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SOME APPLICATIONS OF GROUP-THEORETIC RIPS CONSTRUCTIONS TO THE CLASSIFICATION OF VON NEUMANN ALGEBRAS

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We study various von Neumann algebraic rigidity aspects for the property (T) groups that arise via the Rips construction developed by Belegradek and Osin (*Groups Geom. Dyn.* **2**:1 (2008), 1–12) in geometric group theory. Specifically, developing a new interplay between Popa's deformation/rigidity theory (*Int. Congr. Math, I* (2007), 445–477) and geometric group theory methods, we show that several algebraic features of these groups are completely recognizable from the von Neumann algebraic structure. In particular, we obtain new infinite families of pairwise nonisomorphic property (T) group factors, thereby providing positive evidence towards Connes' rigidity conjecture.

In addition, we use the Rips construction to build examples of property (T) II_1-factors which possess maximal von Neumann subalgebras without property (T), which answers a question raised by Y. Jiang and A. Skalski (arXiv:1903.08190 (2019), version 3).

1. Introduction

The von Neumann algebra $\mathcal{L}(G)$ associated to a countable discrete group G is called *the group von Neumann algebra* and it is defined as the bicommutant of the left regular representation of G computed inside the algebra of all bounded linear operators on the Hilbert space of the square summable functions on G. $\mathcal{L}(G)$ is a II_1-factor (has trivial center) precisely when all nontrivial conjugacy classes of G are infinite (icc), this being the most interesting for study [Murray and von Neumann 1943]. The classification of group factors is a central research theme revolving around the following fundamental question: *What aspects of the group G are remembered by* $\mathcal{L}(G)$? This is a difficult topic as algebraic group properties usually do not survive after passage to the von Neumann algebra regime. Perhaps the best illustration of this phenomenon is Connes' celebrated result [1976] asserting that all amenable icc groups give isomorphic factors. Hence genuinely different groups such as the group of all finite permutations of the positive integers, the lamplighter group, or the wreath product of the integers with itself give rise to isomorphic factors. Ergo, basic algebraic group constructions such as direct products, semidirect products, extensions, inductive limits or classical algebraic invariants such as torsion, rank, or generators and relations in general cannot be recognized from the von Neumann algebraic structure. In this case the only information on G retained by the von Neumann algebra is amenability.

When G is nonamenable, the situation is far more complex and unprecedented progress has been achieved through the emergence of Popa's deformation/rigidity theory [Popa 2007; Vaes 2010;

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Ioana 2013; 2018]. Using this completely new conceptual framework it was shown that various algebraic/analytic properties of groups and their representations can be completely recovered from their von Neumann algebras [Ozawa and Popa 2004; 2010; Ioana et al. 2013; Berbec and Vaes 2014; Chifan et al. 2016b; Drimbe et al. 2019; Chifan and Ioana 2018; Chifan and Udrea 2020]. In this direction an impressive milestone was Ioana, Popa and Vaes's discovery [Ioana et al. 2013] of the first examples of groups G that can be completely reconstructed from $\mathcal{L}(G)$, i.e., *W^*-superrigid groups*.[1] Additional examples were found subsequently in [Berbec and Vaes 2014; Berbec 2015; Chifan and Ioana 2018]. It is worth noting that the general strategies used in establishing these results share a common essential ingredient — the ability to first reconstruct from $\mathcal{L}(G)$ specific given algebraic features of G. For instance, in the examples covered in [Ioana et al. 2013; Berbec and Vaes 2014; Berbec 2015], the first step was to show that whenever $\mathcal{L}(G) \cong \mathcal{L}(H)$, the mystery group H admits a generalized wreath product decomposition exactly as G does; also in the case of [Chifan and Ioana 2018, Theorem A] again the main step was to show that H admits an amalgamated free product splitting exactly as G. These aspects motivate a fairly broad and independent study on this topic — the quest of identifying a comprehensive list of algebraic features of groups which completely pass to the von Neumann algebraic structure. While a couple of works have already appeared in this direction [Chifan et al. 2016b; Chifan and Ioana 2018; Chifan and Udrea 2020], we are still far away from having a satisfactory overview of these properties and a great deal of work remains to be done.

A striking conjecture of Connes predicts that all icc property (T) groups are W^*-superrigid. Despite the fact that this conjecture motivated to great effect a significant portion of the main developments in Popa's deformation/rigidity theory [Popa 2006b; 2006c; Ioana 2011; Ioana et al. 2013], no example of a property (T) W^*-superrigid group is currently known. The first hard evidence towards Connes' conjecture was found in [Cowling and Haagerup 1989], where it was shown that uniform lattices in $\mathrm{Sp}(n, 1)$ give rise to nonisomorphic factors for different values of $n \geqslant 2$. Moreover, for any collection $\{G_k\}_k$ of uniform lattices in $\mathrm{Sp}(n_k, 1)$, $n_k \geqslant 2$, the group algebras $\{\mathcal{L}(\times_{i=1}^n G_i)\}_n$ are pairwise nonisomorphic. Later on, using a completely different approach, Ozawa and Popa [2004] obtained a far-reaching generalization of this result by showing that for any collection $\{G_n\}_n$ of hyperbolic, property (T) groups (e.g., uniform lattices in $\mathrm{Sp}(n, 1)$, $n \geqslant 2$ [Cowling and Haagerup 1989]) the group algebras $\{\mathcal{L}(\times_{i=1}^n G_i)\}_n$ are pairwise nonisomorphic. However, little is known beyond these two classes of examples. Moreover, the current literature offers an extremely limited account on which algebraic features that occur in a property (T) group are completely recognizable at the von Neumann algebraic level. For instance, besides the preservation of the Cowling–Haagerup constant [1989], the amenability of normalizers of infinite amenable subgroups in hyperbolic property (T) groups from [Ozawa and Popa 2010, Theorem 1], and the direct product rigidity for hyperbolic property (T) groups from [Chifan et al. 2016b, Theorem A; Chifan and Udrea 2020, Theorem A] very little is known. Therefore in order to successfully construct property (T) W^*-superrigid groups via a strategy similar to the ones used in [Ioana et al. 2013; Berbec and Vaes 2014; Berbec 2015; Chifan and Ioana 2018] we believe it is imperative to identify new algebraic features of property (T) groups that survive the passage to the von Neumann algebraic regime. Any success in this direction will potentially hint at which group theoretic methods to pursue in order to address Connes' conjecture.

[1] If H is any group such that $\mathcal{L}(G) \cong \mathcal{L}(H)$ then $H \cong G$.

In this paper we make new progress on this study by showing that many algebraic aspects of the Rips constructions developed in geometric group theory by Belegradek and Osin [2008] are entirely recoverable from the von Neumann algebraic structure. To properly introduce the result we briefly describe their construction. Using the prior Dehn filling results from [Osin 2010], Belegradek and Osin [2008, Theorem] showed that for every finitely generated group Q one can find a property (T) group N such that Q can be realized as a finite-index subgroup of $\mathrm{Out}(N)$. This canonically gives rise to an action $Q \curvearrowright^{\sigma} N$ by automorphisms such that the corresponding semidirect product group $N \rtimes_\sigma Q$ is hyperbolic relative to $\{Q\}$. Throughout the document the semidirect products $N \rtimes_\sigma Q$ will be termed Belegradek–Osin group Rips constructions. When Q is torsion-free, one can pick N to be torsion-free as well, and hence both N and $N \rtimes_\sigma Q$ are icc groups. Also when Q has property (T) then $N \rtimes_\sigma Q$ has property (T). Under all these assumptions we will denote by $\mathcal{R}ip_T(Q)$ the class of these Rips construction groups $N \rtimes_\sigma Q$.

The first main result of our paper concerns a fairly large class of canonical fiber products of groups in $\mathcal{R}ip_T(Q)$. Specifically, consider any two groups $N_1 \rtimes_{\sigma_1} Q$, $N_2 \rtimes_{\sigma_2} Q \in \mathcal{R}ip_T(Q)$ and form the canonical fiber product $G = (N_1 \times N_2) \rtimes_\sigma Q$, where $\sigma = (\sigma_1, \sigma_2)$ is the diagonal action. Notice that since property (T) is closed under extensions [Bekka et al. 2008, Section 1.7] it follows that G has property (T). Developing new interplay between geometric group theoretic methods [Rips 1982; Dahmani et al. 2017; Osin 2010; Belegradek and Osin 2008] and deformation/rigidity methods [Ioana 2011; Ioana et al. 2013; Chifan et al. 2016b; 2018; Chifan and Ioana 2018; Chifan and Udrea 2020], for a fairly large family of groups Q, we show that the semidirect product feature of G is an algebraic property completely recoverable from the von Neumann algebraic regime. In addition, we also have a complete reconstruction of the acting group Q. The precise statement is the following:

Theorem A (Theorem 5.1). *Let $Q = Q_1 \times Q_2$, where Q_i are icc, biexact, weakly amenable, property (T), torsion-free, residually finite groups. For $i = 1, 2$, let $N_i \rtimes_{\sigma_i} Q \in \mathcal{R}ip_T(Q)$ and denote by $\Gamma = (N_1 \times N_2) \rtimes_\sigma Q$ the semidirect product associated with the diagonal action $\sigma = \sigma_1 \times \sigma_2 : Q \curvearrowright N_1 \times N_2$. Denote by $\mathcal{M} = \mathcal{L}(\Gamma)$ the corresponding II_1-factor. Assume that Λ is any arbitrary group and $\Theta : \mathcal{L}(\Gamma) \to \mathcal{L}(\Lambda)$ is any $*$-isomorphism. Then there exist group actions by automorphisms $H \curvearrowright^{\tau_i} K_i$ such that $\Lambda = (K_1 \times K_2) \rtimes_\tau H$, where $\tau = \tau_1 \times \tau_2 : H \curvearrowright K_1 \times K_2$ is the diagonal action. Moreover one can find a multiplicative character $\eta : Q \to \mathbb{T}$, a group isomorphism $\delta : Q \to H$ and unitary $w \in \mathcal{L}(\Lambda)$ and $*$-isomorphisms $\Theta_i : \mathcal{L}(N_i) \to \mathcal{L}(K_i)$ such that for all $x_i \in L(N_i)$ and $g \in Q$ we have*

$$\Theta((x_1 \otimes x_2)u_g) = \eta(g) w((\Theta_1(x_1) \otimes \Theta(x_2))v_{\delta(g)})w^*. \tag{1.1}$$

Here $\{u_g : g \in Q\}$ and $\{v_h : h \in H\}$ are the canonical unitaries implementing the actions of $Q \curvearrowright \mathcal{L}(N_1) \mathbin{\bar{\otimes}} \mathcal{L}(N_2)$ and $H \curvearrowright \mathcal{L}(K_1) \mathbin{\bar{\otimes}} \mathcal{L}(K_2)$, respectively.

There are countably infinitely many groups that are residually finite, torsion-free, hyperbolic, and have property (T). A concrete such family is $\{\Lambda_k : k \geqslant 2\}$, where $\Lambda_k < \mathrm{Sp}(k, 1)$ is a uniform lattice. It is well known the Λ_k's are residually finite [Malcev 1940], (virtually) torsion-free [Selberg 1960], hyperbolic [Gromov 1987, Example B], have property (T) (see for instance, [Bekka et al. 2008, Theorem 1.5.3]) and are pairwise nonisomorphic [Cowling and Haagerup 1989]. However, there are infinitely many pairwise nonisomorphic such lattices even in the same Lie group. To see this, fix $k \geqslant 2$ together with a torsion-free, uniform lattice $\Gamma < \mathrm{Sp}(k, 1)$. Since Γ is residually finite there is a sequence of normal, finite-index, proper subgroups $\cdots \lhd \Gamma_{n+1} \lhd \Gamma_n \lhd \cdots \lhd \Gamma_1 \lhd \Gamma$ such that $\bigcap_n \Gamma_n = 1$. Being subgroups, Γ_n are clearly residually finite and torsion-free. Moreover, the finite-index condition implies that all Γ_n's are hyperbolic and have property (T). As the Γ_n's are cohopfian [Prasad 1976] and $\Gamma_n < \Gamma_m$ for every $n < m$, we have $\Gamma_n \ncong \Gamma_m$. Therefore the class $\{\Gamma_n : n \in \mathbb{N}\}$ satisfies our conditions. Finally we note that, since every hyperbolic group is finitely presented and there are only countably many such groups, one cannot built examples of larger cardinality than the ones presented above.

In conclusion, Theorem A provides explicit examples of infinitely many pairwise nonisomorphic group II_1-factors with property (T). Moreover these groups are quite different from the previous classes [Cowling and Haagerup 1989; Ozawa and Popa 2004], as they give rise to factors that are nonsolid ($\mathcal{L}(\Gamma)$ contains two commuting nonamenable subfactors $\mathcal{L}(N_1)$ and $\mathcal{L}(N_2)$), are tensor indecomposable [Das 2020, Lemma 2.3] and do not admit Cartan subalgebras (Corollary 7.2). Moreover, using the Margulis normal subgroup theorem, the factors covered by Theorem A are nonisomorphic to any factor arising from any irreducible lattices in a higher-rank semisimple Lie group (see remarks after the proof of Theorem 5.1). We also mention that Theorem A, or its strong rigidity version Theorem 6.1 (see also Corollary 6.2), provides examples of infinite families of finite-index subgroups $\Gamma_n \leqslant \Gamma$ in a given icc property (T) group Γ such that the corresponding group factors $\mathcal{L}(\Gamma_n)$ and $\mathcal{L}(\Gamma_m)$ are nonisomorphic for $n \neq m$. As the Γ_n's are measure equivalent this provides new counterexamples to D. Shlyakhtenko's question, asking whether measure equivalence of icc groups implies isomorphism of the corresponding group factors (see [Popa 2009, page 18]), which are very different in nature from the ones obtained in [Chifan and Ioana 2011; Chifan et al. 2016b]. We summarize this discussion in the next corollary.

Corollary B (Corollary 6.2). *Assume the same notation as in Theorem A.*

(1) *Let Q_1, Q_2 be uniform lattices in* $\mathrm{Sp}(n, 1)$ *with $n \geqslant 2$ and let $Q := Q_1 \times Q_2$. Also let $\cdots \leqslant Q_1^s \leqslant \cdots \leqslant Q_1^2 \leqslant Q_1^1 \leqslant Q_1$ be an infinite family of finite-index subgroups and define $Q_s := Q_1^s \times Q_2 \leqslant Q$. Then consider $N_1 \rtimes_{\sigma_1} Q, N_2 \rtimes_{\sigma_2} Q \in \mathcal{R}ip_T(Q)$ and let $\Gamma = (N_1 \times N_2) \rtimes_{\sigma_1 \times \sigma_2} Q$. Inside Γ consider the finite-index subgroups $\Gamma_s := (N_1 \times N_2) \rtimes_{\sigma_1 \times \sigma_2} Q_s$. Then the family $\{\mathcal{L}(\Gamma_s) : s \in I\}$ consists of pairwise nonisomorphic finite-index subfactors of $\mathcal{L}(\Gamma)$.*

(2) *Let Γ, Γ_n be as above. Then Γ_n is measure equivalent to Γ for all $n \in \mathbb{N}$, but $\mathcal{L}(\Gamma_n)$ is not isomorphic to $\mathcal{L}(\Gamma_m)$ for $n \neq m$.*

From a different perspective our theorem can be also seen as a von Neumann algebraic superrigidity result regarding conjugacy of actions on noncommutative von Neumann algebras. Notice that very little is known in this direction as most of the known superrigidity results concern algebras arising from actions of groups on probability spaces.

In certain ways one can view Theorem A as a first step towards providing an example of a property (T) superrigid group. While the acting group Q can be completely recovered, as well as certain aspects of the action $Q \curvearrowright N_1 \times N_2$ (e.g., trivial stabilizers) only the product feature of the "core" $\mathcal{L}(N_1 \times N_2)$ can be reconstructed at this point. While the reconstruction of N_1 and N_2 seems to be out of reach momentarily, we believe that a deeper understanding of the Rips construction, along with new von Neumann algebraic techniques are necessary to tackle this problem. We also remark that in a subsequent article it was shown that the group factors as in Theorem A have trivial fundamental group; see [Chifan et al. 2020, Theorem B].

Besides the aforementioned rigidity results we also investigate applications of group Rips constructions to the study of maximal von Neumann algebras. If $\mathcal{M}$ is a von Neumann algebra then a von Neumann subalgebra $\mathcal{N} \subset \mathcal{M}$ is called *maximal* if there is no intermediate von Neumann subalgebra $\mathcal{P}$ so that $\mathcal{N} \subsetneq \mathcal{P} \subsetneq \mathcal{M}$. Understanding the structure of maximal subalgebras of a given von Neumann algebra is a rather difficult problem that is intimately related with the very classification of these objects. Despite a series of remarkable earlier successes on the study of maximal *amenable* subalgebras initiated in [Popa 1983] and continued more recently [Shen 2006; Cameron et al. 2010; Houdayer 2014; Boutonnet and Carderi 2015; 2017; Suzuki 2020; Chifan and Das 2020; Jiang and Skalski 2019a], significantly less is known for the arbitrary maximal ones. For instance Ge's question [2003, Section 3, Question 2] on the existence of nonamenable factors that possess maximal factors that are hyperfinite was settled in the affirmative only very recently by Y. Jiang and A. Skalski [2019a]. In fact in their work they proposed a more systematic approach towards the study of maximal von Neumann subalgebras within various categories, such as the von Neumann algebras with Haagerup's property or with property (T) of Kazhdan. Their investigation also naturally led to several interesting open problems [Jiang and Skalski 2019a, Section 5].

In this paper we explain how in a setting similar to [Jiang and Skalski 2019a] the Belegradek–Osin group Rips construction techniques and Olshanski-type monster groups can be used in conjunction with Galois correspondence results for II_1-factors à la [Choda 1978] to produce many maximal von Neumann subalgebras arising from group/subgroup situation. In particular, through this mix of results we are able to construct many examples of II_1 -actors with property (T) that have maximal von Neumann subalgebras without property (T), thereby answering Problem 5.5 in the first version of [Jiang and Skalski 2019a] (see Theorem 4.4). More specifically, using Olshanskii's small cancellation techniques [2009] in the setting of lacunary hyperbolic groups we explain how one can construct a property (T) monster group Q whose maximal subgroups are all isomorphic to a given rank-1 group[2] Q_m (see Section 2C). Then if one considers the Belegradek–Osin Rips construction $N \rtimes Q$ corresponding to Q then using a Galois correspondence (Lemma 4.2) one can show the following:

Theorem C (Theorem 4.4). *For every maximal rank-*1 *subgroup $Q_m < Q$ consider the subgroup $N \rtimes Q_m < N \rtimes Q$. Then $\mathcal{L}(N \rtimes Q_m) \subset \mathcal{L}(N \rtimes Q)$ is a maximal von Neumann subalgebra.*

Note that since N and Q have property (T), so does $N \rtimes Q$ and therefore the corresponding II_1-factor $\mathcal{L}(N \rtimes Q)$ has property (T) by [Connes and Jones 1985]. However since $N \rtimes Q_m$ surjects onto the infinite abelian group Q_m, it does not have property (T) and hence $\mathcal{L}(N \rtimes Q_m)$ does not have property (T) either. Another solution to the problem of finding maximal subalgebras without property (T) inside factors with property (T) was also obtained independently by Jiang and Skalski [2019b]. Their beautiful solution has a different flavor from ours; even though the Galois correspondence theorem à la Choda is a common ingredient in both of the proofs. Hence we refer the reader to [Jiang and Skalski 2019b, Theorem 4.8] for another solution to the aforementioned problem.

[2]Any group that is isomorphic to a subgroup of $(\mathbb{Q}, +)$ is called rank-1.

2. Preliminaries

2A. ***Notation and terminology.*** We denote by $\mathbb{N}$ and $\mathbb{Z}$ the set of natural numbers and the integers, respectively. For any $k \in \mathbb{N}$ we denote by $\overline{1,k}$ the integers $\{1, 2, \dots, k\}$.

All von Neumann algebras in this document will be denoted by calligraphic letters, e.g., $\mathcal{A}$, $\mathcal{B}$, $\mathcal{M}$, $\mathcal{N}$, etc. Given a von Neumann algebra $\mathcal{M}$ we will denote by $\mathscr{U}(\mathcal{M})$ its unitary group, by $\mathscr{P}(\mathcal{M})$ the set of all its nonzero projections, and by $\mathscr{Z}(\mathcal{M})$ its center. We also denote by $(\mathcal{M})_1$ its unit ball. All algebra inclusions $\mathcal{N} \subseteq \mathcal{M}$ are assumed unital unless otherwise specified. Given an inclusion $\mathcal{N} \subseteq \mathcal{M}$ of von Neumann algebras we denote by $\mathcal{N}' \cap \mathcal{M}$ the relative commutant of $\mathcal{N}$ in $\mathcal{M}$, i.e., the subalgebra of all $x \in \mathcal{M}$ such that $xy = yx$ for all $y \in \mathcal{N}$. We also consider the one-sided quasinormalizer $\mathscr{QN}^{(1)}_{\mathcal{M}}(\mathcal{N})$ (the semigroup of all $x \in \mathcal{M}$ for which there exist $x_1, x_2, \dots, x_n \in \mathcal{M}$ such that $\mathcal{N}x \subseteq \sum_i x_i\mathcal{N}$) and the quasinormalizer $\mathscr{QN}_{\mathcal{M}}(\mathcal{N})$ (the set of all $x \in \mathcal{M}$ for which there exist $x_1, x_2, \dots, x_n \in \mathcal{M}$ such that $\mathcal{N}x \subseteq \sum_i x_i\mathcal{N}$ and $x\mathcal{N} \subseteq \sum_i \mathcal{N}x_i$) and we notice that $\mathcal{N} \subseteq \mathscr{N}_{\mathcal{M}}(\mathcal{N}) \subseteq \mathscr{QN}_{\mathcal{M}}(\mathcal{N}) \subseteq \mathscr{QN}^{(1)}_{\mathcal{M}}(\mathcal{N})$.

All von Neumann algebras $\mathcal{M}$ considered in this article will be tracial, i.e., endowed with a unital, faithful, normal linear functional $\tau : M \to \mathbb{C}$ satisfying $\tau(xy) = \tau(yx)$ for all $x, y \in \mathcal{M}$. This induces a norm on $\mathcal{M}$ by the formula $\|x\|_2 = \tau(x^*x)^{1/2}$ for all $x \in \mathcal{M}$. The $\|\cdot\|_2$-completion of $\mathcal{M}$ will be denoted by $L^2(\mathcal{M})$. For any von Neumann subalgebra $\mathcal{N} \subseteq \mathcal{M}$ we denote by $E_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$ the τ-preserving conditional expectation onto $\mathcal{N}$.

For a countable group G we denote by $\{u_g : g \in G\} \in \mathscr{U}(\ell^2 G)$ its left regular representation given by $u_g(\delta_h) = \delta_{gh}$, where $\delta_h : G \to \mathbb{C}$ is the Dirac function at $\{h\}$. The weak operatorial closure of the linear span of $\{u_g : g \in G\}$ in $\mathscr{B}(\ell^2 G)$ is the so-called group von Neumann algebra and will be denoted by $\mathcal{L}(G)$. $\mathcal{L}(G)$ is a II_1-factor precisely when G has infinite nontrivial conjugacy classes (icc). If $\mathcal{M}$ is a tracial von Neumann algebra and $G \curvearrowright^{\sigma} \mathcal{M}$ is a trace-preserving action we denote by $\mathcal{M} \rtimes_\sigma G$ the corresponding cross product von Neumann algebra [Murray and von Neumann 1937]. For any subset $K \subseteq G$ we denote by $P_{\mathcal{M}K}$ the orthogonal projection from the Hilbert space $L^2(\mathcal{M} \rtimes G)$ onto the closed linear span of $\{xu_g : x \in \mathcal{M}, g \in K\}$. When $\mathcal{M}$ is trivial we will denote this simply by P_K.

Given a subgroup $H \leqslant G$ we denote by $C_G(H)$ the centralizer of H in G and by $N_G(H)$ the normalizer of H in G. Also we will denote by $QN^{(1)}_G(H)$ the one-sided quasinormalizer of H in G; this is the semigroup of all $g \in G$ for which there exist a finite set $F \subseteq G$ such that $Hg \subseteq FH$. Similarly we denote by $QN_G(H)$ the quasinormalizer (or commensurator) of H in G, i.e., the subgroup of all $g \in G$ for which there is a finite set $F \subseteq G$ such that $Hg \subseteq FH$ and $gH \subseteq HF$. We canonically have $HC_G(H) \leqslant N_G(H) \leqslant QN_G(H) \subseteq QN^{(1)}_G(H)$. We often consider the virtual centralizer of H in G, i.e., $vC_G(H) = \{g \in G : |g^H| < \infty\}$. Notice $vC_G(H)$ is a subgroup of G that is normalized by H. When $H = G$, the virtual centralizer is the FC-radical of G. Also one can easily see from definitions that $HvC_G(H) \leqslant QN_G(H)$. For a subgroup $H \leqslant G$ we denote by $\langle\langle H \rangle\rangle$ the normal closure of H in G.

Finally, for any groups G and N and an action $G \curvearrowright^{\sigma} N$ we denote by $N \rtimes_\sigma G$ the corresponding semidirect product group.

2B. ***Popa's intertwining techniques.*** Over fifteen years ago, Sorin Popa introduced [2006b, Theorem 2.1 and Corollary 2.3] a powerful analytic criterion for identifying intertwiners between arbitrary subalgebras of tracial von Neumann algebras. Now this is known in the literature as *Popa's intertwining-by-bimodules technique* and has played a key role in the classification of von Neumann algebras program via Popa's deformation/rigidity theory.

Theorem 2.1 [Popa 2006b]. *Let $(\mathcal{M}, \tau)$ be a separable tracial von Neumann algebra and let $\mathcal{P}, \mathcal{Q} \subseteq \mathcal{M}$ be (not necessarily unital) von Neumann subalgebras. Then the following are equivalent:*

(1) *There exist $p \in \mathscr{P}(\mathcal{P})$, $q \in \mathscr{P}(\mathcal{Q})$, a $*$-homomorphism $\theta : p\mathcal{P}p \to q\mathcal{Q}q$ and a partial isometry $0 \neq v \in q\mathcal{M}p$ such that $\theta(x)v = vx$ for all $x \in p\mathcal{P}p$.*

(2) *For any group $\mathcal{G} \subset \mathscr{U}(\mathcal{P})$ such that $\mathcal{G}'' = \mathcal{P}$, there is no sequence $(u_n)_n \subset \mathcal{G}$ satisfying $\|E_{\mathcal{Q}}(xu_n y)\|_2 \to 0$ for all $x, y \in \mathcal{M}$.*

(3) *There exist finitely many $x_i, y_i \in \mathcal{M}$ and $C > 0$ such that $\sum_i \|E_{\mathcal{Q}}(x_i u y_i)\|_2^2 \geqslant C$ for all $u \in \mathscr{U}(\mathcal{P})$.*

If one of the three equivalent conditions from Theorem 2.1 holds then we say that *a corner of $\mathcal{P}$ embeds into $\mathcal{Q}$ inside $\mathcal{M}$*, and write $\mathcal{P} \prec_{\mathcal{M}} \mathcal{Q}$. If we moreover have $\mathcal{P}p' \prec_{\mathcal{M}} \mathcal{Q}$ for any projection $0 \neq p' \in \mathcal{P}' \cap 1_{\mathcal{P}}\mathcal{M}1_{\mathcal{P}}$ (equivalently, for any projection $0 \neq p' \in \mathscr{Z}(\mathcal{P}' \cap 1_{\mathcal{P}}\mathcal{M}1_P)$), then we write $\mathcal{P} \prec^s_{\mathcal{M}} \mathcal{Q}$.

For further use we record the following result which controls the intertwiners in algebras arising from malnormal subgroups. Its proof is essentially contained in [Popa 2006b, Theorem 3.1] so it will be left to the reader.

Lemma 2.2 [Popa 2006b]. *Assume that $H \leqslant G$ is an almost malnormal subgroup and let $G \curvearrowright \mathcal{N}$ be a trace-preserving action on a tracial von Neumann algebra $\mathcal{N}$. Let $\mathcal{P} \subseteq \mathcal{N} \rtimes H$ be a von Neumann algebra such that $\mathcal{P} \nprec_{\mathcal{N} \rtimes H} \mathcal{N}$. Then for all elements $x, x_1, x_2, \ldots, x_l \in \mathcal{N} \rtimes G$ satisfying $\mathcal{P}x \subseteq \sum_{i=1}^{l} x_i \mathcal{P}$ we must have $x \in \mathcal{N} \rtimes H$.*

We continue with the following intertwining result for group algebras which is a generalization of some previous results obtained under normality assumptions [Drimbe et al. 2019]. For the reader's convenience we also include a brief proof.

Lemma 2.3. *Assume that $H_1, H_2 \leqslant G$ are groups, let $G \curvearrowright \mathcal{N}$ be a trace-preserving action on a tracial von Neumann algebra $\mathcal{N}$ and denote by $\mathcal{M} = \mathcal{N} \rtimes G$ the corresponding crossed product. Also assume that $\mathcal{A} \prec^s \mathcal{N} \rtimes H_1$ is a von Neumann algebra such that $\mathcal{A} \prec_{\mathcal{M}} \mathcal{N} \rtimes H_2$. Then one can find $h \in G$ such that $\mathcal{A} \prec_{\mathcal{M}} \mathcal{N} \rtimes (H_1 \cap hH_2h^{-1})$.*

Proof. Since $\mathcal{A} \prec^s \mathcal{N} \rtimes H_1$, by [Vaes 2013, Lemma 2.6] for every $\varepsilon > 0$ there exists a finite subset $S \subset G$ such that $\|P_{SH_1S}(x) - x\|_2 \leqslant \varepsilon$ for all $x \in (\mathcal{A})_1$. Here for every $K \subseteq G$ we denote by P_K the orthogonal projection from $L^2(\mathcal{M})$ onto the closure of the linear span of $\mathcal{N}u_g$ with $g \in K$. Also since $\mathcal{A} \prec_{\mathcal{M}} \mathcal{N} \rtimes H_2$, by Popa's intertwining techniques there exist a scalar $0 < \delta < 1$ and a finite subset $T \subset G$ so that $\|P_{TH_2T}(x)\|_2 \geqslant \delta$ for all $x \in (\mathcal{A})_1$. Thus, using this in combination with the previous inequality, for every $x \in \mathscr{U}(\mathcal{A})$ and every $\varepsilon > 0$, there are finite subsets $S, T \subset G$ so that $\|P_{TH_2T} \circ P_{SH_1S}(x)\|_2 \geqslant \delta - \varepsilon$. Since there exist finite subsets $R, U \subset G$ such that $TH_2T \cap SH_1S \subseteq U\big(\bigcup_{r \in R} H_2 \cap rH_1r^{-1}\big)U$, we further get that $\|P_{U(\bigcup_{r \in R} H_2 \cap rH_1r^{-1})U}(x)\|_2 \geqslant \delta - \varepsilon$. Then choosing $\varepsilon > 0$ sufficiently small and using Popa's intertwining techniques together with a diagonalization argument (see the proof of [Ioana et al. 2008, Theorem 4.3]) one can find $r \in R$ so that $\mathcal{A} \prec \mathcal{N} \rtimes (H_2 \cap rH_1r^{-1})$, as desired. □

In the sequel we need the following three intertwining lemmas, which establish that under certain conditions, intertwining in a larger algebra implies that the intertwining happens in a "smaller subalgebra".

Lemma 2.4. *Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{N} \subseteq \mathcal{M}$ be von Neumann algebras so that $\mathscr{N}_{\mathcal{M}}(\mathcal{A})'' = \mathcal{M}$. If $\mathcal{B} \prec_{\mathcal{M}} \mathcal{A}$ then $\mathcal{B} \prec_{\mathcal{N}} \mathcal{A}$.*

Proof. Since $\mathcal{B} \prec_{\mathcal{M}} \mathcal{A}$, by Theorem 2.1 one can find $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in \mathcal{M}$ and $c > 0$ such that $\sum_{i=1}^n \|E_{\mathcal{A}}(x_i b y_i)\|_2^2 \geqslant c$ for all $b \in \mathscr{U}(\mathcal{B})$. Since $\mathscr{N}_{\mathcal{M}}(\mathcal{A})'' = \mathcal{M}$, using basic $\|\cdot\|_2$-approximation for x_i and y_i and shrinking $c > 0$ if necessary, one can find $g_1, g_2, \ldots, g_l, h_1, h_2, \ldots, h_l \in \mathscr{N}_{\mathcal{M}}(\mathcal{A})$ and $c' > 0$ such that for all $b \in \mathscr{U}(\mathcal{B})$ we have

$$\sum_{i=1}^n \|E_{\mathcal{A}}(g_i b h_i)\|_2^2 \geqslant c' > 0. \tag{2B.1}$$

Using normalization we see that $E_{\mathcal{A}}(g_i b h_i) = E_{g_i \mathcal{A} g_i^*}(g_i b h_i) = g_i E_{\mathcal{A}}(b h_i g_i) g_i^*$. This, combined with (2B.1) and $\mathcal{A} \subseteq \mathcal{N}$, gives

$$0 < c' \leqslant \sum_{i=1}^l \|E_{\mathcal{A}}(b h_i g_i)\|_2^2 = \sum_{i=1}^l \|E_{\mathcal{A}} \circ E_{\mathcal{N}}(b h_i g_i)\|_2^2 = \sum_{i=1}^l \|E_{\mathcal{A}}(b E_{\mathcal{N}}(h_i g_i))\|_2^2$$

for all $b \in \mathscr{U}(\mathcal{B})$. Since $E_{\mathcal{N}}(h_i g_i) \in \mathcal{N}$, using Theorem 2.1 this clearly shows that $\mathcal{B} \prec_{\mathcal{N}} \mathcal{A}$. $\square$

Lemma 2.5. *Let Q be a group and define $\mathrm{d}(Q) = \{(q, q) : q \in Q\}$. Let $\mathcal{A}$ be a tracial von Neumann algebra and assume $(Q \times Q) \curvearrowright^{\sigma} \mathcal{A}$ is a trace-preserving action. Let $\mathcal{B} \subseteq \mathcal{A}$ be a regular von Neumann subalgebra which is invariant under the action σ. Let $\mathcal{D} \subseteq \mathcal{A} \rtimes_{\sigma} \mathrm{d}(Q)$ be a subalgebra such that $\mathcal{D} \prec_{\mathcal{A} \rtimes_{\sigma} (Q \times Q)} \mathcal{B} \rtimes_{\sigma} \mathrm{d}(Q)$. Then $\mathcal{D} \prec_{\mathcal{A} \rtimes_{\sigma} \mathrm{d}(Q)} \mathcal{B} \rtimes_{\sigma} \mathrm{d}(Q)$.*

Proof. Define $\mathcal{M} := \mathcal{A} \rtimes_{\sigma} (Q \times Q)$, $\mathcal{N} := \mathcal{A} \rtimes_{\sigma} \mathrm{d}(Q)$, and $\mathcal{P} := \mathcal{B} \rtimes_{\sigma} \mathrm{d}(Q)$. Thus $\mathcal{P} \subset \mathcal{N} \subset \mathcal{M}$ and with this notation we establish the following:

Claim 1. *Let $(v_n)_n \subset \mathscr{U}(\mathcal{N})$ be a sequence such that $\lim_{n\to\infty} \|E_{\mathcal{P}}(a v_n b)\|_2 = 0$ for all $a, b \in \mathcal{N}$. Then*

$$\lim_{n\to\infty} \|E_{\mathcal{P}}(x v_n y)\|_2 = 0 \quad \textit{for all } x, y \in \mathcal{M}. \tag{2B.2}$$

Proof of Claim 1. Notice that $(Q \times Q) = (Q \times 1) \rtimes_{\rho} \mathrm{d}(Q)$, where $\mathrm{d}(Q) \curvearrowright^{\rho} (Q \times 1)$ is the action by conjugation. Therefore, using basic $\|\cdot\|_2$-approximations and the $\mathcal{P}$-bimodularity of the conditional expectation $E_{\mathcal{P}}$, it suffices to show (2B.2) only for $x = (u_g \otimes 1)c$ and $y = d(u_h \otimes 1)$ for all $g, h \in Q$ and $c, d \in \mathcal{A}$. Under these assumptions we see that

$$\begin{aligned} E_{\mathcal{P}}((u_g \otimes 1) c v_n d (u_h \otimes 1)) &= E_{\mathcal{P}} \circ P_{(u_g \otimes 1)\mathcal{M}(u_h \otimes 1)}((u_g \otimes 1) c v_n d (u_h \otimes 1)) \\ &= P_{\mathcal{B}(\mathrm{d}(Q) \cap (g,1)\mathrm{d}(Q)(h,1))}((u_g \otimes 1) c v_n d (u_h \otimes 1)). \end{aligned} \tag{2B.3}$$

Here, and throughout the proof, for every set $S \subseteq Q \times Q$ we denote by $P_{\mathcal{B}S}$ the orthogonal projection onto the closed subspace $\overline{\mathrm{span}}\{\mathcal{B} u_g : g \in S\}$.

To this end observe there exists an element $s \in Q$ such that

$$\mathrm{d}(Q) \cap (g, 1)\mathrm{d}(Q)(h, 1) \subseteq [\mathrm{d}(Q) \cap (g, 1)\mathrm{d}(Q)(g^{-1}, 1)]\mathrm{d}(s).$$

Moreover, a basic computation shows that $\mathrm{d}(Q)\cap(g,1)\mathrm{d}(Q)(g^{-1},1)=\mathrm{d}(C_Q(g))$, where $C_Q(g)$ is the centralizer of g in Q. Hence altogether we have $\mathrm{d}(Q)\cap(g,1)\mathrm{d}(Q)(h,1)\subseteq\mathrm{d}(C_Q(g))\mathrm{d}(s)$. Combining this with (2B.3) and using the fact that $u_g\otimes 1$ normalizes $\mathcal{B}\rtimes\mathrm{d}(C_Q(g))$ we see that

$$\begin{aligned}\|E_{\mathcal{P}}((u_g\otimes 1)cv_nd(u_h\otimes 1))\|_2 &\leqslant \|P_{\mathcal{B}(\mathrm{d}(C_Q(g))\mathrm{d}(s))}((u_g\otimes 1)cv_nd(u_h\otimes 1))\|_2\\ &=\|E_{\mathcal{B}\rtimes\mathrm{d}(C_Q(g))}((u_g\otimes 1)cv_nd(u_{hs^{-1}}\otimes u_{s^{-1}}))\|_2\\ &=\|E_{\mathcal{B}\rtimes\mathrm{d}(C_Q(g))}(cv_nd(u_{hs^{-1}g}\otimes u_{s^{-1}}))\|_2\\ &=\|E_{\mathcal{B}\rtimes\mathrm{d}(C_Q(g))}(cv_ndE_{\mathcal{N}}(u_{hs^{-1}g}\otimes u_{s^{-1}}))\|_2\\ &=\delta_{hs^{-1}g,s^{-1}}\|E_{\mathcal{B}\rtimes\mathrm{d}(C_Q(g))}(cv_nd)\|_2\leqslant\|E_{\mathcal{P}}(cv_nd)\|_2.\end{aligned}\tag{2B.4}$$

Letting $n\to\infty$ in (2B.4) and using the assumption, the claim is obtained. □

To show our lemma assume by contradiction that $\mathcal{D}\nprec_{\mathcal{N}}\mathcal{P}$. By Theorem 2.1 there is a sequence of unitaries $(v_n)_n\subset\mathcal{D}\subset\mathcal{N}$ so that $\lim_{n\to\infty}\|E_{\mathcal{P}}(av_nb)\|_2=0$ for all $a,b\in\mathcal{N}$. Using Claim 1 we get $\lim_{n\to\infty}\|E_{\mathcal{P}}(xv_ny)\|_2=0$ for all $x,y\in\mathcal{M}$, which by Theorem 2.1 again implies $\mathcal{D}\nprec_{\mathcal{M}}\mathcal{P}$, a contradiction. □

Lemma 2.6. *Let $\mathcal{C}\subseteq\mathcal{B}$ and $\mathcal{N}\subseteq\mathcal{M}$ be inclusions of tracial von Neumann algebras. If $\mathcal{A}\subseteq\mathcal{N}\,\bar{\otimes}\,\mathcal{B}$ is a von Neumann subalgebra such that $\mathcal{A}\prec_{\mathcal{M}\bar{\otimes}\mathcal{B}}\mathcal{M}\,\bar{\otimes}\,\mathcal{C}$ then $\mathcal{A}\prec_{\mathcal{N}\bar{\otimes}\mathcal{B}}\mathcal{N}\,\bar{\otimes}\,\mathcal{C}$.*

Proof. By Theorem 2.1 one can find $x_i,y_i\in\mathcal{M}\,\bar{\otimes}\,\mathcal{B}$, $i=\overline{1,k}$, and a scalar $c>0$ such that

$$\sum_{i=1}^{n}\|E_{\mathcal{M}\bar{\otimes}\mathcal{C}}(x_iay_i)\|^2\geqslant c\quad\text{for all } d\in\mathscr{U}(\mathcal{A}).\tag{2B.5}$$

Using $\|\cdot\|_2$-approximations of x_i and y_i by finite linear combinations of elements in $\mathcal{M}\,\bar{\otimes}_{\mathrm{alg}}\,\mathcal{B}$ together with the $\mathcal{M}\otimes 1$-bimodularity of $E_{\mathcal{M}\bar{\otimes}\mathcal{C}}$, after increasing k and shrinking $c>0$ if necessary, in (2B.5) we can assume without loss of generality that $x_i,y_i\in 1\otimes\mathcal{B}$. However, since $\mathcal{A}\subseteq\mathcal{N}\,\bar{\otimes}\,\mathcal{B}$, in this situation we have $E_{\mathcal{M}\bar{\otimes}\mathcal{C}}(x_iay_i)=E_{\mathcal{M}\bar{\otimes}\mathcal{C}}\circ E_{\mathcal{N}\bar{\otimes}\mathcal{B}}(x_iay_i)=E_{\mathcal{N}\bar{\otimes}\mathcal{C}}(x_iay_i)$. Thus (2B.5) combined with Theorem 2.1 give $\mathcal{A}\prec_{\mathcal{N}\bar{\otimes}\mathcal{B}}\mathcal{N}\,\bar{\otimes}\,\mathcal{C}$, as desired. □

In the sequel we need the following (minimal) technical variation of [Chifan and Ioana 2018, Lemma 2.6]. The proof is essentially the same with the one presented in that work and we leave the details to the reader.

Lemma 2.7 [Chifan and Ioana 2018, Lemma 2.6]. *Let $\mathcal{P},\mathcal{Q}\subseteq\mathcal{M}$ be inclusions of tracial von Neumann algebras. Assume that $\mathscr{Q}\mathscr{N}^{(1)}_{\mathcal{M}}(\mathcal{P})=P$ and $\mathcal{Q}$ is a II_1-factor. Suppose there is a projection $z\in\mathscr{Z}(\mathcal{P})$ such that $\mathcal{P}z\prec^s\mathcal{Q}$ and a projection $p\in\mathcal{P}z$ such that $p\mathcal{P}p=p\mathcal{Q}p$. Then one can find a unitary $u\in\mathcal{M}$ such that $u\mathcal{P}zu^*=r\mathcal{Q}r$, where $r=uzu^*\in\mathscr{P}(\mathcal{Q})$.*

The next lemma is a mild generalization of [Ioana et al. 2013, Proposition 7.1], using the same techniques (see also the proof of [Krogager and Vaes 2017, Lemma 2.3]).

Lemma 2.8. *Let Λ be an icc group, and let $\mathcal{M}=\mathcal{L}(\Lambda)$. Consider the comultiplication map $\Delta:\mathcal{M}\to\mathcal{M}\,\bar{\otimes}\,\mathcal{M}$ given by $\Delta(v_\lambda)=v_\lambda\otimes v_\lambda$ for all $\lambda\in\Lambda$. Let $\mathcal{A},\mathcal{B}\subseteq\mathcal{M}$ be* (*unital*) *$*$-subalgebras such that $\Delta(\mathcal{A})\subseteq\mathcal{M}\,\bar{\otimes}\,\mathcal{B}$. Then there exists a subgroup $\Sigma<\Lambda$ such that $\mathcal{A}\subseteq\mathcal{L}(\Sigma)\subseteq\mathcal{B}$. In particular, if $\mathcal{A}=\mathcal{B}$, then $\mathcal{A}=\mathcal{L}(\Sigma)$.*

Proof. Let $\Sigma = \{s \in \Lambda : v_s \in \mathcal{B}\}$. Since $\mathcal{B}$ is a unital $*$-subalgebra, Σ is a subgroup, and clearly $\mathcal{L}(\Sigma) \subseteq \mathcal{B}$. We argue that $\mathcal{A} \subseteq \mathcal{L}(\Sigma)$.

Fix $a \in \mathcal{A}$, and let $a = \sum_\lambda a_\lambda v_\lambda$ be its Fourier decomposition. Let $I = \{s \in \Lambda : a_s \neq 0\}$. Fix $s \in I$, and consider the normal linear functional ω on $\mathcal{M}$ given by $\omega(x) = \bar{a}_s \tau(x v_s^*)$. Note that $(\omega \otimes 1)(a) = |a_s|^2 \otimes v_s$. Since $\Delta(\mathcal{A}) \subseteq \mathcal{M} \mathbin{\bar{\otimes}} \mathcal{B}$, we have $(\omega \otimes 1)\Delta(\mathcal{A}) \subseteq \mathbb{C} \mathbin{\bar{\otimes}} \mathcal{B}$. Thus, $v_s \in \mathcal{B}$ implies $s \in \Sigma$. Since this holds for all $s \in I$, we get $a \in \mathcal{L}(\Sigma)$, and hence we are done. □

We end this section with the following elementary result. We are grateful to the referee for suggesting a (much) shorter proof than the one we originally had, which used [Chifan and Das 2018, Proposition 2.3].

Lemma 2.9. *Let $\mathcal{M}$ be a tracial von Neumann algebra and let $\mathcal{N}$ be a II_1-factor, with $\mathcal{N} \subseteq \mathcal{M}$ a unital inclusion. If there is $p \in \mathscr{P}(\mathcal{N})$ so that $p\mathcal{N}p = p\mathcal{M}p$ then $\mathcal{N} = \mathcal{M}$.*

Proof. Shrinking p if necessary we can assume $\tau(p) = 1/n$. Let $v_1, \dots, v_n \in \mathcal{N}$ be partial isometries such that $v_i v_i^* = p$ for all i, and $\sum_{i=1}^n v_i^* v_i = 1$. Fix $x \in \mathcal{M}$. Since for every $1 \leqslant i, j \leqslant n$ we have $v_i x v_j^* \in p\mathcal{M}p = p\mathcal{N}p$, we get $x = \sum_{i,j=1}^n v_i^*(v_i x v_j^*) v_j \in \mathcal{N}$, as desired. □

2C. *Small cancellation techniques.* In this section, we recollect some geometric group theoretic preliminaries that will be used throughout this paper. We refer the reader to [Olshanskii 1991; 1993; Olshanskii et al. 2009] for more details related to the small cancellation techniques. We also refer the reader to [Lyndon and Schupp 1977] for details concerning van Kampen diagrams.

2C1. *van Kampen diagrams.* Given a word W over the alphabet set $\mathcal{S}$, we denote its length by $\|W\|$. We also write $W \equiv V$ to express the letter-for-letter equality for words W, V.

Let G be a group generated by a set of alphabets $\mathcal{S}$. A van Kampen diagram $\triangle$ over a presentation

$$G = \langle S \mid \mathcal{R} \rangle \tag{2C.1}$$

is a finite, oriented, connected, planar 2-complex endowed with a labeling function $\mathrm{Lab} : E(\triangle) \to \mathcal{S}^{\pm 1}$, where $E(\triangle)$ denotes the set of oriented edges of $\triangle$, such that $\mathrm{Lab}(e^{-1}) \equiv (\mathrm{Lab}(e))^{-1}$. Given a cell Π of $\triangle$, $\partial\Pi$ denotes its boundary. Similarly $\partial\triangle$ denotes the boundary of $\triangle$. The labels of $\partial\triangle$ and $\partial\Pi$ are defined up to cyclic permutations. We also stipulate that the label for any cell Π of $\triangle$ is equal to (up to a cyclic permutation) $R^{\pm 1}$, where $R \in \mathcal{R}$.

Using the van Kampen lemma [Lyndon and Schupp 1977, Chapter 5, Theorem 1.1], a word W over the alphabet set $\mathcal{S}$ represents the identity element in the group given by the presentation (2C.1) if and only if there exists a connected, simply connected planar diagram $\triangle$ over (2C.1) satisfying $\mathrm{Lab}(\partial\triangle) \equiv W$.

2C2. *Small cancellation over hyperbolic groups.* Let $G = \langle X \rangle$ be a finitely generated group and X be a finite generating set for G. Recall that the Cayley graph $\Gamma(G, X)$ of a group G with respect to the set of generators X is an oriented labeled 1-complex with vertex set $V(\Gamma(G, X)) = G$ and edge set $E(\Gamma(G, X)) = G \times X^{\pm 1}$. An edge $e = (g, a)$ goes from the vertex g to the vertex ga and has label a. Given a combinatorial path p in the Cayley graph $\Gamma(G, X)$, the length $|p|$ is the number of edges in p. The word length $|g|$ of an element $g \in G$ with respect to the generating set X is defined to be the length of a shortest word in X representing g in the group G, i.e., $|g| := \min_{h =_G g} \|h\|$. The formula

$d(f, g) = |g^{-1} f|$ defines a metric on the group G. The metric on the Cayley graph $\Gamma(G, X)$ is the natural extension of this metric. A word W is called a (λ, c)-quasi geodesic in $\Gamma(G, X)$ for some $\lambda > 0$, $c \geqslant 0$ if $\lambda\|W\| - c \leqslant |W| \leqslant \lambda\|W\| + c$. A word W is called a geodesic if it is a $(1, 0)$-quasigeodesic. A word W in the alphabet $X^{\pm 1}$ is called (λ, c)-quasigeodesic (respectively geodesic) in G if any path in the Cayley graph $\Gamma(G, X)$ labeled by W is (λ, c)-quasigeodesic (respectively geodesic). Throughout this section, $\mathcal{R}$ denotes a symmetric set of words (i.e., it is closed under taking cyclic shifts and inverses of words, and all the words are cyclically reduced) from X^*, the set of words on the alphabet X. A common initial subword of any two distinct words in $\mathcal{R}$ is called a piece. We say that $\mathcal{R}$ satisfies the $C'(\mu)$ condition if any piece contained (as a subword) in a word $R \in \mathcal{R}$ has length smaller than $\mu\|R\|$.

Definition 2.10 [Olshanskii 1993, Section 4]. A subword U of a word $R \in \mathcal{R}$ is called an ϵ-piece of the word R, for $\epsilon \geqslant 0$, if there exists a word $R' \in \mathcal{R}$ satisfying the following conditions:

(1) $R \equiv UV$ and $R' \equiv U'V'$ for some $U', V' \in \mathcal{R}$.

(2) $U' =_G YUZ$ for some $Y, Z \in X^*$, where $\|Y\|, \|Z\| \leqslant \epsilon$.

(3) $YRY^{-1} \neq_G R'$.

We say the system $\mathcal{R}$ satisfies the $C(\lambda, c, \epsilon, \mu, \rho)$-condition for some $\lambda \geqslant 1$, $c \geqslant 0$, $\epsilon \geqslant 0$, $\mu > 0$, $\rho > 0$ if:

(a) $\|R\| \geqslant \rho$ for any $R \in \mathcal{R}$.

(b) Any word $R \in \mathcal{R}$ is a (λ, c)-quasigeodesic.

(c) For any ϵ-piece U of any word $R \in \mathcal{R}$, the inequalities $\|U\|, \|U'\| < \mu\|R\|$ hold.

In practice, we will need some slight modifications of the above definition [Olshanskii 1993, Section 4].

Definition 2.11. A subword U of a word $R \in \mathcal{R}$ is called an ϵ'-piece of the word R, for $\epsilon \geqslant 0$, if:

(1) $R \equiv UVU'V'$ for some $V, U', V' \in X^*$.

(2) $U' =_G YU^{\pm}Z$ for some words $Y, Z \in X^*$, where $\|Y\|, \|Z\| \leqslant \epsilon$.

We say the system $\mathcal{R}$ satisfies the $C'(\lambda, c, \epsilon, \mu, \rho)$-condition for some $\lambda \geqslant 1$, $c \geqslant 0$, $\epsilon \geqslant 0$, $\mu > 0$, $\rho > 0$ if:

(d) $\mathcal{R}$ satisfies the $C(\lambda, c, \epsilon, \mu, \rho)$ condition.

(e) Every ϵ'-piece U of R satisfies $\|U'\| < \mu\|R\|$, where U' is as above.

Let G be a group defined by

$$G = \langle X \mid \mathcal{O} \rangle, \tag{2C.2}$$

where $\mathcal{O}$ is the set of *all* relators (not just the defining relations) of G. Given a symmetrized set of words $\mathcal{R}$ in the alphabet set X, we consider the quotient group

$$H = \langle G \mid \mathcal{R} \rangle = \langle G \mid \mathcal{O} \cup \mathcal{R} \rangle. \tag{2C.3}$$

A cell over a van Kampen diagram over (2C.3) is called an $\mathcal{R}$-cell (respectively, an $\mathcal{O}$-cell) if its boundary label is a word from $\mathcal{R}$ (respectively, $\mathcal{O}$). We always consider a van Kampen diagram over (2C.3) up to some elementary transformations. For example we do not distinguish diagrams if one can be obtained from the other by joining two distinct $\mathcal{O}$-cells having a common edge or by inverse transformations [Olshanskii 1993, Section 5].

3. Some examples of Olshanskii's monster groups in the context of lacunary hyperbolic groups

In this section, we collect some group theoretic results needed for our main theorems in Sections 4 and 5. Readers who are mainly interested in the results in Section 5 may skip ahead to Section 3C. The results in Subsections 3A and 3B shall be required for our main results in Section 4.

In order to derive our main result on the study of maximal von Neumann algebras (i.e., Theorem 4.4) we need to construct a new monster-like group in the same spirit as the famous examples from [Olshanskii 1980]. Specifically, generalizing the geometric methods from [Olshanskii 1993] to the context of lacunary hyperbolic groups [Olshanskii et al. 2009] and using techniques developed in [Khan 2020], we construct a group G such that every maximal subgroup of G is isomorphic to a subgroup of $\mathbb{Q}$, the group of rational numbers. While in our approach we explain in detail how these results are used, the main emphasis will be on the new aspects of these techniques. Therefore we recommend that the interested reader consult beforehand the aforementioned results [Olshanskii 1993; Khan 2020].

3A. ***Elementary subgroups.*** In this section, using methods developed in [Olshanskii 1993], we construct a group Q whose maximal (proper) subgroups are rank-1 abelian groups; see Theorem 3.12. More specifically, we study "special limits" of hyperbolic groups, called lacunary hyperbolic groups, as introduced in [Olshanskii et al. 2009].

Definition 3.1. Let $\alpha : G \to H$ be a group homomorphism and $G = \langle A\rangle,\ H = \langle B\rangle$. The injectivity radius $r_A(\alpha)$ is the radius of largest ball centered at the identity of G in the Cayley graph of G with respect to A on which the restriction of α is injective.

Definition 3.2 [Olshanskii et al. 2009, Theorem 1.2]. A finitely generated group G is called lacunary hyperbolic group if G is the direct limit of a sequence of hyperbolic groups and epimorphisms

$$G_1 \xrightarrow{\eta_1} G_2 \xrightarrow{\eta_2} \cdots \xrightarrow{\eta_{i-1}} G_i \xrightarrow{\eta_i} G_{i+1} \xrightarrow{\eta_{i+1}} G_{i+2} \xrightarrow{\eta_{i+2}} \cdots, \tag{3A.1}$$

where G_i is generated by a finite set S_i and $\eta_i(S_i) = S_{i+1}$. Also the G_i's are δ_i-hyperbolic, where $\delta_i = o(r_{S_i}(\eta_i))$, where $r_{S_i}(\eta_i)$ is the injective radius of η_i with respect to S_i.

Fix ω a nonprincipal ultrafilter. An asymptotic cone $\mathrm{Cone}^\omega(X, e, d)$ of a metric space (X, dist), where $e = \{e_i\}_i$, $e_i \in X$ for all i and $d = \{d_i\}_i$ is an unbounded sequence of nondecreasing positive real numbers, is the ω-limit of the spaces $(X, \mathrm{dist}/d_i)$. The sequence $d = \{d_i\}$ is called a scaling sequence. Following [Olshanskii et al. 2009, Theorem 3.3], G being a lacunary hyperbolic group is equivalent to the existence of a scaling sequence $d = \{d_i\}$ such that the asymptotic cone $\mathrm{Cone}^\omega(\Gamma(G, X), e, d)$ associated with the Cayley graph $\Gamma(G, X)$ for a finite generating set X of G with $e = \{\text{identity}\}$ is an $\mathbb{R}$-tree for any nonprincipal ultrafilter ω. For more details on asymptotic cones and their connection with lacunary hyperbolic groups we refer the reader to [Olshanskii et al. 2009, Section 2.3, Section 3.1].

Our construction relies heavily on the notion of *elementary* subgroups. For the readers' convenience, we collect below some preliminaries regarding elementary subgroups.

Definition 3.3. A group E is called elementary if it is virtually cyclic. Let G be a hyperbolic group and $g \in G$ be an infinite-order element. Then the elementary subgroup containing g is defined as

$$E(g) := \{x \in G : x^{-1} g^n x = g^{\pm n} \text{ for some } n = n(x) \in \mathbb{N}\}.$$

For further use we need the following result describing in depth the structure of elementary subgroups.

Lemma 3.4. (1) [Olshanskii 1991] *If E is a torsion-free elementary group then E is cyclic.*

(2) [Olshanskii 1993, Lemma 1.16] *Let E be an infinite elementary group. Then E contains normal subgroups $T \lhd E^+ \lhd E$ such that $[E : E^+] \leqslant 2$, T is finite and $E^+/T \simeq \mathbb{Z}$. If $E \neq E^+$ then $E/T \simeq D_\infty$ (the infinite dihedral group). For a hyperbolic group G, $E(g)$ is unique maximal elementary subgroup of G containing the infinite-order element $g \in G$.*

In the context of lacunary hyperbolic groups we need to introduce the following definition which generalizes Definition 3.3.

Definition 3.5. Let G be a lacunary hyperbolic group and let $g \in G$ be an infinite-order element. We define $E^{\mathcal{L}}(g) := \{x \in G : x g^n x^{-1} = g^{\pm n} \text{ for some } n = n(x) \in \mathbb{N}\}$.

For future reference we now recall the following structural result regarding torsion elements in a δ-hyperbolic group.

Theorem 3.6 [Gromov 1987, 2.2.B]. *Let $g \in G$ be a torsion element in a δ-hyperbolic group G. Then g is conjugate to an element h in G such that $|h|_G \leqslant 4\delta + 1$.*

The following elementary lemma will be used in the proof of Theorem 3.8. For convenience we include a short proof.

Lemma 3.7. *If G is a torsion-free lacunary hyperbolic group, then one can choose G_i to be torsion-free such that $G = \varinjlim G_i$.*

Proof. Fix a presentation $G = \langle S \mid \mathcal{R} \rangle$. By [Olshanskii et al. 2009, Theorem 3.3], one can choose $G_i := \langle S \mid \mathcal{R}_{c(i)} \rangle$, where $\{c(n)\}_n$ is a strictly increasing sequence such that $\mathcal{R}_{c(i)}$ consists of labels of all cycles in the ball of radius d_i (corresponding to the scaling sequence $\{d_i\}_i$ of the lacunary hyperbolic group) around the identity in $\Gamma(G, S)$. Let r_i be the injectivity radius of the quotient map $\phi_i : G_i \to G_{i+1}$. The lacunary hyperbolic condition implies that $\lim_{i\to\infty} \delta_i / r_i = 0$, where δ_i is the hyperbolic constant for the group G_i for all i. Choose i_0 such that for all $j \geqslant i_0$ we have $r_j > 9\delta_j$. We will show the G_j's are torsion-free for all $j \geqslant i_0$, which proves the lemma.

Fix any $j \geqslant i_0$. Assume by contradiction that $g \in G_j \setminus \{1\}$ is a torsion element. By Theorem 3.6 there is an element $h \in G_j \setminus \{1\}$ such that h is conjugate to g and $|h|_{G_j} \leqslant 4\delta_j + 1$. Thus h is a torsion element of G_j. Since $|h|_{G_j} \leqslant 4\delta_j + 1 < r_i$, h is a nontrivial element of G_k for all $k \geqslant j$. Thus h is a nontrivial torsion element in the limit group G, which is a contradiction! $\square$

The next result generalizes Lemma 3.4, and provides a complete description of the structure of elementary subgroups of a torsion-free lacunary hyperbolic group. This result can be deduced from the main theorem of [Khan 2020]. For the readers' convenience, we include a short proof.

Theorem 3.8. *Let G be a torsion-free lacunary hyperbolic group and let $g \in G$ be an infinite-order element. Then $E^{\mathcal{L}}(g)$ is an abelian group of rank* 1 *(i.e., $E^{\mathcal{L}}(g)$ embeds in* $(\mathbb{Q}, +)$*).*

Proof. From the definition (3A.1) of lacunary hyperbolic group, $E^{\mathcal{L}}(g) = \varinjlim E_i(g)$ for every $e \neq g \in G$, where $E_i(g)$ is the elementary subgroup containing the element g in the hyperbolic group G_i when viewing $g \in G_i$. Since G is torsion-free, one can choose G_i to be torsion-free by Lemma 3.7. By Lemma 3.4 (1) we get that $E_i(g)$ is cyclic for all i. Observe that every surjective homomorphism between hyperbolic groups takes elementary subgroups into elementary subgroups; in particular $E_i(g)$ maps into $E_{i+1}(g)$. We now get the group $E^{\mathcal{L}}(g)$ is equal to $\varinjlim E_i(g)$ as an inductive limit of cyclic groups, which proves the theorem. □

Remark. Let G be a torsion-free lacunary hyperbolic group and let $e \neq g \in G$. Note that $C_G(g) \leqslant E^{\mathcal{L}}(g)$, where $C_G(g)$ is the centralizer of g in G.

3B. ***Maximal subgroups.*** Let $G_0 = \langle X \rangle$ be a torsion-free δ-hyperbolic group with respect to X, where $X = \{x_1, x_2, \dots, x_n\}$ is a finite generating set. Without loss of generality we assume that $E(x_i) \cap E(x_j) = \{e\}$ for $i \neq j$. We define a linear order on X by $x_i^{-1} < x_j^{-1} < x_i < x_j$, whenever $i < j$. Let $F'(X)$ denote the set of all nonempty reduced words on X. Note that the order on X induces the lexicographic order on $F'(X)$. Let $F'(X) = \{w_1, w_2, \dots\}$ be an enumeration with $w_i < w_j$ for $i < j$. Observe that $w_1 = x_1$ and $w_2 = x_2$. We now consider the set $\mathcal{S} := F'(X) \times F'(X) \setminus \{(w, w) : w \in F'(X)\}$ and enumerate the elements of $\mathcal{S}$ as $\mathcal{S} = \{(u_1, v_1), (u_2, v_2), \dots\}$.

Our next goal is to construct the chain

$$G_0 \xhookrightarrow{\beta_0} K_1 \xrightarrow{\alpha_1} G_1' \xrightarrow{\gamma_1} G_1 \xhookrightarrow{\beta_1} K_2 \xrightarrow{\alpha_2} G_2' \xrightarrow{\gamma_1} G_2 \cdots, \tag{3B.1}$$

where K_i, G_i, G_i' are hyperbolic for all i and $\eta_i := \gamma_i \circ \alpha_i \circ \beta_{i-1}$, $i \geqslant 1$, satisfies the conditions in (3A.1).

Let L be a rank-1 abelian group. Then L can be written as $L = \bigcup_{i=0}^{\infty} L_i$, where $L_i = \langle g_i \rangle_\infty$ and $g_i = g_{i+1}^{m_{i+1}}$ for some $m_{i+1} \in \mathbb{N}$. Here $\langle g_i \rangle_\infty$ denotes the infinite cyclic group generated by the infinite-order element g_i.

Since G_0 is nonelementary, there exists a smallest index $j_i \geqslant i$ such that $v_{j_i} \notin E(u_{j_i})$. For $m \in \mathbb{N}$, define

$$H_{i+1}^k := H_{i+1}^{k-1} \underset{u_k = g_{(k,i+1)}^{m_{i+1}}}{*} \langle g_{(k,i+1)} \rangle_\infty, \quad \text{where } H_{i+1}^0 = G_i \text{ and } g_{(k,i+1)} = g_{i+1} \text{ for } k = 1, 2, \dots, j_i. \tag{3B.2}$$

For $i \geqslant 0$ let K_{i+1} be $H_{i+1}^{j_i}$. Note that K_{i+1} is hyperbolic as H_{i+1}^k is hyperbolic for all k by [Mikhajlovskii and Olshanskii 1998, Theorem 3]. Choose $c_i, c_i' \in G_i$ such that $c_i, c_i' \notin E(u_k)$ for all $1 \leqslant k \leqslant j_i$ and $c_i, c_i' \notin E(v_{j_i})$. One can find such c_i and c_i' since there are infinitely many elements in a nonelementary hyperbolic group which are pairwise noncommensurable [Olshanskii 1993, Lemma 3.8]. Let $Y_i := \{g_{(k,i+1)} : 1 \leqslant k \leqslant j_i\}$. Define

$$R_k := g_{(k,i+1)} c_i^{n_{1,k}} c_i' c_i^{n_{2,k}} c_i' \cdots c_i^{n_{s_k,k}} c_i', \tag{3B.3}$$

where $n_{s,k}$, for $1 \leqslant k \leqslant j_i$ are defined as

$$n_{1,k} = 2^{k-1} n_{1,1}, \quad s_k = n_{1,k-1} \quad \text{and} \quad n_{s,k} = n_{1,k} + (s-1).$$

We also denote by $\mathcal{R}_i$ the set of all cyclic shifts of $\{R_k^{\pm 1} : 1 \leqslant k \leqslant j_i\}$.

Lemma 3.9 [Darbinyan 2017, Lemma 5.1]. *There exists a constant K such that the set of words $\mathcal{R}$ defined above by* (3B.3) *are (λ, c)-quasigeodesic in $\Gamma(G, X)$, provided $n_{1,1} \geqslant K$, $c \notin E(g_{(k,i+1)})$, and $c' \notin E(g_{(k,i+1)})$.*

We now define $\widetilde{\mathcal{R}}_{i+1}$ to be the set of words $\mathcal{R}_i$, defined as above, with $n_{1,k} \geqslant K$.

Lemma 3.10 [Darbinyan 2017, Lemma 5.2]. *For any given constant $\epsilon_i \geqslant 0$, $\mu_i > 0$, $\rho_i > 0$, the system of words $\widetilde{\mathcal{R}}_{i+1}$ (defined above) satisfies the $C'(\lambda_i, c_i, \epsilon_i, \mu_i, \rho_i)$ condition over K_{i+1}.*

By construction there is a natural embedding $\beta_i : G_i \hookrightarrow K_{i+1}$. Let $G'_{i+1} := \langle K_{i+1} \mid \widetilde{\mathcal{R}}_{i+1} \rangle$ (where we are using the notation in Section 2C1). The factor group G'_{i+1} is hyperbolic by [Olshanskii 1993, Lemma 7.2]. Now consider the natural quotient map $\alpha_{i+1} : K_{i+1} \twoheadrightarrow G'_{i+1}$. Since $\alpha_{i+1} \circ \beta_i$ takes generators of G_i to generators of G'_{i+1}, the map $\alpha_{i+1} \circ \beta_i$ is surjective.

Consider the set

$$Z_i := \{x \in X : x \notin E(u_{j_i})\}.$$

Let $G_{i+1} := G'_{i+1}/\langle\langle \mathcal{R}(Z_i, u_{j_i}, v_{j_i}, \lambda_i, c_i, \epsilon_i, \mu_i, \rho_i) \rangle\rangle$ and let $\gamma_{i+1} : G'_{i+1} \twoheadrightarrow G_{i+1}$ be the quotient map. Here $\mathcal{R}(Z_i, u_{j_i}, v_{j_i}, \lambda_i, c_i, \epsilon_i, \mu_i, \rho_i)$ is the set of all conjugates and the cyclic shifts of some relations, where we identify the elements of Z_i with words of the form (3B.3) generated by u_{j_i} and v_{j_i}. Since the relators $\mathcal{R}_i$ are generic, we have added all the parameters to indicate these relations satisfy the small cancellation conditions with the parameters and their dependency to the specific set of words. One can choose the powers of u_{j_i} and v_{j_i} such that the small cancellation condition is satisfied by Lemmas 3.9 and 3.10. For more details on how to choose these words, we refer the reader to [Olshanskii 1993, Section 5; Darbinyan 2017, Section 5.4]. Thus it follows that the group G_{i+1} is hyperbolic by [Olshanskii 1993, Lemma 7.2] as one can choose parameters $\lambda_i, c_i, \epsilon_i, \mu_i, \rho_i$ such that $\mathcal{R}(Z_i, u_{j_i}, v_{j_i}, \lambda_i, c_i, \epsilon_i, \mu_i, \rho_i)$ satisfies the $C'(\lambda_i, c_i, \epsilon_i, \mu_i, \rho_i)$ small cancellation condition in Definition 2.11 and the map γ_{i+1} takes generating set to generating set. In particular, $\eta_{i+1} := \gamma_{i+1} \circ \alpha_{i+1} \circ \beta_i$ is a surjective homomorphism which takes the generating set of G_i to the generating set of G_{i+1}. Let $G^L := \varinjlim G_i$. From its definition, it follows that G_{i+1} is the group generated by u_{j_i} and v_{j_i}.

We summarize the above discussion in the following statement.

Lemma 3.11. *The above construction satisfies the following properties*:

(1) *G_i is nonelementary hyperbolic group for all i.*

(2) *Either $u_i \in E(v_i)$ or the group generated by $\{u_i, v_i\}$ in G_{i+1} is equal to all of G_{i+1}.*

(3) *For each element $x \in X$, we have $E(x) = \langle y \rangle$ in G_i, where $x = y^{m_1 m_2 \cdots m_i}$. The exponents m_i are described as follows*: *a rank-1 abelian group L can be written as $L = \bigcup_{i=0}^{\infty} L_i$, where $L_i = \langle g_i \rangle_\infty$ and $g_i = g_{i+1}^{m_{i+1}}$ for some $m_{i+1} \in \mathbb{N}$.*

(4) *$G^L := \varinjlim G_i$ may be chosen to have property (T).*

Proof. Part (1) follows from [Olshanskii 1993, Lemma 7.2]. To see part (2) notice that by definition if $j_i > i$ then $v_i \in E(u_i)$ in G_i. Otherwise if $j_i = i$ then $v_i \notin E(u_i)$ in G_i and G_i is the group generated by $\{u_i, v_i\}$. Part (3) follows immediately from the fact that x is not a proper power in G_0. Finally, for part (4) notice that we may start the above construction with G_0 being a property (T) group. Then G_1 has property (T), as G_0 surjects onto G_1. By induction, each of the groups G_i in the above construction have property (T). Hence G^L has property (T). □

We are now ready to prove the main theorem of this section.

Theorem 3.12. *For any subgroup Q_m of $(\mathbb{Q}, +)$ there exists a nonelementary torsion-free lacunary hyperbolic group G such that all maximal subgroups of G are isomorphic to Q_m. Moreover, we may choose G to have property (T).*

Proof. In the above construction let $L = Q_m$, $G = G^{Q_m}$ and take $d = m_1 m_2 \cdots m_i$ in (3B.2), where $L_i = \langle g_i \rangle_\infty$ and $g_i = g_{i+1}^{m_{i+1}}$ for some $m_{i+1} \in \mathbb{N}$ and $Q_m = \bigcup_{i=1}^{\infty} L_i$. One can choose sparse enough parameters to satisfy the injectivity radius condition in (3A.1), which in turn will ensure that G is lacunary hyperbolic. The above construction also guarantees that $E^{\mathcal{L}}(g) = Q_m$ for all $g \in G \setminus \{1\}$. Suppose $P \nleqslant G$ is a maximal subgroup of G. As P is a proper subgroup, P is abelian by Lemma 3.11 (2). Now let $e \neq h \in G$. Note that as P is abelian, P is contained in the centralizer of h. Now from Definition 3.5 it follows that $g \in P \leqslant E^{\mathcal{L}}(g) (\cong Q_m) \nleqslant G$. By the maximality of P we get $P \cong Q_m$. Thus, all maximal subgroups of G are isomorphic to Q_m and hence any proper subgroup of G is isomorphic to a subgroup of Q_m.

The "moreover" part follows from part (4) of Lemma 3.11. □

We end this section with the following well-known counterexamples to von Neumann's conjecture.

Corollary 3.13 [Olshanskii 1980; 1993]. *For every noncyclic torsion-free hyperbolic group Γ there exists a nonabelian torsion-free quotient $\overline{\Gamma}$ such that all proper subgroups of $\overline{\Gamma}$ are infinite cyclic.*

Proof. Take $Q_m = \mathbb{Z}$ in Theorem 3.12. □

3C. *Belegradek–Osin Rips construction in group theory.* Rips constructions emerged in geometric group theory with [Rips 1982] and represent a rich source of examples for various pathological properties in group theory. This type of construction was used effectively to study automorphisms of property (T) groups. In this direction Ollivier and Wise [2007] were able to construct property (T) groups whose automorphism group contain any given countable group. This answered an important older question of P. de la Harpe and A. Valette about finiteness of outer automorphism groups of property (T) groups. Using the small cancellation methods developed in [Osin 2010; Arzhantseva et al. 2007], Belegradek and Osin discovered the following version of the Rips construction in the context of relatively hyperbolic groups:

Theorem 3.14 [Belegradek and Osin 2008]. *Let H be a nonelementary hyperbolic group, Q be a finitely generated group and S a subgroup of Q. Suppose Q is finitely presented with respect to S. Then there exists a short exact sequence*

$$1 \to N \to G \xrightarrow{\epsilon} Q \to 1$$

and an embedding $\iota : Q \to G$ such that:

(1) *N is isomorphic to a quotient of H.*

(2) *G is hyperbolic relative to the proper subgroup $\iota(S)$.*

(3) $\iota \circ \epsilon = \mathrm{Id}$.

(4) *If H and Q are torsion-free then so is G.*

(5) *The canonical map $\phi : Q \hookrightarrow \mathrm{Out}(N)$ is injective and* $[\mathrm{Out}(N) : \phi(Q)] < \infty$.

This construction is extremely important for our work. We are particularly interested in the case when H is torsion-free and has property (T) and $Q = S$ and is torsion-free. In this situation Theorem 3.14 implies that G is admits a semidirect product decomposition $G = N \rtimes Q$ and it is hyperbolic relative to $\{Q\}$. Notice that the finite conjugacy radical $\mathrm{FC}(N)$ of N is invariant under the action of Q and hence $\mathrm{FC}(N)$ is an amenable normal subgroup G. Since G is relative hyperbolic, it follows that $\mathrm{FC}(N)$ is finite and hence it is trivial as G is torsion-free; in particular N is an icc group. Since G is hyperbolic relative to Q it follows that the stabilizer of any $n \in N$ in Q under the action $Q \curvearrowright^{\sigma} N$ is trivial.

We now introduce the following classes of groups that shall play an extremely important role throughout the rest of the paper.

Definition 3.15. We denote by $\mathcal{R}ip(Q)$ the class of all semidirect products $G = N \rtimes Q$ satisfying the properties of Theorem 3.14, where $Q = S$, Q and H are torsion-free and H has property (T).

Moreover, when Q has property (T), we denote the class $\mathcal{R}ip(Q)$ by $\mathcal{R}ip_T(Q)$.

Since property (T) is closed under extensions, it follows that all groups in $\mathcal{R}ip_T(Q)$ have property (T). Our rigidity results in Section 5 concern this class of groups.

In the second part of this section we recall a powerful method from geometric group theory, termed Dehn filling. We are interested specifically in the group theoretic Dehn filling constructions developed by D. Osin and his collaborators in [Osin 2010; Dahmani et al. 2017]. The next result, which is due to Osin, is a technical variation of [Osin 2010, Theorem 1.1] and [Dahmani et al. 2017, Theorem 7.9] and plays a key role in deriving some of our main rigidity theorems in Section 5 (see Theorems 5.2 and 5.3). For its proof the reader may consult [Chifan et al. 2015, Corollary 5.1].

Theorem 3.16 (Osin). *Let $H \leqslant G$ be infinite groups where H is finitely generated and residually finite. Suppose that G is hyperbolic relative to $\{H\}$. Then there exist a nonelementary hyperbolic group K and an epimorphism $\delta : G \to K$ such that $R = \ker(\delta)$ is isomorphic to a nontrivial (possibly infinite) free product $R = *_{g \in T} R_0^g$, where $T \subset G$ is a subset and $R_0^g = g R_0 g^{-1}$ for a finite-index normal subgroup $R_0 \lhd H$.*

We end this section with an application of Theorem 3.16. The result describes the structure of the normal subgroups N of $N \rtimes Q \in \mathcal{R}ip_T(Q)$. Namely, combining Theorems 3.16 and 3.14 we show that these groups are free-by-hyperbolic. This result will be essential to the proof of Theorem 5.1.

Proposition 3.17. *Let $G = N \rtimes Q \in \mathcal{R}ip_T(Q)$ and assume that Q is an infinite residually finite group. Then N is a $\mathbb{F}_{n+1}$-by-(nonelementary, hyperbolic property (T)) group, where $n \in \mathbb{N} \cup \{\infty\}$.*

Proof. Since G is hyperbolic relative to $\{Q\}$ and Q is residually finite, by Theorem 3.16 there is a nonelementary hyperbolic group K and an epimorphism $\delta : G \to K$ such that $L = \ker(\delta)$ is isomorphic to a nontrivial free product $L = *_{g \in T} Q_0^g$, where $T \subset G$ is a subset and $Q_0 \lhd Q$ is a finite-index, normal subgroup. Since $G = N \rtimes Q$ and Q_0 is normal in Q, one can assume without any loss of generality that

$T \subset N$. Next we show that $N \cap L$ infinite. If it were finite, as G is icc, it would follow that $N \cap L = 1$. As N and L are normal in G, the commutator satisfies $[N, L] \leqslant N \cap L = 1$ and hence $L \leqslant C_G(N)$. To describe this centralizer, fix $g = nq \in C_G(N)$, where $n \in N,\ q \in Q$. Thus for all $m \in N$ we have $nqm = mnq$ and hence $n\sigma_q(m) = mn$, where $\sigma_q(x) = q^{-1}xq$ for all $x \in N$. Therefore $\sigma_q = ad(n)$ and by Theorem 3.14 (5) we must have $q = 1$. This further implies that $m \in Z(N) = 1$ and hence $C_G(N) = 1$; in particular, $L = 1$, which is a contradiction. In conclusion $N \cap L \lhd N$ is an infinite normal subgroup. Using the isomorphism theorem we see that $N/(N \cap L) \cong (NL)/L$. Also from the free product description of L we see that $N \rtimes Q_0 \leqslant NL$ and hence $[G : NL] < \infty$. In particular $(NL)/L$ is a finite-index subgroup of $G/L = K$ and hence $(NL)/L$ is a (nonelementary) hyperbolic, property (T) group. To finish our proof we only need to argue that $N \cap L$ is a free group with at least two generators. Since $L = *_{g \in T} Q_0^g$, by the Kurosh theorem there exist a set $X \subset L$ and a collection of subgroups $Q_i \leqslant Q_0$, together with elements $g_i \in L$ such that $N \cap L = F(X) * (*_{i \in I} Q_i^{g_i})$; here $F(X)$ is a free group with free basis X. In particular, for every $i \in I$ the previous relation implies that $Q_i^{g_i} \leqslant N$ and writing $g_i = n_i q_i$ for some $n_i \in N$, $q_i \in Q$ we see that $Q_i^{q_i} \leqslant N$. As $Q_i^{q_i} \leqslant Q$ we conclude that $Q_i^{q_i} \leqslant N \cap Q = 1$ and hence $Q_i = 1$. Thus $N \cap L = F(X)$ and since G is icc and $N \cap L$ is normal in G, we see that $|X| \geqslant 2$, which finishes the proof. □

4. Maximal von Neumann subalgebras arising from groups Rips construction

If $\mathcal{M}$ is a von Neumann algebra then a von Neumann subalgebra $\mathcal{N} \subset \mathcal{M}$ is called *maximal* if there is no intermediate von Neumann subalgebra $\mathcal{P}$ so that $\mathcal{N} \subsetneq \mathcal{P} \subsetneq \mathcal{M}$. Understanding the structure of maximal subalgebras of a given von Neumann algebra is a rather difficult problem that plays a key role in the very classification of these objects. Despite a series of earlier remarkable successes on the study of maximal *amenable* subalgebras initiated in [Popa 1983] and continued more recently [Shen 2006; Cameron et al. 2010; Houdayer 2014; Boutonnet and Carderi 2015; 2017; Suzuki 2020; Chifan and Das 2020; Jiang and Skalski 2019a], much less is known for the maximal ones. For instance Ge's question [2003, Section 3, Question 2] on the existence of nonamenable factors that possess maximal factors which are amenable was settled in the affirmative only very recently in [Jiang and Skalski 2019a]. We also remark that the study of maximal (or by duality minimal) intermediate subfactors has recently led to the discovery of a rigidity phenomenon for the intermediate subfactor lattice in the case of irreducible finite-index subfactors [Bakshi et al. 2019].

In this section we make new progress in this direction by describing several concrete collections of maximal subalgebras in the von Neumann algebras arising from the groups $\mathcal{R}ip(Q)$ introduced in the previous subsection (see Theorem 4.4 below). In particular, these examples allow construction of property (T) von Neumann algebras which have maximal von Neumann subalgebras without property (T). This answers a question raised in [Jiang and Skalski 2019a, Problem 5.5]. Our arguments rely on the usage of Galois correspondence results for von Neumann algebras à la [Choda 1978] and the classification of maximal subgroups in the monster-type groups provided in Theorem 3.12. We remark that Jiang and Skalski [2019a, Theorem 4.8] independently obtained a different solution, using different techniques.

First we need a couple of basic lemmas concerning automorphisms of groups. For the reader's convenience we include short proofs.

Lemma 4.1. *Let N be a group, let $\mathrm{Id} \neq \alpha \in \mathrm{Aut}(N)$ and denote by $N_1 = \{n \in N : \alpha(n) = n\}$ its fixed point subgroup. Then the following hold*:

(1) *Either $[N : N_1] = \infty$ or there is a subgroup $N_0 \leqslant N_1 \leqslant N$ that is normal in N with $[N : N_0] < \infty$ and such that the induced automorphism $\tilde{\alpha} \in \mathrm{Aut}(N/C_N(N_0))$ given by $\tilde{\alpha}(nC_N(N_0)) = \alpha(n)C_N(N_0)$ is the identity map*; *in particular*, *when N is icc we always have $[N : N_1] = \infty$.*

(2) *Either $[N : N_1] = \infty$, or α has finite order in $\mathrm{Aut}(N)$, or there is a $k \in \mathbb{N}$ and a subgroup $N_0 \leqslant N_1 \leqslant N$ that is normal in N with $[N : N_0] < \infty$ and such that the induced automorphism $\tilde{\alpha} \in \mathrm{Aut}(N/Z(N_0))$ given by $\tilde{\alpha}(nZ(N_0)) = \alpha(n)Z(N_0)$ has order k*; *in particular*, *when all finite-index subgroups of N have trivial center we either have $[N : N_1] = \infty$ or $\tilde{\alpha}$ has finite order.*

Proof. (1) Assume that $2 \leqslant [N : N_1] < \infty$. Then $N_0 := \bigcap_{h \in N} hN_1h^{-1} \leqslant N_1$ is a finite-index normal subgroup of N. Notice that the centralizer $C_N(N_0)$ is also normal in N. Let $n \in N$ and $n_0 \in N_0$. As N_0 is normal, we have $nn_0n^{-1} \in N_0 \leqslant N_1$ and hence $nn_0n^{-1} = \alpha(nn_0n^{-1}) = \alpha(n)n_0\alpha(n^{-1})$. This implies $n_0^{-1}n^{-1}\alpha(n)n_0 = n^{-1}\alpha(n)$ and hence $n^{-1}\alpha(n) \in C_N(N_0)$. Since α acts identically on N_0, one can see that $\alpha(C_N(N_0)) = C_N(N_0)$. Thus one can define an automorphism $\tilde{\alpha} : N/C_N(N_0) \to N/C_N(N_0)$ by letting $\tilde{\alpha}(nC_N(N_0)) = \alpha(n)C_N(N_0)$. However, the previous relations show that $\tilde{\alpha}$ is the identity map, as desired. For the remaining part of the statement, we notice that if $[N : N_1] < \infty$ and N is icc then the centralizer $C_N(N_0)$ is trivial and hence $\alpha = \mathrm{Id}$, which is a contradiction.

(2) Assume $[N : N_1] < \infty$ and α has infinite order in $\mathrm{Aut}(N)$. Also for each $i \geqslant 2$ define $N_i = \{n \in N : \alpha^i(n) = n\}$ and notice that $N_1 \leqslant N_i \leqslant N_{i+1} \leqslant N$. Since $[N : N_1] < \infty$, there is $s \in \mathbb{N}$ so that either $N_s = N_l$ for all $l \geqslant s$, or $N_s = N$. If $N_s = N$ then $\alpha^s = \mathrm{Id}$, contradicting the infinite-order assumption on α. Now assume that $N_s = N_{s+1}$. For every $n \in N_{s+1}$ we have $\alpha^s(n) = \alpha^{s+1}(n)$ and thus $\alpha(n) = n$, which is equivalent to $n \in N_1$. This shows that $N_1 = N_{s+1}$ and combining with the above we conclude that $N_1 = N_i$ for all i.

As $[N : N_1] < \infty$, we have $N_0 := \bigcap_{h \in N} hN_1h^{-1} \leqslant N_1$ is a finite-index normal subgroup of N. The automorphism α induces an automorphism $\tilde{\alpha}$ on the quotient group N/N_0 by $\tilde{\alpha}(nN_0) = \alpha(n)N_0$ for all $n \in N$. Since $[N : N_0] < \infty$, there is $k \in \mathbb{N}$ such that $\tilde{\alpha}^k = \mathrm{Id}$ on N/N_0. Thus for every $n \in N$ we have $n^{-1}\alpha^k(n) \in N_0$.

Let $n \in N$ and $n_0 \in N_0$. By normality we have $nn_0n^{-1} \in N_0 \leqslant N_1$ and hence $nn_0n^{-1} = \alpha^k(nn_0n^{-1}) = \alpha^k(n)n_0\alpha^k(n^{-1})$. This implies $n_0^{-1}n^{-1}\alpha^k(n)n_0 = n^{-1}\alpha^k(n)$ and hence $n^{-1}\alpha^k(n) \in Z(N_0)$. Since N_0 is normal in N, so is $Z(N_0)$. Since α leaves $Z(N_0)$ invariant, the map $\tilde{\alpha} : N/Z(N_0) \to N/Z(N_0)$ given by $\tilde{\alpha}(nZ(N_0)) = \alpha(n)Z(N_0)$ is an automorphism. The previous relations show that it has order k. ☐

Using this we will see that, in the case of icc groups, outer group actions $Q \curvearrowright N$ by automorphisms lift to outer actions $Q \curvearrowright \mathcal{L}(N)$ at the von Neumann algebra level. More precisely we have the following:

Lemma 4.2. *Let N be an icc group and let Q be a group together with an outer action $Q \curvearrowright^{\sigma} N$. Then $\mathcal{L}(N)' \cap \mathcal{L}(N \rtimes_\sigma Q) = \mathbb{C}$.*

Proof. To get $\mathcal{L}(N)' \cap \mathcal{L}(N \rtimes_\sigma Q) = \mathbb{C}$ it suffices to show that for all $g \in (N \rtimes_\sigma Q) \setminus \{e\}$ the N-conjugacy orbit $\mathcal{O}_N(g) = \{ngn^{-1} : n \in N\}$ is infinite. Suppose by contradiction there is $h = n_0q_0 \in (N \rtimes Q) \setminus \{e\}$ with

$n_0 \in N$ and $q_0 \in Q$ such that $|\mathcal{O}_N(h)| < \infty$. Hence there exists a finite-index subgroup $N_1 \leqslant N$ such that $nhn^{-1} = h$ for all $n \in N_1$. This gives that $nn_0q_0n^{-1} = n_0q_0$ and thus $n = n_0q_0nq_0^{-1}n_0^{-1} = \mathrm{ad}(n_0) \circ \sigma_{q_0}(n)$ for all $n \in N_1$. Also, since N is icc, we have $q_0 \neq e$. Let $\alpha = \mathrm{ad}(n_0) \circ \sigma_{q_0}$. Since $Q \curvearrowright N$ is outer it follows that $\mathrm{Id} \neq \alpha \in \mathrm{Aut}(N)$. Since N is icc and $[N : N_1] < \infty$, Lemma 4.1 (1) leads to a contradiction. $\square$

With these results at hand we are now ready to deduce the main result of the section.

Notation 4.3. Fix any rank-1 group Q_m. Consider the lacunary hyperbolic groups Q from Theorem 3.12 where the maximal rank-1 subgroups of Q are isomorphic to Q_m. Also let $N \rtimes Q \in \mathcal{R}ip(Q)$ be the semidirect product obtained via the Rips construction together with the subgroups $N \rtimes Q_m < N \rtimes Q$. Throughout this section we will consider the corresponding von Neumann algebras $\mathcal{M}_m := \mathcal{L}(N \rtimes Q_m) \subset \mathcal{L}(N \rtimes Q) := \mathcal{M}$.

Assuming Notation 4.3, we now show the following:

Theorem 4.4. *$\mathcal{M}_m$ is a maximal von Neumann algebra of $\mathcal{M}$. In particular, if $N \rtimes Q \in \mathcal{R}ip_{\mathcal{T}}(Q)$ then $\mathcal{M}_m$ is a non-property (T) maximal von Neumann subalgebra of a property (T) von Neumann algebra $\mathcal{M}$.*

Proof. Let $\mathcal{P}$ be any intermediate subalgebra $\mathcal{M}_m \subseteq \mathcal{P} \subseteq \mathcal{M}$. Since $\mathcal{M}_m \subset \mathcal{M}$ is spatially isomorphic to the crossed product inclusion $\mathcal{L}(N) \rtimes Q_m \subset \mathcal{L}(N) \rtimes Q$, we have $\mathcal{L}(N) \rtimes Q_m \subseteq \mathcal{P} \subseteq \mathcal{L}(N) \rtimes Q$. By Lemma 4.2 we have $(\mathcal{L}(N) \rtimes Q_m)' \cap (\mathcal{L}(N) \rtimes Q) \subseteq \mathcal{L}(N)' \cap (\mathcal{L}(N) \rtimes Q) = \mathbb{C}$. In particular, $\mathcal{P}$ is a factor. Moreover, by the Galois correspondence theorem [Choda 1978] (see also [Chifan and Das 2020, Corollary 3.8]) there is a subgroup $Q_m \leqslant K \leqslant Q$ so that $\mathcal{P} = \mathcal{L}(N) \rtimes K$. Since by construction Q_m is a maximal subgroup of Q, we must have $K = Q_m$ or Q. Thus we get $\mathcal{P} = \mathcal{M}_m$ or $\mathcal{M}$ and the conclusion follows.

For the remaining part note that $\mathcal{M}$ has property (T) by [Connes and Jones 1985]. Also, since $N \rtimes Q_m$ surjects onto an infinite abelian group, it does not have property (T). Thus by [Connes and Jones 1985] again, $\mathcal{M}_m = \mathcal{L}(N \rtimes Q_m)$ does not have property (T) either. $\square$

As pointed out at the beginning of the section, the above theorem provides a positive answer to [Jiang and Skalski 2019a, Problem 5.5]. Another solution to the problem of finding maximal subalgebras without property (T) inside factors with property (T) was also obtained independently by Jiang and Skalski in a more recent version of that paper. Their beautiful solution has a different flavor from ours; even though the Galois correspondence theorem à la Choda is a common ingredient in both of the proofs. Hence we refer the reader to [Jiang and Skalski 2019b, Theorem 4.8] for another solution to the aforementioned problem. Also note that while the algebras $\mathcal{M}_m$ do not have property (T), they are also nonamenable. In connection with this it would be very interesting if one could find an example of a property (T) II_1-factor which has maximal hyperfinite subfactors. This is essentially Ge's question but for property (T) factors.

In the final part of the section we show that whenever Q_ι is not isomorphic to Q_κ, the resulting maximal von Neumann subalgebras $\mathcal{M}_m$ and $\mathcal{M}_n$ are nonisomorphic. In fact we have the following more precise statement:

Theorem 4.5. *Assume that $Q_\iota, Q_\kappa < (\mathbb{Q}, +)$ and let $\Theta : \mathcal{M}_\iota \to \mathcal{M}_\kappa$ be a $*$-isomorphism. Then there exists a unitary $u \in \mathscr{U}(\mathcal{M}_\kappa)$ such that $\mathrm{ad}(u) \circ \Theta : \mathcal{L}(N_1) \to \mathcal{L}(N_2)$ is a $*$-isomorphism. Moreover* *there exist a group isomorphism* $\delta : Q_\iota \to Q_\kappa$ *and a* 1*-cocycle* $r : Q_\kappa \to \mathscr{U}(\mathcal{L}(N_2))$ *such that for all* $a \in \mathcal{L}(N_1)$ *and* $g \in Q_\iota$ *we have* $\mathrm{ad}(u) \circ \Theta(au_g) = \mathrm{ad}(u) \circ \Theta(a)v_{\delta(g)}r_{\delta(g)}$. *In particular, we have* $\mathrm{ad}(u) \circ \Theta \circ \alpha_g = \mathrm{ad}(r_{\delta(g)}) \circ \beta_{\delta(g)} \circ \mathrm{ad}(u) \circ \Theta$.

Proof. Identify $\mathcal{M}_\iota = \mathcal{L}(N_1) \rtimes Q_\iota$ and $\mathcal{M}_\kappa = \mathcal{L}(N_2) \rtimes Q_\kappa$ and let $\Theta : \mathcal{L}(N_1) \rtimes Q_\iota \to \mathcal{L}(N_2) \rtimes Q_\kappa$ be the $*$-isomorphism. Notice that since $\Theta(\mathcal{L}(N_1))$ has property (T) and Q_κ is amenable, by [Popa 2006a] we have $\Theta(\mathcal{L}(N_1)) \prec_{\mathcal{M}_\kappa} \mathcal{L}(N_2)$. Also by Lemma 4.2 we note that $\Theta(\mathcal{L}(N))$ is a regular irreducible subfactor of $\mathcal{M}_\kappa$, i.e., $\Theta(\mathcal{L}(N_1))' \cap \mathcal{M}_\kappa = \Theta(\mathcal{L}(N_1)' \cap \mathcal{M}_\iota) = \mathbb{C}1$. Similarly, $\mathcal{L}(N_2)$ is a regular irreducible subfactor of $\mathcal{M}_\kappa$ satisfying $\mathcal{L}(N_2) \prec_{\mathcal{M}_\kappa} \Theta(\mathcal{L}(N_1))$. Thus by the proof of [Ioana et al. 2008, Lemma 8.4], since Q_ι's are torsion-free, one can find a unitary $u \in \mathcal{M}_\kappa$ such that $\mathrm{ad}(u) \circ \Theta(\mathcal{L}(N_1)) = \mathcal{L}(N_2)$. So replacing Θ with $\mathrm{ad}(u) \circ \Theta$ we can assume that $\Theta(\mathcal{L}(N_1)) = \mathcal{L}(N_2)$. Hence for every $g \in Q_\iota$ we have $\Theta(\alpha_g(x))\Theta(u_g) = \Theta(u_g)\Theta(x)$ for all $x \in \mathcal{L}(N_1)$. Consider the Fourier decomposition $\Theta(u_g) = \sum_{h \in Q_\kappa} n_h v_h$, where $n_h \in \mathcal{L}(N_2)$. Using the previous relation we get $\Theta(\alpha_g(x))n_h = n_h \beta_h \Theta(x)$ for all $h \in Q_\kappa$ and $x \in \mathcal{L}(N_2)$. Thus $n_h n_h{}^* \in \mathcal{L}(N_2)' \cap \mathcal{M}_\kappa = \mathbb{C}1$ and hence there exist unitary $t_h \in \mathcal{L}(N_2)$ and scalar $s_h \in \mathbb{C}$ so that $n_h = s_h t_h$. Assume there exist $h_1 \neq h_2 \in Q_\kappa$ so that $s_{h_1}, s_{h_2} \neq 0$. This implies that $\Theta(\alpha_g(x)) = t_{h_1}\beta_{h_1}\Theta(x)t^*_{h_1} = t_{h_2}\beta_{h_2}\Theta(x)t^*_{h_2}$ for all $x \in \mathcal{L}(N_2)$. Thus $\beta_{h_1}(t^*_{h_1}t_{h_2})v_{h_1^{-1}h_2} = v^*_{h_1}t^*_{h_1}t_{h_2}v_{h_2} \in \mathcal{L}(N_2)' \cap \mathcal{M}_\kappa = \mathbb{C}1$. Therefore $h_1^{-1}h_2 = 1$ and $h_1 = h_2$, which is a contradiction. In particular there exists a unique $\delta(g) \in Q_\kappa$ so that $s_k = 0$ for all $k \in Q_\kappa \setminus \{\delta(g)\}$. Altogether these show that there is a well-defined map $\delta : Q_\iota \to Q_\kappa$ so that $\Theta(u_g) = n_{\delta(g)}v_{\delta(g)}$ for all $g \in Q_\iota$. It is easy to see that δ is a group isomorphism and the map $r : Q_\kappa \to \mathscr{U}(\mathcal{L}(N_2))$ given by $r(h) = \beta_h(n_h)$ is a 1-cocycle, i.e., $r(hk) = c_h\beta_h(c_k)$. □

Final remarks. We notice that our strategy from the proof of Theorem 4.4 can also be used to produce other examples of non-property (T) subalgebras in property (T) factors. Indeed for Q in the Rips construction one can take in fact *any* torsion-free, property (T) monster group Q in the sense of Olshanskii. If one picks any maximal subgroup $Q_0 < Q$ then, as before, the group von Neumann algebra $\mathcal{L}(N \rtimes Q_0)$ will obviously be maximal in $\mathcal{L}(N \rtimes Q)$. Notice that since $Q_0 < Q$ is maximal, Q_0 is infinite-index in Q. To see this note that if Q_0 is finite-index in Q, then Q_0 has property (T) and hence is finitely generated. Therefore Q_0 would be abelian and hence trivial, which is a contradiction. Therefore Q_0 must have infinite index in Q. In this case it is either finitely generated, in which case is abelian or it is infinitely generated. However, in both scenarios Q_0 does not have property (T) and hence neither does $N \rtimes Q_0$. Thus by [Connes and Jones 1985], $\mathcal{L}(N \rtimes Q_0)$ does not have property (T).

5. Von Neumann algebraic rigidity aspects for groups arising via Rips constructions

An impressive milestone in the classification of von Neumann algebras was the emergence over the past decade of the first examples of groups G that can be completely reconstructed from their von Neumann algebras $\mathcal{L}(G)$, i.e., W^**-superrigid groups* [Ioana et al. 2013; Berbec and Vaes 2014; Chifan and Ioana 2018]. The strategies used in establishing these results share a common key ingredient, namely, the ability to first reconstruct from $\mathcal{L}(G)$ various algebraic features of G such as its (generalized) wreath product decomposition in [Ioana et al. 2013; Berbec and Vaes 2014] and, respectively, its amalgam splitting in [Chifan and Ioana 2018, Theorem A]. This naturally leads to a broad and independent study, specifically identifying canonical group algebraic features of a group that pass to its von Neumann algebra. While several works have emerged recently in this direction [Chifan et al. 2016b; Chifan and Ioana 2018; Chifan and Udrea 2020], the surface has been only scratched and still a great deal of work remains to be done.

A difficult conjecture of Connes predicts that all icc property (T) groups are W^*-superrigid. Unfortunately, not a single example of such group is known at this time. Moreover, in the current literature there is an almost complete lack of examples of algebraic features occurring in a property (T) group that are recognizable at the von Neumann algebraic level. In this section we make progress on this problem for property (T) groups that appear as certain fiber products of Belegradek–Osin Rips-type constructions. Specifically, we have the following result:

Theorem 5.1. *Let $Q = Q_1 \times Q_2$, where Q_i are icc, torsion-free, biexact, property (T), weakly amenable, residually finite groups. For $i = 1, 2$, let $N_i \rtimes_{\sigma_i} Q \in \mathcal{R}ip_T(Q)$ and denote by $\Gamma = (N_1 \times N_2) \rtimes_\sigma Q$ the semidirect product associated with the diagonal action $\sigma = \sigma_1 \times \sigma_2 : Q \curvearrowright N_1 \times N_2$. Denote by $\mathcal{M} = \mathcal{L}(\Gamma)$ the corresponding II_1-factor. Assume that Λ is any arbitrary group and $\Theta : \mathcal{L}(\Gamma) \to \mathcal{L}(\Lambda)$ is any $*$-isomorphism. Then there exist group actions by automorphisms $H \curvearrowright^{\tau_i} K_i$ such that $\Lambda = (K_1 \times K_2) \rtimes_\tau H$, where $\tau = \tau_1 \times \tau_2 : H \curvearrowright K_1 \times K_2$ is the diagonal action. Moreover one can find a multiplicative character $\eta : Q \to \mathbb{T}$, a group isomorphism $\delta : Q \to H$, a unitary $w \in \mathcal{L}(\Lambda)$, and $*$-isomorphisms $\Theta_i : \mathcal{L}(N_i) \to \mathcal{L}(K_i)$ such that for all $x_i \in \mathcal{L}(N_i)$ and $g \in Q$ we have*

$$\Theta((x_1 \otimes x_2)u_g) = \eta(g)w((\Theta_1(x_1) \otimes \Theta(x_2))v_{\delta(g)})w^*. \tag{5.1}$$

Here $\{u_g : g \in Q\}$ and $\{v_h : h \in H\}$ are the canonical unitaries implementing the actions of $Q \curvearrowright \mathcal{L}(N_1) \bar{\otimes} \mathcal{L}(N_2)$ and $H \curvearrowright \mathcal{L}(K_1) \bar{\otimes} \mathcal{L}(K_2)$, respectively.

From a different perspective our theorem can be also seen as a von Neumann algebraic superrigidity result regarding conjugacy of actions on noncommutative von Neumann algebras. Notice that very little is known in this direction as well, as most of the known superrigidity results concern algebras arising from actions of groups on probability spaces.

We continue with a series of preliminary results that are essential to deriving the proof of Theorem 5.1 at the end of the section. First we present a location result for commuting diffuse property (T) subalgebras inside a von Neumann algebra arising from products of relative hyperbolic groups.

Theorem 5.2. *For $i = \overline{1, n}$ let $H_i < G_i$ be an inclusion of infinite groups such that H_i is residually finite and G_i is hyperbolic relative to H_i. Denote by $H = H_1 \times \cdots \times H_n < G_1 \times \cdots \times G_n = G$ the corresponding direct product inclusion. Let $\mathcal{N}_1, \mathcal{N}_2 \subseteq \mathcal{L}(G)$ be two commuting von Neumann subalgebras with property (T). Then for every $i \in \overline{1, n}$ there exists $k \in \overline{1, 2}$ such that $\mathcal{N}_k \prec \mathcal{L}(\widehat{G}_i \times H_i)$, where $\widehat{G}_i := \times_{j \neq i} G_j$.*

Proof. Fix $i \in \overline{1, n}$. Since H_i is residually finite, using Theorem 3.16 there is a short exact sequence

$$1 \to \ker(\pi_i) \hookrightarrow G_i \xrightarrow{\pi_i} F_i \to 1,$$

where F_i is a nonelementary hyperbolic group and $\ker(\pi_i) = \langle H_i^0 \rangle = *_{t\in T_i} (H_i^0)^t$ for some subset $T \subset G_i$ and a finite-index normal subgroup $H_i^0 \lhd H_i$.

Following [Chifan et al. 2015, Notation 3.3] we now consider the von Neumann algebraic embedding corresponding to π_i, i.e., $\Pi_i : \mathcal{L}(G) \to \mathcal{L}(G) \bar{\otimes} \mathcal{L}(F_i)$ given by $\Pi_i(u_g) = u_g \otimes v_{\pi_i(g_i)}$ for all $g = (g_j) \in G$; here the u_g's are the canonical unitaries of $\mathcal{L}(G)$ and the v_h's are the canonical unitaries of $\mathcal{L}(F_i)$. From the hypothesis we have that $\Pi_i(\mathcal{N}_1), \Pi_i(\mathcal{N}_2) \subset \mathcal{L}(G) \bar{\otimes} \mathcal{L}(F_i) =: \widetilde{\mathcal{M}}_i$ are commuting property (T) subalgebras. Let $\mathcal{A} \subset \Pi_i(\mathcal{N}_1)$ be any diffuse amenable von Neumann subalgebra. Using [Popa and Vaes 2014, Theorem 1.4] we have either (a) $\mathcal{A} \prec_{\widetilde{\mathcal{M}}_i} \mathcal{L}(G) \bar{\otimes} 1$ or (b) $\Pi_i(\mathcal{N}_2)$ is amenable relative to $\mathcal{L}(G) \bar{\otimes} 1$ inside $\widetilde{\mathcal{M}}_i$.

Since the $\mathcal{N}_k$'s have property (T), so do the $\Pi_i(\mathcal{N}_k)$'s. Thus using part (b) above we get that $\Pi_i(\mathcal{N}_2) \prec_{\widetilde{\mathcal{M}}} \mathcal{L}(G) \bar{\otimes} 1$. On the other hand, if case (a) above were to hold for all $\mathcal{A}$'s then by [Brown and Ozawa 2008, Corollary F.14] we would get $\Pi_i(\mathcal{N}_1) \prec_{\widetilde{\mathcal{M}}_i} \mathcal{L}(G) \bar{\otimes} 1$. Therefore we can always assume that $\Pi_i(\mathcal{N}_k) \prec_{\widetilde{\mathcal{M}}_i} \mathcal{L}(G) \bar{\otimes} 1$ for $k = 1$ or 2.

Due to symmetry we only treat $k = 1$. Using [Chifan et al. 2015, Proposition 3.4] we get $\mathcal{N}_1 \prec \mathcal{L}(\ker(\Pi_i)) = \mathcal{L}(\widehat{G}_i \times \ker(\pi_i))$. Thus there exist nonzero projections $p \in \mathcal{N}_1$, $q \in \mathcal{L}(\widehat{G}_i \times \ker(\pi_i))$, a nonzero partial isometry $v \in \mathcal{M}$ and a $*$-isomorphism $\phi : p\mathcal{N}_1 p \to \mathcal{B} := \phi(p\mathcal{N}_1 p) \subset q\mathcal{L}(\widehat{G}_i \times \ker(\pi_i))q$ on the image such that

$$\phi(x)v = vx \quad \text{for all } x \in p\mathcal{N}_1 p. \tag{5.2}$$

Also notice that since $\mathcal{N}_1$ has property (T), so does $p\mathcal{N}_1 p$ and therefore $\mathcal{B} \subseteq q\mathcal{L}(\widehat{G}_i \times \ker(\pi_i))q$ is a property (T) subalgebra. Since $\ker(\pi_i) = *_{t\in T} (H_i^0)^t$, by further conjugating q in the factor $\mathcal{L}(\widehat{G}_i \times \ker(\pi_i))$ we can assume that there exists a unitary $u \in \mathcal{L}(\widehat{G}_i \times \ker(\pi_i))$ and a projection $q_0 \in \mathcal{L}(\widehat{G}_i)$ such that $\mathcal{B} \subseteq u(q_0 \mathcal{L}(\widehat{G}_i) q_0) \bar{\otimes} \mathcal{L}(\ker(\pi_i))u^*$. Using property (T) of $\mathcal{B}$ and [Ioana et al. 2008, Theorem] we further conclude that there is $t_0 \in T$ such that $\mathcal{B} \prec_{u(q_0\mathcal{L}(\widehat{G}_i)q_0 \bar{\otimes} \mathcal{L}(\ker(\pi_i)))u^*} u(q_0 \mathcal{L}(\widehat{G}_i) q_0 \otimes \mathcal{L}((H_i^0)^{t_0}))u^*$. Composing this intertwining with ϕ we finally conclude that $\mathcal{N}_1 \prec_{\mathcal{M}} \mathcal{L}(\widehat{G}_i \times H_i^0)$, as desired. ☐

Theorem 5.3. *Under the same assumptions as in Theorem 5.2, for every $k \in \overline{1, n}$ one of the following must hold*:

(1) *There exists $i \in 1, 2$ such that $\mathcal{N}_i \prec_{\mathcal{M}} \mathcal{L}(\widehat{G}_k)$.*

(2) $\mathcal{N}_1 \vee \mathcal{N}_2 \prec_{\mathcal{M}} \mathcal{L}(\widehat{G}_k \times H_k)$.

Proof. From Theorem 5.2 there exists $i \in \overline{1, 2}$ such that $\mathcal{N}_i \prec \mathcal{L}(\widehat{G}_k \times H_k)$. For convenience assume that $i = 1$. Thus there exist nonzero projections $p \in \mathcal{N}_1$, $q \in \mathcal{L}(\widehat{G}_k \times H_k)$, a nonzero partial isometry $v \in \mathcal{M}$ and a $*$-isomorphism $\phi : p\mathcal{N}_1 p \to \mathcal{B} := \phi(p\mathcal{N}_1 p) \subset q\mathcal{L}(\widehat{G}_k \times H_k)q$ on the image such that

$$\phi(x)v = vx \quad \text{for all } x \in p\mathcal{N}_1 p. \tag{5.3}$$

Notice that $q \geqslant vv^* \in \mathcal{B}' \cap q\mathcal{M}q$ and $p \geqslant v^*v \in p\mathcal{N}_i p' \cap p\mathcal{M}p$. Also we can pick v such that $s(E_{\mathcal{L}(\widehat{G}_k \times H_k)}(vv^*)) = q$. Next we assume that $\mathcal{B} \prec_{L(\widehat{G}_k \times H_k)} L(\widehat{G}_k)$. Thus there exist nonzero projections $p' \in \mathcal{B}$, $q' \in \mathcal{L}(\widehat{G}_k)$, a nonzero partial isometry $w \in q'\mathcal{L}(\widehat{G}_k \times H_k)p'$ and a $*$-isomorphism $\psi : p'\mathcal{B}p' \to q'\mathcal{L}(\widehat{G}_k)q'$ on the image such that

$$\psi(x)w = wx \quad \text{for all } x \in p'\mathcal{B}p'. \tag{5.4}$$

Notice that $q \geqslant p' \geqslant ww^* \in (p'\mathcal{B}p')' \cap p'\mathcal{M}p'$ and $q' \geqslant w^*w \in \psi(p'\mathcal{B}p')' \cap q'\mathcal{M}q'$. Using (5.3) and (5.4) we see that

$$\psi(\phi(x))wv = w\phi(x)v = wvx \quad \text{for all } x \in p_0\mathcal{N}_i p_0, \tag{5.5}$$

where $p_0 \in \mathcal{N}_i$ is a projection picked so that $\phi(p_0) = p'$. Also we note that if $0 = wv$ then $0 = wvv^*$, and hence $0 = E_{\mathcal{L}(\widehat{G}_k \times H_k)}(wvv^*) = wE_{\mathcal{L}(\widehat{G}_k \times H_k)}(vv^*)$. This further implies that $0 = ws(E_{\mathcal{L}(\widehat{G}_k \times H_k)}(vv^*)) = wq = w$, which is a contradiction. Thus $wv \neq 0$ and taking the polar decomposition of wv we see that (5.5) gives (1).

Next we assume that $\mathcal{B} \not\prec_{\mathcal{L}(\widehat{G}_k \times H_k)} \mathcal{L}(\widehat{G}_k)$. Since G_k is hyperbolic relative to H_k, by Lemma 2.2 we have that for all $x, x_1x_2, \dots, x_l \in M$ such that $\mathcal{B}x \subseteq \sum_{i=1}^{l} x_i\mathcal{B}$ we must have $x \in \mathcal{L}(\widehat{G}_k \times H_k)$. Hence in particular we have $vv^* \in \mathcal{B}' \cap q\mathcal{M}q \subseteq \mathcal{L}(\widehat{G}_k \times H_k)$ and thus relation (5.3) implies that $\mathcal{B}vv^* = v\mathcal{N}_i v^* \subseteq \mathcal{L}(\widehat{G}_k \times H_k)$. Also for every $c \in \mathcal{N}_{i+1}$ we can see that

$$\begin{aligned}\mathcal{B}vcv^* &= \mathcal{B}vv^*vcv^* = v\mathcal{N}_i v^*vcv^* = vv^*vc\mathcal{N}_i v^*\\ &= vc\mathcal{N}_i v^* = vc\mathcal{N}_i v^*vv^* = vcv^*v\mathcal{N}_i v^* = vcv^*\mathcal{B}vv^* = vcv^*\mathcal{B}.\end{aligned} \tag{5.6}$$

Therefore by Lemma 2.2 again we have $vcv^* \in \mathcal{L}(\widehat{G}_k \times H_k)$ and hence $v\mathcal{N}_{i+1}v^* \subseteq \mathcal{L}(\widehat{G}_k \times H_k)$. Thus $v\mathcal{N}_i\mathcal{N}_{i+1}v^* = vv^*v\mathcal{N}_i\mathcal{N}_{i+1}v^* = v\mathcal{N}_i v^*v\mathcal{N}_{i+1}v^* \subseteq \mathcal{L}(\widehat{G}_k \times H_k)$, which by Popa's intertwining techniques implies that $\mathcal{N}_1 \vee \mathcal{N}_2 \prec \mathcal{L}(\widehat{G}_k \times H_k)$, i.e., (2) holds. $\square$

We now proceed towards proving the main result of this section. To simplify the exposition we first introduce notation that will be used throughout the section.

Notation 5.4. Define $Q = Q_1 \times Q_2$, where Q_i are infinite, residually finite, biexact, property (T), icc groups. Then consider $\Gamma_i = N_i \rtimes Q \in \mathcal{R}ip_T(Q)$ and the semidirect product $\Gamma = (N_1 \times N_2) \rtimes_\sigma Q$ arising from the diagonal action $\sigma = \sigma_1 \times \sigma_2 : Q \to \mathrm{Aut}(N_1 \times N_2)$, i.e., $\sigma_g(n_1, n_2) = ((\sigma_1)_g(n_1), (\sigma_2)_g(n_2))$ for all $(n_1, n_2) \in N_1 \times N_2$. For further use we observe that Γ is the fiber product $\Gamma = \Gamma_1 \times_Q \Gamma_2$ and thus embeds into $\Gamma_1 \times \Gamma_2$, where Q embeds diagonally into $Q \times Q$. In the next proofs when we refer to this copy we will often denote it by $\mathrm{d}(Q)$. Also notice that Γ is a property (T) group as it arises from an extension of property (T) groups. Furthermore, $\Gamma_1, \Gamma_2 \in \mathcal{R}ip_T(Q)$ easily implies that Γ is an icc group.

For future use, use also recall the notion of the comultiplication studied in [Ioana et al. 2013; Ioana 2011]. Let Γ be a group as above, and assume that Λ is a group such that $\mathcal{L}(\Gamma) = \mathcal{L}(\Lambda) = \mathcal{M}$. Then the "comultiplication along Λ" $\Delta : \mathcal{M} \to \mathcal{M} \mathbin{\bar\otimes} \mathcal{M}$ is defined by $\Delta(v_\lambda) = v_\lambda \otimes v_\lambda$ for all $\lambda \in \Lambda$.

Theorem 5.5. *Let Γ be a group as in Notation 5.4 and assume that Λ is a group such that $\mathcal{L}(\Gamma) = \mathcal{L}(\Lambda) = \mathcal{M}$. Let $\Delta : \mathcal{M} \to \mathcal{M} \mathbin{\bar\otimes} \mathcal{M}$ be the comultiplication along Λ as in Notation 5.4. Then the following hold*:

(3) *For all $j \in \overline{1,2}$ there is $i \in \overline{1,2}$ such that $\Delta(\mathcal{L}(N_i)) \prec_{\mathcal{M}\bar\otimes\mathcal{M}} \mathcal{M} \mathbin{\bar\otimes} \mathcal{L}(N_j)$.*

(4) (a) *For all $j \in \overline{1,2}$ there is $i \in \overline{1,2}$ such that $\Delta(\mathcal{L}(Q_i)) \prec_{\mathcal{M}\bar\otimes\mathcal{M}} \mathcal{M} \mathbin{\bar\otimes} \mathcal{L}(N_j)$ or*

(b) $\Delta(\mathcal{L}(Q)) \prec_{\mathcal{M}\bar\otimes\mathcal{M}} \mathcal{M} \mathbin{\bar\otimes} \mathcal{L}(Q)$; *moreover in this case for every $j \in \overline{1,2}$ there is $i \in \overline{1,2}$ such that $\Delta(\mathcal{L}(Q_j)) \prec_{\mathcal{M}\bar\otimes\mathcal{M}} \mathcal{M} \mathbin{\bar\otimes} \mathcal{L}(Q_i)$.*

Proof. Let $\widetilde{\mathcal{M}} = \mathcal{L}(\Gamma_1 \times \Gamma_2)$. Since $\Gamma < \Gamma_1 \times \Gamma_2$, we notice the inclusions $\Delta(\mathcal{L}(N_1)), \Delta(\mathcal{L}(N_2)) \subset \mathcal{M} \bar{\otimes} \mathcal{M} = \mathcal{L}(\Gamma \times \Gamma) \subset \mathcal{L}(\Gamma_1 \times \Gamma_2 \times \Gamma_1 \times \Gamma_2)$. Since Γ_i is hyperbolic relative to Q, using Theorem 5.3 we have either

(5) there exists $i \in 1, 2$ such that $\Delta(\mathcal{L}(N_i)) \prec_{\widetilde{\mathcal{M}} \bar{\otimes} \widetilde{\mathcal{M}}} \mathcal{M} \bar{\otimes} \mathcal{L}(\Gamma_1)$, or

(6) $\Delta(\mathcal{L}(N_1 \times N_2)) \prec_{\widetilde{\mathcal{M}} \bar{\otimes} \widetilde{\mathcal{M}}} \mathcal{M} \bar{\otimes} \mathcal{L}(\Gamma_1 \times Q)$.

Assume (5) holds. Since $\Delta(\mathcal{L}(N_i)) \subset \mathcal{M} \bar{\otimes} \mathcal{L}(\Gamma)$ then by Lemma 2.3 there is an $h \in \Gamma_1 \times \Gamma_2 \times \Gamma_1 \times \Gamma_2$ so that $\Delta(\mathcal{L}(N_i)) \prec_{\widetilde{\mathcal{M}} \bar{\otimes} \widetilde{\mathcal{M}}} \mathcal{L}(\Gamma \times (\Gamma \cap h(\Gamma_1 \times \Gamma_2 \times \Gamma_1)h^{-1})) = \mathcal{L}(\Gamma \times (\Gamma \cap \Gamma_1)) = \mathcal{M} \bar{\otimes} (\mathcal{L}((N_1 \times N_2) \rtimes \mathrm{d}(Q)) \cap (N_1 \rtimes Q \times 1)) = \mathcal{M} \bar{\otimes} \mathcal{L}(N_1)$. Note that since $\Delta(\mathcal{L}(N_i))$ is regular in $\mathcal{M} \bar{\otimes} \mathcal{M}$, using Lemma 2.4, we get that $\Delta(\mathcal{L}(N_i)) \prec_{\mathcal{M} \bar{\otimes} \mathcal{M}} \mathcal{M} \bar{\otimes} \mathcal{L}(\Gamma_1)$, thereby establishing 3).

Assume (6) holds. Since $\Delta(\mathcal{L}(N_1 \times N_2)) \subset \mathcal{L}(\Gamma \times \Gamma)$, by Lemma 2.3 there is an $h \in \Gamma_1 \times \Gamma_2 \times \Gamma_1 \times \Gamma_2$ such that

$$\begin{aligned}\Delta(\mathcal{L}(N_1 \times N_2)) &\prec \mathcal{L}(\Gamma \times (\Gamma \cap h(\Gamma_1 \times \Gamma_2 \times \Gamma_1 \times Q)h^{-1})) \\ &= \mathcal{L}(\Gamma \times (\Gamma \cap (\Gamma_1 \times h_4 Q h_4^{-1}))) \\ &= \mathcal{M} \bar{\otimes} \mathcal{L}((N_1 \times N_2) \rtimes \mathrm{d}(Q)) \cap (N_1 \rtimes Q \times h_4 Q h_4^{-1}).\end{aligned}$$

Since $h_4 \in \Gamma_2 = N_2 \rtimes Q$, we can assume that $h_4 \in N_2$. Notice that

$$\begin{aligned}((N_1 \times N_2) \rtimes \mathrm{d}(Q)) \cap (N_1 \rtimes Q \times h_4 Q h_4^{-1}) &= h_4((N_1 \times N_2) \rtimes \mathrm{d}(Q)) \cap (N_1 \rtimes Q \times Q) h_4^{-1} \\ &= h_4((N_1 \times 1) \rtimes \mathrm{d}(Q)) h_4^{-1}\end{aligned}$$

and hence $\Delta(\mathcal{L}(N_1 \times N_2)) \prec_{\widetilde{\mathcal{M}} \bar{\otimes} \widetilde{\mathcal{M}}} \mathcal{M} \bar{\otimes} \mathcal{L}(N_1 \rtimes \mathrm{d}(Q))$. Moreover using Lemma 2.5 we further have $\Delta(\mathcal{L}(N_1 \times N_2)) \prec_{\mathcal{M} \bar{\otimes} \mathcal{M}} \mathcal{M} \bar{\otimes} \mathcal{L}(N_1 \rtimes \mathrm{d}(Q))$.

In conclusion, there exist a $*$-isomorphism on its image

$$\phi : p\Delta(\mathcal{L}(N_1 \times N_2))p \to \mathcal{B} := \phi(p\Delta(\mathcal{L}(N_1 \times N_2))p) \subseteq q\mathcal{M} \bar{\otimes} \mathcal{L}(N_1 \rtimes \mathrm{d}(Q))$$

and $0 \neq v \in q(\mathcal{M} \bar{\otimes} \mathcal{M})p$ such that

$$\phi(x)v = vx \quad \text{for all } x \in p\Delta(\mathcal{L}(N_1 \times N_2))p. \tag{5.7}$$

Next assume that (3) doesn't hold. Thus proceeding as in the first part of the proof of Theorem 5.3, we get

$$\mathcal{B} \nprec_{\mathcal{M} \bar{\otimes} (N_1 \rtimes \mathrm{d}(Q))} \mathcal{M} \bar{\otimes} \mathcal{L}(N_1) =: \mathcal{M}_1. \tag{5.8}$$

Next we observe the inclusions

$$\begin{aligned}&\mathcal{M}_1 \rtimes_{1 \otimes \sigma} \mathrm{d}(Q) = \mathcal{M} \bar{\otimes} \mathcal{L}(N_1) \rtimes_{1 \otimes \sigma} \mathrm{d}(Q) = \mathcal{M} \bar{\otimes} \mathcal{L}(N_1 \rtimes_\sigma \mathrm{d}(Q)) \\ &\quad \subset \mathcal{M} \bar{\otimes} \mathcal{L}((N_1 \times N_2) \rtimes_\sigma \mathrm{d}(Q)) = \mathcal{M} \bar{\otimes} \mathcal{L}(N_1) \bar{\otimes} \mathcal{L}(N_2) \rtimes \mathrm{d}(Q) = \mathcal{M}_1 \rtimes_{1 \otimes \sigma} N_2 \rtimes \mathrm{d}(Q).\end{aligned} \tag{5.9}$$

Also since Q is malnormal in $N_2 \rtimes Q$ it follows from Lemma 2.2 that $vv^* \in \mathcal{M} \bar{\otimes} \mathcal{L}(N_1 \rtimes \mathrm{d}(Q))$ and hence $\mathcal{B}vv^* \subset \mathcal{M} \bar{\otimes} \mathcal{L}(N_1 \rtimes \mathrm{d}(Q))$. Pick $u \in \mathcal{QN}_{p(\mathcal{M} \bar{\otimes} \mathcal{M})p}(p\Delta(\mathcal{L}(N_1 \times N_2))p)$ and using (5.7) we see that there exist $n_1, n_2, \dots, n_s \in p(\mathcal{M}\mathbin{\bar{\otimes}}\mathcal{M})p$ satisfying

$$\begin{aligned}\mathcal{B}vuv^* &= \mathcal{B}vv^*vuv^* = vp(\Delta(\mathcal{L}(N_1\times N_2)))pv^*vnv^* = vp(\Delta(\mathcal{L}(N_1\times N_2)))pnv^*\\ &\subseteq \sum_{i=1}^{s} vn_i p(\Delta(\mathcal{L}(N_1\times N_2)))pv^* = \sum_{i=1}^{s} vn_i p(\Delta(\mathcal{L}(N_1\times N_2)))pv^*vv^*\\ &= \sum_{i=1}^{s} vn_i pv^*v(\Delta(\mathcal{L}(N_1\times N_2)))pv^* = \sum_{i=1}^{s} vn_i pv^*\mathcal{B}vv^* = \sum_{i=1}^{s} vn_i pv^*\mathcal{B}. \end{aligned}\tag{5.10}$$

Then by Lemma 2.2 again we must have $vuv^* \in \mathcal{M}\mathbin{\bar{\otimes}}\mathcal{L}(N_1 \rtimes \mathrm{d}(Q))$. Hence we have shown that

$$v\mathcal{QN}_{p(\mathcal{M}\bar{\otimes}\mathcal{M})p}(p\Delta(\mathcal{L}(N_1\times N_2))p)v^* \subseteq \mathcal{M}\mathbin{\bar{\otimes}}\mathcal{L}(N_1\rtimes \mathrm{d}(Q)).\tag{5.11}$$

Since $v^*v \in (p\Delta(\mathcal{L}(N_1\times N_2))p)'\cap p(\mathcal{M}\bar{\otimes}\mathcal{M})p \subset \mathcal{QN}_{p(\mathcal{M}\bar{\otimes}\mathcal{M})p}(p(\Delta(\mathcal{L}(N_1\times N_2))))p$, (5.11) further implies

$$v\mathcal{QN}_{p(\mathcal{M}\bar{\otimes}\mathcal{M})p}(p(\Delta(\mathcal{L}(N_1\times N_2)))p)''v^* \subseteq \mathcal{M}\mathbin{\bar{\otimes}}\mathcal{L}(N_1\rtimes \mathrm{d}(Q)).\tag{5.12}$$

Here for every inclusion of von Neumann algebras $\mathcal{R}\subseteq\mathcal{T}$ and projection $p\in\mathcal{R}$ we used the formula $\mathcal{QN}_{p\mathcal{T}p}(p\mathcal{R}p)'' = p\mathcal{QN}_{\mathcal{T}}(\mathcal{R})''p$ [Popa 2006b, Lemma 3.5]. As

$$vp\Delta(\mathcal{M})pv^* \subseteq v\mathcal{QN}_{p(\mathcal{M}\bar{\otimes}\mathcal{M})p}(p(\Delta(\mathcal{L}(N_1\times N_2)))p)''v^*,$$

we conclude that $\Delta(\mathcal{M}) \prec \mathcal{L}(N_1\rtimes Q)$, which contradicts the fact that N_2 is infinite. Thus (3) must always hold.

Next we derive (4). Again we notice that

$$\Delta(\mathcal{L}(Q_1)), \Delta(\mathcal{L}(Q_2)) \subset \Delta(\mathcal{M}) \subset \mathcal{M}\mathbin{\bar{\otimes}}\mathcal{M} = \mathcal{L}(\Gamma\times\Gamma) \subset \mathcal{L}(\Gamma_1\times\Gamma_2\times\Gamma_1\times\Gamma_2).$$

Using Theorem 5.3 we must have either

(7) $\Delta(\mathcal{L}(Q_i)) \prec_{\tilde{\mathcal{M}}\bar{\otimes}\tilde{\mathcal{M}}} \mathcal{M}\mathbin{\bar{\otimes}}\mathcal{L}(\Gamma_1)$, or

(8) $\Delta(\mathcal{L}(Q)) \prec_{\tilde{\mathcal{M}}\bar{\otimes}\tilde{\mathcal{M}}} \mathcal{M}\mathbin{\bar{\otimes}}\mathcal{L}(\Gamma_1\times Q)$.

Proceeding as in the previous case, and using Lemma 2.4, we see that (7) implies $\Delta(\mathcal{L}(Q_i)) \prec_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{M}\mathbin{\bar{\otimes}}\mathcal{L}(N_1)$, which in turn gives (4a). Also proceeding as in the previous case, and using Lemma 2.5, we see that (8) implies

$$\Delta(\mathcal{L}(\mathrm{d}(Q))) \prec_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{M}\mathbin{\bar{\otimes}}\mathcal{L}(N_1\rtimes \mathrm{d}(Q)).\tag{5.13}$$

To show part (4b) we will exploit (5.13). Notice that there exist nonzero projections $r\in\Delta(\mathcal{L}(Q))$, $t\in\mathcal{M}\mathbin{\bar{\otimes}}\mathcal{L}(N_1\rtimes \mathrm{d}(Q))$, a nonzero partial isometry $w\in r(\mathcal{M}\mathbin{\bar{\otimes}}\mathcal{M})t$ and a $*$-isomorphism onto its image $\phi : r\Delta(\mathcal{L}(Q))r \to \mathcal{C} := \phi(r\Delta(\mathcal{L}(Q))r) \subseteq t(\mathcal{M}\mathbin{\bar{\otimes}}\mathcal{L}(N_1\rtimes \mathrm{d}(Q)))t$ such that

$$\phi(x)w = wx \quad \text{for all } x\in r\Delta(\mathcal{L}(Q))r.\tag{5.14}$$

Since $\mathcal{L}(Q)$ is a factor we can assume without loss of generality that $r = \Delta(r_1\otimes r_2)$, where $r_i\in\mathcal{L}(Q_i)$. Hence $\mathcal{C} = \phi(r\Delta(\mathcal{L}(Q))r) = \phi(\Delta(r_1\mathcal{L}(Q_i)r_2))\mathbin{\bar{\otimes}} r_2\mathcal{L}(Q_2)r_2 =: \mathcal{C}_1\vee\mathcal{C}_2$, where $\mathcal{C}_i = \phi(\Delta(r_i\mathcal{L}(Q_i))r_i) \subseteq$

$t(\mathcal{M}\mathbin{\bar{\otimes}}\mathcal{L}(N_1 \rtimes \mathrm{d}(Q)))t$. Notice that the $\mathcal{C}_i$'s are commuting property (T) subfactors of $\mathcal{M}\mathbin{\bar{\otimes}}\mathcal{L}(N_1 \rtimes \mathrm{d}(Q))$. Since $N_i \rtimes Q$ is hyperbolic relative to $\{Q\}$ and seeing

$$\mathcal{C}_1 \vee \mathcal{C}_2 \subseteq \mathcal{M}\mathbin{\bar{\otimes}}\mathcal{L}(N_i \rtimes \mathrm{d}(Q)) \subset \mathcal{L}(\Gamma_1 \times \Gamma_2 \times (N_1 \rtimes \mathrm{d}(Q))),$$

by applying Theorem 5.3 we have that there exists $i \in 1, 2$ such that

(9) $\mathcal{C}_1 \prec_{\tilde{\mathcal{M}}\bar{\otimes}\mathcal{L}(N_1 \rtimes \mathrm{d}(Q))} \mathcal{L}(\Gamma_1 \times \Gamma_2)$ or

(10) $\mathcal{C}_1 \vee \mathcal{C}_2 \prec_{\tilde{\mathcal{M}}\bar{\otimes}\mathcal{L}(N_1 \rtimes \mathrm{d}(Q))} \mathcal{L}(\Gamma_1 \times \Gamma_2 \times \mathrm{d}(Q))$.

Since $\mathcal{C}_1 \subset \mathcal{M}\mathbin{\bar{\otimes}}\mathcal{M}$ then (9) and Lemma 2.6 imply $\mathcal{C}_1 \prec_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{M}\otimes 1$, which by [Ioana 2011, Lemma 9.2] further implies that $\mathcal{C}_1$ is atomic, which is a contradiction. Thus we must have (10). However since $\mathcal{C}_1 \vee \mathcal{C}_2 \subset \mathcal{M}\mathbin{\bar{\otimes}}\mathcal{M}$, part (10) and Lemma 2.6 give $\mathcal{C}_1 \vee \mathcal{C}_2 \prec_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{M}\mathbin{\bar{\otimes}}\mathcal{L}(\mathrm{d}(Q))$ and composing this intertwining with ϕ (as done in the proof of the first case in Theorem 5.3) we get $\Delta(\mathcal{L}(Q)) \prec_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{M}\mathbin{\bar{\otimes}}\mathcal{L}(\mathrm{d}(Q))$. Now we show the "moreover" part. So in particular the above intertwining shows that we can assume from the beginning that $\mathcal{C} = \mathcal{C}_1 \vee \mathcal{C}_2 \subset t(\mathcal{M}\mathbin{\bar{\otimes}}\mathcal{L}(\mathrm{d}(Q)))t$. Since the Q_i are biexact, weakly amenable, by applying [Popa and Vaes 2014, Theorem 1.4] we must have that either $\mathcal{C}_1 \prec \mathcal{M}\mathbin{\bar{\otimes}}\mathcal{L}(\mathrm{d}(Q_1))$ or $\mathcal{C}_2 \prec \mathcal{M}\mathbin{\bar{\otimes}}\mathcal{L}(\mathrm{d}(Q_1))$ or $\mathcal{C}_1 \vee \mathcal{C}_2$ is amenable relative to $\mathcal{M}\mathbin{\bar{\otimes}}\mathcal{L}(\mathrm{d}(Q_1))$ inside $\mathcal{M}\mathbin{\bar{\otimes}}\mathcal{M}$. However since $\mathcal{C}_1 \vee \mathcal{C}_2$ has property (T) the last case above still gives that $\mathcal{C}_1 \vee \mathcal{C}_2 \prec \mathcal{M}\mathbin{\bar{\otimes}}\mathcal{L}(\mathrm{d}(Q_1))$, which completes the proof. □

Theorem 5.6. *Let Γ be a group as in Notation 5.4 and assume that Λ is a group such that $\mathcal{L}(\Gamma) = \mathcal{L}(\Lambda) = \mathcal{M}$. Let $\Delta : \mathcal{M} \to \mathcal{M}\mathbin{\bar{\otimes}}\mathcal{M}$ be the "comultiplication along Λ" as in Notation 5.4. Also assume for every $j \in 1, 2$ there is $i \in 1, 2$ such that either $\Delta(\mathcal{L}(Q_i)) \prec_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{M}\mathbin{\bar{\otimes}}\mathcal{L}(Q_j)$ or $\Delta(\mathcal{L}(Q_i)) \prec_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{M}\mathbin{\bar{\otimes}}\mathcal{L}(N_j)$. Then one can find subgroups $\Phi_1, \Phi_2 \leqslant \Phi \leqslant \Lambda$ such that*:

(1) *Φ_1, Φ_2 are infinite, commuting, property (T), finite-by-icc groups.*

(2) *$[\Phi : \Phi_1\Phi_2] < \infty$ and $\mathrm{QN}^{(1)}_\Lambda(\Phi) = \Phi$.*

(3) *There exist $\mu \in \mathscr{U}(\mathcal{M})$, $z \in \mathscr{P}(\mathscr{Z}(\mathcal{L}(\Phi)))$, $h = \mu z \mu^* \in \mathscr{P}(\mathcal{L}(Q))$ such that*

$$\mu\mathcal{L}(\Phi)z\mu^* = h\mathcal{L}(Q)h. \tag{5.15}$$

Proof. For the proof we use an approach based upon the methods developed in [Chifan et al. 2016b; Chifan and Ioana 2018; Chifan and Udrea 2020]. For the reader's convenience we include all the details.

Since the relative commutants $\mathcal{L}(Q_j)' \cap \mathcal{M}$ and $\mathcal{L}(N_j)' \cap \mathcal{M}$ are nonamenable, in both cases using [Drimbe et al. 2019, Theorem 4.1] (see also [Ioana 2011, Theorem 3.1; Chifan et al. 2016b, Theorem 3.3]), one can find a subgroup $\Sigma < \Lambda$ with $C_\Lambda(\Sigma)$ nonamenable such that $\mathcal{L}(Q_1) \prec_{\mathcal{M}} \mathcal{L}(\Sigma)$. Thus there are $0 \neq p \in \mathscr{P}(\mathcal{L}(Q_1))$, $0 \neq f \in \mathscr{P}(\mathcal{L}(\Sigma))$, a partial isometry $0 \neq v \in f\mathcal{M}p$ and a $*$-isomorphism onto its image $\phi : p\mathcal{L}(Q_1)p \to \mathcal{B} := \phi(p\mathcal{L}(Q_1)p) \subseteq f\mathcal{L}(\Sigma)f$ so that

$$\phi(x)v = vx \quad \text{for all } x \in p\mathcal{L}(Q_1)p. \tag{5.16}$$

Notice that $vv^* \in \mathcal{B}' \cap f\mathcal{M}f$ and $v^*v \in (p\mathcal{L}(Q_1)p)' \cap p\mathcal{M}p = \mathcal{L}(Q_2)p$. Then (5.16) implies that $\mathcal{B}vv^* = v\mathcal{L}(Q_1)v^* = u_1\mathcal{L}(Q_1)v^*vu_1^*$, where $u_1 \in \mathscr{U}(\mathcal{M})$ extends v. Passing to relative commutants we get $vv^*(\mathcal{B}' \cap f\mathcal{M}f)vv^* = u_1v^*v((p\mathcal{L}(Q_1)p)' \cap p\mathcal{M}p)v^*vu_1^* = u_1v^*v(p\mathcal{L}(Q_2))v^*vu_1^*$. These relations further imply $vv^*(\mathcal{B}\vee\mathcal{B}'\cap f\mathcal{M}f)vv^*=\mathcal{B}vv^*\vee vv^*(\mathcal{B}'\cap f\mathcal{M}f)vv^*\subseteq u_1\mathcal{L}(Q)u_1^*$. As $\mathcal{L}(Q)$ is a factor, there is a new $u_2\in\mathscr{U}(\mathcal{M})$, with

$$(\mathcal{B}\vee\mathcal{B}'\cap f\mathcal{M}f)z_2\subseteq u_2\mathcal{L}(Q)u_2^*. \tag{5.17}$$

Here z_2 is the central support of vv^* in $\mathcal{B}\vee\mathcal{B}'\cap f\mathcal{M}f$ and hence $z_2\in\mathscr{Z}(\mathcal{B}'\cap f\mathcal{M}f)$ and $vv^*\leqslant z_2\leqslant f$.

Let $\Omega=C_\Lambda(\Sigma)$ and notice that $\mathcal{L}(\Omega)z_2\subseteq((fL(\Sigma)f)'\cap f\mathcal{M}f)z_2\subseteq(\mathcal{B}'\cap f\mathcal{M}f)z_2\subseteq u_2\mathcal{L}(Q)u_2^*$. Since Q is malnormal in Γ and $z_2\in(L(\Omega)f)'\cap f\mathcal{M}f$, we further have $z_2(\mathcal{L}(\Omega)f\vee((\mathcal{L}(\Omega)f)'\cap f\mathcal{M}f))z_2\subseteq u_2\mathcal{L}(Q)u_2^*$. Again since $\mathcal{L}(Q)$ is a factor, there is $\eta\in\mathscr{U}(\mathcal{M})$ so that

$$(\mathcal{L}(\Omega)f\vee((\mathcal{L}(\Omega)f)'\cap f\mathcal{M}f))z\subseteq\eta^*\mathcal{L}(Q)\eta, \tag{5.18}$$

where z is the central support of z_2 in $\mathcal{L}(\Omega)f\vee((\mathcal{L}(\Omega)f)'\cap f\mathcal{M}f)$. In particular, we have $vv^*\leqslant z_2\leqslant z\leqslant f$. Now since $f\mathcal{L}(\Sigma)f\subseteq(\mathcal{L}(\Omega)f)'\cap f\mathcal{M}f$, by (5.18) we get $(f\mathcal{L}(\Sigma)f\vee\mathcal{L}(\Omega)f)z\subseteq\eta^*\mathcal{L}(Q)\eta$ and hence

$$\eta(\mathcal{L}(\Omega)f\vee f\mathcal{L}(\Sigma)f)z\eta^*\subseteq\mathcal{L}(Q). \tag{5.19}$$

Since $vv^*\leqslant z\in(f\mathcal{L}(\Sigma)f)'\cap f\mathcal{M}f$ and $\mathcal{B}$ is a factor, the map $\phi':p\mathcal{L}(Q)p\to\eta\mathcal{B}z\eta^*\subseteq f\mathcal{L}(\Sigma)fz$ given by $\phi'(x)=\eta\phi(x)z\eta^*$ still defines a $*$-isomorphism that satisfies $\phi'(x)y=yx$ for any $x\in p\mathcal{L}(Q_1)p$, where $0\neq y=\eta zv$ is a partial isometry. Hence, $\mathcal{L}(Q_1)\prec_{\mathcal{M}}u^*f\mathcal{L}(\Sigma)fzu$. Since Q is malnormal in Γ, it follows that $\mathcal{L}(Q_1)\prec_{\mathcal{L}(Q)}\eta f\mathcal{L}(\Sigma)fz\eta^*$.

To this end, using [Chifan et al. 2016a, Proposition 2.4] and its proof, there are $0\neq a\in\mathscr{P}(\mathcal{L}(Q_1))$, $0\neq r=\eta qz\eta^*\in\eta f\mathcal{L}(\Sigma)fz\eta^*$, with $q\in\mathscr{P}(f\mathcal{L}(\Sigma)f)$, and a $*$-isomorphism onto its image ψ : $a\mathcal{L}(Q_1)a\to\mathcal{D}:=\psi(a\mathcal{L}(Q_1)a)\subseteq\eta q\mathcal{L}(\Sigma)qz\eta^*$ satisfying the following properties:

(4) The inclusion $\mathcal{D}\vee(\mathcal{D}'\cap\eta q\mathcal{L}(\Sigma)qz\eta^*)\subseteq\eta q\mathcal{L}(\Sigma)qz\eta^*$ has finite index.

(5) There is a partial isometry $0\neq w\in\mathcal{L}(Q)$ such that $\psi(x)w=wx$ for all $x\in a\mathcal{L}(Q_1)a$.

Now observe the algebras $\mathcal{D}$, $\mathcal{D}'\cap\eta q\mathcal{L}(\Sigma)qz\eta^*$ and $\eta\mathcal{L}(\Omega)qz\eta^*$ are mutually commuting. Also the prior relations show that $\mathcal{D}$ and $\eta\mathcal{L}(\Omega)qz\eta^*$ have no amenable direct summand. Since Q_1 and Q_2 are biexact, it follows that $\mathcal{D}'\cap\eta q\mathcal{L}(\Sigma)qz\eta^*$ must be purely atomic. Therefore, one can find $0\neq e\in\mathscr{P}(\mathscr{Z}(\mathcal{D}'\cap u^*q\mathcal{L}(\Omega)qzu))$ such that after cutting down by q the containment in (4) and replacing $\mathcal{D}$ by $\mathcal{D}e$ one can assume that

(4′) $\mathcal{D}\subseteq\eta q\mathcal{L}(\Sigma)qz\eta^*$ is a finite-index inclusion of nonamenable II_1-factors.

Moreover, replacing w by ew and $\psi(x)$ by $\psi(x)e$ in the intertwining in (5) still holds.

Notice that (5) implies $ww^*\in\mathcal{D}'\cap r\mathcal{L}(Q)r,\ w^*w\in a\mathcal{L}(Q_1)a'\cap a\mathcal{L}(Q)a=\mathbb{C}a\otimes\mathcal{L}(Q_2)$. Thus there exists $0\neq b\in\mathscr{P}(\mathcal{L}(Q_2))$ such that $w^*w=a\otimes b$. Pick $c\in\mathscr{U}(\mathcal{L}(Q))$ such that $w=c(a\otimes b)$. Then (5) gives

$$\mathcal{D}ww^*=w\mathcal{L}(Q_1)w^*=c(a\mathcal{L}(Q_1)a\otimes\mathbb{C}b)c^*. \tag{5.20}$$

Let $\Xi=QN_\Lambda(\Sigma)$. Then using (5.20) and (4′) above we see that

$$c(a\otimes b)\mathcal{L}(Q)(a\otimes b)c^*=ww^*\eta qz\mathscr{QN}_{\mathcal{L}(\Lambda)}(\mathcal{L}(\Sigma))''qz\eta^*ww^*=ww^*\eta qz\mathcal{L}(\Xi)qz\eta^*ww^* \tag{5.21}$$

and also

$$\begin{aligned} c(\mathbb{C}a\otimes b\mathcal{L}(Q_2)b)c^* &= (c(a\mathcal{L}(Q_1)a\otimes \mathbb{C}b)c^*)'\cap c(a\otimes b)\mathcal{L}(Q)(a\otimes b)c^* \\ &= (\mathcal{D}ww^*)'\cap ww^*\eta qz\mathcal{L}(\Xi)qz\eta^*ww^* \\ &= ww^*(\mathcal{D}'\cap \eta qz\mathcal{L}(\Xi)qz\eta^*)ww^*. \end{aligned} \tag{5.22}$$

Using (4′) and [Popa 2002, Lemma 3.1] we also have

$$\mathcal{D}\vee(\eta qz\mathcal{L}(\Sigma)zq\eta^*)'\cap \eta qz\mathcal{L}(\Xi)zq\eta^* \subseteq^f \mathcal{D}\vee\mathcal{D}'\cap \eta qz\mathcal{L}(\Xi)zq\eta^* \subseteq \eta qz\mathcal{L}(\Xi)zq\eta^*, \tag{5.23}$$

where the symbol $\subseteq^f$ above means inclusion of finite index.

Relation (5.20) also shows that

$$\begin{aligned} \mathcal{D}\vee(\eta qz\mathcal{L}(\Sigma)zq\eta^*)'\cap \eta qz\mathcal{L}(\Xi)zq\eta^* &\subseteq^f \eta qz\mathcal{L}(\Sigma)zq\eta^*\vee(\eta qz\mathcal{L}(\Sigma)zq\eta^*)'\cap \eta qz\mathcal{L}(\Xi)zq\eta^* \\ &\subseteq \eta qz\mathcal{L}(\Sigma(vC_\Lambda(\Sigma)))zq\eta^* \subseteq \eta qz\mathcal{L}(\Xi)zq\eta^*. \end{aligned} \tag{5.24}$$

Here $vC_\Lambda(\Sigma)=\{\lambda\in\Lambda : |\lambda^\Sigma|<\infty\}$ is the virtual centralizer of Σ in Λ.

Let $\Phi = QN_\Lambda^{(1)}(\Xi)$. Using (5.21) and the fact that Q is malnormal in Γ, the same argument from [Chifan and Udrea 2020, Claim 5.2, page 26, lines 1–10] shows that $\Xi\leqslant\Phi$ has finite index.

Combining (5.22), (5.20) (5.21) we notice that

$$ww^*(\mathcal{D}\vee\mathcal{D}'\cap \eta qz\mathcal{L}(\Xi)zq\eta^*)ww^* = ww^*\eta qz\mathcal{L}(\Xi)zq\eta^*ww^* = ww^*\eta qz\mathcal{L}(\Phi)zq\eta^*ww^*. \tag{5.25}$$

In particular, (5.25) shows that $\eta qz\mathcal{L}(\Xi)zq\eta^* \prec_{\eta qz\mathcal{L}(\Xi)zq\eta^*} \mathcal{D}\vee\mathcal{D}'\cap \eta qz\mathcal{L}(\Xi)zq\eta^*$ and using the finite-index condition in (5.23) we get $\eta qz\mathcal{L}(\Xi)zq\eta^* \prec_{\eta qz\mathcal{L}(\Xi)zq\eta^*} \mathcal{D}\vee(\eta qz\mathcal{L}(\Sigma)zq\eta^*)'\cap \eta qz\mathcal{L}(\Xi)zq\eta^*$. Thus, by (5.24) we further have $\eta qz\mathcal{L}(\Xi)zq\eta^* \prec_{\eta qz\mathcal{L}(\Xi)zq\eta^*} \eta qz\mathcal{L}(\Sigma(vC_\Lambda(\Sigma)))zq\eta^*$ and since $\Sigma(vC_\Lambda(\Sigma))\leqslant\Phi$ and $[\Phi:\Xi]<\infty$, using [Chifan and Ioana 2018, Lemma 2.6] we get $[\Phi:\Sigma(vC_\Lambda(\Sigma))]<\infty$.

Relation (5.21) also shows that

$$c(a\otimes b)\mathcal{L}(Q)(a\otimes b)c^* = ww^*\eta qz\mathcal{L}(\Xi)zq\eta^*ww^* = ww^*\eta qz\mathcal{L}(\Phi)zq\eta^*ww^*. \tag{5.26}$$

As Q has property (T), by [Chifan and Ioana 2018, Lemma 2.13] so do Φ and Ξ, and hence $\Sigma vC_\Lambda(\Sigma)$ as well. Let $\{\mathcal{O}_n\}_n$ be an enumeration of all the orbits in Λ under conjugation by Σ. Define $\Omega_n := \langle \mathcal{O}_1,\dots,\mathcal{O}_n\rangle$. Clearly $\Omega_n\leqslant\Omega_{n+1}$ and Σ normalizes Ω_n for all n. Notice that $\Omega_n\Sigma\leqslant\Omega_{n+1}\Sigma$ for all n and in fact $\Omega_n\Sigma\nearrow\Sigma(vC_\Lambda(\Sigma))$. Since $\Sigma(vC_\Lambda(\Sigma))$ has property (T), there exists n_0 such that $\Omega_{n_0}\Sigma=\Sigma(vC_\Lambda(\Sigma))$. In particular, there is a finite-index subgroup $\Sigma'\leqslant\Sigma$ such that $[\Sigma',\Omega_{n_0}]=1$, and hence $\Sigma',\Omega_{n_0}\leqslant^f\Sigma(vC_\Lambda(\Sigma))\leqslant^f\Phi$ are commuting subgroups. Moreover if r_1 is the central support of ww^* in $\eta zL(\Phi)qz\eta^*$ then by (5.26) we also have $\eta_0\mathcal{L}(Q)\eta_0^*\supseteq\eta qz\mathcal{L}(\Xi)qz\eta^*r_1$ for some unitary η_0. Now since the Q_i's are biexact, the same argument from [Chifan et al. 2016b] shows that the finite conjugacy radical of Φ is finite. Hence Φ is a finite-by-icc group and this canonically implies that $\Phi_1:=\Sigma'$ and $\Phi_2:=\Omega_{l_0}$ are also finite-by-icc. As Φ has property (T), so do the Φ_i's. Altogether, the above arguments and (5.26) show that there exist subgroups $\Phi_1,\Phi_2\leqslant\Phi<\Lambda$ satisfying the following properties:

(1) Φ_1,Φ_2 are infinite, commuting, property (T), finite-by-icc groups.

(2) $[\Phi:\Phi_1\Phi_2]<\infty$ and $QN_\Lambda^{(1)}(\Phi)=\Phi$.

(3) There exist $\mu \in \mathscr{U}(\mathcal{M}),\ d \in \mathscr{P}(\mathcal{L}(\Phi)),\ h = \mu d\mu^* \in \mathscr{P}(\mathcal{L}(Q))$ such that

$$\mu d\mathcal{L}(\Phi)d\mu^* = h\mathcal{L}(Q)h. \tag{5.27}$$

In the last part of the proof we show that after replacing d with its central support in $\mathcal{L}(Q)$, all the required relations in the statement still hold. Since $\mathcal{L}(Q)$ is a factor, using (5.27) one can find $\xi \in \mathscr{U}(\mathcal{M})$ such that $\xi\mathcal{L}(\Phi)t\xi^* \subseteq \mathcal{L}(Q)$, where t is the central support of d in $\mathcal{L}(Q)$. Hence $\xi\mathcal{L}(\Phi)t\xi^* \subseteq r_2\mathcal{L}(Q)r_2$, where $r_2 = \xi t\xi^*$. Fix $e_o \leqslant t$ and $f_o \leqslant d$ projections in the factor $\mathcal{L}(\Phi)t$ such that $\tau(f_o) \geqslant \tau(e_o)$. From (5.27) we have $\mu f_o\mathcal{L}(\Phi)f_o\mu^* = l\mathcal{L}(Q)l$ and $\xi e_o\mathcal{L}(\Phi)e_o\xi^* \subseteq r_o\mathcal{L}(Q)r_o$, where $r_o = \xi e_o\xi^*$ and $l = \mu f_o\mu^*$. Let $\xi_o \in \mathcal{L}(Q)$ be a unitary such that $r_o \leqslant \xi_o l\xi_o^*$. Thus

$$\xi e_o\mathcal{L}(\Phi)e_o\xi^* \subseteq r_o\mathcal{L}(Q)r_o \subseteq \xi_o l\mathcal{L}(Q)l\xi_o^* = \xi_o\mu f_o\mathcal{L}(\Phi)f_o\mu^*\xi_o^*$$

and hence

$$\mu^*\xi_o^*\xi e_o\mathcal{L}(\Phi)e_o \subseteq f_o\mathcal{L}(\Phi)f_o\mu^*\xi_o^*\xi \subset \mathcal{L}(\Phi)\mu^*\xi_o^*\xi. \tag{5.28}$$

Next let $e_o + p_1 + p_2 + \cdots + p_s = t$, where $p_i \in \mathcal{L}(\Phi)t$ are mutually orthogonal projections such that e_o is von Neumann equivalent (in $\mathcal{L}(\Phi)t$) to p_i for all $i \in \overline{1, s-1}$ and p_s is von Neumann subequivalent to e_o. Now let u_i be unitaries in $\mathcal{L}(\Phi)t$ such that $u_ip_iu_i^* = e_o$ for all $i \in \overline{1, s-1}$ and $u_sp_su_s^* = z_o' \leqslant e_o$. Combining this with (5.28) we get

$$\mu^*\xi_o^*\xi e_o\mathcal{L}(\Phi)p_i = \mu^*\xi_o^*\xi e_o\mathcal{L}(\Phi)u_i^*e_ou_i = \mu^*\xi_o^*\xi e_o\mathcal{L}(\Phi)e_ou_i \subseteq \mathcal{L}(\Phi)\mu^*\xi_o^*\xi u_i$$

for all $i \in \overline{1, s-1}$. Similarly, we get

$$\mu^*\xi_o^*\xi e_o\mathcal{L}(\Phi)p_s = \mu^*\xi_o^*\xi e_o\mathcal{L}(\Phi)u_s^*z_o'u_s = \mu^*\xi_o^*\xi e_o\mathcal{L}(\Phi)z_o'u_s \subseteq \mu^*\xi_o^*\xi e_o\mathcal{L}(\Phi)e_ou_s \subset \mathcal{L}(\Phi)\mu^*\xi_o^*\xi u_s.$$

Using these relations we conclude that

$$\begin{aligned}\mu^*\xi_o^*\xi e_o\mathcal{L}(\Phi) &= \mu^*\xi_o^*\xi e_o\mathcal{L}(\Phi)t = \mu^*\xi_o^*\xi e_o\mathcal{L}(\Phi)\left(e_o + \sum_{i=1}^{s} p_i\right)\\ &\subseteq \mu^*\xi_o^*\xi e_o\mathcal{L}(\Phi)e_o + \sum_{i=1}^{s}\mu^*\xi_o^*\xi e_o\mathcal{L}(\Phi)p_i \subseteq \mathcal{L}(\Phi)\mu^*\xi_o^*\xi + \sum_{i=1}^{s}\mathcal{L}(\Phi)\mu^*\xi_o^*\xi u_i.\end{aligned}$$

In particular, this relation shows that $\mu^*\xi_o^*\xi e_o \in \mathscr{QN}^{(1)}_{\mathcal{L}(\Lambda)}(\mathcal{L}(\Phi))$ and since $\mathscr{QN}^{(1)}_{\mathcal{L}(\Lambda)}(\mathcal{L}(\Phi))'' = \mathcal{L}(\Phi)$ by (2), we conclude that $\mu^*\xi_o^*\xi e_o \in \mathcal{L}(\Phi)$. Thus using this together with (5.28) one can check that

$$\begin{aligned}\xi e_o\mathcal{L}(\Phi)e_o\xi^* &= \xi e_o\xi^*\xi_o\mu(\mu^*\xi_o^*\xi e_o\mathcal{L}(\Phi)e_o\xi^*\xi_o\mu)\mu^*\xi_o^*\xi e_o\xi^*\\ &= \xi e\xi^*\xi_o\mu f_o\mathcal{L}(\Phi)f_o\mu^*\xi_o^*\xi e\xi^*\\ &= \xi e_o\xi^*\xi_o l\mathcal{L}(Q)l\xi_o^*\xi e_o\xi^* = r_o\mathcal{L}(Q)r_o.\end{aligned}$$

In conclusion we have proved that $\xi\mathcal{L}(\Phi)t\xi^* \subseteq r_2\mathcal{L}(Q)r_2$ and for all $e_o \leqslant t$ and $f_o \leqslant d$ projections in the factor $\mathcal{L}(\Phi)t$ such that $\tau(f_o) \geqslant \tau(e_o)$ we have $\xi e_o\mathcal{L}(\Phi)e_o\xi^* = r_o\mathcal{L}(Q)r_o$, where $r_o \leqslant r_2 = \xi t\xi^*$. By Lemma 2.9 this clearly implies $\xi\mathcal{L}(\Phi)t\xi^* = r_2\mathcal{L}(Q)r_2$, which finishes the proof. □

Lemma 5.7. *Let Γ be a group as in Notation 5.4 and assume that Λ is a group such that $\mathcal{L}(\Gamma) = \mathcal{L}(\Lambda) = \mathcal{M}$. Also assume there exists a subgroup $\Phi < \Lambda$, a unitary $\mu \in \mathscr{U}(\mathcal{M})$ and projections $z \in \mathscr{Z}(\mathcal{L}(\Phi))$,*

$r = \mu z \mu^* \in \mathcal{L}(Q)$ such that

$$\mu\mathcal{L}(\Phi)z\mu^* = r\mathcal{L}(Q)r. \tag{5.29}$$

For every $\lambda \in \Lambda \setminus \Phi$ so that $|\Phi \cap \Phi^{\lambda}| = \infty$ we have $zu_\lambda z = 0$. In particular, there is $\lambda_o \in \Lambda \setminus \Phi$ so that $|\Phi \cap \Phi^{\lambda_o}| < \infty$.

Proof. Notice that since $Q < \Gamma = (N_1 \times N_2) \rtimes Q$ is almost malnormal, we have the following property: for every sequence $\mathcal{L}(Q) \ni x_n \to 0$ weakly and every $x, y \in M$ such that $E_{\mathcal{L}(Q)}(x) = E_{\mathcal{L}(Q)}(y) = 0$ we have

$$\|E_{\mathcal{L}(Q)}(xx_k y)\|_2 \to 0 \quad \text{as } k \to \infty. \tag{5.30}$$

Using basic approximations and the $\mathcal{L}(Q)$-bimodularity of the expectation we see that it suffices to check (5.30) only for elements of the form $x = u_n$ and $y = u_m$, where $n, m \in (N_1 \times N_2) \setminus \{1\}$. Consider the Fourier decomposition $x_n = \sum_{h\in Q} \tau(x_k u_{h^{-1}})u_h$ and notice that

$$\begin{aligned}\|E_{\mathcal{L}(Q)}(xx_k y)\|_2^2 &= \Big\|\sum_{h\in Q} \tau(x_k u_{h^{-1}})\delta_{nhm,Q}u_{nhm}\Big\|_2^2 \\ &= \Big\|\sum_{h\in Q} \tau(x_k u_{h^{-1}})\delta_{n\sigma_h(m)h,Q}u_{n\sigma_h(m)h}\Big\|_2^2 = \sum_{h\in Q,\sigma_h(m)=n^{-1}} |\tau(x_k u_{h^{-1}})|^2. \end{aligned}\tag{5.31}$$

Since the action $Q \curvearrowright N_i$ has finite stabilizers one can easily see that the set $\{h \in Q : \sigma_h(m) = n^{-1}\}$ is finite and since $x_n \to 0$ weakly, $\sum_{h\in Q,\sigma_h(m)=n^{-1}} |\tau(x_k u_{h^{-1}})|^2 \to 0$ as $k \to \infty$, which concludes the proof of (5.30). Using the conditional expectation formula for compression we see that (5.30) implies that for every sequence $\mathcal{L}(Q) \ni x_n \to 0$ weakly and every $x, y \in r\mathcal{M}r$ so that $E_{r\mathcal{L}(Q)r}(x) = E_{r\mathcal{L}(Q)r}(y) = 0$ we have $\|E_{r\mathcal{L}(Q)r}(xx_k y)\|_2 \to 0$ as $k \to \infty$. Thus using the formula (5.29) we get that for all $\mu\mathcal{L}(\Phi)z\mu^* \ni x_n \to 0$ weakly and every $x, y \in \mu z\mathcal{M}z\mu^*$ so that $E_{\mu\mathcal{L}(\Phi)z\mu^*}(x) = E_{\mu\mathcal{L}(\Phi)z\mu^*}(y) = 0$ we have $\|E_{\mu\mathcal{L}(\Phi)z\mu^*}(xx_k y)\|_2 \to 0$ as $k \to \infty$. This gives that for all $\mathcal{L}(\Phi)z \ni x_n \to 0$ weakly and every $x, y \in z\mathcal{M}z$ satisfying $E_{\mathcal{L}(\Phi)z}(x) = E_{\mathcal{L}(\Phi)z}(y) = 0$ we have

$$\|E_{\mathcal{L}(\Phi)z}(xx_k y)\|_2 \to 0 \quad \text{as } k \to \infty. \tag{5.32}$$

Fix $\lambda \in \Lambda \setminus \Phi$ so that $|\Phi \cap \Phi^{\lambda}| = \infty$. Hence there are infinite sequences $\lambda_k, \omega_n \in \Lambda$ so that $\lambda\omega_k\lambda^{-1} = \lambda_k$ for all integers k. Since $\lambda \in \Lambda \setminus \Phi$, we have $E_{\mathcal{L}(\Phi)}(u_\lambda z) = E_{\mathcal{L}(\Phi)z}(zu_{\lambda^{-1}}) = 0$. Also we have $u_{\omega_k}z \to 0$ weakly as $k \to \infty$. Using these calculations,

$$\begin{aligned}\|E_{\mathcal{L}(\Phi)}(zu_\lambda zu_{\lambda^{-1}}z)\|_2^2 &= \|E_{\mathcal{L}(\Phi)}(u_\lambda zu_{\lambda^{-1}}z)\|_2^2 = \|u_{\lambda\omega_k\lambda^{-1}}E_{\mathcal{L}(\Phi)}(u_\lambda zu_{\lambda^{-1}}z)\|_2^2 \\ &= \|E_{\mathcal{L}(\Phi)}(u_{\lambda\omega_k}zu_{\lambda^{-1}}z)\|_2^2 = \|E_{\mathcal{L}(\Phi)z}(zu_\lambda zu_{\omega_k}zu_{\lambda^{-1}}z)\|_2^2 \to 0 \quad \text{as } k \to \infty. \end{aligned}\tag{5.33}$$

Also using (5.33) the last quantity above converges to 0 as $k \to \infty$ and hence $E_{\mathcal{L}(\Phi)}(zu_\lambda zu_{\lambda^{-1}}z) = 0$, which gives that $zu_\lambda z = 0$, as desired. For the remaining part notice first that since $[\Gamma : Q] = \infty$, (5.29) implies that $[\Lambda : \Phi] = \infty$. Assume by contradiction that for all $\lambda \in \Lambda \setminus \Phi$ we have $zu_\lambda z = 0$. As $[\Lambda : \Phi] = \infty$, for every positive integer l one can construct inductively $\lambda_i \in \Lambda \setminus \Phi$ with $i \in \overline{1,l}$ such that $\lambda_i\lambda_j^{-1} \in \Lambda \setminus \Phi$ for all $i > j$ such that $i, j \in \overline{1,l}$. But this implies $0 = zu_{\lambda_i\lambda_j^{-1}}z = zu_{\lambda_i}u_{\lambda_j^{-1}}z$ and hence $u_{\lambda_i^{-1}}zu_{\lambda_i}$ are mutually orthogonal projections when $i = \overline{1,l}$. This is obviously false when l is sufficiently large. $\square$

Theorem 5.8. *Assume the same conditions as in Theorem 5.6. Then one can find subgroups* $\Phi_1, \Phi_2 \leqslant \Phi \leqslant \Lambda$ *so that*

(1) Φ_1, Φ_2 *are infinite, icc, property (T) groups so that* $\Phi = \Phi_1 \times \Phi_2$.

(2) $\mathrm{QN}_\Lambda^{(1)}(\Phi) = \Phi$.

(3) *There exists* $\mu \in \mathscr{U}(\mathcal{M})$ *such that* $\mu\mathcal{L}(\Phi)\mu^* = \mathcal{L}(Q)$.

Proof. From Theorem 5.6 there exist subgroups $\Phi_1, \Phi_2 \leqslant \Phi \leqslant \Lambda$ such that:

(1) Φ_1, Φ_2 are, infinite, commuting, finite-by-icc, property (T) groups so that $[\Phi : \Phi_1\Phi_2] < \infty$.

(2) $\mathrm{QN}_\Lambda^{(1)}(\Phi) = \Phi$.

(3) There exist $\mu \in \mathscr{U}(\mathcal{M})$ and $z \in \mathscr{P}(\mathscr{Z}(\mathcal{L}(\Phi)))$ with $h = \mu z\mu^* \in \mathscr{P}(\mathcal{L}(Q))$ satisfying

$$\mu\mathcal{L}(\Phi)z\mu^* = h\mathcal{L}(Q)h. \tag{5.34}$$

Next we show that in (5.34) we can pick $z \in \mathscr{Z}(\mathcal{L}(\Phi))$ maximal with the property that for every projection $t \in \mathscr{Z}(\mathcal{L}(\Phi)z^\perp)$ we have

$$L(\Phi_i)t \nprec_{\mathcal{M}} \mathcal{L}(Q) \quad \text{for } i = 1, 2. \tag{5.35}$$

To see this let $z \in \mathcal{F}$ be a maximal family of mutually orthogonal (minimal) projections $z_i \in \mathscr{Z}(\mathcal{L}(\Phi))$ such that $\mathcal{L}(\Phi)z_i \prec_{\mathcal{M}} \mathcal{L}(Q)$. Note that since Φ has finite conjugacy radical it follows that $\mathcal{F}$ is actually finite. Next let $z \leqslant \sum z_i := a \in \mathscr{Z}(\mathcal{L}(\Phi))$ and we briefly argue that $\mathcal{L}(\Phi)a \prec^s_{\mathcal{M}} \mathcal{L}(Q)$. Indeed since $(\mathcal{L}(\Phi)a)' \cap a\mathcal{M}a = a(\mathcal{L}(\Phi)' \cap \mathcal{M})a = \mathscr{Z}(\mathcal{L}(\Phi))a$ and the latter is finite-dimensional, for every $r \in (\mathcal{L}(\Phi)a)' \cap a\mathcal{M}a$ there is $z_i \in \mathcal{F}$ such that $rz_i = z_i \neq 0$. Since $\mathcal{L}(\Phi)z_i \prec_{\mathcal{M}} \mathcal{L}(Q)$, we have $\mathcal{L}(\Phi)r \prec_{\mathcal{M}} \mathcal{L}(Q)$, as desired. Thus applying Lemma 2.7, after perturbing μ to a new unitary, we get $\mu\mathcal{L}(\Phi)a\mu^* = h_o\mathcal{L}(Q)h_o$. Finally, we show (5.35). Assume by contradiction there is $t_o \in \mathscr{Z}(\mathcal{L}(\Phi)z^\perp)$ so that $\mathcal{L}(\Phi_i)t_o \prec_{\mathcal{M}} \mathcal{L}(Q)$ for some $i = 1, 2$. Thus there exist projections $r \in \mathcal{L}(\Phi)t_o$, $q \in \mathcal{L}(Q)$, a partial isometry $w \in \mathcal{M}$ and a $*$-isomorphism on the image $\phi : r\mathcal{L}(\Phi)r \to \mathcal{B} := \phi(r\mathcal{L}(\Phi)r) \subseteq q\mathcal{L}(Q)q$ such that $\phi(x)w = wx$. Notice that $w^*w \in t_o(\mathcal{L}(\Phi_i)' \cap \mathcal{M})t_o$ and $ww^* \in \mathcal{B}' \cap q\mathcal{M}q$. But since $Q < \Gamma$ is malnormal, it follows that $\mathcal{B}' \cap q\mathcal{M}q \subseteq q\mathcal{L}(Q)q$ and hence $ww^* \in q\mathcal{L}(Q)q$. Using this in combination with previous relations we get $wr\mathcal{L}(\Phi_i)rw^* = \mathcal{B}ww^* \subseteq \mathcal{L}(Q)$ and extending w to a unitary u we have $ur\mathcal{L}(\Phi_i)ru^* \subseteq \mathcal{L}(Q)$. Since $\mathcal{L}(Q)$ is a factor, we can further perturb the unitary u so that $u\mathcal{L}(\Phi_i)r_ou^* \subseteq \mathcal{L}(Q)$, where $r \leqslant r_o \leqslant t_o$ is the central support of r in $\mathcal{L}(\Phi_i)t_o$. Using malnormality of Q again we further get $r_o(\mathcal{L}(\Phi_i) \vee \mathcal{L}(\Phi_i)' \cap \mathcal{M})r_ou^* \subseteq \mathcal{L}(Q)$ and perturbing u we can further assume that $(\mathcal{L}(\Phi_i) \vee \mathcal{L}(\Phi_i)' \cap \mathcal{M})s_ou^* \subseteq \mathcal{L}(Q)$ where $r_o \leqslant s_o$ is the central support of r_o in $\mathcal{L}(\Phi_i) \vee \mathcal{L}(\Phi_i)' \cap \mathcal{M}$. In particular, $u(\mathcal{L}(\Phi)s_ou^* \subseteq \mathcal{L}(Q)$ and hence $\mathcal{L}(\Phi)s_o \subseteq u^*\mathcal{L}(Q)u$. Since $r \leqslant r_o \leqslant s_o$ and $r \leqslant t_o$, the previous containment implies that there is a minimal projection $s' \in \mathcal{L}(\Phi)a^\perp$ so that $\mathcal{L}(\Phi)s' \prec \mathcal{L}(Q)$, which contradicts the maximality assumption on $\mathcal{F}$. Finally replacing z with a in our statement, our claim follows.

Next fix $t \in \mathscr{Z}(\mathcal{L}(\Phi)z^\perp)$. Since $\mathcal{L}(\Phi_1)t$ and $\mathcal{L}(\Phi_2)t$ are commuting property (T) von Neumann algebras, using the same arguments as in the first part of the proof of Theorem 5.5 there are two possibilities: either (i) there exists $j \in 1, 2$ such that $\mathcal{L}(\Phi_j)t \prec_{\mathcal{M}} \mathcal{L}(N_2)$ or (ii) $\mathcal{L}(\Phi)t \prec_{\mathcal{M}} \mathcal{L}(N_2 \rtimes Q)$. Next we briefly argue

(ii) is impossible. Indeed, assuming (ii), Theorem 5.2 for $n = 1$ would imply the existence of $j \in 1, 2$ so that $\mathcal{L}(\Phi_j)t \prec_{\mathcal{M}} \mathcal{L}(Q)$, which obviously contradicts the choice of z. Thus we have (i), and passing to the relative commutants we have $\mathcal{L}(N_1) \prec \mathcal{L}(\Phi_j)t' \cap t\mathcal{M}t = t(\mathcal{L}(\Phi_j)' \cap \mathcal{M})t$. Using the relationships between the Φ_j's we see that $t(\mathcal{L}(\Phi_j)' \cap \mathcal{M})t \subset t\mathcal{L}(\Phi_j) \vee \mathcal{L}(\Phi_j)' \cap \mathcal{M})t \subseteq t\mathcal{L}(\Phi_j(vC_\Lambda(\Phi_j)))t \subseteq t\mathcal{L}(\Phi)t$. In conclusion, we have

$$\mathcal{L}(N_1) \prec_{\mathcal{M}} t\mathcal{L}(\Phi)t \quad \text{for all } t \in \mathscr{P}(\mathcal{L}(\Phi)z^{\perp}). \tag{5.36}$$

Let $A = \{\lambda \in \Lambda : |\Phi \cap \Phi^{\lambda}| < \infty\}$ and $B = \{\lambda \in \Lambda : |\Phi \cap \Phi^{\lambda}| = \infty\}$. Note that $A \cup B = \Lambda$ and $A \neq \varnothing$. Since N_1 is infinite, for every $\lambda \in A$ we have $\mathcal{L}(N_1) \nprec_{\mathcal{M}} \mathcal{L}(\Phi \cap \Phi^{\lambda})z^{\perp}$. Thus using (5.36) together with the same argument from the proof of [Popa and Vaes 2008, Theorem 6.16], working under $z^{\perp}$, we get $z^{\perp}E_{\mathcal{L}(\Phi)}(u_\lambda z^{\perp} x z^{\perp}) = 0$ for all $x \in \mathcal{M}$. This further implies $z^{\perp}u_\lambda z^{\perp} = 0$ for all $\lambda \in A$ and hence $u_\lambda z^{\perp} u_{\lambda^{-1}} \leqslant z$.

On the other hand by Lemma 5.7 for all $\lambda \in B$ we get $zu_\lambda z = 0$, and hence $u_\lambda z u_{\lambda^{-1}} \leqslant z^{\perp}$. So if $B \neq \varnothing$, we obviously have equality in the previous two relations, i.e., $u_\lambda z u_{\lambda^{-1}} = z^{\perp}$ for all $\lambda \in B$ and $u_\lambda z^{\perp} u_{\lambda^{-1}} = z$ for all $\lambda \in A$. These further imply there exist $a_o \in A$ and $b_0 \in B$ such that $A = a_0 C_\Lambda(z^{\perp})$ and $B = b_o C_\Lambda(z)$; here $C_\Lambda(z) \leqslant \Lambda$ is the subgroup of all elements of Λ that commute with z and similarly for $C_\Lambda(z^{\perp})$. Thus $\Lambda = A \cup B = a_o C_\Lambda(z^{\perp}) \cup b_o C_\Lambda(z)$. Thus we can assume, without loss of generality, that $[\Lambda : C_\Lambda(z)] < \infty$. But since Λ is icc this implies $z = 1$. The rest of the statement follows. □

Theorem 5.9. *In Theorem 5.5 we cannot have case* (4a).

Proof. Assume by contradiction that for all $j \in 1, 2$ there is $i \in 1, 2$ such that $\Delta(\mathcal{L}(Q_i)) \prec_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{M}\bar{\otimes}\mathcal{L}(N_j)$. Using [Drimbe et al. 2019, Theorem 4.1] and property (T) on N_j, one can find a subgroup $\Sigma < \Lambda$ such that $\mathcal{L}(Q_i) \prec_{\mathcal{M}} \mathcal{L}(\Sigma)$ and $\mathcal{L}(N_j) \prec_{\mathcal{M}} \mathcal{L}(C_\Lambda(\Sigma))$. Since $\mu\mathcal{L}(\Phi)\mu^* = \mathcal{L}(Q)$ and Q_i are biexact, by the product rigidity results in [Chifan et al. 2016b] one can assume that there is a unitary $u \in \mathcal{L}(Q)$ such that $u\mathcal{L}(Q_1)u^* = \mathcal{L}(\Phi_1)^t$ and $u\mathcal{L}(Q_2)u^* = \mathcal{L}(\Phi_2)^{1/t}$. Thus we get $\mathcal{L}(\Phi_i) \prec_{\mathcal{M}} \mathcal{L}(\Sigma)$, and hence $[\Phi_i : g\Sigma g^{-1} \cap \Phi_i] < \infty$. So working with $g\Sigma g^{-1}$ instead of Σ, we can assume that $[\Phi_i : \Sigma \cap \Phi_i] < \infty$. In particular $\Sigma \cap \Phi_i$ is infinite and since Φ is almost malnormal in Λ, it follows that $C_\Lambda(\Sigma \cap \Phi_i) < \Phi$. Thus we have $\mathcal{L}(N_j) \prec_{\mathcal{M}} \mathcal{L}(C_\Lambda(\Sigma)) \subseteq \mathcal{L}(C_\Lambda(\Sigma \cap \Phi_i)) \subset \mathcal{L}(\Phi) = \mu^*\mathcal{L}(Q)\mu$, which is obviously a contradiction. □

Theorem 5.10. *Let Γ be a group as in Notation 5.4 and assume that Λ is a group such that $\mathcal{L}(\Gamma) = \mathcal{L}(\Lambda) = \mathcal{M}$. Let $\Delta : \mathcal{M} \to \mathcal{M} \bar{\otimes} \mathcal{M}$ be the comultiplication along Λ as in Notation 5.4. Then the following hold*:

(i) $\Delta(\mathcal{L}(N_1)), \Delta(\mathcal{L}(N_2)), \Delta(\mathcal{L}(N_1 \times N_2)) \prec^{s}_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{L}(N_1 \times N_2) \bar{\otimes} \mathcal{L}(N_1 \times N_2)$.

(ii) *There is a unitary $u \in \mathcal{M} \bar{\otimes} \mathcal{M}$ such that $u\Delta(\mathcal{L}(Q))u^* \subseteq \mathcal{L}(Q) \bar{\otimes} \mathcal{L}(Q)$.*

Proof. First we show (i). From Theorem 5.5 we have that for all $j \in 1, 2$ there is $j_i \in 1, 2$ such that $\Delta(\mathcal{L}(N_{j_i})) \prec_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{M} \bar{\otimes} \mathcal{L}(N_j)$. Since $\mathscr{N}_{\mathcal{M}\bar{\otimes}\mathcal{M}}\Delta(\mathcal{L}(N_i))'' \supset \Delta(\mathcal{M})$ and $\Delta(\mathcal{M})' \cap \mathcal{M} \bar{\otimes} \mathcal{M} = \mathbb{C}1$, by [Drimbe et al. 2019, Lemma 2.4 (3)] we actually have $\Delta(\mathcal{L}(N_{j_i})) \prec^{s}_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{M} \bar{\otimes} \mathcal{L}(N_j)$. Notice that for all $i \neq k$ we have $j_i \neq j_k$. Otherwise we would have $\Delta(\mathcal{L}(N_{j_i})) \prec^{s}_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{M} \bar{\otimes} \mathcal{L}(N_1)$ and $\Delta(\mathcal{L}(N_{j_i})) \prec^{s}_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{M} \bar{\otimes} \mathcal{L}(N_2)$, which by [Drimbe et al. 2019, Lemma 2.8 (2)] would imply

$$\Delta(\mathcal{L}(N_{j_i})) \prec^{s}_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{M} \bar{\otimes} \mathcal{L}(N_1 \cap N_2) = \mathcal{M} \otimes 1,$$

which is a contradiction. Furthermore using the same arguments as in [Isono 2020, Lemma 2.6] we have $\Delta(\mathcal{L}(N_1 \times N_2)) \prec^s_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{M} \mathbin{\bar{\otimes}} \mathcal{L}(N_1 \times N_2)$. Then working on the left side of the tensor we get $\Delta(\mathcal{L}(N_1 \times N_2)) \prec^s_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{L}(N_1 \times N_2) \mathbin{\bar{\otimes}} \mathcal{L}(N_1 \times N_2)$.

Finally, notice that part (ii) is a direct consequence of Theorem 5.8. □

5A. *Proof of Theorem 5.1.*

Proof. We divide the proof into separate parts to improve the exposition.

Reconstruction of the acting group Q. To accomplish this we will use the notion of height for elements in group von Neumann algebras as introduced in [Ioana et al. 2013; Ioana 2011]. From the previous theorem recall that $u\Delta(\mathcal{L}(Q))u^* \subseteq \mathcal{L}(Q) \mathbin{\bar{\otimes}} \mathcal{L}(Q)$. Let $\mathcal{A} = u\Delta(\mathcal{L}(N_1))u^*$. Next we claim that

$$h_{Q\times Q}(u\Delta(Q)u^*) > 0. \tag{5A.1}$$

For every $x, y \in \mathcal{L}(Q) \mathbin{\bar{\otimes}} \mathcal{L}(Q)$ and every $a \in \mathcal{A} \mathbin{\bar{\otimes}} \mathcal{A}$ supported on a finite set $F \subset N = N_1 \times N_2$ we have

$$\begin{aligned}
\|E_{\mathcal{A}\bar{\otimes}\mathcal{A}}(xay)\|_2^2 &= \Big\|\sum_{q,l} \tau(xu_{q^{-1}})\tau(yu_l)E_{\mathcal{A}\bar{\otimes}\mathcal{A}}(u_q a u_{l^{-1}})\Big\|_2^2 = \Big\|\sum_{q,l} \tau(xu_{q^{-1}})\tau(yu_l)E_{\mathcal{A}\bar{\otimes}\mathcal{A}}(\sigma_q(a)u_{ql^{-1}})\Big\|_2^2\\
&= \Big\|\sum_{q} \tau(xu_{q^{-1}})\tau(yu_l)\sigma_q(a)\Big\|_2^2 = \Big\|\sum_{q\in Q, n\in N^2} \tau(xu_{q^{-1}})\tau(yu_l)\tau(au_{n^{-1}})u_{\sigma_q(n)}\Big\|_2^2\\
&= \sum_{r\in N^2}\Big|\sum_{\sigma_q(n)=r} \tau(xu_{q^{-1}})\tau(yu_l)\tau(au_{n^{-1}})\Big|^2\\
&\leqslant h^2_{Q\times Q}(x)\sum_{r\in N^2}\Big(\sum_{q\in Q:\sigma_{q^{-1}}(r^{-1})\in F} |\tau(yu_l)|\,|\tau(au_{\sigma_{q^{-1}}(r)})|\Big)^2\\
&\leqslant h^2_{Q\times Q}(x)\|y\|_2^2\,\|a\|_2^2 \max_{r\in N^2} |\{q\in Q : \sigma_{q^{-1}}(r^{-1})\in F\}|.
\end{aligned} \tag{5A.2}$$

This estimate leads to the following property: for all finite sets $K, S \subset Q$, every $a \in \operatorname{span}\{\mathcal{A} \mathbin{\bar{\otimes}} \mathcal{A}u_g : g \in K\}$ and all $\varepsilon > 0$ there exist a scalar $C > 0$ and a finite set $F \subset N^2$ such that, for all $x, y \in \mathcal{L}(Q)\bar{\otimes}\mathcal{L}(Q)$,

$$\begin{aligned}
&\|P_{\sum_{s\in S}\mathcal{A}\bar{\otimes}\mathcal{A}u_s}(xay)\|_2^2\\
&\qquad\leqslant |K|\,|S|C(h^2_{Q\times Q}(x)\|y\|_2^2\,\|a\|_2^2 \max_{r\in N^2}|\{q\in Q : \sigma_{q^{-1}}(r^{-1})\in F\}|) + \varepsilon\|x\|_\infty\|y\|_\infty.
\end{aligned} \tag{5A.3}$$

Note this follows directly from (5A.2) after we decompose the a and the projection $P_{\sum_{s\in S}\mathcal{A}\bar{\otimes}\mathcal{A}u_s}$.

Next we use (5A.3) to prove our claim. Fix $\varepsilon > 0$. Since $\Delta(\mathcal{A}) \nprec \mathcal{M}\otimes 1,\ 1\otimes\mathcal{M}$, by Theorem 2.1 one can find a finite subset $F_o \subset N^2 \setminus ((N\times 1)\cup(1\times N))$ such that $a_{F_o} \in \mathcal{A} \mathbin{\bar{\otimes}} \mathcal{A}$ is supported on F_o and $\|a - a_{F_o}\|_2 \leqslant \varepsilon$. Since $\Delta(\mathcal{A}) \prec^s \mathcal{A} \mathbin{\bar{\otimes}} \mathcal{A}$, there is a finite $S \subseteq Q\times Q$ such that

$$\|P_{\sum_{s\in S}\mathcal{A}\bar{\otimes}\mathcal{A}u_s}(a) - a\|_2 \leqslant \varepsilon \quad \text{for all } a\in\Delta(\mathcal{A}). \tag{5A.4}$$

Assume by contradiction (5A.1) doesn't hold. Thus there is a sequence $t_n \in Q$ such that $h_{Q\times Q}(t_n) = h_{Q\times Q}(u\Delta(u_{t_n})u^*) \to 0$ as $n \to \infty$. As t_n normalizes $\Delta(\mathcal{A})$, one can see that

$$\begin{aligned}1-\varepsilon = \|t_n a t_n^*\|_2^2 - \varepsilon &\leqslant \|P_{\sum_{s\in S}\mathcal{A}\bar{\otimes}\mathcal{A}u_s}(t_n a t_n^*)\|_2^2 \leqslant \|P_{\sum_{s\in S}\mathcal{A}\bar{\otimes}\mathcal{A}u_s}(t_n a t_n^*)\|_2^2 + \varepsilon \\ &\leqslant |F_o||S|C(h^2_{Q\times Q}(t_n)\|t_n\|_2^2\|a_{F_o}\|_2^2 \max_{r\in N^2}|\{q\in Q : \sigma_{q^{-1}}(r^{-1})\in F_o\}|) + \varepsilon\|t_n\|_\infty^2 \\ &\leqslant |F_o||S|C(h^2_{Q\times Q}(t_n)\max_{r\neq 1}|\mathrm{Stab}_Q(r)||F_o|) + 2\varepsilon. \qquad (5A.5)\end{aligned}$$

Since the stabilizer sizes are uniformly bounded, we get a contradiction if $\varepsilon > 0$ is arbitrary small. Now we notice that the height condition, together with Theorem 5.8 and [Chifan and Udrea 2020, Lemmas 2.4, 2.5], already implies $h_Q(\mu\Phi\mu^*) > 0$ and by [Ioana et al. 2013, Theorem 3.1] there is a unitary $\mu_0 \in \mathcal{M}$ such that $\mathbb{T}\mu_0\Phi\mu_0^* = \mathbb{T}Q$.

Reconstruction of a core subgroup and its product feature. From Theorem 5.10, we have

$$\Delta(\mathcal{L}(N_1\times N_2)) \prec^s_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{L}(N_1\times N_2)\,\bar{\otimes}\,\mathcal{L}(N_1\times N_2).$$

Proceeding exactly as in the proof of [Chifan and Udrea 2020, Claim 4.5] we can show that $\Delta(\mathcal{A}) \subseteq \mathcal{A}\bar{\otimes}\mathcal{A}$, where $\mathcal{A} = u\mathcal{L}(N_1\times N_2)u^*$. By Lemma 2.8, there exists a subgroup $\Sigma < \Lambda$ such that $\mathcal{A} = \mathcal{L}(\Sigma)$. The last part of the proof of [Chifan and Udrea 2020, Theorem 5.2] shows that $\Lambda = \Sigma \rtimes \Phi$. In order to reconstruct the product feature of Σ, we need a couple more results.

Claim 2. *For every $i = 1, 2$ there exists $j = 1, 2$ such that*

$$\Delta(\mathcal{L}(N_j)) \prec^s \mathcal{L}(N_1\times N_2)\,\bar{\otimes}\,\mathcal{L}(N_i). \qquad (5A.6)$$

Proof of Claim. We prove this only for $i = 1$ as the other case is similar. We also notice that since $\mathcal{N}_{\mathcal{M}\otimes\mathcal{M}}(\Delta(\mathcal{L}(N_j)))'' \supseteq \Delta(\mathcal{M})$ and $\Delta(\mathcal{M})'\cap\mathcal{M}\,\bar{\otimes}\,\mathcal{M} = \mathbb{C}1$, to establish (5A.6) we only need to show that $\Delta(\mathcal{L}(N_j)) \prec \mathcal{L}(N_1\times N_2)\bar{\otimes}\mathcal{L}(N_i)$. From above we have $\Delta(\mathcal{L}(N_1\times N_2) \prec_{\mathcal{M}\bar{\otimes}\mathcal{M}} \mathcal{L}(N_1\times N_2)\bar{\otimes}\mathcal{L}(N_1\times N_2)$. Hence there exist nonzero projections $a_i \in \Delta(\mathcal{L}(N_i))$ and $b \in \mathcal{L}(N_1\times N_2)\,\bar{\otimes}\,\mathcal{L}(N_1\times N_2)$, a partial isometry $v \in \mathcal{M}\,\bar{\otimes}\,\mathcal{M}$ and a $*$-isomorphism on the image

$$\Psi : a_1\otimes a_2\Delta(\mathcal{L}(N_1\times N_2))a_1\otimes a_2 \to \Psi(a_1\otimes a_2\Delta(\mathcal{L}(N_1\times N_2))a_1\otimes a_2) := \mathcal{R} \subseteq b(\mathcal{L}(N_1\times N_2)\bar{\otimes}\mathcal{L}(N_1\times N_2))b$$

such that $\Psi(x)v = vx$ for all $x \in a_1\otimes a_2\Delta(\mathcal{L}(N_1\times N_2))a_1\otimes a_2$.

Define $\mathcal{D}_i := \Psi(a_i(\Delta(\mathcal{L}(N_i)))a_i) \subseteq b\mathcal{L}(N_1\times N_2)\,\bar{\otimes}\,\mathcal{L}(N_1\times N_2)b$ and notice that $\mathcal{D}_1$ and $\mathcal{D}_2$ are commuting property (T) diffuse subfactors. Since the group N_2 is ($\mathbb{F}_\infty$)-by-(nonelementary hyperbolic group), by [Chifan et al. 2015; Chifan and Kida 2015] it follows that there is $j = 1, 2$ such that $\mathcal{D}_j \prec_{\mathcal{L}(N_1\times N_2)\bar{\otimes}\mathcal{L}(N_1\times N_2)} \mathcal{L}(N_1\times N_2)\,\bar{\otimes}\,\mathcal{L}(N_1\times\mathbb{F}_\infty)$. Since $\mathbb{F}_\infty$ has Haagerup's property and $\mathcal{D}_j$ has property (T) this further implies that $\mathcal{D}_j \prec_{\mathcal{L}(N_1\times N_2)\bar{\otimes}\mathcal{L}(N_1\times N_2)} \mathcal{L}(N_1\times N_2)\,\bar{\otimes}\,\mathcal{L}(N_1)$. Composing this intertwining with Ψ we get $\Delta(\mathcal{L}(N_j)) \prec \mathcal{L}(N_1\times N_2)\,\bar{\otimes}\,\mathcal{L}(N_1)$, as desired.

Also, we note that $j_1 \neq j_2$. Otherwise we would have $\Delta(\mathcal{L}(N_j)) \prec^s \mathcal{L}(N_1\times N_2)\,\bar{\otimes}\,\mathcal{L}(N_1)\cap\mathcal{L}(N_2) = \mathcal{L}(N_1\times N_2)\,\bar{\otimes}\,1$, which obviously contradicts [Ioana et al. 2013, Proposition 7.2.1]. □

Let $\mathcal{A} = u\mathcal{L}(N_1))u^*$. Thus, we get $\Delta(\mathcal{A}) \prec^s \mathcal{L}(N_1\times N_2)\otimes\mathcal{L}(N_i)$ for some $i = 1, 2$. This implies that for every $\varepsilon > 0$ there exists a finite set $S \subset u^*Qu$, containing e, such that $\|d - P_{S\times S}(d)\|_2 \leqslant \varepsilon$ for all

$d \in \Delta(\mathcal{A})$. However, $\Delta(\mathcal{A})$ is invariant under the action of u^*Qu, and hence arguing exactly as in [Chifan and Udrea 2020, Claim 4.5] we get $\Delta(\mathcal{A}) \subset (\mathcal{L}(\Sigma) \bar{\otimes} u\mathcal{L}(N_i)u^*)$. We now separate the argument into two different cases:

Case I: $i = 1$. In this case, $\Delta(\mathcal{A}) \subseteq \mathcal{L}(\Sigma) \bar{\otimes} \mathcal{A}$. Thus by Lemma 2.8 we get that there exists a subgroup $\Sigma_0 < \Sigma$ with $\mathcal{A} = \mathcal{L}(\Sigma_0)$. Now, $\mathcal{A}' \cap \mathcal{L}(\Sigma) = u\mathcal{L}(N_2)u^*$. Thus, $\mathcal{L}(\Sigma_0)' \cap \mathcal{L}(\Sigma) = u\mathcal{L}(N_2)u^*$. Note that Σ and Σ_0 are both icc property (T) groups. This implies $\mathcal{L}(\Sigma_0)' \cap \mathcal{L}(\Sigma) = \mathcal{L}(vC_\Sigma(\Sigma_0))$, where $vC_\Sigma(\Sigma_0)$ denotes the *virtual centralizer* of Σ_0 in Σ. Proceeding as in [Chifan et al. 2018] we can show $\Sigma = \Sigma_0 \times \Sigma_1$.

Case II: $i = 2$. Let $\mathcal{B} = u\mathcal{L}(N_2)u^*$. In this case, $\Delta(\mathcal{A}) \subseteq \mathcal{L}(\Sigma) \bar{\otimes} \mathcal{B}$. However, Lemma 2.8 then implies that $\mathcal{A} \subseteq \mathcal{B}$, which is absurd, as $\mathcal{L}(N_1)$ and $\mathcal{L}(N_2)$ are orthogonal algebras. Hence this case is impossible. $\square$

Remarks. (1) There are several immediate consequences of Theorem 5.1. For instance one can easily see the von Neumann algebras covered by this theorem are nonisomorphic with the ones arising from any irreducible lattice in higher-rank Lie group. Indeed, if Λ is any such lattice satisfying $\mathcal{L}(\Gamma) \cong \mathcal{L}(\Lambda)$, then Theorem 5.1 would imply that Λ must contain an infinite normal subgroup of infinite index which contradicts Margulis' normal subgroup theorem.

(2) While it well known there are uncountably many nonisomorphic group II_1-factors with property (T) [Popa 2007], little is known about producing concrete examples of such families. In fact the only currently known infinite families of pairwise nonisomorphic property (T) groups factors are $\{\mathcal{L}(G_n) : n \geqslant 2\}$ for G_n uniform latices in $\mathrm{Sp}(n, 1)$ [Cowling and Haagerup 1989] and $\{\mathcal{L}(G_1 \times G_2 \times \cdots \times G_k) : k \geqslant 1\}$, where G_k is any icc property (T) hyperbolic group [Ozawa and Popa 2004]. Theorem 5.1 makes new progress in this direction by providing a new explicit infinite family of icc property (T) groups which gives rise to pairwise nonisomorphic II_1-factors. For instance, in the statement one can simply let Q_i vary in any infinite family of nonisomorphic uniform lattices in $\mathrm{Sp}(n, 1)$ for any $n \neq 2$. Unlike the other families, ours consists of factors which are not solid, do not admit tensor decompositions [Chifan et al. 2018], and do not have Cartan subalgebras [Chifan et al. 2015].

(3) We notice that Theorem 5.1 still holds if instead of $\Gamma = (N_1 \times N_2) \rtimes (Q_1 \times Q_2)$ one considers any finite-index subgroup of Γ of the form $\Gamma_{s,r} = (N_1 \times N_2) \rtimes (Q_1^s \times Q_2^r) \leqslant \Gamma$, where $Q_1^s \leqslant Q_1$ and $Q_2^r \leqslant Q_2$ are arbitrary finite-index subgroups. One can verify these groups still enjoy all the algebraic/geometric properties used in the proof of Theorem 5.1 (including the fact that $N_1 \rtimes Q_1^s$ is hyperbolic relative to Q_1^s and $N_1 \rtimes Q_2^r$ is hyperbolic relative to Q_2^r) and hence all the von Neumann algebraic arguments in the proof of Theorem 5.1 apply verbatim. The details are left to the reader.

(4) The group factors considered in Theorem 5.1 have trivial fundamental group by [Chifan et al. 2020, Theorem B]

6. Concrete examples of infinitely many pairwise nonisomorphic group II_1-factors with property (T)

In this section we present several applications of our main techniques to the structural study of property (T) group factors. An earlier result of Popa [2007] shows that the map $\Gamma \mapsto \mathcal{L}(\Gamma)$ is at most countable-to-1. Since there are uncountably many icc property (T) groups, this obviously implies the existence of uncountably many group property (T) factors which are pairwise nonisomorphic. However, currently there are still no explicit constructions of such families in the literature. In this section we make new progress in this direction by showing that the canonical fiber product of Belegradek–Osin Rips construction groups can be successfully used to provide possibly the first such examples (Corollary 6.4). In addition, our methods also yield other interesting consequences. For instance, they can be used to provide an infinite series of finite-index subfactors of a given property (T) II_1-factor that are pairwise nonisomorphic, which is also a novelty in the area (Corollary 6.2). This further gives infinitely many examples of icc, property (T) groups Γ_n measure equivalent to a fixed group Γ such that $\mathcal{L}(\Gamma_n)$ are pairwise mutually nonisomorphic. The first examples of group measure equivalent groups Γ and Λ giving rise to nonisomorphic group von Neumann algebras were given in [Chifan and Ioana 2011], thereby answering a question of Shlyakhtenko. Note that the examples in [Chifan and Ioana 2011] don't have property (T).

The following is the main von Neumann algebraic result of the section. Some of the arguments used in the proof are very similar to the ones used in the proof of Theorem 5.1 and thus we shall just refer the reader to the previous section for these. However, we will include all the details on the new aspects of the proof.

Theorem 6.1. *Let Q_1, Q_2, P_1, P_2 be icc, torsion-free, residually finite property (T) groups. Let $Q = Q_1 \times Q_2$ and $P = P_1 \times P_2$. Assume that $N_1 \rtimes Q$, $N_2 \rtimes Q \in \mathcal{R}ip_T(Q)$ and $M_1 \rtimes P$, $M_2 \rtimes P \in \mathcal{R}ip_T(P)$. Assume that $\Theta : \mathcal{L}((N_1 \times N_2) \rtimes Q) \to \mathcal{L}((M_1 \times M_2) \rtimes P)$ is a $*$-isomorphism.*

Then one can find a $*$-isomorphism, $\Theta_i : \mathcal{L}(N_i) \to \mathcal{L}(M_i)$, a group isomorphism $\delta : Q \to P$, a multiplicative character $\eta : Q \to \mathbb{T}$, and a unitary $u \in \mathscr{U}(\mathcal{L}((M_1 \times M_2) \rtimes P))$ such that for all $\gamma \in Q$, $x_i \in N_i$ we have

$$\Theta((x_1 \otimes x_2)u_\gamma) = \eta(\gamma)u(\Theta_1(x_1) \otimes \Theta_2(x_2)v_{\delta(\gamma)})u^*.$$

Proof. Let $\mathcal{M} = \mathcal{L}((M_1 \times M_2) \rtimes P)$, $\Gamma_i = N_i \rtimes Q$ and $\widetilde{\mathcal{M}} = \mathcal{L}(\Gamma_1 \times \Gamma_2)$. Note that $\Theta(\mathcal{L}(N_1))$ and $\Theta(\mathcal{L}(N_2))$ are commuting property (T) subfactors of $\mathcal{L}((M_1 \times M_2) \rtimes P)$. Hence by Theorem 5.3 we have that either

(1) exists $i \in \{1, 2\}$ such that $\Theta(\mathcal{L}(N_i)) \prec_{\widetilde{\mathcal{M}}} \mathcal{L}(\Gamma_1)$ or

(2) $\Theta(\mathcal{L}(N_1 \times N_2)) \prec_{\widetilde{\mathcal{M}}} \mathcal{L}(\Gamma_1 \times P)$.

Assume (1) holds. Then proceeding as in the first part of proof of Theorem 5.5 we have $\Theta(\mathcal{L}(N_i)) \prec_{\widetilde{\mathcal{M}}} \mathcal{L}(M_1)$. As $\mathcal{L}(M_1)$ is regular in $\mathcal{M}$, we conclude using Lemma 2.4 that $\Theta(\mathcal{L}(N_i)) \prec_{\mathcal{M}} \mathcal{L}(M_1)$.

Assume (2). Then by the same argument as in the second part of the proof of Theorem 5.5 we have $\Theta(\mathcal{L}(N_1 \times N_2)) \prec_{\widetilde{\mathcal{M}}} \mathcal{L}(M_1 \rtimes \mathrm{diag}(P))$. Thus if $\Theta(\mathcal{L}(N_i)) \nprec \mathcal{L}(M_1)$ for all $i = 1, 2$, then the same argument as in the last part of Theorem 5.5 will lead to a contradiction.

In conclusion, we have shown that for all $i = 1, 2$ there exists $j \in 1, 2$ such that $\Theta(\mathcal{L}(N_j)) \prec_{\mathcal{M}} \mathcal{L}(M_i)$. As $\Theta(\mathcal{L}(N_j))$ is regular in $\mathcal{M}$, we actually have $\Theta(\mathcal{L}(N_j)) \prec^s_{\mathcal{M}} \mathcal{L}(M_i)$. Notice that in particular this forces different i's to give rise to different j's. Indeed, otherwise we would have $\Theta(\mathcal{L}(N_j)) \prec^s_{\mathcal{M}} \mathcal{L}(M_1)$ and $\Theta(\mathcal{L}(N_j)) \prec^s_{\mathcal{M}} \mathcal{L}(M_2)$. Then by [Drimbe et al. 2019, Lemma 2.6], this would imply $\Theta(\mathcal{L}(N_j)) \prec_{\mathcal{M}} \mathcal{L}(M_1) \cap \mathcal{L}(M_2) = \mathbb{C}$, which is obviously a contradiction. Therefore we get that either

(4a) $\Theta(\mathcal{L}(N_1)) \prec^s_{\mathcal{M}} \mathcal{L}(M_1)$ and $\Theta(\mathcal{L}(N_2)) \prec^s_{\mathcal{M}} \mathcal{L}(M_2)$ or

(4b) $\Theta(\mathcal{L}(N_1)) \prec^s_{\mathcal{M}} \mathcal{L}(M_2)$ and $\Theta(\mathcal{L}(N_2)) \prec^s_{\mathcal{M}} \mathcal{L}(M_1)$.

Note that both cases imply $\Theta(\mathcal{L}(N_1)), \Theta(\mathcal{L}(N_2)) \prec^s_{\mathcal{M}} \mathcal{L}(M_1 \times M_2)$. Using [Isono 2020, Lemma 2.6], we further get

$$\Theta(\mathcal{L}(N_1 \times N_2)) \prec^s_{\mathcal{M}} \mathcal{L}(M_1 \times M_2). \tag{6.1}$$

Proceeding in a similar manner, we also have the reverse intertwining $\mathcal{L}(M_1 \times M_2) \prec^s_{\mathcal{M}} \Theta(\mathcal{L}(N_1 \times N_2))$. Since $\mathcal{L}(M_1 \times M_2)$, $\mathcal{L}(N_1 \times N_2)$ are irreducible, regular subfactors of $\mathcal{M}$, by [Ioana et al. 2008, Lemma 8.4] one can find $u \in \mathcal{U}(\mathcal{M})$ such that

$$u\mathcal{L}(M_1 \times M_2)u^* = \Theta(\mathcal{L}(N_1 \times N_2)). \tag{6.2}$$

Note that $\Theta(\mathcal{L}(Q_1)), \Theta(\mathcal{L}(Q_2))$ are commuting property (T) subfactors of $\mathcal{L}((M_1 \times M_2) \rtimes P)$. Proceeding exactly as in the first part of the proof, we conclude that either $\Theta(\mathcal{L}(Q_i)) \prec_{\tilde{\mathcal{M}}} \mathcal{L}(\Gamma_1)$ or $\Theta(\mathcal{L}(Q_1 \times Q_2)) \prec_{\tilde{\mathcal{M}}} \mathcal{L}(\Gamma_1 \rtimes P)$. As before, this further implies that either

(7) $\Theta(\mathcal{L}(Q_i)) \prec_{\mathcal{M}} \mathcal{L}(M_1)$ or

(8) $\Theta(\mathcal{L}(Q_1 \times Q_2)) \prec_{\mathcal{M}} \mathcal{L}(M_1 \rtimes \operatorname{diag}(P))$.

Assume (7). Since by (6.2) we also have $\mathcal{L}(M_1) \prec^s_{\mathcal{M}} \Theta(\mathcal{L}(N_1 \times N_2))$ and hence by [Vaes 2009, Lemma 3.7] we conclude $\Theta(\mathcal{L}(Q_i)) \prec_{\mathcal{M}} \Theta(\mathcal{L}(N_1 \times N_2))$. However, this implies Q_i is finite, which is a contradiction.

Hence, we must have (8). Proceeding as in the end of proof of Theorem 5.5, we conclude that $\Theta(\mathcal{L}(Q)) \prec_{\mathcal{M}} \mathcal{L}(P)$. Thus there exists $\Psi : p\Theta(\mathcal{L}(Q))p \to \mathcal{R} := \Psi(p\Theta(\mathcal{L}(Q))p) \subseteq q\mathcal{L}(P)q$ such that $\Psi(x)v = vx$ for all $x \in p\Theta(\mathcal{L}(Q))p$. Also note that $vv^* \in \mathcal{R}' \cap q\mathcal{M}q$ and $v^*v \in p\Theta(\mathcal{L}(Q))p' \cap p\mathcal{M}p$. Since $\mathcal{R} \subseteq q\mathcal{L}(P)q$ is diffuse and $P \leqslant (M_1 \times M_2) \rtimes P$ is a malnormal subgroup, we have $\mathscr{QN}_{q\mathcal{M}q}(\mathcal{R})'' \subseteq q\mathcal{L}(P)q$. Thus $vv^* \in q\mathcal{L}(P)q$ and hence $vp\Theta(\mathcal{L}(Q))pv^* = \mathcal{R}vv^* \subseteq q\mathcal{L}(P)q$. Extending v to a unitary v_0 in $\mathcal{M}$ we have $v_0 p\Theta(\mathcal{L}(Q))pv_0^* \subseteq \mathcal{L}(P)$. As $\mathcal{L}(P)$ and $\mathcal{L}(Q)$ are factors, after perturbing v_0 to a new unitary we may assume that

(9) $v_0\Theta(\mathcal{L}(Q))v_0^* \subseteq \mathcal{L}(P)$.

In a similar manner we have that there exists $w_0 \in \mathscr{U}(\mathcal{M})$ with

(10) $w_0\mathcal{L}(P)w_0^* \subseteq \Theta(\mathcal{L}(Q))$.

Conditions (9) and (10) imply $w_0\mathcal{L}(P)w_0^* \subseteq \Theta(\mathcal{L}(Q)) \subseteq v_0^*\mathcal{L}(P)v_0$. In particular, $v_0w_0\mathcal{L}(P)w_0^*v_0^* \subseteq \mathcal{L}(P)$. Since P is malnormal in $(M_1 \times M_2) \rtimes P$, we have $v_0w_0 \in \mathcal{L}(P)$ and hence $w_0\mathcal{L}(P)w_0^* = v_0^*\mathcal{L}(P)v_0$. Combining this with the above relations we get

(11) $w_0\mathcal{L}(P)w_0^* = \Theta(\mathcal{L}(Q))$.

Since the action $Q \curvearrowright (N_1 \times N_2)$ has trivial stabilizers, using conditions (11) and (6), arguing as in the proof of Theorem 5.1, we get $h_{w_0\mathcal{L}(P)w_0^*}(\Theta(Q)) > 0$. By [Ioana et al. 2013, Theorem 3.3] we get that there exists $w_1 \in \mathscr{U}(\mathcal{M})$ and an isomorphism $\delta : Q \to P$ such that $\Theta(u_g) = w_1 v_{\delta(g)} w_1^*$ for all $g \in Q$.

Finally, this together with relation (4), proceeding exactly as in the proof of Theorem 5.1, implies the desired conclusion. □

The previous theorem can be used to provide an infinite series of finite-index subfactors of a given property (T) II_1-factor that are pairwise nonisomorphic.

Corollary 6.2. (1) *Let Q_1, Q_2 be uniform lattices in* $\mathrm{Sp}(n,1)$ *with $n \geqslant 2$ and let $Q := Q_1 \times Q_2$. Also let $\cdots \leqslant Q_1^s \leqslant \cdots \leqslant Q_1^2 \leqslant Q_1^1 \leqslant Q_1$ be an infinite family of finite-index subgroups and define $Q_s := Q_1^s \times Q_2 \leqslant Q$. Then consider $N_1 \rtimes_{\sigma_1} Q$, $N_2 \rtimes_{\sigma_2} Q \in \mathcal{R}ip_T(Q)$ and let $\Gamma = (N_1 \times N_2) \rtimes_{\sigma_1 \times \sigma_2} Q$. Inside Γ consider the finite-index subgroups $\Gamma_s := (N_1 \times N_2) \rtimes_{\sigma_1 \times \sigma_2} Q_s$. Then the family $\{\mathcal{L}(\Gamma_s) : s \in I\}$ consists of pairwise nonisomorphic finite-index subfactors of $\mathcal{L}(\Gamma)$.*

(2) *Let Γ, Γ_n be as above. Then Γ_n is measure equivalent to Γ for all $n \in \mathbb{N}$, but $\mathcal{L}(\Gamma_n)$ is not isomorphic to $\mathcal{L}(\Gamma_m)$ for $n \neq m$.*

Proof. (1) Assume $\mathcal{L}(\Gamma_s) \cong \mathcal{L}(\Gamma_l)$. Notice that Q_2, Q_1^s, Q_1^l are torsion-free, residually finite property (T) groups. Thus applying Theorem 6.1 we get in particular that $Q_s \cong Q_l$. However since Q_2, Q_1^s, and Q_1^l are icc hyperbolic, this further implies $Q_1^s \cong Q_1^l$. However, by [Prasad 1976] or the cohopfian property of one-ended hyperbolic groups, this implies $s = l$ and the proof follows.

(2) As $[\Gamma : \Gamma_n] < \infty$, Γ_n is measure equivalent to Γ, and hence Γ_n is measure equivalent to Γ_m for all $n, m \in \mathbb{N}$. The rest follows from part (1). □

Notation. Denote by $\mathcal{ST}$ denote the family of all icc, torsion-free, residually finite property (T) groups.

For further use we record the following elementary result. Its proof is left to the reader.

Proposition 6.3. *Fix Q to be an icc, torsion-free, residually finite, hyperbolic property (T) group. For instance, Q can be chosen to be a uniform lattice in* $\mathrm{Sp}(n,1)$ *for $n \geqslant 2$. Then the family $\mathcal{ST}' = \{G \times Q : G \in \mathcal{ST}\}$ consists of pairwise nonisomorphic groups.*

Finally, we present the main application of this section:

Corollary 6.4. *Let $\{Q_\iota\}_{\iota \in \mathcal{I}}$ be an infinite family of pairwise nonisomorphic groups in $\mathcal{ST}'$. Consider the semidirect products $N_{\iota_1} \rtimes_{\sigma_1} Q_\alpha$, $N_{\iota_2} \rtimes_{\sigma_2} Q_\iota \in \mathcal{R}ip_T(Q_\iota)$ for every $\iota \in \mathcal{I}$. Consider the canonical semidirect product $\Gamma_\iota := (N_{\iota_1} \times N_{\iota_2}) \rtimes_{\sigma_1 \times \sigma_2} Q_\iota$ corresponding to the diagonal action $\sigma_1 \times \sigma_2$. Then $\{\mathcal{L}(\Gamma_\iota) : \iota \in \mathcal{I}\}$ is an infinite family of pairwise nonisomorphic group II_1-factors with property (T).*

Proof. This follows directly from Theorem 6.1 and Proposition 6.3 □

We strongly believe the family $\mathcal{ST}$ consists of uncountably many pairwise nonisomorphic groups. In this scenario, Corollary 6.4 would provide an explicit family of uncountably many nonisomorphic property (T) group von Neumann algebras. However, we were unable to find in the literature a reference for whether $\mathcal{ST}$ contains uncountably many nonisomorphic groups. Therefore we leave the following as an open question.

Open Problem. *Find examples of uncountably many nonisomorphic icc property (T) groups G that give nonstably isomorphic II_1-factors $\mathcal{L}(G)$.*

7. Cartan-rigidity for von Neumann algebras of groups in $\mathcal{R}ip(Q)$

In this last section we classify the Cartan subalgebras in II_1-factors associated with the groups in $\mathcal{R}ip_T(Q)$ and their free ergodic pmp actions on probability spaces (see Theorem 7.1, and Corollary 7.2). Our proofs rely in an essential way on the methods introduced in [Popa and Vaes 2014; Chifan et al. 2015], as well as on the group theoretic Dehn filling discussed in Section 3C. For convenience we include detailed proofs.

First we establish the following general intertwining result regarding crossed product algebras arising from groups in $\mathcal{R}ip(Q)$.

Theorem 7.1. *Let $Q = Q_1 \times Q_2$, where Q_i are residually finite groups. For every $i = 1, 2$, let $\Gamma_i = N_i \rtimes_{\sigma_i} Q \in \mathcal{R}ip(Q)$ and denote by $\Gamma = (N_1 \times N_2) \rtimes_\sigma Q$ the semidirect product associated with the diagonal action $\sigma = (\sigma_1, \sigma_2) : Q \to \mathrm{Aut}(N_1 \times N_2)$. Let $\mathcal{P}$ be a von Neumann algebra together with an action $\Gamma \curvearrowright \mathcal{P}$ and define $\mathcal{M} = \mathcal{P} \rtimes \Gamma$. Let $p \in \mathcal{M}$ be a projection and let $\mathcal{A} \subset p\mathcal{M}p$ be a maximal abelian self-adjoint subalgebra (masa) whose normalizer $\mathscr{N}_{p\mathcal{M}p}(\mathcal{A})'' \subseteq p\mathcal{M}p$ has finite index. Then $\mathcal{A} \prec_{\mathcal{M}} \mathcal{P}$.*

Proof. Since $\Gamma_i = N_i \rtimes Q$ is hyperbolic relative to a residually finite group Q, by Theorem 3.16 there exists a nonelementary hyperbolic group H_i, a subset $T_i \subseteq N_i$ with $|T_i| \geqslant 2$ and a normal subgroup $R_i \lhd Q$ of finite index such that we have a short exact sequence

$$1 \to *_{t \in T_i} R_i^t \hookrightarrow \Gamma_i \xrightarrow{\varepsilon_i} H_i \to 1.$$

In particular there are infinite groups K_1, K_2 so that $*_{t \in T_i} R_i^t = K_1 * K_2$.

Denote by $\pi_i : \Gamma \twoheadrightarrow \Gamma_i$ the canonical projection given by $\pi_i((n_1, n_2)q) = n_i q$ for all $(n_1, n_2)q \in (N_1 \times N_2) \rtimes Q = \Gamma$. Then for every $i = 1, 2$ consider the epimorphism $\rho_i = \varepsilon_i \circ \pi_i : \Gamma \to H_i$. Following [Chifan et al. 2015, Section 3], consider the $*$-embedding $\Delta^{\rho_i} : \mathcal{M} \to \mathcal{M} \mathbin{\bar{\otimes}} \mathcal{L}(H_i) := \widetilde{\mathcal{M}}_i$ given by $\Delta^{\rho_i}(x u_g) = x u_g \otimes v_{\rho_i(g)}$ for all $x \in \mathcal{M}$, $g \in \Gamma$. Here $(u_g)_{g \in \Gamma}$ and $(v_h)_{h \in H_i}$ are the canonical group unitaries in $\mathcal{P} \rtimes \Gamma$ and $\mathcal{L}(H_i)$, respectively. As $\mathcal{A}$ is amenable, [Popa and Vaes 2014, Theorem 1.4] implies either (a) $\Delta^{\rho_i}(\mathcal{A}) \prec_{\widetilde{\mathcal{M}}_i} \mathcal{M} \otimes 1$ or (b) the normalizer $\Delta^{\rho_i}(\mathscr{N}_{p\mathcal{M}p}(\mathcal{A})'')$ is amenable relative to $\mathcal{M} \otimes 1$ inside $\widetilde{\mathcal{M}}_i$. Assume (b) holds. As $\mathscr{N}_{p\mathcal{M}p}(\mathcal{A})'' \subseteq p\mathcal{M}p$ has finite index, it follows that $\Delta^{\rho_i}(p\mathcal{M}p)$ is amenable relative to $\mathcal{M} \otimes 1$ inside $\widetilde{\mathcal{M}}_i$. However, using [Chifan et al. 2015, Proposition 3.5] this further gives that H_i is amenable, a contradiction. Thus (a) must hold and using [loc. cit., Proposition 3.4] we get $\mathcal{A} \prec_{\mathcal{M}} \mathcal{P} \rtimes \ker(\rho_i)$. Let $\mathcal{N} = \mathcal{P} \rtimes \ker(\rho_i)$ and using [loc. cit., Proposition 3.6] we can find a projection $0 \neq q \in \mathcal{N}$ such that a masa $\mathcal{B} \subset q\mathcal{N}q$ with $\mathcal{Q} := \mathscr{N}_{q\mathcal{N}q}(\mathcal{B})'' \subseteq q\mathcal{N}q$ has finite index. In addition one can find projections $0 \neq p_0 \in \mathcal{A}$, $0 \neq q_0' \in \mathcal{B}' \cap p\mathcal{M}p$ and a unitary $u \in \mathcal{M}$ such that $u(\mathcal{A}p_0)u^* = \mathcal{B}p_0$.

To this end, observe the restriction homomorphism $\pi_i : \ker(\rho_i) \to K_1 * K_2$ is an epimorphism with $\ker(\pi_i) = N_{\hat{i}}$. As before, consider the $*$-embedding $\Delta^{\pi_i} : \mathcal{N} \to \mathcal{N} \mathbin{\bar{\otimes}} \mathcal{L}(K_1 * K_2)$ given by $\Delta^{\pi_i}(x u_g) = x u_g \otimes v_{\pi_i(g)}$ for all $x \in \mathcal{P}$, $g \in \ker(\rho_i)$. Define $\widetilde{\mathcal{N}}_i := \mathcal{N} \mathbin{\bar{\otimes}} \mathcal{L}(\ker(\rho_i))$. Also fix $0 \neq z \in \mathscr{Z}(\mathcal{Q}' \cap q\mathcal{N}q)$. Since $\Delta^{\pi_i}(\mathcal{B}z) \subset \mathcal{N} \mathbin{\bar{\otimes}} \mathcal{L}(K_1 * K_2)$ is amenable, using [Ioana 2013; Vaes 2014] one of the following must hold: (c) $\Delta^{\pi_i}(\mathcal{Q}z)$ is amenable relative to $\mathcal{N} \otimes 1$ inside $\widetilde{\mathcal{N}}_i$; (d) $\Delta^{\pi_i}(\mathcal{Q}z) \prec_{\widetilde{\mathcal{N}}_i} \mathcal{N} \mathbin{\bar{\otimes}} \mathcal{L}(K_j)$ for some $j = 1, 2$; (e) $\Delta^{\pi_i}(\mathcal{B}z) \prec_{\widetilde{\mathcal{N}}_i} \mathcal{N} \otimes 1$.

Assume (c) holds. As $\mathcal{Q} \subseteq q\mathcal{N}q$ is finite-index so is $\mathcal{Q}z \subseteq z\mathcal{N}z$ and [Chifan et al. 2015, Lemma 2.4] implies $z\mathcal{N}z \prec^s \mathcal{Q}z$. Using [Ozawa and Popa 2010, Proposition 2.3 (3)] we get $\Delta^{\pi_i}(z\mathcal{N}z)$ is amenable relative to $\mathcal{N} \otimes 1$ inside $\widetilde{\mathcal{N}}_i$. Thus [Chifan et al. 2015, Proposition 3.5] implies that $K_1 * K_2$ is amenable, a contradiction. Assume (d) holds. By [loc. cit., Proposition 3.4] we have $\mathcal{Q}z \prec \mathcal{P} \rtimes (\pi_i)^{-1}(K_j)$ and using [Drimbe et al. 2019, Lemma 2.4 (3)] one can find a projection $0 \neq r \in \mathscr{Z}(\mathcal{Q}z' \cap z\mathcal{N}z)$ such that $\mathcal{Q}r \prec^s \mathcal{P} \rtimes (\pi_i)^{-1}(K_j)$. Since $\mathcal{Q}z \subseteq z\mathcal{N}z$ is of finite index, so is $\mathcal{Q}r \subseteq r\mathcal{N}r$ and thus $r\mathcal{N}r \prec_{\mathcal{N}} \mathcal{Q}r$. Therefore using [Drimbe et al. 2019, Lemma 2.4 (1)] (or [Vaes 2009, Remark 3.7]) we conclude that $\mathcal{N} \prec \mathcal{P} \rtimes (\pi_i)^{-1}(K_j)$. However, this implies that $\pi^{-1}(K_j) \leqslant \ker(\rho_i)$ is finite-index, a contradiction. Hence

(e) must hold and using [Chifan et al. 2015, Proposition 3.4] we further get $\mathcal{B}z \prec_{\mathcal{N}} \mathcal{P} \rtimes N_{\hat{i}}$. Since this holds for all z, we conclude that $\mathcal{B} \prec^{s}_{\mathcal{N}} \mathcal{P} \rtimes N_{\hat{i}}$. This combined with the prior paragraph clearly implies $\mathcal{A} \prec \mathcal{P} \rtimes N_{\hat{i}}$.

Since all the arguments above still work and the same conclusion holds if one replaces $\mathcal{A}$ by $\mathcal{A}a$ for any projection $0 \neq a \in \mathcal{A}$, one actually has $\mathcal{A} \prec^{s}_{\mathcal{M}} \mathcal{P} \rtimes N_{\hat{i}}$. Since this holds for all $i = 1, 2$, using [Drimbe et al. 2019, Lemma 2.8 (2)] one concludes that $\mathcal{A} \prec_{\mathcal{M}} \mathcal{P}$, as desired. □

Corollary 7.2. *Let Γ be a group as in the previous theorem and let $\Gamma \curvearrowright X$ be a free ergodic pmp action on a probability space. Then the following hold*:

(1) *The crossed product $L^{\infty}(X) \rtimes \Gamma$ has unique Cartan subalgebra.*

(2) *The group von Neumann algebra $\mathcal{L}(\Gamma)$ has no Cartan subalgebra.*

Proof. (1) Let $\mathcal{A} \subset L^{\infty}(X) \rtimes \Gamma =: \mathcal{M}$ be a Cartan subalgebra. By Theorem 7.1 we have $\mathcal{A} \prec_{\mathcal{M}} L^{\infty}(X)$ and since $L^{\infty}(X) \subseteq \mathcal{M}$ is Cartan then [Popa 2006a, Theorem] gives the conclusion.

(2) If $\mathcal{A} \subset \mathcal{L}(\Gamma)$ is a Cartan subalgebra then Theorem 7.1 implies $\mathcal{A} \prec \mathbb{C}1$, which contradicts that $\mathcal{A}$ is diffuse. □

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