_1} v_{z_2} \chi_0(\xi_1, \xi_2) dz_1 dz_2 + \sum_{N \geq 1} \sum_{Q \in \mathcal{Q}_N} \int_Q u_{z_1} v_{z_2} dz_1 dz_2,$$

and estimate each group of terms in L^q for q between p and 2. For the sum over \mathcal{Q}_N we interpolate between the L^p and L^2 bounds. Writing $\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{2}$, we have

$$\begin{aligned} \left\| \sum_{Q \in \mathcal{Q}_N} \int_Q u_{z_1} v_{z_2} dz_1 dz_2 \right\|_{L^q} &\leq \left\| \sum_{Q \in \mathcal{Q}_N} \int_Q u_{z_1} v_{z_2} dz_1 dz_2 \right\|_{L^p}^{1-\theta} \left\| \sum_{Q \in \mathcal{Q}_N} \int_Q u_{z_1} v_{z_2} dz_1 dz_2 \right\|_{L^2}^{\theta} \\ &\lesssim N^{-\delta(1-\theta)+\alpha\theta} \left[\left(\sup_z |\langle f, \psi_z \rangle| \right)^{\frac{1}{p'}} \left(\sup_z |\langle g, \psi_z \rangle| \right)^{\frac{1}{p'}} \right]^{\theta} (\|f\|_{L_x^2} \|g\|_{L_x^2})^{1-\theta+\frac{\theta}{p}} \end{aligned}$$

and for q sufficiently close to p (hence θ sufficiently small) the exponent of N is negative.

For the “near-diagonal” sum, we interpolate between L^1 and L^2 . For the L^1 bound we simply use Minkowski’s inequality and the estimate $\|U(t)\psi_{z_1} U(t)\psi_{z_2}\|_{L^1} \lesssim_N \langle x_1 - x_2 \rangle^{-N}$ when $|\xi_1 - \xi_2| \leq 1$ to obtain

$$\begin{aligned} \left\| \int u_{z_1} v_{z_2} \chi_0(\xi_1, \xi_2) dx_1 dx_2 d\xi_1 d\xi_2 \right\|_{L_x^1} &\lesssim \int |f_{z_1} g_{z_2}| \langle x_1 - x_2 \rangle^{-N} \chi_0(\xi_1, \xi_2) dz_1 dz_2 \\ &\lesssim \|f_z\|_{L_z^2} \|g_z\|_{L_z^2}, \end{aligned}$$

which when combined with [Proposition 4.1](#) yields

$$\begin{aligned} \left\| \int u_{z_1} v_{z_2} \chi_0(\xi_1, \xi_2) dz_1 dz_2 \right\|_{L^q} &\lesssim \left\| \int u_{z_1} v_{z_2} \chi_0(\xi_1, \xi_2) dz_1 dz_2 \right\|_{L^1}^{1-\theta'} \left\| \int u_{z_1} v_{z_2} \chi_0(\xi_1, \xi_2) dz_1 dz_2 \right\|_{L^2}^{\theta'} \\ &\lesssim \left[\left(\sup_z |f_z| \right)^{\frac{1}{p'}} \left(\sup_z |g_z| \right)^{\frac{1}{p'}} \right]^{\theta'} (\|f_z\|_{L_z^2} \|g_z\|_{L_z^2})^{1-\theta'+\frac{\theta'}{p}} \\ &\lesssim \left(\sup_z |\langle f, \psi_z \rangle| \sup_z |\langle g, \psi_z \rangle| \right)^{\frac{\theta}{p'}} (\|f\|_{L^2} \|g\|_{L^2})^{1-\theta+\frac{\theta}{p}} \end{aligned}$$

for some $1 < p < 2$, where $\frac{1}{q} = 1 - \theta' + \frac{\theta'}{2}$.

Summing in N , we conclude that

$$\|uv\|_{L^q} \lesssim \left[\left(\sup_z |\langle f, \psi_z \rangle| \right)^{\frac{1}{p'}} \left(\sup_z |\langle g, \psi_z \rangle| \right)^{\frac{1}{p'}} \right]^{\theta} (\|f\|_{L_x^2} \|g\|_{L_x^2})^{1-\frac{\theta}{p'}}$$

for some $\theta = \theta(p) \in (1, \frac{d+2}{d})$. Taking $u = v$ we obtain [Proposition 3.2](#).

6. The restriction-type estimate

The purpose of this section is to prove [Theorem 1.5](#).

We shall systematically use the following notation. For $N \geq 1$ and a potential V , we consider the rescaled potentials

$$V_N(t, x) := N^{-2} V(N^{-2}t, N^{-1}x).$$

Let $U(t, s)$ and $U_N(t, s)$ denote the propagators for the corresponding Schrödinger operators $H(t) := -\frac{1}{2}\Delta + V$ and $H_N(t) := -\frac{1}{2}\Delta + V_N$. We will often use the letter U to write the propagators for different potentials $V \in \mathcal{V}$; this ambiguity will not cause any serious issue, however, since all the estimates we

shall need are valid uniformly over \mathcal{V} . Further, due to the time-translation invariance of our assumptions we shall usually just consider the propagator from time 0 and write $U(t) := U(t, 0)$, $U_N(t) := U_N(t, 0)$.

In the sequel, the letter C will denote a constant, depending only on the dimension d , which may change from line to line.

6A. Preliminary reductions. The hypotheses of [Theorem 1.5](#) are invariant under various transformations of u and v :

- Galilei boosts $u(0) \mapsto \pi(z_0)u(0)$, $u \mapsto \pi(z_0^t)u^{z_0}$, where u^{z_0} satisfies $(D_t - \Delta + V^{z_0})u^{z_0} = 0$, $u^{z_0}(0) = u(0)$.
- Spatial rotations: for an orthogonal matrix g , $(g \cdot u)(t, x) := u(t, g^{-1} \cdot x)$ satisfies

$$[D_t(g \cdot u) - \Delta + (g \cdot V)](g \cdot u) = 0.$$

- Rescaling $u \mapsto u_\lambda = \lambda^{-d/2}u(\lambda^{-2}t, \lambda^{-1}x)$ for $\lambda > 1$. Then u_λ satisfies $(D_t - \Delta + V_\lambda)u_\lambda = 0$ with a smoother potential $V_\lambda(t, x) = \lambda^{-2}V(\lambda^{-2}t, \lambda^{-1}x)$.

We may and shall assume hereafter that V vanishes to second order at $x = 0$; that is, $V(t, 0) = 0$ and $\partial_x V(t, 0) = 0$ for all t . Indeed let $z_0^t = (x_0^t, \xi_0^t)$ be the bicharacteristic with $(x_0, \xi_0) = (0, 0)$. Then by [Lemma 1.1](#),

$$\begin{aligned} \|U(t)fU(t)g\|_{L^{(d+3)/(d+1)}} &= \|(\pi(z_0^t)U^{z_0}(t)f)(\pi(z_0^t)U^{z_0}(t)g)\|_{L^{(d+3)/(d+1)}} \\ &= \|U^{z_0}(t)fU^{z_0}(t)g\|_{L^{(d+3)/(d+1)}}, \end{aligned}$$

and the potential $V^{z_0}(t, x) = V(t, x_0^t + x) - V(t, x_0^t) - x\partial_x V(t, x_0^t)$ vanishes to second order at $x = 0$.

[Theorem 1.5](#) is equivalent by rescaling to:

Theorem 6.1. *Given $S_1, S_2 \subset \mathbb{R}_\xi^d$ with $\text{diam}(S_j) \leq 1$ and $c^{-1} \geq \text{dist}(S_1, S_2) \geq c$ for some $0 < c < 1$, there exists a constant $\eta = \eta(c) > 0$ such that if $V \in \mathcal{V}$ and $\tau_0 > 0$ satisfies*

$$(\tau_0 + \tau_0^2)\|\partial_x^2 V\|_{L_{t,x}^\infty} < \eta, \quad (27)$$

then, for any $f, g \in L^2(\mathbb{R}^d)$ with $\text{supp}(\hat{f}) \subset S_1$ and $\text{supp}(\hat{g}) \subset S_2$, the corresponding Schrödinger solutions $u_N = U_N(t)f$ and $v_N = U_N(t)g$ satisfy the estimate

$$\|u_N v_N\|_{L^q([-\tau_0 N^2, \tau_0 N^2] \times \mathbb{R}^d)} \lesssim_\varepsilon N^\varepsilon \|f\|_{L^2} \|g\|_{L^2} \quad \text{for all } \frac{d+3}{d+1} \leq q < \frac{d+2}{d}, \quad (28)$$

for any $\varepsilon > 0$ and $N \geq 1$.

In fact it suffices to take S_1 and S_2 of the form

$$S_1 = \left\{ \xi : \left| \xi - \frac{c}{2}e_1 \right| \leq \frac{c}{100} \right\}, \quad S_2 = \left\{ \xi : \left| \xi + \frac{c}{2}e_1 \right| \leq \frac{c}{100} \right\}. \quad (29)$$

General S_j can be reduced to this case by decomposing $\hat{f} = \sum_j \hat{f}_j$ and $\hat{g} = \sum_k \hat{g}_k$ into pieces supported in small balls and applying an appropriate Galilei boost and rotation for each pair (f_j, g_k) and possibly also a rescaling to bring the Fourier supports closer, which only reduces $\|\partial_x^2 V\|_{L^\infty}$. Henceforth we shall assume (29).

6B. General remarks. We use the induction-on-scales method pioneered in [Wolff 2001] for the cone and adapted in [Tao 2003] to the paraboloid. Our proof is modeled closely on Tao's treatment of the $V = 0$ case, and the reader may find it helpful to read the following exposition in parallel with [Tao 2003]. The main differences are as follows:

- The induction scheme (Section 6E) is complicated by the fact that frequency is not conserved, so one cannot directly apply an induction hypothesis which involves assumptions on the frequency supports at time 0 to a space-time ball at a later time.
- The low regularity of V in time makes the bilinear L^2 estimate (Section 6H) more delicate and we obtain weaker decay from temporal oscillations.
- In the final Kakeya-type estimate, the tubes in the key combinatorial lemma (Lemma 6.11, the analogue of Lemma 8.1 in Tao) are curved. Also, we need to be slightly more precise to compensate for the weaker decay in the L^2 bound.

6C. Discrete wavepacket decomposition. While the first part of this paper employed continuous wavepacket transforms, the following discrete decomposition, taken essentially from [Tao 2003], is more conventional in restriction theory and convenient for the combinatorial arguments involved. To each $z_0 = (x_0, \xi_0)$ in classical phase space with bicharacteristic $\gamma_{z_0}(t) = (x_0^t, \xi_0^t)$, we associate a space-time “tube”

$$T_{z_0} := \{(t, x) : |x - x_0^t| \leq R^{\frac{1}{2}}, |t| \leq R\}.$$

For such a tube T , let $z(T) = (x(T), \xi(T))$ denote the corresponding initial point in phase space. A wavepacket ϕ associated to the bicharacteristic $z_0 \mapsto z_0^t$ is essentially supported in space-time on the tube T_{z_0} , and we shall often emphasize this fact by writing ϕ_T .

Lemma 6.2. *Let $u = U_N(t)f$ be a linear Schrödinger solution with $\text{supp}(\hat{f}) \subset S_1$. For each $1 \leq R \leq N^2$, there exists a collection of tubes \mathbf{T} and a decomposition*

$$u = \sum_{T \in \mathbf{T}} a_T \phi_T$$

into $R \times (R^{1/2})^d$ wave packets with the following properties:

- Each $T \in \mathbf{T}$ satisfies $(x(T), \xi(T)) \in R^{1/2}\mathbb{Z}^d \times R^{-1/2}\mathbb{Z}^d$.
- Each wavepacket ϕ_T is a Schrödinger solution localized near the bicharacteristic $(x(T)^t, \xi(T)^t)$, i.e., it satisfies the pointwise bounds

$$\begin{aligned} |(R^{\frac{1}{2}} \partial_x)^k \phi_T(t)| &\lesssim_{k,M} \left\langle \frac{x - x(T)^t}{R^{\frac{1}{2}}} \right\rangle^{-M} \quad \text{for all } k, M \geq 0, \\ |(R^{-\frac{1}{2}} \partial_\xi)^k \hat{\phi}_T(t)| &\lesssim_{k,M} \left\langle \frac{\xi - \xi(T)^t}{R^{-\frac{1}{2}}} \right\rangle^{-M} \quad \text{for all } k, M \geq 0. \end{aligned} \tag{30}$$

Moreover, $\hat{\phi}_T[0]$ is supported in an $R^{-1/2}$ neighborhood of $\xi(T) \in S_1$.

- The complex coefficients a_T are square-summable:

$$\sum_T |a_T|^2 \lesssim \|f\|_{L^2}^2.$$

Moreover, for any subcollection of tubes $\mathbf{T}' \subset \mathbf{T}$ and complex numbers a_T , one has

$$\left\| \sum_{T \in \mathbf{T}'} a_T \phi_T \right\|_{L^2}^2 \lesssim \sum_{T \in \mathbf{T}'} |a_T|^2.$$

A similar decomposition also holds for $v = U_N(t)g$.

Proof sketch. We outline the main steps as this construction is fairly standard; consult for instance [Tao 2003, Lemma 4.1]. Begin with partitions of unity $1 = \sum_{x_0 \in \mathbb{Z}^d} \eta(x - x_0)$ and $1 = \sum_{\xi_0 \in \mathbb{Z}^d} \chi(\xi - \xi_0)$ such that χ and $\hat{\eta}$ are compactly supported. By rescaling and quantizing, we obtain a pseudodifferential partition of unity used to decompose the initial data

$$f = \sum_{(x_0, \xi_0)} \eta\left(\frac{x - x_0}{R^{\frac{1}{2}}}\right) \chi(R^{\frac{1}{2}}(D - \xi_0)) f.$$

The propagation estimates then follow from the next lemma. \square

Lemma 6.3. *If ϕ_{z_0} is a scale- R wavepacket concentrated at z_0 , and $U_N(t)$ is the propagator for $H(t) = -\frac{1}{2}\Delta + V_N$, then $U_N(t)$ is a scale- R wavepacket concentrated at z_0^t for all $|t| \leq R$.*

Proof. By rescaling we reduce to $R = 1$ and replace V by $V_{N/R^{1/2}}$ which also belongs to \mathcal{V} since $N/R^{1/2} \geq 1$. Then the symbol $a = \frac{1}{2}|\xi|^2 + V_{N/R^{1/2}}(t, x)$ satisfies the estimates (7), and we can appeal to Lemma 2.4. \square

6D. Localization. The proof of Theorem 6.1 begins with the observation that it suffices to establish the same estimate with the space-time norm restricted to a box of the form

$$\Omega_N = [-N^2, N^2] \times [-AN^2, AN^2]^d.$$

Theorem 6.4. *Assume the hypotheses and notation of Theorem 6.1 and replace c by $c/2$ and take $\text{diam}(S_j) \leq \frac{11}{10}$. Then there exists $A = A(c) > 0$ such that*

$$\|u_N v_N\|_{L^{(d+3)/(d+1)}(\Omega_N)} \lesssim \varepsilon N^\varepsilon \|f\|_{L^2} \|g\|_{L^2} \quad (31)$$

for any $\varepsilon > 0$.

Remark. In the wavepacket decomposition of u_N and v_N , the Fourier supports of the wavepackets are contained in a slight dilate $S_j + B(0, CN^{-1})$ of S_j . Hence at various junctures we need to adjust various constants to accommodate this minor enlargement of Fourier supports.

The full theorem then follows from an approximate finite speed of propagation argument:

Lemma 6.5. *Theorem 6.4 implies Theorem 6.1.*

Proof of Lemma 6.5. Partition physical space $\mathbb{R}^d = \bigcup_{j \in \mathbb{Z}^d} Q_j$ into cubes of width $\sim N^2$, where Q_j denotes the cube with center $N^2 j \in N^2 \mathbb{Z}^d$. Decompose $u := u_N$ and $v := v_N$ into $N^2 \times (N)^d$ wavepackets, and group the terms in the product according to their relative initial positions. Write

$$u = \sum_T a_T \phi_T = \sum_{j \in \mathbb{Z}^d} \sum_{T \in \mathbf{T}_j} u_T,$$

$$v = \sum_{T'} b_{T'} \phi_{T'} = \sum_{j' \in \mathbb{Z}^d} \sum_{T' \in \mathbf{T}'_{j'}} v_{T'},$$

where $\mathbf{T}_j = \{T \in \mathbf{T} : x(T) \in Q_j\}$ and similarly for $\mathbf{T}'_{j'}$. Using the triangle inequality we estimate

$$\|uv\|_{L^{(d+3)/(d+1)}} \leq \sum_{k \geq 0} \left\| \sum_{|j-j'| \sim 2^k} \sum_{T \in \mathbf{T}_j, T' \in \mathbf{T}'_{j'}} u_T v_{T'} \right\|_{L^{(d+3)/(d+1)}}. \quad (32)$$

For the k -th sum, note from (19) that if $(x_1, \xi_1) := (x(T), \xi(T))$ and $(x_2, \xi_2) := (x(T'), \xi(T'))$, we have

$$\begin{aligned} |x_1^t - x_2^t| &\geq (1 - C t^2 \|\partial_x^2 V_N\|_{L^\infty}) |x_1 - x_2| - (|t| + C |t|^3 \|\partial_x^2 V_N\|_{L^\infty}) |\xi_1 - \xi_2| \\ &\geq (1 - C \tau_0^2 \|\partial_x^2 V\|_{L^\infty}) |x_1 - x_2| - N^2 (1 + C \tau_0^2 \|\partial_x^2 V\|_{L^\infty}) |\xi_1 - \xi_2| \\ &\geq (1 - C \eta) |x_1 - x_2| - N^2 (1 + C \eta) |\xi_1 - \xi_2|, \end{aligned}$$

where C hides the harmless Gronwall factor. As $|\xi_1 - \xi_2| \leq c^{-1}$, there exists $k(c)$ such that if $|x_1 - x_2| \geq 2^k N^2$ and η is chosen small enough we obtain $|x_1^t - x_2^t| \gtrsim 2^k N^2$ for $k \geq k(c)$. Thus the tubes in \mathbf{T}_j and $\mathbf{T}'_{j'}$ are separated in space by distance $\gtrsim 2^k N^2$, and since each wavepacket ϕ_T decays rapidly away from its tube T in units of N , we have

$$\|\phi_T \phi_{T'}\|_{L^{(d+3)/(d+1)}} \lesssim 2^{-101dk} N^{-101d},$$

and estimate crudely as follows:

$$\begin{aligned} \left\| \sum_{|j-j'| \sim 2^k} \sum_{T \in \mathbf{T}_j, T' \in \mathbf{T}'_{j'}} u_T v_{T'} \right\|_{L^{(d+3)/(d+1)}} &\lesssim 2^{-101dk} N^{-101d} \sum_{|j-j'| \sim 2^k} \sum_{T \in \mathbf{T}_j, T' \in \mathbf{T}'_{j'}} |a_T b_{T'}| \\ &\lesssim 2^{-101dk} N^{-100d} \sum_{|j-j'| \sim 2^k} \left(\sum_{T \in \mathbf{T}_j} |a_T|^2 \right)^{\frac{1}{2}} \left(\sum_{T' \in \mathbf{T}'_{j'}} |b_{T'}|^2 \right)^{\frac{1}{2}} \\ &\lesssim 2^{-100dk} N^{-100d} \left(\sum_j \sum_{T \in \mathbf{T}_j} |a_T|^2 \right)^{\frac{1}{2}} \left(\sum_j \sum_{T' \in \mathbf{T}'_{j'}} |b_{T'}|^2 \right)^{\frac{1}{2}} \\ &\lesssim 2^{-100dk} N^{-100d} \|f\|_{L^2} \|g\|_{L^2}. \end{aligned}$$

For the “near diagonal” part of the sum (32), where $|j - j'| \leq 2^{k(c)}$, we group the terms by their average initial positions:

$$\begin{aligned} \left\| \sum_{|j-j'| \lesssim 1} \sum_{T \in \mathbf{T}_j, T' \in \mathbf{T}'_{j'}} u_T v_{T'} \right\|_{L^{(d+3)/(d+1)}} &\leq \sum_{m \in \mathbb{Z}^d + \mathbb{Z}^d} \sum_{|j-j'| \lesssim 1, j+j'=m} \left\| \sum_{T \in \mathbf{T}_j, T' \in \mathbf{T}'_{j'}} u_T v_{T'} \right\|_{L^{(d+3)/(d+1)}}. \quad (33) \end{aligned}$$

For each pair (j, j') , we translate the initial data by the midpoint

$$x_{jj'} := \frac{j + j'}{2} N^2$$

of Q_j and $Q_{j'}$, using [Lemma 1.1](#) to write

$$u_T = \pi(z_{jj'}^t) a_T \tilde{\phi}_T =: \tilde{u}_T, \quad v_T = b_{T'} \pi(z_{jj'}^t) \tilde{\phi}_{T'} =: \tilde{v}_T,$$

where $z_{jj'} = (x_{jj'}, 0)$ and

$$\tilde{\phi}_T(t) = U^{(x_{jj'}, 0)}(t) \pi(-x_{jj'}, 0) \phi_T[0]$$

is a wavepacket solution for the modified potential $V^{(x_{jj'}, 0)}$. The norm on the right side above therefore can be written as

$$\left\| \sum_{T \in \tilde{\mathbf{T}}_j, T' \in \tilde{\mathbf{T}}'_j} \tilde{u}_T \tilde{v}_{T'} \right\|_{L^{(d+3)/(d+1)}},$$

where the initial positions $x(T)$ and $x(T')$ of the tubes now belong to the translated cubes $\tilde{Q}_j := Q_j - x_{jj'}$, $\tilde{Q}_{j'} - x_{jj'}$, which are now distance $\lesssim N^2$ from the origin (note however that the tubes in $\tilde{\mathbf{T}}_j$ are not simply translates of those in \mathbf{T}_j).

By simple bicharacteristic estimates and the wavepacket bounds [\(30\)](#), for large A the norm outside

$$\Omega_N := [-N^2, N^2] \times [-AN^2, AN^2]^d$$

is negligible:

$$\begin{aligned} \left\| \sum_{T \in \tilde{\mathbf{T}}_j, T' \in \tilde{\mathbf{T}}'_j} \tilde{u}_T \tilde{v}_{T'} \right\|_{L^{(d+3)/(d+1)}([-N^2, N^2] \times [-AN^2, AN^2]^c)} &\lesssim N^{-100d} \left(\sum_{T \in \tilde{\mathbf{T}}_j} |a_T|^2 \right)^{\frac{1}{2}} \left(\sum_{T' \in \tilde{\mathbf{T}}'_j} |b_{T'}|^2 \right)^{\frac{1}{2}} \\ &\lesssim N^{-100d} \left(\sum_{T \in \mathbf{T}_j} |a_T|^2 \right)^{\frac{1}{2}} \left(\sum_{T' \in \mathbf{T}'_j} |b_{T'}|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Inside Ω_N we invoke [6.4](#) using the fact that the $V^{(x_{jj'}, 0)}$ also satisfies the hypothesis [\(27\)](#) and that the wavepacket decompositions of u_N and v_N satisfy the relaxed Fourier support conditions in that proposition. Altogether, the right side of [\(33\)](#) is bounded by

$$\begin{aligned} \sum_{m \in \mathbb{Z}^d + \mathbb{Z}^d} \sum_{|j - j'| \lesssim 1, j + j' = m} N^\varepsilon \left(\sum_{T \in \mathbf{T}_j} |a_T|^2 \right)^{\frac{1}{2}} \left(\sum_{T' \in \mathbf{T}'_j} |b_{T'}|^2 \right)^{\frac{1}{2}} \\ &\lesssim N^\varepsilon \sum_m \left(\sum_{|j - \frac{m}{2}| \lesssim 1} \sum_{T \in \mathbf{T}_j} |a_T|^2 \right)^{\frac{1}{2}} \left(\sum_{|j' - \frac{m}{2}| \lesssim 1} \sum_{T' \in \mathbf{T}'_j} |b_{T'}|^2 \right)^{\frac{1}{2}} \\ &\lesssim N^\varepsilon \left(\sum_T |a_T|^2 \right)^{\frac{1}{2}} \left(\sum_{T'} |b_{T'}|^2 \right)^{\frac{1}{2}} \\ &\lesssim N^\varepsilon \|f\|_{L^2} \|g\|_{L^2}, \end{aligned}$$

thus recovering [Theorem 6.1](#). \square

6E. Induction on scales. Our induction scheme is set up slightly differently from Tao's to accommodate the nonconservation of frequency support of solutions.

In this section, we explicitly display the dependence of the propagator on the potential, and write $U_N^V(t) = U_N^V(t, 0)$ for the propagator with potential V_N .

Let $\text{IH}(\alpha)$ denote the following statement:

There exists $C_\alpha > 0$ such that for each $N \geq 1$ and for all potentials $V \in \mathcal{V}_\eta$, the estimate

$$\|U_N^V(t) f U_N^V(t) g\|_{L^{(d+3)/(d+1)}(\Omega_N)} \leq C_\alpha N^{2\alpha} \|f\|_{L^2} \|g\|_{L^2} \quad (34)$$

holds for all $f, g \in L^2(\mathbb{R}^d)$ with \hat{f}, \hat{g} supported in S_1 and S_2 , respectively.

We prove:

Inductive Step: If $\text{IH}(\alpha)$ holds, then $\text{IH}(\max((1-\delta)\alpha, C\delta) + \varepsilon)$ holds for all $0 < \delta, \varepsilon \ll 1$.

By choosing δ and ε sufficiently small depending on α , we can always arrange that

$$\max((1-\delta)\alpha, C\delta) + C\varepsilon < \alpha - c\alpha^2$$

for some absolute constant c , and [Theorem 6.4](#) follows.

The inductive hypothesis $\text{IH}(\alpha)$ shall be used to improve the estimate (34) over subregions $Q_R \subset \Omega_N$ at smaller scales $\text{diam}(Q_R) \sim N^{2(1-\delta)} \ll N^2$.

Proposition 6.6. *Suppose $\text{IH}(\alpha)$ holds. Then for all $1 \leq R \leq \frac{1}{16}N^2$ and all space-time balls $Q_R \subset 2\Omega_N$ of diameter R , the estimate*

$$\|U_N^V(t) f U_N^V(t) g\|_{L^{(d+3)/(d+1)}(Q_R)} \leq C_\alpha R^\alpha \|f\|_{L^2} \|g\|_{L^2}$$

holds for all $f, g \in L^2(\mathbb{R}^d)$ with \hat{f}, \hat{g} supported in $\tilde{S}_1 := S_1 + B(0, \frac{c}{100})$ and $\tilde{S}_2 := S_2 + B(0, \frac{c}{100})$, respectively.

Proof. We begin by estimating how much the Fourier supports can shift.

Lemma 6.7. *For $1 \leq R \leq N^2$, let $Q_R \subset 2\Omega_N$ be a space-time ball with center (t_Q, x_Q) and diameter R . Suppose the initial data f, g satisfy $\text{supp}(\hat{f}) \subset \tilde{S}_1$ and $\text{supp}(\hat{g}) \subset \tilde{S}_2$. There exist decompositions $u(t_Q) = f_1 + f_2$ and $v(t_Q) = g_1 + g_2$, with the following properties:*

- \hat{f}_1 and \hat{g}_1 are supported in sets S'_1, S'_2 with $\text{diam}(S'_j) \leq \frac{c}{10}$ and $\text{dist}(S'_1, S'_2) \in [\frac{4c}{5}, \frac{5c}{4}]$.
- $\|f_2\|_{L^2} \lesssim N^{-100d} \|f\|_{L^2}$ and $\|g_2\|_{L^2} \lesssim N^{-100d} \|g\|_{L^2}$.

Proof. Begin by decomposing $u = U_N^V f$ and $v = U_N^V g$ into $N^2 \times (N)^d$ wavepackets:

$$u = \sum_{T \in \mathbf{T}_1} a_T \phi_T, \quad v = \sum_{T \in \mathbf{T}_2} b_T \phi_T. \quad (35)$$

By the spatial localization (30), we may ignore in u and v the packets whose tubes $T \in \mathbf{T}_j$ do not intersect $2Q_N := [-N^2, N^2] \times [-2AN^2, 2AN^2]$, as the portion of the sum involving those terms contributes at most $O(N^{-100d}) \|f\|_{L^2} \|g\|_{L^2}$. Thus there are $O(N^{2d})$ remaining terms.

Suppose ϕ_{T_1} and ϕ_{T_2} are wavepackets in the decomposition for u .

Let (x_1^t, ξ_1^t) and (x_2^t, ξ_2^t) be bicharacteristics with $|x_1|, |x_2| \leq 2AN^2$. By (19), for $|t| \leq \tau_0 N^2$ we have

$$\begin{aligned} |\xi_1^t - \xi_2^t - (\xi_1 - \xi_2)| &\leq C \tau_0 N^2 N^{-4} \|\partial_x^2 V\|_{L^\infty} (2AN^2 + \tau_0 N^2 |\xi_1 - \xi_2|) \\ &\leq C(\tau_0 A + \tau_0^2) \|\partial_x^2 V\|_{L^\infty} \leq C\eta. \end{aligned}$$

Therefore, recalling the definitions of \tilde{S}_j , we see that we have $|\xi_1^{t_Q} - \xi_2^{t_Q}| \leq \frac{c}{20} + C\eta$ if ξ_1, ξ_2 both belong to \tilde{S}_1 or \tilde{S}_2 , while $|\xi_1^{t_Q} - \xi_2^{t_Q}| \in [\frac{9c}{10}, \frac{10c}{9}]$ if $\xi_1 \in \tilde{S}_1$ and $\xi_2 \in \tilde{S}_2$. Choose $\eta = \eta(c)$ sufficiently small.

Consequently, if

$$\tilde{S}_j^t := \{\xi^t : \xi \in \tilde{S}_j, |x| \leq AN^2\} \quad (36)$$

denotes the set of frequencies of the wavepackets at time t , then $\text{diam}(\tilde{S}_j^t) \leq \text{diam}(S_j) + C\eta$ and $\text{dist}(\tilde{S}_1^t, \tilde{S}_2^t) \geq \frac{9}{10} \text{dist}(S_1, S_2)$. Now let S'_j denote $O(N^{-9/10})$ neighborhoods of \tilde{S}_j^t , and take the decompositions

$$u(t_Q) = f_1 + f_2, \quad v(t_Q) = g_1 + g_2,$$

where \hat{f}_1 is supported on \tilde{S}_1 and \hat{f}_2 on the complement, and similarly for g_1, g_2 . For N large enough we have $\text{dist}(S'_1 S'_2) \in [\frac{4c}{5}, \frac{5c}{4}]$. The estimates in the second bullet point now follow from the rapid decay of each wavepacket from its central frequency on the N^{-1} scale (the estimates (30) with $R = N^2$). \square

The proof of the proposition concludes with several applications of Lemma 1.1. Write

$$U(t, t_Q) f_1 = U(t, t_Q) \pi(x_Q, 0) \pi(-x_Q, 0) f_1 = \pi(z_Q^t) U^{z_Q}(t, t_Q) \tilde{f}_1 = \pi(z_Q^t) \tilde{u}(t + t_Q),$$

where $z_Q^t = (x_Q, 0)$. For $|t - t_Q| \leq R$ and $|x_Q| \leq AN^2$ we have $|x_Q^t - x_Q^{t_Q}| \leq 2|t - t_Q| \leq 2R$ provided that η is sufficiently small. Therefore, letting $\tilde{Q}_R = 2(Q_R - (t_Q, x_Q))$,

$$\|uv\|_{L^{(d+3)/(d+1)}(Q_R)} \lesssim \|\tilde{u}\tilde{v}\|_{L^{(d+3)/(d+1)}(\tilde{Q}_R)} + N^{-100d} \|f\|_{L^2} \|g\|_{L^2}.$$

It remains to consider the first term on the right side. The initial data \tilde{f}_1, \tilde{g}_1 for \tilde{u} and \tilde{v} have Fourier transforms supported in S'_1, S'_2 . We abuse notation and redefine

$$f := \tilde{f}_1, \quad g := \tilde{g}_1.$$

Cover $S'_j = \bigcup_k B_{j,k}$ by finitely overlapping balls of radius $\frac{c}{200}$. Using a subordinate partition of unity, we reduce to the case where $\text{supp } \hat{f} \subset B_{1,k_1}$ and $\text{supp } \hat{g} \subset B_{2,k_2}$. Again using Lemma 1.1, we may assume $B_{1,k_1} = -B_{2,k_2}$ and that their centers lie on the e_1 -axis.

Since $2c \geq \text{dist}(B_{1,k_1}, B_{2,k_2}) \geq \frac{c}{2}$, there exists some scaling factor $\lambda \in [\frac{1}{2}, 2]$ such that $\lambda^{-1} B_{j,k_j} \subset S_j$. Consider the rescalings

$$u_\lambda = U_{\frac{N}{\lambda}}^V(t) f_\lambda = U_{(2R)^{1/2}}^{\tilde{V}}(t) f_\lambda, \quad v_\lambda = U_{\frac{N}{\lambda}}^V(t) g_\lambda = U_{(2R)^{1/2}}^{\tilde{V}}(t) g_\lambda,$$

where

$$\tilde{V}(t, x) = 2R\lambda^2 N^{-2} V(2R\lambda^2 N^{-2} t, (2R)^{\frac{1}{2}} \lambda N^{-1} x).$$

The potential \tilde{V} satisfies $\|\partial_x^2 \tilde{V}\|_{L^\infty} \leq \|\partial_x^2 V\|_{L^\infty}$ since $2R\lambda^2 N^{-2} \leq 8RN^{-2} \leq \frac{1}{2}$, and $\hat{u}_\lambda(0)$ and $\hat{v}_\lambda(0)$ are supported in S_1 and S_2 . Hence we can apply IH(α) to conclude that

$$\|\tilde{u}\tilde{v}\|_{L^{(d+3)/(d+1)}(\tilde{Q}_R)} \lesssim \|u_\lambda v_\lambda\|_{L^{(d+3)/(d+1)}(\tilde{Q}_{2R})} \leq C_\alpha R^\alpha \|f_\lambda\|_{L^2} \|g_\lambda\|_{L^2}. \quad \square$$

From here on the argument hews closely to Tao's. We recall the following notation: write

$$A \lesssim B$$

if $A \lesssim_\varepsilon N^\varepsilon B$ for all $N \gg 1$ and for all $\varepsilon > 0$.

To reiterate, we want to prove

$$\|U_N^V f U_N^V g\|_{L^{(d+3)/(d+1)}(\Omega_N)} \lesssim N^{2\max((1-\delta)\alpha, C\delta)} \|f\|_{L^2} \|g\|_{L^2}, \quad (37)$$

assuming $\text{supp}(\hat{f}) \subset S_1$ and $\text{supp}(\hat{g}) \subset S_2$ with $\text{diam}(S_j) \leq 1$ and $\text{dist}(S_1, S_2) \geq c$.

Normalize f and g in L^2 , and take the decomposition

$$u := U_N^V f = \sum_T a_T \phi_T, \quad v := U_N^V g = \sum_T b_T \phi_T.$$

As in the proof of [Lemma 6.7](#), we discard all but the $O(N^{2d})$ wavepackets whose tubes intersect $2\Omega_N$. We also throw away the terms where $|a_T| = O(N^{-100d})$ or $|b_T| = O(N^{-100d})$, as that portion of the product can be bounded using the estimates [\(30\)](#) and Cauchy–Schwarz.

Consequently, in the decompositions of u and v we only consider the tubes T with $N^{-100d} \lesssim |a_T|, |b_T| \lesssim 1$. Partitioning the interval $[N^{-100d}, 1]$ into $\log N$ dyadic groups, we may further restrict to the tubes with $|a_T| \sim \gamma_1$ and $|b_T| \sim \gamma_2$ for dyadic numbers $N^{-100d} \lesssim \gamma_1, \gamma_2 \lesssim 1$. Let $\mathbf{T}_1, \mathbf{T}_2$ be the tubes for u and v , respectively, with this property. It therefore suffices to prove

$$\left\| \sum_{T_1 \in \mathbf{T}_1} \phi_{T_1} \sum_{T_2 \in \mathbf{T}_2} \phi_{T_2} \right\|_{L^{(d+3)/(d+1)}(\Omega_N)} \lesssim (N^{2(1-\delta)\alpha} + N^{2C\delta}) \# \mathbf{T}_1^{\frac{1}{2}} \# \mathbf{T}_2^{\frac{1}{2}}$$

(we have absorbed the complex phases into the wavepackets).

We have in effect reduced to considering the region of the phase space $\{(x, \xi) : |x| \lesssim N^2, |\xi| \lesssim 1\}$, where the potential makes only a small perturbation to the Euclidean flow. For if $|x^s| \lesssim N^2$ and $|t - s| \leq N^2$, one has

$$\begin{aligned} |x^t| &\lesssim N^2, \\ |\xi^t - \xi^s| &\leq \int_s^t |\partial_x(V_N)(\tau, x^\tau)| d\tau \lesssim \int_s^t |x^\tau| \int_0^1 |\partial_x^2 V_N(\tau, sx^\tau)| ds d\tau \lesssim \tau_0 \|\partial_x^2 V\|_{L^\infty} \lesssim \eta. \end{aligned}$$

Thus if $\xi \in S_j$, then ξ^t belongs to a small neighborhood of S_j provided that $\eta \ll c$ is a small multiple of c . For concreteness we choose η so that

$$|\xi^t - \xi^s| \leq \frac{c}{100}. \quad (38)$$

6F. Coarse-scale decomposition. Following Tao, for small $\delta > 0$ we decompose $\Omega_N = \bigcup_{B \in \mathcal{B}'} B$ into $O(N^{2\delta d})$ smaller balls of radius $N^{2(1-\delta)}$ and estimate

$$\left\| \sum_{T_1 \in \mathcal{T}_1} \sum_{T_2 \in \mathcal{T}_2} \phi_{T_1} \phi_{T_2} \right\|_{L^{(d+3)/(d+1)}(\Omega_N)} \lesssim \sum_{B \in \mathcal{B}} \left\| \sum_{T_1 \in \mathcal{T}_1} \sum_{T_2 \in \mathcal{T}_2} \phi_{T_1} \phi_{T_2} \right\|_{L^{(d+3)/(d+1)}(B)}.$$

Let \sim be a relation between tubes and balls to be specified later. Estimate the norm by the local part

$$\sum_{B \in \mathcal{B}} \left\| \sum_{T_1 \sim B} \phi_{T_1} \sum_{T_2 \sim B} \phi_{T_2} \right\|_{L^{(d+3)/(d+1)}(B)} \quad (39)$$

and the global part

$$\sum_{B \in \mathcal{B}} \left\| \sum_{T_1 \sim B \text{ or } T_2 \sim B} \phi_{T_1} \phi_{T_2} \right\|_{L^{(d+3)/(d+1)}(B)}. \quad (40)$$

We use [Proposition 6.6](#) with $R = N^{2(1-\delta)} \leq \frac{1}{16} N^2$ to estimate the local term by

$$\begin{aligned} (39) &\lesssim \sum_{B \in \mathcal{B}} N^{2(1-\delta)\alpha} \left(\sum_{T_1 \sim B} 1 \right)^{\frac{1}{2}} \left(\sum_{T_2 \sim B} 1 \right)^{\frac{1}{2}} \\ &\lesssim \left(\sum_{T_1 \in \mathcal{T}_1} \#\{B : T_1 \sim B\} \right)^{\frac{1}{2}} \left(\sum_{T_2 \in \mathcal{T}_2} \#\{B : T_2 \sim B\} \right)^{\frac{1}{2}} \lesssim 1 \end{aligned}$$

if the relation \sim is chosen so that each T is associated to $\lesssim 1$ balls. Note that this step is why the Fourier supports are enlarged in that proposition, as $\text{supp}(\hat{\phi}_{T_1}(0))$ is not quite contained in S_1 .

Heuristically, a judicious choice of \sim allows one to avoid the worst interactions that would otherwise occur in the bilinear L^2 estimate if one were to natively interpolate between L^1 and L^2 . For example, if all the tubes were to intersect in a single ball B , it would be better to bound $L^{(d+3)/(d+1)}(B)$ directly using the inductive hypothesis rather than attempt to estimate $L^2(B)$.

The global piece (40) is controlled by interpolating between L^1 and L^2 . By Cauchy–Schwarz and conservation of L^2 norm,

$$\begin{aligned} &\sum_{B} \left\| \sum_{T_1 \sim B \text{ or } T_2 \sim B} \phi_{T_1} \phi_{T_2} \right\|_{L^1(B)} \\ &\lesssim \sum_{B} \left(\left\| \sum_{T_1 \sim B} \phi_T \right\|_{L^2(B)} + \left\| \sum_{T_2 \sim B} \phi_T \right\|_{L^2(B)} \right) \left(\left\| \sum_{T_2 \sim B} \phi_T \right\|_{L^2(B)} + \left\| \sum_{T_1 \sim B} \phi_T \right\|_{L^2(B)} \right) \\ &\lesssim N^{2\delta} N^2 \#T_1^{\frac{1}{2}} \#T_2^{\frac{1}{2}}. \end{aligned} \quad (41)$$

The remaining sections prove the L^2 estimate

$$\left\| \sum_{T_1 \sim B \text{ or } T_2 \sim B} \phi_{T_1} \phi_{T_2} \right\|_{L^2(B)} \lesssim N^{-\frac{d-1}{2}} N^{C\delta} \#T_1^{\frac{1}{2}} \#T_2^{\frac{1}{2}}. \quad (42)$$

6G. Fine scale decomposition. Cover $\Omega_N = \bigcup_{q \in \mathbf{q}} q$ by a finitely overlapping collection \mathbf{q} of balls of radius N . It suffices to show

$$\sum_{q \in \mathbf{q}: q \subset 2B} \left\| \sum_{T_1 \sim B \text{ or } T_2 \sim B} \phi_{T_1} \phi_{T_2} \right\|_{L^2(q)}^2 \lesssim N^{-(d-1)} N^{C\delta} \# \mathbf{T}_1 \mathbf{T}_2.$$

We adopt the following notation from Tao. Fix $q \in \mathbf{q}$ and let μ_1, μ_2, λ_1 be dyadic numbers:

- $\mathbf{T}_j(q)$ is the set of tubes $T \in \mathbf{T}_j$ such that $T \cap N^\delta q$ is nonempty, where $N^\delta q$ denotes an N^δ neighborhood of q .
- $\mathbf{T}_j^{\sim B}(q) = \{T \in \mathbf{T}_j(q) : T \sim B\}$.
- $\mathbf{q}(\mu_1, \mu_2)$ is the set of balls q such that $\#\{T_j \in \mathbf{T}_j : T_j \cap N^\delta q \neq \emptyset\} \sim \mu_j$.
- $\lambda(T, \mu_1, \mu_2)$ is the number of (N^δ neighborhoods of) balls $q \in \mathbf{q}(\mu_1, \mu_2)$ that T intersects.
- $\mathbf{T}_j[\lambda_1, \mu_1, \mu_2]$ is the set of tubes $T \in \mathbf{T}_j$ such that $\lambda(T, \mu_1, \mu_2) \sim \lambda_1$.

Pigeonholing dyadically in μ_1, μ_2 , and λ_1 , it suffices to show

$$\sum_{q \in \mathbf{q}(\mu_1, \mu_2): q \subset 2B} \left\| \sum_{T_1 \in \mathbf{T}_1^{\sim B}(q) \cap \mathbf{T}_1[\lambda_1, \mu_1, \mu_2]} \sum_{T_2 \in \mathbf{T}_2(q)} \phi_{T_1} \phi_{T_2} \right\|_{L^2(q)}^2 \lesssim N^{C\delta} N^{-(d-1)} \# \mathbf{T}_1 \# \mathbf{T}_2.$$

6H. The L^2 bound. Fix a ball $q = q(t_q, x_q) \in \mathbf{q}(\mu_1, \mu_2)$ centered at (t_q, x_q) . Suppose want to estimate an expression of the form

$$\left\| \sum_{T_1} \sum_{T_2} \phi_{T_1} \phi_{T_2} \right\|_{L^2(q)}^2.$$

There are two main points to keep in mind:

- Only tubes that intersect $N^\delta q$ will make a nontrivial contribution; that is, tubes whose bicharacteristics (x^t, ξ^t) satisfy $|x^{t_q} - x_q| \leq N^{1+\delta}$.
- To decouple the contributions of tubes that all overlap near q , one needs to exploit oscillation in space and time. While Tao employs the space-time Fourier transform, we instead integrate by parts in space and time. Expanding out the L^2 norm

$$\sum_{T_1, T_2} \sum_{T_3, T_4} \langle \phi_{T_1} \phi_{T_2}, \phi_{T_3} \phi_{T_4} \rangle \tag{43}$$

and integrating by parts in both space and time, we shall obtain terms of the form

$$(N |\xi_1^t + \xi_2^t - \xi_3^t - \xi_4^t|)^{-1}, \quad (N ||\xi_1^t - \xi_2^t|^2 - |\xi_3^t - \xi_4^t|^2|)^{-1},$$

where (x_j^t, ξ_j^t) are bicharacteristics with $|x_j^{t_q} - x_q| \leq N^{1+\delta}$. Since, by (19), the relative frequencies $\xi_j^t - \xi_k^t$ vary by at most $O(N^{-2+2\delta})$ during the $O(N^{1+\delta})$ time window when the wavepackets intersect the ball $N^\delta q$, we can freeze $t = t_q$ above; see [Lemma 6.10](#) below.

Hence, the integral (43) will be small unless $|x_j^{t_q} - x_q| \leq N^{1+\delta}$ for all j and the frequencies ξ_j^t satisfy both resonance conditions

$$|\xi_1^{t_q} + \xi_2^{t_q} - \xi_3^{t_q} - \xi_4^{t_q}| = O(N^{-1}), \quad |\xi_1^{t_q} - \xi_2^{t_q}|^2 - |\xi_3^{t_q} - \xi_4^{t_q}|^2 = O(N^{-1}). \quad (44)$$

The preceding discussion motivates the following definition. Let

$$Z_{q,j} := \{(x, \xi) : |x| \leq 2AN^2, \xi \in S_j, |x^{t_q} - x_q| \leq N^{1+\delta}\}.$$

For frequencies ξ_1 and ξ_2' , define the “space-time resonance” set

$$\begin{aligned} Z(\xi_1, \xi_2') &= \{(x'_1, \xi_1') \in Z_{q,1} : \text{there exists } (x_2, \xi_2) \in Z_{q,2} \text{ such that} \\ &\quad \xi_1 + \xi_2^{t_q} = (\xi_1')^{t_q} + \xi_2' \text{ and } |\xi_1 - \xi_2^{t_q}|^2 = |(\xi_1')^{t_q} - \xi_2'|^2\}, \\ \pi(\xi_1, \xi_2') &= \{(\xi_1')^{t_q} : (x'_1, \xi_1') \in Z(\xi_1, \xi_2')\}. \end{aligned}$$

This is a slight modification of Tao’s definition which reflects the time dependence of frequency.

The following lemma follows from elementary geometry.

Lemma 6.8. *The set $\pi(\xi_1, \xi_2')$ is contained in the hyperplane passing through ξ_1 and orthogonal to $\xi_2' - \xi_1$ and is therefore transverse to $\xi_2' - \xi_1$ if ξ_1 and ξ_2' are small perturbations of ξ_1 and ξ_2' , respectively.*

Due to the limited time regularity of the phase, we can actually integrate by parts just once in time. The resulting weaker decay still turns out to be just enough provided that we slightly refine the analogue of Tao’s main combinatorial estimate for tubes (estimate (48) below). Hence we need to account more carefully for the contributions away from the “resonant set” π .

For ξ_1, ξ_2' and $k > 0$, define the “time nonresonance” sets

$$\begin{aligned} Z_0^t(\xi_1, \xi_2') &= \{(x'_1, \xi_1') \in Z_{q,1} : \text{there exists } (x_2, \xi_2) \in Z_{q,2} \text{ such that } |\xi_1 + \xi_2^{t_q} - (\xi_1')^{t_q} - \xi_2'| \leq N^{-1+C\delta} \\ &\quad \text{and } ||\xi_1 - \xi_2^{t_q}|^2 - |(\xi_1')^{t_q} - \xi_2'|^2| \leq N^{-1+C\delta}\}, \end{aligned}$$

$$\begin{aligned} Z_k^t(\xi_1, \xi_2') &= \{(x'_1, \xi_1') \in Z_{q,1} : \text{for all } (x_2, \xi_2) \in Z_{q,2} \text{ with } |\xi_1 + \xi_2^{t_q} - (\xi_1')^{t_q} - \xi_2'| \leq N^{-1+C\delta}, \\ &\quad ||\xi_1 - \xi_2^{t_q}|^2 - |(\xi_1')^{t_q} - \xi_2'|^2| \in (2^{k-1}N^{-1+C\delta}, 2^kN^{-1+C\delta}]\}, \end{aligned}$$

the “space nonresonance” set

$$Z^s(\xi_1, \xi_2') = \{(x'_1, \xi_1') \in Z_{q,1} : |\xi_1 + \xi_2^{t_q} - (\xi_1')^{t_q} + \xi_2'| > N^{-1+C\delta} \text{ for all } (x_2, \xi_2) \in B_{q,2}\},$$

and the corresponding frequencies at time t_q

$$\begin{aligned} \pi_k^t(\xi_1, \xi_2') &= \{(\xi_1')^{t_q} : (x'_1, \xi_1') \in Z_k^t(\xi_1, \xi_2')\}, \\ \pi^s(\xi_1, \xi_2') &= \{(\xi_1')^{t_q} : (x'_1, \xi_1') \in Z^s(\xi_1, \xi_2')\}. \end{aligned}$$

An elementary computation shows that

$$\text{dist}(\pi_k^t, \pi) \lesssim 2^k N^{-1+C\delta}. \quad (45)$$

Indeed, writing $\delta_1 := (\xi'_1)^{t_q} - \xi_1$, $\delta_2 := \xi_2^{t_q} - \xi'_2$, and decomposing $\delta_j = \delta_j^\parallel + \delta_j^\perp$ into the components parallel and orthogonal to $\xi_1 - \xi'_2$, we have

$$\begin{aligned} |\xi_1 - \xi_2^{t_q}|^2 - |(\xi'_1)^{t_q} - \xi'_2|^2 &= |\xi_1 - \xi'_2 - \delta_2|^2 - |\delta_1 + \xi_1 - \xi'_2|^2 \\ &= -2\langle \xi_1 - \xi'_2, \delta_1 + \delta_2 \rangle + \delta_2^2 - \delta_1^2 \\ &= -2\langle \xi_1 - \xi'_2, \delta_1^\parallel + \delta_2^\parallel \rangle + O(N^{-1+C\delta}) \quad (\text{since } |\delta_1 - \delta_2| \leq N^{1+\delta}) \\ &= -4\langle \xi_1 - \xi'_2, \delta_1^\parallel \rangle + O(N^{-1+C\delta}). \end{aligned}$$

Thus $|(\xi'_1)^{t_q} - \xi_1, \xi_1 - \xi'_2| \lesssim 2^k N^{-1+C\delta}$ and the claim follows from Lemma 6.8.

For $q \in \mathbf{q}(\mu_1, \mu_2)$ with $q \subset 2B$, define

$$\mathbf{T}_1^{\sim B}(q, \lambda_1, \mu_1, \mu_2, \xi_1, \xi'_2, k)$$

to be the collection of tubes $T \in \mathbf{T}_1^{\sim B}(q) \cap \mathbf{T}_1[\lambda_1, \mu_1, \mu_2]$ such that $\xi(T)^{t_q} \in \pi_k^t(\xi_1, \xi'_2)$. Set

$$v_k(q, \lambda_1, \mu_1, \mu_2) := \sup_{\xi_1 \in S_1, \xi'_2 \in S_2} \#\mathbf{T}_1^{\sim B}(q, \lambda_1, \mu_1, \mu_2, \xi_1^{t_q}, (\xi'_2)^{t_q}, k), \quad (46)$$

where $|x_1^{t_q} - x_q| + |(x'_2)^{t_q} - x_q| \lesssim N^{1+\delta}$.

Then, the analogue of Tao's Lemma 7.1 is:

Lemma 6.9. *For each $q \in \mathbf{q}(\mu_1, \mu_2)$, we have*

$$\begin{aligned} &\left\| \sum_{T_1 \in \mathbf{T}_1^{\sim B}(q) \cap \mathbf{T}_1[\lambda_1, \mu_1, \mu_2]} \sum_{T_2 \in \mathbf{T}_2(q)} \phi_{T_1} \phi_{T_2} \right\|_{L^2(q)}^2 \\ &\lesssim N^{C\delta} N^{-(d-1)} \sup_k 2^{-k} v_k(q, \lambda_1, \mu_1, \mu_2) \#(\mathbf{T}_1^{\sim B}(q) \cap \mathbf{T}_1[\lambda_1, \mu_1, \mu_2]) \# \mathbf{T}_2(q). \end{aligned}$$

Proof. For conciseness, set

$$\begin{aligned} \mathbf{T}'_1 &:= \mathbf{T}_1^{\sim B}(q) \cap \mathbf{T}_1[\lambda_1, \mu_1, \mu_2], \\ \mathbf{T}_2 &:= \mathbf{T}_2(q). \end{aligned}$$

Then the norm $L^2(q)$ is bounded by the norm $L^2(\eta_N dx dt)$, where $\eta_N(t)$ is a smooth weight equal to 1 on $|t - t_q| \leq N^{1+\delta}$ and supported in $|t - t_q| \leq 2N^{1+\delta}$:

$$\left\| \sum_{T_1 \in \mathbf{T}'_1} \sum_{T_2 \in \mathbf{T}'_2} \phi_{T_1} \phi_{T_2} \right\|_{L^2(\eta_N dx dt)}^2 = \sum_{T_1, T'_1 \in \mathbf{T}'_1} \sum_{T_2, T'_2 \in \mathbf{T}'_2} \langle \phi_{T_1} \phi_{T_2}, \phi_{T'_1} \phi_{T'_2} \rangle_{L^2(\chi_N dx dt)}.$$

By the bounds (30) and the transversality of the tubes in \mathbf{T}'_1 and \mathbf{T}'_2 , the integrand has magnitude N^{-2d} and is essentially supported on a space-time ball of width N . Thus we have the crude bound

$$|\langle \phi_{T_1} \phi_{T_2}, \phi_{T'_1} \phi_{T'_2} \rangle| \lesssim N^{C\delta} N^{-2d} N^{d+1} = N^{C\delta} N^{-(d-1)}.$$

On the other hand, we may integrate by parts to obtain a more refined bound.

Lemma 6.10. *For each $k_1, k_2, \ell \geq 0$ and for all tubes $T_1, T_3 \in \mathbf{T}'_1$, $T_2, T_4 \in \mathbf{T}'_2$, we have*

$$|\langle \phi_{T_1} \phi_{T_2}, \phi_{T_3} \phi_{T_4} \rangle|$$

$$\lesssim_{k_1, k_2} N^{C\delta} N^{-(d-1)} \min[N^{-\ell} |\xi_1^{t_q} + \xi_2^{t_q} - \xi_3^{t_q} - \xi_4^{t_q}|^{-\ell}, N^{-1} ||\xi_1^{t_q} - \xi_2^{t_q}|^2 - |\xi_3^{t_q} - \xi_4^{t_q}|^2|^{-1}].$$

Proof. The proof has a similar flavor to the earlier estimate (23) but takes advantage of oscillation in both space and time.

Let $z_j^t = (x_j^t, \xi_j^t)$ denote the bicharacteristic for ϕ_{T_j} , $j = 1, 2, 3, 4$. By Lemmas 1.1 and 6.2, we can write

$$\langle \phi_{T_1} \phi_{T_2}, \phi_{T_3} \phi_{T_4} \rangle = \int e^{i\Psi} \phi_1 \phi_2 \bar{\phi}_3 \bar{\phi}_4 \eta_N(t) dx dt, \quad (47)$$

where ϕ_j is a Schrödinger solution which satisfies

$$(N \partial_x)^k \phi_j(t, x) \lesssim_{k, M} N^{-\frac{d}{2}} \langle N^{-1} (x - x_j^t) \rangle^{-M},$$

and

$$\Psi = \sum_{j=1}^4 \sigma_j \left[\langle x - x_j^t, \xi_j^t \rangle - \int_0^\tau \frac{1}{2} |\xi_j^\tau|^2 - V(\tau, x_j^\tau) d\tau \right], \quad \sigma = (+, +, -, -).$$

Using the rapid decay of each ϕ_j , we may harmlessly (with $O(N^{-100d})$ error) localize ϕ_j to an N^δ neighborhood of the tube T_j , so that $\phi_j(t)$ is supported in an $O(N^{1+\delta})$ neighborhood of the classical path x_j^t .

Then

$$\partial_x \Psi = \sum_j \sigma_j \xi_j^t, \quad -\partial_t \Psi = \frac{1}{2} \sum_j \sigma_j |\xi_j^t|^2 + \sum_j \sigma_j [V(t, x_j^t) + \langle x - x_j^t, \partial_x V(t, x_j^t) \rangle].$$

The first bound in the statement of the lemma results from integrating by parts in x , as in the proof of (23), to gain factors of $(N |\xi_1^t + \xi_2^t - \xi_3^t - \xi_4^t|)^{-1}$. Since

$$\xi_1^t + \xi_2^t - \xi_3^t - \xi_4^t = \xi_1^{t_q} + \xi_2^{t_q} - \xi_3^{t_q} - \xi_4^{t_q} + O(N^{-2+2\delta})$$

during the time window $|t - t_q| \leq O(N^{1+\delta})$ when $|x_j^t - x_q| \leq N^{1+\delta}$, we may replace t by t_q .

As in our work in one space dimension (more specifically, the proof of [Jao et al. 2019, Lemma 4.4]), instead of integrating by parts purely in time we use a vector field adapted to the average bicharacteristic for the four wavepackets ϕ_{T_j} . Defining

$$\bar{x}^t := \frac{1}{4} \sum_{j=1}^4 x_j^t, \quad \bar{\xi}^t := \sum_{j=1}^4 \xi_j^t, \quad L := \partial_t + \langle \bar{\xi}^t, \partial_x \rangle,$$

we compute as in that paper that

$$-L \Psi = \frac{1}{2} \sum_j \sigma_j |\bar{\xi}_j^t|^2 + \sum_j \sigma_j [V^{\bar{z}}(t, \bar{x}_j^t) + \langle x - x_j^t, \partial_x (V^{\bar{z}})(t, \bar{x}_j^t) \rangle],$$

where

$$\bar{x}_j^t := x_j^t - \bar{x}^t, \quad \bar{\xi}_j^t := \xi_j^t - \bar{\xi}^t$$

denote the coordinates of $\phi_{T_j}(t)$ in phase space relative to $(\bar{x}^t, \bar{\xi}^t)$; see Figure 2.

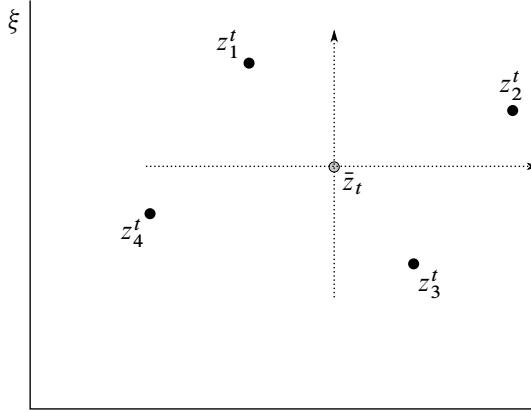


Figure 2. Phase space coordinates relative to the “center of mass”.

We cannot yet integrate by parts since that would require two time derivatives of the phase Ψ , but the assumptions on V only allow Ψ to be differentiated once in time. However, we can decompose Ψ into $\Psi = \Psi_1 + \Psi_2$, where Ψ_2 has two time derivatives and accounts for the majority of the oscillation of $e^{i\Psi}$; indeed, we define Ψ_1 and Ψ_2 via the ODE

$$\begin{aligned} -L\Psi_2 &= \frac{1}{2} \sum_j \sigma_j |\bar{\xi}_j^t|^2 = \frac{1}{4} (|\xi_1^{t_q} - \xi_2^{t_q}|^2 - |\xi_3^{t_q} - \xi_4^{t_q}|^2) + O(N^{-2+2\delta}), \\ -L\Psi_1 &= \sum_j \sigma_j [V^{\bar{z}}(t, \bar{x}_j^t) + \langle x - x_j^t, \partial_x(V^{\bar{z}})(t, \bar{x}_j^t) \rangle] = O(N^{-2+2\delta}). \end{aligned}$$

As before we have frozen $t = t_q$ in the main term with error at most $O(N^{-2+2\delta})$, and also used the estimates $|\bar{x}_j^t| \leq \max_{j,k} |x_j^t - x_k^t| \lesssim N^{1+\delta}$, $|x - x_j^t| \lesssim N^{1+\delta}$ on the support of the integrand (47). Note also that the equation

$$\frac{d}{dt} \xi_j^t = -\partial_x V(t, x_j^t)$$

implies $L^2 \Psi_2 = O(N^{-2})$. Now integrate by parts using the phase Ψ_2 to obtain

$$\begin{aligned} \text{RHS (47)} &= \int e^{i\Psi_2} e^{i\Psi_1} \prod_j \phi_j \eta_N(t) dx dt = i \int e^{i\Psi_2} \left\langle L, \frac{L\Psi_2}{|L\Psi_2|^2} \right\rangle e^{i\Psi_1} \phi_1 \phi_2 \bar{\phi}_3 \bar{\phi}_4 \eta_N(t) dx dt \\ &= i \int e^{i\Psi} \left[-\frac{L^2 \Psi_2}{|L\Psi_2|^2} + \left\langle \frac{L\Psi_2}{|L\Psi_2|^2}, iL\Psi_1 + L \right\rangle \right] \phi_1 \phi_2 \bar{\phi}_3 \bar{\phi}_4 \eta_N(t) dx dt, \end{aligned}$$

and the second bound in the lemma follows. \square

Returning to the proof of Lemma 6.9, we decompose the sum into

$$\sum_{(T_1, T_2') \in \mathbf{T}'_1 \times \mathbf{T}'_2} \left[\sum_{T_1' \in \mathbf{T}'_1} \sum_{T_2 \in \mathbf{T}'_2} + \sum_{0 \leq k \lesssim \log N} \sum_{T_1' \in \mathbf{T}'_{1,k}} \sum_{T_2 \in \mathbf{T}'_2} \right],$$

where \mathbf{T}_1^s is the set of tubes in \mathbf{T}_1' whose bicharacteristic $((x'_1)^t, (\xi'_1)^t)$ satisfies $(\xi'_1)^{t_q} \in \pi^s(\xi_1^{t_q}, (\xi'_2)^{t_q})$, and we abbreviate

$$\mathbf{T}_{1,k}' := \mathbf{T}_1^{\sim B}(q, \lambda_1, \mu_1, \mu_2, \xi_1^{t_q}, (\xi'_2)^{t_q}, k).$$

The contribution from the “space nonresonance” terms \mathbf{T}_1^s is $O(N^{-100d})$.

Now consider the k -th sum. [Lemma 6.10](#) implies

$$|\langle \phi_{T_1} \phi_{T_2}, \phi_{T_1'} \phi_{T_2'} \rangle| \lesssim N^{C\delta} N^{-(d-1)} 2^{-k}.$$

For each $T_1' \in \mathbf{T}_1^{\sim B}(q, \lambda_1, \mu_1, \mu_2, \xi_1^{t_q}, (\xi'_2)^{t_q}, k)$, the possible tubes T_2 correspond to the bicharacteristics (x_2^t, x_2^t) such that

$$|x_2^{t_q} - x_q| \leq N^{1+\delta}, \quad \xi_1^{t_q} + \xi_2^{t_q} - (\xi'_1)^{t_q} - (\xi'_2)^{t_q} = O(N^{-1+C\delta}).$$

The preimage of this set under the time- t_q Hamiltonian flow map is an $(N^{1+C\delta})^d \times (N^{-1+C\delta})^{-d}$ box, so there are $O(N^{C\delta})$ choices of tubes T_2 . Therefore, the k -th sum is at most

$$N^{C\delta} N^{-(d-1)} 2^{-k} v_k \# \mathbf{T}_1' \# \mathbf{T}_2',$$

whereupon the sum over k is replaced by the supremum at the cost of a $\log N$ factor. \square

It remains to show that

$$\sum_{q \in \mathbf{q}(\mu_1, \mu_2): q \subset 2B} 2^{-k} v_k(q, \lambda_1, \mu_1, \mu_2) \# (\mathbf{T}_1^{\sim B}(q) \cap \mathbf{T}_1[\lambda_1, \mu_1, \mu_2]) \# \mathbf{T}_2(q) \lesssim N^{C\delta} \# \mathbf{T}_1 \# \mathbf{T}_2. \quad (48)$$

6I. Tube combinatorics. This section begins exactly as in [\[Tao 2003, Section 8\]](#). We define the relation \sim between tubes and radius $N^{2(1-\delta)}$ balls. For a tube $T \in \mathbf{T}_1[\lambda_1, \mu_1, \mu_2]$, let $B(T, \lambda_1, \mu_1, \mu_2)$ be a ball $B \in \mathcal{B}$ that maximizes

$$\#\{q \in \mathbf{q}(\mu_1, \mu_2) : T \cap N^\delta q \neq \phi; q \cap B \neq \phi\}.$$

As T intersects roughly λ_1 (neighborhoods of) balls $q \in \mathbf{q}(\mu_1, \mu_2)$ in total and there are $O(N^{2\delta})$ many balls in \mathcal{B} , $B(T, \lambda_1, \mu_1, \mu_2)$ must intersect at least $N^{-2\delta} \lambda_1$ of those balls.

Declare $T \sim_{\lambda_1, \mu_1, \mu_2} B'$ if $T \in \mathbf{T}_1[\lambda_1, \mu_1, \mu_2]$ and $B' \subset 10B(T, \lambda_1, \mu_1, \mu_2)$. Finally, for $T \in \mathbf{T}_1$ set $T \sim B$ if $T \sim_{\lambda_1, \mu_1, \mu_2} B$ for some λ_1, μ_1, μ_2 . Evidently $T \sim B$ for at most $(\log N)^3 \lesssim 1$ many balls. The relation between tubes in \mathbf{T}_2 and balls in \mathcal{B} is defined similarly.

Now we begin the proof of (48). On one hand,

$$\begin{aligned} \sum_{q \in \mathbf{q}(\mu_1, \mu_2)} \# (\mathbf{T}_1[\lambda_1, \mu_1, \mu_2] \cap \mathbf{T}_1(q)) &= \sum_{q \in \mathbf{q}(\mu_1, \mu_2)} \sum_{T_1 \in \mathbf{T}_1[\lambda_1, \mu_1, \mu_2] \cap \mathbf{T}_1(q)} 1_{T_1 \cap N^\delta q \neq \phi} \\ &= \sum_{T \in \mathbf{T}_1[\lambda_1, \mu_1, \mu_2]} \sum_{q \in \mathbf{q}(\mu_1, \mu_2)} 1_{T \cap N^\delta q \neq \phi} \lesssim \sum_{T \in \mathbf{T}_1} \lambda_1 = \lambda_1 \# \mathbf{T}_1. \end{aligned}$$

On the other hand, by definition $\# \mathbf{T}_2(q) \lesssim \mu_2$. The claim (48) would therefore follow if we could show

$$v_k(q_0, \lambda_1, \mu_1, \mu_2) \lesssim 2^k N^{C\delta} \frac{\# \mathbf{T}_2}{\lambda_1 \mu_2} \quad (49)$$

for all $q_0 \in \mathbf{q}(\mu_1, \mu_2)$ such that $q_0 \subset 2B$.

Fix $\xi_1 \in S_1$, $\xi'_2 \in S_2$, and a ball $q_0 = q_0(t_q, x_q)$. Recalling the definition (46) of ν_k , we need to show

$$\#\mathbf{T}_1^{\sim B}(q_0, \lambda_1, \mu_1, \mu_2, \xi_1^{t_q}, (\xi'_2)^{t_q}, k) \lesssim 2^k N^{C\delta} \frac{\#\mathbf{T}_2}{\lambda_1 \mu_2}.$$

For brevity write $\mathbf{T}'_1 := \mathbf{T}_{1,k}^{\sim B}(q_0, \lambda_1, \mu_1, \mu_2, \xi_1^{t_q}, (\xi'_2)^{t_q}, k)$.

Fix $T_1 \in \mathbf{T}'_1$. Since $T_1 \sim B$, the ball $2B(T_1, \lambda_1, \mu_1, \mu_2)$ has distance $\gtrsim N^{2(1-\delta)}$ from q_0 . Thus

$$\#\{q \in \mathbf{q}(\mu_1, \mu_2) : T_1 \cap N^\delta q \neq \emptyset, \text{dist}(q, q_0) \gtrsim N^{2(1-\delta)}\} \gtrsim N^{-2\delta} \lambda_1.$$

As each $q \in \mathbf{q}(\mu_1, \mu_2)$ intersects approximately μ_2 (N^δ -neighborhoods of) tubes in \mathbf{T}_2 ,

$$\#\{(q, T_2) \in \mathbf{q}(\mu_1, \mu_2) \times \mathbf{T}_2 : T_1 \cap N^\delta q \neq \emptyset, T_2 \cap N^\delta q \neq \emptyset, \text{dist}(q, q_0) \gtrsim N^{2(1-\delta)}\} \gtrsim N^{-2\delta} \lambda_1 \mu_2.$$

Therefore

$$\#\{(q, T_1, T_2) \in \mathbf{q} \times \mathbf{T}'_1 \times \mathbf{T}_2 : T_1 \cap N^\delta q \neq \emptyset, T_2 \cap N^\delta q \neq \emptyset, \text{dist}(q, q_0) \gtrsim N^{-2\delta} N^2\} \gtrsim N^{-2\delta} \lambda_1 \mu_2 \#\mathbf{T}'_1.$$

On the other hand, the cardinality can be bounded above by the following analogue of Tao's Lemma 8.1:

Lemma 6.11. *For each $T_2 \in \mathbf{T}_2$,*

$$\#\{(q, T_1) \in \mathbf{q} \times \mathbf{T}'_1 : T_1 \cap N^\delta q \neq \emptyset, T_2 \cap N^\delta q \neq \emptyset, \text{dist}(q, q_0) \gtrsim N^{-2\delta} N^2\} \lesssim 2^k N^{C\delta}.$$

Proof. We estimate in two steps:

- For any tubes $T_1 \in \mathbf{T}'_1$ and $T_2 \in \mathbf{T}_2$, the intersection $N^\delta T_1 \cap N^\delta T_2$ is contained in a ball of radius $N^{C\delta}$.
- The number of tubes $T_1 \in \mathbf{T}'_1$ such that T_1 intersects $N^\delta T_2$ at distance $\gtrsim N^{-2\delta} N^2$ from q_0 bounded above by $2^k N^{C\delta}$.

The first is evident from transversality. Hence we turn to the second claim.

In Tao's situation, the tubes in \mathbf{T}'_1 are all constrained to an $O(N^{-1+C\delta})$ neighborhood of a space-time hyperplane transverse to the tube T_2 (basically because of Lemma 6.8), and there are $O(N^{C\delta})$ many such tubes that intersect T_2 at distance $\gtrsim N^{-2\delta} N^2$ from q_0 . The extra 2^k factor results from the fact that we allow the tubes to deviate from that hyperplane by distance $2^k N^{-1+C\delta}$. Also, since our tubes are curved it is more convenient to work with their associated bicharacteristics instead of using Euclidean geometry in space-time.

Fix a tube $T_2 \in \mathbf{T}_2$ with ray $t \mapsto (x_2^t, \xi_2^t)$. Then, the tubes $T_1 \in \mathbf{T}'_1$ such that $N^\delta T_1 \cap N^\delta T_2$ are characterized by the property that

$$|x(T_1)^t - x_2^t| \lesssim N^{1+\delta} \quad \text{for some } |t - t_q| \gtrsim N^{-2\delta} N^2.$$

We need to count the tubes in \mathbf{T}'_1 with this property. The bicharacteristics for such tubes emanate from the region

$$\begin{aligned} \Sigma := \{(x, \xi) : \text{dist}(\xi, S_1) \leq N^{-1+C\delta}, \xi^{t_q} \in \pi_k^t, \\ |x^{t_q} - x_q| \leq N^{1+\delta}, |x^t - x_2^t| \leq N^{1+\delta} \text{ for some } |t - t_q| \gtrsim N^{-2\delta} N^2\}; \end{aligned}$$

hence it suffices to bound the cardinality of the intersection $(N\mathbb{Z}^d \times N^{-1}\mathbb{Z}^d) \cap \Sigma$.

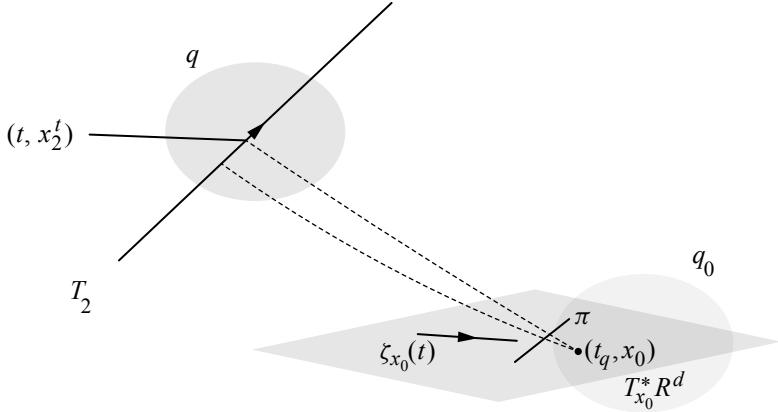


Figure 3. $\xi_{x_0}(t) \in T_{x_0}^* \mathbb{R}^d$ is the set of tangent (covectors) for rays passing through (t_q, x_0) that intersect the ray (t, x_2^t) for the tube T_2 at times $|t - t_q| \gtrsim N^{2-2\delta}$.

Denote by Σ^t the image of Σ under the time- t Hamiltonian flow map $(x, \xi) \mapsto (x^t, \xi^t)$. Recall from (36) that S_j^t denotes the image of the initial frequency set S_j for initial positions x with $|x| \lesssim N^2$; we saw earlier in (38) that S_j^t is a small perturbation of S_j .

Fix a basepoint x_0 with $|x_0 - x_q| \leq N^{1+\delta}$. By Lemma 2.1 and the Hadamard global inverse function theorem, when $t \neq t_q$ we can parametrize the graph of the flow map $(x^{t_q}, \xi^{t_q}) \mapsto (x^t, \xi^t)$ by the variables

$$(x^{t_q}, x^t) \mapsto ((x^{t_q}, \xi^{t_q}(x^t, x^{t_q})) \mapsto (x^t, \xi^t(x^{t_q}, x^t))).$$

Let $\xi(t, x) := \xi^{t_q}(x_0, x) \in T_{x_0}^* \mathbb{R}^d$ be the initial momentum $\xi(t, x) \in T_{x_0}^* \mathbb{R}^d$ such that the bicharacteristic with $x^{t_q} = x_0$ and $\xi^{t_q} = \xi(t, x)$ satisfies $x^t = x$.

Lemma 6.12. *Suppose at least one $T_1 \in \mathbf{T}'_1$ intersects $N^\delta T_2$. For $|t - t_q| \gtrsim N^{-2\delta} N^2$, the curve $t \mapsto \xi_{x_0}(t) := \xi(t, x_2^t) \in T_{x_0}^* \mathbb{R}^d$ is transverse to the hyperplane containing $\pi(\xi_1, \xi_2')$ for all $\xi_1 \in S_1^{t_q}$ and $\xi_2' \in S_2^{t_q}$ (see Figure 3). More precisely there exists $C(\eta) > 0$ such that*

$$\angle(\dot{\xi}_{x_0}(t), \pi(\xi_1, \xi_2')) > C(\eta) \quad \text{for all } \xi_1 \in S_1^{t_q}, \xi_2' \in S_2^{t_q},$$

where the angle $\angle(v, W)$ between a vector v and a subspace W is defined in the usual manner. Moreover, for each t the image of an $N^{1+\delta}$ neighborhood of x_2^t under the map $x \mapsto \xi(t, x)$ belongs to an $N^{-1+C\delta}$ neighborhood of $\xi_{x_0}(t)$.

Proof. By a slight abuse of notation we write $(x^t(y, \xi), \xi^t(y, \xi))$ for the bicharacteristic passing through (y, ξ) at time $t = t_q$ instead of $t = 0$. Both claims are consequences of Lemma 2.1, which yields

$$\begin{aligned} x_2^t &= x^t(x_0, \xi_{x_0}(t)), \quad \xi^{t_q}(x_0, \xi_{x_0}(t)) = \xi_{x_0}(t), \\ \xi_2^t &= \frac{d}{dt} x_2^t = \xi^t(x_0, \xi_{x_0}(t)) + \frac{\partial x^t}{\partial \xi_{x_0}} \dot{\xi}_{x_0}(t) = \xi^t(x_0, \xi_{x_0}(t)) + (t - t_q)(I + O(\eta)) \dot{\xi}_{x_0}(t). \end{aligned}$$

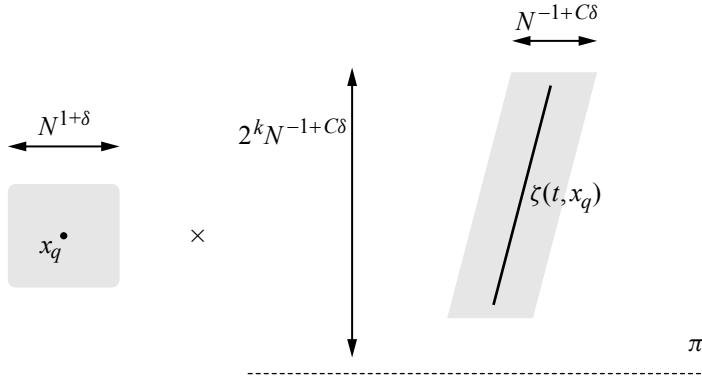


Figure 4. The phase-space region Σ^{t_q} .

Therefore

$$\dot{\xi}_{x_0}(t) = (t - t_q)^{-1} (I + O(\eta))(\xi_2^t - \xi^t(x_0, \xi_{x_0}(t))). \quad (50)$$

We claim that for any $C > 1$,

$$\text{dist}(\xi_{x_0}(t_q), S_1^{t_q}) \lesssim_C N^{-1+C\delta}. \quad (51)$$

Otherwise, as $|t - t_q| \gtrsim N^{2(1-\delta)}$, for any ray (x_1^t, ξ_1^t) with $\xi_1 \in S_1$ and $|x_1^{t_q} - x_q| \leq N^{1+\delta}$, the estimates (19) would imply

$$\begin{aligned} |x_1^t - x_2^t| &\gtrsim |t - t_q| |\xi_1^{t_q} - \xi_{x_0}(t)| - |x_1^{t_q} - x_0| \\ &\gtrsim N^{-1+C\delta} - N^{1+\delta} \gtrsim N^{1+C\delta}, \end{aligned}$$

so we get the contradiction that every $T_1 \in \mathcal{T}'_1$ misses T_2 by at least $N^{C\delta}$.

By the near-constancy (38) of the frequency variable and the definition (29) of S_j , the covector $\xi_2^t - \xi^t(x_0, \xi_{x_0}(t))$ belongs a small perturbation (say, of magnitude at most $\frac{c}{50}$) of the difference set $S_2 - S_1 = -2ce_1 + B(0, \frac{c}{50})$, and hence by Lemma 6.8 is transverse to the hyperplane containing $\pi(\xi_1, \xi_2')$. The first claim now follows from (50).

The argument just given also implies the second statement: a ray with $x^{t_q} = x_0$ and $|x_2^t - x^t| \leq N^{1+\delta}$ must satisfy $|\xi^{t_q} - \xi_{x_0}(t)| \lesssim N^{-1+C\delta}$. \square

By the second part of the lemma, the fiber of Σ^{t_q} in $T_{x_0}^* \mathbb{R}^d$ is contained in a “frequency tube”

$$\Theta(x_0) := \bigcup_{|t - t_q| \gtrsim N^{2(1-\delta)}} B(\xi_{x_0}(t), N^{-1+C\delta}).$$

As the basepoint x_0 varies in an $N^{1+\delta}$ neighborhood of x_q , the estimate (19) implies that the curve $\xi_{x_0}(t)$ shifts by at most $O(N^{-1+3\delta})$. Hence the tubes $\Theta(x_0)$ are all contained in a dilate of $\Theta(x_q)$, which we denote by

$$\tilde{\Theta}(x_q) := \bigcup_t B(\xi_{x_q}(t), N^{-1+C\delta})$$

with a larger C .

Therefore, Σ^{t_q} is contained in the region

$$\tilde{\Sigma}^{t_q} := \{(x, \xi) : |x - x_q| \leq N^{1+\delta}, \xi \in \pi_k^t \cap \tilde{\Theta}(x_q) \subset \{\xi \in \tilde{\Theta}(x_q) : \text{dist}(\xi, \pi) \lesssim 2^k N^{-1+C\delta}\}\},$$

where for the last containment we recall the estimate (45). The region $\tilde{\Sigma}^{t_q}$ is sketched in Figure 4. Using the previous lemma for the central curve ζ_{x_q} , the frequency projection $(x, \xi) \mapsto \xi$ of $\tilde{\Sigma}^{t_q}$ can be covered by approximately 2^k finitely overlapping cubes $\bigcup_{1 \leq j \lesssim 2^k} Q_j$ of width $N^{-1+C\delta}$. By (19), the preimage of each box

$$B(x_q, N^{1+\delta}) \times Q_j$$

under the flow map $(x, \xi) \mapsto (x^{t_q}, \xi^{t_q})$ is contained in a $(CN^{1+C\delta})^d \times (CN^{-1+C\delta})^d$ box. The union of these preimages covers Σ and contains at most $O(2^k N^{C\delta})$ points in $N\mathbb{Z}^d \times N^{-1}\mathbb{Z}^d$. \square

7. Remarks on magnetic potentials

We sketch the modifications needed to prove Theorem 1.6. The symbol for $H(t)$ is

$$a = \frac{1}{2}|\xi|^2 + \langle A, \xi \rangle + V(t, x),$$

where $A = A_j(t, x)dx^j$ and A_j are linear functions in the space variables with bounded time-dependent coefficients.

- Easy computation shows that the symbol map $a \mapsto a^{z_0}$ in Lemma 1.1 is

$$a^{z_0} = \frac{1}{2}|\xi|^2 + \langle A_{(1)}^{z_0}(t, x), \xi \rangle + \langle A_{(2)}^{z_0}(t, x), \xi_0^t \rangle + V_{(2)}^{z_0}(t, x),$$

where $A_{(1)}^{z_0}(t, x) = A(t, x_0^t + x) - A(t, x_0^t)$ and $A_{(2)}^{z_0}(t, x) = A(t, x_0^t + x) - \langle x, \partial_x A(t, x_0^t) \rangle - A(x_0^t)$, and similarly for V . Thus when A is linear, the first-order component of the symbol is exactly ‘‘Galilei-invariant’’, preserved by the transformation $a \mapsto a^{z_0}$ in Lemma 1.1.

- After rescaling, the inequality (15) takes the form

$$\|U_N f U_N g\|_{L^{(d+3)/(d+1)}([-t_0 N^2, t_0 N^2] \times \mathbb{R}^d)} \lesssim_\varepsilon N^\varepsilon \|f\|_{L^2} \|g\|_{L^2},$$

where $U_N(t)$ is the propagator for the rescaled symbol

$$a_N := N^{-2}a(N^{-2}t, N^{-1}x, N\xi) = \frac{1}{2}|\xi|^2 + N^{-2}\langle A(x), \xi \rangle + N^{-2}V(N^{-2}t, N^{-1}x).$$

- Exploiting Galilei-invariance, we may reduce to a spatially localized estimate as in Theorem 6.4. Note that in the region of phase space corresponding to that estimate $\{(x, \xi) : |x| \leq N^2, |\xi| \lesssim 1\}$, and over an $O(N^2)$ time interval, both potential terms have strength $O(1)$ when integrated over the time interval $|t| \lesssim N^2$. However the magnetic term dominates near $x = 0$.
- Then, the rest of the previous proof can be mimicked with essentially no change except for Lemma 6.10. There, one argues essentially as before except the vector field L for integrating by parts should be replaced by

$$L := \partial_t + \langle \overline{a_\xi(z_j^t)}, \partial_x \rangle,$$

where $z_j^t = (x_j^t, \xi_j^t)$ and $\overline{a_\xi(z_j^t)} = \frac{1}{4} \sum_k a_\xi(z_k^t)$. Then one finds that

$$-L\Psi = \frac{1}{2} \sum_j \sigma_j |\bar{\xi}_j^t|^2 + \sum_j \sigma_j \langle A(\bar{x}_j^t), \xi_j^t \rangle + \sum_j \sigma_j [V^{\bar{z}}(t, \bar{x}_j^t) + \langle x - x_j^t, \partial_x(V^{\bar{z}})(t, \bar{x}_j^t) \rangle],$$

and decomposes as before $\Psi = \Psi_1 + \Psi_2$, where

$$-L\Psi_1 = \frac{1}{2} \sum_j \sigma_j |\bar{\xi}_j^t|^2 = |\xi_1^{t_q} - \xi_2^{t_q}|^2 - |\xi_3^{t_q} - \xi_4^{t_q}|^2 + O(N^{-1+\delta}),$$

$$-L\Psi_2 = \sum_j \sigma_j \langle A(\bar{x}_j^t), \xi_j^t \rangle + \sum_j \sigma_j [V^{\bar{z}}(t, \bar{x}_j^t) + \langle x - x_j^t, \partial_x(V^{\bar{z}})(t, \bar{x}_j^t) \rangle] = O(N^{-1+\delta}).$$

As in the proof of [Lemma 6.10](#) the error terms are computed from the estimates [\(19\)](#), $|t - t_q| \lesssim N^{1+\delta}$, and $|\bar{x}_j^t| \lesssim N^{1+\delta}$. The errors are larger than before due to the magnetic term $a_{x\xi} = O(N^{-2})$ but are still acceptable.

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