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GROWTH OF HIGH L^p NORMS FOR EIGENFUNCTIONS AN APPLICATION OF GEODESIC BEAMS

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This work concerns L^p norms of high energy Laplace eigenfunctions: $(-\Delta_g - \lambda^2)\phi_\lambda = 0$, $\|\phi_\lambda\|_{L^2} = 1$. Sogge (1988) gave optimal estimates on the growth of $\|\phi_\lambda\|_{L^p}$ for a general compact Riemannian manifold. Here we give general dynamical conditions guaranteeing quantitative improvements in L^p estimates for $p > p_c$, where p_c is the critical exponent. We also apply results of an earlier paper (Canzani and Galkowski 2018) to obtain quantitative improvements in concrete geometric settings including all product manifolds. These are the first results giving quantitative improvements for estimates on the L^p growth of eigenfunctions that only require dynamical assumptions. In contrast with previous improvements, our assumptions are local in the sense that they depend only on the geodesics passing through a shrinking neighborhood of a given set in M. Moreover, we give a structure theorem for eigenfunctions which saturate the quantitatively improved L^p bound. Modulo an error, the theorem describes these eigenfunctions as finite sums of quasimodes which, roughly, approximate zonal harmonics on the sphere scaled by $1/\sqrt{\log \lambda}$.

1. Introduction

Let (M, g) be a smooth, compact, Riemannian manifold of dimension n and consider normalized Laplace eigenfunctions, i.e., solutions to

$$(-\Delta_g - \lambda^2)\phi_\lambda = 0, \quad \|\phi_\lambda\|_{L^2(M)} = 1.$$

This article studies the growth of L^p norms of the eigenfunctions ϕ_λ as $\lambda \to \infty$. Since the work of Sogge [1988], it has been known that there is a change of behavior in the growth of L^p norms for eigenfunctions at the *critical exponent* $p_c := 2(n+1)/(n-1)$. In particular,

$$\|\phi_\lambda\|_{L^p(M)} \le C\lambda^{\delta(p)}, \quad \delta(p) := \begin{cases} \frac{n-1}{2} - \frac{n}{p}, & p_c \le p, \\ \frac{n-1}{4} - \frac{n-1}{2p}, & 2 \le p \le p_c. \end{cases} \tag{1-1}$$

For $p \ge p_c$, (1-1) is saturated by the zonal harmonics on the round sphere S^n. On the other hand, for $p \le p_c$, these bounds are saturated by the highest weight spherical harmonics on S^n, also known as Gaussian beams. In a very strong sense, the authors showed in [Canzani and Galkowski 2021, page 4] that any eigenfunction saturating (1-1) for $p > p_c$ behaves like a zonal harmonic, while Blair and Sogge [2015b; 2017] showed that for $p < p_c$ such eigenfunctions behave like Gaussian beams. In the case $p \le p_c$, Blair and Sogge [2015a; 2018; 2019] have made substantial progress on improved L^p estimates on manifolds with nonpositive curvature.

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This article concerns the behavior of L^p norms for high p; that is, for $p > p_c$. While there has been a great deal of work on L^p norms of eigenfunctions [Hezari and Rivière 2016; Koch et al. 2007; Sogge et al. 2011; Sogge and Zelditch 2002; 2016; Tacy 2018; 2019; Toth and Zelditch 2002; 2003], this article departs from the now standard approaches. We both adapt the geodesic beam methods developed by the authors in [Canzani and Galkowski 2023; 2019; 2021; Canzani et al. 2018; Galkowski 2018; 2019; Galkowski and Toth 2018; 2020] and develop a new second microlocal calculus used to understand the number of points at which $|u_\lambda|$ can be large (see Section 1A for details on the new ideas here). By doing this, we give general dynamical conditions guaranteeing quantitative improvements over (1-1) for $p > p_c$. In order to work in compact subsets of phase space, we semiclassically rescale our problem. Let $h = \lambda^{-1}$, and, abusing notation slightly, write $\phi_\lambda = \phi_h$, so that

$$(-h^2\Delta_g - 1)\phi_h = 0, \quad \|\phi_h\|_{L^2(M)} = 1.$$

We also work with the semiclassical Sobolev spaces $H_h^s(M)$, with $s \in \mathbb{R}$, defined by the norm

$$\|u\|^2_{H_h^s(M)} := \langle (-h^2\Delta_g + 1)^s u, u\rangle_{L^2(M)}.$$

We start by stating a consequence of our main theorem. Let Ξ denote the collection of maximal unit speed geodesics for (M, g). For m a positive integer, $r > 0$, $t \in \mathbb{R}$, and $x \in M$, define

$$\Xi_x^{m,r,t} := \{\gamma \in \Xi : \gamma(0) = x \text{ and there exists at least } m \text{ conjugate points to } x \text{ in } \gamma(t-r, t+r)\},$$

where we count conjugate points with multiplicity. Next, for a set $V \subset M$, write

$$\mathcal{C}_V^{m,r,t} := \bigcup_{x \in V} \{\gamma(t) : \gamma \in \Xi_x^{m,r,t}\}.$$

Note that if $r_t \to 0^+$ as $|t| \to \infty$, then saying $y \in \mathcal{C}_x^{n-1,r_t,t}$ for t large indicates that y behaves like a point that is maximally conjugate to x. This is the case for every point x on the sphere when y is either equal to x or its antipodal point. The following result applies under the assumption that points are not maximally conjugate and obtains quantitative improvements.

Theorem 1.1. *Let $p > p_c$ and $U \subset M$, and assume there exist $t_0 > 0$ and $a > 0$ such that*

$$\inf_{x_1, x_2 \in U} d(x_1, \mathcal{C}_{x_2}^{n-1,r_t,t}) \geq r_t \quad \textit{for } t \geq t_0,$$

with $r_t = \frac{1}{a}e^{-at}$. Then, there exist $C > 0$ and $h_0 > 0$ such that, for $0 < h < h_0$ and $u \in \mathcal{D}'(M)$,

$$\|u\|_{L^p(U)} \leq Ch^{-\delta(p)}\bigg(\frac{\|u\|_{L^2(M)}}{\sqrt{\log h^{-1}}} + \frac{\sqrt{\log h^{-1}}}{h}\|(-h^2\Delta_g - 1)u\|_{H_h^{(n-3)/2-n/p}(M)}\bigg).$$

The assumption in Theorem 1.1 rules out maximal conjugacy of any two points $x, y \in U$ uniformly up to time ∞, and we expect it to hold for a dense set of metrics on any smooth manifold M with $U = M$. Since Theorem 1.1 includes the case of manifolds without conjugate points, it generalizes the work of Hassell and Tacy [2015], where it was shown that logarithmic improvements in L^p norms for $p > p_c$ are possible on manifolds with nonpositive curvature. One family of examples where the assumptions of Theorem 1.1 hold is that of product manifolds [Canzani and Galkowski 2021, Lemma 1.1], i.e.,

$(M_1 \times M_2, g_1 \oplus g_2)$, where the (M_i, g_i) are nontrivial compact Riemannian manifolds. Note that this family of examples includes manifolds with large numbers of conjugate points, e.g., $S^2 \times M$ for any nontrivial M.

The proof of Theorem 1.1 gives a great deal of information about eigenfunctions which saturate L^p bounds ($p > p_c$). Indeed, its proof yields Theorem 3.8 (see Section 3G), which describes the profile of these functions modulo an error in L^p. It shows that, under the assumptions of Theorem 1.1, an eigenfunction can saturate the *logarithmically improved* L^∞ norm near at most *boundedly many* points (it actually shows the same for the L^p norm when $p > p_c$). That is, for $\varepsilon > 0$, there is $N_\varepsilon > 0$ such that

$$\#\left\{\alpha \in \mathcal{I}(h) : \|u\|_{L^\infty(B(x_\alpha, R(h)))} \geq \frac{\varepsilon h^{(1-n)/2}\sqrt{t_0}}{\sqrt{\log h^{-1}}}\|u\|_{L^2(M)},\ B(x_\alpha, R(h)) \cap U \neq \varnothing\right\} \leq N_\varepsilon, \tag{1-2}$$

where $\{x_\alpha\}_{\alpha\in\mathcal{I}(h)}$ is a maximal $R(h) := h^{1/2-\delta}$ separated collection of points with $\delta > 0$.

Moreover, modulo an error small in L^p, near each of these points the eigenfunction u can be decomposed as a sum of quasimodes which are similar to the highest weight spherical harmonics scaled by $h^{(n-1)/4}/\sqrt{\log h^{-1}}$ whose number is nearly proportional to $h^{(1-n)/2}$. Indeed, Theorem 3.8 (see Section 3G) shows that there is a collection of geodesic tubes $\{\mathcal{T}_j\}_{j\in\mathcal{L}(\varepsilon,u)} \subset S^*M$ of radius $R(h)$ (see Definition 1.3) with indices in the set $\mathcal{L}(\varepsilon, u) = \bigcup_{i=1}^{C} \mathcal{J}_i$ and with pairwise disjoint tubes $\mathcal{T}_k \cap \mathcal{T}_\ell = \varnothing$ for $k, \ell \in \mathcal{J}_i$ with $k \neq \ell$, such that

$$u = u_e + \frac{1}{\sqrt{\log h^{-1}}}\sum_{j\in\mathcal{L}(\varepsilon,u)} v_j.$$

Here, u_e should be understood as an error term satisfying, for all $p \leq q \leq \infty$,

$$\|u_e\|_{L^q} \leq \varepsilon h^{-\delta(q)}(\log h^{-1})^{-1/2}\|u\|_{L^2}.$$

Each v_j is microsupported in the geodesic tube $\mathcal{T}_j$ and is a quasimode with

$$\|(-h^2\Delta_g - 1)v_j\|_{L^2} \leq C\varepsilon^{-1} h R(h)^{(n-1)/2}\|u\|_{L^2} \quad \text{and} \quad \|v_j\|_{L^2} \leq C\varepsilon^{-1} R(h)^{(n-1)/2}\|u\|_{L^2}. \tag{1-3}$$

While similar to highest weight spherical harmonics (also known as Gaussian beams), they are not as tightly localized to a geodesic segment and do not have Gaussian profiles. We refer to these quasimodes as *geodesic beams* (see Remark 3.2 and Figure 1 for an illustration).

Furthermore, in Theorem 3.8 we prove that near each point x_α on which u nearly saturates the L^p bound, i.e., for α that belongs to the set displayed in (1-2), we have

$$c\varepsilon^2 R(h)^{1-n} \leq |\mathcal{L}(\varepsilon, u, \alpha)| \leq C R(h)^{1-n}, \tag{1-4}$$

where $\mathcal{L}(\varepsilon, u, \alpha) := \{j \in \mathcal{L}(\varepsilon, u) : \pi_M(\mathcal{T}_j) \cap B(x_\alpha, 3R(h)) \neq \varnothing\}$ and $\pi_M : S^*M \to M$ is the natural projection. Since $\dim S^*_{x_\alpha}M = n - 1$, this means that at points x_α at which u nearly saturates its L^p norm there must be a full measure set of directions on which u is microsupported. In addition, we also prove that the collection of geodesic beams v_j on which u has its microsupport carries a positive portion of the total L^2 mass:

$$\sum_{j\in\mathcal{L}(\varepsilon,u,\alpha)} \|v_j\|^2_{L^2} \geq c^2\varepsilon^2\|u\|^2_{L^2}.$$

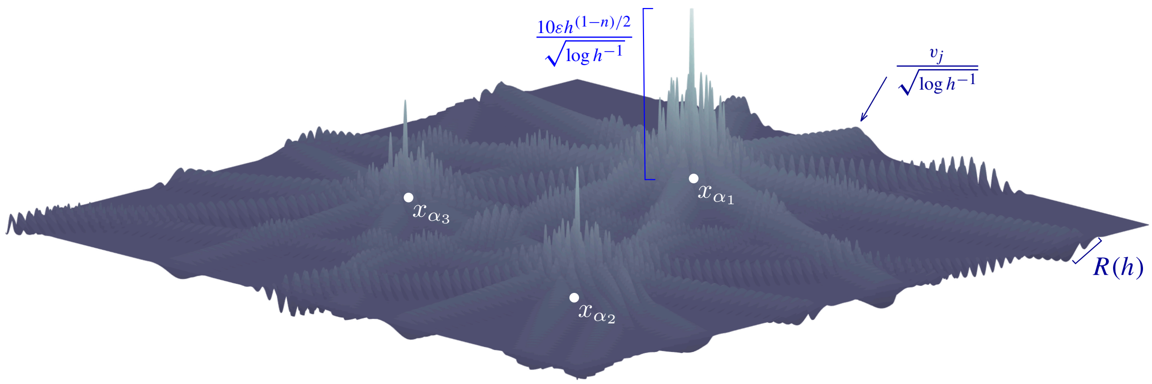

Figure 1. The figure illustrates a function u that saturates the L^∞ bound at three points $x_{\alpha_1}, x_{\alpha_2}, x_{\alpha_3}$ viewed as a superposition of geodesic beams v_j. Each ridge corresponds to a beam v_j and is microsupported on a tube $\mathcal{T}_j$ of radius $R(h)$.

Note that, together with (1-3) and (1-4), this implies that most of the geodesic beams carry mass *exactly* proportional to $R(h)^{(n-1)/2}\|u\|_{L^2}$, and hence that the mass is nearly uniform over all possible directions. For the precise statement of these estimates, see Section 3G.

Remark 1.2. Note that we do *not* use the bound (1-2) to prove our main theorem. Instead, this decomposition is a consequence of the proof of Theorem 1.1, which, in principle describes much more about the profile of eigenfunctions (see the outline of the proof in Section 1A for more details).

The proofs of Theorems 1.1 and 3.8 hinge on a much more general theorem, Theorem 1.4, which does not require global geometric assumptions on (M, g). As far as the authors are aware, Theorem 1.4 is the first result giving quantitative estimates for the L^p growth of eigenfunctions that *only* requires dynamical assumptions. We emphasize that, in contrast with previous improvements on Sogge's L^p estimates, the assumptions in Theorem 1.4 are purely dynamical and, moreover, are local in the sense that they depend only on the geodesics passing through a shrinking neighborhood of a given set in M. Moreover, the techniques do not require long-time wave parametrices.

Theorem 1.4 controls $\|u\|_{L^p(U)}$ using an assumption on the maximal volume of long geodesics joining any two given points in U. For our proof, it is necessary to control the number of points in U where the L^∞ norm of u can be large (see Step 4 in Section 1A). This is a very delicate and technical part of the argument, as the points in question may be approaching one another at rates $\sim h^\delta$ as $h \to 0^+$ with $0 < \delta < \frac{1}{2}$. To state our theorem, we need to introduce a few geometric objects. First, consider the Hamiltonian function $p \in C^\infty(T^*M\backslash\{0\})$,

$$p(x, \xi) = |\xi|_g - 1,$$

and let $\varphi_t : T^*M \setminus 0 \to T^*M \setminus 0$ denote the Hamiltonian flow for p at time t, which coincides with the geodesic flow in this case. We also define the *maximal expansion rate* and the *Ehrenfest time* at frequency h^{-1}, respectively, as

$$\Lambda_{\max} := \limsup_{|t|\to\infty} \frac{1}{|t|} \log \sup_{S^*M} \|d\varphi_t(x,\xi)\| \quad \text{and} \quad T_e(h) := \frac{\log h^{-1}}{2\Lambda_{\max}}, \tag{1-5}$$

where $\|\cdot\|$ denotes the norm in any metric on $T(T^*M)$. Note that $\Lambda_{\max} \in [0,\infty)$, and if $\Lambda_{\max} = 0$ we may replace it by an arbitrarily small positive constant. We next describe a cover of S^*M by geodesic tubes.

For each $\rho_0 \in S^*M$, the cosphere bundle to M, let $H_{\rho_0} \subset M$ be a hypersurface such that $\rho_0 \in SN^*H_{\rho_0}$, the unit conormal bundle to H_{ρ_0}. Then, let

$$\mathcal{H}_{\rho_0} \subset T^*_{H_{\rho_0}} M = \{(x,\xi) \in T^*M : x \in H_{\rho_0}\}$$

be a hypersurface containing $SN^*H_{\rho_0}$. Next, for $q \in \mathcal{H}_{\rho_0}$ and $\tau > 0$, we define the tube through q of radius $R(h) > 0$ and "length" $\tau + R(h)$ as

$$\begin{aligned} \Lambda_q^\tau(R(h)) &:= \bigcup_{|t|\le \tau + R(h)} \varphi_t(B_{\mathcal{H}_{\rho_0}}(q, R(h))), \\ B_{\mathcal{H}_{\rho_0}}(q, R(h)) &:= \{\rho \in \mathcal{H}_{\rho_0} : d(\rho, q) \le R(h)\}, \end{aligned} \tag{1-6}$$

where d is the distance induced by the Sasaki metric on T^*M (see e.g., [Blair 2010, Chapter 9] for a description of the Sasaki metric). Note that the tube runs along the geodesic through $q \in \mathcal{H}_{\rho_0}$. Similarly, for $A \subset S^*M$, we define $\Lambda_A^\tau(R(h))$ in the same way, replacing q with A in (1-6).

Definition 1.3. Let $A \subset S^*M$, $r > 0$, and $\{\rho_j(r)\}_{j=1}^{N_r} \subset A$ for some $N_r > 0$. We say the collection of tubes $\{\Lambda_{\rho_j}^\tau(r)\}_{j=1}^{N_r}$ is a (τ, r) *cover* of a set $A \subset S^*M$ provided

$$\Lambda_A^\tau\big(\tfrac{1}{2}r\big) \subset \bigcup_{j=1}^{N_r} \mathcal{T}_j, \quad \mathcal{T}_j := \Lambda_{\rho_j}^\tau(r).$$

Given a (τ, r) cover $\{\mathcal{T}_j\}_{j\in\mathcal{J}}$ for S^*M, for each $x \in M$ we define

$$\mathcal{J}_x := \{j \in \mathcal{J} : \pi(\mathcal{T}_j) \cap B(x, r) \neq \varnothing\}.$$

We are now ready to state Theorem 1.4, where we give *explicit dynamical conditions* guaranteeing quantitative improvements in L^p norms.

Theorem 1.4. *There exists $\tau_M > 0$ such that for all $p > p_c$ and $\varepsilon_0 > 0$ the following holds. Let $U \subset M$ and $0 < \delta_1 < \delta_2 < \frac{1}{2}$, and let $h^{\delta_2} \le R(h) \le h^{\delta_1}$ for all $h > 0$. Let $1 \le T(h) \le (1 - 2\delta_2) T_e(h)$ and let $t_0 > 0$ be h-independent. Let $\{\mathcal{T}_j\}_{j\in\mathcal{J}}$ be a $(\tau, R(h))$ cover for S^*M for some $0 < \tau < \tau_M$.*

Suppose that, for any pair of points $x_1, x_2 \in U$, the tubes over x_1 can be partitioned into a disjoint union

$$\mathcal{J}_{x_1} = \mathcal{B}_{x_1,x_2} \sqcup \mathcal{G}_{x_1,x_2},$$

where

$$\bigcup_{j\in\mathcal{G}_{x_1,x_2}} \varphi_t(\mathcal{T}_j) \cap S^*_{B(x_2,R(h))} M = \varnothing, \quad t \in [t_0, T(h)].$$

Then, there are $h_0 > 0$ and $C > 0$ such that, for all $u \in \mathcal{D}'(M)$ and $0 < h < h_0$,

$$\|u\|_{L^p(U)} \le Ch^{-\delta(p)}\biggl(\frac{\sqrt{t_0}}{\sqrt{T(h)}} + \bigl[\sup_{x_1,x_2\in U} |\mathcal{B}_{x_1,x_2}|R(h)^{n-1}\bigr]^{(6+\varepsilon_0)^{-1}(1-p_c/p)}\biggr) \times \biggl(\|u\|_{L^2} + \frac{T(h)}{h}\|(-h^2\Delta_g - 1)u\|_{H_h^{(n-3)/2-n/p}}\biggr). \quad (1\text{-}7)$$

In order to interpret (1-7), note that we think of the tubes $\mathcal{G}_{x_1,x_2}$ and $\mathcal{B}_{x_1,x_2}$ as good (or nonlooping) and bad (or looping), respectively. Then, observe that

$$|\mathcal{B}_{x_1,x_2}|R(h)^{n-1} \sim \mathrm{vol}\Bigl(\bigcup_{j\in\mathcal{B}_{x_1,x_2}} \mathcal{T}_j \cap S^*_{x_1}M\Bigr)$$

and that $\bigcup_{j\in\mathcal{B}_{x_1,x_2}} \mathcal{T}_j$ is the set of directions over x_1 which may loop through x_2 in time $T(h)$. Therefore, if the volume of points in $S^*_{x_1}M$ looping through x_2 is bounded by $T(h)^{-(3+\varepsilon_0)(1-p_c/p)^{-1}}$, (1-7) provides $T(h)^{-1/2}$ improvements over the standard L^p bounds. We expect these nonlooping-type assumptions to be valid for a dense set of metrics on any smooth manifold M.

Theorem 1.4 can be used to obtain improved L^p resolvent bounds [Cuenin 2020, Theorem 2.21] which, as shown there, are stable by certain rough perturbations. These estimates in turn can be used to construct complex geometric optics solutions and solve certain inverse problems [Dos Santos Ferreira et al. 2013].

One can check using a similar argument to that in [Canzani and Galkowski 2021, Lemma 5.1 (see also Theorem 5, Section 1.5.3)] that in certain integrable situations

$$\bigl(\sup_{x_1,x_2\in U} |\mathcal{B}_{x_1,x_2}|R(h)^{n-1}\bigr)^{(6+\varepsilon_0)^{-1}(1-p_c/p)} \le \frac{C}{\sqrt{T(h)}},$$

with $T(h) \gg \log h^{-1}$ and U a nontrivial open subset of M, thus producing $o((\log h^{-1})^{-1/2})$ improvements on the L^p norms over U after an application of Theorem 1.4. One example of such an integrable system is the spherical pendulum where U can be taken to be any set that lies at a positive distance from the poles.

For other examples, where one can understand these types of good and bad tubes, we refer the reader to [Canzani and Galkowski 2023], where they are used to understand averages and L^∞ norms under various assumptions on M, including that it has Anosov geodesic flow or nonpositive curvature. Since our results do not require parametrices for the wave group, we expect that the arguments leading to Theorem 1.4 will provide *polynomial* improvements over Sogge's estimates on manifolds where Egorov-type theorems hold for longer than logarithmic times.

Note that Theorem 1.4 addresses L^p norms with $p_c < p \le \infty$, while the authors' previous work in [Canzani and Galkowski 2021] considers $p = \infty$ alone. Moreover, for $p = \infty$, the estimate in Theorem 1.4 is actually *weaker* than those in that previous work in that it requires an assumption about geodesics passing near two distinct points, while those in that previous work require only a nonrecurrent assumption on geodesics passing through a small neighborhood of a single point. This is because describing the L^p norm for $p < \infty$ requires understanding the behavior at many points simultaneously, while the L^∞ norm cares only about a single point with maximal growth.

Remark 1.5. The proofs below could be adapted to the case of quasimodes for real principal type semiclassical pseudodifferential operators of Laplace type. That is, to operators with principal symbol p satisfying both that $\partial_\xi p \neq 0$ on $\{p = 0\}$ and that $\{p = 0\} \cap T_x^*M$ has positive definite second fundamental form. This is the case, for example, for Schrödinger operators away from the forbidden region. However, for concreteness and simplicity of exposition, we have chosen to consider only the Laplace operator.

1A. ***Outline of the proof of Theorem 1.4.*** Our method for proving Theorem 1.4 differs from the standard approaches for treating L^p norms in two major ways: it hinges on adapting the geodesic beam techniques constructed by the authors [Canzani and Galkowski 2021] and on the development of a new second microlocal calculus. We now give a detailed sketch of the argument used in this proof.

To simplify the presentation in this outline, we suppose u is a Laplace eigenfunction and $U = M$, and sketch the proof of Theorem 1.4.

Step 1: We first write $u = \sum_j \chi_{\mathcal{T}_j} u$, where the $\mathcal{T}_j$ are as in Definition 1.3 and $\chi_{\mathcal{T}_j}$ is a microlocal cutoff to $\mathcal{T}_j$ which approximately commutes with $P = -h^2\Delta_g - 1$; see Section 3A. We also cover M by balls $\{B(x_\alpha, R)\}_{\alpha \in \mathcal{I}}$ such that $\mathcal{I}$ consists of a union of boundedly many collections of disjoint balls. We next organize the tubes $\mathcal{T}_j$ by the L^2 mass of $\chi_{\mathcal{T}_j} u$, writing

$$\mathcal{A}_k := \{j : 2^{-(k+1)}\|u\|_{L^2} \leq \|\chi_{\mathcal{T}_j} u\|_{L^2} \leq 2^{-k}\|u\|_{L^2}\};$$

see Section 3B. For each k, we then organize the balls $B(x_\alpha, R)$ by the L^∞ norm of $\sum_{j \in \mathcal{A}_k} \chi_{\mathcal{T}_j} u$, writing

$$\mathcal{I}_{k,m} := \left\{\alpha \in \mathcal{I} : 2^{m-k-1}\|u\|_{L^2} \leq h^{(n-1)/2} R^{(1-n)/2} \Big\| \sum_{j \in \mathcal{A}_k} \chi_{\mathcal{T}_j} u \Big\|_{L^\infty(B(x_\alpha, R))} \leq 2^{m-k}\|u\|_{L^2}\right\}; \tag{1-8}$$

see Section 3C. The reason for this choice comes from the geodesic beam estimate (see [Canzani and Galkowski 2021])

$$\Big\| \sum_{j \in \mathcal{A}_k} \chi_{\mathcal{T}_j} u \Big\|_{L^\infty(B(x_\alpha, R))} \leq C h^{(1-n)/2} R^{(n-1)/2} \sum_{j \in \mathcal{A}_k(\alpha)} \|\chi_{\mathcal{T}_j} u\|_{L^2}, \tag{1-9}$$

where $\mathcal{A}_k(\alpha)$ denotes those tubes $\mathcal{T}_j$ such that $j \in \mathcal{A}_k$ and $\mathcal{T}_j$ passes over the ball $B(x_\alpha, R)$. Because of the definition of $\mathcal{A}_k$, we have that 2^m is a lower bound for the number of tubes in $\mathcal{A}_k(\alpha)$ for $\alpha \in \mathcal{I}_{k,m}$; see (3-20).

With this bookkeeping completed, we record the estimate on the L^p norm:

$$\|u\|_{L^p} \leq C \sum_k \bigg(\sum_m \Big\| \sum_{j \in \mathcal{A}_{k,m}} \chi_{\mathcal{T}_j} u \Big\|^p_{L^p(\bigcup_{\alpha \in \mathcal{I}_{k,m}} B(x_\alpha, R))} \bigg)^{1/p}, \tag{1-10}$$

where $\mathcal{A}_{k,m} = \bigcup_{\alpha \in \mathcal{I}_{k,m}} \mathcal{A}_k(\alpha)$, i.e., those tubes in $\mathcal{A}_k$ which pass over a ball in $\mathcal{I}_{k,m}$.

Step 2: We control each L^p norm in (1-10) by using interpolation between the L^∞ estimate analogous to (1-9) and the standard L^{p_c} estimate:

$$\Big\| \sum_{j \in \mathcal{A}_{k,m}} \chi_{\mathcal{T}_j} u \Big\|_{L^{p_c}} \leq C h^{-1/p_c} \Big\| \sum_{j \in \mathcal{A}_{k,m}} \chi_{\mathcal{T}_j} u \Big\|_{L^2} \leq C h^{-1/p_c} 2^{-k} |\mathcal{A}_{k,m}|^{1/2} \|u\|_{L^2}.$$

In Section 3D, we start by handling the "easy" piece where the L^∞ norm is smaller than $T(h)^{-N}h^{-(n-1)/2}$ for some very large N. This piece can be neglected since the standard interpolation estimate shows that it has L^p norm $\ll h^{-\delta(p)}/\sqrt{T(h)}\|u\|_{L^2}$.

Next, in Section 3E, we write $\mathcal{A}_{k,m} = \mathcal{G}_{k,m} \sqcup \mathcal{B}_{k,m}$, where $\bigcup_{j\in\mathcal{G}_{k,m}} \mathcal{T}_j$ is non-self-looping in the sense that

$$\bigcup_{t\in[t_0,T(h)]} \varphi_t\big(\bigcup_{j\in\mathcal{G}_{k,m}}\mathcal{T}_j\big) \bigcap \bigcup_{j\in\mathcal{G}_{k,m}}\mathcal{T}_j = \varnothing.$$

Using non-self-looping estimates from [Canzani and Galkowski 2021] (see also Lemma 3.6) and summing carefully, we are able to show that

$$C\sum_k\Bigg(\sum_m\bigg\|\sum_{j\in\mathcal{G}_{k,m}}\chi_{\mathcal{T}_j}u\bigg\|^p_{L^p(\bigcup_{\alpha\in\mathcal{I}_{k,m}}B(x_\alpha,R))}\Bigg)^{1/p} \le \frac{h^{-\delta(p)}}{\sqrt{T(h)}}\|u\|_{L^2}.$$

This is done in Section 3E2.

Our final task is to estimate the sum over the bad tubes. For this, we again use the geodesic beam estimate to control the L^∞ norm of $\sum_{j\in\mathcal{B}_{k,m}}\chi_{\mathcal{T}_j}u$ by the maximal number, $|\mathcal{B}^{\max}_{k,m}|$, of "bad" tubes passing over a ball $B(x_\alpha, R)$ with $\alpha\in\mathcal{I}_{k,m}$. In addition, we control the L^2 norm of this sum by $|\mathcal{B}_{k,m}|^{1/2}2^{-k}$. The numbers of "bad" tubes are estimated in the next step.

Step 3: We first estimate $|\mathcal{B}^{\max}_{k,m}|$ using the dynamical hypothesis. In particular, we check that

$$|\mathcal{B}^{\max}_{k,m}| \le |\mathcal{I}_{k,m}|\,|\mathcal{B}_{x_1,x_2}|.$$

This estimate comes from imagining the worst case scenario that every tube connecting some ball $B(x_\alpha, R)$ with $\alpha\in\mathcal{I}_{k,m}$ to another ball $B(x_\beta, R)$ with $\beta\in\mathcal{I}_{k,m}$ lies in $\mathcal{A}_k$ and that no such tube connects $B(x_\alpha, R)$ to $B(x_\beta, R)$ and $B(x_{\beta'}, R)$ for $\beta\ne\beta'$; see (3-46). Using a similar argument, we can see that

$$|\mathcal{B}_{k,m}| \le |\mathcal{I}_{k,m}|^2 \sup_{x_1,x_2}|\mathcal{B}_{x_1,x_2}|;$$

see (3-39). Thus, it remains only to estimate $|\mathcal{I}_{k,m}|$.

Step 4: To estimate the size of $\mathcal{I}_{k,m}$, we need to estimate the number of balls on which the combination of beams $w_{k,m} := \sum_{j\in\mathcal{A}_{k,m}}\chi_{\mathcal{T}_j}u$ with L^2 mass 2^{-k} has L^∞ norm $2^{m-k}R^{(n-1)/2}h^{(1-n)/2}\|u\|_{L^2}$.

To do this, we aim to understand both the minimal amount of L^2 mass needed for an eigenfunction to have a certain (large) L^∞ norm and where that mass must be located in phase space. The standard Hörmander-type L^∞ bound (as presented in [Koch et al. 2007]) answers the first question: for $x\in M$,

$$h^{(n-1)/2}|w(x)| \le C(\|w\|_{L^2} + h^{-1}\|Pw\|_{L^2}). \tag{1-11}$$

To answer the second question, we need to understand to what extent this inequality can be microlocalized. Because of the invariance of eigenfunctions under the geodesic flow we localize to the coisotropic submanifolds

$$\Gamma_x := \bigcup_{|t|\le 1}\varphi_t(T^*_xM). \tag{1-12}$$

We want three properties for X_{Γ_x}, our localizers to Γ_x; see (3-25) for the precise requirements and Theorem 6.3 for their construction. First, they should localize tightly (h^ρ with $\rho \sim 1$) to Γ_x. Second, they must nearly maintain the value of a function at x:

$$w(x) = (X_{\Gamma_x} w)(x) + O(h^\infty). \tag{1-13}$$

Third, they must preserve quasimodes for P so that, using the inequality (1-11), we have

$$h^{(n-1)/2}|(X_{\Gamma_x} w_{k,m})(x)| \le C\|X_{\Gamma_x} w_{k,m}\|_{L^2}. \tag{1-14}$$

Thus, from (1-13) and (1-14) it follows that, for $\alpha \in \mathcal{I}_{k,m}$, there is $\tilde{x}_\alpha \in B(x_\alpha, R)$ with

$$R^{(n-1)/2} 2^{m-k} \|u\|_{L^2} \le h^{(n-1)/2}\|X_{\Gamma_{\tilde{x}_\alpha}} w_{k,m}\|_{L^\infty(B(x_\alpha,R))} \le \|X_{\Gamma_{\tilde{x}_\alpha}} w_{k,m}\|_{L^2}. \tag{1-15}$$

Note that we use Γ_x as defined above, as opposed to the flowout of $S^*_x M$, precisely so that (1-13) is possible.

Finally, we will bound $|\mathcal{I}_{k,m}|$ by summing (1-15) over all balls in $\mathcal{I}_{k,m}$ to obtain

$$R^{n-1} 2^{2(m-k)} |\mathcal{I}_{k,m}| \le \sum_{\alpha \in \mathcal{I}_{k,m}} \|X_{\Gamma_{\tilde{x}_\alpha}} w_{k,m}\|^2_{L^2}. \tag{1-16}$$

We produce an upper bound on (1-16) of the form

$$\sum_{\alpha \in \mathcal{I}_{k,m}} \|X_{\Gamma_{\tilde{x}_\alpha}} w_{k,m}\|^2_{L^2} \le \|w_{k,m}\|^2_{L^2}. \tag{1-17}$$

This follows from Proposition 6.6 (see the analysis leading to (3-31)) and controls the minimal L^2 mass necessary for $w_{k,m}$ to have a large value at *all* the points in $\mathcal{I}_{k,m}$. We view this estimate as an uncertainty principle type of result in which we prove that, for $\tilde{x}_\alpha \ne \tilde{x}_\beta$, localization to $\Gamma_{\tilde{x}_\alpha}$ and $\Gamma_{\tilde{x}_\beta}$ are incompatible in the sense that

$$\|X_{\Gamma_{\tilde{x}_\alpha}} X_{\Gamma_{\tilde{x}_\beta}}\|_{L^2 \to L^2} \ll 1, \tag{1-18}$$

with uniform estimates in $d(\tilde{x}_\alpha, \tilde{x}_\beta)$. Combining (1-16) with (1-17) yields the bound needed on $|\mathcal{I}_{k,m}|$ to finish the analysis in the proof of Theorem 1.4. This is done in Section 3E1.

Remark 1.6 (uncertainty principle). Note that, if the function $w_{k,m}$ could be localized simultaneously on all the manifolds $\Gamma_{\tilde{x}_\alpha}$, then we would have

$$\sum_{\alpha \in \mathcal{I}_{k,m}} \|X_{\Gamma_{\tilde{x}_\alpha}} w_{k,m}\|^2_{L^2} \ge c|\mathcal{I}_{k,m}|\|w_{k,m}\|^2_{L^2} \gg \|w_{k,m}\|^2_{L^2}.$$

This contradicts (1-17). Hence, if one more carefully quantifies this argument by assigning weights to the localized masses $\|X_{\Gamma_{\tilde{x}_\alpha}} w_{k,m}\|_{L^2}$, we can understand how much of the L^2 mass of $w_{k,m}$ can be localized to many $\Gamma_{\tilde{x}_\alpha}$. This is a type of uncertainty principle. Since (1-15) shows that $\Gamma_{\tilde{x}_\alpha}$ must carry mass in order for $w_{k,m}(\tilde{x}_\alpha)$ to be large, this can be thought of as an estimate on how much a "single unit" of L^2 mass can be used to produce a large L^∞ norm at multiple points.

Remark 1.7 (zonal harmonics). Another way to think of the estimate (1-18) is on the round sphere S^2, where the natural enemy is a zonal harmonic Z_x at a point $x \in S^2$. Recall that the zonal harmonic Z_x is localized h close to Γ_x, in the sense that in a fixed size neighborhood of x,

$$X_{\Gamma_x} Z_x = Z_x + O(h^\infty).$$

The estimate (1-18), or more precisely Corollary 6.5, can be used to give lower bounds on

$$\Big\| \sum_{x_\alpha \in \mathcal{I}} Z_{x_\alpha} \Big\|_{L^2}^2 = \sum_{x_\alpha \in \mathcal{I}} \|Z_{x_\alpha}\|_{L^2}^2 + \sum_{x_\alpha \neq x_\beta} \langle X^*_{\Gamma_{x_\beta}} X_{\Gamma_{x_\alpha}} Z_{x_\alpha}, Z_{x_\beta} \rangle_{L^2},$$

where $d(x_\alpha, x_\beta) > R$ for $\alpha \neq \beta$. Equation (1-18) shows that, for $\alpha \neq \beta$,

$$\|X^*_{\Gamma_{x_\beta}} X_{\Gamma_{x_\alpha}}\| \ll 1$$

and hence quantifies the amount of cancellation in such a sum. This cancellation is easy to see with $d(x_\alpha, x_\beta) > c > 0$, but becomes much more subtle when this distance is small.

Remark 1.8 (second microlocal calculus). In order to build the localizers X_{Γ_x} satisfying (1-13) and (1-14), we develop a new second microlocal calculus associated to a Lagrangian foliation L over a coisotropic submanifold $\Gamma \subset T^*M$. In the case of the Γ_x defined in (1-12), the leaves of L will be given by $\varphi_t(T^*_x M)$ for a fixed time t. The calculus allows for simultaneous h^ρ localization (with ρ close to 1) along a leaf of L and along Γ. Because of this and the fact that $T^*_x M$ is one such leaf, we can find localizers with the property (1-13). We note that other works on L^p norms, especially [Blair and Sogge 2015b; 2017], use localizers to $h^{1/2}$ neighborhoods of geodesic segments. However, when two cutoffs X_1 and X_2 localizing at scale $h^{1/2}$ have overlapping support, we always have

$$\|X_1 X_2\|_{L^2 \to L^2} \sim 1,$$

and hence (1-18) does not hold. Therefore, in our framework it is necessary to localize in some directions at scales below $h^{1/2}$ and hence to develop a special calculus associated to the pairs (L, Γ). The calculus, which is developed in Section 5, can be seen as an interpolation between those in [Dyatlov and Zahl 2016; Sjöstrand and Zworski 1999].

Outline of the paper. In Section 2, we construct the covers of S^*M and T^*M consisting of tubes and balls, respectively, which are necessary in the rest of the article. Section 3 contains the proof of Theorems 1.4 and 3.8. This proof uses the anisotropic calculus developed in Section 5 and the almost-orthogonality results from Section 6. Section 4 contains the necessary dynamical arguments to prove Theorem 1.1 using Theorem 1.4.

2. Tube lemmas

The next few lemmas are aimed at constructing (τ, r)-good covers and partitions of various subsets of T^*M; see also [Canzani and Galkowski 2021, Section 3.2]. Before we proceed, we recall the symbol classes $S^m_\delta(T^*M)$; see also, e.g., [Zworski 2012, Chapters 4, 9]. We say that $a \in C^\infty(T^*M)$ is in $S^m_\delta(T^*M)$ if, for all $\alpha, \beta \in \mathbb{N}^d$, there is $C_{\alpha\beta} > 0$ such that, for $0 < h < 1$,

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha\beta} h^{-\delta(|\alpha|+|\beta|)} \langle\xi\rangle^{m-|\beta|}, \quad \langle\xi\rangle := (1+|\xi|^2)^{1/2}.$$

We sometimes write $S_\delta(T^*M) = S^0_\delta(T^*M)$, and we write $a \in S^m_\delta(T^*M; A)$ if $a \in C^\infty(T^*M; A)$ is also in $S^m_\delta(T^*M)$.

Definition 2.1 (good covers and partitions). Let $A \subset T^*M$, $r > 0$, and $\{\rho_j(r)\}_{j=1}^{N_r} \subset A$ be a collection of points for some $N_r > 0$. Let $\mathfrak{D}$ be a positive integer. We say that the collection of tubes $\{\Lambda^\tau_{\rho_j}(r)\}_{j=1}^{N_r}$ is a $(\mathfrak{D}, \tau, r)$-*good cover* of $A \subset T^*M$ provided it is a (τ, r) cover of A and there exists a partition $\{\mathcal{J}_\ell\}_{\ell=1}^{\mathfrak{D}}$ of $\{1, \dots, N_r\}$ such that for every $\ell \in \{1, \dots, \mathfrak{D}\}$,

$$\Lambda^\tau_{\rho_j}(3r) \cap \Lambda^\tau_{\rho_i}(3r) = \varnothing, \quad i, j \in \mathcal{J}_\ell, \ \ i \ne j.$$

In addition, for $0 \le \delta \le \frac{1}{2}$ and $R(h) \ge 8h^\delta$, we say that a collection $\{\chi_j\}_{j=1}^{N_h} \subset S_\delta(T^*M; [0, 1])$ is a δ-*good partition for A associated to a* $(\mathfrak{D}, \tau, R(h))$-*good cover* if $\{\chi_j\}_{j=1}^{N_h}$ is bounded in S_δ and

$$\operatorname{supp} \chi_j \subset \Lambda^\tau_{\rho_j}(R(h)) \quad \text{and} \quad \sum_{j=1}^{N_h} \chi_j \ge 1 \quad \text{on } \Lambda_A^{\tau/2}\big(\tfrac{1}{2}R(h)\big).$$

Remark 2.2. We show below that for any compact Riemannian manifold M, there are $\mathfrak{D}_M, R_0, \tau_0 > 0$, depending only on (M, g), such that, for $0 < \tau < \tau_0$ and $0 < r < R_0$, there exists a $(\mathfrak{D}_M, \tau, r)$-good cover for S^*M.

We start by constructing a useful cover of any Riemannian manifold with bounded curvature.

Lemma 2.3. *Let $\widetilde{M}$ be a compact Riemannian manifold. There exist $\mathfrak{D}_n > 0$, depending only on n, and $R_0 > 0$, depending only on n and a lower bound for the sectional curvature of $\widetilde{M}$, so that the following holds: for $0 < r < R_0$, there exists a finite collection of points $\{x_\alpha\}_{\alpha\in\mathcal{I}} \subset \widetilde{M}$, $\mathcal{I} = \{1, \dots, N_r\}$, and a partition $\{\mathcal{I}_i\}_{i=1}^{\mathfrak{D}_n}$ of $\mathcal{I}$ such that*

$$\widetilde{M} \subset \bigcup_{\alpha\in\mathcal{I}} B(x_\alpha, r), \quad B(x_{\alpha_1}, 3r) \cap B(x_{\alpha_2}, 3r) = \varnothing \quad \textit{for } \alpha_1, \alpha_2 \in \mathcal{I}_i, \ \ \alpha_1 \ne \alpha_2,$$

$$\{x_\alpha\}_{\alpha\in\mathcal{I}} \text{ is a maximal } \tfrac{1}{2}r\text{-separated set in } \widetilde{M}.$$

Proof. Let $\{x_\alpha\}_{\alpha\in\mathcal{I}}$ be a maximal $\frac{1}{2}r$-separated set in $\widetilde{M}$. Fix $\alpha_0 \in \mathcal{I}$ and suppose $B(x_{\alpha_0}, 3r) \cap B(x_\alpha, 3r) \ne \varnothing$ for all $\alpha \in \mathcal{K}_{\alpha_0} \subset \mathcal{I}$. Then, for all $\alpha \in \mathcal{K}_{\alpha_0}$, we have $B\big(x_\alpha, \frac{1}{2}r\big) \subset B(x_{\alpha_0}, 8r)$. In particular,

$$\sum_{\alpha\in\mathcal{K}_{\alpha_0}} \operatorname{vol}\big(B\big(x_\alpha, \tfrac{1}{2}r\big)\big) \le \operatorname{vol}(B(x_{\alpha_0}, 8r)).$$

Now, there exist $R_0 > 0$, depending on n and a lower bound on the sectional curvature of $\widetilde{M}$, and $\mathfrak{D}_n > 0$, depending only on n, such that, for all $0 < r < R_0$,

$$\operatorname{vol}(B(x_{\alpha_0}, 8r)) \le \operatorname{vol}(B(x_\alpha, 14r)) \le \mathfrak{D}_n \operatorname{vol}\big(B\big(x_\alpha, \tfrac{1}{2}r\big)\big). \tag{2-1}$$

Hence, it follows from (2-1) that

$$\sum_{\alpha\in\mathcal{K}_{\alpha_0}} \mathrm{vol}\big(B\big(x_\alpha, \tfrac{1}{2}r\big)\big) \le \mathrm{vol}(B(\rho_{\alpha_0}, 8r)) \le \frac{\mathfrak{D}_n}{|\mathcal{K}_{\alpha_0}|} \sum_{\alpha\in\mathcal{K}_{\alpha_0}} \mathrm{vol}\big(B\big(x_\alpha, \tfrac{1}{2}r\big)\big).$$

In particular, $|\mathcal{K}_{\alpha_0}| \le \mathfrak{D}_n$.

At this point we have proved that each of the balls $B(x_\alpha, 3r)$ intersects at most $\mathfrak{D}_n - 1$ other balls. We now construct the sets $\mathcal{I}_1, \dots, \mathcal{I}_{\mathfrak{D}_n}$ using a greedy algorithm. We will say that the index α_1 *intersects* the index α_2 if

$$B(x_{\alpha_1}, 3r) \cap B(x_{\alpha_2}, 3r) \ne \varnothing.$$

We place the index $1 \in \mathcal{I}_1$. Then suppose we have placed the indices $\{1, \dots, \alpha\}$ in $\mathcal{I}_1, \dots, \mathcal{I}_{\mathfrak{D}_n}$ so each of the $\mathcal{I}_i$ consists of disjoint indices. Then, since $\alpha + 1$ intersects at most $\mathfrak{D}_n - 1$ indices, it is disjoint from $\mathcal{I}_i$ for some i. We add the index α to $\mathcal{I}_i$. By induction we obtain the partition $\mathcal{I}_1, \dots, \mathcal{I}_{\mathfrak{D}_n}$.

Now, suppose that there exists $x \in \widetilde{M}$ such that $x \notin \bigcup_{\alpha\in\mathcal{I}} B(x_\alpha, r)$. Then, $\min_{\alpha\in\mathcal{I}} d(x, x_\alpha) \ge r$, a contradiction of the $\frac{1}{2}r$ maximality of x_α. ☐

In order to construct our microlocal partition, we first fix a smooth hypersurface $H \subset M$, and choose Fermi normal coordinates $x = (x_1, x')$ in a neighborhood of $H = \{x_1 = 0\}$. We write $(\xi_1, \xi') \in T_x^*M$ for the dual coordinates. Let

$$\Sigma_H := \big\{(x, \xi) \in S_H^*M \;\big|\; |\xi_1| \ge \tfrac{1}{2}\big\}. \tag{2-2}$$

We then consider

$$\mathcal{H}_{\Sigma_H} := \big\{(x, \xi) \in T_H^*M \;\big|\; |\xi_1| \ge \tfrac{1}{2},\ \tfrac{1}{2} < |\xi|_{g(x)} < \tfrac{3}{2}\big\}. \tag{2-3}$$

Then $\mathcal{H}_{\Sigma_H}$ is transverse to the geodesic flow and there is $0 < \tau_{\mathrm{inj}H} < 1$ such that the map

$$\Psi : [-\tau_{\mathrm{inj}H}, \tau_{\mathrm{inj}H}] \times \mathcal{H}_{\Sigma_H} \to T^*M, \quad \Psi(t, \rho) := \varphi_t(\rho), \tag{2-4}$$

is injective. Our next lemma shows that there is $\mathfrak{D}_n > 0$ depending only on n such that one can construct a $(\mathfrak{D}_n, \tau, r)$-good cover of Σ_H.

Lemma 2.4. *There exist $\mathfrak{D}_n > 0$ depending only on n and $R_0 = R_0(n, H) > 0$ such that, for $0 < r_1 < R_0$, $0 < r_0 \le \frac{1}{2}r_1$, there exist points $\{\rho_j\}_{j=1}^{N_{r_1}} \subset \Sigma_H$ and a partition $\{\mathcal{J}_i\}_{i=1}^{\mathfrak{D}_n}$ of $\{1, \dots, N_{r_1}\}$ such that, for all $0 < \tau < \frac{1}{2}\tau_{\mathrm{inj}H}$,*

$$\Lambda^\tau_{\Sigma_H}(r_0) \subset \bigcup_{j=1}^{N_{r_1}} \Lambda^\tau_{\rho_j}(r_1), \qquad \Lambda^\tau_{\rho_j}(3r_1) \cap \Lambda^\tau_{\rho_\ell}(3r_1) = \varnothing, \quad \textit{for } j, \ell \in \mathcal{J}_i,\ \ j \ne \ell.$$

Proof. We first apply Lemma 2.3 to $\widetilde{M} = \Sigma_H$ to obtain $R_0 > 0$ depending only on n and the sectional curvature of H and that of M near H such that, for $0 < r_1 < R_0$, there exist $\{\rho_j\}_{j=1}^{N_{r_1}} \subset \Sigma_H$ and a partition

$\{\mathcal{J}_i\}_{i=1}^{\mathfrak{D}_n}$ of $\{1, \dots, N_{r_1}\}$ such that

$$\Sigma_H \subset \bigcup_{j=1}^{N_{r_1}} B(\rho_j, r_1), \quad B(\rho_j, 3r_1) \cap B(\rho_\ell, 3r_1) = \varnothing \quad \text{for } j, \ell \in \mathcal{J}_i, \quad j \neq \ell,$$

$$\{\rho_j\}_{j=1}^{N_{r_1}} \text{ is a maximal } \tfrac{1}{2}r_1\text{-separated set in } \Sigma_H.$$

Now, suppose that $j, \ell \in \mathcal{J}_i$ and

$$\Lambda^\tau_{\rho_\ell}(3r_1) \cap \Lambda^\tau_{\rho_j}(3r_1) \neq \varnothing.$$

Then, there exist

$$q_\ell \in B(\rho_\ell, 3r_1) \cap \mathcal{H}_{\Sigma_H}, \quad q_j \in B(\rho_j, 3r_1) \cap \mathcal{H}_{\Sigma_H},$$

and $t_\ell, t_j \in [-\tau, \tau]$ such that $\varphi_{t_\ell - t_j}(q_\ell) = q_j$. Here, $\mathcal{H}_\Sigma$ is the hypersurface defined in (2-3). In particular, for $\tau < \frac{1}{2}\tau_{\text{inj}H}$, this implies that $q_\ell = q_j$, $t_\ell = t_j$, and hence $B(\rho_\ell, 3r_1) \cap B(\rho_j, 3r_1) \neq \varnothing$, a contradiction.

Now, suppose $r_0 \leq r_1$ and that there exists $\rho \in \Lambda^\tau_{\Sigma_H}(r_0)$ so that $\rho \notin \bigcup_{j=1,\dots,N_{r_1}} \Lambda^\tau_{\rho_j}(r_1)$. Then, there are $|t| < \tau + r_0$ and $q \in \mathcal{H}_{\Sigma_H}$ such that

$$\rho = \varphi_t(q), \quad d(q, \Sigma_H) < r_0, \quad \min_{j=1,\dots,N_{r_1}} d(q, \rho_j) \geq r_1.$$

In particular, there exists $\tilde{\rho} \in \Sigma_H$ with $d(q, \tilde{\rho}) < r_0$ such that for all $j = 1, \dots, N_{r_1}$,

$$d(\tilde{\rho}, \rho_j) \geq d(q, \rho_j) - d(q, \tilde{\rho}) > r_1 - r_0.$$

This contradicts the maximality of $\{\rho_j\}_{j=1}^{N_{r_1}}$ if $r_0 \leq \frac{1}{2}r_1$. □

We proceed to build a δ-good partition of unity associated to the cover we constructed in Lemma 2.4. The key feature in this partition is that it is invariant under the geodesic flow. Indeed, the partition is built so that its quantization commutes with the operator $P = -h^2\Delta - I$ in a neighborhood of Σ_H.

Proposition 2.5. *There exist $\tau_1 = \tau_1(\tau_{\text{inj}H}) > 0$ and $\varepsilon_1 = \varepsilon_1(\tau_1) > 0$, and given $0 < \delta < \frac{1}{2}$ and $0 < \varepsilon \leq \varepsilon_1$ there exists $h_1 > 0$ such that, for any $0 < \tau \leq \tau_1$ and $R(h) \geq 2h^\delta$, the following holds.*

There exist $C_1 > 0$ such that for all $0 < h \leq h_1$ and every $(\tau, R(h))$ cover of Σ_H there exists a partition of unity

$$\chi_j \in S_\delta \cap C_c^\infty(T^*M; [-C_1 h^{1-2\delta}, 1 + C_1 h^{1-2\delta}])$$

on $\Lambda^\tau_{\Sigma_H}\big(\frac{1}{2}R(h)\big)$ for which

$$\operatorname{supp} \chi_j \subset \Lambda^{\tau+\varepsilon}_{\rho_j}(R(h)), \quad \operatorname{MS}_h([P, \operatorname{Op}_h(\chi_j)]) \cap \Lambda^\tau_{\Sigma_H}(\varepsilon) = \varnothing, \quad \sum_j \chi_j \equiv 1 \ \text{ on } \Lambda^\tau_{\Sigma_H}\big(\tfrac{1}{2}R(h)\big),$$

$\{\chi_j\}_j$ is bounded in S_δ, and $[-h^2\Delta_g, \operatorname{Op}_h(\chi_j)]$ is bounded in Ψ_δ.

Proof. The proof is identical to that of [Canzani and Galkowski 2021, Proposition 3.4]. Although the claim that $\sum_j \chi_j \equiv 1$ on $\Lambda^\tau_{\Sigma_H}\big(\frac{1}{2}R(h)\big)$ does not appear in its statement, it is included in its proof. □

3. Proof of Theorem 1.4

For each $q \in S^*M$, choose a hypersurface $H_q \subset M$ with $q \in SN^*H_q$ and $\tau_{\mathrm{inj}\, H_q} > \frac{1}{2}\,\mathrm{inj}(M)$, where $\tau_{\mathrm{inj}\, H_q}$ is defined in (2-4) and $\mathrm{inj}(M)$ is the injectivity radius of M. We next use Lemma 2.4 to generate a cover of Σ_{H_q}. Lemma 2.4 yields the existence of $\mathfrak{D}_n > 0$ depending only on n and $R_0 = R_0(n, H_q) > 0$ such that the following holds: Since by assumption $R(h) \le h^{\delta_1}$, there is $h_0 > 0$ such that $h^{\delta_2} \le R(h) \le R_0$ for all $0 < h < h_0$. Also, set $r_1 := R(h)$ and $r_0 := \frac{1}{2}R(h)$. Then, by Lemma 2.4 there exist

$$N_{R(h)} = N_{R(h)}(q, R(h)) > 0, \quad \{\rho_j\}_{j\in\mathcal{J}_q} \subset \Sigma_{H_q} with \mathcal{J}_q = \{1, \dots, N_{R(h)}\},$$

and a partition $\{\mathcal{J}_{q,i}\}_{i=1}^{\mathfrak{D}_n}$ of $\mathcal{J}_q$, such that, for all $0 < \tau < \frac{1}{2}\tau_{\mathrm{inj} H_q}$,

$$\Lambda^\tau_{\Sigma_{H_q}}\big(\tfrac{1}{2}R(h)\big) \subset \bigcup_{j\in\mathcal{J}_q} \Lambda^\tau_{\rho_j}(R(h)), \tag{3-1}$$

$$\bigcup_{i=1}^{\mathfrak{D}_n} \mathcal{J}_{q,i} = \mathcal{J}_q, \tag{3-2}$$

$$\Lambda^\tau_{\rho_{j_1}}(3R(h)) \cap \Lambda^\tau_{\rho_{j_2}}(3R(h)) = \varnothing \quad \text{for } j_1, j_2 \in \mathcal{J}_{q,i}, \quad j_1 \neq j_2. \tag{3-3}$$

By (3-1) there is an h-independent open neighborhood of q, $V_q \subset S^*M$, covered by tubes as in Lemma 2.4. Since S^*M is compact, we may choose $\{q_\ell\}_{\ell=1}^L$ with L independent of h such that $S^*M \subset \bigcup_{\ell=1}^L V_{q_\ell}$. In particular, if $0 < \tau \le \min_{1\le\ell\le L} \tau_{H_{q_\ell}}$ and for each $\ell \in \{1, \dots, L\}$ we let

$$\mathcal{T}_{q_\ell, j} = \Lambda^\tau_{\rho_j}(R(h)),$$

then there is $\mathfrak{D}_M > 0$ such that

$$\bigcup_{\ell=1}^{L} \{\mathcal{T}_{q_\ell,j}\}_{j\in\mathcal{J}_{q_\ell}}$$

is a $(\mathfrak{D}_M, \tau, R(h))$-good cover for S^*M. Let $\{\psi_{q_\ell}\}_{\ell=1}^L \subset C_c^\infty(T^*M)$ satisfy

$$\operatorname{supp} \psi_{q_\ell} \subset \left\{ (x,\xi) \in T^*M \setminus \{0\} \;\middle|\; \left(x, \frac{\xi}{|\xi|_g}\right) \in V_{q_\ell} \right\} \quad \text{for all } \ell = 1, \dots, L,$$

$$\sum_{\ell=1}^{L} \psi_{q_\ell} \equiv 1 \text{ in an } h\text{-independent neighborhood of } S^*M.$$

We split the analysis of u in two parts: near and away from the characteristic variety $\{p = 0\} = S^*M$. In what follows we use C to denote a positive constant that may change from line to line.

3A. ***It suffices to study u near the characteristic variety.*** In this section we reduce the study of $\|u\|_{L^p(U)}$ to an h-dependent neighborhood of the characteristic variety $\{p = 0\} = S^*M$. We will use repeatedly the following result.

Lemma 3.1. *For all $\varepsilon > 0$ and all $p \ge 2$, there exists $C > 0$ such that*

$$\|u\|_{L^p} \le C h^{n(1/p-1/2)} \|u\|_{H_h^{n(1/2-1/p)+\varepsilon}}. \tag{3-4}$$

Proof. By [Galkowski 2019, Lemma 6.1] (or more precisely its proof), for any $\varepsilon > 0$, there exists $C_\varepsilon \geq 1$ so that $\|\mathrm{Id}\|_{H_h^{n/2+\varepsilon} \to L^\infty} \leq C_\varepsilon h^{-n/2}$. By complex interpolation of $\mathrm{Id} : L^2 \to L^2$ and $\mathrm{Id} : H_h^{n/2+\varepsilon} \to L^\infty$ with $\theta = 2/p$, we obtain $\|\mathrm{Id}\|_{H_h{}^{(n/2+\varepsilon)(1-\theta)} \to L^p} \leq C_\varepsilon^{1-\theta} h^{-n(1-\theta)/2}$, and this yields (3-4). □

Observe that

$$u = \sum_{\ell=1}^{L} \mathrm{Op}_h(\psi_{q_\ell})u + \biggl(1 - \sum_{\ell=1}^{L} \mathrm{Op}_h(\psi_{q_\ell})\biggr)u.$$

Since $1 - \sum_{\ell=1}^{L} \psi_{q_\ell} = 0$ in an h-independent neighborhood of $S^*M = \{p = 0\}$, by the standard elliptic parametrix construction (e.g., [Dyatlov and Zworski 2019, Appendix E]) there is $E \in \Psi^{-2}(M)$ with

$$1 - \sum_{\ell=1}^{L} \mathrm{Op}_h(\psi_{q_\ell}) = EP + O(h^\infty)_{\Psi^{-\infty}}. \tag{3-5}$$

Next, combining (3-5) with Lemma 3.1 and using that $h^{n(1/p-1/2)} = h^{-\delta(p)+1/2}h^{-1}$, we have

$$\begin{aligned}\biggl\|\biggl(1 - \sum_{\ell=1}^{L} \mathrm{Op}_h(\psi_{q_\ell})\biggr)u\biggr\|_{L^p} &\leq Ch^{n(1/p-1/2)}\|EPu\|_{H_h^{n(1/2-1/p)+\varepsilon}} + O(h^\infty)\|u\|_{L^2} \\ &\leq Ch^{-\delta(p)+1/2}h^{-1}\|Pu\|_{H_h^{n(1/2-1/p)+\varepsilon-2}} + O(h^\infty)\|u\|_{L^2}. \end{aligned}\tag{3-6}$$

It remains to understand the terms $\mathrm{Op}_h(\psi_{q_\ell})u$. Since there are finitely many such terms,

$$\biggl\|\sum_{\ell=1}^{L} \mathrm{Op}_h(\psi_{q_\ell})u\biggr\|_{L^p} \leq \sum_{\ell=1}^{L} \|\mathrm{Op}_h(\psi_{q_\ell})u\|_{L^p}, \tag{3-7}$$

and we consider each term $\|\mathrm{Op}_h(\psi_{q_\ell})u\|_{L^p}$ individually.

By Proposition 2.5, for each $\ell = 1, \dots, L$, there exist $\tau_1(q_\ell) > 0$ and $\varepsilon_1(q_\ell) > 0$ and a family of cutoffs $\{\tilde{\chi}_{\mathcal{T}_{q_\ell,j}}\}_{j \in \mathcal{J}_{q_\ell}}$ with $\tilde{\chi}_{\mathcal{T}_{q_\ell,j}}$ supported in $\Lambda_{\rho_j}^{\tau+\varepsilon_1(q_\ell)}(R(h))$ such that, for all $0 < \tau < \tau_1(q_\ell)$,

$$\sum_{j \in \mathcal{J}_{q_\ell}} \tilde{\chi}_{\mathcal{T}_{q_\ell,j}} \equiv 1 \quad \text{on } \Lambda^\tau_{\Sigma_{H_{q_\ell}}}\bigl(\tfrac{1}{2}R(h)\bigr). \tag{3-8}$$

Let $\tau_0(q_\ell)$ be as in [Canzani and Galkowski 2021, Theorem 10]. Then, set

$$\tau_M := \min_{1 \leq \ell \leq L}\bigl\{\tfrac{1}{4}\,\mathrm{inj}(M),\ \tau_0(q_\ell),\ \tau_1(q_\ell),\ \tfrac{1}{2}\tau_{\mathrm{inj}\,H_{q_\ell}}\bigr\}.$$

From now on we work with tubes $\mathcal{T}_{q_\ell,j} = \Lambda^\tau_{\rho_j}(R(h))$ for some $0 < \tau < \tau_M$. Next, we localize u near and away from $\Lambda^\tau_{\Sigma_{H_{q_\ell}}}(h^\delta)$:

$$\mathrm{Op}_h(\psi_{q_\ell})u = \sum_{j \in \mathcal{J}_{q_\ell}} \mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_{q_\ell,j}})\,\mathrm{Op}_h(\psi_{q_\ell})u + \biggl(1 - \sum_{j \in \mathcal{J}_{q_\ell}} \mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_{q_\ell,j}})\biggr)\mathrm{Op}_h(\psi_{q_\ell})u.$$

Remark 3.2. We refer to functions of the form $\mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_{q_\ell,j}})u$ as *geodesic beams*. One can check using Proposition 2.5 that if u solves $Pu = O(h)_{L^2}$, then the geodesic beams solve

$$P\mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_{q_\ell,j}})u = O(h)_{H_h^k}$$

for any k and are localized to an $R(h)$ neighborhood of a length ~ 1 segment of a geodesic.

In particular, by (3-8), $\frac{1}{2}R(h) \geq \frac{1}{2}h^{\delta_2}$, and [Canzani and Galkowski 2021, Lemma 3.6], there is $E \in h^{-\delta_2}\Psi^{\mathrm{comp}}_{\delta_2}$ so that

$$\Big(1 - \sum_{j \in \mathcal{J}_{q_\ell}} \mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_{q_\ell,j}})\Big) \mathrm{Op}_h(\psi_{q_\ell}) = EP + O_{\Psi^{-\infty}}(h^\infty). \tag{3-9}$$

Since $h^{n(1/p-1/2)-\delta_2} = h^{-\delta(p)+1/2-\delta_2}h^{-1}$, combining (3-9) with Lemma 3.1 yields

$$\Big\| \Big(1 - \sum_{j \in \mathcal{J}_{q_\ell}} \mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_{q_\ell,j}})\Big) \mathrm{Op}_h(\psi_{q_\ell})u \Big\|_{L^p} \leq Ch^{-\delta(p)-1/2-\delta_2}\|Pu\|_{H_h^{n(1/2-1/p)+\varepsilon-2}} + O(h^\infty)\|u\|_{L^2}. \tag{3-10}$$

Combining (3-6), (3-7), and (3-10), we have proved that for $U \subset M$,

$$\begin{aligned} \|u\|_{L^p(U)} \leq \sum_{\ell=1}^{L} \Big\| \sum_{j \in \mathcal{J}_{q_\ell}} \mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_{q_\ell,j}}) \mathrm{Op}_h(\psi_{q_\ell})u \Big\|_{L^p(U)} \\ + Ch^{-\delta(p)+1/2-\delta_2}h^{-1}\|Pu\|_{H_h^{n(1/2-1/p)+\varepsilon-2}} + O(h^\infty)\|u\|_{L^2}. \end{aligned} \tag{3-11}$$

3B. ***Filtering tubes by L^2 mass.*** By (3-11) it only remains to control terms of the form

$$\Big\| \sum_{j \in \mathcal{J}_{q_\ell}} \mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_{q_\ell,j}}) \mathrm{Op}_h(\psi_{q_\ell})u \Big\|_{L^p},$$

where u is localized to V_{q_ℓ} within the characteristic variety S^*M and, more importantly, to the tubes $\mathcal{T}_{q_\ell,j}$. We fix ℓ and, abusing notation slightly, write

$$\psi := \psi_{q_\ell}, \quad \mathcal{J} = \mathcal{J}_{q_\ell}, \quad \mathcal{T}_j = \mathcal{T}_{q_\ell,j}, \quad \tilde{\chi}_{\mathcal{T}_j} := \tilde{\chi}_{\mathcal{T}_{q_\ell,j}}, \quad v := \sum_{j \in \mathcal{J}} \mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_j}) \mathrm{Op}_h(\psi)u. \tag{3-12}$$

Let $T = T(h) \geq 1$. For each $j \in \mathcal{J}$ let

$$\chi_{\mathcal{T}_j} \in C_c^\infty(T^*M; [0,1]) \cap S_\delta \tag{3-13}$$

be a smooth cut-off function with $\operatorname{supp} \chi_{\mathcal{T}_j} \subset \mathcal{T}_j$ and $\chi_{\mathcal{T}_j} \equiv 1$ on $\operatorname{supp} \tilde{\chi}_{\mathcal{T}_j}$, and such that $\{\chi_j\}_j$ is bounded in S_δ. We shall work with the modified norm

$$\|u\|_{P,T} := \|u\|_{L^2} + \frac{T}{h}\|Pu\|_{L^2}.$$

Note that this norm is the natural norm for obtaining $T^{-1/2}$ improved estimates in L^p bounds since the fact that u is an $o(T^{-1}h)$ quasimode implies, roughly, that u is an accurate solution to $(hD_t + P)u = 0$ for times $t \leq T$. For each integer $k \geq -1$, we consider the set

$$\mathcal{A}_k = \Big\{ j \in \mathcal{J} : \frac{1}{2^{k+1}}\|u\|_{P,T} \leq \|\mathrm{Op}_h(\chi_{\mathcal{T}_j})u\|_{L^2} + h^{-1}\|\mathrm{Op}_h(\chi_{\mathcal{T}_j})Pu\|_{L^2} \leq \frac{1}{2^k}\|u\|_{P,T} \Big\}. \tag{3-14}$$

It follows that $\mathcal{A}_k$ consists of those tubes $\mathcal{T}_j$ with L^2 mass comparable to 2^{-k}.

Remark 3.3. Note that if $A \in \Psi_\delta$ and $\mathrm{MS}_h(A) \subset \{\chi_{\mathcal{T}_j} \equiv 1\}$, then the elliptic estimate implies

$$\|Av\|_{L^2} \le C\|\mathrm{Op}_h(\chi_{\mathcal{T}_j})v\|_{L^2} + O(h^\infty)\|v\|_{L^2}.$$

In particular, if $j \in \mathcal{A}_k$ and $\mathrm{MS}_h(A) \subset \{\chi_{\mathcal{T}_j} \equiv 1\}$, then

$$\|Au\|_{L^2} + h^{-1}\|APu\|_{L^2} \le C2^{-k}\|u\|_{P,T} + O(h^\infty)\|u\|_{P,T}.$$

Observe that since $|\chi_{\mathcal{T}_j}| \le 1$, for h small enough depending on finitely many seminorms of χ_j, $\|\mathrm{Op}_h(\chi_{\mathcal{T}_j})\|_{L^2\to L^2} \le 2$. In particular, this together with $T \ge 1$ implies that

$$\mathcal{J} = \bigcup_{k\ge -1} \mathcal{A}_k. \tag{3-15}$$

Lemma 3.4. *There exists $C_n > 0$ so that for all $k \ge -1$*

$$|\mathcal{A}_k| \le C_n 2^{2k}. \tag{3-16}$$

Proof. According to (3-2), the collection $\{\mathcal{T}_j\}_{j\in\mathcal{J}}$ can be partitioned into $\mathfrak{D}_n$ sets of disjoint tubes. Thus, we have $\sum_{j\in\mathcal{J}} |\chi_{\mathcal{T}_j}|^2 \le \mathfrak{D}_n$ and there is $C_n > 0$ depending only on n such that

$$\Big\|\sum_{j\in\mathcal{J}} \mathrm{Op}_h(\chi_{\mathcal{T}_j})^* \,\mathrm{Op}_h(\chi_{\mathcal{T}_j})\Big\|_{L^2\to L^2} \le C_n.$$

In particular,

$$\sum_{j\in\mathcal{J}} \|\mathrm{Op}_h(\chi_{\mathcal{T}_j})u\|_{L^2}^2 \le C_n\|u\|_{L^2}^2,$$
$$\sum_{j\in\mathcal{J}} \|\mathrm{Op}_h(\chi_{\mathcal{T}_j})Pu\|_{L^2}^2 \le C_n\|Pu\|_{L^2}^2.$$

Therefore,

$$|\mathcal{A}_k|2^{-2k-2}\|u\|_{P,T}^2 \le 2\bigg(\sum_{j\in\mathcal{A}_k} \|\mathrm{Op}_h(\chi_{\mathcal{T}_j})u\|_{L^2}^2 + h^{-2}\,\|\mathrm{Op}_h(\chi_{\mathcal{T}_j})Pu\|_{L^2}^2\bigg) \le C_n\|u\|_{P,T}^2. \qquad \square$$

Next, let

$$w_k := \sum_{j\in\mathcal{A}_k} \mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_j})\mathrm{Op}_h(\psi)u. \tag{3-17}$$

Then, by (3-12) and (3-15) we have

$$v = \sum_{k=-1}^{\infty} w_k. \tag{3-18}$$

The goal is therefore to control $\|w_k\|_{L^p(U)}$ for each k since the triangle inequality yields

$$\|v\|_{L^p(U)} \le \sum_{k=-1}^{\infty} \|w_k\|_{L^p(U)}.$$

3C. ***Filtering tubes by L^∞ weight on shrinking balls.*** By Lemma 2.3, there are points $\{x_\alpha\}_{\alpha\in\mathcal{I}}\subset M$ such that there exists a partition $\{\mathcal{I}_i\}_{i=1}^{\mathfrak{D}_n}$ of $\mathcal{I}$ such that

$$M\subset\bigcup_{\alpha\in\mathcal{I}}B(x_\alpha,R(h)),$$
$$B(x_{\alpha_1},3R(h))\cap B(x_{\alpha_2},3R(h))=\varnothing\quad\text{for }\alpha_1,\alpha_2\in\mathcal{I}_i,\ \ \alpha_1\neq\alpha_2.$$

Then, for $m\in\mathbb{Z}$, define

$$\mathcal{I}_{k,m}:=\left\{\alpha\in\mathcal{I}_U:2^{m-1}\le h^{(n-1)/2}R(h)^{(1-n)/2}2^k\frac{\|w_k\|_{L^\infty(B(x_\alpha,R(h)))}}{\|u\|_{P,T}}\le 2^m\right\},\tag{3-19}$$

where $\mathcal{I}_U:=\{\alpha\in\mathcal{I}:B(x_\alpha,R(h))\cap U\neq\varnothing\}$. For each $k\in\mathbb{Z}_+$ and $\alpha\in\mathcal{I}$, consider the sets

$$\mathcal{A}_k(\alpha):=\{j\in\mathcal{A}_k:\pi_M(\mathcal{T}_j)\cap B(x_\alpha,2R(h))\neq\varnothing\},$$

where $\pi_M:T^*M\to M$ is the standard projection. The indices in $\mathcal{A}_k$ are those that correspond to tubes with mass comparable to $\frac{1}{2^k}\|u\|_{P,T}$, while indices in $\mathcal{A}_k(\alpha)$ correspond to tubes of mass $\frac{1}{2^k}\|u\|_{P,T}$ that run over the ball $B(x_\alpha,2R(h))$. In particular, we claim that Lemma 3.4 and [Canzani and Galkowski 2021, Lemma 3.7] yield the existence of $C_n,c_M>0$ such that

$$c_M2^m\le|\mathcal{A}_k(\alpha)|\le C_n2^{2k}\quad\text{for }\alpha\in\mathcal{I}_{k,m}.\tag{3-20}$$

The upper bound follows directly from Lemma 3.4, while, to obtain the lower bound, we first observe that for $\alpha\in\mathcal{I}_{k,m}$,

$$2^{m-1}h^{(1-n)/2}R(h)^{(n-1)/2}2^{-k}\|u\|_{P,T}\le\|w_k\|_{L^\infty(B(x_\alpha,R(h)))}.\tag{3-21}$$

In addition, (3-14) and [Canzani and Galkowski 2021, Lemma 3.7] imply that there exist $c_M>0$, $\tau_M>0$, and $C_n>0$, depending on M and n respectively, such that for all $N>0$ there exists $C_N>0$ with

$$\begin{aligned}&\|w_k\|_{L^\infty(B(x_\alpha,R(h)))}\\&\quad\le\frac{C_nR(h)^{(n-1)/2}}{\tau_M^{1/2}h^{(n-1)/2}}\sum_{j\in\mathcal{A}_k(\alpha)}\|\mathrm{Op}_h(\tilde\chi_{\mathcal{T}_j})\,\mathrm{Op}_h(\psi)u\|_{L^2}+h^{-1}\|\mathrm{Op}_h(\tilde\chi_{\mathcal{T}_j})P\,\mathrm{Op}_h(\psi)u\|_{L^2}+C_Nh^N\|u\|_{P,T}\\&\quad\le c_M^{-1}h^{-(n-1)/2}R(h)^{(n-1)/2}2^{-k}\|u\|_{P,T}|\mathcal{A}_k(\alpha)|+C_Nh^N\|u\|_{P,T},\end{aligned}$$

which, combined with (3-21), proves the lower bound in (3-20). To obtain the second bound we used Remark 3.3. To simplify notation, let

$$\mathcal{A}_{k,m}:=\bigcup_{\alpha\in\mathcal{I}_{k,m}}\mathcal{A}_k(\alpha).\tag{3-22}$$

Note that for each $\alpha\in\mathcal{I}_{k,m}$, there is $\tilde{x}_\alpha\in B(x_\alpha,R(h))$ such that

$$|w_k(\tilde{x}_\alpha)|\ge2^{m-1}h^{(1-n)/2}R(h)^{(n-1)/2}2^{-k}\|u\|_{P,T}.\tag{3-23}$$

We finish this section with a result that controls the size of $\mathcal{I}_{k,m}$ in terms of that of $\mathcal{A}_{k,m}$. Let

$$\tfrac12(\delta_2+1)<\rho<1,\tag{3-24}$$

$0 < \varepsilon < \delta$, $\tilde{\chi} \in C_c^\infty((-1,1))$, and define the operator

$$\chi_{h,\tilde{x}_\alpha} u(x) := \tilde{\chi}\Big(\frac{1}{\varepsilon} h^{-\rho} d(x,\tilde{x}_\alpha)\Big)\Big[\mathrm{Op}_h\Big(\tilde{\chi}\Big(\frac{1}{\varepsilon}(|\xi|_g - 1)\Big)\Big) u\Big](x).$$

In Lemma 6.2 we prove that $\chi_{h,\tilde{x}_\alpha} \in \Psi^{-\infty}_{\Gamma_{\tilde{x}_\alpha}, L_{\tilde{x}_\alpha},\rho}$, where

$$\Omega_{\tilde{x}_\alpha} = \{\xi \in T^*_{\tilde{x}_\alpha} M : |1 - |\xi|_{g(\tilde{x}_\alpha)}| < \delta\}, \qquad \Gamma_{\tilde{x}_\alpha} = \bigcup_{|t| < \frac{1}{2}\operatorname{inj}(M)} \varphi_t(\Omega_{\tilde{x}_\alpha}),$$

and $\Psi^{-\infty}_{\Gamma_{\tilde{x}_\alpha}, L_{\tilde{x}_\alpha},\rho}$ is a class of smoothing pseudodifferential operators that allows for localization to h^ρ neighborhoods of $\Gamma_{\tilde{x}_\alpha}$ and is compatible with localization to h^ρ neighborhoods of the foliation $L_{\tilde{x}_\alpha}$ of $\Gamma_{\tilde{x}_\alpha}$ generated by $\Omega_{\tilde{x}_\alpha}$.

In Theorem 6.3 for $\varepsilon > 0$ we explain how to build a cut-off operator $X_{\tilde{x}_\alpha} \in \Psi^{-\infty}_{\Gamma_{\tilde{x}_\alpha}, L_{\tilde{x}_\alpha},\rho}$ such that

$$\begin{cases} \chi_{h,\tilde{x}_\alpha} X_{\tilde{x}_\alpha} = \chi_{h,\tilde{x}_\alpha} + O(h^\infty)_{\Psi^{-\infty}}, \\ \mathrm{WF}_h{}'([P, X_{\tilde{x}_\alpha}]) \cap \big\{(x,\xi) : x \in B\big(\tilde{x}_\alpha, \tfrac{1}{2}\operatorname{inj} M\big),\ \xi \in \Omega_x\big\} = \varnothing, \end{cases} \tag{3-25}$$

where $\operatorname{inj} M$ denotes the injectivity radius of M. Moreover, $X_{\tilde{x}_\alpha}$ acts microlocally in the sense that if $a, b \in S(T^*M)$ with $\operatorname{supp} a \cap \operatorname{supp} b = \varnothing$, then

$$\mathrm{Op}_h(a) X_{\tilde{x}_\alpha} \mathrm{Op}_h(b) = O(h^\infty)_{\Psi^{-\infty}}. \tag{3-26}$$

Lemma 3.5. *Let $\frac{1}{2}(\delta_2 + 1) < \rho \le 1$. There exists $C > 0$ such that for every $k \ge -1$ and $m \in \mathbb{Z}$ the following holds*: *if*

$$|\mathcal{A}_{k,m}| \le C\, 2^{2m} R(h)^{n-1} (h^{\rho - 1/2} R(h)^{-1/2})^{-2n(n-1)/(3n+1)},$$

then

$$|\mathcal{I}_{k,m}| \le C |\mathcal{A}_{k,m}| 2^{-2m} R(h)^{1-n}. \tag{3-27}$$

Proof. We claim that by (3-17), for $\alpha \in \mathcal{I}_{k,m}$,

$$\chi_{h,\tilde{x}_\alpha} w_k = \chi_{h,\tilde{x}_\alpha} w_{k,m} + O(h^\infty \|u\|_{L^2}) \quad \text{and} \quad w_{k,m} := \sum_{j \in \mathcal{A}_{k,m}} \mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_j}) \mathrm{Op}_h(\psi) u. \tag{3-28}$$

Indeed, it suffices to show that $\chi_{h,\tilde{x}_\alpha} \mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_j}) \mathrm{Op}_h(\psi) u = O(h^\infty \|u\|_{L^2})$ for $\alpha \in \mathcal{I}_{k,m}$ and $j \notin \mathcal{A}_{k,m}$. Note that for such indices $\pi_M(\mathcal{T}_j) \cap B(\tilde{x}_\alpha, 2R(h)) = \varnothing$, while

$$\operatorname{supp} \tilde{\chi}\Big(\frac{1}{\varepsilon} h^{-\rho} d(x,\tilde{x}_\alpha)\Big) \subset B(\tilde{x}_\alpha, C\varepsilon h^\rho) \subset B\big(x_\alpha, \tfrac{3}{2} R(h)\big)$$

for some $C > 0$ and all h small enough.

Our next goal is to produce a lower bound for $|\mathcal{A}_{k,m}|$ in terms of $|\mathcal{I}_{k,m}|$ by using the lower bound (3-23) on $\|\chi_{h,\tilde{x}_\alpha} w_{k,m}\|_{L^\infty}$ for indices $\alpha \in \mathcal{I}_{k,m}$. By (3-25), we have

$$\chi_{h,\tilde{x}_\alpha} w_{k,m} = \chi_{h,\tilde{x}_\alpha} X_{\tilde{x}_\alpha} w_{k,m} + O(h^\infty)_{L^\infty}$$

for $\alpha \in \mathcal{I}_{k,m}$.

Next, note that since $\mathrm{MS}_h(\tilde{\chi}_{\mathcal{T}_j}) \subset \{||\xi|_g - 1| \ll \varepsilon\}$, using (3-26) we have

$$\begin{aligned}\mathrm{Op}_h\Big(\tilde{\chi}\Big(\frac{1}{\varepsilon}(|\xi|_g-1)\Big)i\Big)X_{\tilde{x}_\alpha}w_{k,m}&\\ &= \mathrm{Op}_h\Big(\tilde{\chi}\Big(\frac{1}{\varepsilon}(|\xi|_g-1)\Big)\Big)X_{\tilde{x}_\alpha}\,\mathrm{Op}_h\Big(\tilde{\chi}\Big(\frac{10}{\varepsilon}(|\xi|_g-1)\Big)\Big)w_{k,m}+O(h^\infty\|u\|_{P,T})_{L^\infty}\\ &= X_{\tilde{x}_\alpha}w_{k,m}+O(h^\infty\|u\|_{P,T})_{L^\infty}.\end{aligned}$$

In particular, using this with (3-23) and (3-28),

$$\begin{aligned}2^{m-1}h^{(1-n)/2}R(h)^{(n-1)/2}2^{-k}\|u\|_{P,T} &\le \|\chi_{h,\tilde{x}_\alpha}w_k\|_{L^\infty}\\ &\le \Big\|\mathrm{Op}_h\Big(\tilde{\chi}\Big(\frac{1}{\varepsilon}(|\xi|_g-1)\Big)\Big)X_{\tilde{x}_\alpha}w_{k,m}\Big\|_{L^\infty}+O(h^\infty)\|u\|_{P,T}\\ &= \|X_{\tilde{x}_\alpha}w_{k,m}\|_{L^\infty}+O(h^\infty)\|u\|_{P,T}.\end{aligned}\tag{3-29}$$

Therefore, applying the standard L^∞ bound for quasimodes of the Laplacian (see, e.g., [Zworski 2012, Theorem 7.12]) and using, by (3-25), that $X_{\tilde{x}_\alpha}$ nearly commutes with P on $B\big(\tilde{x}_\alpha, \frac{1}{2}\,\mathrm{inj}\,M\big)$, we have

$$\begin{aligned}2^{m-1}R(h)^{(n-1)/2}2^{-k}\|u\|_{P,T} &\le C(\|X_{\tilde{x}_\alpha}w_{k,m}\|_{L^2}+h^{-1}\|PX_{\tilde{x}_\alpha}w_{k,m}\|_{L^2(B)})+O(h^\infty\|u\|_{P,T}).\\ &\le C(\|X_{\tilde{x}_\alpha}w_{k,m}\|_{L^2}+h^{-1}\|X_{\tilde{x}_\alpha}Pw_{k,m}\|_{L^2})+O(h^\infty\|u\|_{P,T}).\end{aligned}\tag{3-30}$$

Note that we have canceled the factor $h^{(1-n)/2}$ which appears both in (3-29) and the standard L^∞ bounds for quasimodes. Using that $h^{2\rho-1}R(h)^{-1}=o(1)$, Proposition 6.6 proves that, for all $\widetilde{\mathcal{I}} \subset \mathcal{I}_{k,m}$ and $v \in L^2(M)$,

$$\sum_{\alpha\in\widetilde{\mathcal{I}}}\|X_{\tilde{x}_\alpha}v\|_{L^2}^2 \le C(1+a_h|\widetilde{\mathcal{I}}|^{(3n+1)/(2n)})\|v\|_{L^2}^2,$$

where $a_h=(h^{\rho-1/2}R(h)^{-1/2})^{n-1}$. As a consequence, (3-30) gives

$$\begin{aligned}|\widetilde{\mathcal{I}}|R(h)^{n-1}2^{-2k}2^{2(m-1)}\|u\|_{P,T}^2 &\le C\Big(\sum_{\alpha\in\widetilde{\mathcal{I}}}\|X_{\tilde{x}_\alpha}w_{k,m}\|_{L^2}^2+h^{-2}\sum_{\alpha\in\widetilde{\mathcal{I}}}\|X_{\tilde{x}_\alpha}Pw_{k,m}\|_{L^2}^2\Big)\\ &\le C(1+a_h|\widetilde{\mathcal{I}}|^{(3n+1)/(2n)})(\|w_{k,m}\|_{L^2}^2+h^{-2}\|Pw_{k,m}\|_{L^2}^2)\\ &\le C(1+a_h|\widetilde{\mathcal{I}}|^{(3n+1)/(2n)})2^{-2k}|\mathcal{A}_{k,m}|\|u\|_{P,T}^2.\end{aligned}$$

The last inequality follows from the definition of $w_{k,m}$ together with the definition of $\mathcal{A}_k$ in (3-14).

In particular, we have proved that there is $C>0$ such that for all $\widetilde{\mathcal{I}} \subset \mathcal{I}_{k,m}$,

$$|\widetilde{\mathcal{I}}|R(h)^{n-1}2^{2m} \le C\max(1, a_h|\widetilde{\mathcal{I}}|^{(3n+1)/(2n)})|\mathcal{A}_{k,m}|.\tag{3-31}$$

Now, suppose that $a_h|\mathcal{I}_{k,m}|^{(3n+1)/(2n)} \ge 1$. Then, there exists $\widetilde{\mathcal{I}} \subset \mathcal{I}_{k,m}$ such that $a_h|\widetilde{\mathcal{I}}|^{(3n+1)/(2n)}=1$. In particular, $|\widetilde{\mathcal{I}}|R(h)^{n-1}2^{2m} \le C|\mathcal{A}_{k,m}|$. This implies that if

$$|\mathcal{A}_{k,m}| \le \frac{1}{C}a_h^{-(2n)/(3n+1)}R(h)^{n-1}2^{2m},$$

then $a_h|\mathcal{I}_{k,m}|^{(3n+1)/(2n)} \le 1$, and so by (3-31),

$$|\mathcal{I}_{k,m}|R(h)^{n-1}2^{2m} \le C|\mathcal{A}_{k,m}|. \qquad \square$$

Note that for $w_{k,m}$ defined as in (3-28),

$$\|w_k\|^p_{L^p(U)} \le \mathfrak{D}_n \sum_{m=-\infty}^{\infty} \|w_k\|^p_{L^p(U_{k,m})} = \mathfrak{D}_n \sum_{m=-\infty}^{\infty} \|w_{k,m}\|^p_{L^p(U_{k,m})} + O(h^\infty \|u\|_{P,T}), \tag{3-32}$$

where

$$U_{k,m} := \bigcup_{\alpha \in \mathcal{I}_{k,m}} B(x_\alpha, R(h)). \tag{3-33}$$

Finally, we split the study of $\|w_k\|_{L^p(U)}$ into two regimes: tubes with low or high L^∞ mass. Fix $N > 0$ large, to be determined later. (Indeed, we will see that it suffices to take $N > \frac{1}{2}(1 - p_c/p)^{-1}$.) Then, we claim that, for each $k \ge -1$,

$$\|w_k\|^p_{L^p(U)} \le \mathfrak{D}_n \sum_{m=-\infty}^{m_{1,k}} \|w_{k,m}\|^p_{L^p(U_{k,m})} + \mathfrak{D}_n \sum_{m=m_{1,k}+1}^{m_{2,k}} \|w_{k,m}\|^p_{L^p(U_{k,m})} + O(h^\infty \|u\|_{P,T}), \tag{3-34}$$

where $m_{1,k}$ and $m_{2,k}$ are defined by

$$\begin{aligned} 2^{m_{1,k}} &= \min\biggl(\frac{2^k R(h)^{(1-n)/2}}{T^N}, c_n 2^{2k}, c_0 R(h)^{1-n}\biggr), \\ 2^{m_{2,k}} &= \min(c_n 2^{2k}, c_0 R(h)^{1-n}), \end{aligned}$$

where c_0 and c_n are described in what follows. Indeed, note that the bound (3-20) yields that 2^m is bounded by $|\mathcal{A}_k(\alpha)|$ for all $\alpha \in \mathcal{I}_{k,m}$, and the latter is controlled by $c_0 R(h)^{n-1}$ for some $c_0 > 0$, depending only on (M, g). Also, note that by (3-20) the $w_{k,m}$ are only defined for m satisfying $2^m \le c_n 2^{2k}$. These observations justify that the second sum in (3-34) runs only up to $m_{2,k}$.

3D. ***Control of the low L^∞ mass term, $m \le m_{1,k}$.*** We first estimate the small m term in (3-34). The estimates here essentially amount to interpolation between L^{p_c} and L^∞. From the definition of $\mathcal{I}_{k,m}$ in (3-19), together with $\frac{1}{2}(1-n)(p-p_c) - 1 = -p\delta(p)$ and using Sogge's L^{p_c} estimate

$$\begin{aligned} \|w_{k,m}\|_{L^{p_c}(U_{k,m})} &\le C h^{-1/p_c} (\|w_{k,m}\|_{L^2} + h^{-1} \|P w_{k,m}\|_{L^2}) \\ &\le C h^{-1/p_c} \|u\|_{P,T}, \end{aligned}$$

we obtain

$$\begin{aligned} \sum_{m=-\infty}^{m_{1,k}} \|w_{k,m}\|^p_{L^p(U_{k,m})} &\le C \sum_{m=-\infty}^{m_{1,k}} \|w_{k,m}\|^{p-p_c}_{L^\infty(U_{k,m})} \|w_{k,m}\|^{p_c}_{L^{p_c}(U_{k,m})} \\ &\le C h^{-p\delta(p)} R(h)^{(n-1)(p-p_c)/2} 2^{-k(p-p_c)} \sum_{m=-\infty}^{m_{1,k}} 2^{m(p-p_c)} \|u\|^p_{P,T} \\ &\le C h^{-p\delta(p)} R(h)^{(n-1)(p-p_c)/2} 2^{(m_{1,k}-k)(p-p_c)} \|u\|^p_{P,T}. \end{aligned}$$

It follows that

$$\sum_{k \ge -1} \biggl(\sum_{m=-\infty}^{m_{1,k}} \|w_{k,m}\|^p_{L^p(U_{k,m})}\biggr)^{1/p} \le C h^{-\delta(p)} R(h)^{(n-1)(1-p_c/p)/2} \|u\|_{P,T} \sum_{k \ge -1} 2^{(m_{1,k}-k)(1-p_c/p)}. \tag{3-35}$$

Finally, define k_1 and k_2 such that

$$2^{k_1} = \frac{R(h)^{(1-n)/2}}{c_n T^N} \quad \text{and} \quad 2^{k_2} = c_0 R(h)^{(1-n)/2} T^N. \tag{3-36}$$

If $k \le k_1$, then $2^{m_{1,k}} = c_n 2^{2k}$, so there exists $C_{n,p} > 0$ such that

$$\sum_{k=-1}^{k_1} 2^{(m_{1,k}-k)(1-p_c/p)} \le C_{n,p} \frac{R(h)^{(1-n)(1-p_c/p)/2}}{T^{N(1-p_c/p)}}.$$

If $k_1 \le k \le k_2$, then $2^{m_{1,k}} = 2^k R(h)^{(1-n)/2}/T^N$. Therefore, since $|k_2 - k_1| \le cN \log T$ for some $c > 0$, there exists $C > 0$ such that

$$\sum_{k=k_1}^{k_2} 2^{(m_{1,k}-k)(1-p_c/p)} \le CN \log T \frac{R(h)^{(1-n)(1-p_c/p)/2}}{T^{N(1-p_c/p)}}.$$

Last, if $k \ge k_2$, then $2^{m_{1,k}} = c_0 R(h)^{1-n}$, so there exists $C_p > 0$ such that

$$\sum_{k=k_2}^{\infty} 2^{(m_{1,k}-k)(1-p_c/p)} \le C_p \frac{R(h)^{(1-n)(1-p_c/p)/2}}{T^{N(1-p_c/p)}}.$$

Putting these three bounds together with (3-35), we obtain

$$\sum_{k\ge -1} \Bigg(\sum_{m=-\infty}^{m_{1,k}} \|w_{k,m}\|^p_{L^p(U_{k,m})} \Bigg)^{1/p} \le C h^{-\delta(p)} \frac{N \log T}{T^{N(1-p_c/p)}} \|u\|_{P,T}. \tag{3-37}$$

3E. ***Control of the high L^∞ mass term, $m \le m_{1,k}$.*** In this section we estimate the large m term in (3-34). To do this we write

$$\mathcal{A}_{k,m} = \mathcal{G}_{k,m} \sqcup \mathcal{B}_{k,m},$$

where the set of "good" tubes $\bigcup_{j\in\mathcal{G}_{k,m}} \mathcal{T}_j$ is $[t_0, T]$ non-self-looping and the number of "bad" tubes $|\mathcal{B}_{k,m}|$ is small. To do this, let

$$\mathcal{B}_U(\alpha,\beta) := \Bigg\{ j \in \bigcup_k \mathcal{A}_k(\alpha) : \bigcup_{t=t_0}^{T} \varphi_t(\mathcal{T}_j) \cap S^*_{B(x_\beta,2R(h))} M \neq \varnothing \Bigg\}. \tag{3-38}$$

Then, we define

$$\mathcal{B}_{k,m} := \bigcup_{\alpha,\beta\in\mathcal{I}_{k,m}} \mathcal{B}_U(\alpha,\beta) \cap \mathcal{A}_k(\alpha).$$

Let $\mathcal{G}_{k,m} := \mathcal{A}_{k,m} \setminus \mathcal{B}_{k,m}$. Then, by construction, $\bigcup_{j\in\mathcal{G}_{k,m}} \mathcal{T}_j$ is $[t_0, T]$ non-self-looping and we have

$$|\mathcal{B}_{k,m}| \le c |\mathcal{I}_{k,m}|^2 |\mathcal{B}_U| \tag{3-39}$$

for some $c > 0$, where

$$|\mathcal{B}_U| := \sup\{ |\mathcal{B}_U(\alpha,\beta)| : \alpha,\beta \in \mathcal{I} \}. \tag{3-40}$$

That is, $|\mathcal{B}_U|$ is the maximum number of loops of length in $[t_0, T]$ joining any two points in U.

Then, define

$$w^{\mathcal{G}}_{k,m} := \sum_{j\in\mathcal{G}_{k,m}} \mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_j})\,\mathrm{Op}_h(\psi)u \quad\text{and}\quad w^{\mathcal{B}}_{k,m} := \sum_{j\in\mathcal{B}_{k,m}} \mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_j})\,\mathrm{Op}_h(\psi)u. \tag{3-41}$$

Next, consider

$$\biggl(\sum_{m=m_{1,k}}^{m_{2,k}} \|w_{k,m}\|^p_{L^p(U_{k,m})}\biggr)^{1/p} \le \biggl(\sum_{m=m_{1,k}}^{m_{2,k}} \|w^{\mathcal{G}}_{k,m}\|^p_{L^p(U_{k,m})}\biggr)^{1/p} + \biggl(\sum_{m=m_{1,k}}^{m_{2,k}} \|w^{\mathcal{B}}_{k,m}\|^p_{L^p(U_{k,m})}\biggr)^{1/p}. \tag{3-42}$$

3E1. *Bound on the looping piece.* We start by estimating the "bad" piece

$$\sum_{k\ge -1}\biggl(\sum_{m=m_{1,k}}^{m_{2,k}} \|w^{\mathcal{B}}_{k,m}\|^p_{L^p(U_{k,m})}\biggr)^{1/p}.$$

Observe that if $2^{m_{1,k}} = \min(c_0 R(h)^{1-n}, c_n 2^{2k})$, then $m_{1,k} = m_{2,k}$ and we need not consider this part of the sum. Therefore, the high L^∞ mass term has

$$2^{m_{1,k}} = \frac{2^k R(h)^{(1-n)/2}}{T^N} \tag{3-43}$$

and $k_1 \le k \le k_2$. Hence, for $m_{1,k} < m \le m_{2,k}$, Lemma 3.4 gives that there is $C_n > 0$ with

$$|\mathcal{A}_{k,m}| \le C_n 2^{2k} \le C_n R(h)^{n-1} 2^{2m} T^{2N}.$$

Furthermore, since $R(h) \ge h^{\delta_2}$ with $\delta_2 < \frac{1}{2}$, (3-24) yields that there is $\varepsilon = \varepsilon(n, N) > 0$ such that $h^{\rho-1/2}R(h)^{-1/2} < h^{\varepsilon}$, and hence, since $T = O(\log h^{-1})$,

$$|\mathcal{A}_{k,m}| = o(R(h)^{n-1} 2^{2m} (h^{\rho-1/2} R(h)^{-1/2})^{-2n(n-1)/(3n+1)}).$$

In particular, a consequence of Lemma 3.5 is the existence of $h_0 > 0$ and $C > 0$ such that

$$|\mathcal{I}_{k,m}| \le C R(h)^{1-n} 2^{-2m} |\mathcal{A}_{k,m}| \tag{3-44}$$

$$\le C R(h)^{1-n} 2^{2k-2m} \tag{3-45}$$

for all $0 < h \le h_0$, where we have used again Lemma 3.4 to bound $|\mathcal{A}_{k,m}|$.

Next, note that for each point in $\mathcal{I}_{k,m}$ there are at most $c|\mathcal{I}_{k,m}||\mathcal{B}_U|$ tubes in $\mathcal{B}_{k,m}$ touching it. Therefore, we may apply [Canzani and Galkowski 2021, Lemma 3.7] to obtain $C > 0$ such that

$$\|w^{\mathcal{B}}_{k,m}\|_{L^\infty(U_{k,m})} \le C h^{(1-n)/2} R(h)^{(n-1)/2} |\mathcal{I}_{k,m}||\mathcal{B}_U| 2^{-k} \|u\|_{P,T}. \tag{3-46}$$

Using (3-46) and interpolating between L^∞ and L^{p_c} we obtain

$$\|w^{\mathcal{B}}_{k,m}\|^p_{L^p(U_{k,m})} \le C h^{-p\delta(p)} (R(h)^{(n-1)/2} |\mathcal{I}_{k,m}||\mathcal{B}_U| 2^{-k} \|u\|_{P,T})^{p-p_c} \|w^{\mathcal{B}}_{k,m}\|^{p_c}_{L^2(U_{k,m})}. \tag{3-47}$$

In addition, since combining (3-14) with (3-39) yields

$$\|w^{\mathcal{B}}_{k,m}\|_{L^2(U_{k,m})} \le C|\mathcal{B}_{k,m}|^{1/2} 2^{-k} \|u\|_{P,T} \le C 2^{-k} |\mathcal{I}_{k,m}||\mathcal{B}_U|^{1/2} \|u\|_{P,T},$$

the bounds in (3-47) and (3-45) together with the definition of $m_{1,k}$ in (3-43) yield

$$\begin{aligned}\sum_{m=m_{1,k}}^{m_{2,k}} \|w^{\mathcal{B}}_{k,m}\|^p_{L^p(U_{k,m})} &\le Ch^{-p\delta(p)} R(h)^{(n-1)(p-p_c)/2} \sum_{m=m_{1,k}}^{m_{2,k}} |\mathcal{I}_{k,m}|^p |\mathcal{B}_U|^{p-p_c/2} 2^{-kp} \|u\|^p_{P,T} \\ &\le Ch^{-p\delta(p)} R(h)^{(n-1)(-p-p_c)/2} 2^{kp} |\mathcal{B}_U|^{p-p_c/2} \|u\|^p_{P,T} \sum_{m=m_{1,k}}^{m_{2,k}} 2^{-2mp} \\ &\le Ch^{-p\delta(p)} R(h)^{(n-1)(p-p_c)/2} |\mathcal{B}_U|^{p-p_c/2} T^{2Np}\, 2^{-kp} \|u\|^p_{P,T}.\end{aligned}$$

Then, with k_1 and k_2 defined as in (3-36), we have

$$\begin{aligned}\sum_{k=k_1}^{k_2} \biggl(\sum_{m=m_{1,k}}^{m_{2,k}} \|w^{\mathcal{B}}_{k,m}\|^p_{L^p(U_{k,m})}\biggr)^{1/p} &\le Ch^{-\delta(p)} R(h)^{(n-1)(1-p_c/p)/2} |\mathcal{B}_U|^{1-p_c/(2p)} T^{2N} \|u\|_{P,T} \sum_{k=k_1}^{k_2} 2^{-k} \\ &\le Ch^{-\delta(p)} (R(h)^{n-1} |\mathcal{B}_U|)^{1-p_c/(2p)} T^{3N} \|u\|_{P,T}.\end{aligned}$$

Finally, since we only need to consider $k_1 \le k \le k_2$,

$$\sum_{k\ge -1} \biggl(\sum_{m=m_{1,k}}^{m_{2,k}} \|w^{\mathcal{B}}_{k,m}\|^p_{L^p(U_{k,m})}\biggr)^{1/p} \le Ch^{-\delta(p)} (R(h)^{n-1} |\mathcal{B}_U|)^{1-p_c/(2p)} T^{3N} \|u\|_{P,T}. \tag{3-48}$$

3E2. *Bound on the non-self-looping piece.* In this section we aim to control the "good" piece,

$$\sum_{k\ge -1} \biggl(\sum_{m=m_{1,k}}^{m_{2,k}} \|w^{\mathcal{G}}_{k,m}\|^p_{L^p(U_{k,m})}\biggr)^{1/p}. \tag{3-49}$$

So far all L^p bounds appearing have been $\ll h^{(1-n)/2}/\sqrt{T}$. The reason for this is that the bounds were obtained by interpolation with an L^∞ estimate which is substantially stronger than $h^{(1-n)/2}/\sqrt{T}$.

We now estimate the number of non-self-looping tubes $\mathcal{T}_j$ with $j \in \mathcal{A}_k$. That is, tubes on which the L^2 mass of u is comparable to $2^{-k}\|u\|_{P,T}$.

Lemma 3.6. *Let $k \in \mathbb{Z}$, $k \ge -1$, and $t_0 > 1$. Suppose that $\mathcal{G} \subset \mathcal{A}_k$ is such that*

$$\bigcup_{j\in\mathcal{G}} \mathcal{T}_j \text{ is } [t_0, T] \text{ non-self-looping}.$$

Then, there exists a constant $C_n > 0$, depending only on n, such that $|\mathcal{G}| \le (C_n t_0/T) 2^{2k}$.

Proof. Using that $\mathcal{G} \subset \mathcal{A}_k$, we have

$$|\mathcal{G}| \frac{\|u\|^2_{P,T}}{2^{2(k+1)}} \le 2 \sum_{j\in\mathcal{G}} (\|\mathrm{Op}_h(\chi_{\mathcal{T}_j})u\|^2_{L^2} + h^{-2} \|\mathrm{Op}_h(\chi_{\mathcal{T}_j})Pu\|^2_{L^2}). \tag{3-50}$$

Since $\{\mathcal{T}_j\}_{j\in\mathcal{G}}$ is $(\mathfrak{D}_n, \tau, R(h))$-good, there are $\{\mathcal{G}_i\}_{i=1}^{\mathfrak{D}_n} \subset \mathcal{G}$, such that, for each $i = 1, \dots, \mathfrak{D}_n$,

$$\mathcal{T}_j \cap \mathcal{T}_k = \varnothing, \quad j, k \in \mathcal{G}_i, \quad j \ne k.$$

By [Canzani and Galkowski 2021, Lemma 4.1] with $t_\ell = t_0$ and $T_\ell = T$ for all ℓ,

$$\sum_{j\in\mathcal{G}} \|\mathrm{Op}_h(\chi_{\mathcal{T}_j})u\|_{L^2}^2 \le \sum_{i=1}^{\mathfrak{D}_n}\sum_{j\in\mathcal{G}_i} \|\mathrm{Op}_h(\chi_{\mathcal{T}_j})u\|_{L^2}^2 \le \frac{\mathfrak{D}_n 4t_0}{T}\|u\|_{P,T}^2. \tag{3-51}$$

On the other hand, since $\sum_{j\in\mathcal{G}_i}\|\mathrm{Op}_h(\chi_{\mathcal{T}_j})\|^2 \le 2$ for each i,

$$\sum_{j\in\mathcal{G}} \|\mathrm{Op}_h(\chi_{\mathcal{T}_j})Pu\|_{L^2}^2 \le 2\mathfrak{D}_n\|Pu\|_{L^2}^2. \tag{3-52}$$

Combining (3-50), (3-51), and (3-52) yields

$$|\mathcal{G}|\frac{\|u\|_{P,T}^2}{2^{2(k+1)}} \le \frac{8\mathfrak{D}_n t_0}{T}\|u\|_{P,T}^2 + \frac{4\mathfrak{D}_n}{h^2}\|Pu\|_{L^2}^2 \le \frac{8\mathfrak{D}_n t_0 + 4\mathfrak{D}_n/T}{T}\|u\|_{P,T}^2. \qquad \square$$

We may now proceed to estimate the L^p norm of the nonlooping piece (3-49). The first step is to notice that we only need to sum up to $m \le m_{3,k}$, where $m_{3,k}$ is defined by

$$2^{m_{3,k}} := \min\left(\frac{C_n t_0 2^{2k}}{c_M T}, c_0 R(h)^{1-n}\right),$$

$c_M > 0$ is as defined in (3-20), and $C_n > 0$ is the constant in Lemma 3.6. To see this, first observe that, using (3-19), (3-44), and (3-46), for each $\alpha \in \mathcal{I}_{k,m}$,

$$\begin{aligned}\|w_{k,m}^{\mathcal{G}}\|_{L^\infty(B(x_\alpha,R(h)))} &\le \|w_{k,m}\|_{L^\infty(B(x_\alpha,R(h)))} + \|w_{k,m}^{\mathcal{B}}\|_{L^\infty(B(x_\alpha,R(h)))}\\ &\le C(2^m + |\mathcal{I}_{k,m}||\mathcal{B}_U|)2^{-k}h^{(1-n)/2}R(h)^{(n-1)/2}\|u\|_{P,T}\\ &\le C(1+R(h)^{1-n}2^{-3m}|\mathcal{A}_{k,m}||\mathcal{B}_U|)2^{m-k}h^{(1-n)/2}R(h)^{(n-1)/2}\|u\|_{P,T}. \end{aligned}\tag{3-53}$$

Furthermore, since $|\mathcal{G}_{k,m}| \ge |\mathcal{A}_{k,m}| - |\mathcal{I}_{k,m}|^2|\mathcal{B}_U|$ and $\mathcal{G}_{k,m}$ is $[t_0, T]$ non-self-looping, Lemma 3.6 yields the existence of $C_n > 0$ such that

$$|\mathcal{A}_{k,m}| - |\mathcal{I}_{k,m}|^2|\mathcal{B}_U| \le C_n\frac{t_0}{T}2^{2k}.$$

Next, since $m_{1,k} \le m \le m_{2,k}$, we may apply Lemma 3.5 to bound $|\mathcal{I}_{k,m}|$ as in (3-44) to obtain that for some $C > 0$,

$$|\mathcal{A}_{k,m}|(1 - CR(h)^{2(1-n)}2^{-4m}|\mathcal{A}_{k,m}||\mathcal{B}_U|) \le C_n\frac{t_0}{T}2^{2k}. \tag{3-54}$$

In addition, provided

$$|\mathcal{B}_U|R(h)^{n-1} \ll T^{-6N}, \tag{3-55}$$

we have that, for $m \ge m_{1,k}$ and $k_1 \le k \le k_2$,

$$R(h)^{2(1-n)}2^{-4m}|\mathcal{A}_{k,m}||\mathcal{B}_U| \le R(h)^{2(1-n)}2^{-4m+2k}|\mathcal{B}_U| \le 2^{-2k}T^{4N}|\mathcal{B}_U| \le R(h)^{n-1}T^{6N}|\mathcal{B}_U| \ll 1, \tag{3-56}$$

where we used that, by (3-20), $|A_{k,m}|$ is controlled by 2^{2k} to get the first inequality, that $m \ge m_{1,k}$ to get the second, and that $k \ge k_1$ to get the third. Combining (3-54) and the bound in (3-56) we obtain $|\mathcal{A}_{k,m}| \le C_n t_0 2^{2k}/T$, and so, by (3-20), $2^m \le C_n t_0 2^{2k}/(c_M T)$. As claimed, this shows that to deal with (3-49) we only need to sum up to $m \le m_{3,k}$.

The next step is to use interpolation to control the first sum in (3-49) by

$$\sum_{m=m_{1,k}}^{m_{2,k}} \|w^{\mathcal{G}}_{k,m}\|^p_{L^p(U_{k,m})} = \sum_{m=m_{1,k}}^{m_{3,k}} \|w^{\mathcal{G}}_{k,m}\|^p_{L^p(U_{k,m})}. \tag{3-57}$$

We claim that (3-53) yields

$$\|w^{\mathcal{G}}_{k,m}\|_{L^\infty(B(x_\alpha,R(h)))} \le C2^{m-k}h^{(1-n)/2}R(h)^{(n-1)/2}\|u\|_{P,T}. \tag{3-58}$$

Indeed, using the bound (3-55) on $|\mathcal{B}_U|$, that $|\mathcal{A}_{k,m}|$ is controlled by 2^{2k}, that $m \ge m_{1,k}$ as in (3-43), and that $k_1 \le k \le k_2$, we have

$$R(h)^{1-n}2^{-3m}|\mathcal{A}_{k,m}||\mathcal{B}_U| \ll R(h)^{2(1-n)}2^{-3m+2k}T^{-6N} \le T^{-2N}.$$

Note that

$$\begin{aligned}
\|w^{\mathcal{G}}_{k,m}\|_{L^{p_c}(U_{k,m})} &\le Ch^{-1/p_c}(\|w^{\mathcal{G}}_{k,m}\|_{L^2}+h^{-1}\|Pw^{\mathcal{G}}_{k,m}\|_{L^2})\\
&\le Ch^{-1/p_c}\Biggl(\|w^{\mathcal{G}}_{k,m}\|_{L^2}+h^{-1}\Bigl\|\sum_{j\in\mathcal{G}_{k,m}}[P,\mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_j})]w^{\mathcal{G}}_{k,m}\Bigr\|_{L^2}+h^{-1}\Bigl\|\sum_{j\in\mathcal{G}_{k,m}}\mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_j})Pu\Bigr\|_{L^2}\Biggr)\\
&\le Ch^{-1/p_c}2^{-k}|\mathcal{G}_{k,m}|^{1/2}\|u\|_{P,T}+O(h^\infty\|u\|_{P,T}),
\end{aligned}$$

where the last line follows from the definition of $\mathcal{A}_{k,m}$, the fact that $[P,\mathrm{Op}_h(\tilde{\chi}_{\mathcal{T}_j})] \in h\Psi_\delta$ with its microsupport contained in $\operatorname{supp}\tilde{\chi}_{\mathcal{T}_j}$, and Remark 3.3. Finally, by Lemma 3.6, $|\mathcal{G}_{k,m}| \le (C_nt_0/T)2^{2k}$, and hence

$$\|w^{\mathcal{G}}_{k,m}\|_{L^{p_c}(U_{k,m})} \le C\sqrt{\frac{t_0}{T}}h^{-1/p_c}\|u\|_{P,T}+O(h^\infty\|u\|_{P,T}).$$

Using this together with interpolation and (3-58) we obtain

$$\begin{aligned}
\|w^{\mathcal{G}}_{k,m}\|^p_{L^p(U_{k,m})} &\le \|w^{\mathcal{G}}_{k,m}\|^{p-p_c}_{L^\infty(U_{k,m})}\|w^{\mathcal{G}}_{k,m}\|^{p_c}_{L^{p_c}(U_{k,m})}\\
&\le Ch^{-p\delta(p)}(R(h)^{(n-1)/2}2^{m-k})^{p-p_c}\frac{t_0^{p_c/2}}{T^{p_c/2}}\|u\|^p_{P,T}+O(h^\infty\|u\|^p_{P,T}).
\end{aligned} \tag{3-59}$$

Using this, we estimate (3-57):

$$\sum_{m=m_{1,k}}^{m_{2,k}}\|w^{\mathcal{G}}_{k,m}\|^p_{L^p(U_{k,m})} \le Ch^{-p\delta(p)}(R(h)^{(n-1)/2}2^{(m_{3,k}-k)})^{p-p_c}\|u\|^p_{P,T}\frac{t_0^{p_c/2}}{T^{p_c/2}}+O(h^\infty\|u\|^p_{P,T}). \tag{3-60}$$

Then, summing in k, and again using that only $k_1 \le k \le k_2$ contribute,

$$\begin{aligned}
\sum_{k=-1}^{\infty}\Biggl(\sum_{m=m_{1,k}}^{m_{2,k}}\|w^{\mathcal{G}}_{k,m}\|^p_{L^p(U_m)}\Biggr)^{1/p}\\
&\le Ch^{-\delta(p)}\|u\|_{P,T}\frac{t_0^{p_c/(2p)}}{T^{p_c/(2p)}}\sum_{k=k_1}^{k_2}(R(h)^{(n-1)/2}2^{(m_{3,k}-k)})^{1-p_c/p}+O(h^\infty\|u\|_{P,T})\\
&\le Ch^{-\delta(p)}\frac{t_0^{1/2}}{T^{1/2}}\|u\|_{P,T}+O(h^\infty\|u\|_{P,T}).
\end{aligned} \tag{3-61}$$

Note that the sum over k in (3-61) is controlled by the value of k for which $C_nt_02^{2k}/(c_MT)=c_0R(h)^{1-n}$, since the sum is geometrically increasing before such k and geometrically decreasing afterward.

3F. ***Wrapping up the proof of Theorem 1.4.*** Combining (3-37), (3-48), and (3-61) with (3-42) and (3-34), and taking $N > \frac{1}{2}(1-p_c/p)^{-1}$ provided $R(h)^{n-1}|\mathcal{B}_U| \le CT^{-6N}$ for some $C>0$, we obtain

$$\|v\|_{L^p(U)} \le \sum_{k=-1}^{\infty} \|w_k\|_{L^p(U)} \le Ch^{-\delta(p)}\biggl(\frac{t_0^{1/2}}{T^{1/2}} + (R(h)^{n-1}|\mathcal{B}_U|)^{1-p_c/(2p)}T^{3N}\biggr)\|u\|_{P,T}$$

as requested in (3-55). Since this estimate holds only when $|\mathcal{B}_U|R(h)^{n-1} \le CT^{-6N}$, we replace T by $T_0 := \min\bigl\{\frac{1}{C}(R(h)^{n-1}|\mathcal{B}_U|)^{-1/6N}, T\bigr\}$, so that

$$\begin{aligned}
\|v\|_{L^p(U)} &\le Ch^{-\delta(p)}\biggl(\frac{t_0^{1/2}}{T_0{}^{1/2}} + (R(h)^{n-1}|\mathcal{B}_U|)^{1-p_c/(2p)}T_0^{3N}\biggr)\|u\|_{P,T} \\
&\le Ch^{-\delta(p)}\biggl(\frac{t_0^{1/2}}{T^{1/2}} + t_0^{1/2}(R(h)^{n-1}|\mathcal{B}_U|)^{1/(12N)} + (R(h)^{n-1}|\mathcal{B}_U|)^{(1-p_c/p)/2}\biggr)\|u\|_{P,T} \\
&\le Ch^{-\delta(p)}\biggl(\frac{t_0^{1/2}}{T^{1/2}} + (R(h)^{n-1}|\mathcal{B}_U|)^{1/(12N)}\biggr)\|u\|_{P,T},
\end{aligned} \tag{3-62}$$

where the constant C is adjusted from line to line.

Next, combining (3-62) with (3-11) and the definition of v in (3-12), we obtain

$$\|u\|_{L^p(U)} \le Ch^{-\delta(p)}\biggl(\frac{t_0^{1/2}}{T^{1/2}} + (R(h)^{n-1}|\mathcal{B}_U|)^{1/(12N)}\biggr)\|u\|_{P,T} + Ch^{-\delta(p)+1/2-\delta_2}h^{-1}\|Pu\|_{H_h^{n(1/2-1/p)+\varepsilon-2}}.$$

Putting $\varepsilon = \frac{1}{2}$ and setting $N = \frac{1}{2}\bigl(1+\frac{1}{6}\varepsilon_0\bigr)(1-p_c/p)^{-1}$, estimate (1-7) will follow once we relate $|\mathcal{B}_U|$ for a given $(\tau, R(h))$ cover to $|\mathcal{B}_U|$ for the $(\mathfrak{D}, \tau, R(h))$ cover used in our proof.

Finally, to finish the proof of Theorem 1.4, we need to show that for any $(\tau, R(h))$ cover $\{\mathcal{T}_j\}_j$ of S^*M, up to a constant depending only on M, $|\mathcal{B}_U|$ can be bounded by $|\widetilde{\mathcal{B}}_U|$ where $\widetilde{\mathcal{B}}_U$ is defined as in (3-40) using a $(\widetilde{\mathfrak{D}}, \tau, R(h))$-good cover $\{\widetilde{\mathcal{T}}_k\}_k$ of S^*M.

Lemma 3.7. *There exists $C_M > 0$ depending only on M such that if $\{\mathcal{T}_j\}_{j\in\mathcal{J}}$ and $\{\widetilde{\mathcal{T}}_k\}_{k\in\mathcal{K}}$ are a $(\tau, R(h))$ cover of S^*M and a $(\widetilde{\mathfrak{D}}, \tau, R(h))$-good cover of S^*M, respectively, and $|\mathcal{B}_U|$ and $|\widetilde{\mathcal{B}}_U|$ are defined as in* (3-40) *for the covers $\{\mathcal{T}_j\}_{j\in\mathcal{J}}$ and $\{\widetilde{\mathcal{T}}_k\}_{k\in\mathcal{K}}$, respectively, then*

$$|\widetilde{\mathcal{B}}_U| \le C_M\widetilde{\mathfrak{D}}|\mathcal{B}_U|.$$

Proof. Fix α, β such that $x_\alpha, x_\beta \in U$. Suppose that $j \in \mathcal{B}_U(\alpha,\beta)$, where $\mathcal{B}_U(\alpha,\beta)$ is as in (3-38). Then, there is $k \in \widetilde{\mathcal{B}}_U(\alpha,\beta)$ such that $\widetilde{\mathcal{T}}_k \cap \mathcal{T}_j \neq \varnothing$. Now, fix $j \in \mathcal{J}$ and let

$$\mathcal{C}_j := \{k \in \mathcal{K} : \mathcal{T}_j \cap \widetilde{\mathcal{T}}_k \neq \varnothing\}.$$

We claim that there is $c_M > 0$ such that for each $k \in \mathcal{C}_j$,

$$\widetilde{\mathcal{T}}_k \subset \Lambda_{\rho_j}^{c_M\tau}(c_M R(h)). \tag{3-63}$$

Assuming (3-63) for now, there exists $C_M > 0$ such that

$$|\mathcal{C}_j| \le \widetilde{\mathfrak{D}}\frac{\operatorname{vol}(\Lambda_{\rho_j}^{c_M\tau}(c_M R(h))}{\inf_{k\in\mathcal{K}}\operatorname{vol}(\widetilde{\mathcal{T}}_k)} \le \widetilde{\mathfrak{D}}C_M.$$

Thus, for each $j \in \mathcal{B}_U(\alpha, \beta)$, there are at most $C_M \widetilde{\mathfrak{D}}$ elements in $\widetilde{\mathcal{B}}_U(\alpha, \beta)$, and hence

$$|\mathcal{B}_U(\alpha, \beta)| \geq \frac{|\widetilde{\mathcal{B}}_U(\alpha, \beta)|}{C_M \widetilde{\mathfrak{D}}}$$

as claimed.

We now prove (3-63). Let $q \in \widetilde{\mathcal{T}}_k$. Then, there are $\rho'_k, \rho'_j, q' \in S^*M$ and $t_k, t_j, s \in [\tau - R(h), \tau + R(h)]$ such that

$$d(\rho_k, \rho'_k) < R(h), \quad d(\rho_j, \rho'_j) < R(h), \quad d(\rho_k, q') < R(h),$$
$$\varphi_{t_k}(\rho'_k) = \varphi_{t_j}(\rho'_j), \quad \varphi_s(q') = q.$$

In particular, $d(q', \rho'_k) < 2R(h)$, so there is $c_M > 0$ such that $d(\varphi_{t_k}(\rho'_k), \varphi_{t_k - s}(q)) < c_M R(h)$. Applying φ_{-t_j}, and adjusting c_M in a way depending only on M, we have $d(\rho'_j, \varphi_{t_k - t_j - s}(q)) < c_M R(h)$. In particular, adjusting c_M again, $d(\rho_j, \varphi_{t_k - t_j - s}(q)) < c_M R(h)$ and the claim follows. □

3G. *Profiles of near-saturating functions.* As explained in the introduction, our next theorem describes the profiles of functions which extremize the improved bounds from Theorem 1.4.

Theorem 3.8. *Let $p > p_c$, $T(h) \to \infty$, and $\delta > 0$. Let $0 < \delta_1 < \delta_2 < \frac{1}{2}$, $h^{\delta_2} \leq R(h) \leq h^{\delta_1}$, and $\{x_\alpha\}_{\alpha \in \mathcal{I}(h)} \subset M$ be a maximal $R(h)$-separated set. Let $\mathcal{B}_U$ be as in* (3-40), *and suppose that*

$$|\mathcal{B}_U| R(h)^{n-1} T(h)^{3p/(p-p_c)+\delta} = o(1)$$

and $u \in \mathcal{D}'(M)$ with

$$\|Pu\|_{H_h^{(n-3)/2}} = o\Big(\frac{h}{T}\|u\|_{L^2}\Big). \tag{3-64}$$

For $\varepsilon > 0$, set

$$\mathcal{S}_U(h, \varepsilon, u) := \left\{\alpha \in \mathcal{I}(h) : \|u\|_{L^\infty(B(x_\alpha, R(h)))} \geq \frac{\varepsilon h^{(1-n)/2}\sqrt{t_0}}{\sqrt{T(h)}}\|u\|_{L^2(M)},\ B(x_\alpha, R(h)) \cap U \neq \varnothing\right\}.$$

Then, there are $c, C > 0$ such that, for all $\varepsilon > 0$, there are $N_\varepsilon > 0$ and $h_0 > 0$ such that $|\mathcal{S}_U(h, \varepsilon, u)| \leq N_\varepsilon$ for all $0 < h \leq h_0$.

Moreover, there is a collection of geodesic tubes $\{\mathcal{T}_j\}_{j \in \mathcal{L}(\varepsilon, u)}$ of radius $R(h)$ (see Definition 1.3) with indices satisfying $\mathcal{L}(\varepsilon, u) = \bigcup_{i=1}^C \mathcal{J}_i$ and $\mathcal{T}_k \cap \mathcal{T}_\ell = \varnothing$ for $k, \ell \in \mathcal{J}_i$ with $k \neq \ell$, such that

$$u = u_e + \frac{1}{\sqrt{T(h)}} \sum_{j \in \mathcal{L}(\varepsilon, u)} v_j,$$

where v_j is microsupported in $\mathcal{T}_j$, $|\mathcal{L}(\varepsilon, u)| \leq C\varepsilon^{-2} R(h)^{1-n}$, and, for all $p \leq q \leq \infty$,

$$\|u_e\|_{L^q} \leq \varepsilon h^{-\delta(q)} (T(h))^{-1/2} \|u\|_{L^2},$$
$$\|v_j\|_{L^2} \leq C\varepsilon^{-1} R(h)^{(n-1)/2} \|u\|_{L^2}, \quad \|Pv_j\|_{L^2} \leq C\varepsilon^{-1} R(h)^{(n-1)/2} h \|u\|_{L^2}.$$

Finally, with $\mathcal{L}(\varepsilon, u, \alpha) := \{j \in \mathcal{L}(\varepsilon, u) : \pi(\mathcal{T}_j) \cap B(x_\alpha, 3R(h)) \neq \varnothing\}$, for every $\alpha \in \mathcal{S}_U(h, \varepsilon, u)$,

$$c\varepsilon^2 R(h)^{1-n} \leq |\mathcal{L}(\varepsilon, u, \alpha)| \leq C R(h)^{1-n} \quad \text{and} \quad \sum_{j \in \mathcal{L}(\varepsilon, u, \alpha)} \|v_j\|_{L^2}^2 \geq c^2 \varepsilon^2 \|u\|_{L^2}^2.$$

The proof of Theorem 3.8 is completed in the following three subsections.

3G1. *Proof of the bound on* $|\mathcal{S}_U(h,\varepsilon,u)|$. We claim that there is $c>0$ such that, for $\alpha\in S_U(h,\varepsilon,u)$,

$$\frac{c\varepsilon\sqrt{t_0}}{\sqrt{T}}h^{-1/p}\|u\|_{P,T}\le\|u\|_{L^p(B(x_\alpha,2R(h)))}. \tag{3-65}$$

To see (3-65), first let $\chi_0,\chi_1\in C_c^\infty(-2,2)$ with $\chi_0\equiv 1$ on $\left[-\frac{3}{2},\frac{3}{2}\right]$ and $\chi_1\equiv 1$ on $\operatorname{supp}\chi_0$ and note that, by Lemma 3.1, the elliptic parametrix construction for P, and (3-64),

$$\|(1-\chi_0(-h^2\Delta_g))u\|_{L^p}\le Ch^{-\delta(p)-1/2}\|Pu\|_{H_h^{(n-3)/2}}=o\biggl(\frac{h^{-\delta(p)+1/2}}{T}\biggr)\|u\|_{L^2}. \tag{3-66}$$

Therefore, for $\alpha\in S_U(h,\varepsilon,u)$, we have

$$\|\chi_0(-h^2\Delta_g)u\|_{L^\infty(B(x_\alpha,R(h)))}\ge\frac{\varepsilon h^{(1-n)/2}}{2\sqrt{T}}\|u\|_{L^2(M)} \tag{3-67}$$

for h small enough. Next, set $\chi_{\alpha,h}(x):=\chi_0(R(h)^{-1}d(x,x_\alpha))$ and note

$$\chi_1(-h^2\Delta_g)\chi_{\alpha,h}\chi_0(-h^2\Delta_g)u=\chi_{\alpha,h}\chi_0(-h^2\Delta_g)u+O(h^\infty\|u\|_{L^2})_{C^\infty}.$$

Then, by (3-67) and [Zworski 2012, Theorem 7.15],

$$\begin{aligned}\frac{\varepsilon h^{(1-n)/2}}{2\sqrt{T}}\|u\|_{L^2(M)}&\le\|\chi_0(-h^2\Delta_g)u\|_{L^\infty(B(x_\alpha,R(h)))}\\&=\|\chi_{\alpha,h}\chi_0(-h^2\Delta_g)u\|_{L^\infty(B(x_\alpha,R(h)))}\\&=\|\chi_1(-h^2\Delta_g)\chi_{\alpha,h}\chi_0(-h^2\Delta_g)u\|_{L^\infty(B(x_\alpha,R(h)))}+O(h^\infty)\|u\|_{L^2}\\&\le Ch^{-n/p}(\|\chi_0(-h^2\Delta_g)u\|_{L^p(B(x_\alpha,2R(h)))}+O(h^\infty)\|u\|_{L^2}).\end{aligned} \tag{3-68}$$

Combining (3-68) and (3-66) yields the claim in (3-65). It then follows from Theorem 1.4 that, if $\{\alpha_i\}_{i=1}^N\subset S_U(h,\varepsilon,u)$ with $B(x_{\alpha_i},2R(h))\cap B(x_{\alpha_j},2R(h))=\varnothing$ for $i\ne j$, we have

$$N^{1/p}\frac{c\varepsilon\sqrt{t_0}}{\sqrt{T}}h^{-1/p}\|u\|_{P,T}\le\|u\|_{L^p}\le Ch^{-1/p}\|u\|_{L^2}\le Ch^{-1/p}\frac{\sqrt{t_0}}{\sqrt{T}}\|u\|_{P,T}.$$

Then, $N^{1/p}\le C\varepsilon^{-1}$. Since at most $\mathfrak{D}_n$ balls $B(x_\alpha,2R(h))$ intersect, $|S_U(h,\varepsilon,u)|\le C\mathfrak{D}_n\varepsilon^{-p}$.

3G2. *Preliminaries for the decomposition of* u. Let $q\in\mathbb{R}$ such that $p\le q\le\infty$. Below, all implicit constants are uniform for $p\le q\le\infty$. As above, it suffices to prove the statement for v as in (3-12) instead of u. Then, we write $v=\sum_{k=-1}^\infty w_k$ as in (3-18). For $V\subset U$, by the same analysis that led to (3-34),

$$\|w_k\|_{L^q(V)}^q\le\mathfrak{D}_n\sum_{m=-\infty}^{m_{2,k}}\|w_{k,m}\|_{L^q(V\cap U_{k,m})}^q+O(h^\infty)\|u\|_{P,T},$$

where $w_{k,m}$ is as in (3-28). Then, by (3-37) with $N=\frac{1}{2}q/(q-p_c)+\frac{1}{6}\delta$,

$$\sum_{k\ge -1}\biggl(\sum_{m=-\infty}^{m_1,k}\|w_{k,m}\|_{L^q(U_{k,m})}^q\biggr)^{1/q}\le Ch^{-\delta(q)}\frac{\log T}{T^{1/2+\delta(q-p_c)/(6q)}}\|u\|_{P,T} \tag{3-69}$$

for h small enough. Then, splitting $w_{k,m} = w^{\mathcal{B}}_{k,m} + w^{\mathcal{G}}_{k,m}$, as in (3-41), we have by (3-48) that

$$\sum_{k\geq -1}\biggl(\sum_{m=m_{1,k}}^{m_{2,k}} \|w^{\mathcal{B}}_{k,m}\|^q_{L^q(U_{k,m})}\biggr)^{1/q} \leq Ch^{-\delta(q)}(R(h)^{n-1}|\mathcal{B}_U|)^{1-p_c/(2q)}T^{3q/(2(q-p_c))+\delta/2}\|u\|_{P,T}. \tag{3-70}$$

Define k_1^ε and k_2^ε, respectively, by

$$2^{2k_1^\varepsilon} = \frac{C^{-2}\mathfrak{D}_n^{-2}\varepsilon^2 R(h)^{1-n}c_M T}{4C_n t_0} \quad\text{and}\quad 2^{2k_2^\varepsilon} = \frac{C^{2}\mathfrak{D}_n^{2}\varepsilon^{-2} R(h)^{1-n}c_M T}{4C_n t_0}, \tag{3-71}$$

where C is as in (3-61). Then, define $\mathcal{K}(\varepsilon) := \{k : k_1^\varepsilon \leq k \leq k_2^\varepsilon\}$ and note that, since $2^{(k_2^\varepsilon - k_1^\varepsilon)} = C^2\mathfrak{D}_n^2\varepsilon^{-2}$, $|\mathcal{K}(\varepsilon)| \leq \log_2(4C^2\mathfrak{D}_n^2\varepsilon^{-2}) =: K_\varepsilon$. Using (3-59) and summing over $k \notin \mathcal{K}(\varepsilon)$, it follows that

$$\sum_{k\notin\mathcal{K}(\varepsilon)}\biggl(\sum_{m=m_{1,k}}^{m_{3,k}} \|w^{\mathcal{G}}_{k,m}\|^q_{L^q(U_{k,m})}\biggr)^{1/q} \leq \frac{\varepsilon}{4\mathfrak{D}_n}\frac{h^{-\delta(q)}\sqrt{t_0}}{\sqrt{T}}\|u\|_{P,T}. \tag{3-72}$$

Next, for $k \in \mathcal{K}(\varepsilon)$, let

$$\mathcal{M}(k,\varepsilon) := \{m : m^\varepsilon_{3,k} \leq m \leq m_{3,k}\}, \quad m^\varepsilon_{3,k} := m_{3,k} - \frac{q}{q-p_c}\log_2(\varepsilon^{-1}2C\mathfrak{D}_n),$$

and note $|\mathcal{M}(k,\varepsilon)| \leq (q/(q-p_c))\log_2(\varepsilon^{-1}2C\mathfrak{D}_n) := M_\varepsilon$. Using (3-59) and summing over $k \in \mathcal{K}(\varepsilon)$ and $m \notin \mathcal{M}(k,\varepsilon)$, it follows that

$$\begin{aligned}\sum_{k\in\mathcal{K}(\varepsilon)}\biggl(\sum_{m\notin\mathcal{M}(k,\varepsilon)} \|w^{\mathcal{G}}_{k,m}\|^q_{L^q(U_{k,m})}\biggr)^{1/q} &\leq Ch^{-\delta(q)}\frac{t_0^{p_c/(2q)}}{T^{p_c/(2q)}}\sum_{k\in\mathcal{K}(\varepsilon)}(R(h)^{(n-1)/2}2^{m^\varepsilon_{3,k}-k})^{1-p_c/q}\|u\|_{P,T} + O(h^\infty\|u\|_{P,T}) \\ &\leq \frac{\varepsilon}{4\mathfrak{D}_n}\frac{h^{-\delta(q)}t_0^{1/2}}{T^{1/2}}\|u\|_{P,T}. \end{aligned} \tag{3-73}$$

Let

$$\mathcal{N}_{k,m}(\varepsilon) := \biggl\{\alpha \in \mathcal{I}_{k,m} : \|w^{\mathcal{G}}_{k,m}\|_{L^\infty(B(x_\alpha,R(h)))} \geq \frac{\varepsilon}{4\mathfrak{D}_n M_\varepsilon K_\varepsilon}\frac{h^{(1-n)/2}\sqrt{t_0}}{\sqrt{T}}\|u\|_{P,T}\biggr\}. \tag{3-74}$$

We claim

$$\mathcal{S}_U(h,\varepsilon,u) \subset \bigcup_{k\in\mathcal{K}(\varepsilon)}\bigcup_{m\in\mathcal{M}(k,\varepsilon)}\mathcal{N}_{k,m}(\varepsilon). \tag{3-75}$$

To prove (3-75), suppose $\alpha \notin \bigcup_{k\in\mathcal{K}(\varepsilon)}\bigcup_{m\in\mathcal{M}(k,\varepsilon)}\mathcal{N}_{k,m}(\varepsilon)$. Then, using (3-69) with $q=\infty$ and $N = \frac{1}{2}+\frac{\delta}{6}$,

$$\frac{1}{\mathfrak{D}_n}\|v\|_{L^\infty(B(x_\alpha,R(h)))} \leq \frac{Ch^{(1-n)/2}\log T}{T^{1/2+\delta/6}}\|u\|_{P,T} + \sum_{k\geq -1}\sum_{m=m_{1,k}}^{m_{2,k}}\|w_{k,m}\|_{L^\infty(U_{k,m})}. \tag{3-76}$$

Next, for the second term in the right-hand side of (3-76), we write the decomposition

$$\sum_{k\geq -1}\sum_{m=m_{1,k}}^{m_{2,k}}\|w^{\mathcal{B}}_{k,m}\|_{L^\infty(U_{k,m})} + \sum_{k\notin\mathcal{K}(\varepsilon)}\sum_{m=m_{1,k}}^{m_{3,k}}\|w^{\mathcal{G}}_{k,m}\|_{L^\infty(U_{k,m})} + \sum_{k\in\mathcal{K}(\varepsilon)}\sum_{m=m_{1,k}}^{m_{2,k}}\|w^{\mathcal{G}}_{k,m}\|_{L^\infty(U_{k,m})}. \tag{3-77}$$

Note that in the term with the sum over $k \notin \mathcal{K}(\varepsilon)$ we only sum over $m \le m_{3,k}$ for the same reason as in (3-57). We bound the three terms in (3-77) using (3-70), (3-72), (3-73), and (3-74) with $q = \infty$ and $N = \frac{1}{2} + \frac{\delta}{6}$. Combining with (3-76) this yields

$$\frac{1}{\mathfrak{D}_n}\|v\|_{L^\infty(B(x_\alpha,R(h)))} \le Ch^{(1-n)/2}\|u\|_{P,T}\left(\frac{\log T}{T^{1/2+\delta/6}} + R(h)^{n-1}|\mathcal{B}_U|T^{3/2+\delta/2} + \frac{3\varepsilon}{4\mathfrak{D}_n}\frac{\sqrt{t_0}}{\sqrt{T}} + O(h^\infty)\right).$$

Thus, if $\alpha \notin \bigcup_{k\in\mathcal{K}(\varepsilon)}\bigcup_{m\in\mathcal{M}(k,\varepsilon)}\mathcal{N}_{k,m}(\varepsilon)$, then $\|v\|_{L^\infty(B(x_\alpha,R(h)))} \le \varepsilon h^{(1-n)/2}\frac{\sqrt{t_0}}{\sqrt{T}}\|u\|_{P,T}$ for h small enough. In particular, $\alpha \notin \mathcal{S}_U(h,\varepsilon,u)$. This proves the claim (3-75).

3G3. *Decomposition of u.* We next decompose u as described in the theorem. First, put

$$\begin{aligned} u_{e,1} &:= \sum_{k\ge -1}\sum_{m=-\infty}^{m_{1,k}} w_{k,m} + \sum_{k\ge -1}\sum_{m=m_{1,k}}^{m_{2,k}} w^{\mathcal{B}}_{k,m} + \sum_{k\notin\mathcal{K}(\varepsilon)}\sum_{m=m_{1,k}}^{m_{3,k}} w^{\mathcal{G}}_{k,m} + \sum_{k\in\mathcal{K}(\varepsilon)}\sum_{m\notin\mathcal{M}(k,\varepsilon)} w^{\mathcal{G}}_{k,m},\\ u_{\text{big}} &:= \sum_{k\in\mathcal{K}(\varepsilon)}\sum_{m\in\mathcal{M}(k,\varepsilon)} w^{\mathcal{G}}_{k,m}, \end{aligned}$$

and $u_{e,2} := u - u_{\text{big}} - u_{e,1}$. Note that

$$\begin{aligned} \|u_{e,1}\|_{L^q} &\le \frac{3\varepsilon}{4}h^{-\delta(q)}\frac{\sqrt{t_0}}{\sqrt{T}}\|u\|_{P,T},\\ \|u_{e,2}\|_{L^q} &\le Ch^{-\delta(q)+1/2-\delta_2}h^{-1}\|Pu\|_{H_h^{(n-3)/2}}, \end{aligned}$$

where we use (3-70), (3-72), (3-73), (3-76), and (3-77) to obtain the first estimate, and (3-11) to obtain the second. These two estimates prove the claim on $\|u_e\|_{L^q}$ after combining them with (3-64). Next, observe that

$$u_{\text{big}} = \sum_{j\in\mathcal{L}(\varepsilon)} u_j,\quad u_j := \operatorname{Op}_h(\tilde\chi_{\mathcal{T}_j})\operatorname{Op}_h(\psi)u,\quad\text{and}\quad \mathcal{L}(\varepsilon) := \bigcup_{k\in\mathcal{K}(\varepsilon)}\bigcup_{m\in\mathcal{M}(k,\varepsilon)}\mathcal{G}_{k,m}.$$

We claim that the statement of the theorem holds with $v_j = \sqrt{T}u_j$. Note that the v_j are manifestly microsupported inside $\mathcal{T}_j$.

Let $\alpha \in \mathcal{S}_U(h,\varepsilon,u)$. Then by definition,

$$\|u_{\text{big}}\|_{L^\infty(B(x_\alpha,R(h)))} \ge \frac{\varepsilon}{4}h^{(1-n)/2}\frac{\sqrt{t_0}}{\sqrt{T}}\|u\|_{P,T}. \tag{3-78}$$

Note that for all $j \in \mathcal{L}(\varepsilon)$, the estimate

$$\|\operatorname{Op}_h(\tilde\chi_{\mathcal{T}_j})\operatorname{Op}_h(\psi)u\|_{L^2} + h^{-1}\|\operatorname{Op}_h(\tilde\chi_{\mathcal{T}_j})\operatorname{Op}_h(\psi)Pu\|_{L^2} \le 2^{-k_1^\varepsilon+1}\|u\|_{P,T} \tag{3-79}$$

follows from the definition of $\mathcal{A}_k$ in (3-14) and the fact that $\chi_{\mathcal{T}_j} \equiv 1$ on $\operatorname{supp}\tilde\chi_{\mathcal{T}_j}$. To see that u_j is a quasimode, we use the definition of $\mathcal{A}_k$ again, together with Proposition 2.5, and obtain

$$\|Pu_j\|_{L^2} \le \|[-h^2\Delta_g, \operatorname{Op}_h(\tilde\chi_{\mathcal{T}_j})]u_j\|_{L^2} + \|\operatorname{Op}_h(\tilde\chi_{\mathcal{T}_j})Pu\|_{L^2} \le C2^{-k_1^\varepsilon}h\|u\|_{P,T}. \tag{3-80}$$

The definition of k_1^ε together with (3-79) and (3-80) give the required bounds on v_j and Pv_j.

Next, define

$$\mathcal{L}(\varepsilon,u,\alpha):=\{j\in\mathcal{L}:\pi_M(\mathcal{T}_j)\cap B(x_\alpha,3R(h))\neq\varnothing\},$$

and note that by [Canzani and Galkowski 2021, Lemma 3.7],

$$\begin{aligned}
&\|u_{\mathrm{big}}\|_{L^\infty(B(x_\alpha,R(h)))}\\
&\le Ch^{(1-n)/2}R(h)^{(n-1)/2}\sum_{j\in\mathcal{L}(\varepsilon,u,\alpha)}\|\operatorname{Op}_h(\tilde{\chi}_{\mathcal{T}_j})\operatorname{Op}_h(\psi)u\|_{L^2}+h^{-1}\|\operatorname{Op}_h(\tilde{\chi}_{\mathcal{T}_j})\operatorname{Op}_h(\psi)Pu\|_{L^2}+O(h^\infty)\|u\|_{L^2}\\
&\le Ch^{(1-n)/2}R(h)^{(n-1)/2}2^{-k_1^\varepsilon}|\mathcal{L}(\varepsilon,u,\alpha)|\|u\|_{P,T}+O(h^\infty)\|u\|_{P,T}. \qquad (3\text{-}81)
\end{aligned}$$

(Note that in [Canzani and Galkowski 2021, Lemma 3.7], the number $\tau|H_pr_H(\rho_\gamma)|$ appears in the prefactor. In our circumstance, one can check that $|H_pr_H(\rho_\gamma)|=2$ and $\tau>0$ is a number uniformly bounded below by $c\operatorname{inj}(M)$ for some $c>0$.) Therefore, combining (3-78) with (3-81) yields

$$\varepsilon\frac{\sqrt{t_0}}{\sqrt{T}}\le CR(h)^{(n-1)/2}2^{-k_1^\varepsilon}|\mathcal{L}(\varepsilon,\alpha,u)|+O(h^\infty).$$

Moreover, $\bigcup_{j\in\mathcal{L}(\varepsilon,u)}\mathcal{T}_j$ is $[t_0,T]$ non-self-looping and so by Lemma 3.6, $|\mathcal{L}(\varepsilon,u)|\le(C_nt_0/T)2^{2k_2^\varepsilon}$. Using the definition of k_1^ε and k_2^ε in (3-71), we have, for h small enough,

$$c\varepsilon^2R(h)^{1-n}=\varepsilon\frac{\sqrt{t_0}}{\sqrt{T}}R(h)^{(1-n)/2}2^{k_1^\varepsilon}\le|\mathcal{L}(\varepsilon,u,\alpha)|\le|\mathcal{L}(\varepsilon,u)|\le\frac{C_nt_0}{T}2^{2k_2^\varepsilon}\le C\varepsilon^{-2}R(h)^{1-n},$$

which yields the upper bound on $|\mathcal{L}(\varepsilon,u)|$ and the lower bound on $|\mathcal{L}(\varepsilon,u,\alpha)|$. Note that the upper bound on $|\mathcal{L}(\varepsilon,u,\alpha)|$ follows from the fact that the total number of tubes over $B(x_\alpha,3R(h))$ is bounded by $CR(h)^{1-n}$. Next, we note that the fact that at most $\mathfrak{D}_n$ tubes $\mathcal{T}_j$ overlap implies

$$\sum_{j\in\mathcal{L}(\varepsilon,u,\alpha)}\|\operatorname{Op}_h(\tilde{\chi}_{\mathcal{T}_j})\operatorname{Op}_h(\psi)Pu\|_{L^2}^2\le C\|Pu\|_{L^2}^2+O(h^\infty\|u\|_{L^2}).$$

Therefore, using the first inequality in (3-81) again, applying Cauchy–Schwarz, and using that there is $C>0$ such that $|\mathcal{L}(\varepsilon,u,\alpha)|\le CR(h)^{1-n}$, we have

$$\begin{aligned}
\frac{\varepsilon}{4}\frac{\sqrt{t_0}}{\sqrt{T}}\|u\|_{P,T}&\le CR(h)^{(n-1)/2}|\mathcal{L}(\varepsilon,u,\alpha)|^{1/2}\Bigg(\sum_{j\in\mathcal{L}(\varepsilon,u,\alpha)}\|u_j\|_{L^2}^2\Bigg)^{1/2}+Ch^{-1}\|Pu\|_{L^2}+O(h^\infty)\|u\|_{L^2}\\
&\le C\Bigg(\sum_{j\in\mathcal{L}(\varepsilon,u,\alpha)}\|u_j\|_{L^2}^2\Bigg)^{1/2}+o(T^{-1}\|u\|_{L^2}). \qquad (3\text{-}82)
\end{aligned}$$

Here, the $o(T^{-1}\|u\|_{L^2})$ term comes from using (3-64). In particular, for h small enough,

$$c\frac{\sqrt{t_0}}{\sqrt{T}}\|u\|_{P,T}\le\Bigg(\sum_{j\in\mathcal{L}(\varepsilon,u,\alpha)}\|u_j\|^2\Bigg)^{1/2}.$$

This completes the proof of Theorem 3.8. □

4. Proof of Theorem 1.1

In order to finish the proof of Theorem 1.1, we need to verify that the hypotheses of Theorem 1.4 hold with $T(h) = b \log h^{-1}$ for some $b > 0$, such that, for all $x_1, x_2 \in U$, there is some splitting $\mathcal{J}_{x_1} = \mathcal{G}_{x_1,x_2} \cup \mathcal{B}_{x_1,x_2}$ of the set of tubes over $x_1 \in M$ with a set of "bad" tubes $\mathcal{B}_{x_1,x_2}$ satisfying

$$(|\mathcal{B}_{x_1,x_2}| R(h)^{n-1})^{(1-p_c/p)/(6+\varepsilon_0)} \leq T(h)^{-1/2}$$

and $\varepsilon_0 > 0$. Fix $x_1, x_2 \in U$ and let $F_1, F_2 : T^*M \to \mathbb{R}^{n+1}$ be smooth functions such that, for $i = 1, 2$,

$$\begin{gathered} S^*_{x_i}M = F_i^{-1}(0), \quad \tfrac{1}{2} d(q, S^*_{x_i}M) \leq |F_i(q)| \leq 2d(q, S^*_{x_i}M), \quad \max_{|\alpha|\leq 2}(|\partial^\alpha F_i(q)|) \leq 2, \\ dF_i(q) \text{ has a right inverse } R_{F_i}(q) \text{ with } \|R_{F_i}(q)\| \leq 2. \end{gathered} \tag{4-1}$$

Define also $\psi_i : \mathbb{R} \times T^*M \to \mathbb{R}^{n+1}$ by $\psi_i(t, \rho) = F_i \circ \varphi_t(\rho)$.

To find $\mathcal{B}_{x_1,x_2}$, we apply the arguments from [Canzani and Galkowski 2023, Sections 2, 4]. In particular, fix $a > 0$ and let $r_t := a^{-1} e^{-a|t|}$, $\Lambda > \Lambda_{\max}$, and $\Lambda_{\max}$ be as in (1-5). Suppose that $d(x_2\,, \mathcal{C}_{x_1}^{n-1, r_{t_0}, t_0}) > r_{t_0}$. Then for $\rho_0 \in S^*_{x_1}M$ with $d(S^*_{x_2}M, \varphi_{t_0}(\rho_0)) < r_{t_0}$, we have by [Canzani and Galkowski 2023, Lemma 4.1] that there exists $\boldsymbol{w} \in T_{\rho_0} S^*_{x_1}M$ such that

$$d(\psi_2)_{(t_0,\rho_0)} : \mathbb{R}\partial_t \times \mathbb{R}\boldsymbol{w} \to T_{\psi_2(t_0,\rho_0)}\mathbb{R}^{n+1}$$

has a left inverse $L_{(t_0,\rho_0)}$ satisfying

$$\|L_{(t_0,\rho_0)}\| \leq C_M \max(a e^{C_M(a+\Lambda)|t_0|}, 1),$$

Next, let $\{\Lambda^\tau_{\rho_j}(r_1)\}$ be a $(\mathfrak{D}_M, \tau, r_1)$-good cover for S^*M. We apply [Canzani and Galkowski 2023, Proposition 2.2] to construct $\mathcal{B}_{x_1,x_2}$ and $\mathcal{G}_{x_1,x_2}$.

Remark 4.1. We must point out that we are applying the proof of that proposition rather than the proposition as stated. The only difference here is that the loops we are interested in go from a point x_1 to a point x_2, where x_1 and x_2 are not necessarily equal. This does not affect the proof.

We use [Canzani and Galkowski 2023, Proposition 2.2] to see that there exist $\alpha_1 = \alpha_1(M) > 0$, $\alpha_2 = \alpha_2(M, a)$, and $\boldsymbol{C}_0 = \boldsymbol{C}_0(M, a)$ such that the following holds. Let $r_0, r_1, r_2 > 0$ satisfy

$$r_0 < r_1, \quad r_1 < \alpha_1 r_2, \quad r_2 \leq \min\{R_0, 1, \alpha_2 e^{-\gamma T}\}, \quad r_0 < \tfrac{1}{3} e^{-\Lambda T} r_2, \tag{4-2}$$

where $\gamma = 5\Lambda + 2a$ and $\Lambda > \Lambda_{\max}$ where $\Lambda_{\max}$ is as in (1-5). Then, for all balls $B \subset S^*_{x_1}M$ of radius $R_0 > 0$, there is a family of points $\{\rho_j\}_{j \in \mathcal{B}_B} \subset S^*_{x_1}M$ such that

$$|\mathcal{B}_B| \leq \boldsymbol{C}_0 \mathfrak{D}_n r_2 \frac{R_0^{n-1}}{r_1^{n-1}} T\, e^{4(2\Lambda+a)T},$$

and for $j \in \mathcal{G}_B := \{j \in \mathcal{J}_{x_1} : B(\rho_j, 2r_1) \cap B \neq \varnothing\} \setminus \mathcal{B}_B\}$,

$$\bigcup_{t \in [t_0, T]} \varphi_t(\Lambda^\tau_{\rho_j}(r_1)) \cap \Lambda^\tau_{S^*_{x_2}M}(r_1) = \varnothing.$$

We proceed to apply [Canzani and Galkowski 2023, Proposition 2.2]. There is $c_M r^{1-n} \geq N_r > 0$ such that, for all $x_1 \in M$, we can cover $S^*_{x_1}M$ by N_r balls. Let $0 < R_0 < 1$ and $\{B_i\}_{i=1}^{N_{R_0}}$ be such a cover. Fix $0 < \varepsilon < \varepsilon_1 < \frac{1}{4}$ and set

$$r_0 := h^{\varepsilon_1}, \quad r_1 := h^{\varepsilon}, \quad r_2 := \frac{2}{\alpha_1} h^{\varepsilon}.$$

Let $T(h) = b \log h^{-1}$ with

$$0 < b < \frac{1}{4\Lambda_{\max}} < \frac{1-2\varepsilon_1}{2\Lambda_{\max}}$$

to be chosen later. Then, the assumptions in (4-2) hold provided

$$h^{\varepsilon} < \min\left\{\tfrac{1}{2}\alpha_1\alpha_2 e^{-\gamma T},\, \tfrac{1}{2}\alpha_1 R_0\right\} \quad \text{and} \quad h^{\varepsilon_1-\varepsilon} < \frac{2}{3\alpha_1} e^{-\Lambda T}.$$

In particular, if we set $\alpha_3 = \frac{1}{2}\alpha_1\alpha_2$ and $\alpha_4 = \frac{2}{3}\alpha_1^{-1}$, the assumptions in (4-2) hold provided $h < \left(\frac{1}{2}\alpha_1 R(h)\right)^{1/\varepsilon}$ and

$$T(h) < \min\left\{\frac{\varepsilon}{\gamma}\log h^{-1} + \frac{\log \alpha_3}{\gamma},\ \frac{\varepsilon_1-\varepsilon}{\Lambda}\log h^{-1} + \frac{\log(\alpha_4)}{\Lambda}\right\}. \tag{4-3}$$

Fix $b > 0$ and $h_0 > 0$ such that $b < \frac{1}{12}\min(\varepsilon, \varepsilon_1 - \varepsilon)/(2\Lambda + a)$ and (4-3) is satisfied for all $h < h_0$. Note that this implies that $b = b(M, a)$ and $h_0 = h_0(M, a)$. Let $\mathcal{B}_{x_1,x_2} := \bigcup_{i=1}^{N_{R_0}} \mathcal{B}_{B_i}$. For $j \in \mathcal{G}_{x_1,x_2} := \mathcal{J}_{x_1} \setminus \mathcal{B}_{x_1,x_2}$, we then have

$$\bigcup_{t\in[t_0,T]} \varphi_t(\Lambda^{\tau}_{\rho_j}(r_1)) \cap \Lambda^{\tau}_{S^*_{x_2}M}(r_1) = \varnothing.$$

Moreover, shrinking h_0 in a way depending only on (M, a, ε), we have, for $0 < h < h_0$,

$$r_1^{n-1}|\mathcal{B}_{x_1,x_2}| \leq C_M \boldsymbol{C}_0 \mathfrak{D}_n r_2 T e^{4(2\Lambda+a)T} \leq h^{\varepsilon/3}.$$

Therefore, putting $R(h) = r_1 = h^{\varepsilon}$ and $T = T(h) = b \log h^{-1}$ in Theorem 1.4 proves Theorem 1.1.

5. Anisotropic pseudodifferential calculus

In this section, we develop the second microlocal calculi necessary to understand "effective sharing" of L^2 mass between two nearby points. That is, to answer the question: how much L^2 mass is necessary to produce high L^∞ growth at two nearby points? To that end, we develop a calculus associated to the coisotropic

$$\Gamma_x := \bigcup_{|t|<\frac{1}{2}\operatorname{inj}(M)} \varphi_t(\Omega_x), \quad \Omega_x := \{\xi \in T^*_x M : |1 - |\xi|_g| < \delta\},$$

which allows for localization to the Lagrangian leaves $\varphi_t(\Omega_x)$. In Section 6B we will see, using a type of uncertainty principle, that the calculi associated to two distinct points, $x_\alpha, x_\beta \in M$, are incompatible in the sense that, despite the fact that Γ_{x_α} and Γ_{x_β} intersect in a dimension 2 submanifold, for operators X_{x_α} and X_{x_β} localizing to Γ_{x_α} and Γ_{x_β}, respectively,

$$\|X_{x_\alpha} X_{x_\beta}\|_{L^2\to L^2} \ll \|X_{x_\alpha}\|_{L^2\to L^2} \|X_{x_\beta}\|_{L^2\to L^2}.$$

Let $\Gamma \subset T^*M$ be a coisotropic submanifold and $L = \{L_q\}_{q\in\Gamma}$ be a family of Lagrangian subspaces $L_q \subset T_q\Gamma$ that is integrable in the sense that if U is a neighborhood of Γ, and V and W are smooth vector fields on T^*M such that $V_q, W_q \in L_q$ for all $q \in \Gamma$, then $[V, W]_q \in L_q$ for all $q \in \Gamma$. The aim of this section is to introduce a calculus of pseudodifferential operators associated to (L, Γ) that allows for localization to h^ρ neighborhoods of Γ with $0 \le \rho < 1$ and is compatible with localization to h^ρ neighborhoods of the foliation of Γ generated by L. This calculus is close in spirit to those developed in [Dyatlov and Zahl 2016; Sjöstrand and Zworski 1999]. To see the relationships between these calculi, note that the calculus in [Dyatlov and Zahl 2016] allows for localization to any leaf of a Lagrangian foliation defined over an open subset of T^*M, while that in [Sjöstrand and Zworski 1999] allows for localization to a single hypersurface. The calculus developed in this paper is designed to allow localization along leaves of a Lagrangian foliation defined only over a coisotropic submanifold of T^*M. In the case that the coisotropic is a whole open set, this calculus is the same as the one developed in [Dyatlov and Zahl 2016]. Similarly, in the case that the coisotropic is a hypersurface and no Lagrangian foliation is prescribed, the calculus becomes that developed in [Sjöstrand and Zworski 1999].

Definition 5.1. Let Γ be a coisotropic submanifold and L a Lagrangian foliation on Γ. Fix $0 \le \rho < 1$ and let k be a positive integer. We say that $a \in S^k_{\Gamma,L,\rho}$ if $a \in C^\infty(T^*M)$, a is supported in an h-independent compact set, and

$$V_1 \cdots V_{\ell_1} W_1 \cdots W_{\ell_2} a = O(h^{-\rho\ell_2} \langle h^{-\rho} d(\Gamma, \cdot)\rangle^{k-\ell_2}), \tag{5-1}$$

where $W_1, \dots, W_{\ell_2}$ are any vector fields on T^*M, $V_1, \dots, V_{\ell_1}$ are vector fields on T^*M with

$$(V_1)_q, \dots, (V_{\ell_1})_q \in L_q$$

for $q \in \Gamma$, and $q \mapsto d(\Gamma, q)$ is the distance from q to Γ induced by the Sasaki metric on T^*M.

We also define symbol classes associated to only to the coisotropic submanifold Γ.

Definition 5.2. Let Γ be a coisotropic submanifold. We say that $a \in S^k_{\Gamma,\rho}$ if $a \in C^\infty(T^*M)$, a is supported in an h-independent compact set, and

$$V_1 \cdots V_{\ell_1} W_1 \cdots W_{\ell_2} a = O(h^{-\rho\ell_2} \langle h^{-\rho} d(\Gamma, \cdot)\rangle^{k-\ell_2})$$

where $V_1, \dots, V_{\ell_1}$ are tangent vector fields to Γ and $W_1, \dots, W_{\ell_2}$ are any vector fields.

5A. *Model case.*

The goal of this section is to define the quantization of symbols in $S^k_{\Gamma_0,L_0,\rho}$, where Γ_0 and L_0 are a model pair of coisotropic and Lagrangian foliation defined below. The model coisotropic submanifold of dimension $2n - r$ is

$$\Gamma_0 := \{(x', x'', \xi', \xi'') \in \mathbb{R}^r \times \mathbb{R}^{n-r} \times \mathbb{R}^r \times \mathbb{R}^{n-r} : x' = 0\}$$

with Lagrangian foliation

$$L_0 := \{L_{0,q}\}_{q\in\Gamma_0}, \quad L_{0,q} = \operatorname{span}\{\partial_{\xi_i}, i = 1, \dots, n\} \subset T_q\Gamma_0.$$

Note that in this model case the distance from a point (x,ξ) to Γ_0 is controlled by $|x'|$. Therefore, $a \in S^k_{\Gamma_0,L_0,\rho}$ if and only if a is supported in an h-independent compact set and, for all $(\alpha,\beta)\in\mathbb{N}^n\times\mathbb{N}^n$, there exists $C_{\alpha,\beta}>0$ such that

$$|\partial_x^\alpha\partial_\xi^\beta a| \le C_{\alpha,\beta}h^{-\rho|\alpha|}\langle h^{-\rho}|x'|\rangle^{k-|\alpha|}.$$

In the model case, it will be convenient to define $\tilde{a}\in C^\infty(\mathbb{R}^n_x\times\mathbb{R}^n_\xi\times\mathbb{R}^r_\lambda)$ such that

$$a(x,\xi)=\tilde{a}(x,\xi,h^{-\rho}x'),$$

and, for all $(\alpha',\alpha'',\beta,\gamma)\in\mathbb{N}^r\times\mathbb{N}^{n-r}\times\mathbb{N}^n\times\mathbb{N}^r$, there exists $C_{\alpha,\beta,\gamma}>0$ such that

$$|\partial_{x'}^{\alpha'}\partial_{x''}^{\alpha''}\partial_\xi^\beta\partial_\lambda^\gamma\tilde{a}(x,\xi,\lambda)| \le C_{\alpha,\beta,\gamma}h^{-\rho|\alpha''|}\langle\lambda\rangle^{k-|\gamma|-|\alpha''|}. \tag{5-2}$$

Similarly, if $a\in S^k_{\Gamma_0,\rho}$, then, for $(\alpha',\alpha'',\beta,\gamma)\in\mathbb{N}^r\times\mathbb{N}^{n-r}\times\mathbb{N}^n\times\mathbb{N}^r$, there exists $C_{\alpha,\beta,\gamma}>0$ such that

$$|\partial_{x'}^{\alpha'}\partial_{x''}^{\alpha''}\partial_\xi^\beta\partial_\lambda^\gamma\tilde{a}(x,\xi,\lambda)| \le C_{\alpha,\beta,\gamma}\langle\lambda\rangle^{k-|\gamma|}. \tag{5-3}$$

Definition 5.3. The symbols associated with this submanifold are as follows: We say $a\in\widetilde{S}^k_{\Gamma_0,L_0,\rho}$ if $a\in C^\infty(\mathbb{R}^n_x\times\mathbb{R}^n_\xi\times\mathbb{R}^r_\lambda)$ satisfies (5-2) and a is supported in an h-independent compact set in (x,ξ). If we have the improved estimates (5-3) then we say that $a\in\widetilde{S}^k_{\Gamma_0,\rho}$.

Remark 5.4. While there is no ρ in the definition of $\widetilde{S}^k_{\Gamma_0,\rho}$, we keep it in the notation for consistency.

Let $a\in\widetilde{S}^k_{\Gamma_0,L_0,\rho}$. We then define

$$[\widetilde{\mathrm{Op}}_h(a)]u(x) := \frac{1}{(2\pi h)^n}\int e^{i\langle x-y,\xi\rangle/h}a(x,\xi,h^{-\rho}x')u(y)\,dy\,d\xi.$$

Since $a\in\widetilde{S}^k_{\Gamma_0,L_0,\rho}$ is compactly supported in x, there exists $C>0$ such that on the support of the integrand $|\lambda|\le Ch^{-\rho}$, and hence $h\le Ch^{1-\rho}\langle\lambda\rangle^{-1}$. This will be important when computing certain asymptotic expansions.

Lemma 5.5. *Let $k\in\mathbb{R}$ and $a\in\widetilde{S}^k_{\Gamma_0,L_0,\rho}$. Then,*

$$\|\widetilde{\mathrm{Op}}_h(a)\|_{L^2\to L^2} \le C\sup_{\mathbb{R}^{2n}}|a(x,\xi,h^{-\rho}x')| + O(h^{-\rho\max(k,0)+(1-\rho)/2}).$$

Proof. Define $T_\delta : L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)$ by

$$T_\delta u(x) := h^{n\delta/2}u(h^\delta x). \tag{5-4}$$

Then T_δ is unitary and, for $a\in\widetilde{S}^k_{\Gamma_0,L_0,\rho}$,

$$\widetilde{\mathrm{Op}}_h(a)u = T^{-1}_{(1+\rho)/2}\,\mathrm{Op}_1(a_h)T_{(1+\rho)/2}u,\quad a_h(x,\xi):=a(h^{(1+\rho)/2}x,h^{(1-\rho)/2}\xi,h^{(1-\rho)/2}x').$$

Then, for all $\alpha,\beta\in\mathbb{N}^n$, there exists $C_{\alpha,\beta}$ such that

$$|\partial_x^\alpha\partial_\xi^\beta a_h| \le C_{\alpha,\beta}h^{(1-\rho)(|\alpha|+|\beta|)/2}\langle h^{(1-\rho)/2}x'\rangle^{k-|\alpha|}.$$

Now, since $a_h \in S_{(1-\rho)/2}$, by [Zworski 2012, Theorem 4.23] there is a universal constant $M > 0$ with

$$\|\widetilde{\operatorname{Op}}_1(a_h)\|_{L^2\to L^2} \le C \sum_{|\alpha|\le Mn} \sup_{\mathbb{R}^{2n}} |\partial^\alpha a_h| \le C \sup|a| + C_a h^{-\max(\rho k,0)+(1-\rho)/2}.$$

(To see that [Zworski 2012, Theorem 4.23] applies equally well to the left quantization, we apply the change of quantization formula [Zworski 2012, Theorem 4.13] and the boundedness of $e^{i\langle QD,D\rangle/2}$ on symbol classes [Zworski 2012, Theorem 4.17].) ☐

Lemma 5.6. *Suppose that $a \in \widetilde{S}^{k_1}_{\Gamma_0,L_0,\rho}$ and $b \in \widetilde{S}^{k_2}_{\Gamma_0,L_0,\rho}$. Then*

$$\widetilde{\operatorname{Op}}_h(a)\widetilde{\operatorname{Op}}_h(b) = \widetilde{\operatorname{Op}}_h(c) + O(h^\infty)_{L^2\to L^2},$$

where $c \in \widetilde{S}^{k_1+k_2}_{\Gamma_0,L_0,\rho}$ satisfies

$$c = ab + O(h^{1-\rho})_{\widetilde{S}^{k_1+k_2-1}_{\Gamma_0,L_0,\rho}}. \tag{5-5}$$

In particular,

$$c \sim \sum_j \sum_{|\alpha|=j} \frac{i^j}{j!}((hD_{x''})^{\alpha''}(hD_{x'} + h^{1-\rho}D_\lambda)^{\alpha'} b) D_\xi^\alpha a. \tag{5-6}$$

If instead $a \in \widetilde{S}^{k_1}_{\Gamma_0,\rho}$ and $b \in \widetilde{S}^{k_2}_{\Gamma_0,\rho}$, then the remainder in (5-5) *lies in $h^{1-\rho}\widetilde{S}^{k_1+k_2-1}_{\Gamma_0,\rho}$.*

Proof. With T_δ as in (5-4), we have $\widetilde{\operatorname{Op}}_h(a)\widetilde{\operatorname{Op}}_h(b) = T^{-1}_{\rho/2}\operatorname{Op}_h(a_h)\operatorname{Op}_h(b_h)T_{\rho/2}$, where

$$a_h = a(h^{\rho/2}x, h^{-\rho/2}\xi, h^{-\rho/2}x') \quad \text{and} \quad b_h = b(h^{\rho/2}x, h^{-\rho/2}\xi, h^{-\rho/2}x').$$

Now, for all $\alpha, \beta \in \mathbb{N}^n$, there exists $C_{\alpha,\beta}$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a_h| \le C_{\alpha,\beta} h^{-\rho(|\alpha|+|\beta|)/2}\langle h^{-\rho/2}x'\rangle^{k_1-|\alpha|} \quad \text{and} \quad |\partial_x^\alpha \partial_\xi^\beta b_h| \le C_{\alpha,\beta} h^{-\rho(|\alpha|+|\beta|)/2}\langle h^{-\rho/2}x'\rangle^{k_2-|\alpha|}.$$

In particular, using that a and b are compactly supported, we have that $a_h \in h^{-\max(\rho k_1,0)}S_{\rho/2}$ and $b_h \in h^{-\max(\rho k_2,0)}S_{\rho/2}$, and hence [Zworski 2012, Theorems 4.14, 4.17] apply. In particular, if we let $M > 0$ and $\tilde{k} := \max(k_1,0) + \max(k_2,0)$, we obtain $\operatorname{Op}_h(a_h)\operatorname{Op}_h(b_h) = \operatorname{Op}_h(c_h)$, where, for any $N > 0$,

$$\begin{aligned} c_h(x,\xi) &= \sum_{j=0}^{N-1} \sum_{|\alpha|=j} \frac{h^j i^j}{j!}(D_\xi^\alpha a_h(x,\xi))(D_x^\alpha b_h(x,\xi)) + O(h^{-\rho\tilde{k}+N(1-\rho)})_{S_{\rho/2}} \\ &= \sum_{j=0}^{N-1} \sum_{|\alpha|=j} \sum_{\alpha'+\alpha''=\alpha} \frac{h^{(1-\rho)j} i^j}{j!}(D_\xi^\alpha a)_h[(h^\rho D_{x''})^{\alpha''}(h^\rho D_{x'} + D_\lambda)^{\alpha'} b]_h + O(h^{-\rho\tilde{k}+N(1-\rho)})_{S_{\rho/2}}. \end{aligned}$$

Choosing

$$N = \max\left(k_1 + k_2, \frac{\rho\tilde{k} + M}{1-\rho}\right),$$

the remainder is $O(h^M)_{S_{\rho/2}}$. Moreover, since a and b were compactly supported, we may assume, introducing an h^∞ error, that the remainder is supported in $\{(x,\xi) : |(x,\xi)| \le Ch^{-\rho/2}\}$. Putting

$$c = \sum_{j=0}^{N-1} \sum_{|\alpha|=j} \sum_{\alpha'+\alpha''=\alpha} \frac{i^j}{j!}(D_\xi^\alpha a)[(hD_{x''})^{\alpha''}(hD_{x'} + h^{1-\rho}D_\lambda)^{\alpha'} b],$$

we thus have $T^{-1}_{\rho/2}\operatorname{Op}_h(c_h)T_{\rho/2} = \widetilde{\operatorname{Op}}_h(c) + O(h^M)_{\mathcal{D}'\to C^\infty}$ as claimed. ☐

Lemma 5.7. *Suppose that $a \in \widetilde{S}^{m_1}_{\Gamma_0,L_0,\rho}$ and $b \in \widetilde{S}^{m_2}_{\Gamma_0,L_0,\rho}$. Then,*

$$[\widetilde{\mathrm{Op}}_h(a), \widetilde{\mathrm{Op}}_h(b)] = -ih^{1-\rho}\widetilde{\mathrm{Op}}_h(c) + O(h^\infty)_{L^2\to L^2},$$

where $c \in \widetilde{S}^{m_1+m_2-2}_{\Gamma_0,L_0,\rho}$ satisfies

$$c = h^\rho \sum_{i=1}^{n}(\partial_{\xi_i} a\partial_{x_i} b - \partial_{\xi_i} b\partial_{x_i} a) + \sum_{i=1}^{r}(\partial_{\xi_i} a\partial_{\lambda_i} b - \partial_{\lambda_i} a\partial_{\xi_i} b) + O(h^{1-\rho})_{\widetilde{S}^{m_1+m_2-2}_{\Gamma_0,L_0,\rho}}.$$

If instead $a \in \widetilde{S}^{m_1}_{\Gamma_0,\rho}$ and $b \in \widetilde{S}^{m_2}_{\Gamma_0,\rho}$, then the remainder lies in $h^{1-\rho}\widetilde{S}^{m_1+m_2-2}_{\Gamma_0,\rho}$. Moreover, if $a \in S^{\mathrm{comp}}(\mathbb{R}^{2n})$ is independent of λ and $\partial_{\xi'} a = e(x,\xi)x'$ with $e(x,\xi) : \mathbb{R}^r \to \mathbb{R}^r$ for all (x,ξ), then

$$[\widetilde{\mathrm{Op}}_h(a), \widetilde{\mathrm{Op}}_h(b)] = -ih\widetilde{\mathrm{Op}}_h(c) + O(h^\infty)_{\Psi^{-\infty}}$$

with $c = H_a b + \sum_{i=1}^{r}(e\lambda)_i\partial_{\lambda_i} b + O(h^{1-\rho})_{\widetilde{S}^{m_2-1}_{\Gamma_0,L_0,\rho}}$. Similarly, the same conclusion holds if $b \in \widetilde{S}^{m_2}_{\Gamma_0,\rho}$ with the error term in c being $O(h^{1-\rho})_{\widetilde{S}^{m_2-1}_{\Gamma_0,\rho}}$.

Proof. In each case, we need only apply formula (5-6). □

5B. *Reduction to normal form.* In order to define the quantization of symbols in $S_{\Gamma,L,\rho}$ for general (Γ, L), we first explain how to reduce the problem to the model case (Γ_0, L_0).

Lemma 5.8. *Let L be a Lagrangian foliation over a coisotropic submanifold $\Gamma \subset \mathbb{R}^{2n}$ of dimension $2n-r$. Then, there is a neighborhood U_0 of (x_0, ξ_0) and a symplectomorphism $\kappa : U_0 \to V_0 \subset T^*\mathbb{R}^n$ for each $(x_0, \xi_0) \in \Gamma$ such that*

$$\kappa(\Gamma \cap U_0) = \Gamma_0 \cap V_0 \qquad \text{and} \qquad (\kappa_*)_q L_q = L_{0,q} \quad \text{for } q \in \Gamma \cap U_0.$$

Proof. We first put Γ in normal form. That is, we build symplectic coordinates (y, η) such that

$$\Gamma = \{(y,\eta) : y_1 = \cdots = y_r = 0\}. \tag{5-7}$$

First, assume $r = 1$ and let $f_1 \in C^\infty(T^*M)$ define Γ. By Darboux's theorem (see e.g., [Zworski 2012, Theorem 12.1]) there are symplectic coordinates such that $y_1 = f_1$, and the proof of (5-7) is complete for $r = 1$.

Next, assume that we can put any coisotropic of codimension $r-1$ in normal form. Let $f_1, \dots, f_r \in C^\infty(T^*M)$ define Γ. Then, for $X \in T\Gamma$ and $i = 1, \dots, r$,

$$\sigma(X, H_{f_i}) = df_i(X) = 0.$$

In addition, since Γ is coisotropic, $(T\Gamma)^\perp \subset T\Gamma$, and so $H_{f_i} \in T\Gamma$ for all $i = 1, \dots, r$. In particular,

$$\{f_i, f_j\} = H_{f_i} f_j = df_j(H_{f_i}) = 0 \quad \text{on } \Gamma.$$

Now, using Darboux' theorem, choose symplectic coordinates $(y,\eta) = (y_1, y', \eta_1, \eta')$ such that $y_1 = f_1$ and $(x_0, \xi_0) \mapsto (0, 0)$. Then, $\partial_{\eta_1} f_j = \{f_j, y_1\} = 0$ on Γ for $j = 2, \dots, r$. Next, we will observe that $\Gamma = \{(y,\eta) : y_1 = f_2 = \cdots = f_r = 0\}$ and dy_1 and $\{df_j\}_{j=2}^r$ are independent. Thus, since $\partial_{\eta_1} f_j = 0$ on Γ,

$$\Gamma = \{(y,\eta) : y_1 = 0,\ f_j(0, y', 0, \eta') = 0,\ j = 2, \dots, r\}.$$

Now, $\{y_1=\eta_1=0\}\cap\Gamma$ is a coisotropic submanifold of codimension $r-1$ in $T^*\{y_1=0\}$. Hence, by induction, there are symplectic coordinates $(y_2,\dots,y_n,\eta_2,\dots,\eta_n)$ on $T^*\{y_1=0\}$ such that

$$\Gamma\cap\{y_1=\eta_1=0\}=\{y_1=\eta_1=0,\ y_2=\cdots=y_r=0\}.$$

In particular,

$$\{(y',\eta'):f_j(0,y',0,\eta')=0,\ j=2,\dots,r\}=\{y_2=\cdots=y_r=0\}.$$

Thus, extending $(y_2,\dots,y_n,\eta_2,\dots,\eta_n)$ to be independent of (y_1,η_1) puts Γ in the form (5-7).

Next, we adjust the coordinates to be adapted to L along Γ. First, define $\tilde{y}_i:=y_i$ for $i=1,\dots,r$. Then, since $L\subset T\Gamma$, for every $i=1,\dots,r$, we have that $d\tilde{y}_i(X)|_\Gamma$ is well defined for $X\in L$ and $d\tilde{y}_i(X)|_\Gamma=0$. Next, since L is integrable, the Frobenius theorem [Lee 2013, Theorem 19.21] shows that there are coordinates $(\tilde{y}_{r+1},\dots,\tilde{y}_n,\tilde{\xi}_1,\dots,\tilde{\xi}_n)$ on Γ, defined in a neighborhood of $(0,0)$, such that L is the annihilator of $d\tilde{y}$. Since we know that for every $X\in L$,

$$\sigma(X,H_{\tilde{y}_i})=d\tilde{y}_i(X)=0$$

and L is Lagrangian, we conclude that $H_{\tilde{y}_i}\in L$. In particular, since L is the annihilator of $d\tilde{y}$,

$$\{\tilde{y}_i,\tilde{y}_j\}=H_{\tilde{y}_i}\tilde{y}_j=d\tilde{y}_j(H_{\tilde{y}_i})=0.$$

Now, extend $(\tilde{y}_{r+1},\dots,\tilde{y}_n,\tilde{\xi}_1,\dots,\tilde{\xi}_n)$ outside Γ to be independent of $(\tilde{y}_1,\dots,\tilde{y}_r)$. Then, $\{\tilde{y}_i,\tilde{y}_j\}=0$ in a neighborhood of (x_0,ξ_0), and hence, by Darboux's theorem, there are functions $\{\tilde{\eta}_j\}_{j=1}^n$ such that $\{\tilde{y}_i,\tilde{\eta}_j\}=\delta_{ij}$ and $\{\tilde{\eta}_i,\tilde{\eta}_j\}=0$. In particular, in the $(\tilde{y},\tilde{\eta})$ coordinates, $\Gamma=\{(\tilde{y},\tilde{\eta}):\tilde{y}_1=\cdots=\tilde{y}_r=0\}$ and $d\tilde{y}(L)|_\Gamma=0$. In particular, $L=\operatorname{span}\{\partial\tilde{\eta}_i\}$ as claimed. □

In order to create a well-defined global calculus of pseudodifferential operators associated to (Γ,L), we will need to show invariance under conjugation by Fourier integral operators (FIOs) preserving the pair (L_0,Γ_0).

Proposition 5.9. *Suppose that U_0 and V_0 are neighborhoods of $(0,0)$ in $T^*\mathbb{R}^n$ and $\kappa:U_0\to V_0$ is a symplectomorphism such that*

$$\kappa(0,0)=(0,0),\quad \kappa(\Gamma_0\cap U_0)=\Gamma_0\cap V_0,\quad \kappa_*|_{\Gamma_0}L_0=L_0|_{\Gamma_0}. \tag{5-8}$$

Next, let T be a semiclassically elliptic FIO microlocally defined in a neighborhood of

$$((0,0),(0,0))\in T^*\mathbb{R}^n\times T^*\mathbb{R}^n$$

quantizing κ. Then, for $a\in\widetilde{S}^k_{\Gamma_0,L_0,\rho}$, there are $b\in\widetilde{S}^k_{\Gamma_0,L_0,\rho}$ and $c\in\widetilde{S}^{k-1}_{\Gamma_0,L_0,\rho}$ such that

$$T^{-1}\widetilde{\mathrm{Op}}_h(a)T=\widetilde{\mathrm{Op}}_h(b)\quad\textit{and}\quad b=a\circ K_\kappa+h^{1-\rho}c,$$

where $K_\kappa:T^*\mathbb{R}^n\times\mathbb{R}^r\to T^*\mathbb{R}^n\times\mathbb{R}^r$ is defined by

$$K_\kappa(y,\eta,\mu)=\left(\kappa(y,\eta),\pi_{x'}(\kappa(y,\eta))\frac{|\mu|}{|y'|}\right),$$

and $\pi_{x'}:T^*\mathbb{R}^n\to\mathbb{R}^r$ is the projection onto the first r-spatial coordinates. In addition, if $a\in\widetilde{S}^k_{\Gamma_0,\rho}$, then $c\in\widetilde{S}^{k-1}_{\Gamma_0,\rho}$ and $b\in\widetilde{S}^k_{\Gamma_0,\rho}$.

To prove Proposition 5.9, we follow [Sjöstrand and Zworski 1999]. First, observe that the proposition holds with $\kappa = \mathrm{Id}$ since then T is a standard pseudodifferential operator. In addition, the proposition also holds whenever, for a given $j \in \{1, \dots, n\}$, we work with

$$\kappa(y, \eta) := (y_1, \dots, y_{j-1}, -y_j, y_{j+1}, \dots, y_n, \eta_1, \dots, \eta_{j-1}, -\eta_j, \eta_{j+1}, \dots, \eta_n).$$

Indeed, this follows from the fact that in this case an FIO quantizing κ is

$$Tu(x) = u(x_1, \dots, x_{j-1}, -x_j, x_{j+1}, \dots, x_n),$$

and so the conclusion of the proposition follows from a direct computation together with the identity case. Thus, we may assume that

$$\kappa(y, \eta) = (x, \xi) \quad \Rightarrow \quad x_i y_i \geq 0, \quad i = 1, \dots, n. \tag{5-9}$$

Lemma 5.10. *Let κ be a symplectomorphism satisfying* (5-8) *and* (5-9). *Then, there is a piecewise smooth family of symplectomorphisms* $[0, 1] \ni t \mapsto \kappa_t$ *such that κ_t satisfies* (5-8), (5-9), $\kappa_0 = \mathrm{Id}$, *and* $\kappa_1 = \kappa$.

Proof. In what follows we assume that $\kappa(y, \eta) = (x, \xi)$ but reorder the coordinates: $(y', y'', \eta', \eta'') \in T^*\mathbb{R}^n$ is written as $(y', \eta', y'', \eta'') \in \mathbb{R}^{2r} \times \mathbb{R}^{2(n-r)}$. Let ξ' and $\kappa'' = (x''(y', \eta), \xi''(y', \eta))$ with

$$\kappa|_{\Gamma_0} : (0, \eta', y'', \eta'') \mapsto (0, \xi'(y'', \eta), \kappa''(y'', \eta)).$$

Now, since $(\kappa_*)|_{\Gamma_0} L_0 = L_0$, we have, for $i = 1, \dots, n$,

$$\kappa_* \partial_{\eta_i} = \frac{\partial x_j}{\partial \eta_i} \partial_{x_j} + \frac{\partial \xi_j}{\partial \eta_i} \partial_{\xi_j} \in L_0, \tag{5-10}$$

and hence

$$\partial_\eta x|_{\Gamma_0} \equiv 0. \tag{5-11}$$

Next, since κ preserves Γ_0, $\{\kappa^* x_i\}_{i=1}^r$ defines Γ_0, and $\mathrm{span}\{d\kappa^* x_i|_{\Gamma_0}\}_{i=1}^r = \mathrm{span}\{dy_i|_{\Gamma_0}\}_{i=1}^r$, we have

$$\mathrm{span}\{H_{\kappa^* x_i}|_{\Gamma_0}\}_{i=1}^r = \mathrm{span}\{H_{y_i}|_{\Gamma_0}\}_{i=1}^r.$$

By Jacobi's theorem, $\kappa_* H_{\kappa^* x^i} = H_{x_i}$. Therefore,

$$(\kappa|_{\Gamma_0})_*(\mathrm{span}\{H_{y_i}\}_{i=1}^r|_{\Gamma_0}) = \mathrm{span}\{H_{x_i}\}_{i=1}^r|_{\Gamma_0},$$

and we conclude from (5-10) that $\xi''|_{\Gamma_0}$ is independent of η', and hence that κ'' is independent of η'. In particular, κ'' is a symplectomorphism on $T^*\mathbb{R}^{n-r}$. This also implies that, for each fixed (y'', η''), the map $\eta' \mapsto \xi'(y'', \eta', \eta'')$ is a diffeomorphism. Writing

$$\kappa''(y'', \eta'') = (x''(y'', \eta''), \xi''(y'', \eta'')),$$

we have by (5-11) that $\partial_{\eta''} x'' = 0$, and hence $x'' = x''(y'')$. Now, since κ'' is symplectic,

$$(\partial_{\eta''} \xi'' d\eta'' + \partial_{y''} \xi'' dy'') \wedge \partial_{y''} x'' dy'' = d\eta'' \wedge dy'',$$

and so we conclude that

$$(\partial_{y''} x'')^t \partial_{\eta''} \xi'' = \mathrm{Id}, \quad (\partial_{y''} x'')^t \partial_{y''} \xi'' \text{ is diagonal.} \tag{5-12}$$

The first equality in (5-12) gives that $\partial_{\eta''}\xi''$ is a function of y'' only, and hence there exists a function $F = F(y'')$ such that

$$\xi''(y'', \eta'') = [(\partial x''(y''))^t]^{-1}(\eta'' - F(y'')).$$

Therefore, calculating on $\eta'' = F(y'')$, the second statement in (5-12) implies that $-\partial_{y''}F(y'')\,dy''\wedge dy'' = 0$. In particular, $d(F(y'')\cdot dy'') = 0$. It follows from the Poincaré lemma that, shrinking the neighborhood of $(0,0)$ to be simply connected if necessary, $F(y'')\cdot dy'' = d\psi(y'')$ for some function $\psi = \psi(y'')$. Hence,

$$\kappa''(y'', \eta'') = (x''(y''), [(dx''(y''))^t]^{-1}(\eta'' - \partial\psi(y''))). \tag{5-13}$$

Now, every symplectomorphism of the form (5-13) preserves L_0. Hence, we can deform κ'' to the identity by putting $\psi_t = t\psi$ and deforming x'' to the identity. Since the assumption in (5-9) implies $\partial_{y''}x'' > 0$, this can be done simply by taking $x''_t = (1-t)\,\mathrm{Id} + tx''$. Putting $\kappa''_t = (x''_t, \xi''_t)$, there is κ''_t such that $\kappa''_0 = \mathrm{Id}$ and $\kappa''_1 = \kappa''$. Now, composing κ with

$$(y', \eta'; y'', \eta'') \mapsto (y', \eta'; (\kappa''_t)^{-1}(y'', \eta'')),$$

we reduce to the case that $\kappa'' = \mathrm{Id}$. In particular, we need only consider the case in which

$$\kappa(y', \eta', y'', \eta'') = (f(y,\eta)y', \xi'(y'', \eta) + h_0(y,\eta)y', (y'', \eta'') + h_1(y,\eta)y'), \tag{5-14}$$

where $f(y,\eta) \in \mathbb{GL}_r$, $h_0(y,\eta)$ is an $r\times r$ matrix, and $h_1(y,\eta)$ is an $2(n-r)\times r$ matrix. Next, we claim that the projection map from $\operatorname{graph}(\kappa)$ to $\mathbb{R}^{2n}$ defined as $(x,\xi; y,\eta) \mapsto (x,\eta)$ is a local diffeomorphism. To see this, note that, for $|y'|$ small, the map $(x'', \eta'') \mapsto (y'', \xi'')$ is a diffeomorphism, that, for each fixed (y'', η''), the map $\eta' \mapsto \xi'$ is a diffeomorphism, and that $\det \partial_{y'}x'|_{\Gamma_0} \neq 0$. Thus, κ has a generating function ϕ:

$$\kappa : (\partial_\eta\phi(x,\eta), \eta) \mapsto (x, \partial_x\phi(x,\eta))$$

such that

$$\det \partial^2_{x\eta}\phi(0,0) \neq 0 \quad \text{and} \quad \partial_{\eta'}\phi(0, x'', \eta) = 0.$$

Now, writing $\kappa = (\kappa', \kappa'')$, we have $\kappa'' = \mathrm{Id}$ at $x' = 0$. Therefore,

$$\partial_{\eta''}\phi(0, x'', \eta) = x'' \quad \text{and} \quad \partial_{x''}\phi(0, x'', \eta) = \eta'',$$

and we have $\phi(0, x'', \eta) = \langle x'', \eta''\rangle + C$ for some $C \in \mathbb{R}$. We may choose $C = 0$ to obtain

$$\phi(x,\eta) = \langle x'', \eta''\rangle + g(x,\eta)x' \tag{5-15}$$

for some $g : \mathbb{R}^{2n} \to \mathbb{M}_{1\times r}$. Finally, since $\kappa(0,0) = (0,0)$ and $\partial^2_{x\eta}\phi$ is nondegenerate, we have that $\partial_{x'}\phi(0,0) = g(0,0) = 0$ and $\partial_{\eta'}g$ is nondegenerate. In fact (5-9) implies that, as a quadratic form,

$$\partial_{\eta'}g > 0. \tag{5-16}$$

Observe next that every ϕ satisfying (5-15) for some g satisfying (5-16) and $g(0,0) = 0$ generates a canonical transformation satisfying (5-14) and (5-9). In particular, the symplectomorphism satisfies (5-8). Thus, we can deform from the identity by putting $g_t = (1-t)\eta' + tg$. □

Finally, we proceed with the proof of Proposition 5.9.

Proof of Proposition 5.9. Let κ_t be as in Lemma 5.10. That is, a piecewise smooth deformation from $\kappa_0 = \mathrm{Id}$ to $\kappa_1 = \kappa$ such that κ_t preserves Γ_0 and $(\kappa_t)_*|_{\Gamma_0}$ preserves L_0. Let T_t be a piecewise smooth family of elliptic FIOs defined microlocally near (0, 0), quantizing κ_t, and satisfying

$$hD_tT_t + T_tQ_t = 0 \quad \text{and} \quad T_0 = \mathrm{Id}. \tag{5-17}$$

Here, Q_t is a smooth family of pseudodifferential operators with symbol q_t satisfying $\partial_t\kappa_t = (\kappa_t)_*H_{q_t}$. (Such an FIO exists, for example, by [Zworski 2012, Chapter 10], and q_t exists by [Zworski 2012, Thoerems 11.3, 11.4].) Next, define

$$A_t := T_t^{-1}\widetilde{\mathrm{Op}}_h(a)T_t.$$

Note that $T^{-1}\widetilde{\mathrm{Op}}(a)T = T^{-1}T_1T_1^{-1}\widetilde{\mathrm{Op}}(a)T_1T_1^{-1}T + O(h^\infty)_{\Psi^{-\infty}}$. Hence, since the proposition follows by direct calculation when $\kappa = \mathrm{Id}$, we may assume that $T = T_1$.

In that case, our goal is to find a symbol b such that $A_1 = \mathrm{Op}_h(b)$. First, observe that (5-17) implies that $hD_tT_t^{-1} - Q_tT_t^{-1} = 0$ and so

$$hD_tA_t = [Q_t, A_t] \quad \text{and} \quad A_0 = \widetilde{\mathrm{Op}}_h(a).$$

We will construct $b_t \in \widetilde{S}^k_{\Gamma_0,L_0,\rho}$ such that $B_t := \widetilde{\mathrm{Op}}_h(b_t)$ satisfies

$$hD_tB_t = [Q_t, B_t] + O(h^\infty)_{\Psi^{-\infty}} \quad \text{and} \quad B_0 = \widetilde{\mathrm{Op}}_h(a). \tag{5-18}$$

This would yield that $B_t - A_t = O(h^\infty)_{L^2\to L^2}$ and the argument would then be finished by setting $b = b_1$. Indeed, that $B_t - A_t = O(h^\infty)_{L^2\to L^2}$ would follow from the fact that, by (5-18),

$$hD_t(T_tB_tT_t^{-1}) = O(h^\infty)_{\Psi^{-\infty}},$$

and hence, since $T_0 = \mathrm{Id}$ and $B_0 = \widetilde{\mathrm{Op}}_h(a)$, we have $T_tB_tT_t^{-1} - \widetilde{\mathrm{Op}}_h(a) = O(h^\infty)_{\Psi^{-\infty}}$. Combining this with the fact that both T_t and T_t^{-1} are bounded on H_h^k completes the proof.

To find b_t as in (5-18), note that since κ_t preserves Γ_0 and L_0, $\partial_t\kappa_t = H_{q_t}$ and H_{q_t} is tangent to L_0 on Γ_0. Therefore, $\partial_{\eta'}q_t = 0$ on $y' = 0$, and so there exists $r_t(y,\eta)$ such that $\partial_{\eta'}q_t(y,\eta) = r_t(y,\eta)y'$. Hence, by Lemma 5.7, for any $b \in \widetilde{S}^k_{\Gamma_0,L_0,\rho}$,

$$[Q_t, \widetilde{\mathrm{Op}}_h(b)] = -ih\widetilde{\mathrm{Op}}_h(f) + O(h^\infty)_{\Psi^{-\infty}} \quad \text{and} \quad f = H_{q_t}b + \sum_{j=1}^r (r_t\lambda)_j(\partial_\lambda b)_j + O(h^{1-\rho})_{\widetilde{S}^{k-2}_{\Gamma_0,L_0,\rho}}.$$

Then, letting $b_t^0 := a \circ K_{\kappa_t} \in \widetilde{S}^k_{\Gamma_0,L_0,\rho}$ and $B_t^0 = \widetilde{\mathrm{Op}}_h(b_t^0)$ yields

$$hD_tB_t^0 = -ih\widetilde{\mathrm{Op}}_h(H_{q_t}b_t^0 + (r_t\mu)\cdot\partial_\mu b_t^0) = [Q_t, B_t^0] + h^{2-\rho}\widetilde{\mathrm{Op}}_h(e_t^0),$$

where $e_t^0 \in \widetilde{S}^{k-2}_{\Gamma_0,L_0,\rho}$. This follows from the fact that if we set $\mu(y) = y'h^{-\rho}$, then

$$\partial_t(b_t^0(y,\eta,\mu(y))) = H_{q_t}b_t^0(y,\eta,\mu(y)) + \partial_\mu b_t^0(y,\eta,\mu(y))H_{q_t}(\mu(y))$$

and $H_{q_t}\mu(y) = r_t(y,\eta)\mu(y)$.

Iterating this procedure and solving away successive errors finishes the proof of Proposition 5.9. If $a \in \widetilde{S}^k_{\Gamma_0,\rho}$, then we need only use that $\partial_{\xi'}q_t = r_tx'$ and we obtain the remaining results. $\square$

Our next lemma follows [Sjöstrand and Zworski 1999, Lemma 4.1] and gives a characterization of our second microlocal calculus in terms of the action of an operator. In what follows, given operators A and B, we define the operator ad_A by $\mathrm{ad}_A B = [A, B]$.

Lemma 5.11 (Beal's criteria). *Let $A_h : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ and $k \in \mathbb{Z}$. Then, $A_h = \widetilde{\mathrm{Op}}_h(a)$ for some $a \in \widetilde{S}^k_{\Gamma_0, L_0, \rho}$ if and only if, for any $\alpha, \beta \in \mathbb{R}^n$, there exists $C > 0$ with*

$$\|\mathrm{ad}^{\alpha}_{h^{-\rho}x}\, \mathrm{ad}^{\beta}_{hD_x} A_h u\|_{|\beta|-\min(k,0)} \le C h^{(1-\rho)(|\alpha|+|\beta|)} \|u\|_{\max(k,0)},$$

where $\|u\|_r := \|u\|_{L^2} + \|h^{-\rho r}|x'|^r u\|_{L^2}$ for $r \ge 0$. Similarly, $A_h = \mathrm{Op}_h(a)$ for some $a \in \widetilde{S}^k_{\Gamma_0,\rho}$ if and only if

$$\|\mathrm{ad}^{\alpha'}_{h^{-\rho}x'}\, \mathrm{ad}^{\alpha''}_{x''}\, \mathrm{ad}^{\beta'}_{hD_{x'}}\, \mathrm{ad}^{\beta''}_{hD_{x''}} A_h u\|_{|\beta'|-\min(k,0)} \le C h^{(1-\rho)(|\alpha'|+|\beta'|)+|\alpha''|+|\beta''|} \|u\|_{\max(k,0)}.$$

Proof. The fact that $A_h = \widetilde{\mathrm{Op}}_h(a)$ for some $a \in \widetilde{S}^k_{\Gamma_0,L_0,\rho}$ implies the estimates above follow directly from the model calculus. Let U_h be the unitary (on L^2) operator, $U_h u(x) = h^{n/2} u(hx)$, and note that

$$\|U_h^{-1} u\|_r = \|u\|_{L^2} + \|h^{(1-\rho)r}|x'|^r u\|_{L^2}.$$

Then, consider $\tilde{A}_h := U_h A_h U_h^{-1}$. For fixed h, we can use Beal's criteria (see e.g., [Zworski 2012, Theorem 8.3]) to see that there is a_h such that $\tilde{A}_h = a_h(x, D)$. Define a such that $a(hx, \xi; h) = a_h(x, \xi)$, and hence $A_h = \mathrm{Op}_h(a)$. Note that, for $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle \tilde{A}_h \psi, \phi\rangle = \frac{1}{(2\pi)^n} \iint e^{i\langle x,\xi\rangle} a_h(x,\xi) \hat{\psi}(\xi) \overline{\phi(x)}\, dx\, d\xi, \tag{5-19}$$

where $\hat{\psi}(\xi) = (\mathcal{F}\psi)(\xi) = \int e^{-i\langle y,\xi\rangle}\psi(y)\, dy$. Next, define

$$B_h := U_h\, \mathrm{ad}^{\alpha}_{h^{-\rho}x}(\mathrm{ad}^{\beta}_{hD_x}(A_h)) U_h^{-1}.$$

Since $D_x U_h = U_h h D_x$ and $U_h^{-1} D_x = h D_x U_h^{-1}$, we have

$$B_h = \mathrm{ad}^{\alpha}_{h^{1-\rho}x}\, \mathrm{ad}^{\beta}_{D_x} \tilde{A}_h = (-i)^{|\alpha|+|\beta|} h^{(1-\rho)|\alpha|} b_h(x, D),$$

where $b_h(x,\xi) = (-\partial_\xi)^\alpha \partial_x^\beta a_h(x,\xi)$. Our goal is then to understand the behavior of $b_h(x,\xi)$ in terms of h and $\langle h^{1-\rho}x'\rangle$. Let τ_{x_0} and $\hat{\tau}_{\xi_0}$ be the physical and frequency shift operators

$$\tau_{x_0} u(x) = u(x - x_0) \quad \text{and} \quad \hat{\tau}_{\xi_0} u(x) = e^{i\langle x,\xi_0\rangle} u(x)$$

with $\mathcal{F}\hat{\tau}_{\xi_0} = \tau_{\xi_0}\mathcal{F}$ and $\mathcal{F}\tau_{x_0} = \hat{\tau}_{-x_0}$. In addition, write $\|u\|_{(-r)} := \|\langle h^{1-\rho}x'\rangle^{-r} u\|_{L^2}$ for the dual norm to $\|u\|_{(r)} := \|U_h^{-1} u\|_r$.

Assume that $k \ge 0$. Then, the definition of B_h combined with the assumptions yields

$$|\langle B \tau_{x_0}\hat{\tau}_{\xi_0}\psi, \tau_{y_0}\hat{\tau}_{\eta_0}\phi\rangle| \le h^{(1-\rho)(|\alpha|+|\beta|)} \|\tau_{x_0}\hat{\tau}_{\xi_0}\psi\|_{(k)} \|\tau_{y_0}\hat{\tau}_{\eta_0}\phi\|_{-|\beta|}. \tag{5-20}$$

In addition, note that, for fixed $\psi, \phi \in \mathcal{S}$,

$$\|\tau_{x_0}\hat{\tau}_{\xi_0}\psi\|_{(k)} \sim \langle h^{1-\rho}(x_0)'\rangle^k \quad \text{and} \quad \|\tau_{y_0}\hat{\tau}_{\eta_0}\psi\|_{(-|\beta|)} \sim \langle h^{1-\rho}(y_0)'\rangle^{-|\beta|}.$$

Therefore, (5-20) leads to

$$|\langle B \tau_{x_0}\hat{\tau}_{\xi_0}\psi, \tau_{y_0}\hat{\tau}_{\eta_0}\phi\rangle| \le C h^{(1-\rho)(|\alpha|+|\beta|)} \langle h^{1-\rho}(x_0)'\rangle^k \langle h^{1-\rho}(y_0)'\rangle^{-|\beta|}. \tag{5-21}$$

On the other hand, we have by (5-19) that

$$|\langle B\tau_{x_0}\hat{\tau}_{\xi_0}\psi,\, \tau_{y_0}\hat{\tau}_{\eta_0}\phi\rangle| = \frac{h^{(1-\rho)|\alpha|}}{(2\pi)^n}\left|\iint e^{i\langle x,\xi\rangle} b_h(x,\xi)\hat{\psi}(\xi-\xi_0)e^{-i\langle x_0,\xi-\xi_0\rangle - i\langle \eta_0, x-y_0\rangle}\bar{\phi}(x-y_0)\,dx\,d\xi\right|$$
$$= h^{(1-\rho)|\alpha|}|\mathcal{F}((\tau_{y_0,\xi_0}\chi)b_h)(\eta_0-\xi_0, x_0-y_0)|, \tag{5-22}$$

where $\chi(x,\xi) = e^{i\langle x,\xi\rangle}\hat{\psi}(\xi)\bar{\phi}(x)$. Combining (5-22) with (5-21) we then have

$$|\mathcal{F}((\tau_{y_0,\xi_0}\chi)\partial_\xi^\alpha\partial_x^\beta a_h)(\eta_0-\xi_0, x_0-y_0)| \le Ch^{(1-\rho)|\beta|}\langle h^{1-\rho}(x_0)'\rangle^k\langle h^{1-\rho}(y_0)'\rangle^{-|\beta|}.$$

Next, note that χ can be replaced by any fixed function in C_c^∞ by taking ψ and ϕ with $\hat{\psi}(\xi)\phi(x)\neq 0$ on $\operatorname{supp}\chi$. Putting $\zeta = \eta_0-\xi_0$ and $z = x_0-y_0$, we obtain that, for every $\tilde{\alpha},\tilde{\beta}\in\mathbb{N}^n$,

$$|\mathcal{F}(\partial_\xi^{\tilde{\alpha}}\partial_x^{\tilde{\beta}}(\tau_{y_0,\xi_0}\chi)\partial_\xi^\alpha\partial_x^\beta a_h)(\zeta,z)| \le Ch^{(1-\rho)|\beta|}\langle h^{1-\rho}(x_0)'\rangle^k\langle h^{1-\rho}(x_0-z)'\rangle^{-|\beta|}.$$

Hence,

$$|z^{\tilde{\alpha}}\zeta^{\tilde{\beta}}\mathcal{F}((\tau_{y_0,\xi_0}\chi)\partial_\xi^\alpha\partial_x^\beta a_h)(\zeta,z)| \le Ch^{(1-\rho)|\beta|}\langle h^{1-\rho}(x_0)'\rangle^k\langle h^{1-\rho}(x_0-z)'\rangle^{-|\beta|}.$$

In particular, for every $N>0$,

$$|\mathcal{F}((\tau_{y_0,\xi_0}\chi)\partial_\xi^\alpha\partial_x^\beta a_h)(\zeta,z)| \le Ch^{(1-\rho)|\beta|}\langle h^{1-\rho}(x_0)'\rangle^{k-|\beta|}\langle\zeta\rangle^{-N}\langle z\rangle^{-N},$$

and, as a consequence, we obtain

$$\partial_\xi^\alpha\partial_x^\beta a_h(x,\xi) = \partial_\xi^\alpha\partial_x^\beta(a(hx,\xi)) = O(h^{(1-\rho)|\beta|}\langle h^{1-\rho}x'\rangle^{k-|\beta|}).$$

This gives the first claim of the lemma for $k\ge 0$. For $k\le 0$, we consider $\langle h^{-\rho}x'\rangle^{-k}A$ and use the composition formulae. A nearly identical argument yields the second claim. □

5C. ***Definition of the second microlocal class.*** With Proposition 5.9 in place, we are now in a position to define the class of operators with symbols in $S^k_{\Gamma,L,\rho}$.

Definition 5.12. Let $\Gamma\subset U\subset T^*M$ be a coisotropic submanifold, U an open set, and L a Lagrangian foliation on Γ. A *chart for* (Γ, L) is a symplectomorphism

$$\kappa: U_0\to V,\quad U_0\subset U,\quad V\subset T^*\mathbb{R}^n,$$

such that $\kappa(U_0\cap\Gamma)\subset V\cap\Gamma_0$ and $\kappa_{*,q}L_q = (L_0)_{\kappa(q)}$ for $q\in\Gamma\cap U$.

We now define the pseudodifferential operators associated to (Γ, L).

Definition 5.13. Let M be a smooth, compact manifold and $U\subset T^*M$ open, $\Gamma\subset U$ a coisotropic submanifold, L a Lagrangian foliation on Γ, and $\rho\in[0,1)$. We say that $A:\mathcal{D}'(M)\to C_c^\infty(M)$ is a *semiclassical pseudodifferential operator with symbol class* $S^k_{\Gamma,L,\rho}(U)$ (and write $A\in\Psi^k_{\Gamma,L,\rho}(U)$) if there are charts $\{\kappa_\ell\}_{\ell=1}^N$ for (Γ, L) and symbols $\{a_\ell\}_{\ell=1}^N\subset\widetilde{S}^k_{\Gamma,L,\rho}(U)$ such that A can be written in the form

$$A = \sum_{\ell=1}^N T'_\ell\,\widetilde{\mathrm{Op}}_h(a_\ell)\,T_\ell + O(h^\infty)_{\mathcal{D}'\to C^\infty}, \tag{5-23}$$

where T_ℓ and T'_ℓ are FIOs quantizing κ_ℓ and κ_ℓ^{-1} for $\ell = 1,\dots,N$.

We say that A is a *semiclassical pseudodifferential operator with symbol class* $S^k_{\Gamma,\rho}(U)$ (and write $A\in\Psi^k_{\Gamma,\rho}(U)$) if there are symbols $\{a_\ell\}_{\ell=1}^N\subset\widetilde{S}^k_{\Gamma,\rho}(U)$ such that A can be written in the form (5-23).

Lemma 5.14. *Suppose that $\kappa:U\to T^*\mathbb{R}^n$ is a chart for (Γ,L), T quantizes κ, and T' quantizes κ^{-1}. If $A\in\Psi^k_{\Gamma,L,\rho}(U)$, then there is $a\in\widetilde{S}^k_{\Gamma,L,\rho}(U)$ with* $\operatorname{supp} a(\,\cdot\,,\cdot\,,\lambda)\subset\kappa(U)$ *such that*

$$TAT'=\widetilde{\operatorname{Op}}_h(a)+O(h^\infty)_{\mathcal{D}'\to C^\infty}.$$

Moreover, if A is given by (5-23), *then*

$$a\circ K_\kappa=\sigma(T'T)\sum_{\ell=1}^N\sigma(T'_\ell T_\ell)\,(a_\ell\circ K_{\kappa_\ell})+O(h^{1-\rho})_{\widetilde{S}^{k-1}_{\Gamma,L,\rho}}.$$

Proof. Note that we can write

$$TAT'=\sum_{\ell=1}^N TT'_\ell\widetilde{\operatorname{Op}}_h(a_\ell)T_\ell T'+O(h^\infty)_{\mathcal{D}'\to C^\infty}.$$

Next, note that TT'_ℓ quantizes $\kappa\circ\kappa_\ell^{-1}$ and that $T_\ell T'$ quantizes $\kappa_\ell\circ\kappa^{-1}$. Letting F_ℓ be a microlocally unitary FIO quantizing $\kappa_\ell\circ\kappa^{-1}$, we have that F_ℓ satisfies the hypotheses of Proposition 5.9 and we can write

$$T_\ell T'=C_LF_\ell\quad\text{and}\quad TT'_\ell=F_\ell^{-1}C_R$$

with $C_L,C_R\in\Psi(M)$ satisfying $\sigma(C_RC_L)=(\sigma(T'_\ell T_\ell)\circ\kappa_\ell^{-1})(\sigma(T'T)\circ\kappa_\ell^{-1})$. Therefore,

$$\begin{aligned}TT'_\ell\widetilde{\operatorname{Op}}_h(a_\ell)T_\ell T'&=F_\ell^{-1}C_R\widetilde{\operatorname{Op}}_h(a_\ell)C_LF_\ell=\operatorname{Op}_h(b_\ell)+(h^\infty)_{\mathcal{D}'\to C^\infty},\\ b_\ell&=(\sigma(C_RC_L)\circ\kappa_\ell\circ\kappa^{-1})(a_\ell\circ K_{\kappa_\ell\circ\kappa^{-1}})+O(h^{1-\rho})_{\widetilde{S}^{k-1}_{\Gamma,L,\rho}}.\end{aligned}$$ □

Lemma 5.15. *Let $\Gamma\subset U\subset T^*M$ be a coisotropic submanifold, U be an open set, and L be a Lagrangian foliation on Γ. There is a principal symbol map*

$$\sigma_{\Gamma,L}:\Psi^k_{\Gamma,L,\rho}(U)\to S^k_{\Gamma,L,\rho}(U)/h^{1-\rho}S^{k-1}_{\Gamma,L,\rho}(U)$$

such that, for $A\in\Psi^{k_1}_{\Gamma,L,\rho}(U)$ and $B\in\Psi^{k_2}_{\Gamma,L,\rho}(U)$,

$$\sigma_{\Gamma,L}(AB)=\sigma_{\Gamma,L}(A)\sigma_{\Gamma,L}(B)\quad\textit{and}\quad\sigma_{\Gamma,L}([A,B])=-ih\{\sigma_{\Gamma,L}(A),\sigma_{\Gamma,L}(B)\}.\tag{5-24}$$

Furthermore, the sequence

$$0\to h^{1-\rho}\Psi^{k-1}_{\Gamma,L,\rho}(U)\xrightarrow{\iota}\Psi^k_{\Gamma,L,\rho}(U)\xrightarrow{\sigma_{\Gamma,L}}S^k_{\Gamma,L,\rho}(U)/h^{1-\rho}S^{k-1}_{\Gamma,L,\rho}(U)\to 0$$

is exact. The same holds with σ_Γ, $\Psi_{\Gamma,\rho}$, and $S^k_{\Gamma,\rho}$.

Proof. For A as in (5-23), we define

$$\sigma_{\Gamma,L}(A)=\sum_{\ell=1}^N\sigma(T_\ell T'_\ell)(\tilde a_\ell\circ\kappa),$$

where $\tilde a_\ell(x,\xi):=a_\ell(x,\xi,h^{-\rho}x')$. The fact that σ is well defined then follows from Lemma 5.14, and the formulae (5-24) follow from Lemma 5.6.

To see that the sequence is exact, we only need to check that if $A \in \Psi^k_{\Gamma,L,\rho}$ and $\sigma_{\Gamma,L}(A) = 0$, then $A \in h^{1-\rho}\Psi^{k-1}_{\Gamma,L,\rho}$. To do this, we may assume that $\mathrm{WF}_h{}'(A) \subset U$ such that there is a chart (κ, U) for (Γ, L). Let T be a microlocally unitary FIO quantizing κ and suppose that $\sigma_{\Gamma,L}(A) \in h^{1-\rho}S^{k-1}_{\Gamma,L,\rho}$. Then, by the first part of Lemma 5.14, we know $TAT^{-1} = \widetilde{\mathrm{Op}}_h(a) + O(h^\infty)$ for some $a \in \widetilde{S}^k_{\Gamma,L,\rho}$. Then, by the second part of Lemma 5.14, since $\sigma_{\Gamma,L}(A) \in h^{1-\rho}S^{k-1}_{\Gamma,L,\rho}$, we have that $a \in h^{1-\rho}\widetilde{S}^{k-1}_{\Gamma,L,\rho}$ and, in particular, $A \in h^{1-\rho}\Psi^{k-1}_{\Gamma,L,\rho}$. □

Note that if $A \in \Psi^{\mathrm{comp}}(M)$, then $A \in \Psi^0_{\Gamma,L,\rho}$ and $\sigma(A) = \sigma_\Gamma(A)$. Furthermore, if $A \in \Psi^k_{\Gamma,\rho}$, then $A \in \Psi^k_{\Gamma,L,\rho}$ and $\sigma_\Gamma(A) = \sigma_{\Gamma,L}(A)$.

Lemma 5.16. *Let $\Gamma \subset U \subset T^*M$ be a coisotropic submanifold, U be an open set, and L be a Lagrangian foliation on Γ. There is a noncanonical quantization procedure*

$$\mathrm{Op}_h^{\Gamma,L} : S^k_{\Gamma,L,\rho}(U) \to \Psi^k_{\Gamma,L,\rho}(U)$$

such that, for all $A \in \Psi^k_{\Gamma,L,\rho}(U)$, there is $a \in S^k_{\Gamma,L,\rho}(U)$ such that $\mathrm{Op}_h^{\Gamma,L}(a) = A + O(h^\infty)_{\mathcal{D}'\to C^\infty}$ and

$$\sigma_{\Gamma,L} \circ \mathrm{Op}_h^{\Gamma,L} : S^k_{\Gamma,L,\rho}(U) \to S^k_{\Gamma,L,\rho}(U)/h^{1-\rho}S^{k-1}_{\Gamma,L,\rho}(U)$$

is the natural projection map.

Proof. Let $\{(\kappa_\ell, U_\ell)\}_{\ell=1}^N$ be charts for (Γ, L) such that $\{U_\ell\}_{\ell=1}^N$ is a locally finite cover for U, T_ℓ and T'_ℓ quantize κ_ℓ and κ_ℓ^{-1}, respectively, and $\sigma(T'_\ell T_\ell) \in C^\infty_c(U_\ell)$ is a partition of unity on U. Let $a \in S^k_{\Gamma,L,\rho}(U)$. Then, define $a_\ell \in \widetilde{S}^k_{\Gamma_0,L_0,\rho}$ such that $a_\ell(x, \xi, h^{-\rho}x') := (\chi_\ell a) \circ \kappa^{-1}(x, \xi)$, where $\chi_\ell \equiv 1$ on $\operatorname{supp} \sigma(T'_\ell T_\ell)$. We then define the quantization map

$$\mathrm{Op}_h^{\Gamma,L}(a) := \sum_{\ell=1}^N T'_\ell \widetilde{\mathrm{Op}}_h(a_\ell) T_\ell.$$

The fact that $\sigma_{\Gamma,L} \circ \mathrm{Op}_h^{\Gamma,L}$ is the natural projection follows immediately. Now, fix $A \in \Psi^k_{\Gamma,L,\rho}(U)$. Put $a_0 = \sigma_{\Gamma,L}(A)$. Then, $A = \mathrm{Op}_h^{\Gamma,L}(a_0) + h^{1-\rho}A_1$, where $A_1 \in \Psi^{k-1}_{\Gamma,L,\rho}$. We define $a_k = \sigma_{\Gamma,L}(A_k)$ inductively for $k \geq 1$ by

$$h^{(k+1)(1-\rho)}A_{k+1} = A - \sum_{k=0}^k h^{k(1-\rho)} \mathrm{Op}_h^{\Gamma,L}(a_k).$$

Then, letting $a \sim \sum_k h^{k(1-\rho)}a_k$, we have $A = \mathrm{Op}_h^{\Gamma,L}(a) + O(h^\infty)_{\mathcal{D}'\to C^\infty}$ as claimed. □

Remark 5.17. Note that $E := \sum_{\ell=1}^N T_\ell T'_\ell$ is an elliptic pseudodifferential operator with symbol 1. Therefore, there is $E' \in \Psi^0$ with $\sigma(E') = 1$ such that $E'EE' = \mathrm{Id}$. Replacing T_ℓ by $E'T_\ell$ and T'_ℓ by $T'_\ell E'$, we may (and will) ask for $\sum_{\ell=1}^N T_\ell T'_\ell = \mathrm{Id}$, and so $\mathrm{Op}_h^{\Gamma,L}(1) = \mathrm{Id}$.

Lemma 5.18. *Let $\Gamma \subset U \subset T^*M$ be a coisotropic submanifold. If $A \in \Psi^k_{\Gamma,\rho}(U)$ and $P \in \Psi^m(U)$ with symbol p such that, for every $q \in \Gamma$, we have $H_p(q) \in T_q\Gamma$. Then,*

$$\frac{i}{h}[P, A] = \mathrm{Op}_h^\Gamma(H_p a) + O(h^{1-\rho})_{\Psi^{k-1}_{\Gamma,\rho}},$$

where $a(x, \xi; h) = \sigma_\Gamma(A)(x, \xi, h^{-\rho}x')$.

Proof. Suppose that $\mathrm{WF}_h{}'(A) \subset U_\ell$ for $U_\ell \subset U$ open, and suppose that $\kappa : U_\ell \to T^*\mathbb{R}^n$ is a chart for (Γ, L). Note that we may assume that $\mathrm{WF}_h(A)' \subset U_\ell$ and then use a partition of unity to cover U with a family $\{U_\ell\}_\ell$. Therefore, there exist a Fourier integral operator T that is microlocally elliptic on U_ℓ and quantizes κ and $a \in \widetilde{S}^k_{\Gamma,\rho}$ such that $A = T^{-1}\widetilde{\mathrm{Op}}_h(a)T + O(h^\infty)_{\mathcal{D}'\to C^\infty}$. Then, on $\mathrm{WF}_h{}'(A)$,

$$T[P, A]T^{-1} = [TPT^{-1}, \widetilde{\mathrm{Op}}_h(a)] + O(h^\infty)_{\mathcal{D}'\to C^\infty}.$$

Now, $TPT^{-1} = \mathrm{Op}_h(p\circ\kappa^{-1}) + O(h)_{\Psi^{m-1}}$. Hence, a direct computation using Lemma 5.7 gives

$$[TPT^{-1}, \widetilde{\mathrm{Op}}_h(a)] = -ih\widetilde{\mathrm{Op}}_h(c) + O(h^{2-\rho})_{\widetilde{\Psi}^{k-2}_{\Gamma_0,\rho}}$$

with $c(x, \xi, h^{-\rho}x') = H_{p\circ\kappa^{-1}}(a(x, \xi, h^{-\rho}x')) \in S^{k-1}_{\Gamma,\rho}(U_\ell)$. In particular,

$$[P, A] = -ihT^{-1}\widetilde{\mathrm{Op}}_h(c)T + O(h^{2-\rho})_{\Psi^{k-2}_{\Gamma,\rho}}.$$

Therefore, $[P, A] \in h\Psi^{k-1}_{\Gamma,\rho}$ with symbol $\sigma_\Gamma(ih^{-1}[P, A]) = H_p(a(x, \xi, h^{-\rho}x'))$. □

6. An uncertainty principle for coisotropic localizers

The first goal of this section is to build a family of cut-off operators X_y with $y \in M$ that act as the identity on the shrinking ball $B(y, h^\rho)$ and such that they commute with P in a fixed-size neighborhood of y. This is the content of Section 6A. The second goal is to control $\|X_{y_1}X_{y_2}\|_{L^2\to L^2}$ in terms of the distance $d(y_1, y_2)$ as this distance shrinks to 0. We do this in Section 6B. Finally, in Section 6C, we study the consequences of these estimates for the almost-orthogonality of X_{y_i}.

In order to localize to the ball $B(y, h^\rho)$ in a way compatible with microlocalization we need to make sense of

$$\chi_y(x) = \tilde{\chi}\Big(\frac{1}{\varepsilon}h^{-\rho}d(x, y)\Big), \quad \tilde{\chi} \in C^\infty_c((-1, 1)),$$

as an operator in some anisotropic pseudodifferential calculus. As a function, χ_y is in the symbol class $S^{-\infty}_{\Gamma_y,L_y}$, where Γ_y and L_y are the coisotropic submanifold and Lagrangian foliation defined as follows: fix $\delta > 0$, to be chosen small later, and, for each $x \in M$, let

$$\Gamma_y := \bigcup_{|t|<\frac{1}{2}\mathrm{inj}(M)} \varphi_t(\Omega_y), \quad \Omega_y := \{\xi \in T^*_yM : |1 - |\xi|_g| < \delta\}. \tag{6-1}$$

In this section, we construct localizers to Γ_y adapted to the Laplacian and study the incompatibility between localization to Γ_{y_1} and Γ_{y_2} as a function of the distance between $y_1, y_2 \in M$. Let $y \in M$. In what follows we work with the Lagrangian foliation L_y of Γ_y given by

$$L_y = \{L_{y,\tilde{q}}\}_{\tilde{q}\in\Gamma_y}, \quad L_{y,\tilde{q}} = (\varphi_t)_*(T_qT^*_yM),$$

where $\tilde{q} = \varphi_t(q)$ for some $|t| < \frac{1}{2}\mathrm{inj}(M)$ and $q \in \Omega_y$.

Remark 6.1. In fact, it will be enough for us to show that $\chi_y(x)\tilde{\chi}(\delta^{-1}(|hD|_g - 1)) \in \Psi_{\Gamma_y,L_y,\rho}$ since we will be working near the characteristic variety for the Laplacian.

6A. ***Coisotropic cutoffs adapted to the Laplacian.***

Lemma 6.2. *Let $y \in M$, $0 < \varepsilon < \delta$, $0 \le \rho < 1$, $\tilde{\chi} \in C_c^\infty((-1,1))$, and define the operator $\chi_{h,y}$ by*

$$\chi_{h,y} u(x) := \tilde{\chi}\Big(\frac{1}{\varepsilon} h^{-\rho} d(x,y)\Big)\Big[\mathrm{Op}_h\Big(\tilde{\chi}\Big(\frac{1}{\varepsilon}(|\xi|_g - 1)\Big)\Big)u\Big](x). \tag{6-2}$$

Then, $\chi_{h,y} \in \Psi^{-\infty}_{\Gamma_y, L_y, \rho}$.

Proof. We will use Lemma 5.11 to prove the claim. First, observe that we may work in a single chart for (Γ_y, L_y) by using a partition of unity. Therefore, suppose that $B \in \Psi^0$ and $\kappa : U_0 \to T^*\mathbb{R}^n$ is a chart for (Γ_y, L_y), $V_0 \Subset U_0$, and T is an FIO quantizing κ that is microlocally unitary on V_0. Furthermore, since $\kappa_* L_y = L_0$, we may assume that $\kappa(U_0 \cap T_y^*M) \subset T_0^*\mathbb{R}^n$. Denote the microlocal inverse of T by T'. Then, observe that, for A and B with wavefront set in V_0,

$$\mathrm{ad}_A(TBT') = T\,\mathrm{ad}_{T'AT}(B)T' + O(h^\infty)_{\mathcal{D}' \to C^\infty}.$$

By a partition of unity, we will work as though $\chi_{h,y}$ were microsupported in U_0. We then consider, for all $N > 0$ and $\alpha, \beta \in \mathbb{N}^n$,

$$\begin{aligned} h^{-2N\rho}&|x'|^{2N}\,\mathrm{ad}^\alpha_{h^{-\rho}x}\,\mathrm{ad}^\beta_{hD_x}(T\chi_{h,y}T') \\ &= h^{-2\rho N}T(T'|x'|^2T)^N\,\mathrm{ad}^\alpha_{h^{-\rho}T'xT}\,(\mathrm{ad}^\beta_{T'hD_xT}\,(\chi_{h,y}))T' + O(h^\infty)_{\mathcal{D}' \to C^\infty}. \end{aligned}$$

In order to prove the requisite estimates, we will first view $\chi_{h,y}$ as an element of the model microlocal class. In particular, we work with $x \in M$ written in geodesic normal coordinates centered at y, so that

$$\chi_{h,y} u(x) = \tilde{\chi}\Big(\frac{1}{\varepsilon} h^{-\rho}|x|\Big)\Big[\mathrm{Op}_h\Big(\tilde{\chi}\Big(\frac{1}{\varepsilon}(|\xi|_g - 1)\Big)\Big)u\Big](x).$$

Then,

$$\chi_{h,y} = \widetilde{\mathrm{Op}}_h\Big(\frac{1}{\varepsilon}\tilde{\chi}(\lambda)\Big)\,\mathrm{Op}_h\Big(\tilde{\chi}\Big(\frac{1}{\varepsilon}(|\xi| - 1)\Big)\Big)$$

is an element of $\widetilde{\Psi}^{-\infty}_{\Gamma_0, L_0, \rho}$ with $r = n$, and so we can apply Lemma 5.7 to compute $\mathrm{ad}_A(\chi_{h,y})$ for $A \in \Psi^{-\infty}(M)$. In particular,

$$\mathrm{ad}_{T'hD_xT}(\chi_{h,y}) = \widetilde{\mathrm{Op}}_h(c) + O(h^\infty), \tag{6-3}$$

where $c \in h^{1-\rho}\widetilde{S}^{-\infty}_{\Gamma_0, L_0, \rho}$ is supported on $\{(x,\xi,\lambda) : |x| \le \varepsilon h^\rho,\ |\lambda| \le \varepsilon\}$. Now, suppose $c \in \widetilde{S}^{-\infty}_{\Gamma_0, L_0, \rho}$ is supported on $\{(x,\xi,\lambda) : |x| \le \varepsilon h^\rho,\ |\lambda| \le \varepsilon\}$ and $B \in \Psi^{-\infty}$ with $\sigma(B)(0,\xi) = 0$. Then, again using Lemma 5.7 and the fact that $\partial_{\xi'}\sigma(B)|_{x'=0} = 0$,

$$\mathrm{ad}_B(\widetilde{\mathrm{Op}}_h(c)) = \widetilde{\mathrm{Op}}_h(c') + O(h^\infty), \tag{6-4}$$

where $c' \in h\widetilde{S}^{-\infty}_{\Gamma_0, L_0, \rho}$ is supported on $\{(x,\xi,\lambda) : |x| \le \varepsilon h^\rho,\ |\lambda| \le \varepsilon\}$.

Now, note that since $\kappa(T_y^*M) \subset T_0^*\mathbb{R}^n$, then, for all $i = 1, \dots, n$, we have that $B = T'x_iT$ has symbol $\sigma(B) = [b(x,\xi)x]_i$ for some $b \in C^\infty(T^*M; \mathbb{M}_{n\times n})$. Therefore, (6-3) and (6-4) yield

$$\mathrm{ad}^\alpha_{h^{-\rho}T'xT}(\mathrm{ad}^\beta_{T'hD_xT}(\chi_{h,y})) = h^{(1-\rho)(|\alpha|+|\beta|)}\widetilde{\mathrm{Op}}_h(c') + O(h^\infty),$$

where $c' \in \widetilde{S}^{-\infty}_{\Gamma_0, L_0, \rho}$ is supported on $\{(x, \xi, \lambda) : |x| \le \varepsilon h^\rho,\ |\lambda| \le \varepsilon\}$. Finally, using again that $T' x_i T$ has symbol $[b(x, \xi) x]_i$, we have that (6-4) gives

$$\|h^{-2N\rho} |x'|^{2N} \operatorname{ad}^\alpha_{h^{-\rho} T' x T} (\operatorname{ad}^\beta_{T' h D_x T} (\chi_{h,y}))\|_{L^2 \to L^2} \le C h^{(1-\rho)(|\alpha|+|\beta|)}. \qquad \square$$

We next construct a pseudodifferential cutoff, $X_y \in \Psi^{-\infty}_{\Gamma_y, \rho}$, which is microlocally the identity near $S^*_y M$ and which essentially commutes with $P = -h^2 \Delta_g - 1$ near y. In particular, we will have

$$\chi_{h,y} X_y = \chi_{h,y} + O(h^\infty)_{\Psi^{-\infty}}.$$

When considering the value of a quasimode u that is h^ρ close to the point y, this will allow us to effectively work with $X_y u$ instead.

Theorem 6.3. *Let $y \in M$, $0 < \varepsilon < \delta$, and $0 \le \rho < 1$. Then, there exists $X_y \in \Psi^{-\infty}_{\Gamma_y, \rho} \subset \Psi^{-\infty}_{\Gamma_y, L_y, \rho}$ satisfying*

(1) *If $\chi_{h,y}$ is defined as in* (6-2), *then*

$$\chi_{h,y} X_y = \chi_{h,y} + O(h^\infty)_{\Psi^{-\infty}}. \tag{6-5}$$

(2) $\mathrm{WF}_h'([P, X_y]) \cap \{(x, \xi) : x \in B(y, \tfrac{1}{2} \operatorname{inj}(M)),\ \xi \in \Omega_x\} = \varnothing$.

Proof. First, we note that we will actually prove that $X_y \in \Psi^{-\infty}_{\Gamma_y, \rho}$, and so the result will follow since $\Psi^{-\infty}_{\Gamma_y, \rho} \subset \Psi^{-\infty}_{\Gamma_y, L_y, \rho}$. Let $\mathcal{H} \subset T^*M$ be transverse to the Hamiltonian flow H_p such that $\Omega_y \subset \mathcal{H}$. Next, let $\varkappa \in C^\infty_c((-2, 2))$ with $\varkappa \equiv 1$ on $[-1, 1]$, and define $\varkappa_0 \in C^\infty_c(\mathcal{H})$ by

$$\varkappa_0 = \varkappa(h^{-\rho} d(x, y)) \varkappa\Big(\frac{2}{\delta}(1 - |\xi|_g)\Big),$$

where δ is as in the definition of Ω_y. Let $\psi \in C^\infty_c(T^*M)$ with

$$\psi \equiv 1 \quad \text{on } B(y, \tfrac{1}{2} \operatorname{inj}(M)) \cap \{|\xi|_g < 2\}, \qquad \operatorname{supp} \psi \subset B(y, \tfrac{3}{4} \operatorname{inj}(M)).$$

Then, let χ_0 be defined locally by $H_p \chi_0 \equiv 0$ and $\chi_0|_{\mathcal{H}} = \varkappa_0$ such that $\chi_0 \in S^{-\infty}_{\Gamma_y, \rho}$. That is, $\chi_0(\varphi_t(q)) = \psi(\varphi_t(q)) \chi_0(q)$ for $|t| < \operatorname{inj}(M)$ and $q \in \mathcal{H}$. Next, observe that by Lemma 5.7 there is $e_0 \in S^{-\infty}_{\Gamma_y, \rho}$ such that

$$-\frac{i}{h}[P, \operatorname{Op}^{\Gamma_y}_h(\chi_0)] = h^{1-\rho} \operatorname{Op}^{\Gamma_y}_h(e_0), \quad \operatorname{supp} e_0 \cap B(y, \tfrac{1}{2} \operatorname{inj}(M)) \subset \bigcup_{|t| < \frac{3}{4} \operatorname{inj}(M)} \varphi_t(\mathcal{H} \cap \operatorname{supp} \partial \varkappa_0).$$

(Here and below $\partial \varkappa_0$ denotes the gradient of $\varkappa_0$.) Suppose that there exist $\chi_{k-1}, e_{k-1} \in S^{-\infty}_{\Gamma_y, \rho}$ such that

$$-\frac{i}{h}[P, \operatorname{Op}^{\Gamma_y}_h(\chi_{k-1})] = h^{k(1-\rho)} \operatorname{Op}_h(e_{k-1}), \quad \operatorname{supp} e_{k-1} \cap B(y, \tfrac{1}{2} \operatorname{inj}(M)) \subset \bigcup_{|t| < \frac{3}{4} \operatorname{inj}(M)} \varphi_t(\mathcal{H} \cap \operatorname{supp} \partial \varkappa_0).$$

Then, define $\tilde{\chi}_k \in S^{-\infty}_{\Gamma_y, \rho}$ by solving locally $H_p \tilde{\chi}_k = e_{k-1}$ and $\tilde{\chi}_k|_{\mathcal{H}} = 0$. Note that then

$$\operatorname{supp} \tilde{\chi}_k \cap B(y, \tfrac{1}{2} \operatorname{inj}(M)) \subset \bigcup_{|t| < \frac{3}{4} \operatorname{inj}(M)} \varphi_t(\mathcal{H} \cap \operatorname{supp} \partial \varkappa_0)$$

and

$$h^{-k(1-\rho)} \sigma\Big(\frac{i}{h}[P, \operatorname{Op}^{\Gamma_y}_h(\chi_{k-1} + h^{k(1-\rho)} \tilde{\chi}_k)]\Big) = H_p \tilde{\chi}_k - e_{k-1} = 0.$$

In particular, with $\chi_k := \chi_{k-1} + h^{k(1-\rho)}\tilde{\chi}_k$, we obtain $-\frac{i}{h}[P, \mathrm{Op}_h^{\Gamma_y}(\chi_k)] = h^{(k+1)(1-\rho)}\,\mathrm{Op}_h(e_k)$ with $e_k \in S^{-\infty}_{\Gamma_y,\rho}$ and

$$\operatorname{supp} e_k \cap B\big(y, \tfrac{1}{2}\operatorname{inj}(M)\big) \subset \bigcup_{|t|<\frac{3}{4}\operatorname{inj}(M)} \varphi_t(\mathcal{H} \cap \operatorname{supp} \partial \varkappa_0).$$

Setting

$$X_y = \mathrm{Op}_h^{\Gamma_y}(\chi_\infty) \quad \text{and} \quad \chi_\infty \sim \Big(\chi_0 + \sum_k (\chi_{k+1} - \chi_k)\Big),$$

we have that X_y satisfies the second claim and, moreover, $\chi_\infty \equiv 1$ on

$$\bigcup_{|t|\le\frac{1}{4}\operatorname{inj}(M)} \varphi_t\big(\mathcal{H} \cap \{d(x,y) < h^\rho\} \cap \big\{||\xi|_g - 1| < \tfrac{1}{2}\delta\big\}\big).$$

To see the first claim, observe that, for $\varepsilon > 0$ small enough,

$$B(y, \varepsilon h^\rho) \cap \{||\xi|_g - 1| < \delta\} \subset \bigcup_{|t|\le\frac{1}{4}\operatorname{inj}(M)} \varphi_t\big(\mathcal{H} \cap \{d(x,y) < h^\rho\} \cap \big\{||\xi|_g - 1| < \tfrac{1}{2}\delta\big\}\big),$$

and hence, by Lemma 5.6,

$$\chi_{h,y} X_y = \chi_{h,y}\,\mathrm{Op}_h^{\Gamma,L}(1) + O(h^\infty)_{\Psi^{-\infty}} = \chi_{h,y} + O(h^\infty)_{\Psi^{-\infty}}. \qquad \square$$

6B. ***An uncertainty principle for coisotropic localizers.*** Let $\Gamma(t) \subset T^*\mathbb{R}^n$, $t \in (-\varepsilon_0, \varepsilon_0)$, be a smooth family of coisotropic submanifolds of dimension $n+1$ with

$$\Gamma(0) = \{(0, x_n, \xi', \xi_n) : x_n \in \mathbb{R},\ \xi' \in \mathbb{R}^{n-1},\ \xi_n \in \mathbb{R}\}.$$

Assume that for each t, we define $\Gamma(t)$ by the functions $\{q_i(t)\}_{i=1}^{n-1} \subset C^\infty(\mathbb{R}^{2n})$ with $q_i(0) = x_i$ (note that $q_i(t)$ should be thought of as a function in $C^\infty(\mathbb{R}^{2n})$ parametrized by t). Moreover, assume that there are $c, C > 0$ such that for $i = 1, \dots, n-1$,

$$|\{q_i(t), x_i\}| \ge c|t| \quad \text{on } \Gamma(0) \cap \Gamma(t), \quad |t| > 0, \tag{6-6}$$

and, for all $i, j = 1, \dots, n-1$ and all $t \in (-\varepsilon_0, \varepsilon_0)$,

$$\{q_i(t), q_j(t)\} = 0, \quad \{q_i(t), \xi_n\} = 0, \quad |\{q_i(t), x_j\}| \le Ct^2 \quad \text{on } \Gamma(0) \cap \Gamma(t), \quad i \ne j. \tag{6-7}$$

The main goal of this section is to prove the following proposition.

Proposition 6.4. *Let $0 < \rho < 1$ and $\{\Gamma(t) : t \in (-\varepsilon_0, \varepsilon_0)\}$ be as above. Suppose that $X(t) \in \Psi^{-\infty}_{\Gamma(t),\rho}$ for all $t \in (-\varepsilon_0, \varepsilon_0)$ and that there is $\varepsilon > 0$ such that $h^{\rho-\varepsilon} \le |t| < \varepsilon_0$. Then,*

$$\|X(0)X(t)\|_{L^2 \to L^2} \le C h^{(n-1)(2\rho-1)/2} t^{(1-n)/2}.$$

Proof. We begin by finding a convenient chart for $\Gamma(t)$. By Darboux's theorem (see, e.g., [Zworski 2012, Theorem 12.1]), there is a smooth family of symplectomorphisms $\kappa_t : V_1 \to V_2$ such that, for $j = 1, \dots, n-1$,

$$\kappa_t^*(q_j(t)) = y_j \quad \text{and} \quad \kappa_t^* \xi_n = \eta_n, \tag{6-8}$$

where V_1 and V_2 are simply connected neighborhoods of 0. Note that $\kappa_t(\Gamma(0)) = \Gamma(t)$ with this setup, so κ_t^{-1} is a chart for $\Gamma(t)$. By [Zworski 2012, Theorem 11.4], the symplectomorphism κ_t can be extended to a family of symplectomorphisms on $T^*\mathbb{R}^n$ that is the identity outside a compact set, and such that there is a smooth family of symbols $p_t \in C^\infty(T^*\mathbb{R}^n)$ satisfying $\partial_t \kappa_t = (\kappa_t)_* H_{p_t}$.

Now, let $U(t) : L^2 \to L^2$ solve

$$(hD_t + \mathrm{Op}_h(p_t))U(t) = 0, \quad U(0) = \mathrm{Id}.$$

Then, $U(t)$ is microlocally unitary from V_1 to V_2 in the sense that if $a \in C_c^\infty(V_1)$ and $b \in C_c^\infty(V_2)$ then

$$[U(t)]^* U(t)\,\mathrm{Op}_h(a) = \mathrm{Op}_h(a) + O(h^\infty)_{\Psi^{-\infty}} \quad \text{and} \quad U(t)[U(t)]^*\,\mathrm{Op}_h(b) = \mathrm{Op}_h(b) + O(h^\infty)_{\Psi^{-\infty}},$$

and $U(t)$ quantizes κ_t. Moreover,

$$U(t) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{i(\phi(t,x,\eta) - \langle y,\eta\rangle)/h} b(t,x,\eta;h)\,d\eta,$$

where $b(t,\cdot\,) \in S^{\mathrm{comp}}(T^*\mathbb{R}^n)$ and the phase function $\phi(t,\cdot\,) \in C^\infty(T^*\mathbb{R}^n;\mathbb{R})$ satisfies

$$\partial_t \phi + p_t(x, \partial_x \phi) = 0, \quad \phi(0,x,\eta) = \langle x,\eta\rangle$$

for all $t \in (-\varepsilon_0, \varepsilon_0)$. Since $U(t)$ is microlocally unitary, it is enough to estimate the operator

$$A(t) := X(0)X(t)U(t).$$

First, note that since $X(t) \in \Psi^{-\infty}_{\Gamma(t),\rho}$ and $U(t)$ quantizes κ_t, there exists $a(t) \in \widetilde{S}^{-\infty}_{\Gamma_0,\rho}$ with $t \in (-\varepsilon_0, \varepsilon_0)$ such that $X(t) = U(t)\widetilde{\mathrm{Op}}_h(a(t))[U(t)]^* + O(h^\infty)_{L^2\to L^2}$, and so

$$A(t) = \widetilde{\mathrm{Op}}_h(a(0))U(t)\widetilde{\mathrm{Op}}_h(a(t)) + O(h^\infty)_{L^2\to L^2}.$$

Fix $N > n-1$ and let $\chi = \chi(\lambda) \in \widetilde{S}^{-N}_{\Gamma_0,\rho}$ be such that $|\chi(\lambda)| \geq c\langle\lambda\rangle^{-N}$. Now, since $a(t) \in \widetilde{S}^{-\infty}_{\Gamma_0,\rho}$, by the elliptic parametrix construction there are $e_L(t), e_R(t) \in \widetilde{S}^{-\infty}_{\Gamma_0,\rho}$ such that

$$\widetilde{\mathrm{Op}}_h(e_L(t))\widetilde{\mathrm{Op}}_h(\chi) = \widetilde{\mathrm{Op}}_h(a(t)) + O(h^\infty)_{L^2\to L^2}, \quad \widetilde{\mathrm{Op}}_h(\chi)\widetilde{\mathrm{Op}}_h(e_R(t)) = \widetilde{\mathrm{Op}}_h(a(t)) + O(h^\infty)_{L^2\to L^2}$$

for all $t \in (-\varepsilon_0, \varepsilon_0)$. Note that we are implicitly using the fact that $a(t)$ is compactly supported in (x,ξ) to handle the fact that χ is not compactly supported in (x,ξ). Thus,

$$A(t) = \widetilde{\mathrm{Op}}_h(e_L(0))\widetilde{\mathrm{Op}}_h(\chi)U(t)\widetilde{\mathrm{Op}}_h(\chi)\widetilde{\mathrm{Op}}_h(e_R(t)) + O(h^\infty)_{L^2\to L^2}.$$

Since $\widetilde{\mathrm{Op}}_h(e_L(t))$ and $\widetilde{\mathrm{Op}}_h(e_R(t))$ are L^2 bounded uniformly in $t \in (-\varepsilon_0, \varepsilon_0)$, we estimate

$$\tilde{A}(t) := \widetilde{\mathrm{Op}}_h(\chi)U(t)\widetilde{\mathrm{Op}}_h(\chi).$$

In fact, we estimate $B(t) := \tilde{A}(t)(\tilde{A}(t))^*$ by considering its kernel:

$$\begin{aligned} B(t;x,y) &= \int U(t)(x,w)U(t)^*(w,y)\chi(h^{-\rho}x')\chi(h^{-\rho}y')\chi(h^{-\rho}w')^2\,dw \\ &= \frac{1}{(2\pi h)^{2n}} \int e^{i\Phi(t,x,w,y,\eta,\xi)/h} b(t,x,\eta)\bar{b}(t,y,\xi)\chi(h^{-\rho}x')\chi(h^{-\rho}y')\chi(h^{-\rho}w')^2\,dw\,d\eta\,d\xi \end{aligned}$$

with $\Phi(t,x,w,y,\eta,\xi)=\phi(t,x,\eta)-\phi(t,y,\xi)+\langle w,\xi-\eta\rangle$. First, performing stationary phase in (w_n,η_n) yields

$$B(t;x,y)=\frac{1}{(2\pi h)^{2n-1}}\int F(t,x,w',\xi_n)\overline{F(t,y,w',\xi_n)}\,dw'\,d\xi_n,$$
$$F(t,x,w',\xi_n):=\int e^{i(\phi(t,x,\eta',\xi_n)-\langle w',\eta'\rangle)/h}b_1(t,x,\eta',\xi_n)\chi(h^{-\rho}x')\chi(h^{-\rho}w')^2\,d\eta'$$

for some $b_1\in S^{\mathrm{comp}}(T^*\mathbb{R}^n)$. Next, note that since $\phi(0,x,\eta)=\langle x,\eta\rangle$,

$$\phi(t,x,\eta)-\langle x,\eta\rangle=t\tilde{\phi}(t,x,\eta)$$

with $\tilde{\phi}$ such that, for every multi-index α, there exists $C_\alpha>0$ with $|\partial^\alpha_{t,x,\eta}\tilde{\phi}|\le C_\alpha$.

Next, we claim that there exists $C>0$ such that

$$\|(\partial^2_{\eta'}\tilde{\phi}(t,x,\eta))^{-1}\|\le C\quad\text{if }\ (x,\eta)\in\Gamma(0),\ \ \partial_{\eta'}\phi(t,x,\eta)=0. \tag{6-9}$$

We postpone the proof of (6-9) and proceed to finish the proof of the lemma.

To continue the proof, note that, modulo an $O(h^{N\varepsilon})$ error, we may assume that the integrand of $B(t;x,y)$ is supported in $\{(x,y,w'):|x'|\le h^{\rho-\varepsilon},\ |y'|\le h^{\rho-\varepsilon},\ |w'|\le h^{\rho-\varepsilon}\}$ and $h^{\rho-\varepsilon}\le|t|$. Therefore, the bound in (6-9) continues to hold on the support of the integrand. By (6-9) and

$$\partial^2_{\eta'}(\phi(t,x,\eta)-t\tilde{\phi}(t,x,\eta))=0, \tag{6-10}$$

there is a unique critical point $\eta'_c(t,x,w',\xi_n)$ for the map $\eta'\mapsto\phi(t,x,\eta',\xi_n)-\langle w',\eta'\rangle$, in an $O(1)$ neighborhood of η'_c. Indeed, since $|\partial^3_{\eta'}\phi|\le Ct$,

$$\partial_{\eta'}\phi=t(\langle\partial^2_{\eta'}\tilde{\phi}(t,x,\eta'_c,\xi_n)(\eta'-\eta'_c),\eta'-\eta'_c\rangle+O(|\eta-\eta'_c|^3)).$$

In particular, η'_c is the unique solution to $\partial_{\eta'}\phi(t,x,\eta'_c,\xi_n)-w'=0$.

Next, again using (6-10), by applying the method of stationary phase in η' to F with small parameter h/t, we obtain

$$B(t;x,y)=\frac{1}{(2\pi h)^n t^{n-1}}\int e^{i\Phi_1(t,x,w',y,\xi_n)/h}B_1(t;x,y,w',\eta'_c,\xi)\,dw'\,d\xi_n,$$
$$\Phi_1(t,x,w',y,\xi_n):=\Psi(t,x,w',\xi_n)-\Psi(t,y,w',\xi_n),$$
$$\Psi(t,x,w',\xi_n):=\phi(t,x,\eta'_c(t,x,w',\xi_n),\xi_n)-\langle w',\eta'_c(t,x,w',\xi_n)\rangle,$$
$$B_1(t;x,y,w',\eta',\xi):=b_2(t,x,\eta',\xi_n)\bar{b}(t,y,\xi',\xi_n)\chi(h^{-\rho}x')\chi(h^{-\rho}y')\chi(h^{-\rho}w')^2$$

for some $b_2\in S^{\mathrm{comp}}(T^*\mathbb{R}^n)$. Next, observe that

$$\begin{aligned}\partial_{x_n}\partial_{\xi_n}\Psi(t,x,w',\xi_n)&=\partial_{x_n}\partial_{\xi_n}(\langle x'-w',\eta'_c\rangle+x_n\xi_n+O(t)_{C^\infty}\\&=\langle x'-w',\partial_{x_n}\partial_{\xi_n}\eta'_c\rangle+1+O(t)\\&=1+O(t)+O(h^\rho)=1+O(t),\end{aligned}$$

where in the last line we use the fact that $|t|\ge h^{\rho-\varepsilon}$, and therefore, there exist $c>0$ and a function $g=g(x',y,w',\xi_n)$ such that $|\partial_{\xi_n}\Phi_1|\ge c|x_n-g|$. In particular, integration by parts in ξ_n (with the operator

$L=(h^2+\partial_{\xi_n}\Phi_1 hD_{\xi_n})/(h^2+|\partial_{\xi_n}\Phi_1|^2))$ shows that for any $N>0$ there is $C_N>0$ such that

$$|B(t;x,y)|\le C_N h^{-n}t^{1-n}h^{\rho(n-1)}\chi(h^{-\rho}y')\chi(h^{-\rho}x')\frac{h^{2N}+h^N|x_n-g|^N}{(h^2+|x_n-g|^2)^N}.$$

Applying Schur's lemma together with the fact that there exists $C>0$ such that, for all t,

$$\sup_x\int|B(t;x,y)|\,dy+\sup_y\int|B(t;x,y)|\,dx\le Ch^{(2\rho-1)(n-1)}t^{1-n}$$

yields that $\|B(t)\|_{L^2\to L^2}\le Ch^{(2\rho-1)(n-1)}t^{1-n}$ for all $t\in(-\varepsilon_0,\varepsilon_0)$, and hence

$$\|X(0)X(t)\|_{L^2\to L^2}\le Ch^{(n-1)(2\rho-1)/2}t^{(1-n)/2},$$

as claimed.

Proof of the bound in (6-9). Let $\phi_t(x,\eta):=\phi(t,x,\eta)$ and $\varphi_t(x,y,\eta):=\phi_t(x,\eta)-\langle y,\eta\rangle$. Then we have $C_{\varphi_t}=\{(x,y,\eta):\partial_\eta\phi_t(x,\eta)=y\}$, and so

$$\Lambda_{\varphi_t}=\{(x,\ \partial_x\phi_t(x,\eta),\ \partial_\eta\phi_t(x,\eta),\ -\eta)\}\subset T^*\mathbb{R}^n\times T^*\mathbb{R}^n.$$

In particular, since Λ_{φ_t} is the twisted graph of κ_t, we have that κ_t is characterized by

$$\kappa_t(\partial_\eta\phi_t(x,\eta),\eta)=(x,\partial_x\phi_t(x,\eta)).$$

Furthermore, since $\kappa_t(\Gamma(0))=\Gamma(t)$, we have

$$\Gamma(t)=\{(x,\xi):\kappa_t(y,\eta)=(x,\xi),\ y=\partial_\eta\phi_t(x,\eta),\ \xi=\partial_x\phi_t(x,\eta),\ (y,\eta)\in\Gamma(0)\}.$$

Then, using $\kappa_t^*\xi_n=\eta_n$,

$$\Gamma(t)=\{(x,\xi):\xi'=\partial_{x'}\phi_t(x,\eta),\ \partial_{\eta'}\phi_t(x,\eta)=0,\ \xi_n=\eta_n,\ \eta\in\mathbb{R}^n\}.$$

Next, let $\tilde p:=(\tilde x,\tilde\eta)\in\Gamma(0)$ be such that $\partial_{\eta'}\phi_t(\tilde x,\tilde\eta)=0$. Without loss of generality, in what follows we assume that $\tilde x_n=0$. Letting $\Gamma_0(t):=\Gamma(t)|_{\{x_n=0\}}$ we have that

$$\Gamma_0(t)=\{(x,\xi):\xi'=\partial_{x'}\phi_t(x,\eta),\ \partial_{\eta'}\phi_t(x,\eta)=0,\ x_n=0,\ \xi_n=\eta_n,\ \eta\in\mathbb{R}^n\}.$$

In particular, letting $\tilde\xi:=(\partial_{x'}\phi_t(\tilde p),\tilde\eta_n)$ and $\tilde p_0:=(\tilde x,\tilde\xi)$, we have $\tilde p_0\in\Gamma_0(t)\cap\Gamma_0(0)$ and

$$\begin{aligned}T_{\tilde p_0}\Gamma_0(t)=\Big\{(\delta_x,\delta_\xi):\delta_{\xi'}&=\partial_x\partial_{x'}\phi_t(\tilde p)\delta_x+\partial_\eta\partial_{x'}\phi_t(\tilde p)\delta_\eta,\\ &\partial_x\partial_{\eta'}\phi_t(\tilde p)\delta_x+\partial_\eta\partial_{\eta'}\phi_t(\tilde p)\delta_\eta=0,\ \delta_{x_n}=0,\ \delta_{\xi_n}=\delta_{\eta_n},\ \delta_\eta\in\mathbb{R}^n\Big\}.\end{aligned}$$

Next, we note that $\partial_{x_n}\in T_{\tilde p_0}\Gamma(t)$ and $H_{q_i(t)}\in T_{\tilde p_0}\Gamma(t)$ for all $i=1,\dots,n-1$. Therefore, since $\partial_{x_n}q_i(t)=0$, we also know that $H'_{q_i(t)}:=(\partial_{\xi'}q_i(t),0,-\partial_{x'}q_i(t),0)\in T_{\tilde p_0}\Gamma_0(t)$ for all $i=1,\dots,n-1$. We claim that there exists $C>0$ such that, for all $v=(\delta_{x'},0,\delta_{\xi'},0)\in\operatorname{span}\{H'_{q_i(t)}\}_{i=1}^{n-1}\subset T_{\tilde p_0}\Gamma(t)$, we have

$$\|\delta_{x'}\|\ge Ct\|\delta_{\xi'}\|.\tag{6-11}$$

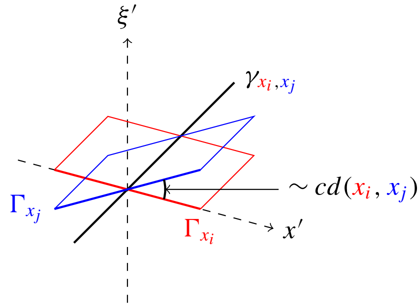

Figure 2. A pictorial representation of the coisotropics involved in Corollary 6.5, where γ_{x_i,x_j} is the geodesic from x_i to x_j. Localization to both Γ_{x_i} and Γ_{x_j} implies localization in the nonsymplectically orthogonal directions x' and ξ'. The uncertainty principle then rules this behavior out.

Suppose that the claim in (6-11) holds. Then, note that for each such v, since $\delta_{x_n} = 0$ and $\delta_{\xi_n} = 0$, we have that there is $\delta_{\eta'} \in \mathbb{R}^{n-1}$ such that

$$\delta_{\xi'} = \partial^2_{x'}\phi_t(\tilde{p})\delta_{x'} + \partial^2_{\eta'x'}\phi_t(\tilde{p})\delta_{\eta'}, \quad \partial^2_{x'\eta'}\phi_t(\tilde{p})\delta_{x'} + \partial^2_{\eta'}\phi_t(\tilde{p})\delta_{\eta'} = 0.$$

Using that $\partial^2_{x'\eta'}\phi_t(\tilde{p}) = \mathrm{Id} + O(t)$ and $\partial^2_{x'}\phi_t(\tilde{p}) = O(t)$, we conclude that

$$\partial^2_{\eta'}\phi_t(\tilde{p})[\partial^2_{\eta'x'}\phi_t(\tilde{p})]^{-1}\delta_{\xi'} = (\partial^2_{\eta'}\phi_t(\tilde{p})[\partial^2_{\eta'x'}\phi_t(\tilde{p})]^{-1}\partial^2_{x'}\phi_t(\tilde{p}) - \partial^2_{x'\eta'}\phi_t(\tilde{p}))\delta_{x'},$$

and so

$$\partial^2_{\eta'}\phi_t(\tilde{p})(\mathrm{Id} + O(t))\delta_{\xi'} = (-\mathrm{Id} + O(t))\delta_{x'}. \tag{6-12}$$

Let $H'_{q_i(t)} = (\delta^{(i)}_{x'}, 0, \delta^{(i)}_{\xi'}, 0)$. Since $\tilde{p}_0 \in \Gamma(t) \cap \Gamma(0)$, assumptions (6-6) and (6-7) yield that the vectors $\{\delta^{(i)}_{x'}\}_{i=1}^{n-1}$ are linearly independent. Indeed, setting $e_i := (\delta_{ij})_{j=1}^{n-1} \in \mathbb{R}^{n-1}$,

$$\delta^{(i)}_{x'} = \partial_{\xi_i} q_i(t) e_i + O(t^2), \quad |\partial_{\xi_i} q_i(t)| \geq Ct \tag{6-13}$$

for t small enough. Furthermore, (6-12) then yields that the $\{\delta^{(i)}_{\xi'}\}_{i=1}^{n-1}$ are linearly independent. Then, combining (6-12) with (6-11) yields (6-9) as claimed. □

To finish it only remains to prove (6-11). Let $v = (\delta_{x'}, 0, \delta_{\xi'}, 0) \in \operatorname{span}\{H'_{q_i(t)}\}_{i=1}^{n-1}$. Then, there is $a \in \mathbb{R}^{n-1}$ such that $\delta_{x'} = \sum_{i=1}^{n-1} a_i \delta^{(i)}_{x'}$ and $\delta_{\xi'} = \sum_{i=1}^{n-1} a_i \delta^{(i)}_{\xi'}$. Next, note that by (6-13) we have $\|\delta_{x'}\| \geq \|a\|(Ct + O(t^2))$. Since $\|\delta_{\xi'}\| \leq C_0\|a\|$ for some $C_0 > 0$, the claim in (6-11) follows. □

For each $x \in M$, let Γ_x be as in (6-1). (See Figure 2 for a schematic representation of these two coisotropic submanifolds.) Then we have the following result.

Corollary 6.5. *Let $0 < \rho < 1$, $0 < \varepsilon < \rho$, and $\gamma(t) : (-\varepsilon_0, \varepsilon_0) \to M$ be a unit speed geodesic. Then, for $X(t) \in \Psi^{-\infty}_{\Gamma_{\gamma(t)},\rho}$ and h such that $h^{\rho-\varepsilon} \leq |t| < \varepsilon_0$,*

$$\|X(0)X(t)\|_{L^2 \to L^2} \leq Ch^{(n-1)(2\rho-1)/2} t^{(1-n)/2}.$$

Proof. To do this, we study the geometry of the flow-out coisotropics $\Gamma_{\gamma(t)}$. Namely, we prove that $\Gamma_{\gamma(t)}$ is defined by some functions $\{q_i(t)\}_{i=1}^n$ with $q_i(0) = x_i$ that satisfy (6-6) and (6-7). We then apply Proposition 6.4 to $\Gamma(t) = \kappa^{-1}(\Gamma_{\gamma(t)})$ for a suitable symplectomorphism κ.

Fix coordinates $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ on M such that $\gamma(t) = (0, t)$ and

$$\partial^2_{\xi'} |\xi|_{g(x)}\big|_{x=0,\xi=(0,1)} = \mathrm{Id}\,.$$

For each $t \in (-\varepsilon_0, \varepsilon_0)$, let $\mathcal{H}_t$ be the submanifold transverse to the Hamiltonian vector field H_p defined by

$$\mathcal{H}_t := \{(x', t, \xi', \xi_n) : 2\xi_n > |\xi|_g,\ |x'| \le \delta_0\},$$

where $\delta_0 > 0$ is chosen such that $\Gamma_{\gamma(t)} \cap \mathcal{H}_t = \{(0, t, \xi', \xi_n) : 2\xi_n > |\xi|_g,\ ||\xi|_g - 1| < \delta\}$.

In particular, as a subset of $\{||\xi|_g - 1| < \delta\}$, we define $\Gamma_{\gamma(t)} \cap \mathcal{H}_t$ by the coordinate functions $\{x_i\}_{i=1}^{n-1}$. For each $t \in (-\varepsilon_0, \varepsilon_0)$ let $\tilde{q}_i(t) : \mathcal{H}_t \to \mathbb{R}$ be given by $\tilde{q}_i(t) = x_i$ for $i = 1, \dots, n-1$. Then, define $\{q_i(t)\}_{i=1}^{n-1}$ on T^*M by

$$H_p q_i(t) = 0, \quad q_i(t)|_{\mathcal{H}_t} = \tilde{q}_i(t).$$

For all t, we note that $H_p(H_{q_i(t)} q_j(t)) = 0$ and

$$\{q_i(t), q_j(t)\}|_{\mathcal{H}_t} = \partial_{\xi_n} q_i(t) \partial_{x_n} q_j(t) - \partial_{\xi_n} q_j(t) \partial_{x_n} q_i(t) + \widetilde{H}_{q_i(t)} q_j(t),$$

where $\widetilde{H}$ is the Hamiltonian vector field in $T^*\{x_n = t\}$. In particular, since $\partial_{\xi_n} \tilde{q}_i(t) = 0$ and $\widetilde{H}_{q_i(t)}$ is tangent to $\mathcal{H}_t$, we have $\{q_i(t), q_j(t)\}|_{\mathcal{H}_t} = 0$. Hence, $\{q_i(t), q_j(t)\} \equiv 0$, $\{q_i(t), p\} = 0$, $q_i(0) = x_i$, and $\{q_i(t)\}_{i=1}^{n-1}$ define $\Gamma_{\gamma(t)}$. Next, observe that there exists $s \in \mathbb{R}$ such that, for each $i = 1, \dots, n-1$, $q_i(0)(x, \xi) = x_i(\varphi_s(x, \xi))$ with $\varphi_s(x, \xi) \in \mathcal{H}_0$. Since $\partial_{\xi_n} p \neq 0$ on $\mathcal{H}_0$, for E near 0 there exist a_E and e_E such that

$$p(x, \xi) - E = e_E(x, \xi)(\xi_n - a_E(x, \xi')) \tag{6-14}$$

with $e_E > c$ for some constant $c > 0$. In particular, $\varphi_s = e^{sH_p}$ is a reparametrization of $\tilde{\varphi}_s := e^{s(H_{\xi_n - a_E})}$ on $\{p = E\}$, and we have that, for $(x, \xi) \in \{p = E\}$ and all $i = 1, \dots, n-1$,

$$q_i(0)(x, \xi) = x_i(\tilde{\varphi}_{-x_n}(x, \xi)) = x_i + x_n \partial_{\xi_i} a_E(x, \xi') + O(x_n^2)_{C^\infty}.$$

In particular, on $\mathcal{H}_t \cap \{p = E\}$, using this together with the fact that since $H_{q_j(t)}$ is tangent to $\{p = E\}$ and $x_n = t$, $\partial_{\xi_n} q_i(t) = \partial_{\xi_n} \tilde{q}_i(t) = 0$, we have

$$\begin{aligned}\{q_j(t), q_i(0)\}|_{\mathcal{H}_t \cap \{p=E\}} &= \partial_{\xi_n} q_j(t) \partial_{x_n} q_i(0) - \partial_{x_n} q_j(t) \partial_{\xi_n} q_i(0) + \widetilde{H}_{q_j(t)} q_i(0)\\ &= \partial_{\xi_n} \tilde{q}_j(t) \partial_{x_n} q_i(0) - \partial_{x_n} q_j(t) O(t^2) + \widetilde{H}_{\tilde{q}_j(t)} q_i(0)\\ &= O(t^2) + \partial_{\xi_j}(t \partial_{\xi_i} a_E)(0, \xi').\end{aligned}$$

Now, since $\partial^2_\xi p|_{T\{p=E\}} > 0$ and, for all $i, j = 1, \dots, n$,

$$\partial_{\xi_i \xi_j} p = \partial_{\xi_j} \partial_{\xi_i} e_E(\xi_n - a_E) + \partial_{\xi_i} e_E(\delta_{nj} - \partial_{\xi_j} a_E) + \partial_{\xi_j} e_E(\delta_{ni} - \partial_{\xi_i} a_E) - e_E \partial_{\xi_j} \partial_{\xi_i} a_E, \tag{6-15}$$

we have, as quadratic forms, $\partial^2_\xi p|_{T\{p=E\}} = -e_E \partial^2_\xi a_E|_{T\{p=E\}}$. Indeed, if $V = \sum_j V^j \partial_{\xi_j} \in T\{p = E\}$, then

$$0 = V(p - E)|_{p=E} = e_E V(\xi_n - a_E) + (V e_E)(\xi_n - E)|_{p=E} = e_E V(\xi_n - a_E),$$

and therefore, since $e_E \neq 0$, we have $V(\xi_n - a_E)|_{p=E} = 0$. Next, observe that, on $\{p = E\}$,

$$\partial_{\xi_i} e_E \sum_j (\delta_{nj} - \partial_{\xi_j} a_E) V^j = \partial_{\xi_i} e_E (V(\xi_n - a_E)) = 0.$$

In particular, the first three terms in (6-15) vanish on $T\{p = E\}$.

Hence, since $\partial^2_{\xi'} p|_{x=0,\xi=(0,1)} = \mathrm{Id}$, we have that $\partial^2_{\xi'} a_E(0, \xi') < 0$ is a multiple of the identity at $x = 0$, $\xi' = 0$, and $p = E$. Next, observe that

$$\Gamma_{\gamma(0)} \cap \Gamma_{\gamma(t)} \subset \{(0, s, 0, \xi_n) : s \in \mathbb{R},\ \xi' = 0\}.$$

Therefore, there are $c, C > 0$ with

$$c\delta_{ij} t + O(t^2) \leq \left|\{q_i(t), q_j(0)\}|_{\mathcal{H}_t \cap \{p=E\} \cap \Gamma_{\gamma(0)} \cap \Gamma_{\gamma(t)}}\right| \leq C\delta_{ij} t + O(t^2)$$

on $\Gamma_{\gamma(0)} \cap \Gamma_{\gamma(t)}$. Then, $c\delta_{ij} t + O(t^2) \leq \left|\{q_i(t), q_j(0)\}|_{\{p=E\}}\right| \leq C\delta_{ij} t + O(t^2)$ by invariance under H_p. Since E small is arbitrary, this holds on $\Gamma_{\gamma(0)} \cap \Gamma_{\gamma(t)}$.

Now, by Darboux's theorem, there is a symplectomorphism κ such that, for all $i = 1, \dots, n-1$, $\kappa^* q_i(0) = x_i$ and $\kappa^* p = \xi_n$. In particular, $\kappa^{-1}(\Gamma_{\gamma(0)}) \subset \Gamma(0) = \{(0, x_n, \xi', \xi_n) : x_n \in \mathbb{R},\ \xi \in \mathbb{R}^{n-1} \times \mathbb{R}\}$ and, abusing notation slightly by relabeling $q_i(t) = \kappa^* q_i(t)$, we have that (6-6) and (6-7) hold. In particular, Proposition 6.4 applies to $\Gamma(t) = \kappa^{-1}(\Gamma_{\gamma(t)})$.

Now, let U be a microlocally unitary quantization of κ and $X(t) \in \Psi^{-\infty}_{\Gamma_{\gamma(t)},\rho}$. Then, $U^{-1} X(t) U \in \Psi^{-\infty}_{\Gamma(t),\rho}$ and hence the corollary is proved. $\square$

6C. ***Almost orthogonality for coisotropic cutoffs.*** In this section, we finally prove an estimate which shows that coisotropic cutoffs associated with Γ_{x_i} for many x_i are almost orthogonal. This, together with the fact that these cutoffs respect pointwise values near x_i, is what allows us to control the number of points at which a quasimode may be large.

Proposition 6.6. *Let $\{B(x_i, R)\}_{i=1}^{N(h)}$ be a $(\mathfrak{D}, R)$-good cover for M, and $X_i \in \Psi^{-\infty}_{\Gamma_{x_i},\rho}$, $i = 1, \dots, N(h)$, with uniform symbol estimates. Then, there are $C > 0$ and $h_0 > 0$ such that, for all $0 < h < h_0$, $\mathcal{J} \subset \{1, \dots, N(h)\}$ and $u \in L^2(M)$, we have*

$$\sum_{j \in \mathcal{J}} \|X_j u\|^2_{L^2} \leq C(1 + (h^{2\rho-1} R^{-1})^{(n-1)/2} |\mathcal{J}|^{(3n+1)/(2n)} (1 + (h^{2\rho-1} R^{-1})^{(n-1)/4})) \|u\|^2_{L^2}. \tag{6-16}$$

Proof. To prove this bound we will decompose the sum in (6-16) as

$$\sum_{i \in \mathcal{J}} \|X_i u\|^2_{L^2} = \Big\|\sum_{i \in \mathcal{J}} X_i u\Big\|^2_{L^2} - \Big\langle \sum_{\substack{i,j \in \mathcal{J} \\ i \neq j}} X_j^* X_i u, u\Big\rangle. \tag{6-17}$$

First, we note that by Corollary 6.5, (once with $X(0) = X_j^*$ and $X(t) = X_i$, and once with $X(0) = X_j$ and $X(t) = X_i^*$) there exists $C > 0$ such that, for $i \neq j$,

$$\|X_j^* X_i\| + \|X_j X_i^*\| \leq C h^{(n-1)(\rho-1/2)} d(x_i, x_j)^{(1-n)/2}.$$

Therefore, by the Cotlar–Stein lemma,

$$\Big\|\sum_{j\in\mathcal{J}} X_j\Big\| \le \sup_{j\in\mathcal{J}}\Big(\|X_j\| + \sum_{i\in\mathcal{J}\setminus\{j\}} \|X_j^*X_i\|^{1/2} + \|X_jX_i^*\|^{1/2}\Big)$$
$$\le 2 + Ch^{(n-1)(\rho-1/2)/2}\sup_{j\in\mathcal{J}}\sum_{i\in\mathcal{J}\setminus\{j\}} d(x_i,x_j)^{(1-n)/4}.$$

To estimate the sum, observe that there exists $C>0$ such that, for any $j\in\mathcal{J}$ and any positive integer k,

$$\frac{2^{kn}}{C} \le \#\{i : 2^kR \le d(x_i,x_j) \le 2^{k+1}R\} \le C2^{(k+1)n}.$$

In particular, there is $C>0$ such that, for any $j\in\mathcal{J}$,

$$\sum_{i\in\mathcal{J}\setminus\{j\}} d(x_i,x_j)^{(1-n)/4} \le C\sum_{k=0}^{\frac{1}{n}\log_2|\mathcal{J}|} 2^{kn}(2^kR)^{(1-n)/4} \le C|\mathcal{J}|^{(3n+1)/(4n)}R^{(1-n)/4}. \tag{6-18}$$

Therefore, we shall bound the first term in (6-17) using

$$\Big\|\sum_{j\in\mathcal{J}} X_j\Big\| \le C + Ch^{(n-1)(\rho-1/2)/2}R^{(1-n)/4}|\mathcal{J}|^{(3n+1)/(4n)}. \tag{6-19}$$

We next proceed to control the second term in (6-17). Let $\widetilde{X}_j\in\Psi^{-\infty}_{\Gamma_{x_j},\rho}$ such that

$$\widetilde{X}_jX_j = X_j + O(h^\infty)_{L^2\to L^2}.$$

By the Cotlar–Stein Lemma,

$$\Big\|\sum_{\substack{i,j\in\mathcal{J}\\ i\neq j}} X_j^*X_i\Big\| \le \sup_{\substack{k,\ell\in\mathcal{J}\\ k\neq\ell}}\sum_{\substack{i,j\in\mathcal{J}\\ i\neq j}} \|X_k^*\widetilde{X}_\ell X_\ell X_j^*\widetilde{X}_j^*X_i\|^{1/2} + \|X_\ell^*\widetilde{X}_kX_kX_i^*\widetilde{X}_i^*X_j\|^{1/2} + O(h^\infty|\mathcal{J}|^2). \tag{6-20}$$

By Corollary 6.5 there exists $C>0$ such that, for $k\neq\ell,\ i\neq j$,

$$\|X_k^*\widetilde{X}_\ell X_\ell X_j^*\widetilde{X}_j^*X_i\| \le Ch^{(n-1)(2\rho-1)}\min\{1, h^{(n-1)(2\rho-1)/2}d(x_j,x_\ell)^{-(n-1)/2}\}(d(x_k,x_\ell)d(x_j,x_i))^{(1-n)/2}.$$

Using that

$$\sup_{\substack{k,\ell\in\mathcal{J}\\ k\neq\ell}} d(x_k,x_\ell)^{(1-n)/4} \le R^{(1-n)/4},$$

adding in (6-20), and combining with the bound in (6-18), we get

$$\Big\|\sum_{\substack{i,j\in\mathcal{J}\\ i\neq j}} X_j^*X_i\Big\| \le Ch^{(n-1)(2\rho-1)/2}(1+h^{(n-1)(2\rho-1)/4}|\mathcal{J}|^{(3n+1)/(4n)}R^{(1-n)/4})|\mathcal{J}|^{(3n+1)/(4n)}R^{(1-n)/2}. \tag{6-21}$$

In particular, combining (6-19) and (6-21) into (6-17) we obtain

$$\sum_{i\in\mathcal{J}}\|X_iu\|^2 \le C(1+h^{(n-1)(\rho-1/2)}R^{(1-n)/2}|\mathcal{J}|^{(3n+1)/(2n)} + h^{3(n-1)(2\rho-1)/4}R^{3(1-n)/4}|\mathcal{J}|^{(3n+1)/(2n)})\|u\|^2_{L^2}$$
$$\le C(1+h^{(n-1)(\rho-1/2)}R^{(1-n)/2}(1+(h^{2\rho-1}R^{-1})^{(n-1)/4})|\mathcal{J}|^{(3n+1)/(2n)})\|u\|^2_{L^2}. \qquad\square$$

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