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PARAMETRIX FOR A SEMICLASSICAL SUBELLIPTIC OPERATOR

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We demonstrate a parametrix construction, together with associated pseudodifferential operator calculus, for an operator of sum-of-squares type with semiclassical parameter. The form of operator we consider includes the generator of kinetic Brownian motion on the cosphere bundle of a Riemannian manifold. Regularity estimates in semiclassical Sobolev spaces are proven by establishing mapping properties for the parametrix.

1. Introduction

We deal in this paper with a class of second order, subelliptic partial differential operators of the sum-of-squares form

$$P_h = X_0 - h\sum_{j=1}^{d} X_j^2 - h\sum_{j=1}^{d} c_j X_j, \quad h \in (0,1], \tag{1-1}$$

where the X_j for $0 \le j \le d$ are smooth vector fields, the c_j are smooth functions, and $h > 0$ is considered as a semiclassical parameter. We work in $2d+1$ dimensions, either on a compact manifold or an open subset of $\mathbb{R}^{2d+1}$, and make the following assumptions throughout this paper.

Assumption 1.
- The collection of $2d+1$ vectors $\{X_0, X_1, \dots, X_d, [X_0, X_1], \dots, [X_0, X_d]\}$ spans the tangent space at each base point.
- The collection $\{X_1, \dots, X_d\}$ is involutive (closed under commutation of vector fields).

For each $h > 0$ the operator P_h is subelliptic by a result of [Hörmander 1967], and by [Rothschild and Stein 1976] the operator P_h controls $\frac{2}{3}$-derivatives in the Sobolev space sense. In the semiclassical setting it is natural to work with a semiclassical notion of Sobolev spaces; we refer to [Zworski 2012] for a treatment of semiclassical analysis. The question of interest in this paper is the dependence on h of the various constants in a priori inequalities for P_h, both in L^2 and semiclassical Sobolev spaces.

Our work is motivated by that of Alexis Drouot [2017], who studied such an operator on the cosphere bundle $S^*(M)$ of a $(d{+}1)$-dimensional Riemannian manifold M. The paper [Drouot 2017] considers the operator $P_h = H + h\Delta_{\mathbb{S}}$, with H the generator of the Hamiltonian/geodesic flow and $\Delta_{\mathbb{S}}$ the nonnegative Laplace–Beltrami operator along the fibers of the cosphere bundle. In local coordinate charts this operator can be represented in the form (1-1), where $X_0 = H$, and $\{X_j\}_{j=1}^d$ is any local orthonormal frame for the

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tangent space of the fibers of $S^*(M)$. In [Drouot 2017] it is shown that, if M is negatively curved, then as $h \to 0$ the eigenvalues of $-iP_h$ converge to the Pollicott–Ruelle resonances of M. The analogous result was proven in [Dyatlov and Zworski 2015] for $P_h = H + h\Delta$, where Δ is the Laplacian on $S^*(M)$. The interest in taking $P_h = H + h\Delta_{\mathbb{S}}$ is that this operator generates what is known as kinetic Brownian motion on M. For a treatment of this process we refer to [Franchi and Le Jan 2007; Grothaus and Stilgenbauer 2013; Angst, Bailleul, and Tardif 2015; Li 2016].

A key step in the proof of convergence in [Drouot 2017] was controlling the subelliptic estimates for P_h as $h \to 0$. We emphasize that the estimates we prove are the same as in that paper, with an occasional improvement in the remainder terms. The aim here is to obtain a finer microlocal understanding of the parametrix. We obtain a parametrix valid on the region $h\Delta \geq 1$, strictly larger than the semiclassical region $h^2\Delta \geq 1$. The restriction $h\Delta \geq 1$ arises from the largest region of phase space on which the uncertainty principle holds for the parametrix. The estimates in [Drouot 2017] were obtained through commutator methods, analogous to the work of [Hörmander 1967]. Our approach is more similar to that of [Rothschild and Stein 1976], in that we use an approximation to the operator at each point by a model nilpotent Lie group, and construct a parametrix from the inverse of the model operator on that group. Estimates are then obtained from mapping properties for the parametrix. In contrast to [Rothschild and Stein 1976], which lifted the operator to a higher-dimensional Lie group on which the parametrix is represented as a singular integral kernel, we construct the parametrix in pseudodifferential form on the space itself. This procedure is motivated by the author's work [Smith 1994] on the $\bar{\partial}_b$ problem on three-dimensional CR manifolds of finite type.

When constructing a parametrix for P_h of the form (1-1), it is more natural from the semiclassical viewpoint to consider $hP_h = hX_0 + \sum_{j=1}^{d}(hX_j)^2$, and quantize symbols in terms of $h\eta$. This leads to placing an extra factor of h on the variables η'' dual to X_j for $d+1 \leq j \leq 2d$, since $[hX_0, hX_j] \sim h^2 X_{j+d}$. The quantization of symbols is naturally carried out using exponential coordinates with respect to an extension of $\{X_j\}_{j=0}^{d}$ to a frame $\{X_j\}_{j=0}^{2d}$. We will require that:

Assumption 2. If $1 \leq i \leq d$, then $[X_0, X_i] - 2X_{i+d} \in \operatorname{span}(X_0, \ldots, X_d)$.

This can of course be arranged by setting $X_{i+d} = 2[X_0, X_i]$. In the model nilpotent Lie group setting where all other commutators vanish, there is a natural nonisotropic dilation structure using powers (2, 1, 3). Precisely, we split $\eta \in \mathbb{R}^{2d+1} = \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ into (η_0, η', η''), and similarly use X' as abbreviation for the collection $(X_1, \ldots, X_d)$, and $X'' = (X_{d+1}, \ldots, X_{2d})$. Then the dilation that respects the fundamental solution for the model operator is

$$\delta_r(\eta) = (r^2\eta_0, r\eta', r^3\eta'').$$

We now summarize the main result of this paper, leaving details to be expanded upon in later sections. For simplicity consider an open set $U \subset \mathbb{R}^{2d+1}$. For a multi-index $\alpha \in \mathbb{N}^{2d+1}$, let

$$\operatorname{order}(\alpha) = 2\alpha_0 + |\alpha'| + 3|\alpha''|.$$

We use $\exp_x(y)$ to denote the time-1 flow of x along $\sum_{j=0}^{2d} y_j X_j$.

Proposition 3. *Given $\rho(x) \in C_c^\infty(U)$, there is $\chi_0 \in C_c^\infty(\mathbb{R}^{2d+1})$ and an h-dependent family of symbols $a(x,\eta)$ satisfying*

$$|\partial_x^\beta \partial_\eta^\alpha a(x,\eta)| \le C_{\alpha,\beta} h (h^{\frac{1}{2}} + |\eta_0|^{\frac{1}{2}} + |\eta'| + |\eta''|^{\frac{1}{3}})^{-2-\mathrm{order}(\alpha)},$$

with $C_{\alpha,\beta}$ independent of $h \in (0,1]$, so that the operator $a_h(x,hD)$ defined by

$$a_h(x,hD) = \frac{1}{(2\pi)^{2d+1}} \int e^{-i\langle y,\eta\rangle} a(x, h\eta_0, h\eta', h^2\eta'') f(\exp_x(y)) \chi_0(y)\, dy\, d\eta$$

satisfies

$$a_h(x,hD) \circ P_h = \rho(x) + r_h(x,hD),$$

where $r_h(x,hD)$ is an operator that satisfies the following with C_{p_1,p_2} independent of $h \in (0,1]$, for any polynomials $p_j(\eta)$ on $\mathbb{R}^{2d+1}$:

$$\| p_1(X_0, h^{\frac{1}{2}} X', h^{\frac{1}{2}} X'') \circ r_h(x,hD) \circ p_2(X_0, h^{\frac{1}{2}} X', h^{\frac{1}{2}} X'') f \|_{L^2} \le C_{p_1,p_2} \|f\|_{L^2}.$$

For example, one can take p_1 or p_2 to yield the operator $(1 + X_0^* X_0)^{N_1} (1 + h\Delta)^{N_2}$, where Δ is the Laplacian on $\mathbb{R}^{2d+1}$. These bounds roughly say that the parametrix inverts P_h on the region

$$\{\Delta \ge h^{-1}\} \cup \{|X_0| \ge 1\}.$$

In particular, the remainder term r_h will be of order h^∞ if the solution is localized to a region where $\Delta \ge h^{-1-\epsilon}$ for some $\epsilon > 0$.

We remark that in the calculus developed here P_h is of order 2, and thus distinct from the standard semiclassical calculus where hP_h is of order 2. This is related to the fact that we are working on the region $|\eta| \ge h^{1/2}$ rather than $|\eta| \ge 1$. Symbols of order j are weighted by a factor $h^{-j/2}$ to ensure that symbols of negative order (but not necessarily their derivatives) remain bounded as $h \to 0$. With this accounting, X_0 is an operator of order 2, $h^{1/2} X_j$ is of order 1 for $1 \le j \le d$, and $h^{1/2} X_j$ is of order 3 for $d+1 \le j \le 2d$.

Together with the composition calculus, pseudolocality arguments, and L^2 mapping bounds for operators, we deduce the regularity results on $S^*(M)$ for P_h that were established in [Drouot 2017]. These are stated in Theorems 20 and 21.

The outline of this paper is as follows. In Section 2 we introduce a model operator of P_h on a step-2 nilpotent group, and discuss the homogeneous fundamental solution in this setting. In Section 3 we study the degree to which the model operator, attached to M by exponential coordinates, approximates P_h. Careful estimates of the Taylor expansion of vector fields and exponential coordinates are needed to obtain uniform estimates as $h \to 0$. In Section 4 we prove that operators of the form $a_h(x,hD)$ form an algebra under composition. This allows for the construction of parametrices from the inverse for the model operator on the nilpotent Lie group. In Section 5 we establish L^2 boundedness of order-0 operators in local coordinates, using a nonisotropic Littlewood–Paley decomposition of the operator and the Cotlar–Stein lemma. Finally, in Section 6 we establish the main regularity estimates for P_h in h-Sobolev spaces, leading to the proof of the bounds in [Drouot 2017].

2. Operators on model domains

In this section we consider a nilpotent Lie group structure on $\mathbb{R}^{2d+1}$ that captures the commutation relations of the vector fields X_j, and we introduce a left-invariant model of P_h. The top-order term in the parametrix for P_h at a point $x \in U$ will be given by the fundamental solution for the model operator, attached to U via exponential coordinates at x relative to the frame $\{X_j\}_{j=0}^{2d}$. In subsequent sections we show that the model operator agrees to leading order with the expression of P_h in exponential coordinates, and develop a graded pseudodifferential calculus that allows us to produce a parametrix for P_h modulo a smoothing operator. We start by considering $h = 1$, and then obtain the fundamental solution for all h by a suitable dilation.

We use the variables $y = (y_0, y', y'') \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$, with dual variables $\eta = (\eta_0, \eta', \eta'')$, and introduce the dilation structure

$$\delta_r(\eta) = (r^2\eta_0, r\eta', r^3\eta''), \quad \delta_{r^{-1}}(y) = (r^{-2}y_0, r^{-1}y', r^{-3}y'').$$

We also introduce a corresponding nonisotropic homogeneous weight $m \in C^\infty(\mathbb{R}^{2d+1}\backslash\{0\})$,

$$m(\eta) = (|\eta_0|^6 + |\eta'|^{12} + |\eta''|^4)^{\frac{1}{12}},$$

so that $m(\delta_r(\eta)) = rm(\eta)$, and $3^{-12/5} \le m(\eta) \le 1$ when $|\eta| = 1$.

Consider the frame of vector fields on $\mathbb{R}^{2d+1}$ given by

- $Y_0 = \partial_0 - \sum_{j=1}^d y_j \partial_{j+d}$,
- $Y_j = \partial_j + y_0\partial_{j+d}$ for $1 \le j \le d$,
- $Y_j = \partial_j$ for $j \ge d+1$,

and observe that

$$[Y_0, Y_j] = 2Y_{j+d} \quad \text{if } 1 \le j \le d,$$

with all other commutators equal to 0. The collection $\{Y_j\}_{j=0}^{2d}$ forms a nilpotent (step-2) Lie algebra. These are left-invariant vector fields associated to the nilpotent Lie group structure on $\mathbb{R}^{2d+1}$ with product

$$y \times w = (y_0 + w_0, y' + w', y'' + w'' + y_0w' - w_0y').$$

The exponential map at base point y, and corresponding exponential coordinates, are given by

$$\begin{aligned} \overline{\exp}_y(w) &= (y_0 + w_0, y' + w', y'' + w'' + y_0w' - w_0y'), \\ \overline{\Theta}_y(z) &= (z_0 - y_0, z' - y', z'' - y'' - y_0z' + z_0y'), \end{aligned} \tag{2-1}$$

so in particular $\overline{\Theta}_0(w) = w$.

The vector field Y_0 is homogeneous of order 2 under δ_r in that

$$Y_0(f \circ r^{-1}) = r^{-2}(Y_0 f) \circ \delta_{r^{-1}},$$

which we summarize by writing $\operatorname{order}(Y_0) = 2$. Similarly, $\operatorname{order}(Y_j) = 1$ for $1 \le j \le d$, and $\operatorname{order}(Y_j) = 3$ for $d + 1 \le j \le 2d$. More generally, if we define the order of a multi-index α by

$$\operatorname{order}(\alpha) = 2\alpha_0 + \alpha_1 + \cdots + \alpha_d + 3\alpha_{d+1} + \cdots + 3\alpha_{2d} = 2\alpha_0 + |\alpha'| + 3|\alpha''|,$$

then the monomial differential operator $y^\beta \partial_y^\alpha$ will be homogeneous, with order given by

$$\operatorname{order}(y^\beta \partial_y^\alpha) = \operatorname{order}(\alpha) - \operatorname{order}(\beta). \tag{2-2}$$

The left-invariant differential operator $Y_0 - \sum_{j=1}^d Y_j^2$ is subelliptic and homogeneous of order 2. By [Folland 1975], it has a unique homogeneous fundamental solution $K(y) \in C^\infty(\mathbb{R}^{2d+1}\setminus\{0\})$,

$$\Bigl(Y_0 - \sum_{j=1}^d Y_j^2\Bigr) K(y) = \delta(y), \quad K(\delta_{r^{-1}}(y)) = r^{(2+4d)-2} K(y).$$

The homogeneous inverse for $Y_0 - \sum_{j=1}^d Y_j^2$ is given by convolution with K, which we will express in pseudodifferential form. Precisely, if we let $q_0(\eta) = \widehat{K}$, then $q_0(\delta_r \eta) = r^{-2} q_0(\eta)$, and the operator

$$q_0(D) f(y) = \frac{1}{(2\pi)^{2d+1}} \int_{\mathbb{R}^{2d+1}} e^{-i\langle \overline{\Theta}_y(z), \eta\rangle} q_0(\eta) f(z)\, dz\, d\eta$$

is a left and right inverse for $Y_0 - \sum_{j=1}^d Y_j^2$ on the space of Schwartz functions.

To conclude this section we consider the semiclassical subelliptic operator $hY_0 - \sum_{j=1}^d h^2 Y_j^2$. This is naturally associated to dilating y_0 and y' by h, and y'' by h^2, in that

$$\Bigl(Y_0 - \sum_{j=1}^d Y_j^2\Bigr)(f(hy_0, hy', h^2 y'')) = \Bigl(hY_0 f - \sum_{j=1}^d h^2 Y_j^2 f\Bigr)(hy_0, hy', h^2 y'').$$

Consequently, if we introduce the operation on symbols

$$a_h(\eta) = a(\eta_0, \eta', h\eta''),$$

then the inverse for $hY_0 - \sum_{j=1}^d h^2 Y_j^2$ is given by the semiclassical quantization of q_h,

$$\begin{aligned} q_{0,h}(hD) f(y) &= \frac{1}{(2\pi h)^{2d+1}} \int_{\mathbb{R}^{4d+2}} e^{-i\langle \overline{\Theta}_y(z), \zeta\rangle / h} q_{0,h}(\eta) f(z)\, dz\, d\zeta \\ &= \frac{1}{(2\pi)^{2d+1}} \int_{\mathbb{R}^{4d+2}} e^{-i\langle z, \zeta\rangle} q_{0,h}(\eta) f(\overline{\exp}_y(hz))\, dz\, d\zeta. \end{aligned}$$

3. Approximation by the model domain

Recall that we consider a spanning collection $\{X_0, X_1, \dots, X_{2d}\}$ of vector fields on an open subset U of $\mathbb{R}^{2d+1}$ satisfying the following conditions:

- The collection $\{X_1, \dots, X_d\}$ is involutive (closed under commutation of vector fields).
- If $1 \le i \le d$, then $[X_0, X_i] - 2X_{i+d} \in \operatorname{span}(X_0, \dots, X_d)$.

We will use $x, \tilde{x}$ to denote variables in U and y, z to denote variables in $\mathbb{R}^{2d+1}$.

Let $\exp_x(y)$ be exponential coordinates with base point x in the frame $\{X_0, \dots, X_{2d}\}$. That is, $\exp_x(y) = \gamma(1)$, where $\gamma(0) = x$ and $\gamma'(t) = \sum_{j=0}^{2d} y_j X_j(\gamma(t))$. Define exponential coordinates Θ_x as the local inverse of $\exp_x$ in a neighborhood of x:

$$\Theta_x(\exp_x(y)) = y, \quad \exp_x(\Theta_x(\tilde{x})) = \tilde{x}.$$

Recall the definition (2-2) of the order of a monomial differential operator in y. Consistent with this we have $\operatorname{order}(Y_0)=2$, $\operatorname{order}(Y_j)=1$ if $1\le j\le d$, and $\operatorname{order}(Y_j)=3$ if $d+1\le j\le 2d$.

Lemma 4. *For $0\le j\le 2d$, we can write*

$$(X_j f)(\exp_x(y)) = Y_j(f(\exp_x(y))) + R_j(x,y,\partial_y)f(\exp_x(y)),$$

where $\operatorname{order}(R_j)<\operatorname{order}(Y_j)$, *in the sense that the Taylor expansion*

$$R_j(x,y,\partial_y)=\sum_{\alpha,k} c_{j,\alpha,k}(x)y^\alpha\partial_k$$

includes only terms with $\operatorname{order}(y^\alpha\partial_k)<\operatorname{order}(Y_j)$.

Additionally, $c_{0,\alpha,k}(x)\equiv 0$ *unless there is at least one factor of y_j with $j\ge 1$ occurring in y^α.*

Proof. Any term $y^\alpha\partial_k$ with $|\alpha|>2$ is of order ≤ 0, so we need examine the Taylor expansion of X_j in exponential coordinates only to second power in y. Additionally, $\operatorname{order}(y_iy_j\partial_k)\le 1$, and equals 1 only if $1\le i,\, j\le d$ and $k\ge d+1$. To see that such a term cannot arise in R_j for $1\le j\le d$, the only case where $\operatorname{order}(Y_j)\le 1$, we use involutivity of $\{X_1,\dots,X_d\}$ and the Frobenius theorem to see that this collection remains tangent to the flowout of the subspace $y_0=y''=0$, and hence we can write $X_j=\sum_{k=1}^d c_k(x,y)\partial_k$ if $y_0=y''=0$ and $1\le j\le d$.

Thus, we need show that in the expansion of R_j about $y=0$ the terms linear in y are of order strictly less than $\operatorname{order}(Y_j)$. For $j\ge d+1$ this is immediate, since $X_j=\partial_j=Y_j$ at $y=0$, and any vector field that vanishes at 0 includes terms of order at most 2. For $0\le j\le d$ we expand

$$X_j=\partial_j+\sum_{i,k}c_{jik}(x)y_i+(y^2)\partial_y.$$

Since radial lines in exponential coordinates are integral curves of $\sum_{j=0}^{2d}y_jX_j$, we have

$$\sum_{j=0}^{2d}y_jX_j=\sum_{j=0}^{2d}y_j\partial_j, \tag{3-1}$$

from which we deduce

$$c_{ijk}=-c_{jik}.$$

Also, since $[X_0,X_j]-2X_{j+d}\in\operatorname{span}(X_0,\dots,X_d)$, we deduce for $j=1,\dots,d$ that

$$c_{j0k}=\begin{cases}1, & k=j+d,\\ 0, & k>d \text{ and } k\ne j+d.\end{cases}$$

Since $\operatorname{order}(y_i\partial_k)<2$ unless $k>d$, we deduce $\operatorname{order}(R_0(x,y,\partial_y))<2$.

By involutivity of $\{X_1,\dots,X_d\}$, if $1\le i,\, j\le d$ then $c_{jik}=0$ unless also $1\le k\le d$, in which case $\operatorname{order}(y_i\partial_k)=0$. And if $i>d$ then $\operatorname{order}(y_i\partial_k)\le 0$ for all k. So if $1\le j\le d$ then all terms $c_{jik}y_i\partial_k$ for $i\ne 0$ have order ≤ 0, and since $c_{j0k}=\delta_{k,j+d}$ we conclude $\operatorname{order}(R_j(x,y,\partial_y))\le 0$ if $1\le j\le d$.

To conclude the lemma, we note by (3-1) that if $y'=y''=0$ then $X_0=\partial_{y_0}$, from which we obtain $R_0\equiv 0$ along $y'=y''=0$. □

For $x \in U$, and y, z in a neighborhood of 0 in $\mathbb{R}^{2d+1}$, we introduce the functions

$$\Theta(x, y, z) = \Theta_{\exp_x(y)}(\exp_x(z)), \quad \widetilde{\Theta}(x, y, w) = \Theta_x(\exp_{\exp_x(y)}(w)), \tag{3-2}$$

where we recall $\Theta_x(\tilde{x})$ denotes exponential coordinates in X_j centered at x. For fixed x and y these are inverse functions of each other on their domains:

$$z = \widetilde{\Theta}(x, y, w) \quad \Longleftrightarrow \quad w = \Theta(x, y, z).$$

To invert in the y-variable we note that $v = \widetilde{\Theta}(x, y, w)$ implies $y = \widetilde{\Theta}(x, v, -w)$.

Observe that $\Theta(x, y, z) = -\Theta(x, z, y)$, and $\Theta(x, y, z) = z - y + \mathcal{O}(y, z)^2$. For more precise estimates on Θ and $\widetilde{\Theta}$ we consider their Taylor expansions in exponential coordinates at x. We first assign a notion of order to a smooth function $f(x, y, z)$. Consistent with (2-2), we make the following definition.

Definition 5. For a smooth function $f(x, y, z)$ defined on an open subset of $U \times \mathbb{R}^{2d+1} \times \mathbb{R}^{2d+1}$ containing $U \times \{0, 0\}$, we say that $\operatorname{order}(f) < -j$ if for all $x \in U$

$$(\partial_y^\alpha \partial_z^\beta f)(x, 0, 0) = 0 \quad \text{for all } \alpha, \beta : \operatorname{order}(\alpha + \beta) \le j.$$

Equivalently, the Taylor expansion of f in y, z about $y = z = 0$ contains only monomials $y^\alpha z^\beta$ with $\operatorname{order}(\alpha + \beta) > j$. We let $\operatorname{order}(f)$ be the least $n \in \mathbb{Z}$ such that $\operatorname{order}(f) < n + 1$.

Recalling the definition (2-1) of $\overline{\Theta}_y(z)$, which are exponential coordinates in the frame Y_j on the model domain, we have the following.

Lemma 6. *We have* $\Theta(x, y, z) = \overline{\Theta}_y(z) + R(x, y, z)$, *where* $\operatorname{order}(R_j) < \operatorname{order}(y_j)$ *for each* j. *Similarly*, $\widetilde{\Theta}(x, y, w) = \overline{\Theta}_{-y}(w) + \widetilde{R}(x, y, w)$, *where* $\operatorname{order}(\widetilde{R}_j) < \operatorname{order}(y_j)$ *for each* j.

Proof. We work in exponential coordinates $y = \Theta_x(\,\cdot\,)$ centered at x, and use Lemma 4 to consider X_j as a vector field in y. Then $z = \widetilde{\Theta}(x, y, w)$ means that $z = \gamma(1)$, where $\gamma(t)$ is the integral curve of $w \cdot X \equiv \sum w_k X_k$ with $\gamma(0) = y$. Taking the Taylor expansion of $\gamma(t)$ about $t = 0$ and evaluating at $t = 1$ gives the following expansion of $z = \widetilde{\Theta}(x, y, w)$ in terms of w:

$$z_j = y_j + (w \cdot X)_j(y) + \sum_{k=1}^{\infty} \frac{1}{(k+1)!} (w \cdot X)^k (w \cdot X)_j(y), \tag{3-3}$$

where $(w \cdot X)(y, \partial_y)$ acts on y and $(w \cdot X)_j(y)$ is its ∂_j coefficient as a function of y. It is seen from Lemma 4 that $w \cdot X$ does not increase the order of a function $f(x, y, w)$, and $w \cdot X - w \cdot Y$ decreases the order of $f(x, y, w)$ by at least 1. Also, as functions of (y, w)

$$\operatorname{order}((w \cdot Y)_j(y)) = \operatorname{order}(y_j), \quad \operatorname{order}((w \cdot X)_j(y) - (w \cdot Y)_j(y)) < \operatorname{order}(y_j).$$

Thus, if we replace $w \cdot X$ by $w \cdot Y$ in the expansion (3-3) then the right-hand side is changed by terms of strictly lower order than y_j. It follows that we can write

$$(z_0, z', z'') = (y_0 + w_0,\, y' + w',\, y'' + w'' + y_0 w' - w_0 y') + (\widetilde{R}_0, \widetilde{R}', \widetilde{R}''), \tag{3-4}$$

where $\operatorname{order}(\widetilde{R}_0) < -2$, $\operatorname{order}(\widetilde{R}') < -1$, and $\operatorname{order}(\widetilde{R}'') < -3$ as functions of (y, w). Recalling the formula (2-1), this completes the second statement of the lemma.

We next invert the map $w \to z$ to express $w = w(y, z) = \Theta(x, y, z)$, and use (3-4) to write

$$w(y, z) = \overline{\Theta}_y(z) - (\widetilde{R}_0, \widetilde{R}', -y_0\widetilde{R}' + y'\widetilde{R}_0 + \widetilde{R}'') \equiv \overline{\Theta}_y(z) - R(x, y, z),$$

where $\widetilde{R} = \widetilde{R}(y, w(y, z))$. Since w is equal to $z - y$ plus quadratic terms in (y, z), we see that $R(x, y, z)$ has no linear terms in y or z, and hence $\operatorname{order}(w_0) \le -2$ and $\operatorname{order}(w') \le -1$, since quadratic terms are of order at most -2. This also shows that $\operatorname{order}(-y_0\widetilde{R}' + y'\widetilde{R}_0) \le -3$.

To conclude the lemma it suffices to show that $\operatorname{order}(\widetilde{R}''(y, z)) \le -3$, since together with the preceding this shows that $\operatorname{order}(w_j(y, z)) \le \operatorname{order}(y_j)$ for all j, from which it follows that $\operatorname{order}(\widetilde{R}_j(y, w(y, z))) \le \operatorname{order}(\widetilde{R}_j(y, w)) < \operatorname{order}(y_j)$. We know that $\operatorname{order}(w''(y, z)) \le -2$ since quadratic terms are order ≤ -2, and by the above that $\operatorname{order}(w_j(y, z)) \le \operatorname{order}(y_j)$ for $j \le d$. Since $\operatorname{order}(\widetilde{R}'') < -3$ as a function of (y, w), it is easy to see by examining terms in (y, w) of order ≤ -4 that $\operatorname{order}(\widetilde{R}'') \le -3$ as a function of (y, z), concluding the proof. □

We make a few important additional observations about the terms that can occur in the Taylor expansion of $\Theta(x, y, z)$ about $y = z = 0$ and $\widetilde{\Theta}(x, y, w)$ about $y = w = 0$. First, we have

$$\Theta(x, y, z) = z_0 - y_0 \quad \text{if } y' = z' = y'' = z'' = 0,$$
$$\widetilde{\Theta}(x, y, w) = y_0 + w_0 \quad \text{if } y' = w' = y'' = w'' = 0.$$

Consequently, every nonvanishing term in the Taylor expansion of $R(x, y, z)$ about $y = z = 0$ must include a factor of either y', z', y'', or z''. Similarly, every nonvanishing term in the Taylor expansion of $\widetilde{R}(x, y, w)$ about $y = w = 0$ must include a factor of either y', w', y'', or w''.

Additionally, since the collection $\{X_j\}_{j=1}^d$ is involutive it follows that R_0 and R'' vanish if $y_0 = z_0 = y'' = z'' = 0$, and hence every nonvanishing term in the Taylor expansions of R_0 and R'' must contain a factor other than (y', z'). Similarly $\widetilde{R}_0$ and $\widetilde{R}''$ must each contain a factor other than (y', w'). Combining this with the fact that $R(x, y, z) = 0$ if $z = y$, we can write

$$R_j(x, y, z) = \sum_{\substack{|\alpha|+|\beta|=2\\ |\beta|\ge 1}} c_{j,\alpha,\beta}(x) y^\alpha (z-y)^\beta + \sum_{\substack{|\alpha|+|\beta|=3\\ |\beta|\ge 1}} c_{j,\alpha,\beta}(x, y, z) y^\alpha (z-y)^\beta \tag{3-5}$$

for smooth functions $c_{j,\alpha,\beta}$, where $c_{j,\alpha,\beta} \equiv 0$ unless $\operatorname{order}(y^\alpha z^\beta) < \operatorname{order}(y_j)$, and also unless one of α', β', α'', or β'' is nonzero. Additionally, if $j = 0$ or $j \ge d+1$ then $c_{j,\alpha,\beta} \equiv 0$ unless one of α_0, β_0, α'', or β'' is nonzero.

The same conditions also hold on $\tilde{c}_{j,\alpha,\beta}$ in the following expansion of $\widetilde{R}(x, y, w)$:

$$\widetilde{R}_j(x, y, w) = \sum_{\substack{|\alpha|+|\beta|=2\\ |\beta|\ge 1}} \tilde{c}_{j,\alpha,\beta}(x) y^\alpha w^\beta + \sum_{\substack{|\alpha|+|\beta|=3\\ |\beta|\ge 1}} \tilde{c}_{j,\alpha,\beta}(x, y, w) y^\alpha w^\beta.$$

4. The semiclassical calculus on U

In this section we introduce the nonisotropic semiclassical quantization and h-dependent symbol classes that we use to construct the parametrix for P_h. As seen for the model operator, the phase variables associated to X'' need to be scaled by h^2, as opposed to the h-scaling for variables associated to X_0

and X'. The symbol classes are naturally associated to the nonisotropic dilation structure δ_r. We will define them using the nonisotropic norm

$$m(\eta) = (|\eta_0|^6 + |\eta'|^{12} + |\eta''|^4)^{\frac{1}{12}} \approx |\eta_0|^{\frac{1}{2}} + |\eta'| + |\eta''|^{\frac{1}{3}},$$

which is smooth for $\eta \neq 0$ and homogeneous of degree 1, in that $m(\delta_r(\eta)) = rm(\eta)$.

We assume that K is compactly contained in U, and choose r_1 so that the exponential map $y \to \exp_x(y)$ is a diffeomorphism on the ball $\{|y| \le r_1\}$ for all $x \in K$. We also fix $r_0 < r_1$ such that

$$\bigcup_{\tilde{x} \in \exp_x(\bar{B}_{r_0})} \exp_{\tilde{x}}(\bar{B}_{r_0}) \subset \exp_x(B_{r_1}).$$

We fix functions $\chi_j \in C_c^\infty(B_{r_j})$ with $\chi_0(y) = 1$ for $|y| \le \frac{1}{2} r_0$ and

$$\chi_1(\Theta_x(\cdot)) = 1 \text{ on a neighborhood of } \bigcup_{\tilde{x} \in \exp_x(\bar{B}_{r_0})} \exp_{\tilde{x}}(\bar{B}_{r_0}).$$

Given a symbol $a(x, \eta) \in C^\infty(U \times \mathbb{R}^{2d+1})$ supported where $x \in K$, we let

$$a_h(x, \eta) = a(x, \eta_0, \eta', h\eta''),$$

and define a nonisotropic semiclassical quantization of a by the rule

$$\begin{aligned} a_h(x, hD) f(x) &= \frac{1}{(2\pi h)^{2d+1}} \int_{\mathbb{R}^{4d+2}} e^{-i\langle y,\eta\rangle/h} a_h(x, \eta) \chi_0(y) f(\exp_x(y))\, dy\, d\eta \\ &= \frac{1}{(2\pi)^{2d+1}} \int_{\mathbb{R}^{4d+2}} e^{-i\langle y,\eta\rangle} a_h(x, \eta) \chi_0(hy) f(\exp_x(hy))\, dy\, d\eta. \end{aligned} \tag{4-1}$$

Thus the Schwartz kernel of $a_h(x, hD)$ is supported in $K \times K_{r_0}$, where K_{r_0} is the image of $K \times \bar{B}_{r_0}$ under $(x, y) \to \exp_x(y)$. In contrast to the usual semiclassical scaling $\eta \to h\eta$, the nonisotropic scaling $(h\eta_0, h\eta', h^2\eta'')$ arises from the missing directions X'' being obtained from commutators of X_0 and X'.

If $p(x, \eta) = \sum_{|\alpha| \le n} c_\alpha(x) \eta^\alpha$ is a polynomial in η, then

$$(p_h(x, hD) f)(x) = p_h(x, (-i\partial_y)) f(\exp_x(hy))\big|_{y=0}.$$

In particular, we have the following correspondence of symbols to operators:

$$i\eta_j : hX_j, \quad 0 \le j \le d, \qquad i\eta_j : h^2 X_j, \quad d+1 \le j \le 2d. \tag{4-2}$$

Suppose that the symbol a satisfies homogeneous order-0 type estimates of the form

$$|\partial_x^\beta \partial_\eta^\alpha a(x, \eta)| \le C_{\alpha,\beta} m(\eta)^{-\operatorname{order}(\alpha)}.$$

The uncertainty principle, needed for example for proving L^2 continuity of $a_h(x, hD)$, requires uniform bounds on $\partial_x^\alpha (h\partial_\eta)^\alpha a_h(x, \eta)$. On the other hand,

$$\begin{aligned} |\partial_x^\alpha (h\partial_\eta)^\alpha a_h(x, \eta)| &= h^{|\alpha_0| + |\alpha'| + 2|\alpha''|} |(\partial_x^\alpha \partial_\eta^\alpha a)_h(x, \eta)| \\ &\le C_\alpha h^{|\alpha_0| + |\alpha'| + 2|\alpha''|} m(\eta_0, \eta', h\eta'')^{-2|\alpha_0| - |\alpha'| - 3|\alpha''|}. \end{aligned}$$

To have uniform bounds as $h \to 0$ for every α would require truncating $a(x,\eta)$ to where $m(\eta) \geq h^{1/2}$. It is convenient to work with bounded symbols; hence for symbols of order n we will multiply by a factor of $h^{-n/2}$ to ensure that symbols of any order be of size $\lesssim 1$ when $m(\eta) \leq h^{1/2}$.

Definition 7. Let $m(h,\eta) = (h^{1/2} + m(\eta))$. A h-dependent family of symbols $a(x,\eta)$ belongs to $S^n(m)$ if, for all α, β, there is $C_{\alpha,\beta}$ independent of h such that, for $0 < h \leq 1$,

$$|\partial_x^\beta \partial_\eta^\alpha a(x,\eta)| \leq C_{\alpha,\beta} h^{-\frac{n}{2}} m(h,\eta)^{n-\mathrm{order}(\alpha)}.$$

We let $\Psi_h^n(m)$ denote the collection of operators $a_h(x, hD)$ as in (4-1) with $a \in S_h^n(m)$.

We also define $S^{-\infty}(m) = \bigcap_{k\in\mathbb{N}} S^{-k}(m)$, and let $\Psi_h^{-\infty}(m)$ denote operators that can be written in the form (4-1) with χ_0 replaced by χ_1, and $a \in S^{-\infty}(m)$.

Remark 8. We define $\Psi_h^{-\infty}(m)$ using χ_1 in the quantization rule (4-1) since the composition of operators defined using χ_0 need not have Schwartz kernel supported inside B_{r_0} (in local exponential coordinates). We also note that results below concerning continuity and composition of symbols are independent of the particular choice of χ_0. We show in Lemma 13 that replacing χ_0 by another function in $C_c^\infty(B_{r_0})$ that equals 1 on $B_{r_0/2}$ changes $a_h(x, hD)$ by a term in $\Psi_h^{-\infty}(m)$.

For polynomial symbols we note that

$$h^{-\frac{1}{2}\,\mathrm{order}(\alpha)} \eta^\alpha \in S^{\mathrm{order}(\alpha)}(m). \tag{4-3}$$

By (4-2) we then have the following examples, which will show that $P_h \in \Psi_h^2(m)$:

$$\begin{aligned} X_0 &\in \Psi_h^2(m), \\ h^{\frac{1}{2}} X_j &\in \Psi_h^1(m), \quad 1 \leq j \leq d, \\ h^{\frac{1}{2}} X_j &\in \Psi_h^3(m), \quad d+1 \leq j \leq 2d. \end{aligned} \tag{4-4}$$

A more general example of a symbol in $S^n(m)$ is $h^{-n/2} a(\eta)(1 - \phi(h^{-1/2} m(\eta)))$, where $\phi \in C_c^\infty(\mathbb{R}^{2d+1})$ equals 1 on a neighborhood of 0, and $a \in C^\infty(\mathbb{R}^{2d+1} \backslash \{0\})$ satisfies $a(\delta_r \eta) = r^n a(\eta)$.

It is easy to verify the following properties:

$$\begin{aligned} S^n(m) \cdot S^{n'}(m) &\subset S^{n+n'}(m), \\ S^n(m) &\supset S^{n'}(m) \quad \text{if } n' < n, \\ a \in S^n(m) \quad \Longrightarrow \quad h^{\frac{1}{2}\,\mathrm{order}(\alpha)} \partial_\eta^\alpha \partial_x^\beta a &\in S^{n-\mathrm{order}(\alpha)}. \end{aligned} \tag{4-5}$$

Definition 9. Given a sequence of symbols $a_j \in S^{n-j}(m)$ we say that $a \sim \sum_j a_j$ if for all N

$$a - \sum_{j=0}^{N-1} a_j \in S^{n-N}(m).$$

Consequently, a is uniquely determined up to a symbol in $S^{-\infty}(m)$.

We note the following simple example of a symbol in $S^{-\infty}(m)$:

$$\text{If } \phi \in \mathcal{S}(\mathbb{R}) \text{ and } \phi(s) = 1 \text{ when } |s| \le 1 \text{ then } \phi(h^{-\frac{1}{2}} m(\eta)) \in S^{-\infty}(m). \tag{4-6}$$

That this symbol belongs to $S^{-\infty}(m)$ is seen by noting that

$$(1 + h^{-\frac{1}{2}} m(\eta))^{-N} = h^{\frac{N}{2}} m(h, \eta)^{-N},$$

together with the bounds $|\partial_\eta^\alpha m(\eta)| \le C_\alpha m(\eta)^{1-\operatorname{order}(\alpha)}$, where we use that all derivatives vanish unless $m(\eta) \ge h^{1/2}$; hence $m(\eta) \approx m(h, \eta)$, since ϕ is assumed constant near 0.

Lemma 10. *Suppose that $a_j \in S^{n-j}(m)$, $j \in \mathbb{N}$. Then there exists $a \in S^n(m)$ with $a \sim \sum_j a_j$.*

Proof. Fix $\phi \in C_c^\infty((-2, 2))$ with $\phi = 1$ for $|s| < 1$. We will construct a sequence of real numbers $R_j \ge 1$ with $R_j \to \infty$ such that for all N

$$\sum_{j=N}^{\infty} (1 - \phi(R_j^{-1} h^{-\frac{1}{2}} m(\eta))) a_j(x, \eta) \text{ converges in } S^{n-N}(m). \tag{4-7}$$

Defining a to be this sum for $N = 0$ then gives the result since by (4-6), for each j,

$$\phi(R_j^{-1} h^{-\frac{1}{2}} m(\eta)) a_j(x, \eta) \in S^{-\infty}(m).$$

The proof of (4-6) shows that the $S^0(m)$ seminorms of $\phi(R^{-1} h^{-1/2} m(\eta))$ are uniformly bounded independent of R for $R \ge 1$. The result (4-7) follows if we choose R_j so that for all $|\alpha| + |\beta| \le j$

$$|\partial_x^\beta \partial_\eta^\alpha (1 - \phi(R_j^{-1} h^{-\frac{1}{2}} m(\eta))) a_j(x, \eta)| \le 2^{-j} h^{-\frac{n+1-j}{2}} m(h, \eta)^{n+1-j-\operatorname{order}(\alpha)}.$$

Such R_j can be chosen by observing that on the support of $1 - \phi(R_j^{-1} h^{-1/2} m(\eta))$ we have the bound $h^{1/2} m(h, \eta)^{-1} \le (1 + R_j)^{-1}$. □

We now turn to the composition result for operators. Due to support considerations of the Schwartz kernels involved, expressing the composition of two operators quantized using the cutoff χ_0 requires quantizing the symbol of the composition using the cutoff χ_1, but we shall later see that the difference is an operator with symbol in $S^{-\infty}$. For simplicity we consider the case where the order of the composition is negative, which is the case needed to produce an inverse for P_h modulo $\Psi_h^{-\infty}(m)$.

In the proof we decompose an operator $a_h(x, hD)$ into a sum of nonisotropic dilates of unit-scale convolution kernels. This decomposition is also used in establishing L^2 bounds for order 0 operators. Let ϕ and ψ generate a smooth Littlewood–Paley decomposition of $[0, \infty)$:

$$1 = \phi(s) + \sum_{j=1}^{\infty} \psi(2^{-j} s), \quad \operatorname{supp}(\phi) \subset [0, 2), \ \operatorname{supp}(\psi) \subset \left(\tfrac{1}{2}, 2\right). \tag{4-8}$$

Given a symbol $a \in S^n(m)$, we make the decomposition

$$a(x, \eta) = \phi(h^{-\frac{1}{2}} m(\eta)) a(x, \eta) + \sum_{j=1}^{\infty} \psi(h^{-\frac{1}{2}} 2^{-j} m(\eta)) a(x, \eta) = \sum_{j=0}^{\infty} a_j(x, \eta). \tag{4-9}$$

Then a_j is supported where $m(h,\eta)\approx 2^j h^{1/2}$, and thus

$$|\partial_x^\beta\partial_\eta^\alpha a_j(x,\eta)|\le C_{\alpha,\beta}2^{jn}(2^j h^{\frac12})^{-\operatorname{order}(\alpha)}.$$

It follows that $a_0(x,\delta_{h^{1/2}}(\eta))\in C_c^\infty(K\times\{|\eta|<8\})$ with bounds uniform over h, and for $j\ge 1$ that $2^{-jn}a_j(x,\delta_{2^j h^{1/2}}(\eta))$ is uniformly bounded in $C_c^\infty\bigl(K\times\bigl\{\frac18<|\eta|<8\bigr\}\bigr)$ over h and j.

Theorem 11. *Given $a\in S^n(m)$ and $b\in S^{n'}(m)$, with $n+n'<0$, there is $c\in S^{n+n'}(m)$ so that*

$$a_h(x,hD)\circ b_h(x,hD)f(x)=\frac{1}{(2\pi h)^{2d+1}}\int e^{-i\langle y,\xi\rangle/h}c_h(x,\xi)\chi_1(y)f(\exp_x(y))\,dy\,d\xi. \tag{4-10}$$

Proof. For $x\in K$ and $h>0$ we can write

$$\chi_1(\Theta_x(\tilde x))f(\tilde x)=\frac{1}{(2\pi h)^{2d+1}}\int e^{i\langle\Theta_x(\tilde x),\xi\rangle/h-i\langle y,\xi\rangle/h}\chi_1(y)f(\exp_x(y))\,dy\,d\xi.$$

Since $a_h(x,hD)b_h(x,hD)f(x)=a_h(x,hD)b_h(x,hD)\bigl(\chi_1(\Theta_x(\,\cdot\,))f\bigr)(x)$, we know (4-10) holds with

$$c_h(x,\xi)=\bigl(a_h(x,hD)b_h(x,hD)e^{i\langle\Theta_x(\,\cdot\,),\xi\rangle/h}\bigr)(x).$$

We thus need to show that $c(x,\xi)=c_h(x,\xi_0,\xi',h^{-1}\xi'')\in S^{n+n'}(m)$.

Let a_i and b_j be the nonisotropic Littlewood–Paley decomposition of a and b as in (4-9), and define c_{ij} by

$$(c_{ij})_h(x,\xi)=\bigl((a_i)_h(x,hD)(b_j)_h(x,hD)e^{i\langle\Theta_x(\,\cdot\,),\xi\rangle/h}\bigr)(x),$$

so that $c=\sum_{ij}c_{ij}$. From (4-1) we can write $(c_{ij})_h(x,\xi)$ as

$$\frac{1}{(2\pi)^{4d+2}}\int e^{-i\langle y,\eta\rangle-i\langle w,\zeta\rangle+ih^{-1}\langle\widetilde\Theta(x,hy,hw),\xi\rangle}(a_i)_h(x,\eta)(b_j)_h(\exp_x(hy),\zeta)\chi_0(hy)\chi_0(hw)\,dw\,d\zeta\,dy\,d\eta.$$

Consider first the case $i\ge j$. We substitute $w=h^{-1}\Theta(x,hy,hz)$, defined in (3-2), to write this as

$$\frac{1}{(2\pi)^{4d+2}}\int e^{-i\langle y,\eta\rangle-ih^{-1}\langle\Theta(x,hy,hz),\zeta\rangle+i\langle z,\xi\rangle}a_i(x,\eta_0,\eta',h\eta'')b_j(\exp_x(hy),\zeta_0,\zeta',h\zeta'')$$
$$\times\chi_0(hy)\chi_0(\Theta(x,hy,hz))|D_z\Theta|(x,hy,hz)\,dz\,d\zeta\,dy\,d\eta.$$

By the comments following (4-9) applied to b_j, we write

$$b_j(\exp_x(hy),\zeta_0,\zeta',h\zeta'')\chi_0(hy)\chi_0(\Theta(x,hy,hz))|D_z\Theta|(x,hy,hz)$$
$$=2^{jn'}\tilde b_j(x,hy,hz,\delta_{2^{-j}h^{-1/2}}(\zeta_0,\zeta',h\zeta'')),$$

where $\tilde b_j\in C_c^\infty(K\times B_{r_0}\times B_{r_1}\times B_8)$, with bounds uniform over h and j, and a similar representation holds for a_i with 2^j replaced by 2^i and n' replaced by n. We make a nonisotropic dilation of ζ and η by the factors $(2^{2j}h,2^jh^{1/2},2^{3j}h^{1/2})$, and of z and y by the reciprocal factors, to write

$$c_{ij}(x,\xi)=2^{j(n+n')}\tilde c_{ij}(x,\delta_{2^{-j}h^{-1/2}}(\xi)),$$

where $\tilde{c}_{ij}(x,\xi)$ is given by

$$2^{(i-j)n}\frac{1}{(2\pi)^{4d+2}}\int e^{-i\langle y,\eta\rangle-i\langle\overline{\Theta}_y(z)+R(h,x,y,z),\zeta\rangle+i\langle z,\xi\rangle}\tilde{a}_i(x,\delta_{2^{j-i}}(\eta)) \\ \times\tilde{b}_j\big(x,\delta_{2^{-j}}(y_0,h^{\frac12}y',h^{\frac12}y''),\delta_{2^{-j}}(z_0,h^{\frac12}z',h^{\frac12}z''),\zeta\big)\,dz\,d\zeta\,dy\,d\eta, \tag{4-11}$$

where, recalling Lemma 6,

$$\begin{aligned}\langle R(h,x,y,z),\zeta\rangle =&2^{2j}R_0\big(x,\delta_{2^{-j}}(y_0,h^{\frac12}y',h^{\frac12}y''),\delta_{2^{-j}}(z_0,h^{\frac12}z',h^{\frac12}z'')\big)\zeta_0\\ &+2^jh^{-\frac12}R'\big(x,\delta_{2^{-j}}(y_0,h^{\frac12}y',h^{\frac12}y''),\delta_{2^{-j}}(z_0,h^{\frac12}z',h^{\frac12}z'')\big)\cdot\zeta'\\ &+2^{3j}h^{-\frac12}R''\big(x,\delta_{2^{-j}}(y_0,h^{\frac12}y',h^{\frac12}y''),\delta_{2^{-j}}(z_0,h^{\frac12}z',h^{\frac12}z'')\big)\cdot\zeta''.\end{aligned}$$

By the support condition on $\tilde{b}_j$ we have $|\zeta|\le 8$. Also, if $i\ge 1$ then $\tilde{a}_i(x,\eta)=0$ when $|\eta|\le\frac18$.

We next apply the expansion (3-5) to the right-hand side. The condition on order($y^\alpha z^\beta$) ensures that we bring out strictly more powers of 2^{-j} than needed to cancel the powers of 2^j in front, and since there is at least one factor of (y',z',y'',z'') we also bring out a factor $h^{1/2}$ to cancel off the $h^{-1/2}$ in front. We conclude that, on the support of the integrand,

$$|R(h,x,y,z)|\le C2^{-j}|z-y|(|y|+|z-y|+|y|^2+|z-y|^2),$$

and also

$$|\partial_x^\gamma\partial_y^\alpha\partial_z^\beta R(h,x,y,z)|\le C_{\gamma,\alpha,\beta}2^{-j}(1+|y|^3+|z-y|^3). \tag{4-12}$$

Additionally, if we let $w=\overline{\Theta}(x,y,z)+R(h,x,y,z)$, then with analogous notation we see from (3-2) and Lemma 6 that $z=\overline{\Theta}_{-y}(w)+\widetilde{R}(h,x,y,w)$, where

$$|\widetilde{R}(h,x,y,w)|\le C2^{-j}|w|(|y|+|w|+|y|^2+|w|^2).$$

Consequently, since $\widetilde{\Theta}$ is the inverse function to Θ for fixed y, uniformly over j we have

$$\begin{aligned}&|\overline{\Theta}_y(z)+R(h,x,y,z)|\le C|z-y|(1+|y|^2+|z-y|^2),\\ &\qquad|z-y|\le C|\overline{\Theta}_y(z)+R(h,x,y,z)|(1+|y|^2+|\overline{\Theta}_y(z)+R(h,x,y,z)|^2),\end{aligned}$$

and hence

$$(1+|y|^2)^{-1}|z-y|\le C|\overline{\Theta}_y(z)+R(h,x,y,z)|(1+|\overline{\Theta}_y(z)+R(h,x,y,z)|^2). \tag{4-13}$$

Considering the function

$$g_{ij}(x,y)=\frac{1}{(2\pi)^{4d+2}}\int e^{-i\langle y,\eta\rangle}\tilde{a}_i(x,\delta_{2^{j-i}}(\eta))\,d\eta,$$

simple estimates show that

$$|\partial_x^\alpha g_{ij}(x,y)|\le C_{N,\alpha,\beta}2^{(4d+2)(i-j)}(1+2^{2(i-j)}|y_0|+2^{i-j}|y'|+2^{3(i-j)}|y''|)^{-N}. \tag{4-14}$$

Additionally, if $i > j$, hence $i \ge 1$, then $\tilde a_i(x,\eta)$ vanishes for $|\eta| \le \frac18$, and thus can be assumed to be of the form $|\eta|^{2k}\tilde a_i(x,\eta)$ for similar $\tilde a_i(x,\eta)$. Thus, if $i > j$ then for all $k \in \mathbb{N}$ we can write

$$g_{ij}(x,y) = \sum_{|\gamma|=2k} 2^{(j-i)\,\mathrm{order}(\gamma)} \partial_y^\gamma g_{ij,\gamma}(x,y), \tag{4-15}$$

where $g_{ij,\gamma}(x,y)$ satisfies the same estimates (4-14) as $g_{ij}(x,y)$. On the other hand, if we set

$$f_j(x,y,z) = \int e^{-i\langle \overline{\Theta}_y(z)+R(h,x,y,z),\zeta\rangle} \tilde b_j\bigl(x, \delta_{2^{-j}}(y_0, h^{\frac12}y', h^{\frac12}y''), \delta_{2^{-j}}(z_0, h^{\frac12}z', h^{\frac12}z''), \zeta\bigr)\, d\zeta,$$

then

$$f_j(x,y,z) = \rho(x,y,z,\overline{\Theta}_y(z) + R(h,x,y,z)),$$

where $\rho(x,y,z,w)$ is smooth in (x,y,z) and Schwartz in w. By (4-12) and (4-13) we have

$$|\partial_x^\alpha \partial_y^\beta \partial_z^\theta f_j(x,y,z)| \le C_{N,\alpha,\beta,\theta}(1+|y|+|y-z|)^{3(|\alpha|+|\beta|+|\theta|)}(1+(1+|y|^2)^{-1}|y-z|)^{-N}.$$

Applying (4-15) and integrating by parts in y leads to the bound, for all N, α, β,

$$\left|\partial_x^\alpha \partial_z^\beta \int g_{ij}(x,y) f_j(x,y,z)\, dy\right| \le C_{N,\alpha,\beta} 2^{2k(j-i)}(1+|z|)^{-N}.$$

Since $\tilde c_{ij}(x,\xi)$, defined in (4-11), is $2^{(i-j)n}$ times the Fourier transform in z of this integral, we obtain uniform (over i and j) Schwartz bounds on $2^{i-j}\tilde c_{ij}(x,\xi)$, with compact support in x.

In the case $j \ge i$, we can similarly write $c_{ij}(x,\xi) = 2^{i(n+n')}\tilde c_{ij}(x, \delta_{2^{-i}h^{-1/2}}(\xi))$, where we have uniform Schwartz bounds over i and j on $2^{j-i}\tilde c_{ij}(x,\xi)$. The analysis is similar to the case $i \ge j$, using instead the following representation for $c_{ij}(x,\xi)$:

$$\frac{1}{(2\pi)^{4d+2}} \int e^{-ih^{-1}\langle \widetilde\Theta(x,hv,-hw),\eta\rangle - i\langle w,\zeta\rangle + i\langle v,\xi\rangle} a_i(x,\eta_0,\eta',h\eta'') b_j(\exp_{\exp_x(hv)}(-hw), \zeta_0, \zeta', h\zeta'') \\ \times \chi_0(\widetilde\Theta(x,hv,-hw))\chi_0(hw)|D_v\widetilde\Theta|(x,hv,-hw)\, dw\, d\zeta\, dv\, d\eta.$$

It thus suffices to show that $\sum_{i\ge j} 2^{j(n+n')}\tilde c_{ij}(x,\delta_{2^{-j}h^{-1/2}}(\xi)) \in S^{n+n'}(m)$. We prove that

$$\left|\sum_{i\ge j} 2^{j(n+n')}\tilde c_{ij}(x,\delta_{2^{-j}h^{-1/2}}(\xi))\right| \le C(1+h^{-\frac12}m(\xi))^{n+n'}.$$

Estimates on derivatives will follow similarly since applying ∂_ξ^α has the effect of multiplying the j-th term by $(2^{-j}h^{-1/2})^{\mathrm{order}(\alpha)}$. We use the uniform Schwartz bounds on $\tilde c_{ij}$ to bound the sum by

$$C_N \sum_{i\ge j\ge 0} 2^{j(n+n')} 2^{j-i}(1+2^{-j}h^{-\frac12}m(\xi))^{-N}.$$

The sum over i is trivial. Given ξ, take j_0 so that $2^{j_0} = h^{-1/2}m(\xi)$. We then split

$$\sum_{j\ge 0} 2^{j(n+n')}(1+2^{-j}h^{-\frac12}m(\xi))^{-N} \le \sum_{j\ge j_0} 2^{j(n+n')} + \sum_{j<j_0} 2^{j(n+n'+N)}(h^{-\frac12}m(\xi))^{-N}.$$

Recall that we assume $n+n'<0$. We take N so $N+n+n'>0$. If $h^{-1/2}m(\xi)\le 1$, we have only the first sum, which is bounded by a constant. If $h^{-1/2}m(\xi)>1$, then the two terms are convergent geometric sums that both are bounded by $(h^{-1/2}m(\xi))^{n+n'}$. □

Remark 12. The result of Theorem 11 still holds if one replaces the function $\chi_0(y)$ used in quantizing a or b by any function $\chi(x,y)\in C_c^\infty(K\times B_{r_0})$, since this is harmlessly absorbed into $\tilde{b}_j$ without changing the estimates for $\tilde{b}_j$ nor the condition on the support of the Schwartz kernel.

Lemma 13. *Suppose that $\beta,\chi\in C_c^\infty(B_{r_1})$, and $\beta(y)=0$ for $|y|\le\delta$, where $\delta>0$. Suppose also that $\chi=1$ on $\operatorname{supp}(\beta)$. Then if $a\in S^n(m)$ for some n, one can write*

$$\frac{1}{(2\pi h)^{2d+1}}\int e^{-i\langle y,\xi\rangle/h}a_h(x,\xi)\beta(y)f(\exp_x(y))\,dy\,d\xi = \frac{1}{(2\pi h)^{2d+1}}\int e^{-i\langle y,\xi\rangle/h}r_h(x,\xi)\chi(y)f(\exp_x(y))\,dy\,d\xi,$$

where $r\in S^{-\infty}(m)$.

Proof. We write $\beta(y)=|y|^{2N}\beta_N(y)$ for $\beta_N\in C_c^\infty(B_{r_1})$. Since $\chi\beta_N=\beta_N$, following the first part of Theorem 11 we have equality of the two sides if r_h is the symbol

$$r_h(x,\xi)=\frac{1}{(2\pi h)^{2d+1}}\int e^{-i\langle y,\xi-\eta\rangle/h}((h^2\Delta_{\eta_0,\eta'}+h^4\Delta_{\eta''})^N a)_h(x,\eta)\beta_N(y)\,dy\,d\eta.$$

By (4-5), $a_N=(h^2\Delta_{\eta_0,\eta'}+h^4\Delta_{\eta''})^N a\in S^{n-2N}(m)$. We then write

$$r(x,\xi)=\frac{1}{(2\pi)^{2d+1}h^{3d+1}}\int\hat{\beta}_N\Big(\frac{\xi_0-\eta_0}{h},\frac{\xi'-\eta'}{h},\frac{\xi''-\eta''}{h^2}\Big)a_N(x,\eta)\,d\eta.$$

We have $|a_N(x,\eta)|\le C_N(1+m(\delta_{h^{-1/2}}(\eta)))^{n-2N}$, and Peetre's inequality yields

$$(1+|\delta_{h^{-1/2}}(\xi-\eta)|)^{-|n|-2N}(1+m(\delta_{h^{-1/2}}(\eta)))^{n-2N}\le C_N(1+m(\delta_{h^{-1/2}}(\xi)))^{n-2N},$$

which shows that $|r(x,\xi)|\le C_N h^N m(h,\xi)^{-2N}$ for all N. The term $\partial_x^\beta\partial_\xi^\alpha r(x,\xi)$ comes from the same convolution applied to $\partial_x^\beta\partial_\eta^\alpha a_N(x,\eta)$, and we conclude $r\in S^{-\infty}$. □

Corollary 14. *Suppose P_h is as in* (1-1). *Given $\rho\in C_c^\infty(K^o)$, there is a symbol $q(x,\xi)\in S^{-2}(m)$, with principal symbol $h\rho(x)(1-\phi(h^{-1/2}m(\xi)))q_0(\xi)$, so that*

$$q_h(x,hD)\circ P_h=\rho(x)+R,\qquad R\in\Psi_h^{-\infty}(m).$$

Proof. Fix $\tilde{\rho}(x)\in C_c^\infty(K)$ with $\tilde{\rho}=1$ on a neighborhood of $\operatorname{supp}(\rho)$. Define

$$\tilde{q}_0(\xi)=h(1-\phi(h^{-\frac{1}{2}}m(\xi)))q_0(\xi),$$

where $q_0(\xi)$ is the Fourier transform of the fundamental solution for $Y_0-\sum_{j=1}^d Y_j^2$, as defined in Section 2, and ϕ is as in (4-8). Then $\tilde{\rho}(x)\tilde{q}_0(\xi)\in S^{-2}(m)$. We first show that

$$\tilde{\rho}(x)\tilde{q}_{0,h}(hD)\circ P_h=\tilde{\rho}(x)-r_h^1(x,hD),\qquad r^1\in S^{-1}(m).$$

By the construction of $q_0(\xi)$ we have

$$\frac{1}{(2\pi h)^{2d+1}}\int e^{-i\langle y,\xi\rangle/h}hq_{0,h}(\xi)\Bigl(Y_0-h\sum_{j=1}^{d}Y_j^2\Bigr)\chi_0(y)f(\exp_x(y))\,dy\,d\xi=f(x),$$

where the Y_j are the model vector fields acting in the y-variable. Replacing hq_0 by $\tilde{q}_0$ changes the composition by an order-0 symbol supported where $m(\xi)\le 2h^{1/2}$, hence by a symbol in $S^{-\infty}$.

Generally, we see that for $f(x,y)$ compactly supported in y we can write

$$\int e^{-i\langle y,\xi\rangle/h}\tilde{q}_{0,h}(\xi)y^\alpha\partial_y^\beta f(x,y)\,dy\,d\xi=\int e^{-i\langle y,\xi\rangle/h}b_h(\xi)f(x,y)\,dy\,d\xi,$$

where

$$b(x,\xi)=i^{|\beta|-|\alpha|}h^{\alpha_0+|\alpha'|+2|\alpha''|-|\beta|}\partial_\xi^\alpha\xi^\beta\tilde{q}_0(\xi).$$

By (4-3) and (4-5), we know that

$$h^{\frac{1}{2}\operatorname{order}(\alpha)-\frac{1}{2}\operatorname{order}(\beta)}\partial_\xi^\alpha\xi^\beta\tilde{q}_0(\xi)\in S^{\operatorname{order}(\beta)-\operatorname{order}(\alpha)-2}(m).$$

Recall that

$$(X_jf)(\exp_x(y))=(Y_j+R_j(x,y,\partial_y))f(\exp_x(y)),$$

where the Taylor expansion of R_j contains terms $y^\alpha\partial_y^\beta$ of order strictly less than $\operatorname{order}(Y_j)$, and where $|\beta|=1$. Since commutators of X_j with χ_0 lead to terms of order $-\infty$, we need show that

$$\frac{1}{(2\pi h)^{2d+1}}\int e^{-i\langle y,\xi\rangle/h}hq_{0,h}(\xi)R_0(x,y,\partial_y)\chi_0(y)f(\exp_x(y))\,dy\,d\xi$$

is an operator of order -1 in f. For the terms that arise in the Taylor expansion of R_0 we have $\operatorname{order}(\beta)-\operatorname{order}(\alpha)\le 1$, so we need check for such terms we also have

$$\alpha_0+|\alpha'|+2|\alpha''|-|\beta|\ge\tfrac{1}{2}\operatorname{order}(\alpha)-\tfrac{1}{2}\operatorname{order}(\beta),$$

in order to match up the powers of h. Since $|\beta|=1$ and $\operatorname{order}(\alpha)=2\alpha_0+|\alpha'|+3|\alpha''|$, this holds provided that $|\alpha'|+|\alpha''|\ge 1$, which is the case for R_0 by Lemma 4.

We similarly need check that this is an operator of order -2 if R_0 is replaced by $h^{1/2}R_j$ with $1\le j\le d$. Since $\operatorname{order}(\beta)-\operatorname{order}(\alpha)\le 0$ in this case, this reduces to verifying that

$$\alpha_0+|\alpha'|+2|\alpha''|-|\beta|+\tfrac{1}{2}\ge\tfrac{1}{2}\operatorname{order}(\alpha)-\tfrac{1}{2}\operatorname{order}(\beta),$$

which always holds if $|\beta|=1$.

We note that the remainder term in the Taylor expansion will also be of the desired order, but with $\chi_0(y)$ replaced by $c_{j,\alpha,k}(x,y)\chi_0(y)$. By Remark 12 this does not affect the conclusion of the corollary, since the form for q_h will involve composition with $r_h^1(x,hD)$.

By Theorem 11 we can recursively define symbols $r^j\in S^{-j}(m)$ for $j\ge 2$ by the rule

$$r_h^j(x,hD)\circ r_h^1(x,hD)f(x)=\frac{1}{(2\pi h)^{2d+1}}\int e^{-i\langle y,\xi\rangle/h}r_h^{j+1}(x,\xi)\chi_1(y)f(\exp_x(y))\,dy\,d\xi,$$

where we recall that $r_h^j(x, hD)$ is quantized using χ_0. Let $r \sim \sum_{j=0}^{\infty} r^j$, so $r \in S^{-1}(m)$. Also define $q \in S^{-2}(m)$ so that

$$\rho(x)(I + r_h(x, hD))\tilde{\rho}(x)\tilde{q}_{0,h}(hD) f(x) = \frac{1}{(2\pi h)^{2d+1}} \int e^{-i\langle y,\xi\rangle/h} q_h(x, \xi)\chi_1(y) f(\exp_x(y))\, dy\, d\xi.$$

By the above and Lemma 13, the following operator is in $\Psi_h^{-\infty}(m)$:

$$Rf(x) = \frac{1}{(2\pi h)^{2d+1}} \int e^{-i\langle y,\xi\rangle/h} q_h(x, \xi)(\chi_1(y) - \chi_0(y))(P_h f)(\exp_x(y))\, dy\, d\xi.$$

Thus, modulo $\Psi_h^{-\infty}(m)$ we have

$$q_h(x, hD) \circ P_h = \rho(x)(I + r_h(x, hD))(\tilde{\rho}(x) - r_h^1(x, hD)).$$

Next we choose $\delta > 0$ so that $\tilde{\rho}(\exp_x(y)) = 1$ if $x \in \mathrm{supp}(\rho)$ and $|y| \le \delta$, and take $\chi_\delta \in C_c^\infty(B_\delta)$ with $\chi_\delta = 1$ on $B_{\delta/2}$. Then

$$\begin{aligned}\rho(x)r_h(x, hD)((1 - \tilde{\rho}) f)(x) &\\ = \frac{1}{(2\pi h)^{2d+1}} &\int e^{-i\langle y,\xi\rangle/h} \rho(x) r_h(x, \xi)(\chi_0(y) - \chi_\delta(y))((1 - \tilde{\rho}) f)(\exp_x(y))\, dy\, d\xi,\end{aligned}$$

so by Lemma 13 we have, modulo $\Psi_h^{-\infty}(m)$,

$$q_h(x, hD) \circ P_h = \rho(x)(I + r_h(x, hD))(I - r_h^1(x, hD)).$$

Finally, since the difference between using χ_1 instead of χ_0 in the quantization of r^{j+1} gives a term in $\Psi_h^{-\infty}(m)$, we see that $q_h(x, hD) \circ P_h = \rho(x)$ modulo $\Psi_h^{-\infty}(m)$. □

Remark 15. The above proof shows the following composition result concerning partial differential operators. Suppose

$$P_h = \sum_{\mathrm{order}(\alpha) \le n'} c_\alpha(x) X_0^{\alpha_0} (h^{\frac{1}{2}} X')^{\alpha'} (h^{\frac{1}{2}} X'')^{\alpha''}, \quad c_\alpha(x) \in C_c^\infty(K).$$

Then if $a \in S^n(m)$, we can write $a_h(x, hD) \circ P_h f$ and $P_h \circ a_h(x, hD) f$ in the form

$$\frac{1}{(2\pi h)^{2d+1}} \int e^{-i\langle y,\xi\rangle/h} b_h(x, \xi)\chi(y) f(\exp_x(y))\, dy\, d\xi$$

for $\chi \in C_c^\infty(B_{r_0})$ and $b \in S^{n+n'}(m)$.

5. L^2 boundedness for order-0 operators

Given a symbol in $S^n(m)$ we decompose $a = \sum_j a_j$ as in (4-9). The operator $a_{j,h}(x, hD)$ is given by the following integral kernel on $U \times U$ with respect to the measure $dm(\tilde{x})$, where $w(x, \tilde{x})dm(\tilde{x}) = \exp_x^*(dy)$:

$$K_j(x, \tilde{x}) = w(x, \tilde{x})\chi_0(\Theta_x(\tilde{x})) \int e^{-i\langle\Theta_x(\tilde{x}),\eta\rangle} a_{j,h}(x, h\eta)\, d\eta.$$

We can write $a_{j,h}(x, h\eta) = 2^{jn}\tilde a_j(x, 2^{-2j}\eta_0, 2^{-j}h^{1/2}\eta', 2^{-3j}h^{1/2}\eta'')$, where $\tilde a_j(x,\eta) \in C_c^\infty(K \times B_8)$, with uniform bounds over j. Furthermore, $\tilde a_j$ vanishes for $|\eta| \le \frac{1}{8}$ if $j \ge 1$.

Consequently, there are Schwartz functions $\rho_j(x, y)$, supported for $x \in K$ with Schwartz norms independent of j, so that

$$(w^{-1}K_j)(x, \exp_x(y)) = 2^{jn}2^{j(2+4d)}h^{-d}\rho_j(x, 2^{2j}y_0, 2^j h^{-\frac12}y', 2^{3j}h^{-\frac12}y'')\chi_0(y), \tag{5-1}$$

and in particular, for all N,

$$|K_j(x,\tilde x)| \le C_N 2^{jn}2^{j(2+4d)}h^{-d}\bigl(1 + 2^{2j}|\Theta_x(\tilde x)_0| + 2^j h^{-\frac12}|\Theta_x(\tilde x)'| + 2^{3j}h^{-\frac12}|\Theta_x(\tilde x)''|\bigr)^{-N}. \tag{5-2}$$

If $a \in S^{-\infty}$ then (5-1) holds for all $n \in \mathbb{Z}$, and summing over j we obtain the following.

Corollary 16. *If $a \in S^{-\infty}(m)$, then $a_h(x, hD)$ is given by a smooth integral kernel $K(x,\tilde x)$ in the measure $dm(\tilde x)$, so that for some Schwartz function $\rho(x, y)$, supported for $x \in \mathrm{supp}(a)$,*

$$(w^{-1}K)(x, \exp_x(y)) = h^{-d}\rho(x, y_0, h^{-\frac12}y', h^{-\frac12}y'')\chi_0(y).$$

We next observe that the vector fields $2^{-2j}Y_0$, $2^{-j}h^{1/2}Y'$, and $2^{-3j}h^{1/2}Y''$ acting as differential operators in y all preserve the form (5-1) of $w^{-1}K_j$; that is, they give an expression of the same form with ρ_j uniformly bounded over j in each Schwartz seminorm.

The same holds for the operators $2^{-2j}X_0$, $2^{-j}h^{1/2}X'$, and $2^{-3j}h^{1/2}X''$, acting on $K_j(x,\tilde x)$ as differential operators in either the x- or $\tilde x$-variable. For action in the $\tilde x$-variable, this follows by Lemma 4, where we use that there is at least one factor of y' or y'' in the expansion of $R_0(x, y, \partial_y)$ to compensate for the factor of $h^{-1/2}$ coming from the $\partial_{y'}$ and $\partial_{y''}$ terms in the expansion of X_0. For action in the x-variable we work in coordinates $x = \exp_{\tilde x}(y)$, hence $\tilde x = \exp_x(-y)$, to write

$$(w^{-1}K_j)(\exp_{\tilde x}(y), \tilde x) = 2^{jn}2^{j(2+4d)}h^{-d}\rho_j\bigl(\exp_{\tilde x}(y), -2^{2j}y_0, -2^j h^{-\frac12}y', -2^{3j}h^{-\frac12}y''\bigr)\chi_0(-y). \tag{5-3}$$

To summarize, for $a \in S^n(m)$, we can write

$$\begin{aligned}(2^{-2j}X_0)^{\alpha_0}&(2^{-j}h^{\frac12}X')^{\alpha'}(2^{-3j}h^{\frac12}X'')^{\alpha''}K_j(x,\tilde x)\\ &= 2^{jn}2^{j(2+4d)}h^{-d}\rho_{j,\alpha}\bigl(x, 2^{2j}\Theta_x(\tilde x)_0, 2^j h^{-\frac12}\Theta_x(\tilde x)', 2^{3j}h^{-\frac12}\Theta_x(\tilde x)''\bigr)\chi_\alpha(x,\tilde x),\end{aligned} \tag{5-4}$$

where the functions $\rho_{j,\alpha}$ and χ_α satisfy seminorm bounds that depend on α, but are uniform over j and h. This holds with any given vector X in the product acting as a vector field in x or $\tilde x$.

Conversely, suppose that $j \ge 1$, so that $\tilde a_j(x,\eta) \in C_c^\infty\bigl(K \times \bigl\{\frac18 \le |\eta| \le 8\bigr\}\bigr)$. Then for any ℓ, dividing $\tilde a_j$ by $|\eta|^{2\ell}$ shows that we can write

$$\begin{aligned}&(w^{-1}K_j)(x, \exp_x(y))\\ &= 2^{jn}2^{j(2+4d)}h^{-d}\sum_{|\alpha|=2\ell}\chi_\alpha(x,y)(2^{-2j}\partial_{y_0})^{\alpha_0}(2^{-j}h^{\frac12}\partial_{y'})^{\alpha'}(2^{-3j}h^{\frac12}\partial_{y''})^{\alpha''}\rho_{j,\alpha}(x, 2^{2j}y_0, 2^j h^{-\frac12}y', 2^{3j}h^{-\frac12}y'')\end{aligned}$$

for Schwartz functions $\rho_{j,\alpha}$ that are uniformly bounded over j, and $\chi_\alpha \in C_c^\infty(K \times B_{r_0})$.

Using Lemma 4, we write

$$\begin{aligned}\partial_{y_0} &= X_0 + y'\cdot X'' - R_0(x, y, \partial_y) - y'\cdot R''(x, y, \partial_y),\\ \partial_{y'} &= X' - y_0 X'' - R'(x, y, \partial_y) + y_0 R''(x, y, \partial_y),\\ \partial_{y''} &= X'' - R''(x, y, \partial_y),\end{aligned}$$

where the X_j act in y. Substituting this into $R(x, y, \partial_y)$, and using that the X_j form a smooth frame, we can expand each ∂_{y_j} as a finite sum over $2 \le |\alpha| \le 3$:

$$\begin{aligned}\partial_{y_0} &= X_0 + y'\cdot X'' + \sum_{\alpha,k} c_{0,\alpha,k}(x, y) y^\alpha X_k, && \text{order}(Y_k) - \text{order}(\alpha) < 2,\\ \partial_{y_j} &= X_j - y_0 X_{j+d} + \sum_{\alpha,k} c_{j,\alpha,k}(x, y) y^\alpha X_k, && \text{order}(Y_k) - \text{order}(\alpha) < 1,\ \ 1 \le j \le d,\\ \partial_{y_j} &= X_j + \sum_{\alpha,k} c_{j,\alpha,k}(x, y) y^\alpha X_k, && \text{order}(Y_k) - \text{order}(\alpha) < 3,\ \ d+1 \le j \le 2d.\end{aligned}$$

Additionally, $c_{0,\alpha,k} \equiv 0$ unless either $\alpha' \neq 0$ or $\alpha'' \neq 0$.

Let $\overline{X}_j$ denote the transpose of the differential operator X_j with respect to dy. Taking the transpose of the above identities, it follows that, with the $\overline{X}_j$ acting on y, we can write

$$\begin{aligned}&(w^{-1}K_j)(x, \exp_x(y))\\ &= 2^{jn} 2^{j(2+4d)} h^{-d} \sum_{|\alpha|=2\ell} \chi_\alpha(x, y)(2^{-2j}\overline{X}_0)^{\alpha_0}(2^{-j}h^{\frac{1}{2}}\overline{X}')^{\alpha'}(2^{-3j}h^{\frac{1}{2}}\overline{X}'')^{\alpha''}\rho_{j,\alpha}(x, 2^{2j}y_0, 2^j h^{-\frac{1}{2}}y', 2^{3j}h^{-\frac{1}{2}}y''),\end{aligned}$$

where the $\rho_{j,\alpha}$ may depend on h, but with uniform Schwartz bounds over $0 \le h \le 1$ and $j \in \mathbb{N}$. Expressing the action of $\overline{X}$ in terms of $\tilde{x}$, this leads to the expansion

$$K_j(x, \tilde{x}) = \sum_{|\alpha|=2\ell} \sum_{\beta\le\alpha} 2^{-j\,\text{order}(\alpha)}(\overline{X}_0)^{\beta_0}(h^{\frac{1}{2}}\overline{X}')^{\beta'}(h^{\frac{1}{2}}\overline{X}'')^{\beta''} K_{j,\alpha,\beta}(x, \tilde{x})$$

for kernels $K_{j,\alpha,\beta}$ satisfying (5-2) with C_N depending on ℓ but uniform over j, α, β. Here we can take $\overline{X}_j$ to be the transpose of X_j with respect to $dm(\tilde{x})$, since that differs from the transpose with respect to dy by a smooth function.

Theorem 17. *If $a \in S^0(m)$, then $a_h(x, hD)$ is a bounded linear operator on $L^2(U)$, with operator norm depending on only a finite number of seminorm bounds for $a(x, \xi)$. In particular, the operator norm is uniformly bounded over $0 < h \le 1$.*

Proof. We decompose $a_h(x, hD) = \sum_{j=0}^\infty a_{j,h}(x, hD)$. Using (5-1) and (5-3) it is easily verified that the kernel $K_j(x, \tilde{x})$ of $a_{j,h}(x, hD)$ satisfies the Schur test,

$$\sup_x \int K_j(x, \tilde{x})\, dm(\tilde{x}) \le C, \quad \sup_{\tilde{x}} \int K_j(x, \tilde{x})\, dm(x) \le C.$$

We deduce L^2 boundedness from the Cotlar–Stein lemma (see [Knapp and Stein 1971; Stein 1993]), by showing that, for any $N \in \mathbb{N}$,

$$\|a_{i,h}(x,hD)^* a_{j,h}(x,hD)\|_{L^2\to L^2} + \|a_{i,h}(x,hD) a_{j,h}(x,hD)^*\|_{L^2\to L^2} \le C2^{-N|i-j|} \tag{5-5}$$

for a constant C uniform over h and j. If $i = j$ this follows from L^2 boundedness of each term, so without loss of generality we consider $j > i \ge 0$, and in particular $j \ge 1$. Given $\ell \in \mathbb{N}$ we then write the integral kernel of $a_{i,h}(x,hD)a_{j,h}(x,hD)^*$ as

$$\begin{aligned}\int K_i(x,w)\overline{K_j(\tilde{x},w)}\,dm(w)&\\ &= \int K_i(x,w)\sum_{|\alpha|=2\ell}\sum_{\beta\le\alpha} 2^{-j\,\mathrm{order}(\alpha)}(\overline{X}_0)^{\beta_0}(h^{\frac12}\overline{X}')^{\beta'}(h^{\frac12}\overline{X}'')^{\beta''}\,\overline{K_{j,\alpha,\beta}(x,w)}\,dm(w)\\ &= \sum_{|\alpha|=2\ell}\sum_{\beta\le\alpha} 2^{i\,\mathrm{order}(\beta)-j\,\mathrm{order}(\alpha)}\int K_{i,\beta}(x,w)\overline{K_{j,\alpha,\beta}(x,w)}\,dm(w),\end{aligned}$$

where $K_{i,\beta}(x,w) = (2^{-2i}X_0)^{\beta_0}(2^{-i}h^{1/2}X')^{\beta'}(2^{-3i}h^{1/2}X'')^{\beta''}K_i(x,w)$, and in all cases X acts on w. Since $i\,\mathrm{order}(\beta) - j\,\mathrm{order}(\alpha) \le 2\ell(i-j)$, by using (5-4) and the Schur test on the composition we obtain the bound (5-5) with $N = 2\ell$ for the term $a_{i,h}a^*_{j,h}$. To handle the term $a^*_{i,h}a_{j,h}$ we use the same argument, together with symmetry of the derivative estimates in x and $\tilde{x}$. □

We note the following result for $a \in S^n(m)$, which holds since $2^{-jn}a_j(x,\eta) \in S^0(m)$,

$$\sup_{j\ge 0} 2^{-jn}\|a_{j,h}(x,hD)f\|_{L^2(U)} \le C\|f\|_{L^2(U)}, \quad a \in S^n(m). \tag{5-6}$$

6. Estimates on $S^*(M)$

Let (M, g) be a compact Riemannian manifold of dimension $d+1$, and $S^*(M) \subset T^*(M)$ its unit cosphere bundle. We consider the Hamiltonian function $\frac12|\zeta|^2_{\mathrm{g}(z)} = \frac12\sum_{i,k=1}^{d+1}\mathrm{g}^{ik}(z)\zeta_i\zeta_k$, and recall that $S^*(M)$ is the level set $|\zeta|_{\mathrm{g}(z)} = 1$. We use $X_0 = H$ to denote the Hamiltonian field for $\frac12|\zeta|^2_{\mathrm{g}(z)}$,

$$X_0 = \sum_{i,k=1}^{d+1}\mathrm{g}^{ik}(z)\zeta_i\partial_{z_k} - \frac12\sum_{i,j,k=1}^{d+1}\partial_{z_j}\mathrm{g}^{ik}(z)\zeta_i\zeta_k\partial_{\zeta_j},$$

which is tangent to $S^*(M)$.

We cover $S^*(M)$ by a finite collection of open coordinate charts as follows. Let $\{V_\alpha\}$ form a finite covering of M by coordinate charts, over which we can identity $T^*(M)$ with $V_\alpha \times \mathbb{R}^{d+1}$ and $S^*(M)$ with $V_\alpha \times \mathbb{S}^d$. We cover $\mathbb{S}^d$ by two coordinate charts $W^\pm$ over each of which there is a section of the frame bundle. We thus obtain a cover of $S^*(M)$ by open charts $\{V_\alpha \times W^\pm\}$, which by counting each V_α twice we can label as U_α, such that on U_α there is an orthonormal collection $\{X_j\}_{j=1}^d$ of vertical vector fields that span the tangent space to $S^*_z(M)$ over each $z \in V_\alpha$. The collection $\{X_j\}_{j=1}^d$ is involutive, since it spans the vertical vector fields on U_α.

There is a natural isometric identification $T_\zeta(T_z^*(M)) \sim T_z(M)$, which identifies $\{X_j|_{(z,\zeta)}\}_{j=1}^d$ with an orthonormal collection of vectors $\{\widetilde{X}_j\}_{j=1}^d \subset T_z(M)$, which are also orthogonal to $\pi_*(X_0|_{(z,\zeta)})$. We let $X_{j+d}|_{(z,\zeta)}$ be $-\frac{1}{2}$ times the horizontal lift of $\widetilde{X}_j$. We observe that

$$\pi_*[X_j, X_0] = \sum_{i,k=1}^{d+1} \mathrm{g}^{ik}(z) X_j(\zeta_i) \partial_{z_k} = \widetilde{X}_j,$$

so that $[X_0, X_j] - 2X_{j+d} \in \operatorname{span}\{X_j\}_{j=1}^d$. Thus the assumptions of the Introduction are satisfied for the collection $\{X_j\}_{j=0}^{2d}$.

Let $\Delta_{\mathbb{S}}$ be the induced nonnegative Laplacian acting on the fibers $S_z^*(M)$ of the bundle, and let Δ be the nonnegative Laplacian on $S^*(M)$. See for example [Drouot 2017, Section 2.1] for details, where it is shown that Δ and $\Delta_{\mathbb{S}}$ commute. One verifies that, over each U_α, one has

$$\Delta_{\mathbb{S}} = -\sum_{j=1}^d X_j^2 + \sum_{j=1}^d c_j(z,\zeta) X_j.$$

We now use $x \in \mathbb{R}^{2d+1}$ to denote the variables on U_α, and define

$$P_h = H + h\Delta_{\mathbb{S}} = X_0 - \sum_{j=1}^d hX_j^2 + \sum_{j=1}^d c_j(x) hX_j.$$

Thus on each U_α, the operator P_h differs from the sum of squares considered previously by an operator in $h^{1/2}\Psi_h^1(m)$, and the pseudodifferential calculus shows that, given $\chi_\alpha \in C_c^\infty(U_\alpha)$, there exists a symbol $q_\alpha \in S^{-2}(m)$, the quantization of which depends on χ_α through the choice of χ_0 in (4-1), so that on U_α we have

$$q_{\alpha,h}(x, hD) \circ P_h u = \chi_\alpha(x) u + R_\alpha u, \qquad R_\alpha \in \Psi_h^{-\infty}(m).$$

Note that both $q_{\alpha,h}(x, hD)$ and R_α are properly supported in U_α. We now take a partition of unity χ_α subordinate to the cover U_α, and define

$$Q_h v = \sum_\alpha q_{\alpha,h}(x, hD) v, \qquad Rv = \sum_\alpha R_\alpha v.$$

Then $Q_h \circ P_h = I + R$, and for all N_1, N_2 we have

$$\|(h\Delta)^{N_1} R (h\Delta)^{N_2} u\|_{L^2(S^*(M))} \le C_{N_1,N_2} \|u\|_{L^2(S^*(M))}. \tag{6-1}$$

This follows from Theorems 11 and 17 and the fact that $h\Delta \in \Psi_h^6(U_\alpha)$ for each α, which follows from (4-4).

More generally, we define $\Psi_h^\sigma(m)$ on $S^*(M)$ as sums $\sum_\alpha a_{\alpha,h}(x, hD)$ with $a_\alpha \in S^\sigma(m)$ on U_α. The function χ_0 in the quantization (4-1) depends on the x-support of $a_\alpha(x,\eta)$, which is always assumed to be a compact subset of U_α.

The semiclassical Sobolev spaces are defined on $S^*(M)$ using the spectral decomposition of Δ, with norm

$$\|f\|_{H_h^\sigma} = \|(1 + h^2\Delta)^{\sigma/2} f\|_{L^2}.$$

We will consider cutoffs $\rho(s)$ satisfying, for some $c' > c > 0$,

$$\rho(s) \in C^\infty(\mathbb{R}), \qquad \rho(s) = 0 \quad \text{if } s \le c, \qquad \rho(s) = 1 \quad \text{if } s \ge c'. \tag{6-2}$$

The operator $\rho(h^2\Delta)$ is then defined as a spectral multiplier. We observe the following simple result for $R \in \Psi_h^{-\infty}(m)$ on $S^*(M)$. For all N and σ we have

$$\|\rho(h^2\Delta)Ru\|_{H_h^\sigma} + \|R\rho(h^2\Delta)u\|_{H_h^\sigma} \le C_{N,\sigma}h^N\|u\|_{L^2}. \tag{6-3}$$

This follows by writing $\rho(h^2\Delta)(1+h^2\Delta)^\sigma = f(h^2\Delta)\circ(h^2\Delta)^N$, where the function $f(s)$ is a bounded function provided $N > \sigma$, and using (6-1).

Theorem 18. *Suppose that $\sigma \le 0$, that $A_h \in \Psi_h^\sigma(m)$, and that ρ satisfies (6-2). Then*

$$\|\rho(h^2\Delta)A_hu\|_{H_h^{-\sigma/3}} + \|A_h\rho(h^2\Delta)u\|_{H_h^{-\sigma/3}} \le Ch^{-\sigma/6}\|u\|_{L^2}.$$

Proof. Choose k so $6k+\sigma > 0$. For each $h \in (0, 1]$, we show that $A_h = A_{0,h} + A_{1,h}$, where

$$\|(h^2\Delta)^kA_{0,h}u\|_{L^2} + \|A_{0,h}(h^2\Delta)^ku\|_{L^2} + \|A_{1,h}u\|_{H_h^{-\sigma/3}} \le Ch^{-\sigma/6}\|u\|_{L^2}.$$

The result then follows since $\rho(s) \le \min(s^k, 1)$. Using the Littlewood–Paley decomposition as in the proof of Theorem 17, applied to each a_α in the sum defining a, we let

$$A_{0,h} = \sum_{2^j \le h^{-1/6}} a_{j,h}(x, hD), \quad A_{1,h} = \sum_{2^j > h^{-1/6}} a_{j,h}(x, hD).$$

Recalling the form (5-4), we see that applying $h^2\Delta$ to $a_{j,h}(x, hD)$ is equivalent to multiplying it by at most $2^{6j}h$. As in the proof of (5-5) we conclude that

$$\|(1+h^2\Delta)^ka_{j,h}(x, hD)a_{i,h}(x, hD)^*(1+h^2\Delta)^k\|_{L^2\to L^2} \le (1+2^{6i}h)^k(1+2^{6j}h)^k2^{\sigma(i+j)-|i-j|}.$$

For $2^j, 2^i \ge h^{-1/6}$, we interpolate with the L^2 bounds (5-6) to obtain

$$\|(1+h^2\Delta)^{-\sigma/6}a_{j,h}(x, hD)a_{i,h}(x, hD)^*(1+h^2\Delta)^{-\sigma/6}\|_{L^2\to L^2} \le Ch^{-\sigma/3}2^{-|i-j|}.$$

This estimate also holds for the transposed operators. The Cotlar–Stein lemma then implies the bounds for $A_{1,h}$.

Similarly, we have

$$\|(h^2\Delta)^ka_{j,h}(x, hD)\|_{L^2\to L^2} + \|a_{j,h}(x, hD)(h^2\Delta)^k\|_{L^2\to L^2} \le C(2^{6j}h)^k2^{\sigma j},$$

which we may sum over $2^j \le h^{-1/6}$ to conclude the bounds involving $A_{0,h}$. □

Corollary 19. *Suppose that $\sigma \le 0$ and $A_h \in \Psi_h^\sigma(m)$. Then*

$$\|(1+h\Delta)^{-\sigma/6}A_hu\|_{L^2} \le C\|u\|_{L^2}.$$

Proof. As in the proof of Theorem 18 we observe that, for $k = 0, 1, 2, \dots,$

$$\|(1+h\Delta)^ka_{j,h}(x, hD)a_{i,h}(x, hD)^*(1+h\Delta)^k\|_{L^2\to L^2} \le 2^{6k(i+j)}2^{\sigma(i+j)-|i-j|}.$$

We interpolate between $k = 0$ and any $k > -\sigma/6$ to obtain

$$\|(1+h\Delta)^{-\sigma/6}a_{j,h}(x, hD)a_{i,h}(x, hD)^*(1+h\Delta)^{-\sigma/6}\|_{L^2\to L^2} \le C2^{-|i-j|}.$$

This estimate also holds for the transposed operators. The Cotlar–Stein lemma then implies the result. □

Theorem 20. *The following bound holds for $h \in (0,1]$ and all $N \in \mathbb{N}$:*

$$\|Hu\|_{L^2} + h\|\Delta_{\mathbb{S}} u\|_{L^2} + \|(1+h\Delta)^{\frac{1}{3}} u\|_{L^2} \le C\|P_h u\|_{L^2} + C_N \|(1+h\Delta)^{-N} u\|_{L^2}.$$

Proof. Write $u = Q_h P_h u + Ru$, where $Q_h \in \Psi_h^{-2}(m)$, and note that HQ_h, $h\Delta_{\mathbb{S}} Q_h \in \Psi_h^0(m)$ by Remark 15. Also, for all N we have $HR(1+h\Delta)^N$, $h\Delta_{\mathbb{S}} R(1+h\Delta)^N \in \Psi_h^0(m)$; hence

$$\|HRu\|_{L^2} + h\|\Delta_{\mathbb{S}} Ru\|_{L^2} \le C_N \|(1+h\Delta)^{-N} u\|_{L^2}.$$

Since Q_h, $R(1+h\Delta)^N \in \Psi_h^{-2}(m)$, the result then follows by Corollary 19. □

Theorem 21. *Suppose that ρ_1 and ρ_2 satisfy* (6-2), *and $\rho_2 = 1$ on a neighborhood of* $\operatorname{supp}(\rho_1)$. *Given $\lambda_0 > 0$, the following holds for all N, and all $|\lambda| \le \lambda_0$ and $h \in (0,1]$:*

$$h^{-\frac{1}{3}} \|\rho_1(h^2\Delta)u\|_{H_h^{2/3}} + h^{\frac{1}{3}} \sum_{j=1}^{d} \|X_j \rho_1(h^2\Delta)u\|_{H_h^{1/3}} + \|X_0\rho_1(h^2\Delta)u\|_{L^2} + \|h\Delta_{\mathbb{S}}\rho_1(h^2\Delta)u\|_{L^2} \\ \le C_{N,\lambda_0}(\|\rho_2(h^2\Delta)(P_h-\lambda)u\|_{L^2} + h^N\|u\|_{L^2}).$$

Proof. We follow the scheme of the proof of Theorem 2 of [Drouot 2017], using the parametrix Q_h of P_h to replace the positive commutator arguments. Write

$$\rho_1(h^2\Delta)u = Q_h\rho_1(h^2\Delta)(P_h-\lambda)u + Q_h[P_h, \rho_1(h^2\Delta)]u + \lambda Q_h \rho_1(h^2\Delta)u + R\rho_1(h^2\Delta)u.$$

To handle the commutator term, we use that $[\Delta_{\mathbb{S}}, \rho_1(h^2\Delta)] = 0$; hence $[P_h, \rho_1(h^2\Delta)] = [X_0, \rho_1(h^2\Delta)]$. Now let $\tilde\rho_1(s)$ be any function satisfying (6-2) which equals 1 on a neighborhood of $\operatorname{supp}(\rho_1)$. Then following [Drouot 2017], we use that the essential support of $[X_0, \rho_1(h^2\Delta)]$ is contained within the elliptic set of $\tilde\rho(h^2\Delta)$, and we can thus bound

$$\|[P_h, \rho_1(h^2\Delta)]u\|_{L^2} \le C\|\tilde\rho_1(h^2\Delta)]u\|_{L^2} + C_N h^N \|u\|_{L^2}.$$

Applying Theorem 18 and (6-3) we obtain

$$h^{-\frac{1}{3}} \|\rho_1(h^2\Delta)u\|_{H_h^{2/3}} + h^{-\frac{1}{6}} \sum_{j=1}^{d} \|h^{\frac{1}{2}} X_j \rho_1(h^2\Delta)u\|_{H_h^{1/3}} + \|X_0\rho_1(h^2\Delta)u\|_{L^2} + \|h\Delta_{\mathbb{S}}\rho_1(h^2\Delta)u\|_{L^2} \\ \le C\big(\|\rho_1(h^2\Delta)(P_h-\lambda)u\|_{L^2} + \|\tilde\rho_1(h^2\Delta)u\|_{L^2} + (1+|\lambda|)\|\rho_1(h^2\Delta)u\|_{L^2}\big) + C_N h^N \|u\|_{L^2}.$$

For h bounded away from 0 we can absorb the term $(1+|\lambda|)\|\rho_1(h^2\Delta)u\|_{L^2}$ into $C_N h^N\|u\|_{L^2}$, and for h small we can subtract it from both sides.

From this we deduce the following bound for any such $\tilde\rho_1$:

$$\|\rho_1(h^2\Delta)u\|_{L^2} \le C_{N,\lambda_0}\big(h^{\frac{1}{3}}\|\rho_2(h^2\Delta)(P_h-\lambda)u\|_{L^2} + h^{\frac{1}{3}}\|\tilde\rho_1(h^2\Delta)u\|_{L^2} + h^N\|u\|_{L^2}\big).$$

We now choose a sequence of cutoffs $\tilde\rho_j$ for $1 \le j \le 3N$, satisfying (6-2), such that for all j we have $\tilde\rho_{j+1} = 1$ on a neighborhood of $\operatorname{supp}(\tilde\rho_j)$, and $\rho_2 = 1$ on a neighborhood of $\operatorname{supp}(\tilde\rho_j)$. Then replacing ρ_1 by $\tilde\rho_j$, the preceding estimate shows that

$$\|\tilde\rho_j(h^2\Delta)u\|_{L^2} \le C_{N,\lambda_0}\big(h^{\frac{1}{3}}\|\rho_2(h^2\Delta)(P_h-\lambda)u\|_{L^2} + h^{\frac{1}{3}}\|\tilde\rho_{j+1}(h^2\Delta)u\|_{L^2} + h^N\|u\|_{L^2}\big).$$

We conclude by iteration that

$$\begin{aligned}\|\tilde\rho_1(h^2\Delta)u\|_{L^2} &\le C_{N,\lambda_0}\big(h^{\frac13}\|\rho_2(h^2\Delta)(P_h-\lambda)u\|_{L^2}+h^N\|\rho_2(h^2\Delta)u\|_{L^2}+h^N\|u\|_{L^2}\big)\\ &\le C_{N,\lambda_0}\big(h^{\frac13}\|\rho_2(h^2\Delta)(P_h-\lambda)u\|_{L^2}+h^N\|u\|_{L^2}\big).\end{aligned}$$

Together with the above this yields the statement of the theorem. □

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References

[Angst, Bailleul, and Tardif 2015] J. Angst, I. Bailleul, and C. Tardif, "Kinetic Brownian motion on Riemannian manifolds", *Electron. J. Probab.* **20** (2015), art. id. 110, 40 pp. MR Zbl

[Drouot 2017] A. Drouot, "Stochastic stability of Pollicott–Ruelle resonances", *Comm. Math. Phys.* **356**:2 (2017), 357–396. MR Zbl

[Dyatlov and Zworski 2015] S. Dyatlov and M. Zworski, "Stochastic stability of Pollicott–Ruelle resonances", *Nonlinearity* **28**:10 (2015), 3511–3533. MR Zbl

[Folland 1975] G. B. Folland, "Subelliptic estimates and function spaces on nilpotent Lie groups", *Ark. Mat.* **13**:2 (1975), 161–207. MR Zbl

[Franchi and Le Jan 2007] J. Franchi and Y. Le Jan, "Relativistic diffusions and Schwarzschild geometry", *Comm. Pure Appl. Math.* **60**:2 (2007), 187–251. MR Zbl

[Grothaus and Stilgenbauer 2013] M. Grothaus and P. Stilgenbauer, "Geometric Langevin equations on submanifolds and applications to the stochastic melt-spinning process of nonwovens and biology", *Stoch. Dyn.* **13**:4 (2013), art. id. 1350001, 34 pp. MR Zbl

[Hörmander 1967] L. Hörmander, "Hypoelliptic second order differential equations", *Acta Math.* **119** (1967), 147–171. MR Zbl

[Knapp and Stein 1971] A. W. Knapp and E. M. Stein, "Intertwining operators for semisimple groups", *Ann. of Math.* (2) **93** (1971), 489–578. MR Zbl

[Li 2016] X.-M. Li, "Random perturbation to the geodesic equation", *Ann. Probab.* **44**:1 (2016), 544–566. MR Zbl

[Rothschild and Stein 1976] L. P. Rothschild and E. M. Stein, "Hypoelliptic differential operators and nilpotent groups", *Acta Math.* **137**:3-4 (1976), 247–320. MR Zbl

[Smith 1994] H. F. Smith, "A calculus for three-dimensional CR manifolds of finite type", *J. Funct. Anal.* **120**:1 (1994), 135–162. MR Zbl

[Stein 1993] E. M. Stein, *Harmonic analysis*: *real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series **43**, Princeton University Press, 1993. MR Zbl

[Zworski 2012] M. Zworski, *Semiclassical analysis*, Graduate Studies in Mathematics **138**, American Mathematical Society, Providence, RI, 2012. MR Zbl

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