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CONVEX SETS EVOLVING BY VOLUME-PRESERVING FRACTIONAL MEAN CURVATURE FLOWS

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We consider the volume-preserving geometric evolution of the boundary of a set under fractional mean curvature. We show that smooth convex solutions maintain their fractional curvatures bounded for all times, and the long-time asymptotics approach round spheres. The proofs are based on a priori estimates on the inner and outer radii of the solutions.

1. Introduction

Let $E_0 \subset \mathbb{R}^n$ be a smooth compact convex set, and let $\mathcal{M}_0 = \partial E_0$. For a fixed $s \in (0, 1)$, we consider the evolution of $\mathcal{M}_0$ by volume-preserving fractional mean curvature flow, that is, the family of immersions $F : \mathcal{M}_0 \times [0, T) \to \mathbb{R}^n$ which satisfies

$$\begin{cases} \partial_t F(p,t) = [-H_s(p,t) + h(t)]\nu(p,t), & p \in \mathcal{M}_0,\ t \geqslant 0, \\ F(p,0) = p & p \in \mathcal{M}_0. \end{cases} \tag{1}$$

Here $H_s(p,t)$ and $\nu(p,t)$ denote respectively the fractional mean curvature of order s and the normal vector of the hypersurface $\mathcal{M}_t := F(\mathcal{M}_0, t)$ at the point $F(p,t)$, while the function $h(t)$ is defined as

$$h(t) = \frac{1}{|\mathcal{M}_t|} \int_{\mathcal{M}_t} H_s(x)\, d\mu, \tag{2}$$

where $d\mu$ denotes the surface measure on $\mathcal{M}_t$. With this choice of $h(t)$, the set E_t enclosed by $\mathcal{M}_t$ has constant volume. An interesting feature of this flow is that the fractional s-perimeter of E_t is decreasing, and the monotonicity is strict unless E_t is a sphere.

Fractional (or nonlocal) mean curvature was first defined by Caffarelli, Roquejoffre and Savin [Caffarelli et al. 2010]. It arises naturally when performing the first variation of the fractional perimeter, a nonlocal notion of perimeter introduced in the same paper. We will recall the definitions of these quantities in Section 2. Minimizers of the fractional perimeter are usually called nonlocal minimal sets, and their boundaries nonlocal minimal surfaces. Fractional perimeter and mean curvature have also found application in other contexts, such as image reconstruction and nonlocal capillarity models; see, e.g., [Bosch and Stoll 2015; Maggi and Valdinoci 2017].

Nonlocal minimal surfaces have attracted the interest of many researchers in the last years. One of the main issues is the study of their regularity and the classification of nonlocal minimal cones: many

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results have been obtained, see [Caffarelli et al. 2010; Caffarelli and Valdinoci 2013; Barrios et al. 2014; Savin and Valdinoci 2013; Cinti et al. 2019; Cabré et al. 2020], which exhibit interesting analogies and differences with respect to the classical case. Among the important differences, we mention in particular the fact that fractional minimal surfaces can stick at the boundary of (even smooth and convex) domains, and occupy all the domain for small values of the fractional parameter; see [Dipierro et al. 2017]: these features are in sharp contrast with the classical case and they reveal the important role of the contributions coming from infinity in the geometric displacements of nonlocal minimal surfaces.

A related topic of investigation consists in the study of sets which are stationary for the fractional perimeter, i.e., sets having vanishing nonlocal mean curvature. This is a weaker notion than minimality, and some examples are helicoids and a nonlocal version of catenoids; see, e.g., [Dávila et al. 2018; Cinti et al. 2016]. Sets with constant nonlocal mean curvature, such as Delaunay-type surfaces, have been studied in [Cabré et al. 2018a; 2018b; 2018c; Dávila et al. 2016]. In addition, in [Cabré et al. 2018a; Ciraolo et al. 2018] an analogue of the Alexandrov theorem in the nonlocal setting was proved, which will be crucial for our purposes: any, regular enough, bounded set having constant fractional mean curvature is necessarily a ball.

Before introducing our results, let us recall some properties of the *classical* mean curvature flow, where the speed of the hypersurface is given by the usual mean curvature. This flow has been widely studied in the last decades, both for its geometric interest and for its relevance in physical models describing the dynamics of interfaces. The equation satisfied by the immersion is a parabolic PDE, and smooth solutions exist locally; however, they can become singular in finite time due to curvature blowup. For this reason, various notions of weak solutions have been introduced during the years which allow for the continuation of the evolution after the formation of singularities; see, e.g., [Chen et al. 1991; Evans and Spruck 1991].

An important feature of classical mean curvature flow is that, roughly speaking, it deforms general hypersurfaces into some canonical profiles, possibly after rescaling near the singularities. Such a behavior is related to the diffusive character of the flow and is of great interest for geometric applications. The first result on asymptotic convergence was obtained in [Huisken 1984] in the $h(t) \equiv 0$ case. He proved that *convex hypersurfaces remain smooth up to a finite maximal time at which they shrink to a point, and that they converge to a round sphere after rescaling*. Shortly afterwards, in [Huisken 1987], he obtained an analogous result for the *volume-preserving flow*: in this case, the solution exists for all times and converges to a sphere as $t \to +\infty$. In later years, many researchers have studied the convergence to a sphere for other kinds of geometric flows, with a speed driven by more general functions of the (classical) principal curvatures; see, e.g., [Andrews et al. 2013; Andrews and Wei 2017]. As a possible application of these results, we point out that the convergence to a sphere along a suitable flow can be used to obtain generalizations or alternative proofs of classical geometric inequalities, such as the isoperimetric inequality, or inequalities in convex analysis like the ones by Minkowski or Alexandrov and Fenchel; see, e.g., [McCoy 2005; Schulze 2008; Guan and Li 2009; Andrews et al. 2018].

By contrast, the study of fractional mean curvature flow has started only recently and very few results are known. The existence and uniqueness of weak solutions in the viscosity sense for the flow in the $h(t) \equiv 0$ case have been obtained by various authors with different approaches [Imbert 2009;

Caffarelli and Souganidis 2010; Chambolle et al. 2015]. In particular, Caffarelli and Souganidis [2010] proved convergence to motion by fractional mean curvature of a threshold dynamics scheme. After this, Chambolle, Novaga and Ruffini [Chambolle et al. 2017] extended the results in [Caffarelli and Souganidis 2010] to the anisotropic case and to the presence of an external driving force (that is $h(t) \neq 0$) and proved that the scheme preserves convexity, and, as a consequence, also the limit geometric evolution is convexity-preserving. On the other hand, the existence of smooth solutions has been established only recently in [Julin and La Manna 2020]. Their main result states the short-time existence of a unique classical solution for both the fractional mean curvature flow and the volume-preserving flow, starting from a $C^{1,1}$ initial datum.

Some qualitative properties of smooth solutions were analyzed in [Sáez and Valdinoci 2019], while the formation of neckpinch singularities was studied in [Cinti et al. 2018]. The occurrence of fattening for the fractional mean curvature flow and its generalizations were studied in [Cesaroni et al. 2019].

The aim of this paper is to study the *convergence to a sphere of the solutions of the nonlocal flow* (1) *with convex initial data*. This can be regarded as the first attempt to investigate the asymptotic behavior of solutions to fractional flows, in a similar spirit to the above-mentioned works in the classical case. Our main results are some a priori estimates on smooth solutions, which give a uniform control on the geometry of the evolving surfaces, and establish that the fractional curvature remains uniformly bounded along the flow. As a consequence, we can show that any smooth solution, satisfying suitable regularity assumptions, exists for all times and converges to a sphere. The method is inspired by the one of [Andrews 2001; Sinestrari 2015] in the classical setting and is based on the monotonicity along the flow of the fractional isoperimetric ratio, i.e., the ratio between suitable powers of the fractional perimeter and the enclosed volume. This monotonicity property is specific to the volume-preserving case, and so the approach used here does not apply when $h(t) \equiv 0$, although we expect that case to exhibit a similar behavior, at least if s is suitably close to 1. On the other hand, we include in this paper the treatment of more general flows in the volume-preserving setting, with a *nonlinear speed* of the form $\Phi(H_s)$, with $\Phi(\,\cdot\,)$ a positive increasing function satisfying suitable structural assumptions.

Let us describe our results in more detail. For this, let us denote by $\underline{\rho}_E$ and $\bar{\rho}_E$ the inner radius and the outer radius of a set $E \subset \mathbb{R}^n$, namely

$$\begin{aligned} \underline{\rho}_E &:= \sup\{r > 0 : \text{there exists } x_o \in \mathbb{R}^n \text{ such that } B_r(x_o) \subset E\}, \\ \bar{\rho}_E &:= \inf\{r > 0 : \text{there exists } x_o \in \mathbb{R}^n \text{ such that } B_r(x_o) \supset E\}. \end{aligned} \tag{3}$$

Then our main estimates can be stated as follows:

Theorem 1.1. *Let E_0 be a smooth compact convex set of $\mathbb{R}^n$ and let $\mathcal{M}_0 = \partial E_0$. Let $F : \mathcal{M}_0 \times [0, T) \to \mathbb{R}^n$, with $0 < T \leqslant +\infty$, be a solution of* (1) *of class $C^{2,\beta}$ for some $\beta > s$. Then there exist positive constants $0 < R_1 \leqslant R_2,\ 0 < K_1 \leqslant K_2$, only depending on E_0, such that*

$$\begin{gathered} R_1 \leqslant \underline{\rho}_{E_t} \leqslant \bar{\rho}_{E_t} \leqslant R_2, \\ K_1 \leqslant H_s(p, t) \leqslant K_2, \quad p \in \mathcal{M}_0, \end{gathered}$$

for all $t \in [0, T)$.

As mentioned above, in [Chambolle et al. 2017] it is proven that the nonlocal mean curvature flow with forcing term ($h(t) \not\equiv 0$) preserves convexity. As a consequence, we know that solutions of problem (1) starting from a convex initial datum stay convex for all times.

The proof of Theorem 1.1 relies on a series of delicate estimates based on a nonlocal analysis of geometric flavor, which turns out to be significantly different with respect to the classical case.

Let us describe some intermediate steps in the proof of Theorem 1.1, which we believe to be of interest on their own. One of these results, Proposition 3.1, shows that a bound on the fractional isoperimetric ratio of a convex set implies a bound on the ratio between the outer and inner radii. A similar result was known in the classical case, but the proof in the nonlocal setting is quite different. Another crucial step of our argument is provided by Proposition 4.2, where we estimate the fractional mean curvature in terms of another nonlocal quantity, which has some formal analogy with the norm of the second fundamental form in the classical case. However, since there is no fractional analogue of the second fundamental form, as shown in [Abatangelo and Valdinoci 2014], there is no obvious relation as in the classical case. By suitable estimates of the surface integrals involved, we obtain an inequality which suffices for the purposes of this paper; on the other hand, it would be interesting to investigate further these topics and to derive sharper inequalities in the future.

Theorem 1.1 easily implies that a solution of (1) exists for all times and converges to a sphere as $t \to +\infty$, provided it satisfies suitable regularity and continuation properties. Roughly speaking, we need to know that the solution remains smooth and does not develop singularities as long as the fractional curvature is bounded. More precisely, we assume that there exists a smooth solution of (1) satisfying the following property for some $\beta > s$:

(R) If H_s is bounded on $\mathcal{M}_t$ for all $t \in [0, T_0)$ for some $T_0 \leqslant T$, where T is the maximal time of existence, then the $C^{2,\beta}$-norm of $\mathcal{M}_t$, up to translations, is also bounded for $t \in [0, T)$ by a constant only depending on the supremum of H_s. In addition, either $T_0 = T = +\infty$, or $T_0 < T$.

By "up to translations", we mean that $\mathcal{M}_t$ is not assumed to remain in a bounded set of $\mathbb{R}^n$, and that the $C^{2,\beta}$ bound applies after possibly composing the flow with a suitable, time-dependent, translation (e.g., the one fixing the barycenter). We give below more comments on the possibility of this behavior. For solutions satisfying (R), the following result holds.

Theorem 1.2. *Let E_0 be a smooth compact convex set of $\mathbb{R}^n$ and let $\mathcal{M}_0 = \partial E_0$. Let $F : \mathcal{M}_0 \times [0, T) \to \mathbb{R}^n$, with $0 < T \leqslant +\infty$, be a solution of* (1) *of class $C^{2,\beta}$ for some $\beta > s$ which satisfies property* (R). *Then $T = +\infty$, and $\mathcal{M}_t$ converges to a round sphere as $t \to +\infty$ in $C^{2,\beta}$ norm, possibly up to translations.*

Regarding assumption (R), we observe that it is a natural analogue of some properties which are well known in the classical case, see, e.g., [Huisken 1984, Sections 7–8], and are consequences of the standard parabolic theory. In the fractional setting, the validity of such an assumption is an open problem at the current stage. The only available results in this direction [Julin and La Manna 2020] imply, roughly speaking, that the last claim in (R) is true: if the $C^{1,\beta}$ norm of the solution remains bounded, for some $\beta > s$, then the smooth solution exists for all times. On the other hand, the boundedness of the fractional curvature gives directly $C^{1,\beta}$ bounds only for $\beta \leqslant s$. It can be hoped that solutions of the flow enjoy further regularity, in analogy with some regularity studies on elliptic and parabolic nonlocal problems; see, e.g., [Barrios et al. 2014; Chang-Lara and Dávila 2014a; 2014b; Dipierro et al. 2020]. In this respect, this paper should be regarded as a part of a broader program, which we plan to pursue further in future work.

As observed above, in Theorem 1.2 the convergence to a sphere is in principle only "up to translations", in the sense that the limit set, which is geometrically a sphere, could keep translating indefinitely. In the classical case, the possibility of the additional translation is ruled out either as a consequence of additional estimates on the convergence rate, see, e.g., [Bertini and Pipoli 2017], or by maximum-principle techniques based on reflection methods [Chow and Gulliver 1996; McCoy 2004; Andrews and Wei 2017]. We think that it would be interesting to understand whether these methods can be extended to the nonlocal setting.

The paper is organized as follows:

- In Section 2, we give some preliminaries and we recall the evolution laws of some geometric quantities associated to $\mathcal{M}_t$.
- Section 3 contains our a priori estimates on the inner and outer radii of convex solutions and a lower bound for H_s.
- Section 4 deals with some integral estimates which allow us to bound the fractional mean curvature with the nonlocal analogue of the norm of the nonlocal second fundamental form.
- In Section 5 we prove our key result, which gives an upper bound on the fractional mean curvature.
- In Section 6, we treat the more general case of a flow whose speed is of the form $\Phi(H_s)$, proving an upper bound on the fractional mean curvature.
- Finally, in Section 7, we prove convergence to a sphere in both the standard and the general cases.

2. Preliminaries

Consider a set $E \subset \mathbb{R}^n$, with boundary $\mathcal{M} := \partial E$, and let $s \in (0,1)$. Given $x \in \mathcal{M}$, the *fractional mean curvature* of order s of E (equivalently, of $\mathcal{M}$) at x is defined by

$$H_s(x) = s(1-s) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{\tilde{\chi}_E(y)}{|x-y|^{n+s}}\, dy, \tag{4}$$

where

$$\tilde{\chi}_E(y) = \begin{cases} 1 & \text{if } y \in E^c, \\ -1 & \text{if } y \in E. \end{cases}$$

If $\mathcal{M}$ is smooth, then the fractional mean curvature is well-defined at each point and is a regular function. In fact, the following result is known; see [Figalli et al. 2015, Proposition 6.3; Cabré et al. 2018a, Proposition 2.1].

Theorem 2.1. *Suppose ∂E is of class $C^{1,\beta}$, with $\beta > s$. Then the right-hand side of* (4) *is well-defined and finite for all $x \in \partial E$ and defines a continuous function on ∂E. If in addition ∂E is of class $C^{2,\beta}$, with $\beta > s$, then $H_s \in C^1(\partial E)$ and its derivative in a tangential direction $v \in T_x\mathcal{M}$ is given by*

$$\frac{\partial H_s}{\partial v}(x) = s(1-s)(n+s) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \tilde{\chi}_E(y) \frac{\langle y-x, v\rangle}{|x-y|^{n+s+2}}\, dy. \tag{5}$$

By using the divergence theorem and estimating the boundary terms on $\partial B_\varepsilon(x)$ with techniques similar to the proof of [Cabré et al. 2018a, Proposition 2.1], we can prove that, under the hypotheses of the previous theorem, H_s and its gradient can be written as boundary integrals on $\mathcal{M}$ as follows:

$$H_s(x) = 2(1-s) \lim_{\varepsilon\to 0^+} \int_{\mathcal{M}\setminus B_\varepsilon(x)} \frac{\langle y-x, \nu(y)\rangle}{|x-y|^{n+s}}\, d\mu(y), \tag{6}$$

$$\frac{\partial H_s}{\partial v}(x) = 2s(1-s) \lim_{\varepsilon\to 0^+} \int_{\mathcal{M}\setminus B_\varepsilon(x)} \frac{\nu(y)\cdot v}{|x-y|^{n+s}}\, d\mu(y). \tag{7}$$

We also recall that the *fractional perimeter* of E, as introduced in [Caffarelli et al. 2010], is defined as

$$\mathrm{Per}_s(E) = s(1-s) \int_E \int_{E^c} \frac{dx\, dy}{|x-y|^{n+s}}.$$

Then fractional mean curvature arises as the first variation of the fractional perimeter along a deformation of E; see (8) later.

We state a general criterion for the convergence of singular integrals on the boundary of a smooth compact set E. Suppose that ∂E is of class $C^{1,\beta}$, for some $\beta > s$, and that $f \in C^2(\partial E)$. Then, for any given $x \in \partial E$, the quantity

$$\lim_{\varepsilon\to 0^+} \int_{\mathcal{M}\setminus B_\varepsilon(x)} \frac{f(y)-f(x)}{|x-y|^{n+s}}\, d\mu(y)$$

exists and is finite.This can be proved by standard arguments. Roughly speaking, the contribution of the first-order approximation of $f(y) - f(x)$ around x cancels by symmetry reasons. The remaining terms are of order $O(|y-x|)^{1+\beta}$, by the smoothness of ∂E and of f, and this ensures convergence of the integral. In the following, for simplicity of notation, we will write singular integrals as the ones above as if they were ordinary integrals, with the implicit meaning that they are taken in the principal value sense.

We now recall some notation and general results about geometric evolutions of sets and hypersurfaces. Let us consider a time-dependent family of sets E_t evolving smoothly from a given initial set E_0. We can consider the corresponding evolution of the boundaries, and study the map $F : \mathcal{M}_0 \times [0, T) \to \mathbb{R}^n$, where $\mathcal{M}_0 = \partial E_0$ and $\mathcal{M}_t := \partial E_t$. Let us denote by $V(p,t) := \langle \partial_t F(p,t), \nu(p,t)\rangle$ the normal component of the speed of our flow.

We first recall the properties of the evolution of the classical geometric quantities associated to the hypersurfaces $\mathcal{M}_t$. As in [Huisken 1984], we denote by g_{ij} the components of the metric tensor in a given coordinate system, by g^{ij} its inverse, by h_{ij} the second fundamental form, by $H = h_{ij}g^{ij}$ the mean curvature and by $|A|^2 = h_{ij}g^{jl}h_{lk}g^{ki}$ the squared norm of the second fundamental form. If $\lambda_1 \leqslant \cdots \leqslant \lambda_{n-1}$ denote the principal curvatures at a given point, then $H = \lambda_1 + \cdots + \lambda_{n-1}$, while $|A|^2 = \lambda_1^2 + \cdots + \lambda_{n-1}^2$. We also denote by $\nabla^{\mathcal{M}_t}$, $\Delta^{\mathcal{M}_t}$ respectively the tangential gradient and the Laplace–Beltrami operator defined on $\mathcal{M}_t$.

We denote by $p, q, \ldots$ the points on $\mathcal{M}_0$ and by $x, y, \ldots$ the points on $\mathcal{M}_t$ for positive t, as well as the general points in $\mathbb{R}^n$. For simplicity of notation, when considering the speed V on $\mathcal{M}_t$ for a fixed t, we will usually write $V(x)$ with $x \in \mathcal{M}_t$ instead of $V(p,t)$, with $x = F(p,t)$. We will use similar conventions for all other quantities defined on the evolving hypersurfaces. We also denote by $d\mu$ the surface measure along $\mathcal{M}_t$. In this notation, we recall [Huisken and Polden 1999, Theorem 3.2 and Lemmata 7.4, 7.5 and 7.6] and we have:

Lemma 2.2. *The geometric quantities associated to $\mathcal{M}_t$ satisfy the following equations*:

(i) $\partial_t g_{ij} = 2V h_{ij}$ *and* $\partial_t g^{ij} = -2V h^{ij}$.

(ii) $\partial_t d\mu = V H \, d\mu$.

(iii) $\partial_t \nu = -\nabla^{\mathcal{M}_t} V$.

(iv) $\partial_t h_{ij} = -\nabla_i^{\mathcal{M}_t} \nabla_j^{\mathcal{M}_t} V + h_{ik} g^{km} h_{mj} V$.

(v) $\partial_t H = -\Delta^{\mathcal{M}_t} V - |A|^2 V$.

(vi) $\frac{d}{dt}|E_t| = \int_{\mathcal{M}_t} V(x)\, d\mu$ *and* $\frac{d}{dt}|M_t| = \int_{\mathcal{M}_t} V(x) H(x)\, d\mu$.

Next we recall the evolution of some nonlocal quantities; see [Caffarelli et al. 2010; Dávila et al. 2018, Appendix B, Proposition B.2; Sáez and Valdinoci 2019, Theorem 14].

Lemma 2.3. (i) *The fractional perimeter evolves according to*

$$\frac{d}{dt} \operatorname{Per}_s(E_t) = \int_{\mathcal{M}_t} H_s(x) V(x)\, d\mu. \tag{8}$$

(ii) *The fractional mean curvature satisfies the equation*

$$\frac{\partial_t H_s}{2s(1-s)} = -\int_{\mathcal{M}_t} \frac{V(y) - V(x)}{|y-x|^{n+s}}\, d\mu(y) - V(x) \int_{\mathcal{M}_t} \frac{1 - \nu(y)\cdot\nu(x)}{|y-x|^{n+s}}\, d\mu(y). \tag{9}$$

We remark that there is a clear analogy between these equations and their classical counterparts. Indeed, as proved in [Dávila et al. 2018, Appendix A], we have, for a general smooth function f defined on a (fixed) hypersurface $\mathcal{M}$,

$$\lim_{s\to 1^-} 2s(1-s) \int_{\mathcal{M}} \frac{f(y) - f(x)}{|y-x|^{n+s}}\, d\mu(y) = \omega_n \Delta^{\mathcal{M}} f(x), \tag{10}$$

where ω_n is the volume of the unit ball of $\mathbb{R}^n$. In addition,

$$\lim_{s\to 1^-} 2s(1-s) \int_{\mathcal{M}} \frac{1 - \nu(y)\cdot\nu(x)}{|y-x|^{n+s}}\, d\mu(y) = \omega_n |A|^2. \tag{11}$$

From now on, we assume that the map $F : \mathcal{M}_0 \times [0, T) \to \mathbb{R}^n$ satisfies (1). This corresponds to the normal speed

$$V(p,t) = -H_s(p,t) + h(t),$$

with $h(t)$ defined as in (2).

Then Lemma 2.2(vi) implies the enclosed volume E_t remains constant in time, while by Lemma 2.3(i) the fractional perimeter decreases according to

$$\partial_t \operatorname{Per}_s(E_t) = \int_{\mathcal{M}_t} [-H_s(x) + h(t)] H_s(x)\, d\mu = -\int_{\mathcal{M}_t} [H_s(x) - h(t)]^2\, d\mu \leqslant 0. \tag{12}$$

We conclude this section by recalling the analogue of the Alexandrov theorem in the nonlocal setting.

Theorem 2.4 [Cabré et al. 2018a, Theorem 1.1; Ciraolo et al. 2018, Theorem 1.1]. *Let E be a bounded open set of class $C^{1,s}$ with constant nonlocal mean curvature. Then, E is a ball.*

We point out that, by (12) and Theorem 2.4, the monotonicity of $\mathrm{Per}_s(E_t)$ is strict unless E_t is a sphere.

3. Bounds on inner and outer radii

Given a bounded set $E \subset \mathbb{R}^n$ with nonempty interior and $\omega \in \partial B_1$, we denote by $w_E(\omega)$ the width of the set E in direction ω; i.e.,

$$w_E(\omega) := \sup_{x,y\in E} (x-y)\cdot\omega. \tag{13}$$

Notice that w_E is the distance between the two hyperplanes orthogonal to ω touching E from outside. We also set

$$\underline{w}_E := \inf_{\omega\in\partial B_1} w_E(\omega) \quad \text{and} \quad \overline{w}_E := \sup_{\omega\in\partial B_1} w_E(\omega).$$

By construction, we have

$$\overline{w}_E = \mathrm{diam}(E). \tag{14}$$

Recalling the notation in (3), if E is convex, it is known that

$$\underline{\rho}_E \geqslant \frac{\underline{w}_E}{n+1} \quad \text{and} \quad \bar{\rho}_E \leqslant \frac{\overline{w}_E}{\sqrt{2}}; \tag{15}$$

see, e.g., [Andrews 1994, Lemma 5.4].

Using this notation, the following result holds true:

Proposition 3.1. *For any bounded, convex set $E \subset \mathbb{R}^n$ with nonempty interior, we have*

$$\underline{w}_E \geqslant c\left(\frac{|E|}{\mathrm{Per}_s(E)}\right)^{1/s}, \tag{16}$$

$$\underline{\rho}_E \geqslant c\left(\frac{|E|}{\mathrm{Per}_s(E)}\right)^{1/s}, \tag{17}$$

$$\overline{w}_E \leqslant C(\mathrm{Per}_s(E))^{(n-1)/s}|E|^{(1+s-n)/s}, \tag{18}$$

$$\bar{\rho}_E \leqslant C(\mathrm{Per}_s(E))^{(n-1)/s}|E|^{(1+s-n)/s}, \tag{19}$$

$$\frac{\bar{\rho}_E}{\underline{\rho}_E} \leqslant C(\mathrm{Per}_s(E))^{n/s}|E|^{(s-n)/s} \tag{20}$$

for suitable constants $C > c > 0$ only depending on n, s.

Proof. First of all, we observe that

$$\text{it is enough to prove (16),} \tag{21}$$

since, after that, the claims in (17), (18), (19) and (20) would follow. Indeed, if (16) holds true, then (17) follows directly from (15).

Now we prove (18) assuming that (16) (and so (17)) holds true. To this aim, we observe that we can suppose that

$$\overline{w}_E \geqslant 4\underline{\rho}_E. \tag{22}$$

Indeed, suppose instead that the opposite inequality holds. Then, we use the nonlocal isoperimetric inequality (see [Frank et al. 2008]) to see that

$$|E|^{1/n} = |E|^{(n-s)(n-1)/(ns)}|E|^{(1+s-n)/s} \leqslant C_1(\mathrm{Per}_s(E))^{(n-1)/s}|E|^{(1+s-n)/s}$$

for some $C_1 > 0$. Accordingly, since $|E|^{1/n} \geqslant |B_{\underline{\rho}_E}|^{1/n} = C_2\underline{\rho}_E$, for some $C_2 > 0$, we obtain

$$C_2\underline{\rho}_E \leqslant C_1(\mathrm{Per}_s(E))^{(n-1)/s}|E|^{(1+s-n)/s},$$

and so, if the opposite inequality holds in (22),

$$\frac{C_2\overline{w}_E}{4} \leqslant C_1(\mathrm{Per}_s(E))^{(n-1)/s}|E|^{(1+s-n)/s},$$

which says that (18) is satisfied.

Consequently, we may assume that (22) holds true. Thus, after a translation we may suppose that $B_{\underline{\rho}_E} \subseteq E$ and there exists $p \in \overline{E}$ with $|p| \geqslant \overline{w}_E/2 - \underline{\rho}_E$. We stress that, in view of (22),

$$|p| \geqslant \frac{\overline{w}_E}{4} =: \ell.$$

Since E is convex, the convex hull of p with $B_{\underline{\rho}_E}$ lies in $\overline{E}$ and therefore $|E| \geqslant \tilde{c}\underline{\rho}_E{}^{n-1}\ell$ for some $\tilde{c} > 0$. This and (17) imply

$$\overline{w}_E = 4\ell \leqslant \frac{4|E|}{\tilde{c}\underline{\rho}_E{}^{n-1}} \leqslant \frac{4|E|(\mathrm{Per}_s(E))^{(n-1)/s}}{\tilde{c}c^{n-1}|E|^{(n-1)/s}},$$

which gives (18), as desired.

Then, from (18) and (15), one obtains (19). Finally, (20) clearly follows from (17) and (19). This completes the proof of (21).

In view of (21), from now on we focus on the proof of (16). To this aim, after a rigid motion, we may suppose that $\underline{w}_E$ is realized in the vertical direction, and, more precisely, that

$$E \subseteq \{x_n \in [-\underline{w}_E, 0]\}. \tag{23}$$

We denote by π the projection onto $\mathbb{R}^{n-1} \times \{0\}$ and $E' := \pi(E)$. We consider a nonoverlapping tiling of $\mathbb{R}^{n-1} \times \{0\}$ by cubes $\{Q_i\}_{i\in\mathbb{N}}$ which have side length equal to $\underline{w}_E/\sqrt{n-1}$ (hence, their diagonal is equal to $\underline{w}_E$). We denote by $\mathbb{N}_\star$ the set of indices $i \in \mathbb{N}$ for which Q_i intersects E'. Let also

$$Q := \bigcup_{i\in\mathbb{N}_\star} Q_i \quad \text{and} \quad F := Q \times (0, \underline{w}_E].$$

Due to (23), we know that F lies outside E and therefore

$$\begin{aligned}\mathrm{Per}_s(E) &\geqslant \iint_{E\times F} \frac{dx\,dy}{|x-y|^{n+s}}\\ &= \int_{-\underline{w}_E}^{0} dx_n \int_Q dx' \int_0^{\underline{w}_E} dy_n \int_Q dy' \frac{\chi_E(x',x_n)}{|x-y|^{n+s}}\\ &\geqslant \sum_{i\in\mathbb{N}_\star} \int_{-\underline{w}_E}^{0} dx_n \int_{Q_i} dx' \int_0^{\underline{w}_E} dy_n \int_{Q_i} dy' \frac{\chi_E(x',x_n)}{|x-y|^{n+s}}.\end{aligned}$$

Now we remark that if $x', y' \in Q$, $x_n \in [-\underline{w}_E, 0]$ and $y_n \in (0, \underline{w}_E]$, we have

$$|x-y|^2 = |x'-y'|^2 + |x_n-y_n|^2 \leqslant \underline{w}_E{}^2 + (2\underline{w}_E)^2 = 5\underline{w}_E{}^2.$$

As a consequence,

$$\begin{aligned}\mathrm{Per}_s(E) &\geqslant \frac{1}{5^{(n+s)/2}\underline{w}_E{}^{n+s}} \sum_{i\in\mathbb{N}_\star} \int_{-\underline{w}_E}^{0} dx_n \int_{Q_i} dx' \int_0^{\underline{w}_E} dy_n \int_{Q_i} dy' \chi_E(x',x_n)\\ &= \frac{1}{5^{(n+s)/2}\underline{w}_E{}^{n+s}} \left(\frac{\underline{w}_E}{\sqrt{n-1}}\right)^{n-1} \underline{w}_E \sum_{i\in\mathbb{N}_\star} \int_{-\underline{w}_E}^{0} dx_n \int_{Q_i} dx' \chi_E(x',x_n)\\ &= \frac{1}{5^{(n+s)/2}\underline{w}_E{}^{n+s}} \left(\frac{\underline{w}_E}{\sqrt{n-1}}\right)^{n-1} \underline{w}_E |E|,\end{aligned}$$

where we used (23) once again in the last identity. This estimate plainly implies (16), as desired. □

For completeness, we point out an interesting geometric consequence of the estimate in (20) in terms of the nonlocal isoperimetric ratio

$$\mathcal{I}_s(E) := \frac{(\mathrm{Per}_s(E))^n}{|E|^{n-s}}.$$

Indeed, formula (20) states that if the nonlocal isoperimetric ratio of E is bounded, then so is the ratio between the inner and outer radius of E and, more precisely,

$$\frac{\bar\rho_E}{\underline{\rho}_E} \leqslant C(\mathcal{I}_s(E))^{1/s}.$$

In the local case when $s = 1$, this formula was already known; see, e.g., [Andrews 2001, Proposition 5.1; Sinestrari 2015, Proposition 2.1].

As an immediate consequence of the results of this section, we obtain:

Corollary 3.2. *Let E_0 be a convex subset of $\mathbb{R}^n$ and $\mathcal{M}_0 = \partial E_0$. Let $F : \mathcal{M}_0 \times [0, T) \to \mathbb{R}^n$, with $0 < T \leqslant +\infty$, be a solution of* (1). *Then there exist positive constants $0 < R_1 \leqslant R_2$, only depending on E_0, such that*

$$R_1 \leqslant \underline{\rho}_{E_t} \leqslant \bar\rho_{E_t} \leqslant R_2 \quad \textit{for all } t \in [0, T).$$

In addition, there exists $K_1 > 0$ such that $H_s(p, t) \geqslant K_1$ for all $(p, t) \in \mathcal{M}_0 \times [0, T)$.

Proof. As already mentioned in the Introduction, we know that the evolution given by (1) preserves convexity, as established in [Chambolle et al. 2017]; hence we have that E_t is convex for all $0 < t < T$.

By definition, we have

$$\omega_n \underline{\rho}_{E_t}^n \leqslant |E_t| \leqslant \omega_n \bar{\rho}_{E_t}^n.$$

Since $|E_t|$ is constant, this gives an upper bound on $\underline{\rho}_{E_t}$ and a lower bound on $\bar{\rho}_{E_t}$ in terms of $|E_0|$. On the other hand, since $\mathrm{Per}_s(E_t)$ is decreasing in time, inequality (20) gives a uniform bound on the ratio $\bar{\rho}_{E_t}/\underline{\rho}_{E_t}$. These properties together yield the first assertion.

To prove the lower bound on H_s, let us consider an arbitrary point $x \in \mathcal{M}_t$. Since E_t is convex, it is contained in the half-space $\{y \in \mathbb{R}^n : (y-x)\cdot\nu(x) \leqslant 0\}$. Moreover, by definition, the diameter of E_t is not greater than $2\bar{\rho}_{E_t}$, which is less than $2R_2$. Therefore, if we introduce the half-balls

$$B_+ = \{y \in B_{2R_2}(x) : (y-x)\cdot\nu(x) \geqslant 0\}, \quad B_- = \{y \in B_{2R_2}(x) : (y-x)\cdot\nu(x) \leqslant 0\},$$

we have that $E_t \subset B_-$. It follows that

$$\begin{aligned} \frac{1}{s(1-s)} H_s(x) &= \int_{E_t^c} \frac{dy}{|x-y|^{n+s}} - \int_{E_t} \frac{dy}{|x-y|^{n+s}} \\ &\geqslant \int_{\mathbb{R}^n \setminus B_{2R_2}(x)} \frac{dy}{|x-y|^{n+s}} + \int_{B_+} \frac{dy}{|x-y|^{n+s}} - \int_{B_-} \frac{dy}{|x-y|^{n+s}} \\ &= \int_{\mathbb{R}^n \setminus B_{2R_2}(x)} \frac{dy}{|x-y|^{n+s}} = \int_{|z|\geqslant 2R_2} \frac{dz}{|z|^{n+s}}, \end{aligned}$$

where the last integral is independent of x, t. □

The previous result contains the first part of the statement of Theorem 1.1 (the bounds on inner and outer radii and the lower bound for H_s). To conclude the proof of Theorem 1.1 it remains to establish the upper bound for the fractional mean curvature, which will be done in Section 5.

We conclude this section with the following observation. Corollary 3.2 ensures that, at any given time, there exists a ball of radius R_1 contained in E_t. However, the center of the ball may be different at different times. We want to show that, by choosing a smaller radius, we can find a ball with fixed center which remains inside E_t for a time interval with fixed length.

Lemma 3.3. *For any $t_0 \geqslant 0$, we can find $x_0 \in \mathbb{R}^n$ such that*

$$B_{R_1/2}(x_0) \subset E_t \quad \text{for all } t \in [t_0, t_0 + t^*],$$

where $t^* > 0$ only depends on n, s, R_1.

Proof. As in [Andrews 2001; McCoy 2004], we use a comparison argument. Volume-preserving curvature flows in general do not satisfy an avoidance principle. However, if E_t evolves by (1) and F_t evolves by the standard fractional mean curvature flow (corresponding to $h(t) \equiv 0$) then an easy maximum principle argument shows that if $F_{t_0} \subset E_{t_0}$ at a certain time t_0, then we also have $F_t \subset E_t$ for all $t \geqslant t_0$.

In our case, we can use comparison with a shrinking ball. From the previous corollary, there exists x_0 such that $B_{R_1}(x_0) \subset E_t$. We set $F_{t_0} = B_{R_1}(x_0)$ and we denote by F_t the evolution of F_{t_0} for $t \geqslant t_0$ by standard fractional mean curvature flow, which is a shrinking sphere. We let t^* be the time such that $F_{t_0+t^*} = B_{R_1/2}(x_0)$, whose value only depends on n, s, R_1. Then the comparison argument yields the conclusion. □

4. Integral surface estimates for convex sets

We collect in this section some estimates on weighted integrals along the boundary of a convex set. We start with a uniform estimate of the weighted surface of a convex set only dependent on its inner and outer radii.

Lemma 4.1. *Let $\beta > 1$ and let $E \subset \mathbb{R}^n$ be a bounded, convex set with nonempty interior. Then, there exists a constant $C > 0$, depending on n, such that, for any $\in \partial E$, we have*

$$\int_{\partial E} \frac{d\mu(y)}{|x-y|^{n-\beta}} \leqslant C \frac{\bar{\rho}_E}{\underline{\rho}_E}\left[\frac{1}{\beta-1} + \left(\frac{\bar{\rho}_E}{\underline{\rho}_E}\right)^{n-2}\right](\mathrm{diam}(E))^{\beta-1}.$$

Proof. We can suppose that x is the origin. By definition, there exists $p \in E$ such that $B_{\underline{\rho}_E}(p) \subseteq E$. By convexity, the convex envelope of 0 and $B_{\underline{\rho}_E}(p)$ lies in $\bar{E}$. Up to a rotation, we can assume that $p = (0, \dots, |p|)$. This easily implies, again by convexity, that $B_{\underline{\rho}_E/2}(0) \cap \partial E$ is the graph of a Lipschitz function f, with Lipschitz constant bounded by $2|p|/\underline{\rho}_E \leqslant 4\bar{\rho}_E/\underline{\rho}_E$.

Let us set $\delta := \underline{\rho}_E/2$ and $M := \bar{\rho}_E/\underline{\rho}_E$. In addition, let us denote by C', C'', ... constants depending only on n. We can estimate, using the fact that $\beta > 1$,

$$\int_{\partial E \cap B_\delta} \frac{d\mu(y)}{|y|^{n-\beta}} \leqslant \int_{\substack{y' \in \mathbb{R}^{n-1} \\ |y'| \leqslant \delta}} \frac{\sqrt{1+|\nabla f(y')|^2}}{|y'|^{n-\beta}}\, dy' \leqslant C'M \int_0^\delta \frac{\tau^{n-2}}{\tau^{n-\beta}}\, d\tau = \frac{C'M}{\beta-1}\delta^{\beta-1}. \tag{24}$$

The remaining part of the integral satisfies

$$\int_{\partial E \setminus B_\delta} \frac{d\mu(y)}{|y|^{n-\beta}} \leqslant \frac{1}{\delta^{n-\beta}} \int_{\partial E \setminus B_\delta} d\mu(y) \leqslant \frac{\mu(\partial E)}{\delta^{n-\beta}}. \tag{25}$$

Now we observe that

$$\mu(\partial E) \leqslant \mu(B_{\bar{\rho}_E}). \tag{26}$$

Indeed, we know that there exists $q \in E$ such that $B_{\bar{\rho}_E}(q) \supseteq E$. Let us denote by $\Pi_E : \mathbb{R}^n \to E$ the projection on the convex set E. Then Π_E maps $\partial B_{\bar{\rho}_E}(q)$ onto ∂E and is nonexpansive, from which (26) follows.

As a consequence of (25) and (26), we obtain that

$$\int_{\partial E \setminus B_\delta} \frac{d\mu(y)}{|y|^{n-\beta}} \leqslant \frac{C'' \bar{\rho}_E{}^{n-1}}{\delta^{n-\beta}} = C''' M^{n-1} \delta^{\beta-1}.$$

This and (24) imply the desired result (recall also (14) and (15)). □

Now we obtain a bound on the fractional mean curvature in terms of the integral quantity which appears in the last term of (9). In view of (11), one can consider this estimate as the fractional counterpart of the elementary property that the classical mean curvature is bounded by the norm of the second fundamental form. An estimate of this kind is more delicate to obtain in the nonlocal case, since the fractional mean curvature cannot be realized by the average of finitely many directional curvatures, and so methods involving linear algebra cannot be applied; see [Abatangelo and Valdinoci 2014]. We give here a proof in the case of convex sets, but it is natural to expect that a similar property should hold in a more general setting.

Proposition 4.2. *Let $E \subset \mathbb{R}^n$ be a convex set with $C^{1,\alpha}$ boundary, with $\alpha \in (s, 1)$. Then, there exists $C > 0$, depending on n and on the ratio $\bar\rho_E/\underline{\rho}_E$, such that, for every $x \in \partial E$, we have*

$$H_s(x) \leqslant C(\operatorname{diam}(E))^{(1-s)/2}\left((1-s)\int_{\partial E} \frac{1-\nu(y)\cdot\nu(x)}{|x-y|^{n+s}}\,d\mu(y)\right)^{1/2}.$$

Proof. Given $x \in \partial E$, with exterior normal $\nu(x)$, from the convexity of E we have that $\{p \in \mathbb{R}^n : (p-x)\cdot\nu(x) > 0\}$ touches E from outside at p. As a consequence, if $y \in \partial E$, we have that $(y-x)\cdot\nu(x) \leqslant 0$ and therefore, recalling (6), we have

$$\begin{aligned}
\frac{1}{2(1-s)}H_s(x) &= \int_{\partial E} \frac{(y-x)\cdot\nu(y)}{|x-y|^{n+s}}\,d\mu(y)\\
&= \int_{\partial E} \frac{(y-x)\cdot\nu(x)}{|x-y|^{n+s}}\,d\mu(y) + \int_{\partial E} \frac{(y-x)\cdot(\nu(y)-\nu(x))}{|x-y|^{n+s}}\,d\mu(y)\\
&\leqslant \int_{\partial E} \frac{(y-x)\cdot(\nu(y)-\nu(x))}{|x-y|^{n+s}}\,d\mu(y)\\
&\leqslant \int_{\partial E} \frac{|\nu(y)-\nu(x)|}{|x-y|^{n+s-1}}\,d\mu(y)\\
&= \int_{\partial E} \frac{|\nu(y)-\nu(x)|}{|x-y|^{(n+s)/2}}\,\frac{d\mu(y)}{|x-y|^{(n+s-2)/2}}.
\end{aligned}$$

Hence, exploiting Hölder's inequality,

$$\frac{1}{2(1-s)}H_s(x) \leqslant \sqrt{\int_{\partial E} \frac{|\nu(y)-\nu(x)|^2}{|x-y|^{n+s}}\,d\mu(y)}\ \ \sqrt{\int_{\partial E} \frac{d\mu(y)}{|x-y|^{n+s-2}}}.$$

Since we have $|\nu(y)-\nu(x)|^2 = 2(1-\nu(y)\cdot\nu(x))$, the desired result follows easily from Lemma 4.1 with $\beta := 2-s > 1$. □

5. Upper bound on the fractional curvature

In this section, we show that the bounds on the inner and outer radii imply that the fractional mean curvature of our solution is bounded from above. This, together with Corollary 3.2, will conclude the proof of Theorem 1.1.

To this purpose, we adapt to the nonlocal setting a technique originally introduced in [Tso 1985]. We consider the support function on the evolving hypersurface

$$u(p,t) = \langle F(p,t), \nu(p,t)\rangle.$$

By Lemma 2.2(iii) and the representation (7) of the gradient of H_s, we find that u evolves according to

$$\begin{aligned}
\partial_t u &= \langle \partial_t F, \nu\rangle + \langle F, \partial_t \nu\rangle\\
&= -H_s + h + \langle F, \nabla^{\mathcal{M}} H_s\rangle = -H_s + h + 2s(1-s)\int_{\mathcal{M}_t} \frac{x^T\cdot\nu(y)}{|y-x|^{n+s}}\,d\mu(y).
\end{aligned} \tag{27}$$

From Lemma 3.3, we know that for any t_0 there exists $x_0 \in \mathbb{R}^n$ such that $B_{R_1/2}(x_0) \subset E_t$ for any $t \in [t_0, t_0+t^*]$. For simplicity, we perform our computations in the case $x_0 = 0$. By the convexity of E_t, we deduce that $u \geqslant R_1/2$ on $\mathcal{M}_t$ for all $t \in [t_0, t_0+t^*]$. We then set $\alpha = R_1/4$ and we consider the function

$$W = \frac{H_s}{u-\alpha}.$$

Since

$$\alpha \leqslant u-\alpha \leqslant \operatorname{diam}(\mathcal{M}_t) - \alpha \leqslant 2\bar{\rho}_{E_t} - \alpha,$$

we deduce from Corollary 3.2 that

$$\frac{1}{C} \leqslant \frac{W}{H_s} \leqslant C \tag{28}$$

for some C only depending on n, s and the initial data.

Let us now analyze the evolution equation satisfied by W. By Lemma 2.3(ii), the fractional mean curvature satisfies the equation

$$\frac{\partial_t H_s}{2s(1-s)} = \int_{\mathcal{M}_t} \frac{H_s(y)-H_s(x)}{|y-x|^{n+s}}\, d\mu(y) + (H_s(x)-h(t)) \int_{\mathcal{M}_t} \frac{1-\nu(y)\cdot\nu(x)}{|y-x|^{n+s}}\, d\mu(y).$$

Recalling (27) and neglecting the positive terms containing $h(t)$, we find

$$\begin{aligned}
\frac{\partial_t W(x,t)}{2s(1-s)} &= \frac{1}{u(x)-\alpha}\int_{\partial E_t} \frac{H_s(y)-H_s(x)}{|y-x|^{n+s}}\, d\mu(y) + \frac{H_s(x)-h(t)}{u(x)-\alpha}\int_{\partial E_t} \frac{1-\nu(y)\cdot\nu(x)}{|y-x|^{n+s}}\, d\mu(y) \\
&\qquad - \frac{H_s(x)}{(u(x)-\alpha)^2}\left(\frac{-H_s(x)+h(t)}{2s(1-s)} + \int_{\partial E_t} \frac{x^T\cdot\nu(y)}{|y-x|^{n+s}}\, d\mu(y)\right) \\
&< \frac{1}{u(x)-\alpha}\int_{\partial E_t} \frac{H_s(y)-H_s(x)}{|y-x|^{n+s}}\, d\mu(y) + \frac{H_s(x)}{u(x)-\alpha}\int_{\partial E_t} \frac{1-\nu(y)\cdot\nu(x)}{|y-x|^{n+s}}\, d\mu(y) \\
&\qquad - \frac{H_s(x)}{(u(x)-\alpha)^2}\left(\frac{-H_s(x)}{2s(1-s)} + \int_{\partial E_t} \frac{x^T\cdot\nu(y)}{|y-x|^{n+s}}\, d\mu(y)\right).
\end{aligned} \tag{29}$$

We can write

$$\int_{\partial E_t} \frac{H_s(y)-H_s(x)}{|y-x|^{n+s}}\, d\mu(y) = \int_{\partial E_t} (u(y)-\alpha)\frac{W(y)-W(x)}{|y-x|^{n+s}}\, d\mu(y) + W(x)\int_{\partial E_t} \frac{u(y)-u(x)}{|y-x|^{n+s}}\, d\mu(y). \tag{30}$$

Observe also

$$\begin{aligned}
\int_{\partial E_t} \frac{u(y)-u(x)}{|y-x|^{n+s}}\, d\mu(y) &= \int_{\partial E_t} \frac{y\cdot\nu(y)-x\cdot\nu(x)}{|y-x|^{n+s}}\, d\mu(y) \\
&= \int_{\partial E_t} \frac{(y-x)\cdot\nu(y)}{|y-x|^{n+s}}\, d\mu(y) + \int_{\partial E_t} \frac{(x^T+u(x)\nu(x))\cdot(\nu(y)-\nu(x))}{|y-x|^{n+s}}\, d\mu(y) \\
&= \frac{1}{2(1-s)}H_s(x) + \int_{\partial E_t} \frac{x^T\cdot\nu(y)}{|y-x|^{n+s}}\, d\mu(y) - u(x)\int_{\partial E_t} \frac{1-\nu(x)\cdot\nu(y)}{|y-x|^{n+s}}\, d\mu(y).
\end{aligned} \tag{31}$$

From (30) and (31) we deduce that

$$\frac{1}{u(x)-\alpha}\int_{\partial E_t}\frac{H_s(y)-H_s(x)}{|y-x|^{n+s}}\,d\mu(y)-\frac{H_s(x)}{(u(x)-\alpha)^2}\int_{\partial E_t}\frac{x^T\cdot\nu(y)}{|y-x|^{n+s}}\,d\mu(y)$$
$$=\frac{1}{u(x)-\alpha}\int_{\partial E_t}\frac{(W(y)-W(x))(u(y)-\alpha)}{|y-x|^{n+s}}\,d\mu(y)+\frac{1}{2(1-s)}W^2-u(x)\frac{W(x)}{u(x)-\alpha}\int_{\partial E_t}\frac{1-\nu(x)\cdot\nu(y)}{|y-x|^{n+s}}\,d\mu(y).$$

We then conclude from (29)

$$\frac{\partial_t W}{2s(1-s)}<\frac{1}{u(x)-\alpha}\int_{\partial E_t}\frac{(W(y)-W(x))(u(y)-\alpha)}{|y-x|^{n+s}}\,d\mu(y)$$
$$+\frac{1+s}{2s(1-s)}W^2-\alpha\frac{W(x)}{u(x)-\alpha}\int_{\partial E_t}\frac{1-\nu(x)\cdot\nu(y)}{|y-x|^{n+s}}\,d\mu(y). \tag{32}$$

Recalling the estimate of Proposition 4.2 and (28), we immediately obtain:

Corollary 5.1. *At any point where the spatial maximum for $W(\cdot,t)$ is attained, we have*

$$\partial_t W < C_1W^2 - C_2W^3 \tag{33}$$

for constants C_1, C_2 only depending on n, s and the initial data.

We are now ready to prove the upper bound on the fractional mean curvature.

Theorem 5.2. *Let E_0 be a convex subset of $\mathbb{R}^n$ and $\mathcal{M}_0=\partial E_0$. Let $F:\mathcal{M}_0\times[0,T)\to\mathbb{R}^n$, with $0<T\leqslant+\infty$, be a solution of (1) of class $C^{2,\beta}$ for some $\beta>s$. Then there exists $K_2>0$, only depending on n, s, E_0, such that*

$$H_s(p,t)\leqslant K_2,\qquad p\in\mathcal{M}_0,$$

for all $t\in[0,T)$.

Proof. Let us take an arbitrary $t_0\in[0,T)$. We know from Lemma 3.3 that there exists $x_0\in\mathbb{R}^n$ such that $B_{R_1/2}(x_0)\subset E_t$ for any $t\in[t_0,t_0+t^*]$. In addition, setting $W=H_s(\langle x-x_0,\nu\rangle-R_1/4)^{-1}$, we know that the maximum of W satisfies inequality (33) in this time interval. We need a little care because the point x_0 depends on t_0 and therefore the function W is defined differently in different intervals.

Let us set for simplicity $F(w)=C_1w^2-C_2w^3$ to denote the right-hand side of (33). We observe that $F(w)<0$ for $w>C_1/C_2$. Let us denote by $\tilde{w}(t)$ the solution of the equation $\tilde{w}'(t)=F(\tilde{w}(t))$ defined for $t>0$ and satisfying $\tilde{w}(t)\to+\infty$ as $t\to 0^+$. It is easily seen that such a function exists and is implicitly defined by the formula

$$\int_{\tilde{w}(t)}^{+\infty}\frac{dw}{C_2w^3-C_1w^2}=t.$$

In addition, $\tilde{w}(t)$ is defined for all $t\in(0,+\infty)$ and decreases monotonically from $+\infty$ to C_1/C_2.

We now treat differently the cases $t_0=0$ and $t_0>0$. If $t_0=0$, using the sign properties of the right-hand side of (33), we obtain

$$W(p,t)\leqslant\max\left\{\max_{\mathcal{M}_0}W,\,\frac{C_1}{C_2}\right\},\qquad p\in\mathcal{M},\ t\in[0,t^*].$$

Taking into account (28), this implies

$$H_s(p,t) \leqslant C', \quad p \in \mathcal{M},\ t \in [0, t^*], \tag{34}$$

for a suitable constant C'. If $t_0 > 0$, we observe instead that, again by (33),

$$W(p, t_0 + \tau) \leqslant \tilde{w}(\tau), \quad \tau \in [0, t^*].$$

In particular, since $\tilde{w}$ is monotone,

$$W(p, t_0 + \tau) \leqslant \tilde{w}(t^*/2), \quad \tau \in [t^*/2, t^*].$$

Using (28), it follows that

$$H_s(p,t) \leqslant C'', \quad p \in \mathcal{M},\ t \in [t_0 + t^*/2, t_0 + t^*]. \tag{35}$$

By the arbitrariness of t_0, we conclude from (34)–(35) that $H_s(p,t) \leqslant K_2 := \max\{C', C''\}$ for all p, t. □

6. The case of a nonlinear speed

In this section we study a generalization of problem (1) in which the velocity is given by a general function of the fractional mean curvature. More precisely, we consider

$$\begin{cases} \partial_t F(p,t) = [-\Phi(H_s(p,t)) + \varphi(t)]\nu(p,t), & p \in \mathcal{M}_0,\ t \geqslant 0, \\ F(p,0) = p, & p \in \mathcal{M}_0, \end{cases} \tag{36}$$

where

$$\varphi(t) = \frac{1}{|\mathcal{M}_t|} \int_{\mathcal{M}_t} \Phi(H_s(x))\, d\mu.$$

We assume that $\Phi : [0, +\infty) \to [0, +\infty)$ is a C^2 function, satisfying the following properties:

(i) $\lim_{a \to +\infty} \Phi(a) = +\infty$.

(ii) $\Phi'(a) > 0$ for every $a > 0$.

(iii) $\lim_{a \to +\infty} \Phi'(a) a^2 / \Phi(a) = +\infty$.

Typical examples are functions of the form $\Phi(a) = a^p$ with $p > 0$, but hold in many other cases, e.g., $\Phi(a) = e^a$ or $\Phi(a) = \ln(a+1)$. Assumption (ii) ensures that $\Phi(H_s)$ satisfies the monotonicity assumption (A) in [Chambolle et al. 2015, Section 2] (monotonicity with respect to set inclusion). Hence, by [Chambolle et al. 2015, Theorem 2.21], problem (36) is well-posed and admits a viscosity solution, at least in the case $\varphi \equiv 0$ considered in that paper. In the case of a general $\Phi(H_s)$, the local existence result of smooth solutions is not yet known; there is also no result on the invariance of convexity, since the result in [Chambolle et al. 2017] does not apply. In the classical case, convexity is preserved under some additional structural hypotheses on Φ, see [Bertini and Sinestrari 2018; Andrews and Wei 2017], and it is likely that similar results hold in the fractional case. We will not address these issues here and we will assume instead a priori the existence of a convex smooth solution.

The aim of this section is to prove that Theorem 1.1 holds also for the more general problem (36). We first have the following lemma. As before, we denote by E_t the set enclosed by $\mathcal{M}_t$.

Lemma 6.1. *Flow* (36) *keeps the volume of E_t constant and decreases its fractional perimeter* $\mathrm{Per}_s(E_t)$.

Proof. The first part of the statement is an easy consequence of the choice of $\varphi(t)$. The second part follows exactly as in the proof of [Bertini and Sinestrari 2018, Lemma 3.1] in the local case. □

The uniform bounds on inner and outer radii and the lower bound for H_s are obtained exactly as for the $\Phi(H_s) = H_s$ case (see Section 3), since they just rely on convexity and on the fact that the flow preserves volume and decreases the s-perimeter. Hence, we immediately have the following:

Proposition 6.2. *Let $F : \mathcal{M}_0 \times [0, T) \to \mathbb{R}^n$, with $0 < T \leqslant +\infty$, be a smooth convex solution of* (36). *Then there exist positive constants $0 < R_1 \leqslant R_2$, only depending on E_0, such that*

$$R_1 \leqslant \underline{\rho}_{E_t} \leqslant \bar{\rho}_{E_t} \leqslant R_2 \quad \text{for all } t \in [0, T).$$

In addition, there exists $K_1 > 0$ such that $H_s(p, t) \geqslant K_1$ for all $(p, t) \in \mathcal{M}_0 \times [0, T)$.

From the previous proposition, we deduce again that, by choosing a smaller radius, we can find a ball with fixed center which remains inside E_t for a time interval with fixed length; that is, Lemma 3.3 holds also for solutions of the nonlinear flow (36). The proof of this fact is again by a comparison argument and we refer to [Bertini and Sinestrari 2018, Lemma 3.6] for the details.

In order to prove the analogue of Theorem 1.1 for the flow (36), it remains to establish the upper bound on H_s.

Proposition 6.3. *Let $F : \mathcal{M}_0 \times [0, T) \to \mathbb{R}^n$, with $0 < T \leqslant +\infty$, be a smooth convex solution of* (36). *We have that, at any time $t \in [0, T)$,*

$$\Phi(H_s) \leqslant K_3,$$

where K_3 is a positive constant depending only on n, s, and E_0.

Proof. The proof is similar to the one given in Section 5. We show in detail how the argument is adapted to the case of a general speed, for the sake of clarity. We consider again the support function

$$u(p, t) = \langle F(p, t), \nu(p, t)\rangle.$$

If now F evolves according to (36), recalling Lemma 2.2(iii) and the expression for $\nabla^{\mathcal{M}} H_s$, we have

$$\begin{aligned}\partial_t u &= \langle \partial_t F, \nu\rangle + \langle F, \partial_t \nu\rangle \\ &= -\Phi(H_s) + \varphi(t) + \Phi'(H_s)\langle F, \nabla^{\mathcal{M}} H_s\rangle \\ &= -\Phi(H_s) + \varphi(t) + 2s(1-s)\Phi'(H_s) \int_{\mathcal{M}_t} \frac{x^T \cdot \nu(y)}{|x-y|^{n+s}}\, d\mu(y). \end{aligned} \tag{37}$$

Moreover, using Lemma 2.2(ii), we have that the fractional mean curvature satisfies

$$\frac{\partial_t H_s(x)}{2s(1-s)} = \int_{\mathcal{M}_t} \frac{\Phi(H_s(y)) - \Phi(H_s(x))}{|x-y|^{n+s}}\, d\mu(y) + (\Phi(H_s(x) - \varphi(t)) \int_{\mathcal{M}_t} \frac{1 - \nu(x)\cdot\nu(y)}{|x-y|^{n+s}}\, d\mu(y). \tag{38}$$

We define, much as before, but with the new velocity $\Phi(H_s)$,

$$W = \frac{\Phi(H_s)}{u(x) - \alpha},$$

where α is chosen in the same way as in Section 5. We have that

$$\begin{aligned}
\frac{\partial_t W}{2s(1-s)} &= \frac{1}{2s(1-s)}\left[\frac{\Phi'(H_s)\partial_t H_s}{u(x)-\alpha} - \frac{\Phi(H_s)\partial_t u}{(u(x)-\alpha)^2}\right] \\
&= \frac{\Phi'(H_s(x))}{u(x)-\alpha}\left[\int_{\mathcal{M}_t} \frac{\Phi(H_s(y))-\Phi(H_s(x))}{|x-y|^{n+s}}\,d\mu(y) + (\Phi(H_s(x))-\varphi(t))\int_{\mathcal{M}_t} \frac{1-\nu(x)\cdot\nu(y)}{|x-y|^{n+s}}\,d\mu(y)\right] \\
&\qquad - \frac{\Phi(H_s(x))}{(u(x)-\alpha)^2}\left[\frac{-\Phi(H_s(x))+\varphi(t)}{2s(1-s)} + \Phi'(H_s(x))\int_{\mathcal{M}_t} \frac{x^T\cdot\nu(y)}{|x-y|^{n+s}}\,d\mu(y)\right] \\
&< \frac{\Phi'(H_s(x))}{u(x)-\alpha}\left[\int_{\mathcal{M}_t} \frac{\Phi(H_s(y))-\Phi(H_s(x))}{|x-y|^{n+s}}\,d\mu(y) + \Phi(H_s(x))\int_{\mathcal{M}_t} \frac{1-\nu(x)\cdot\nu(y)}{|x-y|^{n+s}}\,d\mu(y)\right] \\
&\qquad - \frac{\Phi(H_s(x))}{(u(x)-\alpha)^2}\left[\frac{-\Phi(H_s(x))}{2s(1-s)} + \Phi'(H_s(x))\int_{\mathcal{M}_t} \frac{x^T\cdot\nu(y)}{|x-y|^{n+s}}\,d\mu(y)\right]. \qquad (39)
\end{aligned}$$

By the definition of W we have that

$$\begin{aligned}
&\int_{\mathcal{M}_t} \frac{\Phi(H_s(y)) - \Phi(H_s(x))}{|x-y|^{n+s}}\,d\mu(y) \\
&\qquad = \int_{\mathcal{M}_t} (u(y)-\alpha)\frac{(W(y)-W(x))}{|x-y|^{n+s}}\,d\mu(y) + W(x)\int_{\mathcal{M}_t} \frac{u(y)-u(x)}{|x-y|^{n+s}}\,d\mu(y). \qquad (40)
\end{aligned}$$

Moreover, formula (31) holds unchanged, since it is independent of the velocity:

$$\int_{\mathcal{M}_t} \frac{u(y)-u(x)}{|x-y|^{n+s}}\,d\mu(y) = \frac{1}{2(1-s)}H_s(x) + \int_{\mathcal{M}_t} \frac{x^T\cdot\nu(y)}{|x-y|^{n+s}}\,d\mu(y) - u(x)\int_{\mathcal{M}_t} \frac{1-\nu(x)\cdot\nu(y)}{|x-y|^{n+s}}\,d\mu(y). \qquad (41)$$

We combine now (40) and (41) to get

$$\begin{aligned}
&\frac{1}{u(x)-\alpha}\int_{\mathcal{M}_t} \frac{\Phi(H_s(y)) - \Phi(H_s(x))}{|x-y|^{n+s}}\,d\mu(y) \\
&= \frac{1}{u(x)-\alpha}\int_{\mathcal{M}_t} (u(y)-\alpha)\frac{W(y)-W(x)}{|x-y|^{n+s}}\,d\mu(y) \\
&\qquad + \frac{W(x)}{u(x)-\alpha}\left[\frac{1}{2(1-s)}H_s(x) + \int_{\mathcal{M}_t} \frac{x^T\cdot\nu(y)}{|x-y|^{n+s}}\,d\mu(y) - u(x)\int_{\mathcal{M}_t} \frac{1-\nu(x)\cdot\nu(y)}{|x-y|^{n+s}}\,d\mu(y)\right] \\
&= \frac{1}{u(x)-\alpha}\int_{\mathcal{M}_t} (u(y)-\alpha)\frac{W(y)-W(x)}{|x-y|^{n+s}}\,d\mu(y) + \frac{H_s(x)\Phi(H_s(x))}{2(1-s)(u(x)-\alpha)^2} \\
&\qquad + \frac{W(x)}{u(x)-\alpha}\int_{\mathcal{M}_t} \frac{x^T\cdot\nu(y)}{|x-y|^{n+s}}\,d\mu(y) - \frac{W(x)}{u(x)-\alpha}u(x)\int_{\mathcal{M}_t} \frac{1-\nu(x)\cdot\nu(y)}{|x-y|^{n+s}}\,d\mu(y). \qquad (42)
\end{aligned}$$

Finally, plugging (42) into (39), we obtain

$$\frac{\partial_t W}{2s(1-s)} < \Phi'(H_s)\left[\frac{1}{u(x)-\alpha}\int_{\mathcal{M}_t}\frac{(W(y)-W(x))(u(y)-\alpha)}{|x-y|^{n+s}}\,d\mu(y)\right] + \frac{W^2}{2s(1-s)} + \Phi'(H_s)W\left[\frac{H_s(x)}{2(1-s)(u(x)-\alpha)} - \frac{\alpha}{u(x)-\alpha}\int_{\mathcal{M}_t}\frac{1-\nu(x)\cdot\nu(y)}{|x-y|^{n+s}}\,d\mu(y)\right]. \quad (43)$$

This inequality is the analogue of estimate (32) in the presence of a nonlinear speed Φ. Again, we use Proposition 4.2 to bound the last term and we get

$$\frac{\partial_t W}{2s(1-s)} < \Phi'(H_s)\left[\frac{1}{u(x)-\alpha}\int_{\mathcal{M}_t}\frac{(W(y)-W(x))(u(y)-\alpha)}{|x-y|^{n+s}}\,d\mu(y)\right] + C_1 W^2 + \frac{W\Phi'(H_s)H_s}{(u(x)-\alpha)}[C_2 - C_3 H_s]. \quad (44)$$

Setting $\widetilde{W}(t) = \sup_{\mathcal{M}_t} W(x,t)$, we have

$$\partial_t \widetilde{W}(t) \leqslant C_1 \widetilde{W}^2 + \frac{\widetilde{W}\Phi'(H_s)H_s}{u-\alpha}[C_2 - C_3 H_s],$$

where $H_s = H_s(\tilde{x}, t)$ for a suitable $\tilde{x}$ such that $W(\tilde{x}, t) = \widetilde{W}(t)$.

We choose now $K > 3C_2/C_3$, so that $H_s \geqslant K$ implies $C_2 - C_3 H_s \leqslant -2C_3 H_s/3$. Suppose now that there exists t^* such that $\widetilde{W}(t^*) \geqslant \Phi(K)/\alpha$. Recalling that $u - \alpha \geqslant \alpha$ and using the monotonicity of Φ, we deduce that $H_s(x^*, t^*) \geqslant K$ for any x^* such that $W(x^*, t^*) = \widetilde{W}(t^*)$. Hence, at $t = t^*$, we have

$$\partial_t \widetilde{W} \leqslant C_1 \widetilde{W}^2 - \frac{2C_3 \widetilde{W}\Phi'(H_s)H_s^2}{3(u-\alpha)} \leqslant \widetilde{W}^2\left[C_1 - \frac{2C_3}{3}\frac{\Phi'(H_s)H_s^2}{\Phi(H_s)}\right].$$

By property (iii) of Φ, we can choose K large enough so that if $H_s \geqslant K$ we have

$$C_1 - \frac{2C_3}{3}\frac{\Phi'(H_s)H_s^2}{\Phi(H_s)} < -1,$$

which gives

$$\partial_t \widetilde{W} \leqslant -\widetilde{W}^2.$$

From this last estimate, the conclusion follows by a comparison argument, exactly as in the proof of [Bertini and Sinestrari 2018, Proposition 3.7]. □

As a consequence of the boundedness of the speed $\Phi(H_s)$ and of property (i) satisfied by Φ, and recalling Proposition 6.2, we deduce the following:

Corollary 6.4. *We have that H_s is uniformly bounded in* $(0, T)$.

7. Convergence to a sphere

In this section we prove our convergence result (Theorem 1.2 for the case $\Phi(H_s) = H_s$), which for the general problem (36) reads as follows:

Theorem 7.1. *Let $F : \mathcal{M}_0 \times [0, T) \to \mathbb{R}^n$, with $0 < T \leqslant +\infty$, be a smooth convex solution of* (36) *of class $C^{2,\beta}$ for some $\beta > s$ which satisfies property* (R). *Then $T = +\infty$, and $\mathcal{M}_t$ converges to a round sphere as $t \to +\infty$ in $C^{2,\beta}$ norm, possibly up to translations.*

We first observe that, by the lower and upper bounds on H_s, we have that $\Phi'(H_s)$ is bounded from above and below by positive constants for every $t \in [0, +\infty)$.

The crucial step in the proof of Theorem 7.1 is the following result.

Proposition 7.2. *Under our assumption, we have that*

$$\lim_{t\to+\infty} \max_{\mathcal{M}_t} |\Phi(H_s)(x) - \varphi(t)| = 0.$$

Proof. The proof follows the one in [Bertini and Sinestrari 2018, Proposition 4.4]. For any t, let $\overline{H}_s(t)$ be such that $\Phi(\overline{H}_s(t)) = \varphi(t)$. Then, recalling (8), we have

$$\begin{aligned}\frac{d}{dt}\operatorname{Per}_s(E_t) &= \int_{\mathcal{M}_t} H_s\varphi\, d\mu - \int_{\mathcal{M}_t} H_s\Phi(H_s)\, d\mu \\ &= \int_{\mathcal{M}_t} (H_s - \overline{H}_s)(\Phi(\overline{H}_s) - \Phi(H_s))\, d\mu \\ &= -\int_{\mathcal{M}_t} |H_s - \overline{H}_s||\Phi(\overline{H}_s) - \Phi(H_s)|\, d\mu.\end{aligned}$$

Hence, using the boundedness of Φ', we deduce that

$$\frac{d}{dt}\operatorname{Per}_s(E_t) \leqslant -\frac{1}{\sup\Phi'}\int_{\mathcal{M}_t} |\Phi(\overline{H}_s) - \Phi(H_s)|^2\, d\mu = -\frac{1}{\sup\Phi'}\int_{\mathcal{M}_t} |\Phi(\overline{H}_s) - \varphi|^2\, d\mu.$$

Suppose now, by contradiction, that there exists $\varepsilon > 0$ such that $|\Phi(H_s) - \varphi| = \varepsilon$ at some point $(\bar{p}, \bar{t})$. By our regularity assumption and using Theorem 2.1, we have that H_s is uniformly Lipschitz; therefore there exists a uniform radius $r(\varepsilon) > 0$ for which

$$|\Phi(H_s) - \varphi| > \frac{\varepsilon}{2} \quad \text{in } B((\bar{p}, \bar{t}), r(\varepsilon)),$$

which implies

$$\frac{d}{dt}\operatorname{Per}_s(E_t) \leqslant -\eta(\varepsilon) \quad \text{for any } t \in [\bar{t} - r(\varepsilon), \bar{t} + r(\varepsilon)],$$

for some $\eta > 0$. The fact that $\operatorname{Per}_s(E_t) > 0$ and is decreasing in time implies that the above property cannot hold for $\bar{t}$ arbitrarily large. This shows that $|\Phi(H_s) - \varphi|$ tends to zero uniformly. □

We are now ready to give the proof of our convergence result.

Proof of Theorem 7.1. Using our regularity assumption (R) and the uniform bounds for H_s of Corollary 6.4 and Theorem 1.1, we deduce that the flow exists for all $t \in [0, \infty)$ and that the hypersurfaces $\mathcal{M}_t$, possibly up to translations, are bounded in the $C^{2,\beta}$ norm uniformly in t. Hence, the $\mathcal{M}_t$ are precompact in $C^{2,\beta'}$ for $\beta' < \beta$. By Proposition 7.2 and the stability results of [Cozzi 2015], we have that any possible subsequential limit as $t \to +\infty$ has constant fractional curvature. Then Theorem 2.4 ensures that the limit is a ball, with radius uniquely determined by the volume constraint. The uniqueness of the subsequential limit easily implies that the whole family $\mathcal{M}_t$ converges to a sphere as $t \to +\infty$. □

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