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THE STRONG TOPOLOGY OF ω-PLURISUBHARMONIC FUNCTIONS

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On a compact Kähler manifold (X, ω), given a model-type envelope $\psi \in \mathrm{PSH}(X, \omega)$ (i.e., a singularity type) we prove that the Monge–Ampère operator is a homeomorphism between the set of ψ-relative finite energy potentials and the set of ψ-relative finite energy measures endowed with their strong topologies given as the coarsest refinements of the weak topologies such that the relative energies become continuous. Moreover, given a totally ordered family $\mathcal{A}$ of model-type envelopes with positive total mass representing different singularity types, the sets $X_{\mathcal{A}}$ and $Y_{\mathcal{A}}$, given as the union of all ψ-relative finite energy potentials and of all ψ-relative finite energy measures with varying $\psi \in \bar{\mathcal{A}}$, respectively, have two natural strong topologies which extend the strong topologies on each component of the unions. We show that the Monge–Ampère operator produces a homeomorphism between $X_{\mathcal{A}}$ and $Y_{\mathcal{A}}$.

As an application we also prove the strong stability of a sequence of solutions of complex Monge–Ampère equations when the measures have uniformly L^p-bounded densities for $p > 1$ and the prescribed singularities are totally ordered.

1. Introduction

Let (X, ω) be a compact Kähler manifold where ω is a fixed Kähler form, and let $\mathcal{H}_\omega$ denote the set of all Kähler potentials, i.e., all $\varphi \in C^\infty$ such that $\omega + dd^c\varphi$ is a Kähler form. The pioneering work of Yau [1978] shows that the Monge–Ampère operator

$$\begin{aligned} \mathrm{MA}_\omega : \mathcal{H}_{\omega,\mathrm{norm}} &\to \left\{ dV \text{ volume form} : \int_X dV = \int_X \omega^n \right\}, \\ \mathrm{MA}_\omega(\varphi) &:= (\omega + dd^c\varphi)^n, \end{aligned} \tag{1}$$

is a bijection, where for any subset $A \subset \mathrm{PSH}(X, \omega)$ of all ω-plurisubharmonic functions, we use the notation $A_{\mathrm{norm}} := \{u \in A : \sup_X u = 0\}$. Note that the assumption on the total mass of the volume forms in (1) is necessary since $\mathcal{H}_{\omega,\mathrm{norm}}$ represents all Kähler forms in the cohomology class $\{\omega\}$ and the quantity $\int_X \omega^n$ is cohomological.

In [Guedj and Zeriahi 2007] the authors extended the Monge–Ampère operator using the *nonpluripolar product* (as defined successively in [Boucksom et al. 2010]) and the bijection (1) to

$$\mathrm{MA}_\omega : \mathcal{E}_{\mathrm{norm}}(X, \omega) \to \left\{ \mu \text{ nonpluripolar positive measure} : \mu(X) = \int_X \omega^n \right\}, \tag{2}$$

where $\mathcal{E}(X, \omega) := \left\{ u \in \mathrm{PSH}(X, \omega) : \int_X \mathrm{MA}_\omega(u) = \int_X \mathrm{MA}_\omega(0) \right\}$ is the set of all ω-psh functions with full Monge–Ampère mass.

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The set $\mathrm{PSH}(X,\omega)$ is naturally endowed with the L^1-topology which we will call *weak*, but the Monge–Ampère operator in (2) is not continuous even if the set of measures is endowed with the weak topology. Thus in [Berman et al. 2019], setting $V_0 := \int_X \mathrm{MA}_\omega(0)$, strong topologies were introduced for

$$\mathcal{E}^1(X,\omega) := \{u \in \mathcal{E}(X,\omega) : E(u) > -\infty\}$$

and

$$\mathcal{M}^1(X,\omega) := \{V_0\mu : \mu \text{ is a probability measure satisfying } E^*(\mu) < +\infty\},$$

as the coarsest refinements of the weak topologies such that the Monge–Ampère energy $E(u)$ [Aubin 1984; Berman and Boucksom 2010; Boucksom et al. 2010] and the energy for probability measures E^* [Berman et al. 2013; 2019], respectively, become continuous. The map

$$\mathrm{MA}_\omega : (\mathcal{E}^1_{\mathrm{norm}}(X,\omega), \text{strong}) \to (\mathcal{M}^1(X,\omega), \text{strong}) \tag{3}$$

is then a homeomorphism. Later Darvas [2015] showed that $(\mathcal{E}^1(X,\omega), \text{strong})$ actually coincides with the metric closure of $\mathcal{H}_\omega$ endowed with the Finsler metric $|f|_{1,\varphi} := \int_X |f| \,\mathrm{MA}_\omega(\varphi)$ with $\varphi \in \mathcal{H}_\omega$, $f \in T_\varphi \mathcal{H}_\omega \simeq C^\infty(X)$ and associated distance

$$d(u,v) := E(u) + E(v) - 2E(P_\omega(u,v)),$$

where $P_\omega(u,v)$ is the rooftop envelope given basically as the largest ω-psh function bounded above by $\min(u,v)$ [Ross and Witt Nyström 2014]. This metric topology has played an important role in the last decade to characterize the existence of special metrics [Berman et al. 2020; Chen and Cheng 2021a; 2021b; Darvas and Rubinstein 2017].

It is also important and natural to solve complex Monge–Ampère equations requiring that the solutions have some prescribed behavior, for instance along a divisor.

We first recall that on $\mathrm{PSH}(X,\omega)$ there is a natural partial order $\preccurlyeq$ given as $u \preccurlyeq v$ if $u \le v + O(1)$, and the total mass through the Monge–Ampère operator respects such partial order, i.e., $V_u := \int_X \mathrm{MA}_\omega(u) \le V_v$ if $u \preccurlyeq v$ [Boucksom et al. 2010; Witt Nyström 2019]. Thus in [Darvas et al. 2018], the authors introduced the *ψ-relative* analogs of the sets $\mathcal{E}(X,\omega)$ and $\mathcal{E}^1(X,\omega)$, for $\psi \in \mathrm{PSH}(X,\omega)$ fixed, as

$$\mathcal{E}(X,\omega,\psi) := \{u \in \mathrm{PSH}(X,\omega) : u \preccurlyeq \psi \text{ and } V_u = V_v\},$$
$$\mathcal{E}^1(X,\omega,\psi) := \{u \in \mathcal{E}(X,\omega,\psi) : E_\psi(u) > -\infty\},$$

where E_ψ is the ψ-relative energy. They then proved that

$$\mathrm{MA}_\omega : \mathcal{E}_{\mathrm{norm}}(X,\omega,\psi) \to \{\mu \text{ nonpluripolar positive measure} : \mu(X) = V_\psi\} \tag{4}$$

is a bijection if and only if ψ, up to a bounded function, is a *model-type envelope*, or in other words, $\psi = (\lim_{C\to+\infty} P(\psi + C, 0))^*$ satisfies $V_\psi > 0$ (the star is for the upper semicontinuous regularization). There are plenty of these functions, for instance, to any ω-psh function ψ with analytic singularities is associated a unique model-type envelope. We denote by $\mathcal{M}$ the set of all model-type envelopes and by $\mathcal{M}^+$ those elements ψ such that $V_\psi > 0$.

Letting $\psi \in \mathcal{M}^+$, in [Trusiani 2022], we proved that $\mathcal{E}^1(X, \omega, \psi)$ can be endowed with a natural metric topology given by the complete distance $d(u, v) := E_\psi(u) + E_\psi(v) - 2E_\psi(P_\omega(u, v))$.

Analogously to E^*, we introduce in Section 5 a natural ψ-relative energy for probability measures E^*_ψ; thus the set

$$\mathcal{M}^1(X, \omega, \psi) := \{V_\psi \mu : \mu \text{ is a probability measure satisfying } E^*_\psi(\mu) < +\infty\}$$

can be endowed with its strong topology given as the coarsest refinement of the weak topology such that E^*_ψ becomes continuous.

Theorem A. *Let $\psi \in \mathcal{M}^+$. Then*

$$\mathrm{MA}_\omega : (\mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi), d) \to (\mathcal{M}^1(X, \omega, \psi), \text{strong}) \tag{5}$$

is a homeomorphism.

It is natural to wonder if one can extend the bijections (2) and (4) to bigger subsets of $\mathrm{PSH}(X, \omega)$.

Given $\psi_1, \psi_2 \in \mathcal{M}^+$ such that $\psi_1 \neq \psi_2$, the sets $\mathcal{E}(X, \omega, \psi_1)$ and $\mathcal{E}(X, \omega, \psi_2)$ are disjoint ([Darvas et al. 2018, Theorem 1.3] quoted below as Theorem 2.1), but it may happen that $V_{\psi_1} = V_{\psi_2}$. So in these situations, at least one of $\mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi_1)$ or $\mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi_2)$ must be ruled out to extend (4). However, given a totally ordered family $\mathcal{A} \subset \mathcal{M}^+$ of model-type envelopes, the map $\mathcal{A} \ni \psi \to V_\psi$ is injective (again by [Darvas et al. 2018, Theorem 1.3]), i.e.,

$$\mathrm{MA}_\omega : \bigsqcup_{\psi \in \mathcal{A}} \mathcal{E}_{\mathrm{norm}}(X, \omega, \psi) \to \{\mu \text{ nonpluripolar positive measure} : \mu(X) = V_\psi \text{ for } \psi \in \mathcal{A}\}$$

is a bijection.

In [Trusiani 2022] we introduced a complete distance $d_\mathcal{A}$ on

$$X_\mathcal{A} := \bigsqcup_{\psi \in \bar{\mathcal{A}}} \mathcal{E}^1(X, \omega, \psi),$$

where $\bar{\mathcal{A}} \subset \mathcal{M}$ is the weak closure of $\mathcal{A}$ and where we identify $\mathcal{E}^1(X, \omega, \psi_{\min})$ with a point $P_{\psi_{\min}}$ if $\psi_{\min} \in \mathcal{M} \setminus \mathcal{M}^+$ (since in this case $E_\psi \equiv 0$, see Remark 2.7). Here $\psi_{\min}$ is given as the smallest element in $\bar{\mathcal{A}}$, observing that the Monge–Ampère operator $\mathrm{MA}_\omega : \bar{\mathcal{A}} \to \mathrm{MA}_\omega(\bar{\mathcal{A}})$ is a homeomorphism when the range is endowed with the weak topology (Lemma 3.12). We call the strong topology on $X_\mathcal{A}$ the metric topology given by $d_\mathcal{A}$ since $d_{\mathcal{A}|\mathcal{E}^1(X,\omega,\psi)\times\mathcal{E}^1(X,\omega,\psi)} = d$. The precise definition of $d_\mathcal{A}$ is quite technical (in Section 2 we will recall many of its properties), but the strong topology is natural since it is the coarsest refinement of the weak topology such that $E_\cdot(\cdot)$ becomes continuous as Theorem 6.2 shows. In particular the strong topology is independent of the set $\mathcal{A}$ chosen.

Also the set

$$Y_\mathcal{A} := \bigsqcup_{\psi \in \bar{\mathcal{A}}} \mathcal{M}^1(X, \omega, \psi)$$

has a natural strong topology given as the coarsest refinement of the weak topology such that $E^*_\cdot(\cdot)$ becomes continuous.

Theorem B. *The Monge–Ampère map*

$$\mathrm{MA}_\omega : (X_{\mathcal{A},\mathrm{norm}}, d_{\mathcal{A}}) \to (Y_{\mathcal{A}}, \text{strong})$$

is a homeomorphism.

Obviously in Theorem B we define $\mathrm{MA}_\omega(P_{\psi_{\min}}) := 0$ if $V_{\psi_{\min}} = 0$.

Note that by Hartogs' lemma and Theorem 6.2 the metric subspace $X_{\mathcal{A},\mathrm{norm}}$ is complete and represents the set of all closed and positive (1, 1)-currents $T = \omega + dd^c u$ such that $u \in X_{\mathcal{A}}$, where $P_{\psi_{\min}}$ encases all currents whose potentials u are more singular than $\psi_{\min}$ if $V_{\psi_{\min}} = 0$.

Finally, as an application of Theorem B we study an example of the stability of solutions of complex Monge–Ampère equations. Other important situations will be dealt with in a future work.

Theorem C. *Let* $\mathcal{A} := \{\psi_k\}_{k\in\mathbb{N}} \subset \mathcal{M}^+$ *be totally ordered, and let* $\{f_k\}_{k\in\mathbb{N}} \subset L^1 \setminus \{0\}$ *be a sequence of nonnegative functions such that* $f_k \to f \in L^1 \setminus \{0\}$ *and such that* $\int_X f_k \omega^n = V_{\psi_k}$ *for any* $k \in \mathbb{N}$. *Assume also that there exists* $p > 1$ *such that* $\|f_k\|_{L^p}$ *and* $\|f\|_{L^p}$ *are uniformly bounded. Then* $\psi_k \to \psi \in \mathcal{M}^+$ *weakly, and the sequence* $\{u_k\}_{k\in\mathbb{N}}$ *of solutions of*

$$\mathrm{MA}_\omega(u_k) = f_k \omega^n, \quad u_k \in \mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi_k), \tag{6}$$

converges strongly to $u \in X_{\mathcal{A}}$ *(i.e.,* $d_{\mathcal{A}}(u_k, u) \to 0$*), which is the unique solution of*

$$\mathrm{MA}_\omega(u) = f \omega^n, \quad u \in \mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi).$$

In particular, $u_k \to u$ *in capacity.*

The existence of the solutions of (6) follows by Theorem A in [Darvas et al. 2021a], while the fact that the strong convergence implies the convergence in capacity is our Theorem 6.3. Note also that the convergence in capacity of Theorem C was already obtained in [Darvas et al. 2021b]; see Remark 7.1.

1A. ***Structure of the paper.*** Section 2 is dedicated to introducing preliminaries, and, in particular, all necessary results presented in [Trusiani 2022]. In Section 3 we extend some known uniform estimates for $\mathcal{E}^1(X, \omega)$ to the relative setting, and we prove the key upper-semicontinuity of the relative energy functional $E_\cdot(\,\cdot\,)$ in $X_{\mathcal{A}}$. Section 4 regards the properties of the action of measures on $\mathrm{PSH}(X, \omega)$ and, in particular, their continuity. Then Section 5 is dedicated to proving Theorem A. We use a variational approach to show the bijection, then we need some further important properties of the strong topology on $\mathcal{E}^1(X, \omega, \psi)$ to conclude the proof. Section 6 is the heart of the article where we extend the results proved in the previous section to $X_{\mathcal{A}}$, and we present our main Theorem B. Finally in Section 7 we show Theorem C.

1B. ***Future developments.*** As mentioned above, in a future work we will present some strong stability results of more general solutions of complex Monge–Ampère equations with prescribed singularities than Theorem C, starting the study of a kind of *continuity method* where the singularities will also vary. As an application we will study the existence of (log) Kähler–Einstein metrics with prescribed singularities, with a particular focus on the relationships among them varying the singularities.

2. Preliminaries

We recall that given a Kähler complex compact manifold (X, ω), the set $\mathrm{PSH}(X, \omega)$ is the set of all ω-plurisubharmonic functions (ω-psh), i.e., all $u \in L^1$ given locally as the sum of a smooth function and a plurisubharmonic function such that $\omega + dd^c u \geq 0$ as a $(1, 1)$-current. Here $d^c := \frac{i}{2\pi}(\bar\partial - \partial)$ so that $dd^c = \frac{i}{\pi}\partial\bar\partial$. For any pair of ω-psh functions u, v, the function

$$P_\omega[u](v) := \Big(\lim_{C\to\infty} P_\omega(u + C, v)\Big)^* = (\sup\{w \in \mathrm{PSH}(X, \omega) : w \preccurlyeq u,\ w \leq v\})^*$$

is ω-psh, where the star is for the upper semicontinuous regularization and

$$P_\omega(u, v) := (\sup\{w \in \mathrm{PSH}(X, \omega) : w \leq \min(u, v)\})^*.$$

Then the set of all model-type envelopes is defined as

$$\mathcal{M} := \{\psi \in \mathrm{PSH}(X, \omega) : \psi = P_\omega[\psi](0)\}.$$

We also recall that $\mathcal{M}^+$ denotes the elements $\psi \in \mathcal{M}$ such that $V_\psi > 0$ where, as said in the Introduction, $V_\psi := \int_X \mathrm{MA}_\omega(\psi)$.

The class of ψ-relative full mass functions $\mathcal{E}(X, \omega, \psi)$ complies with the following characterization.

Theorem 2.1 [Darvas et al. 2018, Theorem 1.3]. *Suppose $v \in \mathrm{PSH}(X, \omega)$ such that $V_v > 0$ and v is less singular than $u \in \mathrm{PSH}(X, \omega)$. Then the following are equivalent*:

(i) $u \in \mathcal{E}(X, \omega, v)$.

(ii) $P_\omega[u](v) = v$.

(iii) $P_\omega[u](0) = P_\omega[v](0)$.

The clear inclusion $\mathcal{E}(X, \omega, v) \subset \mathcal{E}(X, \omega, P_\omega[v](0))$ may be strict, and it seems more natural in many cases to consider only functions $\psi \in \mathcal{M}$. For instance, as shown in [Darvas et al. 2018], ψ being a model-type envelope is a necessary assumption to make the equation

$$\mathrm{MA}_\omega(u) = \mu, \quad u \in \mathcal{E}(X, \omega, \psi),$$

always solvable where μ is a nonpluripolar measure such that $\mu(X) = V_\psi$. It is also worth recalling that there are plenty of elements in $\mathcal{M}$, since $P_\omega[P_\omega[\psi]] = P_\omega[\psi]$ for any $\psi \in \mathrm{PSH}(X, \omega)$ with $\int_X \mathrm{MA}_\omega(\psi) > 0$, see [Darvas et al. 2018, Theorem 3.12]. Indeed, $v \to P_\omega[v]$ may be thought of as a projection from the set of negative ω-psh functions with positive Monge–Ampère mass to $\mathcal{M}^+$.

We also retrieve the following useful result.

Theorem 2.2 [Darvas et al. 2018, Theorem 3.8]. *Let $u, \psi \in \mathrm{PSH}(X, \omega)$ such that $u \succcurlyeq \psi$. Then*

$$\mathrm{MA}_\omega(P_\omega[\psi](u)) \leq \mathbb{1}_{\{P_\omega[\psi](u)=u\}}\,\mathrm{MA}_\omega(u).$$

In particular, if $\psi \in \mathcal{M}$ then $\mathrm{MA}_\omega(\psi) \leq \mathbb{1}_{\{\psi=0\}}\,\mathrm{MA}_\omega(0)$.

Note also, in Theorem 2.2 the equality holds if u is continuous with bounded distributional Laplacian with respect to ω as a consequence of [Di Nezza and Trapani 2021]. In particular, for any $\psi \in \mathcal{M}$, $\mathrm{MA}_\omega(\psi) = \mathbb{1}_{\{\psi=0\}}\,\mathrm{MA}_\omega(0)$.

2A. ***The metric space*** $(\mathcal{E}^1(X,\omega,\psi), d)$. In this subsection we assume $\psi \in \mathcal{M}^+ := \{\psi \in \mathcal{M} : V_\psi > 0\}$.

As in [Darvas et al. 2018], we also denote by $\mathrm{PSH}(X,\omega,\psi)$ the set of all ω-psh functions which are more singular than ψ, and we recall that a function $u \in \mathrm{PSH}(X,\omega,\psi)$ has *ψ-relative minimal singularities* if $|u-\psi|$ is globally bounded on X. We also use the notation

$$\mathrm{MA}_\omega(u_1^{j_1},\dots,u_l^{j_l}) := (\omega + dd^c u_1)^{j_1} \wedge \cdots \wedge (\omega + dd^c u_l)^{j_l}$$

for $u_1,\dots,u_l \in \mathrm{PSH}(X,\omega)$ where $j_1,\dots,j_l \in \mathbb{N}$ such that $j_1+\cdots+j_l = n$.

Definition 2.3 [Darvas et al. 2018, Section 4.2]. The *ψ-relative energy functional* $E_\psi : \mathrm{PSH}(X,\omega,\psi) \to \mathbb{R}\cup\{-\infty\}$ is defined as

$$E_\psi(u) := \frac{1}{n+1}\sum_{j=0}^{n}\int_X (u-\psi)\,\mathrm{MA}_\omega(u^j,\psi^{n-j})$$

if u has ψ-relative minimal singularities, and as

$$E_\psi(u) := \inf\{E_\psi(v) : v \in \mathcal{E}(X,\omega,\psi) \text{ with } \psi\text{-relative minimal singularities}, v \geq u\}$$

otherwise. The subset $\mathcal{E}^1(X,\omega,\psi) \subset \mathcal{E}(X,\omega,\psi)$ is defined as

$$\mathcal{E}^1(X,\omega,\psi) := \{u \in \mathcal{E}(X,\omega,\psi) : E_\psi(u) > -\infty\}.$$

When $\psi = 0$, the ψ-relative energy functional is the *Aubin–Mabuchi energy functional*, also called the *Monge–Ampère energy*; see [Aubin 1984; Mabuchi 1986].

Proposition 2.4. *The following properties from* [Darvas et al. 2018] *hold*:

(i) [Theorem 4.10] E_ψ *is nondecreasing.*

(ii) [Lemma 4.12] $E_\psi(u) = \lim_{j\to\infty} E_\psi(\max(u,\psi-j))$.

(iii) [Lemma 4.14] E_ψ *is continuous along decreasing sequences.*

(iv) [Theorem 4.10 and Corollary 4.16] E_ψ *is concave along affine curves.*

(v) [Lemma 4.13] $u \in \mathcal{E}^1(X,\omega,\psi)$ *if and only if* $u \in \mathcal{E}(X,\omega,\psi)$ *and* $\int_X (u-\psi)\,\mathrm{MA}_\omega(u) > -\infty$.

(vi) [Proposition 4.19] $E_\psi(u) \geq \limsup_{k\to\infty} E_\psi(u_k)$ *if* $u_k, u \in \mathcal{E}^1(X,\omega,\psi)$ *and* $u_k \to u$ *with respect to the weak topology.*

(vii) [Proposition 4.20] *Letting* $u \in \mathcal{E}^1(X,\omega,\psi)$, $\chi \in \mathcal{C}^0(X)$ *and* $u_t := \sup\{v \in \mathrm{PSH}(X,\omega)\; v \leq u + t\chi\}^*$ *for any* $t > 0$, *then* $t \to E_\psi(u_t)$ *is differentiable and its derivative is given by*

$$\frac{d}{dt}E_\psi(u_t) = \int_X \chi\,\mathrm{MA}_\omega(u_t).$$

(viii) [Theorem 4.10] *If* $u, v \in \mathcal{E}^1(X, \omega, \psi)$, *then*

$$E_\psi(u) - E_\psi(v) = \frac{1}{n+1} \sum_{j=0}^{n} \int_X (u-v)\,\mathrm{MA}_\omega(u^j, v^{n-j})$$

and the function $\mathbb{N} \ni j \to \int_X (u-v)\,\mathrm{MA}_\omega(u^j, v^{n-j})$ *is decreasing. In particular,*

$$\int_X (u-v)\,\mathrm{MA}_\omega(u) \le E_\psi(u) - E_\psi(v) \le \int_X (u-v)\,\mathrm{MA}_\omega(v).$$

(ix) [Theorem 4.10] *If* $u \le v$, *then*

$$E_\psi(u) - E_\psi(v) \le \frac{1}{n+1} \int_X (u-v)\,\mathrm{MA}_\omega(u).$$

Remark 2.5. All the properties of Proposition 2.4 are shown in [Darvas et al. 2018] assuming ψ has *small unbounded locus*, but [Trusiani 2022, Proposition 2.7] and the general integration by parts formula proved in [Xia 2019] allow us to extend these properties to the general case as described in [Trusiani 2022, Remark 2.10].

Recalling that for any $u, v \in \mathcal{E}^1(X, \omega, \psi)$ the function $P_\omega(u, v) = \sup\{w \in \mathrm{PSH}(X, \omega) : w \le \min(u, v)\}^*$ belongs to $\mathcal{E}^1(X, \omega, \psi)$ (see [Trusiani 2022, Proposition 2.13]), then we also have that the function $d : \mathcal{E}^1(X, \omega, \psi) \times \mathcal{E}^1(X, \omega, \psi) \to \mathbb{R}_{\ge 0}$ defined as

$$d(u, v) = E_\psi(u) + E_\psi(v) - 2E_\psi(P_\omega(u, v))$$

assumes finite values. Moreover, it is a complete distance as the next result shows.

Theorem 2.6 [Trusiani 2022, Theorem A]. *$(\mathcal{E}^1(X, \omega, \psi), d)$ is a complete metric space.*

We call the *strong topology* on $\mathcal{E}^1(X, \omega, \psi)$ the metric topology given by the distance d. Note that, by construction, $d(u_k, u) \to 0$ as $k \to \infty$ if $u_k \searrow u$, and $d(u, v) = d(u, w) + d(w, v)$ if $u \le w \le v$; see [Trusiani 2022, Lemma 3.1].

Moreover, as a consequence of Proposition 2.4, it follows that for any $C \in \mathbb{R}_{>0}$ the set

$$\mathcal{E}^1_C(X, \omega, \psi) := \left\{u \in \mathcal{E}^1(X, \omega, \psi) : \sup_X u \le C \text{ and } E_\psi(u) \ge -C\right\}$$

is a weakly compact convex set.

Remark 2.7. If $\psi \in \mathcal{M} \setminus \mathcal{M}^+$, then $\mathcal{E}^1(X, \omega, \psi) = \mathrm{PSH}(X, \omega, \psi)$ since $E_\psi \equiv 0$ by definition; see [Trusiani 2022, Remark 3.10]. In particular, $d \equiv 0$, and it is natural to identify $(\mathcal{E}^1(X, \omega, \psi), d)$ with a point P_ψ. Moreover, we recall that $\mathcal{E}^1(X, \omega, \psi_1) \cap \mathcal{E}^1(X, \omega, \psi_2) = \varnothing$ if $\psi_1, \psi_2 \in \mathcal{M}$, $\psi_1 \ne \psi_2$ and $V_{\psi_2} > 0$.

2B. ***The space*** $(X_{\mathcal{A}}, d_{\mathcal{A}})$**.** From now on we assume $\mathcal{A} \subset \mathcal{M}^+$ to be a totally ordered set of model-type envelopes, and we denote by $\bar{\mathcal{A}}$ its closure as a subset of $\mathrm{PSH}(X, \omega)$ endowed with the weak topology. Note that $\bar{\mathcal{A}} \subset \mathrm{PSH}(X, \omega)$ is compact by [Trusiani 2022, Lemma 2.6]. Indeed, we will prove in Lemma 3.12 that $\bar{\mathcal{A}}$ is actually homeomorphic to its image through the Monge–Ampère operator MA_ω when the set of measures is endowed with the weak topology. This yields that $\bar{\mathcal{A}}$ is also homeomorphic to a closed set contained in $\left[0, \int_X \omega^n\right]$ through the map $\psi \to V_\psi$.

Definition 2.8. We define the set

$$X_{\mathcal{A}} := \bigsqcup_{\psi \in \bar{\mathcal{A}}} \mathcal{E}^1(X, \omega, \psi)$$

if $\psi_{\min} := \inf \mathcal{A}$ satisfies $V_{\psi_{\min}} > 0$, and

$$X_{\mathcal{A}} := P_{\psi_{\min}} \sqcup \bigsqcup_{\psi' \in \bar{\mathcal{A}}, \psi \neq \psi_{\min}} \mathcal{E}^1(X, \omega, \psi)$$

if $V_{\psi_{\min}} = 0$, where $P_{\psi_{\min}}$ is a singleton.

$X_{\mathcal{A}}$ can be endowed with a natural metric structure as [Trusiani 2022, Section 4] shows.

Theorem 2.9 [Trusiani 2022, Theorem B]. *$(X_{\mathcal{A}}, d_{\mathcal{A}})$ is a complete metric space such that*

$$d_{\mathcal{A}|\mathcal{E}^1(X,\omega,\psi)\times\mathcal{E}^1(X,\omega,\psi)} = d$$

for any $\psi \in \bar{\mathcal{A}} \cap \mathcal{M}^+$.

We call the *strong topology* on $X_{\mathcal{A}}$ the metric topology given by the distance $d_{\mathcal{A}}$. Note that the definition is coherent with that of Section 2A since the induced topology on $\mathcal{E}^1(X, \omega, \psi) \subset X_{\mathcal{A}}$ coincides with the strong topology given by d.

We will also need the following contraction property which is the starting point to construct $d_{\mathcal{A}}$.

Proposition 2.10 [Trusiani 2022, Lemma 4.2 and Proposition 4.3]. *Let $\psi_1, \psi_2, \psi_3 \in \mathcal{M}$ such that $\psi_1 \preccurlyeq \psi_2 \preccurlyeq \psi_3$. Then $P_\omega[\psi_1](P_\omega[\psi_2](u)) = P_\omega[\psi_1](u)$ for any $u \in \mathcal{E}^1(X, \omega, \psi_3)$ and $|P_\omega[\psi_1](u) - \psi_1| \le C$ if $|u - \psi_3| \le C$. Moreover, the map*

$$P_\omega[\psi_1](\,\cdot\,) : \mathcal{E}^1(X, \omega, \psi_2) \to \mathrm{PSH}(X, \omega, \psi_1)$$

has image in $\mathcal{E}^1(X, \omega, \psi_1)$ and is a Lipschitz map of constant 1 *when the sets $\mathcal{E}^1(X, \omega, \psi_i)$, $i = 1, 2$, are endowed with the d distances, i.e.,*

$$d(P_\omega[\psi_1](u), P_\omega[\psi_1](v)) \le d(u, v)$$

for any $u, v \in \mathcal{E}^1(X, \omega, \psi_2)$.

Here we report some properties of the distance $d_{\mathcal{A}}$ and some consequences which will be useful later.

Proposition 2.11. *The following properties from* [Trusiani 2022] *hold*:

(i) [Proposition 4.14] *If $u \in \mathcal{E}^1(X, \omega, \psi_1)$ and $v \in \mathcal{E}^1(X, \omega, \psi_2)$ for $\psi_1, \psi_2 \in \bar{\mathcal{A}}$ and $\psi_1 \succcurlyeq \psi_2$, then*

$$d_A(u, v) \ge d(P_\omega[\psi_2](u), v).$$

(ii) [Lemma 4.6] *If $\{\psi_k\}_{k\in\mathbb{N}} \subset \mathcal{M}^+$, $\psi \in \mathcal{M}$, with $\psi_k \searrow \psi$ (resp. $\psi_k \nearrow \psi$ a.e.), $u_k \searrow u$ and $v_k \searrow v$ (resp. $u_k \nearrow u$ a.e. and $v_k \nearrow v$ a.e.), for $u_k, v_k \in \mathcal{E}^1(X, \omega, \psi_k)$ and $u, v \in \mathcal{E}^1(X, \omega, \psi)$ and $|u_k - v_k|$ is uniformly bounded, then*

$$d(u_k, v_k) \to d(u, v).$$

(iii) [Proposition 4.5] *If* $\{\psi_k\}_{k\in\mathbb{N}} \subset \mathcal{M}^+$, $\psi \in \mathcal{M}$, *such that* $\psi_k \to \psi$ *monotonically a.e., then for any* $\psi' \in \mathcal{M}$ *such that* $\psi' \succcurlyeq \psi_k$ *for any* $k \gg 1$ *big enough and for any strongly compact set* $K \subset (\mathcal{E}^1(X, \omega, \psi'), d)$,

$$d(P_\omega[\psi_k](\varphi_1), P_\omega[\psi_k](\varphi_2)) \to d(P_\omega[\psi](\varphi_1), P_\omega[\psi](\varphi_2))$$

uniformly on $K \times K$, *i.e., varying* $(\varphi_1, \varphi_2) \in K \times K$. *In particular, if* $\psi_k, \psi \in \bar{\mathcal{A}}$, *then*

$$d_{\mathcal{A}}(P_\omega[\psi](u), P_\omega[\psi_k](u)) \to 0,$$
$$d(P_\omega[\psi_k](u), P_\omega[\psi_k](v)) \to d(P_\omega[\psi](u), P_\omega[\psi](v))$$

monotonically for any $(u, v) \in \mathcal{E}^1(X, \omega, \psi') \times \mathcal{E}^1(X, \omega, \psi')$.

(iv) [Section 4.2] $d_{\mathcal{A}}(u_1, u_2) \geq |V_{\psi_1} - V_{\psi_2}|$ *if* $u_1 \in \mathcal{E}^1(X, \omega, \psi_1)$ *and* $u_2 \in \mathcal{E}^1(X, \omega, \psi_2)$, *and the equality holds if* $u_1 = \psi_1$ *and* $u_2 = \psi_2$ *(by definition of* $d_{\mathcal{A}}$*).*

The following lemma is a special case of [Xia 2019, Theorem 2.2]; see also [Darvas et al. 2018, Lemma 4.1].

Lemma 2.12 [Trusiani 2022, Proposition 2.7]. *Let* $\{\psi_k\}_{k\in\mathbb{N}} \subset \mathcal{M}^+$, $\psi \in \mathcal{M}$, *such that* $\psi_k \to \psi$ *monotonically almost everywhere. Let also* $u_k, v_k \in \mathcal{E}^1(X, \omega, \psi_k)$ *converge in capacity to* $u, v \in \mathcal{E}^1(X, \omega, \psi)$, *respectively. Then for any* $j = 0, \dots, n$,

$$\mathrm{MA}_\omega(u_k^j, v_k^{n-j}) \to \mathrm{MA}_\omega(u^j, v^{n-j})$$

weakly. Moreover, if $|u_k - v_k|$ *is uniformly bounded, then for any* $j = 0, \dots, n$,

$$(u_k - v_k)\,\mathrm{MA}_\omega(u_k^j, v_k^{n-j}) \to (u - v)\,\mathrm{MA}_\omega(u^j, v^{n-j})$$

weakly.

It is well known that the set of Kähler potentials $\mathcal{H}_\omega := \{\varphi \in \mathrm{PSH}(X, \omega) \cap C^\infty(X) : \omega + dd^c\varphi > 0\}$ is dense in $(\mathcal{E}^1(X, \omega), d)$. The same holds for $P_\omega[\psi](\mathcal{H}_\omega)$ in $(\mathcal{E}^1(X, \omega, \psi), d)$.

Lemma 2.13 [Trusiani 2022, Lemma 4.8]. *The set* $\mathcal{P}_{\mathcal{H}_\omega}(X, \omega, \psi) := P_\omega[\psi](\mathcal{H}) \subset \mathcal{P}(X, \omega, \psi)$ *is dense in* $(\mathcal{E}^1(X, \omega, \psi), d)$.

The following lemma shows that, for $u \in \mathrm{PSH}(X, \omega)$ fixed, the map $\mathcal{M}^+ \ni \psi \to P_\omega[\psi](u)$ is weakly continuous over any totally ordered set of model-type envelopes that are more singular than u.

Lemma 2.14. *Let* $u \in \mathrm{PSH}(X, \omega)$, *and let* $\{\psi_k\}_{k\in\mathbb{N}} \subset \mathcal{M}^+$ *be a totally ordered sequence of model-type envelopes converging to* $\psi \in \mathcal{M}$. *Assume also that* $\psi_k \preccurlyeq u$ *for any* $k \gg 1$ *big enough. Then* $P_\omega[\psi_k](u) \to P_\omega[\psi](u)$ *weakly.*

Proof. As $\{\psi_k\}_{k\in\mathbb{N}}$ is totally ordered, without loss of generality we may assume that $\psi_k \to \psi$ monotonically almost everywhere. Set $\tilde{u} := \lim_{k\to\infty} P_\omega[\psi_k](u)$. We want to prove that $\tilde{u} = P_\omega[\psi](u)$.

Suppose $\psi_k \searrow \psi$. We can immediately check that $P_\omega[\psi_k](u) \leq P_\omega[\psi_k](\sup_X u) = \psi_k + \sup_X u$, which implies $\tilde{u} \leq \psi + \sup_X u$ letting $k \to +\infty$. Thus $\tilde{u} \leq P_\omega[\psi](u)$, as the inequality $\tilde{u} \leq u$ is trivial. Moreover, since $\psi \le \psi_k$ we also have $P_\omega[\psi](u) \le P_\omega[\psi_k](u)$, which clearly yields $P_\omega[\psi](u) \le \tilde{u}$ and concludes this part.

Suppose $\psi_k \nearrow \psi$. Then the inequality $\tilde{u} \le P_\omega[\psi](u)$ is immediate. Next, combining Theorem 2.2 and Proposition 2.10, we have

$$\begin{aligned} \mathrm{MA}_\omega(P_\omega[\psi_k](u)) &= \mathrm{MA}_\omega(P_\omega[\psi_k](P_\omega[\psi](u))) \\ &\le \mathbb{1}_{\{P_\omega[\psi_k](u)=P_\omega[\psi](u)\}} \mathrm{MA}_\omega(P_\omega[\psi](u)) \\ &\le \mathbb{1}_{\{\tilde{u}=P_\omega[\psi](u)\}} \mathrm{MA}_\omega(P_\omega[\psi](u)), \end{aligned}$$

where the last inequality follows from $P_\omega[\psi_k](u) \le \tilde{u} \le P_\omega[\psi](u)$. Thus, as $\mathrm{MA}_\omega(P_\omega[\psi_k](u)) \to \mathrm{MA}_\omega(\tilde{u})$ weakly by [Darvas et al. 2018, Theorem 2.3], we deduce that $\tilde{u} \in \mathcal{E}(X, \omega, \psi)$ and

$$\mathrm{MA}_\omega(\tilde{u}) \le \mathbb{1}_{\{\tilde{u}=P_\omega[\psi](u)\}} \mathrm{MA}_\omega(P_\omega[\psi](u)).$$

Moreover, we also have $P_\omega[\psi](u) \in \mathcal{E}(X, \omega, \psi)$. Indeed, $P_\omega[\psi](u) \le P_\omega[\psi](\sup_X u) = \psi + \sup_X$, i.e., $P_\omega[\psi](u) \preccurlyeq \psi$, while $P_\omega[\psi](u) \ge P_\omega[\psi](\psi_k - C_k) = \psi_k - C_k$ for nonnegative constants C_k and for any $k \gg 1$ big enough as u, ψ are less singular than ψ_k. Thus $P_\omega[\psi](u) \succcurlyeq \psi_k$ for any k, which yields $\int_X \mathrm{MA}_\omega(P_\omega[\psi](u)) \ge V_\psi > 0$ and gives $P_\omega[\psi](u) \in \mathcal{E}(X, \omega, \psi)$. Hence

$$\begin{aligned} 0 &\le \int_X (P_\omega[\psi](u) - \tilde{u}) \mathrm{MA}_\omega(\tilde{u}) \\ &\le \int_{\{\tilde{u}=P_\omega[\psi](u)\}} (P_\omega[\psi](u) - \tilde{u}) \mathrm{MA}_\omega(P_\omega[\psi](u)) = 0, \end{aligned}$$

which by the domination principle of [Darvas et al. 2018, Proposition 3.11] implies $\tilde{u} \ge P_\omega[\psi](u)$. □

3. Tools

In this section we collect some uniform estimates on $\mathcal{E}^1(X, \omega, \psi)$ for $\psi \in \mathcal{M}^+$, we recall the ψ-relative capacity and we prove the upper semicontinuity of $E_\cdot(\cdot)$ on $X_\mathcal{A}$.

3A. *Uniform estimates.* Let $\psi \in \mathcal{M}^+$.

We first define in the ψ-relative setting the analogs of some well-known functionals of the variational approach; see [Berman et al. 2013].

We define the *ψ-relative I- and J-functionals*,

$$I_\psi, J_\psi : \mathcal{E}^1(X, \omega, \psi) \times \mathcal{E}^1(X, \omega, \psi) \to \mathbb{R}, \quad \text{where } \psi \in \mathcal{M}^+,$$

as

$$\begin{aligned} I_\psi(u, v) &:= \int_X (u - v)(\mathrm{MA}_\omega(v) - \mathrm{MA}_\omega(u)), \\ J_\psi(u, v) &:= J_u^\psi(v) := E_\psi(u) - E_\psi(v) + \int_X (v - u) \mathrm{MA}_\omega(u), \end{aligned}$$

respectively; see also [Aubin 1984]. They assume nonnegative values by Proposition 2.4, and I_ψ is clearly symmetric while J_ψ is convex, again by Proposition 2.4. Moreover, the ψ-relative I- and J-functionals are related to each other by the following result.

Lemma 3.1. *Let $u, v \in \mathcal{E}^1(X, \omega, \psi)$. Then*

(i) $$\frac{1}{n+1} I_\psi(u, v) \le J_u^\psi(v) \le \frac{n}{n+1} I_\psi(u, v),$$

(ii) $$\frac{1}{n} J_u^\psi(v) \le J_v^\psi(u) \le n J_u^\psi(v).$$

In particular,

$$d(\psi, u) \le n J_u^\psi(\psi) + (\|\psi\|_{L^1} + \|u\|_{L^1})$$

for any $u \in \mathcal{E}^1(X, \omega, \psi)$ such that $u \le \psi$.

Proof. By Proposition 2.4 it follows that

$$n \int_X (u-v)\,\mathrm{MA}_\omega(u) + \int_X (u-v)\,\mathrm{MA}_\omega(v) \le (n+1)(E_\psi(u) - E_\psi(v)) \le \int_X (u-v)\,\mathrm{MA}_\omega(u) + n \int_X (u-v)\,\mathrm{MA}_\omega(v)$$

for any $u, v \in \mathcal{E}^1(X, \omega, \psi)$, which yields (i) and (ii).

Next, considering $v = \psi$ and assuming $u \le \psi$ from the second inequality in (ii), we obtain

$$d(u, \psi) = -E_\psi(u) \le n J_u^\psi(\psi) + \int_X (\psi - u)\,\mathrm{MA}_\omega(\psi),$$

which implies the assertion since $\mathrm{MA}_\omega(\psi) \le \mathrm{MA}_\omega(0)$ by Theorem 2.2. □

We can now proceed to show the uniform estimates, adapting some results in [Berman et al. 2013].

Lemma 3.2 [Trusiani 2022, Lemma 3.7]. *Let $\psi \in \mathcal{M}^+$. Then there exists positive constants $A > 1$, $B > 0$ depending only on n, ω such that for any $u \in \mathcal{E}^1(X, \omega, \psi)$,*

$$-d(\psi, u) \le V_\psi \sup_X (u - \psi) = V_\psi \sup_X u \le A\, d(\psi, u) + B$$

Remark 3.3. As a consequence of Lemma 3.2, if $d(\psi, u) \le C$, then $\sup_X u \le (AC + B)/V_\psi$ while

$$-E_\psi(u) = d(\psi + (AC + B)/V_\psi, u) - (AC + B) \le d(\psi, u) \le C,$$

i.e., $u \in \mathcal{E}^1_D(X, \omega, \psi)$ where $D := \max(C, (AC + B)/V_\psi)$. Conversely, using the definitions and the triangle inequality, it is easy to check that $d(u, \psi) \le C(2V_\psi + 1)$ for any $u \in \mathcal{E}^1_C(X, \omega, \psi)$.

Proposition 3.4. *Let $C \in \mathbb{R}_{>0}$. Then there exists a continuous increasing function $f_C : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ depending only on C, ω, n with $f_C(0) = 0$ such that*

$$\left| \int_X (u - v)(\mathrm{MA}_\omega(\varphi_1) - \mathrm{MA}_\omega(\varphi_2)) \right| \le f_C(d(u, v)) \tag{7}$$

for any $u, v, \varphi_1, \varphi_2 \in \mathcal{E}^1(X, \omega, \psi)$ with $d(u, \psi), d(v, \psi), d(\varphi_1, \psi), d(\varphi_2, \psi) \le C$.

Proof. As said in Remark 3.3, if $w \in \mathcal{E}^1(X, \omega, \psi)$ with $d(\psi, w) \le C$, then $\tilde{w} := w - (AC + B)/V_\psi$ satisfies $\sup_X \tilde{w} \le 0$ and

$$-E_\psi(\tilde{w}) = d(\psi, \tilde{w}) \le d(\psi, w) + d(w, \tilde{w}) \le C + AC + B =: D.$$

Therefore, setting $\tilde{u} := u - (AC + B)/V_\psi$ and $\tilde{v} := v - (AC + B)/V_\psi$, we can proceed exactly as in [Berman et al. 2013, Lemma 5.8] using the integration by parts formula in [Xia 2019] (see also [Boucksom et al. 2010, Theorem 1.14]) to get

$$\left| \int_X (\tilde{u} - \tilde{v})(\mathrm{MA}_\omega(\varphi_1) - \mathrm{MA}_\omega(\varphi_2)) \right| \le I_\psi(\tilde{u}, \tilde{v}) + h_D(I_\psi(\tilde{u}, \tilde{v})), \tag{8}$$

where $h_D : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is an increasing continuous function depending only on D such that $h_D(0) = 0$. Furthermore, by definition,

$$d(\psi, P_\omega(\tilde{u}, \tilde{v})) \le d(\psi, \tilde{u}) + d(\tilde{u}, P_\omega(\tilde{u}, \tilde{v})) \le d(\psi, \tilde{u}) + d(\tilde{u}, \tilde{v}) \le 3D,$$

so by the triangle inequality and (8) we have

$$\begin{aligned} &\left| \int_X (u - v)(\mathrm{MA}_\omega(\varphi_1) - \mathrm{MA}_\omega(\varphi_2)) \right| \\ &\quad \le I_\psi(\tilde{u}, P_\omega(\tilde{u}, \tilde{v})) + I_\psi(\tilde{v}, P_\omega(\tilde{u}, \tilde{v})) + h_{3D}(I_\psi(\tilde{u}, P_\omega(\tilde{u}, \tilde{v}))) + h_{3D}(I_\psi(\tilde{v}, P_\omega(\tilde{u}, \tilde{v}))). \end{aligned} \tag{9}$$

On the other hand, if $w_1, w_2 \in \mathcal{E}^1(X, \omega, \psi)$ with $w_1 \ge w_2$, then by Proposition 2.4

$$I_\psi(w_1, w_2) \le \int_X (w_1 - w_2)\,\mathrm{MA}_\omega(w_2) \le (n + 1)d(w_1, w_2).$$

Hence from (9) it is sufficient to set $f_C(x) := (n + 1)x + 2h_{3D}((n + 1)x)$ to conclude the proof since clearly $d(\tilde{u}, \tilde{v}) = d(u, v)$. ☐

Corollary 3.5. *Let $\psi \in \mathcal{M}^+$ and let $C \in \mathbb{R}_{>0}$. Then there exists a continuous increasing function $f_C : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ depending only on C, ω, n with $f_C(0) = 0$ such that*

$$\int_X |u - v|\,\mathrm{MA}_\omega(\varphi) \le f_C(d(u, v))$$

for any $u, v, \varphi \in \mathcal{E}^1(X, \omega, \psi)$ with $d(\psi, u), d(\psi, v), d(\psi, \varphi) \le C$.

Proof. Since $d(\psi, P_\omega(u, v)) \le 3C$, letting $g_{3C} : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ be the map (7) of Proposition 3.4, it follows that

$$\begin{aligned} \int_X (u - P_\omega(u, v))\,\mathrm{MA}_\omega(\varphi) &\le \int_X (u - P_\omega(u, v))\,\mathrm{MA}_\omega(P_\omega(u, v)) + g_{3C}(d(u, P_\omega(u, v))) \\ &\le (n + 1)d(u, P_\omega(u, v)) + g_{3C}(d(u, v)), \end{aligned}$$

where in the last inequality we used Proposition 2.4. Hence by the triangle inequality we get

$$\begin{aligned} \int_X |u - v|\,\mathrm{MA}_\omega(\varphi) &\le (n + 1)d(u, P_\omega(u, v)) + (n + 1)d(v, P_\omega(u, v)) + 2g_{3C}(d(u, v)) \\ &= (n + 1)d(u, v) + 2g_{3C}(d(u, v)). \end{aligned}$$

Defining $f_C(x) := (n + 1)x + 2g_{3C}(x)$ concludes the proof. ☐

As a first important consequence we obtain that the strong convergence in $\mathcal{E}^1(X, \omega, \psi)$ implies the weak convergence.

Proposition 3.6. *Let $\psi \in \mathcal{M}^+$ and let $C \in \mathbb{R}_{>0}$. Then there exists a continuous increasing function $f_{C,\psi} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ depending on C, ω, n, ψ with $f_{C,\psi}(0) = 0$ such that*

$$\|u - v\|_{L^1} \leq f_{C,\psi}(d(u, v))$$

for any $u, v \in \mathcal{E}^1(X, \omega, \psi)$ with $d(\psi, u), d(\psi, v) \leq C$. In particular, $u_k \to u$ weakly if $u_k \to u$ strongly.

Proof. Theorem A in [Darvas et al. 2021a] (see also Theorem 1.4 in [Darvas et al. 2018]) implies that there exists $\phi \in \mathcal{E}^1(X, \omega, \psi)$ with $\sup_X \phi = 0$ such that

$$\mathrm{MA}_\omega(\phi) = c\, \mathrm{MA}_\omega(0),$$

where $c := V_\psi / V_0 > 0$. Therefore it follows that

$$\|u - v\|_{L^1} \leq \frac{1}{c} g_{\hat{C}}(d(u, v)),$$

where $\hat{C} := \max(d(\psi, \phi), C)$ and $g_{\hat{C}}$ is the continuous increasing function with $g_{\hat{C}}(0) = 0$ given by Corollary 3.5. Setting $f_{C,\psi} := \frac{1}{c} g_{\hat{C}}$ concludes the proof. □

Finally we also get the following useful estimate.

Proposition 3.7. *Let $\psi \in \mathcal{M}^+$ and let $C \in \mathbb{R}_{>0}$. Then there exists a constant $\tilde{C}$ depending only on C, ω, n such that*

$$\left| \int_X (u - v)(\mathrm{MA}_\omega(\varphi_1) - \mathrm{MA}_\omega(\varphi_2)) \right| \leq \tilde{C} I_\psi(\varphi_1, \varphi_2)^{1/2} \tag{10}$$

for any $u, v, \varphi_1, \varphi_2 \in \mathcal{E}^1(X, \omega, \psi)$ with $d(u, \psi), d(v, \psi), d(\varphi_1, \psi), d(\varphi_2, \psi) \leq C$.

Proof. As in Proposition 3.4 and with the same notation, the function $\tilde{u} := u - (AC + B)/V_\psi$ satisfies $\sup_X u \leq 0$ (by Lemma 3.2) and $-E_\psi(u) \leq C + AC + B =: D$ (and similarly for v, φ_1, φ_2). Therefore by integration by parts and using Lemma 3.8 below, it follows exactly as in [Berman et al. 2013, Lemma 3.13] that there exists a constant $\tilde{C}$ depending only on D, n such that

$$\left| \int_X (\tilde{u} - \tilde{v})(\mathrm{MA}_\omega(\tilde{\varphi}_1) - \mathrm{MA}_\omega(\tilde{\varphi}_2)) \right| \leq \tilde{C} I_\psi(\tilde{\varphi}_1, \tilde{\varphi}_2)^{1/2},$$

which clearly implies (10). □

Lemma 3.8. *Let $C \in \mathbb{R}_{>0}$. Then there exists a constant $\tilde{C}$ depending only on C, ω, n such that*

$$\int_X |u_0 - \psi| (\omega + dd^c u_1) \wedge \cdots \wedge (\omega + dd^c u_n) \leq \tilde{C}$$

for any $u_0, \ldots, u_n \in \mathcal{E}^1(X, \omega, \psi)$, with $d(u_j, \psi) \leq C$ for any $j = 0, \ldots, n$.

Proof. As in Proposition 3.4 and with the same notation, $v_j := u_j - (AC+B)/V_\psi$ satisfies $\sup_X v_j \leq 0$, and setting $v := (v_0 + \cdots + v_n)/(n+1)$ we obtain $\psi - u_0 \leq (n+1)(\psi - v)$. Thus by Proposition 2.4,

$$\int_X (\psi - v_0)\,\mathrm{MA}_\omega(v) \leq (n+1)\int_X (\psi - v)\,\mathrm{MA}_\omega(v) \leq (n+1)^2|E_\psi(v)| \leq (n+1)\sum_{j=0}^{n}|E_\psi(v_j)| \leq (n+1)\sum_{j=0}^{n}(d(\psi,u_j)+D) \leq (n+1)^2(C+D),$$

where $D := AC + B$. On the other hand, $\mathrm{MA}_\omega(v) \geq E(\omega + dd^c u_1) \wedge \cdots \wedge (\omega + dd^c u_n)$, where the constant E depends only on n. Finally we get

$$\int_X |u_0 - \psi|(\omega + dd^c u_1) \wedge \cdots \wedge (\omega + dd^c u_n) \leq D + \frac{1}{E}\int_X (\psi - v_0)\,\mathrm{MA}_\omega(v) \leq D + \frac{(n+1)^2(C+D)}{E}. \qquad \square$$

3B. ψ-relative Monge–Ampère capacity.

Definition 3.9 [Darvas et al. 2018, Section 4.1; Darvas et al. 2021a, Definition 3.1]. Let $B \subset X$ be a Borel set, and let $\psi \in \mathcal{M}^+$. Then its *ψ-relative Monge–Ampère capacity* is defined as

$$\mathrm{Cap}_\psi(B) := \sup\left\{\int_B \mathrm{MA}_\omega(u) : u \in \mathrm{PSH}(X,\omega),\ \psi - 1 \leq u \leq \psi\right\}.$$

In the absolute setting the Monge–Ampère capacity is very useful for studying the existence and regularity of solutions of the degenerate complex Monge–Ampère equation [Kołodziej 1998], and the analog holds in the relative setting [Darvas et al. 2018, 2021a]. We refer to these articles for many properties of the Monge–Ampère capacity.

For any fixed constant A, write $\mathcal{C}_{A,\psi}$ for the set of all probability measures μ on X such that

$$\mu(B) \leq A\,\mathrm{Cap}_\psi(B)$$

for any Borel set $B \subset X$ [Darvas et al. 2018, Section 4.3].

Proposition 3.10. *Let $u \in \mathcal{E}^1(X,\omega,\psi)$ with ψ-relative minimal singularities. Then $\mathrm{MA}_\omega(u)/V_\psi \in \mathcal{C}_{A,\psi}$ for a constant $A > 0$.*

Proof. Let $j \in \mathbb{R}$ such that $u \geq \psi - j$ and assume without loss of generality that $u \leq \psi$ and $j \geq 1$. Then the function $v := j^{-1}u + (1 - j^{-1})\psi$ is a candidate in the definition of Cap_ψ, which implies that $\mathrm{MA}_\omega(v) \leq \mathrm{Cap}_\psi$. Hence, since $\mathrm{MA}_\omega(u) \leq j^n\,\mathrm{MA}(v)$, we get that $\mathrm{MA}_\omega(u) \in \mathcal{C}_{A,\psi}$ for $A = j^n$ and the result follows. $\square$

Lemma 3.11 [Darvas et al. 2018, Lemma 4.18]. *If $\mu \in \mathcal{C}_{A,\psi}$, then there is a constant $B > 0$ depending only on A, n such that*

$$\int_X (u - \psi)^2 \mu \leq B(|E_\psi(u)| + 1)$$

for any $u \in \mathrm{PSH}(X,\omega,\psi)$ such that $\sup_X u = 0$.

Similar to the case $\psi = 0$ (see [Guedj and Zeriahi 2017]), we say that a sequence $u_k \in \mathrm{PSH}(X, \omega)$ converges to $u \in \mathrm{PSH}(X, \omega)$ in ψ-relative capacity for $\psi \in \mathcal{M}$ if

$$\mathrm{Cap}_\psi(\{|u_k - u| \geq \delta\}) \to 0$$

as $k \to \infty$ for any $\delta > 0$.

By [Guedj and Zeriahi 2017, Theorem 10.37] (see also [Berman et al. 2013, Theorem 5.7]) the convergence in $(\mathcal{E}^1(X, \omega), d)$ implies the convergence in capacity. The analog holds for $\psi \in \mathcal{M}^+$, i.e., the strong convergence in $\mathcal{E}^1(X, \omega, \psi)$ implies the convergence in ψ-relative capacity. Indeed, in Proposition 5.7 we will prove the strong convergence implies the convergence in ψ'-relative capacity for any $\psi' \in \mathcal{M}^+$.

3C. *(Weak) upper semicontinuity of $u \to E_{P_\omega[u]}(u)$ over $X_{\mathcal{A}}$*. One of the main features of E_ψ for $\psi \in \mathcal{M}$ is its upper semicontinuity with respect to the weak topology. Here we prove the analog for $E_\cdot(\,\cdot\,)$ over $X_{\mathcal{A}}$.

Lemma 3.12. *The map*

$$\mathrm{MA}_\omega : \bar{\mathcal{A}} \to \mathrm{MA}_\omega(\bar{\mathcal{A}}) \subset \{\mu \text{ a positive measure on } X\}$$

is a homeomorphism considering the weak topologies. In particular, $\bar{\mathcal{A}}$ is homeomorphic to a closed set contained in $\big[0, \int_X \mathrm{MA}_\omega(0)\big]$ through the map $\psi \to V_\psi$.

Proof. The map is well-defined and continuous by [Trusiani 2022, Lemma 2.6]. Moreover, the injectivity follows from the fact that $V_{\psi_1} = V_{\psi_2}$ for $\psi_1, \psi_2 \in \bar{\mathcal{A}}$ implies $\psi_1 = \psi_2$ using Theorem 2.1 and the fact that $\mathcal{A} \subset \mathcal{M}^+$.

Finally, to conclude the proof it is enough to prove that $\psi_k \to \psi$ weakly assuming $V_{\psi_k} \to V_\psi$, and it is clearly sufficient to show that any subsequence of $\{\psi_k\}_{k\in\mathbb{N}}$ admits a subsequence weakly convergent to ψ. Moreover, since $\bar{\mathcal{A}}$ is totally ordered and $\succcurlyeq$ coincides with $\geq$ on $\mathcal{M}$, we may assume $\{\psi_k\}_{k\in\mathbb{N}}$ is a monotonic sequence. Then, up to considering a further subsequence, ψ_k converges almost everywhere to an element $\psi' \in \bar{\mathcal{A}}$ by compactness, and Lemma 2.12 implies that $V_{\psi'} = V_\psi$, i.e., $\psi' = \psi$. □

In the case $\mathcal{A} := \{\psi_k\}_{k\in\mathbb{N}} \subset \mathcal{M}^+$, we say that the $u_k \in \mathcal{E}^1(X, \omega, \psi_k)$ converge weakly to $P_{\psi_{\min}}$ where $\psi_{\min} \in \mathcal{M} \setminus \mathcal{M}^+$ if $|\sup_X u_k| \leq C$ for any $k \in \mathbb{N}$ and any weak accumulation point u of $\{u_k\}_{k\in\mathbb{N}}$ satisfies $u \preccurlyeq \psi_{\min}$. This definition is the most natural since $\mathrm{PSH}(X, \omega, \psi) = \mathcal{E}^1(X, \omega, \psi_{\min})$.

Lemma 3.13. *Let $\{u_k\}_{k\in\mathbb{N}} \subset X_{\mathcal{A}}$ be a sequence converging weakly to $u \in X_{\mathcal{A}}$. If $E_{P_\omega[u_k]}(u_k) \geq C$ uniformly, then $P_\omega[u_k] \to P_\omega[u]$ weakly.*

Proof. By Lemma 3.12 the convergence requested is equivalent to $V_{\psi_k} \to V_\psi$, where we set

$$\psi_k := P_\omega[u_k], \quad \psi := P_\omega[u].$$

Moreover, by a simple contradiction argument it is enough to show that any subsequence $\{\psi_{k_h}\}_{h\in\mathbb{N}}$ admits a subsequence $\{\psi_{k_{h_j}}\}_{j\in\mathbb{N}}$ such that $V_{\psi_{k_{h_j}}} \to V_\psi$. Thus up to considering a subsequence, by abuse of notation and by the lower semicontinuity $\liminf_{k\to\infty} V_{\psi_k} \geq V_\psi$ of [Darvas et al. 2018, Theorem 2.3], we may suppose by contradiction that $\psi_k \searrow \psi'$ for $\psi' \in \mathcal{M}$ such that $V_{\psi'} > V_\psi$. In particular, $V_{\psi'} > 0$ and $\psi' \succcurlyeq \psi$. Then by Proposition 2.10 and Remark 3.3, the sequence $\{P_\omega[\psi'](u_k)\}_{k\in\mathbb{N}}$ is bounded in $(\mathcal{E}^1(X,\omega,\psi'),d)$ and it belongs to $\mathcal{E}^1_{C'}(X,\omega,\psi')$ for some $C'\in\mathbb{R}$. Therefore, up to considering a subsequence, we have that $\{u_k\}_{k\in\mathbb{N}}$ converges weakly to an element $v\in\mathcal{E}^1(X,\omega,\psi)$ (which is the element u itself when $u\neq P_{\psi_{\min}}$), while the sequence $P_\omega[\psi'](u_k)$ converges weakly to an element $w\in\mathcal{E}^1(X,\omega,\psi')$. Thus the contradiction follows from $w\leq v$ since $\psi'\succcurlyeq\psi$, $V_{\psi'}>0$ and $\mathcal{E}^1(X,\omega,\psi')\cap\mathcal{E}^1(X,\omega,\psi)=\varnothing$. □

Proposition 3.14. *Let $\{u_k\}_{k\in\mathbb{N}}\subset X_{\mathcal{A}}$ be a sequence converging weakly to $u\in X_{\mathcal{A}}$. Then*

$$\limsup_{k\to\infty} E_{P_\omega[u_k]}(u_k)\leq E_{P_\omega[u]}(u). \tag{11}$$

Proof. Let $\psi_k:=P_\omega[u_k]$ and $\psi:=P_\omega[u]\in\bar{\mathcal{A}}$. We may assume $\psi_k\neq\psi_{\min}$ for any $k\in\mathbb{N}$ if $\psi=\psi_{\min}$ and $V_{\psi_{\min}}=0$.

Moreover, we can suppose that $E_{\psi_k}(u_k)$ is bounded from below, which implies that $u_k\in\mathcal{E}^1_C(X,\omega,\psi_k)$ for a uniform constant C and that $\psi_k\to\psi$ weakly by Lemma 3.13. Thus since

$$E_{\psi_k}(u_k)=E_{\psi_k}(u_k-C)+CV_{\psi_k}$$

for any $k\in\mathbb{N}$, Lemma 3.12 implies that we may assume that $\sup_X u_k\leq 0$. Furthermore, since $\mathcal{A}$ is totally ordered, it is enough to show (11) when $\psi_k\to\psi$ a.e. monotonically.

If $\psi_k\searrow\psi$, setting $v_k:=(\sup\{u_j:j\geq k\})^*\in\mathcal{E}^1(X,\omega,\psi_k)$, we easily have

$$\limsup_{k\to\infty}E_{\psi_k}(u_k)\leq\limsup_{k\to\infty}E_{\psi_k}(v_k)\leq\limsup_{k\to\infty}E_\psi(P_\omega[\psi](v_k))$$

using the monotonicity of E_{ψ_k} and Proposition 2.10. Hence if $\psi=\psi_{\min}$ and $V_{\psi_{\min}}=0$, then

$$E_\psi(P_\omega[\psi](v_k))=0=E_\psi(u),$$

while otherwise the conclusion follows from Proposition 2.4 since $P_\omega[\psi](v_k)\searrow u$ by construction.

If instead $\psi_k\nearrow\psi$, fix $\epsilon>0$ and for any $k\in\mathbb{N}$ let $j_k\geq k$ such that

$$\sup_{j\geq k}E_{\psi_j}(u_j)\leq E_{\psi_{j_k}}(u_{j_k})+\epsilon.$$

Thus again by Proposition 2.10, $E_{\psi_{j_k}}(u_{j_k})\leq E_{\psi_l}(P_\omega[\psi_l](u_{j_k}))$ for any $l\leq j_k$. Moreover, assuming $E_{\psi_{j_k}}(u_{j_k})$ is bounded from below, $-E_{\psi_l}(P_\omega[\psi_l](u_{j_k}))=d(\psi_l,P_\omega[\psi_l](u_{j_k}))$ is uniformly bounded in l,k, which implies that $\sup_X P_\omega[\psi_l](u_{j_k})$ is uniformly bounded by Remark 3.3 since $V_{\psi_{j_k}}\geq a>0$ for $k\gg 0$ big enough. By compactness, up to considering a subsequence, we obtain $P_\omega[\psi_l](u_{j_k})\to v_l$ weakly where $v_l\in\mathcal{E}^1(X,\omega,\psi_l)$ by the upper semicontinuity of $E_{\psi_l}(\cdot)$ on $\mathcal{E}^1(X,\omega,\psi_l)$. Hence

$$\limsup_{k\to\infty}E_{\psi_k}(u_k)\leq\limsup_{k\to\infty}E_{\psi_l}(P_\omega[\psi_l](u_{j_k}))+\epsilon=E_{\psi_l}(v_l)+\epsilon$$

for any $l\in\mathbb{N}$. Moreover, by construction, $v_l\leq P_\omega[\psi_l](u)$ since $P_\omega[\psi_l](u_{j_k})\leq u_{j_k}$ for any k such that $j_k\geq l$ and $u_{j_k}\to u$ weakly. Therefore by the monotonicity of $E_{\psi_l}(\cdot)$ and by Proposition 2.11 (ii), we conclude that

$$\limsup_{k\to\infty}E_{\psi_k}(u_k)\leq\lim_{l\to\infty}E_{\psi_l}(P_\omega[\psi_l](u))+\epsilon=E_\psi(u)+\epsilon$$

letting $l\to\infty$. □

As a consequence, defining

$$X_{\mathcal{A},C} := \bigsqcup_{\psi\in\bar{\mathcal{A}}} \mathcal{E}^1_C(X,\omega,\psi),$$

we get the following compactness result.

Proposition 3.15. *Let $C, a \in \mathbb{R}_{>0}$. The set*

$$X^a_{\mathcal{A},C} := X_{\mathcal{A},C} \cap \Bigg(\bigsqcup_{\psi\in\bar{\mathcal{A}}:V_\psi\geq a} \mathcal{E}^1(X,\omega,\psi)\Bigg)$$

is compact with respect to the weak topology.

Proof. It follows directly from the definition that

$$X^a_{\mathcal{A},C} \subset \big\{u \in \mathrm{PSH}(X,\omega) : \big|\sup_X u\big| \leq C'\big\},$$

where $C' := \max(C, C/a)$. Therefore by Proposition 8.5 in [Guedj and Zeriahi 2017], $X^a_{\mathcal{A},C}$ is weakly relatively compact. Finally Proposition 3.14 and Hartogs' lemma imply that $X^a_{\mathcal{A},C}$ is also closed with respect to the weak topology, concluding the proof. □

Remark 3.16. The whole set $X_{\mathcal{A},C}$ may not be weakly compact. Indeed, assuming $V_{\psi_{\min}} = 0$ and letting $\psi_k \in \bar{\mathcal{A}}$ such that $\psi_k \searrow \psi_{\min}$, the functions $u_k := \psi_k - 1/\sqrt{V_{\psi_k}}$ belong to $X_{\mathcal{A},V}$ for $V = \int_X \mathrm{MA}_\omega(0)$ since $E_{\psi_k}(u_k) = -\sqrt{V_{\psi_k}}$ but $\sup_X u_k = -1/\sqrt{V_{\psi_k}} \to -\infty$.

4. The action of measures on PSH(X,ω)

In this section we want to replace the action on PSH(X,ω) defined in [Berman et al. 2013] given by a probability measure μ with an action which assumes finite values on elements $u \in \mathrm{PSH}(X,\omega)$ with ψ-relative minimal singularities, where $\psi = P_\omega[u]$ for almost all $\psi \in \mathcal{M}$. On the other hand, for any $\psi \in \mathcal{M}$ we want there to exist many measures μ whose action over $\{u \in \mathrm{PSH}(X,\omega) : P_\omega[u] = \psi\}$ is well-defined. The problem is that μ varies among *all probability measures* while ψ varies among *all model-type envelopes*. So it may happen that μ takes mass on nonpluripolar sets and that the unbounded locus of $\psi \in \mathcal{M}$ is very nasty.

Definition 4.1. Let μ be a probability measure on X. Then μ acts on PSH(X,ω) through the functional $L_\mu : \mathrm{PSH}(X,\omega) \to \mathbb{R}\cup\{-\infty\}$ defined as $L_\mu(u) = -\infty$ if μ charges $\{P_\omega[u] = -\infty\}$, as

$$L_\mu(u) := \int_X (u - P_\omega[u])\mu$$

if u has $P_\omega[u]$-relative minimal singularities and μ does not charge $\{P_\omega[u] = -\infty\}$ and otherwise as

$$L_\mu(u) := \inf\{L_\mu(v) : v \in \mathrm{PSH}(X,\omega) \text{ with } P_\omega[u]\text{-relative minimal singularities}, v \geq u\}.$$

Proposition 4.2. *The following properties hold*:

(i) *L_μ is affine, i.e., it satisfies the scaling property $L_\mu(u+c) = L_\mu(u)+c$ for any $c \in \mathbb{R}$, $u \in \mathrm{PSH}(X,\omega)$.*

(ii) *L_μ is nondecreasing on $\{u \in \mathrm{PSH}(X,\omega) : P_\omega[u] = \psi\}$ for any $\psi \in \mathcal{M}$.*

(iii) $L_\mu(u)=\lim_{j\to\infty}L_\mu(\max(u, P_\omega[u]-j))$ *for any* $u\in \mathrm{PSH}(X,\omega)$.

(iv) *If* μ *is nonpluripolar, then* L_μ *is convex.*

(v) *If* μ *is nonpluripolar and* $u_k\to u$ *and* $P_\omega[u_k]\to P_\omega[u]$ *weakly as* $k\to\infty$, *then*

$$L_\mu(u)\geq \limsup_{k\to\infty} L_\mu(u_k).$$

(vi) *If* $u\in\mathcal{E}^1(X,\omega,\psi)$ *for* $\psi\in\mathcal{M}^+$, *then* $L_{\mathrm{MA}_\omega(u)/V_\psi}$ *is finite on* $\mathcal{E}^1(X,\omega,\psi)$.

Proof. The first two properties follow by definition.

For the third property, setting $\psi:=P_\omega[u]$, clearly $L_\mu(u)\leq \lim_{j\to\infty}L_\mu(\max(u,\psi-j))$. Conversely, for any $v\geq u$ with ψ-relative minimal singularities $v\geq \max(u,\psi-j)$ for $j\gg 0$ big enough, by (ii) we get $L_\mu(v)\geq \lim_{j\to\infty}L_\mu(\max(u,\psi-j))$ which implies (iii) by definition.

Next we prove (iv). Let $v=\sum_{l=1}^m a_l u_l$ be a convex combination of elements $u_l\in\mathrm{PSH}(X,\omega)$. Without loss of generality we may assume $\sup_X v, \sup_X u_l\leq 0$. In particular, we have $L_\mu(v), L_\mu(u_l)\leq 0$.

Suppose $L_\mu(v)>-\infty$ (otherwise it is trivial) and let $\psi:=P_\omega[v]$, $\psi_l:=P_\omega[u_l]$. Then for any $C\in\mathbb{R}_{>0}$ it is easy to see that

$$\sum_{l=1}^m a_l P_\omega(u_l+C,0)\leq P_\omega(v+C,0)\leq\psi,$$

which leads to $\sum_{l=1}^m a_l\psi_l\leq\psi$ letting $C\to\infty$. Hence (iii) yields

$$-\infty<L_\mu(v)=\int_X(v-\psi)\mu\leq\sum_{l=1}^n a_l\int_X(u_l-\psi_l)\mu=\sum_{l=1}^n a_l L_\mu(u_l).$$

Property (v) easily follows from $\limsup_{k\to\infty}\max(u_k, P_\omega[u_k]-j)\leq \max(u,P_\omega[u]-j)$ and (iii), while the last property is a consequence of Lemma 3.8. □

Next, since for any $t\in[0,1]$ and any $u,v\in\mathcal{E}^1(X,\omega,\psi)$

$$\int_X(u-v)\mathrm{MA}_\omega(tu+(1-t)v)=(1-t)^n\int_X(u-v)\mathrm{MA}_\omega(v)+\sum_{j=1}^n\binom{n}{j}t^j(1-t)^{n-j}\int_X(u-v)\mathrm{MA}_\omega(u^j,v^{n-j})$$
$$\geq(1-t)^n\int_X(u-v)\mathrm{MA}_\omega(v)+(1-(1-t)^n)\int_X(u-v)\mathrm{MA}_\omega(u),$$

we can proceed exactly as in [Berman et al. 2013, Proposition 3.4] (see also [Guedj and Zeriahi 2007, Lemma 2.11]), replacing V_θ with ψ, to get the following result.

Proposition 4.3. *Let* $A\subset\mathrm{PSH}(X,\omega)$ *and let* $L:A\to\mathbb{R}\cup\{-\infty\}$ *be a convex and nondecreasing function satisfying the scaling property* $L(u+c)=L(u)+c$ *for any* $c\in\mathbb{R}$.

(i) *If* L *is finite-valued on a weakly compact convex set* $K\subset A$, *then* $L(K)$ *is bounded.*

(ii) *If* $\mathcal{E}^1(X,\omega,\psi)\subset A$ *and* L *is finite-valued on* $\mathcal{E}^1(X,\omega,\psi)$, *then*

$$\sup_{\{u\in\mathcal{E}^1_C(X,\omega,\psi):\sup_X u\leq 0\}}|L|=O(C^{1/2})\quad\text{as } C\to\infty.$$

4A. ***When is L_μ continuous?*** The continuity of L_μ is a hard problem. However, we can characterize its continuity on some weakly compact sets as the next theorem shows.

Theorem 4.4. *Let μ be a nonpluripolar probability measure, and let $K \subset \mathrm{PSH}(X,\omega)$ be a compact convex set such that L_μ is finite on K, the set $\{P_\omega[u] : u \in K\} \subset \mathcal{M}$ is totally ordered and its closure in $\mathrm{PSH}(X,\omega)$ has at most one element in $\mathcal{M} \setminus \mathcal{M}^+$. Suppose also that there exists $C \in \mathbb{R}$ such that $|E_{P_\omega[u]}(u)| \le C$ for any $u \in K$. Then the following properties are equivalent*:

(i) *L_μ is continuous on K.*

(ii) *The map $\tau : K \to L^1(\mu)$, $\tau(u) := u - P_\omega[u]$ is continuous.*

(iii) *The set $\tau(K) \subset L^1(\mu)$ is uniformly integrable, i.e.,*

$$\int_{t=m}^{\infty} \mu\{u \le P_\omega[u] - t\} \to 0$$

as $m \to \infty$, uniformly for $u \in K$.

Proof. We first observe that if $u_k \in K$ converges to $u \in K$, then by Lemma 3.13, $\psi_k \to \psi$, where we set $\psi_k := P_\omega[u_k]$ and $\psi := P_\omega[u]$.

Then we can proceed exactly as in [Berman et al. 2013, Theorem 3.10] to get the equivalence between (i) and (ii), (ii) $\Rightarrow$ (iii) and the fact that the graph of τ is closed. It is important to emphasize that (iii) is equivalent to saying that $\tau(K)$ is *weakly* relative compact by the Dunford–Pettis theorem, i.e., with respect to the weak topology on $L^1(\mu)$ induced by $L^\infty(\mu) = L^1(\mu)^*$.

Finally, assuming that (iii) holds it remains to prove (i). So, letting $u_k, u \in K$ such that $u_k \to u$, we have to show that $\int_X \tau(u_k)\mu \to \int_X \tau(u)\mu$. Since $\tau(K) \subset L^1(\mu)$ is bounded, unless considering a subsequence, we may suppose $\int_X \tau(u_k) \to L \in \mathbb{R}$. By Fatou's lemma,

$$L = \lim_{k\to\infty} \int_X \tau(u_k)\mu \le \int_X \tau(u)\mu. \tag{12}$$

Then for any $k \in \mathbb{N}$ the closed convex envelope

$$C_k := \overline{\mathrm{Conv}\{\tau(u_j) : j \ge k\}}$$

is weakly closed in $L^1(\mu)$ by the Hahn–Banach theorem, which implies that C_k is weakly compact since it is contained in $\tau(K)$. Thus since C_k is a decreasing sequence of nonempty weakly compact sets, there exists $f \in \bigcap_{k\ge 1} C_k$ and there exist elements $v_k \in \mathrm{Conv}(u_j : j \ge k)$ given as finite convex combinations such that $\tau(v_k) \to f$ in $L^1(\mu)$. Moreover, by the closed graph property, $f = \tau(u)$ since $v_k \to u$ as a consequence of $u_k \to u$. On the other hand, by Proposition 4.2 (iv) we get

$$\int_X \tau(v_k)\mu \le \sum_{l=1}^{m_k} a_{l,k} \int_X \tau(u_{k_l})\mu$$

if $v_k = \sum_{l=1}^{m_k} a_{l,k} u_{k_l}$. Hence $L \ge \int_X \tau(u)\mu$, which together with (12) implies $L = \int_X \tau(u)\mu$. ☐

Corollary 4.5. *Let $\psi \in \mathcal{M}^+$ and $\mu \in \mathcal{C}_{A,\psi}$. Then L_μ is continuous on $\mathcal{E}^1_C(X,\omega,\psi)$ for any $C \in \mathbb{R}_{>0}$. In particular, if $\mu = \mathrm{MA}_\omega(u)/V_\psi$ for $u \in \mathcal{E}^1(X,\omega,\psi)$ with ψ-relative minimal singularities, then L_μ is continuous on $\mathcal{E}^1_C(X,\omega,\psi)$ for any $C \in \mathbb{R}_{>0}$.*

Proof. With the notation of Theorem 4.4, $\tau(\mathcal{E}^1_C(X,\omega,\psi))$ is bounded in $L^2(\mu)$ by Lemma 3.11. Hence by Holder's inequality $\tau(\mathcal{E}^1_C(X,\omega,\psi))$ is uniformly integrable and Theorem 4.4 yields the continuity of L_μ on $\mathcal{E}^1_C(X,\omega,\psi)$ for any $C \in \mathbb{R}_{>0}$.

The last assertion follows directly from Proposition 3.10. □

The following lemma will be essential to prove Theorem A and Theorem B.

Lemma 4.6. *Let $\varphi \in \mathcal{H}_\omega$ and let $\mathcal{A} \subset \mathcal{M}$ be a totally ordered subset. Set also $v_\psi := P_\omega[\psi](\varphi)$ for any $\psi \in \mathcal{A}$. Then the actions $\{V_\psi L_{\mathrm{MA}_\omega(v_\psi)/V_\psi}\}_{\psi\in\mathcal{A}}$ take finite values and they are equicontinuous on any compact set $K \subset \mathrm{PSH}(X,\omega)$ such that $\{P_\omega[u] : u \in K\}$ is a totally ordered set whose closure in $\mathrm{PSH}(X,\omega)$ has at most one element in $\mathcal{M}\setminus\mathcal{M}^+$ and such that $|E_{P_\omega[u]}(u)| \le C$ uniformly for any $u \in K$. If $\psi \in \mathcal{M}\setminus\mathcal{M}^+$, for the action $V_\psi L_{\mathrm{MA}_\omega(v_\psi)/V_\psi}$ we mean the null action. In particular, if $\psi_k \to \psi$ monotonically almost everywhere and $\{u_k\}_{k\in\mathbb{N}} \subset K$ converges weakly to $u \in K$, then*

$$\int_X (u_k - P_\omega[u_k])\,\mathrm{MA}_\omega(v_{\psi_k}) \to \int_X (u - P_\omega[u])\,\mathrm{MA}_\omega(v_\psi). \tag{13}$$

Proof. By Theorem 2.2,

$$|V_\psi L_{\mathrm{MA}_\omega(v_\psi)/V_\psi}(u)| \le \int_X |u - P_\omega[u]|\,\mathrm{MA}_\omega(\varphi)$$

for any $u \in \mathrm{PSH}(X,\omega)$ and any $\psi \in \mathcal{A}$, so the actions in the statement assume finite values. Then the equicontinuity on any weak compact set $K \subset \mathrm{PSH}(X,\omega)$ satisfying the assumptions of the lemma follows from

$$V_\psi \left| L_{\mathrm{MA}_\omega(v_\psi)/V_\psi}(w_1) - L_{\mathrm{MA}_\omega(v_\psi)/V_\psi}(w_2) \right| \le \int_X |w_1 - P_\omega[w_1] - w_2 + P_\omega[w_2]|\,\mathrm{MA}_\omega(\varphi)$$

for any $w_1, w_2 \in \mathrm{PSH}(X,\omega)$ since $\mathrm{MA}_\omega(\varphi)$ is a volume form on X and $P_\omega[w_k] \to P_\omega[w]$ if $\{w_k\}_{k\in\mathbb{N}} \subset K$ converges to $w \in K$ under our hypothesis by Lemma 3.13.

For the second assertion, if $\psi_k \searrow \psi$ (resp. $\psi_k \nearrow \psi$ almost everywhere), letting $f_k, f \in L^\infty$ such that $\mathrm{MA}_\omega(v_{\psi_k}) = f_k\,\mathrm{MA}_\omega(\varphi)$ and $\mathrm{MA}_\omega(v_\psi) = f\,\mathrm{MA}_\omega(\varphi)$ (Theorem 2.2), we have $0 \le f_k \le 1$, $0 \le f \le 1$ and $\{f_k\}_{k\in\mathbb{N}}$ is a monotone sequence. Therefore $f_k \to f$ in L^p for any $p > 1$ as $k \to \infty$, which implies

$$\int_X (u - P_\omega[u])\,\mathrm{MA}_\omega(v_{\psi_k}) \to \int_X (u - P_\omega[u])\,\mathrm{MA}_\omega(v_\psi)$$

as $k \to \infty$ since $\mathrm{MA}_\omega(\varphi)$ is a volume form. Hence (13) follows since by the first part of the proof,

$$\int_X (u_k - P_\omega[u_k] - u + P_\omega[u])\,\mathrm{MA}_\omega(v_{\psi_k}) \to 0.$$

□

5. Theorem A

In this section we fix $\psi \in \mathcal{M}^+$ and, using a variational approach, we first prove the bijectivity of the Monge–Ampère operator between $\mathcal{E}^1_{\mathrm{norm}}(X,\omega,\psi)$ and $\mathcal{M}^1(X,\omega,\psi)$, and then we prove that it is actually a homeomorphism considering the strong topologies.

5A. ***Degenerate complex Monge–Ampère equations.*** Letting μ be a probability measure and $\psi \in \mathcal{M}$, we define the functional $F_{\mu,\psi} : \mathcal{E}^1(X, \omega, \psi) \to \mathbb{R} \cup \{-\infty\}$ as

$$F_{\mu,\psi}(u) := (E_\psi - V_\psi L_\mu)(u),$$

where we recall from Section 4 that

$$\begin{aligned} L_\mu(u) &= \lim_{j\to\infty} L_\mu(\max(u, \psi - j)) \\ &= \lim_{j\to\infty} \int_X (\max(u, \psi - j) - \psi)\mu. \end{aligned}$$

$F_{\mu,\psi}$ is clearly a translation invariant functional, and $F_{\mu,\psi} \equiv 0$ for any μ if $V_\psi = 0$.

Proposition 5.1. *Let μ be a probability measure, $\psi \in \mathcal{M}^+$ and let $F := F_{\mu,\psi}$. If L_μ is continuous then F is upper semicontinuous on $\mathcal{E}^1(X, \omega, \psi)$. Moreover, if L_μ is finite-valued on $\mathcal{E}^1(X, \omega, \psi)$, then there exist $A, B > 0$ such that*

$$F(v) \le -A\, d(\psi, v) + B$$

*for any $v \in \mathcal{E}^1_{\text{norm}}(X, \omega, \psi)$, i.e., F is **d-coercive**. In particular, F is upper semicontinuous on $\mathcal{E}^1(X, \omega, \psi)$ and d-coercive on $\mathcal{E}^1_{\text{norm}}(X, \omega, \psi)$ if $\mu = \text{MA}_\omega(u)/V_\psi$ for $u \in \mathcal{E}^1(X, \omega, \psi)$.*

Proof. If L_μ is continuous then F is easily upper semicontinuous by Proposition 2.4.

Then, since $d(\psi, v) = -E_\psi(v)$ on $\mathcal{E}^1_{\text{norm}}(X, \omega, \psi)$, it is easy to check that the coercivity requested is equivalent to

$$\sup_{\mathcal{E}^1_C(X,\omega,\psi)\cap\mathcal{E}^1_{\text{norm}}(X,\omega,\psi)} |L_\mu| \le \frac{(1-A)}{V_\psi} C + O(1),$$

which holds by Proposition 4.3 (ii).

Next assuming $\mu = \text{MA}_\omega(u)/V_\psi$, it is sufficient to check the continuity of L_μ since L_μ is finite-valued on $\mathcal{E}^1(X, \omega, \psi)$ by Proposition 4.2. We may suppose without loss of generality that $u \le \psi$. By Proposition 3.7 and Remark 3.3, for any $C \in \mathbb{R}_{>0}$, L_μ restricted to $\mathcal{E}^1_C(X, \omega, \psi)$ is the uniform limit of L_{μ_j}, where $\mu_j := \text{MA}_\omega(\max(u, \psi - j))$, since $I_\psi(\max(u, \psi - j), u) \to 0$ as $j \to \infty$. Therefore L_μ is continuous on $\mathcal{E}^1_C(X, \omega, \psi)$ because of the uniform limit of continuous functionals L_{μ_j} (Corollary 4.5). □

Because of the concavity of E_ψ, if $\mu = \text{MA}_\omega(u)/V_\psi$ for $u \in \mathcal{E}^1(X, \omega, \psi)$ where $V_\psi > 0$, then

$$J^\psi_u(\psi) = F_{\mu,\psi}(u) = \sup_{\mathcal{E}^1(X,\omega,\psi)} F_{\mu,\psi},$$

i.e., u is a maximizer of $F_{\mu,\psi}$. The other way around also holds as the next result shows.

Proposition 5.2. *Let $\psi \in \mathcal{M}^+$ and let μ be a probability measure such that L_μ is finite-valued on $\mathcal{E}^1(X, \omega, \psi)$. Then $\mu = \text{MA}_\omega(u)/V_\psi$ for $u \in \mathcal{E}^1(X, \omega, \psi)$ if and only if u is a maximizer of $F_{\mu,\psi}$.*

Proof. As said before, it is clear that $\mu = \text{MA}_\omega(u)/V_\psi$ implies that u is a maximizer of $F_{\mu,\psi}$. Conversely, if u is a maximizer of $F_{\mu,\psi}$, then by [Darvas et al. 2018, Theorem 4.22], $\mu = \text{MA}_\omega(u)/V_\psi$. □

Similarly to [Berman et al. 2013] we thus define the *ψ-relative energy* for $\psi \in \mathcal{M}$ of a probability measure μ as

$$E^*_\psi(\mu) := \sup_{u \in \mathcal{E}^1(X,\omega,\psi)} F_{\mu,\psi}(u),$$

i.e., essentially as the Legendre transform of E_ψ. It takes nonnegative values ($F_{\mu,\psi}(\psi) = 0$), and it is easy to check that E^*_ψ is a convex function.

Moreover, defining

$$\mathcal{M}^1(X, \omega, \psi) := \{V_\psi \mu : \mu \text{ is a probability measure satisfying } E^*_\psi(\mu) < \infty\},$$

we note that $\mathcal{M}^1(X, \omega, \psi)$ consists only of the null measure if $V_\psi = 0$, while if $V_\psi > 0$, any probability measure μ such that $V_\psi \mu \in \mathcal{M}^1(X, \omega, \psi)$ is nonpluripolar as the next lemma shows.

Lemma 5.3. *Let $A \subset X$ be a (locally) pluripolar set. Then there exists $u \in \mathcal{E}^1(X, \omega, \psi)$ such that $A \subset \{u = -\infty\}$. In particular, if $V_\psi \mu \in \mathcal{M}^1(X, \omega, \psi)$ for $\psi \in \mathcal{M}^+$, then μ is nonpluripolar.*

Proof. By [Berman et al. 2013, Corollary 2.11], there exists $\varphi \in \mathcal{E}^1(X, \omega)$ such that $A \subset \{\varphi = -\infty\}$. Therefore setting $u := P_\omega[\psi](\varphi)$ proves the first part.

Next, let $V_\psi \mu \in \mathcal{M}^1(X, \omega, \psi)$ for $\psi \in \mathcal{M}^+$ and let μ be a probability measure, and assume by contradiction that μ takes mass on a pluripolar set A. Then by the first part of the proof there exists $u \in \mathcal{E}^1(X, \omega, \psi)$ such that $A \subset \{u = -\infty\}$. On the other hand, since $V_\psi \mu \in \mathcal{M}^1(X, \omega, \psi)$, by definition μ does not charge $\{\psi = -\infty\}$. Thus by Proposition 4.2 (iii) we obtain $L_\mu(u) = -\infty$, a contradiction. □

We now prove that the Monge–Ampère operator is a bijection between $\mathcal{E}^1(X, \omega, \psi)$ and $\mathcal{M}^1(X, \omega, \psi)$.

Lemma 5.4. *Let $\psi \in \mathcal{M}^+$ and $\mu \in \mathcal{C}_{A,\psi}$, where $A \in \mathbb{R}$. Then there exists $u \in \mathcal{E}^1_{\text{norm}}(X, \omega, \psi)$ maximizing $F_{\mu,\psi}$.*

Proof. By Lemma 3.11, L_μ is finite-valued on $\mathcal{E}^1(X, \omega, \psi)$, and it is continuous on $\mathcal{E}^1_C(X, \omega, \psi)$ for any $C \in \mathbb{R}$ thanks to Corollary 4.5. Therefore it follows from Proposition 5.1 that $F_{\mu,\psi}$ is upper semicontinuous and d-coercive on $\mathcal{E}^1_{\text{norm}}(X, \omega, \psi)$. Hence $F_{\mu,\psi}$ admits a maximizer $u \in \mathcal{E}^1_{\text{norm}}(X, \omega, \psi)$ as an easy consequence of the weak compactness of $\mathcal{E}^1_C(X, \omega, \psi)$. □

Proposition 5.5. *Let $\psi \in \mathcal{M}^+$. Then the Monge–Ampère map* $\text{MA} : \mathcal{E}^1_{\text{norm}}(X, \omega, \psi) \to \mathcal{M}^1(X, \omega, \psi)$, $u \to \text{MA}(u)$, *is bijective. Furthermore, if $V_\psi \mu = \text{MA}_\omega(u) \in \mathcal{M}^1(X, \omega, \psi)$ for $u \in \mathcal{E}^1(X, \omega, \psi)$, then any maximizing sequence $u_k \in \mathcal{E}^1_{\text{norm}}(X, \omega, \psi)$ for $F_{\mu,\psi}$ necessarily converges weakly to u.*

Proof. The proof is inspired by [Berman et al. 2013, Theorem 4.7].

The map is well-defined as a consequence of Proposition 5.1, i.e., $\text{MA}_\omega(u) \in \mathcal{M}^1(X, \omega, \psi)$ for any $u \in \mathcal{E}^1(X, \omega, \psi)$. Moreover, the injectivity follows from [Darvas et al. 2021a, Theorem 4.8].

Let $u_k \in \mathcal{E}^1_{\text{norm}}(X, \omega, \psi)$ be a sequence such that $F_{\mu,\psi}(u_k) \nearrow \sup_{\mathcal{E}^1(X,\omega,\psi)} F_{\mu,\psi}$, where $\mu = \text{MA}_\omega(u)/V_\psi$ is a probability measure and $u \in \mathcal{E}^1_{\text{norm}}(X, \omega, \psi)$. Up to considering a subsequence, we may also assume that $u_k \to v \in \text{PSH}(X, \omega)$. Then, by the upper semicontinuity and d-coercivity of $F_{\mu,\psi}$ (Proposition 5.1), it follows that $v \in \mathcal{E}^1_{\text{norm}}(X, \omega, \psi)$ and $F_{\mu,\psi}(v) = \sup_{\mathcal{E}^1(X,\omega,\psi)} F_{\mu,\psi}$. Thus by Proposition 5.2 we get $\mu = \text{MA}_\omega(v)/V_\psi$. Hence $v = u$ since $\sup_X v = \sup_X u = 0$.

Then let μ be a probability measure such that $V_\psi \mu \in \mathcal{M}^1(X, \omega, \psi)$. Again by Proposition 5.2, to prove the existence of $u \in \mathcal{E}^1_{\text{norm}}(X, \omega, \psi)$ such that $\mu = \text{MA}_\omega(u)/V_\psi$ it is sufficient to check that $F_{\mu,\psi}$ admits a maximum over $\mathcal{E}^1_{\text{norm}}(X, \omega, \psi)$. Moreover by Proposition 5.1, we also know that $F_{\mu,\psi}$ is d-coercive on $\mathcal{E}^1_{\text{norm}}(X, \omega, \psi)$. Thus if there exists a constant $A > 0$ such that $\mu \in \mathcal{C}_{A,\psi}$, then Corollary 4.5 leads to the upper semicontinuity of $F_{\mu,\psi}$, which clearly implies that $V_\psi \mu = \text{MA}_\omega(u)$ for $u \in \mathcal{E}^1(X, \omega, \psi)$ since $\mathcal{E}^1_C(X, \omega, \psi) \subset \text{PSH}(X, \omega)$ is compact for any $C \in \mathbb{R}_{>0}$.

In the general case, by [Darvas et al. 2018, Lemma 4.26] (see also [Cegrell 1998]), μ is absolutely continuous with respect to $\nu \in \mathcal{C}_{1,\psi}$ using also that μ is a nonpluripolar measure (Lemma 5.3). Therefore, letting $f \in L^1(\nu)$ such that $\mu = f\nu$, we define for any $k \in \mathbb{N}$

$$\mu_k := (1 + \epsilon_k) \min(f, k)\nu,$$

where the $\epsilon_k > 0$ are chosen such that μ_k is a probability measure, noting that $(1 + \epsilon_k) \min(f, k) \to f$ in $L^1(\nu)$. Then by Lemma 5.4 it follows that $\mu_k = \text{MA}_\omega(u_k)/V_\psi$ for $u_k \in \mathcal{E}^1_{\text{norm}}(X, \omega, \psi)$.

Moreover, by weak compactness we may also assume that $u_k \to u \in \text{PSH}(X, \omega)$, without loss of generality. Note that $u \le \psi$ since $u_k \le \psi$ for any $k \in \mathbb{N}$. Then by [Darvas et al. 2021a, Lemma 2.8] we obtain

$$\text{MA}_\omega(u) \ge V_\psi f \nu = V_\psi \mu,$$

which implies $\text{MA}_\omega(u) = V_\psi \mu$ by [Witt Nyström 2019] since u is more singular than ψ and μ is a probability measure. It remains to prove that $u \in \mathcal{E}^1(X, \omega, \psi)$.

It is not difficult to see that $\mu_k \le 2\mu$ for $k \gg 0$, thus Proposition 4.3 implies that there exists a constant $B > 0$ such that

$$\sup_{\mathcal{E}^1_C(X,\omega,\psi)} |L_{\mu_k}| \le 2 \sup_{\mathcal{E}^1_C(X,\omega,\psi)} |L_\mu| \le 2B(1 + C^{1/2})$$

for any $C \in \mathbb{R}_{>0}$. Therefore

$$J^\psi_{u_k}(\psi) = E_\psi(u_k) + V_\psi |L_{\mu_k}(u_k)| \le \sup_{C>0} (2V_\psi B(1 + C^{1/2}) - C),$$

and Lemma 3.1 yields $d(\psi, u_k) \le D$ for a uniform constant D, i.e., $u_k \in \mathcal{E}^1_{D'}(X, \omega, \psi)$ for any $k \in \mathbb{N}$ for a uniform constant D'; see Remark 3.3. Hence since $\mathcal{E}^1_{D'}(X, \omega, \psi)$ is weakly compact we obtain $u \in \mathcal{E}^1_{D'}(X, \omega, \psi)$. □

5B. *Proof of Theorem A.* We further explore the properties of the strong topology on $\mathcal{E}^1(X, \omega, \psi)$.

By Proposition 3.6, the strong convergence implies the weak convergence. Moreover, the strong topology is the coarsest refinement of the weak topology such that $E_\psi(\cdot)$ becomes continuous.

Proposition 5.6. *Let $\psi \in \mathcal{M}^+$ and $u_k, u \in \mathcal{E}^1(X, \omega, \psi)$. Then $u_k \to u$ strongly if and only if $u_k \to u$ weakly and $E_\psi(u_k) \to E_\psi(u)$.*

Proof. Assume $u_k \to u$ weakly and $E_\psi(u_k) \to E_\psi(u)$. Then $w_k := (\sup\{u_j : j \ge k\})^* \in \mathcal{E}^1(X, \omega, \psi)$ and it decreases to u. Thus by Proposition 2.4, $E_\psi(w_k) \to E_\psi(u)$ and

$$d(u_k, u) \le d(u_k, w_k) + d(w_k, u) = 2E_\psi(w_k) - E_\psi(u_k) - E_\psi(u) \to 0.$$

Conversely, assuming that $d(u_k, u) \to 0$, we immediately get that $u_k \to u$ weakly as said above; see Proposition 3.6. Moreover, $\sup_X u_k, \sup_X u \leq A$ uniformly for a constant $A \in \mathbb{R}$. Thus

$$|E_\psi(u_k) - E_\psi(u)| = |d(\psi + A, u_k) - d(\psi + A, u)| \leq d(u_k, u) \to 0. \qquad \square$$

We also observe that the strong convergence implies the convergence in ψ'-capacity for any $\psi' \in \mathcal{M}^+$.

Proposition 5.7. *Let $\psi \in \mathcal{M}^+$ and $u_k, u \in \mathcal{E}^1(X, \omega, \psi)$ such that $d(u_k, u) \to 0$. Then there exists a subsequence $\{u_{k_j}\}_{j\in\mathbb{N}}$ such that $w_j := (\sup\{u_{k_h} : h \geq j\})^*$ and $v_j := P_\omega(u_{k_j}, u_{k_{j+1}}, \dots)$ belong to $\mathcal{E}^1(X, \omega, \psi)$ and converge monotonically almost everywhere to u. In particular, $u_k \to u$ in ψ'-capacity for any $\psi' \in \mathcal{M}^+$, and $\mathrm{MA}_\omega(u_k^j, \psi^{n-j}) \to \mathrm{MA}_\omega(u^j, \psi^{n-j})$ weakly for any $j = 0, \dots, n$.*

Proof. Since the strong convergence implies the weak convergence by Proposition 5.6, it is clear that $w_k \in \mathcal{E}^1(X, \omega, \psi)$ and that it decreases to u. In particular, up to considering a subsequence we may assume that $d(u_k, w_k) \leq 1/2^k$ for any $k \in \mathbb{N}$.

Next for any $j \geq k$, set $v_{k,j} := P_\omega(u_k, \dots, u_j) \in \mathcal{E}^1(X, \omega, \psi)$ and $v^u_{k,j} := P_\omega(v_{k,j}, u) \in \mathcal{E}^1(X, \omega, \psi)$. Then it follows from Proposition 2.4 and [Darvas et al. 2018, Lemma 3.7] that

$$\begin{aligned} d(u, v^u_{k,j}) &\leq \int_X (u - v^u_{k,j}) \,\mathrm{MA}_\omega(v^u_{k,j}) \leq \int_{\{v^u_{k,j} = v_{k,j}\}} (u - v_{k,j}) \,\mathrm{MA}_\omega(v_{k,j}) \\ &\leq \sum_{s=k}^{j} \int_X (w_s - u_s) \,\mathrm{MA}_\omega(u_s) \leq (n+1) \sum_{s=k}^{j} d(w_s, u_s) \leq \frac{n+1}{2^{k-1}}. \end{aligned}$$

Therefore by Proposition 3.15, $v^u_{k,j}$ decreases (hence converges strongly) to a function $\phi_k \in \mathcal{E}^1(X, \omega, \psi)$ as $j \to \infty$. Similarly we also observe that

$$d(v_{k,j}, v^u_{k,j}) \leq \int_{\{v^u_{k,j} = u\}} (v_{k,j} - u) \,\mathrm{MA}_\omega(u) \leq \int_X |v_{k,1} - u| \,\mathrm{MA}_\omega(u) \leq C$$

uniformly in j by Corollary 3.5. Hence by definition, $d(u, v_{k,j}) \leq C + (n+1)/2^{k-1}$, i.e., $v_{k,j}$ decreases and converges strongly as $j \to \infty$ to the function $v_k = P_\omega(u_k, u_{k+1}, \dots) \in \mathcal{E}^1(X, \omega, \psi)$, again by Proposition 3.15. Moreover, by construction, $u_k \geq v_k \geq \phi_k$ since $v_k \leq v_{k,j} \leq u_k$ for any $j \geq k$. Hence

$$d(u, v_k) \leq d(u, \phi_k) \leq \frac{n+1}{2^{k-1}} \to 0$$

as $k \to \infty$, i.e., $v_k \nearrow u$ strongly.

The convergence in ψ'-capacity for $\psi' \in \mathcal{M}^+$ is now clearly an immediate consequence. Indeed by an easy contradiction argument it is enough to prove that any arbitrary subsequence, which we will keep denoting by $\{u_k\}_{k\in\mathbb{N}}$ for the sake of simplicity, admits a further subsequence $\{u_{k_j}\}_{j\in\mathbb{N}}$ converging in ψ'-capacity to u. Thus taking the subsequence satisfying $v_j \leq u_{k_j} \leq w_j$, where v_j, w_j are the monotonic sequences of the first part of the proposition, the convergence in ψ'-capacity follows from the inclusions

$$\{|u - u_{k_j}| > \delta\} = \{u - u_{k_j} > \delta\} \cup \{u_{k_j} - u > \delta\} \subset \{u - v_j > \delta\} \cup \{w_j - u > \delta\}$$

for any $\delta > 0$. Finally Lemma 2.12 gives the weak convergence of the measures. $\square$

We now endow the set $\mathcal{M}^1(X,\omega,\psi)=\{V_\psi\mu : \mu \text{ is a probability measure satisfying } E^*_\psi(\mu)<+\infty\}$ (Section 5A) with its natural strong topology given as the coarsest refinement of the weak topology such that $E^*_\psi(\,\cdot\,)$ becomes continuous and prove Theorem A.

Theorem A. *Let $\psi\in\mathcal{M}^+$. Then*

$$\mathrm{MA}_\omega : (\mathcal{E}^1_{\mathrm{norm}}(X,\omega,\psi),d)\to(\mathcal{M}^1(X,\omega,\psi),\text{strong})$$

is a homeomorphism.

Proof. The map is bijective as an immediate consequence of Proposition 5.5.

Next, letting the $u_k\in\mathcal{E}^1_{\mathrm{norm}}(X,\omega,\psi)$ converge strongly to $u\in\mathcal{E}^1_{\mathrm{norm}}(X,\omega,\psi)$, Proposition 5.7 gives the weak convergence of $\mathrm{MA}_\omega(u_k)\to\mathrm{MA}_\omega(u)$ as $k\to\infty$. Moreover, since $E^*_\psi(\mathrm{MA}_\omega(v)/V_\psi)=J^\psi_v(\psi)$ for any $v\in\mathcal{E}^1(X,\omega,\psi)$, we get

$$\begin{aligned}|E^*_\psi(\mathrm{MA}_\omega(u_k)/V_\psi)&-E^*_\psi(\mathrm{MA}_\omega(u)/V_\psi)|\\ &\le|E_\psi(u_k)-E_\psi(u)|+\left|\int_X(\psi-u_k)\,\mathrm{MA}_\omega(u_k)-\int_X(\psi-u)\,\mathrm{MA}_\omega(u)\right|\\ &\le|E_\psi(u_k)-E_\psi(u)|+\left|\int_X(\psi-u_k)(\mathrm{MA}_\omega(u_k)-\mathrm{MA}_\omega(u))\right|+\int_X|u_k-u|\,\mathrm{MA}_\omega(u).\end{aligned}\tag{14}$$

Hence $\mathrm{MA}_\omega(u_k)\to\mathrm{MA}_\omega(u)$ strongly in $\mathcal{M}^1(X,\omega,\psi)$ since each term on the right-hand side of (14) goes to 0 as $k\to+\infty$, combining Proposition 5.6, Proposition 3.7 and Corollary 3.5, and recalling that by Proposition 3.4, $I_\psi(u_k,u)\to 0$ as $k\to\infty$.

Conversely, suppose that $\mathrm{MA}_\omega(u_k)\to\mathrm{MA}_\omega(u)$ strongly in $\mathcal{M}^1(X,\omega,\psi)$, where $u_k,u\in\mathcal{E}^1_{\mathrm{norm}}(X,\omega,\psi)$. Then, letting $\{\varphi_j\}_{j\in\mathbb{N}}\subset\mathcal{H}_\omega$ such that $\varphi_j\searrow u$ [Błocki and Kołodziej 2007] and setting $v_j:=P_\omega[\psi](\varphi_j)$, by Lemma 3.1,

$$\begin{aligned}(n+1)I_\psi(u_k,v_j)&\le E_\psi(u_k)-E_\psi(v_j)+\int_X(v_j-u_k)\,\mathrm{MA}_\omega(u_k)\\ &=E^*_\psi(\mathrm{MA}_\omega(u_k)/V_\psi)-E^*_\psi(\mathrm{MA}_\omega(v_j)/V_\psi)+\int_X(v_j-\psi)(\mathrm{MA}_\omega(u_k)-\mathrm{MA}_\omega(v_j)).\end{aligned}\tag{15}$$

By construction and the first part of the proof, it follows that $E^*_\psi(\mathrm{MA}_\omega(u_k)/V_\psi)-E^*_\psi(\mathrm{MA}_\omega(v_j)/V_\psi)\to 0$ as $k,j\to\infty$. Setting $f_j:=v_j-\psi$, we want to prove

$$\limsup_{k\to\infty}\int_X f_j\,\mathrm{MA}_\omega(u_k)=\int_X f_j\,\mathrm{MA}_\omega(u),$$

which would imply $\limsup_{j\to\infty}\limsup_{k\to\infty}I_\psi(u_k,v_j)=0$ since $\int_X f_j(\mathrm{MA}_\omega(u)-\mathrm{MA}_\omega(v_j))\to 0$ as a consequence of Propositions 3.7 and 3.4.

We observe that $\|f_j\|_{L^\infty}\le\|\varphi_j\|_{L^\infty}$ by Proposition 2.10, and we denote by $\{f^s_j\}_{s\in\mathbb{N}}\subset C^\infty$ a sequence of smooth functions converging in capacity to f_j such that $\|f^s_j\|_{L^\infty}\le 2\|f_j\|_{L^\infty}$. Here we briefly recall how to construct such a sequence. Let $\{g^s_j\}_{s\in\mathbb{N}}$ be the sequence of bounded functions converging in capacity to f_j defined as $g^s_j:=\max(v_j,-s)-\max(\psi,-s)$. We have that $\|g^s_j\|_{L^\infty}\le\|f_j\|_{L^\infty}$ and that $\max(v_j,-s),\ \max(\psi,-s)\in\mathrm{PSH}(X,\omega)$. By a regularization process (see [Błocki and Kołodziej 2007]) and a diagonal argument we can now construct a sequence $\{f_j^s\}_{j\in\mathbb{N}} \subset C^\infty$ converging in capacity to f_j such that $\|f_j^s\|_{L^\infty} \leq 2\|g_j^s\| \leq 2\|f_j\|_{L^\infty}$, where $f_j^s = v_j^s - \psi^s$ with v_j^s, ψ^s quasi-psh functions decreasing to v_j, ψ, respectively.

Then letting $\delta > 0$ we have

$$\begin{aligned}\int_X (f_j - f_j^s)\,\mathrm{MA}_\omega(u_k) &\leq \delta V_\psi + 3\|\varphi_j\|_{L^\infty} \int_{\{f_j - f_j^s > \delta\}} \mathrm{MA}_\omega(u_k)\\ &\leq \delta V_\psi + 3\|\varphi_j\|_{L^\infty} \int_{\{\psi^s - \psi > \delta\}} \mathrm{MA}_\omega(u_k)\end{aligned}$$

from the trivial inclusion $\{f_j - f_j^s > \delta\} \subset \{\psi^s - \psi > \delta\}$. Therefore

$$\begin{aligned}\limsup_{s\to\infty}\limsup_{k\to\infty} \int_X (f_j - f_j^s)\,\mathrm{MA}_\omega(u_k) &\leq \delta V_\psi + \limsup_{s\to\infty}\limsup_{k\to\infty} \int_{\{\psi^s - \psi \geq \delta\}} \mathrm{MA}_\omega(u_k)\\ &\leq \delta V_\psi + \limsup_{s\to\infty} \int_{\{\psi^s - \psi \geq \delta\}} \mathrm{MA}_\omega(u) = \delta V_\psi,\end{aligned}$$

where we used that $\{\psi^s - \psi \geq \delta\}$ is a closed set in the plurifine topology. Hence since $f_j^s \in C^\infty$ we obtain

$$\begin{aligned}\limsup_{k\to\infty} \int_X f_j\,\mathrm{MA}_\omega(u_k) &= \limsup_{s\to\infty}\limsup_{k\to\infty}\biggl(\int_X (f_j - f_j^s)\,\mathrm{MA}_\omega(u_k) + \int_X f_j^s\,\mathrm{MA}_\omega(u_k)\biggr)\\ &\leq \limsup_{s\to\infty} \int_X f_j^s\,\mathrm{MA}_\omega(u) = \int_X f_j\,\mathrm{MA}_\omega(u),\end{aligned}$$

which as said above implies $I_\psi(u_k, v_j) \to 0$ letting $k, j \to \infty$ in this order.

Next we obtain $u_k \in \mathcal{E}^1_C(X, \omega, \psi)$ for some $C \in \mathbb{N}$ big enough since $J^\psi_{u_k}(\psi) = E^*_\psi(\mathrm{MA}_\omega(u_k)/V_\psi)$, again by Lemma 3.1. In particular, up to considering a subsequence, $u_k \to w \in \mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi)$ weakly by Proposition 3.15. Observe also that by Proposition 3.7,

$$\biggl|\int_X (\psi - u_k)(\mathrm{MA}_\omega(v_j) - \mathrm{MA}_\omega(u_k))\biggr| \to 0 \tag{16}$$

as $k, j \to \infty$ in this order. Moreover, by Proposition 3.14 and Lemma 4.6,

$$\begin{aligned}&\limsup_{k\to\infty}\biggl(E^*_\psi(\mathrm{MA}_\omega(u_k)/V_\psi) + \int_X (\psi - u_k)(\mathrm{MA}_\omega(v_j) - \mathrm{MA}_\omega(u_k))\biggr)\\ &\qquad = \limsup_{k\to\infty}\biggl(E_\psi(u_k) + \int_X (\psi - u_k)\,\mathrm{MA}_\omega(v_j)\biggr) \leq E_\psi(w) + \int_X (\psi - w)\,\mathrm{MA}_\omega(v_j).\end{aligned} \tag{17}$$

Therefore combining (16) and (17) with the strong convergence of v_j to u we obtain

$$\begin{aligned}E_\psi(u) + \int_X (\psi - u)\,\mathrm{MA}_\omega(u) &= \lim_{k\to\infty} E^*_\psi(\mathrm{MA}_\omega(u_k)/V_\psi)\\ &\leq \limsup_{j\to\infty}\biggl(E_\psi(w) + \int_X (\psi - w)\,\mathrm{MA}_\omega(v_j)\biggr)\\ &= E_\psi(w) + \int_X (\psi - w)\,\mathrm{MA}_\omega(u),\end{aligned}$$

i.e., w is a maximizer of $F_{\mathrm{MA}_\omega(u)/V_\psi,\psi}$. Hence $w=u$ (Proposition 5.5), i.e., $u_k\to u$ weakly. Furthermore, again by Lemma 3.1 and Lemma 4.6,

$$\begin{aligned}\limsup_{k\to\infty}(E_\psi(v_j)-E_\psi(u_k)) &\le \limsup_{k\to\infty}\biggl(\frac{n}{n+1}I_\psi(u_k,v_j)+\biggl|\int_X(u_k-v_j)\,\mathrm{MA}_\omega(v_j)\biggr|\biggr)\\ &\le \biggl|\int_X(u-v_j)\,\mathrm{MA}_\omega(v_j)\biggr|+\limsup_{k\to\infty}\frac{n}{n+1}I_\psi(u_k,v_j). \qquad (18)\end{aligned}$$

Finally letting $j\to\infty$, since $v_j\searrow u$ strongly, we obtain $\liminf_{j\to\infty}E_\psi(u_k)\ge\lim_{j\to\infty}E_\psi(v_j)=E_\psi(u)$, which implies that $E_\psi(u_k)\to E_\psi(u)$ and that $u_k\to u$ strongly by Proposition 5.6. □

The main difference between the proof of Theorem A and the proof of the same result in the absolute setting, i.e., when $\psi=0$, is that for fixed $u\in\mathcal{E}^1(X,\omega,\psi)$ the action

$$\mathcal{M}^1(X,\omega,\psi)\ni\mathrm{MA}_\omega(v)\to\int_X(u-\psi)\,\mathrm{MA}_\omega(v)$$

is not a priori continuous with respect to the weak topologies of measures even if we restrict the action on $\mathcal{M}^1_C(X,\omega,\psi):=\{V_\psi\mu:E^*_\psi(\mu)\le C\}$ for $C\in\mathbb{R}$, while in the absolute setting this is given by [Berman et al. 2019, Proposition 1.7], where the authors used the fact that any $u\in\mathcal{E}^1(X,\omega)$ can be approximated inside the class $\mathcal{E}^1(X,\omega)$ by a sequence of continuous functions.

6. Strong topologies

In this section we investigate the strong topology on $X_{\mathcal{A}}$ in detail, proving that it is the coarsest refinement of the weak topology such that $E_\cdot(\cdot)$ becomes continuous (Theorem 6.2) and proving that the strong convergence implies the convergence in ψ-capacity for any $\psi\in\mathcal{M}^+$ (Theorem 6.3), i.e., we extend all the typical properties of the L^1-metric geometry to the bigger space $X_{\mathcal{A}}$, justifying further the construction of the distance $d_{\mathcal{A}}$ [Trusiani 2022] and its naturality. Moreover, we define the set $Y_{\mathcal{A}}$ and prove Theorem B.

6A. *About* $(X_{\mathcal{A}}, d_{\mathcal{A}})$. First we prove that the strong convergence in $X_{\mathcal{A}}$ implies the weak convergence, recalling that for the weak convergence of $u_k\in\mathcal{E}^1(X,\omega,\psi_k)$ to $P_{\psi_{\min}}$, where $\psi_{\min}\in\mathcal{M}$ with $V_{\psi_{\min}}=0$, we mean that $|\sup_X u_k|\le C$ and that any weak accumulation point of $\{u_k\}_{k\in\mathbb{N}}$ is more singular than $\psi_{\min}$.

Proposition 6.1. *Let $u_k,u\in X_{\mathcal{A}}$ such that $u_k\to u$ strongly. If $u\neq P_{\psi_{\min}}$, then $u_k\to u$ weakly. If instead $u=P_{\psi_{\min}}$, then the following dichotomy holds*:

(i) *$u_k\to P_{\psi_{\min}}$ weakly.*

(ii) $\limsup_{k\to\infty}|\sup_X u_k|=+\infty$.

Proof. The dichotomy for the case $u=P_{\psi_{\min}}$ follows by definition. Indeed, if $|\sup_X u_k|\le C$ and $d_{\mathcal{A}}(u_k,u)\to 0$ as $k\to\infty$, then $V_{\psi_k}\to V_{\psi_{\min}}=0$ by Proposition 2.11 (iv), which implies that $\psi_k\to\psi_{\min}$ by Lemma 3.12. Hence any weak accumulation point u of $\{u_k\}_{k\in\mathbb{N}}$ satisfies $u\le\psi_{\min}+C$.

Thus, let $\psi_k,\psi\in\mathcal{A}$ such that $u_k\in\mathcal{E}^1(X,\omega,\psi_k)$ and $u\in\mathcal{E}^1(X,\omega,\psi)$ where $\psi\in\mathcal{M}^+$. Observe that

$$d(u_k,\psi_k)\le d_{\mathcal{A}}(u_k,u)+d(u,\psi)+d_{\mathcal{A}}(\psi,\psi_k)\le A$$

for a uniform constant $A>0$ by Proposition 2.11 (iv).

On the other hand, by [Błocki and Kołodziej 2007], for any $j \in \mathbb{N}$ there exists $h_j \in \mathcal{H}_\omega$ such that $h_j \geq u$, $\|h_j - u\|_{L^1} \leq 1/j$ and $d(u, P_\omega[\psi](h_j)) \leq 1/j$. In particular, by the triangle inequality and Proposition 2.11, we have

$$\begin{aligned}\limsup_{k\to\infty} d(P_\omega[\psi_k](h_j), \psi_k) &\leq \limsup_{k\to\infty}\Big(d_{\mathcal{A}}(P_\omega[\psi_k](h_j), P_\omega[\psi](h_j)) + \frac{1}{j} + d(u, \psi) + d(\psi, \psi_k)\Big)\\ &\leq d(u, \psi) + \frac{1}{j}, \end{aligned}\tag{19}$$

Similarly, again by the triangle inequality and Proposition 2.11,

$$\limsup_{k\to\infty} d(u_k, P_\omega[\psi_k](h_j)) \leq \limsup_{k\to\infty}\Big(d_{\mathcal{A}}(P_\omega[\psi_k](h_j), P_\omega[\psi](h_j)) + \frac{1}{j} + d_{\mathcal{A}}(u, u_k)\Big) \leq \frac{1}{j} \tag{20}$$

and

$$\begin{aligned}\limsup_{k\to\infty}\|u_k - u\|_{L^1} &\leq \limsup_{k\to\infty}(\|u_k - P_\omega[\psi_k](h_j)\|_{L^1} + \|P_\omega[\psi_k](h_j) - P_\omega[\psi](h_j)\|_{L^1} + \|P_\omega[\psi](h_j) - u\|_{L^1})\\ &\leq \frac{1}{j} + \limsup_{k\to\infty}\|u_k - P_\omega[\psi_k](h_j)\|_{L^1}, \end{aligned}\tag{21}$$

where we also used Lemma 2.14. In particular, we deduce that $d(\psi_k, P_\omega[\psi_k](h_j))$, $d(\psi_k, u_k) \leq C$ for a uniform constant $C \in \mathbb{R}$ from (19) and (20). Next let $\phi_k \in \mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi)$ be the unique solution of $\mathrm{MA}_\omega(\phi_k) = (V_{\psi_k}/V_0)\,\mathrm{MA}_\omega(0)$, and observe that by Proposition 2.4,

$$d(\psi_k, \phi_k) = -E_{\psi_k}(\phi_k) \leq \int_X (\psi_k - \phi_k)\,\mathrm{MA}_\omega(\phi_k) \leq \frac{V_{\psi_k}}{V_0}\int_X |\phi_k|\,\mathrm{MA}_\omega(0) \leq \|\phi_k\|_{L^1} \leq C',$$

since ϕ_k belongs to a compact (hence bounded) subset of $\mathrm{PSH}(X, \omega) \subset L^1$. Therefore, since $V_{\psi_k} \geq a > 0$ for $k \gg 0$ big enough, by Proposition 3.6 it follows that there exists a continuous increasing function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $f(0) = 0$ such that

$$\|u_k - P_\omega[\psi_k](h_j)\|_{L^1} \leq f(d(u_k, P_\omega[\psi_k](h_j)))$$

for any k, j big enough. Hence, combining (20) and (21), the convergence requested follows letting $k, j \to +\infty$ in this order. □

We can now prove the important characterization of the strong convergence as the coarsest refinement of the weak topology such that $E_\cdot(\,\cdot\,)$ becomes continuous.

Theorem 6.2. *Let $u_k \in \mathcal{E}^1(X, \omega, \psi_k)$ and $u \in \mathcal{E}^1(X, \omega, \psi)$ for $\{\psi_k\}_{k\in\mathbb{N}}$, $\psi \in \bar{\mathcal{A}}$. If $\psi \neq \psi_{\min}$ or $V_{\psi_{\min}} > 0$, then the following are equivalent*:

(i) *$u_k \to u$ strongly.*

(ii) *$u_k \to u$ weakly and $E_{\psi_k}(u_k) \to E_\psi(u)$.*

In the case $\psi = \psi_{\min}$ and $V_{\psi_{\min}} = 0$, if $u_k \to P_{\psi_{\min}}$ weakly and $E_{\psi_k}(u_k) \to 0$, then $u_k \to P_{\psi_{\min}}$ strongly. Finally, if $d_{\mathcal{A}}(u_k, P_{\psi_{\min}}) \to 0$ as $k \to \infty$, then the following dichotomy holds:

(a) *$u_k \to P_{\psi_{\min}}$ weakly and $E_{\psi_k}(u_k) \to 0$.*

(b) *$\limsup_{k\to\infty}|\sup_X u_k| = \infty$.*

Proof. (ii) $\Rightarrow$ (i): Assume that (ii) holds where we include the case $u = P_{\psi_{\min}}$ setting $E_\psi(P_{\psi_{\min}}) := 0$. Clearly it is enough to prove that any subsequence of $\{u_k\}_{k\in\mathbb{N}}$ admits a subsequence which is $d_{\mathcal{A}}$-convergent to u. For the sake of simplicity we denote by $\{u_k\}_{k\in\mathbb{N}}$ the arbitrary initial subsequence, and since $\mathcal{A}$ is totally ordered by Lemma 3.13 we may also assume either $\psi_k \searrow \psi$ or $\psi_k \nearrow \psi$ almost everywhere. In particular, even if $u = P_{\psi_{\min}}$ we may suppose that u_k converges weakly to a proper element $v \in \mathcal{E}^1(X, \omega, \psi)$ up to considering a further subsequence by definition of the weak convergence to the point $P_{\psi_{\min}}$. In this case by abuse of notation we denote the function v, which depends on the subsequence chosen, by u. Note also that by Hartogs' lemma we have $u_k \le \psi_k + A$ and $u \le \psi + A$ for a uniform constant $A \in \mathbb{R}_{\ge 0}$ since $|\sup_X u_k| \le A$.

In the case of $\psi_k \searrow \psi$, we have that $v_k := (\sup\{u_j : j \ge k\})^* \in \mathcal{E}^1(X, \omega, \psi_k)$ decreases to u. Thus $w_k := P_\omega[\psi](v_k) \in \mathcal{E}^1(X, \omega, \psi)$ decreases to u, which implies $d(u, w_k) \to 0$ as $k \to \infty$. (If $u = P_{\psi_{\min}}$, we immediately have $w_k = P_{\psi_{\min}}$.)

Moreover, by Propositions 2.4 and 2.10,

$$\begin{aligned} E_\psi(u) = \lim_{k\to\infty} E_\psi(w_k) &= AV_\psi - \lim_{k\to\infty} d(\psi + A, w_k) \\ &\ge \lim_{k\to\infty} (AV_{\psi_k} - d(\psi_k + A, v_k)) \\ &= \limsup_{k\to\infty} E_{\psi_k}(v_k) \ge \lim_{k\to\infty} E_{\psi_k}(u_k) = E_\psi(u) \end{aligned}$$

since $\psi_k + A = P_\omega[\psi_k](A)$. Hence

$$\limsup_{k\to\infty} d(v_k, u_k) = \limsup_{k\to\infty} (d(\psi_k + A, u_k) - d(v_k, \psi_k + A)) = \lim_{k\to\infty} (E_{\psi_k}(v_k) - E_{\psi_k}(u_k)) = 0.$$

Thus by the triangle inequality it is sufficient to show that $\limsup_{k\to\infty} d_{\mathcal{A}}(u, v_k) = 0$.

Next, for any $C \in \mathbb{R}$ we set $v_k^C := \max(v_k, \psi_k - C)$ and $u^C := \max(u, \psi - C)$, and we observe that $d(\psi_k + A, v_k^C) \to d(\psi + A, u^C)$ by Proposition 2.11 since $v_k^C \searrow u^C$. This implies that

$$\begin{aligned} d(v_k, v_k^C) = d(\psi_k + A, v_k) - d(\psi_k + A, v_k^C) &= AV_{\psi_k} - E_{\psi_k}(v_k) - d(\psi_k + A, v_k^C) \\ \to AV_\psi - E_\psi(u) - d(\psi + A, u^C) &= d(\psi + A, u) - d(\psi + A, u^C) = d(u, u^C). \end{aligned}$$

Thus, since $u^C \to u$ strongly, again by the triangle inequality it remains to estimate $d_{\mathcal{A}}(u, v_k^C)$. Fix $\epsilon > 0$ and $\phi_\epsilon \in \mathcal{P}_{\mathcal{H}_\omega}(X, \omega, \psi)$ such that $d(\phi_\epsilon, u) \le \epsilon$ (by Lemma 2.13). Then letting $\varphi \in \mathcal{H}_\omega$ such that $\phi_\epsilon = P_\omega[\psi](\varphi)$ and setting $\phi_{\epsilon,k} := P_\omega[\psi_k](\varphi)$, by Proposition 2.11 we have

$$\begin{aligned} \limsup_{k\to\infty} d_{\mathcal{A}}(u, v_k^C) &\le \limsup_{k\to\infty} (d(u, \phi_\epsilon) + d_{\mathcal{A}}(\phi_\epsilon, \phi_{\epsilon,k}) + d(\phi_{\epsilon,k}, v_k^C)) \\ &\le \epsilon + d(\phi_\epsilon, u^C) \\ &\le 2\epsilon + d(u, u^C), \end{aligned}$$

which concludes the first case of (ii) $\Rightarrow$ (i) by the arbitrariness of ϵ since $u^C \to u$ strongly in $\mathcal{E}^1(X, \omega, \psi)$.

Next assume that $\psi_k \nearrow \psi$ almost everywhere. In this case we may assume $V_{\psi_k} > 0$ for any $k \in \mathbb{N}$. Then $v_k := (\sup\{u_j : j \ge k\})^* \in \mathcal{E}^1(X, \omega, \psi)$ decreases to u. Moreover, setting $w_k := P_\omega[\psi_k](v_k) \in \mathcal{E}^1(X, \omega, \psi_k)$ and combining with the monotonicity of $E_{\psi_k}(\cdot)$, the upper semicontinuity of $E_\cdot(\cdot)$ (Proposition 3.14) and the contraction property of Proposition 2.10, we obtain

$$\begin{aligned} E_\psi(u) = \lim_{k\to\infty} E_\psi(v_k) &= AV_\psi - \lim_{k\to\infty} d(v_k, \psi + A) \\ &\le \liminf_{k\to\infty}(AV_{\psi_k} - d(w_k, \psi_k + A)) \\ &= \liminf_{k\to\infty} E_{\psi_k}(w_k) \le \limsup_{k\to\infty} E_{\psi_k}(w_k) \le E_\psi(u), \end{aligned}$$

i.e., $E_{\psi_k}(w_k) \to E_\psi(u)$ as $k \to \infty$. As an easy consequence we get $d(w_k, u_k) = E_{\psi_k}(w_k) - E_{\psi_k}(u_k) \to 0$, thus it is sufficient to prove that

$$\limsup_{k\to\infty} d_{\mathcal{A}}(u, w_k) = 0.$$

Similar to the previous case, fix $\epsilon > 0$ and let $\phi_\epsilon = P_\omega[\psi](\varphi_\epsilon)$ for $\varphi \in \mathcal{H}_\omega$ such that $d(u, \phi_\epsilon) \le \epsilon$. Again Propositions 2.10 and 2.11 yield

$$\begin{aligned} \limsup_{k\to\infty} d_{\mathcal{A}}(u, w_k) &\le \epsilon + \limsup_{k\to\infty}(d_{\mathcal{A}}(\phi_\epsilon, P_\omega[\psi_k](\phi_\epsilon)) + d(P_\omega[\psi_k](\phi_\epsilon), w_k)) \\ &\le \epsilon + \limsup_{k\to\infty}(d_{\mathcal{A}}(\phi_\epsilon, P_\omega[\psi_k](\phi_\epsilon)) + d(\phi_\epsilon, v_k)) \le 2\epsilon, \end{aligned}$$

which concludes the first part.

(i) $\Rightarrow$ (ii) if $u \ne P_{\psi_{\min}}$, while (i) implies the dichotomy if $u = P_{\psi_{\min}}$: If $u \ne P_{\psi_{\min}}$, then Proposition 6.1 implies that $u_k \to u$ weakly and, in particular, that $|\sup_X u_k| \le A$. Thus it remains to prove that $E_{\psi_k}(u_k) \to E_\psi(u)$.

If $u = P_{\psi_{\min}}$, then again by Proposition 6.1 it remains to show that $E_{\psi_k}(u_k) \to 0$ assuming $u_{k_h} \to P_{\psi_{\min}}$ strongly and weakly. Note that we also have $|\sup_X u_k| \le A$ for a uniform constant $A \in \mathbb{R}$ by definition of the weak convergence to $P_{\psi_{\min}}$.

Since by an easy contradiction argument it is enough to prove that any subsequence of $\{u_k\}_{k\in\mathbb{N}}$ admits a further subsequence such that the convergence of the energies holds, without loss of generality we may assume that $u_k \to u \in \mathcal{E}^1(X, \omega, \psi)$ weakly even in the case $V_\psi = 0$ (i.e., when, with abuse of notation, $u = P_{\psi_{\min}}$).

So we want to show the existence of a further subsequence $\{u_{k_h}\}_{h\in\mathbb{N}}$ such that $E_{\psi_{k_h}}(u_{k_h}) \to E_\psi(u)$ (note that if $V_\psi = 0$, then $E_\psi(u) = 0$). It easily follows that

$$\begin{aligned} |E_{\psi_k}(u_k) - E_\psi(u)| &\le |d(\psi_k + A, u_k) - d(\psi + A, u)| + A|V_{\psi_k} - V_\psi| \\ &\le d_{\mathcal{A}}(u, u_k) + d(\psi_k + A, \psi + A) + A|V_{\psi_k} - V_\psi|, \end{aligned}$$

and this leads to $\lim_{k\to\infty} E_{\psi_k}(u_k) = E_\psi(u)$ by Proposition 2.11, since we have $\psi_k + A = P_\omega[\psi_k](A)$ and $\psi + A = P_\omega[\psi](A)$. Hence $E_{\psi_k}(u_k) \to E_\psi(u)$ as desired. □

Note that in Theorem 6.2, case (b) may happen (Remark 3.16), but obviously one can consider

$$X_{\mathcal{A},\mathrm{norm}} = \bigsqcup_{\psi\in\bar{\mathcal{A}}} \mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi)$$

to exclude such pathology.

The strong convergence also implies the convergence in ψ'-capacity for any $\psi' \in \mathcal{M}^+$, as our next result shows.

Theorem 6.3. *Let $\psi_k, \psi \in \mathcal{A}$ and let $u_k \in \mathcal{E}^1(X, \omega, \psi_k)$ strongly converge to $u \in \mathcal{E}^1(X, \omega, \psi)$. Assume also that $V_\psi > 0$. Then there exists a subsequence $\{u_{k_j}\}_{j\in\mathbb{N}}$ such that the sequences $w_j := (\sup\{u_{k_s} : s \geq j\})^*$ and $v_j := P_\omega(u_{k_j}, u_{k_{j+1}}, \dots)$ belong to $X_{\mathcal{A}}$, satisfying $v_j \leq u_{k_j} \leq w_j$ and converging strongly and monotonically to u. In particular, $u_k \to u$ in ψ'-capacity for any $\psi' \in \mathcal{M}^+$ and $\mathrm{MA}_\omega(u_k^j, \psi_k^{n-j}) \to \mathrm{MA}_\omega(u^k, \psi^{n-j})$ weakly for any $j \in \{0, \dots, n\}$.*

Proof. We first observe that by Theorem 6.2, $u_k \to u$ weakly and $E_{\psi_k}(u_k) \to E_\psi(u)$. In particular, $\sup_X u_k$ is uniformly bounded and the sequence of ω-psh $w_k := (\sup\{u_j : j \geq k\})^*$ decreases to u.

Up to considering a subsequence we may assume either $\psi_k \searrow \psi$ or $\psi_k \nearrow \psi$ almost everywhere. We treat the two cases separately.

Assume first that $\psi_k \searrow \psi$. Since clearly $w_k \in \mathcal{E}^1(X, \omega, \psi_k)$ and $E_{\psi_k}(w_k) \geq E_{\psi_k}(u_k)$, Theorem 6.2 and Proposition 3.14 yield

$$E_\psi(u) = \lim_{k\to\infty} E_{\psi_k}(u_k) \leq \limsup_{k\to\infty} E_{\psi_k}(w_k) \leq E_\psi(u),$$

i.e., $w_k \to u$ strongly. Thus up to considering a further subsequence we can suppose that $d(u_k, w_k) \leq 1/2^k$ for any $k \in \mathbb{N}$.

Next, similar to the proof of Proposition 5.7, we define $v_{j,l} := P_\omega(u_j, \dots, u_{j+l})$ for any $j, l \in \mathbb{N}$, observing that $v_{j,l} \in \mathcal{E}^1(X, \omega, \psi_{j+l})$. Thus the function $v_{j,l}^u := P_\omega(u, v_{j,l}) \in \mathcal{E}^1(X, \omega, \psi)$ satisfies

$$\begin{aligned} d(u, v_{j,l}^u) &\leq \int_X (u - v_{j,l}^u)\,\mathrm{MA}_\omega(v_{j,l}^u) \leq \int_{\{v_{j,l}^u = v_{j,l}\}} (u - v_{j,l})\,\mathrm{MA}_\omega(v_{j,l}) \\ &\leq \sum_{s=j}^{j+l} \int_X (w_s - u_s)\,\mathrm{MA}_\omega(u_s) \leq (n+1) \sum_{s=j}^{j+l} d(w_s, u_s) \leq \frac{n+1}{2^{j-1}}, \end{aligned} \tag{22}$$

where we combined Proposition 2.4 and [Darvas et al. 2018, Lemma 3.7]. Therefore by Proposition 3.15, $v_{j,l}^u$ converges decreasingly and strongly in $\mathcal{E}^1(X, \omega, \psi)$ to a function ϕ_j which satisfies $\phi_j \leq u$.

Similarly,

$$\int_{\{P_\omega(u, v_{j,l}^u) = u\}} (v_{j,l}^u - u)\,\mathrm{MA}_\omega(u) \leq \int_X |v_{j,1}^u - u|\,\mathrm{MA}_\omega(u) < \infty$$

by Corollary 3.5, which implies that $v_{j,l}$ converges decreasingly to $v_j \in \mathcal{E}^1(X, \omega, \psi)$ such that $u \geq v_j \geq \phi_j$, since $v_j \leq u_s$ for any $s \geq j$ and $v_{j,l} \geq v_{j,l}^u$. Hence from (22) we obtain

$$d(u, v_j) \leq d(u, \phi_j) = \lim_{l\to\infty} d(u, v_{j,l}^u) \leq \frac{n+1}{2^{j-1}},$$

i.e., v_j converges increasingly and strongly to u as $j \to \infty$.

Next assume $\psi_k \nearrow \psi$ almost everywhere. In this case, $w_k \in \mathcal{E}^1(X, \omega, \psi)$ for any $k \in \mathbb{N}$, and clearly w_k converges strongly and decreasingly to u. On the other hand, letting $w_{k,k} := P_\omega[\psi_k](w_k)$ we observe by Theorem 6.2 and Proposition 3.14 that $w_{k,k} \to u$ weakly since $w_k \geq w_{k,k} \geq u_k$ and

$$E_\psi(u) = \lim_{k\to\infty} E_{\psi_k}(u_k) \leq \limsup_{k\to\infty} E_{\psi_k}(w_{k,k}) \leq E_\psi(u),$$

i.e., $w_{k,k}\to u$ strongly, again by Theorem 6.2. As in the previous case, we assume that $d(u_k, w_{k,k})\leq 1/2^k$ up to considering a further subsequence. Therefore, setting $v_{j,l} := P_\omega(u_j,\dots,u_{j+l})\in\mathcal{E}^1(X,\omega,\psi_j)$, $u^j := P_\omega[\psi_j](u)$ and $v_{j,l}^{u^j} := P_\omega(v_{j,l}, u^j)$ we obtain

$$d(u^j, v_{j,l}^{u^j}) \leq \int_X (u^j - v_{j,l}^{u^j})\,\mathrm{MA}_\omega(v_{j,l}^{u^j}) \leq \sum_{s=j}^{j+l}\int_X (w_{s,s}-u_s)\,\mathrm{MA}_\omega(u_s)\leq \frac{n+1}{2^{j-1}}, \tag{23}$$

proceeding as in the previous case. This implies that $v_{j,l}^{u^j}$ and $v_{j,l}$ converge decreasingly and strongly to functions $\phi_j, v_j\in\mathcal{E}^1(X,\omega,\psi_j)$, respectively, as $l\to+\infty$ which satisfy $\phi_j\leq v_j\leq u^j$. Therefore combining (23), Proposition 2.11 and the triangle inequality we get

$$\limsup_{j\to\infty} d_{\mathcal{A}}(u, v_j)\leq \limsup_{j\to\infty}(d_{\mathcal{A}}(u,u^j)+d(u^j,\phi_j))\leq \limsup_{j\to\infty}\left(d_{\mathcal{A}}(u,u^j)+\frac{n+1}{2^{j-1}}\right)=0.$$

Hence v_j converges strongly and increasingly to u, so $v_j\nearrow u$ almost everywhere (Proposition 6.1) and the first part of the proof is concluded.

The convergence in ψ'-capacity and the weak convergence of the mixed Monge–Ampère measures follow exactly as in the proof of Proposition 5.7. □

We observe that the assumption $u\neq P_{\psi_{\min}}$ if $V_{\psi_{\min}}=0$ in Theorem 6.3 is obviously necessary as the counterexample of Remark 3.16 shows. On the other hand, if $d_{\mathcal{A}}(u_k, P_{\psi_{\min}})\to 0$, then trivially $\mathrm{MA}_\omega(u_k^j, \psi_k^{n-j})\to 0$ weakly as $k\to\infty$ for any $j\in\{0,\dots,n\}$ as a consequence of $V_{\psi_k}\searrow 0$.

6B. *Proof of Theorem B.*

Definition 6.4. We define $Y_{\mathcal{A}}$ as

$$Y_{\mathcal{A}} := \bigsqcup_{\psi\in\bar{\mathcal{A}}}\mathcal{M}^1(X,\omega,\psi),$$

and we endow it with its natural *strong topology* given as the coarsest refinement of the weak topology such that $E^*_\cdot$ becomes continuous, i.e., $V_{\psi_k}\mu_k$ converges strongly to $V_\psi\mu$ if and only if $V_{\psi_k}\mu_k\to V_\psi\mu$ weakly and $E^*_{\psi_k}(\mu_k)\to E^*_\psi(\mu)$ as $k\to\infty$.

Observe that $Y_{\mathcal{A}}\subset\{$nonpluripolar measures of total mass belonging to $[V_{\psi_{\min}}, V_{\psi_{\max}}]\}$, where clearly $\psi_{\max}:=\sup\mathcal{A}$. As stated in the Introduction, the definition is coherent with [Berman et al. 2019] since if $\psi=0\in\bar{\mathcal{A}}$, then the induced topology on $\mathcal{M}^1(X,\omega)$ coincides with the strong topology as defined in that paper.

We also recall that

$$X_{\mathcal{A},\mathrm{norm}} := \bigsqcup_{\psi\in\bar{\mathcal{A}}}\mathcal{E}^1_{\mathrm{norm}}(X,\omega,\psi),$$

where $\mathcal{E}^1_{\mathrm{norm}}(X,\omega,\psi):=\{u\in\mathcal{E}^1(X,\omega,\psi):\sup_X u=0\}$ (if $V_{\psi_{\min}}=0$, then we can assume $P_{\psi_{\min}}\in X_{\mathcal{A},\mathrm{norm}}$).

Theorem B. *The Monge–Ampère map*

$$\mathrm{MA}_\omega : (X_{\mathcal{A},\mathrm{norm}}, d_{\mathcal{A}})\to (Y_{\mathcal{A}}, \text{strong})$$

is a homeomorphism.

Proof. The map is a bijection as a consequence of Lemma 3.12 and Proposition 5.5, where we clearly define $\mathrm{MA}_\omega(P_{\psi_{\min}}) := 0$, i.e., the null measure.

Step 1: continuity. Assume first that $V_{\psi_{\min}} = 0$ and that $d_{\mathcal{A}}(u_k, P_{\psi_{\min}}) \to 0$ as $k \to \infty$. Then clearly $\mathrm{MA}_\omega(u_k) \to 0$ weakly. Moreover, assuming $u_k \neq P_{\psi_{\min}}$ for any k, it follows from Proposition 2.4 that

$$\begin{aligned} E^*_{\psi_k}(\mathrm{MA}_\omega(u_k)/V_{\psi_k}) &= E_{\psi_k}(u_k) + \int_X (\psi_k - u_k)\,\mathrm{MA}_\omega(u_k) \\ &\leq \frac{n}{n+1}\int_X (\psi_k - u_k)\,\mathrm{MA}_\omega(u_k) \leq -nE_{\psi_k}(u_k) \to 0 \end{aligned}$$

as $k \to \infty$ where the convergence is given by Theorem 6.2. Hence $\mathrm{MA}_\omega(u_k) \to 0$ strongly in $Y_{\mathcal{A}}$.

We can now assume that $u \neq P_{\psi_{\min}}$.

Theorem 6.3 immediately gives the weak convergence of $\mathrm{MA}_\omega(u_k)$ to $\mathrm{MA}_\omega(u)$. Let $\varphi_j \in \mathcal{H}_\omega$ be a decreasing sequence converging to u such that $d(u, P_\omega[\psi](\varphi_j)) \leq 1/j$ for any $j \in \mathbb{N}$ [Błocki and Kołodziej 2007], and set $v_{k,j} := P_\omega[\psi_k](\varphi_j)$ and $v_j := P_\omega[\psi](\varphi_j)$. Observe also that as a consequence of Proposition 2.11 and Theorem 6.2, for any $j \in \mathbb{N}$ there exists $k_j \gg 0$ big enough such that

$$d(\psi_k, v_{k,j}) \leq d_{\mathcal{A}}(\psi_k, \psi) + d(\psi, v_j) + d_{\mathcal{A}}(v_j, v_{k,j}) \leq d(\psi, v_j) + 1 \leq C$$

for any $k \geq k_j$, where C is a uniform constant independent of $j \in \mathbb{N}$. Therefore, again combining Theorem 6.2 with Lemma 4.6 and Proposition 3.7, we obtain

$$\begin{aligned} &\limsup_{k\to\infty} |E^*_{\psi_k}(\mathrm{MA}_\omega(u_k)/V_{\psi_k}) - E^*_{\psi_k}(\mathrm{MA}_\omega(v_{k,j})/V_{\psi_k})| \\ &\leq \limsup_{k\to\infty}\left(|E_{\psi_k}(u_k) - E_{\psi_k}(v_{k,j})| + \left|\int_X (\psi_k - u_k)(\mathrm{MA}_\omega(u_k) - \mathrm{MA}_\omega(v_{k,j}))\right| + \left|\int_X (v_{k,j} - u_k)\,\mathrm{MA}_\omega(v_{k,j})\right|\right) \\ &\leq |E_\psi(u) - E_\psi(v_j)| + \limsup_{k\to\infty} C I_{\psi_k}(u_k, v_{k,j})^{1/2} + \int_X (v_j - u)\,\mathrm{MA}_\omega(v_j), \end{aligned} \tag{24}$$

since clearly we may assume that either $\psi_k \searrow \psi$ or $\psi_k \nearrow \psi$ almost everywhere, up to considering a subsequence. On the other hand, if $k \geq k_j$, Proposition 3.4 implies $I_{\psi_k}(u_k, v_{k,j}) \leq 2f_{\tilde{C}}(d(u_k, v_{k,j}))$, where $\tilde{C}$ is a uniform constant independent of j, k and $f_{\tilde{C}} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a continuous increasing function such that $f_{\tilde{C}}(0) = 0$. Hence continuing the estimates in (24) we get

$$(24) \leq |E_\psi(u) - E_\psi(v_j)| + 2Cf_{\tilde{C}}(d(u, v_j)) + d(v_j, u), \tag{25}$$

using also Propositions 2.4 and 2.11. Letting $j \to \infty$ in (25), it follows that

$$\limsup_{j\to\infty}\limsup_{k\to\infty} |E^*_{\psi_k}(\mathrm{MA}_\omega(u_k)/V_{\psi_k}) - E^*_{\psi_k}(\mathrm{MA}_\omega(v_{k,j})/V_{\psi_k})| = 0$$

since $v_j \searrow u$. Furthermore, it is easy to check that $E^*_{\psi_k}(\mathrm{MA}_\omega(v_{k,j})/V_{\psi_k}) \to E^*_\psi(\mathrm{MA}_\omega(v_j)/V_\psi)$ as $k \to \infty$ for j fixed by Lemma 4.6 and Proposition 2.11. Therefore the convergence

$$E^*_\psi(\mathrm{MA}_\omega(v_j)/V_\psi) \to E^*_\psi(\mathrm{MA}_\omega(u)/V_\psi) \tag{26}$$

as $j \to \infty$ given by Theorem A concludes this step.

Step 2: continuity of the inverse. We will assume $u_k \in \mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi_k)$ and $u \in \mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi)$ such that $\mathrm{MA}_\omega(u_k) \to \mathrm{MA}_\omega(u)$ strongly. Note that when $\psi = \psi_{\min}$ and $V_{\psi_{\min}} = 0$, the assumption does not depend on the function u chosen. Clearly this implies $V_{\psi_k} \to V_\psi$ which leads to $\psi_k \to \psi$ as $k \to \infty$ by Lemma 3.12 since $\mathcal{A} \subset \mathcal{M}^+$ is totally ordered. Hence, up to considering a subsequence, we may assume that $\psi_k \to \psi$ monotonically almost everywhere. We keep the same notation of the previous step for $v_{k,j}, v_j$. We may also suppose that $V_{\psi_k} > 0$ for any $k \in \mathbb{N}$ big enough otherwise it would be trivial.

The strategy is to proceed similarly to the proof of Theorem A, i.e., we first prove that $I_{\psi_k}(u_k, v_{k,j}) \to 0$ as $k, j \to \infty$ in this order. Then we will use this to prove that the unique weak accumulation point of $\{u_k\}_{k\in\mathbb{N}}$ is u. Finally we will deduce the convergence of the ψ_k-relative energies to conclude that $u_k \to u$ strongly thanks to Theorem 6.2.

By Lemma 3.1,

$$\begin{aligned}(n+1)^{-1} & I_{\psi_k}(u_k, v_{k,j}) \\ &\le E_{\psi_k}(u_k) - E_{\psi_k}(v_{k,j}) + \int_X (v_{k,j} - u_k)\,\mathrm{MA}_\omega(u_k) \\ &= E^*_{\psi_k}(\mathrm{MA}_\omega(u_k)/V_{\psi_k}) - E^*_{\psi_k}(\mathrm{MA}_\omega(v_{k,j})/V_{\psi_k}) + \int_X (v_{k,j} - \psi_k)(\mathrm{MA}_\omega(u_k) - \mathrm{MA}_\omega(v_{k,j})) \end{aligned} \tag{27}$$

for any j, k. Moreover, by Step 1 and Proposition 2.11 we know that $E^*_{\psi_k}(\mathrm{MA}_\omega(v_{k,j})/V_{\psi_k})$ converges, as $k \to +\infty$, to 0 if $V_\psi = 0$ and to $E^*_\psi(\mathrm{MA}_\omega(v_j)/V_\psi)$ if $V_\psi > 0$. Next by Lemma 4.6,

$$\int_X (v_{k,j} - \psi_k)\,\mathrm{MA}_\omega(v_{k,j}) \to \int_X (v_j - \psi)\,\mathrm{MA}_\omega(v_j)$$

letting $k \to \infty$. So if $V_\psi = 0$, then from

$$\lim_{k\to\infty} \sup_X (v_{k,j} - \psi_k) = \sup_X (v_j - \psi) = \sup_X v_j$$

we easily get $\limsup_{k\to\infty} I_{\psi_k}(u_k, v_{k,j}) = 0$. Thus we may assume $V_\psi > 0$, and it remains to estimate $\int_X (v_{k,j} - \psi_k)\,\mathrm{MA}_\omega(u_k)$ from above.

We set $f_{k,j} := v_{k,j} - \psi_k$, and as in the proof of Theorem A we construct a sequence of smooth functions $f^s_j := v^s_j - \psi^s$ converging in capacity to $f_j := v_j - \psi$ and satisfying $\|f^s_j\|_{L^\infty} \le 2\|f_j\|_{L^\infty} \le 2\|\varphi_j\|_{L^\infty}$. Here v^s_j and ψ^s are sequences of ω-psh functions decreasing to v_j and ψ, respectively. Then we write

$$\int_X f_{k,j}\,\mathrm{MA}_\omega(u_k) = \int_X (f_{k,j} - f^s_j)\,\mathrm{MA}_\omega(u_k) + \int_X f^s_j\,\mathrm{MA}_\omega(u_k), \tag{28}$$

and we observe that

$$\limsup_{s\to\infty} \limsup_{k\to\infty} \int_X f^s_j\,\mathrm{MA}_\omega(u_k) = \int_X f_j\,\mathrm{MA}_\omega(u),$$

since $\mathrm{MA}_\omega(u_k) \to \mathrm{MA}_\omega(u)$ weakly, $f^s_j \in C^\infty$, f^s_j converges to f_j in capacity and $\|f^s_j\|_{L^\infty} \le 2\|f_j\|_{L^\infty}$. We also claim that the first term on the right-hand side of (28) goes to 0 letting $k, s \to \infty$ in this order.

Indeed, for any $\delta > 0$,

$$\begin{aligned}\int_X (f_{k,j} - f_j)\,\mathrm{MA}_\omega(u_k) &\le \delta V_{\psi_k} + 2\|\varphi_j\|_{L^\infty} \int_{\{f_{k,j}-f_j>\delta\}} \mathrm{MA}_\omega(u_k)\\ &\le \delta V_{\psi_k} + 2\|\varphi_j\|_{L^\infty} \int_{\{|h_{k,j}-h_j|>\delta\}} \mathrm{MA}_\omega(u_k),\end{aligned} \tag{29}$$

where we set $h_{k,j} := v_{k,j}$, $h_j := v_j$ if $\psi_k \searrow \psi$ and $h_{k,j} := \psi_k$, $h_j := \psi$ if instead $\psi_k \nearrow \psi$ almost everywhere. Moreover, since $\{|h_{k,j} - h_j| > \delta\} \subset \{|h_{l,j} - h_j| > \delta\}$ for any $l \le k$, from (29) we obtain

$$\begin{aligned}\limsup_{k\to\infty} \int_X (f_{k,j} - f_j)\,\mathrm{MA}_\omega(u_k) &\le \delta V_\psi + \limsup_{l\to\infty}\limsup_{k\to\infty} 2\|\varphi_j\|_{L^\infty} \int_{\{|h_{l,j}-h_j|\ge\delta\}} \mathrm{MA}_\omega(u_k)\\ &\le \delta V_\psi + \limsup_{l\to\infty} 2\|\varphi_j\|_{L^\infty} \int_{\{|h_{l,j}-h_j|\ge\delta\}} \mathrm{MA}_\omega(u) = \delta V_\psi,\end{aligned}$$

where we also used that $\{|h_{l,j} - h_j| \ge \delta\}$ is a closed set in the plurifine topology since it is equal to $\{v_{l,j} - v_j \ge \delta\}$ if $\psi_l \searrow \psi$ and to $\{\psi - \psi_l \ge \delta\}$ if $\psi_l \nearrow \psi$ almost everywhere. Hence

$$\limsup_{k\to\infty} \int_X (f_{k,j} - f_j)\,\mathrm{MA}_\omega(u_k) \le 0.$$

Similarly we also get

$$\limsup_{s\to\infty}\limsup_{k\to\infty} \int_X (f_j - f_j^s)\,\mathrm{MA}_\omega(u_k) \le 0;$$

see also the proof of Theorem A.

Summarizing from (27), we obtain

$$\begin{aligned}&\limsup_{k\to\infty}(n+1)^{-1} I_{\psi_k}(u_k, v_{k,j})\\ &\le E^*_\psi(\mathrm{MA}_\omega(u)/V_\psi) - E^*_\psi(\mathrm{MA}_\omega(v_j)/V_\psi) + \int_X (v_j - \psi)\,\mathrm{MA}_\omega(u) - \int_X (v_j - \psi)\,\mathrm{MA}_\omega(v_j) =: F_j,\end{aligned} \tag{30}$$

and $F_j \to 0$ as $j \to \infty$ by Step 1 and Proposition 3.7, since $\mathcal{E}^1(X, \omega, \psi) \ni v_j \searrow u \in \mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi)$, hence strongly.

Next by Lemma 3.1, $u_k \in X_{\mathcal{A},C}$ for $C \gg 1$ since $E^*(\mathrm{MA}_\omega(u_k)/V_{\psi_k}) = J^{\psi}_{u_k}(\psi)$ and $\sup_X u_k = 0$, thus, up to considering a further subsequence, $u_k \to w \in \mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi)$ weakly where $d(w, \psi) \le C$. Indeed, if $V_\psi > 0$ this follows from Proposition 3.15 while it is trivial if $V_\psi = 0$. In particular, by Lemma 4.6,

$$\int_X (\psi_k - u_k)\,\mathrm{MA}_\omega(v_{k,j}) \to \int_X (\psi - w)\,\mathrm{MA}_\omega(v_j), \tag{31}$$

$$\int_X (v_{k,j} - u_k)\,\mathrm{MA}_\omega(v_{k,j}) \to \int_X (v_j - w)\,\mathrm{MA}_\omega(v_j) \tag{32}$$

as $j \to \infty$. Therefore if $V_\psi = 0$, then combining $I_{\psi_k}(u_k, v_{k,j}) \to 0$ as $k \to \infty$ with (32) and Lemma 3.1, we obtain

$$\limsup_{k\to\infty}(-E_{\psi_k}(u_k) + E_{\psi_k}(v_{k,j})) \le \limsup_{k\to\infty}\left(\frac{n}{n+1} I_{\psi_k}(u_k, v_{k,j}) + \left|\int_X (v_{k,j} - u_k)\,\mathrm{MA}_\omega(v_{k,j})\right|\right) = 0.$$

This implies that $d(\psi_k, u_k) = -E_{\psi_k}(u_k) \to 0$ as $k \to \infty$, i.e., that $d_{\mathcal{A}}(P_{\psi_{\min}}, u_k) \to 0$ using Theorem 6.2. We may assume from now until the end of the proof that $V_\psi > 0$.

By (31) and Proposition 3.14 it follows that

$$\limsup_{k\to\infty}\biggl(E^*_{\psi_k}(\mathrm{MA}_\omega(u_k)/V_{\psi_k}) + \int_X (\psi_k - u_k)(\mathrm{MA}_\omega(v_{k,j}) - \mathrm{MA}_\omega(u_k))\biggr)$$
$$= \limsup_{k\to\infty}\biggl(E_{\psi_k}(u_k) + \int_X (\psi_k - u_k)\,\mathrm{MA}_\omega(v_{k,j})\biggr) \le E_\psi(w) + \int_X (\psi - w)\,\mathrm{MA}_\omega(v_j). \quad (33)$$

On the other hand, by Proposition 3.7 and (30),

$$\limsup_{k\to\infty}\biggl|\int_X (\psi_k - u_k)(\mathrm{MA}_\omega(v_{k,j}) - \mathrm{MA}_\omega(u_k))\biggr| \le C F_j^{1/2}. \quad (34)$$

In conclusion, by the triangle inequality and combining (33) and (34) we get

$$\begin{aligned} E_\psi(u) + \int_X (\psi - u)\,\mathrm{MA}_\omega(u) &= \lim_{k\to\infty} E^*(\mathrm{MA}_\omega(u_k)/V_{\psi_k}) \\ &\le \limsup_{j\to\infty}\biggl(E_\psi(w) + \int_X (\psi - w)\,\mathrm{MA}_\omega(v_j) + C F_j^{1/2}\biggr) \\ &= E_\omega(w) + \int_X (\psi - w)\,\mathrm{MA}_\omega(u) \end{aligned}$$

since $F_j \to 0$, i.e., $w \in \mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi)$ is a maximizer of $F_{\mathrm{MA}_\omega(u)/V_\psi, \psi}$. Hence $w = u$ (Proposition 5.5), i.e., $u_k \to u$ weakly. Furthermore, similar to the case $V_\psi = 0$, Lemma 3.1 and (32) imply

$$\begin{aligned} E_\psi(v_j) - \liminf_{k\to\infty} E_{\psi_k}(u_k) &= \limsup_{k\to\infty}(-E_{\psi_k}(u_k) + E_{\psi_k}(v_{k,j})) \\ &\le \limsup_{k\to\infty}\biggl(\frac{n}{n+1} I_{\psi_k}(u_k, v_{k,j}) + \biggl|\int_X (u_k - v_{j,k})\,\mathrm{MA}_\omega(v_{k,j})\biggr|\biggr) \\ &\le \frac{n}{n+1} F_j + \biggl|\int_X (u - v_j)\,\mathrm{MA}_\omega(v_j)\biggr|. \end{aligned}$$

Finally, letting $j \to \infty$, since $v_j \to u$ strongly, we obtain $\liminf_{k\to\infty} E_{\psi_k}(u_k) \ge \lim_{j\to\infty} E_\psi(v_j) = E_\psi(u)$. Hence $E_{\psi_k}(u_k) \to E_\psi(u)$ by Proposition 3.14, which implies $d_{\mathcal{A}}(u_k, u) \to 0$ by Theorem 6.2. □

7. Stability of complex Monge–Ampère equations

As stated in the Introduction, we want to use the homeomorphism of Theorem B to deduce the strong stability of solutions of complex Monge–Ampère equations with prescribed singularities when the measures have uniformly bounded L^p density for $p > 1$.

Theorem C. *Let $\mathcal{A} := \{\psi_k\}_{k\in\mathbb{N}} \subset \mathcal{M}^+$ be totally ordered, and let $\{f_k\}_{k\in\mathbb{N}} \subset L^1$ be a sequence of nonnegative functions such that $f_k \to f \in L^1 \setminus \{0\}$ and such that $\int_X f_k \omega^n = V_{\psi_k}$ for any $k \in \mathbb{N}$. Assume also that there exists $p > 1$ such that $\|f_k\|_{L^p}$ and $\|f\|_{L^p}$ are uniformly bounded. Then $\psi_k \to \psi \in \bar{\mathcal{A}} \subset \mathcal{M}^+$,* *and the sequence of solutions of*

$$\mathrm{MA}_\omega(u_k) = f_k\omega^n, \quad u_k \in \mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi_k), \tag{35}$$

converges strongly to $u \in X_{\mathcal{A}}$, which is the unique solution of

$$\mathrm{MA}_\omega(u) = f\omega^n, \quad u \in \mathcal{E}^1_{\mathrm{norm}}(X, \omega, \psi). \tag{36}$$

In particular, $u_k \to u$ in capacity.

Proof. We first observe that the existence of the unique solutions of (35) follows by [Darvas et al. 2021a, Theorem A].

Moreover, letting u be any weak accumulation point for $\{u_k\}_{k\in\mathbb{N}}$ (there exists at least one by compactness), [Darvas et al. 2021a, Lemma 2.8] yields $\mathrm{MA}_\omega(u) \geq f\omega^n$ and by the convergence of f_k to f we also obtain $\int_X f\omega^n = \lim_{k\to\infty} V_{\psi_k}$. Moreover, since $u_k \leq \psi_k$ for any $k \in \mathbb{N}$, by [Witt Nyström 2019] we obtain $\int_X \mathrm{MA}_\omega(u) \leq \lim_{k\to\infty} V_{\psi_k}$. Hence $\mathrm{MA}_\omega(u) = f\omega^n$ which, in particular, means that there is a unique weak accumulation point for $\{u_k\}_{k\in\mathbb{N}}$ and that $\psi_k \to \psi$ as $k \to \infty$ since $V_{\psi_k} \to V_\psi$ (by Lemma 3.12). Then it easily follows by combining Fatou's lemma with Proposition 2.10 and Lemma 2.12 that for any $\varphi \in \mathcal{H}_\omega$,

$$\begin{aligned}\liminf_{k\to\infty} E^*_{\psi_k}(\mathrm{MA}_\omega(u_k)/V_{\psi_k}) &\geq \liminf_{k\to\infty}\left(E_{\psi_k}(P_\omega[\psi_k](\varphi)) + \int_X (\psi_k - P_\omega[\psi_k](\varphi)) f_k\omega^n\right) \\ &\geq E_\psi(P_\omega[\psi](\varphi)) + \int_X (\psi - P_\omega[\psi](\varphi)) f\omega^n,\end{aligned} \tag{37}$$

since $(\psi_k - P_\omega[\psi_k](\varphi)) f_k \to (\psi - P_\omega[\psi](\varphi)) f$ almost everywhere by Lemma 2.14. Thus, for any $v \in \mathcal{E}^1(X, \omega, \psi)$, letting $\varphi_j \in \mathcal{H}_\omega$ be a decreasing sequence converging to v [Błocki and Kołodziej 2007], from inequality (37) we get

$$\begin{aligned}\liminf_{k\to\infty} E^*_{\psi_k}(\mathrm{MA}_\omega(u_k)/V_{\psi_k}) &\geq \limsup_{j\to\infty}\left(E_\psi(P_\omega[\psi](\varphi_j)) + \int_X (\psi - P_\omega[\psi](\varphi_j)) f\omega^n\right) \\ &= E_\psi(v) + \int_X (\psi - v) f\omega^n,\end{aligned}$$

using Proposition 2.4 and the monotone convergence theorem. Hence by definition,

$$\liminf_{k\to\infty} E^*_{\psi_k}(\mathrm{MA}_\omega(u_k)/V_{\psi_k}) \geq E^*_\psi(f\omega^n/V_\psi). \tag{38}$$

On the other hand, since $\|f_k\|_{L^p}$ and $\|f\|_{L^p}$ are uniformly bounded for $p > 1$ and $u_k \to u$, $\psi_k \to \psi$ in L^q for any $q \in [1, +\infty)$ (see [Guedj and Zeriahi 2017, Theorem 1.48]), we also have

$$\int_X (\psi_k - u_k) f_k\omega^n \to \int_X (\psi - u) f\omega^n < +\infty,$$

which implies that $\int_X (\psi - u)\,\mathrm{MA}_\omega(u) < +\infty$, i.e., $u \in \mathcal{E}^1(X, \omega, \psi)$ by Proposition 2.4. Moreover, by Proposition 3.14 we also get

$$\limsup_{k\to\infty} E^*_{\psi_k}(\mathrm{MA}_\omega(u_k)/V_{\psi_k}) \leq E^*_\psi(\mathrm{MA}_\omega(u)/V_\psi),$$

which together with (38) leads us to $\mathrm{MA}_\omega(u_k) \to \mathrm{MA}_\omega(u)$ strongly in $Y_{\mathcal{A}}$ by definition (observe that $\mathrm{MA}_\omega(u_k) = f_k\omega^n \to \mathrm{MA}_\omega(u) = f\omega^n$ weakly). Hence $u_k \to u$ strongly by Theorem B while the convergence in capacity follows from Theorem 6.3. □

Remark 7.1. As said in the Introduction, the convergence in capacity of Theorem C was already obtained in [Darvas et al. 2021b, Theorem 1.4]. Indeed, under the hypotheses of Theorem C it follows from Lemma 2.12 and [Darvas et al. 2021b, Lemma 3.4] that $d_S(\psi_k, \psi) \to 0$ where d_S is the pseudometric on $\{[u] : u \in \mathrm{PSH}(X, \omega)\}$ introduced in [Darvas et al. 2021b], where the class $[u]$ is given by the partial order $\preccurlyeq$.

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