ANALYSIS & PDE

Volume 16 No. 2 2023

JOSEPH FENEUIL

THE GREEN FUNCTION WITH POLE AT INFINITY APPLIED TO THE STUDY OF THE ELLIPTIC MEASURE

msp

ANALYSIS AND PDE
Vol. 16 (2023), No. 2, pp. 545–570
DOI: 10.2140/apde.2023.16.545

THE GREEN FUNCTION WITH POLE AT INFINITY APPLIED TO THE STUDY OF THE ELLIPTIC MEASURE

JOSEPH FENEUIL

In $\mathbb{R}^{d+1}_+$ or in $\mathbb{R}^n \setminus \mathbb{R}^d$ $(d < n-1)$, we study the Green function with pole at infinity defined for instance by David, Engelstein, and Mayboroda. In two cases, we deduce the equivalence between the elliptic measure and the Lebesgue measure on $\mathbb{R}^d$. We further prove the A_∞-absolute continuity of the elliptic measure for operators that can be related to the two previous cases via Carleson measures, extending the range of operators for which the A_∞-absolute continuity of the elliptic measure is known.

1. Introduction

History and motivation. Over the past decades, a considerable number of articles have studied the relationship between the geometry of the boundary of a domain Ω and the L^p-solvability of the Dirichlet problem $-\Delta u = 0$ in Ω. The L^p-solvability of the Dirichlet problem for large p is equivalent to the absolute continuity of the harmonic measure, and we shall focus our presentation on the latter. The theory was pioneered in 1916 by the Riesz brothers (see [Riesz and Riesz 1920]), who established the absolute continuity of the harmonic measure for simply connected domains in the complex plane with a rectifiable boundary. The quantitative and local analogues are stated in [Lavrentev 1963] and [Bishop and Jones 1990], respectively. The development of the theory in $\mathbb{R}^n$, for $n \geq 2$, started in [Dahlberg 1977] and treated Lipschitz domains. Many works were then devoted to finding the optimal conditions on Ω and $\partial\Omega$ to guarantee the absolute continuity of the harmonic measure. It was finally understood that a quantitative version of absolute continuity of the harmonic measure holds if and only if the boundary $\partial\Omega$ is uniformly rectifiable and the domain Ω has enough access to its boundary. A nonexhaustive list of articles that lead to this conclusion includes [Azzam et al. 2016; 2017; David and Jerison 1990; Hofmann et al. 2014; Hofmann and Martell 2014; Semmes 1990], and the minimal access condition to the boundary was recently obtained in [Azzam et al. 2020].

One of the strategies for studying the absolute continuity of the harmonic measure, and by extension the L^p-solvability of the Dirichlet problem, is to make a change of variable in order to obtain an equivalent problem for simpler sets but for more complicated elliptic operators. So instead of studying $-\Delta u = 0$ on a general domain Ω, many works focused their interest on the study of elliptic operators of the form

MSC2020: 35J25, 42B37.
Keywords: Green function with pole at infinity, elliptic measure, A_∞-absolute continuity.

$L = -\operatorname{div} \mathcal{A}\nabla$ on, for instance, $\Omega = \mathbb{R}^{n-1}_+ := \{(x,t) \in \mathbb{R}^{n-1} \times (0,+\infty)\}$. Here, $\mathcal{A}$ is a matrix satisfying the ellipticity and boundedness conditions

$$\mathcal{A}(x,t)\xi \cdot \xi \geq C_L|\xi|^2 \quad \text{for } (x,t) \in \Omega \text{ and } \quad \xi \in \mathbb{R}^n, \tag{1.1}$$

$$|\mathcal{A}(x,t)\xi \cdot \zeta| \leq C_L|\xi||\zeta| \quad \text{for } (x,t) \in \Omega \text{ and } \xi,\zeta \in \mathbb{R}^n, \tag{1.2}$$

for some constant $C_L > 0$. As shown in [Caffarelli et al. 1981b; Modica and Mortola 1980], the conditions (1.1) and (1.2) are not sufficient to ensure that the elliptic measure associated to L is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^{n-1}$, and thus some extra assumptions are needed on $\mathcal{A}$ to obtain our absolute continuity. Two situations that give positive results are heavily studied: the first situation focuses on t-independent matrices $\mathcal{A}$ and are studied in [Jerison and Kenig 1981] (use a Rellich identity), [Auscher et al. 2008] (perturbations), [Hofmann et al. 2015] ($\mathcal{A}$ is nonsymmetric), [Hofmann et al. 2019] (Dirichlet problem in weighted L^p), or [Hofmann et al. 2022] (the antisymmetric part of $\mathcal{A}$ can be unbounded); while in the second situation, the coefficients of $\mathcal{A}$ satisfy some conditions described with the help of Carleson measures and Carleson measure perturbations, and are considered, for instance, in [Dindoš and Pipher 2019; Dindoš et al. 2017; 2007; Fefferman et al. 1991; Hofmann and Martell 2012; Hofmann et al. 2021; Kenig and Pipher 2001].

When the domain is the complement of a thin set, for instance $\Omega = \mathbb{R}^n \setminus \mathbb{R}^d := \{(x,t) \in \mathbb{R}^d \times \mathbb{R}^{n-d},\ t \neq 0\}$ with $d < n-1$, studying the solutions to $-\Delta u = 0$ in Ω does not make sense. Indeed, the solutions to $-\Delta u = 0$ in Ω are the same as the solutions to $-\Delta u = 0$ in $\mathbb{R}^n$, which means that the boundary $\mathbb{R}^d$ is not "seen" by the Laplacian or, in term of harmonic measure, it means that the Brownian motion has zero probability to hit the boundary $\mathbb{R}^d$. In [David et al. 2021b; 2020], G. David, S. Mayboroda, and the author developed an elliptic theory for domains with thin boundaries by using appropriate degenerate operators. If $\Omega = \mathbb{R}^n \setminus \mathbb{R}^d$ is considered, we assume that the elliptic operator $L = -\operatorname{div} A\nabla$ satisfies

$$A(x,t)\xi \cdot \xi \geq C_L|t|^{d+1-n}|\xi|^2 \quad \text{for } (x,t) \in \Omega \text{ and } \quad \xi \in \mathbb{R}^n, \tag{1.3}$$

$$|A(x,t)\xi \cdot \zeta| \leq C_L|t|^{d+1-n}|\xi||\zeta| \quad \text{for } (x,t) \in \Omega \text{ and } \xi,\zeta \in \mathbb{R}^n, \tag{1.4}$$

for some constant $C_L > 0$. The operator L can thus be written as $-\operatorname{div} |t|^{d+1-n}\mathcal{A}\nabla$ where $\mathcal{A}$ satisfies conditions (1.1) and (1.2). Under those conditions, the elliptic measure with pole in $X \in \Omega$ associated to L, denoted by ω_L^X, is the probability measure on $\mathbb{R}^d$ so that the function u_f on Ω constructed for any $f \in C_0^\infty(\mathbb{R}^d)$ as

$$u_f(X) = \int_{\mathbb{R}^d} f(y)\, d\omega_L^X(y) \tag{1.5}$$

is a weak solution to $Lu = 0$, is continuous on $\overline{\Omega}$, and has trace on $\mathbb{R}^d$ equal to f. The articles [David and Mayboroda 2022b; David et al. 2019a; Feneuil 2022; Feneuil et al. 2021; Mayboroda and Poggi 2021; Mayboroda and Zhao 2019] tackled the absolute continuity of the elliptic measure (or L^p-solvability of the Dirichlet problem) in the case where the boundary of Ω is a low dimensional set.

We finish the subsection with the following observation made in [David et al. 2019b]. Let $L = -\operatorname{div} \mathcal{A}\nabla$ be an elliptic operator defined on $\mathbb{R}^{d+1}_+$ that satisfies (1.1)–(1.2). We define $\mathcal{A}_1$ as the top left $d \times d$

submatrix of $\mathcal{A}$, and $\mathcal{A}_2$, $\mathcal{A}_3$, a_4 so that we have the block matrix

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ \mathcal{A}_3 & a_4 \end{pmatrix}. \tag{1.6}$$

For $n > d+1$, we construct the elliptic operator $\widetilde{L} = -\operatorname{div} |t|^{d+1-n} \widetilde{\mathcal{A}} \nabla$ defined on $\mathbb{R}^n \setminus \mathbb{R}^d$ as

$$\widetilde{\mathcal{A}}(x,t) := \begin{pmatrix} \mathcal{A}_1(x,|t|) & \mathcal{A}_2(x,|t|)\dfrac{t}{|t|} \\ \dfrac{t^T}{|t|}\mathcal{A}_3(x,|t|) & a_4(x,|t|) I_{n-d} \end{pmatrix}, \tag{1.7}$$

where t is seen here as a horizontal vector in $\mathbb{R}^{n-d}$, which means that $\mathcal{A}_2 t$ and $t^T \mathcal{A}_3$ are matrices of dimensions $d \times (n-d)$ and $(n-d) \times d$, respectively, and I_{n-d} is the identity matrix of order $n-d$. Then the elliptic measures on $\mathbb{R}^d$ associated to L and $\widetilde{L}$ — we call them $\omega^{(x,r)}$ and $\widetilde{\omega}^{(x,t)}$ — satisfy

$$\widetilde{\omega}^{(x,t)} = \omega^{(x,|t|)} \quad \text{for } (x,t) \in \mathbb{R}^n \setminus \mathbb{R}^d. \tag{1.8}$$

More generally, any solution u to $Lu = 0$ in $\mathbb{R}^{d+1}_+$ yields a solution $\widetilde{u}(x,t) := u(x,|t|)$ to $\widetilde{L}u = 0$ in $\mathbb{R}^n \setminus \mathbb{R}^d$. As a consequence, the construction from [Caffarelli et al. 1981b; Modica and Mortola 1980] can be adapted to provide, for any $1 \leq d < n$, examples of operators whose elliptic measures are not absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$. It also means that if an operator $\widetilde{L} = -\operatorname{div} |t|^{d+1-n} \widetilde{\mathcal{A}} \nabla$ can be written as (1.7) and if the elliptic measure of the original operator $L = -\operatorname{div} \mathcal{A} \nabla$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$, then the elliptic measure associated to $\widetilde{L}$ is also absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$. The above construction provides, for any dimension and codimension of the boundary, a wide range of elliptic operators that satisfy the absolute continuity of the elliptic measure. However, the (relevant) solutions of those operators are radial, i.e., they depend only on the distance to the boundary $\mathbb{R}^d$ and their projection on $\mathbb{R}^d$.

The goal of this article is to go beyond the matrices that can be written as (1.7). Of course, as we shall discuss in the next subsection, we already know of some cases where the first d lines do not matter for the A_∞-absolute continuity of the elliptic measure (see [David et al. 2019a; Feneuil et al. 2021]), and we also know that the A_∞ property is stable under Carleson perturbations (see [Mayboroda and Poggi 2021]). However, we do not know, for instance, whether it is possible that the bottom right corner of $\mathcal{A}$ is not a Carleson perturbation of a submatrix of the form $b(x,|t|) I_{n-d}$.

Most of the earlier literature focused on elliptic operators that are "close" to an operator for which $|t|$ (or t in the codimension 1 case) is a solution. In this article, we show that we are justified in replacing $|t|$ by any x-independent "Green function with pole at infinity". We shall first construct the Green function with pole at infinity in the spirit of [David et al. 2021a]. The Green function (and the Green function with pole at infinity) has a deep connection with the harmonic measure (see Lemma 2.9 below); some recent works [David and Mayboroda 2022a; David et al. 2023; 2022] even started to link the geometry of $\partial\Omega$ directly to bounds on the Green function (instead of estimates on the harmonic measure). We shall thus study the Green function with pole at infinity in a few easy cases and deduce that the harmonic measure and the Lebesgue measure are comparable (hence A_∞-absolute continuous with respect to each other). Then we will use the Green function with pole at infinity as a substitute of $|t|$ in a now classical argument that establishes the stability of the A_∞-absolute continuity of the harmonic measure under some transformations on the elliptic operator. This will enlarge the class of operators for which the A_∞-absolute continuity of the harmonic measure is known, especially in the case where $d < n-1$.

Presentation of the results. In the rest of the article d is an integer in $\{0, \dots, n-1\}$. If $d = n-1$, then $\Omega = \mathbb{R}^n_+ = \mathbb{R}^{d+1}_+ = \{(x,t) \in \mathbb{R}^d \times (0,+\infty)\}$. If $d < n-1$, then $\Omega = \mathbb{R}^n \setminus \mathbb{R}^d = \{(x,t) \in \mathbb{R}^d \times \mathbb{R}^{n-d},\ t \neq 0\}$. When we write that $0 < |t| < r$, we understand $t \in (0,r)$ if $n-d = 1$ and $t \in B(0,r) \subset \mathbb{R}^{n-d}$ otherwise.

If $L = -\operatorname{div} A\nabla$ satisfies (1.3)–(1.4), then the elliptic measure defined in (1.5) is nondegenerate, is doubling, and satisfies the change of pole property (respectively Lemmas 11.10, 11.12, and 11.16 in [David et al. 2021b]), and those conditions are the ones needed to prove the following result from [David et al. 2019a].

Theorem 1.9 (Theorem 8.9 in [David et al. 2019a]). *Let $L = -\operatorname{div} A\nabla$, where the real matrix-valued function A satisfies the ellipticity and boundedness conditions* (1.1)–(1.2). *Assume that there exists $M > 0$ such that, for any Borel set $H \subset \mathbb{R}^d$, the solution u_H defined by $u_H(X) = \omega_L^X(H)$ satisfies the Carleson measure estimate*

$$\sup_{x\in\mathbb{R}^d, r>0} \fint_{B_{\mathbb{R}^d}(x,r)} \int_{|t|<r} |t\nabla u_H|^2 \frac{dy\,dt}{|t|^{n-d}} \le M. \tag{1.10}$$

Then the elliptic measure is A_∞ with respect to the Lebesgue measure on $\mathbb{R}^d$, i.e., for every $\epsilon > 0$ there exists a $\delta > 0$ (that depends only on ϵ, d, n, C_L, and M) such that for every ball $B := B(x,r) \subset \mathbb{R}^d$, every t that satisfies $|t| = r$, and any Borel set $E \subset B$, one has

$$\text{if } \frac{\omega_L^{(x,t)}(E)}{\omega_L^{(x,t)}(B)} < \delta, \quad \text{then } \frac{|E|}{|B|} < \epsilon. \tag{1.11}$$

For a proof when $d = n-1$, see Corollary 3.2 in [Kenig et al. 2016]. The condition (1.10) is closely related to another characterization of the A_∞-absolute continuity of the elliptic measure on $\mathbb{R}^d$ called BMO-solvability, which can be found in [Dindoš et al. 2011] for the codimension 1 case and in [Mayboroda and Zhao 2019] when $d < n-1$.

The condition (1.10) means that $(|t||\nabla u_H|)^2 |t|^{d-n}\,dt\,dx$ is a Carleson measure. In order to lighten the presentation, we introduce a notation for inequalities like (1.10). We say that a quantity f satisfies the *Carleson measure condition* if there exists $C > 0$ such that

$$\|f\|_{L^\infty} \le C \quad \text{and} \quad \sup_{x\in\mathbb{R}^d, r>0} \fint_{B_{\mathbb{R}^d}(x,r)} \int_{|t|<r} |f|^2 \frac{dt\,dy}{|t|^{n-d}} \le C. \tag{1.12}$$

In short, we write $f \in CM_2$ or $f \in CM_2(C)$ when we want to refer to the constant on the right side of the bound (1.10). So to conclude, in order to apply Theorem 1.9, we need to assume that there exists $K > 0$ such that for any Borel set H, the function u_H exists in $CM_2(K)$. It will also be useful to write the variant $f \in \widetilde{CM}_2(C)$ when

$$\sup_{x\in\mathbb{R}^d, r>0} \fint_{B_{\mathbb{R}^d}(x,r)} \int_{|t|<r} \Big(\sup_{|Z-(y,t)|<|t|/4} |f(Z)|^2 \Big) \frac{dt\,dy}{|t|^{n-d}} \leq C. \tag{1.13}$$

To the best of the author's knowledge, in our setting of high codimensional boundaries, the most general condition on the coefficients of the matrix A that ensures the A_∞-absolute continuity of the elliptic measure with respect to the d-dimensional Hausdorff measure is given in [Feneuil et al. 2021].

Theorem 1.14 (Theorem 1.9 (1) in [Feneuil et al. 2021] for $p = 2$). *Let $L = -\operatorname{div} |t|^{d+1-n} \mathcal{A}\nabla$, where the real matrix-valued function $\mathcal{A}$ satisfies the ellipticity and boundedness conditions* (1.1)–(1.2). *Assume that $\mathcal{A}$ can be decomposed as*

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ \mathcal{B}_3 & b \cdot I_{n-d} \end{pmatrix} + \mathcal{C}, \tag{1.15}$$

where I_{n-d} is the identity matrix, $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{B}_3$ are $d \times d$, $d \times (n-d)$, and $(n-d) \times d$ matrix-valued functions, respectively, and b is a scalar function, all of which satisfy

- $K^{-1} \leq b \leq K$,
- $|t|\,|\nabla b| + |t|\,|\nabla_x \mathcal{B}_3| + |t|^{n-d} \operatorname{div}_t(|t|^{d+1-n} \mathcal{B}_3) + |\mathcal{C}| \in CM_2(K)$,

for a constant $K > 0$. Then the hypothesis (1.10) *of Theorem 1.9 is satisfied* (*with a constant M that depends only on d, n, C_L, and K*) *and therefore the elliptic measure ω_L^X is A_∞-absolutely continuous with respect to the Lebesgue measure.*

Remarks. (i) In codimension 1, that is when $d = n - 1$, Theorem 1.14 requires that the last line $\mathfrak{a}_{d+1}$ of the matrix $\mathcal{A}$ can be decomposed as $\mathfrak{a}_{d+1} = \mathfrak{b}_{d+1} + \mathfrak{c}_{d+1}$ with $|t|\,|\nabla \mathfrak{b}_{d+1}| + |\mathfrak{c}_{d+1}| \in CM_2$. This condition is thus weaker than the one found in [Kenig and Pipher 2001], where one assumes that $|t|\,|\nabla\mathcal{A}| \in CM$, and the conditions are the same if we add to that result the perturbation theory from [Hofmann and Martell 2012]. However, to the best of the author's knowledge, the first time where no conditions on the first d lines were assumed is in [David et al. 2019a; Feneuil et al. 2021].

(ii) Observe that if $\mathcal{A}$ is a $(d+1)\times(d+1)$ matrix-valued function on $\mathbb{R}^{d+1}_+$ that satisfies the assumptions of the above theorem, then the $n \times n$ matrix-valued function $\widetilde{\mathcal{A}}$ defined from $\mathcal{A}$ on $\mathbb{R}^n \setminus \mathbb{R}^d$ as in (1.7) also satisfies the assumptions of Theorem 1.14.

(iii) With the same argument as the one used in [Dindoš et al. 2007, Corollary 2.3], one can show that if $\mathcal{B}_3$ and b satisfy

$$(x,t) \mapsto \operatorname*{osc}_{B((x,t),|t|/4)} \mathcal{B}_3 + \operatorname*{osc}_{B((x,t),|t|/4)} b \in CM_2(K), \tag{1.16}$$

where $\operatorname{osc}_B f = \sup_B f - \inf_B f$, then we can find $\widehat{\mathcal{B}}_3$ and $\hat{b}$ such that

$$(x,t) \mapsto \sup_{B((x,t),|t|/4)} |\mathcal{B}_3 - \widehat{\mathcal{B}}_3| + \operatorname*{osc}_{B((x,t),|t|/4)} |b - \hat{b}| \in CM_2(K')$$

and

$$\nabla \widehat{\mathcal{B}}_3 + \nabla \hat{b} \in CM_2(K').$$

So assuming the apparently weaker condition (1.16) is enough to satisfy the assumptions of Theorem 1.14 and therefore obtain the A_∞-absolute continuity of the elliptic measure.

When $d < n-1$, the operator $L = -\operatorname{div} A\nabla$ will necessarily depend on $|t|$ as long as it satisfies (1.3)–(1.4). However, once the weight $|t|^{d+1-n}$ is removed, we can see that Theorem 1.14 does not even consider the simple case where $\mathcal{A} = |t|^{n-d-1}A$ is an arbitrary constant elliptic matrix.

Let $\mathcal{T}_3$ be an $(n-d)\times d$ matrix-valued function and $\mathcal{T}_4$ be an $(n-d)\times(n-d)$ matrix-valued function. We say that $(\mathcal{T}_3, \mathcal{T}_4)$ satisfies (H1) if

$$\mathcal{T}_3 \text{ and } \mathcal{T}_4 \text{ are } x\text{-independent,} \tag{H1}$$

and we say that $(\mathcal{T}_3, \mathcal{T}_4)$ satisfies (H2) if

$$\begin{cases} (\mathcal{T}_3)^T \nabla|t| \ \text{ is } x\text{-independent,} \\ \text{there exists } h : (0, +\infty) \mapsto \mathbb{R} \text{ such that } (\mathcal{T}_4)^T \nabla|t| = h(|t|)\nabla|t|. \end{cases} \tag{H2}$$

In addition, we say that $\mathcal{T}_4$ satisfies (H1)/(H2) if $(0, \mathcal{T}_4)$ satisfies (H1)/(H2). Note that when $d = n-1$, $\mathcal{T}_4$ is a scalar function, and (H1) and (H2) are the same hypothesis.

The condition (H2) for $\mathcal{T}_4$ is neither weaker nor stronger than (H1). Roughly, if $\mathcal{T}_4$ satisfies (H2), then $\nabla|t|$ is an eigenvalue of $\mathcal{T}_4$ and $\mathcal{T}_4$ may depend on x.

Example 1.17. If we set $v(t)$ to be a horizontal vector orthogonal to t and independent of x, for instance $v(t) = (-t_2, t_1, 0, \dots, 0)$, and $a(x)$ to be a vertical vector in $\mathbb{R}^{n-d}$ independent of t, for instance $a(x) = (\cos(x), 0, \dots, 0)^T$, then

$$\mathcal{T}_4 := I_{n-d} + \frac{1}{2|t|} a(x) v(t)$$

satisfies (H2) but not (H1). On the other hand, a matrix $\mathcal{T}_4$ which is constant will satisfy (H1) but not (H2) except if $\mathcal{T}_4$ is actually a scalar multiple of the identity matrix. Also, observe that $\mathcal{T}_4$ can go beyond $b \cdot I_{n-d}$ and still stabilize $\nabla|t|$. Remember that t is seen as a horizontal vector and hence

$$\mathcal{T}_4 := I_{n-d} + \frac{1}{2}\frac{t^T t}{|t|^2}$$

is a matrix that satisfies both (H1) and (H2) but is not the multiplication of the identity matrix by a scalar function.

Our first result states that, if the last $n-d$ lines of $\mathcal{A}$ satisfy either (H1) or (H2), then the elliptic measure and the Lebesgue measure on $\mathbb{R}^d$ are equivalent. Taking matrices as given in Example 1.17 will already allow us to obtain control of the harmonic measure for some elliptic operators not considered in the previous literature (for instance when $\mathcal{A}$ is a constant matrix where $\mathcal{T}_3 \neq 0$ and $\mathcal{T}_4$ is not a scalar multiple of the identity).

Theorem 1.18. *Let $L = -\operatorname{div} |t|^{d+1-n}\mathcal{A}\nabla$ be an elliptic operator satisfying* (1.1)–(1.2). *Assume that L is such that*

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ \mathcal{T}_3 & \mathcal{T}_4 \end{pmatrix}, \quad \textit{where } (\mathcal{T}_3, \mathcal{T}_4) \textit{ satisfies either } \text{(H1)} \textit{ or } \text{(H2)}. \tag{1.19}$$

Then, for any $Y_0 = (y_0, t_0)$ and any Borel set $E \subset \Delta_{Y_0} := B_{\mathbb{R}^d}(y_0, |t_0|)$, we have

$$C^{-1}\frac{|E|}{|\Delta_{Y_0}|} \leq \omega^{Y_0}(E) \leq \frac{|E|}{|\Delta_{Y_0}|}, \tag{1.20}$$

where $|E|$ denotes the d-dimensional Lebesgue measure and $C > 0$ depends only on n, d, and C_L.

For our second result, we consider elliptic operators whose coefficients are close to a matrix of the form in (1.19). We shall show that for such operators the bound in Theorem 1.9 holds by adapting an $S < N$ argument (see [Kenig et al. 2000] and ensuing literature). Our contribution will be the use of the Green function as a substitute for $|t|$, a bit like in [Akman et al. 2023], but we handle the (possible) roughness of the Green function with a much simpler Caccioppoli-type argument.

Theorem 1.21. *Let $L = -\operatorname{div} |t|^{d+1-n}\mathcal{A}\nabla$ be an elliptic operator satisfying* (1.1)–(1.2), *and write the decomposition*

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ \mathcal{B}_3 + \mathcal{C}_3 & b\mathcal{T}_4 + \mathcal{C}_4 \end{pmatrix}, \tag{1.22}$$

where b is a scalar function, $\mathcal{A}_1$ is a $d \times d$ matrix, and the dimensions of $\mathcal{A}_2$, $\mathcal{B}_3$, $\mathcal{C}_3$, $\mathcal{T}_4$, $\mathcal{C}_4$ are such that the matrices complete the $n \times n$ matrix $\mathcal{A}$. Assume that the submatrices of $\mathcal{A}$ satisfy the following:

(a) *$\mathcal{T}_4$ satisfies either* (H1) *or* (H2),

and there exists a constant $K > 0$ such that

(b) $K^{-1} \leq b \leq K$,

(c) $|\mathcal{C}_3| + |\mathcal{C}_4| \in \widetilde{CM}_2(K)$,

(d) $|t||\nabla b| + |t||\operatorname{div}_x(\mathcal{B}_3)^T| + |t|^{n-d}|\operatorname{div}_t(|t|^{d+1-n}\mathcal{B}_3)| \in CM_2(K)$.

Then the hypothesis (1.10) *of Theorem 1.9 is true and thus the elliptic measure ω_L^X is A_∞-absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$.*

Remarks. (i) Theorem 1.14 is a consequence of Theorem 1.21 when $\mathcal{T}_4$ is the identity matrix.

(ii) In the above theorem, when $M = (M_{ij})_{ij}$ is an $(n-d) \times d$ matrix, then the quantity $\operatorname{div}_x M^T$ is a vector in $\mathbb{R}^{n-d}$ whose k-th component is $\sum_{j=1}^{d} \partial_{x_j} M_{kj}$, and similarly the quantity $\operatorname{div}_t M$ is a vector in $\mathbb{R}^d$ whose k-th component is $\sum_{i=1}^{n-d} \partial_{t_i} M_{ik}$.

(iii) When $d = n-1$, $\mathcal{T}_4$ is a scalar function, and (H2) should read "there exists $h : (0, +\infty) \mapsto \mathbb{R}$ such that $\mathcal{T}_4\nabla t = h(t)\nabla t$ for all $t \in (0, +\infty)$", but the later just means that $\mathcal{T}_4$ is x-independent, and thus (H1) and (H2) are the same hypothesis.

(iv) We actually prove a stronger estimate than (1.10); we prove a local $S < N$ L^2-estimate which is stated in (4.10) below. We see a priori no big obstacles in our methods that will stop us from obtaining $N < S$ estimates under the assumptions of Theorem 1.21, and hence from studying the solvability of the Dirichlet problem.

In the next result, we assume a stronger condition on $\mathcal{T}_4$ which will allow us to be slightly more flexible on the bottom left corner of $\mathcal{A}$. In the next lemma, $\mathcal{B}_3$ can satisfy either $|t|^{n-d}|\operatorname{div}_t |t|^{d+1-n}\mathcal{B}_3| \in CM_2$ as in Theorem 1.21, or simply $|t||\operatorname{div}_t \mathcal{B}_3| \in CM_2$.

Theorem 1.23. *Assume that $d < n-2$. Let $L = -\operatorname{div} |t|^{d+1-n}\mathcal{A}\nabla$ be an elliptic operator satisfying* (1.1)–(1.2). *Write the decomposition*

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ \mathcal{B}_3 + \widetilde{\mathcal{B}}_3 + \mathcal{C}_3 & b\mathcal{T}_4 + \mathcal{C}_4 \end{pmatrix}, \tag{1.24}$$

and assume that

(a) $(\mathcal{T}_4)^T \nabla |t| = \nabla |t|$,

and there exists a constant $K > 0$ such that

(b) $K^{-1} \leq b \leq K$,

(c) $|\mathcal{C}_3| + |\mathcal{C}_4| \in \widetilde{CM}_2(K)$,

(d) $|t||\nabla b| + |t||\operatorname{div}_x (\mathcal{B}_3)^T| + |t|^{n-d}|\operatorname{div}_t(|t|^{d+1-n}\mathcal{B}_3)| \in CM_2(K)$,

(e) $|t||\operatorname{div}_x (\widetilde{\mathcal{B}}_3)^T| + |t||\operatorname{div}_t \widetilde{\mathcal{B}}_3| \in CM_2(K)$.

Then the hypothesis (1.10) *of Theorem 1.9 is true and thus the elliptic measure ω_L^X is A_∞-absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$.*

Remark 1.25. The last theorem is a bit unmotivated at the moment. One classical strategy to deal with nonflat boundaries is to make changes of variable. One can thus obtain an equivalent problem where the boundary is better (e.g., flat) but the coefficients of the operators are much worse. See for instance [Kenig and Pipher 2001] in the classical case and [David et al. 2019a] in higher codimension. That is why it is key to obtain, in the flat case, the largest possible set of operators for which the harmonic measure is A_∞-absolute continuous with respect to the Lebesgue measure. The term $\mathcal{B}_3$ is the one that we can treat if we adapt the proof of [Kenig and Pipher 2001] in higher codimension, however $\mathcal{B}_3$ is not well adapted to a change of variable and we would much prefer to use $\mathcal{B}_3$.

In [David et al. 2019a], the authors had to introduce a new (and more complicated) change of variable in order to deal with the case where the boundary is the graph of a Lipschitz function. Still, the construction is limited to graphs of Lipschitz functions with small Lipschitz constant. I claim here that we can deal with big Lipschitz constant if we can allow terms in the form of $\widetilde{\mathcal{B}}_3$ in the bottom left corner of $\mathcal{A}$, as we do in Theorem 1.23.

The full construction of the change of variable that maps the graph of an arbitrary Lipschitz function $\varphi : \mathbb{R}^d \to \mathbb{R}^{n-d}$ to $\mathbb{R}^d$ and that turns the elliptic operator from [David et al. 2019a] into one in the form of (1.24) will not be done here, since it would be too long and technical (and we do not have a new result to prove with it). We will only give a rough idea via an example. If the Lipschitz function is

$$\varphi : x \in \mathbb{R} \mapsto (ax, 0) \in \mathbb{R}^2$$

and its graph is given by $\Phi(x) = (x, \varphi(x))$, then the change of variable that maps $\mathbb{R}$ to the graph of φ constructed in [Kenig and Pipher 2001] would be

$$\rho_1(x, t_1, t_2) = (x, ax + t_1, t_2) = \Phi(x) + (0, t),$$

while the one in [David et al. 2019a] would be

$$\rho_2(x, t_1, t_2) = (x - c_1 t_1, ax + c_2 t_1, t_2) = \Phi(x) + (-c_1 t_1, c_2 t_1, t_2),$$

where $c_1 = a/\sqrt{1+a^2}$ and $c_2 = 1/\sqrt{1+a^2}$ are such that $\Phi(x)$ is orthogonal to $(-c_1 t_1, c_2 t_1, t_2)$. Our alternative is to take

$$\rho_3(x, t_1, t_2) = (x, ax + ct_1, t_2) = \Phi(x) + (0, ct_1, t_2), \quad \text{with } c = \sqrt{1+a^2},$$

which is constructed so that the distance between $\rho_3(x, t_1, t_2)$ and the graph of φ is $|t|$ (like for ρ_2 but where $\rho_3(x, t_1, t_2)$ lies in the plane $\{(x, s_1, s_2),\ (s_1, s_2) \in \mathbb{R}^2\}$, like for ρ_1). If we consider the operator $L = -\operatorname{div} \delta(X)^{-1} \nabla$, where $\delta(X)$ is the distance between X and the graph of φ, then using the change of variable ρ_3 will turn L into $L_3 = -\operatorname{div} |t|^{-1} \mathcal{A}^3 \nabla$ where

$$\mathcal{A}^3 = \begin{pmatrix} c^{-4} & -ac & 0 \\ -ac & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is in the form (1.24), but not in the form (1.15) or (1.22). Replacing an affine function φ by a Lipschitz function included in a plane will already complicate the computations, but if we change Φ to a mollified version $\Phi_{|t|}$ in the construction of ρ_3, then we pretend that it stays "fairly short". Adding the torsion (i.e., when the Lipschitz function is not anymore included in a plane) will complicate the construction even more.

The article is divided as follows. Section 2 introduces the notion of a Green function with pole at infinity and will deduce a relation between this Green function and the elliptic measure that holds whenever L satisfies the ellipticity and boundedness conditions (1.1)–(1.2). Section 3 is devoted to the study of operators of the form (1.19) and proves Theorem 1.18. In Section 4, we demonstrate Theorem 1.21 and 1.23 by establishing a local $S < N$ estimate that implies (1.10).

In the rest of the article, $A \lesssim B$ means that $A \leq CB$ for a constant C whose dependence on the parameters will be stated or will be obvious from context. In addition, $A \approx B$ means $A \lesssim B$ and $B \lesssim A$.

2. General results on the Green function with pole at infinity

In this section, we consider an elliptic operator $L = -\operatorname{div} |t|^{d+1-n} \mathcal{A} \nabla$ satisfying (1.1)–(1.2). Even if this article applies when $\Omega = \mathbb{R}^{d+1}_+$ (if $d = n-1$) or $\Omega = \mathbb{R}^n \setminus \mathbb{R}^d$ (if $d < n-1$), the definitions and results of this section can be easily generalized to domains (and elliptic operators) that enter the scope of the elliptic theory developed in [David et al. 2021b; 2020]. In particular, we only need Ω to satisfy the Harnack chain condition and the corkscrew point condition (see [David et al. 2021b; 2020] for these definitions).

We need a bit of functional theory, which is only needed for the precise statement of Definition 2.4 and Proposition 2.5 below, and can be overlooked. The space

$$W := \left\{ u \in L^1_{\mathrm{loc}}(\Omega) \int_\Omega |\nabla u| \frac{dt\,dx}{|t|^{n-d-1}} < +\infty \right\} \tag{2.1}$$

is equipped with the seminorm $\|u\|_W := \|\nabla u\|_{L^2(\Omega)}$. Observe that $\|\cdot\|_W$ is a norm for $C_0^\infty(\Omega)$ and we write W_0 for the completion of $C_0^\infty(\Omega)$ under $\|\cdot\|_W$. We also define

$$W_0(\overline{\Omega}) := \{u \in W^{1,2}_{\mathrm{loc}}(\Omega) : u\varphi \in W_0 \text{ for any } \varphi \in C_0^\infty(\mathbb{R}^n) < +\infty\}. \tag{2.2}$$

The proof of the properties of W, W_0, and $W_0(\overline{\Omega})$ can be found in [David et al. 2021b; 2020], but let us give a few comments to build the reader's intuition. The spaces W and W_0 are the ones where we find the solutions to the Dirichlet problem $Lu = 0$ in Ω, $u = f \in H^{1/2}(\mathbb{R}^d)$ by using the Lax–Milgram theorem; here $H^{1/2}(\mathbb{R}^d) = W^{1/2,2}(\mathbb{R}^d) = B^{1/2}_{2,2}(\mathbb{R}^d)$ is the (classical) Besov space of traces. The space W_0 is the subspace of W containing the functions with zero trace. The space $W_0(\overline{\Omega})$ is a space bigger than W_0 that possess the same local properties as W_0 but does not have any control when $|(x,t)| \to \infty$.

We recall that $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a weak solution to $Lu = 0$ in Ω if

$$\int_\Omega \mathcal{A}\nabla u \cdot \nabla\varphi \frac{dt\,dx}{|t|^{n-d-1}} = 0 \quad \text{for } \varphi \in C_0^\infty(\Omega). \tag{2.3}$$

Definition 2.4. A Green function (associated to L^*) with pole at infinity is a *positive* weak solution $G := G_{L^*} \in W_0(\overline{\Omega})$ to $L^*u = -\operatorname{div} |t|^{d+1-n}\mathcal{A}^T\nabla u = 0$ in Ω.

Be aware that, in the above definition, the function G is a solution to the adjoint operator $L^* = -\operatorname{div} |t|^{d+1-n}\mathcal{A}^T\nabla$. We prefer here to associate G to the adjoint right away, because it is the appropriate tool we ultimately need for our proofs. But since L and L^* satisfy the same properties (1.1)–(1.2), we have the following.

Proposition 2.5 [David et al. 2021a, Lemma 6.5]. *A Green function with pole at infinity G enjoys the following properties*:

- $G \in C(\overline{\Omega})$, *i.e.*, *$G$ is continuous up to the boundary $\mathbb{R}^d$.*
- $G = 0$ *on* $\mathbb{R}^d$.
- *G is unique up to a constant. We write G_X for the only Green function with pole at infinity which satisfies $G_X(X) = 1$, and the uniqueness gives*

$$G_X(Y)G_Y(X) = 1 \quad \textit{for } X, Y \in \Omega. \tag{2.6}$$

- *Let $G^Y(X)$ be the Green function* (*associated to L^**) *with pole at Y as defined in Chapter* 10 *of* [David et al. 2021b]. *Take $Y_0 = (y_0, t_0) \in \Omega$, and define for $j \in \mathbb{N}$ the point $Y_j = (y_0, 2^j t_0)$. There exists a subsequence $j_n \to \infty$ such that*

$$\frac{G^{Y_{j_n}}}{G^{Y_{j_n}}(Y_0)} \textit{ converges uniformly on compact sets of } \overline{\Omega} \textit{ to } G_{Y_0}. \tag{2.7}$$

Proof. The first two points are a consequence of the De Giorgi–Nash–Moser estimates on weak solutions that can be found (for instance) in [David et al. 2021b, Chapter 8]. The last two points are in Lemma 6.5 from [David et al. 2021a] or in its proof. □

We assign to any point $Z = (z, s) \in \Omega$ the boundary ball

$$\Delta_Z := B(z, |s|) \subset \mathbb{R}^d. \tag{2.8}$$

We apply the comparison principle (see [Caffarelli et al. 1981a] for the codimension 1 case and [David et al. 2021b] for the higher codimension) to compare the Green function with pole at infinity and the elliptic measure.

Lemma 2.9. *Let $Y_0 = (y_0, t_0) \in \Omega$. If $X = (x, t) \in \Omega$ satisfies $x \in B(y_0, 2|t_0|)$ and $0 < |t| < 2|t_0|$, we have that*

$$C^{-1} G_{Y_0}(X) \leq \left(\frac{|t|}{|t_0|}\right)^{1-d} \omega^{Y_0}(\Delta_X) \leq C G_{Y_0}(X), \tag{2.10}$$

where $C > 0$ depends only on n, d, and C_L. Here $G_{Y_0} = G_{L^*,Y_0}$ is defined in Proposition 2.5 and is the Green function associated to L^* with pole at infinity, and $\omega^{Y_0} = \omega_L^{Y_0}$ is the elliptic measure associated to L with pole at Y_0 defined in (1.5).

Remark 2.11. We can use the uniqueness of the Green function (2.6) to get an estimate of $G_{Y_0}(X)$ using the elliptic measure when X is far from Y_0.

Proof. We need to invoke some results from [David et al. 2021b]. The classical case $d = n - 1$ is not included in that work but is either already known to the reader or can be found in the last section of [David et al. 2020].

Let $Y_1 = (y_0, 4t_0)$. The change of pole property [David et al. 2021b, Lemma 11.16] states that, for any Borel set $E \subset \Delta_{Y_1} = 4\Delta_{Y_0}$ and any $Y \in \Omega$ satisfying $|Y - y_0| > 8t_0$, we have

$$\omega^{Y_1}(E) \approx \frac{\omega^Y(E)}{\omega^Y(\Delta_{Y_1})}, \tag{2.12}$$

with constants that depend only on n, d, and C_L. Together with the doubling property of the elliptic measure [David et al. 2021b, Lemma 11.12] and the Harnack inequality [David et al. 2021b, Lemma 8.9], we deduce that, for the same set E, point Y, and with constants that depend on the same parameters, we have

$$\omega^{Y_0}(E) \approx \frac{\omega^Y(E)}{\omega^Y(\Delta_{Y_0})}. \tag{2.13}$$

For our second result, we want to compare the Green function and the elliptic measure. Let $g^X(Y)$ be the Green function associated to L with pole in X. Then [David et al. 2021b, Lemma 10.6] implies that

$$G^Y(X) = g^X(Y) \quad \text{for } X, Y \in \Omega. \tag{2.14}$$

Moreover, [David et al. 2021b, Lemma 11.11] gives, for $X = (x, t) \in \Omega$ and $Y \in \Omega \setminus B_{\mathbb{R}^n}(x, 2|t|)$,

$$|t|^{d-1} g^X(Y) \approx \omega^Y(\Delta_X), \tag{2.15}$$

with constants that depend only on n, d, and C_L. So the combination of (2.14) and (2.15) implies, for $X = (x,t) \in \Omega$, that

$$|t|^{d-1} G^Y(X) \approx \omega^Y(\Delta_X) \quad \text{for } Y \in \Omega \setminus B_{\mathbb{R}^n}(x, 2|t|). \tag{2.16}$$

The proof of the lemma is then pretty easy. Let Y_0 and X be as in the assumptions of the lemma. For any Y far enough from Y_0, we use (2.16) to obtain

$$\frac{G^Y(X)}{G^Y(Y_0)} \approx \frac{|t|^{1-d}\omega^Y(\Delta_X)}{|t_0|^{1-d}\omega^Y(\Delta_{Y_0})},$$

but, since the conditions on X and Y_0 imply $E := \Delta_X \subset 4\Delta_{Y_0}$, the estimate (2.13) yields

$$\frac{G^Y(X)}{G^Y(Y_0)} \approx \left(\frac{|t|}{|t_0|}\right)^{1-d} \omega^{Y_0}(\Delta_X).$$

The above bounds on $G^Y/G^Y(Y_0)$ are uniform in Y, therefore, by (2.7), those bounds are transferred to G_{Y_0}. The lemma follows. □

3. x-independent Green functions with pole at infinity

In this section, we shall make two easy observations: first, that the Green function, associated to

$$L^* = -\operatorname{div} |t|^{n-d-1} \mathcal{A}^T \nabla$$

as in Section 2, with pole at infinity is independent of x whenever $\mathcal{A}$ is x-independent; and second, if both $\mathcal{A}$ and the Green function G with pole at infinity are x-independent, then G does not depend on the first $n-d$ lines of $\mathcal{A}$. We shall invoke, in addition, the uniqueness of the Green function with pole at infinity and (2.12) to deduce that the elliptic measure and the Lebesgue measure are equivalent on $\mathbb{R}^d$ whenever the last $n-d$ lines of $\mathcal{A}$ are x-independent.

Lemma 3.1. *Let $L = -\operatorname{div} |t|^{d+1-n} \mathcal{A}\nabla$ be an elliptic operator satisfying* (1.1)–(1.2) *and where $\mathcal{A}$ is as in* (1.19). *Then the Green function* (*associated to L^**) *with pole at infinity is x-independent and satisfies, for any $Y_0 = (y_0, t_0)$ and $X = (x,t)$ in Ω,*

$$C^{-1} \frac{|t|}{|t_0|} \le G_{Y_0}(X) \le C \frac{|t|}{|t_0|}, \tag{3.2}$$

where the constants depend only on $n-d$ and C_L.

Proof. The proof is similar under either assumption, (H1) or (H2). We know that the Green function with pole at infinity has to depend on $|t|$, but it does not need to depend on x or $t/|t|$. When $(\mathcal{T}_3, \mathcal{T}_4)$ satisfies (H2), L stabilizes the space of functions that depend on $|t|$, and thus by uniqueness the Green function will depend only on $|t|$. When $(\mathcal{T}_3, \mathcal{T}_4)$ satisfies (H1), L stabilizes the space of functions that are x-independent, and hence the Green function will be independent of x.

Case 1: $(\mathcal{T}_3, \mathcal{T}_4)$ satisfies (H1). Define

$$L_0 := -\operatorname{div} |t|^{d+1-n} \mathcal{T}_4 \nabla. \tag{3.3}$$

The operator L_0 is an elliptic operator on $\mathbb{R}^{n-d} \setminus \{0\}$ satisfying the ellipticity and boundedness conditions (1.1)–(1.2) with the same constant C_L as L.

When $d < n-1$, the space $\mathbb{R}^{n-d} \setminus \{0\}$ and the operator L_0 enter the scope of the elliptic theory developed in [David et al. 2021b] or [David et al. 2020],[1] and so all the results in Section 2 hold. Of course, the study of the elliptic measure of L_0, where the boundary is reduced to the point $\{0\}$, is trivial and hence not very interesting. But using this easy case will allow us to find a good candidate for the Green function with pole at infinity for L^*. Let $\omega^X_{L_0}$ be the elliptic measure on $\{0\}$ and $G_{(L_0)^*,t_0}$ be the Green function with pole at infinity (associated to $(L_0)^*$) which takes the value 1 at t_0. Lemma 2.9 implies, for $|t| < 2|t_0|$, that

$$G_{(L_0)^*,t_0}(t) \approx \frac{|t|}{|t_0|} \omega^{Y_0}(\Delta_t) = \frac{|t|}{|t_0|} \omega^{Y_0}(\{0\}) = \frac{|t|}{|t_0|}.$$

The probability measure $\omega^X_{L_0}$ on $\{0\}$ obviously satisfies $\omega^X_{L_0}(\{0\}) = 1$, hence

$$G_{(L_0)^*,t_0}(t) \approx \frac{|t|}{|t_0|} \quad \text{for } |t| < 2|t_0|. \tag{3.4}$$

When $|t| \geq 2|t_0|$, we use (2.6) and (3.4) to write

$$G_{(L_0)^*,t_0}(t) = [G_{(L_0)^*,t}(t_0)]^{-1} \approx \left(\frac{|t_0|}{|t|}\right)^{-1} = \frac{|t|}{|t_0|}.$$

We conclude, for any $t, t_0 \in \mathbb{R}^{n-d}$, that

$$G_{(L_0)^*,t_0}(t) \approx \frac{|t|}{|t_0|}. \tag{3.5}$$

When $d = n-1$, the result (3.5) holds without the need of Lemma 2.9. The operator L_0 is defined on the half line, and there exists $f(t)$ defined on $(0, +\infty)$ such that $L_0 = \partial_t f(t) \partial_t$ and $f(t) \approx 1$ in order to satisfy the ellipticity and boundedness conditions. A simple exercise of integration shows that the Green functions with pole at infinity of $(L_0)^* = L_0$ are

$$G_{(L_0)^*}(t) = K \int_0^t \frac{dt}{f(t)} \approx C|t|, \tag{3.6}$$

where K is any positive constant, and thus (3.5) follows easily.

We set, for $Y_0 = (y_0, t_0) \in \Omega$ and $X = (x, t) \in \Omega$,

$$H_{Y_0}(X) := G_{(L_0)^*,t_0}(t). \tag{3.7}$$

[1] Maybe also when $d = n-1$, but let us not take any risks.

Because of the x-independence of H_{Y_0} and $\mathcal{T}_3$, we have for $\varphi \in C_0^\infty(\Omega)$ that

$$\int_\Omega \mathcal{A}^T \nabla H_{Y_0} \cdot \nabla \varphi \frac{dt\, dx}{|t|^{n-d-1}}$$
$$= \int_{\mathbb{R}^{n-d}} (\mathcal{T}_3)^T \nabla_t G_{(L_0)^*,t_0} \cdot \left(\int_{\mathbb{R}^d} \nabla_x \varphi\, dx \right) \frac{dt}{|t|^{n-d-1}} + \int_{\mathbb{R}^d} \int_{\mathbb{R}^{n-d}} (\mathcal{T}_4)^T \nabla_t G_{(L_0)^*,t_0} \cdot \nabla_t \varphi \frac{dt}{|t|^{n-d-1}}\, dx.$$

The first integral on the right-hand side above is 0 because $\int_{\mathbb{R}^d} \nabla_x \varphi(x,t)\, dx = 0$ for all t. The second integral is also 0 because $G_{(L_0)^*,t_0}$ is a weak solution to $(L_0)^*$. So H_{Y_0} is a weak solution to L^*. Moreover, $H_{Y_0} \in W_0(\mathbb{R}^n)$ because $G_{(L_0)^*,t_0} \in W_0(\mathbb{R}^{n-d})$. By the uniqueness given in Proposition 2.5, we necessarily have

$$G_{Y_0}(X) = H_{Y_0}(X) := G_{(L_0)^*,t_0}(t). \tag{3.8}$$

As a consequence, G_{Y_0} is x-independent, and the conclusion (3.2) of the lemma follows from (3.5).

Case 2: $(\mathcal{T}_3, \mathcal{T}_4)$ satisfies (H2). In this case, the proof is a simple exercise of integration. By (1.1) and (1.2), we have

$$(C_L)^{-1} \big|\nabla|t|\big|^2 \le \mathcal{T}_4 \nabla|t| \cdot \nabla|t| \le C_L \big|\nabla|t|\big|^2 \quad \text{for all } t \in \mathbb{R}^{n-d} \setminus \{0\}.$$

Since $\big|\nabla|t|\big| = 1$, our assumption on $\mathcal{T}_4$ implies that

$$(C_L)^{-1} \le h(|t|) \le C_L \quad \text{for all } t \in \mathbb{R}^{n-d} \setminus \{0\}. \tag{3.9}$$

We define g_{r_0} as

$$g_{r_0} = K_{r_0} \int_0^r \frac{1}{h(r)}\, dr, \tag{3.10}$$

where K is chosen such that $g_{r_0}(r_0) = 1$. Our bounds on h yield

$$g_{r_0} \approx \frac{r}{r_0}. \tag{3.11}$$

We construct now $H_{Y_0}(X)$ for $Y_0 = (y_0, t_0) \in \Omega$ and $X = (x,t) \in \Omega$ as

$$H_{Y_0}(X) := g_{|t_0|}(|t|). \tag{3.12}$$

Observe that since H_{Y_0} depends only on $|t|$, we have

$$L^* H_{Y_0}(X) = g'_{|t_0|}(|t|) \operatorname{div}_x (\mathcal{T}_3)^T \nabla|t| + \operatorname{div}_t [g'_{|t_0|}(|t|)(\mathcal{T}_4)^T \nabla|t|]$$
$$= 0 + K_{|t_0|} \operatorname{div}_t \frac{1}{h(|t|)} h(|t|) = 0,$$

thanks to the conditions (H2) and the definition (3.10). In addition, H_{Y_0} is Lipschitz by (3.10)–(3.9) and is 0 on the boundary, therefore it lies in $W_0(\overline{\Omega})$. So again by uniqueness of the Green function with pole at infinity (see Proposition 2.5), we have $G_{Y_0} = H_{Y_0}$. The conclusion (3.2) is then an easy consequence of (3.12) and (3.11). □

Remark 3.13. An interesting consequence of the above proof, for instance (3.6), is that for a general operator of the form $L = -\operatorname{div} |t|^{d+1-n} \mathcal{A}\nabla$, knowing that the elliptic measure is A_∞-absolute continuous with respect to the Lebesgue measure (or even equivalent to the Lebesgue measure) will not help us to get a lot of control on t-derivatives of the Green functions with pole at infinity. Indeed, it is possible to take h to be any arbitrary function in L^∞ that stays between 1 and 2. In this case, g_{r_0} defined in (3.10) and G_{Y_0} are only Lipschitz. In particular, the nontangential limit of $|\nabla G|$ at the boundary may not exist in any reasonable sense (only inferior and superior limits exist). It means that the estimates on the Green function obtained in [David et al. 2023; 2022] are not equivalent to the A_∞-absolute continuity of the harmonic measure without any restriction on the elliptic operator $\mathcal{L}$.

Corollary 3.14. *Let $L = -\operatorname{div} |t|^{d+1-n} \mathcal{A}\nabla$ be an elliptic operator satisfying* (1.1)–(1.2). *Assume that $\mathcal{A}$ can be written as*

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ 0 & \mathcal{T}_4 \end{pmatrix}, \tag{3.15}$$

where $\mathcal{T}_4 \nabla |t| = \nabla |t|$ for all $t \in \mathbb{R}^{n-d}$.

Then, for $X = (x, t) \in \Omega$ and $Y_0 = (y_0, t_0) \in \Omega$, the Green function with pole at infinity satisfies

$$G_{Y_0}(X) = \frac{|t|}{|t_0|}. \tag{3.16}$$

Proof. Under our assumptions, $\mathcal{T}_4$ satisfies (H2) with $h(r) \equiv 1$. From the proof of Lemma 3.1, we have $G_{Y_0}(X) = g_{|t_0|}(|t|)$ where $g_{r_0}(r)$ is given by (3.10). The lemma follows. □

Proof of Theorem 1.18. Lemma 3.1 easily implies the equivalence between the harmonic measure and the surface measure. It was already done in the proof of Theorem 6.7 in [David et al. 2021a], but let us repeat it for completeness. Take $x \in \Delta_{Y_0}$. The combination of Lemma 2.9 and Lemma 3.1 requires, for any $0 < r < |t_0|$ and any $X = (x, t)$ satisfying $|t| = r$, that

$$\omega^{Y_0}(B_{\mathbb{R}^d}(x, r)) \approx G_{Y_0}(X) \left(\frac{|t|}{|t_0|} \right)^{d-1} \approx \left(\frac{|t|}{|t_0|} \right)^{d} = \frac{|B_{\mathbb{R}^d}(x, r)|}{|\Delta_{Y_0}|}. \tag{3.17}$$

In particular, the measure is absolutely continuous with respect to the d-dimensional Lebesgue measure on $\mathbb{R}^d$, and, by the Lebesgue differentiation theorem, the Poisson kernel $k^{Y_0} := d\omega^{Y_0}/d\mathcal{L}^d$ satisfies, for almost any $x \in \Delta_{Y_0}$,

$$k^{Y_0}(x) = \lim_{r \to 0} \frac{\omega^{Y_0}(B_{\mathbb{R}^d}(x, r))}{|B_{\mathbb{R}^d}(x, r)|} \approx \frac{1}{|\Delta_{Y_0}|}.$$

The theorem follows by integrating k^{Y_0} over E. □

4. Proof of Theorems 1.21 and 1.23

The proof of Theorems 1.21 and 1.23 will rely on an S vs N argument, where S is the square function (which will not be introduced here but is related to the left-hand side of (4.10)) and N is the nontangential maximal function. The importance of the two functionals S and N for the A_∞-absolute continuity of the harmonic measure was noted in [Kenig et al. 2000], and the general method to compare S and N (when Carleson measures are involved) was first found in [Kenig and Pipher 2001]. In [Kenig et al. 2016], it was observed that being able to bound the L^p-norm of S by the L^p-norm of N is enough to guarantee the absolute continuity of the harmonic measure, which is basically our Theorem 1.9. The adaptation of the methods to higher codimensional boundaries can be found in Sections 7 and 8 of [David et al. 2019a] and in [Feneuil et al. 2021].

Let $1 \leq d < n$ be integers, and let $\Omega = \mathbb{R}^n_+ := \{(x,t) \in \mathbb{R}^d \times (0,+\infty)\}$ if $d = n-1$ and $\Omega = \mathbb{R}^n \setminus \mathbb{R}^d := \{(x,t) \in \mathbb{R}^d \times \mathbb{R}^{n-d},\ t \neq 0\}$ if $d < n-1$. The nontangential maximal functions N and $\widetilde{N}$ are defined for any continuous function v on Ω and any $x \in \mathbb{R}^d$ as

$$N(v)(x) = \sup_{(y,t)\in\gamma(x)} |v| \quad \text{and} \quad \widetilde{N}(v)(x) = \sup_{(y,t)\in\gamma(x)} \left(\fint_{|Z-(y,t)|<|t|/4} |v|^2\, dZ \right)^{1/2}, \tag{4.1}$$

where

$$\gamma(x) = \{(y,t) \in \Omega,\ |y-x| < |t|\}. \tag{4.2}$$

We shall introduce here the variants $\gamma_{10}(x) := \{(y,t) \in \Omega,\ |y-x| < 10|t|\}$ and $N_{10}(v)(x) := \sup_{\gamma_{10}(x)} |v|$. They will be used to compare $\widetilde{N}$ and N. Indeed, we have the pointwise bound $\widetilde{N}(v)(x) \leq N_{10}(v)(x)$ and it is well known (see [Stein 1993], Chapter II, Section 2.5.1) that $\|N_{10}(v)\|_2 \approx \|N(v)\|_2$. Altogether,

$$\|\widetilde{N}(v)\|_{L^2(\mathbb{R}^d)} \leq \|N_{10}(v)\|_{L^2(\mathbb{R}^d)} \approx \|N(v)\|_{L^2(\mathbb{R}^d)}. \tag{4.3}$$

We recall that the nontangential maximal functions behave well with the Carleson measure condition (1.12) and (1.13). Indeed, if v is a continuous function on Ω and $f \in CM_2(K)$, then we have the Carleson inequality

$$\int_\Omega f^2 v^2 \frac{dx\,dt}{|t|^{n-d}} \lesssim K \|N(v)\|^2_{L^2(\mathbb{R}^d)}, \tag{4.4}$$

and similarly, if $g \in \widetilde{CM}_2(K)$, then

$$\int_\Omega g^2 v^2 \frac{dx\,dt}{|t|^{n-d}} \lesssim K \|\widetilde{N}(v)\|^2_{L^2(\mathbb{R}^d)} \lesssim K \|N(v)\|^2_{L^2(\mathbb{R}^d)}. \tag{4.5}$$

Combining (4.4) and (4.5) with the Cauchy–Schwarz inequality, for all $w \in L^2_{\mathrm{loc}}(\Omega)$, one has

$$\int_\Omega f v w \frac{dx\,dt}{|t|^{n-d}} \leq C K^{1/2} \|N(v)\|_{L^2(\mathbb{R}^d)} \left(\int_\Omega w^2 \frac{dx\,dt}{|t|^{n-d}} \right)^{1/2}, \tag{4.6}$$

$$\int_\Omega g v w \frac{dx\,dt}{|t|^{n-d}} \leq C K^{1/2} \|\widetilde{N}(v)\|_{L^2(\mathbb{R}^d)} \left(\int_\Omega w^2 \frac{dx\,dt}{|t|^{n-d}} \right)^{1/2}. \tag{4.7}$$

We also introduce cut-off functions associated to tent sets. Choose a smooth function $\phi \in C_0^\infty(\mathbb{R})$ such that $0 \leq \phi \leq 1$, $\phi \equiv 1$ on $(-1,1)$, $\phi \equiv 0$ on $(2,+\infty)$, and $|\phi'| \leq 2$. For a ball $B := B(x,r) \subset \mathbb{R}^d$, we define Ψ_B as

$$\Psi_B(y,t) = \phi\left(\frac{\operatorname{dist}(x,B)}{|t|} \right) \phi\left(\frac{|t|}{r} \right). \tag{4.8}$$

We also associate to B the tent set $T_B := \{(x,t) \in \Omega : x \in B,\ |t| \leq r\}$. The function Ψ_B is such that $\Psi \equiv 1$ on T_B and $\Psi \equiv 0$ on $\Omega \subset T_{2B}$. Note that, if a different definition of tent sets is used, we can easily change the definition of Ψ_B so that Ψ_B is adapted to the other definition of tent sets.

Theorems 1.21 and 1.23 are consequences of the following lemma.

Lemma 4.9. *If $L = -\operatorname{div} |t|^{d+1-n} \nabla$ satisfies the assumptions of either Theorem 1.21 or Theorem 1.23, then, for any ball $B = B(x,r) \subset \mathbb{R}^d$ and for any bounded weak solution u to $Lu = 0$, we have*

$$\int_\Omega |\nabla u|^2 \Psi_B^4 \frac{dt\,dx}{|t|^{n-d-2}} \leq C(1+K) \|N(u\Psi_B)\|^2_{L^2(\mathbb{R}^d)}, \tag{4.10}$$

where $C > 0$ depends only on n, d, and C_L.

Proof of Theorems 1.21 and 1.23 from Lemma 4.9. We only need to show that (4.10) implies (1.10). Take the function $u_H(X) := \omega_L^X(H)$, which is a weak solution to $Lu = 0$ bounded by 1. Pick $x \in \mathbb{R}^d$ and $r > 0$, and define $B := B(x,r) \subset \mathbb{R}^d$. The function Ψ_B is 1 on $B(x,r) \times \{t \in \mathbb{R}^{n-d},\ 0 < |t| < r\}$, so Lemma 4.9 gives

$$\int_{B(x,r)} \int_{|t|<r} |t \nabla u_H|^2 \frac{dy\,dt}{|t|^{n-d}} \lesssim \|N(u_H \Psi_B)\|^2_{L^2(\mathbb{R}^d)}.$$

The function $N(u_H\Psi_B)$is bounded by 1 and is supported on $4B$ (since $u_H\Psi_B$ is supported on T_{2B}). As a consequence, the above bound becomes

$$\int_{B_{\mathbb{R}^d}(x,r)} \int_{|t|<r} |t \nabla u_H|^2 \frac{dy\,dt}{|t|^{n-d}} \lesssim |B(x,4r)| \lesssim |B(x,r)|.$$

The bound (1.10) and thus the theorems follow. □

Proof of Lemma 4.9. The proof will be largely identical under the two kinds of assumptions that we have (the ones from Theorem 1.21 and the ones from Theorem 1.23). The proof will split at the very end (in Step 5), when we consider terms involving $\mathcal{B}_3$ and $\widetilde{\mathcal{B}}_3$ (the bottom left corner of $\mathcal{A}$), which need to be addressed in a different (yet somehow related) manner.

Our proof will follow the outline of the one of Theorem 7.10 in [David et al. 2019a], but will be significantly different on two occasions. First, in Step 3, we give a simple Caccioppoli-type argument to deal with the possible nonsmoothness of the Green function with pole at infinity, which will replace here what was $|t|$ in [David et al. 2019a]. And in Step 5, we will deal with the terms $\widetilde{\mathcal{B}}_3$, which were considered in neither [David et al. 2019a] nor [Feneuil et al. 2021].

Step 1: Carleson estimates on the cut-off functions. In order to deal with finite quantities, we need to refine our cut-off function Ψ_B. We define $\Psi_{B,\epsilon}$ as

$$\Psi_{B,\epsilon}(y,t) = \Psi_B(y,t)\phi\left(\frac{\epsilon}{|t|}\right), \tag{4.11}$$

where ϕ is the smooth function introduced above (4.8) and was already used to define Ψ_B. We first gather some properties of the cut-off function $\Psi_{B,\epsilon}$. Observe that

$$|\nabla \Psi_{B,\epsilon}(y,t)| \lesssim \frac{1}{|t|} \quad \text{for } (y,t) \in \Omega, \tag{4.12}$$

and $\nabla\Psi_{B,\epsilon}$ is supported on $E_1 \cup E_2 \cup E_3$, where

$$E_1 := \{(y,t) \in \Omega,\ \operatorname{dist}(y,B) \le 2|t| \le 2\operatorname{dist}(y,B)\},$$
$$E_2 := \{(y,t) \in \Omega,\ r(B) \le |t| \le 2r(B)\},$$

with $r(B)$ being the radius of B, and

$$E_3 := \{(y,t) \in \Omega,\ |t| \le \epsilon \le 2|t|\}.$$

So we deduce that

$$|t||\nabla\Psi_{B,\epsilon}(y,t)| + |t|^2|\nabla\Psi_{B,\epsilon}(y,t)|^2 \lesssim \mathbb{1}_{E_1\cup E_2\cup E_3}(y,t). \tag{4.13}$$

We will need the fact that $|t||\nabla\Psi_{B,\epsilon}(y,t)|$ and $(|t||\nabla\Psi_{B,\epsilon}(y,t)|)^{1/2}$ satisfy the Carleson measure condition $\widetilde{CM}_2(M)$ for some uniform constant M which, combined with (4.4), implies, for any continuous function v, that

$$\int_\Omega |t||\nabla\Psi_{B,\epsilon}(y,t)|v^2\frac{dt\,dx}{|t|^{n-d}} + \int_\Omega |t||\nabla\Psi_{B,\epsilon}(y,t)|v^2\frac{dt\,dx}{|t|^{n-d}} \lesssim \|\widetilde{N}(v)\|^2_{L^2(\mathbb{R}^d)}. \tag{4.14}$$

Of course, thanks to (4.3), if (4.14) is true, then we also have the analogue estimate where $\widetilde{N}$ is replaced by N. Thanks to (4.13), the claim (4.14) will be then proven if we can show that $\mathbb{1}_{E_1\cup E_2\cup E_3} \in \widetilde{CM}_2(M)$, that is

$$\sup_{x\in\mathbb{R}^d,\, r>0} \fint_{B_{\mathbb{R}^d}(x,r)} \int_{|t|<r} \sup_{|Z-(y,t)|<|t|/4} |\mathbb{1}_{E_1\cup E_2\cup E_3}(Z)|^2 \frac{dy\,dt}{|t|^{n-d}} \lesssim 1. \tag{4.15}$$

However, (4.15) is an immediate consequence of the fact that, for each $y \in \mathbb{R}^d$,

$$\begin{aligned}\int_{t\in\mathbb{R}^{n-d}} &\sup_{|Z-(y,t)|<|t|/4} |\mathbb{1}_{E_1\cup E_2\cup E_3}(Z)|^2 \frac{dt}{|t|^{n-d}} \\ &\le \int_{\operatorname{dist}(y,B)/4\le|t|\le 2\operatorname{dist}(y,B)} \frac{dt}{|t|^{n-d}} + \int_{r(B)/2\le|t|\le 4r(B)} \frac{dt}{|t|^{n-d}} + \int_{\epsilon/4\le|t|\le 2\epsilon} \frac{dt}{|t|^{n-d}} \lesssim 1.\end{aligned}$$

The claim (4.14) follows.

Step 2: introduction of G. First, we decompose L as

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ \mathcal{B}_3+\widetilde{\mathcal{B}}_3+\mathcal{C}_3 & b\mathcal{T}_4+\mathcal{C}_4 \end{pmatrix}, \tag{4.16}$$

so that it includes the assumptions of both Theorem 1.21 and Theorem 1.23. In particular, we have

$$\begin{gathered} |\mathcal{C}_3| + |\mathcal{C}_4| \in \widetilde{CM}_2(K), \\ |t||\nabla b|+|t||\operatorname{div}_x(\mathcal{B}_3)^T|+|t|^{n-d}|\operatorname{div}_t(|t|^{d+1-n}\mathcal{B}_3)|+|t||\operatorname{div}_x(\widetilde{\mathcal{B}}_3)^T|+|t||\operatorname{div}_t\widetilde{\mathcal{B}}_3| \in CM_2(3K). \end{gathered} \tag{4.17}$$

We set $L_0 := -\operatorname{div}|t|^{d+1-n}\mathcal{A}_0\nabla$, where

$$\mathcal{A}_0 := \begin{pmatrix} \frac{1}{b}\mathcal{A}_1 & \frac{1}{b}\mathcal{A}_2 \\ 0 & \mathcal{T}_4 \end{pmatrix}, \tag{4.18}$$

which satisfies

$$\frac{1}{b}\mathcal{A} = \mathcal{A}_0 + \frac{1}{b}\begin{pmatrix} 0 & 0 \\ \mathcal{B}_3 + \widetilde{\mathcal{B}}_3 + \mathcal{C}_3 & \mathcal{C}_4 \end{pmatrix}. \tag{4.19}$$

Let $Y_0 = (y_0, t_0) \in \Omega$ be such that $|t_0| = 1$, and write G for G_{Y_0}, the Green function associated to $(L_0)^*$ with pole at infinity. The important properties of G for this proof are first that $G \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a weak solution to $(L_0)^* u = 0$ in Ω, that is

$$\int_\Omega \mathcal{A}_0 \nabla\varphi \cdot \nabla G \, \frac{dt\,dx}{|t|^{n-d-1}} = 0 \quad \text{for any compactly supported } \varphi \in W^{1,2}(\Omega), \tag{4.20}$$

and second, that Lemma 3.1 requires that

$$G \text{ is } x\text{-independent and } G(X) \approx |t| \quad \text{for all } X = (x,t) \in \Omega. \tag{4.21}$$

Step 3: estimation of $\|\widetilde{N}(u\Psi^2_{B,\epsilon}\nabla G)\|_2$. If the goal were to only obtain (1.10), we would not need to go through the same computations, we would just have to prove

$$\|\widetilde{N}(u_H \Psi^2_{B,\epsilon}\nabla G)\|_{L^2(\mathbb{R}^d)} \lesssim |B|. \tag{4.22}$$

Since G is a weak solution to $L_0 u = 0$, Caccioppoli's inequality yields

$$\fint_{|Z-(y,t)|<|t|/4} |\nabla G|^2 \, dZ \lesssim \frac{1}{|t|^2} \fint_{|Z-(y,t)|<|t|/2} |G|^2 \, dZ \quad \text{for } (y,t) \in \Omega.$$

But since $G \approx |t|$ by (4.21), the above inequality becomes

$$\fint_{|Z-(y,t)|<|t|/4} |\nabla G|^2 \, dZ \lesssim 1.$$

We take the supremum on $(y,t) \in \gamma(x)$ and then integrate on $x \in 100B$, and we get

$$|B| \gtrsim \|\widetilde{N}(\nabla G)\|_{L^2(100B)} \gtrsim \|\widetilde{N}(u_H \Psi^2_{B,\epsilon}\nabla G)\|_{L^2(\mathbb{R}^d)}$$

because $u_H \leq 1$ by construction. The claim (4.22) follows.

However, what we really need in order to prove the inequality (4.10) is

$$\|\widetilde{N}(u\Psi^2_{B,\epsilon}\nabla G)\|_{L^2(\mathbb{R}^d)} \lesssim \|N(u\Psi_{B,\epsilon})\|_{L^2(\mathbb{R}^d)}, \tag{4.23}$$

where u is any weak solution of $Lu = 0$ which is bounded on T_{2B}. To reach this goal, we first need the following Caccioppoli inequality. Let $D \subset \mathbb{R}^n$ be a ball of radius r such that $4D \subset \Omega$ and $5D \cap \partial\Omega \neq \varnothing$. In particular, we have

$$G(X) \approx |t| \approx r \quad \text{for } X = (x,t) \in 2D \tag{4.24}$$

by (4.21). Let Ψ be a function such that $0 \leq \Psi \leq 1$ and $|\nabla\Psi| \lesssim 1/|t|$, and let u be a weak solution to $Lu = 0$. We claim that

$$\fint_D |\nabla G|^2 u^2 \Psi^4 \, dX \lesssim \frac{1}{r^2} \fint_{2D} |u|^2 \Psi^2 \, dX. \tag{4.25}$$

Let Φ be such that $0 \leq \Phi \leq 1$, $\Phi \equiv 1$ on D, $\Phi \equiv 0$ outside $\frac{4}{3}D$, and $|\nabla \Phi| \leq 5r$. Then

$$\int_D |\nabla G|^2 u^2 \Psi^4 \, dX \leq T := \int_D |\nabla G|^2 u^2 \Psi^4 \Phi^2 \, dX. \tag{4.26}$$

The function G is a weak solution of $L_0 u = 0$, so, by the ellipticity of $\mathcal{A}_0$ and since the weight satisfies $|t|^{d+1-n} \approx r^{d+1-n}$ on $2D$, we have

$$\begin{aligned} T &\lesssim \iint_\Omega \mathcal{A}_0 \nabla G \cdot \nabla G u^2 \Psi^4 \Phi^2 \, \frac{dt\,dx}{|t|^{n-d-1}} \\ &= \iint_\Omega \mathcal{A}_0 \nabla[G u^2 \Psi^4 \Phi^2] \cdot \nabla G \, \frac{dt\,dx}{|t|^{n-d-1}} - 2 \iint_\Omega \mathcal{A}_0 \nabla u \cdot \nabla G \, (G u \Psi^4 \Phi^2) \, \frac{dt\,dx}{|t|^{n-d-1}} \\ &\qquad - 2 \iint_\Omega \mathcal{A}_0 \nabla \Phi \cdot \nabla G \, (G u^2 \Psi^4 \Phi) \, \frac{dt\,dx}{|t|^{n-d-1}} - 4 \iint_\Omega \mathcal{A}_0 \nabla \Psi \cdot \nabla G \, (G u^2 \Psi^3 \Phi^2) \, \frac{dt\,dx}{|t|^{n-d-1}} \\ &:= T_1 + T_2 + T_3 + T_4. \end{aligned}$$

The functions G, u, Φ, and Ψ all belong to $L^\infty(2D) \cap W^{1,2}(2D)$, so $G u^2 \Psi^4 \Phi^2$ is a valid test function and (4.20) gives that $T_1 = 0$. By the boundedness of $\mathcal{A}_0$ and Cauchy–Schwarz's inequality, the terms T_2, T_3, and T_4 can be bounded as follows. We have

$$|T_3| \lesssim T^{1/2} \bigg(\iint_\Omega |\nabla \Phi|^2 G^2 u^2 \Psi^4 \frac{dt\,dx}{|t|^{n-d-1}} \bigg)^{1/2} \lesssim T^{1/2} \bigg(\iint_{4D/3} u^2 \Psi^4 \frac{dt\,dx}{|t|^{n-d-1}} \bigg)^{1/2}$$

because $|\nabla \Phi| \lesssim 1/r \approx 1/G$ on $2D$. Similarly

$$|T_4| \lesssim T^{1/2} \bigg(\iint_\Omega |\nabla \Psi| G^2 u^2 \Psi^2 \Phi^2 \frac{dt\,dx}{|t|^{n-d-1}} \bigg)^{1/2} \lesssim T^{1/2} \bigg(\iint_{4D/3} u^2 \Psi^2 \frac{dt\,dx}{|t|^{n-d-1}} \bigg)^{1/2}$$

because $|\nabla \Psi| \lesssim 1/|t| \approx 1/G$ on $2D$. At last

$$|T_2| \lesssim T^{1/2} \bigg(\iint_\Omega |\nabla u|^2 G^2 \Psi^4 \Phi^2 \frac{dt\,dx}{|t|^{n-d-1}} \bigg)^{1/2} \lesssim T^{1/2} \bigg(r^2 \iint_{4D/3} |\nabla u|^2 \Psi^4 \frac{dt\,dx}{|t|^{n-d-1}} \bigg)^{1/2}.$$

We deduce that

$$T \lesssim T^{1/2} \bigg(\iint_{4D/3} u^2 \Psi^2 \frac{dt\,dx}{|t|^{n-d-1}} + r^2 \iint_{4D/3} |\nabla u|^2 \Psi^4 \frac{dt\,dx}{|t|^{n-d-1}} \bigg)^{1/2}$$

and then

$$\iint_D |\nabla G|^2 u^2 \Psi^4 \, \frac{dt\,dx}{|t|^{n-d-1}} \lesssim \iint_{4D/3} u^2 \Psi^2 \frac{dt\,dx}{|t|^{n-d-1}} + r^2 \iint_{4D/3} |\nabla u|^2 \Psi^4 \frac{dt\,dx}{|t|^{n-d-1}}. \tag{4.27}$$

We repeat the process for the last integral of the right-hand side above, using the fact that u is a weak solution to $Lu = 0$, and we obtain[2]

$$r^2 \iint_{4D/3} |\nabla u|^2 \Psi^4 \frac{dt\,dx}{|t|^{n-d-1}} \lesssim \iint_{2D} u^2 \Psi^2 \frac{dt\,dx}{|t|^{n-d-1}}.$$

[2] The estimate below can also be seen as a variant of Caccioppoli's inequality, and is a consequence of Lemma 3.1 (i) in [Feneuil et al. 2021].

We combine the last estimate with (4.27) and get that

$$\iint_D |\nabla G|^2 u^2 \Psi^4 \frac{dt\,dx}{|t|^{n-d-1}} \lesssim \iint_{2D} u^2 \Psi^2 \frac{dt\,dx}{|t|^{n-d-1}}. \tag{4.28}$$

The claim (4.25) follows after we recall that $|t| \approx r$ on $2D$.

We now apply (4.25) and have

$$\fint_{|Z-(y,t)|<|t|/4} |\nabla G|^2 u^2 \Psi_{B,\epsilon}^4 \, dZ \lesssim \frac{1}{|t|^2} \fint_{|Z-(y,t)|<|t|/2} u^2 \Psi_{B,\epsilon}^2 \, dZ \quad \text{for } (y,t) \in \Omega.$$

As a consequence, for any $x \in \mathbb{R}^d$,

$$\widetilde{N}(u\Psi_{B,\epsilon}^2 \nabla G)(x) \lesssim N_{10}(u\Psi_{B,\epsilon})(x).$$

The claim (4.23) follows from (4.3).

Step 4: proof of (4.10). We define

$$J = J_{B,\epsilon} := \int_\Omega |\nabla u|^2 \Psi_{B,\epsilon}^4 \frac{dt\,dx}{|t|^{n-d-2}},$$

and we want to show that

$$J_{B,\epsilon} \lesssim (1+K)\|N(u\Psi_{B,\epsilon})\|_{L^2(\mathbb{R}^d)}^2 + (1+K^{1/2}) J_{B,\epsilon}^{1/2} \|N(u\Psi_{B,\epsilon})\|_{L^2(\mathbb{R}^d)}, \tag{4.29}$$

where K is the constant used in the assumptions of the theorem under proof. Since $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$, all the quantities in (4.29) are finite, and therefore (4.29) improves itself in

$$J_{B,\epsilon} \lesssim (1+K)\|N(u\Psi_{B,\epsilon})\|_{L^2(\mathbb{R}^d)}^2. \tag{4.30}$$

We assumed that the solution u is bounded, so the left-hand side above is uniformly bounded in ϵ. We take then the limit as ϵ goes to 0 to obtain the desired bound (4.10).

To lighten the notation, we shall write until the end of the proof Ψ for $\Psi_{B,\epsilon}$ and J for $J_{B,\epsilon}$. Since b is bounded from above (assumption (b) of both Theorems 1.21 and 1.23), $G \gtrsim |t|$ by (4.21), and $\mathcal{A}$ is elliptic by (1.1), we deduce that

$$J \lesssim I := \iint_\Omega \mathcal{A}\nabla u \cdot \nabla u \frac{\Psi^4 G}{b} \frac{dt\,dy}{|t|^{n-d-1}}.$$

Using the product rule, we insert $\Psi^4 G/b$ into the second gradient, and we obtain

$$\begin{aligned} I = \iint_\Omega \mathcal{A}\nabla u \cdot \nabla\left(\frac{u\Psi^4 G}{b}\right) \frac{dt\,dy}{|t|^{n-d-1}} - 4 \iint_\Omega \mathcal{A}\nabla u \cdot \nabla \Psi \, \frac{u\Psi^3 G}{b} \frac{dt\,dy}{|t|^{n-d-1}} \\ + \iint_\Omega \mathcal{A}\nabla u \cdot \nabla b \, \frac{u\Psi^4 G}{b^2} \frac{dt\,dy}{|t|^{n-d-1}} - \iint_\Omega \mathcal{A}\nabla u \cdot \nabla G \, \frac{u\Psi^4}{b} \frac{dt\,dy}{|t|^{n-d-1}} \\ := I_0 + I_1 + I_2 + I_3. \end{aligned}$$

The term I_0 equals 0 because u is a weak solution to $Lu = 0$ (and the compactly supported function $u\Psi^4 G/b \in W^{1,2}(\Omega)$ is a valid test function thanks to Lemma 8.3 in [David et al. 2021b]). The terms I_1 and I_2 are bounded in a similar manner. Since $b \gtrsim 1$, $G \approx |t|$, $\mathcal{A}$ is bounded (due to (1.2)), and $0 \le \Psi \le 1$, the Cauchy–Schwarz inequality infers that

$$|I_1 + I_2| \lesssim \iint_\Omega |t|(|\nabla\Psi| + |\nabla b|)u\Psi^3|\nabla u| \frac{dt\,dy}{|t|^{n-d-1}} \lesssim J^{1/2}\biggl(\iint_\Omega |t|^2(|\nabla\Psi|^2 + |\nabla b|^2)u^2\Psi^2 \frac{dt\,dy}{|t|^{n-d}}\biggr)^{1/2}.$$

We know that $|t||\nabla b| \in CM(K)$ by assumption (4.17) and that $|t||\nabla\Psi| \in CM$ by (4.14), so the Carleson inequality (4.4) requires that

$$|I_1 + I_2| \lesssim (1 + K^{1/2})J^{1/2}\|N(u\Psi)\|_{L^2(\mathbb{R}^d)}.$$

As for I_3, we use the decomposition of $\mathcal{A}$ given in (4.19) to obtain

$$\begin{aligned}I_3 = -\iint_\Omega (\mathcal{C}_3\nabla_x u + \mathcal{C}_4\nabla_t u)\cdot\nabla_t G \frac{u\Psi^4}{b}\frac{dt\,dy}{|t|^{n-d-1}} \\ - \iint_\Omega \mathcal{A}_0\nabla u\cdot\nabla G(u\Psi^4)\frac{dt\,dy}{|t|^{n-d-1}} - \iint_\Omega (\mathcal{B}_3 + \widetilde{\mathcal{B}}_3)\nabla_x u\cdot\nabla_t G\frac{u\Psi^4}{b}\frac{dt\,dy}{|t|^{n-d-1}} \\ := I_{31} + I_{32} + I_{33}.\end{aligned}$$

Recall that $b \gtrsim 1$, and combined with the fact that $|\mathcal{C}_3| + |\mathcal{C}_4| \in \widetilde{CM}_2(K)$ and (4.7), we deduce

$$\begin{aligned}|I_{31}| &\lesssim \iint_\Omega (|\mathcal{C}_3| + |\mathcal{C}_4|)|\nabla u||u||\nabla G|\Psi^4\frac{dt\,dy}{|t|^{n-d-1}} \lesssim J^{1/2}K^{1/2}\|\widetilde{N}(u\Psi^2\nabla G)\|_{L^2(\mathbb{R}^d)} \\ &\lesssim J^{1/2}K^{1/2}\|N(u\Psi)\|_{L^2(\mathbb{R}^d)}\end{aligned}$$

by (4.6) and then (4.23). We force $(u\Psi^4)$ into the first gradient and I_{32} becomes

$$\begin{aligned}I_{32} &= -\frac{1}{2}\iint_\Omega \mathcal{A}_0\nabla(u^2\Psi^4)\cdot\nabla G\frac{dt\,dy}{|t|^{n-d-1}} + 2\iint_\Omega \mathcal{A}_0\nabla\Psi\cdot\nabla G(u^2\Psi^3)\frac{dt\,dy}{|t|^{n-d-1}} \\ &:= I_{321} + I_{322}.\end{aligned}$$

The term I_{321} equals 0 thanks to (4.20). As for I_{322}, we use the boundedness of $\mathcal{A}_0$ and the inequality $2ab \le a^2 + b^2$ to write

$$I_{322} \lesssim \iint_\Omega |\nabla\Psi||\nabla G|^2u^2\Psi^4\frac{dt\,dy}{|t|^{n-d-1}} + \iint_\Omega |\nabla\Psi|u^2\Psi^2\frac{dt\,dy}{|t|^{n-d-1}},$$

and then, by (4.14) and (4.23),

$$I_{322} \lesssim \|\widetilde{N}(u\Psi^2\nabla G)\|^2_{L^2(\mathbb{R}^d)} + \|\widetilde{N}(u\Psi)\|^2_{L^2(\mathbb{R}^d)} \lesssim \|N(u\Psi)\|^2_{L^2(\mathbb{R}^d)}.$$

Step 5: bound of I_{33}, which is the only difference between Theorems 1.21 and 1.23. Recall that $\mathcal{B}_3$ and $\widetilde{\mathcal{B}}_3$ satisfy the same condition on the x-derivative, that is

$$|t||\mathrm{div}_x(\mathcal{B}_3)^T| + |t||\mathrm{div}_x(\widetilde{\mathcal{B}}_3)^T| \in CM_2(K),$$

but differ on the condition on the t-derivative, which is

$$|t|^{n-d}|\operatorname{div}_t(|t|^{d+1-n}\mathcal{B}_3)| + |t||\operatorname{div}_t \widetilde{\mathcal{B}}_3| \in CM_2(K).$$

The goal is to permute the gradients ∇_x and ∇_t on I_{33}. We define the part of I_{33} that contains $\mathcal{B}_3$ as

$$S := -\iint_\Omega \mathcal{B}_3 \nabla_x u \cdot \nabla_t G \frac{u\Psi^4}{b} \frac{dt\,dy}{|t|^{n-d-1}}. \tag{4.31}$$

Using integration by parts in t, S becomes

$$\begin{aligned} S &= -\frac{1}{2}\int_\Omega \mathcal{B}_3 \nabla_x[u^2]\cdot \nabla_t G \frac{\Psi^4}{b}\frac{dt\,dy}{|t|^{n-d-1}} \\ &= \frac{1}{2}\int_\Omega \operatorname{div}_t(|t|^{d+1-n}\mathcal{B}_3\nabla_x[u^2])\frac{G\Psi^4}{b}dt\,dy \\ &\qquad + 2\int_\Omega \mathcal{B}_3\nabla_x[u^2]\cdot\nabla_t\Psi\frac{G\Psi^3}{b}\frac{dt\,dy}{|t|^{n-d-1}} - \int_\Omega \mathcal{B}_3\nabla_x[u^2]\cdot\nabla_t b\frac{G\Psi^4}{b^2}\frac{dt\,dy}{|t|^{n-d-1}} \\ &:= S_0 + S_1 + S_2. \end{aligned}$$

We write the term S_0 as a sum on the coefficients of $\mathcal{B}_3$, we permute the x and the t-derivatives on u^2, and then we integrate by parts in x. Recall that, in this paper, when M is a matrix-valued function, div M is a vector-valued function whose j-th entry is the divergence of the j-th column of M.

$$\begin{aligned} S_0 &:= \frac{1}{2}\sum_{1\le j\le d<i\le n}\iint_\Omega \partial_{t_i}[|t|^{d+1-n}(\mathcal{B}_3)_{ij}\partial_{x_j}u^2]\frac{G\Psi^4}{b}\,dt\,dy \\ &= \frac{1}{2}\iint_\Omega \operatorname{div}_t(|t|^{d+1-n}\mathcal{B}_3)\cdot\nabla_x(u^2)\frac{G\Psi^4}{b}\,dt\,dy + \frac{1}{2}\sum_{1\le j\le d<i\le n}\iint_\Omega (\mathcal{B}_3)_{ij}\partial_{t_i}[\partial_{x_j}u^2]\frac{G\Psi^4}{b}\frac{dt\,dy}{|t|^{n-d-1}} \\ &= \frac{1}{2}\iint_\Omega \operatorname{div}_t(|t|^{d+1-n}\mathcal{B}_3)\cdot\nabla_x(u^2)\frac{G\Psi^4}{b}\,dt\,dy - \frac{1}{2}\iint_\Omega \operatorname{div}_x(\mathcal{B}_3)^T\cdot\nabla_t[u^2]\frac{G\Psi^4}{b}\frac{dt\,dy}{|t|^{n-d-1}} \\ &\qquad -2\iint_\Omega (\mathcal{B}_3)^T\nabla_t[u^2]\cdot\nabla_x\Psi\frac{G\Psi^3}{b}\frac{dt\,dy}{|t|^{n-d-1}} + \frac{1}{2}\iint_\Omega(\mathcal{B}_3)^T\nabla_t[u^2]\cdot\nabla_x b\frac{G\Psi^3}{b^2}\frac{dt\,dy}{|t|^{n-d-1}} \\ &:= S_3+S_4+S_5+S_6. \end{aligned}$$

We do not have x-derivatives on G because G is x-independent; see (4.21). We deal with S_1, S_2, S_3, S_4, S_5, and S_6 in a similar manner as I_2+I_3 earlier. We have $G \lesssim |t|$, $1/b \lesssim 1$, and $\mathcal{B}_3$ is bounded, hence, if

$$f := |t||\nabla\Psi| + |t||\nabla b| + |t||\operatorname{div}_x(\mathcal{B}_3)^T| + |t|^{n-d}|\operatorname{div}_t(|t|^{d+1-n}\mathcal{B}_3)|,$$

the sum of the S_i can be bounded by

$$|S| \le \sum_{i=1}^{6}|S_i| \lesssim \iint_\Omega f|\nabla(u^2)|\Psi^3\frac{dt\,dy}{|t|^{n-d-1}} \lesssim \iint_\Omega f|\nabla u|u\Psi^3\frac{dt\,dy}{|t|^{n-d-1}}.$$

But, since $f \in CM_2(1+K)$ by (4.17) and (4.14), the Carleson estimate (4.6) yields

$$|S| \leq \sum_{i=1}^{6} |S_i| \lesssim J^{1/2}(1+K^{1/2})\|N(u\Psi)\|_{L^2(\mathbb{R}^d)}, \tag{4.32}$$

as desired. Theorem 1.21 is now proven, because $\widetilde{\mathcal{B}}_3 = 0$ in its assumption.

In order to establish Theorem 1.23, it remains to treat the part of I_{33} that contains $\widetilde{\mathcal{B}}_3$. If $\widetilde{S} := I_{33} - S$, and if, for $i \in \{0, \dots, 6\}$, $\widetilde{S}_i$ is obtained from S_i by substituting $\mathcal{B}_3$ for $\widetilde{\mathcal{B}}_3$, for $i \neq 3$, we can bound $\widetilde{S}_i$ as we bound S_i, because the assumptions on $\widetilde{\mathcal{B}}_3$ match those of $\mathcal{B}_3$. So, similarly to (4.32), we have that

$$|\widetilde{S} - \widetilde{S}_3| \lesssim J^{1/2}(1+K^{1/2})\|N(u\Psi)\|_{L^2(\mathbb{R}^d)}. \tag{4.33}$$

We do not know how to estimate $\widetilde{S}_3$, but instead we know how to estimate

$$\begin{aligned} \widetilde{S}_7 &:= \frac{1}{2} \iint_\Omega \operatorname{div}_t(\widetilde{\mathcal{B}}_3) \cdot \nabla_x (u^2) \frac{G\Psi^4}{b} \frac{dt\,dy}{|t|^{n-d-1}} \\ &= \iint_\Omega \operatorname{div}_t(\widetilde{\mathcal{B}}_3) \cdot \nabla_x u \frac{Gu\Psi^4}{b} \frac{dt\,dy}{|t|^{n-d-1}}. \end{aligned} \tag{4.34}$$

Indeed, we use $G \lesssim |t|$, $1/b \lesssim 1$, $|t||\operatorname{div}_t(\mathcal{B}_3)| \in CM_2(K)$, and the Carleson estimate (4.6), to get, similarly to the S_i's, that

$$|\widetilde{S}_7| \lesssim J^{1/2}(1+K^{1/2})\|N(u\Psi)\|_{L^2(\mathbb{R}^d)}. \tag{4.35}$$

So, in order to bound $\widetilde{S}$ and prove Theorem 1.23 we only have to write $\widetilde{S}$ as a linear combination of $\widetilde{S}_7$ and $\widetilde{S}_3$. Since we are currently under the assumptions of Theorem 1.23, Corollary 3.14 requires that $G = |t|$. With this in mind, we have

$$G|t|^{n-d-1} \operatorname{div}_t(|t|^{d+1-n}\mathcal{B}_3) = G \operatorname{div}_t(\mathcal{B}_3) + (d+1-n)(\nabla_t G)^T \mathcal{B}_3,$$

which can be reformulated as

$$\widetilde{S}_3 = \widetilde{S}_7 + (n-d-1)\widetilde{S}.$$

We conclude that

$$|\widetilde{S}| = \frac{1}{n-d-2}|(\widetilde{S}_3 - \widetilde{S}) + \widetilde{S}_7| \lesssim J^{1/2}(1+K^{1/2})\|N(u\Psi)\|_{L^2(\mathbb{R}^d)}$$

by (4.33) and (4.35). The lemma follows. □

Acknowledgements

Part of this article was written during the author's stay at the Université Paris-Saclay in France, where he was supported by Simons Foundation grant 601941, GD. The author also thanks the referee for carefully reading the manuscript and making valuable suggestions that improved the accessibility and quality of the article.

References

[Akman et al. 2023] M. Akman, S. Hofmann, J. M. Martell, and T. Toro, "Perturbation of elliptic operators in 1-sided NTA domains satisfying the capacity density condition", *Forum Math.* **35**:1 (2023), 245–295. MR Zbl

[Auscher et al. 2008] P. Auscher, A. Axelsson, and S. Hofmann, "Functional calculus of Dirac operators and complex perturbations of Neumann and Dirichlet problems", *J. Funct. Anal.* **255**:2 (2008), 374–448. MR Zbl

[Azzam et al. 2016] J. Azzam, S. Hofmann, J. M. Martell, S. Mayboroda, M. Mourgoglou, X. Tolsa, and A. Volberg, "Rectifiability of harmonic measure", *Geom. Funct. Anal.* **26**:3 (2016), 703–728. MR Zbl

[Azzam et al. 2017] J. Azzam, S. Hofmann, J. M. Martell, K. Nyström, and T. Toro, "A new characterization of chord-arc domains", *J. Eur. Math. Soc.* **19**:4 (2017), 967–981. MR Zbl

[Azzam et al. 2020] J. Azzam, S. Hofmann, J. M. Martell, M. Mourgoglou, and X. Tolsa, "Harmonic measure and quantitative connectivity: geometric characterization of the L^p-solvability of the Dirichlet problem", *Invent. Math.* **222**:3 (2020), 881–993. MR Zbl

[Bishop and Jones 1990] C. J. Bishop and P. W. Jones, "Harmonic measure and arclength", *Ann. of Math.* (2) **132**:3 (1990), 511–547. MR Zbl

[Caffarelli et al. 1981a] L. Caffarelli, E. Fabes, S. Mortola, and S. Salsa, "Boundary behavior of nonnegative solutions of elliptic operators in divergence form", *Indiana Univ. Math. J.* **30**:4 (1981), 621–640. MR Zbl

[Caffarelli et al. 1981b] L. A. Caffarelli, E. B. Fabes, and C. E. Kenig, "Completely singular elliptic-harmonic measures", *Indiana Univ. Math. J.* **30**:6 (1981), 917–924. MR Zbl

[Dahlberg 1977] B. E. J. Dahlberg, "Estimates of harmonic measure", *Arch. Ration. Mech. Anal.* **65**:3 (1977), 275–288. MR Zbl

[David and Jerison 1990] G. David and D. Jerison, "Lipschitz approximation to hypersurfaces, harmonic measure, and singular integrals", *Indiana Univ. Math. J.* **39**:3 (1990), 831–845. MR Zbl

[David and Mayboroda 2022a] G. David and S. Mayboroda, "Approximation of Green functions and domains with uniformly rectifiable boundaries of all dimensions", *Adv. Math.* **410** (2022), art. id. 108717. MR Zbl

[David and Mayboroda 2022b] G. David and S. Mayboroda, "Harmonic measure is absolutely continuous with respect to the Hausdorff measure on all low-dimensional uniformly rectifiable sets", *Int. Math. Res. Not.* **2022** (online publication May 2022).

[David et al. 2019a] G. David, J. Feneuil, and S. Mayboroda, "Dahlberg's theorem in higher co-dimension", *J. Funct. Anal.* **276**:9 (2019), 2731–2820. MR Zbl

[David et al. 2019b] G. David, J. Feneuil, and S. Mayboroda, "A new elliptic measure on lower dimensional sets", *Acta Math. Sin.* (*Engl. Ser.*) **35**:6 (2019), 876–902. MR Zbl

[David et al. 2020] G. David, J. Feneuil, and S. Mayboroda, "Elliptic theory in domains with boundaries of mixed dimension", preprint, 2020. To appear in *Astérisque*. arXiv 2003.09037

[David et al. 2021a] G. David, M. Engelstein, and S. Mayboroda, "Square functions, nontangential limits, and harmonic measure in codimension larger than 1", *Duke Math. J.* **170**:3 (2021), 455–501. MR Zbl

[David et al. 2021b] G. David, J. Feneuil, and S. Mayboroda, *Elliptic theory for sets with higher co-dimensional boundaries*, Mem. Amer. Math. Soc. **1346**, Amer. Math. Soc., Providence, RI, 2021. MR Zbl

[David et al. 2022] G. David, L. Li, and S. Mayboroda, "Carleson measure estimates for the Green function", *Arch. Ration. Mech. Anal.* **243**:3 (2022), 1525–1563. MR Zbl

[David et al. 2023] G. David, J. Feneuil, and S. Mayboroda, "Green function estimates on complements of low-dimensional uniformly rectifiable sets", *Math. Ann.* **385** (2023), 1797–1821.

[Dindoš and Pipher 2019] M. Dindoš and J. Pipher, "Regularity theory for solutions to second order elliptic operators with complex coefficients and the L^p Dirichlet problem", *Adv. Math.* **341** (2019), 255–298. MR Zbl

[Dindoš et al. 2007] M. Dindos, S. Petermichl, and J. Pipher, "The L^p Dirichlet problem for second order elliptic operators and a p-adapted square function", *J. Funct. Anal.* **249**:2 (2007), 372–392. MR Zbl

[Dindoš et al. 2011] M. Dindos, C. Kenig, and J. Pipher, "BMO solvability and the A_∞ condition for elliptic operators", *J. Geom. Anal.* **21**:1 (2011), 78–95. MR Zbl

[Dindoš et al. 2017] M. Dindoš, J. Pipher, and D. Rule, "Boundary value problems for second-order elliptic operators satisfying a Carleson condition", *Comm. Pure Appl. Math.* **70**:7 (2017), 1316–1365. MR Zbl

[Fefferman et al. 1991] R. A. Fefferman, C. E. Kenig, and J. Pipher, "The theory of weights and the Dirichlet problem for elliptic equations", *Ann. of Math.* (2) **134**:1 (1991), 65–124. MR Zbl

[Feneuil 2022] J. Feneuil, "Absolute continuity of the harmonic measure on low dimensional rectifiable sets", *J. Geom. Anal.* **32**:10 (2022), art. id. 247. MR Zbl

[Feneuil et al. 2021] J. Feneuil, S. Mayboroda, and Z. Zhao, "The Dirichlet problem in domains with lower dimensional boundaries", *Rev. Mat. Iberoam.* **37**:3 (2021), 821–910. MR Zbl

[Hofmann and Martell 2012] S. Hofmann and J. M. Martell, "A_∞ estimates via extrapolation of Carleson measures and applications to divergence form elliptic operators", *Trans. Amer. Math. Soc.* **364**:1 (2012), 65–101. MR Zbl

[Hofmann and Martell 2014] S. Hofmann and J. M. Martell, "Uniform rectifiability and harmonic measure, I: Uniform rectifiability implies Poisson kernels in L^p", *Ann. Sci. École Norm. Sup.* (4) **47**:3 (2014), 577–654. MR Zbl

[Hofmann et al. 2014] S. Hofmann, J. M. Martell, and I. Uriarte-Tuero, "Uniform rectifiability and harmonic measure, II: Poisson kernels in L^p imply uniform rectifiability", *Duke Math. J.* **163**:8 (2014), 1601–1654. MR Zbl

[Hofmann et al. 2015] S. Hofmann, C. Kenig, S. Mayboroda, and J. Pipher, "Square function/non-tangential maximal function estimates and the Dirichlet problem for non-symmetric elliptic operators", *J. Amer. Math. Soc.* **28**:2 (2015), 483–529. MR Zbl

[Hofmann et al. 2019] S. Hofmann, P. Le, and A. J. Morris, "Carleson measure estimates and the Dirichlet problem for degenerate elliptic equations", *Anal. PDE* **12**:8 (2019), 2095–2146. MR Zbl

[Hofmann et al. 2021] S. Hofmann, J. M. Martell, S. Mayboroda, T. Toro, and Z. Zhao, "Uniform rectifiability and elliptic operators satisfying a Carleson measure condition", *Geom. Funct. Anal.* **31**:2 (2021), 325–401. MR Zbl

[Hofmann et al. 2022] S. Hofmann, L. Li, S. Mayboroda, and J. Pipher, "The Dirichlet problem for elliptic operators having a BMO anti-symmetric part", *Math. Ann.* **382**:1–2 (2022), 103–168. MR Zbl

[Jerison and Kenig 1981] D. S. Jerison and C. E. Kenig, "The Dirichlet problem in nonsmooth domains", *Ann. of Math.* (2) **113**:2 (1981), 367–382. MR

[Kenig and Pipher 2001] C. E. Kenig and J. Pipher, "The Dirichlet problem for elliptic equations with drift terms", *Publ. Mat.* **45**:1 (2001), 199–217. MR Zbl

[Kenig et al. 2000] C. Kenig, H. Koch, J. Pipher, and T. Toro, "A new approach to absolute continuity of elliptic measure, with applications to non-symmetric equations", *Adv. Math.* **153**:2 (2000), 231–298. MR Zbl

[Kenig et al. 2016] C. Kenig, B. Kirchheim, J. Pipher, and T. Toro, "Square functions and the A_∞ property of elliptic measures", *J. Geom. Anal.* **26**:3 (2016), 2383–2410. MR Zbl

[Lavrentev 1963] M. Lavrentev, "Boundary problems in the theory of univalent functions", pp. 1–35 in *Seventeen papers on functions of complex variables*, Amer. Math. Soc. Transl. (2) **32**, Amer. Math. Soc., Providence, RI, 1963. MR Zbl

[Mayboroda and Poggi 2021] S. Mayboroda and B. Poggi, "Carleson perturbations of elliptic operators on domains with low dimensional boundaries", *J. Funct. Anal.* **280**:8 (2021), art. id. 108930. MR Zbl

[Mayboroda and Zhao 2019] S. Mayboroda and Z. Zhao, "Square function estimates, the BMO Dirichlet problem, and absolute continuity of harmonic measure on lower-dimensional sets", *Anal. PDE* **12**:7 (2019), 1843–1890. MR Zbl

[Modica and Mortola 1980] L. Modica and S. Mortola, "Construction of a singular elliptic-harmonic measure", *Manuscripta Math.* **33**:1 (1980), 81–98. MR Zbl

[Riesz and Riesz 1920] F. Riesz and M. Riesz, "Über die Randwerte einer analytischen Funktion", pp. 27–44 in *Compte rendu du Quatrième Congrès des Mathématiciens Scandinaves* (Stockholm, 1916), Almqvist & Wiksells, Uppsala, Sweden, 1920. Zbl

[Semmes 1990] S. Semmes, "Analysis vs. geometry on a class of rectifiable hypersurfaces in $\mathbb{R}^n$", *Indiana Univ. Math. J.* **39**:4 (1990), 1005–1035. MR Zbl

[Stein 1993] E. M. Stein, *Harmonic analysis*: *real-variable methods*, *orthogonality*, *and oscillatory integrals*, Princeton Math. Ser. **43**, Princeton Univ. Press, 1993. MR Zbl

Received 22 Oct 2020. Revised 26 May 2021. Accepted 15 Jul 2021.

JOSEPH FENEUIL: joseph.feneuil@gmail.com
Laboratoire de mathématiques d'Orsay, Université Paris-Saclay, Orsay, France

Analysis & PDE
msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2023 is US $405/year for the electronic version, and $630/year (+$65, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Analysis & PDE

Volume 16 No. 2 2023