Analysis & PDE

Volume 16 No. 1 2023

Ljudevit Palle

Strichartz estimates for mixed homogeneous surfaces in three dimensions

msp

ANALYSIS AND PDE
Vol. 16 (2023), No. 1, pp. 173–232
DOI: 10.2140/apde.2023.16.173

STRICHARTZ ESTIMATES FOR MIXED HOMOGENEOUS SURFACES IN THREE DIMENSIONS

LJUDEVIT PALLE

We obtain sharp mixed-norm Strichartz estimates associated to mixed homogeneous surfaces in $\mathbb{R}^3$. Cases with and without a damping factor are both considered. In the case when a damping factor is considered our results yield a wide generalization of a result of Carbery, Kenig, and Ziesler for homogeneous polynomial surfaces in $\mathbb{R}^3$. The approach we use is to first classify all possible singularities locally, after which one can tackle the problem by appropriately modifying the methods from a paper of Ginibre and Velo, and by using the recently developed methods by Ikromov and Müller.

1. Introduction

Let us fix a pair $\alpha = (\alpha_1, \alpha_2) \in (0, \infty)^2$, define $|\alpha| := \alpha_1 + \alpha_2$, and introduce its associated α-mixed homogeneous dilations in $\mathbb{R}^2$ by

$$\delta_t(x_1, x_2) = (t^{\alpha_1} x_1, t^{\alpha_2} x_2), \quad t > 0.$$

The main goal of this article is to study Strichartz estimates for a fixed mixed homogeneous surface S, i.e., a surface given as the graph of a fixed smooth function $\phi : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ which is α-mixed homogeneous of degree ρ:

$$\phi \circ \delta_t(x_1, x_2) = t^{\rho} \phi(x_1, x_2), \quad t > 0. \tag{1-1}$$

We may and shall assume without loss of generality that $\rho \in \{-1, 0, 1\}$. Both α and ρ shall be fixed throughout the article. Note that when $\rho = -1$ the function ϕ has a singularity at the origin.

The author was supported by Deutsche Forschungsgemeinschaft, project number 237750060, *Fragen der Harmonischen Analysis im Zusammenhang mit Hyperflächen.*
MSC2020: 35B45, 42B15.
Keywords: Strichartz estimates, Fourier restriction, oscillatory integrals, classification of singularities, mixed homogeneous, mixed-norm.

As is well known, Strichartz estimates are directly related to Fourier restriction estimates and we are in particular interested in the mixed-norm estimate

$$\|\hat{f}\|_{L^2(\mathrm{d}\mu)} \leq C\|f\|_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})}, \quad f \in \mathcal{S}(\mathbb{R}^3), \tag{1-2}$$

where μ is the surface measure

$$\langle \mu, f\rangle = \int_{\mathbb{R}^2\setminus\{0\}} f(x_1, x_2, \phi(x_1, x_2))\, \mathcal{W}(x_1, x_2)\, \mathrm{d}x \tag{1-3}$$

and $p = (p_1, p_3) \in (1,2)^2$. Note that we skip the p_2-exponent which corresponds to the integration in the x_2-variable — here we consider the case $p_1 = p_2$, i.e., we have one exponent $p_1 = p_2$ in the "tangential" direction and another, namely p_3, in the "normal" direction to the surface S at $(0, 0, \phi(0,0))$ (this will be formally true only when ϕ is smooth at the origin).

The weight $\mathcal{W} \geq 0$ is added in order to ensure that the measure has a scaling invariance which will enable us to reduce global estimates to local ones by a Littlewood–Paley argument. We take $\mathcal{W}$ to be α-mixed homogeneous of degree 2ϑ and consider two particular cases. The function $\mathcal{W}$ will be either equal to

$$|x|_\alpha^{2\vartheta} = (|x_1|^{1/\alpha_1} + |x_2|^{1/\alpha_2})^{2\vartheta} \tag{1-4}$$

or equal to the Hessian determinant of ϕ (denoted by $\mathcal{H}_\phi$) raised to the power $|\cdot|^\sigma$, $\sigma \in \big[0, \frac{1}{2}\big)$, i.e.,

$$|\mathcal{H}_\phi(x)|^\sigma = \left|\det\begin{bmatrix} \partial_{x_1}^2\phi & \partial_{x_1}\partial_{x_2}\phi \\ \partial_{x_1}\partial_{x_2}\phi & \partial_{x_2}^2\phi \end{bmatrix}\right|^\sigma. \tag{1-5}$$

The first weight (1-4) is of interest as a type of mixed homogeneous Sobolev weight, while the second one (1-5) was considered originally in [Sjölin 1974] and turns out to be a natural choice when studying Fourier restriction estimates for surfaces with vanishing Gaussian curvature. One can easily show that the Hessian determinant of ϕ is α-mixed homogeneous of degree $2(\rho - |\alpha|)$, and so in the case when $\mathcal{W}$ equals (1-5) the relation between ϑ and σ is $\vartheta = \sigma(\rho - |\alpha|)$. We shall later determine ϑ in Section 2A (and in particular in Proposition 2.1) so that the Fourier restriction estimate for μ is invariant under scaling. This choice depends in general on $p = (p_1, p_3)$.

Oscillatory integrals, Fourier restriction estimates, and other problems related to homogeneous and mixed homogeneous surfaces have been previously studied in works such as [Dendrinos and Zimmermann 2019; Schwend 2020; Greenblatt 2018; Ikromov et al. 2005; Ikromov and Müller 2011; Iosevich and Sawyer 1996; Ferreyra et al. 2004; Ferreyra and Urciuolo 2009; Carbery et al. 2013].

The case of general L^p-L^2 Fourier restriction in $\mathbb{R}^3$ with respect to the Euclidean measure was recently solved in [Ikromov and Müller 2016] for a wide class of smooth surfaces in $\mathbb{R}^3$, including all the analytic ones. Mixed-norm estimates were shown in [Palle 2021] for surfaces given as graphs of functions ϕ in adapted coordinates and also for analytic functions ϕ satisfying $h_{\mathrm{lin}}(\phi) < 2$ (see below for the definition of linear height $h_{\mathrm{lin}}(\phi)$).

In [Carbery et al. 2013] Carbery, Kenig, and Ziesler considered the case with the weight (1-5) for "isotropically" homogeneous (i.e., when $\alpha_1 = \alpha_2$) polynomials ϕ. Since the weight (1-5) has roots at the degenerate points, the estimate (1-2) holds for a wider range of exponents compared to the case when the weight (1-4) is used. As already mentioned, the use of this so-called mitigating or damping factor goes back to [Sjölin 1974] (see also [Cowling et al. 1990; Drury 1990; Kenig et al. 1991]). Its naturalness stems from the fact that it is equiaffine invariant as is the Fourier transformation. In fact, the mitigating factor can be expressed in a parametrization-independent way through the use of so-called affine fundamental forms (see, e.g., [Su 1983; Nomizu and Sasaki 1994]). When one uses the above damping factor (1-5) one can even obtain estimates for certain classes of flat surfaces [Carbery and Ziesler 2002; Abi-Khuzam and Shayya 2006; Carbery et al. 2007]. On the other hand, weak-type $L^{4/3}$-$L^{4(n-1)/(n+1)}$ estimates were obtained in [Oberlin 2012] for a wide class of surfaces having a bounded generic multiplicity (see also [Oberlin 2004]). In the three-dimensional case ($n = 3$) they correspond to precisely the Tomas–Stein range, but otherwise are a strict subset of it. Let us also mention a recent result of [Gressman 2016] where he obtained decay estimates for damping oscillatory integrals for a certain class of singularities.

In this article we shall first classify the possible local singularities for mixed homogeneous surfaces (see Proposition 1.4 below) and then either apply the Fourier restriction estimates obtained in [Ikromov and Müller 2016; Palle 2021] or use the techniques from these articles, and also from [Ginibre and Velo 1992] (see also [Keel and Tao 1998]), to obtain sharp estimates. In particular, we obtain a wide generalization of the Fourier restriction estimate in [Carbery et al. 2013] with methods which are more elementary and avoiding any use of results from algebraic topology or algebraic geometry. Namely, in [Carbery et al. 2013] a result of [Milnor 1964] on Betti numbers is used in order to control the number of connected components of a set given by polynomial inequalities.

In order to state the main results of this paper (namely, Theorem 1.1, Theorem 1.2, Proposition 1.4, and Corollary 1.5) we first recall certain concepts and introduce a few conditions. Recall that the *Taylor support* of a smooth function $\varphi : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ at $P \in \Omega$ is defined as the set $\mathcal{T}(\varphi, P) := \{\tau \in \mathbb{N}_0^2 : \partial^\tau \varphi(P) \neq 0\}$. We call φ a function of *finite type* at P if its Taylor support at P is nonempty. If φ is of finite type at P, then one defines its *Newton polyhedron* $\mathcal{N}(\varphi, P)$ at P as the convex hull of the union of sets $\{(t_1, t_2) \in \mathbb{R}^2 : t_1 \geq \tau_1,\ t_2 \geq \tau_2\}$, where $\tau = (\tau_1, \tau_2)$ goes over the Taylor support of φ at P.

We can now recall the definitions of some very important quantities from the theory of oscillatory integrals which go back to V. I. Arnold and A. N. Varchenko (see, e.g., [Varchenko 1976]). Let us assume for a function φ of finite type at P that $\varphi(P) = 0$ and $\nabla\varphi(P) = 0$. If this is not the case we simply subtract the constant and linear terms of the Taylor series of φ at P. The *Newton distance* $d(\varphi, P)$ of φ at P is then defined as the minimum of the set $\{t \in \mathbb{R} : (t, t) \in \mathcal{N}(\varphi, P)\}$. The face (i.e., a vertex or an edge) where the line $\{(t, t) : t \in \mathbb{R}\}$ intersects the Newton polyhedron $\mathcal{N}(\varphi, P)$ is called the *principal face* and it is denoted by $\pi(\varphi, P)$. Note that $d(\varphi, P) \geq 1$. The *Newton height* $h(\varphi, P)$ of φ at P is defined as the supremum of the set $\{d(\varphi \circ \Phi, P) : \Phi \text{ is a local diffeomorphism at } P\}$. We define the *linear height* $h_{\text{lin}}(\varphi, P)$ analogously — the only difference is that one considers linear coordinate changes centered at P instead of local diffeomorphisms. Note that $h(\varphi, P) \geq h_{\text{lin}}(\varphi, P) \geq d(\varphi, P)$. One says that φ is *adapted* at P if $d(\varphi, P) = h(\varphi, P)$ and that it is *linearly adapted* at P if $d(\varphi, P) = h_{\text{lin}}(\varphi, P)$. Similarly, one says that φ is *adapted* in the Φ coordinates if $d(\varphi \circ \Phi, P) = h(\varphi, P)$ and one defines what it means to be *linearly adapted* in the Φ coordinates analogously. The existence of a coordinate system in which an analytic function is adapted was shown in [Varchenko 1976]. This was generalized to smooth functions of finite type in [Ikromov and Müller 2011]. For the existence of a linear coordinate change in which a function is linearly adapted see [Ikromov and Müller 2016].

Finally, for a function φ of finite type at P satisfying $\varphi(P) = 0$ and $\nabla\varphi(P) = 0$ we recall the definition of *Varchenko's exponent*, denoted by $\nu(\varphi, P)$. It is defined to be 1 if $h(\varphi, P) \geq 2$ and if there exists a coordinate change Φ in which ϕ is adapted and so that the bisectrix $\{(t,t) : t \in \mathbb{R}\}$ intersects the Newton polyhedron $\mathcal{N}(\varphi \circ \Phi, P)$ at a vertex. Otherwise one defines $\nu(\varphi, P) := 0$.

The relation to oscillatory integrals is as follows. If one is given a smooth amplitude a localized at P, then the decay rate of the oscillatory integral $\int a(x)e^{i\lambda\varphi(x)}\,\mathrm{d}x$ is $\lambda^{-1/h(\varphi,P)}(\log\lambda)^{\nu(\varphi,P)}$ for large λ. This also holds when one considers small linear perturbations of φ.

Let us mention that one often translates P to 0, in which case one uses the notation $\mathcal{T}(\varphi)$, $d(\varphi)$, $\nu(\varphi)$, etc., and it is implicitly understood that everything is considered at the origin.

In this article we shall consider either of the following two conditions on our fixed α-mixed homogeneous function ϕ:

(H1) At any given point $(x_1, x_2) \neq (0,0)$ where the Hessian determinant of ϕ vanishes at least one of the mappings $t \mapsto \partial_1^2\phi(t, x_2)$ or $t \mapsto \partial_2^2\phi(x_1, t)$ is of finite type at $t = x_1$ (resp. $t = x_2$), i.e., at least one of them or their derivatives is nonzero when evaluated at the respective points.

(H2) The Hessian determinant $\mathcal{H}_\phi$ is not flat at any point $x \neq 0$.

It actually suffices to check the conditions only at points (x_1, x_2) in, say, a unit circle by homogeneity. Furthermore, we remark that the condition (H2) is stronger than the condition (H1) (this follows from the calculations in Section 3B below).

Let us now introduce a further condition and two new quantities. For a point $v \in \mathbb{R}^2 \setminus \{0\}$ let us define the function

$$\phi_v(x) := \phi(x+v) - \phi(v) - x \cdot \nabla\phi(v).$$

Then we shall often consider whether the following condition is satisfied at v:

(LA) There is a linear coordinate change which is adapted to ϕ_v at the origin.

Compare with the negation of this condition in [Ikromov and Müller 2016, Section 1.2]. Note that the (LA) condition is not the same as linear adaptedness of ϕ_v at 0.

As mentioned, the linear height of ϕ_v and the Newton height of ϕ_v are respectively denoted by $h_{\mathrm{lin}}(\phi, v)$ and $h(\phi, v)$. We define the *global linear height* $h_{\mathrm{lin}}(\phi)$ and the *global Newton height* $h(\phi)$ by the respective expressions

$$h_{\mathrm{lin}}(\phi) = \sup_{v \in \mathbb{S}^1} h_{\mathrm{lin}}(\phi, v), \quad h(\phi) = \sup_{v \in \mathbb{S}^1} h(\phi, v). \tag{1-6}$$

It will be clear from Section 3 that $h_{\mathrm{lin}}(\phi, v)$ and $h(\phi, v)$ do not change along the *homogeneity curve through* v defined as the curve

$$t \mapsto (t^{\alpha_1}v_1, t^{\alpha_2}v_2), \quad t > 0,$$

and therefore in the above definitions of global linear height and global Newton height one could have taken the supremum over the set $\mathbb{R}^2 \setminus \{0\}$ too.

Theorem 1.1. *Let ϕ be mixed homogeneous satisfying condition* (H2). *Let μ be the measure defined as in* (1-3) *with $\mathcal{W}(x) = |\mathcal{H}_\phi(x)|^\sigma$ for some fixed $\sigma \ge 0$. If $\sigma \in \left[0, \frac{1}{3}\right]$, then the Fourier restriction estimate* (1-2) *holds true for*

$$\left(\frac{1}{p_1'}, \frac{1}{p_3'}\right) = \left(\frac{1}{2} - \sigma, \sigma\right).$$

If (LA) *is satisfied at all points $v \ne 0$, then the estimate holds true even if $\sigma \in \left[0, \frac{1}{2}\right)$. In particular, if $\alpha_1 = \alpha_2$, then* (LA) *is automatically satisfied at all points $v \ne 0$ and the estimate holds true for any $\sigma \in \left[0, \frac{1}{2}\right)$.*

Several comments are in order. Firstly, precise conditions for when the (LA) condition is satisfied at $v \ne 0$ can be checked by using the normal-form tables in Section 3 (note that in the Proposition 1.4 below, where the normal forms are listed, only the normal form (vi) is not in adapted coordinates). That one is restricted to $0 \le \sigma \le \frac{1}{3}$ in the case when (LA) is not satisfied is a consequence of a Knapp-type example, as we shall show in Section 4F1. That the result in the above theorem is sharp is well known — as soon as one knows that the Hessian determinant of ϕ does not vanish identically we can apply the classical Knapp example to a point where the Hessian does not vanish which then yields the necessary condition

$$\frac{1}{p_1'} + \frac{1}{p_3'} \le \frac{1}{2}.$$

Secondly, in the case when $\rho = 1 = |\alpha|$, one can extend the above estimate to the range where

$$\frac{1}{p_1'} + \frac{1}{p_3'} = \frac{1}{2}, \qquad \frac{1}{p_3'} \le \sigma.$$

The reason for this is that $\rho = 1 = |\alpha|$ implies that the weight $\mathcal{W} = |\mathcal{H}_\phi|^\sigma$ (and the Hessian determinant) are α-mixed homogeneous of degree 0, and hence bounded on $\mathbb{R}^2$, and so the estimate for $(p_1, p_3) = (2, 1)$ follows trivially by Plancherel.

Finally, let us mention that the most interesting part of the proof of the above theorem is the proof of Fourier restriction for the normal form (v) from Proposition 1.4, which is to be found in Section 4E. There we need to estimate the Fourier transform of a certain measure, and for this we perform a natural decomposition of this measure. What is remarkable is that at the critical frequencies one initially has an infinite number of pieces which are not summable absolutely, but, after a delicate analysis, only $\mathcal{O}(1)$ decomposition pieces turn out to have a "bad" estimate. Interestingly, a similar thing happens in the much easier case of normal form (iv).

In the case of the other weight (which has no roots away from the origin) we have:

Theorem 1.2. *Let ϕ be mixed homogeneous satisfying condition* (H1). *Let μ be the measure defined as in* (1-3) *with $\mathcal{W}(x) = |x|_\alpha^{2\vartheta}$. If the exponents $(p_1, p_3) \in (1, 2)^2$ and $\vartheta \in \mathbb{R}$ satisfy* (see Figure 1)

$$\frac{1}{p_1'} + \frac{h_{\text{lin}}(\phi)}{p_3'} \le \frac{1}{2}, \qquad \frac{1}{p_3'} \le \frac{1}{2h(\phi)}, \qquad \vartheta = \frac{|\alpha|}{p_1'} + \frac{\rho}{p_3'} - \frac{|\alpha|}{2},$$

then the Fourier restriction estimate (1-2) *holds true.*

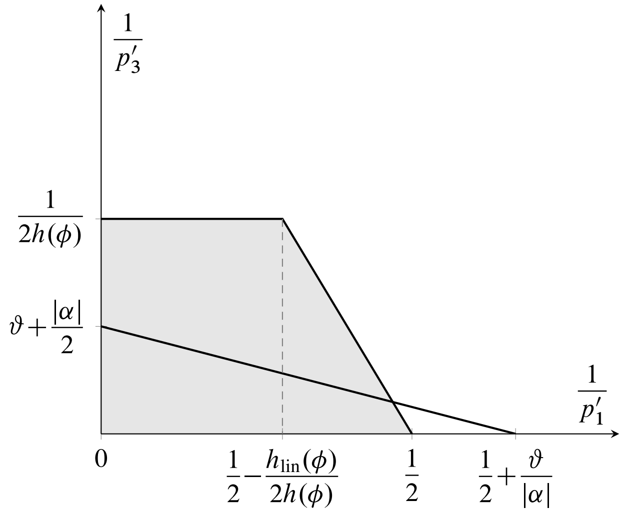

Figure 1. The Riesz diagram for the range of exponents given in Theorem 1.2. The line given by ϑ is drawn for the case when $\rho = 1$, $\vartheta > 0$, and both $|\alpha|$ and ϑ are small.

We remark that the quantity ϑ in the above theorem is allowed to be negative. This theorem is sharp since the corresponding local estimates are sharp — this was shown in [Palle 2021]. We discuss this in more detail at the beginning of Section 5.

As a special case of Theorem 1.1 we obtain:

Corollary 1.3. *Let ϕ be any mixed homogeneous polynomial in $\mathbb{R}^2$ and let μ be the measure defined as in* (1-3) *with $\mathcal{W}(x) = |\mathcal{H}_\phi(x)|^{1/4}$. Then the Fourier restriction estimate* (1-2) *holds true for $p_1' = p_3' = 4$.*

In the case of the above corollary we note that the Hessian determinant can either vanish identically, or it does not vanish to infinite order anywhere, since it is necessarily a nonzero mixed homogeneous polynomial. But the case when the Hessian determinant vanishes identically is trivial, so we are indeed within the scope of Theorem 1.1.

When one considers "isotropically" homogeneous polynomials (i.e., when $\alpha_1 = \alpha_2$), Corollary 1.3 recovers the main result of [Carbery et al. 2013]. The strategy of proof in that work was to first perform certain decompositions of the surface measure in order to get appropriate control over the size of $\nabla\phi$ and the Hessian determinant $\mathcal{H}_\phi$, after which one applies an L^4 argument, as the $L^{4/3}(\mathbb{R}^3) \to L^2(\mathrm{d}\mu)$ Fourier restriction estimate is equivalent to the $L^2(\mathrm{d}\mu) \to L^4(\mathbb{R}^3)$ extension estimate.

Our proofs of Theorems 1.1 and 1.2 are based on the following intermediary result:

Proposition 1.4. *Let $v \in \mathbb{R}^2 \setminus \{0\}$, let ϕ be as above α-mixed homogeneous of degree ρ, and let us assume that it satisfies condition* (H1) *and that its Hessian determinant vanishes at v. Then after a linear transformation of coordinates the function $\phi_v(x) := \phi(x+v) - \phi(v) - x \cdot \nabla\phi(v)$ and its Hessian determinant $\mathcal{H}_{\phi_v}$ assume precisely one of the normal forms in Table 1. In all the cases the appearing functions are smooth and do not vanish at the origin, i.e., $r(0), r_0(0), r_1(0), r_2(0), q(0), \psi(0) \neq 0$, except for the function φ which is flat at the origin.*

case	normal form	additional conditions
(i)	$\phi_v(x) = x_2^k r(x) + \varphi(x)$, $\mathcal{H}_{\phi_v}(x) = x_2^{\tilde{k}+2k-2} r_0(x)$ or $\mathcal{H}_{\phi_v}$ flat at 0	$k \geq 2,\ \tilde{k} \geq 0$
(ii)	$\phi_v(x) = x_1^2 q(x_1) + x_2^k r(x)$, $\mathcal{H}_{\phi_v}(x) = x_2^{k-2} r_0(x)$	$k \geq 3$
(iii)	$\phi_v(x) = x_1^2 r_1(x) + x_2^k r_2(x)$, $\mathcal{H}_{\phi_v}(x) = x_2^{k-2} r_0(x)$	$k \geq 3$, $\partial_2^j r_1(0) = c(\phi, v) j \partial_2^{j-1} r_1(0)$ for $c(\phi, v) \neq 0,\ j = 1, \dots, k-1$
(iv)	$\phi_v(x) = x_1^2 q(x_1) + (x_2 - x_1^2 \psi(x_1))^k r(x)$, $\mathcal{H}_{\phi_v}(x) = (x_2 - x_1^2 \psi(x_1))^{k-2} r_0(x)$	$k \geq 3$
(v)	$\phi_v(x) = x_1^2 r_1(x) + (x_2 - x_1^2 \psi(x_1))^k r_2(x)$, $\mathcal{H}_{\phi_v}(x) = (x_2 - x_1^2 \psi(x_1))^{k-2} r_0(x)$	$k \geq 3$, $\partial_2^j r_1(0) = c(\phi, v) j \partial_2^{j-1} r_1(0)$ for $c(\phi, v) \neq 0,\ j = 1, \dots, k-1$
(vi)	$\phi_v(x) = (x_2 - x_1^2 \psi(x_1))^k r(x)$, $\mathcal{H}_{\phi_v}(x) = (x_2 - x_1^2 \psi(x_1))^{2k-3} r_0(x)$	$k \geq 2$

Table 1. Normal forms for Proposition 1.4.

In the case of normal form (i) *one additionally knows that if the Hessian determinant* $\mathcal{H}_{\phi_v}$ *is not flat at the origin*, *then* φ *vanishes identically. In particular*, *if condition* (H2) *is satisfied*, *then the function* φ *in case* (i) *always vanishes identically and the Hessian determinant is nowhere flat. In the case when* $\alpha_1 = \alpha_2$ *the functions* ϕ_v *and* $\mathcal{H}_{\phi_v}$ *can only take the forms* (i) *or* (ii). *Finally*, *the root of the function* $x \mapsto x_2 - x_1^2 \psi(x_1)$ *corresponds to the homogeneity curve through* v, *though in the coordinate system in which the normal form is given.*

In cases (i) and (ii) one has further subcases (see Section 3A) of a technical nature, so we left them out of the above proposition. We also note that only in case (vi) the function ϕ_v is not in adapted coordinates (and the adapted coordinates can be achieved only through a nonlinear transformation such as $(x_1, x_2) \mapsto (x_1, x_2 + x_1^2 \psi(x_1))$), but it is linearly adapted.

The idea to apply Fourier restriction estimates to obtain a priori estimate for PDEs goes back to [Strichartz 1977]. In our case one can apply the above results to obtain Strichartz estimates for the nonhomogeneous initial problem

$$\begin{cases} (\partial_t - i\phi(D))u(x,t) = F(x,t), & (x,t) \in \mathbb{R}^2 \times (0,\infty), \\ u(x,0) = G(x), & x \in \mathbb{R}^2, \end{cases}$$

where $F \in \mathcal{S}(\mathbb{R}^3)$, $G \in \mathcal{S}(\mathbb{R}^2)$. Namely, by an application of the Christ–Kiselev lemma [2001] one gets the following result:

Corollary 1.5. *Let ϕ, $\mathcal{W}$, and $(p_1, p_3) \in (1,2)^2$ be either as in Theorem 1.1 or 1.2, and let us furthermore assume that $\rho \in \{0,1\}$. Then for the above nonhomogeneous PDE one has the a priori estimate*

$$\|u\|_{L_t^{p_3'}(L_{(x_1,x_2)}^{p_1'})} \le C_1 \|\mathcal{W}^{-1/2}\mathcal{F}G\|_{L^2(\mathbb{R}^2)} + C_2 \|\mathcal{F}_{(x_1,x_2)}^{-1}(\mathcal{W}^{-1}\mathcal{F}_{(x_1,x_2)}F)\|_{L_t^{p_3}(L_{(x_1,x_2)}^{p_1})},$$

where $\mathcal{F}_{(x_1,x_2)}$ is the partial Fourier transformation in the $x = (x_1, x_2)$-direction.

In the case when $\mathcal{W}$ is the function $|\cdot|_\alpha^{2\vartheta}$ the norms on the right-hand side are a type of homogeneous anisotropic Sobolev norms [Triebel 2006, Chapter 5] (in particular, note that $\|\mathcal{W}^{-1/2}\mathcal{F}G\|_{L^2(\mathbb{R}^2)} = \|\mathcal{F}^{-1}\mathcal{W}^{-1/2}\mathcal{F}G\|_{L^2(\mathbb{R}^2)}$).

Since the procedure of how to obtain the corresponding Strichartz estimate from a Fourier restriction estimate is mostly standard we have deferred the sketch of the proof of Corollary 1.5 to the Appendix.

The article is structured in the following way. In Section 2 we first perform some elementary reductions. Since the proofs of Theorems 1.1 and 1.2 are essentially based on Proposition 1.4, we first prove this proposition (and even obtain slightly more precise results) in Section 3. Subsequently we prove Theorems 1.1 and 1.2 in Sections 4 and 5 respectively. In the Appendix we then give a sketch of the proof of Corollary 1.5.

In this paper we use the symbols $\sim$, $\lesssim$, $\gtrsim$, $\ll$, $\gg$ with the following meanings. If two nonnegative quantities A and B are given, then by $A \ll B$ we mean that there exists a sufficiently small positive constant c such that $A \le cB$, and by $A \lesssim B$ we mean that there exists a (possibly large) positive constant C such that $A \le CB$. The relation $A \sim B$ means that there exist positive constants $C_1 \le C_2$ such that $C_1 A \le B \le C_2 A$ is satisfied. Relations $A \gg B$ and $A \gtrsim B$ are defined analogously. Sometimes the implicit constants c, C, C_1, and C_2 depend on certain parameters p, and in order to emphasize this dependence we shall write for example $\lesssim_p$, $\sim_p$, and so on.

We also use the symbols χ_0, χ_1, r, and q generically in the following way. We require χ_0 to be supported in a neighborhood of the origin and identically equal to 1 near the origin. On the other hand, we require χ_1 to be supported away from the origin and identically equal to 1 on an open neighborhood of $\pm 1 \in \mathbb{R}$. Sometimes, when several χ_0 or χ_1 appear within the same formula, they may designate different functions. The functions r and q (also used with subscripts and tildes) shall be used generically as smooth functions which are nonvanishing at the origin, where the function q shall denote a function of one variable, whereas the function r shall denote a function which may generally depend on two variables. Occasionally both of them can also be flat at the origin, in which case we state this explicitly.

2. Preliminary reductions

2A. ***Rescaling and reduction to local estimates.*** As mentioned, the measure we consider is

$$\langle \mu, f \rangle = \int_{\mathbb{R}^2 \setminus \{0\}} f(x_1, x_2, \phi(x_1, x_2))\, \mathcal{W}(x_1, x_2)\, \mathrm{d}x,$$

where $\mathcal{W}$ is nonnegative, continuous on $\mathbb{R}^2 \setminus \{0\}$, and α-mixed homogeneous of degree 2ϑ. In this subsection we determine the degree of homogeneity 2ϑ so that the global Fourier restriction estimate (1-2) becomes equivalent to the local one. By this we mean the following. Let us take a partition of unity $(\eta_j)_{j\in\mathbb{Z}}$ in $\mathbb{R}^2 \setminus \{0\}$,

$$\sum_{j\in\mathbb{Z}} \eta_j(x) = 1, \quad x \neq 0, \tag{2-1}$$

such that $\eta_j = \eta \circ \delta_{2^{-j}}$ for some $\eta = \eta_0 \in C_c^\infty(\mathbb{R}^2)$ supported away from the origin. Let us consider the measures

$$\langle \mu_j, f\rangle := \int_{\mathbb{R}^2} f(x, \phi(x))\, \eta_j(x)\, \mathcal{W}(x)\, \mathrm{d}x, \tag{2-2}$$

which now satisfy $\mu = \sum_{j\in\mathbb{Z}} \mu_j$, and let us furthermore assume that we have the local estimate for some $j_0 \in \mathbb{Z}$:

$$\|\hat{f}\|_{L^2(\mathrm{d}\mu_{j_0})} \le C\|f\|_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})}.$$

We want to determine the degree of homogeneity of $\mathcal{W}$ so that the Fourier restriction estimate is invariant under the dilations δ_t, i.e., that we have

$$\|\hat{f}\|_{L^2(\mathrm{d}\mu_j)} \le C\|f\|_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})} \tag{2-3}$$

for all $j \in \mathbb{Z}$ whenever the estimate is true for some $j_0 \in \mathbb{Z}$. In this case, and if $(p_1, p_3) \in (1,2]^2$, a standard Littlewood–Paley argument (presented below) will then yield

$$\|\hat{f}\|_{L^2(\mathrm{d}\mu)} \le C\|f\|_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})}.$$

To summarize, we have:

Proposition 2.1. *Let $\mathcal{W}$ be α-mixed homogeneous of degree 2ϑ, not identically zero, and continuous on $\mathbb{R}^2 \setminus \{0\}$, let μ be defined as in* (1-3), *and let $p_1, p_3 \in (1,2]$. Then the Fourier restriction estimate* (1-2) *for μ is equivalent to the Fourier restriction estimate* (2-3) *for the measure μ_j for any $j \in \mathbb{Z}$ (as defined in* (2-2)) *if and only if*

$$\vartheta = \frac{|\alpha|}{p_1'} + \frac{\rho}{p_3'} - \frac{|\alpha|}{2} \tag{2-4}$$

is satisfied.

Proof. Let us first determine what 2ϑ, the degree of homogeneity of $\mathcal{W}$, needs to be in order for (2-3) to hold true for all $j \in \mathbb{Z}$ whenever it holds true for some $j_0 \in \mathbb{Z}$. Recall that $|\delta_t x|_\alpha = t|x|_\alpha$. Inspecting the definition (2-2) of μ_j one gets

$$\langle \mu_j, f\rangle = 2^{j|\alpha|+2j\vartheta} \langle \mu_0, \mathrm{Dil}_{(2^{-j\alpha_1}, 2^{-j\alpha_2}, 2^{-j\rho})} f\rangle,$$

where $(\mathrm{Dil}_{(\lambda_1,\lambda_2,\lambda_3)} f)(x_1, x_2, x_3) = f(\lambda_1^{-1}x_1, \lambda_2^{-1}x_2, \lambda_3^{-1}x_3)$. Let us assume that we have for some $j \in \mathbb{Z}$ the estimate

$$\langle \mu_j, |\hat{f}|^2\rangle = \|\hat{f}\|^2_{L^2(\mathrm{d}\mu_j)} \le C^2\|f\|^2_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})}.$$

Since the Fourier transform behaves well with respect to dilations $\mathrm{Dil}_{(\lambda_1,\lambda_2,\lambda_3)}$, we may rescale the above estimate and get

$$\|\hat{f}\|_{L^2(\mathrm{d}\mu_0)} \le C 2^{-j|\alpha|/2-j\vartheta+j(\alpha_1/p_1'+\alpha_2/p_1'+\rho/p_3')}\|f\|_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})}.$$

From this one sees that we need precisely (2-4) in order for the constant in (2-3) to be independent of j. If (2-4) does not hold, then the constant blows up in one of the cases $j\to\infty$ or $j\to-\infty$, and in particular, the Fourier restriction estimate (1-2) for μ cannot hold (here we use that the restriction operators for μ and the μ_j are nonzero since $\mathcal{W}$ is not identically zero).

Let us now assume that we indeed have (2-4). It is obvious that the Fourier restriction estimate for μ implies the Fourier restriction estimate for μ_j for any j. Let us therefore assume that the estimate (2-3) holds true for any $j\in\mathbb{Z}$, and thus for all $j\in\mathbb{Z}$.

Before proceeding further let us denote by $(\tilde{\eta}_j)_{j\in\mathbb{Z}}$ a family of $C_c^\infty(\mathbb{R}^2\setminus\{0\})$ functions such that

$$\tilde{\eta}_j = \tilde{\eta}_0\circ\delta_{2^{-j}} \quad \text{for all } j\in\mathbb{Z},$$

and such that $\tilde{\eta}_j$ is equal to 1 on the support of η_j. One can for example take $\tilde{\eta}_j=\sum_{|k-j|\le N}\eta_k$ for some sufficiently large N. Let us furthermore denote by S_j the Fourier multiplier operator in $\mathbb{R}^3$ with multiplier $(\tilde{\eta}_j\otimes 1)(\xi_1,\xi_2,\xi_3)=\tilde{\eta}_j(\xi_1,\xi_2)$.

Now (2-3) implies

$$\|\widehat{S_j f}\|_{L^2(\mathrm{d}\mu_j)} = \|\hat{f}\|_{L^2(\mathrm{d}\mu_j)} \le C\|S_j f\|_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})}.$$

Therefore

$$\begin{aligned}\|\hat{f}\|^2_{L^2(\mathrm{d}\mu)} &= \langle\mu,|\hat{f}|^2\rangle = \sum_{j\in\mathbb{Z}}\langle\mu_j,|\hat{f}|^2\rangle = \sum_{j\in\mathbb{Z}}\langle\mu_j,|\widehat{S_j f}|^2\rangle\\ &\le C^2\sum_{j\in\mathbb{Z}}\|S_j f\|^2_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})} = C^2\big\|\|S_j f\|_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})}\big\|^2_{l^2_j},\end{aligned}$$

where l^2_j denotes the norm of the Hilbert space of l^2 sequences on $\mathbb{Z}$. Since both $p_1\le 2$ and $p_3\le 2$, we may use Minkowski's inequality to interchange the l^2_j norm with the $L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})$ norm, and subsequently apply Littlewood–Paley theory in the (x_1,x_2)-variable (in particular, we do not need to use mixed-norm Littlewood–Paley theory) to get

$$\begin{aligned}\big\|\|S_j f\|_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})}\big\|^2_{l^2_j} &\le \big\|\|S_j f\|_{l^2_j}\big\|^2_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})}\\ &= \left\|\Big(\sum_{j\in\mathbb{Z}}|S_j f|^2\Big)^{1/2}\right\|^2_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})} \sim \|f\|_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})}. \qquad \square\end{aligned}$$

Remark 2.2 (scaling in the case of Hessian determinant). Using the homogeneity condition of ϕ one easily obtains that the Hessian determinant is also α-mixed homogeneous of degree $2\rho-2|\alpha|$. Thus, when we take $\mathcal{W}=|\mathcal{H}_\phi|^\sigma$, $\mathcal{W}$ is homogeneous of degree $2\vartheta=2\sigma(\rho-|\alpha|)$. Recall that in this case (i.e., as in the assumptions of Theorem 1.1) we assume that

$$\frac{1}{p_1'}=\frac{1}{2}-\sigma, \quad \frac{1}{p_3'}=\sigma,$$

and so by (2-4) the equality $2\vartheta = 2\sigma(\rho - |\alpha|)$ is indeed satisfied, i.e., the desired relation between the exponents if one wants scaling invariance.

Remark 2.3 (a general sufficient condition for local integrability of $\mathcal{W}$). Since $\mathcal{W}$ is mixed homogeneous of degree 2ϑ, $\mathcal{W}|x|_\alpha^{-2\vartheta}$ is mixed homogeneous of degree 0, and in particular a bounded function. Thus $|\mathcal{W}| \lesssim |x|_\alpha^{2\vartheta}$, and so it is sufficient to check when $|x|_\alpha^{2\vartheta}$ is locally integrable in $\mathbb{R}^2$. By symmetry it is sufficient to integrate over $\{(x_1, x_2) : x_1, x_2 > 0\}$. We have

$$\begin{aligned}\int_{x_1,x_2>0,|x|\lesssim 1} |x|_\alpha^{2\vartheta}\,\mathrm{d}x &= \int_{x_1,x_2>0,|x|\lesssim 1} (x_1^{1/\alpha_1} + x_2^{1/\alpha_2})^{2\vartheta}\,\mathrm{d}x \\ &\sim \int_{y_1,y_2>0,|y|\lesssim 1} (y_1^2 + y_2^2)^{2\vartheta} y_1^{2\alpha_1-1} y_2^{2\alpha_2-1}\,\mathrm{d}y \\ &\sim \int_{0<r\lesssim 1}\int_0^{\pi/2} r^{4\vartheta+2|\alpha|-1}(\cos\theta)^{2\alpha_1-1}(\sin\theta)^{2\alpha_2-1}\,\mathrm{d}\theta\,\mathrm{d}r.\end{aligned}$$

Therefore, we must have $4\vartheta + 2|\alpha| - 1 > -1$, i.e.,

$$2\vartheta + |\alpha| > 0.$$

Note that this holds if $\rho \geq 0$, $p_1 > 1$, and ϑ is given by (2-4).

Remark 2.4. When ϕ is smooth at the origin and a nonconstant function, then $\rho = 1$, and the necessary condition obtained by a Knapp-type example associated to the principal face of $\mathcal{N}(\phi)$ in the initial coordinate system (see [Palle 2021, Proposition 2.1]) tells us that

$$\frac{|\alpha|}{p_1'} + \frac{1}{p_3'} \leq \frac{|\alpha|}{2}$$

is necessary for (1-2) if $\mathcal{W} \equiv 1$ (i.e., $\vartheta = 0$). On the other hand, if we define $l_\alpha = \{(t_1, t_3) \in \mathbb{R}^2 : |\alpha| t_1 + t_3 = |\alpha|/2\}$, then the expression (2-4) for ϑ implies that

$$|\vartheta| = \sqrt{1 + |\alpha|^2}\,\mathrm{dist}\biggl(\Bigl(\frac{1}{p_1'}, \frac{1}{p_3'}\Bigr), l_\alpha\biggr).$$

2B. ***Some further reductions.*** According to Proposition 2.1, under the conditions of Theorem 1.1 or Theorem 1.2, we have to prove the Fourier restriction estimate for a measure defined by the mapping

$$f \mapsto \int_{\mathbb{R}^2} f(x, \phi(x))\,\eta(x)\,\mathcal{W}(x)\,\mathrm{d}x,$$

where $\eta \in C_c^\infty(\mathbb{R}^2 \setminus \{0\})$ is supported in a compact annulus centered at the origin. Note that in the case of the weight $\mathcal{W} = |\mathcal{H}_\phi|^\sigma$ (the case of Theorem 1.1) the degree of homogeneity $2\vartheta = 2\sigma(\rho - |\alpha|)$ satisfies the relation (2-4) by Remark 2.2.

Reductions for the amplitude η. One can easily show that in the context of the Fourier restriction problem we may make the following reductions. First, by reordering coordinates and/or changing their sign, and by splitting the amplitude η into functions with smaller support, we may restrict ourselves to amplitudes η with support contained in the half-plane $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \gtrsim 1\}$. Then, by compactness, we may localize to small neighborhoods of points $v \neq 0$ having $v_1 \gtrsim 1$. Thus, one may assume that the support of η is contained in a small neighborhood of some generic point v satisfying $v_1 \sim 1$ and $|v| \lesssim 1$. In fact, compactness and changing signs if necessary implies that we may further assume that either $v_2 = 0$ or $v_2 \sim 1$.

Changing the affine terms of the phase. By the previous discussion it suffices to consider the measure

$$f \mapsto \int_{\mathbb{R}^2} f(x, \phi(x))\, \eta_v(x)\, \mathcal{W}(x)\, \mathrm{d}x, \tag{2-5}$$

where η_v is a smooth function supported in a small neighborhood of a point $v \neq 0$. We now recall the fact that we can freely add or remove linear and constant terms in the expression for ϕ in the context of the Fourier restriction problem. For the constant term this is obvious. For the linear terms this can be achieved by using a linear transformation of the form $(x_1, x_2, x_3) \mapsto (x_1, x_2, b_1 x_1 + b_2 x_2 + x_3)$ (for more details see [Palle 2021, Section 3.1] and note that here the situation is slightly simpler since no Jacobian factor appears). In particular, instead of considering the measure (2-5), we may consider the measure

$$f \mapsto \int_{\mathbb{R}^2} f(x, \phi_v(x - v))\, \eta_v(x)\, \mathcal{W}(x)\, \mathrm{d}x,$$

where we recall that

$$\phi_v(x) := \phi(x + v) - \phi(v) - x \cdot \nabla\phi(v).$$

The strategy for the proofs of Theorems 1.1 and 1.2 should now be clear. The above discussion reduces the problem to proving a local Fourier restriction estimate in the vicinity of a point v, and so one needs to determine the local normal form of ϕ at v, and in the case $\mathcal{W}(x) = |\mathcal{H}_\phi(x)|^\sigma$ one needs to additionally determine the order of vanishing of the Hessian determinant at v in the x_2-direction (after which the normal form of $\mathcal{W}$ will be clear by homogeneity).

3. Local normal forms

In this section we derive the local normal forms for ϕ and for the Hessian determinant $\mathcal{H}_\phi$ at a fixed point $v \neq 0$ (as a consequence we prove Proposition 1.4). The discussion in Section 2B implies that we may assume that $v_1 \sim 1$, and either $v_2 = 0$ or $v_2 \sim 1$.

The structure of this section is as follows. In Section 3A we fix the notation for this section, introduce relevant quantities, and define the coordinate systems y, z, and w (the coordinate systems z and w will not be described precisely until Section 3E though). In Subsections 3B, 3C, and 3D tables with normal forms of ϕ_v are given. It turns out that in most cases y-coordinates suffice and when we use them one obtains the normal forms easily. We deal with the case when y-coordinates do not suffice in Section 3E. In Section 3F we sketch how to calculate what is the order of vanishing of the Hessian determinant for the respective normal forms.

We assume that the (H1) condition is satisfied throughout this section. In fact, in Section 3B we shall explicitly determine the local normal form of ϕ when $t \mapsto \partial_2^2\phi(v_1, t)$ is flat at v_2. In this case it turns out that the Hessian determinant either does not vanish at v, or that it is flat at v. In all the other subsections we shall assume that $t \mapsto \partial_2^2\phi(v_1, t)$ is of finite type at v_2.

3A. ***Notation and some general considerations.*** Let us begin by introducing the notation. It will be useful to define

$$\gamma := \frac{\alpha_2}{\alpha_1} > 0,$$

and for the point $v = (v_1, v_2)$ (recall $v_1 \sim 1$) we define

$$t_0 := v_2 v_1^{-\gamma}.$$

Let us denote the ∂_2-derivatives of ϕ at $(1, t_0)$ by

$$b_j := \partial_2^j \phi(1, t_0) = g^{(j)}(t_0), \quad j \in \mathbb{N}_0,$$

where

$$g(t) := \phi(1, t).$$

We furthermore define

$$k := \inf\{j \geq 2 : b_j \neq 0\}, \tag{3-1}$$

where we take $k = \infty$ if $b_j = 0$ for all $j \geq 2$. The equality $k = \infty$ is equivalent to $g^{(2)}$ being flat at 0. What precisely happens when $g^{(2)}$ is flat at 0 shall be explained in Section 3B, and in the rest of the section (including this subsection) we assume that $k < \infty$, unless explicitly stated otherwise.

General form of mixed homogeneous ϕ. Recall that we denote by $\rho \in \{-1, 0, 1\}$ the degree of homogeneity of ϕ. Then we have for any x satisfying $x_1 > 0$:

$$\phi(x_1, x_2) = x_1^{\rho/\alpha_1} \phi(1, x_2 x_1^{-\gamma}). \tag{3-2}$$

Let us consider the Taylor expansion of $t \mapsto \phi(1, t)$ at t_0:

$$g(t) = \phi(1, t) = b_0 + (t - t_0) b_1 + \frac{1}{k!}(t - t_0)^k g_k(t),$$

where g_k is a smooth function such that $b_k = g_k(0)$. Thus, we get

$$\begin{aligned} \phi(x) &= x_1^{\rho/\alpha_1}\Big(b_0 + (x_2 x_1^{-\gamma} - t_0) b_1 + \frac{1}{k!}(x_2 x_1^{-\gamma} - t_0)^k g_k(x_2 x_1^{-\gamma})\Big) \\ &= x_1^{\rho/\alpha_1}(b_0 - t_0 b_1) + x_2 x_1^{(\rho - \alpha_2)/\alpha_1} b_1 + \frac{1}{k!} x_1^{(\rho - k\alpha_2)/\alpha_1}(x_2 - t_0 x_1^{\gamma})^k g_k(x_2 x_1^{-\gamma}). \end{aligned} \tag{3-3}$$

More generally, we have the formal series expansion:

$$\begin{aligned} \phi(x) &\approx \sum_{j=0}^{\infty} \frac{b_j}{j!}(x_2 - t_0 x_1^{\gamma})^j \, x_1^{\rho/\alpha_1 - j\gamma} \\ &= b_0 x_1^{\rho/\alpha_1} + b_1(x_2 - t_0 x_1^{\gamma}) x_1^{\rho/\alpha_1 - \gamma} + \sum_{j=k}^{\infty} \frac{b_j}{j!}(x_2 - t_0 x_1^{\gamma})^j \, x_1^{\rho/\alpha_1 - j\gamma}. \end{aligned} \tag{3-4}$$

If $\gamma = 1$ (i.e., $\alpha_1 = \alpha_2$) it will usually be better to write

$$\phi(x) = x_1^{\rho/\alpha_1} b_0 + (x_2 - t_0 x_1) x_1^{\rho/\alpha_1 - 1} b_1 + \frac{1}{k!}(x_2 - t_0 x_1)^k \, x_1^{\rho/\alpha_1 - k} \, g_k(x_2 \, x_1^{-1}). \tag{3-5}$$

Since $v_1 \sim 1$, we may assume

$$|x_1^{1/\alpha_1} - v_1^{1/\alpha_1}| \ll 1, \quad |x_2 \, x_1^{-\gamma} - v_2 \, v_1^{-\gamma}| \ll 1.$$

The second condition is equivalent to $|x_2 - t_0 x_1^{\gamma}| \ll 1$. Note that the points on the homogeneity curve through v satisfy the equation $x_2 = t_0 x_1^{\gamma}$.

In order to determine the normal forms it will suffice to introduce three additional coordinate systems, which we shall denote by y, z, and w respectively, each having the point v as their origin. The original coordinate system is denoted by x. The function ϕ in the coordinate system y (resp. z, w) shall be denoted by ϕ^y (resp. ϕ^z, ϕ^w). For the original coordinate system x we simply use ϕ, or ϕ^x for emphasis.

The function ϕ in the coordinate system y (resp. z, w) but without the affine terms at v shall be denoted by ϕ_v^y (resp. ϕ_v^z, ϕ_v^w). This means

$$\phi_v^y(y) := \phi^y(y) - \phi^y(0) - y \cdot \nabla \phi^y(0),$$

and similarly for ϕ_v^z and ϕ_v^w.

The coordinate system y. It is defined through the following affine coordinate change having $v = (v_1, v_2)$ as the origin:

$$\begin{aligned} y_1 &= x_1 - v_1, \\ y_2 &= x_2 - v_2 - \gamma v_2 v_1^{-1}(x_1 - v_1) \\ &= x_2 - (1-\gamma) v_2 - \gamma v_2 v_1^{-1} x_1. \end{aligned}$$

The reverse transformation is

$$\begin{aligned} x_1 &= y_1 + v_1, \\ x_2 &= y_2 + v_2 + \gamma v_2 v_1^{-1} y_1. \end{aligned} \tag{3-6}$$

One can easily check that in these coordinates we can write

$$\begin{aligned} x_2 - t_0 x_1^{\gamma} &= y_2 + v_2 + \gamma v_2 v_1^{-1} y_1 - v_2 (1 + v_1^{-1} y_1)^{\gamma} \\ &= y_2 + v_2 + \gamma v_2 v_1^{-1} y_1 - v_2 \Big(1 + \gamma v_1^{-1} y_1 + \binom{\gamma}{2} v_1^{-2} y_1^2 + \mathcal{O}(y_1^3)\Big) \\ &= y_2 - y_1^2 \omega(y_1); \end{aligned}$$

i.e., the points on the homogeneity curve through v satisfy the equation $y_2 = y_1^2 \omega(y_1)$ in y-coordinates. Above (and in the following) we use the notation

$$\binom{c}{m} = c(c-1) \cdots \frac{c-m+1}{m!}$$

for $c \in \mathbb{R}$ and m a nonnegative integer. Furthermore, we obviously have:

Remark 3.1. It holds that $\omega(0) \neq 0$ if and only if ω is not identically 0 if and only if $v_2 \neq 0$ (i.e., $t_0 \neq 0$) and $\gamma \neq 1$.

The coordinate system y will be used in most of the normal forms below which shall follow directly from the expression

$$\phi^y(y) = (v_1 + y_1)^{\rho/\alpha_1}(b_0 - t_0 b_1) \\ + (v_2 + y_2 + \gamma v_2 v_1^{-1} y_1)\,(v_1 + y_1)^{(\rho-\alpha_2)/\alpha_1} b_1 + (y_2 - y_1^2\omega(y_1))^k r(y), \quad (3\text{-}7)$$

which one obtains from (3-3) and (3-6). When $\gamma = 1$, one uses (3-5) instead and gets

$$\phi^y(y) = (v_1 + y_1)^{\rho/\alpha_1} b_0 + y_2(v_1 + y_1)^{\rho/\alpha_1 - 1} b_1 + y_2^k r(y). \quad (3\text{-}8)$$

In both (3-7) and (3-8) the function r is smooth and nonvanishing at the origin. Let us also note that the expansion (3-4) can be rewritten in y coordinates as

$$\phi^y(y) \approx b_0(v_1 + y_1)^{\rho/\alpha_1} \\ + b_1(y_2 - y_1^2\omega(y_1))(v_1 + y_1)^{\rho/\alpha_1 - \gamma} + \sum_{j=k}^{\infty} \frac{b_j}{j!}(y_2 - y_1^2\omega(y_1))^j\,(v_1 + y_1)^{\rho/\alpha_1 - j\gamma}. \quad (3\text{-}9)$$

The following simple lemma shall be useful later:

Lemma 3.2. *From* (3-7) *and* (3-8) *we get the following information on the second-order derivatives of* ϕ^y:

(1) *It always holds*

$$k = 2 \iff b_2 \neq 0 \iff \partial_2^2 \phi^y(0) \neq 0.$$

(2.a) *If* $\rho \neq 1$ *or* $\alpha_2 \neq 1$ (*i.e.*, $\rho - \alpha_2 \neq 0$), *then*

$$b_1 \neq 0 \iff \partial_1\partial_2\phi^y(0) \neq 0.$$

(2.b) *If* $\rho = \alpha_2 = 1$ *or if* $b_1 = 0$, *then* $\partial_1\partial_2\phi^y(0) = 0$.

(3.a) *If* $\rho = 0$ *and* $\alpha_1 \neq \alpha_2$ (*i.e.*, $\gamma \neq 1$), *or if* $\rho = \alpha_1 = 1$ *and* $\alpha_2 \neq 1$ (*and in particular* $\gamma \neq 1$), *then*

$$b_1 \neq 0,\ \ t_0 \neq 0 \iff \partial_1^2\phi^y(0) \neq 0,$$

and recall that $v_2 \neq 0$ *if and only if* $t_0 \neq 0$.

(3.b) *If* $\rho = \alpha_2 = 1$ *and* $\alpha_1 \neq 1$ (*and in particular* $\gamma \neq 1$), *then*

$$b_0 - t_0 b_1 \neq 0 \iff \partial_1^2\phi^y(0) \neq 0.$$

(3.c) *If* $\gamma = 1$ (*i.e.*, $\alpha_1 = \alpha_2$) *or if* $b_1 = 0$, *then*

$$b_0 \neq 0,\ \frac{\rho}{\alpha_1} \notin \{0, 1\} \iff \partial_1^2\phi^y(0) \neq 0.$$

Note that $\rho/\alpha_1 = 0$ *if and only if* $\rho = 0$, *and* $\rho/\alpha_1 = 1$ *if and only if* $\rho = \alpha_1 = 1$.

Proof. The only not completely trivial case is (3.a). Since in this case $\rho/\alpha_1 \in \{0, 1\}$, the first term in (3-7) is an affine term, and so we can ignore it. Since $k \geq 2$, the third term also does not contribute to the

y_1^2-term in the Taylor series of ϕ^y, and so we can ignore it too. We therefore only need to consider the term

$$(v_2 + y_2 + \gamma v_2 v_1^{-1} y_1)\,(v_1 + y_1)^{(\rho-\alpha_2)/\alpha_1} b_1,$$

and in fact, we may even reduce ourselves to

$$(v_2 + \gamma v_2 v_1^{-1} y_1)\,(v_1 + y_1)^{(\rho-\alpha_2)/\alpha_1} b_1 = b_1 v_2 (1 + \gamma v_1^{-1} y_1)\,(v_1 + y_1)^{(\rho-\alpha_2)/\alpha_1}.$$

Now if $t_0 = 0$ (i.e., $v_2 = 0$) or if $b_1 = 0$, then $\partial_1^2 \phi^y(0) = 0$ follows. Let us now assume $v_2 \neq 0$ and $b_1 \neq 0$. We note that in our case we may rewrite $(\rho - \alpha_2)/\alpha_1 = \rho - \gamma$, and so it suffices to show that

$$\partial_{y_1}^2|_{y_1=0}((1 + \gamma v_1^{-1} y_1)\,(1 + v_1^{-1} y_1)^{\rho-\gamma}) \neq 0.$$

Calculating the second derivative one gets

$$2\gamma v_1^{-2}(\rho - \gamma) + v_1^{-2}(\rho - \gamma)(\rho - \gamma - 1).$$

This is not zero since in this case we have $\rho \in \{0, 1\}$ and $\gamma \notin \{0, 1\}$. □

The coordinate systems z and w. These are defined through affine coordinate changes of the form

$$\begin{aligned} x_1 &= v_1 + z_1, & w_1 &= z_1 + \frac{1}{B} z_2, \\ x_2 &= v_2 + z_2 + A z_1, & w_2 &= z_2, \end{aligned} \tag{3-10}$$

having (v_1, v_2) as their origin, where we shall have $B := A - \gamma v_2 v_1^{-1} \neq 0$ so that the coordinate system y never coincides with the coordinate system z, and the coordinate system z never coincides with the coordinate system w. The constant A shall depend on v and the first few derivatives of ϕ at v (note that $A = B \neq 0$ if $v_2 = t_0 = 0$). These coordinate systems will be described more precisely in Section 3E. There we shall also introduce a smooth function $\tilde{\omega}$ such that

$$x_2 - t_0 x_1^\gamma = y_2 - y_1^2 \omega(y_1) = (w_1 - w_2^2 \tilde{\omega}(w_2)) r_0(w)$$

for some smooth function r_0 satisfying $r_0(0) \neq 0$. Note that we have

$$\begin{aligned} y_1 &= z_1 = w_1 - \frac{1}{B} w_2, \\ y_2 &= z_2 + B z_1 = B w_1. \end{aligned} \tag{3-11}$$

As we shall see in Section 3E below the z-coordinates are only used in the intermediate steps and the normal forms are expressed exclusively in y- or w-coordinates.

Some general considerations regarding the Hessian determinant $\mathcal{H}_\phi$. Recall that

$$\phi(t^{\alpha_1} x_1, t^{\alpha_2} x_2) = t^\rho \phi(x_1, x_2).$$

Taking derivatives in x_1 and x_2 we get

$$(\partial_1^{\tau_1} \partial_2^{\tau_2} \phi)(t^{\alpha_1} x_1, t^{\alpha_2} x_2) = t^{\rho - \tau_1 \alpha_1 - \tau_2 \alpha_2} (\partial_1^{\tau_1} \partial_2^{\tau_2} \phi)(x_1, x_2).$$

Thus, we have for the Hessian determinant of ϕ:

$$\mathcal{H}_\phi(t^{\alpha_1}x_1, t^{\alpha_2}x_2) = t^{2(\rho-|\alpha|)}\mathcal{H}_\phi(x_1, x_2).$$

From this it follows that if $\mathcal{H}_\phi$ vanishes at the point v, then it also vanishes along the homogeneity curve through v, which we recall is parametrized by $t \mapsto (t^{\alpha_1}v_1, t^{\alpha_2}v_2)$.

We are interested in the order of vanishing of $\mathcal{H}_\phi$ in directions transversal to this curve. In particular, if we have $\partial_2^{\tau_2}\mathcal{H}_\phi(v) = 0$ for $\tau_2 < N$ and $\partial_2^N\mathcal{H}_\phi(v) \neq 0$, then by using homogeneity and a Taylor expansion (as we did for ϕ) we get

$$\mathcal{H}_\phi(x) = (x_2 - t_0 x_1^\gamma)^N r_0(x)$$

for some smooth function r_0 satisfying $r_0(v) \neq 0$. Calculating N shall be done in Section 3F by using the normal forms of ϕ. Recall that the Hessian determinant is equivariant under affine coordinate changes, and so we can freely change to y-, z-, or w-coordinates.

Preliminary comments on the normal forms. Let us introduce the following notation for the nondegenerate case (i.e., the case when the Hessian determinant of ϕ does not vanish at v):

(ND) The function ϕ_v is nondegenerate at the origin.

When ϕ_v does not satisfy (ND), we note that Proposition 1.4 implies in particular that after a linear change of coordinates the function ϕ_v takes one of the following three forms:

$$\begin{aligned}
\phi_v^u(u) &= u_1^{k_0} r(u) + \varphi(u),\\
\phi_v^u(u) &= u_1^2 r_1(u) + u_2^{k_0} r_2(u),\\
\phi_v^u(u) &= (u_2 - u_1^2\psi(u_1))^{k_0} r(u),
\end{aligned}$$

where $r(0), \psi(0), r_1(0), r_2(0) \neq 0$, φ is flat at 0, and $k_0 \geq 2$ in the first and third cases, while $k_0 \geq 3$ in the second. Note that the first case corresponds to the normal form (i) of Proposition 1.4, the second case is a reduced version of normal forms (ii), (iii), (iv), (v), and the third corresponds to the normal form (vi).

However, the above three forms do not contain sufficient information to obtain restriction estimates. In this section we shall obtain the much more detailed classification given in Table 2.

All the appearing functions are smooth and do not vanish at the origin, except the function φ, which is always flat at the origin. The number k is as defined in (3-1) and it is always finite in the above normal forms (when it is infinite it turns out that one is necessarily in the case of normal form (i.y2)). On the other hand, the definition of the number $\tilde{k}$ changes from case to case, and we allow $\tilde{k}$ to be infinite only in normal form (i.y1), in which case we consider the Hessian determinant to be flat at the origin. The quantities v_1, γ, A, B appearing in the conditions column and the functions ω and $\tilde{\omega}$ are the same ones as previously defined in this subsection. Let us furthermore remark that normal forms (i.w1) and (i.w2) stem from normal forms (ii.w), (iii), and (v), in the sense that they correspond to $\tilde{k} = \infty$.

Two remarks before we continue. First, note that the normal forms listed in Proposition 1.4 are a compressed version of Table 2 — in the proposition we ignored the subcases, e.g., the normal forms (i.y1), (i.y2), (i.w1), (i.w2) were all compressed in Proposition 1.4 to a single normal form (i). Second,

case	normal form	additional conditions
(i.y1)	$\phi_v^y(y) = y_2^k r(y)$, $\mathcal{H}_{\phi^y}(y) = y_2^{\tilde{k}+2k-2} r_0(y)$	$k \geq 2$, $\tilde{k} \geq 0$ or $\tilde{k} = \infty$
(i.y2)	$\phi_v^y(y) = y_1^{\tilde{k}} q(y_1) + \varphi(y)$, $\mathcal{H}_{\phi^y}$ is flat at 0	$\tilde{k} \geq 2$
(i.w1)	$\phi_v^w(w) = w_2^2 q(w_2) + \varphi(w)$, $\mathcal{H}_{\phi^w}$ is flat at 0	-
(i.w2)	$\phi_v^w(w) = w_2^2 r(w) + \varphi(w)$, $\mathcal{H}_{\phi^w}$ is flat at 0	$v_1 B \partial_1^j r(0) = jA(\gamma - 1)\partial_1^{j-1} r(0)$ for all $j \geq 1$
(ii.y)	$\phi_v^y(y) = y_1^2 q(y_1) + y_2^k r(y)$, $\mathcal{H}_{\phi^y}(y) = y_2^{k-2} r_0(y)$	$k \geq 3$
(ii.w)	$\phi_v^w(w) = w_1^{\tilde{k}} r(w) + w_2^2 q(w_2)$, $\mathcal{H}_{\phi^w}(w) = w_1^{\tilde{k}-2} r_0(w)$	$\tilde{k} \geq 3$
(iii)	$\phi_v^w(w) = w_1^{\tilde{k}} r_1(w) + w_2^2 r_2(w)$, $\mathcal{H}_{\phi^w}(w) = w_1^{\tilde{k}-2} r_0(w)$	$\tilde{k} \geq 3$, $v_1 B \partial_1^j r_2(0) = jA(\gamma - 1)\partial_1^{j-1} r_2(0)$ for all $1 \leq j \leq \tilde{k} - 1$
(iv)	$\phi_v^y(y) = y_1^2 q(y_1) + (y_2 - y_1^2 \omega(y_1))^k r(y)$, $\mathcal{H}_{\phi^y}(y) = (y_2 - y_1^2 \omega(y_1))^{k-2} r_0(y)$	$k \geq 3$
(v)	$\phi_v^w(w) = (w_1 - w_2^2 \tilde{\omega}(w_2))^{\tilde{k}} r_1(w) + w_2^2 r_2(w)$, $\mathcal{H}_{\phi^w}(w) = (w_1 - w_2^2 \tilde{\omega}(w_2))^{\tilde{k}-2} r_0(w)$	$\tilde{k} \geq 3$, $v_1 B \partial_1^j r_2(0) = jA(\gamma - 1)\partial_1^{j-1} r_2(0)$ for $1 \leq j \leq \tilde{k} - 1$
(vi)	$\phi_v^y(y) = (y_2 - y_1^2 \omega(y_1))^k r(y)$, $\mathcal{H}_{\phi^y}(y) = (y_2 - y_1^2 \omega(y_1))^{2k-3} r_0(y)$	$k \geq 2$

Table 2. Detailed classification of normal forms (an uncompressed version of Table 1).

note that the above “uncompressed” table of normal forms is not mutually exclusive in the sense that the forms themselves differ from each other — for example in this sense the normal form (i.y2) obviously contains the case of the normal form (i.w1), the main difference being only the coordinate system which one needs to use in order to obtain them. On the other hand, the normal forms in the compressed table in Proposition 1.4 are in this sense indeed mutually exclusive.

The first step in deriving the above normal forms is to switch to y-coordinates. In most cases (see the tables of cases below for the precise list) this will suffice and the normal form will be obvious, and so in the following subsections we shall leave out most of the details for them. In particular, as a consequence of considerations in Subsections 3C and 3D, we shall obtain:

Lemma 3.3. *If $k \geq 3$ and if we are not in the* (ND) *case, then the function ϕ_v^y is always in one of the normal forms* (i.y1), (i.y2), (ii.y), (iv), *or* (vi).

If $k = 2,\ b_1 \neq 0,\ \rho \neq \alpha_2$, and we are not in the (ND) case, then we shall either need to

(FP) flip coordinates (i.e., exchange x_1 and x_2) and use the y-coordinates associated to the flipped coordinates,

or we shall need w- (and the intermediary z-) coordinates. Details can be found in Section 3E below.

Note that flipping coordinates makes sense only when $v_2 \neq 0$ (and indeed, we shall flip coordinates only when $A = 0$, which, as it turns out, never happens when $v_2 = 0$). After flipping coordinates it will always suffice to use the y-coordinates (associated to the flipped x, v, and α), and in particular, we shall be able to apply Lemma 3.3. Note that these y coordinates are not in general equal to flipped y-coordinates associated to the original x, v, and α.

3B. ***Normal form when $t \mapsto \partial_2^2\phi(1,t)$ is flat at t_0 (i.e., $k = \infty$).*** Let us assume that

$$\partial_2^j\phi(1,t_0) = 0 \quad \text{for all } j \geq 2, \tag{3-12}$$

and so we have $\partial_2^j\phi(v) = 0$ for all v (with $v_1 > 0$) satisfying $v_2 v_1^{-\gamma} = t_0$ by (3-2). By the Euler homogeneous function theorem ϕ satisfies the equation

$$\rho\phi(x) = \alpha_1 x_1 \partial_1\phi(x) + \alpha_2 x_2 \partial_2\phi(x).$$

Taking the derivative $\partial^\tau = \partial_1^{\tau_1}\partial_2^{\tau_2}$ we get at (v_1, v_2) that

$$(\rho - \alpha_1\tau_1 - \alpha_2\tau_2)\partial^\tau\phi(v) = \alpha_1 v_1 \partial^{\tau+(1,0)}\phi(v) + \alpha_2 v_2 \partial^{\tau+(0,1)}\phi(v).$$

From this, the fact that $\alpha_1 v_1 \neq 0$, and the flatness assumption (3-12) it follows by induction in τ_1 that $\partial^\tau\phi(v) = 0$ for all $\tau_1 \geq 0$ and $\tau_2 \geq 2$.

If now $\partial_1\partial_2\phi(v) \neq 0$, then the Hessian determinant does not vanish and we are in the (ND) case (this always happens for example when $\phi(x_1, x_2) = x_1 x_2$). On the other hand, if $\partial_1\partial_2\phi(v) = 0$, then we get in the same way as above that $\partial^\tau\phi(v) = 0$ for all $\tau_1 \geq 1$ and $\tau_2 = 1$. Thus, by using a Taylor expansion at v and by switching to y-coordinates (recall $x_1 = y_1 + v_1$) we may write

$$\phi_v^y(y) = y_1^2 q(y_1) + \varphi(y),$$

where q is a smooth function and φ is a smooth function flat at the origin. In particular, in this case the Hessian determinant vanishes of infinite order at $x = v$ and therefore the condition (H2) cannot hold. This also shows that (H2) is a stronger condition than (H1). Since we assume that at least (H1) holds, we necessarily have that $t \mapsto \partial_1^2\phi(t, v_2)$ is not flat at v_1, and so q cannot be flat at the origin either, i.e., we can write

$$\phi_v^y(y) = y_1^{\tilde{k}}\tilde{q}(y_1) + \varphi(y)$$

for some smooth function $\tilde{q}$ satisfying $\tilde{q}(0) \neq 0$ and $\tilde{k} \geq 2$. This is precisely the normal form (i.y2).

3C. *Normal form tables for ϕ mixed homogeneous of degree $\rho = 0$.* Recall that we assume $k < \infty$ in this and the following subsections. In this case (3-7) becomes

$$\phi^y(y) - (b_0 - t_0 b_1) = (v_2 + y_2 + \gamma v_2 v_1^{-1} y_1)\,(v_1 + y_1)^{-\gamma} b_1 + (y_2 - y_1^2\omega(y_1))^k r(y)$$

if $\gamma \neq 1$, and in the case $\gamma = 1$ we have by (3-8) that

$$\phi^y(y) - b_0 = y_2(v_1 + y_1)^{-1} b_1 + y_2^k r(y). \tag{3-13}$$

We have put the constant terms on the left-hand side since we may freely ignore them. Note that in the case $\gamma = 1$ we have $\partial_1^2\phi^y(0) = 0$ by Lemma 3.2 (3.c).

Case $\gamma = 1$.

conditions	case
$b_1 = 0$	normal form (i.y1)
$b_1 \neq 0$	(ND)

Here we actually have in the case when $b_1 = 0$ a precise order of vanishing of the Hessian determinant: it is always $2k - 2$. This follows from Section 3F (see in particular (3-31)).

If $b_1 \neq 0$, then from (3-13) we obviously have $\partial_1\partial_2\phi^y(0) \neq 0$, and it follows that the Hessian determinant at 0 is nonzero.

Case $\gamma \neq 1$.

conditions	case
$t_0 = 0,\ b_1 = 0$	normal form (i.y1)
$t_0 = 0,\ b_1 \neq 0$	(ND)
$t_0 \neq 0,\ b_1 = 0$	normal form (vi)
$t_0 \neq 0,\ b_1 \neq 0,\ k \geq 3$	(ND)
$t_0 \neq 0,\ b_1 \neq 0,\ k = 2$	(ND), or (FP), or normal form (v), or normal form (i.w2)

In the case $t_0 = 0,\ b_1 \neq 0$ we apply Lemma 3.2 (2.a) and (3.a), and get respectively that $\partial_1\partial_2\phi^y(0) \neq 0$ and $\partial_1^2\phi^y(0) = 0$, from which it indeed follows that we are in the (ND) case. Similarly, in the case $t_0 \neq 0$, $b_1 \neq 0,\ k \geq 3$ we use Lemma 3.2 (1) and (2.a), and obtain that $\partial_2^2\phi^y(0) = 0$ and $\partial_1\partial_2\phi^y(0) \neq 0$, from which we again get that the Hessian determinant of ϕ^y does not vanish.

As the case $t_0 \neq 0,\ b_1 \neq 0,\ k = 2$ shall be treated in the same way as certain other cases which appear later and where w-coordinates may be needed, we have postponed its discussion to Section 3E.

3D. ***Normal form tables for ϕ mixed homogeneous of degree $\rho = \pm 1$.*** Recall that by (3-3) here we have

$$\phi(x) = x_1^{\rho/\alpha_1}(b_0 - t_0 b_1) + x_2\, x_1^{(\rho-\alpha_2)/\alpha_1} b_1 + \frac{1}{k!} x_1^{(\rho-k\alpha_2)/\alpha_1}(x_2 - t_0 x_1^{\gamma})^k g_k(x_2\, x_1^{-\gamma})$$

and according to (3-7) in y-coordinates this is

$$\phi^y(y) = (v_1+y_1)^{\rho/\alpha_1}(b_0 - t_0 b_1) + (v_2 + y_2 + \gamma v_2 v_1^{-1} y_1)\,(v_1+y_1)^{(\rho-\alpha_2)/\alpha_1} b_1 + (y_2 - y_1^2\omega(y_1))^k r(y).$$

In this subsection (where $\rho = \pm 1$) we need to consider five possible subcases. The cases we first consider are when $\rho = \alpha_1$, or $\rho = \alpha_2$, or both. Since α_1 and α_2 are strictly positive, these cases are only possible for $\rho = 1$. The penultimate case is when $\alpha_1 = \alpha_2 \neq \rho$, and the last case is when all of α_1, α_2, and ρ are different from each other.

Case $\rho = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$. In this case the first two terms in (3-7) become affine, and by Remark 3.1 we have $\omega \equiv 0$. As a consequence we have only one case.

conditions	case
-	normal form (i.y1)

Furthermore, we note that initially we know that the order of vanishing of the Hessian determinant is at least $2k-2$, which is always greater than or equal to 2. Since this is true at every point, the Hessian determinant vanishes identically in this case.

Case $\rho = 1$, $\alpha_1 \neq 1$, $\alpha_2 = 1$. Here we first note that by Lemma 3.2 (2.b), we always have $\partial_1\partial_2\phi^y(0) = 0$. This is a simple consequence of the fact that in this case the second term in (3-7) is linear.

conditions	case
$b_0 - t_0 b_1 = 0,\ t_0 = 0$	normal form (i.y1)
$b_0 - t_0 b_1 = 0,\ t_0 \neq 0$	normal form (vi)
$b_0 - t_0 b_1 \neq 0,\ k = 2$	(ND)
$b_0 - t_0 b_1 \neq 0,\ k \geq 3,\ t_0 = 0$	normal form (ii.y)
$b_0 - t_0 b_1 \neq 0,\ k \geq 3,\ t_0 \neq 0$	normal form (iv)

The first two cases in the table are now clear since the first two terms in (3-7) can be ignored. The (ND) case follows from Lemma 3.2 (1) and (3.b) (and as previously mentioned (2.b)). The last two cases follow simply by developing the first term in (3-7) in a Taylor series in y_1 and ignoring the constant and the linear term.

Case $\rho = 1$, $\alpha_1 = 1$, $\alpha_2 \neq 1$. Here we note that the first term in (3-7) becomes linear, and therefore does not influence the normal form of ϕ_v^y.

conditions	case
$t_0 = 0,\ b_1 = 0$	normal form (i.y1)
$t_0 = 0,\ b_1 \neq 0$	(ND)
$t_0 \neq 0,\ b_1 = 0$	normal form (vi)
$t_0 \neq 0,\ b_1 \neq 0,\ k \geq 3$	(ND)
$t_0 \neq 0,\ b_1 \neq 0,\ k = 2$	(ND) or (FP)

The cases $t_0 = 0,\ b_1 \neq 0$ and $t_0 \neq 0,\ b_1 \neq 0, k \geq 3$ are (ND) by the same argumentation as in the table above for $\rho = 0$, $\gamma \neq 1$ (namely, by applying Lemma 3.2 (2.a) and (3.a), in the case $t_0 = 0,\ b_1 \neq 0$, and by applying Lemma 3.2 (1) and (2.a), in the case $t_0 \neq 0,\ b_1 \neq 0,\ k \geq 3$).

Let us note the following for the last case where $t_0 \neq 0,\ b_1 \neq 0$, and $k = 2$. The expression in (3-3) can be rewritten as (after ignoring the first term, which is linear in this case):

$$b_1 x_2 x_1^{1-\gamma} + \tfrac{1}{2} b_2 x_1^{1-2\gamma}(x_2 - t_0 x_1^\gamma)^2 + \mathcal{O}((x_2 - t_0 x_1^\gamma)^3).$$

We want to calculate what the Hessian determinant of $\phi_v^x = \phi_v$ at v is (or equivalently, the Hessian determinant of ϕ at v). For this we only need the second derivatives of ϕ at v, and so we can freely ignore the last term of size $(x_2 - t_0 x_1^\gamma)^3$. After expanding the second term in the above expression and ignoring the linear terms and the term $\mathcal{O}((x_2 - t_0 x_1^\gamma)^3)$ we get

$$(b_1 - t_0 b_2) x_1^{1-\gamma} x_2 + \tfrac{1}{2} b_2 x_1^{1-2\gamma} x_2^2.$$

From this it follows by a direct calculation that

$$\partial_1^2 \phi(v) = -\gamma \frac{v_2}{v_1} \partial_1 \partial_2 \phi(v),$$

and so

$$\mathcal{H}_\phi(v) = -\partial_1 \partial_2 \phi(v) \Big(\partial_1 \partial_2 \phi(v) + \gamma \frac{v_2}{v_1} \partial_2^2 \phi(v) \Big),$$

which we note can be rewritten as

$$\mathcal{H}_\phi(v) = -\partial_1 \partial_2 \phi^x(v)\, \partial_1 \partial_2 \phi^y(0),$$

by (3-6). This implies in particular that $\mathcal{H}_\phi(v) = 0$ if and only if $\partial_1 \partial_2 \phi(v) = 0$ if and only if $\partial_1^2 \phi(v) = 0$ since by Lemma 3.2 (2.a), we know that $\partial_1 \partial_2 \phi^y(0) \neq 0$.

Thus, in the last case where $t_0 \neq 0$, $b_1 \neq 0$, and $k = 2$, we are either in the (ND) case, and otherwise we have $\partial_1^2 \phi(v) = 0$. This means precisely that the "k" associated to the flipped coordinates (and we can flip coordinates since $t_0 \neq 0$, i.e., $v_2 \neq 0$) is necessarily ≥ 3. For the flipped coordinates we may now use the previous table where we have $\rho = 1,\ \alpha_1 \neq 1,\ \alpha_2 = 1$ (or apply Lemma 3.3).

Case $\rho = \pm 1,\ \alpha_1 = \alpha_2 \neq \rho$. Here one uses (3-8):

$$\phi^y(y) = (v_1 + y_1)^{\rho/\alpha_1} b_0 + y_2 (v_1 + y_1)^{\rho/\alpha_1 - 1} b_1 + y_2^k r(y).$$

conditions	case
$b_0 = 0,\ b_1 = 0$	normal form (i.y1)
$b_0 = 0,\ b_1 \neq 0$	(ND)
$b_0 \neq 0,\ b_1 = 0,\ k \geq 3$	normal form (ii.y)
$b_0 \neq 0,\ b_1 = 0,\ k = 2$	(ND)
$b_0 \neq 0,\ b_1 \neq 0,\ k \geq 3$	(ND)
$b_0 \neq 0,\ b_1 \neq 0,\ k = 2$	(ND), or (FP), or normal form (ii.w), or normal form (i.w1)

The first (ND) case $b_0 = 0,\ b_1 \neq 0$ follows from Lemma 3.2 (2.a) and (3.c), the second (ND) case $b_0 \neq 0,\ b_1 = 0,\ k = 2$ follows from Lemma 3.2 (2.a), (3.c), and (1), and the third (ND) case $b_0 \neq 0,\ b_1 \neq 0,\ k \geq 3$ follows from Lemma 3.2 (1) and (2.a). For the last case $b_0 \neq 0,\ b_1 \neq 0,\ k = 2$ we again refer the reader to Section 3E.

We give two further remarks. Firstly, one can show that in the case $b_0 = 0$, $b_1 = 0$ the order of vanishing of the Hessian determinant is precisely equal to $2k - 2$ if and only if we additionally have

$$\frac{\rho}{\alpha_1} \notin \{1, k\},$$

as is shown in Section 3F. Note that here we cannot have $\rho/\alpha_1 = 1$, and when $\rho/\alpha_1 = k$ from Section 3F we see that the Hessian determinant vanishes of order $2k + \tilde{k} - 2$, where $\tilde{k}$ is the smallest positive integer such that $b_{k+\tilde{k}} \neq 0$ (it is also possible $\tilde{k} = \infty$ with the obvious interpretation).

Secondly, here we can calculate explicitly from the derivatives $b_{\tau_2} = g^{(\tau_2)}(t_0)$ the number $\tilde{k}$ in the normal form (ii.w) (see (3-26) in Section 3E). This is already known for homogeneous polynomials [Ferreyra et al. 2004].

Case $\boldsymbol{\rho = \pm 1,\ \alpha_1 \neq \rho,\ \alpha_2 \neq \rho,\ \alpha_1 \neq \alpha_2}$.

conditions	case
$b_1 = 0,\ b_0 = 0,\ t_0 = 0$	normal form (i.y1)
$b_1 = 0,\ b_0 = 0,\ t_0 \neq 0$	normal form (vi)
$b_1 = 0,\ b_0 \neq 0,\ k = 2$	(ND)
$b_1 = 0,\ b_0 \neq 0,\ k \geq 3,\ t_0 = 0$	normal form (ii.y)
$b_1 = 0,\ b_0 \neq 0,\ k \geq 3,\ t_0 \neq 0$	normal form (iv)
$b_1 \neq 0,\ k \geq 3$	(ND)
$b_1 \neq 0,\ k = 2,\ t_0 = 0$	(ND), or normal form (iii), or normal form (i.w2)
$b_1 \neq 0,\ k = 2,\ t_0 \neq 0$	(ND), or (FP), or normal form (v), or normal form (i.w2)

The first (ND) case $b_1 = 0,\ b_0 \neq 0,\ k = 2$ follows from Lemma 3.2 (1), (2.a), and (3.c), and the second (ND) case $b_1 \neq 0,\ k \geq 3$ from Lemma 3.2 (1) and (2.a). For the very last two cases (namely, $b_1 \neq 0,\ k = 2,\ t_0 = 0$ and $b_1 \neq 0,\ k = 2,\ t_0 \neq 0$) we refer the reader, as usual, to Section 3E.

Note that at this point our considerations have proven Lemma 3.3, except for the Hessian part.

3E. ***The case when $\rho \neq \alpha_2$, $b_1 \neq 0$, $k = 2$.*** In this subsection we shall discuss the remaining cases where y-coordinates did not suffice and all of which (as one easily sees from the tables in the previous two subsections) satisfy $\rho \neq \alpha_2,\ b_1 \neq 0,\ k = 2$. Here it will turn out that we are either in the (ND) case, or the (FP) case, or that we need to use the w-coordinates. In this case the form of the function ϕ in y-coordinates is according to (3-7) equal to

$$\phi^y(y) = (v_1+y_1)^{\rho/\alpha_1}(b_0-t_0b_1)+(v_2+y_2+\gamma v_2 v_1^{-1} y_1)\,(v_1+y_1)^{(\rho-\alpha_2)/\alpha_1}b_1+(y_2-y_1^2\omega(y_1))^2 r(y),$$

where $r(0) \neq 0$, and, as noted in Remark 3.1, $\omega \equiv 0$ if and only if $\gamma = 1$ or $t_0 = 0$, and otherwise $\omega(0) \neq 0$. By Lemma 3.2 (1) and (2.a), we have

$$\partial_2^2\phi^y(0) \neq 0 \quad \text{and} \quad \partial_1\partial_2\phi^y(0) \neq 0,$$

i.e., the y_2^2-term and the y_1y_2-term in Taylor expansion of ϕ^y do not vanish. Therefore, depending on what the coefficient of the y_1^2-term is, it can happen that the Hessian determinant vanishes or not.

Case* (ND) *and the definition of z-coordinates. If the Hessian determinant does not vanish, we are in the nondegenerate case. Otherwise, if the Hessian determinant does vanish, then since $\partial_2^2\phi(v) \neq 0$ (which is by definition equivalent to $k = 2$), there is a coordinate system of the form

$$\begin{aligned} x_1 &= v_1 + z_1, \\ x_2 &= v_2 + z_2 + Az_1, \end{aligned}$$

with A unique, such that $\phi^x(x) = \phi^z(z)$, and such that the z_1^2- and z_1z_2-terms in Taylor expansion of ϕ^z at the origin vanish, i.e.,

$$\partial_1^2\phi^z(0) = 0 \quad \text{and} \quad \partial_1\partial_2\phi^z(0) = 0.$$

In particular, the coordinate systems y and z cannot coincide since the term y_1y_2 does not vanish. This implies $B := A - \gamma v_2 v_1^{-1} \neq 0$ (compare (3-6) and (3-10)).

Case* (FP) *and the reduction to $A \neq 0$. Let us now prove that we may reduce ourselves to the case

$$A \neq 0.$$

If $t_0 = 0$ (i.e., $v_2 = 0$), then we always have $A = B \neq 0$. The second possibility is $t_0 \neq 0$, and if in this case we would have $A = 0$, then z- and x-coordinates would coincide (up to a translation) which implies $\partial_{x_1}^2\phi^x(v) = \partial_{z_1}^2\phi^z(0) = 0$. Thus, by flipping coordinates, we would have that the k associated to the flipped coordinates is ≥ 3, and so we would be in the case where the y-coordinates associated to the flipped coordinates would suffice; i.e., we could apply Lemma 3.3.

This is also the reason why in the case when $\rho = 1,\ \alpha_1 = 1$, and $\alpha_2 \neq 1$, it always sufficed to flip coordinates. The calculation below the corresponding table in Section 3D shows that $\mathcal{H}_\phi(v) = 0$ implies $\partial_1^2\phi(v) = \partial_1\partial_2\phi(v) = 0$, which in turn implies that one always has $A = 0$.

The normal form in z-coordinates. Now that we may assume $A \neq 0$, our first step is to write down the Euler equation for homogeneous functions in z-coordinates. The Euler equation is

$$\rho\phi(x) = \alpha_1 x_1 \partial_1 \phi(x) + \alpha_2 x_2 \partial_2 \phi(x).$$

By the definition of z-coordinates we have

$$\partial_{x_1} = \partial_{z_1} - A\partial_{z_2} \quad \text{and} \quad \partial_{x_2} = \partial_{z_2}.$$

Thus, the Euler equation in z-coordinates is

$$\begin{aligned} \rho\phi^z(z) &= \alpha_1(v_1+z_1)\partial_1\phi^z(z) - \alpha_1 A(v_1+z_1)\partial_2\phi^z(z) + \alpha_2(v_2+z_2+Az_1)\partial_2\phi^z(z) \\ &= \alpha_1(v_1+z_1)\partial_1\phi^z(z) + (-\alpha_1 v_1 B + A(-\alpha_1+\alpha_2)z_1 + \alpha_2 z_2)\partial_2\phi^z(z). \end{aligned} \tag{3-14}$$

We now claim that if $\partial_1^{\tau_1+1}\phi^z(0) = \partial_1^{\tau_1}\partial_2\phi^z(0) = 0$ for all $1 \leq \tau_1 < N$ for some $N \geq 2$, then $\partial_1^{N+1}\phi^z(0) = 0$ if and only if $\partial_1^N\partial_2\phi^z(0) = 0$. But this is almost obvious. Namely, we just take the derivative ∂_1^N at 0 in the above Euler equation and get

$$\rho\partial_1^N\phi^z(0) = \alpha_1 v_1 \partial_1^{N+1}\phi^z(0) + \alpha_1 N \partial_1^N\phi^z(0) - \alpha_1 v_1 B \partial_1^N\partial_2\phi^z(0) + AN(-\alpha_1+\alpha_2)\partial_1^{N-1}\partial_2\phi^z(0).$$

Using the assumption on vanishing derivatives we get

$$\partial_1^{N+1}\phi^z(0) = B\partial_1^N\partial_2\phi^z(0). \tag{3-15}$$

As we noted above $B \neq 0$ and our claim follows.

Now recall that $\partial_1^2\phi^z(0) = 0$ and $\partial_1\partial_2\phi^z(0) = 0$. Thus, the previously proved claim implies in particular by an inductive argument in N that either there is a $\tilde{k} \in \mathbb{N}$ such that $3 \leq \tilde{k} < \infty$, satisfying

$$\tilde{k} = \min\{j \geq 2 : \partial_1^j\phi^z(0) \neq 0\} = \min\{j \geq 2 : \partial_1^{j-1}\partial_2\phi^z(0) \neq 0\},$$

and

$$\phi_v^z(z) = z_1^{\tilde{k}} r_1(z) + z_1^{\tilde{k}-1} z_2 r_2(z) + z_2^2 r_3(z), \tag{3-16}$$

where $r_i(0) \neq 0,\ i = 1, 2, 3$, or that

$$\phi_v^z(z) = z_1^N r_{N,1}(z) + z_1^{N-1} z_2 r_{N,2}(z) + z_2^2 r_3(z)$$

for any $N \in \mathbb{N}$, which we shall consider as the case when $\tilde{k} = \infty$. Here the $r_{N,*}$ are zero at the origin.

The normal form in w-coordinates. It will be advantageous to use w-coordinates where, unlike in (3-16), the $w_1^{\tilde{k}-1}w_2$-term is no longer present; i.e., we may write

$$\phi_v^w(w) = w_1^{\tilde{k}} r_1(w) + w_2^2 r_2(w). \tag{3-17}$$

This fact follows directly from (3-15) and from

$$\partial_{w_1} = \partial_{z_1} \quad \text{and} \quad \partial_{w_2} = \partial_{z_2} - \frac{1}{B}\partial_{z_1},$$

which we get from the definition of w coordinates (3-10). Actually, we can gain more information, especially in the case when $\gamma = 1$. To see this let us rewrite the Euler equation in w-coordinates by using (3-14):

$$\begin{aligned}\frac{\rho}{\alpha_1}\phi^w(w) &= \Big(v_1+w_1-\frac{1}{B}w_2\Big)\partial_1\phi^w(w)+\Big(-v_1B+A(\gamma-1)\Big(w_1-\frac{1}{B}w_2\Big)+\gamma w_2\Big)\Big(\partial_2+\frac{1}{B}\partial_1\Big)\phi^w(w)\\ &=\bigg(\frac{B+A(\gamma-1)}{B}w_1+\frac{(B-A)(\gamma-1)}{B^2}w_2\bigg)\partial_1\phi^w(w)\\ &\qquad+\bigg(-v_1B+A(\gamma-1)w_1+\frac{B\gamma-A(\gamma-1)}{B}w_2\bigg)\partial_2\phi^w(w).\end{aligned}$$

Case $\gamma = 1$. Here the Euler equation reduces to

$$\frac{\rho}{\alpha_1}\phi^w(w) = w_1\partial_1\phi^w(w) + (-v_1B + w_2)\partial_2\phi^w(w). \tag{3-18}$$

Taking the $\partial^\tau = \partial_1^{\tau_1}\partial_2^{\tau_2}$-derivative and evaluating at 0 one gets

$$\frac{\rho}{\alpha_1}\partial^\tau\phi^w(0) = \tau_1\partial^\tau\phi^w(0) - v_1B\partial_1^{\tau_1}\partial_2^{\tau_2+1}\phi^w(0) + \tau_2\partial^\tau\phi^w(0),$$

which can be rewritten as

$$\Big(\frac{\rho}{\alpha_1} - |\tau|\Big)\partial^\tau\phi^w(0) = -v_1B\partial_1^{\tau_1}\partial_2^{\tau_2+1}\phi^w(0).$$

From this and the fact from (3-17) that $\partial^\tau\phi^w(0) = 0$ for all τ satisfying $|\tau| = \tau_1 + \tau_2 \geq 2,\ 0 \leq \tau_1 \leq \tilde{k} - 1$, and $0 \leq \tau_2 \leq 1$, one easily gets by induction on τ_2 that

$$\partial_1^{\tau_1}\partial_2^{\tau_2}\phi^w(0) = 0, \quad \text{when } |\tau| = \tau_1 + \tau_2 \geq 2,\ 1 \leq \tau_1 \leq \tilde{k} - 1. \tag{3-19}$$

We now prove a stronger claim, namely that

$$\begin{aligned}\partial_1^{\tau_1}\phi^w(0, w_2) &\equiv 0 \quad \text{for } 2 \leq \tau_1 \leq \tilde{k} - 1,\\ \partial_1\phi^w(0, w_2) &\equiv \partial_1\phi^w(0).\end{aligned} \tag{3-20}$$

In order to obtain this we take the $\partial_1^{\tau_1}$-derivative in (3-18) and evaluate it at $(0, w_2)$ to get

$$\Big(\frac{\rho}{\alpha_1} - \tau_1\Big)\partial_1^{\tau_1}\phi^w(0, w_2) = (-v_1B + w_2)\partial_2\partial_1^{\tau_1}\phi^w(0, w_2).$$

We note that this is a simple ordinary differential equation in w_2 of first order. It has a unique solution for $2 \leq \tau_1 \leq \tilde{k} - 1$ since $-v_1B + w_2 \neq 0$ for small w_2, and since we can take (3-19) as initial conditions. The claim for $2 \leq \tau_1 \leq \tilde{k} - 1$ follows since $\partial_1^{\tau_1}\phi^w(0, w_2) \equiv 0$ is obviously a solution. For $\tau_1 = 1$ we note that the case $\rho/\alpha_1 - \tau_1 = 0$ is trivial, and the solution is a unique constant function (necessarily equal to $\partial_1\phi^w(0)$). When $\tau_1 = 1$ and $\rho/\alpha_1 - \tau_1 \neq 0$, the differential equation evaluated at $w_2 = 0$ gives us that $\partial_1\partial_2\phi^w(0) = 0$ implies $\partial_1\phi^w(0) = 0$, which again means that $\partial_1\phi^w(0, w_2) \equiv 0$ is the unique solution of the given differential equation. We have thus proven (3-20).

Now by using Taylor approximation in w_1 for a fixed w_2, and the just-proven fact for the mapping $w_2 \mapsto \partial_1^{\tau_1}\phi^w(0, w_2)$ for $1 \le \tau_1 \le \tilde{k}-1$, we obtain that the normal form of ϕ^w (3-17) in the case $\gamma = 1$ can be rewritten as

$$\phi_v^w(w) = w_1^{\tilde{k}} r_1(w) + w_2^2 r_2(w_2),$$

where $r_1(0), r_2(0) \neq 0$. Note that now r_2 depends only on w_2. This corresponds to normal form (ii.w) when $\tilde{k}$ is finite and to normal form (i.w1) otherwise.

Case $\gamma \neq 1$. In this case we use our assumption that $A \neq 0$ in a critical way. Here it will be important to know what happens with $\partial_1^{\tau_1}\partial_2^2\phi^w(0)$ for $0 \le \tau_1 \le \tilde{k}-1$, and also how one can rewrite the normal form of the Hessian determinant $\mathcal{H}_{\phi^w}$ (and in particular its root).

Let us begin by taking the $\partial_1^{\tau_1}\partial_2$-derivative of the Euler equation in w-coordinates and evaluating it at $w = 0$. One gets

$$\begin{aligned}\frac{\rho}{\alpha_1}\partial_1^{\tau_1}\partial_2\phi^w(0) = \tau_1\frac{B + A(\gamma-1)}{B}\partial_1^{\tau_1}\partial_2\phi^w(0) + \frac{(B-A)(\gamma-1)}{B^2}\partial_1^{\tau_1+1}\phi^w(0) \\ - v_1 B\partial_1^{\tau_1}\partial_2^2\phi^w(0) + \tau_1 A(\gamma-1)\partial_1^{\tau_1-1}\partial_2^2\phi^w(0) + \frac{B\gamma - A(\gamma-1)}{B}\partial_1^{\tau_1}\partial_2\phi^w(0).\end{aligned}$$

Now recall again from (3-17) that $\partial^\tau\phi^w(0) = 0$ holds for any τ satisfying $|\tau| = \tau_1 + \tau_2 \ge 2$, $0 \le \tau_1 \le \tilde{k}-1$, and $0 \le \tau_2 \le 1$. Thus, if $1 \le \tau_1 \le \tilde{k}-2$ then we get

$$v_1 B\partial_1^{\tau_1}\partial_2^2\phi^w(0) = \tau_1 A(\gamma-1)\partial_1^{\tau_1-1}\partial_2^2\phi^w(0), \tag{3-21}$$

and if $\tau_1 = \tilde{k}-1$, then

$$v_1 B\partial_1^{\tilde{k}-1}\partial_2^2\phi^w(0) = \frac{(B-A)(\gamma-1)}{B^2}\partial_1^{\tilde{k}}\phi^w(0) + (\tilde{k}-1)A(\gamma-1)\partial_1^{\tilde{k}-2}\partial_2^2\phi^w(0);$$

i.e., since $B - A = -\gamma v_2 v_1^{-1}$, we can rewrite this as

$$v_1 B\partial_1^{\tilde{k}-1}\partial_2^2\phi^w(0) + \frac{v_2\gamma(\gamma-1)}{v_1 B^2}\partial_1^{\tilde{k}}\phi^w(0) = (\tilde{k}-1)A(\gamma-1)\partial_1^{\tilde{k}-2}\partial_2^2\phi^w(0). \tag{3-22}$$

Now since $A, B, v_1 \neq 0$, and $\gamma \neq 1$, from (3-21) we may conclude by induction on τ_1 that for $0 \le \tau_1 \le \tilde{k}-2$ one has

$$\partial_1^{\tau_1}\partial_2^2\phi^w(0) \neq 0.$$

In order to unravel what is happening with $\partial_1^{\tilde{k}-1}\partial_2^2\phi^w(0)$ we need to investigate the root of $\mathcal{H}_{\phi^w}$. For this we want to solve the equation

$$x_2 - t_0 x_1^\gamma = y_2 - \binom{\gamma}{2} v_1^{-2} v_2 y_1^2 + \mathcal{O}(y_1^3) = 0,$$

in the w-coordinates, representing the homogeneity curve through v. Recall that by (3-11) we have $y_1 = w_1 - w_2/B$, $y_2 = Bw_1$, and so we want to solve

$$Bw_1 - \binom{\gamma}{2} v_1^{-2} v_2\Big(w_1 - \frac{1}{B}w_2\Big)^2 + \mathcal{O}\Big(\Big(w_1 - \frac{1}{B}w_2\Big)^3\Big) = 0$$

for the w_1-variable in terms of the w_2-variable when $|w_1|$, $|w_2|$ are small numbers. Using the above equation one gets by a simple calculation that

$$w_1 = \frac{v_2\gamma(\gamma-1)}{2v_1^2B^3}w_2^2 + \mathcal{O}(w_2^3) = w_2^2\tilde{\omega}(w_2), \tag{3-23}$$

and $\tilde{\omega} \equiv 0$ if and only if $v_2 = 0 = t_0$. Note that we have the precise value of $\tilde{\omega}(0)$. Using this we can now write down the normal form of w as

$$\begin{aligned} \phi_v^w(w) &= w_1^{\tilde{k}}r_1(w) + w_2^2r_2(w) \\ &= (w_1 - w_2^2\tilde{\omega}(w_2))^{\tilde{k}}r_1(w) + w_2^2\left(r_2(w) + \tilde{k}w_1^{\tilde{k}-1}\frac{v_2\gamma(\gamma-1)}{2v_1^2B^3}r_1(w)\right) + \mathcal{O}(w_2^4) \\ &= (w_1 - w_2^2\tilde{\omega}(w_2))^{\tilde{k}}\tilde{r}_1(w) + w_2^2\tilde{r}_2(w), \end{aligned} \tag{3-24}$$

where one can easily check by using (3-21), (3-22), (3-23), and (3-24) that $\partial_1^{\tau_1}\tilde{r}_2(0) \neq 0$ for all $0 \leq \tau_1 \leq \tilde{k}-1$, and that in fact one has the relations

$$v_1B\partial_1^{\tau_1}\tilde{r}_2(0) = \tau_1A(\gamma-1)\partial_1^{\tau_1-1}\tilde{r}_2(0)$$

for $1 \leq \tau_1 \leq \tilde{k}-1$. If $\tilde{k} = \infty$, then the above normal form in (3-24) corresponds to normal form (i.w2). Otherwise we have $3 \leq \tilde{k} < \infty$ and two subcases. Namely, if $t_0 \neq 0$ (i.e., $\tilde{\omega}(0) \neq 0$), then the above normal form corresponds to normal form (v), and if $t_0 = 0$ (and therefore $\tilde{\omega} \equiv 0$), then it corresponds to normal form (iii).

Determining $\tilde{k}$ in the special case when $\rho = \pm 1$ and $\alpha_1 = \alpha_2 \neq \rho$. According to the last line of the corresponding table for this case in Section 3D here we may assume $b_0, b_1 \neq 0$, and note that here $\gamma = 1$. We prove that the Hessian determinant of ϕ vanishes at v if and only if

$$b_2 = (1-\rho\alpha_1)\frac{b_1^2}{b_0} = \left(1 - \frac{\alpha_1}{\rho}\right)\frac{b_1^2}{b_0}. \tag{3-25}$$

In this case we furthermore have that if $\tilde{k} < \infty$ (corresponding to the case (ii.w)), then

$$\begin{aligned} b_j &= (\rho\alpha_1)^j j!\binom{\rho/\alpha_1}{j}\frac{b_1^j}{b_0^{j-1}}, \quad \text{for } j = 2, \dots, \tilde{k}-1, \\ b_{\tilde{k}} &\neq (\rho\alpha_1)^{\tilde{k}}\tilde{k}!\binom{\rho/\alpha_1}{\tilde{k}}\frac{b_1^{\tilde{k}}}{b_0^{\tilde{k}-1}}, \end{aligned} \tag{3-26}$$

and if $\tilde{k} = \infty$ (corresponding to the case (i.w1)), then

$$b_j = (\rho\alpha_1)^j j!\binom{\rho/\alpha_1}{j}\frac{b_1^j}{b_0^{j-1}} \quad \text{for } j \in \{2, 3, \dots\}.$$

These formulae have already been shown for homogeneous polynomials in [Ferreyra et al. 2004, Lemma 2.2]. Therefore, we only sketch how one can prove them in our slightly more general case.

Recall from (3-9) that we have the formal series for ϕ at $y = 0$:

$$\begin{aligned}\phi^y(y) &\approx (v_1+y_1)^{\rho/\alpha_1} b_0 + y_2(v_1+y_1)^{\rho/\alpha_1-1} b_1 + \frac{1}{2!}(v_1+y_1)^{\rho/\alpha_1-2} y_2^2 b_2 + \cdots \\ &= \sum_{j=0}^{\infty} \frac{b_j}{j!}(v_1+y_1)^{\rho/\alpha_1-j} y_2^j.\end{aligned}$$

From this one gets

$$\partial_1^2\phi^y(0) = b_0\frac{\rho}{\alpha_1}\left(\frac{\rho}{\alpha_1}-1\right)v_1^{\rho/\alpha_1-2}, \quad \partial_1\partial_2\phi^y(0) = b_1\left(\frac{\rho}{\alpha_1}-1\right)v_1^{\rho/\alpha_1-2}, \quad \partial_2^2\phi^y(0) = b_2 v_1^{\rho/\alpha_1-2},$$

and (3-25) follows by a direct computation (recall that $\mathcal{H}_{\phi^y}(0) = 0$ if and only if $\mathcal{H}_\phi(v) = 0$). More generally, we have

$$\partial^\tau\phi^y(0) = \tau_1!\binom{\rho/\alpha_1-\tau_2}{\tau_1} v_1^{\rho/\alpha_1-|\tau|} b_{\tau_2}. \tag{3-27}$$

Let us now determine the relation between y and z when the Hessian determinant vanishes. We may write

$$\begin{aligned} z_1 &= y_1, & \partial_{z_1} &= \partial_{y_1} + B\partial_{y_2}, \\ z_2 &= y_2 - By_1, & \partial_{z_2} &= \partial_{y_2}.\end{aligned}$$

Then by (3-25) one gets that $\partial_1^2\phi^z(0) = \partial_1\partial_2\phi^z(0) = 0$ if and only if

$$B = -\frac{b_0}{b_1}\frac{\rho}{\alpha_1}.$$

From this we can determine the constant A since it is equal to $t_0 + B$, i.e., $A = v_2/v_1 - (\rho b_0)/(\alpha_1 b_1)$.

One can now directly prove (3-26) by induction in j by using (3-27), and the fact that $\partial_1^j\phi^z(0) = 0$ for $2 \le j < \tilde{k}$ and $\partial_1^{\tilde{k}}\phi^z(0) \ne 0$ is equivalent to

$$\begin{aligned}&\left(\partial_1 - \frac{b_0}{b_1}\frac{\rho}{\alpha_1}\partial_2\right)^j \phi^y(0) = 0, \quad j = 2, \dots, \tilde{k}-1, \\ &\left(\partial_1 - \frac{b_0}{b_1}\frac{\rho}{\alpha_1}\partial_2\right)^{\tilde{k}} \phi^y(0) \ne 0.\end{aligned}$$

We have already checked the induction base $j = 2$.

3F. ***Order of vanishing of the Hessian determinant.*** In this subsection we determine the normal forms of the Hessian determinant of ϕ (or more precisely, the order of vanishing of the Hessian determinant of ϕ), as listed in Section 3A. We recall from Section 3A that if $v_1 > 0$, then one can write

$$\mathcal{H}_\phi(x) = (x_2 - t_0 x_1^\gamma)^N r_0(x),$$

where either r_0 is flat in v (which we consider as the case $N = \infty$), or $r_0(v) \ne 0$ and $0 \le N < \infty$. It remains to determine N from the information provided by the normal forms of ϕ. We note that

$$N = \min\{j \ge 0 : (\partial_2^j\mathcal{H})(v) \ne 0\}.$$

***Normal form* (i.y1).** First we note by the normal form tables above that this normal form appears only in cases when either $\gamma = 1$ or $t_0 = v_2 = 0$, and so we have $\omega \equiv 0$. Thus, by (3-9) the function ϕ_v^y has the formal expansion

$$\phi_v^y(y) = \frac{1}{k!} y_2^k (y_1 + v_1)^{\rho/\alpha_1 - k\gamma} g_k(y_2(y_1+v_1)^{-1} + t_0) \\ \approx \sum_{j=k}^{\infty} \frac{b_j}{j!} y_2^j (y_1+v_1)^{\rho/\alpha_1 - j\gamma}, \tag{3-28}$$

and the Hessian determinant vanishes along $y_2 = 0$, which means we need to determine what is the least N such that $(\partial_2^N \mathcal{H}_{\phi^y})(0) \neq 0$. From the above expansion one obtains

$$\partial_1^{\tau_1}\partial_2^{\tau_2}\phi^y(0) = 0, \quad |\tau| = \tau_1 + \tau_2 \geq 2,\ 0 \leq \tau_2 \leq k-1, \\ \partial^\tau \phi^y(0) = \tau_1! \binom{\rho/\alpha_1 - \gamma\tau_2}{\tau_1} v_1^{\rho/\alpha_1 - \tau_1 - \gamma\tau_2} b_{\tau_2}, \quad \tau_2 \geq k. \tag{3-29}$$

By applying the general Leibniz rule to the definition of the Hessian determinant we get

$$\partial_2^N \mathcal{H}_{\phi^y} = \partial_2^N (\partial_1^2\phi^y \partial_2^2\phi^y - (\partial_1\partial_2\phi^y)^2) \\ = \sum_{n=0}^{N} \binom{N}{n} (\partial_1^2\partial_2^n\phi^y\, \partial_2^{N+2-n}\phi^y - \partial_1\partial_2^{n+1}\phi^y\, \partial_1\partial_2^{N+1-n}\phi^y), \tag{3-30}$$

and one can easily check by using (3-29) that $\partial_2^N \mathcal{H}_{\phi^y}(0) = 0$ for $N < 2k-2$. For $N = 2k-2$ we get

$$\partial_2^{2k-2}\mathcal{H}_{\phi^y}(0) = \binom{2k-2}{k} \partial_1^2\partial_2^k\phi^y(0)\partial_2^k\phi^y(0) - \binom{2k-2}{k-1} (\partial_1\partial_2^k\phi^y)^2(0) \\ = \left[\binom{2k-2}{k}\left(\frac{\rho}{\alpha_1} - k\gamma\right)\left(\frac{\rho}{\alpha_1} - k\gamma - 1\right) - \binom{2k-2}{k-1}\left(\frac{\rho}{\alpha_1} - k\gamma\right)^2\right] b_k^2 v_1^{2\rho/\alpha_1 - 2k\gamma - 2} \\ = \left[\frac{k-1}{k}\left(\frac{\rho}{\alpha_1} - k\gamma - 1\right) - \left(\frac{\rho}{\alpha_1} - k\gamma\right)\right]\binom{2k-2}{k-1}\left(\frac{\rho}{\alpha_1} - k\gamma\right) b_k^2 v_1^{2\rho/\alpha_1 - 2k\gamma - 2}.$$

Thus, $\partial_2^{2k-2}\mathcal{H}_{\phi^y}(0) \neq 0$ if and only if

$$\frac{\rho}{\alpha_1} \notin \{k\gamma,\ k\gamma + 1 - k\}. \tag{3-31}$$

Let us now denote by $\tilde{k}$ the smallest positive integer such that $b_{k+\tilde{k}} \neq 0$; i.e., we have

$$b_{k+j} = 0, \quad 0 < j < \tilde{k}, \\ b_{k+\tilde{k}} \neq 0.$$

Case when $\rho/\alpha_1 = k\gamma$. By examining the term $j = k$ in (3-28) we note that in this case we additionally have

$$\partial_2^k \phi^y(0) \neq 0 \\ \partial_1^{\tau_1}\partial_2^k\phi^y(0) = 0, \quad \tau_1 \geq 1.$$

Now by using the information in (3-29), the above additional assumption that $b_{k+j}=0$ for $0<j<\tilde{k}$, $b_{k+\tilde{k}}\neq 0$, and the Leibniz formula (3-30) a straightforward calculation yields that $\partial_2^N \mathcal{H}_{\phi^y}(0)=0$ for $N<2k+\tilde{k}-2$ and $\partial_2^{2k+\tilde{k}-2}\mathcal{H}_{\phi^y}(0)\neq 0$; i.e., we have the precise order of vanishing of the Hessian determinant.

Case when $\rho/\alpha_1 = k\gamma+1-k$. Again, by a straightforward calculation using the Leibniz formula one gets that $\partial_2^N \mathcal{H}_{\phi^y}(0)=0$ for $N<2k+\tilde{k}-2$ and we have for $N=2k+\tilde{k}-2$

$$\partial_2^{2k+\tilde{k}-2}\mathcal{H}_{\phi^y}(0)=\binom{2k+\tilde{k}-2}{k}\partial_1^2\partial_2^k\phi^y(0)\partial_2^{k+\tilde{k}}\phi^y(0) + \binom{2k+\tilde{k}-2}{k-2}\partial_1^2\partial_2^{k+\tilde{k}}\phi^y(0)\partial_2^k\phi^y(0)-2\binom{2k+\tilde{k}-2}{k-1}\partial_1\partial_2^k\phi^y(0)\partial_1\partial_2^{k+\tilde{k}}\phi^y(0).$$

Thus

$$\left(\binom{2k+\tilde{k}-2}{k-2}\right)^{-1}\partial_2^{2k+\tilde{k}-2}\mathcal{H}_{\phi^y}(0)=\frac{(k+\tilde{k})(k+\tilde{k}-1)}{(k-1)k}\partial_1^2\partial_2^k\phi^y(0)\partial_2^{k+\tilde{k}}\phi^y(0) + \partial_1^2\partial_2^{k+\tilde{k}}\phi^y(0)\partial_2^k\phi^y(0)-\frac{2(k+\tilde{k})}{k-1}\partial_1\partial_2^k\phi^y(0)\,\partial_1\partial_2^{k+\tilde{k}}\phi^y(0).$$

This is equal to zero when the expression

$$(k+\tilde{k})(k+\tilde{k}-1)\partial_1^2\partial_2^k\phi^y(0)\partial_2^{k+\tilde{k}}\phi^y(0) + (k-1)k\partial_1^2\partial_2^{k+\tilde{k}}\phi^y(0)\partial_2^k\phi^y(0)-2k(k+\tilde{k})\partial_1\partial_2^k\phi^y(0)\,\partial_1\partial_2^{k+\tilde{k}}\phi^y(0)$$

equals zero. Plugging in the values of the derivatives from (3-29) one obtains that the above expression is equal to

$$(k+\tilde{k})(k+\tilde{k}-1)(1-k)(-k)+(k-1)k(1-k-\tilde{k}\gamma)(-k-\tilde{k}\gamma)-2k(k+\tilde{k})(1-k)(1-k-\tilde{k}\gamma),$$

up to the nonzero constant factor $v_1^{-2k-\tilde{k}\gamma}b_k b_{k+\tilde{k}}$. Factoring out $(1-k)(-k)$ we get

$$(k+\tilde{k})(k+\tilde{k}-1)+(k+\tilde{k}\gamma-1)(k+\tilde{k}\gamma)-2(k+\tilde{k})(k+\tilde{k}\gamma-1)$$

and this equals zero if and only if $\gamma\in\{1,(\tilde{k}+1)/\tilde{k}\}$.

The condition $\rho/\alpha_1=k\gamma+1-k$ tells us that if $\gamma=1$ then $\rho=\alpha_1=\alpha_2=1$, and from the normal form tables we see that this is precisely when the Hessian determinant vanishes to infinite order.

In the case $\gamma=(\tilde{k}+1)/\tilde{k}$ we get that $\rho=1$, $\alpha_1=\tilde{k}/(k+\tilde{k})$ and $\alpha_2=(\tilde{k}+1)/(k+\tilde{k})$. Here the order of vanishing of the Hessian determinant depends explicitly on the values b_j, and so, in contrast to the previous cases, one cannot relate in an easy way the order of vanishing of the Hessian determinant and the form of ϕ in (3-28). As we shall not need the precise order of vanishing of the Hessian determinant in this case, we do not pursue this question further.

Other normal forms. First we recall that normal form (i.y2) was dealt with in Section 3B, and there it was already determined that the Hessian vanishes of infinite order (i.e., it is flat).

In all the remaining normal forms we use either y- or w-coordinates, and so (as already noted in Section 3A) the Hessian determinant in these coordinates has the normal form

$$\mathcal{H}_{\phi^u}(u) = (u_2 - u_1^2 \psi(u_1))^N r_0(u),$$

where u can represent either y- or w-coordinates, and where either N is finite and $r_0(0) \neq 0$, or the Hessian determinant is flat (in which case we consider N to be infinite). The function ψ is equal to either ω or $\tilde{\omega}$. Our goal is to determine $N = \min\{j \geq 0 : (\partial_2^j \mathcal{H}_{\phi^u})(0) \neq 0\}$.

We first note that we can rewrite all the remaining normal forms as either

$$\phi_v^u(u) = (u_2 - u_1^2 \psi(u_1))^{k_0} r(u) \tag{3-32}$$

or

$$\phi_v^u(u) = u_1^2 r_1(u) + u_2^{k_0} r_2(u), \tag{3-33}$$

where $r(0), \psi(0), r_1(0), r_2(0) \neq 0$, and $k_0 \geq 2$ in the first case and $k_0 \geq 3$ in the second. In the second case $k_0 = \infty$ is allowed with an obvious interpretation. Note that the second case (3-33) includes normal forms (ii), (iii), (iv), (v), and also subcases of (i) where the w-coordinates are used. Also note that this is slightly different compared to the three forms mentioned before the detailed table of normal forms in Section 3A.

For both cases (3-32) and (3-33) one can use the Leibniz rule (3-30) and the information on the Taylor series of ϕ_v^u gained from these normal forms to obtain the order of vanishing of the Hessian determinant (in the ∂_{u_2}-direction) by a direct calculation. In the first case (3-32) one gets that the order of vanishing is $N = 2k_0 - 3$ and in the second case (3-33) one gets that $N = k_0 - 2$ (or that the Hessian determinant is flat if $k_0 = \infty$).

4. Fourier restriction when a mitigating factor is present

In this section we prove Theorem 1.1, i.e., the Fourier restriction estimate

$$\|\hat{f}\|_{L^2(\mathrm{d}\mu)} \leq C \|f\|_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})},$$

where μ is the surface measure

$$\langle \mu, f \rangle = \int_{\mathbb{R}^2 \setminus \{0\}} f(x, \phi(x)) \, |\mathcal{H}_\phi(x)|^\sigma \, \mathrm{d}x$$

and the exponents are

$$\left(\frac{1}{\mathfrak{p}_1'}, \frac{1}{\mathfrak{p}_3'}\right) = \left(\frac{1}{2} - \sigma, \sigma\right).$$

The gothic letters are used in order to distinguish the endpoint exponents from the dummy ones. We assume $0 \leq \sigma < \frac{1}{2}$ when only adapted normal forms appear, and $0 \leq \sigma \leq \frac{1}{3}$ if a nonadapted normal form appears. Since the case $\sigma = 0$ follows directly by Plancherel, we may assume $\sigma > 0$.

Our assumptions in this case are that the Hessian determinant $\mathcal{H}_\phi$ does not vanish of infinite order anywhere (i.e., condition (H2) is satisfied). According to Section 2B we may restrict our attention to the localized measure

$$\langle \mu_{0,v}, f\rangle = \int_{\mathbb{R}^2\setminus\{0\}} f(x, \phi_v(x-v))\eta_v(x)|\mathcal{H}_\phi(x)|^\sigma \,\mathrm{d}x,$$

where $v=(v_1,v_2)$ satisfies $v_1\sim 1$, and either $v_2=0$ or $v_2\sim 1$, and where η_v is a smooth nonnegative function with support in a small neighborhood of v.

After changing to y- or w-coordinates from Section 3 we get that $\mu_{0,v}$ can be rewritten as

$$\langle \nu, f\rangle = \int f(x, \phi_{\mathrm{loc}}(x))\, a(x)\, |\mathcal{H}_{\phi_{\mathrm{loc}}}(x)|^\sigma \,\mathrm{d}x,$$

where now a is smooth, nonnegative, and supported in a small neighborhood of the origin, and where we have for ϕ_{loc} the normal form cases (i)–(vi) from Proposition 1.4. Recall that since we assume (H2), in case (i) of Proposition 1.4 the function φ vanishes identically.

The strategy will be to appropriately localize and rescale the problem, and then to use the associated "R^*R" operator. Let us begin by proving modifications of two essentially known results.

Lemma 4.1. *Let $\phi:\Omega\to\mathbb{R}$ be a smooth function on an open set $\Omega\subseteq\mathbb{R}^2$ contained in a ball of radius $\lesssim 1$, and let $\mathcal{H}_\phi=\partial_1^2\phi\partial_2^2\phi-(\partial_1\partial_2\phi)^2$ denote the Hessian determinant of ϕ. We consider the measure defined by*

$$\langle \mu, f\rangle := \int f(x_1,x_2,\phi(x))\, a(x)\,\mathrm{d}x,$$

where $a\in C_c^\infty(\Omega)$ satisfies $\|\partial^\tau a\|_{L^\infty(\Omega)}\lesssim_\tau 1$ for all multiindices τ. If we assume that on Ω we have $|\partial_1^2\phi|\sim 1$, $|\partial^\tau\phi|\lesssim_\tau 1$ for all multiindices τ, and that $|\mathcal{H}_\phi|\sim\varepsilon$ for a bounded, strictly positive (but possibly small) constant ε, then

$$|\hat{\mu}(\xi)|\lesssim \varepsilon^{-1/2}(1+|\xi|)^{-1}.$$

The claim also holds if ϕ and a depend on ε, assuming that the implicit constants appearing in the lemma can be taken to be independent of ε.

Proof. By compactness and translating we may assume that a is supported on a small neighborhood of the origin. We also assume for simplicity that $|\partial_1\phi|\sim 1$, which can be achieved by applying a linear transformation to μ. The Fourier transform of μ is by definition

$$\hat{\mu}(\xi) = \int e^{-i\Phi(x,\xi)}\, a(x)\,\mathrm{d}x,$$

where the phase function is of the form

$$\Phi(x,\xi) = x_1\xi_1 + x_2\xi_2 + \phi(x)\xi_3,$$

from which one easily sees that unless $|\xi_1|\sim|\xi_3|\gtrsim|\xi_2|$, we have very fast decay independent of ε. Let us define

$$s_1=\frac{\xi_1}{\xi_3},\quad s_2=\frac{\xi_2}{\xi_3},\quad \lambda=\xi_3,$$

and rewrite the phase as

$$\Phi(x,\xi) = \lambda(s_1x_1+s_2x_2+\phi(x)),$$

where now $|s_1|\sim 1$ and $|s_2|\lesssim 1$.

Now either the x_1-derivative of Φ has no zeros on the domain of integration (e.g., when s_1 and $\partial_1\phi(0)$ are of the same sign), in which case we get a fast decay by integrating by parts, or there is a unique zero $x_1^c = x_1^c(x_2; s_1, s_2)$ of the equation $\partial_1\Phi(x,\xi) = 0$ in x_1, depending smoothly on $(x_2; s_1, s_2)$ by the implicit function theorem, i.e., we have the relation

$$s_1 + (\partial_1\phi)(x_1^c, x_2) = 0. \tag{4-1}$$

In this case we apply the stationary phase method and get that

$$\hat{\mu}(\xi) = \lambda^{-1/2} \int e^{-i\lambda\Psi(x_2;s_1,s_2)}\, a(x_2, s_1, s_2; \lambda)\, \mathrm{d}x_2,$$

where a is a smooth function in (x_2, s_1, s_2) and a classical symbol of order 0 in λ, and where

$$\Psi(x_2; s_1, s_2) := s_1 x_1^c + s_2 x_2 + \phi(x_1^c, x_2) = \lambda^{-1}\Phi(x_1^c, x_2, \xi).$$

Taking the x_2-derivative of (4-1) we get that

$$\partial_{x_2} x_1^c(x_2; s_1, s_2) = -\frac{\partial_1\partial_2\phi(x_1^c, x_2)}{\partial_1^2\phi(x_1^c, x_2)},$$

and the x_2-derivative of the new phase is by (4-1):

$$\begin{aligned}\lambda\partial_{x_2}\Psi(x_2; s_1, s_2) &= \lambda(s_1\partial_{x_2}x_1^c + s_2 + \partial_{x_2}x_1^c\,\partial_1\phi(x_1^c, x_2) + \partial_2\phi(x_1^c, x_2)) \\ &= \lambda(s_2 + \partial_2\phi(x_1^c, x_2)).\end{aligned}$$

From this and the expression for $(x_1^c)'$ it follows that

$$\lambda\partial_{x_2}^2\Psi(x_2; s_1, s_2) = \lambda\frac{\mathcal{H}_\phi(x_1^c, x_2)}{\partial_1^2\phi(x_1^c, x_2)} \sim \lambda\varepsilon.$$

Thus, we may apply the van der Corput lemma, which then delivers the claim of the lemma. ☐

The following lemma for obtaining mixed-norm Fourier restriction estimates goes back essentially to [Ginibre and Velo 1992] (see also [Keel and Tao 1998]).

Lemma 4.2. *Assume that we are given a bounded open set $\Omega \subseteq \mathbb{R}^2$ and functions $\Phi \in C^\infty(\Omega; \mathbb{R}^2)$, $\phi \in C^\infty(\Omega; \mathbb{R})$, $a \in L^\infty(\Omega)$. Let us consider the measure*

$$\langle \mu, f\rangle := \int f(\Phi(x), \phi(x))\, a(x)\, \mathrm{d}x$$

and the operator $T : f \mapsto f * \hat{\mu}$. If Φ is injective and its Jacobian is of size $|J_\Phi| \sim A_1$, then the $L^1_{x_3}(\mathbb{R}; L^2_{(x_1,x_2)}(\mathbb{R}^2)) \to L^\infty_{x_3}(\mathbb{R}; L^2_{(x_1,x_2)}(\mathbb{R}^2))$ operator norm of T is bounded (up to a universal constant) by $A_1^{-1}\|a\|_{L^\infty}$. If one has furthermore the estimate

$$|\hat{\mu}(\xi)| \le A_2(1 + |\xi_3|)^{-1},$$

then for any $\sigma \in [0, \frac{1}{2})$ and

$$\left(\frac{1}{p_1'}, \frac{1}{p_3'}\right) = \left(\frac{1}{2} - \sigma, \sigma\right)$$

the $L^{p_3}_{x_3}(\mathbb{R}; L^{p_1}_{(x_1,x_2)}(\mathbb{R}^2)) \to L^{p_3'}_{x_3}(\mathbb{R}; L^{p_1'}_{(x_1,x_2)}(\mathbb{R}^2))$ *operator norm of* T *is bounded* (*up to a constant depending on* σ) *by* $(A_1^{-1}\|a\|_{L^\infty})^{1-2\sigma} A_2^{2\sigma}$.

Proof. Let us first introduce the operator $T_{\xi_3} g := g * \hat{\mu}(\,\cdot\,, \xi_3)$ defined for functions g on $\mathbb{R}^2$ and a fixed $\xi_3 \in \mathbb{R}$. Note that then if one writes a function f on $\mathbb{R}^3$ as $f(\xi_1, \xi_2, \xi_3) = f(\xi', \xi_3) = f_{\xi_3}(\xi')$, then

$$Tf(\xi', \xi_3) = \int (f_{\eta_3-\xi_3} * \hat{\mu}(\,\cdot\,, \eta_3))(\xi')\, \mathrm{d}\eta_3 = \int (T_{\eta_3} f_{\eta_3-\xi_3})(\xi')\, \mathrm{d}\eta_3. \tag{4-2}$$

Now note that the $L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ norm of the convolution operator T_{η_3} is bounded by the L^∞ norm of the function $(x_1, x_2) \mapsto (\mathcal{F}^{-1}_{(x_1,x_2)} \hat{\mu}(\,\cdot\,, \eta_3))(x_1, x_2)$, where for functions on $\mathbb{R}^3$ we denote by $\mathcal{F}^{-1}_{(x_1,x_2)}$ the inverse Fourier transform in the first two variables. Afterwards we can estimate the $L^1 \to L^\infty$ norm of the remaining convolution operator in η_3 by the L^∞ norm in η_3 of the kernel. Thus, for the first claim it suffices to prove that the L^∞ norm of $(\mathcal{F}^{-1}_{(x_1,x_2)} \hat{\mu})(x_1, x_2, \eta_3)$ in all three variables is bounded by $A_1^{-1}\|a\|_{L^\infty}$. In order to obtain this estimate note that $\mathcal{F}^{-1}_{(x_1,x_2)} \hat{\mu}$ is equal by Fourier inversion to the Fourier transform of μ in the third coordinate only, i.e., the distribution given by

$$\begin{aligned}
\langle \mathcal{F}_{x_3}\mu, f\rangle &= \langle \mu, \mathcal{F}_{x_3} f\rangle \\
&= \int (\mathcal{F}_{x_3} f)(\Phi(y), \phi(y))\, a(y)\, \mathrm{d}y \\
&= \iint e^{-i\eta_3\phi(y)}\, f(\Phi(y), \eta_3)\, a(y)\, \mathrm{d}\eta_3\, \mathrm{d}y \\
&= \iint e^{-i\eta_3\phi\circ\Phi^{-1}(x)}\, f(x, \eta_3)\, a\circ\Phi^{-1}(x)\, |J_\Phi(x)|^{-1}\, \mathrm{d}\eta_3\, \mathrm{d}x.
\end{aligned}$$

Thus $(\mathcal{F}_{x_3}\mu)(x_1, x_2, \eta_3)$ coincides a.e. with the function

$$(x, \eta_3) \mapsto e^{-i\eta_3\phi\circ\Phi^{-1}(x)}\, a\circ\Phi^{-1}(x)\, |J_\Phi(x)|^{-1},$$

which is now obviously bounded by $A_1^{-1}\|a\|_{L^\infty}$ up to a constant.

For the second claim note that the $L^1(\mathbb{R}^2) \to L^\infty(\mathbb{R}^2)$ norm of T_{ξ_3} is bounded by $A_2(1 + |\xi_3|)^{-1}$, and as just shown the $L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ norm is bounded up to a constant by $A_1^{-1}\|a\|_{L^\infty}$. Interpolating one gets that the $L^{p_1}(\mathbb{R}^2) \to L^{p_1'}(\mathbb{R}^2)$ norm is bounded by

$$(A_1^{-1}\|a\|_{L^\infty})^{1-2\sigma} A_2^{2\sigma} (1 + |\xi_3|)^{-2\sigma}$$

for $p_1' = \left(\frac{1}{2} - \sigma\right)$ and $\sigma \in \left[0, \frac{1}{2}\right]$. For $\sigma < \frac{1}{2}$ the claim now follows by first applying this bound to the expression (4-2) and subsequently using the (weak) Young inequality in the η_3-variable. □

4A. ***Normal form* (i).** In this case the local form of the phase is

$$\phi_{\mathrm{loc}}(x) = x_2^k r(x),$$

where $r(0) \neq 0$ and the Hessian determinant vanishes of order $2k + k_0 - 2$ for some $k_0 \geq 0$, i.e., it has the normal form

$$\mathcal{H}_{\phi_{\mathrm{loc}}}(x) = x_2^{2k+k_0-2} r_0(x)$$

for some smooth function r_0 satisfying $r_0(0) \neq 0$.

We begin by a dyadic decomposition $\nu = \sum_{j \gg 1} \nu_j$ in x_2 followed by scaling $x_2 \mapsto 2^{-j} x_2$. Namely, for a $j \gg 1$ we define

$$\langle \nu_j, f \rangle = \int f(x, \phi_{\mathrm{loc}}(x))\, a(x)\, \chi_1(2^j x_2)\, |\mathcal{H}_{\phi_{\mathrm{loc}}}(x)|^\sigma \, \mathrm{d}x,$$

where $\chi_1(x_2)$ is supported where $|x_2| \sim 1$ and is such that $\sum_{j \in \mathbb{Z}} \chi_1(x_2) = 1$. Thus, by a Littlewood–Paley argument it suffices to prove

$$\|\hat{f}\|^2_{L^2(\mathrm{d}\nu_j)} \lesssim \|f\|^2_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})}, \tag{4-3}$$

with the implicit constant independent of j. We rescale this as follows. First we note that by substituting $x_2 \mapsto 2^{-j} x_2$ we have

$$\begin{aligned}
\|\hat{f}\|^2_{L^2(\mathrm{d}\nu_j)} &= \int |\mathcal{F} f|^2(x, \phi_{\mathrm{loc}}(x))\, a(x)\, \chi_1(2^j x_2)\, |x_2|^{\sigma(2k+k_0-2)}\, |r_0(x)|^\sigma \, \mathrm{d}x \\
&= 2^{-j-j\sigma(2k+k_0-2)} \int |\mathcal{F} f|^2(x_1, 2^{-j} x_2, 2^{-jk} 2^{jk} \phi_{\mathrm{loc}}(x_1, 2^{-j} x_2)) \\
&\qquad\qquad \times a(x_1, 2^{-j} x_2)\, \chi_1(x_2)\, |x_2|^{\sigma(2k+k_0-2)}\, |r_0(x_1, 2^{-j} x_2)|^\sigma \, \mathrm{d}x \\
&= 2^{-j-j\sigma(2k+k_0-2)} \int |\mathcal{F} f|^2(x_1, 2^{-j} x_2, 2^{-jk} \tilde{\phi}(x, 2^{-j})))\, a(x, 2^{-j}) \, \mathrm{d}x.
\end{aligned}$$

The last expression can be rewritten as

$$2^{-j-j\sigma(2k+k_0-2)} \langle \tilde{\nu}_j, |\operatorname{Dil}_{(1,2^j,2^{jk})}(\mathcal{F} f)|^2 \rangle,$$

where

$$\langle \tilde{\nu}_j, f \rangle = \int f(x, \tilde{\phi}(x, 2^{-j}))\, a(x, 2^{-j}) \, \mathrm{d}x.$$

The amplitude $a(x, 2^{-j})$ is now supported so that $|x_1| \ll 1$ and $|x_2| \sim 1$, and it is C^∞ having derivatives uniformly bounded. The phase is

$$\tilde{\phi}(x, 2^{-j}) = 2^{jk} \phi_{\mathrm{loc}}(x_1, 2^{-j} x_2) = x_2^k\, r(x_1, 2^{-j} x_2).$$

Now the inequality (4-3) can be rewritten as

$$\langle \tilde{\nu}_j, |\operatorname{Dil}_{(1,2^j,2^{jk})}(\mathcal{F} f)|^2 \rangle \lesssim 2^{j+j\sigma(2k+k_0-2)} \|f\|^2_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})}.$$

Interchanging the dilation and the Fourier transform we get

$$2^{2j+2jk} \langle \tilde{\nu}_j, |\mathcal{F}(\operatorname{Dil}_{(1,2^{-j},2^{-jk})} f)|^2 \rangle \lesssim 2^{j+j\sigma(2k+k_0-2)} \|f\|^2_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})},$$

and this is equivalent to

$$\begin{aligned}
2^{2j+2jk} \langle \tilde{\nu}_j, |\hat{f}|^2 \rangle &\lesssim 2^{j+j\sigma(2k+k_0-2)} \|\operatorname{Dil}_{(1,2^j,2^{jk})} f\|^2_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})} \\
&= 2^{j+j\sigma(2k+k_0-2)} 2^{2j/\mathfrak{p}_1 + 2jk/\mathfrak{p}_3} \|f\|^2_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})}.
\end{aligned}$$

Plugging in the values of $\mathfrak{p}_1$ and $\mathfrak{p}_3$ we finally obtain

$$\|\hat{f}\|^2_{L^2(\mathrm{d}\tilde{\nu}_j)} \lesssim 2^{\sigma j k_0} \|f\|^2_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})}; \tag{4-4}$$

i.e., this is the rescaled form of the (4-3) inequality.

Now note that from the expression for $\tilde{\phi}(x, 2^{-j})$ we have $|\partial_2\tilde{\phi}| \sim 1 \sim |\partial_2^2\tilde{\phi}|$ and one easily gets by using the definition of the Hessian determinant that

$$\begin{aligned}\mathcal{H}_{\tilde{\phi}}(x, 2^{-j}) &= 2^{j(2k-2)}\mathcal{H}_{\phi_{\mathrm{loc}}}(x_1, 2^{-j}x_2)\\ &= 2^{-jk_0}x_2^{2k+k_0-2}r_0(x_1, 2^{-j}x_2).\end{aligned}$$

Thus $|\mathcal{H}_{\tilde{\phi}}(x, 2^{-j})| \sim 2^{-jk_0}$, from which the estimate (4-4) follows by an application of Lemma 4.1 and subsequently Lemma 4.2.

4B. ***Preliminary rescaling for cases* (ii)–(vi).** In normal form cases (ii)–(vi) the principal face of $\mathcal{N}(\phi_{\mathrm{loc}})$ is compact and so we use the scaling associated to it:

$$\delta_t^\kappa(x) = (t^{\kappa_1}x_1, t^{\kappa_2}x_2),$$

where in cases (ii)-(v) we have

$$\kappa = \left(\frac{1}{2}, \frac{1}{k}\right)$$

and in case (vi) we have

$$\kappa = \left(\frac{1}{2k}, \frac{1}{k}\right).$$

In particular, for $j \gg 1$ we define

$$\langle \nu_j, f\rangle = \int f(x, \phi_{\mathrm{loc}}(x))\, a(x)\, \eta(\delta^\kappa_{2^j}x)\, |\mathcal{H}_{\phi_{\mathrm{loc}}}(x)|^\sigma\, \mathrm{d}x,$$

where η is supported on an annulus and is such that $\sum_{j\in\mathbb{Z}} \eta(\delta^\kappa_{2^j}x) = 1$. By using Littlewood–Paley theory we get that it is sufficient to prove

$$\|\hat{f}\|^2_{L^2(\mathrm{d}\nu_j)} \lesssim \|f\|^2_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})}.$$

Rescaling similarly as in the case of normal form (i), the above estimate is equivalent to

$$\|\hat{f}\|^2_{L^2(\mathrm{d}\tilde{\nu}_j)} \lesssim \|f\|^2_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})}, \tag{4-5}$$

where

$$\langle \tilde{\nu}_j, f\rangle = \int f(x, \tilde{\phi}(x,\delta))\, |\mathcal{H}_{\tilde{\phi}}(x,\delta)|^\sigma\, a(x,\delta)\, \mathrm{d}x. \tag{4-6}$$

Here the amplitude $a(x, \delta)$ is supported on a fixed annulus around the origin,

$$\delta = (\delta_0, \delta_1, \delta_2) := (2^{-j(k-1)/k}, 2^{-j/2}, 2^{-j/k}) \tag{4-7}$$

in cases (ii)–(v), and

$$\delta = (\delta_1, \delta_2) := (2^{-j/(2k)}, 2^{-j/k})$$

in case (vi). The phase which one obtains in (4-6) is

$$\tilde{\phi}(x, \delta) := 2^j \phi_{\mathrm{loc}}(\delta_1 x_1, \delta_2 x_2).$$

The quantity δ_0 will be appear only later when we use the explicit normal forms. From the above phase form it follows that

$$\mathcal{H}_{\tilde{\phi}}(x, \delta) = 2^{j(k-2)/k} \mathcal{H}_{\phi_{\mathrm{loc}}}(\delta_1 x_1, \delta_2 x_2)$$

in cases (ii)–(v), and

$$\mathcal{H}_{\tilde{\phi}}(x, \delta) = 2^{j(2k-3)/k} \mathcal{H}_{\phi_{\mathrm{loc}}}(\delta_1 x_1, \delta_2 x_2)$$

in case (vi).

4C. *Normal forms* (ii) *and* (iii). Using the normal forms for ϕ_{loc} one gets in these cases

$$\begin{aligned}\tilde{\phi}(x, \delta) &= x_1^2 r_1(\delta_1 x_1, \delta_2 x_2) + x_2^k r_2(\delta_1 x_1, \delta_2 x_2),\\ \mathcal{H}_{\tilde{\phi}}(x, \delta) &= x_2^{k-2} r_0(\delta_1 x_1, \delta_2 x_2),\end{aligned}$$

where $r_0(0), r_1(0), r_2(0) \neq 0$, and $k \geq 3$. Hence, for the part where $|x_2| \gtrsim 1$ in (4-6) the Hessian is nondegenerate, and so we may localize to $|x_1| \sim 1$ and $|x_2| \ll 1$, and subsequently perform a dyadic decomposition in the x_2-coordinate; i.e., we define

$$\begin{aligned}\langle \nu_l, f \rangle &:= \int f(x, \tilde{\phi}(x, \delta))\, |x_2|^{\sigma(k-2)}\, \chi_1(2^l x_2)\, a(x, \delta)\, \mathrm{d}x\\ &= 2^{-l-l\sigma(k-2)} \int f(x_1, 2^{-l} x_2, \tilde{\phi}(x_1, 2^{-l} x_2, \delta))\, a(x, \delta, 2^{-l})\, \mathrm{d}x,\end{aligned}$$

where now the amplitude is supported in a domain where $|x_1| \sim 1 \sim |x_2|$ and has uniformly bounded C^N norm for any N. Applying the Littlewood–Paley theorem again and rescaling, it is sufficient for us to prove

$$\|\hat{f}\|^2_{L^2(\mathrm{d}\tilde{\nu}_{j,l})} \lesssim 2^{kl\sigma} \|f\|^2_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})}, \tag{4-8}$$

where the rescaled measure is

$$\langle \tilde{\nu}_{j,l}, f \rangle = \int f(x, \tilde{\phi}(x_1, 2^{-l} x_2, \delta))\, a(x, \delta, 2^{-l})\, \mathrm{d}x.$$

The phase has now the form

$$x_1^2 r_1(\delta_1 x_1, \delta_2 x_2) + 2^{-kl} x_2^k r_2(\delta_1 x_1, 2^{-l} \delta_2 x_2) \tag{4-9}$$

on the domain $|x_1| \sim 1$ and $|x_2| \sim 1$, and its Hessian determinant is of size 2^{-kl}. By Lemma 4.1 we have

$$|\hat{\tilde{\nu}}_{j,l}(\xi)| \lesssim 2^{kl/2}(1 + |\xi|)^{-1}.$$

And so the estimate (4-8) follows by Lemma 4.2.

4D. ***Normal form*** **(iv).** In this case we get

$$\tilde{\phi}(x,\delta) = x_1^2 q(\delta_1 x_1) + (x_2 - \delta_0 x_1^2 \psi(\delta_1 x_1))^k r(\delta_1 x_1, \delta_2 x_2),$$
$$\mathcal{H}_{\tilde{\phi}}(x,\delta) = (x_2 - \delta_0 x_1^2 \psi(\delta_1 x_1))^{k-2} r_0(\delta_1 x_1, \delta_2 x_2),$$

where $q(0), r(0), r_0(0), \psi(0) \neq 0$, and $k \geq 3$. Therefore again, if $|x_2| \gtrsim 1$ the Hessian is nondegenerate and therefore we may concentrate on $|x_1| \sim 1$ and $|x_2| \ll 1$ in (4-6). We perform a dyadic decomposition, though this time depending on how close we are to the root of the Hessian determinant, i.e., we define

$$\langle \nu_l, f \rangle := \int f(x, \tilde{\phi}(x,\delta))\, |x_2 - \delta_0 x_1^2 \psi(\delta_1 x_1)|^{\sigma(k-2)}\, \chi_1(2^l (x_2 - \delta_0 x_1^2 \psi(\delta_1 x_1)))\, a(x,\delta)\, \mathrm{d}x.$$

Next, after changing coordinates from x_2 to $x_2 + \delta_0 x_1^2 \psi(\delta_1 x_1)$ we may write

$$\langle \nu_l, f \rangle = \int f(x_1, x_2 + \delta_0 x_1^2 \psi(\delta_1 x_1), \phi_1(x,\delta))\, |x_2|^{\sigma(k-2)}\, \chi_1(2^l x_2)\, a_1(x,\delta)\, \mathrm{d}x, \tag{4-10}$$

where

$$\begin{aligned} \phi_1(x,\delta) &= x_1^2 q(\delta_1 x_1) + x_2^k r(\delta_1 x_1, \delta_2 x_2 + \delta_0 \delta_2 x_1^2 \psi(\delta_1 x_1, \delta_2 x_2)) \\ &= x_1^2 q(\delta_1 x_1) + x_2^k r(\delta_1 x_1, \delta_2 x_2 + (\delta_1 x_1)^2 \psi(\delta_1 x_1, \delta_2 x_2)) \\ &= x_1^2 q(\delta_1 x_1) + x_2^k \tilde{r}(\delta_1 x_1, \delta_2 x_2). \end{aligned}$$

The function $\tilde{r}$ is a smooth and nonzero at the origin. Finally, we rescale in x_2 as $x_2 \mapsto 2^{-l} x_2$ and may write

$$\begin{aligned} &\langle \nu_l, f \rangle \\ &= 2^{-l-l\sigma(k-2)} \int f(x_1, 2^{-l} x_2 + \delta_0 x_1^2 \psi(\delta_1 x_1), \phi_{j,l}(x,\delta,2^{-l}))\chi_1(x_1)\, \chi_1(x_2)\, a(x,\delta,2^{-l})\, \mathrm{d}x, \end{aligned} \tag{4-11}$$

where the amplitude is a smooth function and the phase is

$$\phi_{j,l}(x,\delta) = x_1^2 q(\delta_1 x_1) + 2^{-kl} x_2^k \tilde{r}(\delta_1 x_1, 2^{-l} \delta_2 x_2).$$

In order to obtain the estimate (4-5) we shall need essentially a variant of Lemma 4.2. Namely, we shall consider the analytic family of operators T_ζ defined by convolution against the Fourier transform of the measure

$$\mu_\zeta := \sum_{2^l \gg 1} 2^{l\sigma(k-2)} 2^{-l\zeta(k-2)} \nu_l, \tag{4-12}$$

where ζ has real part between 0 and $\frac{1}{2}$, and in particular, for a fixed $\xi_3 \in \mathbb{R}^3$, we shall consider the operator $T_\zeta^{\xi_3} : f \mapsto f * \hat{\mu}_\zeta(\cdot, \xi_3)$. Note that we are interested in μ_σ since this is precisely the sum of measures ν_l.

When the real part of ζ is 0 (i.e., $\zeta = it,\ t \in \mathbb{R}$) one considers the $L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ estimate for which we use (4-10). In (4-10) we see that the amplitude is of size $2^{-l\sigma(k-2)}$, which is precisely what we need in (4-12). Since the supports are disjoint when varying l, we get by an argument similar to that in Lemma 4.2 that the operator $L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ norm of $T_{it}^{\xi_3}$ is $\lesssim 1$ (uniform in ξ_3 and t).

When the real part of ζ is $\frac{1}{2}$ we need to prove

$$|\hat{\mu}_{1/2+it}(\xi)| \lesssim (1 + |\xi_3|)^{-1} \tag{4-13}$$

with implicit constant independent of t and ξ_3, since this would give us that the operator norm of $T^{\xi_3}_{1/2+it}$ for mapping $L^1(\mathbb{R}^2) \to L^\infty(\mathbb{R}^2)$ is bounded by $(1+|\xi_3|)^{-1}$.

Thus, under the assumption that we have the estimate (4-13) we may apply complex interpolation for each fixed ξ_3 to the analytic family of operators $T^{\xi_3}_\zeta$ and obtain that the operator norm of $T^{\xi_3}_\sigma$ between spaces $L^{\mathfrak{p}_1}(\mathbb{R}^2) \to L^{\mathfrak{p}_1'}(\mathbb{R}^2)$ is $\lesssim (1+|\xi_3|)^{-2\sigma}$, and so in the same way as in the proof of Lemma 4.2 the (weak) Young inequality in the x_3-direction implies (4-5).

In proving (4-13) it suffices to show that

$$\sum_{2^l \gg 1} 2^{-l(1/2-\sigma)(k-2)} |\hat{\nu}_l(\xi)| \lesssim (1+|\xi_3|)^{-1}$$

for all $\xi \in \mathbb{R}^3$. By (4-11) the Fourier transform of a summand is

$$2^{-l(1/2-\sigma)(k-2)} \hat{\nu}_l(\xi) = 2^{-kl/2} \int e^{-i\Phi(x,\xi,\delta,2^{-l})}\, \chi_1(x_1)\, \chi_1(x_2)\, a(x,\delta,2^{-l})\, \mathrm{d}x,$$

where the phase function is

$$\Phi(x,\xi,\delta,2^{-l}) := \xi_1 x_1 + \xi_2 \delta_0 x_1^2 \psi(\delta_1 x_1) + \xi_3 x_1^2 q(\delta_1 x_1) + 2^{-l} \xi_2 x_2 + 2^{-kl} \xi_3 x_2^k \tilde{r}(\delta_1 x_1, 2^{-l}\delta_2 x_2).$$

We see that when either $|\xi_1| \gg \max\{|\xi_2|, |\xi_3|\}$ or $|\xi_3| \gg \max\{|\xi_1|, |\xi_2|\}$ we can use integration by parts in the x_1-variable and get a very fast decay. This is also the case when $|\xi_1| \sim |\xi_2|$ are much greater than $|\xi_3|$, or when $|\xi_2| \sim |\xi_3|$ are much greater than $|\xi_1|$. If we have $|\xi_2| \gtrsim |\xi_3|$, then we may use integration by parts in x_2 and get

$$|2^{-l(1/2-\sigma)(k-2)} \hat{\nu}_l(\xi)| \lesssim 2^{-kl/2}(1+2^{-l}|\xi_2|)^{-1} \lesssim 2^{-kl/2}(1+2^{-l}|\xi_3|)^{-1},$$

from which (4-13) follows since $k \geq 3$. We are thus left with the case when $|\xi_1| \sim |\xi_3| \gg |\xi_2|$.

Case 1: $2^{-kl}|\xi_3| \lesssim 1$. Here we use the van der Corput lemma in x_1 only and get

$$|2^{-l(1/2-\sigma)(k-2)} \hat{\nu}_l(\xi)| \lesssim 2^{-kl/2} |\xi_3|^{-1/2}.$$

Summation in l then gives precisely (4-13).

Case 2: $2^{-l}|\xi_2| \not\sim 2^{-kl}|\xi_3|$ and $2^{-kl}|\xi_3| \gg 1$. We may use in this case integration by parts in x_2 and then the van der Corput lemma in x_1 and get

$$|2^{-l(1/2-\sigma)(k-2)} \hat{\nu}_l(\xi)| \lesssim 2^{-kl/2} |\xi_3|^{-1/2}\, (2^{-kl}|\xi_3|)^{-1} \lesssim 2^{kl/2} |\xi_3|^{-3/2}.$$

We may now sum in l.

Case 3: $2^{-l}|\xi_2| \sim 2^{-kl}|\xi_3| \gg 1$. Here we have by iterative stationary phase (first in x_2 and then in x_1) that

$$|2^{-l(1/2-\sigma)(k-2)} \hat{\nu}_l(\xi)| \lesssim 2^{-kl/2} |\xi_3|^{-1/2}\, (2^{-kl}|\xi_3|)^{-1/2} = |\xi_3|^{-1}.$$

Here we note that $2^{l(k-1)} \sim |\xi_3||\xi_2|^{-1}$, and so we sum only over finitely many (i.e., $\mathcal{O}(1)$) l for each fixed ξ. Thus, here we also have the estimate (4-13).

4E. ***Normal form* (v).** Recall that here

$$\phi_{\mathrm{loc}}(x) = x_1^2 r_1(x) + (x_2 - x_1^2\psi(x_1))^k r_2(x),$$
$$\mathcal{H}_{\phi_{\mathrm{loc}}}(x) = (x_2 - x_1^2\psi(x_1))^{k-2} r_0(x),$$

where we know that $k \geq 3,\ r_0(0), r_1(0), r_2(0), \psi(0) \neq 0$. Furthermore, recall that this corresponded to the w-coordinates when deriving the normal forms, and we have shown that we additionally have in this case

$$\partial_2^{\tau_2} r_1(0) \neq 0 \quad \text{for all } \tau_2 \in \{0, 1, \dots, k-1\}.$$

In fact, one has the relationship

$$c\tau_2 \partial_2^{\tau_2 - 1} r_1(0) = \partial_2^{\tau_2} r_1(0) \quad \text{for all } \tau_2 \in \{1, \dots, k-1\},$$

where c is some fixed nonzero constant (see Section 3E). This implies for example the relation

$$r_1(0)\, \partial_2^2 r_1(0) - 2(\partial_2 r_1)^2(0) = 0. \tag{4-14}$$

From the above normal form we have

$$\tilde{\phi}(x, \delta) = x_1^2 r_1(\delta_1 x_1, \delta_2 x_2) + (x_2 - \delta_0 x_1^2 \psi(\delta_1 x_1))^k r_2(\delta_1 x_1, \delta_2 x_2),$$
$$\mathcal{H}_{\tilde{\phi}}(x, \delta) = (x_2 - \delta_0 x_1^2 \psi(\delta_1 x_1))^{k-2} r_0(\delta_1 x_1, \delta_2 x_2).$$

We may as usual localize to $|x_1| \sim 1$ and $|x_2| \ll 1$. We shall abuse the notation a bit and denote this localized measure again by $\tilde{\nu}_j$. After changing coordinates from x_2 to $x_2 + \delta_0 x_1^2 \psi(\delta_1 x_1)$ we may write

$$\langle \tilde{\nu}_j, f \rangle = \int f(x_1, x_2 + \delta_0 x_1^2 \psi(\delta_1 x_1), \phi_1(x, \delta))\, |x_2|^{\sigma(k-2)}\, a_1(x, \delta)\, \chi_1(x_1)\, \chi_0(x_2)\, \mathrm{d}x,$$

with the phase being

$$\phi_1(x, \delta) = x_1^2 \tilde{r}_1(\delta_1 x_1, \delta_2 x_2) + x_2^k \tilde{r}_2(\delta_1 x_1, \delta_2 x_2),$$

where $\tilde{r}_1, \tilde{r}_2$ are smooth functions, nonzero at the origin, and satisfy the same properties and relations as r_1 and r_2 mentioned at the beginning of this subsection. As in the case (iv), we also decompose the measure $\tilde{\nu}_j$ as $\tilde{\nu}_j = \sum_l \nu_l$, where

$$\langle \nu_l, f \rangle = \int f(x_1, x_2 + \delta_0 x_1^2 \psi(\delta_1 x_1), \phi_1(x, \delta))\, |x_2|^{\sigma(k-2)}\, a_1(x, \delta)\, \chi_1(x_1)\, \chi_1(2^l x_2)\, \mathrm{d}x.$$

Next, we shall be interested in the rescaled phase

$$\phi_l(x, \delta, 2^{-l}) = \phi_1(x_1, 2^{-l} x_2, \delta) = \tilde{\phi}(x_1, 2^{-l} x_2 + \delta_0 x_1^2 \psi(\delta_1 x_1), \delta).$$

Now we need a relation between the Hessian determinant of ϕ_l and the Hessian determinant of $\tilde{\phi}$. For this let us define for simplicity

$$\varphi(x_1, \delta_1) := \delta_1^2 x_1^2 \psi(\delta_1 x_1).$$

The reason why we have not included the factor δ_2^{-1} will be clear later (recall from (4-7) that $\delta_0 = \delta_1^2\delta_2^{-1}$). A direct calculation shows then

$$\mathcal{H}_{\phi_l} = 2^{-2l}\mathcal{H}_{\tilde{\phi}} + \delta_2^{-1}2^l\partial_1^2\varphi\partial_2\phi_l\partial_2^2\phi_l, \tag{4-15}$$

and due to our localization we have $|\mathcal{H}_{\tilde{\phi}}| \sim 2^{-l(k-2)}$.

We use the same complex interpolation idea as in (iv) according to which it suffices to prove

$$\sum_{2^l \gg 1} 2^{-l(1/2-\sigma)(k-2)}|\hat{\nu}_l(\xi)| \lesssim (1+|\xi_3|)^{-1},$$

where after rescaling $x_2 \mapsto 2^{-l}x_2$ we have

$$2^{-l(1/2-\sigma)(k-2)}\hat{\nu}_l(\xi) = 2^{-kl/2}\int e^{i\Phi_0(x,\xi,\delta,2^{-l})}a(x,\delta,2^{-l})\,\mathrm{d}x,$$

where the phase function for the Fourier transform of ν_l is

$$\begin{aligned}\Phi_0(x,\xi,\delta,2^{-l})\\ &:= \xi_1x_1 + \xi_2\delta_0x_1^2\psi(\delta_1x_1) + \xi_3x_1^2\tilde{r}_1(\delta_1x_1, 2^{-l}\delta_2x_2) + \xi_22^{-l}x_2 + \xi_32^{-kl}x_2^k\tilde{r}_2(\delta_1x_1, 2^{-l}\delta_2x_2)\\ &= \xi_1x_1 + \xi_2\delta_2^{-1}\varphi(x_1,\delta_1) + \xi_22^{-l}x_2 + \xi_3\phi_l(x,\delta,2^{-l}).\end{aligned}$$

The amplitude localizes the integration to $|x_1| \sim 1 \sim |x_2|$.

Using the same argumentation as in the case (iv) we can reduce ourselves to the case when $|\xi_1| \sim |\xi_3|$, $|\xi_2| \ll |\xi_3|$, and $|\xi_3|2^{-kl} \gg 1$ are satisfied.

Now let us make some further reductions using the fact that $\partial_2\tilde{r}_1(0), \partial_2^2\tilde{r}_1(0) \neq 0$. The x_2-derivative of the phase Φ_0 contains three terms of respective sizes $\sim |2^{-l}\delta_2\xi_3|$, $\sim |2^{-l}\xi_2|$, and $\sim |2^{-kl}\xi_3|$. If we may integrate by parts in x_2 (i.e., if one of the above terms is much larger than the other two), we can get an admissible estimate and sum in l. If $|2^{-kl}\xi_3|$ is comparable to the larger of the other two terms, then one easily sees that the second derivative in x_2 is necessarily of size $|2^{-kl}\xi_3|$, and so in this case we get by iterative stationary phase the estimate

$$2^{-l(1/2-\sigma)(k-2)}|\hat{\nu}_l(\xi)| \lesssim (1+|\xi_3|)^{-1}.$$

Note that we do not need to sum in l since there are only finitely many l satisfying one of the relations $|2^{-kl}\xi_3| \sim |2^{-l}\delta_2\xi_3|$ or $|2^{-kl}\xi_3| \sim |2^{-l}\xi_2|$.

We are thus now reduced to the case when

$$|2^{-l}\xi_2| \sim |2^{-l}\delta_2\xi_3| \gg |2^{-kl}\xi_3|, \quad |\xi_1| \sim |\xi_3| \quad \text{and} \quad |\xi_3|2^{-kl} \gg 1.$$

At this point we introduce some further notation,

$$\lambda := \xi_3, \quad s_1 := \frac{\xi_1}{\xi_3}, \quad s_2 := \frac{\xi_2}{\delta_2\xi_3}, \quad \varepsilon := 2^{-l}\delta_2,$$

and so we have $|s_1| \sim 1 \sim |s_2|$, $\lambda 2^{-kl} \gg 1$, and $\varepsilon \gg 2^{-kl}$. The phase Φ_0 can now be rewritten as $\lambda\Phi$, where Φ is

$$\Phi(x, s_1, s_2, \delta_1, \varepsilon, 2^{-kl}) = s_1 x_1 + s_2 \delta_1^2 x_1^2 \psi(\delta_1 x_1) + s_2 \varepsilon x_2 + \phi_l(x, \delta, 2^{-l}),$$

since we note from the form of ϕ_l that ϕ_l can also be taken to depend on $(x_1, x_2, \delta_1, \varepsilon, 2^{-kl})$.

Let us now apply the stationary phase method in x_1. We may rewrite the phase as

$$\Phi(x, s_1, s_2, \delta_1, \varepsilon, 2^{-kl}) = s_1 x_1 + s_2 \varphi + s_2 \varepsilon x_2 + \phi_l,$$

where we recall that $\varphi(x_1, \delta) = \delta_1^2 x_1^2 \psi(\delta_1 x_1)$. We may assume that there is a stationary point for the x_1-derivative since $|\partial_1^2 \phi_l| \sim 1$ and $|s_1| \sim 1$, and as otherwise we may use integration by parts.

We denote by $x_1^c = x_1^c(x_2, s_1, s_2, \delta_1, \varepsilon, 2^{-kl})$ the function such that

$$(\partial_1 \Phi)(x_1^c, x_2, s_1, s_2, \delta_1, \varepsilon, 2^{-kl}) = s_1 + s_2 \partial_1 \varphi + \partial_1 \phi_l = 0. \tag{4-16}$$

Taking the x_2-derivative we get

$$s_2 (x_1^c)' \partial_1^2 \varphi + (x_1^c)' \partial_1^2 \phi_l + \partial_1 \partial_2 \phi_l = 0. \tag{4-17}$$

After applying the stationary phase method in x_1 we gain a decay factor of $\lambda^{-1/2}$; i.e., we have

$$2^{-l(1/2-\sigma)(k-2)} \hat{\nu}_l(\xi) = \lambda^{-1/2} 2^{-kl/2} \int e^{-i\lambda \tilde{\Phi}(x_2, s_1, s_2, \delta_1, \varepsilon, 2^{-kl})} a(x_2, s_1, s_2, \delta, 2^{-l}; \lambda) \, \mathrm{d}x_2,$$

where the new phase is

$$\tilde{\Phi}(x_2, s_1, s_2, \delta_1, \varepsilon, 2^{-kl}) = s_1 x_1^c + s_2 \varphi(x_1^c, \delta_1) + s_2 \varepsilon x_2 + \phi_l(x_1^c, x_2, \delta, 2^{-l}),$$

and the amplitude a is a classical symbol in λ of order 0.

Taking the x_2-derivative of the expression for the new phase $\tilde{\Phi}$ and using (4-16) we get

$$\tilde{\Phi}' = s_2 \varepsilon + \partial_2 \phi_l. \tag{4-18}$$

Therefore, the second derivative of the new phase is

$$\tilde{\Phi}'' = (\partial_2 \phi_l)' = \partial_2^2 \phi_l + (x_1^c)' \partial_1 \partial_2 \phi_l. \tag{4-19}$$

Now using in order (4-17), the definition of $\mathcal{H}_{\phi_l}$ (4-15), (4-18), and (4-19), we obtain

$$\begin{aligned}
(\partial_1^2 \phi_l) \tilde{\Phi}'' &= \partial_1^2 \phi_l \, \partial_2^2 \phi_l + \partial_1 \partial_2 \phi_l (-\partial_1 \partial_2 \phi_l - s_2 (x_1^c)' \partial_1^2 \varphi) \\
&= \mathcal{H}_{\phi_l} - s_2 (x_1^c)' \partial_1^2 \varphi \, \partial_1 \partial_2 \phi_l \\
&= 2^{-2l} \mathcal{H}_{\tilde{\phi}} + \delta_2^{-1} 2^l \partial_1^2 \varphi \partial_2 \phi_l \partial_2^2 \phi_l - s_2 (x_1^c)' \partial_1^2 \varphi \, \partial_1 \partial_2 \phi_l \\
&= 2^{-2l} \mathcal{H}_{\tilde{\phi}} + \varepsilon^{-1} \partial_1^2 \varphi \, \partial_2^2 \phi_l (\tilde{\Phi}' - \varepsilon s_2) - s_2 (x_1^c)' \partial_1^2 \varphi \, \partial_1 \partial_2 \phi_l \\
&= 2^{-2l} \mathcal{H}_{\tilde{\phi}} - s_2 \partial_1^2 \varphi \, \partial_2^2 \phi_l - s_2 (x_1^c)' \partial_1^2 \varphi \, \partial_1 \partial_2 \phi_l + \varepsilon^{-1} \partial_1^2 \varphi \, \partial_2^2 \phi_l \, \tilde{\Phi}' \\
&= 2^{-2l} \mathcal{H}_{\tilde{\phi}} - s_2 \partial_1^2 \varphi \, \tilde{\Phi}'' + \varepsilon^{-1} \partial_1^2 \varphi \, \partial_2^2 \phi_l \, \tilde{\Phi}'.
\end{aligned}$$

Thus, we get

$$(s_2\partial_1^2\varphi+\partial_1^2\phi_l)\widetilde{\Phi}''=2^{-2l}\mathcal{H}_{\tilde{\phi}}+\varepsilon^{-1}\partial_1^2\varphi\,\partial_2^2\phi_l\,\widetilde{\Phi}'. \tag{4-20}$$

Note that we have $|\varepsilon^{-1}\partial_1^2\varphi\,\partial_2^2\phi_l|\ll\delta_1^2\ll 1$ and $|s_2\partial_1^2\varphi+\partial_1^2\phi_l|\sim 1$, and recall that $|2^{-2l}\mathcal{H}_{\tilde{\phi}}|\sim 2^{-kl}$. We claim that either $|\widetilde{\Phi}'|\lesssim 2^{-kl}$ on the whole domain of integration (i.e., for $|x_2|\sim 1$), or that $|\widetilde{\Phi}'|\gtrsim 2^{-kl}$ on the whole domain of integration. This can be shown by using the formula for the solution of a linear first-order ODE (considering $\widetilde{\Phi}'$ as the unknown), or by arguing by contradiction.

Let us argue by contradiction in the following way. Let us assume that there exists a point $|x_2^0|\sim 1$ such that $|\widetilde{\Phi}'(x_2^0)|\leq 2^{-kl}$. Furthermore, let us assume that there exists a point $|x_2^1|\sim 1$ where one has $|\widetilde{\Phi}'|=C_1 2^{-kl}$ for some sufficiently large C_1, and let us assume that x_2^1 is the closest point to x_2^0 satisfying this condition in the sense that $|\widetilde{\Phi}'|<C_1 2^{-kl}$ between x_2^0 and x_2^1. Then the mean value theorem implies that there is a point between x_2^0 and x_2^1 where we have $|\widetilde{\Phi}''|\geq C_2 2^{-kl}$, where C_2 can be taken to tend to ∞ as C_1 tends to ∞. On the other hand, (4-20) implies that on the interval between x_2^0 and x_2^1 we have $|\widetilde{\Phi}''|\leq C_3 2^{-kl}$, where we can take C_3 to be a fixed constant if δ_1 is taken to be sufficiently small when C_1 and C_2 are large (we can always take say C_1 of size δ_1^{-1}). This is a contradiction, i.e., the point x_2^1 where one has $|\widetilde{\Phi}'|\geq C_1 2^{-kl}$ for a too-large C_1 cannot exist within the integration domain.

Now in the case that $|\widetilde{\Phi}'|\gtrsim 2^{-kl}$ we may apply integration by parts and get an estimate summable in l. Let us therefore assume $|\widetilde{\Phi}'|\lesssim 2^{-kl}$, in which case we have $|\widetilde{\Phi}''|\sim 2^{-kl}$ by (4-20). Then the van der Corput lemma implies that

$$2^{-l(1/2-\sigma)(k-2)}|\hat{\nu}_l(\xi)|\lesssim(1+|\xi_3|)^{-1}.$$

The problem is now that a priori we may not sum this estimate in l. Luckily, it turns out that one can pin down the size of 2^{-l}, which in turn will pin down the number l to a finite set of size $\mathcal{O}(1)$. In order to prove this we use the expression (4-18) and the normal form of ϕ_l,

$$\phi_l(x,\delta,2^{-l})=x_1^2\tilde{r}_1(\delta_1x_1,\varepsilon x_2)+2^{-kl}x_2^k\tilde{r}_2(\delta_1x_1,\varepsilon x_2),$$

from which one has

$$(\partial_2\phi_l)(x,\delta,2^{-l})=\varepsilon x_1^2(\partial_2\tilde{r}_1)(\delta_1x_1,\varepsilon x_2)+2^{-kl}x_2^{k-1}\tilde{r}_3(\delta_1x_1,\varepsilon x_2), \tag{4-21}$$

where $\tilde{r}_3(0)\neq 0$ is a smooth function.

The idea is as follows. First, by compactness we may assume that we integrate in x_2 over a sufficiently small neighborhood of a point x_2^0 satisfying $|x_2^0|\sim 1$. In particular, we may write

$$\begin{aligned}\widetilde{\Phi}'(x_2,s_1,s_2,\delta_1,\varepsilon,2^{-kl})&=\widetilde{\Phi}'(x_2^0,s_1,s_2,\delta_1,\varepsilon,2^{-kl})+\mathcal{O}(|\widetilde{\Phi}''|)\\&=\widetilde{\Phi}'(x_2^0,s_1,s_2,\delta_1,\varepsilon,2^{-kl})+\mathcal{O}(2^{-kl}).\end{aligned}$$

Thus, it suffices to prove that

$$|\widetilde{\Phi}'(x_2^0,s_1,s_2,\delta_1,\varepsilon,2^{-kl})|=|s_2\varepsilon+\partial_2\phi_l(x_1^c,x_2^0,s_1,s_2,\delta_1,\varepsilon,2^{-kl})|\ll 2^{-kl}$$

can happen only for finitely many l. If the above inequality does not hold, then we may simply integrate by parts and are able to simply sum in l afterwards.

If we now develop both terms in $\partial_2\phi_l$ in the ε and 2^{-kl} variables (recall that x_1^c depends on both ε and 2^{-kl}), then one gets that the expression for $\widetilde{\Phi}'$ is of the form

$$s_2\varepsilon+\sum_{i=1}^{k-1}\varepsilon^i f_i(x_2^0,s_1,s_2,\delta_1)+2^{-kl}g_0(x_2^0,s_1,s_2,\delta_1)+\mathcal{O}(2^{-kl}),$$

where we used the fact that $\varepsilon^k=(\delta_2 2^{-l})^k\ll 2^{-kl}$. Note that we have $|g_0|\sim 1$ by (4-21) (and also $|f_1|\sim 1$, but this is not important). We have to find out how many l's satisfy

$$\left|\tilde{f}_1(x_2^0,s_1,s_2,\delta_1)+\sum_{i=2}^{k-1}\varepsilon^{i-1}f_i(x_2^0,s_1,s_2,\delta_1)+\varepsilon^{-1}2^{-kl}g_0(x_2^0,s_1,s_2,\delta_1)+\mathcal{O}(\varepsilon^{-1}2^{-kl})\right|\ll\varepsilon^{-1}2^{-kl},$$

where $\tilde{f}_1(x_2^0,s_1,s_2,\delta_1):=s_2+f_1(x_2^0,s_1,s_2,\delta_1)$. But now one easily shows that this inequality is possible only if at least two of the terms are comparable in size (precisely because $|g_0|\sim 1$). This implies in particular that we can determine l in terms of (x_2^0,s_1,s_2,δ_1), which finishes the proof.

We mention that, interestingly, one can prove $f_2(x_2^0,s_1,s_2,0)=0$, a consequence of the relation (4-14).

4F. *Normal form* (vi). Here we obtain

$$\tilde{\phi}(x,\delta)=(x_2-x_1^2\psi(\delta_1x_1))^k r(\delta_1x_1,\delta_2x_2),$$
$$\mathcal{H}_{\tilde{\phi}}(x,\delta)=(x_2-x_1^2\psi(\delta_1x_1))^{2k-3}r_0(\delta_1x_1,\delta_2x_2),$$

where $r(0),r_0(0),\psi(0)\neq 0$. Thus, we may localize to the part where $|x_2-x_1^2\psi(\delta_1x_1)|\ll 1$; i.e., it is sufficient to consider the measure

$$f\mapsto\int f(x,\tilde{\phi}(x,\delta))\,|(x_2-x_1^2\psi(\delta_1x_1))^{2k-3}r_0(\delta_1x_1,\delta_2x_2)|^\sigma\,\chi_0(\tilde{\phi}(x,\delta))\,a(x,\delta)\,\mathrm{d}x$$

since $|\tilde{\phi}(x,\delta)|\sim|x_2-x_1^2\psi(\delta_1x_1)|^k$. Note that here we have $|x_1|\sim 1\sim|x_2|$.

Now, the next idea is to use, as in [Ikromov and Müller 2016], a Littlewood–Paley decomposition in the x_3-direction (for the mixed-norm Littlewood–Paley theory see [Lizorkin 1970]) and reduce ourselves to proving the Fourier restriction estimate for the measure piece

$$\langle\nu_l,f\rangle=\int f(x,\tilde{\phi}(x,\delta))\,|(x_2-x_1^2\psi(\delta_1x_1))^{2k-3}r_0(\delta_1x_1,\delta_2x_2)|^\sigma\,\chi_1(2^{kl}(\tilde{\phi}(x,\delta)))\,a(x,\delta)\,\mathrm{d}x.$$

Using the coordinate transformation $x_2\mapsto x_2+x_1^2\psi(\delta_1x_1)$ we may write

$$\begin{aligned}\langle\nu_l,f\rangle=\int f(x_1,x_2+x_1^2\psi(\delta_1x_1),x_2^k\tilde{r}(\delta_1x_1,\delta_2x_2))&\\ \times|x_2^{2k-3}\tilde{r}_0(\delta_1x_1,\delta_2x_2)|^\sigma\,\chi_1(2^{kl}x_2^k\tilde{r}(\delta_1x_1,\delta_2x_2))\,\tilde{a}(x,\delta)\,\mathrm{d}x,&\end{aligned}$$

where $|\tilde{r}|\sim 1$ is a smooth function. Finally, we use the coordinate transformation $x_2\mapsto 2^{-l}x_2$ and rescale f in the third coordinate. Then we are reduced to proving the Fourier restriction estimate

$$\|\hat{f}\|^2_{L^2(\mathrm{d}\tilde{\nu}_{j,l})}\leq C2^{l(1-3\sigma)}\|f\|^2_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})}\tag{4-22}$$

for the measure

$$\langle \tilde{\nu}_{j,l}, f \rangle = \int f(x_1, 2^{-l} x_2 + x_1^2 \psi(\delta_1 x_1), x_2^k \tilde{r}(\delta_1 x_1, 2^{-l} \delta_2 x_2))\, a(x, \delta, 2^{-l})\, \mathrm{d}x, \tag{4-23}$$

where a is supported so that $|x_1| \sim 1$ and $|x_2| \sim 1$. Now we note that the estimate for $\sigma = 0$ follows by Plancherel, while the estimate for $\sigma = \frac{1}{3}$ is going to be shown in Section 5 since the form of the measure $\tilde{\nu}_{j,l}$ coincides with the form in (5-11) below. Interpolating, we obtain the estimate for all $0 \le \sigma \le \frac{1}{3}$.

Note that when

$$\frac{1}{p_1'} = \frac{1}{p_3'} = \frac{1}{4},$$

one can simplify the proof by a modification of Lemma 4.2, i.e., by using the Fourier decay of $\tilde{\nu}_{j,l}$, which is easily seen to be

$$|\hat{\tilde{\nu}}_{j,l}(\xi)| \lesssim 2^{l/2}(1 + |\xi|)^{-1},$$

and by using the Plancherel theorem, but this time in the (x_1, x_3)-plane (which is why it works only for $1/p_1' = 1/p_3'$) since the mapping $(x_1, x_2) \mapsto (x_1, x_2^k \tilde{r}(\delta_1 x_1, 2^{-l} \delta_2 x_2))$ has Jacobian of size ~ 1. In fact, in Section 5 we shall combine this idea of using Lemma 4.2 with the methods used in [Ikromov and Müller 2016] (and [Palle 2021]).

4F1. *A Knapp-type example.* Let us now show by using a Knapp-type example that one cannot get the estimate

$$\|\hat{f}\|_{L^2(\mathrm{d}\nu)} \le C \|f\|_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})}$$

for $\sigma > \frac{1}{3}$ where ν is the surface measure

$$\langle \nu, f \rangle = \int_{\mathbb{R}^2 \setminus \{0\}} f(x, \phi_{\mathrm{loc}}(x))\, a(x)\, |\mathcal{H}_\phi(x)|^\sigma \, \mathrm{d}x$$

and ϕ_{loc} is given by the normal form (vi). Let us consider the function $f = \varphi_\varepsilon$ defined by

$$\hat{\varphi}_\varepsilon(x) = \chi_0\left(\frac{x_1}{\varepsilon^\delta}\right) \chi_0\left(\frac{x_2}{\varepsilon^{2\delta}}\right) \chi_0\left(\frac{x_3}{\varepsilon}\right)$$

for some small ε and δ. Its mixed L^p norm is

$$\|\varphi_\varepsilon\|_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})} \sim \varepsilon^{3\delta/p_1' + 1/p_3'}.$$

Now, in the integral

$$\int |\hat{\varphi}_\varepsilon|^2 \, \mathrm{d}\nu = \int |\hat{\varphi}_\varepsilon|^2(x, \phi_{\mathrm{loc}}(x)) a(x) |\mathcal{H}_{\phi_{\mathrm{loc}}}(x)|^\sigma \, \mathrm{d}x$$

we integrate over the set

$$D_\varepsilon^0 := \{x \in \mathbb{R}^2 : |x_1| \lesssim \varepsilon^\delta, \quad |x_2| \lesssim \varepsilon^{2\delta}, \quad |\phi_{\mathrm{loc}}(x)| \sim |x_2 - x_1^2 \psi(x_1)|^k \lesssim \varepsilon\}$$

by the definition of φ_ε. If δ is sufficiently small, D_ε^0 contains the set

$$D_\varepsilon := \{x \in \mathbb{R}^2 : |x_1| \lesssim \varepsilon^\delta, \quad |\phi_{\mathrm{loc}}(x)| \sim |x_2 - x_1^2 \psi(x_1)| \lesssim \varepsilon^{1/k}\},$$

and so if the Fourier restriction estimate holds, one has

$$\varepsilon^{6\delta/p_1'+2/p_3'} \sim \|\varphi_\varepsilon\|^2_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})} \gtrsim \int |\hat{\varphi}_\varepsilon|^2 \,\mathrm{d}\nu \gtrsim \int_{D_\varepsilon} |x_2 - x_1^2\psi(x_1)|^{\sigma(2k-3)}\,\mathrm{d}x$$
$$\sim \varepsilon^{\delta} \int_{|y|\lesssim \varepsilon^{1/k}} |y|^{\sigma(2k-3)}\,\mathrm{d}y \sim \varepsilon^{\delta+(\sigma(2k-3)+1)/k}.$$

Letting ε and then δ tend to 0 we obtain the condition

$$\frac{1}{p_3'} \le \frac{\sigma(2k-3)+1}{2k} = \sigma + \frac{1-3\sigma}{2k}.$$

Since we are interested in

$$\frac{1}{p_1'} = \frac{1}{2} - \sigma, \qquad \frac{1}{p_3'} = \sigma,$$

the above inequality implies precisely $\sigma \le \frac{1}{3}$.

5. Fourier restriction without a mitigating factor

Here we prove Theorem 1.2, i.e., the estimate

$$\|\hat{f}\|_{L^2(\mathrm{d}\mu)} \le C\|f\|_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})}$$

for μ the surface measure of the form

$$\langle \mu, f\rangle = \int_{\mathbb{R}^2\setminus\{0\}} f(x,\phi(x))\,|x|_\alpha^{2\vartheta}\,\mathrm{d}x,$$

where

$$\vartheta = \frac{|\alpha|}{p_1'} + \frac{\rho}{p_3'} - \frac{|\alpha|}{2}.$$

Recall that this ϑ is chosen (depending on $(p_1, p_3) \in (1,2]^2$) precisely so that the above restriction estimate is equivalent to the local estimate

$$\|\hat{f}\|_{L^2(\mathrm{d}\mu_0)} \le C\|f\|_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})},$$

where μ_0 is the surface measure

$$\langle \mu_0, f\rangle = \int_{\mathbb{R}^2\setminus\{0\}} f(x,\phi(x))\,\eta(x)\,|x|_\alpha^{2\vartheta}\,\mathrm{d}x \tag{5-1}$$

for $\eta \in C_c^\infty(\mathbb{R}^2 \setminus \{0\})$ identically equal to 1 in an annulus.

Note that $|x|_\alpha^{2\vartheta}$ is not smooth near the axes. Luckily, we shall be able to circumvent this problem by using the Littlewood–Paley theorem to localize away from the axes, as was done in the case with the mitigating factor.

Now we recall the necessary conditions from [Palle 2021, Proposition 2.1] obtained through the Knapp-type examples. Let us fix a point v such that $\eta(v) \neq 0$ and let η_v be a smooth cutoff function identically equal to η on a small neighborhood of v. It suffices to consider the measure

$$\langle \mu_{0,v}, f\rangle = \int_{\mathbb{R}^2\setminus\{0\}} f(x, \phi_v(x-v))\, \eta_v(x)\, |x|_\alpha^{2\vartheta}\, \mathrm{d}x, \tag{5-2}$$

where we recall from the Introduction that

$$\phi_v(x) = \phi(x+v) - \phi(v) - x\cdot\nabla\phi(v).$$

We recall also that $h_{\mathrm{lin}}(\phi, v)$ is the linear height of ϕ_v at its origin, and that $h(\phi, v)$ is its Newton height.

Proposition 2.1 of [Palle 2021] tells us what the necessary conditions on the exponents p_1 and p_3 are if the $L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)}) \to L^2(\mathrm{d}\mu_{0,v})$ Fourier restriction estimate were to hold true. The input data one needs is the Newton polyhedron of the phase function ϕ_v at the origin in both its linearly adapted and adapted coordinates. When the linearly adapted and adapted coordinates do not coincide, one constructs from the two Newton polyhedra the so-called augmented Newton polyhedron. When the linearly adapted and adapted coordinates do coincide, then one obtains a single condition, namely,

$$\frac{1}{p_1'} + \frac{h_{\mathrm{lin}}(\phi, v)}{p_3'} \le \frac{1}{2}. \tag{5-3}$$

Otherwise, in the proposition it is shown that to each edge of the augmented Newton polyhedron, say contained in the line $\{(t_1, t_2) \in \mathbb{R}^2 : \tilde{\kappa}_1 t_1 + \tilde{\kappa}_2 t_2 = 1\}$, one can associate the necessary condition

$$\frac{(1+m)\tilde{\kappa}_1}{p_1'} + \frac{1}{p_3'} \le \frac{\tilde{\kappa}_1 + \tilde{\kappa}_2}{2},$$

where m is the negative reciprocal of the slope of the principal face of the Newton polyhedron of ϕ_v in its linearly adapted coordinates. As shown in [Palle 2021, Proposition 2.1], this set of conditions always contains the condition (5-3) and the condition

$$\frac{1}{p_3'} \le \frac{1}{2h(\phi, v)}. \tag{5-4}$$

Thus, if ϕ satisfies (LA) at v, then $h_{\mathrm{lin}}(\phi, v) = h(\phi, v)$, and the only necessary condition is given by (5-3). If ϕ does not satisfy (LA) at v, then from Proposition 1.4 we deduce that out of all the normal forms this is only possible for the normal form

$$\phi_{v,y}(y) := (y_2 - y_1^2\psi(y_1))^k r(y),$$

where $r(0) \neq 0$, $\psi(0) \neq 0$, and $2 \le k < \infty$, since all the normal forms are linearly adapted and this is the only nonadapted normal form (see [Ikromov and Müller 2011], or the Introduction of [Ikromov and Müller 2016] to find precise conditions for whether a function is in linearly adapted or adapted coordinates). Using this normal form one can now determine its augmented Newton polyhedron, which turns out to have only two edges. Its two associated conditions turn out to be precisely the conditions (5-3) and (5-4), i.e.,

$$\frac{1}{p_1'} + \frac{h_{\mathrm{lin}}(\phi, v)}{p_3'} \le \frac{1}{2} \quad \text{and} \quad \frac{h(\phi, v)}{p_3'} \le \frac{1}{2}. \tag{5-5}$$

One also easily shows $h(\phi, v) = k$ and $h_{\text{lin}}(\phi, v) = 2k/3$. Note that in the case $h_{\text{lin}}(\phi, v) = h(\phi, v)$ the second condition in (5-5) would be redundant. Thus, if we now vary v over the points where $\eta(v) \neq 0$, then we obtain the conditions

$$\frac{1}{p_1'} + \frac{h_{\text{lin}}(\phi)}{p_3'} \le \frac{1}{2} \quad \text{and} \quad \frac{h(\phi)}{p_3'} \le \frac{1}{2},$$

where we remind that $h_{\text{lin}}(\phi)$ and $h(\phi)$ are respectively global linear height and global Newton height defined as in (1-6).

At all points v where (LA) is satisfied and where $|x|_\alpha^{2\vartheta}$ is smooth (i.e., v is not on an axis) we get the local Fourier restriction estimate in the range (5-3) directly from [Palle 2021, Proposition 4.2]. We shall briefly touch upon what happens in the case when v is situated on the axis in Section 5A. In this case one has to only slightly adjust the proofs in Section 4.

In the case when (LA) is not satisfied at v let us call the pair $(\mathfrak{p}_1, \mathfrak{p}_3) = (\mathfrak{p}_1(v), \mathfrak{p}_3(v))$ given by

$$\left(\frac{1}{\mathfrak{p}_1'}, \frac{1}{\mathfrak{p}_3'}\right) = \left(\frac{1}{2} - \frac{h_{\text{lin}}(\phi, v)}{2h(\phi, v)}, \frac{1}{2h(\phi, v)}\right)$$

the critical exponent of ϕ at v. It is obtained as the intersection of the lines

$$\frac{1}{p_1'} + \frac{h_{\text{lin}}(\phi, v)}{p_3'} = \frac{1}{2} \quad \text{and} \quad \frac{1}{p_3'} = \frac{1}{2h(\phi, v)}$$

in the $(1/p_1', 1/p_3')$ plane. Thus, for the local estimate in this case it suffices to prove the inequality

$$\|\hat{f}\|_{L^2(\mathrm{d}\mu_{0,v})} \le C\|f\|_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})},$$

where

$$\langle \mu_{0,v}, f\rangle = \int f(x_1, x_2, \phi_v(x_1, x_2))\, \eta_v(x_1, x_2)\, \mathrm{d}x_1\, \mathrm{d}x_2$$

and

$$\left(\frac{1}{p_1'}, 1p_3'\right) \in \left\{\left(0, \frac{1}{2h(\phi, v)}\right), \left(\frac{1}{2}, 0'\right), \left(\frac{1}{\mathfrak{p}_1'(v)}, \frac{1}{\mathfrak{p}_3'(v)}\right)\right\},$$

since then we get the full range from the necessary conditions by interpolation. We shall only give a sketch of the proof in this case too in Subsections 5B and 5C, since it is almost identical to a type of singularity considered in [Palle 2021, Section 5.5].

5A. ***Fourier restriction for the adapted case.*** As mentioned, in the adapted case one needs to prove the Fourier restriction estimate for $(p_1, p_3) \in (1, 2)^2$ satisfying

$$\frac{1}{p_1'} + \frac{h_{\text{lin}}(\phi, v)}{p_3'} = \frac{1}{2},$$

and the part of the measure where the amplitude in (5-1) is smooth the restriction estimate is already proven in [Palle 2021].

Now the amplitude in (5-1) (in particular the function $x \mapsto |x|_\alpha^{2\vartheta}$) is in general not smooth along the axes $x_1 = 0$ and $x_2 = 0$. Namely, on the $x_1 = 0$ axis one can take only the derivatives (of the amplitude) in the x_2-direction, and analogously on the $x_2 = 0$ axis one can take only derivatives in the x_1-direction. Note that the only possible nonadapted normal form appears only away from the axes.

Let us consider without loss of generality what happens for the point $v = (v_1, 0)$ on the axis $x_2 = 0$ and its associated measure $\mu_{0,v}$ defined in (5-2). We shall only briefly sketch what one needs to do in order to prove the Fourier restriction estimate when the amplitude is not smooth in the x_2-direction at v. Since we are dealing only with adapted normal forms, it suffices to obtain an appropriate estimate on the Fourier transform, after which one can apply Lemma 4.2 or its modification such as [Palle 2021, Lemma 3.8]. For the reader's convenience we state explicitly the result we need (the proof is essentially the same as for Lemma 4.2 — in fact it is even simpler since one can use the usual Young's inequality instead of the weak one).

Lemma 5.1. *Assume that we are given a bounded open set $\Omega \subseteq \mathbb{R}^2$ and functions $\Phi \in C^\infty(\Omega; \mathbb{R}^2)$, $\phi \in C^\infty(\Omega; \mathbb{R})$, $a \in L^\infty(\Omega)$. Let us consider the measure*

$$\langle \mu, f \rangle := \int f(\Phi(x), \phi(x))\, a(x)\, \mathrm{d}x$$

and the operator $T : f \mapsto f * \hat{\mu}$. If Φ is injective, its Jacobian is of size $|J_\Phi| \sim A_1$, and if one has furthermore the estimate

$$|\hat{\mu}(\xi)| \le A_2 (1 + |\xi_3|)^{-1/h}$$

for some $h \in (0, 1)$, then for any $\theta \in [0, 1]$ and

$$\left(\frac{1}{p_1'}, \frac{1}{p_3'} \right) = \left(\frac{1-\theta}{2}, \frac{\theta}{2h} \right)$$

the $L^{p_3}_{x_3}(\mathbb{R}; L^{p_1}_{(x_1,x_2)}(\mathbb{R}^2)) \to L^{p_3'}_{x_3}(\mathbb{R}; L^{p_1'}_{(x_1,x_2)}(\mathbb{R}^2))$ operator norm of T is bounded by $(A_1^{-1} \|a\|_{L^\infty})^{1-\theta} A_2^\theta$.

Often we shall also need to use the Littlewood–Paley theorem in order to localize away from the axis.

According to the normal forms listed at the end of Section 3A, and under the condition (H1), we have the following cases.

<u>Case 1</u>: If (under the notation of Section 3) we have $k = \infty$, then by the considerations from Section 3B the phase at v is

$$\phi_v(x - v) = (x_1 - v_1)^{\tilde{k}} q(x_1 - v_1) + \varphi(x_1, x_2),$$

where $2 \le \tilde{k} < \infty$, $q(0) \ne 0$, and φ is a flat function at v. This corresponds to normal form (i.y2) and we have $h_{\mathrm{lin}}(\phi, v) = \tilde{k}$. Since $|x|_\alpha^{2\vartheta}$ is still smooth in the x_1-direction, one can use the van der Corput lemma in the x_1-direction and get that the decay of the Fourier transform of $\mu_{0,v}$ is $(1 + |\xi|)^{-1/\tilde{k}}$. This now implies the desired estimate by Lemma 5.1.

If $2 \le k < \infty$, then we have three further cases.

<u>Case 2</u>: Let us consider the phase

$$\phi_v(x) = x_2^k r(x),$$

where $r(v) \neq 0$ and $k \geq 2$. In this case the linear height is $h_{\text{lin}}(\phi, v) = k$. Here the idea is to apply the Littlewood–Paley theorem in order to localize away from the axis $x_2 = 0$, and rescale afterwards. Since essentially the same thing was done in Section 4 for this type of singularity (see the proof for normal form (i) in Section 4A), let us just briefly mention the main differences compared to there. Obviously, one scales differently the measure pieces away from the axis obtained by applying the Littlewood–Paley theorem since here we consider different exponents (p_1, p_3). The main difference is that we do not use the Hessian determinant to obtain a decay on the Fourier transformation of the rescaled measure piece (since the Hessian determinant may vanish of infinite order as only (H1) is assumed and not the stronger condition (H2)), but rather determine it directly from the form of the phase above. This we may now do since the new amplitude for the rescaled measure pieces is now smooth.

Case 3: Let us now consider the case when the phase is nondegenerate, i.e., the Hessian determinant does not vanish at v (and in particular $h_{\text{lin}}(\phi, v) = 1$). Here we use the Littlewood–Paley theorem as in Case 2, but after rescaling we use the size of the Hessian determinant of the new phase to get a decay on the Fourier transform of the measure (as was done in Section 4 for normal forms (i), (ii), and (iii)).

Case 4: The final case is when (after an affine change to y- or w-coordinates from Section 3) we have

$$\phi_{v,u}(u) = u_1^2 r_1(u) + u_2^{k_0} r_2(u),$$

where $3 \leq k_0 \leq \infty$, $r_1(0) \neq 0$, and in the case when $k_0 < \infty$ we have $r_2(0) \neq 0$ and $h_{\text{lin}}(\phi, v) = 2k_0/(2+k_0)$. If $k_0 = \infty$ then $h_{\text{lin}}(\phi, v) = 2$, and the above equality holds in the sense that we can take any $k_0 \geq 0$ and r_2 flat at the origin. Inspecting the y- and w-coordinates from Section 3 we see that the $x_2 = 0$ axis corresponds to the $u_2 = 0$ axis.

If $k_0 = \infty$, we can argue in the same way as in the case $k = \infty$ above (here it is critical that $\partial_{u_1} = c\,\partial_{x_1}$, $c \neq 0$, in order to be able to apply the van der Corput lemma in the smooth direction).

Otherwise, if k_0 is finite, we proceed again with a Littlewood–Paley decomposition in the u_2-direction (as was done in Section 4C for normal forms (ii) and (iii)) in order to get a smooth amplitude. At this point one gets that the estimate on the decay of the Fourier transform is $2^{k_0 l/2}(1 + |\xi|)^{-1}$ by using the size of the Hessian determinant. Since the new rescaled phase is (compare with (4-9))

$$u_1^2 r_1(u_1, 2^{-l} u_2) + 2^{-k_0 l} u_2^{k_0} r_2(u_1, 2^{-l} u_2),$$

by applying the van der Corput lemma in u_1 we also have the decay estimate $(1 + |\xi|)^{-1/2}$. Interpolating these two estimates gives the decay $2^l (1 + |\xi|)^{-(2+k_0)/(2k_0)}$, which turns out to be precisely what one needs when applying Lemma 5.1.

5B. ***Fourier restriction for the nonadapted case: preliminaries.*** Let us fix a phase function ϕ_{loc} of the form

$$\phi_{\text{loc}}(x) = (x_2 - x_1^2 \psi(x_1))^k r(x),$$

where $\psi(0), r(0) \neq 0$ and $k \in \mathbb{N}$, $k \geq 2$. The adapted coordinates are obtained by the smooth transformation $y_1 = x_1,\ y_2 = x_2 - x_1^2 \psi(x_1)$:

$$\phi_{\text{loc}}^a(y) := y_2^k r^a(y),$$

where $r^a(0) \neq 0$. Thus, the Newton height of ϕ_{loc} is k and the Newton distance is $d := 2k/3$ (which coincides with the linear height h_{lin}). The Varchenko exponent is 0 since in adapted coordinates the principal face is noncompact. Then from, e.g., [Palle 2021, Section 3.3] we know that we automatically have the Fourier restriction estimate

$$\|\mathcal{F}f\|_{L^2(\mathrm{d}\nu)} \lesssim \|f\|_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})}, \quad f \in \mathcal{S}(\mathbb{R}^3), \tag{5-6}$$

for the exponents

$$\left(\frac{1}{p_1'}, \frac{1}{p_3'}\right) = \left(0, \frac{1}{2k}\right) \quad \text{and} \quad \left(\frac{1}{p_1'}, \frac{1}{p_3'}\right) = \left(\frac{1}{2}, 0\right),$$

and where the measure ν is defined through

$$\langle \nu, f\rangle = \int f(x_1, x_2, \phi_{\mathrm{loc}}(x_1, x_2))\, a(x_1, x_2)\, \mathrm{d}x_1\, \mathrm{d}x_2, \tag{5-7}$$

where $a \in C_c^\infty(\mathbb{R}^2)$ is a nonnegative function supported in a small neighborhood of the origin. It remains to obtain the Fourier restriction estimate for the critical exponent, which in this case is

$$\left(\frac{1}{\mathfrak{p}_1'}, \frac{1}{\mathfrak{p}_3'}\right) = \left(\frac{1}{6}, \frac{1}{2k}\right). \tag{5-8}$$

The case $k = 2$ has been solved in [Palle 2021]. In the case $k = 3$ the critical exponent lies on the diagonal and so this case has already been solved in [Ikromov and Müller 2016].

In the case $k \geq 4$ we have $1/\mathfrak{p}_1' > 1/\mathfrak{p}_3'$ and so one would need to slightly modify the methods used in [Palle 2021] (i.e., the methods for the case $h_{\mathrm{lin}}(\phi) < 2$) since there one interpolated between two points of the form

$$\left(\frac{1}{p_1'}, \frac{1}{p_3'}\right) = \left(0, \frac{s}{2}\right) \quad \text{and} \quad \left(\frac{1}{p_1'}, \frac{1}{p_3'}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

for some $0 < s < 1/k$. In the case $1/\mathfrak{p}_1' > 1/\mathfrak{p}_3'$ in general one would need to interpolate between three points

$$\left(\frac{1}{p_1'}, \frac{1}{p_3'}\right) = (0, 0), \quad \left(\frac{1}{p_1'}, \frac{1}{p_3'}\right) = \left(\frac{1}{2}, \frac{1}{2}\right), \quad \text{and} \quad \left(\frac{1}{p_1'}, \frac{1}{p_3'}\right) = \left(\frac{1}{2}, 0\right).$$

In particular, if one has an operator $T : L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)}) \to L^{p_3'}_{x_3}(L^{p_1'}_{(x_1,x_2)})$ satisfying the estimates

$$\begin{aligned}
\|T\|_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)}) \to L^{p_3'}_{x_3}(L^{p_1'}_{(x_1,x_2)})} &\lesssim A_1 \quad \text{for } \left(\frac{1}{p_1'}, \frac{1}{p_3'}\right) = (0, 0),\\
\|T\|_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)}) \to L^{p_3'}_{x_3}(L^{p_1'}_{(x_1,x_2)})} &\lesssim A_2 \quad \text{for } \left(\frac{1}{p_1'}, \frac{1}{p_3'}\right) = \left(\frac{1}{2}, \frac{1}{2}\right),\\
\|T\|_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)}) \to L^{p_3'}_{x_3}(L^{p_1'}_{(x_1,x_2)})} &\lesssim A_3 \quad \text{for } \left(\frac{1}{p_1'}, \frac{1}{p_3'}\right) = \left(\frac{1}{2}, 0\right),
\end{aligned} \tag{5-9}$$

then by interpolation one has the estimate

$$\|T\|_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)}) \to L^{p_3'}_{x_3}(L^{p_1'}_{(x_1,x_2)})} \lesssim A_1^{2/3}\, A_2^{1/k}\, A_3^{(k-3)/(3k)} \quad \text{for } \left(\frac{1}{p_1'}, \frac{1}{p_3'}\right) = \left(\frac{1}{6}, \frac{1}{2k}\right).$$

In our special case we shall not use the above general approach since we recall that when we considered the case when the mitigating factor was present (to be more precise, the case of normal form (vi) considered in Section 4F), after performing some decompositions and rescalings one got measure pieces for which one needed to prove the Fourier restriction estimate for the exponent

$$\left(\frac{1}{p_1'}, \frac{1}{p_3'}\right) = \left(\frac{1}{6}, \frac{1}{3}\right). \tag{5-10}$$

In the current case without the mitigating factor it turns out that we shall get the same measure pieces, but for which we need to prove the Fourier restriction estimate for the exponent (5-8). Thus, if we have the Fourier restriction estimate for the exponent (5-10), then the Fourier restriction for (5-8) is obtained by interpolating with the result for

$$\left(\frac{1}{p_1'}, \frac{1}{p_3'}\right) = \left(\frac{1}{6}, 0\right),$$

which one can obtain by applying the 2-dimensional Fourier restriction result for curves with nonvanishing curvature.

These stronger estimates for the rescaled measure pieces do not contradict the necessary conditions obtained by Knapp-type examples in [Palle 2021] since the information on the exponents and the Newton height of ϕ is consumed in the rescaling procedure (which is different in this section and in Section 4F).

Let us begin with some preliminary reductions. By the results from [Palle 2021, Section 4.2], instead of considering the whole measure (5-7), we may reduce ourselves to considering the part near the principal root jet in the half-plane $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0\}$:

$$\langle \nu^{\rho_1}, f \rangle = \int_{x_1 \geq 0} f(x, \phi_{\mathrm{loc}}(x))\, a(x)\, \rho_1(x)\, \mathrm{d}x,$$

where

$$\rho_1(x) = \chi_0\left(\frac{x_2 - \psi(0)x_1^2}{\varepsilon x_1^2}\right)$$

for an ε which we can take to be as small as we want.

The next step is to use a Littlewood–Paley argument in the (x_1, x_2)-plane and the scaling by κ dilations

$$\delta_t^\kappa(x) = (t^{\kappa_1} x_1, t^{\kappa_2} x_2),$$

where $\kappa := (1/(2k), 1/k)$ is the weight associated to the principal face of ϕ_{loc}. Then one is reduced to proving (5-6) for the measures

$$\langle \nu_j, f \rangle = \int f(x, \phi(x, \delta))\, a(x, \delta)\, \mathrm{d}x,$$

uniformly in j, where the function $\phi(x, \delta)$ has the form

$$\phi(x, \delta) := (x_2 - x_1^2 \psi(\delta_1 x_1))^k r(\delta_1 x_1, \delta_2 x_2),$$

where

$$\delta = (\delta_1, \delta_2) := (2^{-\kappa_1 j}, 2^{-\kappa_2 j}).$$

Note that we can take $|\delta| \ll 1$. The amplitude $a(x,\delta) \geq 0$ is a smooth function of (x,δ) supported where

$$x_1 \sim 1 \sim |x_2|.$$

We may additionally assume $|x_2 - x_1^2\psi(0)| \ll 1$ due to ρ_1, and by compactness we may in fact reduce ourselves to assuming $|(x_1,x_2)-(v_1^0,v_2^0)| \ll 1$ for some $(v_1^0,v_2^0) \in \mathbb{R}^2$ with $v_1^0 \sim 1$.

The following step is to again apply the Littlewood–Paley theorem, but this time in the x_3-direction (again, for the mixed-norm Littlewood–Paley theory see [Lizorkin 1970]), and reduce the Fourier restriction problem for ν_j to the Fourier restriction for the measures

$$\langle \nu_{\delta,l}, f\rangle = \int f(x,\phi(x,\delta))\,\chi_1(2^{kl}\phi(x,\delta))\,a(x,\delta)\,\mathrm{d}x;$$

i.e., we need to prove

$$\|\mathcal{F}f\|_{L^2(\mathrm{d}\nu_{\delta,l})} \lesssim \|f\|_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})}, \quad f \in \mathcal{S}(\mathbb{R}^3),$$

uniformly in l and δ, where $l \gg 1$ and $|\delta| \ll 1$.

Finally, we perform a change of coordinates and a rescaling. Namely, after substituting $(x_1,x_2) \mapsto (x_1, 2^{-l}x_2 + x_1^2\psi(\delta_1 x_1))$ we get

$$\langle \nu_{\delta,l}, f\rangle = 2^{-l}\int f(x_1, 2^{-l}x_2 + x_1^2\psi(\delta_1x_1), 2^{-kl}\phi^a(x,\delta,l))\,a(x,\delta,l)\,\mathrm{d}x,$$

where

$$a(x,\delta,l) := \chi_1(\phi^a(x,\delta,l))\,a(x_1, 2^{-l}x_2 + x_1^2\psi(\delta_1x_1),\delta),$$
$$\phi^a(x,\delta,l) := x_2^k r(\delta_1x_1, \delta_2(2^{-l}x_2 + x_1^2\psi(\delta_1x_1))).$$

Note that $a(x,\delta,l)$ is again supported in a domain where $x_1 \sim 1 \sim |x_2|$. Rescaling we obtain that the Fourier restriction estimate for $\nu_{\delta,l}$ is equivalent to the estimate

$$\|\mathcal{F}f\|_{L^2(\mathrm{d}\tilde{\nu}_{\delta,l})} \lesssim \|f\|_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})}, \quad f \in \mathcal{S}(\mathbb{R}^3),$$

for the measure

$$\langle \tilde{\nu}_{\delta,l}, f\rangle = \int f(x_1, 2^{-l}x_2 + x_1^2\psi(\delta_1x_1), \phi^a(x,\delta,l))\,a(x,\delta,l)\,\mathrm{d}x. \tag{5-11}$$

As mentioned, since this measure is of the same form as (4-23), we are interested in proving the stronger estimate

$$\|\mathcal{F}f\|_{L^2(\mathrm{d}\tilde{\nu}_{\delta,l})} \lesssim \|f\|_{L^{\tilde{\mathfrak{p}}_3}_{x_3}(L^{\tilde{\mathfrak{p}}_1}_{(x_1,x_2)})}, \quad f \in \mathcal{S}(\mathbb{R}^3),$$

where

$$\left(\frac{1}{\tilde{\mathfrak{p}}_1'}, \frac{1}{\tilde{\mathfrak{p}}_3'}\right) := \left(\frac{1}{6}, \frac{1}{3}\right).$$

Note that we automatically have the estimate for

$$\left(\frac{1}{p_1'}, \frac{1}{p_3'}\right) = \left(\frac{1}{6}, 0\right)$$

by a classical result of Fefferman and Stein [Fefferman 1970] (or [Zygmund 1974]), since $x_1 \mapsto (x_1, 2^{-l}x_2 + x_1^2\psi(\delta_1 x_1))$ is a curve with curvature bounded from below uniformly in $|x_2| \sim 1$, $2^{-l} \ll 1$, and $\delta_1 \ll 1$.

5C. ***Fourier restriction for the nonadapted case*: *spectral decomposition.*** We begin by performing a spectral decomposition of the measure $\tilde{\nu}_{\delta,l}$. For $(\lambda_1, \lambda_2, \lambda_3)$ dyadic numbers with $\lambda_i \geq 1$, $i = 1, 2, 3$, we consider localized measures ν_l^λ defined through

$$\hat{\nu}_l^\lambda(\xi) = \chi_1\left(\frac{\xi_1}{\lambda_1}\right)\chi_1\left(\frac{\xi_2}{\lambda_2}\right)\chi_1\left(\frac{\xi_3}{\lambda_3}\right)\int e^{-i\Phi(x,\delta,l,\xi)}\, a(x,\delta,l)\,\chi_1(x_1)\,\chi_1(x_2)\,\mathrm{d}x, \tag{5-12}$$

where the phase function is

$$\Phi(x,\delta,l,\xi) := \xi_3\phi^a(x,\delta,l) + 2^{-l}\xi_2 x_2 + \xi_2 x_1^2\psi(\delta_1 x_1) + \xi_1 x_1. \tag{5-13}$$

By an abuse of notation, above whenever $\lambda_i = 1$, we consider the cutoff function $\chi_1(\xi_i/\lambda_i)$ to be actually $\chi_0(\xi_1/\lambda_1)$; i.e., it localizes so that $|\xi_i| \lesssim 1$.

Let us introduce the convolution operators $\widetilde{T}_{\delta,l} f := f * \hat{\tilde{\nu}}_{\delta,l}$ and $T_l^\lambda f := f * \hat{\nu}_l^\lambda$. Then we need to show

$$\|\widetilde{T}_{\delta,l}\|_{L_{x_3}^{\tilde{\mathfrak{p}}_3}(L_{(x_1,x_2)}^{\tilde{\mathfrak{p}}_1}) \to L_{x_3}^{\tilde{\mathfrak{p}}_3'}(L_{(x_1,x_2)}^{\tilde{\mathfrak{p}}_1'})} \lesssim 1,$$

since $\widetilde{T}_{\delta,l}$ is the "R^*R" operator, i.e., one has $\widetilde{T}_{\delta_l} = (\widetilde{R}_{\delta,l})^*\widetilde{R}_{\delta,l}$ if $\widetilde{R}_{\delta,l}$ denotes the Fourier restriction operator with respect to the surface measure $\tilde{\nu}_{\delta,l}$. Therefore, the boundedness of $\widetilde{T}_{\delta,l}$ is equivalent to the boundedness of $\widetilde{R}_{\delta,l}$ by Hölder's inequality.

Our first step shall be to reduce the problem to the case when $\lambda_2 \ll 2^l$. In order to achieve this we split the Fourier transform of $\tilde{\nu}_{\delta,l}$ as

$$\hat{\tilde{\nu}}_{\delta,l} = (1 - \chi_0(2^{-l}\xi_2))\hat{\tilde{\nu}}_{\delta,l} + \chi_0(2^{-l}\xi_2)\hat{\tilde{\nu}}_{\delta,l}, \tag{5-14}$$

where we assume that χ_0 is supported in a sufficiently small neighborhood of the origin, and we denote the respective operators for the respective terms by T_I and T_{II}.

For the first term in (5-14) and its operator T_I one uses Lemma 4.2 above, though with a slight modification. First, since on the support of $(1 - \chi_0(2^{-l}\xi_2))\hat{\tilde{\nu}}_{\delta,l}$ we have $|\xi_2| \gtrsim 2^l$, one can easily show by using (5-13) that now

$$|(1 - \chi_0(2^{-l}\xi_2))\hat{\tilde{\nu}}_{\delta,l}| \lesssim 2^{-l/2}(1 + |\xi_3|)^{-1},$$

as the "worst case" is when $|\xi_1| \sim |\xi_2|$ and $|\xi_3| \sim |2^{-l}\xi_2|$, in which case we use stationary phase in both x_1 and x_2 (and in other cases we get a better decay by integrating by parts). In order to obtain the Plancherel estimate $L^1(\mathbb{R}; L^2(\mathbb{R}^2)) \to L^\infty(\mathbb{R}; L^2(\mathbb{R}^2))$ in Lemma 4.2 for T_I it suffices to prove it for T_{II} and $\widetilde{T}_{\delta,l}$ (formally, one needs to actually consider the $L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ estimate for a fixed ξ_3). For the operator $\widetilde{T}_{\delta,l}$ we get the bound 2^l in the same way as in Lemma 4.2. The main fact to notice is that in (5-11) the Jacobian of $(x_1, x_2) \mapsto (x_1, 2^{-l}x_2 + x_1^2\psi(\delta_1 x_1))$ is of size 2^{-l}. One now gets the same estimate automatically for T_{II} since the L^1 norm of the Fourier transform of the cutoff function

$\chi_0(2^{-l}\xi_2)$ is of size ~ 1. The $L^{\tilde{\mathrm{p}}_3}_{x_3}(L^{\tilde{\mathrm{p}}_1}_{(x_1,x_2)}) \to L^{\tilde{\mathrm{p}}_3'}_{x_3}(L^{\tilde{\mathrm{p}}_1'}_{(x_1,x_2)})$ estimate for T_I follows with constant of size $\sim 1 = (2^{-l/2})^{2/3}(2^l)^{1/3}$.

For the operator T_{II} we shall use the spectral decomposition (5-12) where we may now assume $\lambda_2 \ll 2^l$. Recall that for an operator of the form $Tf = f * \hat{g}$ the A_1-constant from (5-9) is bounded by the L^∞ norm of $\hat{g}$, and the A_2-constant is bounded by the L^∞ norm of g. If we now furthermore have that $\hat{g}$ has its support in the ξ_3-coordinate localized at $|\xi_3| \lesssim \lambda_3$, then by [Palle 2021, Lemma 3.9] we have the estimate

$$\|T\|_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)}) \to L^{p_3'}_{x_3}(L^{p_1'}_{(x_1,x_2)})} \lesssim A_1 \lambda_3^{1/2} \quad \text{for } \left(\frac{1}{p_1'}, \frac{1}{p_3'}\right) = \left(0, \frac{1}{4}\right),$$

and so by interpolation we get

$$\|T\|_{L^{\tilde{\mathrm{p}}_3}_{x_3}(L^{\tilde{\mathrm{p}}_1}_{(x_1,x_2)}) \to L^{\tilde{\mathrm{p}}_3'}_{x_3}(L^{\tilde{\mathrm{p}}_1'}_{(x_1,x_2)})} \lesssim A_1^{2/3} A_2^{1/3} \lambda_3^{1/3}. \tag{5-15}$$

The inverse Fourier transform of (5-12) is

$$\nu_l^\lambda(x) = \lambda_1\lambda_2\lambda_3 \int \check{\chi}_1(\lambda_1(x_1 - y_1))\, \check{\chi}_1(\lambda_2(x_2 - 2^{-l} y_2 - y_1^2 \psi(\delta_1 y_1))) \\ \times \check{\chi}_1(\lambda_3(x_3 - \phi^a(y, \delta, l)))\, a(y, \delta, l)\, \chi_1(y_1)\, \chi_1(y_2)\, \mathrm{d}y. \tag{5-16}$$

One can consider either the substitution $(z_1, z_2) = (\lambda_1 y_1, \lambda_2 2^{-l} y_2)$, or the substitution $(z_1, z_2) = (\lambda_1 y_1, \lambda_3 \phi^a(y, \delta, l))$ (in order to carry this out one needs to consider the cases $y_2 \sim 1$ and $y_2 \sim -1$ separately), and get

$$\|\nu_j^\lambda\|_{L^\infty} \lesssim \min\{2^l \lambda_3, \lambda_2\}.$$

But now since $\lambda_2 \ll 2^l$ we may take $A_2 := \lambda_2$.

It remains to calculate the L^∞ bound for the $\hat{\nu}_l^\lambda$ function. This we can do by estimating the oscillatory integral in (5-12). As the calculations for the oscillatory integral in this case are almost identical to the ones in [Palle 2021, Section 5.5], we shall only briefly explain the case when $\lambda_1 \sim \lambda_2,\ 2^{-l}\lambda_2 \ll \lambda_3 \ll \lambda_2$, corresponding to Case 6 in [Palle 2021, Section 5.5]. In all the other cases one gets that one can sum absolutely in the operator norm the operator pieces T_l^λ.

Let us remark that since $\lambda_2 \ll 2^l$, the case when $\lambda_1 \sim \lambda_2,\ 2^{-l}\lambda_2 \sim \lambda_3$, corresponding to Case 4 in [Palle 2021, Section 5.5], does not appear anymore. This is critical since in this case one would not have absolute summability, nor would the complex interpolation method developed in [Ikromov and Müller 2016] work. This is the reason why we needed to consider T_I and T_{II} separately.

Case $\lambda_1 \sim \lambda_2$ and $2^{-l}\lambda_2 \ll \lambda_3 \ll \lambda_2$. As was obtained in [Palle 2021, Section 5.5], we have

$$\|\hat{\nu}_l^\lambda\|_{L^\infty} \lesssim \lambda_1^{-1/2} \lambda_3^{-N} \tag{5-17}$$

for any $N > 0$, that is, we have $A_1 = \lambda_1^{-1/2}\lambda_3^{-N}$, and recall that $A_2 = \lambda_2$, Therefore (5-15) gives

$$\|T_l^\lambda\|_{L^{\tilde{\mathrm{p}}_3}_{x_3}(L^{\tilde{\mathrm{p}}_1}_{(x_1,x_2)}) \to L^{\tilde{\mathrm{p}}_3'}_{x_3}(L^{\tilde{\mathrm{p}}_1'}_{(x_1,x_2)})} \lesssim \lambda_3^{-N}.$$

In order to be able to sum in $\lambda_1 \sim \lambda_2$ we need to use the complex interpolation method from [Ikromov and Müller 2016]. For a fixed λ_3 and ζ a complex number we define the measure $\mu_\zeta^{\lambda_3}$ by

$$\mu_\zeta^{\lambda_3} := \gamma(\zeta) \sum_{\lambda_1,\lambda_2} \lambda_1^{(1-3\zeta)/2} \nu_l^\lambda,$$

where the sum is over $\lambda_3 \ll \lambda_2 \ll 2^l$ and $\lambda_1 \sim \lambda_2$, and where $\gamma(\zeta) = 2^{-3(\zeta-1)/2} - 1$. We denote the associated convolution operator by $T_\zeta^{\lambda_3}$ and we recover with $\zeta = \frac{1}{3}$ the operator we want to estimate.

By a complex interpolation argument it suffices to show that

$$\|T_{it}^{\lambda_3}\|_{L_{x_3}^{p_3}(L_{(x_1,x_2)}^{p_1}) \to L_{x_3}^{p_3'}(L_{(x_1,x_2)}^{p_1'})} \lesssim \lambda_3^{-N} \quad \text{for } \left(\frac{1}{p_1'}, \frac{1}{p_3'}\right) = \left(0, \frac{1}{4}\right),$$

$$\|T_{1+it}^{\lambda_3}\|_{L_{x_3}^{p_3}(L_{(x_1,x_2)}^{p_1}) \to L_{x_3}^{p_3'}(L_{(x_1,x_2)}^{p_1'})} \lesssim 1 \qquad \text{for } \left(\frac{1}{p_1'}, \frac{1}{p_3'}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

for some $N > 0$, with constants uniform in $t \in \mathbb{R}$. The first estimate follows directly from the fact that $\hat{\nu}_l^\lambda$ have essentially disjoint supports with respect to λ and the estimate (5-17) (see [Palle 2021, Lemma 3.8(i)]), and for the other bound we need to estimate the L^∞ norm of the corresponding sum of the expressions (5-16). The proof is the same as in [Palle 2021, Section 5.5, Case 6], up to the formal difference in the function ϕ^a, which here behaves like y_2^k, and there like y_2^2. Since the domain of integration in (5-16) is $|y_2| \sim 1$, this is not essential. This finishes (the sketch of) the proof of the Fourier restriction for the nonadapted case, and also the proof of Theorem 1.2.

Appendix: Application of the Christ–Kiselev lemma

Recall that we consider the nonhomogeneous initial problem

$$\begin{cases} (\partial_t - i\phi(D))u(x,t) = F(x,t), & (x,t) \in \mathbb{R}^2 \times (0,\infty), \\ \qquad\qquad u(x,0) = G(x), & x \in \mathbb{R}^2, \end{cases}$$

for $F \in \mathcal{S}(\mathbb{R}^3)$, $G \in \mathcal{S}(\mathbb{R}^2)$, where ϕ, $\mathcal{W}$, and $(p_1, p_3) \in (1,2)^2$ are either as in Theorem 1.1 or 1.2, and where we additionally assume $\rho \in \{0,1\}$. Note that ϕ is locally bounded and has polynomial growth at infinity, and note that according to Remark 2.3 the weight $\mathcal{W}$ is locally integrable in $\mathbb{R}^2$. The formula for a solution of the above equation is obtained through the Duhamel principle:

$$u(x,t) = (e^{i\phi(D)t} G)(x) + \int_0^t (e^{i\phi(D)(t-s)} F(\,\cdot\,, s))(x,s)\, \mathrm{d}s. \tag{A-1}$$

Note that $u \in C^\infty(\mathbb{R}^2 \times \mathbb{R}) \cap L_t^\infty((C_0)_{(x_1,x_2)}(\mathbb{R}^2))$, where C_0 denotes the space of continuous functions which tend to 0 at infinity.

We consider the following two surface measures (the second defined as in (1-3)):

$$\langle \mu_\phi, f \rangle = \int_{\mathbb{R}^2 \setminus \{0\}} f(x_1, x_2, \phi(x_1,x_2))\, \mathrm{d}x,$$

$$\langle \mu, f \rangle = \int_{\mathbb{R}^2 \setminus \{0\}} f(x_1, x_2, \phi(x_1,x_2))\, \mathcal{W}(x_1,x_2)\, \mathrm{d}x,$$

and we assume that the Fourier restriction estimate (1-2) for μ holds true for $(p_1, p_3) \in (1,2)^2$. One can easily check that

$$(e^{i\phi(D)t}G)(x) = \mathcal{F}^{-1}((\mathcal{F}G)\,\mathrm{d}\mu_\phi)(x,t) = \mathcal{F}^{-1}(\mathcal{W}^{-1}(\mathcal{F}G)\,\mathrm{d}\mu)(x,t),$$

and so this is precisely the Fourier extension operator of μ applied to the function $\mathcal{W}^{-1}\mathcal{F}G$. We can therefore bound the $L_t^{p_3'}(L_{(x_1,x_2)}^{p_1'})$ norm of this expression by the $L^2(\mathrm{d}\mu)$ norm of $\mathcal{W}^{-1}\mathcal{F}G$.

It remains to estimate the $L_t^{p_3'}(L_{(x_1,x_2)}^{p_1'})$ norm of the second term in (A-1). It turns out that the operator associated to this second term is closely related to the operator $f \mapsto f * \mathcal{F}^{-1}\mu$ (which we know is bounded from $L_t^{p_3}(L_{(x_1,x_2)}^{p_1})$ to $L_t^{p_3'}(L_{(x_1,x_2)}^{p_1'})$ since this is the corresponding R^*R operator). Namely, one can check that

$$\int_0^\infty (e^{i\phi(D)(t-s)}F(\,\cdot\,,s))(x,s)\,\mathrm{d}s = ((F\chi_{(0,\infty)}(s)) * (\mathcal{F}^{-1}\mu_\phi))(x,t),$$

and therefore it remains to pass from μ_ϕ to μ and to pass from integrating over $(0,\infty)$ in s to integrating over $(0,t)$ in s.

In order to do this, our first step is to use the Littlewood–Paley theorem in the x-direction so that our problem is reduced to proving the boundedness of the operator

$$\int_0^t (e^{i\phi(D)(t-s)}\,\eta_j(D)\,F(\,\cdot\,,s))(x,s)\,\mathrm{d}s, \tag{A-2}$$

where $(\eta_j)_{j\in\mathbb{Z}}$, $\eta_j = \eta \circ \delta_{2^{-j}}$, constitutes a partition of unity in $\mathbb{R}^2 \setminus \{0\}$ (as in (2-1) in Section 2A) respecting the α-mixed homogeneous dilation $\delta_{2^{-j}}$ defined in (1-1). By unwinding the definition of the operator in (A-2) and inserting the $\mathcal{W}$-factor, one obtains the expression (up to a universal constant)

$$\int_0^t \int_{\mathbb{R}^2} \biggl(\int_{\mathbb{R}^2} e^{i(x-y)\cdot\xi + i(t-s)\phi(\xi)} \eta_j(\xi)\mathcal{W}(\xi)\,\mathrm{d}\xi \biggr) F_{\mathcal{W}^{-1}}(y,s)\,\mathrm{d}y\,\mathrm{d}s, \tag{A-3}$$

where $F_{\mathcal{W}^{-1}} = \mathcal{F}^{-1}_{(x_1,x_2)}(\mathcal{W}^{-1}\mathcal{F}_{(x_1,x_2)}F)$. The expression within the brackets defines a convolution kernel $K_j(t-s; x-y)$ whose associated operator $T_j(t-s)$ in the x-variable is a bounded mapping from $L^{p_0}(\mathbb{R}^2)$ to $L^{p_0'}(\mathbb{R}^2)$ for any $p_0 \in [1,2]$ (since the integrand in the brackets is an $L_c^\infty(\mathbb{R}^2)$ function). Using the dominated convergence theorem one can get strong continuity of the operator-valued function $T_j : \mathbb{R} \to \mathcal{L}(L^{p_0}(\mathbb{R}^2); L^{p_0'}(\mathbb{R}^2))$ (which in turn, by the uniform boundedness principle, implies joint continuity $T_j : \mathbb{R} \times L^{p_0}(\mathbb{R}^2) \to L^{p_0'}(\mathbb{R}^2)$).

We may now apply the Christ–Kiselev lemma (for a proof of this variant see, e.g., [Sogge 1995, Chapter IV, Lemma 2.1]):

Lemma A.1. *Let Y and Z be separable Banach spaces and let $K : \mathbb{R} \to \mathcal{L}(Y,Z)$ be a continuous function from the real numbers to the space of bounded linear mappings $Y \to Z$ equipped with the strong operator topology. If the operator defined by*

$$(Tf)(t) := \int_{\mathbb{R}} K(t-s)\,f(s)\,\mathrm{d}s$$

is a bounded mapping from $L^{p_0}(\mathbb{R}, Y)$ *to* $L^{p_0'}(\mathbb{R}, Z)$ *for some* $p_0 \in (1,2)$, *then the operator defined by*

$$(Wf)(t) := \int_{-\infty}^{t} K(t-s) f(s)\,\mathrm{d}s$$

is also a bounded mapping from $L^{p_0}(\mathbb{R}, Y)$ *to* $L^{p_0'}(\mathbb{R}, Z)$, *and in particular*

$$\|W\|_{L^{p_0}(\mathbb{R},Y)\to L^{p_0'}(\mathbb{R},Z)} \lesssim_{p_0} \|T\|_{L^{p_0}(\mathbb{R},Y)\to L^{p_0'}(\mathbb{R},Z)}.$$

Then we get that the $L_t^{p_3}(L_{(x_1,x_2)}^{p_1}) \to L_t^{p_3'}(L_{(x_1,x_2)}^{p_1'})$ boundedness of the operator in (A-3) (acting on $F_{\mathcal{W}^{-1}}$) is implied by the $L_t^{p_3}(L_{(x_1,x_2)}^{p_1}) \to L_t^{p_3'}(L_{(x_1,x_2)}^{p_1'})$ boundedness of the operator

$$\int_0^\infty \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} e^{i(x-y)\cdot\xi + i(t-s)\phi(\xi)} \eta_j(\xi) \mathcal{W}(\xi)\,\mathrm{d}\xi \right) F_{\mathcal{W}^{-1}}(y,s)\,\mathrm{d}y\,\mathrm{d}s = ((F_{\mathcal{W}^{-1}} \chi_{(0,\infty)}(s)) * (\mathcal{F}^{-1}\mu_j))(x,t),$$

with essentially the same operator constant bound (up to a multiplicative factor which depends only on $p_3 \in (1,2)$). Here μ_j is the localized measure defined in the same way as in (2-2), and recall that this convolution operator is bounded (uniformly in j). This finishes the proof of Corollary 1.5.

References

[Abi-Khuzam and Shayya 2006] F. Abi-Khuzam and B. Shayya, "Fourier restriction to convex surfaces of revolution in $\mathbb{R}^3$", *Publ. Mat.* **50**:1 (2006), 71–85. MR Zbl

[Carbery and Ziesler 2002] A. Carbery and S. Ziesler, "Restriction and decay for flat hypersurfaces", *Publ. Mat.* **46**:2 (2002), 405–434. MR Zbl

[Carbery et al. 2007] A. Carbery, C. Kenig, and S. Ziesler, "Restriction for flat surfaces of revolution in $\mathbb{R}^3$", *Proc. Amer. Math. Soc.* **135**:6 (2007), 1905–1914. MR Zbl

[Carbery et al. 2013] A. Carbery, C. E. Kenig, and S. N. Ziesler, "Restriction for homogeneous polynomial surfaces in $\mathbb{R}^3$", *Trans. Amer. Math. Soc.* **365**:5 (2013), 2367–2407. MR Zbl

[Christ and Kiselev 2001] M. Christ and A. Kiselev, "Maximal functions associated to filtrations", *J. Funct. Anal.* **179**:2 (2001), 409–425. MR Zbl

[Cowling et al. 1990] M. Cowling, S. Disney, G. Mauceri, and D. Müller, "Damping oscillatory integrals", *Invent. Math.* **101**:2 (1990), 237–260. MR Zbl

[Dendrinos and Zimmermann 2019] S. Dendrinos and E. Zimmermann, "On L^p-improving for averages associated to mixed homogeneous polynomial hypersurfaces in $\mathbb{R}^3$", *J. Anal. Math.* **138**:2 (2019), 563–595. MR Zbl

[Drury 1990] S. W. Drury, "Degenerate curves and harmonic analysis", *Math. Proc. Cambridge Philos. Soc.* **108**:1 (1990), 89–96. MR Zbl

[Fefferman 1970] C. Fefferman, "Inequalities for strongly singular convolution operators", *Acta Math.* **124** (1970), 9–36. MR Zbl

[Ferreyra and Urciuolo 2009] E. Ferreyra and M. Urciuolo, "Fourier restriction estimates to mixed homogeneous surfaces", *J. Inequal. Pure Appl. Math.* **10**:2 (2009), art. id. 35. MR Zbl

[Ferreyra et al. 2004] E. Ferreyra, T. Godoy, and M. Urciuolo, "Restriction theorems for the Fourier transform to homogeneous polynomial surfaces in $\mathbb{R}^3$", *Studia Math.* **160**:3 (2004), 249–265. MR Zbl

[Ginibre and Velo 1992] J. Ginibre and G. Velo, "Smoothing properties and retarded estimates for some dispersive evolution equations", *Comm. Math. Phys.* **144**:1 (1992), 163–188. MR Zbl

[Greenblatt 2018] M. Greenblatt, "Fourier transforms of irregular mixed homogeneous hypersurface measures", *Math. Nachr.* **291**:7 (2018), 1075–1087. MR Zbl

[Gressman 2016] P. T. Gressman, "Damping oscillatory integrals by the Hessian determinant via Schrödinger", *Math. Res. Lett.* **23**:2 (2016), 405–430. MR Zbl

[Ikromov and Müller 2011] I. A. Ikromov and D. Müller, "On adapted coordinate systems", *Trans. Amer. Math. Soc.* **363**:6 (2011), 2821–2848. MR Zbl

[Ikromov and Müller 2016] I. A. Ikromov and D. Müller, *Fourier restriction for hypersurfaces in three dimensions and Newton polyhedra*, Ann. of Math. Stud. **194**, Princeton Univ. Press, 2016. MR Zbl

[Ikromov et al. 2005] I. A. Ikromov, M. Kempe, and D. Müller, "Damped oscillatory integrals and boundedness of maximal operators associated to mixed homogeneous hypersurfaces", *Duke Math. J.* **126**:3 (2005), 471–490. MR Zbl

[Iosevich and Sawyer 1996] A. Iosevich and E. Sawyer, "Oscillatory integrals and maximal averages over homogeneous surfaces", *Duke Math. J.* **82**:1 (1996), 103–141. MR Zbl

[Keel and Tao 1998] M. Keel and T. Tao, "Endpoint Strichartz estimates", *Amer. J. Math.* **120**:5 (1998), 955–980. MR Zbl

[Kenig et al. 1991] C. E. Kenig, G. Ponce, and L. Vega, "Oscillatory integrals and regularity of dispersive equations", *Indiana Univ. Math. J.* **40**:1 (1991), 33–69. MR Zbl

[Lizorkin 1970] P. I. Lizorkin, "Multipliers of Fourier integrals and bounds of convolution in spaces with mixed norms: applications", *Izv. Akad. Nauk SSSR Ser. Mat.* **34**:1 (1970), 218–247. In Russian; translated in *Math. USSR-Izv.* **4**:1 (1970), 225–255. MR Zbl

[Milnor 1964] J. Milnor, "On the Betti numbers of real varieties", *Proc. Amer. Math. Soc.* **15**:2 (1964), 275–280. MR Zbl

[Nomizu and Sasaki 1994] K. Nomizu and T. Sasaki, *Affine differential geometry*: *geometry of affine immersions*, Cambridge Tracts Math. **111**, Cambridge Univ. Press, 1994. MR Zbl

[Oberlin 2004] D. M. Oberlin, "A uniform Fourier restriction theorem for surfaces in $\mathbb{R}^3$", *Proc. Amer. Math. Soc.* **132**:4 (2004), 1195–1199. MR Zbl

[Oberlin 2012] D. M. Oberlin, "A uniform Fourier restriction theorem for surfaces in $\mathbb{R}^d$", *Proc. Amer. Math. Soc.* **140**:1 (2012), 263–265. MR Zbl

[Palle 2021] L. Palle, "Mixed norm Strichartz-type estimates for hypersurfaces in three dimensions", *Math. Z.* **297**:3-4 (2021), 1529–1599. MR Zbl

[Schwend 2020] J. Schwend, "Optimal $L^p \to L^q$ estimates for Euclidean averages over prototypical hypersurfaces in $\mathbb{R}^3$", preprint, 2020. arXiv 2012.15789

[Sjölin 1974] P. Sjölin, "Fourier multipliers and estimates of the Fourier transform of measures carried by smooth curves in R^2", *Studia Math.* **51** (1974), 169–182. MR Zbl

[Sogge 1995] C. D. Sogge, *Lectures on nonlinear wave equations*, Monogr. Anal. **2**, International, Boston, 1995. MR Zbl

[Strichartz 1977] R. S. Strichartz, "Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations", *Duke Math. J.* **44**:3 (1977), 705–714. MR Zbl

[Su 1983] Su B., *Affine differential geometry*, Sci. Press, Beijing, 1983. MR Zbl

[Triebel 2006] H. Triebel, *Theory of function spaces, III*, Monogr. Math. **100**, Birkhäuser, Basel, 2006. MR Zbl

[Varchenko 1976] A. N. Varchenko, "Newton polyhedra and estimation of oscillating integrals", *Funkt. Anal. i Prilozhen.* **10**:3 (1976), 13–38. In Russian; translated in *Funct. Anal. Appl.* **10**:3 (1976), 175–196. MR Zbl

[Zygmund 1974] A. Zygmund, "On Fourier coefficients and transforms of functions of two variables", *Studia Math.* **50** (1974), 189–201. MR Zbl

Received 19 Apr 2020. Revised 27 Mar 2021. Accepted 19 May 2021.

LJUDEVIT PALLE: palle@math.uni-kiel.de
Christian-Albrechts-Universität zu Kiel, Kiel, Germany

Analysis & PDE

msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2023 is US $405/year for the electronic version, and $630/year (+$65, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ANALYSIS & PDE

Volume 16 No. 1 2023