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PROPAGATION PROPERTIES OF REACTION-DIFFUSION EQUATIONS IN PERIODIC DOMAINS

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We study the phenomenon of *invasion* for heterogeneous reaction-diffusion equations in periodic domains with monostable and combustion reaction terms. We give an answer to a question raised by Berestycki, Hamel and Nadirashvili concerning the connection between the speed of invasion and the critical speed of fronts. To do so, we extend the classical Freidlin–Gärtner formula to such equations and we derive some bounds on the speed of invasion using estimates on the heat kernel. We also give geometric conditions on the domain that ensure that the spreading occurs at the critical speed of fronts.

1. Introduction and results

1.1. *Introduction.* This paper deals with the spreading properties of the reaction-diffusion equation

$$\begin{cases} \partial_t u = \operatorname{div}(A(x)\nabla u) + q(x)\cdot\nabla u + f(x,u), & t>0,\ x\in\Omega, \\ \nu\cdot A(x)\nabla u = 0, & t>0,\ x\in\partial\Omega. \end{cases} \tag{1}$$

Throughout the paper, the domain Ω and the coefficients are assumed to be periodic. Here, ν stands for the exterior normal. Reaction-diffusion equations arise in the study of various phenomena in biology (propagation of genes, epidemics), physics (combustion), and more recently in social sciences (rioting models). A particular emphasis is given here to the case where the equation is homogeneous but the domain is not the whole space:

$$\begin{cases} \partial_t u = \Delta u + f(u), & t>0,\ x\in\Omega, \\ \partial_\nu u = 0, & t>0,\ x\in\partial\Omega. \end{cases}$$

In such a case, we provide an answer to a question asked by Berestycki, Hamel and Nadirashvili [Berestycki et al. 2005] concerning the relation between the speed of invasion and the speed of fronts for this problem.

Reaction-diffusion equations have been extensively studied since the seminal paper of Kolmogorov, Petrovski and Piskunov [Kolmogorov et al. 1937]. There, the authors dealt with the homogeneous equation

$$\partial_t u = \Delta u + f(u), \quad t>0,\ x\in\mathbb{R}^N, \tag{2}$$

with $f(u)=u(1-u)$. The results of [Kolmogorov et al. 1937] were extended in [Aronson and Weinberger 1978] to more general *reaction terms* f. The basic assumption is that $f(0)=f(1)=0$, so that the constant

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states $u \equiv 0$ and $u \equiv 1$ are stationary solutions. We shall pay a particular attention to the following two types of nonlinearities:

(*monostable*) $f > 0$ in $(0, 1)$.

(*combustion*) There exists $\theta \in (0, 1)$ such that $f = 0$ in $[0, \theta]$, $f > 0$ in $(\theta, 1)$.

These two notions extend to the case where f can depend on x; see Definition 1 below. Two important features of reaction-diffusion equations were derived in [Aronson and Weinberger 1978]. First, (2) admits particular solutions called *traveling fronts*. These are positive entire (i.e., defined for all $t \in \mathbb{R}$) solutions of the form $u(t, x) = \phi(x \cdot e - ct)$, for some $e \in \mathbb{S}^{N-1}$, $c \in \mathbb{R}$, ϕ decreasing and satisfying $\phi(s) \to 1$ as $s \to -\infty$ and $\phi(s) \to 0$ as $s \to +\infty$. The unit vector e is the *direction of propagation*, c is the *speed of propagation* and ϕ is the *profile* of the traveling front. More specifically, there exists a quantity $c^\star$ such that there are fronts with speed c for every $c \geq c^\star$ if f is of *monostable* type, whereas there are traveling fronts only with speed $c = c^\star$ if f is of *combustion* type. Of course, the homogeneity of (2) implies that the quantity $c^\star$ does not depend on the direction of the fronts e. We mention that, if f is of *KPP* type (i.e., if it is monostable and satisfies $f'(0) > 0$ and $f(u) \leq f'(0)u$ for $u \in [0, 1]$), then it is proved in [Kolmogorov et al. 1937] that $c^\star = 2\sqrt{f'(0)}$. The quantity $c^\star$ is called the *critical* (*or minimal*) *speed of fronts*. We consider this quantity in a more general context in Section 1.2.

The second important feature of reaction-diffusion equations is the property of *invasion*. If $u(t, x)$ is a solution of (2) arising from the initial datum u_0 such that

$$u(t, x) \xrightarrow[t \to +\infty]{} 1 \quad \text{locally uniformly in } x,$$

we say that invasion occurs for the initial datum u_0. Of course, this depends on the nonlinearity f. For instance, if f is of combustion type, and if u_0 is a compactly supported nonnegative initial datum and is such that $u_0 \leq \theta$, then the problem (2) boils down to the heat equation, and then $u(t, x) \to 0$ as t goes to $+\infty$ uniformly in x. However, it is shown in [Aronson and Weinberger 1978] that, for every $\eta \in (\theta, 1)$, there is $R > 0$ such that any initial datum such that $u_0(x) \geq \eta \mathbb{1}_{B_R}$ (where B_R is the ball of center 0 and of radius R) satisfies the invasion property. In contrast, if f is of KPP type, then invasion occurs for any nonnegative nonzero initial datum.

Once we know that invasion occurs for some initial data, we can define the *speed of invasion*. We say that $w(e) > 0$ is the speed of invasion for (2) in the direction $e \in \mathbb{S}^{N-1}$ if, for any solution $u(t, x)$ of (2) emerging from a compactly supported nonnegative initial datum that converges to 1 as t goes to $+\infty$, locally uniformly in x, the following holds:

$$\begin{aligned} &\text{for all } c > w(e), && u(t, x + cte) \to 0 \quad \text{as } t \to +\infty, \\ &\text{for all } c \in [0, w(e)), && u(t, x + cte) \to 1 \quad \text{as } t \to +\infty, \end{aligned}$$

locally uniformly in $x \in \mathbb{R}^N$. The homogeneity of (2) yields that the speed of invasion is actually independent of the direction e. Moreover, if f is of KPP type, it is proved in [Kolmogorov et al. 1937] that $w(e) = 2\sqrt{f'(0)}$ for all $e \in \mathbb{S}^{N-1}$. Hence, in this case $c^\star \equiv w$. In other terms, this means that the invasion occurs at the critical speed of fronts in every direction.

One of the main motivations behind the present paper is to understand to what extent this is still satisfied in more general domains, in which case closed formulas for the speeds are not available. Berestycki, Hamel and Nadirashvili [Berestycki et al. 2005] conjectured that the geometry of the domain could give that the invasion does not occur at the critical speed of fronts in every direction (see Question 4 below). We shall construct such domains. We shall also give geometric conditions on the domain that ensure that the invasion speed coincides with the critical speed of fronts in some directions.

In order to state our main results, we first present how the notions of fronts and invasion extend to the case of spatially periodic heterogeneous equations.

1.2. ***Pulsating traveling fronts.*** The notion of *pulsating traveling fronts* was first introduced in dimension $N = 1$ in periodic media by Shigesada, Kawasaki and Teramoto [Shigesada et al. 1986] to generalize the notion of traveling fronts available in the homogenous case. Berestycki and Hamel [2002] extended this notion to the more general framework of (1). Throughout the paper, we assume that A, q, f, Ω are periodic, with the same period; i.e, there are $L_1, \dots, L_N > 0$ such that

$$\text{for all } k \in \prod_{i=1}^{N} L_i\mathbb{Z}, \quad \Omega + \{k\} = \Omega,$$

$$\text{for all } k \in \prod_{i=1}^{N} L_i\mathbb{Z}, \quad f(\,\cdot\, + k, \cdot\,) = f, \quad q(\,\cdot\, + k) = q, \quad A(\,\cdot\, + k) = A.$$

We shall denote by $\mathcal{C} := \prod_{i=1}^{N}[0, L_i)$ the periodicity cell. Typical examples of such domains Ω are domains with "obstacles": if $K \subset \mathbb{R}^N$ is a smooth compact set, we can define the periodic domain $\Omega := (K + L\mathbb{Z}^N)^c$, with $L > 0$ large enough so that the resulting domain is smooth and connected. This domain can be seen as the whole space with K-shaped obstacles periodically distributed.

To simplify the notation, unless otherwise stated, we shall assume that the period is 1, i.e., $L_1 = \cdots = L_N = 1$. In order to apply the results of [Berestycki and Hamel 2002], we make the following assumptions on the domain:

$$\Omega \text{ is a periodic, connected open subset of } \mathbb{R}^N \text{of class } C^3, \tag{3}$$

and the following hypotheses on the coefficients:

$$\begin{gathered} A \in C^3(\overline{\Omega}) \text{ is symmetric and uniformly elliptic and periodic,} \\ q \in C^{1,\alpha}(\overline{\Omega}) \text{ for some } \alpha \in (0,1), \quad \operatorname{div} q = 0, \quad \textstyle\fint_{\mathcal{C}\cap\Omega} q = 0, \quad q \text{ is periodic,} \\ f : \overline{\Omega} \times [0,1] \mapsto \mathbb{R} \text{ is of class } C^{1,\alpha} \text{ for some } \alpha \in (0,1). \end{gathered} \tag{4}$$

We also assume that the nonlinearity f satisfies the following:

$$\begin{aligned} \text{for all } x \in \Omega, &\quad f(x,0) = f(x,1) = 0, \\ \text{there exists } S \in (0,1) \text{ such that for all } x \in \overline{\Omega}, &\quad f(x,\cdot\,) \text{ is nonincreasing in } [S,1], \\ \text{for all } s \in (0,1), &\quad f(\,\cdot\,, s) \text{ is periodic.} \end{aligned} \tag{5}$$

By analogy with the homogeneous case $f = f(u)$, we define monostable, KPP and combustion nonlinearities $f(x, u)$:

Definition 1. We say that f is of *monostable* type if

$$\text{for all } s \in (0,1), \quad \min_{x\in\overline{\Omega}} f(x,s) \geq 0, \quad \max_{x\in\overline{\Omega}} f(x,s) > 0. \tag{6}$$

Among monostable nonlinearities, there is the special class of *KPP* nonlinearities. In addition to being monostable, they satisfy

$$\text{for all } x \in \overline{\Omega}, \text{ for all } s \in [0,1], \quad f(x,s) \leq \partial_s f(x,0)s. \tag{7}$$

We say that f is of *combustion* type if

$$\begin{aligned} &\text{there exists } \theta \in (0,1) \text{ such that for all } (x,s) \in \Omega \times [0,\theta], \quad f(x,s) = 0, \\ &\text{for all } s \in (\theta,1), \quad \min_{x\in\overline{\Omega}} f(x,s) \geq 0, \quad \max_{x\in\overline{\Omega}} f(x,s) > 0. \end{aligned} \tag{8}$$

The important difference between combustion and monostable nonlinearities (from which stems the nonuniqueness of speeds of fronts for monostable equation) is that, when f is of combustion type,

$$\text{there exists } \theta \in (0,1] \text{ such that for all } x \in \Omega, \quad f(x,\cdot) \text{ is nonincreasing in } [0,\theta]. \tag{9}$$

In the periodic framework, the notion of traveling fronts can be generalized by *pulsating traveling fronts*.

Definition 2. A pulsating traveling front in the direction $e \in \mathbb{S}^{N-1}$ of speed $c \in \mathbb{R}\backslash\{0\}$ connecting 1 to 0 is an entire (i.e., defined for all $t \in \mathbb{R}$) solution v of (1) satisfying

$$\begin{cases} \text{for all } k \in \mathbb{Z}^N \quad \text{for all } x \in \Omega, & v(t + (k\cdot e)/c, x) = v(t, x-k), \\ v(t,x) \to 1 \quad \text{as } x\cdot e \to -\infty, & v(t,x) \to 0 \quad \text{as } x \cdot e \to +\infty. \end{cases}$$

Such fronts are known to exist in several situations. For instance, it is proved in [Berestycki and Hamel 2002] that, under hypotheses (4)–(5), for every $e \in \mathbb{S}^{N-1}$, there is $c^\star(e) > 0$, called again the *critical* (*or minimal*) *speed of fronts* in direction e, such that pulsating traveling fronts in the direction e with speed c exist if, and only if, $c \geq c^\star(e)$ when f is of monostable type (6) or only if $c = c^\star(e)$ when f is of combustion type (8); see [Berestycki and Hamel 2002, Theorems 1.13–1.14].

1.3. ***The speed of invasion.*** The results of Kolmogorov, Petrovski, Piskunov [Kolmogorov et al. 1937] and Aronson and Weinberger [1978] concerning the invasion have also been extended to a more general framework than the homogeneous one. First, consider the periodic equation on $\mathbb{R}^N$

$$\partial_t u = \operatorname{div}(A(x)\nabla u) + q(x)\cdot\nabla u + f(x,u), \quad t > 0, \ x \in \mathbb{R}^N. \tag{10}$$

Then, one can define the speed of invasion w as a function from the unit sphere $\mathbb{S}^{N-1}$ to $\mathbb{R}^+$ such that, for every u solution of (1) arising from a compactly supported nonnegative initial datum which converges to 1 as t goes to $+\infty$, locally uniformly in $x \in \mathbb{R}^N$, we have, for $e \in \mathbb{S}^{N-1}$,

$$\begin{aligned} &\text{for all } c > w(e), && u(t, x + cte) \to 0 \quad \text{as} \quad t \to +\infty, \\ &\text{for all } c \in [0, w(e)), && u(t, x+cte) \to 1 \quad \text{as} \quad t \to +\infty, \end{aligned}$$

locally uniformly in $x \in \mathbb{R}^N$.

Using probabilistic techniques, Gärtner and Freidlin [1979] showed the existence of a speed of invasion for (10) when f is of KPP type (7) and A, q, f are x-periodic. They showed that invasion occurs for every nonnegative nonnull compactly supported initial datum and proved what is now known as the *Freidlin–Gärtner formula*:

$$w(e) := \min_{\substack{\xi \in \mathbb{R}^N \\ e\cdot\xi > 0}} \frac{k(\xi)}{e \cdot \xi}, \tag{11}$$

where $k(\xi)$ is the periodic principal eigenvalue of the operator

$$L_\xi u := \operatorname{div}(A\nabla u) - 2\xi \cdot A\nabla u + q \cdot \xi u + (-\operatorname{div}(A\xi) - q \cdot \xi + \xi \cdot A\xi + \partial_u f(x, 0))u.$$

This formula was also proved by Berestycki, Hamel and Nadin [Berestycki et al. 2008] using a PDE approach. Similar properties of spreading for heterogeneous reaction-diffusion equations have been studied with other approaches: the viscosity solution/singular perturbation method was adopted by Evans and Souganidis [1989] and Barles, Soner and Souganidis [1993]. Weinberger [2002] used an abstract discrete system approach.

Berestycki, Hamel and Nadirashvili [Berestycki et al. 2005] showed that, if one considers KPP nonlinearities, the critical speed of pulsating traveling fronts in the direction e for (10) is given by $c^\star(e) = \min_{\lambda>0} k(\lambda e)/\lambda$, where k is the principal eigenvalue introduced before (if the equation were set on a periodic domain Ω instead of $\mathbb{R}^N$, this relation still holds true with k being the periodic principal eigenvalue of the same operator but with the additional boundary condition $\nu \cdot A\nabla u = \lambda(\nu \cdot e)u$ on $\partial\Omega$, see [Berestycki et al. 2005] for the details). Consequently, in the KPP case, the Freidlin–Gärtner formula (11) can be rewritten as

$$w(e) = \min_{e\cdot\xi>0} \frac{c^\star(\xi)}{e \cdot \xi}. \tag{12}$$

The fact that pulsating traveling fronts exist not only in the KPP case but also for other reaction terms, and hence that the formula (12) could make sense in more general frameworks than the KPP one, led Rossi [2017] to extend the Freidlin–Gärtner formula to much more general equations in the whole space, essentially, all those for which pulsating traveling fronts are known to exist.

In this paper, we deal with invasion in domains Ω that are not necessarily $\mathbb{R}^N$. In this case, it is convenient to introduce the notion of *asymptotic set of spreading*.

Definition 3. Let $\mathcal{W} \subset \mathbb{R}^N$ be a closed set coinciding with the closure of its interior. We say that $\mathcal{W}$ is the asymptotic set of spreading for a reaction-diffusion equation if, for any bounded solution $u(t, x)$ emerging from a nonnegative compactly supported initial datum such that $u(t, x) \to 1$ as $t \to +\infty$, locally uniformly in $x \in \overline{\Omega}$, we have

$$\text{for all } K \text{ compact, } K \subset \operatorname{int}(\mathcal{W}), \quad \inf_{x \in tK} u(t, x) \to 1 \quad \text{as } t \to +\infty, \tag{13}$$

$$\text{for all } C \text{ closed, } C \cap \mathcal{W} = \varnothing, \quad \sup_{x \in tC} u(t, x) \to 0 \quad \text{as } t \to +\infty. \tag{14}$$

If only (13) holds, $\mathcal{W}$ is said to be an asymptotic subset of spreading, and if only (14) holds, $\mathcal{W}$ is said to be an asymptotic superset of spreading.

The asymptotic set of spreading relates to the notion of speed of invasion previously described. Indeed, assume that $\mathcal{W}$ is an asymptotic set of spreading and that we can write $\mathcal{W} = \{r\xi : \xi \in \mathbb{S}^{N-1},\ 0 \le r \le w(\xi)\}$ with w a continuous function. Then, if $\Omega = \mathbb{R}^N$, $w(e)$ is the speed of spreading in the direction e, as defined before. For example, if f is a KPP nonlinearity independent of x, then the asymptotic set of spreading associated with the homogeneous equation (2) is the ball of center 0 and of radius $2\sqrt{f'(0)}$.

Observe that, for the definition of the asymptotic set of spreading to be meaningful, it is necessary that there are compactly supported initial data u_0 for which the invasion property holds. Rossi and the author [Ducasse and Rossi 2018] gave necessary and sufficient conditions to have invasion for (1). In particular, we showed there that, if f is of monostable or combustion type, in the sense of Definition 1, and if the drift term q is "not too large" (see [Ducasse and Rossi 2018] for the details), then, setting

$$\theta := \max\{s \in [0, 1) : \text{there exists } x \in \overline{\Omega} \text{ such that } f(x, s) = 0\},$$

we have that, for all $\eta \in (\theta, 1)$, there is $r > 0$ such that any solution of (1) with an initial datum u_0 satisfying

$$u_0 > \eta \quad \text{in } \Omega \cap B_r$$

converges to 1 as t goes to $+\infty$, locally uniformly in $x \in \overline{\Omega}$.

1.4. ***Statement of the main results.*** One of the main motivations behind the present paper is to answer the following question, raised by Berestycki, Hamel and Nadirashvili [Berestycki et al. 2005]:

Question 4. *Consider the homogeneous equation set on a periodic domain* Ω

$$\begin{cases} \partial_t u - \Delta u = f(u), & t > 0,\ x \in \Omega, \\ \partial_\nu u = 0, & t > 0,\ x \in \partial\Omega. \end{cases} \tag{15}$$

Are there domains Ω *such that* $c^\star \not\equiv w$?

We recall that $c^\star$ is the critical speed of pulsating traveling fronts and w is the speed of invasion. Originally, this question was asked for f of KPP type (7), but it also makes sense if f is a monostable (6) or a combustion (8) nonlinearity.

As we already mentioned, if the domain Ω is the whole space $\mathbb{R}^N$ and if f is of KPP type, then w and $c^\star$ are independent of the direction and are both equal to $2\sqrt{df'(0)}$. In general periodic domains, the propagation may not be isotropic anymore: w and $c^\star$ can depend on the direction.

Let us mention that, if we considered the equation with general coefficients (1), it would be actually much easier to have $c^\star \not\equiv w$ with $\Omega = \mathbb{R}^N$. For instance, in dimension 2, when the Laplace operator is replaced by $a\partial^2_{xx} + b\partial^2_{yy}$, with $a, b > 0$, and when the nonlinearity f is of KPP type, one could explicitly compute $c^\star$ and w, see [Berestycki et al. 2005, Remark 1.12], and one could observe that, if $a \ne b$, then $c^\star \not\equiv w$. What was not known is whether the geometry of the domain alone could give that $c^\star \not\equiv w$. We prove that this is the case.

Theorem 5. *Let* f *be a monostable* (6) *or a combustion* (8) *nonlinearity independent of* x. *There are smooth periodic domains* Ω *such that the critical speed of pulsating traveling fronts* $c^\star$ *and the invasion speed* w *for* (15) *are not everywhere equal*, *that is*, $c^\star \not\equiv w$.

This provides a positive answer to Question 4. When the nonlinearity f is of monostable or combustion type, then the domains we exhibit are L-periodic, with L large enough. If f is a KPP nonlinearity, then we can construct domains with any periodicity.

Let us emphasize that Theorem 5 does not say that we can construct domains where $c^\star(e) \neq w(e)$ for every $e \in \mathbb{S}^{N-1}$: we shall explain after why this is actually impossible. Finding directions where the two speeds coincide is the object of Theorem 7 below.

A first step in proving Theorem 5 will be to give a formula for the speed of invasion. We show that the Freidlin–Gärtner formula (12) still holds true for the general equation (1) in the periodic domain Ω with combustion and monostable nonlinearities.

Theorem 6. *Let A, q, f, Ω be periodic, satisfying* (4)–(5). *Assume that f is a monostable* (6) *or a combustion* (8) *nonlinearity. Then,* (1) *has the asymptotic set of spreading*

$$\mathcal{W} = \{r\xi : \xi \in \mathbb{S}^{N-1},\ 0 \le r \le w(\xi)\}, \tag{16}$$

where $w(\xi) := \inf_{e\cdot\xi>0} c^\star(e)/(e\cdot\xi)$, and $c^\star(e)$ is the critical speed of pulsating traveling fronts in the direction e.

This result extends the one by Rossi to the case where the domain is not $\mathbb{R}^N$ anymore. We shall follow the same strategy of proof. As this result is crucial to carry through our investigations, and as the result is of independent interest, we shall prove it in detail in Section 2.

Once Theorem 6 is established, we employ it to derive a simple criterion ensuring that $c^\star \not\equiv w$. We show that if $c^\star \equiv w$, then $c^\star$ and w are necessarily constant; see Proposition 12 below. To answer Question 4 is then tantamount to finding domains where w or $c^\star$ are not constant. Intuitively, we may think that, if a domain is "very obstructed" in a direction, then the speed should be small in this direction.

In order to make this intuition rigorous, we derive new estimates on the invasion speed that take into account the geometry of the domain. This is the subject of Section 3.3. The main tool is an upper bound on the heat kernel in Ω. Once we have these estimates at hand, we are able to construct domains where $c^\star$ and w are not constant, and hence are different. This is done in Section 3.4, proving then Theorem 5.

The remainder of the paper is dedicated to giving conditions under which $c^\star$ and w coincide in some directions. Indeed, observe that, though we can construct domains Ω where $c^\star \not\equiv w$, there is always at least one direction $e \in \mathbb{S}^{N-1}$ such that $c^\star(e) = w(e)$: it is readily seen from the Freidlin–Gärtner formula (12) that any direction $e_{\min}$ that minimizes $c^\star$ satisfies $c^\star(e_{\min}) = w(e_{\min})$. The only other characterization of directions e where $c^\star(e) = w(e)$ we are aware of holds true in the KPP case: it is proved in [Berestycki et al. 2005] that $c^\star(e_{\text{inv}}) = w(e_{\text{inv}})$ if Ω is invariant in the direction e_{inv} (i.e., $\Omega + \{\lambda e_{\text{inv}}\} = \Omega$ for every $\lambda \in \mathbb{R}$).

Our next theorem, proved in Section 4, shows that, if the domain Ω is symmetric, then there are directions where $c^\star$ and w coincide. This result requires $u \mapsto f(u)/u$ to be nonincreasing (strong KPP property).

Theorem 7. *Assume that f satisfies* (4), (5) *and that $u \mapsto f(u)/u$ is nonincreasing. Let $c^\star$ and w be the critical speed of fronts and the speed of invasion for* (15). *Then, assume that there is an orthogonal transformation T such that*:

- *T leaves Ω invariant*; *i.e.*, $T\Omega = \Omega$.
- *There is $e \in \mathbb{S}^{N-1}$ such that $Te = e$, and* $\mathrm{Ker}(T - I_N) = \mathbb{R}e$.

Then

$$c^\star(e) = w(e).$$

This result implies for instance that, if a periodic domain in $\mathbb{R}^2$ is symmetric with respect to an axis, then $c^\star$ and w coincide in the direction of this axis; see Corollary 24 for more examples.

As we shall explain in Remark 25, the hypothesis that $\mathrm{Ker}(T - I_N) = \mathbb{R}e$ is necessary.

Let us conclude this section with some questions that are still open. The set $\mathcal{W}$ given by (16) is sometimes called the *Wulff shape* associated with the surface tension $c^\star$. It appears in crystallography and in isoperimetric problems. A natural question is whether the function w parametrizing the boundary of $\mathcal{W}$ is regular. Rossi [2017] proved that it is continuous. We are not aware of further regularity results. We conjecture that, at least in the KPP case (where $c^\star$ is known to be smooth), w is smooth.

Theorem 5 states that there are domains Ω such that $c^\star \not\equiv w$. One may wonder on the contrary if there are periodic domains $\Omega \neq \mathbb{R}^N$ such that $c^\star \equiv w$. Thanks to our Proposition 12 below, this is equivalent to finding domains where $c^\star$ is constant. As far as we know, the existence of such domains is still open.

Let us also mention that, although the construction of the domain Ω where $c^\star \not\equiv w$ in Theorem 5 will be explicit, our proof will not tell in which direction(s) the two speeds indeed differ. We leave this as an open question.

Remark 8. In addition to the monostable and combustion cases, there is another class of reaction terms f that is widely studied in the literature, namely the *bistable nonlinearities*. The prototype is $f(u) = u(1-u)(u-a)$, with $a \in (0, 1)$. In this paper, we *do not* consider such nonlinearities; indeed, the main tool we use is the existence of pulsating traveling fronts with *positive speed*. If $\Omega = \mathbb{R}^N$, there are results in some particular cases; see [Ducrot 2016; Xin 1991a; 1991b] for instance. If $\Omega \neq \mathbb{R}^N$, the situation is yet to be explored, and the geometry of the domain can yield phenomena that do not appear in the combustion or monostable case. For instance, Rossi and the author showed in [Ducasse and Rossi 2018] that invasion can occur in some directions but not in others. However, we mention that the strategy used to derive Theorem 5 still applies if f is bistable, provided there exist pulsating traveling fronts with positive speeds in every direction $e \in \mathbb{S}^N$.

2. Freidlin–Gärtner formula for a periodic domain

This section is dedicated to the proof of Theorem 6; i.e., we show that the Freidlin–Gärtner formula (12) relating the critical speeds of fronts to the speed of invasion still holds true when the domain is not $\mathbb{R}^N$ but a periodic domain Ω and with monostable or combustion nonlinearities. Our proof is based on the same strategy as the one used in [Rossi 2017]. We start by stating some preliminary technical results. For simplicity, we assume throughout this section that the domain and the coefficients are 1-periodic, i.e., $L_1 = \cdots = L_N = 1$.

2.1. ***Preliminary results.*** In the proof of Theorem 6, we will need some technical lemmas. They generalize those of [Rossi 2017, Section 2.1] to the case where the domain is not $\mathbb{R}^N$ anymore. The main technical difficulty is that Ω is not invariant under translations in general. The proofs follow the same lines as in [Rossi 2017] and can be found for completeness in the Appendix. We say that u is a subsolution (respectively supersolution) if it satisfies (1) with the symbols $=$ replaced by $\leq$ (respectively $\geq$).

The first lemma states that every entire solution that is "large enough" in some direction is actually "front-like" in this direction.

Lemma 9. *Let $\gamma > 0$. Assume that* (4) *and* (5) *hold. Let $u \in C^{1+\alpha/2,2+\alpha}(\mathbb{R} \times \Omega)$ for some $\alpha \in (0,1)$ be an entire supersolution of* (1) *such that*

$$\inf_{\substack{t<0\\ x\cdot e<\gamma t\\ x\in\Omega}} u(t,x) > S,$$

where S is defined in (5). *Then*

$$\liminf_{\delta\to+\infty} \inf_{\substack{t<0\\ x\cdot e<\gamma t-\delta\\ x\in\Omega}} u(t,x) \geq 1.$$

The following lemma is a comparison principle for front-like solutions.

Lemma 10. *Assume that* (4) *and* (5) *hold. Let $\bar{u}$, $\underline{u} \in C^{1+\alpha/2,2+\alpha}(\mathbb{R} \times \Omega)$, for some $\alpha \in (0,1)$, be respectively an entire supersolution and subsolution of* (1). *Assume that there are $e \in \mathbb{S}^{N-1}$, $\gamma > 0$ such that*

$$\bar{u} > 0, \quad \liminf_{\delta\to+\infty} \inf_{\substack{t<0\\ x\cdot e<\gamma t-\delta\\ x\in\Omega}} \bar{u}(t,x) \geq 1. \tag{17}$$

Moreover, assume that $\underline{u} \leq 1$ and that there is $\eta > 0$ such that the following hold:

- *The nonlinearity f is of combustion type* (8) *and*

$$\text{for all } s > 0, \text{ there exists } L \in \mathbb{R},\ \underline{u}(t,x) \leq s \text{ such that if } t \leq 0, \text{ then } x\cdot e \geq (\gamma+\eta)t + L,\ x \in \Omega, \tag{18}$$

  or

- *the nonlinearity f is of monostable type* (6) *and*

$$\text{there exists } L \in \mathbb{R},\ \underline{u}(t,x) \leq 0 \text{ such that if } t \leq 0, \text{ then } x\cdot e \geq (\gamma+\eta)t + L,\ x \in \Omega. \tag{19}$$

Then, the following comparison result holds:

$$\underline{u}(t,x) \leq \bar{u}(t,x), \quad \text{for all } t \in \mathbb{R}, \text{ for all } x \in \Omega.$$

In addition to those two technical lemmas, we shall need the following result, stating that, in our framework, the speed of invasion w is a continuous function:

Lemma 11. *Let A, q, f, Ω be periodic, satisfying* (4)–(5). *Assume that f is of monostable type* (6) *or of combustion type* (8). *Let w be defined by* (12). *Then w is a continuous function from the sphere $\mathbb{S}^{N-1}$ to $\mathbb{R}_+$.*

This lemma is proved in [Rossi 2017] in the case $\Omega = \mathbb{R}^N$, but the proof directly works in our case, so we omit it. It relies on the fact that $c^\star$ is lower semicontinuous. Additionally, we mention that it is proved in [Alfaro and Giletti 2016] that $c^\star$ is actually continuous.

2.2. ***Proof of Theorem 6.*** This section is dedicated to the proof Theorem 6. We show that the Freidlin–Gärtner formula (12) still holds in the context of periodic domains Ω considered in this paper. The proof is divided into several steps. We use a geometric argument, introduced in [Rossi 2017]. The idea is to argue by contradiction: we will consider a solution that invades space, and we will translate our solution in time and space to keep track with the transition zone. Our solution will converge to a "fast" front-like solution, which we shall compare to a pulsating traveling front to get a contradiction.

Proof. We start to prove that $\mathcal{W}$, defined by (16), is an asymptotic subset of spreading. We argue by contradiction. We assume that $\mathcal{W}$ is not an asymptotic subset of spreading; then, there is a compact set $K \subset \operatorname{int}(W)$ such that (13) does not hold. Now, we take $W \subset \mathcal{W}$, W star-shaped with respect to the origin, compact and C^∞ such that $K \subset \operatorname{int}(W)$. We assume that W is the graph of a function $\tilde{w}$, i.e., $W = \{r\xi : \xi \in \mathbb{S}^{N-1},\ 0 \le r \le \tilde{w}(\xi)\}$, with $\tilde{w}$ smooth and $\tilde{w} < w$, so that W is strictly contained in $\mathcal{W}$. We take $\tilde{w}$ strictly positive. This is possible because the function w is continuous thanks to Lemma 11.

The set W satisfies the uniform interior ball estimates

there exists $\rho > 0$ such that for all $x \in \partial W$, there exists $y \in W$ such that $\bar{B}_\rho(y) \subset W$ and $x \in \partial B_\rho(y)$,

where $B_\rho(y)$ is the ball of center y and of radius ρ. In the course of the proof, $u(t, x)$ denotes a solution of (1) arising from a nonnegative, compactly supported initial datum such that invasion occurs; i.e., $u(t, x) \to 1$ as t goes to $+\infty$, locally uniform in $x \in \bar{\Omega}$.

Step 1: Definition of $\mathcal{R}^\eta$. Let $0 < \eta < 1$. We define

$$\mathcal{R}^\eta(t) := \sup\{r \ge 0 : \text{for all } x \in (rW) \cap \bar{\Omega},\ u(t, x) > \eta\}.$$

For $t \ge 0$, this quantity is well-defined because $u(t, x)$ decays to zero as $|x|$ goes to $+\infty$ (this is readily seen by comparison with pulsating traveling fronts) implying that $\mathcal{R}^\eta(t) < +\infty$. Moreover, we have that $\mathcal{R}^\eta(t) \to +\infty$ as t goes to $+\infty$ (because of the assumption that $u(t, x) \to 1$ locally uniformly in x when $t \to +\infty$).

Remembering that we assumed, by contradiction, that there is a compact set $K \subset \operatorname{int}(W)$ such that (13) does not hold, we can infer that there are $\eta, k \in (0, 1)$ such that

$$\liminf_{t \to +\infty} \frac{\mathcal{R}^\eta(t)}{t} < k. \tag{20}$$

Indeed, if this were not the case, then for every $\eta \in (0, 1)$, we would have $\liminf_{t\to+\infty} \mathcal{R}^\eta(t)/t \ge 1$. Hence, taking $h \in (0, 1)$ such that $K \subset hW$, we have

$$\eta \le \liminf_{t\to+\infty} \inf_{x \in R^\eta(t) W} u(t, x) \le \liminf_{t\to+\infty} \inf_{x \in htW} u(t, x) \le \liminf_{t\to+\infty} \inf_{x \in tK} u(t, x).$$

If this were true for each $\eta \in (0, 1)$, it would yield that K satisfies (13), which we assumed not to be the case. Hence, (20) holds. Observe that (20) still holds if we increase η. We do so, and in the following we assume that $\eta \in (S, 1)$, where S is defined in (5).

From now on, we simplify our notation by writing $\mathcal{R}$ instead of $\mathcal{R}^\eta$. Observe that $\mathcal{R}$ is lower semicontinuous. Indeed, let t_n be a sequence such that $t_n \to t_0$ as n goes to $+\infty$ and such that $\mathcal{R}(t_n) \to R \in \mathbb{R}$.

Consider $r > R$. Then, for n large enough, we have that $r > \mathcal{R}(t_n)$, and, by the definition of $\mathcal{R}(t_n)$, there is $x_n \in (rW) \cap \overline{\Omega}$ such that $u(t_n, x_n) \le \eta$. By the continuity of u, there is some $x_0 \in (rW) \cap \overline{\Omega}$ such that $u(t_0, x_0) \le \eta$. This implies that $\mathcal{R}(t_0) \le r$, and then that $\mathcal{R}(t_0) \le R$ by the arbitrariness of $r > R$, and hence the semicontinuity.

Step 2: Shifting the function. By definition of $\mathcal{R}$ we have that $\liminf_{t\to+\infty}(\mathcal{R}(t) - kt) = -\infty$. We define, for $n \in \mathbb{N}$,

$$t_n := \inf\{t \ge 0 : \mathcal{R}(t) - kt \le -n\}.$$

The lower semicontinuity of $\mathcal{R}$, proved in the first step, gives us that the above infimum is a minimum, i.e., that $\mathcal{R}(t_n) - kt_n \le -n < \mathcal{R}(t) - kt$ for all $t < t_n$, and that $t_n \to +\infty$ as n goes to $+\infty$. Hence, the sequence $(t_n)_{n\in\mathbb{N}}$ satisfies

$$\lim_{n\to+\infty} t_n = +\infty \qquad \text{and} \qquad \text{for all } n \in \mathbb{N}, \text{ for all } t \in [0, t_n), \quad \mathcal{R}(t_n) - k(t_n - t) < \mathcal{R}(t).$$

Now, by the definition of $\mathcal{R}(t)$, we have that for all $r > \mathcal{R}(t)$ there exists $x_r \in (rW \cap \overline{\Omega}) \setminus ((\mathcal{R}(t)W) \cap \overline{\Omega})$ such that $u(t, x_r) \le \eta$. Up to extraction, we can assume that $x_r \to x_\infty$ as r goes to $\mathcal{R}(t)$, where $x_\infty \in \overline{\Omega} \cap \partial(\mathcal{R}(t)W)$. By continuity, we have that $u(t, x_\infty) = \eta$.

Hence, we can consider a sequence $(x_n)_{n\in\mathbb{N}} \in \overline{\Omega}$ such that $u(t_n, x_n) = \eta$, with the additional property that $x_n \in \partial(\mathcal{R}(t_n)W)$. Clearly, $|x_n| \to +\infty$ as n goes to $+\infty$. If $x \in \partial W$, let $\tilde{\nu}(x)$ be the outer unit normal to W at the point x. We define

$$\hat{x}_n = \frac{x_n}{\mathcal{R}(t_n)} \quad \text{and} \quad y_n = \hat{x}_n - \rho\tilde{\nu}(\hat{x}_n).$$

By definition, $\hat{x}_n \in \partial W$ and y_n is the center of the interior ball tangent to W at the point $\hat{x}_n$, of radius ρ (we recall that W satisfies the uniform interior ball estimate with radius ρ).

For every n, we define $k_n \in \mathbb{Z}^N$ and $z_n \in [0, 1)^N$ by $x_n = k_n + z_n$. Up to extraction, we can assume that there is $z \in [0, 1]^N$ such that $z_n \to z$ as $n \to +\infty$. We also assume that there is $\hat{x}$ such that $\hat{x}_n$ converges to $\hat{x}$, whence $\tilde{\nu}(\hat{x}_n)$ converges to $\tilde{\nu}(\hat{x})$. We now define, for $n \in \mathbb{N}$, the translated functions

$$u_n(t, x) = u(t + t_n, x + k_n).$$

Thanks to the periodicity and regularity hypotheses on Ω, we can apply the usual interior and portion boundary parabolic estimates (see, for instance [Ladyzhenskaya et al. 1968, Theorems 5.2, 5.3]) to get that u_n converges uniformly locally to an entire solution $u^\star$ of (1). Moreover $u^\star(0, z) = \eta$.

Step 3: Properties of $u^\star$. We show here that $u^\star$ is a front-like solution, in the sense that it satisfies, writing $H_T := \{x \in \Omega : x \cdot \tilde{\nu}(\hat{x}) < -k\hat{x} \cdot \tilde{\nu}(\hat{x})T\}$,

$$\text{for all } T \ge 0, \text{ for all } x \in H_T + \{z\}, \quad u^\star(-T, x) \ge \eta. \tag{21}$$

To show this, take $T \in [0, t_n]$ and $x \in (\mathcal{R}(t_n) - kT)W \cap \Omega$. As $\mathcal{R}(t_n) - kT \le \mathcal{R}(t_n - T)$, we have that $x \in \mathcal{R}(t_n - T)W \cap \Omega$. Therefore, by the definition of $\mathcal{R}$, we have $u(t_n - T, x) \ge \eta$. Then, we have

$$\text{for all } T \in [0, t_n], \text{ for all } x \in ((\mathcal{R}(t_n) - kT)W) \cap \Omega - \{k_n\}, \quad u_n(-T, x) \ge \eta.$$

From that, we infer

$$\text{for all } T \geq 0, \text{ for all } x \in \Omega \cap \bigcup_{M\in\mathbb{N}} \bigcap_{n\geq M} ((\mathcal{R}(t_n) - kT)W - \{k_n\}), \quad u^\star(-T, x) \geq \eta.$$

To prove (21), it suffices to show that $H_T + \{z\} \subset \Omega \cap \bigcup_{M\in\mathbb{N}} \bigcap_{n\geq M}((\mathcal{R}(t_n) - kT)W - \{k_n\})$. To see this, take $x \in H_T + \{z\}$. We start by computing

$$\begin{aligned}\left|\frac{x+k_n}{\mathcal{R}(t_n)-kT} - y_n\right| &= \left|\frac{x+k_n - (\mathcal{R}(t_n)-kT)(\hat{x}_n - \rho\tilde{\nu}(\hat{x}_n))}{\mathcal{R}(t_n)-kT}\right| \\ &= \left|\frac{x+kT\hat{x}_n + (k_n - x_n) + (\mathcal{R}(t_n)-kT)\rho\tilde{\nu}(\hat{x}_n)}{\mathcal{R}(t_n)-kT}\right| \\ &= \left|\rho\tilde{\nu}(\hat{x}_n) + \frac{x+kT\hat{x}_n - z_n}{\mathcal{R}(t_n)-kT}\right|.\end{aligned}$$

Let us write

$$w_n := \frac{x+kT\hat{x}_n - z_n}{\mathcal{R}(t_n)-kT}.$$

This goes to zero as n goes to infinity. The last term in the above equality can be rewritten

$$|\rho\tilde{\nu}(\hat{x}_n) + w_n| = \sqrt{\rho^2 + |w_n|\left(\frac{2\rho\tilde{\nu}(\hat{x}_n)\cdot w_n}{|w_n| + |w_n|}\right)}.$$

Now, observe that

$$\lim_{n\to+\infty} \frac{2\rho\tilde{\nu}(\hat{x}_n)\cdot w_n}{|w_n|+|w_n|} = 2\rho\tilde{\nu}(\hat{x}) \cdot \frac{x+kT\hat{x} - z}{|x+kT\hat{x}-z|}.$$

This limit is strictly negative. Indeed, if $x \in H_T + \{z\}$, then $(x - z) \cdot \tilde{\nu}(\hat{x}) < -kT\hat{x} \cdot \tilde{\nu}(\hat{x})$. Therefore, we have, for n large enough,

$$\left|\frac{x+k_n}{\mathcal{R}(t_n)-kT} - y_n\right| < \rho,$$

which means $(x+k_n)/(\mathcal{R}(t_n)-kT) \in W$ by the definition of y_n and ρ. That is, $x \in (\mathcal{R}(t_n)-kT)W-\{k_n\}$, which concludes this step.

Step 4: Comparison. We now compare the function $u^\star$ constructed in the previous steps to the pulsating traveling front in the direction $\tilde{\nu}(\hat{x})$ with critical speed $c^\star(\tilde{\nu}(\hat{x}))$. Combining Lemma 9 and (21), we have

$$\liminf_{\delta\to+\infty} \inf_{\substack{t<0 \\ x\cdot\tilde{\nu}(\hat{x})<\gamma t-\delta \\ x\in\Omega}} u^\star(t, x) \geq 1,$$

with $\gamma := k\hat{x} \cdot \tilde{\nu}(\hat{x}) > 0$. Hence $u^\star$ satisfies the hypotheses of Lemma 10. Observe that we have

$$\gamma = k\hat{x} \cdot \tilde{\nu}(\hat{x}) = k\frac{\hat{x}}{|\hat{x}|} \cdot \tilde{\nu}(\hat{x})\tilde{w}\left(\frac{\hat{x}}{|\hat{x}|}\right) < \frac{\hat{x}}{|\hat{x}|} \cdot \tilde{\nu}(\hat{x})w\left(\frac{\hat{x}}{|\hat{x}|}\right) \leq c^\star(\tilde{\nu}(\hat{x})),$$

where the last inequality follows from the definition of w in Theorem 6.

Assume first that f is of combustion type (8). Let v be a pulsating traveling front in the direction $\nu(\hat{x})$, with critical speed $c^\star(\nu(\hat{x}))$. Up to a time translation, we normalize it so that $v(0, 0) > u^\star(0, 0)$. Then, v

satisfies the hypotheses of Lemma 10 (with $\eta = c^\star(\nu(\hat{x})) - \gamma$ in the hypotheses of Lemma 10), giving $v \le u^\star$, which is in contradiction with the fact that $v(0,0) > u^\star(0,0)$.

Now, if the nonlinearity is of monostable type (6), we have to construct a function v satisfying (19) to apply Lemma 10. This can be done exactly as in [Rossi 2017, Proposition 2.6]; the fact that the domain is not $\mathbb{R}^N$ adds no difficulty here. This proves that W is an asymptotic subset of spreading, and then, so is $\mathcal{W}$. Now, we show that it is an asymptotic superset of spreading.

Step 5: Superset of spreading. Let C be a closed set such that $\mathcal{W} \cap C = \varnothing$. Then, because w is continuous, we can find $\varepsilon > 0$ so that $\mathcal{W}_\varepsilon := \{r\xi : \xi \in \mathbb{S}^{N-1},\ 0 \le r \le w(\xi) + \varepsilon\}$ is such that $\mathcal{W}_\varepsilon \cap C = \varnothing$. To prove that $\mathcal{W}$ is an asymptotic superset of spreading, it is sufficient to show that $\sup_{x \in t\mathcal{W}_\varepsilon^c} u(t,x) \to 0$ as t goes to $+\infty$. To do so, we take a sequence $(t_n)_{n\in\mathbb{N}} \in (\mathbb{R}^+)^{\mathbb{N}}$ such that t_n goes to infinity as n goes to infinity and a sequence $x_n \in t_n\mathcal{W}_\varepsilon^c$ such that

$$u(t_n, x_n) \ge \tfrac{1}{2} \sup_{x \in t_n \mathcal{W}_\varepsilon^c} u(t_n, x).$$

Up to extraction, we take $e \in \mathbb{S}^{N-1}$ such that $x_n/|x_n| \to e$ as n goes to $+\infty$. Let $\xi \in \mathbb{S}^{N-1}$ be such that $w(e) = c^\star(\xi)/(\xi \cdot e)$, and let v be a pulsating traveling front in the direction ξ with critical speed $c^\star(\xi)$. Up to some translation in time, we can assume, thanks to the parabolic comparison principle, that $u(t,x) \le v(t,x)$ for all $t \ge 0$, for all $x \in \Omega$. Let us show that $v(t_n, x_n)$ goes to zero as $n \to +\infty$.

We write $x_n := (x_n/|x_n| \cdot \xi)|x_n|\xi + d_n$, where d_n is orthogonal to ξ. Because $x_n/|x_n| \to e$ as n goes to $+\infty$, using the continuity of w, for n large enough, we have

$$\left(\frac{x_n}{|x_n|} \cdot \xi\right)|x_n| \ge \left(\frac{x_n}{|x_n|} \cdot \xi\right)\left(w\left(\frac{x_n}{|x_n|}\right) + \varepsilon\right)t_n \ge \left(c^\star(\xi) + (e \cdot \xi)\frac{\varepsilon}{2}\right)t_n.$$

So, we get that, for $n \in \mathbb{N}$ large enough, there is some λ_n such that $\lambda_n \ge c^\star(\xi) + (e \cdot \xi)(\varepsilon/2)$ and $x_n = \lambda_n \xi t_n + d_n$. Now, observe that the definition of the pulsating traveling fronts, Definition 2, implies that $v(t_n, \lambda_n t_n \xi + d_n) \to 0$ as n goes to $+\infty$; hence

$$\lim_{n\to+\infty} \tfrac{1}{2} \sup_{x \in t_n \mathcal{W}_\varepsilon^c} u(t_n, x) \le \lim_{n\to+\infty} u(t_n, x_n) \le \lim_{n\to+\infty} v(t_n, x_n) = 0,$$

which implies the result. □

Now that we have the Freidlin–Gärtner formula (12) at our disposal, we use it to answer Question 4.

3. Invasion and the critical speed of fronts

This whole section is dedicated to the proof of Theorem 5. We consider here the problem (15), with nonlinearity f independent of x of monostable or combustion type. In the following, for f and Ω given, we denote by $c^\star$ and w the critical speed of fronts and the speed of invasion respectively, for (15).

The proof of Theorem 5 is done in several steps: first, we show that $w \equiv c^\star$ is equivalent to saying that w and $c^\star$ are actually constant. This is the object of Section 3.1. Then, we exhibit in Section 3.3 some estimates on the spreading speed that take into account the geometry of the domain. Gathering all this, we will be able to prove Theorem 5.

3.1. ***Comparison between w and $c^\star$.*** This section is dedicated to proving that, if the critical speed of fronts $c^\star$ and the speed of invasion w are everywhere equal, then they are constant. This uses only the Freidlin–Gärtner formula (12) proved in Section 2.

Proposition 12. *Assume that Ω is a smooth periodic domain satisfying* (3) *and that f is a nonlinearity satisfying* (5) *of the monostable* (6) *or combustion* (8) *type. Assume that $c^\star \equiv w$. Then, the functions w and $c^\star$ are constant.*

Proof. Because of the hypotheses on Ω and f, we can apply Theorem 6 to get that for all $e \in \mathbb{S}^{N-1}$ $w(e) = \inf_{\xi \cdot e > 0} c^\star(\xi)/(\xi \cdot e)$. Assume that $w \equiv c^\star$ and take ξ_0, $\xi \in \mathbb{S}^{N-1}$ so that $\xi_0 \cdot \xi > 0$, and let ω be the angle between those two vectors. Let us take $M \in \mathbb{N}$. We define a sequence $(\xi_k)_{k \in [\![0,M]\!]} \in \mathbb{S}^{N-1}$ to be equidistributed on the arc joining ξ_0 to ξ on the sphere; i.e., $\xi_k \cdot \xi_{k+1} = \cos(\omega/M)$ and $\xi_M = \xi$. Then, we have

$$w(\xi_0) \leq w(\xi_1) \frac{1}{\xi_0 \cdot \xi_1} \leq w(\xi_2) \frac{1}{\xi_2 \cdot \xi_1} \frac{1}{\xi_1 \cdot \xi_0}.$$

Iterating and using that $\xi_k \cdot \xi_{k+1} = \cos(\omega/M)$, we get

$$w(\xi_0) \leq w(\xi) \prod_{k=0}^{M-1} \frac{1}{\xi_{k+1} \cdot \xi_k} = w(\xi) \frac{1}{\cos(\omega/M)^M}.$$

Because $1/\cos(\omega/M)^M \to 1$ as M goes to $+\infty$, passing to the limit yields

$$w(\xi_0) \leq w(\xi).$$

Inverting the roles of ξ_0 and ξ, we get $w(\xi_0) = w(\xi)$. Hence, w is constant, and so is $c^\star$. □

Observe that, in the course of the proof, we did not use the particular form of (15), only the Freidlin–Gärtner formula; hence Proposition 12 holds true also for the general equation (1).

As mentioned in the Introduction, we shall use this result to construct domains where $c^\star \not\equiv w$. Indeed, Proposition 12 reduces the problem to finding domains where w or $c^\star$ are not constants. Intuitively, it seems that, if in a certain direction e there are many obstacles, then the speeds w and $c^\star$ should be small. On the contrary, if in a certain direction, there are few obstacles, then the speeds should be larger. Hence, if the domain Ω is very "obstructed" in some direction and not in another, then the speeds should not be constants, and so they would be different.

To construct such domains is actually quite easy if f is KPP and if the dimension is greater than or equal to 3: in this case, domains that are invariants in one direction provide an answer to Question 4. We shall focus on such domains in the next Section 3.2. There, we shall also prove a lemma that will be useful in Section 4. If the nonlinearity is not KPP or if the dimension is equal to 2, things are much more involved. To overcome this difficulty, we introduce estimates for w that do take into account the geometry of the domain. This is done in Section 3.3.

3.2. ***Invasion in domains that are invariant in one direction.*** In this section, f is a KPP nonlinearity independent of x and Ω is invariant in the direction $e \in \mathbb{S}^{N-1}$; i.e., for all $\lambda \in \mathbb{R}$, we have $\Omega + \lambda e = \Omega$. Let us answer Question 4 in this specific case by proving the following:

Proposition 13. *Let Ω be a periodic domain in $\mathbb{R}^N$, $N \geq 3$, satisfying* (3) *and suppose that there is $e \in \mathbb{S}^{N-1}$ such that Ω is invariant in the direction e. Let f satisfying* (5) *be a KPP nonlinearity independent of x. Denoting by $c^\star$ and w the critical speed of fronts and the speed of invasion respectively for problem* (15), *we have*

$$w \equiv c^\star \quad \Longleftrightarrow \quad \Omega = \mathbb{R}^N.$$

This comes directly by combining our Proposition 12 with the following result from [Berestycki et al. 2005]:

Theorem 14. *Let $c^\star$ be the critical speed of fronts for the problem* (15) *with f KPP independent of x. Then $c^\star(e) \leq 2\sqrt{f'(0)}$ and the equality holds if and only if Ω is invariant in the direction e.*

If Ω is a periodic domain satisfying hypothesis (3) and invariant in a direction $\Omega \neq \mathbb{R}^N$, then this theorem implies that $c^\star$ is not a constant function of the direction. Then, Proposition 12 implies that $c^\star \not\equiv w$. This answers Question 4 in the special case where f is KPP and the dimension greater than 3. The general setting is more involved and is addressed after.

Before ending this section, we prove a result concerning domains invariant in a direction that will be useful in Section 4. When considering such domains, we can actually give further information about the shape of the asymptotic set of spreading $\mathcal{W}$. The next result shows that, if Ω is invariant in the direction e, then the spreading speed in a direction orthogonal to e only depends on the part of the domain orthogonal to e.

Proposition 15. *Let Ω be a periodic domain satisfying* (3), *invariant in the direction $e \in \mathbb{S}^{N-1}$. Let $\mathcal{W}$ be the asymptotic set of spreading of* (15) *set on Ω with f satisfying* (5) *and such that $u \mapsto f(u)/u$ is decreasing* (*this implies that f is KPP*)*. Let $\mathcal{H}$ be the hyperplane in $\mathbb{R}^N$ orthogonal to e. Then, if $\mathcal{W}_{\mathcal{H}\cap\Omega}$ is the asymptotic set of spreading for the equation restricted to $\mathcal{H} \cap \Omega$, i.e.*,

$$\begin{cases} \partial_t u - \Delta u = f(u), & t > 0,\ x \in \mathcal{H} \cap \Omega, \\ \partial_{\nu'} u = 0, & t > 0,\ x \in \partial(\mathcal{H} \cap \Omega), \end{cases} \tag{22}$$

where $\nu' \in \mathbb{S}^{N-2}$ denotes the exterior normal to $\mathcal{H} \cap \Omega$, we have

$$\mathcal{W}_{\mathcal{H}\cap\Omega} = \mathcal{W} \cap \mathcal{H}.$$

Proof. To simplify the notation, we denote by w_N the spreading speed for the Fisher-KPP equation (15) set on $\Omega \subset \mathbb{R}^N$ and w_{N-1} the spreading speed for (22) set on $\mathcal{H} \cap \Omega \subset \mathbb{R}^{N-1}$. Similarly, we denote by $c_N^\star$ and $c_{N-1}^\star$ the critical speeds of fronts for (15) and (22) respectively. Up to some rotation of the coordinates, we write the points of Ω in the form (x, y), where $x \in \mathcal{H} \cap \Omega$ and $y \in \mathbb{R}$.

Step 1: $\mathcal{W} \cap \mathcal{H} \subset \mathcal{W}_{\mathcal{H}\cap\Omega}$. We start by showing that, for each $\zeta \in \mathbb{S}^{N-2}$, we have $w_N((\zeta, 0)) \leq w_{N-1}(\zeta)$. To do so, take $\xi \in \mathbb{S}^{N-2}$ such that $\xi \cdot \zeta > 0$. Let $\phi_\xi(t, x)$ be a pulsating traveling front solution of (22) in the direction ξ with critical speed $c_{N-1}^\star(\xi)$. For $(x, y) \in \Omega$, we define $\Phi(t, x, y) := \phi_\xi(t, x)$. Then Φ is solution of (15) on the whole of Ω. If $u_0(x, y)$ is a nonnegative compactly supported initial datum and if $u(t, x, y)$ is the solution of (15) arising from it, we can assume that (up to translation) $u_0(x, y) \leq \Phi(0, x, y)$. Hence, the parabolic comparison principle gives us that

$$u(t, x, y) \leq \Phi(t, x, y) \quad \text{for all } t \geq 0, \text{ for all } (x, y) \in \Omega.$$

Observe that Φ moves in the direction $(\zeta, 0) \in \mathbb{S}^{N-1}$ with speed $c^\star_{N-1}(\xi)/(\xi \cdot \zeta)$. This means that $w_N((\zeta, 0)) \le c^\star_{N-1}(\xi)/(\xi \cdot \zeta)$, and because this is true for all ξ such that $\xi \cdot \zeta > 0$, Theorem 6 implies that $w_N((\zeta, 0)) \le w_{N-1}(\zeta)$.

Step 2: $\mathcal{W}_{\mathcal{H}\cap\Omega} \subset \mathcal{W} \cap \mathcal{H}$. We now prove the reverse inequality. To start, let $\varepsilon > 0$ be fixed such that $\varepsilon^2 < f'(0)$. We define a KPP nonlinearity $f_\varepsilon(u) := f(u) - \varepsilon^2 u$. Let $u_0(x)$ be a smooth, nonnegative, compactly supported function in $\mathcal{H}\cap\Omega$. Let $u_\varepsilon(t, x)$ be the solution arising from u_0 of (15) but *with f replaced by f_ε*.

Define the cut-off function

$$\phi(y) := \begin{cases} \cos(\varepsilon y) & \text{for } |y| \le \pi/(2\varepsilon), \\ 0 & \text{for } |y| \ge \pi/(2\varepsilon). \end{cases}$$

Now, let $v(t, x, y) := u_\varepsilon(t, x)\phi(y)$. Let us show that v is a (generalized) subsolution. An easy computation shows that, for $(x, y) \in \Omega$ such that $v(t, x, y) > 0$, we have

$$\begin{aligned} \partial_t v - \Delta v - f(v) &= f_\varepsilon(u_\varepsilon)\phi(y) + \varepsilon^2 u_\varepsilon \phi(y) - f(u_\varepsilon\phi) \\ &= \left(\frac{f_\varepsilon(u_\varepsilon)}{u_\varepsilon} - \frac{f(u_\varepsilon\phi)}{u_\varepsilon\phi} + \varepsilon^2\right) u_\varepsilon\phi \le 0. \end{aligned}$$

The last inequality comes from the fact that $z \mapsto f(z)/z$ is decreasing. One can then check that $\partial_\nu v = 0$ on $\partial\Omega$. This comes from $\partial_{\nu'} u_\varepsilon = 0$ on $\partial(\Omega\cap\mathcal{H})$ together with the fact that Ω is invariant in the direction e.

Hence, $u_\varepsilon\phi$ is a (generalized) subsolution of (15) (with nonlinearity f). We can observe that u_ε spreads in $\Omega \cap \mathcal{H}$ in the direction $\zeta \in \mathbb{S}^{N-2}$ with speed $w_{N-1}(\zeta) - \varepsilon^2$. Hence, by comparison, we get that $w_{N-1}(\zeta) - \varepsilon^2 \le w_N((\zeta, 0))$. Taking the limit $\varepsilon \to 0$ yields the result. □

Observe that the same result holds in what concerns the critical speed of fronts: using the same notation as in the proof, we can prove that $c^\star_N((e, 0)) = c^\star_{N-1}(e)$ for every $e \in \mathbb{S}^{N-2}$: one inequality is proved in the first step, and the second inequality can be proved as in the second step just by taking u_ε to be front.

Now, we turn to the full proof of Theorem 5, answering then Question 4.

3.3. *Geodesic estimates.* This aim of this section is to establish estimates on $w(e)$ that do take into account the geometry of the domain. The key tool is an estimate on the heat kernel from [Berestycki et al. 2010], following from general results on the heat kernel from [Davies 1989; Grigoryan 1997]. This estimate is valid for domains satisfying the *extension property*. Denoting by $W^{1,p}(\Omega)$ the usual Sobolev space over Ω, a nonempty subset of $\mathbb{R}^N$ satisfies the extension property if, for all $1 \le p \le +\infty$, there is a bounded linear map E from $W^{1,p}(\Omega)$ to $W^{1,p}(\mathbb{R}^N)$ such that $E(f)$ is an extension of f from Ω to $\mathbb{R}^N$ for all $f \in W^{1,p}(\Omega)$. For our purpose, we mention that the smooth periodic domains we consider here satisfy the extension property; see [Stein 1970].

Proposition 16. *Let Ω be a locally C^2 nonempty connected open subset of $\mathbb{R}^N$ satisfying the extension property. Let $p(t, x, y)$ be the heat kernel in $\overline{\Omega}$ with Neumann boundary condition on $\partial\Omega$. Then, for every $\varepsilon > 0$, there are two positive constants C and δ such that*

$$\text{for all } t > 0, \text{ for all } (z, x) \in \overline{\Omega} \times \overline{\Omega}, \quad p(t, z, x) \le C(1 + \delta t^{-N/2}) \exp\left(-\frac{d_\Omega(z, x)^2}{(4+\varepsilon)t}\right), \tag{23}$$

where $d_\Omega(z, x)$ denotes the geodesic distance in $\overline{\Omega}$.

See [Berestycki et al. 2010, Proposition 2.5] for the proof. We use this to get upper estimates on the spreading speed $w(e)$. To do so, we introduce the following coefficient for $e \in \mathbb{S}^{N-1}$:

$$C_\Omega(e) := \liminf_{\lambda \to +\infty} \frac{\lambda}{d_\Omega(0, \lambda e)}. \tag{24}$$

For notational simplicity and without loss of generality, we assume in the following that the point 0 is in Ω (this is always possible up to translation).

This coefficient represents *how much the domain is obstructed* in the direction e. The geodesic distance d_Ω is always greater than the euclidean distance; hence $C_\Omega(e) \le 1$.

Proposition 17. *Let Ω be a domain satisfying* (3) *and f a monostable* (6) *or a combustion* (8) *nonlinearity independent of x. We denote by w the speed of invasion associated to problem* (15). *Then, we have*

$$w(e) \le 2C_\Omega(e)\sqrt{\max_{u\in[0,1]} \frac{f(u)}{u}}. \tag{25}$$

Observe that, if f is a KPP nonlinearity, then this formula boils down to $w(e) \le 2C_\Omega(e)\sqrt{f'(0)}$. In the case where $\Omega = \mathbb{R}^N$, the upper bound is actually the KPP speed $2\sqrt{f'(0)}$.

Proof. Let us observe that it is sufficient to prove the result in the KPP case. Indeed, if f is a monostable or a combustion nonlinearity, then we can find a KPP nonlinearity $\overline{f}$ such that $\overline{f}'(0) = \max_{u\in[0,1]} f(u)/u$ and $\overline{f} \ge f$. If u_0 is an initial datum, denoting by u, respectively $\bar{u}$, the solution of (15) with nonlinearity f, respectively $\overline{f}$, arising from u_0, the parabolic comparison principle tells us that

$$u(t, x) \le \bar{u}(t, x) \quad \text{for all } t \ge 0, \text{ for all } x \in \Omega.$$

Then, $w(e) \le \overline{w}(e)$, for all $e \in \mathbb{S}^{N-1}$, where w, respectively $\overline{w}$, is the invasion speed for (15) with nonlinearity f, respectively $\overline{f}$. Then, it is sufficient to prove the estimate (25) for $\overline{f}$ because $\max_{u\in[0,1]} \overline{f}(u)/u = \max_{u\in[0,1]} f(u)/u$. Hence, in the rest of the proof, we assume that f is KPP, and then $\max_{u\in[0,1]} f(u)/u = f'(0)$.

Let $u(t, x)$ be the solution of the parabolic problem (15) arising from a compactly supported nonnegative initial smooth datum u_0. Let K be a compact set of Ω such that the support of u_0 is in K. We denote by $p(t, x, z)$ the heat kernel with Neumann condition on Ω. Then, we first observe that

$$u(t, x) \le e^{f'(0)t} \int_\Omega p(t, x, z) u_0(z)\, dz. \tag{26}$$

Indeed, $e^{f'(0)t} \int_\Omega p(t, x, z) u_0(z)\, dz$ is the solution of the linearized problem

$$\begin{cases} \partial_t v - \Delta v = f'(0) v, & t > 0,\ x \in \Omega, \\ \partial_\nu v = 0, & t > 0,\ x \in \partial\Omega, \\ v(0, x) = u_0(x), & x \in \Omega, \end{cases} \tag{27}$$

and hence is a supersolution of (15), thanks to the KPP property. Then, the inequality (26) follows by the parabolic comparison principle. Now, let $\varepsilon > 0$ be fixed. Using the estimate (23) in (26), we get

$$u(t, x) \le C(1 + \delta t^{-N/2}) e^{f'(0)t} \int_\Omega \exp\Big(-\frac{d_\Omega(z, x)^2}{(4+\varepsilon)t}\Big) u_0(z)\, dz \tag{28}$$

for some positive constants C and δ (depending on ε). This gives us

$$u(t,x) \le C\|u_0\|_{L^1}(1+\delta t^{-N/2})\exp\left(\left(f'(0) - \frac{(\min_{z\in K} d_\Omega(z,x))^2}{(4+\varepsilon)t^2}\right)t\right). \tag{29}$$

Now, take $e \in \mathbb{S}^{N-1}$ and $\omega > 0$ such that $\omega < w(e)$. Then, $u(t, \omega te) \to 1$ as $t \to +\infty$, by the definition of $w(e)$. Then, necessarily, we have

$$\limsup_{t\to+\infty} \frac{\min_{z\in K} d_\Omega(z, \omega te)}{t} \le \sqrt{(4+\varepsilon)f'(0)},$$

if this were not the case, up to subsequence the right-hand term of (29) would go to zero along some time sequence $(t_n)_{n\in\mathbb{N}}$, $t_n \to +\infty$ as n goes to $+\infty$, which would be in contradiction with the fact that $u(t_n, \omega t_n e)$ goes to 1. Using the triangular inequality for d_Ω and the fact that K is compact we eventually get

$$\omega \le \frac{\sqrt{(4+\varepsilon)f'(0)}}{\limsup_{t\to+\infty} d_\Omega(0,\omega te)/(\omega t)}.$$

Recalling the definition of $C_\Omega(e)$ and that the above inequality is true for every $\varepsilon > 0$, we have

$$\omega \le 2C_\Omega(e)\sqrt{f'(0)},$$

and the result follows. □

We are now in position to answer Question 4.

3.4. ***Domains where $c^\star \not\equiv w$.*** In this section, we construct periodic domains Ω such that $c^\star \not\equiv w$. If f is a KPP nonlinearity, we exhibit a 1-periodic domain (but the periodicity can be chosen arbitrary). If f is a monostable or a combustion nonlinearity, we construct an L-periodic domain, where $L > 0$ can be large. For clarity, we do this in dimension $N = 2$, but these constructions can be easily generalized to larger dimensions.

In the following, we define $e_x := (1, 0)$, $e_y := (0, 1) \in \mathbb{S}^1$ the unit vectors of the canonical basis of $\mathbb{R}^2$. Moreover, we define $e_d := (1/\sqrt{2})(1, 1) \in \mathbb{S}^1$.

3.4.1. *The KPP case.* We show here the following:

Proposition 18. *Let f be a KPP nonlinearity* (7). *There is a smooth periodic domain $\Omega \subset \mathbb{R}^2$ such that*

$$c^\star(e_x) > w(e_d),$$

where $c^\star$ and w are the critical speed of fronts and the speed of invasion respectively for (15) *set in Ω with nonlinearity f.*

We see that in this domain, it is not possible that $w = c^\star$, thanks to Proposition 12. Hence, this answers Question 4 in the KPP case.

Proof. For $\alpha \in \left(\frac{1}{2}, 1\right)$, $\beta \in \left(0, \frac{1}{2}\right)$, we define $\Omega_{\alpha,\beta}$ to be a smooth periodic domain such that

$$\mathbb{Z}^2 + (1-\alpha, \alpha) \times [\beta, 1-\beta] \subset \Omega^c_{\alpha,\beta} \subset \mathbb{Z}^2 + \left(\frac{1-\alpha}{2}, \frac{1+\alpha}{2}\right) \times [\beta, 1-\beta]. \tag{30}$$

This domain is simply $\mathbb{R}^2$ with "almost square" obstacles. For α, β given we denote by $c^\star_{\alpha,\beta}(e)$ the critical speed of fronts in this domain in the direction e. If β is fixed and if we let $\alpha \to 1$, then the domain "converges" in some sense to an array of parallel disconnected strips in the direction e_x. This observation is made rigorous by [Berestycki et al. 2005, Theorem 1.4], where it is proved that

$$c^\star_{\alpha,\beta}(e_x) \xrightarrow[\alpha\to 1]{} 2\sqrt{f'(0)}.$$

Now, let $\kappa \in (1, \sqrt{2})$ and take α close enough to 1 so that $c^\star_{\alpha,\beta}(e_x) > (1/\kappa)2\sqrt{f'(0)}$.

Take $n \in \mathbb{N}$. Denoting by $d_{\Omega_{\alpha,\beta}}$ the geodesic distance in $\Omega_{\alpha,\beta}$, it is easy to see that $d_{\Omega_{\alpha,\beta}}(0, n\sqrt{2}e_d) \geq 2n(\alpha-\beta)$. Plotting this in (24) yields $C_{\Omega_{\alpha,\beta}}(e_d) \leq 1/(\sqrt{2}(\alpha-\beta))$. Taking β small enough, and increasing α if needed, we can assume that $C_{\Omega_{\alpha,\beta}}(e_d) \leq 1/\kappa$. Denoting by $w_{\alpha,\beta}$ the speed of invasion in the domain $\Omega_{\alpha,\beta}$, Proposition 17 implies that $w_{\alpha,\beta}(e_d) \leq (1/\kappa)2\sqrt{f'(0)}$. Hence, $c^\star_{\alpha,\beta}(e_x) > w_{\alpha,\beta}(e_d)$ when α is close enough to 1 and β close enough to 0. This yields the result. □

3.4.2. *Combustion and monostable case.* Now, we answer Question 4 in the case where f is a combustion or a monostable nonlinearity. We do it for f combustion first, and then we explain how this yields the result for monostable nonlinearities.

Proposition 19. *Let f be a combustion nonlinearity* (8). *Then, there are $L > 0$ and a family of smooth L-periodic domains $(\Omega_\alpha)_{\alpha\in(0,1)}$ such that $w_\alpha(e_x) \geq K$, where $K > 0$ is independent of α, and $w_\alpha(e_y) \to 0$ as α goes to* 0.

If $\alpha > 0$ is chosen small enough so that $w_\alpha(e_y) < K$, we see that w_α cannot be constant, and then Proposition 12 implies that $c^\star \not\equiv w$ on Ω_α for α small. This answers Question 4 and proves Theorem 5 when f is a combustion nonlinearity.

Before turning to the proof of Proposition 19, we state the following technical lemma. We recall that we denote by B_R the ball of radius R and of center 0.

Lemma 20. *Let f be a combustion nonlinearity* (8) *independent of x. Then, there are $R, c > 0$ and $\phi \in W^{2,\infty}(\mathbb{R}^2)$, $\phi > 0$ in B_R and $\phi = 0$ on ∂B_R such that, on B_R we have*

$$\Delta\phi + c\partial_x\phi + f(\phi) \geq 0.$$

Proof. We construct ϕ to be radial. We set $\phi(x) := h(|x|)$. Now, take R_1, R_2, $R_3 > 0$ to be chosen after, such that $R_1 < R_2 < R_3$. We set $\tilde{c} := c + 1/R_1$. Let $C \in (\theta, 1)$, and $\alpha, \beta > 0$. We define h as follows:

$$h(r) = \begin{cases} C, & r \in [0, R_1], \\ h(r) - (\alpha/2)(r - R_1)^2 + C, & r \in [R_1, R_2], \\ h(r)\beta(e^{-\tilde{c}(r-R_3)} - 1), & r \in [R_2, R_3]. \end{cases} \tag{31}$$

We can choose $R_1, R_2, R_3, c, \alpha, \beta, C$ such that

$$\begin{gathered} h \in W^{2,\infty}(\mathbb{R}^+), \\ h(R_2) = K, \quad \text{where } K \in (\theta, C) \text{ will be chosen after}, \\ h''(r) + \tilde{c}h'(r) + f(h(r)) \geq 0 \quad \text{for } r \geq 0. \end{gathered} \tag{32}$$

The existence of such a function proves our result, indeed

$$\begin{aligned}\Delta\phi + c\partial_x\phi + f(\phi) &\geq h'' + \left(c + \frac{1}{r}\right)h' + f(h)\\ &\geq h'' + \left(c + \frac{1}{R_1}\right)h' + f(h).\end{aligned}$$

We used the fact that h is nonincreasing and $h'(r) = 0$ if $r \in [0, R_1]$ here.

Let us define

$$F := \inf_{s\in(K,C)} f(s) > 0.$$

Because $h(R_2) = K$, we can bound from behind $f(h(r))$ by F when $r \in [R_1, R_2]$ and by 0 elsewhere. Some easy computations show that (32) boils down to verifying the following algebraic system:

$$\begin{cases}\beta(e^{\tilde{c}(R_3-R_2)} - 1) = K,\\ (\alpha/2)(R_2 - R_1)^2 = C - K,\\ \alpha(R_2 - R_1) = \beta\tilde{c}e^{\tilde{c}(R_3-R_2)},\\ F \geq \alpha(1 + \tilde{c}(R_2 - R_1)).\end{cases} \tag{33}$$

Up to some computations, it is readily seen that (33) admits positive solutions, for instance

$$\begin{aligned}\alpha &= \frac{F}{1 + (C-K)/(2K)}, & R_1 &= \frac{1}{c},\\ c &= \frac{1}{8}\frac{\sqrt{2\alpha(C-K)}}{K}, & R_2 &= \sqrt{\frac{2(C-K)}{\alpha}} + R_1,\\ \beta &= \frac{\sqrt{\alpha(C-K)}}{2c\sqrt{2}} - K, & R_3 &= \frac{1}{2c}\ln\left(1 + \frac{K}{\beta}\right) + R_2.\end{aligned}$$

Hence, $\phi(x) := h(|x|)$ satisfies the lemma with $R := R_3$. □

Now, we use this lemma to prove Proposition 19.

Proof of Proposition 19. Step 1: Construction of the domain. Let $R > 0$ be large enough, so that we can apply Lemma 20. Let $\alpha \in (0, 1)$, $\varepsilon \in [0, \alpha R/2]$ and define

$$\widetilde{K}_\alpha^\varepsilon := \{(x, y) \in \mathbb{R}^2 \text{ such that } \alpha x + R + \varepsilon \leq y \leq \alpha x + (1+\alpha)R - \varepsilon,\ y \in [R, 2R]\}.$$

Now, let K_α be a smooth connected compact set such that

$$\widetilde{K}_\alpha^{\alpha R/4} \subset K_\alpha \subset \widetilde{K}_\alpha^0.$$

We define Ω_α to be the smooth $3R$-periodic domain

$$\Omega_\alpha := (K_\alpha + 3R\mathbb{Z}^2)^c;$$

see Figure 1. Observe that, if $k, l \in \mathbb{Z}^2$ are such that $k \neq l$, then $(K_\alpha + 3Rk) \cap (K_\alpha + 3Rl) = \varnothing$.

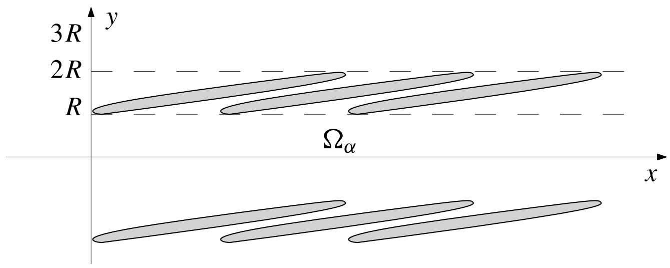

Figure 1. The domain Ω_α used in the proof of Proposition 19.

Step 2: Lower bound on $w_\alpha(e_x)$. For $\alpha > 0$ given, we denote by w_α the speed of invasion for (15) set on the smooth periodic domain Ω_α. Let us show that there is $K > 0$ independent of α such that

$$w_\alpha(e_x) \geq K. \tag{34}$$

Because of the choice of R, we can apply Lemma 20 to find $c > 0$ and $\phi \in W^{2,\infty}(B_R)$, $\phi > 0$ on B_R and $\phi = 0$ on ∂B_R such that $\Delta\phi + c\partial_x\phi + f(\phi) \geq 0$. Now, we define

$$v(t,x,y) := \begin{cases} \phi(x-ct,y) & \text{if } (x,y) \in B_R(cte_x), \\ 0 & \text{elsewhere.} \end{cases}$$

Then, the support of $v(t,\cdot,\cdot)$ never intersects the boundary of Ω_α, and because

$$\partial_t v - \Delta v - f(v) = -c\partial_x\phi - \Delta\phi - f(\phi) \leq 0 \text{ for } (x,y) \in \operatorname{supp} v(t,\cdot,\cdot),$$

we have that v is a nonnegative compactly supported generalized subsolution of (15).

Now, take u_0 a compactly supported initial datum such that $u_0(x,y) \geq \phi(x,y)$ and such that the solution arising from it, say $u(t,x,y)$, converges to 1 (as we mentioned earlier, such initial datum always exists, see [Ducasse and Rossi 2018]). The parabolic comparison principle yields

$$u(t,x,y) \geq \phi(x-ct,y) \quad \text{for all } t \geq 0, \text{ for all } (x,y) \in \Omega_\alpha.$$

By the definition of $w_\alpha(e_x)$, this implies that $w_\alpha(e_x) \geq c$, where c, given by Lemma 20, is independent of α. Hence, (34) holds with $K := c$.

Step 3: Upper bound on $w_\alpha(e_y)$. We now show that $w_\alpha(e_y) \to 0$ as α goes to 0. To do so, we first apply Proposition 17, to get

$$w_\alpha(e_y) \leq 2C_{\Omega_\alpha}(e_y)\sqrt{\max_{u\in[0,1]} \frac{f(u)}{u}}.$$

Let us estimate $C_{\Omega_\alpha}(e_y)$. If we take $n \in \mathbb{N}$, we see that, if α is small enough, $d_{\Omega_\alpha}(0, 4Rne_y) \geq 2Rn\sqrt{1+(1-1/\alpha)^2}$. Then, if α is small enough, $C_{\Omega_\alpha}(e_y) \leq 3\alpha$. Thus

$$w_\alpha(e_y) \leq 6\alpha\sqrt{\max_{u\in[0,1]} \frac{f(u)}{u}} \xrightarrow[\alpha\to 0]{} 0,$$

and hence the result. ☐

Now, Proposition 19 is proved, and answers Question 4 in the combustion case: in Ω_α, $c^\star \not\equiv w$, for $\alpha > 0$ small enough.

Let us now explain how this also answers Question 4 in the monostable case. Take f to be a monostable nonlinearity and let $\underline{f}$ be a combustion nonlinearity and let $\overline{f}$ be a KPP nonlinearity, both independent of x, such that

$$\underline{f} \leq f \leq \overline{f}.$$

Let $\overline{w}_\alpha$, w_α, $\underline{w}_\alpha$ be the invasion speed for the problem (15) with nonlinearity $\overline{f}$, f, $\underline{f}$ respectively. Then, by comparison, we have

$$\text{for all } e \in \mathbb{S}^1, \quad \underline{w}_\alpha(e) \leq w_\alpha(e) \leq \overline{w}_\alpha(e). \tag{35}$$

Now, we can apply Lemma 20 to find $c > 0$ and $\phi \in W^{2,\infty}(\mathbb{R}^2)$, $\phi > 0$ in B_R and $\phi = 0$ on ∂B_R such that, on B_R we have

$$\Delta\phi + c\partial_x\phi + \underline{f}(\phi) \geq 0.$$

Then, consider the domain Ω_α constructed in the proof of Proposition 19, but with this new $R > 0$.

On this domain, we have a lower bound on $\underline{w}_\alpha(e_x)$ independent of α. Moreover, we can show that $\overline{w}_\alpha(e_y)$ goes to zero as α goes to 0, as in the proof of Proposition 19.

Hence, (35) yields that there is $K > 0$ independent of α such that $w_\alpha(e_x) \geq K$, and $w_\alpha(e_y) \to 0$ as α goes to 0. This means that Proposition 19 still holds if f is monostable; hence this answers Question 4 in the monostable case and concludes the proof of Theorem 5.

4. Symmetries of the domain and relation with $c^\star$ and w

This section is dedicated to the proof of Theorem 7. As we mentioned earlier, even in a domain Ω where $c^\star \not\equiv w$, the Freidlin–Gärtner formula yields that any direction $e \in \mathbb{S}^{N-1}$ minimizing $c^\star$ satisfies the equality $c^\star(e) = w(e)$. Theorem 7 gives a geometrical condition that ensures the existence of directions where $c^\star$ and w coincide. To prove it, we first state the following lemma:

Lemma 21. *Let $c^\star$ and w be respectively the critical speed of fronts and the speed of invasion for* (15) *with the nonlinearity f satisfying* (4), (5) *and such that $u \mapsto f(u)/u$ is nonincreasing. For any $k \in \mathbb{N}$ and $e \in \mathbb{S}^{N-1}$, $(\xi_i)_{i\in[\![1,k]\!]} \in (\mathbb{S}^{N-1})^k$ such that*

$$e \in \left\{ x \in \mathbb{R}^N : x = \sum_{i=1}^{k} \lambda_i \xi_i,\ \lambda_i \geq 0 \right\},$$

the following holds:

$$c^\star(e) \leq \max_{i\in[\![1,k]\!]} \frac{c^\star(\xi_i)}{e \cdot \xi_i}.$$

Proof. For $i \in [\![1,k]\!]$, we denote by $\phi_{\xi_i}(t,x)$ a pulsating traveling front solution of (1) in the direction ξ_i with critical speed $c^\star(\xi_i)$. Let

$$v(t,x) := \sum_{i=1}^{k} \phi_{\xi_i}(t,x).$$

Now, the hypotheses on f imply that $f(v) \leq \sum_{i=1}^{k} f(\phi_{\xi_i})$, and then v is a supersolution of (1).

Now, for $\varepsilon > 0$, let f_ε be a combustion nonlinearity satisfying

$$\begin{aligned} 0 \le f_\varepsilon(x,u) \le f(x,u) &\quad \text{for all } u \in [0,1], \text{ for all } x \in \Omega, \\ f_\varepsilon(x,u) = f(x,u) &\quad \text{for all } u \in [0, 1-2\varepsilon], \text{ for all } x \in \Omega, \\ f_\varepsilon(x,u) = 0 &\quad \text{for all } u \in [-\varepsilon, 0], \text{ for all } x \in \Omega, \\ f_\varepsilon(x, 1-\varepsilon) = 0 &\quad \text{for all } x \in \Omega. \end{aligned}$$

Now, take $e \in \mathbb{S}^{N-1}$ such that $e = \sum_{i=1}^k \lambda_i \xi_i$, with $\lambda_i \ge 0$ for all $i \in [\![1,k]\!]$, and let ϕ_e^ε be a pulsating traveling front connecting $1-\varepsilon$ to $-\varepsilon$, a solution of (1) with the combustion nonlinearity f_ε, in the direction e with critical speed $c_\varepsilon^\star(e)$.

Up to some translation in time, we can assume that $\phi_e^\varepsilon(0,x) < 0$ if $x \cdot e > 0$ and, for all $i \in [\![1,k]\!]$, $\phi_{\xi_i}(0,x) \ge 1-\varepsilon$ if $x \cdot \xi_i < 0$.

Moreover, if $x \in \Omega$ is such that $x \cdot e < 0$, then there is at least one of the ξ_i such that $x \cdot \xi_i < 0$. Hence, $v(0,x) > 1-\varepsilon$ if $x \cdot e < 0$. If $x \cdot e \ge 0$, we have $v(0,x) > 0 \ge \phi_e^\varepsilon(0,x)$. Hence

$$v(0,x) \ge \phi_e^\varepsilon(0,x) \quad \text{for all } x \in \Omega.$$

Because $f_\varepsilon \le f$, the parabolic comparison principle yields

$$v(t,x) \ge \phi_e^\varepsilon(t,x) \quad \text{for all } t \ge 0, \text{ for all } x \in \Omega. \tag{36}$$

Now, if we take $\bar{c} > \max_{i \in [\![1,N]\!]} c^\star(\xi_i)/(e \cdot \xi_i)$, we have that $v(t, \bar{c}te) \to 0$ as t goes to $+\infty$. It then follows from (36) that $c_\varepsilon^\star(e) \le \max_{i \in [\![1,N]\!]} c^\star(\xi_i)/(e \cdot \xi_i)$. Now, it is classical that $c_\varepsilon^\star(e) \to c^\star(e)$ as ε goes to 0 (see, for instance, [Rossi 2017, Proposition 2.6]). Taking the limit $\varepsilon \to 0$ then yields the result. □

Remark 22. Lemma 21 yields a very strong geometrical condition on $c^\star$, and prevents it from being any arbitrary function. Consider

$$\mathcal{C} := \{ r(\xi)\xi \in \mathbb{R}^2 : r(\xi) \in [0, c^\star(\xi)] \}.$$

In the case of (15) with $\Omega = \mathbb{R}^N$, $c^\star$ is constant and then $\mathcal{C}$ is a ball. In general, it is not clear what "shapes" $\mathcal{C}$ can adopt. Lemma 21 prevents it from being some natural candidates; for instance, $\mathcal{C}$ cannot be an ellipse with eccentricity larger than $1/\sqrt{2}$. We recall that an ellipse of equation $x^2/a^2 + y^2/b^2 = 1$, with $a > b$, has eccentricity $\sqrt{1 - b^2/a^2}$.

Before turning to the proof of Theorem 7, we need another technical lemma.

Lemma 23. *Let Ω be a periodic domain, and let T be an orthogonal transformation that leaves T invariant; i.e., $T\Omega = \Omega$. Then, at least one of the two possibilities below holds true:*

(i) *T is of finite order; i.e., there is $m \in \mathbb{N}^\star$ such that $T^m = I_N$, where I_N is the identity matrix.*

(ii) *The domain Ω is invariant in a direction orthogonal to the eigenvectors associated with the eigenvalue 1.*

Proof. Assume that T leaves the domain Ω invariant and that it is not of finite order. Then, there is at least one vector e of the canonical basis of $\mathbb{R}^N$ such that

$$T^k(e) \neq e \quad \text{for all } k \in \mathbb{Z}^\star.$$

It is then readily seen that each point of the set $\{T^k(e) : k \in \mathbb{Z}\}$ is a point of accumulation. Therefore, up to extraction,

$$u_k := T^k(e) - e \xrightarrow[k\to+\infty]{} 0.$$

Moreover, because Ω is left invariant by T and because $\Omega + e = \Omega$, there holds

$$\Omega + T^k(e) = \Omega + e \quad \text{for all } k \in \mathbb{N},$$

i.e.,

$$\Omega + u_k = \Omega \quad \text{for all } k \in \mathbb{N}.$$

Now, we can find $v \in \mathbb{S}^{N-1}$ such that $u_k/|u_k|$ converges up to another extraction to v. It is then readily seen that

$$\Omega + \lambda v = \Omega \quad \text{for all } \lambda \in \mathbb{R};$$

i.e., Ω is invariant in the direction v. Observe now that, if y is an eigenvector associated to the eigenvalue 1, then $u_k \cdot y = 0$, from which we get that $v \cdot y = 0$, and this concludes the result. ☐

We are now in position to prove Theorem 7.

Proof of Theorem 7. Let T be an orthogonal transformation as in the theorem and let $e \in \mathbb{S}^{N-1}$ be such that $Te = e$.

Step 1: Reduction to the case of a finite-order orthogonal transformation. Assume that T is not of finite order. Then, owing to Lemma 23, the domain Ω is invariant in at least one direction orthogonal to e. We denote by $\mathcal{S}$ the set of all such directions. It is a subspace of $\mathbb{R}^N$ orthogonal to e such that $T(S) = S$. We define

$$\widetilde{\Omega} := \Omega \cap \mathcal{S}^\perp,$$

where $\mathcal{S}^\perp$ denotes the orthogonal of $\mathcal{S}$. Then, $e \in \widetilde{\Omega}$ and

$$T(\widetilde{\Omega}) = \widetilde{\Omega}.$$

Consider now the problem

$$\begin{cases} \partial_t u - \Delta u = f(u), & t > 0,\ x \in \widetilde{\Omega}, \\ \partial_\nu u = 0, & t > 0,\ x \in \partial\widetilde{\Omega}. \end{cases} \tag{37}$$

We denote by $\mathcal{W}$ the asymptotic set of spreading for (15). Then, owing to Proposition 15, the asymptotic set of spreading for (37) is $\mathcal{W} \cap \mathcal{S}^\perp$. In particular, the speed of invasion and the critical speed of fronts in the direction e for (15) and for (37) are the same. It is then sufficient to prove our result in the domain $\widetilde{\Omega}$, which is not invariant in any direction orthogonal to e. Because $\widetilde{\Omega}$ is left invariant by T and owing to Lemma 23, the restriction of T to $\widetilde{\Omega}$ is of finite order.

Step 2: Proof when T is of finite order. Let us now restrict our attention to the case where T is of finite order; i.e, there is $m \in \mathbb{N}^\star$ such that $T^m = I_N$.

Let ξ_0 be such that $w(e) = c^\star(\xi_0)/(\xi_0 \cdot e)$ and $\xi_0 \cdot e > 0$. If $\xi_0 = e$, then $w(e) = c^\star(e)$ and we are done. If not, we define

$$\xi_k := T^k \xi_0 \quad \text{for } k \in [\![0, m-1]\!].$$

Let us show that the vector e is in the positive cone spanned by the $(\xi_k)_{k\in[\![0,m-1]\!]}$. To do so, observe first that

$$(T^m - I_N)(\xi_0) = (T - I_N)\Bigg(\sum_{k=0}^{k=m-1} T^k \xi_0\Bigg) = (T - I_N)\Bigg(\sum_{k=0}^{k=m-1} \xi_k\Bigg) = 0.$$

Owing to the hypotheses on T, this implies that there is $\lambda \in \mathbb{R}$ such that

$$\sum_{k=0}^{k=m-1} \xi_k = \lambda e. \tag{38}$$

Moreover, because T is orthogonal, we see that, for $k \in [\![0, m-1]\!]$,

$$\xi_k \cdot e = \xi_0 \cdot e > 0, \tag{39}$$

and then, (38) yields that $\lambda > 0$; i.e., e is in the positive cone spanned by the $(\xi_k)_{k\in[\![0,m-1]\!]}$.

Now, owing to Lemma 21, we have

$$c^\star(e) \le \max_{k\in[\![0,m-1]\!]} \frac{c^\star(\xi_k)}{\xi_k \cdot e}. \tag{40}$$

Observe that, because $T\Omega = \Omega$, we have

$$c^\star(\xi_0) = c^\star(\xi_1) = \cdots = c^\star(\xi_{m-1}). \tag{41}$$

Indeed, if $\phi(t, x)$ is a pulsating traveling front solution of (15) in the direction $\xi \in \mathbb{S}^{N-1}$ with speed $c^\star(\xi)$, then $\phi(t, Tx)$ is a pulsating traveling front solution of (15) in the direction $T\xi$ with speed $c^\star(\xi)$. Then, by definition of the critical speed, $c^\star(T\xi) \le c^\star(\xi)$. The same reasoning but with $T\xi$ instead of ξ and with T^{-1} instead of T yields the reverse inequality and then $c^\star(\xi) = c^\star(T\xi)$. Hence, (41) follows from the definition of the ξ_k, $k \in [\![0, m-1]\!]$.

Now, combining (39) and (41) with (40), we see that

$$c^\star(e) \le \frac{c^\star(\xi_0)}{\xi_0 \cdot e} = w(e).$$

Because $w(e) \le c^\star(e)$, thanks to the Freidlin–Gärtner formula (12), we finally get

$$c^\star(e) = w(e),$$

and hence the result. □

We can deduce from this theorem the following:

Corollary 24. *Assume that f satisfies* (4)–(5) *and that $u \mapsto f(u)/u$ is nonincreasing. Let $\Omega \subset \mathbb{R}^N$ be a periodic domain. Then*, $c^\star(e) = w(e)$ *in the following cases*:

- *If $N=2$ and if Ω is symmetric with respect to the line $\mathbb{R}e$.*
- *If $N=3$ and if Ω is stable with respect to the rotation of angle π and of axis directed by e.*
- *If $N\in\mathbb{N}$ and if Ω is symmetric with respect to $N-1$ hyperplanes whose intersection is the line directed by e.*

The cases $N=2$ and $N=3$ are straightforward. For the general case $N\in\mathbb{N}$, one may observe that the composition of $N-1$ symmetries whose stable hyperplanes have a one-dimensional intersection satisfies the hypotheses of Theorem 7.

A typical domain to which we could apply Corollary 24 is the whole space with ball-shaped obstacles, i.e.,

$$\Omega := (\bar{B}_{1/4}+\mathbb{Z}^N)^c.$$

In this domain, Corollary 24 yields that $w(e)=c^\star(e)$ for any $e\in\mathbb{S}^{N-1}$ in the canonical basis. We conclude this section with two remarks.

Remark 25. Let us observe that the hypothesis $\mathrm{Ker}(T-I_N)=\mathbb{R}e$ is necessary in Theorem 7. Indeed, consider $\Omega\subset\mathbb{R}^2$ to be the periodic domain constructed in Proposition 18 and define

$$\widetilde{\Omega} := \Omega\times\mathbb{R}.$$

Let T be the symmetry with respect to the hyperplane orthogonal to $u:=(0,0,1)$. The domain $\widetilde{\Omega}$ is invariant in the direction $u:=(0,0,1)$; therefore we can apply Proposition 15 to see that there are directions orthogonal to u where $c^\star$ and w do not coincide, although these directions are left invariant by T.

Remark 26. Observe that, if one considers the general equation (1), then Theorem 7 still holds provided the coefficients satisfy the same symmetry as the domain; i.e., we need the coefficients to satisfy

$$A(Tx)=TA(x)T^\star,\quad q(Tx)=Tq(x)\quad\text{and}\quad f(Tx,\cdot)=f(x,\cdot),$$

where T is the transformation considered in Theorem 7.

Appendix

Proof of Lemma 9. As we mentioned, Lemma 9 is the natural extension of [Rossi 2017, Lemma 2.1], in the case of a periodic domain.

Proof. Let u be taken as in the lemma. We define

$$h := \liminf_{\delta\to+\infty} \inf_{\substack{t<0\\ x\cdot e<\gamma t-\delta\\ x\in\Omega}} u(t,x).$$

Assume that, by contradiction, $h\in(S,1)$. We can find two sequences $(x_n)_n\in\Omega^{\mathbb{N}}$, $(t_n)_n\in(-\infty,0)^{\mathbb{N}}$ such that $x_n\cdot e-\gamma t_n\to-\infty$ and $u(t_n,x_n)\to h$ as n goes to $+\infty$. Let us define $k_n\in\mathbb{Z}^N$, $z_n\in[0,1)^N$

so that $x_n = k_n + z_n$. Up to extraction, we assume that $z_n \to z$ as n goes to $+\infty$ for some $z \in [0, 1]^N$. Consider the sequence of translated functions

$$u_n := u(\cdot + t_n, \cdot + k_n).$$

These functions are supersolutions of (1), by periodicity of the domain. As before, we can use the usual parabolic estimates to get local uniform convergence of the sequence $(u_n)_n$ to a function u_∞ supersolution of (1). Moreover, we have

$$u_\infty(0, z) = h \le u_\infty(t, x) \quad \text{for all } t \le 0, \text{ for all } x \in \Omega. \tag{42}$$

Indeed, for $t \le 0,\ x \in \Omega$, we have

$$u_n(t, x) = u(t + t_n, x + k_n) \ge \inf_{\substack{\tau<0 \\ y\cdot e-\gamma\tau\le\tilde{\delta}_n}} u(\tau, y), \tag{43}$$

where $\tilde{\delta}_n := x \cdot e - \gamma t - z_n \cdot e + x_n \cdot e - \gamma t_n$ goes to $-\infty$ as n goes to $+\infty$. Hence, passing to the limit $n \to +\infty$ in (43) yields (42). Because $f \ge 0$, it follows from the parabolic maximum principle and Hopf principle that u_∞ is actually equal to h if $t \le 0$ and $x \in \Omega$. This implies that $f(x, h) = 0$, which is in contradiction with the fact that $h \in (S, 1)$ together with hypothesis (5); hence the result. □

Proof of Lemma 10. We now turn to the proof of Lemma 10. Again, it is the natural extension of [Rossi 2017, Lemma 2.2] to the case of a periodic domain.

Proof. Let us define $\bar{u}_\varepsilon := \bar{u} + \varepsilon$, where $\varepsilon > 0$. The hypotheses on $\bar{u}$ yield that there is $\delta > 0$ such that $\bar{u}_\varepsilon(t, x) \ge 1 + \varepsilon/2$ if $t < 0$ and $x \cdot e < \gamma t - \delta,\ x \in \Omega$. The hypotheses on $\underline{u}$ give us that there is $L > 0$ such that $\underline{u}(t, x) \le \varepsilon$ if $t < 0$ and $x \cdot e \ge (\gamma + \eta)t + L$. Hence, there is $T_\varepsilon \le 0$ such that $\bar{u}_\varepsilon(t, x) > \underline{u}(t, x)$ for $t < T_\varepsilon$, for all $x \in \Omega$. Indeed, if t is negative enough, we have $\eta t + L < -\delta$; hence we can take $T_\varepsilon := (-\delta - L)/\eta$.

In order to prove the result, we shall argue by contradiction. Hence, we will assume that there is $\varepsilon_0 > 0$ such that

$$\text{for all } \varepsilon \in (0, \varepsilon_0) \text{ there exist } \tau \in (T_\varepsilon, 0) \text{ and } x_\tau \in \Omega \text{ such that } \bar{u}_\varepsilon(\tau, x_\tau) < \underline{u}(\tau, x_\tau). \tag{44}$$

Indeed, if (44) does not hold, our result follows by letting $\varepsilon \to 0$. Now, we define $t_\varepsilon \in [T_\varepsilon, 0)$ to be the infimum of all the τ such that (44) holds true. Hence

$$\bar{u}_\varepsilon(t, x) \ge \underline{u}(t, x) \quad \text{for all } t \le t_\varepsilon, \text{ for all } x \in \Omega,$$

and by continuity we have

$$\inf_{x\in\Omega} (\bar{u}_\varepsilon - \underline{u})(t_\varepsilon, x) = 0.$$

Thanks to the hypotheses, we can find $\rho_\varepsilon \in \mathbb{R}$ such that

$$\inf_{x\cdot e=\rho_\varepsilon} (\bar{u}_\varepsilon - \underline{u})(t_\varepsilon, x) = 0.$$

Depending on the behavior of ρ_ε, we now consider three cases.

First case: $(\rho_\varepsilon)_{\varepsilon\in(0,\varepsilon_0)}$ is bounded. We can find a sequence of points $(x_\varepsilon)_{\varepsilon\in(0,\varepsilon_0)}$, with $x_\varepsilon \in \Omega$ such that

$$x_\varepsilon \cdot e = \rho_\varepsilon \quad \text{and} \quad \bar{u}_\varepsilon(t_\varepsilon, x_\varepsilon) - \underline{u}(t_\varepsilon, x_\varepsilon) < \varepsilon.$$

We define $k_\varepsilon \in \mathbb{Z}^N$, $y_\varepsilon \in [0,1)^N$ to be such that $x_\varepsilon = k_\varepsilon + y_\varepsilon$. Up to extraction,we can find $y \in [0,1)^N$ such that $y_\varepsilon \to y$ as ε goes to 0.

We now consider the translated functions $\bar{u}_\varepsilon(t+t_\varepsilon, x+k_\varepsilon)$, $\underline{u}(t+t_\varepsilon, x+k_\varepsilon)$. Using parabolic estimates and extracting, these functions converge locally uniformly as ε goes to 0 to $\bar{u}_\infty$, $\underline{u}_\infty$, a supersolution and a subsolution respectively of (1).

Moreover, $\bar{u}_\infty$, $\underline{u}_\infty$ satisfy

$$\bar{u}_\infty(0, y) = \underline{u}_\infty(0, y) \qquad \text{and} \qquad \bar{u}_\infty(t, x) \geq \underline{u}_\infty(t, x) \quad \text{for } t \leq 0,\ x \in \Omega.$$

Hence, the strong comparison principle and the Hopf lemma (see [Protter and Weinberger 1967, Chapter 3]) imply that $\bar{u}_\infty = \underline{u}_\infty$ for $t \leq 0$. But the boundedness of $x_\varepsilon \cdot e = \rho_\varepsilon$ implies that we still have

$$\liminf_{\delta\to+\infty} \inf_{\substack{t<0\\ x\cdot e<\gamma t-\delta\\ x\in\Omega}} \bar{u}_\infty(t, x) \geq 1.$$

However, the hypotheses on $\underline{u}$ yield that there is $K \in \mathbb{R}$ such that

$$\underline{u}_\infty(t, x) \leq \tfrac{1}{2} \quad \text{for all } t < 0, \text{ for all } x \in \Omega \text{ such that } x \cdot e \geq (\gamma+\eta)t + K.$$

Taking $t < 0$ small enough yields a contradiction.

Second case: $\inf_{\varepsilon\in(0,\varepsilon_0)} \rho_\varepsilon = -\infty$. Let us take ε such that $-\rho_\varepsilon$ is large enough to have

$$\inf_{\substack{t<0\\ x\cdot e-\gamma t<\rho_\varepsilon}} \bar{u}(t, x) > S.$$

Because $f(x,\cdot)$ is decreasing in $(S, 1)$, we have that $\bar{u}_\varepsilon = \bar{u} + \varepsilon$ is a supersolution of (1) for $\{(t, x) \in \mathbb{R}\times\Omega : x\cdot e - \gamma t < \rho_\varepsilon\}$.

We can find a sequence $(x_n)_n \in \Omega^{\mathbb{N}}$ such that $x_n \cdot e = 0$ and

$$\lim_{n\to+\infty} (\bar{u}_\varepsilon - \underline{u})(t_\varepsilon, \rho_\varepsilon e + x_n) = 0.$$

We write as before $x_n = k_n + y_n$, where $k_n \in \mathbb{Z}^N$ and $y_n \in [0,1)^N$, and up to extraction we can find $y \in [0,1]^N$ such that $y_n \to y$ as n goes to $+\infty$.

We define $\bar{u}_n^\varepsilon(t, x) := \bar{u}_\varepsilon(t, x+k_n)$ and $\underline{u}_n(t, x) := \underline{u}(t, x+k_n)$. Observe that $\bar{u}_n^\varepsilon$ is a supersolution in $\{(t, x) \in \mathbb{R}\times\Omega : x\cdot e - \gamma t < \rho_\varepsilon - 1\}$. Again, using parabolic estimates and extracting as n goes to $+\infty$, we get two functions $\bar{u}_\infty^\varepsilon$ and $\underline{u}_\infty$ that are respectively a supersolution and a subsolution of (1) on the same set. Moreover, they satisfy $\bar{u}_\infty^\varepsilon(t_\varepsilon, \rho_\varepsilon e + y) = \underline{u}_\infty(t_\varepsilon, \rho_\varepsilon e + y)$; we have a contact point.

Observe that $(t_\varepsilon, \rho_\varepsilon e + y)$ is in $\{(t, x) \in \mathbb{R}\times\Omega : x\cdot e - \gamma t < \rho_\varepsilon - 2\}$. Hence, we can apply the Hopf lemma [Protter and Weinberger 1967, Theorem 6] to (1) on $\{(t, x) \in \mathbb{R}\times\Omega : x\cdot e - \gamma t < \rho_\varepsilon - 1\}$ to get that $(t_\varepsilon, \rho_\varepsilon e + y)$ is not on a boundary point. Therefore, it is an interior contact point and the parabolic comparison principle yields that $\bar{u}_\infty^\varepsilon(t, x) = \underline{u}_\infty(t, x)$ on $\{(t, x) \in \mathbb{R}\times\Omega : x\cdot e - \gamma t < \rho_\varepsilon - 1\}$. But this is not possible, because the hypotheses on $\bar{u}$ imply that there is δ large enough so that $\bar{u}^{\varepsilon}_{\infty}(t,x) \geq 1 + \varepsilon/2$ if $x \cdot e - \gamma t < -\delta$. Because $\underline{u} \leq 1$, we are led to a contradiction.

Third case: $\sup_{\varepsilon \in (0,\varepsilon_0)} \rho_\varepsilon = +\infty$. If we are in the case (19), this cannot happen because $\bar{u}_\varepsilon \geq 0$ and $\underline{u}(t_\varepsilon, x) < 0$ if $x \cdot e$ is large enough. Then, we are left to assume that f satisfies (9) and $\underline{u}$ satisfies (18). In particular, we can take ε small enough so that ρ_ε is large enough to have $\underline{u}(t,x) \leq \theta$ on $\{(t,x) \in \mathbb{R} \times \Omega : x \cdot e - \gamma t > \rho_\varepsilon\}$, where θ is from (9). Hence, $\underline{u}_\varepsilon := \underline{u} - \varepsilon$ is a subsolution of (1) on this set. Arguing as in the previous case, we get a contradiction, and hence the result. □

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AN ELEMENTARY APPROACH TO FREE ENTROPY THEORY FOR CONVEX POTENTIALS

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We present an alternative approach to the theory of free Gibbs states with convex potentials. Instead of solving SDEs, we combine PDE techniques with a notion of asymptotic approximability by trace polynomials for a sequence of functions on $M_N(\mathbb{C})_{\mathrm{sa}}^m$ to prove the following. Suppose μ_N is a probability measure on $M_N(\mathbb{C})_{\mathrm{sa}}^m$ given by uniformly convex and semiconcave potentials V_N, and suppose that the sequence DV_N is asymptotically approximable by trace polynomials. Then the moments of μ_N converge to a noncommutative law λ. Moreover, the free entropies $\chi(\lambda)$, $\underline{\chi}(\lambda)$, and $\chi^*(\lambda)$ agree and equal the limit of the normalized classical entropies of μ_N.

1. Introduction

1A. ***Motivation and main ideas.*** Since Voiculescu [1993; 1994; 1998] introduced the free entropy of a noncommutative law, a number of open problems have prevented a satisfying unification of the theory (as explained in [Voiculescu 2002]). The free entropy χ was defined by taking the lim sup as $N \to \infty$ of the normalized log volume of the space of microstates, where the microstates are certain tuples of $N \times N$ self-adjoint matrices having approximately the correct distribution. It is unclear whether using the lim inf instead of the lim sup would yield the same quantity. Voiculescu also defined a nonmicrostates free entropy χ^* by integrating the free Fisher information of $X + t^{1/2} S$, where S is a free semicircular family free from X, and conjectured that $\chi = \chi^*$.

Biane, Capitaine, and Guionnet [Biane et al. 2003] showed that $\chi \leq \chi^*$ as a consequence of their large deviation principle for the GUE (see also [Cabanal Duvillard and Guionnet 2001]). The proof relied

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Keywords: free entropy, free Fisher information, free Gibbs state, trace polynomials, invariant ensembles.

on stochastic differential equations relative to Hermitian Brownian motion and analyzed exponential functionals of Brownian motion. Recent work of Dabrowski [2016] combined these ideas with stochastic control theory and ultraproduct analysis in order to show that $\chi = \chi^*$ for free Gibbs states defined by a convex and sufficiently regular potential. This resolves this part of the unification problem for a significant class of noncommutative laws.

This paper will prove a result similar to Dabrowski's using deterministic rather than stochastic methods. We want to argue as directly as possible that the classical entropy and Fisher's information of a sequence of random matrix models converge to their free counterparts. Let us motivate and sketch the main ideas, beginning with the heuristics behind Voiculescu's nonmicrostates entropy χ^*.

Consider a noncommutative law λ of an m-tuple and suppose λ is the limit of a sequence of random $N \times N$ matrix distributions μ_N given by convex, semiconcave potentials $V_N : M_N(\mathbb{C})_{\mathrm{sa}}^m \to \mathbb{R}$. Let $\sigma_{t,N}$ be the distribution of m independent GUE matrices which each have normalized variance t, and let σ_t be the noncommutative law of m free semicircular variables which each have variance t. Let $V_{N,t}$ be the potential corresponding to the convolution $\mu_N * \sigma_{t,N}$. The classical Fisher information $\mathcal{I}$ satisfies

$$\frac{d}{dt}\frac{1}{N^2}h(\mu_N * \sigma_{t,N}) = \frac{1}{N^3}\mathcal{I}(\mu_N * \sigma_{t,N}) = \int \|DV_{N,t}(x)\|_2^2 \, d(\mu_N * \sigma_{t,N})(x),$$

and from this we deduce that

$$\frac{1}{N^2}h(\mu_N) + \frac{m}{2}\log N = \frac{1}{2}\int \Big(\frac{m}{1+t} - \frac{1}{N^3}\mathcal{I}(\mu_N * \sigma_{t,N})\Big)\, dt + \frac{m}{2}\log 2\pi e.$$

As $N \to \infty$, we expect the left-hand side to converge to the microstates free entropy $\chi(\lambda)$ because the distribution μ_N should be concentrated on the microstate spaces of the law λ. On the other hand, we expect the right-hand side to converge to the Voiculescu's nonmicrostates free entropy $\chi^*(\lambda)$ defined by

$$\chi^*(\lambda) = \frac{1}{2}\int \Big(\frac{m}{1+t} - \Phi^*(\lambda \boxplus \sigma_t)\Big)\, dt + \frac{m}{2}\log 2\pi e,$$

where Φ^* is the free Fisher information and $\boxplus$ denotes the free convolution [Voiculescu 1998].

Under suitable assumptions on V_N, the microstates free entropy $\chi(\lambda)$ is the $\limsup$ of normalized classical entropies of μ_N. On the right-hand side, we want to show that $N^{-3}\mathcal{I}(\mu_N * \sigma_{t,N}) \to \Phi^*(\lambda \boxplus \sigma_t)$ for all $t \geq 0$. Since the Fisher information is the $L^2(\mu_N)$ norm squared of the score function or (classical) conjugate variable $DV_{N,t}(x)$, we want to prove that the classical conjugate variables $DV_{N,t}(x)$ behave asymptotically like the free conjugate variables for $\lambda \boxplus \sigma_t$ for all t.

This would not be surprising because classical objects associated to invariant random matrix ensembles often behave asymptotically like their free counterparts. For instance, Biane [1997] showed that the entrywise Segal–Bargmann transform of noncommutative functions evaluated on $N \times N$ matrices can be approximated by the free Segal–Bargmann transform computed through analytic functional calculus. Similarly, Guionnet and Shlyakhtenko [2014, Theorem 4.7] showed that classical monotone transport maps for certain random matrix models approximate the free monotone transport. Moreover, Dabrowski's approach [2016] to proving $\chi = \chi^*$ involved constructing solutions to free SDEs as ultraproducts of the solutions to classical SDEs.

In Section 3D, we make precise the idea that a sequence of functions on $M_N(\mathbb{C})^m_{\mathrm{sa}}$ has a "well-defined, noncommutative asymptotic behavior" by defining *asymptotic approximability by trace polynomials* (Definition 3.24). We assume that DV_N at time zero has the approximation property and must show that the same is true for $DV_{N,t}$ for all t.

First, we show that this property is preserved under several operations on sequences, including composition and convolution with the Gaussian law $\sigma_{N,t}$ (see Section 3D). Then in Section 6 we analyze the PDE that describes the evolution of $V_{N,t}$. We show that for all t the solution $V_{N,t}$ can be approximated in a dimension-independent way by applying a sequence of simpler operations, each of which preserves asymptotic approximability by trace polynomials. In other words, if the initial data DV_N is asymptotically approximable by trace polynomials, then so is $DV_{N,t}$, and hence we obtain convergence of the classical Fisher information to the free Fisher information.

This proves the equality $\chi(\lambda) = \chi^*(\lambda)$ whenever a sequence of log-concave random matrix models μ_N converges to λ in an appropriate sense (Theorem 7.1). Another result (Theorem 4.1), proved by similar techniques, establishes sufficient conditions for a sequence of log-concave random matrix models μ_N to converge in moments to a noncommutative law λ, so that Theorem 7.1 can be applied. As a consequence, we show that $\chi = \chi^*$ for a class of free Gibbs states.

1B. ***Main results.*** To fix notation, let $M_N(\mathbb{C})^m_{\mathrm{sa}}$ be space of m-tuples $x = (x_1, \dots, x_m)$ of self-adjoint $N \times N$ matrices and let $\|x\|_2 = \left(\sum_j \tau_N(x_j^2)\right)^{1/2}$, where $\tau_N = (1/N)\operatorname{Tr}$. We denote by $\|x\|_\infty$ the maximum of the operator norms $\|x_j\|$. Recall that a trace polynomial $f(x_1, \dots, x_m)$ is a linear combination of terms of the form

$$p(x) \prod_{j=1}^{n} \tau(p_j(x)),$$

where p and p_j are noncommutative polynomials in $x_1, \dots, x_m$ (see Section 3A).

Consider a sequence of potentials $V_N : M_N(\mathbb{C})^m_{\mathrm{sa}} \to \mathbb{R}$ such that $V_N(x) - (c/2)\|x\|_2^2$ is convex and $V_N(x) - (C/2)\|x\|_2^2$ is concave for some $0 < c < C$. Define the associated probability measure μ_N by

$$d\mu_N(x) = \frac{1}{Z_N} e^{-N^2 V_N(x)}\, dx, \quad Z_N = \int_{M_N(\mathbb{C})^m_{\mathrm{sa}}} e^{-N^2 V_N(x)}\, dx.$$

Assume that the sequence of normalized gradients $DV_N(x) = N \nabla V_N(x)$ is asymptotically approximable by trace polynomials in the sense that for every $\epsilon > 0$ and $R > 0$ there exists a trace polynomial $f(x)$ such that

$$\limsup_{N \to \infty} \sup_{\|x\|_\infty \le R} \|DV_N(x) - f(x)\|_2 \le \epsilon,$$

where $\|x\|_\infty$ denotes the maximum of the operator norms of the x_j. Also, assume that $\int (x - \tau_N(x))\, d\mu_N(x)$ is bounded in operator norm as $N \to \infty$ (it will be zero if μ_N is unitarily invariant or has expectation zero). In this case, we have the following:

(1) There exists a constant R_0 such that $\mu_N(\|x\|_\infty \ge R_0 + \delta) \le m e^{-cN\delta^2/2}$ for $\delta > 0$.

(2) There exists a noncommutative law λ such that

$$\lim_{N\to\infty} \int \tau_N(p(x))\, d\mu_N(x) = \lambda(p)$$

for every noncommutative polynomial p.

(3) The measures μ_N exhibit exponential concentration around λ in the sense that

$$\lim_{N\to\infty} \frac{1}{N^2} \log \mu_N(\|x\|_\infty \le R, |\tau_N(p(x)) - \lambda(p)| \ge \delta) < 0$$

for every $R > 0$ and every noncommutative polynomial p.

(4) The law λ has finite free entropy and we have

$$\chi(\lambda) = \underline{\chi}(\lambda) = \chi^*(\lambda) = \lim_{N\to\infty} \frac{1}{N^2}\Big(h(\mu_N) + \frac{m}{2}\log N\Big),$$

where χ and $\underline{\chi}$ are respectively the lim sup and lim inf versions of microstates free entropy, χ^* is the nonmicrostates free entropy, and h is the classical entropy.

(5) The same holds for $\mu_N * \sigma_{t,N}$ and $\lambda \boxplus \sigma_t$, where $\sigma_{t,N}$ is the law of m independent GUE matrices with variance t and σ_t is the law of m free semicircular variables with variance t.

(6) The law λ has finite free Fisher information. If $\mathcal{I}$ is the classical Fisher information and Φ^* is the free Fisher information, then

$$\lim_{N\to\infty} \frac{1}{N^3}\mathcal{I}(\mu_N * \sigma_{t,N}) = \Phi^*(\lambda \boxplus \sigma_t).$$

(7) The functions $t \mapsto (1/N^3)\mathcal{I}(\mu_N * \sigma_{t,N})$ and $t \mapsto \Phi^*(\lambda \boxplus \sigma_t)$ are decreasing and Lipschitz in t with the absolute value of the derivative bounded by $C^2 m(1+Ct)^{-2}$.

Claims (1) and (3) are standard concentration estimates (see Section 2E), which we do not prove in this paper, but we include them in the statement to clarify the big picture. Claim (2) is proved in Theorem 4.1, which is similar to the earlier results [Guionnet and Shlyakhtenko 2009, Theorem 4.4; Dabrowski et al. 2016, Proposition 50 and Theorem 51; Dabrowski 2016, Theorem 4.4]. Claims (4) through (7) come from Theorem 7.1, which is similar to [Dabrowski 2016, Theorem A].

In particular, we recover [Dabrowski 2016, Theorem A] that $\chi(\lambda) = \underline{\chi}(\lambda) = \chi^*(\lambda)$ when the law λ is a free Gibbs state given by a sufficiently regular convex noncommutative potential $V(X)$, because taking $V_N = V$ will define a sequence of random matrix models μ_N which concentrate around the noncommutative law λ.

Unlike Dabrowski, we do not provide an explicit formula for $(d/dt)\Phi(\lambda \boxplus \sigma_t)$. However, we are able to prove that $\Phi(\lambda \boxplus \sigma_t)$ is Lipschitz in t rather than merely having a derivative in $L^2(dt)$ (and hence being $\frac{1}{2}$-Hölder continuous) as shown by Dabrowski. Our results also allow slightly more flexibility in the choice of random matrix models, so that we do not have to assume that V_N is given by exactly the same formula for every N or that V_N is exactly unitarily invariant.

1C. ***Organization of paper.*** Section 2 establishes notation and reviews basic facts from noncommutative probability and random matrix theory.

Section 3 defines the algebra of trace polynomials and describes how they behave under differentiation and convolution with Gaussians. We then introduce the notion that a sequence $\{\phi_N\}$ of functions $M_N(\mathbb{C})_{\mathrm{sa}}^m \to M_N(\mathbb{C})_{\mathrm{sa}}^m$ of $\mathbb{C}$ is asymptotically approximable by trace polynomials. We show that this approximation property is preserved under several operations including composition and Gaussian convolution.

Section 4 proves Theorem 4.1 concerning the convergence of moments for the measure μ_N (claims (1)–(3) of Section 1B). We evaluate $\int u \, d\mu_N$ for a Lipschitz function u as $\lim_{t\to\infty} T_t^{V_N} u$, where $T_t^{V_N}$ is the semigroup such that $u_t = T_t^{V_N} u$ solves the equation $\partial_t u_t = (2N)^{-1} \Delta u_t - DV \cdot \nabla u_t$. We approximate $T_t^{V_N}$ by iterating simpler operations in order to show that if $N \nabla V_n$ and u_N are asymptotically approximable by trace polynomials, then so is $T_t^{V_N} u_N$, and hence that $\lim_{N\to\infty} \int u_N \, d\mu_N$ exists.

Section 5 reviews the definitions of free entropy and Fisher's information. We also show that the microstates free entropies $\chi(\lambda)$ and $\underline{\chi}(\lambda)$ are the lim sup and lim inf of normalized classical entropies of μ_N, provided that μ_N concentrates around λ and satisfies some mild operator norm tail bounds, and that $\{V_N\}$ is asymptotically approximable by trace polynomials. Similarly, if $\{DV_N\}$ is asymptotically approximable by trace polynomials, then the normalized classical Fisher information converges to the free Fisher information.

Section 6 considers the evolution of the potential $V_N(x, t)$ corresponding to $\mu_N * \sigma_{t,N}$, where $\sigma_{t,N}$ is the law of m independent GUE of variance t. Our goal is to show that if $DV_N(x, 0)$ is asymptotically approximable by trace polynomials, then so is $N \nabla V_N(x, t)$ for all $t > 0$, so that we can apply our previous result that the classical Fisher information converges to the free Fisher information. As in Section 4, we construct the semigroup R_t which solves the PDE as a limit of iterates of simpler operations which are known to preserve asymptotic approximation by trace polynomials.

In Section 7 we conclude the proof of our main theorem on free entropy and Fisher's information (Theorem 7.1), which establishes claims (4)–(7) of Section 1B, assuming a weakened version of the hypothesis and conclusion of Theorem 4.1.

In Section 8, we characterize the limiting noncommutative laws λ which arise in Theorem 4.1 as the free Gibbs states for a certain class of potentials. In particular, we apply Theorem 7.1 to show that $\chi = \chi^*$ for several types of free Gibbs states considered in previous literature.

2. Preliminaries

Here we fix notation and discuss background results that will be used throughout the paper.

2A. ***Notation for matrix algebras.*** Let $M_N(\mathbb{C})$ denote the $N \times N$ matrices over $\mathbb{C}$ and let $M_N(\mathbb{C})_{\mathrm{sa}}$ be the self-adjoint elements. Note that $M_N(\mathbb{C})_{\mathrm{sa}}^m$ is a real inner product space with the inner product $\langle x, y\rangle_{\mathrm{Tr}} := \sum_{j=1}^m \mathrm{Tr}(x_j y_j)$ for $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_m)$. Moreover, $M_N(\mathbb{C})^m$ can be canonically identified with the complexification $\mathbb{C} \otimes_{\mathbb{R}} M_N(\mathbb{C})_{\mathrm{sa}}^m$ by decomposing each matrix into its self-adjoint and anti-self-adjoint parts.

Being a real inner product space, $M_N(\mathbb{C})_{\mathrm{sa}}$ is isomorphic to $\mathbb{R}^{mN^2}$. An explicit choice of coordinates can be made using the following orthonormal basis for $M_N(\mathbb{C})_{\mathrm{sa}}$:

$$\mathcal{B}_N = \{E_{k,k}\}_{k=1}^N \cup \left\{ \frac{1}{\sqrt{2}} E_{k,\ell} + \frac{1}{\sqrt{2}} E_{\ell,k} \right\}_{k<\ell} \cup \left\{ \frac{i}{\sqrt{2}} E_{k,\ell} - \frac{i}{\sqrt{2}} E_{\ell,k} \right\}_{k<\ell}. \tag{2-1}$$

This basis has the property that for all $x, y, z \in M_N(\mathbb{C})$, we have

$$\sum_{b \in \mathcal{B}_N} xbybz = xz \operatorname{Tr}(y), \tag{2-2}$$

which follows from an elementary computation.

We denote the norm corresponding to Tr by $|\cdot|$ (essentially the Euclidean norm). We denote the normalized trace by $\tau_N = (1/N)\operatorname{Tr}$. We denote the corresponding inner product by $\langle x, y\rangle_2 = \sum_{j=1}^m \tau_N(x_j y_j)$ and the norm by $\|\cdot\|_2$. For $x \in M_N(\mathbb{C})$, we denote the operator norm by $\|x\|$. Similarly, if $x = (x_1, \dots, x_m) \in M_N(\mathbb{C})^m$, we write $\|x\|_\infty = \max_j \|x_j\|$.

The symbols ∇ and Δ will represent the gradient and Laplacian operators with respect to the coordinates of $M_N(\mathbb{C})_{\mathrm{sa}}$ in the nonnormalized inner product $\langle\cdot,\cdot\rangle_{\mathrm{Tr}}$. The symbols D and L_N will denote the normalized versions $N\nabla$ and $(1/N)\Delta$ respectively, as well as the corresponding linear transformations on the algebra of trace polynomials. This normalization and notation will be explained and justified in Section 3B.

2B. ***Noncommutative probability spaces and laws.*** The following are standard definitions and facts in noncommutative probability. For further background, see [Voiculescu et al. 1992; Nica and Speicher 2006; Anderson et al. 2010, §5].

Definition 2.1. A *von Neumann algebra* is a unital $\mathbb{C}$-algebra $\mathcal{M}$ of bounded operators on a Hilbert space $\mathcal{H}$ which is closed under adjoints and closed in the weak operator topology.

Definition 2.2. A *tracial von Neumann algebra* or *noncommutative probability space* is a von Neumann algebra $\mathcal{M}$ together with a bounded linear map $\tau : \mathcal{M} \to \mathbb{C}$ which is continuous in the weak operator topology and satisfies $\tau(1) = 1$, $\tau(xy) = \tau(yx)$, and $\tau(x^*x) \geq 0$. The map τ is called a *trace*.

Definition 2.3. For $m \geq 1$, we denote by $\mathrm{NCP}_m = \mathbb{C}\langle X_1, \dots, X_m\rangle$ the algebra of noncommutative polynomials in $X_1, \dots, X_m$, equipped with conjugate-linear involution $*$ such that $X_j^* = X_j$ and $(pq)^* = q^* p^*$. A *noncommutative law* (for an m-tuple) is a map $\lambda : \mathrm{NCP}_m \to \mathbb{C}$ such that

(1) λ is linear,

(2) λ is unital (that is, $\lambda(1) = 1$),

(3) λ is positive, that is, for every $p(X) \in \mathbb{C}\langle X_1, \dots, X_m\rangle$, we have $\lambda(p(X)^* p(X)) \geq 0$,

(4) λ is tracial, that is, $\lambda(p(X)q(X)) = \lambda(q(X)p(X))$.

We denote by Σ_m the space of noncommutative laws equipped with the topology of pointwise convergence on $\mathbb{C}\langle X_1, \dots, X_m\rangle$, that is, convergence in noncommutative moments.

Definition 2.4. We say that a noncommutative law λ is *bounded by R* if we have

$$|\lambda(X_{i_1}, \dots, X_{i_n})| \leq R^n.$$

We denote the space of such laws by $\Sigma_{m,R}$.

Definition 2.5. Suppose that $x_1, \dots x_m$ are bounded self-adjoint elements of a tracial von Neumann algebra $(\mathcal{M}, \tau)$. Then the *law of* $x = (x_1, \dots, x_m)$ is the map

$$\lambda_x : \mathbb{C}\langle X_1, \dots, X_n \rangle \to \mathbb{C} : p(X) \mapsto \tau(p(x)).$$

Definition 2.6. Let $M_N(\mathbb{C})$ be the algebra of $N \times N$ matrices over $\mathbb{C}$. Let $\tau_N = (1/N)\,\mathrm{Tr}$ be the normalized trace. Then $(M_N(\mathbb{C}), \tau_N)$ is a tracial von Neumann algebra, and hence, for every m-tuple of self-adjoint matrices $x = (x_1, \dots, x_m)$, the law λ_x is defined by Definition 2.5.

Proposition 2.7. *The space $\Sigma_{m,R}$ is compact, separable, and metrizable. Moreover, every $\mu \in \Sigma_{m,R}$ can be realized as λ_x for some tuple $x = (x_1, \dots, x_m)$ of self-adjoint elements of a tracial von Neumann algebra $(\mathcal{M}, \tau)$ with $\|x\|_\infty \le R$.*

For the proof of the claim that every noncommutative law can be realized by operators, see [Anderson et al. 2010, Proposition 5.2.14].

2C. *Noncommutative L^α-norms.* On several occasions, we will need to use the noncommutative L^α norms for $\alpha \in [1, +\infty]$. (Here we use α rather than p since the letter p will often be used for a polynomial.) If y is any element of a tracial von Neumann algebra $(\mathcal{M}, \tau)$, then we define $|y| = (y^* y)^{1/2}$ using continuous functional calculus. For $\alpha \in [0, +\infty)$, we define $\|y\|_\alpha = \tau(|y|^\alpha)^{1/\alpha}$. We also define $\|y\|_\infty$ to be the operator norm.

Proposition 2.8. *If $(\mathcal{M}, \tau)$ is a tracial von Neumann algebra and $\alpha \in [1, +\infty]$, then $\|\cdot\|_\alpha$ defines a norm. Moreover, we have the noncommutative Hölder's inequality*

$$\|x_1 \cdots x_n\|_\alpha \le \|x_1\|_{\alpha_1} \cdots \|x_n\|_{\alpha_n}$$

whenever

$$\alpha, \alpha_1, \dots, \alpha_n \in [1, +\infty], \qquad \frac{1}{\alpha_1} + \dots + \frac{1}{\alpha_n} = \frac{1}{\alpha}.$$

Moreover, we have $|\tau(y)| \le \|y\|_1$.

A standard proof of the Hölder inequality uses polar decomposition, complex interpolation, and the three lines lemma. We will in fact only need this inequality for the trace τ_N on $M_N(\mathbb{C})$. Modulo renormalization of the trace, the inequality for matrices follows from the treatment of trace-class operators in [Simon 2005]; see especially Theorems 1.15 and 2.8, as well as the references cited on p. 31. For the setting of von Neumann algebras, a convenient proof can be found in [Correa da Silva 2018, Theorems 2.4–2.6]; for an overview and further history see [Pisier and Xu 2003, §2].

Remark 2.9. One can define the noncommutative L^α norm for a tuple $(y_1, \dots, y_m)$ as

$$\|(y_1, \dots, y_m)\|_\alpha = \begin{cases} \tau(|y_1|^\alpha + \dots + |y_m|^\alpha)^{1/\alpha}, & \alpha \in [1, +\infty), \\ \max_j \|y_j\|, & \alpha = +\infty. \end{cases}$$

However, for tuples, we will only need to use the 2 and ∞ norms.

2D. *Free independence, semicircular law, and GUE.* We will use the following standard definitions and facts from free probability. For further background, refer to [Voiculescu 1986; 1991; Voiculescu et al. 1992; Nica and Speicher 2006; Anderson et al. 2010].

Let $(\mathcal{M}, \tau)$ be a tracial von Neumann algebra, and let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be unital $*$-subalgebras of $\mathcal{M}$. Then we say that $\mathcal{A}_1, \dots, \mathcal{A}_m$ are *freely independent* if given $a_1, \dots, a_k$ with $a_j \in \mathcal{A}_{i_j}$ and $i_j \neq i_{j+1}$ and $\tau(a_j) = 0$ for each j, we have also $\tau(a_1 \cdots a_k) = 0$.

In particular, if $S_1, \dots, S_n$ are *subsets* of $\mathcal{M}$, then we say that they are freely independent if the unital $*$-subalgebras they generate are freely independent. Thus, for instance, self-adjoint elements $x_1, \dots, x_m$ of $\mathcal{M}$ are freely independent if given polynomials $f_1, \dots, f_k$ and indices $i_1, \dots, i_k$ with $i_j \neq i_{j+1}$ such that $\tau(f_j(X_{i_j})) = 0$, we have also $\tau(f_1(X_{i_1}) \cdots f_k(X_{i_k})) = 0$.

The *free convolution* of two noncommutative laws μ and ν (of self-adjoint m-tuples) is defined as the noncommutative law of $(x_1 + y_1, \dots, x_m + y_m)$, given that $\{x_1, \dots, x_m\}$ and $\{y_1, \dots, y_m\}$ are freely independent and the noncommutative law of $(x_1, \dots, x_m)$ is μ, and the noncommutative law of $(y_1, \dots, y_m)$ is ν. Then $\boxplus$ is well-defined, independent of the particular choice of operators that realize the laws μ and ν. Moreover, $\boxplus$ is commutative and associative.

If $X_1, \dots, X_m$ are freely independent, then their joint law is determined by the individual laws of the X_j, each of which is represented by a compactly supported probability measure on $\mathbb{R}$. The *semicircle law* (of mean zero and variance 1) is the probability measure given by density

$$\tfrac{1}{2\pi}\sqrt{4 - x^2}\, \mathbf{1}_{[-2,2]}(x)\, dx.$$

We denote by σ_t the noncommutative law of m freely independent semicircular random variables which each have mean zero and variance t (that is, $\sigma_t(X_j) = 0$ and $\sigma_t(X_j^2) = t$).

These free semicircular families play the role of multivariable Gaussians in free probability. Moreover, the noncommutative laws $\{\sigma_t\}_{t \geq 0}$ form a semigroup under free convolution, that is, $\sigma_s \boxplus \sigma_t = \sigma_{s+t}$ for $s, t \geq 0$.

We denote by $\sigma_{t,N}$ the probability distribution on $M_N(\mathbb{C})_{\text{sa}}^m$ for m independent GUE matrices of normalized variance t, that is,

$$d\sigma_{t,N}(x) = \frac{1}{Z_{N,t}} \exp\left(-N \sum_{j=1}^m \frac{\operatorname{Tr}(x_j^2)}{2t}\right) dx,$$

where $Z_{N,t}$ is chosen so that $\sigma_{t,N}$ is a probability measure. It is well known that the independent GUE matrices behave in the large-N limit like freely independent semicircular random variables; in Section 3, we shall directly state and prove the specific results we will use.

2E. *Concentration and operator norm tail bounds.* The following is a standard concentration estimate for uniformly log-concave random matrix models. The best known proof goes through the log-Sobolev inequality and Herbst's argument (see [Anderson et al. 2010, §4.4.2]), although it can also be proved by directly using the heat semigroup associated to V as in [Ledoux 1992]. We state the theorem here with free probabilistic normalizations.

Theorem 2.10. *Suppose that $V : M_N(\mathbb{C})_{\text{sa}}^m \to \mathbb{R}$ is a potential such that $V(x) - (c/2)\|x\|_2^2$ is convex. Define*

$$d\mu(x) = \frac{1}{Z} \exp(-N^2 V(x))\, dx, \quad Z = \int \exp(-N^2 V(x))\, dx.$$

Suppose that $f : M_N(\mathbb{C})_{\mathrm{sa}}^m \to \mathbb{R}$ *is* K*-Lipschitz with respect to* $\|\cdot\|_2$. *Then*

$$\mu\left(x : f(x) - \int f\,d\mu \geq \delta\right) \leq e^{-cN^2\delta^2/2K^2},$$

and since the same estimate can be applied to $-f$, *we have also*

$$\mu\left(x : \left|f(x) - \int f\,d\mu\right| \geq \delta\right) \leq 2e^{-cN^2\delta^2/2K^2}.$$

In particular, this concentration estimate applies to the GUE law $\sigma_{t,N}$ with $c = 1/t$. In addition to the concentration estimate, we will also use the fact that such uniformly convex random matrix models have subgaussian moments and therefore have good tail bounds on the probability of large operator norm. The following theorem is a special case of [Hargé 2004, Theorem 1.1] and the application to random matrix models is taken from the proof of [Guionnet and Maurel-Segala 2006, Theorem 3.4].

Theorem 2.11. *Let* V *and* μ *be as in Theorem 2.10, and suppose that* $f : M_N(\mathbb{C})_{\mathrm{sa}}^m \to \mathbb{R}$ *is convex. Let* $a = \int x\,d\mu(x)$. *Then*

$$\int f(x-a)\,d\mu(x) \leq \int f(y)\,d\sigma_{c^{-1},N}(y).$$

In particular, if $\|x\|_\alpha$ *denotes the* L^α *norm from Section 2C, then for every* $\alpha \in [1,+\infty]$ *and* $\beta \in [1,+\infty)$ *we have*

$$\int \|x_j - a_j\|_\alpha^\beta\,d\mu(x) \leq \int \|y_j\|_\alpha^\beta\,d\sigma_{c^{-1},N}(y).$$

Proof. The convexity assumption on V means that μ has a log-concave density with respect to the Gaussian measure $\sigma_{c^{-1},N}(y)$. Therefore, the first claim follows from [Hargé 2004, Theorem 1.1]. The second claim follows because norms on vector spaces are convex functions, and the function $t \mapsto t^\beta$ on $[0,+\infty)$ is convex for $\beta \geq 1$. □

Corollary 2.12. *Let* $V_N : M_N(\mathbb{C})_{\mathrm{sa}} \to \mathbb{R}$ *be a function such that* $V_N(x) - (c/2)\|x\|_2^2$ *is convex and let* μ_N *be the corresponding measure. Let* $a_{N,j} = \int x_j\,d\mu_N(x)$. *Then*

$$\limsup_{N\to\infty} \int \|x_j - a_{N,j}\|\,d\mu_N(x) \leq 2c^{-1/2},$$

and

$$\mu_N\left(x : \|x_j\| \geq \int \|y_j\|\,d\mu_N(y_j) + \delta\right) \leq e^{-c\delta^2 N/2}.$$

Proof. In light of Theorem 2.11, for the first claim of the corollary, it suffices to check the special case $\sigma_{c^{-1},N}$. This special case is a standard result in random matrix theory; see for instance the proof of [Anderson et al. 2010, Theorem 2.1.22]. The second claim follows from Theorem 2.10 after we observe that the function on $M_N(\mathbb{C})_{\mathrm{sa}}^N$ given by $x \mapsto \|x_j\|_\infty$ is $N^{1/2}$-Lipschitz with respect to $\|\cdot\|_2$. □

2F. ***Semiconvex and semiconcave functions.*** We recall the following terminology and facts about semiconvex and semiconcave functions. These results are typically applied to functions from $\mathbb{R}^n \to \mathbb{R}$, but of course they hold equally well if $\mathbb{R}^n$ is replaced by a finite-dimensional real inner product space. In particular, we focus on the case of $M_N(\mathbb{C})_{\text{sa}}^m$.

A function $u : M_N(\mathbb{C})_{\text{sa}}^m \to \mathbb{R}$ is *semiconvex* if there exists some $c \in \mathbb{R}$ such that $u(x) - (c/2)\|x\|_2^2$ is convex. If this holds for some $c > 0$, then u is said to be *uniformly convex*. Similarly, $u : M_N(\mathbb{C})_{\text{sa}}^m \to \mathbb{R}$ is said to be *semiconcave* if there exists $C \in \mathbb{R}$ such that $u(x) - (C/2)\|x\|_2^2$ is concave, and it is *uniformly concave* if this holds for some $C < 0$.

Fix m and N. Let $c \leq C$ be real numbers. Then we define

$$\mathcal{E}_{m,N}(c, C) = \left\{ u : M_N(\mathbb{C})_{\text{sa}}^m \to \mathbb{R} : u(x) - \frac{c}{2}\|x\|_2^2 \text{ is convex and } u(x) - \frac{C}{2}\|x\|_2^2 \text{ is concave} \right\}.$$

We will often suppress m and N in the notation and simply write $\mathcal{E}(c, C)$. Throughout the paper, we rely on the following basic properties of functions in $\mathcal{E}(c, C)$.

Proposition 2.13. (1) *The space $\mathcal{E}(c, C)$ is closed under translation, averaging with respect to probability measures, and pointwise limits.*

(2) *A function u is in $\mathcal{E}(c, C)$ if and only if for every point $x_0 \in M_N(\mathbb{C})_{\text{sa}}^m$ there exists some $p \in M_N(\mathbb{C})_{\text{sa}}^m$ such that*

$$u(x_0) + \langle p, x - x_0 \rangle_2 + \tfrac{1}{2}c\|x - x_0\|_2^2 \leq u(x) \leq u(x_0) + \langle p, x - x_0 \rangle_2 + \tfrac{1}{2}C\|x - x_0\|_2^2.$$

(3) *In particular, if $u \in \mathcal{E}(c, C)$, then u is differentiable everywhere.*

(4) *If $u \in \mathcal{E}(c, C)$, then the gradient Du is* $\max(|c|, |C|)$*-Lipschitz with respect to $\|\cdot\|_2$.*

(5) *If $u \in \mathcal{E}(c, C)$, then*

$$c\|x - y\|_2^2 \leq \langle Du(x) - Du(y), x - y \rangle_2 \leq C\|x - y\|_2^2.$$

(6) *If $u \in \mathcal{E}(c, C)$ for some $c > 0$, then u is bounded below and achieves a global minimum at its unique critical point.*

Sketch of proof. (1) This follows from elementary computation and the fact that the same holds for the class of convex functions.

(2), (3) Suppose that $u \in \mathcal{E}(c, C)$. The convex functions $u(x) - (c/2)\|x\|_2^2$ and $(C/2)\|x\|_2^2 - u(x)$ must have supporting hyperplanes at x_0. This yields one vector p which satisfies the left inequality of (2) and another vector p' satisfying the right inequality. Then one checks that p must equal p' and this implies that u is differentiable at x_0. The converse direction of (2) follows again from the characterization of convexity using supporting hyperplanes.

(4), (5) For smooth functions in $\mathcal{E}(c, C)$, one can check these properties directly using calculus. Now consider a general $u \in \mathcal{E}(c, C)$. Let $u_n = u * \rho_n$, where ρ_n is a smooth probability density supported in the ball of radius $1/n$ around 0. Then u_n is smooth and $u_n \to u$ locally uniformly. Also, $u_n \in \mathcal{E}(c, C)$ by (1); hence Du_n is $\max(|c|, |C|)$-Lipschitz. By the Arzelà–Ascoli theorem, after passing to a subsequence, we may assume that Du_n converges locally uniformly to some F. It follows from this local uniform convergence that $F = Du$. Moreover, since (4) and (5) hold for Du_n, they also hold for Du.

(6) This is left as an exercise. □

3. Trace polynomials

In this section, we consider the algebra of trace polynomials in noncommutative variables $X_1, \dots, X_m$, first defined in [Razmyslov 1974; 1985]. As in [Rains 1997; Cébron 2013; Driver et al. 2013], we describe how trace polynomials behave under differentiation (Section 3B) and convolution with Gaussian (Section 3C). Finally, in Section 3D, we define the property of asymptotic approximability by trace polynomials for a sequence of functions on $M_N(\mathbb{C})^m_{\text{sa}}$, which is one of the key technical tools in our proof.

3A. *Definitions.*

Definition 3.1. We define the *$*$-algebra of scalar-valued trace polynomials*, or TrP^0_m, as follows. Let $\mathcal{V}$ be the vector space $\text{NCP}_m \,/\, \text{Span}(pq - qp : p, q \in \text{NCP}_m)$. We define the vector space

$$\text{TrP}^0_m = \bigoplus_{n=0}^{\infty} \mathcal{V}^{\odot n}, \tag{3-1}$$

where $\odot$ is the symmetric tensor power over $\mathbb{C}$. Then TrP^0_m forms a commutative algebra with the tensor operator $\odot$ as the multiplication. We denote the element $p_1 \odot \cdots \odot p_n$ by $\tau(p_1) \cdots \tau(p_n)$, where τ is a formal symbol.

To state the definition more suggestively, an element of TrP^0_m is a linear combination of terms of the form $\tau(p_1(X)) \cdots \tau(p_n(X))$, where $p_1, \dots, p_n$ are noncommutative polynomials in $X_1, \dots, X_m$ and τ is a formal symbol thought of as the trace. By forming a quotient vector space, we identify $\tau(pq)$ with $\tau(qp)$. The trace polynomials form a commutative $*$-algebra TrP^0_m over $\mathbb{C}$ where the $*$-operation is

$$(\tau(p_1(X)) \cdots \tau(p_n(X)))^* = \tau(p_1(X)^*) \cdots \tau(p_n(X)^*) \tag{3-2}$$

and the multiplication operation is the one suggested by the notation.

We define TrP^k_m to be the vector space

$$\text{TrP}^k_m := \text{TrP}^0_m \otimes \mathbb{C}\langle X_1, \dots, X_m \rangle^{\otimes k}.$$

We call the elements of TrP^1_m *operator-valued trace polynomials*. We use the term *trace polynomials* more generally to describe elements of TrP^k_m or tuples of elements from TrP^k_m. Note that TrP^1_m forms a $*$-algebra because it is the tensor product of two $*$-algebras.

Definition 3.2. Suppose that $\mathcal{M}$ is a von Neumann algebra with trace σ. Given $f \in \text{TrP}^1_m$ and a self-adjoint tuple $x = (x_1, \dots, x_m)$ of elements of $\mathcal{M}$, we define $f(x)$ to be the element of $\mathcal{M}$ given by replacing the formal symbols X_j and τ in f by the operator x_j and the trace σ on $\mathcal{M}$. For instance, if $f(X) = p_0(X) \otimes \tau(p_1(X)) \cdots \tau(p_n(X))$ in TrP^1_m, then

$$f(x) = p_0(x)\sigma(p_1(x)) \cdots \sigma(p_n(x)).$$

In particular, we define $f(x)$ when x is an m-tuple of self-adjoint $N \times N$ matrices by setting $\tau = \tau_N$.

Definition 3.3. If $f \in \mathrm{TrP}_m^0$ and λ is a noncommutative law, we define the evaluation $\lambda(f)$ to be the number obtained by replacing the symbol τ with λ everywhere in f. For example, if $f(X_1, X_2, X_3) = \tau(X_1)\tau(X_2X_3) + \tau(X_2^2)$, then we define

$$\lambda(f) = \lambda(f(X_1, \dots, X_m)) = \lambda(X_1)\lambda(X_2X_3) + \lambda(X_2^2).$$

Definition 3.4. We define the *degree* for elements of NCP_m and TrP_m^k as follows. If $p \in \mathrm{NCP}_m$ is a monomial $p(X_1, \dots, X_m) = X_{i_1} \cdots X_{i_d}$, then we define $\deg'(p) = d$. If $p_1, \dots, p_\ell$ and $q_1, \dots, q_k$ are noncommutative monomials, then consider the element $\tau(p_1) \cdots \tau(p_\ell) q_1 \otimes \cdots \otimes q_k \in \mathrm{TrP}_m^k$, and define

$$\deg'(\tau(p_1) \cdots \tau(p_\ell) q_1 \otimes \cdots \otimes q_k) = \deg'(p_1) \cdots \deg'(p_\ell) \deg'(q_1) \cdots \deg'(q_k).$$

For general $f \in \mathrm{TrP}_m^k$, we define the *degree*, $\deg(f)$, as the infimum of $\max(\deg'(f_1), \dots, \deg'(f_\ell))$, where $f = f_1 + \cdots + f_\ell$ and each f_j is a product of noncommutative monomials and traces of noncommutative monomials as above. Similarly, for general $f \in \mathrm{NCP}_m$, we define $\deg(f)$ as the infimum of $\max(\deg'(f_1), \dots, \deg'(f_\ell))$, where $f = f_1 + \cdots + f_\ell$ and each f_j is a noncommutative monomial.

Remark 3.5. One can check that if f is a product of monomials as above, then $\deg(f) = \deg'(f)$. Moreover, the degree makes TrP_m^0 and TrP_m^1 into graded algebras. Finally, we observe that if $f \in \mathrm{TrP}_m^0$ or TrP_m^1, then the function on $M_N(\mathbb{C})_{\mathrm{sa}}^m$ defined by $x \mapsto f(x)$ is a polynomial in the entries of $x_1, \dots, x_m$, and the degree of $x \mapsto f(x)$ with respect to the entries is bounded above by the degree of f in TrP_m^0 or TrP_m^1. None of these facts will be used in what follows, so we omit the proofs.

We also observe that there is a composition operation $(\mathrm{TrP}_m^1)^m \times (\mathrm{TrP}_m^1)^m \to (\mathrm{TrP}_m^1)^m$ defined just as one would expect from manipulations in $M_N(\mathbb{C})$. If $f, g \in (\mathrm{TrP}_m^1)^m$, we define $f(g(x))$ by substituting $g_j(x)$ as the j-th argument of f. Then we multiply elements out by treating the terms of the form $\tau(p)$ like scalars. For instance, if $f(Y_1, Y_2) = (\tau(Y_1Y_2)Y_2, Y_1 + \tau(Y_1^2)Y_2)$ and $g(X_1, X_2) = (\tau(X_1)X_2 + X_1, X_1)$, then $f \circ g(X_1, X_2) = (Z_1, Z_2)$, where

$$Z_1 = \tau([\tau(X_1)X_2 + X_1]X_1)X_1 = \tau(X_1)\tau(X_2X_1)X_1 + \tau(X_1^2)X_1$$

and

$$\begin{aligned} Z_2 &= \tau(X_1)X_2 + X_1 + X_1\tau[(\tau(X_1)X_2 + X_1)^2] \\ &= \tau(X_1)X_2 + X_1 + \tau[\tau(X_1)^2X_2^2 + \tau(X_1)X_2X_1 + \tau(X_1)X_1X_2 + X_1^2]X_1 \\ &= \tau(X_1)X_2 + X_1 + [\tau(X_1)^2\tau(X_2^2) + \tau(X_1)\tau(X_2X_1) + \tau(X_1)\tau(X_1X_2) + \tau(X_1^2)]X_1 \\ &= \tau(X_1)X_2 + X_1 + \tau(X_1)^2\tau(X_2^2)X_1 + 2\tau(X_1)\tau(X_2X_1)X_1 + \tau(X_1^2)X_1. \end{aligned}$$

One can check that composition on $(\mathrm{TrP}_m^1)^m$ is well-defined and associative. Moreover, if f and g are self-adjoint elements of $(\mathrm{TrP}_m^1)^m$, then they define functions $M_N(\mathbb{C})_{\mathrm{sa}}^m \to M_N(\mathbb{C})_{\mathrm{sa}}^m$, and the element $f \circ g \in (\mathrm{TrP}_m^1)^m$ defined abstractly will produce a function $M_N(\mathbb{C})_{\mathrm{sa}}^m \to M_N(\mathbb{C})_{\mathrm{sa}}^m$ which is the composition of the corresponding functions for f and g.

3B. ***Differentiation of trace polynomials.*** In this section, we give explicit formulas for the gradient and Laplacian of trace polynomials and in particular show that these operations have a well-defined limit as $N \to \infty$ (see [Rains 1997; Cébron 2013; Driver et al. 2013, §3]). We first recall the free difference quotients of [Voiculescu 1998].

Definition 3.6. We define the *free difference quotient* (or simply *noncommutative derivative*) $\mathcal{D}_j : \mathrm{NCP}_m \to \mathrm{NCP}_m \otimes \mathrm{NCP}_m$ by

$$\mathcal{D}_j[X_{i_1} \cdots X_{i_n}] = \sum_{k : i_k = j} X_{i_1} \cdots X_{i_{k-1}} \otimes X_{i_{k+1}} \cdots X_{i_n}.$$

We also define $\mathcal{D}_j : \mathrm{NCP}_m^{\otimes n} \to \mathrm{NCP}_m^{\otimes n+1}$ by

$$\mathcal{D}_j[p_1 \otimes \cdots \otimes p_n] = \sum_{k=1}^{n} p_1 \otimes \cdots \otimes p_{k-1} \otimes \mathcal{D}_j p_k \otimes p_{k+1} \otimes \cdots \otimes p_n.$$

Then of course $\mathcal{D}_j^k$ is a well-defined map $\mathrm{NCP}_m^{\otimes n} \to \mathrm{NCP}_m^{\otimes n+k}$.

Remark 3.7. We caution the reader that the notation used in Voiculescu's papers is ∂_j rather than $\mathcal{D}_j$. Moreover, the normalization for $\mathcal{D}_j^n f$ here differs from that of [Voiculescu 1998] by a factor of $n!$.

Definition 3.8. We define the *cyclic derivative* $\mathcal{D}_j^\circ : \mathrm{NCP}_m \to \mathrm{NCP}_m$ as the linear map given by

$$\mathcal{D}_j^\circ[X_{i_1} \cdots X_{i_n}] = \sum_{k : i_k = j} X_{i_{k+1}} \cdots X_{i_n} X_{i_1} \cdots X_{i_{k-1}}.$$

Definition 3.9. Given an algebra $\mathcal{A}$ (e.g., NCP_m), we define the n-th *hash operation* as the multilinear map $\mathcal{A}^{\otimes(n+1)} \times \mathcal{A}^{\otimes n} \to \mathcal{A}$ given by

$$(a_0 \otimes \cdots \otimes a_n) \# (b_1 \otimes \cdots \otimes b_n) = a_0 b_1 a_1 \cdots b_n a_n.$$

Example 3.10. Let $X = (X_1, X_2, X_3)$ and define $f(X) = X_1 X_2 X_1^2 X_3 X_2$. Then

$$\begin{aligned} \mathcal{D}_1 f(X) &= 1 \otimes X_2 X_1^2 X_3 X_2 + X_1 X_2 \otimes X_1 X_3 X_2 + X_1 X_2 X_1 \otimes X_3 X_2, \\ \mathcal{D}_1^\circ f(X) &= X_2 X_1^2 X_3 X_2 + X_1 X_3 X_2 X_1 X_2 + X_3 X_2 X_1 X_2 X_1, \\ \mathcal{D}_1 f(X) \# Y &= Y X_2 X_1^2 X_3 X_2 + X_1 X_2 Y X_1 X_3 X_2 + X_1 X_2 X_1 Y X_3 X_2. \end{aligned}$$

To compute $\mathcal{D}_1^2 f(X) = \mathcal{D}_1[\mathcal{D}_1 f(X)]$, we would add together the three terms

$$\begin{aligned} \mathcal{D}_1[1 \otimes X_2 X_1^2 X_3 X_2] &= 1 \otimes X_2 \otimes X_1 X_3 X_2 + 1 \otimes X_2 X_1 \otimes X_3 X_2, \\ \mathcal{D}_1[X_1 X_2 \otimes X_1 X_3 X_2] &= 1 \otimes X_2 \otimes X_1 X_3 X_2 + X_1 X_2 \otimes 1 \otimes X_3 X_2, \\ \mathcal{D}_1[X_1 X_2 X_1 \otimes X_3 X_2] &= 1 \otimes X_2 X_1 \otimes X_3 X_2 + X_1 X_2 \otimes 1 \otimes X_3 X_2. \end{aligned}$$

Now we will define several "derivative" operators on the spaces of scalar-valued and noncommutative trace polynomials which will correspond to differentiation with respect to the standard coordinates on $M_N(\mathbb{C})_{\mathrm{sa}}^m$. We begin with the gradient.

To fix notation, recall that in Section 2A we gave a canonical orthonormal basis for $M_N(\mathbb{C})_{\mathrm{sa}}$ with respect to the inner product $\langle x, y \rangle = \mathrm{Tr}(x^* y)$. Using these coordinates, we may identify $M_N(\mathbb{C})_{\mathrm{sa}}$ with $\mathbb{R}^{N^2}$

and hence identify $M_N(\mathbb{C})_{\text{sa}}^m$ with $\mathbb{R}^{mN^2}$. Similarly, we identify the complexification $\mathbb{C} \otimes M_N(\mathbb{C})_{\text{sa}}^m$ with $M_N(\mathbb{C})^m$ and with $\mathbb{C}^{mN^2}$. For $f : M_N(\mathbb{C})_{\text{sa}}^m \to \mathbb{C}$ and $x = (x_1, \dots, x_m) \in M_N(\mathbb{C})_{\text{sa}}^m$, we denote by $\nabla f(x) \in M_N(\mathbb{C})^m$ the gradient computed in these coordinates; similarly, we denote by $\nabla_j f(x) \in M_N(\mathbb{C})$ the gradient with respect to x_j computed in these coordinates.

Definition 3.11. Define the *j-th gradient operator* $\text{TrP}_m^0 \to \text{TrP}_m^1$ by

$$D_j\left[\prod_{k=1}^n \tau(p_k)\right] = \sum_{k=1}^n \mathcal{D}_j^\circ p_k \prod_{\ell \neq k} \tau(p_\ell). \tag{3-3}$$

Note that D_j is defined so as to obey the Leibniz rule (that is, it is a derivation).

Lemma 3.12. *If $f \in \text{TrP}_m^0$ is viewed as a function $M_N(\mathbb{C})_{\text{sa}}^m \to \mathbb{C}$, then we have*

$$\nabla_j[f(x)] = \frac{1}{N}[D_j f](x). \tag{3-4}$$

Similarly, for $F : M_N(\mathbb{C})_{\text{sa}}^m \to M_N(\mathbb{C})^m$, let $J_j F$ denote the Jacobian linear transformation (a.k.a. Fréchet derivative) with respect to x_j. Then for a noncommutative polynomial p, we have

$$[J_j p(x)](y) = [\mathcal{D}_j p](x) \# y, \tag{3-5}$$

and hence by the product rule for $p \in \text{NCP}_m$ and $f \in \text{TrP}_m^0$, we have

$$[J_j(pf)(x)](y) = ([\mathcal{D}_j p](x) \# y) f(x) + p(x) \tau_N([D_j f](x) y). \tag{3-6}$$

Proof. By standard computations, for a noncommutative polynomial p and $y \in M_N(\mathbb{C})_{\text{sa}}$, we have

$$[J_j p(x)](y) = [\mathcal{D}_j p](x) \# y,$$
$$\nabla_j[\tau_N(p)](x) = \frac{1}{N}[\mathcal{D}_j^\circ p](x).$$

The claims (3-4) and (3-6) now follow from the product rule. □

Next, we can define the algebraic Laplacian operators on TrP_m^0 and TrP_m^1, which correspond to computing the Laplacian on scalar-valued or vector-valued functions on $M_N(\mathbb{C})_{\text{sa}}^m$, still using the coordinates given in Section 2A.

For $f : M_N(\mathbb{C})_{\text{sa}}^m \to \mathbb{C}$, let $\Delta_j f$ be the Laplacian with respect to the coordinates of the j-th matrix x_j. Note that $\Delta f = \sum_{j=1}^m \Delta_j f$. Similarly, if $f : M_N(\mathbb{C})_{\text{sa}}^m \to M_N(\mathbb{C})$ is an operator-valued function, we define $\Delta_j f$ and Δf by applying Δ_j and Δ entrywise (as is standard notation for the Laplacian of a vector-valued function).

Motivated by (2-2) and the computation in Lemma 3.18 below, we define the map $\eta : \text{NCP}_m^{\otimes 3} \to \text{TrP}_m^1$ by

$$\eta(p_1 \otimes p_2 \otimes p_3) = p_1 p_3 \tau(p_2).$$

Definition 3.13. We define L_j and $L_{N,j} : \text{TrP}_m^0 \to \text{TrP}_m^0$ to be the unique linear operators such that

$$L_j[\tau(p)] = L_{N,j}[\tau(p)] = \tau \circ \eta[\mathcal{D}_j^2 p] \quad \text{for } p \in \text{NCP}_m \tag{3-7}$$

and such that the following product rule is satisfied:

$$L_j[f \cdot g] = L_j[f] \cdot g + f \cdot L_j[g], \tag{3-8}$$

$$L_{N,j}[f \cdot g] = L_{N,j}[f] \cdot g + f \cdot L_{N,j}[g] + \frac{2}{N^2}\tau(D_j f \cdot D_j g). \tag{3-9}$$

Then we define $L = \sum_{j=1}^m L_j$ and $L_N = \sum_{j=1}^m L_{N,j}$.

Remark 3.14. To show the existence of operators $L_{N,j}$ and L_j satisfying (3-7) and the product rule, one can define $L_{N,j}$ more explicitly as the linear operator $\mathrm{TrP}_m^0 \to \mathrm{TrP}_m^0$ given by

$$L_{N,j}[\tau(p_1)\cdots\tau(p_n)] = \sum_{k=1}^n \tau\circ\eta[\mathcal{D}^2 p_k]\cdot\prod_{i\neq k}\tau(p_i) + \frac{1}{N^2}\sum_{k=1}^n\sum_{\ell\neq k}\tau(\mathcal{D}_j^\circ p_k\cdot\mathcal{D}_j^\circ p_\ell)\prod_{i\neq k,\ell}\tau(p_i),$$

and check that this operator is well-defined and satisfies the product rule. Moreover, the uniqueness of the operator $L_{N,j}$ satisfying (3-7) and the product rule follows from the fact that TrP_m^0 is spanned by products of terms of the form $\tau(p)$ for $p \in \mathrm{NCP}_m$. The argument for the existence and uniqueness of L_j is the same.

Example 3.15. Let $X = (X_1, X_2)$. Consider $f(X) = \tau(f_1(X))\tau(f_2(X))$, where $f_1(X) = X_1X_2X_1X_3$ and $f_2(X) = X_2^2 X_1$. Then

$$D_1[\tau(f_1)] = \mathcal{D}_1^\circ f_1 = X_2X_1X_3 + X_3X_1X_2,$$
$$D_1[\tau(f_2)] = \mathcal{D}_2^\circ f_2 = X_2^2,$$

and

$$L_1[\tau(f_1)] = L_{N,1}[\tau(f_1)] = \tau\circ\eta[\mathcal{D}_1^2 f_1] = \tau[\eta[1\otimes X_2\otimes X_3]] = \tau[1\cdot X_3]\cdot\tau[X_2],$$
$$L_1[\tau(f_2)] = L_{N,1}[\tau(f_2)] = 0.$$

Therefore, we have

$$L_1[f] = L_1[\tau(f_1)]\tau(f_2) + \tau(f_1)L_1[\tau(f_2)] = \tau(X_3)\tau(X_2)\tau(X_2^2X_1) + 0,$$

$$\begin{aligned} L_{N,1}[f] &= L_{N,1}[\tau(f_1)]\tau(f_2) + \tau(f_1)L_{N,1}[\tau(f_2)] + \frac{2}{N^2}\tau[\mathcal{D}_1^\circ f_1\mathcal{D}_1^\circ f_2] \\ &= \tau(X_3)\tau(X_2)\tau(X_2^2X_1) + \frac{2}{N^2}\tau[(X_2X_1X_3 + X_3X_1X_2)X_2^2]. \end{aligned}$$

One can carry out a similar computation for $L_2[f]$ and $L_{N,2}[f]$ and thus find $L[f]$ and $L_{N,2}[f]$.

Since we will also deal with the Laplacians of matrix-valued functions on matrices, we also need to define the algebraic Laplacian on *operator-valued* trace polynomials.

Definition 3.16. We also define L_j and $L_{N,j} : \mathrm{TrP}_m^1 \to \mathrm{TrP}_m^1$ to be the unique linear operators on the space of operator-valued trace polynomials such that

$$L_j[p] = L_{N,j}[p] = \eta[\mathcal{D}_j^2 p] \quad \text{for } p \in \mathrm{NCP}_m \tag{3-10}$$

and the following product rule is satisfied for $p \in \mathrm{NCP}_m$ and $f \in \mathrm{TrP}_m^0$:

$$L_j[p \cdot f] = L_j[p] \cdot f + p \cdot L_j[f], \tag{3-11}$$

$$L_{N,j}[p \cdot f] = L_{N,j}[p] \cdot f + p \cdot L_{N,j}[f] + \frac{2}{N^2} \mathcal{D}_j p \# D_j f, \tag{3-12}$$

where $L_j[f]$ and $L_{N,j}[f]$ are given by Definition 3.13. Then we define $L = \sum_{j=1}^m L_j$ and $L_N = \sum_{j=1}^m L_{N,j}$.

Remark 3.17. The argument for the existence and uniqueness of the operators L_j and $L_{N,j}$ on TrP_m^1 is similar to the argument for TrP_m^0, only it relies on the previous scalar-valued case since the scalar-valued case was used in the product rule.

Lemma 3.18. *Let $f \in \mathrm{TrP}_m^0$. Viewing f as a function $M_N(\mathbb{C})_{\mathrm{sa}}^m \to \mathbb{C}$, we have*

$$\Delta_j f(x) = N[L_{N,j} f](x), \quad \Delta f(x) = N[L_N f](x). \tag{3-13}$$

The same formula holds if $f \in \mathrm{TrP}_m^1$ and f is viewed as a function $M_N(\mathbb{C})_{\mathrm{sa}}^m \to M_N(\mathbb{C})$.

Proof. We begin with the special case of computing the Laplacian of $p \in \mathrm{NCP}_m$ (as a *matrix-valued* function). To differentiate, we use the basis $\mathcal{B}_N$ given by (2-1). Note that

$$\begin{aligned}\Delta_j p(x) &= \sum_{b \in \mathcal{B}_N} \frac{d^2}{dt^2}\bigg|_{t=0} f(x_1, \dots, x_{j-1}, x_j + tb, x_{j+1}, \dots, x_m) \\ &= \sum_{b \in \mathcal{B}_N} \mathcal{D}_j^2 p(x) \# (b \otimes b) = N[\eta(\mathcal{D}_j^2 p)](x) = [L_{N,j} p](x),\end{aligned}$$

where the second-to-last equality follows from (2-2).

Next, we consider the case of computing the Laplacian of $\tau_N(p)$ (as a *scalar-valued* function) for $p \in \mathrm{NCP}_m$. Since τ_N is a linear map $M_N(\mathbb{C}) \to \mathbb{C}$, we have

$$\Delta_j[\tau_N(p(x))] = \tau_N(\Delta_j p(x)),$$

where the Laplacian Δ_j on the left-hand side is applied to a scalar-valued function and on the right-hand side it is applied to a matrix-valued function. Therefore, it follows from the previous computation that

$$\Delta_j[\tau_N(p(x))] = N \tau_N([\eta(\mathcal{D}_j^2 p)](x)) = [L_{N,j}[\tau(p)]](x).$$

For the general case of scalar-valued trace polynomials, recall that the vector space of trace polynomials is spanned by elements of the form $f = \tau(p_1) \cdots \tau(p_N)$, where $p_j \in \mathrm{NCP}_m^0$. Let $f_j = \tau(p_j) \in \mathrm{TrP}_m^0$. The Laplacian Δ_j of a product of functions can be computed using the product rule of differentiation as

$$\Delta_j f(x) = \sum_{j=1}^n N[L_{N,j} f_k](x) \prod_{i \neq k} \tau_N(f_i(x)) + \sum_{k=1}^n \sum_{\ell \neq k} \mathrm{Tr}(\nabla_j f_k(x) \nabla_j f_\ell(x)) \prod_{i \neq k, \ell} f_i(x).$$

The special case proved above shows that $\Delta_j[f_k(x)] = N[L_{N,j} f](x)$. Moreover, by (3-4), we have $\nabla_j[f_k(x)] = (1/N)[D_j f_k](x)$. Thus, we have

$$\Delta_j f(x) = \sum_{k=1}^n N[L_{N,j} f_k](x) \prod_{i \neq k} f_i(x) + \frac{1}{N} \sum_{k=1}^n \sum_{\ell \neq k} \tau_N([D_j f_k](x)[D_j f_\ell](x)) \prod_{i \neq k, \ell} f_i(x).$$

Because of the product rule in the definition of $L_{N,j}$, the right-hand side equals $N[L_{N,j} f](x)$. This completes the proof of (3-13) in the scalar-valued case. The proof for the operator-valued case is similar, using the cases proved above, as well as (3-4) and (3-6). □

Corollary 3.19. *Let $f \in \mathrm{TrP}_m^0$ or TrP_m^1. If we view f as a function on $M_N(\mathbb{C})_{\mathrm{sa}}^m$, then $(1/N)\Delta f$ is a trace polynomial of lower degree than f, and we have coefficientwise*

$$\lim_{N\to\infty} \frac{1}{N}\Delta f(x) = \lim_{N\to\infty} L_N f(x) = Lf(x).$$

Remark 3.20. We have shown that if f is a scalar-valued trace polynomial, then viewed as a map $M_N(\mathbb{C})_{\mathrm{sa}}^m \to \mathbb{C}$, we have

$$Du = N\nabla f, \quad L_N f = \frac{1}{N}\Delta f.$$

Therefore, in the rest of the paper, we will freely write Df and $L_N f$ for $N\nabla f$ and $(1/N)\Delta f$ for general functions $f : M_N(\mathbb{C})_{\mathrm{sa}}^m \to \mathbb{C}$. The same considerations apply to the Laplacian for operator-valued trace polynomials, viewed as maps $M_N(\mathbb{C})_{\mathrm{sa}}^m \to M_N(\mathbb{C})$.

3C. *Convolution of trace polynomials and gaussians.* Let $f \in \mathrm{TrP}_m^0$ or $f \in \mathrm{TrP}_m^1$. Then viewing f as a function defined on $M_N(\mathbb{C})_{\mathrm{sa}}^m$, we may define the convolution of f with the probability measure $\sigma_{t,N}$ (the law of an m-tuple of independent GUE). This is equivalent to the classical convolution of f with the function $M_N(\mathbb{C})_{\mathrm{sa}}^m \to \mathbb{R}$ giving the density of the measure $\sigma_{t,N}$. Moreover, $f_t = f * \sigma_{t,N}$ is the solution to the heat equation with initial condition f, or more precisely

$$\partial_t f_t = \frac{1}{2N}\Delta f_t.$$

(The integral formula for the solution to the heat equation with the Laplacian Δ is well known [Evans 2010, §2.3], and to solve the equation with $(2N)^{-1}\Delta$ one renormalizes time by a factor of $(2N)^{-1}$, and this corresponds precisely to our normalizations in the definition of $\sigma_{N,t}$. We leave this computation to the reader.)

We showed in the last subsection that $L_N = (1/N)\Delta$ on trace polynomials is given by a purely algebraic computation. Moreover, examining the construction of L_N, one can see that it maps trace polynomials of degree $\leq d$ to trace polynomials of degree $\leq d$. We can view L_N and L as linear transformations on the finite-dimensional vector space of trace polynomials of degree $\leq d$ and define $\exp(tL_N/2)$ and $\exp(tL/2)$ by the matrix exponential.

Because this holds for any d, we know that $\exp(tL_N/2)$ and $\exp(tL/2)$ define linear transformations $\mathrm{TrP}_m^0 \to \mathrm{TrP}_m^0$ and $\mathrm{TrP}_m^1 \to \mathrm{TrP}_m^1$. Moreover, a standard computation shows that $f_t = \exp(tL_N/2)f$ satisfies the heat equation $\partial_t f_t = L_N f_t/2$. These observations, together with Corollary 3.19 yield the following.

Lemma 3.21. *Let f be a trace polynomial in TrP_m^0 or TrP_m^1. Then we have*

$$\sigma_{t,N} * f(x) = \left[\exp\left(\frac{tL_N}{2}\right) f\right](x), \tag{3-14}$$

with $\deg(\exp(tL_N/2)f) \leq \deg(f)$, *and we have*

$$\lim_{N\to\infty} \exp\left(\frac{tL_N}{2}\right) f = \exp\left(\frac{tL}{2}\right) f \quad \textit{coefficientwise.} \tag{3-15}$$

Example 3.22. Let $X = (X_1, \dots, X_m)$ and define $f(X) = \sum_{j=1}^m X_j^2$. Note that $\mathcal{D}_j^2[f(X)] = 2(1 \otimes 1 \otimes 1)$ for each j, and hence $L[\tau(f)] = 2m = L_N[\tau(f)]$. We also have $\mathcal{D}_j^\circ f = 2X_j$. Hence,

$$L[\tau(f)^2] = 2L[\tau(f)]\tau(f) = 4m\tau(f),$$
$$L_N[\tau(f)^2] = 2L[\tau(f)]\tau(f) + 2\sum_{j=1}^m \tau(\mathcal{D}_j^\circ f \cdot \mathcal{D}_j^\circ f) = 4m\tau(f) + \frac{8m}{N^2}\tau(f).$$

Therefore, $(L/2)[\tau(f)^2] = 2m\tau(f)$ and $(L/2)[\tau(f)] = m$. Thus, the span of $\tau(f)^2$, $\tau(f)$, and 1 is invariant under the operator $(L/2)$, and $(L/2)$ is given by a nilpotent matrix on this subspace. Direct computation then shows that

$$e^{-tL/2}[\tau(f)^2] = \tau(f)^2 + 2mt\tau(f) + m^2t^2.$$

A similar computation shows that

$$e^{-tL_N/2}[\tau(f)^2] = \tau(f)^2 + 2m\Big(1 + \frac{2}{N^2}\Big)t\tau(f) + m^2\Big(1 + \frac{2}{N^2}\Big)\frac{t^2}{2}.$$

Thus, as $N \to +\infty$, we have $e^{-tL_N/2}[\tau(f)^2] \to e^{-tL/2}[\tau(f)^2]$.

The probabilistic interpretation of $f * \sigma_{t,N} = \exp(tL_N/2)f$, which follows from a standard computation, is that $\sigma_{t,N} * f(x)$ is the expectation of $f(x + t^{1/2}Y)$, where Y is an m-tuple of independent GUE of variance 1. Moreover, for every probability measure μ on $M_N(\mathbb{C})_{\mathrm{sa}}^m$ with finite moments, we have

$$\int f(x)\, d(\mu * \sigma_{t,N})(x) = \int (\sigma_{t,N} * f)(x)\, d\mu(x) = \int \Big[\exp\Big(\frac{tL_N}{2}\Big) f\Big](x)\, d\mu(x). \tag{3-16}$$

In the free setting, the operator $\exp(tL/2)$ has a similar relationship with the free convolution with σ_t. This fact is standard in free probability, but because we need it for Lemmas 3.28 and 7.4 below, we include a sketch of the proof here.

Lemma 3.23. *Let $\lambda \in \Sigma_{n,R}$ be a noncommutative law. Then for any trace polynomial $f \in \mathrm{TrP}_m^0$, we have*

$$\lambda \boxplus \sigma_t(f) = \lambda\Big(\exp\Big(\frac{tL}{2}\Big) f\Big). \tag{3-17}$$

Proof. Because free convolution with σ_t forms a semigroup and $\exp(tL/2)$ is also a semigroup, it suffices to prove that

$$\frac{d}{dt}\Big|_{t=0} \lambda \boxplus \sigma_t(f) = \frac{\lambda}{2}(Lf).$$

By the product rule, it suffices to handle the case of $f = \tau(p)$ for $p \in \mathrm{NCP}_m$ by showing that

$$\frac{d}{dt}\Big|_{t=0} \lambda \boxplus \sigma_t(p) = \frac{\lambda}{2}(\eta(\mathcal{D}_j^2 p)).$$

Let $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_m)$ be freely independent m-tuples of self-adjoint elements of a tracial von Neumann algebra $(\mathcal{M}, \tau)$, such that the law of x is λ and the law of y is σ_1. We want to compute $(d/dt)|_{t=0}\tau(p(x+t^{1/2}y))$. But note that

$$p(x+t^{1/2}y) = p(x) + t^{1/2}\sum_{j=1}^m \mathcal{D}_j p(x) \# y_j + \frac{t}{2}\sum_{j,k=1}^m \mathcal{D}_j\mathcal{D}_k p(x) \# (y_j \otimes y_k) + O(t^{3/2}).$$

A moment computation with free independence shows that the terms of order $t^{1/2}$ have expectation zero, and so do the terms of order t with $j \neq k$. We are left with

$$\frac{d}{dt}\bigg|_{t=0} \tau(p(x+t^{1/2}y)) = \frac{1}{2}\sum_{j=1}^n \tau(\mathcal{D}_j^2 p(x) \# (y_j \otimes y_j)),$$

which using freeness evaluates to $\frac{1}{2}\sum_{j=1}^n \tau(\eta(\mathcal{D}_j^2 p(x))) = \tau(Lp(x)/2)$. □

3D. ***Asymptotic approximation by trace polynomials.*** Now we are ready to define the approximation property which captures the asymptotic behavior of functions on $M_N(\mathbb{C})_{\mathrm{sa}}^m$.

Definition 3.24. A sequence of functions $\phi_N : M_N(\mathbb{C})_{\mathrm{sa}}^m \to M_N(\mathbb{C})^m$ is said to be *asymptotically approximable by trace polynomials* if for every $\epsilon > 0$ and $R > 0$ there exists some $f \in (\mathrm{TrP}_m^1)^m$ (an m-tuple of operator-valued trace polynomials) such that

$$\limsup_{N\to\infty} \sup_{\|x\|_\infty \le R} \|\phi_N(x) - f(x)\|_2 \le \epsilon.$$

In this case, we call f an (ϵ, R)*-approximation of* $\{\phi_N\}$. We make the same definitions for functions $\phi_N : M_N(\mathbb{C})_{\mathrm{sa}}^m \to \mathbb{C}$, except that we use scalar-valued trace polynomials (elements of TrP_m^0) and apply the absolute value rather than the 2-norm.

Observation 3.25. If $f \in (\mathrm{TrP}_m^1)^m$ and if f_N denotes the map $M_N(\mathbb{C})_{\mathrm{sa}}^m \to M_N(\mathbb{C})^m$ given by $x \mapsto f(x)$, then f_N is asymptotically approximable by trace polynomials. Also, asymptotically approximable sequences form a vector space over $\mathbb{C}$.

Observation 3.26. Let $\{\phi_N^{(\ell)}\}_{N,\ell\in\mathbb{N}}$ be a sequence of functions where $\phi_N^{(\ell)} : M_N(\mathbb{C})_{\mathrm{sa}}^m \to M_N(\mathbb{C})^m$. Suppose that $\{\phi_N\}$ is another sequence such that for every $R > 0$

$$\lim_{\ell\to\infty} \limsup_{N\to\infty} \sup_{\|x\|_\infty \le R} \|\phi_N^{(\ell)}(x) - \phi_N(x)\|_2 = 0.$$

If $\{\phi_N^{(\ell)}\}_{N\in\mathbb{N}}$ is asymptotically approximable by trace polynomials for each ℓ, then so is $\{\phi_N\}_{N\in\mathbb{N}}$. The same holds in the case of scalar-valued functions and scalar-valued trace polynomials.

Lemma 3.27. *Let $\phi_N, \psi_N : M_N(\mathbb{C})_{\mathrm{sa}}^m \to M_N(\mathbb{C})_{\mathrm{sa}}^m$. Suppose that $\{\phi_N\}$ and $\{\psi_N\}$ are both asymptotically approximable by trace polynomials, and furthermore suppose that $\{\phi_N\}_{N\in\mathbb{N}}$ is uniformly Lipschitz in $\|\cdot\|_2$, that is, for some $K > 0$,*

$$\|\phi_N(x) - \phi_N(y)\|_2 \le K\|x-y\|_2 \quad \textit{for all } x, y, \textit{ for all } N.$$

Then $\{\phi_N \circ \psi_N\}$ is asymptotically approximable by trace polynomials.

Proof. It is straightforward to check that if $\{\phi_N\}$ is a sequence of functions that map self-adjoint tuples to *self-adjoint* tuples and if $(f_1, \dots, f_m)$ is an (ϵ, R)-approximation of $\{\phi_N\}$, then so is $\frac{1}{2}(f_1+f_1^*, \dots, f_m+f_m^*)$. Thus, we may assume without loss of generality that the operator-valued trace polynomials used in our approximations for $\{\phi_N\}$ and $\{\psi_N\}$ are self-adjoint, so it makes sense to compose them with ϕ_N or ψ_N.

Choose $\epsilon > 0$ and $R > 0$. Choose an m-tuple of self-adjoint trace polynomials g which is an $(\epsilon/(2K), R)$-approximation of $\{\psi_N\}$. Since g is a trace polynomial, there exists some $R' > 0$ such that for any tuple x of self-adjoint matrices of any size, we have

$$\|x\|_\infty \le R \quad \Longrightarrow \quad \|g(x)\|_\infty \le R'.$$

Now because ϕ_N is asymptotically approximable by trace polynomials, we can choose a polynomial f which is an $(\epsilon/2, R')$-approximation of $\{\phi_N\}$. Now we observe that when $\|x\|_\infty \le R$ (hence $\|g(x)\|_\infty \le R'$), we have

$$\begin{aligned}\|\phi_N \circ \psi_N(x) - f \circ g(x)\|_2 &\le \|\phi_N \circ \psi_N(x) - \phi_N \circ g(x)\|_2 + \|\phi_N \circ g(x) - f \circ g(x)\|_2 \\ &\le K \sup_{\|x\|_\infty \le R} \|\psi_N(x) - g(x)\|_2 + \sup_{\|y\|_\infty \le R'} \|\phi_N(y) - f(y)\|_2.\end{aligned}$$

Therefore,

$$\limsup_{N\to\infty} \sup_{\|x\|_\infty \le R} \|\phi_N \circ \psi_N(x) - f \circ g(x)\|_2 \le K \cdot \frac{\epsilon}{2K} + \frac{\epsilon}{2} = \epsilon. \qquad \square$$

Lemma 3.28. *Suppose that $\phi_N : M_N(\mathbb{C})_{\text{sa}}^m \to M_N(\mathbb{C})_{\text{sa}}^m$ is asymptotically approximable by trace polynomials and that*

$$\|\phi_N(x)\|_2 \le A\left(1 + \sum_j \tau_N(x_j^{2n})\right) \tag{3-18}$$

for some $A > 0$ and some integer $n \ge 0$. If $\{\phi_N\}$ is asymptotically approximable by trace polynomials, then so is $\{\phi_N * \sigma_{t,N}\}$.

Proof. Fix $R > 0$ and $\epsilon > 0$. Choose a trace polynomial f which is an $(\epsilon, R + 3t^{1/2})$ approximation for $\{\phi_N\}$. Now for x with $\|x\|_\infty \le R$, we estimate

$$\|\sigma_{t,N} * \phi_N(x) - \sigma_{t,N} * f(x)\|_2 \le \int \|\phi_N(x+y) - f(x+y)\|_2 \, d\sigma_{t,N}(y).$$

We break this integral into two pieces: The integral over the region where $\|y\|_\infty \le 3t^{1/2}$ is bounded by ϵ as $N \to \infty$ by our choice of f. Furthermore, we claim that the integral over the region where $\|y\|_\infty > 3t^{1/2}$ vanishes as $N \to \infty$. Using assumption (3-18) and the fact that f is a trace polynomial, we see that there exists a $C > 0$ and integer $d > 0$, depending only on R, A, n, and f, such that

$$\sup_{\|x\|_\infty \le R} [\|\phi_N(x+y)\|_2 + \|f(x+y)\|_2] \le C\left(1 + \sum_j \tau_N(y_j^{2d})\right).$$

Therefore, we have

$$\int_{\|y\|_\infty \ge 3t^{1/2}} \|\phi_N(x+y) - f(x+y)\|_2 \, d\sigma_{t,N}(y) \le C \int_{\|y\|_\infty \ge 3t^{1/2}} \left(1 + \sum_j \tau_N(y_j^{2d})\right) d\sigma_{t,N}(y).$$

This vanishes as $N \to \infty$ by Corollary 2.12 applied to the GUE. Therefore, we have

$$\limsup_{N\to\infty} \sup_{\|x\|_\infty \le R} \|\sigma_{t,N} * \phi_N(x) - \sigma_{t,N} * f(x)\|_2 \le \epsilon.$$

On the other hand, by Lemma 3.21, we have $\sigma_{t,N} * f = \exp(tL_N/2) f \to \exp(tL/2) f$ coefficientwise, and therefore,

$$\limsup_{N\to\infty} \sup_{\|x\|_\infty \le R} \|\sigma_{t,N} * f(x) - \Big[\exp\Big(\frac{tL}{2}\Big) f\Big](x)\|_2 = 0,$$

so that

$$\limsup_{N\to\infty} \sup_{\|x\|_\infty \le R} \|\sigma_{t,N} * \phi_N(x) - \Big[\exp\Big(\frac{tL}{2}\Big) f\Big](x)\|_2 \le \epsilon. \qquad \square$$

Lemma 3.29. *Suppose that $\phi_N : M_N(\mathbb{C})_{\mathrm{sa}}^m \to \mathbb{C}$ and suppose that $\{D\phi_N\} = \{N\nabla\phi_N\}$ is asymptotically approximable by trace polynomials and that $\phi_N(0) = 0$. Then $\{\phi_N\}$ is asymptotically approximable by trace polynomials.*

Proof. Given a trace polynomial $F \in (\mathrm{TrP}_m^1)^m$, we can define

$$f(X) = \int_0^1 \tau(F(tX)X)\, dt$$

in TrP_m^0. Then we have

$$\begin{aligned} \sup_{\|x\|_\infty \le R} |\phi_N(x) - f(x)| &= \sup_{\|x\|_\infty \le R} \left| \int_0^1 \langle D\phi_N(tx) - F(tx), x\rangle_2 \, dt \right| \\ &\le R \sup_{\|y\|_\infty \le R} \|N\nabla\phi_N(y) - F(y)\|_2. \qquad \square \end{aligned}$$

4. Convergence of moments

Our goal in this section is prove the following theorem. The convergence of moments is related to [Guionnet and Shlyakhtenko 2009, Theorem 4.4; Dabrowski et al. 2016, Proposition 50 and Theorem 51; Dabrowski 2016, Theorem 4.4], and we include versions of standard concentration estimates (see Section 2E) in the statement.

Theorem 4.1. *Let $V_N : M_N(\mathbb{C})_{\mathrm{sa}}^m \to \mathbb{R}$ be a sequence of potentials such that $V_N(x) - (c/2)\|x\|_2^2$ is convex and $V_N(x) - (C/2)\|x\|_2^2$ is concave. Let μ_N be the associated measure. Suppose that the sequence $\{DV_N\}$ is asymptotically approximable by trace polynomials, and assume that*

$$M = \limsup_{N\to\infty} \max_j \left\| \int (x_j - \tau_N(x_j)1)\, d\mu_N(x) \right\| < +\infty, \tag{4-1}$$

where 1 denotes the $N \times N$ identity matrix.

(1) *We have the following bounds on the operator norm: if $R_N = \max_j \int \|x_j\|\, d\mu_N(x)$, then*

$$\begin{aligned} \limsup_{N\to\infty} R_N &\le \frac{2}{c^{1/2}} + \frac{1}{c} \limsup_{N\to\infty} \max_j \left| \int \tau_N(x_j)\, d\mu_N(x) \right| + M \\ &\le \frac{2}{c^{1/2}} + \frac{1}{c} \limsup_{N\to\infty} \|DV_N(0)\|_2 + \frac{C-c}{2c^{3/2}} + M, \end{aligned}$$

and as a consequence of concentration we have for each j that

$$\mu_N(\|x_j\| \geq R_N + \delta) \leq e^{-cN\delta^2/2}.$$

(2) *There exists a noncommutative law $\lambda \in \Sigma_{m,R_*}$, where $R_* = \limsup_{N\to\infty} R_N$, such that for every noncommutative polynomial p*

$$\lim_{N\to\infty} \int \tau_N(p(x))\, d\mu_N(x) = \lambda(p).$$

(3) *The sequence $\{\mu_N\}$ exhibits exponential concentration around λ in the sense that, for every $R > 0$ and every neighborhood $\mathcal{U}$ of λ in Σ_m,*

$$\limsup_{N\to\infty} \frac{1}{N^2} \log \mu_N(x \in M_N(\mathbb{C})_{\mathrm{sa}}^m : \|x\|_\infty \leq R, \lambda_x \notin \mathcal{U}) < 0.$$

Remark 4.2. The rather artificial hypothesis that $\limsup_{N\to\infty} \max_j \left\| \int (x_j - \tau_N(x_j))\, d\mu_k(x) \right\| < +\infty$ is trivially satisfied if either μ_N has expectation zero or μ_N is invariant under unitary conjugation and hence $\int x_j\, d\mu_N(x)$ is equal to $\int \tau_N(x_j)\, d\mu_N(x)$ times the identity matrix.

We have already seen in Section 2E that concentration estimates and operator norm tail bounds are standard. To prove that the moments converge, something more is needed; indeed, the only assumption relating the measures μ_N for different values of N is the fact that DV_N is asymptotically approximable by trace polynomials. But even if DV_N is given by the same "trace analytic-function" for different values of N, it is not immediate that the measure would concentrate in the same regions for matrices of different sizes.

To prove convergence of moments, we want to express $\int u\, d\mu_N$ in terms of DV_N for a Lipschitz function u. One of the standard techniques is to show μ_N is the unique stationary distribution for a process X_t that satisfies the SDE

$$dX_t = dY_t - \frac{DV_N(X_t)}{2}\, dt, \tag{4-2}$$

where Y_t is a GUE Brownian motion. This machinery lies behind the log-Sobolev inequality and concentration results, as well as earlier theorems about convergence of moments for general convex potentials.

Specifically, Dabrowski, Guionnet, and Shylakhtenko [Dabrowski et al. 2016, Proposition 5] used the free version of this SDE to show that for a noncommutative potential V satisfying certain convexity assumptions, there exists a free Gibbs law for V which is the unique stationary distribution. As an application, they showed convergence of moments for random matrix models given by $V_N = V$ [Dabrowski et al. 2016, Proposition 50 and Theorem 51], essentially a special case of our Theorem 4.1.

Dabrowski [2016, Theorem 4.4] was able to show convergence of moments under weaker convexity assumptions by constructing the solution to the free SDE as an ultralimit of the finite-dimensional solutions. Our theorem has convexity assumptions similar to Dabrowski's, but we consider a more general sequence of potentials V_N. Like Dabrowski, we analyze the free case by taking the limit of finite-dimensional results, but we use deterministic rather than stochastic methods.

Instead of the solving the SDE, we study the associated semigroup $T_t^{V_N}$, acting on Lipschitz functions u, given by

$$T_t^{V_N} u(x) = E_x[u(X_t)],$$

where X_t is the process solving the SDE (4-2) with initial condition x. The semigroup provides the solution to a certain PDE; that is, if $u(x,t) = T_t u_0(x)$, then we have

$$\partial_t u = \frac{1}{2N}\Delta u - \frac{N}{2}\nabla V_N \cdot \nabla u = \frac{L_N u}{2} - \frac{\langle DV_N, Du\rangle_2}{2}.$$

The semigroup $T_t^{V_N}$ will decrease the Lipschitz norms of functions and thus, if u is Lipschitz, then $T_t^{V_N} u$ will converge to $\int u \, d\mu_N$ as $t \to \infty$.

Solving the differential equation and taking $t \to \infty$ provides a way to evaluate $\int u \, d\mu_N$ in terms of DV_N. We will describe a construction of the semigroup T_t^V through iterating simpler operations (Section 4A), and then we will show (Lemma 4.10) that the iteration procedure preserves approximability by trace polynomials and hence conclude that $\lim_{N\to\infty} \int u \, d\mu_N$ exists.

4A. ***Iterative construction of the semigroup.*** To simplify notation in this section, we fix N and fix a potential $V : M_N(\mathbb{C})_{\text{sa}}^m \to \mathbb{R}$ such that $V(x) - (c/2)\|x\|_2^2$ is convex and $V(x) - (C/2)\|x\|_2^2$ is concave for some $0 < c \le C$. We also write T_t rather than T_t^V.

We will construct T_t by combining two simpler semigroups corresponding to the stochastic and deterministic terms of $dY_t - (DV/2)(X_t)\,dt$. Recall that the solution to the heat equation $\partial_t u = (2N)^{-1}\Delta u$ with initial data u_0 is given by the heat semigroup:

$$P_t u_0(x) = \int u_0(x+y)\, d\sigma_{t,N}(y).$$

Meanwhile, the solution to $\partial_t u = -\frac{1}{2}\langle DV, Du\rangle_2$ with initial data u_0 is given by

$$S_t u_0(x) = u_0(W(x,t)),$$

where $W(x,t)$ is the solution to the ODE

$$\partial_t W(x,t) = -\tfrac{1}{2} DV(W(x,t)), \quad W(x,0) = x. \tag{4-3}$$

We want to define $T_t = \lim_{n\to\infty}(P_{t/n} S_{t/n})^n$. This is motivated by Trotter's product formula which asserts that $e^{t(A+B)} = \lim_{n\to\infty}(e^{tA/n} e^{tB/n})^n$ for nice enough self-adjoint operators A and B (see [Trotter 1959; Kato 1978; Simon 1979, pp. 4–6]). In our case, we must show that $(P_{t/n} S_{t/n})^n$ converges as $n \to \infty$ and derive dimension-independent error bounds.

We use the following basic properties of the semigroups P_t and S_t. Here if $u : M_N(\mathbb{C})_{\text{sa}}^m \to \mathbb{C}$, then $\|u\|_{\text{Lip}}$ denotes the Lipschitz norm with respect to the *normalized* L^2 metric $\|\cdot\|_2$ on $M_N(\mathbb{C})_{\text{sa}}^m$ and $\|u\|_{L^\infty}$ denotes the standard L^∞ norm. We are only concerned with Lipschitz functions, so in the following estimates, the reader may always assume u is Lipschitz, but of course $\|u\|_{L^\infty}$ may be infinite for Lipschitz functions.

Lemma 4.3. (1) $\|P_t u\|_{L^\infty} \le \|u\|_{L^\infty}$.

(2) $\|P_t u\|_{\text{Lip}} \le \|u\|_{\text{Lip}}$.

(3) $\|P_t u - u\|_{L^\infty} \le m^{1/2} t^{1/2} \|u\|_{\text{Lip}}$.

Proof. (1) and (2) follow from the fact that $P_t u$ is u convolved with a probability measure. To prove (3), suppose $\|u\|_{\mathrm{Lip}} < +\infty$. Then

$$\begin{aligned}|P_t u(x) - u(x)| &= \left|\int (u(x+y) - u(x))\, d\sigma_{t,N}(y)\right| \\ &\leq \int |u(x+y) - u(x)|\, d\sigma_{t,N}(y) \\ &\leq \|u\|_{\mathrm{Lip}} \int \|y\|_2\, d\sigma_{t,N}(y).\end{aligned}$$

Meanwhile,

$$\int \|y\|_2\, d\sigma_{t,N}(y) \leq \left(\int 1\, d\sigma_{t,N}(y)\right)\left(\int \|y\|_2^2\, d\sigma_{t,N}(y)\right)^{1/2} = (mt)^{1/2},$$

since y is an m-tuple $(y_1, \dots, y_m)$ and $\int \tau_N(y_j^2)\, d\sigma_{t,N}(y) = t$ for each j. □

Lemma 4.4. (1) *The solution $W(x,t)$ to* (4-3) *exists for all t.*

(2) $\|W(x,t) - W(y,t)\|_2 \leq e^{-ct/2}\|x - y\|_2$.

(3) $\|W(x,t) - x\|_2 \leq (t/2)\|DV(x)\|_2$.

(4) $\|(W(x,t) - x) - (W(y,t) - y)\|_2 \leq (C/c)(1 - e^{-ct/2})\|x - y\|_2$.

(5) $\|S_t u\|_{\mathrm{Lip}} \leq e^{-ct/2}\|u\|_{\mathrm{Lip}}$.

(6) $\|S_t u\|_{L^\infty} \leq \|u\|_{L^\infty}$ *if u is continuous.*

Proof. (1) The convexity and semiconcavity assumptions on V imply that DV is C-Lipschitz (see Proposition 2.13(4)) and therefore global existence of the solution follows from the Picard–Lindelöf theorem.

(2) Let $\widetilde{V}(x) = V(x) - (c/2)\|x\|_2^2$. By Proposition 2.13(5),

$$\langle DV(x) - DV(y), x - y\rangle_2 \geq c\|x - y\|_2^2.$$

Now observe that

$$\begin{aligned}\frac{d}{dt}\|W(x,t) - W(y,t)\|_2^2 &= -\langle DV(W(x,t)) - DV(W(y,t)), W(x,t) - W(y,t)\rangle_2 \\ &\leq -c\|W(x,t) - W(y,t)\|_2^2,\end{aligned}$$

and hence by Grönwall's inequality, $\|W(x,t) - W(y,t)\|_2^2 \leq e^{-ct}\|W(x,0) - W(y,0)\|_2^2 = e^{-ct}\|x - y\|_2^2$.

(3) Note that

$$\begin{aligned}\frac{d}{dt}\|W(x,t) - x\|_2^2 &= -\langle DV(W(x,t)), W(x,t) - x\rangle_2 \\ &= -\langle DV(W(x,t)) - DV(x), W(x,t) - x\rangle_2 - \langle DV(x), W(x,t) - x\rangle_2 \\ &\leq \|DV(x)\|_2\|W(x,t) - x\|_2.\end{aligned}$$

Meanwhile, $\|W(x,t) - x\|_2$ is Lipschitz in t and hence differentiable almost everywhere and we have

$$\frac{d}{dt}\|W(x,t) - x\|_2^2 = 2\|W(x,t) - x\|_2 \frac{d}{dt}\|W(x,t) - x\|_2.$$

Thus, we have

$$\frac{d}{dt}\|W(x,t)-x\|_2 \leq \tfrac{1}{2}\|DV(x)\|_2,$$

which proves (3).

(4) We observe that

$$\begin{aligned}\|(W(x,t)-x)-(W(y,t)-y)\|_2 &\leq \frac{1}{2}\int_0^t \|DV(W(x,s))-DV(W(y,s))\|_2\,ds\\ &\leq \frac{C}{2}\int_0^t \|W(x,s)-W(y,s)\|_2\,ds\\ &\leq \frac{C}{2}\int_0^t e^{-cs/2}\|x-y\|_2\,ds = \frac{C}{c}(1-e^{-ct/2})\|x-y\|_2.\end{aligned}$$

(5) This follows from (2).

(6) This is immediate because $S_t u$ is u precomposed with another function. □

Now we combine P_t and S_t as in Trotter's formula, except that for technical convenience we define our approximations using dyadic time intervals rather than subdividing $[0,t]$ into intervals of size t/n. We set $\mathbb{N}=\{1,2,3,\dots\}$ and $\mathbb{N}_0=\{0,1,2,\dots\}$. We also denote by $\mathbb{Q}_2^+=\bigcup_{\ell\geq 0}2^{-\ell}\mathbb{N}_0$ the nonnegative dyadic rationals.

Lemma 4.5. *For $t\in 2^{-\ell}\mathbb{N}_0$, define*

$$T_{t,\ell}u=(P_{2^{-\ell}}S_{2^{-\ell}})^{2^\ell t}u.$$

For $t\in\mathbb{Q}_2^+$, the limit $T_t u := \lim_{\ell\to\infty} T_{t,\ell}u$ exists and we have

$$\|T_{t,\ell}u - T_t u\|_{L^\infty} \leq \frac{Cm^{1/2}}{c(2-2^{1/2})}2^{-\ell/2}\|u\|_{\mathrm{Lip}}.$$

We also have $\|T_t u\|_{\mathrm{Lip}} \leq e^{-ct/2}\|u\|_{\mathrm{Lip}}$.

Proof. We want to show that the sequence $\{T_{t,\ell}u\}_\ell$ is Cauchy by estimating the difference between consecutive terms. Suppose that $t\in 2^{-\ell}\mathbb{N}_0$ and write $t=n/2^\ell$ and $\delta=2^{-\ell-1}$. Note the telescoping series identity

$$T_{t,\ell+1}u - T_{t,\ell}u = \sum_{j=0}^{n-1}(P_\delta S_\delta)^{2j}P_\delta(S_\delta P_\delta - P_\delta S_\delta)S_\delta(P_{2\delta}S_{2\delta})^{n-1-j}u. \tag{4-4}$$

Thus, we want to estimate $S_\delta P_\delta - P_\delta S_\delta$ and then control the propagation of the errors through the applications of the other operators. Note that for a Lipschitz function v, we have using Lemma 4.4(4) that

$$\begin{aligned}|S_\delta P_\delta v(x) - P_\delta S_\delta v(x)| &\leq \int |v(W(x,\delta)+y)-v(W(x+y,\delta))|\,d\sigma_{\delta,N}(y)\\ &\leq \|v\|_{\mathrm{Lip}}\int \|(W(x,\delta)-x)-(W(x+y,\delta)-(x+y))\|_2\,d\sigma_{\delta,N}(y)\\ &\leq \|v\|_{\mathrm{Lip}}\frac{C}{c}(1-e^{-c\delta/2})\int\|y\|_2\,d\sigma_{\delta,N}(y)\\ &\leq \|v\|_{\mathrm{Lip}}\frac{C}{c}(1-e^{-c\delta/2})(m\delta)^{1/2},\end{aligned}$$

where the last inequality follows by the same reasoning as Lemma 4.3(3). Therefore,

$$\|S_\delta P_\delta v - P_\delta S_\delta v\|_{L^\infty} \le \frac{C}{c} m^{1/2} \delta^{1/2} (1 - e^{-c\delta/2}) \|v\|_{\mathrm{Lip}}. \tag{4-5}$$

Therefore, we can estimate a single term in the telescoping series identity (4-4) by

$$\begin{aligned}\|(P_\delta S_\delta)^{2j} P_\delta (S_\delta P_\delta - P_\delta S_\delta) S_\delta (P_{2\delta} S_{2\delta})^{n-1-j} u\|_{L^\infty} &\le \|(S_\delta P_\delta - P_\delta S_\delta) S_\delta (P_{2\delta} S_{2\delta})^{n-1-j} u\|_{L^\infty} \\ &\le \frac{C}{c} m^{1/2} \delta^{1/2} (1 - e^{-c\delta/2}) \|S_\delta (P_{2\delta} S_{2\delta})^{n-1-j} u\|_{\mathrm{Lip}} \\ &\le \frac{C}{c} m^{1/2} \delta^{1/2} (1 - e^{-c\delta/2}) e^{-c\delta/2} e^{-c\delta(n-j-1)/2} \|u\|_{\mathrm{Lip}}.\end{aligned}$$

Here we have first applied the fact that P_δ and S_δ are contractions with respect to the L^∞ norm from Lemmas 4.3(1) and 4.4(6); second, we used the estimate (4-5) for $S_\delta P_\delta - P_\delta S_\delta$; and third we used the estimates $\|P_\delta u\|_{\mathrm{Lip}} \le \|u\|_{\mathrm{Lip}}$ and $\|S_\delta u\|_{\mathrm{Lip}} \le e^{-c\delta/2} \|u\|_{\mathrm{Lip}}$ found in Lemmas 4.3(2) and 4.4(5). Now summing up the telescoping series, we get

$$\begin{aligned}\|T_{t,\ell+1} u - T_{t,\ell} u\|_{L^\infty} &\le \sum_{j=0}^{n-1} \frac{C}{c} m^{1/2} \delta^{1/2} (1 - e^{-c\delta/2}) e^{-c\delta/2} e^{-c\delta(n-j-1)/2} \|u\|_{\mathrm{Lip}} \\ &\le \frac{C}{c} m^{1/2} \delta^{1/2} (1 - e^{-c\delta/2}) e^{-c\delta/2} \frac{1}{1 - e^{-c\delta/2}} \|u\|_{\mathrm{Lip}} \\ &= \frac{C}{c} m^{1/2} \delta^{1/2} e^{-c\delta/2} \|u\|_{\mathrm{Lip}} \le \frac{C}{2c} m^{1/2} \delta^{1/2} \|u\|_{\mathrm{Lip}}.\end{aligned}$$

In other words, we have

$$\|T_{t,\ell+1} u - T_{t,\ell} u\|_{L^\infty} \le \frac{C m^{1/2}}{2c} 2^{-(\ell+1)/2} \|u\|_{\mathrm{Lip}}.$$

It follows that the sequence is Cauchy with respect to $\|\cdot\|_{L^\infty}$ and we have the desired estimate on $\|T_{t,\ell} u - T_t u\|_{L^\infty}$ from summing the geometric series.

The estimate $\|T_{t,\ell} u\|_{\mathrm{Lip}} \le e^{-ct/2} \|u\|_{\mathrm{Lip}}$ follows from Lemmas 4.3(2) and 4.4(5), and then by taking the limit as $\ell \to +\infty$, we obtain $\|T_t u\|_{\mathrm{Lip}} \le e^{-ct/2} \|u\|_{\mathrm{Lip}}$. □

Lemma 4.6. *The semigroup T_t defined above extends to a semigroup defined for positive t such that for $s \le t$*

$$|T_t u(x) - T_s u(x)| \le e^{-cs/2} \Big(\frac{C}{c} (3\sqrt{2} + 5)(t-s)^{1/2} + \|DV(x)\|_2 (t-s) \Big) \|u\|_{\mathrm{Lip}},$$

and $\|T_t u\|_{\mathrm{Lip}} \le e^{-ct/2} \|u\|_{\mathrm{Lip}}$.

Proof. We first prove the estimate on $|T_t u - T_s u|$ for dyadic values of s and t. First, consider the case where $t = 2^{-\ell}$ and $s = 0$. Note that

$$(T_t - 1)u = (T_t - P_t S_t)u + (P_t - 1) S_t u + (S_t - 1)u.$$

The first term can be estimated by Lemma 4.5 with ℓ=1, the second term can be estimated by Lemmas 4.3(3) and 4.4(5) as

$$\|(P_t - 1) S_t u\|_{L^\infty} \le m^{1/2} t^{1/2} \|S_t u\|_{\mathrm{Lip}} \le m^{1/2} t^{1/2} \|u\|_{\mathrm{Lip}},$$

and the third term can be estimated by Lemma 4.4(3). Altogether, we obtain

$$|T_t u(x) - u(x)| \leq \left(\frac{Cm^{1/2}}{c(2-2^{1/2})} t^{1/2} + m^{1/2} t^{1/2} + \frac{t}{2} \|DV(x)\|_2 \right) \|u\|_{\text{Lip}}.$$

In the case of general dyadic s and t, suppose $t > s$ and write $t - s$ in a binary expansion to obtain

$$t = s + \sum_{j=n+1}^{\infty} a_j 2^{-j},$$

where $a_j \in \{0, 1\}$ and $a_{n+1} = 1$. Note that $2^{-n-1} \leq |s - t| \leq 2^{-n}$. Let $t_k = s + \sum_{j=n+1}^{k} a_j 2^{-j}$. Then

$$\begin{aligned}
|T_t u(x) - T_s u(x)| &\leq \sum_{j=n+1}^{\infty} |T_{t_j} u(x) - T_{t_{j-1}} u(x)| \\
&\leq \sum_{j=n+1}^{\infty} \left(\frac{Cm^{1/2}}{c(2-2^{1/2})} 2^{-j/2} + m^{1/2} 2^{-j/2} + \frac{2^{-j}}{2} \|DV(x)\|_2 \right) \|T_{t_{j-1}} u\|_{\text{Lip}} \\
&\leq \left(\left(\frac{Cm^{1/2}}{c(2-2^{1/2})} + 1 \right) \frac{1}{1-2^{-1/2}} \cdot 2^{-(n+1)/2} + \|DV(x)\|_2 \cdot 2^{-(n+1)} \right) \|T_s u\|_{\text{Lip}} \\
&\leq \left(\left(\frac{Cm^{1/2}}{c(2-2^{1/2})} + 1 \right) \frac{1}{1-2^{-1/2}} (t-s)^{1/2} + \|DV(x)\|_2 (t-s) \right) e^{-cs/2} \|u\|_{\text{Lip}} \\
&\leq e^{-cs/2} \left(\frac{Cm^{1/2}}{c} (3\sqrt{2}+5)(t-s)^{1/2} + \|DV(x)\|_2 (t-s) \right) \|u\|_{\text{Lip}},
\end{aligned}$$

where we used the crude estimate that $1 \leq Cm^{1/2}/c$ to combine the first two terms. Because this continuity estimate holds for dyadic values of s and t, we can extend the definition of $T_t u$ to all positive t. Furthermore, because $\|T_t u\|_{\text{Lip}} \leq e^{-ct/2} \|u\|_{\text{Lip}}$ for dyadic t, the same must hold for real values of t.

Now let us verify that $T_s T_t = T_{s+t}$ for all real t. Choose dyadic $s_n \searrow s$ and $t_n \searrow t$ and let u be a Lipschitz function. We know that $T_{s_n} T_{t_n} u = T_{s_n + t_n} u$ and that $T_{s_n+t_n} u \to T_{s+t} u$ locally uniformly, so it suffices to show that $T_{s_n} T_{t_n} u \to T_s T_t u$. Observe that

$$|T_{s_n} T_{t_n} u - T_s T_t u| \leq |(T_{s_n} - T_s) T_{t_n} u| + |T_s (T_{t_n} - T_t) u|.$$

The first term can be estimated by

$$|(T_{s_n} - T_s) T_{t_n} u(x)| \leq e^{-s/2} \left(\frac{C}{c} (3\sqrt{2}+5)(s_n - s)^{1/2} + \|DV(x)\|_2 (s_n - s) \right) \|T_{t_n} u\|_{\text{Lip}},$$

which goes to zero as $n \to \infty$. For the second term, we first note that

$$|(T_{t_n} - T_t) u(x)| \leq e^{-t/2} \left(\frac{C}{c} (3\sqrt{2}+5)(t_n - t)^{1/2} + \|DV(x)\|_2 (t_n - t) \right) \|u\|_{\text{Lip}}.$$

Let $h_n(x)$ be the right-hand side. Note that $u \leq v$ implies that $T_s u \leq T_s v$ because this holds for the operators P_s and S_s (since P_s is given by convolution and S_s is given by composition). Therefore,

$$|T_s (T_{t_n} - T_t) u(x)| \leq T_s |(T_{t_n} - T_t) u|(x) \leq T_s h_n(x).$$

Because DV is C-Lipschitz, we know that h_n is an $e^{-t/2}(t_n - t)C\|u\|_{\mathrm{Lip}}$-Lipschitz function and hence

$$\begin{aligned}|T_s h_n(x)| &\le h_n(x) + |(T_s - 1)h_n(x)| \\ &\le h_n(x) + e^{-t/2}(t_n - t)C\|u\|_{\mathrm{Lip}}\Big(\frac{C}{c}(3\sqrt{2}+5)s^{1/2} + \|DV(x)\|_2 s\Big),\end{aligned}$$

which goes to zero as $n \to \infty$. □

Lemma 4.7. *Let $u(x)$ be Lipschitz. Then $T_t u$ is a weak solution of the equation*

$$\partial_t T_t u = \frac{1}{2N}\Delta(T_t u) - \frac{N}{2}\nabla V \cdot \nabla(T_t u)$$

in the sense that for $\phi \in C_c^\infty(M_N(\mathbb{C})_{\mathrm{sa}}^m)$, we have

$$\int_{M_N(\mathbb{C})_{\mathrm{sa}}^m} [(T_{t_1}u)\phi - (T_{t_0}u)\phi] = \int_{t_0}^{t_1}\int_{M_N(\mathbb{C})_{\mathrm{sa}}^m}\Big[-\frac{1}{2N}\nabla(T_s u)\cdot\nabla\phi - \frac{N}{2}(\nabla V\cdot\nabla(T_s u))\phi\Big]\,ds.$$

Proof. Recall that by Rademacher's theorem if u is Lipschitz, then ∇u exists almost everywhere and it is in $L^\infty(M_N(\mathbb{C})_{\mathrm{sa}}^m)$. In particular, because ∇V is Lipschitz, we also know that the second derivatives of V exist almost everywhere and are in $L^\infty(M_N(\mathbb{C})_{\mathrm{sa}}^m)$.

We begin by considering $\int(S_\delta P_\delta - 1)u\cdot\phi$ for a Lipschitz $u : M_N(\mathbb{C})_{\mathrm{sa}}^m \to \mathbb{R}$ and a $\phi \in C_c^\infty(M_N(\mathbb{C})_{\mathrm{sa}}^m)$ and $\delta > 0$. Note that

$$(S_\delta P_\delta - 1)u = (S_\delta - 1)P_\delta u + (P_\delta - 1)u.$$

Now $P_\delta u$ is the convolution of u with the Gaussian and so $\nabla(P_\delta u) = P_\delta(\nabla u)$. Because the gradient of the Gaussian is $O(\delta^{-1/2})$, we see that the first derivatives of $P_\delta(\nabla u)$ are $O(\delta^{-1/2})$ in L^∞ (here our estimates may depend on N):

$$P_\delta u(y) - P_\delta u(x) = \nabla P_\delta u(x)\cdot(x-y) + O(\delta^{-1/2}\|x-y\|_2^2).$$

Now using (4-3) and Lemma 4.4(3), we have $W(x,\delta) - x = (N\delta/2)\nabla V(x) + O(\delta^2)$ uniformly on any compact set K. Therefore,

$$(S_\delta - 1)P_\delta u(x) = P_\delta u(W(x,\delta)) - P_\delta u(x) = -\frac{N\delta}{2}\nabla(P_\delta u)(x)\cdot\nabla V(x) + O(\delta^{3/2}).$$

Now we have

$$\begin{aligned}\int(S_\delta P_\delta - 1)u\cdot\phi &= \int(S_\delta - 1)P_\delta u\,\phi + \int(P_\delta - 1)u\,\phi \\ &= -\frac{N\delta}{2}\int[\nabla(P_\delta u)\cdot\nabla V]\phi + \int u\,(P_\delta - 1)\phi + O(\delta^{3/2}) \\ &= -\frac{N\delta}{2}\int P_\delta u[(\Delta V)\phi + \nabla V\cdot\nabla\phi] + \int u\frac{\delta}{2N}\Delta\phi + O(\delta^{3/2}) \\ &= -\frac{N\delta}{2}\int u\,P_\delta[(\Delta V)\phi + \nabla V\cdot\nabla\phi] + \frac{\delta}{2N}\int u\,\Delta\phi + O(\delta^{3/2}),\end{aligned}$$

where the error estimates depend only on C, N, $\|u\|_{\mathrm{Lip}}$, the support of ϕ, and the L^∞ norms of its derivatives. We also know from (4-5) that $(S_\delta P_\delta - P_\delta S_\delta)u$ is bounded by $\|u\|_{\mathrm{Lip}}(Cm^{1/2}/c)(1-e^{-c\delta})\delta^{1/2}$, which is $O(\delta^{3/2})$. Therefore,

$$\int (P_\delta S_\delta - 1)u \cdot \phi = -\frac{N\delta}{2} \int u \, P_\delta[\Delta V \phi + \nabla V \cdot \nabla \phi] + \frac{\delta}{2N} \int u \, \Delta \phi + O(\delta^{3/2}).$$

Now suppose that t is a dyadic rational and write $t = n\delta$, where $\delta = 2^{-\ell}$ for some integer ℓ. Recall that $T_{t,\ell} = (P_\delta S_\delta)^n$. Then by a telescoping series argument

$$\int (T_{t,\ell} - 1)u \cdot \phi = \sum_{j=0}^{n-1} \Big(-\frac{N\delta}{2} \int T_{j\delta,\ell} u \, P_\delta[(\Delta V)\phi + \nabla V \cdot \nabla \phi] + \frac{\delta}{2N} \int T_{j\delta,\ell} u \, \Delta \phi \Big) + O(\delta^{1/2}).$$

We fix a dyadic t and take $\ell \to \infty$ (and hence $\delta \to 0$). The above sum over j may be viewed as a Riemann sum for an integral from 0 to t, where δ is the mesh size. Using Lemma 4.6, we know that $T_t u$ is Hölder continuous in t. Also, by Lebesgue differentiation theory,

$$P_\delta[(\Delta V)\phi + \nabla V \cdot \nabla \phi] \to (\Delta V)\phi + \nabla V \cdot \nabla \phi$$

in $L^1_{\mathrm{loc}}(M_N(\mathbb{C})^m_{\mathrm{sa}})$. There is no difficulty in taking the limit as $\delta \to 0$ inside the integral because ϕ has compact support and all the functions we are integrating are bounded on compact sets. Thus, we obtain

$$\int (T_t - 1)u \cdot \phi \, dx = \int_0^t \int \Big(-\frac{N}{2} T_s u[(\Delta V)\phi + \nabla V \cdot \nabla \phi] + \frac{1}{2N} T_s u \, \Delta \phi \Big) \, dx \, ds.$$

We may extend this equality from dyadic t to all positive t using Lemma 4.6. Finally, after another integration by parts (which is justified by approximation by smooth functions in the appropriate Sobolev spaces), we have

$$\int (T_t - 1)u \cdot \phi \, dx = \int_0^t \int \Big(-\frac{N}{2} [\nabla(T_s u) \cdot \nabla V]\phi - \frac{1}{2N} \nabla(T_s u) \cdot \nabla \phi \Big) \, dx \, ds.$$

The asserted formula then follows by replacing u with $T_{t_0} u$ and t with $t_1 - t_0$. □

Lemma 4.8. *If μ is the measure given by the potential V and if u is Lipschitz, then we have*

$$\int T_t u \, d\mu = \int u \, d\mu.$$

Proof. By applying Lemma 4.7 and approximating $(1/Z)\exp(-N^2 V(x))$ by compactly supported smooth functions, we see that

$$\int T_t u \, d\mu - \int u \, d\mu = \frac{1}{Z} \iint_0^t \Big[-\frac{1}{2N} \nabla(T_s u) \cdot \nabla[e^{-N^2 V}] - \frac{N}{2} (\nabla V \cdot \nabla(T_s u)) e^{-N^2 V} \Big] \, ds \, dx = 0. \quad \square$$

Lemma 4.9. *We have $T_t u(x) \to \int u \, d\mu$ as $t \to \infty$ and more precisely*

$$\Big| T_t u(x) - \int u \, d\mu \Big| \le e^{-ct/2} \Big(\frac{4Cm^{1/2}}{c^2} (6 + 5\sqrt{2}) t^{-1/2} + \frac{2}{c} \| V(x) \|_2 \Big) \|u\|_{\mathrm{Lip}}.$$

Proof. Fix t and fix $r \geq t$. Let n be an integer. Then using Lemma 4.6,

$$\begin{aligned}|T_{t+r}u(x) - T_t u(x)| &\leq \sum_{j=0}^{n-1} |T_{t+r(j+1)/n}u(x) - T_{t+rj/n}u(x)| \\ &\leq \sum_{j=0}^{n-1} e^{-ct/2}e^{-crj/2n}\Big(\frac{Cm^{1/2}}{c}(3\sqrt{2}+5)\Big(\frac{r}{n}\Big)^{1/2} + \|V(x)\|_2\Big(\frac{r}{n}\Big)\Big)\|u\|_{\mathrm{Lip}} \\ &\leq e^{-ct/2}\frac{1}{1-e^{-cr/2n}}\Big(\frac{Cm^{1/2}}{c}(3\sqrt{2}+5)\Big(\frac{r}{n}\Big)^{1/2} + \|V(x)\|_2\Big(\frac{r}{n}\Big)\Big)\|u\|_{\mathrm{Lip}} \\ &\leq e^{-ct/2}\frac{2n}{cr}\Big(\frac{Cm^{1/2}}{c}(3\sqrt{2}+5)\Big(\frac{r}{n}\Big)^{1/2} + \|V(x)\|_2\Big(\frac{r}{n}\Big)\Big)\|u\|_{\mathrm{Lip}}.\end{aligned}$$

Since $r \geq t$, we can choose n such that $t/4 \leq r/n \leq t/2$. Then we have

$$|T_{t+r}u(x) - T_t u(x)| \leq e^{-ct/2}\Big(\frac{4Cm^{1/2}}{c^2}(6+5\sqrt{2})t^{-1/2} + \frac{2}{c}\|V(x)\|_2\Big)\|u\|_{\mathrm{Lip}}.$$

Because this holds for all sufficiently large r, this shows that $\lim_{t\to\infty} T_t u(x)$ exists. Because $\|T_t u\|_{\mathrm{Lip}} \leq e^{-ct/2}\|u\|_{\mathrm{Lip}}$, the limit must be constant and therefore equals $\int u\, d\mu$. Moreover, we have the asserted rate of convergence by taking $r \to \infty$ in the above estimate. □

4B. ***Approximability and convergence of moments.*** Now we are ready to show that the sequence of semigroups $T_t^{V_N}$ associated to a sequence of potentials V_N will preserve asymptotic approximability by trace polynomials and as a consequence we will show that the moments of the associated measures μ_N converge.

Lemma 4.10. *Let $V_N : M_N(\mathbb{C})_{\mathrm{sa}}^m \to \mathbb{R}$ be a sequence of potentials such that $V_N(x) - (c/2)\|x\|_2^2$ is convex and $V_N(x) - (C/2)\|x\|_2^2$ is concave, where $0 < c \leq C$. For each N, let μ_N be the associated measure. Let $S_t^{V_N}$ and $T_t^{V_N}$ denote the semigroups defined in the previous section. Suppose that the sequence $\{DV_N\}$ is asymptotically approximable by trace polynomials. Suppose that $\{u_N\}$ is a sequence of scalar-valued K-Lipschitz functions which is asymptotically approximable by (scalar-valued) trace polynomials. Then*:

(1) $\{S_t^{V_N}u_N\}$ *is asymptotically approximable by trace polynomials for each $t \geq 0$.*

(2) $\{T_t^{V_N}u_N\}$ *is asymptotically approximable by trace polynomials for each $t \geq 0$.*

(3) $\lim_{N\to\infty}\int u_N\, d\mu_N$ *exists.*

Proof. (1) Recall that $S_t^{V_N}u_N = u_N(W_N(x,t))$, where W_N is the solution to (4-3). Thus, by Lemma 3.27, it suffices to show that $W_N(x,t)$ is asymptotically approximable by trace polynomials for each t. To this end, we write $W_N(x,t)$ as the limit as $\ell \to \infty$ of Picard iterates $W_{N,\ell}$ given by

$$W_{N,0}(x,t) = x, \qquad W_{N,\ell+1}(x,t) = x - \frac{1}{2}\int_0^t DV_k(W_N(x,s))\, ds.$$

Because DV_N is C-Lipschitz, the standard Picard–Lindelöf arguments show that

$$\|W_{N,\ell}(x,t) - W_N(x,t)\|_2 \leq \sum_{n=\ell+1}^{\infty} \frac{C^{n-1}t^n}{2^n n!}\|DV_N(x)\|_2.$$

Because DV_N is asymptotically approximable by trace polynomials, we know that $\|DV_N(x)\|_2$ is uniformly bounded on $\|x\| \leq R$ for any given $R > 0$, and therefore, for each T and $R > 0$, the convergence of $W_{N,\ell}$ to W_N as $\ell \to \infty$ is uniform for all $\|x\| \leq R$ and $t \leq T$ and $N \in \mathbb{N}$. Thus, by Observation 3.26, it suffices to show that each Picard iterate $\{W_{N,\ell}(x, t)\}_N$ is asymptotically approximable by trace polynomials.

Fix $T > 0$. We claim that, for every ℓ, for every $R > 0$ and $\epsilon > 0$, there exists a trace polynomial $f(X, t)$ with coefficients that are polynomial functions of t such that

$$\limsup_{N\to\infty} \sup_{t\in[0,T]} \sup_{\|x\|_\infty \leq R} \|W_{N,\ell}(x, t) - f(x, t)\|_2 \leq \epsilon.$$

We proceed by induction on ℓ, with the base case $\ell = 0$ being trivial. For the inductive step, fix ϵ and R, and choose a trace polynomial $f(X, t)$ which provides an $(\epsilon/(CT), R)$ approximation for $W_{N,\ell}$ for all $t \leq T$. Let

$$R' = \sup_{t\in[0,T]} \sup_{N} \sup_{x\in M_N(\mathbb{C})^m_{\text{sa}}:\|x\|_\infty \leq R} \|f(x, t)\| < +\infty.$$

Choose another trace polynomial $g(X)$ which is an $(\epsilon/T, R')$ approximation for $\{DV_N\}$, and let $h(X, t) = X - \frac{1}{2}\int_0^t g(f(X, s))\, ds$. Then arguing as in Lemma 3.27, we have for $\|x\| \leq R$ and $t \in [0, T]$ that

$$\begin{aligned}\|W_{N,\ell+1}(x, t) - h(x, t)\| &\leq \frac{1}{2}\int_0^t \|DV_N(W_{N,\ell}(x, s)) - g(f(x, s))\|_2\, ds \\ &\leq \frac{t}{2}\sup_{\|y\|\leq R'} \|DV_N(y) - g(y)\|_2 + \frac{Ct}{2}\sup_{s\in[0,T]}\sup_{\|x\|\leq R} \|W_{N,\ell}(x, s) - f(x, s)\|_2.\end{aligned}$$

Taking $N \to \infty$, we see that $h(x, t)$ is an (ϵ, R) approximation for $\{W_{N,\ell}(x, t)\}_N$ for all $t \leq T$.

(2) We have shown that $S_t^{V_k}$ preserves asymptotic approximability. Moreover, if the sequence $u_N : M_N(\mathbb{C})^m_{\text{sa}} \to \mathbb{C}$ is asymptotically approximable by trace polynomials and u_N is K-Lipschitz, then the sequence $P_t u_N$ is also asymptotically approximable by trace polynomials by Lemma 3.28 (the hypothesis (3-18) is satisfied since $|u_N(x)| \leq |u_N(0)| + K\|x\|_2$ and $|u_N(0)|$ is bounded as $N \to +\infty$ because u_N is asymptotically approximable by trace polynomials). Therefore, the iterated operator $T_{t,\ell}^{V_N} = (P_{2^{-\ell}} S_{2^{-\ell}}^{V_N})^{2^\ell t}$ preserves asymptotic approximability for dyadic values of t. Taking $\ell \to \infty$, we see by Observation 3.26 and Lemma 4.5 that $T_t^{V_N}$ preserves asymptotic approximability for dyadic values of t. Finally, we extend the approximability property to $T_t^{V_N}$ for all real t using Observation 3.26 and Lemma 4.6.

(3) We know by Lemma 4.9 that $T_t^{V_N} u_N(x) \to \int u_N\, d\mu_N$ as $t \to \infty$ with estimates that are independent of N. It follows by Observation 3.26 that the sequence of constant functions $\left\{\int u_N\, d\mu_N\right\}$ is asymptotically approximable by trace polynomials. But since these functions are constant, this simply means that the limit of $\int u_N\, d\mu_N$ as $N \to \infty$ exists. □

Proof of Theorem 4.1. (1) Let $a_N = \int x\, d\mu_N(x)$ and $a_{N,j} = \int x_j\, d\mu_N(x)$. Note that

$$R_N \leq \max_j \int \|x_j - a_{N,j}\|\, d\mu_N(x) + \max_j \left|\int \tau_N(x_j)\, d\mu_N(x)\right| + \max_j \left\|\int (x_j - \tau_N(x_j))\, d\mu(x)\right\|.$$

When we take the lim sup as $N \to \infty$, the first term is bounded by $2/c^{1/2}$ by Corollary 2.12, while the last term is bounded by M. It remains to estimate $\int \tau_N(x_j)\, d\mu_N(x)$.

Using integration by parts, we see that

$$\int DV_N(x)\,d\mu_N(x) = -\frac{1}{Z}\int \frac{1}{N}\nabla[e^{-V_N}(x)]\,dx = 0.$$

On the other hand, we may estimate $\|DV_N(x) - (DV_N(0) + ((C+c)/2)x)\|_2$ as follows. We assumed that $V_N(x) - (c/2)\|x\|_2^2$ is convex and $V_N(x) - (C/2)\|x\|_2^2$ is concave. Let $\widetilde{V}_N(x) = V_N(x) - ((C+c)/2)\|x\|_2^2$. Then $\widetilde{V}_N(x) + ((C-c)/2)\|x\|_2^2$ is convex and $\widetilde{V}_N(x) - ((C-c)/2)\|x\|_2^2$ is concave. Therefore, $D\widetilde{V}_N$ is $(C-c)/2$-Lipschitz with respect to $\|\cdot\|_2$ by Proposition 2.13(4). It follows that

$$\Big\|DV_N(x) - \Big(DV_N(0) + \frac{C+c}{2}x\Big)\Big\|_2 = \|D\widetilde{V}_N(x) - D\widetilde{V}_N(0)\|_2 \le \frac{C-c}{2}\|x\|_2.$$

Therefore,

$$\begin{aligned}\Big\|DV_N(0) + \frac{C+c}{2}a_N\Big\|_2 &= \Big\|-DV_N(x) + \Big(DV_N(0) + \frac{C+c}{2}x\Big)\Big\|_2 \\ &\le \frac{C-c}{2}\int \|x\|_2\,d\mu_N(x) \\ &\le \frac{C-c}{2}\bigg(\|a_N\|_2 + \bigg(\int \|x-a_N\|_2^2\,d\mu(x)\bigg)^{1/2}\bigg) \\ &\le \frac{C-c}{2}(\|a_N\|_2 + c^{-1/2}),\end{aligned}$$

where the last step follows from Theorem 2.11. Altogether,

$$\frac{C+c}{2}\|a_N\|_2 \le \frac{C-c}{2}\|a_N\|_2 + \|DV_N(0)\|_2 + \frac{C-c}{2c^{1/2}}.$$

Then we move $((C-c)/2)\|a_N\|_2$ to the left-hand side and divide the equation by c to obtain

$$\bigg|\int \tau_N(x_j)\,d\mu_N(x)\bigg| \le \|a_N\|_2 \le \frac{1}{c}\|DV_N(0)\|_2 + \frac{C-c}{2c^{3/2}},$$

which proves the asserted estimate on R_N. The tail estimate on $\mu_N(\|x_j\| \ge R_N + \delta)$ follows from Corollary 2.12.

(2) Fix a noncommutative polynomial p. Let $R_* = \limsup_{N\in\mathbb{N}} R_N$, which we know is finite because of (1) and suppose that $R' > R_*$. Let $\psi \in C_c^\infty(\mathbb{R})$ be such that $\psi(t) = t$ for $|t| \le R'$, and define $\Psi(x_1, \dots, x_m) = (\psi(x_1), \dots, \psi(x_m))$, where $\psi(x_j)$ is defined through the continuous functional calculus for self-adjoint operators. Now $x \mapsto \psi(x)$ is Lipschitz in $\|\cdot\|_2$ for $x \in M_N(\mathbb{C})_{\mathrm{sa}}$ with constants independent of N (see for instance Proposition 8.8 below). It follows that $p(\Psi(x))$ is globally Lipschitz in $\|\cdot\|_2$ and it equals $p(x)$ when $\|x\| \le R'$.

Furthermore, we claim that the sequence $\tau_N(p(\Psi(x)))$ is asymptotically approximable by trace polynomials. To see this, choose some radius r and $\delta > 0$. By the Weierstrass approximation theorem, there exists a polynomial $\hat{\psi}(t)$ such that $|\psi(t) - \hat{\psi}(t)| \le \delta$ for $t \in [-r, r]$. By the spectral mapping theorem, we have $\|\psi(y) - \hat{\psi}(y)\| \le \delta$ if $y \in M_N(\mathbb{C})_{\mathrm{sa}}$ and $\|y\| \le r$. In particular, if we let $\widehat{\Psi}(x) = (\hat{\psi}(x_1), \dots, \hat{\psi}(x_m))$ for $x \in M_N(\mathbb{C})_{\mathrm{sa}}^m$, then we have $\|\Psi(x) - \widehat{\Psi}(x)\| \le \delta$ when $\|x\|_\infty \le r$. Given $\epsilon > 0$, we may choose δ small enough to guarantee that $|\tau_N(p(\Psi(x))) - \tau_N(p(\widehat{\Psi}(x)))| \leq \epsilon$ for $\|x\|_\infty \leq r$, and clearly $\tau_N(p(\widehat{\Psi}(x)))$ is a trace polynomial. Thus, $\tau_N(p(\Psi(x)))$ is asymptotically approximable by trace polynomials.

Therefore, by Lemma 4.10, the limit

$$\lambda(p) = \lim_{N\to\infty} \int \tau_N(p(\Psi(x)))\, d\mu_N(x)$$

exists. Clearly, λ satisfies all the conditions to be a noncommutative law (Definition 2.3). Furthermore, because of the operator norm bounds (1), we know that $\int_{\|x\|\geq R'} \tau_N(p(x))\, d\mu_N(x)$ is finite and approaches zero as $N \to \infty$ and the same holds for the integral of $\tau_N(p(\Psi(x)))$. Therefore,

$$\lim_{N\to\infty} \int \tau_N(p(x))\, d\mu_N(x) = \lim_{N\to\infty} \int \tau_N(p(\Psi(x)))\, d\mu_N(x) = \lambda(p).$$

Also, we have $\lambda(p) = \lim_{N\to\infty} \int_{\|x\|\leq R'} \tau_N(p(x))\, d\mu_N(x)$ and hence $\lambda \in \Sigma_{m,R'}$. But since this holds for every $R' > R_*$, we have $\lambda \in \Sigma_{m,R_*}$.

(3) It suffices to prove the concentration claim (3) for sufficiently large R, say $R > 2R_*$. Because the topology of $\Sigma_{m,R}$ is generated by the functions $\lambda \mapsto \lambda(p)$ for noncommutative polynomials p, it suffices to consider the case where $\mathcal{U} = \{\lambda' : |\lambda'(p) - \lambda(p)| < \epsilon\}$ for some noncommutative polynomial p. Choose a function $\psi \in C_c^\infty(\mathbb{R})$ with $\psi(t) = t$ for $|t| \leq R$, and let Ψ be as above. Then by Theorem 2.10,

$$\mu_N\left(\left|\tau_N(p(\Psi(x))) - \int \tau_N(p\circ\Psi)\, d\mu_N\right| \geq \frac{\epsilon}{2}\right) \leq 2e^{-N^2\epsilon^2/8\|\tau_N(p\circ\Psi)\|_{\mathrm{Lip}}^2}.$$

But by the same reasoning as in part (2), we know that large enough N, we have

$$\left|\int \tau_N(p\circ\Psi)\, d\mu_N - \lambda(p)\right| \leq \frac{\epsilon}{2},$$

and hence

$$\limsup_{N\to\infty} \frac{1}{N^2} \log \mu_N(\|x\| \leq R,\ |\tau_N(p(x)) - \lambda(p)| \geq \epsilon) < 0. \qquad \square$$

5. Entropy and Fisher's information

5A. *Classical entropy.* In this section, we will state sufficient conditions for the microstates free entropies χ and $\underline{\chi}$ to be evaluated as the lim sup and lim inf of renormalized classical entropies. Recall that the (classical, continuous) entropy of a measure $d\mu(x) = \rho(x)\, dx$ on $\mathbb{R}^n$ is defined as

$$h(\mu) := \int_{\mathbb{R}^n} -\rho \log \rho,$$

whenever the integral makes sense. If μ does not have a density, then we set $h(\mu) = -\infty$. We will later use the following basic facts about the classical entropy, so for convenience we provide a proof.

Lemma 5.1. *Assume that μ is a probability measure on $\mathbb{R}^n$ with density ρ and that $\int |x|^2\, d\mu(x) < +\infty$.*

(1) *The positive part of $-\rho\log\rho$ has finite integral and hence $\int -\rho\log\rho$ is well-defined in $[-\infty, +)$.*

(2) *We have $h(\mu) \leq (n/2)\log 2\pi a e$, where $a = \int |x|^2\, d\mu(x)/n$, and equality is achieved in the case of a centered Gaussian with covariance matrix aI.*

(3) *Suppose* $\{\mu_k\}$ *is a sequence of probability measures with density* ρ_k. *Suppose* $\rho_k \to \rho$ *pointwise almost everywhere and that* $\int |x|^2 \, d\mu_k(x) \to \int |x|^2 \, d\mu(x) < +\infty$. *Then* $\limsup_{k\to\infty} h(\mu_k) \le h(\mu)$.

(4) *If* ν *is a probability measure with finite second moments*, *then* $h(\mu * \nu) \ge h(\mu)$.

Proof. (1) Let $a = \int |x|^2 \, d\mu(x)/n$. Let $g(x) = (2\pi a)^{-n/2} e^{-|x|^2/(2a)}$ be the Gaussian of variance a, and let γ be the corresponding Gaussian measure. Let $\tilde{\rho} = \rho/f$ be the density of μ relative to the Gaussian. We write

$$\begin{aligned} -\rho(x)\log\rho(x) &= -\tilde{\rho}(x)\log\tilde{\rho}(x)\cdot g(x) - \tilde{\rho}(x)\log g(x)\cdot g(x) \\ &= -\tilde{\rho}(x)\log\tilde{\rho}(x)\cdot g(x) + \Big(\frac{|x|^2}{2a} + \frac{n}{2}\log 2\pi a\Big)\rho(x). \end{aligned}$$

The second term has a finite integral by assumption. The function $-t\log t$ is bounded above for $t \in \mathbb{R}$, and $g(x)$ is a probability density; thus, the positive part of $-\tilde{\rho}\log\tilde{\rho}\cdot g$ has finite integral. Hence, $\int -\rho\log\rho$ is well-defined.

(2) The function $-t\log t$ is concave and its tangent line at $t = 0$ is $1 - t$, and hence $-t\log t \le 1 - t$. Thus,

$$\int -\tilde{\rho}\log\tilde{\rho}\, d\gamma \le \int (1 - \tilde{\rho})\, d\gamma = 0,$$

so

$$h(\mu) \le \int \Big(\frac{|x|^2}{2a} + \frac{n}{2}\log 2\pi a\Big)\rho(x)\, dx = \frac{n}{2} + \frac{n}{2}\log 2\pi a = \frac{n}{2}\log 2\pi e.$$

In the case where $\mu = \gamma$, we have $\tilde{\rho} = 1$ and hence $\int -\tilde{\rho}\log\tilde{\rho} = 0$.

(3) Let γ be the Gaussian of covariance matrix I and g its density. Let $\tilde{\rho}_k = \rho_k/g$. As before,

$$h(\mu_k) = \int -\tilde{\rho}_k\log\tilde{\rho}_k\, d\gamma + \int \Big(\frac{|x|^2}{2} + \frac{n}{2}\log 2\pi\Big)\, d\mu_k.$$

By assumption, the second term converges to $\int (|x|^2/2 + (n/2)\log 2\pi)\, d\mu$ as $k \to \infty$. Since the function $-t\log t$ is bounded above and γ is a probability measure, the integral of the positive part of $-\tilde{\rho}_k\log\tilde{\rho}_k$ converges to the corresponding quantity for ρ. For the negative part, we can apply Fatou's lemma. This yields $\limsup_{k\to\infty} h(\mu_k) = h(\mu)$.

(4) We can assume without loss of generality that one of the measures, say μ, has finite entropy. Then $\mu * \nu$ has a density given almost everywhere by $\tilde{\rho}(x) = \int \rho(x - y)\, d\nu(y)$. Since $-t\log t$ is concave, Jensen's inequality implies that

$$-\tilde{\rho}(x)\log\tilde{\rho}(x) \ge \int -\rho(x - y)\log\rho(x - y)\, d\nu(y).$$

The right-hand side is

$$\iint -\rho(x - y)\log\rho(x - y)\, d\nu(y)\, dx = \iint -\rho(x - y)\log\rho(x - y)\, dx\, d\nu(y) = h(\mu),$$

where the exchange of order is justified because we know that $-\rho\log\rho$ is integrable since $h(\mu) > -\infty$. Therefore, $h(\mu * \nu) = \int -\tilde{\rho}\log\tilde{\rho} \ge h(\mu)$. □

5B. ***Microstates free entropy.*** Because there is no integral formula known for free entropy of multiple noncommuting variables as in the classical case, Voiculescu [1993; 1994] defined the free analogue of entropy using Boltzmann's microstates viewpoint on entropy.

Definition 5.2. For $\mathcal{U} \subseteq \Sigma_m$, we define the *microstate space*

$$\Gamma_N(\mathcal{U}) = \{x \in M_N(\mathbb{C})_{\mathrm{sa}}^m : \lambda_x \in \mathcal{U}\},$$
$$\Gamma_{N,R}(\mathcal{U}) = \{x \in M_N(\mathbb{C})_{\mathrm{sa}}^m : \lambda_x \in \mathcal{U},\ \|x\|_\infty \le R\}.$$

The *microstates free entropy* of a noncommutative law λ is defined as

$$\chi_R(\lambda) = \inf_{\mathcal{U} \ni \lambda} \limsup_{N \to \infty} \Big(\frac{1}{N^2} \log \operatorname{vol} \Gamma_{N,R}(\mathcal{U}) + \frac{m}{2} \log N \Big),$$
$$\chi(\lambda) = \sup_{R>0} \chi_R(\lambda).$$

Here vol denotes the Lebesgue measure with respect to the identification of $M_N(\mathbb{C})_{\mathrm{sa}}^m$ with $\mathbb{R}^{mN^2}$ as in Section 2A, and $\mathcal{U}$ ranges over all open neighborhoods of λ in Σ_m. Similarly, we denote

$$\underline{\chi}_R(\lambda) = \inf_{\mathcal{U} \ni \lambda} \liminf_{N \to \infty} \Big(\frac{1}{N^2} \log \operatorname{vol} \Gamma_{N,R}(\mathcal{U}) + \frac{m}{2} \log N \Big),$$
$$\underline{\chi}(\lambda) = \sup_{R>0} \chi_R(\lambda).$$

Remark 5.3. Note that $\mathcal{U} \subseteq \mathcal{V}$ implies that

$$\limsup_{N \to \infty} \Big(\frac{1}{N^2} \log \operatorname{vol} \Gamma_{N,R}(\mathcal{U}) + \frac{m}{2} \log N \Big) \le \limsup_{N \to \infty} \Big(\frac{1}{N^2} \log \operatorname{vol} \Gamma_{N,R}(\mathcal{V}) + \frac{m}{2} \log N \Big).$$

Hence, to estimate the infimum over $\mathcal{U}$ (that is, $\chi_R(\lambda)$), we can always restrict our attention to neighborhoods $\mathcal{U}$ contained inside some fixed $\mathcal{V}$. The same holds for the lim inf variant of entropy.

Definition 5.4. A sequence of probability measures μ_N on $M_N(\mathbb{C})_{\mathrm{sa}}^m$ is said to *concentrate around the noncommutative law* λ if $\lambda_x \to \lambda$ in probability when x is chosen according to μ_N, that is, for any neighborhood $\mathcal{U}$ of λ in Σ_m, we have

$$\lim_{k \to \infty} \mu_N(x \in \Gamma_N(\mathcal{U})) = 1.$$

Proposition 5.5. *Let $V_N : M_N(\mathbb{C})_{\mathrm{sa}}^m \to \mathbb{R}$ be a potential with $\int \exp(-N^2 V_N(x))\, dx < +\infty$ and let μ_N be the associated measure. Assume*:

(A) *The sequence $\{\mu_N\}$ concentrates around a noncommutative law λ.*

(B) *The sequence $\{V_N\}$ is asymptotically approximable by scalar-valued trace polynomials.*

(C) *For some $n \ge 1$ and $a, b > 0$ we have $|V_N| \le a + b \sum_{j=1}^m \tau_N(x_j^{2n})$.*

(D) *There exists $R_0 > 0$ such that*

$$\lim_{N \to \infty} \int_{\|x\|_\infty \ge R_0} \Big(1 + \sum_{j=1}^m \tau_N(x_j^{2n}) \Big)\, d\mu_N(x) = 0,$$

where n is the same number as in (C).

Then λ can be realized as the law of noncommutative random variables $X = (X_1, \dots, X_m)$ in a von Neumann algebra $(\mathcal{M}, \tau)$ with $\|X_j\| \leq R_0$. Moreover, we have

$$\chi(\lambda) = \chi_{R_0}(\lambda) = \limsup_{N\to\infty}\Big(\frac{1}{N^2}h(\mu_N) + \frac{m}{2}\log N\Big), \tag{5-1}$$

$$\underline{\chi}(\lambda) = \underline{\chi}_{R_0}(\lambda) = \liminf_{N\to\infty}\Big(\frac{1}{N^2}h(\mu_N) + \frac{m}{2}\log N\Big). \tag{5-2}$$

Proof. It follows from assumptions (A) and (D) that for every noncommutative polynomial p

$$\lim_{N\to\infty}\int_{\|x\|_\infty\leq R_0}\tau_N(p(x))\,d\mu_N(x) = \lambda(p).$$

In particular, this implies that $|\lambda(X_{i_1}\dots X_{i_k})| \leq R_0^k$ for every $i_1, \dots, i_k$, and hence $\lambda \in \Sigma_{m,R_0}$. The fact that λ can be realized by operators in a von Neumann algebra is standard (Proposition 2.7).

Now let us evaluate χ_R and $\underline{\chi}_R$ for $R \geq R_0$. Recall that

$$d\mu_N(x) = \frac{1}{Z_N}\exp(-N^2V_N(x))\,dx, \quad Z_N = \int\exp(-N^2V_N(x))\,dx,$$

and note that

$$h(\mu_N) = N^2\int V_N(x)\,d\mu_N(x) + \log Z_N.$$

The assumptions (C) and (D) imply that

$$\lim_{N\to\infty}\int_{\|x\|_\infty\geq R}|V_N(x)|\,d\mu_N(x) = 0 \quad\text{and}\quad \lim_{N\to\infty}\mu_N(x : \|x\|_\infty \geq R) = 0.$$

Therefore, if we let

$$d\mu_{N,R}(x) = \frac{1}{Z_{N,R}}\mathbf{1}_{\|x\|_\infty\leq R}\exp(-N^2V_N(x))\,dx, \quad Z_{N,R} = \int_{\|x\|_\infty\leq R}\exp(-N^2V_N(x))\,dx,$$

then, as $N \to \infty$, we have

$$\int V_N\,d\mu_N - \int V_N\,d\mu_{N,R} \to 0, \quad \log Z_N - \log Z_{N,R} \to 0,$$

and hence

$$\frac{1}{N^2}h(\mu_N) - \frac{1}{N^2}h(\mu_{N,R}) \to 0.$$

Fix $\epsilon > 0$. By assumption (B), there is a scalar-valued trace polynomial f such that $|V_N(x) - f(x)| \leq \epsilon/2$ for $\|x\|_\infty \leq R$ and for sufficiently large N. Now because the trace polynomial f is continuous with respect to convergence in noncommutative moments, the set $\mathcal{U} = \{\lambda' : |\lambda'(f) - \lambda(f)| < \epsilon/2\}$ is open. Now suppose that $\mathcal{V} \subseteq \mathcal{U}$ is a neighborhood of λ. Note that

$$\lim_{N\to\infty}\mu_{N,R}(\Gamma_{N,R}(\mathcal{V})) = \lim_{N\to\infty}\frac{Z_N}{Z_{N,R}}\mu_N(\Gamma_N(\mathcal{V}) \cap \{x : \|x\|_\infty \leq R\}) = 1,$$

where we have used that $Z_N/Z_{N,R} \to 1$ as shown above, that $\mu_N(\Gamma_N(\mathcal{V})) \to 1$ by assumption (A), and that $\mu_N(\|x\|_\infty \leq R) \to 1$ by assumption (D). Moreover, by our choice of f and $\mathcal{U}$, we have

$$x \in \Gamma_{N,R}(\mathcal{V}) \quad\Longrightarrow\quad |V_N(x) - \lambda(f)| \leq \epsilon.$$

Therefore,

$$Z_{N,R}\mu_{N,R}(\Gamma_{N,R}(\mathcal{V})) = \int_{\Gamma_{N,R}(\mathcal{V})} \exp(-N^2 V_N(x))\,dx$$
$$= \operatorname{vol}\Gamma_{N,R}(\mathcal{V}) \exp(-N^2(\lambda(f) + O(\epsilon))).$$

Thus,

$$\log Z_{N,R} + \log \mu_{N,R}(\Gamma_{N,R}(\mathcal{V})) = \log \operatorname{vol}\Gamma_{N,R}(\mathcal{V}) - N^2(\lambda(f) + O(\epsilon)).$$

Meanwhile, note that $|f(x)|$ is bounded by some constant K whenever $\|x\|_\infty \le R$ (where K is independent of N). Therefore,

$$\int V_N\,d\mu_{N,R} = \int_{\Gamma_{N,R}(\mathcal{V})} V_N\,d\mu_{N,R} + \int_{\Gamma_{N,R}(\mathcal{V}^c)} V_N\,d\mu_{N,R}$$
$$= \int_{\Gamma_{N,R}(\mathcal{V})} \lambda[f]\,d\mu_{N,R} + \int_{\Gamma_{N,R}(\mathcal{V}^c)} \lambda_x[f]\,d\mu_{N,R} + O(\epsilon)$$
$$= \lambda(f)\mu_{N,R}(\Gamma_{N,R}(\mathcal{V})) + O(\epsilon) + O\big(K\,\mu_N(\Gamma_{N,R}(\mathcal{V}^c))\big).$$

Altogether,

$$\frac{1}{N^2} h(\mu_{N,R}) = \int V_N\,d\mu_{N,R} + \frac{1}{N^2}\log Z_{N,R}$$
$$=\lambda(f)(\mu_{N,R}(\Gamma_{N,R}(\mathcal{V})) - 1) + \frac{1}{N^2}\log\operatorname{vol}\Gamma_{N,R}(\mathcal{V})$$
$$+ O(\epsilon) + O(K\,\mu_N(\Gamma_{N,R}(\mathcal{V}^c))) - \frac{1}{N^2}\log\mu_{N,R}(\Gamma_{N,R}(\mathcal{V})).$$

Now we apply the fact that $\mu_{N,R}(\Gamma_{N,R}(\mathcal{V})) \to 1$ to obtain

$$\limsup_{N\to\infty} \frac{1}{N^2}|h(\mu_{N,R}) - \log\operatorname{vol}\Gamma_{N,R}(\mathcal{V})| = O(\epsilon).$$

In light of Remark 5.3, because this holds for all sufficiently small neighborhoods $\mathcal{V} \subseteq \mathcal{U}$ with the error $O(\epsilon)$ only depending on $\mathcal{U}$, we have

$$\chi_R(\lambda) = \limsup_{N\to\infty}\Big(\frac{1}{N^2}h(\mu_{N,R}) + \frac{m}{2}\log N\Big) + O(\epsilon)$$
$$= \limsup_{N\to\infty}\Big(\frac{1}{N^2}h(\mu_N) + \frac{m}{2}\log N\Big) + O(\epsilon).$$

Next, we take $\epsilon \to 0$ and obtain $\chi_R(\lambda) = \limsup_{N\to\infty}(N^{-2}\log h(\mu_N) + (m/2)\log N)$ for $R \ge R_0$. Now $\chi(\lambda) = \sup_R \chi_R(\lambda)$ and $\chi_R(\lambda)$ is an increasing function of R. Since our claim about $\chi_R(\lambda)$ holds for sufficiently large R, it also holds for $\chi(\lambda)$, so (5-1) is proved. The proof of (5-2) is identical. □

5C. ***Classical Fisher information.*** The classical Fisher information of a probability measure μ on $\mathbb{R}^n$ describes how the entropy changes when μ is convolved with a Gaussian. Suppose μ is given by the smooth density $\rho > 0$ on $\mathbb{R}^n$, and let γ_t be the multivariable Gaussian measure on $\mathbb{R}^n$ with covariance matrix tI. Then the density ρ_t for $\mu_t = \mu * \gamma_t$ evolves according to the heat equation $\partial_t\rho_t = \Delta/2\rho_t$. Integration by parts shows that $\partial_t h(\mu_t) = \frac{1}{2}\int |\nabla\rho_t/\rho_t|^2\,d\mu_t$ (which we justify in more detail below).

The *Fisher information* of μ represents the derivative at time zero and it is defined as

$$\mathcal{I}(\mu) := \int \left| \frac{\nabla \rho}{\rho} \right|^2 d\mu.$$

The Fisher information is the $L^2(\mu)$ norm of the function $-\nabla \rho(x)/\rho(x)$, which is known as the *score function*. If X is a random variable with smooth density ρ, then the $\mathbb{R}^n$-valued random variable $\Xi = -\nabla \rho(X)/\rho(X)$ satisfies the integration-by-parts relation

$$E[\Xi \cdot f(X)] = -\int \frac{\nabla \rho(x)}{\rho(x)} f(x)\rho(x)\,dx = \int \rho(x) \nabla f(x)\,dx = E[\nabla f(X)] \quad \text{for } f \in C_c^\infty(\mathbb{R}^n), \tag{5-3}$$

or equivalently $E[\Xi_j f(X)] = E[\partial_j f(X)]$ for each j.

In fact, the integration-by-parts relation $E[\Xi \cdot f(X)] = E[\nabla f(X)]$ makes sense even if we do not assume that X has a smooth density. Following the terminology used by Voiculescu in the free case, if X is an $\mathbb{R}^n$-valued random variable on the probability space (Ω, P), we say that an $\mathbb{R}^n$-valued random variable $\Xi \in L^2(\Omega, P)$ is a (*classical*) *conjugate variable for* X if $E[\Xi \cdot f(X)] = E[\nabla f(X)]$ and if each Ξ_j is in the closure of $\{f(X) : f \in C_c^\infty(\mathbb{R}^n)\}$ in $L^2(\Omega, P)$.

In other words, this means that Ξ is a function of X (up to almost sure equivalence) and satisfies the integration-by-parts relation. Since the integration-by-parts relation uniquely determines the $L^2(\Omega, P)$ inner product of Ξ_j and $f(X)$ for all $f \in C_c^\infty(\mathbb{R}^n)$, it follows that the conjugate variable is unique (up to almost sure equivalence), and that it is given by $f(X)$ for some f that only depends on the law of X. Thus, we may unambiguously define the *Fisher information* $\mathcal{I}(\mu) = E[|\Xi|^2]$ if $X \sim \mu$ and Ξ is a conjugate variable to X, and $\mathcal{I}(\mu) = +\infty$ if no conjugate variable exists.

The probabilistic viewpoint enables us to produce conjugate variables and estimate Fisher information using conditional expectations. (See [Voiculescu 1998, Proposition 3.7] for the free case.)

Lemma 5.6. *Suppose that X and Y are independent $\mathbb{R}^n$-valued random variables with $X \sim \mu$ and $Y \sim \nu$. If Ξ is a conjugate variable for X, then $E[\Xi | X+Y]$ is a conjugate variable for $X+Y$. In particular,*

$$\mathcal{I}(\mu * \nu) \leq \min(\mathcal{I}(\mu), \mathcal{I}(\nu)).$$

Proof. Because X and Y are independent, we have for $g \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ that $E[\Xi_j g(X,Y)] = E[\partial_{X_j} g(X,Y)]$. In particular, if $f \in C_c^\infty(\mathbb{R}^n)$, then

$$E[\Xi_j f(X+Y)] = E[\partial_{X_j}(f(X+Y))] = E[(\partial_j f)(X+Y)].$$

But $E[\Xi_j | X+Y]$ is the orthogonal projection onto the closed span of $\{f(X+Y) : f \in C_c^\infty(\mathbb{R}^n)\}$ and hence

$$E[E[\Xi_j | X+Y] f(X+Y)] = E[\partial_j f(X+Y)].$$

So $\mathcal{I}(\mu * \nu) = E[|E[\Xi | X+Y]|^2] \leq E[|\Xi|^2] = \mathcal{I}(\mu)$. By symmetry, $\mathcal{I}(\mu * \nu) \leq \mathcal{I}(\nu)$. □

The entropy of a measure μ can be recovered by integrating the Fisher information of $\mu * \gamma_t$. The following integral formula was the motivation for Voiculescu's definition of nonmicrostates free entropy χ^*. For the reader's convenience, we include a statement and proof in the random matrix setting with free probabilistic normalizations. See also [Barron 1986, Lemma 1; Voiculescu 1998, Proposition 7.6]. Recall that we identify $M_N(\mathbb{C})_{\mathrm{sa}}^m$ with $\mathbb{R}^{mN^2}$ using the orthonormal basis given in Section 2A rather than entrywise coordinates (since some entries are real and some are complex).

Lemma 5.7. *Let μ be a probability measure on $M_N(\mathbb{C})_{\mathrm{sa}}^m$ with finite variance and with density ρ, and let $\sigma_{t,N}$ be the law of m independent GUEs of normalized variance t. If $a = (1/m)\int \|x\|_2^2 \, d\mu(x) = (1/(mN))\int |x|^2 \, d\mu(x)$, then we have for $t \geq 0$ that*

$$\frac{m}{a+t} \leq \frac{1}{N^3}\mathcal{I}(\mu * \sigma_{t,N}) \leq \min\Big(\frac{m}{t}, \frac{1}{N^3}\mathcal{I}(\mu)\Big). \tag{5-4}$$

Moreover,

$$\frac{1}{N^2}h(\mu * \sigma_{t,N}) - \frac{1}{N^2}h(\mu) = \frac{1}{2}\int_0^t \frac{1}{N^3}\mathcal{I}(\mu * \sigma_{s,N}) \, ds \tag{5-5}$$

and

$$\frac{1}{N^2}h(\mu) + \frac{m}{2}\log N = \frac{1}{2}\int_0^\infty \Big(\frac{m}{1+s} - \frac{1}{N^3}\mathcal{I}(\mu * \sigma_{s,N})\Big) \, ds + \frac{m}{2}\log 2\pi e. \tag{5-6}$$

Proof. To prove (5-4), suppose $t \geq 0$ and let X and Y be random variables with the laws μ and $\sigma_{t,N}$ respectively. The lower bound is trivial if $\mathcal{I}(\mu * \sigma_{t,N}) = +\infty$, so suppose that $X + Y$ has a conjugate variable Ξ. Then after some computation, the integration-by-parts relation shows that $E\langle \Xi, X+Y\rangle_{\mathrm{Tr}} = mN^2$. Thus,

$$E[|\Xi|^2] \geq \frac{|E\langle \Xi, X+Y\rangle_{\mathrm{Tr}}|^2}{E|X+Y|^2} = \frac{(mN^2)^2}{N(ma+mt)} = \frac{N^3}{a+t}$$

since the variance of Y with respect to the nonnormalized inner product is Nmt and the variance of X is Na. The upper bound is trivial in the case where $t = 0$. If $t > 0$, then by the previous lemma $\mathcal{I}(\mu * \sigma_{t,N}) \leq \min(\mathcal{I}(\mu), \mathcal{I}(\sigma_{t,N}))$. Moreover, a direct computation shows that if $Y \sim \sigma_{t,N}$, then the conjugate variable is $(N/t)Y$ and the Fisher information is mN^3/t.

Next, to prove (5-5), let $\mu_t := \mu * \sigma_{t,N}$. By basic properties of convolving positive functions with the Gaussian, μ_t has a smooth density ρ_t. We claim that if $0 < \delta < t$, then

$$h(\mu_t) - h(\mu_\delta) = \frac{1}{2N}\int_\delta^t \mathcal{I}(\mu_s) \, ds = \frac{1}{2N}\int_\delta^t \int_{M_N(\mathbb{C})_{\mathrm{sa}}^m} \frac{|\nabla \rho_s(x)|^2}{\rho_s(x)} \, dx \, ds. \tag{5-7}$$

This will follow from integration by parts, but to give a complete justification, we first introduce a smooth compactly supported "cutoff" function $\psi_R : M_N(\mathbb{C})_{\mathrm{sa}}^m \to \mathbb{R}$ such that $0 \leq \psi_R \leq 1$ and $\psi_R(x) = 1$ when $|x| \leq R$ and $\psi_R(x) = 0$ when $|x| \geq 2R$. By taking ψ_R to be rescaling by R of some fixed function, we can arrange that $\|\nabla \psi_R(x)\|_2 \leq C/R$ for some constant C. Because $\partial_s \rho_s = (2N)^{-1}\Delta \rho_s$, we have

$$\begin{aligned}\frac{d}{dt}\Big[-\int \psi_R \rho_s \log \rho_s\Big] &= -\frac{1}{2N}\int \psi_R \cdot (\Delta \rho_s \log \rho_s + \Delta \rho_s) \\ &= \frac{1}{2N}\int \psi_R \frac{|\nabla \rho_s|^2}{\rho_s} + \frac{1}{2N}\int \nabla \psi_R \cdot \nabla \rho_s \cdot (1 + \log \rho_s),\end{aligned}$$

where all the integrals are taken over $M_N(\mathbb{C})_{\mathrm{sa}}^m$ with respect to dx. This implies

$$-\int \psi_R \rho_t \log \rho_t + \int \psi_R \rho_\delta \log \rho_\delta = \frac{1}{2N} \int_\delta^t \int \psi_R \left| \frac{\nabla \rho_s}{\rho_s} \right|^2 d\mu_s - \frac{1}{2N} \int_\delta^t \int \left(\nabla \psi_R \cdot \frac{\nabla \rho_s}{\rho_s} \right) (1 + \log \rho_s) \, d\mu_s. \quad \text{(5-8)}$$

We must now take the limit of each term as $R \to +\infty$. For the first term on the right-hand side, the monotone convergence theorem yields

$$\lim_{R \to +\infty} \int_\delta^t \int \psi_R \left| \frac{\nabla \rho_s}{\rho_s} \right|^2 d\mu_s \, ds = \int_\delta^t \mathcal{I}(\mu_s) \, ds.$$

The second term on the right-hand side of (5-8) can be estimated as follows. Note that $\mu_s = \mu * \sigma_{s,N}$ and that $\sigma_{s,N}$ has a density that is bounded uniformly for $s \in [\delta, t]$ and $x \in M_N(\mathbb{C})_{\mathrm{sa}}^m$. Therefore, ρ_s is uniformly bounded for $s \in [\delta, t]$ and $x \in M_N(\mathbb{C})_{\mathrm{sa}}^m$ and hence $\log \rho_s$ is uniformly bounded above. To obtain a lower bound on $\log \rho_s$, first note that there is a $K > 0$ such that

$$\mu(x : |x| \leq K) \geq \tfrac{1}{2}.$$

Now if $x \in M_N(\mathbb{C})_{\mathrm{sa}}^m$ and $|y| \leq K$, then $|x - y| \geq |x| - K$ and hence

$$|x - y|^2 \leq |x|^2 - 2K|x| + K^2 \geq 2|x|^2 + 2K^2,$$

where the last inequality follows because $2K|x| \leq |x|^2/2 + 2K^2$ by the arithmetic-geometric mean inequality. Therefore, letting Z be the normalizing constant for $\sigma_{t,N}$, we have

$$\begin{aligned} \rho_s(x) &= \frac{1}{Z} \int e^{-(N/(2t))|x-y|^2} \, d\mu(y) \geq \frac{1}{Z} \int_{|y| \leq K} \int e^{-(N/(2t))|x-y|^2} \, d\mu(y) \\ &\geq \frac{1}{Z} \int_{|y| \leq K} e^{-(N/t)(|x|^2 + K^2)} \, d\mu(y) \geq \frac{e^{-NK^2/t}}{2Z} e^{-(N/t)|x|^2}, \end{aligned}$$

so that $\log \rho_s \geq K' - |x|^2$ for some constant K'. In particular, combining our upper and lower bounds, there is a constant α such that for sufficiently large x, we have $|1 + \log \rho_s| \leq \alpha |x|^2$. Recall that $\nabla \psi_R(x)$ is supported when $R \leq |x| \leq 2R$ and bounded by C/R and thus $|\nabla \psi_R(x)| \leq C/|x|$. Altogether we have $|\nabla \psi_R (1 + \log \rho_s)| \leq \beta |x|$ for some constant β when $|x|$ is large enough. Thus, the second term on the right-hand side of (5-8) is bounded by

$$\begin{aligned} \int_\delta^t \int |(\nabla \psi_R \cdot \Xi_s)(1 + \log \rho_s)| \, d\mu_s \, ds &\leq \beta \int_\delta^t \int_{|x| \geq R} |x| \left| \frac{\nabla \rho_s(x)}{\rho_s(x)} \right| d\mu_s(x) \, ds \\ &\leq \frac{\beta}{2} \int_\delta^t \int_{|x| \geq R} \left(|x|^2 + \left| \frac{\nabla \rho_s(x)}{\rho_s(x)} \right|^2 \right) d\mu_s(x) \, ds. \end{aligned}$$

The right-hand side is the tail of the convergent integral

$$\int_\delta^t \int \left(|x|^2 + \left| \frac{\nabla \rho_s(x)}{\rho_s(x)} \right|^2 \right) d\mu_s(x) \, ds = \int_\delta^t [(a + ms) + \mathcal{I}(\mu_s)] \, ds < +\infty,$$

and therefore it goes to zero as $R \to +\infty$ by the dominated convergence theorem.

As for the left-hand side of (5-8), we can apply the dominated convergence theorem to $-\int \psi_R \rho_t \log \rho_t$ and $-\int \psi_R \rho_\delta \log \rho_\delta$ given our earlier estimate that ρ_s is subquadratic for each s. Thus, after taking $R \to \infty$ in (5-8), we obtain (5-7).

To complete the proof of (5-5), we must take $\delta \searrow 0$ in (5-7). We can take the limit of the right-hand side of (5-7) by the monotone convergence theorem. For the left-hand side of (5-7), Lemma 5.1(3) implies that $\limsup_{\delta \searrow 0} h(\mu_\delta) \leq h(\mu)$ because $\rho_\delta \to \rho$ almost everywhere by Lebesgue differentiation theory. On the other hand, $h(\mu_\delta) \geq h(\mu)$ by Lemma 5.1(4); hence $h(\mu_\delta) \to h(\mu)$, so (5-5) is proved.

To prove (5-6), we follow [Voiculescu 1998, Proposition 7.6]. First, suppose that $h(\mu) > -\infty$. Note that

$$h(\mu) = \frac{1}{2} \int_0^t \Big(\frac{mN^2}{1+s} - \frac{1}{N} \mathcal{I}(\mu_s) \Big) \, ds - \frac{mN^2}{2} \log(1+t) + h(\mu_t).$$

If $h(\mu) > -\infty$, then

$$\int_0^1 \Big(\frac{mN^2}{1+s} - \frac{1}{N} \mathcal{I}(\mu_s) \Big) \, ds$$

is finite. In light of (5-4), the integral from 1 to ∞ is also finite and by the dominated convergence theorem

$$\lim_{t \to \infty} \frac{1}{2} \int_0^t \Big(\frac{mN^2}{1+s} - \frac{1}{N} \mathcal{I}(\mu_s) \Big) \, ds = \frac{1}{2} \int_0^\infty \Big(\frac{mN^2}{1+s} - \frac{1}{N} \mathcal{I}(\mu_s) \Big) \, ds.$$

It remains to understand the behavior of $h(\mu_t) - (mN^2/2) \log(1+t)$. By Lemma 5.1(4) and (2),

$$h(\mu_t) \geq h(\sigma_{t,N}) = \frac{mN^2}{2} \log \frac{2\pi e t}{N} = \frac{mN^2}{2} \log \frac{2\pi e}{N} + \frac{mN^2}{2} \log t.$$

On the other hand, by Lemma 5.1(2), since $\int |x|^2 \, d\mu_t(x) = N(a + tm)$, we have

$$h(\mu_t) \leq \frac{mN^2}{2} \log \frac{2\pi e(a+t)}{N} = \frac{mN^2}{2} \log \frac{2\pi e}{N} + \frac{mN^2}{2} \log(a+t).$$

As $t \to \infty$, we have $\log(1+t) - \log(a+t) \to 0$ and $\log(1+t) - \log t \to 0$ and therefore

$$h(\mu_t) - \frac{mN^2}{2} \log(1+t) \to \frac{mN^2}{2} \log \frac{2\pi e}{N} = \frac{mN^2}{2} \log 2\pi e - \frac{mN^2}{2} \log N.$$

Hence,

$$h(\mu) = \frac{1}{2} \int_0^\infty \Big(\frac{mN^2}{1+s} - \frac{1}{N} \mathcal{I}(\mu_s) \Big) \, ds + \frac{mN^2}{2} \log 2\pi e - \frac{mN^2}{2} \log N,$$

which is equivalent to the asserted formula (5-6). In the case where $h(\mu) = -\infty$, we also have

$$\int_0^1 \Big(\frac{mN^2}{1+s} - \frac{1}{N} \mathcal{I}(\mu_s) \Big) \, ds = -\infty$$

by (5-5), but the integral from 1 to ∞ is finite as shown above. So both sides of (5-6) are $-\infty$. □

5D. ***Free Fisher information.*** The starting point for the definition of free Fisher information is the integration-by-parts formula (5-3). Indeed, if we formally apply this to a noncommutative polynomial p and renormalize, we obtain

$$\int \tau_N \Big(\frac{1}{N} \Xi_j(x) p(x) \Big) \, d\mu(x) = \int \tau_N \otimes \tau_N (\mathcal{D}_j p(x)) \, d\mu(x), \tag{5-9}$$

(and this integration by parts is justified under sufficient assumptions of finite moments). Voiculescu therefore made the following definitions:

Definition 5.8 [Voiculescu 1998, §3]. Let $X = (X_1, \dots, X_m)$ be a tuple of self-adjoint random variables in a tracial von Neumann algebra $(\mathcal{M}, \tau)$ and assume that $\mathcal{M}$ is generated by X as a von Neumann algebra. We say that $\xi = (\xi_1, \dots, \xi_m) \in L^2(\mathcal{M}, \tau)^m$ is a *(free) conjugate variable* of X if

$$\tau(\xi_j p(X)) = \tau \otimes \tau(\mathcal{D}_j p(X)) \tag{5-10}$$

for every noncommutative polynomial p. The free conjugate variable, if it exists, is unique. If it exists, we say that X (or equivalently the law of X) has finite free Fisher information and define $\Phi^*(X) := \Phi^*(\lambda_X) := \sum_j \tau(\xi_j^2)$. We also denote the conjugate variable ξ by $J(X)$.

Definition 5.9 [Voiculescu 1998, Definition 7.1]. The *nonmicrostates free entropy* of a noncommutative law λ is

$$\chi^*(\lambda) := \frac{1}{2} \int_0^\infty \Big(\frac{m}{1+t} - \Phi^*(\lambda \boxplus \sigma_t) \Big) + \frac{1}{2} \log 2\pi e.$$

Now we are ready to state conditions under which the classical Fisher information of a sequence of measures μ_N converges to the free Fisher information of the law λ. First, to clarify the normalization, note that if $d\mu_N(x) = (1/Z_N) \exp(-N^2 V_N(x))\, dx$, then the classical conjugate variable is given by $\Xi_N = N^2 \nabla V_N$. The normalized conjugate variable used in (5-9) is $(1/N)\Xi_N = N \nabla V_N = DV_N$. The corresponding normalized Fisher information is then

$$\int \|DV_N\|_2^2 \, d\mu_N = \int \frac{1}{N} \left| \frac{1}{N} \Xi_N \right|^2 d\mu = \frac{1}{N^3} \mathcal{I}(\mu_N),$$

which is the same normalization as in Lemma 5.7.

Proposition 5.10. *Let $V_N : M_N(\mathbb{C})_{\mathrm{sa}}^m \to \mathbb{R}$ be a potential with $\int \exp(-N^2 V_N(x))\, dx < +\infty$ and let μ_N be the associated measure. Assume*:

(A) *The sequence μ_N concentrates around a noncommutative law λ.*

(B) *The sequence $\{DV_N\}$ is asymptotically approximable by trace polynomials.*

(C) *For some $n \geq 0$ and $a, b > 0$ we have $\|DV_N\|_2^2 \leq a + b \sum_{j=1}^m \tau_N(x_j^{2n})$.*

(D) *There exists $R_0 > 0$ such that*

$$\lim_{N \to \infty} \int_{\|x\|_\infty \geq R_0} \Big(1 + \sum_{j=1}^m \tau_N(x_j^{2n}) \Big) \, d\mu_N(x) = 0.$$

Then:

(1) *The law λ can be realized by self-adjoint random variables $X = (X_1, \dots, X_m)$ in a tracial von Neumann algebra $(\mathcal{M}, \tau)$ with $\|X_j\| \leq R_0$.*

(2) *There exists a sequence of trace polynomials $f^{(k)} \in (\mathrm{TrP}_m^1)^m$ such that*

$$\lim_{k \to \infty} \limsup_{N \to \infty} \sup_{\|x\|_\infty \leq R_0} \|DV_N(x) - f^{(k)}(x)\|_2 = 0.$$

(3) *If $\{f^{(k)}\}$ is any sequence as in* (2), *then $\{f_k(X)\}$ converges in $L^2(\mathcal{M},\tau)$ and the limit is the conjugate variable $J(X)$.*

(4) *The law λ has finite free Fisher information and $N^{-3}\mathcal{I}(\mu_N) \to \Phi^*(\lambda)$ as $N \to \infty$.*

Proof. (1) This follows from the same argument as Proposition 5.5.

(2) This follows from the definition of asymptotic approximability by trace polynomials.

(3) Let $\{f^{(k)}\}$ be a sequence as in (2). Because μ_N concentrates around λ and $\mu_N(\{x : \|x\|_\infty \le R_0\}) \to 1$ as $N \to +\infty$ by (D), we have

$$\lambda[(f^{(j)} - f^{(k)})^*(f^{(j)} - f^{(k)})] = \lim_{N\to\infty} \int_{\|x\|_\infty \le R_0} \tau_N[(f^{(j)} - f^{(k)})^*(f^{(j)} - f^{(k)})(x)]\, d\mu_N(x).$$

For every $\epsilon > 0$, if j and N are large enough, then $\sup_{\|x\|_\infty \le R_0} \|DV_N(x) - f^{(j)}(x)\|_2 < \epsilon$ by our assumption on $f^{(j)}$. In particular, if j and k are sufficiently large, then $\lambda[(f^{(j)} - f^{(k)})^*(f^{(j)} - f^{(k)})] < (2\epsilon)^2$. This shows that $\{f^{(k)}(X)\}$ is Cauchy in $L^2(M,\lambda)$ since X has the law λ.

Let $\xi = \lim_{k\to\infty} f^{(k)}(X)$. We must show that ξ is the conjugate variable for X. Let $\psi \in C_c^\infty(\mathbb{R})$ such that $\psi(y) = y$ when $|y| \le R_0$. For $x \in M_N(\mathbb{C})_{\text{sa}}^m$, let $\Psi(x) = (\psi(x_1), \dots, \psi(x_m))$. By (5-9), because $DV_N(x)$ is the classical conjugate variable for X, we have for every noncommutative polynomial p that

$$\int \tau_N[D_j V_N(x) \cdot p(\Psi(x))]\, d\mu_N(x) = \int D_j[\tau_N(p(\Psi(x)))]\, d\mu_N(x).$$

It follows from our assumptions (C) and (D) that

$$\lim_{N\to\infty} \int_{\|x\|_\infty \ge R_0} \|DV_N(x)\|_2^2\, d\mu_N(x) = 0.$$

Because $p(\Psi(x))$ and $D_j[\tau_N(p(\Psi(x)))]$ are globally bounded in operator norm, the integral of these quantities over $\|x\|_\infty \ge R_0$ will vanish as $N \to \infty$ and therefore

$$\int_{\|x\|_\infty \le R_0} \tau_N[D_j V_N(x) p(\Psi(x))]\, d\mu_N(x) - \int_{\|x\|_\infty \le R_0} D_j[\tau(p(\Psi(x)))]\, d\mu_N(x) \to 0.$$

But since $p(\Psi(x)) = p(x)$ on this region, we have

$$\int_{\|x\|_\infty \le R_0} \tau_N[D_j V_N(x) p(x)]\, d\mu_N(x) - \int_{\|x\|_\infty < R_0} \tau_N \otimes \tau_N[\mathcal{D}_j p(x)]\, d\mu_N(x) \to 0.$$

Now the second term converges to $\lambda \otimes \lambda[\mathcal{D}_j p] = \tau \otimes \tau[\mathcal{D}_j p(X)]$ by our concentration assumption (A). For the first term, we can replace $D_j V_N(x)$ by $f_j^{(k)}(x)$ with an error bounded by

$$\sup_{\|x\|_\infty \le R_0} \|f^{(k)}(x) - DV_N(x)\|_2.$$

Then we apply concentration to conclude that

$$\int \tau_N[f_j^{(k)}(x)^* p(x)]\, d\mu_N(x) \to \lambda[(f_j^{(k)})^* p].$$

Overall,

$$\left|\lambda[(f_j^{(k)})^* p] - \lambda \otimes \lambda[\mathcal{D}_j p]\right| \leq \limsup_{N\to\infty} \sup_{\|x\|_\infty \leq R_0} \|f^{(k)}(x) - DV_N(x)\|_2.$$

Taking $k \to \infty$, we obtain $\tau[\xi_j p(X)] - \tau \otimes \tau[\mathcal{D}_j p(X)] = 0$ as desired.

(4) We know from (3) that λ has finite Fisher information. Assumptions (C) and (D) imply that

$$\frac{1}{N^3}\mathcal{I}(\mu_N) - \int_{\|x\|_\infty \leq R_0} \|DV_N(x)\|_2^2 \, d\mu_N(x) \to 0.$$

By arguments similar to those before, we can approximate DV_N by $f^{(k)}$ on $\|x\|_\infty \leq R_0$, approximate $\int_{\|x\|_\infty \leq R_0} \|f^{(k)}\|_2^2 \, d\mu_N$ by $\lambda((f^{(k)})^* f^{(k)})$, and then approximate $\lambda((f^{(k)})^* f^{(k)})$ by $\tau(\xi^*\xi) = \Phi^*(\lambda)$, where the error terms vanish as $N \to \infty$ and then $k \to \infty$. This implies that $N^{-3}\mathcal{I}(\mu_N) \to \Phi^*(\lambda)$. □

6. Evolution of the conjugate variables

6A. ***Motivation and statement of the equation.*** In the last section, we stated conditions under which the classical entropy and Fisher information of μ_N converge to their free counterparts for the limiting noncommutative law λ. In order to prove that $\chi(\lambda) = \chi^*(\lambda)$, we want to take the limit in the integral formula (5-6), and therefore, we want $N^{-3}\mathcal{I}(\mu_N * \sigma_{t,N}) \to \Phi^*(\lambda \boxplus \sigma_t)$ for all $t \geq 0$. In order to apply Proposition 5.10 to $\mu_N * \sigma_{t,N}$, we need to show that $\{DV_{N,t}\}_N$ is asymptotically approximable by trace polynomials, where $V_{N,t}$ is the potential corresponding to $\mu_N * \sigma_{t,N}$.

By adding a constant to each V_N if necessary, we may assume without loss of generality that $Z_N = 1$. We recall that $V_{N,t}(x)$ is given by

$$\exp(-N^2 V_{N,t}(x)) = \int \exp(-N^2 V_N(x+y)) \, d\sigma_{t,N}(y). \tag{6-1}$$

Then $\exp(-N^2 V_{N,t}(x))$ solves the normalized heat equation

$$\partial_t[\exp(-N^2 V_{N,t}(x))] = \frac{1}{2N}\Delta[\exp(-N^2 V_{N,t}(x))], \tag{6-2}$$

where $(1/N)\Delta = L_N$ is the normalized Laplacian. However, we do not know how to show that $DV_N(\,\cdot\,, t)$ is asymptotically approximable by trace polynomials from a direct analysis of the heat equation because of the dimension-dependent factor of N^2 in the exponent. What we want is a dimension-independent and "hands-on" way of producing $V_{N,t}$ from V_N.

As in Section 4, we will analyze the PDE which describes the evolution of the function $V_{N,t}$. First, let us derive the equation by rewriting (6-2) in terms of $V_{N,t}$ rather than $e^{-N^2 V_{N,t}}$. By the chain rule,

$$\partial_t[\exp(-N^2 V_{N,t})] = -N^2 \partial_t V_{N,t} \cdot \exp(-N^2 V_{N,t})$$

and

$$\begin{aligned}\Delta[\exp(-N^2 V_{N,t})] &= [\Delta(-N^2 V_{N,t}) + |\nabla(-N^2 V_{N,t})|^2] \exp(-N^2 V_{N,t}) \\ &= (-N^2 \Delta V_{N,t} + N^4 |\nabla V_{N,t}|^2) \exp(-N^2 V_{N,t}),\end{aligned}$$

where Δ and ∇ denote the classical (nonnormalized) Laplacian and gradient, where $M_N(\mathbb{C})_{\mathrm{sa}}^m$ has been identified with $\mathbb{R}^{mN^2}$ using the coordinates in Section 2A. Thus, our equation becomes

$$-N^2 \partial_t V_{N,t} = \frac{1}{2N}(-N^2 \Delta V_{N,t} + N^4 |\nabla V_{N,t}|^2),$$
$$\partial_t V_{N,t} = \frac{1}{2N} \Delta V_{N,t} - \frac{N}{2} |\nabla V_{N,t}|^2.$$

Recall that $(1/N)\Delta$ is the normalized Laplacian discussed in Section 3B. The normalized gradient is $DV_{N,t} = N \nabla V_{N,t}$, and the normalized Euclidean norm is

$$\|x\|_2^2 = \sum_{j=1}^m \tau_N(x_j^2) = \frac{1}{N} \sum_{j=1}^m \mathrm{Tr}(x_j^2) = \frac{1}{N}|x|^2.$$

Then

$$N|\nabla V_{N,t}|^2 = \frac{1}{N}|N \nabla V_{N,t}|^2 = \frac{1}{N}|DV_{N,t}|^2 = \|DV_{N,t}\|_2^2,$$

and therefore we obtain the following equation that is normalized in a dimension-independent way:

$$\partial_t V_{N,t} = \tfrac{1}{2} L_N V_{N,t} - \tfrac{1}{2}\|DV_{N,t}\|_2^2. \tag{6-3}$$

In the remainder of this section, we study a semigroup R_t acting on convex and semiconcave functions on $M_N(\mathbb{C})_{\mathrm{sa}}^m$ such that $V_{N,t} = R_t V_N$ (here R_t depends implicitly on N). In Sections 6B–6F, we construct R_t from scratch by iterating the heat semigroup and Hopf–Lax semigroup. Next, in Section 6G, we verify that $R_t V_N$ solves (6-3) in the *viscosity sense* (for background, see [Crandall et al. 1992]), and deduce that $R_t V_N$ must agree with the smooth solution $V_{N,t}$ defined by (6-1). Finally, in Section 6H, we show that if $\{DV_N\}$ is asymptotically approximable by trace polynomials, then so is $\{D(R_t V_N)\}$.

6B. ***Strategy to approximate solutions.*** To construct the semigroup R_t that solves (6-3), we view the equation as a hybrid between the heat equation $\partial_t u = (2N)^{-1}\Delta u$ and the Hamilton–Jacobi equation with quadratic potential $\partial_t u = -\frac{1}{2}\|Du\|_2^2$. The heat equation can be solved by the heat semigroup

$$P_t u(x) := \int u(x+y)\, d\sigma_{t,N}(y), \tag{6-4}$$

while the Hamilton–Jacobi equation can be solved using the inf-convolution semigroup

$$Q_t u(x) := \inf\Big[u(x+y) + \frac{1}{2t}\|y\|_2^2\Big] \tag{6-5}$$

as a special case of the Hopf–Lax formula (see [Evans 2010, Chapter 3.3]).

In Dabrowski's approach, the solution to (6-3) was expressed through a formula of Boué, Dupuis, and Üstunel as the infimum of $E\big[u\big(x + B_t + \int_0^t Y_s\, ds\big) + \frac{1}{2}\int_0^t \|Y_s\|_2^2\, ds\big]$ over a certain class of stochastic processes Y_t adapted to a standard Brownian motion B_t (see [Dabrowski 2016, Theorem 3.1]). This formula, roughly speaking, combines the Gaussian convolution and inf-convolution operations by replacing the y in the definition of Q_t by a stochastic process and allowing it to evolve with B_t. Dabrowski [2016, Section 5] then identifies the minimizing process Y_t as a Brownian bridge and analyzes it using a forward-backward SDE. Through the Picard iteration for solving the SDE, he shows that the solution is well-approximated by noncommutative functions.

We instead give a deterministic proof following the same strategy as in Section 4 that is motivated by Trotter's formula, we define a semigroup $R_t u$ at dyadic times t by alternating between $P_{2^{-\ell}}$ and $Q_{2^{-\ell}}$ and then letting $\ell \to \infty$. We establish convergence through a telescoping series argument after showing that $P_t Q_t - Q_t P_t = o(t)$. Then we show that $R_t u$ depends continuously on t in order to extend its definition to all positive real t.

In contrast to Section 4, we must understand how the semigroups P_t, Q_t, and R_t affect Du as well as u, and we want $D(R_t u)$ to be Lipschitz for all t. We therefore view these operators as acting on spaces of the form

$$\mathcal{E}(c, C) = \{u : M_N(\mathbb{C})_{\text{sa}}^m \to \mathbb{R},\, u(x) - (c/2)\|x\|_2^2 \text{ is convex and } u(x) - (C/2)\|x\|_2^2 \text{ is concave}\},$$

where $0 \le c \le C < +\infty$, where we suppress the dependence on m and N in the notation. These spaces have the virtue that if $u \in \mathcal{E}(c, C)$, then $\|Du\|_{\text{Lip}} \le C$ automatically (see Proposition 2.13(4)).

At every step of the proof, we include estimates both for u and for Du. In addition, controlling the error propagation requires more work because Q_t and R_t are not contractions with respect to $\|Du\|_{L^\infty}$.

The following theorem summarizes the results of the construction. To clarify the notation, for a measurable function $u : M_N(\mathbb{C})_{\text{sa}}^m \to \mathbb{R}$, the notation $\|u\|_{L^\infty}$ is the standard L^∞ norm. If $F : M_N(\mathbb{C})_{\text{sa}}^m \to M_N(\mathbb{C})_{\text{sa}}^m$ (for instance $F = Du$ for some $u : M_N(\mathbb{C})_{\text{sa}}^m \to \mathbb{R}$), then $\|F\|_{L^\infty} = \sup_{x \in M_N(\mathbb{C})_{\text{sa}}^m} \|F(x)\|_2$; similarly, $\|F\|_{\text{Lip}}$ is the Lipschitz norm of F when using $\|\cdot\|_2$ in both the domain and the target space. Note that $\|F\|_2$ does *not* denote the L^2 norm of F with respect to any measure, but rather $\big(\sum_{j=1}^m \tau(F_j^2)\big)^{1/2}$, which is a function of x. Recall that $\mathbb{Q}_2^+$ denotes the nonnegative dyadic rationals, and $\mathbb{N}_0$ denote the natural numbers with zero included. Moreover, we assume throughout the section that $0 \le c \le C < +\infty$.

Theorem 6.1. *There exists a semigroup of nonlinear operators $R_t : \bigcup_{C>0} \mathcal{E}(0, C) \to \bigcup_{C>0} \mathcal{E}(0, C)$ with the following properties*:

(1) <u>Change in convexity</u>: *If $u \in \mathcal{E}(c, C)$ where $0 \le c \le C$, then $R_t u \in \mathcal{E}(c(1+ct)^{-1}, C(t+Ct)^{-1})$.*

(2) <u>Approximation by iteration</u>: *For $\ell \in \mathbb{Z}$ and $t \in 2^{-\ell}\mathbb{N}_0$, write $R_{t,\ell} u = (P_{2^{-\ell}} Q_{2^{-\ell}})^{2^\ell t} u$. Fix such a value of t and fix $u \in \mathcal{E}(0, C)$.*

   (a) *If $2^{-\ell-1} C \le 1$, then*

$$|R_t u - R_{t,\ell} u| \le \left(\frac{3}{2}\frac{C^2 m t}{1+Ct} + \log(1+Ct)(m + Cm + \|Du\|_2^2)\right) 2^{-\ell}.$$

   (b) $\|D(R_{t,\ell} u) - D(R_t u)\|_{L^\infty} \le [t/2 + C(t/2)^2] C^2 m^{1/2} (2 \cdot 2^{-\ell/2} + 2^{-3\ell/2} C)$.

(3) <u>Continuity in time</u>: *Suppose $s \le t \in \mathbb{R}_+$ and $u \in \mathcal{E}(0, C)$.*

   (a) $R_t u \le R_s u + (m/2)[\log(1+Ct) - \log(1+Cs)]$.

   (b) $R_t u \ge R_s u - ((t-s)/2)(Cm + \|Du\|_2^2)$.

   (c) *If $C(t-s) \le 1$, then* $\|D(R_t u) - D(R_s u)\|_2 \le 5Cm^{1/2} 2^{1/2} (t-s)^{1/2} + C(t-s)\|Du\|_2$.

(4) <u>Error estimates</u>: *Let $t \in \mathbb{R}_+$ and $u, v \in \mathcal{E}(0, C)$. Then*:

   (a) $\|D(R_t u) - D(R_t v)\|_{L^\infty} \le (1+Ct)\|Du - Dv\|_{L^\infty}$.

(b) *If* $u \leq v + a + b\|Dv\|_2^2$, *where* $a \in \mathbb{R}$ *and* $b \geq 0$, *then*

$$R_t u \leq R_t v + a + b\frac{C^2 mt}{1+Ct} + b\|D(R_t v)\|_2^2.$$

(c) *We have*

$$\|D(R_t u)\|_2^2 \leq \frac{C^2 mt}{1+Ct} + \|Du\|_2^2.$$

Remark 6.2. Knowing that $\exp(-N^2(R_t u)) = P_t \exp(-N^2 u)$, one can deduce (1) from the Brascamp–Lieb and Hölder inequalities, as in [Brascamp and Lieb 1976, Theorem 4.3]. But the proof of (1) given here is independent of that work.

We also point out that semigroups and discrete-time approximation schemes have been employed to study Hamilton–Jacobi equations in Hilbert space (e.g., by [Barbu 1986]), another setting that requires dimension-independent estimates.

6C. ***The Hopf–Lax semigroup, the heat semigroup, and convexity.*** We remind the reader of our standing assumption that $0 \leq c \leq C$.

Lemma 6.3. *Suppose* $u \in \mathcal{E}(c, C)$. *Then*:

(1) $P_t u \in \mathcal{E}(c, C)$.

(2) $\|D(P_t u) - Du\|_{L^\infty} \leq Cm^{1/2}t^{1/2}$.

Proof. (1) This follows because $\mathcal{E}(c, C)$ is closed under translation and averaging, and hence convolution by a probability measure.

(2) We know that Du is C-Lipschitz and thus

$$\begin{aligned}\|D(P_t u)(x) - Du(x)\|_2 &\leq \int \|Du(x+y) - Du(x)\|_2 \, d\sigma_{t,N}(y)\\ &\leq \int C\|y\|_2 \, d\sigma_{t,N}(y) \leq Cm^{1/2}t^{1/2}.\end{aligned}$$

□

The following lemma gives basic properties of Q_t from the PDE literature; see for instance [Ekeland and Lasry 1980, pp. 309–311; Lasry and Lions 1986; Crandall et al. 1992, Lemma A.5; Evans 2010, Section 3.3.2]. For completeness and convenience, we include a proof of all the facts we will use.

Lemma 6.4. (1) *If* $u, v : M_N(\mathbb{C})_{\mathrm{sa}}^m \to \mathbb{R}$ *and* $u \leq v$, *then* $P_t u \leq P_t v$ *and* $Q_t u \leq Q_t v$.

(2) *Suppose that* $v(x) = a + \frac{1}{2}\langle p, x\rangle_2 \langle Ax, x\rangle_2$, *where* $a \in \mathbb{R}$, $p \in M_N(\mathbb{C})_{\mathrm{sa}}^m$, *and* A *is a positive semidefinite linear map* $M_N(\mathbb{C})_{\mathrm{sa}}^m \to M_N(\mathbb{C})_{\mathrm{sa}}^m$. *Then*

$$\begin{aligned}P_t v(x) &= a + \frac{t}{2N^2}\operatorname{Tr}(A) + \langle p, x\rangle + \frac{1}{2}\langle Ax, x\rangle_2,\\ Q_t v(x) &= a - \frac{t}{2}\|p\|^2 + \langle p, x\rangle + \frac{1}{2}\langle A(1+tA)^{-1}(x-tp), x-tp\rangle.\end{aligned}$$

Remark 6.5. Here $\operatorname{Tr}(A)$ denotes the trace of A as a linear transformation of the vector space $M_N(\mathbb{C})_{\mathrm{sa}}^m$, which is well-defined because the trace of a matrix is similarity-invariant. In particular, we may compute

$\mathrm{Tr}(A)$ using an orthonormal basis of $M_N(\mathbb{C})_{\mathrm{sa}}^m$, and the choice of basis and of the normalization of the inner product does not matter. Note that the trace of the identity is mN^2, which makes the normalization in the above formula dimension-independent.

Proof of Lemma 6.4. (1) is immediate to check from the definition. We leave the first formula of (2) as an exercise. To prove the last formula, fix $t > 0$ and $x \in M_N(\mathbb{C})_{\mathrm{sa}}^m$ and note that $u(y) + (1/(2t))\|y - x\|_2^2$ is a uniformly convex function of y and therefore it has a unique minimizer. The minimizer y must be a critical point and hence

$$0 = Du(y) + \frac{1}{t}(y - x) = p + Ay + \frac{1}{t}(y - x).$$

Thus, $(1 + tA)y = x - tp$ and $y - x = -t(p + Ay)$, so that

$$Q_t u(x) = u(y) + \frac{1}{2t}\|y - x\|_2^2,$$

which reduces after several lines of manipulation to the claimed formula. □

Lemma 6.6. *Let $u \in \mathcal{E}(c, C)$.*

(1) *The operators $\{Q_t\}_{t\geq 0}$ form a semigroup; that is, $Q_s Q_t u = Q_{s+t} u$ for $s, t \geq 0$.*
(2) *For each $x_0 \in M_N(\mathbb{C})_{\mathrm{sa}}^m$ and $t \geq 0$, the infimum $Q_t u(x_0) = \inf_y [u(y) + (t/2)\|y - x_0\|_2^2]$ is achieved at a unique point y_0 satisfying $y_0 = x_0 - t Du(y_0)$.*
(3) *If $x_0 \in M_N(\mathbb{C})_{\mathrm{sa}}^m$ and y_0 is the minimizer from* (2), *then $D(Q_t u)(x_0) = Du(y_0)$.*
(4) *We have $Q_t u \in \mathcal{E}(c(1 + ct)^{-1}, C(1 + Ct)^{-1})$.*
(5) $\|D(Q_t u)(x_0)\|_2 = \|Du(y_0)\|_2 \leq (1 + ct)^{-1}\|Du(x_0)\|_2$.

Proof. (1) By definition

$$\begin{aligned} Q_s Q_t u(x) &= \inf_y \Big[Q_t u(y) + \frac{1}{2s}\|x - y\|_2^2 \Big] = \inf_y \inf_z \Big[u(z) + \frac{1}{2t}\|y - z\|_2^2 + \frac{1}{2s}\|x - y\|_2^2 \Big] \\ &= \inf_z \Big[u(z) + \inf_y \Big[\frac{1}{2t}\|y - z\|_2^2 + \frac{1}{2s}\|x - y\|_2^2 \Big]\Big]. \end{aligned}$$

But note that

$$\inf_y \Big[\frac{1}{2t}\|y - z\|_2^2 + \frac{1}{2s}\|x - y\|_2^2 \Big]$$

is by definition $Q_s f(z)$, where $f(x) = (1/(2t))\|x - z\|_2^2$. If $g(x) = (1/(2t))\|x\|_2^2$, then by the previous lemma, we have

$$Q_s g(x) = \frac{1}{2}\frac{t^{-1}}{1+t^{-1}s}\|x\|_2^2 = \frac{1}{2(s+t)}\|x\|_2^2.$$

Since Q_s is clearly translation-invariant, $Q_s f(x) = ((s + t)/2)\|x - z\|_2^2$. Therefore,

$$Q_s Q_t u(x) = \inf_z \Big[u(z) + \frac{1}{2(s+t)}\|x - z\|_2^2 \Big] = Q_{s+t} u(x).$$

(2) Fix x_0. Note that the function $y \mapsto [u(y) + (1/(2t))\|y - x_0\|_2^2]$ is in $\mathcal{E}(c + 1/(2t), C + 1/(2t))$ and hence it achieves a global minimum at the unique critical point. Thus, the infimum is achieved at the point y_0 satisfying $Du(y_0) = (1/t)(y_0 - x_0)$, or in other words $y_0 = x_0 - t\,Du(y_0)$.

(3), (4) Let x_0 and y_0 be as above. Let $p = Du(y_0)$. Because $u \in \mathcal{E}(c, C)$, we have for all x that

$$u(y_0) + \langle p, x - y_0\rangle_2 + \frac{c}{2}\langle x - y_0\rangle \le u(x) \le u(y_0) + \langle p, x - y_0\rangle_2 + \frac{C}{2}\|x - y_0\|_2^2.$$

Let $\underline{v}(y)$ and $\bar{v}(y)$ be the functions on the left and right-hand sides. Then by Lemma 6.4 (1), we have $Q_t\underline{v} \le Q_t u \le Q_t\bar{v}$. To compute $Q_t\underline{v}$, we apply Lemma 6.4 (2) with $A = cI$ and with a change of coordinates to translate y_0 to the origin, and we obtain

$$Q_t\underline{v}(x) = u(y_0) - \frac{t}{2}\|p\|_2^2 + \langle p, x - y_0\rangle + \frac{c}{2}(1 + ct)^{-1}\|x - y_0 - tp\|^2.$$

Since $y_0 + tp = x_0$ and $p = (y_0 - x_0)/t$, this becomes

$$\begin{aligned} Q_t\underline{v}(x) &= u(y_0) - \frac{t}{2}\|p\|_2^2 + t\|p\|_2^2 + \langle p, x - x_0\rangle + \frac{c}{2}(1 + ct)^{-1}\|x - x_0\|_2^2 \\ &= u(y_0) + \frac{1}{2t}\|y_0 - x_0\|_2^2 + \langle p, x - x_0\rangle + \frac{c}{2}(1 + ct)^{-1}\|x - x_0\|_2^2 \\ &= Q_t u(x_0) + \langle p, x - x_0\rangle + \frac{c}{2}(1 + ct)^{-1}\|x - x_0\|_2^2. \end{aligned}$$

The analogous computation holds for $Q_t\bar{v}$ as well. Thus, we have

$$Q_t u(x_0) + \langle p, x - x_0\rangle + \frac{c}{2}(1 + ct)^{-1}\|x - x_0\|_2^2 \le Q_t u(x) \le Q_t u(x_0) + \langle p, x - x_0\rangle + \frac{C}{2}(1 + Ct)^{-1}\|x - x_0\|_2^2.$$

This inequality implies that $D(Q_t u)(x_0) = p = Du(y_0)$. Since the above inequality holds for every x_0, we see that $Q_t u \in \mathcal{E}(c(1 + ct)^{-1}, C(1 + Ct)^{-1})$ by Proposition 2.13(2).

(5) Let x_0, y_0, and p be as above. Then we have

$$\langle Du(y_0) - Du(x_0), y_0 - x_0\rangle_2 \ge c\|y_0 - x_0\|_2^2.$$

But recall that $y_0 - x_0 = -t\,Du(y_0)$ and hence

$$-t\langle Du(y_0) - Du(x_0), Du(y_0)\rangle \ge ct^2\|Du(y_0)\|_2^2.$$

Rearranging produces

$$(1 + ct)\|Du(y_0)\|_2^2 \le \langle Du(x_0), Du(y_0)\rangle_2 \le \|Du(x_0)\|_2\|Du(y_0)\|_2,$$

and hence $(1 + ct)\|Du(y_0)\|_2 \le \|Du(x_0)\|_2$ as desired. □

Corollary 6.7. *Let $u \in \mathcal{E}(c, C)$ and $s, t \ge 0$.*

(1) *For each x, the gradient $D(Q_t u)(x)$ is the unique vector p satisfying $p = Du(x - tp)$.*

(2) *We have $Q_t u(x) = u(x - t D(Q_t u)(x)) + (t/2)\|D(Q_t u)(x)\|_2^2$.*

(3) $u(x) - (t/2)(1 + Ct)\|D(Q_t u)(x)\|^2 \le Q_t u(x) \le u(x) - (t/2)(1 + ct)\|D(Q_t u)(x)\|_2^2$.

Proof. (1) and (2) follow from Lemma 6.6(2) and (3).

To prove (3), fix x and let $y = x - tD(Q_t u)(x)$. By Proposition 2.13(2),

$$u(y) + \langle Du(y), x - y\rangle_2 + \frac{c}{2}\|x - y\|_2^2 \le u(x) \le u(y) + \langle Du(y), x - y\rangle + \frac{C}{2}\|x - y\|_2^2.$$

Hence,

$$u(x) - \langle Du(y), x - y\rangle_2 - \frac{C}{2}\|x - y\|_2^2 \le u(y) \le u(x) - \langle Du(y), x - y\rangle_2 - \frac{c}{2}\|x - y\|_2^2.$$

But from the previous lemma, we know that $Du(y) = D(Q_t u)(x)$ and $x - y = tD(Q_t u)(x)$, so that

$$u(x) - t\|D(Q_t u)(x)\|_2^2 - \frac{C}{2}t^2\|D(Q_t u)(x)\|_2^2 \le u(y) \le u(x) - t\|D(Q_t u)(x)\|_2^2 - \frac{c}{2}\|D(Q_t u)(x)\|_2^2.$$

Finally, we substitute $Q_t u(x) = u(y) + (t/2)\|D(Q_t u)(x)\|_2^2$ and obtain (3). □

6D. ***Estimates for error propagation.*** To prepare for our iteration procedure, we first prove some estimates to control the propagation of errors.

Lemma 6.8. *If $u, v \in \mathcal{E}(c, C)$, then we have*:

(1) $\|D(P_t u) - D(P_t v)\|_{L^\infty} \le \|Du - Dv\|_{L^\infty}$.

(2) $\|D(Q_t u) - D(Q_t v)\|_{L^\infty} \le (1 + Ct)\|Du - Dv\|_{L^\infty}$.

Proof. The first inequality follows because $D(P_t u) - D(P_t v)$ is the convolution of $Du - Dv$ with the Gaussian density. To prove the second inequality, note that

$$\begin{aligned}\|D(Q_t u)(x) - D(Q_t v)(x)\|_2 &= \|Du(x - tD(Q_t u)(x)) - Dv(x - tD(Q_t v)(x))\|_2 \\ &\le \|Du(x - tD(Q_t v)(x)) - Dv(x - tD(Q_t v)(x))\|_2 \\ &\qquad + \|Du(x - tD(Q_t u)(x)) - Du(x - tD(Q_t v)(x))\|_2 \\ &\le \|Du - Dv\|_{L^\infty} + Ct\|D(Q_t u)(x) - D(Q_t v)(x)\|_2,\end{aligned}$$

where the last inequality follows because Du is C-Lipschitz. This implies that for $t < 1/C$

$$\|D(Q_t u) - D(Q_t v)\|_{L^\infty} \le (1 - Ct)^{-1}\|Du - Dv\|_{L^\infty}.$$

Now we improve the estimate using the semigroup property of Q_t. Fix a positive integer k and for $j = 1, \dots, k$, let $t_j = tj/k$, and let $C_j = C(1 + Ct_j)^{-1}$. Then $Q_{t_j}u$ and $Q_{t_j}v$ are in $\mathcal{E}(0, C_j)$. Thus, we have

$$\|D(Q_{t_{j+1}}u) - D(Q_{t_{j+1}}v)\|_{L^\infty} \le \left(1 - \frac{C_j t}{k}\right)^{-1}\|D(Q_{t_j}u) - D(Q_{t_j}v)\|_{L^\infty},$$

and hence

$$\|D(Q_t u) - D(Q_t v)\|_{L^\infty} \le \|Du - Dv\|_{L^\infty}\prod_{j=0}^{k-1}\frac{1}{1 - C_j t/k}.$$

Now

$$\begin{aligned}\log \prod_{j=0}^{k-1} \frac{1}{1-C_j t/k} &= \sum_{j=0}^{k-1} -\log\left(1-\frac{C_j t}{k}\right) = \sum_{j=0}^{k-1}\left(\frac{C_j t}{k} + O\left(\frac{1}{k^2}\right)\right) \\ &= \sum_{j=0}^{k-1} \frac{C}{1+Ct_j}(t_{j+1}-t_j) + O\left(\frac{1}{k}\right) \\ &= \int_0^t \frac{C}{1+Cs}\,ds + O\left(\frac{1}{k}\right) = \log(1+Ct) + O\left(\frac{1}{k}\right).\end{aligned}$$

Hence,

$$\|D(Q_t u) - D(Q_t v)\|_{L^\infty} \leq \left(1 + Ct + O\left(\frac{1}{k}\right)\right)\|Du - Dv\|_{L^\infty},$$

and the proof is completed by taking $k \to \infty$. □

Lemma 6.9. *Suppose that $u : M_N(\mathbb{C})_{\mathrm{sa}}^m \to \mathbb{R}$ is convex and let $v \in \mathcal{E}(c, C)$ and $u \leq v + a + b\|Dv\|_2^2$ for some $a \in \mathbb{R}$ and $b \geq 0$.*

(1) $P_t u \leq P_t v + a + bC^2 mt + b\|D(P_t v)\|_2^2$.

(2) $Q_t u \leq Q_t v + a + b\|D(Q_t v)\|_2^2$.

Proof. (1) Using monotonicity and linearity of P_t, we have

$$P_t u \leq P_t v + a + b\int \|Dv(x+y)\|_2^2\,d\sigma(y).$$

So it suffices to show that

$$\int \|Dv(x+y)\|_2^2\,d\sigma_{t,N}(y) - \|D(P_t v)(x)\|_2^2 \leq C^2 mt.$$

In probabilistic terms, the left-hand side is the variance of the random variable $Dv(x+Y)$, where $Y \sim \sigma_{t,N}$. Since the variance is translation-invariant, this is the same as the variance of $Dv(x+Y) - Dv(x)$, and this is bounded above by the second moment

$$E\|Dv(x+Y) - Dv(x)\|_2^2 \leq C^2 \cdot E\|Y\|_2^2 = C^2 mt.$$

(2) Note that

$$\begin{aligned}Q_t u(x) &= \inf_y\left[u(y) + \frac{1}{2t}\|y-x\|_2^2\right] \\ &\leq u(x - tD(Q_t v)(x)) + \frac{t}{2}\|D(Q_t v)(x)\|_2^2 \\ &\leq v(x - tD(Q_t v)(x)) + \frac{t}{2}\|D(Q_t v)(x)\|_2^2 + a + b\|Dv(x - tD(Q_t v)(x))\|_2^2 \\ &= Q_t v(x) + a + b\|D(Q_t v)(x)\|_2^2,\end{aligned}$$

where the last equality follows from Corollary 6.7(1) and (2). □

Lemma 6.10. *Let $u \in \mathcal{E}(0, C)$. Then*:

(1) $\|D(Q_t u)\|_2^2 \leq \|Du\|_2^2$.

(2) $\|D(P_t u)\|_2^2 \leq C^2 mt + \|Du\|_2^2$.

Proof. The first claim follows from Lemma 6.6(5). To prove the second claim, note that by Minkowski's inequality

$$\begin{aligned}\|D(P_t u)(x)\|_2^2 &= \left\| \int Du(x+y)\, d\sigma_{t,N}(y) \right\|_2^2 \\ &\leq \int \|Du(x+y)\|_2^2\, d\sigma_{t,N}(y) \leq C^2 mt + \|Du(x)\|_2^2,\end{aligned}$$

where the last inequality was shown in the proof of Lemma 6.9(1). □

Next, we iterate the previous inequalities to obtain our main lemma on error propagation.

Lemma 6.11. *Let* $t_1, \dots, t_n > 0$ *and write*

$$\begin{aligned} t^* &= t_1 + \cdots + t_n, \\ R &= P_{t_n} Q_{t_n} \cdots P_{t_1} Q_{t_1}. \end{aligned}$$

Let $u, v \in \mathcal{E}(c, C)$.

(1) $Ru, Rv \in \mathcal{E}(c(1+ct^*)^{-1}, C(1+Ct^*)^{-1})$.

(2) $\|D(Ru) - D(Rv)\|_{L^\infty} \leq (1+Ct^*)\|Du - Dv\|_{L^\infty}$.

(3) *If* $u \leq v + a + b\|Dv\|_2^2$ *with* $a \in \mathbb{R}$ *and* $b \geq 0$, *then we have*

$$Ru \leq Rv + a + b\frac{C^2 mt^*}{1+Ct^*} + b\|D(Rv)\|_2^2.$$

In particular, $u \leq v$ *implies* $Ru \leq Rv$.

(4) *We have*

$$\|D(Ru)\|_2^2 \leq \frac{C^2 mt^*}{1+Ct^*} + \|Du\|_2^2 \leq Cm + \|Du\|_2^2.$$

Proof. (1) Let $u \in \mathcal{E}(c, C)$. Let $t_j^* = t_1 + \cdots + t_j$ and $u_j = P_{s_j} Q_{t_j} \cdots P_{s_1} Q_{t_1} u$. We show by induction that $u_j \in \mathcal{E}(c(1+ct_j^*)^{-1}, C(1+Ct_j^*)^{-1})$. The base case $j = 0$ is trivial. For the induction step, note that

$$\frac{c(1+ct_j^*)^{-1}}{1+[c(1+ct_j^*)^{-1}]t_{j+1}} = \frac{c}{(1+ct_j^*)+ct_{j+1}} = c(1+ct_{j+1}^*)^{-1}$$

and the same holds with c replaced by C. Hence, by Lemma 6.6(4), if $u_j \in \mathcal{E}(c(1+ct_j^*)^{-1}, C(1+Ct_j^*)^{-1})$, then $Q_{t_{j+1}} u_j \in \mathcal{E}(c(1+ct_{j+1}^*)^{-1}, C(1+Ct_{j+1}^*)^{-1})$. By Lemma 6.3, this implies that $u_{j+1} = P_{t_{j+1}} Q_{t_{j+1}} u_j \in \mathcal{E}(c(1+ct_{j+1}^*)^{-1}, C(1+Ct_{j+1}^*)^{-1})$. The same argument of course applies to v.

(2) Let t_j^* and u_j be as in the proof of (1) and define v_j similarly to u_j. We show by induction that $\|Du_j - Dv_j\|_{L^\infty} \leq (1+Ct_j^*)\|Du - Dv\|_{L^\infty}$. The base case $j = 0$ is trivial. For the induction step, recall that $u_j, v_j \in \mathcal{E}(c(1+ct_j^*)^{-1}, C(1+Ct_j^*)^{-1})$ and hence by Lemma 6.8 and the induction hypothesis

$$\begin{aligned}\|D(Q_{t_{j+1}} u_j) - D(Q_{t_{j+1}} v_j)\|_{L^\infty} &\leq (1 + C(1+Ct_j^*)^{-1} t_{j+1})\|Du_j - Dv_j\|_{L^\infty} \\ &\leq (1 + C(1+Ct_j^*)^{-1} t_{j+1})(1+Ct_j^*)\|Du - Dv\|_{L^\infty} \\ &= (1+Ct_{j+1}^*)\|Du - Dv\|_{L^\infty}.\end{aligned}$$

Then by Lemma 6.8 again, since $u_{j+1} = P_{t_{j+1}} Q_{t_{j+1}} u_j$ and $v_{j+1} = P_{t_{j+1}} Q_{t_{j+1}} v_j$, we have

$$\|Du_{j+1} - Dv_{j+1}\|_{L^\infty} \le (1 + Ct_{j+1}^*)\|Du - Dv\|_{L^\infty}.$$

This completes the induction and the case $j = n$ is the claim (2).

(3) First, we show by induction on j that

$$u_j \le v_j + a + b \sum_{i=1}^{j} \frac{C^2 m t_i}{(1 + Ct_i^*)^2} + b\|Dv_j\|_2^2.$$

The base case $j = 0$ is trivial. If the claim holds for u_j and v_j, then it also holds for $Q_{t_{j+1}} u_j$ and $Q_{t_{j+1}} v_j$ by Lemma 6.9(2). Then we apply Lemma 6.9(1) together with the fact that $Q_{t_{j+1}} u_j$ and $Q_{t_{j+1}} v_j$ are in $\mathcal{E}(c(1 + ct_{j+1}^*)^{-1}, C(t + Ct_{j+1}^*)^{-1})$ to conclude that

$$u_{j+1} \le v_{j+1} + a + b \sum_{i=1}^{j+1} \frac{C^2 m t_i}{(1 + Ct_i^*)^2} + b\|Dv_{j+1}\|_2^2.$$

This completes the induction. Finally, we observe that $\sum_{i=1}^n C^2 m t_i/(1 + Ct_i^*)^2$ is the lower Riemann sum for the function $C^2 m/(1 + Ct)^2$ on the interval $[0, t^*]$ with respect to the partition $\{0, t_1^*, \ldots, t_n^*\}$. Thus,

$$\sum_{i=1}^{n} \frac{C^2 m t_i}{(1 + Ct_i^*)^2} \le \int_0^{t^*} \frac{C^2 m}{(1 + Ct)^2}\, dt = Cm\left(1 - \frac{1}{1 + Ct^*}\right) = \frac{C^2 m t^*}{1 + Ct^*}.$$

This shows the main claim of (3), and the claim that $u \le v$ implies $Ru \le Rv$ is the special case when $a = 0$ and $b = 0$.

(4) By Lemma 6.10, we have $\|D(Q_{t_{j+1}} u_j)\|_2^2 \le \|Du_j\|_2^2$ and

$$\|Du_{j+1}\|_2^2 \le \frac{C^2 m t_{j+1}}{1 + Ct_{j+1}^*} + \|D(Q_{t_{j+1}} u_j)\|_2^2 \le \frac{C^2 m t_{j+1}}{1 + Ct_{j+1}^*} + \|Du_j\|_2^2.$$

We sum from $j = 0, \ldots, n - 1$ and obtain the same lower Riemann sum as in the proof of (3). The final estimate $Cm + \|Du\|_2^2$ follows because $C^2 mt/(1 + Ct) \le Cm$. □

6E. ***Iterative construction of R_t for dyadic t.*** We are now ready to carry out the Trotter's formula strategy and construct the semigroup for dyadic values of t. The next step is to show that the operators P_t and Q_t almost commute when t is small.

Lemma 6.12. *Let $u \in \mathcal{E}(c, C)$ and $t > 0$.*

(1) $\|D(Q_t P_t u) - D(P_t Q_t u)\|_{L^\infty} \le C^2 m^{1/2}(2 + Ct)t^{3/2}$.

(2) $P_t Q_t u \le Q_t P_t u$.

(3) *If $Ct \le 1$, then* $Q_t P_t u \le P_t Q_t u + 2C^2 mt^2 + 2Ct^2\|D(P_t Q_t u)\|_2^2$.

Proof. (1) Applying Corollary 6.7(1) to $P_t u$ yields

$$D(Q_t P_t u)(x) = D(P_t u)(x - tD(Q_t P_t u)(x)) = \int Du(x + y - tD(Q_t P_t u)(x))\, d\sigma_{t,n}(y).$$

On the other hand,

$$D(P_t Q_t u)(x) = \int D(Q_t u)(x+y)\, d\sigma_{t,n}(y) = \int Du(x+y-tD(Q_t u)(x+y))\, d\sigma_{t,n}(y).$$

Because Du is C-Lipschitz, we have

$$\|D(Q_t P_t u)(x) - D(P_t Q_t u)(x)\|_2 \leq Ct \int \|D(Q_t u)(x+y) - D(Q_t P_t u)(x)\|_2\, d\sigma_{t,n}(y).$$

We can estimate the integrand by

$$\|D(Q_t u)(x+y) - D(Q_t u)(x)\|_2 + \|D(Q_t u)(x) - D(Q_t P_t u)(x)\|_2.$$

Integrating the first term and using the fact that $D(Q_t u)$ is C-Lipschitz (since $Q_t u \in \mathcal{E}(0, C)$ by Lemma 6.6(4)), we have

$$\int \|D(Q_t u)(x+y) - D(Q_t u)(x)\|_2\, d\sigma_{t,n}(y) \leq C \int \|y\|_2\, d\sigma_{t,n} \leq Cm^{1/2} t^{1/2}.$$

Meanwhile, the second term is independent of y and thus it is unchanged when we integrate it against the probability measure $\sigma_{t,N}$, and this quantity can be estimated using Lemmas 6.8(2) and 6.3(2) as

$$\|D(Q_t u)(x) - D(Q_t P_t u)(x)\|_2 \leq (1+Ct)\|Du - D(P_t u)\|_{L^\infty} \leq (1+Ct)Cm^{1/2} t^{1/2}.$$

Altogether, we obtain

$$\|D(Q_t P_t u)(x) - D(P_t Q_t u)(x)\|_2 \leq C^2 m^{1/2}(2+Ct)t^{3/2}.$$

(2) The idea is that the average of the infimum is less than or equal to the infimum of the average. More precisely,

$$\begin{aligned} P_t Q_t u(x) &= \int \inf_y \Big(u(y) + \frac{1}{2t}\|(x+z) - y\|_2^2 \Big)\, d\sigma_{t,N}(z) \\ &= \int \inf_y \Big(u(y-z) + \frac{1}{2t}\|x-y\|_2^2 \Big)\, d\sigma_{t,N}(z) \\ &\leq \inf_y \int \Big(u(y-z) + \frac{1}{2t}\|x-y\|_2^2 \Big)\, d\sigma_{t,N}(z) \\ &= \inf_y \Big(P_t u(y) + \frac{1}{2t}\|x-y\|_2^2 \Big) = Q_t P_t u(x). \end{aligned}$$

(3) By Corollary 6.7(3),

$$Q_t P_t u \leq P_t u - \frac{t}{2}\|D(Q_t P_t u)\|_2^2. \tag{6-6}$$

Also by Corollary 6.7(3),

$$u \leq Q_t u + \frac{t}{2}(1+Ct)\|D(Q_t u)\|_2^2.$$

Hence, by Lemma 6.9, since $Q_t u \in \mathcal{E}(c(1+ct)^{-1}, C(1+Ct)^{-1}) \subseteq \mathcal{E}(0, C)$, we have

$$P_t u \leq P_t Q_t u + \frac{C^2 m t^2}{2}(1+Ct) + \frac{t}{2}(1+Ct)\|D(P_t Q_t)\|_2^2. \tag{6-7}$$

Plugging (6-7) into (6-6), we obtain

$$Q_t P_t u \leq P_t Q_t u + \frac{C^2 m t^2}{2}(1+Ct) - \frac{t}{2}\|D(Q_t P_t u)\|_2^2 + \frac{t}{2}(1+Ct)\|D(P_t Q_t)\|_2^2. \tag{6-8}$$

By using part (1), we have

$$\begin{aligned}\|D(Q_t P_t u)\|_2^2 &\geq [\|D(P_t Q_t)u\|_2 - C^2 m^{1/2} t^{3/2}(2+Ct)]^2 \\ &\geq \|D(P_t Q_t u)\|_2^2 - 2C^2 m^{1/2} t^{3/2}(2+Ct)\|D(P_t Q_t u)\|_2 \\ &\geq \|D(P_t Q_t u)\|_2^2 - (2+Ct)[C^3 m t^2 + Ct\|D(P_t Q_t u)\|_2^2],\end{aligned}$$

where the last step follows from the arithmetic-geometric mean inequality

$$2Cm^{1/2}t^{1/2}\|D(P_t Q_t u)\|_2 \leq C^2 m t + \|D(P_t Q_t u)\|_2^2.$$

So substituting our estimate for $\|D(Q_t P_t u)\|_2^2$ into (6-8), we see that $P_t Q_t u - Q_t P_t u$ is bounded by

$$\frac{C^2 m t^2}{2} + \frac{t}{2}(2+Ct)[C^3 m t^2 + Ct\|D(P_t Q_t u)\|_2^2] - \frac{t}{2}\|D(P_t Q_t u)\|_2^2 + \frac{t}{2}(1+Ct)\|D(P_t Q_t)\|_2^2.$$

Now we cancel the first-order terms $(t/2)\|D(P_t Q_t u)\|_2^2$ and we estimate $2+Ct$ by 3 using our assumption that $Ct \leq 1$. Thus, this is bounded by

$$\frac{C^2 m t^2}{2} + \frac{3t}{2}[C^3 m t^2 + Ct\|D(P_t Q_t u)\|_2^2] + \frac{Ct^2}{2}\|D(P_t Q_t u)\|_2^2 \leq 2C^2 m t^2 + 2Ct^2\|D(P_t Q_t u)\|_2^2,$$

where we have again used our assumption $Ct \leq 1$ to cancel a factor of Ct from the term $t \cdot C^3 m t^2$. □

Finally, we can construct the semigroup R_t for dyadic values of t. As in the statement of Theorem 6.1, we define $R_{t,\ell} u = (P_{2^{-\ell}} Q_{2^{-\ell}})^{2^\ell t} u$ whenever $\ell \in \mathbb{Z}$ and $t \in 2^{-\ell}\mathbb{N}_0$.

Lemma 6.13. *Let $C \geq 0$. For $t \in \mathbb{Q}_2^+$ and $u \in \mathcal{E}(0, C)$, the limit $R_t u = \lim_{\ell\to\infty} R_{t,\ell} u$ exists. Moreover, we have for $t \in 2^{-\ell}\mathbb{N}_0$ that:*

(1) $R_{t,\ell} u \leq R_t u$.

(2) *If $C/2^{\ell+1} \leq 1$, then*

$$R_t u \leq R_{t,\ell} u + \left(\frac{3}{2}\frac{C^2 m t}{1+Ct} + \log(1+Ct)(m + Cm + \|Du\|_2^2)\right)2^{-\ell}.$$

(3) $\|D(R_{t,\ell} u) - D(R_t u)\|_{L^\infty} \leq [t/2 + C(t/2)^2] C^2 m^{1/2}(2 \cdot 2^{-\ell/2} + 2^{-3\ell/2} C)$.

Proof. First, we prove some intermediate claims relating $R_{t,\ell} u$ and $R_{t,\ell+1} u$. To this end, we fix $\ell \in \mathbb{Z}$ and suppose $t = 2^{-\ell} n$ for some $n \in \mathbb{N}_0$. Let $\delta = 2^{-\ell-1}$. For $j = 0, \dots, n$, define

$$u_j = (P_\delta Q_\delta)^{2(n-j)}(P_{2\delta} Q_{2\delta})^j u.$$

and note that

$$u_0 = R_{t,\ell+1} u, \quad u_n = R_{t,\ell} u.$$

Let

$$v_j = Q_\delta (P_{2\delta} Q_{2\delta})^j u.$$

Then for $j = 1, \dots, n$, we have

$$u_{j-1} = [(P_\delta Q_\delta)^{2(n-j)} P_\delta](Q_\delta P_\delta v_{j-1}),$$
$$u_j = [(P_\delta Q_\delta)^{2(n-j)} P_\delta](P_\delta Q_\delta v_{j-1}).$$

We also define for $k = 1, \dots, 2n$,

$$C_k = C(1 + Ck\delta)^{-1}, \quad c_k = c(1 + ck\delta)^{-1}.$$

Thus, by Lemmas 6.11(1) and 6.6(4), we have $v_{j-1} \in \mathcal{E}(c_{2j-1}, C_{2j-1})$.

First, we claim that

$$R_{t,\ell} u \le R_{t,\ell+1} u. \tag{6-9}$$

Now by Lemma 6.12(2), we have

$$P_\delta Q_\delta v_{j-1} \le Q_\delta P_\delta v_{j-1}.$$

Hence, by monotonicity of P_t and Q_t (Lemma 6.11(3)), we have $u_j \le u_{j-1}$. Hence, $R_{t,\ell} u = u_n \le u_0 = R_{t,\ell+1} u$, proving (6-9).

For an inequality in the other direction, we claim that

$$R_{t,\ell+1} u \le R_{t,\ell} u + \left(\frac{3}{2} \frac{C^2 mt}{1 + Ct} + \log(1 + Ct)(m + Cm + \|Du\|_2^2) \right) 2^{-\ell-1}. \tag{6-10}$$

By Lemma 6.12(3), since $v_{j-1} \in \mathcal{E}(c_{2j-1}, C_{2j-1})$, we obtain

$$Q_\delta P_\delta v_{j-1} \le P_\delta Q_\delta v_{j-1} + 2C_{2j-1}^2 m\delta^2 + 2C_{2j-1}\delta^2 \|D(P_\delta Q_\delta v_{j-1})\|_2^2.$$

Thus, by Lemma 6.9(1), since $Q_\delta P_\delta v_{j-1}$ and $P_\delta Q_\delta v_{j-1}$ are in $\mathcal{E}(c_{2j}, C_{2j})$, we have

$$P_\delta Q_\delta P_\delta v_{j-1} \le P_{2\delta} Q_\delta v_{j-1} + 2C_{2j-1} m\delta^2 + 2C_{2j-1}\delta^2 (C_{2j}^2 m\delta + \|D(P_{2\delta} Q_\delta v_{j-1})\|_2^2).$$

Recalling that u_{j-1} and u_j are obtained by applying $(P_\delta Q_\delta)^{2(n-j)}$ to $P_\delta Q_\delta P_\delta v_{j-1}$ and $P_{2\delta} Q_\delta v_{j-1}$, and that $P_\delta Q_\delta P_\delta v_{j-1}$ and $P_{2\delta} Q_\delta v_{j-1}$ are in $\mathcal{E}(c_{2j}, C_{2j})$, we may apply Lemma 6.11(3) and to conclude that

$$u_{j-1} \le u_j + 2C_{2j-1} m\delta^2 + 2C_{2j-1}\delta^2 \left(C_{2j}^2 m\delta + \frac{C_{2j}^2 m(n-j)\delta}{1 + 2C_{2j}(n-j)\delta} + \|Du_j\|_2^2 \right).$$

By our assumption, $C_{2j}\delta \le C\delta \le 1$, and thus

$$C_{2j}^2 m\delta + \frac{C_{2j}^2 m(n-j)\delta}{1 + 2C_{2j}(n-j)\delta} \le C_{2j} m + \frac{C_{2j} m}{2} = \frac{3C_{2j} m}{2} \le \frac{3C_{2j-1} m}{2}.$$

Therefore,

$$u_{j-1} - u_j \le 2C_{2j-1} m\delta^2 + 3C_{2j-1}^2 m\delta^2 + 2C_{2j-1}\delta^2 \|Du_j\|_2^2.$$

By Lemma 6.11(4), we have $\|Du_j\|_2 \le Cm + \|Du\|_2^2$, and hence

$$u_{j-1} - u_j \le 3C_{2j-1}^2 m\delta^2 + 2C_{2j-1}\delta^2 (m + Cm + \|Du\|_2^2).$$

Therefore, summing from $j = 1, \dots, n$, we have

$$\begin{aligned} R_{t,\ell+1}u - R_{t,\ell}u &\le 3m\delta^2 \sum_{j=1}^n C_{2j-1}^2 + 2\delta^2(m + Cm + \|Du\|_2^2)\bigg(\sum_{j=1}^n C_{2j-1}\bigg) \\ &= \frac{3m\delta}{2}\bigg(\sum_{j=1}^n C_{2j-1}^2(2\delta)\bigg) + \delta(m + Cm + \|Du\|_2^2)\bigg(\sum_{j=1}^n C_{2j-1}(2\delta)\bigg). \end{aligned}$$

Recalling the definition of C_{2j-1}, two times the first sum is $\sum_{j=1}^n C^2(2\delta)/(1 + C(2j-1)\delta)^2$, which is the Riemann sum for the function $\phi(s) = C^2/(1 + Cs)^2$ on the interval $[0, t] = [0, 2n\delta]$, where we use a partition into subintervals of length 2δ and evaluate ϕ at the midpoint of each interval. Because ϕ is convex, the value of ϕ at the midpoint is less than or equal to the average value over the subinterval and therefore

$$\sum_{j=1}^n \frac{C^2(2\delta)}{(1 + C(2j-1)\delta)^2} \le \int_0^t \frac{C^2}{(1 + Cs)^2}\, ds = \frac{C^2 t}{1 + Ct}.$$

By similar reasoning,

$$\sum_{j=1}^n C_{2j-1}(2\delta) = \sum_{j=1}^n \frac{C\delta}{(1 + C(2j-1)\delta)} \le \int_0^t \frac{C}{1 + Cs}\, ds = \log(1 + Ct).$$

Therefore,

$$R_{t,\ell+1}u - R_{t,\ell}u \le \bigg(\frac{3}{2}\frac{C^2 mt}{1 + Ct} + \log(1 + Ct)(m + Cm + \|Du\|_2^2)\bigg)\delta,$$

which proves (6-10).

Together, (6-9) and (6-10) show that

$$|R_{t,\ell+1}u - R_{t,\ell}u| \le \bigg(\frac{3}{2}\frac{C^2 mt}{1 + Ct} + \log(1 + Ct)(m + Cm + \|Du\|_2^2)\bigg)2^{-\ell-1}.$$

Because the right-hand side is summable in ℓ, we see that the sequence $\{R_{t,\ell}u(x)\}_{\ell\in\mathbb{N}}$ is Cauchy and hence converges. Thus, $\lim_{\ell\to\infty} R_{t,\ell}u$ exists. Also, by (6-9) the convergence is monotone and thus $R_{t,\ell}u \le R_t u$, establishing (1). On the other hand, we obtain (2) by summing up the estimate (6-10) from ℓ to ∞ using the geometric series formula.

It remains to prove (3). We first claim that

$$\|D(R_{t,\ell+1}u) - D(R_{t,\ell}u)\|_{L^\infty} \le \bigg[\frac{t}{2} + C\Big(\frac{t}{2}\Big)^2\bigg]C^2 m^{1/2}(2 + 2^{-(\ell+1)}C)2^{-(\ell+1)/2}. \tag{6-11}$$

By Lemma 6.11(1), we know $Q_\delta P_\delta v_{j-1}$ and $P_\delta Q_\delta v_{j-1}$ are in $\mathcal{E}(c(1 + 2cj\delta)^{-1}, C(1 + 2Cj\delta)^{-1})$, and hence in $\mathcal{E}(0, C)$. Therefore, by Lemmas 6.11(2) and 6.12(1), we have

$$\begin{aligned} \|Du_j - Du_{j-1}\|_{L^\infty} &\le [1 + 2C(n-j)\delta]\|D(Q_\delta P_\delta v_j) - D(P_\delta Q_\delta v_j)\|_{L^\infty} \\ &\le [1 + 2C(n-j)\delta]C^2 m^{1/2}(2 + C\delta)\delta^{3/2}. \end{aligned}$$

Therefore,

$$\begin{aligned}\|D(R_{t,\ell+1}u)-D(R_{t,\ell}u)\|_{L^\infty} &\le \sum_{j=1}^{n}\|Du_j-Du_{j-1}\|_{L^\infty} \le \sum_{j=1}^{n}[1+2C(n-j)\delta]C^2m^{1/2}(2+C\delta)\delta^{3/2}\\ &= [n+Cn(n-1)\delta]C^2m^{1/2}(2+C\delta)\delta^{3/2}\\ &\le \left[\frac{t}{2}+C\left(\frac{t}{2}\right)^2\right]C^2m^{1/2}(2+C\delta)\delta^{1/2}\\ &= \left[\frac{t}{2}+C\left(\frac{t}{2}\right)^2\right]C^2m^{1/2}(2+2^{-(\ell+1)}C)2^{-(\ell+1)/2}\end{aligned}$$

since $2n\delta = t$. This proves (6-11).

Because $[t/2+C(t/2)^2]C^2m^{1/2}(2+2^{-(\ell+1)}C)2^{-(\ell+1)/2}$ is summable with respect to ℓ, we see that $\{D(R_{t,\ell}u)\}_{\ell\in\mathbb{N}}$ is Cauchy with respect to the L^∞ norm (even though the individual functions may not be in L^∞) and hence converges uniformly to some function. We already know that $R_{t,\ell}u$ converges to R_tu, so the limit of $D(R_{t,\ell}u)$ must be $D(R_tu)$. We obtain the estimate (3) by summing (6-11) from ℓ to ∞ using the geometric series formula. □

Corollary 6.14. *Let $0 \le c \le C$. Let $u, v \in \mathcal{E}(c, C)$ and let $t \ge 0$ be a dyadic rational.*

(1) $R_tu, R_tv \in \mathcal{E}(c(1+ct)^{-1}, C(1+Ct)^{-1})$.

(2) $\|D(R_tu)-D(R_tv)\|_{L^\infty} \le (1+Ct)\|Du-Dv\|_{L^\infty}$.

(3) *If $u \le v+a+b\|Dv\|_2^2$ for some $a \in \mathbb{R}$ and $b \ge 0$, then*

$$R_tu \le R_tv + a + b\frac{C^2mt}{1+Ct} + b\|D(R_tv)\|_2^2.$$

(4) $\|D(R_tu)\|_2^2 \le (C^2mt)/(1+Ct)+\|Du\|_2^2$.

Proof. We know that these properties hold for $R_{t,\ell}$ by Lemma 6.11. By Lemma 6.13, they also hold in the limit taking $\ell\to\infty$. (For (1), we rely on Proposition 2.13(1).) □

6F. ***Continuity and semigroup property.*** In order to extend R_t to all real $t \ge 0$, we prove estimates that show that R_t depends continuously on t. We begin with some simple estimates for P_t and Q_t.

Lemma 6.15. *Let $\ell \in \mathbb{Z}$ and suppose that $t \in 2^{-\ell}\mathbb{N}_0$ and $u \in \mathcal{E}(0, C)$. Then*:

(1) $u \le P_tu \le u+(C/2)mt$.

(2) $u-(t/2)\|Du\|_2^2 \le Q_tu \le u$.

(3) $\|D(Q_tu)-Du\|_2 \le Ct\|Du\|_2$.

Proof. (1) Because u is convex and $u(x)-(C/2)\|x\|_2^2$ is concave, we have

$$u(x)+\langle Du(x), y\rangle \le u(x+y) \le u(x)+\langle Du(x), y\rangle + \frac{C}{2}\|y\|_2^2.$$

Integrating with respect to $d\sigma_{t,N}(y)$ yields

$$u(x) \le P_tu(x) \le u(x)+\frac{Cmt}{2} \quad \text{for } u \in \mathcal{E}(0, C).$$

(2) As for the operator Q_t, it is immediate from the definition that $Q_t u \leq u$. On the other hand, using Corollary 6.7(2) and the convexity of u,

$$\begin{aligned} Q_t u(x) &= u(x - tD(Q_t u)(x)) + \frac{t}{2}\|D(Q_t u)(x)\|_2^2 \\ &\geq u(x) - t\langle D(Q_t u)(x), Du(x)\rangle_2 + \frac{t}{2}\|D(Q_t u)(x)\|_2^2 \\ &\geq u(x) - \frac{t}{2}\|Du(x)\|_2^2, \end{aligned}$$

where the last inequality follows because $\langle D(Q_t u)(x), Du(x)\rangle_2 \leq \frac{1}{2}\|D(Q_t u)(x)\|_2^2 + \frac{1}{2}\|Du(x)\|_2^2$.

(3) Using the fact that Du is C-Lipschitz, together with Corollary 6.7(1) and Lemma 6.6(5),

$$\begin{aligned} \|D(Q_t u)(x) - Du(x)\|_2 &= \|Du(x - tD(Q_t u)(x)) - Du(x)\|_2 \\ &\leq Ct\|D(Q_t u)(x)\|_2 \\ &\leq Ct\|Du(x)\|_2. \end{aligned}$$

□

Lemma 6.16. *Let $s \leq t$ be two numbers in $\mathbb{Q}_2^+$, and let $u \in \mathcal{E}(0, C)$.*

(1) $R_t u \leq R_s u + (m/2)[\log(1 + Ct) - \log(1 + Cs)]$.

(2) $R_t u \geq R_s u - ((t - s)/2)(Cm + \|Du\|_2^2)$.

(3) *If $C(t - s) \leq 1$, then* $\|D(R_t u) - D(R_s u)\|_2 \leq 5Cm^{1/2}2^{1/2}(t - s)^{1/2} + C(t - s)\|Du\|_2$.

Moreover, if $\ell \in \mathbb{Z}$ and if $s, t \in 2^{-\ell}\mathbb{N}_0$, then the same estimates hold with R_t replaced by $R_{t,\ell}$.

Proof. (1) Fix $\ell \in \mathbb{Z}$ and let $\delta = 2^{-\ell}$. Suppose $s = n\delta$ and $t = n'\delta$, where $n, n' \in \mathbb{N}_0$. By the previous lemma,

$$\begin{aligned} R_{(j+1)\delta,\ell}u &= P_\delta Q_\delta R_{j\delta,\ell}u \\ &\leq Q_\delta R_{j\delta,\ell}u + \frac{Cm\delta}{2(1 + C(j+1)\delta)} \\ &\leq R_{j\delta,\ell}u + \frac{Cm\delta}{2(1 + C(j+1)\delta)}, \end{aligned}$$

where we have used the fact that $Q_\delta R_{j\delta,\ell}u \in \mathcal{E}(0, C(1 + C(j+1)\delta)^{-1})$. Therefore,

$$R_{n'\delta,\ell}u \leq R_{n\delta,u} + \sum_{j=n}^{n'-1} \frac{Cm\delta}{2(1 + C(j+1)\delta)}.$$

Since the sum on the right-hand side is a lower Riemann sum for the function $Cm\delta/(2(1 + C\tau))$ for $\tau \in [s, t]$, we obtain

$$R_{t,\ell}u \leq R_{s,\ell}u + \frac{m}{2}[\log(1 + Ct) - \log(1 + Cs)].$$

We obtain (1) by letting $\ell \to +\infty$ and using Lemma 6.13.

(2) Let $\ell, \delta, s, t, n, n'$ be as above. By the previous lemma,

$$\begin{aligned} R_{(j+1)\delta,\ell} u = P_\delta Q_\delta R_{j\delta,\ell} u &\geq Q_\delta R_{j\delta,\ell} u \\ &\geq R_{j\delta,\ell} u - \frac{\delta}{2} \|D(R_{j\delta,\ell} u)\|_2^2 \\ &\geq R_{j\delta,\ell} u - \frac{\delta}{2}(Cm + \|Du\|_2^2), \end{aligned}$$

where the last inequality follows from Lemma 6.11(4). So when we sum from $j = n$ to $n' - 1$, we obtain

$$R_t u \geq R_s u - \frac{t-s}{2}(Cm + \|Du\|_2^2).$$

Then (2) follows by taking $\ell \to +\infty$.

(3) Assume that $s, t \in 2^{-\ell}\mathbb{N}_0$. Choose $k \in \mathbb{Z}$ such that $2^{-k-1} \leq t - s \leq 2^{-k}$. Then we may write $t - s$ in a binary expansion

$$t - s = \sum_{j=k+1}^{\ell} a_j 2^{-j},$$

where $a_j \in \{0, 1\}$ for each j and $a_{k+1} = 1$. Let

$$t_j = s + a_{k+1} 2^{-k-1} + \cdots + a_j 2^{-j}.$$

Let $u_j = R_{t_j,\ell} u$. We will estimate $\|Du_j(x) - Du_{j-1}(x)\|_2$ for each j. Of course, if $a_j = 0$, then $u_j = u_{j-1}$, so there is nothing to prove. On the other hand, suppose that $a_j = 1$. Now we estimate (at our given point x, suppressed in the notation)

$$\begin{aligned} \|D(R_{2^{-j},\ell} u_{j-1}) - Du_{j-1}\|_2 \leq{}& \|D(R_{2^{-j},\ell} u_{j-1}) - D(P_{2^{-j}} Q_{2^{-j}} u_{j-1})\|_2 \\ &+ \|D(P_{2^{-j}} Q_{2^{-j}} u_{j-1}) - D(Q_{2^{-j}} u_{j-1})\|_2 \\ &+ \|D(Q_{2^{-j}} u_{j-1}) - Du_{j-1}\|_2. \end{aligned} \tag{6-12}$$

The first term on the right-hand side may be estimated as follows. Recall that we proved Lemma 6.13(3) from the estimate (6-11) by summing the geometric series. The same reasoning shows that if $\ell \geq j$ and $\delta \in 2^{-\ell}\mathbb{N}_0$, then

$$\|D(R_{\delta,\ell} u_{j-1}) - D(R_{\delta,j} u_{j-1})\|_{L^\infty} \leq \left[\frac{\delta}{2} + C\left(\frac{\delta}{2}\right)^2\right] C^2 m^{1/2} (2 \cdot 2^{-j/2} + 2^{-3j/2} C)$$

since $u_{j-1} \in \mathcal{E}(0, C)$. If we substitute $\delta = 2^{-j}$, then $R_{2^{-j},j}$ is simply equal to $P_{2^{-j}} Q_{2^{-j}}$. Thus, at the point x,

$$\|D(R_{2^{-j},\ell} u_{j-1}) - D(P_{2^{-j}} Q_{2^{-j}} u_{j-1})\|_2 \leq C^2 m^{1/2} \left[\frac{2^{-j}}{2} + \frac{C 2^{-2j}}{4}\right] [2 \cdot 2^{-j/2} + 2^{-3j/2} C].$$

By our assumption $C 2^{-j} \leq C(t - s) \leq 1$ and hence we may replace $C 2^{-2j}/4$ by $2^{-j}/2$ and replace $2^{-3j/2} C$ by $2^{-j/2}$ and hence

$$\|D(R_{2^{-j},\ell} u_{j-1}) - D(P_{2^{-j}} Q_{2^{-j}} u_{j-1})\|_2 \leq 3 C^2 m^{1/2} 2^{-3j/2} \leq 3 C m^{1/2} 2^{-j/2}.$$

The second term on the right-hand side of (6-12) can be estimated by Lemma 6.3(2) by

$$\|D(P_{2^{-j}}Q_{2^{-j}}u_{j-1}) - D(Q_{2^{-j}}u_{j-1})\|_2 \le Cm^{1/2}2^{-j/2}$$

since $Q_{2^{-j}}u_{j-1} \in \mathcal{E}(0,C)$. The third term on the right-hand side of (6-12) can be estimated using Lemma 6.15(3) by

$$\|D(Q_{2^{-j}}u_{j-1}) - Du_{j-1}\|_2 \le C2^{-j}\|Du_{j-1}\|_2.$$

Meanwhile, by Lemma 6.11(4) and the triangle inequality

$$\|Du_{j-1}\|_2 \le \sqrt{Cm + \|Du\|_2^2} \le C^{1/2}m^{1/2} + \|Du\|_2.$$

So using the fact $C2^{-j} \le 1$, we have

$$\|D(Q_{2^{-j}}u_{j-1}) - Du_{j-1}\|_2 \le C^{3/2}m^{1/2}2^{-j} + C2^{-j}\|Du\|_2 \le Cm^{1/2}2^{-j/2} + C(t_j - t_{j-1})\|Du\|_2.$$

Therefore, plugging all our estimates into (6-12), we get

$$\|Du_j - Du_{j+1}\|_2 \le 5Cm^{1/2}2^{-j/2} + C(t_j - t_{j-1})\|Du\|_2.$$

Then summing from $j = k+1$ to ℓ we obtain

$$\begin{aligned}\|Du_\ell - Du_k\|_2 &\le 5Cm^{1/2}2^{-k/2} + C(t-s)\|Du\|_2 \\ &\le 5Cm^{1/2}2^{1/2}(t-s)^{1/2} + C(t-s)\|Du\|_2.\end{aligned}$$

Because $u_\ell = R_{t,\ell}u$ and $u_k = R_{s,\ell}u$, we have shown that (3) holds for $R_{s,\ell}$ and $R_{t,\ell}$ instead of R_s and R_t. Thus, (3) follows by taking $\ell \to +\infty$. □

Proof of Theorem 6.1. Lemma 6.16 shows that if $t \ge 0$ and if t_ℓ is a sequence of dyadic rationals converging to t as $\ell \to \infty$, then $R_{t_\ell}u$ converges to some function v and this function is independent of the approximating sequence, so we define $R_t u = v$. Claims (1), (3), and (4) of the theorem were proved for dyadic t in Corollary 6.14(1), Lemma 6.16, and Corollary 6.14(2)–(4) respectively, and each of these claims can be extended to real $t \ge 0$ in light of the continuity estimates Lemma 6.16. Claim (2) of the theorem is Lemma 6.13.

Thus, it remains to show that R_t is a semigroup. That is, we must show that $R_sR_tu = R_{s+t}u$ for $u \in \mathcal{E}(0,C)$ (and we have not even checked this for dyadic s, t yet). First, we check this property for real $s, t \ge 0$ under the additional restriction that $Ct \le \frac{1}{2}$. For each $\ell \in \mathbb{Z}$, there exist s_ℓ and $t_\ell \in 2^{-\ell}\mathbb{N}_0$ such that $s - 2^{-\ell} < s_\ell \le s$ and $t - 2^{-\ell} < t_\ell \le t$. By Lemma 6.16(1) and (2) we have

$$|R_{t_\ell}u - R_tu| \le \frac{|t_\ell - t|}{2}(Cm + \|Du\|_2^2) \le 2^{-\ell}\frac{1}{2}(Cm + \|Du\|_2^2),$$

since $|\log(1+Ct_\ell) - \log(1+Ct)| \le C|t_\ell - t|$ (from computation of the derivative of $\log(1+Ct)$). By Lemma 6.13(1) and (2), if $C2^{-\ell-1} \le 1$, then

$$|R_{t_\ell,\ell}u - R_{t_\ell}u| \le 2^{-\ell}\left(\frac{3}{2}\frac{C^2mt}{1+Ct} + \log(1+Ct_\ell)(m + Cm + \|Du\|_2^2)\right).$$

Since $t_\ell \leq t$, we can replace t_ℓ by t on the right-hand side. By the triangle inequality, we obtain

$$|R_{t_\ell,\ell}u - R_t u| \leq 2^{-\ell} K_t (1 + \|Du\|_2^2) \tag{6-13}$$

for some constant K_t depending on t (and C). Using Lemma 6.16(3), or rather its extension to real values of t,

$$\begin{aligned}\|D(R_t u) - Du\|_2 &\leq 5Cm^{1/2}2^{1/2}t^{1/2} + Ct\|Du\|_2 \\ &\leq 5Cm^{1/2}2^{1/2}t^{1/2} + Ct\|D(R_t u) - Du\|_2 + Ct\|D(R_t u)\|_2.\end{aligned}$$

Hence,

$$\|D(R_t u) - Du\|_2 \leq (1 - Ct)^{-1}[5Cm^{1/2}2^{1/2}t^{1/2} + Ct\|D(R_t u)\|_2],$$

so by the triangle inequality,

$$\|Du\|_2 \leq \|D(R_t u)\|_2 + (1 - Ct)^{-1}[5Cm^{1/2}2^{1/2}t^{1/2} + Ct\|D(R_t u)\|_2.$$

By squaring and applying the arithmetic-geometric mean inequality, we get

$$\|Du\|_2^2 \leq A_t + B_t\|D(R_t u)\|_2^2$$

for some constants A_t and B_t depending on t. The same reasoning applies to $R_{t_\ell,\ell}$ since Lemma 6.16(3) holds for $R_{t_\ell,\ell}$ also. We thus obtain

$$\begin{aligned}\|Du\|_2 &\leq \|D(R_{t_\ell,\ell}u)\|_2 + (1 - Ct_\ell)^{-1}[5Cm^{1/2}2^{1/2}t_\ell^{1/2} + Ct_\ell\|D(R_{t_\ell,\ell}u)\|_2 \\ &\leq \|D(R_{t_\ell,\ell}u)\|_2 + (1 - Ct)^{-1}[5Cm^{1/2}2^{1/2}t^{1/2} + Ct\|D(R_{t_\ell,\ell}u)\|_2\end{aligned}$$

and so

$$\|Du\|_2^2 \leq A_t + B_t\|D(R_{t_\ell,\ell}u)\|_2^2.$$

Overall,

$$\begin{aligned}R_t u &\leq R_{t_\ell,\ell}u + 2^{-\ell}K_t(1 + A_t + B_t\|D(R_{t_\ell,\ell}u)\|_2^2), \\ R_{t_\ell,\ell}u &\leq R_t u + 2^{-\ell}K_t(1 + A_t + B_t\|D(R_t u)\|_2^2).\end{aligned}$$

So by Lemma 6.11(3) and (4)

$$\begin{aligned}R_{s_\ell,\ell}R_t u &\leq R_{s_\ell,\ell}R_{t_\ell,\ell}u + 2^{-\ell}K_t(1 + A_t + B_t\|D(R_{s_\ell,\ell}R_{t_\ell,\ell}u)\|_2^2) \\ &\leq R_{s_\ell,\ell}R_{t_\ell,\ell}u + 2^{-\ell}K_t(1 + A_t + CmB_t + B_t\|Du\|_2^2),\end{aligned}$$

and the same holds with R_t and $R_{t_\ell,\ell}$ switched, so that

$$|R_{s_\ell,\ell}R_t u - R_{s_\ell+t_\ell,\ell}u| \leq 2^{-\ell}K_t(1 + A_t + CmB_t + B_t\|Du\|_2^2), \tag{6-14}$$

where we have used that $R_{s_\ell+t_\ell,\ell}u = R_{s_\ell,\ell}R_{t_\ell,\ell}u$.

By the same token as (6-13), since $R_t u \in \mathcal{E}(0, C)$, we have

$$|R_{s_\ell,\ell}R_t u - R_s R_t u| \leq 2^{-\ell}K_s(1 + \|D(R_t u)\|_2^2). \tag{6-15}$$

Similarly, since $(s + t) - (s_\ell + t_\ell) \leq 2 \cdot 2^{-\ell}$, we have

$$|R_{s_\ell+t_\ell,\ell}u - R_{s+t}u| \leq 2^{-\ell} \cdot 2K_{s+t}(1 + \|Du\|_2^2). \tag{6-16}$$

Combining these with (6-14) using the triangle inequality, we get

$$|R_s R_t u - R_{s+t} u| \leq 2^{-\ell} K_t (1 + A_t + CmB_t + B_t \|Du\|_2^2) \\ + 2^{-\ell} K_s (1 + \|D(R_t u)\|_2^2) + 2^{-\ell} \cdot 2K_{s+t}(1 + \|Du\|_2^2).$$

Taking $\ell \to \infty$, we get $R_s R_t u = R_{s+t} u$ as desired. This completes the case when $Ct \leq \frac{1}{2}$.

In the general case, suppose $s, t \geq 0$ and $u \in \mathcal{E}(0, C)$. Choose n large enough that $Ct/n \leq \frac{1}{2}$. Then for $j = 1, \dots, n-1$, we have $R_{t/n}^{n-j} u \in \mathcal{E}(0, C)$. Therefore, by the previous argument

$$R_{s+jt/n} R_{t/n}^{n-j} u = (R_{s+jt/n} R_{t/n})(R_{t/n}^{n-j-1} u) = R_{s+(j+1)t/n} R_{t/n}^{n-j-1} u,$$

so by induction $R_{s+t} u = R_s R_{t/n}^n u$. Since this also holds with s replaced by 0, we have $R_{t/n}^n u = R_t u$. Thus, $R_{s+t} u = R_s R_t u$. □

6G. *Solution to the differential equation.* It remains to show that the semigroup R_t produces solutions to the differential equation $\partial_t u = (2N)^{-1} \Delta u - \frac{1}{2} \|Du\|_2^2$, and that the result agrees with the solution produced by solving the heat equation for $\exp(-N^2 u)$. More precisely, we will prove the following.

Theorem 6.17. *Let $u_0 : M_N(\mathbb{C})_{sa}^m \to \mathbb{R}$ be a given function in $\mathcal{E}(c, C)$ for some $c \geq 0$. Let $u(x, t) = R_t u(x)$. Then u is a smooth function on $M_N(\mathbb{C})_{sa}^m \times (0, +\infty)$ and it solves the equation $\partial_t u = (2N)^{-1} \Delta u - \frac{1}{2} \|Du\|_2^2$. Moreover*, $\exp(-N^2 \cdot R_t u_0) = P_t[\exp(-N^2 u_0)]$.

At this point, we have not proved enough smoothness for $R_t u$ to show that it solves the equation in the classical sense. Therefore, as an intermediate step, we show that u solves the equation in the *viscosity sense* (for background on viscosity solutions, see [Crandall et al. 1992]). We will then deduce that $\exp(-N^2 u)$ is a viscosity solution of the heat equation and hence show it agrees with the smooth solution of the heat equation.

The definition of viscosity solution for parabolic equations is as follows. Here we continue to use the vector space $M_N(\mathbb{C})_{sa}^m$ with the normalized inner product (rather than $\mathbb{R}^n$ for some n). For smooth $u : M_N(\mathbb{C})_{sa}^m \to \mathbb{R}$, we denote by Du and Hu the gradient and Hessian with respect to the inner product $\langle \cdot, \cdot \rangle_2$; in other words, if $x_0 \in M_N(\mathbb{C})_{sa}^m$, then $Du(x_0)$ is the vector in $M_N(\mathbb{C})_{sa}^m$ and $Hu(x_0)$ is the linear transformation $M_N(\mathbb{C})_{sa}^m \to M_N(\mathbb{C})_{sa}^m$ such that

$$u(x) = u(x_0) + \langle Du(x_0), x - x_0 \rangle_2 + \tfrac{1}{2} \langle Hu(x_0)[x - x_0], x - x_0 \rangle_2 + o(\|x - x_0\|_2^2).$$

We denote the space of linear transformations $M_N(\mathbb{C})_{sa}^m \to M_N(\mathbb{C})_{sa}^m$ by $B(M_N(\mathbb{C})_{sa}^m)$, and we denote the self-adjoint elements by $B(M_N(\mathbb{C})_{sa}^m)_{sa}$.

Definition 6.18. Let $F : B(M_N(\mathbb{C})_{sa}^m)_{sa} \times M_N(\mathbb{C})_{sa} \times \mathbb{R} \times M_N(\mathbb{C})_{sa} \to \mathbb{R}$ be continuous, and consider the partial differential equation

$$\partial_t u = F(Hu, Du, u, x). \tag{6-17}$$

We say that a function $u : M_N(\mathbb{C})_{sa}^m \times [0, +\infty) \to \mathbb{R}$ is a *viscosity subsolution* if it is upper semicontinuous and if the following condition holds: Suppose that

$$x_0 \in M_N(\mathbb{C})_{sa}^m, \quad t_0 > 0, \quad A \in B(M_N(\mathbb{C})_{sa}^m)_{sa}, \quad p \in M_N(\mathbb{C})_{sa}^m, \quad \alpha \in \mathbb{R},$$

and suppose that u satisfies

$$u(x,t) \le u(x_0,t_0) + \alpha(t-t_0) + \langle p, x-x_0\rangle_2 + \tfrac{1}{2}\langle A(x-x_0), x-x_0\rangle_2 + o(|t-t_0| + \|x-x_0\|_2^2). \quad (6\text{-}18)$$

Then we also have

$$\alpha \le F(A, p, u(x_0), x_0). \quad (6\text{-}19)$$

Definition 6.19. With the same setup as above, we say that $u : M_N(\mathbb{C})_{\mathrm{sa}}^m \times [0,+\infty) \to \mathbb{R}$ is a *viscosity supersolution* if it is lower semicontinuous and the following condition holds: if x_0, t_0, A, p, α are as above and if

$$u(x,t) \ge u(x_0,t_0) + \alpha(t-t_0) + \langle p, x-x_0\rangle_2 + \tfrac{1}{2}\langle A(x-x_0), x-x_0\rangle_2 + o(|t-t_0| + \|x-x_0\|_2^2), \quad (6\text{-}20)$$

then

$$\alpha \ge F(A, p, u(x_0), x_0). \quad (6\text{-}21)$$

Definition 6.20. We say that u is a *viscosity solution* if it is both a subsolution and a supersolution.

Remark 6.21. Roughly speaking, being a viscosity solution means that whenever there exist upper or lower second-order Taylor approximations to u, then we can evaluate the differential operator F on the Taylor approximation and get an inequality in one direction.

Example 6.22. The heat equation $\partial_t u = (2N)^{-1}\Delta u$ is obtained by taking

$$F(A, p, u, x) = \frac{1}{2N^2}\operatorname{Tr}(A).$$

To understand why $1/N^2$ is the correct normalization on the right-hand side, suppose that u is smooth and $A = Hu(x_0)$ and $p = Du(x_0)$, so that

$$u(x) = u(x_0) + \langle p, x-x_0\rangle_2 + \tfrac{1}{2}\langle A(x-x_0), x-x_0\rangle_2 + o(\|x-x_0\|_2^2).$$

In terms of the nonnormalized inner product (which we denote by the dot product), this means that

$$u(x) = u(x_0) + \frac{1}{N} p\cdot(x-x_0) + \frac{1}{2N}(A(x-x_0))\cdot(x-x_0).$$

Thus, the Hessian with respect to the nonnormalized inner product is $(1/N)A$. Hence, $(1/N)\Delta u(x_0) = (1/N^2)\operatorname{Tr}(A)$. Similarly, the equation $\partial_t u = (2N)^{-1}\Delta u - \frac{1}{2}\|Du\|_2^2$ is obtained by taking

$$F(A, p, u, x) = \frac{1}{2N^2}\operatorname{Tr}(A) - \frac{1}{2}\|p\|_2^2.$$

Proposition 6.23. *Let $u_0 \in \mathcal{E}(0, C)$ and define $u(x,t) = R_t u_0(x)$. Then u is a viscosity solution of the equation $\partial_t u = (2N)^{-1}\Delta u - \frac{1}{2}\|Du\|_2^2$.*

Proof. First, note that u is continuous. Indeed, by Theorem 6.1(3), u is continuous in t with a modulus of continuity that is uniform for x in a bounded region (this follows because the term $\|Du_0\|_2^2$ on the right-hand side of Lemma 6.16(2) is bounded on bounded regions since Du_0 is C-Lipschitz). Also, $u(\,\cdot\,, t)$ is continuous for each t since it is in $\mathcal{E}(0, C)$. Together, this implies u is jointly continuous in (x, t).

To show that u is a viscosity supersolution, suppose that we have a lower second-order approximation at the point (x_0, t_0), where $x_0 \in M_N(\mathbb{C})^m_{\mathrm{sa}}$ and $t_0 > 0$, given by

$$u(x,t) \geq u(x_0,t_0) + \alpha(t-t_0) + \langle p, x-x_0\rangle + \tfrac{1}{2}\langle A(x-x_0), x-x_0\rangle_2 + o(|t-t_0| + \|x-x_0\|_2^2).$$

Then we must show that $\alpha \geq 1/(2N^2)\operatorname{Tr}(A) - \frac{1}{2}\|p\|_2^2$.

Our first goal is to replace the soft bound $o(|t-t_0| + \|x-x_0\|_2^2)$ by a more explicit error bound, at the cost of modifying α and A by some positive ϵ. Pick $\epsilon > 0$. Then there exists $r > 0$ such that if $|t-t_0| + \|x-x_0\|_2^2 < 2r$, then we have

$$u(x,t) \geq u(x_0,t_0) + \alpha(t-t_0) - \epsilon|t-t_0| + \langle p, x-x_0\rangle + \tfrac{1}{2}\langle (A-\epsilon I)(x-x_0), x-x_0\rangle_2. \tag{6-22}$$

Let us assume that $t_0 - r < t \leq t_0$, so that the above inequality holds for $\|x-x_0\|^2 < r$ and we have $\alpha(t-t_0) - \epsilon|t-t_0| = (\alpha+\epsilon)(t-t_0)$. For x such that $\|x-x_0\|_2^2 \geq r$, we may use Theorem 6.1(3b), the fact that Du is C-Lipschitz, and the convexity of u to conclude that

$$\begin{aligned} u(x,t) &\geq u_0(x) - \frac{t}{2}(Cm + \|Du\|_2^2) \\ &\geq u_0(x_0) + \langle Du(x_0), x-x_0\rangle_2 - \frac{t}{2}(Cm + (\|Du(x_0)\|_2 + C\|x-x_0\|_2)^2). \end{aligned}$$

In other words, u is bounded below by a quadratic in $x - x_0$, and the estimate holds uniformly for t in a bounded interval. Moreover, the right-hand side of (6-22) is also bounded by a quadratic in $x - x_0$ uniformly for $t \in [t_0 - r, t_0 + r]$. It follows that for a large enough constant K_ϵ, we have

$$u(x_0,t_0) + (\alpha+\epsilon)(t-t_0) + \langle p, x-x_0\rangle + \tfrac{1}{2}\langle (A-\epsilon I)(x-x_0), x-x_0\rangle_2 - u(x,t) \leq K_\epsilon \|x-x_0\|_2^4$$

whenever $t \in (t_0 - t, t_0]$ and $\|x-x_0\|_2 \geq r$. Therefore, overall, assuming that $t \in (t_0 - r, t_0]$, we have

$$u(x,t) \geq u(x_0,t_0) + (\alpha+\epsilon)(t-t_0) + \langle p, x-x_0\rangle + \tfrac{1}{2}\langle (A-\epsilon I)(x-x_0), x-x_0\rangle_2 - K_\epsilon\|x-x_0\|_2^4. \tag{6-23}$$

For $t \in \mathbb{R}$, let us write $u_t(x) = u(x,t) = R_t u_0(x)$. Now the strategy for proving that $\alpha + \epsilon \geq (1/(2N^2))\operatorname{Tr}(A-\epsilon I) - \frac{1}{2}\|p\|_2^2$ is roughly to use the fact that $u_{t_0}(x_0) = R_\delta u_{t_0-\delta}(x_0)$ and estimate $u_{t_0-\delta}(x_0)$ from above using the upper Taylor approximation for small $\delta > 0$. However, for the sake of computation, it is easier to estimate $Q_\delta P_\delta u_{t_0-\delta}$ rather than R_δ (and then we will control the error between R_δ and $Q_\delta P_\delta$ using Lemmas 6.12 and 6.13).

Let $\delta \in (0, r)$. Then using the above inequality and monotonicity of P_δ, we have

$$\begin{aligned} P_\delta u_{t_0-\delta}(x) \geq{}& u_{t_0}(x_0) - (\alpha+\epsilon)\delta + \langle p, x-x_0\rangle \\ &+ \frac{1}{2N^2}\operatorname{Tr}(A-\epsilon I)\delta + \tfrac{1}{2}\langle (A-\epsilon I)(x-x_0), x-x_0\rangle \\ &- K_\epsilon\Big(\|x-x_0\|_2^4 + 2\Big(1+\frac{2}{N^2}\Big)m\delta\|x-x_0\|_2^2 + m^2\Big(1+\frac{2}{N^2}\Big)\delta^2\Big). \end{aligned}$$

Here we have evaluated P_δ applied to $\|x-x_0\|_2^4$ using Example 3.22 and the translation-invariance of P_δ. Now recall that $Q_\delta P_\delta u_{t_0-\delta}(x_0)$ is obtained by evaluating $P_\delta u_{t_0-\delta}$ at $x_0 - \delta D(Q_\delta P_\delta u_{t_0-\delta})(x_0)$. Also, in light of Lemma 6.11(4) and Corollary 6.14(4), $\|D(Q_\delta P_\delta u_{t_0-\delta})(x_0)\|_2^2$ is bounded by $\|Du_0(x_0)\|_2^2$ plus a constant. In particular, $\|D(Q_\delta P_\delta u_{t_0-\delta})(x_0)\|_2$ is bounded as $\delta \to 0$. Therefore,

$$\begin{aligned}Q_\delta P_\delta u_{t_0-\delta}(x_0) &= P_\delta u_{t_0-\delta}(x_0 - \delta D(Q_\delta P_\delta u_{t_0-\delta})(x_0)) + \frac{\delta}{2}\|D(Q_\delta P_\delta u_{t_0-\delta})(x_0)\|_2^2 \\ &\geq u_{t_0}(x_0) + \frac{\delta}{2}\|D(Q_\delta P_\delta u_{t_0-\delta})(x_0)\|_2^2 + (\alpha+\epsilon)(-\delta) \\ &\qquad - \langle p, D(Q_\delta P_\delta u_{t_0-\delta})(x_0)\rangle\delta + \frac{1}{2N^2}\operatorname{Tr}(A-\epsilon I)\delta + O(\delta^2). \quad (6\text{-}24)\end{aligned}$$

(Here the implicit constant in $O(\delta^2)$ depends on ϵ.)

Because $u_{t_0-\delta} \in \mathcal{E}(0, C)$, Lemma 6.12(2) and (3) imply that if $C\delta \leq 1$, then

$$|Q_\delta P_\delta u_{t_0-\delta}(x_0) - P_\delta Q_\delta u_{t_0-\delta}(x_0)| \leq 2C^2 m\delta^2 + 2C\delta^2\|D(P_\delta Q_\delta u_{t_0-\delta})(x_0)\|_2.$$

Again by Lemma 6.11(4) and Theorem 6.1(4c), $\|D(Q_\delta P_\delta u_{t_0-\delta})(x_0)\|_2^2$ is bounded by $\|Du_0(x_0)\|_2^2$ plus a constant, so that

$$Q_\delta P_\delta u_{t_0-\delta}(x_0) = P_\delta Q_\delta u_{t_0-\delta}(x_0) + O(\delta^2).$$

Also, if we let $\delta_\ell = 2^{-\ell}$ for $\ell \in \mathbb{Z}$, then Lemma 6.13 implies that when $2C\delta_\ell \leq 1$ and $\delta_\ell < r$, we have

$$\begin{aligned}|P_{\delta_\ell} Q_{\delta_\ell} u_{t_0-\delta}(x_0) - R_{\delta_\ell} u_{t_0-\delta_\ell}(x_0)| &= |R_{\delta_\ell,\ell} u_{t_0-\delta_\ell}(x_0) - R_{\delta_\ell} u_{t_0-\delta_\ell}(x_0)| \\ &\leq \left(\frac{3}{2}\frac{C^2 m\delta_\ell}{1-C\delta_\ell} + \log(1+C\delta_\ell)(m + Cm + \|Du(x_0)\|_2^2)\right)2^{-\ell} = O(\delta_\ell^2).\end{aligned}$$

So overall

$$Q_{\delta_\ell} P_{\delta_\ell} u_{t_0-\delta_\ell}(x_0) = R_{\delta_\ell} u_{t_0-\delta_\ell}(x_0) + O(\delta_\ell^2) = u_{t_0}(x_0) + O(\delta_\ell^2). \quad (6\text{-}25)$$

Using similar reasoning, Lemma 6.12(1) shows that

$$D(Q_{\delta_\ell} P_{\delta_\ell} u_{t_0-\delta_\ell})(x_0) = D(P_{\delta_\ell} Q_{\delta_\ell} u_{t_0-\delta_\ell})(x_0) + O(\delta_\ell^{3/2}).$$

Then using Lemma 6.13(3), we obtain

$$D(P_{\delta_\ell} Q_{\delta_\ell} u_{t_0-\delta_\ell})(x_0) = D(R_{\delta_\ell} u_{t_0-\delta_\ell})(x_0) + O(\delta^{3/2}).$$

Finally, because $u_{t_0-\delta} \in \mathcal{E}(0, C)$, it is differentiable everywhere; the upper Taylor approximation (6-22) implies that $u_{t_0}(x) \leq u_{t_0}(x_0) + \langle p, x - x_0\rangle_2 + o(\|x - x_0\|_2)$ and therefore p must equal $Du_{t_0}(x_0)$. Thus, overall

$$D(Q_{\delta_\ell} P_{\delta_\ell} u_{t_0-\delta_\ell})(x_0) = p + O(\delta_\ell^{3/2}). \quad (6\text{-}26)$$

Substituting (6-25) and (6-26) into (6-24), we obtain

$$u_{t_0}(x_0) \geq u_{t_0}(x_0) + \frac{1}{2}\|p\|_2^2\delta_\ell + (\alpha+\epsilon)(-\delta_\ell) - \|p\|_2^2\delta_\ell + \frac{1}{2N^2}\operatorname{Tr}(A-\epsilon I)\delta_\ell + O(\delta_\ell^2).$$

We cancel $u_{t_0}(x_0)$ from both sides, divide by δ_ℓ, and move $\alpha+\epsilon$ to the left-hand side to conclude that

$$\alpha + \epsilon \geq \frac{1}{2N^2}\operatorname{Tr}(A-\epsilon I)\delta_\ell - \frac{1}{2}\|p\|_2^2 + O(\delta_\ell).$$

Then taking $\ell \to \infty$, we get $\alpha + \epsilon \geq (1/(2N^2))\operatorname{Tr}(A-\epsilon I) - \frac{1}{2}\|p\|_2^2$. Since ϵ was arbitrary, we have $\alpha \geq (1/(2N^2))\operatorname{Tr}(A) - \frac{1}{2}\|p\|_2^2$. This shows that u is a viscosity supersolution.

To show that the u is a viscosity subsolution, the argument is symmetrical for the most part. However, to obtain the constant K_ϵ in (6-23), we used the one-sided estimate Theorem 6.1(3b) to show that u is bounded below by a quadratic in $x - x_0$ that is independent of t, so long as $t \in (t_0 - r, t_0]$. To show that u is a viscosity subsolution, we want to prove an analogous quadratic upper bound. But by Theorem 6.1(3a) and semiconcavity of u_0, we have for $t \leq t_0$ that

$$\begin{aligned} u_t(x) &\leq u_0(x) + \frac{m}{2}\log(1 + Ct_0) \\ &\leq u_0(x_0) + \langle Du_0(x_0), x - x_0\rangle + \frac{C}{2}\|x - x_0\|_2^2 + \frac{m}{2}\log(1 + Ct_0), \end{aligned}$$

which is the desired upper bound. The rest of the argument is symmetrical except that $\alpha + \epsilon$ is replaced by $\alpha - \epsilon$ and $A - \epsilon I$ is replaced by $A + \epsilon I$. □

Lemma 6.24. *Let $u : M_N(\mathbb{C})_{\mathrm{sa}}^m \times [0, +\infty) \to \mathbb{R}$. Then u is a viscosity solution to $\partial_t u = (2N)^{-1}\Delta u - \frac{1}{2}\|Du\|_2^2$ if and only if $\exp(-N^2 u)$ is a viscosity solution to $\partial_t u = (2N)^{-1}\Delta u$.*

Proof. More precisely, we claim that u is a viscosity subsolution if and only if $\exp(-N^2u)$ is viscosity supersolution and vice versa. Suppose that u is a subsolution, and let us show that $v = \exp(-N^2u)$ is a supersolution. If u is upper semicontinuous, then v is lower semicontinuous. Now suppose that we have a lower Taylor approximation at (x_0, t_0)

$$v(x, t) \geq v(x_0, t_0) + \alpha(t - t_0) + \langle p, x - x_0\rangle_2 + \tfrac{1}{2}\langle A(x - x_0), x - x_0\rangle_2 + o(|t - t_0| + \|x - x_0\|_2^2).$$

Note that $v > 0$ and $u = (-1/N^2)\log v$. The function $h \mapsto \log h$ is increasing and analytic for $h > 0$ and we have

$$\log(h + \delta) = \log(h) + \log\Big(1 + \frac{\delta}{h}\Big) = \log(h) + \frac{\delta}{h} - \frac{1}{2}\Big(\frac{\delta}{h}\Big)^2 + O(\delta^3).$$

Substituting $h = v(x_0, t_0) = \exp(-N^2u(x_0, t_0))$ and

$$\delta = v(x, t) - v(x_0, t_0) = \alpha(t - t_0) + \langle p, x - x_0\rangle_2 + \tfrac{1}{2}\langle A(x - x_0), x - x_0\rangle_2 + o(|t - t_0| + |x - x_0|^2),$$

we get

$$\begin{aligned} -N^2u(x, t) \geq -N^2u(x_0, t_0) &+ \frac{\alpha}{v(x_0, t_0)}(t - t_0) + \frac{1}{v(x_0, t_0)}\langle p, x - x_0\rangle_2 \\ &+ \frac{1}{2v(x_0, t_0)}\langle A(x - x_0), x - x_0\rangle_2 - \frac{1}{2v(x_0, t_0)^2}\langle p, x - x_0\rangle_2^2 + o(|t - t_0| + \|x - x_0\|_2^2), \end{aligned}$$

since $\langle p, x - x_0\rangle_2 / v(x_0, t_0)^2$ is the only term from $-(\delta/h)^2/2 + O(\delta^3)$ that is not $o(|t - t_0| + \|x - x_0\|_2^2)$ (here we use the fact that $|t - t_0|\|x - x_0\|_2 \leq \frac{2}{3}|t - t_0|^{3/2} + \frac{1}{3}\|x - x_0\|_2^3$). Let us denote by P the linear map $P(x - x_0) = p\langle p, x - x_0\rangle_2$. Then the above inequality becomes

$$\begin{aligned} u(x, t) \leq u(x_0, t_0) &- \frac{\alpha}{N^2 v(x_0, t_0)}(t - t_0) - \frac{1}{N^2 v(x_0, t_0)}\langle p, x - x_0\rangle_2 \\ &- \frac{1}{2N^2 v(x_0, t_0)}\langle A(x - x_0), x - x_0\rangle_2 + \frac{1}{2N^2 v(x_0, t_0)^2}\langle P(x - x_0), (x - x_0)\rangle + o(|t - t_0| + \|x - x_0\|_2^2). \end{aligned}$$

Because u is a subsolution, we have

$$-\frac{\alpha}{N^2 v(x_0, t_0)} \leq -\frac{1}{2N^4} \operatorname{Tr}(A) + \frac{1}{2N^4 v(x_0, t_0)^2} \operatorname{Tr}(P) - \frac{1}{2N^4 v(x_0, t_0)^2} \|p\|_2^2.$$

But $\operatorname{Tr}(P) = \|p\|_2^2$, so the last two terms cancel. Thus,

$$\alpha \geq \frac{1}{2N^2} \operatorname{Tr}(A)$$

as desired. So v is a supersolution.

A symmetrical argument shows that if v is a supersolution, then u is a subsolution. The other two claims are proved in the same way except using the Taylor expansion of the exponential function instead of the logarithm. □

Now we are ready to prove Theorem 6.17 in the special case where u_0 is bounded below.

Lemma 6.25. *Suppose that $u_0 \in \mathcal{E}(0, C)$ is bounded below. Then*

$$\exp(-N^2 R_t u_0) = P_t[\exp(-N^2 u_0)].$$

Proof. Let $v(x,t) = \exp(-N^2 R_t u_0(x))$ and let $w(x,t) = P_t[\exp(-N^2 u_0)](x)$. Since u_0 is bounded below by some constant K, we have $R_t u_0 \geq K$ by monotonicity of R_t (see Corollary 6.14(3)) and the fact that it does not affect constant functions (since the same is true of P_t and Q_t). Hence, $v = \exp(-N^2 R_t u_0) \leq \exp(-N^2 K)$. We also have $\exp(-N^2 u_0) \leq \exp(-N^2 K)$ and hence $w \leq \exp(-N^2 K)$.

Thus, v and w are both bounded, w is a smooth solution to the heat equation, and v is a viscosity solution by the previous lemma. We will conclude from this that $v = w$ (and this is nothing but a standard argument for the maximum principle together with the basic philosophy of viscosity solutions).

To show that $v \leq w$, choose $\epsilon > 0$, and consider the function

$$\phi(x,t) = v(x,t) - w(x,t) - \frac{\epsilon}{2}\|x\|_2^2 - 2m\epsilon t.$$

Suppose for contradiction that $\phi > 0$ at some point. Since ϕ is continuous on $M_N(\mathbb{C})_{\text{sa}}^m \times [0, +\infty)$ and since w and v are bounded, ϕ achieves a maximum at some (x_0, t_0). Since the maximum is strictly positive, we have $t_0 > 0$. Let

$$\psi(x,t) = w(x,t) + \frac{\epsilon}{2}\|x\|_2^2 + 2m\epsilon t,$$

so that $\phi(x,t) = v(x,t) - \psi(x,t)$. Then $\phi(x,t) \leq \phi(x_0, t_0)$ implies that

$$\begin{aligned} v(x,t) &\leq v(x_0,t_0) + \psi(x,t) - \psi(x_0,t_0) \\ &= v(x_0,t_0) + \partial_t \psi(x_0,t_0)(t-t_0) + \langle D\psi(x_0,t_0), x - x_0\rangle_2 \\ &\qquad + \tfrac{1}{2}\langle H\psi(x_0,t_0)(x-x_0), x-x_0\rangle_2 + o(|t-t_0| + \|x-x_0\|_2^2), \end{aligned}$$

where the last step follows because ψ is smooth. Because v is a viscosity subsolution,

$$\partial_t \psi(x_0,t_0) \leq \frac{1}{2N} \Delta\psi(x_0,t_0).$$

However, this is a contradiction because at every point (x, t), we have

$$\partial_t \psi = \partial_t w + 2m\epsilon > \frac{1}{2N} \Delta w + m\epsilon = \frac{1}{2N} \Delta \psi,$$

by computation and the fact that w solves the heat equation. It follows that $\phi \leq 0$ and hence $v(x,t) \leq w(x,t) + (\epsilon/2)\|x\|_2^2 + 2m\epsilon t$. Since ϵ was arbitrary, $v \leq w$. Then a symmetrical argument shows that $v \geq w$. □

Thus, to prove Theorem 6.17, it only remains to remove the boundedness assumption on u_0. We achieve this by replacing u_0 with the function

$$\tilde{u}_0(x) = u_0(x) - \langle Du_0(0), x \rangle_2, \tag{6-27}$$

which is nonnegative by the convexity of u_0 and hence it is bounded below.

Lemma 6.26. *Let $u_0 \in \mathcal{E}(0, C)$ and let $\tilde{u}_0$ be given by* (6-27). *Let $v_0 = \exp(-N^2 u_0)$ and $\tilde{v}_0 = \exp(-N^2 \tilde{u}_0)$. Then the integral defining $P_t \exp(-N^2 u_0)$ is well-defined and also*

$$P_t v_0(x) = \exp(-N^2 \langle Du_0(0), x \rangle + \frac{N^2 t}{2} \|Du_0(0)\|_2^2) P_t \tilde{v}_0(x - t Du_0(0)).$$

Proof. We can write

$$d\sigma_{t,N}(y) = \frac{1}{Z_N} \exp\Big(-\frac{N^2}{2t} \|y\|_2^2\Big)\, dy.$$

Also, set $p = Du_0(0)$. Then

$$\begin{aligned}
P_t v_0(x) &= \frac{1}{Z_N} \int \exp(-N^2 u_0(x+y)) \exp\Big(-\frac{N^2}{2t} \|y\|_2^2\Big)\, dy \\
&= \frac{1}{Z_N} \int \exp\Big(-N^2 \tilde{u}_0(x+y) - N^2 \langle p, x+y \rangle - \frac{N^2}{2t} \|y\|_2^2\Big)\, dy \\
&= \frac{1}{Z_N} \int \exp\Big(-N^2 \tilde{u}_0(x+y) - N^2 \langle p, x \rangle + \frac{N^2 t}{2} \|p\|_2^2 - \frac{N^2}{2t} \|y + tp\|_2^2\Big)\, dy \\
&= \frac{1}{Z_N} \int \exp\Big(-N^2 \tilde{u}_0(x - tp + z) - N^2 \langle p, x \rangle + \frac{N^2 t}{2} \|p\|_2^2 - \frac{N^2}{2t} \|z\|_2^2\Big)\, dz \\
&= \exp\Big(-N^2 \langle p, x \rangle + \frac{N^2 t}{2} \|p\|_2^2\Big) P_t \tilde{v}_0(x - tp).
\end{aligned}$$

□

Lemma 6.27. *Let $u_0 \in \mathcal{E}(0, C)$, let $p \in M_N(\mathbb{C})_{\text{sa}}^m$, and let $\tilde{u}_0(x) = u_0(x) - \langle p, x \rangle_2$. Then*:

(1) $P_t u_0(x) = P_t \tilde{u}_0(x) + \langle p, x \rangle_2$.

(2) $Q_t u_0(x) = Q_t \tilde{u}_0(x - tp) + \langle p, x \rangle_2 - (t/2)\|p\|_2^2$.

(3) $R_t u_0(x) = R_t \tilde{u}_0(x - tp) + \langle p, x \rangle_2 - (t/2)\|p\|_2^2$.

Proof. (1) holds because P_t is a linear operator and it also does not affect linear functions. To prove (2), fix x and let y be the point where the infimum defining $Q_t u_0(x)$ is achieved and let $\tilde{y}$ be the point where the infimum defining $Q_t \tilde{u}_0(x - tp)$ is achieved. By Corollary 6.7(1), the points y and $\tilde{y}$ are characterized respectively by the relations

$$y = x - t Du_0(y), \quad \tilde{y} = x - tp - t D\tilde{u}_0(\tilde{y}).$$

But $D\tilde{u}_0(\tilde{y}) = Du_0(\tilde{y}) - p$. Thus, $x - tDu_0(\tilde{y}) = \tilde{y}$, so that $y = \tilde{y}$. Then

$$\begin{aligned} Q_t u_0(x) &= u_0(y) + \frac{1}{2t}\|y - x\|_2^2 \\ &= \tilde{u}_0(y) + \langle p, y\rangle_2 + \frac{1}{2t}\|y - x\|_2^2 \\ &= \tilde{u}_0(y) + \langle p, x\rangle_2 - \frac{t}{2}\|p\|_2^2 + \frac{1}{2t}\|y - (x - tp)\|_2^2 \\ &= Q_t\tilde{u}_0(x - tp) + \langle p, x\rangle_2 - \frac{t}{2}\|p\|_2^2. \end{aligned}$$

(3) It follows by iteration (after some computation) that for $t \in 2^{-\ell}\mathbb{N}_0$, we have

$$R_{t,\ell}u_0(x) = R_t\tilde{u}_0(x - tp) + \langle p, x\rangle_2 - \frac{t}{2}\|p\|_2^2.$$

Then by Lemma 6.13, we may take $\ell \to \infty$, and by Theorem 6.1(3), we may extend the inequality to all real t. □

Proof of Theorem 6.17. We have already proved the case where u_0 is bounded. For the general case, let $u_0 \in \mathcal{E}(0, C)$. Define $p = Du_0(0)$ and $\tilde{u}_0(x) = u_0(x) - \langle p, x\rangle_2$. As remarked above, $\tilde{u}_0$ is bounded below by zero. By Lemma 6.26, the bounded case, and Lemma 6.27,

$$\begin{aligned} P_t \exp(-N^2 u_0)(x) &= \exp\Big(-N^2\langle p, x\rangle + \frac{N^2 t}{2}\|p\|_2^2\Big)[P_t\exp(-N^2\tilde{u}_0)](x - tp) \\ &= \exp\Big(-N^2\langle p, x\rangle + \frac{N^2 t}{2}\|p\|_2^2\Big)\exp(-N^2 R_t\tilde{u}_0(x - tp)) \\ &= \exp\Big(-N^2\Big(R_t\tilde{u}_0(x - tp) + \langle p, x\rangle - \frac{t}{2}\|p\|_2^2\Big)\Big) \\ &= \exp(-N^2 R_t u_0(x)). \end{aligned}$$

In particular, since $P_t\exp(-N^2\tilde{u}_0)$ is smooth for $t > 0$, we see that all the functions in the above equation are smooth for $t > 0$, and hence $R_t u_0(x)$ is smooth function of (x, t). Also, $P_t[\exp(-N^2 u_0)] = \exp(-N^2 R_t u_0)$ as desired. □

6H. *Approximation by trace polynomials.* Now we are ready to prove that R_t preserves asymptotic approximability by trace polynomials.

Proposition 6.28. *Let $\{V_N\}$ be a sequence of functions $M_N(\mathbb{C})_{\mathrm{sa}}^m \to \mathbb{R}$ such that V_N is convex and $V_N(x) - (C/2)\|x\|_2^2$ is concave, and $\{DV_N\}$ is asymptotically approximable by trace polynomials. Then for every $t \geq 0$, the sequences $\{D(P_t V_N)\}$, $\{D(Q_t V_N)\}$, and $\{D(R_t V_N)\}$ are asymptotically approximable by trace polynomials.*

Proof. The fact that $\{D(P_t V_N)\}$ is asymptotically approximable by trace polynomials follows from Lemma 3.28.

Now consider $D(Q_t V_N)$. Note that by Corollary 6.7(1), $D(Q_t V_N)(x)$ is the solution of the fixed point equation

$$y = DV_N(x - ty).$$

But if $t < 1/C$, then $y \mapsto DV_N(x - ty)$ is a contraction and thus iterates of this function will converge to the fixed point. Let us define $\phi_{N,0}(x) = 0$ and $\phi_{N,\ell+1}(x) = DV_N(x - t\phi_{N,\ell}(x))$. By Lemma 6.6(5), the distance from 0 to the fixed point $D(Q_t V_N)(x)$ is bounded by $\|DV_N(x)\|_2$; hence

$$\|\phi_{N,\ell}(x) - D(Q_t V_N)(x)\|_2 \leq C^\ell t^\ell \|DV_N(x)\|_2.$$

Because $DV_{N,t}$ is C-Lipschitz, Lemma 3.27 implies that $\{\phi_{N,\ell}\}_N$ is asymptotically approximable by trace polynomials.

Now $\|DV_N(0)\|_2$ is bounded by some constant A as $N \to \infty$ because DV_N is asymptotically approximable by trace polynomials. Since DV_N is also C-Lipschitz, $\|DV_N(x)\|_2 \leq A + C\|x\|_2$. In particular, $\|\phi_{N,\ell}(x) - D(Q_t V_N)(x)\|_2 \leq C^\ell t^\ell (A + C\|x\|_2)$. Thus, by Observation 3.26, $\{D(Q_t V_N)\}$ is asymptotically approximable by trace polynomials.

This holds whenever $t < 1/C$. But for general t, we can write $Q_t = Q_{t/n}^n$, where n is large enough that $t/n < 1/C$, and then iterating the previous statement shows that $\{Q_t V_N\}$ is asymptotically approximable by trace polynomials.

For the sequence $\{D(R_t V_N)\}$, first note that when $t \in \mathbb{Q}_2^+$, we know $\{D(R_{t,\ell} V_N)\}$ is asymptotically approximable by trace polynomials (where ℓ is large enough that $R_{t,\ell}$ is defined). By Theorem 6.1(1c) and Observation 3.26, the sequence $\{D(R_t V_N)\}$ is asymptotically approximable by trace polynomials for $t \in \mathbb{Q}_2^+$. Finally, by Theorem 6.1(2d) and Observation 3.26, the sequence $\{D(R_t V_N)\}$ is asymptotically approximable by trace polynomials for all $t \in \mathbb{R}^+$. □

7. Main theorem on free entropy

We are now ready to prove the following theorem which shows that $\chi = \chi^*$ for a law which is the limit of log-concave random matrix models.

Theorem 7.1. *Let μ_N be a sequence of probability measures on $M_N(\mathbb{C})_{\mathrm{sa}}^m$ given by the potential V_N. Assume*:

(A) *The potential $V_N(x)$ is convex and $V_N(x) - (C/2)\|x\|_2^2$ is concave for some $C > 0$ independent of N.*

(B) *The sequence μ_N concentrates around some noncommutative law λ with $\lambda(X_j^2) > 0$.*

(C) *For some $R_0 > 0$, we have $\lim_{N\to\infty} \int_{\|x\| \geq R_0} (1 + \|x\|_2^2)\, d\mu_N(x) = 0$.*

(D) *The sequence $\{DV_N\}$ is asymptotically approximable by trace polynomials.*

Then $\lambda \in \Sigma_{m,R_0}$ and moreover:

(1) *The law λ has finite Fisher information $\Phi^*(\lambda)$, and for all $t \geq 0$, we have*

$$\lim_{N\to\infty} \frac{1}{N^3} \mathcal{I}(\mu_N * \sigma_{t,N}) \to \Phi^*(\lambda \boxplus \sigma_t).$$

(2) *We have for all $t \geq 0$*

$$\chi(\lambda \boxplus \sigma_t) = \underline{\chi}(\lambda \boxplus \sigma_t) = \lim_{N\to\infty} \frac{1}{N^2}\Big(h(\mu_N * \sigma_{t,N}) + \frac{m}{2}\log N\Big) = \chi^*(\lambda \boxplus \sigma_t).$$

(3) *The functions* $t \mapsto (1/N^3)\mathcal{I}(\mu_N * \sigma_{t,N})$ *and* $t \mapsto \Phi^*(\lambda \boxplus \sigma_t)$ *are decreasing and Lipschitz and the absolute value of the derivative* (*where defined*) *is bounded by* $C^2 m(1+Ct)^{-2}$.

Remark 7.2. If $V_N(x) - (c/2)\|x\|_2^2$ is convex and $V_N(x) - (C/2)\|x\|_2^2$ is concave and if $\{DV_N\}$ is asymptotically approximable by trace polynomials, then Theorem 4.1 implies that μ_N satisfies the hypotheses of Theorem 7.1 for some noncommutative law λ.

However, Theorem 7.1 holds in a slightly more general situation than Theorem 4.1 in that we do not have to assume uniform convexity, finite moments, or exponential concentration.

In preparation for the proof of Theorem 7.1, we have already verified that the hypotheses (A), (C), and (D) are preserved under Gaussian convolution. Now we show that (B) is preserved in Lemma 7.4. This is straightforward apart from one subtlety — although we have assumed that for every noncommutative polynomial p, the noncommutative moment $\tau_N(p(x))$ concentrates around $\lambda(p)$ under μ_N, we have *not* assumed that $|\tau_N(p(x))|$ has finite expectation. To deal with this issue, we first prove an auxiliary lemma.

Lemma 7.3. *Let* λ *be a noncommutative law in* Σ_m, *let* $p(X,Y) = p(X_1,\dots,X_m,Y_1,\dots,Y_m)$ *be a noncommutative polynomial of* $2m$ *variables*, *and let* $R > 0$. *Then there exists a neighborhood* $\mathcal{V}$ *of* λ *in* Σ_m *and a constant* K *such that*, *for all* $N \in \mathbb{N}$, *for all* $x \in \Gamma_N(\mathcal{V})$, *the function* $y \mapsto \tau_N(p(x,y))$ *is* K*-Lipschitz with respect to* $\|\cdot\|_2$ *for self-adjoint tuples* y *in the operator-norm ball* $\{y : \|y_j\| \le R\}$.

Proof. To prove the lemma, it suffices to consider the case of a noncommutative monomial. Indeed, if $p = \sum_{j=1}^n p_j$, where p_j is a monomial, and if we find neighborhoods $\mathcal{V}_j$ and Lipschitz constants K_j for each p_j, then the result will also hold for p with $\mathcal{V} = \bigcap_{j=1}^n \mathcal{V}_j$ and $K = \sum_{j=1}^n K_j$.

Thus, assume without loss of generality that $p(X,Y)$ is a noncommutative monomial. Then it can be written in the form

$$p(X,Y) = q_0(X) Y_{i_1} q_1(X) Y_{i_2} \cdots q_{\ell-1}(X) Y_{i_\ell} q_\ell(X),$$

where $i_j \in \{1,\dots,m\}$ and $q_j(X)$ is a noncommutative monomial in X (which of course is allowed to be 1). Consider $x, y, y' \in M_N(\mathbb{C})_{\text{sa}}^m$, and suppose that $\|y_i\|_\infty \le R$ and $\|y_i'\|_\infty \le R$ for each i. Then

$$p(x,y) - p(x,y') = \sum_{j=1}^{\ell} q_0(x) y_{i_1} \cdots y_{i_{j-1}} q_{j-1}(x)(y_{i_j} - y_{i_j}') q_i(x) y_{i_{j+1}} \cdots y_{i_\ell} q_\ell(x).$$

Recalling the noncommutative L^α norms and Hölder's inequality (see Section 2C), we have

$$\|p(x,y) - p(x,y')\|_1 \le \Bigg(\sum_{j=1}^{\ell} \prod_{k \ne j} \|q_j(x)\|_{2(\ell+1)} \prod_{k<j} \|y_{i_k}\|_\infty \prod_{k>j} \|y_{i_k}'\|_\infty \Bigg) \|y_j - y_j'\|_2.$$

This implies that

$$|\tau_N(p(x,y)) - \tau_N(p(x,y'))| \le \Bigg(\sum_{j=1}^{\ell} \prod_{k \ne j} \|q_j(x)\|_{2(\ell+1)} \Bigg) R^{\ell-1} \|y - y'\|_2.$$

Now

$$\|q_j(x)\|_{2(\ell+1)} = (\tau_N[(q_j(x)^* q_j(x))^{\ell+1}])^{1/(2(\ell+1))}.$$

We can define

$$\mathcal{V} = \{\lambda' : \lambda'[(q_j^* q_j)^{\ell+1}] < \lambda[(q_j^* q_j)^{\ell+1}] + 1 \text{ for } j = 0, \dots, \ell\}.$$

Then $\|q_j(x)\|_{2(\ell+1)}$ is uniformly bounded for $x \in \Gamma_N(\mathcal{V})$ for each $j = 0, \dots, \ell$. Suppose that each of these quantities is bounded by K. Then the above estimate shows that

$$|\tau_N(p(x, y)) - \tau_N(p(x, y))| \le \ell K^{\ell+1} R^{\ell-1} \|y - y'\|_2$$

whenever $x \in \Gamma_N(\mathcal{V})$ and y, y' are in the operator-norm ball of radius R. □

Lemma 7.4. *Suppose that $\{\mu_N\}$ concentrates around a noncommutative law λ. Then $\{\mu_N * \sigma_{t,N}\}$ concentrates around $\lambda \boxplus \sigma_t$ for every $t > 0$.*

Proof. Fix t. Let $X_N = (X_{N,1}, \dots, X_{N,m})$ and $Y_N = (Y_{N,1}, \dots, Y_{N,m})$ be independent random variables with the laws μ_N and $\sigma_{t,N}$ respectively. Because the topology on the space Σ_m of noncommutative laws is generated by noncommutative moments, it suffices to show that for each noncommutative polynomial p and $\delta > 0$

$$\lim_{N\to\infty} P(|\tau_N(p(X_N + Y_N)) - \lambda \boxplus \sigma_t(p)| \ge \delta) = 0.$$

Fix p and let d be its degree. By the previous lemma, there is a neighborhood $\mathcal{V}$ of λ and a constant K such that for every $x \in \Gamma_N(\mathcal{V})$, the function $y \mapsto \tau_N(p(x + y))$ is K-Lipschitz with respect to $\|\cdot\|_2$ on the operator-norm ball $\{y : \|y\|_\infty \le 4t^{1/2}\}$. By shrinking $\mathcal{V}$ if necessary, we may also assume that $\tau_N(q(x))$ is uniformly bounded for every noncommutative monomial $q(x)$ of degree less than or equal to d.

Choose a C_c^∞ function $\psi : \mathbb{R} \to \mathbb{R}$ such that $\psi(z) = z$ for $|z| \le 3t^{1/2}$ and $|\psi(z)| \le 4t^{1/2}$. Then

$$\Psi : (y_1, \dots, y_m) \mapsto (\psi(y_1), \dots, \psi(y_m))$$

is globally Lipschitz in $\|\cdot\|_2$ and it also maps $M_N(\mathbb{C})_{\mathrm{sa}}^m$ into the operator-norm ball of radius $4t^{1/2}$ (which is the region where $z \mapsto \tau_N(p(x, z))$ was assumed to be K-Lipschitz with respect to $\|\cdot\|_2$ whenever $x \in \Gamma_N(\mathcal{V})$). This implies that there is some constant K' such that $y \mapsto \tau_N(p(x, \Psi(y)))$ is K'-Lipschitz for all $x \in \Gamma_N(\mathcal{V})$.

Let

$$\begin{aligned}
\alpha_N(x) &= E[\tau_N(p(x + \Psi(Y_N)))],\\
\beta_N(x) &= E[\tau_N(p(x + Y + N))] = \exp\Big(\frac{tL_N}{2}\Big)[\tau(p)](x),\\
\beta(x) &= \exp\Big(\frac{tL}{2}\Big)[\tau(p)](x).
\end{aligned}$$

By Theorem 2.10 applied to Y_N,

$$x \in \Gamma_N(\mathcal{V}) \quad \Longrightarrow \quad P\Big(|\tau_N(p(x + \Psi(Y_N))) - \alpha_N(x)| \ge \frac{\delta}{3}\Big) \le 2e^{-\delta^2 N^2/(18t(K')^2)}.$$

On the other hand, we know by standard tail estimates on the GUE (see Corollary 2.12) that

$$\lim_{N\to\infty} E[\tau_N(q(Y_N))\mathbf{1}_{\|Y_N\| \ge 3t^{1/2}}] = 0$$

for every noncommutative polynomial q. This implies that $|\alpha_N(x)-\beta_N(x)| \to 0$ uniformly for $x \in \Gamma_N(\mathcal{V})$. On the other hand, by Lemma 3.21,

$$\beta_N(x) = \exp\Big(\frac{tL_N}{2}\Big)[\tau(p)](x) \to \exp\Big(\frac{tL}{2}\Big)[\tau(p)](x) = \beta(x)$$

where the convergence occurs coefficientwise. Now $\exp(tL_N/2)[\tau(p)]$ is a sum of products of traces of noncommutative monomials q of degree $\leq d$ and for every such q, we know $\tau_N(q(x))$ is uniformly bounded on $\Gamma_N(\mathcal{V})$ by our choice of $\mathcal{V}$. Thus, coefficientwise convergence of $\beta_N \to \beta$ implies uniform convergence for $x \in \Gamma_N(\mathcal{V})$. Therefore, for sufficiently large N we have $|\beta_N(x) - \beta(x)| \leq \delta/3$ for $x \in \Gamma_N(\mathcal{V})$, and hence

$$P\Big(|\tau_N(p(X_N+Y_N)) - \tau(\beta(X_N))| \geq \frac{2\delta}{3},\ X_N \in \Gamma_N(\mathcal{V}),\ \|Y_N\| \leq 3t^{1/2}\Big) \leq 2e^{-\delta^2N^2/(18t(K')^2)},$$

where we have applied the Fubini–Tonelli theorem for the product measure $\mu_N \otimes \sigma_{t,N}$. By our concentration assumption,

$$P\Big(|\tau_N(\beta(X_N)) - \lambda(\beta)| \geq \frac{\delta}{3}\Big) \to 0, \quad P(X_N \in \Gamma_N(\mathcal{V})) \to 1,$$

and by Corollary 2.12 also $P(\|Y_k\| \geq 3t^{1/2}) \to 0$. Altogether, we have

$$P(|\tau_N(p(X_N+Y_N)) - \lambda(\beta)| \geq \delta) \to 0.$$

But note that $\lambda(\beta) = \lambda(\exp(tL/2)[\tau(p)]) = (\lambda \boxplus \sigma_t)(p)$ by Lemma 3.23. Thus, the proof is complete. □

Proof of Theorem 7.1. Let $V_{N,t} = R_t V_N$ be the potential associated to $\mu_N * \sigma_{t,N}$. Let us verify that $V_{N,t}$ satisfies the assumptions (A)–(D) for every $t > 0$.

(A) This follows from Theorem 6.1(1) because $V_{N,t} = R_t V_N$; hence $V_{N,t} \in \mathcal{E}(0, C)$.

(B) This follows from Lemma 7.4.

(C) This follows from tail bounds on the GUE (Corollary 2.12).

(D) This follows from Proposition 6.28.

Next, the fact that $\lambda \in \Sigma_{m,R_0}$ follows from Proposition 5.5 with $n = 1$.

Claim (1) of the theorem follows by applying Proposition 5.10 to $\mu_N * \sigma_{t,N}$ with $n = 1$.

For claim (2), recall that by Lemma 5.7, (5-6),

$$\frac{1}{N^2}h(\mu_N) + \frac{m}{2}\log N = \frac{1}{2}\int_0^\infty \Big(\frac{m}{1+t} - \frac{1}{N^3}\mathcal{I}(\mu_N * \sigma_{t,N})\Big)\,ds + \frac{m}{2}\log 2\pi e. \tag{7-1}$$

Because $N^{-3}\mathcal{I}(\mu_N)$ converges as $N \to \infty$, there is some constant K with $N^{-3}\mathcal{I}(\mu_N) \leq K$ for all N. Also, because of assumptions (B) and (C), we have $\int \|x\|_2^2\,d\mu_N(x) \to \sum_{j=1}^m \lambda_j(X_j^2) > 0$. Therefore, there is a constant a such that $\int \|x\|_2^2\,d\mu_N(x) \geq ma$ for large enough N. Thus, (5-4), we have for sufficiently large N that

$$\frac{m}{a+t} \leq \frac{1}{N^3}\mathcal{I}(\mu_N * \sigma_{t,N}) \leq \min\Big(M, \frac{m}{t}\Big).$$

Thus, we can apply the dominated convergence theorem to take the limit as $N \to \infty$ inside the integral on the right-hand side of (7-1) and apply claim (1) to conclude that

$$\lim_{N\to\infty}\Big(\frac{1}{N^2}h(\mu_N) + \frac{m}{2}\log N\Big) \to \chi^*(\lambda).$$

On the left-hand side of (7-1), we will apply Proposition 5.5 with $n = 1$. We may replace V_N by $V_N - V_N(0)$ without changing μ_N (because the definition of μ_N includes the normalizing constant Z_N anyway). Then because $\{DV_N\}$ is asymptotically approximable by trace polynomials, we know that $\{V_N\}$ is asymptotically approximable by trace polynomials (Lemma 3.29). Therefore, the hypotheses of Proposition 5.5 are satisfied and so

$$\chi(\lambda) = \limsup_{N\to\infty}\Big(\frac{1}{N^2}h(\mu_N) + \frac{m}{2}\log N\Big) = \chi^*(\lambda)$$

and the same holds for $\underline{\chi}(\lambda)$. Moreover, this holds for $\mu_N * \sigma_{t,N}$ just as well as μ_N because $\mu_N * \sigma_{t,N}$ satisfies the same assumptions (A)–(D).

For claim (3), first fix N and let X be a random variable with law μ_N, and let Y_t be an independent Hermitian Brownian motion (here $Y_t \sim \sigma_{t,N}$). Let $\Xi_t = DV_{N,t}(X + Y_t)$, which is the conjugate variable of $X + Y_t$. Then

$$\frac{1}{N^3}\mathcal{I}(\mu_N * \sigma_{t,N}) = E\|\Xi_t\|_2^2$$

Suppose $0 \le s \le t \le T$. Then $X + Y_t$ is the sum of the independent random variables $X + Y_s$ and $Y_t - Y_s$, and thus $\Xi_t = E[\Xi_s | X + Y_t]$ by Lemma 5.6. In other words, Ξ_t is the orthogonal projection of $DV_{N,s}(X + Y_s)$ onto the space of L^2 random variables that are functions of $X + Y_t$, or in other words it is the function of $X + Y_t$ that is closest to Ξ_s in L^2. This implies that

$$\begin{aligned}[\|\Xi_s - \Xi_t\|_2^2] &\le E[\|DV_{N,s}(X+Y_s) - DV_{N,s}(X+Y_t)\|_2^2] \\ &\le E\Big[\frac{C^2}{(1+Cs)^2}\|Y_s - Y_t\|_2^2\Big] \\ &= \frac{C^2}{(1+Cs)^2}m(t-s)\end{aligned}$$

using the fact that $V_{N,s} \in \mathcal{E}(0, C(1+Cs)^{-1})$ and hence $DV_{N,s}$ is $C(1+Cs)^{-1}$-Lipschitz. Since Ξ_t is the orthogonal projection of Ξ_s onto this subspace, we know $\Xi_s - \Xi_t$ is orthogonal to Ξ_t and hence

$$E[\|\Xi_s\|_2^2] - E[\|\Xi_t\|_2^2] = E[\|\Xi_s - \Xi_t\|_2^2].$$

Overall,

$$0 \le \frac{1}{N^3}\mathcal{I}(\mu_N * \sigma_{s,N}) - \frac{1}{N^3}\mathcal{I}(\mu_N * \sigma_{t,N}) \le \frac{C^2}{(1+Cs)^2}m(t-s).$$

This immediately proves that $t \mapsto N^{-3}\mathcal{I}(\mu_N * \sigma_{t,N})$ is a decreasing function of t, it is Lipschitz, and the absolute value of the derivative is bounded by $C^2m/(1+Ct)^2$. The same holds for $\Phi^*(\lambda \boxplus \sigma_t)$ by taking the limit as $N \to \infty$. □

8. Free Gibbs laws

In the situation of Theorem 4.1, we want to interpret the law λ as the free Gibbs state for a potential which is the limit of the V_N. To this end, we will define a noncommutative function space where each element is a limit of functions on $M_N(\mathbb{C})^m_{\mathrm{sa}}$. We will then give several characterizations of the closure of trace polynomials in this space, as well as the class of potentials to which our previous results apply.

8A. ***Asymptotic approximation and function spaces.*** Let $Y_\bullet = \{Y_N\}$ be a sequence of normed vector spaces. We define a (possibly infinite) seminorm on sequences $\phi_\bullet = \{\phi_N\}$ of functions $M_N(\mathbb{C})^m_{\mathrm{sa}} \to Y_N$ by

$$\|\phi_\bullet\|_{R,Y_\bullet} = \limsup_{N\to\infty} \sup_{\|x\|\le R} \|\phi_N(x)\|_{Y_N}.$$

Let $\mathcal{F}_m(Y_\bullet)$ be the vector space

$$\{\phi_\bullet : \|\phi_\bullet\|_{R,Y_\bullet} < +\infty \text{ for all } R\}/\{\phi_\bullet : \|\phi_\bullet\|_{R,Y_\bullet} = 0 \text{ for all } R\}.$$

For a sequence $\phi_\bullet$, we denote its equivalence class by $[\phi_\bullet]$.

We equip $\mathcal{F}_m(Y_\bullet)$ with the topology generated by the seminorms $\|\cdot\|_{R,Y_\bullet}$, or equivalently given by the metric

$$d_{\mathcal{F}_m(Y_\bullet)}(\phi_\bullet, \psi_\bullet) = \sum_{n=1}^\infty \frac{1}{2^n} \min(\|\phi_\bullet - \psi_\bullet\|_{n,Y}, 1). \tag{8-1}$$

Note that $\mathcal{F}_m(Y_\bullet)$ is a complete metric space in this metric and is a locally convex topological vector space.

There is a canonical map from the vector space of scalar-valued trace polynomials TrP^0_m into $\mathcal{F}^0_m := \mathcal{F}_m(\mathbb{C})$ by the map that sends a trace polynomial to the corresponding sequence of functions it defines on $M_N(\mathbb{C})^m_{\mathrm{sa}}$. A sequence $\phi_\bullet$ is asymptotically approximable by trace polynomials if and only if $[\phi_\bullet]$ is in the closure of the image of TrP^0_m in $\mathcal{F}^0_m$, which we will denote by $\mathcal{T}^0_m$. (Unfortunately, we do not know whether the map $\mathrm{TrP}^0_m \to \mathcal{F}^0_m$ is injective, but this point is irrelevant for our purposes.)

Similarly, let $M_\bullet(\mathbb{C})^m$ be the sequence $\{M_N(\mathbb{C})^m\}$ equipped with $\|\cdot\|_2$. There is a canonical map from TrP^1_m into $\mathcal{F}^1_m := \mathcal{F}_m(M_\bullet(\mathbb{C}))$ given by mapping a trace polynomial to the corresponding sequence of functions on matrices. A sequence $\phi_\bullet$ of functions $M_N(\mathbb{C})^m_{\mathrm{sa}} \to M_N(\mathbb{C})_{\mathrm{sa}}$ is asymptotically approximable by trace polynomials if and only if $[\phi_\bullet]$ is in the closure of the image of TrP^1_m, which we denote by $\mathcal{T}^1_m$.

The spaces $\mathcal{T}^0_m$ and $\mathcal{T}^1_m$ can be viewed as noncommutative function spaces through the following alternative characterization. Here $\mathcal{R}$ denotes the hyperfinite II_1 factor and $\mathcal{R}^\omega$ denotes its ultrapower; for an explanation, see [Anantharaman and Popa 2016, §1.6 and §5.4] or [Capraro 2010, pp. 5–7].

Lemma 8.1. *Let $f \in \mathrm{TrP}^0_m$. Then we have*

$$\limsup_{N\to\infty} \sup_{\substack{x\in M_N(\mathbb{C})^m_{\mathrm{sa}} \\ \|x\|_\infty \le R}} |f(x)| = \sup_N \sup_{\substack{x\in M_N(\mathbb{C})^m_{\mathrm{sa}} \\ \|x\|_\infty \le R}} |f(x)| = \sup_{\substack{x\in (\mathcal{R}^\omega_{\mathrm{sa}})^m \\ \|x\|_\infty \le R}} |f(x)|. \tag{8-2}$$

If we denote the common value by $\|f\|_{\mathcal{T}^0_m,R}$, then this family of seminorms defines a metrizable topology on TrP^0_m *with the metric given as in* (8-1), *and $\mathcal{T}^0_m$ is the completion of* TrP^0_m *in this metric. The same* *result holds for $\mathcal{T}_m^1$ using the seminorm*

$$\limsup_{N\to\infty} \sup_{\substack{x\in M_N(\mathbb{C})_{\mathrm{sa}}^m \\ \|x\|_\infty\le R}} \|f(x)\|_2 = \sup_N \sup_{\substack{x\in M_N(\mathbb{C})_{\mathrm{sa}}^m \\ \|x\|_\infty\le R}} \|f(x)\|_2 = \sup_{\substack{x\in(\mathcal{R}_{\mathrm{sa}}^\omega)^m \\ \|x\|_\infty\le R}} \|f(x)\|_2. \tag{8-3}$$

Proof. Fix f and let A, B, and C be the three quantities in (8-2) from left to right. It is clear that $A \le B$. Moreover, $B \le C$ because there is an isometric trace-preserving embedding of $M_N(\mathbb{C})$ into $\mathcal{R}^\omega$. To show that $C \le A$, pick $x \in (\mathcal{R}_{\mathrm{sa}}^\omega)^m$ with $\|x\| \le R$. Then there exists $x_n \in \mathcal{R}_{\mathrm{sa}}^m$ with $\|x_n\| \le R$ and $x = \lim_{n\to\omega} x_n$. For each n, we can choose an N_n, an embedding $M_{N_n}(\mathbb{C}) \to \mathcal{R}$ and a $y_n \in M_{N_n}(\mathbb{C})$ such that $\|y_n\| \le R$ and $\|x_n - y_n\|_2 \le 1/2^n$ and $\lim_{n\to\infty} N_n = +\infty$. Then $x = \lim_{n\to\omega} y_n$ and $|f(x)| = \lim_{n\to\omega} |f(y_n)| \le A$. This shows that the three seminorms in (8-2) are equal, and the other claims follow because these seminorms are the same as the seminorms for $\mathcal{F}_m^0$. □

From this point of view, every $f \in \mathcal{T}_m^0$ has a canonical sequence that represents its equivalence class in $\mathcal{F}_m^0$, constructed as follows. If we write f as the limit of a sequence of trace polynomials $f^{(k)}$, then $f_{(k)}|_{M_N(\mathbb{C})_{\mathrm{sa}}^m}$ converges locally uniformly on $M_N(\mathbb{C})_{\mathrm{sa}}^m$ as $k \to \infty$ and the limit is independent of the approximating sequence $f^{(k)}$. We can therefore define $f|_{M_N(\mathbb{C})_{\mathrm{sa}}^m}$ to be this limit.

Similarly, f defines a function on $(\mathcal{R}_{\mathrm{sa}}^\omega)^m$. Moreover, if $(\mathcal{M}, \tau)$ is a tracial von Neumann algebra and there is a trace-preserving embedding $\iota : \mathcal{M} \to \mathcal{R}^\omega$, then we may define $f|_{\mathcal{M}} = f \circ \iota$. It is easy to see that this is independent of the choice of trace-preserving embedding if f is a trace polynomial, and this holds for general $f \in \mathcal{T}_m^0$ or $\mathcal{T}_m^1$ by density of trace polynomials. In this sense, $\mathcal{T}_m^0$ and $\mathcal{T}_m^1$ represent spaces of universal scalar- or operator-valued functions that can be applied to self-adjoint operators in every $\mathcal{R}_{\mathrm{sa}}^\omega$-embeddable tracial von Neumann algebra.

In the scalar-valued case, we have yet another characterization of $\mathcal{T}_m^0$:

Lemma 8.2. *Let $\Sigma_{m,\mathrm{bdd}} = \bigcup_{R>0} \Sigma_{m,R}$. Let $C(\Sigma_{m,\mathrm{bdd}})$ be the space of functions $g : \Sigma_{m,\mathrm{bdd}} \to \mathbb{C}$ such that $g \in C(\Sigma_{m,R})$ for every R, equipped with the family of seminorms $\|\cdot\|_{C(\Sigma_{m,R})}$. Then $\mathcal{T}_m^0$ is isomorphic to $C(\Sigma_{m,\mathrm{bdd}})$ as a topological vector space.*

Proof. For a scalar-valued trace polynomial f, the value $f(x)$ only depends on the law of x, so that $f(x) = g(\lambda_x)$ for some function $g : \Sigma_m \to \mathbb{R}$ such that $g \in C(\Sigma_{m,R})$ for all R, and we have

$$\|f\|_{\mathcal{T}_m^0,R} = \|g\|_{C(\Sigma_{m,R})}.$$

Passing to the completion with respect to the metric defined as in (8-1), we have a map $\iota : \mathcal{T}_m^0 \to C(\Sigma_{m,\mathrm{bdd}})$ which is an isomorphism onto its image. To show that ι is surjective, note the algebra of trace polynomials is self-adjoint and separates points in $\Sigma_{m,R}$, and hence by the Stone–Weierstrass theorem, trace polynomials are dense in $C(\Sigma_{m,R})$ for every R. Therefore, if $g \in C(\Sigma_{m,R})$, we can choose a trace polynomial $g^{(k)}(\lambda_x) = f^{(k)}(x)$ such that $\|g - g^{(k)}\|_{C(\Sigma_{m,k})} \le 1/2^k$. Then $f^{(k)}$ converges to some f in $\mathcal{T}_m^0$, and we have $\iota(f) = g$. □

8B. ***Convex differentiable functions.*** Now we are ready to characterize the type of convex functions which occur in Theorem 7.1. First of all, we let $\mathcal{T}_m^{0,1}$ be the completion of the trace polynomials with respect to the metric

$$d(f,g)=\sum_{n=1}^{\infty}\frac{1}{2^n}[\min(1,\|f-g\|_{\mathcal{T}_m^0,n})+\min(1,\|Df-Dg\|_{(\mathcal{T}_m^1)^m,n})].$$

Observe that if $f\in\mathcal{T}_m^{0,1}$ and $f^{(k)}$ is a sequence of trace polynomials converging to f in $\mathcal{T}_m^{0,1}$ as $k\to\infty$, then $Df^{(k)}$ converges in $(\mathcal{T}_m^1)^m$ and the limit is independent of the choice of approximating sequence. We denote this limit by Df.

Remark 8.3. If f and $f^{(k)}$ are as above, then since $Df^{(k)}$ is a tuple of trace polynomials, it is continuous on the operator norm ball $\{y\in M_N(\mathbb{C})_{\text{sa}}^m:\|y\|_\infty\le R\}$ with a modulus of continuity that only depends on R and does not depend on N. Because $Df^{(k)}\to Df$ uniformly on the operator-norm ball (with rate of convergence independent of N), we know Df is also continuous on this operator-norm ball with modulus of continuity independent of N.

It follows that for every $x,y\in M_N(\mathbb{C})_{\text{sa}}^m$ with $\|x\|,\|y\|\le R$ we have

$$f(y)-f(x)=\langle Df(x),y-x\rangle_2+o(\|y-x\|_2),$$

where the error estimate only depends on R and not on N. In particular, this shows Df is uniquely determined by f. Also, it shows that $Df|_{M_N(\mathbb{C})_{\text{sa}}^m}$ is equal to the normalized gradient of $f|_{M_N(\mathbb{C})_{\text{sa}}^m}$ in the ordinary sense of functions on $M_N(\mathbb{C})_{\text{sa}}^m\cong\mathbb{R}^{mN^2}$.

Lemma 8.4. *Let $f\in\mathcal{T}_m^{0,1}$ be real-valued. The following are equivalent*:

(1) *The function $f|_{M_N(\mathbb{C})_{\text{sa}}^m}$ is convex for every N.*

(2) *The function f is convex as a function on $(\mathcal{R}_{\text{sa}}^\omega)^m$.*

(3) *There exists a sequence of differentiable convex functions $V_N:M_N(\mathbb{C})_{\text{sa}}^m\to\mathbb{R}$ such that $[V_\bullet]=f$ and $[DV_\bullet]=Df$. (Here $DV_\bullet$ denotes the sequence $(DV_N)_{N\in\mathbb{N}}$, where D is the normalized gradient understood in the standard sense of calculus.)*

Proof. The implication (1) $\Longrightarrow$ (2) follows from an argument similar to the proof of Lemma 8.1.

The implication (1) $\Longrightarrow$ (3) holds because we can take $V_N=f|_{M_N(\mathbb{C})_{\text{sa}}^m}$.

Now we will prove (3) $\Longrightarrow$ (1). Fix N. To prove that $f|_{M_N(\mathbb{C})_{\text{sa}}^m}$ is convex, it suffices to show that $\langle Df(x)-Df(y),x-y\rangle_2\ge 0$ for every $x,y\in M_N(\mathbb{C})_{\text{sa}}^m$. For $k\in\mathbb{N}$, consider $x\otimes I_k$ and $y\otimes I_k$ in $M_{Nk}(\mathbb{C})_{\text{sa}}^m$. Then, as $k\to\infty$,

$$\langle Df(x)-Df(y),x-y\rangle_2=\langle Df(x\otimes I_k)-Df(y\otimes I_k),x\otimes I_k-y\otimes O_k\rangle_2;$$

meanwhile, if $R=\max(\|x\|,\|y\|)$, then since $DV_N-Df\to 0$ in $\|\cdot\|_2$ uniformly on the operator norm ball of radius R, we have as $k\to\infty$ that

$$\langle Df(x\otimes I_k)-Df(y\otimes I_k),x\otimes I_k-y\otimes I_k\rangle_2-\langle DV_{Nk}(x\otimes I_k)-DV_{Nk}(y\otimes I_k),x\otimes I_k-y\otimes I_k\rangle_2\to 0.$$

Because V_{Nk} is convex, the second inner product is ≥ 0 and therefore $\langle Df(x)-Df(y),x-y\rangle_2\ge 0$. □

Let $\mathcal{E}_m(c, C)^{0,1}$ denote the class of $V \in \mathcal{T}_m^{0,1}$ such that $V(x) - (c/2)\|x\|_2^2$ is convex and $V(x) - (C/2)\|x\|_2^2$ is concave. If $0 < c < C$ and $V \in \mathcal{E}_m(c, C)^{0,1}$ and $V_N = V|_{M_N(\mathbb{C})_{\mathrm{sa}}^m}$, then the sequence of normalized gradients DV_N is asymptotically approximable by trace polynomials. If we let μ_N be the corresponding measure on $M_N(\mathbb{C})_{\mathrm{sa}}^m$, then Theorem 4.1 (the hypothesis (4-1) being trivially satisfied by unitary invariance) implies that μ_N concentrates around a noncommutative law λ_V, which we will call the *free Gibbs state for the potential V*.

Furthermore, the free Gibbs state λ_V is independent of the choice of representative sequence in the following sense. Let μ_N be the measure on $M_N(\mathbb{C})_{\mathrm{sa}}^m$ given by the potential $V_N = V|_{M_N(\mathbb{C})_{\mathrm{sa}}^m}$. Let W_N be another sequence of potentials satisfying the hypotheses of Theorem 4.1 such that $[W_\bullet] = V$ in $\mathcal{T}_m^{0,1}$, and let ν_N be the sequence of random matrix measures given by W_N. By Theorem 4.1, ν_N concentrates around some noncommutative law λ. We claim that $\lambda = \lambda_V$. To prove this, consider the sequence $\widetilde{V}_N$ which equals V_N for odd N and W_N for even N. Then $[\widetilde{V}_\bullet] = V$ in $\mathcal{T}_m^{0,1}$, which means that $\{D\widetilde{V}_N\}_{N\in\mathbb{N}}$ is asymptotically approximable by trace polynomials. Therefore,

$$\lambda_V(p) = \lim_{\substack{N \text{ even} \\ N\to\infty}} \int \tau_N(p)\, d\mu_N = \lim_{\substack{N \text{ odd} \\ N\to\infty}} \int \tau_N(p)\, d\nu_N = \lambda(p).$$

In fact, Lemma 8.4 implies that the noncommutative laws λ which occur as limits in Theorem 4.1 are precisely the free Gibbs laws for potentials $V \in \mathcal{E}_m(c, C)^{0,1}$. In particular, Theorem 7.1 implies that $\chi = \underline{\chi} = \chi^*$ for every such law.

Remark 8.5. We have *not* proved that the law λ_V is uniquely characterized by the Schwinger–Dyson equation $\lambda[DV(X)f(X)] = \lambda\otimes\lambda[\mathcal{D}f(X)]$, although something like this is implied by [Dabrowski 2016]. One could hope to prove this by letting the semigroup T_t^V act on an abstract space of Lipschitz functions which is the completion of trace polynomials (where the metric now allows x to come from any tracial von Neumann algebra rather than only the $\mathcal{R}^\omega$-embeddable algebras). We would want to show that if λ satisfies the Schwinger–Dyson equation, then $\lambda(T_t^V u) = \lambda(u)$, but to justify the computation, we need to show more regularity of $T_t^V u$ than we have done in this paper. In the SDE approach as well, the proof that λ_V is characterized by Schwinger–Dyson is subtle when we do not assume more regularity for V (see [Dabrowski 2010; 2016]).

8C. ***Examples of convex potentials.*** A natural class of examples of functions in $\mathcal{E}_m(c, C)^{0,1}$ are those of the form

$$V(x) = \tfrac{1}{2}\|x\|_2^2 + \epsilon f(u)$$

where ϵ is a small positive parameter,

$$u = (u_1, \dots, u_m), \quad u_j = \frac{x_j + 4i}{x_j - 4i},$$

and f is a real-valued trace polynomial in u and u^*. Computations similar to those of Section 3B show that the normalized Hessian of $\mathrm{Jac}(Df(u(x)))$ with respect to x is bounded uniformly in N. Therefore, $V \in \mathcal{E}_m\left(\tfrac{1}{2}, \tfrac{3}{2}\right)^{0,1}$ for sufficiently small ϵ. Similar examples are described in the introduction of [Dabrowski

2016]. More generally, we can replace the trace polynomial $f(u)$ by a power series where the individual terms are trace monomials in u.

The class $\mathcal{E}_m(c, C)^{0,1}$ does not include trace polynomials in x because if g is a trace polynomial of degree ≥ 3, then we cannot have $g(x)$ convex and $g(x) - (C/2)\|x\|_2^2$ concave (globally). However, if we consider a potential which is a small perturbation of a quadratic (as considered in [Guionnet and Maurel-Segala 2006; Guionnet and Shlyakhtenko 2014]), we can fix this problem by introducing an operator-norm cut-off as follows.

Let f be a scalar-valued trace polynomial and let us define

$$V^{(\epsilon)}(x) = \|x\|_2^2 + \epsilon f(x). \tag{8-4}$$

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a C_c^∞ function such that $\phi(t) = t$ for $|t| \leq R$ and $\phi(t) = 0$ for $|t| \geq 2R$. Let $\Phi : M_N(\mathbb{C})_{\text{sa}}^m \to M_N(\mathbb{C})_{\text{sa}}^m$ be given by $\Phi_N(x) = (\phi(x_1), \dots, \phi(x_m))$.

$$\widetilde{V}_N^{(\epsilon)}(x) = \|x\|_2^2 + \epsilon f_N(\Phi_N(x)). \tag{8-5}$$

We will prove the following.

Proposition 8.6. *Let $\widetilde{V}_N^{(\epsilon)}$ be given as above. Then $[\widetilde{V}_\bullet^{(\epsilon)}] \in \mathcal{T}_m^{0,1}$. Moreover, given $\delta > 0$, we have $[\widetilde{V}_\bullet^{(\epsilon)}] \in \mathcal{E}_m(1-\delta, 1+\delta)^{0,1}$ for sufficiently small ϵ (depending on f, R, and δ).*

As a consequence, we will deduce the following result about measures defined by $V^{(\epsilon)}$ restricted to an operator-norm ball (without the smooth cut-off Φ).

Proposition 8.7. *Let $2 < R' < R$, let f be a trace polynomial, and let $V^{(\epsilon)}$ be as in* (8-4). *Let*

$$d\mu_N^{(\epsilon)}(x) = \frac{1}{Z_N} \exp(-N^2 V_N^{(\epsilon)}(x)) \mathbf{1}_{\|x\| \leq R} \, dx.$$

For sufficiently small ϵ (depending on f, R, and R'), we have the following. The measure $\mu_N^{(\epsilon)}$ exhibits exponential concentration around a noncommutative law $\lambda^{(\epsilon)} \in \Sigma_{m,R'}$. If $X \in (\mathcal{M}, \tau)$ is a noncommutative m-tuple realizing the law $\lambda^{(\epsilon)}$, then the conjugate variable is given by $DV^{(\epsilon)}(X)$. Moreover, we have

$$\chi(\lambda^{(\epsilon)}) = \underline{\chi}(\lambda^{(\epsilon)}) = \chi^*(\lambda^{(\epsilon)}) = \lim_{N \to \infty} \left(\frac{1}{N^2} h(\mu_N^{(\epsilon)}) + \frac{m}{2} \log N \right).$$

To fix notation for the remainder of this section, functions without a subscript, such as f, will denote elements of $\mathcal{T}_m^0$ or $\mathcal{T}_m^{0,1}$, and Df will denote the "gradient" defined in the abstract space $\mathcal{T}_m^{0,1}$ as the limit of the "gradients" of trace polynomials approximating f. However, f_N will denote $f|_{M_N(\mathbb{C})_{\text{sa}}^m}$, and Df_N will denote the normalized gradient $N \nabla f_N$ defined in the usual sense of calculus with respect to $\langle \cdot, \cdot \rangle_2$ on $M_N(\mathbb{C})_{\text{sa}}^m$. Moreover, $Hf_N = \operatorname{Jac}(Df_N)$ will denote the Hessian of f_N with respect to $\langle \cdot, \cdot \rangle_2$.

In order to prove Proposition 8.6, we must understand $D[f_N \circ \Phi_N]$ and $H[f_N \circ \Phi_N]$. To this end, we recall some results of [Peller 2006] on noncommutative derivatives of $\phi(x)$, where ϕ is a smooth function on the real line.

For a polynomial ϕ in one variable, the noncommutative derivative $\mathcal{D}\phi \in \mathbb{C}\langle X\rangle \otimes \mathbb{C}\langle X\rangle$ defined by Definition 3.6 can be written as the difference quotient

$$\mathcal{D}\phi(s,t) = \frac{\phi(s)-\phi(t)}{s-t},$$

where we view $\mathbb{C}\langle X\rangle \otimes \mathbb{C}\langle X\rangle$ as a subset of functions on $\mathbb{R}^2$ with the variables s and t. However, the above difference quotient makes sense whenever $\phi : \mathbb{R} \to \mathbb{C}$ is smooth. Thus, it defines an extension of $\mathcal{D}$ to continuously differentiable functions ϕ of one variable.

Similarly, if ϕ is a polynomial, then the higher-order noncommutative derivatives $\mathcal{D}^n\phi$ can be viewed as functions of $n+1$ variables, which are obtained through iterated difference quotients and thus their definition can be extended to smooth functions ϕ. (However, beware that we have *not* defined $\mathcal{D}_j^n\phi$ if ϕ is a nonpolynomial function of *multiple* variables.)

If ϕ is a polynomial, then to estimate $\phi(X)-\phi(Y)$ for operators X and Y with norm bounded by R, one seeks to control the norm of $\mathcal{D}\phi$ in the projective tensor product $L^\infty[-R,R] \mathbin{\widehat{\otimes}} L^\infty[-R,R]$. Similarly, if ϕ is a smooth function and $\phi(X)$ and $\phi(Y)$ are defined through functional calculus, one can estimate the operator norm $\|\phi(X)-\phi(Y)\|$ by representing ϕ as an integral of simpler functions (e.g., by Fourier analysis) whose noncommutative derivatives are easier to analyze. In this case, it is convenient to write $\mathcal{D}\phi$ as an *integral* rather than a *sum* of simple tensors.

We thus consider the integral projective tensor powers of the space of bounded Borel functions $\mathcal{B}(\mathbb{R})$. The *integral projective tensor product* $\mathcal{B}(\mathbb{R})^{\widehat{\otimes}_i n}$ consists of Borel functions G on $\mathbb{R}^n$ which admit a representation

$$G(x_1,\dots,x_n) = \int_\Omega G_1(x_1,\omega)\cdots G_n(x_n,\omega)\,d\mu(\omega) \tag{8-6}$$

for some measure space (Ω,μ) such that

$$\int_\Omega \|G_1(\,\cdot\,,\omega)\|_{\mathcal{B}(\mathbb{R})}\cdots\|G_1(\,\cdot\,,\omega)\|_{\mathcal{B}(\mathbb{R})}\,d\mu(\omega) < +\infty \tag{8-7}$$

and we define $\|G\|_{\mathcal{B}(\mathbb{R})^{\widehat{\otimes}_i n}}$ to be the infimum of (8-7) over all representations (8-6).

Given $G \in \mathcal{B}(\mathbb{R})^{\hat{\omega}_i n}$, bounded self-adjoint operators $x_0,\dots,x_n$ and bounded operators $y_1,\dots,y_n$, we define

$$G(x_0,\dots,x_n)\,\#\,(y_1\otimes\cdots\otimes y_n) = \int_\Omega G_0(x_0,\omega)y_1G_1(x_1,\omega)\cdots y_nG_n(x_n,\omega)\,d\mu(\omega), \tag{8-8}$$

where $G_0,\dots,G_n$ satisfy (8-6). This is well-defined by [Peller 2006, Lemma 3.1]. If the x_j and y_j are elements of a tracial von Neumann algebra $(\mathcal{M},\tau)$, we have by the noncommutative Hölder's inequality (see Section 2C) that if $1/\alpha = 1/\alpha_1+\cdots+1/\alpha_n$, then

$$\|G(x_0,\dots,x_n)\,\#\,(y_1\otimes\cdots\otimes y_n)\|_\alpha \le \|G\|_{\mathcal{B}(\mathbb{R})^{\widehat{\otimes}_i(n+1)}}\|y_1\|_{\alpha_1}\cdots\|y_n\|_{\alpha_n}. \tag{8-9}$$

Moreover, we have the following bounds on the noncommutative derivatives of ϕ as a corollary of the results of [Peller 2006].

Proposition 8.8. *There exists a constant K_n such that for all $\phi \in C_c^\infty(\mathbb{R})$*

$$\|\mathcal{D}^n \phi\|_{\mathcal{B}(\mathbb{R})^{\widehat{\otimes}_i (n+1)}} \leq K_n \int_{\mathbb{R}} |\hat{\phi}(\xi)\xi^n| \, d\xi. \tag{8-10}$$

Proof. As in [Peller 2006, §2], choose $w \in C_c^\infty$ such that $0 \leq w \leq \chi_{[-1/2,2]}$ and $\sum_{k\in\mathbb{Z}} w(2^{-k}\xi) = 1$ for $\xi > 0$. Let W_k and $W_k^\#$ be given by $\widehat{W}_k(\xi) = w(2^{-n}\xi)$ and $\widehat{W}_k^\#(\xi) = w(-2^{-k}x)$, where $\hat{}$ denotes the Fourier transform. It is shown in [Peller 2006, Theorem 5.5] that

$$\|\mathcal{D}^n \phi\|_{\mathcal{B}(\mathbb{R})^{\widehat{\otimes}_i (n+1)}} \leq K_n \sum_{k\in\mathbb{Z}} 2^{nk} (\|W_k * \phi\|_{L^\infty(\mathbb{R})} + \|W_k^\# * \phi\|_{L^\infty(\mathbb{R})}).$$

This can be estimated by the right-hand side of (8-10) (for a possibly different constant) by a standard Fourier analysis computation. □

Proof of Proposition 8.6. Recall that $\widetilde{V}_N^{(\epsilon)}(x) = \frac{1}{2}\|x\|_2^2 + \epsilon f_N \circ \Phi_N$. Thus, to show that the sequence $V_N^{(\epsilon)}$ defines an element of $\mathcal{T}_0^m$, it suffices to prove this for $f_N \circ \Phi_N$. To this end, it is sufficient to show that for each $r > 0$ there is a sequence of trace polynomials $\{g^{(k)}\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \sup_{N\in\mathbb{N}} \sup_{x\in M_N(\mathbb{C})_{\text{sa}}^m : \|x\|_\infty \leq r} |g^{(k)}(x) - f_N \circ \Phi_N(x)| = 0$$

and

$$\lim_{k\to\infty} \sup_{N\in\mathbb{N}} \sup_{x\in M_N(\mathbb{C})_{\text{sa}}^m : \|x\|_\infty \leq r} \|Dg^{(k)}(x) - D[f_N \circ \Phi_N(x)]\|_2.$$

Fix $r > 0$. By standard approximation techniques, there exist Schwarz functions $\phi^{(k)} : \mathbb{R} \to \mathbb{R}$ such that $\phi^{(k)}|_{[-r,r]}$ is a polynomial and $\phi^{(k)} \to \phi$ in the Schwarz space as $k \to \infty$. By Proposition 8.8, we have $\mathcal{D}^n \phi^{(k)} \to \mathcal{D}^n \phi$ in $\mathcal{B}(\mathbb{R})^{\widehat{\otimes}_i (n+1)}$ as $k \to \infty$ for every n.

Let $\Phi_N^{(k)}(x_1, \dots, x_m) = (\phi^{(k)}(x_1), \dots, \phi^{(k)}(x_m))$. Then $f_N \circ \Phi_N^{(k)}$ is given by a trace polynomial $g^{(k)}$ on $\{\|x\|_\infty \leq r\}$. Because of the spectral mapping theorem,

$$\sup_{\|x\|\leq r} \|\Phi_N^{(k)}(x) - \Phi_N(x)\|_\infty \leq m \sup_{t\in[-r,r]} |\phi^{(k)}(t) - \phi(t)|,$$

which is independent of N and vanishes as $k \to \infty$. Thus, our trace polynomials $g^{(k)}$ approximate $f_N \circ \Phi_N$ uniformly on the operator norm ball $\{x : \|x\|_\infty \leq r\}$.

Next, we must show that $Dg^{(k)}$ approximates $D[f_N \circ \Phi_N]$ uniformly in $\|\cdot\|_2$ on the operator-norm ball $\{\|x\|_\infty \leq r\}$. By the chain rule, we have

$$D_j[f_N \circ \Phi_N] = \operatorname{Jac}_j(\Phi_N)^t [D_j f_N],$$

where D_j and Jac_j are the normalized gradient and Jacobian with respect to the variable $x_j \in M_N(\mathbb{C})_{\text{sa}}$. Now

$$\operatorname{Jac}_j(\Phi_N)(x)y = \mathcal{D}\phi(x_j) \# y.$$

Now $\mathcal{D}\phi$ viewed as an element of the tensor product $\mathbb{C}[X] \otimes \mathbb{C}[X]$ is invariant under the flip map that switches the order of the tensorands; this is because $\mathcal{D}\phi$ is represented as a difference quotient for one-variable functions. Flip invariance implies that

$$\tau_N[(\mathcal{D}\phi(x_j) \# y)z] = \tau_N[y(\mathcal{D}\phi(x_j) \# z)],$$

which means that the operator $\operatorname{Jac}_j(\Phi_N)(x)$ on $M_N(\mathbb{C})_{\mathrm{sa}}$ is self-adjoint. Hence,

$$D_j[f_N \circ \Phi_N](x) = \operatorname{Jac}_j(\Phi_N(x))[D_j f_N](x) = \mathcal{D}\phi(x_j) \# D_j f_N(\Phi_N(x)).$$

This function is given by a trace polynomial on $\{\|x\|_\infty \leq r\}$, and it is also equal to the trace polynomial $D_j g^{(k)}$ when evaluated on any tuple of matrices because both functions are equal to the gradient of $g^{(k)}|_{M_N(\mathbb{C})_{\mathrm{sa}}^m}$. Moreover, for $\|x\|_\infty \leq r$, we have

$$\mathcal{D}\phi_k(x_j) \# D_j f(\Phi_k(x)) = \mathcal{D}\phi_k(x_j) \# D_j f(\Phi(x)) + \mathcal{D}\phi_k(x_j) \# [D_j f(\Phi_k(x)) - D_j f(\Phi(x))].$$

The first term converges to $\mathcal{D}\phi(x_j) \# D_j f(\Phi(x))$ in $\|\cdot\|_2$ uniformly on $\{\|x\|_\infty \leq r\}$ using (8-9) with estimates independent of N. Similarly, because the images of Φ_k and Φ are contained in an operator norm ball and $D_j f$ is K-Lipschitz in $\|\cdot\|_2$ on this ball for some $K > 0$, we have $D_j f(\Phi_k(x)) - D_j f(\Phi(x)) \to 0$ uniformly. This in turn implies that the second term goes to zero because $\mathcal{D}\phi_k(x_j)$ is uniformly bounded in $\mathcal{B}(\mathbb{R}) \widehat{\otimes}_i \mathcal{B}(\mathbb{R})$. Thus, for every $r > 0$, there is a sequence of trace polynomials $g^{(k)}$ such that $g_k \to f \circ \Phi$ and $Dg^{(k)} \to D(f \circ \Phi)$ uniformly on $\{\|x\|_\infty \leq r\}$. This means that $f \circ \Phi \in \mathcal{T}_m^{1,0}$.

It follows that the sequence $\widetilde{V}_N^{(\epsilon)}$ defines a function in $\mathcal{T}_m^{0,1}$ for every ϵ. It remains to show that this function is in $\mathcal{E}_m(1-\delta, 1+\delta)^{0,1}$ for sufficiently small ϵ. To this end, it suffices to show that $f_N \circ \Phi_N$ defines a function in $\mathcal{E}_m(-a, a)^{0,1}$ for some real $a > 0$. Thus, we only need to obtain some upper and lower bounds on the operator norm of $H[f_N \circ \Phi_N]$ that are independent of N. However, this is equivalent to showing that $D_j(f_N \circ \Phi_N) = \mathcal{D}\phi_N(x_j) \# D_j f_N(\Phi_N(x))$ is Lipschitz in $\|\cdot\|_2$ for each j (uniformly in N). Because $\mathcal{D}^2\phi$ is bounded in $\mathcal{B}(\mathbb{R}) \widehat{\otimes}_i \mathcal{B}(\mathbb{R}) \widehat{\otimes}_i \mathcal{B}(\mathbb{R})$, we see that

$$\|\mathcal{D}\phi(x_j) \# y - \mathcal{D}\phi(x_j') \# y\|_2 \leq K\|x_j - x_j'\|_2 \|y\|_\infty$$

for some constant K; we may apply this to $y = D_j f_N(\Phi_N(x))$, which is bounded in $\|\cdot\|_\infty$ because $D_j f_N$ is a trace polynomial and $\Phi_N(x)$ is bounded in $\|\cdot\|_\infty$. Together with the fact that $D_j f_N(\Phi_N(x))$ is Lipschitz in $\|\cdot\|_2$, this implies that $D_j(f_N \circ \Phi_N)$ is Lipschitz in $\|\cdot\|_2$ as desired. □

Proof of Proposition 8.7. Let $\tilde{\mu}_N^{(\epsilon)}$ be the measure on $M_N(\mathbb{C})_{\mathrm{sa}}^m$ given by the potential $\widetilde{V}_N^{(\epsilon)}$. Let δ be a number in $(0, 1)$ to be chosen later. By Proposition 8.6, we have that $\widetilde{V}^{(\epsilon)} \in \mathcal{E}_m(1-\delta, 1+\delta)^{0,1}$ for sufficiently small ϵ. By Theorem 4.1, the laws $\tilde{\mu}_N$ concentrate around a noncommutative law λ. Furthermore, in Theorem 4.1(1), we can take $M = 0$ and $c = 1-\delta$ and $C = 1+\delta$, so that

$$\limsup_{N\to\infty} R_N \leq \frac{2}{(1-\delta)^{1/2}} + \frac{\|D\widetilde{V}^{(\epsilon)}(0)\|_2}{1-\delta} + \frac{\delta}{(1-\delta)^{3/2}}.$$

Note that $D\widetilde{V}^{(\epsilon)}(0) = DV^{(\epsilon)}(0) = \epsilon Df(0)$ is a scalar multiple of the identity matrix since f is a trace polynomial. Because $R' > 2$, we may choose δ sufficiently small that

$$\frac{2}{(1-\delta)^{1/2}} + \frac{\delta}{(1-\delta)^{3/2}} < R'.$$

Then by choosing ϵ (and hence $\|D\widetilde{V}^{(\epsilon)}(0)\|_2$) sufficiently small, we can arrange that

$$R_* = \limsup_{N\to\infty} R_N < R'.$$

This implies that the measures $\tilde{\mu}_N^{(\epsilon)}$concentrate on the ball $\{\|x\|_\infty \leq R'\}$. For $\|x\|_\infty \leq R$, we have $\widetilde{V}^{(\epsilon)}(x) = V^{(\epsilon)}(x)$, and therefore $\mu_N^{(\epsilon)}$ is the (normalized) restriction of $\tilde{\mu}_N^{(\epsilon)}$ to $\{\|x\|_\infty \leq R\}$. It follows that $\mu_N^{(\epsilon)}$ concentrates around the law $\lambda^{(\epsilon)}$ as well.

If $X \in (\mathcal{M}, \tau)$ realizes the law $\lambda^{(\epsilon)}$, then $\|X\|_\infty \leq R'$ since $\lambda \in \Sigma_{m,R_*} \subseteq \Sigma_{m,R'}$ by Theorem 4.1(2). Moreover, by Proposition 5.10, the conjugate variables for λ are given by $D\widetilde{V}(X) = DV(X)$. Moreover, by Theorem 7.1 applied to $\tilde{\mu}_N^{(\epsilon)}$, we have

$$\chi(\lambda^{(\epsilon)}) = \underline{\chi}(\lambda^{(\epsilon)}) = \chi^*(\lambda^{(\epsilon)}) = \lim_{N\to\infty}\left(\frac{1}{N^2}h(\tilde{\mu}_N^{(\epsilon)}) + \frac{m}{2}\log N\right).$$

In the last equality, we can replace $\tilde{\mu}_N$ by μ_N as in the proof of Proposition 5.5 because $\tilde{\mu}_N$ concentrates on $\{\|x\|_\infty \leq R'\}$. □

Remark 8.9. The approach given here probably does not give the optimal range of ϵ for Proposition 8.7. To get the best result, one would want a more direct way to extend the potential $V^{(\epsilon)} : \{\|x\|_\infty \leq R\} \to \mathbb{R}$ to a potential $\widetilde{V}^{(\epsilon)}$ defined everywhere. This leads us to ask the following question.

Suppose that V is a real-valued function in the closure of trace polynomials with respect to the norm $\|f\|_{\mathcal{T}_m^0,R} + \|Df\|_{\mathcal{T}_m^1,R}$, and hence V defines a function $\{x : \|x\|_\infty \leq R\} \to \mathbb{R}$ for $x \in M_N(\mathbb{C})_{\mathrm{sa}}^m$. If $V(x) - (c/2)\|x\|_2^2$ is convex and $V(x) - (C/2)\|x\|_2^2$ is concave on $\{\|x\| \leq R\}$, then does V extend to a potential $\widetilde{V} \in \mathcal{E}_m(c, C)^{0,1}$? What if we allow $\widetilde{V}$ to have slightly worse constants c and C?

The construction of extensions that preserve the convexity properties is not difficult, but it is less obvious how to construct an extension that one can verify preserves the approximability by trace polynomials.

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Note added in proof

Since this paper was first submitted, the author has extended the techniques to cover conditional expectations and entropy in [Jekel 2020a] and in particular obtained an alternative proof of Theorem 7.1.

Moreover, the Ph.D. thesis [Jekel 2020b] contains the results of this paper and [Jekel 2020a] with more detail and historical background.

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PARAMETRIX FOR A SEMICLASSICAL SUBELLIPTIC OPERATOR

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We demonstrate a parametrix construction, together with associated pseudodifferential operator calculus, for an operator of sum-of-squares type with semiclassical parameter. The form of operator we consider includes the generator of kinetic Brownian motion on the cosphere bundle of a Riemannian manifold. Regularity estimates in semiclassical Sobolev spaces are proven by establishing mapping properties for the parametrix.

1. Introduction

We deal in this paper with a class of second order, subelliptic partial differential operators of the sum-of-squares form

$$P_h = X_0 - h\sum_{j=1}^{d} X_j^2 - h\sum_{j=1}^{d} c_j X_j, \quad h \in (0, 1], \tag{1-1}$$

where the X_j for $0 \le j \le d$ are smooth vector fields, the c_j are smooth functions, and $h > 0$ is considered as a semiclassical parameter. We work in $2d+1$ dimensions, either on a compact manifold or an open subset of $\mathbb{R}^{2d+1}$, and make the following assumptions throughout this paper.

Assumption 1.
- The collection of $2d+1$ vectors $\{X_0, X_1, \dots, X_d, [X_0, X_1], \dots, [X_0, X_d]\}$ spans the tangent space at each base point.
- The collection $\{X_1, \dots, X_d\}$ is involutive (closed under commutation of vector fields).

For each $h > 0$ the operator P_h is subelliptic by a result of [Hörmander 1967], and by [Rothschild and Stein 1976] the operator P_h controls $\frac{2}{3}$-derivatives in the Sobolev space sense. In the semiclassical setting it is natural to work with a semiclassical notion of Sobolev spaces; we refer to [Zworski 2012] for a treatment of semiclassical analysis. The question of interest in this paper is the dependence on h of the various constants in a priori inequalities for P_h, both in L^2 and semiclassical Sobolev spaces.

Our work is motivated by that of Alexis Drouot [2017], who studied such an operator on the cosphere bundle $S^*(M)$ of a $(d+1)$-dimensional Riemannian manifold M. The paper [Drouot 2017] considers the operator $P_h = H + h\Delta_{\mathbb{S}}$, with H the generator of the Hamiltonian/geodesic flow and $\Delta_{\mathbb{S}}$ the nonnegative Laplace–Beltrami operator along the fibers of the cosphere bundle. In local coordinate charts this operator can be represented in the form (1-1), where $X_0 = H$, and $\{X_j\}_{j=1}^{d}$ is any local orthonormal frame for the

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tangent space of the fibers of $S^*(M)$. In [Drouot 2017] it is shown that, if M is negatively curved, then as $h \to 0$ the eigenvalues of $-iP_h$ converge to the Pollicott–Ruelle resonances of M. The analogous result was proven in [Dyatlov and Zworski 2015] for $P_h = H + h\Delta$, where Δ is the Laplacian on $S^*(M)$. The interest in taking $P_h = H + h\Delta_{\mathbb{S}}$ is that this operator generates what is known as kinetic Brownian motion on M. For a treatment of this process we refer to [Franchi and Le Jan 2007; Grothaus and Stilgenbauer 2013; Angst, Bailleul, and Tardif 2015; Li 2016].

A key step in the proof of convergence in [Drouot 2017] was controlling the subelliptic estimates for P_h as $h \to 0$. We emphasize that the estimates we prove are the same as in that paper, with an occasional improvement in the remainder terms. The aim here is to obtain a finer microlocal understanding of the parametrix. We obtain a parametrix valid on the region $h\Delta \geq 1$, strictly larger than the semiclassical region $h^2\Delta \geq 1$. The restriction $h\Delta \geq 1$ arises from the largest region of phase space on which the uncertainty principle holds for the parametrix. The estimates in [Drouot 2017] were obtained through commutator methods, analogous to the work of [Hörmander 1967]. Our approach is more similar to that of [Rothschild and Stein 1976], in that we use an approximation to the operator at each point by a model nilpotent Lie group, and construct a parametrix from the inverse of the model operator on that group. Estimates are then obtained from mapping properties for the parametrix. In contrast to [Rothschild and Stein 1976], which lifted the operator to a higher-dimensional Lie group on which the parametrix is represented as a singular integral kernel, we construct the parametrix in pseudodifferential form on the space itself. This procedure is motivated by the author's work [Smith 1994] on the $\bar{\partial}_b$ problem on three-dimensional CR manifolds of finite type.

When constructing a parametrix for P_h of the form (1-1), it is more natural from the semiclassical viewpoint to consider $hP_h = hX_0 + \sum_{j=1}^d (hX_j)^2$, and quantize symbols in terms of $h\eta$. This leads to placing an extra factor of h on the variables η'' dual to X_j for $d+1 \leq j \leq 2d$, since $[hX_0, hX_j] \sim h^2 X_{j+d}$. The quantization of symbols is naturally carried out using exponential coordinates with respect to an extension of $\{X_j\}_{j=0}^d$ to a frame $\{X_j\}_{j=0}^{2d}$. We will require that:

Assumption 2. If $1 \leq i \leq d$, then $[X_0, X_i] - 2X_{i+d} \in \operatorname{span}(X_0, \ldots, X_d)$.

This can of course be arranged by setting $X_{i+d} = 2[X_0, X_i]$. In the model nilpotent Lie group setting where all other commutators vanish, there is a natural nonisotropic dilation structure using powers (2, 1, 3). Precisely, we split $\eta \in \mathbb{R}^{2d+1} = \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ into (η_0, η', η''), and similarly use X' as abbreviation for the collection $(X_1, \ldots, X_d)$, and $X'' = (X_{d+1}, \ldots, X_{2d})$. Then the dilation that respects the fundamental solution for the model operator is

$$\delta_r(\eta) = (r^2\eta_0, r\eta', r^3\eta'').$$

We now summarize the main result of this paper, leaving details to be expanded upon in later sections. For simplicity consider an open set $U \subset \mathbb{R}^{2d+1}$. For a multi-index $\alpha \in \mathbb{N}^{2d+1}$, let

$$\operatorname{order}(\alpha) = 2\alpha_0 + |\alpha'| + 3|\alpha''|.$$

We use $\exp_x(y)$ to denote the time-1 flow of x along $\sum_{j=0}^{2d} y_j X_j$.

Proposition 3. *Given $\rho(x) \in C_c^\infty(U)$, there is $\chi_0 \in C_c^\infty(\mathbb{R}^{2d+1})$ and an h-dependent family of symbols $a(x,\eta)$ satisfying*

$$|\partial_x^\beta \partial_\eta^\alpha a(x,\eta)| \le C_{\alpha,\beta} h (h^{\frac{1}{2}} + |\eta_0|^{\frac{1}{2}} + |\eta'| + |\eta''|^{\frac{1}{3}})^{-2-\mathrm{order}(\alpha)},$$

with $C_{\alpha,\beta}$ independent of $h \in (0,1]$, so that the operator $a_h(x, hD)$ defined by

$$a_h(x, hD) = \frac{1}{(2\pi)^{2d+1}} \int e^{-i\langle y,\eta\rangle} a(x, h\eta_0, h\eta', h^2\eta'') f(\exp_x(y)) \chi_0(y)\, dy\, d\eta$$

satisfies

$$a_h(x, hD) \circ P_h = \rho(x) + r_h(x, hD),$$

where $r_h(x, hD)$ is an operator that satisfies the following with C_{p_1,p_2} independent of $h \in (0,1]$, for any polynomials $p_j(\eta)$ on $\mathbb{R}^{2d+1}$:

$$\|p_1(X_0, h^{\frac{1}{2}}X', h^{\frac{1}{2}}X'') \circ r_h(x, hD) \circ p_2(X_0, h^{\frac{1}{2}}X', h^{\frac{1}{2}}X'') f\|_{L^2} \le C_{p_1,p_2} \|f\|_{L^2}.$$

For example, one can take p_1 or p_2 to yield the operator $(1 + X_0^* X_0)^{N_1}(1 + h\Delta)^{N_2}$, where Δ is the Laplacian on $\mathbb{R}^{2d+1}$. These bounds roughly say that the parametrix inverts P_h on the region

$$\{\Delta \ge h^{-1}\} \cup \{|X_0| \ge 1\}.$$

In particular, the remainder term r_h will be of order h^∞ if the solution is localized to a region where $\Delta \ge h^{-1-\epsilon}$ for some $\epsilon > 0$.

We remark that in the calculus developed here P_h is of order 2, and thus distinct from the standard semiclassical calculus where hP_h is of order 2. This is related to the fact that we are working on the region $|\eta| \ge h^{1/2}$ rather than $|\eta| \ge 1$. Symbols of order j are weighted by a factor $h^{-j/2}$ to ensure that symbols of negative order (but not necessarily their derivatives) remain bounded as $h \to 0$. With this accounting, X_0 is an operator of order 2, $h^{1/2}X_j$ is of order 1 for $1 \le j \le d$, and $h^{1/2}X_j$ is of order 3 for $d+1 \le j \le 2d$.

Together with the composition calculus, pseudolocality arguments, and L^2 mapping bounds for operators, we deduce the regularity results on $S^*(M)$ for P_h that were established in [Drouot 2017]. These are stated in Theorems 20 and 21.

The outline of this paper is as follows. In Section 2 we introduce a model operator of P_h on a step-2 nilpotent group, and discuss the homogeneous fundamental solution in this setting. In Section 3 we study the degree to which the model operator, attached to M by exponential coordinates, approximates P_h. Careful estimates of the Taylor expansion of vector fields and exponential coordinates are needed to obtain uniform estimates as $h \to 0$. In Section 4 we prove that operators of the form $a_h(x, hD)$ form an algebra under composition. This allows for the construction of parametrices from the inverse for the model operator on the nilpotent Lie group. In Section 5 we establish L^2 boundedness of order-0 operators in local coordinates, using a nonisotropic Littlewood–Paley decomposition of the operator and the Cotlar–Stein lemma. Finally, in Section 6 we establish the main regularity estimates for P_h in h-Sobolev spaces, leading to the proof of the bounds in [Drouot 2017].

2. Operators on model domains

In this section we consider a nilpotent Lie group structure on $\mathbb{R}^{2d+1}$ that captures the commutation relations of the vector fields X_j, and we introduce a left-invariant model of P_h. The top-order term in the parametrix for P_h at a point $x \in U$ will be given by the fundamental solution for the model operator, attached to U via exponential coordinates at x relative to the frame $\{X_j\}_{j=0}^{2d}$. In subsequent sections we show that the model operator agrees to leading order with the expression of P_h in exponential coordinates, and develop a graded pseudodifferential calculus that allows us to produce a parametrix for P_h modulo a smoothing operator. We start by considering $h = 1$, and then obtain the fundamental solution for all h by a suitable dilation.

We use the variables $y = (y_0, y', y'') \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$, with dual variables $\eta = (\eta_0, \eta', \eta'')$, and introduce the dilation structure

$$\delta_r(\eta) = (r^2\eta_0, r\eta', r^3\eta''), \quad \delta_{r^{-1}}(y) = (r^{-2}y_0, r^{-1}y', r^{-3}y'').$$

We also introduce a corresponding nonisotropic homogeneous weight $m \in C^\infty(\mathbb{R}^{2d+1}\backslash\{0\})$,

$$m(\eta) = (|\eta_0|^6 + |\eta'|^{12} + |\eta''|^4)^{\frac{1}{12}},$$

so that $m(\delta_r(\eta)) = rm(\eta)$, and $3^{-12/5} \le m(\eta) \le 1$ when $|\eta| = 1$.

Consider the frame of vector fields on $\mathbb{R}^{2d+1}$ given by

- $Y_0 = \partial_0 - \sum_{j=1}^d y_j \partial_{j+d}$,
- $Y_j = \partial_j + y_0\partial_{j+d}$ for $1 \le j \le d$,
- $Y_j = \partial_j$ for $j \ge d+1$,

and observe that

$$[Y_0, Y_j] = 2Y_{j+d} \quad \text{if } 1 \le j \le d,$$

with all other commutators equal to 0. The collection $\{Y_j\}_{j=0}^{2d}$ forms a nilpotent (step-2) Lie algebra. These are left-invariant vector fields associated to the nilpotent Lie group structure on $\mathbb{R}^{2d+1}$ with product

$$y \times w = (y_0 + w_0, y' + w', y'' + w'' + y_0 w' - w_0 y').$$

The exponential map at base point y, and corresponding exponential coordinates, are given by

$$\begin{aligned} \overline{\exp}_y(w) &= (y_0 + w_0, y' + w', y'' + w'' + y_0 w' - w_0 y'), \\ \overline{\Theta}_y(z) &= (z_0 - y_0, z' - y', z'' - y'' - y_0 z' + z_0 y'), \end{aligned} \tag{2-1}$$

so in particular $\overline{\Theta}_0(w) = w$.

The vector field Y_0 is homogeneous of order 2 under δ_r in that

$$Y_0(f \circ r^{-1}) = r^{-2}(Y_0 f) \circ \delta_{r^{-1}},$$

which we summarize by writing $\operatorname{order}(Y_0) = 2$. Similarly, $\operatorname{order}(Y_j) = 1$ for $1 \le j \le d$, and $\operatorname{order}(Y_j) = 3$ for $d+1 \le j \le 2d$. More generally, if we define the order of a multi-index α by

$$\operatorname{order}(\alpha) = 2\alpha_0 + \alpha_1 + \cdots + \alpha_d + 3\alpha_{d+1} + \cdots + 3\alpha_{2d} = 2\alpha_0 + |\alpha'| + 3|\alpha''|,$$

then the monomial differential operator $y^\beta \partial_y^\alpha$ will be homogeneous, with order given by

$$\operatorname{order}(y^\beta \partial_y^\alpha) = \operatorname{order}(\alpha) - \operatorname{order}(\beta). \tag{2-2}$$

The left-invariant differential operator $Y_0 - \sum_{j=1}^d Y_j^2$ is subelliptic and homogeneous of order 2. By [Folland 1975], it has a unique homogeneous fundamental solution $K(y) \in C^\infty(\mathbb{R}^{2d+1} \backslash \{0\})$,

$$\Bigl(Y_0 - \sum_{j=1}^d Y_j^2\Bigr) K(y) = \delta(y), \quad K(\delta_{r^{-1}}(y)) = r^{(2+4d)-2} K(y).$$

The homogeneous inverse for $Y_0 - \sum_{j=1}^d Y_j^2$ is given by convolution with K, which we will express in pseudodifferential form. Precisely, if we let $q_0(\eta) = \widehat{K}$, then $q_0(\delta_r \eta) = r^{-2} q_0(\eta)$, and the operator

$$q_0(D) f(y) = \frac{1}{(2\pi)^{2d+1}} \int_{\mathbb{R}^{2d+1}} e^{-i\langle \overline{\Theta}_y(z), \eta\rangle} q_0(\eta) f(z)\, dz\, d\eta$$

is a left and right inverse for $Y_0 - \sum_{j=1}^d Y_j^2$ on the space of Schwartz functions.

To conclude this section we consider the semiclassical subelliptic operator $hY_0 - \sum_{j=1}^d h^2 Y_j^2$. This is naturally associated to dilating y_0 and y' by h, and y'' by h^2, in that

$$\Bigl(Y_0 - \sum_{j=1}^d Y_j^2\Bigr)(f(hy_0, hy', h^2 y'')) = \Bigl(hY_0 f - \sum_{j=1}^d h^2 Y_j^2 f\Bigr)(hy_0, hy', h^2 y'').$$

Consequently, if we introduce the operation on symbols

$$a_h(\eta) = a(\eta_0, \eta', h\eta''),$$

then the inverse for $hY_0 - \sum_{j=1}^d h^2 Y_j^2$ is given by the semiclassical quantization of q_h,

$$\begin{aligned} q_{0,h}(hD) f(y) &= \frac{1}{(2\pi h)^{2d+1}} \int_{\mathbb{R}^{4d+2}} e^{-i\langle \overline{\Theta}_y(z), \zeta\rangle / h} q_{0,h}(\eta) f(z)\, dz\, d\zeta \\ &= \frac{1}{(2\pi)^{2d+1}} \int_{\mathbb{R}^{4d+2}} e^{-i\langle z, \zeta\rangle} q_{0,h}(\eta) f(\overline{\exp}_y(hz))\, dz\, d\zeta. \end{aligned}$$

3. Approximation by the model domain

Recall that we consider a spanning collection $\{X_0, X_1, \dots, X_{2d}\}$ of vector fields on an open subset U of $\mathbb{R}^{2d+1}$ satisfying the following conditions:

- The collection $\{X_1, \dots, X_d\}$ is involutive (closed under commutation of vector fields).
- If $1 \le i \le d$, then $[X_0, X_i] - 2X_{i+d} \in \operatorname{span}(X_0, \dots, X_d)$.

We will use $x, \tilde{x}$ to denote variables in U and y, z to denote variables in $\mathbb{R}^{2d+1}$.

Let $\exp_x(y)$ be exponential coordinates with base point x in the frame $\{X_0, \dots, X_{2d}\}$. That is, $\exp_x(y) = \gamma(1)$, where $\gamma(0) = x$ and $\gamma'(t) = \sum_{j=0}^{2d} y_j X_j(\gamma(t))$. Define exponential coordinates Θ_x as the local inverse of $\exp_x$ in a neighborhood of x:

$$\Theta_x(\exp_x(y)) = y, \quad \exp_x(\Theta_x(\tilde{x})) = \tilde{x}.$$

Recall the definition (2-2) of the order of a monomial differential operator in y. Consistent with this we have $\mathrm{order}(Y_0) = 2$, $\mathrm{order}(Y_j) = 1$ if $1 \le j \le d$, and $\mathrm{order}(Y_j) = 3$ if $d+1 \le j \le 2d$.

Lemma 4. *For* $0 \le j \le 2d$, *we can write*

$$(X_j f)(\exp_x(y)) = Y_j(f(\exp_x(y))) + R_j(x, y, \partial_y) f(\exp_x(y)),$$

where $\mathrm{order}(R_j) < \mathrm{order}(Y_j)$, *in the sense that the Taylor expansion*

$$R_j(x, y, \partial_y) = \sum_{\alpha,k} c_{j,\alpha,k}(x) y^\alpha \partial_k$$

includes only terms with $\mathrm{order}(y^\alpha \partial_k) < \mathrm{order}(Y_j)$.

Additionally, $c_{0,\alpha,k}(x) \equiv 0$ *unless there is at least one factor of* y_j *with* $j \ge 1$ *occurring in* y^α.

Proof. Any term $y^\alpha \partial_k$ with $|\alpha| > 2$ is of order ≤ 0, so we need examine the Taylor expansion of X_j in exponential coordinates only to second power in y. Additionally, $\mathrm{order}(y_i y_j \partial_k) \le 1$, and equals 1 only if $1 \le i, j \le d$ and $k \ge d+1$. To see that such a term cannot arise in R_j for $1 \le j \le d$, the only case where $\mathrm{order}(Y_j) \le 1$, we use involutivity of $\{X_1, \dots, X_d\}$ and the Frobenius theorem to see that this collection remains tangent to the flowout of the subspace $y_0 = y'' = 0$, and hence we can write $X_j = \sum_{k=1}^d c_k(x, y)\partial_k$ if $y_0 = y'' = 0$ and $1 \le j \le d$.

Thus, we need show that in the expansion of R_j about $y = 0$ the terms linear in y are of order strictly less than $\mathrm{order}(Y_j)$. For $j \ge d+1$ this is immediate, since $X_j = \partial_j = Y_j$ at $y = 0$, and any vector field that vanishes at 0 includes terms of order at most 2. For $0 \le j \le d$ we expand

$$X_j = \partial_j + \sum_{i,k} c_{jik}(x) y_i + (y^2)\partial_y.$$

Since radial lines in exponential coordinates are integral curves of $\sum_{j=0}^{2d} y_j X_j$, we have

$$\sum_{j=0}^{2d} y_j X_j = \sum_{j=0}^{2d} y_j \partial_j, \tag{3-1}$$

from which we deduce

$$c_{ijk} = -c_{jik}.$$

Also, since $[X_0, X_j] - 2X_{j+d} \in \mathrm{span}(X_0, \dots, X_d)$, we deduce for $j = 1, \dots, d$ that

$$c_{j0k} = \begin{cases} 1, & k = j+d, \\ 0, & k > d \text{ and } k \ne j+d. \end{cases}$$

Since $\mathrm{order}(y_i \partial_k) < 2$ unless $k > d$, we deduce $\mathrm{order}(R_0(x, y, \partial_y)) < 2$.

By involutivity of $\{X_1, \dots, X_d\}$, if $1 \le i, j \le d$ then $c_{jik} = 0$ unless also $1 \le k \le d$, in which case $\mathrm{order}(y_i \partial_k) = 0$. And if $i > d$ then $\mathrm{order}(y_i \partial_k) \le 0$ for all k. So if $1 \le j \le d$ then all terms $c_{jik} y_i \partial_k$ for $i \ne 0$ have order ≤ 0, and since $c_{j0k} = \delta_{k,j+d}$ we conclude $\mathrm{order}(R_j(x, y, \partial_y)) \le 0$ if $1 \le j \le d$.

To conclude the lemma, we note by (3-1) that if $y' = y'' = 0$ then $X_0 = \partial_{y_0}$, from which we obtain $R_0 \equiv 0$ along $y' = y'' = 0$. □

For $x \in U$, and y, z in a neighborhood of 0 in $\mathbb{R}^{2d+1}$, we introduce the functions

$$\Theta(x, y, z) = \Theta_{\exp_x(y)}(\exp_x(z)), \quad \widetilde{\Theta}(x, y, w) = \Theta_x(\exp_{\exp_x(y)}(w)), \tag{3-2}$$

where we recall $\Theta_x(\tilde{x})$ denotes exponential coordinates in X_j centered at x. For fixed x and y these are inverse functions of each other on their domains:

$$z = \widetilde{\Theta}(x, y, w) \iff w = \Theta(x, y, z).$$

To invert in the y-variable we note that $v = \widetilde{\Theta}(x, y, w)$ implies $y = \widetilde{\Theta}(x, v, -w)$.

Observe that $\Theta(x, y, z) = -\Theta(x, z, y)$, and $\Theta(x, y, z) = z - y + \mathcal{O}(y, z)^2$. For more precise estimates on Θ and $\widetilde{\Theta}$ we consider their Taylor expansions in exponential coordinates at x. We first assign a notion of order to a smooth function $f(x, y, z)$. Consistent with (2-2), we make the following definition.

Definition 5. For a smooth function $f(x, y, z)$ defined on an open subset of $U \times \mathbb{R}^{2d+1} \times \mathbb{R}^{2d+1}$ containing $U \times \{0, 0\}$, we say that $\mathrm{order}(f) < -j$ if for all $x \in U$

$$(\partial_y^\alpha \partial_z^\beta f)(x, 0, 0) = 0 \quad \text{for all } \alpha, \beta : \mathrm{order}(\alpha + \beta) \le j.$$

Equivalently, the Taylor expansion of f in y, z about $y = z = 0$ contains only monomials $y^\alpha z^\beta$ with $\mathrm{order}(\alpha + \beta) > j$. We let $\mathrm{order}(f)$ be the least $n \in \mathbb{Z}$ such that $\mathrm{order}(f) < n + 1$.

Recalling the definition (2-1) of $\overline{\Theta}_y(z)$, which are exponential coordinates in the frame Y_j on the model domain, we have the following.

Lemma 6. *We have* $\Theta(x, y, z) = \overline{\Theta}_y(z) + R(x, y, z)$, *where* $\mathrm{order}(R_j) < \mathrm{order}(y_j)$ *for each* j. *Similarly*, $\widetilde{\Theta}(x, y, w) = \overline{\Theta}_{-y}(w) + \widetilde{R}(x, y, w)$, *where* $\mathrm{order}(\widetilde{R}_j) < \mathrm{order}(y_j)$ *for each* j.

Proof. We work in exponential coordinates $y = \Theta_x(\,\cdot\,)$ centered at x, and use Lemma 4 to consider X_j as a vector field in y. Then $z = \widetilde{\Theta}(x, y, w)$ means that $z = \gamma(1)$, where $\gamma(t)$ is the integral curve of $w \cdot X \equiv \sum w_k X_k$ with $\gamma(0) = y$. Taking the Taylor expansion of $\gamma(t)$ about $t = 0$ and evaluating at $t = 1$ gives the following expansion of $z = \widetilde{\Theta}(x, y, w)$ in terms of w:

$$z_j = y_j + (w \cdot X)_j(y) + \sum_{k=1}^{\infty} \frac{1}{(k+1)!} (w \cdot X)^k (w \cdot X)_j(y), \tag{3-3}$$

where $(w \cdot X)(y, \partial_y)$ acts on y and $(w \cdot X)_j(y)$ is its ∂_j coefficient as a function of y. It is seen from Lemma 4 that $w \cdot X$ does not increase the order of a function $f(x, y, w)$, and $w \cdot X - w \cdot Y$ decreases the order of $f(x, y, w)$ by at least 1. Also, as functions of (y, w)

$$\mathrm{order}((w \cdot Y)_j(y)) = \mathrm{order}(y_j), \quad \mathrm{order}((w \cdot X)_j(y) - (w \cdot Y)_j(y)) < \mathrm{order}(y_j).$$

Thus, if we replace $w \cdot X$ by $w \cdot Y$ in the expansion (3-3) then the right-hand side is changed by terms of strictly lower order than y_j. It follows that we can write

$$(z_0, z', z'') = (y_0 + w_0, y' + w', y'' + w'' + y_0 w' - w_0 y') + (\widetilde{R}_0, \widetilde{R}', \widetilde{R}''), \tag{3-4}$$

where $\mathrm{order}(\widetilde{R}_0) < -2$, $\mathrm{order}(\widetilde{R}') < -1$, and $\mathrm{order}(\widetilde{R}'') < -3$ as functions of (y, w). Recalling the formula (2-1), this completes the second statement of the lemma.

We next invert the map $w \to z$ to express $w = w(y,z) = \Theta(x,y,z)$, and use (3-4) to write

$$w(y,z) = \overline{\Theta}_y(z) - (\widetilde{R}_0, \widetilde{R}', -y_0\widetilde{R}' + y'\widetilde{R}_0 + \widetilde{R}'') \equiv \overline{\Theta}_y(z) - R(x,y,z),$$

where $\widetilde{R} = \widetilde{R}(y, w(y,z))$. Since w is equal to $z - y$ plus quadratic terms in (y,z), we see that $R(x,y,z)$ has no linear terms in y or z, and hence $\mathrm{order}(w_0) \le -2$ and $\mathrm{order}(w') \le -1$, since quadratic terms are of order at most -2. This also shows that $\mathrm{order}(-y_0\widetilde{R}' + y'\widetilde{R}_0) \le -3$.

To conclude the lemma it suffices to show that $\mathrm{order}(\widetilde{R}''(y,z)) \le -3$, since together with the preceding this shows that $\mathrm{order}(w_j(y,z)) \le \mathrm{order}(y_j)$ for all j, from which it follows that $\mathrm{order}(\widetilde{R}_j(y, w(y,z))) \le \mathrm{order}(\widetilde{R}_j(y,w)) < \mathrm{order}(y_j)$. We know that $\mathrm{order}(w''(y,z)) \le -2$ since quadratic terms are order ≤ -2, and by the above that $\mathrm{order}(w_j(y,z)) \le \mathrm{order}(y_j)$ for $j \le d$. Since $\mathrm{order}(\widetilde{R}'') < -3$ as a function of (y,w), it is easy to see by examining terms in (y,w) of order ≤ -4 that $\mathrm{order}(\widetilde{R}'') \le -3$ as a function of (y,z), concluding the proof. □

We make a few important additional observations about the terms that can occur in the Taylor expansion of $\Theta(x,y,z)$ about $y = z = 0$ and $\widetilde{\Theta}(x,y,w)$ about $y = w = 0$. First, we have

$$\Theta(x,y,z) = z_0 - y_0 \quad \text{if } y' = z' = y'' = z'' = 0,$$
$$\widetilde{\Theta}(x,y,w) = y_0 + w_0 \quad \text{if } y' = w' = y'' = w'' = 0.$$

Consequently, every nonvanishing term in the Taylor expansion of $R(x,y,z)$ about $y = z = 0$ must include a factor of either y', z', y'', or z''. Similarly, every nonvanishing term in the Taylor expansion of $\widetilde{R}(x,y,w)$ about $y = w = 0$ must include a factor of either y', w', y'', or w''.

Additionally, since the collection $\{X_j\}_{j=1}^d$ is involutive it follows that R_0 and R'' vanish if $y_0 = z_0 = y'' = z'' = 0$, and hence every nonvanishing term in the Taylor expansions of R_0 and R'' must contain a factor other than (y', z'). Similarly $\widetilde{R}_0$ and $\widetilde{R}''$ must each contain a factor other than (y', w'). Combining this with the fact that $R(x,y,z) = 0$ if $z = y$, we can write

$$R_j(x,y,z) = \sum_{\substack{|\alpha|+|\beta|=2\\|\beta|\ge 1}} c_{j,\alpha,\beta}(x)y^\alpha(z-y)^\beta + \sum_{\substack{|\alpha|+|\beta|=3\\|\beta|\ge 1}} c_{j,\alpha,\beta}(x,y,z)y^\alpha(z-y)^\beta \tag{3-5}$$

for smooth functions $c_{j,\alpha,\beta}$, where $c_{j,\alpha,\beta} \equiv 0$ unless $\mathrm{order}(y^\alpha z^\beta) < \mathrm{order}(y_j)$, and also unless one of α', β', α'', or β'' is nonzero. Additionally, if $j = 0$ or $j \ge d+1$ then $c_{j,\alpha,\beta} \equiv 0$ unless one of α_0, β_0, α'', or β'' is nonzero.

The same conditions also hold on $\tilde{c}_{j,\alpha,\beta}$ in the following expansion of $\widetilde{R}(x,y,w)$:

$$\widetilde{R}_j(x,y,w) = \sum_{\substack{|\alpha|+|\beta|=2\\|\beta|\ge 1}} \tilde{c}_{j,\alpha,\beta}(x)y^\alpha w^\beta + \sum_{\substack{|\alpha|+|\beta|=3\\|\beta|\ge 1}} \tilde{c}_{j,\alpha,\beta}(x,y,w)y^\alpha w^\beta.$$

4. The semiclassical calculus on U

In this section we introduce the nonisotropic semiclassical quantization and h-dependent symbol classes that we use to construct the parametrix for P_h. As seen for the model operator, the phase variables associated to X'' need to be scaled by h^2, as opposed to the h-scaling for variables associated to X_0

and X'. The symbol classes are naturally associated to the nonisotropic dilation structure δ_r. We will define them using the nonisotropic norm

$$m(\eta) = (|\eta_0|^6 + |\eta'|^{12} + |\eta''|^4)^{\frac{1}{12}} \approx |\eta_0|^{\frac{1}{2}} + |\eta'| + |\eta''|^{\frac{1}{3}},$$

which is smooth for $\eta \neq 0$ and homogeneous of degree 1, in that $m(\delta_r(\eta)) = rm(\eta)$.

We assume that K is compactly contained in U, and choose r_1 so that the exponential map $y \to \exp_x(y)$ is a diffeomorphism on the ball $\{|y| \leq r_1\}$ for all $x \in K$. We also fix $r_0 < r_1$ such that

$$\bigcup_{\tilde{x} \in \exp_x(\bar{B}_{r_0})} \exp_{\tilde{x}}(\bar{B}_{r_0}) \subset \exp_x(B_{r_1}).$$

We fix functions $\chi_j \in C_c^\infty(B_{r_j})$ with $\chi_0(y) = 1$ for $|y| \leq \frac{1}{2} r_0$ and

$$\chi_1(\Theta_x(\,\cdot\,)) = 1 \text{ on a neighborhood of } \bigcup_{\tilde{x} \in \exp_x(\bar{B}_{r_0})} \exp_{\tilde{x}}(\bar{B}_{r_0}).$$

Given a symbol $a(x, \eta) \in C^\infty(U \times \mathbb{R}^{2d+1})$ supported where $x \in K$, we let

$$a_h(x, \eta) = a(x, \eta_0, \eta', h\eta''),$$

and define a nonisotropic semiclassical quantization of a by the rule

$$\begin{aligned} a_h(x, hD) f(x) &= \frac{1}{(2\pi h)^{2d+1}} \int_{\mathbb{R}^{4d+2}} e^{-i\langle y, \eta\rangle/h} a_h(x, \eta) \chi_0(y) f(\exp_x(y))\, dy\, d\eta \\ &= \frac{1}{(2\pi)^{2d+1}} \int_{\mathbb{R}^{4d+2}} e^{-i\langle y, \eta\rangle} a_h(x, \eta) \chi_0(hy) f(\exp_x(hy))\, dy\, d\eta. \end{aligned} \tag{4-1}$$

Thus the Schwartz kernel of $a_h(x, hD)$ is supported in $K \times K_{r_0}$, where K_{r_0} is the image of $K \times \bar{B}_{r_0}$ under $(x, y) \to \exp_x(y)$. In contrast to the usual semiclassical scaling $\eta \to h\eta$, the nonisotropic scaling $(h\eta_0, h\eta', h^2\eta'')$ arises from the missing directions X'' being obtained from commutators of X_0 and X'.

If $p(x, \eta) = \sum_{|\alpha| \leq n} c_\alpha(x) \eta^\alpha$ is a polynomial in η, then

$$(p_h(x, hD) f)(x) = p_h(x, (-i\partial_y)) f(\exp_x(hy))\big|_{y=0}.$$

In particular, we have the following correspondence of symbols to operators:

$$i\eta_j : hX_j, \quad 0 \leq j \leq d, \qquad i\eta_j : h^2 X_j, \quad d+1 \leq j \leq 2d. \tag{4-2}$$

Suppose that the symbol a satisfies homogeneous order-0 type estimates of the form

$$|\partial_x^\beta \partial_\eta^\alpha a(x, \eta)| \leq C_{\alpha,\beta} m(\eta)^{-\operatorname{order}(\alpha)}.$$

The uncertainty principle, needed for example for proving L^2 continuity of $a_h(x, hD)$, requires uniform bounds on $\partial_x^\alpha (h\partial_\eta)^\alpha a_h(x, \eta)$. On the other hand,

$$\begin{aligned} |\partial_x^\alpha (h\partial_\eta)^\alpha a_h(x, \eta)| &= h^{|\alpha_0| + |\alpha'| + 2|\alpha''|} |(\partial_x^\alpha \partial_\eta^\alpha a)_h(x, \eta)| \\ &\leq C_\alpha h^{|\alpha_0| + |\alpha'| + 2|\alpha''|} m(\eta_0, \eta', h\eta'')^{-2|\alpha_0| - |\alpha'| - 3|\alpha''|}. \end{aligned}$$

To have uniform bounds as $h \to 0$ for every α would require truncating $a(x,\eta)$ to where $m(\eta) \geq h^{1/2}$. It is convenient to work with bounded symbols; hence for symbols of order n we will multiply by a factor of $h^{-n/2}$ to ensure that symbols of any order be of size $\lesssim 1$ when $m(\eta) \leq h^{1/2}$.

Definition 7. Let $m(h,\eta) = (h^{1/2} + m(\eta))$. A h-dependent family of symbols $a(x,\eta)$ belongs to $S^n(m)$ if, for all α, β, there is $C_{\alpha,\beta}$ independent of h such that, for $0 < h \leq 1$,

$$|\partial_x^\beta \partial_\eta^\alpha a(x,\eta)| \leq C_{\alpha,\beta} h^{-\frac{n}{2}} m(h,\eta)^{n-\mathrm{order}(\alpha)}.$$

We let $\Psi_h^n(m)$ denote the collection of operators $a_h(x,hD)$ as in (4-1) with $a \in S_h^n(m)$.

We also define $S^{-\infty}(m) = \bigcap_{k\in\mathbb{N}} S^{-k}(m)$, and let $\Psi_h^{-\infty}(m)$ denote operators that can be written in the form (4-1) with χ_0 replaced by χ_1, and $a \in S^{-\infty}(m)$.

Remark 8. We define $\Psi_h^{-\infty}(m)$ using χ_1 in the quantization rule (4-1) since the composition of operators defined using χ_0 need not have Schwartz kernel supported inside B_{r_0} (in local exponential coordinates). We also note that results below concerning continuity and composition of symbols are independent of the particular choice of χ_0. We show in Lemma 13 that replacing χ_0 by another function in $C_c^\infty(B_{r_0})$ that equals 1 on $B_{r_0/2}$ changes $a_h(x,hD)$ by a term in $\Psi_h^{-\infty}(m)$.

For polynomial symbols we note that

$$h^{-\frac{1}{2}\,\mathrm{order}(\alpha)} \eta^\alpha \in S^{\mathrm{order}(\alpha)}(m). \tag{4-3}$$

By (4-2) we then have the following examples, which will show that $P_h \in \Psi_h^2(m)$:

$$\begin{aligned} X_0 &\in \Psi_h^2(m), \\ h^{\frac{1}{2}} X_j &\in \Psi_h^1(m), \quad 1 \leq j \leq d, \\ h^{\frac{1}{2}} X_j &\in \Psi_h^3(m), \quad d+1 \leq j \leq 2d. \end{aligned} \tag{4-4}$$

A more general example of a symbol in $S^n(m)$ is $h^{-n/2} a(\eta)(1 - \phi(h^{-1/2} m(\eta)))$, where $\phi \in C_c^\infty(\mathbb{R}^{2d+1})$ equals 1 on a neighborhood of 0, and $a \in C^\infty(\mathbb{R}^{2d+1} \backslash \{0\})$ satisfies $a(\delta_r \eta) = r^n a(\eta)$.

It is easy to verify the following properties:

$$\begin{gathered} S^n(m) \cdot S^{n'}(m) \subset S^{n+n'}(m), \\ S^n(m) \supset S^{n'}(m) \quad \text{if } n' < n, \\ a \in S^n(m) \quad \Longrightarrow \quad h^{\frac{1}{2}\,\mathrm{order}(\alpha)} \partial_\eta^\alpha \partial_x^\beta a \in S^{n-\mathrm{order}(\alpha)}. \end{gathered} \tag{4-5}$$

Definition 9. Given a sequence of symbols $a_j \in S^{n-j}(m)$ we say that $a \sim \sum_j a_j$ if for all N

$$a - \sum_{j=0}^{N-1} a_j \in S^{n-N}(m).$$

Consequently, a is uniquely determined up to a symbol in $S^{-\infty}(m)$.

We note the following simple example of a symbol in $S^{-\infty}(m)$:

$$\text{If } \phi \in \mathcal{S}(\mathbb{R}) \text{ and } \phi(s) = 1 \text{ when } |s| \le 1 \text{ then } \phi(h^{-\frac{1}{2}} m(\eta)) \in S^{-\infty}(m). \tag{4-6}$$

That this symbol belongs to $S^{-\infty}(m)$ is seen by noting that

$$(1 + h^{-\frac{1}{2}} m(\eta))^{-N} = h^{\frac{N}{2}} m(h, \eta)^{-N},$$

together with the bounds $|\partial_\eta^\alpha m(\eta)| \le C_\alpha m(\eta)^{1-\mathrm{order}(\alpha)}$, where we use that all derivatives vanish unless $m(\eta) \ge h^{1/2}$; hence $m(\eta) \approx m(h, \eta)$, since ϕ is assumed constant near 0.

Lemma 10. *Suppose that $a_j \in S^{n-j}(m)$, $j \in \mathbb{N}$. Then there exists $a \in S^n(m)$ with $a \sim \sum_j a_j$.*

Proof. Fix $\phi \in C_c^\infty((-2, 2))$ with $\phi = 1$ for $|s| < 1$. We will construct a sequence of real numbers $R_j \ge 1$ with $R_j \to \infty$ such that for all N

$$\sum_{j=N}^{\infty} (1 - \phi(R_j^{-1} h^{-\frac{1}{2}} m(\eta))) a_j(x, \eta) \ \text{ converges in } S^{n-N}(m). \tag{4-7}$$

Defining a to be this sum for $N = 0$ then gives the result since by (4-6), for each j,

$$\phi(R_j^{-1} h^{-\frac{1}{2}} m(\eta)) a_j(x, \eta) \in S^{-\infty}(m).$$

The proof of (4-6) shows that the $S^0(m)$ seminorms of $\phi(R^{-1} h^{-1/2} m(\eta))$ are uniformly bounded independent of R for $R \ge 1$. The result (4-7) follows if we choose R_j so that for all $|\alpha| + |\beta| \le j$

$$|\partial_x^\beta \partial_\eta^\alpha (1 - \phi(R_j^{-1} h^{-\frac{1}{2}} m(\eta))) a_j(x, \eta)| \le 2^{-j} h^{-\frac{n+1-j}{2}} m(h, \eta)^{n+1-j-\mathrm{order}(\alpha)}.$$

Such R_j can be chosen by observing that on the support of $1 - \phi(R_j^{-1} h^{-1/2} m(\eta))$ we have the bound $h^{1/2} m(h, \eta)^{-1} \le (1 + R_j)^{-1}$. □

We now turn to the composition result for operators. Due to support considerations of the Schwartz kernels involved, expressing the composition of two operators quantized using the cutoff χ_0 requires quantizing the symbol of the composition using the cutoff χ_1, but we shall later see that the difference is an operator with symbol in $S^{-\infty}$. For simplicity we consider the case where the order of the composition is negative, which is the case needed to produce an inverse for P_h modulo $\Psi_h^{-\infty}(m)$.

In the proof we decompose an operator $a_h(x, hD)$ into a sum of nonisotropic dilates of unit-scale convolution kernels. This decomposition is also used in establishing L^2 bounds for order 0 operators. Let ϕ and ψ generate a smooth Littlewood–Paley decomposition of $[0, \infty)$:

$$1 = \phi(s) + \sum_{j=1}^{\infty} \psi(2^{-j} s), \quad \mathrm{supp}(\phi) \subset [0, 2),\ \mathrm{supp}(\psi) \subset \left(\tfrac{1}{2}, 2\right). \tag{4-8}$$

Given a symbol $a \in S^n(m)$, we make the decomposition

$$a(x, \eta) = \phi(h^{-\frac{1}{2}} m(\eta)) a(x, \eta) + \sum_{j=1}^{\infty} \psi(h^{-\frac{1}{2}} 2^{-j} m(\eta)) a(x, \eta) = \sum_{j=0}^{\infty} a_j(x, \eta). \tag{4-9}$$

Then a_j is supported where $m(h,\eta)\approx 2^j h^{1/2}$, and thus

$$|\partial_x^\beta\partial_\eta^\alpha a_j(x,\eta)|\le C_{\alpha,\beta}2^{jn}(2^j h^{\frac12})^{-\operatorname{order}(\alpha)}.$$

It follows that $a_0(x,\delta_{h^{1/2}}(\eta))\in C_c^\infty(K\times\{|\eta|<8\})$ with bounds uniform over h, and for $j\ge 1$ that $2^{-jn}a_j(x,\delta_{2^j h^{1/2}}(\eta))$ is uniformly bounded in $C_c^\infty\big(K\times\big\{\frac18<|\eta|<8\big\}\big)$ over h and j.

Theorem 11. *Given $a\in S^n(m)$ and $b\in S^{n'}(m)$, with $n+n'<0$, there is $c\in S^{n+n'}(m)$ so that*

$$a_h(x,hD)\circ b_h(x,hD)f(x)=\frac{1}{(2\pi h)^{2d+1}}\int e^{-i\langle y,\xi\rangle/h}c_h(x,\xi)\chi_1(y)f(\exp_x(y))\,dy\,d\xi. \tag{4-10}$$

Proof. For $x\in K$ and $h>0$ we can write

$$\chi_1(\Theta_x(\tilde x))f(\tilde x)=\frac{1}{(2\pi h)^{2d+1}}\int e^{i\langle\Theta_x(\tilde x),\xi\rangle/h-i\langle y,\xi\rangle/h}\chi_1(y)f(\exp_x(y))\,dy\,d\xi.$$

Since $a_h(x,hD)b_h(x,hD)f(x)=a_h(x,hD)b_h(x,hD)\big(\chi_1(\Theta_x(\cdot))f\big)(x)$, we know (4-10) holds with

$$c_h(x,\xi)=\big(a_h(x,hD)b_h(x,hD)e^{i\langle\Theta_x(\cdot),\xi\rangle/h}\big)(x).$$

We thus need to show that $c(x,\xi)=c_h(x,\xi_0,\xi',h^{-1}\xi'')\in S^{n+n'}(m)$.

Let a_i and b_j be the nonisotropic Littlewood–Paley decomposition of a and b as in (4-9), and define c_{ij} by

$$(c_{ij})_h(x,\xi)=\big((a_i)_h(x,hD)(b_j)_h(x,hD)e^{i\langle\Theta_x(\cdot),\xi\rangle/h}\big)(x),$$

so that $c=\sum_{ij}c_{ij}$. From (4-1) we can write $(c_{ij})_h(x,\xi)$ as

$$\frac{1}{(2\pi)^{4d+2}}\int e^{-i\langle y,\eta\rangle-i\langle w,\zeta\rangle+ih^{-1}\langle\widetilde\Theta(x,hy,hw),\xi\rangle}(a_i)_h(x,\eta)(b_j)_h(\exp_x(hy),\zeta)\chi_0(hy)\chi_0(hw)\,dw\,d\zeta\,dy\,d\eta.$$

Consider first the case $i\ge j$. We substitute $w=h^{-1}\Theta(x,hy,hz)$, defined in (3-2), to write this as

$$\frac{1}{(2\pi)^{4d+2}}\int e^{-i\langle y,\eta\rangle-ih^{-1}\langle\Theta(x,hy,hz),\zeta\rangle+i\langle z,\xi\rangle}a_i(x,\eta_0,\eta',h\eta'')b_j(\exp_x(hy),\zeta_0,\zeta',h\zeta'')$$
$$\times\chi_0(hy)\chi_0(\Theta(x,hy,hz))|D_z\Theta|(x,hy,hz)\,dz\,d\zeta\,dy\,d\eta.$$

By the comments following (4-9) applied to b_j, we write

$$b_j(\exp_x(hy),\zeta_0,\zeta',h\zeta'')\chi_0(hy)\chi_0(\Theta(x,hy,hz))|D_z\Theta|(x,hy,hz)$$
$$=2^{jn'}\tilde b_j(x,hy,hz,\delta_{2^{-j}h^{-1/2}}(\zeta_0,\zeta',h\zeta'')),$$

where $\tilde b_j\in C_c^\infty(K\times B_{r_0}\times B_{r_1}\times B_8)$, with bounds uniform over h and j, and a similar representation holds for a_i with 2^j replaced by 2^i and n' replaced by n. We make a nonisotropic dilation of ζ and η by the factors $(2^{2j}h,2^jh^{1/2},2^{3j}h^{1/2})$, and of z and y by the reciprocal factors, to write

$$c_{ij}(x,\xi)=2^{j(n+n')}\tilde c_{ij}(x,\delta_{2^{-j}h^{-1/2}}(\xi)),$$

where $\tilde{c}_{ij}(x,\xi)$ is given by

$$2^{(i-j)n}\frac{1}{(2\pi)^{4d+2}}\int e^{-i\langle y,\eta\rangle - i\langle \overline{\Theta}_y(z)+R(h,x,y,z),\zeta\rangle + i\langle z,\xi\rangle}\tilde{a}_i(x,\delta_{2^{j-i}}(\eta)) \\ \times \tilde{b}_j\big(x,\delta_{2^{-j}}(y_0,h^{\frac12}y',h^{\frac12}y''),\delta_{2^{-j}}(z_0,h^{\frac12}z',h^{\frac12}z''),\zeta\big)\,dz\,d\zeta\,dy\,d\eta, \quad (4\text{-}11)$$

where, recalling Lemma 6,

$$\begin{aligned}\langle R(h,x,y,z),\zeta\rangle =& 2^{2j}R_0\big(x,\delta_{2^{-j}}(y_0,h^{\frac12}y',h^{\frac12}y''),\delta_{2^{-j}}(z_0,h^{\frac12}z',h^{\frac12}z'')\big)\zeta_0\\ &+2^jh^{-\frac12}R'\big(x,\delta_{2^{-j}}(y_0,h^{\frac12}y',h^{\frac12}y''),\delta_{2^{-j}}(z_0,h^{\frac12}z',h^{\frac12}z'')\big)\cdot\zeta'\\ &+2^{3j}h^{-\frac12}R''\big(x,\delta_{2^{-j}}(y_0,h^{\frac12}y',h^{\frac12}y''),\delta_{2^{-j}}(z_0,h^{\frac12}z',h^{\frac12}z'')\big)\cdot\zeta''.\end{aligned}$$

By the support condition on $\tilde{b}_j$ we have $|\zeta|\le 8$. Also, if $i\ge 1$ then $\tilde{a}_i(x,\eta)=0$ when $|\eta|\le\frac18$.

We next apply the expansion (3-5) to the right-hand side. The condition on order$(y^\alpha z^\beta)$ ensures that we bring out strictly more powers of 2^{-j} than needed to cancel the powers of 2^j in front, and since there is at least one factor of (y',z',y'',z'') we also bring out a factor $h^{1/2}$ to cancel off the $h^{-1/2}$ in front. We conclude that, on the support of the integrand,

$$|R(h,x,y,z)|\le C2^{-j}|z-y|(|y|+|z-y|+|y|^2+|z-y|^2),$$

and also

$$|\partial_x^\gamma\partial_y^\alpha\partial_z^\beta R(h,x,y,z)|\le C_{\gamma,\alpha,\beta}2^{-j}(1+|y|^3+|z-y|^3). \quad (4\text{-}12)$$

Additionally, if we let $w=\overline{\Theta}(x,y,z)+R(h,x,y,z)$, then with analogous notation we see from (3-2) and Lemma 6 that $z=\overline{\Theta}_{-y}(w)+\widetilde{R}(h,x,y,w)$, where

$$|\widetilde{R}(h,x,y,w)|\le C2^{-j}|w|(|y|+|w|+|y|^2+|w|^2).$$

Consequently, since $\widetilde{\Theta}$ is the inverse function to Θ for fixed y, uniformly over j we have

$$\begin{aligned}&|\overline{\Theta}_y(z)+R(h,x,y,z)|\le C|z-y|(1+|y|^2+|z-y|^2),\\ &\qquad |z-y|\le C|\overline{\Theta}_y(z)+R(h,x,y,z)|(1+|y|^2+|\overline{\Theta}_y(z)+R(h,x,y,z)|^2),\end{aligned}$$

and hence

$$(1+|y|^2)^{-1}|z-y|\le C|\overline{\Theta}_y(z)+R(h,x,y,z)|(1+|\overline{\Theta}_y(z)+R(h,x,y,z)|^2). \quad (4\text{-}13)$$

Considering the function

$$g_{ij}(x,y)=\frac{1}{(2\pi)^{4d+2}}\int e^{-i\langle y,\eta\rangle}\tilde{a}_i(x,\delta_{2^{j-i}}(\eta))\,d\eta,$$

simple estimates show that

$$|\partial_x^\alpha g_{ij}(x,y)|\le C_{N,\alpha,\beta}2^{(4d+2)(i-j)}(1+2^{2(i-j)}|y_0|+2^{i-j}|y'|+2^{3(i-j)}|y''|)^{-N}. \quad (4\text{-}14)$$

Additionally, if $i > j$, hence $i \geq 1$, then $\tilde{a}_i(x,\eta)$ vanishes for $|\eta| \leq \frac{1}{8}$, and thus can be assumed to be of the form $|\eta|^{2k}\tilde{a}_i(x,\eta)$ for similar $\tilde{a}_i(x,\eta)$. Thus, if $i > j$ then for all $k \in \mathbb{N}$ we can write

$$g_{ij}(x,y) = \sum_{|\gamma|=2k} 2^{(j-i)\,\mathrm{order}(\gamma)} \partial_y^\gamma g_{ij,\gamma}(x,y), \tag{4-15}$$

where $g_{ij,\gamma}(x,y)$ satisfies the same estimates (4-14) as $g_{ij}(x,y)$. On the other hand, if we set

$$f_j(x,y,z) = \int e^{-i\langle \overline{\Theta}_y(z)+R(h,x,y,z),\zeta\rangle} \tilde{b}_j\bigl(x, \delta_{2^{-j}}(y_0, h^{\frac12}y', h^{\frac12}y''), \delta_{2^{-j}}(z_0, h^{\frac12}z', h^{\frac12}z''), \zeta\bigr)\, d\zeta,$$

then

$$f_j(x,y,z) = \rho(x,y,z,\overline{\Theta}_y(z) + R(h,x,y,z)),$$

where $\rho(x,y,z,w)$ is smooth in (x,y,z) and Schwartz in w. By (4-12) and (4-13) we have

$$|\partial_x^\alpha \partial_y^\beta \partial_z^\theta f_j(x,y,z)| \leq C_{N,\alpha,\beta,\theta}(1+|y|+|y-z|)^{3(|\alpha|+|\beta|+|\theta|)}(1+(1+|y|^2)^{-1}|y-z|)^{-N}.$$

Applying (4-15) and integrating by parts in y leads to the bound, for all N, α, β,

$$\left|\partial_x^\alpha \partial_z^\beta \int g_{ij}(x,y) f_j(x,y,z)\, dy\right| \leq C_{N,\alpha,\beta} 2^{2k(j-i)}(1+|z|)^{-N}.$$

Since $\tilde{c}_{ij}(x,\xi)$, defined in (4-11), is $2^{(i-j)n}$ times the Fourier transform in z of this integral, we obtain uniform (over i and j) Schwartz bounds on $2^{i-j}\tilde{c}_{ij}(x,\xi)$, with compact support in x.

In the case $j \geq i$, we can similarly write $c_{ij}(x,\xi) = 2^{i(n+n')}\tilde{c}_{ij}(x, \delta_{2^{-i}h^{-1/2}}(\xi))$, where we have uniform Schwartz bounds over i and j on $2^{j-i}\tilde{c}_{ij}(x,\xi)$. The analysis is similar to the case $i \geq j$, using instead the following representation for $c_{ij}(x,\xi)$:

$$\frac{1}{(2\pi)^{4d+2}} \int e^{-ih^{-1}\langle \widetilde{\Theta}(x,hv,-hw),\eta\rangle - i\langle w,\zeta\rangle + i\langle v,\xi\rangle} a_i(x,\eta_0,\eta',h\eta'') b_j(\exp_{\exp_x(hv)}(-hw), \zeta_0, \zeta', h\zeta'')$$
$$\times\, \chi_0(\widetilde{\Theta}(x,hv,-hw))\chi_0(hw)|D_v\widetilde{\Theta}|(x,hv,-hw)\, dw\, d\zeta\, dv\, d\eta.$$

It thus suffices to show that $\sum_{i\geq j} 2^{j(n+n')}\tilde{c}_{ij}(x, \delta_{2^{-j}h^{-1/2}}(\xi)) \in S^{n+n'}(m)$. We prove that

$$\left|\sum_{i\geq j} 2^{j(n+n')}\tilde{c}_{ij}(x, \delta_{2^{-j}h^{-1/2}}(\xi))\right| \leq C(1+h^{-\frac12}m(\xi))^{n+n'}.$$

Estimates on derivatives will follow similarly since applying ∂_ξ^α has the effect of multiplying the j-th term by $(2^{-j}h^{-1/2})^{\mathrm{order}(\alpha)}$. We use the uniform Schwartz bounds on $\tilde{c}_{ij}$ to bound the sum by

$$C_N \sum_{i\geq j\geq 0} 2^{j(n+n')}2^{j-i}(1+2^{-j}h^{-\frac12}m(\xi))^{-N}.$$

The sum over i is trivial. Given ξ, take j_0 so that $2^{j_0} = h^{-1/2}m(\xi)$. We then split

$$\sum_{j\geq 0} 2^{j(n+n')}(1+2^{-j}h^{-\frac12}m(\xi))^{-N} \leq \sum_{j\geq j_0} 2^{j(n+n')} + \sum_{j<j_0} 2^{j(n+n'+N)}(h^{-\frac12}m(\xi))^{-N}.$$

Recall that we assume $n+n'<0$. We take N so $N+n+n'>0$. If $h^{-1/2}m(\xi)\le 1$, we have only the first sum, which is bounded by a constant. If $h^{-1/2}m(\xi)>1$, then the two terms are convergent geometric sums that both are bounded by $(h^{-1/2}m(\xi))^{n+n'}$. □

Remark 12. The result of Theorem 11 still holds if one replaces the function $\chi_0(y)$ used in quantizing a or b by any function $\chi(x,y)\in C_c^\infty(K\times B_{r_0})$, since this is harmlessly absorbed into $\tilde{b}_j$ without changing the estimates for $\tilde{b}_j$ nor the condition on the support of the Schwartz kernel.

Lemma 13. *Suppose that $\beta,\chi\in C_c^\infty(B_{r_1})$, and $\beta(y)=0$ for $|y|\le\delta$, where $\delta>0$. Suppose also that $\chi=1$ on $\operatorname{supp}(\beta)$. Then if $a\in S^n(m)$ for some n, one can write*

$$\frac{1}{(2\pi h)^{2d+1}}\int e^{-i\langle y,\xi\rangle/h}a_h(x,\xi)\beta(y)f(\exp_x(y))\,dy\,d\xi = \frac{1}{(2\pi h)^{2d+1}}\int e^{-i\langle y,\xi\rangle/h}r_h(x,\xi)\chi(y)f(\exp_x(y))\,dy\,d\xi,$$

where $r\in S^{-\infty}(m)$.

Proof. We write $\beta(y)=|y|^{2N}\beta_N(y)$ for $\beta_N\in C_c^\infty(B_{r_1})$. Since $\chi\beta_N=\beta_N$, following the first part of Theorem 11 we have equality of the two sides if r_h is the symbol

$$r_h(x,\xi)=\frac{1}{(2\pi h)^{2d+1}}\int e^{-i\langle y,\xi-\eta\rangle/h}((h^2\Delta_{\eta_0,\eta'}+h^4\Delta_{\eta''})^N a)_h(x,\eta)\beta_N(y)\,dy\,d\eta.$$

By (4-5), $a_N=(h^2\Delta_{\eta_0,\eta'}+h^4\Delta_{\eta''})^N a\in S^{n-2N}(m)$. We then write

$$r(x,\xi)=\frac{1}{(2\pi)^{2d+1}h^{3d+1}}\int\hat{\beta}_N\Big(\frac{\xi_0-\eta_0}{h},\frac{\xi'-\eta'}{h},\frac{\xi''-\eta''}{h^2}\Big)a_N(x,\eta)\,d\eta.$$

We have $|a_N(x,\eta)|\le C_N(1+m(\delta_{h^{-1/2}}(\eta)))^{n-2N}$, and Peetre's inequality yields

$$(1+|\delta_{h^{-1/2}}(\xi-\eta)|)^{-|n|-2N}(1+m(\delta_{h^{-1/2}}(\eta)))^{n-2N}\le C_N(1+m(\delta_{h^{-1/2}}(\xi)))^{n-2N},$$

which shows that $|r(x,\xi)|\le C_N h^N m(h,\xi)^{-2N}$ for all N. The term $\partial_x^\beta\partial_\xi^\alpha r(x,\xi)$ comes from the same convolution applied to $\partial_x^\beta\partial_\eta^\alpha a_N(x,\eta)$, and we conclude $r\in S^{-\infty}$. □

Corollary 14. *Suppose P_h is as in* (1-1). *Given $\rho\in C_c^\infty(K^o)$, there is a symbol $q(x,\xi)\in S^{-2}(m)$, with principal symbol $h\rho(x)(1-\phi(h^{-1/2}m(\xi)))q_0(\xi)$, so that*

$$q_h(x,hD)\circ P_h=\rho(x)+R,\qquad R\in\Psi_h^{-\infty}(m).$$

Proof. Fix $\tilde{\rho}(x)\in C_c^\infty(K)$ with $\tilde{\rho}=1$ on a neighborhood of $\operatorname{supp}(\rho)$. Define

$$\tilde{q}_0(\xi)=h(1-\phi(h^{-\frac{1}{2}}m(\xi)))q_0(\xi),$$

where $q_0(\xi)$ is the Fourier transform of the fundamental solution for $Y_0-\sum_{j=1}^d Y_j^2$, as defined in Section 2, and ϕ is as in (4-8). Then $\tilde{\rho}(x)\tilde{q}_0(\xi)\in S^{-2}(m)$. We first show that

$$\tilde{\rho}(x)\tilde{q}_{0,h}(hD)\circ P_h=\tilde{\rho}(x)-r_h^1(x,hD),\qquad r^1\in S^{-1}(m).$$

By the construction of $q_0(\xi)$ we have

$$\frac{1}{(2\pi h)^{2d+1}}\int e^{-i\langle y,\xi\rangle/h}hq_{0,h}(\xi)\Big(Y_0-h\sum_{j=1}^{d}Y_j^2\Big)\chi_0(y)f(\exp_x(y))\,dy\,d\xi=f(x),$$

where the Y_j are the model vector fields acting in the y-variable. Replacing hq_0 by $\tilde q_0$ changes the composition by an order-0 symbol supported where $m(\xi)\le 2h^{1/2}$, hence by a symbol in $S^{-\infty}$.

Generally, we see that for $f(x,y)$ compactly supported in y we can write

$$\int e^{-i\langle y,\xi\rangle/h}\tilde q_{0,h}(\xi)y^\alpha\partial_y^\beta f(x,y)\,dy\,d\xi=\int e^{-i\langle y,\xi\rangle/h}b_h(\xi)f(x,y)\,dy\,d\xi,$$

where

$$b(x,\xi)=i^{|\beta|-|\alpha|}h^{\alpha_0+|\alpha'|+2|\alpha''|-|\beta|}\partial_\xi^\alpha\xi^\beta\tilde q_0(\xi).$$

By (4-3) and (4-5), we know that

$$h^{\frac12\operatorname{order}(\alpha)-\frac12\operatorname{order}(\beta)}\partial_\xi^\alpha\xi^\beta\tilde q_0(\xi)\in S^{\operatorname{order}(\beta)-\operatorname{order}(\alpha)-2}(m).$$

Recall that

$$(X_jf)(\exp_x(y))=(Y_j+R_j(x,y,\partial_y))f(\exp_x(y)),$$

where the Taylor expansion of R_j contains terms $y^\alpha\partial_y^\beta$ of order strictly less than $\operatorname{order}(Y_j)$, and where $|\beta|=1$. Since commutators of X_j with χ_0 lead to terms of order $-\infty$, we need show that

$$\frac{1}{(2\pi h)^{2d+1}}\int e^{-i\langle y,\xi\rangle/h}hq_{0,h}(\xi)R_0(x,y,\partial_y)\chi_0(y)f(\exp_x(y))\,dy\,d\xi$$

is an operator of order -1 in f. For the terms that arise in the Taylor expansion of R_0 we have $\operatorname{order}(\beta)-\operatorname{order}(\alpha)\le 1$, so we need check for such terms we also have

$$\alpha_0+|\alpha'|+2|\alpha''|-|\beta|\ge\tfrac12\operatorname{order}(\alpha)-\tfrac12\operatorname{order}(\beta),$$

in order to match up the powers of h. Since $|\beta|=1$ and $\operatorname{order}(\alpha)=2\alpha_0+|\alpha'|+3|\alpha''|$, this holds provided that $|\alpha'|+|\alpha''|\ge 1$, which is the case for R_0 by Lemma 4.

We similarly need check that this is an operator of order -2 if R_0 is replaced by $h^{1/2}R_j$ with $1\le j\le d$. Since $\operatorname{order}(\beta)-\operatorname{order}(\alpha)\le 0$ in this case, this reduces to verifying that

$$\alpha_0+|\alpha'|+2|\alpha''|-|\beta|+\tfrac12\ge\tfrac12\operatorname{order}(\alpha)-\tfrac12\operatorname{order}(\beta),$$

which always holds if $|\beta|=1$.

We note that the remainder term in the Taylor expansion will also be of the desired order, but with $\chi_0(y)$ replaced by $c_{j,\alpha,k}(x,y)\chi_0(y)$. By Remark 12 this does not affect the conclusion of the corollary, since the form for q_h will involve composition with $r_h^1(x,hD)$.

By Theorem 11 we can recursively define symbols $r^j\in S^{-j}(m)$ for $j\ge 2$ by the rule

$$r_h^j(x,hD)\circ r_h^1(x,hD)f(x)=\frac{1}{(2\pi h)^{2d+1}}\int e^{-i\langle y,\xi\rangle/h}r_h^{j+1}(x,\xi)\chi_1(y)f(\exp_x(y))\,dy\,d\xi,$$

where we recall that $r_h^j(x,hD)$ is quantized using χ_0. Let $r \sim \sum_{j=0}^{\infty} r^j$, so $r \in S^{-1}(m)$. Also define $q \in S^{-2}(m)$ so that

$$\rho(x)(I+r_h(x,hD))\tilde\rho(x)\tilde q_{0,h}(hD)f(x) = \frac{1}{(2\pi h)^{2d+1}}\int e^{-i\langle y,\xi\rangle/h} q_h(x,\xi)\chi_1(y)f(\exp_x(y))\,dy\,d\xi.$$

By the above and Lemma 13, the following operator is in $\Psi_h^{-\infty}(m)$:

$$Rf(x) = \frac{1}{(2\pi h)^{2d+1}}\int e^{-i\langle y,\xi\rangle/h} q_h(x,\xi)(\chi_1(y)-\chi_0(y))(P_hf)(\exp_x(y))\,dy\,d\xi.$$

Thus, modulo $\Psi_h^{-\infty}(m)$ we have

$$q_h(x,hD)\circ P_h = \rho(x)(I+r_h(x,hD))(\tilde\rho(x)-r_h^1(x,hD)).$$

Next we choose $\delta>0$ so that $\tilde\rho(\exp_x(y))=1$ if $x\in\operatorname{supp}(\rho)$ and $|y|\le\delta$, and take $\chi_\delta \in C_c^\infty(B_\delta)$ with $\chi_\delta = 1$ on $B_{\delta/2}$. Then

$$\begin{aligned}\rho(x)&r_h(x,hD)((1-\tilde\rho)f)(x)\\ &= \frac{1}{(2\pi h)^{2d+1}}\int e^{-i\langle y,\xi\rangle/h}\rho(x)r_h(x,\xi)(\chi_0(y)-\chi_\delta(y))((1-\tilde\rho)f)(\exp_x(y))\,dy\,d\xi,\end{aligned}$$

so by Lemma 13 we have, modulo $\Psi_h^{-\infty}(m)$,

$$q_h(x,hD)\circ P_h = \rho(x)(I+r_h(x,hD))(I-r_h^1(x,hD)).$$

Finally, since the difference between using χ_1 instead of χ_0 in the quantization of r^{j+1} gives a term in $\Psi_h^{-\infty}(m)$, we see that $q_h(x,hD)\circ P_h = \rho(x)$ modulo $\Psi_h^{-\infty}(m)$. □

Remark 15. The above proof shows the following composition result concerning partial differential operators. Suppose

$$P_h = \sum_{\text{order}(\alpha)\le n'} c_\alpha(x)X_0^{\alpha_0}(h^{\frac12}X')^{\alpha'}(h^{\frac12}X'')^{\alpha''}, \quad c_\alpha(x)\in C_c^\infty(K).$$

Then if $a\in S^n(m)$, we can write $a_h(x,hD)\circ P_hf$ and $P_h\circ a_h(x,hD)f$ in the form

$$\frac{1}{(2\pi h)^{2d+1}}\int e^{-i\langle y,\xi\rangle/h} b_h(x,\xi)\chi(y)f(\exp_x(y))\,dy\,d\xi$$

for $\chi\in C_c^\infty(B_{r_0})$ and $b\in S^{n+n'}(m)$.

5. L^2 boundedness for order-0 operators

Given a symbol in $S^n(m)$ we decompose $a=\sum_j a_j$ as in (4-9). The operator $a_{j,h}(x,hD)$ is given by the following integral kernel on $U\times U$ with respect to the measure $dm(\tilde x)$, where $w(x,\tilde x)dm(\tilde x)=\exp_x^*(dy)$:

$$K_j(x,\tilde x) = w(x,\tilde x)\chi_0(\Theta_x(\tilde x))\int e^{-i\langle\Theta_x(\tilde x),\eta\rangle} a_{j,h}(x,h\eta)\,d\eta.$$

We can write $a_{j,h}(x, h\eta) = 2^{jn}\tilde{a}_j(x, 2^{-2j}\eta_0, 2^{-j}h^{1/2}\eta', 2^{-3j}h^{1/2}\eta'')$, where $\tilde{a}_j(x,\eta) \in C_c^\infty(K \times B_8)$, with uniform bounds over j. Furthermore, $\tilde{a}_j$ vanishes for $|\eta| \le \frac{1}{8}$ if $j \ge 1$.

Consequently, there are Schwartz functions $\rho_j(x, y)$, supported for $x \in K$ with Schwartz norms independent of j, so that

$$(w^{-1}K_j)(x, \exp_x(y)) = 2^{jn}2^{j(2+4d)}h^{-d}\rho_j(x, 2^{2j}y_0, 2^j h^{-\frac{1}{2}}y', 2^{3j}h^{-\frac{1}{2}}y'')\chi_0(y), \tag{5-1}$$

and in particular, for all N,

$$|K_j(x,\tilde{x})| \le C_N 2^{jn}2^{j(2+4d)}h^{-d}\big(1 + 2^{2j}|\Theta_x(\tilde{x})_0| + 2^j h^{-\frac{1}{2}}|\Theta_x(\tilde{x})'| + 2^{3j}h^{-\frac{1}{2}}|\Theta_x(\tilde{x})''|\big)^{-N}. \tag{5-2}$$

If $a \in S^{-\infty}$ then (5-1) holds for all $n \in \mathbb{Z}$, and summing over j we obtain the following.

Corollary 16. *If $a \in S^{-\infty}(m)$, then $a_h(x, hD)$ is given by a smooth integral kernel $K(x,\tilde{x})$ in the measure $dm(\tilde{x})$, so that for some Schwartz function $\rho(x, y)$, supported for $x \in \operatorname{supp}(a)$,*

$$(w^{-1}K)(x, \exp_x(y)) = h^{-d}\rho(x, y_0, h^{-\frac{1}{2}}y', h^{-\frac{1}{2}}y'')\chi_0(y).$$

We next observe that the vector fields $2^{-2j}Y_0$, $2^{-j}h^{1/2}Y'$, and $2^{-3j}h^{1/2}Y''$ acting as differential operators in y all preserve the form (5-1) of $w^{-1}K_j$; that is, they give an expression of the same form with ρ_j uniformly bounded over j in each Schwartz seminorm.

The same holds for the operators $2^{-2j}X_0$, $2^{-j}h^{1/2}X'$, and $2^{-3j}h^{1/2}X''$, acting on $K_j(x,\tilde{x})$ as differential operators in either the x- or $\tilde{x}$-variable. For action in the $\tilde{x}$-variable, this follows by Lemma 4, where we use that there is at least one factor of y' or y'' in the expansion of $R_0(x, y, \partial_y)$ to compensate for the factor of $h^{-1/2}$ coming from the $\partial_{y'}$ and $\partial_{y''}$ terms in the expansion of X_0. For action in the x-variable we work in coordinates $x = \exp_{\tilde{x}}(y)$, hence $\tilde{x} = \exp_x(-y)$, to write

$$(w^{-1}K_j)(\exp_{\tilde{x}}(y), \tilde{x}) = 2^{jn}2^{j(2+4d)}h^{-d}\rho_j\big(\exp_{\tilde{x}}(y), -2^{2j}y_0, -2^j h^{-\frac{1}{2}}y', -2^{3j}h^{-\frac{1}{2}}y''\big)\chi_0(-y). \tag{5-3}$$

To summarize, for $a \in S^n(m)$, we can write

$$\begin{aligned}(2^{-2j}X_0)^{\alpha_0}&(2^{-j}h^{\frac{1}{2}}X')^{\alpha'}(2^{-3j}h^{\frac{1}{2}}X'')^{\alpha''}K_j(x,\tilde{x}) \\ &= 2^{jn}2^{j(2+4d)}h^{-d}\rho_{j,\alpha}\big(x, 2^{2j}\Theta_x(\tilde{x})_0, 2^j h^{-\frac{1}{2}}\Theta_x(\tilde{x})', 2^{3j}h^{-\frac{1}{2}}\Theta_x(\tilde{x})''\big)\chi_\alpha(x,\tilde{x}), \end{aligned} \tag{5-4}$$

where the functions $\rho_{j,\alpha}$ and χ_α satisfy seminorm bounds that depend on α, but are uniform over j and h. This holds with any given vector X in the product acting as a vector field in x or $\tilde{x}$.

Conversely, suppose that $j \ge 1$, so that $\tilde{a}_j(x,\eta) \in C_c^\infty\big(K \times \big\{\frac{1}{8} \le |\eta| \le 8\big\}\big)$. Then for any ℓ, dividing $\tilde{a}_j$ by $|\eta|^{2\ell}$ shows that we can write

$$\begin{aligned}&(w^{-1}K_j)(x, \exp_x(y)) \\ &= 2^{jn}2^{j(2+4d)}h^{-d}\sum_{|\alpha|=2\ell}\chi_\alpha(x,y)(2^{-2j}\partial_{y_0})^{\alpha_0}(2^{-j}h^{\frac{1}{2}}\partial_{y'})^{\alpha'}(2^{-3j}h^{\frac{1}{2}}\partial_{y''})^{\alpha''}\rho_{j,\alpha}(x, 2^{2j}y_0, 2^j h^{-\frac{1}{2}}y', 2^{3j}h^{-\frac{1}{2}}y'')\end{aligned}$$

for Schwartz functions $\rho_{j,\alpha}$ that are uniformly bounded over j, and $\chi_\alpha \in C_c^\infty(K \times B_{r_0})$.

Using Lemma 4, we write

$$\begin{aligned}\partial_{y_0} &= X_0 + y' \cdot X'' - R_0(x, y, \partial_y) - y' \cdot R''(x, y, \partial_y),\\ \partial_{y'} &= X' - y_0 X'' - R'(x, y, \partial_y) + y_0 R''(x, y, \partial_y),\\ \partial_{y''} &= X'' - R''(x, y, \partial_y),\end{aligned}$$

where the X_j act in y. Substituting this into $R(x, y, \partial_y)$, and using that the X_j form a smooth frame, we can expand each ∂_{y_j} as a finite sum over $2 \le |\alpha| \le 3$:

$$\begin{aligned}\partial_{y_0} &= X_0 + y' \cdot X'' + \sum_{\alpha,k} c_{0,\alpha,k}(x, y) y^\alpha X_k, &&\text{order}(Y_k) - \text{order}(\alpha) < 2,\\ \partial_{y_j} &= X_j - y_0 X_{j+d} + \sum_{\alpha,k} c_{j,\alpha,k}(x, y) y^\alpha X_k, &&\text{order}(Y_k) - \text{order}(\alpha) < 1,\ 1 \le j \le d,\\ \partial_{y_j} &= X_j + \sum_{\alpha,k} c_{j,\alpha,k}(x, y) y^\alpha X_k, &&\text{order}(Y_k) - \text{order}(\alpha) < 3,\ d+1 \le j \le 2d.\end{aligned}$$

Additionally, $c_{0,\alpha,k} \equiv 0$ unless either $\alpha' \neq 0$ or $\alpha'' \neq 0$.

Let $\overline{X}_j$ denote the transpose of the differential operator X_j with respect to dy. Taking the transpose of the above identities, it follows that, with the $\overline{X}_j$ acting on y, we can write

$$\begin{aligned}&(w^{-1} K_j)(x, \exp_x(y))\\ &= 2^{jn} 2^{j(2+4d)} h^{-d} \sum_{|\alpha|=2\ell} \chi_\alpha(x, y) (2^{-2j} \overline{X}_0)^{\alpha_0} (2^{-j} h^{\frac12} \overline{X}')^{\alpha'} (2^{-3j} h^{\frac12} \overline{X}'')^{\alpha''} \rho_{j,\alpha}(x, 2^{2j} y_0, 2^j h^{-\frac12} y', 2^{3j} h^{-\frac12} y''),\end{aligned}$$

where the $\rho_{j,\alpha}$ may depend on h, but with uniform Schwartz bounds over $0 \le h \le 1$ and $j \in \mathbb{N}$. Expressing the action of $\overline{X}$ in terms of $\tilde{x}$, this leads to the expansion

$$K_j(x, \tilde{x}) = \sum_{|\alpha|=2\ell} \sum_{\beta \le \alpha} 2^{-j\,\text{order}(\alpha)} (\overline{X}_0)^{\beta_0} (h^{\frac12} \overline{X}')^{\beta'} (h^{\frac12} \overline{X}'')^{\beta''} K_{j,\alpha,\beta}(x, \tilde{x})$$

for kernels $K_{j,\alpha,\beta}$ satisfying (5-2) with C_N depending on ℓ but uniform over j, α, β. Here we can take $\overline{X}_j$ to be the transpose of X_j with respect to $dm(\tilde{x})$, since that differs from the transpose with respect to dy by a smooth function.

Theorem 17. *If $a \in S^0(m)$, then $a_h(x, hD)$ is a bounded linear operator on $L^2(U)$, with operator norm depending on only a finite number of seminorm bounds for $a(x, \xi)$. In particular, the operator norm is uniformly bounded over $0 < h \le 1$.*

Proof. We decompose $a_h(x, hD) = \sum_{j=0}^{\infty} a_{j,h}(x, hD)$. Using (5-1) and (5-3) it is easily verified that the kernel $K_j(x, \tilde{x})$ of $a_{j,h}(x, hD)$ satisfies the Schur test,

$$\sup_x \int K_j(x, \tilde{x})\, dm(\tilde{x}) \le C, \quad \sup_{\tilde{x}} \int K_j(x, \tilde{x})\, dm(x) \le C.$$

We deduce L^2 boundedness from the Cotlar–Stein lemma (see [Knapp and Stein 1971; Stein 1993]), by showing that, for any $N\in\mathbb{N}$,

$$\|a_{i,h}(x,hD)^*a_{j,h}(x,hD)\|_{L^2\to L^2}+\|a_{i,h}(x,hD)a_{j,h}(x,hD)^*\|_{L^2\to L^2}\le C2^{-N|i-j|} \tag{5-5}$$

for a constant C uniform over h and j. If $i=j$ this follows from L^2 boundedness of each term, so without loss of generality we consider $j>i\ge 0$, and in particular $j\ge 1$. Given $\ell\in\mathbb{N}$ we then write the integral kernel of $a_{i,h}(x,hD)a_{j,h}(x,hD)^*$ as

$$\begin{aligned}\int K_i(x,w)\overline{K_j(\tilde{x},w)}\,dm(w)&\\ &=\int K_i(x,w)\sum_{|\alpha|=2\ell}\sum_{\beta\le\alpha}2^{-j\,\mathrm{order}(\alpha)}(\overline{X}_0)^{\beta_0}(h^{\frac12}\overline{X}')^{\beta'}(h^{\frac12}\overline{X}'')^{\beta''}\,\overline{K_{j,\alpha,\beta}(x,w)}\,dm(w)\\ &=\sum_{|\alpha|=2\ell}\sum_{\beta\le\alpha}2^{i\,\mathrm{order}(\beta)-j\,\mathrm{order}(\alpha)}\int K_{i,\beta}(x,w)\overline{K_{j,\alpha,\beta}(x,w)}\,dm(w),\end{aligned}$$

where $K_{i,\beta}(x,w)=(2^{-2i}X_0)^{\beta_0}(2^{-i}h^{1/2}X')^{\beta'}(2^{-3i}h^{1/2}X'')^{\beta''}K_i(x,w)$, and in all cases X acts on w. Since $i\,\mathrm{order}(\beta)-j\,\mathrm{order}(\alpha)\le 2\ell(i-j)$, by using (5-4) and the Schur test on the composition we obtain the bound (5-5) with $N=2\ell$ for the term $a_{i,h}a^*_{j,h}$. To handle the term $a^*_{i,h}a_{j,h}$ we use the same argument, together with symmetry of the derivative estimates in x and $\tilde{x}$. ☐

We note the following result for $a\in S^n(m)$, which holds since $2^{-jn}a_j(x,\eta)\in S^0(m)$,

$$\sup_{j\ge 0}2^{-jn}\|a_{j,h}(x,hD)f\|_{L^2(U)}\le C\|f\|_{L^2(U)},\quad a\in S^n(m). \tag{5-6}$$

6. Estimates on $S^*(M)$

Let (M,g) be a compact Riemannian manifold of dimension $d+1$, and $S^*(M)\subset T^*(M)$ its unit cosphere bundle. We consider the Hamiltonian function $\frac12|\zeta|^2_{\mathrm{g}(z)}=\frac12\sum_{i,k=1}^{d+1}\mathrm{g}^{ik}(z)\zeta_i\zeta_k$, and recall that $S^*(M)$ is the level set $|\zeta|_{\mathrm{g}(z)}=1$. We use $X_0=H$ to denote the Hamiltonian field for $\frac12|\zeta|^2_{\mathrm{g}(z)}$,

$$X_0=\sum_{i,k=1}^{d+1}\mathrm{g}^{ik}(z)\zeta_i\partial_{z_k}-\frac12\sum_{i,j,k=1}^{d+1}\partial_{z_j}\mathrm{g}^{ik}(z)\zeta_i\zeta_k\partial_{\zeta_j},$$

which is tangent to $S^*(M)$.

We cover $S^*(M)$ by a finite collection of open coordinate charts as follows. Let $\{V_\alpha\}$ form a finite covering of M by coordinate charts, over which we can identity $T^*(M)$ with $V_\alpha\times\mathbb{R}^{d+1}$ and $S^*(M)$ with $V_\alpha\times\mathbb{S}^d$. We cover $\mathbb{S}^d$ by two coordinate charts $W^\pm$ over each of which there is a section of the frame bundle. We thus obtain a cover of $S^*(M)$ by open charts $\{V_\alpha\times W^\pm\}$, which by counting each V_α twice we can label as U_α, such that on U_α there is an orthonormal collection $\{X_j\}_{j=1}^d$ of vertical vector fields that span the tangent space to $S^*_z(M)$ over each $z\in V_\alpha$. The collection $\{X_j\}_{j=1}^d$ is involutive, since it spans the vertical vector fields on U_α.

There is a natural isometric identification $T_\zeta(T_z^*(M)) \sim T_z(M)$, which identifies $\{X_j|_{(z,\zeta)}\}_{j=1}^d$ with an orthonormal collection of vectors $\{\widetilde{X}_j\}_{j=1}^d \subset T_z(M)$, which are also orthogonal to $\pi_*(X_0|_{(z,\zeta)})$. We let $X_{j+d}|_{(z,\zeta)}$ be $-\frac{1}{2}$ times the horizontal lift of $\widetilde{X}_j$. We observe that

$$\pi_*[X_j, X_0] = \sum_{i,k=1}^{d+1} \mathrm{g}^{ik}(z) X_j(\zeta_i)\partial_{z_k} = \widetilde{X}_j,$$

so that $[X_0, X_j] - 2X_{j+d} \in \operatorname{span}\{X_j\}_{j=1}^d$. Thus the assumptions of the Introduction are satisfied for the collection $\{X_j\}_{j=0}^{2d}$.

Let $\Delta_{\mathbb{S}}$ be the induced nonnegative Laplacian acting on the fibers $S_z^*(M)$ of the bundle, and let Δ be the nonnegative Laplacian on $S^*(M)$. See for example [Drouot 2017, Section 2.1] for details, where it is shown that Δ and $\Delta_{\mathbb{S}}$ commute. One verifies that, over each U_α, one has

$$\Delta_{\mathbb{S}} = -\sum_{j=1}^d X_j^2 + \sum_{j=1}^d c_j(z,\zeta) X_j.$$

We now use $x \in \mathbb{R}^{2d+1}$ to denote the variables on U_α, and define

$$P_h = H + h\Delta_{\mathbb{S}} = X_0 - \sum_{j=1}^d hX_j^2 + \sum_{j=1}^d c_j(x) hX_j.$$

Thus on each U_α, the operator P_h differs from the sum of squares considered previously by an operator in $h^{1/2}\Psi_h^1(m)$, and the pseudodifferential calculus shows that, given $\chi_\alpha \in C_c^\infty(U_\alpha)$, there exists a symbol $q_\alpha \in S^{-2}(m)$, the quantization of which depends on χ_α through the choice of χ_0 in (4-1), so that on U_α we have

$$q_{\alpha,h}(x, hD) \circ P_h u = \chi_\alpha(x) u + R_\alpha u, \quad R_\alpha \in \Psi_h^{-\infty}(m).$$

Note that both $q_{\alpha,h}(x, hD)$ and R_α are properly supported in U_α. We now take a partition of unity χ_α subordinate to the cover U_α, and define

$$Q_h v = \sum_\alpha q_{\alpha,h}(x, hD) v, \quad Rv = \sum_\alpha R_\alpha v.$$

Then $Q_h \circ P_h = I + R$, and for all N_1, N_2 we have

$$\|(h\Delta)^{N_1} R (h\Delta)^{N_2} u\|_{L^2(S^*(M))} \le C_{N_1,N_2} \|u\|_{L^2(S^*(M))}. \tag{6-1}$$

This follows from Theorems 11 and 17 and the fact that $h\Delta \in \Psi_h^6(U_\alpha)$ for each α, which follows from (4-4).

More generally, we define $\Psi_h^\sigma(m)$ on $S^*(M)$ as sums $\sum_\alpha a_{\alpha,h}(x, hD)$ with $a_\alpha \in S^\sigma(m)$ on U_α. The function χ_0 in the quantization (4-1) depends on the x-support of $a_\alpha(x,\eta)$, which is always assumed to be a compact subset of U_α.

The semiclassical Sobolev spaces are defined on $S^*(M)$ using the spectral decomposition of Δ, with norm

$$\|f\|_{H_h^\sigma} = \|(1 + h^2\Delta)^{\sigma/2} f\|_{L^2}.$$

We will consider cutoffs $\rho(s)$ satisfying, for some $c' > c > 0$,

$$\rho(s) \in C^\infty(\mathbb{R}), \qquad \rho(s) = 0 \quad \text{if } s \le c, \qquad \rho(s) = 1 \quad \text{if } s \ge c'. \tag{6-2}$$

The operator $\rho(h^2\Delta)$ is then defined as a spectral multiplier. We observe the following simple result for $R \in \Psi_h^{-\infty}(m)$ on $S^*(M)$. For all N and σ we have

$$\|\rho(h^2\Delta)Ru\|_{H_h^\sigma} + \|R\rho(h^2\Delta)u\|_{H_h^\sigma} \le C_{N,\sigma} h^N \|u\|_{L^2}. \tag{6-3}$$

This follows by writing $\rho(h^2\Delta)(1+h^2\Delta)^\sigma = f(h^2\Delta)\circ(h^2\Delta)^N$, where the function $f(s)$ is a bounded function provided $N > \sigma$, and using (6-1).

Theorem 18. *Suppose that $\sigma \le 0$, that $A_h \in \Psi_h^\sigma(m)$, and that ρ satisfies (6-2). Then*

$$\|\rho(h^2\Delta)A_h u\|_{H_h^{-\sigma/3}} + \|A_h\rho(h^2\Delta)u\|_{H_h^{-\sigma/3}} \le Ch^{-\sigma/6}\|u\|_{L^2}.$$

Proof. Choose k so $6k+\sigma > 0$. For each $h \in (0, 1]$, we show that $A_h = A_{0,h} + A_{1,h}$, where

$$\|(h^2\Delta)^k A_{0,h}u\|_{L^2} + \|A_{0,h}(h^2\Delta)^k u\|_{L^2} + \|A_{1,h}u\|_{H_h^{-\sigma/3}} \le Ch^{-\sigma/6}\|u\|_{L^2}.$$

The result then follows since $\rho(s) \le \min(s^k, 1)$. Using the Littlewood–Paley decomposition as in the proof of Theorem 17, applied to each a_α in the sum defining a, we let

$$A_{0,h} = \sum_{2^j \le h^{-1/6}} a_{j,h}(x, hD), \qquad A_{1,h} = \sum_{2^j > h^{-1/6}} a_{j,h}(x, hD).$$

Recalling the form (5-4), we see that applying $h^2\Delta$ to $a_{j,h}(x, hD)$ is equivalent to multiplying it by at most $2^{6j}h$. As in the proof of (5-5) we conclude that

$$\|(1+h^2\Delta)^k a_{j,h}(x, hD)a_{i,h}(x, hD)^*(1+h^2\Delta)^k\|_{L^2\to L^2} \le (1+2^{6i}h)^k(1+2^{6j}h)^k 2^{\sigma(i+j)-|i-j|}.$$

For $2^j, 2^i \ge h^{-1/6}$, we interpolate with the L^2 bounds (5-6) to obtain

$$\|(1+h^2\Delta)^{-\sigma/6}a_{j,h}(x, hD)a_{i,h}(x, hD)^*(1+h^2\Delta)^{-\sigma/6}\|_{L^2\to L^2} \le Ch^{-\sigma/3}2^{-|i-j|}.$$

This estimate also holds for the transposed operators. The Cotlar–Stein lemma then implies the bounds for $A_{1,h}$.

Similarly, we have

$$\|(h^2\Delta)^k a_{j,h}(x, hD)\|_{L^2\to L^2} + \|a_{j,h}(x, hD)(h^2\Delta)^k\|_{L^2\to L^2} \le C(2^{6j}h)^k 2^{\sigma j},$$

which we may sum over $2^j \le h^{-1/6}$ to conclude the bounds involving $A_{0,h}$. □

Corollary 19. *Suppose that $\sigma \le 0$ and $A_h \in \Psi_h^\sigma(m)$. Then*

$$\|(1+h\Delta)^{-\sigma/6}A_h u\|_{L^2} \le C\|u\|_{L^2}.$$

Proof. As in the proof of Theorem 18 we observe that, for $k = 0, 1, 2, \ldots,$

$$\|(1+h\Delta)^k a_{j,h}(x, hD)a_{i,h}(x, hD)^*(1+h\Delta)^k\|_{L^2\to L^2} \le 2^{6k(i+j)}2^{\sigma(i+j)-|i-j|}.$$

We interpolate between $k = 0$ and any $k > -\sigma/6$ to obtain

$$\|(1+h\Delta)^{-\sigma/6}a_{j,h}(x, hD)a_{i,h}(x, hD)^*(1+h\Delta)^{-\sigma/6}\|_{L^2\to L^2} \le C2^{-|i-j|}.$$

This estimate also holds for the transposed operators. The Cotlar–Stein lemma then implies the result. □

Theorem 20. *The following bound holds for* $h \in (0, 1]$ *and all* $N \in \mathbb{N}$:

$$\|Hu\|_{L^2} + h\|\Delta_{\mathbb{S}} u\|_{L^2} + \|(1+h\Delta)^{\frac{1}{3}} u\|_{L^2} \le C\|P_h u\|_{L^2} + C_N \|(1+h\Delta)^{-N} u\|_{L^2}.$$

Proof. Write $u = Q_h P_h u + Ru$, where $Q_h \in \Psi_h^{-2}(m)$, and note that $HQ_h,\ h\Delta_{\mathbb{S}} Q_h \in \Psi_h^0(m)$ by Remark 15. Also, for all N we have $HR(1+h\Delta)^N,\ h\Delta_{\mathbb{S}} R(1+h\Delta)^N \in \Psi_h^0(m)$; hence

$$\|HRu\|_{L^2} + h\|\Delta_{\mathbb{S}} Ru\|_{L^2} \le C_N \|(1+h\Delta)^{-N} u\|_{L^2}.$$

Since $Q_h,\ R(1+h\Delta)^N \in \Psi_h^{-2}(m)$, the result then follows by Corollary 19. □

Theorem 21. *Suppose that* ρ_1 *and* ρ_2 *satisfy* (6-2), *and* $\rho_2 = 1$ *on a neighborhood of* $\operatorname{supp}(\rho_1)$. *Given* $\lambda_0 > 0$, *the following holds for all N, and all* $|\lambda| \le \lambda_0$ *and* $h \in (0, 1]$:

$$h^{-\frac{1}{3}} \|\rho_1(h^2\Delta)u\|_{H_h^{2/3}} + h^{\frac{1}{3}} \sum_{j=1}^{d} \|X_j \rho_1(h^2\Delta)u\|_{H_h^{1/3}} + \|X_0\rho_1(h^2\Delta)u\|_{L^2} + \|h\Delta_{\mathbb{S}}\rho_1(h^2\Delta)u\|_{L^2} \\ \le C_{N,\lambda_0}(\|\rho_2(h^2\Delta)(P_h-\lambda)u\|_{L^2} + h^N\|u\|_{L^2}).$$

Proof. We follow the scheme of the proof of Theorem 2 of [Drouot 2017], using the parametrix Q_h of P_h to replace the positive commutator arguments. Write

$$\rho_1(h^2\Delta)u = Q_h\rho_1(h^2\Delta)(P_h-\lambda)u + Q_h[P_h, \rho_1(h^2\Delta)]u + \lambda Q_h\rho_1(h^2\Delta)u + R\rho_1(h^2\Delta)u.$$

To handle the commutator term, we use that $[\Delta_{\mathbb{S}}, \rho_1(h^2\Delta)] = 0$; hence $[P_h, \rho_1(h^2\Delta)] = [X_0, \rho_1(h^2\Delta)]$. Now let $\tilde\rho_1(s)$ be any function satisfying (6-2) which equals 1 on a neighborhood of $\operatorname{supp}(\rho_1)$. Then following [Drouot 2017], we use that the essential support of $[X_0, \rho_1(h^2\Delta)]$ is contained within the elliptic set of $\tilde\rho(h^2\Delta)$, and we can thus bound

$$\|[P_h, \rho_1(h^2\Delta)]u\|_{L^2} \le C\|\tilde\rho_1(h^2\Delta)]u\|_{L^2} + C_N h^N \|u\|_{L^2}.$$

Applying Theorem 18 and (6-3) we obtain

$$h^{-\frac{1}{3}} \|\rho_1(h^2\Delta)u\|_{H_h^{2/3}} + h^{-\frac{1}{6}} \sum_{j=1}^{d} \|h^{\frac{1}{2}} X_j \rho_1(h^2\Delta)u\|_{H_h^{1/3}} + \|X_0\rho_1(h^2\Delta)u\|_{L^2} + \|h\Delta_{\mathbb{S}}\rho_1(h^2\Delta)u\|_{L^2} \\ \le C\big(\|\rho_1(h^2\Delta)(P_h-\lambda)u\|_{L^2} + \|\tilde\rho_1(h^2\Delta)u\|_{L^2} + (1+|\lambda|)\|\rho_1(h^2\Delta)u\|_{L^2}\big) + C_N h^N\|u\|_{L^2}.$$

For h bounded away from 0 we can absorb the term $(1+|\lambda|)\|\rho_1(h^2\Delta)u\|_{L^2}$ into $C_N h^N\|u\|_{L^2}$, and for h small we can subtract it from both sides.

From this we deduce the following bound for any such $\tilde\rho_1$:

$$\|\rho_1(h^2\Delta)u\|_{L^2} \le C_{N,\lambda_0}\big(h^{\frac{1}{3}}\|\rho_2(h^2\Delta)(P_h-\lambda)u\|_{L^2} + h^{\frac{1}{3}}\|\tilde\rho_1(h^2\Delta)u\|_{L^2} + h^N\|u\|_{L^2}\big).$$

We now choose a sequence of cutoffs $\tilde\rho_j$ for $1 \le j \le 3N$, satisfying (6-2), such that for all j we have $\tilde\rho_{j+1} = 1$ on a neighborhood of $\operatorname{supp}(\tilde\rho_j)$, and $\rho_2 = 1$ on a neighborhood of $\operatorname{supp}(\tilde\rho_j)$. Then replacing ρ_1 by $\tilde\rho_j$, the preceding estimate shows that

$$\|\tilde\rho_j(h^2\Delta)u\|_{L^2} \le C_{N,\lambda_0}\big(h^{\frac{1}{3}}\|\rho_2(h^2\Delta)(P_h-\lambda)u\|_{L^2} + h^{\frac{1}{3}}\|\tilde\rho_{j+1}(h^2\Delta)u\|_{L^2} + h^N\|u\|_{L^2}\big).$$

We conclude by iteration that

$$\begin{aligned}\|\tilde{\rho}_1(h^2\Delta)u\|_{L^2} &\le C_{N,\lambda_0}\big(h^{\frac{1}{3}}\|\rho_2(h^2\Delta)(P_h-\lambda)u\|_{L^2}+h^N\|\rho_2(h^2\Delta)u\|_{L^2}+h^N\|u\|_{L^2}\big)\\ &\le C_{N,\lambda_0}\big(h^{\frac{1}{3}}\|\rho_2(h^2\Delta)(P_h-\lambda)u\|_{L^2}+h^N\|u\|_{L^2}\big).\end{aligned}$$

Together with the above this yields the statement of the theorem. □

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ON THE PROPAGATION OF REGULARITY FOR SOLUTIONS OF THE DISPERSION GENERALIZED BENJAMIN–ONO EQUATION

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To my parents

We study some properties of propagation of regularity of solutions of the dispersive generalized Benjamin–Ono (BO) equation. This model defines a family of dispersive equations that can be seen as a dispersive interpolation between the Benjamin–Ono equation and the Korteweg–de Vries (KdV) equation.

Recently, it has been shown that solutions of the KdV and BO equations satisfy the following property: if the initial data has some prescribed regularity on the right-hand side of the real line, then this regularity is propagated with infinite speed by the flow solution.

In this case the nonlocal term present in the dispersive generalized Benjamin–Ono equation is more challenging that the one in the BO equation. To deal with this a new approach is needed. The new ingredient is to combine commutator expansions into the weighted energy estimate. This allows us to obtain the property of propagation and explicitly the smoothing effect.

1. Introduction

The aim of this work is to study some special regularity properties of solutions to the initial value problem (IVP) associated to the *dispersive generalized Benjamin–Ono equation*

$$\begin{cases} \partial_t u - D_x^{\alpha+1}\partial_x u + u\,\partial_x u = 0, & x,t\in\mathbb{R},\ 0<\alpha<1,\\ u(x,0)=u_0(x), \end{cases} \tag{1.1}$$

where D_x^s, denotes the homogeneous derivative of order $s\in\mathbb{R}$,

$$D_x^s = (-\partial_x^2)^{\frac{s}{2}} \quad \text{thus} \quad D_x^s f = c_s(|\xi|^s\hat{f}(\xi))^{\vee},$$

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which in its polar form is decomposed as $D_x^s = (\mathcal{H}\partial_x)^s$, where $\mathcal{H}$ denotes the *Hilbert transform*

$$\mathcal{H}f(x) = \frac{1}{\pi} \lim_{\epsilon\to 0^+} \int_{|y|\geq\epsilon} \frac{f(x-y)}{y}\,\mathrm{d}y = (-i\,\operatorname{sgn}(\xi)\hat{f}(\xi))\check{}\,(x),$$

where $\hat{}$ denotes the Fourier transform and $\check{}$ denotes its inverse. These equations model vorticity waves in the coastal zone; see [Molinet et al. 2001].

Our starting point is a property established by Isaza, Linares and Ponce [Isaza et al. 2015] concerning the solutions of the IVP associated to the k-generalized KdV equation

$$\begin{cases} \partial_t u + \partial_x^3 u + u^k \partial_x u = 0, & x, t \in \mathbb{R},\ k \in \mathbb{N}, \\ u(x,0) = u_0(x). \end{cases} \tag{1.2}$$

It was shown in [Isaza et al. 2015] that the unidirectional dispersion of the k-generalized KdV equation gives the following propagation of regularity phenomena.

Theorem 1.3 [Isaza et al. 2015]. *If* $u_0 \in H^{3/4^+}(\mathbb{R})$ *and for some* $l \in \mathbb{Z},\ l \geq 1$ *and* $x_0 \in \mathbb{R}$

$$\|\partial_x^l u_0\|^2_{L^2((x_0,\infty))} = \int_{x_0}^{\infty} |\partial_x^l u_0(x)|^2\,\mathrm{d}x < \infty, \tag{1.4}$$

then the solution of the IVP associated to (1.2) *satisfies that for any* $v > 0$ *and* $\epsilon > 0$

$$\sup_{0\leq t\leq T} \int_{x_0+\epsilon-vt}^{\infty} (\partial_x^j u)^2(x,t)\,\mathrm{d}x < c \tag{1.5}$$

for $j = 0, 1, 2, \ldots, l$ *with* $c = c(l;\ \|u_0\|_{H^{3/4^+}(\mathbb{R})};\ \|\partial_x^l u_0\|_{L^2((x_0,\infty))};\ v;\ \epsilon;\ T)$. *In particular, for all* $t \in (0, T]$, *the restriction of* $u(\cdot, t)$ *to any interval* (x_0, ∞) *belongs to* $H^l((x_0, \infty))$.

Moreover, for any $v \geq 0$, $\epsilon > 0$ *and* $R > 0$

$$\int_0^T \int_{x_0+\epsilon-vt}^{x_0+R-vt} (\partial_x^{l+1} u)^2(x,t)\,\mathrm{d}x\,\mathrm{d}t < c,$$

with $c = c(l;\ \|u_0\|_{H^{3/4^+}(\mathbb{R})};\ \|\partial_x^l u_0\|_{L^2((x_0,\infty))};\ v;\ \epsilon;\ R;\ T)$.

The proof of Theorem 1.3 is based on weighted energy estimates. In particular, the iterative process in the induction argument is based on a property discovered originally by T. Kato [1983] in the context of the KdV equation. More precisely, he showed that solution of the KdV equation satisfies

$$\int_0^T \int_{-R}^{R} (\partial_x u)^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \leq c(R;\ T;\ \|u_0\|_{L^2_x}), \tag{1.6}$$

where this is the fundamental fact in his proof of existence of the global weak solutions of (1.2) for $k = 1$ and initial data in $L^2(\mathbb{R})$.

This result was also obtained for the Benjamin–Ono equation [Isaza et al. 2016a] but it does not follow as the KdV case because of the presence of the Hilbert transform.

Later on, [Kenig et al. 2018] extended the results in Theorem 1.3 to the case when the local regularity of the initial data u_0 in (1.4) is measured with fractional indices. The scope of this case is much more involved, and its proof is mainly based in weighted energy estimates combined with techniques involving pseudodifferential operators and singular integrals. The property described in Theorem 1.3 is intrinsic to suitable solutions of some nonlinear dispersive models; see also [Linares et al. 2017]. In the context of two-dimensional models, analogous results for the Kadomtsev–Petviashvili II equation [Isaza et al. 2016b] and the Zakharov–Kuznetsov [Linares and Ponce 2018] equation were proved.

Before stating our main result we will give an overview of the local well-posedness of the IVP (1.1).

Following [Kato 1983] we have that the initial value problem IVP (1.1) is *locally well-posed* (LWP) in the Banach space X if for every initial condition $u_0 \in X$ there exists $T > 0$ and a unique solution $u(t)$ satisfying

$$u \in C([0, T] : X) \cap A_T, \tag{1.7}$$

where A_T is an auxiliary function space. Moreover, the solution map $u_0 \mapsto u$ is continuous from X into the class (1.7). If T can be taken arbitrarily large, one says that the IVP (1.1) is *globally well-posed* (GWP) in the space X.

It is natural to study the IVP (1.1) in the Sobolev space

$$H^s(\mathbb{R}) = (1 - \partial_x^2)^{-\frac{s}{2}} L^2(\mathbb{R}), \quad s \in \mathbb{R}.$$

There exist remarkable differences between the KdV (1.2) and the IVP (1.1). In case of KdV, e.g., it possesses infinite conserved quantities, defines a Hamiltonian system, has multisoliton solutions and is a completely integrable system by the inverse scattering method [Coifman and Wickerhauser 1990; Fokas and Ablowitz 1983]. Instead, in the case of the IVP (1.1) there is no integrability, but three conserved quantities (see [Sidi et al. 1986]), specifically

$$\mathrm{I}[u](t) = \int_{\mathbb{R}} u \,\mathrm{d}x, \quad \mathrm{M}[u](t) = \int_{\mathbb{R}} u^2 \,\mathrm{d}x, \quad \mathrm{H}[u](t) = \frac{1}{2}\int_{\mathbb{R}} |D_x^{\frac{1+\alpha}{2}} u|^2 \,\mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}} u^3 \,\mathrm{d}x,$$

are satisfied at least for smooth solutions.

Another property in which these two models differ resides in the fact that one can obtain a local existence theory for the KdV equation in $H^s(\mathbb{R})$, based on the contraction principle. On the contrary, this cannot be done in the case of the IVP (1.1). This is a consequence of the fact that dispersion is not enough to deal with the nonlinear term. In this direction, Molinet, Saut and Tzvetkov [Molinet et al. 2001] showed that for $0 \leq \alpha < 1$ the IVP (1.1) with the assumption $u_0 \in H^s(\mathbb{R})$ is not enough to prove local well-posedness by using fixed-point arguments or the Picard iteration method.

Nevertheless, Molinet and Ribaud [2006] proved global well-posedness by considering initial data in a weighted low-frequency Sobolev space. Later, using suitable spaces of Bourgain type, Herr [2007] proved local well-posedness for initial data in $H^s(\mathbb{R}) \cap \dot{H}^{-\omega}(\mathbb{R})$ for any $s > -\frac{3\alpha}{4}$, $\omega = \frac{1}{\alpha+1} - \frac{1}{2}$, where $\dot{H}^{-\omega}(\mathbb{R})$ is a weighted low-frequency Sobolev space (for more details see [Herr 2007]); next by using a conservation law, these results are extended to global well-posedness in $H^s(\mathbb{R}) \cap \dot{H}^{-\omega}(\mathbb{R})$, for $s \geq 0$, $\omega = \frac{1}{\alpha+1} - \frac{1}{2}$. In this sense, an improvement was obtained by Herr, Ionescu, Kenig and Koch [Herr et al. 2010], who showed that the IVP (1.1) is globally well-posed in the space of the real-valued $L^2(\mathbb{R})$-functions by using a renormalization method to control the strong low-high frequency interactions. However, it is not clear that these results described above can be used to establish our main result. Thus a local theory obtained by using energy estimates in addition to dispersive properties of the smooth solutions is required.

In the first step, we obtain the following a priori estimate for solutions of IVP (1.1):

$$\|u\|_{L^\infty_T H^s_x} \lesssim \|u_0\|_{H^s_x} \, \mathrm{e}^{c\|\partial_x u\|_{L^1_T L^\infty_x}};$$

part of this estimate is based on the Kato–Ponce commutator estimate [1988].

The inequality above reads as follows: in order for the solution u to lie in the Sobolev space $H^s(\mathbb{R})$, continuously in time, we must control the term $\|\partial_x u\|_{L^1_T L^\infty_x}$.

First, we use results of Kenig, Ponce and Vega [Kenig et al. 1991a] concerning oscillatory integrals in order to obtain the classical Strichartz estimates associated to the group $S(t) = \mathrm{e}^{t D_x^{\alpha+1}\partial_x}$, corresponding to the linear part of the equation in (1.1).

Additionally, the technique introduced in [Koch and Tzvetkov 2003] related to the refined Strichartz estimate is fundamental in our analysis. Specifically, their method is mainly based in a decomposition of the time interval into small pieces whose lengths depends on the spatial frequencies of the solution. This approach allowed Koch and Tzvetkov to prove local well-posedness for the Benjamin–Ono equation in $H^{5/4^+}(\mathbb{R})$. Then, Kenig and Koenig [2003] enhanced this estimate, which led to proving local well-posedness for the Benjamin–Ono equation in $H^{9/8^+}(\mathbb{R})$.

Several issues arise when handling the nonlinear part of the equation in (1.1); nevertheless, following the work of Kenig, Ponce and Vega [Kenig et al. 1993], we manage the loss of derivatives by combining the local smoothing effect and a maximal function estimate of the group $S(t) = \mathrm{e}^{t D_x^{\alpha+1}\partial_x}$.

These observations lead us to present our first result.

Theorem A. *Let $0 < \alpha < 1$. Set $s(\alpha) = \frac{9}{8} - \frac{3\alpha}{8}$ and assume that $s > s(\alpha)$. Then, for any $u_0 \in H^s(\mathbb{R})$, there exists a positive time $T = T(\|u_0\|_{H^s(\mathbb{R})}) > 0$ and a unique solution u satisfying* (1.1) *such that*

$$u \in C([0,T] : H^s(\mathbb{R})) \quad \text{and} \quad \partial_x u \in L^1([0,T] : L^\infty(\mathbb{R})). \tag{1.8}$$

Moreover, for any $r > 0$, the map $u_0 \mapsto u(t)$ is continuous from the ball $\{u_0 \in H^s(\mathbb{R}) : \|u_0\|_{H^s(\mathbb{R})} < r\}$ to $C([0,T] : H^s(\mathbb{R}))$.

Theorem A is the base result to describe the propagation of regularity phenomena. As we mentioned above, the propagation of regularity phenomena is satisfied by the BO and KdV equations. These two models correspond to particular cases of the IVP (1.1), specifically by taking $\alpha = 0$ and $\alpha = 1$.

A question that arises naturally is to determine whether the propagation of regularity phenomena is satisfied for a model with an intermediate dispersion between these two models mentioned above.

Our main result gives answer to this problem and it is summarized in the following:

Theorem B. *Let $u_0 \in H^s(\mathbb{R})$, with $s = \frac{3-\alpha}{2}$, and $u = u(x,t)$ be the corresponding solution of the IVP* (1.1) *provided by Theorem A.*

If for some $x_0 \in \mathbb{R}$ and for some $m \in \mathbb{Z}^+$, $m \geq 2$,

$$\partial_x^m u_0 \in L^2(\{x \geq x_0\}), \tag{1.9}$$

then for any $v \geq 0,\ T > 0,\ \epsilon > 0$ *and* $\tau > \epsilon$

$$\sup_{0\leq t\leq T}\int_{x_0+\epsilon-vt}^{\infty}(\partial_x^j u)^2(x,t)\,\mathrm{d}x + \int_0^T\int_{x_0+\epsilon-vt}^{x_0+\tau-vt}(D_x^{\frac{\alpha+1}{2}}\partial_x^j u)^2(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{x_0+\epsilon-vt}^{x_0+\tau-vt}(D_x^{\frac{\alpha+1}{2}}\mathcal{H}\partial_x^j u)^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \leq c \tag{1.10}$$

for $j = 1, 2, \ldots, m$, *with* $c = c(T; \epsilon; v; \alpha; \|u_0\|_{H^s}; \|\partial_x^m u_0\|_{L^2((x_0,\infty))}) > 0$.

If in addition to (1.9) *there exists* $x_0 \in \mathbb{R}^+$ *with*

$$D_x^{\frac{1-\alpha}{2}}\partial_x^m u_0 \in L^2(\{x \geq x_0\}) \tag{1.11}$$

then for any $v \geq 0,\ \epsilon > 0$ *and* $\tau > \epsilon$

$$\sup_{0\leq t\leq T}\int_{x_0+\epsilon-vt}^{\infty}(D_x^{\frac{1-\alpha}{2}}\partial_x^m u)^2(x,t)\,\mathrm{d}x + \int_0^T\int_{x_0+\epsilon-vt}^{x_0+\tau-vt}(\partial_x^{m+1}u)^2(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{x_0+\epsilon-vt}^{x_0+\tau-vt}(\partial_x^{m+1}\mathcal{H}u)^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \leq c, \tag{1.12}$$

with $c = c(T; \epsilon; v; \alpha; \|u_0\|_{H^s}; \|D_x^{(1-\alpha)/2}\partial_x^m u_0\|_{L^2((x_0,\infty))}) > 0$.

Although the argument of the proof of Theorem B follows in spirit that of KdV, i.e., an induction process combined with weighted energy estimates, the presence of the nonlocal operator $D_x^{\alpha+1}\partial_x$ in the term providing the dispersion, makes the proof much harder. More precisely, two difficulties appear, the most important of which is to obtain explicitly the Kato smoothing effect [1983], which in the proof of Theorem 1.3 is fundamental.

In contrast to the KdV equation, the gain of the local smoothing in solutions of the dispersive generalized Benjamin–Ono equation is just $\frac{\alpha+1}{2}$ derivatives, so as occurs in the case of the Benjamin–Ono equation [Isaza et al. 2016a], the iterative argument in the induction process is carried out in two steps, one for positive integers m and another one for $m + \frac{1-\alpha}{2}$ derivatives.

In the case of the BO equation [Isaza et al. 2016a], the authors obtain the smoothing effect basing their analysis on several commutator estimates, such as the extension of Calderón's first commutator for the Hilbert transform [Baishanski and Coifman 1967]. However, their method of proof does not allow them to obtain explicitly the local smoothing as in [Kato 1983].

The advantage of our method is that it allows us to obtain explicitly the smoothing effect for any $\alpha \in (0, 1)$ in the IVP (1.1). Roughly, we rewrite the term modeling the dispersive part of the equation in (1.1) in terms of an expression involving $[\mathcal{H}D_x^{\alpha+2}; \chi_{\epsilon,b}^2]$. At this point, we incorporate results of [Ginibre and Velo 1991] about commutator decomposition. This allows us to obtain explicitly the smoothing effect as in [Kato 1983] at every step of the induction process in the energy estimate. Additionally, this approach allows us to study the propagation of regularity phenomena in models where the dispersion is lower in comparison with that of IVP (1.1). We address this issue in a forthcoming work; specifically we study the propagation of regularity phenomena in real solutions of the model

$$\partial_t u - D_x^{\alpha}\partial_x u + u\partial_x u = 0, \quad x, t \in \mathbb{R},\ 0 < \alpha < 1.$$

As a direct consequence of Theorem B, one has that for an appropriate class of initial data the singularity of the solution travels with infinite speed to the left as time evolves. Also, the time reversibility property implies that the solution cannot have had some regularity in the past.

Concerning the nonlinear part of IVP (1.1) in the weighted energy estimate, several issues arise. Nevertheless, following the approach of [Kenig et al. 2018], combined with [Kato and Ponce 1988; Li 2019] on the generalization of several commutator estimates, allows us to overcome these difficulties.

Remark 1.13. (I) It will be clear from our proof that the requirement on the initial data, that is, $u_0 \in H^{(3-\alpha)/2}(\mathbb{R})$ in Theorem B, can be lowered to $H^{((9-3\alpha)/8)+}(\mathbb{R})$.

(II) Also it is worth highlighting that the proof of Theorem B can be extended to solutions of the IVP

$$\begin{cases} \partial_t u - D_x^{\alpha+1}\partial_x u + u^k\,\partial_x u = 0, \quad x,t\in\mathbb{R},\ 0<\alpha<1,\ k\in\mathbb{Z}^+, \\ u(x,0)=u_0(x). \end{cases} \tag{1.14}$$

(III) The results in Theorem B still hold for solutions of the defocusing generalized dispersive Benjamin–Ono equation

$$\begin{cases} \partial_t u - D_x^{\alpha+1}\partial_x u - u\,\partial_x u = 0, \quad x,t\in\mathbb{R},\ 0<\alpha<1, \\ u(x,0)=u_0(x). \end{cases}$$

This can be seen applying Theorem B to the function $v(x,t)=u(-x,-t)$, where $u(x,t)$ is a solution of (1.1). In short, Theorem B remains valid, backward in time for initial data u_0 satisfying (1.9) and (1.11).

Next, we present some immediate consequences of Theorem B.

Corollary 1.15. *Let $u\in C([-T,T]:H^{(3-\alpha)/2}(\mathbb{R}))$ be a solution of the equation in* (1.1) *described by Theorem B. If there exist $n,m\in\mathbb{Z}^+$ with $m\le n$ such that for some $\tau_1,\tau_2\in\mathbb{R}$ with $\tau_1<\tau_2$*

$$\int_{\tau_2}^{\infty} |\partial_x^n u_0(x)|^2\,dx<\infty \quad \textit{but} \quad \partial_x^m u_0\notin L^2((\tau_1,\infty)),$$

then for any $t\in(0,T)$ and any $v>0$ and $\epsilon>0$

$$\int_{\tau_2+\epsilon-vt}^{\infty} |\partial_x^n u(x,t)|^2\,dx<\infty,$$

and for any $t\in(-T,0)$ and any $\tau_3\in\mathbb{R}$

$$\int_{\tau_3}^{\infty} |\partial_x^m u(x,t)|^2\,dx=\infty.$$

The rest of the paper is organized as follows: in the Section 2 we fix the notation to be used throughout the document. Section 3 contains a brief summary of commutator estimates involving fractional derivatives. Section 4 deals with the local well-posedness. Finally, in Sections 5 and 6 we prove Theorems A and B.

2. Notation

The following notation will be used extensively throughout this article. The operator $J^s=(1-\partial_x^2)^{s/2}$ denotes the Bessel potentials of order $-s$.

For $1 \le p \le \infty$, $L^p(\mathbb{R})$ is the usual Lebesgue space with the norm $\|\cdot\|_{L^p}$; additionally for $s \in \mathbb{R}$, we consider the Sobolev space $H^s(\mathbb{R})$ is defined via its usual norm $\|f\|_{H^s} = \|J^s f\|_{L^2}$. In this context, we define

$$H^\infty(\mathbb{R}) = \bigcap_{s \ge 0} H^s(\mathbb{R}).$$

Let $f = f(x,t)$ be a function defined for $x \in \mathbb{R}$ and t in the time interval $[0,T]$, with $T > 0$, or in the whole line $\mathbb{R}$. Then if A denotes any of the spaces defined above, we define the spaces $L^p_T A_x$ and $L^p_t A_x$ by the norms

$$\|f\|_{L^p_T A_x} = \left(\int_0^T \|f(\cdot,t)\|_A^p \,\mathrm{d}t \right)^{\frac{1}{p}} \quad \text{and} \quad \|f\|_{L^p_t A_x} = \left(\int_{\mathbb{R}} \|f(\cdot,t)\|_A^p \,\mathrm{d}t \right)^{\frac{1}{p}}$$

for $1 \le p \le \infty$ with the natural modification in the case $p = \infty$. Moreover, we use similar definitions for the mixed spaces $L^q_x L^p_t$ and $L^q_x L^p_T$ with $1 \le p, q \le \infty$.

For two quantities A and B, we write $A \lesssim B$ if $A \le cB$ for some constant $c > 0$. Similarly, $A \gtrsim B$ if $A \ge cB$ for some $c > 0$. We write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. The dependence of the constant c on other parameters or constants is usually clear from the context and we will often suppress this dependence whenever possible.

For a real number a we will write a^+ instead of $a + \epsilon$ whenever ϵ is a positive number whose value is small enough.

3. Preliminaries

In this section, we state several inequalities to be used in the next sections.

First, we have an extension of the Calderón commutator theorem [1965] established in [Baishanski and Coifman 1967].

Theorem 3.1. *For any $p \in (1,\infty)$ and any $l, m \in \mathbb{Z}^+ \cup \{0\}$ there exists $c = c(p; l; m) > 0$ such that*

$$\|\partial_x^l [\mathcal{H}; \psi]\, \partial_x^m f\|_{L^p} \le c \|\partial_x^{m+l} \psi\|_{L^\infty} \|f\|_{L^p}. \tag{3.2}$$

For a different proof see [Dawson et al. 2008, Lemma 3.1].

In our analysis the Leibniz rule for fractional derivatives, established in [Grafakos and Oh 2014; Kato and Ponce 1988; Kenig et al. 1994], will be crucial. Even though most of these estimates are valid in several dimensions, we will restrict our attention to the one-dimensional case.

Lemma 3.3. *For $s > 0$, $p \in [1,\infty)$,*

$$\|D^s(fg)\|_{L^p} \lesssim \|f\|_{L^{p_1}} \|D^s g\|_{L^{p_2}} + \|g\|_{L^{p_3}} \|D^s f\|_{L^{p_4}}, \tag{3.4}$$

with

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}, \quad p_j \in (1,\infty],\ j = 1,2,3,4.$$

Also, we will state the fractional Leibniz rule proved by Kenig, Ponce and Vega [Kenig et al. 1993].

Lemma 3.5. *Let $s = s_1 + s_2 \in (0, 1)$, with $s_1, s_2 \in (0, s)$, and $p, p_1, p_2 \in (1, \infty)$, satisfy*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}.$$

Then,

$$\|D^s(fg) - fD^s g - gD^s f\|_{L^p} \lesssim \|D^{s_1} f\|_{L^{p_1}} \|D^{s_2} g\|_{L^{p_2}}. \tag{3.6}$$

Moreover, the case $s_2 = 0$ and $p_2 = \infty$ is allowed.

A natural question about Lemma 3.5 is to investigate the possible generalization of the estimate (3.6) when $s \geq 1$. The answer to this question was given recently by D. Li [2019]; he established new fractional Leibniz rules for the nonlocal operator D^s, $s > 0$, and related ones, including various endpoint situations.

Theorem 3.7. *Let $s > 0$ and $1 < p < \infty$. Then for any $s_1, s_2 \geq 0$ with $s = s_1 + s_2$, and any $f, g \in \mathcal{S}(\mathbb{R}^n)$, the following hold:*

(1) *If $1 < p_1, p_2 < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, then*

$$\left\| D^s(fg) - \sum_{\alpha \leq s_1} \frac{1}{\alpha!} \partial_x^\alpha f D^{s,\alpha} g - \sum_{\beta \leq s_2} \frac{1}{\beta!} \partial_x^\beta g D^{s,\beta} f \right\|_{L^p} \lesssim \|D^{s_1} f\|_{L^{p_1}} \|D^{s_2} g\|_{L^{p_2}}. \tag{3.8}$$

(2) *If $p_1 = p$, $p_2 = \infty$, then*

$$\left\| D^s(fg) - \sum_{\alpha < s_1} \frac{1}{\alpha!} \partial_x^\alpha f D^{s,\alpha} g - \sum_{\beta \leq s_2} \frac{1}{\beta!} \partial_x^\beta g D^{s,\beta} f \right\|_{L^p} \lesssim \|D^{s_1} f\|_{L^p} \|D^{s_2} g\|_{\mathrm{BMO}},$$

where $\|\cdot\|_{\mathrm{BMO}}$ denotes the norm in the BMO space.[1]

(3) *If $p_1 = \infty$, $p_2 = p$, then*

$$\left\| D^s(fg) - \sum_{\alpha \leq s_1} \frac{1}{\alpha!} \partial^\alpha f D^{s,\alpha} g - \sum_{\beta < s_2} \frac{1}{\beta!} \partial^\beta g D^{s,\beta} f \right\|_{L^p} \lesssim \|D^{s_1} f\|_{\mathrm{BMO}} \|D^{s_2} g\|_{L^p}.$$

The operator $D^{s,\alpha}$ is defined via Fourier transform[2]

$$\widehat{D^{s,\alpha} g}(\xi) = \widehat{D}^{s,\alpha}(\xi)\hat{g}(\xi),$$
$$\widehat{D}^{s,\alpha}(\xi) = i^{-\alpha} \partial_\xi^\alpha (|\xi|^s).$$

Remark 3.9. As usual empty summation (such as $\sum_{0 \leq \alpha < 0}$) is defined as zero.

Proof. For a detailed proof of this theorem and related results, see [Li 2019]. □

Next we have the following commutator estimates involving nonhomogeneous fractional derivatives, established by Kato and Ponce.

[1]For any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the BMO seminorm is given by $\|f\|_{\mathrm{BMO}} = \sup_Q \frac{1}{|Q|} \int_Q |f(y) - (f)_Q| \, \mathrm{d}y$, where $(f)_Q$ is the average of f on Q, and the supreme is taken over all cubes Q in $\mathbb{R}^n$.

[2]The precise form of the Fourier transform does not matter.

Lemma 3.10 [Kato and Ponce 1988]. *Let $s>0$ and $p, p_2, p_3 \in (1,\infty)$ and $p_1, p_4 \in (1,\infty]$ be such that*

$$\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}+\frac{1}{p_4}.$$

Then,

$$\|[J^s; f]g\|_{L^p} \lesssim \|\partial_x f\|_{L^{p_1}}\|J^{s-1}g\|_{L^{p_2}}+\|J^s f\|_{L^{p_3}}\|g\|_{L^{p_4}} \tag{3.11}$$

$$\|J^s(fg)\|_{L^p} \lesssim \|J^s f\|_{L^{p_1}}\|g\|_{L^{p_2}}+\|J^s g\|_{L^{p_3}}\|f\|_{L^{p_4}}. \tag{3.12}$$

There are many other reformulations and generalizations of the Kato–Ponce commutator inequalities; see [Bényi and Oh 2014]. Recently Li [2019] has obtained a family of refined Kato–Ponce-type inequalities for the operator D^s. In particular he showed that:

Lemma 3.13. *Let $1<p<\infty$. Let $1<p_1, p_2, p_3, p_4 \le \infty$ satisfy*

$$\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}+\frac{1}{p_4}.$$

Therefore:

(a) *If $0<s\le 1$, then*

$$\|D^s(fg)-fD^s g\|_{L^p} \lesssim \|D^{s-1}\partial_x f\|_{L^{p_1}}\|g\|_{L^{p_2}}.$$

(b) *If $s>1$, then*

$$\|D^s(fg)-fD^s g\|_{L^p} \lesssim \|D^{s-1}\partial_x f\|_{L^{p_1}}\|g\|_{L^{p_2}}+\|\partial_x f\|_{L^{p_3}}\|D^{s-1}g\|_{L^{p_4}}. \tag{3.14}$$

For a more detailed exposition on these estimates see [Li 2019, Section 5].

In addition, we have the following inequality of Gagliardo–Nirenberg type:

Lemma 3.15. *Let $1<q, p<\infty$, $1<r\le\infty$ and $0<\alpha<\beta$. Then*,

$$\|D^\alpha f\|_{L^p} \lesssim c\|f\|_{L^r}^{1-\theta}\|D^\beta f\|_{L^q}^{\theta}$$

with

$$\frac{1}{p}-\alpha=(1-\theta)\frac{1}{r}+\theta\Big(\frac{1}{q}-\beta\Big), \quad \theta\in\Big[\frac{\alpha}{\beta},1\Big].$$

Proof. See [Bergh and Löfström 1976, Chapter 4]. □

Now, we present a result that will help us to establish the propagation of regularity of solutions of (1.1). A previous result [Kenig et al. 2018, Corollary 2.1] was proved using the fact that J^r, $r\in\mathbb{R}$, can be seen as a pseudodifferential operator. Thus, this approach allows us to obtain an expression for J^r in terms of a convolution with a certain kernel $k(x,y)$ which enjoys some properties on localized regions in $\mathbb{R}^2$. In fact, this is known as the singular integral realization of a pseudodifferential operator, whose proof can be found in [Stein 1993, Chapter 4].

The estimate we consider here involves the nonlocal operator D^s instead of J^s.

Lemma 3.16. *Let $m \in \mathbb{Z}^+$ and $s \geq 0$. If $f \in L^2(\mathbb{R})$ and $g \in L^p(\mathbb{R}),\ 2 \leq p \leq \infty$, with*

$$\operatorname{dist}(\operatorname{supp}(f), \operatorname{supp}(g)) \geq \delta > 0. \tag{3.17}$$

Then

$$\|g\, \partial_x^m D^s f\|_{L^2} \lesssim \|g\|_{L^p} \|f\|_{L^2}.$$

Proof. Let f, g be functions in the Schwartz class satisfying (3.17).

Notice that

$$g(x)(D_x^s \partial_x^m f)(x) = \frac{g(x)}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} \mathrm{e}^{ix\xi}\, |\xi|^s \widehat{\partial_x^m f}(\xi)\, \mathrm{d}\xi = \frac{g(x)}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} |\xi|^s (\widehat{\tau_{-x} \partial_x^m f})(\xi)\, \mathrm{d}\xi, \tag{3.18}$$

where τ_h is the translation operator.[3]

Moreover, the last expression in (3.18) defines a tempered distribution for s in a suitable class, which will be specified later. Indeed, for $z \in \mathbb{C}$ with $-1 < \operatorname{Re}(z) < 0$

$$\frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} |\xi|^z (\widehat{\tau_{-x} \partial_x^m \varphi})(\xi)\, \mathrm{d}\xi = c(z) \int_{\mathbb{R}} \frac{(\tau_{-x} \partial_x^m \varphi)(y)}{|y|^{1+z}}\, \mathrm{d}y \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}), \tag{3.19}$$

with $c(z)$ is independent of φ. In fact, evaluating $\varphi(x) = \mathrm{e}^{-x^2/2}$ in (3.19) yields

$$c(z) = \frac{2^z\, \Gamma\left(\frac{z+1}{2}\right)}{\pi^{\frac{1}{2}} \Gamma\left(-\frac{z}{2}\right)}.$$

Thus, for every $\varphi \in \mathcal{S}(\mathbb{R})$ the right-hand side in (3.19) defines a meromorphic function for every test function, which can be extended analytically to a wider range of complex numbers z, specifically z with $\operatorname{Im}(z) = 0$ and $\operatorname{Re}(z) = s > 0$, which is the case that pertains to us. By an abuse of notation, we will denote the meromorphic extension and the original as the same.

Thus, combining (3.17), (3.18) and (3.19) it follows that

$$g(x)(D_x^s \partial_x^m f)(x) = c(s) \int_{\mathbb{R}} \frac{g(x)(\tau_{-x} \partial_x^m f)(y)}{|y|^{1+s}}\, \mathrm{d}y = c(s) g(x) \left(f * \frac{\mathbb{1}_{\{|y| \geq \delta\}}}{|y|^{s+m+1}} \right)(x).$$

Notice that the kernel in the integral expression is not anymore singular due to the condition (3.17). In fact, in the particular case that m is even, we obtain after apply integration by parts

$$g(x)(D_x^s \partial_x^m f)(x) = c(s, m) g(x) \left(f * \frac{\mathbb{1}_{\{|y| \geq \delta\}}}{|y|^{s+m+1}} \right)(x)$$

and in the case m being odd

$$g(x)(D_x^s \partial_x^m f)(x) = c(s, m) g(x) \left(f * \frac{y \mathbb{1}_{\{|y| \geq \delta\}}}{|y|^{s+m+2}} \right)(x).$$

Finally, in both cases combining Young's inequality and Hölder's inequality one gets

$$\|g\, \partial_x^m D_x^s f\|_{L^2} \lesssim \|g\|_{L^p} \|f\|_{L^2} \left\| \frac{\mathbb{1}_{\{|y| \geq \delta\}}}{|\cdot|^{s+m+1}} \right\|_{L^r} \lesssim \|g\|_{L^p} \|f\|_{L^2},$$

where the index p satisfies $\frac{1}{2} = \frac{1}{p} + \frac{1}{r}$, which clearly implies $p \in [2, \infty]$, as was required. □

[3] For $h \in \mathbb{R}$ the translation operator τ_h is defined as $(\tau_h f)(x) = f(x - h)$.

Further in the paper we will use extensively some results about the commutator beyond those presented in this section. Next, we will study the smoothing effect for solutions of the dispersive generalized Benjamin–Ono equation (1.1) following Kato's ideas [1983].

3A. ***Commutator expansions.*** In this section we present several new main tools obtained in [Ginibre and Velo 1989; 1991] which will be the cornerstone of the proof of Theorem B. They include commutator expansions together with their estimates. The basic problem is to handle the nonlocal operator D^s for noninteger s and in particular to obtain representations of its commutator with multiplication operators by functions that exhibit as much locality as possible.

Let $a = 2\mu + 1 > 1$, let n be a nonnegative integer and h be a smooth function with suitable decay at infinity, for instance with $h' \in C_0^\infty(\mathbb{R})$.

We define the operator

$$R_n(a) = [HD^a; h] - \tfrac{1}{2}(P_n(a) - HP_n(a)H), \tag{3.20}$$

$$P_n(a) = a \sum_{0 \le j \le n} c_{2j+1}(-1)^j 4^{-j} D^{\mu-j}(h^{(2j+1)} D^{\mu-j}), \tag{3.21}$$

where

$$c_1 = 1, \quad c_{2j+1} = \frac{1}{(2j+1)!} \prod_{0 \le k < j} (a^2 - (2k+1)^2) \quad \text{and} \quad H = -\mathcal{H}.$$

It was shown in [Ginibre and Velo 1989] that the operator $R_n(a)$ can be represented in terms of anticommutators[4] as follows:

$$R_n(a) = \tfrac{1}{2}([H; Q_n(a)]_+ + [D^a; [H; h]]_+), \tag{3.22}$$

where the operator $Q_n(a)$ is represented in the Fourier space variables by the integral kernel

$$Q_n(a) \to (2\pi)^{\frac{1}{2}} \hat{h}(\xi - \xi')|\xi\xi'|^{\frac{a}{2}} 2aq_n(a,t), \tag{3.23}$$

with $|\xi| = |\xi'|\, \mathrm{e}^{2t}$ and

$$q_n(a,t) = \frac{1}{a}(a^2 - (2n+1)^2)c_{2n+1} \int_0^t \sinh^{2n+1}\tau \, \sinh((a(t-\tau)))\, d\tau. \tag{3.24}$$

Based on (3.22) and (3.23), Ginibre and Velo [1991] obtained the following properties of boundedness and compactness of the operator $R_n(a)$.

Proposition 3.25. *Let n be a nonnegative integer, $a \ge 1$, and $\sigma \ge 0$ be such that*

$$2n+1 \le a + 2\sigma \le 2n+3. \tag{3.26}$$

Then:

(a) *The operator $D^\sigma R_n(a) D^\sigma$ is bounded in L^2 with norm*

$$\|D^\sigma R_n(a) D^\sigma f\|_{L^2} \le C(2\pi)^{-\frac{1}{2}} \|(\widehat{D^{a+2\sigma} h})\|_{L^1_\xi} \|f\|_{L^2}. \tag{3.27}$$

If $a \ge 2n+1$, one can take $C = 1$.

---

[4] For any two operators P and Q we denote the anticommutator by $[P; Q]_+ = PQ + QP$.

(b) *Assume in addition that*

$$2n+1 \leq a+2\sigma < 2n+3.$$

Then the operator $D^{\sigma} R_n(a) D^{\sigma}$ *is compact in* $L^2(\mathbb{R})$.

Proof. See Proposition 2.2 in [Ginibre and Velo 1991]. □

In fact Proposition 3.25 is a generalization of a previous result, where the derivatives of the operator $R_n(a)$ are not considered; see [Ginibre and Velo 1989, Proposition 1].

The estimate (3.27) yields the following identity of localization of derivatives.

Lemma 3.28. *Assume* $0 < \alpha < 1$. *Let be* $\varphi \in C^{\infty}(\mathbb{R})$ *with* $\varphi' \in C_0^{\infty}(\mathbb{R})$. *Then,*

$$\int_{\mathbb{R}} \varphi f \, D^{\alpha+1}\partial_x f \, \mathrm{d}x = \left(\frac{\alpha+2}{4}\right) \int_{\mathbb{R}} (|D^{\frac{\alpha+1}{2}} f|^2 + |D^{\frac{\alpha+1}{2}} \mathcal{H} f|^2)\varphi' \, \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} f \, R_0(\alpha+2) f \, \mathrm{d}x. \quad (3.29)$$

Proof. The proof follows the ideas presented in Proposition 2.12 in [Linares et al. 2014]. □

4. The linear problem

The aim of this section is to obtain Strichartz estimates associated to solutions of the IVP (1.1).

First, consider the linear problem

$$\begin{cases} \partial_t u - D_x^{\alpha+1}\partial_x u = 0, & x, t \in \mathbb{R}, \ 0 < \alpha < 1, \\ u(x,0) = u_0(x), \end{cases} \quad (4.1)$$

whose solution is given by

$$u(x,t) = S(t)u_0 = (\mathrm{e}^{it|\xi|^{\alpha+1}\xi} \, \hat{u}_0)^{\vee}. \quad (4.2)$$

We begin studying estimates of the unitary group obtained in (4.2).

Proposition 4.3. *Assume that* $0 < \alpha < 1$. *Let* q, p *satisfy* $\frac{2}{q} + \frac{1}{p} = \frac{1}{2}$ *with* $2 \leq p \leq \infty$.

Then

$$\|D_x^{\frac{\alpha}{q}} S(t)u_0\|_{L_t^q L_x^p} \lesssim \|u_0\|_{L_x^2} \quad (4.4)$$

for all $u_0 \in L^2(\mathbb{R})$.

Proof. The proof follows as an application of Theorem 2.1 in [Kenig et al. 1991a]. □

Remark 4.5. Notice that the condition on p implies $q \in [4,\infty]$, which in one of the extremal cases $(p,q) = (\infty, 4)$ yields

$$\|D_x^{\frac{\alpha}{4}} S(t)u_0\|_{L_t^4 L_x^{\infty}} \lesssim \|u_0\|_{L_x^2},$$

which shows the gain of $\frac{\alpha}{4}$ derivatives globally in time for solutions of (4.1).

Lemma 4.6. *Assume that* $0 < \alpha < 1$. *Let* ψ_k *be a* $C^{\infty}(\mathbb{R})$ *function supported in the interval* $[2^{k-1}, 2^{k+1}]$, *where* $k \in \mathbb{Z}^+$. *Then, the function* H_k^{α} *defined as*

$$H_k^{\alpha}(x) = \begin{cases} 2^k & \text{if } |x| \leq 1, \\ 2^{\frac{k}{2}}|x|^{-\frac{1}{2}} & \text{if } 1 \leq |x| \leq c2^{k(\alpha+1)}, \\ (1+x^2)^{-1} & \text{if } |x| > c2^{k(\alpha+1)} \end{cases}$$

satisfies

$$\left|\int_{-\infty}^{\infty} e^{i(t\xi|\xi|^{\alpha+1}+x\xi)}\,\psi_k(\xi)\,d\xi\right| \lesssim H_k^\alpha(x) \tag{4.7}$$

for $|t| \le 2$, *where the constant* c *does not depend on* t *or* k.

Moreover, we have

$$\sum_{l=-\infty}^{\infty} H_k^\alpha(|l|) \lesssim 2^{k(\frac{\alpha+1}{2})}. \tag{4.8}$$

Proof. The proof of estimate (4.7) is given in [Kenig et al. 1991b, Proposition 2.6] and it uses arguments of localization and the classical Van der Corput lemma. Meanwhile, (4.8) follows exactly that of Lemma 2.6 in [Linares et al. 2014]. □

Theorem 4.9. *Assume* $0 < \alpha < 1$. *Let* $s > \frac{1}{2}$. *Then,*

$$\|S(t)u_0\|_{L_x^2L_t^\infty([-1,1])} \le \left(\sum_{j=-\infty}^{\infty} \sup_{|t|\le 1}\ \sup_{j\le x<j+1} |S(t)u_0(x)|^2\right)^{\frac{1}{2}} \lesssim \|u_0\|_{H_x^s}$$

for any $u_0 \in H^s(\mathbb{R})$.

Proof. See Theorem 2.7 in [Kenig et al. 1991b]. □

Next, we recall a maximal function estimate proved by Kenig, Ponce and Vega [Kenig et al. 1991b].

Corollary 4.10. *Assume that* $0 < \alpha < 1$. *Then, for any* $s > \frac{1}{2}$ *and any* $\eta > \frac{3}{4}$

$$\left(\sum_{j=-\infty}^{\infty} \sup_{|t|\le T}\ \sup_{j\le x<j+1} |S(t)v_0|^2\right)^{\frac{1}{2}} \lesssim (1+T)^\eta \|v_0\|_{H_x^s}.$$

Proof. See Corollary 2.8 in [Kenig et al. 1991b]. □

4A. *The nonlinear problem.* This section is devoted to studying general properties of solutions of the nonlinear problem

$$\begin{cases} \partial_t u - D_x^{\alpha+1}\partial_x u + u\,\partial_x u = 0, & x, t \in \mathbb{R},\ 0 < \alpha < 1, \\ u(x,0) = u_0(x). \end{cases} \tag{4.11}$$

We begin this section by stating the following local existence theorem proved in [Kato 1975; Saut and Temam 1976].

Theorem 4.12. (1) *For any* $u_0 \in H^s(\mathbb{R})$ *with* $s > \frac{3}{2}$ *there exists a unique solution* u *to* (4.11) *in the class* $C([-T,T] : H^s(\mathbb{R}))$ *with* $T = T(\|u_0\|_{H^s}) > 0$.

(2) *For any* $T' < T$ *there exists a neighborhood* V *of* u_0 *in* $H^s(\mathbb{R})$ *such that the map* $\tilde{u}_0 \mapsto \tilde{u}(t)$ *from* V *into* $C([-T',T'] : H^s(\mathbb{R}))$ *is continuous.*

(3) *If* $u_0 \in H^{s'}(\mathbb{R})$ *with* $s' > s$, *then the time of existence* T *can be taken to depend only on* $\|u_0\|_{H^s}$.

Our first goal will be to obtain some energy estimates satisfied by smooth solutions of the IVP (4.11). We firstly present a result that arises as a consequence of commutator estimates.

Lemma 4.13. *Suppose that* $0 < \alpha < 1$. *Let* $u \in C([0,T] : H^\infty(\mathbb{R}))$ *be a smooth solution of* (4.11). *If* $s > 0$ *is given*, *then*

$$\|u\|_{L^\infty_T H^s_x} \lesssim \|u_0\|_{H^s_x} \, \mathrm{e}^{c\|\partial_x u\|_{L^1_T L^\infty_x}}. \tag{4.14}$$

Proof. Let $s > 0$. By a standard energy estimate argument we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} (J^s_x u)^2 \, \mathrm{d}x + \int_{\mathbb{R}} [J^s_x; u]\partial_x u \, J^s_x u \, \mathrm{d}x + \int_{\mathbb{R}} u \, J^s_x u \, J^s_x \partial_x u \, \mathrm{d}x = 0.$$

Hence integration by parts, Gronwall's inequality and the commutator estimate (3.11) lead to (4.14). □

Remark 4.15. In view of the energy estimate (4.14), the key point to obtaining a priori estimates in $H^s_x(\mathbb{R})$ is to control $\|\partial_x u\|_{L^1_T L^\infty_x}$ at the $H^s_x(\mathbb{R})$-level.

In addition to this estimate, we will present the smoothing effect provided by solutions of the dispersive generalized Benjamin–Ono equation. In fact, the smoothing effect was first observed by Kato [1983] in the context of the Korteweg–de Vries equation. Following Kato's approach joint with the commutator expansions, we present a result proved by Kenig, Ponce and Vega [Kenig et al. 1991b, Lemma 5.1].

Proposition 4.16. *Let* φ *denote a nondecreasing smooth function such that* $\operatorname{supp}\varphi' \subset (-1,2)$ *and* $\varphi'|_{[0,1)} = 1$. *For* $j \in \mathbb{Z}$, *we define* $\varphi_j(\cdot) = \varphi(\cdot - j)$. *Let* $u \in C([0,T] : H^\infty(\mathbb{R}))$ *be a real smooth solution of* (1.1) *with* $0 < \alpha < 1$. *Assume also that* $s \geq 0$ *and* $r > \frac{1}{2}$. *Then*,

$$\left(\int_0^T \int_{\mathbb{R}} \left(|D^{s+\frac{\alpha+1}{2}}_x u(x,t)|^2 + |D^{s+\frac{\alpha+1}{2}}_x \mathcal{H}u(x,t)|^2\right)\varphi'_j(x) \, \mathrm{d}x \, \mathrm{d}t\right)^{\frac{1}{2}} \lesssim (1 + T + \|\partial_x u\|_{L^1_T L^\infty_x} + T\|u\|_{L^\infty_T H^r_x})^{\frac{1}{2}} \|u\|_{L^\infty_T H^s_x}. \tag{4.17}$$

In addition to the smoothing effect presented above, we will need the following localized version of the $H^s(\mathbb{R})$-norm. For this purpose we will consider a cutoff function φ with the same characteristics as those in Proposition 4.16.

Proposition 4.18. *Let* $s \geq 0$. *Then*, *for any* $f \in H^s(\mathbb{R})$

$$\|f\|_{H^s(\mathbb{R})} \sim \left(\sum_{j=-\infty}^{\infty} \|f\varphi'_j\|^2_{H^s(\mathbb{R})}\right)^{\frac{1}{2}}.$$

Hence our first goal in establishing the local well-posedness of (4.11) is to obtain Strichartz estimates associated to solutions of

$$\partial_t u - D^{1+\alpha}_x \partial_x u = F. \tag{4.19}$$

Proposition 4.20. *Assume that* $0 < \alpha < 1$, $T > 0$ *and* $\sigma \in [0,1]$. *Let* u *be a smooth solution to* (4.19) *defined on the time interval* $[0,T]$. *Then there exist* $0 \leq \mu_1, \mu_2 < \frac{1}{2}$ *such that*

$$\|\partial_x u\|_{L^2_T L^\infty_x} \lesssim T^{\mu_1} \|J^{1-\frac{\alpha}{4}+\frac{\sigma}{4}+\epsilon} u\|_{L^\infty_T L^2_x} + T^{\mu_2} \|J^{1-\frac{\alpha}{4}-\frac{3\sigma}{4}+\epsilon} F\|_{L^2_T L^2_x} \tag{4.21}$$

for any $\epsilon > 0$.

Remark 4.22. The optimal choice in the parameters present in the estimate (4.21) corresponds to $\sigma = \frac{1-\alpha}{2}$. Indeed, as is pointed out by Kenig and Koenig [2003, Proposition 2.8] in the case of the Benjamin–Ono equation (case $\alpha = 0$) given a linear estimate of the form

$$\|\partial_x u\|_{L^2_T L^\infty_x} \lesssim T^{\mu_1} \|J^a u\|_{L^\infty_T L^2_x} + T^{\mu_2} \|J^b F\|_{L^2_T L^2_x}$$

the idea is to apply the smoothing effect (4.17) and absorb as many as derivatives as possible of the function F. Concerning to our case, the approach requires the choice $a = b + \frac{1-\alpha}{2}$; this particular choice, $\sigma = \frac{1-\alpha}{2}$, in the estimate (4.21) provides the regularity $s > \frac{9}{8} - \frac{3\alpha}{8}$ in Theorem A.

Proof. Let $f = \sum_k f_k$ denote the Littlewood–Paley decomposition of a function f. More precisely we choose functions $\eta, \chi \in C^\infty(\mathbb{R})$ with $\operatorname{supp}(\eta) \subseteq \left\{\xi : \frac{1}{2} < |\xi| < 2\right\}$ and $\operatorname{supp}(\chi) \subseteq \{\xi : |\xi| < 2\}$ such that

$$\sum_{k=1}^{\infty} \eta\left(\frac{\xi}{2^k}\right) + \chi(\xi) = 1$$

and $f_k = P_k(f)$, where $\widehat{(P_0 f)}(\xi) = \chi(\xi)\hat{f}(\xi)$ and $\widehat{(P_k f)}(\xi) = \eta(\xi/2^k)\hat{f}(\xi)$ for all $k \geq 1$.

Fix $\epsilon > 0$. Let $p > \frac{1}{\epsilon}$. By Sobolev embedding and the Littlewood–Paley theorem it follows that

$$\|f\|_{L^\infty_x} \lesssim \|J^\epsilon f\|_{L^p_x} \sim \left\| \left(\sum_{k=0}^{\infty} |J^\epsilon P_k f|^2 \right)^{\frac{1}{2}} \right\|_{L^p_x} = \left\| \sum_{k=0}^{\infty} |J^\epsilon P_k f|^2 \right\|^{\frac{1}{2}}_{L^{p/2}_x} \lesssim \left(\sum_{k=0}^{\infty} \|J^\epsilon P_k f\|^2_{L^p_x} \right)^{\frac{1}{2}}.$$

Therefore, to obtain (4.21) it enough to prove that for $p > 2$

$$\|\partial_x u_k\|_{L^2_T L^p_x} \lesssim \|D_x^{1-\frac{\alpha}{4}+\frac{\sigma}{4}+\frac{\alpha-\sigma}{2p}} u_k\|_{L^\infty_T L^2_x} + \|D_x^{1-\frac{\alpha}{4}-\frac{3\sigma}{4}+\frac{\alpha-\sigma}{2p}} F_k\|_{L^2_T L^2_x}, \quad k \geq 1.$$

The estimate for the case $k = 0$ follows using Hölder's inequality and (4.4). For such reason we fix $k \geq 1$, and at this level of frequencies we have

$$\partial_t u_k - D_x^{\alpha+1} \partial_x u_k = F_k.$$

Consider a partition of the interval $[0, T] = \bigcup_{j \in J} I_j$, where $I_j = [a_j, b_j]$, and $T = b_j$ for some j. Indeed, we choose a quantity $\sim 2^{k\sigma} T^{1-\mu}$ of intervals, with length $|I_j| \sim 2^{-k\sigma} T^\mu$, where μ is a positive number to be fixed.

Let q be such that

$$\frac{2}{q} + \frac{1}{p} = \frac{1}{2}.$$

Using that u solves the integral equation

$$u(t) = S(t)u_0 + \int_0^t S(t-t')F(t')\,\mathrm{d}t', \tag{4.23}$$

we deduce that

$$\|\partial_x u_k\|_{L^2_T L^p_x} \lesssim (T^\mu 2^{-k\sigma})^{\left(\frac{1}{2}-\frac{1}{q}\right)} \left(\sum_{j \in J} \|S(t-a_j)\partial_x u_k(a_j)\|^2_{L^q_{I_j} L^p_x} + \left\| \int_{a_j}^t S(t-s)\partial_x F_k(s)\,\mathrm{d}s \right\|^2_{L^q_{I_j} L^p_x} \right)^{\frac{1}{2}}.$$

In this sense, it follows from (4.4) that

$$
\begin{aligned}
\|\partial_x u_k\|_{L^2_T L^p_x} &\lesssim (T^{\mu}2^{-k\sigma})^{(\frac{1}{2}-\frac{1}{q})}\Big\{\sum_{j\in J}\|D_x^{-\frac{\alpha}{q}}\partial_x u_k(a_j)\|^2_{L^2_x}+\sum_{j\in J}\Big(\int_{I_j}\|D_x^{-\frac{\alpha}{q}}\partial_x F_k(t)\|_{L^2_x}\,\mathrm{d}t\Big)^2\Big\}^{\frac{1}{2}}\\
&\lesssim (T^{\mu}2^{-k\sigma})^{(\frac{1}{2}-\frac{1}{q})}\Big\{\Big(\sum_{j\in J}\|D_x^{1-\frac{\alpha}{q}}u_k\|^2_{L^\infty_T L^2_x}\Big)^{\frac{1}{2}}+\Big(\sum_{j\in J}T^{\mu}2^{-k\sigma}\int_{I_j}\|D_x^{1-\frac{\alpha}{q}}F_k(t)\|^2_{L^2_x}\,\mathrm{d}t\Big)^{\frac{1}{2}}\Big\}\\
&\lesssim (T^{\mu}2^{-k\sigma})^{(\frac{1}{2}-\frac{1}{q})}(T^{1-\mu}2^{k\sigma})^{\frac{1}{2}}\|D_x^{1-\frac{\alpha}{q}}u_k\|_{L^\infty_T L^2_x}\\
&\qquad\qquad + (T^{\mu}2^{-k\sigma})^{(\frac{1}{2}-\frac{1}{q})}(T^{\mu}2^{-k\sigma})^{\frac{1}{2}}\Big(\int_0^T\|D_x^{1-\frac{\alpha}{q}}F_k(t)\|^2_{L^2_x}\,\mathrm{d}t\Big)^{\frac{1}{2}}\\
&\lesssim T^{\frac{1}{2}-\frac{\mu}{q}}\|D_x^{1-\frac{\alpha}{q}+\frac{\sigma}{q}}u_k\|_{L^\infty_T L^2_x}+T^{\mu(1-\frac{1}{q})}\|D_x^{1-\frac{\alpha}{q}+\frac{\sigma}{q}-\sigma}F_k\|_{L^2_T L^2_x},
\end{aligned}
$$

since

$$1-\frac{\alpha}{q}+\frac{\sigma}{q}=1-\frac{\alpha}{4}+\frac{\sigma}{4}+\frac{\alpha-\sigma}{2p}\quad\text{and}\quad 1-\frac{\alpha}{q}+\frac{\sigma}{q}-\sigma=1-\frac{\alpha}{4}-\frac{3\sigma}{4}+\frac{\alpha-\sigma}{2p}.$$

We recall that $\epsilon>\frac{1}{p}$, $\sigma\in[0,1]$ and $\alpha\in(0,1)$; then

$$\epsilon+\frac{\alpha-\sigma}{2p}>\frac{\alpha-\sigma+2}{2p}>0.$$

Next, we choose $\mu_1=\frac{1}{2}-\frac{\mu}{q}$, $\mu_2=\mu\big(1-\frac{1}{q}\big)$ with the particular choice $\mu=\frac{1}{2}$.

Gathering the inequalities above, the proposition follows. $\square$

Now we turn our attention to the proof of Theorem A. Our starting point will be the energy estimate (4.14), where, as was remarked above, the key point is to establish a priori control of $\|\partial_x u\|_{L^1_T L^\infty_x}$.

5. Proof of Theorem A

5A. ***A priori estimates.*** First notice that by scaling, it is enough to deal with small initial data in the H^s-norm. Indeed, if $u(x,t)$ is a solution of (1.1) defined on a time interval $[0,T]$, for some positive time T, then, for all $\lambda>0$, $u_\lambda(x,t)=\lambda^{1+\alpha}u(\lambda x,\lambda^{2+\alpha}t)$ is also solution with initial data $u_{0,\lambda}(x)=\lambda^{1+\alpha}u_0(\lambda x)$, and time interval $[0,T/\lambda^{2+\alpha}]$.

For any $\delta>0$, we define $B_\delta(0)$ as the ball with center at the origin in $H^s(\mathbb{R})$ and radius δ.

Since

$$\|u_{0,\lambda}\|_{L^2_x}=\lambda^{\frac{1+2\alpha}{2}}\|u_0\|_{L^2_x}\quad\text{and}\quad\|D^s_x u_{0,\lambda}\|_{L^2_x}=\lambda^{\frac{1}{2}+\alpha+s}\|D^s_x u_0\|_{L^2_x},$$

we have

$$\|u_{0,\lambda}\|_{H^s_x}\lesssim\lambda^{\frac{1}{2}+\alpha}(1+\lambda^s)\|u_0\|_{H^s_x},$$

so we can force $u_\lambda(\cdot,0)$ to belong to the ball $B_\delta(0)$ by choosing the parameter λ with the condition

$$\lambda\sim\min\{\delta^{\frac{2}{1+2\alpha}}\|u_0\|_{H^s_x}^{-\frac{2}{1+2\alpha}},1\}.$$

Thus, the existence and uniqueness of a solution to (1.1) on the time interval $[0,1]$ for small initial data $\|u_0\|_{H^s_x}$ will ensure the existence and uniqueness of a solution to (1.1) for arbitrary large initial data on a time interval $[0,T]$ with

$$T\sim\min\{1,\|u_0\|_{H^s_x}^{-\frac{2(2+\alpha)}{1+2\alpha}}\}.$$

Thus, without loss of generality we will assume that $T \leq 1$ and that

$$\Lambda := \|u_0\|_{L^2} + \|D^s u_0\|_{L^2} \leq \delta,$$

where δ is a small positive number to be fixed later.

We fix s such that $s(\alpha) = \frac{9}{8} - \frac{3\alpha}{8} < s < \frac{3}{2} - \frac{\alpha}{2}$ and set $\epsilon = s - s(\alpha) > 0$.

Next, taking $\sigma = \frac{1-\alpha}{2} > 0$, $F = -u\,\partial_x u$ in (4.21) together with (4.14) yields

$$\begin{aligned} \|\partial_x u\|_{L^2_T L^\infty_x} &\lesssim T^{\mu_1} \|J^s u\|_{L^\infty_T L^2_x} + T^{\mu_2} \|J^{1-\frac{\alpha}{4}-\frac{3\sigma}{4}+\epsilon}(u\partial_x u)\|_{L^2_T L^2_x} \\ &\lesssim \Lambda + \Lambda\, \mathrm{e}^{c\|\partial_x u\|_{L^2_T L^\infty_x}} + \|D_x^{s+\frac{\alpha-1}{2}}(u\partial_x u)\|_{L^2_T L^2_x}. \end{aligned} \tag{5.1}$$

Now, to analyze the product coming from the nonlinear term we use the Leibniz rule for fractional derivatives (3.6) together with the energy estimate (4.14) as follows:

$$\begin{aligned} \|D_x^{s+\frac{\alpha-1}{2}}(u\partial_x u)\|_{L^2_T L^2_x} &\lesssim \|u D_x^{s+\frac{\alpha-1}{2}}\partial_x u\|_{L^2_T L^2_x} + \big\| \|\partial_x u(t)\|_{L^\infty_x} \|D_x^{s+\frac{\alpha-1}{2}} u(t)\|_{L^2_x} \big\|_{L^2_T} \\ &\lesssim \|u D_x^{s+\frac{\alpha-1}{2}}\partial_x u\|_{L^2_T L^2_x} + \Lambda \|\partial_x u\|_{L^2_T L^\infty_x}\, \mathrm{e}^{c\|\partial_x u\|_{L^2_T L^\infty_x}}. \end{aligned} \tag{5.2}$$

To handle the first term in the right-hand side above, we incorporate Kato's smoothing effect estimate obtained in (4.17) in the following way:

$$\begin{aligned} \|u D_x^{s+\frac{\alpha-1}{2}}\partial_x u\|_{L^2_T L^2_x} &\leq \bigg(\sum_{j=-\infty}^{\infty} \int_0^T \|u(t)\|^2_{L^\infty_{[j,j+1)}} \|D_x^{s+\frac{\alpha+1}{2}} \mathcal{H} u(t)\|^2_{L^2_{[j,j+1)}}\,\mathrm{d}t \bigg)^{\frac{1}{2}} \\ &\lesssim \bigg(\sum_{j=-\infty}^{\infty} \|u\|^2_{L^\infty_T L^\infty_{[j,j+1)}} \bigg)^{\frac{1}{2}} (1+\Lambda)\Lambda\, \mathrm{e}^{c\|\partial_x u\|_{L^2_T L^\infty_x}}. \end{aligned} \tag{5.3}$$

In summary, gathering the estimates (5.1)–(5.3) yields

$$\|\partial_x u\|_{L^2_T L^\infty_x} \lesssim \Lambda(1+\Lambda)\, \mathrm{e}^{c\|\partial_x u\|_{L^2_T L^\infty_x}} \bigg(\sum_{j=-\infty}^{\infty} \|u\|^2_{L^\infty_T L^\infty_{[j,j+1)}} \bigg)^{\frac{1}{2}} + \Lambda + \Lambda\, \mathrm{e}^{c\|\partial_x u\|_{L^2_T L^\infty_x}}. \tag{5.4}$$

Since u is a solution to (4.11), by Duhamel's formula it follows that

$$u(t) = S(t)u_0 - \int_0^t S(t-s)(u\,\partial_x u)(s)\,\mathrm{d}s,$$

where $S(t) = \mathrm{e}^{tD_x^{\alpha+1}\partial_x}$.

Now, we fix $\eta > 0$ such that $\eta < \frac{1+\alpha}{8}$; this choice implies that $\eta + \frac{1}{2} < s + \frac{\alpha-1}{2}$. Hence, Sobolev's embedding, Hölder's inequality and Corollary 4.10 produce

$$\begin{aligned} \bigg(\sum_{j=-\infty}^{\infty} \|u\|^2_{L^\infty_T L^\infty_{[j,j+1)}} \bigg)^{\frac{1}{2}} &\lesssim \bigg(\sum_{j=-\infty}^{\infty} \|S(t)u_0\|^2_{L^\infty_T L^\infty_{[j,j+1)}} \bigg)^{\frac{1}{2}} + \bigg(\sum_{j=-\infty}^{\infty} \bigg\| \int_0^t S(t-s)(u\partial_x u)(s)\,\mathrm{d}s \bigg\|^2_{L^\infty_T L^\infty_{[j,j+1)}} \bigg)^{\frac{1}{2}} \\ &\lesssim (1+T)\Lambda + (1+T)\|u\partial_x u\|_{L^1_T H^{\eta+1/2}_x} \\ &\lesssim \Lambda + \|u\partial_x u\|_{L^1_T L^2_x} + \|D_x^{\eta+\frac{1}{2}}(u\,\partial_x u)\|_{L^1_T L^2_x} \\ &\lesssim \Lambda + \Lambda\|\partial_x u\|_{L^2_T L^\infty_x} + \|D_x^{\eta+\frac{1}{2}}(u\,\partial_x u)\|_{L^2_T L^2_x}. \end{aligned} \tag{5.5}$$

Employing an argument similar to the one applied in (5.2) and (5.4) it is possible to bound the last term in the right-hand side as follows:

$$\|D_x^{\eta+\frac{1}{2}}(u\partial_x u)\|_{L^2_T L^2_x} \lesssim \bigg(\sum_{j=-\infty}^{\infty} \|u\|^2_{L^\infty_T L^\infty_{[j,j+1)}}\bigg)^{\frac{1}{2}} \Lambda(\Lambda+1)\, \mathrm{e}^{c\|\partial_x u\|_{L^2_T L^\infty_x}} + \Lambda\, \mathrm{e}^{c\|\partial_x u\|_{L^2_T L^\infty_x}}. \tag{5.6}$$

Next, we define

$$\phi(T) = \bigg(\int_0^T \|\partial_x u(s)\|^2_{L^\infty_x}\,\mathrm{d}s\bigg)^{\frac{1}{2}} + \bigg(\sum_{j=-\infty}^{\infty} \|u\|^2_{L^\infty_T L^\infty_{[j,j+1)}}\bigg)^{\frac{1}{2}},$$

which is a continuous, nondecreasing function of T.

From (5.4), (5.5) and (5.6) it follows that

$$\phi(T) \lesssim \Lambda(\Lambda+1)\phi(T)\,\mathrm{e}^{c\,\phi(T)} + \Lambda\,\mathrm{e}^{c\,\phi(T)}\,\phi(T) + \Lambda\,\mathrm{e}^{c\,\phi(T)} + \Lambda + \Lambda\phi(T).$$

Now, if we suppose that $\Lambda \le \delta \le 1$, we obtain

$$\phi(T) \le c\Lambda + c\Lambda\,\mathrm{e}^{c\phi(T)}$$

for some constant $c > 0$.

To complete the proof we will show that if there exists $\delta > 0$ such that $\Lambda \le \delta$, then $\phi(1) \le A$ for some constant $A > 0$.

To do this, we define the function

$$\Psi(x, y) = x - cy - cy\,\mathrm{e}^{cx}. \tag{5.7}$$

First notice that $\Psi(0,0) = 0$ and $\partial_x\Psi(0,0) = 1$. Then the implicit function theorem asserts that there exists $\delta > 0$ and a smooth function $\xi(y)$ such that $\xi(0) = 0$, and $\Psi(\xi(y), y) = 0$ for $|y| \le \delta$.

Notice that the condition $\Psi(\xi(y), y) = 0$ implies that $\xi(y) > 0$ for $y > 0$. Moreover, since $\partial_x\Psi(0,0) = 1$, the function $\Psi(\,\cdot\,, y)$ is increasing close to $\xi(y)$ whenever δ is chosen sufficiently small.

Let us suppose that $\Lambda \le \delta$, and set $\lambda = \xi(\Lambda)$. Then, combining interpolation and Proposition 4.18 we obtain

$$\phi(0) = \bigg(\sum_{j=-\infty}^{\infty}\Big(\sup_{x\in[j,j+1)} |u(x,0)|\Big)^2\bigg)^{\frac{1}{2}} \lesssim \|u_0\|_{H^s(\mathbb{R})} \le c_1\|u_0\|_{L^2} + c_1\|D_x^s u_0\|_{L^2_x},$$

where we take $c > c_1$.

Therefore

$$\phi(0) \le c_1\Lambda < c\Lambda + c\Lambda\,\mathrm{e}^{c\xi(\Lambda)} = \lambda.$$

Suppose that $\phi(T) > \lambda$ for some $T \in (0,1)$ and define

$$T_0 = \inf\{T \in (0,1) \mid \phi(T) > \lambda\}.$$

Hence, $T_0 > 0$ and $\phi(T_0) = \lambda$; additionally, there exists a decreasing sequence $\{T_n\}_{n\ge 1}$ converging to T_0 such that $\phi(T_n) > \lambda$. In addition, notice that (5.7) implies $\Psi(\phi(T), \Lambda) \le 0$ for all $T \in [0,1]$.

Since the function $\Psi(\,\cdot\,, \Lambda)$ is increasing near λ, we have

$$\Psi(\phi(T_n), \Lambda) > \Psi(\phi(T_0), \Lambda) = \Psi(\lambda, \Lambda) = \Psi(\xi(\Lambda), \Lambda) = 0$$

for n sufficiently large.

This is a contradiction with the fact that $\phi(T) > \lambda$. So we conclude $\phi(T) \le A$ for all $T \in (0, 1)$, as was claimed. Thus, $\phi(1) \le A$.

In conclusion we have proved that

$$\phi(T) = \left(\int_0^T \|\partial_x u(s)\|_{L_x^\infty}^2 \, \mathrm{d}s\right)^{\frac{1}{2}} + \left(\sum_{j=-\infty}^{\infty} \|u\|_{L_T^\infty L_{[j,j+1)}^\infty}^2\right)^{\frac{1}{2}} \lesssim \|u_0\|_{H_x^s} \quad \text{for all } T \in [0, 1]. \tag{5.8}$$

At this stage, the existence, uniqueness, and continuous dependence on the initial data follows from the standard compactness and Bona–Smith approximation arguments; see for example [Kenig et al. 1991b; Ponce 1991].

6. Proof of Theorem B

The aim of this section is to prove Theorem B. To achieve this goal is necessary to take into account two important aspects of our analysis: first, the ambient space, which in our case is the Sobolev space where the theorem is valid together with the properties satisfied by the real solutions of the dispersive generalized Benjamin–Ono equation; and second, the auxiliary weight functions involved in the energy estimates, which we will describe in detail.

The following is a summary of the local well-posedness and Kato's smoothing effect presented in the previous sections.

Theorem C. *If* $u_0 \in H^s(\mathbb{R})$, $s \ge \frac{3-\alpha}{2}$, $\alpha \in (0, 1)$, *then there exist a positive time* $T = T(\|u_0\|_{H^s}) > 0$ *and a unique solution of the IVP* (1.1) *such that*

(a) $u \in C([-T, T] : H^s(\mathbb{R}))$,

(b) (*Strichartz*) $\partial_x u \in L^1([-T, T] : L^\infty(\mathbb{R}))$,

(c) (*smoothing effect*) *for* $R > 0$,

$$\int_{-T}^{T}\int_{-R}^{R}\left(|\partial_x D_x^{r+\frac{\alpha+1}{2}} u|^2 + |\mathcal{H}\partial_x D_x^{r+\frac{\alpha+1}{2}} u|^2\right) \mathrm{d}x \, \mathrm{d}t \le C \tag{6.1}$$

with $r \in \left(\frac{9-3\alpha}{8}, s\right]$ *and* $C = C(\alpha; R; T; \|u_0\|_{H_x^s}) > 0$.

Since we have set the Sobolev space where we will work, the next step is the description of the cutoff functions to be used in the proof.

In this part we consider families of cutoff functions that will be used systematically in the proof of Theorem B. This collection of weight functions was constructed originally in [Isaza et al. 2015; Kenig et al. 2018] in the proof of Theorem 1.3.

More precisely, for $\epsilon > 0$ and $b \ge 5\epsilon$ define the families of functions

$$\chi_{\epsilon,b},\ \phi_{\epsilon,b},\ \tilde{\phi}_{\epsilon,b},\ \psi_\epsilon,\ \eta_{\epsilon,b} \in C^\infty(\mathbb{R})$$

satisfying the following properties:

(1) $\chi'_{\epsilon,b} \ge 0$.

(2) $\chi_{\epsilon,b}(x) = 0$ if $x \le \epsilon$, and $\chi_{\epsilon,b}(x) = 1$ if $x \ge b$.

(3) $\operatorname{supp}(\chi_{\epsilon,b}) \subseteq [\epsilon, \infty)$.

(4) $\chi'_{\epsilon,b}(x) \geq \frac{1}{10(b-\epsilon)} \mathbb{1}_{[2\epsilon, b-2\epsilon]}(x)$.

(5) $\operatorname{supp}(\chi'_{\epsilon,b}) \subseteq [\epsilon, b]$.

(6) There exists a real number c_j such that

$$|\chi^{(j)}_{\epsilon,b}(x)| \leq c_j \chi'_{\frac{\epsilon}{3}, b+\epsilon}(x) \quad \text{for all } x \in \mathbb{R},\ j \in \mathbb{Z}^+.$$

(7) For $x \in (3\epsilon, \infty)$

$$\chi_{\epsilon,b}(x) \geq \frac{1}{2} \frac{\epsilon}{b-3\epsilon}.$$

(8) For $x \in \mathbb{R}$

$$\chi'_{\frac{\epsilon}{3}, b+\epsilon}(x) \leq \frac{\epsilon}{b-3\epsilon}.$$

(9) Given $\epsilon > 0$ and $b \geq 5\epsilon$ there exist $c_1, c_2 > 0$ such that

$$\chi'_{\epsilon,b}(x) \leq c_1 \chi'_{\frac{\epsilon}{3}, b+\epsilon}(x) \chi_{\frac{\epsilon}{3}, b+\epsilon}(x),$$
$$\chi'_{\epsilon,b}(x) \leq c_2 \chi_{\frac{\epsilon}{5}, \epsilon}(x).$$

(10) For $\epsilon > 0$ given and $b \geq 5\epsilon$, we define the function

$$\eta_{\epsilon,b} = \sqrt{\chi_{\epsilon,b} \chi'_{\epsilon,b}}.$$

(11) $\operatorname{supp}(\phi_{\epsilon,b}),\ \operatorname{supp}(\tilde{\phi}_{\epsilon,b}) \subset \left[\frac{\epsilon}{4}, b\right]$.

(12) $\phi_\epsilon(x) = \tilde{\phi}_{\epsilon,b}(x) = 1,\ x \in \left[\frac{\epsilon}{2}, \epsilon\right]$.

(13) $\operatorname{supp}(\psi_\epsilon) \subseteq \left(-\infty, \frac{\epsilon}{2}\right]$.

(14) For $x \in \mathbb{R}$

$$\chi_{\epsilon,b}(x) + \phi_{\epsilon,b}(x) + \psi_\epsilon(x) = 1,$$
$$\chi^2_{\epsilon,b}(x) + \tilde{\phi}^2_{\epsilon,b}(x) + \psi_\epsilon(x) = 1.$$

The family $\{\chi_{\epsilon,b} \mid \epsilon > 0,\ b \geq 5\epsilon\}$ is constructed as follows: let $\rho \in C_0^\infty(\mathbb{R}),\ \rho(x) \geq 0$, even, with $\operatorname{supp}(\rho) \subseteq (-1, 1)$ and $\|\rho\|_{L^1} = 1$.

Then define

$$\nu_{\epsilon,b}(x) = \begin{cases} 0, & x \leq 2\epsilon, \\ \dfrac{x}{b-3\epsilon} - \dfrac{2\epsilon}{b-3\epsilon}, & 2\epsilon \leq x \leq b-\epsilon, \\ 1, & x \geq b-\epsilon, \end{cases}$$

and

$$\chi_{\epsilon,b}(x) = \rho_\epsilon * \nu_{\epsilon,b}(x),$$

where $\rho_\epsilon(x) = \epsilon^{-1} \rho\left(\frac{x}{\epsilon}\right)$.

Now that we have described all the required estimates and tools necessary, we present the proof of our main result.

Proof of Theorem B. Since the argument is translation invariant, without loss of generality we will consider the case $x_0 = 0$.

First, we will describe the formal calculations assuming as much as regularity as possible; later we provide the justification using a limiting process.

The proof will be established by induction; however, in every step of the induction we will subdivide every case into steps, due to the nonlocal nature of the operator involving the dispersive part in the equation in (1.1).

Case $j = 1$. *Step 1*: First we apply one spatial derivative to the equation in (1.1); after that we multiply by $\partial_x u(x,t)\chi^2_{\epsilon,b}(x+vt)$, and finally we integrate in the x-variable to obtain the identity

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}(\partial_x u)^2\chi^2_{\epsilon,b}\,\mathrm{d}x \underbrace{-\frac{v}{2}\int_{\mathbb{R}}(\partial_x u)^2(\chi^2_{\epsilon,b})'\,\mathrm{d}x}_{A_1(t)} \underbrace{-\int_{\mathbb{R}}(\partial_x D_x^{\alpha+1}\partial_x u)\,\partial_x u\,\chi^2_{\epsilon,b}\,\mathrm{d}x}_{A_2(t)} + \underbrace{\int_{\mathbb{R}}\partial_x(u\partial_x u)\,\partial_x u\,\chi^2_{\epsilon,b}\,\mathrm{d}x}_{A_3(t)} = 0.$$

Step 1.1: Combining the local theory we obtain the following

$$\int_0^T |A_1(t)|\,\mathrm{d}t \le \frac{v}{2}\int_0^T\int_{\mathbb{R}}(\partial_x u)^2(\chi^2_{\epsilon,b})'\,\mathrm{d}x\,\mathrm{d}t \lesssim \|u\|_{L^\infty_T H_x^{(3-\alpha)/2}}.$$

Step 1.2: Integration by parts and Plancherel's identity allow us to rewrite the term A_2 as

$$A_2(t) = \frac{1}{2}\int_{\mathbb{R}}\partial_x u[D_x^{\alpha+1}\partial_x;\,\chi^2_{\epsilon,b}]\partial_x u\,\mathrm{d}x = -\frac{1}{2}\int_{\mathbb{R}}\partial_x u[\mathcal{H}D_x^{\alpha+2};\,\chi^2_{\epsilon,b}]\partial_x u\,\mathrm{d}x. \tag{6.2}$$

Since $\alpha + 2 > 1$, we have by (3.20) that the commutator $[\mathcal{H}D_x^{\alpha+2};\,\chi^2_{\epsilon,b}]$ can be decomposed as

$$[\mathcal{H}D_x^{\alpha+2};\,\chi^2_{\epsilon,b}] = -\tfrac{1}{2}P_n(\alpha+2) + \tfrac{1}{2}\mathcal{H}P_n(\alpha+2)\mathcal{H} - R_n(\alpha+2) \tag{6.3}$$

for some positive integer n, which will be fixed later.

Inserting (6.3) into (6.2)

$$\begin{aligned} A_2(t) &= \frac{1}{2}\int_{\mathbb{R}}\partial_x u\,R_n(\alpha+2)\partial_x u\,\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}}\partial_x u\,P_n(\alpha+2)\partial_x u\,\mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}}\partial_x u\,\mathcal{H}P_n(\alpha+2)\mathcal{H}\partial_x u\,\mathrm{d}x \\ &= A_{2,1}(t) + A_{2,2}(t) + A_{2,3}(t). \end{aligned}$$

Now, we proceed to fix the value of n present in the terms $A_{2,1}$, $A_{2,2}$ and $A_{2,3}$, according to a determinate condition.

First, notice that

$$A_{2,1}(t) = \frac{1}{2}\int_{\mathbb{R}} D_x\mathcal{H}u R_n(\alpha+2)D_x\mathcal{H}u\,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}}\mathcal{H}u D_x\{R_n(\alpha+2)D_x\mathcal{H}u\}\,\mathrm{d}x.$$

Then we fix n such that $2n+1 \le a + 2\sigma \le 2n+3$, which according to the case we are studying ($j = 1$), corresponds to $a = \alpha + 2$ and $\sigma = 1$. This produces $n = 1$.

For this n in particular we have by Proposition 3.25 that $R_1(\alpha+2)$ maps L^2_x into L^2_x.

Hence,

$$A_{2,1}(t) \lesssim \|\mathcal{H}u(t)\|^2_{L^2_x}\|\widehat{D_x^{4+\alpha}\chi^2_{\epsilon,b}}\|_{L^1_\xi} = c\|u_0\|^2_{L^2_x}\|\widehat{D_x^{4+\alpha}\chi^2_{\epsilon,b}}\|_{L^1_\xi},$$

which after integrating in time yields

$$\int_0^T |A_{2,1}(t)|\,\mathrm{d}t \lesssim \|u_0\|_{L^2_x}^2 \sup_{0\le t\le T} \|\widehat{D_x^{4+\alpha}\chi_{\epsilon,b}^2}\|_{L^1_\xi}.$$

Next, we turn our attention to $A_{2,2}$. Replacing $P_1(\alpha+2)$ into $A_{2,2}$

$$\begin{aligned} A_{2,2}(t) &= \tilde{c}_1 \int_{\mathbb{R}} (D_x^{\frac{\alpha+1}{2}} \partial_x u)^2 (\chi_{\epsilon,b}^2)'\,\mathrm{d}x - \tilde{c}_3 \int_{\mathbb{R}} (D_x^{\frac{\alpha+1}{2}} \mathcal{H}u)^2 (\chi_{\epsilon,b}^2)'''\,\mathrm{d}x \\ &= A_{2,2,1}(t) + A_{2,2,2}(t). \end{aligned}$$

We shall underline that $A_{2,2,1}(t)$ is positive; additionally it represents explicitly the smoothing effect for the case $j=1$.

Regarding $A_{2,2,2}$, the local theory combined with interpolation leads to

$$\int_0^T |A_{2,2,2}(t)|\,\mathrm{d}t \lesssim \|u\|_{L^\infty_T H_x^{(3-\alpha)/2}}. \tag{6.4}$$

After substituting (3.21) into $A_{2,3}$ and using the fact that the Hilbert transform is skew-symmetric

$$A_{2,3}(t) = \tilde{c}_1 \int_{\mathbb{R}} (D_x^{1+\frac{\alpha+1}{2}} u)^2 (\chi_{\epsilon,b}^2)'\,\mathrm{d}x - \tilde{c}_3 \int_{\mathbb{R}} (\mathcal{H} D_x^{\frac{\alpha+1}{2}} u)^2 (\chi_{\epsilon,b}^2)'''\,\mathrm{d}x = A_{2,3,1}(t) + A_{2,3,2}(t).$$

Notice that the term $A_{2,3,1}$ is positive and represents the smoothing effect. In contrast, the term $A_{2,3,2}$ is estimated as we did with $A_{2,2,2}$ in (6.4). So, after integration in the time variable

$$\int_0^T |A_{2,3,2}(t)|\,\mathrm{d}t \lesssim \|u\|_{L^\infty_T H_x^{(3-\alpha)/2}}.$$

Finally, after apply integration by parts

$$A_3(t) = \frac{1}{2} \int_{\mathbb{R}} \partial_x u\,(\partial_x u)^2 \chi_{\epsilon,b}^2\,\mathrm{d}x - \frac{1}{2} \int_{\mathbb{R}} u (\partial_x u)^2 (\chi_{\epsilon,b}^2)'\,\mathrm{d}x = A_{3,1}(t) + A_{3,2}(t).$$

On one hand,

$$|A_{3,1}(t)| \lesssim \|\partial_x u(t)\|_{L^\infty_x} \int_{\mathbb{R}} (\partial_x u)^2 \chi_{\epsilon,b}^2\,\mathrm{d}x,$$

where the integral expression on the right-hand side is the quantity to be estimated by means of Gronwall's inequality.

On the other hand,

$$|A_{3,2}(t) \lesssim \|u(t)\|_{L^\infty_x} \int_0^T (\partial_x u)^2 (\chi_{\epsilon,b}^2)'\,\mathrm{d}x.$$

By Sobolev embedding we have after integrating in time

$$\int_0^T |A_{3,2}(t)|\,\mathrm{d}t \lesssim \big(\sup_{0\le t\le T} \|u(t)\|_{H_x^{s(\alpha)+}} \big) \int_0^T \int_{\mathbb{R}} (\partial_x u)^2 (\chi_{\epsilon,b}^2)'\,\mathrm{d}x\,\mathrm{d}t \le c.$$

Since $\|\partial_x u\|_{L^1_T L^\infty_x} < \infty$, after gathering all estimates above and applying Gronwall's inequality we obtain

$$\sup_{0\le t\le T} \|\partial_x u \chi_{\epsilon,b}\|^2_{L^2_x} + \|D_x^{\frac{\alpha+1}{2}} \partial_x u \eta_{\epsilon,b}\|^2_{L^2_T L^2_x} + \|D_x^{1+\frac{\alpha+1}{2}} u \eta_{\epsilon,b}\|^2_{L^2_T L^2_x} \le c^*_{1,1}, \tag{6.5}$$

where $c^*_{1,1} = c^*_{1,1}(\alpha; \epsilon; T; \|u_0\|_{H_x^{(3-\alpha)/2}}; \|\partial_x u_0 \chi_{\epsilon,b}\|_{L^2_x}) > 0$ for any $\epsilon > 0$, $b \ge 5\epsilon$ and $v \ge 0$.

This estimate finishes step 1 corresponding to the case $j = 1$.

The local smoothing effect obtained above is just $\frac{1+\alpha}{2}$ derivatives; see [Isaza et al. 2016a]. So, the iterative argument is carried out in two steps, the first step for positive integers m and the second one for $m + \frac{1-\alpha}{2}$.

Step 2: After applying the operator $D_x^{(1-\alpha)/2}\partial_x$ to the equation in (1.1) and multiplying the result by $D_x^{(1-\alpha)/2}\partial_x u \chi^2_{\epsilon,b}(x+vt)$ one gets

$$D_x^{\frac{1-\alpha}{2}} \partial_x \partial_t u D_x^{\frac{1-\alpha}{2}} \partial_x u \chi^2_{\epsilon,b} - D_x^{\frac{1-\alpha}{2}} \partial_x D_x^{1+\alpha} \partial_x u D_x^{\frac{1-\alpha}{2}} \partial_x u \chi^2_{\epsilon,b} + D_x^{\frac{1-\alpha}{2}} \partial_x (u \partial_x u) D_x^{\frac{1-\alpha}{2}} \partial_x u \chi^2_{\epsilon,b} = 0,$$

which after integrating in the spatial variable becomes

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} (D_x^{\frac{1-\alpha}{2}} \partial_x u)^2 \chi^2_{\epsilon,b}\, \mathrm{d}x \underbrace{-v \int_{\mathbb{R}} (D_x^{\frac{1-\alpha}{2}} \partial_x u)^2 (\chi^2_{\epsilon,b})'\, \mathrm{d}x}_{A_1(t)} \\ \underbrace{-\int_{\mathbb{R}} (D_x^{\frac{1-\alpha}{2}} \partial_x D_x^{1+\alpha} \partial_x u)\, D_x^{\frac{1-\alpha}{2}} \partial_x u \chi^2_{\epsilon,b}\, \mathrm{d}x}_{A_2(t)} + \underbrace{\int_{\mathbb{R}} D_x^{\frac{1-\alpha}{2}} \partial_x (u\partial_x u) D_x^{\frac{1-\alpha}{2}} \partial_x u \chi^2_{\epsilon,b}\, \mathrm{d}x}_{A_3(t)} = 0. \end{aligned}$$

Step 2.1: First observe that by the local theory

$$\int_0^T |A_1(t)|\, \mathrm{d}t \le |v| \int_0^T \int_{\mathbb{R}} (D_x^{\frac{1-\alpha}{2}} \partial_x u)^2 (\chi^2_{\epsilon,b})'\, \mathrm{d}x\, \mathrm{d}t \lesssim \|u\|_{L^\infty_T H_x^{(3-\alpha)/2}}.$$

Step 2.2: Concerning the term A_2, integration by parts and Plancherel's identity yield

$$A_2(t) = -\frac{1}{2} \int_{\mathbb{R}} D_x^{\frac{1-\alpha}{2}+1} \mathcal{H} u [\mathcal{H} D_x^{2+\alpha}; \chi^2_{\epsilon,b}] D_x^{\frac{1-\alpha}{2}+1} \mathcal{H} u\, \mathrm{d}x. \tag{6.6}$$

Since $2+\alpha > 1$, we have by (3.20) that the commutator $[\mathcal{H} D_x^{\alpha+2}; \chi^2_{\epsilon,b}]$ can be decomposed as

$$[\mathcal{H} D_x^{\alpha+2}; \chi^2_{\epsilon,b}] + \tfrac{1}{2} P_n(\alpha+2) + R_n(\alpha+2) = \tfrac{1}{2} \mathcal{H} P_n(\alpha+2) \mathcal{H} \tag{6.7}$$

for some positive integer n, which as in the previous cases will be fixed suitably.

Substituting (6.7) into (6.6)

$$\begin{aligned} A_2(t) &= \frac{1}{2} \int_{\mathbb{R}} D_x^{\frac{3-\alpha}{2}} \mathcal{H} u (R_n(\alpha+2) D_x^{\frac{3-\alpha}{2}} \mathcal{H} u)\, \mathrm{d}x \\ &\quad + \frac{1}{4} \int_{\mathbb{R}} D_x^{\frac{3-\alpha}{2}} \mathcal{H} u (P_n(\alpha+2) D_x^{\frac{3-\alpha}{2}} \mathcal{H} u)\, \mathrm{d}x - \frac{1}{4} \int_{\mathbb{R}} D_x^{\frac{3-\alpha}{2}} \mathcal{H} u (\mathcal{H} P_n(\alpha+2) \mathcal{H} D_x^{\frac{3-\alpha}{2}} \mathcal{H} u)\, \mathrm{d}x \\ &= A_{2,1}(t) + A_{2,2}(t) + A_{2,3}(t). \end{aligned}$$

Fixing the value of n present in the terms $A_{2,1}$, $A_{2,2}$ and $A_{2,3}$ requires an argument almost similar to the one used in step 1. First, we deal with $A_{2,1}$ where a simple computation produces

$$A_{2,1}(t)=\frac{1}{2}\int_{\mathbb{R}}\mathcal{H}u D_x^{\frac{3-\alpha}{2}}\{R_n(\alpha+2)D_x^{\frac{3-\alpha}{2}}\mathcal{H}u\}\,\mathrm{d}x.$$

We fix $n\in\mathbb{Z}^+$ in such a way that

$$2n+1\le a+2\sigma\le 2n+3,$$

where $a=\alpha+2$ and $\sigma=\frac{3-\alpha}{2}$ in order to obtain $n=1$ or $n=2$. For the sake of simplicity we choose $n=1$.

Hence, by construction $R_1(\alpha+2)$ satisfies the hypothesis of Proposition 3.25, and

$$|A_{2,1}(t)|\lesssim\|\mathcal{H}u(t)\|_{L^2_x}\|\widehat{D^5_x(\chi^2_{\epsilon,b})}\|_{L^1_\xi}\lesssim\|u_0\|_{L^2_x}\|\widehat{D^5_x(\chi^2_{\epsilon,b})}\|_{L^1_\xi}.$$

Thus

$$\int_0^T|A_{2,1}(t)|\,\mathrm{d}t\lesssim\|u_0\|_{L^2_x}\sup_{0\le t\le T}\|\widehat{D^5_x(\chi^2_{\epsilon,b})}\|_{L^1_\xi}.$$

Next, after replacing $P_1(\alpha+2)$ in $A_{2,2}$

$$\begin{aligned}A_{2,2}(t)&=\left(\frac{\alpha+2}{4}\right)\int_{\mathbb{R}}(\mathcal{H}\partial_x^2u)^2(\chi^2_{\epsilon,b})'\,\mathrm{d}x-c_3\left(\frac{\alpha+2}{16}\right)\int_{\mathbb{R}}(\partial_xu)^2(\chi^2_{\epsilon,b})'''\,\mathrm{d}x\\&=A_{2,2,1}(t)+A_{2,2,2}(t).\end{aligned}$$

The smoothing effect corresponds to the term $A_{2,2,1}$ and it will be bounded after integrating in time. In contrast, bounding $A_{2,2,2}$ requires only the local theory; in fact

$$\int_0^T|A_{2,2,2}(t)|\,\mathrm{d}t\lesssim\|u\|_{L^\infty_TH_x^{(3-\alpha)/2}}.$$

Concerning the term $A_{2,3}$ we have after replacing $P_1(\alpha+2)$ and using the properties of the Hilbert transform that

$$\begin{aligned}A_{2,3}(t)&=\left(\frac{\alpha+2}{4}\right)\int_{\mathbb{R}}(\partial_x^2u)^2(\chi^2_{\epsilon,b})'\,\mathrm{d}x-c_3\left(\frac{\alpha+2}{16}\right)\int_{\mathbb{R}}(D_xu)^2(\chi^2_{\epsilon,b})'''\,\mathrm{d}x\\&=A_{2,3,1}(t)+A_{2,3,2}(t).\end{aligned}$$

As before, $A_{2,3,1}\ge 0$ and represents the smoothing effect. Additionally, the local theory and interpolation yield

$$\int_0^T|A_{2,3,2}(t)|\,\mathrm{d}t\lesssim\|u\|_{L^\infty_TH_x^{(3-\alpha)/2}}.$$

Step 2.3: It only remains to handle the term A_3. We can write

$$\begin{aligned}D_x^{\frac{1-\alpha}{2}}\partial_x(u\partial_xu)\chi_{\epsilon,b}&=-\tfrac{1}{2}[D_x^{\frac{1-\alpha}{2}}\partial_x;\,\chi_{\epsilon,b}]\,\partial_x((\chi_{\epsilon,b}u)^2+(\tilde{\phi}_{\epsilon,b}u)^2+(\psi_\epsilon u^2))\\&\qquad+[D_x^{\frac{1-\alpha}{2}}\partial_x;\,u\chi_{\epsilon,b}]\,\partial_x((\chi_{\epsilon,b}u)+(u\phi_{\epsilon,b})+(u\psi_\epsilon))+u\chi_{\epsilon,b}D_x^{\frac{1-\alpha}{2}}\partial_x^2u\\&=\widetilde{A}_{3,1}(t)+\widetilde{A}_{3,2}(t)+\widetilde{A}_{3,3}(t)+\widetilde{A}_{3,4}(t)+\widetilde{A}_{3,5}(t)+\widetilde{A}_{3,6}(t)+\widetilde{A}_{3,7}(t).\end{aligned}\tag{6.8}$$

First, we rewrite $\widetilde{A}_{3,1}$ as

$$\widetilde{A}_{3,1}(t) = c_\alpha \mathcal{H}[D_x^{1+\frac{1-\alpha}{2}};\, \chi_{\epsilon,b}]\, \partial_x((u\chi_{\epsilon,b})^2) + c_\alpha[\mathcal{H};\, \chi_{\epsilon,b}]\, D_x^{1+\frac{1-\alpha}{2}}\, \partial_x((\chi_{\epsilon,b}u)^2),$$

where c_α denotes a non-null constant. Next, combining (3.4), (3.14) and Lemma 3.15 one gets

$$\|\widetilde{A}_{3,1}(t)\|_{L^2_x} \lesssim \|D_x^{1+\frac{1-\alpha}{2}}(u\chi_{\epsilon,b})\|_{L^2_x}\|u\|_{L^\infty_x} + \|u_0\|_{L^2_x}\|u\|_{L^\infty_x},$$
$$\|\widetilde{A}_{3,2}(t)\|_{L^2_x} \lesssim \|D_x^{1+\frac{1-\alpha}{2}}(u\tilde{\phi}_{\epsilon,b})\|_{L^2_x}\|u\|_{L^\infty_x} + \|u_0\|_{L^2_x}\|u\|_{L^\infty_x}.$$

Next, we recall that by construction

$$\operatorname{dist}(\operatorname{supp}(\chi_{\epsilon,b}), \operatorname{supp}(\psi_\epsilon)) \geq \frac{\epsilon}{2},$$

so, by Lemma 3.16

$$\|\widetilde{A}_{3,3}(t)\|_{L^2_x} = \|[D_x^{\frac{1-\alpha}{2}}\partial_x;\, \chi_\epsilon]\, \partial_x(\psi_\epsilon u^2)\|_{L^2_x} \lesssim \|u_0\|_{L^2_x}\|u\|_{L^\infty_x}.$$

We can rewrite $\widetilde{A}_{3,4}$ as

$$\widetilde{A}_{3,4}(t) = c\mathcal{H}[D_x^{1+\frac{1-\alpha}{2}};\, u\chi_{\epsilon,b}]\, \partial_x(u\chi_{\epsilon,b}) - c[\mathcal{H};\, u\chi_{\epsilon,b}]\, \partial_x D_x^{1+\frac{1-\alpha}{2}}(u\chi_{\epsilon,b})$$

for some non-null constant c.

Thus, by the commutator estimates (3.2) and Lemma 3.13

$$\|\widetilde{A}_{3,4}(t)\|_{L^2_x} \lesssim \|\partial_x(u\chi_{\epsilon,b})\|_{L^\infty_x}\|D_x^{1+\frac{1-\alpha}{2}}(u\chi_{\epsilon,b})\|_{L^2_x}.$$

Applying the same procedure to $\widetilde{A}_{3,5}$ yields

$$\|\widetilde{A}_{3,5}(t)\|_{L^2_x} \lesssim \|\partial_x(u\chi_{\epsilon,b})\|_{L^\infty_x}\|D_x^{1+\frac{1-\alpha}{2}}(u\phi_{\epsilon,b})\|_{L^2_x} + \|\partial_x(u\phi_{\epsilon,b})\|_{L^\infty_x}\|D_x^{1+\frac{1-\alpha}{2}}(u\chi_{\epsilon,b})\|_{L^2_x}.$$

Since the supports of $\chi_{\epsilon,b}$ and ψ_ϵ are separated, we obtain by Lemma 3.16

$$\|\widetilde{A}_{3,6}(t)\|_{L^2_x} = \|u\chi_{\epsilon,b}\partial_x^2 D_x^{1+\frac{1-\alpha}{2}}(u\psi_\epsilon)\|_{L^2_x} \lesssim \|u_0\|_{L^2_x}\|u\|_{L^\infty_x}.$$

To finish with the estimates above we use the relation

$$\chi_{\epsilon,b}(x) + \phi_{\epsilon,b}(x) + \psi_\epsilon(x) = 1 \quad \text{for all } x \in \mathbb{R}.$$

Then

$$\begin{aligned} D_x^{1+\frac{1-\alpha}{2}}(u\chi_{\epsilon,b}) &= D_x^{1+\frac{1-\alpha}{2}}u\chi_{\epsilon,b} + [D_x^{1+\frac{1-\alpha}{2}};\, \chi_{\epsilon,b}](u\chi_{\epsilon,b} + u\phi_{\epsilon,b} + u\psi_\epsilon) \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Notice that $\|I_1\|_{L^2_x}$ is the quantity to estimate. In contrast, $\|I_2\|_{L^2_x}$ and $\|I_3\|_{L^2_x}$ can be handled by Lemma 3.13 combined with the local theory. Meanwhile I_3 can be bounded by using Lemma 3.16.

We notice that the gain of regularity obtained in the step 1 implies that $\|D_x^{1+(1+\alpha)/2}(u\phi_{\epsilon,b})\|_{L^2_x} < \infty$. To show this we use Theorem 3.7 and Hölder's inequality as follows:

$$\|D_x^{1+\frac{1+\alpha}{2}}(u\phi_{\epsilon,b})\|_{L^2_x} \lesssim \|u_0\|_{L^2_x} + \|u\|_{L^\infty_x} + \|\mathbb{1}_{[\frac{\epsilon}{4},b]}D_x^{1+\frac{1+\alpha}{2}}u\|_{L^2_x} + \|\mathbb{1}_{[\frac{\epsilon}{4},b]}\mathcal{H}D_x^{\frac{1+\alpha}{2}}u\|_{L^2_x}. \tag{6.9}$$

The second term on the right-hand side after we integrate in time is controlled by using Sobolev's embedding. Meanwhile, the third term can be handled after integrating in time and using (6.5) with $(\epsilon, b) = \left(\frac{\epsilon}{24}, b + \frac{7\epsilon}{24}\right)$.

The fourth term in the right-hand side can be bounded by combining the local theory and interpolation.

Hence, after integration in time

$$\|D_x^{1+\frac{1+\alpha}{2}}(u\phi_{\epsilon,b})\|_{L_T^2 L_x^2} < \infty, \tag{6.10}$$

which clearly implies $\|D_x^{1+(1-\alpha)/2}(u\phi_{\epsilon,b})\|_{L_T^2 L_x^2} < \infty$, as required. We can handle $\|D_x^{1+(1-\alpha)/2}(u\tilde{\phi}_{\epsilon,b})\|_{L_x^2}$ similarly.

Finally,

$$\begin{aligned} A_{3,7}(t) &= -\frac{1}{2}\int_{\mathbb{R}} \partial_x u \chi_{\epsilon,b}^2 (D_x^{\frac{1-\alpha}{2}} \partial_x u)^2 \,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}} u (\chi_{\epsilon,b}^2)' (D_x^{\frac{1-\alpha}{2}} \partial_x u)^2 \,\mathrm{d}x \\ &= A_{3,7,1}(t) + A_{3,7,2}(t). \end{aligned}$$

We have

$$|A_{3,7,1}(t)| \lesssim \|\partial_x u(t)\|_{L_x^\infty} \int_{\mathbb{R}} (D_x^{\frac{1-\alpha}{2}} \partial_x u)^2 \chi_{\epsilon,b}^2 \,\mathrm{d}x,$$

where the right-hand side can be estimated using Gronwall's inequality and the local theory $\|\partial_x u\|_{L_T^1 L_x^\infty} < \infty$.

Sobolev's embedding leads us to

$$\int_0^T |A_{3,7,2}(t)| \,\mathrm{d}t \lesssim \Big(\sup_{0\le t\le T} \|u(t)\|_{H_x^{s(\alpha)+}} \Big) \int_0^T \int_{\mathbb{R}} \chi_{\epsilon,b} \chi_{\epsilon,b}' (D_x^{\frac{1-\alpha}{2}} \partial_x u)^2 \,\mathrm{d}x \,\mathrm{d}t.$$

Gathering all the information corresponding to this step combined with Gronwall's inequality yields

$$\sup_{0\le t\le T} \|D_x^{\frac{1-\alpha}{2}} \partial_x u \chi_{\epsilon,b}\|_{L_x^2}^2 + \|\partial_x^2 u \eta_{\epsilon,b}\|_{L_T^2 L_x^2}^2 + \|\mathcal{H}\partial_x u \eta_{\epsilon,b}\|_{L_T^2 L_x^2}^2 \le c_{1,2}^*, \tag{6.11}$$

with $c_{1,2}^* = c_{1,2}^*(\alpha; \epsilon; T; v; \|u_0\|_{H_x^{(3-\alpha)/2}}; \|D_x^{(1-\alpha)/2} \partial_x u_0 \chi_{\epsilon,b}\|_{L_x^2})$ for any $\epsilon > 0$, $b \ge 5\epsilon$ and $v \ge 0$.

This finishes step 2, corresponding to the case $j = 1$ in the induction process.

Next, we present the case $j = 2$, to show how we proceed in the case j even.

Case $j = 2$. *Step 1*: First we apply two spatial derivatives to the equation in (1.1); after that we multiply by $\partial_x^2 u(x,t) \chi_{\epsilon,b}^2(x + vt)$, and finally we integrate in the x-variable to obtain the identity

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} (\partial_x^2 u)^2 \chi_{\epsilon,b}^2 \,\mathrm{d}x \\ &\quad \underbrace{- \frac{v}{2} \int_{\mathbb{R}} (\partial_x^2 u)^2 (\chi_{\epsilon,b}^2)' \,\mathrm{d}x}_{A_1(t)} \underbrace{- \int_{\mathbb{R}} (\partial_x^2 D_x^{\alpha+1} \partial_x u) \, \partial_x^2 u \chi_{\epsilon,b}^2 \,\mathrm{d}x}_{A_2(t)} + \underbrace{\int_{\mathbb{R}} \partial_x^2 (u \partial_x u) \partial_x^2 u \chi_{\epsilon,b}^2 \,\mathrm{d}x}_{A_3(t)} = 0. \end{aligned} \tag{6.12}$$

As was done in the previous steps, we first proceed to estimate A_1.

Step 1.1: By (6.11) it follows that

$$\int_0^T |A_1(t)| \,\mathrm{d}t \le \int_0^T \int_{\mathbb{R}} (\partial_x^2 u)^2 (\chi_{\epsilon,b}^2)' \,\mathrm{d}x \,\mathrm{d}t \le c_{1,2}^*. \tag{6.13}$$

Step 1.2: To extract information from the term A_2 we use integration by parts and Plancherel's identity to obtain

$$A_2(t)=\frac{1}{2}\int_{\mathbb{R}}\partial_x^2u[D_x^{\alpha+1}\partial_x;\,\chi_{\epsilon,b}^2]\,\partial_x^2u\,\mathrm{d}x=-\frac{1}{2}\int_{\mathbb{R}}\partial_x^2u[\mathcal{H}D_x^{\alpha+2};\,\chi_{\epsilon,b}^2]\,\partial_x^2u\,\mathrm{d}x. \tag{6.14}$$

Although this stage of the process is related to the one performed in step 1 (for $j=1$),we will use again the commutator expansion in (3.20), taking into account in this case that $a=\alpha+2>1$ and n is a nonnegative integer whose value will be fixed later.

Then,

$$\begin{aligned}A_2(t)&=\frac{1}{2}\int_{\mathbb{R}}\partial_x^2u\;R_n(s+2)\;\partial_x^2u\,\mathrm{d}x+\frac{1}{4}\int_{\mathbb{R}}\partial_x^2u\;P_n(s+2)\;\partial_x^2u\,\mathrm{d}x-\frac{1}{4}\int_{\mathbb{R}}\partial_x^2u\;\mathcal{H}P_n(s+2)\mathcal{H}\partial_x^2u\,\mathrm{d}x\\&=A_{2,1}(t)+A_{2,2}(t)+A_{2,3}(t).\end{aligned}$$

Essentially, the key term which allows us to fix the value of n is $A_{2,1}$. Indeed, after some integration by parts

$$A_{2,1}(t)=\frac{1}{2}\int_{\mathbb{R}}u\;\partial_x^2R_n(\alpha+2)\;\partial_x^2u\,\mathrm{d}x=\frac{1}{2}\int_{\mathbb{R}}u\;\partial_x^2\{R_n(\alpha+2)\;\partial_x^2u\}\,\mathrm{d}x.$$

We fix n such that it satisfies

$$2n+1\le a+2\sigma\le 2n+3.$$

In this case with $a=\alpha+2>1$ and $\sigma=2$, we obtain $n=2$.

Hence by construction Proposition 3.25 guarantees that $D_x^2R_2(\alpha+2)D_x^2$ is bounded in L_x^2.

Thus

$$|A_{2,1}(t)|\lesssim\|u(t)\|_{L_x^2}^2\|D_x^2R_2(\alpha+2)D_x^2u\|_{L_x^2}\le c\|u_0\|_{L_x^2}^2\|\widehat{D_x^{\alpha+6}(\chi_{\epsilon,b}^2)}\|_{L_\xi^1}.$$

Since we fixed $n=2$, we proceed to handle the contribution coming from $A_{2,2}$ and $A_{2,3}$.

Next,

$$\begin{aligned}A_{2,2}(t)&=\tilde{c}_1\int_{\mathbb{R}}(D_x^{\frac{\alpha+1}{2}}\partial_x^2u)^2(\chi_{\epsilon,b}^2)'\,\mathrm{d}x-\tilde{c}_3\int_{\mathbb{R}}(D_x^{1+\frac{\alpha+1}{2}}u)^2(\chi_{\epsilon,b}^2)^{(3)}\,\mathrm{d}x+c_5\left(\frac{\alpha+2}{64}\right)\int_{\mathbb{R}}(D_x^{\frac{\alpha+1}{2}}u)^2(\chi_{\epsilon,b}^2)^{(5)}\,\mathrm{d}x\\&=A_{2,2,1}(t)+A_{2,2,2}(t)+A_{2,2,3}(t).\end{aligned}$$

Notice that $A_{2,2,1}\ge 0$ represents the smoothing effect.

We recall that

$$|\chi_{\epsilon,b}^{(j)}(x)|\lesssim\chi'_{\frac{\epsilon}{3},b+\epsilon}(x)\quad\text{for all }x\in\mathbb{R},\ j\in\mathbb{Z}^+.$$

Then

$$\int_0^T|A_{2,2,2}(t)|\,\mathrm{d}t\lesssim\int_0^T\!\!\int_{\mathbb{R}}(D_x^{1+\frac{1+\alpha}{2}}u)^2\chi'_{\frac{\epsilon}{3},b+\epsilon}\,\mathrm{d}x\,\mathrm{d}t.$$

Taking $(\epsilon,b)=\left(\frac{\epsilon}{9},b+\frac{10\epsilon}{9}\right)$ in (6.5) combined with the properties of the cutoff function we have

$$\int_0^T|A_{2,2,2}(t)|\,\mathrm{d}t\lesssim c_{1,1}^*.$$

To finish the terms that make A_2 we proceed to estimate $A_{2,2,3}$.

As usual the low regularity is controlled by interpolation and the local theory. Therefore

$$\int_0^T |A_{2,2,3}(t)|\,\mathrm{d}t \lesssim \|u\|_{L^\infty_T H^{(3-\alpha)/2}_x}.$$

Next,

$$\begin{aligned} A_{2,3}(t) = \tilde{c}_1 \int_{\mathbb{R}} (\mathcal{H} D_x^{\frac{\alpha+1}{2}} \partial_x^2 u)^2 (\chi^2_{\epsilon,b})'\,\mathrm{d}x \\ - \tilde{c}_3 \int_{\mathbb{R}} (D_x^{\frac{\alpha+1}{2}} \partial_x u)^2 (\chi^2_{\epsilon,b})^{(3)}\,\mathrm{d}x + \left(\frac{\alpha+2}{64}\right) c_5 \int_{\mathbb{R}} (D_x^{\frac{\alpha+1}{2}} \mathcal{H} u)^2 (\chi^2_{\epsilon,b})^{(5)}\,\mathrm{d}x \\ = A_{2,3,1}(t) + A_{2,3,2}(t) + A_{2,3,3}(t). \end{aligned}$$

$A_{2,3,1}$ is positive and it will provide the smoothing effect after being integrated in time.

The terms $A_{2,3,2}$ and $A_{2,3,3}$ can be handled exactly in the same way that we treated $A_{2,2,2}$ and $A_{2,2,3}$ respectively, so we will omit the proof.

Step 1.3: Finally,

$$\begin{aligned} A_3(t) &= \frac{5}{2} \int_{\mathbb{R}} \partial_x u (\partial_x^2 u)^2 \chi^2_{\epsilon,b}\,\mathrm{d}x - \frac{1}{2} \int_{\mathbb{R}} u (\partial_x^2 u)^2 (\chi^2_{\epsilon,b})'\,\mathrm{d}x \\ &= A_{3,1}(t) + A_{3,2}(t). \end{aligned}$$

First,

$$|A_{3,1}(t)| \lesssim \|\partial_x u(t)\|_{L^\infty_x} \int_{\mathbb{R}} (\partial_x^2 u)^2 \chi^2_{\epsilon,b}\,\mathrm{d}x, \tag{6.15}$$

by the local theory $\partial_x u \in L^1([0,T] : L^\infty_x(\mathbb{R}))$ (see Theorem C(b)), and the integral expression is the quantity we want estimate.

Next,

$$|A_{3,2}(t)| \lesssim \|u(t)\|_{L^\infty_x} \int_{\mathbb{R}} (\partial_x^2 u)^2 (\chi^2_{\epsilon,b})'\,\mathrm{d}x. \tag{6.16}$$

After applying the Sobolev embedding and integrating in the time variable we obtain

$$\int_0^T |A_{3,2}(t)|\,\mathrm{d}t \lesssim \Big(\sup_{0\le t\le T} \|u(t)\|_{H^{s(\alpha)+}_x} \Big) \int_0^T \int_{\mathbb{R}} (\partial_x^2 u)^2 (\chi^2_{\epsilon,b})'\,\mathrm{d}x\,\mathrm{d}t,$$

and the integral term in the right-hand side was estimated previously in (6.13).

Thus, after grouping all the terms and applying Gronwall's inequality we obtain

$$\sup_{0\le t\le T} \|\partial_x^2 u \chi_{\epsilon,b}\|^2_{L^2_x} + \|D_x^{\frac{\alpha+1}{2}} \partial_x^2 u \eta_{\epsilon,b}\|^2_{L^2_T L^2_x} + \|D_x^{\frac{\alpha+1}{2}} \mathcal{H} \partial_x^2 u \eta_{\epsilon,b}\|^2_{L^2_T L^2_x} \le c^*_{2,1}, \tag{6.17}$$

where $c^*_{2,1} = c^*_{2,1}(\alpha; \epsilon; T; v; \|u_0\|_{H^{(3-\alpha)/2}_x}; \|\partial_x^2 u_0 \chi_{\epsilon,b}\|_{L^2_x})$ for any $\epsilon > 0$, $b \ge 5\epsilon$ and $v \ge 0$.

Step 2: From equation in (1.1) one gets after applying the operator $D_x^{(1-\alpha)/2}\partial_x^2$ and multiplying the result by $D_x^{(1-\alpha)/2}\partial_x^2 u \chi^2_{\epsilon,b}(x+vt)$

$$D_x^{\frac{1-\alpha}{2}} \partial_x^2 \partial_t u D_x^{\frac{1-\alpha}{2}} \partial_x^2 u \chi^2_{\epsilon,b} - D_x^{\frac{1-\alpha}{2}} \partial_x^2 D_x^{1+\alpha} \partial_x u D_x^{\frac{1-\alpha}{2}} \partial_x^2 u \chi^2_{\epsilon,b} + D_x^{\frac{1-\alpha}{2}} \partial_x^2 (u \partial_x u) D_x^{\frac{1-\alpha}{2}} \partial_x^2 u \chi^2_{\epsilon,b} = 0,$$

which after integration in the spatial variable becomes

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}(D_x^{\frac{1-\alpha}{2}}\partial_x^2 u)^2\chi_{\epsilon,b}^2\,\mathrm{d}x \underbrace{-\frac{v}{2}\int_{\mathbb{R}}(D_x^{\frac{1-\alpha}{2}}\partial_x^2 u)^2(\chi_{\epsilon,b}^2)'\,\mathrm{d}x}_{A_1(t)}$$
$$\underbrace{-\int_{\mathbb{R}}(D_x^{\frac{1-\alpha}{2}}\partial_x^2 D_x^{1+\alpha}\partial_x u)(D_x^{\frac{1-\alpha}{2}}\partial_x^2 u)\chi_{\epsilon,b}^2\,\mathrm{d}x}_{A_2(t)}+\underbrace{\int_{\mathbb{R}}(D_x^{\frac{1-\alpha}{2}}\partial_x^2(u\partial_x u))(D_x^{\frac{1-\alpha}{2}}\partial_x^2 u)\chi_{\epsilon,b}^2\,\mathrm{d}x}_{A_3(t)}=0.$$

To estimate A_1 we will use different techniques from the ones implemented to bound A_1 in the previous step. The main difficulty we have to face is dealing with the nonlocal character of the operator D_x^s for $s\in\mathbb{R}^+\backslash 2\mathbb{N}$; the case $s\in 2\mathbb{N}$ is less complicated because D_x^s becomes local, so we can integrate by parts.

The strategy to solve this issue will be the following. In (6.17) we proved that u has a gain of $\frac{\alpha+1}{2}$ derivatives (local), which in total sum to $2+\frac{1+\alpha}{2}$. This suggests that if we can find an appropriate channel where we can localize the smoothing effect, we shall be able to recover all the local derivatives $r$with $r\le 2+\frac{1+\alpha}{2}$.

Henceforth we will employ recurrently a technique of localization of the commutator used by Kenig, Linares, Ponce and Vega [Kenig et al. 2018] in the study of propagation of regularity (fractional) for solutions of the k-generalized KdV equation. Indeed, the idea consists in constructing an appropriate system of smooth partitions of unit length, localizing the regions where the information obtained in the previous cases is available.

We recall that for $\epsilon>0$ and $b\ge 5\epsilon$

$$\eta_{\epsilon,b}=\sqrt{\chi_{\epsilon,b}\chi_{\epsilon,b}'}\quad\text{and}\quad \chi_{\epsilon,b}+\phi_{\epsilon,b}+\psi_\epsilon=1. \tag{6.18}$$

Step 2.1: We claim

$$\|D_x^{\frac{1+\alpha}{2}}\partial_x^2(u\eta_{\epsilon,b})\|_{L_T^2L_x^2}<\infty. \tag{6.19}$$

Combining the commutator estimate (3.14), (6.18), Hölder's inequality and (6.17) yields

$$\begin{aligned}
&\|D_x^{\frac{1+\alpha}{2}}\partial_x^2(u\eta_{\epsilon,b})\|_{L_T^2L_x^2}\\
&\le\|D_x^{2+\frac{1+\alpha}{2}}u\eta_{\epsilon,b}\|_{L_T^2L_x^2}+\|[D_x^{2+\frac{1+\alpha}{2}};\eta_{\epsilon,b}](u\chi_{\epsilon,b}+u\phi_{\epsilon,b}+u\psi_\epsilon)\|_{L_T^2L_x^2}\\
&\lesssim(c_{2,1}^*)^2+\underbrace{\|D_x^{1+\frac{1+\alpha}{2}}(u\chi_{\epsilon,b})\|_{L_T^2L_x^2}}_{B_1}+\underbrace{\|D_x^{1+\frac{1+\alpha}{2}}(u\phi_{\epsilon,b})\|_{L_T^2L_x^2}}_{B_2}+\|u_0\|_{L_x^2}+\underbrace{\|\eta_{\epsilon,b}D_x^{2+\frac{1+\alpha}{2}}(u\psi_\epsilon)\|_{L_T^2L_x^2}}_{B_3}.
\end{aligned}\tag{6.20}$$

Since $\chi_{\epsilon/5,\epsilon}=1$ on the support of $\chi_{\epsilon,b}$ we have

$$\chi_{\epsilon,b}(x)\chi_{\frac{\epsilon}{5},\epsilon}(x)=\chi_{\epsilon,b}(x)\quad\text{for all } x\in\mathbb{R}.$$

Thus, combining Lemma 3.15 and Young's inequality we obtain

$$\|D_x^{1+\frac{1+\alpha}{2}}(u\chi_{\epsilon,b})\|_{L_x^2}\lesssim\|\partial_x^2u\chi_{\epsilon,b}\|_{L_x^2}+\|\partial_xu\chi_{\frac{\epsilon}{5},\epsilon}\|_{L_x^2}+\|u_0\|_{L_x^2}. \tag{6.21}$$

Then, an application of (6.17) adapted to every case yields

$$B_1\lesssim\|\partial_x^2u\chi_{\epsilon,b}\|_{L_T^\infty L_x^2}+\|\partial_xu\chi_{\frac{\epsilon}{5},\epsilon}\|_{L_T^\infty L_x^2}+\|u_0\|_{L_x^2}\lesssim c_{2,1}^*+c_{1,1}^*+\|u_0\|_{L_x^2}. \tag{6.22}$$

Notice that B_2 was estimated in the case $j=1$, step 2, see (6.10), so we will omit the proof. Next, we recall that by construction

$$\operatorname{dist}(\operatorname{supp}(\eta_{\epsilon,b}),\operatorname{supp}(\psi_\epsilon)) \geq \frac{\epsilon}{2}.$$

Hence by Lemma 3.16

$$B_3 = \|\eta_{\epsilon,b} D_x^{2+\frac{1+\alpha}{2}}(u\psi_\epsilon)\|_{L^2_T L^2_x} \lesssim \|\eta_{\epsilon,b}\|_{L^\infty_T L^\infty_x}\|u_0\|_{L^2_x}. \tag{6.23}$$

The claim follows by gathering the calculations above.

At this point we have proved that locally in the interval $[\epsilon, b]$ there exist $2+\frac{\alpha+1}{2}$ derivatives. By Lemma 3.15 we get

$$\|D_x^{2+\frac{1-\alpha}{2}}(u\eta_{\epsilon,b})\|_{L^2_T L^2_x} \lesssim \|D_x^{2+\frac{1+\alpha}{2}}(u\eta_{\epsilon,b})\|_{L^2_T L^2_x} + \|u_0\|_{L^2_x} < \infty.$$

As before

$$D_x^{2+\frac{1-\alpha}{2}} u\eta_{\epsilon,b} = D_x^{2+\frac{1-\alpha}{2}}(u\eta_{\epsilon,b}) - [D_x^{2+\frac{1-\alpha}{2}};\eta_{\epsilon,b}](u\chi_{\epsilon,b}+u\phi_{\epsilon,b}+u\psi_\epsilon).$$

The argument used in the proof of the claim yields

$$\|D_x^{2+\frac{1-\alpha}{2}} u\eta_{\epsilon,b}\|_{L^2_T L^2_x} < \infty.$$

Therefore,

$$\int_0^T |A_1(t)|\,\mathrm{d}t \leq |v|\int_0^T\int_{\mathbb{R}} (D_x^{\frac{1-\alpha}{2}}\partial_x^2 u)^2(\chi^2_{\epsilon,b})'\,\mathrm{d}x\,\mathrm{d}t \lesssim \|D_x^{2+\frac{1-\alpha}{2}} u\eta_{\epsilon,b}\|^2_{L^2_T L^2_x} < \infty. \tag{6.24}$$

Step 2.2: Now we focus our attention on the term A_2. Notice that after integration by parts and Plancherel's identity

$$A_2(t) = -\frac{1}{2}\int_{\mathbb{R}} D_x^{\frac{5-\alpha}{2}} u[\mathcal{H}D_x^{2+\alpha};\chi^2_{\epsilon,b}]D_x^{\frac{5-\alpha}{2}} u\,\mathrm{d}x. \tag{6.25}$$

The procedure to decompose the commutator will be similar to that in the previous step; the main difference relies on the fact that the quantity of derivatives is higher in comparison with step 1.

Concerning this, we notice that $2+\alpha > 1$ and by (3.20) the commutator $[\mathcal{H}D_x^{\alpha+2};\chi^2_{\epsilon,b}]$ can be decomposed as

$$[\mathcal{H}D_x^{\alpha+2};\chi^2_{\epsilon,b}] + \tfrac{1}{2}P_n(\alpha+2) + R_n(\alpha+2) = \tfrac{1}{2}\mathcal{H}P_n(\alpha+2)\mathcal{H} \tag{6.26}$$

for some positive integer n. We shall fix the value of n satisfying a suitable condition.

Substituting (6.26) into (6.25) produces

$$\begin{aligned} A_2(t) &= \frac{1}{2}\int_{\mathbb{R}} D_x^{\frac{5-\alpha}{2}} u(R_n(\alpha+2)D_x^{\frac{5-\alpha}{2}} u)\,\mathrm{d}x \\ &\qquad + \frac{1}{4}\int_{\mathbb{R}} D_x^{\frac{5-\alpha}{2}} u(P_n(\alpha+2)D_x^{\frac{5-\alpha}{2}} u)\,\mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}} D_x^{\frac{5-\alpha}{2}} u(\mathcal{H}P_n(\alpha+2)\mathcal{H}D_x^{\frac{5-\alpha}{2}} u)\,\mathrm{d}x \\ &= A_{2,1}(t) + A_{2,2}(t) + A_{2,3}(t). \end{aligned} \tag{6.27}$$

Now we proceed to fix the value of n present in $A_{2,1}$, $A_{2,2}$ and $A_{2,3}$.

First we deal with the term that determines the value n in the decomposition associated to A_2. In this case it corresponds to $A_{2,1}$.

Applying Plancherel's identity, $A_{2,1}$ becomes

$$A_{2,1}(t) = \frac{1}{2}\int_{\mathbb{R}} u D_x^{\frac{5-\alpha}{2}}\{R_n(\alpha+2)D_x^{\frac{5-\alpha}{2}}u\}\,\mathrm{d}x.$$

We fix n such that it satisfies (3.26); i.e.,

$$2n+1 \le a+2\sigma \le 2n+3,$$

with $a=\alpha+2$ and $\sigma=\frac{5-\alpha}{2}$, which produces $n=2$ or $n=3$. Nevertheless, for the sake of simplicity we take $n=2$.

Hence, by construction $R_2(\alpha+2)$ is bounded in L^2_x (see Proposition 3.25).

Thus,

$$\int_0^T |A_{2,1}(t)|\,\mathrm{d}t \le c\int_0^T \|u(t)\|^2_{L^2_x}\|\widehat{D_x^7(\chi^2_{\epsilon,b}(\,\cdot\,+vt))}\|_{L^1_\xi}\,\mathrm{d}t \lesssim \|u_0\|^2_{L^2_x}\sup_{0\le t\le T}\|\widehat{D_x^7(\chi^2_{\epsilon,b})}\|_{L^1_\xi}.$$

Since we have fixed $n=2$, we obtain, after substituting $P_2(\alpha+2)$ into $A_{2,2}$,

$$\begin{aligned}A_{2,2}(t) &= \tilde{c}_1\int_{\mathbb{R}}(\mathcal{H}\partial_x^3u)^2(\chi^2_{\epsilon,b})'\,\mathrm{d}x - \tilde{c}_3\int_{\mathbb{R}}(\partial_x^2u)^2(\chi^2_{\epsilon,b})^{(3)}\,\mathrm{d}x + \tilde{c}_5\int_{\mathbb{R}}(\mathcal{H}\partial_xu)^2(\chi^2_{\epsilon,b})^{(5)}\,\mathrm{d}x\\ &= A_{2,2,1}(t)+A_{2,2,2}(t)+A_{2,2,3}(t).\end{aligned}$$

We underline that $A_{2,2,1}$ is positive and represents the smoothing effect.

On the other hand, by (6.11) with $(\epsilon,b)=\left(\frac{\epsilon}{5},\epsilon\right)$ we have

$$\int_0^T |A_{2,2,2}(t)|\,\mathrm{d}t = c\int_0^T\int_{\mathbb{R}}(\partial_x^2u)^2\chi^2_{\frac{\epsilon}{5},\epsilon}(\chi^2_{\epsilon,b})'''\,\mathrm{d}x\,\mathrm{d}t \lesssim \sup_{0\le t\le T}\int_{\mathbb{R}}(\partial_x^2u)^2\chi^2_{\frac{\epsilon}{5},\epsilon}\,\mathrm{d}x \lesssim c^*_{1,2}. \tag{6.28}$$

Next, by the local theory

$$\int_0^T |A_{2,2,3}(t)|\,\mathrm{d}t \lesssim \|u\|_{L^\infty_T H_x^{(3-\alpha)/2}}. \tag{6.29}$$

After replacing $P_2(\alpha+2)$ into $A_{2,3}$, and using the fact that the Hilbert transform is skew adjoint

$$\begin{aligned}A_{2,3}(t) &= \left(\frac{\alpha+2}{4}\right)\int_{\mathbb{R}}(\partial_x^3u)^2(\chi^2_{\epsilon,b})'\,\mathrm{d}x\\ &\qquad - c_3\left(\frac{\alpha+2}{16}\right)\int_{\mathbb{R}}(\mathcal{H}\partial_x^2u)^2(\chi^2_{\epsilon,b})'''\,\mathrm{d}x + c_5\left(\frac{\alpha+2}{64}\right)\int_{\mathbb{R}}(\partial_xu)^2(\chi^2_{\epsilon,b})^{(5)}\,\mathrm{d}x\\ &= A_{2,3,1}(t)+A_{2,3,2}(t)+A_{2,3,3}(t).\end{aligned}$$

Notice that $A_{2,3,1}\ge 0$ and it represents the smoothing effect. However, $A_{2,3,2}$ can be handled if we take $(\epsilon,b)=\left(\frac{\epsilon}{5},\epsilon\right)$ in (6.5) as follows:

$$A_{2,3,3}(t) = \int_{\mathbb{R}}(\partial_x^2u)^2\chi^2_{\frac{\epsilon}{5},\epsilon}(\chi^2_{\epsilon,b})'''\,\mathrm{d}x \lesssim \int_{\mathbb{R}}(\partial_x^2u)^2\chi^2_{\frac{\epsilon}{5},\epsilon}\,\mathrm{d}x.$$

Thus,

$$\int_0^T |A_{2,3,3}(t)|\,\mathrm{d}t \lesssim \sup_{0\le t\le T}\int_{\mathbb{R}}(\partial_xu)^2\chi^2_{\frac{\epsilon}{5},\epsilon}\,\mathrm{d}x \lesssim c^*_{1,1}.$$

To finish the estimate of A_2 it only remains to bound $A_{2,3,2}$. To do this we recall that

$$|\chi^{(j)}_{\epsilon,b}(x)| \lesssim \chi'_{\frac{\epsilon}{3},b+\epsilon}(x) \quad \text{for all } x\in\mathbb{R},\ j\in\mathbb{Z}^+,$$

which together with property (9) of $\chi_{\epsilon,b}$ yields

$$\int_0^T\int_{\mathbb{R}} (\mathcal{H}\partial_x^2 u)^2 \chi'_{\frac{\epsilon}{3},b+\epsilon}\,\mathrm{d}x\,\mathrm{d}t \lesssim \|\mathcal{H}\partial_x^2 u\eta_{\frac{\epsilon}{9},b+\frac{10\epsilon}{9}}\|^2_{L^2_T L^2_x} \lesssim c^*_{1,2},$$

where the last inequality is obtained taking $(\epsilon,b) = \left(\frac{\epsilon}{9}, b+\frac{10\epsilon}{9}\right)$ in (6.11). The term $A_{2,3,3}$ can be handled by interpolation and the local theory.

Step 2.3: Finally we turn our attention to A_3.We start rewriting the nonlinear part as

$$\begin{aligned} D_x^{\frac{1-\alpha}{2}}\partial_x^2(u\,\partial_x u)\chi_{\epsilon,b} &= -\tfrac{1}{2}[D_x^{\frac{1-\alpha}{2}}\partial_x^2;\,\chi_{\epsilon,b}]\,\partial_x((u\chi_{\epsilon,b})^2 + (u\tilde{\phi}_{\epsilon,b})^2 + (\psi_\epsilon u^2)) \\ &\quad + [D_x^{\frac{1-\alpha}{2}}\partial_x^2;\,u\chi_{\epsilon,b}]\,\partial_x((u\chi_{\epsilon,b}) + (u\phi_{\epsilon,b}) + (u\psi_\epsilon)) + u\chi_{\epsilon,b}\,D_x^{\frac{1-\alpha}{2}}\partial_x^3 u \\ &= \widetilde{A}_{3,1}(t)+\widetilde{A}_{3,2}(t)+\widetilde{A}_{3,3}(t)+\widetilde{A}_{3,4}(t)+\widetilde{A}_{3,5}(t)+\widetilde{A}_{3,6}(t)+\widetilde{A}_{3,7}(t). \end{aligned} \tag{6.30}$$

Hence, after substituting (6.29) into A_3 and applying Hölder's inequality

$$\begin{aligned} A_3(t) &= \sum_{1\le m\le 6}\int_{\mathbb{R}} \widetilde{A}_{3,m}(t)\,D_x^{\frac{1-\alpha}{2}}\partial_x^2 u\chi_{\epsilon,b}\,\mathrm{d}x + \int_{\mathbb{R}}\widetilde{A}_{3,7}(t)\,D_x^{\frac{1-\alpha}{2}}\partial_x^2 u\chi_{\epsilon,b}\,\mathrm{d}x \\ &\le \sum_{1\le m\le 6}\|\widetilde{A}_{3,m}(t)\|_{L^2_x}\|D_x^{2+\frac{1-\alpha}{2}}u(t)\chi_{\epsilon,b}(\,\cdot\,+vt)\|_{L^2_x} + \int_{\mathbb{R}}\widetilde{A}_{3,7}(t)\,D_x^{\frac{1-\alpha}{2}}\partial_x^2 u\chi_{\epsilon,b}\,\mathrm{d}x \\ &= \|D_x^{2+\frac{1-\alpha}{2}}u(t)\chi_{\epsilon,b}(\,\cdot\,+vt)\|_{L^2_x}\sum_{1\le m\le 6}A_{3,m}(t) + A_{3,7}(t). \end{aligned}$$

Notice that the first factor in the right-hand side is the quantity to be estimated by Gronwall's inequality. So, we shall focus on establishing control of the remaining terms.

First,combining (3.4), (3.14) and Lemma 3.15 one gets that

$$\|\widetilde{A}_{3,1}(t)\|_{L^2_x} \lesssim \|D_x^{2+\frac{1-\alpha}{2}}(u\chi_{\epsilon,b})\|_{L^2_x}\|u\|_{L^\infty_x} + \|u_0\|_{L^2_x}\|u\|_{L^\infty_x}, \tag{6.31}$$

$$\|\widetilde{A}_{3,2}(t)\|_{L^2_x} \lesssim \|D_x^{2+\frac{1-\alpha}{2}}(u\tilde{\phi}_{\epsilon,b})\|_{L^2_x}\|u\|_{L^\infty_x} + \|u_0\|_{L^2_x}\|u\|_{L^\infty_x}. \tag{6.32}$$

To finish with the quadratic terms, we employ Lemma 3.16:

$$\|\widetilde{A}_{3,3}(t)\|_{L^2_x} \lesssim \|u_0\|_{L^2_x}\|u\|_{L^\infty_x}.$$

Combining (3.2) and (3.14) we obtain

$$\|\widetilde{A}_{3,4}(t)\|_{L^2_x} \lesssim \|\partial_x(u\chi_{\epsilon,b})\|_{L^\infty_x}\|D_x^{2+\frac{1-\alpha}{2}}(u\chi_{\epsilon,b})\|_{L^2_x}.$$

Meanwhile,

$$\|\widetilde{A}_{3,5}(t)\|_{L^2_x} \lesssim \|\partial_x(u\chi_{\epsilon,b})\|_{L^\infty_x}\|D_x^{2+\frac{1-\alpha}{2}}(u\phi_{\epsilon,b})\|_{L^2_x} + \|\partial_x(u\phi_{\epsilon,b})\|_{L^\infty_x}\|D_x^{2+\frac{1-\alpha}{2}}(u\chi_{\epsilon,b})\|_{L^2_x}.$$

Next, we recall that by construction

$$\operatorname{dist}(\operatorname{supp}(\chi_{\epsilon,b}), \operatorname{supp}(\psi_\epsilon)) \geq \frac{\epsilon}{2}.$$

Thus by Lemma 3.16

$$\|\widetilde{A}_{3,6}(t)\|_{L^2_x} \lesssim \|u_0\|_{L^2_x}\|u\|_{L^\infty_x}.$$

To complete the estimates in (6.31)–(6.32) it only remains for us to bound $\|D_x^{2+(1-\alpha)/2}(u\chi_{\epsilon,b})\|_{L^2_x}$, $\|D_x^{2+(1-\alpha)/2}(u\tilde{\phi}_{\epsilon,b})\|_{L^2_x}$, and $\|D_x^{2+(1-\alpha)/2}(u\phi_{\epsilon,b})\|_{L^2_x}$.

For the first term we proceed by writing

$$\begin{aligned} D_x^{2+\frac{1-\alpha}{2}}(u\chi_{\epsilon,b}) &= D_x^{2+\frac{1-\alpha}{2}}u\chi_{\epsilon,b} + [D_x^{2+\frac{1-\alpha}{2}};\chi_{\epsilon,b}](u\chi_{\epsilon,b}+u\phi_{\epsilon,b}+u\psi_\epsilon) \\ &= I_1+I_2+I_3+I_4. \end{aligned}$$

Notice that $\|I_1\|_{L^2_x}$ is the quantity to be estimated by Gronwall's inequality. Meanwhile, $\|I_2\|_{L^2_x}$, $\|I_3\|_{L^2_x}$ and $\|I_4\|_{L^2_x}$ were estimated previously in the case $j=1$, step 2.

Next, we focus on estimating the term $\|D_x^{2+(1+\alpha)/2}(u\phi_{\epsilon,b})\|_{L^2_x}$, which will be treated by means of Hölder's inequality and Theorem 3.7 as follows:

$$\|D_x^{2+\frac{1+\alpha}{2}}(u\phi_{\epsilon,b})\|_{L^2_x} \lesssim \|u_0\|_{L^2_x}^{\frac{1}{2}}\|u\|_{L^\infty_x}^{\frac{1}{2}} + \|\eta_{\frac{\epsilon}{24},b+\frac{7\epsilon}{24}}D_x^{2+\frac{1+\alpha}{2}}u\|_{L^2_x} + \|\eta_{\frac{\epsilon}{24},b+\frac{7\epsilon}{24}}D_x^{\frac{1+\alpha}{2}}\partial_x u\|_{L^2_x} + \|D_x^{\frac{1+\alpha}{2}}u\|_{L^2_x}.$$

After integrating in time, the second and third terms on the right-hand side can be estimated taking $(\epsilon,b)=\left(\frac{\epsilon}{24}, b+\frac{7\epsilon}{24}\right)$ in (6.17) and (6.5) respectively. Hence, after integrating in time it follows by interpolation that $\|D_x^{2+(1-\alpha)/2}(u\phi_{\epsilon,b})\|_{L^2_TL^2_x} < \infty$.

We can bound $\|D_x^{2+(1-\alpha)/2}(u\tilde{\phi}_{\epsilon,b})\|_{L^2_x}$ analogously.

Finally, after integrating by parts

$$\begin{aligned} A_{3,7}(t) &= -\frac{1}{2}\int_{\mathbb{R}}\partial_x u\chi_{\epsilon,b}^2(D_x^{\frac{1-\alpha}{2}}\partial_x^2 u)^2\,\mathrm{d}x - \int_{\mathbb{R}} u\chi_{\epsilon,b}\chi_{\epsilon,b}'(D_x^{\frac{1-\alpha}{2}}\partial_x^2 u)^2\,\mathrm{d}x \\ &= A_{3,7,1}(t) + A_{3,7,2}(t). \end{aligned}$$

First,

$$|A_{3,7,1}(t)| \lesssim \|\partial_x u(t)\|_{L^\infty_x}\int_{\mathbb{R}}(D_x^{\frac{1-\alpha}{2}}\partial_x^2 u)^2\chi_{\epsilon,b}^2\,\mathrm{d}x,$$

where the last integral is the quantity that will be estimated using Gronwall's inequality, and the other factor will be controlled after integration in time.

After integration in time and Sobolev's embedding it follows that

$$\begin{aligned} \int_0^T |A_{3,7,2}(t)|\,\mathrm{d}t &\lesssim \int_0^T\int_{\mathbb{R}} u\,(\chi_{\epsilon,b}^2)'(D_x^{\frac{1-\alpha}{2}}\partial_x^2 u)^2\,\mathrm{d}x\,\mathrm{d}t \\ &\lesssim \Big(\sup_{0\leq t\leq T}\|u(t)\|_{H_x^{s(\alpha)+}}\Big)\int_0^T\int_{\mathbb{R}}(D_x^{\frac{1-\alpha}{2}}\partial_x^2 u)^2\,(\chi_{\epsilon,b}^2)'\,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

and the last term was already estimated in (6.24).

Thus, after collecting all the information in this step and applying Gronwall's inequality together with hypothesis (1.11), we obtain

$$\sup_{0\le t\le T} \|D_x^{\frac{1-\alpha}{2}}\partial_x^2 u\chi_{\epsilon,b}\|_{L_x^2}^2 + \|\partial_x^3 u\eta_{\epsilon,b}\|_{L_T^2L_x^2}^2 + \|\mathcal{H}\partial_x^3 u\eta_{\epsilon,b}\|_{L_T^2L_x^2}^2 \le c_{2,2}^*,$$

where $c_{2,2}^* = c_{2,2}^*(\alpha;\epsilon;T;v;\|u_0\|_{H_x^{(3-\alpha)/2}};\|D_x^{(1-\alpha)/2}\partial_x^2 u_0\chi_{\epsilon,b}\|_{L_x^2})$ for any $\epsilon>0$, $b\ge 5\epsilon$ and $v>0$.

According to the induction argument we shall assume that (1.12) holds for $j\le m$ with $j\in\mathbb{Z}$ and $j\ge 2$; i.e.,

$$\sup_{0\le t\le T} \|\partial_x^j u\chi_{\epsilon,b}\|_{L_x^2}^2 + \|D_x^{\frac{1+\alpha}{2}}\partial_x^j u\eta_{\epsilon,b}\|_{L_T^2L_x^2}^2 + \|\mathcal{H}D_x^{\frac{1+\alpha}{2}}\partial_x^j u\eta_{\epsilon,b}\|_{L_T^2L_x^2}^2 \le c_{j,1}^* \tag{6.33}$$

for $j=1,2,\dots,m$ with $m\ge 1$, for any $\epsilon>0$, $b\ge 5\epsilon$ $v\ge 0$.

Step 3: We will assume j an even integer. The case where j is odd follows by an argument similar to the case $j=1$.

By reasoning analogous to that employed in the case $j=2$ it follows that

$$D_x^{\frac{1-\alpha}{2}}\partial_x^j\partial_t u\, D_x^{\frac{1-\alpha}{2}}\partial_x^j u\chi_{\epsilon,b}^2 - D_x^{\frac{1-\alpha}{2}}\partial_x^j D_x^{1+\alpha}\partial_x u\, D_x^{\frac{1-\alpha}{2}}\partial_x^j u\chi_{\epsilon,b}^2 + D_x^{\frac{1-\alpha}{2}}\partial_x^j(u\,\partial_x u)\, D_x^{\frac{1-\alpha}{2}}\partial_x^j u\chi_{\epsilon,b}^2 = 0,$$

which after integrating in time yields the identity

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}(D_x^{\frac{1-\alpha}{2}}\partial_x^j u)^2\chi_{\epsilon,b}^2\,\mathrm{d}x \underbrace{-\frac{v}{2}\int_{\mathbb{R}}(D_x^{\frac{1-\alpha}{2}}\partial_x^j u)^2(\chi_{\epsilon,b}^2)'\,\mathrm{d}x}_{A_1(t)}$$
$$\underbrace{-\int_{\mathbb{R}}(D_x^{\frac{1-\alpha}{2}}\partial_x^j D_x^{1+\alpha}\partial_x u)(D_x^{\frac{1-\alpha}{2}}\partial_x^j u\chi_{\epsilon,b}^2)\,\mathrm{d}x}_{A_2(t)} + \underbrace{\int_{\mathbb{R}} D_x^{\frac{1-\alpha}{2}}\partial_x^j(u\,\partial_x u)(D_x^{\frac{1-\alpha}{2}}\partial_x^j u\chi_{\epsilon,b}^2)\,\mathrm{d}x}_{A_3(t)} = 0. \tag{6.34}$$

Step 3.1: We claim that

$$\|D_x^{j+\frac{1+\alpha}{2}}(u\eta_{\epsilon,b})\|_{L_T^2L_x^2} < \infty. \tag{6.35}$$

We proceed as in the case $j=2$. A combination of the commutator estimate (3.14), (6.18), Hölder's inequality and (6.33) yields

$$\begin{aligned}\|D_x^{\frac{1+\alpha}{2}}\partial_x^j(u\eta_{\epsilon,b})\|_{L_T^2L_x^2} &\le \|D_x^{j+\frac{1+\alpha}{2}}u\eta_{\epsilon,b}\|_{L_T^2L_x^2} + \|[D_x^{j+\frac{1+\alpha}{2}};\eta_{\epsilon,b}](u\chi_{\epsilon,b}+u\phi_{\epsilon,b}+u\psi_\epsilon)\|_{L_T^2L_x^2}\\ &\lesssim (c_{j,1}^*)^2 + \underbrace{\|D_x^{j-1+\frac{1+\alpha}{2}}(u\chi_{\epsilon,b})\|_{L_T^2L_x^2}}_{B_1}\\ &\quad + \|u_0\|_{L_x^2} + \underbrace{\|D_x^{j-1+\frac{1+\alpha}{2}}(u\phi_{\epsilon,b})\|_{L_T^2L_x^2}}_{B_2} + \underbrace{\|\eta_{\epsilon,b}D_x^{j+\frac{1+\alpha}{2}}(u\psi_\epsilon)\|_{L_T^2L_x^2}}_{B_3}.\end{aligned} \tag{6.36}$$

Since $\chi_{\epsilon/5,\epsilon}=1$ on the support of $\chi_{\epsilon,b}$ we have

$$\chi_{\epsilon,b}(x)\chi_{\frac{\epsilon}{5},\epsilon}(x) = \chi_{\epsilon,b}(x)\quad\text{for all } x\in\mathbb{R}.$$

Combining Lemma 3.15 and Young's inequality

$$\|D_x^{j+\frac{\alpha-1}{2}}(u\chi_{\epsilon,b})\|_{L_x^2} \lesssim \|\partial_x^j u\chi_{\epsilon,b}\|_{L_x^2}^2 + \sum_{2\le k\le j-1}\gamma_{k,j}\|\chi_{\epsilon,b}^{(j-k)}\|_{L_x^\infty}\|\partial_x^k u\chi_{\frac{\epsilon}{5},\epsilon}\|_{L_x^2} + \|u\|_{L_T^\infty H_x^{(3-\alpha)/2}} + \|u_0\|_{L_x^2}. \tag{6.37}$$

Hence, taking $(\epsilon, b) = \left(\frac{\epsilon}{5}, \epsilon\right)$ in (6.33) yields

$$B_1 \lesssim c^*_{j,1} + \sum_{2\le k\le j-1} \gamma_{k,j} c^*_{k,1} + \|u\|_{L^\infty_T H_x^{(3-\alpha)/2}} + \|u_0\|_{L^2_x}. \tag{6.38}$$

B_2 can be estimated as in step 2 of the case $j = 1$, so it is bounded by the induction hypothesis.

Next, since

$$\operatorname{dist}(\operatorname{supp}(\eta_{\epsilon,b}), \operatorname{supp}(\psi_\epsilon)) \ge \frac{\epsilon}{2},$$

we have by Lemma 3.16

$$\|\eta_{\epsilon,b} D_x^{j+\frac{\alpha+1}{2}}(u\psi_\epsilon)\|_{L^2_x} = \|\eta_{\epsilon,b} D_x^{j+\frac{1+\alpha}{2}}(u\psi_\epsilon)\|_{L^2_x} \lesssim \|\eta_{\frac{\epsilon}{8},b+\epsilon}\|_{L^\infty_x} \|u_0\|_{L^2_x}.$$

Gathering the estimates above, (6.35) follows.

We have proved that locally in the interval $[\epsilon, b]$ there exist $j + \frac{\alpha+1}{2}$ derivatives. So, by Lemma 3.15 we obtain

$$\|D_x^{j+\frac{1-\alpha}{2}}(u\eta_{\epsilon,b})\|_{L^2_T L^2_x} \lesssim \|D_x^{j+\frac{1+\alpha}{2}}(u\eta_{\epsilon,b})\|_{L^2_T L^2_x} + \|u_0\|_{L^2_x};$$

then, as before

$$D_x^{j+\frac{1-\alpha}{2}} u\eta_{\epsilon,b} = c_j D_x^{j+\frac{1-\alpha}{2}}(u\eta_{\epsilon,b}) - c_j [D_x^{j+\frac{1-\alpha}{2}}; \eta_{\epsilon,b}](u\chi_{\epsilon,b} + u\phi_{\epsilon,b} + u\psi_\epsilon),$$

where c_j is a constant depending only on j.

Hence, if we proceed as in the proof of the claim (6.35) above, we have

$$\|D_x^{j+\frac{1-\alpha}{2}} u\eta_{\epsilon,b}\|_{L^2_T L^2_x} < \infty. \tag{6.39}$$

Therefore

$$\int_0^T |A_1(t)|\, \mathrm{d}t = v\|D_x^{j+\frac{1-\alpha}{2}} u\eta_{\epsilon,b}\|^2_{L^2_T L^2_x} < \infty.$$

Step 3.2: To handle the term A_2 we use the same procedure as in the previous steps. First,

$$A_2(t) = -\frac{1}{2}\int_{\mathbb{R}} D_x^{\frac{2j+1-\alpha}{2}} u[\mathcal{H} D_x^{2+\alpha}; \chi^2_{\epsilon,b}] D_x^{\frac{2j+1-\alpha}{2}} u\, \mathrm{d}x \tag{6.40}$$

since

$$[\mathcal{H} D_x^{\alpha+2}; \chi^2_{\epsilon,b}] + \tfrac{1}{2} P_n(\alpha+2) + R_n(\alpha+2) = \tfrac{1}{2}\mathcal{H} P_n(\alpha+2)\mathcal{H} \tag{6.41}$$

for some positive integer n. Substituting (6.41) into (6.40) produces

$$\begin{aligned} A_2(t) &= \frac{1}{2}\int_{\mathbb{R}} D_x^{\frac{2j+1-\alpha}{2}} u(R_n(\alpha+2) D_x^{\frac{2j+1-\alpha}{2}} u)\, \mathrm{d}x \\ &\quad + \frac{1}{4}\int_{\mathbb{R}} D_x^{\frac{2j+1-\alpha}{2}} u(P_n(\alpha+2) D_x^{\frac{2j+1-\alpha}{2}} u)\, \mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}} D_x^{\frac{2j+1-\alpha}{2}} u(\mathcal{H} P_n(\alpha+2)\mathcal{H} D_x^{\frac{2j+1-\alpha}{2}} u)\, \mathrm{d}x \\ &= A_{2,1}(t) + A_{2,2}(t) + A_{2,3}(t). \end{aligned} \tag{6.42}$$

As above we deal first with the crucial term in the decomposition associated to A_2, that is, $A_{2,1}$.

Applying Plancherel's identity yields

$$A_{2,1}(t) = \frac{1}{2}\int_{\mathbb{R}} u D_x^{\frac{2j+1-\alpha}{2}} \{R_n(\alpha+2) D_x^{\frac{2j+1-\alpha}{2}} u\}\, \mathrm{d}x.$$

We fix n such that (3.26) is satisfied. In this case we have to take $a = \alpha + 2$ and $\sigma = \frac{2j+1-\alpha}{2}$ to get $n = j$. As occurs in the previous cases it is possible for $n = j + 1$.

Thus, by construction $R_j(\alpha+2)$ is bounded in L^2_x (see Proposition 3.25).

Then

$$|A_{2,1}(t)| \lesssim \|u_0\|_{L^2_x}^2 \|\widehat{D_x^{2j+3}(\chi_{\epsilon,b}^2)}\|_{L^1_\xi}$$

and

$$\int_0^T |A_{2,1}(t)|\,\mathrm{d}t \lesssim \|u_0\|_{L^2_x}^2 \sup_{0\le t\le T} \|\widehat{D_x^{2j+3}(\chi_{\epsilon,b}^2)}\|_{L^1_\xi}.$$

Substituting $P_j(\alpha+2)$ into $A_{2,2}$

$$\begin{aligned}A_{2,2}(t) &= \left(\frac{\alpha+2}{4}\right)\int_{\mathbb{R}}(\mathcal{H}\partial_x^{j+1}u)^2(\chi_{\epsilon,b}^2)'\,\mathrm{d}x + \left(\frac{\alpha+2}{2}\right)\sum_{l=1}^{j} c_{2l+1}(-1)^l 4^{-l}\int_{\mathbb{R}}(D_x^{j-l+1}u)^2(\chi_{\epsilon,b}^2)^{(2l+1)}\,\mathrm{d}x\\ &= A_{2,2,1}(t) + \sum_{l=1}^{j-1} A_{2,2,l}(t) + A_{2,2,j}(t).\end{aligned}$$

Note that $A_{2,2,1}$ is positive and it gives the smoothing effect after integration in time, and $A_{2,2,j}$ is bounded by using the local theory. To handle the remainder terms we recall that by construction

$$|(\chi_{\epsilon,b})^{(j)}(x)| \lesssim \chi'_{\frac{\epsilon}{3},b+\epsilon}(x) \lesssim \chi_{\frac{\epsilon}{9},b+\frac{10\epsilon}{9}}(x)\chi'_{\frac{\epsilon}{9},b+\frac{10\epsilon}{9}}(x) \tag{6.43}$$

for $x \in \mathbb{R},\ j \in \mathbb{Z}^+$.

Hence for $j > 2$

$$\begin{aligned}\int_0^T |A_{2,2,l}(t)|\,\mathrm{d}t &\lesssim \int_0^T\int_{\mathbb{R}}(D_x^{j-l+1}u)^2\chi'_{\frac{\epsilon}{3},b+\epsilon}\,\mathrm{d}x\,\mathrm{d}t\\ &\lesssim \int_0^T\int_{\mathbb{R}}(D_x^{j-l+1}u)^2\chi_{\frac{\epsilon}{9},b+\frac{10\epsilon}{9}}\chi'_{\frac{\epsilon}{9},b+\frac{10\epsilon}{9}}\,\mathrm{d}x\,\mathrm{d}t;\end{aligned} \tag{6.44}$$

thus if we apply (6.33) with $\left(\frac{\epsilon}{9}, b+\frac{4\epsilon}{3}\right)$ instead of (ϵ, b) we obtain

$$\int_0^T\int_{\mathbb{R}}(D_x^{j-l+1}u)^2(\chi_{\frac{\epsilon}{9},b+\frac{10\epsilon}{9}}\chi'_{\frac{\epsilon}{9},b+\frac{10\epsilon}{9}})\,\mathrm{d}x\,\mathrm{d}t \le c^*_{l,2}$$

for $l = 1, 2, \ldots, j-1$.

Meanwhile,

$$\begin{aligned}A_{2,3}(t) &= \left(\frac{\alpha+2}{4}\right)\int_{\mathbb{R}}(\partial_x^{j+1}u)^2(\chi_{\epsilon,b}^2)'\,\mathrm{d}x + \left(\frac{\alpha+2}{4}\right)\sum_{l=1}^{j} c_{2l+1}(-1)^l 4^{-l}\int_{\mathbb{R}}(\mathcal{H}D_x^{j-l+1}u)^2(\chi_{\epsilon,b}^2)^{(2l+1)}\,\mathrm{d}x\\ &= A_{2,3,1}(t) + \sum_{l=1}^{j-1} A_{2,3,l}(t) + A_{2,3,j}(t).\end{aligned} \tag{6.45}$$

As we can see $A_{2,3,1} \ge 0$ and it represents the smoothing effect. Additionally, applying an argument similar to that employed in (6.43)–(6.44), it is possible to bound the remainder terms in (6.45). Anyway,

$$\int_0^T |A_{2,3,l}(t)|\,\mathrm{d}t \lesssim c^*_{l,2}, \quad 1 \le l \le j-1.$$

Step 3.3: It only remains to estimate A_3 to finish step 3.

$$\begin{aligned}D_x^{\frac{1-\alpha}{2}}\partial_x^j(u\partial_x u)\chi_{\epsilon,b} &= -\tfrac{1}{2}[D_x^{\frac{1-\alpha}{2}}\partial_x^j;\ \chi_{\epsilon,b}]\partial_x((u\chi_{\epsilon,b})^2 + (u\tilde{\phi}_{\epsilon,b})^2 + (\psi_\epsilon u^2))\\ &\quad + [D_x^{\frac{1-\alpha}{2}}\partial_x^j;\ u\chi_{\epsilon,b}]\partial_x((u\chi_{\epsilon,b}) + (u\phi_{\epsilon,b}) + (u\psi_\epsilon)) + u\chi_{\epsilon,b}D_x^{\frac{1-\alpha}{2}}\partial_x^j(\partial_x u)\\ &= \widetilde{A}_{3,1}(t) + \widetilde{A}_{3,2}(t) + \widetilde{A}_{3,3}(t) + \widetilde{A}_{3,4}(t) + \widetilde{A}_{3,5}(t) + \widetilde{A}_{3,6}(t) + \widetilde{A}_{3,7}(t).\end{aligned} \tag{6.46}$$

Substituting (6.46) into A_3 and applying Hölder's inequality

$$\begin{aligned}A_3(t) &= \sum_{1\le k\le 6}\int_{\mathbb{R}} \widetilde{A}_{3,k}(t)\, D_x^{\frac{1-\alpha}{2}}\partial_x^j u\chi_{\epsilon,b}\,\mathrm{d}x + \int_{\mathbb{R}} \widetilde{A}_{3,7}(t)\, D_x^{\frac{1-\alpha}{2}}\partial_x^j u\chi_{\epsilon,b}\,\mathrm{d}x\\ &\le \sum_{1\le k\le 6}\|\widetilde{A}_{3,k}(t)\|_{L^2_x}\|D_x^{j+\frac{1-\alpha}{2}}u(t)\chi_{\epsilon,b}(\,\cdot+vt)\|_{L^2_x} + \int_{\mathbb{R}}\widetilde{A}_{3,7}(t)D_x^{\frac{1-\alpha}{2}}\partial_x^j u\chi_{\epsilon,b}\,\mathrm{d}x.\\ &= \|D_x^{j+\frac{1-\alpha}{2}}u(t)\chi_{\epsilon,b}(\,\cdot+vt)\|_{L^2_x}\sum_{1\le m\le 6}A_{3,k}(t)+A_{3,7}(t).\end{aligned}$$

The first factor on the right-hand side is the quantity to be estimated.

We will start by estimating the easiest term:

$$\begin{aligned}A_{3,7}(t) &= -\frac{1}{2}\int_{\mathbb{R}}\partial_x u\chi_{\epsilon,b}^2\,(D_x^{\frac{1-\alpha}{2}}\partial_x^j u)^2\,\mathrm{d}x - \int_{\mathbb{R}} u\chi_{\epsilon,b}\chi_{\epsilon,b}'(D_x^{\frac{1-\alpha}{2}}\partial_x^j u)^2\,\mathrm{d}x\\ &= A_{3,7,1}(t)+A_{3,7,2}(t).\end{aligned}$$

We have that

$$|A_{3,7,1}(t)| \lesssim \|\partial_x u(t)\|_{L^\infty_x}\int_{\mathbb{R}}(D_x^{\frac{1-\alpha}{2}}\partial_x^j u)^2\chi_{\epsilon,b}^2\,\mathrm{d}x,$$

where the last integral is the quantity that we want to estimate, and the another factor will be controlled after integration in time.

After integration in time and Sobolev's embedding

$$\begin{aligned}\int_0^T |A_{3,7,2}(t)|\,\mathrm{d}t &\lesssim \int_0^T\int_{\mathbb{R}} u\,(\chi_{\epsilon,b}^2)'(D_x^{\frac{1-\alpha}{2}}\partial_x^j u)\mathrm{d}x\,\mathrm{d}t\\ &\lesssim \Big(\sup_{0\le t\le T}\|u(t)\|_{H_x^{s(\alpha)+}}\Big)\int_0^T\int_{\mathbb{R}}(D_x^{\frac{1-\alpha}{2}}\partial_x^j u)^2\,(\chi_{\epsilon,b}^2)'\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where the integral expression on the right-hand side was already estimated in (6.39).

To handle the contribution coming from $\widetilde{A}_{3,1}$ and $\widetilde{A}_{3,2}$, we apply a combination of (3.4), (3.14) and Lemma 3.15 to obtain

$$\begin{aligned}\|\widetilde{A}_{3,1}(t)\|_{L^2_x} &\lesssim \|D_x^{j+\frac{1-\alpha}{2}}(u\chi_{\epsilon,b})\|_{L^2_x}\|u\|_{L^\infty_x} + \|u_0\|_{L^2_x}\|u\|_{L^\infty_x},\\ \|\widetilde{A}_{3,2}(t)\|_{L^2_x} &\lesssim \|D_x^{j+\frac{1-\alpha}{2}}(u\tilde{\phi}_{\epsilon,b})\|_{L^2_x}\|u\|_{L^\infty_x} + \|u_0\|_{L^2_x}\|u\|_{L^\infty_x}.\end{aligned}\tag{6.47}$$

The condition on the supports of $\chi_{\epsilon,b}$ and ψ_ϵ combined with Lemma 3.16 implies

$$\|\widetilde{A}_{3,3}(t)\|_{L^2_x} \lesssim \|u_0\|_{L^2_x}\|u\|_{L^\infty_x}.$$

By using (3.2) and (3.14)

$$\begin{aligned}&\|\widetilde{A}_{3,4}(t)\|_{L^2_x} \lesssim \|\partial_x(u\chi_{\epsilon,b})\|_{L^\infty_x}\|D_x^{j+\frac{1-\alpha}{2}}(u\chi_{\epsilon,b})\|_{L^2_x},\\ &\|\widetilde{A}_{3,5}(t)\|_{L^2_x} \lesssim \|\partial_x(u\chi_{\epsilon,b})\|_{L^\infty_x}\|D_x^{j+\frac{1-\alpha}{2}}(u\phi_{\epsilon,b})\|_{L^2_x} + \|\partial_x(u\phi_{\epsilon,b})\|_{L^\infty_x}\|D_x^{j+\frac{1-\alpha}{2}}(u\chi_{\epsilon,b})\|_{L^2_x}.\end{aligned}$$

An application of Lemma 3.16 leads to

$$\|\widetilde{A}_{3,6}(t)\|_{L^2_x} = \|u\chi_{\epsilon,b}\partial_x D_x^{j+\frac{1-\alpha}{2}}(u\psi_\epsilon)\|_{L^2_x} \lesssim \|u_0\|_{L^2_x}\|u\|_{L^\infty_x}. \tag{6.48}$$

To complete the estimate in (6.47)–(6.48) we write

$$\chi_{\epsilon,b}(x) + \phi_{\epsilon,b}(x) + \psi_\epsilon(x) = 1 \quad \text{for all } x \in \mathbb{R};$$

then

$$\begin{aligned} D_x^{j+\frac{1-\alpha}{2}}(u\chi_{\epsilon,b}) &= D_x^{j+\frac{1-\alpha}{2}}u\chi_{\epsilon,b} + [D_x^{j+\frac{1-\alpha}{2}};\chi_{\epsilon,b}](u\chi_{\epsilon,b} + u\phi_{\epsilon,b} + u\psi_\epsilon) \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Notice that $\|I_1\|_{L^2_x}$ is the quantity to be estimated. In contrast, I_4 is handled by using Lemma 3.16. In regards to $\|I_2\|_{L^2_x}$ and $\|I_3\|_{L^2_x}$, Lemma 3.13 combined with the local theory, and the step 2 in the case $j = 1$ produces the required bounds.

By Theorem 3.7 and Hölder's inequality

$$\begin{aligned} &\|D_x^{j+\frac{1+\alpha}{2}}(u\phi_{\epsilon,b})\|_{L^2_x} \\ &\quad\lesssim \|u\|_{L^4_x}\|D_x^{j+\frac{1+\alpha}{2}}\phi_{\epsilon,b}\|_{L^4_x} + \Big\|\sum_{\beta\le j}\frac{1}{\beta!}\partial_x^\beta\phi_{\epsilon,b}D_x^{s,\beta}u\Big\|_{L^2_x} \\ &\quad\lesssim \|u_0\|_{L^2_x}^{1/2}\|u\|_{L^\infty_x}^{1/2} + \sum_{\beta\in\mathbb{Q}_1(j)}\frac{1}{\beta!}\|\partial_x^\beta\phi_{\epsilon,b}D_x^{j-\beta+\frac{\alpha+1}{2}}u\|_{L^2_x} + \sum_{\beta\in\mathbb{Q}_2(j)}\frac{1}{\beta!}\|\partial_x^\beta\phi_{\epsilon,b}\mathcal{H}D_x^{j-\beta+\frac{\alpha+1}{2}}u\|_{L^2_x}, \end{aligned} \tag{6.49}$$

where $\mathbb{Q}_1(j)$, $\mathbb{Q}_2(j)$ denote odd integers and even integers in $\{0, 1, \dots, j\}$ respectively.

To estimate the second term in (6.49), note that $\partial_x^\beta\phi_{\epsilon,b}$ is supported in $\left[\frac{\epsilon}{4}, b\right]$; then

$$\begin{aligned} \sum_{\beta\in\mathbb{Q}_1(j)}\frac{1}{\beta!}\|\partial_x^\beta\phi_{\epsilon,b}D_x^{j-\beta+\frac{\alpha+1}{2}}u\|_{L^2_x} &\lesssim \sum_{\beta\in\mathbb{Q}_1(j)}\frac{1}{\beta!}\|\mathbb{1}_{[\frac{\epsilon}{8},b]}D_x^{j-\beta+\frac{\alpha+1}{2}}u\|_{L^2_x} \\ &\lesssim \sum_{\beta\in\mathbb{Q}_1(j)}\frac{1}{\beta!}\|\eta_{\frac{\epsilon}{24},b+\frac{7\epsilon}{24}}D_x^{j-\beta+\frac{\alpha+1}{2}}u\|_{L^2_x}. \end{aligned}$$

Hence, after integrating in time and applying (6.33) with $(\epsilon, b) = \left(\frac{\epsilon}{24}, b + \frac{7\epsilon}{24}\right)$ we obtain

$$\sum_{\beta\in\mathbb{Q}_1(j)}\frac{1}{\beta!}\|\eta_{\frac{\epsilon}{24},b+\frac{7\epsilon}{24}}D_x^{j-\beta+\frac{\alpha+1}{2}}u\|_{L^2_TL^2_x} \lesssim \sum_{\beta\in\mathbb{Q}_1(j)}(c^*_{j-\beta,1})^{\frac{1}{2}} < \infty$$

by the induction hypothesis.

Analogously, we can handle the third term in (6.49):

$$\sum_{\beta\in\mathbb{Q}_2(j),\beta\neq j}\frac{1}{\beta!}\|\partial_x^\beta\phi_{\epsilon,b}\mathcal{H}D_x^{j-\beta+\frac{\alpha+1}{2}}u\|_{L^2_TL^2_x} \lesssim \sum_{\beta\in\mathbb{Q}_2(j),\beta\neq j}(c^*_{j-\beta,1})^{\frac{1}{2}} + \|u\|_{L^\infty_TH_x^{(3-\alpha)/2}} < \infty.$$

Therefore, after integrating in time and applying Hölder's inequality we have

$$\|D_x^{j+\frac{1+\alpha}{2}}(u\phi_{\epsilon,b})\|_{L^2_TL^2_x} < \infty.$$

Next, by interpolation and Young's inequality

$$\|D_x^{j+\frac{1-\alpha}{2}}(u\phi_{\epsilon,b})\|_{L^2_TL^2_x} \lesssim \|D_x^{j+\frac{1+\alpha}{2}}(u\phi_{\epsilon,b})\|_{L^2_TL^2_x} + \|u_0\|_{L^2_x} < \infty. \tag{6.50}$$

If we apply (6.49)–(6.50) then

$$\|D_x^{j+\frac{1-\alpha}{2}}(u\tilde{\phi}_{\epsilon,b})\|_{L^2_T L^2_x} < \infty.$$

Finally, after collecting all of the information and applying Gronwall's inequality we obtain

$$\sup_{0\le t\le T} \|D_x^{\frac{1-\alpha}{2}}\partial_x^j u\chi_{\epsilon,b}\|^2_{L^2_x} + \|\partial_x^{j+1}u\eta_{\epsilon,b}\|^2_{L^2_T L^2_x} + \|\mathcal{H}\partial_x^{j+1}u\eta_{\epsilon,b}\|^2_{L^2_T L^2_x} \le c^*_{j,2},$$

where $c^*_{j,2} = c^*_{j,2}(\alpha; \epsilon; T; v; \|u_0\|_{H_x^{(3-\alpha)/2}}; \|D_x^{(1-\alpha)/2}\partial_x^j u_0\chi_{\epsilon,b}\|_{L^2_x})$ for any $\epsilon > 0,\ b \ge 5\epsilon$ and $v \ge 0$.

This finishes the induction process.

To justify the previous estimates we shall follow the following argument of regularization. For arbitrary initial data $u_0 \in H^s(\mathbb{R})\ s > \frac{3-\alpha}{2}$, we consider the regularized initial data $u_0^\mu = \rho_\mu * u_0$ with $\rho \in C_0^\infty(\mathbb{R}),\ \operatorname{supp}\rho \subset (-1,1),\ \rho \ge 0,\ \|\rho\|_{L^1} = 1$ and

$$\rho_\mu(x) = \mu^{-1}\rho\left(\frac{x}{\mu}\right) \quad \text{for } \mu > 0.$$

The solution u^μ of the IVP (1.1) corresponding to the smoothed data $u_0^\mu = \rho_\mu * u_0$ satisfies

$$u^\mu \in C([0,T] : H^\infty(\mathbb{R}));$$

we note that the time of existence is independent of μ.

Therefore, the smoothness of u^μ allows us to conclude that

$$\sup_{0\le t\le T} \|\partial_x^m u^\mu \chi_{\epsilon,b}\|^2_{L^2_x} + \|D_x^{m+\frac{1+\alpha}{2}}u^\mu\|^2_{L^2_T L^2_x} + \|\mathcal{H}D_x^{m+\frac{1+\alpha}{2}}u^\mu\eta_{\epsilon,b}\|^2_{L^2_T L^2_x} \le c^*,$$

where $c^* = c^*(\alpha; \epsilon; T; v; \|u_0^\mu\|_{H_x^{(3-\alpha)/2}}; \|\partial_x^m u_0^\mu \chi_{\epsilon,b}\|_{L^2_x})$. In fact our next task is to prove that the constant c^* is independent of the parameter μ.

The independence from the parameter $\mu > 0$ can be reached first noticing that

$$\|u_0^\mu\|_{H_x^{(3-\alpha)/2}} \le \|u_0\|_{H_x^{(3-\alpha)/2}}\|\hat{\rho}_\mu\|_{L^\infty_\xi} = \|u_0\|_{H_x^{(3-\alpha)/2}}\|\rho_\mu\|_{L^1_x} = \|u_0\|_{H_x^{(3-\alpha)/2}}.$$

Next, since $\chi_{\epsilon,b}(x) = 0$ for $x \le \epsilon$, restricting to $\mu \in (0,\epsilon)$ it follows by Young's inequality

$$\int_\epsilon^\infty (\partial_x^m u_0^\mu)^2\, dx \le \|\rho_\mu\|_{L^1_\xi}\|\partial_x^m u_0\|_{L^2_x((0,\infty))} = \|\partial_x^m u_0\|_{L^2_x((0,\infty))}.$$

Using the continuous dependence of the solution upon the data we have that

$$\sup_{t\in[0,T]} \|u^\mu(t) - u(t)\|_{H_x^{(3-\alpha)/2}} \xrightarrow[\mu\to 0]{} 0.$$

Combining this fact with the independence of the constant c^* from the parameter μ, weak compactness and Fatou's lemma, the theorem holds for all $u_0 \in H^s(\mathbb{R}),\ s > \frac{3-\alpha}{2}$. □

Remark 6.51. The proof of Theorem B remains valid for the defocusing dispersive generalized Benjamin–Ono equation

$$\begin{cases} \partial_t u - D_x^{\alpha+1}\partial_x u - u\,\partial_x u = 0, & x, t \in \mathbb{R},\ 0 < \alpha < 1, \\ u(x,0) = u_0(x). \end{cases}$$

In this direction, the propagation of regularity holds for $u(-x,-t)$, where $u(x,t)$ a solution of (1.1). In other words, this means that for initial data satisfying the conditions (1.9) and (1.11) on the left-hand side of the real line, Theorem B remains valid backward in time.

A consequence of the Theorem B is the following corollary, which describes the asymptotic behavior of the function in (1.10).

Corollary 6.52. *Let* $u \in C([-T,T] : H^{(3-\alpha)/2}(\mathbb{R}))$ *be a solution of the equation in* (1.1) *described by Theorem B.*

Then, for any $t \in (0,T]$ *and* $\delta > 0$

$$\int_{-\infty}^{\infty} \frac{1}{\langle x_-\rangle^{j+\delta}} (\partial_x^j u)^2(x,t)\,\mathrm{d}x \le \frac{c}{t}, \tag{6.53}$$

where $x_- = \max\{0,-x\}$, *c is a positive constant and* $\langle x\rangle := \sqrt{1+x^2}$.

For the proof of (6.53) we use the following lemma provided in [Segata and Smith 2017].

Lemma 6.54. *Let* $f : [0,\infty) \to [0,\infty)$ *be a continuous function. If for* $a > 0$

$$\int_0^a f(x)\,\mathrm{d}x \le ca^p,$$

then for every $\delta > 0$

$$\int_0^{\infty} \frac{f(x)}{\langle x\rangle^{p+\delta}}\,\mathrm{d}x \le c(p).$$

Proof. The proof follows by using a smooth dyadic partition of unit of $\mathbb{R}^+$. □

Remark 6.55. Observe that the lemma also applies when integrating a nonnegative function on the interval $[-(a+\epsilon), -\epsilon]$, implying decay on the left half-line.

Proof of Corollary 6.52. We shall recall that Theorem B with $x_0 = 0$ asserts that any $\epsilon > 0$

$$\sup_{t\in[0,T]} \int_{\epsilon - vt}^{\infty} (\partial_x^j u)^2(x,t)\,\mathrm{d}x \le c^*.$$

For fixed $t \in [0,T]$ we split the integral term as follows:

$$\int_{\epsilon-vt}^{\infty} (\partial_x^j u)^2(x,t)\,\mathrm{d}x = \int_{\epsilon-vt}^{\epsilon} (\partial_x^j u)^2(x,t)\,\mathrm{d}x + \int_{\epsilon}^{\infty} (\partial_x^j u)^2(x,t)\,\mathrm{d}x.$$

The second term in the right-hand side is easily bounded by using Theorem B with $v = 0$. Hence, we just need to estimate the first integral in the right-hand side.

Notice that after making a change of variables,

$$\int_{\epsilon-vt}^{\epsilon} (\partial_x^j u)^2(x,t)\,\mathrm{d}x = \int_{-(\epsilon-vt)}^{-\epsilon} (\partial_x^j u)^2(x+2\epsilon,t)\,\mathrm{d}x \le c^*.$$

Thus by using Lemma 6.54 and Remark 6.55 we find

$$\int_{-\infty}^{-\epsilon} \frac{1}{\langle x+2\epsilon\rangle^{j+\delta}} (\partial_x^j u)^2(x+2\epsilon,t)\,\mathrm{d}x = \int_{-\infty}^{\epsilon} \frac{1}{\langle x\rangle^{j+\delta}} (\partial_x^j u)^2(x,t)\,\mathrm{d}x \le \frac{c^*}{t^j}.$$

In summary, we have proved that for all $j \in \mathbb{Z}^+,\ j \geq 2$ and any $\delta > 0$

$$\int_{-\infty}^{\epsilon} \frac{1}{\langle x \rangle^{j+\delta}} (\partial_x^j u)^2(x,t)\,\mathrm{d}x \leq \frac{c^*}{t}, \tag{6.56}$$

$$\int_{\epsilon}^{\infty} (\partial_x^j u)^2(x,t)\,\mathrm{d}x \leq c^*. \tag{6.57}$$

If we apply the Lemma 6.54 to (6.57) we obtain extra decay in the right-hand side. This allow us to obtain a uniform expression that combines (6.56) and (6.57); that is, there exists a constant c such that for any $t \in (0, T]$ and $\delta > 0$

$$\int_{-\infty}^{\infty} \frac{1}{\langle x_- \rangle^{j+\delta}} (\partial_x^j u)^2(x,t)\,\mathrm{d}x \leq \frac{c}{t}. \qquad \square$$

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OPTIMAL REGULARITY IN TIME AND SPACE FOR THE POROUS MEDIUM EQUATION

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Regularity estimates in time and space for solutions to the porous medium equation are shown in the scale of Sobolev spaces. In addition, higher spatial regularity for powers of the solutions is obtained. Scaling arguments indicate that these estimates are optimal. In the linear limit, the proven regularity estimates are consistent with the optimal regularity of the linear case.

1. Introduction

We prove estimates on the time and space regularity of solutions to porous medium equations

$$\begin{cases} \partial_t u - \Delta u^{[m]} = S & \text{in } (0,T)\times\mathbb{R}^d, \\ u(0) = u_0 & \text{in } \mathbb{R}^d, \end{cases} \tag{1-1}$$

where $u^{[m]} := |u|^{m-1}u$ with $m > 1$, $u_0 \in L^1(\mathbb{R}^d)$ and $S \in L^1((0,T)\times\mathbb{R}^d)$. Solutions to porous medium equations are known to exhibit nonlinear phenomena like slow diffusion or filling up of holes at finite rate: If the initial data is compactly supported, then the support of the solution evolves with a free boundary that has finite speed of propagation. The solution close to the boundary is not smooth even for smooth initial data and zero forcing.

Despite many works on the problem of regularity of solutions to porous medium equations, until recently, established regularity results in the literature in terms of Hölder or Sobolev spaces were restricted to spatial differentiability of order less than 1; see [Ebmeyer 2005; Tadmor and Tao 2007]. For $m \searrow 1$ this is in stark contrast to the limiting case $m = 1$, where u is up to twice weakly differentiable in space. Very recently, the first author has proven optimal spatial regularity for (1-1) in [Gess 2020] for initial data

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$u_0 \in (L^1 \cap L^{1+\varepsilon})(\mathbb{R}^d)$ for some $\varepsilon > 0$. This leaves open three main aspects addressed in the present work: first, the derivation of optimal[1] space-time regularity, second, the limit case $u_0 \in L^1(\mathbb{R}^d)$, which is of particular importance since it covers the case of the Barenblatt solution for which the estimates are shown to be optimal, see Section 3 below, and third, higher-order integrability. Solving these three open problems is the purpose of the present paper.

The first main result provides optimal space-time regularity for L^1 data.

Theorem 1.1. *Let $u_0 \in L^1(\mathbb{R}^d)$, $S \in L^1((0,T) \times \mathbb{R}^d)$ and $m \in (1,\infty)$. Let u be the unique entropy solution to* (1-1) *on* $[0,T] \times \mathbb{R}^d$.

(i) *Let $p \in (1,m]$ and define*

$$\kappa_t := \frac{m-p}{p}\frac{1}{m-1}, \quad \kappa_x := \frac{p-1}{p}\frac{2}{m-1}.$$

Then for all $\sigma_t \in [0,\kappa_t) \cup \{0\}$ and $\sigma_x \in [0,\kappa_x)$ we have

$$u \in W^{\sigma_t,p}(0,T; W^{\sigma_x,p}(\mathbb{R}^d)).$$

Moreover, we have the estimate

$$\|u\|_{W^{\sigma_t,p}(0,T;W^{\sigma_x,p}(\mathbb{R}^d))} \lesssim \|u_0\|_{L^1_x}^m + \|S\|_{L^1_{t,x}}^m + 1. \tag{1-2}$$

(ii) *Suppose $\mathscr{O} \Subset \mathbb{R}^d$. Let $s \in [0,1]$ and define*

$$p := s(m-1)+1, \quad \kappa_t := \frac{1-s}{s(m-1)+1}, \quad \kappa_x := \frac{2s}{s(m-1)+1}.$$

Then for all $\sigma_t \in [0,\kappa_t) \cup \{0\}$, $\sigma_x \in [0,\kappa_x) \cup \{0\}$ and $q \in [1,p]$ we have

$$u \in W^{\sigma_t,q}(0,T; W^{\sigma_x,q}(\mathscr{O})).$$

Moreover, we have the estimate

$$\|u\|_{W^{\sigma_t,q}(0,T;W^{\sigma_x,q}(\mathscr{O}))} \lesssim \|u_0\|_{L^1_x}^m + \|S\|_{L^1_{t,x}}^m + 1. \tag{1-3}$$

In [Tadmor and Tao 2007; Ebmeyer 2005] initial data in $L^1 \cap L^\infty$ was considered. However, the methods employed in these works did not allow a systematic analysis of the order of integrability of the solutions. For example, the results of [Ebmeyer 2005] are restricted to the particular order of integrability $p = 2/(m+1)$, while [Tadmor and Tao 2007] is restricted to $p = 1$. In the second main result we provide a systematic treatment of higher-order integrability. In particular, this includes and generalizes the corresponding results of [Ebmeyer 2005] in terms of regularity in Sobolev spaces.

Noting that the regularity of $u^{[m]}$ contains information on the time regularity of u in light of (1-1), in addition, we analyze the spatial regularity of powers of the solution u^μ for $\mu \in [1,m]$.

Theorem 1.2. *Let $u_0 \in L^1(\mathbb{R}^d) \cap L^\rho(\mathbb{R}^d)$, $S \in L^1([0,T] \times \mathbb{R}^d) \cap L^\rho([0,T] \times \mathbb{R}^d)$ for some $\rho \in (1,\infty)$ and assume $m \in (1,\infty)$. Let u be the unique entropy solution to* (1-1) *on* $[0,T] \times \mathbb{R}^d$.

[1] Optimality is indicated by scaling arguments in Section 3 below, and the derived estimates are consistent with the optimal space-time regularity in the linear case $m = 1$.

(i) *Let* $\mu \in [1, m]$. *Then for all*

$$p \in \left(1, \frac{m-1+\rho}{\mu}\right), \quad \sigma_x \in \left[0, \frac{\mu p - 1}{p} \frac{2}{m-2+\rho}\right)$$

we have

$$u^{[\mu]} \in L^p(0, T; W^{\sigma_x, p}(\mathbb{R}^d)),$$

and we have the estimate

$$\|u^{[\mu]}\|_{L^p(0,T;W^{\sigma_x,p}(\mathbb{R}^d))} \lesssim \|u_0\|^{\mu\rho}_{L^1_x \cap L^\rho_x} + \|S\|^{\mu\rho}_{L^1_{t,x} \cap L^\rho_{t,x}} + 1. \tag{1-4}$$

(ii) *Let* $p \in (\rho, m-1+\rho)$ *and define*

$$\kappa_t := \frac{m-1+\rho-p}{p} \frac{1}{m-1}, \quad \kappa_x := \frac{p-\rho}{p} \frac{2}{m-1}.$$

Then for all $\sigma_t \in [0, \kappa_t)$ *and* $\sigma_x \in [0, \kappa_x)$ *we have*

$$u \in W^{\sigma_t, p}(0, T; W^{\sigma_x, p}(\mathbb{R}^d)).$$

Moreover, *we have the estimate*

$$\|u\|_{W^{\sigma_t,p}(0,T;W^{\sigma_x,p}(\mathbb{R}^d))} \lesssim \|u_0\|^{\rho}_{L^1_x \cap L^\rho_x} + \|S\|^{\rho}_{L^1_{t,x} \cap L^\rho_{t,x}} + 1. \tag{1-5}$$

Much as in Theorem 1.1, if one restricts to estimates that are localized in space, the rigid interdependency of the coefficients in Theorem 1.2 can be relaxed.

Corollary 1.3. *Under the assumptions of Theorem 1.2*, *suppose* $\mathcal{O} \Subset \mathbb{R}^d$.

(i) *Let* $\mu \in [1, m]$. *Then for all* $\sigma_x \in [0, 2\mu/m)$ *and* $q \in [1, m/\mu]$ *we have*

$$u^{[\mu]} \in L^q(0, T; W^{\sigma_x, q}(\mathcal{O})),$$

and we have the estimate

$$\|u^{[\mu]}\|_{L^q(0,T;W^{\sigma_x,q}(\mathcal{O}))} \lesssim \|u_0\|^{\mu\rho}_{L^1_x \cap L^\rho_x} + \|S\|^{\mu\rho}_{L^1_{t,x} \cap L^\rho_{t,x}} + 1. \tag{1-6}$$

(ii) *Let* $s \in [0, 1]$ *and define*

$$p := s(m-1)+1, \quad \kappa_t := \frac{1-s}{s(m-1)+1}, \quad \kappa_x := \frac{2s}{s(m-1)+1}.$$

Then for all $\sigma_t \in [0, \kappa_t) \cup \{0\}$, $\sigma_x \in [0, \kappa_x) \cup \{0\}$ *and* $q \in [1, p]$ *we have*

$$u \in W^{\sigma_t, q}(0, T; W^{\sigma_x, q}(\mathcal{O})).$$

Moreover, *we have the estimate*

$$\|u\|_{W^{\sigma_t,q}(0,T;W^{\sigma_x,q}(\mathcal{O}))} \lesssim \|u_0\|^{\rho}_{L^1_x \cap L^\rho_x} + \|S\|^{\rho}_{L^1_{t,x} \cap L^\rho_{t,x}} + 1. \tag{1-7}$$

The methods employed in this work are inspired by [Tadmor and Tao 2007] and rely on the kinetic form of (1-1), that is, with $f(t, x, v) := 1_{v<u(t,x)} - 1_{v<0}$,

$$\partial_t f - m|v|^{m-1} \Delta_x f = \partial_v q + S(t, x) \delta_{u(t,x)}(v) \tag{1-8}$$

for a nonnegative measure q, which allows the use of averaging lemmas and real interpolation. There is a relatively short yet intense history of applying such velocity-averaging techniques to deduce regularizing effects in nonlinear PDEs — from the early works [DiPerna, Lions, and Meyer 1991; Golse, Lions, Perthame, and Sentis 1988; Lions, Perthame, and Tadmor 1994a; 1994b; 1996] to the more recent [Arsénio and Masmoudi 2019; DeVore and Petrova 2001; Golse and Perthame 2013; Golse and Saint-Raymond 2004; Jabin 2009; Jabin and Vega 2004; Perthame 2002]. An essential difference to purely spatial regularity consists in the necessity to work with anisotropic fractional Sobolev spaces, which only in their homogeneous form are nicely adapted to the Fourier analytic methods of this work, much in contrast to the purely spatial case in [Gess 2020]. This leads to the so-called dominating mixed anisotropic Besov spaces introduced in [Schmeisser and Triebel 1987]. Passing from these homogeneous anisotropic spaces to standard inhomogeneous fractional Sobolev spaces is delicate and treated in detail below. A main ingredient in the proof of optimal regularity in [Gess 2020] was the existence of singular moments $\int_{t,x,v} |v|^{-\gamma} q$ for $\gamma \in (0,1)$. This ceases to be true for general L^1-initial data. This difficulty is overcome in the present work by treating separately the degeneracy at $|v| = 0$ and the singularity at $|v| = \infty$ as they appear in (1-8). This also necessitates making use of (1-8) in the case of small spatial modes ξ in order to obtain optimal time regularity; see Corollary 4.7 below.

Comments on the literature. The (spatial) regularity of solutions to porous medium equations in Sobolev spaces has previously been considered in [Ebmeyer 2005; Gess 2020; Tadmor and Tao 2007]. Since our main focus is on time-space regularity, we refer to [Gess 2020] for a more detailed account on the available literature in this regard.

In the case of nonnegative solutions the spatial regularity of special types of powers of solutions has been investigated in the literature. For example, much work is devoted to the pressure defined by $v := (m/(m-1))u^{m-1}$; see, e.g., [Vázquez 2007]. In the recent work [Gianazza and Schwarzacher 2019] the authors proved higher integrability for nonnegative, local weak solutions to forced porous medium equations in the sense that $u^{(m+1)/2} \in L^{2+\varepsilon}_{\text{loc}}((0,T); W^{1,2+\varepsilon}_{\text{loc}})$ for all $\varepsilon > 0$ small enough. This result was generalized in [Bögelein, Duzaar, Korte, and Scheven 2019].

The analysis of regularity in time of solutions to porous medium equations (without forcing) has a long history initiated in [1979] and continued in [Crandall, Pazy, and Tartar 1979; Bénilan and Crandall 1981], where it was shown that

$$\partial_t u \in L^1_{\text{loc}}((0,\infty); L^1(\mathbb{R}^d)) \tag{1-9}$$

for $u_0 \in L^1(\mathbb{R}^d)$. Subsequently, Crandall and Pierre [1982a; 1982b] devoted considerable effort to relaxing the required assumptions on the nonlinearity ψ in the case of generalized porous medium equations

$$\partial_t u - \Delta\psi(u) = 0 \quad \text{in } (0,T) \times \mathbb{R}^d. \tag{1-10}$$

More precisely, in [Crandall and Pierre 1982a] assuming the global generalized homogeneity condition

$$\nu \frac{\psi(r)\psi''(r)}{(\psi'(r))^2} \in [m, M] \tag{1-11}$$

for some $0 < m < M$, $\nu \in \{\pm 1\}$ and all $r \in \mathbb{R}$, (1-9) was recovered.

It should be noted that the methods developed in these works are restricted to the nonforced case $S \equiv 0$. In fact, for $S \not\equiv 0$, the linear case $m = 1$ demonstrates that (1-9) should not be expected. We are not aware of results proving regularity in time in Sobolev spaces for porous medium equations with nonvanishing forcing. In this sense, restricting to regularity in time alone, the results of the present work can be regarded as the (partial) extension of the results of [Aronson and Bénilan 1979; Bénilan and Crandall 1981; Crandall, Pazy, and Tartar 1979; Crandall and Pierre 1982a; 1982b] to nonvanishing forcing.

We are not aware of previous results on mixed time and space regularity in Sobolev spaces for solutions to porous medium equations.

For simplicity of the presentation we restrict to the nonlinearity $\psi(u) = u^{[m]}$ in this work. However, the methods that we present are not restricted to this case, as long as ψ satisfies a nonlinearity condition as in [Gess 2020]. In addition, by means of a velocity decomposition, i.e., writing

$$u(t,x) = \sum_{i=1}^{K} u^i(t,x) := \sum_{i=1}^{K} \int_v \varphi^i(v) f(t,x,v)\,\mathrm{d}v,$$

where $\varphi^i,\ i = 1, \dots, K$, is a smooth decomposition of the unity, such a nonlinearity condition only needs to be supposed locally at points of degeneracy. This is in contrast to the assumptions, such as (1-11), supposed in the series of works [Aronson and Bénilan 1979; Bénilan and Crandall 1981; Crandall, Pazy, and Tartar 1979; Crandall and Pierre 1982a; 1982b] mentioned above, which can be regarded as *global* generalized homogeneity conditions.

Structure of this work. In Section 2 we collect information on the class of homogeneous and inhomogeneous anisotropic, dominating mixed-derivative spaces employed in this work. The optimality of the obtained estimates is indicated in Section 3 by scaling arguments and by explicit computations in case of the Barenblatt solution. In Section 4 we provide general averaging lemmas (Lemmas 4.2 and 4.4) in the framework of homogeneous dominating mixed-derivative spaces and translate them to more standard inhomogeneous anisotropic fractional Sobolev spaces (Corollaries 4.5, 4.6 and 4.7). In this formulation, they imply the main result by their application to the porous medium equation in Section 5.

2. Preliminaries, notation and function spaces

We use the notation $a \lesssim b$ if there is a universal constant $C > 0$ such that $a \leqslant Cb$. We introduce $a \gtrsim b$ in a similar manner, and write $a \sim b$ if $a \lesssim b$ and $a \gtrsim b$. For a Banach space X and $I \subset \mathbb{R}$ we denote by $C(I; X)$ the space of bounded and continuous X-valued functions endowed with the norm $\|f\|_{C(I;X)} := \sup_{t\in I} \|f(t)\|_X$. If $X = \mathbb{R}$ we write $C(I)$. For $k \in \mathbb{N} \cup \{\infty\}$, the space of k-times continuously differentiable functions is denoted by $C^k(I; X)$. The subspace of $C^k(I; X)$ consisting of compactly supported functions is denoted by $C_c^k(I; X)$. Moreover, we write $\mathscr{M}_{TV}$ for the space of all measures with finite total variation. Throughout this article we use several types of L^p-based function spaces. For a Banach space X and $p \in [1, \infty]$, we endow the Bochner–Lebesgue space $L^p(\mathbb{R}; X)$ with the usual norm

$$\|f\|_{L^p(\mathbb{R};X)} := \left(\int_{\mathbb{R}} \|f(t)\|_X^p \,\mathrm{d}t\right)^{\frac{1}{p}},$$

with the standard modification in the case of $p = \infty$. For $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, the corresponding X-valued Sobolev space is denoted by $W^{k,p}(\mathbb{R}; X)$. If $\sigma \in (0, \infty)$ is noninteger (say $\sigma = k + r$, with $k \in \mathbb{N}_0$ and $r \in (0, 1)$), then we define the X-valued Sobolev–Slobodecki space $W^{\sigma,p}(\mathbb{R}; X)$ as the space of functions in $W^{k,p}(\mathbb{R}; X)$ with

$$\|f\|_{\dot{W}^{\sigma,p}(\mathbb{R};X)} := \left(\int_{\mathbb{R}\times\mathbb{R}} \frac{\|D^k f(t) - D^k f(s)\|_X^p}{|t-s|^{rp+1}} \,\mathrm{d}s\,\mathrm{d}t \right)^{\frac{1}{p}} < \infty, \tag{2-1}$$

again with the usual modification in the case of $p = \infty$. Further, let $\dot{W}^{\sigma,p}(\mathbb{R}; X)$ be the space of all locally integrable X-valued functions f for which (2-1) is finite. If we factor out the equivalence relation $\sim$, where $f \sim g$ if $\|f - g\|_{\dot{W}^{\sigma,p}(\mathbb{R};X)} = 0$, the space $\dot{W}^{\sigma,p}(\mathbb{R}; X)$ equipped with the norm $\|\cdot\|_{\dot{W}^{\sigma,p}(\mathbb{R};X)}$ is a Banach space.

Moreover, in order to treat regularity results in both time and space efficiently, we introduce spaces with dominating mixed derivatives set in the framework of Fourier analysis, that is, corresponding Besov spaces. These spaces have a long history in the literature, beginning with [Nikolsky 1962; 1963a; 1963b]. We refer the reader to [Schmeisser and Triebel 1987]. We adopt the notation of [Schmeisser and Triebel 1987] for the nonhomogeneous spaces. Corresponding homogeneous Besov spaces are treated in [Triebel 1977a; 1977b]; we adapt the notation to be consistent with that of [Schmeisser and Triebel 1987]. We recall from [Triebel 1977a] the definition of the spaces $\mathscr{Z}$ and $\mathscr{Z}'$ replacing the standard Schwartz space $\mathscr{S} = \mathscr{S}(\mathbb{R}^{d+1})$ and the space of tempered distributions $\mathscr{S}' = \mathscr{S}'(\mathbb{R}^{d+1})$ in the definition of homogeneous spaces. As we are concerned with function spaces in the time variable $t \in \mathbb{R}$ and the spatial variable $x \in \mathbb{R}^d$, we introduce, in addition to $\mathbb{R}^{d+1} = \mathbb{R}_t \times \mathbb{R}^d_x$, also the subset

$$\dot{\mathbb{R}}^{d+1} := \{(t, x) \in \mathbb{R}^{d+1} : t|x| \neq 0\}.$$

Note that in [Triebel 1977a], the notation $\overset{+}{\mathbb{R}}{}^2$ is used, which gives a better geometrical intuition of the set taken out of $\mathbb{R}^2$. However, for typesetting reasons, we have decided on the notation $\dot{\mathbb{R}}^{d+1}$. Then we let $\dot{\mathscr{D}}$ be the subset of the standard space of test functions $\mathscr{D}$, consisting of functions with compact support in $\dot{\mathbb{R}}^{d+1}$ and view it as a locally convex space equipped with the canonical topology. Its dual space is denoted by $\dot{\mathscr{D}}'$, and is referred to as distributions over $\dot{\mathbb{R}}^{d+1}$. We define $\mathscr{Z}$ as the image of $\dot{\mathscr{D}} \subset \mathscr{S}$ under the Fourier transform $\mathscr{F}$ in time and space, equipped with the inherited topology from $\dot{\mathscr{D}}$. The corresponding dual space is denoted by $\mathscr{Z}'$. Since $\mathscr{F} : \dot{\mathscr{D}} \to \mathscr{Z}$, we can define by duality the Fourier transform $\mathscr{F} : \mathscr{Z}' \to \dot{\mathscr{D}}'$.

It holds $\mathscr{Z} \subset \mathscr{S}$ with a continuous embedding, but the fact that $\mathscr{Z}$ is not densely embedded in $\mathscr{S}$ prevents one from stating $\mathscr{S}' \subset \mathscr{Z}'$. However, we note that for $p \in (1, \infty)$, the space $L^p(\mathbb{R}^{d+1})$ can be viewed both as subspace of $\mathscr{S}'$ and as a subspace of $\mathscr{Z}'$; see Theorem 3.3 in [Triebel 1977a].

Let φ be a smooth function supported in the annulus $\left\{\xi \in \mathbb{R}^d : \frac{1}{2} \leqslant |\xi| \leqslant 2\right\}$ and such that

$$\sum_{j\in\mathbb{Z}} \varphi_j(\xi) := \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1 \quad \text{for all } \xi \in \mathbb{R}^d \setminus \{0\}.$$

Similarly, let η be a smooth function supported in $\left(-2, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 2\right)$ with

$$\sum_{l\in\mathbb{Z}} \eta_l(\tau) := \sum_{l\in\mathbb{Z}} \eta(2^{-l}\tau) = 1 \quad \text{for all } \tau \in \mathbb{R} \setminus \{0\}.$$

Moreover, define $\phi_j := \varphi_j$ for $j \geqslant 1$ and $\phi_0 := 1 - \sum_{j\geqslant 1} \phi_j$, as well as $\psi_l := \eta_l$ for $l \geqslant 1$ and $\psi_0 := 1 - \sum_{l\geqslant 1} \eta_l$. We will use the shorthand notation $\eta_l\varphi_j$ for the function $(\tau, \xi) \mapsto \eta_l(\tau)\varphi_j(\xi)$, and similarly for combinations of ψ_l and ϕ_j.

Definition 2.1. Let $\sigma_i \in (-\infty, \infty)$, $i = t, x$, and $p \in [1, \infty]$. Set $\bar\sigma := (\sigma_t, \sigma_x)$.

(i) The homogeneous Besov space with dominating mixed derivatives $S^{\bar\sigma}_{p,\infty}\dot B(\mathbb{R}^{d+1})$ is given by

$$S^{\bar\sigma}_{p,\infty}\dot B := S^{\bar\sigma}_{p,\infty}\dot B(\mathbb{R}^{d+1}) := \{f \in \mathscr{Z}' : \|f\|_{S^{\bar\sigma}_{p,\infty}\dot B} < \infty\},$$

with the norm

$$\|f\|_{S^{\bar\sigma}_{p,\infty}\dot B} := \sup_{l,j\in\mathbb{Z}} 2^{\sigma_t l} 2^{\sigma_x j} \|\mathscr{F}^{-1}_{t,x} \eta_l\varphi_j \mathscr{F}_{t,x} f\|_{L^p(\mathbb{R}^{d+1})}.$$

Similarly, the space $S^{\bar\sigma}_{p,\infty,(\infty)}\dot B(\mathbb{R}^{d+1})$ is given via the norm

$$\|f\|_{S^{\bar\sigma}_{p,\infty,(\infty)}\dot B} := \sup_{l,j\in\mathbb{Z}} 2^{\sigma_t l} 2^{\sigma_x j} \|\mathscr{F}^{-1}_{t,x} \eta_l\varphi_j \mathscr{F}_{t,x} f\|_{L^{p,\infty}(\mathbb{R}^{d+1})}.$$

(ii) The homogeneous Chemin–Lerner spaces $\widetilde L^p_t \dot B^{\sigma_x}_{p,\infty}(\mathbb{R}^{d+1})$ and $\widetilde L^p_x \dot B^{\sigma_t}_{p,\infty}(\mathbb{R}^{d+1})$ are given by

$$\widetilde L^p_t \dot B^{\sigma_x}_{p,\infty} := \widetilde L^p_t \dot B^{\sigma_x}_{p,\infty}(\mathbb{R}^{d+1}) := \{f \in \mathscr{S}' : \|f\|_{\widetilde L^p_t \dot B^{\sigma_x}_{p,\infty}} < \infty\},$$
$$\widetilde L^p_x \dot B^{\sigma_t}_{p,\infty} := \widetilde L^p_x \dot B^{\sigma_t}_{p,\infty}(\mathbb{R}^{d+1}) := \{f \in \mathscr{S}' : \|f\|_{\widetilde L^p_x \dot B^{\sigma_t}_{p,\infty}} < \infty\},$$

with the norms

$$\|f\|_{\widetilde L^p_t \dot B^{\sigma_x}_{p,\infty}} := \sup_{j\in\mathbb{Z}} 2^{\sigma_x j} \|\mathscr{F}^{-1}_x \varphi_j \mathscr{F}_x f\|_{L^p(\mathbb{R}^{d+1})},$$
$$\|f\|_{\widetilde L^p_x \dot B^{\sigma_t}_{p,\infty}} := \sup_{l\in\mathbb{Z}} 2^{\sigma_t l} \|\mathscr{F}^{-1}_t \eta_l \mathscr{F}_t f\|_{L^p(\mathbb{R}^{d+1})},$$

respectively.

(iii) The nonhomogeneous Besov space with dominating mixed derivatives $S^{\bar\sigma}_{p,\infty}B(\mathbb{R}^{d+1})$ is given by

$$S^{\bar\sigma}_{p,\infty}B := S^{\bar\sigma}_{p,\infty}B(\mathbb{R}^{d+1}) := \{f \in \mathscr{S}'(\mathbb{R}^{d+1}) : \|f\|_{S^{\bar\sigma}_{p,\infty}B} < \infty\},$$

with the norm

$$\|f\|_{S^{\bar\sigma}_{p,\infty}B} := \sup_{l,j\geqslant 0} 2^{\sigma_t l} 2^{\sigma_x j} \|\mathscr{F}^{-1}_{t,x} \psi_l\phi_j \mathscr{F}_{t,x} f\|_{L^p(\mathbb{R}^{d+1})}.$$

(iv) The nonhomogeneous Chemin–Lerner space $\widetilde L^p_t B^{\sigma_x}_{p,\infty}(\mathbb{R}^{d+1})$ is given by

$$\widetilde L^p_t B^{\sigma_x}_{p,\infty} := \widetilde L^p_t B^{\sigma_x}_{p,\infty}(\mathbb{R}^{d+1}) := \{f \in \mathscr{S}' : \|f\|_{\widetilde L^p_t B^{\sigma_x}_{p,\infty}} < \infty\},$$

with the norm $\|f\|_{\widetilde L^p_t B^{\sigma_x}_{p,\infty}} := \sup_{j\geqslant 0} 2^{\sigma_x j} \|\mathscr{F}^{-1}_x \phi_j \mathscr{F}_x f\|_{L^p(\mathbb{R}^{d+1})}$.

Remark 2.2. All spaces considered in Definition 2.1 are Banach spaces; see [Triebel 1977a]. Note that for $\vartheta \in \mathbb{R}$, we use the notation $\vartheta\bar\sigma = (\vartheta\sigma_t, \vartheta\sigma_x)$. In this note, we restrict ourselves to the third index of the Besov-type space being infinity, in which case the spaces $S^{\bar\sigma}_{p,\infty}B$ are sometimes called *Nikolsky spaces of dominating mixed derivatives* in the literature. However, there is no conceptual limitation to consider also third indices $q \in [1, \infty]$. By the same token, one could also consider spaces with different indices p and q in different directions. We refer the reader to [Schmeisser and Triebel 1987] for more details concerning such spaces.

Lemma 2.3. *Let $\kappa_x \geqslant 0$ and $p \in [1,\infty]$. Then*

$$\widetilde{L}_t^p B_{p,\infty}^{\kappa_x+\varepsilon}(\mathbb{R}^{d+1}) \subset L^p(\mathbb{R}; W^{\kappa_x,p}(\mathbb{R}^d)) \subset \widetilde{L}_t^p B_{p,\infty}^{\kappa_x-\delta}(\mathbb{R}^{d+1}),$$

whenever $\varepsilon > 0$ and $\delta \in (0,\kappa_x]$.

Proof. This follows from [Bahouri, Chemin, and Danchin 2011, p. 98]. □

Lemma 2.4. *Let $\kappa_t, \kappa_x > 0$ and $p \in [1,\infty)$. Then $S_{p,\infty}^{\bar{\kappa}} B \subset W^{\sigma_t,p}(\mathbb{R}; W^{\sigma_x,p}(\mathbb{R}^d))$ whenever $\sigma_t \in [0,\kappa_t)$ and $\sigma_x \in [0,\kappa_x)$.*

Proof. The proof is a combination of results in [Schmeisser and Triebel 1987], which are written for $\mathbb{R} \times \mathbb{R}$ but also true for $\mathbb{R} \times \mathbb{R}^d$ by an inspection of their respective proofs: Without loss of generality, we can assume that σ_t and σ_x are noninteger. By [loc. cit., Section 2.3.4, Remark 4], we have $W^{\sigma_t,p}(\mathbb{R}; W^{\sigma_x,p}(\mathbb{R}^d)) = SB_{p,p}^{\bar{\sigma}}$; see [loc. cit., Section 2.2.1, Definition 2] for a definition of the latter space. Since by [loc. cit., Section 2.2.3, Proposition 2] we have $S_{p,\infty}^{\bar{\kappa}} B \subset SB_{p,p}^{\bar{\sigma}}$, this yields the claim. □

Lemma 2.5. *Let $\sigma_t, \sigma_x > 0$ and $p \in [1,\infty]$. Then*

$$(L^p(\mathbb{R}^{d+1}) \cap \widetilde{L}_x^p \dot{B}_{p,\infty}^{\sigma_t} \cap \widetilde{L}_t^p \dot{B}_{p,\infty}^{\sigma_x} \cap S_{p,\infty}^{\bar{\sigma}} \dot{B}) = S_{p,\infty}^{\bar{\sigma}} B$$

with equivalent norms.

Proof. As smooth and compactly supported functions, ψ_0 and ϕ_0 extend to L^p multipliers for all $p \in [1,\infty]$; see, e.g., [Bergh and Löfström 1976].
For $f \in (L^p(\mathbb{R}^{d+1}) \cap \widetilde{L}_x^p \dot{B}_{p,\infty}^{\sigma_t} \cap \widetilde{L}_t^p \dot{B}_{p,\infty}^{\sigma_x} \cap S_{p,\infty}^{\bar{\sigma}} \dot{B}) \subset \mathscr{S}'(\mathbb{R}^{d+1})$ we obtain

$$\begin{aligned}
\|f\|_{S_{p,\infty}^{\bar{\sigma}} B} &\leqslant \|\mathscr{F}_{t,x}^{-1}\psi_0\phi_0\mathscr{F}_{t,x} f\|_{L_{t,x}^p} + \sup_{l>0} 2^{\sigma_t l} \|\mathscr{F}_{t,x}^{-1}\eta_l\phi_0\mathscr{F}_{t,x} f\|_{L_{t,x}^p} \\
&\qquad + \sup_{j>0} 2^{\sigma_x j} \|\mathscr{F}_{t,x}^{-1}\psi_0\varphi_j\mathscr{F}_{t,x} f\|_{L_{t,x}^p} + \sup_{l,j>0} 2^{\sigma_t l} 2^{\sigma_x j} \|\mathscr{F}_{t,x}^{-1}\eta_l\varphi_j\mathscr{F}_{t,x} f\|_{L_{t,x}^p} \\
&\lesssim \|f\|_{L_{t,x}^p} + \sup_{l>0} 2^{\sigma_t l} \|\mathscr{F}_t^{-1}\eta_l\mathscr{F}_t f\|_{L_{t,x}^p} \\
&\qquad + \sup_{j>0} 2^{\sigma_x j} \|\mathscr{F}_x^{-1}\varphi_j\mathscr{F}_x f\|_{L_{t,x}^p} + \sup_{l,j>0} 2^{\sigma_t l} 2^{\sigma_x j} \|\mathscr{F}_{t,x}^{-1}\eta_l\varphi_j\mathscr{F}_{t,x} f\|_{L_{t,x}^p} \\
&\lesssim \|f\|_{L_{t,x}^p} + \|f\|_{\widetilde{L}_x^p \dot{B}_{p,\infty}^{\sigma_t}} + \|f\|_{\widetilde{L}_t^p \dot{B}_{p,\infty}^{\sigma_x}} + \|f\|_{S_{p,\infty}^{\bar{\sigma}} \dot{B}}.
\end{aligned}$$

Conversely, for $f \in S_{p,\infty}^{\bar{\sigma}} B$, we estimate the four contributions corresponding to $L^p(\mathbb{R}^{d+1})$, $\widetilde{L}_x^p \dot{B}_{p,\infty}^{\sigma_t}$, $\widetilde{L}_t^p \dot{B}_{p,\infty}^{\sigma_x}$, and $S_{p,\infty}^{\bar{\sigma}} \dot{B}$ separately. We start by noting that due to $\sigma_t, \sigma_x > 0$, the invariance of multiplier norms with respect to dilation, $\eta_l = \eta_l \tilde{\psi}_0$ for $l \leqslant 0$ and $\varphi_j = \varphi_j \tilde{\phi}_0$ for $j \leqslant 0$, where $\tilde{\psi}_0 := \psi_0 + \psi_1$ and $\tilde{\phi}_0 := \phi_0 + \phi_1$, we have

$$\begin{aligned}
\sup_{l\leqslant 0} 2^{\sigma_t l} \|\mathscr{F}_t^{-1}\eta_l\mathscr{F}_t f\|_{L_{t,x}^p} &\lesssim \|\mathscr{F}_t^{-1}\tilde{\psi}_0\mathscr{F}_t f\|_{L_{t,x}^p}, \\
\sup_{j\leqslant 0} 2^{\sigma_x j} \|\mathscr{F}_x^{-1}\varphi_j\mathscr{F}_x f\|_{L_{t,x}^p} &\lesssim \|\mathscr{F}_x^{-1}\tilde{\phi}_0\mathscr{F}_x f\|_{L_{t,x}^p}.
\end{aligned}$$

Furthermore we use the fact that for $\sigma > 0$ one has the estimate $\sum_{n\geqslant 0}|a_n| \lesssim \sup_{n\geqslant 0} 2^{\sigma n}|a_n|$ for any sequence $(a_n)\subset\mathbb{R}$ with a constant depending on σ. With this, we obtain

$$\|f\|_{L^p_{t,x}} \leqslant \sum_{l,j\geqslant 0} \|\mathscr{F}^{-1}_{t,x}\psi_l\phi_j\mathscr{F}_{t,x}f\|_{L^p_{t,x}} \lesssim \sup_{l,j\geqslant 0} 2^{\sigma_t l}2^{\sigma_x j}\|\mathscr{F}^{-1}_{t,x}\psi_l\phi_j\mathscr{F}_{t,x}f\|_{L^p_{t,x}} \leqslant \|f\|_{S^{\bar\sigma}_{p,\infty}B}.$$

Next, we compute

$$\begin{aligned}
\|f\|_{\tilde L^p_x \dot B^{\sigma_t}_{p,\infty}} &\leqslant \sup_{l\leqslant 0} 2^{\sigma_t l}\|\mathscr{F}^{-1}_t\eta_l\mathscr{F}_t f\|_{L^p_{t,x}} + \sup_{l>0} 2^{\sigma_t l}\|\mathscr{F}^{-1}_t\psi_l\mathscr{F}_t f\|_{L^p_{t,x}}\\
&\lesssim \|\mathscr{F}^{-1}_t\tilde\psi_0\mathscr{F}_t f\|_{L^p_{t,x}} + \sup_{l>0} 2^{\sigma_t l}\|\mathscr{F}^{-1}_t\psi_l\mathscr{F}_t f\|_{L^p_{t,x}}\\
&\leqslant \sum_{j\geqslant 0}\|\mathscr{F}^{-1}_{t,x}\tilde\psi_0\phi_j\mathscr{F}_{t,x}f\|_{L^p_{t,x}} + \sup_{l>0}\sum_{j\geqslant 0} 2^{\sigma_t l}\|\mathscr{F}^{-1}_{t,x}\psi_l\phi_j\mathscr{F}_{t,x}f\|_{L^p_{t,x}}\\
&\lesssim \sup_{j\geqslant 0} 2^{\sigma_x j}\|\mathscr{F}^{-1}_{t,x}\tilde\psi_0\phi_j\mathscr{F}_{t,x}f\|_{L^p_{t,x}} + \sup_{l>0,j\geqslant 0} 2^{\sigma_t l}2^{\sigma_x j}\|\mathscr{F}^{-1}_{t,x}\psi_l\phi_j\mathscr{F}_{t,x}f\|_{L^p_{t,x}} \lesssim \|f\|_{S^{\bar\sigma}_{p,\infty}B}.
\end{aligned}$$

By analogy, $\|f\|_{\tilde L^p_t \dot B^{\sigma_x}_{p,\infty}} \lesssim \|f\|_{S^{\bar\sigma}_{p,\infty}B}$. Hence, it remains to control $\|f\|_{S^{\bar\sigma}_{p,\infty}\dot B}$. We split this term into the four contributions

$$\begin{aligned}
\|f\|_{S^{\bar\sigma}_{p,\infty}\dot B} = \sup_{l,j>0} 2^{\sigma_t l}2^{\sigma_x j}\|\mathscr{F}^{-1}_{t,x}\psi_l\phi_j\mathscr{F}_{t,x}f\|_{L^p_{t,x}} + \sup_{l>0,j\leqslant 0} 2^{\sigma_t l}2^{\sigma_x j}\|\mathscr{F}^{-1}_{t,x}\psi_l\varphi_j\mathscr{F}_{t,x}f\|_{L^p_{t,x}}\\
+ \sup_{l\leqslant 0,j>0} 2^{\sigma_t l}2^{\sigma_x j}\|\mathscr{F}^{-1}_{t,x}\eta_l\phi_j\mathscr{F}_{t,x}f\|_{L^p_{t,x}} + \sup_{l,j\leqslant 0} 2^{\sigma_t l}2^{\sigma_x j}\|\mathscr{F}^{-1}_{t,x}\eta_l\varphi_j\mathscr{F}_{t,x}f\|_{L^p_{t,x}}.
\end{aligned}$$

The first contribution is immediately estimated by $\|f\|_{S^{\bar\sigma}_{p,\infty}B}$. For the second contribution, we have

$$\sup_{l>0,j\leqslant 0} 2^{\sigma_t l}2^{\sigma_x j}\|\mathscr{F}^{-1}_{t,x}\psi_l\varphi_j\mathscr{F}_{t,x}f\|_{L^p_{t,x}} \lesssim \sup_{l>0} 2^{\sigma_t l}\|\mathscr{F}^{-1}_{t,x}\psi_l\tilde\phi_0\mathscr{F}_{t,x}f\|_{L^p_{t,x}} \leqslant \|f\|_{S^{\bar\sigma}_{p,\infty}B},$$

and a similar estimate holds for the third contribution. For the fourth contribution, we have

$$\sup_{l,j\leqslant 0} 2^{\sigma_t l}2^{\sigma_x j}\|\mathscr{F}^{-1}_{t,x}\eta_l\varphi_j\mathscr{F}_{t,x}f\|_{L^p_{t,x}} \lesssim \|\mathscr{F}^{-1}_{t,x}\tilde\psi_0\tilde\phi_0\mathscr{F}_{t,x}f\|_{L^p_{t,x}}. \qquad \square$$

3. Optimality of estimates via scaling

It is well known that in the linear case $m = 1$ one has estimates of the form

$$\|u\|_{L^1_t \dot W^{\sigma_x,1}_x} \leqslant c(\sigma_x)(\|u_0\|_{L^1_x} + \|S\|_{L^1_{t,x}}), \tag{3-1}$$

for all $\sigma_x < 2$. In the case $m > 1$, such an estimate cannot be true for *any* $\sigma_x > 0$ anymore. Intuitively, this is due to the linear nature of (3-1) (observe that the integrability exponent is equal on both sides of the inequality), which is not compatible with the nonlinear equation (1-1). We will make this intuition more precise by the following lemma based on a scaling argument.

Lemma 3.1. *Let $T > 0$, $m > 1$, $\mu \in [1, m]$, $p \in [1, \infty)$ and $\sigma_t, \sigma_x \geqslant 0$. Assume that there is a constant $c = c(m, \mu, p, \sigma_t, \sigma_x) > 0$ such that*

$$\|u^{[\mu]}\|^p_{\dot W^{\sigma_t,p}(0,T;\dot W^{\sigma_x,p}(\mathbb{R}^d))} \leqslant c(\|u_0\|_{L^1(\mathbb{R}^d)} + \|S\|_{L^1(0,T;L^1(\mathbb{R}^d))}) \tag{3-2}$$

for all solutions u to (1-1). *Then*

$$p \leqslant \frac{m}{\mu + (m-1)\sigma_t} \leqslant \frac{m}{\mu}, \quad \sigma_t \leqslant \frac{m - \mu p}{p(m-1)} \leqslant \frac{m-\mu}{m-1}, \quad \sigma_x = \frac{\mu p - 1}{p}\frac{2}{m-1} \leqslant \frac{2(\mu - \sigma_t)}{m} \leqslant \frac{2\mu}{m}. \tag{3-3}$$

In particular, if $\sigma_t = (m-\mu)/(m-1)$, then $p = 1$ and $\sigma_x = 2(\mu-1)/(m-1)$.

Proof. For positive constants $\eta, \gamma \geqslant 1$ with $\eta^{m-1} = \gamma$ and a fixed triple (u, u_0, S) such that u satisfies (1-1) with initial condition u_0 and forcing S we consider the rescaled quantities $(\tilde u, \tilde u_0, \widetilde S)$ defined via

$$\tilde u(t,x) := \eta u(\gamma t, x), \quad \tilde u_0(x) := \eta u_0(x), \quad \widetilde S(t,x) := \eta^m S(\gamma t, x),$$

where we have tacitly extended S on $(T, \gamma T)$ by 0. Then $\tilde u$ satisfies (1-1) with $\tilde u_0 \in L^1(\mathbb{R}^d)$ and $\widetilde S \in L^1(0,T;L^1(\mathbb{R}^d))$, so that (3-2) gives

$$\|\tilde u^{[\mu]}\|^p_{\dot W^{\sigma_t,p}(0,T;\dot W^{\sigma_x,p}(\mathbb{R}^d))} \leqslant c(\|\tilde u_0\|_{L^1(\mathbb{R}^d)} + \|\widetilde S\|_{L^1(0,T;L^1(\mathbb{R}^d))}). \tag{3-4}$$

We observe

$$\|\tilde u^{[\mu]}\|^p_{\dot W^{\sigma_t,p}(0,T;\dot W^{\sigma_x,p}(\mathbb{R}^d))} = \eta^{\mu p}\gamma^{\sigma_t p - 1}\|u^{[\mu]}\|^p_{\dot W^{\sigma_t,p}(0,\gamma T;\dot W^{\sigma_x,p}(\mathbb{R}^d))},$$

as well as $\|\tilde u_0\|_{L^1(\mathbb{R}^d)} = \eta\|u_0\|_{L^1(\mathbb{R}^d)}$ and $\|\widetilde S\|_{L^1(0,T;L^1(\mathbb{R}^d))} = \eta\|S\|_{L^1(0,\gamma T;L^1(\mathbb{R}^d))}$. Thus, it follows from (3-4) that

$$\begin{aligned}\|u^{[\mu]}\|^p_{\dot W^{\sigma_t,p}(0,T;\dot W^{\sigma_x,p}(\mathbb{R}^d))} &\leqslant c\eta^{1-\mu p}\gamma^{1-\sigma_t p}(\|u_0\|_{L^1(\mathbb{R}^d)} + \|S\|_{L^1(0,\gamma T;L^1(\mathbb{R}^d))}) \\ &= c\eta^{(m-1)(1-\sigma_t p)+1-\mu p}(\|u_0\|_{L^1(\mathbb{R}^d)} + \|S\|_{L^1(0,T;L^1(\mathbb{R}^d))}).\end{aligned} \tag{3-5}$$

As long as u_0 or S are nontrivial and unless

$$(m-1)(1-\sigma_t p) + 1 - \mu p \geqslant 0, \tag{3-6}$$

this gives the contradiction $u = 0$ by sending $\eta \to \infty$ (and consequently also $\gamma \to \infty$). Since $\sigma_t \geqslant 0$, (3-6) gives

$$p \leqslant \frac{m}{\mu + (m-1)\sigma_t} \leqslant \frac{m}{\mu}.$$

By the same token, since $p \geqslant 1$, (3-6) gives

$$\sigma_t \leqslant \frac{m - \mu p}{p(m-1)} \leqslant \frac{m-\mu}{m-1}.$$

Next, we rescale in space. More precisely, for positive constants $\eta, \gamma > 0$ with $\eta^{1-m} = \gamma^2$ and a fixed triple (u, u_0, S) as above we consider the rescaled quantities $(\tilde u, \tilde u_0, \widetilde S)$ defined via

$$\tilde u(t,x) := \eta u(t, \gamma x), \quad \tilde u_0(x) := \eta u_0(\gamma x), \quad \widetilde S(t,x) := \eta S(t, \gamma x).$$

Then $\tilde{u}$ satisfies (1-1) with $\tilde{u}_0 \in L^1(\mathbb{R}^d)$ and $\widetilde{S} \in L^1(0,T;L^1(\mathbb{R}^d))$, so that (3-2) gives

$$\|\tilde{u}^{[\mu]}\|^p_{\dot{W}^{\sigma_t,p}(0,T;\dot{W}^{\sigma_x,p}(\mathbb{R}^d))} \leqslant c(\|\tilde{u}_0\|_{L^1(\mathbb{R}^d)} + \|\widetilde{S}\|_{L^1(0,T;L^1(\mathbb{R}^d))}). \tag{3-7}$$

We have

$$\|\tilde{u}^{[\mu]}\|^p_{\dot{W}^{\sigma_t,p}(0,T;\dot{W}^{\sigma_x,p}(\mathbb{R}^d))} = \eta^{\mu p}\gamma^{\sigma_x p-d}\|u^{[\mu]}\|^p_{\dot{W}^{\sigma_t,p}(0,T;\dot{W}^{\sigma_x,p}(\mathbb{R}^d))},$$

as well as $\|\tilde{u}_0\|_{L^1(\mathbb{R}^d)} = \eta\gamma^{-d}\|u_0\|_{L^1(\mathbb{R}^d)}$ and $\|\widetilde{S}\|_{L^1(0,T;L^1(\mathbb{R}^d))} = \eta\gamma^{-d}\|S\|_{L^1(0,T;L^1(\mathbb{R}^d))}$. Thus, it follows from (3-7) and the relation $\eta^{1-m} = \gamma^2$ that

$$\begin{aligned}\|u^{[\mu]}\|^p_{\dot{W}^{\sigma_t,p}(0,T;\dot{W}^{\sigma_x,p}(\mathbb{R}^d))} &\leqslant c\eta^{1-\mu p}\gamma^{-\sigma_x p}(\|u_0\|_{L^1(\mathbb{R}^d)} + \|S\|_{L^1(0,T;L^1(\mathbb{R}^d))}) \\ &= c\eta^{\frac{\sigma_x p(m-1)}{2}+1-\mu p}(\|u_0\|_{L^1(\mathbb{R}^d)} + \|S\|_{L^1(0,T;L^1(\mathbb{R}^d))}).\end{aligned} \tag{3-8}$$

As long as u_0 or S are nontrivial and unless

$$\frac{\sigma_x p(m-1)}{2} + 1 - \mu p = 0 \quad\Longleftrightarrow\quad \sigma_x = \frac{\mu p - 1}{p}\frac{2}{m-1}, \tag{3-9}$$

this gives the contradiction $u = 0$ by sending $\eta \to 0$ or $\eta \to \infty$ (and consequently $\gamma \to \infty$ or $\gamma \to 0$, respectively). Plugging into (3-9) the restrictions on p and σ_t, we obtain the result. □

Remark 3.2. If one sets $\mu = 1$, $p = 1$ and $\sigma_t = 0$, Lemma 3.1 tells us that σ_x cannot be positive, which is what we claimed following (3-1). Moreover, we emphasize that Lemma 3.1 shows that in the case of the whole space, the regularity exponent $\sigma_x \in [2(\mu-1)/(m-1), 2\mu/m]$ is in a one-to-one correspondence to the integrability exponent $p \in [1, m/\mu]$ via

$$\sigma_x = \frac{\mu p - 1}{p}\frac{2}{m-1} \quad \text{and} \quad p = \frac{2}{2\mu - \sigma_x(m-1)}.$$

The Barenblatt solution. Consider the Barenblatt solution

$$u_{BB}(t,x) := t^{-\alpha}(C - k|xt^{-\beta}|^2)_+^{\frac{1}{m-1}},$$

where

$$m > 1, \quad \alpha := \frac{d}{d(m-1)+2}, \quad k = \frac{\alpha(m-1)}{2md}, \quad \beta = \frac{\alpha}{d},$$

and $C > 0$ is a free constant. Then, for $\mu \in [1, m]$, $u_{BB}^{[\mu]} \in L^{m/\mu}(0,T;\dot{W}^{s,m/\mu}(\mathbb{R}^d))$ implies $s < 2\mu/m$.

Proof. With $F(x) := (C - k|x|^2)_+^{\mu/(m-1)}$ we have $u_{BB}^{[\mu]}(t,x) = t^{-\alpha\mu}F(xt^{-\beta})$. We next observe that, for $s \in (0,1)$ and each $t \geqslant 0$,

$$\begin{aligned}\|u_{BB}^{[\mu]}(t,\cdot)\|^{\frac{m}{\mu}}_{\dot{W}^{s,m/\mu}(\mathbb{R}^d)} &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u_{BB}^{[\mu]}(t,x) - u_{BB}^{[\mu]}(t,y)|^{\frac{m}{\mu}}}{|x-y|^{\frac{sm}{\mu}+d}}\,\mathrm{d}x\,\mathrm{d}y \\ &= t^{-\alpha m - \beta(\frac{sm}{\mu}+d)+2d\beta}\|F\|^{\frac{m}{\mu}}_{\dot{W}^{s,m/\mu}(\mathbb{R}^d)}.\end{aligned}$$

Hence,

$$\|u_{BB}^{[\mu]}\|^{\frac{m}{\mu}}_{L^{m/\mu}(0,T;\dot{W}^{s,\mu/m}(\mathbb{R}^d))} = \|t^{-\alpha m - \beta(\frac{sm}{\mu}+d)+2d\beta}\|_{L^1(0,T)}\|F\|^{\frac{m}{\mu}}_{\dot{W}^{s,m/\mu}(\mathbb{R}^d)},$$

which is finite if and only if

$$-\alpha m-\beta\left(\frac{sm}{\mu}+d\right)+2d\beta>-1 \quad \text{and} \quad F\in \dot{W}^{s,\frac{m}{\mu}}(\mathbb{R}^d).$$

Hence, necessarily

$$m+\frac{1}{d}\left(\frac{sm}{\mu}+d\right)-2<\frac{1}{\alpha}=\frac{d(m-1)+2}{d},$$

which is equivalent to $s<2\mu/m$. In the case $s\in(1,2)$ we observe that it holds

$$\partial_{x_i}u_{BB}^{[\mu]}(t,x)=t^{-\alpha\mu+\beta}\partial_{x_i}F(xt^{-\beta}),$$

so that analogous arguments may be applied. □

4. Averaging lemma approach

In [Gess 2020], an averaging lemma was introduced that can be applied directly to the porous medium equations (1-1) to obtain estimates on the spatial regularity of u, but so far, no corresponding estimates for powers of the solution u^{μ} or its time regularity could be obtained. In this section, we provide an averaging lemma that gives a comprehensive answer to both of these questions. To this end, we recall the definition of the anisotropic and isotropic truncation properties from [Gess 2020], which extend the truncation property introduced in [Tadmor and Tao 2007, Definition 2.1].

Definition 4.1. (i) Let m be a complex-valued Fourier multiplier. We say that m has the truncation property if, for any locally supported bump function ψ on $\mathbb{C}$ and any $1\leqslant p<\infty$, the multiplier with symbol $\psi(m(\xi)/\delta)$ is an L^p-multiplier as well as an $\mathscr{M}_{TV}$-multiplier uniformly in $\delta>0$, that is, its L^p-multiplier norm ($\mathscr{M}_{TV}$-multiplier norm resp.) depends only on the support and C^l size of ψ (for some large l that may depend on m) but otherwise is independent of δ.

(ii) Let $m:\mathbb{R}^d_\xi\times\mathbb{R}_v\to\mathbb{C}$ be a Carathéodory function such that $m(\,\cdot\,,v)$ is radial for all $v\in\mathbb{R}$. Then m is said to satisfy the isotropic truncation property if, for every bump function ψ supported on a ball in $\mathbb{C}$, every bump function φ supported in $\{\xi\in\mathbb{C}:\frac{1}{2}\leqslant|\xi|\leqslant 2\}$ and every $1<p<\infty$,

$$M_{\psi,J}f(x,v):=\mathscr{F}_x^{-1}\varphi\left(\frac{|\xi|^2}{J^2}\right)\psi\left(\frac{m(\xi,v)}{\delta}\right)\mathscr{F}_xf(x)$$

is an L^p_x-multiplier for all $v\in\mathbb{R}$, $J=2^j$, $j\in\mathbb{Z}$, and, for all $r\geqslant 1$,

$$\|\|M_{\psi,J}\|_{\mathscr{M}^p}\|_{L^r_v}\lesssim|\Omega_m(J,\delta)|^{\frac{1}{r}},$$

where

$$\Omega_m(J,\delta):=\left\{v\in\mathbb{R}:\left|\frac{m(J,v)}{\delta}\right|\in\operatorname{supp}\psi\right\}.$$

Here we use an abuse of notation

$$\left|\frac{m(J,v)}{\delta}\right|:=\sup\left\{\left|\frac{m(\xi,v)}{\delta}\right|:\frac{|\xi|^2}{J^2}\in\operatorname{supp}\varphi\right\}.$$

We recall that for $m(\xi, v) := |\xi|^2|v|$, the anisotropic truncation property is satisfied uniformly in v by Example A.2 in [Gess 2020] and the isotropic truncation property is satisfied by Example 3.2 in [Gess 2020], albeit only in the case $J \geqslant 1$. However, the proof given there can be used without any changes to obtain the full assertion for general $J \in \mathbb{Z}$.

Lemma 4.2. *Assume $m \in (1, \infty)$, $\gamma \in (-\infty, m)$, $\mu \in [1, m+1-\gamma)$ and let $f \in L^{\beta}_{t,x,v}$, where $\beta' = 1/\rho$ with $\rho \in (0,1)$, be a solution to*

$$\mathscr{L}(\partial_t, \nabla_x, v) f(t,x,v) = g_0(t,x,v) + \partial_v g_1(t,x,v) \text{ on } \mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{R}_v. \tag{4-1}$$

Here, the differential operator $\mathscr{L}(\partial_t, \nabla_x, v)$ that is given in terms of its symbol

$$\mathscr{L}(i\tau, i\xi, v) := i\tau + |v|^{m-1}|\xi|^2, \tag{4-2}$$

and g_i are Radon measures satisfying

$$|g_0|(t,x,v)|v|^{1-\gamma} + |g_1|(t,x,v)|v|^{-\gamma} \in \mathscr{M}_{TV}(\mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{R}_v).$$

Suppose

$$s \in \left(\frac{\mu - 2 + \gamma}{m-1}, 1 \right] \cap [0,1].$$

Then $\bar{f} \in S^{\bar{\kappa}}_{p,\infty,(\infty)}\dot{B}$, where $\bar{f}(t,x) := \int f(t,x,v)|v|^{\mu-1}\,\mathrm{d}v$, $\bar{\kappa} := (\kappa_t, \kappa_x)$ and

$$p := \frac{s(m-1)+1-\gamma+\rho}{\rho\mu+(1-\rho)(s(m-1)+1-\gamma)}, \quad \kappa_t := \frac{(1-s)(\mu-1+\rho)}{s(m-1)+1-\gamma+\rho}, \quad \kappa_x := \frac{2s(\mu-1+\rho)}{s(m-1)+1-\gamma+\rho}. \tag{4-3}$$

Moreover, we have the estimate

$$\|\bar{f}\|_{S^{\bar{\kappa}}_{p,\infty,(\infty)}\dot{B}} \lesssim \||v|^{1-\gamma} g_0\|_{\mathscr{M}_{TV}} + \||v|^{-\gamma} g_1\|_{\mathscr{M}_{TV}} + \|f\|_{L^{\beta}_{t,x,v}}. \tag{4-4}$$

If additionally $\bar{f} \in L^r_{t,x}$, $p \neq r \in [1,\infty]$, then for all $q \in (\min\{p,r\}, \max\{p,r\})$ it holds $\bar{f} \in S^{\vartheta\bar{\kappa}}_{q,\infty}\dot{B}$, where $\vartheta \in (0,1)$ is such that

$$\frac{1}{q} = \frac{1-\vartheta}{r} + \frac{\vartheta}{p}.$$

In this case we have

$$\|\bar{f}\|_{S^{\vartheta\bar{\kappa}}_{q,\infty}\dot{B}} \lesssim \||v|^{1-\gamma} g_0\|_{\mathscr{M}_{TV}} + \||v|^{-\gamma} g_1\|_{\mathscr{M}_{TV}} + \|f\|_{L^{\beta}_{t,x,v}} + \|\bar{f}\|_{L^r_{t,x}}. \tag{4-5}$$

Finally, if $s = 1$ and consequently $\kappa_t = 0$, then (4-5) remains true if we replace the space $S^{\vartheta\bar{\kappa}}_{q,\infty}\dot{B} = S^{(0,\vartheta\kappa_x)}_{q,\infty}\dot{B}$ by $\widetilde{L}^q_t \dot{B}^{\vartheta\kappa_x}_{q,\infty}$.

Remark 4.3. Observe that for

$$\rho \in \left(\frac{m+1-\gamma-\mu}{m+1-\gamma}, 1 \right)$$

one may prescribe a specific integrability exponent. More precisely, given

$$\tilde{p} \in \left[\frac{1-\gamma+\rho}{\rho\mu+(1-\rho)(1-\gamma)}, \frac{m+1-\gamma}{\mu} \right] \cap \left(1, \frac{m+1-\gamma}{\mu} \right]$$

choose

$$s := \frac{\mu\tilde{p}\rho + \tilde{p}(1-\rho)(1-\gamma) - 1 + \gamma + \rho}{(m-1)(1-\tilde{p}(1-\rho))} \in \left(\frac{\mu - 2 + \gamma}{m-1}, 1\right] \cap [0,1].$$

Then (4-3) reads $p = \tilde{p}$, as well as

$$\kappa_t := \frac{m + \rho - \gamma - \mu p\rho + p(1-\rho)(\gamma - m)}{p\rho}\frac{1}{m-1},$$
$$\kappa_x := \frac{\mu p\rho + p(1-\rho)(1-\gamma) - 1 + \gamma - \rho}{p\rho}\frac{2}{m-1}.$$

Observe that in the limiting case $\rho \to 1$ and $\gamma \to 1$, these orders of differentiability correspond to the ones found in (3-3).

Proof of Lemma 4.2. We first assume that f is compactly supported with respect to the variable v. This condition will enter only qualitatively, and never appears in quantitative form. Therefore, at the end of the proof, we can again remove this additional assumption.

Since we are interested in regularity in terms of homogeneous Besov spaces, we decompose f into Littlewood–Paley blocks with respect to the t-variable and the x-variable. Let $\{\eta_l\}_{l\in\mathbb{Z}}$ be a partition of unity on $\mathbb{R}\setminus\{0\}$ and $\{\varphi_j\}_{j\in\mathbb{Z}}$ a partition of unity on $\mathbb{R}^d\setminus\{0\}$ as in Section 2. Then we define for $l, j \in \mathbb{Z}$

$$f_{l,j} := \mathcal{F}^{-1}_{t,x}[\eta_l\varphi_j\mathcal{F}_{t,x}f],$$

where $\mathcal{F}_{t,x}f_{l,j}(\tau,\xi,v)$ is supported on frequencies $|\xi| \sim 2^j$, $|\tau| \sim 2^l$ for $l, j \in \mathbb{Z}$. Similarly, we define the decompositions $g_{0,l,j}$ and $g_{1,l,j}$ of g_0 and g_1, respectively. We consider a microlocal decomposition of $f_{l,j}$ connected to the degeneracy of the operator $\mathcal{L}(\partial_t, \nabla_x, v)$. Let $\psi_0 \in C_c^\infty(\mathbb{R})$ be a smooth function supported in $B_2(0)$ and set $\psi_1 := 1 - \psi_0$. For $\delta > 0$ to be specified later we write

$$f_{l,j} = \mathcal{F}_x^{-1}\psi_0\left(\frac{|v||\xi|^2}{\delta}\right)\mathcal{F}_x f_{l,j} + \mathcal{F}_x^{-1}\psi_1\left(\frac{|v||\xi|^2}{\delta}\right)\mathcal{F}_x f_{l,j} =: f^0_{l,j} + f^1_{l,j}.$$

Since f is a solution to (4-1), we have

$$\mathcal{F}^{-1}_{t,x}\mathcal{L}(i\tau, i\xi, v)\mathcal{F}_{t,x}f^1_{l,j}(t,x,v) = \mathcal{F}_x^{-1}\psi_1\left(\frac{|v||\xi|^2}{\delta}\right)\mathcal{F}_x\Big(g_{0,l,j}(t,x,v) + \partial_v g_{1,l,j}(t,x,v)\Big) \quad (4\text{-}6)$$

and thus

$$\begin{aligned} f^1_{l,j}(t,x,v) &= \mathcal{F}^{-1}_{t,x}\psi_1\left(\frac{|v||\xi|^2}{\delta}\right)\frac{1}{\mathcal{L}(i\tau, i\xi, v)}\mathcal{F}_{t,x}g_{0,l,j}(t,x,v) \\ &\qquad + \mathcal{F}^{-1}_{t,x}\psi_1\left(\frac{|v||\xi|^2}{\delta}\right)\frac{1}{\mathcal{L}(i\tau, i\xi, v)}\mathcal{F}_{t,x}\partial_v g_{1,l,j}(t,x,v) \\ &=: f^2_{l,j}(t,x,v) + f^3_{l,j}(t,x,v). \end{aligned} \quad (4\text{-}7)$$

In conclusion, we have arrived at the decomposition

$$\bar{f}_{l,j} := \int f_{l,j}|v|^{\mu-1}\,\mathrm{d}v = \int f^0_{l,j}|v|^{\mu-1}\,\mathrm{d}v + \int f^2_{l,j}|v|^{\mu-1}\,\mathrm{d}v + \int f^3_{l,j}|v|^{\mu-1}\,\mathrm{d}v =: \bar{f}^0_{l,j} + \bar{f}^2_{l,j} + \bar{f}^3_{l,j}.$$

We aim to estimate the regularity of these three contributions separately.

Step 1: f^0. We note that we have the estimate $\|\mathscr{F}_t^{-1}\eta_l\mathscr{F}f\|_{L^\beta_{t,x}} \lesssim \|f\|_{L^\beta_{t,x}}$ with a constant independent of l, since $\|\eta_l\|_{\mathscr{M}^\beta} = \|\eta_0\|_{\mathscr{M}^\beta} < \infty$. Let $l, j \in \mathbb{Z}$ be arbitrary, fixed. Then, we have that $|v| \leqslant 2\cdot 2^{-2j}\delta$ on the support of $\varphi(2^{-j}\xi)\psi_0(|v||\xi|^2/\delta)$, so that $|\Omega_m(2^j,\delta)| \lesssim |[-2\cdot 2^{-2j}\delta, 2\cdot 2^{-2j}\delta]| \lesssim 2^{-2j}\delta$. Hence, by the isotropic truncation property and Minkowski's and Hölder's inequality it holds

$$\begin{aligned}
\left\|\int f^0_{l,j}|v|^{\mu-1}\,\mathrm{d}v\right\|_{L^\beta_{t,x}} &= \left\|\int \mathscr{F}_x^{-1}\psi_0\Big(\frac{|v||\xi|^2}{\delta}\Big)|v|^{\mu-1}\mathscr{F}_x f_{l,j}\,\mathrm{d}v\right\|_{L^\beta_{t,x}} \\
&\lesssim \int\left\|\mathscr{F}_x^{-1}\psi_0\Big(\frac{|v||\xi|^2}{\delta}\Big)|v|^{\mu-1}\mathscr{F}_x f_{l,j}\right\|_{L^\beta_{t,x}}\mathrm{d}v \\
&\lesssim \Big(\frac{\delta}{2^{2j}}\Big)^{\mu-1}\int\left\|\mathscr{F}_x^{-1}\psi_0\Big(\frac{|v||\xi|^2}{\delta}\Big)\mathscr{F}_x f_{l,j}\right\|_{L^\beta_{t,x}}\mathrm{d}v \\
&\lesssim \Big(\frac{\delta}{2^{2j}}\Big)^{\mu-1}\int \|M_{\psi_0,2^{-j}}\|_{\mathscr{M}^\beta}\|f\|_{L^\beta_{t,x}}\,\mathrm{d}v \\
&\leqslant \Big(\frac{\delta}{2^{2j}}\Big)^{\mu-1}\|\|M_{\psi_0,2^{-j}}\|_{\mathscr{M}^\beta}\|_{L^{\beta'}_v}\|f\|_{L^\beta_{t,x,v}} \\
&\lesssim \Big(\frac{\delta}{2^{2j}}\Big)^{\mu-1}|\Omega_m(2^j,\delta)|^{\frac{1}{\beta'}}\|f\|_{L^\beta_{t,x,v}} \lesssim \Big(\frac{\delta}{2^{2j}}\Big)^{\mu-1+\rho}\|f\|_{L^\beta_{t,x,v}},
\end{aligned}$$

where we have used $\beta' = 1/\rho$.

Step 2: f^2. Let $l, j \in \mathbb{Z}$ be arbitrary, fixed. Since $s \in [0,1]$, we clearly have

$$|\tau|^{1-s}|v|^{s(m-1)}|\xi|^{2s} \leqslant |\mathscr{L}(i\tau, i\xi, v)|.$$

Moreover, in light of $s > \mu-2+\gamma/(m-1)$ we have on the support of $\eta_l\varphi_j\psi_1(|v||\xi|^2/\delta)$ (so that $|\tau|\sim 2^l$, $|\xi|\sim 2^j$, and $|v| \gtrsim 2^{-2j}\delta$)

$$\frac{|v|^{\mu-2+\gamma}}{|\mathscr{L}(i\tau,i\xi,v)|} \lesssim \frac{|v|^{\mu-2+\gamma}}{|\tau|^{1-s}|v|^{s(m-1)}|\xi|^{2s}} \lesssim \frac{(2^{-2j}\delta)^{\mu-2+\gamma-s(m-1)}}{2^{l(1-s)}2^{2js}} = \frac{2^{2j(s(m-2)-\mu+2-\gamma)}}{\delta^{s(m-1)-\mu+2-\gamma}2^{l(1-s)}}.$$

Hence, by Theorem B.1 and Lemma B.4, $|v|^{\mu-2+\gamma}/\mathscr{L}(i\tau,i\xi,v)$ acts on the support of $\eta_l\varphi_j\psi_1(|v||\xi|^2/\delta)$ as a constant multiplier of order

$$\frac{2^{2j(s(m-2)-\mu+2-\gamma)}}{\delta^{s(m-1)-\mu+2-\gamma}2^{l(1-s)}}.$$

Consequently, by the anisotropic truncation property

$$\begin{aligned}
\left\|\int f^2_{l,j}|v|^{\mu-1}\,\mathrm{d}v\right\|_{L^1_{t,x}} &= \left\|\int \mathscr{F}_{t,x}^{-1}\psi_1\Big(\frac{|v||\xi|^2}{\delta}\Big)\frac{|v|^{\mu-2+\gamma}}{\mathscr{L}(i\tau,i\xi,v)}\mathscr{F}_{t,x}|v|^{1-\gamma}g_{0,l,j}\,\mathrm{d}v\right\|_{L^1_{t,x}} \\
&\lesssim \frac{2^{2j(s(m-2)-\mu+2-\gamma)}}{\delta^{s(m-1)-\mu+2-\gamma}2^{l(1-s)}}\||v|^{1-\gamma}g_0\|_{\mathscr{M}_{TV}}.
\end{aligned}$$

Here, we have used that with $\psi_0(|v||\xi|^2/\delta)$ also

$$\psi_1\left(\frac{|v||\xi|^2}{\delta}\right) = 1 - \psi_0\left(\frac{|v||\xi|^2}{\delta}\right)$$

is a bounded $\mathscr{M}_{TV}$-multiplier independent of $\delta > 0$.

Step 3: f^3. Let $l, j \in \mathbb{Z}$ arbitrary, fixed. We observe (recall $\mathscr{L}(i\tau, i\xi, v) = i\tau + |v|^{m-1}|\xi|^2$)

$$\begin{aligned}
\int f^3_{l,j}|v|^{\mu-1}\,\mathrm{d}v &= -\int \mathscr{F}^{-1}_{t,x}\psi_1'\left(\frac{|v||\xi|^2}{\delta}\right)\frac{\operatorname{sgn}(v)|\xi|^2}{\delta}\frac{|v|^{\mu-1}}{\mathscr{L}(i\tau, i\xi, v)}\mathscr{F}_{t,x}g_{1,l,j}\,\mathrm{d}v \\
&\quad - (\mu-1)\int \mathscr{F}^{-1}_{t,x}\psi_1\left(\frac{|v||\xi|^2}{\delta}\right)\frac{\operatorname{sgn}(v)|v|^{\mu-2}}{\mathscr{L}(i\tau, i\xi, v)}\mathscr{F}_{t,x}g_{1,l,j}\,\mathrm{d}v \\
&\quad + \int \mathscr{F}^{-1}_{t,x}\psi_1\left(\frac{|v||\xi|^2}{\delta}\right)\frac{|v|^{\mu-1}\partial_v\mathscr{L}(i\tau, i\xi, v)}{\mathscr{L}(i\tau, i\xi, v)^2}\mathscr{F}_{t,x}g_{1,l,j}\,\mathrm{d}v \\
&= -\int \mathscr{F}^{-1}_{t,x}\psi_1'\left(\frac{|v||\xi|^2}{\delta}\right)\frac{|v||\xi|^2}{\delta}\frac{\operatorname{sgn}(v)|v|^{\mu-2+\gamma}}{\mathscr{L}(i\tau, i\xi, v)}\mathscr{F}_{t,x}|v|^{-\gamma}g_{1,l,j}\,\mathrm{d}v \\
&\quad - (\mu-1)\int \mathscr{F}^{-1}_{t,x}\psi_1\left(\frac{|v||\xi|^2}{\delta}\right)\frac{\operatorname{sgn}(v)|v|^{\mu-2+\gamma}}{\mathscr{L}(i\tau, i\xi, v)}\mathscr{F}_{t,x}|v|^{-\gamma}g_{1,l,j}\,\mathrm{d}v \\
&\quad + (m-1)\int \mathscr{F}^{-1}_{t,x}\psi_1\left(\frac{|v||\xi|^2}{\delta}\right)\frac{|v|^{\mu+m-3+\gamma}|\xi|^2}{\mathscr{L}(i\tau, i\xi, v)^2}\mathscr{F}_{t,x}|v|^{-\gamma}g_{1,l,j}\,\mathrm{d}v.
\end{aligned}$$

Observe that ψ_1' is supported on an annulus. Therefore, we have as before $|\tau| \sim 2^l$, $|\xi| \sim 2^j$ and $|v| \gtrsim 2^{-2j}\delta$ on the support of $\eta_l\varphi_j\psi_1(|v||\xi|^2/\delta)$, and additionally also $|v| \sim 2^{-2j}\delta$ on the support of $\eta_l\varphi_j\psi_1'(|v||\xi|^2/\delta)$. This last observation allows us to estimate the expression $|v||\xi|^2/\delta$ appearing in the first integral on the right-hand side by

$$\frac{|v||\xi|^2}{\delta} \lesssim 1.$$

As in Step 2, we obtain

$$\frac{|v|^{\mu-2+\gamma}}{|\mathscr{L}(i\tau, i\xi, v)|} \lesssim \frac{2^{2j(s(m-2)-\mu+2-\gamma)}}{\delta^{s(m-1)-\mu+2-\gamma}2^{l(1-s)}},$$

and, similarly,

$$\begin{aligned}
\frac{|v|^{\mu+m-3+\gamma}|\xi|^2}{|\mathscr{L}(i\tau, i\xi, v)|^2} &= \frac{|v|^{\mu-2+\gamma}}{|\mathscr{L}(i\tau, i\xi, v)|}\frac{|v|^{m-1}|\xi|^2}{|\mathscr{L}(i\tau, i\xi, v)|} \\
&\lesssim \frac{|v|^{\mu-2+\gamma}}{|\mathscr{L}(i\tau, i\xi, v)|} \lesssim \frac{2^{2j(s(m-2)-\mu+2-\gamma)}}{\delta^{s(m-1)-\mu+2-\gamma}2^{l(1-s)}}.
\end{aligned}$$

In light of these estimates, the expressions

$$\frac{|v||\xi|^2}{\delta}\frac{\operatorname{sgn}(v)|v|^{\mu-2+\gamma}}{\mathscr{L}(i\tau, i\xi, v)}, \quad \frac{\operatorname{sgn}(v)|v|^{\mu-2+\gamma}}{\mathscr{L}(i\tau, i\xi, v)}, \quad \frac{|v|^{\mu+m-3+\gamma}|\xi|^2}{\mathscr{L}(i\tau, i\xi, v)^2}$$

extend by Theorem B.1 and Lemma B.4 to constant multipliers of order

$$\frac{2^{2j(s(m-2)-\mu+2-\gamma)}}{\delta^{s(m-1)-\mu+2-\gamma}2^{l(1-s)}}$$

on supports of $\eta_l\varphi_j\psi_1'(|v||\xi|^2/\delta)$ and $\eta_l\varphi_j\psi_1(|v||\xi|^2/\delta)$, respectively. Hence, by the anisotropic truncation property, we obtain

$$\left\| \int f^3_{l,j}|v|^{\mu-1}\,\mathrm{d}v \right\|_{L^1_{t,x}} \lesssim \frac{2^{2j(s(m-2)-\mu+2-\gamma)}}{\delta^{s(m-1)-\mu+2-\gamma}2^{l(1-s)}} \| |v|^{-\gamma} g_{1,j}\|_{\mathcal{M}_{TV}}.$$

Step 4: Conclusion. We aim to conclude by real interpolation. We set, for $z > 0$,

$$K(z,\bar f_{l,j}) := \inf\{\|\bar f^1_{l,j}\|_{L^1_{t,x}} + z\|\bar f^0_{l,j}\|_{L^\beta_{t,x}} : \bar f^0_{l,j}\in L^\beta_{t,x},\ \bar f^1_{l,j}\in L^1_{t,x},\ \bar f_{l,j} = \bar f^0_{l,j}+\bar f^1_{l,j}\}.$$

By the above estimates we obtain

$$K(z,\bar f_{l,j}) \lesssim \frac{2^{2j(s(m-2)-\mu+2-\gamma)}}{\delta^{s(m-1)-\mu+2-\gamma}2^{l(1-s)}}(\||v|^{1-\gamma}g_0\|_{\mathcal{M}_{TV}} + \||v|^{-\gamma}g_1\|_{\mathcal{M}_{TV}}) + z\left(\frac{\delta}{2^{2j}}\right)^{\mu-1+\rho}\|f\|_{L^\beta_{t,x,v}}.$$

We now equilibrate the first and the second term on the right-hand side: we choose $\delta > 0$ such that

$$\frac{2^{2j(s(m-2)-\mu+2-\gamma)}}{\delta^{s(m-1)-\mu+2-\gamma}2^{l(1-s)}} = z\left(\frac{\delta}{2^{2j}}\right)^{\mu-1+\rho};$$

that is,

$$\delta^{-a}c^{1-s}d^{-a+s} = z\delta^b d^b,$$

with $a := s(m-1)-\mu+2-\gamma,\ b := \mu-1+\rho,\ c := 2^{-l}$ and $d := 2^{-2j}$. This yields

$$\delta = z^{-\frac{1}{a+b}}c^{\frac{1-s}{a+b}}d^{\frac{s-a-b}{a+b}},$$

and further

$$\delta^{-a}c^{1-s}d^{-a+s} = z^{\frac{a}{a+b}}c^{\frac{(1-s)b}{a+b}}d^{\frac{sb}{a+b}}.$$

Hence, with

$$\theta := \frac{a}{a+b} = \frac{s(m-1)-\mu+2-\gamma}{s(m-1)+1-\gamma+\rho}$$

we obtain

$$\begin{aligned} z^{-\theta}K(z,\bar f_{l,j}) &\lesssim 2^{-l\frac{(1-s)(\mu-1+\rho)}{s(m-1)+1-\gamma+\rho}}2^{-2j\frac{s(\mu-1+\rho)}{s(m-1)+1-\gamma+\rho}}(\||v|^{1-\gamma}g_0\|_{\mathcal{M}_{TV}} + \||v|^{-\gamma}g_1\|_{\mathcal{M}_{TV}} + \|f\|_{L^\beta_{t,x,v}}) \\ &= 2^{-l\kappa_t}2^{-j\kappa_x}(\||v|^{1-\gamma}g_0\|_{\mathcal{M}_{TV}} + \||v|^{-\gamma}g_1\|_{\mathcal{M}_{TV}} + \|f\|_{L^\beta_{t,x,v}}). \end{aligned}$$

Observe that $1-\theta+\theta/\beta = 1-\theta+\theta(1-\rho) = 1-\theta\rho$, so that $(L^1_{t,x}, L^\beta_{t,x})_{\theta,\infty} = L^{p,\infty}_{t,x}$ with

$$p = \frac{1}{1-\theta\rho} = \frac{a+b}{a(1-\rho)+b} = \frac{s(m-1)+1-\gamma+\rho}{\rho\mu+(1-\rho)(s(m-1)+1-\gamma)}.$$

Hence, we may take the supremum over $z > 0$ to obtain

$$\|\bar f_{l,j}\|_{L^{p,\infty}_{t,x}} \lesssim 2^{-l\kappa_t}2^{-j\kappa_x}(\||v|^{1-\gamma}g_0\|_{\mathcal{M}_{TV}} + \||v|^{-\gamma}g_1\|_{\mathcal{M}_{TV}} + \|f\|_{L^\beta_{t,x,v}}). \tag{4-8}$$

Multiplying by $2^{l\kappa_t}2^{j\kappa_x}$ and taking the supremum over $j, l \in \mathbb{Z}$ yields (4-4).

If we assume additionally $\bar{f} \in L^r_{t,x}$, $r \neq p$, we choose for $q \in (\min\{p, r\}, \max\{p, r\})$ a corresponding $\vartheta \in (0,1)$ subject to $1/q = (1-\vartheta)/r + \vartheta/p$. Then using $(L^r_{t,x}, L^{p,\infty}_{t,x})_{\vartheta,q} = L^q_{t,x}$, together with (4-8), we obtain

$$\begin{aligned}\|\bar{f}_{l,j}\|_{L^q_{t,x}} &\lesssim \|\bar{f}_{l,j}\|^{1-\vartheta}_{L^r_{t,x}} \|\bar{f}_{l,j}\|^{\vartheta}_{L^{p,\infty}_{t,x}} \\ &\lesssim \|\bar{f}\|^{1-\vartheta}_{L^r_{t,x}} 2^{-l\vartheta\kappa_t} 2^{-j\vartheta\kappa_x} (\||v|^{1-\gamma} g_0\|_{\mathcal{M}_{TV}} + \||v|^{-\gamma} g_1\|_{\mathcal{M}_{TV}} + \|f\|_{L^\infty_{t,x,v}})^{\vartheta} \\ &\leqslant 2^{-l\vartheta\kappa_t} 2^{-j\vartheta\kappa_x} (\||v|^{1-\gamma} g_0\|_{\mathcal{M}_{TV}} + \||v|^{-\gamma} g_1\|_{\mathcal{M}_{TV}} + \|f\|_{L^\beta_{t,x,v}} + \|\bar{f}\|_{L^r_{t,x}}).\end{aligned}$$

Multiplying by $2^{l\vartheta\kappa_t} 2^{j\vartheta\kappa_x}$ and taking the supremum over $j, l \in \mathbb{Z}$ yields (4-5).

Finally we note that if $s = 1$ and consequently $\kappa_t = 0$, then the partition of unity $\{\eta_l\}_{l\in\mathbb{Z}}$ in the Fourier space connected to the time variable t is not necessary. Hence, if we set $\alpha_\tau = 0$ whenever Lemma B.4 is invoked and replace Theorem B.1 by its isotropic variant (see Remark B.3), we obtain

$$\|\bar{f}_j\|_{L^q_{t,x}} \lesssim 2^{-j\vartheta\kappa_x} (\||v|^{1-\gamma} g_0\|_{\mathcal{M}_{TV}} + \||v|^{-\gamma} g_1\|_{\mathcal{M}_{TV}} + \|f\|_{L^\beta_{t,x,v}} + \|\bar{f}\|_{L^r_{t,x}}),$$

which shows $\bar{f} \in \widetilde{L}^q \dot{B}^{\vartheta\kappa_x}_{q,\infty}$.

It remains to consider the case when f is not localized in v. We observe that for a smooth cut-off function $\psi \in C^\infty_c(\mathbb{R})$, the function $(t,x,v) \to f(t,x,v)\psi(v) =: f^\psi(t,x,v)$ is a solution to

$$\mathcal{L}(\partial_t, \nabla_x, v) f^\psi(t,x,v) = g^\psi_0(t,x,v) + g^{\psi'}_1(t,x,v) + \partial_v g^\psi_1(t,x,v) \quad \text{on } \mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{R}_v,$$

where g^ψ_0, $g^{\psi'}_1$ and g^ψ_1 are defined analogously. Hence, estimate (4-8) reads in this case

$$\|\bar{f}^\psi_{l,j}\|_{L^{p,\infty}_{t,x}} \leqslant 2^{-l\vartheta\kappa_t} 2^{-j\vartheta\kappa_x} (\||v|^{1-\gamma}(g^\psi_0 + g^{\psi'}_1)\|_{\mathcal{M}_{TV}} + \||v|^{-\gamma} g^\psi_1\|_{\mathcal{M}_{TV}} + \|f^\psi\|_{L^\beta_{t,x,v}}).$$

Since $|v|^{-\gamma} g_1 \in \mathcal{M}_{TV}$ by assumption, there exists for $\varepsilon_n \downarrow 0$ a sequence $r_n \uparrow \infty$ such that

$$\int_{\mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{R}_v} \chi_{\{r_n \leqslant |v|\}} |v|^{-\gamma} g_1 \,\mathrm{d}v \,\mathrm{d}x \,\mathrm{d}t \leqslant \varepsilon_n$$

for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$ and a smooth cut-off function $\psi \in C^\infty_c(\mathbb{R})$ with $\psi = 1$ on $B_1(0)$ and $\operatorname{supp} \psi \subset B_2(0)$, we define ψ_n via $\psi_n(v) := \psi(v/r_n)$. Hence ψ'_n is supported on $r_n \leqslant |v| \leqslant 2r_n$ and takes values in $[0, 1/r_n]$, so that we may estimate

$$\begin{aligned}\||v|^{1-\gamma} g^{\psi'_n}_1\|_{\mathcal{M}_{TV}} &= \int_{\mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{R}_v} |\psi'_n(v)|\,|v|(|v|^{-\gamma} g_1) \,\mathrm{d}v \,\mathrm{d}x \,\mathrm{d}t \\ &= \int_{\mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{R}_v} \chi_{\{r_n \leqslant |v| \leqslant 2r_n\}} |\psi'_n(v)|\,|v|(|v|^{-\gamma} g_1) \,\mathrm{d}v \,\mathrm{d}x \,\mathrm{d}t \\ &\lesssim \int \chi_{r_n \leqslant |v| \leqslant 2r_n} |v|^{-\gamma} g_1 \,\mathrm{d}v \leqslant \varepsilon_n.\end{aligned}$$

Thus, taking the limit $n \to \infty$ and using Fatou's lemma, we obtain (4-8) also for general f. Multiplying by $2^{l\vartheta\kappa_t} 2^{j\vartheta\kappa_x}$ and taking the supremum over $j, l \in \mathbb{Z}$, we may conclude as before. □

Lemma 4.4. *Assume $\gamma \in (-\infty, 1)$, $m \in (1, \infty)$, $\mu \in [1, 2-\gamma)$, $\rho \in (0, 1]$, $\beta' = 1/\rho$, and let f, g_0, g_1, and $\bar{f}$ be as in Lemma 4.2. Define*

$$p := \frac{1-\gamma+\rho}{\rho\mu + (1-\rho)(1-\gamma)}, \quad \kappa_t := \frac{\mu-1+\rho}{1-\gamma+\rho}. \tag{4-9}$$

If $\bar{f} \in L^r_{t,x}$, $p \neq r \in [1, \infty]$, then for all $q \in (\min\{p, r\}, \max\{p, r\})$ we have $\bar{f} \in \widetilde{L}^q_x \dot{B}^{\vartheta\kappa_t}_{q,\infty}$, where $\vartheta \in (0, 1)$ is such that

$$\frac{1}{q} = \frac{1-\vartheta}{r} + \frac{\vartheta}{p}.$$

Moreover,

$$\|\bar{f}\|_{\widetilde{L}^q_x \dot{B}^{\vartheta\kappa_t}_{q,\infty}} \lesssim \||v|^{1-\gamma} g_0\|_{\mathscr{M}_{TV}} + \||v|^{-\gamma} g_1\|_{\mathscr{M}_{TV}} + \|f\|_{L^\beta_{t,x,v}} + \|\bar{f}\|_{L^r_{t,x}}. \tag{4-10}$$

Proof. By the same arguments as in the proof of Lemma 4.2, we may assume that f is localized in v. In fact, the whole proof of Lemma 4.4 is similar to the one of Lemma 4.2, with the modification that here we consider a microlocal decomposition of f depending on the size of v only and do not localize in the Fourier space connected to the spatial variable x. More precisely, let $\{\eta_l\}_{l\in\mathbb{Z}}$ be a partition of unity on $\mathbb{R} \setminus \{0\}$ as in Section 2. Then we define for $l \in \mathbb{Z}$

$$f_l := \mathscr{F}_x^{-1}[\eta_l \mathscr{F}_t f],$$

where $\mathscr{F}_t f_l(\tau, x, v)$ is supported on frequencies $|\tau| \sim 2^l$ for $l \in \mathbb{Z}$. Similarly, we define the decompositions $g_{0,l}$ and $g_{1,l}$ of g_0 and g_1, respectively. Moreover, we again consider a smooth function $\psi_0 \in C_c^\infty(\mathbb{R})$ supported in $B_2(0)$ and set $\psi_1 := 1 - \psi_0$. For $\delta > 0$ to be specified later we write

$$f_l = \psi_0\left(\frac{|v|}{\delta}\right) f_l + \psi_1\left(\frac{|v|}{\delta}\right) f_l =: f_l^0 + f_l^1.$$

Since f is a solution to (4-1), we have

$$\mathscr{F}^{-1}_{t,x} \mathscr{L}(i\tau, i\xi, v) \mathscr{F}_{t,x} f_l^1(t, x, v) = \psi_1\left(\frac{|v|}{\delta}\right)\big(g_{0,l}(t, x, v) + \partial_v g_{1,l}(t, x, v)\big)$$

and thus

$$\begin{aligned} f_l^1(t, x, v) &\\ = \mathscr{F}^{-1}_{t,x} \psi_1\left(\frac{|v|}{\delta}\right) &\frac{1}{\mathscr{L}(i\tau, i\xi, v)} \mathscr{F}_{t,x} g_{0,l}(t, x, v) + \mathscr{F}^{-1}_{t,x} \psi_1\left(\frac{|v|}{\delta}\right) \frac{1}{\mathscr{L}(i\tau, i\xi, v)} \mathscr{F}_{t,x} \partial_v g_{1,l}(t, x, v) \\ =: f_l^2(t, x, v) &+ f_l^3(t, x, v), \end{aligned}$$

so that we arrive at the decomposition

$$\begin{aligned} \bar{f}_l := \int f_l |v|^{\mu-1}\, \mathrm{d}v &= \int f_l^0 |v|^{\mu-1}\, \mathrm{d}v + \int f_l^2 |v|^{\mu-1}\, \mathrm{d}v + \int f_l^3 |v|^{\mu-1}\, \mathrm{d}v \\ &=: \bar{f}_l^0 + \bar{f}_l^2 + \bar{f}_l^3. \end{aligned}$$

Again, we treat the three contributions separately.

Step 1: f^0. Let $l \in \mathbb{Z}$ be arbitrary, fixed. Since $|v| \lesssim \delta$ on the support of $\psi_0(|v|/\delta)$, using Minkowski's and Hölder's inequalities, we have

$$\begin{aligned}\left\| \int f_l^0 |v|^{\mu-1} \, \mathrm{d}v \right\|_{L^\beta_{t,x}} &= \left\| \int \psi_0\left(\frac{|v|}{\delta}\right) |v|^{\mu-1} f_l \, \mathrm{d}v \right\|_{L^\beta_{t,x}} \\ &\leqslant \int |\psi_0|\left(\frac{|v|}{\delta}\right) |v|^{\mu-1} \| f_l \|_{L^\beta_{t,x}} \, \mathrm{d}v \\ &\lesssim \delta^{\mu-1} \int |\psi_0|\left(\frac{|v|}{\delta}\right) \| f_l \|_{L^\beta_{t,x}} \, \mathrm{d}v \\ &\lesssim \delta^{\mu-1} \| f \|_{L^\beta_{t,x,v}} \left(\int |\psi_0|\left(\frac{|v|}{\delta}\right)^{\beta'} \mathrm{d}v \right)^{\frac{1}{\beta'}} \\ &\lesssim \delta^{\mu-1+\rho} \| f \|_{L^\beta_{t,x,v}}.\end{aligned}$$

Step 2: f^2. Let $l \in \mathbb{Z}$ be arbitrary, fixed. Since $\mu \leqslant 2-\gamma$, we have on the support of $\eta_l \psi_1(|v|/\delta)$ (so that $|\tau| \sim 2^l$ and $|v| \geqslant \delta$)

$$\frac{|v|^{\mu-2+\gamma}}{|\mathscr{L}(i\tau, i\xi, v)|} \lesssim \frac{|v|^{\mu-2+\gamma}}{|\tau|} \lesssim \frac{\delta^{\mu-2+\gamma}}{2^l}.$$

By Lemma B.4 applied with $\alpha_\xi = 0$ and the isotropic variant of Theorem B.1 (see Remark B.3), $|v|^{\mu-2+\gamma}/|\mathscr{L}(i\tau, i\xi, v)|$ acts as a constant multiplier of order $\delta^{\mu-2+\gamma}/2^l$ on the support of $\eta_l \psi_1(|v|/\delta)$. Consequently

$$\begin{aligned}\left\| \int f_l^2 |v|^{\mu-1} \, \mathrm{d}v \right\|_{L^1_{t,x}} &= \left\| \int \mathscr{F}^{-1}_{t,x} \psi_1\left(\frac{|v|}{\delta}\right) \frac{|v|^{\mu-2+\gamma}}{\mathscr{L}(i\tau, i\xi, v)} \mathscr{F}_{t,x} |v|^{1-\gamma} g_{0,l} \, \mathrm{d}v \right\|_{L^1_{t,x}} \\ &\lesssim \frac{\delta^{\mu-2+\gamma}}{2^l} \| |v|^{1-\gamma} g_0 \|_{\mathscr{M}_{TV}}.\end{aligned}$$

Step 3: f^3. Let $l \in \mathbb{Z}$ be arbitrary, fixed. We observe (recall $\mathscr{L}(i\tau, i\xi, v) = i\tau + |v|^{m-1}|\xi|^2$)

$$\begin{aligned}\int f_l^3 |v|^{\mu-1} \, \mathrm{d}v &= -\int \mathscr{F}^{-1}_{t,x} \psi_1'\left(\frac{|v|}{\delta}\right) \frac{\operatorname{sgn}(v)}{\delta} \frac{|v|^{\mu-1}}{\mathscr{L}(i\tau, i\xi, v)} \mathscr{F}_{t,x} g_{1,l} \, \mathrm{d}v \\ &\quad - (\mu-1) \int \mathscr{F}^{-1}_{t,x} \psi_1\left(\frac{|v|}{\delta}\right) \frac{\operatorname{sgn}(v)|v|^{\mu-2}}{\mathscr{L}(i\tau, i\xi, v)} \mathscr{F}_{t,x} g_{1,l} \, \mathrm{d}v \\ &\quad + \int \mathscr{F}^{-1}_{t,x} \psi_1\left(\frac{|v|}{\delta}\right) \frac{|v|^{\mu-1} \partial_v \mathscr{L}(i\tau, i\xi, v)}{\mathscr{L}(i\tau, i\xi, v)^2} \mathscr{F}_{t,x} g_{1,l} \, \mathrm{d}v \\ &= -\int \mathscr{F}^{-1}_{t,x} \psi_1'\left(\frac{|v|}{\delta}\right) \frac{|v|}{\delta} \frac{\operatorname{sgn}(v)|v|^{\mu-2+\gamma}}{\mathscr{L}(i\tau, i\xi, v)} \mathscr{F}_{t,x} |v|^{-\gamma} g_{1,l} \, \mathrm{d}v \\ &\quad - (\mu-1) \int \mathscr{F}^{-1}_{t,x} \psi_1\left(\frac{|v|}{\delta}\right) \frac{\operatorname{sgn}(v)|v|^{\mu-2+\gamma}}{\mathscr{L}(i\tau, i\xi, v)} \mathscr{F}_{t,x} |v|^{-\gamma} g_{1,l} \, \mathrm{d}v \\ &\quad + (m-1) \int \mathscr{F}^{-1}_{t,x} \psi_1\left(\frac{|v|}{\delta}\right) \frac{|v|^{\mu+m-3+\gamma} |\xi|^2}{\mathscr{L}(i\tau, i\xi, v)^2} \mathscr{F}_{t,x} |v|^{-\gamma} g_{1,l} \, \mathrm{d}v\end{aligned}$$

Observe that ψ_1' is supported on an annulus. Therefore, we have as before $|\tau| \sim 2^l$ and $|v| \geqslant \delta$ on the support of $\eta_l \psi_1(|v|/\delta)$, and additionally also $|v| \sim \delta$ on the support of $\eta_l \psi_1'(|v|/\delta)$. This last observation allows us to estimate the expression $|v|/\delta$ appearing in the first integral on the right-hand side by $|v|/\delta \lesssim 1$. As in Step 2, we obtain

$$\frac{|v|^{\mu-2+\gamma}}{|\mathscr{L}(i\tau, i\xi, v)|} \lesssim \frac{\delta^{\mu-2+\gamma}}{2^l},$$

and, similarly,

$$\frac{|v|^{\mu+m-3+\gamma}|\xi|^2}{|\mathscr{L}(i\tau, i\xi, v)|^2} = \frac{|v|^{\mu-2+\gamma}}{|\mathscr{L}(i\tau, i\xi, v)|}\frac{|v|^{m-1}|\xi|^2}{|\mathscr{L}(i\tau, i\xi, v)|} \lesssim \frac{|v|^{\mu-2+\gamma}}{|\mathscr{L}(i\tau, i\xi, v)|} \lesssim \frac{\delta^{\mu-2+\gamma}}{2^l}.$$

In light of these estimates, Lemma B.4 applied with $\alpha_\xi = 0$ and the isotropic variant of Theorem B.1 (see Remark B.3) show that the expressions

$$\frac{|v|}{\delta}\frac{\operatorname{sgn}(v)|v|^{\mu-2+\gamma}}{\mathscr{L}(i\tau, i\xi, v)}, \quad \frac{\operatorname{sgn}(v)|v|^{\mu-2+\gamma}}{\mathscr{L}(i\tau, i\xi, v)}, \quad \frac{|v|^{\mu+m-3+\gamma}|\xi|^2}{\mathscr{L}(i\tau, i\xi, v)^2}$$

extend to constant multipliers of order $\delta^{\mu-2+\gamma}/2^l$ on the supports of $\eta_l \psi_1'(|v|/\delta)$ and $\eta_l \psi_1(|v|/\delta)$, respectively. Hence, we obtain

$$\left\| \int f_l^3 |v|^{\mu-1}\,\mathrm{d}v \right\|_{L^1_{t,x}} \lesssim \frac{\delta^{\mu-2+\gamma}}{2^l} \||v|^{-\gamma} g_{1,j}\|_{\mathscr{M}_{TV}}.$$

Step 4: Conclusion. We aim to conclude by real interpolation. We set, for $z > 0$,

$$K(z, \bar{f}_l) := \inf\{\|\bar{f}_l^1\|_{L^1_{t,x}} + z\|\bar{f}_l^0\|_{L^\beta_{t,x}} : \bar{f}_l^0 \in L^\beta_{t,x}, \bar{f}_l^1 \in L^1_{t,x}, \bar{f}_l = \bar{f}_l^0 + \bar{f}_l^1\}.$$

By the above estimates we obtain

$$K(z, \bar{f}_l) \lesssim \frac{\delta^{\mu-2+\gamma}}{2^l}(\||v|^{1-\gamma} g_0\|_{\mathscr{M}_{TV}} + \||v|^{-\gamma} g_1\|_{\mathscr{M}_{TV}}) + z\delta^{\mu-1+\rho}\|f\|_{L^\beta_{t,x,v}}.$$

We now equilibrate the first and the second term on the right-hand side: we choose $\delta > 0$ such that

$$\frac{\delta^{\mu-2+\gamma}}{2^l} = z\delta^{\mu-1+\rho};$$

that is,

$$\delta := z^{-\frac{1}{1-\gamma+\rho}} 2^{-\frac{l}{1-\gamma+\rho}}.$$

Hence, with

$$\theta := \frac{-\mu + 2 - \gamma}{1 - \gamma + \rho}$$

we obtain

$$\begin{aligned} z^{-\theta} K(z, \bar{f}_l) &\lesssim 2^{-l\frac{\mu-1+\rho}{1-\gamma+\rho}}(\||v|^{1-\gamma} g_0\|_{\mathscr{M}_{TV}} + \||v|^{-\gamma} g_1\|_{\mathscr{M}_{TV}} + \|f\|_{L^\beta_{t,x,v}}) \\ &= 2^{-l\kappa_t}(\||v|^{1-\gamma} g_0\|_{\mathscr{M}_{TV}} + \||v|^{-\gamma} g_1\|_{\mathscr{M}_{TV}} + \|f\|_{L^\beta_{t,x,v}}). \end{aligned}$$

As in Step 4 of the proof of Lemma 4.2 we use $(L^1_{t,x}, L^\beta_{t,x})_{\theta,\infty} = L^{p,\infty}_{t,x}$ with

$$p = \frac{1}{1-\theta\rho} = \frac{1-\gamma+\rho}{\rho\mu+(1-\rho)(1-\gamma)}$$

to obtain

$$\|\bar{f}_l\|_{L^{p,\infty}_{t,x}} \lesssim 2^{-l\kappa_t}(\||v|^{1-\gamma}g_0\|_{\mathscr{M}_{TV}} + \||v|^{-\gamma}g_1\|_{\mathscr{M}_{TV}} + \|f\|_{L^\beta_{t,x,v}}). \tag{4-11}$$

For $q \in (\min\{p,r\}, \max\{p,r\})$ we choose a corresponding $\vartheta \in (0,1)$ subject to $1/q = (1-\vartheta)/r + \vartheta/p$. Then using $(L^r_{t,x}, L^{p,\infty}_{t,x})_{\vartheta,q} = L^q_{t,x}$, together with (4-11), we obtain

$$\begin{aligned}\|\bar{f}_l\|_{L^q_{t,x}} &\lesssim \|\bar{f}_l\|^{1-\vartheta}_{L^r_{t,x}}\|\bar{f}_l\|^{\vartheta}_{L^{p,\infty}_{t,x}} \\ &\lesssim \|\bar{f}\|^{1-\vartheta}_{L^r_{t,x}} 2^{-l\vartheta\kappa_t}(\||v|^{1-\gamma}g_0\|_{\mathscr{M}_{TV}} + \||v|^{-\gamma}g_1\|_{\mathscr{M}_{TV}} + \|f\|_{L^\infty_{t,x,v}})^{\vartheta} \\ &\leqslant 2^{-l\vartheta\kappa_t}(\||v|^{1-\gamma}g_0\|_{\mathscr{M}_{TV}} + \||v|^{-\gamma}g_1\|_{\mathscr{M}_{TV}} + \|f\|_{L^\infty_{t,x,v}} + \|\bar{f}\|_{L^r_{t,x}}).\end{aligned}$$

Multiplying by $2^{l\vartheta\kappa_t}$ and taking the supremum over $l \in \mathbb{Z}$ yields (4-10). □

Corollary 4.5. *Let $m \in (1,\infty)$, $\gamma \in (-\infty, m)$, $\mu \in [1, m+1-\gamma)$, $f \in L^1_{t,x,v} \cap L^\infty_{t,x,v}$ be a solution to* (4-1), *and let g_0, g_1 and $\bar{f}$ be as in Lemma 4.2. Let $q \in (1, (m+1-\gamma)/\mu)$ and define*

$$\tilde{\kappa}_x := \frac{\mu q - 1}{q}\frac{2}{m-\gamma}.$$

If $\bar{f} \in L^1(\mathbb{R}^{d+1}) \cap L^q(\mathbb{R}; L^1(\mathbb{R}^d))$, then $\bar{f} \in L^q(\mathbb{R}; W^{\sigma_x,q}(\mathbb{R}^d))$ for all $\sigma_x \in [0, \tilde{\kappa}_x)$. Furthermore,

$$\|\bar{f}\|_{L^q_t(W^{\sigma_x,q}_x)} \lesssim \||v|^{1-\gamma}g_0\|_{\mathscr{M}_{TV}} + \||v|^{-\gamma}g_1\|_{\mathscr{M}_{TV}} + \|f\|_{L^1_{t,x,v}\cap L^\infty_{t,x,v}} + \|\bar{f}\|_{L^1_{t,x}\cap L^q_t L^1_x}. \tag{4-12}$$

Proof. We recall the decomposition $f_j = \mathscr{F}^{-1}_x \varphi_j \mathscr{F}_x f$ introduced in the proof of Lemma 4.2. We argue that it suffices to consider the case when $f_j = 0$ for all $j < 0$. Indeed, the part $f_< := \sum_{j<0} f_j$ can be estimated in view of Bernstein's lemma, see [Bahouri, Chemin, and Danchin 2011, Lemma 2.1], via

$$\|\bar{f}_<\|_{L^q_t(W^{\sigma_x,q}_x)} \lesssim \|\bar{f}\|_{L^q_t L^1_x}.$$

We aim to control $\bar{f}$ in $\tilde{L}^q_t \dot{B}^{\vartheta\kappa_x}_{q,\infty}$ where $\vartheta \in (0,1)$ is sufficiently large such that $\sigma_x < \vartheta\kappa_x$, and then use Lemma 2.3 to the effect of

$$\|\bar{f}\|_{L^q_t(W^{\sigma_x,q}_x)} \lesssim \|\bar{f}\|_{\tilde{L}^q_t B^{\vartheta\kappa_x}_{q,\infty}} = \|\bar{f}\|_{\tilde{L}^q_t \dot{B}^{\vartheta\kappa_x}_{q,\infty}},$$

where the last equality is apparent from the definition of the homogeneous and nonhomogeneous Chemin–Lerner spaces and the fact that the low frequencies of f vanish. Thus, it remains to establish

$$\|\bar{f}\|_{\tilde{L}^q \dot{B}^{\vartheta\kappa_x}_{q,\infty}} \lesssim \||v|^{1-\gamma}g_0\|_{\mathscr{M}_{TV}} + \||v|^{-\gamma}g_1\|_{\mathscr{M}_{TV}} + \|f\|_{L^1_{t,x,v}\cap L^\infty_{t,x,v}} + \|\bar{f}\|_{L^1_{t,x}}. \tag{4-13}$$

For $\tilde{p} \in (1, (m+1-\gamma)/\mu)$, choose

$$\rho := \frac{(\tilde{p}-1)(m-\gamma)}{1+\tilde{p}(m-\mu-\gamma)}.$$

We claim that ρ is positive and well-defined: Since the numerator is positive due to $\tilde p > 1$ and $m > \gamma$, it remains to check that the denominator is positive. This is obvious for $\mu \leqslant m-\gamma$. For $\mu > m-\gamma$, we observe that due to $\mu < m+1-\gamma$ we have

$$\tilde p < \frac{m+1-\gamma}{\mu} < \frac{1}{\mu+\gamma-m},$$

which implies $1+\tilde p(m-\mu-\gamma) > 0$. Moreover, $\tilde p < (m+1-\gamma)/\mu$ can be rewritten as $(\tilde p-1)(m-\gamma) < 1+\tilde p(m-\mu-\gamma)$, so that $\rho \in (0,1)$. Hence, we may apply Lemma 4.2 with this choice of ρ and with $s=1$. One checks that in this case the integrability and differentiability exponents in (4-3) read

$$p = \tilde p, \quad \kappa_t = 0, \quad \kappa_x = \frac{\mu\tilde p - 1}{\tilde p}\frac{2}{m-\gamma}.$$

Choose $\tilde p \in (q, (m+1-\gamma)/\mu)$ so that $\tilde\kappa_x < \kappa_x$ and define $\vartheta \in (0,1)$ through

$$\frac{1}{q} = 1-\vartheta+\frac{\vartheta}{\tilde p}.$$

We may choose $\tilde p \in (q,(m+1-\gamma)/\mu)$ sufficiently small so that $\vartheta \in (0,1)$ is so large that $\sigma_x < \vartheta\tilde\kappa_x < \vartheta\kappa_x$. In view of (4-5) (with the space $S^{\vartheta\tilde\kappa}_{q,\infty}\dot B = S^{(0,\vartheta\kappa_x)}_{q,\infty}\dot B$ replaced by $\widetilde L^q_t \dot B^{\vartheta\kappa_x}_{q,\infty}$) we obtain

$$\|\bar f_j\|_{L^q_{t,x}} \lesssim 2^{-j\vartheta\kappa_x}(\||v|^{1-\gamma}g_0\|_{\mathscr{M}_{TV}} + \||v|^{-\gamma}g_1\|_{\mathscr{M}_{TV}} + \|f\|_{L^\beta_{t,x,v}} + \|\bar f\|_{L^1_{t,x}}),$$

where we recall the notation $\bar f_j := \int \mathscr{F}_x^{-1}[\varphi_j\mathscr{F}_x f]|v|^{\mu-1}\,\mathrm{d}v$. If we multiply by $2^{j\vartheta\kappa_x}$ and take the supremum over $j \in \mathbb{Z}$, this yields

$$\|\bar f\|_{\widetilde L^q_t \dot B^{\vartheta\kappa_x}_{q,\infty}} \lesssim \||v|^{1-\gamma}g_0\|_{\mathscr{M}_{TV}} + \||v|^{-\gamma}g_1\|_{\mathscr{M}_{TV}} + \|f\|_{L^\beta_{t,x,v}} + \|\bar f\|_{L^1_{t,x}}.$$

By the estimate $\|f\|_{L^\beta_{t,x,v}} \lesssim \|f\|_{L^1_{t,x,v}} + \|f\|_{L^\infty_{t,x,v}}$, this gives (4-13). □

Corollary 4.6. *Let $m \in (1,\infty)$, $\gamma \in (-\infty,1)$, $f \in L^1_{t,x,v} \cap L^\infty_{t,x,v}$ be a solution to* (4-1), *and let g_0 and g_1 be as in Lemma 4.2. Assume $\bar f \in L^r_{t,x}$ for all $r \in [1, m+1-\gamma)$, where $\bar f(t,x) := \int f(t,x,v)\,\mathrm{d}v$. Let $\tilde p \in (2-\gamma, m+1-\gamma)$ and define*

$$\tilde\kappa_t := \frac{m+1-\gamma-\tilde p}{\tilde p}\frac{1}{m-1}, \quad \tilde\kappa_x := \frac{\tilde p - 2+\gamma}{\tilde p}\frac{2}{m-1}.$$

Then $\bar f \in W^{\sigma_t,\tilde p}(\mathbb{R}; W^{\sigma_x,\tilde p}(\mathbb{R}^d))$ for all $\sigma_t \in [0,\tilde\kappa_t)$ and $\sigma_x \in [0,\tilde\kappa_x)$. Furthermore, there is an $r \in (\tilde p, m+1-\gamma)$ such that

$$\|\bar f\|_{W^{\sigma_t,\tilde p}(W^{\sigma_x,\tilde p})} \lesssim \||v|^{1-\gamma}g_0\|_{\mathscr{M}_{TV}} + \||v|^{-\gamma}g_1\|_{\mathscr{M}_{TV}} + \|f\|_{L^1_{t,x,v}\cap L^\infty_{t,x,v}} + \|\bar f\|_{L^r_{t,x}}. \tag{4-14}$$

Proof. As we need to pass from homogeneous spaces (the output of Lemmas 4.2 and 4.4) to a nonhomogeneous space, our strategy is to invoke Lemmas 2.5 and 2.4. The input to Lemma 2.5 requires four pieces of information, namely control of $\bar f$ in $L^{\tilde p}(\mathbb{R}^{d+1})$, $\widetilde L^{\tilde p}_x \dot B^{\sigma_t}_{\tilde p,\infty}$, $\widetilde L^{\tilde p}_t \dot B^{\sigma_x}_{\tilde p,\infty}$ and $S^{\bar\sigma}_{\tilde p,\infty}\dot B$. Since the control of $\bar f$ in $L^{\tilde p}(\mathbb{R}^{d+1})$ is ensured by assumption, we concentrate on the other three contributions. Note that the main difficulty lies in the condition that both the integrability exponent and the orders of differentiability have to match exactly.

Step 1: $\bar{f} \in S^{\bar{\sigma}}_{\tilde{p},\infty}\dot{B}$. Let $r \in (\tilde{p}, m+1-\gamma)$ to be chosen in Step 3. We claim that there exist functions $k_t, k_x : (0,\infty) \to (0,\infty)$ with $k_t(\varepsilon), k_x(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that it holds for all $\varepsilon \ll 1$

$$\|\bar{f}\|_{S^{\bar{\sigma}}_{\tilde{p},\infty}\dot{B}} \lesssim \||v|^{1-\gamma} g_0\|_{\mathscr{M}_{TV}} + \||v|^{-\gamma} g_1\|_{\mathscr{M}_{TV}} + \|f\|_{L^1_{t,x,v} \cap L^\infty_{t,x,v}} + \|\bar{f}\|_{L^r_{t,x}}, \tag{4-15}$$

where we have used the notation $\sigma_t := \tilde{\kappa}_t - k_t(\varepsilon)$ and $\sigma_x := \tilde{\kappa}_x - k_x(\varepsilon)$.

We apply Lemma 4.2 with $\mu = 1$, $\rho = 1-\varepsilon$, and $s := s_\varepsilon \in (0,1)$, where s_ε is chosen so that the integrability assertion in (4-3) reads $p = \tilde{p}$; this is possible for ρ close to 1 in view of Remark 4.3. Moreover, we may choose $\vartheta \in (0,1)$ such that κ_t and κ_x defined through (4-3) satisfy $\vartheta\kappa_t = \tilde{\kappa}_t - k_t(\varepsilon)$ and $\vartheta\kappa_x = \tilde{\kappa}_x - k_x(\varepsilon)$ for some functions k_t and k_x as above. Then for $1 < q_0 < \tilde{p} < q_1 < m+1-\gamma$ so that

$$\frac{1}{q_0} = 1 - \vartheta + \frac{\vartheta}{\tilde{p}}, \quad \frac{1}{q_1} = \frac{1-\vartheta}{r} + \frac{\vartheta}{\tilde{p}},$$

we obtain in view of (4-5) that

$$\|\bar{f}_{l,j}\|_{L^{q_i}_{t,x}} \lesssim 2^{-l\vartheta\kappa_t} 2^{-j\vartheta\kappa_x} (\||v|^{1-\gamma} g_0\|_{\mathscr{M}_{TV}} + \||v|^{-\gamma} g_1\|_{\mathscr{M}_{TV}} + \|f\|_{L^\beta_{t,x,v}} + \|\bar{f}\|_{L^1_{t,x} \cap L^r_{t,x}}),$$

for $i = 0, 1$, where we recall the notation $\bar{f}_{l,j} := \int \mathscr{F}^{-1}_{t,x}[\eta_l \varphi_j \mathscr{F}_{t,x} f]\,\mathrm{d}v$. Since $(L^{q_0}_{t,x}, L^{q_1}_{t,x})_{\theta,\tilde{p}} = L^{\tilde{p}}_{t,x}$ for an appropriate $\theta \in (0,1)$, we thus obtain

$$\|\bar{f}_{l,j}\|_{L^{\tilde{p}}_{t,x}} \lesssim 2^{-l\vartheta\kappa_t} 2^{-j\vartheta\kappa_x} (\||v|^{1-\gamma} g_0\|_{\mathscr{M}_{TV}} + \||v|^{-\gamma} g_1\|_{\mathscr{M}_{TV}} + \|f\|_{L^\beta_{t,x,v}} + \|\bar{f}\|_{L^1_{t,x}} + \|\bar{f}\|_{L^r_{t,x}}),$$

which after multiplying by $2^{l\vartheta\kappa_t} 2^{j\vartheta\kappa_x}$ and taking the supremum over $l, j \in \mathbb{Z}$ yields

$$\|\bar{f}\|_{S^{\vartheta(\kappa_t,\kappa_x)}_{\tilde{p},\infty}\dot{B}} \lesssim \||v|^{1-\gamma} g_0\|_{\mathscr{M}_{TV}} + \||v|^{-\gamma} g_1\|_{\mathscr{M}_{TV}} + \|f\|_{L^\beta_{t,x,v}} + \|\bar{f}\|_{L^1_{t,x}} + \|\bar{f}\|_{L^r_{t,x}}. \tag{4-16}$$

By the estimate $\|f\|_{L^\beta_{t,x,v}} + \|\bar{f}\|_{L^1_{t,x}} \lesssim \|f\|_{L^1_{t,x,v}} + \|f\|_{L^\infty_{t,x,v}}$, this gives (4-15).

Step 2: $\bar{f} \in \tilde{L}^{\tilde{p}}_t \dot{B}^{\sigma_x}_{\tilde{p},\infty}$. In this step we establish

$$\|\bar{f}\|_{\tilde{L}^{\tilde{p}}\dot{B}^{\sigma_x}_{\tilde{p},\infty}} \lesssim \||v|^{1-\gamma} g_0\|_{\mathscr{M}_{TV}} + \||v|^{-\gamma} g_1\|_{\mathscr{M}_{TV}} + \|f\|_{L^1_{t,x,v} \cap L^\infty_{t,x,v}} + \|\bar{f}\|_{L^r_{t,x}}. \tag{4-17}$$

Choose

$$\rho := \frac{(\tilde{p}-1)(m-\gamma)}{1+\tilde{p}(m-1-\gamma)}.$$

We claim that ρ is positive and well-defined: Since the numerator is positive due to $\tilde{p} > 1$ and $m > \gamma$, it remains to check that the denominator is positive. This is obvious for $\gamma \leqslant m-1$. For $\gamma > m-1$, we observe that

$$\tilde{p} < m+1-\gamma < \frac{1}{1+\gamma-m},$$

which implies $1 + \tilde{p}(m-1-\gamma) > 0$. Moreover, $\tilde{p} < m+1-\gamma$ can be rewritten as $(\tilde{p}-1)(m-\gamma) < 1 + \tilde{p}(m-1-\gamma)$, so that $\rho \in (0,1)$. Hence, we may apply Lemma 4.2 with this choice of ρ and with $s = 1$. One checks that in this case the integrability and differentiability exponents in (4-3) read

$$p = \tilde{p}, \quad \kappa_t = 0, \quad \kappa_x = \frac{p-1}{p}\frac{2}{m-\gamma}.$$

We observe that $\kappa_x \geqslant \tilde{\kappa}_x$ and hence we find $\vartheta \in (0,1)$ such that $\vartheta \kappa_x = \tilde{\kappa}_x - k_x(\varepsilon)$. The same interpolation argument as in Step 1 gives now the estimate (4-17).

Step 3: $\bar{f} \in \widetilde{L}^{\tilde{p}}_x \dot{B}^{\sigma_t}_{\tilde{p},\infty}$. In this step we show that there is some $r \in (\tilde{p}, m+1-\gamma)$ such that

$$\|\bar{f}\|_{\widetilde{L}^{\tilde{p}}_x \dot{B}^{\sigma_t}_{\tilde{p},\infty}} \lesssim \||v|^{1-\gamma} g_0\|_{\mathcal{M}_{TV}} + \||v|^{-\gamma} g_1\|_{\mathcal{M}_{TV}} + \|f\|_{L^\infty_{t,x,v}} + \|\bar{f}\|_{L^r_{t,x}}. \tag{4-18}$$

We apply Lemma 4.4 with $\mu = 1$ and $\rho = 1$. In this case, (4-9) reads $p = 2-\gamma$ and $\kappa_t = 1/(2-\gamma)$. Since $\tilde{p} > 2-\gamma$, we have $\tilde{\kappa}_t < \kappa_t$. Hence, we can choose $\vartheta \in (0,1)$ such that $\vartheta \kappa_t = \tilde{\kappa}_t - k_t(\varepsilon)$. In particular,

$$\vartheta < \frac{\tilde{\kappa}_t}{\kappa_t} = \frac{m+1-\gamma-\tilde{p}}{\tilde{p}} \frac{2-\gamma}{m-1} < \frac{2-\gamma}{\tilde{p}},$$

so that

$$r = \frac{\tilde{p}(2-\gamma)(1-\vartheta)}{2-\gamma-\vartheta\tilde{p}}$$

is well-defined. Since r is increasing in ϑ due to $\tilde{p} > 2-\gamma$, we see that $r \in (\tilde{p}, m+1-\gamma)$. We have $1/\tilde{p} = (1-\vartheta)/r + \vartheta/p$, and hence Lemma 4.4 gives estimate (4-18).

Step 4: Conclusion. Since $\bar{f} \in L^{\tilde{p}}_{t,x}$ by assumption, Lemma 2.5 combined with Lemma 2.4 yields the result. □

Corollary 4.7. *Let $m \in (1,\infty)$, $\gamma \in (-\infty, m)$, and let $f \in L^1_{t,x,v} \cap L^\infty_{t,x,v}$ be a solution to* (4-1). *Let g_0 and g_1 be as in Lemma 4.2 and assume additionally*

$$|g_0|(t,x,v) \in \mathcal{M}_{TV}(\mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{R}_v).$$

Assume $\tilde{p} \in (2-\gamma, m+1-\gamma) \cap (1, m+1-\gamma)$ and define

$$\tilde{\kappa}_t := \frac{m+1-\gamma-\tilde{p}}{\tilde{p}} \frac{1}{m-1}, \quad \tilde{\kappa}_x := \frac{\tilde{p}-2+\gamma}{\tilde{p}} \frac{2}{m-1}.$$

If $\bar{f} \in L^r(\mathbb{R}^{d+1}) \cap L^1(\mathbb{R}; L^{\tilde{p}}(\mathbb{R}^d))$ for all $r \in [1, m+1-\gamma)$, where $\bar{f}(t,x) := \int f(t,x,v)\,\mathrm{d}v$, and if $\int |v|^{m-1} f \,\mathrm{d}v \in L^1(\mathbb{R}^{d+1})$, then $\bar{f} \in W^{\sigma_t,\tilde{p}}(\mathbb{R}; W^{\sigma_x,\tilde{p}}(\mathbb{R}^d))$ for all $\sigma_t \in [0, \tilde{\kappa}_t)$ and $\sigma_x \in [0, \tilde{\kappa}_x)$. Furthermore, there is an $r \in (\tilde{p}, m+1-\gamma)$ such that

$$\begin{aligned}\|\bar{f}\|_{W^{\sigma_t,\tilde{p}}(W^{\sigma_x,\tilde{p}})} \lesssim{}& \|g_0\|_{\mathcal{M}_{TV}} + \||v|^{1-\gamma} g_0\|_{\mathcal{M}_{TV}} + \||v|^{-\gamma} g_1\|_{\mathcal{M}_{TV}} \\ &+ \|f\|_{L^1_{t,x,v} \cap L^\infty_{t,x,v}} + \|\bar{f}\|_{L^1_t L^{\tilde{p}}_x \cap L^r_{t,x}} + \left\| \int |v|^{m-1} f \,\mathrm{d}v \right\|_{L^1_{t,x}}.\end{aligned} \tag{4-19}$$

Proof. It suffices to adapt Step 3 of the proof of Corollary 4.6, that is, the control of $\bar{f}$ in $\widetilde{L}^{\tilde{p}}_x \dot{B}^{\sigma_t}_{\tilde{p},\infty}$.

Step 3: $\bar{f} \in \widetilde{L}^{\tilde{p}}_x \dot{B}^{\sigma_t}_{\tilde{p},\infty}$. In this step we show that there is some $r \in (\tilde{p}, m+1-\gamma)$ such that

$$\begin{aligned}\|\bar{f}\|_{\widetilde{L}^{\tilde{p}}_x \dot{B}^{\sigma_t}_{\tilde{p},\infty}} \lesssim{}& \|g_0\|_{\mathcal{M}_{TV}} + \||v|^{1-\gamma} g_0\|_{\mathcal{M}_{TV}} + \||v|^{-\gamma} g_1\|_{\mathcal{M}_{TV}} \\ &+ \|f\|_{L^1_{t,x,v} \cap L^\infty_{t,x,v}} + \|\bar{f}\|_{L^1_t L^{\tilde{p}}_x \cap L^r_{t,x}} + \left\| \int |v|^{m-1} f \,\mathrm{d}v \right\|_{L^1_{t,x}}.\end{aligned} \tag{4-20}$$

We split f into three contributions

$$f = \mathscr{F}_t^{-1}\psi_0(\tau)\mathscr{F}_t f + \mathscr{F}_{t,x}^{-1}(1-\psi_0(\tau))(1-\phi_0(\xi))\mathscr{F}_{t,x} f + \mathscr{F}_{t,x}^{-1}(1-\psi_0(\tau))\phi_0(\xi)\mathscr{F}_{t,x} f \\ =: f^1 + f^2 + f^3.$$

The low time-frequency part f^1 can be estimated in view of Lemma 2.3 and Bernstein's lemma, see [Bahouri, Chemin, and Danchin 2011, Lemma 2.1], via

$$\|\bar{f}^1\|_{\tilde{L}_x^{\tilde{p}} \dot{B}^{\sigma_t}_{\tilde{p},\infty}} \lesssim \|\bar{f}^1\|_{\tilde{L}_x^{\tilde{p}} B^{\sigma_t}_{\tilde{p},\infty}} \lesssim \|\bar{f}^1\|_{W^{\sigma_t+\varepsilon,\tilde{p}}(L_x^{\tilde{p}})} \lesssim \|\bar{f}\|_{L_t^1 L_x^{\tilde{p}}}. \tag{4-21}$$

Next, we apply Lemma 4.2 with $\mu = 1$, sufficiently large $\rho \in (0,1)$ and sufficiently small $s \in ((\gamma-1)/(m-1), 1]$ so that (4-3) implies $p < \tilde{p}$ and $\kappa_t > \tilde{\kappa}_t$. Hence, we can choose $\vartheta \in (0,1)$ such that $\tilde{\kappa}_t > \vartheta\kappa_t > \tilde{\kappa}_t - k_t(\varepsilon)$. In particular, in light of Remark 4.3

$$\vartheta < \frac{\tilde{\kappa}_t}{\kappa_t} = \frac{m+1-\gamma-\tilde{p}}{m+\rho-\gamma-p\rho+p(1-\rho)(\gamma-m)}\frac{p\rho}{\tilde{p}} < \frac{p}{\tilde{p}} \quad \text{if } 1-\rho \ll 1,$$

so that $r = \tilde{p}p(1-\vartheta)/(p-\vartheta\tilde{p})$ is well-defined. Since r is increasing in ϑ due to $\tilde{p} > p$, we see that $r \in (\tilde{p}, m+1-\gamma)$. We have $1/\tilde{p} = (1-\vartheta)/r + \vartheta/p$, and hence Lemma 4.2 gives

$$\|\bar{f}\|_{S^{\vartheta\bar{\kappa}}_{\tilde{p},\infty}\dot{B}} \lesssim \||v|^{1-\gamma} g_0\|_{\mathscr{M}_{TV}} + \||v|^{-\gamma} g_1\|_{\mathscr{M}_{TV}} + \|f\|_{L^1_{t,x,v}\cap L^\infty_{t,x,v}} + \|\bar{f}\|_{L^r_{t,x}}.$$

Thus, since f^2 is supported only on $\eta_l\varphi_j$ for nonnegative $l, j \in \mathbb{Z}$, Lemmas 2.3 and 2.4 show, in view of the definition of the homogeneous and nonhomogeneous Besov spaces and $\sigma_t < \vartheta\kappa_t$ as well as $0 < \vartheta\kappa_x$,

$$\|\bar{f}^2\|_{\tilde{L}_x^{\tilde{p}} \dot{B}^{\sigma_t}_{\tilde{p},\infty}} = \|\bar{f}^2\|_{\tilde{L}_x^{\tilde{p}} B^{\sigma_t}_{\tilde{p},\infty}} \lesssim \|\bar{f}^2\|_{S^{\vartheta\bar{\kappa}}_{\tilde{p},\infty}B} = \|\bar{f}\|_{S^{\vartheta\bar{\kappa}}_{\tilde{p},\infty}\dot{B}}.$$

Thus,

$$\|\bar{f}^2\|_{\tilde{L}_x^{\tilde{p}} \dot{B}^{\sigma_t}_{\tilde{p},\infty}} \lesssim \||v|^{1-\gamma} g_0\|_{\mathscr{M}_{TV}} + \||v|^{-\gamma} g_1\|_{\mathscr{M}_{TV}} + \|f\|_{L^1_{t,x,v}\cap L^\infty_{t,x,v}} + \|\bar{f}\|_{L^r_{t,x}}. \tag{4-22}$$

It remains to estimate the contribution of f^3. For $l \in \mathbb{Z}$, we introduce $f_l^3 := \mathscr{F}_t^{-1}\eta_l(\tau)\mathscr{F}_t f^3$. Since $f_l^3 = 0$ for $l < 0$, we may concentrate on the case $l \geqslant 0$. Observe that f_l^3 solves the equation

$$f_l^3 = -m|v|^{m-1}\mathscr{F}_{t,x}^{-1}\frac{|\xi|^2}{i\tau}\eta_l(\tau)\phi_0(\xi)\mathscr{F}_{t,x} f + \mathscr{F}_{t,x}^{-1}\frac{\phi_0(\xi)}{i\tau}\mathscr{F}_{t,x} g_{0,l} + \mathscr{F}_{t,x}^{-1}\frac{\phi_0(\xi)}{i\tau}\mathscr{F}_{t,x}\partial_v g_{1,l}.$$

Integrating in v, we obtain

$$\bar{f}_l^3 = -m\int |v|^{m-1}\mathscr{F}_{t,x}^{-1}\frac{|\xi|^2}{i\tau}\eta_l(\tau)\phi_0(\xi)\mathscr{F}_{t,x} f \,\mathrm{d}v + \mathscr{F}_{t,x}^{-1}\frac{1}{i\tau}\phi_0(\xi)\mathscr{F}_{t,x}\int g_{0,l,j}\,\mathrm{d}v.$$

Since $|\xi|^2$ acts as a constant multiplier on the support of ϕ_0 and τ^{-1} acts as a constant multiplier of order 2^{-l} on the support of η_l, it follows by Bernstein's lemma

$$\|\bar{f}_l^3\|_{L^{\tilde{p}}_{t,x}} \lesssim 2^{l(1-\frac{1}{\tilde{p}})}\|\bar{f}_l^3\|_{L^1_{t,x}} \lesssim 2^{-l\frac{1}{\tilde{p}}}\left(\left\|\int |v|^{m-1} f\,\mathrm{d}v\right\|_{L^1_{t,x}} + \|g_0\|_{\mathscr{M}_{TV}}\right).$$

Since $\tilde{p} > 2-\gamma$, we have

$$\sigma_t < \tilde{\kappa}_t = \frac{m+1-\gamma-\tilde{p}}{\tilde{p}} \frac{1}{m-1} < \frac{1}{\tilde{p}}.$$

In view of $l \geqslant 0$ this yields

$$\|\bar{f}_l^3\|_{L^{\tilde{p}}_{t,x}} \lesssim 2^{-l\sigma_t}\left(\left\|\int |v|^{m-1} f \,\mathrm{d}v\right\|_{L^1_{t,x}} + \|g_0\|_{\mathcal{M}_{TV}}\right).$$

Multiplying by $2^{l\sigma_t}$ and taking the supremum over $l \in \mathbb{Z}$, we conclude

$$\|\bar{f}^3\|_{\tilde{L}^{\tilde{p}}_x \dot{B}^{\sigma_t}_{\tilde{p},\infty}} \lesssim \left\|\int |v|^{m-1} f \,\mathrm{d}v\right\|_{L^1_{t,x}} + \|g_0\|_{\mathcal{M}_{TV}}. \tag{4-23}$$

Collecting (4-21), (4-22) and (4-23), we arrive at (4-20). □

5. Application to porous medium equations

In this section, we provide proofs of our main results by applying the averaging lemmas obtained in the previous section to entropy solutions to (1-1).

Proof of Theorem 1.2. We first argue that we have $u \in L^s_{t,x}$ for all $s \in [1, m-1+\rho)$. Since $T < \infty$, Theorem A.2 gives

$$\|u\|_{L^1_{t,x}} \lesssim \sup_{t\in[0,T]} \|u(t)\|_{L^1_x} \lesssim \|u_0\|_{L^1_x} + \|S\|_{L^1_{t,x}}, \tag{5-1}$$

so that we may concentrate on $s > 1$. Let f be the kinetic function corresponding to u and solving (1-8). In order to apply Corollary 4.5 with $\mu = 1$ and $\sigma_x = 0$, we need to extend (1-8) to all times $t \in \mathbb{R}$, which can be achieved by multiplication with a smooth cut-off function $\varphi \in C^\infty_c(0,T)$ with $0 \leqslant \varphi \leqslant 1$. Hence, we set $g_0 := \delta_{v=u(t,x)} S + \partial_t \varphi f$ and $g_1 := q$. Let $\gamma := 2-\rho$, so that $s \in (1, m+1-\gamma)$. From (4-12) we obtain

$$\begin{aligned}\|\varphi u\|_{L^s_{t,x}} &\lesssim \||v|^{\rho-1} g_0\|_{\mathcal{M}_{TV}} + \||v|^{\rho-2} g_1\|_{\mathcal{M}_{TV}} + \|\varphi f\|_{L^1_{t,x,v}\cap L^\infty_{t,x,v}} + \|\varphi u\|_{L^1_{t,x}\cap L^s_t L^1_x} \\ &\lesssim \||v|^{\rho-1} g_0\|_{\mathcal{M}_{TV}} + \||v|^{\rho-2} g_1\|_{\mathcal{M}_{TV}} + \|f\|_{L^1_{t,x,v}\cap L^\infty_{t,x,v}} + \sup_{t\in[0,T]} \|u(t)\|_{L^1_x}.\end{aligned}$$

We note that since trivially $f \in L^\infty_{t,x,v}$ with norm bounded by 1, estimate (5-1) gives

$$\|f\|_{L^1_{t,x,v}\cap L^\infty_{t,x,v}} + \sup_{t\in[0,T]} \|u(t)\|_{L^1_x} \lesssim \|u\|_{L^1_{t,x}} + 1 + \sup_{t\in[0,T]} \|u(t)\|_{L^1_x} \lesssim \|u_0\|_{L^1_x} + \|S\|_{L^1_{t,x}} + 1.$$

Next, we check that $|v|^{\rho-1} g_0 \in \mathcal{M}_{TV}$. Indeed, we observe that $(\rho-1)\rho' := \rho$, and hence, applying Lemma A.3,

$$\begin{aligned}\||v|^{\rho-1} g_0\|_{\mathcal{M}_{TV}} &= \||v|^{\rho-1}(\delta_{v=u(t,x)} S + \partial_t \varphi f)\|_{\mathcal{M}_{TV}} \lesssim \||u|^{\rho-1} S\|_{L^1_{t,x}} + \|\partial_t \varphi |u|^\rho\|_{L^1_{t,x}} \\ &\lesssim \||u|^{(\rho-1)\rho'}\|_{L^1_{t,x}} + \||S|^\rho\|_{L^1_{t,x}} + \|\partial_t \varphi |u|^\rho\|_{L^1_{t,x}} \\ &\lesssim \|u_0\|^\rho_{L^\rho_x} + \|S\|^\rho_{L^\rho_{t,x}} + \|\partial_t \varphi |u|^\rho\|_{L^1_{t,x}}.\end{aligned}$$

Utilizing Lemma A.3 once more to the effect of

$$\||v|^{\rho-2}g_1\|_{\mathscr{M}_{TV}} = \||v|^{\rho-2}q\|_{\mathscr{M}_{TV}} \lesssim \|u_0\|^{\rho}_{L^{\rho}_x} + \|S\|^{\rho}_{L^{\rho}_{t,x}},$$

we obtain

$$\|\varphi u\|_{L^s_{t,x}} \lesssim \|u_0\|^{\rho}_{L^1_x\cap L^{\rho}_x} + \|S\|^{\rho}_{L^1_{t,x}\cap L^{\rho}_{t,x}} + \|\partial_t\varphi|u|^{\rho}\|_{L^1_{t,x}} + 1.$$

We may set $\varphi_n(t) = \psi(nt) - \psi(nt - T/2)$, where $\psi \in C^\infty(\mathbb{R})$ with $0 \leqslant \psi \leqslant 1$, $\operatorname{supp}\psi \subset (0,\infty)$, $\psi(t) = 1$ for $t > T/2$ and $\|\partial_t\psi\|_{L^1} = 1$. For $n \to \infty$, φ_n converges to $1_{[0,T]}$ in the supremum norm, while $\partial_t\varphi_n$ is a smooth approximation of $\delta_{\{t=0\}} - \delta_{\{t=T\}}$. Therefore, $\|\varphi_n u\|_{L^s_{t,x}} \to \|u\|_{L^s_{t,x}}$ and by an application of Lemma A.3

$$\|\partial_t\varphi_n|u|^{\rho}\|_{L^1_{t,x}} \to \||u|(0)^{\rho} - |u|(T)^{\rho}\|_{L^1_x} \lesssim \|u_0\|^{\rho}_{L^{\rho}_x} + \|S\|^{\rho}_{L^{\rho}_{t,x}},$$

so that $u \in L^s([0,T]\times\mathbb{R}^d)$ and

$$\|u\|_{L^s_{t,x}} \lesssim \|u_0\|^{\rho}_{L^1_x\cap L^{\rho}_x} + \|S\|^{\rho}_{L^1_{t,x}\cap L^{\rho}_{t,x}} + 1. \tag{5-2}$$

(i) We apply Corollary 4.5 once more. Let f, φ, g_0, g_1 and γ be as above. Then, in particular $p \in (1, (m+1-\gamma)/\mu)$. From (4-12) we obtain

$$\|\varphi u^{[\mu]}\|_{L^p_t(W^{\sigma_x,p})} \lesssim \||v|^{1-\gamma}g_0\|_{\mathscr{M}_{TV}} + \||v|^{-\gamma}g_1\|_{\mathscr{M}_{TV}} + \|f\|_{L^1_{t,x,v}\cap L^\infty_{t,x,v}} + \|u^{[\mu]}\|_{L^1_{t,x}\cap L^p_t L^1_x}.$$

The first three contributions on the right-hand side are estimated as above. For the last contribution, we note $1 \leqslant \mu < p\mu$ and thus

$$\begin{aligned}\|u^{[\mu]}\|_{L^1_{t,x}\cap L^p_t L^1_x} &\lesssim \|u^{[\mu]}\|_{L^p_t L^1_x} = \|u\|^{\mu}_{L^{p\mu}_t L^{\mu}_x} \lesssim (\|u\|_{L^{p\mu}_t L^1_x} + \|u\|_{L^{p\mu}_{t,x}})^{\mu}\\ &\lesssim (\sup_{t\in[0,T]}\|u(t)\|_{L^1_x} + \|u\|_{L^{p\mu}_{t,x}})^{\mu} \lesssim \sup_{t\in[0,T]}\|u(t)\|^{\mu}_{L^1_x} + \|u\|^{\mu}_{L^{p\mu}_{t,x}} + 1.\end{aligned}$$

Furthermore, (5-1) together with (5-2) applied with $s = p\mu \in (1, m-1+\rho)$ shows

$$\sup_{t\in[0,T]}\|u(t)\|^{\mu}_{L^1_x} + \|u\|^{\mu}_{L^{p\mu}_{t,x}} + 1 \lesssim \|u_0\|^{\mu\rho}_{L^1_x\cap L^{\rho}_x} + \|S\|^{\mu\rho}_{L^1_{t,x}\cap L^{\rho}_{t,x}} + 1.$$

Hence, arguing as above by taking the limit $\varphi_n \to 1_{[0,T]}$, we obtain $u^{[\mu]} \in L^p(\mathbb{R}; W^{\sigma_x,p}(\mathbb{R}^d))$ and (1-4).

(ii) The proof is similar to the first part, but we use Corollary 4.6 instead of Corollary 4.5. Again we localize in time by multiplying with a smooth cut-off function $\varphi \in C^\infty_c(0,T)$ with $0 \leqslant \varphi \leqslant 1$ and set g_0 and g_1 as before. Choose $\gamma := 2-\rho$, so that $p \in (2-\gamma, m+1-\gamma)$. From (4-14) in Corollary 4.6 we obtain

$$\|\varphi u\|_{W^{\sigma_t,p}(W^{\sigma_x,p})} \lesssim \||v|^{1-\gamma}g_0\|_{\mathscr{M}_{TV}} + \||v|^{-\gamma}g_1\|_{\mathscr{M}_{TV}} + \|f\|_{L^1_{t,x,v}\cap L^\infty_{t,x,v}} + \|u\|_{L^r_{t,x}},$$

where $r \in (\rho, m-1+\rho)$. The terms involving g_0, g_1 and f can be estimated as above, while the $L^r_{t,x}$-norm of u can be estimated by (5-2). Choosing φ_n as above, we hence infer that $\varphi_n u$ is bounded in

$W^{\sigma_t,p}(0,T;W^{\sigma_x,p}(\mathbb{R}^d))$ and

$$\sup_{n\in\mathbb{N}}\|\varphi_n u\|_{W^{\sigma_t,p}(W^{\sigma_x,p})} \lesssim \|u_0\|^{\rho}_{L^1_x\cap L^{\rho}_x} + \|S\|^{\rho}_{L^1_{t,x}\cap L^{\rho}_{t,x}} + 1.$$

Since $\varphi_n u \to u1_{[0,T]}$ in the sense of distributions, we obtain the result by the weak lower semicontinuity of the norm in $W^{\sigma_t,p}(0,T;W^{\sigma_x,p}(\mathbb{R}^d))$. □

Proof of Corollary 1.3. (i) Let $\sigma_x \in [0, 2\mu/m)$. We apply Theorem 1.2(i) with $p = m/\mu$ for sufficiently small $\eta \in (1,\rho]$ so that

$$\sigma_x < \frac{\mu p - 1}{p}\frac{2}{m-2+\eta} = \frac{2\mu}{m}\frac{m-1}{m-2+\eta}$$

and observe that for all $q \in [1,p]$ we have the embedding $L^p(0,T;W^{\sigma_x,p}(\mathbb{R}^d)) \subset L^q(0,T;W^{\sigma_x,q}(\mathscr{O}))$.

(ii) For $s>0$ we have, with $p = s(m-1)+1 \in (1,m]$,

$$\kappa_t = \frac{1-s}{s(m-1)+1} = \frac{m-p}{p}\frac{1}{m-1}, \quad \kappa_x = \frac{2s}{s(m-1)+1} = \frac{p-1}{p}\frac{2}{m-1}.$$

Hence, in this case the assertion follows by an application of Theorem 1.2(ii) with sufficiently small $\eta \in (1,\rho]$ such that $p > \rho$ and

$$\sigma_x < \frac{p-\rho}{p}\frac{2}{m-1}$$

combined with the embedding

$$W^{\sigma_t,p}(0,T;W^{\sigma_x,p}(\mathbb{R}^d)) \subset W^{\sigma_t,q}(0,T;W^{\sigma_x,q}(\mathscr{O})).$$

If $s=0$ and $\sigma_t \in [0,1)$, we may choose $s_0 > 0$ such that

$$\sigma_t < \frac{1-s_0}{s_0(m-1)+1} =: \kappa_t(s_0),$$

and the result follows by the embedding

$$W^{\kappa_t(s_0),s_0(m-1)+1}(0,T;L^{s_0(m-1)+1}(\mathscr{O})) \subset W^{\sigma_t,1}(0,T;L^1(\mathscr{O})). \qquad \square$$

Proof of Theorem 1.1. The proof is similar to that of Theorem 1.2(ii), but we discriminate between small and large velocity contributions to the kinetic function. Let f be the kinetic function corresponding to u and solving (1-8). We extend again to all times $t \in \mathbb{R}$ by multiplying with a smooth cut-off function $\varphi \in C_c^\infty(0,T)$ with $0 \leqslant \varphi \leqslant 1$. Further, we split $f =: f^< + f^>$ and $q =: q^< + q^>$ into a small-velocity and a large-velocity part by multiplying with a smooth cut-off function ψ_0 respectively $\psi_1 := 1-\psi_0$ in v. This gives rise to the two equations

$$\partial_t(\varphi f^<) - m|v|^{m-1}\Delta_x(\varphi f^<) = \varphi\psi_0\delta_{v=u(t,x)}S + \partial_v(\varphi q^<) - \varphi q\partial_v\psi_0 + \partial_t\varphi f^<,$$
$$\partial_t(\varphi f^>) - m|v|^{m-1}\Delta_x(\varphi f^>) = \varphi\psi_1\delta_{v=u(t,x)}S + \partial_v(\varphi q^>) + \varphi q\partial_v\psi_0 + \partial_t\varphi f^>,$$

Integrating $f^<$ and $f^>$ in v, we obtain a decomposition of $u = u^< + u^>$.

The proof proceeds in several steps: In first the three steps, we argue that $u \in L^s(0,T;L^s(\mathbb{R}^d))$ for all $s \in [1, m+2/d)$ if $d \geqslant 2$ and $s \in [1, m+1)$ if $d=1$. With this additional bound, we can conclude the higher-order estimates in the last three steps of the proof. We only detail the proof for $d \geqslant 2$, the case $d = 1$ being similar.

Step 1: In this step we establish for $\rho \in (m, md/(d-2))$ the bound

$$\|u^<\|_{L_t^m L_x^\rho} \lesssim \|u_0\|_{L_x^1} + \|S\|_{L_{t,x}^1} + 1. \tag{5-3}$$

Set $g_0 := \varphi\psi_0\delta_{v=u(t,x)}S + \partial_t\varphi f^< - \varphi q \partial_v \psi_0$, $g_1 := \varphi q^<$, and

$$\sigma_x := \frac{d}{m} - \frac{d}{\rho} \in \left(0, \frac{2}{m}\right).$$

Consequently, we may choose $\gamma \in (0,1)$ so large that

$$\sigma_x \in \left[0, \frac{m-1}{m}\frac{2}{m-\gamma}\right).$$

From Corollary 4.5 applied with $\mu = 1$ and $q = m$ we obtain

$$\begin{aligned}\|\varphi u^<\|_{L_t^m W_x^{\sigma_x,m}} &\lesssim \||v|^{1-\gamma} g_0\|_{\mathscr{M}_{TV}} + \||v|^{-\gamma} g_1\|_{\mathscr{M}_{TV}} + \|\varphi f^<\|_{L^1_{t,x,v}\cap L^\infty_{t,x,v}} + \|\varphi u^<\|_{L^1_{t,x}\cap L_t^m L_x^1} \\ &\lesssim \||v|^{1-\gamma} g_0\|_{\mathscr{M}_{TV}} + \||v|^{-\gamma} g_1\|_{\mathscr{M}_{TV}} + \|f\|_{L^1_{t,x,v}\cap L^\infty_{t,x,v}} + \sup_{t\in[0,T]} \|u(t)\|_{L_x^1}.\end{aligned}$$

We note that since trivially $f^< \in L^\infty_{t,x,v}$ with norm bounded by 1 we have by Theorem A.2

$$\|f\|_{L^1_{t,x,v}\cap L^\infty_{t,x,v}} + \sup_{t\in[0,T]} \|u(t)\|_{L_x^1} \lesssim \|u\|_{L^1_{t,x}} + 1 + \sup_{t\in[0,T]} \|u(t)\|_{L_x^1} \lesssim \|u_0\|_{L_x^1} + \|S\|_{L^1_{t,x}} + 1.$$

Next, we check that $|v|^{1-\gamma} g_0 \in \mathscr{M}_{TV}$. Indeed, since $|v|^{1-\gamma}$ can be estimated by a constant on the supports of ψ_0 and $\partial_v\psi_0$, we may apply Lemma A.4 to the effect of

$$\begin{aligned}\||v|^{1-\gamma} g_0\|_{\mathscr{M}_{TV}} &= \||v|^{1-\gamma}(\varphi\psi_0\delta_{v=u(t,x)}S + \partial_t\varphi f^< - \varphi q\partial_v\psi_0)\|_{\mathscr{M}_{TV}} \\ &\lesssim \|S\|_{L^1_{t,x}} + \|\partial_t\varphi|u|\|_{L^1_{t,x}} + \|q\partial_v\psi_0\|_{\mathscr{M}_{TV}} \\ &\lesssim \|\partial_t\varphi|u|\|_{L^1_{t,x}} + \|u_0\|_{L_x^1} + \|S\|_{L^1_{t,x}}.\end{aligned}$$

Utilizing Lemma A.4 once more to the effect of

$$\||v|^{-\gamma} g_1\|_{\mathscr{M}_{TV}} \lesssim \||v|^{-\gamma} q^<\|_{\mathscr{M}_{TV}} \lesssim \|u_0\|_{L_x^1} + \|S\|_{L^1_{t,x}},$$

we obtain by Sobolev embedding

$$\|\varphi u^<\|_{L_t^m L_x^\rho} \lesssim \|\varphi u^<\|_{L_t^m W_x^{\sigma_x,m}} \lesssim \|u_0\|_{L_x^1} + \|\partial_t\varphi|u|\|_{L^1_{t,x}} + \|S\|_{L^1_{t,x}} + 1. \tag{5-4}$$

With the same construction $\varphi_n \to 1_{[0,T]}$ as in the proof of Theorem 1.2, this gives (5-3).

Step 2: Next, we investigate $u^>$ and establish for $\eta \in (1, m)$ and

$$\eta^* = \frac{\eta d(m-1)}{d(m-1) - 2(\eta - 1)}$$

the bound

$$\|u^>\|_{L_t^\eta L_x^{\eta^*}} \lesssim \|u_0\|_{L_x^1} + \|S\|_{L^1_{t,x}} + 1. \tag{5-5}$$

Set $g_0 := \varphi\psi_1\delta_{v=u(t,x)}S + \partial_t\varphi f^> + \varphi q\partial_v\psi_0$ and $g_1 := \varphi q^>$. Choose $\gamma \in (1, m)$ sufficiently small, so that $\eta \in (1, m+1-\gamma)$, and define

$$\sigma_x := \frac{\eta-1}{\eta}\frac{2}{m-1} \in \left(0, \frac{\eta-1}{\eta}\frac{2}{m-\gamma}\right).$$

We apply Corollary 4.5 with $\mu = 1$ and $q = \eta$, which gives

$$\begin{aligned}\|\varphi u^>\|_{L_t^\eta W_x^{\sigma_x,\eta}} &\lesssim \||v|^{1-\gamma}g_0\|_{\mathcal{M}_{TV}} + \||v|^{-\gamma}g_1\|_{\mathcal{M}_{TV}} + \|\varphi f^>\|_{L^1_{t,x,v}\cap L^\infty_{t,x,v}} + \|\varphi u^>\|_{L^1_{t,x}\cap L^\eta_t L^1_x}\\ &\lesssim \||v|^{1-\gamma}g_0\|_{\mathcal{M}_{TV}} + \||v|^{-\gamma}g_1\|_{\mathcal{M}_{TV}} + \|f\|_{L^1_{t,x,v}\cap L^\infty_{t,x,v}} + \sup_{t\in[0,T]}\|u(t)\|_{L^1_x}.\end{aligned}$$

The terms involving f and u are estimated as in Step 1. Further, since $|v|^{1-\gamma}$ can be estimated by a constant on the support of ψ_1 and $\partial_v\psi_0$, we have by Lemma A.4

$$\begin{aligned}\||v|^{1-\gamma}g_0\|_{\mathcal{M}_{TV}} &= \||v|^{1-\gamma}(\varphi\psi_1\delta_{v=u(t,x)}S + \partial_t\varphi f^> + \varphi q\partial_v\psi_0)\|_{\mathcal{M}_{TV}}\\ &\lesssim \|S\|_{L^1_{t,x}} + \|\partial_t\varphi|u|\|_{L^1_{t,x}} + \|q\partial_v\psi_0\|_{\mathcal{M}_{TV}}\\ &\lesssim \|\partial_t\varphi|u|\|_{L^1_{t,x}} + \|u_0\|_{L^1_x} + \|S\|_{L^1_{t,x}},\end{aligned}$$

and, again due to Lemma A.4,

$$\||v|^{-\gamma}g_1\|_{\mathcal{M}_{TV}} \lesssim \||v|^{-\gamma}q^>\|_{\mathcal{M}_{TV}} \lesssim \|u_0\|_{L^1_x} + \|S\|_{L^1_{t,x}}.$$

Since $\eta^* = \eta d/(d-\sigma_x\eta)$, we have by Sobolev embedding $W_x^{\sigma_x,\eta} \subset L_x^{\eta^*}$, and hence

$$\|\varphi u^>\|_{L_t^\eta L_x^{\eta^*}} \lesssim \|\varphi u^>\|_{L_t^\eta W_x^{\sigma_x,\eta}} \lesssim \|u_0\|_{L^1_x} + \|\partial_t\varphi|u|\|_{L^1_{t,x}} + \|S\|_{L^1_{t,x}} + 1.$$

With the same construction $\varphi_n \to 1_{[0,T]}$ as before, this yields (5-5).

Step 3: In this step, we show that for $s \in [1, m+2/d)$ we have

$$\|u\|_{L^s_{t,x}} \lesssim \|u_0\|_{L^1_x} + \|S\|_{L^1_{t,x}} + 1. \tag{5-6}$$

Observe that it suffices to show the assertion for $s > m$, since $u \in L^1(0,T;L^1(\mathbb{R}^d))$ is already established by Theorem A.2.

Define

$$\rho := \frac{m}{m+1-s} \in \left(m, \frac{md}{d-2}\right).$$

For $\vartheta \in (0,1)$, it holds $[L_t^\infty L_x^1, L_t^m L_x^\rho]_\vartheta = L_t^{p_\vartheta} L_x^{q_\vartheta}$ with

$$\frac{1}{p_\vartheta} = \frac{\vartheta}{m} \quad \text{and} \quad \frac{1}{q_\vartheta} = 1 - \vartheta + \frac{\vartheta}{\rho}.$$

Choosing

$$\vartheta := \frac{m\rho}{m\rho + \rho - m} \in (0,1),$$

we obtain $p_\vartheta = q_\vartheta = s$, and hence by (5-3) and Theorem A.2

$$\|u^<\|_{L^s_{t,x}} \lesssim \|u^<\|_{L_t^\infty L_x^1} + \|u^<\|_{L_t^m L_x^\rho} \lesssim \|u_0\|_{L^1_x} + \|S\|_{L^1_{t,x}} + 1. \tag{5-7}$$

Next, we define

$$\eta := \frac{sd(m-1)+2}{d(m-1)+2} \in (1,m) \quad \text{and} \quad \eta^* = \frac{\eta d}{d-2(\eta-1)/(m-1)}$$

and observe that for $\vartheta \in (0,1)$, it holds $[L_t^\infty L_x^1, L_t^\eta L_x^{\eta^*}]_\vartheta = L_t^{p_\vartheta} L_x^{q_\vartheta}$ with

$$\frac{1}{p_\vartheta} = \frac{\vartheta}{\eta} \quad \text{and} \quad \frac{1}{q_\vartheta} = 1-\vartheta+\frac{\vartheta}{\eta^*}.$$

Choosing

$$\vartheta := \frac{\eta d(m-1)}{\eta d(m-1)+2(\eta-1)} \in (0,1),$$

we obtain $p_\vartheta = q_\vartheta = s$, and hence by (5-5) and Theorem A.2

$$\|u^>\|_{L^s_{t,x}} \lesssim \|u^>\|_{L_t^\infty L_x^1} + \|u^>\|_{L_t^\eta L_x^{\eta^*}} \lesssim \|u_0\|_{L_x^1} + \|S\|_{L^1_{t,x}} + 1. \tag{5-8}$$

Combining (5-7) and (5-8), we obtain (5-6).

Step 4: In this step we argue that

$$\|\varphi u^<\|_{W^{\sigma_t,p}(W^{\sigma_x,p})} \lesssim \|\partial_t\varphi|u|\|_{L^1_{t,x}} + \|u_0\|^m_{L^1_x} + \|S\|^m_{L^1_{t,x}} + 1.$$

Indeed, we choose $\gamma \in (0,1)$ so large that

$$\sigma_x < \frac{p-2+\gamma}{p}\frac{2}{m-1}$$

and $m+1-\gamma < m+2/d$. Then we apply Corollary 4.7 with $g_0 := \varphi\psi_0\delta_{v=u(t,x)}S + \partial_t\varphi f^< - \varphi q\partial_v\psi_0$, $g_1 := \varphi q^<$ and $\tilde{p} = p$. We obtain by (4-19) some $r \in (p, m+1-\gamma)$ such that

$$\|\varphi u^<\|_{W^{\sigma_t,p}(W^{\sigma_x,p})} \lesssim \|g_0\|_{\mathcal{M}_{TV}} + \||v|^{1-\gamma}g_0\|_{\mathcal{M}_{TV}} + \||v|^{-\gamma}g_1\|_{\mathcal{M}_{TV}} + \|f\|_{L^1_{t,x,v}\cap L^\infty_{t,x,v}} + \|u\|_{L^1_tL^p_x\cap L^r_{t,x}} + \||u|^m\|_{L^1_{t,x}}.$$

The first four terms on the right-hand side can be estimated as in Step 1 (indeed, we did not use the coefficient $|v|^{1-\gamma}$ in the estimate of g_0) via

$$\|g_0\|_{\mathcal{M}_{TV}} + \||v|^{1-\gamma}g_0\|_{\mathcal{M}_{TV}} + \||v|^{-\gamma}g_1\|_{\mathcal{M}_{TV}} + \|f\|_{L^1_{t,x,v}\cap L^\infty_{t,x,v}} \lesssim \|\partial_t\varphi|u|\|_{L^1_{t,x}} + \|u_0\|_{L^1_x} + \|S\|_{L^1_{t,x}} + 1,$$

while the last two terms are estimated in light of $r < m+1-\gamma < m+2/d$ through (5-6) as

$$\|u\|_{L^1_tL^p_x\cap L^r_{t,x}} + \||u|^m\|_{L^1_{t,x}} \lesssim \|u\|_{L^p_{t,x}\cap L^r_{t,x}} + \|u\|^m_{L^m_{t,x}} \lesssim \|u_0\|^m_{L^1_x} + \|S\|^m_{L^1_{t,x}} + 1.$$

Step 5: In this step we establish

$$\|\varphi u^>\|_{W^{\sigma_t,p}(W^{\sigma_x,p})} \lesssim \|\partial_t\varphi|u|\|_{L^1_{t,x}} + \|u_0\|^m_{L^1_x} + \|S\|^m_{L^1_{t,x}} + 1. \tag{5-9}$$

Assume first $p < m$. Choose $\gamma \in (1,m)$ so small that $p \in (1, m+1-\gamma)$ and

$$\sigma_t < \frac{m+1-\gamma-p}{p}\frac{1}{m-1}$$

and apply Corollary 4.7 with $g_0 := \varphi\psi_1\delta_{v=u(t,x)}S + \partial_t\varphi f^{>} + \varphi q\partial_v\psi_0$, $g_1 := \varphi q^{>}$ and $\tilde{p} = p$. Estimate (4-19) gives

$$\|\varphi u^{>}\|_{W^{\sigma_t,p}(W^{\sigma_x,p})} \lesssim \|g_0\|_{\mathcal{M}_{TV}} + \||v|^{1-\gamma}g_0\|_{\mathcal{M}_{TV}} + \||v|^{-\gamma}g_1\|_{\mathcal{M}_{TV}} + \|f\|_{L^1_{t,x,v}\cap L^\infty_{t,x,v}} + \|u\|_{L^1_tL^p_x\cap L^r_{t,x}} + \||u|^m\|_{L^1_{t,x}}.$$

The first four terms on the right-hand side are estimated as in Step 2 via

$$\|g_0\|_{\mathcal{M}_{TV}} + \||v|^{1-\gamma}g_0\|_{\mathcal{M}_{TV}} + \||v|^{-\gamma}g_1\|_{\mathcal{M}_{TV}} + \|f\|_{L^1_{t,x,v}\cap L^\infty_{t,x,v}} \lesssim \|\partial_t\varphi|u|\|_{L^1_{t,x}} + \|u_0\|_{L^1_x} + \|S\|_{L^1_{t,x}} + 1,$$

while the last two terms are estimated through (5-6) as

$$\|u\|_{L^1_tL^p_x\cap L^r_{t,x}} + \||u|^m\|_{L^1_{t,x}} \lesssim \|u\|_{L^p_{t,x}\cap L^r_{t,x}} + \|u\|^m_{L^m_{t,x}} \lesssim \|u_0\|^m_{L^1_x} + \|S\|^m_{L^1_{t,x}} + 1.$$

Hence, we have shown (5-9) in the case $p \in (1,m)$. If $p = m$, we choose $p_0 \in (1,m)$ sufficiently large such that for

$$\kappa_x(p_0) := \frac{p_0-1}{p_0}\frac{2}{m-1}$$

it holds

$$\kappa_x(p_0) - \frac{d}{p_0} > \sigma_x - \frac{d}{m}.$$

We observe that for

$$\kappa_t(p_0) := \frac{m-p_0}{p_0}\frac{1}{m-1}$$

it holds

$$\kappa_t(p_0) - \frac{1}{p_0} > \sigma_t - \frac{1}{m}$$

due to $p_0 < m$ (indeed, we have necessarily $\sigma_t = 0$). Choosing sufficiently large $\sigma_x(p_0) < \kappa_x(p_0)$ and $\sigma_t(p_0) < \kappa_t(p_0)$, we conclude by Sobolev embedding

$$\begin{aligned}\|\varphi u^{>}\|_{L^m_t(W^{\sigma_x,m}_x)} &\lesssim \|\varphi u^{>}\|_{W^{\sigma_t(p_0),p_0}(W^{\sigma_x(p_0),p_0})}\\ &\lesssim \|\partial_t\varphi|u|\|_{L^1_{t,x}} + \|u_0\|^m_{L^1_x} + \|S\|^m_{L^1_{t,x}} + 1,\end{aligned}$$

which is (5-9) in the case $p = m$.

Step 6: Conclusion. With the same construction $\varphi_n \to 1_{[0,T]}$ as in the proof of Theorem 1.2, Steps 4 and 5 combine to

$$\begin{aligned}\sup_{n\in\mathbb{N}}\|\varphi_n u\|_{W^{\sigma_t,p}(W^{\sigma_x,p})} &\lesssim \sup_{n\in\mathbb{N}}\|\varphi_n u^{<}\|_{W^{\sigma_t,p}(W^{\sigma_x,p})} + \sup_{n\in\mathbb{N}}\|\varphi_n u^{>}\|_{W^{\sigma_t,p}(W^{\sigma_x,p})}\\ &\lesssim \|u_0\|^m_{L^1_x} + \|S\|^m_{L^1_{t,x}} + 1.\end{aligned}$$

Since $\varphi_n u \to u\,1_{[0,T]}$ in the sense of distributions, we obtain (1-2) by the weak lower semicontinuity of the norm in $W^{\sigma_t,p}(0,T;W^{\sigma_x,p}(\mathbb{R}^d))$. Estimate (1-3) follows analogously to the proof of Corollary 1.3(ii). □

Appendix A: Kinetic solutions

In this section we recall some details of the concept of entropy/kinetic solutions and their well-posedness for partial differential equations of the type

$$\begin{aligned} \partial_t u + \operatorname{div} A(u) &= \operatorname{div}(b(u)\nabla u) + S(t,x) \quad \text{on } (0,T)\times\mathbb{R}^d_x \\ u(0) &= u_0 \quad \text{on } \mathbb{R}^d_x, \end{aligned} \tag{A-1}$$

where

$$\begin{gathered} u_0 \in L^1(\mathbb{R}^d_x), \quad S \in L^1([0,T]\times\mathbb{R}^d_x), \quad T \geqslant 0, \\ a := A' \in C(\mathbb{R};\mathbb{R}^d) \cap C^1(\mathbb{R}\setminus\{0\};\mathbb{R}^d), \\ b = (b_{jk})_{j,k=1,\dots,d} \in C(\mathbb{R}; S^{d\times d}_+) \cap C^1(\mathbb{R}\setminus\{0\}; S^{d\times d}_+). \end{gathered} \tag{A-2}$$

Here, $S^{d\times d}_+$ denotes the space of symmetric, nonnegative definite matrices. For $b = (b)_{i,j=1,\dots,d} \in S^{d\times d}_+$ we set $\sigma = b^{1/2}$, that is, $b_{i,j} = \sum_{k=1}^d \sigma_{i,k}\sigma_{k,j}$. For a locally bounded function $b:\mathbb{R}\to S^{d\times d}_+$ we let $\beta_{i,k}$ be such that $\beta'_{i,k}(v) = \sigma_{i,k}(v)$. Similarly, for $\psi \in C^\infty_c(\mathbb{R}_v)$ we let $\beta^\psi_{i,j}$ be such that $(\beta^\psi_{i,k})'(v) = \psi(v)\sigma_{i,k}(v)$. The corresponding kinetic form of (A-1) reads, see [Chen and Perthame 2003],

$$\begin{aligned} \mathscr{L}(\partial_t, \nabla_x, v) f(t,x,v) &= \partial_t f + a(v)\cdot\nabla_x f - \operatorname{div}(b(v)\nabla_x f) \\ &= \partial_v q + S(t,x)\delta_{u(t,x)=v}(v), \end{aligned}$$

where $q \in \mathscr{M}^+$ and $\mathscr{L}$ is identified with the symbol

$$\mathscr{L}(i\tau, i\xi, v) := i\tau + a(v)\cdot i\xi - (b(v)\xi,\xi). \tag{A-3}$$

We will use the terms kinetic and entropy solution synonymously. From [Chen and Perthame 2003] we recall the definition of entropy/kinetic solutions to (A-1).

Definition A.1. We say that $u \in C([0,T]; L^1(\mathbb{R}^d))$ is an entropy solution to (A-1) if the corresponding kinetic function f satisfies:

(i) For any nonnegative $\psi \in \mathscr{D}(\mathbb{R})$, $k = 1,\dots,d$,

$$\sum_{i=1}^d \partial_{x_i}\beta^\psi_{ik}(u) \in L^2([0,T]\times\mathbb{R}^d).$$

(ii) For any two nonnegative functions $\psi_1, \psi_2 \in \mathscr{D}(\mathbb{R})$, $k = 1,\dots,d$,

$$\sqrt{\psi_1(u(t,x))}\sum_{i=1}^d \partial_{x_i}\beta^{\psi_2}_{ik}(u(t,x)) = \sum_{i=1}^d \partial_{x_i}\beta^{\psi_1\psi_2}_{ik}(u(t,x)) \quad \text{a.e.}$$

(iii) There are nonnegative measures $m, n \in \mathscr{M}^+$ such that, in the sense of distributions,

$$\partial_t f + a(v)\cdot\nabla_x f - \operatorname{div}(b(v)\nabla_x f) = \partial_v(m+n) + \delta_{v=u(t,x)}S \quad \text{on } (0,T)\times\mathbb{R}^d_x\times\mathbb{R}_v,$$

where n is defined by

$$\int \psi(v) n(t,x,v)\, \mathrm{d}v = \sum_{k=1}^{d} \left(\sum_{i=1}^{d} \partial_{x_i} \beta_{ik}^{\psi}(u(t,x)) \right)^2$$

for any $\psi \in \mathscr{D}(\mathbb{R})$ with $\psi \geqslant 0$.

(iv) We have

$$\int (m+n)\, \mathrm{d}x\, \mathrm{d}t \leqslant \mu(v) \in L_0^{\infty}(\mathbb{R}),$$

where L_0^{∞} is the space of L^{∞}-functions vanishing for $|v| \to \infty$.

The well-posedness of entropy solutions to (A-1) follows along the same lines as [Chen and Perthame 2003]. In this form, it can be found in [Gess 2020].

Theorem A.2. *Let $u_0 \in L^1(\mathbb{R}^d)$ and $S \in L^1([0,T] \times \mathbb{R}^d)$. Then there is a unique entropy solution u to* (A-1) *satisfying $u \in C([0,T]; L^1(\mathbb{R}^d))$. For two entropy solutions u^1, u^2 with initial conditions u_0^1, u_0^2 and forcing S^1, S^2 we have*

$$\sup_{t \in [0,T]} \|u^1(t) - u^2(t)\|_{L^1(\mathbb{R}^d)} \leqslant \|u_0^1 - u_0^2\|_{L^1(\mathbb{R}^d)} + \|S^1 - S^2\|_{L^1([0,T]\times\mathbb{R}^d)}.$$

Furthermore, the following a priori estimate was given in Lemma 2.3 in [Gess 2020].

Lemma A.3. *Let u be the unique entropy solution to* (A-1) *with $u_0 \in (L^1 \cap L^{2-\gamma})(\mathbb{R}_x^d)$ and $S \in (L^1 \cap L^{2-\gamma})([0,T] \times \mathbb{R}_x^d)$ for some $\gamma \in (-\infty, 1)$. Then, there is a constant $C = C(T,g) \geqslant 0$ such that*

$$\sup_{t\in[0,T]} \|u(t)\|_{L_x^{2-\gamma}}^{2-\gamma} + (1-\gamma) \int_0^T \int_{\mathbb{R}^{d+1}} |v|^{-\gamma} q \,\, \mathrm{d}v\, \mathrm{d}x\, \mathrm{d}r \leqslant C(\|u_0\|_{L_x^{2-\gamma}}^{2-\gamma} + \|S\|_{L_{t,x}^{2-\gamma}}^{2-\gamma}).$$

In the case of L^1 initial data a different proof for the existence of singular moments of the kinetic measure q is needed.

Lemma A.4. *Let u be the unique entropy solution to* (A-1) *with $u_0 \in L^1(\mathbb{R}_x^d)$ and $S \in L^1([0,T] \times \mathbb{R}_x^d)$. Then*, *the map*

$$v \mapsto \int_0^T \int_{\mathbb{R}_x^d} q(r,x,v)\, \mathrm{d}x\, \mathrm{d}r$$

is continuous and, *for all $v_0 \in \mathbb{R}_v$*, *we have*

$$\begin{aligned}\int_0^T \int_{\mathbb{R}_x^d} q(r,x,v_0)\, \mathrm{d}x\, \mathrm{d}r &\leqslant \int_{\mathbb{R}_x^d} (\operatorname{sgn}(v_0)(u_0 - v_0))_+ \, \mathrm{d}x + \int_0^T \int_{\mathbb{R}_x^d} \operatorname{sgn}_+(\operatorname{sgn}(v_0)(u - v_0)) S \, \mathrm{d}x\, \mathrm{d}r \\ &\leqslant \int_{\mathbb{R}_x^d} |u_0|\, \mathrm{d}x + \int_0^T \int_{\mathbb{R}_x^d} |S|\, \mathrm{d}x\, \mathrm{d}r. \qquad \text{(A-4)}\end{aligned}$$

Proof. In the proof, we use the short-hand notation $\bar{g}(v) := \int_0^T \int_{\mathbb{R}_x^d} g(r,x,v)\, \mathrm{d}x\, \mathrm{d}r$ for a generic $g : (r,x,v) \mapsto g(r,x,v)$. We first argue that $\bar{q}$ has left and right limits. Indeed, by a standard approximation argument, the kinetic formulation yields, for every $\eta \in C_c^\infty(\mathbb{R}_v)$,

$$\begin{aligned}\int_{\mathbb{R}_v} \eta' \bar{q}\,\mathrm{d}v &= -\int_{\mathbb{R}_v} \eta\Big(\int_{\mathbb{R}_x^d} f\,\mathrm{d}x\big|_0^T\Big)\,\mathrm{d}v + \overline{\eta(u)S}\\ &= -\int_{\mathbb{R}_v} \eta\Big(\int_{\mathbb{R}_x^d} f\,\mathrm{d}x\big|_0^T\Big)\,\mathrm{d}v + \int_{\mathbb{R}_v} \eta'\,\overline{1_{v<u}S}\,\mathrm{d}v. \end{aligned} \tag{A-5}$$

Since $v \mapsto \int_{\mathbb{R}_x^d} f(r,x,v)\,\mathrm{d}x|_0^T$ is in $L^1(\mathbb{R}_v)$, this implies $\bar{q} - \overline{1_{v<u}S} \in \dot{W}^{1,1}(\mathbb{R}_v)$. Since $\overline{1_{v<u}S} \in BV_{\mathrm{loc}}(\mathbb{R}_v)$, this shows $\bar{q} \in BV_{\mathrm{loc}}(\mathbb{R}_v)$ and thus the existence of left and right limits.

Next we claim that (A-5) continues to holds for all $\eta \in C^\infty(\mathbb{R}_v)$ with $\eta' \in C_c^\infty(\mathbb{R}_v)$. For $R > 0$ let $\varphi_R \in C_c^\infty(\mathbb{R}_v)$ be such that $\varphi_R(v) = 1$ for $|v| \leqslant R$, $\operatorname{supp}\varphi_R \subset [-(R+1), R+1]$ and $|\varphi_R| + |\varphi_R'| \lesssim 1$. Defining $\eta_R := \eta\varphi_R$, we have by (A-5)

$$\int_{\mathbb{R}_v} (\eta'\varphi_R + \eta\varphi_R')\bar{q}\,\mathrm{d}v = -\int_{\mathbb{R}_v} \eta_R\Big(\int_{\mathbb{R}_x^d} f\,\mathrm{d}x\big|_0^T\Big)\,\mathrm{d}v + \overline{\eta_R(u)S}.$$

Since η_R is uniformly bounded in R, $\eta_R \to \eta$ locally uniformly, $v \mapsto \int_{\mathbb{R}_x^d} f(r,x,v)\,\mathrm{d}x|_0^T$ is in $L^1(\mathbb{R}_v)$ and $S \in L^1([0,T]\times\mathbb{R}_x^d)$, we may take the limit $R \to \infty$ on the right-hand side by dominated convergence. Again by dominated convergence the contribution from the term $\eta'\varphi_R$ to the left-hand side converges, since η' has compact support and $\bar{q} \in BV_{\mathrm{loc}}(\mathbb{R}_v) \subset L^1_{\mathrm{loc}}(\mathbb{R}_v)$. Moreover, the contribution from the term $\eta\varphi_R'$ vanishes for $R \to \infty$, since both η and φ_R' are bounded, $\operatorname{supp}\varphi_R' \subset [-(R+1), -R] \cup [R, R+1]$ and $\bar{q} \in L_0^\infty(\mathbb{R}_v)$ by Definition A.1(iv).

We are now in the position to conclude. Assume first $v_0 \in \mathbb{R}_+$. Let $\phi_\pm \in C_c^\infty(\mathbb{R}_v)$ with $\phi_\pm \geqslant 0$, $\operatorname{supp}\phi_+ \subset [0,1]$, $\operatorname{supp}\phi_- \subset [-1,0]$, $\int_{\mathbb{R}_v}\phi_\pm\,\mathrm{d}v = 1$ and define $\phi_\pm^\varepsilon(v) = \varepsilon^{-1}\phi_\pm(\varepsilon^{-1}v)$ for $\varepsilon > 0$. Moreover let $\eta_\pm^\varepsilon$ be such that $(\eta_\pm^\varepsilon)'(v) = \phi_\pm^\varepsilon(v - v_0)$ and $(\eta_\pm^\varepsilon)(v_0) = 0$. Observe that $(\eta_\pm^\varepsilon)' \to \delta_{v=v_0}$ and $\eta_\pm^\varepsilon(v) \to \operatorname{sgn}_+(v - v_0)$ as $\varepsilon \searrow 0$ independent of the choice of $\pm$. Choosing now $\eta := \eta_\pm^\varepsilon$ in (A-5) and using dominated convergence to take the limit $\varepsilon \searrow 0$, we obtain

$$\begin{aligned}\bar{q}(v_0\pm) &= -\int_{\mathbb{R}_x^d} (u - v_0)_+\,\mathrm{d}x\big|_0^T + \int_0^T\int_{\mathbb{R}_x^d} \operatorname{sgn}_+(u - v_0)S\,\mathrm{d}x\,\mathrm{d}r\\ &\leqslant \int_{\mathbb{R}_x^d} (u_0 - v_0)_+\,\mathrm{d}x + \int_0^T\int_{\mathbb{R}_x^d} \operatorname{sgn}_+(u - v_0)S\,\mathrm{d}x\,\mathrm{d}r.\end{aligned}$$

In particular $\bar{q}(v_0-) = \bar{q}(v_0+)$, so that $\bar{q}$ is continuous. The case $v_0 \in \mathbb{R}_-$ is treated analogously replacing the conditions $\phi_\pm \geqslant 0$ and $\int_{\mathbb{R}_v}\phi_\pm\,\mathrm{d}v = 1$ by $\phi_\pm \leqslant 0$ and $\int_{\mathbb{R}_v}\phi_\pm\,\mathrm{d}v = -1$, respectively, so that $\eta_\pm^\varepsilon(v) \to \operatorname{sgn}_+(v - v_0)$ is replaced by $\eta_\pm^\varepsilon(v) \to \operatorname{sgn}_+(-(v - v_0))$. □

Appendix B: Fourier multipliers

In this section, we provide some Fourier multiplier results well-adapted to our averaging lemma, Lemma 4.2. We recall the definition of $\dot{\mathbb{R}}^{d+1}$ and of the functions η_l and φ_j given in Section 2, and define $\tilde{\eta}_l := \eta_{l-1} + \eta_l + \eta_{l+1}$ and $\tilde{\varphi}_j := \varphi_{j-1} + \varphi_j + \varphi_{j+1}$. We observe $\tilde{\eta}_l(2^l\,\cdot) = \tilde{\eta}_0$ and $\tilde{\varphi}(2^j\,\cdot) = \tilde{\varphi}_0$. Moreover, $\tilde{\eta}_l$ and $\tilde{\varphi}_j$ are identically unity on the support of η_l and φ_j, respectively.

Theorem B.1. *Let $k = 2 + 2[1 + d/2]$. Let $m : \dot{\mathbb{R}}^{d+1} \to \mathbb{C}$ be k-times differentiable and such that for all $\alpha = (\alpha_\tau, \alpha_\xi) \in \mathbb{N}_0 \times \mathbb{N}_0^d$ with $|\alpha| \leqslant k$ there is a constant C_α such that for all $(\tau, \xi) \in \dot{\mathbb{R}}^{d+1}$*

$$|\partial_\tau^{\alpha_\tau} \partial_\xi^{\alpha_\xi} m(\tau, \xi)| \leqslant C_\alpha |\tau|^{-\alpha_\tau} |\xi|^{-|\alpha_\xi|}. \tag{B-1}$$

Then there is a constant $C > 0$, depending only on the constants C_α, such that for any $p \in [1, \infty]$ and all $l, j \in \mathbb{Z}$, we have

$$\|\tilde{\eta}_l \tilde{\varphi}_j m\|_{\mathscr{M}^p} \leqslant C; \tag{B-2}$$

i.e., $\tilde{\eta}_l \tilde{\varphi}_j m$ (more precisely the mapping $(\tau, \xi) \mapsto \tilde{\eta}_l(\tau) \tilde{\varphi}_j(\xi) m(\tau, \xi)$) extends to an $L^p_{t,x}$-multiplier with a norm independent of l and j. Furthermore, this mapping extends to an $\mathscr{M}_{TV}$-multiplier with the same norm bound.

Proof. Since $\|\cdot\|_{\mathscr{M}^p} \leqslant \|\cdot\|_{\mathscr{M}^1}$, it suffices to estimate the L^1 multiplier norm of $\tilde{\eta}_l \tilde{\varphi}_j m$ in order to obtain (B-2). Since multiplier norms are invariant under dilation and since $\|m\|_{\mathscr{M}^1}$ is equal to the total mass of $\mathscr{F}^{-1} m$, see [Bergh and Löfström 1976, Theorem 6.1.2], we have

$$\|\tilde{\eta}_l \tilde{\varphi}_j m\|_{\mathscr{M}^1} = \|\tilde{\eta}_0 \tilde{\varphi}_0 m_{l,j}\|_{\mathscr{M}^1} = \|\mathscr{F}^{-1}_{t,x} \tilde{\eta}_0 \tilde{\varphi}_0 m_{l,j}\|_{L^1_{t,x}},$$

where $m_{l,j}(\tau, \xi) := m(2^l \tau, 2^j \xi)$. Let $M := [1 + d/2]$. We observe

$$\begin{aligned}
&(1+t^2)(1+|x|^2)^M \mathscr{F}^{-1}_{t,x}[\tilde{\eta}_0 \tilde{\varphi}_0 m_{l,j}](t,x) \\
&\qquad = c_d \int_{\mathbb{R}_t \times \mathbb{R}^d_x} (\mathrm{id} - \partial_\tau^2)(\mathrm{id} - \Delta_\xi)^M (e^{it\tau + ix\cdot\xi}) \tilde{\eta}_0(\tau) \tilde{\varphi}_0(\xi) m(2^l \tau, 2^j \xi) \,\mathrm{d}\xi \,\mathrm{d}\tau \\
&\qquad = c_d \int_{\mathbb{R}_t \times \mathbb{R}^d_x} e^{it\tau + ix\cdot\xi} (\mathrm{id} - \partial_\tau^2)(\mathrm{id} - \Delta_\xi)^M (\tilde{\eta}_0(\tau) \tilde{\varphi}_0(\xi) m(2^l \tau, 2^j \xi)) \,\mathrm{d}\xi \,\mathrm{d}\tau \\
&\qquad = \sum_{\substack{\alpha_\tau + \beta_\tau \leqslant 2 \\ |\alpha_\xi| + |\beta_\xi| \leqslant 2M}} c_{d,\alpha,\beta} 2^{l\beta_\tau} 2^{j|\beta_\xi|} \int_{\mathbb{R}_t \times \mathbb{R}^d_x} e^{it\tau + ix\cdot\xi} \partial_\tau^{\alpha_\tau} \tilde{\eta}_0(\tau) \partial_\xi^{\alpha_\xi} \tilde{\varphi}_0(\xi) \partial_\tau^{\beta_\tau} \partial_\xi^{\beta_\xi} m(2^l \tau, 2^j \xi) \,\mathrm{d}\xi \,\mathrm{d}\tau,
\end{aligned}$$

where c_d and $c_{d,\alpha,\beta}$ are constants that do not depend on l and j. On $\operatorname{supp} \tilde{\eta}_0 \times \operatorname{supp} \tilde{\varphi}_0$ we have $|\partial_\tau^{\beta_\tau} \partial_\xi^{\beta_\xi} m(2^l \tau, 2^j \xi)| \leqslant C_\beta 2^{-l\beta_\tau} 2^{-j|\beta_\xi|}$, and hence we obtain

$$(1+t^2)(1+|x|^2)^M |\mathscr{F}^{-1}_{t,x}[\tilde{\eta}_0 \tilde{\varphi}_0 m_{l,j}](t,x)| \leqslant c.$$

Since $2M > d$, it follows $\|\mathscr{F}^{-1}_{t,x}[\tilde{\eta}_0 \tilde{\varphi}_0 m_{l,j}]\|_{L^1_{t,x}} \leqslant C$, which yields (B-2). In particular, $\tilde{\eta}_l \tilde{\varphi}_j m$ is an L^1-multiplier with a norm bound independent of l and j, and as such extends to a multiplier on $\mathscr{M}_{TV}$ with the same norm bound. □

Remark B.2. In Theorem B.1, the assumptions on the differentiability of m may be relaxed: Indeed, the proof shows that it suffices to assume that m is a continuous function such that $\partial_\tau^{\alpha_\tau} m$, $\partial_\xi^{\alpha_x} m$ and $\partial_\tau^{\alpha_\tau} \partial_\xi^{\alpha_x} m$ exist for all $\alpha = (\alpha_\tau, \alpha_\xi)$ with $\alpha_\tau \leqslant 2$ and $|\alpha_\xi| \leqslant 2[1 + d/2]$, and that (B-1) holds for these choices of α.

Remark B.3. Clearly, Theorem B.1 has an isotropic variant; see [Bahouri, Chemin, and Danchin 2011, Lemma 2.2]. More precisely, a simple adaptation of the proof shows the following: Let $k = 2[1 + d/2]$. Let $m : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ be k-times differentiable and such that for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leqslant k$ there is a constant C_α such that for all $\xi \in \mathbb{R}^d \setminus \{0\}$ we have $|\partial^\alpha m(\xi)| \leqslant C_\alpha |\xi|^{-|\alpha|}$. Then there is a constant $C > 0$, depending only on the constants C_α such that for any $p \in [1,\infty]$ and all $j \in \mathbb{Z}$ we have $\|\tilde{\varphi}_j m\|_{\mathscr{M}^p} \leqslant C$. Again, $\tilde{\varphi}_j m$ extends to an $\mathscr{M}_{TV}$-multiplier (in ξ) with the same norm bound.

Lemma B.4. *Let $\mathscr{L}$ be defined as in* (4-2) *and fix $\alpha = (\alpha_\tau, \alpha_\xi) \in \mathbb{N}_0 \times \mathbb{N}_0^d$. Then we have for all $(\tau, \xi, v) \in \dot{\mathbb{R}}^{d+1} \times \mathbb{R}$ the estimate*

$$\left| \partial_\tau^{\alpha_\tau} \partial_\xi^{\alpha_\xi} \frac{1}{\mathscr{L}(i\tau, i\xi, v)} \right| \lesssim \frac{1}{|\mathscr{L}(i\tau, i\xi, v)|} |\tau|^{-\alpha_\tau} |\xi|^{-|\alpha_\xi|}.$$

Proof. The proof rests on the identity

$$\partial_\xi^{\alpha_\xi} \frac{1}{\mathscr{L}(i\tau, i\xi, v)} = \sum_\beta c_\beta \frac{\xi^\beta |v|^{(m-1)N_\beta}}{\mathscr{L}(i\tau, i\xi, v)^{1+N_\beta}},$$

where c_β are constants, $N_\beta := (|\alpha_\xi| + |\beta|)/2$, and the sum runs over those $\beta \in \mathbb{N}_0^d$ with $|\beta| \leqslant |\alpha_\xi|$ such that $|\alpha_\xi| + |\beta|$ is even. The identity can be proven easily by induction on the order of α_ξ. From this and $\partial_\tau \mathscr{L}(i\tau, i\xi, v) = i$, it immediately follows

$$\left| \partial_\tau^{\alpha_\tau} \partial_\xi^{\alpha_\xi} \frac{1}{\mathscr{L}(i\tau, i\xi, v)} \right| \lesssim \sum_\beta \left| \frac{\xi^\beta |v|^{(m-1)N_\beta}}{\mathscr{L}(i\tau, i\xi, v)^{1+\alpha_\tau+N_\beta}} \right|,$$

which in view of

$$\frac{|\xi|^{|\beta|} |v|^{(m-1)N_\beta}}{|\mathscr{L}(i\tau, i\xi, v)|^{N_\beta}} \leqslant \frac{|\xi|^{|\beta|} |v|^{(m-1)N_\beta}}{(|v|^{m-1} |\xi|^2)^{N_\beta}} = |\xi|^{-(2N_\beta - |\beta|)} = |\xi|^{-|\alpha_\xi|}$$

and

$$\frac{1}{|\mathscr{L}(i\tau, i\xi, v)|^{\alpha_\tau}} \leqslant |\tau|^{-\alpha_\tau}$$

yields the assertion. □

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