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NONEXISTENCE OF THE BOX DIMENSION FOR DYNAMICALLY INVARIANT SETS

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One of the key challenges in the dimension theory of smooth dynamical systems lies in establishing whether or not the Hausdorff, lower and upper box dimensions coincide for invariant sets. For sets invariant under conformal dynamics, these three dimensions always coincide. On the other hand, considerable attention has been given to examples of sets invariant under nonconformal dynamics whose Hausdorff and box dimensions do not coincide. These constructions exploit the fact that the Hausdorff and box dimensions quantify size in fundamentally different ways, the former in terms of covers by sets of varying diameters and the latter in terms of covers by sets of fixed diameters. In this article we construct the first example of a dynamically invariant set with distinct lower and upper box dimensions. Heuristically, this says that if size is quantified in terms of covers by sets of equal diameters, a dynamically invariant set can appear bigger when viewed at certain resolutions than at others.

1. Introduction

The dimension theory of dynamical systems is the study of the complexity of sets and measures which remain invariant under dynamics, from a dimension theoretic point of view. This branch of dynamical systems has its foundations in the seminal work [Bowen 1979] on the dimension of quasicircles and [Ruelle 1982] on the dimension of conformal repellers, and has since developed into an independent field of research which continues to receive noteworthy attention in the literature [Bárány et al. 2019; Cao et al. 2019; Das and Simmons 2017]. For an overview of this extensive field, see the monographs [Barreira 2008; Pesin 1997] and the surveys [Barreira and Gelfert 2011; Chen and Pesin 2010; Schmeling 2001].

The most common ways of measuring the dimension of invariant sets are through the Hausdorff dimension and the lower and upper box dimensions, which quantify the complexity of the set in related but subtly distinct ways. Roughly speaking, the Hausdorff dimension measures how efficiently the set can be covered by sets of arbitrarily small size, whereas the lower and upper box dimensions measure this in terms of covers by sets of uniform size, along the scales for which this can be done in the most and least efficient way, respectively. Given a subset E of a separable metric space X, the lower and upper box dimensions are defined by

$$\underline{\dim}_{\mathrm{B}} E = \liminf_{\delta\to 0} \frac{\log N_\delta(E)}{-\log\delta} \quad \text{and} \quad \overline{\dim}_{\mathrm{B}} E = \limsup_{\delta\to 0} \frac{\log N_\delta(E)}{-\log\delta},$$

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respectively, where $N_\delta(E)$ denotes the smallest number of sets of diameter $\delta > 0$ required to cover E. If the lower and upper box dimensions coincide we call the common value the box dimension, written $\dim_{\mathrm{B}}$, otherwise we say that the box dimension does not exist.

For any subset $E \subseteq X$,

$$\dim_{\mathrm{H}} E \le \underline{\dim}_{\mathrm{B}} E \le \overline{\dim}_{\mathrm{B}} E, \tag{1}$$

where $\dim_{\mathrm{H}}$ denotes the Hausdorff dimension. A priori each inequality may or may not be strict. However, when E is invariant under a smooth mapping f, the additional structure imposed by the dynamical invariance of E means that certain properties of f can either force some degree of homogeneity or, on the contrary, inhomogeneity across the set, forcing equalities or strict inequalities in (1), respectively. Characterising which properties of f imply or preclude equalities in (1) is one of the key challenges in dimension theory.

A common feature in the dimension theory of smooth *conformal* dynamics is the coincidence of the Hausdorff and lower and upper box dimensions for invariant sets. For example, in the setting of smooth expanding maps, the following result pertains to a more general result which was obtained independently by Gatzouras and Peres [1997] and Barreira [1996], generalising previous results of Falconer [1989].

Theorem 1.1 [Barreira 1996; Gatzouras and Peres 1997]. *Suppose $f : M \to M$ is a C^1 map of a Riemannian manifold M and that $\Lambda = f(\Lambda)$ is a compact set such that $f^{-1}(\Lambda) \cap U \subset \Lambda$ for some open neighbourhood U of Λ. Additionally, assume that*

- *f is **conformal**: for each $x \in M$, the derivative $d_x f$ is a scalar multiple of an isometry,*
- *f is **expanding** on Λ: there exist constants $C > 0$ and $\lambda > 1$ such that, for all $x \in \Lambda$ and u in the tangent space $T_x M$,*

$$\|d_x f^n u\| \ge C\lambda^n \|u\|.$$

Then, for any compact set $F = f(F) \subset \Lambda$,

$$\underline{\dim}_{\mathrm{B}} F = \overline{\dim}_{\mathrm{B}} F = \dim_{\mathrm{H}} F.$$

Similar results hold in the setting of smooth diffeomorphisms. For example, if $f : M \to M$ is a topologically transitive C^1 diffeomorphism with a basic set Λ and f is conformal on Λ, then we have $\dim_{\mathrm{H}} \Lambda = \underline{\dim}_{\mathrm{B}} \Lambda = \overline{\dim}_{\mathrm{B}} \Lambda$ [Barreira 1996; Pesin 1997], and an analogous statement holds for the dimensions of the intersections of Λ with its local stable and unstable manifolds [Palis and Viana 1988; Takens 1988].

In contrast, in the realm of smooth *nonconformal* dynamical systems, coincidence of the Hausdorff and box dimensions is no longer a universal trait of invariant sets. Indeed, examples of invariant sets with distinct Hausdorff and box dimensions have attracted enormous attention [Bedford 1984; Kenyon and Peres 1996; Lalley and Gatzouras 1992; McMullen 1984; Neunhäuserer 2002; Pollicott and Weiss 1994] and discussion in surveys [Barreira and Gelfert 2011; Chen and Pesin 2010; Fraser 2021]. This type of dimension gap result exploits the fact that the Hausdorff dimension quantifies the size of the set in terms of covers by sets of varying diameters rather than fixed diameters which are used by the box dimension. Indeed invariant sets of certain nonconformal dynamics will contain long, thin and well-aligned copies of itself, meaning that covering by sets of varying diameter is often more efficient, inducing this type of dimension gap. However, surprisingly there seems to be no mention in the literature of the possibility of a dynamically invariant set with *distinct lower and upper box dimensions*. Our main result demonstrates the existence of such sets.

Theorem 1.2. *There exist integers $n > m \geq 2$ and a compact subset of the torus $F \subset \mathbb{T}^2$ such that F is invariant, $F = T(F)$ under the expanding toral endomorphism*

$$T(x, y) = (mx \mod 1, ny \mod 1)$$

and

$$\underline{\dim}_{\mathrm{B}} F < \overline{\dim}_{\mathrm{B}} F.$$

In particular, the box dimension of F does not exist.

Since $n > m$, we have that T is a nonconformal map. Well-known examples from the literature, such as Bedford–McMullen carpets [Fraser 2021], demonstrate that equality of the Hausdorff and box dimensions is not guaranteed in Theorem 1.1 if the assumption of conformality is dropped. Furthermore, Theorem 1.2 indicates that the lower and upper box dimensions need not coincide either in Theorem 1.1 if the assumption of conformality is dropped. This is arguably a more striking type of dimension gap since, while it is easy to see that sets invariant under nonconformal dynamics may cease to be homogeneous in space, which is captured by the possibility of distinct Hausdorff and box dimensions, one would expect the dynamical invariance to at least force homogeneity in scale, but our result demonstrates that this too can fail. In particular Theorem 1.2 describes that, when measuring size in terms of covers by sets of equal diameter, a dynamically invariant set can sometimes appear bigger and at other times appear smaller depending on the "resolution" we are viewing it at. We highlight that our construction is also significantly more involved than standard examples of invariant sets with distinct Hausdorff and box dimensions, such as Bedford–McMullen carpets.

The dynamics of T on the invariant set F, which will be constructed in Section 2, has two key features which in conjunction induce distinct box dimensions. Firstly, the nonconformality of T causes the box dimensions of F to be sensitive to the length of time it takes for an orbit of T to move from a subset $A \subset F$ which is "entropy maximising" for the dynamics of T to a subset B which is "entropy maximising" for the dynamics of the projection $x \mapsto mx \mod 1$ of T. Secondly, the dynamics on F, which can be modelled by a topologically mixing *coded subshift* [Blanchard and Hansel 1986] on an appropriate symbolic space, has the property that the length of time it takes an orbit of T to move from A to B is highly dependent on how long the orbit has spent in A. In particular, the dynamics fails to satisfy most forms of specification [Kwietniak et al. 2016]. The resolution at which F is viewed determines how long the orbits of points of interest (for the dimension estimates at that particular resolution) spend in A, and combined with the properties mentioned above this forces distinct box dimensions.

Finally, we discuss some connections between Theorem 1.2 and the literature on self-affine and sub-self-affine sets. Let $\{S_i : \mathbb{R}^d \to \mathbb{R}^d\}_{i=1}^N$ be a collection of affine contractions, i.e., $S_i(\cdot) = A_i(\cdot) + \boldsymbol{t}_i$ for each $1 \leq i \leq N$, where $A_i \in GL(d, \mathbb{R})$ with Euclidean norm $\|A_i\| < 1$ and $\boldsymbol{t}_i \in \mathbb{R}^d$. We call $\{S_i\}_{i=1}^N$ an affine iterated function system. A *sub-self-affine set* [Käenmäki and Vilppolainen 2010] is a nonempty, compact set $E \subset \mathbb{R}^d$ such that

$$E \subseteq \bigcup_{i=1}^{N} S_i(E). \tag{2}$$

If (2) is an equality then E is called a *self-affine set*, in particular every self-affine set is an example of a sub-self-affine set. Every affine iterated function system admits a unique self-affine set. However, there are infinitely many sub-self-affine sets which are not self-affine. Indeed, the unique self-affine set is the image of the full shift $\{1, \dots, N\}^{\mathbb{N}}$ under an appropriate projection induced from the family $\{S_i\}_{i=1}^N$, whereas sub-self-affine sets are in one-to-one correspondence with the projections of subshifts of the full shift. Under suitable "separation conditions" on $\{S_i\}_{i=1}^N$, any sub-self-affine set E satisfies $f(E) \subseteq E$ for an appropriate piecewise expanding map f given by the inverses of the contractions. The set F which will be constructed in Section 2 to prove Theorem 1.2 is a sub-self-affine set (which is not self-affine) for the affine iterated function system induced from the inverse branches of T.

The dimension theory of self-affine sets has been an active topic of research since the 1980s and substantial progress has been made in recent years. Sub-self-affine sets were introduced by Käenmäki and Vilppolainen [2010] as natural analogues of sub-self-similar sets which were studied earlier by Falconer [1995]. It is known by the results of Falconer [1988] and Käenmäki and Vilppolainen [2010] that the box dimension of a generic sub-self-affine sets exists, moreover this has been verified for large explicit families of planar self-affine sets [Bárány et al. 2019]. However, the following question was open until now.

Question 1.3. Does the box dimension of every (sub-)self-affine set exist?

The version of the above question for self-affine sets is a folklore open question within the fractal geometry community, to which the answer is widely conjectured to be affirmative. In contrast, a corollary of our main result is that the answer to Question 1.3 for general sub-self-affine sets is negative.

Corollary 1.4. *There exist sub-self-affine sets whose box dimension does not exist.*

Organisation of paper. In Section 2 we construct the set F and its underlying subshift Σ and offer some heuristic reasoning behind Theorem 1.2. Section 3 contains entropy estimates. In Section 4 we introduce the scales for the lower and upper box dimension computations and prove Theorem 1.2. Section 5 contains some questions for further investigation.

2. Construction of a $(\times m, \times n)$-invariant set

Fix $m = 2$ and $n = 12$. Let

$$\Delta = \{(a, b) : 1 \le a \le 2,\ 1 \le b \le 12,\ a, b \in \mathbb{N}\}.$$

For any $(a, b) \in \Delta$, define the contraction $S_{(a,b)} : [0, 1]^2 \to [0, 1]^2$ as

$$S_{(a,b)}(x, y) = \left(\tfrac{1}{2}x + \tfrac{1}{2}(a-1),\ \tfrac{1}{12}y + \tfrac{1}{12}(b-1)\right).$$

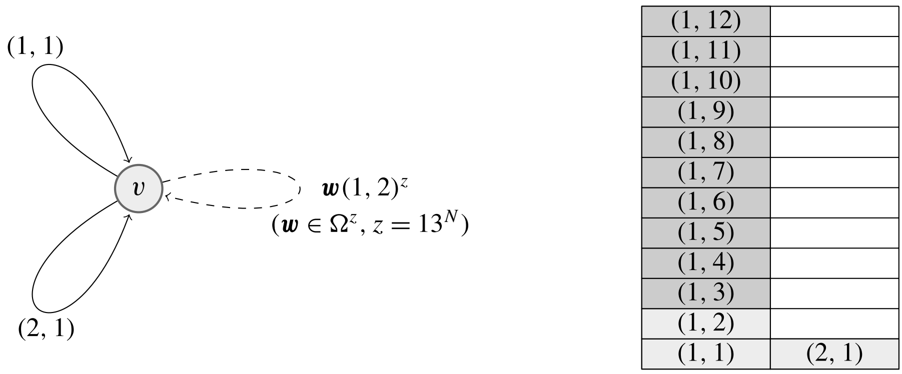

Figure 1. Left: The presentation G of Σ. The dashed loop indicates that, for each $N \in \mathbb{N}$ and $\boldsymbol{w} \in \Omega^{13^N}$, there is a path of length $2 \cdot 13^N$ which begins and ends at v such that its sequence of labels reads $\boldsymbol{w}(1,2)^{13^N}$. Right: Images of $[0,1]^2$ under $S_{(a,b)}$ for each (a,b) that labels some edge in G. The darker of the shaded rectangles correspond to $S_{(a,b)}([0,1]^2)$ for $(a,b) \in \Omega$.

These are the partial inverses of T. If $\mathtt{i}, \mathtt{j} \in \Delta^{\mathbb{N}}$ with $\mathtt{i} \neq \mathtt{j}$, we let $\mathtt{i} \wedge \mathtt{j}$ denote the longest common prefix to $\mathtt{i}$ and $\mathtt{j}$ and denote its length by $|\mathtt{i} \wedge \mathtt{j}|$. We equip $\Delta^{\mathbb{N}}$ with the metric

$$d(\mathtt{i}, \mathtt{j}) = \begin{cases} 1/2^{|\mathtt{i} \wedge \mathtt{j}|} & \text{if } \mathtt{i} \neq \mathtt{j}, \\ 0 & \text{if } \mathtt{i} = \mathtt{j}. \end{cases}$$

The set F that satisfies Theorem 1.2 will be the projection of a set $\Sigma \subseteq \Delta^{\mathbb{N}}$ under the continuous and surjective (but not injective) coding map $\Pi : \Delta^{\mathbb{N}} \to [0,1]^2$ given by

$$\Pi((a_1, b_1)(a_2, b_2) \cdots) := \lim_{n \to \infty} S_{(a_1,b_1)\cdots(a_n,b_n)}(0),$$

where $S_{(a_1,b_1)\cdots(a_n,b_n)}$ denotes the composition $S_{(a_1,b_1)} \circ \cdots \circ S_{(a_n,b_n)}$.

Let $\Omega = \{(1,i)\}_{i=3}^{12}$. For each $N \in \mathbb{N}$, let Ω^N denote words of length N with symbols in Ω, and $\Omega^{\mathbb{N}}$ the set of infinite sequences with symbols in Ω. Given any $(a,b) \in \Delta$, we denote by $(a,b)^n$ the word $(a,b)(a,b)\cdots(a,b)$ of length n. Define $\mathcal{C}$ to be the collection of words

$$\mathcal{C} := \{(1,1), (2,1)\} \cup \bigcup_{N=1}^{\infty} \bigcup_{\boldsymbol{w} \in \Omega^{13^N}} \{\boldsymbol{w}(1,2)^{13^N}\}$$

and

$$B := \{\boldsymbol{u}\boldsymbol{u}_1\boldsymbol{u}_2\boldsymbol{u}_3 \cdots : \boldsymbol{u}_i \in \mathcal{C} \text{ for all } i \in \mathbb{N},\ \boldsymbol{u} \text{ is a suffix of some word in } \mathcal{C}\}. \tag{3}$$

Then we define the sequence space $\Sigma = \bar{B}$.[1] Equivalently B can be understood as the set of all infinite sequences which label a one-sided infinite path on the directed graph G in Figure 1. G is called the presentation of Σ.

[1]The set of accumulation points $\Sigma \setminus B$ will turn out to be unimportant for our analysis, but for the reader's convenience we provide a description of this set in (4).

It is easy to check that $\sigma(\Sigma) = \Sigma$, where $\sigma : \Sigma \to \Sigma$ denotes the left shift map. In particular, Σ is an example of a *coded subshift*, meaning a subshift which can be expressed as the closure of the space of all infinite paths on a path-connected (possibly infinite) graph, which were first introduced by Blanchard and Hansel [1986]. Note that whenever this graph is finite, its coded subshift is necessarily sofic, and that any $(\times m, \times n)$-invariant set which can be modelled by a sofic shift has a well-defined box dimension which can be explicitly computed [Fraser and Jurga 2020; Kenyon and Peres 1996]. Finally, we set $F = \Pi(\Sigma)$, noting that $F = T(F)$ since $\sigma(\Sigma) = \Sigma$ and $\Pi \circ \sigma = T \circ \Pi$. From this it is easy to see that F is a sub-self-affine set for the iterated function system $\{S_{(a,b)} : (a, b) \in \Delta\}$.

While of course it will be necessary to cover the entirety of F and obtain bounds on the size of this cover at different scales, the proof of Theorem 1.2 will essentially boil down to the asymptotic difference that emerges between

(a) the size of the cover — by squares of side 12^{-13^N} — of the intersection of F with the collection of rectangles $\{S_{\boldsymbol{i}}([0, 1]^2) : \boldsymbol{i} \in \Omega^{13^N}\}$, and

(b) the size of the cover — by squares of side $12^{-13^{N-1/2}}$ — of the intersection of F with the collection of rectangles $\{S_{\boldsymbol{i}}([0, 1]^2) : \boldsymbol{i} \in \Omega^{13^{N-1/2}}\}$.

Roughly speaking, F occupies a large proportion of the width of each rectangle $S_{\boldsymbol{i}}([0, 1]^2)$ in case (a). Such a rectangle has width 2^{-13^N} and height 12^{-13^N} (which equals the sidelength of squares in the cover). For any $\boldsymbol{i} \in \Omega^{13^N}$ and $\boldsymbol{j} \in \{(1, 1), (2, 1)\}^{13^N(\log 12/\log 2-2)}$, we have that $\boldsymbol{i}(1, 2)^{13^N}\boldsymbol{j}$ constitutes a legal word in Σ and each $S_{\boldsymbol{i}(1,2)^{13^N}\boldsymbol{j}}([0, 1]^2)$ has width roughly 12^{-13^N} (which equals the sidelength of squares in the cover), therefore $S_{\boldsymbol{i}}([0, 1]^2)$ requires roughly $2^{13^N(\log 12/\log 2-2)}$ squares to cover it. Importantly, this is a positive power of 12^{13^N}, which indicates "growth" in dimension.

In case (b), F occupies a very thin proportion of the width of each rectangle $S_{\boldsymbol{i}}([0, 1]^2)$. Each such rectangle has width $2^{-13^{N-1/2}}$ and height $12^{-13^{N-1/2}}$ (which is equal to the sidelength of squares in this cover). Any $\mathtt{i} \in \Sigma$ which begins with a word in $\Omega^{13^{N-1/2}}$ can be written as $\mathtt{i} = \boldsymbol{i}\boldsymbol{j}\mathtt{j}$ for $\boldsymbol{i} \in \Omega^{13^{N-1/2}}$, $\boldsymbol{j} = (1, b_1)\cdots(1, b_{13^N})$ and some infinite word $\mathtt{j} \in \Sigma$. In particular, any point in $F \cap S_{\boldsymbol{i}}([0, 1]^2)$ belongs to $S_{\boldsymbol{i}\boldsymbol{j}}([0, 1]^2)$ which has width *less than* $12^{-13^{N-1/2}}$. In particular, only one square of sidelength $12^{-13^{N-1/2}}$ is required to cover $S_{\boldsymbol{i}}([0, 1]^2)$, meaning no further "growth" in dimension at this scale.

Notation. For any $N \in \mathbb{N}$, we let Σ_N denote the subwords of sequences in Σ of length N. Finite words in $\bigcup_{N=1}^{\infty} \Sigma_N$ will be denoted in bold using notation such as $\boldsymbol{i}$ or $\boldsymbol{j}$, whereas infinite words in Σ will be denoted using typewriter notation such as $\mathtt{i}$ and $\mathtt{j}$. For infinite sequences $\mathtt{i} = (a_1, b_1)(a_2, b_2)\cdots$ and integers $n \geq 1$, we write $\mathtt{i}|n$ for the truncation of $\mathtt{i}$ to its first n symbols: $\mathtt{i}|n = (a_1, b_1)\cdots(a_n, b_n)$. The same notation is used for the truncation of a finite word $\boldsymbol{i} = (a_1, b_1)\cdots(a_m, b_m)$ to its first n symbols: $\boldsymbol{i}|n = (a_1, b_1)\cdots(a_n, b_n)$ when $m \geq n$. For any finite word $\boldsymbol{i} = (a_1, b_1)\cdots(a_n, b_n)$, its length is denoted by $|\boldsymbol{i}| = n$. Given any $(a, b) \in \Delta$, we write $(a, b)^{\infty}$ for the infinite word $(a, b)(a, b)\cdots$. For any finite word $\boldsymbol{i}$, we denote the cylinder set by $[\boldsymbol{i}] := \{\mathtt{i} \in \Sigma : \mathtt{i}|n = \boldsymbol{i}\}$. We let $\varnothing$ denote the empty word.

To avoid a profusion of constants, we write $A \lesssim B$ if $A \leq cB$ for some universal constant $c > 0$. We write $A \lesssim_{\varepsilon} B$ if $A \leq c_{\varepsilon}B$ for all $\varepsilon > 0$, where the constant c_{ε} depends on ε. We write $A \gtrsim B$ if $B \lesssim A$ and write $A \approx B$ if both $A \lesssim B$ and $B \lesssim A$, and we define the notation $A \gtrsim_{\varepsilon} B$ and $A \approx_{\varepsilon} B$ analogously.

3. Entropy estimates

In this section we obtain estimates on the entropy of important subsets of Σ. Let $\mathcal{G}_N$ be the words in Σ_N which label a path that starts and ends at the vertex v of the graph G in Figure 1. Define

$$h(\mathcal{G}) := \limsup_{N\to\infty} \frac{1}{N} \log \#\mathcal{G}_N,$$

where $\#\mathcal{G}_N$ denotes the cardinality of $\mathcal{G}_N$.

Lemma 3.1. $$h(\mathcal{G}) \le \log 4.$$

Proof. Fix $N \in \mathbb{N}$. Given a word in $\mathcal{G}_N$, let c denote the number of symbols belonging to Ω and a denote the number of symbols belonging to $\{(1, 1), (2, 1)\}$, noting that

(a) $2c + a = N$ and

(b) $c = \sum_{i=1}^{j} 13^{n_i}$ for some integers $n_1, \dots, n_j$.

Fix $0 \le a \le N$ and let $\mathcal{S}_c$ be the set of possible ways that $c = \frac{1}{2}(N - a)$ can be written as an ordered sum $c = \sum_{i=1}^{j} 13^{n_i}$. By ordered sum, we mean that if $(n'_1, \dots, n'_j)$ is a permutation of $(n_1, \dots, n_j)$ such that $(n'_1, \dots, n'_j) \ne (n_1, \dots, n_j)$, then $\sum_{i=1}^{j} 13^{n_i}$ is considered a distinct way of writing c as a sum of powers of 13. Observe that $j \le \frac{1}{13}c$ (for example, consider writing $c = 13 \cdot \frac{1}{13}c$ when c is a multiple of 13).

We begin by bounding $\#\mathcal{S}_c \le 2^{c/13-1}$. Recall that any $n \in \mathbb{N}$ can be expressed in 2^{n-1} ways as an ordered sum of one or more positive integers. Moreover, $\#\mathcal{S}_c$ is clearly bounded above by the number of ways that $\frac{1}{13}c$ can be decomposed into an ordered sum $\sum_{i=1}^{\ell} p_i$ for some positive integers $p_1, \dots, p_\ell$. Hence $\#\mathcal{S}_c \le 2^{c/13-1}$.

Now let us return to considering a word in $\mathcal{G}_N$. Following each substring of symbols from Ω, there is a tail of the same length consisting of $(1, 2)$'s. The a symbols from $\{(1, 1), (2, 1)\}$ can either be placed directly after any of these tails or at the beginning of the word. Therefore assuming that the string contains $c = \frac{1}{2}(N - a)$ symbols from Ω in blocks of lengths $13^{n_1}, \dots, 13^{n_j}$ — so that $c = \sum_{i=1}^{j} 13^{n_i}$ — it follows that there are $\binom{a+j}{j}$ ways in which the a symbols from $\{(1, 1), (2, 1)\}$ can be distributed. Bounding this above by the central binomial term and using the bounds $\binom{2K}{K} \le 4^K$ and $j \le \frac{1}{13}c$ we obtain $\binom{a+j}{j} \le 2^{a+(N-a)/(2\cdot 13)}$. Hence

$$\begin{aligned}\#\mathcal{G}_N &\le \sum_{a=0}^{N} \#\mathcal{S}_{(N-a)/2} 2^{a+(N-a)/(2\cdot 13)} 10^{(N-a)/2} 2^a \le \sum_{a=0}^{N} 2^{2a+(N-a)(2/13+\log_2 10)/2} \\ &= \frac{2^{2(N+1)} - 2^{(2/13+\log_2 10)(N+1)/2}}{2^2 - 2^{(2/13+\log_2 10)/2}} \lesssim 4^N\end{aligned}$$

since $\frac{1}{2}\left(\frac{2}{13} + \log_2 10\right) < 2$, completing the proof of the lemma. □

Let $\mathcal{I}_N$ be the words in Σ_N which label a path that ends at v in the graph G in Figure 1. Clearly $\mathcal{G}_N \subseteq \mathcal{I}_N$. Writing $\mathcal{I}^* = \bigcup_{N=1}^{\infty} \mathcal{I}_N$ and $\Omega^* = \bigcup_{N=1}^{\infty} \Omega^N$, observe that

$$\Sigma \setminus B = \{\boldsymbol{u}\mathtt{w} : \boldsymbol{u} \in \mathcal{I}^* \cup \varnothing,\ \mathtt{w} \in \Omega^{\mathbb{N}}\} \cup \{\boldsymbol{w}(1, 2)^\infty : \boldsymbol{w} \in \Omega^* \cup \varnothing\}. \tag{4}$$

Define

$$h(\mathcal{I}) = \limsup_{N\to\infty} \frac{1}{N} \log \#\mathcal{I}_N.$$

Lemma 3.2. $$h(\mathcal{I}) \leq \log 4.$$

Proof. Fix $N \in \mathbb{N}$. Note that any word in $\mathcal{I}_N \setminus \mathcal{G}_N$ is either of the form

(a) $(1,2)^z \boldsymbol{g}$ for $\boldsymbol{g} \in \mathcal{G}_{N-z}$ or

(b) $\boldsymbol{w}(1,2)^z \boldsymbol{g}$ for $z = 13^k$ for some $k \in \mathbb{N}$ and $\boldsymbol{w} \in \Omega^w$, where $0 < w < z$ and $\boldsymbol{g} \in \mathcal{G}_{N-z-w}$.

Fix any $\varepsilon > 0$. The number of words of the form (a) is

$$\sum_{z=1}^{N} \#\mathcal{G}_{N-z} \lesssim_\varepsilon e^{N(h(\mathcal{G})+\varepsilon)} = (4e^\varepsilon)^N.$$

The number of words of the form (b) is

$$\sum_{z=13^k<N} \sum_{w=1}^{\min\{z-1,N-z\}} 10^w \#\mathcal{G}_{N-z-w} \lesssim_\varepsilon \sum_{z=13^k<N} \sum_{w=1}^{\min\{z-1,N-z\}} 10^w (4e^\varepsilon)^{N-z-w}$$
$$\lesssim \sum_{z=13^k<N} \left(\tfrac{10}{4}\right)^{\min\{z-1,N-z\}} (4e^\varepsilon)^{N-z}.$$

Since

$$\sum_{z=13^k<N/2} \left(\tfrac{10}{4}\right)^{\min\{z-1,N-z\}} (4e^\varepsilon)^{N-z} = \sum_{z=13^k<N/2} 10^{z-1} 4^{N-2z+1} e^{\varepsilon(N-z)} \lesssim_\varepsilon (4e^{2\varepsilon})^N$$

and

$$\sum_{N/2\leq z=13^k<N} \left(\tfrac{10}{4}\right)^{\min\{z-1,N-z\}} (4e^\varepsilon)^{N-z} = \sum_{N/2\leq z=13^k<N} (10e^\varepsilon)^{N-z} \lesssim_\varepsilon (10e^{2\varepsilon})^{N/2} < 4^N$$

for sufficiently small ε, we have that

$$\#\mathcal{I}_N \lesssim_\varepsilon (4e^{2\varepsilon})^N.$$

Since $\varepsilon > 0$ was arbitrary, the proof is complete. □

4. Dimension estimates

In this section, we introduce the sequences of scales which will be used for the lower and upper box dimension estimates and prove Theorem 1.2. We also show how the proof of Theorem 1.2 can be used to construct an infinitely generated self-affine set whose box dimension does not exist.

Let $\delta > 0$. We let $k(\delta)$ denote the unique positive integer satisfying $12^{-k(\delta)} \leq \delta < 12^{1-k(\delta)}$ and $l(\delta)$ denote the unique positive integer satisfying $2^{-l(\delta)} \leq \delta < 2^{1-l(\delta)}$, noting that $k(\delta) < l(\delta)$ for sufficiently small δ. By definition $l(\delta) = \lceil -\log\delta/\log 2\rceil$ and $k(\delta) = \lceil -\log\delta/\log 12\rceil$.

Define the projection $\pi : \Delta^{\mathbb{N}} \to \{1,2\}^{\mathbb{N}}$ by $\pi((a_1,b_1)(a_2,b_2)\cdots) = (a_1a_2\cdots)$. For $\boldsymbol{i} \in \Sigma_k$ and $l > k$, define

$$M(\boldsymbol{i}, l) = \#\pi(\boldsymbol{j} \in \Sigma_l : \boldsymbol{j}|k = \boldsymbol{i}). \tag{5}$$

Our general covering strategy at each scale δ can now be described as follows. For each $\boldsymbol{i} \in \Sigma_{k(\delta)}$, observe that $S_{\boldsymbol{i}}([0,1]^2)$ is a rectangle of height $1/12^{k(\delta)} \approx \delta$. In particular, $N_\delta(\Pi(\Sigma)) \approx \sum_{\boldsymbol{i}\in\Sigma_{k(\delta)}} N_\delta(\Pi([\boldsymbol{i}]))$. For each $\boldsymbol{j} \in \Sigma_{l(\delta)}$, we note that $S_{\boldsymbol{j}}([0,1]^2)$ has width $1/2^{l(\delta)} \approx \delta$. Therefore, for each $\boldsymbol{i} \in \Sigma_{k(\delta)}$, we cover each projected cylinder $\Pi([\boldsymbol{i}])$ independently by considering how many level $l(\delta)$ columns contain part of the set $\Pi(\Sigma)$ inside $\Pi([\boldsymbol{i}])$. Since by definition the number of such columns is given by $M(\boldsymbol{i}, l(\delta))$, we obtain

$$N_\delta(\Pi(\Sigma)) \approx \sum_{\boldsymbol{i} \in \Sigma_{k(\delta)}} N_\delta(\Pi([\boldsymbol{i}])) \approx \sum_{\boldsymbol{i} \in \Sigma_{k(\delta)}} M(\boldsymbol{i}, l(\delta)).$$

Define the null sequence $\{\delta_N\}_{N\in\mathbb{N}}$ by $\delta_N = 1/12^{13^N}$, noting that $l(\delta_N) = \lceil 13^N \log 12/\log 2 \rceil$ and $k(\delta_N) = 13^N$. Also define the null sequence $\{\delta'_N\}_{N\in\mathbb{N}}$ by $\delta'_N = 1/12^{13^{N-1/2}}$, noting that $k(\delta'_N) = \lceil 13^{N-1/2} \rceil$ and $l(\delta'_N) = \lceil 13^{N-1/2} \log 12/\log 2 \rceil$.

In this section we will prove that

$$\limsup_{N\to\infty} \frac{\log N_{\delta_N}(\Pi(\Sigma))}{-\log \delta_N} > \liminf_{N\to\infty} \frac{\log N_{\delta'_N}(\Pi(\Sigma))}{-\log \delta'_N}. \tag{6}$$

Theorem 1.2 will follow from (6) since it implies that $\overline{\dim}_{\mathrm{B}}\, \Pi(\Sigma) > \underline{\dim}_{\mathrm{B}}\, \Pi(\Sigma)$.

Lemma 4.1 (scales with large dimension).

$$\limsup_{N\to\infty} \frac{\log N_{\delta_N}(\Pi(\Sigma))}{-\log \delta_N} \geq \frac{\log 10}{\log 12} + \log 2\left(\frac{1}{\log 2} - \frac{2}{\log 12}\right).$$

Proof. For all $\boldsymbol{w} \in \Omega^{k(\delta_N)}$ and $\boldsymbol{u} \in \{(1,1),(2,1)\}^{l(\delta_N)-2k(\delta_N)}$, we have that $\boldsymbol{w}(1,2)^{k(\delta_N)}\boldsymbol{u} \in \Sigma_{l(\delta_N)}$. In particular, for any $\boldsymbol{w} \in \Omega^{k(\delta_N)}$,

$$M(\boldsymbol{w}, l(\delta_N)) = 2^{l(\delta_N)-2k(\delta_N)} \approx 2^{(\log 12/\log 2 - 2)13^N}, \tag{7}$$

noting that $\log 12/\log 2 > 2$. Hence

$$\begin{aligned} N_{\delta_N}(\Pi(\Sigma)) \geq N_{\delta_N}\left(\bigcup_{\boldsymbol{w}\in\Omega^{k(\delta_N)}} \Pi([\boldsymbol{w}])\right) &\approx \sum_{\boldsymbol{w}\in\Omega^{k(\delta_N)}} N_{\delta_N}(\Pi([\boldsymbol{w}])) \\ &\approx \sum_{\boldsymbol{w}\in\Omega^{k(\delta_N)}} M(\boldsymbol{w}, l(\delta_N)) \approx 10^{13^N} 2^{(\log 12/\log 2-2)13^N}. \end{aligned}$$

Hence for some uniform constant $c > 0$,

$$\begin{aligned} \frac{\log N_{\delta_N}(\Pi(\Sigma))}{-\log \delta_N} &\geq \frac{13^N \log 10}{13^N \log 12} + \frac{13^N\left(\frac{\log 12}{\log 2} - 2\right)\log 2}{13^N \log 12} + \frac{\log c}{-13^N \log 12} \\ &= \frac{\log 10}{\log 12} + \log 2\left(\frac{1}{\log 2} - \frac{2}{\log 12}\right) + \frac{\log c}{-13^N \log 12}. \end{aligned}$$

The result follows by letting $N \to \infty$. □

Lemma 4.2 (scales with small dimension).

$$\liminf_{N\to\infty} \frac{\log N_{\delta'_N}(\Pi(\Sigma))}{-\log \delta'_N} \leq \frac{\frac{1}{\sqrt{13}}\log 10 + \left(1 - \frac{1}{\sqrt{13}}\right)\log 4}{\log 12} + \log 2\left(\frac{1}{\log 2} - \frac{1+\frac{1}{\sqrt{13}}}{\log 12}\right).$$

Proof. Let $\varepsilon > 0$. Recall that for all $N \in \mathbb{N}$, we have $-\log \delta'_N = 13^{N-1/2} \log 12$, $k(\delta'_N) = \lceil 13^{N-1/2} \rceil$ and $l(\delta'_N) = \lceil 13^{N-1/2} \log 12/\log 2 \rceil$. Recall that $\Sigma = \bar{B}$, where B is the set of all infinite sequences which label a one-sided infinite path on the graph G given in Figure 1, and where the set of points $\bar{B} \setminus B$ are characterised in (4). Therefore, any word $\boldsymbol{i} \in \Sigma_{k(\delta'_N)}$ has one of the following forms:

(a) $\boldsymbol{i} = \boldsymbol{u}$ for $\boldsymbol{u} \in \mathcal{I}_{k(\delta'_N)}$.

(b) $\boldsymbol{i} = \boldsymbol{u}\boldsymbol{w}$ for $\boldsymbol{u} \in \mathcal{I}_u$ and $\boldsymbol{w} \in \Omega^w$, where $u + w = k(\delta'_N)$.

(c) $\boldsymbol{i} = \boldsymbol{w}$ for $\boldsymbol{w} \in \Omega^{k(\delta'_N)}$.

(d) $\boldsymbol{i} = \boldsymbol{w}(1, 2)^z$ for $\boldsymbol{w} \in \Omega^w$, where $w + z = k(\delta'_N)$.

(e) $\boldsymbol{i} = \boldsymbol{u}\boldsymbol{w}(1, 2)^z$ for $\boldsymbol{u} \in \mathcal{I}_u$ and $\boldsymbol{w} \in \Omega^w$, where $u + w + z = k(\delta'_N)$ and $z \le w$.

Let $Y_a \subset \Sigma_{k(\delta'_N)}$ be the set of words which are of the form (a), and let $X_a \subset \Sigma$ be the subset

$$X_a := \{\mathtt{i} \in \Sigma : \mathtt{i}|k(\delta'_N) \in Y_a\}.$$

Define X_b, X_c, X_d, X_e and Y_b, Y_c, Y_d, Y_e analogously. We note that these sets are not all mutually exclusive, for example $Y_a \cap Y_e \neq \varnothing$, but this will not affect our bounds.

Upper bound on $N_{\delta'_N}(\Pi(X_a))$. For any $\boldsymbol{j} \in \{(1, 1), (2, 1)\}^{l(\delta'_N)-k(\delta'_N)}$ and $\boldsymbol{u} \in \mathcal{I}_{k(\delta'_N)}$, we have $\boldsymbol{u}\boldsymbol{j} \in \Sigma^{l(\delta'_N)}$. Therefore, for each $\boldsymbol{u} \in \mathcal{I}_{k(\delta'_N)}$,

$$M(\boldsymbol{u}, l(\delta'_N)) = 2^{l(\delta'_N)-k(\delta'_N)} \approx 2^{13^{N-1/2}(\log 12/\log 2-1)}. \tag{8}$$

Hence

$$N_{\delta'_N}(\Pi(X_a)) \approx \sum_{\boldsymbol{u} \in Y_a} N_{\delta'_N}(\Pi([\boldsymbol{u}])) \approx \sum_{\boldsymbol{u} \in \mathcal{I}_{k(\delta'_N)}} M(\boldsymbol{u}, l(\delta'_N)) \lesssim_\varepsilon (4e^\varepsilon)^{13^{N-1/2}} 2^{13^{N-1/2}(\log 12/\log 2-1)}$$

by Lemma 3.2 and (8). Since $\varepsilon > 0$ was chosen arbitrarily and $-\log \delta'_N = 13^{N-1/2} \log 12$, we deduce that

$$\liminf_{N\to\infty} \frac{\log N_{\delta'_N}(\Pi(X_a))}{-\log \delta'_N} \le \frac{\log 4}{\log 12} + \log 2 \left(\frac{1}{\log 2} - \frac{1}{\log 12} \right). \tag{9}$$

Upper bound on $N_{\delta'_N}(\Pi(X_c))$. Suppose $\mathtt{i} \in X_c$, so that $\mathtt{i}|k(\delta'_N) = \boldsymbol{w} \in \Omega^{k(\delta'_N)}$. By definition of Σ, either $\mathtt{i} \in \Omega^{\mathbb{N}}$ or $\mathtt{i}$ begins with $\boldsymbol{u}(1, 2)^z$ for some $\boldsymbol{u} \in \Omega^*$, where $|\boldsymbol{u}| \ge k(\delta'_N) = \lceil 13^{N-1/2} \rceil$ and $z \ge 13^N$. For N sufficiently large,

$$z + |\boldsymbol{u}| \ge 13^N + 13^{N-1/2} > 13^{1/2} 13^{N-1/2} > \left\lceil \frac{\log 12}{\log 2} 13^{N-1/2} \right\rceil = l(\delta'_N).$$

In particular, for any $\boldsymbol{w} \in \Omega^{k(\delta'_N)}$,

$$M(\boldsymbol{w}, l(\delta'_N)) = 1. \tag{10}$$

By (10),

$$N_{\delta'_N}(\Pi(X_c)) \approx \sum_{\boldsymbol{w} \in Y_c} N_{\delta'_N}(\Pi([\boldsymbol{w}])) \approx \sum_{\boldsymbol{w} \in \Omega^{k(\delta'_N)}} M(\boldsymbol{w}, l(\delta'_N)) = 10^{k(\delta'_N)} \approx 10^{13^{N-1/2}}.$$

Therefore, since $-\log \delta'_N = 13^{N-1/2} \log 12$,

$$\liminf_{N\to\infty} \frac{\log N_{\delta'_N}(\Pi(X_c))}{-\log \delta'_N} \leq \frac{\log 10}{\log 12}. \tag{11}$$

Upper bound on $N_{\delta'_N}(\Pi(X_d))$. For $x > 0$ we let $\mathcal{T}(x)$ denote the smallest power of 13 which is greater than or equal to x. Suppose $\mathtt{i} \in X_d$, so that $\mathtt{i}|k(\delta'_N) = \boldsymbol{w}(1,2)^z$ for $\boldsymbol{w} \in \Omega^w$, where $w + z = k(\delta'_N)$. Either $\mathtt{i} = \boldsymbol{w}(1,2)^\infty$ or $\mathtt{i}$ begins with $\boldsymbol{w}(1,2)^{z'}\boldsymbol{j}$ for some $\boldsymbol{j} \in \Sigma_1 \setminus \{(1,2)\}$ and

$$z' \geq \mathcal{T}(\max\{w, z\}) = \mathcal{T}(\max\{w, k(\delta'_N) - w\}) = 13^N,$$

where the final equality is because, for sufficiently large N,

$$\max\{w, k(\delta'_N) - w\} \geq \tfrac{1}{2}k(\delta'_N) = \tfrac{1}{2}\lceil 13^{N-1/2}\rceil > 13^{N-1}.$$

Moreover, for sufficiently large N,

$$w + z' \geq 13^N > \left\lceil 13^{N-1/2}\frac{\log 12}{\log 2}\right\rceil = l(\delta'_N).$$

In particular, for any $\boldsymbol{w}(1,2)^z \in Y_d$,

$$M(\boldsymbol{w}(1,2)^z, l(\delta'_N)) = 1. \tag{12}$$

By (12),

$$\begin{aligned} N_{\delta'_N}(\Pi(X_d)) &\approx \sum_{\boldsymbol{i}\in Y_d} N_{\delta'_N}(\Pi([\boldsymbol{i}])) \\ &\approx \sum_{w=1}^{k(\delta'_N)-1} \sum_{\boldsymbol{w}\in\Omega^w} M(\boldsymbol{w}(1,2)^{k(\delta'_N)-w}, l(\delta'_N)) \\ &\lesssim_\varepsilon (10e^\varepsilon)^{k(\delta'_N)} \approx (10e^\varepsilon)^{13^{N-1/2}}. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary and $-\log \delta'_N = 13^{N-1/2} \log 12$,

$$\liminf_{N\to\infty} \frac{\log N_{\delta'_N}(\Pi(X_d))}{-\log \delta'_N} \leq \frac{\log 10}{\log 12}. \tag{13}$$

Upper bound on $N_{\delta'_N}(\Pi(X_b))$. Suppose $\mathtt{i} \in X_b$, so that $\mathtt{i}|k(\delta'_N) = \boldsymbol{u}\boldsymbol{w}$ for $\boldsymbol{u} \in \mathcal{I}_u$ and $\boldsymbol{w} \in \Omega^w$, where $u + w = k(\delta'_N)$. Either $\mathtt{i} = \boldsymbol{u}\mathtt{j}$, where $\mathtt{j} \in \Omega^{\mathbb{N}}$, or $\mathtt{i}$ begins with $\boldsymbol{u}\boldsymbol{v}(1,2)^z$, where $\boldsymbol{v} \in \Omega^z$ and we have $z = |\boldsymbol{v}| = \mathcal{T}(|\boldsymbol{v}|) \geq \mathcal{T}(w)$. In particular, for any $\boldsymbol{u}\boldsymbol{w} \in Y_b$,

$$\begin{aligned} M(\boldsymbol{u}\boldsymbol{w}, l(\delta'_N)) &\leq 2^{l(\delta'_N)-|\boldsymbol{u}|-|\boldsymbol{v}|-z} \\ &= 2^{l(\delta'_N)-|\boldsymbol{u}|-2z} \\ &= 2^{l(\delta'_N)-k(\delta'_N)+w-2z} \\ &\leq 2^{l(\delta'_N)-k(\delta'_N)+w-2\cdot\mathcal{T}(w)}. \end{aligned} \tag{14}$$

By (14) and Lemma 3.2,

$$
\begin{aligned}
&N_{\delta'_N}(\Pi(X_b)) \\
&\quad\approx \sum_{\boldsymbol{i}\in Y_b} N_{\delta'_N}(\Pi([\boldsymbol{i}])) \approx \sum_{w=1}^{k(\delta'_N)-1} \sum_{\boldsymbol{u}\in\mathcal{I}^{k(\delta'_N)-w}} \sum_{\boldsymbol{w}\in\Omega^w} M(\boldsymbol{uw}, l(\delta'_N)) \\
&\quad\lesssim_\varepsilon \sum_{w=1}^{k(\delta'_N)-1} 10^w (4e^\varepsilon)^{k(\delta'_N)-w} 2^{l(\delta'_N)-k(\delta'_N)+w-2\mathcal{T}(w)} \\
&\quad\le \sum_{w=1}^{13^{N-1}} \left(\frac{10\cdot 2}{4e^\varepsilon\cdot 2^2}\right)^w (4e^\varepsilon)^{k(\delta'_N)} 2^{l(\delta'_N)-k(\delta'_N)} + \sum_{w=13^{N-1}+1}^{k(\delta'_N)-1} \left(\frac{10\cdot 2}{4e^\varepsilon\cdot 2^{2\sqrt{13}}}\right)^w (4e^\varepsilon)^{k(\delta'_N)} 2^{l(\delta'_N)-k(\delta'_N)}, \qquad (15)
\end{aligned}
$$

where in the last line of (15) we have used the trivial lower bound $\mathcal{T}(x)\ge x$ in the first sum and, in the second sum, that, for all $13^{N-1}+1\le x\le k(\delta'_N)-1=\lceil 13^{N-1/2}\rceil-1$,

$$
\sqrt{13}x \le \sqrt{13}(13^{N-1/2}-1)\le 13^N=\mathcal{T}(x). \qquad (16)
$$

For sufficiently small $\varepsilon>0$, the first sum of the last line of (15) can be bounded above by

$$
\sum_{w=1}^{13^{N-1}} \left(\frac{10\cdot 2}{4e^\varepsilon\cdot 2^2}\right)^w (4e^\varepsilon)^{k(\delta'_N)} 2^{l(\delta'_N)-k(\delta'_N)} \lesssim_\varepsilon 10^{13^{N-1}} (4e^{2\varepsilon})^{k(\delta'_N)-13^{N-1}} 2^{l(\delta'_N)-k(\delta'_N)-13^{N-1}}.
$$

For sufficiently small $\varepsilon>0$,

$$
\frac{10\cdot 2}{4e^\varepsilon\cdot 2^{2\sqrt{13}}} = \frac{5}{e^\varepsilon 4^{\sqrt{13}}} < 1;
$$

hence the second sum of the last line of (15) can be bounded above by

$$
\begin{aligned}
\sum_{w=13^{N-1}+1}^{k(\delta'_N)-1} \left(\frac{10\cdot 2}{4e^\varepsilon\cdot 2^{2\sqrt{13}}}\right)^w (4e^\varepsilon)^{k(\delta'_N)} 2^{l(\delta'_N)-k(\delta'_N)} &\lesssim_\varepsilon 10^{13^{N-1}} (4e^{2\varepsilon})^{k(\delta'_N)-13^{N-1}} 2^{l(\delta'_N)-k(\delta'_N)+13^{N-1}-2\cdot 13^{N-1/2}} \\
&< 10^{13^{N-1}} (4e^{2\varepsilon})^{k(\delta'_N)-13^{N-1}} 2^{l(\delta'_N)-k(\delta'_N)-13^{N-1}}.
\end{aligned}
$$

In particular,

$$
\begin{aligned}
N_{\delta'_N}(\Pi(X_b)) &\lesssim_\varepsilon 10^{13^{N-1}} (4e^{2\varepsilon})^{k(\delta'_N)-13^{N-1}} 2^{l(\delta'_N)-k(\delta'_N)-13^{N-1}} \\
&\approx 10^{13^{N-1}} (4e^{2\varepsilon})^{13^{N-1/2}-13^{N-1}} 2^{(\log 12/\log 2-1)13^{N-1/2}-13^{N-1}}.
\end{aligned}
$$

Since $\varepsilon>0$ was chosen arbitrarily and $-\log\delta'_N = 13^{N-1/2}\log 12$,

$$
\liminf_{N\to\infty} \frac{\log N_{\delta'_N}(\Pi(X_b))}{-\log\delta'_N} \le \frac{\frac{1}{\sqrt{13}}\log 10 + \big(1-\frac{1}{\sqrt{13}}\big)\log 4}{\log 12} + \log 2\left(\frac{1}{\log 2} - \frac{1+\frac{1}{\sqrt{13}}}{\log 12}\right). \qquad (17)
$$

Upper bound on $N_{\delta'_N}(\Pi(X_e))$. If $\boldsymbol{uw}(1,2)^z\in Y_e$ with $|\boldsymbol{w}|=w$ and $|\boldsymbol{u}|=u$, then since $u+w\le k(\delta'_N)=\lceil 13^{N-1/2}\rceil$ we have

$$
l(\delta'_N)-2w-u \ge l(\delta'_N)-2\lceil 13^{N-1/2}\rceil > l(\delta'_N) - \left\lceil \frac{\log 12}{\log 2} 13^{N-1/2}\right\rceil = 0.
$$

In particular,

$$M(\boldsymbol{uw}(1,2)^z, l(\delta'_N)) = 2^{l(\delta'_N)-2w-u}. \tag{18}$$

By (18) and Lemma 3.2,

$$\begin{aligned}
N_{\delta'_N}(\Pi(X_e)) \approx \sum_{\boldsymbol{i}\in Y_e} N_{\delta'_N}(\Pi([\boldsymbol{i}])) &\approx \sum_{w=13^r\le 13^{N-1}} \sum_{u=1}^{k(\delta'_N)-w-1} \sum_{\boldsymbol{u}\in\mathcal{I}^u} \sum_{\boldsymbol{w}\in\Omega^w} M(\boldsymbol{uw}(1,2)^{k(\delta'_N)-u-w}, l(\delta'_N)) \\
&\lesssim_\varepsilon \sum_{w=13^r\le 13^{N-1}} \sum_{u=1}^{k(\delta'_N)-w-1} (4e^\varepsilon)^u 10^w 2^{l(\delta'_N)-2w-u} \\
&\lesssim_\varepsilon \sum_{w=13^r\le 13^{N-1}} \left(\frac{10\cdot 2}{4e^{2\varepsilon}\cdot 2^2}\right)^w \left(\frac{4e^{2\varepsilon}}{2}\right)^{k(\delta'_N)} 2^{l(\delta'_N)} \\
&\lesssim_\varepsilon 10^{13^{N-1}} (4e^{3\varepsilon})^{k(\delta'_N)-13^{N-1}} 2^{l(\delta'_N)-k(\delta'_N)-13^{N-1}} \\
&\approx 10^{13^{N-1}} (4e^{3\varepsilon})^{13^{N-1/2}-13^{N-1}} 2^{(\log 12/\log 2-1)13^{N-1/2}-13^{N-1}}.
\end{aligned}$$

Since $\varepsilon > 0$ was chosen arbitrarily and $-\log\delta'_N = 13^{N-1/2}\log 12$,

$$\liminf_{N\to\infty} \frac{\log N_{\delta'_N}(\Pi(X_e))}{-\log\delta'_N} \le \frac{\frac{1}{\sqrt{13}}\log 10 + \left(1-\frac{1}{\sqrt{13}}\right)\log 4}{\log 12} + \log 2\left(\frac{1}{\log 2} - \frac{1+\frac{1}{\sqrt{13}}}{\log 12}\right). \tag{19}$$

Since the upper bounds in (17) and (19) are strictly greater than the upper bounds in (9), (11) and (13) the proof is complete. □

Proof of Theorem 1.2. $\Pi(\Sigma)$ is invariant under the smooth expanding map

$$T(x,y) = (mx \mod 1, ny \mod 1).$$

Note that to four decimal places

$$\frac{\log 10}{\log 12} + \log 2\left(\frac{1}{\log 2} - \frac{2}{\log 12}\right) \approx 1.3687$$

and

$$\frac{\frac{1}{\sqrt{13}}\log 10 + \left(1-\frac{1}{\sqrt{13}}\right)\log 4}{\log 12} + \log 2\left(\frac{1}{\log 2} - \frac{1+\frac{1}{\sqrt{13}}}{\log 12}\right) \approx 1.3038.$$

By Lemmas 4.1 and 4.2,

$$\overline{\dim}_{\mathrm{B}}\,\Pi(\Sigma) \ge \limsup_{N\to\infty} \frac{\log N_{\delta_N}(\Pi(\Sigma))}{-\log\delta_N} > \liminf_{N\to\infty} \frac{\log N_{\delta'_N}(\Pi(\Sigma))}{-\log\delta'_N} \ge \underline{\dim}_{\mathrm{B}}\,\Pi(\Sigma).$$

In particular, the box dimension of $\Pi(\Sigma)$ does not exist. □

Remark 4.3. Lemmas 4.1 and 4.2 can also be used to demonstrate the existence of infinitely generated self-affine sets whose box dimensions are distinct. Consider the countable family of affine contractions

$$\{S_{(1,1)}\} \cup \{S_{(1,2)}\} \cup \bigcup_{N=1}^{\infty} \bigcup_{\boldsymbol{w}\in\Omega^{13^N}} \{S_{\boldsymbol{w}(1,2)^{13^N}}\}$$

which generates the infinitely generated self-affine set $E = \Pi(\widetilde{\Sigma})$, where

$$\widetilde{\Sigma} := \{\boldsymbol{u}_1\boldsymbol{u}_2\cdots : \boldsymbol{u}_i \in \mathcal{C} \text{ for all } i \in \mathbb{N}\}.$$

Since $E \subset F$, we have that $\underline{\dim}_{\mathrm{B}} E \leq \underline{\dim}_{\mathrm{B}} F$. On the other hand, for all $N \in \mathbb{N}$, $\boldsymbol{w} \in \Omega^{k(\delta_N)}$ and $\boldsymbol{u} \in \{(1,1),(2,1)\}^{l(\delta_N)-2k(\delta_N)}$,

$$[\boldsymbol{w}(1,2)^{13^N}\boldsymbol{u}] \cap \widetilde{\Sigma} \neq \varnothing.$$

Therefore by bounding $N_{\delta_N}(E)$ in the same way as in Lemma 4.1 we deduce that $\underline{\dim}_{\mathrm{B}} E < \overline{\dim}_{\mathrm{B}} E$.

5. Further questions

Here we suggest possible directions for future work.

Question 5.1. Does there exist an expanding repeller whose box dimension does not exist? Namely, does there exist a smooth expanding map $f : M \to M$ of a Riemannian manifold M and compact set $\Lambda = f(\Lambda)$ such that $\Lambda = \{x \in U : f^n(x) \in U, \forall n \in \mathbb{N}\}$ for some open neighbourhood U of Λ?

Question 5.2. Given a smooth diffeomorphism $f : M \to M$, does the box dimension of its basic set (or intersections of the basic set with local stable and unstable manifolds) always exist?

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