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Kuang-Ru Wu

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A WESS–ZUMINO–WITTEN TYPE EQUATION IN THE SPACE OF KÄHLER POTENTIALS IN TERMS OF HERMITIAN–YANG–MILLS METRICS

KUANG-RU WU

We prove that the solution of a Wess–Zumino–Witten type equation from a domain D in $\mathbb{C}^m$ to the space of Kähler potentials can be approximated uniformly by Hermitian–Yang–Mills metrics on certain vector bundles. The key is a new version of Berndtsson's theorem on the positivity of direct image bundles.

1. Introduction

Let L be a positive line bundle over a compact complex manifold X of dimension n, and let h be a positively curved metric on L with curvature ω. The space of Kähler potentials is

$$\mathcal{H}_\omega = \{\phi \in C^\infty(X, \mathbb{R}) : \omega + i\partial\bar{\partial}\phi > 0\},$$

and for a positive integer k we denote by $\mathcal{H}_k$ the space of inner products on $H^0(X, L^k)$. Starting from a question asked by Yau [1987] and the work of Tian [1990], Zelditch [1998], Catlin [1999], and many others, it is well known that a given Kähler potential $\phi \in \mathcal{H}_\omega$ can be approximated by $\phi_k \in \mathcal{H}_\omega$ associated with $\mathcal{H}_k$ as $k \to \infty$. Furthermore, Mabuchi [1987], Semmes [1992], and Donaldson [1999] discovered that $\mathcal{H}_\omega$ carries a Riemannian metric which allows one to talk about geometry, especially geodesics, of $\mathcal{H}_\omega$. Thanks to Phong and Sturm [2006], Chen and Sun [2012], Berndtsson [2018], and Darvas, Lu, and Rubinstein [Darvas et al. 2020], geodesics in $\mathcal{H}_\omega$ can be approximated by geodesics in $\mathcal{H}_k$ as $k \to \infty$. More generally, one may wonder if harmonic maps into $\mathcal{H}_\omega$ can also be approximated by harmonic maps associated with $\mathcal{H}_k$. A version of this was confirmed by Rubinstein and Zelditch [2010] when X is toric, and the maps take values in toric Kähler metrics; see also [Song and Zelditch 2007; 2010].

Here we focus on a Wess–Zumino–Witten (WZW) type equation for a map from $D \subset \mathbb{C}^m$ to $\mathcal{H}_\omega$, and we show that the solution to such an equation can be approximated by Hermitian–Yang–Mills metrics on certain direct image bundles. We will also see how this result recovers some of those mentioned in the first paragraph.

We first explain how to derive this WZW equation. Recall that the tangent space $T_\phi\mathcal{H}_\omega$ at $\phi \in \mathcal{H}_\omega$ can be canonically identified with $C^\infty(X, \mathbb{R})$, and following [Donaldson 1999; Mabuchi 1987; Semmes 1992],

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the Mabuchi metric g_M on $\mathcal{H}_\omega$ is

$$g_M(\xi,\eta)=\int_X \xi\eta\omega_\phi^n, \quad \text{for } \phi\in\mathcal{H}_\omega \text{ and } \xi,\eta\in T_\phi\mathcal{H}_\omega.$$

Let D be a bounded smooth strongly pseudoconvex domain in $\mathbb{C}^m$. A map $\Phi : D\to\mathcal{H}_\omega$ will be identified as $\Phi : D\times X\to\mathbb{R}$ with $\Phi(z,\cdot)\in\mathcal{H}_\omega$ for $z\in D$. A map $\Phi : D\to\mathcal{H}_\omega$ is said to be harmonic if it is a critical point of the functional $E(\Phi)=\int_D|\Phi_*|^2\,dV$, where dV is the Euclidean volume form on D, Φ_* is the differential of Φ, and $|\Phi_*|$ is the Hilbert–Schmidt norm of Φ_*, measured by the Mabuchi metric g_M and the Euclidean metric of D. A straightforward computation gives the harmonic map equation

$$\sum_{j=1}^m |\nabla\Phi_{z_j}|^2-2\Phi_{z_j\bar z_j}=0, \tag{1}$$

where $\{z_j\}$ are coordinates on D and $|\nabla\Phi_{z_j}(z)|^2$ is computed using the metric $\omega_{\Phi(z)}$. On the other hand, there is a perturbed functional $\mathscr{E}$, whose Euler–Lagrange equation is also of interest. The construction of this perturbed functional is similar to that of [Donaldson 1999, Section 5] (see also [Witten 1983]), where one-dimensional D were considered. In order to define $\mathscr{E}$, we first define a three-form θ on $\mathcal{H}_\omega$: for $\phi\in\mathcal{H}_\omega$ and $\xi_1,\xi_2,\xi_3\in T_\phi\mathcal{H}_\omega$,

$$\theta(\xi_1,\xi_2,\xi_3):=g_M(\{\xi_1,\xi_2\}_{\omega_\phi},\xi_3)=\int_X\{\xi_1,\xi_2\}_{\omega_\phi}\xi_3\omega_\phi^n, \tag{2}$$

where $\{\cdot,\cdot\}_{\omega_\phi}$ is the Poisson bracket determined by the symplectic form ω_ϕ. This three-form θ is d-closed by Lemma 4.5 below, and by Lemma 4.4 there is a two-form α on $\mathcal{H}_\omega$ such that $d\alpha=\theta$. For a map $\Phi : D\to\mathcal{H}_\omega$, we define

$$\mathscr{E}(\Phi):=E(\Phi)+4i\sum_j\int_D\alpha(\Phi_{\bar z_j},\Phi_{z_j})\,dV.$$

We will show in Lemma 4.6 that the Euler–Lagrange equation of $\mathscr{E}$ is

$$\sum_{j=1}^m |\nabla\Phi_{z_j}|^2-2\Phi_{z_j\bar z_j}+i\{\Phi_{\bar z_j},\Phi_{z_j}\}_{\omega_\Phi}=0. \tag{3}$$

Following [Witten 1983] and [Donaldson 1999], we call (3) the WZW equation for a map $\Phi : D\to\mathcal{H}_\omega$.

Donaldson [1999] showed, when $m=1$, that the WZW equation is equivalent to a homogeneous complex Monge–Ampère equation. We have the following extended equivalence for $m\geq 1$ by a similar computation. Let $\pi : D\times X\to X$ be the projection onto X. Then the extended equivalence is

$$\Phi \text{ solves (3)} \iff (i\partial\bar\partial\Phi+\pi^*\omega)^{n+1}\wedge\left(i\sum_{j=1}^m dz_j\wedge d\bar z_j\right)^{m-1}=0. \tag{4}$$

This suggests that the proper generality of the WZW equation is for maps from a Kähler manifold D to $\mathcal{H}_\omega$. Nevertheless, in this paper we restrict to $D\subset\mathbb{C}^m$.

The next step is to construct a solution of the WZW equation, and then we will show it can be approximated by the solutions of Hermitian–Yang–Mills equations.

Definition 1.1. We will say that a function $u : D \times X \to [-\infty, \infty)$ is ω-subharmonic on graphs if, for any holomorphic map f from an open subset of D to X, we have that $\psi(f(z)) + u(z, f(z))$ is subharmonic, where ψ is a local potential of ω.

This definition does not depend on the choice of ψ since any two local potentials differ by a pluriharmonic function. (This definition has its origin in the works of Slodkowski [1988; 1990a; 1990b], and Coifman and Semmes [1993]; however, they focus on functions u defined on $D \times V$ with a vector space V where $u(z, \cdot\,)$ are norms or quasinorms, whereas we consider simply functions on $D \times X$. There is also a notion of k-subharmonicity, see [Błocki 2005], but it is not equivalent to subharmonicity on graphs.)

Let v be a real-valued smooth function on $\partial D \times X$ and $\partial D \ni z \mapsto v(z, \cdot\,) = v_z \in \mathcal{H}_\omega$. We simply write $v \in C^\infty(\partial D, \mathcal{H}_\omega)$. Consider the Perron family

$$G_v := \Big\{ u \in \operatorname{usc}(D \times X) : u \text{ is } \omega\text{-subharmonic on graphs}, \ \limsup_{D \ni z \to \zeta \in \partial D} u(z, x) \le v(\zeta, x) \Big\}.$$

As we will later see, the upper envelope $V = \sup\{u : u \in G_v\}$ is a weak solution of the WZW equation from D to $\mathcal{H}_\omega$. The above setup is for $\mathcal{H}_\omega$. As for $\mathcal{H}_k$, we recall first the two maps that connect $\mathcal{H}_\omega$ and $\mathcal{H}_k$. The Hilbert map $H_k : \mathcal{H}_\omega \to \mathcal{H}_k$ is

$$H_k(\phi)(s, s) = \int_X h^k(s, s) e^{-k\phi} \omega^n, \quad \text{for } \phi \in \mathcal{H}_\omega \text{ and } s \in H^0(X, L^k).$$

In the other direction, the Fubini–Study map $FS_k : \mathcal{H}_k \to \mathcal{H}_\omega$ is given by

$$FS_k(G)(x) = \frac{1}{k} \log \sup_{\substack{s \in H^0(X, L^k) \\ G(s,s) \le 1}} h^k(s, s)(x), \quad \text{for } G \in \mathcal{H}_k \text{ and } x \in X.$$

Now following the definitions from [Coifman and Semmes 1993], let $\mathcal{N}_k^*$ be the set of norms on $H^0(X, L^k)^*$. Then a norm function $D \ni z \mapsto U_z \in \mathcal{N}_k^*$ is said to be subharmonic if $\log U_z(f(z))$ is subharmonic for any holomorphic section $f : W \subset D \to H^0(X, L^k)^*$. The second Perron family we consider is

$$G_v^k := \Big\{ D \ni z \to U_z \in \mathcal{N}_k^* \text{ is subharmonic}, \ \limsup_{D \ni z \to \zeta \in \partial D} U_z^2(s) \le H_k^*(v_\zeta)(s, s) \text{ for any } s \in H^0(X, L^k)^* \Big\},$$

where $H_k^*(v)$ is the inner product dual to $H_k(v)$. We note a remarkable theorem about the upper envelope $V^k = \sup\{U : U \in G_v^k\}$ from [Coifman and Semmes 1993], which shows that V^k is not only a norm but an inner product (see [Slodkowski 1990a, Corollary 2.7] for a different proof); moreover, it solves the Hermitian–Yang–Mills equation (see also [Donaldson 1992])

$$\begin{cases} \Lambda\Theta(V^k) = 0, \\ V^k|_{\partial D} = H_k^*(v). \end{cases}$$

Here we view V^k as a Hermitian metric on the bundle $\bar{D} \times H^0(X, L^k)^* \to \bar{D}$, and $\Theta(V^k)$ is its curvature. Further, Λ is the trace with respect to the Euclidean metric of D, so in general $\Lambda\Theta(V^k)$ takes values in endomorphisms of $H^0(X, L^k)^*$. Denoting the dual metric by $(V^k)^*$, our main result is that the upper envelope V of G_v is the limit of Hermitian–Yang–Mills metrics.

Theorem 1.2. *$FS_k((V^k)^*)$ converges to V uniformly on $D \times X$, as $k \to \infty$.*

Now we turn to the interpretation of the upper envelope V and its relation to the WZW equation. The next theorem shows that V solves the WZW equation under a regularity assumption.

Theorem 1.3. *If the upper envelope V of G_v is in $C^2(D \times X)$, then*

$$(i\partial\bar{\partial}V + \pi^*\omega)^{n+1} \wedge \left(i \sum_{j=1}^{m} dz_j \wedge d\bar{z}_j\right)^{m-1} = 0.$$

As a result, Theorems 1.2 and 1.3 together show that the solution of the WZW equation can be approximated by the Hermitian–Yang–Mills metrics. (The equation in Theorem 1.3 is similar to the complex Hessian equation, which has been studied extensively in [Błocki 2005; Collins and Picard 2019; Dinew and Kołodziej 2014; Dinew et al. 2019; Lu and Nguyen 2015; 2019], and we hope to return to it in the future.)

The C^2 assumption in Theorem 1.3 is somewhat artificial. At this point, we are able to show V is continuous by Corollary 3.3, and it is desirable to prove higher regularity of V either through PDE techniques or pluripotential theory which we will pursue in a different paper. The guiding example is when $m = 1$. In that case the WZW equation is the much studied complex Monge–Ampère equation; it is known that V is not smooth in general (see [Darvas 2014; Darvas and Lempert 2012; Lempert and Vivas 2013]), and $C^{1,1}$ is the best one can hope for (see [Błocki 2012; Chen 2000; Chu et al. 2017]).

We mention briefly works related to our result. If $m = 1$ and $D \subset \mathbb{C}$ is an annulus, and v is invariant under rotation of the annulus, then Theorems 1.2 and 1.3 recover the geodesic approximation result of Phong and Sturm [2006] and Berndtsson [2018]. When X is toric, these theorems are reduced to the harmonic approximation of Rubinstein and Zelditch [2010], except that C^2 convergence is proved in their paper (see also [Song and Zelditch 2007; 2010]).

The proof of Theorem 1.2 hinges on Theorem 2.1, a result regarding the positivity of direct image bundles. Although Berndtsson's theorem [2009] has played a crucial role in approximation theorems similar to Theorem 1.2 (for example [Berman and Keller 2012; Berndtsson 2018; Darvas and Wu 2019; Darvas et al. 2020]), when it comes to approximating by Hermitian–Yang–Mills metrics, a subharmonic analogue of Berndtsson's theorem is desired. It is Theorem 2.1, where we prove a version of positivity of direct image bundles for weights that are subharmonic on graphs. This is perhaps the crux of this paper. A corresponding result on Stein manifolds can be proved easily following the proof of Theorem 2.1.

The WZW equation (3) is the harmonic map (1) perturbed with Poisson's bracket, which is closely related to the geometry of $\mathcal{H}_\omega$, an infinite-dimensional nonpositively curved manifold. Since the theory of harmonic maps into nonpositively curved manifolds is well developed by Eells and Sampson [1964], Hamilton [1975], and many others, a possible future direction is to see if one can combine the classical results with those of this paper to study $\mathcal{H}_\omega$. Yet another possible but remote direction is to use slope stability in the Donaldson–Uhlenbeck–Yau theorem to study the K-stability by Theorem 1.2. This is a vast subject and we only mention a few papers that are closer to our study. See [Chen et al. 2015a; 2015b; 2015c; Dervan and Keller 2019; Donaldson 1985; Li 2012; Székelyhidi 2014; Uhlenbeck and Yau 1986; Zhang 2021].

Before we end this introduction, a few words about the structure of this paper. In Section 2, the subharmonic version of positivity of direct image bundles is proved, except we put off a technical lemma to Section 5. Section 3 is devoted to Theorem 1.2 and Section 4 to Theorem 1.3. In Section 6, we draw parallels with [Darvas and Wu 2019].

2. Positivity of direct image bundles

Consider a Hermitian holomorphic line bundle $(E, g) \to X^n$ over a compact complex manifold, and assume the curvature η of the metric g is positive. For two sections $s, t \in H^0(X, E \otimes K_X)$, we write locally

$$s = \sigma \otimes s', \quad t = \tau \otimes t',$$

where $\sigma, \tau \in E$ and $s', t' \in K_X$. (Such an expression is possible as long as one of the bundles is of rank 1. In the current case, E and K_X are both line bundles.) We extend the metric g to acting on sections of $E \otimes K_X$ by setting $g(s,t) = g(\sigma, \tau) s' \wedge \bar{t}'$, which is an (n,n)-form. It is not hard to see this (n,n)-form is globally defined on X.

We define a variant of the Hilbert map: $\mathrm{Hilb}_{E\otimes K_X}(u)$, for a function $u : D \times X \to \mathbb{R}$, is given by

$$\mathrm{Hilb}_{E\otimes K_X}(u)(s,s) = \int_X g(s,s) e^{-u(z,\cdot)}$$

with $s \in H^0(X, E \otimes K_X)$. Since the integrand on the right is already an (n,n)-form, the integral makes sense. In the following, suitable assumptions will be made on u to make sure the integral converges. Then the map $z \mapsto \mathrm{Hilb}_{E\otimes K_X}(u)$ is a Hermitian metric on the bundle $D \times H^0(X, E \otimes K_X) \to D$. The main result of this section is the following positivity theorem.

Theorem 2.1. *If u is bounded and upper semicontinuous (usc) on $D \times X$, and η-subharmonic on graphs, then the dual metric $\mathrm{Hilb}^*_{E\otimes K_X}(u)$ is a subharmonic norm function.*

The following approximation lemma is somewhat technical and we postpone its proof to Section 5.

Lemma 2.2. *Let u be a bounded usc function on $D \times X$ which is η-subharmonic on graphs. Then, for D' relatively compact open in D, there exist $\varepsilon_j \searrow 0$ and $u_j \in C^\infty(D' \times X)$ decreasing to u, where u_j is $(1-\varepsilon_j)\eta$-subharmonic on graphs. Namely, for any holomorphic map f from an open subset of D' to X, $\Delta(\psi(f(z)) + u_j(z, f(z))) \geq \varepsilon_j \Delta(\psi(f(z))$, where $\eta = i\partial\bar{\partial}\psi$ locally.*

Proof of Theorem 2.1. Since being a subharmonic norm function is a local property, we focus on D', a relatively compact open set in D. Take ε_j and u_j as in Lemma 2.2. Assuming the theorem holds for such a u_j (namely, the dual metric $\mathrm{Hilb}^*_{E\otimes K_X}(u_j)$ is a subharmonic norm function), it follows that $\mathrm{Hilb}^*_{E\otimes K_X}(u)$ is also a subharmonic norm function because $\mathrm{Hilb}^*_{E\otimes K_X}(u_j)$ decreases to $\mathrm{Hilb}^*_{E\otimes K_X}(u)$ as $j \to \infty$.

As a result, we only need to prove the theorem for $u \in C^\infty(D' \times X)$ with the property that there exists $\varepsilon > 0$ such that for any holomorphic function f from an open subset of D' to X,

$$\Delta(\psi(f(z)) + u(z, f(z))) \geq \varepsilon \Delta(\psi(f(z)), \quad \text{where } \eta = i\partial\bar{\partial}\psi \text{ locally.} \tag{5}$$

In a coordinate system $\Omega \subset \mathbb{C}^n$ on X, we will not write out the coordinate map. We will use Greek letters μ, λ for indices of coordinates on X, and Roman letters i, j for indices of coordinates on D; moreover, f^μ means the μ-th component of f, whereas $\psi_{\mu\bar\lambda}$, $u_{i\bar i}$, and $u_{i\bar\lambda}$ mean partial derivatives $\partial^2\psi/\partial x_\mu \partial\bar x_\lambda$, $\partial^2 u/\partial z_i\partial\bar z_i$, and $\partial^2 u/\partial z_i\partial\bar x_\lambda$, respectively. In this coordinate system $\Omega\subset\mathbb{C}^n$ on X, we first show that the matrix $(\psi_{\mu\bar\lambda}+u_{\mu\bar\lambda})(z_0, x_0)$ is positive definite, for any given $(z_0, x_0)\in D'\times\Omega$. Inequality (5) about $\psi+u$ is unchanged after a translation in coordinates of $D'\times\Omega$, so we can assume $(z_0, x_0) = (0, 0)$. In terms of local coordinates, inequality (5) becomes

$$\varepsilon\sum_{i,\lambda,\mu}\psi_{\mu\bar\lambda}\frac{\partial f^\mu}{\partial z_i}\frac{\partial\bar f^\lambda}{\partial\bar z_i}\le\sum_{i,\lambda,\mu}\psi_{\mu\bar\lambda}\frac{\partial f^\mu}{\partial z_i}\frac{\partial\bar f^\lambda}{\partial\bar z_i}+\sum_i u_{i\bar i}+\sum_{i,\lambda}u_{i\bar\lambda}\frac{\partial\bar f^\lambda}{\partial\bar z_i}+\sum_{i,\mu}u_{\bar i\mu}\frac{\partial f^\mu}{\partial z_i}+\sum_{i,\lambda,\mu}u_{\mu\bar\lambda}\frac{\partial f^\mu}{\partial z_i}\frac{\partial\bar f^\lambda}{\partial\bar z_i}. \tag{6}$$

Fix $(\xi_1, \xi_2, \dots, \xi_n)\in\mathbb{C}^n$. For N a positive number, we consider $f(z) = N(\xi_1, \xi_2, \dots, \xi_n)z_1$; note that $f(z)$ is in Ω by restricting z in a small neighborhood of 0 in D'. With such a choice of f, we deduce from (6) that

$$\begin{aligned}\varepsilon\sum_{\lambda,\mu}\psi_{\mu\bar\lambda}(0,0)\xi_\mu\bar\xi_\lambda N^2 &\le\sum_{\lambda,\mu}\psi_{\mu\bar\lambda}(0,0)\xi_\mu\bar\xi_\lambda N^2+\sum_i u_{i\bar i}(0,0)\\ &\quad+\sum_\lambda u_{1\bar\lambda}(0,0)\bar\xi_\lambda N+\sum_\mu u_{\bar 1\mu}(0,0)\xi_\mu N+\sum_{\lambda,\mu}u_{\mu\bar\lambda}(0,0)\xi_\mu\bar\xi_\lambda N^2. \end{aligned}\tag{7}$$

For larger N, we have to restrict f to a smaller domain in D', but since inequality (7) is evaluated at $(0, 0)$, it holds for any N. Divide (7) by N^2 and send N to infinity, to obtain $(\psi_{\mu\bar\lambda}+u_{\mu\bar\lambda})(0,0)\ge\varepsilon(\psi_{\mu\bar\lambda})(0,0)$ as matrices, and hence $(\psi_{\mu\bar\lambda}+u_{\mu\bar\lambda})(0,0)$ is positive definite.

Let $L^2(X, E\otimes K_X)$ be the space of measurable sections s whose L^2-norm $\int_X g(s,s)e^{-u(z,\cdot)}$ is finite. Since different z will give rise to comparable L^2-norms, the space $L^2(X, E\otimes K_X)$ does not change with z, and so we have a Hermitian Hilbert bundle $D'\times L^2(X, E\otimes K_X)\to D'$ which has $D'\times H^0(X, E\otimes K_X)\to D'$ as a subbundle. Denote the curvature of the subbundle by $\Theta=\sum\Theta_{j\bar k}dz_j\wedge d z_{\bar k}$. This setup is almost identical to [Berndtsson 2009, Theorem 1.1], where the author observed that the second fundamental form of the subbundle $D'\times H^0(X, E\otimes K_X)\to D'$ can be controlled by L^2-estimates. Following the computations in Section 3 of the same work, we deduce

$$\sum_j(\Theta_{j\bar j}s, s)\ge\int_X K(z,\cdot)g(s,s)e^{-u(z,\cdot)}, \tag{8}$$

where $s\in H^0(X, E\otimes K_X)$ and $K: D'\times X\to\mathbb{R}$ is a smooth function, given in local coordinates on X by

$$K=\sum_j\biggl(u_{j\bar j}-\sum_{\lambda,\mu}(\psi+u)^{\bar\lambda\mu}u_{j\bar\lambda}u_{\bar j\mu}\biggr);$$

here $(\psi+u)^{\bar\lambda\mu}$ stands for the inverse matrix of $(\psi+u)_{\bar\lambda\mu}$; cf. [Berndtsson 2009, Formula (3.1)].

We claim that $K\ge 0$. Fix $(z_0, x_0)\in D'\times X$ with a coordinate system Ω around x_0. First notice that ψ is independent of z, so if we denote $\psi(x)+u(z,x)$ by $\phi(z,x)$, then

$$K=\sum_j\biggl(\phi_{j\bar j}-\sum_{\lambda,\mu}\phi_{j\bar\lambda}\phi^{\bar\lambda\mu}\phi_{\bar j\mu}\biggr).$$

Since the matrix $(\phi_{\mu\bar{\lambda}})$ is positive definite, we can assume local coordinates in Ω are such that $(\phi_{\mu\bar{\lambda}})$ is the identity matrix at (z_0, x_0), and therefore $K(z_0, x_0) = \sum_j (\phi_{j\bar{j}} - \sum_\lambda |\phi_{j\bar{\lambda}}|^2)(z_0, x_0)$. For a holomorphic function f from an open subset of D' to Ω, the subharmonicity of $\phi(z, f(z))$ reads as

$$\sum_i \phi_{i\bar{i}} + \sum_{i,\lambda} \phi_{i\bar{\lambda}} \frac{\partial \bar{f}^\lambda}{\partial \bar{z}_i} + \sum_{i,\mu} \phi_{\bar{i}\mu} \frac{\partial f^\mu}{\partial z_i} + \sum_{i,\lambda,\mu} \phi_{\mu\bar{\lambda}} \frac{\partial f^\mu}{\partial z_i} \frac{\partial \bar{f}^\lambda}{\partial \bar{z}_i} \geq 0. \tag{9}$$

Without loss of generality, we assume $(z_0, x_0) = (0, 0)$ and choose $f^\lambda = -\sum_i \phi_{i\bar{\lambda}}(0, 0) z_i$ in (9) with z small so that $f(z)$ is in Ω. Inequality (9) becomes $\sum_j \big(\phi_{j\bar{j}} - \sum_\lambda |\phi_{j\bar{\lambda}}|^2\big)(0, 0) \geq 0$. Therefore $K \geq 0$. See also the remark after Lemma 4.1 for a slightly different proof of this claim and an invariant meaning of K.

As a result, (8) implies $\sum_j (\Theta_{j\bar{j}} s, s) \geq 0$, and hence the curvature of the dual metric $\mathrm{Hilb}^*_{E\otimes K_X}(u)$ satisfies the opposite inequality; according to [Coifman and Semmes 1993, Theorem 4.1] this implies $\mathrm{Hilb}^*_{E\otimes K_X}(u)$ is a subharmonic norm function. □

Now we replace (E, g) by $(L^k \otimes K_X^*, h^k \otimes \omega^n)$, which is positively curved for large k since

$$\Theta(h^k \otimes \omega^n) = k\omega + \mathrm{Ric}\,\omega.$$

We have the following proposition regarding the metric $H_k(u)$ on the bundle $D \times H^0(X, L^k)$.

Proposition 2.3. *Suppose u is a bounded usc function on $D \times X$ and with some $\varepsilon \in (0, 1)$ we have that u is $(1-\varepsilon)\omega$-subharmonic on graphs. Then there exists $k_0 = k_0(\varepsilon, \omega)$, independent of u, such that, for $k \geq k_0$, the dual metric $H_k^*(u)$ is a subharmonic norm function.*

Proof. In order to use Theorem 2.1, we must check if ku is $(k\omega + \mathrm{Ric}\,\omega)$-subharmonic on graphs. Suppose that $\omega = i\partial\bar{\partial}\psi$ and $\mathrm{Ric}\,\omega = i\partial\bar{\partial}\phi$ locally. Then we want to see if $k\psi(f(z)) + \phi(f(z)) + ku(z, f(z))$ is subharmonic for any holomorphic map f. Note that

$$k\psi + \phi + ku = k(1-\varepsilon)\psi + ku + \varepsilon k\psi + \phi,$$

and $k(1-\varepsilon)\psi(f(z)) + ku(z, f(z))$ is subharmonic by assumption. On the other hand, there exists k_0 depending on ε and ω such that $\varepsilon k\psi + \phi$ is plurisubharmonic (psh) for $k \geq k_0$. Therefore, for $k \geq k_0$, ku is $(k\omega + \mathrm{Ric}\,\omega)$-subharmonic on graphs. By Theorem 2.1, the metric $\mathrm{Hilb}^*_{L^k}(ku)$ is a subharmonic norm function for $k \geq k_0$. The proposition follows since $\mathrm{Hilb}_{L^k}(ku) = H_k(u)$. □

3. Approximation by Hermitian–Yang–Mills metrics

Recall that D is in $\mathbb{C}^m$ and $(L, h) \to X^n$ is a positive line bundle with curvature ω.

Lemma 3.1. *Let u be an usc function on $D \times X$ and ω-subharmonic on graphs. Then for any fixed $z \in D$, $u(z, x)$ is ω-psh on X, and for any fixed $x \in X$, $u(z, x)$ is subharmonic on D.*

This can be seen as a special case of an abstract theorem in [Slodkowski 1990a, Section 1], whose proof we translate to our setting.

Proof. By choosing the holomorphic map f constant in the definition of ω-subharmonic on graphs, it follows immediately that $u(z, x)$ is subharmonic in z.

For a fixed $z_0 \in D$, we want to show $x \mapsto \psi(x) + u(z_0, x)$ is psh in a coordinate system $\Omega \subset \mathbb{C}^n$ on X, where ψ is a local potential of ω. Let P be the complex line $\{\lambda e_1 : \lambda \in \mathbb{C}, e_1 = (1, 0, \dots, 0) \in \mathbb{C}^n\}$. Without loss of generality, it suffices to prove that, for $\lambda e_1 \in P \cap \Omega$, the function $\lambda \mapsto \psi(\lambda e_1) + u(0, \lambda e_1)$ is subharmonic. Let U be a disc in $P \cap \Omega$, and we simply write $U = \{\lambda \in \mathbb{C} : |\lambda - a| < R\}$. Let $h(\lambda)$ be harmonic on U and continuous up to the boundary. We will be done if

$$\psi(ae_1) + u(0, ae_1) + h(a) \leq \max_{\lambda \in \partial U} (u\psi(\lambda e_1) + u(0, \lambda e_1) + h(\lambda)).$$

Suppose the inequality is not true. By [Slodkowski 1986, Lemma 4.5] with $\partial U \subset \bar{U}$ as the two compact sets in that lemma, there is an $\mathbb{R}$-linear function $l : \mathbb{C} \to \mathbb{R}$ and $b \in U$ such that, if we write

$$v(z, \lambda) = \psi(\lambda e_1) + u(z, \lambda e_1) + h(\lambda) + l(\lambda), \tag{10}$$

then

$$v(0, b) > v(0, \lambda), \quad \text{for } \lambda \in U - \{b\}.$$

Now define $W(z, \lambda_1, \dots, \lambda_m) := v(z, \lambda_1) + \cdots + v(z, \lambda_m)$ in a neighborhood of $(0, b^*) := (0, b, \dots, b)$ in $\mathbb{C}^m \times \mathbb{C}^m$. As $W(0, b^*) > W(0, \lambda_1, \dots, \lambda_m)$ for $(\lambda_1, \dots, \lambda_m) \neq b^*$, there exists a ball $B \subset \mathbb{C}^m$ of radius r centered at b^* such that

$$W(0, b^*) > \max_{\{0\} \times \partial B} W.$$

Since W is usc, there exists $\varepsilon > 0$ such that $W(z, \lambda_1, \dots, \lambda_m) < W(0, b^*)$, for $|z| \leq \varepsilon$ and $(\lambda_1, \dots, \lambda_m) \in \partial B$. Let $S = (r/\varepsilon)\,\mathrm{Id}_{\mathbb{C}^m}$. We have $W(z, b^* + S(z)) < W(0, b^*)$ for $|z| = \varepsilon$, which contradicts the maximum principle because $W(z, b^* + S(z)) = \sum_{i=1}^m v(z, b + (r/\varepsilon)z_i)$ is subharmonic by (10). □

Although in the Introduction the boundary data v is in $C^\infty(\partial D, \mathcal{H}_\omega)$, we will prove a lemma for a broader class of boundary data ν. Let $\nu : \partial D \times X \to \mathbb{R}$ be a continuous map such that $\nu_z(\cdot) := \nu(z, \cdot) \in \mathrm{PSH}(X, \omega)$ for $z \in \partial D$. Let

$$G_\nu = \Big\{ u \in \mathrm{usc}(D \times X) : u \text{ is } \omega\text{-subharmonic on graphs}, \limsup_{D \ni z \to \zeta \in \partial D} u(z, x) \leq \nu(\zeta, x) \Big\}.$$

In order to study the properties of the upper envelope $\mathcal{V}$ of G_ν, we introduce a closely related family. With the projection $\pi : D \times X \to X$, let

$$F_\nu := \Big\{ u : u \in \mathrm{PSH}(D \times X, \pi^*\omega), \limsup_{D \ni z \to \zeta \in \partial D} u(z, x) \leq \nu(\zeta, x) \Big\}.$$

The upper envelope of F_ν extends to a solution $\mathcal{U} \in C(\bar{D} \times X)$ of

$$\begin{cases} (\pi^*\omega + i\partial\bar{\partial}\mathcal{U})^{n+m} = 0 & \text{on } D \times X, \\ \pi^*\omega + i\partial\bar{\partial}\mathcal{U} \geq 0 & \text{on } D \times X, \\ \mathcal{U}|_{\partial D \times X} = \nu; \end{cases}$$

see for example [Boucksom 2012; Darvas and Wu 2019]. In addition, we also need the solution h to the Dirichlet problem

$$\begin{cases} \sum_j h_{j\bar{j}} + \Delta_\omega h + 2n = 0 \quad \text{on } D \times X, \\ h|_{\partial D \times X} = \nu. \end{cases}$$

Lemma 3.2. *If we denote the upper envelopes of G_ν and F_ν by $\mathcal{V}$ and $\mathcal{U}$, respectively, then $\mathcal{U} \leq \mathcal{V} \leq h$ and*

$$\lim_{(z,x)\to(z_0,x_0)\in\partial D\times X} \mathcal{V}(z,x) = \nu(z_0,x_0).$$

Moreover, if ν is negative, then so is $\mathcal{V}$.

Proof. Unraveling the definitions of F_ν and G_ν, we see $F_\nu \subset G_\nu$, so $\mathcal{U} \leq \mathcal{V}$. For any $u \in G_\nu$, $u(z,\cdot)$ is ω-psh for fixed z by Lemma 3.1, hence $\Delta_\omega u + 2n \geq 0$; in addition, $u(\cdot\,,x)$ is subharmonic for fixed x. By the maximum principle, $u \leq h$ and hence $\mathcal{V} \leq h$ also. $\mathcal{U}$ and h are both equal to ν on $\partial D \times X$, and so is $\mathcal{V}$.

For a fixed $x_0 \in X$, let $H_0(z)$ be the harmonic function on D with boundary values $\nu(z,x_0)$. For $u \in G_\nu$, we have $u(z,x_0) \leq H_0(z)$, and therefore $\mathcal{V}(z,x_0) \leq H_0(z)$. The second statement follows at once. □

With Proposition 2.3 at hand, we can start to prove Theorem 1.2. The following envelope will be used in the proof: for an usc function F on X, we introduce

$$P(F) := \sup\{h \in \mathrm{PSH}(X,\omega) \mid h \leq F\} \in \mathrm{PSH}(X,\omega);$$

see [Berman 2019; Ross and Witt Nyström 2017].

Proof of Theorem 1.2. Without loss of generality, we will assume $\upsilon \leq 0$. Fix $\delta > 1$, and for $z \in \partial D$, define $\upsilon_z^\delta = P(\delta \upsilon_z)$. By [Darvas and Wu 2019, Lemma 4.9], $\partial D \times X \ni (z,x) \mapsto \upsilon_z^\delta(x)$ is continuous. Let V^δ be the upper envelope of G_{υ^δ}. By Lemma 3.2, $V^\delta \leq 0$, and so $u \leq 0$ for $u \in G_{\upsilon^\delta}$. The next step is to have a better upper bound for $u \in G_{\upsilon^\delta}$. To that end, we can look instead at $\max\{u,c\}$, which is still in G_{υ^δ} as long as the constant $c \leq \min \upsilon^\delta$. Since $\max\{u,c\}$ is bounded, we will assume u is bounded. Moreover, u/δ is ω/δ-subharmonic on graphs. According to Proposition 2.3, there exists $k_0 = k_0(\delta)$ such that for $k \geq k_0$, $H_k^*(u/\delta)$ is a subharmonic norm function. Because $\limsup_{\partial D} H_k^*(u/\delta) \leq H_k^*(\upsilon)$, it follows that $H_k^*(u/\delta) \in G_\upsilon^k$ and therefore $H_k^*(u/\delta) \leq V^k$ on D and $FS_k(H_k(u/\delta)) \leq FS_k((V^k)^*)$. By Lemma 3.1, we have $\omega + i\partial\bar{\partial} u/\delta \geq (1-1/\delta)\omega$, (the operator $i\partial\bar{\partial}$ here is with respect to variables in X). The Ohsawa–Takegoshi extension theorem implies (see [Darvas et al. 2020, Theorem 2.11] or [Darvas and Wu 2019, Lemma 4.10]) that there exist $C > 0$ and $k_0(\delta)$ such that, for $k \geq k_0$,

$$\frac{1}{\delta} u - \frac{C}{k} \leq FS_k \circ H_k\Big(\frac{1}{\delta} u\Big) \leq FS_k((V^k)^*).$$

Since $\delta \upsilon \leq 0$, both V^δ and u are negative by Lemma 3.2, and as a result we have $u - C/k \leq FS_k((V^k)^*)$; this statement is true for any $u \in G_{\upsilon^\delta}$, so we actually have $V^\delta - C/k \leq FS_k((V^k)^*)$. In addition, since $\upsilon_z + (\delta-1)\inf_{\partial D\times X}(\upsilon_z)$ is a competitor in $P(\delta \upsilon_z)$,

$$V + (\delta - 1) \inf_{\partial D \times X}(\upsilon) \leq V^\delta.$$

Putting things together, we conclude

$$V + (\delta - 1) \inf_{\partial D \times X} (v) - \frac{C}{k} \le FS_k((V^k)^*), \quad \text{for } k \ge k_0(\delta). \tag{11}$$

Next we claim that $FS_k((V_z^k)^*)(x)$ is ω-subharmonic on graphs. Some preparation is needed. Let s be a nonvanishing holomorphic section of L^k over an open set $Y \subset X$. Let $e^{-k\phi} := h^k(s, s)$ and $s_k^* : Y \to (L^k)^*$ be defined by $s_k^*(x)(\cdot) = h^k(\cdot, e^{k\phi(x)/2} s(x))$ for $x \in Y$. Suppose $\hat{s}_k^* : Y \to H^0(X, L^k)^*$ is the pointwise evaluation map of s_k^*, namely $\hat{s}_k^*(x)(\sigma) := s_k^*(x)(\sigma(x))$ for $\sigma \in H^0(X, L^k)$. Then we have the following formula, which is taken from [Darvas and Wu 2019, Lemma 4.1]:

$$FS_k((V_z^k)^*)(x) = \frac{2}{k} \log[V_z^k(\hat{s}_k^*(x))], \quad x \in Y. \tag{12}$$

Meanwhile, for $\sigma \in H^0(X, L^k)$, we have $e^{k\phi(x)/2} \hat{s}_k^*(x)(\sigma) = \sigma(x)/s(x)$ is holomorphic, so $e^{k\phi/2}\hat{s}_k^*$ is holomorphic. Hence for any holomorphic map g from an open subset of D to X,

$$\Delta(\phi(g(z)) + FS_k((V_z^k)^*)(g(z))) = \Delta\Big(\frac{1}{k} \log[V_z^k((e^{k\phi/2}\hat{s}_k^*) \circ g(z))]^2\Big). \tag{13}$$

By [Coifman and Semmes 1993, Theorem 4.1], the Hermitian–Yang–Mills metric V_z^k is a subharmonic norm function, so the last term of (13) is nonnegative, which means $FS_k((V^k)^*)$ is ω-subharmonic on graphs as we claimed. Further, according to the Tian–Catlin–Zelditch asymptotic theorem or by [Darvas and Wu 2019, Lemma 4.10], we have an easier but cruder estimate

$$FS_k((V_z^k)^*|_{\partial D}) = FS_k(H_k(v)) \le v + O(\log k/k),$$

so

$$FS_k((V^k)^*) \in G_{v+O(\log k/k)}$$

and

$$FS_k((V^k)^*) \le V + O(\log k/k).$$

This last inequality together with (11) concludes the proof. □

It is natural to ask if V belongs to G_v. A standard approach to show that the envelope belongs to a family is to take upper regularization, and the case at hand is very similar to [Coifman and Semmes 1993, Lemma 11.11], where upper regularization is taken in the z-variables. The reason it works in their lemma is because their function in the x-variables is a norm, but ours is not and regularization does not seem to work. However, with Theorem 1.2 one can easily show $V \in G_v$. It would be interesting to prove $V \in G_v$ directly without using Theorem 1.2; after all, G_v and V can be defined on any Kähler manifold (X, ω) without reference to a line bundle.

Corollary 3.3. *The upper envelope V is continuous, and $V \in G_v$.*

Proof. The first statement is a direct consequence of Theorem 1.2. As to the second statement, let ψ be a local potential of ω and f a holomorphic map from an open subset of D to X. For any $u \in G_v$, $\psi(f(z)) + u(z, f(z))$ is subharmonic; hence $\psi(f(z)) + V(z, f(z))$, the supremum over $u \in G_v$, is also subharmonic since V is continuous. By Lemma 3.2, it follows that $V \in G_v$. □

4. The WZW equation

We will prove Theorem 1.3 and compute the Euler–Lagrange equation of $\mathscr{E}$ in this section. We begin with an observation. Suppose u is a C^2 function on $D \times X$ and ψ is a local potential of ω. Consider the complex Hessian of $u+\psi$ with respect to a fixed coordinate z_j in D and local coordinates x in X where ψ is defined,

$$\begin{pmatrix} (u+\psi)_{z_j\bar{z}_j} & (u+\psi)_{z_j\bar{x}_1} & \cdots & (u+\psi)_{z_j\bar{x}_n} \\ (u+\psi)_{x_1\bar{z}_j} & (u+\psi)_{x_1\bar{x}_1} & \cdots & (u+\psi)_{x_1\bar{x}_n} \\ \vdots & \vdots & \ddots & \vdots \\ (u+\psi)_{x_n\bar{z}_j} & (u+\psi)_{x_n\bar{x}_1} & \cdots & (u+\psi)_{x_n\bar{x}_n} \end{pmatrix}, \tag{14}$$

which we will denote by $(u+\psi)_j$. Then

$$\begin{aligned}(i\partial\bar{\partial}u+\pi^*\omega)^{n+1} &\wedge \left(i\sum_{j=1}^m dz_j\wedge d\bar{z}_j\right)^{m-1} \\ &= (n+1)!(m-1)!\sum_{j=1}^m \det(u+\psi)_j\left(\bigwedge_{k=1}^m i\,dz_k\wedge d\bar{z}_k\wedge\bigwedge_{l=1}^n i\,dx_l\wedge d\bar{x}_l\right).\end{aligned} \tag{15}$$

Lemma 4.1. *Suppose u is a C^2 function on $D\times X$ and $\omega+i\partial\bar{\partial}u(z,\cdot)>0$ on X for all $z\in D$. Then u is ω-subharmonic on graphs if and only if*

$$(i\partial\bar{\partial}u+\pi^*\omega)^{n+1}\wedge\left(i\sum_{j=1}^m dz_j\wedge d\bar{z}_j\right)^{m-1}\geq 0.$$

Proof. Let ψ be a local potential of ω and denote the complex Hessian of $u+\psi$ with respect to z_j and x by $(u+\psi)_j$, as in the matrix (14). Due to (15), we will focus on $\sum_{j=1}^m \det(u+\psi)_j$.

Let f be a holomorphic function from an open subset of D to X. Then in a coordinate system on X,

$$\Delta(\psi(f(z))+u(z,f(z)))=\sum_{i,\lambda,\mu}\psi_{\mu\bar{\lambda}}\frac{\partial f^\mu}{\partial z_i}\frac{\partial\bar{f}^\lambda}{\partial\bar{z}_i}+\sum_i u_{i\bar{i}}+\sum_{i,\lambda}u_{i\bar{\lambda}}\frac{\partial\bar{f}^\lambda}{\partial\bar{z}_i}+\sum_{i,\mu}u_{\bar{i}\mu}\frac{\partial f^\mu}{\partial z_i}+\sum_{i,\lambda,\mu}u_{\mu\bar{\lambda}}\frac{\partial f^\mu}{\partial z_i}\frac{\partial\bar{f}^\lambda}{\partial\bar{z}_i}.$$

If we denote the matrix $(\psi_{\mu\bar{\lambda}}+u_{\mu\bar{\lambda}})$ by A and the column vector $(u_{i\bar{\lambda}})$ by B_i, then the right side of the equation above can be written as

$$\sum_i\left(\left\langle A\frac{\partial f}{\partial z_i},\frac{\partial f}{\partial z_i}\right\rangle+\left\langle B_i,\frac{\partial f}{\partial z_i}\right\rangle+\overline{\left\langle B_i,\frac{\partial f}{\partial z_i}\right\rangle}+u_{i\bar{i}}\right), \tag{16}$$

where the angled inner product is the usual Euclidean inner product and $\partial f/\partial z_i$ is the column vector $(\partial f^\mu/\partial z_i)$. The matrix form can be further written as

$$\sum_i\left(\left\|\sqrt{A}\frac{\partial f}{\partial z_i}+\sqrt{A}^{-1}B_i\right\|^2-\|\sqrt{A}^{-1}B_i\|^2+u_{i\bar{i}}\right). \tag{17}$$

Notice that

$$\sum_i(-\|\sqrt{A}^{-1}B_i\|^2+u_{i\bar{i}})=\sum_i(u_{i\bar{i}}-\langle A^{-1}B_i,B_i\rangle)=\sum_i\Big(u_{i\bar{i}}-\sum_{\lambda,\mu}u_{i\bar{\lambda}}(\psi+u)^{\bar{\lambda}\mu}u_{\bar{i}\mu}\Big)$$
$$=\sum_i\frac{\det(u+\psi)_i}{\det(\psi_{\mu\bar{\lambda}}+u_{\mu\bar{\lambda}})},\tag{18}$$

where the last equality can be deduced from Schur's formula for determinants of block matrices as follows (see also [Semmes 1992; Berndtsson 2009] for a different computation). We examine the complex Hessian of $u+\psi$,

$$(u+\psi)_j=\begin{pmatrix}(u+\psi)_{z_j\bar{z}_j} & (u+\psi)_{z_j\bar{x}_1} & \cdots & (u+\psi)_{z_j\bar{x}_n}\\ (u+\psi)_{x_1\bar{z}_j} & (u+\psi)_{x_1\bar{x}_1} & \cdots & (u+\psi)_{x_1\bar{x}_n}\\ \vdots & \vdots & \ddots & \vdots\\ (u+\psi)_{x_n\bar{z}_j} & (u+\psi)_{x_n\bar{x}_1} & \cdots & (u+\psi)_{x_n\bar{x}_n}\end{pmatrix},$$

and find that the Schur complement of the trailing $n\times n$ minor $((u+\psi)_{\mu\bar{\lambda}})$ is precisely

$$u_{j\bar{j}}-\sum_{\lambda,\mu}u_{j\bar{\lambda}}(u+\psi)^{\bar{\lambda}\mu}u_{\bar{j}\mu},$$

which is also equal to $\det(u+\psi)_j/\det((u+\psi)_{\mu\bar{\lambda}})$ by Schur's formula; see [Horn and Zhang 2005].

Now u is ω-subharmonic on graphs if and only if (17) is nonnegative for any holomorphic maps f, and it is equivalent to the last summation in (18) being nonnegative. The lemma follows from the positivity of the matrix $(\psi_{\mu\bar{\lambda}}+u_{\mu\bar{\lambda}})$ and (15). □

From (15) and (18), the function K in the proof of Theorem 2.1 has the invariant expression

$$K=\frac{m!n!}{(m-1)!(n+1)!}\frac{(\pi^*\omega+i\partial\bar{\partial}u)^{n+1}\wedge\big(i\sum_{j=1}^m dz_j\wedge d\bar{z}_j\big)^{m-1}}{(\omega+i\partial\bar{\partial}u)^n\wedge\big(i\sum_{j=1}^m dz_j\wedge d\bar{z}_j\big)^m},$$

and one can see $K\geq 0$ if u is ω-subharmonic on graphs. See also [Campana et al. 2019, Section 4.1, Formula (85)].

Proof of Theorem 1.3. By (15), the equation

$$(i\partial\bar{\partial}V+\pi^*\omega)^{n+1}\wedge\Big(i\sum_{j=1}^m dz_j\wedge d\bar{z}_j\Big)^{m-1}=0$$

is equivalent to $\sum_j\det(\psi+V)_j=0$, so we will prove the latter equation.

By Corollary 3.3, the function V is ω-subharmonic on graphs, and hence $V(z,x)$ is ω-psh on X by Lemma 3.1. Take a coordinate chart Ω of X. Then for $\varepsilon>0$ and $x\in\Omega$, the function $V(z,x)+\varepsilon|x|^2$ satisfies the assumption of Lemma 4.1, so $\sum_i\det(\psi+V+\varepsilon|x|^2)_i\geq 0$ and $\sum_i\det(\psi+V)_i\geq 0$.

Suppose $\sum_i\det(\psi+V)_i$ is positive at a point p in $D\times X$. We may assume $\det(\psi+V)_1$ is positive at p. We digress here to prove the following lemma.

Lemma 4.2. *Let A be an $(n+1)\times(n+1)$ Hermitian matrix partitioned as*

$$A = \begin{pmatrix} a_{11} & a_{12} \cdots & a_{1(n+1)} \\ a_{21} & & \\ \vdots & B & \\ a_{(n+1)1} & & \end{pmatrix},$$

where B has size $n \times n$. If $\det A > 0$ *and the matrix* $B \geq 0$, *then the matrix* $A > 0$.

Proof. The semipositivity of B implies that A has at least n nonnegative eigenvalues, and actually it has at least n positive eigenvalues since A is invertible. The last eigenvalue of A must also be positive because $\det A > 0$. □

By the above lemma, the matrix $(\psi + V)_1$ is actually positive at p. Its $n\times n$ trailing minor $(\psi+V)_{\mu\bar{\lambda}}(p)$ is also positive. Since V is assumed to be C^2, we can find a neighborhood N of p in $D \times X$ such that the matrix $(\psi + V)_{\mu\bar{\lambda}} > \delta$ in N, for some positive number δ. By possibly shrinking N, we also have $\sum_i \det(\psi + V)_i > 0$ in N.

For the last step in the proof of Theorem 1.3, choose a smooth cutoff function ρ supported in N with $-\delta/2 \leq (\rho_{\mu\bar{\lambda}}) \leq \delta/2$ and such that $\sum_i \det(\psi + V + \rho)_i > 0$ in N. We see the function $V + \rho$ satisfies the assumption of Lemma 4.1 on N, and hence $V + \rho$ is ω-subharmonic on graphs and is in G_v, which contradicts $V = \sup G_v$. Therefore, $\sum_j \det(\psi + V)_j = 0$. □

As in the Introduction, θ on $\mathcal{H}_\omega$ is given by

$$\theta(\xi_1, \xi_2, \xi_3) := g_M(\{\xi_1, \xi_2\}_{\omega_\phi}, \xi_3) = \int_X \{\xi_1, \xi_2\}_{\omega_\phi} \xi_3 \omega_\phi^n, \tag{19}$$

where $\phi \in \mathcal{H}_\omega$ and $\xi_1, \xi_2, \xi_3 \in T_\phi \mathcal{H}_\omega$. We have $\{\xi_1, \xi_2\}_{\omega_\phi} \omega_\phi^n = n d\xi_1 \wedge d\xi_2 \wedge \omega_\phi^{n-1}$, and using integration by parts we deduce that

$$\int_X \{\xi_1, \xi_2\}_{\omega_\phi} \xi_3 \omega_\phi^n = \int_X \xi_1 \{\xi_2, \xi_3\}_{\omega_\phi} \omega_\phi^n,$$

and therefore θ is indeed skew-symmetric and a three-form. Moreover, θ is smooth in the sense that, for smooth vector fields X_1, X_2, X_3, the function $\theta(X_1, X_2, X_3) : \mathcal{H}_\omega \to \mathbb{R}$ is smooth. The rest of this section is devoted to proving that the three-form θ is d-closed on $\mathcal{H}_\omega$ and showing the derivation of the Euler–Lagrange equation of $\mathscr{E}$.

The exterior derivative and the Poincaré lemma over a Banach manifold are discussed in detail in [Abraham et al. 1988, Supplement 6.4A], and although $\mathcal{H}_\omega$ is a Fréchet manifold, we can still derive the following two lemmas by similar approaches. See [Hamilton 1982] for a discussion of Fréchet manifolds.

We define first the exterior derivative on $\mathcal{H}_\omega$. Given a smooth k-form β on $\mathcal{H}_\omega$ and tangent vectors $\xi_0, \dots, \xi_k$ at $T_\phi \mathcal{H}_\omega$, in order to define

$$d\beta(\xi_0, \dots, \xi_k),$$

we extend ξ_i to vector fields on $\mathcal{H}_\omega$, which are constant in the canonical trivialization $T\mathcal{H}_\omega \approx \mathcal{H}_\omega \times C^\infty(X)$. Still denoting the constant vector fields by ξ_i, the exterior derivative is given by the well-known formula

$$d\beta(\xi_0, \dots, \xi_k) = \sum_{j=0}^{k} (-1)^j L_{\xi_j}(\beta(\xi_0, \dots, \hat{\xi}_j, \dots, \xi_k)) + \sum_{i<j} (-1)^{i+j} \beta(L_{\xi_i}\xi_j, \xi_0, \dots, \hat{\xi}_i, \dots, \hat{\xi}_j, \dots, \xi_k),$$

where $\hat{\xi}_j$ means ξ_j is to be omitted and L_{ξ_i} is the Lie derivative along ξ_i. Since the flow that ξ_i generates is simply the translation $t \mapsto \phi + t\xi_i$, the Lie derivative $L_{\xi_i}\xi_j$ equals 0. We summarize the discussion in the following lemma.

Lemma 4.3. *Let β be a smooth k-form on $\mathcal{H}_\omega$, and let $\xi_0, \dots, \xi_k$ be vector fields on $\mathcal{H}_\omega$ which are constant in the canonical trivialization $T\mathcal{H}_\omega \approx \mathcal{H}_\omega \times C^\infty(X)$. Then*

$$d\beta(\xi_0, \dots, \xi_k) = \sum_{j=0}^{k} (-1)^j L_{\xi_j}(\beta(\xi_0, \dots, \hat{\xi}_j, \dots, \xi_k)) \tag{20}$$

$$= \sum_{j=0}^{k} (-1)^j \frac{d}{dt}\Big|_{t=0} \beta(\xi_0, \dots, \hat{\xi}_j, \dots, \xi_k)(\phi + t\xi_j), \tag{21}$$

where $\hat{\xi}_j$ means ξ_j is to be omitted. (*This formula is true if $\mathcal{H}_\omega \subset C^\infty(X)$ is replaced by an open subset of a Fréchet space.*)

Lemma 4.4. *If β is a d-closed smooth k-form on $\mathcal{H}_\omega$, then there exists a $(k-1)$-form $H\beta$ on $\mathcal{H}_\omega$ such that $d(H\beta) = \beta$.*

Proof. The proof is similar to the finite-dimensional case. Recall that $\mathcal{H}_\omega$ is convex and that $0 \in \mathcal{H}_\omega$. Given $\xi_1, \dots, \xi_{k-1} \in T_\phi\mathcal{H}_\omega$, we define the $(k-1)$-form $H\beta$ by

$$H\beta(\xi_1, \dots, \xi_{k-1}) = \int_0^1 t^{k-1} \beta(\phi, \xi_1, \dots, \xi_{k-1})(t\phi)\, dt.$$

Here $\phi, \xi_1, \dots, \xi_{k-1}$ are regarded as constant vector fields through $T\mathcal{H}_\omega \approx \mathcal{H}_\omega \times C^\infty(X)$.

To find $d(H\beta)(\xi_1, \dots, \xi_k)$, let us compute

$$\begin{aligned}
&\frac{d}{dt}\Big|_{t=0} H\beta(\xi_1, \dots, \hat{\xi}_j, \dots, \xi_k)(\phi + t\xi_j) \\
&\quad = \lim_{h\to 0} \frac{H\beta(\xi_1, \dots, \hat{\xi}_j, \dots, \xi_k)(\phi + h\xi_j) - H\beta(\xi_1, \dots, \hat{\xi}_j, \dots, \xi_k)(\phi)}{h} \\
&\quad = \lim_{h\to 0} \int_0^1 t^{k-1} \frac{\beta(\phi + h\xi_j, \xi_1, \dots, \hat{\xi}_j, \dots, \xi_k)(t\phi + th\xi_j) - \beta(\phi, \xi_1, \dots, \hat{\xi}_j, \dots, \xi_k)(t\phi)}{h}\, dt \\
&\quad = \lim_{h\to 0} \int_0^1 \biggl(t^{k-1} \frac{\beta(\phi, \xi_1, \dots, \hat{\xi}_j, \dots, \xi_k)(t\phi + th\xi_j) - \beta(\phi, \xi_1, \dots, \hat{\xi}_j, \dots, \xi_k)(t\phi)}{h} \\
&\qquad\qquad\qquad + t^{k-1} \beta(\xi_j, \xi_1, \dots, \hat{\xi}_j, \dots, \xi_k)(t\phi + th\xi_j) \biggr)\, dt.
\end{aligned}$$

As $(t, h) \mapsto \beta(\phi, \xi_1, \dots, \hat{\xi}_j, \dots, \xi_k)(t\phi + th\xi_j)$ and $(t, h) \mapsto \beta(\xi_j, \xi_1, \dots, \hat{\xi}_j, \dots, \xi_k)(t\phi + th\xi_j)$ are smooth, we can exchange the limit and integral and obtain

$$\begin{aligned}
&\int_0^1 t^{k-1} \frac{d}{dh}\Big|_{h=0} \beta(\phi, \xi_1, \dots, \hat{\xi}_j, \dots, \xi_k)(t\phi + th\xi_j) + t^{k-1} \beta(\xi_j, \xi_1, \dots, \hat{\xi}_j, \dots, \xi_k)(t\phi)\, dt \\
&\qquad = \int_0^1 t^k \frac{d}{dh}\Big|_{h=0} \beta(\phi, \xi_1, \dots, \hat{\xi}_j, \dots, \xi_k)(t\phi + h\xi_j) + (-1)^{j-1} t^{k-1} \beta(\xi_1, \dots, \xi_k)(t\phi)\, dt.
\end{aligned}$$

As a result, by Lemma 4.3,

$$d(H\beta)(\xi_1,\dots,\xi_k) = \sum_{j=1}^{k}(-1)^{j+1}\int_0^1 t^k \frac{d}{dh}\Big|_{h=0}\beta(\phi,\xi_1,\dots,\hat{\xi}_j,\dots,\xi_k)(t\phi+h\xi_j)+(-1)^{j-1}t^{k-1}\beta(\xi_1,\dots,\xi_k)(t\phi)\,dt.$$

On the other hand,

$$\begin{aligned}H(d\beta)(\xi_1,\dots,\xi_k) &= \int_0^1 t^k(d\beta)(\phi,\xi_1,\dots,\xi_k)(t\phi)\,dt\\ &= \int_0^1 t^k\Big(\sum_{j=1}^{k}(-1)^j\frac{d}{dh}\Big|_{h=0}\beta(\phi,\xi_1,\dots,\hat{\xi}_j,\dots,\xi_k)(t\phi+h\xi_j)+\frac{d}{dh}\Big|_{h=0}\beta(\xi_1,\dots,\xi_k)(t\phi+h\phi)\Big)\,dt,\end{aligned}$$

where the last equality is due to Lemma 4.3. Therefore

$$\begin{aligned}[d(H\beta)+H(d\beta)](\xi_1,\dots,\xi_k) &= \int_0^1 kt^{k-1}\beta(\xi_1,\dots,\xi_k)(t\phi)+t^k\frac{d}{dh}\Big|_{h=0}\beta(\xi_1,\dots,\xi_k)(t\phi+h\phi)\,dt\\ &= \int_0^1\frac{d}{dt}(t^k\beta(\xi_1,\dots,\xi_k)(t\phi))\,dt = \beta(\xi_1,\dots,\xi_k),\end{aligned}$$

and the lemma follows since $d\beta = 0$. □

Lemma 4.5. *The three-form θ is d-closed.*

Proof. This is similar to the derivation of the Aubin–Yau functional and the Mabuchi energy; see e.g., [Błocki 2013, Section 4]. Consider four vector fields ξ_1,ξ_2,ξ_3,ξ_4 on $\mathcal{H}_\omega$ which are constant in the canonical trivialization $T\mathcal{H}_\omega \approx \mathcal{H}_\omega\times C^\infty(X)$. By Lemma 4.3,

$$d\theta(\xi_1,\xi_2,\xi_3,\xi_4) = \xi_1\theta(\xi_2,\xi_3,\xi_4)-\xi_2\theta(\xi_1,\xi_3,\xi_4)+\xi_3\theta(\xi_1,\xi_2,\xi_4)-\xi_4\theta(\xi_1,\xi_2,\xi_3). \tag{22}$$

Using

$$\{\xi_3,\xi_4\}_{\omega_\phi}\omega_\phi^n = nd\xi_3\wedge d\xi_4\wedge\omega_\phi^{n-1} \quad\text{and}\quad \frac{d}{dt}\Big|_{t=0}\omega_{\phi+t\xi_1}^{n-1} = (n-1)i\partial\bar{\partial}\xi_1\wedge\omega_\phi^{n-2},$$

we have

$$\begin{aligned}\xi_1\theta(\xi_2,\xi_3,\xi_4) = \xi_1\theta(\xi_3,\xi_4,\xi_2) &= \frac{d}{dt}\Big|_{t=0}\theta(\xi_3,\xi_4,\xi_2)(\phi+t\xi_1)\\ &= \frac{d}{dt}\Big|_{t=0}\int_X\{\xi_3,\xi_4\}_{\omega_{\phi+t\xi_1}}\xi_2\omega_{\phi+t\xi_1}^n\\ &= \frac{d}{dt}\Big|_{t=0}\int_X\xi_2 nd\xi_3\wedge d\xi_4\wedge\omega_{\phi+t\xi_1}^{n-1}\\ &= \int_X\xi_2 nd\xi_3\wedge d\xi_4\wedge(n-1)i\partial\bar{\partial}\xi_1\wedge\omega_\phi^{n-2}\\ &= \int_X\xi_1 nd\xi_3\wedge d\xi_4\wedge(n-1)i\partial\bar{\partial}\xi_2\wedge\omega_\phi^{n-2} = \xi_2\theta(\xi_1,\xi_3,\xi_4),\end{aligned}$$

where the second to last equality is due to integration by parts. Because of the symmetry in index, all terms on the right side of (22) equal 0, and therefore $d\theta = 0$. $\square$

Since θ is d-closed, there exists a two-form α on $\mathcal{H}_\omega$ such that $d\alpha = \theta$ by Lemma 4.4. For a map $\Phi : D \to \mathcal{H}_\omega$, the derivative $\Phi_{z_j} = \frac{1}{2}(\Phi_{\operatorname{Re} z_j} - i\Phi_{\operatorname{Im} z_j})$ is a section of $\mathbb{C} \otimes T\mathcal{H}_\omega$ along Φ, and $\alpha(\Phi_{\bar{z}_j}, \Phi_{z_j})$ is a function on D. We define

$$\begin{aligned}\mathscr{E}(\Phi) :&= E(\Phi) + 4i \sum_j \int_D \alpha(\Phi_{\bar{z}_j}, \Phi_{z_j})\, dV \\ &= \int_D |\Phi_*|^2 dV + 4i \sum_j \int_D \alpha(\Phi_{\bar{z}_j}, \Phi_{z_j})\, dV,\end{aligned}$$

with dV the Euclidean volume form on D.

Lemma 4.6. *The Euler–Lagrange equation of $\mathscr{E}$ is*

$$\sum_{j=1}^{m} |\nabla \Phi_{z_j}|^2 - 2\Phi_{z_j\bar{z}_j} + i\{\Phi_{\bar{z}_j}, \Phi_{z_j}\}_{\omega_\Phi} = 0, \tag{23}$$

where $\nabla\Phi_{z_j}$ is the gradient of Φ_{z_j} with respect to the metric ω_Φ.

Proof. Let Ψ be a smooth map from D to $C^\infty(X)$ with compact support. The variational equation is

$$0 = \frac{d}{dt}\Big|_{t=0} \left(\int_D |(\Phi + t\Psi)_*|^2\, dV + 4i \sum_j \int_D \alpha((\Phi + t\Psi)_{\bar{z}_j}, (\Phi + t\Psi)_{z_j})\, dV \right). \tag{24}$$

An extension of the computation in [Donaldson 1999, Section 2] shows that the first term in (24) equals

$$\frac{d}{dt}\Big|_{t=0} \int_D |(\Phi + t\Psi)_*|^2\, dV = \int_D \int_X 4\Big(\sum_j |\nabla \Phi_{z_j}|^2 - 2 \sum_j \Phi_{z_j\bar{z}_j} \Big) \Psi \omega_\Phi^n\, dV. \tag{25}$$

The remaining task is to compute the second term in (24).

To that end, we denote $C^\infty(X, \mathbb{C})$ by $C^\infty_{\mathbb{C}}(X)$ and introduce $A : \mathcal{H}_\omega \times C^\infty_{\mathbb{C}}(X) \times C^\infty_{\mathbb{C}}(X) \to \mathbb{C}$ as follows. If $(u, \xi), (u, \eta) \in \mathcal{H}_\omega \times C^\infty_{\mathbb{C}}(X) \approx \mathbb{C} \otimes T\mathcal{H}_\omega$, then $A(u, \xi, \eta) := \alpha((u, \xi), (u, \eta))$. Therefore, for fixed small $t \in \mathbb{R}$, $\alpha((\Phi + t\Psi)_{\bar{z}_j}, (\Phi + t\Psi)_{z_j}) = A(\Phi + t\Psi, (\Phi + t\Psi)_{\bar{z}_j}, (\Phi + t\Psi)_{z_j})$ maps from D to $\mathbb{C}$. By the chain rule,

$$\begin{aligned}&\frac{d}{dt}\Big|_{t=0} A(\Phi + t\Psi, (\Phi + t\Psi)_{\bar{z}_j}, (\Phi + t\Psi)_{z_j}) \\ &\qquad = d_1 A(\Phi, \Phi_{\bar{z}_j}, \Phi_{z_j})(\Psi) + d_2 A(\Phi, \Phi_{\bar{z}_j}, \Phi_{z_j})(\Psi_{\bar{z}_j}) + d_3 A(\Phi, \Phi_{\bar{z}_j}, \Phi_{z_j})(\Psi_{z_j}),\end{aligned} \tag{26}$$

where $d_1 A$, $d_2 A$, and $d_3 A$ are partial differentials of A. Since A is linear in the second and third variables, $d_2 A(\Phi, \Phi_{\bar{z}_j}, \Phi_{z_j})(\Psi_{\bar{z}_j}) = A(\Phi, \Psi_{\bar{z}_j}, \Phi_{z_j})$ and $d_3 A(\Phi, \Phi_{\bar{z}_j}, \Phi_{z_j})(\Psi_{z_j}) = A(\Phi, \Phi_{\bar{z}_j}, \Psi_{z_j})$. Hence the right side of (26) becomes

$$d_1 A(\Phi, \Phi_{\bar{z}_j}, \Phi_{z_j})(\Psi) + A(\Phi, \Psi_{\bar{z}_j}, \Phi_{z_j}) + A(\Phi, \Phi_{\bar{z}_j}, \Psi_{z_j}). \tag{27}$$

By similar computations,

$$\begin{aligned}\frac{\partial}{\partial \bar z_j}A(\Phi,\Psi,\Phi_{z_j}) &= d_1A(\Phi,\Psi,\Phi_{z_j})(\Phi_{\bar z_j})+A(\Phi,\Psi_{\bar z_j},\Phi_{z_j})+A(\Phi,\Psi,\Phi_{z_j\bar z_j}),\\ \frac{\partial}{\partial z_j}A(\Phi,\Phi_{\bar z_j},\Psi) &= d_1A(\Phi,\Phi_{\bar z_j},\Psi)(\Phi_{z_j})+A(\Phi,\Phi_{\bar z_j z_j},\Psi)+A(\Phi,\Phi_{\bar z_j},\Psi_{z_j}).\end{aligned}\tag{28}$$

So integration by parts gives

$$\begin{aligned}\int_D A(\Phi,\Psi_{\bar z_j},\Phi_{z_j})\,dV &= -\int_D (d_1A(\Phi,\Psi,\Phi_{z_j})(\Phi_{\bar z_j})+A(\Phi,\Psi,\Phi_{z_j\bar z_j}))\,dV,\\ \int_D A(\Phi,\Phi_{\bar z_j},\Psi_{z_j})\,dV &= -\int_D (d_1A(\Phi,\Phi_{\bar z_j},\Psi)(\Phi_{z_j})+A(\Phi,\Phi_{\bar z_j z_j},\Psi))\,dV.\end{aligned}\tag{29}$$

Combining (27) and (29), we find

$$\begin{aligned}&\frac{d}{dt}\Big|_{t=0}\int_D \alpha((\Phi+t\Psi)_{\bar z_j},(\Phi+t\Psi)_{z_j})\,dV\\ &\qquad=\int_D d_1A(\Phi,\Phi_{\bar z_j},\Phi_{z_j})(\Psi)-d_1A(\Phi,\Psi,\Phi_{z_j})(\Phi_{\bar z_j})-d_1A(\Phi,\Phi_{\bar z_j},\Psi)(\Phi_{z_j})\,dV.\end{aligned}\tag{30}$$

For a fixed point $z_0\in D$, let $\Psi(z_0)$, $\Phi_{\bar z_j}(z_0)$, and $\Phi_{z_j}(z_0)$ define three constant vector fields on $\mathcal{H}_\omega$ denoted by ξ_1, ξ_2, and ξ_3, respectively. By Lemma 4.3,

$$d\alpha(\xi_1,\xi_2,\xi_3)=\xi_1\alpha(\xi_2,\xi_3)-\xi_2\alpha(\xi_1,\xi_3)+\xi_3\alpha(\xi_1,\xi_2).$$

Meanwhile, for constant vector fields ξ_1, ξ_2, and ξ_3, the function $\xi_1\alpha(\xi_2,\xi_3)$ evaluated at $u\in\mathcal{H}_\omega$ is $d_1A(u,\xi_2,\xi_3)(\xi_1)$. So at $\Phi(z_0)\in\mathcal{H}_\omega$,

$$\begin{aligned}d\alpha(\xi_1,\xi_2,\xi_3) &= d_1A(\Phi(z_0),\xi_2,\xi_3)(\xi_1)-d_1A(\Phi(z_0),\xi_1,\xi_3)(\xi_2)+d_1A(\Phi(z_0),\xi_1,\xi_2)(\xi_3)\\ &= d_1A(\Phi(z_0),\xi_2,\xi_3)(\xi_1)-d_1A(\Phi(z_0),\xi_1,\xi_3)(\xi_2)-d_1A(\Phi(z_0),\xi_2,\xi_1)(\xi_3).\end{aligned}\tag{31}$$

Hence (30) becomes

$$\int_D d\alpha(\Psi,\Phi_{\bar z_j},\Phi_{z_j})\,dV=\int_D \theta(\Psi,\Phi_{\bar z_j},\Phi_{z_j})\,dV=\int_D\int_X \{\Phi_{\bar z_j},\Phi_{z_j}\}_{\omega_\Phi}\Psi\omega_\Phi^n\,dV.\tag{32}$$

Finally, with (25) and (32), the variational equation (24) becomes

$$0=\int_D\int_X\Big(4\Big(\sum_j|\nabla\Phi_{z_j}|^2-2\sum_j\Phi_{z_j\bar z_j}\Big)+4i\sum_j\{\Phi_{\bar z_j},\Phi_{z_j}\}_{\omega_\Phi}\Big)\Psi\omega_\Phi^n\,dV,\tag{33}$$

and we obtain the Euler–Lagrange equation

$$\sum_j|\nabla\Phi_{z_j}|^2-2\sum_j\Phi_{z_j\bar z_j}+i\sum_j\{\Phi_{\bar z_j},\Phi_{z_j}\}_{\omega_\Phi}=0.$$ □

5. Lemma 2.2

This section is mainly devoted to the proof of Lemma 2.2, and we will follow closely the ideas in [Błocki and Kołodziej 2007]. The first two lemmas, concerning smooth approximation of continuous η-subharmonic functions, are based on the exposition in [Demailly 2012, Chapter I, Section 5E] of [Richberg 1968]. See also [Demailly 1992].

Let $\theta \in C^\infty(\mathbb{R}, \mathbb{R})$ be a nonnegative function having support in $[-1, 1]$ with $\int_{\mathbb{R}} \theta(h)\, dh = 1$ and $\int_{\mathbb{R}} h\theta(h)\, dh = 0$. For arbitrary $\xi = (\xi_1, \ldots, \xi_p) \in (0, \infty)^p$, the regularized maximal function is

$$M_\xi(t_1, \ldots, t_p) := \int_{\mathbb{R}^n} \max\{t_1 + h_1, \ldots, t_p + h_p\} \prod_{j=1}^{n} \theta\left(\frac{h_j}{\xi_j}\right) \frac{dh_1}{\xi_1} \cdots \frac{dh_p}{\xi_p}.$$

Lemma 5.1. *Fix a closed smooth positive $(1, 1)$-form η on X. Let $\Omega_\alpha \Subset D \times X$ be a locally finite open cover of $D \times X$, let c be a real number, and let $u_\alpha \in C^\infty(\overline{\Omega}_\alpha)$ such that $u_\alpha(z, x) + c|z|^2$ is η-subharmonic on graphs. Assume that there exists a family $\{\xi_\alpha\}$ of positive numbers such that, for all β and $(z, x) \in \partial\Omega_\beta$,*

$$u_\beta(z, x) + \xi_\beta \le \max_{\alpha:(z,x)\in\Omega_\alpha} \{u_\alpha(z, x) - \xi_\alpha\}.$$

Define a function $\tilde{u}$ on $D \times X$ as follows. Given $(z, x) \in D \times X$, let $A = \{\alpha : (z, x) \in \Omega_\alpha\}$, $\xi_A = (\xi_\alpha)_{\alpha\in A}$, $u_A(z, x) = \{u_\alpha(z, x) : \alpha \in A\}$, and

$$\tilde{u}(z, x) := M_{\xi_A}(u_A(z, x)).$$

Then $\tilde{u}$ is in $C^\infty(D \times X)$ and $\tilde{u}(z, x) + c|z|^2$ is η-subharmonic on graphs.

Proof. As in the proof of [Demailly 2012, Chapter I, Lemma 5.17 and Corollary 5.19], one can deduce that for a fixed point in $D \times X$, there exist a neighborhood V and a finite set I of indices α such that $V \subset \bigcap_{\alpha\in I} \Omega_\alpha$ and on which $\tilde{u} = M_{\xi_I}(u_I)$. As a result, by [Demailly 2012, Lemma 5.18 (a)], $\tilde{u}$ is smooth on $D \times X$. Now for a holomorphic map f from an open subset of D to X, we have

$$\begin{aligned}\tilde{u}(z, f(z)) + c|z|^2 + \psi(f(z)) &= c|z|^2 + \psi(f(z)) + M_{\xi_I}(u_I(z, f(z))) \\ &= M_{\xi_I}(c|z|^2 + \psi(f(z)) + u_I(z, f(z))),\end{aligned}$$

where $\eta = i\partial\bar{\partial}\psi$ and we use [Demailly 2012, Lemma 5.18 (d)] in the last equality. Furthermore, since $c|z|^2 + \psi(f(z)) + u_\alpha(z, f(z))$ is subharmonic by assumption, so is $M_{\xi_I}(c|z|^2 + \psi(f(z)) + u_I(z, f(z)))$ by [Demailly 2012, Lemma 5.18 (a)], and therefore $\tilde{u} + c|z|^2$ is η-subharmonic on graphs. □

We introduce here some notation that will be used later. Let ρ_1 and ρ_2 be kernels (i.e., nonnegative radial smooth functions with support in the unit ball and having integral one) in $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively. For $\varepsilon > 0$, write $\rho_{1,\varepsilon}(\cdot) := \varepsilon^{-m}\rho_1(\cdot/\varepsilon)$, and let $\rho_{2,\varepsilon}$ be similarly defined.

The proof of the following lemma is very similar to that of [Demailly 2012, Chapter 1, Theorem 5.21].

Lemma 5.2. *Let $u \in C(D \times X)$ be η-subharmonic on graphs. For any number $\lambda > 0$, there exists $\tilde{u} \in C^\infty(D \times X)$ such that $u \le \tilde{u} \le u + M\lambda$, where M depends only on the diameter of D and $\tilde{u}$ is $(1 + \lambda)\eta$-subharmonic on graphs.*

Proof. Let $\{\Omega_\alpha\}$ be a locally finite open cover of $D \times X$ by relatively compact open balls contained in coordinate patches of $D \times X$. Choose concentric balls $\Omega''_\alpha \subset \Omega'_\alpha \subset \Omega_\alpha$ of radii $r''_\alpha < r'_\alpha < r_\alpha$ and center $(c_\alpha, 0)$ in the given coordinates (z, x) near $\overline{\Omega}_\alpha$, such that the Ω''_α still cover $D \times X$ and η has a local potential ψ_α in a neighborhood of $\overline{\Omega}_\alpha$. For small $\varepsilon_\alpha > 0$ and $\delta_\alpha > 0$, we set

$$u_\alpha(z, x) = ((u + \psi_\alpha) * \rho_{\varepsilon_\alpha})(z, x) - \psi_\alpha(x) + \delta_\alpha(r'^2_\alpha - |z - c_\alpha|^2 - |x|^2) \quad \text{on } \overline{\Omega}_\alpha,$$

where $*\rho_{\varepsilon_\alpha}$ is the convolution with $\rho_{\varepsilon_\alpha} := \rho_{1,\varepsilon_\alpha}\rho_{2,\varepsilon_\alpha}$. Since $\psi_\alpha(x) + u(z, x)$ is subharmonic in z and psh in x by Lemma 3.1, the functions $(\psi_\alpha + u) * \rho_{\varepsilon_\alpha}$ decrease to $\psi_\alpha + u$ as ε_α goes to 0, locally uniformly because u is continuous. For ε_α and δ_α small enough, we have $u_\alpha \le u + \frac{1}{2}\lambda$ on $\overline{\Omega}_\alpha$. Moreover, for any holomorphic map f from an open subset of D to X,

$$\begin{aligned}\Delta(u_\alpha(z, f(z)) + \psi_\alpha(f(z))) &= \Delta((u + \psi_\alpha) * \rho_{\varepsilon_\alpha})(z, f(z)) - \delta_\alpha \Delta(|z - c_\alpha|^2 + |f(z)|^2) \\ &\ge -\delta_\alpha \Delta(|z - c_\alpha|^2 + |f(z)|^2) \\ &\ge -\lambda\Delta|z|^2 - \lambda\Delta\psi_\alpha(f(z)),\end{aligned}$$

where the first inequality is due to the fact that $(u + \psi_\alpha) * \rho_{\varepsilon_\alpha}$ is subharmonic on holomorphic graphs, which can be verified easily because $(u + \psi_\alpha)$ is subharmonic on holomorphic graphs (or see the proof of Lemma 2.2 where we provide such verification). So $u_\alpha(z, x) + \lambda|z|^2$ is $(1 + \lambda)\eta$-subharmonic on graphs. Set

$$\xi_\alpha = \delta_\alpha \min\bigl\{r'^2_\alpha - r''^2_\alpha, \tfrac{1}{2}(r^2_\alpha - r'^2_\alpha)\bigr\}.$$

Choose first δ_α such that $\xi_\alpha < \frac{1}{2}\lambda$, and then ε_α so small that $u \le (u + \psi_\alpha) * \rho_{\varepsilon_\alpha}(z, x) - \psi_\alpha(x) < u + \xi_\alpha$ on $\overline{\Omega}_\alpha$. As $\delta_\alpha(r'^2_\alpha - |z - c_\alpha|^2 - |x|^2)$ is less than or equal to $-2\xi_\alpha$ on $\partial\Omega_\alpha$ and greater than ξ_α on Ω''_α, we have $u_\alpha < u - \xi_\alpha$ on $\partial\Omega_\alpha$ and $u_\alpha > u + \xi_\alpha$ on Ω''_α, so that the assumption in Lemma 5.1 is satisfied. Also, the function

$$U(z, x) := M_{\xi_A}(u_A(z, x)), \quad \text{for } A = \{\alpha : \Omega_\alpha \ni (z, x)\},$$

is in $C^\infty(D \times X)$ and $U(z, x) + \lambda|z|^2$ is $(1 + \lambda)\eta$-subharmonic on graphs. Then we have $u \le U \le u + \lambda$ by [Demailly 2012, Lemma 5.18 (b)], and the function defined by $\tilde{u} := U + \lambda|z|^2$ is what we need. □

The following lemma is proved in the same way as Lemmas 4 and 5 in [Błocki and Kołodziej 2007]. The only issue is keeping track of uniformity.

Lemma 5.3. *Let U, V be two open sets in $\mathbb{C}^n$ and F a biholomorphic map from U to V. Let u be usc, bounded, and subharmonic on holomorphic graphs in $D \times U$. Define the convolution*

$$u_{\delta_1,\delta_2}(z, x) = \int_{\mathbb{C}^n} \int_{\mathbb{C}^m} u(z - a, x - b)\rho_{1,\delta_1}(a)\rho_{2,\delta_2}(b)\, da\, db,$$

where ρ_{1,δ_1} and ρ_{2,δ_2} are kernels in $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively. On the other hand, define

$$u^F_{\delta_1,\delta_2}(z, x) = (u \circ (\mathrm{Id} \times F^{-1}))_{\delta_1,\delta_2} \circ (\mathrm{Id} \times F). \tag{34}$$

Then $(u^F_{\delta_1,\delta_2} - u_{\delta_1,\delta_2})(z, x) \to 0$ locally uniformly in z, x, and δ_1 as $\delta_2 \to 0$.

Proof. Define

$$\hat{u}_{\delta_2}(z,x) = \max_{\{z\}\times\overline{B(x,\delta_2)}} u,$$

$$\tilde{u}_{\delta_2}(z,x) = ⨍_{\partial B(x,\delta_2)} u(z,b)\,db,$$

$$u_{\delta_2}(z,x) = \int_{\mathbb{C}^n} u(z,x-b)\rho_{2,\delta_2}(b)\,db,$$

where ⨍ means the average. Their counterparts under $\mathrm{Id}\times F^{-1}$ and $\mathrm{Id}\times F$ as in (34) are denoted by $\hat{u}^F_{\delta_2}(z,x)$, $\tilde{u}^F_{\delta_2}(z,x)$, and $u^F_{\delta_2}(z,x)$, respectively.

By Lemma 3.1, $u(z,\cdot)$ is psh in U, so $\hat{u}_{\delta_2}(z,x)$ is a convex function of $\log\delta_2$. Fixing $a \ge 1$ and $r > 0$, choose δ_2 so small that $0 \le (\log a)/(\log(r/\delta_2)) \le 1$. Then by convexity,

$$0 \le \hat{u}_{a\delta_2}(z,x) - \hat{u}_{\delta_2}(z,x) \le \frac{\log a}{\log(r/\delta_2)}(\hat{u}_r(z,x) - \hat{u}_{\delta_2}(z,x)).$$

Since u is assumed to be bounded, it follows that for any $a > 0$ (for the case $1 > a > 0$, use $1/a$ instead), $\hat{u}_{a\delta_2}(z,x) - \hat{u}_{\delta_2}(z,x)$ goes to 0 as $\delta_2 \to 0$, locally uniformly in z and x. Then following the same argument as in [Błocki and Kołodziej 2007, Lemma 4], we see $\hat{u}^F_{\delta_2} - \hat{u}_{\delta_2}$ goes to 0 locally uniformly in z and x, as $\delta_2 \to 0$.

Since $u(z,\cdot)$ is psh in U, $\tilde{u}_{\delta_2}(z,x)$ is convex in $\log\delta_2$. By the argument of [Błocki and Kołodziej 2007, Lemma 5] and the fact that u is bounded, we see both $\hat{u}_{\delta_2} - \tilde{u}_{\delta_2}$ and $\tilde{u}_{\delta_2} - u_{\delta_2}$ go to 0 locally uniformly in z, x, as $\delta_2 \to 0$, and as a result, so does $u^F_{\delta_2} - u_{\delta_2}$. Since $(u^F_{\delta_1,\delta_2} - u_{\delta_1,\delta_2})$ is the convolution of $(u^F_{\delta_2} - u_{\delta_2})$ in z, we see at once the conclusion of the lemma. □

Proof of Lemma 2.2. Fix a finite number of charts $U_\alpha \Supset V_\alpha$ such that V_α covers X, and η has a local potential ψ_α in a neighborhood of $\overline{U}_\alpha$. For each α, let $f_\alpha : U_\alpha \to \mathbb{C}^n$ be the coordinate map, we consider the convolution $((\psi_\alpha + u)\circ f_\alpha^{-1})_{\delta_1,\delta_2} \circ f_\alpha$, which we simply denote by $(\psi_\alpha + u)_{\delta_1,\delta_2}$ on $D \times U_\alpha$. Because u added by a constant still satisfies the same assumption in Lemma 2.2, we will assume u is so negative that $(\psi_\alpha + u)_{\delta_1,\delta_2} - \psi_\alpha < -a$ for some $a > 0$ and all α. At the same time, we consider the convolution of $(\psi_\alpha + u)$ under f_β, namely $((\psi_\alpha + u)\circ f_\beta^{-1})_{\delta_1,\delta_2} \circ f_\beta$, which can be written as

$$((\psi_\alpha + u)\circ f_\alpha^{-1} \circ F^{-1})_{\delta_1,\delta_2} \circ F \circ f_\alpha, \tag{35}$$

if $F^{-1} = f_\alpha \circ f_\beta^{-1}$. We denote (35) by $(\psi_\alpha + u)^F_{\delta_1,\delta_2}$ (the notation is consistent with Lemma 5.3 except we do not write out the identity map of D here). By Lemma 5.3 on $D \times (U_\alpha \cap U_\beta)$

$$(\psi_\alpha + u)_{\delta_1,\delta_2} - (\psi_\beta + u)_{\delta_1,\delta_2} = (\psi_\alpha + u)_{\delta_1,\delta_2} - (\psi_\alpha + u)^F_{\delta_1,\delta_2} + (\psi_\alpha + u - (\psi_\beta + u))^F_{\delta_1,\delta_2} \to \psi_\alpha - \psi_\beta \tag{36}$$

locally uniformly in z and x, as $\delta_2, \delta_1 \to 0$.

Let χ_α be a smooth function in U_α that is 0 in V_α and -1 near ∂U_α. We have $i\partial\bar{\partial}\chi_\alpha \ge -C\eta$ for some constant C. For $0 < \varepsilon < 1$, according to (36) we can find δ_1, δ_2 small enough such that for any β and for any $(z,x) \in \overline{D}' \times \partial U_\beta$,

$$\Big((\psi_\beta + u)_{\delta_1,\delta_2} - \psi_\beta + \frac{\varepsilon}{C}\chi_\beta\Big)(z,x) < \max_{(z,x)\in\overline{D}'\times U_\alpha} \Big((\psi_\alpha + u)_{\delta_1,\delta_2} - \psi_\alpha + \frac{\varepsilon}{C}\chi_\alpha\Big)(z,x),$$

where the maximum is taken over all $\bar{D}' \times U_\alpha$ that contain (z,x). Let $\delta = \min\{\delta_1, \delta_2\}$. Then by [Demailly 2012, Chapter I, Lemma 5.17], the function

$$u_\delta^\varepsilon(z,x) := \max_{(z,x)\in \bar{D}'\times U_\alpha} \Big((\psi_\alpha + u)_{\delta,\delta} - \psi_\alpha + \frac{\varepsilon}{C}\chi_\alpha\Big)(z,x)$$

is continuous on $\bar{D}' \times X$. Notice that $u_\delta^\varepsilon(z,x) < -a$ for any $0 < \varepsilon < 1$. Since $\psi_\alpha(x) + u(z,x)$ is subharmonic in z and psh in x by Lemma 3.1, the function $(\psi_\alpha + u)_{\delta,\delta}$ is decreasing to $\psi_\alpha + u$ as $\delta \to 0$, and hence u_δ^ε is decreasing to u as $\delta \to 0$.

We already know that $\psi_\alpha + u$ is subharmonic on holomorphic graphs, and in this paragraph we will show this is also true for $(\psi_\alpha + u)_{\delta,\delta}$. Let us denote $\psi_\alpha + u$ by G momentarily: We want to show that, for any holomorphic map g from an open subset of D to U_α, the function $G_{\delta,\delta}(z, g(z))$ is subharmonic. Indeed, since G is bounded on $D \times U_\alpha$, the convolution $G_{\delta,\delta}$ is smooth and so $G_{\delta,\delta}(z, g(z))$ is usc. The map $w \mapsto G(w, g(w+a) - b)$ is subharmonic, therefore the mean-value inequality says

$$G(z-a, g(z)-b) \le \fint_{B(z-a,r)} G(w, g(w+a)-b)\, dw.$$

So,

$$\begin{aligned} G_{\delta,\delta}(z, g(z)) &\le \int_{\mathbb{C}^n}\int_{\mathbb{C}^m}\fint_{B(z-a,r)} G(w, g(w+a)-b)\, dw \rho_{1,\delta}(a)\rho_{2,\delta}(b)\, da\, db \\ &= \int_{\mathbb{C}^n}\int_{\mathbb{C}^m}\fint_{B(z,r)} G(W-a, g(W)-b)\, dW \rho_{1,\delta}(a)\rho_{2,\delta}(b)\, da\, db \\ &= \fint_{B(z,r)} G_{\delta,\delta}(W, g(W))\, dW; \end{aligned}$$

the use of Fubini's theorem is justified since G is bounded on $D \times U_\alpha$. As a result, $G_{\delta,\delta}(z, g(z))$ is subharmonic.

The fact that $(\psi_\alpha + u)_{\delta,\delta}$ is subharmonic on holomorphic graphs together with $(\chi_\alpha)_{\lambda\bar{\mu}} \ge -C(\psi_\alpha)_{\lambda\bar{\mu}}$ as matrices, shows, for any holomorphic function f from an open subset of D' to X,

$$\Delta\Big((\psi_\alpha + u)_{\delta,\delta} - \psi_\alpha + \frac{\varepsilon}{C}\chi_\alpha\Big)(z, f(z)) \ge (-1-\varepsilon)\Delta\psi_\alpha(f(z)),$$

so u_δ^ε is $(1+\varepsilon)\eta$-subharmonic on graphs.

So far we have shown that given $1 < p \in \mathbb{N}$, there exists $q_0 \in \mathbb{N}$ such that, for $q > q_0$, the functions $u_{1/q}^{1/p}$ are in $C(\bar{D}' \times X)$, $(1+1/p)\eta$-subharmonic on graphs, and decrease to u as $q \to \infty$. For simplicity, we will denote $u_{1/q}^{1/p}$ by u_q^p. Let M be the constant in Lemma 5.2. We will construct $u_{j_k}^k$ inductively with $j_k > k^2$ and $\tilde{u}_k \in C^\infty(D' \times X)$ such that

$$u_{j_k}^k + \frac{1}{j_k} \le \tilde{u}_k \le u_{j_k}^k + \frac{1}{j_k} + \frac{M}{j_k}. \tag{37}$$

Moreover, $\tilde{u}_k$ is $(1+1/k)(1+1/j_k)\eta$-subharmonic on graphs, and $u_{j_k}^k + 1/j_k + M/j_k$ is less than both $u_{j_{k-1}}^{k-1} + 1/j_{k-1}$ and $u_{j_{k-1}}^2 + 1/j_{k-1}$.

Suppose that this is true at the $(k-1)$-th step. As $u_{j_{k-1}}^{k-1} + 1/j_{k-1}$ and $u_{j_{k-1}}^2 + 1/j_{k-1}$ are both greater than u, we can find $j_k > \max\{j_{k-1}, k^2\}$ such that $u_{j_k}^k + 1/j_k + M/j_k$ is less than both $u_{j_{k-1}}^{k-1} + 1/j_{k-1}$

and $u^2_{j_{k-1}} + 1/j_{k-1}$ by continuity on the compact set $\bar{D}' \times X$. We can then find a function $\tilde{u}_k \in C^\infty(D' \times X)$ with $u^k_{j_k} + 1/j_k \le \tilde{u}_k \le u^k_{j_k} + 1/j_k + M/j_k$ and where $\tilde{u}_k$ is $(1+1/k)(1+1/j_k)\eta$-subharmonic on graphs by applying Lemma 5.2 with $\lambda = 1/j_k$. So the induction process is true at the k-th step. (One can begin the induction process with $u^2_{j_2} + 1/j_2$ with j_2 large enough such that $u^2_{j_2} + 1/j_2 < 0$.)

One can see that $\tilde{u}_k$ is decreasing to u. Since $\tilde{u}_k < 0$, we have that $(1-1/k)\tilde{u}_k$ is still decreasing to u. The function $(1-1/k)\tilde{u}_k$ is $(1-1/k^2)(1+1/j_k)\eta$-subharmonic on graphs, and, because $j_k > k^2$, is also $(1-1/k^2 j_k)\eta$-subharmonic on graphs. So the $(1-1/k)\tilde{u}_k$ are the desired approximants. □

6. A remark

In this final section, we compare results in this paper to those in [Darvas and Wu 2019], where the author and Darvas consider two other families closely related to G_v and G^k_v. For $\pi : D \times X \to X$, define

$$F_v := \Big\{ u : u \in \mathrm{PSH}(D \times X, \pi^*\omega),\ \limsup_{D \ni z \to \zeta \in \partial D} u(z, x) \le v(\zeta, x) \Big\},$$

$$F^k_v := \Big\{ D \ni z \to U_z \in \mathcal{N}^*_k \text{ is Griffiths negative},\ \limsup_{D \ni z \to \zeta \in \partial D} U^2_z(s) \le H^*_k(v_\zeta)(s, s) \text{ for any } s \in H^0(X, L^k)^* \Big\},$$

where a norm function U_z is called Griffiths negative if $\log U_z(f(z))$ is psh for any holomorphic section $f : W \subset D \to H^0(X, L^k)^*$. Denote the upper envelopes of F_v and F^k_v by U and U^k, respectively. Then one result in [Darvas and Wu 2019] is that $FS_k((U^k_z)^*)$ converges to U uniformly.

The transition from the aforementioned paper to this paper is the change of plurisubharmonicity to subharmonicity, as one can see when comparing the definitions of F^k_v and G^k_v. Such a change between F_v and G_v is a little more subtle, and it can be seen as follows. Let ψ be a local potential of ω. Then a function $u \in \mathrm{PSH}(D \times X, \pi^*\omega)$ is equivalent to $\psi(x) + u(z, x)$ being psh in z and x jointly, which is also equivalent to $\psi(f(z)) + u(z, f(z))$ being psh for any holomorphic function $f : U \subset D \to X$ (see Lemma 6.1 below); therefore we see the change from F_v to G_v is again plurisubharmonicity to subharmonicity. Also notice that when $\dim D = 1$, Theorem 1.2 and the result in [Darvas and Wu 2019] are the same because $F_v = G_v$ and $F^k_v = G^k_v$.

Lemma 6.1. *Let Ω_1 and Ω_2 be open sets in $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively. If $u(z, \xi)$ is an usc function on $\Omega_1 \times \Omega_2$ such that $u(z, s(z))$ is psh for any holomorphic map s from an open subset of Ω_1 to Ω_2, then u is psh on $\Omega_1 \times \Omega_2$.*

Proof. We want to show that u is subharmonic on any complex line in $\Omega_1 \times \Omega_2$, and it suffices to consider the line $\mathbb{C} \ni \lambda \mapsto (\lambda z_0, \lambda \xi_0)$ where $(z_0, \xi_0) \in \Omega_1 \times \Omega_2$. In the case when z_0 and ξ_0 are both nonzero, we may assume $z_0 = (1, 0, \ldots, 0)$ and $\xi_0 = (1, 0, \ldots, 0)$. Let $G : \Omega_1 \to \mathbb{C}$ be the projection on the first coordinate, and let $F : \mathbb{C} \to \Omega_2$ be the injection to the first coordinate. By assumption, $u(z, F \circ G(z))$ is psh, so the function $\lambda \mapsto u(\lambda z_0, F \circ G(\lambda z_0)) = u(\lambda z_0, \lambda \xi_0)$ is subharmonic.

If $\xi_0 = 0$, then the function $\lambda \mapsto u(\lambda z_0, 0)$ is of course subharmonic. The final case is $z_0 = 0$ and $\xi_0 = (1, 0, \ldots, 0)$, and we need to show the function

$$\lambda \mapsto u(0, \ldots, 0; \lambda, 0, \ldots, 0)$$

is subharmonic, where the semicolon “;” in the argument is to separate the variables of $\mathbb{C}^m$ and $\mathbb{C}^n$. Given $\varepsilon > 0$ and $a \in \mathbb{C}$, the function $z \mapsto u(z_1, \dots, z_m; z_1/\varepsilon + a, 0, \dots, 0)$ is psh, so its restriction to the complex line $\lambda \mapsto ((\lambda - a)\varepsilon, 0, \dots, 0)$ is subharmonic; namely, $\lambda \mapsto u((\lambda - a)\varepsilon, 0, \dots, 0; \lambda, 0, \dots, 0)$ is subharmonic. Hence,

$$u(0, \dots, 0; a, 0, \dots, 0) \le \fint_{\partial B(a,r)} u((\lambda - a)\varepsilon, 0, \dots, 0; \lambda, 0, \dots, 0)\, d\lambda,$$

for $r > 0$. By Fatou's lemma and the fact that u is usc,

$$\limsup_{\varepsilon \to 0} \int_{\partial B(a,r)} u((\lambda - a)\varepsilon, 0, \dots, 0; \lambda, 0, \dots, 0)\, d\lambda \le \int_{\partial B(a,r)} u(0, 0, \dots, 0; \lambda, 0, \dots, 0)\, d\lambda.$$

As a result,

$$u(0, \dots, 0; a, 0, \dots, 0) \le \fint_{\partial B(a,r)} u(0, \dots, 0; \lambda, 0, \dots, 0)\, d\lambda. \qquad \square$$

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KUANG-RU WU: krwu@gate.sinica.edu.tw
Institute of Mathematics, Academia Sinica, Taipei, Taiwan

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