Björn Ivarsson, Frank Kutzschebauch and Erik Løw

HOLOMORPHIC FACTORIZATION OF MAPPINGS INTO Sp$_4$(ℂ)

msp

HOLOMORPHIC FACTORIZATION OF MAPPINGS INTO Sp$_4$(ℂ)

BJÖRN IVARSSON, FRANK KUTZSCHEBAUCH AND ERIK LØW

We prove that any null-homotopic holomorphic map from a Stein space X to the symplectic group Sp$_4$(ℂ) can be written as a finite product of elementary symplectic matrices with holomorphic entries.

1. Introduction

The continuous or holomorphic parameter dependence of classical linear algebra results over the fields ℝ or ℂ form a circle of very natural questions of general mathematical interest. For example the factorization of continuous matrices as a product of continuous elementary matrices has been studied and solved by Vaserstein [1988]. The corresponding holomorphic problem for the special linear group SL$_n$ was posed by Gromov [1989] and was finally solved by the first two authors in [Ivarsson and Kutzschebauch 2012]. The study of algebraic dependence is connected with famous work by Suslin [1977], Cohn [1966], Bass, Milnor, and Serre [Bass et al. 1967] and many others.

These parameter dependence questions are a part of algebraic K-theory and the study of linear algebra over general rings. Factorization of Chevalley groups over $\mathbb{R}$ and $\mathbb{C}$ into elementary matrices is classically well known. For Chevalley groups over general rings this is much more difficult and studied a lot. For an overview, see, for example, [Vavilov and Stepanov 2011].

We are mainly interested in the rings of holomorphic functions on Stein spaces. The only known holomorphic result is the existence for the special linear groups in [Ivarsson and Kutzschebauch 2012], where Gromov's problem is solved in full generality. In the special case of an open Riemann surface the problem was solved earlier (absolutely unnoticed) by Klein and Ramspott [1987]. The authors [Ivarsson et al. 2020] also proved the main result of this paper for any size of symplectic matrices in the special case of an open Riemann surface.

In the present paper we consider the symplectic groups over rings of holomorphic functions on Stein spaces. The main result is (see Section 2 for notation)

Main Theorem (Theorem 3.1). *Let X be a finite-dimensional reduced Stein space and $f : X \to \mathrm{Sp}_4(\mathbb{C})$ be a holomorphic mapping that is null-homotopic. Then there exist a natural number K, depending only on the dimension of X, and holomorphic mappings*

$$G_1, \ldots, G_K : X \to \mathbb{C}^3$$

such that

$$f(x) = M_1(G_1(x)) \cdots M_K(G_K(x)).$$

We remind the reader that a mapping is null-homotopic if it is homotopic to a constant map. By the Oka–Grauert principle it is equivalent for a holomorphic map from a Stein space into a complex Lie group to be null-homotopic via holomorphic maps or via continuous maps (see Theorem 5.3.2 in the standard reference [Forstnerič 2017]).

Our main tool is the Oka principle for stratified elliptic submersions, the most elaborate result in modern Oka theory. In order to apply an Oka principle one needs a topological solution which we take from our previous work on symplectic groups over rings of continuous functions on topological spaces. The Oka principle lets us homotope the topological solution to a holomorphic one. The technical details needed to prove that certain fibrations are stratified elliptic are considerable and we have so far only been able to complete these details for Sp_4. We expect that a similar result holds for Sp_{2n}.

Factorization of symplectic groups over other rings (of mainly algebraic nature) has been considered before for example by Kopeiko [1978], and Grunewald, Mennicke and Vaserstein [Grunewald et al. 1991].

The paper is organized as follows. In Section 2 we recall our results on factorization of continuous matrices and prove a slight extension about the number of factors. In Section 3 we state our main results and give an overview over the proof. In Section 4 we explain how our results can be reformulated in the language used in algebraic K-theory. In Section 5 we recall the theorems from Oka theory which we use in our proof.

In Section 6 we give the proofs of Lemmas 3.3 and 3.4, where we prove that the most important fibrations in this paper, the projections of products of elementary symplectic matrices onto their last row, are surjective and we determine where they are submersive. This is done for symplectic matrices of all sizes, since we hope to be able to prove in the future that these fibrations are stratified elliptic for all sizes.

The rest of the paper is devoted to proving that our fibration (for (4×4)-matrices) is stratified elliptic in order to be able to apply Oka theory. In Section 7 we describe the stratification with respect to which we want to prove that the important fibration is stratified elliptic. This has to do with how the set of $2n$ algebraic equations defining a fiber in the fibration can be reduced to n equations. In the case of the special linear group in [Ivarsson and Kutzschebauch 2012] we were able to reduce to one single equation independent of the size of the matrices. This was the crucial trick to prove ellipticity by finding complete vector fields, which corresponds to Gromov's example of a spray. This inability to reduce to fewer equations is the main difference between the situation of the symplectic group and the special linear group. It leads to all the difficult technical work in the rest of the paper. In Section 8 we introduce our method to find complete vector fields tangent to the fibration. However not all of them are complete and we deduce that the Gromov-spray produced by them is not dominating. We determine which of them are complete. In Section 9 we explain our strategy to enlarge the set of complete vector fields so that this enlarged collection now spans the tangent space at all points and thus gives a fiber-dominating spray. The realization of this strategy takes Section 10, where we introduce useful quantities, Sections 11, 12, and 13, where we prove the result for three, four, and five (elementary symplectic) factors, and finally we can give an inductive (over the number of factors) proof in Section 14. The reason for dealing with the low numbers of factors separately is that some of the fibers of our fibration are reducible when there are a small number of factors, and from five factors on all fibers are irreducible. In Section 15 we end the paper with an application to the problem of a product of exponentials and formulate some open questions.

2. Continuous factorization

Let

$$\omega = \sum_{j=1}^{n} dz_j \wedge dz_{j+n}$$

be the symplectic form in $\mathbb{C}^{2n}$. With respect to ω, symplectic matrices are those that can be written in block form as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C and D are complex $n \times n$ matrices satisfying

$$A^T C = C^T A, \tag{2.0.1}$$

$$B^T D = D^T B, \tag{2.0.2}$$

$$A^T D - C^T B = I_n, \tag{2.0.3}$$

where I_n is the $n \times n$ identity matrix. In the case $B = C = 0$ this means $D = (A^T)^{-1}$, and in the case $A = D = I_n$ this means B and C are symmetric and $C^T B = 0$. Let U_n denote an $n \times n$ matrix satisfying $U_n = U_n^T$ and 0_n the $n \times n$ zero matrix. We call those matrices that are written in block form as

$$\begin{pmatrix} I_n & 0_n \\ U_n & I_n \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} I_n & U_n \\ 0_n & I_n \end{pmatrix}$$

elementary symplectic matrices. Let

$$U_n(x_1, \ldots, x_{n(n+1)/2}) = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ x_2 & x_{n+1} & \ldots & x_{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_n & x_{2n-1} & \ldots & x_{n(n+1)/2} \end{pmatrix}.$$

Given a map $G : X \to \mathbb{C}^{n(n+1)/2}$ let

$$U_n(G(x)) = U_n(G_1(x), \ldots, G_{n(n+1)/2}(x)),$$

where the G_j's are components of the map G. For odd k let

$$M_k(G(x)) = \begin{pmatrix} I_n & 0_n \\ U_n(G(x)) & I_n \end{pmatrix},$$

and for even k

$$M_k(G(x)) = \begin{pmatrix} I_n & U_n(G(x)) \\ 0_n & I_n \end{pmatrix}.$$

The following result is a refinement of [Ivarsson et al. 2020, Theorem 1.3].

Theorem 2.1 (continuous Vaserstein problem for symplectic matrices). *There exists a natural number $K(n, d)$ such that given any finite-dimensional normal topological space X of (covering) dimension d and any null-homotopic continuous mapping $M : X \to \mathrm{Sp}_{2n}(\mathbb{C})$ there exist K continuous mappings*

$$G_1, \ldots, G_K : X \to \mathbb{C}^{n(n+1)/2}$$

such that

$$M(x) = M_1(G_1(x)) \cdots M_K(G_K(x)).$$

Proof. Theorem 1.3 in [Ivarsson et al. 2020] does not give a uniform bound on the number of factors depending on n and d. Suppose such a bound does not exist; i.e., for all natural numbers i there are normal topological spaces X_i of dimension d and null-homotopic continuous maps $f_i : X_i \to \mathrm{Sp}_{2n}(\mathbb{C})$ such that f_i does not factor over a product of less than i elementary symplectic matrices. Let X equal $\bigcup_{i=1}^{\infty} X_i$, the disjoint union of the spaces X_i, and $F : X \to \mathrm{Sp}_{2n}(\mathbb{C})$ be the map that is equal to f_i on X_i. By Theorem 1.3 in [Ivarsson et al. 2020] F factors over a finite number of elementary symplectic matrices. Consequently all f_i factor over the same number of elementary symplectic matrices, which contradicts the assumption on f_i. $\square$

3. Statement of the main result and overview of proof

We state the main result of this paper which is a holomorphic version of Theorem 2.1 for $\mathrm{Sp}_4(\mathbb{C})$.

Theorem 3.1. *There exists a natural number $N(d)$ such that given any finite-dimensional reduced Stein space X of dimension d and any null-homotopic holomorphic mapping $f : X \to \mathrm{Sp}_4(\mathbb{C})$ there exist N holomorphic mappings*

$$G_1, \ldots, G_N : X \to \mathbb{C}^3$$

such that

$$f(x) = M_1(G_1(x)) \cdots M_N(G_N(x)).$$

We have the following corollary.

Corollary 3.2. *Let X be a finite-dimensional reduced Stein space that is topologically contractible and $f : X \to \mathrm{Sp}_4(\mathbb{C})$ be a holomorphic mapping. Then there exist a natural number N and holomorphic mappings*

$$G_1, \ldots, G_N : X \to \mathbb{C}^3$$

such that

$$f(x) = M_1(G_1(x)) \cdots M_N(G_N(x)).$$

The strategy for proving Theorem 3.1 is as follows. Define

$$\Psi_K : (\mathbb{C}^3)^K \to \mathrm{Sp}_4(\mathbb{C})$$

as

$$\Psi_K(x_1, \ldots, x_{3K}) = M_1(x_1, x_2, x_3) \cdots M_K(x_{3K-2}, x_{3K-1}, x_{3K}). \tag{3.0.1}$$

We want to show the existence of a holomorphic map

$$G = (G_1, \ldots, G_K) : X \to (\mathbb{C}^3)^K$$

such that

$$\begin{array}{ccc}
 & & (\mathbb{C}^3)^K \\
 & \overset{G}{\nearrow} & \big\downarrow {\scriptstyle \Psi_K} \\
X & \underset{f}{\longrightarrow} & \mathrm{Sp}_4(\mathbb{C})
\end{array}$$

is commutative. Theorem 2.1 shows the existence of a continuous map such that the diagram above is commutative.

We will prove Theorem 3.1 using the Oka–Grauert–Gromov principle for sections of holomorphic submersions over X. One candidate submersion would be to use the pull-back of $\Psi_K : (\mathbb{C}^3)^K \to \mathrm{Sp}_4(\mathbb{C})$. It turns out that Ψ_K is not a submersion at all points in $(\mathbb{C}^3)^K$. It is a surjective holomorphic submersion if one removes a certain subset from $(\mathbb{C}^3)^K$. Unfortunately the fibers of this submersion are quite difficult to analyze and we therefore elect to study

$$\begin{array}{ccc}
 & & (\mathbb{C}^3)^K \\
 & \overset{F}{\nearrow} & \big\downarrow {\scriptstyle \pi_4 \circ \Psi_K} \\
X & \underset{\pi_4 \circ f}{\longrightarrow} & \mathbb{C}^4 \setminus \{0\}
\end{array}$$

where we define the projection $\pi_4 : \mathrm{Sp}_4(\mathbb{C}) \to \mathbb{C}^4 \setminus \{0\}$ to be the projection of a matrix to its last row:

$$\pi_4 \begin{pmatrix} z_{11} & \cdots & z_{14} \\ \vdots & \ddots & \vdots \\ z_{41} & \cdots & z_{44} \end{pmatrix} = (z_{41}, \ldots, z_{44}).$$

However, even the map $\Phi_K = \pi_4 \circ \Psi_K : (\mathbb{C}^3)^K \to \mathbb{C}^4 \setminus \{0\}$ is not submersive everywhere. We have the three results below (Lemmas 3.3 and 3.4 and Proposition 3.6) about that map which will be proved in later sections.

We introduce some notation. Projecting to the last row introduces an asymmetry between upper and lower triangular elementary matrices and therefore we will denote by z's the variables in the lower triangular matrices and by w's the variables in the upper triangular matrices. For example, the right-hand side of (3.0.1) becomes

$$
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ z_1 & z_2 & 1 & 0 \\ z_2 & z_3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & w_1 & w_2 \\ 0 & 1 & w_2 & w_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 & w_{3k-2} & w_{3k-1} \\ 0 & 1 & w_{3k-1} & w_{3k} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

for even $K = 2k$.

Let

$$
\vec{Z}_K = \begin{cases} (z_1, z_2, z_3, w_1, w_2, w_3, \ldots, w_{3k-2}, w_{3k-1}, w_{3k}) & \text{if } K = 2k, \\ (z_1, z_2, z_3, w_1, w_2, w_3, \ldots, z_{3k+1}, z_{3k+2}, z_{3k+3}) & \text{if } K = 2k+1 \end{cases}
$$

and

$$
W_K = \begin{cases} \begin{pmatrix} w_1 & w_2 & w_4 & w_5 & \ldots & w_{3k-5} & w_{3k-4} \\ w_2 & w_3 & w_5 & w_6 & \ldots & w_{3k-4} & w_{3k-3} \end{pmatrix} & \text{if } K = 2k, \\[2mm] \begin{pmatrix} w_1 & w_2 & w_4 & w_5 & \ldots & w_{3k-2} & w_{3k-1} \\ w_2 & w_3 & w_5 & w_6 & \ldots & w_{3k-1} & w_{3k} \end{pmatrix} & \text{if } K = 2k+1. \end{cases}
$$

Also, when $K = 2k$ or $K = 2k+1$, let

$$
A_K = \bigcap_{1 \le j \le k} \{\vec{Z}_K \in (\mathbb{C}^3)^K : z_{3j-1} = z_{3j} = 0\},
$$

$$
B_K = \{\vec{Z}_K \in (\mathbb{C}^3)^K : \operatorname{Rank} W_K < 2\}
$$

and

$$
S_K = A_K \cap B_K. \tag{3.0.2}
$$

We have Lemma 3.3, which follows from a simple calculation.

Lemma 3.3. *The mapping*

$$
\Phi_K = \pi_4 \circ \Psi_K : (\mathbb{C}^3)^K \setminus S_K \to \mathbb{C}^4 \setminus \{0\}
$$

is surjective when $K \ge 3$.

Lemma 3.4. *For $K \ge 3$ the mapping*

$$
\Phi_K = \pi_4 \circ \Psi_K : (\mathbb{C}^3)^K \to \mathbb{C}^4 \setminus \{0\}
$$

is a holomorphic submersion exactly at points $\vec{Z}_K \in (\mathbb{C}^3)^K \setminus S_K$, where S_K is defined by (3.0.2) above. That is, S_K is the set of points where the entries in the last row of each lower triangular matrix are zero, except for the K-th matrix where no conditions are imposed, and the rank of the matrix W_K, which does not involve entries from the K-th matrix, is strictly less than 2.

Remark 3.5. Lemmas 3.3 and 3.4 both generalize to $2n \times 2n$ matrices and the proofs are identical. In Section 6 we therefore consider the general case.

Proposition 3.6. *For $n = 1$ and $n = 2$ the map*

$$
\begin{array}{c}
(\mathbb{C}^{n(n+1)/2})^K \setminus S_K \\
\downarrow {\scriptstyle \pi_{2n} \circ \Psi_K} \\
\mathbb{C}^{2n} \setminus \{0\}
\end{array}
\tag{3.0.3}
$$

is a stratified elliptic submersion.

Corollary 3.7. *Let $n = 1$ or $n = 2$. Let X be a finite-dimensional reduced Stein space and $f : X \to \mathrm{Sp}_{2n}(\mathbb{C})$ be a holomorphic map. Assume that there exists a natural number K and a continuous map $F : X \to (\mathbb{C}^{n(n+1)/2})^K \setminus S_K$ such that*

is commutative. Then there exists a holomorphic map $G : X \to (\mathbb{C}^{n(n+1)/2})^K \setminus S_K$, homotopic to F via continuous maps $F_t : X \to (\mathbb{C}^{n(n+1)/2})^K \setminus S_K$, such that the diagram above is commutative for all F_t.

Proof. The pull-back of (3.0.3) by $\pi_{2n} \circ f$ is a stratified elliptic submersion over the Stein base X. Thus by Theorem 5.6 there is a homotopy from the given continuous section to a holomorphic section. This is equivalent to the desired homotopy F_t. An even better way to perform this proof is to say that the map (3.0.3) is an Oka map, see [Forstnerič 2017, Corollary 7.4.5(i)], which yields the desired conclusion. $\square$

Remark 3.8. The fact that the map (3.0.3) is an Oka map yields a parametric version of Corollary 3.7. This means that the holomorphic map can be replaced by a continuous map $f_P : X \times P \to \mathrm{Sp}_{2n}(\mathbb{C})$, which is holomorphic for each fixed parameter $p \in P$, where P is a compact Hausdorff topological space.

We need the following version of the Whitehead lemma:

$$
\begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -a & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\tag{3.0.4}
$$

Proof of Theorem 3.1. We will prove the theorem for a single map. The existence of a uniform bound $N(d)$ follows as in the proof of Theorem 2.1. Since a finite-dimensional Stein space is finite-dimensional as a topological space there are $K - 2$ continuous mappings

$$
G_1, \ldots, G_{K-2} : X \to \mathbb{C}^3
$$

such that

$$
f(x) = M_1(G_1(x)) \cdots M_{K-2}(G_{K-2}(x)).
$$

Let $H : X \to \mathbb{C}^3$ be a constant map such that $U_2(H)$ has nonzero second row, let $\mathbb{O} : X \to \mathbb{C}^3$ be the zero map, and replace the above factorization by

$$f(x) = M_1(H)M_2(\mathbb{O})M_3(G_1(x) - H)M_4(G_2(x)) \cdots M_K(G_{K-2}(x))$$

(suppressing the variables in the constant maps H and $\mathbb{O}$). Notice that the second factor is the identity matrix.

This factorization by K continuous elementary symplectic matrices avoids the singularity set S_K and thus we find $F : X \to (\mathbb{C}^3)^K \setminus S_K$ with $\Psi_K(F) = f$.

Using Corollary 3.7 we know that $F_0 := F$ is homotopic to a holomorphic map $G = F_1$, via continuous maps F_t, such that

$$\pi_4(f(x)) = \pi_4 \circ \Psi_K(F_t(x)), \quad 0 \le t \le 1,$$

that is, the last row of the matrices $\Psi_K(F_t(x))$ is constant. Therefore

$$\Psi_K(F_t(x))f(x)^{-1} = \begin{pmatrix} \tilde{f}_{11,t}(x) & \tilde{f}_{12,t}(x) & \tilde{f}_{13,t}(x) & \tilde{f}_{14,t}(x) \\ \tilde{f}_{21,t}(x) & \tilde{f}_{22,t}(x) & \tilde{f}_{23,t}(x) & \tilde{f}_{24,t}(x) \\ \tilde{f}_{31,t}(x) & \tilde{f}_{32,t}(x) & \tilde{f}_{33,t}(x) & \tilde{f}_{34,t}(x) \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since these matrices are symplectic, it automatically follows that $\tilde{f}_{12,t}(x) \equiv 0$, $\tilde{f}_{22,t}(x) \equiv 1$, and $\tilde{f}_{32,t}(x) \equiv 0$ so that

$$\Psi_K(F_t(x))f(x)^{-1} = \begin{pmatrix} \tilde{f}_{11,t}(x) & 0 & \tilde{f}_{13,t}(x) & \tilde{f}_{14,t}(x) \\ \tilde{f}_{21,t}(x) & 1 & \tilde{f}_{23,t}(x) & \tilde{f}_{24,t}(x) \\ \tilde{f}_{31,t}(x) & 0 & \tilde{f}_{33,t}(x) & \tilde{f}_{34,t}(x) \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.0.5}$$

and in addition

$$\tilde{f}_t(x) = \begin{pmatrix} \tilde{f}_{11,t}(x) & \tilde{f}_{13,t}(x) \\ \tilde{f}_{31,t}(x) & \tilde{f}_{33,t}(x) \end{pmatrix} \in \mathrm{Sp}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C}). \tag{3.0.6}$$

Since $\Psi_K(F_0(x)) = f(x)$, we see that $\tilde{f}_0 = \mathrm{Id}$, and thus the holomorphic map $\tilde{f} := \tilde{f}_1 : X \to \mathrm{SL}_2(\mathbb{C})$ is null-homotopic. Let ψ be the standard inclusion of Sp_2 in Sp_4; see for example [Grunewald et al. 1991]. By the main result from [Ivarsson and Kutzschebauch 2012] the matrix

$$\psi(\tilde{f}(x)^{-1}) = \begin{pmatrix} \tilde{f}_{33}(x) & 0 & -\tilde{f}_{13}(x) & 0 \\ 0 & 1 & 0 & 0 \\ -\tilde{f}_{31}(x) & 0 & \tilde{f}_{11}(x) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.0.7}$$

is a product of holomorphic elementary symplectic matrices. Therefore it suffices to show that

$$\Psi_K(G(x))f(x)^{-1} \cdot \psi(\tilde{f}(x)^{-1}) = \begin{pmatrix} 1 & 0 & 0 & \tilde{f}_{14}(x) \\ -\tilde{f}_{34}(x) & 1 & \tilde{f}_{14}(x) & \tilde{f}_{24}(x) \\ 0 & 0 & 1 & \tilde{f}_{34}(x) \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.0.8}$$

is a product of elementary symplectic matrices. In order to deduce that the right-hand side of (3.0.8) has the claimed form one has to use the fact that (3.0.5) is symplectic. Since

$$
\begin{pmatrix}
1 & 0 & 0 & \tilde{f}_{14}(x) \\
-\tilde{f}_{34}(x) & 1 & \tilde{f}_{14}(x) & \tilde{f}_{24}(x) \\
0 & 0 & 1 & \tilde{f}_{34}(x) \\
0 & 0 & 0 & 1
\end{pmatrix}
=
\begin{pmatrix}
1 & 0 & 0 & 0 \\
-\tilde{f}_{34}(x) & 1 & 0 & 0 \\
0 & 0 & 1 & \tilde{f}_{34}(x) \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & \tilde{f}_{14}(x) \\
0 & 1 & \tilde{f}_{14}(x) & \tilde{f}_{14}(x)\tilde{f}_{34}(x)+\tilde{f}_{24}(x) \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
$$

the result follows by the Whitehead lemma, (3.0.4). $\qquad\square$

Analyzing this proof and using Remark 3.8 one sees that we can actually prove a parametric version of our main theorem.

Theorem 3.9. *Let X be a finite-dimensional reduced Stein space, P be a compact Hausdorff topological (parameter) space, and $f : P \times X \to \mathrm{Sp}_4(\mathbb{C})$ be a continuous mapping, holomorphic for each fixed $p \in P$, that is null-homotopic. Then there exist a natural number K and continuous mappings, holomorphic for each fixed parameter $p \in P$,*

$$
G_1, \ldots, G_K : P \times X \to \mathbb{C}^3
$$

such that

$$
f(p, x) = M_1(G_1(p, x)) \cdots M_K(G_K(p, x)).
$$

To complete the proof of the theorem we need to establish Proposition 3.6 and Lemmas 3.4 and 3.3.

Remark 3.10. Proposition 3.6 is the crucial ingredient in the proof of Theorem 3.1. Its proof is by far the most difficult part of the paper. As pointed out in Remark 3.5, Lemma 3.4 holds for general n. Also if Proposition 3.6 holds for some n then Corollary 3.7 also holds for that n. Moreover the reduction of the size of the symplectic matrix from Sp_4 to Sp_2 done in the proof of Theorem 3.1 generalizes easily to a reduction from Sp_{2n} to Sp_{2n-2} if Corollary 3.7 holds for n (see for example the proof of Lemma 4.4 in [Grunewald et al. 1991]). Therefore if Proposition 3.6 can be proven for $n = 1, \ldots, m$ then the following holds true.

Conjecture 3.11. *Let X be a finite-dimensional reduced Stein space and $f : X \to \mathrm{Sp}_{2m}(\mathbb{C})$ be a holomorphic mapping that is null-homotopic. Then there exist a natural number K and holomorphic mappings*

$$
G_1, \ldots, G_K : X \to \mathbb{C}^{m(m+1)/2}
$$

such that

$$
f(x) = M_1(G_1(x)) \cdots M_K(G_K(x)).
$$

In the case of a 1-dimensional Stein space, i.e., an open Riemann surface, this conjecture was established in [Ivarsson et al. 2020]. The condition of null-homotopy is automatically satisfied in this case, since an open Riemann surface is homotopy equivalent to a 1-dimensional CW-complex and the group $\mathrm{Sp}_{2m}(\mathbb{C})$ is simply connected. The proof uses the analytic ingredient that the Bass stable rank of $\mathcal{O}(X)$ is 1 for an open Riemann surface and proceeds then by linear algebra arguments.

4. Formulation in algebraic terms

We relate our results to algebraic K-theory and reformulate them in those terms. The following is a standard notion:

Definition 4.1. For a commutative ring R the set $U_m(R)$ of unimodular rows of length m is defined as

$$\{(r_1, r_2, \ldots, r_m) \in R^m : r_1, r_2, \ldots r_m \text{ generate } R \text{ as an ideal}\}.$$

In our main example, if $\mathcal{O}(X)$ is the ring of holomorphic functions on a Stein space X, a row $(f_1, f_2, \ldots, f_m) \in \mathcal{O}^m(X)$ is unimodular if and only if the functions $f_1, f_2, \ldots, f_m$ have no common zeros, a well-known application of Cartan's Theorem B.

Since null-homotopy is an important assumption in our studies we denote the set of null-homotopic unimodular rows in $U_m(\mathcal{O}(X))$ by $U_m^0(\mathcal{O}(X))$. This set can be seen as the path-connected component of the space of holomorphic maps from X to $\mathbb{C}^m \setminus \{0\}$ containing the constant map $(0, 0, , \ldots, 0, 1) = e_m$. By the Oka–Grauert principle $\mathbb{C}^m \setminus \{0\} = \mathrm{GL}_m(\mathbb{C})/\mathrm{GL}_{m-1}(\mathbb{C})$ is an Oka manifold; therefore the path-connected components of continuous and holomorphic maps $X \to \mathbb{C}^m \setminus \{0\}$ are in bijection. This says that unimodular rows in $U_m(\mathcal{O}(X))$ are null-homotopic in the holomorphic sense if and only if they are null-homotopic in the continuous sense.

Algebraic K-theorists consider Chevalley groups over rings; in our example we consider the null-homotopic elements of them.

Definition 4.2. $\mathrm{Sp}_{2n}^0(\mathcal{O}(X))$ denotes the group of null-homotopic holomorphic maps from a Stein space X to the symplectic group $\mathrm{Sp}_{2n}(\mathbb{C})$, which in other words is the path-connected component of the group $\mathrm{Sp}_{2n}(\mathcal{O}(X))$ containing the identity.

Again by the Oka–Grauert principle holomorphic maps $X \to \mathrm{Sp}_{2n}(\mathbb{C})$ are homotopic via holomorphic maps if and only if they are homotopic via continuous maps.

Clearly the last row of a matrix in $\mathrm{Sp}_{2n}(\mathcal{O}(X))$ is unimodular, i.e., an element of $U_{2n}(\mathcal{O}(X))$. Whether a unimodular row in $U_{2n}(\mathcal{O}(X))$ is the last row of a matrix in $\mathrm{Sp}_{2n}(\mathcal{O}(X))$ is by Oka theory a purely topological problem. Let us illustrate this by an example.

Extending a unimodular row to an invertible matrix can be reformulated as follows: given a trivial line subbundle of the trivial bundle $X \times \mathbb{C}^n$ of rank n over X, can it be complemented by a trivial bundle?

This of course is not always the case: The (nontrivial) tangent bundle T of the sphere S^{2n+1} ($n \geq 4$) is the complement of the trivial normal bundle N to the sphere S^{2n+1} in $\mathbb{R}^{2n+2}$. To make this a holomorphic example consider X to be a Grauert tube around S^{2n+1}, i.e., a Stein manifold which has a strong deformation retraction ρ onto its totally real maximal-dimensional submanifold S^{2n+1}. The bundle T is replaced by the complexified tangent bundle to the sphere pulled back onto X by the retraction ρ and equipped with its unique structure of holomorphic vector bundle (which is still not a trivial bundle). The pull-back of the complexified trivial bundle N is still a trivial line subbundle of $X \times \mathbb{C}^{2n}$. Thus we have found an example of a holomorphic row which cannot be completed to an invertible matrix in $\mathrm{GL}_{2n}(\mathcal{O}(X))$ and thus not to a matrix in $\mathrm{Sp}_{2n}(\mathcal{O}(X))$ either.

For null-homotopic rows the situation is better.

Lemma 4.3. *Every element $U_{2n}^0(\mathcal{O}(X))$ extends to a null-homotopic matrix $A \in \mathrm{Sp}_{2n}^0(\mathcal{O}(X))$.*

Proof. Let $F = (f_1, \ldots, f_{2n}) : X \to \mathbb{C}^{2n} \setminus \{0\}$ be a null-homotopic holomorphic map, and the homotopy to the constant map $F_1(x) = e_{2n}$ be denoted by F_t, $t \in [0, 1]$. The map $\pi_{2n} : \mathrm{Sp}_{2n}(\mathbb{C}) \to \mathbb{C}^{2n} \setminus \{0\}$ is a locally trivial holomorphic fiber bundle with typical fiber $F \cong \mathrm{Sp}_{2n-2}(\mathbb{C}) \times \mathbb{C}^{4n-1}$ which is an Oka manifold. Our problem is to find a global section of the pull-back of this fibration by the map $F = F_0$. Since a locally trivial bundle is a Serre fibration and the constant last row can be extended to a constant (thus null-homotopic) symplectic matrix, we find a continuous section of this pull-back bundle over the whole homotopy. Thus the restriction to $X \times \{0\}$ is a null-homotopic continuous symplectic matrix. Since the fiber F is Oka, we find a homotopy to a holomorphic symplectic matrix, which is still null-homotopic. $\square$

The notion of elementary symplectic matrices over a ring R is the same as explained in Section 2.

Let W_n denote an $n \times n$ matrix with entries in the ring R satisfying $W_n = W_n^T$ and 0_n the $n \times n$ zero matrix. We call those matrices that are written in block form as

$$\begin{pmatrix} I_n & 0_n \\ W_n & I_n \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} I_n & W_n \\ 0_n & I_n \end{pmatrix}$$

elementary symplectic matrices over R. The group generated by them, the elementary symplectic group, is denoted by $\mathrm{Ep}_{2n}(R)$. We consider the group $\mathrm{Ep}_{2n}(\mathcal{O}(X))$ which is easily seen to be a subgroup of $\mathrm{Sp}_{2n}^0(\mathcal{O}(X))$ (multiply the symmetric matrices W_n by a real number $t \in [0, 1]$).

The meaning of Corollary 3.7 in K-theoretic terms is now the following:

Proposition 4.4. *Let $n = 1$ or $n = 2$. For a Stein space X the group $\mathrm{Ep}_{2n}(\mathcal{O}(X))$ acts transitively on the set of null-homotopic unimodular rows $U_{2n}^0(\mathcal{O}(X))$.*

Proof. Let $u \in U_{2n}^0(\mathcal{O}(X))$ be a null-homotopic unimodular row. By the above lemma we can extend it to a null-homotopic symplectic matrix $A \in \mathrm{Sp}_{2n}^0(\mathcal{O}(X))$. Now we just follow the beginning of the proof of Theorem 3.1. By Theorem 2.1 we can factorize $A(x)$ as a product of elementary symplectic matrices with continuous entries. Adding two more elementary symplectic matrices we can achieve that the factorization avoids the singularity set S_K. Applying Corollary 3.7 we know that $A_0 := A$ is homotopic to a holomorphic map $G = A_1$, via continuous maps A_t, such that

$$\pi_4(A(x)) = \pi_4 \circ \Psi_K(A_t(x)), \quad 0 \le t \le 1,$$

that is, the last row of the matrices $\Psi_K(A_t(x))$ is constant. Therefore

$$\Psi_K(A_t(x))A(x)^{-1} = \begin{pmatrix} \tilde{a}_{11,t}(x) & \tilde{a}_{12,t}(x) & \tilde{a}_{13,t}(x) & \tilde{a}_{14,t}(x) \\ \tilde{a}_{21,t}(x) & \tilde{a}_{22,t}(x) & \tilde{a}_{23,t}(x) & \tilde{a}_{24,t}(x) \\ \tilde{a}_{31,t}(x) & \tilde{a}_{32,t}(x) & \tilde{a}_{33,t}(x) & \tilde{a}_{34,t}(x) \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This shows that the element $\Psi_K(G(x))$ of $\mathrm{Ep}_{2n}(\mathcal{O}(X))$ has the last row equal to u or equivalently moves the constant row e_{2n} to u. $\square$

Let $\psi : \mathrm{SL}_2 \to \mathrm{Sp}_4$ be the standard embedding given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.0.1}$$

Continuing like in the proof of Theorem 3.1 we see that it gives the following "inductive step".

Proposition 4.5. *For a Stein space X we have*

$$\mathrm{Sp}_4^0(\mathcal{O}(X)) = \mathrm{Ep}_4(\mathcal{O}(X)) \cdot \psi(\mathrm{Sp}_2^0(\mathcal{O}(X))).$$

In a similar way one can deduce from our earlier results (Proposition 2.8 and the proof of Theorem 2.3 in [Ivarsson and Kutzschebauch 2012]) the corresponding statements for the special linear groups. The definition of the elementary group E_n and the inclusion ψ of SL_{n-1} into SL_n are the usual ones.

Proposition 4.6. *For a Stein space X and any $n \geq 2$ the group $\mathrm{E}_n(\mathcal{O}(X))$ acts transitively on the set of null-homotopic unimodular rows $U_n^0(\mathcal{O}(X))$.*

Proposition 4.7. *For a Stein space X and any $n \geq 2$ we have*

$$\mathrm{SL}_n^0(\mathcal{O}(X)) = \mathrm{E}_n(\mathcal{O}(X)) \cdot \psi(\mathrm{SL}_{n-1}^0(\mathcal{O}(X))).$$

5. Stratified sprays

We will introduce the concept of a spray associated with a holomorphic submersion following [Gromov 1989; Forstnerič and Prezelj 2002]. First we introduce some notation and terminology. Let $h : Z \to X$ be a holomorphic submersion of a complex manifold Z onto a complex manifold X. For any $x \in X$ the fiber over x of this submersion will be denoted by Z_x. At each point $z \in Z$ the tangent space $T_z Z$ contains *the vertical tangent space* $V T_z Z = \ker Dh$. For holomorphic vector bundles $p : E \to Z$ we denote the zero element in the fiber E_z by 0_z.

Definition 5.1. Let $h : Z \to X$ be a holomorphic submersion of a complex manifold Z onto a complex manifold X. A spray on Z associated with h is a triple (E, p, s), where $p : E \to Z$ is a holomorphic vector bundle and $s : E \to Z$ is a holomorphic map such that for each $z \in Z$ we have

(i) $s(E_z) \subset Z_{h(z)}$,

(ii) $s(0_z) = z$, and

(iii) the derivative $Ds(0_z) : T_{0_z} E \to T_z Z$ maps the subspace $E_z \subset T_{0_z} E$ surjectively onto the vertical tangent space $V T_z Z$.

Remark 5.2. We will also say that the submersion admits a spray. A spray associated with a holomorphic submersion is sometimes called a (fiber-)dominating spray.

One way of constructing dominating sprays, as pointed out by Gromov, is to find finitely many $\mathbb{C}$-complete vector fields that are tangent to the fibers and span the tangent space of the fibers at all

points in Z. One can then use the flows φ_j^t of these vector fields V_j to define $s : Z \times \mathbb{C}^N \to Z$ via $s(z, t_1, \ldots, t_N) = \varphi_1^{t_1} \circ \cdots \circ \varphi_N^{t_N}(z)$, which gives a spray.

Definition 5.3. Let X and Z be complex spaces. A holomorphic map $h : Z \to X$ is said to be a submersion if for each point $z_0 \in Z$ it is locally equivalent via a fiber-preserving biholomorphic map to a projection $p : U \times V \to U$, where $U \subset X$ is an open set containing $h(z_0)$ and V is an open set in some $\mathbb{C}^d$.

We will need to use stratified sprays, which are defined as follows.

Definition 5.4. We say that a submersion $h : Z \to X$ admits stratified sprays if there is a descending chain of closed complex subspaces $X = X_m \supset \cdots \supset X_0$ such that each stratum $Y_k = X_k \setminus X_{k-1}$ is regular and the restricted submersion $h : Z|_{Y_k} \to Y_k$ admits a spray over a small neighborhood of any point $x \in Y_k$.

Remark 5.5. We say that the stratification $X = X_m \supset \cdots \supset X_0$ is associated with the stratified spray.

In [Forstnerič and Prezelj 2001], see also [Forstnerič 2010, Theorem 8.3], the following is proved.

Theorem 5.6. *Let X be a Stein space with a descending chain of closed complex subspaces $X = X_m \supset \cdots \supset X_0$ such that each stratum $Y_k = X_k \setminus X_{k-1}$ is regular. Assume that $h : Z \to X$ is a holomorphic submersion which admits stratified sprays. Then any continuous section $f_0 : X \to Z$ such that $f_0|_{X_0}$ is holomorphic can be deformed to a holomorphic section $f_1 : X \to Z$ by a homotopy that is fixed on X_0.*

6. Proofs of Lemmas 3.3 and 3.4

Lemmas 3.3 and 3.4 hold for square matrices of any size. In this section we therefore look at $2n \times 2n$ matrices. Given two vectors $\vec{a}$ and $\vec{b}$ in $\mathbb{C}^n$ (i.e., $n \times 1$ matrices), we denote by

$$\begin{pmatrix} \vec{a} \\ \vec{b} \end{pmatrix}$$

the obvious vector in $\mathbb{C}^{2n}$.

We shall consider products of $2n \times 2n$ matrices

$$\begin{pmatrix} I_n & 0 \\ Z_1 & I_n \end{pmatrix} \begin{pmatrix} I_n & W_1 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ Z_2 & I_n \end{pmatrix} \begin{pmatrix} I_n & W_2 \\ 0 & I_n \end{pmatrix} \cdots,$$

where $Z_1, Z_2, \ldots$ and $W_1, W_2, \ldots$ are $n \times n$ matrices of variables

$$Z_k = (z_{k,ij}), \quad W_k = (w_{k,ij}), \quad 1 \le i, j \le n.$$

They are symmetric, i.e., $z_{k,ij} = z_{k,ji}$ and $w_{k,ij} = w_{k,ji}$. We call the variables $z_{k,n1}, \ldots, z_{k,nn}$ *last row variables* (this term does not apply to the w-variables). If we have K factors, there are $Kn(n+1)/2$ variables. We will also think of the K-tuple $(Z_1, W_1, Z_2, W_2, \ldots)$ as a point in $\mathbb{C}^{Kn(n+1)/2}$. We will study the last row of this product, which is a map $\Phi_K : \mathbb{C}^{Kn(n+1)/2} \to \mathbb{C}^{2n} \setminus \{0\}$. We prefer to work with the transpose of this row, which we denote by P^K, a vector in $\mathbb{C}^{2n}$. It follows that

$$P^1 = \begin{pmatrix} \vec{z} \\ \vec{e_n} \end{pmatrix},$$

where $\vec{z} = (z_{1,n1}, \ldots, z_{1,nn})^T$ and $\vec{e_n}$ is the last standard basis vector of $\mathbb{C}^n$.

The set S_K for $K \geq 2$ is now defined as the set of K-tuples of symmetric matrices $(Z_1, W_1, \ldots)$ such that in the first $K - 1$ matrices all the last row variables (of the Z's) are 0 and the set of all columns of the W's does not span $\mathbb{C}^n$. (This means that the augmented matrix $W_1|W_2|\cdots$ has rank less than n.)

Lemma 6.1. $P^K : \mathbb{C}^{Kn(n+1)/2} \setminus S_K \to \mathbb{C}^{2n} \setminus \{0\}$ *is surjective for* $K \geq 3$.

Proof. We prove the result for $K = 3$. For $K > 3$, simply put $W_2 = Z_3 = W_3 = \cdots = 0$. The proof uses an easy fact from linear algebra; given two vectors $\vec{c}$ and $\vec{d}$ in $\mathbb{C}^n$ with $\vec{c} \neq \vec{0}$ there is a symmetric matrix M such that $M\vec{c} = \vec{d}$. Now let

$$\begin{pmatrix} \vec{a} \\ \vec{b} \end{pmatrix} \in \mathbb{C}^{2n} \setminus \{0\}.$$

Pick any symmetric matrix Z_2 such that $\vec{z} = \vec{a} - Z_2\vec{b} \neq \vec{0}$ and let Z_1 be any symmetric matrix whose last row is $\vec{z}$ and W_1 a symmetric matrix such that $W_1\vec{z} = \vec{b} - \vec{e}_n$. Then $(Z_1, W_1, Z_2) \notin S_3$ and for this choice we have

$$P^3 = \begin{pmatrix} I_n & Z_2 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ W_1 & I_n \end{pmatrix} \begin{pmatrix} \vec{z} \\ \vec{e}_n \end{pmatrix} = \begin{pmatrix} I_n & Z_2 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} \vec{z} \\ \vec{b} \end{pmatrix} = \begin{pmatrix} \vec{a} \\ \vec{b} \end{pmatrix}. \qquad \square$$

Slightly abusing notation, we denote the Jacobian matrix of Φ_K by JP^K. This is a $(2n \times Kn(n+1)/2)$-matrix whose columns are the derivatives of P^K with respect to one particular variable. We denote the components of P^K by P_i^K, $1 \leq i \leq 2n$. It follows that

$$P^{2k+1} = \begin{pmatrix} I_n & Z_{k+1} \\ 0 & I_n \end{pmatrix} P^{2k}, \tag{6.0.1}$$

$$P^{2k+2} = \begin{pmatrix} I_n & 0 \\ W_{k+1} & I_n \end{pmatrix} P^{2k+1}. \tag{6.0.2}$$

We shall look at the final part of JP^{2k+1}, the part where we differentiate with respect to the new variables $z_{k+1,11}, \ldots, z_{k+1,n1}, z_{k+1,22}, \ldots, z_{k+1,n2}, \ldots, z_{k+1,nn}$. This is a $(2n \times n(n+1)/2)$-matrix. The column where we differentiate with respect to $z_{k+1,ij}$ will consist of P_{n+i}^{2k} in row number j and P_{n+j}^{2k} in row number i. Hence the bottom half of this matrix is zero and we only look at the upper half, an $(n \times n(n+1)/2)$-matrix which we denote by A_{k+1}. If we consider just the columns which contain one particular P_{n+i}^{2k}, we get a square $n \times n$-matrix whose i-th row is $(P_{n+1}^{2k}, \ldots, P_{2n}^{2k})$, has P_{n+i}^{2k} along the diagonal and is otherwise zero. The determinant of this submatrix is $(P_{n+i}^{2k})^n$.

The situation is similar for the final part of JP^{2k+2}, except now the top half is zero and the bottom half B_{k+1} contains $P_1^{2k+1}, \ldots, P_n^{2k+1}$ in the same pattern as for A_{k+1}.

In the proof of the next lemma it will be convenient to use the following notation: if A and B are two matrices with the same column length, we let $A|B$ denote the matrix obtained by augmenting A with B to the right. By e_{2n} we denote the last vector in the standard basis of $\mathbb{C}^{2n}$.

Lemma 6.2. P^K *is a submersion exactly on the set* $\mathbb{C}^{Kn(n+1)/2} \setminus S_K$. *If* $K = 2k$ *and all the last row variables are zero, then* $P^{2k} = e_{2n}$ *and the span of the bottom half of the* JP^{2k} *columns equals the span of the columns of* $W_1, W_2, \ldots, W_k$.

Proof. For $N = 1$ the theorem is empty. $P^1 = (z_{1,n1}, \ldots, z_{1,nn}, 0, \ldots, 0, 1)$ and

$$JP^1 = \begin{pmatrix} I_n \\ 0 \end{pmatrix},$$

where we have removed all zero columns. For $N = 2$ we have

$$P^2 = \begin{pmatrix} I_n & 0 \\ W_1 & I_n \end{pmatrix} P^1.$$

This implies

$$JP^2 = \begin{pmatrix} I_n & 0 \\ W_1 & I_n \end{pmatrix} \begin{pmatrix} I_n \\ 0 \end{pmatrix} \,\bigg|\, \begin{pmatrix} 0 \\ B_1 \end{pmatrix} = \begin{pmatrix} I_n \\ W_1 \end{pmatrix} \,\bigg|\, \begin{pmatrix} 0 \\ B_1 \end{pmatrix},$$

which has full rank if and only if B_1 has full rank. Since $P_i^1 = z_{1,ni}$, by the discussion preceding the lemma, B_1 has full rank if and only if at least one $z_{1,ni}$ is nonzero.

If all $z_{1,ni}$ are zero, then $P^1 = e_{2n}$ and $B_1 = 0$. Hence the statement about the span is trivially true.

We now assume that the theorem is true for $N = 2k$. We have

$$JP^{2k+1} = \begin{pmatrix} I_n & Z_{k+1} \\ 0 & I_n \end{pmatrix} JP^{2k} \,\bigg|\, \begin{pmatrix} A_{k+1} \\ 0 \end{pmatrix}. \tag{6.0.3}$$

If at least one of the previous last row variables is nonzero, then JP^{2k} has full rank by the induction hypothesis and so does JP^{2k+1}. If not, then $P^{2k} = e_{2n}$ and $A_{k+1} = I_n$, after removing zero columns. If $JP^{2k} = \begin{pmatrix} A \\ B \end{pmatrix}$, then

$$JP^{2k+1} = \begin{pmatrix} A + Z_{k+1}B & I_n \\ B & 0 \end{pmatrix},$$

which has full rank if and only if B has full rank. But the column span of B equals the column span of $W_1, \ldots, W_k$. This proves the first part of the lemma for $N = 2k + 1$.

If all the previous last row variables are zero, it also follows that

$$P^{2k+1} = (z_{k+1,n1}, \ldots, z_{k+1,nn}, 0, \ldots, 0, 1)^t.$$

Finally

$$JP^{2k+2} = \begin{pmatrix} I_n & 0 \\ W_{k+1} & I_n \end{pmatrix} JP^{2k+1} \,\bigg|\, \begin{pmatrix} 0 \\ B_{k+1} \end{pmatrix}, \tag{6.0.4}$$

which has full rank if JP^{2k+1} does.

If not, then by the above all the previous last row variables are zero and

$$JP^{2k+2} = \begin{pmatrix} A + Z_{k+1}B & I_n \\ B + W_{k+1}(A + Z_{k+1}B) & W_{k+1} \end{pmatrix} JP^{2k+1} \,\bigg|\, \begin{pmatrix} 0 \\ B_{k+1} \end{pmatrix},$$

which has full rank if and only if at least one $z_{k+1,ni}$ is nonzero by the discussion preceding the lemma. This proves the first part of the lemma for $N = 2k + 2$.

If all the $z_{k+1,ni}$ also are zero, then $P^{2k+1} = e_{2n}$ and so $P^{2k+2} = e_{2n}$. Also $B_{k+1} = 0$ and since the columns of $W_{k+1}(A + Z_{k+1}B)$ are linear combinations of the columns of W_{k+1}, the span of the bottom half of JP^{2k+2} equals the span of the columns of $W_1, \ldots, W_{k+1}$ by the induction hypothesis. $\qquad\square$

7. The stratification

The goal in this section is to describe the stratification needed to understand that the submersion $\pi_4 \circ \Psi_K :$
$(\mathbb{C}^3)^K \setminus S_K \to \mathbb{C}^4 \setminus \{0\}$ is a stratified elliptic submersion. Let

$$\vec{Z}_K = \begin{cases} (z_1, z_2, z_3, w_1, w_2, w_3, \dots, w_{3k-2}, w_{3k-1}, w_{3k}) & \text{if } K = 2k, \\ (z_1, z_2, z_3, w_1, w_2, w_3, \dots, z_{3k+1}, z_{3k+2}, z_{3k+3}) & \text{if } K = 2k+1 \end{cases}$$

and

$$\pi_4 \circ \Psi_K(\vec{Z}_K) = (P_1^K(\vec{Z}_K), P_2^K(\vec{Z}_K), P_3^K(\vec{Z}_K), P_4^K(\vec{Z}_K)).$$

Remark 7.1. We will abuse notation in the following way in the paper. A polynomial not containing a variable can be interpreted as a polynomial of that variable. More precisely, let $L < K$. We have the projection $\pi : \mathbb{C}^K \to \mathbb{C}^L$, $\pi(x_1, \dots, x_L, \dots, x_K) = (x_1, \dots, x_L)$ and $\pi^* : \mathbb{C}[\mathbb{C}^L] \to \mathbb{C}[\mathbb{C}^K]$. For $p \in \mathbb{C}[\mathbb{C}^L]$ we still write p instead of $\pi^*(p)$.

We want to study the fibers

$$\mathcal{F}^K_{(a_1, a_2, a_3, a_4)} = (\pi_4 \circ \Psi_K)^{-1}(a_1, a_2, a_3, a_4).$$

Assume first that $K = 2k + 1 \geq 3$ is odd. We see that

$$\pi_4 \circ \Psi_K(\vec{Z}_K) = \pi_4 \circ \Psi_{K-1}(\vec{Z}_{K-1}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ z_{3k+1} & z_{3k+2} & 1 & 0 \\ z_{3k+2} & z_{3k+3} & 0 & 1 \end{pmatrix}$$

and we get

$$\begin{aligned}
P_1^K(\vec{Z}_K) &= P_1^{K-1}(\vec{Z}_{K-1}) + z_{3k+1} P_3^{K-1}(\vec{Z}_{K-1}) + z_{3k+2} P_4^{K-1}(\vec{Z}_{K-1}), \\
P_2^K(\vec{Z}_K) &= P_2^{K-1}(\vec{Z}_{K-1}) + z_{3k+2} P_3^{K-1}(\vec{Z}_{K-1}) + z_{3k+3} P_4^{K-1}(\vec{Z}_{K-1}), \\
P_3^K(\vec{Z}_K) &= P_3^{K-1}(\vec{Z}_{K-1}), \\
P_4^K(\vec{Z}_K) &= P_4^{K-1}(\vec{Z}_{K-1}).
\end{aligned}$$

We are led to the equations

$$\begin{aligned}
a_1 &= P_1^K(\vec{Z}_K) = P_1^{K-1}(\vec{Z}_{K-1}) + z_{3k+1} P_3^{K-1}(\vec{Z}_{K-1}) + z_{3k+2} P_4^{K-1}(\vec{Z}_{K-1}), \\
a_2 &= P_2^K(\vec{Z}_K) = P_2^{K-1}(\vec{Z}_{K-1}) + z_{3k+2} P_3^{K-1}(\vec{Z}_{K-1}) + z_{3k+3} P_4^{K-1}(\vec{Z}_{K-1}), \\
a_3 &= P_3^K(\vec{Z}_K) = P_3^{K-1}(\vec{Z}_{K-1}), \\
a_4 &= P_4^K(\vec{Z}_K) = P_4^{K-1}(\vec{Z}_{K-1}).
\end{aligned} \tag{7.0.1}$$

Notice that these equations simplify to

$$\begin{aligned}
a_1 &= P_1^{K-1}(\vec{Z}_{K-1}) + a_3 z_{3k+1} + a_4 z_{3k+2}, \\
a_2 &= P_2^{K-1}(\vec{Z}_{K-1}) + a_3 z_{3k+2} + a_4 z_{3k+3}, \\
a_3 &= P_3^{K-1}(\vec{Z}_{K-1}), \\
a_4 &= P_4^{K-1}(\vec{Z}_{K-1}).
\end{aligned}$$

If $(a_3, a_4) \neq (0, 0)$ then we can solve the two first equations for two of the three variables z_{3k+1}, z_{3k+2}, z_{3k+3} and we see that the fiber is a graph over $\mathcal{G}^{K-1}_{(a_3,a_4)} \times \mathbb{C}$, where

$$\mathcal{G}^{K-1}_{(a_3,a_4)} = \{\vec{Z}_{K-1} \in \mathbb{C}^{3K-3} : a_3 = P_3^{K-1}(\vec{Z}_{K-1}),\ a_4 = P_4^{K-1}(\vec{Z}_{K-1})\}.$$

If $(a_3, a_4) = (0, 0)$, we get $\mathcal{F}^K_{(a_1,a_2,0,0)} = \mathcal{F}^{K-1}_{(a_1,a_2,0,0)} \times \mathbb{C}^3$. We see that we get two main cases, namely $(a_3, a_4) = (0, 0)$ and $(a_3, a_4) \neq (0, 0)$. The last case will break into the two subcases, namely $(a_3, a_4) \neq (0, 1)$ and $(a_3, a_4) = (0, 1)$. We need these subcases because $\mathcal{G}^{K-1}_{(0,1)}$ is not smooth. We list the strata below:

• The strata of generic fibers: When $(a_3, a_4) \neq (0, 0)$, the fibers are graphs over $\mathcal{G}^{K-1}_{(a_3,a_4)} \times \mathbb{C}$. This set is divided into two strata as follows:

  – Smooth generic fibers: When $(a_3, a_4) \neq (0, 1)$, the fibers are smooth.

  – Singular generic fibers: When $(a_3, a_4) = (0, 1)$, the fibers are nonsmooth.

• The stratum of nongeneric fibers: When $(a_3, a_4) = (0, 0)$, the fibers are $\mathcal{F}^K_{(a_1,a_2,0,0)} = \mathcal{F}^{K-1}_{(a_1,a_2,0,0)} \times \mathbb{C}^3$. Moreover the fibers are smooth.

We now analyze the case when $K = 2k \geq 3$ is even. Now we have

$$\pi_4 \circ \Psi_K(\vec{Z}_K) = \pi_4 \circ \Psi_{K-1}(\vec{Z}_{K-1}) \begin{pmatrix} 1 & 0 & w_{3k-2} & w_{3k-1} \\ 0 & 1 & w_{3k-1} & w_{3k} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and $\mathcal{F}^K_{(a_1,a_2,a_3,a_4)}$ is the solution set of the equations

$$\begin{aligned}
a_1 &= P_1^K(\vec{Z}_K) = P_1^{K-1}(\vec{Z}_{K-1}), \\
a_2 &= P_2^K(\vec{Z}_K) = P_2^{K-1}(\vec{Z}_{K-1}), \\
a_3 &= P_3^K(\vec{Z}_K) = P_3^{K-1}(\vec{Z}_{K-1}) + w_{3k-2}P_1^{K-1}(\vec{Z}_{K-1}) + w_{3k-1}P_2^{K-1}(\vec{Z}_{K-1}), \\
a_4 &= P_4^K(\vec{Z}_K) = P_4^{K-1}(\vec{Z}_{K-1}) + w_{3k-1}P_1^{K-1}(\vec{Z}_{K-1}) + w_{3k}P_2^{K-1}(\vec{Z}_{K-1}).
\end{aligned} \tag{7.0.2}$$

As in the previous case these equations simplify:

$$\begin{aligned}
a_1 &= P_1^{K-1}(\vec{Z}_{K-1}), \\
a_2 &= P_2^{K-1}(\vec{Z}_{K-1}), \\
a_3 &= P_3^{K-1}(\vec{Z}_{K-1}) + a_1 w_{3k-2} + a_2 w_{3k-1}, \\
a_4 &= P_4^{K-1}(\vec{Z}_{K-1}) + a_1 w_{3k-1} + a_2 w_{3k}.
\end{aligned}$$

Let

$$\mathcal{H}^{K-1}_{(a_1,a_2)} = \{\vec{Z}_{K-1} \in \mathbb{C}^{3K-3} : a_1 = P_1^{K-1}(\vec{Z}_{K-1}),\ a_2 = P_2^{K-1}(\vec{Z}_{K-1})\}.$$

An analysis similar to that above gives us the following strata:

• The stratum of generic fibers: When $(a_1, a_2) \neq (0, 0)$, the fibers are graphs over $\mathcal{H}^{K-1}_{(a_1,a_2)} \times \mathbb{C}$. Moreover the fibers are smooth.

• The strata of nongeneric fibers: When $(a_1, a_2) = (0, 0)$, the fibers are $\mathcal{F}_{(0,0,a_3,a_4)}^K = \mathcal{F}_{(0,0,a_3,a_4)}^{K-1} \times \mathbb{C}^3$. This set is divided into two strata as follows:

  – Smooth nongeneric fibers: When $(a_3, a_4) \neq (0, 1)$, the fibers are smooth.

  – Singular nongeneric fibers: When $(a_3, a_4) = (0, 1)$, the fibers are nonsmooth.

8. Determination of complete vector fields

The description of the fibers in Section 7 leads us to study vector fields simultaneously tangent to the level sets $\{P = c_1\}$, $\{Q = c_2\}$ of two functions $P, Q : \mathbb{C}^N \to \mathbb{C}$. Such fields can be constructed in the following way. Pick three variables x, y, z from the variables $x_1, \ldots, x_N$ on $\mathbb{C}^N$ and consider the vector fields

$$D_{xyz}(P, Q) = \det \begin{pmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial P/\partial x & \partial P/\partial y & \partial P/\partial z \\ \partial Q/\partial x & \partial Q/\partial y & \partial Q/\partial z \end{pmatrix}, \tag{8.0.1}$$

which are simultaneously tangent to the level sets. As mentioned in Section 5 we want to use a finite collection of complete vector fields spanning tangent space at every point to prove (stratified) ellipticity. It is an easy exercise to show that the collection of these vector fields over all possible triples spans the tangent space at smooth points of the variety $\{P = c_1\} \cap \{Q = c_2\}$. It turns out that many of the vector fields we get by this method are complete but unfortunately not all of them. The complete vector fields from this collection will not span the tangent space at all points for all level sets. To overcome this difficulty and still producing dominating sprays from this collection of available complete fields is the main technical part of our paper explained in Section 9.

Now we will begin to describe the complete vector fields tangent to the fibers of $\pi_4 \circ \Psi_K = (P_1^K, P_2^K, P_3^K, P_4^K)$ that we get using (8.0.1). It will be convenient to group the variables as in Section 6, $Z_1, W_1, Z_2, W_2, \ldots$, where

$$Z_k = \begin{pmatrix} z_{3k-2} & z_{3k-1} \\ z_{3k-1} & z_{3k} \end{pmatrix}$$

and similarly for W_k. Since the variable z_1 never enters in P^K, we omit it from the first group Z_1. Note that $P^1 = (z_2, z_3, 0, 1)^T$. We are going to study the vector fields

$$V_{ij}^K(x, y, z) = D_{xyz}(P_i^K, P_j^K).$$

The 2×2 minors occurring as coefficients are denoted by $C_{ij}^K(\cdot, \cdot)$, i.e.,

$$V_{ij}^K(x, y, z) = C_{ij}^K(y, z) \frac{\partial}{\partial x} - C_{ij}^K(x, z) \frac{\partial}{\partial y} + C_{ij}^K(x, y) \frac{\partial}{\partial z}.$$

The description of the complete vector fields will be done inductively. We start with $K = 2$. We have to study $\mathcal{G}_{(a_3,a_4)}^2$, or equivalently, the equations

$$\begin{aligned} a_3 &= P_3^2(z_1, \ldots, w_3) = z_2 w_1 + z_3 w_2, \\ a_4 &= P_4^2(z_1, \ldots, w_3) = 1 + z_2 w_2 + z_3 w_3. \end{aligned} \tag{8.0.2}$$

We are interested in which triples (x, y, z) of variables from the list z_2, z_3, w_1, w_2, w_3 give complete vector fields $V_{34}^2(x, y, z)$ and we denote the set of these triples by $\mathcal{T}_2$. By definition $\mathcal{T}_1 = \varnothing$.

An easy computation gives that

$$\mathcal{T}_2 = \{(w_1, w_2, w_3), (z_2, w_2, w_3), (z_3, w_1, w_2),$$
$$(z_2, w_1, w_3), (z_3, w_1, w_3), (z_2, z_3, w_1), (z_2, z_3, w_3)\}. \quad (8.0.3)$$

For all the remaining noncomplete triples there is a variable such that the equation is quadratic for that variable. We are now interested in determining at every stage the triples of variables (x, y, z) such that $V_{12}^{2k+1}(x, y, z)$, for $K = 2k + 1$ odd, and $V_{34}^{2k+2}(x, y, z)$, for $K = 2k + 2$ even, are complete. We shall denote the set of such triples by $\mathcal{T}_K$. The terms occurring in P^K are of degree 1 in the occurring variables; hence the coefficients C_{ij}^K are either of degree 1 or 2 in the occurring variables. A triple giving a coefficient which is quadratic in the integration variable (for instance if $C_{ij}^K(y, z)$ is quadratic in the x-variable) will not be complete, and we shall refer to such a triple as a quadratic triple and the corresponding vector field as a quadratic vector field. The content of the next lemma is that all the remaining triples give complete vector fields. The variables that do not occur in a triple will have constant solutions and are therefore treated as such in the proof.

Lemma 8.1. *For $k \geq 1$, we have $\mathcal{T}_{2k} \subset \mathcal{T}_{2k+1} \subset \mathcal{T}_{2k+2}$. Moreover*

$$\mathcal{T}_{2k+1} \setminus \mathcal{T}_{2k} = \{(z_{3k+1}, z_{3k+2}, z_{3k+3})\}$$
$$\cup \{(w_{3k-2}, z_{3k+1}, z_{3k+3}), (w_{3k-2}, z_{3k+2}, z_{3k+3})\}$$
$$\cup \{(w_{3k}, z_{3k+1}, z_{3k+2}), (w_{3k}, z_{3k+1}, z_{3k+3})\}$$
$$\cup \{(a, b, z_{3k+1}), (a, b, z_{3k+3}) : a \text{ and } b \text{ are from the same group}\}$$
$$\cup \{(a, b, z_{3k+1}) : a \text{ the last variable of one group and } b \text{ the first of the next}\}$$
$$\cup \{(a, b, z_{3k+3}) : a \text{ the last variable of one group and } b \text{ the first of the next}\}$$
$$\cup \{(a, b, z_{3k+1}) : a \text{ the first variable of one group and } b \text{ the last of the next}\}$$
$$\cup \{(a, b, z_{3k+3}) : a \text{ the first variable of one group and } b \text{ the last of the next}\} \quad (8.0.4)$$

and

$$\mathcal{T}_{2k+2} \setminus \mathcal{T}_{2k+1} = \{(w_{3k+1}, w_{3k+2}, w_{3k+3})\}$$
$$\cup \{(z_{3k+1}, w_{3k+1}, w_{3k+3}), (z_{3k+1}, w_{3k+2}, w_{3k+3})\}$$
$$\cup \{(z_{3k+3}, w_{3k+1}, w_{3k+2}), (z_{3k+3}, w_{3k+1}, w_{3k+3})\}$$
$$\cup \{(a, b, w_{3k+1}), (a, b, w_{3k+3}) : a \text{ and } b \text{ are from the same group}\}$$
$$\cup \{(a, b, w_{3k+1}) : a \text{ the last variable of one group and } b \text{ the first of the next}\}$$
$$\cup \{(a, b, w_{3k+3}) : a \text{ the last variable of one group and } b \text{ the first of the next}\}$$
$$\cup \{(a, b, w_{3k+1}) : a \text{ the first variable of one group and } b \text{ the last of the next}\}$$
$$\cup \{(a, b, w_{3k+3}) : a \text{ the first variable of one group and } b \text{ the last of the next}\}. \quad (8.0.5)$$

In combination with (8.0.3) this gives us a complete description of the sets $\mathcal{T}_L, \ L \geq 2$.

Proof. The result is true for $\mathcal{T}_2$. The first group is interpreted as $\{z_2, z_3\}$ and z_{3k+1} must be replaced by z_2. The missing triplets are precisely the quadratic triples.

We shall prove (8.0.4), the proof of (8.0.5) being identical. There is a lot of symmetry in the proof and we will not repeat arguments already given in a situation symmetric to a proven statement. We first consider triples (x, y, z) not containing any variables from the new group Z_{k+1}, i.e., z_{3k+1}, z_{3k+2} and z_{3k+3}. It then follows from (6.0.1) (omitting variables for shorter notation) that

$$V_{12}^{2k+1} = V_{12}^{2k} + z_{3k+1} V_{32}^{2k} - z_{3k+2} V_{24}^{2k} - z_{3k+2} V_{31}^{2k} + z_{3k+3} V_{14}^{2k} + (z_{3k+1} z_{3k+3} - z_{3k+2}^2) V_{34}^{2k}. \quad (8.0.6)$$

A quadratic triple will still be quadratic since V_{34}^{2k} is. For a triple in $\mathcal{T}_{2k}$, notice that in all of the first five terms the V_{ij}^{2k} is obtained by replacing one or two of the functions P_3^{2k} and P_4^{2k} by P_1^{2k} and/or P_2^{2k}. By (6.0.2) all of the terms occurring in P_1^{2k} or P_2^{2k} divide a term occurring in P_3^{2k} and also a term occurring in P_4^{2k}. This means that all terms occurring in the first five vector fields above are already present in V_{34}^{2k} and completeness is not destroyed. We also notice that for any pair x, y of previous variables, the coefficient $C_{12}^{2k+1}(x, y)$ will also satisfy (8.0.6).

We next consider triples containing some of the new variables z_{3k+1}, z_{3k+2} and z_{3k+3}. The Jacobian matrix is now given by (6.0.3), where

$$A^{k+1} = \begin{pmatrix} P_3^{2k} & P_4^{2k} & 0 \\ 0 & P_3^{2k} & P_4^{2k} \end{pmatrix}. \quad (8.0.7)$$

If the triple contains all three variables, then

$$V_{12}^{2k+1}(z_{k+1}, z_{k+2}, z_{k+3}) = (P_4^{2k})^2 \frac{\partial}{\partial z_{3k+1}} - (P_3^{2k})(P_4^{2k}) \frac{\partial}{\partial z_{3k+2}} + (P_3^{2k})^2 \frac{\partial}{\partial z_{3k+3}},$$

and the coefficients do not contain any of the Z_{k+1}-variables; hence this is complete. (The solutions are just affine functions.)

We now consider the case of two new variables. The first possibility is (x, z_{3k+1}, z_{3k+2}). The coefficient of $\partial/\partial x$ is $(P_3^{2k})^2$. Since P_3^{2k} contains all previous variables except w_{3k}, this is quadratic in all those variables and $x = w_{3k}$ is the only possibility. The solution for w_{3k} is affine. The coefficient of $\partial/\partial z_{3k+2}$ is now

$$-\left(\frac{\partial P_2^{2k}}{\partial w_{3k}} + z_{k+3} \frac{\partial P_4^{2k}}{\partial w_{3k}} \right),$$

which is just a constant and the solution is again affine. Finally the coefficient of $\partial/\partial z_{3k+1}$ is given by

$$\frac{\partial P_1^{2k}}{\partial w_{3k}} + z_{k+2} \frac{\partial P_4^{2k}}{\partial w_{3k}},$$

which is an affine function and the solution is entire. Hence this field is complete.

Precisely the same logic applies to the triple (x, z_{3k+2}, z_{3k+3}) except now w_{3k-2} is the only missing variable (now in P_4^{2k}).

The final possibility of two new variables is the triple (x, z_{3k+1}, z_{3k+3}). The coefficient of $\partial/\partial x$ is now $P_3^{2k} P_4^{2k}$ which is of degree 1 in w_{3k-2} and w_{3k} and quadratic in all other previous variables. We consider the case of $x = w_{3k-2}$, the case $x = w_{3k}$ being identical. The coefficient is an affine function of w_{3k-2}; hence the solution is entire. The coefficient of $\partial/\partial z_{3k+1}$ is $-z_{3k+1} P_1^{2k-1} P_4^{2k}$, which is just a

linear function of z_{3k+1} and the solution is entire. The coefficient of $\partial/\partial z_{3k+3}$ is $-z_{3k+2}P_2^{2k-1}P_4^{2k}$, which is just a constant and the solution is affine.

We finally consider the case of one new variable and two previous variables x, y. It follows that $C_{12}^{2k+1}(x, y)$ satisfies (8.0.6), and hence is quadratic in z_{3k+2}, so this cannot be the new variable. In order to investigate z_{3k+1} and z_{3k+3} we need to understand which variables are involved in the coefficients. To do this we look at each previous group of variables Z_j and W_j for $1 \le j \le k$ and see which variables are involved in the first two rows of the Jacobian with respect to these variables at level $2k+1$. For a Z_j-group we need to consider the matrix

$$\begin{pmatrix} \partial P_1^{2k+1}/\partial z_{3j-2} & \partial P_1^{2k+1}/\partial z_{3j-1} & \partial P_1^{2k+1}/\partial z_{3j} \\ \partial P_2^{2k+1}/\partial z_{3j-2} & \partial P_2^{2k+1}/\partial z_{3j-1} & \partial P_2^{2k+1}/\partial z_{3j} \end{pmatrix}$$

and the same for a W_j-group. The Z_1-group only consists of z_2 and z_3. The Z_j-variables do not occur in the above matrix. There is a simple formula for the above matrix which follows from (6.0.3) and (6.0.4). The matrix is the first two rows of the matrix $(I = I_2)$:

$$\begin{pmatrix} I & Z_{k+1} \\ 0 & I \end{pmatrix} \dots \begin{pmatrix} I & 0 \\ W_j & I \end{pmatrix} \begin{pmatrix} A_j \\ 0 \end{pmatrix}$$

and this formula makes it easy to track which variables are missing at each step, in addition to the Z_j-variables. We arrive at the following matrix of missing variables:

$$\begin{pmatrix} w_{3j-3}, w_{3j}, z_{3k+3} & z_{3k+3} & w_{3j-5}, w_{3j-2}, z_{3k+3} \\ w_{3j-3}, w_{3j}, z_{3k+1} & z_{3k+1} & w_{3j-5}, w_{3j-2}, z_{3k+1} \end{pmatrix}.$$

In the case $j = 1$ the missing-variable matrix is

$$\begin{pmatrix} w_3, z_{3k+3} & w_1, z_{3k+3} \\ w_3, z_{3k+1} & w_1, z_{3k+1} \end{pmatrix}.$$

We now consider a W_j-group. Again the W_j-variables do not enter. We now have to consider the first two rows of the matrix

$$\begin{pmatrix} I & Z_{k+1} \\ 0 & I \end{pmatrix} \dots \begin{pmatrix} I & Z_{j+1} \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 \\ B_j \end{pmatrix},$$

and this leads to the following missing-variable matrix for $j < k$:

$$\begin{pmatrix} z_{3j}, z_{3j+3}, z_{3k+3} & z_{3k+3} & z_{3j-2}, z_{3j+1}, z_{3k+3} \\ z_{3j}, z_{3j+3}, z_{3k+1} & z_{3k+1} & z_{3j-2}, z_{3j+1}, z_{3k+1} \end{pmatrix}.$$

For $j = 1$ we replace z_{3j-2} by z_2. For $j = k$ the middle entries in the upper-left and the lower-right corners are replaced by z_{3k+2}.

We first investigate triples (x, y, z_{3k+1}), where x and y are not from Z_{k+1}. If x and y are from the same group, then since z_{3k+1} occurs in every entry in the second row of the missing-variable matrix, $C_{12}^{2k+1}(x, z_{3k+1})$ and $C_{12}^{2k+1}(y, z_{3k+1})$ do not depend on any of the variables x, y, z_{3k+1}; hence x and y are both affine functions. $C_{12}^{2k+1}(x, y)$ does not depend on x, y and is of degree 1 in z_{3k+1}; hence the solution is entire.

Now assume that x and y are from different groups. If x is not a missing variable in $\partial P_2^{2k+1}/\partial y$, then y is not a missing variable in $\partial P_2^{2k+1}/\partial x$. The variables x and y are not both w_{3k}; let's say x is. Then

$$C_{12}^{2k+1}(y, z_{3k+1}) = -\left(\frac{\partial P_2^{2k+1}}{\partial y}\right) P_3^{2k}$$

is quadratic in x and the field is not complete. Hence x and y must both appear in the second row of the missing-variable matrix of each other.

We now look at possibilities for x and y. Assume first that x is in Z_j group with $1 < j \leq k$. There are now four possibilities:

- $x = z_{3j-2}$ in which case $y = w_{3j-3}$ or $y = w_{3j}$, or

- $x = z_{3j}$ in which case $y = w_{3j-5}$ or $y = w_{3j-2}$.

We consider the first case. Then

$$C_{12}^{2k+1}(w_{3j-3}, z_{3j-2}) = \frac{\partial P_1^{2k+1}}{\partial w_{3j-3}} \frac{\partial P_2^{2k+1}}{\partial z_{3j-2}} - \frac{\partial P_2^{2k+1}}{\partial w_{3j-3}} \frac{\partial P_1^{2k+1}}{\partial z_{3j-2}},$$

and from the missing-variable matrix we see that this does not depend on z_{3j-2} and w_{3j-3} and is of degree 1 in z_{3k+1}; hence we have an entire solution for z_{3k+1}. We also have

$$C_{12}^{2k+1}(w_{3j-3}, z_{3k+1}) = -\frac{\partial P_2^{2k+1}}{\partial w_{3j-3}} P_3^{2k},$$

$$C_{12}^{2k+1}(z_{3j-2}, z_{3k+1}) = -\frac{\partial P_2^{2k+1}}{\partial z_{3j-2}} P_3^{2k}.$$

The partial derivatives on the right-hand sides do not depend on any of the variables in the triple, and hence are just constants. It also follows from the missing-variable matrix that P_3^{2k} does not contain the product of z_{3j-2} and w_{3j-3}; hence the equations for these two variables form a linear system with constant coefficients. This has an entire solution. The three other cases all have similar structure and have entire solutions. In the case $j = 1$, we either have $x = z_2$ and $y = w_3$, or $x = z_3$ and $y = w_1$ and the discussion is the same. It also follows from the missing-variable matrix that x and y cannot come from different W-groups. This proves the result in the case of picking z_{3k+1} from the last group. The proof in the case of picking z_{3k+3} from the last group is completely symmetric. This provides the final detail in the proof. $\square$

In order to produce complete fields that are also tangential to fibers of the submersion, we introduce the following notation and terminology.

Definition 8.2. Let $\Xi_3 = \mathcal{T}_2$. For $K \geq 4$ let

$$\Xi_K = \mathcal{T}_{K-1} \setminus \mathcal{T}_{K-2}.$$

We say that the triples in Ξ_K are *introduced on level K*.

We will now use these complete fields to produce complete fields which are tangential to the fibers $\mathcal{F}_{(a_1,a_2,a_3,a_4)}^K$. Here we will use triples introduced on level K to produce complete tangential fields.

First consider the case $K = 2k + 1 \geq 3$ odd.

If $a_3 \neq 0$, we use (7.0.1) to get

$$z_{3k+2} = \frac{1}{a_3}(a_2 - P_2^{2k}(\vec{Z}_{2k}) - a_4 z_{3k+3})$$

and

$$
\begin{aligned}
z_{3k+1} &= \frac{1}{a_3}(a_1 - P_1^{2k}(\vec{Z}_{2k}) - a_4 z_{3k+2}) \\
&= \frac{1}{a_3}\left(a_1 - P_1^{2k}(\vec{Z}_{2k}) - \frac{a_4}{a_3}(a_2 - P_2^{2k}(\vec{Z}_{2k}) - z_{3k+3} P_4^{2k}(\vec{Z}_{2k}))\right) \\
&= \frac{1}{a_3^2}(a_1 a_3 - a_3 P_1^{2k}(\vec{Z}_{2k}) - a_2 a_4 + a_4 P_2^{2k}(\vec{Z}_{2k}) + a_4^2 z_{3k+3}).
\end{aligned}
$$

Using this we define a biholomorphism

$$\alpha : \mathcal{G}_{(a_3,a_4)}^{2k} \times \mathbb{C}_{z_{3k+3}} \to \mathcal{F}_{(a_1,a_2,a_3,a_4)}^{K}.$$

On

$$\mathcal{G}_{(a_3,a_4)}^{2k} \times \mathbb{C}_{z_{3k+3}}$$

we have the complete fields $\partial_{x_1 x_2 x_3}^{2k}$ for x_1, x_2, x_3 in Ξ_{2k+1} and also the complete field $\partial/\partial z_{3k+3}$. Using the biholomorphism α we get complete fields on $\mathcal{F}_{(a_1,a_2,a_3,a_4)}^{K}$ for $a_3 \neq 0$ of the form

$$\theta_{x_1 x_2 x_3}^{2k+1,*} = \partial_{x_1 x_2 x_3}^{2k} + \frac{1}{a_3}\partial_{x_1 x_2 x_3}^{2k}(P_2^{2k}(\vec{Z}_{2k}))\frac{\partial}{\partial z_{3k+2}} + \frac{1}{a_3^2}\partial_{x_1 x_2 x_3}^{2k}(a_3 P_1^{2k}(\vec{Z}_{2k}) - a_4 P_2^{2k}(\vec{Z}_{2k}))\frac{\partial}{\partial z_{3k+1}} \quad (8.0.8)$$

and

$$\gamma^{2k+1,*} = \frac{\partial}{\partial z_{3k+3}} + \frac{a_4^2}{a_3^2}\frac{\partial}{\partial z_{3k+1}} - \frac{a_4}{a_3}\frac{\partial}{\partial z_{3k+2}}. \quad (8.0.9)$$

Since $P_3^{2k} = a_3$ and $P_4^{2k} = a_4$ on the fiber, we get meromorphic fields on $(\mathbb{C}^3)^K$

$$
\begin{aligned}
\theta_{x_1 x_2 x_3}^{2k+1,*} = \partial_{x_1 x_2 x_3}^{2k} &+ \frac{\partial_{x_1 x_2 x_3}^{2k}(P_2^{2k}(\vec{Z}_{2k}))}{P_3^{2k}(\vec{Z}_{2k})}\frac{\partial}{\partial z_{3k+2}} \\
&+ \left(\frac{\partial_{x_1 x_2 x_3}^{2k}(P_1^{2k}(\vec{Z}_{2k}))}{P_3^{2k}(\vec{Z}_{2k})} - \frac{P_4^{2k}(\vec{Z}_{2k})\partial_{x_1 x_2 x_3}^{2k}(P_2^{2k}(\vec{Z}_{2k}))}{P_3^{2k}(\vec{Z}_{2k})^2}\right)\frac{\partial}{\partial z_{3k+1}} \quad (8.0.10)
\end{aligned}
$$

and

$$\gamma^{2k+1,*} = \frac{\partial}{\partial z_{3k+3}} + \frac{P_4^{2k}(\vec{Z}_{2k})^2}{P_3^{2k}(\vec{Z}_{2k})^2}\frac{\partial}{\partial z_{3k+1}} - \frac{P_4^{2k}(\vec{Z}_{2k})}{P_3^{2k}(\vec{Z}_{2k})}\frac{\partial}{\partial z_{3k+2}} \quad (8.0.11)$$

(abusing notation), with poles on $P_3^{2k} = 0$. Since P_3^{2k} is in the kernel of these fields, we can multiply the fields by $(P_3^{2k})^2$ and get the following complete fields that are globally defined on $(\mathbb{C}^3)^K$ and preserve the fibers of $\pi_4 \circ \Psi_K$:

$$
\begin{aligned}
\theta_{x_1 x_2 x_3}^{2k+1} &= P_3^{2k}(\vec{Z}_{2k})^2 \theta_{x_1 x_2 x_3}^{2k+1,*} \\
&= P_3^{2k}(\vec{Z}_{2k})^2 \partial_{x_1 x_2 x_3}^{2k} + P_3^{2k}(\vec{Z}_{2k})\partial_{x_1 x_2 x_3}^{2k}(P_2^{2k}(\vec{Z}_{2k}))\frac{\partial}{\partial z_{3k+2}} \\
&+ \left[P_3^{2k}(\vec{Z}_{2k})\partial_{x_1 x_2 x_3}^{2k}(P_1^{2k}(\vec{Z}_{2k})) - P_4^{2k}(\vec{Z}_{2k})\partial_{x_1 x_2 x_3}^{2k}(P_2^{2k}(\vec{Z}_{2k}))\right]\frac{\partial}{\partial z_{3k+1}} \quad (8.0.12)
\end{aligned}
$$

for $x_1, x_2, x_3 \in \Xi_{2k+1}$, and

$$
\begin{aligned}
\gamma^{2k+1} &= P_3^{2k}(\vec{Z}_{2k})^2 \gamma^{2k+1,*} \\
&= P_3^{2k}(\vec{Z}_{2k})^2 \frac{\partial}{\partial z_{3k+3}} + P_4^{2k}(\vec{Z}_{2k})^2 \frac{\partial}{\partial z_{3k+1}} - P_3^{2k}(\vec{Z}_{2k}) P_4^{2k}(\vec{Z}_{2k}) \frac{\partial}{\partial z_{3k+2}}.
\end{aligned} \tag{8.0.13}
$$

If $a_4 \neq 0$ we can define a biholomorphism

$$
\beta : \mathcal{G}_{(a_3,a_4)}^{2k} \times \mathbb{C}_{z_{3k+1}} \to \mathcal{F}_{(a_1,a_2,a_3,a_4)}^K
$$

using (7.0.1) and

$$
z_{3k+2} = \frac{1}{a_4}(a_1 - P_1^{2k}(\vec{Z}_{2k}) - a_3 z_{3k+1})
$$

and

$$
\begin{aligned}
z_{3k+3} &= \frac{1}{a_4}(a_2 - P_2^{2k}(\vec{Z}_{2k}) - a_3 z_{3k+2}) \\
&= \frac{1}{a_4}\left(a_2 - P_2^{2k}(\vec{Z}_{2k}) - \frac{a_3}{a_4}(a_1 - P_1^{2k}(\vec{Z}_{2k}) - a_3 z_{3k+1})\right) \\
&= \frac{1}{a_4^2}(a_2 a_4 - a_4 P_2^{2k}(\vec{Z}_{2k}) - a_1 a_3 + a_3 P_1^{2k}(\vec{Z}_{2k}) + a_3^2 z_{3k+1}).
\end{aligned}
$$

On

$$
\mathcal{G}_{(a_3,a_4)}^{2k} \times \mathbb{C}_{z_{3k+1}}
$$

we have the complete fields $\partial_{x_1 x_2 x_3}^{2k}$ for x_1, x_2, x_3 in Ξ_{2k+1} and $\partial/\partial z_{3k+1}$. Proceeding as above we get the complete fields

$$
\begin{aligned}
\phi_{x_1 x_2 x_3}^{2k+1} &= P_4^{2k}(\vec{Z}_{2k})^2 \partial_{x_1 x_2 x_3}^{2k} + P_4^{2k}(\vec{Z}_{2k}) \partial_{x_1 x_2 x_3}^{2k}(P_1^{2k}(\vec{Z}_{2k})) \frac{\partial}{\partial z_{3k+2}} \\
&\quad + \left[P_4^{2k}(\vec{Z}_{2k}) \partial_{x_1 x_2 x_3}^{2k}(P_2^{2k}(\vec{Z}_{2k})) - P_3^{2k}(\vec{Z}_{2k}) \partial_{x_1 x_2 x_3}^{2k}(P_1^{2k}(\vec{Z}_{2k})) \right] \frac{\partial}{\partial z_{3k+3}}
\end{aligned} \tag{8.0.14}
$$

for $x_1, x_2, x_3 \in \Psi_{2k+1}$. The field γ^{2k+1} is the same as in the case $a_3 \neq 0$.

For the case $K = 2k \geq 3$ even, an analogous procedure leads to the following complete fields on $(\mathbb{C}^3)^K$ tangent to the fibers of $\pi_4 \circ \Psi_K$:

$$
\begin{aligned}
\theta_{x_1 x_2 x_3}^{2k} &= P_1^{2k-1}(\vec{Z}_{2k-1})^2 \partial_{x_1 x_2 x_3}^{2k-1} + P_1^{2k-1}(\vec{Z}_{2k-1}) \partial_{x_1 x_2 x_3}^{2k-1}(P_4^{2k-1}(\vec{Z}_{2k-1})) \frac{\partial}{\partial w_{3k-1}} \\
&\quad + \left[P_1^{2k-1}(\vec{Z}_{2k-1}) \partial_{x_1 x_2 x_3}^{2k-1}(P_3^{2k-1}(\vec{Z}_{2k-1})) - P_2^{2k-1}(\vec{Z}_{2k-1}) \partial_{x_1 x_2 x_3}^{2k-1}(P_4^{2k-1}(\vec{Z}_{2k-1})) \right] \frac{\partial}{\partial w_{3k-2}}
\end{aligned} \tag{8.0.15}
$$

for $x_1, x_2, x_3 \in \Xi_{2k}$,

$$
\begin{aligned}
\phi_{x_1 x_2 x_3}^{2k} &= P_2^{2k-1}(\vec{Z}_{2k-1})^2 \partial_{x_1 x_2 x_3}^{2k-1} + P_2^{2k-1}(\vec{Z}_{2k-1}) \partial_{x_1 x_2 x_3}^{2k-1}(P_3^{2k-1}(\vec{Z}_{2k-1})) \frac{\partial}{\partial w_{3k-1}} \\
&\quad + \left[P_2^{2k-1}(\vec{Z}_{2k-1}) \partial_{x_1 x_2 x_3}^{2k-1}(P_4^{2k-1}(\vec{Z}_{2k-1})) - P_1^{2k-1}(\vec{Z}_{2k-1}) \partial_{x_1 x_2 x_3}^{2k-1}(P_3^{2k-1}(\vec{Z}_{2k-1})) \right] \frac{\partial}{\partial w_{3k}}
\end{aligned} \tag{8.0.16}
$$

for $x_1, x_2, x_3 \in \Xi_{2k}$, and

$$
\gamma^{2k} = P_1^{2k-1}(\vec{Z}_{2k-1})^2 \frac{\partial}{\partial w_{3k}} + P_2^{2k-1}(\vec{Z}_{2k-1})^2 \frac{\partial}{\partial w_{3k-2}} - P_1^{2k-1}(\vec{Z}_{2k-1}) P_2^{2k-1}(\vec{Z}_{2k-1}) \frac{\partial}{\partial w_{3k-1}}. \tag{8.0.17}
$$

Remark 8.3. It follows from the inductive formulas (7.0.1) and (7.0.2) that $\theta^K_{x_1 x_2 x_3}$, $\phi^K_{x_1 x_2 x_3}$ and γ^K, considered as vector fields on $(\mathbb{C}^3)^L$, are tangent to the fibers $\mathcal{F}^L_{(a_1, a_2, a_3 a_4)}$ for $L \geq K$. In other words, the fields associated with triples introduced on level K are tangential to all fibers $\mathcal{F}^L$ for $L \geq K$.

9. Strategy of proof of stratified ellipticity

We outline the strategy for proving that the submersion is a stratified elliptic submersion. We have seen that the fibers are given by four polynomial equations. We have also seen that these four equations can be reduced to two equations. We then use the exact form of these two equations to find Ξ_K so that $\partial^K_{x_1 x_2 x_3}$ are complete vector fields exactly when $x_1, x_2, x_3 \in \Xi_K$. This leads us to the globally defined complete vector fields $\theta^K_{x_1 x_2 x_3}$, $\phi^K_{x_1 x_2 x_3}$ and γ^K described in Section 8. Find a big (a complement of an analytic subset) "good" set on the fibers where the collection of these vector fields spans the tangent space of the fiber. For points outside the good set find a complete field V whose orbit through the point intersects the good set. At points along the orbit that are also in the good set, the collection of complete vector fields above spans. Now pull back the collection of vector fields by suitable flow automorphisms of V and add these fields to the collection (see Definition 10.7). This enlarged collection of complete vector fields spans in a bigger set, thus enlarging the good set. Continue this enlarging of the collection of vector fields until it spans the tangent space at every point of every fiber in the stratum. To accomplish this strategy we need the following technical results.

Lemma 9.1. *Let M be a Stein manifold and $N_0 \subset N \subset M$ analytic subvarieties. Given a finite collection $\theta_1, \ldots, \theta_k$ of complete holomorphic vector fields on M which span the tangent space $T_x M$ at all points $x \in M \setminus N$ and given another complete holomorphic vector field ϕ on M (whose flow we denote by $\alpha_t \in \mathrm{Aut}_{\mathrm{hol}}(M)$, $t \in \mathbb{C}$) with the property that the orbit through points of $N \setminus N_0$ is leaving N; i.e., for all $x \in N \setminus N_0$ we have $\{\alpha_t(x) : t \in \mathbb{C}\} \not\subset N$. Then there are finitely many times $t_i \in \mathbb{C}$, $i = 1, \ldots, l$, such that for all $x \in N \setminus N_0$ we have $\{\alpha_{t_i}(x)\}^l_{i=1} \not\subset N$. In particular the finite collection $\{\alpha^\star_{t_i}(\theta_m)\}^{l,k}_{i=1, m=1}$ of complete holomorphic vector fields on M spans the tangent space $T_x M$ at all points $x \in M \setminus N_0$.*

Proof. The analytic subset N has at most countably many components. Denote by B_i those components which are not entirely contained in N_0. Define a_0 to be the maximal dimension of them. Choose a point x_i from each of those B_i. For every i the set $A_i := \{t \in \mathbb{C} : \alpha_t(x_i) \in N\}$ is discrete. Since a countable union of discrete sets is meager in $\mathbb{C}$, we can find t_1 such that $t_1 \notin A_i$ for all i. Denote by $\widetilde{B}_i$ those components of the analytic subset $N_1 := \{y \in N : \alpha_{t_1}(y) \in N\}$ which are not entirely contained in N_0 and define a_1 to be the maximal dimension of them. By construction $a_1 < a_0$. Choose a point $\tilde{x}_i$ from each of those $\widetilde{B}_i$. For every i the set $\tilde{A}_i := \{t \in \mathbb{C} : \alpha_t(\tilde{x}_i) \in N\}$ is discrete. Since a countable union of discrete sets is meager in $\mathbb{C}$, we can find t_2 such that $t_2 \notin \tilde{A}_i$ for all i.

Let a_2 be the maximal dimension of those components of the analytic subset $N_2 := \{y \in N : \alpha_{t_1}(y) \in N \text{ and } \alpha_{t_2}(y) \in N\}$ which are not entirely contained in N_0. By construction $a_2 < a_1$ and continuing the construction after finitely steps we reach our conclusion. $\square$

The next lemma is a generalized and parametrized version of the previous one. It is adapted to the stratified spray situation. Namely, we have to produce sprays not on a single fiber but in a neighborhood

of the fiber in each stratum (see Definition 5.4). In our case it will be on the whole stratum. The following definitions are straightforward. The first one was introduced in [Andrist and Kutzschebauch 2018, page 918].

Definition 9.2. Let $\pi : X \to Y$ be a holomorphic map between complex manifolds and denote by $d\pi : TX \to TY$ the tangent map. We call a holomorphic vector field θ on X *fiber-preserving* if $d\pi(\theta) = 0$.

Definition 9.3. A subset N of a complex manifold M is called invariant with respect to a collection of vector fields on M if for each of the vector fields we have: for each starting point $x \in N$ the local flow of the field (which is defined in a neighborhood of time 0) remains contained in N.

Lemma 9.4. *Consider a submersion $\pi : M \to Y$ with connected fibers $M_y := \pi^{-1}(y)$ and a finite collection of complete fiber-preserving holomorphic vector fields on M such that in each fiber M_y there is a point $x \in M_y$ where they span the tangent space $T_x M_y$. Suppose there is no analytic subset N of M contained in a fiber M_y which is invariant under the flows of $\theta_1, \dots, \theta_k$. Then a finite subset of the set $\Gamma(\theta_1, \dots, \theta_k)$ is spanning $T_x M_{\pi(x)}$ for all $x \in M$.*

Proof. Let $N \subset M$ be the set of points x where $\mathrm{span}\{(\theta_1, \dots, \theta_k)\} \neq T_x M_{\pi(x)}$. By assumption $N \cup M_y$ is a proper analytic subset of M_y for each $y \in Y$. Since there is no invariant analytic subset different from the fibers for each $x_0 \in N$, there is a field θ_i whose flow starting in x_0 will leave N, i.e., go through points where $(\theta_1, \dots, \theta_k)$ span $T_x M_{\pi(x)}$. Now choose (at most countably many) points, one from each component of N. As in the proof of the proceeding lemma find finitely many times t_i and enlarge the collection $\theta_1, \dots, \theta_k$ by the pullbacks $(\alpha_i(t_i))^*(\theta_m)i$, $m = 1, \dots, k$. We then get a new finite collection of complete fields where the set of points where this new collection does not span the tangent space of the π-fiber has smaller dimension. By finite induction on the dimension we get the desired result. $\square$

10. Auxiliary quantities and results

Define

$$
\mathcal{M}^K_{x_1 x_2 x_3} = \begin{pmatrix} \partial P_1^K/\partial x_1 & \partial P_1^K/\partial x_2 & \partial P_1^K/\partial x_3 \\ \partial P_2^K/\partial x_1 & \partial P_2^K/\partial x_2 & \partial P_2^K/\partial x_3 \\ \partial P_3^K/\partial x_1 & \partial P_3^K/\partial x_2 & \partial P_3^K/\partial x_3 \\ \partial P_4^K/\partial x_1 & \partial P_4^K/\partial x_2 & \partial P_4^K/\partial x_3 \end{pmatrix} \tag{10.0.1}
$$

for any triple x_1, x_2, x_3 from $\vec{Z}_K$. Removing the j-th row from $\mathcal{M}^K_{x_1 x_2 x_3}$ gives us 3×3 matrices which we denote by $\mathcal{M}^{K,j}_{x_1 x_2 x_3}$. Let

$$
\mathcal{R}^{K,j}_{x_1 x_2 x_3} = \det \mathcal{M}^{K,j}_{x_1 x_2 x_3}.
$$

The significance of the functions $\mathcal{R}^{K,j}_{x_1 x_2 x_3}$ is understood if one notices, because of (8.0.1), that

$$
\mathcal{R}^{2k+1,1}_{x_1 x_2 x_3} = \partial^{2k}_{x_1 x_2 x_3} P_2^{2k}, \tag{10.0.2}
$$

$$
\mathcal{R}^{2k+1,2}_{x_1 x_2 x_3} = \partial^{2k}_{x_1 x_2 x_3} P_1^{2k} \tag{10.0.3}
$$

and that

$$\mathcal{R}^{2k,3}_{x_1x_2x_3} = \partial^{2k-1}_{x_1x_2x_3} P^{2k-1}_4, \tag{10.0.4}$$

$$\mathcal{R}^{2k,4}_{x_1x_2x_3} = \partial^{2k-1}_{x_1x_2x_3} P^{2k-1}_3. \tag{10.0.5}$$

From (7.0.1) and (7.0.2) we get the relations

$$\begin{pmatrix} \mathcal{R}^{2k+1,1}_{x_1x_2x_3} \\ \mathcal{R}^{2k+1,2}_{x_1x_2x_3} \\ \mathcal{R}^{2k+1,3}_{x_1x_2x_3} \\ \mathcal{R}^{2k+1,4}_{x_1x_2x_3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -z_{3k+1} & z_{3k+2} & 1 & 0 \\ z_{3k+2} & -z_{3k+3} & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{R}^{2k,1}_{x_1x_2x_3} \\ \mathcal{R}^{2k,2}_{x_1x_2x_3} \\ \mathcal{R}^{2k,3}_{x_1x_2x_3} \\ \mathcal{R}^{2k,4}_{x_1x_2x_3} \end{pmatrix} \tag{10.0.6}$$

and

$$\begin{pmatrix} \mathcal{R}^{2k,1}_{x_1x_2x_3} \\ \mathcal{R}^{2k,2}_{x_1x_2x_3} \\ \mathcal{R}^{2k,3}_{x_1x_2x_3} \\ \mathcal{R}^{2k,4}_{x_1x_2x_3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -w_{3k-2} & w_{3k-1} \\ 0 & 1 & w_{3k-1} & -w_{3k} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{R}^{2k-1,1}_{x_1x_2x_3} \\ \mathcal{R}^{2k-1,2}_{x_1x_2x_3} \\ \mathcal{R}^{2k-1,3}_{x_1x_2x_3} \\ \mathcal{R}^{2k-1,4}_{x_1x_2x_3} \end{pmatrix}. \tag{10.0.7}$$

Consider the vector fields $\theta^L_{x_1x_2x_3}$ and $\phi^L_{x_1x_2x_3}$, where $(x_1, x_2, x_3) \in \mathcal{T}_L$ and $3 \le L \le K$. Rewriting (8.0.12), (8.0.14), (8.0.15) and (8.0.16) using these functions we get

$$\theta^{2k+1}_{x_1x_2x_3} = P^{2k}_3(\vec{Z}_{2k})^2 \partial^{2k}_{x_1x_2x_3} + P^{2k}_3(\vec{Z}_{2k})\mathcal{R}^{2k+1,1}_{x_1x_2x_3}(\vec{Z}_{2k})\frac{\partial}{\partial z_{3k+2}}$$
$$+ \left(P^{2k}_3(\vec{Z}_{2k})\mathcal{R}^{2k+1,2}_{x_1x_2x_3}(\vec{Z}_{2k}) - P^{2k}_4(\vec{Z}_{2k})\mathcal{R}^{2k+1,1}_{x_1x_2x_3}(\vec{Z}_{2k}) \right) \frac{\partial}{\partial z_{3k+1}}, \tag{10.0.8}$$

$$\phi^{2k+1}_{x_1x_2x_3} = P^{2k}_4(\vec{Z}_{2k})^2 \partial^{2k}_{x_1x_2x_3} + P^{2k}_4(\vec{Z}_{2k})\mathcal{R}^{2k+1,2}_{x_1x_2x_3}(\vec{Z}_{2k})\frac{\partial}{\partial z_{3k+2}}$$
$$+ \left(P^{2k}_4(\vec{Z}_{2k})\mathcal{R}^{2k+1,1}_{x_1x_2x_3}(\vec{Z}_{2k}) - P^{2k}_3(\vec{Z}_{2k})\mathcal{R}^{2k+1,2}_{x_1x_2x_3}(\vec{Z}_{2k}) \right) \frac{\partial}{\partial z_{3k+3}}, \tag{10.0.9}$$

$$\theta^{2k}_{x_1x_2x_3} = P^{2k-1}_1(\vec{Z}_{2k-1})^2 \partial^{2k-1}_{x_1x_2x_3} + P^{2k-1}_1(\vec{Z}_{2k-1})\mathcal{R}^{2k,3}_{x_1x_2x_3}(\vec{Z}_{2k-1})\frac{\partial}{\partial w_{3k-1}}$$
$$+ \left(P^{2k-1}_1(\vec{Z}_{2k-1})\mathcal{R}^{2k,4}_{x_1x_2x_3}(\vec{Z}_{2k-1}) - P^{2k-1}_2(\vec{Z}_{2k-1})\mathcal{R}^{2k,3}_{x_1x_2x_3}(\vec{Z}_{2k-1}) \right) \frac{\partial}{\partial w_{3k-2}}, \tag{10.0.10}$$

$$\phi^{2k}_{x_1x_2x_3} = P^{2k-1}_2(\vec{Z}_{2k-1})^2 \partial^{2k-1}_{x_1x_2x_3} + P^{2k-1}_2(\vec{Z}_{2k-1})\mathcal{R}^{2k,4}_{x_1x_2x_3}(\vec{Z}_{2k-1})\frac{\partial}{\partial w_{3k-1}}$$
$$+ \left(P^{2k-1}_2(\vec{Z}_{2k-1})\mathcal{R}^{2k,3}_{x_1x_2x_3}(\vec{Z}_{2k-1}) - P^{2k-1}_1(\vec{Z}_{2k-1})\mathcal{R}^{2k,4}_{x_1x_2x_3}(\vec{Z}_{2k-1}) \right) \frac{\partial}{\partial w_{3k}}. \tag{10.0.11}$$

We see that half of the functions $\mathcal{R}^{K,j}_{x_1x_2x_3}$ occur in the coefficients of the last three directions. As already observed the fields $\theta^L_{x_1x_2x_3}$ and $\phi^L_{x_1x_2x_3}$ for $L < K$ have zero components along the last three directions. We have to make sure that the projection onto the last three variables of the collection of fields $\theta^K_{x_1x_2x_3}$ and $\phi^K_{x_1x_2x_3}$ spans a 3-dimensional space. If this is true for a point, we will say that the fields span all new directions in the point. In order to determine if our fields span all new directions in a point $\vec{Z}_K \in \mathcal{F}^K_{a_1a_2a_3a_4}$

we will use the following. Let $N_K = |\mathcal{T}_K|$ be the number of complete triples. Define the $(2 \times N_{K-1})$-matrices

$$\Omega_{x_1x_2x_3}^K(\vec{Z}_K) = \begin{cases} \begin{pmatrix} \mathcal{R}_{x_1x_2x_3}^{K-1,1}(\vec{Z}_K) & \cdots \\ \mathcal{R}_{x_1x_2x_3}^{K-1,2}(\vec{Z}_K) & \cdots \end{pmatrix} & \text{when } K \text{ odd,} \\[2em] \begin{pmatrix} \mathcal{R}_{x_1x_2x_3}^{K-1,3}(\vec{Z}_K) & \cdots \\ \mathcal{R}_{x_1x_2x_3}^{K-1,4}(\vec{Z}_K) & \cdots \end{pmatrix} & \text{when } K \text{ even,} \end{cases}$$

where (x_1, x_2, x_3) run over all triples in $\mathcal{T}_{K-1}$. Using the formulas (10.0.8), (10.0.9), (10.0.10), (10.0.11), and remembering that a fiber $\mathcal{F}_{(a_1,a_2,a_3,a_4)}^K$ is called generic if $(a_1, a_2) \neq (0, 0)$ when K is even and if $(a_3, a_4) \neq (0, 0)$ when K is odd, it is an exercise in linear algebra to prove the lemma below.

Lemma 10.1. *If in a point $\vec{Z}_K \in \mathcal{F}_{a_1a_2a_3a_4}^K$ in a generic fiber*

$$\text{Rank } \Omega_{x_1x_2x_3}^K(\vec{Z}_K) = 2$$

then

$$\{\theta_{x_1x_2x_3}^K : (x_1, x_2, x_3) \in \mathcal{T}_{K-1}\} \cup \{\phi_{x_1x_2x_3}^K : (x_1, x_2, x_3) \in \mathcal{T}_{K-1}\} \cup \{\gamma^K\}$$

span all three new directions. If

$$\text{Rank } \Omega_{x_1x_2x_3}^K(\vec{Z}_K) = 1$$

then

$$\{\theta_{x_1x_2x_3}^K : (x_1, x_2, x_3) \in \mathcal{T}_{K-1}\} \cup \{\phi_{x_1x_2x_3}^K : (x_1, x_2, x_3) \in \mathcal{T}_{K-1}\} \cup \{\gamma^K\}$$

span two out of three new directions.

Because of the formulas (10.0.6) and (10.0.7) we have the lemma below.

Lemma 10.2. *Let $K \leq L$ and put*

$$\mathcal{M}_K^L(\vec{Z}_L) = \begin{pmatrix} \mathcal{R}_{x_1x_2x_3}^{L,1}(\vec{Z}_L) & \cdots \\ \mathcal{R}_{x_1x_2x_3}^{L,2}(\vec{Z}_L) & \cdots \\ \mathcal{R}_{x_1x_2x_3}^{L,3}(\vec{Z}_L) & \cdots \\ \mathcal{R}_{x_1x_2x_3}^{L,4}(\vec{Z}_L) & \cdots \end{pmatrix},$$

where (x_1, x_2, x_3) run over all triples in $\mathcal{T}_K$. For all $L \geq K$

$$\text{Rank } \mathcal{M}_K^K(\vec{Z}_L) = \text{Rank } \mathcal{M}_K^L(\vec{Z}_L).$$

The importance of Lemmas 10.1 and 10.2 is seen in the following corollary.

Corollary 10.3. *Let $L > K$ and $\vec{Z}_K$ be a point where $\text{Rank } \mathcal{M}_K^K(\vec{Z}_K) = 4$. Then for all points $\vec{Z}_L$ contained in a generic fiber $\mathcal{F}_{(a_1,a_2,a_3,a_4)}^L$ such that $\pi(\vec{Z}_L) = \vec{Z}_K$, the complete fields*

$$\{\theta_{x_1x_2x_3}^L : (x_1, x_2, x_3) \in \mathcal{T}_{L-1}\} \cup \{\phi_{x_1x_2x_3}^L : (x_1, x_2, x_3) \in \mathcal{T}_{L-1}\} \cup \{\gamma^L\}$$

span all new directions (the directions along the last three variables in $(\mathbb{C}^3)^L$).

Proof. Two rows of the rank-4 matrix $\mathcal{M}_K^L(\vec{Z}_L)$ are linearly independent.   □

$\partial^2_{x_1 x_2 x_3}$	$\mathcal{R}^{2,1}_{x_1 x_2 x_3}$	$\mathcal{R}^{2,2}_{x_1 x_2 x_3}$	$\mathcal{R}^{2,3}_{x_1 x_2 x_3}$	$\mathcal{R}^{2,4}_{x_1 x_2 x_3}$
$\partial^2_{w_1 w_2 w_3}$	0	0	0	0
$\partial^2_{z_2 w_2 w_3}$	0	z_3^2	0	0
$\partial^2_{z_3 w_1 w_2}$	z_2^2	0	0	0
$\partial^2_{z_2 w_1 w_3}$	0	$z_2 z_3$	0	0
$\partial^2_{z_3 w_1 w_3}$	$z_2 z_3$	0	0	0
$\partial^2_{z_2 z_3 w_1}$	$z_2 w_2$	$-z_2 w_3$	0	z_2
$\partial^2_{z_2 z_3 w_3}$	$-z_3 w_1$	$z_3 w_2$	z_3	0

Table 1. The expressions for $\mathcal{R}^{2,i}_{x_1 x_2 x_3}$.

Corollary 10.4. *Let $L \geq 3$ and $\vec{Z}_L$ be a point that is contained in a generic fiber $\mathcal{F}^L_{(a_1, a_2, a_3, a_4)}$ and such that $z_2 z_3 \neq 0$. Then*

$$\{\theta^L_{x_1 x_2 x_3} : (x_1, x_2, x_3) \in \mathcal{T}_{L-1}\} \cup \{\phi^L_{x_1 x_2 x_3} : (x_1, x_2, x_3) \in \mathcal{T}_{L-1}\} \cup \{\gamma^L\}$$

span all new directions in $\vec{Z}_L$.

Proof. The corresponding matrix for $L = 3$ is contained in Table 1. From this table the claim is an easy exercise in linear algebra. $\qquad\square$

In order to use this corollary we need the following lemma.

Lemma 10.5. *We have the following cases for the function $P = z_2 z_3$ and the fibers $\mathcal{F}^K$:*

(1) *P is not identically zero on $\mathcal{F}^K_{(a_1, a_2, a_3, a_4)}$ for $K \geq 5$. For these K the fibers $\mathcal{F}^K_{(a_1, a_2, a_3, a_4)}$ are irreducible.*

(2) *The fibers $\mathcal{F}^4_{(a_1, a_2, a_3, a_4)}$ are irreducible except when*

$$(a_1, a_2, a_3, a_4) = (0, 0, 0, 1).$$

The function P is not identically zero on fibers except for one component of $\mathcal{F}^4_{(0,0,0,1)}$.

(3) *The fibers $\mathcal{F}^3_{(a_1, a_2, a_3, a_4)}$ are irreducible except when*

$$(a_1, a_2, a_3, a_4) = (a_1, a_2, 0, 1).$$

The function P is not identically zero on fibers except on $\mathcal{F}^3_{(0, a_2, 0, 0)}$ or $\mathcal{F}^3_{(a_1, 0, 0, 0)}$ or on one component of the reducible fiber $\mathcal{F}^3_{(a_1, a_2, 0, 1)}$ where it is identically zero.

Proof. We first prove (3). The fibers $\mathcal{F}^3_{(a_1, a_2, 0, 0)}$ are just biholomorphic to $\mathbb{C}^5$ and z_2, z_3 are constantly equal to a_1, a_2. This shows that they are irreducible and that the assertion about P is true. The fibers $\mathcal{F}^3_{(a_1, a_2, 0, 1)}$ are isomorphic to the variety $\mathcal{G}^2_{(0,1)}$ given by two equations which can be written in matrix form as

$$\begin{pmatrix} w_1 & w_2 \\ w_2 & w_3 \end{pmatrix} \begin{pmatrix} z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{10.0.12}$$

From this it can be seen that $\mathcal{G}^2_{(0,1)}$ has two irreducible components. One is

$$A_1 = \{z_2, z_3 : z_2 = z_3 = 0\} \cong \mathbb{C}^3_{w_1 w_2 w_3} \tag{10.0.13}$$

and the other is

$$A_2 = \left\{ \begin{pmatrix} w_1 & w_2 \\ w_2 & w_3 \end{pmatrix} : \begin{pmatrix} w_1 & w_2 \\ w_2 & w_3 \end{pmatrix} \begin{pmatrix} z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \det \begin{pmatrix} w_1 & w_2 \\ w_2 & w_3 \end{pmatrix} = 0 \right\}. \tag{10.0.14}$$

The singularity set of $\mathcal{G}^2_{(0,1)}$ is $A_1 \cap A_2$. Clearly P is identically zero on A_1 and not identically zero on A_2. Observe that $\mathcal{F}^3_{(a_1,a_2,0,1)}$ are connected, their smooth part consists of the two connected components $A_1 \setminus A_2$ and $A_2 \setminus A_1$.

The smooth generic fibers $\mathcal{F}^3_{(a_1,a_2,a_3,a_4)}$ for $(a_3, a_4) \notin \{(0, 0), (0, 1)\}$ are isomorphic to the variety $\mathcal{G}^2_{(a_3,a_4)}$ given by the two equations

$$z_2 w_1 + z_3 w_2 = a_3, \tag{10.0.15}$$

$$z_2 w_2 + z_3 w_3 + 1 = a_4. \tag{10.0.16}$$

In the case $z_2 \neq 0$ these equations can be used to express w_2 and w_3 by the other variables and we get a chart isomorphic to $\mathbb{C}^\star_{z_2} \times \mathbb{C}_{z_3} \times \mathbb{C}_{w_3}$. In the case $z_3 \neq 0$ we can express w_2 and w_3 by other variables, which gives us a similar chart. Thus $\mathcal{G}^2_{(a_3,a_4)}$ is covered by two connected charts with nonempty intersection which shows that it is connected. Thus the smooth generic fibers $\mathcal{F}^3_{(a_1,a_2,a_3,a_4)}$ are irreducible. The function P is not identically zero on both charts. The assertion (3) is completely proven.

Next we prove assertion (2). The nongeneric fibers $\mathcal{F}^4_{(0,0,a_3,a_4)}$ are isomorphic to $\mathcal{F}^3_{(0,0,a_3,a_4)} \times \mathbb{C}^3$, where $\mathbb{C}^3$ corresponds to the new variables w_4, w_5, w_6. All assumptions about these fibers follow therefore from the corresponding assumptions about $\mathcal{F}^3_{(0,0,a_3,a_4)}$.

In the case of generic fibers which are known to be smooth (see Section 7) we just have to prove that they are connected. For this consider

$$\mathcal{F}^4_{(a_1,a_2,a_3,a_4)} = \bigcup_{(w_4,w_5,w_6)\in\mathbb{C}^3} \mathcal{F}^3_{(a_1,a_2,b_3,b_4)}, \tag{10.0.17}$$

where $b_3 = a_3 - w_4 a_1 - w_5 a_2$ and $b_4 = a_4 - w_5 a_1 - w_6 a_2$. In other words we consider the surjective projection $\rho : \mathcal{F}^4_{(a_1,a_2,a_3,a_4)} \to \mathbb{C}^3$, mapping a point to its last three coordinates (w_4, w_5, w_6), where the ρ-fibers are just fibers $\mathcal{F}^3_{(a_1,a_2,b_3,b_4)}$. Connectedness of the ρ-fibers implies that a connected component of $\mathcal{F}^4_{(a_1,a_2,a_3,a_4)}$ has to be ρ-saturated. Since ρ is a submersion in generic points of the fiber (it is not a submersion only in singular points of an $\mathcal{F}^3$-fiber), any connected component of $\mathcal{F}^4_{(a_1,a_2,a_3,a_4)}$ is equal to $\rho^{-1}(U)$, where U is some open subset of the base $\mathbb{C}^3$. Since the base is connected and ρ is surjective, connectedness of $\mathcal{F}^4_{(a_1,a_2,a_3,a_4)}$ follows. The function P is not identically zero on any $\mathcal{F}^3$-fiber contained in $\mathcal{F}^4_{(a_1,a_2,a_3,a_4)}$, and thus not identically zero on $\mathcal{F}^4_{(a_1,a_2,a_3,a_4)}$ itself. This concludes the proof of (2).

Last we prove assertion (1). The connectedness of the fibers $\mathcal{F}^K_{(a_1,a_2,a_3,a_4)}$ for $K \geq 5$ can be proven by induction in a way similar to the connectedness of the generic $\mathcal{F}^4$-fibers is deduced from the properties of $\mathcal{F}^3$-fibers. As above we consider the surjective projection $\rho : \mathcal{F}^K_{(a_1,a_2,a_3,a_4)} \to \mathbb{C}^3$ onto the last three variables whose fibers are $\mathcal{F}^{K-1}$-fibers. Since again $\mathcal{F}^{K-1}$-fibers are connected and ρ is a submersion in smooth points of the $\mathcal{F}^{K-1}$-fibers, any connected component of $\mathcal{F}^K_{(a_1,a_2,a_3,a_4)}$ is of the form $\rho^{-1}(U)$, where U is some open subset of the base $\mathbb{C}^3$.

In addition we will prove by induction that the smooth part of the singular fibers

$$\mathcal{F}^K_{(a_1,a_2,a_3,a_4)} \setminus \mathrm{Sing}\, \mathcal{F}^K_{(a_1,a_2,a_3,a_4)}$$

is connected for $K \geq 5$. Together with connectedness of the fibers this implies the irreducibility of the fibers.

For even K, the singular fibers are the singular $\mathcal{F}^{K-1}$-fibers times $\mathbb{C}^3$ and therefore the connectedness of the smooth part follows by the induction hypothesis.

For odd $K = 2k+1$, we are faced with the following situation: The singular fiber is $\mathcal{F}^K_{(a_1,a_2,0,1)}$ and it is fibered by $\mathcal{F}^{K-1}$-fibers all of which are smooth except for the fibers $\mathcal{F}^{K-1}_{(0,0,0,1)}$. The union of those fibers forms a codimension-2 subvariety of $\mathcal{F}^K_{(a_1,a_2,0,1)}$ (given by the equations $z_{3k+2} = z_{3k+3} = 0$). By the argument above, the complement, call it W, of this union in $\mathcal{F}^K_{(a_1,a_2,0,1)}$ is connected. The singular points of $\mathcal{F}^K_{(a_1,a_2,0,1)}$ are contained in that union and are contained in (but not equal to) the union of the singular points of the fibers $\mathcal{F}^{K-1}_{(0,0,0,1)}$. We want to prove that any smooth point p of $\mathcal{F}^K_{(a_1,a_2,0,1)}$ which is contained in a fiber $\mathcal{F}^{K-1}_{(0,0,0,1)}$ is contained in the connected component containing W. Since the complement of W has codimension 2 in $\mathcal{F}^K_{(a_1,a_2,0,1)}$, an open neighborhood of p in $\mathcal{F}^K_{(a_1,a_2,0,1)}$ has to intersect W, which gives the desired conclusion.

As in the proof of (2), the function P cannot be identically zero on any fiber $\mathcal{F}^K_{(a_1,a_2,a_3,a_4)}$ since this fiber contains $\mathcal{F}^{K-1}$-fibers on which, by the induction hypothesis, P is not identically zero. $\qquad\square$

Remark 10.6. The fact that after a certain number of factors the fibers of the fibration all become irreducible is very general. It was proven by J. Draisma as an outcome of an interesting discussion with the second author. The irreducibility statement in our lemma is just an example of a much more general property. We refer the interested reader to [Draisma 2022]. The exact number at and past which irreducibility of the fibers holds (in our case 5) is not known in general, although Draisma gives a bound.

Definition 10.7. Let M be a manifold and A be a set of complete vector fields on M. The flows of elements of A give one-parameter subgroups of $\mathrm{Aut}(M)$. Denote by S the group generated by elements of those one-parameter subgroups (finite compositions of time maps of vector fields of elements from A). Define

$$\Gamma(A) = \{\alpha^* X : \alpha \in S \text{ and } X \in A\}.$$

Obviously $\Gamma(A)$ consists of complete vector fields and we call it *the collection generated by A*.

Definition 10.8. Let $L \geq 3$. We define

$$\mathcal{Q}_L = \Gamma\left(\bigcup_{J=3}^{L} \left\{ \{\theta^J_{x_1 x_2 x_3} : (x_1, x_2, x_3) \in \Xi_J\} \cup \{\phi^J_{x_1 x_2 x_3} : (x_1, x_2, x_3) \in \Xi_J\} \cup \{\gamma^J\} \right\} \right).$$

At each step of the induction we will prove the following proposition, which plays a crucial role in the inductive proof of Proposition 3.6.

Proposition 10.9. *For each $L \geq 4$ we have*: *There are finitely many (complete) fields from $\mathcal{Q}_L$ which span the tangent space $T_x \mathcal{F}^L$ at each smooth point of any generic fiber $\mathcal{F}^L$. For $L = 3$ there are finitely many (complete) fields from $\mathcal{Q}_3$ which span the tangent space $T_x \mathcal{F}^3$ at each point of any smooth generic fiber $\mathcal{F}^3$.*

Remark 10.10. For $L = 3$ singular generic fibers $\mathcal{F}^3_{(a_1, a_2, 0, 1)}$ have two irreducible components and we can prove the statement about smooth points on generic fibers only for one of those components. It is false for the other component.

11. Proof of Proposition 3.6: three matrix factors

In Table 2 we list the coefficients of the fields $\partial^2_{x_1 x_2 x_3}$ for all $x_1, x_2, x_3 \in \mathcal{T}_2$.

We first consider the stratum of *smooth generic fibers*, where we have

$$(a_3, a_4) \notin \{(0, 0), (0, 1)\}.$$

Notice that $z_2 = z_3 = 0$ is contained in $\mathcal{F}^3_{(z_5, z_6, 0, 1)}$ and therefore z_2 and z_3 are never simultaneously zero on any fiber in this stratum. It is enough to show that $\mathcal{G}^2_{(a_3, a_4)}$ is elliptic. We see from the table that the fields $\partial^2_{z_3 w_1 w_3}, \partial^2_{z_2 w_1 w_3}, \partial^2_{w_1 w_2 w_3}$ span the tangent space $T_{\vec{Z}_2} \mathcal{G}^2_{(a_3, a_4)}$ for all points $\vec{Z}_2$ where $z_2 z_3 \neq 0$. The complement of this good set is the disjoint union of the analytic subsets $\mathcal{A} = \{\vec{Z}_2 : z_2 = 0\}$ and $\mathcal{B} = \{\vec{Z}_2 : z_3 = 0\}$. From the table we see that $\partial^2_{z_2 w_2 w_3}(z_2) = z_3^2$, which is nowhere-zero on $\mathcal{A}$. Also $\partial^2_{z_3 w_1 w_2}(z_3) = z_2^2$, which is nowhere-zero on $\mathcal{B}$. By Lemma 9.1 there exist finitely many complete fields from

$$\Gamma(\{\partial^2_{x_1 x_2 x_3} : (x_1, x_2, x_3) \in \mathcal{T}_2\})$$

that span the tangent space $T_{\vec{Z}_2} \mathcal{G}^2_{(a_3, a_4)}$ for all points in the stratum. Therefore $\mathcal{G}^2_{(a_3, a_4)}$ is elliptic. It follows that there are finitely many complete fields from

$$\Gamma\left(\{\theta^3_{x_1 x_2 x_3} : (x_1, x_2, x_3) \in \mathcal{T}_2\} \cup \{\phi^3_{x_1 x_2 x_3} : (x_1, x_2, x_3) \in \mathcal{T}_2\} \cup \{\gamma^3\}\right)$$

that span the tangent space $T_{\vec{Z}_3} \mathcal{F}^3_{(a_1, a_2, a_3, a_4)}$ for all points in the stratum.

Now we consider the stratum of *nonsmooth generic fibers* $(a_3, a_4) = (0, 1)$. The two equations defining $\mathcal{G}^2_{(0, 1)}$ can be written in matrix form as

$$\begin{pmatrix} w_1 & w_2 \\ w_2 & w_3 \end{pmatrix} \begin{pmatrix} z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{11.0.1}$$

	$\partial/\partial z_2$	$\partial/\partial z_3$	$\partial/\partial w_1$	$\partial/\partial w_2$	$\partial/\partial w_3$
$\partial^2_{w_1 w_2 w_3}$	0	0	z_3^2	$-z_2 z_3$	z_2^2
$\partial^2_{z_2 w_2 w_3}$	z_3^2	0	0	$-w_1 z_3$	$w_1 z_2 - w_2 z_3$
$\partial^2_{z_3 w_1 w_2}$	0	z_2^2	$z_3 w_3 - w_2 z_2$	$-z_2 w_3$	0
$\partial^2_{z_2 w_1 w_3}$	$z_2 z_3$	0	$-w_1 z_3$	0	$-z_2 w_2$
$\partial^2_{z_3 w_1 w_3}$	0	$z_2 z_3$	$-w_2 z_3$	0	$-z_2 w_3$
$\partial^2_{z_2 z_3 w_1}$	$-z_2 w_3$	$z_2 w_2$	$w_1 w_3 - w_2^2$	0	0
$\partial^2_{z_2 z_3 w_3}$	$w_2 z_3$	$w_1 z_3$	0	0	$w_1 w_3 - w_2^2$

Table 2. Coefficients of complete vector fields. For example, $\partial^2_{w_1 w_2 w_3} = z_3^2 (\partial/\partial w_1) - z_2 z_3 (\partial/\partial w_2) + z_2^2 (\partial/\partial w_3)$.

Recall that $\mathcal{G}^2_{(0,1)}$ has two irreducible components. The components are given by (see (10.0.13) and (10.0.14))

$$A_1 = \{z_2, z_3 : z_2 = z_3 = 0\} \cong \mathbb{C}^3_{w_1 w_2 w_3} \tag{11.0.2}$$

and

$$A_2 = \left\{ \begin{pmatrix} w_1 & w_2 \\ w_2 & w_3 \end{pmatrix} : \begin{pmatrix} w_1 & w_2 \\ w_2 & w_3 \end{pmatrix} \begin{pmatrix} z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \det \begin{pmatrix} w_1 & w_2 \\ w_2 & w_3 \end{pmatrix} = 0 \right\}. \tag{11.0.3}$$

The singularity set of $\mathcal{G}^2_{(0,1)}$ is $A_1 \cap A_2$. We have to show that the smooth part of $\mathcal{G}^2_{(0,1)}$, that is, the disjoint union of $A_1 \setminus A_2$ and $A_2 \setminus A_1$, is elliptic. In the proof for the smooth generic case it is shown that on the set where z_2 and z_3 are not both zero, there exists a collection of complete spanning vector fields. Since $A_2 \setminus A_1$ is contained in that set, we need only consider $A_1 \setminus A_2$. The set $A_1 \setminus A_2$ is biholomorphic to $\mathbb{C}^3 \setminus \{w_1 w_3 - w_2^2 = 0\}$. The vector fields $(w_1 w_3 - w_2^2)(\partial/\partial w_1)$, $(w_1 w_3 - w_2^2)(\partial/\partial w_3)$, $2w_2(\partial/\partial w_1) + w_3(\partial/\partial w_2)$, $2w_2(\partial/\partial w_3) + w_1(\partial/\partial w_2)$ are complete on $\mathbb{C}^3 \setminus \{w_1 w_3 - w_2^2 = 0\}$ and span the tangent space in all points outside the analytic set $A' = \{w_1 = w_3 = 0\} \cap (A_1 \setminus A_2)$. Since w_2 is nowhere-zero on A' any of the four complete fields points out of A'. By Lemma 9.1 the proof is complete. Observe that we also have proved Proposition 10.9 for $L = 3$. Notice that the fields $2w_2(\partial/\partial w_1) + w_3(\partial/\partial w_2)$, $2w_2(\partial/\partial w_3) + w_1(\partial/\partial w_2)$ are not in $\mathcal{Q}_3$ and this explains the difference between $L = 3$ and $L \geq 4$ in Proposition 10.9. See Remark 10.10.

The stratum of *nongeneric fibers* is a locally trivial bundle with fibers $\mathbb{C}^5 (\cong \mathcal{F}^3_{(a_1, a_2, 0, 0)})$ which is an elliptic submersion.

12. Proof of Proposition 3.6: four matrix factors

We begin the proof by studying the stratum of *generic fibers*, $(a_1, a_2) \neq (0, 0)$. We write

$$\mathcal{F}^4_{(a_1, a_2, a_3, a_4)} = \bigcup_{(w_4, w_5, w_6) \in \mathbb{C}^3} \mathcal{F}^3_{(a_1, a_2, b_3, b_4)}, \tag{12.0.1}$$

where $b_3 = a_3 - w_4 a_1 - w_5 a_2$ and $b_4 = a_4 - w_5 a_1 - w_6 a_2$. We need to find finitely many complete vector fields spanning $T_{\vec{Z}_4} \mathcal{F}^4_{(a_1, a_2, a_3, a_4)}$ for points $\vec{Z}_4$ in the stratum of generic fibers. Because of (12.0.1) there are b_3 and b_4 so that $\vec{Z}_3 \in \mathcal{F}^3_{(a_1, a_2, b_3, b_4)}$ and $\vec{Z}_4 = (\vec{Z}_3, w_4, w_5, w_6)$. We first consider the set of points in these fibers having the property that $(b_3, b_4) \neq (0, 0)$ or $(0, 1)$. Under these assumptions, $\vec{Z}_4$ lies in a generic smooth fiber $\mathcal{F}^3_{(a_1, a_2, b_3, b_4)}$ and we know from Section 11 that there is a finite collection of fields from $\mathcal{Q}_3$ which spans

$$T_{\vec{Z}_4} \mathcal{F}^3_{(a_1, a_2, b_3, b_4)} \subset T_{\vec{Z}_4} \mathcal{F}^4_{(a_1, a_2, a_3, a_4)}.$$

Corollary 10.4 together with Lemma 10.5(3) shows that for the set defined by $z_2 z_3 \neq 0$ (which is a Zariski open and dense set of points of the generic fiber $\mathcal{F}^4_{(a_1, a_2, a_3, a_4)}$) the fields

$$\{\theta^4_{x_1 x_2 x_3} : (x_1, x_2, x_3) \in \mathcal{T}_3\} \cup \{\phi^4_{x_1 x_2 x_3} : (x_1, x_2, x_3) \in \mathcal{T}_3\} \cup \{\gamma^4\}$$

span the new directions w_4, w_5, w_6. Since these new directions are complementary to

$$T_{\vec{Z}_4} \mathcal{F}^3_{(a_1, a_2, b_3, b_4)} \subset T_{\vec{Z}_4} \mathcal{F}^4_{(a_1, a_2, a_3, a_4)},$$

we have found finitely many complete fields spanning $T_{\vec{Z}_4}\mathcal{F}^4_{(a_1,a_2,a_3,a_4)}$ for points in a Zariski open dense set in all smooth generic fibers $\mathcal{F}^3_{(a_1,a_2,b_3,b_4)}$. Using Lemma 9.1 we get finitely many complete fields spanning $T_{\vec{Z}_4}\mathcal{F}^4_{(a_1,a_2,a_3,a_4)}$ for all points in all generic fibers $\mathcal{F}^4_{(a_1,a_2,a_3,a_4)}$ with the property that $(b_3, b_4) \neq (0, 0)$ or $(0, 1)$. Next we consider points $\vec{Z}_4$ where $(b_3, b_4) = (0, 1)$, i.e,

$$\vec{Z}_4 \in \mathcal{F}^3_{(a_1,a_2,0,1)} \subset \mathcal{F}^4_{(a_1,a_2,a_3,a_4)}.$$

Remember that

$$\mathcal{F}^3_{(a_1,a_2,0,1)} = A_1 \cup A_2 = A_1 \dot{\cup}(A_2 \setminus A_1) \tag{12.0.2}$$

(see (10.0.13) and (10.0.14)), where A_1 and A_2 are irreducible components. In the proof for $K = 3$ we saw that there is a finite collection from $\mathcal{Q}_3$ which spans all tangent spaces

$$T_{\vec{Z}_4}\mathcal{F}^3_{(a_1,a_2,0,1)} \subset T_{\vec{Z}_4}\mathcal{F}^4_{(a_1,a_2,a_3,a_4)}$$

for all points in $A_2 \setminus A_1$. Lemma 10.5(3) gives that z_2z_3 is not identically zero on $A_2 \setminus A_1$ and as above, appealing to Lemma 9.1, we get spanning fields for the fiber $\mathcal{F}^4_{(a_1,a_2,a_3,a_4)}$ in all points of $A_2 \setminus A_1$. Our aim is to exclude the existence of a subset of the fiber invariant under the flows of fields from $\mathcal{Q}_4$. By the reasoning above, such a subset must be contained in A_1 or the set of points $\vec{Z}_4$ where $(b_3, b_4) = (0, 0)$. Next we show that such a subset is disjoint from A_1. A calculation shows that

$$\partial^3_{z_2z_3z_6} = (z_4w_2 + z_5w_3)\frac{\partial}{\partial z_2} - (1 + z_4w_1 + z_5w_2)\frac{\partial}{\partial z_3} + \cdots.$$

Therefore the complete fields

$$\theta^4_{z_2z_3z_6} = a_1^2\partial^3_{z_2z_3z_6} + \cdots,$$
$$\phi^4_{z_2z_3z_6} = a_2^2\partial^3_{z_2z_3z_6} + \cdots$$

move points out of A_1 (into the big orbit) unless, in addition to $z_2 = z_3 = 0$, also

$$1 + z_4w_1 + z_5w_2 = z_4w_2 + z_5w_3 = 0. \tag{12.0.3}$$

Points in an invariant subset must also satisfy these equations. A calculation gives that $\partial^3_{z_4z_5z_6} = \partial/\partial z_4$ when $z_2 = z_3 = 0$. Therefore the complete fields

$$\theta^4_{z_4z_5z_6} = a_1^2\partial^3_{z_4z_5z_6} + \cdots,$$
$$\phi^4_{z_4z_5z_6} = a_2^2\partial^3_{z_4z_5z_6} + \cdots$$

move points out of this set since

$$\theta^4_{z_4z_5z_6}(1 + z_4w_1 + z_5w_2) = a_1^2w_1,$$
$$\theta^4_{z_4z_5z_6}(z_4w_2 + z_5w_3) = a_1^2w_2,$$
$$\phi^4_{z_4z_5z_6}(1 + z_4w_1 + z_5w_2) = a_2^2w_1,$$
$$\phi^4_{z_4z_5z_6}(z_4w_2 + z_5w_3) = a_2^2w_2$$

cannot all be zero, because this would contradict (12.0.3). We now turn to points $\vec{Z}_4$ where $(b_3, b_4) = (0, 0)$ and again show that these points are not contained in an invariant subset and hence no such invariant subset

exists. We will find fields $\theta^4_{x_1 x_2 x_3}$ or $\phi^4_{x_1 x_2 x_3}$ such that $\mathcal{R}^{3,3}_{x_1 x_2 x_3} \neq 0$ or $\mathcal{R}^{3,4}_{x_1 x_2 x_3} \neq 0$. We begin by noticing that at points $\vec{Z}_4$ with $z_2 z_3 \neq 0$ we can leave the invariant set. Also $z_2 = 0 = z_3$ cannot occur in $\mathcal{F}^3_{(a_1,a_2,0,0)}$. Two cases, $z_2 \neq 0 = z_3$ and $z_2 = 0 \neq z_3$, remain. Assume first that $z_2 \neq 0 = z_3$ (and $b_3 = b_4 = 0$). Here we begin by choosing the triple (z_3, w_1, w_2). Since $\mathcal{R}^{3,3}_{z_3 w_1 w_2} = -z_2^2 z_4$ and $\mathcal{R}^{3,4}_{z_3 w_1 w_2} = z_2^2 z_5$, we move out of $\mathcal{F}^3_{(a_1,a_2,0,0)}$ unless $z_4 = z_5 = 0$. Assuming in addition that $z_4 = z_5 = 0$ we choose the triple (z_2, z_3, w_1). For such points, $\mathcal{R}^{3,4}_{z_2 z_3 w_1} = z_2 + z_2 w_3 z_6$ (and $\mathcal{R}^{3,3}_{z_2 z_3 w_1} = 0$) so if $1 + w_3 z_6 \neq 0$ we move out of $\mathcal{F}^3_{(a_1,a_2,0,0)}$. Choose (z_2, z_3, z_6). Notice that

$$\theta^4_{z_2 z_3 z_6} = a_1^2 \frac{\partial}{\partial z_6} \quad \text{and} \quad \phi^4_{z_2 z_3 z_6} = a_2^2 \frac{\partial}{\partial z_6}$$

at these points and $\theta^4_{z_2 z_3 z_6}(1 + w_3 z_6) = a_1^2 w_3$, $\phi^4_{z_2 z_3 z_6}(1 + w_3 z_6) = a_2^2 w_3$ which both cannot be zero since $1 + w_3 z_6 = 0$ implies $w_3 \neq 0$ and we assume that $(a_1, a_2) \neq (0, 0)$.

Now assume that $z_2 = 0 \neq z_3$ (and also $b_3 = b_4 = 0$) and choose the triple (z_2, w_2, w_3). Since $\mathcal{R}^{3,3}_{z_2 w_2 w_3} = z_3^2 z_5$ and $\mathcal{R}^{3,4}_{z_2 w_2 w_3} = -z_3^2 z_6$ we move out of $\mathcal{F}^3_{(a_1,a_2,0,0)}$ unless $z_5 = z_6 = 0$. Assuming in addition that $z_5 = z_6 = 0$ we choose the triple (z_2, z_3, w_3). For such points $\mathcal{R}^{3,3}_{z_2 z_3 w_3} = z_3 + z_3 w_1 z_4$ (and $\mathcal{R}^{3,4}_{z_2 z_3 w_3} = 0$) so if $1 + w_1 z_4 \neq 0$ we move out of $\mathcal{F}^3_{(a_1,a_2,0,0)}$. Therefore assume also that $1 + w_1 z_4 = 0$ Choose (z_2, z_3, z_4). Notice that

$$\theta^4_{z_2 z_3 z_4} = a_1^2 \frac{\partial}{\partial z_4} \quad \text{and} \quad \phi^4_{z_2 z_3 z_4} = a_2^2 \frac{\partial}{\partial z_4}$$

at these points and $\theta^4_{z_2 z_3 z_4}(1 + w_1 z_4) = a_1^2 w_1$, $\phi^4_{z_2 z_3 z_4}(1 + w_1 z_4) = a_2^2 w_1$, which both cannot be zero since $1 + w_1 z_4 = 0$ implies $w_1 \neq 0$ and we assumed that $(a_1, a_2) \neq (0, 0)$. This lets us conclude that there is no invariant subset with respect to $\mathcal{Q}_4$ and we have handled the stratum of generic fibers. Note that this proves Proposition 10.9 for $K = 4$.

We need to study the stratum of *nongeneric fibers*. This stratum consists of those fibers where $a_1 = a_2 = 0$. We notice that these fibers satisfy

$$\mathcal{F}^4_{(0,0,a_3,a_4)} = \mathcal{F}^3_{(0,0,a_3,a_4)} \times \mathbb{C}^3$$

and since $\mathcal{F}^3_{(0,0,a_3,a_4)}$ is elliptic we have proven Proposition 3.6 for $K = 4$.

13. Proof of Proposition 3.6: five matrix factors

We assume that $K = 5$ and we have seen that the submersions $\Phi_L = \pi_4 \circ \Psi_L$ are stratified elliptic submersions when $3 \leq L \leq 4$ and that Proposition 10.9 is true when $3 \leq L \leq 4$.

We study

$$\mathcal{F}^5_{(a_1,a_2,a_3,a_4)} = \bigcup_{(z_7,z_8,z_9)\in\mathbb{C}^3} \mathcal{F}^4_{(b_1,b_2,a_3,a_4)}, \tag{13.0.1}$$

where $b_1 = a_1 - z_7 a_3 - z_8 a_4$ and $b_2 = a_2 - z_8 a_3 - z_9 a_4$. Let $\vec{Z}_5 \in \mathcal{F}^5_{(a_1,a_2,a_3,a_4)}$. Because of (13.0.1) there are b_1 and b_2 so that $\vec{Z}_4 \in \mathcal{F}^4_{(b_1,b_2,a_3,a_4)}$ and $\vec{Z}_5 = (\vec{Z}_4, z_7, z_8, z_9)$.

First we study the stratum of *smooth generic fibers*. Fibers in this stratum are those satisfying $(a_3, a_4) \notin \{(0, 0), (0, 1)\}$. First notice that if $(b_1, b_2) \neq (0, 0)$ then $\mathcal{F}^4_{(b_1,b_2,a_3,a_4)}$ is a generic smooth fiber

	$\partial/\partial z_2$	$\partial/\partial z_3$	$\partial/\partial z_5$	$\partial/\partial z_6$
P_1^4	$1+w_1 z_4$	$w_3 z_4$	1	0
P_2^4	0	1	0	1
P_3^4	$w_1+w_4+w_1 w_4 z_4$	$w_2+w_5+w_2 w_4 z_4$	w_4	w_5
P_4^4	$w_2+w_5+w_1 w_5 z_4$	$w_3+w_6+w_2 w_5 z_4$	w_5	w_6

Table 3. The nonzero partial derivatives of P_1^4, P_2^4, P_3^4, and P_4^4.

for Φ_4 and as above, Proposition 10.9 for $L = 4$, Corollary 10.4 and Lemma 10.5(2) show that for these points we have spanning fields. If $(b_1, b_2) = (0, 0)$ then $\mathcal{F}_{(0,0,a_3,a_4)}^4$ is a nongeneric smooth fiber for Φ_4 and

$$\mathcal{F}_{(0,0,a_3,a_4)}^4 = \mathcal{F}_{(0,0,a_3,a_4)}^3 \times \mathbb{C}^3.$$

Since we assume that $(a_3, a_4) \neq (0, 1)$ (in this case $(a_3, a_4) \neq (0, 0)$ is automatic), we know by Corollary 10.4, Lemma 10.5(3) and Proposition 10.9 that we have spanning fields. This also shows that Proposition 10.9 holds for these fibers when $L = 5$.

We now study the stratum of *singular generic fibers*. Here $(a_3, a_4) = (0, 1)$. Again notice that if $(b_1, b_2) \neq (0, 0)$ then

$$\mathcal{F}_{(b_1,b_2,0,1)}^4$$

is a generic smooth fiber for Φ_4, and Proposition 10.9 (for $L = 4$), Corollary 10.4 and Lemma 10.5 show that for these points we have spanning fields as above. Next we study the case $(b_1, b_2) = (0, 0)$. In this case we see that

$$\mathcal{F}_{(0,0,0,1)}^4 \cong \mathcal{F}_{(0,0,0,1)}^3 \times \mathbb{C}^3.$$

We write, as in Section 11,

$$\mathcal{F}_{(0,0,0,1)}^3 = A_1 \cup A_2.$$

In $A_2 \setminus A_1$ we can use the argument as in the smooth generic case in Section 11: $z_2 z_3 \not\equiv 0$ and $\partial_{z_2 w_2 w_3}^2(z_2) = z_3^2$ make it possible to leave the set where $z_2 = 0$, and $\partial_{z_3 w_1 w_2}^2(z_3) = z_2^2$ makes it possible to leave the set where $z_3 = 0$.

Now we need to deal with points in $A_1 \times \mathbb{C}^3 \subset \mathcal{F}_{(0,0,0,1)}^4$. Because of the inclusion we find $z_5 = z_6 = 0$. Define $C = \{z_2 = z_3 = z_5 = z_6 = 0\} \subset \mathcal{F}_{(0,0,0,1)}^4 \subset \mathcal{F}_{(0,0,0,1)}^5$, which contains the set of singularities

$$\text{Sing}(\mathcal{F}_{(0,0,0,1)}^5) = C \cap \left\{ \text{Rank} \begin{pmatrix} w_1 & w_2 & w_4 & w_5 \\ w_2 & w_3 & w_5 & w_6 \end{pmatrix} < 2 \right\}.$$

In order to prove Propositions 10.9 and 3.6 we need to show that fields from $\mathcal{Q}_5$ move out from $C \setminus \text{Sing}(\mathcal{F}_{(0,0,0,1)}^5)$. Calculating the partial derivatives of $P_1^4, \ldots, P_4^4$ in points of C we find that the ones that are nonzero are those listed in Table 3. We examine the complete field $\phi_{z_2 z_3 z_6}^5$. This field has some complicated components which on C take the form

$$\phi_{z_2 z_3 z_6}^5 = \mathcal{D}_1 \frac{\partial}{\partial z_2} + \mathcal{D}_2 \frac{\partial}{\partial z_3} + \mathcal{D}_3 \frac{\partial}{\partial z_6} + \cdots,$$

where

$$\mathcal{D}_1 = \det \begin{pmatrix} w_2+w_5+w_2 w_4 z_4 & w_5 \\ w_3+w_6+w_2 w_5 z_4 & w_6 \end{pmatrix},$$

$$\mathcal{D}_2 = \det \begin{pmatrix} w_1+w_4+w_1 w_4 z_4 & w_5 \\ w_2+w_5+w_1 w_5 z_4 & w_6 \end{pmatrix},$$

$$\mathcal{D}_3 = \det \begin{pmatrix} w_1+w_4+w_1 w_4 z_4 & w_2+w_5+w_2 w_4 z_4 \\ w_2+w_5+w_1 w_5 z_4 & w_3+w_6+w_2 w_5 z_4 \end{pmatrix}.$$

Whenever at least one of $\mathcal{D}_1$, $\mathcal{D}_2$ or $\mathcal{D}_3$ is nonzero we can move out of C. Suppose we are in a point of $C \setminus \mathrm{Sing}(\mathcal{F}^5_{(0,0,0,1)})$ where $\mathcal{D}_1 = \mathcal{D}_2 = \mathcal{D}_3 = 0$. Observe that

$$\mathrm{Rank} \begin{pmatrix} w_1 & w_2 & w_4 & w_5 \\ w_2 & w_3 & w_5 & w_6 \end{pmatrix} = \mathrm{Rank} \begin{pmatrix} w_1+w_4+w_1 w_4 z_4 & w_2+w_5+w_2 w_4 z_4 & w_4 & w_5 \\ w_2+w_5+w_1 w_5 z_4 & w_3+w_6+w_2 w_5 z_4 & w_5 & w_6 \end{pmatrix}$$

(in this case it is 2) since

$$\begin{pmatrix} w_1+w_4+w_1 w_4 z_4 \\ w_2+w_5+w_1 w_5 z_4 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + (1 + w_1 z_4) \begin{pmatrix} w_4 \\ w_5 \end{pmatrix},$$

$$\begin{pmatrix} w_2+w_5+w_2 w_4 z_4 \\ w_3+w_6+w_2 w_5 z_4 \end{pmatrix} = \begin{pmatrix} w_2 \\ w_3 \end{pmatrix} + \begin{pmatrix} w_5 \\ w_6 \end{pmatrix} + w_2 z_4 \begin{pmatrix} w_4 \\ w_5 \end{pmatrix}.$$

The fact that $\mathcal{D}_1 = \mathcal{D}_2 = \mathcal{D}_3 = 0$ means that the rank drops when we remove the third column from these matrices. This implies that the third column is nonzero and the other columns are multiples of a nonzero vector v which moreover is linearly independent of the third column. Now we use the field γ^3 (see (8.0.13)) to show that the set

$$I = C \setminus \mathrm{Sing}(\mathcal{F}^5_{(0,0,0,1)}) \cap \{\mathcal{D}_1 = \mathcal{D}_2 = \mathcal{D}_3 = 0\}$$

does not contain an invariant subset under fields from $\mathcal{Q}_5$. In the points of I we have that $\gamma^3 = \partial/\partial z_4$. We consider two cases.

<u>Case 1</u>: $(w_5, w_6) \neq (0, 0)$. In this case

$$\det \begin{pmatrix} w_4 & w_5 \\ w_5 & w_6 \end{pmatrix} \neq 0.$$

We have

$$\gamma^3(\mathcal{D}_1) = w_2 \det \begin{pmatrix} w_4 & w_5 \\ w_5 & w_6 \end{pmatrix}.$$

Thus γ^3 moves points out of I unless $w_2 = 0$. Looking at

$$\gamma^3(\mathcal{D}_2) = w_1 \det \begin{pmatrix} w_4 & w_5 \\ w_5 & w_6 \end{pmatrix}$$

we see that $w_1 = 0$ for I to be invariant. Assuming in addition $w_1 = w_2 = 0$ we find that

$$\mathcal{D}_2 = \det \begin{pmatrix} w_4 & w_5 \\ w_5 & w_6, \end{pmatrix}$$

which is a contradiction since $\mathcal{D}_2 = 0$ on I.

<u>Case 2</u>: $(w_5, w_6) = (0, 0)$. This implies $w_4 \neq 0$. On these assumptions

$$\mathcal{D}_3 = (w_1 w_3 - w_2^2)(1 + z_4 w_4) + w_3 w_4$$

and

$$\gamma^3(\mathcal{D}_3) = (w_1 w_3 - w_2^2) w_4.$$

Now $\gamma^3(\mathcal{D}_3) = 0$ implies that $w_1 w_3 - w_2^2 = 0$, which in combination with $\mathcal{D}_3 = 0$ implies that $w_3 = 0$. This in turn gives $w_2 = 0$ and

$$\begin{pmatrix} w_1 & w_2 & w_4 & w_5 \\ w_2 & w_3 & w_5 & w_6 \end{pmatrix} = \begin{pmatrix} w_1 & 0 & w_4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

which contradicts the assumption that

$$\operatorname{Rank} \begin{pmatrix} w_1 & w_2 & w_4 & w_5 \\ w_2 & w_3 & w_5 & w_6 \end{pmatrix} = 2.$$

Finally we study the stratum of *nongeneric fibers*, that is, $(a_3, a_4) = (0, 0)$. Here all fibers are smooth. Also

$$\mathcal{F}_{(a_1, a_2, 0, 0)}^5 = \mathcal{F}_{(a_1, a_2, 0, 0)}^4 \times \mathbb{C}^3$$

and since $\mathcal{F}_{(a_1, a_2, 0, 0)}^4$ is elliptic we are done.

14. Proof of Proposition 3.6: induction steps

Recall the description of the stratification for the submersion $\Phi_M = \pi_4 \circ \Psi_M$ given in Section 7. When M is *odd* we have the following strata:

- The strata of *generic fibers*: When $(a_3, a_4) \neq (0, 0)$, the fibers are graphs over $\mathcal{G}_{(a_3, a_4)}^{M-1} \times \mathbb{C}$. This set is divided into two strata as follows:

  – Smooth generic fibers: When $(a_3, a_4) \neq (0, 1)$, the fibers are smooth.

  – Singular generic fibers: When $(a_3, a_4) = (0, 1)$, the fibers are nonsmooth.

- The stratum of *nongeneric fibers*: When $(a_3, a_4) = (0, 0)$ the fibers are $\mathcal{F}_{(a_1, a_2, 0, 0)}^M = \mathcal{F}_{(a_1, a_2, 0, 0)}^{M-1} \times \mathbb{C}^3$. Moreover the fibers are smooth.

When M is *even* we have the following strata:

- The stratum of *generic fibers*: When $(a_1, a_2) \neq (0, 0)$, the fibers are graphs over $\mathcal{H}_{(a_1, a_2)}^{M-1} \times \mathbb{C}$. Moreover the fibers are smooth.

- The strata of *nongeneric fibers*: When $(a_1, a_2) = (0, 0)$, the fibers are $\mathcal{F}_{(0, 0, a_3, a_4)}^M = \mathcal{F}_{(0, 0, a_3, a_4)}^{M-1} \times \mathbb{C}^3$. This set is divided into two strata as follows:

  – Smooth nongeneric fibers: When $(a_3, a_4) \neq (0, 1)$, the fibers are smooth.

  – Singular nongeneric fibers: When $(a_3, a_4) = (0, 1)$, the fibers are nonsmooth.

We will now complete the proof by doing the induction steps necessary.

14.1. *Even number of factors.* We begin by showing that the stratified submersion is elliptic when the number of matrix factors is even. This case is easier than the case when the number of factors is odd, which we will deal with in Section 14.2. Assume that $K = 2k - 1 \geq 5$ and that the submersions $\Phi_L = \pi_4 \circ \Psi_L$ are stratified elliptic submersions when $3 \leq L \leq K$ and that Proposition 10.9 is true when $3 \leq L \leq K$.

We study

$$\mathcal{F}^{K+1}_{(a_1,a_2,a_3,a_4)} = \mathcal{F}^{2k}_{(a_1,a_2,a_3,a_4)} = \bigcup_{(w_{3k-2},w_{3k-1},w_{3k})\in\mathbb{C}^3} \mathcal{F}^{2k-1}_{(a_1,a_2,b_3,b_4)}, \tag{14.1.1}$$

where $b_3 = a_3 - w_{3k-2}a_1 - w_{3k-1}a_2$ and $b_4 = a_4 - w_{3k-1}a_1 - w_{3k}a_2$. That is, we use the new group of variables w_{3k-2}, w_{3k-1} and w_{3k} to present $\mathcal{F}^{2k}_{(a_1,a_2,a_3,a_4)}$ as a fibration over $\mathbb{C}^3$ with fibers $\mathcal{F}^{2k-1}$.

Let us describe the strategy similar to the cases of four and five matrix factors. We want to use Proposition 10.9 for $K = 2k - 1$, which gives us complete fields that span along that fibration. Next we want to find complete fields among those that are tangential to $\mathcal{F}^{2k}_{(a_1,a_2,a_3,a_4)}$ that also are transversal to the fibers in the fibration. We will appeal to Corollary 10.4 and Lemma 10.5(1) to find these fields. Taken together this will show that a subset A in the fiber $\mathcal{F}^{2k}_{(a_1,a_2,a_3,a_4)}$ that is invariant with respect to vector fields from $\mathcal{Q}_{2k}$ must be contained in the union of nongeneric fibers $\mathcal{F}^{2k-1}$ and singular points of generic fibers $\mathcal{F}^{2k-1}$. Call this union $\mathcal{U}^{2k}_{(a_1,a_2,a_3,a_4)}$. Our aim will then be to show that there cannot exist such an invariant set A by showing that every point in $\mathcal{U}^{2k}_{(a_1,a_2,a_3,a_4)}$ can be moved into $\mathcal{F}^{2k}_{(a_1,a_2,a_3,a_4)} \setminus \mathcal{U}^{2k}_{(a_1,a_2,a_3,a_4)}$ by vector fields in $\mathcal{Q}_{2k}$.

We now take care of the details. By (14.1.1) there are b_3 and b_4 so that $\vec{Z}_{2k-1} \in \mathcal{F}^{2k-1}_{(a_1,a_2,b_3,b_4)}$ and

$$\vec{Z}_{2k} = (\vec{Z}_{2k-1}, w_{3k-2}, w_{3k-1}, w_{3k}).$$

We begin by studying the stratum of *generic fibers*, that is, $(a_1, a_2) \neq (0, 0)$. For points where $(b_3, b_4) \notin \{(0, 0), (0, 1)\}$ we have that $\mathcal{F}^{2k-1}_{(a_1,a_2,b_3,b_4)}$ is a smooth generic fiber for the submersion Φ_{2k-1}, and Proposition 10.9 (for $L = 2k - 1$) together with Corollary 10.4 and Lemma 10.5(1) let us conclude that we have complete vector fields spanning the tangent space of $\mathcal{F}^{2k}_{(a_1,a_2,a_3,a_4)}$ at these points. For points where $(b_3, b_4) = (0, 0)$ we have

$$\mathcal{F}^{2k-1}_{(a_1,a_2,0,0)} = \mathcal{F}^{2k-2}_{(a_1,a_2,0,0)} \times \mathbb{C}^3$$

and Proposition 10.9 (for $L = 2k - 2$ applied to the first factor) together with Corollary 10.4 and Lemma 10.5(1) (Lemma 10.5(2) when $2k - 2 = 4$) show that we have spanning fields in these points. If $(b_3, b_4) = (0, 1)$ then $\mathcal{F}^{2k-1}_{(a_1,a_2,0,1)}$ is a singular generic fiber for Φ_{2k-1} and at smooth points of the fiber we have complete spanning fields by Proposition 10.9 (for $L = 2k - 1$), Corollary 10.4 and Lemma 10.5. It remains to study

$$\vec{Z}_{2k-1} \in \mathrm{Sing}(\mathcal{F}^{2k-1}_{(a_1,a_2,0,1)}),$$

which is given by

$$z_2 = z_3 = z_5 = z_6 = \cdots = z_{3k-4} = z_{3k-3} = 0 \tag{14.1.2}$$

and

$$\mathrm{Rank} \begin{pmatrix} w_1 & w_2 & \dots & w_{3k-5} & w_{3k-4} \\ w_2 & w_3 & \dots & w_{3k-4} & w_{3k-3} \end{pmatrix} < 2. \tag{14.1.3}$$

A calculation assuming (14.1.2) shows that

$$\partial_{z_{3k-4}z_{3k-3}z_{3k}}^{2k-1} = (z_{3k-2}w_{3k-4} + z_{3k-1}w_{3k-3})\frac{\partial}{\partial z_{3k-4}} - (1 + z_{3k-2}w_{3k-5} + z_{3k-1}w_{3k-4})\frac{\partial}{\partial z_{3k-3}} + \cdots.$$

Therefore the complete fields

$$\theta_{z_{3k-4}z_{3k-3}z_{3k}}^{2k} = a_1^2 \partial_{z_{3k-4}z_{3k-3}z_{3k}}^{2k-1} + \cdots,$$

$$\phi_{z_{3k-4}z_{3k-3}z_{3k}}^{2k} = a_2^2 \partial_{z_{3k-4}z_{3k-3}z_{3k}}^{2k-1} + \cdots$$

move points out of $\mathrm{Sing}(\mathcal{F}_{(a_1,a_2,0,1)}^{2k-1})$ (into the big orbit) unless in addition to (14.1.2) and (14.1.3) also

$$z_{3k-2}w_{3k-4} + z_{3k-1}w_{3k-3} = 1 + z_{3k-2}w_{3k-5} + z_{3k-1}w_{3k-4} = 0. \tag{14.1.4}$$

Points in an invariant subset must satisfy also these equations. A calculation assuming (14.1.2) gives that $\partial_{z_{3k-2}z_{3k-1}z_{3k}}^{2k-1} = \partial/\partial z_{3k-2}$. Therefore the complete fields

$$\theta_{z_{3k-2}z_{3k-1}z_{3k}}^{2k} = a_1^2 \partial_{z_{3k-2}z_{3k-1}z_{3k}}^{2k-1} + \cdots,$$

$$\theta_{z_{3k-2}z_{3k-1}z_{3k}}^{2k} = a_2^2 \partial_{z_{3k-2}z_{3k-1}z_{3k}}^{2k-1} + \cdots$$

move points out of this set since

$$\theta_{z_{3k-2}z_{3k-1}z_{3k}}^{2k}(1 + z_{3k-2}w_{3k-5} + z_{3k-1}w_{3k-4}) = a_1^2 w_{3k-5},$$

$$\theta_{z_{3k-2}z_{3k-1}z_{3k}}^{2k}(z_{3k-2}w_{3k-4} + z_{3k-1}w_{3k-3}) = a_1^2 w_{3k-4},$$

$$\phi_{z_{3k-2}z_{3k-1}z_{3k}}^{2k}(1 + z_{3k-2}w_{3k-5} + z_{3k-1}w_{3k-4}) = a_2^2 w_{3k-5},$$

$$\phi_{z_{3k-2}z_{3k-1}z_{3k}}^{2k}(z_{3k-2}w_{3k-4} + z_{3k-1}w_{3k-3}) = a_2^2 w_{3k-4}$$

cannot all be zero, because this would contradict (14.1.4). Notice that this proves Proposition 10.9 for $L = 2k$.

Now we study the stratum of *nongeneric fibers*, that is, $a_1 = a_2 = 0$. In this case we know that

$$\mathcal{F}_{(0,0,a_3,a_4)}^{2k} = \mathcal{F}_{(0,0,a_3,a_4)}^{2k-1} \times \mathbb{C}^3$$

and by the induction assumption we are done. This finishes the induction step for an even number of factors.

14.2. Odd number of factors. We assume that $K = 2k \geq 6$ and that the submersions $\Phi_L = \pi_4 \circ \Psi_L$ are stratified elliptic submersions when $3 \leq L \leq K$ and that Proposition 10.9 is true when $3 \leq L \leq K$.

We study

$$\mathcal{F}_{(a_1,a_2,a_3,a_4)}^{K+1} = \mathcal{F}_{(a_1,a_2,a_3,a_4)}^{2k+1} = \bigcup_{(z_{3k+1},z_{3k+2},z_{3k+3})\in\mathbb{C}^3} \mathcal{F}_{(b_1,b_2,a_3,a_4)}^{2k}, \tag{14.2.1}$$

where $b_1 = a_1 - z_{3k+1}a_3 - z_{3k+2}a_4$ and $b_2 = a_2 - z_{3k+2}a_3 - z_{3k+3}a_4$. Let $\vec{Z}_{2k+1} \in \mathcal{F}_{(a_1,a_2,a_3,a_4)}^{2k+1}$. Because of (14.2.1) there are b_1 and b_2 so that $\vec{Z}_{2k} \in \mathcal{F}_{(b_1,b_2,a_3,a_4)}^{2k}$ and

$$\vec{Z}_{2k+1} = (\vec{Z}_{2k}, z_{3k+1}, z_{3k+2}, z_{3k+3}).$$

Begin with the stratum of *smooth generic fibers*, that is,

$$(a_3, a_4) \notin \{(0, 0), (0, 1)\}.$$

First notice that if $(b_1, b_2) \neq (0, 0)$ then $\mathcal{F}^{2k}_{(b_1, b_2, a_3, a_4)}$ is a generic smooth fiber for Φ_{2k} and as above, Proposition 10.9 (for $L = 2k$), Corollary 10.4 and Lemma 10.5 show that for these points we have spanning fields. If $(b_1, b_2) = (0, 0)$ then

$$\mathcal{F}^{2k}_{(0, 0, a_3, a_4)} \cong \mathbb{C}^3 \times \mathcal{F}^{2k-1}_{(0, 0, a_3, a_4)}$$

is a nongeneric smooth fiber for Φ_{2k} and, since $\mathcal{F}^{2k-1}_{(0, 0, a_3, a_4)}$ is a generic smooth fiber, Proposition 10.9 (for $L = 2k - 1$), Corollary 10.4 and Lemma 10.5 show that for these points we have spanning fields.

We now study the stratum of *singular generic fibers*. Here $(a_3, a_4) = (0, 1)$. Again notice that if $(b_1, b_2) \neq (0, 0)$ then $\mathcal{F}^{2k}_{(b_1, b_2, 0, 1)}$ is a generic smooth fiber for Φ_{2k}, and Proposition 10.9 (for $L = 2k$), Corollary 10.4 and Lemma 10.5 show that for these points we have spanning fields as above. Next we study the case $(b_1, b_2) = (0, 0)$. In this case we see that $\mathcal{F}^{2k}_{(0, 0, 0, 1)}$ is a singular nongeneric fiber of Φ_{2k} and

$$\mathcal{F}^{2k}_{(0, 0, 0, 1)} \cong \mathcal{F}^{2k-1}_{(0, 0, 0, 1)} \times \mathbb{C}^3_{w_{3k-2} w_{3k-1} w_{3k}}.$$

The smooth points of $\mathcal{F}^{2k-1}_{(0, 0, 0, 1)}$ (which is generic) are handled using Proposition 10.9 (for $L = 2k - 1$), Corollary 10.4 and Lemma 10.5. We have the chain of inclusions

$$\mathcal{F}^{2k-1}_{(a_1, a_2, 0, 1)} \supset \mathcal{F}^{2k}_{(0, 0, 0, 1)} = \mathcal{F}^{2k-1}_{(0, 0, 0, 1)} \times \mathbb{C}^3 \supset \mathrm{Sing}(\mathcal{F}^{2k}_{(0, 0, 0, 1)}) \times \mathbb{C}^3 \supset \mathrm{Sing}(\mathcal{F}^{2k+1}_{(a_1, a_2, 0, 1)}).$$

By the arguments above any possible invariant subset must be contained in

$$J = (\mathrm{Sing}(\mathcal{F}^{2k}_{(0, 0, 0, 1)}) \times \mathbb{C}^3) \setminus \mathrm{Sing}(\mathcal{F}^{2k+1}_{(a_1, a_2, 0, 1)}).$$

Points in J are characterized by $z_2 = z_3 = \cdots = z_{3k-4} = z_{3k-3} = z_{3k-1} = z_{3k} = 0$,

$$\mathrm{Rank} \begin{pmatrix} w_1 & w_2 & \dots & w_{3k-5} & w_{3k-4} \\ w_2 & w_3 & \dots & w_{3k-4} & w_{3k-3} \end{pmatrix} < 2$$

and

$$\mathrm{Rank} \begin{pmatrix} w_1 & w_2 & \dots & w_{3k-5} & w_{3k-4} & w_{3k-2} & w_{3k-1} \\ w_2 & w_3 & \dots & w_{3k-4} & w_{3k-3} & w_{3k-1} & w_{3k} \end{pmatrix} = 2.$$

Take the largest $l < k$ such that

$$\mathrm{Rank} \begin{pmatrix} w_{3l-2} & w_{3l-1} \\ w_{3l-1} & w_{3l} \end{pmatrix} = 1.$$

Let $\widehat{Z} = \sum_{j=l+1}^{k} z_{3j-2}$. We examine the complete field $\phi^{2k+1}_{z_{3l-1} z_{3l} z_{3k}}$. This field has some complicated components which on J take the form

$$\phi^{2k+1}_{z_{3l-1} z_{3l} z_{3k}} = \mathcal{D}_1 \frac{\partial}{\partial z_{3l-1}} + \mathcal{D}_2 \frac{\partial}{\partial z_{3l}} + \mathcal{D}_3 \frac{\partial}{\partial z_{3k}} + \cdots,$$

where

$$\mathcal{D}_1 = \det \begin{pmatrix} w_{3l-1}+w_{3k-1}+w_{3l-1}w_{3k-2}\widehat{Z} & w_{3k-1} \\ w_{3l}+w_{3k}+w_{3l-1}w_{3k-1}\widehat{Z} & w_{3k} \end{pmatrix},$$

$$\mathcal{D}_2 = \det \begin{pmatrix} w_{3l-2}+w_{3k-2}+w_{3l-2}w_{3k-2}\widehat{Z} & w_{3k-1} \\ w_{3l-1}+w_{3k-1}+w_{3l-2}w_{3k-1}\widehat{Z} & w_{3k} \end{pmatrix},$$

$$\mathcal{D}_3 = \det \begin{pmatrix} w_{3l-2}+w_{3k-2}+w_{3l-2}w_{3k-2}\widehat{Z} & w_{3l-1}+w_{3k-1}+w_{3l-1}w_{3k-2}\widehat{Z} \\ w_{3l-1}+w_{3k-1}+w_{3l-2}w_{3k-1}\widehat{Z} & w_{3l}+w_{3k}+w_{3l-1}w_{3k-1}\widehat{Z} \end{pmatrix}.$$

Whenever at least one of $\mathcal{D}_1$, $\mathcal{D}_2$ or $\mathcal{D}_3$ is nonzero we can move out of J. Now suppose we are in a point of J where $\mathcal{D}_1 = \mathcal{D}_2 = \mathcal{D}_3 = 0$.

Let

$$\mathcal{C} = \begin{pmatrix} w_{3l-2}+w_{3k-2}+w_{3l-2}w_{3k-2}\widehat{Z} & w_{3l-1}+w_{3k-1}+w_{3l-1}w_{3k-2}\widehat{Z} & w_{3k-2} & w_{3k-1} \\ w_{3l-1}+w_{3k-1}+w_{3l-2}w_{3k-1}\widehat{Z} & w_{3l}+w_{3k}+w_{3l-1}w_{3k-1}\widehat{Z} & w_{3k-1} & w_{3k} \end{pmatrix}$$

and observe that

$$2 = \mathrm{Rank} \begin{pmatrix} w_{3l-2} & w_{3l-1} & w_{3k-2} & w_{3k-1} \\ w_{3l-1} & w_{3l} & w_{3k-1} & w_{3k} \end{pmatrix} = \mathrm{Rank}\,\mathcal{C}$$

by column operations.

The fact that $\mathcal{D}_1 = \mathcal{D}_2 = \mathcal{D}_3 = 0$ means that the rank drops when we remove the third column from these matrices. This implies that the third column is nonzero and the other columns are multiples of a nonzero vector v which, moreover, is linearly independent of the third column. Now we use the field γ^{3l} (see (8.0.13) or (8.0.17)) to show that the set

$$I = J \cap \{\mathcal{D}_1 = \mathcal{D}_2 = \mathcal{D}_3 = 0\}$$

does not contain an invariant subset under fields from $\mathcal{Q}_{2k+1}$. In the points that we are considering, $\gamma^{3l} = \partial/\partial z_{3l+1}$. We consider two cases.

<u>Case 1</u>: $(w_{3k-1}, w_{3k}) \neq (0, 0)$. In this case

$$\det \begin{pmatrix} w_{3k-2} & w_{3k-1} \\ w_{3k-1} & w_{3k} \end{pmatrix} \neq 0.$$

We have

$$\gamma^{3l}(\mathcal{D}_1) = w_{3l-1} \det \begin{pmatrix} w_{3k-2} & w_{3k-1} \\ w_{3k-1} & w_{3k} \end{pmatrix}.$$

Thus γ^{3l} moves points out of I unless $w_{3l-1} = 0$. Looking at

$$\gamma^{3l}(\mathcal{D}_2) = w_{3l-2} \det \begin{pmatrix} w_{3k-2} & w_{3k-1} \\ w_{3k-1} & w_{3k} \end{pmatrix},$$

we see that $w_{3l-2} = 0$ for I to be invariant. Assuming in addition $w_{3l-2} = w_{3l-1} = 0$ we find that

$$\mathcal{D}_2 = \det \begin{pmatrix} w_{3k-2} & w_{3k-1} \\ w_{3k-1} & w_{3k} \end{pmatrix} = 0,$$

which is a contradiction.

<u>Case 2</u>: $(w_{3k-1}, w_{3k}) = (0, 0)$. This implies $w_{3k-2} \neq 0$. By these assumptions

$$\mathcal{D}_3 = (w_{3l-2}w_{3l} - w_{3l-1}^2)(1 + w_{3k-2}\widehat{Z}) + w_{3l}w_{3k-2}$$

and

$$\gamma^{3l}(\mathcal{D}_3) = (w_{3l-2}w_{3l} - w_{3l-1}^2)w_{3k-2}.$$

Now $\mathcal{D}_3 = \gamma^{3l}(\mathcal{D}_3) = 0$ implies that $(w_{3l-2}w_{3l} - w_{3l-1}^2) = 0$ and $w_{3l} = 0$. The first equality gives $w_{3l-1} = 0$, which altogether contradicts the assumption that

$$\mathrm{Rank} \begin{pmatrix} w_{3l-2} & w_{3l-1} & w_{3k-2} & w_{3k-1} \\ w_{3l-1} & w_{3l} & w_{3k-1} & w_{3k} \end{pmatrix} = 2.$$

Finally we study the stratum of *nongeneric fibers*, that is, $(a_3, a_4) = (0, 0)$. Here all fibers are smooth. Also

$$\mathcal{F}_{(a_1,a_2,0,0)}^{2k+1} = \mathcal{F}_{(a_1,a_2,0,0)}^{2k} \times \mathbb{C}^3$$

and since $\mathcal{F}_{(a_1,a_2,0,0)}^{2k}$ is elliptic, by the induction hypothesis we are done.

15. Product of exponentials and open questions

For a Stein space X, a complex Lie group G and its exponential map $\exp: \mathfrak{g} \to G$, we say that a holomorphic map $f: X \to G$ is a product of k exponentials if there are holomorphic maps $f_1, \ldots, f_k: X \to \mathfrak{g}$ such that

$$f = \exp(f_1) \cdots \exp(f_k).$$

It is easy to see that any map f which is a product of exponentials (for some sufficiently large k) is null-homotopic. In the case where G is the special linear group $\mathrm{SL}_n(\mathbb{C})$ the converse follows from [Ivarsson and Kutzschebauch 2012] as explained in [Doubtsov and Kutzschebauch 2019]. In the same way we prove:

Theorem 15.1. *For a Stein space X there is a number N depending on the dimension of X such that any null-homotopic holomorphic map $f: X \to \mathrm{Sp}_4(\mathbb{C})$ can be factorized as*

$$f(x) = \exp(G_1(x)) \cdots \exp(G_K(x)).$$

where $G_i: X \to \mathfrak{sp}_4(\mathbb{C})$ are holomorphic maps.

Proof. By Theorem 3.1 we find K elementary symplectic matrices $A_i(x) \in \mathrm{Sp}_4(\mathcal{O}(X))$, $i = 1, 2, \ldots K$, such that

$$f(x) = A_1(x) \cdots A_K(x).$$

Now remark that the logarithmic series

$$\ln(\mathrm{Id} + B) = \sum \frac{1}{n} B^n$$

is finite for the nilpotent matrices $B_i = A_i - \mathrm{Id}$. $\qquad \square$

Open Problem 15.2. *Determine the optimal number K in Theorem 15.1.*

Open Problem 15.3. *Determine the optimal numbers of factors in Theorem 3.1.*

The smooth fibers

$$\mathcal{F}^K_{(a_1,a_2,a_3,a_4)} = (\pi_4 \circ \Psi_K)^{-1}(a_1, a_2, a_3, a_4)$$

of the fibration projecting the product of K elementary symplectic matrices to its last row are smooth affine algebraic varieties. They are new examples of Oka manifolds, since we prove as a by-product of Proposition 3.6 that they are holomorphically flexible (for a definition see the work of Arzhantsev, Flenner, Kaliman, Kutzschebauch and Zaidenberg [Arzhantsev et al. 2013]). Our proof does not give the algebraic flexibility of them, even if our initial complete fields obtained in Section 8 are algebraic. The problem is that their flows are not always algebraic (not all of them are locally nilpotent). Therefore the pull-backs by their flows are merely holomorphic vector fields.

Open Problem 15.4. *Which other* (*stronger*) *flexibility properties like algebraic flexibility*, *algebraic* (*volume*) *density property*, *or* (*volume*) *density property do the fibers* $\mathcal{F}^K_{(a_1,a_2,a_3,a_4)}$ *admit?*

For the definition of these flexibility properties we refer to the overview article [Kutzschebauch 2014]. Let us remark that the fibers of the fibration for five elementary factors in [Ivarsson and Kutzschebauch 2012] have been thoroughly studied in [Kaliman and Kutzschebauch 2011; 2016, Section 7]. They were the starting point for the introduction of the class of generalized Gizatullin surfaces whose final classification was achieved by Kaliman, Kutzschebauch and Leuenberger [Kaliman et al. 2020]. The topology of these fibers for any number of elementary factors has been studied in [De Vito 2020], where it was also proven that they admit the algebraic volume density property. Such studies are interesting since the possible topological types of Oka manifolds or manifolds with the density property are not understood at the moment.

Open Problem 15.5. *Determine the homology groups of the fibers* $\mathcal{F}^K_{(a_1,a_2,a_3,a_4)}$.

And finally:

Open Problem 15.6. *Prove Conjecture 3.11.*

References

[Andrist and Kutzschebauch 2018] R. B. Andrist and F. Kutzschebauch, "The fibred density property and the automorphism group of the spectral ball", *Math. Ann.* **370**:1-2 (2018), 917–936. MR Zbl

[Arzhantsev et al. 2013] I. Arzhantsev, H. Flenner, S. Kaliman, F. Kutzschebauch, and M. Zaidenberg, "Flexible varieties and automorphism groups", *Duke Math. J.* **162**:4 (2013), 767–823. MR Zbl

[Bass et al. 1967] H. Bass, J. Milnor, and J.-P. Serre, "Solution of the congruence subgroup problem for SL_n ($n \geq 3$) and Sp_{2n} ($n \geq 2$)", *Inst. Hautes Études Sci. Publ. Math.* **33** (1967), 59–137. MR Zbl

[Cohn 1966] P. M. Cohn, "On the structure of the GL_2 of a ring", *Inst. Hautes Études Sci. Publ. Math.* **30** (1966), 5–53. MR Zbl

[De Vito 2020] G. De Vito, "New examples of Stein manifolds with volume density property", *Complex Anal. Synerg.* **6**:2 (2020), art. id. 9. MR Zbl

[Doubtsov and Kutzschebauch 2019] E. Doubtsov and F. Kutzschebauch, "Factorization by elementary matrices, null-homotopy and products of exponentials for invertible matrices over rings", *Anal. Math. Phys.* **9**:3 (2019), 1005–1018. MR Zbl

[Draisma 2022] J. Draisma, "The irreducible control property in matrix groups", *Linear Algebra Appl.* **634** (2022), 15–29. MR

[Forstnerič 2010] F. Forstnerič, "The Oka principle for sections of stratified fiber bundles", *Pure Appl. Math. Q.* **6**:3 (2010), 843–874. MR Zbl

[Forstnerič 2017] F. Forstnerič, *Stein manifolds and holomorphic mappings*: *the homotopy principle in complex analysis*, 2nd ed., Ergebnisse der Mathematik (3) **56**, Springer, 2017. MR Zbl

[Forstnerič and Prezelj 2001] F. Forstnerič and J. Prezelj, "Extending holomorphic sections from complex subvarieties", *Math. Z.* **236**:1 (2001), 43–68. MR Zbl

[Forstnerič and Prezelj 2002] F. Forstnerič and J. Prezelj, "Oka's principle for holomorphic submersions with sprays", *Math. Ann.* **322**:4 (2002), 633–666. MR Zbl

[Gromov 1989] M. Gromov, "Oka's principle for holomorphic sections of elliptic bundles", *J. Amer. Math. Soc.* **2**:4 (1989), 851–897. MR Zbl

[Grunewald et al. 1991] F. Grunewald, J. Mennicke, and L. Vaserstein, "On symplectic groups over polynomial rings", *Math. Z.* **206**:1 (1991), 35–56. MR Zbl

[Ivarsson and Kutzschebauch 2012] B. Ivarsson and F. Kutzschebauch, "Holomorphic factorization of mappings into SL$_n(\mathbb{C})$", *Ann. of Math.* (2) **175**:1 (2012), 45–69. MR Zbl

[Ivarsson et al. 2020] B. Ivarsson, F. Kutzschebauch, and E. Løw, "Factorization of symplectic matrices into elementary factors", *Proc. Amer. Math. Soc.* **148**:5 (2020), 1963–1970. MR Zbl

[Kaliman and Kutzschebauch 2011] S. Kaliman and F. Kutzschebauch, "On the present state of the Andersén–Lempert theory", pp. 85–122 in *Affine algebraic geometry* (Montreal, 2009), edited by D. Daigle et al., CRM Proc. Lect. Notes **54**, Amer. Math. Soc., Providence, RI, 2011. MR Zbl

[Kaliman and Kutzschebauch 2016] S. Kaliman and F. Kutzschebauch, "On algebraic volume density property", *Transform. Groups* **21**:2 (2016), 451–478. MR Zbl

[Kaliman et al. 2020] S. Kaliman, F. Kutzschebauch, and M. Leuenberger, "Complete algebraic vector fields on affine surfaces", *Int. J. Math.* **31**:3 (2020), art. id. 2050018. MR Zbl

[Klein and Ramspott 1987] M. Klein and K. J. Ramspott, "Ein Transformationssatz für Idealbasen holomorpher Funktionen", *Bayer. Akad. Wiss. Math.-Natur. Kl. Sitzungsber.* **1987**:4 (1987), 93–100. MR Zbl

[Kopeiko 1978] V. I. Kopeiko, "The stabilization of symplectic groups over a polynomial ring", *Mat. Sb.* (*N.S.*) **106(148)**:1 (1978), 94–107. In Russian; translated in *Math. USSR-Sb.* **34**:5 (1978), 655–669. MR Zbl

[Kutzschebauch 2014] F. Kutzschebauch, "Flexibility properties in complex analysis and affine algebraic geometry", pp. 387–405 in *Automorphisms in birational and affine geometry* (Levico Terme, Italy, 2012), edited by I. Cheltsov et al., Springer Proc. Math. Stat. **79**, Springer, 2014. MR Zbl

[Suslin 1977] A. A. Suslin, "The structure of the special linear group over rings of polynomials", *Izv. Akad. Nauk SSSR Ser. Mat.* **41**:2 (1977), 235–252. In Russian; translated in *Math. USSR-Izv.* **11**:2 (1977), 221–238. MR Zbl

[Vaserstein 1988] L. N. Vaserstein, "Reduction of a matrix depending on parameters to a diagonal form by addition operations", *Proc. Amer. Math. Soc.* **103**:3 (1988), 741–746. MR Zbl

[Vavilov and Stepanov 2011] N. A. Vavilov and A. V. Stepanov, "Linear groups over general rings, I: Generalities", *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov.* (*POMI*) **394**:22 (2011), 33–139. In Russian; translated in *J. Math. Sci.* (*N.Y.*) **188**:5 (2013), 490–550. MR Zbl

BJÖRN IVARSSON: bjorn.ivarsson@aalto.fi
Department of Mathematics and Systems Analysis, Aalto University, Aalto, Finland

FRANK KUTZSCHEBAUCH: frank.kutzschebauch@math.unibe.ch
Departement Mathematik, Universität Bern, Bern, Switzerland

ERIK LØW: elow@math.uio.no
Department of Mathematics, University of Oslo, Oslo, Norway

Analysis & PDE

msp.org/apde

Analysis & PDE

Volume 16   No. 1   2023