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GLOBAL REGULARITY FOR THE NONLINEAR WAVE EQUATION WITH SLIGHTLY SUPERCRITICAL POWER

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We consider the defocusing nonlinear wave equation $\Box u = |u|^{p-1}u$ in $\mathbb{R}^3 \times [0,\infty)$. We prove that for any initial datum with a scaling-subcritical norm bounded by M_0 the equation is globally well-posed for $p = 5+\delta$, where $\delta \in (0, \delta_0(M_0))$.

1. Introduction

We consider the Cauchy problem for the nonlinear defocusing wave equation on $\mathbb{R}^3$, that is,

$$\begin{cases} \Box u = |u|^{p-1}u, \\ (u, \partial_t u)(\,\cdot\,, 0) = (u_0, u_1) \in (\dot H^1 \cap \dot H^2) \times H^1, \end{cases} \tag{1}$$

where $u : \mathbb{R}^3 \times I \to \mathbb{R}$, $p > 1$ and $\Box = -\partial_{tt} + \Delta$ is the d'Alembertian. For sufficiently regular solutions of (1) the energy

$$E(u)(t) := \int \frac{1}{2}|\partial_t u|^2 + \frac{1}{2}|\nabla u|^2 + \frac{|u|^{p+1}}{p+1}\,\mathrm{d}x$$

is conserved, i.e., $E(t) = E$. Moreover, there is a natural scaling associated to (1): for $\lambda > 0$ the map

$$u \mapsto u_\lambda(x,t) = \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda t)$$

preserves solutions of (1). Correspondingly, the energy rescales like $E(u_\lambda)(t) = \lambda^{(5-p)/(p-1)} E(u)(t)$ and hence the equation is energy-supercritical for $p > 5$. Our goal is to show that given any (possibly large) initial data (u_0, u_1), the supercritical nonlinear defocusing wave equation (1) is globally well-posed at least for an open interval of exponents $p \in [5, 5+\delta_0)$.

Theorem 1.1. *Let $\|(u_0,u_1)\|_{\dot H^1 \cap \dot H^2 \times H^1} \le M_0$. Then there exists $\delta_0 = \delta_0(M_0) > 0$ such that for any $\delta \in (0,\delta_0)$ there exists a global solution u of* (1) *with $p = 5+\delta$ from the initial data (u_0,u_1). Moreover, there exists a universal constant $C > 1$ such that for any time t*

$$\|(u,\partial_t u)(t)\|_{\dot H^1 \cap \dot H^2 \times H^1} \le \|(u_0,u_1)\|_{\dot H^1 \cap \dot H^2 \times H^1} e^{C(1+(CE(u))^{CE(u)^{352}})} \tag{2}$$

and we have the global spacetime bound

$$\|u\|_{L^{2(p-1)}(\mathbb{R}^3 \times \mathbb{R})} \le C(1 + (CE(u))^{CE(u)^{352}}).$$

In particular, the solution scatters as $t \to \pm\infty$.

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Global regularity and scattering for the energy-critical regime was established in [Struwe 1988; Grillakis 1990]. The classical results in the critical case were recently improved to obtain explicit double exponential bounds [Tao 2006b] and to allow a critical nonlinearity with an extra logarithmic factor $f(u) = u^5 \log(2+u^2)$ in the case of spherical symmetric data [Tao 2007]. Exploiting the method introduced in [Tao 2006b; Roy 2009] could remove the assumption of spherical symmetry for slightly log log-supercritical growth. In two-dimensions, global regularity has also been established for the slightly supercritical nonlinearity $f(u) = ue^{u^2}$ in [Struwe 2011]. For the classical supercritical nonlinearity $f(u) = |u|^{p-1}u$ with $p > 5$, global existence and scattering of solutions still holds for small data in scaling-invariant spaces, for instance in $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$, where

$$s_p := 1 + \frac{\delta}{2(p-1)}$$

is the critical Sobolev exponent. For general large data, however, the problem of global regularity and scattering is still open: apart from conditional regularity results in terms of the critical Sobolev regularity [Kenig and Merle 2011; Killip and Visan 2011], global solutions have been built only from particular classes of initial data [Krieger and Schlag 2017; Beceanu and Soffer 2018] or for a nonlinearity satisfying the null condition as in [Wang and Yu 2016; Miao et al. 2019].

Our result should be seen in line with [Tao 2006b; Roy 2009], pushing global regularity in a slightly supercritical regime. Although the nonlinearity considered in those papers has a logarithmically supercritical growth at infinity, it still comes, up to lower-order terms, with the scaling associated to the critical case $p = 5$. Correspondingly, both the scaling-invariant quantities of the critical regime, as well as some logarithmically higher integrability, are controlled by the energy. Instead, we consider the supercritical nonlinearity (1) and achieve global existence and scattering by paying the price of working on bounded sets of initial data, as previously done for other equations, such as SQG [Coti Zelati and Vicol 2016] and Navier–Stokes [Colombo and Haffter 2021]. As in [Roy 2009; Coti Zelati and Vicol 2016; Colombo and Haffter 2021], the crucial ingredient of the proof of Theorem 1.1 is a (quantitative) long-time estimate. In the spherically symmetric case, the classical Morawetz inequality gives an a priori spacetime bound as long as the solution exists. The following result replaces this long-time estimate in the absence of symmetry assumptions.

Theorem 1.2 (a priori spacetime bound). *There exists a universal constant $C \geq 1$ such that, for any solution $(u, \partial_t u) \in L^\infty(J, (\dot{H}^1 \cap \dot{H}^2 \times H^1)(\mathbb{R}^3))$ of (1) with $p = 5+\delta$, $\delta \in (0,1)$, defining $M := \|u\|_{L^\infty(\mathbb{R}^3 \times J)}$, $E := E(u)$ and $L := \|(u, \partial_t u)\|_{L^\infty(J,(\dot{H}^{s_p} \times \dot{H}^{s_p-1})(\mathbb{R}^3))}$ the following hold:*

- *If $\min\{EM^{\delta/2}, L\} < c_0$, then $\|u\|_{L^{2(p-1)}(\mathbb{R}^3 \times J)} \leq 1$.*
- *If $\min\{EM^{\delta/2}, L\} \geq c_0$ and $(CEM^{\delta/2}L)^{C(EM^{\delta/2}L)^{176}} \leq 2^{1/\delta}$, then*

$$\|u\|_{L^{2(p-1)}(\mathbb{R}^3 \times J)} \leq (CEM^{\frac{\delta}{2}}L)^{C(EM^{\delta/2}L)^{176}}. \tag{3}$$

Corollary 1.3. *There exists a universal constant $C \geq 1$ such that the following holds. Let $M_0 > 0$ be given. Then there exists $\delta_0 = \delta_0(M_0) > 0$ such that, for any solution $(u, \partial_t u) \in L^\infty(J, (\dot{H}^1 \cap \dot{H}^2 \times H^1)(\mathbb{R}^3))$*

of (1) *with* $p = 5 + \delta$ *for* $\delta \in (0, \delta_0]$ *and with* $\|(u, \partial_t u)\|_{L^\infty(J,(\dot H^1 \cap \dot H^2 \times H^1)(\mathbb{R}^3))} \le M_0$, *we have the a priori spacetime bound*

$$\|u\|_{L^{2(p-1)}(\mathbb{R}^3 \times J)} \le \max\{1, (CE(u)M_0^{\frac{\delta}{2}})^{C(E(u)M_0^{\delta/2})^{352}}\}. \tag{4}$$

Remark 1.4. From the proof, we observe that δ_0 has the following dependence as $M_0 \to \infty$: there exists $C' \ge 1$ such that

$$\delta_0 := \min\left\{1, \frac{\ln 2}{\ln M_0}, \frac{\ln 2}{\ln(C'E)(C'E)^{352}}\right\}.$$

Theorem 1.1 follows from Corollary 1.3 and a continuity argument, taking advantage of the fact that, if the estimate (4) involves in the right-hand side higher-order norms of the solution itself, which we a priori don't control for large times, on the other side they appear only to the power δ and hence can be kept under control for δ small.

The proof of Theorem 1.2 follows instead the scheme introduced in [Tao 2006b] to obtain double exponential bounds on critical Strichartz norms based on Bourgain's "induction on energy" method [1999]. In [Roy 2009], the scheme has been successfully applied to a log-supercritical equation assuming a (subcritical) a priori bound M on $\|u\|_{L^\infty(\mathbb{R}^3 \times J)}$: indeed, it was noticed that the induction on the energy, which does not allow the inclusion of the a priori bound M, can actually be bypassed by a simpler ad-hoc argument. We will use the latter strategy also in our case. Rather than controlling an L^4L^{12} norm as performed in the mentioned papers, we estimate an $L^{2(p-1)}$ norm, which is scaling-critical for every p. To follow their line of proof, we need to overcome some issues related to the supercritical nature of our equation: for instance, a fundamental use of the equation in all critical global regularity results is the localized energy equality and the subsequent potential energy decay, first used in [Struwe 1988; Grillakis 1990; Shatah and Struwe 1993]. In the supercritical regime, the localized energy inequality becomes less powerful, since the nonlinear term is estimated this time in terms of a power of the length of the time interval besides the energy itself (see Lemma 4.5). To be able to still take advantage of this localized energy inequality, we need a control on the length of the so-called unexceptional intervals, which was not derived before in [Tao 2006b; Roy 2009] and seems to work in the supercritical case only. To achieve this control, we introduce another scaling-invariant norm of u accounting for more differentiability, namely $L^\infty \dot H^{s_p}$. This quantity, which appears in the final estimate (3), was not needed in [Tao 2006b; Roy 2009]. It turns out to be fundamental to bound the length of unexceptional intervals by performing a mass concentration in $\dot H^{s_p}$, rather than in $\dot H^1$ (see Lemma 6.2), and thereby obtaining an upper bound on the mass concentration radius.

The strategy of proof of Theorem 1.1 is very flexible and we plan to apply it in a future work to the radial supercritical Schrödinger equation. For instance, as regards the initial data, the statement of Theorem 1.1 is written with $(u_0, u_1) \in \dot H^1 \cap \dot H^2 \times H^1$ and in the proof we take advantage of the embedding of $H^{3/2+\epsilon}$ in L^∞. However, we will investigate whether a similar result holds just above the critical threshold, namely for $(u_0, u_1) \in \dot H^1 \cap \dot H^{1+\epsilon} \times H^\epsilon$ for some $\epsilon > 0$, with δ_0 depending on ϵ.

2. Preliminaries

2A. ***Energy-flux equality.*** With the notation of [Shatah and Struwe 1998], we introduce the forward-in-time wave cone, the truncated cone and their boundaries centered at $z_0 = (x_0, t_0) \in \mathbb{R}^3 \times \mathbb{R}$ defined by

$$\begin{aligned} K(z_0) &:= \{z = (x,t) \in \mathbb{R}^4 : |x - x_0| \le t - t_0\}, \\ K_s^t(z_0) &:= K(z_0) \cap (\mathbb{R}^3 \times [s,t]), \\ M_s^t(z_0) &:= \{z = (x,r) \in \mathbb{R}^3 \times (s,t) : |x - x_0| = r - t_0\}, \\ D(t; z_0) &:= K(z_0) \cap (\mathbb{R}^3 \times t). \end{aligned}$$

Correspondingly, we introduce the localized energy as well as the energy flux

$$\begin{aligned} E(u; D(t;z_0)) &:= \int_{D(t;z_0)} \frac{1}{2}|\partial_t u|^2 + \frac{1}{2}|\nabla u|^2 + \frac{|u|^{p+1}}{p+1}\,\mathrm{d}x, \\ \mathrm{Flux}(u; M_s^t(z_0)) &:= \int_{M_s^t(z_0)} \frac{1}{2}\left|\nabla u - \frac{x - x_0}{|x - x_0|}\partial_t u\right|^2 + \frac{|u|^{p+1}}{p+1}\,\frac{\mathrm{d}\sigma}{\sqrt{2}}. \end{aligned}$$

Let us recall that for any sufficiently regular solution we have the energy-flux identity

$$E(u; D(t;z_0)) + \mathrm{Flux}(u; M_s^t(z_0)) = E(u; D(s;z_0)) \tag{5}$$

for any $0 < s < t$. Indeed, (5) is obtained by integrating $(\Box u - |u|^{p-1}u)\,\partial_t u$ on $K_s^t(z_0)$; see for instance [Shatah and Struwe 1998]. Whenever $z_0 = (0,0)$, we will not write the dependence on z_0; we will write $\Gamma_+(I)$ for the forward wave cone centered at 0 and truncated by I,

$$\Gamma_+(I) := \{(x,t) \in \mathbb{R}^3 \times \mathbb{R} : |x| < t,\ t \in I\},$$

and we define $e(t) := E(u; D(t))$. We can then rewrite (5) for any $0 < s < t$ as

$$e(t) - e(s) = \int_{M_s^t} \frac{1}{2}\left|\nabla u - \frac{x}{t}\,\partial_t u\right|^2 + \frac{|u|^{p+1}}{p+1}\,\frac{\mathrm{d}\sigma}{2}.$$

2B. ***Strichartz estimates.*** Let $u : \mathbb{R}^3 \times I \to \mathbb{R}$ solve the linear wave equation $\Box u = F$. Let $m \in \left[1, \frac{3}{2}\right)$. Then for any $(q,r) \in (2,\infty] \times [1,\infty)$ wave-m-admissible and for any conjugate pair $(\tilde q, \tilde r) \in [1,+\infty] \times [1,+\infty]$ with

$$\frac{1}{\tilde q} + \frac{3}{\tilde r} - 2 = \frac{1}{q} + \frac{3}{r} = \frac{3}{2} - m, \tag{6}$$

we have

$$\|u\|_{L^q(I,L^r)} + \|(u,\partial_t u)\|_{L^\infty(I,\dot H^m \times \dot H^{m-1})} \le C(\|(u,\partial_t u)(t_0)\|_{\dot H_x^m \times \dot H_x^{m-1}} + \|F\|_{L^{\tilde q}(I,L^{\tilde r})}), \tag{7}$$

where $t_0 \in I$ is a generic time. The above Strichartz estimates are classical and we refer for instance to [Ginibre and Velo 1995; Keel and Tao 1998; Lindblad and Sogge 1995; Sogge 1995]. Notice that $(q,r) = (2(p-1), 2(p-1))$ is wave-s_p-admissible and all (q,r) wave-s_p-admissible are scaling-critical. Moreover, the constant C can be taken independent of $m \in \left[1, \frac{5}{4}\right]$.

2C. ***Localized Strichartz estimates.*** By the finite speed of propagation, we can localize the above Strichartz estimates on wave cones. Let $I = [a, b]$ and $m \in \left[1, \frac{3}{2}\right)$. For any solution $u : \mathbb{R}^3 \times I \to \mathbb{R}$ of a linear wave equation $\Box u = F$, we have for any (q, r) wave-m-admissible and any conjugate pair $(\tilde{q}, \tilde{r})$ satisfying (6) the localized estimate

$$\|u\|_{L^q L^r(\Gamma_+(I))} \lesssim \|(u, \partial_t u)(b)\|_{(\dot{H}^m \times \dot{H}^{m-1})(\mathbb{R}^3)} + \|F\|_{L^{\tilde{q}} L^{\tilde{r}}(\Gamma_+(I))}. \tag{8}$$

As a consequence, if $I = [a, b] = J_1 \cup J_2$, we have

$$\|u\|_{L^q L^r(\Gamma_+(J_1))} \lesssim \|(u, \partial_t u)(b)\|_{(\dot{H}^m \times \dot{H}^{m-1})(\mathbb{R}^3)} + \|F\|_{L^{\tilde{q}} L^{\tilde{r}}(\Gamma_+(J_1 \cup J_2))}.$$

2D. ***Littlewood–Paley projection.*** We follow the presentation of [Tao 2006a]. Fix $\phi \in C_c^\infty(\mathbb{R}^d)$ radially symmetric, $0 \le \phi \le 1$ such that $\operatorname{supp} \phi \subseteq B_2(0)$ and $\phi \equiv 1$ on $B_1(0)$. For $N \in 2^{\mathbb{Z}}$, introduce the Fourier multipliers

$$\begin{aligned} \widehat{P_{\le N} f}(\xi) &:= \phi(\xi/N) \hat{f}(\xi), \\ \widehat{P_{> N} f}(\xi) &:= (1 - \phi(\xi/N)) \hat{f}(\xi), \\ \widehat{P_N f}(\xi) &:= (\phi(\xi/N) - \phi(2\xi/N)) \hat{f}(\xi). \end{aligned}$$

The above projections can equivalently be written as convolution operators and the Young inequality shows that the Littlewood–Paley projections are bounded on L^p for any $1 \le p \le +\infty$. Moreover, we have the Bernstein inequalities

$$\|P_{\le N} f\|_{L_x^q(\mathbb{R}^d)} \lesssim_{p,q} N^{d\left(\frac{1}{p} - \frac{1}{q}\right)} \|P_{\le N} f\|_{L_x^p(\mathbb{R}^d)} \tag{9}$$

for $1 \le p \le q \le +\infty$ and the same holds with $P_N f$ in place of $P_{\le N} f$. Moreover, for $1 < p < +\infty$ we also recall the fundamental Littlewood–Paley inequality

$$\|f\|_{L^p(\mathbb{R}^d)} \sim \left\| \left(\sum_{N \in 2^{\mathbb{Z}}} |P_N f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^d)}. \tag{10}$$

2E. ***Dependence of constants.*** In the rest of the paper, all constants will be independent of the choice of $\delta \in [0, 1)$. We keep the estimates in scaling-invariant form (for instance, in all the statements of the lemmas in Sections 3–6). We write the terms in the estimate in terms of simpler scaling-invariant quantities, such as $E\|u\|_{L^\infty}^{\delta/2}$, $\|u\|_{L^{2(p-1)}}$, $\|u\|_{L^\infty \dot{H}^{s_p}}$, $ET^{-\delta/(p-1)}$ (see for instance (16)).

3. Spacetime norm bound under a scaling-invariant smallness assumption

In this section, we recall that the Strichartz estimates give a universal control on the critical $L^{2(p-1)}$ spacetime norm, which is in particular independent of the length of the time interval of existence, provided that the solution satisfies a suitable scaling-invariant smallness assumption. In our context, we formulate the smallness assumption in terms of the critical $\dot{H}^{s_p}$ norm as well as a scaling-invariant combination of the energy and the L^∞ norm.

Lemma 3.1. *Let* $p = 5 + \delta$ *for* $\delta \in (0, 1)$ *and consider a solution* $(u, \partial_t u) \in L^\infty(I, \dot H^1 \cap \dot H^2 \times H^1)$ *to* (1). *Assume additionally that* $\|u\|_{L^\infty(\mathbb{R}^3 \times I)} \le M$. *There exists a universal* $0 < c_0 < 1$ *such that if*

$$EM^{\frac{\delta}{2}} \le c_0 \quad \textit{or} \quad \|(u, \partial_t u)\|_{L^\infty(I,(\dot H^{s_p} \times \dot H^{s_p-1})(\mathbb{R}^3))} \le c_0,$$

then

$$\|u\|_{L^{2(p-1)}(\mathbb{R}^3 \times I)} \le 1. \tag{11}$$

Proof. Let us first assume that $EM^{\delta/2} \le c_0$ for a $c_0 < 1$ yet to be chosen. By interpolation

$$\|u\|_{L^{2(p-1)}} \le \|u\|_{L^\infty}^{\frac{\delta}{p-1}} \|u\|_{L^8}^{\frac{4}{p-1}}.$$

We notice that $(8, 8)$ is wave-1-admissible. By the Strichartz estimate (7) $\bigl($with $m = 1$ and $(\tilde q, \tilde r) = \bigl(2, \frac{3}{2}\bigr)\bigr)$, Hölder and the Sobolev embedding $\dot H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ we have

$$\|u\|_{L^8_{t,x}} \lesssim E^{\frac{1}{2}} + \||u|^{p-1}u\|_{L^2 L^{3/2}} \lesssim E^{\frac{1}{2}} + \||u|^{p-1}\|_{L^2_{t,x}} \|u\|_{L^\infty L^6} \lesssim E^{\frac{1}{2}}(1 + \|u\|_{L^{2(p-1)}}^{p-1}).$$

Summarizing, we have obtained that for a $C \ge 1$

$$\|u\|_{L^{2(p-1)}} \le C(M^{\frac{\delta}{2}} E)^{\frac{2}{p-1}} (1 + \|u\|_{L^{2(p-1)}}^4),$$

from which (11) follows setting $c_0 := (4C)^{-(p-1)/2} < 1$.

Let us now assume $\|(u, \partial_t u)\|_{L^\infty(\dot H^{s_p} \times \dot H^{s_p-1})} \le c_0'$ for a $0 < c_0' < 1$. Observing that $(2(p-1), 2(p-1))$ is wave-s_p-admissible, by the Strichartz estimate (7) (with $m = s_p$ and $(\tilde q, \tilde r) = (2, 6(p-1)/(3p+1))$), Hölder and the Sobolev embedding $\dot H^{s_p}(\mathbb{R}^3) \hookrightarrow L^{3(p-1)/2}(\mathbb{R}^3)$, we have

$$\begin{aligned}
\|u\|_{L^{2(p-1)}} &\lesssim \|(u, \partial_t u)\|_{L^\infty(\dot H^{s_p} \times \dot H^{s_p-1})} + \||u|^{p-1}u\|_{L^2 L^{6(p-1)/(3p+1)}} \\
&\lesssim \|(u, \partial_t u)\|_{L^\infty(\dot H^{s_p} \times \dot H^{s_p-1})} + \||u|^{p-1}\|_{L^2_{t,x}} \|u\|_{L^\infty L^{3(p-1)/2}} \\
&\lesssim \|(u, \partial_t u)\|_{L^\infty(\dot H^{s_p} \times \dot H^{s_p-1})}(1 + \|u\|_{L^{2(p-1)}}^{p-1}).
\end{aligned}$$

Calling C' the constant in the above inequality, (11) follows by setting $c_0' := (4C')^{-1}$. □

4. Spacetime norm decay in forward wave cones

The goal of this section is to prove the following proposition, which identifies a subinterval J (of quantified length) with small $L^{2(p-1)}$ norm of u in any sufficiently large given interval $I = [T_1, T_2]$. The main difference to the energy-critical case $p = 5$ [Tao 2006b, Corollary 4.11] lies in the fact that the largeness requirement on I can no longer be reached by simply choosing T_2 big enough (see Remark 4.3).

Proposition 4.1 (spacetime-norm decay). *Let* $p = 5 + \delta$ *with* $\delta \in (0, 1)$, $I = [T_1, T_2] \subset (0, \infty)$ *and consider a solution* $(u, \partial_t u) \in L^\infty(I, \dot H^1 \cap \dot H^2 \times H^1)$ *to* (1). *Assume that* $\|u\|_{L^\infty(\mathbb{R}^3 \times I)} \le M$. *There exists a universal constant* $0 < C_2 < 1$ *such that if* $0 < \eta < 1$ *is such that*

$$\eta < C_2(EM^{\frac{\delta}{2}})^{\frac{7}{6(p-1)}} \tag{12}$$

then the following holds for any A satisfying

$$A > (C_2\eta^{-1})^{\frac{12(p-1)}{5}}(EM^{\frac{\delta}{2}})^{\frac{14}{5}} : \tag{13}$$

if T_1 and T_2 are such that

$$\frac{T_2}{T_1} \geq A^{3(C_2\eta^{-1})^{6(p-1)(p+1)/5}(EM^{\delta/2})^{(9p+19)/10}\max\{(C_2\eta^{-1})^{-6(p-1)^2/5}(EM^{\delta/2})^{9(p-1)/10},(M^{(p-1)/2}T_2)^{\delta/2}\}}, \tag{14}$$

then there exists a subinterval $J = [t', At'] \subseteq I$ with

$$\|u\|_{L^{2(p-1)}(\Gamma_+(J))} \leq \eta.$$

Remark 4.2 (simplified assumptions in the large energy regime). In the large energy regime $EM^{\delta/2} \geq c_0$, with c_0 defined through Lemma 3.1, the hypothesis (12) can be simplified to

$$\eta < C_2 c_0^{\frac{7}{6(p-1)}} := c_0',$$

where we observe that $0 < c_0' \leq 1$. Moreover, the assumption (14) can be replaced by the stronger condition

$$\frac{T_2}{T_1} \geq A^{3(C_2\eta^{-1})^{6(p-1)(p+1)/5}(EM^{\delta/2})^{9p+19/10}\max\{c_0^{(p-1)/2},(M^{(p-1)/2}T_2)^{\delta/2}\}}. \tag{15}$$

Remark 4.3. The assumptions of Proposition 4.1 comprise an upper bound on T_1 for any fixed η satisfying (12), A satisfying (13) and T_2 satisfying (14). However, this will not be the spirit of the application of this proposition: we will rather fix T_1 and consider (14) as a condition on T_2 and δ. This condition may sound strange since, when all other parameters are fixed, (14) is not verified for large T_2. On the other hand, we will instead fix

$$T_2 := T_1 A^{3(C_2\eta^{-1})^{6(p-1)(p+1)/5}(EM^{\delta/2})^{(9p+19)/10}}$$

and notice that (14) is verified for δ sufficiently small.

As a first step to the proof of Proposition 4.1, we show that if the $L^{2(p-1)}$ norm of u in a strip is bounded from below, the Strichartz estimates imply a lower bound on the $L^\infty L^{p+1}$ norm in the same interval.

Lemma 4.4 (lower bound on global and local potential energy). *Let $p = 5+\delta$ with $\delta \in (0, 1)$ and $\eta \in (0, 1]$. Consider a solution $(u, \partial_t u) \in L^\infty(I, \dot{H}^1 \cap \dot{H}^2 \times H^1)$ to (1). Assume that $\|u\|_{L^{2(p-1)}(\mathbb{R}^3\times I)} \geq \eta$ and $\|u\|_{L^\infty(\mathbb{R}^3\times I)} \leq M$. Then there exists $0 < C_1 \leq 1$ universal such that*

$$\|u\|^{p+1}_{L^\infty(I,L^{p+1})} \geq C_1\eta^{\frac{12}{5}(p-1)}(M^{\frac{\delta}{2}}E)^{-\frac{9}{5}}M^{-\frac{\delta}{2}}. \tag{16}$$

Moreover, by finite speed of propagation the same estimate can be obtained by replacing $\mathbb{R}^3 \times I$ by any truncated forward wave cone $\Gamma_+(I)$.

Proof. Let $0 < \eta \le 1$. By shrinking I, we can assume without loss of generality that $\|u\|_{L^{2(p-1)}(\mathbb{R}^3\times I)} = \eta$. We observe that we control all wave-1-admissible spacetime norms with the energy. Indeed, fix (q, r) wave-1-admissible. By the Strichartz estimate (7) with $m = 1$ and Hölder

$$\|u\|_{L^qL^r} \lesssim E^{\frac12} + \||u|^{p-1}u\|_{L^2L^{3/2}} \lesssim E^{\frac12} + \|u\|_{L^\infty L^6}\||u|^{p-1}\|_{L^2_{t,x}} \lesssim E^{\frac12} + E^{\frac12}\eta^{p-1} \lesssim E^{\frac12}. \tag{17}$$

We observe that the pair $(3, 18)$ is wave-1-admissible and that $(3, 18)$ and $(\infty, p+1)$ interpolate to $\big(\big(\frac56(p+1)+3, \frac56(p+1)+3\big) = \big(8+\frac56\delta, 8+\frac56\delta\big)$. By interpolation and (17), we thus have

$$\|u\|^{2(p-1)}_{L^{2(p-1)}} \le \|u\|^{\frac76\delta}_{L^\infty_{t,x}}\|u\|^{8+\frac56\delta}_{L^{8+(5/6)\delta}_{t,x}} \le M^{\frac76\delta}\|u\|^{\frac56(p+1)}_{L^\infty L^{p+1}}\|u\|^3_{L^3L^{18}} \lesssim (M^{\frac{\delta}{2}}E)^{\frac32}M^{\frac{5}{12}\delta}\|u\|^{\frac56(p+1)}_{L^\infty L^{p+1}}. \quad \square$$

We now come to a localized energy inequality of Morawetz-type which, in the critical case $p = 5$, implies the potential energy decay and hence it is crucial for the global regularity in the critical case [Grillakis 1990; Struwe 1988]. In the supercritical case, the former localized energy inequality degenerates and will only lead to some decay estimate on bounded intervals: indeed the presence of the extra term $b^{\delta/(p+1)}$ in the right-hand side of (18) below makes the inequality interesting only when an estimate on the length of the interval is at hand.

Lemma 4.5. *Let $\delta \in [0, 1)$ and $p = 5+\delta$. For any $0 < a < b$ and any weak finite energy solution $(u, \partial_t u) \in C([a, b], \dot H^1 \cap L^{p+1}) \cap L^p([a, b], L^{2p}) \times C([a, b], L^2)$ of (1), we have*

$$\int_{|x|\le b} |u(x, b)|^{p+1}\,\mathrm{d}x \lesssim \frac{a}{b}E + e(b) - e(a) + b^{\frac{\delta}{p+1}}(e(b) - e(a))^{\frac{2}{p+1}}. \tag{18}$$

Proof. Let us first assume that $u \in C^2(\mathbb{R}^3 \times [a, b])$ is a classical solution of (1). We follow the notation of [Shatah and Struwe 1993; Bahouri and Shatah 1998] and introduce the quantities

$$\begin{aligned}
Q_0 &:= \frac12((\partial_t u)^2 + |\nabla u|^2) + \frac{|u|^{p+1}}{p+1} + \partial_t u\Big(\frac{x}{t}\cdot\nabla u\Big),\\
P_0 &:= \frac{x}{t}\Big(\frac{(\partial_t u)^2}{2} - \frac{|\nabla u|^2}{2} - \frac{|u|^{p+1}}{p+1}\Big) + \nabla u\Big(\partial_t u + \frac{x}{t}\cdot\nabla u + \frac{u}{t}\Big),\\
R_0 &:= \Big(1 - \frac{4}{p+1}\Big)|u|^{p+1}.
\end{aligned}$$

Observe $R_0 \ge 0$. Multiplying (1) by $(t\,\partial_t u + x\cdot\nabla u + u)$ one obtains $\partial_t(t\,Q_0 + \partial_t u\,u) - \operatorname{div}(tP_0) + R_0 = 0$; see [Shatah and Struwe 1998, Chapter 2.3]. Integrating on K_a^b (recall the definitions in Section 2), we obtain

$$\begin{aligned}
b\int_{D(b)} Q_0\,\mathrm{d}x - a\int_{D(a)} Q_0\,\mathrm{d}x + \int_{K_a^b} R_0\,\mathrm{d}x\,\mathrm{d}t&\\
&= -\int_{D(b)} \partial_t u\,u\,\mathrm{d}x + \int_{D(a)} \partial_t u\,u\,\mathrm{d}x + \int_{M_a^b}\Big(t\,Q_0 + \partial_t u\,u + t\,P_0\cdot\frac{x}{|x|}\Big)\frac{\mathrm{d}\sigma}{\sqrt2}\\
&= \int_{M_a^b} t\Big(\partial_t u + \frac{x}{t}\cdot\nabla u + \frac{u}{t}\Big)^2\frac{\mathrm{d}\sigma}{\sqrt2},
\end{aligned} \tag{19}$$

where in the second equality we used the computations of [Bahouri and Shatah 1998, Section 2] for $p = 5$ to rewrite the last addend on the right-hand side. Indeed, on M_a^b the integrand

$$t\,Q_0 + \partial_t u + P_0 \cdot \frac{x}{|x|} = t(\partial_t u)^2 + 2\partial_t u\, x \cdot \nabla u + \partial_t u\, u$$

is now independent of p. Proceeding as in [Bahouri and Gérard 1999], we estimate on K_a^b

$$\partial_t u\, \frac{x}{t} \cdot \nabla u \le \frac{(\partial_t u)^2}{2} + \frac{1}{2}\left|\frac{x}{t} \cdot \nabla u\right|^2 \le \frac{(\partial_t u)^2}{2} + \frac{1}{2}|\nabla u|^2. \tag{20}$$

We infer from (19)–(20), the positivity of R_0 and the conservation of the energy that

$$\begin{aligned}
\int_{D(b)} \frac{|u|^{p+1}}{p+1}\,\mathrm{d}x &\le \frac{a}{b}\int_{D(a)} Q_0\,\mathrm{d}x + \frac{1}{b}\int_{M_a^b} t\left(\partial_t u + \frac{x}{t}\cdot\nabla u + \frac{u}{t}\right)^2 \frac{\mathrm{d}\sigma}{\sqrt{2}} \\
&\le \frac{a}{b}\int_{D(a)}\left(\frac{|u|^{p+1}}{p+1} + (\partial_t u)^2 + |\nabla u|^2\right)\mathrm{d}x + \frac{1}{b}\int_{M_a^b} t\left(\partial_t u + \frac{x}{t}\cdot\nabla u + \frac{u}{t}\right)^2 \frac{\mathrm{d}\sigma}{\sqrt{2}} \\
&\le \frac{a}{b}E + \frac{1}{b}\int_{M_a^b} t\left(\partial_t u + \frac{x}{t}\cdot\nabla u + \frac{u}{t}\right)^2 \frac{\mathrm{d}\sigma}{\sqrt{2}}.
\end{aligned}$$

We estimate the last term on the right-hand side as in [Bahouri and Gérard 1999]: we use (5) to bound

$$\frac{1}{b}\int_{M_a^b} t\left(\partial_t u + \frac{x}{t}\cdot\nabla u + \frac{u}{t}\right)^2 \frac{\mathrm{d}\sigma}{\sqrt{2}} \le 2(e(b)-e(a)) + 2\int_{M_a^b} \frac{u^2}{t^2}\,\frac{\mathrm{d}\sigma}{\sqrt{2}}.$$

The main difference with respect to the energy-critical regime is the estimate of the second addend which now deteriorates with b. Indeed, we estimate by Hölder

$$\int_{M_a^b} \frac{u^2}{t^2}\,\frac{\mathrm{d}\sigma}{\sqrt{2}} \le b^{\frac{\delta}{p+1}}\left(\int_{M_a^b} \frac{|u|^{p+1}}{p+1}\,\frac{\mathrm{d}\sigma}{\sqrt{2}}\right)^{\frac{2}{p+1}} \lesssim b^{\frac{\delta}{p+1}}(e(b)-e(a))^{\frac{2}{p+1}}.$$

Collecting terms, we have obtained (18) for classical solutions $u \in C^2(\mathbb{R}^3 \times [a,b])$.

If u is a weak finite-energy solution of (1) as in the statement, we proceed as in [Bahouri and Gérard 1999]: we fix a family of mollifiers $\{\rho_\epsilon\}_{\epsilon>0}$ in space and define $u_\epsilon := u * \rho_\epsilon$. Then, setting

$$f_\epsilon = -|u_\epsilon|^{p-1}u_\epsilon + (|u|^{p-1}u) * \rho_\epsilon,$$

$u_\epsilon \in C^2(\mathbb{R}^3 \times [a,b])$ is a classical solution of

$$\Box u_\epsilon = |u_\epsilon|^{p-1}u_\epsilon + f_\epsilon. \tag{21}$$

By assumption, $f_\epsilon \in L^1([a,b], L^2)$ can be treated as a source term. We then deduce (18) by proving the analogous local energy inequality for a nonlinear wave equation with right-hand side (21) and pass to the limit $\epsilon \to 0$. We refer to [Bahouri and Gérard 1999, Lemma 2.3] for details. □

Lemma 4.5 can be viewed as decay estimate for the potential energy. Again, when compared to the critical case [Tao 2006b, Corollary 4.10], the supercriticality of the equation weakens the decay by introducing a new dependence on T_2, the endpoint of the interval to which the decay estimate is applied, which deteriorates as $T_2 \to +\infty$.

Proposition 4.6 (potential energy decay in forward wave cones). *Let $I = [T_1, T_2] \subset (0, +\infty)$ and consider a solution $(u, \partial_t u) \in L^\infty(I, \dot{H}^1 \cap \dot{H}^2 \times H^1)$ to (1) with $p = 5 + \delta$ for some $\delta \in (0, 1)$. Let $0 < \theta$ such that*

$$ET_2^{-\frac{\delta}{p-1}} \theta^{-(p+1)} > 1. \tag{22}$$

Let $A > 0$ be such that

$$A \geq ET_2^{-\frac{\delta}{p-1}} \theta^{-(p+1)} \quad \text{and} \quad A^{3ET_2^{-\delta/(p-1)}\theta^{-(p+1)}\max\{1,\theta^{-(p+1)(p-1)/2}\}} T_1 \leq T_2. \tag{23}$$

Then there exists a subinterval of the form $J = [t', At']$ such that

$$\|u\|_{L^\infty L^{p+1}(\Gamma_+(J))} \lesssim T_2^{\frac{\delta}{(p-1)(p+1)}} \theta.$$

Notice that θ in the previous statement is not dimensional.

Proof. Let $\theta > 0$ be as in (22) and fix $A \geq ET_2^{-\delta/(p-1)} \theta^{-(p+1)}$. Let N to be chosen later be such that $A^{2N} T_1 \leq T_2$, namely

$$\bigcup_{i=1}^{N} [A^{2(n-1)} T_1, A^{2n} T_1] \subseteq I.$$

Since e is nondecreasing in time (see (5)), we have $e(A^{2n}t) - e(A^{2(n-1)}t) \geq 0$ for all n and

$$0 \leq \sum_{n=1}^{N} e(A^{2n} T_1) - e(A^{2(n-1)} T_1) = e(A^{2N} T_1) - e(T_1) \leq E.$$

Hence there exists $n_0 \in \{1, \dots, N\}$ such that $e(A^{2n_0} T_1) - e(A^{2(n_0-1)} T_1) \leq EN^{-1}$. Splitting the interval as

$$[A^{2(n_0-1)} T_1, A^{2n_0} T_1] = [A^{2(n_0-1)} T_1, A^{2n_0-1} T_1] \cup [A^{2n_0-1} T_1, A^{2n_0} T_1],$$

we have, applying Lemma 4.5 with $a := A^{2(n_0-1)} T_1$ and varying $b \in [A^{2n_0-1} T_1, A^{2n_0} T_1]$, that

$$\begin{aligned} \|u\|^{p+1}_{L^\infty L^{p+1}(\Gamma_+([A^{2n_0-1}T_1, A^{2n_0}T_1])} &\lesssim \frac{1}{A} E + EN^{-1} + (A^{2n_0} T_1)^{\frac{\delta}{p+1}} (EN^{-1})^{\frac{2}{p+1}} \\ &\lesssim T_2^{\frac{\delta}{p-1}} \theta^{p+1} + EN^{-1} + T_2^{\frac{\delta}{p+1}} (EN^{-1})^{\frac{2}{p+1}} \lesssim T_2^{\frac{\delta}{p-1}} \theta^{p+1}, \end{aligned}$$

provided

$$(EN^{-1})^{\frac{2}{p+1}} \leq T_2^{\frac{2\delta}{(p-1)(p+1)}} \theta^{p+1} \quad \text{and} \quad EN^{-1} \leq T_2^{\frac{\delta}{p-1}} \theta^{p+1},$$

or equivalently,

$$ET_2^{-\frac{\delta}{p-1}} \theta^{-(p+1)} \max\{1, \theta^{-\frac{(p+1)(p-1)}{2}}\} \leq N.$$

For the latter, we have to ask that $[T_1, A^{2N} T_1] \subseteq [T_1, T_2]$, which is enforced by the second requirement in (23). □

Proof of Proposition 4.1. Fix $0 < \theta$ yet to be determined such that $ET_2^{-\delta/(p-1)}\theta^{-(p+1)} > 1$. Fix $A \geq ET_2^{-\delta/(p-1)}\theta^{-(p+1)}$ and assume that (23) holds. By Proposition 4.6, there exists a subinterval J of the form $J := [t', At']$ and $C' \geq 1$ such that

$$\|u\|_{L^\infty L^{p+1}(\Gamma_+(J))} \leq C' T_2^{\frac{\delta}{(p-1)(p+1)}} \theta. \tag{24}$$

We claim that if we choose θ appropriately, we have $\|u\|_{L^{2(p-1)}(\Gamma_+(J))} \leq \eta$. Indeed, assume by contradiction that $\|u\|_{L^{2(p-1)}(\Gamma_+(J))} \geq \eta$. Then we have from Lemma 4.4

$$\|u\|_{L^\infty L^{p+1}(\Gamma_+(J))} \geq C_1 \eta^{\frac{12(p-1)}{5(p+1)}} (M^{\frac{\delta}{2}} E)^{-\frac{9}{5(p+1)}} M^{-\frac{\delta}{2(p+1)}}.$$

Choosing θ to be

$$\theta := \frac{C_1}{2C'} \eta^{\frac{12(p-1)}{5(p+1)}} (M^{\frac{\delta}{2}} E)^{-\frac{9}{5(p+1)}} M^{-\frac{\delta}{2(p+1)}} T_2^{-\frac{\delta}{(p+1)(p-1)}},$$

we reach a contradiction with (24). Let us now verify the hypothesis on θ: We observe that

$$ET_2^{-\frac{\delta}{p-1}} \theta^{-(p+1)} = (C_1(2C')^{-1})^{-(p+1)} \eta^{-\frac{12(p-1)}{5}} (EM^{\frac{\delta}{2}})^{\frac{14}{5}}$$

such that hypothesis (22) is enforced if

$$0 < \eta < (C_1^{-1} 2C')^{\frac{5(p+1)}{12(p-1)}} (EM^{\frac{\delta}{2}})^{\frac{7}{6(p-1)}}.$$

This explains the hypotheses (12) and (13) with the choice

$$C_2 := (C_1^{-1} 2C')^{\frac{5(p+1)}{12(p-1)}}.$$

We also rewrite the largeness hypothesis on I, namely the second formula in (23), in terms of η,

$$\begin{aligned}\theta^{-\frac{(p+1)(p-1)}{2}} &= (C_1(2C')^{-1})^{-\frac{(p+1)(p-1)}{2}} \eta^{-\frac{6(p-1)^2}{5}} (EM^{\frac{\delta}{2}})^{\frac{9(p-1)}{10}} M^{\frac{\delta(p-1)}{4}} T_2^{\frac{\delta}{2}} \\ &= (C_2\eta^{-1})^{\frac{6(p-1)^2}{5}} (M^{\frac{p-1}{2}} T_2)^{\frac{\delta}{2}} (EM^{\frac{\delta}{2}})^{\frac{9(p-1)}{10}},\end{aligned}$$

so that

$$\begin{aligned}&\max\{1, \theta^{-\frac{(p+1)(p-1)}{2}}\} \\ &\quad = (C_2\eta^{-1})^{\frac{6(p-1)^2}{5}} (EM^{\frac{\delta}{2}})^{\frac{9(p-1)}{10}} \max\{(C_2\eta^{-1})^{\frac{-6(p-1)^2}{5}} (EM^{\frac{\delta}{2}})^{\frac{-9(p-1)}{10}}, (M^{\frac{p-1}{2}} T_2)^{\frac{\delta}{2}}\}.\end{aligned}$$

This shows that (14) implies the second inequality in (23). □

5. Asymptotic stability

Let $u : \mathbb{R}^3 \times I \to \mathbb{R}$ solve an inhomogeneous wave equation $\Box u = F$. We now introduce the free evolution u_{l,t_0} from time t_0, that is, the unique solution of the free wave equation $\Box u_{l,t_0} = 0$ which agrees with u at time t_0, i.e., $(u_{l,t_0}, \partial_t u_{l,t_0})(t_0) = (u, \partial_t u)(t_0)$. We recall that, from solving the linear wave equation in Fourier space, we have the representation formula

$$u_{l,t_0}(t) = \cos(t\sqrt{-\Delta}) u(t_0) + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} \partial_t u(t_0),$$

where we use Fourier multiplier notation; see for instance [Sogge 1995]. From this representation as well as the Strichartz estimate (7), it follows that for any $m \in \left[1, \frac{3}{2}\right)$ and any (p, q) satisfying (6) we have the estimate

$$\|(u_{l,t_0}, \partial_t u_{l,t_0})\|_{L^\infty(I,\dot H^m\times \dot H^{m-1})} + \|u_{l,t_0}\|_{L^{2(p-1)}(\mathbb{R}^3\times I)} \lesssim \|(u,\partial_t u)(t_0)\|_{\dot H^m\times\dot H^{m-1}}. \tag{25}$$

From Duhamel's principle it follows that we can write for $t \in I$

$$u(t) = u_{l,t_0}(t) + \int_{t_0}^{t} \frac{\sin((t-t')\sqrt{-\Delta})}{\sqrt{-\Delta}} F(t')\,\mathrm{d}t'. \tag{26}$$

We recall from [Shatah and Struwe 1998, Chapter 4] that for $t \neq t'$ we have the explicit expression

$$\frac{\sin((t-t')\sqrt{-\Delta})}{\sqrt{-\Delta}} F(t') = \frac{1}{4\pi(t-t')} \int_{|x-x'|=|t-t'|} F(t',x')\,\mathrm{d}\mathcal{H}^2(x').$$

We recall that the linear evolution enjoys asymptotic stability in the following sense.

Lemma 5.1 (asymptotic stability for the linear evolution). *Let $p = 5+\delta$ with $\delta \in (0,1)$. Let u be a solution to* (1) *on $\mathbb{R}^3\times I'$ with $\|u\|_{L^\infty(\mathbb{R}^3\times I')} \le M$. Then for any $I = [t_1,t_2] \subseteq I'$ and any $t \in I'\setminus I$ we have*

$$\|u_{l,t_2}(t) - u_{l,t_1}(t)\|_{L^\infty(\mathbb{R}^3)} \lesssim (EM^{\frac{\delta}{2}})^{\frac{2p}{3(p-1)}} \operatorname{dist}(t,I)^{-\frac{2}{p-1}}.$$

Proof. From (5) we deduce that

$$\partial_t e(t) \ge \int_{|x|=t} \frac{|u(y,t)|^{p+1}}{p+1}\,\mathrm{d}\mathcal{H}^2(y).$$

Integrating in time, by translation invariance and time reversibility, we have

$$\int_I \int_{|x'-x|=|t'-t|} |u(x',t')|^{p+1}\,\mathrm{d}\mathcal{H}^2(x')\,\mathrm{d}t' \lesssim E$$

for any $(x,t) \in \mathbb{R}^3 \times I'$. Using (26), we write for $t \in I'\setminus I$

$$u_{l,t_2}(t) - u_{l,t_1}(t) = -\frac{1}{4\pi}\int_{t_1}^{t_2} \frac{1}{|t-t'|}\int_{|x-x'|=|t-t'|} |u(x',t')|^p\,\mathrm{d}\mathcal{H}^2(x')\,\mathrm{d}t'.$$

We apply Hölder with

$$\left(\frac{3(p-1)}{2p}, \frac{3(p-1)}{p-3}\right) = \left(\frac{p+1+\frac{\delta}{2}}{p}, \frac{p+1+\frac{\delta}{2}}{1+\frac{\delta}{2}}\right)$$

to estimate for any $x \in \mathbb{R}^3$

$$\begin{aligned}
&|u_{l,t_2}(x,t) - u_{l,t_1}(x,t)| \\
&\quad\lesssim \int_{t_1}^{t_2} \frac{1}{|t-t'|}\int_{|x-x'|=|t-t'|} |u(x',t')|^p\,\mathrm{d}\mathcal{H}^2(x')\,\mathrm{d}t' \\
&\quad\lesssim \left(\int_{t_1}^{t_2}\int_{|x-x'|=|t-t'|} |u|^{p+1+\frac{\delta}{2}}(x',t')\,\mathrm{d}\mathcal{H}^2(x')\,\mathrm{d}t'\right)^{\frac{2p}{3(p-1)}} \left(\int_{t_1}^{t_2} \frac{\mathrm{d}t'}{|t-t'|^{\frac{3(p-1)}{p-3}-2}}\right)^{\frac{p-3}{3(p-1)}} \\
&\quad\lesssim \left(\|u\|^{\frac{\delta}{2}}_{L^\infty(\mathbb{R}^3\times I)} \int_{t_1}^{t_2}\int_{|x-x'|=|t-t'|} |u|^{p+1}(x',t')\mathrm{d}\mathcal{H}^2(x')\,\mathrm{d}t'\right)^{\frac{2p}{3(p-1)}} \operatorname{dist}(t,I)^{-\frac{2}{p-1}} \\
&\quad\lesssim (M^{\frac{\delta}{2}}E)^{\frac{2p}{3(p-1)}} \operatorname{dist}(t,I)^{-\frac{2}{p-1}}. \qquad \square
\end{aligned}$$

The importance of the above asymptotic stability lies in the following corollary.

Corollary 5.2. *Let $p = 5 + \delta$ with $\delta \in (0, 1)$ and $I = [t_-, t_+]$. Consider a solution $(u, \partial_t u) \in L^\infty(I, \dot{H}^1 \cap \dot{H}^2 \times H^1)$ to* (1) *and assume that $\|u\|_{L^\infty(\mathbb{R}^3 \times I)} \le M$. Consider $I_1 = [t_1, t_2]$ and $I_2 = [t_2, t_3]$ for any $t_- \le t_1 < t_2 < t_3 \le t_+$. Then*

$$\|u_{l,t_3} - u_{l,t_+}\|_{L^{2(p-1)}(\Gamma_+(I_1))} \lesssim \frac{|I_1|^{\frac{1}{2(p-1)}}}{|I_2|^{\frac{1}{2(p-1)}}} (EM^{\frac{\delta}{2}})^{\frac{p}{6(p-1)}} \|u\|^{\frac{3}{4}}_{L^\infty(I,(\dot{H}^{s_p} \times \dot{H}^{s_p-1}))}.$$

Proof. We observe that the pair $\big(\infty, \frac{3}{2}(p-1)\big)$ is wave-s_p-admissible, where we recall that $s_p := 1 + \delta/(2(p-1))$ is the critical Sobolev regularity of (1). We estimate by Hölder

$$\|u_{l,t_3} - u_{l,t_+}\|_{L^{2(p-1)}(\Gamma_+(I_1))} \lesssim |I_1|^{\frac{1}{2(p-1)}} \|u_{l,t_2} - u_{l,t_3}\|^{\frac{1}{4}}_{L^\infty(\mathbb{R}^3 \times I_1)} \|u_{l,t_3} - u_{l,t_+}\|^{\frac{3}{4}}_{L^\infty L^{3(p-1)/2}(\Gamma_+(I_1))}.$$

Observe that $v := u_{l,t_3} - u_{l,t_+}$ solves $\Box v = 0$ with $v(t_3) = u(t_3) - u_{l,t_+}(t_3)$. Hence by the Strichartz estimate (7) and (25) we have

$$\begin{aligned}\|v\|_{L^\infty L^{3(p-1)/2}(\Gamma_+(I_1))} &\lesssim \|(v, \partial_t v)(t_3)\|_{(\dot{H}^{s_p} \times \dot{H}^{s_p-1})(\mathbb{R}^3)} \\ &\lesssim \|(u, \partial_t u)(t_3)\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}} + \|(u_{l,t_+}, \partial_t u_{l,t_+})(t_3)\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}} \\ &\lesssim \|(u, \partial_t u)(t_3)\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}} + \|(u, \partial_t u)(t_+)\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}} \\ &\lesssim \|(u, \partial_t u)\|_{L^\infty(I,(\dot{H}^{s_p} \times \dot{H}^{s_p-1}))}.\end{aligned}$$ □

6. A reverse Sobolev inequality and mass concentration

The section is devoted to proving that, if u solves (1), then there exists a suitable ball with controlled size which contains an amount of L^2 norm, quantified in terms of $\|u\|_{L^{2(p-1)}}$ and $\|u\|_{H^s}$. A key ingredient in the proof is the reverse Sobolev inequality of Tao, generalized for any $s \in \big(0, \frac{3}{2}\big)$. We present the proof for completeness, since the original argument used the fact that p was integer.

Proposition 6.1. *Let $0 < s < \frac{3}{2}$ and $\frac{1}{q} := \frac{1}{2} - \frac{s}{3}$. Let $f \in \dot{H}^s(\mathbb{R}^3)$. Then there exists $x \in \mathbb{R}^3$ and $0 < r \le \frac{2}{N}$ such that*

$$\left(\frac{1}{r^{2s}} \int_{B(x,r)} f^2(y)\,\mathrm{d}y\right)^{\frac{1}{2}} \gtrsim \|P_{\ge N} f\|^{\left(\frac{3}{2s}\right)^2}_{L^q(\mathbb{R}^3)} \|f\|^{1-\left(\frac{3}{2s}\right)^2}_{\dot{H}^s}. \tag{27}$$

Proof. By replacing f with $\tilde{f}(x) := (1/\|f\|_{\dot{H}^s}) f(x)$ we can assume without loss of generality that $\|f\|_{\dot{H}^s} = 1$.

Step 1: *Let $g \in \dot{H}^s$ with $\|g\|_{\dot{H}^s} \le 1$. Then there exists $\overline{N} \in 2^{\mathbb{Z}}$ such that*

$$\|g\|^{\frac{3}{2s}}_{L^q} \lesssim \|P_{\overline{N}} g\|_{L^q}, \tag{28}$$

and as a consequence

$$\|g\|^{\left(\frac{3}{2s}\right)^2}_{L^q} \overline{N}^{\frac{3}{q}} \lesssim \|P_{\overline{N}} g\|_{L^\infty}. \tag{29}$$

From (10), Plancherel's theorem and the hypothesis $\|g\|_{\dot H^s} \le 1$, we infer that

$$\sum_{N\in 2^{\mathbb{Z}}} N^{2s}\|P_N g\|_{L^2}^2 \lesssim 1. \tag{30}$$

By interpolation, (30) and the definition of q we see that (29) is a consequence of (28); indeed

$$\|P_{\overline{N}}g\|_{L^q} \le \|P_{\overline{N}}g\|_{L^2}^{\frac{2}{q}}\|P_{\overline{N}}g\|_{L^\infty}^{1-\frac{2}{q}} = \overline{N}^{-\frac{2s}{q}}(\overline{N}^{2s}\|P_{\overline{N}}g\|_{L^2}^2)^{\frac{1}{q}}\|P_{\overline{N}}g\|_{L^\infty}^{1-\frac{2}{q}} \lesssim \overline{N}^{-\frac{2s}{q}}\|P_{\overline{N}}g\|_{L^\infty}^{\frac{2s}{3}}.$$

We are left to prove (28). Let us fix $M \in \mathbb{N}$ big enough such that $\frac{q}{2} \in (M-1, M]$. With this choice of M, we ensure the subadditivity of the map $x \mapsto x^{q/(2M)}$. We then write, using the hypothesis, (10), the aforementioned subadditivity, a reordering and Hölder,

$$\begin{aligned}
\|g\|_{L^q}^q &\lesssim \int\Big(\sum_{M\in 2^{\mathbb{Z}}}|P_M g(x)|^2\Big)^{\frac{q}{2}}\,\mathrm{d}x = \int\prod_{i=1}^{M}\Big(\sum_{N_i\in 2^{\mathbb{Z}}}|P_{N_i}g(x)|^2\Big)^{\frac{q}{2M}}\,\mathrm{d}x\\
&\le \int\prod_{i=1}^{M}\sum_{N_i\in 2^{\mathbb{Z}}}|P_{N_i}g(x)|^{\frac{q}{M}}\,\mathrm{d}x \lesssim \sum_{N_1\le\cdots\le N_M}\int\prod_{i=1}^{M}|P_{N_i}g(x)|^{\frac{q}{M}}\,\mathrm{d}x\\
&\lesssim \Big(\sup_{N\in 2^{\mathbb{Z}}}\|P_N g\|_{L^q}\Big)^{\frac{q(M-2)}{M}}\sum_{N_1\le\cdots\le N_M}\Big(\int|P_{N_1}g(x)|^{\frac{q}{2}}|P_{N_M}g(x)|^{\frac{q}{2}}\,\mathrm{d}x\Big)^{\frac{2}{M}}.
\end{aligned}$$

In all sums on $N_1 \le \cdots \le N_M$, we intend that each N_i belongs to $2^{\mathbb{Z}}$. We claim that the second factor is bounded by a constant. Indeed, we estimate the last integral for fixed N_1 and N_M using Hölder by

$$\begin{aligned}
&\Big(\int|P_{N_1}g(x)|^{\frac{q}{2}}|P_{N_M}g(x)|^{\frac{q}{2}}\,\mathrm{d}x\Big)^{\frac{2}{M}}\\
&\quad\le \Big(\|P_{N_1}g\|_{L^\infty}^{\frac{M}{2}}\int|P_{N_1}g(x)|^{\frac{q-M}{2}}|P_{N_M}g(x)|^{\frac{q-M}{2}}|P_{N_M}g(x)|^{\frac{M}{2}}\,\mathrm{d}x\Big)^{\frac{2}{M}}\\
&\quad\le \|P_{N_1}g\|_{L^\infty}\|P_{N_1}g\|_{L^q}^{\frac{q-M}{M}}\|P_{N_M}g\|_{L^q}^{\frac{q-M}{M}}\|P_{N_M}g\|_{L^{q/2}}.
\end{aligned}$$

By Bernstein's inequality (9) and the definition of q, we have

$$\|P_{N_1}g\|_{L^\infty}\|P_{N_M}g\|_{L^{\frac{q}{2}}} \lesssim N_1^{\frac{3}{2}}N_M^{\frac{3}{2}-\frac{6}{q}}\|P_{N_1}g\|_{L^2}\|P_{N_M}g\|_{L^2} = N_1^{\frac{3}{2}}N_M^{2s-\frac{3}{2}}\|P_{N_1}g\|_{L^2}\|P_{N_M}g\|_{L^2}.$$

Combining the three estimates, we deduce that

$$\begin{aligned}
\|g\|_{L^q}^q &\lesssim \Big(\sup_{N\in 2^{\mathbb{Z}}}\|P_N g\|_{L^q}\Big)^{q-2}\sum_{N_1\le\cdots\le N_M}\|P_{N_1}g\|_{L^\infty}\|P_{N_M}g\|_{L^{q/2}}\\
&\lesssim \Big(\sup_{N\in 2^{\mathbb{Z}}}\|P_N g\|_{L^q}\Big)^{q-2}\sum_{N_1\le\cdots\le N_M}N_1^{\frac{3}{2}-s}N_M^{s-\frac{3}{2}}(N_1^{2s}\|P_{N_1}g\|_{L^2}^2+N_M^{2s}\|P_{N_M}g\|_{L^2}^2).
\end{aligned}$$

Let us consider the first addend on the right-hand side (the second is handled analogously):

$$\begin{aligned}
\sum_{N_1\le\cdots\le N_M}N_1^{\frac{3}{2}-s}N_M^{s-\frac{3}{2}}N_1^{2s}\|P_{N_1}g\|_{L^2}^2 &\le \sum_{n_1\in\mathbb{Z}}2^{2n_1 s}\|P_{2^{n_1}}g\|_{L^2}^2\sum_{n_M=n_1}^{\infty}(n_M-n_1)^{M-2}2^{-(\frac{3}{2}-s)(n_M-n_1)}\\
&\lesssim \sum_{n_1\in\mathbb{Z}}2^{2n_1 s}\|P_{2^{n_1}}g\|_{L^2}^2 \lesssim 1,
\end{aligned}$$

where we used that for fixed $s \in \left(0, \frac{3}{2}\right)$ the series $\|P_{2^{n_1}} g\|_{L^2}^2 \sum_{n=0}^{\infty} n^{M-2} 2^{-(3/2-s)n}$ converges for every $M \in \mathbb{N}$ as well as (30). We conclude from (31) that

$$\|g\|_{L^q}^{\frac{3}{2s}} = \|g\|_{L^q}^{\frac{q}{q-2}} \lesssim \sup_{N \in 2^{\mathbb{Z}}} \|P_N\|_{L^q},$$

which implies (28).

Step 2: *Let $\overline{N}, N \in 2^{\mathbb{Z}}$ and define $\psi_{\overline{N}} := \overline{N}^3 \psi(\overline{N} x)$, where ψ is a bump function supported in $B_1(0)$ whose Fourier transform has magnitude ~ 1 on $B_{100}(0)$. Then we can rewrite*

$$P_{\overline{N}} P_{\geq N} f = \widetilde{P}_{\overline{N}}(f * \psi_{\overline{N}}),$$

where $\widetilde{P}_{\overline{N}}$ is a Fourier multiplier which is bounded on L^∞.

The claimed identity of Fourier multipliers follows by setting $\mathcal{F}(\widetilde{P}_{\overline{N}})(\xi) := \Psi(\xi/\overline{N})$, where

$$\Psi(\xi) := (\varphi(\xi) - \varphi(2\xi))(1 - \varphi(\xi \overline{N}/N))\hat{\psi}(\xi)^{-1}.$$

To verify that $\widetilde{P}_{\overline{N}}$ is bounded on L^∞, for $g \in L^\infty$ we estimate by Young and a change of variables

$$\|\widetilde{P}_{\overline{N}} g\|_{L^\infty} \lesssim \|\mathcal{F}^{-1}(\Psi(\xi/\overline{N}))\|_{L^1} \|g\|_{L^\infty} = \|\mathcal{F}^{-1}(\Psi)\|_{L^1} \|g\|_{L^\infty}.$$

Observe that $\Psi \in C_c^\infty(\mathbb{R}^3) \subseteq \mathcal{S}(\mathbb{R}^3)$, so that $\|\mathcal{F}^{-1}(\Psi)\|_{L^1} < +\infty$.

Step 3: *Conclusion of the proof.*

We apply Step 1 to $g = P_{\geq N} f$ to deduce that there exist $\overline{N} \in 2^{\mathbb{Z}}$ such that

$$\|P_{\geq N} f\|_{L^q}^{\left(\frac{3}{2s}\right)^2} \overline{N}^{\frac{3}{q}} \lesssim \|P_{\overline{N}} P_{\geq N} f\|_{L^\infty}.$$

We observe that $\overline{N} \geq \frac{N}{2}$ because otherwise $P_{\overline{N}} P_{\geq N} f = 0$. By Step 2, we deduce that there exists $x \in \mathbb{R}^3$ such that

$$\|P_{\geq N} f\|_{L^q}^{\left(\frac{3}{2s}\right)^2} \overline{N}^{\frac{3}{q}} \lesssim |\psi_{\overline{N}} * f(x)| \leq \overline{N}^{\frac{3}{2}} \left(\int_{B\left(x, \frac{1}{\overline{N}}\right)} f^2(y) \, \mathrm{d}y \right)^{\frac{1}{2}} \|\psi\|_{L^2}.$$

Combining the two inequalities, we obtain the claimed inequality (27) with $r := \frac{1}{\overline{N}} \in \left(0, \frac{2}{N}\right]$. □

The proposition above will be applied with $s = s_p$; the choice of $s \neq 1$ is in turn fundamental in the main theorem, since it allows us to give an upper bound on the r_0 given by the mass concentration only in terms of E, M, $\|u\|_{L^\infty \dot{H}^{s_p}}$.

Lemma 6.2 (mass concentration). *Let $p = 5 + \delta$ for $\delta \in (0, 1)$ and let $0 < \eta \leq 1$. Assume*

$$\|u\|_{L^{2(p-1)}(\mathbb{R}^3 \times I)} \geq \eta \quad \textit{and} \quad \|u\|_{L^\infty(\mathbb{R}^3 \times I)} \leq M.$$

Then, for any $1 \leq s \leq s_p := 1 + \delta/(2(p-1))$ there exists $(x, t) \in \mathbb{R}^3 \times I$ and $r > 0$ such that

$$\frac{1}{r^{2s}} \int_{B(x,r)} u^2(y, t) \, \mathrm{d}y \gtrsim \|u\|_{L^\infty(I, \dot{H}^{s_p}(\mathbb{R}^3))}^{-\alpha_0} (M^{\frac{\delta}{2}} E)^{-\alpha_1} M^{-(s_p - s)(p-1)} \eta^{\alpha_2}, \tag{31}$$

where $\alpha_i = \alpha_i(s) \geq 0$ are defined as

$$\alpha_0 := (\gamma-2)\frac{s-1}{s_p-1}, \quad \alpha_1 := \frac{3}{10}\gamma(3-2s)+\frac{\gamma-2}{2}\frac{s_p-s}{s_p-1} \quad \textit{and} \quad \alpha_2 := \frac{3-2s}{5}2(p-1)\gamma \quad \textit{for } \gamma := \frac{9}{2s^2}.$$

Moreover,

$$|I| \gtrsim \eta^{2(p-1)} \|u\|^{-\alpha_0'}_{L^\infty(I,\dot H^{s_p}(\mathbb{R}^3))} (EM^{\frac{\delta}{2}})^{-\alpha_1'} M^{\frac{(s-1)(p-1)}{2}} r^s, \tag{32}$$

where $\alpha_i'(s) \geq 0$ are defined as

$$\alpha_0' := 2(p-1) - \frac{(s-1)(p-1)(p+1)}{\delta} \quad \textit{and} \quad \alpha_1' := \frac{(s-1)(p-1)}{\delta}.$$

Proof. Fix $1 \leq s \leq s_p = 1+\delta/(2(p-1))$ and set

$$\frac{1}{q} := \frac{1}{2} - \frac{s}{3},$$

the conjugate Sobolev exponent. By shrinking I, we can always assume that $\|u\|_{L^{2(p-1)}(\mathbb{R}^3\times I)} = \eta$. Recalling the proof of Lemma 4.4, we have that for any wave-1-admissible (q,r)

$$\|u\|_{L^q L^r} \lesssim E^{\frac{1}{2}}. \tag{33}$$

Step 1: *We find a frequency scale $N \in 2^{\mathbb{Z}}$, where $\|P_{\geq N} f\|_{L^{2(p-1)}(\mathbb{R}^3\times I)} \gtrsim \eta$.*

By Hölder and Bernstein (9) with exponents $2(p-1)$ and $6(p-1)/(s+3) \in [6,q^*]$ we estimate

$$\|P_{<N}u\|_{L^{2(p-1)}} \lesssim |I|^{\frac{1}{2(p-1)}} \|P_{<N}u\|_{L^\infty L^{2(p-1)}} \lesssim |I|^{\frac{1}{2(p-1)}} N^{\frac{s}{2(p-1)}} \|u\|_{L^\infty L^{6(p-1)/(s+3)}}.$$

We observe that by interpolation and the Sobolev embedding of $\dot H^{s_p} \hookrightarrow L^{3(p-1)/2}$

$$\|u\|_{L^\infty L^{6(p-1)/(s+3)}} \leq \|u\|^{1-\frac{(s-1)(p+1)}{2\delta}}_{L^\infty \frac{3(p-1)}{2}} \|u\|^{\frac{(s-1)(p+1)}{2\delta}}_{L^\infty L^{p+1}} \lesssim \|u\|^{1-\frac{(s-1)(p+1)}{2\delta}}_{L^\infty \dot H^{s_p}} (EM^{\frac{\delta}{2}})^{\frac{(s-1)}{2\delta}} M^{-\frac{(s-1)}{4}}.$$

Thus if we choose the frequency scale $N \in 2^{\mathbb{Z}}$ such that

$$|I|^{\frac{1}{2(p-1)}} N^{\frac{s}{2(p-1)}} \|u\|^{1-\frac{(s-1)(p+1)}{2\delta}}_{L^\infty \dot H^{s_p}} (EM^{\frac{\delta}{2}})^{\frac{(s-1)}{2\delta}} M^{-\frac{(s-1)}{4}} = c\eta \tag{34}$$

for a universal small constant $0 < c \ll 1$, we can ensure that $\|P_{\geq N}u\|_{L^{2(p-1)}(\mathbb{R}^3\times I)} \gtrsim \eta$.

Step 2: *We deduce a lower bound of $\|P_{\geq N}u\|_{L^\infty(I,L^q(\mathbb{R}^3))}$ in terms of η, E, M.*

Observe that the pair $(3,18)$ is wave-1-admissible and $(3,18)$ and (∞,q) interpolate to $\left(\frac{5}{6}q+3, \frac{5}{6}q+3\right)$. Using (33) and (34), we have by Hölder

$$\begin{aligned}
\eta^{2(p-1)} &\lesssim \|P_{\geq N}u\|^{2(p-1)}_{L^{2(p-1)}_{t,x}} \lesssim \|P_{\geq N}u\|^{2(p-1)-(\frac{5}{6}q+3)}_{L^\infty_{t,x}} \|P_{\geq N}u\|^{\frac{5}{6}q+3}_{L^{(5/6)q+3}_{t,x}} \\
&\lesssim M^{5+2\delta-\frac{5}{6}q} \|P_{\geq N}u\|^3_{L^3L^{18}} \|P_{\geq N}u\|^{\frac{5}{6}q}_{L^\infty L^q} \\
&\lesssim M^{\frac{5}{6}q(\frac{6}{q}+\frac{3}{2q}\delta-1)} (M^{\frac{\delta}{2}}E)^{\frac{3}{2}} \|P_{\geq N}u\|^{\frac{5}{6}q}_{L^\infty L^q};
\end{aligned}$$

hence after some easy algebraic manipulations

$$\begin{aligned}\|P_{\geq N} u\|_{L^\infty L^q} &\gtrsim \eta^{\frac{12}{5q}(p-1)} (M^{\frac{\delta}{2}} E)^{-\frac{9}{5q}} M^{-(\frac{6}{q}+\frac{3}{2q}\delta-1)} \\ &= \eta^{\frac{(3-2s)}{5}2(p-1)} (M^{\frac{\delta}{2}} E)^{-\frac{3}{10}(3-2s)} M^{-\frac{1}{2}(s_p-s)(p-1)}.\end{aligned}$$

Step 3: We apply the reverse Sobolev of Proposition 6.1 to conclude that there exists $(x,t) \in \mathbb{R}^3 \times I$ and $0 < r \leq \frac{2}{N}$ such that

$$\frac{1}{r^{2s}} \int_{B(x,r)} u^2(y,t)\,\mathrm{d}y \gtrsim \|u\|^{2-\gamma}_{L^\infty(I,\dot{H}^s(\mathbb{R}^3))} (\eta^{\frac{(3-2s)}{5}2(p-1)} (M^{\frac{\delta}{2}} E)^{-\frac{3}{10}(3-2s)} M^{-\frac{1}{2}(s_p-s)(p-1)})^\gamma, \quad (35)$$

where $\gamma := 9/(2s^2)$. Moreover from (34) we get

$$\begin{aligned}|I| &= \frac{(c\eta)^{2(p-1)} M^{\frac{(s-1)(p-1)}{2}}}{\|u\|^{2(p-1)-\frac{(s-1)(p-1)(p+1)}{\delta}}_{L^\infty \dot{H}^{s_p}} (EM^{\frac{\delta}{2}})^{\frac{(s-1)(p-1)}{\delta}} N^s} \\ &\gtrsim \eta^{2(p-1)} \frac{M^{\frac{(s-1)(p-1)}{2}}}{\|u\|^{2(p-1)-\frac{(s-1)(p-1)(p+1)}{\delta}}_{L^\infty \dot{H}^{s_p}} (EM^{\frac{\delta}{2}})^{\frac{(s-1)(p-1)}{\delta}}} r^s.\end{aligned}$$

We now rewrite (35): by interpolation and energy conservation,

$$\|u\|_{L^\infty \dot{H}^s} \leq E^{\frac{(s_p-s)(p-1)}{\delta}} \|u\|^{\frac{2(s-1)(p-1)}{\delta}}_{L^\infty \dot{H}^{s_p}}.$$

Observe that $\gamma \geq 2$ for $s \in \left(0, \frac{3}{2}\right)$. Thus we have

$$\|u\|^{2-\gamma}_{L^\infty \dot{H}^s} \gtrsim (M^{\frac{\delta}{2}} E)^{\frac{(s_p-s)(p-1)(2-\gamma)}{\delta}} \|u\|^{\frac{2(s-1)(p-1)(2-\gamma)}{\delta}}_{L^\infty \dot{H}^{s_p}} M^{\frac{(s_p-s)(p-1)(\gamma-2)}{2}},$$

so that

$$\begin{aligned}\frac{1}{r^{2s}} \int_{B(x,r)} & u^2(y,t)\,\mathrm{d}y \\ &\gtrsim \|u\|^{-(\gamma-2)\frac{s-1}{s_p-1}}_{L^\infty \dot{H}^{s_p}} (M^{\frac{\delta}{2}} E)^{-[\frac{3}{10}\gamma(3-2s)+\frac{\gamma-2}{2}\frac{s_p-s}{s_p-1}]} M^{-(s_p-s)(p-1)} \eta^{\frac{3-2s}{5}2(p-1)\gamma}. \quad \square\end{aligned}$$

Remark 6.3 (optimization of exponents on η, $\|u\|_{L^\infty \dot{H}^{s_p}}$ and $EM^{\delta/2}$). Whilst the free powers of M in (31) and (32) are fixed by scaling, the other powers come from interpolation and can be optimized. Since we are not aiming at an optimal double exponential bound, we can take in Step 2 of the proof of Lemma 6.2 any Strichartz-1-pair (q', r') (here $(3, 18)$) such that (∞, q) and (q', r') interpolate to $(\tilde{r}, \tilde{r})$ with $\tilde{r} \leq 2(p-1)$. Alternatively, to optimize the exponents α_1 and α_2, we first suppose that the endpoint $(2, \infty)$ was Strichartz-1-admissible, interpolate in Step 2 between $(2, \infty)$ and (∞, q) and conclude in Step 3 as before. We then approximate $(2, \infty)$ by wave-1-admissible pairs $(2+\epsilon, 6(2+\epsilon)/\epsilon)$. Letting $\epsilon \to 0$,

$$\frac{3-2s}{6}(\gamma) + \frac{\gamma-2}{2}\frac{s_p-s}{s_p-1}$$

and $\alpha_2(s)$ approaches

$$\frac{3-2s}{3}(p-1).$$

In the very same way, the free exponents in Lemma 4.4 can be optimized. Proceeding in this way, we would obtain the lower bound, for any $\omega > 0$ (and an implicit constant depending on ω),

$$\|u\|_{L^\infty L^{p+1}}^{p+1} \gtrsim \eta^{2(p-1)+\omega}(EM^{\frac{\delta}{2}})^{-(1+\omega)}M^{-\frac{\delta}{2}}.$$

7. Proof of Theorem 1.2 and Corollary 1.3

We have now assembled all necessary tools to prove Theorem 1.2. We outline now its main steps which are based on the scheme of [Tao 2006b] and its adaptation in [Roy 2009].

Let $(u, \partial_t u) \in L^\infty(J, \dot{H}^1 \cap \dot{H}^2 \times H^1)$ solve (1). Whenever the scaling-invariant smallness assumption of the first item of Theorem 1.2 holds, then Lemma 3.1 gives the desired spacetime bound. Otherwise, we split J into subintervals J_i such that on each subinterval the $L^{2(p-1)}$ spacetime is completely under control and substantial, i.e., $\|u\|_{L^{2(p-1)}(\mathbb{R}^3\times J_i)} = 1$ for all but eventually the last subinterval. The estimate (3) is then equivalent to estimating the number of such subintervals and relies on the following key arguments:

(i) Using Lemma 4.4, we deduce that on each J_i also the potential energy $L^\infty(J_i, L^{p+1})$ is substantial: it has a quantitative lower bound in terms of E, M and L.

(ii) Lemma 6.2 allows us to identify a ball $B = B(x_i, r_i)$ such that mass concentrates on B at time $t = t_i \in J_i$. The mass concentration can be extended to a neighborhood of t_i using that the local mass is Lipschitz in time. At the same time, the size of intervals J_i where such concentration happens is bounded from below in terms of E, M, L and r_i.

(iii) In the scheme of [Tao 2006b], the previous observation together with the finite speed of propagation is used to remove a cone in spacetime, containing the mass-concentration "bubble", and to construct a new solution $\tilde{u}$ with smaller energy than u which coincides with u outside the cone. This allows us to perform an induction on the level sets of the energy since for sufficiently small energy the claimed estimate holds by Lemma 3.1. In our setting, such an induction argument seems not applicable, since the solution $\tilde{u}$ does not need to obey the same a priori bounds on the $L^\infty(J, \dot{H}^1 \cap \dot{H}^2 \times H^1)$ norm as u.

(iv) As in [Roy 2009] we bypass the induction on the energy by an ad-hoc argument. By time reversal and translation symmetry and the lower bound on the length of (ii), it is enough to estimate the length of $K_+ = J \cap [t_0, +\infty)$, where $(x_0 = 0, t_0)$ is the point where mass concentration occurs at the minimal mass concentration radius (among those individuated before). As in (ii), the mass concentration at minimal radius extends to a neighborhood $\tilde{J}_0$ of t_0. We then look at the truncated-in-time cone $\Gamma_+(K_+)$ and we define a new splitting of K_+ in subintervals $\tilde{J}_i$ such that the $L^{2(p-1)}$ norm on every truncated-in-time cone $\Gamma_+(\tilde{J}_i)$ is substantial and such that $\tilde{J}_1 \subset \tilde{J}_0$. The asymptotic stability of Section 5 controls inductively the size $|\tilde{J}_{j+1}| \lesssim |\tilde{J}_j|$. Moreover, the size of $\tilde{J}_0$ is controlled from below by the mass concentration argument in (ii) and from above by an upper bound on the mass concentration radius (which was not needed in [Roy 2009]). If $|K_+|$ was too large, then by the decay of the potential energy Proposition 4.6 there must be a subinterval such that on the truncated-in-time cone the $L^{2(p-1)}$ spacetime norm is small. By construction, such subinterval cannot be covered by many $\tilde{J}_i$, which means that one of them has to be sufficiently large, contradicting the upper bound on $|\tilde{J}_0|$.

Proof of Theorem 1.2. Let $p = 5+\delta$ with $\delta \in (0,1)$, $J = [t_-, t_+]$ and consider a solution $(u, \partial_t u) \in L^\infty(J, ((\dot{H}^1 \cap \dot{H}^2) \times H^1)(\mathbb{R}^3))$ to (1) as in the statement. If either $EM^{\delta/2} < c_0$ or $L < c_0$, then we conclude by Lemma 3.1 that $\|u\|_{L^{2(p-1)}(\mathbb{R}^3 \times J)} \le 1$. For the rest of the argument, we thus may assume the lower bound

$$\min\{EM^{\frac{\delta}{2}}, L\} \ge c_0,$$

where $c_0 > 0$ is the universal constant given by Lemma 3.1.

Let $C > 2c_0^{-2}$ be a universal constant that will be fixed at the end of the proof. The inequality imposed on C guarantees that $CLEM^{\delta/2} > 2$.

Moreover, we may assume without loss of generality that $\|u\|_{L^{2(p-1)}(\mathbb{R}^3 \times J)} \ge 1$. We then split J into subintervals $J_1, \dots, J_l$ such that

- $\|u\|_{L^{2(p-1)}(\mathbb{R}^3 \times J_i)} = 1$ for $i = 1, \dots, l-1$,
- $\|u\|_{L^{2(p-1)}(\mathbb{R}^3 \times J_l)} \le 1$.

We call J_i exceptional if

$$\|u_{l,t_+}\|_{L^{2(p-1)}(\mathbb{R}^3 \times J_i)} + \|u_{l,t_-}\|_{L^{2(p-1)}(\mathbb{R}^3 \times J_i)} \ge B_{\mathrm{exc}}^{-1}$$

for some $B_{\mathrm{exc}} \ge 1$ yet to be defined. We have by Strichartz estimates (7) that

$$\|u_{l,t_+}\|_{L^{2(p-1)}(\mathbb{R}^3 \times J)},\ \|u_{l,t_-}\|_{L^{2(p-1)}(\mathbb{R}^3 \times J)} \lesssim L.$$

In particular, J cannot consist of too many exceptional intervals. More precisely, calling the number of exceptional intervals $N_{\mathrm{exc}} := |\{i \in \{1, \dots, l\} : J_i \text{ exceptional}\}|$, we have the bound

$$N_{\mathrm{exc}} \lesssim LB_{\mathrm{exc}}.$$

Between two exceptional intervals there can lie a chain $K = J_{i_0} \cup \dots \cup J_{i_1}$ of unexceptional intervals. However, since a chain K of unexceptional intervals has to be confined between two exceptional intervals (or one of its endpoints is t_- or t_+), the number of chains of unexceptional intervals N_{chain} is comparable to N_{exc}, that is,

$$N_{\mathrm{chain}} \lesssim N_{\mathrm{exc}}.$$

For a chain $K = J_{i_0} \cup \dots \cup J_{i_1}$ of unexceptional intervals, we define $N(K) := i_1 + 1 - i_0$ to be the number of intervals it is made of. Summarizing, we have

$$\|u\|^{2(p-1)}_{L^{2(p-1)}(\mathbb{R}^3 \times J)} \le N_{\mathrm{exc}} + N_{\mathrm{chain}} \sup_K N(K) \lesssim LB_{\mathrm{exc}}(1 + \sup_K N(K)).$$

The proof is thus concluded with the following lemma and with the choice of B_{exc} in (36) below. $\square$

Lemma 7.1. *There exists a universal constant $C \ge 1$ such that the following holds: Consider a solution $(u, \partial_t u) \in L^\infty(J, (\dot{H}^1 \cap \dot{H}^2 \times H^1)(\mathbb{R}^3))$ of (1) with $p = 5+\delta$, $\delta \in (0,1)$. Define $M := \|u\|_{L^\infty(\mathbb{R}^3 \times J)}$, $E := E(u)$ and $L := \|(u, \partial_t u)\|_{L^\infty(J, (\dot{H}^{s_p} \times \dot{H}^{s_p - 1})(\mathbb{R}^3))}$ on $J = [t_-, t_+]$ and set*

$$B_{\mathrm{exc}} := (CEM^{\frac{\delta}{2}}L)^{C(EM^{\delta/2}L)^{176}}. \tag{36}$$

Assume that $B_{\mathrm{exc}}^{\delta/2} \le 2$ *and that*

$$\min\{EM^{\frac{\delta}{2}}, L\} \ge c_0. \tag{37}$$

Then for any chain of unexceptional intervals, that is, for any $K = J_{i_0} \cup \cdots \cup J_{i_1} \subseteq J$ *with*

$$\begin{aligned} \|u\|_{L^{2(p-1)}(\mathbb{R}^3\times J_i)} &= 1, \\ \|u_{l,t_+}\|_{L^{2(p-1)}(\mathbb{R}^3\times J_i)} + \|u_{l,t_-}\|_{L^{2(p-1)}(\mathbb{R}^3\times J_i)} &\le B_{\mathrm{exc}}^{-1}, \end{aligned} \tag{38}$$

for all $i \in \{i_0, \dots, i_1\}$, *we have the estimate*

$$N(K) \lesssim B_{\mathrm{exc}}.$$

Proof of Lemma 7.1. Step 0: *Let* α_0, α_0', α_1 *and* α_1' *be defined through Lemma 6.2 for* $s = s_p$, *that is, for* $\gamma := 2(3/(2s_p))^2 \in \left[\frac{7}{2}, \frac{9}{2}\right]$,

$$\alpha_0 = \gamma - 2 \in \left[\tfrac{3}{2}, \tfrac{5}{2}\right], \quad \alpha_1 = \frac{6\gamma}{5(p-1)} \in \left[\tfrac{3}{4}, \tfrac{3}{2}\right], \quad \alpha_0' = 5 + \tfrac{3}{2}\delta \in \left[5, \tfrac{13}{2}\right] \quad \textit{and} \quad \alpha_1' = \tfrac{1}{2}. \tag{39}$$

We prove that there exists $(t_0, x_0, r_0) \in K \times \mathbb{R}^3 \times (0, +\infty)$ *such that*

(i) *mass concentrates in* $B(x_0, r_0)$ *at time* t_0, *i.e.*,

$$\frac{1}{r_0^{2s_p}} \int_{B(x_0,r_0)} u^2(y, t_0)\,\mathrm{d}y \ge C_6 L^{-\alpha_0} (EM^{\frac{\delta}{2}})^{-\alpha_1}, \tag{40}$$

(ii) *the length of the* J_i *is uniformly bounded from below in terms of* r_0, *i.e.*, *for all* $i = i_0, \dots, i_1$

$$|J_i| \ge C_7 L^{-\alpha_0'} (EM^{\frac{\delta}{2}})^{-\alpha_1'} M^{\frac{\delta}{4}} r_0^{s_p}. \tag{41}$$

From (i), *we immediately also deduce the lower bound on the mass concentration radius*

$$r_0 \gtrsim (L^{-\alpha_0} (EM^{\frac{\delta}{2}})^{-\alpha_1})^{\frac{p-1}{4}} M^{-\frac{p-1}{2}}. \tag{42}$$

By (38), we can apply the mass concentration Lemma 6.2 with $\eta = 1$ and $s = s_p$ to find that for any $i \in \{i_0, \dots, i_1\}$ there exists $(t_i, x_i, r_i) \in J_i \times \mathbb{R}^3 \times (0, +\infty)$ such that

$$\begin{aligned} \frac{1}{r_i^{2s_p}} \int_{B(x_i,r_i)} u^2(y, t_i)\,\mathrm{d}y &\ge C_6 L^{-\alpha_0} (EM^{\frac{\delta}{2}})^{-\alpha_1}, \\ |J_i| &\ge C_7 L^{-\alpha_0'} (EM^{\frac{\delta}{2}})^{-\alpha_1'} M^{\frac{\delta}{4}} r_i^{s_p}. \end{aligned}$$

Defining the minimal mass concentration radius $r_0 := \min_{i\in\{i_0,\dots,i_1\}} r_i$ and calling the associated point in spacetime (x_0, t_0), we reach (i) and (ii). The lower bound on the mass concentration radius (42) is a consequence of the simple observation that the left-hand side of (40) can be bounded from above, up to constants, by $r_0^{3-2s_p} M^2 = r_0^{4/(p-1)} M^2$. By time and space translation symmetry, we can assume without loss of generality that $x_0 = 0$ and that $t_0 = r_0$ such that $B(x_0, r_0) \times \{t_0\}$ lies in the forward wave cone centered in $(0, 0)$. In view of (ii) it is enough to prove

$$|K| \lesssim L^{-\alpha_0'} (EM^{\frac{\delta}{2}})^{-\alpha_1'} M^{\frac{\delta}{4}} B_{\mathrm{exc}} r_0^{s_p}.$$

Moreover, by time reversal symmetry, it is enough to estimate $K_+ := K \cap [t_0, +\infty)$, i.e., to show

$$|K_+| \lesssim L^{-\alpha_0'}(EM^{\frac{\delta}{2}})^{-\alpha_1'} M^{\frac{\delta}{4}} B_{\text{exc}} r_0^{s_p}. \tag{43}$$

Step 1: *We find a cylinder $B(x_0, r_0) \times \tilde{J}_0 \subseteq \Gamma_+(K_+)$ in spacetime such that*:

(i) *Mass still concentrates in $B(x_0, r_0)$ for any $t \in \tilde{J}_0$, i.e., for $t \in \tilde{J}_0$ it holds*

$$\frac{1}{r_0^{2s_p}} \int_{B(x_0,r_0)} u^2(y,t)\,\mathrm{d}y \geq \frac{C_6}{2} L^{-\alpha_0}(M^{\frac{\delta}{2}} E)^{-\alpha_1}. \tag{44}$$

(ii) *$\tilde{J}_0$ has controlled length, i.e.*,

$$L^{-\frac{\alpha_0}{2}}(M^{\frac{\delta}{2}} E)^{-\frac{\alpha_1+1}{2}} M^{\frac{\delta}{4}} r_0^{s_p} \lesssim |\tilde{J}_0| \leq M^{\frac{\delta}{4}} r_0^{s_p}.$$

(iii) *$\tilde{J}_0$ does not carry too much of the spacetime norm. More precisely*,

$$\|u\|_{L^{2(p-1)}(\mathbb{R}^3 \times \tilde{J}_0)}^{2(p-1)} \lesssim L^{\alpha_0' - \frac{\alpha_0}{2}}. \tag{45}$$

The local mass is Lipschitz in time with Lipschitz constant at most $\|\partial_t u\|_{L^\infty(J, L^2(\mathbb{R}^3))} \lesssim E^{1/2}$. More precisely, we have

$$\left| \left(\int_{B(x_0,r_0)} u^2(y,t)\,\mathrm{d}y \right)^{\frac{1}{2}} - \left(\int_{B(x_0,r_0)} u^2(y,t_0)\,\mathrm{d}y \right)^{\frac{1}{2}} \right| \lesssim E^{\frac{1}{2}} |t - t_0|.$$

In particular, if

$$E^{\frac{1}{2}} |t - t_0| \leq c_1 L^{-\frac{\alpha_0}{2}} (M^{\frac{\delta}{2}} E)^{-\frac{\alpha_1}{2}} r_0^{s_p}$$

for a universal $0 < c_1 \ll 1$ yet to be chosen sufficiently small, then we still have the mass concentration on the bubble $B(x_0, r_0) \times \tilde{J}_0$, where

$$\tilde{J}_0 := [t_0, t_0 + c_1 L^{-\frac{\alpha_0}{2}} (M^{\frac{\delta}{2}} E)^{-\frac{\alpha_1+1}{2}} M^{\frac{\delta}{4}} r_0^{s_p}].$$

More precisely, for any $t \in \tilde{J}_0$, (44) holds. We observe that

$$|\tilde{J}_0| = c_1 M^{\frac{\delta}{4}} L^{-\frac{\alpha_0}{2}} (EM^{\frac{\delta}{2}})^{-\frac{1}{2}(\alpha_1+1)} r_0^{s_p} \leq c_1 c_0^{-\frac{1}{2}(\alpha_0+\alpha_1+1)} M^{\frac{\delta}{4}} r_0^{s_p} \tag{46}$$

such that we can choose $c_1 < c_0^{5/2}$ to ensure (ii). Finally, if $K_+ \subset \tilde{J}_0$ is a strict subset, then $|K_+| \leq |\tilde{J}_0|$ and (43) holds (for big enough constants in the definition of B_{exc}). Thus we can assume that $\tilde{J}_0 \subseteq K_+$ and hence $B(x_0, r_0) \times \tilde{J}_0 \subseteq \Gamma_+(K_+)$. Finally, let us argue that $\tilde{J}_0$ cannot be covered by too many unexceptional intervals and thus cannot carry too much spacetime norm. Indeed, from (41), (46) and (37) we deduce that $\tilde{J}_0$ can be covered by at most

$$\frac{c_1 L^{-\frac{\alpha_0}{2}} (EM^{\frac{\delta}{2}})^{-\frac{1}{2}(\alpha_1+1)} M^{\frac{\delta}{4}} r_0^{s_p}}{C_7 L^{-\alpha_0'} (EM^{\frac{\delta}{2}})^{-\alpha_1'} M^{\frac{\delta}{4}} r_0^{s_p}} \lesssim L^{\alpha_0' - \frac{\alpha_0}{2}}$$

many intervals of the family $\{J_i\}_{i=i_0}^{i_1}$. Hence by (38) we deduce (45).

Step 2: *Let*

$$\tilde{\eta} := c_2(LEM^{\frac{\delta}{2}})^{-\frac{3}{2}} \in (0, c_0'), \tag{47}$$

with c_0' defined through Remark 4.2 (so that $\tilde{\eta}$ is admissible for the spacetime norm decay on large intervals). For a suitable choice of the universal constant c_2, we truncate $\Gamma_+(K_+)$ into wave cones $\{\Gamma_+(\tilde{J}_i)\}_{i=1}^k$ such that:

(i) *Each of them carries substantial spacetime norm $\tilde{\eta}$, i.e., $\|u\|_{L^{2(p-1)}(\Gamma_+(\tilde{J}_i))} = \tilde{\eta}$ for $i = 1, \dots, k-1$ and $\|u\|_{L^{2(p-1)}(\Gamma_+(\tilde{J}_k))} \le \tilde{\eta}$.*

(ii) *The first interval is not too long, that is, $\tilde{J}_1 \subseteq \tilde{J}_0$.*

For an $\tilde{\eta}$ yet to be chosen, we will truncate $\Gamma_+(K_+)$ into wave cones $\{\Gamma_+(\tilde{J}_i)\}_{i=1}^k$ such that

$$\|u\|_{L^{2(p-1}(\Gamma_+(\tilde{J}_i))} = \tilde{\eta} \quad \text{for } i = 1, \dots, k-1 \qquad \text{and} \qquad \|u\|_{L^{2(p-1)}(\Gamma_+(\tilde{J}_k))} \le \tilde{\eta}.$$

We come to the choice of $\tilde{\eta}$. Let us estimate the spacetime norm on the mass concentration cylinder from above

$$\int_{\tilde{J}_0}\int_{B(x_0,r_0)} u^2(y,t)\,\mathrm{d}y\,\mathrm{d}t \lesssim \left(\int_{\Gamma_+(\tilde{J}_0)} |u|^{2(p-1)}(y,t)\,\mathrm{d}y\,\mathrm{d}t\right)^{\frac{1}{p-1}} |\tilde{J}_0|^{\frac{p-2}{p-1}} r_0^{\frac{3(p-2)}{p-1}}$$

and from below, using (44),

$$\int_{\tilde{J}_0}\int_{B(x_0,r_0)} u^2(y,t)\,\mathrm{d}y\,\mathrm{d}t \gtrsim |\tilde{J}_0| L^{-\alpha_0}(M^{\frac{\delta}{2}}E)^{-\alpha_1} r_0^{2s_p}.$$

We have obtained, using the definition of $\tilde{J}_0$ from Step 1, that

$$\|u\|_{L^{2(p-1)}(\Gamma_+(\tilde{J}_0))} \gtrsim (L^{-\alpha_0}(EM^{\frac{\delta}{2}})^{-\alpha_1})^{\frac{2p-1}{4(p-1)}} (E^{-1} r_0^{\frac{\delta}{p-1}})^{\frac{1}{4(p-1)}}.$$

Using (45), we obtain an upper bound on r_0, that is,

$$\begin{aligned} r_0^{\delta} &\lesssim (L^{\alpha_0}(EM^{\frac{\delta}{2}})^{\alpha_1})^{(2p-1)(p-1)} E^{p-1} \|u\|^{4(p-1)^2}_{L^{2(p-1)}(\Gamma_+(\tilde{J}_0))} \\ &\lesssim (L^{\alpha_0}(EM^{\frac{\delta}{2}})^{\alpha_1})^{(2p-1)(p-1)} E^{p-1} L^{(\alpha_0' - \frac{\alpha_0}{2})2(p-1)} \\ &= M^{-\frac{\delta(p-1)}{2}} L^{2(p-1)(\alpha_0(p-1)+\alpha_0')} (EM^{\frac{\delta}{2}})^{(p-1)(\alpha_1(2p-1)+1)}. \end{aligned} \tag{48}$$

On the other hand, using the lower bound on r_0 given by (42), we can estimate furthermore, recalling (37) and (39), that

$$\begin{aligned} \|u\|_{L^{2(p-1)}(\Gamma_+(\tilde{J}_0))} &\gtrsim (L^{-\alpha_0}(EM^{\frac{\delta}{2}})^{-\alpha_1})^{\frac{2p-1}{4(p-1)} + \frac{\delta}{16(p-1)}} (EM^{\frac{\delta}{2}})^{-\frac{1}{4(p-1)}} \\ &= L^{-\frac{9}{16}\alpha_0} (EM^{\frac{\delta}{2}})^{-(\frac{9}{16}\alpha_1 + \frac{1}{4(p-1)})} \gtrsim (LEM^{\frac{\delta}{2}})^{-\frac{3}{2}}. \end{aligned}$$

Thus choosing $\tilde{\eta} := c_2(LEM^{\delta/2})^{-3/2}$, for a small universal constant $0 < c_2 < 1$, we ensure that $\tilde{J}_1 \subseteq \tilde{J}_0$. Choosing c_2 even smaller, namely $c_2 \le c_0' c_0^3$, we ensure that $\tilde{\eta} \in (0, c_0')$, with c_0' given by Remark 4.2.

Step 3: *We prove the following dichotomy* (*analogous to* [Tao 2006b, Lemma 5.2]). *Let* $j \in \{1, \dots, k-1\}$. *Then*, *for some universal constants* $C_8 > 8$ *and* $C_9 < 1$, *either*

$$|\tilde{J}_{j+1}| \le C_8 \tilde{\eta}^{-15} |\tilde{J}_j| \quad or \quad |\tilde{J}_j| \ge C_9 \tilde{\eta}^5 M^{\frac{\delta}{4}} B_{\mathrm{exc}} r_0^{s_p}.$$

Consider two subsequent intervals $\tilde{J}_j = [t_{j-1}, t_j]$ and $\tilde{J}_{j+1} = [t_j, t_{j+1}]$ for some $j \in \{1, \dots, k-1\}$. We have by the localized Strichartz estimates (8) (with $(\tilde{q}, \tilde{r}) = (2, 6(p-1)/(3p+1))$ and $v := u - u_{l,t_{j+1}}$ solving $\Box v = |u|^{p-1} u$ with initial datum $(v, \partial_t v)(t_{j+1}) = (0, 0)$) and Hölder that

$$\begin{aligned} \|u - u_{l,t_{j+1}}\|_{L^{2(p-1)}(\Gamma_+(\tilde{J}_j))} &\lesssim \| |u|^{p-1} u \|_{L^{\tilde{q}} L^{\tilde{r}}(\Gamma_+(\tilde{J}_j \cup \tilde{J}_{j+1}))} \\ &\lesssim \|u\|_{L^\infty L^{3(p-1)/2}(\Gamma_+(\tilde{J}_j \cup \tilde{J}_{j+1}))} \|u\|^{p-1}_{L^{2(p-1)}(\Gamma_+(\tilde{J}_j \cup \tilde{J}_{j+1}))} \\ &\lesssim \|u\|^{\frac{\delta}{3(p-1)}}_{L^\infty(\mathbb{R}^3 \times J)} \|u\|^{\frac{2(p+1)}{3(p-1)}}_{L^\infty L^{p+1}(\mathbb{R}^3 \times J)} \tilde{\eta}^{p-1} \lesssim (E M^{\frac{\delta}{2}})^{\frac{2}{3(p-1)}} \tilde{\eta}^{p-1}. \end{aligned}$$

Using (37) and (47), we have

$$\tilde{\eta}^{p-2} (E M^{\frac{\delta}{2}})^{\frac{2}{3(p-1)}} \le c_2^{\frac{4}{9(p-1)}} L^{-\frac{4}{9(p-1)}} \tilde{\eta}^{p-2-\frac{4}{9(p-1)}} \le (c_2 c_0^{-1})^{\frac{4}{9(p-1)}} \le (c_0')^{\frac{4}{9(p-1)}} c_0^{\frac{8}{9(p-1)}} \le c_0^{\frac{8}{9(p-1)}},$$

where we recall that from the choice of c_0 in Lemma 3.1, it is clear that it beats also the constant arising from Strichartz estimates. We infer $\|u - u_{l,t_{j+1}}\|_{L^{2(p-1)}(\Gamma_+(\tilde{J}_j))} \le \tilde{\eta}$. Since $\|u\|_{L^{2(p-1)}(\Gamma_+(\tilde{J}_j))} = \tilde{\eta}$ by construction, the triangular inequality implies

$$\|u_{l,t_{j+1}}\|_{L^{2(p-1)}(\Gamma_+(\tilde{J}_j))} \gtrsim \tilde{\eta}.$$

This now gives rise to a dichotomy: either $\|u_{l,t_{j+1}} - u_{l,t_+}\|_{L^{2(p-1)}(\Gamma_+(\tilde{J}_j))} \gtrsim \tilde{\eta}$ or the scattering solution u_{l,t_+} is nonnegligible $\|u_{l,t_+}\|_{L^{2(p-1)}(\Gamma_+(\tilde{J}_j))} \gtrsim \tilde{\eta}$.

Case 1: Assume $\|u_{l,t_{j+1}} - u_{l,t_+}\|_{L^{2(p-1)}(\Gamma_+(\tilde{J}_j))} \gtrsim \tilde{\eta}$. Then in view of Corollary 5.2, we have

$$|\tilde{J}_{j+1}| \lesssim \tilde{\eta}^{-2(p-1)} (E M^{\frac{\delta}{2}})^{\frac{p}{3}} L^{\frac{3(p-1)}{2}} |\tilde{J}_j| \lesssim \tilde{\eta}^{-2(p-1)} (E M^{\frac{\delta}{2}} L)^{\frac{15}{2}} |\tilde{J}_j| \lesssim \tilde{\eta}^{-15} |\tilde{J}_j|,$$

where in the second inequality we used (37) and in the last the definition (47).

Case 2: Assume $\|u_{l,t_+}\|_{L^{2(p-1)}(\Gamma_+(\tilde{J}_j))} \gtrsim \tilde{\eta}$. Recall that K_+ consists of unexceptional intervals. Hence we need at least $\tilde{\eta} B_{\mathrm{exc}}$ many of them to cover $\tilde{J}_j$. Recalling the lower bound on the length of unexceptional intervals, the definition of $\tilde{\eta}$, (37) and that $\alpha_0' > \alpha_1'$ from (39), we have

$$\begin{aligned} |\tilde{J}_j| &\ge C_7 \tilde{\eta} L^{-\alpha_0'} (E M^{\frac{\delta}{2}})^{-\alpha_1'} M^{\frac{\delta}{4}} B_{\mathrm{exc}} r_0^{s_p} = C_7 \tilde{\eta} (E M^{\frac{\delta}{2}} L)^{-\alpha_0'} (E M^{\frac{\delta}{2}})^{\alpha_0' - \alpha_1'} M^{\frac{\delta}{4}} B_{\mathrm{exc}} r_0^{s_p} \\ &\ge C_7 \tilde{\eta}^{1 + \frac{2}{3}\alpha_0'} c_2^{-\frac{2\alpha_0'}{3}} c_0^{\alpha_0' - \alpha_1'} M^{\frac{\delta}{4}} B_{\mathrm{exc}} r_0^{s_p} \ge C_9 \tilde{\eta}^{\frac{11}{2}} M^{\frac{\delta}{4}} B_{\mathrm{exc}} r_0^{s_p}, \end{aligned}$$

where in the last inequality we introduced a universal constant $C_9 \le C_7 c_2^{-2\alpha_0'/3} c_0^{\alpha_0' - \alpha_1'}$.

Step 4: *We show that*

$$|K_+| \le C_9 \tilde{\eta}^{\frac{11}{2}} M^{\frac{\delta}{4}} B_{\mathrm{exc}} r_0^{s_p}.$$

Since $0 < \tilde{\eta} \le 1$, *this implies in particular that* $|K_+| \le C_9 M^{\delta/4} B_{\mathrm{exc}} r_0^{s_p}$ *and we achieved* (43), *thereby concluding the proof.*

Let us therefore assume by contradiction that $|K_+| > C_9\tilde{\eta}^{11/2}M^{\delta/4}B_{\rm exc}r_0^{s_p}$. We call $\tilde{J}_{j_1}$ the first interval for which $|\tilde{J}_1\cup\cdots\cup\tilde{J}_{j_1}| > C_9\tilde{\eta}^{11/2}M^{\delta/4}B_{\rm exc}r_0^{s_p}$. We observe that up to choosing the constant C in the definition of $B_{\rm exc}$ big enough, we may assume that

$$\tilde{\eta}^{\frac{11}{2}}B_{\rm exc} > \max\left\{\frac{2}{C_9},1\right\}. \tag{49}$$

By the definition of j_1, we then have:

(i) $j_1\neq 1$. Indeed, by Steps 1 and 2, $|\tilde{J}_1|\le|\tilde{J}_0|\le M^{\delta/4}r_0^{s_p}$.

(ii) For every $j\in\{1,\dots,j_1-1\}$ we have $|\tilde{J}_{j+1}|\le C_8\tilde{\eta}^{-15}|\tilde{J}_j|$. This follows from Step 3 since the second option in the dichotomy is ruled out.

Let us call $[T_1,T_2]:=\tilde{J}_2\cup\dots\tilde{J}_{j_1-1}$. We want to apply the spacetime norm decay result of Proposition 4.1 on $I=[T_1,T_2]$ with $\eta=\frac{\tilde{\eta}}{4}$. Recall that by choice of $\tilde{\eta}$ in Step 2, we have that $\frac{\tilde{\eta}}{4}\in(0,c_0')$ is admissible for the spacetime norm decay. We need thus a lower bound on the length of I. By construction, Step 2 and (ii)

$$C_9\tilde{\eta}^{\frac{11}{2}}M^{\frac{\delta}{4}}B_{\rm exc}r_0^{s_p}\le|\tilde{J}_1|+\cdots+|\tilde{J}_{j_1}|\le M^{\frac{\delta}{4}}r_0^{s_p}+(T_2-T_1)+C_8\tilde{\eta}^{-15}(T_2-T_1),$$

so that

$$T_2-T_1\ge\frac{1}{2C_8}\tilde{\eta}^{\frac{41}{2}}M^{\frac{\delta}{4}}B_{\rm exc}r_0^{s_p}. \tag{50}$$

On the other hand, we have from Step 2 and the lower bound on r_0 (42)

$$\begin{aligned}T_1&\le r_0+M^{\frac{\delta}{4}}r_0^{s_p}=M^{\frac{\delta}{4}}r_0^{s_p}(1+r_0^{1-s_p}M^{\frac{-\delta}{4}})\lesssim M^{\frac{\delta}{4}}r_0^{s_p}(1+(L^{\alpha_0}(EM^{\frac{\delta}{2}})^{\alpha_1})^{\frac{2\delta}{(p-1)^2}})\\&\lesssim M^{\frac{\delta}{4}}r_0^{s_p}\tilde{\eta}^{-\frac{2(\alpha_0+\alpha_1)\delta}{\gamma(p-1)^2}}\lesssim\tilde{\eta}^{-\frac{1}{4}}M^{\frac{\delta}{4}}r_0^{s_p}.\end{aligned}$$

Summarizing, we have obtained

$$\frac{T_2}{T_1}\ge\frac{T_2-T_1}{T_1}\ge C_{10}\tilde{\eta}^{21}B_{\rm exc}. \tag{51}$$

We now claim that to reach a contradiction, it is enough to find A and a constant $C\ge 1$ such that we can verify the following three requirements:

(R1) A satisfies the hypothesis (13) of Proposition 4.1, that is,

$$A>(4C_2\tilde{\eta}^{-1})^{\frac{12(p-1)}{5}}(EM^{\frac{\delta}{2}})^{\frac{14}{5}}.$$

(R2) The interval I is sufficiently large to apply Proposition 4.1, i.e., (14) is satisfied. In view of (51), we can enforce (15) if

$$\begin{aligned}B_{\rm exc}&=(CEM^{\frac{\delta}{2}}L)^{C(EM^{\delta/2}L)^{176}}\\&\ge C_{10}^{-1}\tilde{\eta}^{-21}A^{3(4C_2\tilde{\eta}^{-1})^{6(p-1)(p+1)/5}(EM^{\delta/2})^{(9p+19)/10}\max\{c_0^{(p-1)/2},(M^{(p-1)/2}T_2)^{\delta/2}\}}.\end{aligned}$$

(R3) Moreover $\sqrt{A}>2C_8\tilde{\eta}^{-15}$.

Observe that (R3) ensures in particular that $A > 4$. If (R1)–(R3) hold, we are in the position to conclude the proof following [Roy 2009]. The difficulty in the supercritical case instead lies in verifying the requirements (R1)–(R3). Indeed, if (R1)–(R3) hold, we infer from Proposition 4.1 that there exists $[t_1', At_1'] \subseteq \tilde{J}_2 \cup \dots \tilde{J}_{j_1-1}$ such that

$$\|u\|_{L^{2(p-1)}(\Gamma_+([t_1',At_1']))} \leq \frac{\tilde{\eta}}{4}.$$

In particular, $[t_1', At_1']$ is covered by at most two consecutive intervals of the family $\{J_j\}_{j=2}^{j_1-1}$. We claim that then there exists $j \in \{2, \dots, j_1-1\}$ such that

$$|\tilde{J}_j| \geq \frac{\sqrt{A}}{2} |\tilde{J}_{j-1}|. \tag{52}$$

Notice that in view of (R3), the claim contradicts (ii) such that we reached a contradiction. Indeed, assume first, that $[t_1', At_1']$ is covered by one interval $\tilde{J}_j$ for some $j \in \{2, \dots, j_1-1\}$. Then, recalling that $A > 4$, we have

$$|\tilde{J}_j| \geq t_1'(A-1) \geq \frac{A}{2} t_1' \geq \frac{A}{2} |\tilde{J}_{j-1}| \geq \frac{\sqrt{A}}{2} |\tilde{J}_{j-1}|.$$

Assume now that $[t_1', At_1']$ is covered by two intervals $\tilde{J}_j = [a_j, b_j]$ and $\tilde{J}_{j+1} = [a_{j+1}, b_{j+1}]$ for some $j \in \{2, \dots, j_1-2\}$. We consider two cases. First, if $b_j \leq \sqrt{A} t_1'$, then $|\tilde{J}_{j+1}| \geq t_1'(A - \sqrt{A})$ and $|\tilde{J}_j| \leq \sqrt{A} t_1'$ such that

$$|\tilde{J}_{j+1}| \geq (\sqrt{A}-1)|\tilde{J}_j| \geq \frac{\sqrt{A}}{2} |\tilde{J}_j|.$$

Second, if $b_j > \sqrt{A} t_1'$, then $|\tilde{J}_j| \geq (\sqrt{A}-1) t_1'$ and $|\tilde{J}_{j-1}| \leq t_1'$ such that

$$|\tilde{J}_j| \geq (\sqrt{A}-1)|\tilde{J}_{j-1}| \geq \frac{\sqrt{A}}{2} |\tilde{J}_{j-1}|.$$

This proves (52).

To conclude the proof, we are left to verify the requirements (R1)-(R3) by choosing A and C. We observe that the right-hand side of (R1) can be bounded from above using (47) and (37) by

$$(4C_2 \tilde{\eta}^{-1})^{\frac{12(p-1)}{5}} (EM^{\frac{\delta}{2}})^{\frac{14}{5}} \leq C_{11} \tilde{\eta}^{-14}$$

such that (R1) and (R3) are enforced if we set

$$A := C_{12} \tilde{\eta}^{-30}$$

for $C_{12} := \max\{3C_8, C_{11}\}^2$. We are left to verify (R2). We observe that from (49)

$$T_2 = T_1 + (T_2 - T_1) \lesssim \tilde{\eta}^{-1} M^{\frac{\delta}{4}} r_0^{s_p} + \tilde{\eta}^{\frac{11}{2}} M^{\frac{\delta}{4}} B_{\mathrm{exc}} r_0^{s_p} \lesssim M^{\frac{\delta}{4}} B_{\mathrm{exc}} r_0^{s_p}.$$

Combining this with the upper bound on r_0 in (48) and using (39), we obtain

$$\begin{aligned}(M^{\frac{p-1}{2}} T_2)^{\frac{\delta}{2}} &\lesssim (M^{\frac{8+3\delta}{4}} B_{\mathrm{exc}} r_0^{s_p})^{\frac{\delta}{2}} \lesssim B_{\mathrm{exc}}^{\frac{\delta}{2}} L^{s_p(p-1)(\alpha_0(p-1)+\alpha_0')} (EM^{\frac{\delta}{2}})^{\frac{s_p}{2}(p-1)(\alpha_1(2p-1)+1)} \\ &\lesssim B_{\mathrm{exc}}^{\frac{\delta}{2}} (EM^{\frac{\delta}{2}} L)^{105} \leq C_{13} B_{\mathrm{exc}}^{\frac{\delta}{2}} \tilde{\eta}^{-70}.\end{aligned}$$

We now bound the right-hand side of (R2) from above using again (47) and (37) by

$$\begin{aligned} C_{10}^{-1}\tilde{\eta}^{-21}(C_{12}\tilde{\eta}^{-30})^{3(4C_2\tilde{\eta}^{-1})^{42}(EM^{\delta/2})^{(9p+19)/10}\max\{c_0^{(p-1)/2},(M^{\delta(p-1)/2}T_2)^{\delta/2}\}} \\ \le C_{10}^{-1}\tilde{\eta}^{-21}(C_{12}\tilde{\eta}^{-30})^{3C_{13}(4C_2\tilde{\eta}^{-1})^{42}(c_2c_0^{-1}\tilde{\eta}^{-1})^{(9p+19)/15}\tilde{\eta}^{-70}B_{\mathrm{exc}}^{\delta/2}} \\ \le (C'EM^{\frac{\delta}{2}}L)^{C'\tilde{\eta}^{-117}B_{\mathrm{exc}}^{\delta/2}} \le (CEM^{\frac{\delta}{2}}L)^{(C/2)(EM^{\delta/2}L)^{176}B_{\mathrm{exc}}^{\delta/2}} \end{aligned}$$

for a big enough constant $C, C' \ge 1$. We now define B_{exc} to be

$$B_{\mathrm{exc}} := (CEM^{\frac{\delta}{2}}L)^{C(EM^{\delta/2}L)^{176}}$$

for the same constant C. With this definition, (R2) is enforced since we assumed $B_{\mathrm{exc}}^{\delta/2} \le 2$. $\square$

Proof of Corollary 1.3. Consider a solution $(u, \partial_t u) \in L^\infty(J, (\dot{H}^1 \cap \dot{H}^2 \times H^1)(\mathbb{R}^3))$ of (1) with $p = 5+\delta$ for $\delta \in [0, 1)$ and with

$$\|(u, \partial_t u)\|_{L^\infty(J,(\dot{H}^1\cap\dot{H}^2\times H^1)(\mathbb{R}^3))} \le M_0.$$

By interpolation, conservation of the energy and the Sobolev embeddings $(\dot{H}^1 \cap \dot{H}^2)(\mathbb{R}^3) \hookrightarrow W^{1,6}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$, we observe

$$\begin{aligned} L &:= \|(u, \partial_t u)\|_{L^\infty(J,\dot{H}^{s_p}\times\dot{H}^{s_p-1})} \le E^{1-\frac{\delta}{2(p-1)}} M_0^{\frac{\delta}{2(p-1)}}, \\ M &:= \|u\|_{L^\infty(\mathbb{R}^3\times J)} \le C_S M_0. \end{aligned}$$

By Theorem 1.2, if $\min\{EM^{\delta/2}, L\} < c_0$, then $\|u\|_{L^{2(p-1)}(\mathbb{R}^3\times J)} \le 1$. Otherwise, we may assume $\min\{EM^{\delta/2}, L\} \ge c_0$ and we fix $0 \le \delta \le \min\{1, (\ln 2)/(\ln M_0)\}$. We estimate as above

$$EM^{\frac{\delta}{2}}L \le C_S^{\frac{\delta}{2}(1+\frac{\delta}{2(p-1)})} c_0^{-\frac{\delta}{2(p-1)}} E^2 M_0^{\delta(1-\frac{p+1}{4(p-1)})} \le 2C_S c_0^{-1} E^2 =: (C'E)^2$$

for $C' := (2C_S c_0^{-1})^1/2$. Thus the corollary follows, if we can meet the smallness requirement of Theorem 1.2 which now reads, setting $\overline{C} := \sqrt{C}C'$,

$$((\overline{C}E)^{2C(C'E)^{352}})^\delta \le 2.$$

The latter holds defining

$$\delta_0 := \min\left\{1, \frac{\ln 2}{\ln M_0}, \frac{\ln 2}{\ln(\overline{C}E)2C(\overline{C}E)^{352}}\right\}.$$

Observe that δ_0 depends on M_0 only, since $E = E(u_0, u_1)$ depends on the initial data only. $\square$

8. Proof of Theorem 1.1

By time reversibility, it is enough to consider forward-in-time solutions. Thanks to classical local well-posedness and existence theory [Sogge 1995], the proof of Theorem 1.1 consists in establishing an a priori bound on $\|(u, \partial_t u)\|_{L^\infty([0,T],\dot{H}^1\cap\dot{H}^2\times H^1)}$ which is uniform in T.

Lemma 8.1 (local boundedness). *Let $\delta \in (0,1)$, $p = 5+\delta$ and consider a solution*

$$(u, \partial_t u) \in L^\infty(I, \dot H^1 \cap \dot H^2 \times H^1)$$

to (1) *on $I = [t_0, t_1]$. Then there exists a universal constant $C_l \geq 1$ such that if*

$$\|u\|^{p-1}_{L^{2(p-1)}(\mathbb{R}^3 \times I)} < C_l^{-1}, \tag{53}$$

then

$$\|(u, \partial_t u)\|_{L^\infty(I, \dot H^1 \cap \dot H^2 \times H^1)} \leq C_l \|(u, \partial_t u)(t_0)\|_{H^1 \cap \dot H^2 \times H^1}.$$

Proof. For $t \in I$, define $Z(t) := \|(u, \partial_t u)(t)\|_{H^1 \cap \dot H^2 \times H^1}$. By Strichartz estimate (7), Hölder and the Sobolev embedding of $\dot H^1 \hookrightarrow L^6$ we have

$$\begin{aligned} Z(t) &\lesssim Z(t_0) + \||u|^{p-1} u\|_{L^2([t_0,t], L^{3/2})} + \|\nabla(|u|^{p-1} u)\|_{L^2([t_0,t], L^{3/2})} \\ &\lesssim Z(t_0) + \||u|^{p-1}\|_{L^2(\mathbb{R}^3 \times [t_0,t])} (\|u\|_{L^\infty([t_0,t], L^6)} + \|\nabla u\|_{L^\infty([t_0,t], L^6)}) \\ &\lesssim Z(t_0) + \|u\|^{p-1}_{L^{2(p-1)}(\mathbb{R}^3 \times [t_0,t])} \sup_{t' \in [t_0,t]} Z(t'). \end{aligned}$$

We set $Y(t) := \sup_{t' \in [t_0,t]} Z(t')$. Observe that Y is nondecreasing, continuous, $Y(t_0) = Z(t_0)$ and

$$Y(t) \leq C(Z(t_0) + \|u\|^{p-1}_{L^{2(p-1)}(\mathbb{R}^3 \times I)} Y(t)) \tag{54}$$

for any $t \in I$. Setting $C_l := 2C$, we have by monotonicity that $Y(t) \leq C_l Z(t_0)$ for all $t \in [t_0, \bar t]$, where $\bar t := \sup\{t \in [t_0, t_1] : Y(t) \leq C_l Z(t_0)\}$. We claim that if $\|u\|^{p-1}_{L^{2(p-1)}(\mathbb{R}^3 \times I)} \leq C_l^{-1}$, then $\bar t = t_1$. Assume by contradiction that $\bar t < t_1$. By continuity $Y(\bar t) = C_l Z(t_0)$ and by the validity of (54) at $\bar t$, we obtain

$$C_l Z(t_0) = Y(\bar t) \leq C Z(t_0) + C \|u\|^{p-1}_{L^{2(p-1)}(\mathbb{R}^3 \times I)} Y(\bar t) < 2C Z(t_0) = C_l Z(t_0),$$

which is a contradiction. □

We achieve an a priori bound on $(u, \partial_t u)$ in $L^\infty([0,T], \dot H^1 \cap \dot H^2 \times H^1)$, uniform in T, by iterating Lemma 8.1 on a partition $\{I_n\}_{n=1}^N$ of $[0,T]$, where the smallness assumption (53)

$$\|u\|_{L^{2(p-1)}(\mathbb{R}^3 \times I_n)} < C_l^{-\frac{1}{p-1}}$$

is satisfied by construction. Corollary 1.3 is crucial to control N, independent of T, in terms of a double exponential in E and $\|(u, \partial_t u)\|^\delta_{L^\infty \dot H^1 \cap H^2 \times H^1}$. The crucial observation is that in the limit as $\delta \to 0$, N is a double exponential of the energy which in turn is controlled by the initial data only. This will allow us to iterate the local bound obtained in Lemma 8.1 on bounded sets of initial data for δ small enough.

Proof of Theorem 1.1. Fix $(u_0, u_1) \in \dot H^1 \cap \dot H^2 \times H^1$. Consider $(u, \partial_t u)$ a solution to (1) with $p = 5+\delta$ for $\delta \in (0,1)$. We introduce the set

$$\mathcal{F} := \{T \in [0, +\infty) : \|(u, \partial_t u)\|_{L^\infty([0,T], \dot H^1 \cap \dot H^2 \times H^1)} \leq M_0\}$$

for some $M_0 = M_0(\|(u_0, u_1)\|_{\dot H^1 \cap \dot H^2 \times H^1})$ yet to be chosen large enough. We claim that $\mathcal{F} = [0, +\infty)$. For $M_0 \geq \|(u_0, u_1)\|_{\dot H^1 \cap \dot H^2 \times H^1}$, it is clear that $0 \in \mathcal{F}$ and, by continuity, that $\mathcal{F}$ is a closed set. We show openness. Let $T \in \mathcal{F}$. By continuity, there exists $\epsilon > 0$ such that for all $T' \in [0, T+\epsilon)$ we have

$$\|(u, \partial_t u)\|_{L^\infty([0,T'],\dot H^1\cap \dot H^2\times H^1)} \le 2M_0.$$

Fix such a T' and let us show that $T' \in \mathcal{F}$. If $\delta \le \delta_0(2M_0)$, with δ_0 given through Corollary 1.3, then

$$\|u\|_{L^{2(p-1)}(\mathbb{R}^3\times[0,T'])} \le \max\{1, (CE(2M_0)^{\frac{\delta}{2}})^{C(E(2M_0)^{\delta/2})^{352}}\}. \tag{55}$$

We can split $[0, T']$ into subintervals $\{J_i\}_{i=1}^N$ such that

- $\|u\|_{L^{2(p-1)}(\mathbb{R}^3\times J_i)} = \frac{1}{2} C_l^{-1/(p-1)}$ for $i = 1, \dots, N-1$,
- $\|u\|_{L^{2(p-1)}(\mathbb{R}^3\times J_N)} \le \frac{1}{2} C_l^{-1/(p-1)}$,

and we deduce by iterating Lemma 8.1 that

$$\|(u, \partial_t u)\|_{L^\infty([0,T'],\dot H^1\cap \dot H^2\times H^1)} \le C_l^N \|(u_0, u_1)\|_{H^1\cap \dot H^2\times H^1}. \tag{56}$$

Moreover, from (55) we have the upper bound

$$N \le 2C_l^{\frac{1}{p-1}} \max\{1, (CE(2M_0)^{\frac{\delta}{2}})^{C(E(2M_0)^{\delta/2})^{352}}\}. \tag{57}$$

We want to show that with an appropriate choice of $M_0 = M_0(\|(u_0, u_1)\|_{H^1\cap \dot H^2\times H^1})$ and of $\delta = \delta(\|(u_0, u_1)\|_{H^1\cap \dot H^2\times H^1})$, we have

$$N \le (\ln C_l)^{-1} \ln(M_0/\|(u_0, u_1)\|_{H^1\cap \dot H^2\times H^1}), \tag{58}$$

which in view of (56) implies $\|(u, \partial_t u)\|_{L^\infty([0,T'],\dot H^1\cap \dot H^2\times H^1)} \le M_0$ concluding the proof. Observe that for M_0 fixed, we have that the right-hand side of (57) as $\delta \to 0$ converges, more precisely

$$\lim_{\delta\to 0} 2C_l^{\frac{1}{p-1}} \max\{1, (CE(2M_0)^{\frac{\delta}{2}})^{C(E(2M_0)^{\delta/2})^{352}}\} = 2C_l^{\frac{1}{4}} \max\{1, (CE)^{CE^{352}}\}. \tag{59}$$

We now choose M_0 such that the right-hand side of (58) exceeds (59) by a factor 2; that is, we choose $M_0(E, \|(u_0, u_1)\|_{H^1\cap \dot H^2\times H^1})$ such that

$$(\ln C_l)^{-1} \ln(M_0/\|(u_0, u_1)\|_{H^1\cap \dot H^2\times H^1}) \ge 4C_l^{\frac{1}{4}} \max\{1, (CE)^{CE^{352}}\}$$

or, equivalently,

$$M_0 \ge \|(u_0, u_1)\|_{H^1\cap \dot H^2\times H^1} e^{4C_l^{1/4} \ln C_l \max\{1,(CE)^{CE^{352}}\}}.$$

Finally, by (57) we can choose $\bar\delta_0 = \bar\delta_0(M_0) < \delta_0(2M_0)$ even smaller such that for all $\delta \in (0, \bar\delta_0)$ we have

$$N \le 4C_l^{\frac{1}{4}} \max\{1, (CE)^{CE^{352}}\}. \tag{60}$$

This finishes the proof that $F = [0, +\infty)$ and in particular the solution $(u, \partial_t u)$ cannot blow up. Recalling the choice of M_0, we then obtain (2). As a byproduct of the upper bound (60) on N, independent of the size of the interval, we also obtain

$$\|u\|_{L^{2(p-1)}(\mathbb{R}^3\times[0,+\infty)} \le \frac{1}{2} C_l^{-\frac{1}{p-1}} 4C_l^{\frac{1}{4}} \max\{1, (CE)^{CE^{352}}\} \le 2\max\{1, (CE)^{CE^{352}}\},$$

where we used that $C_l \ge 1$. $\square$

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SUBELLIPTIC WAVE EQUATIONS ARE NEVER OBSERVABLE

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It is well known that observability (and, by duality, controllability) of the elliptic wave equation, i.e., with a Riemannian Laplacian, in time T_0 is almost equivalent to the geometric control condition (GCC), which stipulates that any geodesic ray meets the control set within time T_0. We show that in the subelliptic setting, the GCC is never satisfied, and that subelliptic wave equations are never observable in finite time. More precisely, given any subelliptic Laplacian $\Delta = -\sum_{i=1}^{m} X_i^* X_i$ on a manifold M, and any measurable subset $\omega \subset M$ such that $M \backslash \omega$ contains in its interior a point q with $[X_i, X_j](q) \notin \mathrm{Span}(X_1, \dots, X_m)$ for some $1 \leqslant i, j \leqslant m$, we show that, for any $T_0 > 0$, the wave equation with subelliptic Laplacian Δ is not observable on ω in time T_0.

The proof is based on the construction of sequences of solutions of the wave equation concentrating on geodesics (for the associated sub-Riemannian distance) spending a long time in $M \backslash \omega$. As a counterpart, we prove a positive result of observability for the wave equation in the Heisenberg group, where the observation set is a well-chosen part of the phase space.

1. Introduction

1.1. *Setting.* Let $n \in \mathbb{N}^*$ and let M be a smooth connected compact manifold of dimension n with a nonempty boundary ∂M. Let μ be a smooth volume on M. We consider $m \geqslant 1$ smooth vector fields $X_1, \dots, X_m$ on M which are not necessarily independent, and we assume that the following Hörmander condition [1967] holds:

The vector fields $X_1, \dots, X_m$ and their iterated brackets $[X_i, X_j]$, $[X_i, [X_j, X_k]]$, etc.
span the tangent space $T_q M$ at every point $q \in M$.

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We consider the sub-Laplacian Δ defined by

$$\Delta = -\sum_{i=1}^{m} X_i^* X_i = \sum_{i=1}^{m} X_i^2 + \mathrm{div}_\mu(X_i) X_i,$$

where the star designates the transpose in $L^2(M, \mu)$ and the divergence with respect to μ is defined by $L_X \mu = (\mathrm{div}_\mu X)\mu$, where L_X stands for the Lie derivative. Then Δ is hypoelliptic; see [Hörmander 1967, Theorem 1.1].

We consider Δ with Dirichlet boundary conditions and the domain $D(\Delta)$ which is the completion in $L^2(M, \mu)$ of the set of all $u \in C_c^\infty(M)$ for the norm $\|(\mathrm{Id} - \Delta)u\|_{L^2}$. We also consider the operator $(-\Delta)^{1/2}$ with domain $D((-\Delta)^{1/2})$ which is the completion in $L^2(M, \mu)$ of the set of all $u \in C_c^\infty(M)$ for the norm $\|(\mathrm{Id} - \Delta)^{1/2} u\|_{L^2}$.

Consider the wave equation

$$\begin{cases} \partial_{tt}^2 u - \Delta u = 0 \quad \text{in } (0, T) \times M, \\ u = 0 \quad \text{on } (0, T) \times \partial M, \\ (u_{|t=0}, \partial_t u_{|t=0}) = (u_0, u_1), \end{cases} \tag{1}$$

where $T > 0$. It is well known (see for example [Garetto and Ruzhansky 2015, Theorem 2.1; Engel and Nagel 2000, Chapter II, Section 6]) that for any $(u_0, u_1) \in D((-\Delta)^{1/2}) \times L^2(M)$, there exists a unique solution

$$u \in C^0(0, T; D((-\Delta)^{\frac{1}{2}})) \cap C^1(0, T; L^2(M)) \tag{2}$$

to (1) (in a mild sense).

We set

$$\|v\|_{\mathcal{H}} = \left(\int_M \sum_{j=1}^{m} (X_j v(x))^2 \, d\mu(x) \right)^{\frac{1}{2}}. \tag{3}$$

Note that $\|v\|_{\mathcal{H}} = \|(-\Delta)^{1/2} v\|_{L^2(M,\mu)}$.

The natural energy of a solution is

$$E(u(t, \cdot)) = \tfrac{1}{2}(\|\partial_t u(t, \cdot)\|_{L^2(M,\mu)}^2 + \|u(t, \cdot)\|_{\mathcal{H}}^2).$$

If u is a solution of (1), then

$$\frac{d}{dt} E(u(t, \cdot)) = 0,$$

and therefore the energy of u at any time is equal to

$$\|(u_0, u_1)\|_{\mathcal{H} \times L^2}^2 = \|u_0\|_{\mathcal{H}}^2 + \|u_1\|_{L^2(M,\mu)}^2.$$

In this paper, we investigate exact observability for the wave equation (1).

Definition 1. Let $T_0 > 0$ and $\omega \subset M$ be a μ-measurable subset. The subelliptic wave equation (1) is exactly observable on ω in time T_0 if there exists a constant $C_{T_0}(\omega) > 0$ such that, for any $(u_0, u_1) \in D((-\Delta)^{1/2}) \times L^2(M)$, the solution u of (1) satisfies

$$\int_0^{T_0} \int_\omega |\partial_t u(t, x)|^2 \, d\mu(x) \, dt \geqslant C_{T_0}(\omega) \|(u_0, u_1)\|_{\mathcal{H} \times L^2}^2. \tag{4}$$

1.2. ***Main result.*** Our main result is the following.

Theorem 2. *Let $T_0 > 0$ and let $\omega \subset M$ be a measurable subset. We assume that there exist $1 \leqslant i, j \leqslant m$ and q in the interior of $M \backslash \omega$ such that $[X_i, X_j](q) \notin \mathrm{Span}(X_1(q), \dots, X_m(q))$. Then the subelliptic wave equation* (1) *is not exactly observable on ω in time T_0.*

Consequently, using a duality argument (see Section 4.2), we obtain that exact controllability also does not hold in any finite time.

Definition 3. Let $T_0 > 0$ and $\omega \subset M$ be a measurable subset. The subelliptic wave equation (1) is exactly controllable on ω in time T_0 if for any $(u_0, u_1) \in D((-\Delta)^{1/2}) \times L^2(M)$ there exists $g \in L^2((0, T_0) \times M)$ such that the solution u of

$$\begin{cases} \partial_{tt}^2 u - \Delta u = \mathbf{1}_\omega g & \text{in } (0, T_0) \times M, \\ u = 0 \quad \text{on } (0, T_0) \times \partial M, \\ (u_{|t=0}, \partial_t u_{|t=0}) = (u_0, u_1), \end{cases} \tag{5}$$

satisfies $u(T_0, \cdot) = 0$.

Corollary 4. *Let $T_0 > 0$ and let $\omega \subset M$ be a measurable subset. We assume that there exist $1 \leqslant i, j \leqslant m$ and q in the interior of $M \backslash \omega$ such that $[X_i, X_j](q) \notin \mathrm{Span}(X_1(q), \dots, X_m(q))$. Then the subelliptic wave equation* (1) *is not exactly controllable on ω in time T_0.*

In what follows, we denote by $\mathcal{D}$ the set of all vector fields that can be decomposed as linear combinations with smooth coefficients of the X_i:

$$\mathcal{D} = \mathrm{Span}(X_1, \dots, X_m) \subset TM.$$

$\mathcal{D}$ is called the *distribution* associated to the vector fields $X_1, \dots, X_m$. For $q \in M$, we denote by $\mathcal{D}_q \subset T_q M$ the distribution $\mathcal{D}$ taken at point q.

The assumptions of Theorem 2 are satisfied as soon as the interior U of $M \setminus \omega$ is nonempty and $\mathcal{D}$ has constant rank $< n$ in U. Indeed, under these conditions, we can argue by contradiction: assume that for any $q \in U$ and any $1 \leqslant i, j \leqslant m$, it holds $[X_i, X_j](q) \in \mathrm{Span}(X_1(q), \dots, X_m(q)) = \mathcal{D}_q$. Then we have $[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}$ in U, i.e., $\mathcal{D}$ is involutive. By Frobenius's theorem, $\mathcal{D}$ is then completely integrable, which contradicts Hörmander's condition.

The following examples show that the assumptions of Theorem 2 are also satisfied in some nonconstant-rank cases:

Example 5. In the Baouendi–Grushin case, for which $X_1 = \partial_{x_1}$ and $X_2 = x_1 \partial_{x_2}$ are vector fields on $(-1, 1)_{x_1} \times \mathbb{T}_{x_2}$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, the corresponding sub-Laplacian $\Delta = X_1^2 + X_2^2$ (here, $\mu = dx_1\, dx_2$ for simplicity) is elliptic outside of the singular submanifold $S = \{x_1 = 0\}$. Therefore, the corresponding subelliptic wave equation is observable on any open subset containing S (with some finite minimal time of observability, see [Bardos et al. 1992]), but according to Theorem 2, it is not observable in any finite time on any subset ω such that the interior of $M \setminus \omega$ has a nonempty intersection with S.

Example 6. In the Martinet case, the vector fields are $X_1 = \partial_{x_1}$ and $X_2 = \partial_{x_2} + x_1^2 \partial_{x_3}$ on $(-1, 1)_{x_1} \times \mathbb{T}_{x_2} \times \mathbb{T}_{x_3}$, and the corresponding sub-Laplacian is $\Delta = X_1^2 + X_2^2$ (again, $\mu = dx_1\, dx_2\, dx_3$ for simplicity).

Then, we have $[X_1, X_2] = 2x_1\partial_{x_3}$. The only points at which this bracket belongs to the distribution $\mathrm{Span}(X_1, X_2)$ are the points for which $x_1 = 0$. Since this set of points has empty interior, the assumptions of Theorem 2 are satisfied as soon as $M \setminus \omega$ has nonempty interior.

Remark 7. The assumption of compactness on M is not necessary; we may remove it and just require that the subelliptic wave equation (1) in M is well-posed. It is for example the case if M is complete for the sub-Riemannian distance induced by $X_1, \dots, X_m$ since Δ is then essentially self-adjoint [Strichartz 1986].

Remark 8. Theorem 2 remains true if M has no boundary. In this case, (1) is well-posed in a space slightly smaller than (2): a condition of null average has to be added since nonzero constant functions on M are solutions of (1); see Section 1.5. The observability inequality of Theorem 2 remains true in this space of solutions; anticipating the proof, we notice that the spiraling normal geodesics of Proposition 17 still exist (since their construction is purely local), and we subtract from the initial datum u_0^k of the localized solutions constructed in Proposition 16 their spatial average $\int_M u_0^k \, d\mu$.

Remark 9. Thanks to abstract results (see for example [Miller 2012]), Theorem 2 remains true when the subelliptic wave equation (1) is replaced by the subelliptic half-wave equation $\partial_t u + i\sqrt{-\Delta}u = 0$ with Dirichlet boundary conditions.

1.3. ***Ideas of the proof.*** In the sequel, we define a normal geodesic[1] to be the projection on M of a bicharacteristic (parametrized by time) for the principal symbol of the wave equation (1). We will give a more detailed definition in Section 1.4.

The proof of Theorem 2 mainly requires two ingredients:

(1) There exist solutions of the free subelliptic wave equation (1) whose energy concentrates along any given normal geodesic.

(2) There exist normal geodesics which "spiral" around curves transverse to $\mathcal{D}$, and which therefore remain arbitrarily close to their starting point on arbitrarily large time intervals.

Combining the two above facts, the proof of Theorem 2 is straightforward (see Section 4.1). Note that the first point follows from the general theory of propagation of complex Lagrangian spaces, while the second point is the main novelty of this paper.

Since our construction is purely local (meaning that it does not "feel" the boundary and only relies on the local structure of the vector fields), we can focus on the case where there is a (small) open neighborhood V of the origin O such that $V \subset M \setminus \omega$, and $[X_i, X_j](O) \notin \mathcal{D}_O$ for some $1 \leqslant i, j \leqslant m$. In the sequel, we assume it is the case.

Let us give an example of vector fields where the spiraling normal geodesics used in the proof of Theorem 2 are particularly simple. We consider the three-dimensional manifold with boundary $M_1 = (-1, 1)_{x_1} \times \mathbb{T}_{x_2} \times \mathbb{T}_{x_3}$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z} \approx (-1, 1)$ is the one-dimensional torus. We endow M_1 with

[1]This terminology is common in sub-Riemannian geometry, and it is justified by the fact that we can naturally associate to the vector fields $X_1, \dots, X_m$ a metric structure on M for which these projected paths are geodesics; see [Montgomery 2002].

the vector fields $X_1 = \partial_{x_1}$ and $X_2 = \partial_{x_2} - x_1\partial_{x_3}$. This is the Heisenberg manifold with boundary. We endow M_1 with an arbitrary smooth volume μ. The normal geodesics we consider are given by

$$\begin{aligned} x_1(t) &= \varepsilon \sin(t/\varepsilon), \\ x_2(t) &= \varepsilon \cos(t/\varepsilon) - \varepsilon, \\ x_3(t) &= \varepsilon(t/2 - \varepsilon \sin(2t/\varepsilon)/4). \end{aligned} \tag{6}$$

They spiral around the x_3-axis $x_1 = x_2 = 0$.

Here, one should think of ε as a small parameter. In the sequel, we denote by x_ε the normal geodesic with parameter ε.

Clearly, given any $T_0 > 0$, for ε sufficiently small, we have $x_\varepsilon(t) \in V$ for every $t \in (0, T_0)$. Our objective is to construct solutions u^k of the subelliptic wave equation (1) such that $\|(u_0^k, u_1^k)\|_{\mathcal{H}\times L^2} = 1$ and the energy of $u^k(t, \cdot)$ concentrates outside of an open set V_t containing $x_\varepsilon(t)$, i.e.,

$$\int_{M_1\setminus V_t} \big(|\partial_t u^k(t, x)|^2 + (X_1 u_k(t, x))^2 + (X_2 u_k(t, x))^2\big)\, d\mu(x)$$

tends to 0 as $k \to +\infty$ uniformly with respect to $t \in (0, T_0)$. As a consequence, the observability inequality (4) fails.

The construction of solutions of the free wave equation whose energy concentrates on geodesics is classical in the elliptic (or Riemannian) case; these are the so-called Gaussian beams, for which a construction can be found for example in [Ralston 1982]. Here, we adapt this construction to our subelliptic (sub-Riemannian) setting, which does not raise any problem since the normal geodesics we consider stay in the elliptic part of the operator Δ. It may also be directly justified with the theory of propagation of complex Lagrangian spaces (see Section 2).

In the case of general vector fields $X_1, \dots, X_m$, the existence of spiraling normal geodesics also has to be justified. For that purpose, we first approximate $X_1, \dots, X_m$ by their *nilpotent approximations*, and we then prove that, for these approximations, such a family of spiraling normal geodesics exists, as in the Heisenberg case.

1.4. ***Normal geodesics.*** In this section, we explain in more details what *normal geodesics* are. As said before, they are natural extensions of Riemannian geodesics since they are projections of bicharacteristics.

We denote by $S^m_{\mathrm{phg}}(T^*((0, T) \times M))$ the set of polyhomogeneous symbols of order m with compact support and by $\Psi^m_{\mathrm{phg}}((0, T) \times M)$ the set of associated polyhomogeneous pseudodifferential operators of order m whose distribution kernel has compact support in $(0, T) \times M$ (see Appendix A).

We set $P = \partial^2_{tt} - \Delta \in \Psi^2_{\mathrm{phg}}((0, T) \times M)$, whose principal symbol is

$$p_2(t, \tau, x, \xi) = -\tau^2 + g^*(x, \xi),$$

with τ the dual variable of t and g^* the principal symbol of $-\Delta$. For $\xi \in T^*M$, we have (see Appendix A)

$$g^* = \sum_{i=1}^m h^2_{X_i}.$$

Here, given any smooth vector field X on M, we denote by h_X the Hamiltonian function (momentum map) on T^*M associated with X defined in local (x,ξ)-coordinates by $h_X(x,\xi)=\xi(X(x))$.

In $T^*(\mathbb{R}\times M)$, the Hamiltonian vector field $\vec{p}_2$ associated with p_2 is given by $\vec{p}_2 f=\{p_2, f\}$, where $\{\cdot,\cdot\}$ denotes the Poisson bracket (see Appendix A). Since $\vec{p}_2 p_2=0$, we get that p_2 is constant along the integral curves of $\vec{p}_2$. Thus, the characteristic set $\mathcal{C}(p_2)=\{p_2=0\}$ is preserved by the flow of $\vec{p}_2$. Null-bicharacteristics are then defined as the maximal integral curves of $\vec{p}_2$ which live in $\mathcal{C}(p_2)$. In other words, the null-bicharacteristics are the maximal solutions of

$$\begin{cases} \dot{t}(s)=-2\tau(s), \\ \dot{x}(s)=\nabla_\xi g^*(x(s),\xi(s)), \\ \dot{\tau}(s)=0, \\ \dot{\xi}(s)=-\nabla_x g^*(x(s),\xi(s)), \\ \tau^2(0)=g^*(x(0),\xi(0)). \end{cases} \tag{7}$$

This definition needs to be adapted when the null-bicharacteristic meets the boundary ∂M, but in the sequel, we only consider solutions of (7) on time intervals where $x(t)$ does not reach ∂M.

In the sequel, we take $\tau=-\frac{1}{2}$, which gives $g^*(x(s),\xi(s))=\frac{1}{4}$. This also implies that $t(s)=s+t_0$ and, taking t as a time parameter, we are led to solve

$$\begin{cases} \dot{x}(t)=\nabla_\xi g^*(x(t),\xi(t)), \\ \dot{\xi}(t)=-\nabla_x g^*(x(t),\xi(t)), \\ g^*(x(0),\xi(0))=\frac{1}{4}. \end{cases} \tag{8}$$

In other words, the t-variable parametrizes null-bicharacteristics in a way that they are traveled at speed 1.

Remark 10. In the subelliptic setting, the cosphere bundle S^*M can be decomposed as $S^*M=U^*M\cup S\Sigma$, where $U^*M=\left\{g^*=\frac{1}{4}\right\}$ is a cylinder bundle, $\Sigma=\{g^*=0\}$ is the characteristic cone and $S\Sigma$ is the sphere bundle of Σ; see [Colin de Verdière et al. 2018, Section 1].

We denote by $\phi_t: S^*M\to S^*M$ the (normal) geodesic flow defined by $\phi_t(x_0,\xi_0)=(x(t),\xi(t))$, where $(x(t),\xi(t))$ is a solution of the system given by the first two lines of (8) and initial conditions (x_0,ξ_0). Note that any point in $S\Sigma$ is a fixed point of ϕ_t and that the other normal geodesics are traveled at speed 1 since we took $g^*=\frac{1}{4}$ in U^*M (see Remark 10).

The curves $x(t)$ which solve (8) are geodesics (i.e., local minimizers) for a sub-Riemannian metric g; see [Montgomery 2002, Theorem 1.14].

1.5. ***Observability in some regions of phase-space.*** We have explained in Section 1.3 that the existence of solutions of the subelliptic wave equation (1) concentrated on spiraling normal geodesics is an obstruction to observability in Theorem 2. Our goal in this section is to state a result ensuring observability if one "removes" in some sense these normal geodesics.

For this result, we focus on a version of the Heisenberg manifold described in Section 1.3 which has *no boundary*. This technical assumption avoids us using boundary microlocal defect measures in the proof, which, in this sub-Riemannian setting, are difficult to handle. As a counterpart, we need to consider solutions of the wave equation with null initial average, in order to get well-posedness.

We consider the Heisenberg group G, that is, $\mathbb{R}^3$ with the composition law

$$(x_1, x_2, x_3) \star (x_1', x_2', x_3') = (x_1 + x_1', x_2 + x_2', x_3 + x_3' - x_1 x_2').$$

Then $X_1 = \partial_{x_1}$ and $X_2 = \partial_{x_2} - x_1 \partial_{x_3}$ are left-invariant vector fields on G. Since $\Gamma = \sqrt{2\pi}\mathbb{Z} \times \sqrt{2\pi}\mathbb{Z} \times 2\pi\mathbb{Z}$ is a co-compact subgroup of G, the left quotient $M_H = \Gamma \backslash G$ is a compact three-dimensional manifold and, moreover, X_1 and X_2 are well-defined as vector fields on the quotient. We call M_H endowed with the vector fields X_1 and X_2 the "Heisenberg manifold without boundary". Finally, we define the Heisenberg Laplacian $\Delta_H = X_1^2 + X_2^2$ on M_H. Since $[X_1, X_2] = -\partial_{x_3}$, it is a hypoelliptic operator. We endow M_H with an arbitrary smooth volume μ.

We introduce the space

$$L_0^2 = \left\{ u_0 \in L^2(M_H),\ \int_{M_H} u_0 \, d\mu = 0 \right\}$$

and we consider the operator Δ_H whose domain $D(\Delta_H)$ is the completion in L_0^2 of the set of all $u \in C_c^\infty(M_H)$ with null-average for the norm $\|(\mathrm{Id} - \Delta_H)u\|_{L^2}$. Then, $-\Delta_H$ is positive definite and we consider $(-\Delta_H)^{1/2}$ with domain $D((-\Delta_H)^{1/2}) = \mathcal{H}_0 := L_0^2 \cap \mathcal{H}(M_H)$. The wave equation

$$\begin{cases} \partial_{tt}^2 u - \Delta_H u = 0 & \text{in } \mathbb{R} \times M_H, \\ (u_{|t=0}, \partial_t u_{|t=0}) = (u_0, u_1) \in D((-\Delta_H)^{\frac{1}{2}}) \times L_0^2, \end{cases} \tag{9}$$

admits a unique solution $u \in C^0(\mathbb{R}; D((-\Delta_H)^{1/2})) \cap C^1(\mathbb{R}; L_0^2)$.

We note that $-\Delta_H$ is invertible in L_0^2. The space $\mathcal{H}_0$ is endowed with the norm $\|u\|_{\mathcal{H}}$ (defined in (3) and also equal to $\|(-\Delta_H)^{1/2}u\|_{L^2}$), and its topological dual $\mathcal{H}_0'$ is endowed with the norm $\|u\|_{\mathcal{H}_0'} := \|(-\Delta_H)^{-1/2}u\|_{L^2}$.

We note that $g^*(x, \xi) = \xi_1^2 + (\xi_2 - x_1\xi_3)^2$ and hence the null-bicharacteristics are solutions of

$$\begin{aligned} \dot{x}_1(t) &= 2\xi_1, & \dot{\xi}_1(t) &= 2\xi_3(\xi_2 - x_1\xi_3), \\ \dot{x}_2(t) &= 2(\xi_2 - x_1\xi_3), & \dot{\xi}_2(t) &= 0, \\ \dot{x}_3(t) &= -2x_1(\xi_2 - x_1\xi_3), & \dot{\xi}_3(t) &= 0. \end{aligned} \tag{10}$$

The spiraling normal geodesics described in Section 1.3 correspond to $\xi_1 = \cos(t/\varepsilon)/2$, $\xi_2 = 0$ and $\xi_3 = 1/(2\varepsilon)$. In particular, the constant ξ_3 is a kind of rounding number reflecting the fact that the normal geodesic spirals at a certain speed around the x_3-axis. Moreover, ξ_3 is preserved under the flow (somehow, the Heisenberg flow is completely integrable), and this property plays a key role in the proof of Theorem 11 below and justifies that we state it only for the Heisenberg manifold (without boundary).

As said above, normal geodesics corresponding to a large momentum ξ_3 are precisely the ones used to contradict observability in Theorem 2. We expect to be able to establish observability if we consider only solutions of (1) whose ξ_3 (in a certain sense) is not too large. This is the purpose of our second main result.

Set

$$V_\varepsilon = \left\{ (x, \xi) \in T^*M_H : |\xi_3| > \frac{1}{\varepsilon}(g_x^*(\xi))^{\frac{1}{2}} \right\}.$$

Note that since ξ_3 is constant along null-bicharacteristics, V_ε and its complement V_ε^c are invariant under the bicharacteristic equations (10).

In the next statement, we define a horizontal strip to be the periodization under the action of the co-compact subgroup Γ of a set of the form

$$\{(x_1, x_2, x_3) : (x_1, x_2) \in [0, \sqrt{2\pi})^2,\ x_3 \in I\},$$

where I is a strict open subinterval of $[0, 2\pi)$.

Theorem 11. *Let $B \subset M_H$ be an open subset and suppose that B is sufficiently small, so that $\omega = M_H \backslash B$ contains a horizontal strip. Let $a \in S^0_{\mathrm{phg}}(T^*M_H)$, $a \geqslant 0$, such that, denoting by $j : T^*\omega \to T^*M_H$ the canonical injection,*

$$j(T^*\omega) \cup V_\varepsilon \subset \mathrm{Supp}(a) \subset T^*M_H,$$

and in particular a does not depend on time. There exists $\kappa > 0$ such that, for any $\varepsilon > 0$ and any $T \geqslant \kappa\varepsilon^{-1}$, it holds

$$C\|(u(0), \partial_t u(0))\|^2_{\mathcal{H}_0 \times L^2_0} \leqslant \int_0^T |(\mathrm{Op}(a)\partial_t u, \partial_t u)_{L^2}|\, dt + \|(u(0), \partial_t u(0))\|^2_{L^2_0 \times \mathcal{H}'_0} \tag{11}$$

for some $C = C(\varepsilon, T) > 0$ and for any solution $u \in C^0(\mathbb{R}; D((-\Delta_H)^{1/2})) \cap C^1(\mathbb{R}; L^2_0)$ of (9).

The term $\|(u_0, u_1)\|^2_{L^2 \times \mathcal{H}'_0}$ in the right-hand side of (11) cannot be removed; i.e., our statement only consists of a *weak* observability inequality. Indeed, the usual way to remove such terms is to use a unique continuation argument for eigenfunctions φ of Δ, but here it does not work since $\mathrm{Op}(a)\varphi = 0$ does not imply in general that $\varphi \equiv 0$ in the whole manifold, even if the support of a contains $j(T^*\omega)$ for some nonempty open set ω: in some sense, there is no “pseudodifferential unique continuation argument”.

1.6. *Comments on the existing literature.*

Elliptic and subelliptic waves. The exact controllability/observability of the elliptic wave equation is known to be almost equivalent to the so-called geometric control condition (GCC) (see [Bardos et al. 1992]) that any geodesic enters the control set ω within time T. In some sense, our main result is that GCC is not satisfied in the subelliptic setting, as soon as $M\backslash\omega$ contains in its interior a point x at which Δ is “truly subelliptic”. For the elliptic wave equation, in many geometrical situations, there exists a minimal time $T_0 > 0$ such that observability holds only for $T \geqslant T_0$: when there exists a geodesic $\gamma : (0, T_0) \to M$ traveled at speed 1 which does not meet $\bar{\omega}$, one constructs a sequence of initial data $(u_0^k, u_1^k)_{k\in\mathbb{N}^*}$ of the wave equation whose associated microlocal defect measure is concentrated on $(x_0, \xi_0) \in S^*M$ taken to be the initial conditions for the null-bicharacteristic projecting onto γ. Then, the associated sequence of solutions $(u^k)_{k\in\mathbb{N}^*}$ of the wave equation has an associated microlocal defect measure ν which is invariant under the geodesic flow: $\vec{p}\nu = 0$, where $\vec{p}$ is the Hamiltonian flow associated to the principal symbol p of the wave operator. In particular, denoting by $\pi : T^*M \to M$ the canonical projection, $\pi_*\nu$ gives no mass to ω since γ is contained in $M \setminus \bar{\omega}$, and this proves that observability cannot hold.

In the subelliptic setting, the invariance property $\vec{p}\nu = 0$ does not give any information on ν on the characteristic manifold Σ, since $\vec{p} = -2\tau\partial_t + \vec{g}^*$ vanishes on Σ. This is related to the lack of information on propagation of singularities in this characteristic manifold; see the main theorem of [Lascar 1982]. If one instead tries to use the propagation of the microlocal defect measure for subelliptic half-wave equations, one is immediately confronted with the fact that $\sqrt{-\Delta}$ is not a pseudodifferential operator near Σ.

This is why, in this paper, we used only the elliptic part of the symbol g^* (or, equivalently, the strictly hyperbolic part of p_2), where the propagation properties can be established, and then the problem is reduced to proving geometric results on normal geodesics.

Subelliptic Schrödinger equations. The recent article [Burq and Sun 2019] deals with the same observability problem, but for subelliptic Schrödinger equations: namely, the authors consider the Baouendi–Grushin Schrödinger equation $i\partial_t u - \Delta_G u = 0$, where $u \in L^2((0,T)\times M_G)$, $M_G = (-1,1)_x \times \mathbb{T}_y$ and $\Delta_G = \partial_x^2 + x^2\partial_y^2$ is the Baouendi–Grushin Laplacian. Given a control set of the form $\omega = (-1,1)_x \times \omega_y$, where ω_y is an open subset of $\mathbb{T}$, the authors prove the existence of a minimal time of control $\mathcal{L}(\omega)$ related to the maximal height of a horizontal strip contained in $M_G \backslash \omega$. The intuition is that there are solutions of the Baouendi–Grushin Schrödinger equation which travel along the degenerate line $x = 0$ at a finite speed; in some sense, along this line, the Schrödinger equation behaves like a classical (half)-wave equation. What we want here is to explain in a few words why there is a minimal time of observability for the Schrödinger equation, while the wave equation is never observable in finite time as shown by Theorem 2.

The plane $\mathbb{R}^2_{x,y}$ endowed with the vector fields ∂_x and $x\partial_y$ also admits normal geodesics similar to the 1-parameter family q_ε, namely, for $\varepsilon > 0$,

$$\begin{aligned} x(t) &= \varepsilon \sin(t/\varepsilon), \\ y(t) &= \varepsilon(t/2 - \varepsilon \sin(2t/\varepsilon)/4). \end{aligned}$$

These normal geodesics, denoted by γ_ε, also "spiral" around the line $x = 0$ more and more quickly as $\varepsilon \to 0$, and so we might expect to construct solutions of the Baouendi–Grushin Schrödinger equation with energy concentrated along γ_ε, which would contradict observability when $\varepsilon \to 0$ as above for the Heisenberg wave equation.

However, we can convince ourselves that it is not possible to construct such solutions: in some sense, the dispersion phenomena of the Schrödinger equation exactly compensate for the lengthening of the normal geodesics γ_ε as $\varepsilon \to 0$ and explain that even these Gaussian beams may be observed in ω from a certain minimal time $\mathcal{L}(\omega) > 0$ which is uniform in ε.

To put this argument into a more formal form, we consider the solutions of the bicharacteristic equations for the Baouendi–Grushin Schrödinger equation $i\partial_t u - \Delta_G u = 0$ given by

$$\begin{aligned} x(t) &= \varepsilon \sin(\xi_y t), & \xi_x(t) &= \varepsilon \xi_y \cos(\xi_y t), \\ y(t) &= \varepsilon^2 \xi_y \left(\frac{t}{2} - \frac{\sin(2\xi_y t)}{4\xi_y}\right), & \xi_y(t) &= \xi_y. \end{aligned}$$

It follows from the hypoellipticity of Δ_G (see [Burq and Sun 2019, Section 3] for a proof) that

$$|\xi_y|^{1/2} \lesssim \sqrt{-\Delta_G} = (|\xi_x|^2 + x^2|\xi_y|^2)^{1/2} = \varepsilon|\xi_y|.$$

Therefore $\varepsilon^2|\xi_y| \gtrsim 1$, and hence $|y(t)| \gtrsim t$, independently from ε and ξ_y. This heuristic gives the intuition that a minimal time $\mathcal{L}(\omega)$ is required to detect all solutions of the Baouendi–Grushin Schrödinger equation from ω, but that for $T_0 > \mathcal{L}(\omega)$, no solution is localized enough to stay in $M\backslash\omega$ during the time interval $(0, T_0)$. Roughly speaking, the frequencies of order ξ_y travel at speed $\sim \xi_y$, which is typical for a dispersion phenomenon. This picture is very different from the one for the wave equation (which we consider in this paper) for which no dispersion occurs.

With similar ideas, in [Letrouit and Sun 2021], the interplay between the subellipticity effects measured by the nonholonomic order of the distribution $\mathcal{D}$ (see Section 3.1) and the strength of dispersion of Schrödinger-type equations was investigated. More precisely, for $\Delta_\gamma = \partial_x^2 + |x|^{2\gamma}\partial_y^2$ on $M = (-1, 1)_x \times \mathbb{T}_y$, and for $s \in \mathbb{N}$, the observability properties of the Schrödinger-type equation $(i\partial_t - (-\Delta_\gamma)^s)u = 0$ were shown to depend on the value $\kappa = 2s/(\gamma + 1)$. In particular it is proved that, for $\kappa < 1$, observability fails for any time, which is consistent with the present result, and that for $\kappa = 1$, observability holds only for sufficiently large times, which is consistent with the result of [Burq and Sun 2019]. The results of [Letrouit and Sun 2021] are somehow Schrödinger analogues of the results of [Beauchard et al. 2014] which deal with a similar problem for the Baouendi–Grushin heat equation.

General bibliographical comments. Control of subelliptic PDEs has attracted much attention in the last decade. Most results in the literature deal with subelliptic parabolic equations, either the Baouendi–Grushin heat equation [Koenig 2017; Duprez and Koenig 2020; Beauchard et al. 2020] or the heat equation in the Heisenberg group [Beauchard and Cannarsa 2017]. The paper [Burq and Sun 2019] was the first to deal with a subelliptic Schrödinger equation and the present work is the first to handle exact controllability of subelliptic wave equations.

A slightly different problem is the *approximate* controllability of hypoelliptic PDEs, which was studied in [Laurent and Léautaud 2022] for hypoelliptic wave and heat equations. Approximate controllability is weaker than exact controllability, and it amounts to proving "quantitative" unique continuation results for hypoelliptic operators. For the hypoelliptic wave equation, it is proved in [Laurent and Léautaud 2022] that for $T > 2\sup_{x\in M}(\mathrm{dist}(x, \omega))$ (here, dist is the sub-Riemannian distance), the observation of the solution on $(0, T) \times \omega$ determines the initial data, and therefore the whole solution.

1.7. *Organization of the paper.* In Section 2, we construct exact solutions of the subelliptic wave equation (1) concentrating on any given normal geodesic. First, in Section 2.1, we show that, given any normal geodesic $t \mapsto x(t)$ which does not hit ∂M in the time interval $(0, T)$, it is possible to construct a sequence $(v_k)_{k\in\mathbb{N}}$ of *approximate solutions* of (1) whose energy concentrates along $t \mapsto x(t)$ during the time interval $(0, T)$ as $k \to +\infty$. By "approximate", we mean here that $\partial_{tt}^2 v_k - \Delta v_k$ is small, but not necessarily exactly equal to 0. In Section 2.1, we provide a first proof for this construction using the classical propagation of complex Lagrangian spaces. Another proof using a Gaussian beam approach is provided in Appendix B. Then, in Section 2.2, using this sequence $(v_k)_{k\in\mathbb{N}}$, we explain how to construct a sequence $(u_k)_{k\in\mathbb{N}}$ of *exact* solutions of $(\partial_{tt}^2 - \Delta)u = 0$ in M with the same concentration property along the normal geodesic $t \mapsto x(t)$.

In Section 3, we prove the existence of normal geodesics which spiral in M, spending an arbitrarily large time in $M\backslash\omega$. These normal geodesics generalize the example described in Section 1.3 for the Heisenberg manifold with boundary. The proof proceeds in two steps: first, we show that it is sufficient to prove the result in the so-called “nilpotent case” (Section 3.2), and then we prove it in the nilpotent case (Section 3.3).

In Section 4.1, we use the results of Sections 2 and 3 to conclude the proof of Theorem 2. In Section 4.2, we deduce Corollary 4 by a duality argument. Finally, in Section 4.3, we prove Theorem 11.

2. Gaussian beams along normal geodesics

2.1. ***Construction of sequences of approximate solutions.*** We consider a solution $(x(t), \xi(t))_{t\in[0,T]}$ of (8) on M. We shall describe the construction of solutions of

$$\partial_{tt}^2 u - \Delta u = 0 \tag{12}$$

on $[0, T] \times M$ with energy

$$E(u(t,\cdot)) := \tfrac{1}{2}(\|\partial_t u(t,\cdot)\|^2_{L^2(M,\mu)} + \|u(t,\cdot)\|^2_{\mathcal{H}})$$

concentrated along $x(t)$ for $t \in [0, T]$. The following proposition, which is inspired by [Ralston 1982; Macià and Zuazua 2002], shows that it is possible, at least for approximate solutions of (12).

Proposition 12. *Fix $T > 0$ and let $(x(t), \xi(t))_{t\in[0,T]}$ be a solution of* (8) *$\big($in particular $g^*(x(0), \xi(0)) = \frac{1}{4}\big)$ which does not hit the boundary ∂M in the time interval $(0, T)$. Then there exist $a_0, \psi \in C^2((0, T) \times M)$ such that, setting, for $k \in \mathbb{N}$,*

$$v_k(t, x) = k^{\frac{n}{4}-1} a_0(t, x) e^{ik\psi(t,x)}$$

the following properties hold:

- *v_k is an approximate solution of* (12), *meaning that*

$$\|\partial_{tt}^2 v_k - \Delta v_k\|_{L^1((0,T);L^2(M))} \leqslant C k^{-\frac{1}{2}}. \tag{13}$$

- *The energy of v_k is bounded below with respect to k and $t \in [0, T]$*:

$$\textit{there exists } A > 0 \textit{ such that, for all } t \in [0, T], \quad \liminf_{k\to+\infty} E(v_k(t,\cdot)) \geqslant A. \tag{14}$$

- *The energy of v_k is small off $x(t)$: For any $t \in [0, T]$, we fix V_t an open subset of M for the initial topology of M, containing $x(t)$, so that the mapping $t \mapsto V_t$ is continuous (V_t is chosen sufficiently small so that this makes sense in a chart). Then*

$$\sup_{t\in[0,T]} \int_{M\setminus V_t} \Big(|\partial_t v_k(t, x)|^2 + \sum_{j=1}^m (X_j v_k(t, x))^2 \Big)\, d\mu(x) \xrightarrow[k\to+\infty]{} 0. \tag{15}$$

Remark 13. The construction of approximate solutions such as the ones provided by Proposition 12 is usually done for strictly hyperbolic operators, that is, operators with a principal symbol p_m of order m such that the polynomial $f(s) = p_m(t, q, s, \xi)$ has m distinct real roots when $\xi \neq 0$; see for example [Ralston 1982]. The operator $\partial_{tt}^2 - \Delta$ is not strictly hyperbolic because g^* is degenerate, but our proof shows that the same construction may be adapted without difficulty to this operator *along normal bicharacteristics*. This is due to the fact that along normal bicharacteristics, $\partial_{tt}^2 - \Delta$ is indeed strictly hyperbolic (or equivalently, Δ is elliptic). It was already noted by [Ralston 1982] that the construction of Gaussian beams could be done for more general operators than strictly hyperbolic ones, and that the differences between the strictly hyperbolic case and more general cases arise while dealing with propagation of singularities. Also, in [Hörmander 1985, Chapter 24.2], it was noticed that "since only microlocal properties of p_2 are important, it is easy to see that hyperbolicity may be replaced by $\nabla_\xi p_2 \neq 0$."

Hereafter we provide two proofs of Proposition 12. The first proof is short and is actually quite straightforward for readers acquainted with the theory of propagation of complex Lagrangian spaces, once one has noticed that the solutions of (8) which we consider live in the *elliptic part* of the principal symbol of $-\Delta$. For the sake of completeness, and because this also has its own interest, we provide in Appendix B a second proof, longer but more elementary and accessible without any knowledge of complex Lagrangian spaces; it relies on the construction of Gaussian beams in the subelliptic context. The two proofs follow parallel paths, and indeed, the computations which are only sketched in the first proof are written in full detail in the second proof, given in Appendix B.

First proof of Proposition 12. The construction of Gaussian beams, or more generally of a WKB approximation, is related to the transport of complex Lagrangian spaces along bicharacteristics, as reported for example in [Hörmander 1985, Chapter 24.2; Ivrii 2019, Volume I, Part I, Chapter 1.2]. Our proof follows the lines of [Hörmander 1985, pages 426–428].

A usual way to solve (at least approximately) evolution equations of the form

$$Pu = 0, \tag{16}$$

where P is a hyperbolic second-order differential operator with real principal symbol and C^∞ coefficients, is to search for oscillatory solutions

$$v_k(x) = k^{\frac{n}{4}-1} a_0(x) e^{ik\psi(x)}. \tag{17}$$

In this expression as in the rest of the proof, we suppress the time variable t. Thus, we use $x = (x_0, x_1, \dots, x_n)$, where $x_0 = t$ in the earlier notation, and we set $x' = (x_1, \dots, x_n)$. Similarly, we take the notation $\xi = (\xi_0, \xi_1, \dots, \xi_n)$, where $\xi_0 = \tau$ previously, and $\xi' = (\xi_1, \dots, \xi_n)$. The bicharacteristics are parametrized by s as in (7), and without loss of generality, we only consider bicharacteristics with $x(0) = 0$ at $s = 0$, which implies in particular $x_0(s) = s$ because of our choice $\tau^2(s) = g^*(x(s), \xi(s)) = \frac{1}{4}$.

Taking charts of M, we can assume $M \subset \mathbb{R}^n$. The precise argument for reducing to this case is written at the end of Appendix B. Also, in the sequel, $P = \partial_{tt}^2 - \Delta$.

Plugging the ansatz (17) into (16), we get

$$Pv_k = (k^{\frac{n}{4}+1} A_1 + k^{\frac{n}{4}} A_2 + k^{\frac{n}{4}-1} A_3) e^{ik\psi}, \tag{18}$$

with

$$A_1(x) = p_2(x, \nabla\psi(x)) a_0(x), \quad A_2(x) = La_0(x), \quad A_3(x) = \partial_{tt}^2 a_0(x) - \Delta a_0(x),$$

and L is a transport operator given by

$$La_0 = \frac{1}{i} \sum_{j=0}^{n} \frac{\partial p_2}{\partial \xi_j}(x, \nabla\psi(x)) \frac{\partial a_0}{\partial x_j} + \frac{1}{2i} \left(\sum_{j,k=0}^{n} \frac{\partial^2 p_2}{\partial \xi_j \partial \xi_k}(x, \nabla\psi(x)) \frac{\partial^2 \psi}{\partial x_j \partial x_k} \right) a_0. \tag{19}$$

For v_k to be an approximate solution of P, we are first led to cancel the higher-order term in (18), i.e.,

$$f(x) := p_2(x, \nabla\psi(x)) = 0, \tag{20}$$

which we solve for initial conditions

$$\psi(0, x') = \psi_0(x'), \quad \nabla\psi_0(0) = \xi'(0) \quad \text{and} \quad \psi_0(0) = 0 \tag{21}$$

(i.e., we fix such a ψ_0, and then we solve (20) for ψ). Indeed, it will be sufficient for our purpose for (20) to be satisfied at second order along the curve $x(s)$; i.e., $D_x^\alpha f(x(s)) = 0$ for any $|\alpha| \leqslant 2$ and any s. For that, we first notice that the choice $\nabla\psi(x(s)) = \xi(s)$ ensures that (20) holds at orders 0 and 1 along the curve $s \mapsto x(s)$ (see Appendix B for detailed computations). Now, we explain how to choose $D^2\psi(x(s))$ adequately in order for (20) to hold at order 2.

We use the decomposition of p_2 into

$$p_2(x_0, x', \xi_0, \xi') = -(\xi_0 - r(x', \xi'))(\xi_0 + r(x', \xi')) + R(x', \xi'),$$

where $r = \sqrt{g^*}$ in a conic neighborhood of $(0, \xi(0))$. Note that $\sqrt{g^*}$ is smooth in small conic neighborhoods of $(0, \xi(0))$ since $g^*(0, \xi(0)) = \frac{1}{4} \neq 0$. Indeed, g^* is elliptic along the whole bicharacteristic since $g^*(x(t), \xi(t)) = \frac{1}{4}$ is preserved by the bicharacteristic flow. The rest term $R(x', \xi')$ is smooth and microlocally supported far from the bicharacteristic; i.e., $R(x', \xi') = 0$ for any $(x', \xi') \in T^*M$ in a conic neighborhood of $(x'(s), \xi'(s))$ for $s \in [0, T]$.

We consider the bicharacteristic γ_+ starting at $(0, 0, r(0, \xi'(0)), \xi'(0))$ and the bicharacteristic γ_- starting at $(0, 0, -r(0, \xi'(0)), \xi'(0))$.

We denote by $\Phi^\pm(x_0, y', \eta')$ the solution of the Hamilton equations with Hamiltonian $H_\pm(x_0, x', \xi') = \xi_0 \mp r(x', \xi')$ and initial datum $(x', \xi') = (y', \eta')$ at $x_0 = 0$. In other words, $\Phi^\pm(x_0, y', \eta') = e^{x_0 \vec{H}_\pm}(0, y', \eta')$. Then, for any s, $\Phi(s, \cdot)$ is well-defined and symplectic from a neighborhood of $(0, \xi'(0))$ to a neighborhood of $H_\pm(s, 0, \xi'(0))$.

The solution $\psi(s, \cdot)$ of (20) and (21) is equal to 0 on $\gamma_\pm$ and $\nabla\psi(s, \cdot)$ is obtained by the transport of the values of $\nabla\psi_0$ by $\Phi^\pm(s, \cdot)$. In other words, to compute $\nabla\psi(s, \cdot)$, one transports the Lagrangian subspace $\Lambda_0 = \{(x', \nabla\psi_0(x'))\}$ along the Hamiltonian flow $\vec{H}_\pm$ during a time s, which yields $\Lambda_s \subset T^*M$, and then, if possible, one writes Λ_s under the form $\{(x', \nabla_{x'}\psi(s, x'))\}$, which gives $\nabla_{x'}\psi(s, x')$. The trouble is that the solution is only local in time: when $x' \mapsto \pi(\Phi^\pm(s, x', \nabla\psi_0(x')))$ ceases to be a diffeomorphism (conjugate point), where $\pi : T^*M \to M$ is the canonical projection, we see that the process described above does not work (appearance of caustics). In the language of Lagrangian spaces, $\Lambda_0 = \{(x', \nabla\psi_0(x'))\} \subset T^*M$ is a Lagrangian subspace and, since $\Phi^\pm(s, \cdot)$ is a symplectomorphism, $\Lambda_s = \Phi^\pm(s, \Lambda_0)$ is Lagrangian as well. If $\pi_{|\Lambda_s}$ is a local diffeomorphism, one can locally describe Λ_s by $\Lambda_s = \{(x', \nabla_{x'}\psi(s, x'))\} \subset T^*M$ for some function $\psi(s, \cdot)$, but blow-up happens when $\operatorname{rank}(d\pi_{|\Lambda_s}) < n$ (classical conjugate point theory), and such a $\psi(s, \cdot)$ may not exist.

However, if the phase ψ_0 is complex, quadratic, and satisfies the condition $\operatorname{Im}(D^2\psi_0) > 0$, where $D^2\psi_0$ denotes the Hessian, no blow-up happens, and the solution is global-in-time. Let us explain why. Indeed, $\Lambda_0 = \{(x', \nabla\psi_0(x'))\}$ then lives in the complexification of the tangent space T^*M, which may be thought of as $\mathbb{C}^{2(n+1)}$. We take coordinates (y,η) on $T^*\mathbb{R}^{n+1}$ or $T^*\mathbb{C}^{n+1}$ and we consider the symplectic forms defined by $\sigma=\sum dy_j\wedge d\eta_j$ and $\sigma_{\mathbb{C}}=\sum dy_j\wedge \overline{d\eta_j}$.

Because of the condition $\operatorname{Im}(D^2\psi_0)>0$, Λ_0 is called a "strictly positive Lagrangian space" (see [Hörmander 1985, Definition 21.5.5]), meaning that $i\sigma_{\mathbb{C}}(v,v)>0$ for v in the tangent space to Λ_0. For any s, the symplectic forms σ and $\sigma_{\mathbb{C}}$ are preserved by $\Phi(s,\cdot)$, meaning that $\Phi(s,\cdot)_*\sigma=\sigma$ and $\Phi(s,\cdot)_*\sigma_{\mathbb{C}}=\sigma_{\mathbb{C}}$; therefore $\sigma=0$ on the tangent space to Λ_s, and $i\sigma_{\mathbb{C}}(v,v)>0$ for v tangent to Λ_s. It precisely means that Λ_s is also a strictly positive Lagrangian space. Then, by [Hörmander 1985, Proposition 21.5.9], we know that there exists $\psi(s,\cdot)$ complex and quadratic with $\operatorname{Im}(D^2\psi(s,\cdot))>0$ such that $\Lambda_s=\{(x',\nabla_{x'}\psi(s,x'))\}$ (to apply [Hörmander 1985, Proposition 21.5.9], recall that, for $\varphi(x')=\frac{1}{2}(Ax',x')$, it holds $\nabla\varphi(x')=Ax'$). In other words, the key point in using complex phases is that strictly positive Lagrangian spaces are parametrized by complex quadratic phases φ with $\operatorname{Im}(D^2\varphi)>0$, whereas real Lagrangian spaces were not parametrized by real phases (see explanations above). This parametrization is a diffeomorphism from the Grassmannian of strictly positive Lagrangian spaces to the space of complex quadratic phases with φ with $\operatorname{Im}(D^2\varphi)>0$. Hence, the phase

$$\psi(s,y')=\nabla_{x'}\psi(x(s))\cdot(y'-x'(s))+\tfrac{1}{2}(y'-x'(s))\cdot D^2_{x'}\psi(s,x'(s))(y'-x'(s))$$

for $s\in[0,T]$ and $y'\in\mathbb{R}^n$ is smooth and for this choice (20) is satisfied at second order along $s\mapsto x(s)$ (the rest $R(x',\xi')$ plays no role since it vanishes in a neighborhood of $s\mapsto x(s)$).

Then, we note that A_2 vanishes along the bicharacteristic if and only if $La_0(x(s))=0$ (see also [Hörmander 1985, equation (24.2.9)]). According to (19), this turns out to be a linear transport equation on $a_0(x(s))$, with leading coefficient $\nabla_\xi p_2(x(s),\xi(s))$ different from 0. Given $a\neq 0$ at $(t=0,\ x'=x'(0))$, this transport equation has a solution $a_0(x(s))$ with initial datum a, and, by Cauchy uniqueness, $a_0(x(s))\neq 0$ for any s. We can choose a_0 in a smooth (and arbitrary) way outside the bicharacteristic. We choose it to vanish outside a small neighborhood of this bicharacteristic, so that no boundary effect happens.

With these choices of ψ and a_0, the bound (13) then follows from the following result whose proof is given in [Ralston 1982, Lemma 2.8].

Lemma 14. *Let $c(x)$ be a function on $\mathbb{R}^{n+1}$ which vanishes at order $S-1$ on a curve Γ for some $S\geqslant 1$. Suppose that* $\operatorname{Supp} c\cap\{|x_0|\leqslant T\}$ *is compact and that* $\operatorname{Im}\psi(x)\geqslant ad(x)^2$ *on this set for some constant $a>0$, where $d(x)$ denotes the distance from the point $x\in\mathbb{R}^{d+1}$ to the curve Γ. Then there exists a constant C such that*

$$\int_{|x_0|\leqslant T}|c(x)e^{ik\psi(x)}|^2\,dx\leqslant Ck^{-S-\frac{n}{2}}.$$

Let us now sketch the end of the proof, which is given in Appendix B in full detail. We apply Lemma 14 to $S=3$, $c=A_1$ and to $S=1$, $c=A_2$, and we get

$$\|\partial_{tt}^2 v_k-\Delta v_k\|_{L^1(0,T;L^2(M))}\leqslant C(k^{-\frac{1}{2}}+k^{-\frac{1}{2}}+k^{-1}),$$

which implies (13). The bounds (14) and (15) follow from the facts that $\operatorname{Im}(D^2\psi(s,\cdot))>0$ and $v_k(x)=k^{n/4-1}a_0(x)e^{ik\psi(x)}$. ∎

Remark 15. An interesting question would be to understand the delocalization properties of the Gaussian beams constructed along normal geodesics in Proposition 12. Compared with the usual Riemannian case done for example in [Ralston 1982], there is a new phenomenon in the sub-Riemannian case since the normal geodesic $x(t)$ (or, more precisely, its lift to S^*M) may approach the characteristic manifold $\Sigma=\{g^*=0\}$, which is the set of directions in which Δ is not elliptic. In finite time T as in our case, the lift of the normal geodesic remains far from Σ, but it may happen as $T\to+\infty$ that it goes closer and closer to Σ. The question is then to understand the link between the delocalization properties of the Gaussian beams constructed along such a normal geodesic, and notably the interplay between the time T and the semiclassical parameter $\frac{1}{k}$.

2.2. ***Construction of sequences of exact solutions in M.*** In this section, using the approximate solutions of Section 2.1, we construct *exact* solutions of (12) whose energy concentrates along a given normal geodesic of M which does not meet the boundary ∂M during the time interval $[0,T]$.

Proposition 16. *Let $(x(t),\xi(t))_{t\in[0,T]}$ be a solution of* (8) *in M $\big($in particular $g^*(x(0),\xi(0))=\frac{1}{4}\big)$ which does not meet ∂M. Let $\theta\in C_c^\infty([0,T]\times M)$ with $\theta(t,\cdot)\equiv 1$ in a neighborhood of $x(t)$ and such that the support of $\theta(t,\cdot)$ stays at positive distance of ∂M.*

Suppose $(v_k)_{k\in\mathbb{N}}$ is constructed along $x(t)$ as in Proposition 12 and u_k is the solution of the Cauchy problem

$$\begin{cases}(\partial_{tt}^2-\Delta)u_k=0 & \text{in } (0,T)\times M,\\ u_k=0 & \text{in } (0,T)\times\partial M,\\ u_{k|t=0}=(\theta v_k)_{|t=0}, & \partial_t u_{k|t=0}=[\partial_t(\theta v_k)]_{|t=0}.\end{cases}$$

Then:

- *The energy of u_k is bounded below with respect to k and $t\in[0,T]$:*

$$\text{there exists } A>0 \text{ such that, for all } t\in[0,T],\quad \liminf_{k\to+\infty} E(u_k(t,\cdot))\geqslant A. \tag{22}$$

- *The energy of u_k is small off $x(t)$: For any $t\in[0,T]$, we fix V_t an open subset of M for the initial topology of M, containing $x(t)$, so that the mapping $t\mapsto V_t$ is continuous (V_t is chosen sufficiently small so that this makes sense in a chart). Then*

$$\sup_{t\in[0,T]}\int_{M\setminus V_t}\Big(|\partial_t u_k(t,x)|^2+\sum_{j=1}^m (X_j u_k(t,x))^2\Big)\,d\mu(x)\xrightarrow[k\to+\infty]{}0. \tag{23}$$

Proof of Proposition 16. Set $h_k=(\partial_{tt}^2-\Delta)(\theta v_k)$. We consider w_k the solution of the Cauchy problem

$$\begin{cases}(\partial_{tt}^2-\Delta)w_k=h_k & \text{in } (0,T)\times M,\\ w_k=0 & \text{in } (0,T)\times\partial M,\\ (w_{k|t=0},\partial_t w_{k|t=0})=(0,0).\end{cases} \tag{24}$$

Differentiating $E(w_k(t,\cdot))$ and using Gronwall's lemma, we get the energy inequality

$$\sup_{t\in[0,T]} E(w_k(t,\cdot))\leqslant C(E(w_k(0,\cdot))+\|h_k\|_{L^1(0,T;L^2(M))}).$$

Therefore, using (13), we get $\sup_{t\in[0,T]} E(w_k(t,\cdot)) \leqslant Ck^{-1}$. Since $u_k = \theta v_k - w_k$, we obtain that

$$\lim_{k\to+\infty} E(u_k(t,\cdot)) = \lim_{k\to+\infty} E((\theta v_k)(t,\cdot)) = \lim_{k\to+\infty} E(v_k(t,\cdot))$$

for every $t \in [0, T]$, where the last equality comes from the fact that θ and its derivatives are bounded and $\|v_k\|_{L^2} \leqslant Ck^{-1}$ when $k \to +\infty$. Using (14), we conclude that (22) holds.

To prove (23), we observe similarly that

$$\sup_{t\in[0,T]} \int_{M\setminus V_t} \left(|\partial_t u_k(t,x)|^2 + \sum_{j=1}^m (X_j u_k(t,x))^2 \right) d\mu(x) \\ \leqslant C \sup_{t\in[0,T]} \left(\int_{M\setminus V_t} (|\partial_t v_k(t,x)|^2 + \sum_{j=1}^m (X_j v_k(t,x))^2)\, d\mu(x) \right) + Ck^{-\frac{1}{2}} \to 0$$

as $k \to +\infty$, according to (15). It concludes the proof of Proposition 16. □

3. Existence of spiraling normal geodesics

The goal of this section is to prove the following proposition, which is the second building block of the proof of Theorem 2, after the construction of localized solutions of the subelliptic wave equation (1) done in Section 2.

We say that $X_1, \dots, X_m$ satisfies the property (P) at $q \in M$ if the following holds:

(P) *For any open neighborhood V of q, for any $T_0 > 0$, there exists a nonstationary normal geodesic $t \mapsto x(t)$, traveled at speed 1, such that $x(t) \in V$ for any $t \in [0, T_0]$.*

Proposition 17. *At any point $q \in M$ such that there exist $1 \leqslant i, j \leqslant m$ with $[X_i, X_j](q) \notin \mathcal{D}_q$, property (P) holds.*

In Section 3.1, we define the so-called nilpotent approximations $\widehat{X}_1^q, \dots, \widehat{X}_m^q$ at a point $q \in M$, which are first-order approximations of $X_1, \dots, X_m$ at $q \in M$ such that the associated Lie algebra $\mathrm{Lie}(\widehat{X}_1^q, \dots, \widehat{X}_m^q)$ is nilpotent. Roughly, we have $\widehat{X}_i^q \approx X_i(q)$, but low-order terms of $X_i(q)$ are not taken into account for defining $\widehat{X}_i^q$, so that the high-order brackets of the $\widehat{X}_i^q$ vanish (which is not generally the case for the X_i). These nilpotent approximations are good local approximations of the vector fields $X_1, \dots, X_m$, and their study is much simpler.

The proof of Proposition 17 splits into two steps: first, we show that it is sufficient to prove the result in the nilpotent case (Section 3.2), then we handle this simpler case (Section 3.3).

3.1. ***Nilpotent approximation.*** In this section, we recall the construction of the nilpotent approximations $\widehat{X}_1^q, \dots, \widehat{X}_m^q$. The definitions we give are classical, and the reader can refer to [Agrachev et al. 2020, Chapter 10; Jean 2014, Chapter 2] for more material on this section. This construction is related to the notion of tangent space in the Gromov–Hausdorff sense of a sub-Riemannian structure $(M, \mathcal{D}, g)$ at a point $q \in M$; the tangent space is defined intrinsically (meaning that it does not depend on a choice of coordinates or of local frame) as an equivalence class under the action of sub-Riemannian isometries; see [Bellaïche 1996; Jean 2014].

Sub-Riemannian flag. We define the sub-Riemannian flag as follows: we set $\mathcal{D}^0 = \{0\}$, $\mathcal{D}^1 = \mathcal{D}$, and, for any $j \geqslant 1$, $\mathcal{D}^{j+1} = \mathcal{D}^j + [\mathcal{D}, \mathcal{D}^j]$. For any point $q \in M$, it defines a flag

$$\{0\} = \mathcal{D}^0_q \subset \mathcal{D}^1_q \subset \cdots \subset \mathcal{D}^{r-1}_q \subsetneq \mathcal{D}^{r(q)}_q = T_q M.$$

The integer $r(q)$ is called the nonholonomic order of $\mathcal{D}$ at q, and it is equal to 2 everywhere in the Heisenberg manifold for example. Note that it depends on q; see Example 5 in Section 1.2 (the Baouendi–Grushin example).

For $0 \leqslant i \leqslant r(q)$, we set $n_i(q) = \dim \mathcal{D}^i_q$, and the sequence $(n_i(q))_{0 \leqslant i \leqslant r(q)}$ is called the growth vector at point q. We set $\mathcal{Q}(q) = \sum_{i=1}^{r(q)} i(n_i(q) - n_{i-1}(q))$, which is generically the Hausdorff dimension of the metric space given by the sub-Riemannian distance on M; see [Mitchell 1985]. Finally, we define the nondecreasing sequence of weights $w_i(q)$ for $1 \leqslant i \leqslant n$ as follows. Given any $1 \leqslant i \leqslant n$, there exists a unique $1 \leqslant j \leqslant n$ such that $n_{j-1}(q) + 1 \leqslant i \leqslant n_j(q)$. We set $w_i(q) = j$. For example, for any q in the Heisenberg manifold, $w_1(q) = w_2(q) = 1$ and $w_3(q) = 2$; indeed, the coordinates x_1 and x_2 have "weight 1", while the coordinate x_3 has "weight 2" since ∂_{x_3} requires a bracket to be generated.

Regular and singular points. We say that $q \in M$ is regular if the growth vector $(n_i(q'))_{0 \leqslant i \leqslant r(q')}$ at q' is constant for q' in a neighborhood of q. Otherwise, q is said to be singular. If any point $q \in M$ is regular, we say that the structure is equiregular. For example, the Heisenberg manifold is equiregular, but not the Baouendi–Grushin example.

Nonholonomic orders. The nonholonomic order of a smooth germ of function is given by the formula

$$\operatorname{ord}_q(f) = \min\{s \in \mathbb{N} : \text{there exists } i_1, \ldots, i_s \in \{1, \ldots, m\} \text{ such that } (X_{i_1} \cdots X_{i_s} f)(q) \neq 0\},$$

where we adopt the convention that $\min \varnothing = +\infty$.

The nonholonomic order of a smooth germ of vector field X at q, denoted by $\operatorname{ord}_q(X)$, is the real number defined by

$$\operatorname{ord}_q(X) = \sup\{\sigma \in \mathbb{R} : \operatorname{ord}_q(Xf) \geqslant \sigma + \operatorname{ord}_q(f) \text{ for all } f \in C^\infty(q)\}.$$

For example, it holds $\operatorname{ord}_q([X, Y]) \geqslant \operatorname{ord}_q(X) + \operatorname{ord}_q(Y)$ and $\operatorname{ord}_q(fX) \geqslant \operatorname{ord}_q(f) + \operatorname{ord}_q(X)$. As a consequence, every X which has the property that $X(q') \in \mathcal{D}^i_{q'}$ for any q' in a neighborhood of q is of nonholonomic order $\geqslant -i$.

Privileged coordinates. Locally around $q \in M$, it is possible to define a set of so-called "privileged coordinates" of M; see [Bellaïche 1996].

A family $(Z_1, \ldots, Z_n)$ of n vector fields is said to be adapted to the sub-Riemannian flag at q if it is a frame of $T_q M$ at q and if $Z_i(q) \in \mathcal{D}^{w_i(q)}_q$ for any $i \in \{1, \ldots, n\}$. In other words, for any $i \in \{1, \ldots, r(q)\}$, the vectors $Z_1, \ldots, Z_{n_i(q)}$ at q span $\mathcal{D}^i_q$.

A system of privileged coordinates at q is a system of local coordinates $(x_1, \ldots, x_n)$ such that

$$\operatorname{ord}_q(x_i) = w_i \quad \text{for } 1 \leqslant i \leqslant n. \tag{25}$$

In particular, for privileged coordinates, we have $\partial_{x_i} \in \mathcal{D}^{w_i(q)}_q \setminus \mathcal{D}^{w_i(q)-1}_q$ at q, meaning that privileged coordinates are adapted to the flag.

Example: *exponential coordinates of the second kind.* Choose an adapted frame $(Z_1, \dots, Z_n)$ at q. It is proved in [Jean 2014, Appendix B] that the inverse of the local diffeomorphism

$$(x_1, \dots, x_n) \mapsto \exp(x_1 Z_1) \circ \cdots \circ \exp(x_n Z_n)(q)$$

defines privileged coordinates at q, called exponential coordinates of the second kind.

Dilations. We consider a chart of privileged coordinates at q given by a smooth mapping $\psi_q : U \to \mathbb{R}^n$, where U is a neighborhood of q in M, with $\psi_q(q) = 0$. For every $\varepsilon \in \mathbb{R}\setminus\{0\}$, we consider the dilation $\delta_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\delta_\varepsilon(x) = (\varepsilon^{w_i(q)} x_1, \dots, \varepsilon^{w_n(q)} x_n)$$

for every $x = (x_1, \dots, x_n)$. A dilation δ_ε acts also on functions and vector fields on $\mathbb{R}^n$ by pull-back: $\delta_\varepsilon^* f = f \circ \delta_\varepsilon$ and $\delta_\varepsilon^* X$ is the vector field such that $(\delta_\varepsilon^* X)(\delta_\varepsilon^* f) = \delta_\varepsilon^*(Xf)$ for any $f \in C^1(\mathbb{R}^n)$. In particular, for any vector field X of nonholonomic order k, it holds $\delta_\varepsilon^* X = \varepsilon^{-k} X$.

Nilpotent approximation. Fix a system of privileged coordinates $(x_1, \dots, x_n)$ at q. Given a sequence of integers $\alpha = (\alpha_1, \dots, \alpha_n)$, we define the weighted degree of $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ to be

$$w(\alpha) = w_1(q)\alpha_1 + \cdots + w_n(q)\alpha_n.$$

Coming back to the vector fields $X_1, \dots, X_m$, we can write the Taylor expansion

$$X_i(x) \sim \sum_{\alpha, j} a_{\alpha,j} x^\alpha \partial_{x_j}. \tag{26}$$

Since $X_i \in \mathcal{D}$, its nonholonomic order is necessarily -1; hence it holds $w(\alpha) \geqslant w_j(q) - 1$ if $a_{\alpha,j} \neq 0$. Therefore, we may write X_i as a formal series

$$X_i = X_i^{(-1)} + X_i^{(0)} + X_i^{(1)} + \cdots,$$

where $X_i^{(s)}$ is a homogeneous vector field of degree s, meaning that

$$\delta_\varepsilon^* (\psi_q)_* X_i^{(s)} = \varepsilon^s (\psi_q)_* X_i^{(s)}.$$

We set $\widehat{X}_i^q = (\psi_q)_* X_i^{(-1)}$ for $1 \leqslant i \leqslant m$. Then $\widehat{X}_i^q$ is homogeneous of degree -1 with respect to dilations, i.e., $\delta_\varepsilon^* \widehat{X}_i^q = \varepsilon^{-1} \widehat{X}_i^q$ for any $\varepsilon \neq 0$. Each $\widehat{X}_i^q$ may be seen as a vector field on $\mathbb{R}^n$ thanks to the coordinates $(x_1, \dots, x_n)$. Moreover,

$$\widehat{X}_i^q = \lim_{\varepsilon \to 0} \varepsilon \delta_\varepsilon^* (\psi_q)_* X_i$$

in the C^∞ topology; all derivatives uniformly converge on compact subsets. For $\varepsilon > 0$ small enough we have

$$X_i^\varepsilon := \varepsilon \delta_\varepsilon^* (\psi_q)_* X_i = \widehat{X}_i^q + \varepsilon R_i^\varepsilon,$$

where R_i^ε depends smoothly on ε for the C^∞ topology; see also [Agrachev et al. 2020, Lemma 10.58]. An important property is that $(\widehat{X}_1^q, \dots, \widehat{X}_m^q)$ generates a nilpotent Lie algebra of step $r(q)$; see [Jean 2014, Proposition 2.3].

The nilpotent approximation of $X_1, \dots, X_m$ at q is then defined as $\widehat{M}^q \simeq \mathbb{R}^n$ endowed with the vector fields $\widehat{X}_1^q, \dots, \widehat{X}_m^q$. It is important to note that the nilpotent approximation depends on the initial choice of privileged coordinates. For an explicit example of computation of nilpotent approximation; see [Jean 2014, Example 2.8].

3.2. ***Reduction to the nilpotent case.*** In this section, we show the following.

Lemma 18. *Let $X_1, \dots, X_m$ be smooth vector fields on M satisfying Hörmander's condition, and let $q \in M$. If the property* (P) *holds at point $0 \in \mathbb{R}^n$ for the nilpotent approximation $\widehat{X}_1^q, \dots, \widehat{X}_m^q$, then the property* (P) *holds at point q for $X_1, \dots, X_m$.*

Note that the above lemma is true for any nilpotent approximation $\widehat{X}_1^q, \dots, \widehat{X}_m^q$ at q, i.e., for any choice of privileged coordinates (see Section 3.1).

Proof of Lemma 18. We use the notation h_Z for the momentum map associated with the vector field Z (see Section 1.4). We use the notation of Section 3.1, in particular the coordinate chart ψ_q.

We set $Y_i = (\psi_q)_* X_i$ and $X_i^\varepsilon = \varepsilon \delta_\varepsilon^* Y_i$ which is a vector field on $\mathbb{R}^n$. Recall that

$$X_i^\varepsilon = \widehat{X}_i^q + \varepsilon R_i^\varepsilon,$$

where R_i^ε depends smoothly on ε for the C^∞ topology. Therefore, using the homogeneity of $\widehat{X}_i^q$, we get, for any $\varepsilon > 0$,

$$Y_i = \frac{1}{\varepsilon}(\delta_\varepsilon)_* X_i^\varepsilon = \frac{1}{\varepsilon}(\delta_\varepsilon)_*(\widehat{X}_i^q + \varepsilon R_i^\varepsilon) = \widehat{X}_i^q + (\delta_\varepsilon)_* R_i^\varepsilon. \tag{27}$$

The vector field $(\delta_\varepsilon)_* R_i^\varepsilon(x)$ does not depend on ε and has a size which tends uniformly to 0 as $x \to 0 \in \widehat{M}^q \simeq \mathbb{R}^n$. Recall that the Hamiltonian $\widehat{H}$ associated to the vector fields $\widehat{X}_i^q$ is given by

$$\widehat{H} = \sum_{i=1}^m h_{\widehat{X}_i^q}^2.$$

Similarly, we set

$$H = \sum_{i=1}^m h_{Y_i}^2.$$

We note that (27) gives

$$h_{Y_i} = h_{\widehat{X}_i^q} + h_{(\delta_\varepsilon)_* R_i^\varepsilon}.$$

Hence

$$\vec{H} = 2\sum_{i=1}^m h_{Y_i} \vec{h}_{Y_i} = \vec{\widehat{H}} + \vec{\Theta}, \tag{28}$$

where $\vec{\Theta}$ is a smooth vector field on $T^*\mathbb{R}^n$ such that

$$\|(d\pi \circ \vec{\Theta})(x,\xi)\| \leqslant C\|x\| \tag{29}$$

when $\|x\| \to 0$ (independently of ξ), where $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$ is the canonical projection. This last point comes from the smooth dependence of R_i^ε on ε for the C^∞ topology (uniform convergence of all derivatives on compact subsets of $\mathbb{R}^n$).

Given the projection of an integral curve $c(\cdot)$ of $\vec{H}$, we denote by $\hat{c}(\cdot)$ the projection of the integral curve of $\vec{\widehat{H}}$ with same initial covector. Combining (28) and (29), and using Gronwall's lemma, we obtain the following result:

Fix $T_0 > 0$. For any neighborhood V of 0 in $\mathbb{R}^n$,

there exists another neighborhood V' of 0 such that if $c_{|[0,T_0]} \subset V'$, then $\hat{c}_{|[0,T_0]} \subset V$.

Therefore, if the property (P) holds at $0 \in \mathbb{R}^n$ for $\widehat{X}_1^q, \dots, \widehat{X}_m^q$, then it holds also at $0 \in \mathbb{R}^n$ for the vector fields $Y_1, \dots, Y_m$.

Using that $X_i = \psi_q^* Y_i$, we can pull back the result to M and obtain that the property (P) holds at point q for $X_1, \dots, X_m$, which concludes the proof of Proposition 17. □

Thanks to Lemma 18, it is sufficient to prove the property (P) under the additional assumption that

$$M \subset \mathbb{R}^n \text{ and } \mathrm{Lie}(X_1, \dots, X_m) \text{ is nilpotent.} \tag{30}$$

In all that follows, we assume that this is the case.

3.3. ***End of the proof of Proposition 17.*** Let us finish the proof of Proposition 17. Our ideas are inspired by [Agrachev and Gauthier 2001, Section 6].

First step: *reduction to the constant Goh matrix case.* We consider an adapted frame $Y_1, \dots, Y_n$ at q. We take exponential coordinates of the second kind at q; we consider the inverse ψ_q of the diffeomorphism

$$(x_1, \dots, x_n) \mapsto \exp(x_1 Y_1) \cdots \exp(x_n Y_n)(q).$$

Then we write the Taylor expansion (26) of $X_1, \dots, X_m$ in these coordinates. Thanks to Lemma 18, we can assume that all terms in these Taylor expansions have nonholonomic order -1. We denote by ξ_i the dual variable of x_i. We use the notation $n_1, n_2, \dots$ introduced in Section 3.1, and we make a strong use of (25).

Claim 1. *If a normal geodesic* $(x(t), \xi(t))_{t\in\mathbb{R}}$ *has initial momentum satisfying* $\xi_k(0) = 0$ *for any* $k \geqslant n_2+1$, *then* $\dot{\xi}_k \equiv 0$ *for any* $k \geqslant n_1 + 1$, *and in particular* $\xi_k \equiv 0$ *for any* $k \geqslant n_2 + 1$.

Proof. We write

$$X_j(x) = \sum_{i=1}^{n} a_{ij}(x)\partial_{x_i}, \quad j = 1, \dots, m,$$

where the a_{ij} are homogeneous polynomials. We have

$$g^*(x, \xi) = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij}(x)\xi_i \right)^2. \tag{31}$$

Let $k \geqslant n_2+1$, which means that x_k has nonholonomic order $\geqslant 3$. If $a_{ij}(x)$ depends on x_k, then necessarily $i \geqslant n_3+1$, since $a_{ij}(x)\partial_{x_i}$ has nonholonomic order -1. Thus, writing explicitly $\dot{\xi}_k = -\partial g^*/\partial x_k$ thanks to (31), there is in front of each term a factor ξ_i for some i which is in particular $\geqslant n_2 + 1$. By Cauchy uniqueness, we deduce that $\xi_k \equiv 0$ for any $k \geqslant n_2 + 1$.

Now, let $k \geqslant n_1 + 1$, which means that x_k has nonholonomic order $\geqslant 2$. If $a_{ij}(x)$ depends on x_k, then necessarily $i \geqslant n_2+1$, since $a_{ij}(x)\partial_{x_i}$ has nonholonomic order -1. Thus, writing explicitly $\dot{\xi}_k = -\partial g^*/\partial x_k$ thanks to (31), there is in front of each term a factor ξ_i for some i which is $\geqslant n_2 + 1$. It is null by the previous conclusion; hence $\dot{\xi}_k \equiv 0$. □

The previous claim will help us to reduce the complexity of the vector fields X_i once again (after the first reduction provided by Lemma 18). Let us consider, for any $1 \leqslant j \leqslant m$, the vector field

$$X_j^{\mathrm{red}} = \sum_{i=1}^{n_2} a_{ij}(x)\partial_{x_i}, \tag{32}$$

where the sum is taken only up to n_2. We also consider the reduced Hamiltonian on T^*M

$$g^*_{\mathrm{red}} = \sum_{j=1}^{m} h^2_{X_j^{\mathrm{red}}}.$$

Claim 2. *If $X_1^{\mathrm{red}}, \dots, X_m^{\mathrm{red}}$ satisfy property* (P) *at q, then $X_1, \dots, X_m$ satisfy property* (P) *at q.*

Proof. Let us assume that $X_1^{\mathrm{red}}, \dots, X_m^{\mathrm{red}}$ satisfy property (P) at q. Let $T_0 > 0$ and let $(x^{\mathrm{red},\varepsilon}(0), \xi^{\mathrm{red},\varepsilon}(0))$ be initial data for the Hamiltonian system associated to g^*_{red} which yield speed-1 normal geodesics $(x^{\mathrm{red},\varepsilon}(t), \xi^{\mathrm{red},\varepsilon}(t))$ such that $x^{\mathrm{red},\varepsilon}(t) \to q$ uniformly over $(0, T_0)$ as $\varepsilon \to 0$.

We can assume without loss of generality that $\xi_i^{\mathrm{red},\varepsilon}(0) = 0$ for any $i \geqslant n_2 + 1$, since these momenta (preserved under the reduced Hamiltonian evolution) do not change the projection $x^{\mathrm{red},\varepsilon}(t)$ of the normal geodesic. We consider $(x^\varepsilon(0), \xi^\varepsilon(0)) = (x^{\mathrm{red},\varepsilon}(0), \xi^{\mathrm{red},\varepsilon}(0))$ as initial data for the (nonreduced) Hamiltonian evolution associated to g^*. Then we notice that $\xi_k^\varepsilon \equiv 0$ for $k \geqslant n_2 + 1$ thanks to Claim 1. It follows that when $i \leqslant n_2$, we have $x_i^\varepsilon(t) = x_i^{\mathrm{red},\varepsilon}(t)$; i.e., the coordinate x_i is the same for the reduced and the nonreduced Hamiltonian evolution.

Finally, we take k such that $n_2 + 1 \leqslant k \leqslant n_3$. Since g^* is given by (31), we have

$$\dot{x}_k^\varepsilon = \frac{\partial g^*}{\partial \xi_k} = 2\sum_{j=1}^{m} a_{kj}(x^\varepsilon)\left(\sum_{i=1}^{n} a_{ij}(x^\varepsilon)\xi_i^\varepsilon\right). \tag{33}$$

But a_{kj} has necessarily nonholonomic order 2 since ∂_{x_k} has nonholonomic order -3. Thus, $a_{kj}(x)$ is a nonconstant homogeneous polynomial in $x_1, \dots, x_{n_2}$. Since $x_1^\varepsilon, \dots, x_{n_2}^\varepsilon$ converge to q uniformly over $(0, T_0)$ as $\varepsilon \to 0$, it is also the case of x_k^ε according to (33), noticing that

$$\left|\sum_{i=1}^{n} a_{ij}(x^\varepsilon)\xi_i^\varepsilon\right| \leqslant (g^*)^{\frac{1}{2}} = \tfrac{1}{2}$$

for any j. In other words, $x_{n_2+1}^\varepsilon, \dots, x_{n_3}^\varepsilon$ also converge to q uniformly over $(0, T_0)$ as $\varepsilon \to 0$.

We can repeat this argument successively for $k \in \{n_3 + 1, \dots, n_4\}$, $k \in \{n_4 + 1, \dots, n_5\}$, etc., and we finally obtain the result: for any $1 \leqslant k \leqslant n$, x_k^ε converges to q uniformly over $(0, T_0)$ as $\varepsilon \to 0$. □

Thanks to the previous claim, we are now reduced to proving Proposition 17 for the vector fields $X_1^{\mathrm{red}}, \dots, X_m^{\mathrm{red}}$. In order to keep notation as simple as possible, we simplify to $X_1, \dots, X_m$; i.e., we drop the upper notation “red”. Also, without loss of generality we assume that $q = 0$.

If we choose our normal geodesics so that $x(0)=0$, then $x_i \equiv 0$ for any $i \geqslant n_2+1$ thanks to (32). In other words, we forget the coordinates $x_{n_2+1}, \dots, x_n$ in the sequel, since they all vanish.[2]

Second step: *conclusion of the proof.* Now, we write the normal extremal system in its "control" form. We refer the reader to [Agrachev et al. 2020, Chapter 4]. We have

$$\dot{x}(t)=\sum_{i=1}^{m} u_i(t) X_i(x(t)), \tag{34}$$

where the u_i are the controls, explicitly given by

$$u_i(t)=2h_{X_i}(x(t),\xi(t)) \tag{35}$$

since $(x(t),\xi(t))=e^{t\vec{g}^*}(0,\xi_0)$. Thanks to (32), we rewrite (34) as

$$\dot{x}(t)=F(x(t))u(t), \tag{36}$$

where $F=(a_{ij})$, which has size $n_2\times m$, and $u={}^t(u_1,\dots,u_m)$. Differentiating (35), we have the complementary equation

$$\dot{u}(t)=G(x(t),\xi(t))u(t),$$

where G is the Goh matrix

$$G=(2\{h_{X_j},h_{X_i}\})_{1\leqslant i,j\leqslant m}$$

(it differs from the usual Goh matrix by a factor -2 due to the absence of factor $\frac{1}{2}$ in the Hamiltonian g^* in our notation).

Let us prove that $G(t)$ is constant in t. Fix $1\leqslant j,j'\leqslant m$. We notice that in (32), a_{ij} is a constant (independent of x) as soon as $1\leqslant i\leqslant n_1$ since ∂_{x_i} has weight -1. This implies

$$[X_j,X_{j'}] \text{ is spanned by the vector fields } \partial_{x_{n_1+1}},\partial_{x_{n_1+2}},\dots,\partial_{x_{n_2}}. \tag{37}$$

Putting this into the relation $\{h_{X_j},h_{X_{j'}}\}=h_{[X_j,X_{j'}]}$, and using that the dual variables ξ_k for $n_1+1\leqslant k\leqslant n_2$ are preserved under the Hamiltonian evolution (due to Claim 1), we get that $G(t)\equiv G$ is constant in t.

We know that $G\neq 0$ and that G is antisymmetric. The whole control space $\mathbb{R}^m$ is the direct sum of the image of G and the kernel of G, and G is nondegenerate on its image. We take u_0 in an invariant plane of G; in other words its projection on the kernel of G vanishes (see Remark 19). We denote by $\widetilde{G}$ the restriction of G to this invariant plane. We also assume that u_0, decomposed as $u_0=(u_{01},\dots,u_{0m})\in\mathbb{R}^m$, satisfies $\sum_{i=1}^m u_{0i}^2=\frac{1}{4}$. Then $u(t)=e^{t\widetilde{G}}u_0$ and since $e^{t\widetilde{G}}$ is an orthogonal matrix, we have $\|e^{t\widetilde{G}}u_0\|=\|u_0\|$. We have by integration by parts

$$x(t)=\int_0^t F(x(s))e^{s\widetilde{G}}u_0\,ds=F(x(t))\widetilde{G}^{-1}(e^{t\widetilde{G}}-I)u_0-\int_0^t \frac{d}{ds}(F(x(s)))\widetilde{G}^{-1}(e^{s\widetilde{G}}-I)u_0\,ds. \tag{38}$$

[2] Note that this is the case only because we are now working with the reduced Hamiltonian evolution; otherwise, under the original Hamiltonian evolution associated to (31), the x_i (for $i\geqslant n_2+1$) remain small according to Claim 2, but do not necessarily vanish.

Let us now choose the initial data of our family of normal geodesics (indexed by ε). The starting point $x^\varepsilon(0) = 0$ is the same for any ε; we only have to specify the initial covectors $\xi^\varepsilon = \xi^\varepsilon(0) \in T_0^*\mathbb{R}^m$. For any $i = 1, \ldots, m$, we impose that

$$\langle \xi^\varepsilon, X_i \rangle = u_{0i}. \tag{39}$$

It follows that $g^*(x(0), \xi^\varepsilon(0)) = \sum_{i=1}^m u_{0i}^2 = \frac{1}{4}$ for any $\varepsilon > 0$. Now, we notice that $\mathrm{Span}(X_1, \ldots, X_m)$ is in direct sum with the Span of the $[X_i, X_j]$ for i, j running over $1, \ldots, m$ (this follows from (37)). Fixing $G^0 \neq 0$ an antisymmetric matrix and $\widetilde{G}^0$ its restriction to an invariant plane, we can specify, simultaneously to (39), that

$$\langle \xi^\varepsilon, 2[X_j, X_i] \rangle = \varepsilon^{-1} G^0_{ij}.$$

Then $x^\varepsilon(t)$ is given by (38) applied with $\widetilde{G} = \varepsilon^{-1}\widetilde{G}^0$, which brings a factor ε in front of (38).

Recall finally that the coefficients a_{ij} which compose F have nonholonomic order 0 or 1; thus they are degree-1 (or constant) homogeneous polynomials in $x_1, \ldots, x_{n_1}$. Thus $\frac{d}{ds}(F(x(s)))$ is a linear combination of $\dot{x}_i(s)$ which we can rewrite thanks to (36) as a combination with bounded coefficients $\big(\text{since } \sum_{i=1}^m u_i^2 = \frac{1}{4}\big)$ of the $x_i(s)$. Hence, applying the Gronwall lemma in (38), we get $\|x^\varepsilon(t)\| \leqslant C\varepsilon$, which concludes the proof.

Remark 19. Let us explain why we choose u_0 to be in an invariant plane of G. If the projection of u_0 to the kernel of G is nonzero then the primitive of the exponential of $e^{(t/\varepsilon)G_0}u_0$ contains a linear term that does not depend on ε. Then the corresponding trajectory follows a singular curve; see [Agrachev et al. 2020, Chapter 4] for a definition. This means we find normal geodesics which spiral around a singular curve and do not remain close to their initial point over $(0, T_0)$, although their initial covector is "high in the cylinder bundle U^*M". For example, for the Hamiltonian $\xi_1^2 + (\xi_2 + x_1^2\xi_3)^2$ associated to the "Martinet" vector fields $X_1 = \partial_{x_1}$, $X_2 = \partial_{x_2} + x_1^2\partial_{x_3}$ in $\mathbb{R}^3$, there exist normal geodesics which spiral around the singular curve $(t, 0, 0)$.

Remark 20. The normal geodesics constructed above lose their optimality quickly, in the sense that their first conjugate point and their cut-point are close to q.

4. Proofs

4.1. *Proof of Theorem 2.* In this section, we conclude the proof of Theorem 2.

Fix a point q in the interior of $M \setminus \omega$ and $1 \leqslant i, j \leqslant m$ such that $[X_i, X_j](q) \notin \mathcal{D}_q$. Fix also an open neighborhood V of q in M such that $V \subset M \setminus \omega$. Fix V' an open neighborhood of q in M such that $\overline{V}' \subset V$, and fix also $T_0 > 0$.

As already explained in Section 1.3, to conclude the proof of Theorem 2, we use Proposition 16 applied to the particular normal geodesics constructed in Proposition 17.

By Proposition 17, we know that there exists a normal geodesic $t \mapsto x(t)$ such that $x(t) \in V'$ for any $t \in (0, T_0)$. It is the projection of a bicharacteristic $(x(t), \xi(t))$ and since it is nonstationary and travels at speed 1, it holds $g^*(x(t), \xi(t)) = \frac{1}{4}$. We denote by $(u_k)_{k\in\mathbb{N}}$ a sequence of solutions of (12) as in

Proposition 16 whose energy at time t concentrates on $x(t)$ for $t \in (0, T_0)$. Because of (22), we know that

$$\|(u_k(0), \partial_t u_k(0))\|_{\mathcal{H}\times L^2} \geqslant c > 0$$

uniformly in k.

Therefore, in order to establish Theorem 2, it is sufficient to show that

$$\int_0^{T_0}\int_\omega |\partial_t u_k(t,x)|^2 \, d\mu(x)\, dt \xrightarrow[k\to+\infty]{} 0. \tag{40}$$

Since $x(t) \in V'$ for any $t \in (0, T_0)$, we get that for V_t chosen sufficiently small for any $t \in (0, T_0)$, the inclusion $V_t \subset V$ holds (see Proposition 16 for the definition of V_t). Combining this last remark with (23), we get (40), which concludes the proof of Theorem 2.

4.2. *Proof of Corollary 4.* We endow the topological dual $\mathcal{H}(M)'$ with the norm

$$\|v\|_{\mathcal{H}(M)'} = \|(-\Delta)^{-1/2} v\|_{L^2(M)}.$$

The following proposition is standard; see, e.g., [Tucsnak and Weiss 2009; Le Rousseau et al. 2017].

Lemma 21. *Let $T_0 > 0$ and $\omega \subset M$ be a measurable set. Then the following two observability properties are equivalent*:

(P1) *There exists C_{T_0} such that, for any $(v_0, v_1) \in D((-\Delta)^{1/2}) \times L^2(M)$, the solution*

$$v \in C^0(0, T_0; D((-\Delta)^{\frac12})) \cap C^1(0, T_0; L^2(M))$$

of (1) *satisfies*

$$\int_0^{T_0}\int_\omega |\partial_t v(t,q)|^2 \, d\mu(q)\, dt \geqslant C_{T_0} \|(v_0, v_1)\|_{\mathcal{H}(M)\times L^2(M)}. \tag{41}$$

(P2) *There exists C_{T_0} such that, for any $(v_0, v_1) \in L^2(M) \times D((-\Delta)^{-1/2})$, the solution*

$$v \in C^0(0, T_0; L^2(M)) \cap C^1(0, T_0; D((-\Delta)^{-\frac12}))$$

of (1) *satisfies*

$$\int_0^{T_0}\int_\omega |v(t,q)|^2 \, d\mu(q)\, dt \geqslant C_{T_0} \|(v_0, v_1)\|^2_{L^2\times\mathcal{H}(M)'}. \tag{42}$$

Proof. Let us assume that (P2) holds. Let u be a solution of (1) with initial conditions $(u_0, u_1) \in D((-\Delta)^{1/2}) \times L^2(M)$. We set $v = \partial_t u$, which is a solution of (1) with initial data $v_{|t=0} = u_1 \in L^2(M)$ and $\partial_t v_{|t=0} = \Delta u_0 \in D((-\Delta)^{-1/2})$. Since $\|(v_0, v_1)\|_{L^2\times\mathcal{H}(M)'} = \|(u_1, \Delta u_0)\|_{L^2\times\mathcal{H}(M)'} = \|(u_0, u_1)\|_{\mathcal{H}(M)\times L^2}$, applying the observability inequality (42) to $v = \partial_t u$, we obtain (41). The proof of the other implication is similar. $\square$

Finally, using Theorem 2, Lemma 21 and the standard HUM method [Lions 1988], we get Corollary 4.

4.3. ***Proof of Theorem 11.*** We consider the space of functions $u \in C^\infty([0,T] \times M_H)$ such that

$$\int_{M_H} u(t,\cdot)\,d\mu = 0$$

for any $t \in [0,T]$, and we denote by $\mathcal{H}_T$ its completion for the norm $\|\cdot\|_{\mathcal{H}_T}$ induced by the scalar product

$$(u,v)_{\mathcal{H}_T} = \int_0^T \int_{M_H} (\partial_t u \partial_t v + (X_1 u)(X_1 v) + (X_2 u)(X_2 v))\,d\mu\,dt.$$

We consider also the topological dual $\mathcal{H}_0'$ of the space $\mathcal{H}_0$ (see Section 1.5).

Lemma 22. *The injections $\mathcal{H}_0 \hookrightarrow L^2(M_H)$, $L^2(M_H) \hookrightarrow \mathcal{H}_0'$ and $\mathcal{H}_T \hookrightarrow L^2((0,T) \times M_H)$ are compact.*

Proof. Let $(\varphi_k)_{k\in\mathbb{N}}$ be an orthonormal basis of real eigenfunctions of $L^2(M_H)$, labeled with increasing eigenvalues $0 = \lambda_0 < \lambda_1 \leqslant \cdots \leqslant \lambda_k \to +\infty$, so that $-\Delta_H \varphi_k = \lambda_k \varphi_k$. The fact that $\lambda_1 > 0$, which will be used in the sequel, can be proved as follows: If $-\Delta_H \varphi = 0$ then $\int_{M_H} ((X_1\varphi)^2 + (X_2\varphi)^2)\,d\mu = 0$ and, since $\varphi \in C^\infty(M_H)$ by hypoelliptic regularity, we get $X_1\varphi(x) = X_2\varphi(x) = 0$ for any $x \in M_H$. Hence, $[X_1, X_2]\varphi \equiv 0$, and all together, this proves that φ is constant; thus $\lambda_1 > 0$.

We prove the last injection. Let $u \in \mathcal{H}_T$. Writing $u(t,\cdot) = \sum_{k=1}^\infty a_k(t)\varphi_k(\cdot)$ (note that there is no 0-mode since $u(t,\cdot)$ has null average), we see that

$$\|u\|_{\mathcal{H}_T}^2 \geqslant (-\Delta_H u, u)_{L^2((0,T)\times M_H)} = \sum_{k=1}^\infty \lambda_k \|a_k\|_{L^2((0,T))}^2 \geqslant \lambda_1 \sum_{k=1}^\infty \|a_k\|_{L^2((0,T))}^2 = \lambda_1 \|u\|_{L^2((0,T)\times M_H)}^2,$$

and thus $\mathcal{H}_T$ embeds continuously into $L^2((0,T)\times M_H)$. Then, using a classical subelliptic estimate (see [Hörmander 1967; Rothschild and Stein 1976, Theorem 17]), we know that there exists $C > 0$ such that

$$\|u\|_{H^{1/2}((0,T)\times M_H)} \leqslant C(\|u\|_{L^2((0,T)\times M_H)} + \|u\|_{\mathcal{H}_T}).$$

Together with the previous estimate, we obtain that, for any $u \in \mathcal{H}_T$, $\|u\|_{H^{1/2}((0,T)\times M_H)} \leqslant C\|u\|_{\mathcal{H}_T}$. Then, the result follows from the fact that the injection $H^{1/2}((0,T)\times M_H) \hookrightarrow L^2((0,T)\times M_H)$ is compact.

The proof of the compact injection $\mathcal{H}_0 \hookrightarrow L^2(M_H)$ is similar, and the compact injection $L^2(M_H) \hookrightarrow \mathcal{H}_0'$ follows by duality. □

Proof of Theorem 11. In this proof, we use the notation $P = \partial_{tt}^2 - \Delta_H$. For the sake of a contradiction, suppose that there exists a sequence $(u^k)_{k\in\mathbb{N}}$ of solutions of the wave equation such that $\|(u_0^k, u_1^k)\|_{\mathcal{H}\times L^2} = 1$ for any $k \in \mathbb{N}$ and

$$\|(u_0^k, u_1^k)\|_{L^2\times\mathcal{H}_0'} \to 0, \qquad \int_0^T |(\mathrm{Op}(a)\partial_t u^k, \partial_t u^k)_{L^2(M_H,\mu)}|\,dt \to 0 \tag{43}$$

as $k \to +\infty$. Following the strategy of [Tartar 1990; Gérard 1991], our goal is to associate a defect measure to the sequence $(u^k)_{k\in\mathbb{N}}$. Since the functional spaces involved in our result are unusual, we give the argument in detail.

First, up to extraction of a subsequence which we omit, (u_0^k, u_1^k) converges weakly in $\mathcal{H}_0 \times L^2(M_H)$ and, using the first convergence in (43) and the compact embedding $\mathcal{H}_0 \times L^2(M_H) \hookrightarrow L^2(M_H) \times \mathcal{H}_0'$, we get that $(u_0^k, u_1^k) \rightharpoonup 0$ in $\mathcal{H}_0 \times L_0^2$. Using the continuity of the solution with respect to the initial data, we obtain that $u^k \rightharpoonup 0$ weakly in $\mathcal{H}_T$. Using Lemma 22, we obtain $u^k \to 0$ strongly in $L^2((0,T)\times M_H)$.

Fix $B \in \Psi^0_{\mathrm{phg}}((0,T)\times M_H)$. We have

$$\begin{aligned}(Bu^k, u^k)_{\mathcal{H}_T} &= \int_0^T\int_{M_H} \big((\partial_t Bu^k)(\partial_t u^k) + (X_1Bu^k)(X_1u^k) + (X_2Bu^k)(X_2u^k)\big)\, d\mu(q)\, dt \\ &= \int_0^T\int_{M_H} \big(([\partial_t, B]u^k)(\partial_t u^k) + ([X_1, B]u^k)(X_1u^k) + ([X_2, B]u^k)(X_2u^k)\big)\, d\mu(q)\, dt \\ &\qquad + \int_0^T\int_{M_H} \big((B\partial_t u^k)(\partial_t u^k) + (BX_1u^k)(X_1u^k) + (BX_2u^k)(X_2u^k)\big)\, d\mu(q)\, dt. \qquad (44)\end{aligned}$$

Since $[\partial_t, B]\in\Psi^0_{\mathrm{phg}}((0,T)\times M_H)$, $[X_j, B]\in\Psi^0_{\mathrm{phg}}((0,T)\times M_H)$ and $u^k\to 0$ strongly in $L^2((0,T)\times M_H)$, the first of the two lines in (44) converges to 0 as $k\to+\infty$. Moreover, the last line is bounded uniformly in k since $B\in\Psi^0_{\mathrm{phg}}((0,T)\times M_H)$. Hence $(Bu^k, u^k)_{\mathcal{H}_T}$ is uniformly bounded. By a standard diagonal extraction argument (see [Gérard 1991] for example), there exists a subsequence, which we still denote by $(u^k)_{k\in\mathbb{N}}$ such that (Bu^k, u^k) converges for any B of principal symbol b in a countable dense subset of $C_c^\infty((0,T)\times M_H)$. Moreover, the limit only depends on the principal symbol b, and not on the full symbol.

Let us now prove that

$$\liminf_{k\to+\infty} (Bu^k, u^k)_{\mathcal{H}_T} \geqslant 0 \qquad (45)$$

when $b\geqslant 0$. With a bracket argument as in (44), we see that it is equivalent to proving that the liminf as $k\to+\infty$ of the quantity

$$Q_k(B) = (B\partial_t u^k, \partial_t u^k)_{L^2} + (BX_1u^k, X_1u^k)_{L^2} + (BX_2u^k, X_2u^k)_{L^2} \qquad (46)$$

is $\geqslant 0$. But there exists $B'\in\Psi^0_{\mathrm{phg}}((0,T)\times M_H)$ such that $B'-B\in\Psi^{-1}_{\mathrm{phg}}((0,T)\times M_H)$ and B' is positive (this is the so-called Friedrichs quantization, see for example [Taylor 1974, Chapter VII]). Then, $\liminf_{k\to+\infty} Q_k(B')\geqslant 0$, and $Q_k(B'-B)\to 0$ since $(B'-B)\partial_t\in\Psi^0_{\mathrm{phg}}((0,T)\times M_H)$ and $u^k\to 0$ strongly in $L^2((0,T)\times M_H)$. It immediately implies that (45) holds.

Therefore, setting $p=\sigma_p(P)$ and denoting by $\mathcal{C}(p)$ the characteristic manifold $\mathcal{C}(p)=\{p=0\}$, there exists a nonnegative Radon measure ν on $S^*(\mathcal{C}(p))=\mathcal{C}(p)/(0,+\infty)$ such that

$$(\mathrm{Op}(b)u^k, u^k)_{\mathcal{H}_T} \to \int_{S^*(\mathcal{C}(p))} b\, d\nu$$

for any $b\in S^0_{\mathrm{phg}}((0,T)\times M_H)$.

Let $C\in\Psi^{-1}_{\mathrm{phg}}((0,T)\times M_H)$ of principal symbol c. We have $\vec{p}c=\{p,c\}\in S^0_{\mathrm{phg}}((0,T)\times M_H)$ and, for any $k\in\mathbb{N}$,

$$((CP-PC)u^k, u^k)_{\mathcal{H}_T} = (CPu^k, u^k)_{\mathcal{H}_T} - (Cu^k, Pu^k)_{\mathcal{H}_T} = 0 \qquad (47)$$

since $Pu^k=0$. To be fully rigorous, the identity of the previous line, which holds for any solution $u\in\mathcal{H}_T$ of the wave equation, is first proved for smooth initial data since $Pu\notin\mathcal{H}_T$ in general, and then extended to general solutions $u\in\mathcal{H}_T$. Taking principal symbols in (47), we get $\langle\nu, \vec{p}c\rangle = 0$.

Therefore, denoting by $(\psi_s)_{s\in\mathbb{R}}$ the maximal solutions of

$$\frac{d}{ds}\psi_s(\rho) = \vec{p}(\psi_s(\rho)), \quad \rho \in T^*(\mathbb{R}\times M_H)$$

(see (7)), we get that, for any $s \in (0, T)$,

$$0 = \langle \nu, \vec{p}c \circ \psi_s\rangle = \left\langle \nu, \frac{d}{ds}c \circ \psi_s\right\rangle = \frac{d}{ds}\langle \nu, c\circ\psi_s\rangle$$

and hence

$$\langle \nu, c\rangle = \langle \nu, c\circ\psi_s\rangle. \tag{48}$$

We note here that the precise homogeneity of c (namely $c \in S^{-1}_{\mathrm{phg}}((0, T)\times M_H)$) does not matter since ν is a measure on the sphere bundle $S^*(\mathcal{C}(p))$. The identity (48) means that ν is invariant under the flow $\vec{p}$.

From the second convergence in (43), we can deduce that

$$\nu = 0 \quad \text{in } S^*(\mathcal{C}(p)) \cap T^*((0, T)\times \mathrm{Supp}(a)). \tag{49}$$

The proof of this fact, which is standard (see for example [Burq and Sun 2022, Section 6.2]), is given in Appendix C.

Let us prove that any normal geodesic of M_H with momentum $\xi \in V_\varepsilon^c$ enters ω in time at most $\kappa\varepsilon^{-1}$ for some $\kappa > 0$, which does not depend on ε. Indeed, the solutions of the bicharacteristic equations (10) with $g^* = \frac{1}{4}$ and $\xi_3 \neq 0$ are given by

$$x_1(t) = \frac{1}{2\xi_3}\cos(2\xi_3 t+\phi) + \frac{\xi_2}{\xi_3}, \quad x_2(t) = B - \frac{1}{2\xi_3}\sin(2\xi_3 t+\phi),$$
$$x_3(t) = C + \frac{t}{4\xi_3} + \frac{1}{16\xi_3^2}\sin(2(2\xi_3 t+\phi)) + \frac{\xi_2}{2\xi_3^2}\sin(2\xi_3 t+\phi),$$

where B, C, ξ_2, ξ_3 are constants. Since $\xi \in V_\varepsilon^c$ and $g^* = \frac{1}{4}$, it holds

$$\frac{1}{4|\xi_3|} \geqslant \frac{\varepsilon}{2}.$$

Hence, we can conclude using the expression for x_3 (whose derivative is roughly $(4|\xi_3|)^{-1}$) and the fact that $\omega = M_H\setminus B$ contains a horizontal strip. Note that if $\xi_3 = 0$, the expressions of $x_1(t), x_2(t), x_3(t)$ are much simpler and we can conclude similarly.

Hence, together with (49), the propagation property (48) implies that $\nu \equiv 0$. It follows that $\|u^k\|_{\mathcal{H}_T} \to 0$. By conservation of energy, it is a contradiction with the normalization $\|(u_0^k, u_1^k)\|_{\mathcal{H}\times L^2} = 1$. Hence, (11) holds. □

Appendix A: Pseudodifferential calculus

We denote by Ω an open set of a d-dimensional manifold (typically $d = n$ or $d = n+1$ with the notation of this paper) equipped with a smooth volume μ. We denote by q the variable in Ω, typically $q = x$ or $q = (t, x)$ with our notation.

Let $\omega_0 = dp \wedge dq$ be the canonical symplectic form on $T^*\Omega$ written in canonical coordinates (q, p). The Hamiltonian vector field $\vec{f}$ of a function $f \in C^\infty(T^*\Omega)$ is defined by the relation

$$\omega_0(\vec{f}, \cdot) = -df(\cdot).$$

In the coordinates (q, p), it reads

$$\vec{f} = \sum_{j=1}^{d} (\partial_{p_j} f)\partial_{q_j} - (\partial_{q_j} f)\partial_{p_j}.$$

In these coordinates, the Poisson bracket is

$$\{f, g\} = \omega_0(\vec{f}, \vec{g}) = \sum_{j=1}^{d} (\partial_{p_j} f)(\partial_{q_j} g) - (\partial_{q_j} f)(\partial_{p_j} g),$$

which is also equal to $\vec{f}g$ and $-\vec{g}f$.

Let $\pi : T^*\Omega \to \Omega$ be the canonical projection. We recall briefly some facts concerning pseudodifferential calculus, following [Hörmander 1985, Chapter 18].

We denote by $S^m_{\mathrm{hom}}(T^*\Omega)$ the set of homogeneous symbols of degree m with compact support in Ω. We also write $S^m_{\mathrm{phg}}(T^*\Omega)$ for the set of polyhomogeneous symbols of degree m with compact support in Ω. Hence, $a \in S^m_{\mathrm{phg}}(T^*\Omega)$ if $a \in C^\infty(T^*\Omega)$, $\pi(\mathrm{Supp}(a))$ is a compact of Ω, and there exist $a_j \in S^{m-j}_{\mathrm{hom}}(T^*\Omega)$ such that, for all $N \in \mathbb{N}$, $a - \sum_{j=0}^{N} a_j \in S^{m-N-1}_{\mathrm{phg}}(T^*\Omega)$. We denote by $\Psi^m_{\mathrm{phg}}(T^*\Omega)$ the space of polyhomogeneous pseudodifferential operators of order m on Ω, with a compactly supported kernel in $\Omega \times \Omega$. For $A \in \Psi^m_{\mathrm{phg}}(\Omega)$, we denote by $\sigma_p(A) \in S^m_{\mathrm{phg}}(T^*\Omega)$ the principal symbol of A. The subprincipal symbol is characterized by the action of pseudodifferential operators on oscillating functions: if $A \in \Psi^m_{\mathrm{phg}}(\Omega)$ and $f(q) = b(q)e^{ikS(q)}$ with b, S smooth and real-valued, then

$$\int_\Omega A(f)\bar{f}\, d\mu = k^m \int_\Omega \Big(\sigma_p(A)(q, S'(q)) + \frac{1}{k}\sigma_{\mathrm{sub}}(A)(q, S'(q))\Big)|f(q)|^2\, d\mu(q) + O(k^{m-2}).$$

A quantization is a continuous linear mapping

$$\mathrm{Op} : S^m_{\mathrm{phg}}(T^*\Omega) \to \Psi^m_{\mathrm{phg}}(\Omega)$$

satisfying $\sigma_p(\mathrm{Op}(a)) = a$. An example of quantization is obtained by using partitions of unity and, locally, the Weyl quantization, which is given in local coordinates by

$$\mathrm{Op}^W(a) f(q) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d_{q'} \times \mathbb{R}^d_p} e^{i\langle q-q', p\rangle} a\Big(\frac{q+q'}{2}, p\Big) f(q')\, dq'\, dp.$$

We have the following properties:

(1) If $A \in \Psi^l_{\mathrm{phg}}(\Omega)$ and $B \in \Psi^m_{\mathrm{phg}}(\Omega)$, then $[A, B] \in \Psi^{l+m-1}_{\mathrm{phg}}(\Omega)$ and $\sigma_p([A, B]) = \frac{1}{i}\{\sigma_p(a), \sigma_p(b)\}$.

(2) If X is a vector field on Ω and X^* is its formal adjoint in $L^2(\Omega, \mu)$, then $X^*X \in \Psi^2_{\mathrm{phg}}(\Omega)$, $\sigma_p(X^*X) = h_X^2$ and $\sigma_{\mathrm{sub}}(X^*X) = 0$.

(3) If $A \in \Psi^m_{\mathrm{phg}}(\Omega)$, then A maps continuously the space $H^s(\Omega)$ to the space $H^{s-m}(\Omega)$.

Appendix B: Proof of Proposition 12

In this appendix, we give a second proof of Proposition 12 written in a more elementary form than the one of Section 2.1. Let us first prove the result when $M \subset \mathbb{R}^n$, following the proof of [Ralston 1982]. The general case is addressed at the end of this section.

As in the proof of Section 2.1, we suppress the time variable t. Thus we use $x = (x_0, x_1, \dots, x_n)$, where $x_0 = t$. Similarly, $\xi = (\xi_0, \xi_1, \dots, \xi_n)$, where $\xi_0 = \tau$ previously. Let Γ be the curve given by $x(s) \in \mathbb{R}^{n+1}$. We insist on the fact that in the proof the bicharacteristics are parametrized by s, as in (7). We consider functions of the form

$$v_k(x) = k^{\frac{n}{4}-1} a_0(x) e^{ik\psi(x)}.$$

We would like to choose $\psi(x)$ such that for all $s \in \mathbb{R}$, $\psi(x(s))$ is real-valued and

$$\operatorname{Im} \frac{\partial^2 \psi}{\partial x_i \partial x_j}(x(s))$$

is positive definite on vectors orthogonal to $\dot{x}(s)$. Roughly speaking, $|e^{ik\psi(x)}|$ will then look like a Gaussian distribution on planes perpendicular to Γ in $\mathbb{R}^{n+1}$.

We first observe that $\partial_{tt}^2 v_k - \Delta v_k$ can be decomposed as

$$\partial_{tt}^2 v_k - \Delta v_k = (k^{\frac{n}{4}+1} A_1 + k^{\frac{n}{4}} A_2 + k^{\frac{n}{4}-1} A_3) e^{ik\psi}, \tag{50}$$

with

$$A_1(x) = p_2(x, \nabla\psi(x)) a_0(x), \quad A_2(x) = L a_0(x), \quad A_3(x) = \partial_{tt}^2 a_0(x) - \Delta a_0(x).$$

Here we have set

$$L a_0 = \frac{1}{i} \sum_{j=0}^{n} \frac{\partial p_2}{\partial \xi_j}(x, \nabla\psi(x)) \frac{\partial a_0}{\partial x_j} + \frac{1}{2i} \left(\sum_{j,k=0}^{n} \frac{\partial^2 p_2}{\partial \xi_j \partial \xi_k}(x, \nabla\psi(x)) \frac{\partial^2 \psi}{\partial x_j \partial x_k} \right) a_0. \tag{51}$$

(For general strictly hyperbolic operators, L contains a term with the subprincipal symbol of the operator, but here it is null; see Appendix A.)

In what follows, we construct a_0 and ψ so that $A_1(x)$ vanishes at order 2 along Γ and $A_2(x)$ vanishes at order 0 along the same curve. We will then be able to use Lemma 14 with $S = 3$ and $S = 1$ respectively.

Analysis of $A_1(x)$. Our goal is to show that, if we choose ψ adequately, we can make the quantity

$$f(x) = p_2(x, \nabla\psi(x)) \tag{52}$$

vanish at order 2 on Γ. For the vanishing at order 0, we prescribe that ψ satisfies $\nabla\psi(x(s)) = \xi(s)$, and then $f(x(s)) = 0$ since $(x(s), \xi(s))$ is a null-bicharacteristic. Note that this is possible since $x(s) \neq x(s')$ for any $s \neq s'$, due to $\dot{x}_0(s) = 1$ (bicharacteristics are traveled at speed 1; see Section 1.4). For the vanishing at order 1, using (52) and (7), we remark that, for any $0 \leqslant j \leqslant n$,

$$\begin{aligned}\frac{\partial f}{\partial x_j}(x(s)) &= \frac{\partial p_2}{\partial x_j}(x(s)) + \sum_{k=0}^{n} \frac{\partial p_2}{\partial \xi_k}(x(s)) \frac{\partial \psi}{\partial x_j \partial x_k}(x(s)) \\ &= -\dot{\xi}_j(s) + \sum_{k=0}^{n} \dot{x}_k(s) \frac{\partial \psi}{\partial x_j \partial x_k}(x(s)) \\ &= -\frac{d}{ds}\left(\frac{\partial \psi}{\partial x_j}(x(s))\right) + \sum_{k=0}^{n} \dot{x}_k(s) \frac{\partial \psi}{\partial x_j \partial x_k}(x(s)) = 0. \end{aligned} \tag{53}$$

Therefore, f vanishes automatically at order 1 along Γ (without making any particular choice for ψ): it just follows from (52) and the bicharacteristic equations (7). But for $f(x)$ to vanish at order 2 along Γ, it is required to choose a particular ψ. In the end, we will find that if ψ is given by the formula (59) below, with M being a solution of (54), then f vanishes at order 2 along Γ. Let us explain why.

Using the Einstein summation notation, we want that, for any $0 \leqslant i, j \leqslant n$, it holds

$$0 = \frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 p_2}{\partial x_j \partial x_i} + \frac{\partial^2 p_2}{\partial \xi_k \partial x_i} \frac{\partial^2 \psi}{\partial x_j \partial x_k} + \frac{\partial^2 p_2}{\partial x_j \partial \xi_k} \frac{\partial^2 \psi}{\partial x_i \partial x_k} + \frac{\partial^2 p_2}{\partial \xi_l \partial \xi_k} \frac{\partial^2 \psi}{\partial x_i \partial x_k} \frac{\partial^2 \psi}{\partial x_j \partial x_l} + \frac{\partial p_2}{\partial \xi_k} \frac{\partial^3 \psi}{\partial x_j \partial x_k \partial x_i}$$

along Γ. Introducing the matrices

$$(M(s))_{ij} = \frac{\partial^2 \psi}{\partial x_i \partial x_j}(x(s)), \qquad (A(s))_{ij} = \frac{\partial^2 p_2}{\partial x_i \partial x_j}(x(s), \xi(s)),$$
$$(B(s))_{ij} = \frac{\partial^2 p_2}{\partial \xi_i \partial x_j}(x(s), \xi(s)), \quad (C(s))_{ij} = \frac{\partial^2 p_2}{\partial \xi_i \partial \xi_j}(x(s), \xi(s)),$$

this amounts to solving the matricial Riccati equation

$$\frac{dM}{ds} + MCM + B^T M + MB + A = 0 \tag{54}$$

on a finite-length time interval. While solving (54), we also require $M(s)$ to be symmetric, $\mathrm{Im}(M(s))$ to be positive definite on the orthogonal complement of $\dot{x}(s)$, and $M(s)\dot{x}(s) = \dot{\xi}(s)$ to hold for all s due to (53).

Let M_0 be a symmetric $(n+1) \times (n+1)$ matrix with $\mathrm{Im}(M_0) > 0$ on the orthogonal complement of $\dot{x}(0)$ and $M_0 \dot{x}(0) = \dot{\xi}(0)$ (in particular $\mathrm{Im}(M_0)\dot{x}(0) = 0$). It is shown in [Ralston 1982] that there exists a global solution $M(s)$ on $[0, T]$ of (54) which satisfies all the above conditions and such that $M(0) = M_0$. The proof just requires that A, C are symmetric, but does not need anything special about p_2 (in particular, it applies to our sub-Riemannian case where p_2 is degenerate). For the sake of completeness, we recall the proof here.

We consider $(Y(s), N(s))$ the matrix solution with initial data $(Y(0), N(0)) = (\mathrm{Id}, M_0)$ (where Id is the $(n+1) \times (n+1)$ identity matrix) to the linear system

$$\begin{cases} \dot{Y} = BY + CN, \\ \dot{N} = -AY - B^T N. \end{cases} \tag{55}$$

We note that $(Y(s)\dot{x}(0), N(s)\dot{x}(0))$ then also solves (55), with Y and N being this time vectorial. One can check that $(\dot{x}(s), \dot{\xi}(s))$ is the solution of the same linear system with same initial data, and therefore, for any $s \in \mathbb{R}$,

$$\dot{x}(s) = Y(s)\dot{x}(0), \quad \dot{\xi}(s) = N(s)\dot{x}(0). \tag{56}$$

All the coefficients in (55) are real and A and C are symmetric, and it follows that the flow defined by (55) *on vectors* preserves both the real symplectic form acting on pairs $(y, \eta) \in (\mathbb{R}^{n+1})^2$ and $(y', \eta') \in (\mathbb{R}^{n+1})^2$ given by

$$\sigma((y, \eta), (y', \eta')) = y \cdot \eta' - \eta \cdot y'$$

and the complexified form $\sigma_{\mathbb{C}}((y, \eta), (y', \eta')) = \sigma((y, \eta), (\bar{y}', \bar{\eta}'))$ for $(y, \eta) \in (\mathbb{C}^{n+1})^2$ and $(y', \eta') \in (\mathbb{C}^{n+1})^2$. When we say that $\sigma_{\mathbb{C}}$ is invariant under (55), it means that we allow complex vectorial initial data in (55).

Let us prove that $Y(s)$ is invertible for any s. Let $v \in \mathbb{C}^{n+1}$ and $s_0 \in \mathbb{R}$ be such that $Y(s_0)v = 0$. We set $y(s_0) = Y(s_0)v$ and $\eta(s_0) = N(s_0)v$ and consider $\chi(s_0) = (y(s_0), \eta(s_0))$. From the conservation of $\sigma_{\mathbb{C}}$, we get

$$0 = \sigma_{\mathbb{C}}(\chi(s_0), \chi(s_0)) = \sigma_{\mathbb{C}}(\chi(0), \chi(0)) = v \cdot \overline{M_0 v} - \bar{v} \cdot M_0 v = -2i\bar{v} \cdot (\mathrm{Im}(M_0))v.$$

Since $\mathrm{Im}(M_0)$ is positive definite on the orthogonal complement to $\dot{x}(0)$, it holds $v = \lambda \dot{x}(0)$ for some $\lambda \in \mathbb{C}$. Hence

$$0 = Y(s_0)v = \lambda Y(s_0)\dot{x}(0) = \lambda \dot{x}(s_0),$$

where the last equality comes from (56). Since $\dot{x}_0(s_0) = (\partial p_2/\partial \xi_0)(s_0) = -2\xi_0(s_0) = 1$, it holds $\dot{x}(s_0) \neq 0$; hence $\lambda = 0$. It follows that $v = 0$ and $Y(s_0)$ is invertible.

Now, for any $s \in \mathbb{R}$, we set

$$M(s) = N(s)Y(s)^{-1},$$

which is a solution of (54) with $M(0) = M_0$. It satisfies $M(s)\dot{x}(s) = \dot{\xi}(s)$ thanks to (56). Moreover, it is symmetric: if we denote by $y^i(s)$ and $\eta^i(s)$ the column vectors of Y and N, by preservation of σ, for any $0 \leqslant i, j \leqslant n$, the quantity

$$\sigma((y^i(s), \eta^i(s)), (y^j(s), \eta^j(s))) = y^i(s) \cdot M(s)y^j(s) - y^j(s) \cdot M(s)y^i(s)$$

is equal to the same quantity at $s = 0$, which is equal to 0 since M_0 is symmetric.

Let us finally prove that, for any $s \in \mathbb{R}$, $\mathrm{Im}(M(s))$ is positive definite on the orthogonal complement of $\dot{x}(s)$. Let $y(s_0) \in \mathbb{C}^{n+1}$ be in the orthogonal complement of $\dot{x}(s_0)$. We decompose $y(s_0)$ on the column vectors of $Y(s_0)$:

$$y(s_0) = \sum_{i=0}^{n} b_i y^i(s_0), \quad b_i \in \mathbb{C}.$$

For $s \in \mathbb{R}$, we consider $y(s) = \sum_{i=0}^{n} b_i y^i(s)$ and we set $\chi(s) = \sum_{i=0}^{n} b_i (y^i(s), \eta^i(s))$. Then,

$$\sigma_{\mathbb{C}}(\chi(s), \chi(s)) = -2i\,\overline{y(s)} \cdot \mathrm{Im}(M(s))y(s). \tag{57}$$

By preservation of $\sigma_{\mathbb{C}}$ and using (57), we get that

$$\overline{y(s_0)} \cdot \mathrm{Im}(M(s_0)) y(s_0) = \overline{y(0)} \cdot \mathrm{Im}(M_0) y(0). \tag{58}$$

But $y(0)$ cannot be proportional to $\dot{x}(0)$; otherwise, using (56), we would get that $y(s_0)$ is proportional to $\dot{x}(s_0)$. Hence, the right-hand side in (58) is > 0, which implies that $\mathrm{Im}(M(s_0))$ is positive definite on the orthogonal complement to $\dot{x}(s_0)$.

Therefore, we found a choice for the second-order derivatives of ψ along Γ which meets all our conditions. For $x = (t, x') \in \mathbb{R} \times \mathbb{R}^n$ and s such that $t = t(s)$, we set

$$\psi(x) = \xi'(s) \cdot (x' - x'(s)) + \tfrac{1}{2}(x' - x'(s)) \cdot M(s)(x' - x'(s)), \tag{59}$$

and f vanishes at order 2 along Γ for this choice of ψ.

To sum up, as in the Riemannian (or "strictly hyperbolic") case handled in [Ralston 1982], the key observation is that the invariance of σ and $\sigma_{\mathbb{C}}$ prevents the solutions of (54) with positive imaginary part on the orthogonal complement of $\dot{x}(0)$ from blowing up.

Analysis of $A_2(x)$. We note that A_2 vanishes along Γ if and only if $La_0(x(s)) = 0$. According to (51), this turns out to be a linear transport equation on $a_0(x(s))$. Moreover, the coefficient of the first-order term, namely $\nabla_\xi p_2(x(s), \xi(s))$, is different from 0. Therefore, given $a_0 \neq 0$ at $(t=0, x=x(0))$, this transport equation has a solution $a_0(x(s))$ with initial datum a_0, and, by Cauchy uniqueness, $a_0(x(s)) \neq 0$ for any s. Note that we have prescribed a_0 only along Γ, and we may choose a_0 in a smooth (and arbitrary) way outside Γ. We choose it to vanish outside a small neighborhood of Γ.

***Proof of* (13).** We use (50) and we apply Lemma 14 to $S = 3$, $c = A_1$ and to $S = 1$, $c = A_2$, and we get

$$\|\partial_{tt}^2 v_k - \Delta v_k\|_{L^1(0,T;L^2(M))} \leqslant C(k^{-\frac{1}{2}} + k^{-\frac{1}{2}} + k^{-1}),$$

which implies (13).

***Proof of* (14).** We first observe that since $\mathrm{Im}(M(s))$ is positive definite on the orthogonal complement of $\dot{x}(s)$ and continuous as a function of s, there exist $\alpha, C > 0$ such that, for any $t(s) \in [0, T]$ and any $x' \in M$,

$$|\partial_t v_k(t(s), x')|^2 + \sum_{j=1}^{m} |X_j v_k(t(s), x')|^2 \geqslant (C|a_0(t(s), x')|^2 k^{\frac{n}{2}} + O(k^{2(\frac{n}{2}-1)}))e^{-\alpha k d(x', x'(s))^2},$$

where $d(\cdot, \cdot)$ denotes the Euclidean distance in $\mathbb{R}^n$. We denote by ℓ_n the Lebesgue measure on $\mathbb{R}^n$. Using the observation that, for any function f,

$$\int_M f(x') e^{-\alpha k d(x', x'(s))^2} \, d\mu(x') \sim \frac{\pi^{n/2}}{k^{n/2}\sqrt{\alpha}} f(x'(s)) \frac{d\mu}{d\ell_n}(x'(s)) \tag{60}$$

as $k \to +\infty$, and the fact that $a_0(x(s)) \neq 0$, we obtain (14).

***Proof of* (15).** We observe that since $\mathrm{Im}(M(s))$ is positive definite (uniformly in s) on the orthogonal complement of $\dot{x}(s)$, there exist $C, \alpha' > 0$ such that, for any $t \in [0, T]$, for any $x' \in M$, $|\partial_t v_k(t(s), x')|$ and $|X_j v_k(t(s), x')|$ are both bounded above by $Ck^{n/4}e^{-\alpha' k d(x', x'(s))^2}$. Therefore

$$\begin{aligned}\int_{M\setminus V_{t(s)}} \Bigl(|\partial_t v_k(t(s), x')|^2 + \sum_{j=1}^{m} |X_j v_k(t(s), x')|^2\Bigr)\, d\mu(x') \\ \leqslant Ck^{n/2} \int_{M\setminus V_{t(s)}} e^{-2\alpha' k d(x', x'(s))^2}\, d\mu(x') \\ \leqslant Ck^{n/2} \int_{M\setminus V_{t(s)}} e^{-2\alpha' k d(x', x'(s))^2}\, d\ell_n(x') + \mathrm{o}(1), \qquad (61)\end{aligned}$$

where, in the last line, we used the fact that $|d\mu/d\ell_n| \leqslant C$ in a fixed compact subset of M (since μ is a smooth volume), and the $\mathrm{o}(1)$ comes from the eventual blowup of μ at the boundary of M.

Now, $M \subset \mathbb{R}^n$, and there exists $r > 0$ such that $B_d(x(s), r) \subset V_{t(s)}$ for any s such that $t(s) \in (0, T)$, where $d(\cdot, \cdot)$ still denotes the Euclidean distance in $\mathbb{R}^n$. Therefore, we bound above the integral in (61) by

$$Ck^{n/2} \int_{\mathbb{R}^n \setminus B_d(x(s), r)} e^{-2\alpha' k d(x', x'(s))^2}\, d\ell_n(x'). \qquad (62)$$

Making the change of variables $y = k^{-1/2}(y - x(s))$, we can bound (62) from above by

$$C \int_{\mathbb{R}^n \setminus B_d(0, rk^{1/2})} e^{-2\alpha' \|y\|^2}\, d\ell_n(y),$$

with $\|\cdot\|$ the Euclidean norm. This last expression is bounded above by

$$Ce^{-\alpha' r^2 k} \int_{\mathbb{R}^n} e^{-\alpha' \|y\|^2}\, d\ell_n(y),$$

which implies (15).

Extension of the result to any manifold M. In the case of a general manifold M, not necessarily included in $\mathbb{R}^n$, we use charts together with the above construction. We cover M by a set of charts $(U_\alpha, \varphi_\alpha)$, where (U_α) is a family of open sets of M covering M and $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ is an homeomorphism U_α onto an open subset of $\mathbb{R}^n$. Take a solution $(x(t), \xi(t))_{t\in[0,T]}$ of (8). It visits a finite number of charts in the order $U_{\alpha_1}, U_{\alpha_2}, \ldots$, and we choose the charts and a_0 so that $v_k(t, \cdot)$ is supported in a unique chart at each time t. The above construction shows how to construct a_0 and ψ as long as $x(t)$ remains in the same chart. For any $l \geqslant 1$, we choose t_l so that $x(t_l) \in U_{\alpha_l} \cap U_{\alpha_{l+1}}$ and $a_0(t_l, \cdot)$ is supported in $U_{\alpha_l} \cap U_{\alpha_{l+1}}$. Since there is a (local) solution v_k for any choice of initial $a_0(t_l, x(t_l))$ and $\mathrm{Im}(\partial^2\psi/(\partial x_i\, \partial x_j))(t_l, x(t_l))$ in Proposition 12, we see that v_k may be continued from the chart U_{α_l} to the chart $U_{\alpha_{l+1}}$. This continuation is smooth since the two solutions coincide as long as $a_0(t, \cdot)$ is supported in $U_{\alpha_l} \cap U_{\alpha_{l+1}}$. Patching all solutions on the time intervals $[t_l, t_{l+1}]$ together, it yields a global-in-time solution v_k, as desired.

Appendix C: Proof of (49)

Because of the second convergence in (43) and the nonnegativity of a, it amounts to proving that

$$(X_1 \operatorname{Op}(a)u^k, X_1 u^k)_{L^2((0,T)\times M_H)} + (X_2 \operatorname{Op}(a)u^k, X_2 u^k)_{L^2((0,T)\times M_H)} \to 0.$$

Now, we notice that for any $B \in \Psi^0_{\mathrm{phg}}((0,T)\times M_H)$, it holds

$$(Bu^k, X_1u^k)_{L^2((0,T)\times M_H)} \xrightarrow[k\to+\infty]{} 0 \quad \text{and} \quad (Bu^k, \partial_t u^k)_{L^2((0,T)\times M_H)} \xrightarrow[k\to+\infty]{} 0 \tag{63}$$

since $u^k \to 0$ strongly in $L^2((0,T)\times M_H)$ and both X_1u^k and $\partial_t u^k$ are bounded in $L^2((0,T)\times M_H)$. We apply this to $B = [X_1, \mathrm{Op}(a)]$, and then, also using (63), we see that we can replace $\mathrm{Op}(a)$ by its Friedrichs quantization $\mathrm{Op}^F(a)$, which is positive; see [Taylor 1974, Chapter VII]. In other words, we are reduced to proving

$$(\mathrm{Op}^F(a)X_1u^k, X_1u^k)_{L^2((0,T)\times M_H)} + (\mathrm{Op}^F(a)X_2u^k, X_2u^k)_{L^2((0,T)\times M_H)} \xrightarrow[k\to+\infty]{} 0. \tag{64}$$

Let $\delta > 0$ and $\tilde a \in S^0_{\mathrm{phg}}((-\delta, T+\delta)\times M_H)$, $0 \leqslant \tilde a \leqslant \sup(a)$, and such that $\tilde a(t,\cdot) = a(\cdot)$ for $0 \leqslant t \leqslant T$. Making repeated use of (63) and of integrations by parts (since $\tilde a$ is compactly supported in time), we have

$$\begin{aligned}
\sum_{j=1}^2 (\mathrm{Op}^F(\tilde a)X_ju^k, X_ju^k)_{L^2((0,T)\times M_H)} &= \sum_{j=1}^2 (X_j \,\mathrm{Op}^F(\tilde a)u^k, X_ju^k)_{L^2((0,T)\times M_H)} + o(1)\\
&= -(\mathrm{Op}^F(\tilde a)u^k, \Delta u^k)_{L^2((0,T)\times M_H)} + \mathrm{o}(1)\\
&= -(\mathrm{Op}^F(\tilde a)u^k, \partial_t^2 u^k)_{L^2((0,T)\times M_H)} + \mathrm{o}(1)\\
&= (\partial_t \,\mathrm{Op}^F(\tilde a)u^k, \partial_t u^k)_{L^2((0,T)\times M_H)} + \mathrm{o}(1)\\
&= (\mathrm{Op}^F(\tilde a)\partial_t u^k, \partial_t u^k)_{L^2((0,T)\times M_H)} + \mathrm{o}(1).
\end{aligned}$$

Finally we note that since Op^F is a positive quantization, we have

$$\begin{aligned}
\sum_{j=1}^2 (\mathrm{Op}^F(a)X_ju^k, X_ju^k)_{L^2((0,T)\times M_H)} &\leqslant \sum_{j=1}^2 (\mathrm{Op}^F(\tilde a)X_ju^k, X_ju^k)_{L^2((0,T)\times M_H)}\\
&= (\mathrm{Op}^F(\tilde a)\partial_t u^k, \partial_t u^k)_{L^2((0,T)\times M_H)} + \mathrm{o}(1)\\
&\leqslant C\delta + (\mathrm{Op}^F(a)\partial_t u^k, \partial_t u^k)_{L^2((0,T)\times M_H)} + \mathrm{o}(1)\\
&\leqslant C\delta + o(1),
\end{aligned}$$

where C does not depend on δ. Taking $\delta \to 0$ concludes the proof of (64), and consequently (49) holds.

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QUANTITATIVE ALEXANDROV THEOREM AND ASYMPTOTIC BEHAVIOR OF THE VOLUME PRESERVING MEAN CURVATURE FLOW

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We prove a new quantitative version of the Alexandrov theorem which states that if the mean curvature of a regular set in $\mathbb{R}^{n+1}$ is close to a constant in the L^n sense, then the set is close to a union of disjoint balls with respect to the Hausdorff distance. This result is more general than the previous quantifications of the Alexandrov theorem, and using it we are able to show that in $\mathbb{R}^2$ and $\mathbb{R}^3$ a weak solution of the volume preserving mean curvature flow starting from a set of finite perimeter asymptotically convergences to a disjoint union of equisize balls, up to possible translations. Here by a weak solution we mean a flat flow, obtained via the minimizing movements scheme.

1. Introduction

Here we study the asymptotic behavior of the weak solution of the volume preserving mean curvature flow starting from a set of finite perimeter. In the classical setting we are given a smooth set $E_0 \subset \mathbb{R}^{n+1}$ and we let it evolve into a smooth family of sets $(E_t)_t$ according to the law, where the normal velocity V_t is proportional to the mean curvature of E_t as

$$V_t = -(H_{E_t} - \overline{H}_{E_t}) \quad \text{on } \partial E_t, \tag{1-1}$$

where $\overline{H}_{E_t} = \fint_{\partial E_t} H_{E_t}\, d\mathcal{H}^n$. Equations of mean curvature type are important in geometry, where one usually studies the geometric properties of ∂E_t which are inherited from ∂E_0. Equation (1-1) can also be seen as a volume preserving gradient flow of the surface area. These equations arise naturally in physical models involving surface tension; see [Taylor et al. 1992].

The main issue with (1-1) is that it may develop singularities in finite time even in the plane [Mayer 2001; Mayer and Simonett 2000]. In order to pass over the singular time one may try to do a surgery procedure and restart the flow after a singular time as in [Huisken and Sinestrari 2009] or to define a weak solution of (1-1), which is what we will consider here. For the mean curvature flow one may define a weak solution by using the varifold setting by Brakke [1978], the level set solution developed independently by Chen, Giga and Goto [Chen et al. 1989] and Evans and Spruck [1991], or by using the minimizing movements scheme developed independently by Almgren, Taylor and Wang [Almgren et al. 1993] and Luckhaus and Stürzenhecker [1995]. Since we want the solution of (1-1) to be a family of sets and since (1-1) does not satisfy the comparison principle, the natural choice is to define a weak solution via the

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minimizing movements scheme as in [Almgren et al. 1993; Luckhaus and Sturzenhecker 1995]. This solution is usually called a flat flow, and it is well defined due to [Mugnai et al. 2016] but might not be unique.

The advantage of the flat flow is that it is defined for all times for any bounded initial set with finite perimeter and we may thus study its asymptotic behavior. Heuristically, one may guess that the flat flow converges to a critical point of the static problem, which are classified in [Delgadino and Maggi 2019] as a disjoint union of balls, possibly tangent to each other. The asymptotic convergence of (1-1) has been proved for initial sets with certain geometric properties such as convexity [Huisken 1987], nearly spherical [Escher and Simonett 1998] or sets which are near a stable critical set in the flat torus in low dimensions [Niinikoski 2021]. We note that in these cases the flow does not develop singularities and is thus classically well defined for all times. The result in [Kim and Kwon 2020] shows that the convergence holds also for star-shaped sets, up to possible translations. For the mean curvature flow with forcing, the asymptotic behavior has been studied for the level set solution in [Giga et al. 2019; 2020] and for the flat flow in the plane in [Fusco et al. 2022]. The result closest to ours is the work by Morini, Ponsiglione and Spadaro [Morini et al. 2022], where the authors prove that the discrete-in-time approximation of the flat flow of (1-1) converges exponentially fast to a disjoint union of balls. Here we are able to pass the time discretization to zero and characterize the limit sets for the flat flow of (1-1) in $\mathbb{R}^2$ and $\mathbb{R}^3$. The precise definition of the flat flow is given in Section 4.

Theorem 1.1. *Assume $E_0 \subset \mathbb{R}^{n+1}$, with $n \le 2$ and $|E_0| = |B_1|$, is a bounded set of finite perimeter which is either essentially open or essentially closed, and let $(E_t)_{t\ge 0}$ be a flat flow of* (1-1) *starting from E_0. There is $N \in \mathbb{N}$ such that the following holds: for every $\varepsilon > 0$ there is $T_\varepsilon > 0$ such that for every $t \ge T_\varepsilon$ there are points $x_1, \dots, x_N$, which may depend on time, with $|x_i - x_j| \ge 2r$ for $i \ne j$ and $r = N^{-1/(n+1)}$ such that for $F_t = \bigcup_{i=1}^N B_r(x_i)$,*

$$\sup_{x \in E_t \Delta F_t} d_{\partial F_t}(x) \le \varepsilon.$$

Here $d_{\partial F}$ denotes the distance function. To the best of our knowledge this is the first result on the characterization of the asymptotic limit of (1-1) in $\mathbb{R}^3$. The above result holds for any limit of the approximative flat flow, and we do not need the additional assumption on the convergence of the perimeters as in [Luckhaus and Sturzenhecker 1995; Mugnai et al. 2016]. We note that the assumption on E_0 being either essentially open or closed is only needed to ensure that the flow is continuous up to time zero. It plays no role in the asymptotic analysis.

Concerning the limiting configurations, Theorem 1.1 is sharp since the flow (1-1) may converge to tangent balls as shown in [Fusco et al. 2022]. On the other hand, we believe that one may rule out the possible translations and the flow actually convergences to a disjoint union of balls. The higher dimensional case and the possible speed of convergence are also open problems.

Quantitative Alexandrov theorem. The proof of Theorem 1.1 is based on the dissipation inequality proven in [Mugnai et al. 2016] and stated in Proposition 4.1. This implies that there is a sequence of times $t_j \to \infty$ such that the mean curvatures of the evolving sets E_{t_j} are asymptotically close to a constant with respect to the L^2-norm. Therefore, we need a quantified version of the Alexandrov theorem which enables us to conclude that the sets E_{t_j} are close to a disjoint union of balls.

There is a lot of recent research on generalizing the Alexandrov theorem [Ciraolo and Maggi 2017; Delgadino and Maggi 2019; Delgadino et al. 2018; De Rosa et al. 2020; Krummel and Maggi 2017; Magnanini and Poggesi 2020]. We refer the survey paper [Ciraolo 2021] for the state of the art. Unfortunately, none of the available results is applicable to our problem, and we are also not able to use the characterization of the critical sets in [Delgadino and Maggi 2019, Corollary 2] to identify the limit set. Indeed, even if we know that the sets E_{t_j} converge to a set of finite perimeter and their mean curvatures converge to a constant, it is not clear why the limit set is a set of finite perimeter with weak mean curvature as this class of sets is not in general closed. Our main result is the following quantification of the Alexandrov theorem, which is the main technical tool in the proof of Theorem 1.1.

Theorem 1.2. *Let $E \subset \mathbb{R}^{n+1}$ be a C^2 regular set such that $P(E) \le C_0$ and $|E| \ge 1/C_0$. There are positive constants $q = q(n) \in (0, 1]$, $C = C(C_0, n)$ and $\delta = \delta(C_0, n)$ such that if $\|H_E - \lambda\|_{L^n(\partial E)} \le \delta$ for some $\lambda \in \mathbb{R}$, then $1/C \le \lambda \le C$ and there are points $x_1, \dots, x_N$ with $|x_i - x_j| \ge 2R$, where $R = n/\lambda$, such that for $F = \bigcup_{i=1}^N B_R(x_i)$,*

$$\sup_{x \in E \Delta F} d_{\partial F}(x) \le C \|H_E - \lambda\|^q_{L^n(\partial E)}.$$

Moreover,

$$|P(E) - N(n+1)\omega_{n+1} R^n| \le C \|H_E - \lambda\|^q_{L^n(\partial E)}.$$

The main advantage of Theorem 1.2 with respect to the previous results in the literature is that we do not assume any geometric restriction on E such as mean convexity. Moreover, we assume the mean curvature to be close to a constant only in the L^n sense, which is exactly what we need for the asymptotic analysis in Theorem 1.1. This makes the proof challenging as, for example, we cannot use the estimates from Allard's regularity theory [1972].

Theorem 1.2 is sharp in the sense that $\|H_E - \lambda\|_{L^n(\partial E)}$ cannot be replaced by a weaker L^p-norm. This can be seen by considering a set which is a union of the unit ball and a ball of small radius ε located far away. On the other hand, the dissipation inequality in Proposition 4.1 controls only the L^2-norm of the mean curvature, which is the reason we cannot prove Theorem 1.1 in higher dimensions. The proof of Theorem 1.2 is done in a constructive way and we obtain an explicit bound on the exponent $q = (n+2)^{-3}$. It would be interesting to obtain the sharp bound as it might be crucial in order to obtain the possible exponential convergence of (1-1) as in [Morini et al. 2022]. In the two-dimensional case the optimal power $q = 1$ is proven in [Fusco et al. 2022].

Outline of the proof of Theorem 1.2. Since the proof of Theorem 1.2 is rather long, we outline it here. As in [Delgadino and Maggi 2019], our argument is based on the proof of the Heinze–Karcher inequality by Montiel and Ros [1991], which is an alternative for the proof in [Ros 1987]. In [Delgadino and Maggi 2019], the authors are able to generalize the Montiel–Ros argument to sets of finite perimeter with weak distributional mean curvature. Here we revisit the argument by Montiel and Ros and deduce in Proposition 3.3 that for E and R as in Theorem 1.2 and for $0 < r < R$, the volume of the set

$E_r = \{x \in E : \mathrm{dist}(x, \partial E) > r\}$ satisfies the estimate

$$\left| |E_r| - \frac{|E|}{R^{n+1}}(R-r)^{n+1} \right| \le C \| H_E - \lambda \|_{L^n(\partial E)}.$$

We use this in Step 1 of the proof of Theorem 1.2 to deduce that for r close to R the set E_r is a union of a finite number of components, or clusters, with positive distance to each other.

We note that the above inequality is not enough to conclude the proof as, e.g., the cube $Q = (-1, 1)^{n+1}$ satisfies $|Q_r| = (1-r)^{n+1}|Q|$. Therefore, we need further information from the Montiel–Ros argument and we prove in Proposition 3.3 that the Minkowski sum $E_r + B_\rho = \{x \in \mathbb{R}^{n+1} : \mathrm{dist}(x, E_r) < \rho\}$, with $0 < \rho < r < R$, satisfies

$$\left| |E_r + B_\rho| - \frac{|E|}{R^{n+1}}(R-(r-\rho))^{n+1} \right| \le \frac{C}{(R-r)^{n+1}} \| H_E - \lambda \|_{L^n(\partial E)}.$$

This enables us to prove that the components of $E_r + B_\rho \subset E$, with properly chosen ρ and r, are almost spherical. In particular, if E satisfies the above estimate with $C = 0$, then it is a disjoint union of balls. This, together with the density estimate from [Topping 2008], concludes the proof.

2. Notation and preliminary results

In this section we briefly introduce our notation and recall some results from differential geometry. Given a set $E \subset \mathbb{R}^{n+1}$ the distance function $d_E : \mathbb{R}^{n+1} \to [0, \infty)$ is defined, as usual, as

$$d_E(x) := \inf_{y \in E} |x - y|,$$

and we denote the signed distance function $\bar{d}_E : \mathbb{R}^{n+1} \to \mathbb{R}$ by

$$\bar{d}_E(x) := \begin{cases} -d_{\partial E}(x) & \text{for } x \in E, \\ d_{\partial E}(x) & \text{for } x \in \mathbb{R}^{n+1} \setminus E. \end{cases}$$

Then clearly $d_{\partial E} = |\bar{d}_E|$. We denote the ball with radius r centered at x by $B_r(x)$ and by B_r if it is centered at the origin. Given a set $E \subset \mathbb{R}^{n+1}$ we denote its ρ-enlargement by the Minkowski sum

$$E + B_\rho = \{x + y \in \mathbb{R}^{n+1} : x \in E,\ y \in B_\rho\} = \{x \in \mathbb{R}^{n+1} : d_E(x) < \rho\}.$$

For a measurable set $E \subset \mathbb{R}^{n+1}$ the shorthand notation $|E|$ denotes its Lebesgue measure, and we denote the k-dimensional measure of the unit ball in $\mathbb{R}^k$ by ω_k. In some cases, we may use the shorthand notation $|E|$ more generally for a measurable set $E \subset \mathbb{R}^k$ to denote its k-dimensional Lebesgue measure but this shall be clear from context.

For a set of finite perimeter $E \subset \mathbb{R}^{n+1}$ we denote its reduced boundary by $\partial^* E$ and the perimeter by $P(E)$. Recall that $P(E) = \mathcal{H}^n(\partial^* E)$ and for a regular enough set, $\partial^* E = \partial E$. The relative isoperimetric inequality states that for every set of finite perimeter E and for every ball $B_r(x)$,

$$\mathcal{H}^n(\partial^* E \cap B_r(x))^{(n+1)/n} \ge c_n \min\{|E \cap B_r(x)|, |B_r(x) \setminus E|\},$$

for a dimensional constant c_n. We refer to [Maggi 2012] for an introduction to the topic.

We define the *tangential differential* of $F \in C^1(\mathbb{R}^{n+1}; \mathbb{R}^m)$ on ∂E by

$$\mathrm{D}_\tau F(x) = \mathrm{D}F(x)(I - \nu_E(x) \otimes \nu_E(x)),$$

where ν_E denotes the unit outer normal of E. For a function $f \in C^1(\mathbb{R}^{n+1}; \mathbb{R})$ we denote by $\nabla_\tau f$ its tangential gradient which is a vector in $\mathbb{R}^{n+1}$. We define the tangential divergence of $F \in C^1(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$ by $\operatorname{div}_\tau F = \operatorname{Tr}(\mathrm{D}_\tau F)$. Then the divergence theorem on manifolds generalizes to

$$\int_{\partial^* E} \operatorname{div}_\tau F \, \mathrm{d}\mathcal{H}^n = \int_{\partial^* E} H_E \langle F, \nu_E \rangle \, \mathrm{d}\mathcal{H}^n,$$

where $H_E \in L^1(\partial^* E)$ is the distributional mean curvature. When ∂E is smooth, H_E agrees with the classical definition of the mean curvature, which for us is the sum of the principal curvatures.

We begin by recalling the well-known inequality proven first by Simon [1993] in $\mathbb{R}^3$ and then by Topping [2008] in the general case.

Theorem 2.1. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a compact and connected C^2-hypersurface. Then*

$$\operatorname{diam}(\Sigma) \le C_n \int_\Sigma |H_\Sigma|^{n-1} \, \mathrm{d}\mathcal{H}^n, \tag{2-1}$$

where C_n depends only on the dimension.

We need also the Michael–Simon inequality [Michael and Simon 1973].

Theorem 2.2. *Let $\Sigma \subset \mathbb{R}^{n+1}$, $n \ge 2$, be a compact C^2-hypersurface. Then for every nonnegative $\varphi \in C^1(\mathbb{R}^{n+1})$,*

$$\|\varphi\|_{L^{n/(n-1)}(\Sigma)} \le C_n \int_\Sigma |\nabla_\tau \varphi| + \varphi |H_\Sigma| \, \mathrm{d}\mathcal{H}^n, \tag{2-2}$$

where C_n depends only on the dimension.

The following density-type estimate is essentially proven in [Morini et al. 2022, Lemma 2.1].

Proposition 2.3. *Let $E \subset \mathbb{R}^{n+1}$ be a set of finite perimeter with $P(E) > 0$ and $0 < \beta < 1$. There is a positive constant $c = c(n, \beta)$ such that*

$$r_{E,\beta} := \sup\{r \in \mathbb{R}_+ : \textit{there exists } x \in \mathbb{R}^{n+1} \textit{ with } |B_r(x) \cap E| \ge \beta |B_r(x)|\} \ge c \frac{|E|}{P(E)}.$$

We use the previous results to prove the following lemma, which is useful when we bound the Lagrange multipliers and the number of the components of the flat flow of (1-1).

Lemma 2.4. *Let $E \subset \mathbb{R}^{n+1}$ be a bounded set of finite perimeter with a distributional mean curvature $H_E \in L^1(\partial^* E)$, $\lambda \in \mathbb{R}$ and $1 \le C_0 < \infty$. There is a positive constant $C = C(C_0, n)$ such that*:

(i) *If $P(E) \le C_0$ and $|E| \ge 1/C_0$, then*

$$1/C - C \|H_E - \lambda\|_{L^1(\partial^* E)} \le \lambda \le C + C \|H_E - \lambda\|_{L^1(\partial^* E)}.$$

(ii) *If $P(E) \le C_0$, $|E| \ge 1/C_0$ and E is C^2 regular, then the number of components of E is bounded by $C(1 + \|H_E - \lambda\|^n_{L^n(\partial E)})$ and the diameters of the components are bounded by $C(1 + \|H_E - \lambda\|^{n-1}_{L^{n-1}(\partial E)})$.*

Proof. Our standing assumptions throughout the proof are $P(E) \le C_0$ and $|E| \ge 1/C_0$. The perimeter bound and the global isoperimetric inequality yield

$$|E| \le c_n P(E)^{(n+1)/n} \le c_n C_0^{(n+1)/n}.$$

By the assumptions on E and by the divergence theorems, we compute the following for any vector field $F \in C^1(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$:

$$\begin{aligned}\lambda \int_E \operatorname{div} F \,\mathrm{d}x &= \int_{\partial^* E} \lambda \langle F, \nu_E\rangle \,\mathrm{d}\mathcal{H}^n \\ &= \int_{\partial^* E} H_E \langle F, \nu_E\rangle \,\mathrm{d}\mathcal{H}^n + \int_{\partial^* E} (\lambda - H_E)\langle F, \nu_E\rangle \,\mathrm{d}\mathcal{H}^n \\ &= \int_{\partial^* E} \operatorname{div}_\tau F \,\mathrm{d}\mathcal{H}^n + \int_{\partial^* E} (\lambda - H_E)\langle F, \nu_E\rangle \,\mathrm{d}\mathcal{H}^n. \end{aligned} \tag{2-3}$$

Our goal is to construct a suitable vector field F to obtain (i) from (2-3). To this aim, we use first the relative isoperimetric inequality, Proposition 2.3 and a suitable continuity argument to find positive $r_0 = r_0(C_0, n)$, $R_0 = R_0(C_0, n)$ and r such that $r_0 \le r \le R_0$ and, by possibly translating the coordinates, $|B_r \cap E| = \frac{1}{2}|B_r|$. Again, it follows from the relative isoperimetric inequality that $\mathcal{H}^n(\partial^* E \cap B_r) \ge c$ for some positive $c = c(C_0, n)$. Choose a decreasing C^1 function $f : \mathbb{R} \to \mathbb{R}$ with

$$f(t) = \begin{cases} (2r)^{-1} & \text{for } t \le \frac{3}{2}r \\ t^{-1} & \text{for } t \ge \frac{5}{2}r \end{cases}$$

and for which the conditions $f(t) \le \min\{(2r)^{-1}, t^{-1}\}$ and $|f'(t)| \le (2r)^{-2}$ hold on $\left[\frac{3}{2}r, \frac{5}{2}r\right]$. We define the function $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by setting $F(x) = f(|x|)x$. Then F is a C^1 vector field with

$$\begin{aligned} \mathrm{D}F(x) &= f(|x|)I + \frac{f'(|x|)}{|x|} x \otimes x && \text{for every } x \in \mathbb{R}^{n+1}, \\ \operatorname{div} F(x) &= (n+1) f(|x|) + f'(|x|)|x| && \text{for every } x \in \mathbb{R}^{n+1}, \\ \operatorname{div}_\tau F(x) &= n f(|x|) + f'(|x|)\left(|x| - \frac{\langle x, \nu_E\rangle^2}{|x|}\right) && \text{for every } x \in \partial^* E. \end{aligned}$$

Then $0 < \operatorname{div} F \le (n+1)(2r)^{-1}$ everywhere and $\operatorname{div} F = (n+1)(2r)^{-1}$ in B_r, so by using these and the earlier observations we obtain

$$\begin{aligned} \frac{n+1}{4R_0}|B_{r_0}| \le \frac{n+1}{4r}|B_r| &= \frac{n+1}{2r}|B_r \cap E| \\ &\le \int_E \operatorname{div} F \,\mathrm{d}x \le \frac{n+1}{2r}|E| \le \frac{c_n(n+1)}{2r_0} C_0^{(n+1)/n}. \end{aligned} \tag{2-4}$$

Again, $0 \le \operatorname{div}_\tau F \le n(2r)^{-1}$ on $\partial^* E$ and $\operatorname{div}_\tau F = n(2r)^{-1}$ on $\partial^* E \cap B_r$, and thus

$$\frac{nc}{2R_0} \le \frac{n}{2r}\mathcal{H}^n(\partial^* E \cap B_r) \le \int_{\partial^* E} \operatorname{div}_\tau F \,\mathrm{d}\mathcal{H}^n \le \frac{nP(E)}{2r} \le \frac{nC_0}{2r_0}. \tag{2-5}$$

We use (2-3), (2-4), (2-5) and $|F| \le 1$ to obtain (i).

The claim (ii) is easy to prove in the planar case and therefore we assume that $n \geq 2$. Let $E_1, E_2, \ldots, E_N$ denote the connected components of E. We apply Theorem 2.2 on ∂E_i with $\varphi = 1$ and use Hölder's inequality to obtain

$$C_n^{-1} \leq \|H_{E_i}\|_{L^n(\partial E_i)} \leq \|H_{E_i} - \lambda\|_{L^n(\partial E_i)} + |\lambda| P(E_i)^{1/n},$$

from which we conclude, using (i) and Hölder's inequality, that

$$\begin{aligned} N C_n^{-n} &\leq 2^n \|H_E - \lambda\|^n_{L^n(\partial E)} + 2^n |\lambda|^n P(E) \\ &\leq 2^n \|H_E - \lambda\|^n_{L^n(\partial E)} + 2^{2n} C_0 C^n (1 + \|H_E - \lambda\|^n_{L^1(\partial E)}) \\ &\leq 2^n \|H_E - \lambda\|^n_{L^n(\partial E)} + 2^{2n} C_0 C^n (1 + C_0^{n-1} \|H_E - \lambda\|^n_{L^n(\partial E)}). \end{aligned} \tag{2-6}$$

On the other hand, Theorem 2.1 together with (i) and Hölder's inequality implies

$$\begin{aligned} \sum_i \operatorname{diam}(E_i) &\leq \sum_i C_n \int_{\partial E_i} |H_{E_i}|^{n-1} \, d\mathcal{H}^n \\ &\leq \sum_i 2^{n-1} C_n \biggl(\int_{\partial E_i} |H_{E_i} - \lambda|^{n-1} \, d\mathcal{H}^n + |\lambda|^{n-1} P(E_i) \biggr) \\ &\leq 2^{n-1} C_n \biggl(\int_{\partial E} |H_E - \lambda|^{n-1} \, d\mathcal{H}^n + P(E) |\lambda|^{n-1} \biggr) \\ &\leq 2^{n-1} C_n (\|H_E - \lambda\|^{n-1}_{L^{n-1}(\partial E)} + 2^{n-1} C_0 C^n (1 + \|H_E - \lambda\|^{n-1}_{L^1(\partial E)})) \\ &\leq 2^{n-1} C_n (\|H_E - \lambda\|^{n-1}_{L^{n-1}(\partial E)} + 2^{n-1} C_0 C^n (1 + C_0^{n-2} \|H_E - \lambda\|^{n-1}_{L^{n-1}(\partial E)})). \end{aligned} \tag{2-7}$$

Thus, by possibly increasing C, the second claim follows from (2-6) and (2-7). □

3. Quantitative Alexandrov theorem

We split the proof of Theorem 1.2 into two parts. We first revisit the Montiel–Ros argument in Proposition 3.3 where all the technical heavy lifting is done. The idea of Proposition 3.3 is to transform the (local) information of the mean curvature of E being close to a constant into information on the ρ-enlargement of the level sets of the distance function of ∂E. We note that the statement of Proposition 3.3 is given by the sharp exponent. The proof of Theorem 1.2 is then based on purely geometric arguments.

We first state the following equivalent formulation of the theorem.

Remark 3.1. Once we prove that in Theorem 1.2 the number of component of E is bounded, the statement on the L^∞-distance is equivalent to the fact that, under the assumption $\|H_E - \lambda\|_{L^n(\partial E)} \leq \delta$, there are points $x_1, \ldots, x_N$ such that

$$\bigcup_{i=1}^N B_{\rho_-}(x_i) \subset E \subset \bigcup_{i=1}^N B_{\rho_+}(x_i),$$

where we have $\rho_- = R - C\|H_E - \lambda\|^q_{L^n(\partial E)}$, $\rho_+ = R + C\|H_E - \lambda\|^q_{L^n(\partial E)}$, $R = n/\lambda$ and the balls $B_{\rho_-}(x_1), \ldots, B_{\rho_-}(x_N)$ are disjoint to each other. We leave the details to the reader.

In Theorem 1.2 we assume that the mean curvature is bounded only in the L^n sense and thus the estimates from Allard's regularity theory [1972] are not available for us. Indeed, the L^n-boundedness of the mean curvature is not strong enough to give proper density estimates. Moreover, even in the three dimensional case $\mathbb{R}^3$ we cannot use the results from [Simon 1993], because we do not have a uniform bound on the Euler characteristic of the set E. However, if we know that the mean curvature is close to a constant with respect to the L^n-norm, then the following density estimate holds. The proof is based on [Topping 2008, Lemma 1.2].

Lemma 3.2. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a compact C^2-hypersurface and $\lambda \in \mathbb{R}_+$. There is a positive dimensional constant δ_n such that if $\|H_\Sigma - \lambda\|_{L^n(\Sigma)} \le \delta_n$, then*

$$\delta_n \le \frac{\mathcal{H}^n(B(x,r)\cap\Sigma)}{r^n}$$

for every $x \in \Sigma$ and $0 < r \le \delta_n/\lambda$.

Proof. The planar case $n = 1$ is rather obvious and we leave it to the reader. Assume $n \ge 2$. Fix $x \in \Sigma$ and define $V : [0,\infty) \to [0,\infty)$ as $V(r) = \mathcal{H}^n(B_r(x)\cap\Sigma)$. Since V is increasing, the derivative $V'(r)$ is defined for almost every $r \in [0,\infty)$, and

$$\int_{r_1}^{r_2} V'(\rho)\,\mathrm{d}\rho \le V(r_2) - V(r_1) \quad \text{whenever } 0 \le r_1 < r_2.$$

By a standard foliation argument we have that $\mathcal{H}^n(\partial B_r(x)\cap\Sigma) > 0$ for at most countably many $r \in \mathbb{R}_+$. Thus $V'(r)$ is defined and $\mathcal{H}^n(\partial B_r(x)\cap\Sigma) = 0$ for almost every $r \in [0,\infty)$. Fix such an r and choose $h \in \mathbb{R}_+$ for which $\mathcal{H}^n(\partial B_{r+h}(x)\cap\Sigma) = 0$. Define a cut-off function $f_h : \mathbb{R}^{n+1} \to \mathbb{R}$ by setting

$$f_h(y) = \begin{cases} 1, & y \in B_r(x), \\ 1 - |y-x|/h, & y \in B_{r+h}(x) \setminus B_r(x), \\ 0, & y \in \mathbb{R}^{n+1} \setminus B_{r+h}(x). \end{cases}$$

By using a suitable approximation argument combined with Theorem 2.2 we obtain

$$V(r)^{(n-1)/n} \le C_n\left(\frac{V(r+h)-V(r)}{h} + \|f_h H_\Sigma\|_{L^1(\Sigma)}\right).$$

In turn, we may choose a sequence $(h_k)_k$ such that $h_k \to 0$ and $\mathcal{H}^n(\partial B_{r+h_k}(x)\cap\Sigma) = 0$. Then by letting $k \to \infty$ the previous estimate yields

$$\begin{aligned} V(r)^{(n-1)/n} &\le C_n\left(V'(r) + \int_{\bar{B}_r(x)\cap\Sigma} |H_\Sigma|\,\mathrm{d}\mathcal{H}^n\right) \\ &\le C_n\left(V'(r) + \int_{B_r(x)\cap\Sigma} |H_\Sigma|\,\mathrm{d}\mathcal{H}^n\right) \\ &\le C_n\left(V'(r) + \int_{B_r(x)\cap\Sigma} |H_\Sigma - \lambda|\,\mathrm{d}\mathcal{H}^n + \lambda V(r)\right) \\ &\le C_n(V'(r) + \|H_\Sigma - \lambda\|_{L^n(\Sigma)} V(r)^{(n-1)/n} + \lambda V(r)). \end{aligned}$$

Thus for almost every $r \in (0, \infty)$,

$$\left(\frac{C_n^{-1} - \|H_\Sigma - \lambda\|_{L^n(\Sigma)}}{V(r)^{1/n}} - \lambda\right) V(r) \le V'(r).$$

If $\|H_\Sigma - \lambda\|_{L^n(\Sigma)} \le \delta_n$ for small δ_n, then the above inequality implies

$$\frac{1}{2C_n} V(r)^{1-1/n} - \lambda V(r) \le V'(r).$$

Fix $r < \delta_n/\lambda$. We will assume that $V(r) \le \delta_n r^n$, since otherwise the claim is trivially true. By the monotonicity of V we have

$$V(\rho)^{1/n} \le V(r)^{1/n} \le \delta_n/\lambda$$

for all $0 < \rho < r$. For δ_n small enough the above inequality then yields

$$\frac{1}{4C_n} V(\rho)^{1-1/n} \le V'(\rho)$$

for almost every $0 < \rho < r$. The claim follows by integrating this over $(0, r)$. $\square$

Montiel–Ros argument. We recall that for $E \subset \mathbb{R}^{n+1}$ we write

$$E_r := \{x \in E : \operatorname{dist}(x, \partial E) > r\}. \tag{3-1}$$

We use the fact that E is C^2 regular and say that $x \in \partial E$ satisfies the interior ball condition with radius r if, for $y = x - r\nu_E(x)$, it holds that $B_r(y) \subset E$. For $r > 0$ we define

$$\Gamma_r := \{x \in \partial E : x \text{ satisfies the interior ball condition with radius } r\}. \tag{3-2}$$

Proposition 3.3. *Let $\lambda \in \mathbb{R}$ and suppose that a bounded and C^2 regular set $E \subset \mathbb{R}^{n+1}$ satisfies $P(E) \le C_0$ and $|E| \ge 1/C_0$ with $C_0 \in \mathbb{R}_+$. Then for $0 < r < R$ with $R = n/\lambda$,*

$$\left| |E_r| - \frac{|E|}{R^{n+1}}(R-r)^{n+1} \right| \le C\|H_E - \lambda\|_{L^n(\partial E)}$$

and

$$\mathcal{H}^n(\partial E \setminus \Gamma_r) \le \frac{C}{(R-r)^{n+1}} \|H_E - \lambda\|_{L^n(\partial E)},$$

provided that $\|H_E - \lambda\|_{L^n(\partial E)} \le \delta$, where the constants C and δ depend only on C_0 and on the dimension. Moreover, under the same assumptions, for $0 < \rho < r < R$,

$$\left| |E_r + B_\rho| - \frac{|E|}{R^{n+1}}(R-(r-\rho))^{n+1} \right| \le \frac{C}{(R-r)^{n+1}} \|H_E - \lambda\|_{L^n(\partial E)}.$$

Proof. As we already mentioned the proof is based on the Montiel–Ros argument for the Heinze–Karcher inequality, which we recall shortly. To that aim, we define $\zeta : \partial E \times \mathbb{R} \to \mathbb{R}^{n+1}$ as

$$\zeta(x, t) = x - t\nu_E(x).$$

We denote the principle curvatures of ∂E at x by $k_1(x), \dots, k_n(x)$ and assume that they are pointwise ordered as $k_i(x) \le k_{i+1}(x)$. If we consider $\partial E \times \mathbb{R}$ as a hypersurface embedded in $\mathbb{R}^{n+2}$, then its tangential Jacobian is

$$J_\tau \zeta(x,t) = \prod_{i=1}^n |1 - t k_i(x)| \quad \text{on } \partial E \times \mathbb{R}.$$

For every bounded Borel set $M \subset \partial E \times \mathbb{R}$ we have, by the area formula,

$$\int_{\zeta(M)} \mathcal{H}^0(\zeta^{-1}(y) \cap M)\, \mathrm{d}y = \int_M J_\tau \zeta \, \mathrm{d}\mathcal{H}^{n+1}.$$

In the proof, C denotes a positive constant which may change from line to line, depending only on C_0 and on the dimension.

Step 1: In order to utilize Lemma 2.4, we choose $\delta = \delta(C_0, n)$ to be the same as in the lemma and assume $\|H_E - \lambda\|_{L^n(\partial E)} \le \delta$. Then E has N connected components with $N \le C$. We may thus prove the claim componentwise and assume that E is connected. We write

$$\Sigma := \left\{x \in \partial E : |H_E(x) - \lambda| < \tfrac{1}{2}\lambda\right\}.$$

By Lemma 2.4 we have $\lambda \ge 1/C$, and thus by Hölder's inequality

$$\mathcal{H}^n(\partial E \setminus \Sigma) \le \frac{2}{\lambda} \int_{\partial E} |H_E(x) - \lambda| \, \mathrm{d}\mathcal{H}^n \le C \|H_E - \lambda\|_{L^n(\partial E)}. \tag{3-3}$$

Moreover, we have

$$\begin{aligned} \frac{n}{n+1} \int_\Sigma \frac{1}{H_E} \, \mathrm{d}\mathcal{H}^n &= \frac{n}{n+1} \int_\Sigma \left(\frac{1}{\lambda} + \left(\frac{1}{H_E} - \frac{1}{\lambda} \right) \right) \mathrm{d}\mathcal{H}^n \\ &\le \frac{nP(E)}{(n+1)\lambda} + C \|H_E(x) - \lambda\|_{L^n(\partial E)}. \end{aligned}$$

Since E is connected, Lemma 2.4 yields $\operatorname{diam}(E) \le \tilde{R}$ with $\tilde{R} = \tilde{R}(C_0, n) \ge R$. Choose $x_0 \in E$. Then using (2-3) with $F(x) = x - x_0$ we obtain

$$nP(E) = (n+1)\lambda |E| + \int_{\partial E} (H_E - \lambda) \langle (x - x_0), \nu_E \rangle \, \mathrm{d}\mathcal{H}^n,$$

which in turn implies

$$\big| nP(E) - (n+1)\lambda |E| \big| \le C \|H_E - \lambda\|_{L^n(\partial E)}. \tag{3-4}$$

Hence we deduce

$$\frac{n}{n+1} \int_\Sigma \frac{1}{H_E} \, \mathrm{d}\mathcal{H}^n \le |E| + C \|H_E - \lambda\|_{L^n(\partial E)}. \tag{3-5}$$

Next we define

$$Z = \{(x,t) \in \Sigma \times [0, \infty) : 0 \le t \le 1/k_n(x)\}.$$

Note that this is well defined, since $x \in \Sigma$ implies $k_n(x) \geq H_E(x)/n \geq \lambda/(2n) > 0$. We also set

$$\Sigma_1' = \{x \in \partial E \setminus \Sigma : k_n(x) \leq 1/\tilde{R}\} \quad \text{and} \quad \Sigma_2' = \{x \in \partial E \setminus \Sigma : k_n(x) > 1/\tilde{R}\},$$
$$Z_1' = \Sigma_1' \times [0, \tilde{R}] \quad \text{and} \quad Z_2' = \{(x,t) \in \Sigma_2' \times [0,\infty) : 0 \leq t \leq 1/k_n(x)\},$$

and finally

$$Z' = Z_1' \cup Z_2'.$$

Then Z and Z' are disjoint and bounded Borel sets and $E \subset \zeta(Z \cup Z')$. To see this fix $y \in E$ and let $x \in \partial E$ be such that $r = d_{\partial E}(y) = |x - y|$. Then we may write $y = x - r\nu_E(x)$, and by the maximum principle $k_n(x) \leq 1/r$. Since $\operatorname{diam}(E) \leq \tilde{R}$, we have $r \leq \tilde{R}$ and so we conclude that $(x, r) \in Z \cup Z'$ and $y = \zeta(x, r)$.

We now recall the Montiel–Ros argument. We use the fact that E is a subset of $\zeta(Z \cup Z')$, the area formula, the arithmetic geometric inequality and the fact that $1/k_n(x) \leq n/H_E(x)$ for $x \in \Sigma$ to obtain

$$\begin{aligned}
|E| \leq |\zeta(Z)| + |\zeta(Z')| &\leq \int_{\zeta(Z)} \mathcal{H}^0(\zeta^{-1}(y) \cap Z)\,\mathrm{d}y + |\zeta(Z')| \\
&= \int_Z J_\tau \zeta \,\mathrm{d}\mathcal{H}^{n+1} + |\zeta(Z')| \\
&= \int_\Sigma \int_0^{1/k_n(x)} \prod_{i=1}^n (1 - t k_i(x))\,\mathrm{d}t\,\mathrm{d}\mathcal{H}^n + |\zeta(Z')| \\
&\leq \int_\Sigma \int_0^{1/k_n(x)} \left(1 - \frac{t}{n} H_E(x)\right)^n \mathrm{d}t\,\mathrm{d}\mathcal{H}^n + |\zeta(Z')| \\
&\leq \int_\Sigma \int_0^{n/H_E(x)} \left(1 - \frac{t}{n} H_E(x)\right)^n \mathrm{d}t\,\mathrm{d}\mathcal{H}^n + |\zeta(Z')| \\
&= \frac{n}{n+1} \int_\Sigma \frac{1}{H_E}\,\mathrm{d}\mathcal{H}^n + |\zeta(Z')|.
\end{aligned}$$

Next we quantify the previous four inequalities. To that aim we define the nonnegative numbers R_1, R_2, R_3 and R_4 as

$$R_1 = |\zeta(Z) \setminus E|, \tag{3-6}$$

$$R_2 = \int_{\zeta(Z)} |\mathcal{H}^0(\zeta^{-1}(y) \cap Z) - 1|\,\mathrm{d}y, \tag{3-7}$$

$$R_3 = \int_\Sigma \int_0^{1/k_n(x)} \left| \left(1 - \frac{t}{n} H_E(x)\right)^n - \prod_{i=1}^n (1 - t k_i(x)) \right| \mathrm{d}t\,\mathrm{d}\mathcal{H}^n, \tag{3-8}$$

$$R_4 = \int_\Sigma \int_{1/k_n(x)}^{n/H_E(x)} \left| 1 - \frac{t}{n} H_E(x) \right|^n \mathrm{d}t\,\mathrm{d}\mathcal{H}^n. \tag{3-9}$$

Then by repeating the Montiel–Ros argument we deduce that

$$|E| \leq \frac{n}{n+1} \int_\Sigma \frac{1}{H_E}\,\mathrm{d}\mathcal{H}^n + |\zeta(Z')| - R_1 - R_2 - R_3 - R_4.$$

Therefore, by (3-5),

$$R_1+R_2+R_3+R_4 \le |\zeta(Z')| + C\|H_E-\lambda\|_{L^n(\partial E)},$$

where the R_i are defined in (3-6)–(3-9).

Let us next show that

$$|\zeta(Z')| \le C\|H_E(x)-\lambda\|_{L^n(\partial E)}. \tag{3-10}$$

Indeed, by the area formula we have

$$|\zeta(Z')| \le \int_{Z'} J_\tau\zeta \,\mathrm{d}\mathcal{H}^{n+1} = \int_{\Sigma_1'}\int_0^{\tilde R}\prod_{i=1}^n|1-tk_i(x)|\,\mathrm{d}t\,\mathrm{d}\mathcal{H}^n + \int_{\Sigma_2'}\int_0^{1/k_n(x)}\prod_{i=1}^n|1-tk_i(x)|\,\mathrm{d}t\,\mathrm{d}\mathcal{H}^n. \tag{3-11}$$

By the definition of Σ_1', we have $|1-tk_i(x)| = (1-tk_i(x))$ for every $(x,t)\in\Sigma_1'\times[0,\tilde R]$, and therefore by the arithmetic-geometric inequality we may estimate

$$\prod_{i=1}^n|1-tk_i(x)| \le C(1+|H_E(x)|^n) \quad \text{for } (x,t)\in\Sigma_1'\times[0,\tilde R].$$

Similarly, we deduce that

$$\prod_{i=1}^n|1-tk_i(x)| \le C(1+t^n|H_E(x)|^n) \qquad \text{for } x\in\Sigma_2' \quad\text{and}\quad 0\le t\le 1/k_n(x).$$

On the other hand, by the definition of Σ_2' we have $1/k_n(x) < \tilde R$. Therefore, by (3-11), $\lambda \le C$ and (3-3) we have

$$\begin{aligned}
|\zeta(Z')| &\le C\int_{\Sigma_1'\cup\Sigma_2'}\int_0^{\tilde R}(1+|H_E(x)|^n)\,\mathrm{d}t\,\mathrm{d}\mathcal{H}^n \\
&= C\tilde R\int_{\partial E\setminus\Sigma}(1+|H_E(x)|^n)\,\mathrm{d}\mathcal{H}^n \\
&\le C\int_{\partial E\setminus\Sigma}(1+\lambda^n+|H_E-\lambda|^n)\,\mathrm{d}\mathcal{H}^n \\
&\le C(\mathcal{H}^n(\partial E\setminus\Sigma)+\|H_E-\lambda\|^n_{L^n(\partial E)}) \\
&\le C\|H_E-\lambda\|_{L^n(\partial E)}
\end{aligned}$$

when $\|H_E-\lambda\|_{L^n(\partial E)}\le 1$. Hence by decreasing δ, if needed, we have (3-11). In particular,

$$R_1+R_2+R_3+R_4 \le C\|H_E-\lambda\|_{L^n(\partial E)}, \tag{3-12}$$

where the R_i are defined in (3-6)–(3-9).

Step 2: Here we utilize the estimate (3-12) and prove the following auxiliary result. For a Borel set $\Gamma\subset\partial E$ and $0<r<R$,

$$|E\cap\zeta(Z\cap(\Gamma\times(r,R)))| \ge \frac{\mathcal{H}^n(\Gamma)}{(n+1)R^n}(R-r)^{n+1} - C\|H_E-\lambda\|_{L^n(\partial E)}. \tag{3-13}$$

We prove (3-13) by "backtracking" the Montiel–Ros argument. By the definition of $R_1,\ R_2,\ R_3,\ R_4$ and (3-12) we may estimate

$$\begin{aligned}
|E\cap\zeta(Z\cap(\Gamma\times(r,R)))| &\ge |\zeta(Z\cap(\Gamma\times(r,R)))|-R_1\\
&\ge \int_{\zeta(Z\cap(\Gamma\times(r,R)))}\mathcal{H}^0(\zeta^{-1}(y)\cap Z\cap(\Gamma\times(r,R)))\,\mathrm{d}y-R_1-R_2\\
&= \int_{\Gamma\cap\Sigma}\int_{\min\{r,1/k_n(x)\}}^{\min\{R,1/k_n(x)\}}\prod_{i=1}^{n}(1-tk_i(x))\,\mathrm{d}t\,\mathrm{d}\mathcal{H}^n-R_1-R_2\\
&\ge \int_{\Gamma\cap\Sigma}\int_{\min\{r,1/k_n(x)\}}^{\min\{R,1/k_n(x)\}}\Big(1-\frac{t}{n}H_E(x)\Big)^n\,\mathrm{d}t\,\mathrm{d}\mathcal{H}^n-R_1-R_2-R_3\\
&\ge \int_{\Gamma\cap\Sigma}\int_{\min\{r,1/k_n(x)\}}^{\min\{R,n/H_E(x)\}}\Big(1-\frac{t}{n}H_E\Big)^n\,\mathrm{d}t\,\mathrm{d}\mathcal{H}^n-R_1-R_2-R_3-R_4\\
&\ge \int_{\Gamma\cap\Sigma}\int_{\min\{r,n/H_E(x)\}}^{\min\{R,n/H_E(x)\}}\Big(1-\frac{t}{n}H_E\Big)^n\,\mathrm{d}t\,\mathrm{d}\mathcal{H}^n-R_1-R_2-R_3-R_4.
\end{aligned}$$

Recall that for $x\in\Sigma$, we have $\frac{1}{2}\lambda\le H_E(x)\le 2\lambda$ and $R=n/\lambda$. Therefore, we may estimate

$$\begin{aligned}
\int_{\Gamma\cap\Sigma}\int_{\min\{r,n/H_E(x)\}}^{\min\{R,n/H_E(x)\}}\Big(1-\frac{t}{n}H_E\Big)^n\,\mathrm{d}t\,\mathrm{d}\mathcal{H}^n &\ge \int_{\Gamma\cap\Sigma}\int_{\min\{r,n/H_E(x)\}}^{\min\{R,n/H_E(x)\}}\Big(1-\frac{t}{n}\lambda\Big)^n\,\mathrm{d}t\,\mathrm{d}\mathcal{H}^n-C\|H_E-\lambda\|_{L^n(\partial E)}\\
&\ge \int_{\Gamma\cap\Sigma}\int_{r}^{R}\Big(1-\frac{t}{n}\lambda\Big)^n\,\mathrm{d}t\,\mathrm{d}\mathcal{H}^n-C\|H_E-\lambda\|_{L^n(\partial E)}\\
&= \frac{\mathcal{H}^n(\Gamma\cap\Sigma)n}{(n+1)\lambda}\Big(1-\frac{\lambda}{n}r\Big)^{n+1}-C\|H_E-\lambda\|_{L^n(\partial E)}\\
&= \frac{\mathcal{H}^n(\Gamma\cap\Sigma)R}{(n+1)}\Big(1-\frac{r}{R}\Big)^{n+1}-C\|H_E-\lambda\|_{L^n(\partial E)}.
\end{aligned}$$

Hence we obtain (3-13) from the previous two inequalities, from (3-3) and from (3-12).

Step 3: Here we finally prove the proposition. Recall the definition of E_r in (3-1). Let us first prove that

$$|E_r|\ge\frac{P(E)}{(n+1)R^n}(R-r)^{n+1}-C\|H_E-\lambda\|_{L^n(\partial E)} \tag{3-14}$$

for all $0<r<R$.

To this aim, we claim that

$$E\cap\zeta(Z\cap(\Sigma\times(r,R)))\subset E_r\cup\{y\in\zeta(Z):\mathcal{H}^0(\zeta^{-1}(y)\cap Z)\ge 2\}\cup\zeta(Z'). \tag{3-15}$$

The point of this inclusion is that almost every point which is of the form $y=x-t\nu_E(x)$, for $x\in Z$ and $t\in(r,R)$, belongs to E_r.

To this aim, let $y\in E\cap\zeta(\Sigma\times(r,R))$. Then we may write $y=x-t\nu_E(x)=\zeta(x,t)$ for some $x\in\Sigma$ and $t\in(r,R)$, with $(x,t)\in Z$. If $d_{\partial E}(y)=|y-x|$, then $y\in E_r$ because $|x-y|=t>r$. Otherwise,

$$d_{\partial E}(y)=|y-\tilde{x}|=\tilde{r}<t\quad\text{for }\tilde{x}\in\partial E,$$

so we may write $y = \tilde{x} - \tilde{r}\nu_E(x) = \zeta(\tilde{x}, \tilde{r})$ and $(\tilde{x}, \tilde{r}) \in Z \cup Z'$. Again, if $(\tilde{x}, \tilde{r}) \notin Z'$, then $(\tilde{x}, \tilde{r}) \in Z$ and thus $\mathcal{H}^0(\zeta^{-1}(y) \cap Z) \geq 2$. Hence we have (3-15).

Recall that by the definition of R_2 and by (3-12),

$$|\{y \in \zeta(Z) : \mathcal{H}^0(\zeta^{-1}(y) \cap Z) \geq 2\}| \leq \int_{\zeta(Z)} |\mathcal{H}^0(\zeta^{-1}(y) \cap Z) - 1| \,\mathrm{d}y \\ \leq C\|H_E - \lambda\|_{L^n(\partial E)}. \tag{3-16}$$

We then use (3-15), (3-16), (3-10) and (3-13) with $\Gamma = \Sigma$ to deduce

$$|E_r| \geq |E \cap \zeta(Z \cap (\Sigma \times (r, R)))| - C\|H_E - \lambda\|_{L^n(\partial E)} \\ \geq \frac{\mathcal{H}^n(\Sigma)}{(n+1)R^n}(R-r)^{n+1} - C\|H_E - \lambda\|_{L^n(\partial E)}.$$

The inequality (3-14) then follows from (3-3).

Let us next show that for all $r \in (0, R)$,

$$|E_r| \leq \frac{\mathcal{H}^n(\Gamma_r)}{(n+1)R^n}(R-r)^{n+1} + C\|H_E - \lambda\|_{L^n(\partial E)}, \tag{3-17}$$

where $\Gamma_r \subset \partial E$ is defined in (3-2).

First we show

$$|E_R| \leq C\|H_E - \lambda\|_{L^n(\partial E)}. \tag{3-18}$$

This follows from an already familiar argument, so we only sketch it here. It is easy to see that

$$E_R \subset \zeta(Z') \cup \zeta(Z \cap (\Sigma \times (R, \infty))).$$

Moreover, since $\frac{1}{2}\lambda \leq H_E(x) \leq 2\lambda$ for $x \in \Sigma$,

$$J_\tau \zeta(x, t) = \prod_{i=1}^{n} |1 - t k_i(x)| \leq C(1 + |H_E(x)|^n) \leq C \quad \text{for } (x, t) \in Z \cap (\Sigma \times (R, \infty)).$$

Recall that $R = n/\lambda$. Therefore, we have

$$|\zeta(Z \cap (\Sigma \times (R, \infty)))| \leq \int_\Sigma \int_R^{\max\{n/H_E(x), R\}} J_\tau \zeta(x, t) \,\mathrm{d}t \,\mathrm{d}\mathcal{H}^n \\ \leq C \int_\Sigma \left| \frac{n}{H_E} - R \right| \mathrm{d}t \,\mathrm{d}\mathcal{H}^n \\ \leq C\|H_E - \lambda\|_{L^n(\partial E)}.$$

The estimate (3-18) then follows from $|E_R| \leq |\zeta(Z \cap (\Sigma \times (R, \infty)))| + |\zeta(Z')|$ and (3-10).

Note that for all $\rho \in (r, R)$ we have $\{x \in E : d_{\partial E}(x) = \rho\} = \zeta(\Gamma_\rho, \rho)$ and $\Gamma_\rho \subset \Gamma_r$. We also set $\zeta_\rho = \zeta(\,\cdot\,, \rho) : \partial E \to \mathbb{R}^{n+1}$, and thus $\{x \in E : d_{\partial E}(x) = \rho\} = \zeta_\rho(\Gamma_\rho)$ and

$$J_\tau \zeta_\rho(x) = \prod_{i=1}^{n} |1 - \rho k_i(x)| \leq \left(1 - \frac{H_E}{n}\rho\right)^n \quad \text{for } x \in \Gamma_\rho.$$

Therefore by (3-18) and by coarea and area formulas we obtain

$$
\begin{aligned}
|E_r| \le |E_r| - |E_R| + C\|H_E-\lambda\|_{L^n(\partial E)} &\le \int_r^R \mathcal{H}^n(\{x\in E : d_{\partial E}=\rho\})\,\mathrm{d}\rho + C\|H_E-\lambda\|_{L^n(\partial E)}\\
&= \int_r^R \mathcal{H}^n(\zeta_\rho(\Gamma_\rho))\,\mathrm{d}\rho + C\|H_E-\lambda\|_{L^n(\partial E)}\\
&\le \int_r^R\int_{\Gamma_\rho} J_\tau\zeta_\rho(x)\,\mathrm{d}\mathcal{H}^n\,\mathrm{d}\rho + C\|H_E-\lambda\|_{L^n(\partial E)}\\
&\le \int_r^R\int_{\Gamma_\rho}\Big(1-\frac{H_E}{n}\rho\Big)^n\,\mathrm{d}\mathcal{H}^n\,\mathrm{d}\rho + C\|H_E-\lambda\|_{L^n(\partial E)}\\
&\le \int_r^R \mathcal{H}^n(\Gamma_\rho)\Big(1-\frac{\lambda}{n}\rho\Big)^n\,\mathrm{d}\rho + C\|H_E-\lambda\|_{L^n(\partial E)}\\
&\le \mathcal{H}^n(\Gamma_r)\int_r^R\Big(1-\frac{\rho}{R}\Big)^n\,\mathrm{d}\rho + C\|H_E-\lambda\|_{L^n(\partial E)}\\
&= \frac{\mathcal{H}^n(\Gamma_r)}{(n+1)R^n}(R-r)^{n+1} + C\|H_E-\lambda\|_{L^n(\partial E)}.
\end{aligned}
$$

Hence we have (3-17).

The second claim of the proposition follows immediately from (3-14) and (3-17). These also imply

$$
\left| |E_r| - \frac{P(E)}{(n+1)R^n}(R-r)^{n+1}\right| \le C\|H_E-\lambda\|_{L^n(\partial E)}.
$$

The first claim thus follows from (3-4) and $R=n/\lambda$.

For the last claim we refine the inclusion (3-15) and show that for $0<\rho<r<R$ and $r'\in(r,R)$,

$$
E\cap\zeta(Z\cap(\Gamma_{r'}\times(r'-\rho,R)))\subset(E_r+B_\rho)\cup\{y\in\zeta(Z):\mathcal{H}^0(\zeta^{-1}(y)\cap Z)\ge 2\}\cup\zeta(Z'). \tag{3-19}
$$

Indeed, let $y\in E\cap\zeta(Z\cap(\Gamma_{r'}\times(r'-\rho,R)))$. Then we may write $y=x-t\nu_E(x)$ for some $x\in\Sigma\cap\Gamma_{r'}$ and $t\in(r'-\rho,R)$, with $(x,t)\in Z$. If $t\in(r',R)$, then by (3-15),

$$
\begin{aligned}
y\in E\cap\zeta(Z\cap(\Sigma\times(r,R)))&\subset E_r\cup\{y\in\zeta(Z):\mathcal{H}^0(\zeta^{-1}(y)\cap Z)\ge 2\}\cup\zeta(Z')\\
&\subset(E_r+B_\rho)\cup\{y\in\zeta(Z):\mathcal{H}^0(\zeta^{-1}(y)\cap Z)\ge 2\}\cup\zeta(Z').
\end{aligned}
$$

Let us then assume that $t\in(r'-\rho,r']$. We write $y=x-r'\nu_E(x)+(r'-t)\nu_E(x)$. Since $x\in\Gamma_{r'}$, i.e., ∂E satisfies the interior ball condition at x with radius $r'>r$, necessarily we have $x-r'\nu_E(x)\in E_r$. Therefore, since $0\le r'-t<\rho$, we conclude that $y\in E_r+B_\rho$ and (3-19) follows.

We use (3-10), (3-13), (3-16) and (3-19) to conclude

$$
\begin{aligned}
|E_r+B_\rho| &\ge |E\cap\zeta(Z\cap(\Gamma_{r'}\cap\times(r'-\rho,R)))| - C\|H_E-\lambda\|_{L^n(\partial E)}\\
&\ge \frac{\mathcal{H}^n(\Gamma_{r'})}{(n+1)R^n}(R-(r'-\rho))^{n+1} - C\|H_E-\lambda\|_{L^n(\partial E)}.
\end{aligned}
$$

By using the second claim of the proposition and then letting $r' \to r$, we deduce

$$|E_r + B_\rho| \geq \frac{P(E)}{(n+1)R^n}(R-(r-\rho))^{n+1} - \frac{C}{(R-r)^{n+1}}\|H_E - \lambda\|_{L^n(\partial E)}.$$

On the other hand, clearly $E_r + B_\rho \subset E_{r-\rho}$. Then by (3-17) we have

$$|E_r + B_\rho| \leq |E_{r-\rho}| \leq \frac{P(E)}{(n+1)R^n}(R-(r-\rho))^{n+1} + C\|H_E - \lambda\|_{L^n(\partial E)}.$$

The last claim thus follows from the two previous inequalities and (3-4). □

Proof of Theorem 1.2. Let E, λ and C_0 be as in the formulation of Theorem 1.2. Recall that we write $R = n/\lambda$. As before C denotes a constant which may change from line to line but always depends only on C_0 and n. Let us write

$$\varepsilon := \|H_E - \lambda\|_{L^n(\partial E)}.$$

If $\varepsilon = 0$, then E is a disjoint union of balls by [Delgadino and Maggi 2019]. Let us then assume that $0 < \varepsilon \leq \delta$, where δ is initially set as in Proposition 3.3. We might shrink δ several times but always in such a way that it depends only on C_0 and the dimension n. Indeed, by shrinking δ, if needed, Lemma 2.4 provides the estimates

$$1/C \leq \lambda \quad \text{and} \quad R \leq C,$$

and hence the first claim of Theorem 1.2 is clear. We will use these estimates repeatedly without further mention.

By Lemma 2.4, the number of connected components of E and their diameters are bounded by C. Thus, by applying a similar argument as in the proof of Proposition 3.3 (to obtain (3-4)) on each component and then summing these estimates we obtain

$$\Big| nP(E) - (n+1)\lambda|E| \Big| \leq C\varepsilon. \tag{3-20}$$

By possibly shrinking δ we have $R - \delta^{1/(n+2)} \geq \frac{1}{2}R$. Choose $r_0 = R - \varepsilon^{1/(n+2)}$. Then the volume estimates given by Proposition 3.3 read as

$$\left| |E_r| - \frac{|E|}{R^{n+1}}(R-r)^{n+1} \right| \leq C\varepsilon \tag{3-21}$$

for all $0 \leq r < R$ and

$$\left| |E_r + B_\rho| - \frac{|E|}{R^{n+1}}(R-(r-\rho))^{n+1} \right| \leq C\varepsilon^{1/(n+2)} \tag{3-22}$$

for all $0 \leq \rho \leq r \leq r_0$. We remark that by (3-21) we have

$$|E_{r_0}| \geq \frac{|E|}{R^{n+1}}\varepsilon^{(n+1)/(n+2)} - C\varepsilon \geq \frac{1}{C}\varepsilon^{(n+1)/(n+2)} - C\varepsilon.$$

Hence by decreasing δ, if needed, we may assume that E_{r_0} is nonempty. This implies that $E_{r'}$ is nonempty for $r' > r_0$ when $|r' - r_0|$ is small enough. Since for any $r' > r_0$ it is geometrically clear that $\Gamma_{r'} \subset \partial E_{r_0} + \bar{B}_{r_0}$, and then by using Proposition 3.3 and $r_0 = R - \varepsilon^{1/(n+2)}$ we have

$$\mathcal{H}^n(\partial E \setminus (\bar{E}_{r_0} + \bar{B}_{r_0})) \le \mathcal{H}^n(\partial E \setminus \Gamma_{r'}) \le C \frac{\varepsilon}{(r_0 - r' + \varepsilon^{1/(n+2)})^{n+1}}.$$

Thus by letting $r' \to r_0$ the previous estimate yields

$$\mathcal{H}^n(\partial E \setminus (\bar{E}_{r_0} + \bar{B}_{r_0})) \le C\varepsilon^{1/(n+2)}. \tag{3-23}$$

As previously, we divide the proof into three steps.

Step 1: Recall that $r_0 = R - \varepsilon^{1/(n+2)} \ge \frac{1}{2}R$. We prove that there is a positive constant $d_0 = d_0(C_0, n) \le \frac{1}{4}R$ such that if $x, y \in E_{r_0}$, then either

$$|x - y| < \varepsilon^{1/(2(n+2))} \quad \text{or} \quad |x - y| \ge d_0. \tag{3-24}$$

Let us fix $x, y \in E_{r_0}$. We write $d := |x - y|$ and denote the segment from x to y by

$$J_{xy} := \{tx + (1-t)y : t \in [0, 1]\}.$$

We may assume that d is small, since otherwise the claim (3-24) is trivially true. To be more precise, we assume

$$d \le \min\{\tfrac{1}{4}R, 1\}. \tag{3-25}$$

Let us first show that

$$J_{xy} \subset E_{r_0 - R^{-1}d^2}. \tag{3-26}$$

Note that $r_0 - R^{-1}d^2 > 0$ by $r_0 \ge \frac{1}{2}R$ and (3-25), and hence $E_{r_0 - R^{-1}d^2}$ is well defined and nonempty. Choose $z \in \mathbb{R}^{n+1} \setminus E$ and $z' \in J_{xy}$ such that

$$|z - z'| = \operatorname{dist}(\mathbb{R}^{n+1} \setminus E, J_{xy}).$$

If $z' = x$ or $z' = y$, then it follows from $x, y \in E_{r_0}$ that $|z - z'| > r_0$. If not, then from the fact that z' is the closest point on J_{xy} to z, we deduce that the vector $x - z'$ is orthogonal to $z - z'$, i.e., $\langle x - z', z - z' \rangle = 0$. Note also that $\min\{|x - z'|, |y - z'|\} \le \frac{1}{2}d$ and we may thus assume that $|x - z'| \le \frac{1}{2}d$. Therefore, by the Pythagorean theorem we have

$$|x - z|^2 = |x - z'|^2 + |z - z'|^2 \le \tfrac{1}{4}d^2 + |z - z'|^2.$$

Since $|x - z| > r_0$, the previous estimate gives us

$$|z - z'|^2 > r_0^2 - \tfrac{1}{4}d^2.$$

We deduce from $r_0 \ge \frac{1}{2}R$ and (3-25) that

$$\left(r_0^2 - \tfrac{1}{4}d^2\right)^{1/2} \ge r_0 - R^{-1}d^2.$$

The previous two estimates yield $|z - z'| > r_0 - R^{-1}d^2$, and claim (3-26) follows due to the choice of z and z'.

Again, we use $r_0 \geq \frac{1}{2}R$ and (3-25) to observe that

$$r_0 - (1+R^{-1})d^2 \geq r_0 - d - R^{-1}d^2 \geq \tfrac{1}{2}R - \tfrac{1}{4}R - \tfrac{1}{16}R > 0.$$

Thus $E_{r_0-(1+R^{-1})d^2}$ is well defined and nonempty. Next, we deduce from (3-26) and $E_r + B_\rho \subset E_{r-\rho}$ that

$$J_{xy} + B_{d^2} \subset E_{r_0 - R^{-1}d^2} + B_{d^2} \subset E_{r_0-(1+R^{-1})d^2}. \tag{3-27}$$

Since $J_{xy} + B_{d^2}$ contains the cylinder $J_{xy} \times B^n_{d^2}$, it is clear that

$$|J_{xy} + B_{d^2}| \geq \omega_n d^{1+2n}.$$

On the other hand, (3-21) and $\varepsilon \leq 1$ (we may assume $\delta \leq 1$) imply

$$\begin{aligned} |E_{r_0-(1+R^{-1})d^2}| &\leq \frac{|E|}{R^{n+1}}(R - (r_0 - (1+R^{-1})d^2))^{n+1} + C\varepsilon \\ &= \frac{|E|}{R^{n+1}}(\varepsilon^{1/(n+2)} + (1+R^{-1})d^2)^{n+1} + C\varepsilon \\ &\leq \frac{|E|}{R^{n+1}}(\varepsilon^{1/(n+2)} + (1+R^{-1})d^2)^{n+1} + C\varepsilon^{(n+1)/(n+2)} \\ &\leq Cd^{2(n+1)} + C\varepsilon^{(n+1)/(n+2)}. \end{aligned}$$

Then (3-27) yields

$$\omega_n d^{1+2n} \leq Cd^{2(n+1)} + C\varepsilon^{(n+1)/(n+2)}.$$

If $d \geq \varepsilon^{1/(2(n+2))}$, then

$$\omega_n d^{1+2n} \leq Cd^{2(n+1)}.$$

This implies $d \geq c > 0$ for some $c = c(C_0, n)$. By recalling (3-25), claim (3-24) follows.

Step 2: By (3-24) and possibly replacing δ with $\min\{\delta, (\frac{1}{8}d_0)^{2(n+2)}\}$ we may divide the set E_{r_0} into N clusters $E^1_{r_0}, \dots, E^N_{r_0}$ such that we fix a point $x_i \in E_{r_0}$ and define the corresponding cluster $E^i_{r_0}$ as

$$E^i_{r_0} = \{x \in E_{r_0} : |x - x_i| \leq \tfrac{1}{8}d_0\}.$$

By (3-24), we have $E^i_{r_0} \subset B_{\varepsilon_0}(x_i)$, where $\varepsilon_0 = \varepsilon^{1/(2(n+2))}$ and $|x_i - x_j| \geq d_0$ for $i \neq j$. Therefore, we have for every $\rho > 0$

$$\bigcup_{i=1}^N B_\rho(x_i) \subset E_{r_0} + B_\rho \subset \bigcup_{i=1}^N B_{\rho+\varepsilon_0}(x_i). \tag{3-28}$$

Since $r_0 \geq \frac{1}{2}R > \frac{1}{4}R \geq d_0$ and $|x_i - x_j| \geq d_0$ for $i \neq j$, we have that the balls $B_\rho(x_1), \dots, B_\rho(x_N)$ with $\rho = \frac{1}{4}d_0$ are disjoint and contained in E, which in turn implies there is an upper bound $N_0 = N_0(C_0, n) \in \mathbb{N}$ for the number of clusters N.

Next we improve the lower bound $|x_i - x_j| \geq d_0$ and prove that there is a positive constant $C_1 = C_1(C_0, n)$ such that

$$|x_i - x_j| \geq 2R - 2C_1\varepsilon^{1/(n+2)^2} \quad \text{for all pairs } i \neq j. \tag{3-29}$$

As a byproduct we prove the last statement of the theorem, i.e., we show

$$|P(E) - N(n+1)\omega_{n+1}R^n| \le C\varepsilon^{1/(2(n+2))}. \tag{3-30}$$

Recall that the balls $B_{d_0/4}(x_1), \dots, B_{d_0/4}(x_N)$ are disjoint. Therefore, using $N \le N_0$ and (3-28) with $\rho = \frac{1}{4}d_0$ we deduce

$$\big||E_{r_0} + B_{d_0/4}| - N\omega_{n+1}\big(\tfrac{1}{4}d_0\big)^{n+1}\big| \le C\varepsilon_0 = C\varepsilon^{1/(2(n+2))}.$$

On the other hand, we have $\frac{1}{4}d_0 \le \frac{1}{16}R < \frac{1}{2}R \le r_0$, so we may use (3-22) to obtain

$$\left||E_{r_0} + B_{d_0/4}| - \frac{|E|}{R^{n+1}}\big(\tfrac{1}{4}d_0 + \varepsilon^{1/(n+2)}\big)^{n+1}\right| \le C\varepsilon^{1/(n+2)}.$$

These two estimates and $\varepsilon \le 1$ imply

$$\big||E| - N\omega_{n+1}R^{n+1}\big| \le C\varepsilon^{1/(2(n+2))}. \tag{3-31}$$

Thus (3-20), $R = n/\lambda$ and (3-31) yield (3-30).

To obtain (3-29), let us assume that there is $0 < h < \frac{1}{2}R$ such that $|x_i - x_j| < 2R - 2h$ for some $i \ne j$. This implies that the balls $B_R(x_i)$ and $B_R(x_j)$ intersect each other such that a set enclosed by a spherical cap of height h is included in their intersection. As the volume enclosed by the spherical cap of height h has a lower bound $c_n R^{n+1}h^{(n+2)/2}$, with some dimensional constant c_n, then there is $c = c(C_0, n)$ such that

$$|B_R(x_i) \cap B_R(x_j)| \ge ch^{(n+2)/2}.$$

We use the previous estimate as well as (3-22), (3-28), (3-31), $\varepsilon \le 1$ and $N \le N_0$ to estimate

$$\begin{aligned}
N\omega_{n+1}R^{n+1} &\le |E| + C\varepsilon_0 \\
&\le |E_{r_0} + B_{r_0}| + C\varepsilon_0 + C\varepsilon^{1/(n+2)} \\
&\le \left|\bigcup_{i=1}^N B_{R+\varepsilon_0}(x_i)\right| + C\varepsilon_0 + C\varepsilon^{1/(n+2)} \\
&\le \left|\bigcup_{i=1}^N B_R(x_i)\right| + N\omega_{n+1}((R+\varepsilon_0)^{n+1} - R^{n+1}) + C\varepsilon_0 + C\varepsilon^{1/(n+2)} \\
&\le N\omega_{n+1}R^{n+1} - |B_R(x_i) \cap B_R(x_j)| + C\varepsilon_0 + C\varepsilon^{1/(n+2)} \\
&\le N\omega_{n+1}R^{n+1} - ch^{(n+2)/2} + C\varepsilon_0 + C\varepsilon^{1/(n+2)} \\
&= N\omega_{n+1}R^{n+1} - ch^{(n+2)/2} + C\varepsilon^{1/(2(n+2))} + C\varepsilon^{1/(n+2)} \\
&\le N\omega_{n+1}R^{n+1} - ch^{(n+2)/2} + C\varepsilon^{1/(2(n+2))}.
\end{aligned}$$

Thus $h^{(n+2)/2} \le C\varepsilon^{1/(2(n+2))}$ and (3-29) follows.

<u>Step 3</u>: Let C_1 be as in (3-29). By decreasing δ, if needed, we may assume

$$0 < R - C_1\varepsilon^{1/(n+2)^2} < R - \varepsilon^{1/(n+2)} = r_0.$$

Then by (3-28) and (3-29) we have that the balls $B_\rho(x_1), \dots, B_\rho(x_N)$, with $\rho = R - C_1\varepsilon^{1/(n+2)^2}$, are disjoint and

$$\bigcup_{i=1}^{N} B_\rho(x_i) \subset E_{r_0} + B_\rho \subset E_{r_0-\rho} \subset E. \tag{3-32}$$

This, $\varepsilon \le 1$, $N \le N_0$ and (3-31) imply

$$\left| E \setminus \bigcup_{i=1}^{N} B_\rho(x_i) \right| \le C\varepsilon^{1/(n+2)^2}. \tag{3-33}$$

Set $\varepsilon_1 = \varepsilon^{1/(n+2)^3}$. We prove

$$E \subset \bigcup_{i=1}^{N} B_\eta(x_i) \tag{3-34}$$

for $\eta = R + C_2\varepsilon_1$ with some positive $C_2 = C_2(n, C_0)$. By decreasing δ, if necessary, we deduce from (3-33) that

$$|B_{\varepsilon_1}| > \left| E \setminus \bigcup_{i=1}^{N} B_\rho(x_i) \right|.$$

Thus, if $x \in E_{\varepsilon_1}$, then $B_{\varepsilon_1}(x) \cap \bigcup_{i=1}^{N} B_\rho(x_i)$ must be nonempty. This implies

$$E_{\varepsilon_1} \subset \bigcup_{i=1}^{N} B_{\rho+\varepsilon_1}(x_i). \tag{3-35}$$

Assume that for $x \in \partial E$,

$$d_x := \operatorname{dist}(x, \bar{E}_{r_0} + \bar{B}_{r_0}) > 0.$$

Then by (3-23)

$$\mathcal{H}^n(\partial E \cap B(x, d_x)) \le C\varepsilon^{1/(n+2)}.$$

Let $\delta_n \in \mathbb{R}_+$ be as in Lemma 3.2, and set $r_x = \min\{d_x, \delta_n/\lambda\}$. Again, by possibly decreasing δ so that $\delta \le \delta_n$, Lemma 3.2 yields

$$\delta_n r_x^n \le \mathcal{H}^n(\partial E \cap B_{r_x}(x)).$$

By combining the two previous estimates we have

$$\min\left\{ d_x, \frac{\delta_n}{\lambda} \right\} \le C\varepsilon^{1/(n(n+2))}.$$

Since $\delta_n/\lambda \ge \delta_n/C$, by decreasing δ, if necessary, the previous estimate implies $r_x = d_x$ and further yields

$$d_x \le C\varepsilon^{1/(n(n+2))} \le C\varepsilon^{1/(n+2)^2}. \tag{3-36}$$

On the other hand, by (3-28),

$$\bar{E}_{r_0} + \bar{B}_{r_0} \subset E_{r_0} + B_R \subset \bigcup_{i=1}^{N} B_{R+\varepsilon_0}(x_i), \tag{3-37}$$

where $\varepsilon_0 = \varepsilon^{1/(2(n+2))} \le \varepsilon^{1/(n+2)^2}$. Thus (3-36) and (3-37) imply

$$\partial E \subset \bigcup_{i=1}^{N} B_{\tilde{\eta}}(x_i)$$

with $\tilde{\eta} = R + C\varepsilon^{1/(n+2)^2}$. By combining this observation with (3-35) we obtain (3-34).

Finally, by decreasing δ one more time, if necessary, (3-30), (3-32) and (3-34) yield

$$\bigcup_{i=1}^{N} B_{\rho_-}(x_i) \subset E \subset \bigcup_{i=1}^{N} B_{\rho_+}(x_i),$$

where $\rho_- = R - C\varepsilon^{1/(n+2)^3}$, $\rho_+ = R + C\varepsilon^{1/(n+2)^3}$, the balls $B_{\rho_-}(x_1), \dots, B_{\rho_-}(x_N)$ are mutually disjoint, for N we have

$$|P(E) - N(n+1)\omega_{n+1}R^n| \le C\varepsilon^{1/(n+2)^3}$$

and $C = C(C_0, n) \in \mathbb{R}_+$. The claim of Theorem 1.2 then follows by Remark 3.1. □

4. Asymptotic behavior of the volume preserving mean curvature flow

In this section we first define the flat flow and recall some of its basic properties. We do this in the general dimensional case $\mathbb{R}^{n+1}$ and restrict ourselves to the case $n \le 2$ only in the proof of Theorem 1.1. We begin by defining the flat flow of (1-1).

Assume that $E_0 \subset \mathbb{R}^{n+1}$ is a bounded set of finite perimeter with the volume of the unit ball $|E_0| = \omega_{n+1}$. For a given $h \in \mathbb{R}_+$ we construct a sequence of sets $(E_k^h)_{k=1}^{\infty}$ by an iterative minimizing procedure called minimizing movements, where initially $E_0^h = E_0$ and E_{k+1}^h is a minimizer of the problem

$$\mathcal{F}_h(E, E_k) = P(E) + \frac{1}{h}\int_E \bar{d}_{E_k}\,\mathrm{d}x + \frac{1}{\sqrt{h}}\big||E| - \omega_{n+1}\big|. \tag{4-1}$$

Recall that $\bar{d}_{E_k}$ is the signed distance function from E_k. We then define the approximative flat flow $(E_t^h)_{t\ge 0}$ by

$$E_t^h = E_k^h, \quad \text{for } (k-1)h \le t < kh. \tag{4-2}$$

By [Mugnai et al. 2016] we know that there is a subsequence of the approximative flat flow which converges:

$$(E_t^{h_l})_{t\ge 0} \to (E_t)_{t\ge 0},$$

where for every $t > 0$ the set E_t is a set of finite perimeter with $|E_t| = \omega_{n+1}$. Any such limit is called a flat flow of (1-1). It follows from [Mugnai et al. 2016] that when $n \le 6$ and if the perimeters of E_t^h converge, i.e., $\lim_{h\to 0} P(E_t^h) = P(E_t)$ for every $t > 0$, then the flat flow is a weak solution of the volume preserving mean curvature flow. It is not known if the flat flow coincides with the classical solution of (1-1) when the latter is well defined and smooth, but the result in [Chambolle and Novaga 2008] seems to suggest this (see also [Chambolle et al. 2015]).

Preliminary results. Let us take a more rigorous approach to the concepts heuristically introduced above. We base this mainly on [Mugnai et al. 2016], with the only difference being that the volume constraint has a different value. Obviously, this does not affect the arguments.

First, we take a closer look at the functional $\mathcal{F}_h$ given by (4-1). If $E, F \subset \mathbb{R}^{n+1}$ are bounded sets of finite perimeter, then it is easy to see that modifications of E in a set of measure zero do not affect the value $\mathcal{F}_h(E, F)$, whereas such modifications of F may lead to drastic changes of the value of $\mathcal{F}_h(E, F)$. To eliminate this issue, we use a convention that a topological boundary of a set of finite perimeter is always the support of the corresponding Gauss–Green measure. Thus, we consider $\mathcal{F}_h$ as a functional

$$X_{n+1} \times \{A \in X_{n+1} : A \neq \varnothing\} \to \mathbb{R},$$

where

$$X_{n+1} = \{E \subset \mathbb{R}^{n+1} : E \text{ is a bounded set of finite perimeter with } \partial E = \operatorname{spt} \mu_E\}.$$

We remark that if E_0 is essentially open or closed and $E_0 \in X_{n+1}$, then we may assume X_{n+1} to be open or closed, respectively.

For $F \in X_{n+1}$ nonempty, there is always a minimizer E of the functional $\mathcal{F}_h(\,\cdot\,, F)$ in the class X_{n+1} satisfying the *discrete dissipation inequality*

$$P(E) + \frac{1}{h}\int_{E\Delta F} d_{\partial F}\,\mathrm{d}x + \frac{1}{\sqrt{h}}\big||E| - \omega_{n+1}\big| \le P(F) + \frac{1}{\sqrt{h}}\big||F| - \omega_{n+1}\big|; \tag{4-3}$$

see [Mugnai et al. 2016, Lemma 3.1]. Moreover, there is a dimensional constant C_n such that

$$\sup_{E\Delta F} d_{\partial F} \le C_n\sqrt{h}; \tag{4-4}$$

see [Mugnai et al. 2016, Proposition 3.2]. The minimizer E is always a (Λ, r_0)-*minimizer* in any open neighborhood of E with suitable $\Lambda, r_0 \in \mathbb{R}_+$ satisfying $\Lambda r_0 \le 1$. Thus, by the standard regularity theory [Maggi 2012, Theorem 26.5 and Theorem 28.1] $\partial^* E$ is relatively open in ∂E and $C^{1,\alpha}$ regular with any $0 < \alpha < \frac{1}{2}$ and the Hausdorff dimension of the singular part $\partial E \setminus \partial^* E$ is at most $n-7$. These imply that E can always be chosen as an open set. On the other hand, if E is nonempty, it has a Lipschitz-continuous distributional mean curvature H_E satisfying the Euler–Lagrange equation

$$\frac{\bar{d}_F}{h} = -H_E + \lambda_E, \tag{4-5}$$

where the *Lagrange multiplier* can be written in the case $|E| \neq \omega_{n+1}$ as

$$\lambda_E = \frac{1}{\sqrt{h}}\operatorname{sgn}(\omega_{n+1} - |E|), \tag{4-6}$$

see [Mugnai et al. 2016, Lemma 3.7]. Thus, using standard elliptic estimates one can show that $\partial^* E$ is in fact $C^{2,\alpha}$ regular and (4-5) holds in the classical sense on $\partial^* E$. In particular, E is $C^{2,\alpha}$ regular when $n \le 6$. Moreover, if $x \in \partial E$ satisfies the exterior or interior ball condition with any r, then it must belong to the reduced boundary of E. This is well known and follows essentially from [Delgadino and Maggi 2019, Lemma 3].

Now let us turn our attention back to flat flows. Let $E_0 \in X_{n+1}$ be a set with volume ω_{n+1} and let $0 < h < (\omega_{n+1}/P(E_0))^2$. Then we find a minimizer $E_1^h \in X_{n+1}$ for $\mathcal{F}_h(\,\cdot\,, E_0)$, and by (4-3) we have $\big||E_1^h| - \omega_{n+1}\big| \le \sqrt{h}P(E_0)$ implying, via the condition $h < (\omega_n/P(E_0))^2$, that E_1^h is nonempty. Again we find a minimizer $E_2^h \in X_{n+1}$ for $\mathcal{F}_h(\,\cdot\,, E_1)$, and using (4-3) twice we obtain $\big||E_2^h| - \omega_{n+1}\big| \le \sqrt{h}P(E_0)$ and thus E_2^h is also nonempty. By continuing the procedure we find nonempty sets $E_0^h, E_1^h, E_2^h, \ldots \in X_{n+1}$ as mentioned earlier, i.e., $E_0^h = E_0$ and E_k^h is a minimizer of $\mathcal{F}_h(\,\cdot\,, E_{k-1})$ for every $k \in \mathbb{N}$. Thus we may define an approximate flat flow $(E_t^h)_{t\ge 0}$, with the initial set E_0, defined by (4-2). Further, a flat flow as a limit is defined as before. By iterating (4-3) we obtain

$$P(E_{kh}^h) + \frac{1}{h}\sum_{j=1}^{k}\int_{E_{jh}^h \Delta E_{(j-1)h}^h} d_{\partial E_{(j-1)h}^h}\,\mathrm{d}x + \frac{1}{\sqrt{h}}\big||E_{kh}^h| - \omega_{n+1}\big| \le P(E_0) \quad \text{for every } k \in \mathbb{N}. \tag{4-7}$$

By the earlier discussion we may assume that E_t^h is an open set, for every $t \ge h$, and ∂E_t^h is C^2 regular up to the singular part $\partial E_t^h \setminus \partial^* E_t^h$ with Hausdorff dimension at most $n-7$. We use the shorthand notation λ_t^h for the corresponding Lagrange multiplier.

Next we list some basic properties of the approximative flat flow.

Proposition 4.1. *Let $(E_t^h)_{t\ge 0}$ be an approximative flat flow starting from $E_0 \in X_{n+1}$ with volume ω_{n+1} and $P(E_0) \le C_0$. There is a positive constant $C = C(C_0, n)$ such that the following hold for every $0 < h < (\omega_n/P(E_0))^2$:*

(i) *For every s, t with $h \le s \le t - h$ we have $|E_s^h \Delta E_t^h| \le C\sqrt{t-s}$.*

(ii) *Suppose that for a given $T_1 \ge 0$ we have $|E_{T_1}^h| = \omega_{n+1}$. Then $P(E_{T_1}^h) \ge P(E_t^h)$ for every $t \ge T_1$ and*

$$\int_{T_1+h}^{T_2}\int_{\partial^* E_t^h}(H_{E_t^h} - \lambda_t^h)^2\,\mathrm{d}\mathcal{H}^n\,\mathrm{d}t \le C(P(E_{T_1}^h) - P(E_{T_2}^h))$$

for every $T_2 \ge T_1 + h$. Moreover, for every $h \le T_1 < T_2$,

$$\int_{T_1}^{T_2}\int_{\partial^* E_t^h}(H_{E_t^h} - \lambda_t^h)^2\,\mathrm{d}\mathcal{H}^n\,\mathrm{d}t \le C\,P(E_0).$$

(iii) *For every $T > 0$ there is $R = R(E_0, T)$ such that $E_t^h \subset B_R$ for all $0 \le t \le T$.*

(iv) *If $(h_k)_k$ is a sequence of positive numbers converging to zero, then up to a subsequence there exist approximative flat flows $((E_t^{h_k})_{t\ge 0})_k$ which converge to a flat flow $(E_t)_{t\ge 0}$ in the L^1 sense in space and pointwise in time, where $E_t \in X_{n+1}$, i.e., for every $t \ge 0$,*

$$\lim_{h_k \to 0}|E_t^{h_k} \Delta E_t| = 0.$$

The limit flow also satisfies $|E_s \Delta E_t| \le C\sqrt{t-s}$ for every $0 < s < t$ and $|E_t| = \omega_{n+1}$ for every $t \ge 0$.

(v) *If E_0 is either open or closed, then the sequence in* (iv) *converges to $(E_t)_{t\ge 0}$ in the L^1 sense in space and compactly uniformly in time, i.e., for a fixed T,*

$$\lim_{h_k \to 0}\sup_{t\in[0,T]}|E_t^{h_k} \Delta E_t| = 0.$$

Moreover, $|E_s \Delta E_t| \le C\sqrt{t-s}$ for every $0 \le s < t$.

Proof. Claims (i)–(iv) are essentially proved in [Mugnai et al. 2016]; see the proofs of Proposition 3.5, Lemma 3.6 and Theorem 2.2.

To prove (v) we first show that

$$|E_h^h \Delta E_0| \to 0 \quad \text{as } h \to 0,$$

which immediately implies via (iv) that $|E_0 \Delta E_t| \le C\sqrt{t}$ for every $t \ge 0$ and hence the second claim of (v) holds. Then the compactly uniform convergence in time is a rather direct consequence of this and (i).

To this aim, let $(h_k)_k$ be an arbitrary sequence of positive numbers converging to zero. By (iii) and by the standard compactness property of sets of finite perimeter, there is a bounded set of finite perimeter E_∞ such that, up to extracting a subsequence, $E_{h_k}^{h_k} \to E_\infty$ in the L^1 sense. In particular, by (4-7) we have $|E_\infty| = \omega_{n+1} = |E_0|$. Again, by using $|E_{h_k}^{h_k} \Delta E_\infty| \to 0$ and (4-4) we have

$$|E_\infty \setminus \{y \in \mathbb{R}^n : \bar{d}_{E_0}(y) \le j^{-1}\}| = 0 \quad \text{and} \quad |\{y \in \mathbb{R}^n : \bar{d}_{E_0}(y) \le -j^{-1}\} \setminus E_\infty| = 0$$

for every $j \in \mathbb{N}$. Thus, by letting $j \to \infty$ we obtain $|E_\infty \setminus \bar{E}_0| = 0$ and $|\mathrm{int}(E_0) \setminus E_\infty| = 0$. Since E_0 is open or closed, this means either $|E_\infty \setminus E_0| = 0$ or $|E_0 \setminus E_\infty| = 0$. But now $|E_\infty| = |E_0|$, so the previous yields $|E_\infty \Delta E_0| = 0$. Thus $|E_{h_k}^{h_k} \setminus E_0| \to 0$ up to a subsequence and since $(h_k)_k$ was arbitrarily chosen we have $|E_h^h \Delta E_0| \to 0$. □

We note that claim (v) does not hold for every bounded set of finite perimeter E_0. As an example one may construct a wild set of finite perimeter E_0 such that $|E_h^h \Delta E_0| \ge c_0 > 0$ for all $h > 0$.

By [Mugnai et al. 2016, Corollary 3.10], for a fixed time $T \ge h$, we have that the integral $\int_h^T |\lambda_t^h|^2 \, \mathrm{d}t$ is uniformly bounded in h and hence, via (4-6), that $|\{t \in (h, T) : |E_t^h| \ne \omega_{n+1}\}| \le Ch$, where C depends also on T. We may improve this by using Lemma 2.4.

Proposition 4.2. *Let $C_0 > 0$ and $E_0 \in X_{n+1}$ be a set of finite perimeter with volume ω_{n+1} and $P(E_0) \le C_0$. There are positive constants $C = C(C_0, n)$ and $h_0 = h_0(C_0, n)$ such that if $h \le h_0$ and $(E_t^h)_{t \ge 0}$ is an approximative flat flow starting from E_0, then for every $h \le T_1 \le T_2$*

$$\int_{T_1}^{T_2} |\lambda_t^h|^2 \, \mathrm{d}t \le C(T_2 - T_1 + 1) \quad \textit{and} \quad |\{t \in (T_1, T_2) : |E_t^h| \ne \omega_{n+1}\}| \le Ch(T_2 - T_1 + 1).$$

Proof. By (4-7) we may choose $h_0 = h_0(C_0, n)$ such that $|E_t^h| \ge \frac{1}{2}\omega_{n+1}$ whenever $h \le h_0$. We may also assume $C_0 > 2\omega_{n+1}$ so that $|E_t^h| \ge 1/C_0$ for $h \le h_0$. Thus, by Lemma 2.4 and $P(E_t^h) \le C_0$, we find a positive $C = C(C_0, n)$ such that for every $t \ge h$ and $h \le h_0$

$$|\lambda_t^h|^2 \le C\left(1 + \int_{\partial^* E_t^h} (H_{E_t^h} - \lambda_t^h)^2 \, \mathrm{d}\mathcal{H}^n\right),$$

and therefore

$$\int_{T_1}^{T_2} |\lambda_t^h|^2 \, \mathrm{d}t \le C(T_2 - T_1) + C \int_{T_1}^{T_2} \int_{\partial^* E_t^h} (H_{E_t^h} - \lambda_t^h)^2 \, \mathrm{d}\mathcal{H}^n \, \mathrm{d}t.$$

By Proposition 4.1 (ii) we obtain the first inequality. The first inequality implies, via (4-6), the second inequality with the same constant C. □

We need also the following comparison result for the proof.

Lemma 4.3. *Let $1 \le C_0 < \infty$. Assume $E_0 \in X_{n+1}$ is a set of finite perimeter with volume ω_{n+1} and $P(E_0) \le C_0$, and let $F = \bigcup_{i=1}^N B_r(x_i)$ with $|x_i - x_j| \ge 2r$ and $1/C_0 \le r \le C_0$. There is a positive constant $\varepsilon_0 = \varepsilon_0(C_0, n)$ such that if $(E_t^h)_{t\ge 0}$ is an approximative flat flow starting from E_0 and*

$$\sup_{x \in E_{t_0}^h \Delta F} d_{\partial F}(x) \le \varepsilon \quad \text{with } \varepsilon \le \varepsilon_0$$

for $t_0 \ge 0$, then

$$\sup_{x \in E_t^h \Delta F} d_{\partial F}(x) \le C\varepsilon^{1/9} \quad \text{for all } t_0 < t < t_0 + \sqrt{\varepsilon}$$

provided that $h \le \min\{\sqrt{\varepsilon}, h_0\}$, where $h_0 = h_0(C_0, n)$ is as in Proposition 4.2.

Proof. Our standing assumptions are

$$h \le \min\{\sqrt{\varepsilon}, h_0\} \quad \text{and} \quad \varepsilon \le \min\{1/(2C_0), 1\}.$$

As usual, C denotes a positive constant which may change from line to line but depends only on the parameters C_0 and n.

Without loss of generality we may assume $t_0 = 0$. Fix an arbitrary $x_i \in \{x_1, \dots, x_N\}$. Up to translating the coordinates we may assume that $x_i = 0$. We set for every $k = 0, 1, 2, \dots$

$$\rho_k = \inf\{|x| : x \in \mathbb{R}^{n+1} \setminus E_{kh}^h\}$$

and

$$r_k = \min\{r, \rho_0, \dots, \rho_k\}.$$

We claim that

$$r_{k+1}^2 - r_k^2 \ge -C_1(1 + |\lambda_{(k+1)h}^h|)h, \tag{4-8}$$

with some positive constant $C_1 = C_1(C_0, n)$. First, if $r_{k+1} = r_k$, the claim (4-8) is trivially true. Thus we may assume $r_{k+1} < r_k$ which implies $\rho_{k+1} = r_{k+1} < r_k \le \rho_k$. Then $\rho_k > 0$ which in turn means

$$\rho_k = \min_{\partial E_{kh}^h} |x|.$$

Since $E_{(k+1)h}^h$ is bounded and open, there is a point $x \in \mathbb{R}^{n+1} \setminus E_{(k+1)h}^h$ with $\rho_{k+1} = |x|$. Let x' be a closest point to x on ∂E_{kh}^h. Then

$$r_{k+1} + |\bar{d}_{E_{kh}^h}(x)| = |x| + |\bar{d}_{E_{kh}^h}(x)| \ge |x'| \ge \rho_k \ge r_k.$$

The condition $|x| < \rho_k$ means x exists in E_{kh}^h, so the previous estimate yields

$$r_{k+1} - r_k \ge \bar{d}_{E_{kh}^h}(x). \tag{4-9}$$

Again, $x \in E_{kh}^h \setminus E_{(k+1)h}^h$ so by Equation (4-4), $|\bar{d}_{E_{kh}^h}(x)| \le C_n\sqrt{h}$ and hence

$$r_{k+1} - r_k \ge -C_n\sqrt{h}. \tag{4-10}$$

We split the argument into two cases. First, if $r_{k+1} < C_n\sqrt{h}$, then by (4-10) we have $r_k < 2C_n\sqrt{h}$. Therefore, using (4-10) we obtain

$$r_{k+1}^2 - r_k^2 \geq -C_n(r_{k+1}+r_k)\sqrt{h} \geq -3C_n^2 h. \tag{4-11}$$

If $r_{k+1} \geq C_n\sqrt{h}$, then by (4-10) we have $r_k \leq 2r_{k+1}$. Since $r_{k+1} > 0$, we have $x \in \partial E^h_{(k+1)h}$ and $E^h_{(k+1)h}$ satisfies the interior ball condition of radius r_{k+1} at x. Thus by the discussion in Section 2, x belongs to the reduced boundary of $E^h_{(k+1)h}$ and therefore by the maximum principle $H_{E^h_{(k+1)h}}(x) \leq n/r_{k+1}$. Again, by the previous estimate, (4-9), the Euler–Lagrange equation (4-5) and $r_{k+1} \leq C_0$ we obtain

$$\begin{aligned}\frac{r_{k+1}-r_k}{h} \geq \frac{\bar{d}_{E^h_{kh}}(x)}{h} &\geq -\frac{n}{r_{k+1}} - |\lambda^h_{(k+1)h}| \\ &\geq -\frac{1}{r_{k+1}}(n + C_0|\lambda^h_{(k+1)h}|).\end{aligned}$$

Therefore

$$\frac{r_{k+1}^2 - r_k^2}{h} \geq -\left(1+\frac{r_k}{r_{k+1}}\right)(n + C_0|\lambda^h_{(k+1)h}|) \geq -3(n + C_0|\lambda^h_{(k+1)h}|). \tag{4-12}$$

Thus (4-11) and (4-12) yield the claim (4-8) in the case $r_{k+1} < r_k$.

We iterate (4-8) up to $K \in \mathbb{N}$, which is chosen so that $Kh \in (\sqrt{\varepsilon}, 2\sqrt{\varepsilon})$ (recall $h < \sqrt{\varepsilon}$), and use Proposition 4.2 to obtain

$$\begin{aligned} r_K^2 - r_0^2 &\geq -C_1 \sum_{k=0}^{K-1} (1 + |\lambda^h_{(k+1)h}|)h \\ &= -C_1 K h - C_1 \int_h^{(K+1)h} |\lambda^h_t| \,\mathrm{d}t \\ &\geq -2C_1\sqrt{\varepsilon} - C_1 \int_h^{3\sqrt{\varepsilon}} |\lambda^h_t| \,\mathrm{d}t \\ &\geq -2C_1\sqrt{\varepsilon} - \int_h^{3\sqrt{\varepsilon}} \varepsilon^{-1/4} + \varepsilon^{1/4}|\lambda^h_t|^2 \,\mathrm{d}t \\ &\geq -C\varepsilon^{1/4}\left(1 + \int_h^{3\sqrt{\varepsilon}} |\lambda^h_t|^2 \,\mathrm{d}t\right) \\ &\geq -C\varepsilon^{1/4}. \end{aligned} \tag{4-13}$$

By the assumption $\sup_{x\in E_0 \Delta F} d_{\partial F}(x) \leq \varepsilon$ we have $r - \varepsilon \leq r_0$. Thus we divide $r_K^2 - r_0^2$ by $r_K + r_0$ and use $r_0 \geq r - \varepsilon \geq \frac{1}{2}r \geq 1/(2C_0)$ as well as (4-13) to find a positive constant $C_2 = C_2(C_0, n)$ such that $r_K \geq r - C_2\varepsilon^{1/4}$. This means that

$$\inf_{\mathbb{R}^{n+1}\setminus E^h_t} \bar{d}_{B_r(x_i)} \geq -C_2\varepsilon^{1/4} \quad \text{for all } t < \sqrt{\varepsilon},$$

and again due to the arbitrariness of $x_i \in \{x_1, \dots, x_N\}$, that

$$\inf_{\mathbb{R}^{n+1}\setminus E^h_t} \bar{d}_F \geq -C_2\varepsilon^{1/4} \quad \text{for all } t < \sqrt{\varepsilon}.$$

To conclude the proof, we show that there is a positive constant $\varepsilon_1 = \varepsilon_1(C_0, n)$ such that

$$\sup_{E_t^h} \bar{d}_F \le 2\varepsilon^{1/9} \quad \text{for all } t < \sqrt{\varepsilon} \tag{4-14}$$

provided that $\varepsilon \le \varepsilon_1$. To this aim we choose an arbitrary $x_0 \in \mathbb{R}^{n+1} \setminus \bar{F}$ with $\bar{d}_F(x_0) \ge 2\varepsilon^{1/9}$. For every $k = 0, 1, 2, \dots,$ we set

$$\rho_k = \inf_{x \in E_{kh}^h} |x - x_0|$$

and

$$r_k = \min\{2\varepsilon^{1/9}, \rho_1, \dots, \rho_k\}.$$

In particular, $r_k \le 2C_0^{1/9}$. A slight modification of the procedure we used to obtain (4-13) yields

$$r_K^2 - r_0^2 \ge -C\varepsilon^{1/4},$$

where K is the same as described earlier. Again, the conditions $\sup_{x \in E_0 \Delta F} d_{\partial F}(x) \le \varepsilon$ and $\varepsilon \le 1$ imply $r_0 \ge 2\varepsilon^{1/9} - \varepsilon \ge \varepsilon^{1/9}$. Thus

$$r_K - r_0 \ge -C\frac{\varepsilon^{1/4}}{r_0} \ge -C\varepsilon^{5/36} = -C\varepsilon^{1/36}\varepsilon^{1/9},$$

and thus

$$r_K \ge (1 - C\varepsilon^{1/36})\varepsilon^{1/9} > \tfrac{1}{2}\varepsilon^{1/9},$$

when ε is small enough. Since x_0, with $d_F(x_0) \ge 2\varepsilon^{1/9}$, was arbitrarily chosen we deduce that

$$E_{kh}^h \subset \{x \in \mathbb{R}^{n+1} : d_F(x) \le 2\varepsilon^{1/9}\} \quad \text{for all } k = 0, \dots, K.$$

The claim (4-14) then follows from the choice of K. □

Proof of Theorem 1.1. The proof of Theorem 1.1 is based on Theorem 1.2. We first use it together with the dissipation inequality in Proposition 4.1 (ii) to deduce that there exists a sequence of times $t_j \to \infty$ such that the sets E_{t_j} are close to a disjoint union of balls. Since the perimeter of the approximative flat flow is essentially decreasing, the number of balls is also monotone. In particular, we deduce that after some time, the sets E_{t_j} are close to a fixed number, say N, of balls. We use the second statement of Theorem 1.2 to deduce that the perimeters of E_{t_j} converge to the perimeter of N balls with volume ω_{n+1} and thus the right-hand side of the dissipation inequality converges to zero. This allows us to improve our estimate and use Theorem 1.2 again to deduce that the flat flow E_t is close to a disjoint union of N balls for all large t except a set of times with small measure. The statement then finally follows from Lemma 4.3.

Proof. Assume that the initial set $E_0 \in X_{n+1}$ has the volume of the unit ball $|E_0| = \omega_{n+1}$, fix a positive C_0 with $C_0 \ge \max\{1, P(E_0)\}$ and assume $h < (C_0/\omega_{n+1})^2$. Let $(E_t)_{t\ge0}$ be a flat flow starting from E_0 and let $(E_t^{h_l})_{t\ge0}$ be an approximative flat flow which by Proposition 4.1 converges to $(E_t)_{t\ge0}$ locally uniformly in L^1. We simplify the notation and denote the converging subsequence again by h. Since we are now in the dimensions 2 and 3 ($n = 1, 2$), the sets E_t^h are C^2 regular.

Step 1: Let us denote

$$\Sigma_h := \{t \in (0, \infty) : |E_t^h| \neq \omega_{n+1}\}. \tag{4-15}$$

By (4-7) and Proposition 4.2 we find a constant $h_0 = h_0(C_0, n) < 1$ such that $|E_t^h| \geq 1/C_0$ for every $t \geq 0$ and

$$|(T_1, T_2) \cap \Sigma_h| \leq \tfrac{1}{3}(T_2 - T_1)$$

for every $T_1 \geq 1$ and $T_2 \geq T_1 + 1$ provided that $h \leq h_0$. On the other hand, by Proposition 4.1 (ii) we have, for every $h \leq h_0$ and $l \in \mathbb{N}$, that

$$I_{l,h} := \fint_{l^2}^{(l+1)^2} \|H_{E_t^h} - \lambda_t^h\|^2_{L^2(\partial E_t^h)} \, dt \leq \frac{C}{l}.$$

By Chebysev's inequality,

$$|\{t \in (l^2, (l+1)^2) : \|H_{E_t^h} - \lambda_t^h\|^2_{L^2(\partial E_t^h)} \geq 3I_{l,h}\}| \leq \tfrac{1}{3}((l+1)^2 - l^2).$$

Therefore, by choosing $T_1 = l^2$ and $T_2 = (l+1)^2$ we deduce that the set

$$\{t \in (T_1, T_2) : |E_t^h| = \omega_{n+1}, \, \|H_{E_t^h} - \lambda_t^h\|^2_{L^2(\partial E_t^h)} < 3I_{l,h}\}$$

is nonempty. Thus if $h \leq h_0$, then there is a sequence of times $(T_l^h)_l$, with $l^2 \leq T_l^h \leq (l+1)^2$, such that the corresponding sets satisfy $|E_{T_l^h}^h| = \omega_{n+1}$ and

$$\|H_{E_{T_l^h}^h} - \lambda_{T_l^h}^h\|_{L^2(\partial E_{T_l^h}^h)} \leq C l^{-1/2}. \tag{4-16}$$

By slight abuse of the notation we set $E_l^h := E_{T_l^h}^h$ and $\lambda_{l,h} := \lambda_{T_l^h}^h$ for $h \leq h_0$. Since the sets E_l^h are C^2 regular and bounded and thanks to $P(E_0) \leq C_0$, $|E_l^h| \geq 1/C_0$, (4-16) and Theorem 1.2, we find $l_0 = l_0(C_0, n)$ such that for every $l \geq l_0$ we have $1/C \leq \lambda_{l,h} \leq C$,

$$|P(E_l^h) - N_l^h (n+1)\omega_{n+1}(r_l^h)^n| \leq C l^{-q/2} \quad \text{and} \quad \sup_{E_l^h \Delta F_l^h} d_{\partial F_l^h} \leq C l^{-q/2}, \tag{4-17}$$

where $r_l^h = n/\lambda_{l,h}$ and F_l^h is a union of N_l^h pairwise disjoint (open) balls of radius $r_{l,h}$. Since we have $1/C \leq \lambda_{l,h} \leq C$, we also have $1/C \leq r_{l,h} \leq C$, which together with the perimeter estimate $P(E_l^h) \leq P(E_0) \leq C_0$ implies that there is $N_0 = N_0(C_0, n) \in \mathbb{N}$ such that $N_l^h \leq N_0$. Further, the distance estimate in (4-17), together with $1/C \leq r_{l,h} \leq C$ and $N_l^h \leq N_0$, yields

$$|E_l^h \Delta F_l^h| \leq C l^{-q/2}.$$

Since $|E_l^h| = \omega_{n+1}$, we have that the estimate above implies $|(r_{l,h})^{n+1} N_l^h - 1| \leq C l^{-q/2}$ and further that $|(r_{l,h})^n (N_l^h)^{n/(n+1)} - 1| \leq C l^{-q/2}$. This inequality, the perimeter estimate in (4-17) and $N_l^h \leq N_0$ imply

$$|P(E_l^h) - (n+1)\omega_{n+1}(N_l^h)^{1/(n+1)}| \leq C l^{-q/2}. \tag{4-18}$$

Since by Proposition 4.1 (ii) $(P(E_l^h))_{l \geq l_0}$ is nonincreasing, we have that (4-18) implies there is a positive integer $l_1 = l_1(C_0, n) \geq l_0$ for which $(N_l^h)_{l \geq l_1}$ is nonincreasing for all $h \leq h_0$.

Step 2: For $l \geq l_1$ and $h \leq h_0$ the sets E_l^h are thus close to N_l^h balls. We claim that there are $N \in \mathbb{N}$ and $l_2 \geq l_1$ such that for every integer $L \geq l_2$,

$$N_l^h = N \quad \text{for all } l_2 \leq l \leq L, \tag{4-19}$$

provided that h is small enough.

By using a standard diagonal argument and possibly passing to a subsequence we find a sequence of positive integers $(N_l)_{l\geq l_1}$, with $N_l \leq N_0$, such that $N_l^h \to N_l$ for every $l \geq l_1$. Since $(N_l^h)_{l\geq l_1}$ is nonincreasing, we have that $(N_l)_{l\geq l_1}$ is nonincreasing too and hence there are $N, l_2 \in \mathbb{N}$, $l_2 \geq l_1$, such that $N_l = N$ for every $l \geq l_2$. Hence we have (4-19) by the convergence of N_l^h to N_l.

We obtain from (4-18) and (4-19) that

$$|P(E_l^h) - (n+1)\omega_{n+1}(N)^{1/(n+1)}| \leq Cl^{-q/2} \tag{4-20}$$

for $l_2 \leq l \leq L$, provided that h is small enough. Therefore, it follows from Proposition 4.1 (ii) that

$$\int_{T_l^h+h}^{T_L^h} \|H_{E_t^h} - \lambda_t^h\|^2_{L^2(\partial E_t^h)}\,\mathrm{d}t \leq Cl^{-q/2}.$$

Since $h \leq 1$ and $L > 1$ was arbitrarily chosen, the above yields

$$\sup_{T\geq(l+2)^2}\left[\limsup_{h\to 0}\int_{(l+2)^2}^{T} \|H_{E_t^h} - \lambda_t^h\|^2_{L^2(\partial E_t^h)}\,\mathrm{d}t\right] \leq Cl^{-q/2} \tag{4-21}$$

for every $l \geq l_2$.

Step 3: Let us fix a small δ, the choice of which will be clear later. Then it follows from (4-21), (4-20) and the fact that the map $t \mapsto P(E_t^h)$ is nonincreasing in Σ_h that there is T_δ such that for every $T \geq T_\delta + 1$ there is $h_{\delta,T}$ such that

$$\int_{T_\delta}^{T} \|H_{E_t^h} - \lambda_t^h\|^2_{L^2(\partial E_t^h)}\,\mathrm{d}t \leq \delta \tag{4-22}$$

for all $h \leq h_{\delta,T}$ and

$$|P(E_t^h) - (n+1)\omega_{n+1}N^{1/(n+1)}| \leq \delta \tag{4-23}$$

for all $t \in (T_\delta, T) \setminus \Sigma_h$. On the other hand, by Proposition 4.2 and by decreasing $h_{\delta,T}$ if necessary, we deduce that

$$|\Sigma_h \cap (T_\delta, T)| \leq \delta \quad \text{for all } h \leq h_{\delta,T}. \tag{4-24}$$

Let $\varepsilon > 0$ and let us fix $t \geq T_\delta + 1$. (The time $T_\delta + 1$ will be T_ε in the claim.) We claim that, when δ is chosen small enough, we have

$$\sup_{E_t^h \Delta F_t^h} d_{\partial F_t^h} \leq \varepsilon \tag{4-25}$$

for $h \leq h_{\delta,T}$, where F_t^h is a union of N pairwise disjoint (open) balls of radius $r = N^{-1/(n+1)}$ with volume ω_{n+1}.

Fix $T \geq t+1$. Then it follows from (4-22) that

$$\int_{t-\delta^{1/4}}^{t} \|H_{E^h_\tau} - \lambda^h_\tau\|^2_{L^2(\partial E^h_\tau)} \, d\tau \leq \delta,$$

and from (4-23) and (4-24) that

$$|P(E^h_\tau) - (n+1)\omega_{n+1} N^{1/(n+1)}| \leq \delta \quad \text{for all } \tau \in (t-\delta^{1/4}, t) \setminus \Sigma_h$$

and $|\Sigma_h \cap (t-\delta^{1/4}, t)| \leq \delta$. Using these estimates we deduce that there is $t_0 \in (t-\delta^{1/4}, t)$ such that $|E^h_{t_0}| = \omega_{n+1}$,

$$|P(E^h_{t_0}) - (n+1)\omega_{n+1} N^{1/(n+1)}| \leq \delta \tag{4-26}$$

and

$$\|H_{E^h_{t_0}} - \lambda^h_{t_0}\|_{L^2(\partial E^h_{t_0})} \leq \delta^{1/4}.$$

Theorem 1.2 implies that

$$\sup_{E^h_{t_0} \Delta F^h_{t_0}} d_{\partial F^h_{t_0}} \leq C\delta^{q/4}$$

for all $h \leq h_{\delta,T}$, where $F^h_{t_0}$ is a union of $N_{t_0,h}$ pairwise disjoint (open) balls of radius $r_{t_0,h}$ with volume ω_{n+1}, and

$$|P(E^h_{t_0}) - N_{t_0,h}(n+1)\omega_{n+1} r^n_{t_0,h}| \leq C\delta^{q/4}.$$

Since $1/C \leq r_{t_0,h} \leq C$, as in Step 1 we deduce from the previous two estimates that $|E^h_{t_0} \Delta F^h_{t_0}| \leq C\delta^{q/4}$. Then by (4-26) and $|F^h_{t_0}| = \omega_{n+1}$ we further conclude that $N_{t_0,h} = N$, i.e., $F^h_{t_0}$ is a union of N pairwise disjoint (open) balls with volume ω_{n+1} and radius $r = N^{-1/(n+1)}$.

By Lemma 4.3,

$$\sup_{E^h_\tau \Delta F^h_{t_0}} d_{\partial F^h_{t_0}} \leq C\delta^{q/36} \quad \text{for all } t_0 < \tau < t_0 + \delta^{q/8}$$

and $h \leq h_{\delta,T}$. In particular, since $\delta^{q/8} > \delta^{1/4}$ the above inequality holds for t. This proves (4-25) by choosing $F^h_t = F^h_{t_0}$ and δ small enough. The claim follows by letting $h \to 0$. Note that by Proposition 4.1 (iii) there is $R > 0$ such that $F^h_t \subset B_R$ for all $h \leq h_{\delta,T}$. Therefore, by passing to another subsequence if necessary, we have that $F^h_t \to F_t$, where F_t is a union of N pairwise disjoint (open) balls with volume ω_{n+1}, and by (4-25),

$$\sup_{E_t \Delta F_t} d_{\partial F_t} \leq \varepsilon. \qquad \square$$

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A SIMPLE NUCLEAR C^*-ALGEBRA WITH AN INTERNAL ASYMMETRY

ILAN HIRSHBERG AND N. CHRISTOPHER PHILLIPS

We construct an example of a simple approximately homogeneous C^*-algebra such that its Elliott invariant admits an automorphism which is not induced by an automorphism of the algebra.

Classification theory for simple nuclear C^*-algebras reached a milestone recently. The results of [Elliott et al. 2015; Tikuisis et al. 2017], building on decades of work by many authors, show that simple separable unital C^*-algebras with finite nuclear dimension satisfying the universal coefficient theorem are classified via the Elliott invariant, $\mathrm{Ell}(\,\cdot\,)$, which consists of the ordered K_0-group along with the class of the identity, the K_1-group, the trace simplex, and the pairing between the trace simplex and the K_0-group. Earlier counterexamples due to Toms [2008] and Rørdam [2003], related to ideas of Villadsen [1998], show that one cannot expect to be able to extend this classification theorem beyond the case of finite nuclear dimension, at least not without either extending the invariant or restricting to another class of C^*-algebras. An important facet of the classification theorems is a form of rigidity. Starting with two C^*-algebras A and B and an isomorphism $\Phi : \mathrm{Ell}(A) \to \mathrm{Ell}(B)$, one not only shows that A and B are isomorphic, but rather that there exists an isomorphism from A to B which induces the given isomorphism Φ on the level of the Elliott invariant.

The goal of this paper is to illustrate how this existence property may fail in the infinite nuclear dimension setting, even when restricting to a class consisting of a single C^*-algebra. Namely, we construct an example of a simple unital nuclear separable AH algebra C, along with an automorphism of $\mathrm{Ell}(C)$, which is not induced by any automorphism of C. This can be viewed as a companion of sorts to [Toms 2008, Theorem 1.2], where it was shown that when such automorphisms exist, they need not be unique in the sense described. The mechanism of the example is that if there were such an automorphism φ, there would be projections $p, q \in \mathbb{C}$ such that $\varphi(p) = q$ but such that the corners pCp and qCq have different radii of comparison [Toms 2006] (the definition is recalled at the beginning of Section 1). This further shows that simple unital AH algebras can be quite inhomogeneous. In particular, extending the Elliott invariant by adding something as simple as the radius of comparison will not help for the classification of AH algebras which are not Jiang–Su stable.

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We now give an overview of our construction. We start with the counterexample from [Toms 2008, Theorem 1.1]. We consider two direct systems, described diagrammatically as follows:

$$\begin{array}{c} C(X_0) \rightrightarrows C(X_1) \otimes M_{r(1)} \rightrightarrows C(X_2) \otimes M_{r(2)} \rightrightarrows \cdots \\ C([0,1]) \rightrightarrows C([0,1]) \otimes M_{r(1)} \rightrightarrows C([0,1]) \otimes M_{r(2)} \rightrightarrows \cdots \end{array} \tag{0-1}$$

The ordinary arrows indicate a large (and rapidly increasing) number of embeddings which are carefully chosen, and the dotted arrows indicate a small number of point evaluation maps, thrown in so as to ensure that the resulting direct limit is simple. The spaces in the upper diagram are contractible CW complexes whose dimension increases rapidly compared to the sizes of the matrix algebras. (Toms uses cubes; in our construction we found it easier to use cones over products of spheres, but the underlying idea is similar.) The direct system is constructed so as to have positive radius of comparison. We use [Thomsen 1994] to choose the lower diagram so as to mimic the upper diagram, and produce the same Elliott invariant. As the resulting algebra on the bottom is AI, it has strict comparison, and therefore is not isomorphic to the one on the top. (In [Toms 2008] it isn't important for the two diagrams to match up nicely in terms of the ranks of the matrices involved. However, we will show that it can be done, as it is important for us.)

Our construction involves moving the point evaluations across, so as to merge the two systems:

$$\begin{tikzcd} C(X_0) \arrow[r, shift left] \arrow[r, shift right] \arrow[dr, dotted] & C(X_1) \otimes M_{r(1)} \arrow[r, shift left] \arrow[r, shift right] \arrow[dr, dotted] & C(X_2) \otimes M_{r(2)} \arrow[r, shift left] \arrow[r, shift right] \arrow[dr, dotted] & \cdots \\ C([0,1]) \arrow[r, shift left] \arrow[r, shift right] \arrow[ur, dotted] & C([0,1]) \otimes M_{r(1)} \arrow[r, shift left] \arrow[r, shift right] \arrow[ur, dotted] & C([0,1]) \otimes M_{r(2)} \arrow[r, shift left] \arrow[r, shift right] \arrow[ur, dotted] & \cdots \end{tikzcd} \tag{0-2}$$

With care, one can arrange for the flip between the two levels of the diagram to make sense as an automorphism of the Elliott invariant. The resulting C^*-algebra has positive radius of comparison and behaves roughly as badly as Toms' example. Nevertheless, we can distinguish a part of it which roughly corresponds to the rapid dimension growth diagram on the top from a part which roughly corresponds to the AI part on the bottom. Namely, if at the first level $C(X_0) \oplus C([0,1])$ we denote by q the function which is 1 on X_0 and 0 on $[0,1]$, and we define $q^\perp = 1 - q$, then the K_0-classes of q and $q^\perp$ will be switched by the automorphism of the Elliott invariant we construct. However, we can tell apart the corners qCq and $q^\perp C q^\perp$ by considering their radii of comparison.

Section 1 develops the choices needed to get different radii of comparison in different corners of the algebra we construct. Section 2 contains the work needed to assemble the ingredients of the construction into a simple C^*-algebra whose Elliott invariant admits an appropriate automorphism. The main theorem is in Section 3.

1. Upper and lower bounds on the radius of comparison

We recall the required standard definitions and notation related to the Cuntz semigroup. See Section 2 of [Rørdam 1992] for details. For a unital C^*-algebra A, we denote its tracial state space by $\mathrm{T}(A)$. We take $M_\infty(A) = \bigcup_{n=1}^\infty M_n(A)$, using the usual embeddings $M_n(A) \hookrightarrow M_{n+1}(A)$. For $\tau \in \mathrm{T}(A)$, we define $d_\tau : M_\infty(A)_+ \to [0, \infty)$ by $d_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n})$. If $a, b \in M_\infty(A)_+$, then $a \precsim b$ (a is Cuntz subequivalent to b) if there is a sequence $(v_n)_{n=1}^{\infty}$ in $M_\infty(A)$ such that $\lim_{n\to\infty} v_n b v_n^* = a$.

Following [Toms 2006, Definition 6.1], for $\rho \in [0, \infty)$, we say that A has ρ-comparison if whenever $a, b \in M_\infty(A)_+$ satisfy $d_\tau(a) + \rho < d_\tau(b)$ for all $\tau \in \mathrm{T}(A)$, then $a \precsim b$. The radius of comparison of A, denoted by $\mathrm{rc}(A)$, is

$$\mathrm{rc}(A) = \inf(\{\rho \in [0, \infty) \mid A \text{ has } \rho\text{-comparison}\}).$$

We take $\mathrm{rc}(A) = \infty$ if there is no ρ such that A has ρ-comparison. Since AH algebras are nuclear, all quasitraces on them are traces by [Haagerup 2014, Theorem 5.11]. Thus, we ignore quasitraces. Also, by [Phillips 2014, Proposition 6.12], the radius of comparison remains unchanged if we replace $M_\infty(A)$ by $K \otimes A$ throughout. Thus, we may work only in $M_\infty(A)$.

Our construction uses a specific setup, with a number of parameters of various kinds which must be chosen to satisfy specific conditions. Construction 1.1 lists for reference many of the objects used in it, and some of the conditions they must satisfy. It abstracts the diagram (0-2). Construction 1.6 specifies the choices of spaces and maps needed for the results on Cuntz comparison, and Construction 2.17, together with the additional maps in parts (11), (12), and (13) of Construction 1.1, is used to arrange the existence of a suitable automorphism of the tracial state space of the algebra we construct. Because of the necessity of passing to a subsystem at one stage in this process, we must start the proof of the main theorem with a version of just the top row in the diagram (0-1); this is Construction 3.3. Many of the lemmas use only a few of the objects and their properties, so that the reader can refer back to just the relevant parts of the constructions. In particular, many details are used only in this section or only in Section 2. Some of the details are used for just one lemma each.

Construction 1.1. For much of this paper, we will consider algebras constructed in the following way and using the following notation:

(1) $(d(n))_{n=0,1,2,\ldots}$ and $(k(n))_{n=0,1,2,\ldots}$ are sequences in $\mathbb{Z}_{\geq 0}$, with $d(0) = 1$ and $k(0) = 0$. Moreover, for $n \in \mathbb{Z}_{\geq 0}$,

$$l(n) = d(n) + k(n), \quad r(n) = \prod_{j=0}^{n} l(j), \quad \text{and} \quad s(n) = \prod_{j=0}^{n} d(j).$$

Further define $t(n)$ inductively as follows. Set $t(0) = 0$, and

$$t(n+1) = d(n+1)t(n) + k(n+1)[r(n) - t(n)].$$

(See Lemma 1.14 for the significance of $t(n)$.)

(2) We will assume that $k(n) < d(n)$ for all $n \in \mathbb{Z}_{\geq 0}$.

(3) We define

$$\kappa = \inf_{n\in\mathbb{Z}_{>0}} \frac{s(n)}{r(n)}.$$

For estimates involving the radius of comparison, we will assume $\kappa > \frac{1}{2}$.

(4) The numbers $\omega, \omega' \in (0, \infty]$ are defined by

$$\omega = \frac{k(1)}{k(1)+d(1)} \quad \text{and} \quad \omega' = \sum_{n=2}^{\infty} \frac{k(n)}{k(n)+d(n)}.$$

We will require $\omega' < \omega < \frac{1}{2}$. In particular,

$$\sum_{n=1}^{\infty} \frac{k(n)}{k(n)+d(n)} < \infty.$$

(5) We will also eventually require that κ as in (3) and ω as in (4) are related by $2\kappa - 1 > 2\omega$. This can easily be arranged with a suitable choice of $d(1)$ and $k(1)$.

(6) $(X_n)_{n=0,1,2,\ldots}$ and $(Y_n)_{n=0,1,2,\ldots}$ are sequences of compact metric spaces. (They will be further specified in Construction 1.6.)

(7) For $n \in \mathbb{Z}_{\geq 0}$, the algebra C_n is

$$C_n = M_{r(n)} \otimes (C(X_n) \oplus C(Y_n)).$$

We further make the identifications

$$\begin{aligned} C(X_{n+1},\ M_{r(n+1)}) &= M_{l(n+1)} \otimes C(X_{n+1},\ M_{r(n)}), \\ C(Y_{n+1},\ M_{r(n+1)}) &= M_{l(n+1)} \otimes C(Y_{n+1},\ M_{r(n)}), \\ C(X_n) \oplus C(Y_n) &= C(X_n \amalg Y_n), \\ C(X_n,\ M_{r(n)}) \oplus C(Y_n,\ M_{r(n)}) &= C(X_n \amalg Y_n, M_{r(n)}). \end{aligned}$$

(8) For $n \in \mathbb{Z}_{>0}$, we are given a unital homomorphism

$$\gamma_n : C(X_n) \oplus C(Y_n) \to M_{l(n+1)}(C(X_{n+1}) \oplus C(Y_{n+1})),$$

and the homomorphism

$$\Gamma_{n+1,\,n} : C_n \to C_{n+1}$$

is given by $\Gamma_{n+1,\,n} = \mathrm{id}_{M_{r(n)}} \otimes \gamma_n$. Moreover, for $m, n \in \mathbb{Z}_{\geq 0}$ with $m \leq n$,

$$\Gamma_{n,m} = \Gamma_{n,n-1} \circ \Gamma_{n-1,\,n-2} \circ \cdots \circ \Gamma_{m+1,m} : C_m \to C_n.$$

In particular, $\Gamma_{n,n} = \mathrm{id}_{C_n}$.

(9) We require that the maps

$$\gamma_n : C(X_n \amalg Y_n) \to M_{l(n+1)}(C(X_{n+1} \amalg Y_{n+1}))$$

in (8) be diagonal; that is, that there exist continuous functions

$$S_{n,1},\ S_{n,2},\ \ldots,\ S_{n,\,l(n+1)} : X_{n+1} \amalg Y_{n+1} \to X_n \amalg Y_n$$

such that for all $f \in C(X_n \amalg Y_n)$, we have

$$\gamma_n(f) = \mathrm{diag}(f \circ S_{n,1},\ f \circ S_{n,2},\ \ldots,\ f \circ S_{n,\,l(n+1)}).$$

(These maps will be specified further in Construction 1.6.)

(10) We set $C = \varinjlim_n C_n$, taken with respect to the maps $\Gamma_{n,m}$. The maps associated with the direct limit will be called $\Gamma_{\infty,m} : C_m \to C$ for $m \in \mathbb{Z}_{\geq 0}$.

As we need to work with two diagrams which are similar in most positions, as in diagrams (0-1) and (0-2), we sometimes use additional objects and conditions in the construction, as follows:

(11) For $n \in \mathbb{Z}_{>0}$, we may be given an additional unital homomorphism

$$\gamma_n^{(0)} : C(X_n) \oplus C(Y_n) \to M_{l(n+1)}(C(X_{n+1}) \oplus C(Y_{n+1})).$$

Then the maps $\Gamma_{n+1,\,n}^{(0)} : C_n \to C_{n+1}$, $\Gamma_{n,m}^{(0)} : C_m \to C_n$ are defined analogously to (8), the algebra $C^{(0)}$ is given as $C^{(0)} = \varinjlim_n C_n$, taken with respect to the maps $\Gamma_{n,m}^{(0)}$, and the maps $\Gamma_{\infty,m}^{(0)} : C_m \to C^{(0)}$ are defined analogously to (10).

(12) In (11), analogously to (9), we may require that there be

$$S_{n,1}^{(0)},\ S_{n,2}^{(0)},\ \ldots,\ S_{n,\,l(n+1)}^{(0)} : X_{n+1} \amalg Y_{n+1} \to X_n \amalg Y_n$$

such that for all $f \in C(X_n \amalg Y_n)$ we have

$$\gamma_n^{(0)}(f) = \operatorname{diag}(f \circ S_{n,1}^{(0)},\ f \circ S_{n,2}^{(0)},\ \ldots,\ f \circ S_{n,\,l(n+1)}^{(0)}).$$

(These maps will be specified further in Construction 1.6.)

(13) Assuming diagonal maps as in (9), we may require that they agree in the coordinates $1, 2, \ldots, d(n+1)$; that is, for $n \in \mathbb{Z}_{>0}$ and $k = 1, 2, \ldots, d(n+1)$, we have $S_{n,k}^{(0)} = S_{n,k}$.

Lemma 1.2. *In Construction 1.1*(1), *the sequence* $(s(n)/r(n))_{n=1,2,\ldots}$ *is strictly decreasing.*

Proof. The proof is straightforward. □

Lemma 1.3. *In Construction 1.1*(1), *and assuming Construction 1.1*(2), *we have*

$$0 = \frac{t(0)}{r(0)} < \frac{t(1)}{r(1)} < \frac{t(2)}{r(2)} < \cdots < \frac{1}{2}.$$

Proof. We have $t(0) = 0$ by definition. We prove by induction on $n \in \mathbb{Z}_{>0}$ that

$$\frac{t(n-1)}{r(n-1)} < \frac{t(n)}{r(n)} < \frac{1}{2}. \tag{1-1}$$

This will finish the proof. For $n = 1$, we have

$$\frac{t(1)}{r(1)} = \frac{k(1)}{k(1) + d(1)},$$

which is in $\left(0, \frac{1}{2}\right)$ by Construction 1.1(2). Now assume (1-1); we prove this relation with $n+1$ in place of n. We have $r(n) - t(n) > t(n)$, so

$$\frac{t(n+1)}{r(n+1)} = \frac{d(n+1)t(n) + k(n+1)[r(n) - t(n)]}{[d(n+1) + k(n+1)]r(n)} > \frac{d(n+1)t(n) + k(n+1)t(n)}{[d(n+1) + k(n+1)]r(n)} = \frac{t(n)}{r(n)}. \tag{1-2}$$

Also, with

$$\alpha = \frac{d(n+1)}{d(n+1)+k(n+1)} \quad \text{and} \quad \beta = \frac{t(n)}{r(n)},$$

starting with the first step in (1-2), and at the end using $\alpha > \frac{1}{2}$ (by Construction 1.1(2)) and $\beta < \frac{1}{2}$ (by the induction hypothesis), we have

$$\frac{t(n+1)}{r(n+1)} = \alpha\beta + (1-\alpha)(1-\beta) = \tfrac{1}{2}[1-(2\alpha-1)(1-2\beta)] < \tfrac{1}{2}.$$

This completes the induction, and the proof. □

Lemma 1.4. *With the notation of Constructions 1.1*(1) *and 1.1*(4), *and assuming the conditions in Constructions 1.1*(2) *and 1.1*(4), *for all $n \in \mathbb{Z}_{>0}$ we have*

$$\omega \leq \frac{t(n)}{r(n)} \leq \omega + \omega' < 2\omega.$$

Proof. The third inequality is immediate from Construction 1.1(4).

By Lemma 1.3, the sequence $(t(n)/r(n))_{n=1,2,\ldots}$ is strictly increasing. Also,

$$\frac{t(1)}{r(1)} = \frac{k(1)}{k(1)+d(1)} = \omega. \tag{1-3}$$

The first inequality in the statement now follows.

Next, we claim that

$$\frac{t(n)}{r(n)} \leq \sum_{j=1}^{n} \frac{k(j)}{k(j)+d(j)}$$

for all $n \in \mathbb{Z}_{>0}$. The case $n = 1$ is (1-3). Assume this inequality is known for n. Then

$$\begin{aligned} \frac{t(n+1)}{r(n+1)} &= \left(\frac{d(n+1)}{k(n+1)+d(n+1)}\right)\left(\frac{t(n)}{r(n)}\right) + \left(\frac{k(n+1)}{k(n+1)+d(n+1)}\right)\left(\frac{r(n)-t(n)}{r(n)}\right) \\ &\leq \frac{t(n)}{r(n)} + \frac{k(n+1)}{k(n+1)+d(n+1)} \leq \sum_{j=1}^{n+1} \frac{k(j)}{k(j)+d(j)}, \end{aligned}$$

as desired.

The second inequality in the statement now follows. □

Notation 1.5. For a topological space X, we define

$$\operatorname{cone}(X) = (X \times [0, 1])/(X \times \{0\}).$$

Then $\operatorname{cone}(X)$ is contractible, and $\operatorname{cone}(\,\cdot\,)$ is a covariant functor: if $T : X \to Y$ is a continuous map, then it induces a continuous map $\operatorname{cone}(T) : \operatorname{cone}(X) \to \operatorname{cone}(Y)$. We identify X with the image of $X \times \{1\}$ in $\operatorname{cone}(X)$.

Construction 1.6. We give further details on the spaces X_n and Y_n in Construction 1.1(6).

(14) The space X_n is chosen as follows. First set $Z_0 = S^2$. With $(d(n))_{n=0,1,2,\ldots}$ and $(s(n))_{n=0,1,2,\ldots}$ as in Construction 1.1(1), define inductively

$$Z_n = Z_{n-1}^{d(n)} = (S^2)^{s(n)}.$$

Then set $X_n = \mathrm{cone}(Z_n)$. (In particular, X_n is contractible, and $Z_n \subset X_n$ as in Notation 1.5.) Further, for $n \in \mathbb{Z}_{\geq 0}$ and $j = 1, 2, \ldots, d(n+1)$, we let $P_j^{(n)} : Z_{n+1} \to Z_n$ be the j-th coordinate projection, and we set $Q_j^{(n)} = \mathrm{cone}(P_j^{(n)}) : X_{n+1} \to X_n$.

(15) $Y_n = [0, 1]$ for all $n \in \mathbb{Z}_{>0}$. (In particular, Y_n is contractible.)

(16) We assume we are given points $x_m \in X_m$ for $m \in \mathbb{Z}_{\geq 0}$ such that, using the notation in (14), for all $n \in \mathbb{Z}_{\geq 0}$, the set

$$\Big\{(Q_{\nu_1}^{(n)} \circ Q_{\nu_2}^{(n+1)} \circ \cdots \circ Q_{\nu_{m-n}}^{(m-1)})(x_m) \mid m = n+1,\ n+2,\ \ldots \text{ and } \nu_j = 1, 2, \ldots, d(n+j) \\ \text{for } j = 1, 2, \ldots, m-n\Big\}$$

is dense in X_n.

(17) We assume we are given a sequence $(y_k)_{k=0,1,2,\ldots}$ in $[0, 1]$ such that for all $n \in \mathbb{Z}_{\geq 0}$ the set $\{y_k \mid k \geq n\}$ is dense in $[0, 1]$.

(18) The maps

$$\gamma_n : C(X_n \amalg Y_n) \to M_{l(n+1)}(C(X_{n+1} \amalg Y_{n+1}))$$

will be as in Construction 1.1(9), with the maps $S_{n,j} : X_{n+1} \amalg Y_{n+1} \to X_n \amalg Y_n$ appearing there defined as follows:

(a) With $Q_j^{(n)}$ as in (14), we set $S_{n,j}(x) = Q_j^{(n)}(x)$ for $x \in X_{n+1}$ and $j = 1, 2, \ldots, d(n+1)$.

(b) $S_{n,j}(x) = y_n$ for

$$x \in X_{n+1} \quad \text{and} \quad j = d(n+1)+1,\ d(n+1)+2,\ \ldots,\ l(n+1).$$

(c) There are continuous functions

$$R_{n,1},\ R_{n,2},\ \ldots,\ R_{n,\,d(n+1)} : Y_{n+1} \to Y_n$$

(which will be taken from Proposition 2.14 below) such that $S_{n,j}(y) = R_{n,j}(y)$ for $y \in Y_{n+1}$ and $j = 1, 2, \ldots, d(n+1)$.

(d) $S_{n,j}(y) = x_n$ for

$$y \in Y_{n+1} \quad \text{and} \quad j = d(n+1)+1,\ d(n+1)+2,\ \ldots,\ l(n+1).$$

(19) The maps

$$\gamma_n^{(0)} : C(X_n \amalg Y_n) \to M_{l(n+1)}(C(X_{n+1} \amalg Y_{n+1}))$$

will be as in Construction 1.1(12), with the maps $S_{n,j}^{(0)} : X_{n+1} \amalg Y_{n+1} \to X_n \amalg Y_n$ appearing there given by $S_{n,j}^{(0)} = S_{n,j}$ for $j = 1, 2, \ldots, d(n+1)$ and to be specified later for $j = d(n+1)+1,\ d(n+1)+2,\ \ldots,\ l(n+1)$.

With the choices in Construction 1.6 (18), the map

$$\gamma_n : C(X_n) \oplus C(Y_n) \to C(X_{n+1},\ M_{l(n+1)}) \oplus C(Y_{n+1},\ M_{l(n+1)})$$

in Construction 1.1 (8), as further specified in Construction 1.1 (9), is given as follows. With $\mathbb{C}^{d(n)}$ viewed as embedded in $M_{d(n)}$ as the diagonal matrices, there is a homomorphism

$$\delta_n : C(Y_n) \to C(Y_{n+1},\ \mathbb{C}^{d(n+1)}) \subset C(Y_{n+1}, M_{d(n+1)})$$

such that

$$\gamma_n(f, g) = \Big(\operatorname{diag}\big(f \circ Q_1^{(n)},\ f \circ Q_2^{(n)},\ \dots,\ f \circ Q_{d(n+1)}^{(n)},\ \underbrace{g(y_n),\ g(y_n),\ \dots,\ g(y_n)}_{k(n+1)\text{ times}}\big), \operatorname{diag}\big(\delta_n(g),\ \underbrace{f(x_n),\ f(x_n),\ \dots,\ f(x_n)}_{k(n+1)\text{ times}}\big)\Big). \tag{1-4}$$

For the purposes of this section, we need no further information on the maps δ_n, except that they send constant functions to constant functions.

Lemma 1.7. *Assume the notation and choices in parts* (1), (7), (8), *and* (10) *of Construction 1.1, and in Construction 1.6* (*except part* (19)) *and the parts of Construction 1.1 referred to there. Then the algebra C is simple.*

Proof. Using Construction 1.6 (16), this is easily deduced from [Dădărlat et al. 1992, Proposition 2.1]. □

Notation 1.8. Let $p \in C(S^2, M_2)$ denote the Bott projection, and let L be the tautological line bundle over $S^2 \cong \mathbb{CP}^1$. (Thus, the range of p is the section space of L.) Recalling that $X_0 = \operatorname{cone}(S^2)$, parametrized as in Notation 1.5, define $b \in C(X_0, M_2)$ by $b(\lambda) = \lambda \cdot p$ for $\lambda \in [0, 1]$. Assuming the notation and choices in parts (1), (6), (7), (8), and (10) of Construction 1.1 and in Construction 1.6, for $n \in \mathbb{Z}_{\geq 0}$ set $b_n = (\mathrm{id}_{M_2} \otimes \Gamma_{n,0})(b, 0) \in M_2(C_n)$.

We require the following simple lemma concerning characteristic classes. It gives us a way of estimating the radius of comparison, which is similar to the one used in [Villadsen 1998, Lemma 1], but more suitable for the types of estimates we need here.

Lemma 1.9. *The Cartesian product $L^{\times k}$ does not embed in a trivial bundle over $(S^2)^k$ of rank less than $2k$.*

Proof. We refer the reader to [Milnor and Stasheff 1974, Section 14] for an account of Chern classes. The Chern character $c(L)$ is of the form $1 + \varepsilon$, where ε is a generator of $H^2(S^2, \mathbb{Z})$, and the product operation satisfies $\varepsilon^2 = 0$. Let $P_1, P_2, \dots, P_k : (S^2)^k \to S^2$ be the coordinate projections. For $j = 1, 2, \dots, k$, set $\varepsilon_j = P_j^*(\varepsilon)$. The elements $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k \in H^2((S^2)^k, \mathbb{Z})$, along with $1 \in H^0((S^2)^k, \mathbb{Z})$ (the standard generator) generate the cohomology ring of $(S^2)^k$ and satisfy $\varepsilon_j^2 = 0$ for $j = 1, 2, \dots, k$. By naturality of the Chern character [Milnor and Stasheff 1974, Lemma 14.2] and the product theorem [Milnor and Stasheff 1974, (14.7) on page 164], we have $c(L^{\times k}) = \prod_{j=1}^k (1 + \varepsilon_j)$. Now, suppose $L^{\times k}$ embeds as a subbundle of a trivial bundle E. Let F be the complementary bundle, so that $L^{\times k} \oplus F = E$. By the product theorem, $c(L^{\times k})c(F) = c(L^{\times k} \oplus F) = c(E) = 1$. Thus, $c(F) = c(L^{\times k})^{-1} = \prod_{j=1}^k (1 - \varepsilon_j)$. Since $c(F)$ has a nonzero term in the top cohomology group $H^{2k}((S^2)^k)$, it follows that $\operatorname{rank}(F)$ is at least k. Thus, $\operatorname{rank}(E) = \operatorname{rank}(L^{\times k}) + \operatorname{rank}(F) \geq 2k$, as required. □

Lemma 1.10. *Adopt the assumptions and notation of Notation 1.8. Let $n \in \mathbb{Z}_{>0}$. Then $b_n|_{Z_n}$ is the orthogonal sum of a projection p_n whose range is isomorphic to the section space of the Cartesian product bundle $L^{\times s(n)}$ and a constant function of rank at most $r(n) - s(n) - t(n)$.*

We don't expect $b_n|_{Z_n}$ to be a projection, since some of the point evaluations occurring in the maps of the direct system will be at points $x \in \operatorname{cone}(Z_m) \setminus Z_m$ for values of $m < n$, and $b_m(x)$ is not a projection for such x.

We don't need the estimate on the rank of the second part of the description of $b_n|_{Z_n}$; it is included to make the construction more explicit. If there are no evaluations at the "cone points"

$$(Z_m \times \{0\})/(Z_m \times \{0\}) \in (Z_m \times [0, 1])/(Z_m \times \{0\})$$

(following the parametrization in Notation 1.5), then this rank will be exactly $r(n) - s(n) - t(n)$.

Proof of Lemma 1.10. For $n \in \mathbb{Z}_{\geq 0}$ write $b_n = (c_n, g_n)$, with

$$c_n \in M_2(C(X_n, M_{r(n)})) \quad \text{and} \quad g_n \in M_2(C(Y_n, M_{r(n)})).$$

Further, for $j = 1, 2, \ldots, s(n)$ let $T_j^{(n)} : (S^2)^{s(n)} \to S^2$ be the j-th coordinate projection. We claim that c_n is an orthogonal sum $c_{n,0} + c_{n,1}$, in which $c_{n,0}$ is the direct sum of the functions $b \circ \operatorname{cone}(T_j^{(n)})$ for $j = 1, 2, \ldots, s(n)$ and $c_{n,1}$ is a constant function of rank at most $r(n) - s(n) - t(n)$, and moreover that g_n is a constant function of rank at most $t(n)$. The statement of the lemma follows from this claim.

The proof of the claim is by induction on n. The claim is true for $n = 0$, by the definition of b and since $s(0) = 1$, $t(0) = 0$, and $r(0) - s(0) - t(0) = 0$.

Now assume that the claim is known for n, recall that $\Gamma_{n+1,\, n} = \mathrm{id}_{M_{r(n)}} \otimes \gamma_n$ (see Construction 1.1(8)), and examine the summands in the description (1-4) of the map γ_n (after Construction 1.6). With this convention, first take (f, g) in (1-4) to be $(c_{n,0}, 0)$. The first coordinate $\Gamma_{n+1,n}(c_{n,0}, 0)_1$ is of the form required for $c_{n+1,0}$, while $\Gamma_{n+1,n}(c_{n,0}, 0)_2$ is a constant function of rank $k(n+1)s(n)$ unless $c_n(x_n) = 0$, in which case it is zero. In the same manner, we see that:

- $\Gamma_{n+1,n}(c_{n,1}, 0)_1$ is constant of rank at most $d(n+1)[r(n) - s(n) - t(n)]$.
- $\Gamma_{n+1,n}(c_{n,1}, 0)_2$ is constant of rank at most $k(n+1)[r(n) - s(n) - t(n)]$.
- $\Gamma_{n+1,n}(0, g_n)_1$ is constant of rank at most $k(n+1)t(n)$.
- $\Gamma_{n+1,n}(0, g_n)_2$ is constant of rank at most $d(n+1)t(n)$.

Putting these together, we get in the first coordinate of $\Gamma_{n+1,n}(b_n)$ the direct sum of $c_{n+1,0}$ as described and a constant function of rank at most

$$d(n+1)[r(n) - s(n) - t(n)] + k(n+1)t(n).$$

A computation shows that this expression is equal to $r(n+1) - s(n+1) - t(n+1)$. In the second coordinate we get a constant function of rank at most

$$k(n+1)s(n) + k(n+1)[r(n) - s(n) - t(n)] + d(n+1)t(n) = t(n+1).$$

This completes the induction, and the proof. □

Corollary 1.11. *Adopt the assumptions and notation of Notation 1.8. Let $n \in \mathbb{Z}_{\geq 0}$. Let $e = (e_1, e_2)$ be an element in $M_\infty(C_n) \cong M_\infty(C(X_n) \oplus C(Y_n))$ such that e_1 is a projection which is equivalent to a constant projection. If there exists $x \in M_\infty(C_n)$ such that $\|xex^* - b_n\| < \frac{1}{2}$ then* $\operatorname{rank}(e_1) \geq 2s(n)$.

Proof. Recall from Construction 1.6(14) and Notation 1.5 that

$$Z_n = (S^2)^{s(n)} \quad \text{and} \quad Z_n \subset \operatorname{cone}(Z_n) = X_n \subset X_n \amalg Y_n.$$

Also recall the line bundle L and the projection p from Notation 1.8.

It follows from Lemma 1.10 that there is a projection $q \in M_{2r(n)}(C(Z_n))$ whose range is isomorphic to the section space of the $s(n)$-dimensional vector bundle $L^{\times s(n)}$ and such that $q(b_n|_{Z_n})q = q$. Now $\|xex^* - b_n\| < \frac{1}{2}$ implies $\|q(xex^*|_{Z_n})q - q\| < \frac{1}{2}$. Since $e|_{Z_n}$ and $q|_{Z_n}$ are projections, it follows that $q|_{Z_n}$ is Murray–von Neumann equivalent to a subprojection of $e|_{Z_n} = e_1|_{Z_n}$. Therefore $\operatorname{rank}(e_1|_{Z_n}) \geq 2s(n)$ by Lemma 1.9. So $\operatorname{rank}(e_1) \geq 2s(n)$. □

Although not strictly needed for the sequel, we record the following.

Corollary 1.12. *Assume the notation and choices in parts* (1), (3) $\big($*including* $\kappa > \frac{1}{2}\big)$, (7), (8), *and* (10) *of Construction 1.1, and in Construction 1.6* (*except part* (19)) *and the parts of Construction 1.1 referred to there. Then the algebra C satisfies* $\operatorname{rc}(C) \geq 2\kappa - 1 > 0$.

Proof. Suppose $\rho < 2\kappa - 1$. We show that C does not have ρ-comparison. Choose $n \in \mathbb{Z}_{>0}$ such that $1/r(n) < 2\kappa - 1 - \rho$. Choose $M \in \mathbb{Z}_{\geq 0}$ such that $\rho + 1 < M/r(n) < 2\kappa$. Let $e \in M_\infty(C_n)$ be a trivial projection of rank M. By slight abuse of notation, we use $\Gamma_{m,n}$ to denote the amplified map from $M_\infty(C_n)$ to $M_\infty(C_m)$ as well. For $m > n$, the rank of $\Gamma_{m,n}(e)$ is $Mr(m)/r(n)$, and the choice of M guarantees that this rank is strictly less than $2s(m)$. Now, for any trace τ on C_m (and thus for any trace on C), and justifying the last step afterwards, we have

$$d_\tau(\Gamma_{m,n}(e)) = \tau(\Gamma_{m,n}(e)) = \frac{1}{r(m)} \cdot M \cdot \frac{r(m)}{r(n)} \geq 1 + \rho > d_\tau(b_m) + \rho.$$

To explain the last step, recall b_m from Notation 1.8, and use Lemma 1.10 to see that the ranks of its components $(b_m)_1 \in M_2(C(X_m, M_{r(m)}))$ and $(b_m)_2 \in M_2(C(Y_m, M_{r(m)}))$ are both less than $r(m)$, while the identity element has rank $r(m)$.

On the other hand, if $\Gamma_{\infty,0}(b) \precsim \Gamma_{\infty,n}(e)$ then, in particular, there exists some $m > n$ and $x \in M_\infty(C_m)$ such that $\|x\Gamma_{m,n}(e)x^* - b_m\| < \frac{1}{2}$, which contradicts Corollary 1.11. □

Notation 1.13. We assume the notation and choices in parts (1), (6), (7), (8), and (10) of Construction 1.1. In particular, $C_0 = C(X_0) \oplus C(Y_0)$. Define $q_0 = (1, 0) \in C(X_0) \oplus C(Y_0)$ and $q_0^\perp = 1 - q_0$. For $n \in \mathbb{Z}_{>0}$ define $q_n = \Gamma_{n,0}(q_0) \in C_n$ and $q_n^\perp = 1 - q_n$, and finally, define $q = \Gamma_{\infty,0}(q_0) \in C$ and $q^\perp = 1 - q$.

Lemma 1.14. *Make the assumptions in Notation 1.13. Further assume the notation and choices in Construction 1.6* (*except part* (19))*. Then the projection*

$$1 - q_n \in M_{l(n)}(C(X_n)) \oplus M_{l(n)}(C(Y_n))$$

has the form (e, f) for a constant projection $e \in M_{l(n)}(C(X_n)) = C(X_n, M_{l(n)})$ of rank $t(n)$ and a constant projection $f \in M_{l(n)}(C(Y_n)) = C(Y_n, M_{l(n)})$ of rank $r(n) - t(n)$.

From Construction 1.6, we don't actually need to know anything about the spaces X_n and Y_n, we don't need to know anything about the points x_n and y_n except which spaces they are in, and we don't need to know anything about the maps $Q_j^{(n)}$ and $R_{n,j}$ except their domains and codomains.

Proof of Lemma 1.14. The proof is an easy induction argument, using the fact that the image of a constant function under a diagonal map is again a constant function. □

Lemma 1.15. *Assume the notation and choices in parts* (1)–(10) *of Construction 1.1*, *Construction 1.6* (*except part* (19)), *and Notation 1.13*, *including* $k(n) < d(n)$ *for all* $n \in \mathbb{Z}_{\geq 0}$, $\kappa > \frac{1}{2}$, $\omega > \omega'$, *and* $2\kappa - 1 > 2\omega$. *Then*

$$\operatorname{rc}(q^{\perp} C q^{\perp}) \geq \frac{2\kappa - 1}{2\omega}.$$

Proof. We proceed as in the proof of Corollary 1.12, although the rank computations are somewhat more involved. The difference is in the definition of d_τ. In this corner, d_τ is normalized so that $d_\tau(q^{\perp}) = 1$ for all $\tau \in \mathrm{T}(C)$. To avoid redefining the notation, we will use τ to denote a tracial state on C, and therefore our dimension functions will be of the form $a \mapsto d_\tau(a)/\tau(q^{\perp})$, noting that $\tau(q^{\perp}) = d_\tau(q^{\perp})$ since $q^{\perp}$ is a projection.

It suffices to show that for all $\rho \in (1, (2\kappa - 1)/(2\omega)) \cap \mathbb{Q}$, we have $\operatorname{rc}(q^{\perp} C q^{\perp}) \geq \rho$.

Fix $\delta \in (0, \omega)$ such that

$$\rho < (1 - \delta)\left(\frac{2\kappa - 1}{2\omega}\right). \tag{1-5}$$

Set

$$\varepsilon = \frac{\delta}{2\rho(1 - \delta)} > 0. \tag{1-6}$$

Since the sequence $(s(n)/r(n))_{n=0,1,2,\dots}$ is nonincreasing and converges to a nonzero limit κ, there exists $n_0 \in \mathbb{Z}_{\geq 0}$ such that, for all n and m with $m \geq n \geq n_0$, we have

$$0 \leq 1 - \frac{r(n)}{s(n)} \cdot \frac{s(m)}{r(m)} < \varepsilon.$$

This implies that

$$\frac{r(m)}{r(n)} - \frac{s(m)}{s(n)} < \varepsilon \cdot \frac{r(m)}{r(n)}. \tag{1-7}$$

Using (1-5) and $\delta < \omega$ at the first step, we get

$$1 - \omega + 2\rho\omega < 1 - \delta + 2(1 - \delta)\left(\frac{2\kappa - 1}{2\omega}\right)\omega = 2\kappa(1 - \delta).$$

Now write $\rho = \alpha/\beta$ with $\alpha, \beta \in \mathbb{Z}_{>0}$. Choose $n \geq n_0$ such that

$$\frac{\beta}{r(n)} < 2\kappa(1 - \delta) - (1 - \omega + 2\rho\omega).$$

Then there exists $N_1 \in \mathbb{Z}_{>0}$ such that $\rho N_1 \in \mathbb{Z}_{>0}$ and

$$2\kappa(1 - \delta) > \frac{N_1}{r(n)} > 1 - \omega + 2\rho\omega. \tag{1-8}$$

Set

$$N_2 = \rho N_1. \tag{1-9}$$

Using $\rho > 1$ at the last step, we have

$$\frac{N_2}{r(n)} = \frac{\rho N_1}{r(n)} > \rho(1-\omega+2\rho\omega) > \rho(1-\omega)+2\omega.$$

Now suppose $e \in M_\infty(C_n) = M_\infty(C(X_n) \oplus C(Y_n))$ is an ordered pair whose first component is a trivial projection on X_n of rank N_1 and whose second component is a (trivial) projection on Y_n of rank N_2. Let $m > n$, and let f be the first component of $\Gamma_{m,n}(e)$; we estimate $\operatorname{rank}(f)$. (The second component is a trivial projection over Y_m whose rank we don't care about.) Now f is the direct sum of $r(m)/r(n)$ trivial projections, coming from $C(X_n, M_{r(n)})$ and $C(Y_n, M_{r(n)})$. At least $s(m)/s(n)$ of these summands come from $C(X_n, M_{r(n)})$. So at most $r(m)/r(n) - s(m)/s(n)$ of these summands come from $C(Y_n, M_{r(n)})$. The summands coming from $C(X_n, M_{r(n)})$ have rank N_1 and the summands coming from $C(Y_n, M_{r(n)})$ have rank N_2. Since $N_2 > N_1$, we get

$$\operatorname{rank}(f) \leq \left(\frac{r(m)}{r(n)} - \frac{s(m)}{s(n)}\right)N_2 + \frac{s(m)}{s(n)} \cdot N_1 = \frac{r(m)}{r(n)} \cdot N_1 + \left(\frac{r(m)}{r(n)} - \frac{s(m)}{s(n)}\right)(N_2 - N_1).$$

Combining this with (1-7) at the first step, and using (1-9) at the second step, (1-6) at the third step, (1-8) at the fifth step, and Construction 1.1(3) at the sixth step, we get

$$\begin{aligned}\operatorname{rank}(f) &< \frac{r(m)}{r(n)} \cdot (N_1 + \varepsilon N_2) = \frac{r(m)}{r(n)} \cdot (1+\varepsilon\rho) \cdot N_1 \\ &= \frac{r(m)}{r(n)} \cdot \frac{2-\delta}{2(1-\delta)} \cdot N_1 < \frac{r(m)}{r(n)} \cdot \frac{N_1}{1-\delta} < 2\kappa r(m) \leq 2s(m).\end{aligned}$$

So Corollary 1.11 implies that there is no $x \in M_\infty(C_m)$ for which $\|x\Gamma_{n,m}(e)x^* - b_m\| < \frac{1}{2}$. Since $m > n$ is arbitrary,

$$\Gamma_{\infty,n}(e) \not\precsim b. \tag{1-10}$$

Now let τ be a trace on C, and restrict it to $C_n \cong M_{r(n)}(C(X_n) \oplus C(Y_n))$. Denote by tr the normalized trace on $M_{r(n)}$. There is a probability measure μ on $X_n \amalg Y_n$ such that $\tau(a) = \int_{X_n \amalg Y_n} \operatorname{tr}(a)\, d\mu$ for all $a \in C_n$. Define $\lambda = \mu(X_n)$, so $1 - \lambda = \mu(Y_n)$. Then, using (1-9) at the second step,

$$\tau(e) = \frac{\lambda N_1 + (1-\lambda)N_2}{r(n)} = \frac{[\lambda + \rho(1-\lambda)]N_1}{r(n)}.$$

Using Lemma 1.14 to calculate the ranks of the components of $q_n^\perp$, we get

$$\tau(q_n^\perp) = \frac{\lambda t(n) + (1-\lambda)[r(n) - t(n)]}{r(n)}, \tag{1-11}$$

$$\tau(q_n) = 1 - \tau(q_n^\perp) = \frac{\lambda[r(n) - t(n)] + (1-\lambda)t(n)}{r(n)}. \tag{1-12}$$

It follows from Lemmas 1.10 and 1.14 that $d_\tau(b_n) \leq \tau(q_n)$. Using this at the first step, and (1-11) and (1-12) at the second step, we get

$$\frac{d_\tau(b_n)}{\tau(q_n^\perp)} \leq \frac{\tau(q_n)}{\tau(q_n^\perp)} = \frac{\lambda[r(n)-t(n)]+(1-\lambda)t(n)}{\lambda t(n)+(1-\lambda)[r(n)-t(n)]}.$$

So

$$\frac{\tau(e)-d_\tau(b_n)}{\tau(q_n^\perp)} \geq \frac{(\lambda+\rho(1-\lambda))N_1-(\lambda[r(n)-t(n)]+(1-\lambda)t(n))}{\lambda t(n)+(1-\lambda)[r(n)-t(n)]}.$$

The last expression is a fractional linear function in λ and is defined for all values of λ in the interval $[0, 1]$. Any such function is monotone on $[0, 1]$. In the following calculations, we recall from Lemma 1.4 that $\omega \leq t(n)/r(n) < 2\omega$. If we set $\lambda = 1$ and use (1-8), the value we obtain is

$$\frac{N_1/r(n)-(1-t(n)/r(n))}{t(n)/r(n)} > \frac{(1-\omega+2\rho\omega)-(1-\omega)}{2\omega} = \rho.$$

If we set $\lambda = 0$, we get, using (1-8) at the first step and $\rho > 1$ at the last step,

$$\frac{\rho N_1/r(n)-t(n)/r(n)}{1-t(n)/r(n)} > \frac{\rho(1-\omega+2\rho\omega)-2\omega}{1-\omega} = \rho+\frac{2\rho^2\omega-2\omega}{1-\omega} > \rho.$$

Therefore

$$\frac{d_\tau(\Gamma_{\infty,n}(e))}{d_\tau(q^\perp)} > \frac{d_\tau(b)}{d_\tau(q^\perp)}+\rho$$

for all traces τ on C, so $\mathrm{rc}(q^\perp C q^\perp) > \rho$, as required. □

We now turn to the issue of finding upper bounds on the radius of comparison. For this, we appeal to results from [Niu 2014]. Niu [2014, Definition 3.6] introduced a notion of mean dimension for a diagonal AH-system. Suppose we are given a direct system of homogeneous algebras of the form

$$A_n = C(K_{n,1}) \otimes M_{j_{n,1}} \oplus C(K_{n,2}) \otimes M_{j_{n,2}} \oplus \cdots \oplus C(K_{n,m(n)}) \otimes M_{j_{n,m(n)}},$$

in which each of the spaces involved is a connected finite CW complex, and the connecting maps are unital diagonal maps. Let γ denote the mean dimension of this system, in the sense of Niu. It follows trivially from [Niu 2014, Definition 3.6] that

$$\gamma \leq \lim_{n\to\infty} \max\left(\left\{\frac{\dim(K_{n,l})}{j_{n,l}} \;\middle|\; l = 1, 2, \ldots, m(n)\right\}\right).$$

Theorem 6.2 of [Niu 2014] states that if A is the direct limit of a system as above, and A is simple, then $\mathrm{rc}(A) \leq \gamma/2$. Since the system we are considering here is of this type, Niu's theorem applies. With that at hand, we can derive an upper bound for the radius of comparison of the complementary corner.

Lemma 1.16. *Under the same assumptions as in Lemma 1.15, we have*

$$\mathrm{rc}(qCq) \leq \frac{1}{1-2\omega}.$$

Proof. The algebra C is simple by Lemma 1.7, so qCq is also simple. This fact and Lemma 1.14 allow us to apply the discussion above, getting

$$\mathrm{rc}(qCq) \leq \frac{1}{2} \lim_{n\to\infty} \max\left(\frac{\dim(X_n)}{\mathrm{rank}(q_n|_{X_n})}, \, \frac{\dim(Y_n)}{\mathrm{rank}(q_n|_{Y_n})} \right).$$

As $\dim(Y_n) = 1$ for all n, the second term converges to 0. As for the first term, by Construction 1.6(14), we have $\dim(X_n) = 2s(n)+1$. Also, $\mathrm{rank}(q_n|_{X_n}) = r(n) - t(n)$ by Lemma 1.14. Thus, by Construction 1.1(1) and Lemma 1.4, and using $d(n) \to \infty$ (which follows from Construction 1.1(4)) at the last step,

$$\lim_{n\to\infty} \frac{\dim(X_n)}{\mathrm{rank}(q_n|_{X_n})} = \lim_{n\to\infty} \frac{2s(n)+1}{r(n)-t(n)} \leq \lim_{n\to\infty} \frac{2r(n)+1}{r(n)-t(n)} \leq \frac{2}{1-2\omega}.$$

This gives us the required estimate. □

Lemma 1.17. *Let the assumptions and notation be as in Notation 1.13*, *Construction 1.6*(14), *and Construction 1.6*(15). *If $e \in C$ is a projection which has the same K_0-class as q then e is unitarily equivalent to q. The same holds with $q^{\perp}$ in place of q.*

Proof. This can be seen directly from the construction. For each $n \in \mathbb{Z}_{\geq 0}$, since X_n and Y_n are contractible (Constructions 1.6(14) and (15)), if $e \in M_\infty(C_n)$ is a projection which has the same K_0-class as q, then e is actually unitarily equivalent to q_n. The same holds for $q_n^{\perp}$. It follows that this is the case in C as well. □

We point out that this lemma can also be deduced using cancellation. By [Elliott et al. 2009, Theorem 4.1], simple unital AH algebras which arise from AH systems with diagonal maps have stable rank 1. Rieffel has shown that C^*-algebras with stable rank 1 have cancellation; see [Blackadar 1998, Theorem 6.5.1].

2. The tracial state space

For a compact Hausdorff space X, we will need all of $C(X, \mathbb{R})$ (the space of real-valued continuous functions on X), the tracial state space of $C(X)$ (and of $C(X, M_n)$), and the space of affine functions on the tracial state space. This last space is an order unit space, and much of our work will be done there.

For later reference, we recall some of the definitions, and then describe how to move between these spaces. We begin with the definition of an order unit space from the discussion before Proposition II.1.3 of [Alfsen 1971]. We suppress the order unit in our notation, since (except in several abstract results) our order unit spaces will always be sets of affine continuous functions on compact convex sets with order unit the constant function 1.

Definition 2.1. An *order unit space* V is a partially ordered real Banach space (see page 1 of [Goodearl 1986] for the axioms of a partially ordered real vector space) which is *Archimedean* (if $v \in V$ and $\{\lambda v \mid \lambda \in (0, \infty)\}$ has an upper bound, then $v \leq 0$), with a distinguished element $e \in V$ which is an *order unit* (that is, for every $v \in V$ there is $\lambda \in (0, \infty)$ such that $-\lambda e \leq v \leq \lambda e$), and such that the norm on V satisfies

$$\|v\| = \inf(\{\lambda \in (0, \infty) \mid -\lambda e \leq v \leq \lambda e\})$$

for all $v \in V$.

The morphisms of order unit spaces are the positive linear maps which preserve the order units.

The morphisms of compact convex sets (compact convex subsets of locally convex topological vector spaces) are just the continuous affine maps.

Definition 2.2. If K is a compact convex set, we denote by $\mathrm{Aff}(K)$ the order unit space of continuous affine functions $f : K \to \mathbb{R}$, with the supremum norm and with order unit the constant function 1.

If K and L are compact convex sets and $\lambda : K \to L$ is continuous and affine, we let $\lambda^* : \mathrm{Aff}(L) \to \mathrm{Aff}(K)$ be the positive linear order unit preserving map given by $\lambda^*(f) = f \circ \lambda$ for $f \in \mathrm{Aff}(L)$.

This definition makes $K \mapsto \mathrm{Aff}(K)$ a functor.

Definition 2.3. If V is an order unit space with order unit e, we denote by $S(V)$ (or $S(V, e)$ if e is not understood) its state space (the order unit space morphisms to $(\mathbb{R}, 1)$), which is a compact convex set with the weak* topology.

If W is another order unit space and $\varphi : V \to W$ is positive, linear, and order unit preserving, we let $S(\varphi) : S(W) \to S(V)$ be the continuous affine map given by $S(\varphi)(\omega) = \omega \circ \varphi$ for $\omega \in S(W)$.

This definition makes $V \mapsto S(V)$ a functor.

Theorem 2.4 [Goodearl 1986, Theorem 7.1]. *There is a natural isomorphism $S(\mathrm{Aff}(K)) \cong K$ for compact convex sets K, given by sending $x \in K$ to the evaluation map $\mathrm{ev}_x : \mathrm{Aff}(K) \to \mathbb{R}$ defined by $\mathrm{ev}_x(f) = f(x)$ for $f \in \mathrm{Aff}(K)$.*

Definition 2.5. For a unital C*-algebra A, we denote its tracial state space by $\mathrm{T}(A)$. If A and B are unital C^*-algebras and $\varphi : A \to B$ is a unital homomorphism, we let $\mathrm{T}(\varphi) : \mathrm{T}(B) \to \mathrm{T}(A)$ be the continuous affine map given by $\mathrm{T}(\varphi)(\tau) = \tau \circ \varphi$ for $\tau \in \mathrm{T}(B)$. We let $\hat{\varphi} : \mathrm{Aff}(\mathrm{T}(A)) \to \mathrm{Aff}(\mathrm{T}(B))$ be the positive order unit preserving map given by $\hat{\varphi}(f) = f \circ \mathrm{T}(\varphi)$ for $f \in \mathrm{Aff}(\mathrm{T}(A))$. (Thus, $\hat{\varphi} = \mathrm{T}(\varphi)^*$.)

Lemma 2.6. *Let X be a compact Hausdorff space. Then $C(X, \mathbb{R})$, with the supremum norm and distinguished element the constant function 1, is a complete order unit space. Restriction of tracial states on $C(X)$ is an affine homeomorphism from $\mathrm{T}(C(X))$ to $S(C(X, \mathbb{R}))$. The map from X to $S(C(X, \mathbb{R}))$ which sends $x \in X$ to the point evaluation $\mathrm{ev}_x : C(X, \mathbb{R}) \to \mathbb{R}$ is a homeomorphism onto its image, and the map $R_X : \mathrm{Aff}(S(C(X, \mathbb{R}))) \to C(X, \mathbb{R})$, given by $R_X(f)(x) = f(\mathrm{ev}_x)$ for $f \in \mathrm{Aff}(S(C(X, \mathbb{R})))$ and $x \in X$, is an isomorphism of order unit spaces.*

If Y is another compact Hausdorff space, then the function which sends a positive linear order unit preserving map $Q : C(X, \mathbb{R}) \to C(Y, \mathbb{R})$ to $S(Q) : S(C(Y, \mathbb{R})) \to S(C(X, \mathbb{R}))$, as in Definition 2.3, is a bijection to the continuous affine maps from $S(C(Y, \mathbb{R}))$ to $S(C(X, \mathbb{R}))$. Its inverse is the map E given as follows. For a continuous affine map $\lambda : S(C(Y, \mathbb{R})) \to S(C(X, \mathbb{R}))$, using the notation of Definition 2.2, define $E(\lambda) : C(X, \mathbb{R}) \to C(Y, \mathbb{R})$ by $E(\lambda) = R_Y \circ \lambda^* \circ R_X^{-1}$.

A positive linear order unit preserving map from $C(X, \mathbb{R})$ to $C(Y, \mathbb{R})$ is called a *Markov operator*.

Proof of Lemma 2.6. It is immediate that $C(X, \mathbb{R})$ is a complete order unit space. The identification of $S(C(X, \mathbb{R}))$ is also immediate. The fact that R_X is bijective follows from [Goodearl 1986, Corollary 11.20] using the identification of X with the extreme points of $S(C(X, \mathbb{R}))$.

For the second paragraph, it is immediate that S sends positive linear order unit preserving maps to continuous affine maps, and that E does the reverse. For the rest, we must show that $S \circ E$ and $E \circ S$ are the identity maps on the appropriate sets.

We first claim that for $g \in \mathrm{Aff}(S(C(X, \mathbb{R})))$ and $\rho \in S(C(X, \mathbb{R}))$ we have

$$g(\rho) = \rho(R_X(g)). \tag{2-1}$$

This formula is true by definition when $\rho = \mathrm{ev}_x$ for some $x \in X$. Since, for fixed g, both sides of (2-1) are continuous affine functions of ρ, and since $S(C(X, \mathbb{R}))$ is the closed convex hull of $\{\mathrm{ev}_x \mid x \in X\}$, the claim follows.

We next claim that if $\lambda : S(C(Y, \mathbb{R})) \to S(C(X, \mathbb{R}))$ is continuous and affine, $\omega \in S(C(Y, \mathbb{R}))$, and $g \in \mathrm{Aff}(S(C(X, \mathbb{R})))$, then

$$(\omega \circ R_Y)(g \circ \lambda) = (\lambda(\omega) \circ R_X)(g). \tag{2-2}$$

To prove this claim, for the same reasons as in the proof of the first claim, it suffices to prove this when there is $y \in Y$ such that $\omega = \mathrm{ev}_y$. In this case, using the definition of R_Y at the second step, and the previous claim with $\rho = \lambda(\mathrm{ev}_y)$ at the third step,

$$(\mathrm{ev}_y \circ R_Y)(g \circ \lambda) = R_Y(g \circ \lambda)(y) = (g \circ \lambda)(\mathrm{ev}_y) = (\lambda(\mathrm{ev}_y) \circ R_X)(g),$$

as desired.

Now let $\lambda : S(C(Y, \mathbb{R})) \to S(C(X, \mathbb{R}))$ be continuous and affine; we prove that $S(E(\lambda)) = \lambda$. Let $\omega \in S(C(X, \mathbb{R}))$ and let $f \in C(Y, \mathbb{R})$. Working through the definitions gives

$$S(E(\lambda))(\omega)(f) = (\omega \circ R_Y)(R_X^{-1}(f) \circ \lambda).$$

By (2-2) with $g = R_X^{-1}(f)$, the right-hand side is $\lambda(\omega)(f)$, as desired.

Finally, let $Q : C(X, \mathbb{R}) \to C(Y, \mathbb{R})$ be a positive linear order unit preserving map; we show that $E(S(Q)) = Q$. Let $f \in C(X, \mathbb{R})$ and let $y \in Y$. Working through the definitions gives

$$E(S(Q))(f)(y) = R_X^{-1}(f)(\mathrm{ev}_y \circ Q).$$

Applying (2-1) with $g = R_X^{-1}(f)$ and $\rho = \mathrm{ev}_y \circ Q$, we see that the right-hand side is $(\mathrm{ev}_y \circ Q)(f) = Q(f)(y)$. This proves that $E(S(Q)) = Q$, and the proof is complete. $\square$

Direct limits of direct systems of order unit spaces are constructed at the beginning of Section 3 of [Thomsen 1994], including Lemma 3.1 there.

Proposition 2.7. *Let $((D_n)_{n=0,1,2,\ldots}, (\varphi_{n,m})_{0 \le m \le n})$ be a direct system of unital C^*-algebras and unital homomorphisms. Set $D = \varinjlim_n D_n$. Then there are a natural homeomorphism*

$$\mathrm{T}(D) \to \varprojlim_n \mathrm{T}(D_n)$$

and a natural isomorphism

$$\mathrm{Aff}(\mathrm{T}(D)) \to \varinjlim_n \mathrm{Aff}(\mathrm{T}(D_n))$$

of order unit spaces.

Proof. The first part is Lemma 3.3 of [Thomsen 1994].

The second part is Lemma 3.2 of [Thomsen 1994], combined with the fact (Theorem 2.4) that the state space of $\mathrm{Aff}(K)$ is naturally identified with K. □

Definition 2.8. Let V and W be order unit spaces, with order units $e \in V$ and $f \in W$. We define the *direct sum* $V \oplus W$ to be the vector space direct sum $V \oplus W$ as a real vector space, with the order $(v_1, w_1) \leq (v_2, w_2)$ for $v_1, v_2 \in V$ and $w_1, w_2 \in W$ if and only if $v_1 \leq v_2$ and $w_1 \leq w_2$, with the order unit (e, f), and the norm $\|(v, w)\| = \max(\|v\|, \|w\|)$.

Lemma 2.9. *Let V and W be order unit spaces. Then $V \oplus W$ as in Definition 2.8 is an order unit space, which is complete if V and W are.*

Proof. The proof is straightforward. □

Lemma 2.10. *Let A and B be unital C^*-algebras. Then, taking the direct sum on the right to be as in Definition 2.8, there is an isomorphism*

$$\mathrm{Aff}(\mathrm{T}(A \oplus B)) \cong \mathrm{Aff}(\mathrm{T}(A)) \oplus \mathrm{Aff}(\mathrm{T}(B)),$$

given as follows. Identify $\mathrm{T}(A)$ with a subset of $\mathrm{T}(A \oplus B)$ by, for $\tau \in \mathrm{T}(A)$, defining $i(\tau)(a, b) = \tau(a)$ for all $a \in A$ and $b \in B$, and similarly identify $\mathrm{T}(B)$ with a subset of $\mathrm{T}(A \oplus B)$. Then the map $\mathrm{Aff}(\mathrm{T}(A \oplus B)) \to \mathrm{Aff}(\mathrm{T}(A)) \oplus \mathrm{Aff}(\mathrm{T}(B))$ is $f \mapsto (f|_{\mathrm{T}(A)}, f|_{\mathrm{T}(B)})$.

Proof. It is clear that if $f \in \mathrm{Aff}(\mathrm{T}(A \oplus B))$, then $f|_{\mathrm{T}(A)} \in \mathrm{Aff}(\mathrm{T}(A))$ and $f|_{\mathrm{T}(B)} \in \mathrm{Aff}(\mathrm{T}(B))$, and moreover that the map of the lemma is linear, positive, and preserves the order units. One easily checks that every tracial state on $A \oplus B$ is a convex combination of tracial states on A and B, from which it follows that if $f|_{\mathrm{T}(A)} = 0$ and $f|_{\mathrm{T}(B)} = 0$ then $f = 0$.

It remains to prove that the map of the lemma is surjective. Let $g \in \mathrm{Aff}(\mathrm{T}(A))$ and $h \in \mathrm{Aff}(\mathrm{T}(B))$. Define $f : \mathrm{T}(A \oplus B) \to \mathbb{R}$ by, for $\tau \in \mathrm{T}(A \oplus B)$,

$$f(\tau) = \tau(1, 0) g(\tau(1, 0)^{-1} \tau|_A) + \tau(0, 1) g(\tau(0, 1)^{-1} \tau|_B)$$

(taking the first summand to be zero if $\tau(1, 0) = 0$ and the second summand to be zero if $\tau(0, 1) = 0$). Straightforward but somewhat tedious calculations show that f is weak* continuous and affine, and clearly $f|_{\mathrm{T}(A)} = g$ and $f|_{\mathrm{T}(B)} = h$. □

The following result generalizes Lemma 3.4 of [Thomsen 1994]. It still isn't the most general Elliott approximate intertwining result for order unit spaces, because we assume that the underlying order unit spaces of the two direct systems are the same. The main effect of this assumption is to simplify the notation.

Proposition 2.11. *Let $(V_m)_{m=0,1,2,\ldots}$ be a sequence of separable complete order unit spaces, and let*

$$((V_m)_{m=0,1,2,\ldots}, (\varphi_{n,m})_{0 \leq m \leq n}) \quad \textit{and} \quad ((V_m)_{m=0,1,2,\ldots}, (\varphi'_{n,m})_{0 \leq m \leq n})$$

be two direct systems of order unit spaces, using the same spaces, and with maps $\varphi_{n,m}, \varphi'_{n,m} : V_m \to V_n$ which are linear, positive, and preserve the order units. Let V and V' be the direct limits

$$V = \varinjlim((V_m)_{m=0,1,2,\ldots}, (\varphi_{n,m})_{0 \leq m \leq n}) \quad \textit{and} \quad V' = \varinjlim((V_m)_{m=0,1,2,\ldots}, (\varphi'_{n,m})_{0 \leq m \leq n}),$$

with corresponding maps

$$\varphi_{\infty,n} : V_n \to V \quad \text{and} \quad \varphi'_{\infty,n} : V_n \to V'$$

for $n \in \mathbb{Z}_{\geq 0}$. For $n \in \mathbb{Z}_{\geq 0}$ further let

$$v_0^{(n)},\ v_1^{(n)},\ \ldots \in V_n$$

be a dense sequence in the closed unit ball of V_n, and define $F_n \subset V_n$ to be the finite set

$$F_n = \bigcup_{m=0}^{n} \bigl[\{\varphi_{n,m}(v_k^{(m)}) : 0 \leq k \leq n\} \cup \{\varphi'_{n,m}(v_k^{(m)}) : 0 \leq k \leq n\}\bigr].$$

Suppose that there are $\delta_0, \delta_1, \ldots \in (0, \infty)$ such that

$$\sum_{n=0}^{\infty} \delta_n < \infty \tag{2-3}$$

and for all $n \in \mathbb{Z}_{\geq 0}$ and all $v \in F_n$ we have

$$\|\varphi_{n+1,\,n}(v) - \varphi'_{n+1,\,n}(v)\| < \delta_n.$$

Then there is a unique isomorphism $\rho : V \to V'$ such that for all $m \in \mathbb{Z}_{\geq 0}$ and all $v \in V_m$ we have

$$\rho(\varphi_{\infty,m}(v)) = \lim_{n\to\infty} (\varphi'_{\infty,n} \circ \varphi_{n,m})(v).$$

Its inverse is determined by

$$\rho^{-1}(\varphi'_{\infty,m}(v)) = \lim_{n\to\infty} (\varphi_{\infty,n} \circ \varphi'_{n,m})(v)$$

for $m \in \mathbb{Z}_{\geq 0}$ and $v \in V_m$.

Proof. We first claim that for $m \in \mathbb{Z}_{\geq 0}$ and $v \in F_m$, the sequence $((\varphi'_{\infty,n} \circ \varphi_{n,m})(v))_{n \geq m}$ is a Cauchy sequence in V'. For $n \geq m$, we estimate, using $\|\varphi'_{\infty,n+1}\| \leq 1$, $\|v\| \leq 1$, and $\varphi_{n,m}(v) \in F_n$ at the last step:

$$\begin{aligned}\|(\varphi'_{\infty,n+1} \circ \varphi_{n+1,\,m})(v) - (\varphi'_{\infty,n} \circ \varphi_{n,m})(v)\| &= \|(\varphi'_{\infty,n+1} \circ \varphi_{n+1,\,n} \circ \varphi_{n,m})(v) - (\varphi'_{\infty,n+1} \circ \varphi'_{n+1,\,n} \circ \varphi_{n,m})(v)\| \\ &\leq \|\varphi'_{\infty,n+1}\| \|\varphi_{n+1,\,n}(\varphi_{n,m}(v)) - \varphi'_{n+1,\,n}(\varphi_{n,m}(v))\| \leq \delta_n.\end{aligned}$$

The claim now follows from (2-3).

Next, we claim that for $m \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{Z}_{>0}$, the sequence $((\varphi'_{\infty,n} \circ \varphi_{n,m})(v_k^{(m)}))_{n \geq m}$ is a Cauchy sequence in V'. Indeed, taking $m_0 = \max(m, k)$, this follows from the previous claim and the fact that $\varphi_{m_0,m}(v_k^{(m)}) \in F_{m_0}$.

Now we claim that for $m \in \mathbb{Z}_{\geq 0}$ and $v \in V_m$, the sequence $((\varphi'_{\infty,n} \circ \varphi_{n,m})(v))_{n \geq m}$ is a Cauchy sequence in V'. Without loss of generality $\|v\| \leq 1$. This claim follows from a standard $\varepsilon/3$ argument: to show that

$$\|(\varphi'_{\infty,n_1} \circ \varphi_{n_1,m})(v) - (\varphi'_{\infty,n_2} \circ \varphi_{n_2,m})(v)\| < \varepsilon$$

for all sufficiently large n_1 and n_2, choose $k \in \mathbb{Z}_{>0}$ such that $\|v - v_k^{(m)}\| < \varepsilon/3$, and use the previous claim.

Since V' is complete, it follows that $\lim_{n\to\infty} (\varphi'_{\infty,n} \circ \varphi_{n,m})(v)$ exists for all $m \in \mathbb{Z}_{\geq 0}$ and $v \in V_m$. Since $\|\varphi'_{\infty,n} \circ \varphi_{n,m}\| \leq 1$ whenever $m, n \in \mathbb{Z}_{\geq 0}$ satisfy $m \leq n$, it follows that for $m \in \mathbb{Z}_{>0}$ there is a unique bounded linear map $\rho_m : V_m \to V'$ such that $\|\rho_m\| \leq 1$ and $\rho_m(v) = \lim_{n\to\infty} (\varphi'_{\infty,n} \circ \varphi_{n,m})(v)$ for all $v \in V_m$.

It is clear from the construction that $\rho_n \circ \varphi_{n,m} = \rho_m$ whenever $m, n \in \mathbb{Z}_{\geq 0}$ satisfy $m \leq n$. By the universal property of the direct limit, there is a unique bounded linear map $\rho : V \to V'$ such that $\rho \circ \varphi_{\infty,m} = \rho_m$ for all $m \in \mathbb{Z}_{\geq 0}$. It is clearly contractive, order preserving, order unit preserving, and uniquely determined as in the statement of the proposition.

The same argument shows that there is a unique contractive linear map $\lambda : V' \to V$ determined in the analogous way. For all $m \in \mathbb{Z}_{\geq 0}$, we have

$$\lambda \circ \rho \circ \varphi_{\infty,m} = \lambda \circ \varphi'_{\infty,m} = \varphi_{\infty,m},$$

so the universal property of the direct limit implies $\lambda \circ \rho = \mathrm{id}_V$. Similarly $\rho \circ \lambda = \mathrm{id}_{V'}$. □

Proposition 2.12. *The isomorphism of Proposition 2.11 has the following naturality property. Let the notation be as there, and suppose that, in addition, we are given separable complete order unit spaces W_n for $n \in \mathbb{Z}_{\geq 0}$, direct systems*

$$((W_m)_{m=0,1,2,\ldots}, (\psi_{n,m})_{0 \leq m \leq n}) \quad \text{and} \quad ((W_m)_{m=0,1,2,\ldots}, (\psi'_{n,m})_{0 \leq m \leq n})$$

using the same spaces, with positive linear order unit preserving maps, with direct limits W and W', and with corresponding maps

$$\psi_{\infty,n} : W_n \to W \quad \text{and} \quad \psi'_{\infty,n} : W_n \to W'$$

for $n \in \mathbb{Z}_{\geq 0}$. Also suppose that for $n \in \mathbb{Z}_{>0}$ there is a sequence

$$w_0^{(n)},\ w_1^{(n)},\ \ldots \in W_n$$

which is dense in the closed unit ball of W_n, and that there is a sequence $(\varepsilon_n)_{n=0,1,2,\ldots}$ in $(0, \infty)$ such that $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ and, with

$$G_n = \bigcup_{m=0}^{n} [\{\psi_{n,m}(w_k^{(m)}) \mid 0 \leq k \leq n\} \cup \{\psi'_{n,m}(w_k^{(m)}) \mid 0 \leq k \leq n\}],$$

for all $n \in \mathbb{Z}_{\geq 0}$ and all $w \in G_n$ we have

$$\|\psi_{n+1,\,n}(w) - \psi'_{n+1,\,n}(w)\| < \varepsilon_n.$$

Let $\sigma : W \to W'$ be the isomorphism of Proposition 2.11. Suppose further that we have positive linear order unit preserving maps $\mu_n, \mu'_n : V_n \to W_n$ for $n \in \mathbb{Z}_{\geq 0}$ such that

$$\mu_n \circ \varphi_{n,m} = \psi_{n,m} \circ \mu_m \quad \text{and} \quad \mu'_n \circ \varphi'_{n,m} = \psi'_{n,m} \circ \mu'_m$$

for all $m, n \in \mathbb{Z}_{\geq 0}$ with $m \leq n$. Let $\mu : V \to W$ and $\mu' : V' \to W'$ be the induced maps of the direct limits. Then $\mu' \circ \rho = \sigma \circ \mu$.

Proof. By construction, $\rho : V \to V'$ and $\sigma : W \to W'$ are determined by

$$\rho(\varphi_{\infty,m}(v)) = \lim_{n \to \infty} (\varphi'_{\infty,n} \circ \varphi_{n,m})(v) \tag{2-4}$$

for $m \in \mathbb{Z}_{\geq 0}$ and $v \in V_m$, and

$$\sigma(\psi_{\infty,m}(w)) = \lim_{n\to\infty} (\psi'_{\infty,n} \circ \psi_{n,m})(w) \tag{2-5}$$

for $m \in \mathbb{Z}_{\geq 0}$ and $w \in W_m$. Using (2-4) at the first step and (2-5) at the last step, for $m \in \mathbb{Z}_{\geq 0}$ and $v \in V_m$ we therefore have

$$\begin{aligned}(\mu' \circ \rho)(\varphi_{\infty,m}(v)) &= \mu'\bigl(\lim_{n\to\infty} (\varphi'_{\infty,n} \circ \varphi_{n,m})(v)\bigr) = \lim_{n\to\infty} (\mu' \circ \varphi'_{\infty,n} \circ \varphi_{n,m})(v) \\ &= \lim_{n\to\infty} (\psi'_{\infty,n} \circ \psi_{n,m} \circ \mu_m)(v) = (\sigma \circ \mu)(\varphi_{\infty,m}(v)).\end{aligned}$$

Since $\bigcup_{m=0}^{\infty} \varphi_{\infty,m}(V_m)$ is dense in V, the result follows. □

Proposition 2.14 below can essentially be extracted from the proof of Lemma 3.7 of [Thomsen 1994]. We give here a precise formulation which is needed for our purposes. The difference between our formulation and that of [Thomsen 1994] is that we need more control over the matrix sizes in the construction. In the argument, the following result substitutes for Lemma 3.6 there.

Lemma 2.13 (based on [Thomsen 1994]). *Let X and Y be compact Hausdorff spaces, with X path connected. Let $\lambda : \mathrm{T}(C(Y)) \to \mathrm{T}(C(X))$ be affine and continuous. Let $E(\lambda) : C(X, \mathbb{R}) \to C(Y, \mathbb{R})$ be as in Lemma 2.6. Then for every $\varepsilon > 0$ and every finite set $F \subset C(X, \mathbb{R})$ there exists $N_0 \in \mathbb{Z}_{>0}$ such that for every $N \in \mathbb{Z}_{>0}$ with $N \geq N_0$ there are continuous functions $g_1, g_2, \ldots, g_N : Y \to X$ such that for every $f \in F$ we have*

$$\Bigl\| E(\lambda)(f) - \frac{1}{N} \sum_{j=1}^{N} f \circ g_j \Bigr\|_\infty < \varepsilon.$$

Proof. It suffices to prove the result under the additional assumption that $\|f\| \leq 1$ for all $f \in F$.

Let $\varepsilon > 0$. Since $E(\lambda)$ is a Markov operator, Theorem 2.1 of [Thomsen 1994] provides $n \in \mathbb{Z}_{>0}$, unital homomorphisms $\psi_1, \psi_2, \ldots, \psi_n : C(X) \to C(Y)$, and $\alpha_1, \alpha_2, \ldots, \alpha_n \in [0, 1]$ with $\sum_{l=1}^{n} \alpha_l = 1$ such that

$$\Bigl\| E(\lambda)(f) - \sum_{l=1}^{n} \alpha_l \psi_l(f) \Bigr\|_\infty < \frac{\varepsilon}{2}$$

for all $f \in F$. Note that if $\beta_1, \beta_2, \ldots, \beta_n \in [0, 1]$ satisfy $\sum_{l=1}^{n} |\alpha_l - \beta_l| < \varepsilon/2$ then

$$\Bigl\| E(\lambda)(f) - \sum_{l=1}^{n} \beta_l \psi_l(f) \Bigr\|_\infty < \varepsilon$$

for all $f \in F$. Choose $N_0 \in \mathbb{Z}_{>0}$ such that $N_0 > 4n/\varepsilon$. Let $N \in \mathbb{Z}_{>0}$ satisfy $N \geq N_0$. For $l = 1, 2, \ldots, n-1$ choose $\beta_l \in (\alpha_l - 1/N,\ \alpha_l] \cap (1/N)\mathbb{Z}$, and set $\beta_n = 1 - \sum_{l=1}^{n-1} \beta_l$. Then

$$\beta_1, \beta_2, \ldots, \beta_n \in \frac{1}{N}\mathbb{Z}_{\geq 0}, \quad \sum_{l=1}^{n} \beta_l = 1, \quad \text{and} \quad \sum_{l=1}^{n} |\alpha_l - \beta_l| < \frac{\varepsilon}{2}.$$

Set $m_l = N\beta_l$ for $l = 1, 2, \ldots, n$. Then for all $f \in F$ we have

$$\Bigl\| E(\lambda)(f) - \frac{1}{N} \sum_{l=1}^{n} m_l \psi_l(f) \Bigr\|_\infty < \varepsilon.$$

Now for $l = 1, 2, \ldots, n$ let $h_l : Y \to X$ be the continuous function such that $\psi_l(f) = f \circ h_l$ for all $f \in C(X)$, and for $j = 1, 2, \ldots, N$ define $g_j = h_l$ when

$$\sum_{k=1}^{l-1} m_k < j \leq \sum_{k=1}^{l} m_k.$$

Then

$$\frac{1}{N} \sum_{l=1}^{n} m_l \psi_l(f) = \frac{1}{N} \sum_{j=1}^{N} f \circ g_j$$

for all $f \in C(X)$. □

Proposition 2.14. *Let K be a metrizable Choquet simplex, and let $(l(n))_{n=0,1,2,\ldots}$ be a sequence of integers such that $l(n) \geq 2$ for all $n > 0$. For $n \in \mathbb{Z}_{\geq 0}$ set $r(n) = \prod_{j=1}^{n} l(j)$. Then there exist $n_0 < n_1 < n_2 < \cdots \in \mathbb{Z}_{\geq 0}$, with $n_0 = 0$ and $n_1 = 1$, and a direct system*

$$C([0,1]) \otimes M_{r(n_0)} \xrightarrow{\alpha_{1,0}} C([0,1]) \otimes M_{r(n_1)} \xrightarrow{\alpha_{2,1}} C([0,1]) \otimes M_{r(n_2)} \xrightarrow{\alpha_{3,2}} \cdots$$

with injective maps which are diagonal (in the sense analogous to Construction 1.1(9)) and such that the direct limit A satisfies $\mathrm{T}(A) \cong K$.

It is easy to arrange that the algebra A in this proposition be simple: by Proposition 2.11, replacement of a small enough fraction of the maps $g_{k,l}$ in the proof with suitable point evaluations does not change the tracial state space. However, doing so at this stage does not help with later work.

The conditions $n_0 = 0$ and $n_1 = 1$ are needed because we will later need to pass to a corresponding subsystem of a system as in Construction 1.1 (more accurately, Construction 3.3 below), and we want to avoid later complexity of the argument by preserving the value of ω.

Proof of Proposition 2.14. We mostly follow the proof of Lemma 3.7 of [Thomsen 1994], using Lemma 2.13 in place of Lemma 3.6 of [Thomsen 1994], and slightly changing the order of the steps to accommodate the difference between our conclusion and that of Theorem 3.9 of [Thomsen 1994]. For convenience, we will use Proposition 2.11 in place of Lemma 3.4 of [Thomsen 1994].

For convenience of notation, and following [Thomsen 1994], set $P = \mathrm{T}(C([0,1]))$. Lemma 3.8 of [Thomsen 1994] provides an inverse system $((P_k)_{k=0,1,\ldots}, (\lambda_{j,k})_{0 \leq j \leq k})$ with continuous affine maps $\lambda_{j,k} : P_k \to P_j$ such that $P_k = P$ for all $k \in \mathbb{Z}_{\geq 0}$ and

$$\varprojlim((P_k)_{k=0,1,\ldots}, (\lambda_{j,k})_{0 \leq j \leq k}) \cong K. \tag{2-6}$$

Choose $f_0, f_1, \ldots \in C([0,1], \mathbb{R})$ such that $\{f_0, f_1, \ldots\}$ is dense in $C([0,1], \mathbb{R})$.

We now construct numbers $n_k \in \mathbb{Z}_{>0}$ for $k \in \mathbb{Z}_{\geq 0}$, finite subsets $F_k \subset C([0,1], \mathbb{R})$ for $k \in \mathbb{Z}_{\geq 0}$, positive unital linear maps $\psi_{k+1,k} : C([0,1], \mathbb{R}) \to C([0,1], \mathbb{R})$ for $k \in \mathbb{Z}_{>0}$, and continuous functions

$$g_{k,1}, g_{k,2}, \ldots, g_{k, r(n_{k+1})/r(n_k)} : [0,1] \to [0,1]$$

such that the following conditions are satisfied:

(1) $F_0 = \{f_0\}$ and for $k \in \mathbb{Z}_{\geq 0}$,

$$F_{k+1} = F_k \cup \{f_{k+1}\} \cup E(\lambda_{k,k+1})(F_k \cup \{f_{k+1}\}) \cup \psi_{k+1,k}(F_k \cup \{f_{k+1}\}).$$

(2) $n_0 = 0$, $n_1 = 1$, and $n_2 = 2$, and for $k \in \mathbb{Z}_{>0}$ with $k \geq 2$ we have $n_{k+1} > n_k$ and $r(n_{k+1})/r(n_k) > 2^k$.

(3) For $k \in \mathbb{Z}_{\geq 0}$ and $f \in C([0, 1], \mathbb{R})$,
$$\psi_{k+1,\,k}(f) = \frac{r(n_k)}{r(n_{k+1})} \sum_{l=1}^{r(n_{k+1})/r(n_k)} f \circ g_{k,l}.$$

(4) $\|E(\lambda_{k,\,k+1})(f) - \psi_{k+1,\,k}(f)\| < 2^{-k}$ for $k \geq 2$ and $f \in F_k$.

We carry out the construction by induction on k. Define $F_0 = \{f_0\}$, $n_0 = 0$, and $n_1 = 1$. Take $g_{0,l} : [0, 1] \to [0, 1]$ to be the identity map for $l = 1, 2, \ldots, r(1)$. Then define $\psi_{1,\,0}$ by (3) and define F_1 by (1).

Now suppose $k \geq 1$ and we have F_k and n_k; we construct
$$F_{k+1}, \quad n_{k+1}, \quad g_{k,1}, g_{k,2}, \ldots, g_{k,\,r(n_{k+1})/r(n_k)}, \quad \text{and} \quad \psi_{k+1,\,k}.$$
Apply Lemma 2.13 with $\lambda = \lambda_{k,\,k+1}$, with $\varepsilon = 2^{-k}$, and with $F = F_k$, obtaining $N_0 \in \mathbb{Z}_{>0}$. Choose $n_{k+1} > n_k$ and so large that
$$\frac{r(n_{k+1})}{r(n_k)} > \max(N_0,\ 2^k).$$
This gives (2). Apply the conclusion of Lemma 2.13 with $N = r(n_{k+1})/r(n_k)$, calling the resulting functions $g_{k,1}, g_{k,2}, \ldots, g_{k,\,r(n_{k+1})/r(n_k)}$. Then define $\psi_{k+1,\,k}$ by (3). This gives (4). Finally, define F_{k+1} by (1). This completes the induction.

For $j, k \in \mathbb{Z}_{\geq 0}$ with $j \leq k$, define $\psi_{k,\,j} : C([0, 1], \mathbb{R}) \to C([0, 1], \mathbb{R})$ by
$$\psi_{k,\,j} = \psi_{k,\,k-1} \circ \psi_{k-1,\,k-2} \circ \cdots \circ \psi_{j+1,\,j}.$$

An induction argument shows that for $j, k \in \mathbb{Z}_{\geq 0}$ with $j \leq k$, we have
$$E(\lambda_{j,k})(f_j) \in F_k \quad \text{and} \quad \psi_{k,\,j}(f_j) \in F_k.$$
This condition, together with Proposition 2.11, allows us to conclude that, as order unit spaces, we have
$$\varinjlim((C([0, 1], \mathbb{R}))_{k=0,1,\ldots},\ (E(\lambda_{j,k}))_{0 \leq j \leq k}) \cong \varinjlim((C([0, 1], \mathbb{R}))_{k=0,1,\ldots},\ (\psi_{k,j})_{0 \leq j \leq k}). \tag{2-7}$$

For $k \in \mathbb{Z}_{\geq 0}$ define
$$\alpha_{k+1,\,k} : C([0, 1],\ M_{r(n_k)}) \to C([0, 1],\ M_{r(n_{k+1})}) = M_{r(n_{k+1})/r(n_k)}(C([0, 1],\ M_{r(n_k)}))$$
by
$$\alpha_{k+1,\,k}(f) = \operatorname{diag}(f \circ g_{k,1},\ f \circ g_{k,2},\ \ldots,\ f \circ g_{k,\,r(n_{k+1})/r(n_k)})$$
for $f \in C([0, 1],\ M_{r(n_k)})$. Let A be the resulting direct limit C^*-algebra.

It is easy to check, and is stated as Lemma 3.5 of [Thomsen 1994], that $\widehat{\alpha_{k+1,k}} = \psi_{k+1,k}$. Letting V and W be the order unit spaces
$$V = \varinjlim((C([0, 1],\ \mathbb{R}))_{k=0,1,\ldots}, (E(\lambda_{j,k}))_{0 \leq j \leq k}),$$
$$W = \varinjlim((C([0, 1],\ \mathbb{R}))_{k=0,1,\ldots}, (\widehat{\alpha_{k,j}})_{0 \leq j \leq k}),$$
(2-7) now says $V \cong W$. Lemma 3.2 of [Thomsen 1994] and (2-6) imply that $V \cong \operatorname{Aff}(K)$. Proposition 2.7 implies that $\operatorname{Aff}(T(A)) \cong W$. So $\operatorname{Aff}(T(A)) \cong \operatorname{Aff}(K)$, whence $T(A) \cong K$ by Theorem 2.4. ☐

Proposition 2.15. *Let $(D_n)_{n=0,1,2,\ldots}$ and $(C_n)_{n=0,1,2,\ldots}$ be sequences of unital C^*-algebras. Let*

$$\begin{aligned} &((D_n)_{n=0,1,2,\ldots}, (\varphi_{n,m})_{0\le m\le n}), \quad ((D_n)_{n=0,1,2,\ldots}, (\varphi'_{n,m})_{0\le m\le n}),\\ &((C_n)_{n=0,1,2,\ldots}, (\psi_{n,m})_{0\le m\le n}), \quad ((C_n)_{n=0,1,2,\ldots}, (\psi'_{n,m})_{0\le m\le n}) \end{aligned}$$

be direct systems with unital homomorphisms, and call the direct limits (in order) D, D', C, and C'. Suppose further that we have unital homomorphisms $\mu_n, \mu'_n : D_n \to C_n$ for $n \in \mathbb{Z}_{\ge 0}$ such that

$$\mu_n \circ \varphi_{n,m} = \psi_{n,m} \circ \mu_m \quad \text{and} \quad \mu'_n \circ \varphi'_{n,m} = \psi'_{n,m} \circ \mu'_m$$

for all $m, n \in \mathbb{Z}_{\ge 0}$ with $m \le n$. Let $\mu : D \to C$ and $\mu' : D' \to C'$ be the induced maps of the direct limits. Assume that for all $m \in \mathbb{Z}_{\ge 0}$ we have

$$\sum_{n=m}^{\infty} \|\widehat{\varphi_{n,m}} - \widehat{\varphi'_{n,m}}\| < \infty \quad \text{and} \quad \sum_{n=m}^{\infty} \|\widehat{\psi_{n,m}} - \widehat{\psi'_{n,m}}\| < \infty.$$

Then there exist isomorphisms

$$\rho : \mathrm{Aff}(\mathrm{T}(D)) \to \mathrm{Aff}(\mathrm{T}(D')) \quad \text{and} \quad \sigma : \mathrm{Aff}(\mathrm{T}(C)) \to \mathrm{Aff}(\mathrm{T}(C'))$$

such that $\widehat{\mu'} \circ \rho = \sigma \circ \hat{\mu}$. Moreover, if $C_n = D_n$ for all $n \in \mathbb{Z}_{\ge 0}$ and $\psi_{n,m} = \varphi_{n,m}$ and $\psi'_{n,m} = \varphi_{n,m}$ for all m and n, then we can take $\sigma = \rho$.

Proof. We can apply Propositions 2.11 and 2.12 using arbitrary countable dense subsets of the closed unit balls of $\mathrm{Aff}(\mathrm{T}(D_n))$ and $\mathrm{Aff}(\mathrm{T}(C_n))$ for $n \in \mathbb{Z}_{>0}$. Under the hypotheses of the last statement, the uniqueness statement in Proposition 2.11 implies that $\sigma = \rho$. □

Lemma 2.16. *Adopt the notation of Construction 1.1, including* (11) (*a second set of maps*), *and* (9) *and* (13) (*diagonal maps, agreeing in the coordinates* $1, 2, \ldots, d(n+1)$). *Then*

$$\|\widehat{\Gamma^{(0)}_{n+1,\,n}} - \widehat{\Gamma_{n+1,\,n}}\| \le \frac{2k(n+1)}{d(n+1) + k(n+1)}$$

for all $n \in \mathbb{Z}_{\ge 0}$.

Proof. For a compact metrizable space Z, let $M(Z)$ be the real Banach space consisting of all signed Borel measures on Z. (That is, $M(Z)$ is the dual space of $C(Z, \mathbb{R})$.) Identify Z with the set of point masses in $M(Z)$. For $n \in \mathbb{Z}_{\ge 0}$, we can identify $\mathrm{T}(C_n)$ with the weak* compact convex subset of $M(X_n \amalg Y_n)$ consisting of probability measures. Thus $X_n \amalg Y_n \subset \mathrm{T}(C_n)$. For every function $f \in \mathrm{Aff}(\mathrm{T}(C_n))$, the function $\iota_n(f)(z) = f(z) \cdot 1_{M_{r(n)}}$ for $z \in X_n \amalg Y_n$ is in $C(X_n \amalg Y_n,\ M_{r(n)}) = C_n$; and $\tau(\iota_n(f)) = f(\tau)$ for all $\tau \in X_n \amalg Y_n \subset \mathrm{T}(C_n)$, hence also all $\tau \in \mathrm{T}(C_n)$ by linearity and continuity.

For $f \in \mathrm{Aff}(\mathrm{T}(C_n))$ and $\tau \in \mathrm{T}(C_{n+1})$, we can apply the formula in Construction 1.1(9) to $\iota_n(f)$ and apply τ to everything, to get

$$\widehat{\Gamma^{(0)}_{n+1,\,n}}(f)(\tau) = \frac{1}{l(n+1)} \sum_{k=1}^{l(n+1)} \tau(f \circ S^{(0)}_{n,1}) \quad \text{and} \quad \widehat{\Gamma_{n+1,\,n}}(f)(\tau) = \frac{1}{l(n+1)} \sum_{k=1}^{l(n+1)} \tau(f \circ S_{n,1}).$$

Using (13), we get

$$\begin{aligned}|\widehat{\Gamma^{(0)}_{n+1,\,n}}(f)(\tau)-\widehat{\Gamma_{n+1,\,n}}(f)(\tau)| &= \frac{1}{l(n+1)}\left|\sum_{k=d(n+1)+1}^{l(n+1)}[\tau(f\circ S^{(0)}_{n,1})-\tau(f\circ S_{n,1})]\right| \\ &\leq \frac{l(n+1)-d(n+1)}{l(n+1)}(2\|f\|_\infty).\end{aligned}$$

The conclusion follows. □

We add additional parts to Constructions 1.1 and 1.6.

Construction 2.17. Adopt the assumptions and notation of all parts of Construction 1.1 (except (13)), and in addition make the following assumptions and definitions:

(20) For all $m \in \mathbb{Z}_{\geq 0}$, the maps $S^{(0)}_{m,j}, S_{m,j} : X_{m+1} \amalg Y_{m+1} \to X_m \amalg Y_m$ satisfy

$$S^{(0)}_{m,j}(X_{m+1}) \subset X_m \quad \text{and} \quad S^{(0)}_{m,j}(Y_{m+1}) \subset Y_m$$

for $j = 1, 2, \ldots, l(m)$,

$$S_{m,j}(X_{m+1}) \subset X_m \quad \text{and} \quad S_{m,j}(Y_{m+1}) \subset Y_m$$

for $j = 1, 2, \ldots, d(m)$, and

$$S_{m,j}(X_{m+1}) \subset Y_m \quad \text{and} \quad S_{m,j}(Y_{m+1}) \subset X_m$$

for $j = d(m)+1,\ d(m)+2,\ \ldots,\ l(m)$.

(21) For $m \in \mathbb{Z}_{\geq 0}$, define $D_m = M_{r(m)} \oplus M_{r(m)}$. Define $\varphi^{(0)}_{m+1,m},\ \varphi_{m+1,m} : D_m \to D_{m+1}$ by, for $a, b \in M_{r(m)}$,

$$\begin{aligned}\varphi^{(0)}_{m+1,m}(a,b) &= (\mathrm{diag}(a, a, \ldots, a),\ \mathrm{diag}(b, b, \ldots, b)),\\ \varphi_{m+1,m}(a,b) &= (\mathrm{diag}(a, a, \ldots, a, b, b, \ldots, b),\ \mathrm{diag}(b, b, \ldots, b, a, a, \ldots, a)),\end{aligned}$$

in which a occurs $d(m)$ times in the first entry in the second line on the right and $k(m)$ times in the second entry, while b occurs $k(m)$ times in the first entry and $d(m)$ times in the second entry. For $m, n \in \mathbb{Z}_{\geq 0}$ with $m \leq n$, define

$$\varphi_{n,m} = \varphi_{n,n-1} \circ \varphi_{n-1,\,n-2} \circ \cdots \circ \varphi_{m+1,m} : D_m \to D_n,$$

and define $\varphi^{(0)}_{n,m} : D_m \to D_n$ similarly. Define AF algebras by

$$D = \varinjlim_m (D_m, \varphi_{m+1,m}) \quad \text{and} \quad D^{(0)} = \varinjlim_m (D_m, \varphi^{(0)}_{m+1,m}),$$

and for $m \in \mathbb{Z}_{>0}$ let $\varphi_{\infty,m} : D_m \to D$ and $\varphi^{(0)}_{\infty,m} : D_m \to D^{(0)}$ be the maps associated to these direct limits.

(22) For $m \in \mathbb{Z}_{\geq 0}$, define $\mu_m : D_m \to C_m$ as follows. For $a, b \in M_{r(m)}$ let $f \in C(X_m, M_{r(m)})$ and $g \in C(Y_m, M_{r(m)})$ be the constant functions with values a and b. Then set $\mu_m(a,b) = (f, g)$. Further, following Lemma 2.18(2) below, let $\mu : D \to C$ and $\mu^{(0)} : D^{(0)} \to C^{(0)}$ be the direct limits of the maps μ_m.

(23) For $m \in \mathbb{Z}_{\geq 0}$, define $\theta_m : D_m \to D_m$ by $\theta_m(a,b) = (b,a)$ for $a, b \in M_{r(m)}$. Further, following Lemma 2.18(3) below, let $\theta \in \mathrm{Aut}(D)$ and $\theta^{(0)} \in \mathrm{Aut}(D^{(0)})$ be the direct limits of the maps θ_m.

Lemma 2.18. *Under the assumptions of Constructions 1.1* (*except* (13)), *1.6*, *and 2.17*, *the following hold*:

(1) *The direct system* $((C_n^{(0)})_{n=0,1,2,\ldots}, (\Gamma_{n,m}^{(0)})_{0\leq m\leq n})$ *is the direct sum of two direct systems*

$$((C(X_n,\ M_{r(n)}))_{n=0,1,2,\ldots}, (\Gamma_{n,m}^{(0)}|_{C(X_m,\ M_{r(m)})})_{0\leq m\leq n}),$$
$$((C(Y_n,\ M_{r(n)}))_{n=0,1,2,\ldots}, (\Gamma_{n,m}^{(0)}|_{C(Y_m,\ M_{r(m)})})_{0\leq m\leq n}),$$

and $C^{(0)}$ *is isomorphic to the direct sum of the direct limits* A *and* B *of these systems.*

(2) *For all* $m, n \in \mathbb{Z}_{\geq 0}$ *with* $m \leq n$,

$$\Gamma_{n,m}^{(0)} \circ \mu_m = \mu_n \circ \varphi_{n,m}^{(0)} \quad \textit{and} \quad \Gamma_{n,m} \circ \mu_m = \mu_n \circ \varphi_{n,m}.$$

Moreover, *the maps* μ_m *induce unital homomorphisms* $\mu^{(0)} : D^{(0)} \to C^{(0)}$ *and* $\mu : D \to C$, *and for all* $m \in \mathbb{Z}_{\geq 0}$,

$$\Gamma_{\infty,m}^{(0)} \circ \mu_m = \mu^{(0)} \circ \varphi_{\infty,m}^{(0)} \quad \textit{and} \quad \Gamma_{\infty,m} \circ \mu_m = \mu \circ \varphi_{\infty,m}.$$

(3) *For all* $m, n \in \mathbb{Z}_{\geq 0}$ *with* $m \leq n$,

$$\varphi_{n,m}^{(0)} \circ \theta_m = \theta_n \circ \varphi_{n,m}^{(0)} \quad \textit{and} \quad \varphi_{n,m} \circ \theta_m = \theta_n \circ \varphi_{n,m}.$$

The maps θ_m *induce automorphisms* $\theta : D \to D$ *and* $\theta^{(0)} : D^{(0)} \to D^{(0)}$ *such that*

$$\varphi_{\infty,m} \circ \theta_m = \theta \circ \varphi_{\infty,m} \quad \textit{and} \quad \varphi_{\infty,m}^{(0)} \circ \theta_m = \theta^{(0)} \circ \varphi_{\infty,m}^{(0)}$$

for all $m \in \mathbb{Z}_{\geq 0}$.

(4) *For all* $m \in \mathbb{Z}_{\geq 0}$, $(\mu_m)_* : K_*(D_m) \to K_*(C_m)$ *is an isomorphism*, *and*

$$\mu_* : K_*(D) \to K_*(C) \quad \textit{and} \quad (\mu^{(0)})_* : K_*(D^{(0)}) \to K_*(C^{(0)})$$

are isomorphisms.

Proof. The fact that all the maps in (4) are isomorphisms on K-theory comes from the assumption that the spaces X_m and Y_m are contractible ((14) and (15) in Construction 1.6). Everything else is essentially immediate from the constructions. □

3. The main theorem

We now have the ingredients to deduce the main theorem of this paper, Theorem 3.2.

To state the theorem, we first need to define automorphisms of Elliott invariants, so we need a category in which they lie. For convenience, we restrict to unital C^*-algebras, and we give a very basic list of conditions.

Definition 3.1. An *abstract unital Elliott invariant* is a tuple $G = (G_0, (G_0)_+, g, G_1, K, \rho)$ in which $(G_0, (G_0)_+, g)$ is a preordered abelian group with distinguished positive element g which is an order unit, G_1 is an abelian group, K is a Choquet simplex (possibly empty), and $\rho : G_0 \to \mathrm{Aff}(K)$ is an order preserving group homomorphism such that $\rho(g)$ is the constant function 1. (If $K = \varnothing$, we take $\mathrm{Aff}(K) = \{0\}$, and we take ρ to be the constant function with value 0.)

If

$$G^{(0)} = (G_0^{(0)}, (G_0^{(0)})_+, g^{(0)}, G_1^{(0)}, K^{(0)}, \rho^{(0)}) \quad \text{and} \quad G^{(1)} = (G_0^{(1)}, (G_0^{(1)})_+, g^{(1)}, G_1^{(1)}, K^{(1)}, \rho^{(1)})$$

are abstract unital Elliott invariants, then a *morphism* from $G^{(0)}$ to $G^{(1)}$ is a triple $F = (F_0, F_1, S)$ in which $F_0 : G_0^{(0)} \to G_0^{(1)}$ is a group homomorphism satisfying

$$F_0((G_0^{(0)})_+) \subset (G_0^{(1)})_+ \quad \text{and} \quad F_0(g^{(0)}) = g^{(1)},$$

$F_1 : G_1^{(0)} \to G_1^{(0)}$ is a group homomorphism, and $S : K^{(1)} \to K^{(0)}$ is a continuous affine map satisfying

$$\rho^{(1)}(F_0(\eta)) = \rho^{(0)}(\eta) \circ S \tag{3-1}$$

for all $\eta \in G_0^{(0)}$.

If

$$F^{(0)} : G^{(0)} \to G^{(1)} \quad \text{and} \quad F^{(1)} = (F_0^{(1)}, F_1^{(1)}, S^{(1)}) : G^{(1)} \to G^{(2)}$$

are morphisms of abstract unital Elliott invariants, then define

$$F^{(1)} \circ F^{(0)} = (F_0^{(1)} \circ F_0^{(0)},\ F_1^{(1)} \circ F_1^{(0)},\ S^{(0)} \circ S^{(1)}).$$

(Note: $S^{(0)} \circ S^{(1)}$, not $S^{(1)} \circ S^{(0)}$.)

The *Elliott invariant of a unital C^*-algebra A* is

$$\mathrm{Ell}(A) = (K_0(A),\ K_0(A)_+,\ [1],\ K_1(A),\ \mathrm{T}(A),\ \rho_A),$$

in which $\rho_A : K_0(A) \to \mathrm{Aff}(\mathrm{T}(A))$ is given by $\rho_A(\eta)(\tau) = \tau_*(\eta)$ for $\eta \in K_0(A)$ and $\tau \in \mathrm{T}(A)$.

If A and B are unital C^*-algebras and $\varphi : A \to B$ is a unital homomorphism, then we define $\varphi_* : \mathrm{Ell}(A) \to \mathrm{Ell}(B)$ to consist of the maps φ_* from $K_0(A)$ to $K_0(B)$ and from $K_1(A)$ to $K_1(B)$, together with the map $\mathrm{T}(\varphi)$ of Definition 2.5. We write it as $(\varphi_{*,0},\ \varphi_{*,1},\ \mathrm{T}(\varphi))$.

Definition 3.1 is enough to make the abstract unital Elliott invariants into a category such that $\mathrm{Ell}(\,\cdot\,)$ is a functor from unital C^*-algebras and unital homomorphisms to abstract unital Elliott invariants.

Theorem 3.2. *There exists a simple unital separable AH algebra C with stable rank* 1 *and with the following property. There exists an automorphism F of* $\mathrm{Ell}(C)$ *such that there is no automorphism α of C satisfying $\alpha_* = F$. Moreover, the automorphism F in this example can be chosen so that $F \circ F$ is the identity morphism of* $\mathrm{Ell}(C)$.

We outline the proof. We make a first pass through Constructions 1.1 and 1.6, without the spaces Y_n, and without specifying the point evaluation maps. This is Construction 3.3 below. We get a direct system; call its direct limit $\widetilde{C}$. Apply Proposition 2.14 using the sequence of matrix sizes in this system and $K = \mathrm{T}(\widetilde{C})$. Doing so requires passing to a subsequence of the sequence of matrix sizes. Replace the original system with the corresponding subsystem; Lemma 3.5 below justifies this. Then make a second pass through Constructions 1.1 and 1.6, taking the spaces X_n and the maps between them from this subsystem and the spaces Y_n and the maps between them from the system gotten from Proposition 2.14, as needed substituting appropriate point evaluations for the diagonal entries of the formulas for the maps. This requires sufficiently few changes that, by our work in Section 2, the tracial state space remains the same. Therefore the algebra obtained from these constructions has an order two automorphism of its tracial state space which corresponds to exchanging the two rows in the diagram (0-2). The constructions have been designed so that there is also a corresponding automorphism of the K-theory. Our work in Section 1 rules out the possibility of a corresponding automorphism of the algebra, because such an automorphism would necessarily send a particular corner of the algebra to another one with a different radius of comparison.

We start with the following construction, which is "half" of Construction 1.1, and gives just the top row of the diagram (0-1).

Construction 3.3. We will consider direct systems and their associated direct limits constructed as follows.

(1) The sequences $(d(n))_{n=0,1,2,\ldots}$ and $(k(n))_{n=0,1,2,\ldots}$ in $\mathbb{Z}_{\geq 0}$ are as in Construction 1.1 (1) and satisfy the condition of Construction 1.1 (2). We further define $(l(n))_{n=0,1,2,\ldots}$, $(r(n))_{n=0,1,2,\ldots}$, $(s(n))_{n=0,1,2,\ldots}$, and $(t(n))_{n=0,1,2,\ldots}$ as in Construction 1.1 (1).

(2) Following Constructions 1.1 (3) and (4), we define

$$\kappa = \inf_{n \in \mathbb{Z}_{>0}} \frac{s(n)}{r(n)}, \quad \omega = \frac{k(1)}{k(1)+d(1)}, \quad \text{and} \quad \omega' = \sum_{n=2}^{\infty} \frac{k(n)}{k(n)+d(n)}.$$

(These will not be used directly in connection with this direct system.)

(3) As in Construction 1.6 (14), we define compact metric spaces by $X_n = \mathrm{cone}((S^2)^{s(n)})$ for $n \in \mathbb{Z}_{\geq 0}$, and we define maps $Q_j^{(n)} : X_{n+1} \to X_n$ for $n \in \mathbb{Z}_{\geq 0}$ and $j = 1, 2, \ldots, d(n+1)$ to be the cones over the projection maps

$$(S^2)^{s(n+1)} = ((S^2)^{s(n)})^{d(n+1)} \to (S^2)^{s(n)}.$$

(4) We are given maps $\delta_n : C(X_n) \to C(X_{n+1}, M_{l(n+1)})$ (as in Construction 1.1 (8), but with only one summand) which are diagonal; that is, there are continuous maps

$$T_{n,1}, \ T_{n,2}, \ \ldots, \ T_{n,l(n+1)} : X_{n+1} \to X_n$$

such that

$$\delta_n(f) = \mathrm{diag}(f \circ T_{n,1}, \ f \circ T_{n,2}, \ \ldots, \ f \circ T_{n,l(n+1)})$$

for $f \in C(X_n)$. (Compare with Construction 1.1 (9).) Moreover, $T_{n,j} = Q_j^{(n)}$ for $j = 1, 2, \ldots, d(n+1)$. The maps $T_{n,j}$ are unspecified for $j = d(n+1)+1, \ d(n+1)+2, \ \ldots, \ l(n+1)$.

(5) Set $A_n = M_{r(n)} \otimes C(X_n)$ (like in Construction 1.1 (7) but with only one summand). Following Construction 1.1 (8), set

$$\Delta_{n+1,\,n} = \mathrm{id}_{M_{r(n)}} \otimes \delta_n : A_n \to A_{n+1},$$

and for $m, n \in \mathbb{Z}_{\geq 0}$ with $m \leq n$, take

$$\Delta_{n,m} = \Delta_{n,n-1} \circ \Delta_{n-1,\,n-2} \circ \cdots \circ \Delta_{m+1,m} : A_m \to A_n.$$

(6) Define $A = \varinjlim_n A_n$, taken with respect to the maps $\Delta_{n,m}$. For $n \in \mathbb{Z}_{\geq 0}$, let $\Delta_{\infty,n} : A_n \to A$ be the map associated with the direct limit.

To avoid confusing notation, we isolate the following computation as a lemma.

Lemma 3.4. *Let $n \in \mathbb{Z}_{>0}$ and let $\kappa_1, \kappa_2, \ldots, \kappa_n, \delta_1, \delta_2, \ldots, \delta_n \in (0, \infty)$. Then*

$$\sum_{j=1}^{n} \frac{\kappa_j}{\delta_j + \kappa_j} \geq \frac{\prod_{j=1}^{n}(\delta_j + \kappa_j) - \prod_{j=1}^{n} \delta_j}{\prod_{j=1}^{n}(\delta_j + \kappa_j)}.$$

Proof. For $j = 1, 2, \ldots, n$ define

$$\lambda_j = \frac{\kappa_j}{\delta_j + \kappa_j}.$$

Then $\lambda_j \in (0, 1)$. Some calculation shows that the conclusion of the lemma becomes

$$\sum_{j=1}^{n} \lambda_j \geq 1 - \prod_{j=1}^{n}(1 - \lambda_j). \tag{3-2}$$

We prove (3-2) by induction on n. For $n = 1$ it is trivial. Suppose (3-2) is known for some value of n. Given $\lambda_1, \lambda_2, \ldots, \lambda_{n+1} \in (0, 1)$, set $\mu = 1 - (1 - \lambda_n)(1 - \lambda_{n+1})$. Then

$$\mu \in (0, 1) \quad \text{and} \quad \mu = \lambda_n + \lambda_{n+1} - \lambda_n \lambda_{n+1} \leq \lambda_n + \lambda_{n+1}.$$

Applying the induction hypothesis on $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \mu$ at the second step, we then have

$$\sum_{j=1}^{n+1} \lambda_j \geq \sum_{j=1}^{n-1} \lambda_j + \mu \geq 1 - \left[\prod_{j=1}^{n-1}(1 - \lambda_j)\right](1 - \mu) = 1 - \prod_{j=1}^{n+1}(1 - \lambda_j).$$

This completes the induction, and the proof of the lemma. □

Lemma 3.5. *Let a direct system as in Construction 3.3 be given, but using sequences $(\tilde{d}(n))_{n=0,1,2,\ldots}$ and $(\tilde{k}(n))_{n=0,1,2,\ldots}$ in place of $(d(n))_{n=0,1,2,\ldots}$ and $(k(n))_{n=0,1,2,\ldots}$. Denote the additional sequences analogous to those in Construction 3.3(1) by $\tilde{l}$, $\tilde{r}$, and $\tilde{s}$. Denote the numbers analogous to those in Construction 3.3(2) by $\tilde{\kappa}$, $\tilde{\omega}$, and $\tilde{\omega}'$. Denote the spaces used in the system by $\widetilde{X}_n$. Let $\nu : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ be a strictly increasing function such that $\nu(0) = 0$ and $\nu(1) = 1$. Then the direct system $(C(\widetilde{X}_{\nu(m)}, M_{\tilde{r}(\nu(m))}))_{m=0,1,2,\ldots}$ is isomorphic to a system as in Construction 3.3, with the choices $d(0) = 1$, $k(0) = 0$,*

$$d(m) = \tilde{d}(\nu(m-1)+1)\tilde{d}(\nu(m-1)+2)\cdots\tilde{d}(\nu(m)), \tag{3-3}$$

$$k(m) = \tilde{l}(\nu(m-1)+1)\tilde{l}(\nu(m-1)+2)\cdots\tilde{l}(\nu(m)) - d(m) \tag{3-4}$$

for $m \in \mathbb{Z}_{>0}$. Moreover, following the notation of Construction 3.3,

$$\begin{gathered} l(m) = \tilde{l}(\nu(m-1)+1)\tilde{l}(\nu(m-1)+2)\cdots\tilde{l}(\nu(m)), \\ r(m) = \tilde{r}(\nu(m)), \quad \text{and} \quad s(m) = \tilde{s}(\nu(m)) \end{gathered} \tag{3-5}$$

for $m \in \mathbb{Z}_{\geq 0}$, and

$$\kappa = \tilde{\kappa}, \quad \omega = \tilde{\omega}, \quad \text{and} \quad \omega' \leq \tilde{\omega}'.$$

Proof. Given the definitions of d and k, the proofs of the formulas for l, r, and s are easy.

Using Lemma 1.2 at the first and fourth steps, we now get

$$\tilde{\kappa} = \lim_{n\to\infty} \frac{\tilde{s}(n)}{\tilde{r}(n)} = \lim_{m\to\infty} \frac{\tilde{r}(\nu(m))}{\tilde{s}(\nu(m))} = \lim_{m\to\infty} \frac{s(m)}{r(m)} = \kappa.$$

We have $\omega = \tilde{\omega}$ because $\nu(1) = 1$.

Using Lemma 3.4 at the second step and (3-3), (3-4) and (3-5) at the third step, we have

$$\begin{aligned}\tilde{\omega}' &= \sum_{m=2}^{\infty} \sum_{j=\nu(m-1)+1}^{\nu(m)} \frac{\tilde{k}(j)}{\tilde{k}(j)+\tilde{d}(j)} \\ &\geq \sum_{m=2}^{\infty} \frac{\prod_{j=\nu(m-1)+1}^{\nu(m)}[\tilde{d}(j)+\tilde{k}(j)] - \prod_{j=\nu(m-1)+1}^{\nu(m)} \tilde{d}(j)}{\prod_{j=\nu(m-1)+1}^{\nu(m)}[\tilde{d}(j)+\tilde{k}(j)]} = \sum_{m=2}^{\infty} \frac{k(m)}{k(m)+d(m)} = \omega'.\end{aligned}$$

Define $X_m = \widetilde{X}_{\nu(m)}$ for $m \in \mathbb{Z}_{\geq 0}$. Clearly $X_m = \operatorname{cone}((S^2)^{s(m)})$, as required. Denote the maps in the system of the hypotheses by

$$\tilde{\delta}_n : C(\widetilde{X}_n) \to C(\widetilde{X}_{n+1},\ M_{\tilde{l}(n+1)}) \quad \text{and} \quad \tilde{\Delta}_{n,m} : \widetilde{C}_m \to \widetilde{C}_n,$$

with $\tilde{\delta}_n$ being built using maps

$$\widetilde{T}_{n,1},\ \widetilde{T}_{n,2},\ \dots,\ \widetilde{T}_{n,\,l(n+1)} : \widetilde{X}_{n+1} \to \widetilde{X}_n,$$

as in Construction 3.3(4). For $p = \nu(m),\ \nu(m)+1,\ \dots,\ \nu(m+1)-1$, set

$$j(p) = \frac{\tilde{r}(p)}{\tilde{r}(\nu(m))} = \tilde{l}(\nu(m)+1)\tilde{l}(\nu(m)+2)\cdots\tilde{l}(p).$$

Then define

$$\delta_m^{(0)} : C(\widetilde{X}_{\nu(m)}) \to C(\widetilde{X}_{\nu(m+1)},\ M_{l(n+1)})$$

by

$$\delta_m^{(0)} = \mathrm{id}_{M_{j(\nu(m+1)-1)}} \otimes \tilde{\delta}_{\nu(m+1)-1} \circ \mathrm{id}_{M_{j(\nu(m+1)-2)}} \otimes \tilde{\delta}_{\nu(m+1)-2} \circ \cdots \circ \tilde{\delta}_{\nu(m)}.$$

(In the last term we omit $\mathrm{id}_{M_{j(\nu(m))}}$ since $j(\nu(m)) = 1$.) With this definition, one checks that $\mathrm{id}_{M_{\tilde{r}(\nu(m))}} \otimes \tilde{\delta}_m = \tilde{\Delta}_{\nu(m+1),\,\nu(m)}$, so that the direct system gotten using the maps $\delta_m^{(0)}$ in Construction 3.3 is a subsystem of the system given in the hypotheses.

We claim that $\delta_m^{(0)}$ is unitarily equivalent to a map $\delta_m : C(X_m) \to C(X_{m+1},\ M_{l(n+1)})$ as in Construction 3.3. This will imply isomorphism of the direct systems, and complete the proof of the lemma. First, $\delta_m^{(0)}$ is given as in Construction 3.3(4) using some maps from $\widetilde{X}_{\nu(m+1)}$ to $\widetilde{X}_{\nu(m)}$, namely all possible compositions

$$\widetilde{T}_{\nu(m),\,i_{\nu(m)}} \circ \widetilde{T}_{\nu(m)+1,\,i_{\nu(m)+1}} \circ \cdots \circ \widetilde{T}_{\nu(m+1)-1,\,i_{\nu(m+1)-1}},$$

with $i_p = 1, 2, \dots, \tilde{l}(p+1)$ for $p = \nu(m),\ \nu(m)+1,\ \dots,\ \nu(m+1)-1$. Moreover, since the composition of projection maps is a projection map, restricting to $i_p = 1, 2, \dots, \tilde{d}(p+1)$ for all p gives exactly all the maps $Q_j^{(m)} : X_{m+1} \to X_m$ for $j = 1, 2, \dots, d(n+1)$. Therefore $\delta_m^{(0)}$ is unitarily equivalent to a map as in Construction 3.3 by a permutation matrix. □

Proof of Theorem 3.2. Choose $N \in \mathbb{Z}_{>0}$ such that

$$N > 5 \quad \text{and} \quad \exp\Big(-\frac{1}{N-1}\Big) > \frac{3}{4}. \tag{3-6}$$

(For example, $N = 6$ will work.) In Construction 1.1(1) we make preliminary choices of the numbers $d(n)$ etc., calling them $\tilde{d}(n)$ etc. Take $\tilde{d}(0) = 1$ and $\tilde{k}(0) = 0$, and take $\tilde{d}(n) = N^n$ and $\tilde{k}(n) = 1$ for $n \in \mathbb{Z}_{>0}$. Then

$$\tilde{l}(n) = N^n + 1, \quad \tilde{r}(n) = \prod_{j=1}^{n}(N^j + 1), \quad \text{and} \quad \tilde{s}(n) = \prod_{j=1}^{n} N^j$$

for $n \in \mathbb{Z}_{>0}$. We obtain numbers as in Construction 3.3(2) (equivalently, Constructions 1.1(3) and (4)), which we call $\tilde{\kappa}$, $\tilde{\omega}$, and $\tilde{\omega}'$. Further, adopt the definitions and notation of Construction 3.3, except that we use $\widetilde{X}_n$ instead of X_n and similarly throughout. That is, in Construction 3.3(3) we call the spaces $\widetilde{X}_n$ instead of X_n, the projection maps $\widetilde{Q}_j^{(n)}$, in Construction 3.3(4) we call the maps of algebras $\tilde{\delta}_n$ and the maps of spaces $\widetilde{T}_{n,j} : \widetilde{X}_{n+1} \to \widetilde{X}_n$, in Construction 3.3(5) we call the algebras $\tilde{A}_n$ and the maps $\tilde{\Delta}_{n,m}$, and in Construction 3.3(6) we call the direct limit $\tilde{A}$ and the maps to it $\tilde{\Delta}_{\infty,n}$. As in Construction 3.3(4), we take $\widetilde{T}_{n,j} = \widetilde{Q}_j^{(n)}$ for $j = 1, 2, \ldots, \tilde{d}(n+1)$. For $n \in \mathbb{Z}_{\geq 0}$ choose an arbitrary point $\tilde{x}_n \in \widetilde{X}_n$, and for $j = \tilde{d}(n+1)+1$ let $\widetilde{T}_{n,j}$ be the constant function on $\widetilde{X}_{n+1}$ with value $\tilde{x}_n$. (Note that $\tilde{d}(n+1)+1 = \tilde{l}(n+1)$.)

We claim that the conditions in Constructions 1.1(3), 1.1(4), and 1.1(5) are satisfied, and moreover that

$$\frac{1}{1-2\tilde{\omega}} < \frac{2\tilde{\kappa}-1}{2\tilde{\omega}}.$$

For $n \in \mathbb{Z}_{>0}$ we have, using $\log(m+1) - \log(m) < 1/m$ at the third step,

$$\frac{\tilde{s}(n)}{\tilde{r}(n)} = \prod_{j=1}^{n} \frac{N^j}{N^j+1} = \exp\Big(\sum_{j=1}^{n} -[\log(N^j+1) - \log(N^j)]\Big) \geq \exp\Big(-\sum_{j=1}^{n} \frac{1}{N^j}\Big) > \exp\Big(-\frac{1}{N-1}\Big).$$

So $\tilde{\kappa} \geq \exp(-1/(N-1)) > \frac{3}{4}$ by (3-6). Furthermore,

$$\tilde{\omega} = \frac{1}{N+1} < \frac{1}{4} \quad \text{and} \quad \tilde{\omega}' = \sum_{j=2}^{\infty} \frac{1}{N^j+1} < \sum_{j=2}^{\infty} \frac{1}{N^j} = \frac{1}{N(N-1)},$$

so the conditions $\tilde{\omega}' < \tilde{\omega} < \frac{1}{2}$ in Construction 1.1(4) and $2\tilde{\kappa} - 1 > 2\tilde{\omega}$ in Construction 1.1(5) are satisfied. Moreover,

$$\frac{1}{1-2\tilde{\omega}} = \frac{N+1}{N-1} < \frac{N+1}{4} = \frac{1}{4\tilde{\omega}} = \frac{2\big(\frac{3}{4}\big)-1}{2\tilde{\omega}} < \frac{2\tilde{\kappa}-1}{2\tilde{\omega}}.$$

The claim is proved.

Apply Proposition 2.14 with $K = \mathrm{T}(\tilde{A})$ and with $\tilde{l}(n)$ and $\tilde{r}(n)$ in place of $l(n)$ and $r(n)$, getting a strictly increasing sequence, which we call $(\nu(n))_{n=0,1,2,\ldots}$, with $\nu(j) = j$ for $j = 0, 1$, an AI algebra B_0 (called A in Proposition 2.14) which is the direct limit of a unital system

$$C([0,1]) \otimes M_{r(\nu(0))} \xrightarrow{\alpha_{1,0}} C([0,1]) \otimes M_{r(\nu(1))} \xrightarrow{\alpha_{2,1}} C([0,1]) \otimes M_{r(\nu(2))} \xrightarrow{\alpha_{3,2}} \cdots,$$

with injective diagonal maps $\alpha_{n+1,\,n}$ given by

$$f \mapsto \operatorname{diag}(f \circ R_{n,1},\; f \circ R_{n,2},\; \ldots,\; f \circ R_{n,\,r(\nu_{n+1})/r(\nu_n)})$$

for continuous functions

$$R_{n,1}, R_{n,2}, \ldots, R_{n,\,r(\nu(n+1))/r(\nu(n))} : [0,1] \to [0,1],$$

and an isomorphism $\mathrm{T}(B_0) \to \mathrm{T}(\tilde{A})$.

Apply Lemma 3.5 with this choice of ν. Define the sequences $(d(n))_{n=0,1,2,\ldots}$ and $(k(n))_{n=0,1,2,\ldots}$ as in Lemma 3.5, and then make all the definitions in Constructions 1.1 and 1.6. (Some are also given in the statement of Lemma 3.5.) Then, as in the proof of Lemma 3.5, $X_n = \widetilde{X}_{\nu(n)}$. We make the following choices for the unspecified objects in these constructions. We choose points $x_n \in X_n$ and $y_n \in [0,1]$ for $n \in \mathbb{Z}_{\geq 0}$ such that the conditions in Constructions 1.6(16) and (17) are satisfied. (It is easy to see that this can be done.) Use these points in parts (b) and (d) of Construction 1.6(18). Take the maps

$$R_{n,1},\; R_{n,2},\; \ldots,\; R_{n,\,d(n+1)} : Y_{n+1} \to Y_n$$

in part (c) of Construction 1.6(18) to be those from the application of Proposition 2.14 above. For $j = 1, 2, \ldots, l(n+1)$, let $S_{n,j}^{(0)}|_{X_{n+1}} : X_{n+1} \to X_n$ be the maps in the system obtained from Lemma 3.5, and take $S_{n,j}^{(0)}|_{Y_{n+1}} = R_{n,j}$. The requirement $S_{n,j}^{(0)} = S_{n,j}$ for $j = 1, 2, \ldots, d(n+1)$ in Construction 1.6(19) is then satisfied, so that the condition in Construction 1.1(13) is also satisfied. Moreover, with these choices, the conditions in Construction 2.17(20) are satisfied.

By Lemma 3.5, the numbers κ, ω, and ω' from Constructions 1.1(3) and (4) satisfy

$$\kappa = \tilde{\kappa}, \quad \omega = \tilde{\omega}, \quad \text{and} \quad \omega' \leq \tilde{\omega}'.$$

Therefore $\kappa > \frac{1}{2}$, $\omega' < \omega < \frac{1}{2}$, and $2\kappa - 1 > 2\omega$, as required in Constructions 1.1(3), (4), and (5); moreover

$$\frac{1}{1-2\omega} < \frac{2\kappa - 1}{2\omega}. \tag{3-7}$$

The algebra C is simple by Lemma 1.7.

The algebras A and B of Lemma 2.18(1) are now $A = \tilde{A}$ and $B = B_0$, so $C^{(0)}$, as in Construction 1.1(11), is isomorphic to $\tilde{A} \oplus B_0$. The isomorphism $\mathrm{T}(B_0) \to \mathrm{T}(\tilde{A})$ gives an isomorphism $\zeta_0^{(0)} : \mathrm{Aff}(\mathrm{T}(A)) \to \mathrm{Aff}(\mathrm{T}(B))$. This provides an automorphism of $\mathrm{Aff}(\mathrm{T}(A)) \oplus \mathrm{Aff}(\mathrm{T}(B))$, given by

$$(f, g) \mapsto ((\zeta_0^{(0)})^{-1}(g),\; \zeta_0^{(0)}(f)).$$

Let $\zeta^{(0)}$ be the corresponding automorphism of $\mathrm{Aff}(\mathrm{T}(A \oplus B)) = \mathrm{Aff}(\mathrm{T}(C^{(0)}))$ gotten using Lemma 2.10. Clearly $\zeta^{(0)} \circ \zeta^{(0)}$ is the identity map on $\mathrm{Aff}(\mathrm{T}(C^{(0)}))$.

Adopt the notation of Construction 2.17: C and $C^{(0)}$ are as already described, D and $D^{(0)}$ are the AF algebras from Construction 2.17(21), $\mu : D \to C$ and $\mu^{(0)} : D^{(0)} \to C^{(0)}$ are the maps of Construction 2.17(22) (which are isomorphisms on K-theory by Lemma 2.18(4)), and $\theta \in \mathrm{Aut}(D)$ and $\theta^{(0)} \in \mathrm{Aut}(D^{(0)})$ are as in Construction 2.17(23).

Define $E = \varinjlim_n M_{r(m)}$, with respect to the maps $a \mapsto \mathrm{diag}(a, a, \ldots, a)$, with a repeated $l(n)$ times. The direct system defining $D^{(0)}$ is the direct sum of two copies of the direct system just defined, so

$$D^{(0)} \cong E \oplus E \quad \text{and} \quad \mathrm{Aff}(\mathrm{T}(D^{(0)})) \cong \mathrm{Aff}(\mathrm{T}(E \oplus E)).$$

Since E is a UHF algebra, we have $\mathrm{Aff}(\mathrm{T}(E)) \cong \mathbb{R}$ with the usual order and order unit 1. Using $\mathrm{id}_{\mathrm{Aff}(\mathrm{T}(E))}$ in place of $\zeta_0^{(0)}$ above, we get an automorphism of $\mathrm{Aff}(\mathrm{T}(D^{(0)}))$. But this automorphism is just $\widehat{\theta^{(0)}}$.

We claim that $\zeta^{(0)} \circ \widehat{\mu^{(0)}} = \widehat{\mu^{(0)}} \circ \widehat{\theta^{(0)}}$. To prove the claim, we work with

$$\mathrm{Aff}(\mathrm{T}(E)) \oplus \mathrm{Aff}(\mathrm{T}(E)) \quad \text{and} \quad \mathrm{Aff}(\mathrm{T}(A)) \oplus \mathrm{Aff}(\mathrm{T}(B))$$

in place of $\mathrm{Aff}(\mathrm{T}(D^{(0)}))$ and $\mathrm{Aff}(\mathrm{T}(C^{(0)}))$, but keep the same names for the maps.

Since $\mu^{(0)} : E \oplus E \to A \oplus B$ is the direct sum of unital maps from the first summand to A and the second summand to B, the map $\widehat{\mu^{(0)}}$ is similarly a direct sum of maps $\mathrm{Aff}(\mathrm{T}(E)) \to \mathrm{Aff}(\mathrm{T}(A))$ and $\mathrm{Aff}(\mathrm{T}(E)) \to \mathrm{Aff}(\mathrm{T}(B))$. Let e and f be the order units of $\mathrm{Aff}(\mathrm{T}(A))$ and $\mathrm{Aff}(\mathrm{T}(B))$. The unique positive order unit preserving maps $\mathrm{Aff}(\mathrm{T}(E)) \to \mathrm{Aff}(\mathrm{T}(A))$ and $\mathrm{Aff}(\mathrm{T}(E)) \to \mathrm{Aff}(\mathrm{T}(B))$ are $\alpha \mapsto \alpha e$ and $\beta \mapsto \beta f$ for $\alpha, \beta \in \mathbb{R}$. Therefore $\widehat{\mu^{(0)}}(\alpha, \beta) = (\alpha e, \beta f)$. Since $\zeta_0^{(0)}$ is order unit preserving, we have $\zeta_0^{(0)}(e) = f$, so

$$\zeta^{(0)}(\alpha e, \beta f) = (\beta e, \alpha f) = \widehat{\mu^{(0)}}(\beta, \alpha) = (\widehat{\mu^{(0)}} \circ \widehat{\theta^{(0)}})(\alpha, \beta).$$

The claim follows.

Using conditions (4) and (13) in Construction 1.1, Lemma 2.16, and Proposition 2.15, we get isomorphisms

$$\rho : \mathrm{Aff}(\mathrm{T}(D^{(0)})) \to \mathrm{Aff}(\mathrm{T}(D)) \quad \text{and} \quad \sigma : \mathrm{Aff}(\mathrm{T}(C^{(0)})) \to \mathrm{Aff}(\mathrm{T}(C))$$

such that $\widehat{\mu} \circ \rho = \sigma \circ \widehat{\mu^{(0)}}$. Define

$$\eta = \rho \circ \widehat{\theta^{(0)}} \circ \rho^{-1} \in \mathrm{Aut}(\mathrm{Aff}(\mathrm{T}(D))) \quad \text{and} \quad \zeta = \sigma \circ \zeta^{(0)} \circ \sigma^{-1} \in \mathrm{Aut}(\mathrm{Aff}(\mathrm{T}(C))).$$

A calculation now shows that the claim above implies

$$\zeta \circ \hat{\mu} = \hat{\mu} \circ \eta. \tag{3-8}$$

We also have $\zeta \circ \zeta = \mathrm{id}_{\mathrm{Aff}(\mathrm{T}(C))}$.

We want to apply Proposition 2.15 with D_n and $\varphi_{n,m}$ as in Construction 2.17(21), and $\varphi_{n,m}^{(0)}$ as there in place of $\varphi'_{n,m}$, so that D and $D^{(0)}$ are as already given, with $C_n = D_n$ for all $n \in \mathbb{Z}_{\geq 0}$ and $\psi_{n,m} = \varphi_{n,m}$ and $\psi'_{n,m} = \varphi'_{n,m}$ for all m and n, and with θ_n, $\theta_n^{(0)}$, θ, and $\theta^{(0)}$ from Construction 2.17(23) in place of μ_n, μ'_n, μ, and μ'. As before, this application is justified by conditions (4) and (13) in Construction 1.1, and Lemma 2.16. The outcome is an isomorphism $\rho' : \mathrm{Aff}(\mathrm{T}(D^{(0)})) \to \mathrm{Aff}(\mathrm{T}(D))$ such that

$$\hat{\theta} = \rho' \circ \widehat{\theta^{(0)}} \circ (\rho')^{-1}. \tag{3-9}$$

We claim that $\eta = \hat{\theta}$. The "right" way to do this is presumably to show that $\rho' = \rho$ above, but the following argument is easier to write. We have

$$\mathrm{Aff}(\mathrm{T}(D)) \cong \mathrm{Aff}(\mathrm{T}(D^{(0)})) \cong \mathbb{R}^2,$$

with order $(\alpha, \beta) \geq 0$ if and only if $\alpha \geq 0$ and $\beta \geq 0$ and order unit $(1, 1)$. Since the state space $S(\mathbb{R}^2)$ of $\mathbb{R}^2$ with this order unit space structure is an interval, and automorphisms of order unit spaces preserve the extreme points of the state space, there is only one possible action of a nontrivial automorphism of $\mathbb{R}^2$ on $S(\mathbb{R}^2)$. Theorem 2.4 implies that $\mathbb{R}^2 \cong \mathrm{Aff}(S(\mathbb{R}^2))$, so there is only one nontrivial automorphism of $\mathbb{R}^2$. Since $\widehat{\theta^{(0)}}$ is nontrivial, so is $\hat{\theta}$ by (3-9), and so is η by its definition. The claim follows.

The claim and (3-8) imply

$$\zeta \circ \hat{\mu} = \hat{\mu} \circ \hat{\theta}. \tag{3-10}$$

Passing to state spaces and applying Theorem 2.4, we get an affine homeomorphism $H : \mathrm{T}(C) \to \mathrm{T}(C)$ such that $\zeta(f) = f \circ H$ for all $f \in \mathrm{Aff}(\mathrm{T}(C))$, and moreover $H \circ H = \mathrm{id}_{\mathrm{T}(C)}$. By Lemma 2.18(4), the expression $\mu_* \circ \theta_* \circ (\mu_*)^{-1}$ is a well-defined automorphism of $K_*(C)$, of order 2. We claim that $F = (\mu_* \circ \theta_* \circ (\mu_*)^{-1},\ H)$ is an order two automorphism of $\mathrm{Ell}(C)$. We use the notation of Definition 3.1 for the Elliott invariant of a C^*-algebra; in particular, ρ_C and ρ_D are not related to the maps ρ and ρ' above. The only part needing work is the compatibility condition (3-1) in Definition 3.1, which amounts to showing that

$$\rho_C \circ \mu_* \circ \theta_* \circ (\mu_*)^{-1} = \zeta \circ \rho_C.$$

To see this, we calculate, using at the second and last steps the notation of Definition 2.5 and the fact that the morphisms of Elliott invariants defined by μ and θ satisfy (3-1) in Definition 3.1, and using (3-10) at the third step,

$$\begin{aligned} \zeta \circ \rho_C &= \zeta \circ \rho_C \circ \mu_* \circ (\mu_*)^{-1} = \zeta \circ \hat{\mu} \circ \rho_D \circ (\mu_*)^{-1} \\ &= \hat{\mu} \circ \hat{\theta} \circ \rho_D \circ (\mu_*)^{-1} = \rho_C \circ \mu_* \circ \theta_* \circ (\mu_*)^{-1}, \end{aligned}$$

as desired.

Thus, we have constructed an automorphism F of $\mathrm{Ell}(C)$ of order 2. It remains to show that F is not induced by any automorphism of C.

Using (3-10) on the last components, one easily sees that $F \circ \mu_* = \mu_* \circ \theta_*$. Let q and $q^\perp$ be as in Notation 1.13. In the construction of D as in Construction 2.17(21), set $e = \varphi_{\infty,0}((1, 0))$ and $e^\perp = 1 - e = \varphi_{\infty,0}((0, 1))$. Then $\theta(e) = e^\perp$, $\mu(e) = q$, and $\mu(e^\perp) = q^\perp$. Therefore $F([q]) = [q^\perp]$.

Suppose now that there exists an automorphism α such that $\alpha_* = F$. Then $[\alpha(q)] = [q^\perp]$. By Lemma 1.17, $\alpha(q)$ is unitarily equivalent to $q^\perp$. Let u be a unitary such that $u\alpha(q)u^* = q^\perp$. Thus, since $\alpha(qAq) = \alpha(q)A\alpha(q) = u^*q^\perp Aq^\perp u$, it follows that the qAq and $q^\perp Aq^\perp$ have the same radius of comparison. By (3-7), this contradicts Lemmas 1.15 and 1.16. □

Remark 3.6. One can easily check that, with C as in the proof of Theorem 3.2, there is a unique automorphism of $\mathrm{Ell}(C)$ whose component automorphism of the tracial state space is as in the proof. Therefore the conclusion can be slightly strengthened: there is an automorphism of $\mathrm{T}(C)$ which is compatible with an automorphism of $\mathrm{Ell}(C)$ but which is not induced by any automorphism of C.

Question 3.7. *Does there exist a compact metric space X and a minimal homeomorphism $h : X \to X$ such that the crossed product $C^*(\mathbb{Z}, X, h)$ has the same features as the example we construct here*?

Our construction provides an example of an automorphism of order 2 of the Elliott invariant which is not induced by any automorphism of the C*-algebra. The question of whether there exists an example of such an automorphism of the invariant which is induced by an automorphism of the algebra but not by one of order 2 is an older question by Blackadar, which we record below. For Kirchberg algebras in the UCT class, it is known that any order two automorphism of the Elliott invariant is induced by an order two automorphism of the C*-algebra [Benson et al. 2003]; also see [Katsura 2008] for a generalization to actions of many other finite groups. However, very little seems to be known in the stably finite case, even for classifiable C*-algebras (and in fact even for AF algebras).

Question 3.8 (Blackadar). *Does there exist a simple separable stably finite unital nuclear C^*-algebra C and an automorphism F of* $\mathrm{Ell}(C)$ *such that*:

(1) *$F \circ F$ is the identity morphism of* $\mathrm{Ell}(C)$.

(2) *There is an automorphism α of C such that $\alpha_* = F$.*

(3) *There is no α as in* (2) *which in addition satisfies $\alpha \circ \alpha = \mathrm{id}_C$.*

Can such an algebra be chosen to be AH and have stable rank 1?

Our method of proof suggests that, instead of being just a number, the radius of comparison should be taken to be a function from $V(A)$ to $[0, \infty]$. If one uses the generalization to nonunital algebras in [Blackadar et al. 2012, Section 3.3], one could presumably even get a function from $\mathrm{Cu}(A)$ to $[0, \infty]$.

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PARTIAL REGULARITY OF LERAY–HOPF WEAK SOLUTIONS TO THE INCOMPRESSIBLE NAVIER–STOKES EQUATIONS WITH HYPERDISSIPATION

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We show that if u is a Leray–Hopf weak solution to the incompressible Navier–Stokes equations with hyperdissipation $\alpha \in \left(1, \frac{5}{4}\right)$ then there exists a set $S \subset \mathbb{R}^3$ such that u remains bounded outside of S at each blow-up time, the Hausdorff dimension of S is bounded above by $5-4\alpha$ and its box-counting dimension is bounded by $\frac{1}{3}(-16\alpha^2+16\alpha+5)$. Our approach is inspired by the ideas of Katz and Pavlović (*Geom. Funct. Anal.* **12**:2 (2002), 355–379).

1. Introduction

We are concerned with the incompressible Navier–Stokes equations with hyperdissipation,

$$\begin{aligned} u_t + (-\Delta)^\alpha u + (u\cdot\nabla)u + \nabla p = 0 \quad \text{in } \mathbb{R}^3,\\ \operatorname{div} u = 0, \end{aligned} \tag{1-1}$$

where $\alpha \in \left(1, \frac{5}{4}\right)$. The equations are equipped with an initial condition $u(0)=u_0$, where u_0 is given. We note that the symbol $(-\Delta)^\alpha$ is defined as the pseudodifferential operator with the symbol $(2\pi)^{2\alpha}|\xi|^{2\alpha}$ in the Fourier space, which makes (1-1) a system of pseudodifferential equations.

It is well known that the hyperdissipative Navier–Stokes equations (1-1) are globally well-posed for $\alpha \geq \frac{5}{4}$, which was proved by Lions [1969]; see also [Tao 2009]. The question of well-posedness for $\alpha < \frac{5}{4}$, including the case $\alpha = 1$ of the classical Navier–Stokes equations, remains open.

The first partial regularity result for the hyperdissipative (1-1) model was given by Katz and Pavlović [2002], who proved that the Hausdorff dimension of the singular set in space at the first blow-up time of a local-in-time strong solution is bounded by $5-4\alpha$ for $\alpha \in \left(1, \frac{5}{4}\right)$. Recently Colombo et al. [2020] showed that if $\alpha \in \left(1, \frac{5}{4}\right]$, u is a suitable weak solution of (1-1) on $\mathbb{R}^3 \times (0,\infty)$ and

$$S' := \{(x,t) : u \text{ is unbounded in every neighbourhood of } (x,t)\}$$

denotes the singular set in space-time then $\mathcal{P}^{5-4\alpha}(S')=0$, where $\mathcal{P}^s$ denotes the s-dimensional parabolic Hausdorff measure. This is a stronger result than that of [Katz and Pavlović 2002] since it is concerned with the space-time singular set S' (rather than the singular set in space at the first blow-up), it is a statement about the Hausdorff measure of the singular set (rather than merely the Hausdorff dimension) and it includes the case $\alpha = \frac{5}{4}$ (in which case the statement, $\mathcal{P}^0(S')=0$, means that the singular set is in fact empty, and so (1-1) is globally well-posed). The main ingredient of the notion of a "suitable weak

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solution" in the approach of [Colombo et al. 2020] is a local energy inequality, which is a generalisation of the classical local energy inequality in the Navier–Stokes equations (i.e., when $\alpha = 1$) to the case $\alpha \in \left(1, \frac{5}{4}\right)$. The fractional Laplacian $(-\Delta)^\alpha$ is incorporated in the local energy inequality using a version of the extension operator introduced in [Caffarelli and Silvestre 2007]; see also [Yang 2013; Kwon and Ożański 2022; Colombo et al. 2020, Theorem 2.3]. Colombo et al. [2020] also showed a bound on the box-counting dimension of the singular set

$$d_B(S' \cap (\mathbb{R}^3 \times [t, \infty))) \le \tfrac{1}{3}(-8\alpha^2 - 2\alpha + 15) \tag{1-2}$$

for every $t > 0$. We note that this bound reduces to 0 at $\alpha = \frac{5}{4}$ and converges to $\frac{5}{3}$ as $\alpha \to 1^+$, which is the bound that one can deduce from the classical result of [Caffarelli et al. 1982]; see [Robinson and Sadowski 2007] or Lemma 2.3 in [Ożański 2019] for a proof. We note that this bound (for the Navier–Stokes equations) has recently been improved by [Wang and Yang 2019] $\left(\text{to the bound } d_B(S) \le \frac{7}{6}\right)$.

Here, we build on the work of [Katz and Pavlović 2002], as their ideas offer an entirely different viewpoint on the theory of partial regularity of the Navier–Stokes equations (or the Navier–Stokes equations with hyper- and hypodissipation), as compared to the early work of Scheffer [1976a; 1976b; 1977; 1978; 1980] and the celebrated result of [Caffarelli et al. 1982], as well as alternative approaches of [Vasseur 2007; Lin 1998; Ladyzhenskaya and Seregin 1999] and numerous extensions of the theory, such as [Colombo et al. 2020; Tang and Yu 2015; Kwon and Ożański 2022]. Instead it is concerned with the dynamics (in time) of energy packets that are localised both in the frequency space and the real space $\mathbb{R}^3$, and with studying how these packets move in space, as well as transfer the energy between the high and low frequencies. An important concept in this approach is the so-called *barrier* (see (3-23)), which, in a sense, quarantines a fixed region in space in a way that prevents too much energy flux entering the region. This property is essential in showing regularity at points outside of the singular set.

In order to state our results, we will say that u is a (global-in-time) *Leray–Hopf weak solution* of (1-1) if

(i) it satisfies the equations in a weak sense, namely

$$\int_0^t \int \left(-u\varphi_t + (-\Delta)^{\alpha/2} u \cdot (-\Delta)^{\alpha/2}\varphi + (u\cdot\nabla)u\cdot\varphi\right) = \int u_0 \cdot \varphi - \int u(t)\cdot\varphi(t)$$

holds for all $t > 0$ and all $\varphi \in C_0^\infty([0,\infty)\times\mathbb{R}^3;\mathbb{R}^3)$, with $\operatorname{div}\varphi(s) = 0$ for all $s \ge 0$ (where we wrote $\int \equiv \int_{\mathbb{R}^3}$ for brevity),

(ii) the strong energy inequality,

$$\tfrac{1}{2}\|u(t)\|^2 + \int_s^t \|(-\Delta)^{\alpha/2} u(\tau)\|^2 \,\mathrm{d}\tau \le \tfrac{1}{2}\|u(s)\|^2 \tag{1-3}$$

holds for almost every $s \ge 0$ (including $s = 0$) and every $t > s$. Here $\|\cdot\|$ denotes the $\|\cdot\|_{L^2(\mathbb{R})}$ norm.

We note that Leray–Hopf weak solutions admit *intervals of regularity*; namely for every Leray–Hopf weak solution there exists a family of pairwise disjoint intervals $(a_i, b_i) \subset (0,\infty)$ such that u coincides with some strong solution of (1-1) on each interval and

$$\mathcal{H}^{(5-4\alpha)/2\alpha}\left(\mathbb{R} \setminus \bigcup_i (a_i, b_i)\right) = 0; \tag{1-4}$$

see Theorem 2.6 and Lemma 4.1 in [Jiu and Wang 2014] for a proof. This is a generalisation of the corresponding statement in the case $\alpha = 1$ (i.e., in the case of the Navier–Stokes equations); see Section 6.4.3 in [Ożański and Pooley 2018] and Chapter 8 in [Robinson et al. 2016].

Given $u_0 \in L^2(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$ there exists at least one global-in-time Leray–Hopf weak solution (see Theorem 2.2 in [Colombo et al. 2020], for example). We denote by S the singular set in space of u at single blow-up times, namely

$$S := \bigcup_i S_i, \tag{1-5}$$

where

$$S_i := \left\{x \in \mathbb{R}^3 : u \text{ is unbounded in } U \times \left(\tfrac{1}{2}(a_i + b_i), b_i\right) \text{ for any neighbourhood } U \text{ of } x\right\}$$

denotes the singular set. In particular, if $x \notin S$ then $\limsup_{t\to b_i^-} \|u(t)\|_{L^\infty(U)} \le c_i$ for every i and $U \ni x$. The first of our main results is the following.

Theorem 1.1. *Let u be a Leray–Hopf weak solution of* (1-1) *with $\alpha \in \left(1, \frac{5}{4}\right)$ and an initial condition $u_0 \in H^1(\mathbb{R}^3)$, and let $\varepsilon > 0$. There exists $C > 0$ and a family of collections $\mathcal{B}_j$ of cubes $Q \subset \mathbb{R}^3$ of sidelength $2^{-j(1+\varepsilon)}$ such that*

$$\#\mathcal{B}_j \le C\, 2^{j(5-4\alpha+\varepsilon)}$$

for each $j \in \mathbb{N}$, and

$$S \subset \limsup_{j\to\infty} \bigcup_{Q\in\mathcal{B}_j} Q. \tag{1-6}$$

In particular, $d_H(S) \le 5 - 4\alpha$.

Here d_H stands for the Hausdorff dimension, and we recall that $\limsup_{j\to\infty} G_j := \bigcap_{k\ge 0} \bigcup_{j\ge k} G_j$ denotes the set of points belonging to infinitely many G_j's. It is well known (see Lemma 3.1 in [Katz and Pavlović 2002], for example) that (1-6) implies that $d_H(S) \le 5 - 4\alpha + \varepsilon$, from which the last claim of the theorem follows by sending $\varepsilon \to 0$.

We note that C might depend on ε, but it does not depend on the interval of regularity (a_i, b_i), which gives us a control of the structure of the singular sets S_i that is uniform across blow-ups in time of a Leray–Hopf weak solution. This is an improvement of the result of Katz and Pavlović [2002], who obtained such control for a given strong solution, and so for each interval of regularity (a_i, b_i) of a Leray–Hopf weak solution their result implies existence of $C_i > 0$ such that $S_i \subset \limsup_{j\to\infty} \bigcup_{Q\in\mathcal{B}_j^{(i)}} Q$ for some collections $\mathcal{B}_j^{(i)}$ of cubes of sidelength $2^{-j(1+\varepsilon)}$ satisfying $\mathcal{B}_j^{(i)} \le C_i\, 2^{j(5-4\alpha+\varepsilon)}$ for all j. One could therefore expect that the constants C_i become unbounded as i varies (for example in a scenario of a limit point of the set of blow-up times $\{b_i\}$), and Theorem 1.1 shows that it does not happen.

We note, however, that Theorem 1.1 does not estimate the dimension of the singular set at the blow-up time which is not an endpoint b_i of an interval of regularity (but instead a limit of a sequence of such b_i's). In other words, if $x \notin S$, $U \ni x$ is a small open neighbourhood of x and $\{(a_i, b_i)\}_i$ is a collection of consecutive intervals of regularity of u, we show that $\sup_{U\times((a_i+b_i)/2, b_i)} |u| = c_i < \infty$, but our result does not exclude the possibility that $c_i \to \infty$ as $i \to \infty$. It also does not imply boundedness of $|u(t)|$ at times t near the left endpoint a_i of any interval of regularity (a_i, b_i). These issues are related to the fact that inside the barrier we still have to deal with infinitely many energy packets (i.e., infinitely many frequencies and cubes in $\mathbb{R}^3$). Thus, supposing that the estimate on the energy packets inside the barrier breaks down at some t, we are unable to localize the packet (i.e., the frequency and the cube) on which the growth occurs near t, unless t is located inside an interval of regularity; see Step 1 of the proof of Theorem 3.7 for details.

The proof of Theorem 1.1 is inspired by the strategy of the proof of [Katz and Pavlović 2002], which we extend to the case of Leray–Hopf weak solutions and we use a more robust main estimate. The main estimate controls the time derivative of the L^2 norm of the Littlewood–Paley projection $P_j u$ combined with a cut-off in space (the *energy packet*); see (3-2). We show that such norm is continuous in time (regardless of putative singularities of a Leray–Hopf weak solution), which makes the main estimate valid for all $t > 0$. Inspired by [Katz and Pavlović 2002], we then define *bad cubes* and *good cubes* (see (3-15)) and show that we have a certain more-than-critical decay on a cube that is good and has some good ancestors. We then construct $\mathcal{B}_j$ as a certain cover of bad cubes and prove (1-6).

Our second main result is concerned with the box-counting dimension. We let

$$S^{(k)} := \bigcup_{i \le k} S_i. \tag{1-7}$$

Theorem 1.2. *Let u be as in Theorem 1.1. Then $d_B(S^{(k)}) \le \frac{1}{3}(-16\alpha^2 + 16\alpha + 5)$ for every $k \in \mathbb{N}$.*

We prove the theorem by sharpening the argument outlined below Theorem 1.1. We recall that the box-counting dimension d_B is concerned with covering the given set by a collection of balls of radius r,

$$d_B(K) := \limsup_{r \to 0} \frac{\log N(K, r)}{-\log r}, \tag{1-8}$$

where $N(K, r)$ denotes the minimal number of balls (or boxes) of radius r required to cover K. In this context, one can actually use the families $\mathcal{B}_j$ from (1-6) to deduce that $d_B(S^{(k)}) \le \frac{1}{9}(-64\alpha^3 + 96\alpha^2 - 48\alpha + 35)$ for every k, which we discuss in detail in Section 4. This is however a worse estimate than claimed in Theorem 1.2.

In fact, in Section 4 we improve this estimate by constructing refined families $\mathcal{C}_j$ that, in a sense, give a more robust control of the low modes, which reduces the number of cubes required to cover the singular set and hence improve the bound on $d_B(S^{(k)})$. See the informal discussion following Proposition 4.1 for more insight about this improvement.

We note that we can only estimate $d_B(S^{(k)})$ (rather than $d_B(S)$) because of the localisation issue described above. To be more precise, for each sufficiently small $\delta > 0$ we can construct a family of cubes of sidelength $\delta > 0$ that covers the singular set when t approaches a singular time, and that has cardinality less than or equal to $\delta^{(-16\alpha^2+16\alpha+5)/3+\varepsilon}$ for any given $\varepsilon > 0$. This family can be constructed independently of the interval of regularity, but given x outside of this family we can show that the solution is bounded in a neighbourhood of x if the choice of (sufficiently small) δ is dependent on the interval of regularity. This gives the limitation to only finite number of intervals of regularity in the definition of $S^{(k)}$.

We note that the result of [Colombo et al. 2020] is stronger than our result in the sense that it is concerned with the space-time singular set S' (rather than the singular set S in space), it is concerned with the parabolic Hausdorff measure of S' (rather than merely the bound on $d_H(S')$) and its estimate of $d_B(S')$ is sharper than our estimate on $d_B(S_k)$.

However, our result is stronger than [Colombo et al. 2020] in the sense that it applies to any Leray–Hopf weak solutions (rather than merely suitable weak solutions). In other words we do not use the local energy inequality, which is the main ingredient of [Colombo et al. 2020]. Also, our approach does not include any estimates of the pressure function. In fact we only consider the Leray projection of the first equation in (1-1), which eliminates the pressure. Furthermore, our approach can be thought of as an extension of the global regularity of (1-1) for $\alpha > \frac{5}{4}$. In fact, the following corollary can be proved almost immediately using our main estimate; see Section 3F.

Corollary 1.3. *If* $\alpha > \frac{5}{4}$ *then* (1-1) *is globally well-posed.*

We also point out that our estimate on the box-counting dimension, $d_B(S_k) \leq \frac{1}{3}(-16\alpha^2 + 16\alpha + 5)$, converges to $\frac{5}{3}$ as $\alpha \to 1^+$, just as (1-2).

Finally, we also correct a number of imprecisions appearing in [Katz and Pavlović 2002]; see for example Remark 3.4 and Step 1 of the proof of Theorem 3.7.

The structure of the article is as follows. In Section 2 we introduce some preliminary concepts including the Littlewood–Paley projections, paraproduct decomposition, and Bernstein inequalities, as well as a number of analytic tools that allow us to manipulate quantities involving cut-offs in both the real space and the Fourier space, which includes estimates of the errors when one moves a Littlewood–Paley projection across spatial cut-offs and vice versa. We prove the first result, Theorem 1.1, in Section 3. We prove Corollary 1.3 in Section 3F and we prove the second result, Theorem 1.2, in Section 4.

2. Preliminaries

Unless specified otherwise, all function spaces are considered on the whole space $\mathbb{R}^3$. In particular $L^2 := L^2(\mathbb{R}^3)$. We do not use the summation convention. We will write $\partial_i := \partial_{x_i}$, $B(R) := \{x \in \mathbb{R}^3 : |x| \leq R\}$, $\int := \int_{\mathbb{R}^3}$, and $\|\cdot\|_p := \|\cdot\|_{L^p(\mathbb{R}^3)}$. We reserve the notation "$\|\cdot\|$" for the L^2 norm, that is, $\|\cdot\| := \|\cdot\|_2$.

We denote any positive constant by c (whose value may change at each appearance). We point out that c might depend on u_0 and α, which we consider fixed throughout the article. As for the constants dependent on some parameters, we sometimes emphasise the parameters by using subscripts. For example, $c_{k,q}$ is any constant dependent on k and q.

We denote by $e(j)$ (a *j-negligible error*) any quantity that can be bounded (in absolute value) by $c_K 2^{-Kj}$ for any given $K > 0$.

We say that a differential inequality $f' \leq g$ on a time interval I is satisfied in the *integral sense* if

$$f(t) \leq f(s) + \int_s^t g(\tau)\,\mathrm{d}\tau \quad \text{for every } t, s \in I \text{ with } t > s. \tag{2-1}$$

We recall that Leray–Hopf weak solutions are weakly continuous with values in L^2. Indeed, it follows from part (i) of the definition that

$$\int u(t)\varphi \;\text{ is continuous for every } \varphi \in C_0^\infty(\mathbb{R}^3) \text{ with } \operatorname{div}\varphi = 0.$$

This is also true if $\operatorname{div}\varphi \neq 0$, as in this case one can apply the Helmholtz decomposition to write $\varphi = \phi + \nabla\psi$, where $\operatorname{div}\phi = 0$ (then $\int u(t)\phi$ is continuous and $\int u(t)\nabla\psi = 0$ since $u(t)$ is divergence-free). Thus, since part (ii) gives that $\{u(t)\}_{t\geq 0}$ is bounded in L^2, weak continuity of $u(t)$ follows.

2A. ***Littlewood–Paley projections.*** Given $f \in L^1(\mathbb{R}^3)$, we denote by $\hat{f}$ its Fourier transform, i.e.,

$$\hat{f}(\xi) := \int f(x) \mathrm{e}^{-2\pi i x\cdot\xi}\,\mathrm{d}x, \quad \xi \in \mathbb{R}^3,$$

and by $\check{f}$ its inverse Fourier transform, i.e., $\check{f}(x) := \hat{f}(-x)$. Let $h \in C^\infty(\mathbb{R}; [0, 1])$ be any function such that $h(x) = 1$ for $x < 1$ and $h(x) = 0$ for $x > 2$. We set $p(x) := h(|x|) - h(2|x|)$, where $x \in \mathbb{R}^3$, we let

$$p_j(\xi) := p(2^{-j}\xi) \quad \text{for } j \in \mathbb{Z}, \tag{2-2}$$

and we let P_j (the j-th Littlewood–Paley projection) be the corresponding multiplier operator, that is,

$$\widehat{P_j f}(\xi) := p_j(\xi)\hat{f}(\xi).$$

By construction, $\operatorname{supp} p_j \subset B(2^{j+1}) \setminus B(2^{j-1})$. We note that $\sum_{j\in\mathbb{Z}} p_j = 1$, and so formally $\sum_{j\in\mathbb{Z}} P_j = \mathrm{id}$. We also define

$$\widetilde{P}_j \equiv P_{j\pm 2} := \sum_{k=j-2}^{j+2} P_k, \quad P_{j-4,j+2} := \sum_{k=j-4}^{j+2} P_k, \quad P_{\leq j} := \sum_{k=-\infty}^{j} P_k, \quad P_{\geq j} := \sum_{k=j}^{\infty} P_k, \tag{2-3}$$

and analogously for $\tilde{p}_j,\ p_{j-4,j+2},\ p_{\leq j},\ p_{\geq j}$. By a direct calculation one obtains that

$$\check{p}_j(y) = 2^{3j}\check{p}(2^j y) \tag{2-4}$$

for all $j \in \mathbb{Z},\ y \in \mathbb{R}^3$. In particular $\|\check{p}_j\|_1 = c$ and so, since $P_j f = \check{p}_j * f$ (where "$*$" denotes the convolution), Young's inequality for convolutions gives

$$\|P_j u\|_q \leq c\|u\|_q \tag{2-5}$$

for any $q \in [1, \infty]$. Moreover, given $K > 0$ there exists $c_K > 0$ such that

$$|\check{p}_j(y)| \leq c_K (2^j|y|)^{-2K} 2^{3j}, \tag{2-6}$$

$$|\partial_i \check{p}_j(y)| \leq c_K (2^j|y|)^{-2K} 2^{4j} \tag{2-7}$$

for all $j \in \mathbb{Z},\ y \neq 0$ and $i = 1, 2, 3$. Indeed, the case $j = 0$ follows by noting that

$$\mathrm{e}^{2\pi i y\cdot\xi} = (-4\pi^2|y|^2)^{-K}\Delta_\xi^K \mathrm{e}^{2\pi i y\cdot\xi}$$

and calculating

$$|\check{p}(y)| = \left|\int p(\xi)\mathrm{e}^{2\pi i y\cdot\xi}\,\mathrm{d}\xi\right| = (4\pi^2|y|^2)^{-K}\left|\int \Delta^K p(\xi)\mathrm{e}^{2\pi i y\cdot\xi}\,\mathrm{d}\xi\right| \leq c_K|y|^{-2K}\int_{B(2)}|\Delta^K p| = c_K|y|^{-2K}$$

(and similarly $|\partial_i\check{p}(y)| \leq c_K|y|^{-2K}$), where we have integrated by parts $2K$ times, and the case $j \neq 0$ follows from (2-4). Using (2-6) and (2-7) we also get

$$\|\check{p}_j\|_{L^q(B(d)^c)} \leq C_{K,q}(d2^j)^{-2K+3/q}\,2^{3j(q-1)/q} \tag{2-8}$$

and

$$\|\partial_i \check{p}_j\|_{L^q(B(d)^c)} \le C_{K,q}(d2^j)^{-2K+3/q}\, 2^{j(1+n(q-1)/q)}, \tag{2-9}$$

respectively, for any $K>0,\ d>0,\ i=1,2,3,\ j\in\mathbb{Z}$ and $q\ge 1$. Indeed

$$\int_{\mathbb{R}^3\setminus B(d)} |\check{p}_j(y)|^q \,\mathrm{d}y \le C_{K,q} 2^{-jq(2K-3)} \int_{|y|\ge d} |y|^{-2Kq}\,\mathrm{d}y = C_{K,q}2^{-jq(2K-3)} d^{-2Kq+3},$$

from which (2-8) follows (and (2-9) follows analogously). We note that the same is true when p is replaced by any compactly supported multiplier.

Corollary 2.1. *Let $\lambda\in C_0^\infty(\mathbb{R}^3)$ and, given $j\in\mathbb{Z}$, set $\lambda_j(\xi):=\lambda(2^{-j}\xi)$. Then, given $d>0$,*

$$\|\check{\lambda}_j\|_{L^2(\mathbb{R}^3\setminus B(d))} \le c_K 2^{-j(2K-3)} d^{-2K+3/2}.$$

We will denote by T the Leray projection, that is,

$$\widehat{Tf}(\xi) := \left(I - \frac{\xi\otimes\xi}{|\xi|^2}\right)\hat{f}, \tag{2-10}$$

where $f:\mathbb{R}^3\to\mathbb{R}^3$, and I denotes the 3×3 identity matrix.

2B. *Bernstein inequalities.* Here we point out classical Bernstein inequalities on $\mathbb{R}^3$:

$$\|P_j f\|_q \le c\, 2^{3j(1/p-1/q)} \|P_j f\|_p, \tag{2-11}$$

$$\|P_{\le j} f\|_q \le c\, 2^{3j(1/p-1/q)} \|P_{\le j} f\|_p \tag{2-12}$$

for any $1\le p\le q\le\infty$. We refer the reader to Lemma 2.1 of [Bahouri et al. 2011] for a proof.

2C. *The paraproduct formula.* Here we briefly describe the Bony decomposition formula, that is, we concern ourselves with a structure of a Littlewood–Paley projection of a product of two functions, $P_j(fg)$. One could obviously write $f=\sum_{k\in\mathbb{Z}} P_k f$ (and similarly for g) to obtain that

$$P_j(fg) = P_j\left(\sum_{k,m\in\mathbb{Z}} P_k f\, P_m g\right). \tag{2-13}$$

However, since functions p_j, p_k have pairwise disjoint supports for many pairs $j,k\in\mathbb{Z}$, one could speculate that some of the terms on the right-hand side of (2-13) vanish. This is indeed the case and

$$\begin{aligned} P_j(fg) &= P_j\left(P_{j\pm2} f\, P_{\le j-5} g + P_{\le j-5} f\, P_{j\pm 2} g + P_{j-4,j+2} f\, P_{j\pm4} g + \sum_{k\ge j+3} P_k f\, P_{k\pm 2} g\right) \\ &= P_j(K_{\mathrm{loc,low}} + K_{\mathrm{low,loc}} + K_{\mathrm{loc}} + K_{\mathrm{hh}}), \end{aligned} \tag{2-14}$$

which is also known as *Bony's decomposition formula*. For the sake of completeness we prove the formula below. Heuristically speaking, $K_{\mathrm{loc,low}}$ corresponds to interactions between local (i.e., around j) modes of f and low modes of g, $K_{\mathrm{low,loc}}$ to interactions between low modes of f and local modes of g, K_{loc} to local interactions and K_{hh} to interactions between high modes; see Figure 1 for a geometric interpretation of (2-14). We now prove (2-14). For this it is sufficient to show that

$$P_j(P_k f\, P_m g) = 0 \quad \text{for } (k,m)\in R_1\cup R_2\cup R_3, \tag{2-15}$$

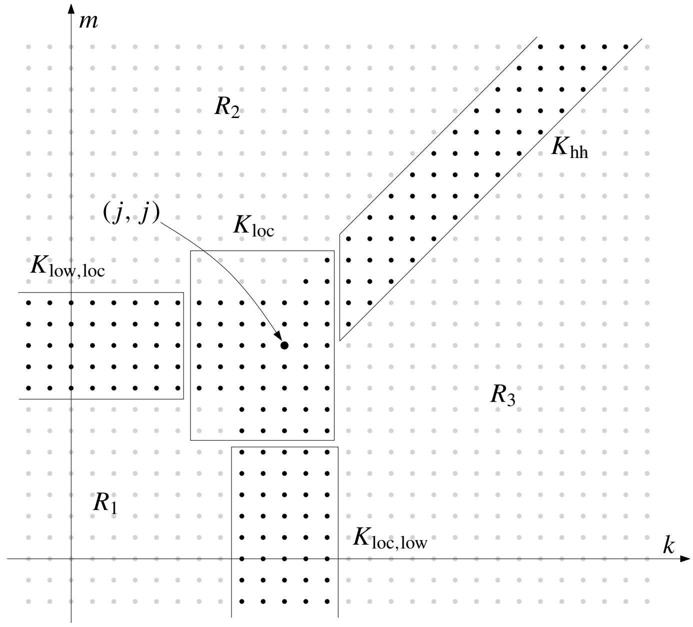

Figure 1. Sketch of the interpretation of the terms on the right-hand side of (2-14). The regions R_1, R_2, R_3 (consisting of grey dots) correspond to pairs (k, m) for which $P_j(P_k f\ P_m g)$ vanishes; see the discussion following (2-15).

where R_1, R_2, R_3 are as sketched in Figure 1. The Fourier transform of $w := P_j(P_k f\ P_m g)$ is

$$\hat{w}(\xi) = p_j(\xi) \int p_k(\eta) \hat{f}(\eta) p_m(\xi - \eta) \hat{g}(\xi - \eta)\, \mathrm{d}\eta.$$

We can assume that $|\xi| \in (2^{j-1}, 2^{j+1})$ (as otherwise $p_j(\xi)$ vanishes) and that $|\eta| \in (2^{k-1}, 2^{k+1})$ (as otherwise $p_k(\eta)$ vanishes).

Case 1: $(k, m) \in R_1$. Suppose that $k \geq m$ (the opposite case is analogous). Then $j \geq k+3$ (see Figure 1) and so

$$|\xi - \eta| \geq |\xi| - |\eta| \geq 2^{j-1} - 2^{k+1} \geq 2^{k+2} - 2^{k+1} = 2^{k+1} \geq 2^{m+1}.$$

Thus $p_m(\xi - \eta)$ vanishes.

Case 2: $(k, m) \in R_2 \cup R_3$. Suppose that $(k, m) \in R_2$ (the case $(k, m) \in R_3$ is analogous). Then $m \geq k+3$ and $m \geq j+3$ (see Figure 1) and so

$$|\xi - \eta| \leq |\xi| + |\eta| \leq 2^{j+1} + 2^{k+1} \leq 2 \cdot 2^{m-2} = 2^{m-1}.$$

Hence $p_m(\xi - \eta)$ vanishes as well, and so (2-15) follows.

2D. *Moving bump functions across Littlewood–Paley projections.* Here we show the following:

Lemma 2.2. *Let $\phi_1, \phi_2 : \mathbb{R}^3 \to [0, 1]$ be such that their supports are separated by at least $d > 2^{-j}$. Then*

$$\|\phi_1 P_j(\phi_2 f)\|_q \leq c_K (d 2^j)^{-2K+3} \|f\|_q$$

for all $q \in [1, \infty]$, $j \in \mathbb{Z}$, $K > 0$ *and* $f \in L^q(\mathbb{R}^3)$. *Furthermore, if* $|\nabla \phi_2| \le c\, d^{-1}$ *then*

$$\|\phi_1 P_j(\phi_2 \nabla f)\|_q \le c_K (d2^j)^{-2K+3} 2^j \|f\|_q.$$

We will only use the lemma (and the corollary below) with $q = 2$ or $q = 1$.

Proof. We note that

$$\begin{aligned}\phi_1 P_j(\phi_2 f)(x) &= \phi_1(x) \int_{\operatorname{supp}\phi_2} \check{p}_j(x-y)\phi_2(y) f(y)\, \mathrm{d}y \\ &= \phi_1(x) \int_{\operatorname{supp}\phi_2} \chi_{|x-y|>d}\, \check{p}_j(x-y)\phi_2(y) f(y)\, \mathrm{d}y \end{aligned} \tag{2-16}$$

since the supports of ϕ_1, ϕ_2 are at least d apart. Thus using Young's inequality for convolutions

$$\|\phi_1 P_j(\phi_2 f)\|_q \le \|\check{p}_j\|_{L^1(B(d)^c)} \|\phi_2 f\|_q \le c_K (d2^j)^{-2K+3} \|f\|_q$$

for any $K > 0$, where we used (2-8). This shows the first claim of the lemma. The second claim follows by replacing f by ∇f in (2-16), integrating by parts, and using Young's inequality for convolutions to give

$$\begin{aligned}\|\phi_1 P_j(\phi_2 \nabla f)\|_q &\le c \|\nabla \check{p}_j\|_{L^1(B(d)^c)} \|\phi_2 f\|_q + \|\check{p}_j\|_{L^1(B(d)^c)} \|\nabla \phi_2 f\|_q \\ &\le c_K (d2^j)^{-2K+3} 2^j \|f\|_q,\end{aligned}$$

where we also used the assumption that $|\nabla \phi_2| \le c\, d^{-1} < c\, 2^j$. $\square$

In fact the same result is valid when P_j is replaced by the composition of P_j with any 0-homogeneous multiplier (e.g., the Leray projector).

Corollary 2.3. *Let M be a bounded, 0-homogeneous multiplier (i.e., $\widehat{Mf}(\xi) = m(\xi)\hat{f}(\xi)$, where $m(\lambda\xi) = m(\xi)$ for any $\lambda > 0$). Let $\phi_1, \phi_2 : \mathbb{R}^3 \to [0, 1]$ be such that their supports are separated by at least $d > 2^{-j}$. Then*

$$\|\phi_1 M P_j(\phi_2 \nabla f)\|_q \le c_K (d2^j)^{-2K+3} 2^j \|f\|_q$$

for all $q \in [1, \infty]$, $j \in \mathbb{Z}$, $K > 0$ *and* $f \in L^q(\mathbb{R}^3)$.

2E. ***Moving Littlewood–Paley projections across spatial cut-offs.*** We say that $\phi \in C_0^\infty(\mathbb{R}^3)$ is a *d-cutoff* if $\operatorname{diam}(\operatorname{supp}\phi) \le c\, d$ and $|D^l \phi| \le c_l d^{-l}$ for any $l \ge 0$.

We denote by $e_d(j)$ any quantity that can be bounded (in absolute value) by $c_K 2^{cj} (d2^j)^{c-K}$ for any given $K > 0$. The point of such notation is that it will articulate the dependence of the size of the error in our main estimate (see Proposition 3.1) on both j and d.

In this section we show that, roughly speaking, we can move Littlewood–Paley projections P_j across d-cutoffs as long as $d > 2^{-j}$.

Lemma 2.4. *Given a d-cutoff ϕ, $q \in [1, \infty]$ and multiindices α, β, with $|\beta|, |\alpha| \le 3$,*

$$\|(1 - \widetilde{P}_j) D^\alpha (\phi P_j D^\beta f)\|_q \le e_d(j) \|f\|_q$$

for every j.

Proof. We write $\phi = \phi_1 + \phi_2$, where

$$\hat{\phi}_1(\xi) := \chi_{|\xi|\le 2^{j-2}}\hat{\phi}(\xi),$$
$$\hat{\phi}_2(\xi) := \chi_{|\xi|> 2^{j-2}}\hat{\phi}(\xi).$$

Note that

$$\widehat{(\phi_1 P_j D^\beta f)}(\xi) = \int \hat{\phi}_1(\xi-\eta)p_j(\eta)(2\pi i)^{|\beta|}\eta^\beta \hat{f}(\eta)\,\mathrm{d}\eta$$

is supported in $|\xi| \in (2^{j-2}, 2^{j+2})$ (as $\hat{\phi}_1(\xi-\eta)$ is supported in $\{|\xi-\eta| \le 2^{j-2}\}$ and $p_j(\eta)$ is supported in $\{2^{j-1} < |\eta| < 2^{j+1}\}$). Since $\tilde{p}_j(\xi) = 1$ for such ξ, we obtain

$$\phi_1 P_j D^\beta f = \widetilde{P}_j \phi_1 P_j D^\beta f, \tag{2-17}$$

and so it suffices to show that

$$\|(1-\widetilde{P}_j)D^\alpha(\phi_2 P_j D^\beta f)\|_q \le e_d(j)\|f\|_q.$$

We will show that

$$\|\widehat{D^\alpha \phi_2}\|_1 \le e_d(j) \tag{2-18}$$

for every $|\alpha| \le 3$. Then the claim follows by writing

$$\begin{aligned}\|(1-\widetilde{P}_j)D^\alpha(\phi_2 P_j D^\beta f)\|_q &\le \sum_{\alpha_1+\alpha_2=\alpha} \|D^{\alpha_1}\phi_2 P_j D^{\alpha_2+\beta} f\|_q \\ &\le \sum_{\alpha_1+\alpha_2=\alpha} \|D^{\alpha_1}\phi_2\|_\infty \|P_j D^{\alpha_2+\beta} f\|_q \\ &\le \sum_{|\alpha_1|\le 3} \|\widehat{D^{\alpha_1}\phi_2}\|_1 \cdot 2^{6j}\|f\|_q \le e_d(j)\|f\|_q.\end{aligned}$$

In order to see (2-18) we first note that

$$\begin{aligned}|\widehat{D^\alpha\phi_2}(\xi)| &\le c|\xi|^{|\alpha|}\left|\int \phi_2(x)\mathrm{e}^{-2\pi \mathrm{i} x\cdot\xi}\,\mathrm{d}x\right| \\ &= c|\xi|^{|\alpha|}(4\pi^2|\xi|^2)^{-K}\left|\int \phi_2(x)\Delta^K \mathrm{e}^{-2\pi \mathrm{i} x\cdot\xi}\,\mathrm{d}x\right| \\ &= c|\xi|^{|\alpha|}(4\pi^2|\xi|^2)^{-K}\left|\int \Delta^K\phi_2(x)\mathrm{e}^{-2\pi \mathrm{i} x\cdot\xi}\,\mathrm{d}x\right| \le c_K|\xi|^{-2K+|\alpha|}d^{-2K+3}.\end{aligned}$$

Thus

$$\|\widehat{D^\alpha\phi_2}\|_1 = c\int_{|\xi|>2^{j-2}} |\widehat{D^\alpha\phi_2}(\xi)| \le c_K d^{-2K+3}\int_{|\xi|>2^{j-2}} |\xi|^{-2K+|\alpha|} = c_K 2^{3j}(d2^j)^{-2K+3},$$

which gives (2-18). □

Similarly one can put the Littlewood–Paley projection "inside the cutoff". In this case one can prove a statement similar to Lemma 2.4, but, since we will only need a version with no derivatives, we state a simplified statement.

Corollary 2.5. *Given a d-cutoff ϕ, $\|P_j(\phi(1-P_{j\pm 2})f)\| \le e_d(j)$ for every j.*

Proof. The claim follows using the same decomposition as above, $\phi = \phi_1 + \phi_2$. Since

$$\widehat{\phi(1-P_{j\pm 2})f}(\xi) = \int \hat{\phi}_1(\xi-\eta)(1-p_{j\pm 2}(\eta))\hat{f}(\eta)\,\mathrm{d}\eta,$$

we see that (since $|\eta| \in (-\infty, 2^{j-2}) \cup (2^{j+2}, \infty)$) either $|\xi| \geq |\eta| - |\xi-\eta| \geq 2^{j+2} - 2^{j-2} \geq 2^{j+1}$ or $|\xi| \leq |\eta| + |\xi - \eta| \leq 2^{j-2} + 2^{j-2} = 2^{j-1}$. In any case $p_j(\xi) = 0$ and so $P_j(\phi_1(1-\widetilde{\widetilde{P}}_j)f) = 0$. The part involving ϕ_2 can be estimated by $e_d(j)$ using the same argument as above. □

2F. *Cubes.* We denote by Q any open cube in $\mathbb{R}^3$. Given $a > 1$, we denote by aQ the cube with the same centre as Q and a times larger sidelength. We sometimes write $Q(x)$ to emphasise that cube Q is centred at a point $x \in \mathbb{R}^3$. Given an open cube Q of sidelength $d > 0$, we let $\phi_Q \in C_0^\infty(\mathbb{R}^3; [0,1])$ be a d-cutoff such that

$$\phi_Q = 1 \text{ on } Q, \quad \operatorname{supp}\phi_Q \subset \tfrac{7}{6}Q, \quad \text{and} \quad \|\nabla^k \phi_Q\|_\infty \leq C_k d^{-k}. \tag{2-19}$$

Note that

$$|\xi|^k |\hat{\phi}_Q(\xi)| \leq c_k d^{3-k} \quad \text{for } \xi \in \mathbb{R}^3, \tag{2-20}$$

which can be shown by a direct computation.

2G. *Localised Bernstein inequalities.* If Q is a cube of sidelength $d > 2^{-j}$ then

$$\|\phi_Q P_j f\|_q \leq c\, 2^{3j(1/2-1/q)} \|\phi_Q P_j f\| + e_d(j)\|f\|_q, \tag{2-21}$$

due to Lemma 2.4 and the classical Bernstein inequality (2-11).

2H. *Absolute continuity.* Here we state two lemmas that will help us (in Step 1 of the proof of Proposition 3.1) in proving the main estimate for Leray–Hopf weak solutions.

Lemma 2.6. *Suppose that $f : [a,b] \to \mathbb{R}$ is continuous and such that $f' \in L^1(a,b)$. Then*

$$f(t) = f(s) + \int_s^t f'(\tau)\,\mathrm{d}\tau$$

for every $s, t \in (a,b)$.

Proof. This is elementary. □

Lemma 2.7. *If $u(x,t)$ is weakly continuous in time on an interval (a,b) with values in $L^2(\mathbb{R}^3)$ then $P_j u$ is strongly continuous in time into $L^2(\Omega)$ on (a,b) for any bounded domain $\Omega \subset \mathbb{R}^3$.*

Proof. We note that

$$\|P_j u(t) - P_j u(s)\|^2_{L^2(\Omega)} = \int_\Omega \left| \int \check{p}_j(x-y)(u(y,t) - u(y,s))\,\mathrm{d}y \right|^2 \mathrm{d}x.$$

Weak continuity of $u(t)$ gives that the integral inside the absolute value converges to 0 as $t \to s$ (for any fixed x). Furthermore it is bounded by

$$\|\check{p}_j\|\,\|u(t) - u(s)\| \leq c_j,$$

where we used the Cauchy–Schwarz inequality and the fact that u is bounded in L^2 (a property of functions weakly continuous in L^2). Since the constant function c_j^2 is integrable on Ω, the claim of the lemma follows from the dominated convergence theorem. □

3. The proof of the main result

In this section we prove Theorem 1.1; namely we will show that $d_H(S) \le 5-4\alpha$, where S is the singular set in space of a Leray–Hopf weak solution (recall (1-5)). We will actually show that

$$d_H(S) \le 5-4\alpha+\varepsilon$$

for any

$$\varepsilon \in \left(0, \min\left(\tfrac{1}{3}(4\alpha-4), \tfrac{1}{20}\right)\right). \tag{3-1}$$

We now fix such ε and we allow every constant (denoted by "c") to depend on ε.

We say that a cube Q is a j-cube if it has sidelength $2^{-j(1-\varepsilon)}$. The reason for considering such "almost dyadic cubes" (rather than the dyadic cubes of sidelength 2^{-j}) is that $e_d(j)=e(j)$ for $d=2^{-j(1-\varepsilon)}$ (which is not true for $d=2^{-j}$). We say that a cover of a set is a j-*cover* if it consists only of j-cubes. We denote by $S_j(\Omega)$ any j-cover of Ω such that $\#S_j(\Omega) \le c\left(\tfrac{1}{2}\operatorname{diam}(\Omega)^{-j(1-\varepsilon)}\right)^3$.

Moreover, given a j-cube and $k\in\mathbb{Z}$, we denote the k-cube concentric with Q by Q_k, that is,

$$Q_k := 2^{(j-k)(1-\varepsilon)}Q.$$

3A. ***The main estimate.*** Given a cube Q and $j\in\mathbb{Z}$ we let

$$u_{Q,j} := \|\phi_Q P_j u\|$$

and we write

$$u_{Q,j\pm2} := \sum_{k=j-2}^{j+2} u_{Q,k}.$$

We point out that $u_{Q,j}$ is a function of time, which we will often skip in our notation.

We start with a derivation of an estimate for $u_{Q,j}$ for any $j\in\mathbb{Z}$ and any cube Q of sidelength $d>16\cdot 2^{-j}$.

Proposition 3.1 (main estimate). *Let u be a Leray–Hopf weak solution of the hyperdissipative Navier–Stokes equations* (1-1) *on the time interval* $[0,\infty)$ *and let $d>16\cdot 2^{-j}$. Then $u_{Q,j}$ is continuous on* $[0,\infty)$ *and*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}u_{Q,j}^2 &\le -c\,2^{2\alpha j}u_{Q,j}^2 + c\,u_{Q,j}\Biggl(2^j u_{3Q/2,j\pm2}\sum_{\theta j\le k\le j-5}2^{3k/2}u_{\max(Q_k,3Q/2),k} \\ &\qquad + 2^{5j/2}u^2_{3Q/2,j\pm4} + 2^{3j/2}\sum_{k\ge j+1}2^k u^2_{3Q/2,k}\Biggr) + e_{\mathrm{diss}} + \sum_{k\ge\theta j}e_d(k) \\ &= -G_{\mathrm{diss}} + c\,u_{Q,j}(G_{\mathrm{low,loc}}+G_{\mathrm{loc}}+G_{\mathrm{hh}}) + e_{\mathrm{diss}} + e_{\mathrm{vl}} + \sum_{k\ge\theta j}e_d(k)\end{aligned} \tag{3-2}$$

is satisfied in the integral sense (*recall* (2-1)) *for any cube Q of side-length d and any $j \in \mathbb{Z}$, where*

$$\theta := \tfrac{2}{3}(2\alpha - 1 - \varepsilon) \tag{3-3}$$

and

$$e_{\text{diss}} := c\, 2^{2\alpha j}(d2^j)^{-1} u^2_{3Q/2,\, j\pm 2},$$
$$e_{\text{vl}} := c\, 2^{2\alpha j} 2^{-\varepsilon j} u^2_{3Q/2,\, j\pm 2}.$$

Here $\max\left(Q_k, \frac{3}{2}Q\right)$ denotes the larger of the cubes Q_k, $\frac{3}{2}Q$, and G_{diss} should be thought of as the *dissipation term*, $G_{\text{low,loc}}$ the interaction between low (i.e., modes $k \le j-5$) and local modes (i.e., modes $j \pm 2$), G_{loc} the local interactions (i.e., including only the modes $j \pm 4$) and G_{hh} the interactions between high modes (i.e., modes $k \ge j$).

The role of the parameter θ is to separate the "very low" Littlewood projections from the "low" Littlewood–Paley projections. That is (roughly speaking), given $j \in \mathbb{N}$ we will not have to worry about the Littlewood–Paley projections P_k with $k < \theta j$ (i.e., they will be effortlessly absorbed by the dissipation at the price of the error term e_{vl} ("vl" here stands for "very low"); see (3-12)–(3-13) below for a detailed explanation), which is the reason why such modes are not included in $G_{\text{low,loc}}$. In fact $G_{\text{low,loc}}$ is (roughly speaking) the most dangerous term, as it represents, in a sense, the injection of energy from low scales to high scales, and we will need to use G_{diss} to counteract it; see Step 5 in the proof of Theorem 3.7.

The error term e_{diss} appearing in the estimate is the error appearing when estimating the dissipation term, and it cannot be estimated by $e_d(j)$. Its appearance is a drawback of the main estimate, but in our applications (in Theorems 3.3 and 3.7) it can be absorbed by G_{diss}.

Proof of Proposition 3.1. Recall (1-4) that a Leray–Hopf weak solution admits intervals of regularity.

<u>Step 1</u>: We show that it is sufficient to show (3-2) on each of the intervals of regularity.

On each interval of regularity (a, b) we apply the Leray projection (recall (2-10) to the first equation of (1-1) to obtain

$$u_t + (-\Delta)^\alpha u + T[(u\cdot\nabla)u] = 0.$$

Multiplying by $P_j(\phi_Q^2 P_j u)$ and integrating in space we obtain (at any given time)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} u^2_{Q,j} = -\int (-\Delta)^\alpha u\, P_j(\phi_Q^2 P_j u) - \int T[(u\cdot\nabla)u]\, P_j(\phi_Q^2 P_j u) =: I + J.$$

We note that $I, J \in L^1(0, T)$ for every $T > 0$. Indeed, by brutal estimates

$$\begin{aligned} |J| &= \left| \int \phi_Q P_j T[(u\cdot\nabla)u]\, \phi_Q^2 P_j u \right| \\ &\le \|P_j T[(u\cdot\nabla)u]\|_1 \|P_j u\|_\infty \le c\|u\|\|\nabla u\| \cdot 2^{3j/2}\|P_j u\| \le c\, 2^{3j/2}\|\nabla u\| \end{aligned}$$

(where we used Bernstein inequality (2-11) in the third line), which is integrable on $(0, T)$ for every $T > 0$. That $I \in L^1(0, T)$ for every $T > 0$ is a consequence of Step 2 below. Thus, since $u(t)$ is weakly continuous with values in L^2 (recall Section 2), Lemma 2.6 gives that (3-2) is valid (in the integral sense) on $[0, \infty)$.

Thus it suffices to show that $I + J$ can be estimated by the right-hand side of (3-2).

Step 2: We show that $I \le -G_{\mathrm{diss}} + e_{\mathrm{diss}} + e_d(j)$. (Note that this gives in particular that $I \in L^1(0,\infty)$, since (trivially) $u_{Q',j} \le c$ for every cube Q' and every j.)

We write

$$\begin{aligned} I &= -\int \phi_Q (-\Delta)^\alpha P_j u \, \phi_Q P_j u \\ &= -\int (-\Delta)^\alpha \widetilde{P}_j(\phi_Q P_j u) \, \phi_Q P_j u - \int (-\Delta)^\alpha (1-\widetilde{P}_j)(\phi_Q P_j u) \, \phi_Q P_j u - \int [\phi_Q, (-\Delta)^\alpha] P_j u \, \phi_Q P_j u \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

Note that, due to the Plancherel theorem

$$\begin{aligned} I_1 &= -c \int |\xi|^{2\alpha} \tilde{p}_j(\xi) |\hat{v}(\xi)|^2 \, \mathrm{d}\xi \le -c\, 2^{2\alpha j} \int \tilde{p}_j(\xi) |\hat{v}(\xi)|^2 \, \mathrm{d}\xi \\ &= -c\, 2^{2\alpha j} \int \widetilde{P}_j v \cdot v = -c\, 2^{2\alpha j} u_{Q,j}^2 + c\, 2^{2\alpha j} \int (1-\widetilde{P}_j) v \cdot v \\ &\le -c\, 2^{2\alpha j} u_{Q,j}^2 + c\, 2^{2\alpha j} \|(1-\widetilde{P}_j) v\| = -G_{\mathrm{diss}} + e_d(j), \end{aligned}$$

where we wrote $v := \phi_Q P_j u$ for brevity, and we used the fact that $\|v\| \le c$ (recall (1-3)) in the last line, as well as Lemma 2.4 in the last equality.

Step 2.1: We show that $I_2 \le e_d(j)$.

We write

$$I_2 \le \|(-\Delta)^\alpha (1-\widetilde{P}_j)(\phi_Q P_j u)\| u_{Q,j},$$

and we will show that

$$\|(-\Delta)^\alpha (1-\widetilde{P}_j)(\phi_Q P_j u)\| \le e_d(j). \tag{3-4}$$

(This completes this step as $u_{Q,j} \le c$, as above.) Indeed, (3-4) follows in a way similar to Lemma 2.4 by taking the decomposition

$$\phi_Q = \phi_1 + \phi_2,$$

where

$$\begin{aligned} \hat{\phi}_1(\xi) &:= \chi_{|\xi| \le 2^{j-2}} \hat{\phi}_Q(\xi), \\ \hat{\phi}_2(\xi) &:= \chi_{|\xi| > 2^{j-2}} \hat{\phi}_Q(\xi). \end{aligned}$$

We see that $\phi_1 P_j u = \widetilde{P}_j(\phi_1 P_j u)$ (because of the supports in Fourier space, see (2-17)) and so it is sufficient to show that

$$\|(-\Delta)^\alpha (\phi_2 P_j u)\| \le e_d(j)$$

(since $\|1-\widetilde{P}_j\| \le 1$). Since the Fourier transform of $(-\Delta)^\alpha(\phi_2 P_j u)$ is

$$\begin{aligned} &c|\xi|^{2\alpha} \int \hat{\phi}_2(\xi-\eta) p_j(\eta) \hat{u}(\eta) \, \mathrm{d}\eta \\ &\qquad \le c \int |\xi-\eta|^{2\alpha} |\hat{\phi}_2(\xi-\eta) p_j(\eta) \hat{u}(\eta)| \, \mathrm{d}\eta + c \int |\eta|^{2\alpha} |\hat{\phi}_2(\xi-\eta) p_j(\eta) \hat{u}(\eta)| \, \mathrm{d}\eta, \end{aligned}$$

we obtain

$$\|(-\Delta)^\alpha (\phi_2 P_j u)\| \le c\|u\| \int_{|\xi| > 2^{j-2}} |\xi|^{2\alpha} |\hat{\phi}_2(\xi)| \, \mathrm{d}\xi + c \|\hat{\phi}_2\|_1 \|(-\Delta)^{2\alpha} P_j u\| \le e_d(j),$$

where we used the Plancherel theorem, (2-18) and the fact that $\| |\cdot|^{2\alpha} \hat{\phi}_2(\cdot)\|_1 \le e_d(j)$ (which follows in the same way as (2-18)).

Step 2.2: We show that $I_3 \le e_{\text{diss}} + e_d(j)$.

We have

$$I_3 \le \|[\phi_Q, (-\Delta)^\alpha] P_j u\| u_{Q,j}.$$

For brevity we let $v := P_j(\phi_{3Q/2} u)$, $\phi := \phi_Q$ and

$$W := [\phi, (-\Delta)^\alpha] v.$$

We will show below that

$$\|W\| \le c\, 2^{2\alpha j} (d\, 2^j)^{-1} u_{3Q/2,j} + e_d(j),$$

and we will show in Step 2.2c that

$$\|W\| = \|[\phi, (-\Delta)^\alpha] P_j u\| + e_d(j), \tag{3-5}$$

from which the claim of this step follows (and so, together with Step 2.1, finishes Step 2). Since

$$\widehat{W}(\xi) = c \int (|\eta|^{2\alpha} - |\xi|^{2\alpha}) \hat{\phi}(\xi - \eta) \hat{v}(\eta)\, \mathrm{d}\eta,$$

we can decompose W by writing $\int = \int_{|\eta-\xi| \le 2^{j-3}} + \int_{|\eta - \xi| > 2^{j-3}}$, that is,

$$W = W_1 + W_2,$$

where

$$\begin{aligned} \widehat{W}_1(\xi) &:= c \int_{|\eta-\xi|\le 2^{j-3}} (|\eta|^{2\alpha} - |\xi|^{2\alpha}) \hat{\phi}(\xi - \eta) \hat{v}(\eta)\, \mathrm{d}\eta, \\ \widehat{W}_2(\xi) &:= c \int_{|\eta-\xi| > 2^{j-3}} (|\eta|^{2\alpha} - |\xi|^{2\alpha}) \hat{\phi}(\xi - \eta) \hat{v}(\eta)\, \mathrm{d}\eta. \end{aligned}$$

We will show (in Step 2.2b below) that $\|W_2\| \le e_d(j)$. As for W_1, since supp $p_j \subset \{|\eta| \in (2^{j-1}, 2^{j+1})\}$, note that

$$\operatorname{supp} \widehat{W}_1 \subset \{|\xi| \in (2^{j-2}, 2^{j+2})\}. \tag{3-6}$$

Setting $f(z) := z^\alpha$ and expanding it in the Taylor series around $|\xi|^2$ we obtain

$$|\eta|^{2\alpha} - |\xi|^{2\alpha} = \sum_{k=1}^{3} \frac{f^{(k)}(|\xi|^2)}{k!} (|\eta|^2 - |\xi|^2)^k + \frac{f^{(4)}(z_0)}{24} (|\eta|^2 - |\xi|^2)^4,$$

where z_0 belongs to the interval with endpoints $|\eta|^2$ and $|\xi|^2$ (and so in particular $z_0 \in [2^{2j-4}, 2^{2j+4}]$). Writing

$$|\eta|^2 - |\xi|^2 = \sum_{i=1}^{3} (\eta_i - \xi_i)(\eta_i + \xi_i)$$

and taking the k-th power we obtain

$$|\eta|^{2\alpha} - |\xi|^{2\alpha} = \sum_{k=1}^{4} c_k f^{(k)}(z) \sum_{|\beta|=k,\ |\gamma_1|+|\gamma_2|=k} c_{\beta\gamma_1\gamma_2} (\eta - \xi)^\beta \eta^{\gamma_1} \xi^{\gamma_2},$$

where $z=|\xi|^2$ (for $k\le 3$) or $z=z_0$ (for $k=4$). Thus, noting that $|f^{(k)}(z)|\le c\,2^{j(2\alpha-2k)}$,

$$
\begin{aligned}
|\widehat{W}_1(\xi)| &\le c\sum_{k=1}^{3}|f^{(k)}(|\xi|^2)|\sum_{|\beta|=k,\,|\gamma_1|+|\gamma_2|=k}|\xi|^{|\gamma_2|}\left|\int_{|\eta-\xi|\le 2^{j-3}}(\xi-\eta)^\beta\hat\phi(\xi-\eta)\eta^{\gamma_1}\hat v(\eta)\,\mathrm{d}\eta\right| \\
&\qquad + c\sum_{|\beta|=4,\,|\gamma_1|+|\gamma_2|=4}|\xi|^{|\gamma_2|}\left|\int_{|\eta-\xi|\le 2^{j-3}}f^{(k)}(z_0)(\xi-\eta)^\beta\hat\phi(\xi-\eta)\eta^{\gamma_1}\hat v(\eta)\,\mathrm{d}\eta\right| \\
&\le c\sum_{k=1}^{3}\sum_{|\beta|=k,\,|\gamma_1|+|\gamma_2|=k}2^{j(2\alpha-2k+|\gamma_2|)}\left|\int_{|\eta-\xi|\le 2^{j-3}}(\xi-\eta)^\beta\hat\phi(\xi-\eta)\eta^{\gamma_1}\hat v(\eta)\,\mathrm{d}\eta\right| \\
&\qquad + c\,2^{j(2\alpha-4)}\int_{|\eta-\xi|\le 2^{j-3}}|\xi-\eta|^4|\hat\phi(\xi-\eta)\hat v(\eta)|\,\mathrm{d}\eta \\
&\le c\sum_{k=1}^{3}\sum_{|\beta|=k,\,|\gamma_1|+|\gamma_2|=k}2^{j(2\alpha-2k+|\gamma_2|)}|\widehat{D^\beta\phi D^{\gamma_1}v}(\xi)| \\
&\qquad + c\sum_{k=1}^{3}\sum_{|\beta|=k,\,|\gamma_1|+|\gamma_2|=k}2^{j(2\alpha-2k+|\gamma_2|)}\left|\int_{|\eta-\xi|>2^{j-3}}(\xi-\eta)^\beta\hat\phi(\xi-\eta)\eta^{\gamma_1}\hat v(\eta)\,\mathrm{d}\eta\right| \\
&\qquad + c\,2^{j(2\alpha-4)}\int_{|\eta-\xi|\le 2^{j-3}}|\xi-\eta|^4|\hat\phi(\xi-\eta)\hat v(\eta)|\,\mathrm{d}\eta \\
&=: c\sum_{k=1}^{3}\sum_{|\beta|=k,\,|\gamma_1|+|\gamma_2|=k}2^{j(2\alpha-2k+|\gamma_2|)}|\widehat{D^\beta\phi D^{\gamma_1}v}(\xi)|+\mathrm{Err}_1(\xi)+\mathrm{Err}_2(\xi).
\end{aligned}
$$

We will show below (in Step 2.2a below) that

$$\|\mathrm{Err}_1\|,\ \|\mathrm{Err}_2\|\le c\,2^{2\alpha j}(d\,2^j)^{-1}u_{3Q/2,j\pm 2}+e_d(j).$$

This, together with the Plancherel identity gives

$$
\begin{aligned}
\|W_1\| &\le c\sum_{k=1}^{3}\sum_{|\beta|=k,\,|\gamma_1|+|\gamma_2|=k}2^{j(2\alpha-2k+|\gamma_2|)}\|D^\beta\phi D^{\gamma_1}v\|+c\,2^{2\alpha j}(d\,2^j)^{-1}u_{3Q/2,j\pm2}+e_d(j) \\
&\le c\sum_{k=1}^{3}2^{2\alpha j}(d\,2^j)^{-k}\|v\|+c\,2^{2\alpha j}(d\,2^j)^{-1}u_{3Q/2,j\pm2}+e_d(j),
\end{aligned}
$$

where we used the facts that $|\nabla^k\phi|\le c\,d^{-k}$ for $k=1,2,3$, and $\|D^{\gamma_1}v\|\le c\,2^{j|\gamma_2|}\|v\|$ (by applying Lemma 2.4). Since $d>2^{-j}$ and

$$\|v\|\le\|\phi_{3Q/2}\widetilde{P}_ju\|+e_d(j)=u_{3Q/2,j\pm2}+e_d(j)$$

(where we applied Corollary 2.5), we thus arrive at

$$\|W_1\|\le c\,2^{2\alpha j}(d\,2^j)^{-1}u_{3Q/2,j\pm2}+e_d(j),$$

as required.

Step 2.2a: We show that $\|\mathrm{Err}_1\|\le e_d(j)$ and $\|\mathrm{Err}_2\|\le c\,2^{2\alpha j}(d\,2^j)^{-1}u_{3Q/2,j\pm2}+e_d(j)$.

We focus on Err_1 first. We have

$$\begin{aligned}
\mathrm{Err}_1(\xi) &= c\sum_{k=1}^{3}\sum_{|\beta|=k,\,|\gamma_1|+|\gamma_2|=k} 2^{j(2\alpha-2k+|\gamma_2|)}\left|\int_{|\eta-\xi|>2^{j-3}}(\xi-\eta)^{\beta}\hat{\phi}(\xi-\eta)\eta^{\gamma_1}\hat{v}(\eta)\,\mathrm{d}\eta\right| \\
&\le c\sum_{k=1}^{3} 2^{j(2\alpha-k)}\int_{|\eta-\xi|>2^{j-3}}|\xi-\eta|^{k}\,|\hat{\phi}(\xi-\eta)\hat{v}(\eta)|\,\mathrm{d}\eta \\
&\le c\,2^{j(2\alpha-K)}\int_{|\eta-\xi|>2^{j-3}}|\xi-\eta|^{K}\,|\hat{\phi}(\xi-\eta)\hat{v}(\eta)|\,\mathrm{d}\eta \\
&\le c_K\,2^{j(2\alpha-K)}d^{1-K}\int_{|\eta-\xi|>2^{j}}|\xi-\eta|^{-2}\,|\hat{v}(\eta)|\,\mathrm{d}\eta \\
&\le c_K\,2^{j(2\alpha-1)}(d\,2^{j})^{(1-K)}\left(\int_{|\eta-\xi|>2^{j-5}}|\xi-\eta|^{-4}\,\mathrm{d}\eta\right)^{1/2}
\end{aligned}$$

for every $K>3$, where we used (2-20) in the fourth line as well as the Cauchy–Schwarz inequality, (2-2) and the fact that $\|v\|\le\|u\|\le c$ (recall (1-3)) in the last line. Thus $\mathrm{Err}_1(\xi)\le e_d(j)$ for every $\xi\in\mathbb{R}^3$, and hence (since $|\xi|\le 2^{j+2}$) also $\|\mathrm{Err}_1\|\le e_d(j)$.

As for Err_2 we write

$$\begin{aligned}
\mathrm{Err}_2(\xi) &= c\,2^{j(2\alpha-4)}\int_{|\eta-\xi|\le 2^{j-3}}|\xi-\eta|^{4}\,|\hat{\phi}(\xi-\eta)\hat{v}(\eta)|\,\mathrm{d}\eta \\
&\le c\,2^{j(2\alpha-4)}d^{-1}\int_{|\eta-\xi|\le 2^{j-3}}|\hat{v}(\eta)|\,\mathrm{d}\eta \\
&\le c\,2^{j(2\alpha-3/2)}(d\,2^{j})^{-1}\|v\| \\
&= c\,2^{j(2\alpha-3/2)}(d\,2^{j})^{-1}\|P_j\phi_{3Q/2}u\| \\
&\le c\,2^{j(2\alpha-3/2)}(d\,2^{j})^{-1}u_{3Q/2,j\pm 2}+e_d(j),
\end{aligned}$$

where we used (2-20) in the second line, the Cauchy–Schwarz inequality (as above) in the third line, and Corollary 2.5 in the last line. Thus

$$\|\mathrm{Err}_2\|\le c\,2^{2\alpha j}(d\,2^{j})^{-1}u_{3Q/2,j\pm 2}+e_d(j),$$

as required.

<u>Step 2.2b</u>: We show that $\|W_2\|\le e_d(j)$.

Indeed, since $|\xi|^{2\alpha}\le c|\eta|^{2\alpha}+c|\xi-\eta|^{2\alpha}$, we obtain for any $K>2\alpha$

$$\begin{aligned}
|\widehat{W}_2(\xi)| &= \left|\int_{|\eta-\xi|>2^{j-5}}(|\eta|^{2\alpha}-|\xi|^{2\alpha})\hat{\phi}(\xi-\eta)\hat{v}(\eta)\,\mathrm{d}\eta\right| \\
&\le c\int_{|\eta-\xi|>2^{j-5}}|\eta|^{2\alpha}\,|\hat{\phi}(\xi-\eta)\hat{v}(\eta)|\,\mathrm{d}\eta + c\int_{|\eta-\xi|>2^{j-5}}|\xi-\eta|^{2\alpha}\,|\hat{\phi}(\xi-\eta)\hat{v}(\eta)|\,\mathrm{d}\eta \\
&\le c_K 2^{j(2\alpha-K)}\int_{|\eta-\xi|>2^{j-5}}|\xi-\eta|^{K}\,|\hat{\phi}(\xi-\eta)\hat{v}(\eta)|\,\mathrm{d}\eta,
\end{aligned}$$

where we used the inequality $1<c_K|\xi-\eta|^{K}2^{-jK}$, as well as $|\eta|\le c\,2^{j}$ inside the first integral in the second line and the inequality $1\le c_K|\xi-\eta|^{K-2\alpha}2^{-j(K-2\alpha)}$ inside the second integral. Thus, using the

Plancherel identity and Young's inequality for convolutions

$$\begin{aligned}\|W_2\| = \|\widehat{W}_2\| &\le c_K 2^{j(2\alpha-K)}\|v\|\int_{|\eta|>2^{j-5}} |\eta|^K |\hat{\phi}(\eta)|\,\mathrm{d}\eta \\ &\le c_K 2^{j(2\alpha-K)}\int_{|\eta|>2^{j-5}} |\eta|^{K+4}|\hat{\phi}(\eta)|\,|\eta|^{-4}\,\mathrm{d}\eta \\ &\le c_K 2^{j(2\alpha-K)} d^{-(K+1)}\int_{|\eta|>2^{j-5}} |\eta|^{-4}\,\mathrm{d}\eta \\ &= c_K 2^{2\alpha j}(d\,2^j)^{-(K+1)},\end{aligned}$$

as required, where we used (2-20) in the third inequality.

Step 2.2c: We show that $\|[\phi,(-\Delta)^\alpha]P_j(1-\phi_{3Q/2})u\| \le e_d(j)$. (This implies (3-5).)

Indeed, letting (for brevity) $w := (1-\phi_{3Q/2})u$ and $q_j(\xi) := |\xi|^{2\alpha} p_j(\xi)$, we can write

$$\phi(-\Delta)^\alpha P_j w(x) = \phi(x)\int_{\{|x-y|\ge d/3\}} \check{q}_j(x-y)w(y)\,\mathrm{d}y,$$

as in (2-16). Thus, since $\|\check{q}_j\|_{L^1(B(d/3)^c)} \le e_d(j)$ (as in (2-8)), we can use Young's inequality for convolutions to obtain

$$\|\phi(-\Delta)^\alpha P_j w\| \le \|\check{q}_j\|_{L^1(B(d/3)^c)}\|w\| \le e_d(j). \tag{3-7}$$

On the other hand

$$\begin{aligned}\|(-\Delta)^\alpha(\phi P_j w)\| &\le \|(-\Delta)^\alpha \widetilde{P}_j(\phi P_j w)\| + \|(-\Delta)^\alpha(1-\widetilde{P}_j)(\phi P_j w)\| \\ &\le c\,2^{2\alpha j}\|\phi P_j w\| + e_d(j) \le e_d(j),\end{aligned}$$

where we used (3-4) (applied with w instead of u) in the second line and Lemma 2.2 in the last line. This and (3-7) prove the claim.

Step 3: We show that $J \le c\,u_{Q,j}(G_{\text{low,loc}} + G_{\text{loc}} + G_{\text{hh}}) + e_{\text{vl}} + \sum_{k\ge\theta j} e_d(k)$. (This together with Step 2 finishes the proof.)

We can rewrite J in the form

$$J = -\int \phi_Q P_j T[(u\cdot\nabla)u]\cdot(\phi_Q P_j u) = -\sum_{i,l,m}\int \phi_Q T_{mi} P_j(u_l\,\partial_l u_m)\phi_Q P_j u_i,$$

where we used the fact that "T_{mi}" and "P_j" are multipliers (so that they commute). (Recall that $\widehat{T}_{mi}(\xi) = (\delta_{mi} - \xi_m\xi_i|\xi|^{-2})$, see (2-10).) We now apply the paraproduct formula (2-14) to $P_j(u_l\partial_l u_m)$ to write

$$J = J_{\text{loc,low}} + J_{\text{low,loc}} + J_{\text{loc}} + J_{\text{hh}},$$

where each of $J_{\text{loc,low}}$, $J_{\text{low,loc}}$, J_{loc}, J_{hh} equals J except for the term $u_l\,\partial_l u_m$, which is replaced by the corresponding combination of the modes of u_l and $\partial_l u_m$, as in the paraproduct formula (see (3-8) and (3-10) below). We estimate J_{hh} in Step 3.1 below and $J_{\text{loc,low}}$, $J_{\text{low,loc}}$, J_{loc} in Step 3.2.

Step 3.1: We show that $J_{\text{hh}} \le c\,u_{Q,j}G_{\text{hh}} + \sum_{k\ge j} e_d(k)$.

We write

$$
\begin{aligned}
J_{\mathrm{hh}} &= -\sum_{i,l,m} \int \phi_Q T_{mi} P_j \Big(\sum_{k\ge j+3} P_k u_l \widetilde{P}_k \partial_l u_m \Big) \phi_Q P_j u_i \\
&\le \|\phi_Q P_j u\|_\infty \sum_{i,l,m} \Big\| \phi_Q T_{mi} P_j \sum_{k\ge j+3} (P_k u_l \widetilde{P}_k \partial_l u_m) \Big\|_1 \\
&\le c\, 2^{3j/2} u_{Q,j} \sum_{i,l,m} \Big\| \phi_Q T_{mi} P_j \sum_{k\ge j+3} (P_k u_l \widetilde{P}_k \partial_l u_m) \Big\|_1 + e_d(j) \\
&\le c\, 2^{3j/2} u_{Q,j} \sum_{i,l,m} \Big\| \phi_Q T_{mi} P_j \phi_{3Q/2}^3 \Big(\sum_{k\ge j+3} P_k u_l \widetilde{P}_k \partial_l u_m \Big) \Big\|_1 + e_d(j) \\
&\le c\, 2^{3j/2} u_{Q,j} \sum_{k\ge j+3} \|\phi_{3Q/2} P_k u\| \|\phi_{3Q/2}^2 \widetilde{P}_k \nabla u\| + e_d(j),
\end{aligned}
\tag{3-8}
$$

where, in the fourth line we applied Corollary 2.3 with $f := \sum_{k\ge j+3} P_k u_l \widetilde{P}_k u_m$ and noted that $\operatorname{supp} \phi \subset \frac{7}{3} Q$ is separated from $\operatorname{supp}(1-\phi_{3Q/2}^3)$ by at least $\frac{1}{3}d$. As for the third line, we used $P_j u = P_j \widetilde{P}_j u$, (2-21) and (2-11) to write

$$
\|\phi_Q P_j u\|_\infty \le c\, 2^{3j/2} u_{Q,j} + e_d(j) \|\widetilde{P}_j u\|_\infty \le c\, 2^{3j/2} u_{Q,j} + e_d(j),
$$

as well as noted that $e_d(j)$ multiplied by the (long) L^1 norm still gives $e_d(j)$, since we can brutally estimate this norm,

$$
\begin{aligned}
\Big\| \phi_Q T_{mi} P_j \sum_{k\ge j+3} (P_k u_l \widetilde{P}_k \partial_l u_m) \Big\|_1 &\le \|\phi_Q\| \Big\| P_j \partial_l T_{mi} \sum_{k\ge j+3} (P_k u_l \widetilde{P}_k u_m) \Big\| \\
&\le c\, d^{3/2} 2^j \Big\| P_j T_{mi} \sum_{k\ge j+3} (P_k u_l \widetilde{P}_k u_m) \Big\| \\
&\le c\, d^{3/2} 2^{5j/2} \Big\| P_j \sum_{k\ge j+3} (P_k u_l \widetilde{P}_k u_m) \Big\|_1 \\
&\le c\, d^{3/2} 2^{5j/2} \sum_{k\ge j+3} \|P_k u_l \widetilde{P}_k u_m\|_1 \le c\, d^{3/2} 2^{5j/2} \sum_{k\ge j+1} \|P_k u\|^2 \\
&\le c\, d^{3/2} 2^{5j/2} \|u\|^2 \le c\, d^{3/2} 2^{5j/2}
\end{aligned}
$$

for each i, l, m, where we used the Cauchy–Schwarz inequality in the first line, boundedness (in L^2) of the Leray projection (i.e., the fact that $|\widehat{T}_{mi}(\xi)| \le 1$) and the Bernstein inequality (2-11) in the third line, (2-5) in the fourth line and the Cauchy–Schwarz inequality (twice) in the fifth line.

Noting that

$$
\begin{aligned}
\|\phi_{3Q/2}^2 \widetilde{P}_k \nabla u\| &= \|P_{k\pm 2}(\phi_{3Q/2}^2 \nabla \widetilde{P}_k u)\| + e_d(k) \\
&\le \|P_{k\pm 2} \nabla(\phi_{3Q/2}^2 \widetilde{P}_k u)\| + 2\|P_{k\pm 2}(\nabla \phi_{3Q/2}\, \phi_{3Q/2} \widetilde{P}_k u)\| + e_d(k) \\
&\le c\, 2^k \|\phi_{3Q/2}^2 \widetilde{P}_k u\| + c\, d^{-1} u_{3Q/2,k\pm 2} + e_d(k) \\
&\le c\, 2^k u_{3Q/2,k\pm 2} + e_d(k),
\end{aligned}
$$

where we used Lemma 2.4 in the first inequality, the fact that $\| \widetilde{P}_k \| \le 1$ and (2-19) in the third inequality, and the assumption $d > 2^{-j} > 2^{-k}$ in the last inequality, we obtain

$$J_{\mathrm{hh}} \le c\, 2^{3j/2} u_{Q,j} \sum_{k \ge j+1} 2^k u_{3Q/2,k}^2 + \sum_{k \ge j} e_d(k), \tag{3-9}$$

as required, where we also applied the Cauchy–Schwarz inequality in the first sum.

Step 3.2: We show that $J_{\mathrm{loc,low}} + J_{\mathrm{low,loc}} + J_{\mathrm{loc}} \le c\, u_{Q,j} (G_{\mathrm{low,loc}} + G_{\mathrm{loc}}) + e_{\mathrm{vl}} + \sum_{k \ge \theta j} e_d(k)$. (This completes the proof of Step 3.)

We set

$$U_{lm} := \widetilde{P}_j u_l \sum_{k \le j-5} P_k u_m + \widetilde{P}_j u_m \sum_{k \le j-5} P_k u_l + \left(\sum_{k=j-4}^{j+2} P_k u_l \right) \left(\sum_{k=j-4}^{j+4} P_k u_m \right)$$

to write

$$\begin{aligned} J_{\mathrm{loc,low}} + J_{\mathrm{low,loc}} + J_{\mathrm{loc}} &= - \sum_{i,l,m} \int \phi_Q T_{mi} P_j \partial_l U_{ml} \phi_Q P_j u_i \le u_{Q,j} \sum_{i,l,m} \| \phi_Q T_{mi} P_j \partial_l U_{ml} \| \\ &= u_{Q,j} \sum_{i,l,m} \| \phi_Q T_{mi} P_j (\phi_{3Q/2}^3 \partial_l U_{ml}) \| + e_d(j) \\ &\le c\, u_{Q,j} \sum_{l,m} \| P_j (\phi_{3Q/2}^3 \partial_l U_{ml}) \| + e_d(j) \\ &\le c\, u_{Q,j} \sum_{l,m} (\| P_j \partial_l (\phi_{3Q/2}^3 U_{ml}) \| + 3 \| P_j (\phi_{3Q/2}^2 \partial_l \phi_{3Q/2} U_{ml}) \|) + e_d(j) \\ &\le c\, 2^j u_{Q,j} \sum_{l,m} \| \phi_{3Q/2}^2 U_{ml} \| + e_d(j), \end{aligned} \tag{3-10}$$

where we applied Corollary 2.3 (with $q := 2$ and $f := U_{ml}$) in the third line, as well as (2-19) (as in the previous calculation) and the assumption $d > 2^{-j}$ in the last line.

We note that for each m, l

$$\| \phi_{3Q/2}^2 U_{ml} \| \le 2 u_{3Q/2, j \pm 2} \left\| \phi_{3Q/2} \sum_{k \le j-5} P_k u \right\|_\infty + \| \phi_{3Q/2} P_{j \pm 4} u \|_\infty u_{3Q/2, j \pm 4}. \tag{3-11}$$

Since we can estimate the above L^∞ norm including the summation by writing

$$\sum_{k \le j-5} = \sum_{k < \theta j} + \sum_{\theta j \le k \le j-5},$$

that is,

$$\begin{aligned} \left\| \phi_{3Q/2} \sum_{k \le j-5} P_k u \right\|_\infty &\le \left\| \phi_{3Q/2} \sum_{k < \theta j} P_k u \right\|_\infty + \sum_{\theta j \le k \le j-5} \| \phi_{\max(Q_k, 3Q/2)} P_k u \|_\infty \\ &\le \| P_{\le \theta j} u \|_\infty + c \sum_{\theta j \le k \le j-5} 2^{3k/2} u_{\max(Q_k, 3Q/2), k} + \sum_{k \ge \theta j} e_d(k) \\ &\le c\, 2^{3\theta j/2} + c \sum_{\theta j \le k \le j-5} 2^{3k/2} u_{\max(Q_k, 3Q/2), k} + \sum_{k \ge \theta j} e_d(k), \end{aligned} \tag{3-12}$$

where we used the localised Bernstein inequality (2-21) in the second line (note that taking $\max\big(Q_k, \frac{3}{2}Q\big)$ is necessary since only then can we guarantee that the sidelength of such cube is greater than 2^{-k}, as required by (2-21)) and the Bernstein inequality (2-12) in the last line, we can plug it in (3-11) to get

$$\|\phi_{3Q/2}^2 U_{ml}\| \le c\, u_{3Q/2,j\pm 2} 2^{3\theta j/2} + c\, u_{3Q/2,j\pm 2} \sum_{\theta j \le k \le j-5} 2^{3k/2} u_{3Q/2,k} + c\, 2^{3j/2} u_{3Q/2,j\pm 4}^2 + \sum_{k\ge\theta j} e_d(k),$$

where we used the assumption $d > 2^{-j+4}$ to apply the localised Bernstein inequality(2-21) again. Inserting this into (3-10) and using the fact that $\frac{3}{2}\theta = 2\alpha - 1 - \varepsilon$, we obtain

$$J_{\text{loc,low}} + J_{\text{low,loc}} + J_{\text{loc}} \le c\, 2^{2\alpha j} 2^{-\varepsilon j} u_{3Q/2,j\pm 2}^2 + c\, 2^j u_{Q,j} u_{3Q/2,j\pm 2} \sum_{\theta j \le k \le j-5} 2^{3k/2} u_{3Q/2,k} \\ + c\, 2^{5j/2} u_{Q,j} u_{3Q/2,j\pm 4}^2 + \sum_{k\ge\theta j} e_d(k), \quad (3\text{-}13)$$

as required (note the first term on the right-hand side the is the "very low modes error", e_{vl}). □

We now constrain ourselves to j-cubes. Given a j-cube Q we will write

$$u_Q := u_{Q,j}$$

for brevity. The above proposition then reduces to the following.

Corollary 3.2. *Let u be a Leray–Hopf weak solution of the Navier–Stokes equations* (1-1) *on the time interval* $[0, \infty)$. *Let Q be a j-cube with j large enough so that* $2^{\varepsilon j} \ge 16$. *Then*

$$\frac{\mathrm{d}}{\mathrm{d}t} u_Q^2 \le -c\, 2^{2\alpha j} u_Q^2 + c\, u_Q \Bigg(u_{3Q/2,j\pm 2} \sum_{\theta j \le k \le j-5} 2^{j+3k/2} u_{Q_k} + 2^{5j/2} u_{3Q/2,j\pm 4}^2 + \sum_{k \ge j+1} 2^{3j/2+k} u_{3Q/2,k}^2 \Bigg) \\ + c\, 2^{j(2\alpha-\varepsilon)} u_{3Q/2,j\pm 2}^2 + e(j). \quad (3\text{-}14)$$

Proof. We apply the estimate from Proposition 3.1 (which is valid due to the assumption $2^{\varepsilon j} > 16$). Since

$$e_{\text{diss}} \le c\, 2^{j(2\alpha-\varepsilon)} u_{3Q/2,j\pm 2}^2$$

and

$$\sum_{k\ge\theta j} e_d(k) \le c_K \sum_{k\ge\theta j} 2^{ck} 2^{\varepsilon k(c-K)} \le c_K 2^{c\theta j + \varepsilon\theta j(c-K)} = e(j),$$

where K is taken large enough (to guarantee the summability of the geometric series), we arrive at (3-14), as required. □

3B. ***Good cubes and bad cubes.*** We now fix $u_0 \in H^1(\mathbb{R}^3)$ and a Leray–Hopf weak solution with initial data u_0. We say that a cube Q is j-good if

$$\int_0^\infty \int_Q \sum_{k\ge j} 2^{2\alpha k} |P_k u|^2 \le 2^{-j(5-4\alpha+\varepsilon)}. \quad (3\text{-}15)$$

We say that a j-cube is good if it is j-good. Otherwise we say that it is bad.

3C. ***Critical regularity on cubes with some good ancestors.*** We show that, for sufficiently large j, goodness of a j-cube and some of its ancestors guarantees critical regularity ($+\varepsilon$) of u_Q on a smaller cube Q.

Theorem 3.3. *There exists $j_0 > 0$ (sufficiently large) such that whenever Q is a j-cube such that $j \geq j_0$ and each Q_{k-10}, $k \in [\theta j, j]$, is good then*

$$u_Q(t) < 2^{-(j/2)(5-4\alpha+\varepsilon)} \quad \textit{for } t \in [0, T).$$

Remark 3.4. The above theorem appears in an imprecise form as Theorem 7.1 in [Katz and Pavlović 2002].[1] This is related to the somewhat unexpected way in which the dissipation error is handled in Lemma 6.3 in the same work. This lemma is in fact not needed, and it seems necessary to incorporate the dissipation error directly into the main estimate (in order to get around the imprecision), as in e_{diss} in (3-2).

Moreover the statement of Theorem 7.1 in [Katz and Pavlović 2002] suggests that goodness of only one cube is sufficient for the critical decay, which is not consistent with its proof (which uses goodness of the ancestors in the third line on p. 375).

Proof. Note that the claim is true for sufficiently small $t > 0$ since $u_0 \in H^1$, so that

$$\|P_j u_0\|^2 = \int p_j^2(\xi)|\hat{u}_0(\xi)|^2 \,\mathrm{d}\xi \leq c\, 2^{-2j} \int |\xi|^2 |\hat{u}_0(\xi)|^2 \,\mathrm{d}\xi \leq c\, 2^{-2j} \|u_0\|_{H^1}^2 < 2^{-j(5-4\alpha+\varepsilon)}$$

for sufficiently large j, and $u(t)$ remains bounded in H^1 for small $t > 0$. Suppose that the theorem is false, and let t_0 be the first time when it fails and Q a j-cube for which it fails. Then

$$u_Q(t) \leq 2^{-(j/2)(5-4\alpha+\varepsilon)} \quad \text{for } t \leq t_0, \tag{3-16}$$

with equality for $t = t_0$. Let $t_1 \in (0, t_0)$ be the last time when $u_Q(t_1) \leq \frac{1}{2} 2^{-(j/2)(5-4\alpha+\varepsilon)}$, so that

$$\tfrac{1}{2} 2^{-(j/2)(5-4\alpha+\varepsilon)} \leq u_Q(t) \leq 2^{-(j/2)(5-4\alpha+\varepsilon)} \quad \text{for } t \in (t_1, t_0). \tag{3-17}$$

Note that, since $\operatorname{supp} \phi_{3Q/2} \subset \frac{7}{4}Q \subset Q_{j-1} \subset Q_{j-10}$ and Q_{j-10} is good,

$$\int_{t_1}^{t_0} \sum_{k \geq j-10} 2^{2\alpha k} u_{3Q/2,k}^2 \leq c \int_{t_1}^{t_0} \int_{Q_{j-10}} \sum_{k \geq j-10} 2^{2\alpha k} |P_k u|^2 \leq c\, 2^{-j(5-4\alpha+\varepsilon)},$$

and so in particular (recalling that $\alpha \in \left(1, \frac{5}{4}\right)$)

$$\int_{t_1}^{t_0} u_{3Q/2,j\pm 4}^2 \leq c\, 2^{-j(5-2\alpha+\varepsilon)} \tag{3-18}$$

and

$$\int_{t_1}^{t_0} \sum_{k \geq j+1} 2^k u_{3Q/2,k}^2 \leq c\, 2^{j(1-2\alpha)} \int_{t_1}^{t_0} \sum_{k \geq j} 2^{2\alpha k} u_{3Q/2,k}^2 \leq c\, 2^{-j(4-2\alpha+\varepsilon)}. \tag{3-19}$$

[1]The claim following "we must have" on p. 374 does not follow, as the assumption of the proof by contradiction is only on Q, rather than on every cube in its nuclear family.

Moreover, since Q_{k-10} is good for every $k \in [\theta j, j]$, we also have

$$\int_{t_1}^{t_0} u_{Q_k}^2 \le c\, 2^{-k(5-2\alpha+\varepsilon)}$$

(as in (3-18)), and so

$$\int_{t_1}^{t_0} \sum_{\theta j \le k \le j-5} 2^{3k} u_{Q_k}^2 \le c \sum_{\theta j \le k \le j-5} 2^{-k(2-2\alpha+\varepsilon)} \le c\, 2^{-j(2-2\alpha+\varepsilon)}, \tag{3-20}$$

where we used the fact that $\alpha > 1$ and the fact that $\varepsilon > 0$ is small (recall (3-1)).[2]

Applying the main estimate (3-14) between t_1 and t_0 (and ignoring the first term on the right-hand side) and then utilizing (3-18)–(3-20) we obtain

$$\begin{aligned} 2^{-j(5-4\alpha+\varepsilon)} &= \tfrac{4}{3}(u_Q(t_0)^2 - u_Q(t_1)^2) \\ &\le c \int_{t_1}^{t_0} u_Q \Big(2^j u_{3Q/2,j\pm 2} \sum_{\theta j \le k \le j-5} 2^{3k/2} u_{3Q/2,k} + 2^{5j/2} u_{3Q/2,j\pm 4}^2 + 2^{3j/2} \sum_{k \ge j+1} 2^k u_{3Q/2,k}^2 \Big) \\ &\qquad\qquad + c\, 2^{j(2\alpha-\varepsilon)} \int_{t_1}^{t_0} u_{3Q/2,j\pm 2}^2 + e(j) \\ &\le c\, 2^{-(j/2)(5-4\alpha+\varepsilon)} \big(2^j 2^{-(j/2)(5-2\alpha+\varepsilon)} 2^{-(j/2)(2-2\alpha+\varepsilon/2)} + 2^{5j/2} 2^{-j(5-2\alpha+\varepsilon)} + 2^{3j/2} 2^{-j(4-2\alpha+\varepsilon)} \big) \\ &\qquad\qquad + c\, 2^{j(2\alpha-\varepsilon)} 2^{-j(5-2\alpha+\varepsilon)} \\ &\le c\, 2^{-j(5-4\alpha+\varepsilon)} (2^{-3j\varepsilon/8} + 2^{-j\varepsilon/2} + 2^{-j\varepsilon/2} + 2^{-3j\varepsilon/2}) \\ &\le c\, 2^{-j(5-4\alpha+\varepsilon)} 2^{-3j\varepsilon/8}, \end{aligned}$$

where, in the second inequality, we also used the Cauchy–Schwarz inequality and used the inequality $j \le c\, 2^{j\varepsilon/4}$, as well as absorbed $e(j)$ (by writing, for example, $e(j) \le c\, 2^{-j(5-4\alpha+2\varepsilon)}$ — recall the beginning of Section 2 for the definition of the j-negligible error $e(j)$). Thus

$$1 \le c\, 2^{-j\varepsilon/4},$$

which gives a contradiction for sufficiently large j. □

3D. ***The singular set.*** Having defined good cubes and bad cubes, and observing that we have a "slightly more than critical" estimate on a cube that has some good ancestors (Theorem 3.3), we now characterize the singular set S in terms of its covers by bad cubes, and (in the next section) we show a much stronger (than critical) estimate outside S.

Let A_j denote the union of all bad j-cubes. Using Vitali covering lemma we can find a cover $\mathcal{A}_j$ that covers A_j and such that

$$\#\mathcal{A}_j \le c\, 2^{j(5-4\alpha+\varepsilon)}. \tag{3-21}$$

Indeed, the Vitali covering lemma gives a sequence of pairwise disjoint bad j-cubes $Q^{(l)}$ such that

$$A_j \subset \bigcup_l 5Q^{(l)}$$

[2]The restriction $\alpha > 1$ is used here, but $\alpha \ge 1$ would be sufficient by noting that $\sum_{k \ge \theta j} 2^{-k\varepsilon} \le c\, 2^{-j\theta\varepsilon}$. Indeed, since $\theta > \frac{5}{8}$ (recall (3-3)), the last inequality of this proof would become $1 \le c\, 2^{-j\varepsilon(\theta-1/2-1/8)}$, which still gives contradiction for large j.

However, since $\int_0^\infty \int |(-\Delta)^{\alpha/2} u|^2 \le c$ (from the energy inequality, recall (1-3)),

$$\begin{aligned} c &\geq \int_0^\infty \int |\xi|^{2\alpha} |\hat{u}(\xi)|^2 = \sum_{k\in\mathbb{Z}} \int_0^\infty \int p_k(\xi) |\xi|^{2\alpha} |\hat{u}(\xi)|^2 \\ &\geq c \sum_{k\geq j} 2^{2\alpha k} \int_0^\infty \int p_k(\xi)^2 |\hat{u}(\xi)|^2 = c \int_0^\infty \int \sum_{k\geq j} 2^{2\alpha k} |P_k u|^2 \\ &\geq c \sum_l \int_0^\infty \int_{Q^{(l)}} \sum_{k\geq j} 2^{2\alpha k} |P_k u|^2 \geq c \sum_l 2^{-j(5-4\alpha+\varepsilon)}, \end{aligned} \tag{3-22}$$

where we used the Plancherel identity (twice, in the first and fourth lines), Tonelli's theorem (twice, in the second and fourth lines), and the fact that $Q^{(l)}$'s are pairwise disjoint in the fifth line. Thus

$$\#\{l\} \leq c\, 2^{j(5-4\alpha+\varepsilon)},$$

and so $\mathcal{A}_j$ can be obtained by covering each of $5Q^{(l)}$ by at most 6^3 j-cubes.

In the remainder of this section we will show that there exists a (larger) j-cover $\mathcal{B}_j$ of all bad j-cubes (i.e., of A_j) with the same cardinality (i.e., satisfying (3-21), but with a larger constant) and the additional property that

$$\begin{aligned} &\text{for any } x \text{ outside of } \mathcal{B}_j \text{ there exists } r \in (0, 2^{-10}) \\ &\qquad\qquad \text{such that } \partial(rQ_j(x)) \text{ does not touch any bad } k\text{-cube for any } k \geq j. \end{aligned} \tag{3-23}$$

(Recall that $Q_j(x)$ denotes the j-cube centred at x.) We will refer to $\partial(rQ_j(x))$ as the *barrier*, and to (3-23) as the *barrier property*. We first discuss a simple geometric lemma.

Lemma 3.5 (geometric lemma). *Let $Q = Q(y)$, $Q' = Q'(x)$ be open cubes with sidelengths $2a$, $2b$, respectively. Then*

$$\partial(rQ) \text{ intersects } Q' \quad \Longrightarrow \quad r \in [r_{Q'} - b/a, r_{Q'} + b/a],$$

where $r_{Q'} > 0$ is such that $x \in \partial(r_{Q'}Q)$.

Proof. We will write $\gamma := b/a$ for brevity. We split the reasoning into cases.

Case 1: $y \in \partial Q'$. Then $r_{Q'} = b/a$ (see Figure 2 (middle)), and so $r \geq r_{Q'} - b/a$ trivially. Moreover $\partial(rQ)$ intersects Q' if and only if $ra < 2b$ (see Figure 2 (middle)), that is, $r < 2b/a = r_{Q'} + b/a$, as required.

Case 2: $y \notin \overline{Q}'$. Then $r_{Q'} > b/a$ (which is clear by comparison with Case 1), and $\partial(rQ)$ intersects Q' if and only if

$$r_{Q'}a - b < ra < r_{Q'}a + b$$

(see Figure 2 (right)), as required.

Case 3: $y \in Q'$. Then $r_{Q'} < b/a$ and $\partial(rQ)$ intersects Q' if and only if

$$b - r_{Q'}a < ra < r_{Q'}a + b$$

(see Figure 2 (left)). The claim follows by ignoring the first of these two inequalities (and writing $r \geq 0 > r_{Q'} - b/a$ instead). □

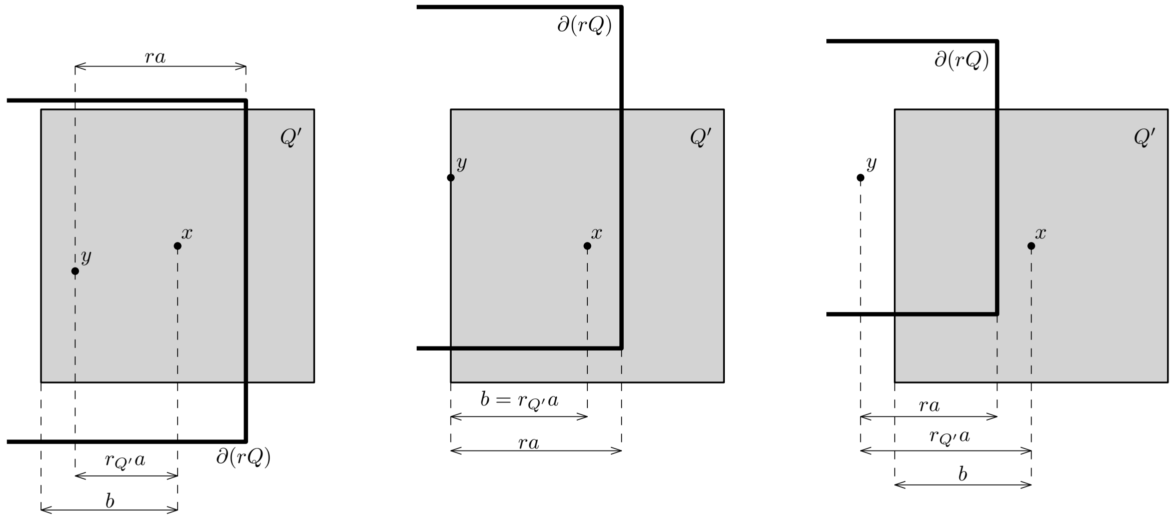

Figure 2. Sketch of the interpretation of Lemma 3.5.

We can now construct the j-cover satisfying the barrier property (3-23).

Lemma 3.6. *For every $j \geq 0$ there exists a j-cover $\mathcal{B}_j$ of A_j such that $\#\mathcal{B}_j \leq c\, 2^{j(5-4\alpha+\varepsilon)}$ and the barrier property* (3-23) *holds.*

Proof. (Here we follow the argument from [Katz and Pavlović 2002, Section 8].) We will find a j-cover (also denoted by $\mathcal{B}_j$) of A_j such that

$$\text{for any } j\text{-cube } Q \text{ outside of } \mathcal{B}_j \text{ there exists } r \in (0, 2^{-10}) \\ \text{such that } \partial(rQ) \text{ does not touch any bad } k\text{-cube for any } k \geq j. \tag{3-24}$$

(Here "outside" is a short-hand notation for "disjoint with every element of".) The barrier property (3-23) is then recovered by replacing every j-cube $Q \in \mathcal{B}_j$ by $3Q$ and covering it by at most 4^3 j-cubes. Indeed, then for any x outside of such set we have that $Q_j(x)$ (the j-cube centred at x) is outside of $\mathcal{B}_j$ and so the barrier property (3-23) follows from (3-24).

Step 1: We define *naughty* j-cubes.

We say that a j-cube Q is *k-naughty*, for $k \geq j$, if it intersects more than $\eta 2^{(k-j)(5-4\alpha+2\varepsilon)}$ elements of $\mathcal{A}_k$. Here $\eta \in (0,1)$ is a universal constant, whose value we fix in Step 4 below. We say that a j-cube is *naughty* if it is k-naughty for any $k \geq j$. (Note that a bad cube is naughty. A good cube is not necessarily naughty, and vice versa.)

Step 2: For each $k \geq j$ we construct a j-cover $\mathcal{B}_{j,k}$ of all k-naughty j-cubes such that

$$\#\mathcal{B}_{j,k} \leq c\eta^{-1} 2^{j(5-4\alpha+\varepsilon)} 2^{\varepsilon(j-k)}. \tag{3-25}$$

(Note that $\mathcal{B}_{j,j}$ covers all j-naughty j-cubes, and so in particular all bad j-cubes.)

Let $Q^{(1)}$ be any k-naughty j-cube. Given $Q^{(1)}, \dots, Q^{(l)}$ let $Q^{(l+1)}$ be any k-naughty j-cube that is disjoint with each of $3Q^{(1)}, \dots, 3Q^{(l)}$. Note that then $3Q^{(1)}, \dots, 3Q^{(l)}$ contain all elements of $\mathcal{A}_k$ that

$Q^{(1)}, \dots, Q^{(l)}$ intersect. This means that $Q^{(l+1)}$ intersects at least $\eta\, 2^{(k-j)(5-4\alpha+2\varepsilon)}$ "new" elements of $\mathcal{A}_k$ (i.e., the elements that none of $Q^{(1)}, \dots, Q^{(l)}$ intersect). This means that such an iterative definition can go on for at most

$$L := \#\mathcal{A}_k \eta^{-1} 2^{(k-j)(5-4\alpha+2\varepsilon)} \le c\eta^{-1} 2^{j(5-4\alpha+\varepsilon)} 2^{\varepsilon(j-k)}$$

steps, and then the family $\{3Q^{(1)}, \dots, 3Q^{(L)}\}$ covers all k-naughty j-cubes. We now cover each of $3Q^{(l)}$ ($l = 1, \dots, L$) by at most 4^3 j-cubes to obtain $\mathcal{B}_{j,k}$. (Note (3-25) then follows from the upper bound on L.)

Step 3: We define $\mathcal{B}_j$.

Let

$$\mathcal{B}_j := \bigcup_{k \ge j} \mathcal{B}_{j,k}.$$

By construction, $\mathcal{B}_j$ covers all naughty j-cubes (and so, in particular, all bad j-cubes) and

$$\#\mathcal{B}_j \le \sum_{k \ge j} \#\mathcal{B}_{j,k} \le c\eta^{-1} 2^{j(5-4\alpha+\varepsilon)} \sum_{k \ge j} 2^{\varepsilon(j-k)} = c\eta^{-1} 2^{j(5-4\alpha+\varepsilon)},$$

as required (given η is fixed).

Step 4: We show that (3-24) holds for sufficiently small $\eta \in (0, 1)$. (This, together with the previous step, finishes the proof.)

Let Q be a j-cube disjoint with all elements of $\mathcal{B}_j$. Let us denote by $\mathcal{C}^k(Q)$ the collection of k-cubes Q' ($k \ge j$) from $\mathcal{A}_k$ intersecting Q. Since Q is not naughty (as otherwise it would be covered by $\mathcal{B}_j$)

$$\#\mathcal{C}^k(Q) \le \eta 2^{(k-j)(5-4\alpha+2\varepsilon)}.$$

Let $r_{Q'} \in (0, \infty)$ be such that $\partial(r_{Q'} Q)$ contains the centre of Q'. Applying Lemma 3.5 with $2a = 2^{-j(1-\varepsilon)}$ and $2b = 2^{-k(1-\varepsilon)}$ we obtain that

$$\partial(rQ) \text{ intersects } Q' \quad \Longrightarrow \quad r \in [r_{Q'} - 2^{(1-\varepsilon)(j-k)}, r_{Q'} + 2^{(1-\varepsilon)(j-k)}].$$

Thus if $f_k(r)$ denotes the number of bad k-cubes that intersect $\partial(rQ)$ then

$$f_k(r) \le \sum_{Q' \in \mathcal{C}^k(Q)} \chi_{[r_{Q'} - 2^{(1-\varepsilon)(j-k)}, r_{Q'} + 2^{(1-\varepsilon)(j-k)}]}(r).$$

Thus

$$\|f_k\|_{L^1(0,2^{-10})} \le 2\, \#\mathcal{C}^k(Q) 2^{(1-\varepsilon)(j-k)} \le 2\eta 2^{(4\alpha-4-3\varepsilon)(j-k)},$$

and so letting $f := \sum_{k \ge j} f_k$ and recalling that $\alpha > 1$ and ε is small enough so that $4\alpha - 4 - 3\varepsilon > 0$ (see (3-1)) we obtain

$$\|f\|_{L^1(0,2^{-10})} \le \sum_{k \ge j} \|f_k\|_{L^1(0,2^{-10})} \le c\eta.$$

(This is the only place in the article where we need the assumption $\alpha > 1$; otherwise $\alpha \ge 1$ would be sufficient.) By choosing $\eta \in (0, 1)$ sufficiently small such that $c\eta < \frac{1}{2} 2^{-10}$ we see that $\|f\|_{L^1(0,2^{-10})} < 2^{-10}$, and so there exists $r \in (0, 2^{-10})$ such that $f(r) = 0$ (recall that f takes only integer values). In other words there exists r such that $\partial(rQ)$ does not intersect any element of $\mathcal{A}_k$ for any $k \ge j$, and so in particular any bad k-cube. □

We now let

$$E := \limsup_{j\to\infty} \bigcup_{Q\in\mathcal{B}_j} Q.$$

Observe that, since $\#\mathcal{B}_j \le c\, 2^{j(5-4\alpha+\varepsilon)}$,

$$d_H(E) \le 5 - 4\alpha + \varepsilon;$$

see, for example, Lemma 3.1 in [Katz and Pavlović 2002] for a proof.

3E. ***Regularity outside E.*** We now show that for every $x \notin E$ and every interval of regularity (a_i, b_i) there exists an open neighbourhood of x on which $u(t)$ remains bounded $\big(\text{as } t \in \big(\tfrac{1}{2}(a_i+b_i), b_i\big)\big)$. This together with the above bound on $d_H(S)$ finishes the proof of Theorem 1.1.

Note that if $x \notin E$ then for sufficiently large j_0

$$x \notin Q \quad \text{for any } Q \in \mathcal{B}_j, \text{ for } j \ge j_0.$$

In particular

$$x \text{ does not belong to any bad } j\text{-cube for } j \ge j_0 \tag{3-26}$$

(since $\mathcal{B}_j$ is a cover of all bad j-cubes), and for any $j_1 \ge j_0$ there exists $r = r(x, j_1) \in (0, 2^{-10})$ such that

$$\partial(rQ_{j_1}(x)) \text{ does not intersect any bad } k\text{-cube with } k \ge j_1 \tag{3-27}$$

(by the barrier property, (3-23)). The point is that the barrier can be constructed for any $j_1 \ge j_0$. This will be relevant for us, since in the proof of regularity below we will consider a j-cube with $j \ge j_1 \ge j_0/\theta^2$. Thus we will be able to deal with some of the low modes ($k \in [\theta j, j-5]$)) using (3-26) and others using (3-27). Indeed, for such modes we will have “cubes larger than j-cube” (i.e., Q_k with $k < j$) and we will obtain the critical decay on such cubes by either utilising the barrier property (3-27) (for cubes that are only “a little bit larger”, see Case 1 in Step 2 for details) or the fact that distant ancestors are large enough to contain x so that we can use (3-26). As for local and high modes (i.e., $k \ge j-5$), we will use the barrier property (3-27) to obtain critical regularity for cubes located near the barrier, with more and more regularity on cubes located further away from the barrier towards the interior. In fact we can guarantee an arbitrary strong estimate for cubes located sufficiently far from the barrier, but we limit ourselves to the estimate $\lesssim 2^{-j(5-4\alpha+10)/2}$.

We now proceed to a rigorous version of the above explanation.

Theorem 3.7 (regularity outside E). *Let $x \notin E$. Given an interval of regularity (a_i, b_i), there exists $c_i > 1$ and $j_1 = j_1(c_i) \in \mathbb{N}$ such that*

$$u_Q(t) < c_i 2^{-j\rho(Q)/2} \tag{3-28}$$

for all $t \in \big(\tfrac{1}{2}(a_i+b_i), b_i\big)$ and for every j-cube $Q \subset rQ_{j_1}(x)$, where $r \in (0, 2^{-10})$ is as in (3-27),

$$\rho(Q) := 5 - 4\alpha + \min\big(10, \tfrac{1}{10}\varepsilon\delta(Q)\big)$$

and $\delta(Q)$ denotes the smallest $k \in \mathbb{N}$ such that Q_{j-k} intersects $\partial(rQ_{j_1}(x))$.

Note that the theorem gives no restriction on the range of j's, but it is clear from the inclusion $Q \subset rQ_{j_1}(x)$ that $j \ge j_1 + 10$ (as $r < 2^{-10}$).

Proof. Since u is a strong solution in (a_i, b_i), it is continuous in time in (a_i, b_i) with values in H^6 (recall (1-4)). Thus letting

$$c_i := 1 + c\big\| u\big(\tfrac{1}{2}(a_i+b_i)\big)\big\|_{H^6}$$

we see that, for any j-cube Q, $u_Q\big(\tfrac{1}{2}(a_i+b_i)\big) \le \| P_j u\big(\tfrac{1}{2}(a_i+b_i)\big)\| < c_i 2^{-6j}$, and hence also $u_Q(t) < c_i$ for some $t > \tfrac{1}{2}(a_i+b_i)$ (due to the continuity of the H^6 norm). Thus the claim remains valid on some nonempty time interval following $\tfrac{1}{2}(a_i+b_i)$ (since $\rho(Q) \le 5 - 4\alpha + 10 \le 11$).

Since the interval of regularity (a_i, b_i) is fixed, from now on we will suppress the subindex "i", for brevity.

We take j_0 sufficiently large so that (3-26) and the claims of Corollary 3.2 and Theorem 3.3 are valid (we will let j_0 even larger below). We let j_1 be the smallest integer such that

$$j_1 \ge (j_0 + 10)/\theta^2. \tag{3-29}$$

We also note that

$$\text{if } Q'(y) \text{ is a } k\text{-cube centred at } y \in rQ_{j_1}(x) \text{ and touching the barrier } \partial(rQ_{j_1}(x)) \text{ then } Q' \text{ is good if } k \ge j_0. \tag{3-30}$$

Indeed, if $k \ge j_1$ then Q' is good by the barrier property (3-27). If $k < j_1$ then $Q' \supset rQ_{j_1}(x) \ni x$ (as the sidelength of $Q'(y)$ is more than 2^{10} times larger than the sidelength of $rQ_{j_1}(x) \ni y$), and so Q' is good by (3-26).

Suppose that the theorem is false and let $t_0 > \tfrac{1}{2}(a+b)$ be the first time when it fails. Then

$$u_{Q'}(t) \le c\, 2^{-k\rho(Q')/2} \qquad \text{for all } t \in [0, t_0] \text{ and all } k\text{-cubes } Q' \subset rQ_{j_1}(x) \tag{3-31}$$

and there exists a j-cube $Q \subset rQ_{j_1}(x)$ (for some $j \ge 0$) such that

$$u_Q(t_0) \ge 2^{-j\rho(Q)/2}. \tag{3-32}$$

We note that the existence of such Q is nontrivial, since there are infinitely many functions $u_{Q'}(t)$ for $Q' \subset rQ_{j_1}(x)$. In fact one can think of a scenario when all such $u_{Q'}$'s remain close to zero until t_0 with a sequence of $u_{Q'}$'s growing faster and faster past t_0 (in such scenario (3-31) holds but not (3-32)). We verify in Step 1 below that such a scenario does not happen (i.e., that such Q exists) as long as t_0 lies inside (a, b).[3]

We now let $t_1 \in (0, t_0)$ be the last time such that $u_Q(t_1) = \tfrac{1}{2} 2^{-j\rho(Q)/2}$. Then

$$u_Q(t) \in \big[\tfrac{1}{2}2^{-j\rho(Q)/2}, 2^{-j\rho(Q)/2}\big] \qquad \text{for } t \in [t_1, t_0]. \tag{3-33}$$

The main estimate (3-14) gives

$$\begin{aligned} 2^{-j\rho(Q)} &= \tfrac{4}{3}(u_Q(t_0)^2 - u_Q(t_1)^2) \\ &\le -c\, 2^{2\alpha j}\int_{t_1}^{t_0} u_Q^2 + c\int_{t_1}^{t_0} u_Q\Big(2^j u_{3Q/2,j\pm 2}\sum_{\theta j \le k \le j-5} 2^{3k/2} u_{Q_k} + 2^{5j/2} u_{3Q/2,j\pm 4}^2 + 2^{3j/2}\sum_{k \ge j+1} 2^k u_{3Q/2,k}^2\Big) \\ &\qquad + c\int_{t_1}^{t_0} 2^{2\alpha j} 2^{-j\varepsilon} u_{3Q/2,j\pm 2}^2 + e(j), \end{aligned} \tag{3-34}$$

[3] This is the localisation issue that we referred to in the Introduction. This issue was ignored in [Katz and Pavlović 2002].

where we omitted the time argument in our notation. Note that we can write

$$e(j) \le c\, 2^{-20j}$$

(recall the beginning of Section 2 for the definition of $e(j)$, the j-negligible error), so that it can be ignored (i.e., it can be absorbed into the left-hand side for sufficiently large j). We will estimate the terms appearing on the right-hand side of (3-34) in Steps 2–4 below, and we will conclude the proof in Step 5.

Step 1: We verify (3-32).

Let $m \in \mathbb{N}$. By definition of t_0 there exists $\tau \in (t_0, t_0+1/m)$ and a j-cube Q such that $u_Q(\tau) \ge c\, 2^{-j\rho(Q)/2}$. We claim that (3-32) holds for such Q if m is taken sufficiently large. Indeed, if it does not, then $2^{j\rho(Q)/2} u_Q(t_0) \le 1$ for each m, and so

$$c - 1 \le 2^{j\rho(Q)/2}(u_Q(\tau) - u_Q(t_0)) \le 2^{11j/2} \|\phi_Q^2 P_j(u(\tau) - u(t_0))\| \le c\|u(\tau) - u(t_0)\|^2_{H^6(\mathbb{R}^3)}$$

for all m, uniformly in j, and so continuity of u in time (on (a, b)) with values in H^6 gives a contradiction for sufficiently large m. (Note that, for simplicity, we have omitted the dependence of τ and Q on m in the notation above.)

Step 2: We observe that $\delta(Q) \ge 11$, so that in particular

$$\rho(Q) \ge 5 - 4\alpha + \varepsilon. \tag{3-35}$$

In order to see this, note that if $\delta(Q) \le 10$ then Q_{j-10} touches $\partial(r Q_{j_1}(x))$. Thus (3-30) implies that Q_{k-10} is good for every $k \in [\theta j, j]$, since

$$k - 10 \ge \theta j - 10 \ge \theta j_1 - 10 \ge j_0$$

by our choice (3-29) of j_1. Hence Theorem 3.3 gives that

$$2u_Q(t_0) < 2^{-j(5-4\alpha+\varepsilon)/2} \le 2^{-j(5-4\alpha+\varepsilon\delta(Q)/10)/2} = 2^{-j\rho(Q)/2},$$

which contradicts (3-32).

Step 3: We show that

$$\begin{aligned} u_{Q_k}(t) &\le c\, 2^{-k(5-4\alpha+\varepsilon)/2}, & k &\in [\theta j, j-5], \\ u_{3Q/2,k}(t) &\le \begin{cases} c\, 2^{-j(\rho(Q)-2\varepsilon/5)/2}, & k \in [j-4, \dots, j+100/\varepsilon], \\ c\, 2^{-3j} 2^{-k(9-4\alpha)/2}, & k \ge j + 100/\varepsilon, \end{cases} \end{aligned} \tag{3-36}$$

for $t \in (t_1, t_0)$.

Case 1: $k \in [\theta j, j-5]$. If $\delta(Q_k) \ge 11$ then in particular $Q_k \subset r Q_{j_1}(x)$ and $\rho(Q_k) \ge 5 - 4\alpha + \varepsilon$, and so the claim follows from (3-31). If $\delta(Q_k) \le 10$ then Q_{l-10} is good for every $l \in [\theta k, k]$ due to (3-30), since

$$l - 10 \ge \theta k - 10 \ge \theta^2 j - 10 \ge \theta^2 j_1 - 10 \ge j_0. \tag{3-37}$$

Therefore the claim follows from Theorem 3.3.

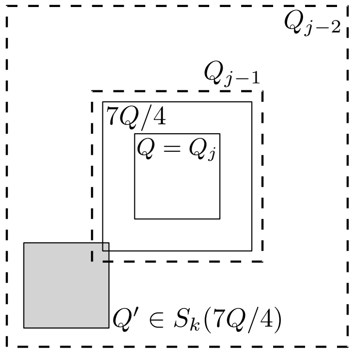

Figure 3. An illustration of (3-40) - note that each $Q' \in S_k\big(\frac{7}{4}Q\big)$ is of the same size as Q (as in the illustration) or smaller (as $k \geq j$).

Case 2: $k \in [j-4, \dots, j+100/\varepsilon)$. Then

$$\delta(Q_k) = \delta(Q) + k - j \geq \delta(Q) - 4 \geq 7, \tag{3-38}$$

where we used Step 2 in the last inequality. Hence $Q_k \subset r\,Q_{j_1}(x)$ and

$$\rho(Q_k) \geq \rho(Q) - \tfrac{2}{5}\varepsilon.$$

Thus since for $k \in [j-4, j-1]$ we have $\frac{3}{2}Q \subset Q_k$, (3-31) gives

$$u_{3Q/2,k} \leq 2^{-k\rho(Q_k)/2} \leq 2^{-k(\rho(Q)-2\varepsilon/5)/2} \leq c\,2^{-j(\rho(Q)-2\varepsilon/5)/2},$$

as required. If $k \geq j$ we note that

$$u_{3Q/2,k} \leq \sum_{Q' \in S_k(7Q/4)} u_{Q'}, \tag{3-39}$$

where $S_k\big(\frac{7}{4}Q\big)$ denotes a cover of $\frac{7}{4}Q$ by k-cubes with $\#S_k\big(\frac{7}{4}Q\big) \leq c\,2^{3(k-j)(1-\varepsilon)}$ (recall the beginning of Section 3). Since

$$Q'_j = 2^{-(j-k)(1-\varepsilon)}Q' \subset Q_{j-2} \quad \text{for every } Q' \in S_k\big(\tfrac{7}{4}Q\big), \tag{3-40}$$

see Figure 3, we obtain

$$\delta(Q') = \delta(Q'_j) + k - j \geq \delta(Q_{j-2}) = \delta(Q) - 2, \tag{3-41}$$

and so $\rho(Q') \geq \rho(Q) - \frac{1}{5}\varepsilon$. Therefore (3-31) gives

$$u_{Q'} \leq 2^{-k\rho(Q')/2} \leq 2^{-k(\rho(Q)-\varepsilon/5)/2} \leq c\,2^{-j(\rho(Q)-2\varepsilon/5)/2},$$

and since $\#S_k\big(\frac{7}{4}Q\big) \leq c\,2^{300(1-\varepsilon)/\varepsilon} = c$ (recall our constants may depend on ε) the claim follows by applying (3-39) above.

Case 3: $k \geq j + 100/\varepsilon$. For such k we improve (3-41) by writing

$$\delta(Q') = \delta(Q'_j) + k - j \geq \delta(Q_{j-2}) + 100/\varepsilon = \delta(Q) + 100/\varepsilon - 2 > 100/\varepsilon \tag{3-42}$$

for any $Q' \in S_k\left(\frac{7}{4}Q\right)$ where we used Step 2 in the last inequality. This gives $\rho(Q') = 15 - 4\alpha$. Thus using (3-39) and the estimate $\# S_k\left(\frac{7}{4}Q\right) \le c\, 2^{3(k-j)(1-\varepsilon)} \le c\, 2^{3(k-j)}$ we arrive at

$$u_{3Q/2,k} \le \sum_{Q' \in S_k(7Q/4)} u_{Q'} \le \sum_{Q' \in S_k(7Q/4)} 2^{-k\rho(Q')/2} \le c\, 2^{3(k-j)} 2^{-k\rho(Q')/2} = c\, 2^{-3j} 2^{-k(9-4\alpha)/2},$$

as required.

Step 4: We use the previous step to estimate the terms appearing on the right-hand side of the main estimate (3-34). Namely we show that

$$\begin{aligned} \sum_{\theta j \le k \le j-5} 2^{3k/2} u_{3Q/2,k} &\le c\, 2^{3j/2} 2^{-j(5-4\alpha)/2} 2^{-j\varepsilon/2}, \\ u^2_{3Q/2,j\pm 4} &\le c\, 2^{-j\rho(Q)/2} 2^{-j(5-4\alpha)/2} 2^{-j\varepsilon/10}, \\ \sum_{k \ge j+1} 2^k u^2_{3Q/2,k} &\le c\, 2^j 2^{-j\rho(Q)/2} 2^{-j(5-4\alpha)/2} 2^{-j\varepsilon/10}. \end{aligned} \tag{3-43}$$

We note that, although the terms appearing on the right-hand side might look complicated, we write them in this form to articulate their roles. As for the factors $2^{3j/2}$ or 2^j, these are "bad factors" which, together with the corresponding factor in the main estimate (3-34), give $2^{5j/2}$. This should be compared against the factor $2^{2\alpha j}$ which is a "good factor" given by the dissipation (i.e., by the first term on the right-hand side of (3-34), which comes with a minus). This brings us to the factors of the form $2^{-j(5-4\alpha)}$ whose role is exactly to balance the "bad factor" against the "good factors".

As for the factors $2^{-j\rho(Q)/2}$, we point out that together with the corresponding factor u_Q (which is bounded above and below by $2^{-j\rho(Q)/2}$ due to (3-33)) appearing in the basic estimate, one obtains $2^{-j\rho(Q)}$ as the common factor of all terms in (3-34).

Finally, the role of any factor involving ε is to make sure that the balance falls in our favour, namely that the resulting constant at all terms on the right-hand side of (3-34) (except for the first term), is smaller than the constant at the first term (the dissipation term). Writing the estimates in the form (3-43) also exposes the value of $5-4\alpha$, which is our desired bound on the Hausdorff dimension.

We now briefly verify (3-43). The first two of them follow from Step 3 by a simple calculation,

$$\sum_{\theta j \le k \le j-5} 2^{3k/2} u_{3Q/2,k} \le c \sum_{\theta j \le k \le j-5} 2^{-k(2-4\alpha+\varepsilon)/2} \le c\, 2^{-j(2-4\alpha+\varepsilon)/2} \tag{3-44}$$

and

$$u^2_{3Q/2,j\pm 4} \le c\, 2^{-j(\rho(Q)-2\varepsilon/5)} = c\, 2^{-j\rho(Q)/2} 2^{-j(\rho(Q)-4\varepsilon/5)/2} \le c\, 2^{-j\rho(Q)/2} 2^{-j(5-4\alpha)/2} 2^{-j\varepsilon/10},$$

as required, where we used (3-35) in the last inequality. As for the third estimate in (3-43) we write

$$\sum_{k \ge j+1} = \sum_{j+1 \le k \le j+100/\varepsilon} + \sum_{k > j+100/\varepsilon},$$

and estimate each of the two sums separately,

$$\sum_{j+1 \le k \le j+100/\varepsilon} 2^k u^2_{3Q/2,k} \le c\, 2^j 2^{-j(\rho(Q)-2\varepsilon/5)} \le c\, 2^j 2^{-j\rho(Q)/2} 2^{-j(5-4\alpha+\varepsilon/5)/2}$$

(recall that c might depend on ε), where we used (3-35) in the last inequality, and

$$\sum_{k>j+100/\varepsilon} 2^k u_{3Q/2,k}^2 \le c\, 2^{-3j} \sum_{k>j+100/\varepsilon} 2^{-k(8-4\alpha)} \le c\, 2^{-j(11-4\alpha)} \le c\, 2^j 2^{-j\rho(Q)/2} 2^{-j(5-4\alpha+\varepsilon/5)/2},$$

where we used the inequality $11-4\alpha \ge -1+\frac{1}{2}\rho(Q)+\frac{1}{2}(5-4\alpha)+\frac{1}{10}\varepsilon$ (a trivial consequence of the fact that $\rho(Q)\le 5-4\alpha+10$) in the last inequality.

Step 5: We conclude the proof.

Applying the estimates from the previous step into the main estimate (3-34) and recalling that $u_{3Q/2,j\pm 2}^2 \le c\, 2^{-j(\rho(Q)-2\varepsilon/5)}$ (from Step 3) we obtain

$$\begin{aligned}
&2^{-j\rho(Q)}\\
&\le -c\, 2^{2\alpha j}\int_{t_1}^{t_0} u_Q^2 + c\int_{t_1}^{t_0} u_Q\big(2^j 2^{-j(\rho(Q)-2\varepsilon/5)/2} 2^{3j/2} 2^{-j(5-4\alpha)/2} 2^{-j\varepsilon/2} + 2^{5j/2} 2^{-j\rho(Q)/2} 2^{-j(5-4\alpha)/2} 2^{-j\varepsilon/10}\\
&\qquad\qquad + 2^{3j/2} 2^j 2^{-j\rho(Q)/2} 2^{-j(5-4\alpha)/2} 2^{-j\varepsilon/10}\big)\\
&\quad + 2^{2\alpha j} 2^{-j\varepsilon}\int_{t_1}^{t_0} 2^{-j(\rho(Q)-2\varepsilon/5)}\\
&= -c\, 2^{2\alpha j}\int_{t_1}^{t_0} u_Q^2 + c\, 2^{2\alpha j}\int_{t_1}^{t_0} u_Q (2^{-j\rho(Q)/2}(2^{-3j\varepsilon/10} + 2^{-j\varepsilon/10} + 2^{-j\varepsilon/10})) + c\, 2^{2\alpha j} 2^{-3j\varepsilon/5}\int_{t_1}^{t_0} 2^{-j\rho(Q)}\\
&\le -c\, 2^{j(2\alpha-\rho(Q))}(t_0-t_1)(1-c\, 2^{-j\varepsilon/10}),
\end{aligned}$$

where we used the lower bound $u_Q \ge \frac{1}{2}\, 2^{-j\rho(Q)/2}$ (see (3-33)) in the last line. Therefore if j_0 is sufficiently large so that

$$1 - c\, 2^{-j_1\varepsilon/10} > 0$$

(where c is the last constant appearing in the calculation above; recall also that j_1 is given by (3-29)), we obtain

$$1 \le 0,$$

a contradiction. □

Corollary 3.8. *Given $x \notin E$ and an interval of regularity (a_i, b_i) there exists an open neighbourhood U of x such that*

$$\|u(t)\|_{L^\infty(U)} \text{ remains bounded for } t \in \left(\tfrac{1}{2}(a_i+b_i), b_i\right).$$

Proof. We fix an interval of regularity. By Theorem 3.7 there exists j_1 and $r \in (0, 2^{-10})$ such that

$$u_Q(t) \le 2^{-j\rho(Q)/2}$$

for all $t \in [0, T)$ and all j-cubes $Q \subset r Q_{j_1}(x)$. Let $j_2 \in \mathbb{N}$ be the smallest number such that $\delta(Q) \ge 100/\varepsilon$ for every j-cube $Q \subset Q_{j_2}(x)$. (Note that the last condition implies also that $j \ge j_2$.) Then $\rho(Q) \ge 10$ for any such j-cube Q and so $u_Q \le c\, 2^{-5j}$. We let

$$U := Q_{j_2+2}(x).$$

To show that $\|u(t)\|_{L^\infty(U)}$ remains bounded, we note that the localised Bernstein inequality (2-21) gives

$$\|\phi_Q P_j u\|_\infty \le c\, 2^{3j/2} u_Q + e(j) \le c\, 2^{-7j/2}$$

for every j-cube $Q \in S_j(U)$ with $j \geq j_2 + 2$. Hence

$$\|P_j u\|_{L^\infty(U)} \leq \sum_{Q \in S_j(U)} \|\phi_Q P_j u\|_\infty \leq c\, 2^{3(1-\varepsilon)(j-(j_2+2))} 2^{-7j/2} = c_{j_2} 2^{-j/2}$$

for such j and so

$$\begin{aligned}\|u\|_{L^\infty(U)} &\leq \|P_{\leq j_2+1} u\|_\infty + \sum_{j \geq j_2+2} \|P_j u\|_{L^\infty(U)} \\ &\leq c\, 2^{3j_2/2} \|P_{\leq j_2+1} u\| + c_{j_2} \sum_{j \geq j_2+2} 2^{-j/2} \leq c_{j_2},\end{aligned}$$

as required, where we used the Bernstein inequality (2-12) in the second inequality. □

3F. ***Regularity for $\alpha > \frac{5}{4}$.*** Here we briefly verify Corollary 1.3. Letting $\varepsilon \in (0, 4\alpha - 5)$ we see that any j-cube ($j \geq 0$) satisfies

$$u_Q(t) \leq c \leq c\, 2^{-j(5-4\alpha+\varepsilon)}$$

for all $t \geq 0$. Thus any closed and sufficiently small surface $\mathcal{S} \subset \mathbb{R}^3$ can be used as a barrier, and Theorem 3.7 (with $\partial(r Q_{j_1}(x))$ replaced by $\mathcal{S}$) gives that $u_Q(t) < 2^{-j\rho(Q)/2}$ for all j-cubes Q located inside $\mathcal{S}$ and all $t \geq 0$ (provided u_0 is sufficiently smooth). Furthermore j_2 (from the proof of Corollary 3.8) can be chosen independently of x (i.e., depending only on how small $\mathcal{S}$ is), and consequently Corollary 3.8 gives boundedness of $\|u(t)\|_\infty$ in $t > 0$.

4. The box-counting dimension

Here we prove Theorem 1.2; namely that $d_B(S^{(k)}) \leq \frac{1}{3}(-16\alpha^2 + 16\alpha + 5)$, where $S^{(k)} := \bigcup_{i \leq k} S_i$ (recall (1-7)).

A bound on $d_B(S^{(k)})$ can in fact be obtained by examining the proof of Theorem 3.7 above. Namely, observing that the only consequence of $x \notin E$ that we used in its proof was that

$$x \notin Q \quad \text{for any } Q \in \mathcal{B}_k, \quad k \in [\theta^2 j_1 - 10, j_1], \tag{4-1}$$

where j_1 is taken sufficiently large. In fact, this allowed us to deduce that for a given j-cube $Q \subset r Q_{j_1}(x)$ the cube $Q_k = 2^{(j-k)(1-\varepsilon)} Q$ is good for such k's (take $j_0 := \lfloor \theta^2 j_1 - 10 \rfloor$ and recall (3-26), (3-27) and (3-30)). This, in light of Theorem 3.3, gave us the "slightly more than critical" decay, which in turn enabled us to deduce better decay for cubes located further inside the barrier $r Q_{j_1}(x)$. Corollary 3.8 then deduced that $x \notin S$.

Using (4-1) we see that for sufficiently large j

$$\bigcup_{k \in \{\lfloor \theta^2 j - 10 \rfloor, \dots, j\}} \bigcup_{Q \in \mathcal{B}_k} Q$$

contains the singular set in space at a given blow-up time. Thus, covering each of the covers $\mathcal{B}_k$ ($k \in \{\lfloor \theta^2 j - 10 \rfloor, \dots, j\}$) by at most

$$c\, 2^{3(j-k)(1-\varepsilon)} \# \mathcal{B}_k \leq c\, 2^{3(j-k)(1-\varepsilon)} 2^{k(5-4\alpha+\varepsilon)} = c\, 2^{3j(1-\varepsilon)} 2^{k(2-4\alpha+2\varepsilon)}$$

j-cubes we obtain a cover of the singular set by at most

$$
\begin{aligned}
c \sum_{k=\lfloor \theta^2 j-10\rfloor}^{j} 2^{3j(1-\varepsilon)} 2^{k(2-4\alpha+2\varepsilon)} &\le c\, 2^{j(3-3\varepsilon+\theta^2(2-4\alpha+2\varepsilon))} \\
&= c\, 2^{j(-64\alpha^3+96\alpha^2(1+\varepsilon)-48\alpha(1+\varepsilon)^2+35+8\varepsilon^3+8\varepsilon^2-3\varepsilon)/9}
\end{aligned} \tag{4-2}
$$

j-cubes, where we substituted $\theta = \frac{2}{3}(2\alpha - 1 - \varepsilon)$ (recall (3-3)) in the last line. In other words $N(S^{(m)}, r)$, the minimal number of r-balls required to cover $S^{(m)}$ (recall the definition (1-8) of the box-counting dimension), satisfies

$$
N(S^{(m)}, r) \le c\, r^{(-64\alpha^3+96\alpha^2(1+\varepsilon)-48\alpha(1+\varepsilon)^2+35+8\varepsilon^3+8\varepsilon^2-3\varepsilon)/9(1-\varepsilon)} \tag{4-3}
$$

for sufficiently small r. This gives that

$$
d_B(S^{(m)}) \le \tfrac{1}{9}(-64\alpha^3 + 96\alpha^2 - 48\alpha + 35) \tag{4-4}
$$

for every $m \in \mathbb{N}$. As noted in the Introduction, we point out that the required smallness of r for (4-3) to hold depends on the interval of regularity (a_i, b_i). This is the reason why we only estimate $d_B(S^{(m)})$, rather than $d_B(S)$.

In what follows we present a sharper argument that allows one to get rid of one of θ's in the first line of (4-2) to yield the following.

Proposition 4.1. *Given the interval of regularity (a_i, b_i) the set*

$$
\bigcup_{k\in\{\lfloor \theta j-10\rfloor,\dots,j\}} \; \bigcup_{Q\in\mathcal{B}_k} Q
$$

covers the singular set in space at time b_i if j is sufficiently large.

Assuming this proposition and letting $\mathcal{C}_j$ be a j-cover of all elements of $\mathcal{B}_k$ for $k = \lfloor \theta j - 10\rfloor, \dots, j$, we obtain a j-cover of the singular set with

$$
\#\mathcal{C}_j \le c \sum_{k=\lfloor \theta j-10\rfloor}^{j} 2^{3(j-k)(1-\varepsilon)} \#\mathcal{B}_k \le c\, 2^{j(3-3\varepsilon+\theta(2-4\alpha+2\varepsilon))} = c\, 2^{j(-16\alpha^2+16\alpha(1+\varepsilon)+5-17\varepsilon-4\varepsilon^2)/3},
$$

which shows that $d_B(S^{(m)}) \le \frac{1}{3}(-16\alpha^2 + 16\alpha + 5)$ for all $m \in \mathbb{N}$, by an argument analogous to that above. This is sharper than (4-4), and it proves Theorem 1.2. We note that if one was able to get rid of the other θ in (4-2), then one would obtain $d_B(S) \le 5 - 4\alpha$, i.e., the same bound as for $d_H(S)$.

Before proceeding to the proof of Proposition 4.1, we comment on the main idea of Proposition 4.1 in an informal way.

Recall (3-37) that for each $k \in [\theta j, j-5]$ we needed Q_{l-10} to be good for $l \in [\theta k, k]$, and deduced from the "ε-better than critical" decay for u_{Q_k} (in Case 1 of Step 3 of the proof of Theorem 3.7, by using Theorem 3.3), which we have then plugged into the sum of the low modes of the main estimate (3-34) (in (3-44) above). However, looking closely at this term of the main estimate,

$$
2^j \int_{t_1}^{t_0} u_Q u_{3Q/2,\,j\pm 2} \sum_{\theta j \le k \le j-5} 2^{3k/2} u_{Q_k},
$$

we observe that it has a structure similar to the definition of a good cube (3-15). Indeed, ignoring u_Q and $u_{3Q/2,j\pm 2}$ for a moment we see that we could use (3-15) to estimate it. If that were possible, we would only need to require that Q_k (or rather Q_{k-10}) is good for $k\in[\theta j,\ j-5]$, and so we would end up with a saving of one θ. The only problem is that (3-15) is concerned with the time integral of a squared function, rather than the function itself, and so, applying the Cauchy–Schwarz inequality in the time integral we would obtain an additional factor of $(t_0-t_1)^{-1/2}$; see the last term in (4-8) below. It turns out that this additional factor can be taken care of by absorbing a part of this term by the left-hand side (as in (4-9) below).

Proof of Proposition 4.1.. We will show that if j_1 is sufficiently large then every x outside of $\mathcal{C}_{j_1}$ is a regular point in the given interval of regularity (a,b). We set

$$j_0 := \lfloor \theta j_1 - 10\rfloor. \tag{4-5}$$

As in Theorem 3.7 we show that, for sufficiently large $j_1 = j_1(c_i)$,

$$u_Q(t) < c_i 2^{-j\rho(Q)} \tag{4-6}$$

for every $x\notin\bigcup_{Q\in\mathcal{C}_{j_1}} Q$, where c_i depends on the interval of regularity (a_i,b_i). In fact, we can copy the entire proof of Theorem 3.7, except for Step 4, where we replace the estimate on the low modes (i.e., the first inequality in (3-43)) by

$$\sum_{k\in[\theta j,j-5]} 2^{3k/2}\int_{t_1}^{t_0} u_{Q_k} \le c(t_0-t_1)2^{-j(2-4\alpha+\varepsilon)/2} + c(t_0-t_1)^{1/2}2^{-j(2-2\alpha+\varepsilon)/2}, \tag{4-7}$$

which we prove below. Given (4-7), we can plug it, together with the remaining two inequalities in (3-43), into the main estimate (3-34) (just as we did in Step 5 of the proof of Theorem 3.7 above) to yield

$$\begin{aligned}
2^{-j\rho(Q)} &= c(u_Q(t_0)^2 - u_Q(t_1)^2)\\
&\le -c\,2^{2\alpha j}\int_{t_1}^{t_0} u_Q^2 + c\int_{t_1}^{t_0} u_Q\Big(2^j u_{3Q/2,j\pm 2}\sum_{\theta j\le k\le j-5} 2^{3k/2}u_{Q_k} + 2^{5j/2}u_{3Q/2,j\pm 4}^2 + 2^{3j/2}\sum_{k\ge j+1} 2^k u_{3Q/2,k}^2\Big)\\
&\qquad\qquad + 2^{2\alpha j}2^{-j\varepsilon}\int_{t_1}^{t_0} u_{3Q/2,j\pm 2}^2 + e(j)\\
&\le -c\,2^{2\alpha j}(t_0-t_1)2^{-j\rho(Q)} + c\,2^{-j\rho(Q)}2^{j(1+\varepsilon/5)}\int_{t_1}^{t_0}\sum_{\theta j\le k\le j-5} 2^{3k/2}u_{Q_k}\\
&\qquad\qquad + c(t_0-t_1)2^{2\alpha j}2^{-j\rho(Q)}(2^{j\varepsilon/10}+2^{j\varepsilon/10}+2^{3j\varepsilon/5})\\
&\le 2^{2\alpha j}(t_0-t_1)2^{-j\rho(Q)}(-c+c\,2^{-j\varepsilon/10}) + c(t_0-t_1)^{1/2}2^{-j\rho(Q)}2^{\alpha j}2^{-3j\varepsilon/10},
\end{aligned}\tag{4-8}$$

where, in the last step, we applied (4-7) to estimate the low modes. At this point we obtain the same inequality as before (i.e., as in Step 5 of the proof of Theorem 3.7), except for the last term, which can be estimated using Young's inequality $ab\le\frac12 a^2+\frac12 b^2$ to give

$$\tfrac12\, 2^{-j\rho(Q)} + c\,2^{2\alpha j}(t_0-t_1)2^{-j\rho(Q)}2^{-3j\varepsilon/5}. \tag{4-9}$$

Absorbing the first term above on the left-hand side we obtain

$$1 \leq 2^{2\alpha j}(t_0 - t_1)(-c + c\,2^{-j\varepsilon/10}),$$

which gives a contradiction for sufficiently large j.

It remains to verify (4-7). To this end, if $\delta(Q_k) \geq 11$ then, as before, we can use the fact that the claim (4-6) remains valid until t_0 to obtain that

$$\sum_{\substack{k\in[\theta j, j-5]\\ \delta(Q_k)\geq 11}} 2^{3k/2}\int_{t_1}^{t_0} u_{Q_k} \leq c(t_0 - t_1) \sum_{\substack{k\in[\theta j, j-5]\\ \delta(Q_k)\geq 11}} 2^{3k/2}2^{-k\rho(Q_k)/2} \leq c(t_0 - t_1)2^{-j(2-4\alpha+\varepsilon)/2},$$

where we used the fact that $\rho(Q_k) \geq 5 - 4\alpha + \varepsilon$ in the last inequality.

If $\delta(Q_k) \leq 10$ then Q_{k-10} intersects the barrier $\partial(r Q_{j_1}(x))$, and so it is good as $k - 10 \geq \theta j - 10 \geq j_0$ (recall (3-30) and (4-5)). Thus since $\phi_{Q_k} \leq 1_{Q_{k-10}}$ (recall (2-19)) the definition (3-15) of a good cube gives

$$\int_{t_1}^{t_0} u_{Q_k}^2 \leq \int_{t_1}^{t_0}\int_{Q_{k-10}} |P_k u|^2 \leq c\,2^{-k(5-2\alpha+\varepsilon)}.$$

Hence

$$\begin{aligned}\sum_{k\in[\theta j, j-5],\delta(Q_k)\leq 10} 2^{3k/2}\int_{t_1}^{t_0} u_{Q_k} &\leq (t_0 - t_1)^{1/2} \sum_{\theta j,\dots,j-5,\delta(Q_k)\leq 10} 2^{3k/2}\Big(\int_{t_1}^{t_0} u_{Q_k}^2\Big)^{1/2}\\ &\leq c(t_0 - t_1)^{1/2}\sum_{k\leq j-5} 2^{-k(2-2\alpha+\varepsilon)/2} = c(t_0 - t_1)^{1/2}2^{-j(2-2\alpha+\varepsilon)/2},\end{aligned}$$

as required. □

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THE PESKIN PROBLEM WITH VISCOSITY CONTRAST

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The Peskin problem models the dynamics of a closed elastic filament immersed in an incompressible fluid. We consider the case when the inner and outer viscosities are possibly different. This viscosity contrast adds further nonlocal effects to the system through the implicit nonlocal relation between the net force and the free interface. We prove the first global well-posedness result for the Peskin problem in this setting. The result applies for *medium-size* initial interfaces in critical spaces and shows instant analytic smoothing. We carefully calculate the medium-size constraint on the initial data. These results are new even without viscosity contrast.

1. Introduction

Fluid structure interaction (FSI) problems in which an elastic structure interacts with a surrounding fluid are found in many areas of science and engineering. Many numerical algorithms have been developed for such problems, and the scientific computing of FSI problems continues to be a very active area of research [Li and Ito 2006; Peskin 2002; Tryggvason et al. 2001; Richter 2017]. The *Peskin problem*, considered in this paper, is arguably one of the simplest FSI problems and has been used extensively in physical modeling as well as in the development of numerical algorithms as a prototypical test problem.

1A. ***Formulation.*** Consider the following fluid problem in $\mathbb{R}^2$. A closed elastic string Γ encloses a simply connected bounded domain $\Omega_1 \subset \mathbb{R}^2$ filled with a Stokes fluid with viscosity μ_1. The outside region $\Omega_2 = \mathbb{R}^2 \backslash (\Omega_1 \cup \Gamma)$ is filled with a Stokes fluid of viscosity μ_2. The equations satisfied are

$$\mu_1 \Delta \boldsymbol{u} - \nabla p = \boldsymbol{0} \quad \text{in } \Omega_1, \tag{1-1}$$

$$\mu_2 \Delta \boldsymbol{u} - \nabla p = \boldsymbol{0} \quad \text{in } \Omega_2, \tag{1-2}$$

$$\nabla \cdot \boldsymbol{u} = 0 \quad \text{in } \mathbb{R}^2 \backslash \Gamma. \tag{1-3}$$

Here $\boldsymbol{u}$ is the velocity field and p is the pressure.

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We must specify the interface conditions at Γ. Parametrize Γ by the material or Lagrangian coordinate $\theta \in \mathbb{S} = \mathbb{R}/(2\pi\mathbb{Z})$, and let $\boldsymbol{\mathcal{X}}(\theta, t)$ denote the coordinate position of Γ at time t. The parametrization is in the counterclockwise direction, so that the interior region Ω_1 is on the left-hand side of the tangent vector $\partial\boldsymbol{\mathcal{X}}/\partial\theta$. For any quantity w defined on Ω_1 and Ω_2, we set

$$[w] = w|_{\Gamma_1} - w|_{\Gamma_2},$$

where $w|_{\Gamma_1}$ and $w|_{\Gamma_2}$ are the trace values of w at Γ evaluated from Ω_1 (interior) and Ω_2 (exterior) sides of Γ. Let $\boldsymbol{n}$ be the outward-pointing unit normal vector on Γ:

$$\boldsymbol{n} = -\frac{\partial_\theta \boldsymbol{\mathcal{X}}^\perp}{|\partial_\theta \boldsymbol{\mathcal{X}}|}, \quad \partial_\theta \boldsymbol{\mathcal{X}} = \frac{\partial \boldsymbol{\mathcal{X}}}{\partial \theta}, \quad \partial_\theta \boldsymbol{\mathcal{X}}^\perp = \mathcal{R}\partial_\theta \boldsymbol{\mathcal{X}}, \quad \mathcal{R} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

where $\mathcal{R}$ is the $\frac{\pi}{2}$-rotation matrix. The interface conditions are

$$\frac{\partial \boldsymbol{\mathcal{X}}}{\partial t} = \boldsymbol{u}(\boldsymbol{\mathcal{X}}, t), \tag{1-4}$$

$$[\boldsymbol{u}] = 0, \tag{1-5}$$

$$[\Sigma \boldsymbol{n}] = \boldsymbol{F}_{\mathrm{el}} |\partial_\theta \boldsymbol{\mathcal{X}}|^{-1}, \quad \Sigma = \begin{cases} \mu_1(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}}) - pI & \text{in } \Omega_1, \\ \mu_2(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}}) - pI & \text{in } \Omega_2, \end{cases} \tag{1-6}$$

where I is the 2×2 identity matrix. The first condition is the no-slip boundary condition and the second is the stress balance condition, where Σ is the fluid stress and $\boldsymbol{F}_{\mathrm{el}}$ is the elastic force exerted by the string Γ. We let

$$\boldsymbol{F}_{\mathrm{el}} = k_0 \partial_\theta^2 \boldsymbol{\mathcal{X}}, \quad k_0 > 0, \tag{1-7}$$

where k_0 is the elasticity constant of the string Γ.

In the far field, $\boldsymbol{x} \to \infty$, we impose the condition that $\boldsymbol{u} \to 0$ and $p \to 0$. This completes the specification of the Peskin problem.

Let us rewrite the above problem using boundary integral equations. Given some function $\boldsymbol{F}$ defined on Γ, we express the solution to our problem as the following single-layer potential on $\mathbb{S} = [-\pi, \pi]$:

$$\boldsymbol{u}(\boldsymbol{x}, t) = \int_{\mathbb{S}} G(\boldsymbol{x} - \boldsymbol{\mathcal{X}}(\eta)) \boldsymbol{F}(\eta)\, d\eta, \tag{1-8}$$

$$G(\boldsymbol{x}) = \frac{1}{4\pi}\left(-\log|\boldsymbol{x}| I + \frac{\boldsymbol{x} \otimes \boldsymbol{x}}{|\boldsymbol{x}|^2}\right), \quad \boldsymbol{x} = (x_1, x_2)^{\mathrm{T}} \in \mathbb{R}^2, \tag{1-9}$$

where G is the stokeslet, the fundamental solution of the two-dimensional Stokes problem. Additionally for $\boldsymbol{y} = (y_1, y_2)^{\mathrm{T}} \in \mathbb{R}^2$ we use the notation

$$\boldsymbol{x} \otimes \boldsymbol{y} = \begin{bmatrix} x_1 y_1 & x_1 y_2 \\ x_2 y_1 & x_2 y_2 \end{bmatrix}.$$

We note that $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{F}$ (and other variables) depend on t, but we will often suppress this dependence to avoid cluttered notation. We note that the single-layer potential does not have a velocity jump across the interface, and thus the boundary condition (1-5) is automatically satisfied. We then have

$$\frac{\partial \boldsymbol{\mathcal{X}}}{\partial t}(\theta) = \int_{\mathbb{S}} G(\Delta \boldsymbol{\mathcal{X}}) \boldsymbol{F}(\eta)\, d\eta, \tag{1-10}$$

where we use the notation

$$\Delta \boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{X}}(\theta) - \boldsymbol{\mathcal{X}}(\eta).$$

On the other hand, the stress interface condition (1-6) is not automatically satisfied, and this will lead to an equation for $\boldsymbol{F}$. Let us compute the stress associated with the single-layer expression (1-8). The stress Σ in Ω_2 is given by

$$\Sigma_{ij}(\boldsymbol{x}) = \mu_2 \int_{\mathbb{S}} \mathcal{T}_{ijk}(\boldsymbol{x} - \boldsymbol{\mathcal{X}}(\eta)) F_k(\eta)\, d\eta,$$

with

$$\mathcal{T}_{ijk} = -\frac{1}{\pi} \frac{x_i x_j x_k}{|\boldsymbol{x}|^4}, \tag{1-11}$$

where the subscripts denote the components of the respective tensors/vectors, such as $\boldsymbol{F} = (F_1, F_2)^{\mathrm{T}}$, and the summation convention is in effect for repeated indices. We refer to Chapter 2 of [Pozrikidis 1992] for further details on the derivation of the stokeslet and the stresslet tensors. In Ω_1, the stress is given by

$$\Sigma_{ij}(\boldsymbol{x}, t) = \mu_1 \int_{\mathbb{S}} \mathcal{T}_{ijk}(\boldsymbol{x} - \boldsymbol{\mathcal{X}}(\eta)) F_k(\eta)\, d\eta.$$

Thus, the trace values of the normal stresses are given by the equations

$$\Sigma_{ij}(\boldsymbol{\mathcal{X}}(\theta)) n_j(\theta)|_{\Gamma_2} = \mu_2 \left(-\frac{1}{2} F_i |\partial_\theta \boldsymbol{\mathcal{X}}|^{-1} + \mathrm{pv}\!\int_{\mathbb{S}} \mathcal{T}_{ijk}(\Delta \boldsymbol{\mathcal{X}}) F_k(\eta) n_j(\theta)\, d\eta \right),$$

$$\Sigma_{ij}(\boldsymbol{\mathcal{X}}(\theta)) n_j(\theta)|_{\Gamma_1} = \mu_1 \left(\frac{1}{2} F_i |\partial_\theta \boldsymbol{\mathcal{X}}|^{-1} + \mathrm{pv}\!\int_{\mathbb{S}} \mathcal{T}_{ijk}(\Delta \boldsymbol{\mathcal{X}}) F_k(\eta) n_j(\theta)\, d\eta \right).$$

The stress jump condition (1-6) thus reduces to (for $i = 1, 2$)

$$F_i(\theta) + 2A_\mu \int_{\mathbb{S}} \mathcal{T}_{ijk}(\Delta \boldsymbol{\mathcal{X}}) F_k(\eta) \partial_\theta \mathcal{X}_j^{\perp}(\theta)\, d\eta = \frac{2}{\mu_1 + \mu_2} F_{\mathrm{el},i}(\theta),$$

where

$$A_\mu = \frac{\mu_2 - \mu_1}{\mu_1 + \mu_2}. \tag{1-12}$$

We define

$$\mathcal{S}_i(\boldsymbol{F}, \boldsymbol{\mathcal{X}})(\theta) = -\partial_\theta \mathcal{X}_j^{\perp}(\theta) \int_{\mathbb{S}} \mathcal{T}_{ijk}(\Delta \boldsymbol{\mathcal{X}}) F_k(\eta)\, d\eta.$$

We will frequently write it in vector notation as

$$\boldsymbol{F}(\theta) = 2A_\mu \boldsymbol{\mathcal{S}}(\boldsymbol{F}, \boldsymbol{\mathcal{X}})(\theta) + 2A_e \widetilde{\boldsymbol{F}}_{\mathrm{el}}(\theta), \tag{1-13}$$

where

$$\boldsymbol{\mathcal{S}}(\boldsymbol{F}, \boldsymbol{\mathcal{X}})(\theta) = -\partial_\theta \boldsymbol{\mathcal{X}}(\theta)^{\perp} \cdot \int_{\mathbb{S}} \mathcal{T}(\boldsymbol{\mathcal{X}}(\theta) - \boldsymbol{\mathcal{X}}(\eta)) \cdot \boldsymbol{F}(\eta)\, d\eta, \tag{1-14}$$

with

$$\widetilde{\boldsymbol{F}}_{\mathrm{el}} = \frac{1}{k_0} \boldsymbol{F}_{\mathrm{el}}, \quad A_e = \frac{k_0}{\mu_2 + \mu_1}. \tag{1-15}$$

We point out that the above boundary integral equation has a unique solution $\boldsymbol{F}$ given $\boldsymbol{F}_{\mathrm{el}}$ for sufficiently smooth $\boldsymbol{\mathcal{X}}$.

The Peskin problem thus reduces to the integral equations (1-10) and (1-13) for $\boldsymbol{\mathcal{X}}$, where G, $\mathcal{T}$, A_μ, $\boldsymbol{\mathcal{S}}$, A_e, and $\widetilde{\boldsymbol{F}}_{\mathrm{el}}$ are given by (1-9), (1-11), (1-12), (1-14), and (1-15), with $\boldsymbol{F}_{\mathrm{el}}$ given by (1-7). Note also that, when $A_\mu = 0$, i.e., $\mu_1 = \mu_2$, equation (1-13) reduces to $\boldsymbol{F} = 2A_e\boldsymbol{F}_{\mathrm{el}}$, and we may just work with the single equation (1-10).

Assuming that the stationary solutions are sufficiently smooth, it can be shown by an easy calculation that the only stationary solutions are those in which $\boldsymbol{\mathcal{X}}$ is a uniformly parametrized circle and the velocity field is $\boldsymbol{u} = 0$; see Section 5.1 of [Mori et al. 2019]. Thus, all of the equilibrium configurations of (1-10) and (1-13) are spanned by

$$\boldsymbol{e}_r(\theta) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \quad \boldsymbol{e}_t(\theta) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}, \quad \boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{1-16}$$

1B. ***Critical regularity and related results.*** A general guideline for seeking the most natural and largest class of initial data for a given problem is to identify its scaling and consider a function space that is critical (invariant) with respect to this scaling. The Peskin problem given above by (1-10) and (1-13) is invariant under dilation, and thus to make proper sense of scaling one must first fix a reference scale. Consider the scaling parameter $\lambda > 0$. The domain scales accordingly from the torus $\mathbb{S} = [-\pi, \pi]$ to $\mathbb{S}/\lambda = [-\pi/\lambda, \pi/\lambda]$. Then, we choose as the reference scale the length of uniformly parametrized circles, which we pick to be 2π. Given the additional rotation and translation invariance of the problem, let us consider the particular choice

$$\boldsymbol{X}_{*,\lambda}(\theta) = \lambda^{-1}\boldsymbol{X}_*(\lambda\theta),$$

where $\boldsymbol{X}_*(\theta) = \boldsymbol{e}_r(\theta)$. Then, the system (1-10), (1-13) is written in terms of the difference $\boldsymbol{X}(\theta, t) = \boldsymbol{\mathcal{X}}(\theta, t) - \boldsymbol{X}_*(\theta)$. One can check that the following dilation invariance holds: if $\boldsymbol{X}(\theta, t)$ is a solution, then $\boldsymbol{X}_\lambda(\theta, t) = \lambda^{-1}\boldsymbol{X}(\lambda\theta, \lambda t)$ is also a solution.

More generally, if the elastic force $\boldsymbol{F}_{\mathrm{el}}$ is given by (1-7), then (1-10) has an additional scaling invariance given by $\boldsymbol{X}_{\lambda,\tau}(\theta, t) = \tau\boldsymbol{X}(\lambda\theta, \lambda t)$ and $\boldsymbol{X}_{*,\lambda,\tau}(\theta) = \tau\boldsymbol{X}_*(\lambda\theta)$ for any $\lambda, \tau > 0$. The stress jump condition (1-13) then scales as $\boldsymbol{F}_{\lambda,\tau}(\theta, t) = \lambda^2\tau\boldsymbol{F}(\lambda\theta, \lambda t)$. This more general scaling leaves the equation invariant with τ unrelated to λ. We note however that the chord arc condition, defined below in (1-17), is only invariant under the dilation rescaling where $\tau = \lambda^{-1}$.

The analytical study of the Peskin problem was initiated in [Lin and Tong 2019; Mori et al. 2019], in which the case of equal viscosity $\mu_1 = \mu_2$ was studied. In [Lin and Tong 2019], well-posedness was established in $\boldsymbol{\mathcal{X}} \in C([0, T]; H^{5/2}(\mathbb{S}))$, $T > 0$, with initial data $\boldsymbol{\mathcal{X}}_0$ in $H^{5/2}(\mathbb{S})$, whereas in [Mori et al. 2019], the solution resides in $\boldsymbol{\mathcal{X}} \in C([0, T]; C^{1,\alpha}(\mathbb{S}))$, $\alpha > 0$, $T > 0$, with initial data $\boldsymbol{\mathcal{X}}_0$ in $h^{1,\alpha}(\mathbb{S})$, $\alpha > 0$ (this space is the completion of smooth functions in the $C^{1,\alpha}$ norm). These spaces are subcritical with respect to the above scaling. Indeed, in the L^2 Sobolev scale, $H^{3/2}(\mathbb{S})$ (or $C([0, T]; H^{3/2}(\mathbb{S}))$) is the critical space, whereas in the scale of (Hölder) continuous functions, $C^1(\mathbb{S})$ (or $C([0, T]; C^1(\mathbb{S}))$) is the critical scale. In this sense, the results in [Mori et al. 2019] are only barely subcritical. The semilinear parabolic methods [Lunardi 1995] that are used in [Mori et al. 2019] rely crucially on subcriticality, however, and do not seem to be readily extendible to the critical regularity exponent.

In this paper, we consider the Peskin problem in which the viscosities μ_1 and μ_2 are not necessarily equal. Furthermore, we establish a solution theory with initial data $\boldsymbol{\mathcal{X}}_0$ in the Wiener space $\mathcal{F}^{1,1}(\mathbb{S})$, the space of functions whose derivatives have a Fourier series that is absolutely summable (see Section 1C). This space is critical with respect to the scaling of the Peskin problem identified above.

In contrast to [Lin and Tong 2019; Mori et al. 2019], our theory is restricted to initial data that is sufficiently close to the stationary states, i.e., the uniformly parametrized circles. The papers [Lin and Tong 2019; Mori et al. 2019] establish local-in-time well-posedness in their respective function spaces subject to the following arc-chord condition on the initial data:

$$|\boldsymbol{\mathcal{X}}_0|_* \equiv \inf_{\theta,\eta\in\mathbb{S},\theta\neq\eta} \frac{|\boldsymbol{\mathcal{X}}_0(\theta)-\boldsymbol{\mathcal{X}}_0(\eta)|}{|\theta-\eta|} > 0. \tag{1-17}$$

In this sense, our results might be better compared to the results on asymptotic stability of the uniformly parametrized circle obtained in [Lin and Tong 2019; Mori et al. 2019]. The uniformly parametrized circle is proved to be exponentially stable in the above L^2 Sobolev and Hölder scales respectively, and in the latter paper, it is proved that the solution is in $C^\infty(\mathbb{S})$ for all positive time. In this paper, we improve upon this result to prove that the solution is analytic for positive time.

Local-in-time well-posedness for initial data in $\mathcal{F}^{1,1}$ merely satisfying condition (1-17) is an open question that we do not address in this paper. It is notable, however, that the arc-chord condition (1-17) is invariant under the dilation scaling described above. In [Mori et al. 2019], it is shown that, if the solution ceases to exist as t approaches $t_* < \infty$, then following must hold:

$$\lim_{t\to t_*} \varrho_\alpha(\boldsymbol{\mathcal{X}}) = \infty, \quad \varrho_\alpha(\boldsymbol{\mathcal{X}}) = \frac{\|\partial_\theta \boldsymbol{\mathcal{X}}\|_{C^\alpha}}{|\boldsymbol{\mathcal{X}}|_*} \quad \text{for any } \alpha > 0.$$

On the other hand, if $\varrho_\alpha(\boldsymbol{\mathcal{X}})$ remains bounded for all time for some $\alpha > 0$, then $\boldsymbol{\mathcal{X}}$ must converge to a uniformly parametrized circle. A similar criterion, in which the numerator of ϱ_α is replaced with a critical norm such as the $\mathcal{F}^{1,1}$ norm, would be a major improvement that should lead to a better understanding of the global-in-time dynamics of the Peskin problem.

Another extension of the Peskin problem is to consider the following elastic force in place of (1-7):

$$\boldsymbol{F}_{\mathrm{el}} = \partial_\theta \left(\mathcal{T}(|\partial_\theta \boldsymbol{\mathcal{X}}|) \frac{\partial_\theta \boldsymbol{\mathcal{X}}}{|\partial_\theta \boldsymbol{\mathcal{X}}|} \right), \tag{1-18}$$

where $\mathcal{T}(s)$ is a tension coefficient that must satisfy the structure condition $\mathcal{T} > 0$ and $d\mathcal{T}/ds > 0$. Note that the above expression is reduced to (1-7) if we take $\mathcal{T}(s) = k_0 s$, hence $k_0 = \mathcal{T}(1) = d\mathcal{T}/ds$. In the case of equal viscosity $\mu_1 = \mu_2$, a local-in-time well-posedness theory for initial data satisfying (1-17) under the more general force (1-18) is established in [Rodenberg 2018] in the Hölder scale similarly to [Mori et al. 2019], using nonlinear parabolic methods [Lunardi 1995]. It is expected that the results and methods of this paper can be extended to this more general case.

Finally, we mention [Tong 2021] in which the author considers a regularization of the Peskin problem inspired by the immersed boundary method, extending the techniques in [Lin and Tong 2019]. Such studies may form the basis for numerical analysis of the Peskin problem.

The surface tension problem, in which the interface is not elastic but only exerts a surface tension, may be the most closely related class of problems for which there are extensive analytical studies. We note that our problem is distinct from the surface tension problem; in contrast to an elastic interface considered in the Peskin problem, an interface with surface tension only does not resist stretching. This difference manifests itself in the different energy dissipation laws satisfied by the respective problems; see Section 1.1 of [Mori et al. 2019]. We refer the reader to [Prüss and Simonett 2009; 2016; Shimizu 2009] for an extensive survey of the analytical study of the surface tension problem.

There is also an increasing number of analytical studies on fluid-structure interaction problems in which an elastic structure interacts with a fluid, related to the Peskin problem considered here [Ambrose and Siegel 2017; Cheng et al. 2007; Cheng and Shkoller 2010; Liu and Ambrose 2017; Muha and Canić 2013; Plotnikov and Toland 2011; 2012; Li 2021; Boulakia et al. 2012]. The equations dealt with in these studies are typically more complicated than those of the Peskin problem; the sharp results obtained for the simpler Peskin problem should serve as a guide to what is possibly true for the more complicated model problems.

From an analytical perspective, the Muskat problem is perhaps the closest nonlinear PDE to our problem for which there is a large body of analytical studies. However, it models a very different physical setting: two immiscible and incompressible fluids in a porous media governed by Darcy's law. On the other hand, for a nearly flat interface in the presence of gravity both problems have the same symbol at the linear level. The authors of [Constantin et al. 2013] introduced the use of the Wiener algebra to obtain global well-posedness results for the Muskat problem at critical regularity. Moreover, the size restriction on the initial data was given by an explicit constant that is independent of any physical parameter. These techniques were extended in [Constantin et al. 2016; Gancedo et al. 2019a] to deal with the three-dimensional setting and the case of viscosity jumps, respectively. Other results for the Muskat problem that only require *medium-size* initial data in critical spaces (as opposed to the more standard arbitrarily small data condition) [Cameron 2019; 2020] rely on the maximum principle; these methods have thus far not been shown to be well-suited to deal with viscosity contrasts.

In this paper, we will use spaces related to the Wiener algebra that allow us to perform careful and detailed estimates on the nonlinear terms to control explicitly the size constraint on the initial data (see Figure 1). As opposed to the Muskat problem, here the problem is not only described by the shape of the interface: the parametrization corresponds to the distribution of material points, and thus it matters. As a consequence, we have to develop further techniques to deal with a system of equations (for both components of the curve). Interestingly, a careful understanding of the linear system, together with an appropriate change of framework, allows us to decouple the frequencies associated to the projection of the interface onto the space of equilibria from the others. Indeed, we overcome a major difficulty of the very recent result in [Gancedo et al. 2019b] that deals with the Muskat problem for closed interfaces (i.e., bubbles), and obtain the global existence and uniqueness result for the Peskin problem with viscosity contrast at critical regularity.

1C. ***Notation and functional spaces.*** We summarize here the notation and functional spaces that will be used throughout the paper.

For a vector $\boldsymbol{x} = (x_1, x_2)^{\mathrm{T}} \in \mathbb{C}^2$ we define

$$\boldsymbol{x}^{\perp} \stackrel{\text{def}}{=} \mathcal{R}\boldsymbol{x}, \quad \mathcal{R} \stackrel{\text{def}}{=} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \mathcal{R}^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

We denote the Euclidean norm as

$$|\boldsymbol{x}| = \sqrt{\boldsymbol{x}^T \bar{\boldsymbol{x}}} = \sqrt{|x_1|^2 + |x_2|^2},$$

and for a matrix $A = (a_{ij})_{1 \le i,j \le 2}$ we use the induced matrix norm

$$\|A\| = \sigma_{\max}(A), \tag{1-19}$$

where $\sigma_{\max}(A)$ is the largest singular value of A. For a vector such as $\boldsymbol{X}_*$ we will write $\boldsymbol{X}_{*,j}$ to be the j-th component of that vector.

We now define the periodic Hilbert transform of a function f with period $2P$ as

$$\mathcal{H}(f)(\theta) \stackrel{\text{def}}{=} \frac{1}{2P} \operatorname{pv}\int_{-P}^{P} \frac{f(\theta-\eta)}{\tan\left(\frac{\eta}{2P/\pi}\right)}\, d\eta = \frac{1}{4P} \operatorname{pv}\int_{-P}^{P} \frac{f(\theta-\eta) - f(\theta+\eta)}{\tan\left(\frac{\eta}{2P/\pi}\right)}\, d\eta. \tag{1-20}$$

Unless stated otherwise, throughout the paper we will use the case $P = \pi$. In this case, we also define the Fourier transform of a periodic function f with domain $\mathbb{S} = [-\pi, \pi]$ as

$$\mathcal{F}(f)(k) \stackrel{\text{def}}{=} \hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-ik\theta}\, d\theta, \quad k \in \mathbb{Z}.$$

Further $\mathcal{F}(\mathcal{H}(f))(k) = -i \operatorname{sgn}(k) \hat{f}(k)$. Then we define the operator Λ using the Fourier transform as $\mathcal{F}(\Lambda f)(k) \stackrel{\text{def}}{=} |k| \hat{f}(k)$. And we observe that $\mathcal{H}(\partial_\theta f)(\theta) = \Lambda f$.

We denote by $f * g$ the standard convolution of f and g. We use the iterated convolution notation

$$*^k f = \underbrace{f * \cdots * f}_{k-1 \text{ convolutions of } k \text{ copies of } f} \tag{1-21}$$

Thus for instance $*^2 f = f * f$.

We also use the following notation for the discrete delta function, $\delta_a(k)$, which is the function that is equal to 1 when $k = a$ and equal to 0 elsewhere. Throughout the paper we will further define

$$\delta_{1,-1}(k) = \delta_1(k) + \delta_{-1}(k). \tag{1-22}$$

We further define the high-frequency cut-off operator $\mathcal{J}_M$ for $M \ge 0$ by

$$\widehat{\mathcal{J}_M X}(k) \stackrel{\text{def}}{=} 1_{|k| \le M} \widehat{X}(k), \tag{1-23}$$

where 1_A is the standard indicator function of the set A, so that $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ if $x \notin A$.

For two vectors $\boldsymbol{X}(\theta), \boldsymbol{Y}(\theta) \in \mathbb{R}^2$ we define

$$\langle \boldsymbol{X}, \boldsymbol{Y} \rangle = \int_{\mathbb{S}} \boldsymbol{X}(\theta) \cdot \boldsymbol{Y}(\theta)\, d\theta. \tag{1-24}$$

Generalizing the Wiener algebra of functions with absolutely convergent Fourier series as in [Gancedo et al. 2019b], we further define the homogeneous $\dot{\mathcal{F}}^{s,1}_{\nu}$ and nonhomogeneous $\mathcal{F}^{s,1}_{\nu}$ norms as

$$\|X\|_{\dot{\mathcal{F}}^{s,1}_{\nu}} = \sum_{k\in\mathbb{Z}\setminus\{0\}} e^{\nu(t)|k|}|k|^s|\widehat{X}(k)|, \qquad s\in\mathbb{R}, \tag{1-25}$$

$$\|X\|_{\mathcal{F}^{s,1}_{\nu}} = |\widehat{X}(0)| + \sum_{k\in\mathbb{Z}\setminus\{0\}} e^{\nu(t)|k|}|k|^s|\widehat{X}(k)|, \quad s\geq 0, \tag{1-26}$$

with

$$\nu(t) = \nu_\infty \frac{t}{1+t} \geq 0, \tag{1-27}$$

and $\nu_\infty > 0$ is chosen sufficiently small. Note that $\nu(0)=0$, $\nu(t)>0$ for all $t>0$. Further $\nu'(t)\leq\nu_\infty$ and $\nu(t)\leq\nu_\infty$ are bounded for all time. When $\nu\equiv 0$, we write $\dot{\mathcal{F}}^{s,1}_0=\dot{\mathcal{F}}^{s,1}$ and $\mathcal{F}^{s,1}_0=\mathcal{F}^{s,1}$. These are the main norms that we will use in this paper. Note that when $s=1$, the $\dot{\mathcal{F}}^{s,1}$ norm is critical for the Peskin problem.

In this paper we write $A\lesssim B$ if $A\leq CB$ for some inessential constant $C>0$. We also write $A\approx B$ if both $A\lesssim B$ and $B\lesssim A$ hold. Throughout the paper, we will define

$$C_i = C_i(\|X\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}) = C_i(\|X\|_{\dot{\mathcal{F}}^{1,1}_{\nu}};\nu_\infty) > 0, \quad i=1,2,\ldots, \tag{1-28}$$

as functions that are increasing in $\|X\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}\geq 0$ and might depend on the analyticity constant ν_∞, with the properties that $C_i(\|X\|_{\dot{\mathcal{F}}^{1,1}_{\nu}})\approx 1$ for all $\nu_\infty\geq 0$ and $\lim_{\|X\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}\to 0^+} C_i(\|X\|_{\dot{\mathcal{F}}^{1,1}_{\nu}};0)=1$. We will also define

$$D_i = D_i(\|X\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}) = D_i(\|X\|_{\dot{\mathcal{F}}^{1,1}_{\nu}};A_\mu,\nu_\infty) > 0, \quad i=1,2,\ldots,$$

as functions that are increasing in $\|X\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}\geq 0$ and might depend on the physical parameter A_μ and the analyticity constant ν_∞, with the properties that $D_i(\|X\|_{\dot{\mathcal{F}}^{1,1}_{\nu}})\approx 1$ for all $A_\mu\in(-1,1)$ and all $\nu_\infty\geq 0$, and $\lim_{\|X\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}\to 0^+} D_i(\|X\|_{\dot{\mathcal{F}}^{1,1}_{\nu}};0,0)=1$.

1D. *Main results.* In this section we will state the main result of this paper: namely, that membranes whose initial interface has critical regularity (in terms of the scaling of the problem), and that are not too far from an equilibrium configuration, become instantaneously analytic and converge exponentially fast to the equilibrium. Without loss of generality, we assume that the initial area enclosed by the membrane is π. We get the result under an explicit medium-size condition for the initial deviation and for general viscosity contrast $A_\mu\in(-1,1)$.

Definition 1.1 (strong solution). Let

$$\boldsymbol{\mathcal{X}}\in C([0,T];\mathcal{F}^{1,1})\cap C^1((0,T];\mathcal{F}^{0,1})$$

and

$$|\boldsymbol{\mathcal{X}}|_*(t) = \inf_{\theta,\eta\in\mathbb{S},\theta\neq\eta} \frac{|\boldsymbol{\mathcal{X}}(\theta,t)-\boldsymbol{\mathcal{X}}(\eta,t)|}{|\theta-\eta|} > 0$$

for $0\leq t\leq T$. Then, $\boldsymbol{\mathcal{X}}$ is a *strong solution* to the viscosity-contrast Peskin problem with initial value $\boldsymbol{\mathcal{X}}(0)=\boldsymbol{\mathcal{X}}_0$ if it satisfies (1-10), (1-13) for $0<t\leq T$ and $\boldsymbol{\mathcal{X}}(t)\to\boldsymbol{\mathcal{X}}_0$ in $\mathcal{F}^{1,1}$ as $t\to 0$.

Theorem 1.2 (main result). *Let* $A_\mu \in (-1, 1)$ *and* $\boldsymbol{\mathcal{X}}_0 \in \mathcal{F}^{1,1}$. *Let* $\boldsymbol{X}_{0,c}$ *be the projection of* $\boldsymbol{\mathcal{X}}_0$ *onto the vector space spanned by* (1-16) *and* $\boldsymbol{X}_0 = \boldsymbol{\mathcal{X}}_0 - \boldsymbol{X}_{0,c}$; *thus* $\boldsymbol{X}_0$ *is mean zero and* $\widehat{\boldsymbol{\mathcal{X}}}_0(0) = \widehat{\boldsymbol{X}}_{0,c}(0)$. *Assume that initially the deviation* $\boldsymbol{X}_0$ *satisfies the medium-size condition*

$$\|\boldsymbol{X}_0\|_{\dot{\mathcal{F}}^{1,1}} < k(A_\mu), \tag{1-29}$$

where $k(A_\mu) > 0$ *is defined in* (4-9) (*see also* (4-10) *and Figure 1*), *and that the area enclosed by* $\boldsymbol{\mathcal{X}}_0$ *is* π. *Then, for any* $T > 0$, *there exists a constant* $\nu_\infty > 0$ *such that there exists a* ***unique global strong solution*** $\boldsymbol{\mathcal{X}}(t)$ *to the system* (1-10) *and* (1-13), *which lies in the space*

$$\boldsymbol{\mathcal{X}} \in C([0, T]; \mathcal{F}^{1,1}_\nu) \cap C^1((0, T]; \mathcal{F}^{0,1}_\nu) \cap L^1([0, T]; \dot{\mathcal{F}}^{2,1}_\nu),$$

with $\nu(t)$ *given by* (1-27). *In particular, it becomes instantaneously analytic. Moreover, the following energy inequality is satisfied for* $0 \le t \le T$:

$$\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}(t) + \frac{A_e}{4}\mathcal{C}\int_0^t \|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{2,1}_\nu}(\tau)\, d\tau \le \|\boldsymbol{X}_0\|_{\dot{\mathcal{F}}^{1,1}}, \tag{1-30}$$

with $\mathcal{C} = \mathcal{C}(\|\boldsymbol{X}_0\|_{\dot{\mathcal{F}}^{1,1}}, A_\mu, \nu_\infty) > 0$ *defined in* (4-12). *In addition,*

$$\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}(t) \le \|\boldsymbol{X}_0\|_{\dot{\mathcal{F}}^{1,1}} e^{-(A_e/4)\mathcal{C}t}. \tag{1-31}$$

The zero frequency $\widehat{\boldsymbol{X}}_c(0)$ *remains uniformly bounded for all times as*

$$|\widehat{\boldsymbol{X}}_c(0)| \le |\widehat{\boldsymbol{X}}_{0,c}(0)| + \widetilde{C}\|\boldsymbol{X}_0\|^2_{\dot{\mathcal{F}}^{1,1}},$$

with $\widetilde{C} = \widetilde{C}(\|\boldsymbol{X}_0\|_{\dot{\mathcal{F}}^{1,1}}, A_\mu) > 0$ *given in* (4-15), *while*

$$1 - \tfrac{1}{2}\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}^{1,1}_\nu} \le |\widehat{\boldsymbol{X}}_c(1)|^2 \le 1 + \tfrac{1}{2}\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}^{1,1}_\nu}. \tag{1-32}$$

We remark that the decay to zero of the deviation $\boldsymbol{X}$ in (1-31) together with (1-32) shows the exponentially fast convergence to a uniformly parametrized circle with the same area as the initial one.

Remark 1.3. The size of $\nu_\infty > 0$ is limited by the size of the initial data. This can be seen in (4-7). Because we are only interested in having any fixed but arbitrarily small ν_∞ to ensure analyticity, we stated the size condition as in (1-29).

Remark 1.4. In our results, we assume that both viscosities μ_1 and μ_2 are positive and hence $-1 < A_\mu < 1$. We remark on the endpoint cases of $A_\mu = \pm 1$, which formally correspond to the cases when $\mu_1 = 0$ or $\mu_2 = 0$. As can be seen from Figure 1, the allowed size of the deviation from $\boldsymbol{X}_0$ tends to 0 as $A_\mu \to \pm 1$, which may indicate potential difficulties in formulating a well-posed mathematical problem for the endpoint cases. From a physical standpoint, it does not make sense to set the viscosity to 0 in either Ω_1 or Ω_2, and thus a proper treatment of these endpoint cases will require a rethinking of the physical situation under consideration. The case $A_\mu = -1$ or $\mu_2 = 0$ may be thought of as corresponding to the problem in which a droplet of Stokesian fluid is floating in vacuum. One significant difference between this and the Peskin problem is that in the former problem a droplet in linear translation or rigid rotation experiences no external forces. The force balance and continuity equations will thus have to be

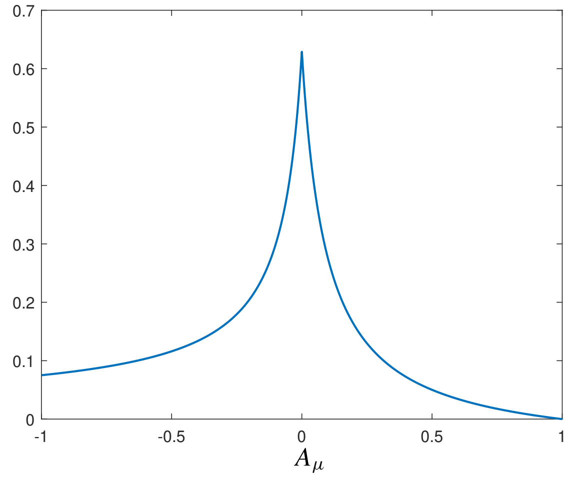

Figure 1. $10^3 k(A_\mu)$

supplemented by auxiliary conditions that assure uniqueness, after which this problem is likely to be well-posed. In the case $A_\mu = 1$ or $\mu_1 = 0$, Ω_1 might be considered to be vacuum. It is not clear if this problem is well-posed. We will not pursue these issues further.

1E. ***Outline.*** The rest of the paper is structured as follows. In Section 2, we first decompose in Section 2A the system (1-10), (1-13) into zero-order, linear, and nonlinear parts around the equilibrium configuration, and then in Section 2B we perform the linearization of the problem and show its parabolic structure. Section 2C shows how this structure leads to dissipation and in Section 2D we summarize the system of equations in its final form. Section 3 contains the crucial nonlinear estimates needed to prove Theorem 1.2. Finally, Section 4 is dedicated to the proof of Theorem 1.2 via a regularization argument and also shows the uniqueness of the solutions.

2. Linearization around the steady state

We will linearize the system (1-9)-(1-15), with $\boldsymbol{F}_{\text{el}}$ given by (1-7), around a time-dependent uniformly parametrized circle with center $(c(t), d(t))$ and radius $R(t)$:

$$\begin{aligned} \boldsymbol{X}_c(\theta, t) &= a(t)\boldsymbol{e}_r(\theta) + b(t)\boldsymbol{e}_t(\theta) + c(t)\boldsymbol{e}_1 + d(t)\boldsymbol{e}_2, \\ R^2(t) &= a^2(t) + b^2(t), \end{aligned} \tag{2-1}$$

where $a(t)$, $b(t)$, $c(t)$ and $d(t)$ are arbitrary time-dependent functions and $\boldsymbol{e}_r(\theta)$, $\boldsymbol{e}_t(\theta)$, $\boldsymbol{e}_1(\theta)$, $\boldsymbol{e}_2(\theta)$ are defined in (1-16). For notational convenience, we will suppress the time dependence of the coefficients.

2A. ***Nonlinear expansion.*** We will denote by $\boldsymbol{X}(\theta)$ the deviation from the circle $\boldsymbol{X}_c(\theta)$ as $\boldsymbol{X}(\theta) = \boldsymbol{\mathcal{X}}(\theta) - \boldsymbol{X}_c(\theta)$. We define further

$$\Delta \boldsymbol{X} \stackrel{\text{def}}{=} \boldsymbol{X}(\theta) - \boldsymbol{X}(\eta)$$

and

$$\Delta_\eta \boldsymbol{X}(\theta) \overset{\text{def}}{=} \frac{\boldsymbol{X}(\theta)-\boldsymbol{X}(\eta)}{2\sin\left(\frac{\theta-\eta}{2}\right)}. \tag{2-2}$$

In particular, we have

$$\Delta_\eta \boldsymbol{X}_c(\theta) = a\boldsymbol{e}_t\Big(\frac{\theta+\eta}{2}\Big) - b\boldsymbol{e}_r\Big(\frac{\theta+\eta}{2}\Big),$$

since

$$\Delta_\eta \boldsymbol{e}_r(\theta) = \begin{bmatrix} -\sin\left(\frac{\theta+\eta}{2}\right) \\ \cos\left(\frac{\theta+\eta}{2}\right)\end{bmatrix} = \boldsymbol{e}_t\Big(\frac{\theta+\eta}{2}\Big),$$

$$\Delta_\eta \boldsymbol{e}_t(\theta) = \begin{bmatrix} -\cos\left(\frac{\theta+\eta}{2}\right) \\ -\sin\left(\frac{\theta+\eta}{2}\right)\end{bmatrix} = -\boldsymbol{e}_r\Big(\frac{\theta+\eta}{2}\Big),$$

where we have used the trigonometric identities

$$\sin(\theta)-\sin(\eta) = 2\sin\Big(\frac{\theta-\eta}{2}\Big)\cos\Big(\frac{\theta+\eta}{2}\Big),$$

$$\cos(\theta)-\cos(\eta) = -2\sin\Big(\frac{\theta-\eta}{2}\Big)\sin\Big(\frac{\theta+\eta}{2}\Big).$$

Recalling (2-1) and using the identities

$$\partial_\theta \boldsymbol{e}_r(\theta) = \boldsymbol{e}_r(\theta)^\perp = \boldsymbol{e}_t(\theta),$$

$$\partial_\theta \boldsymbol{e}_t(\theta) = \boldsymbol{e}_t(\theta)^\perp = -\boldsymbol{e}_r(\theta),$$

one has

$$\partial_\theta \boldsymbol{X}_c(\theta) = a\boldsymbol{e}_t(\theta) - b\boldsymbol{e}_r(\theta),$$

$$\partial_\theta \boldsymbol{X}_c(\theta)^\perp = -a\boldsymbol{e}_r(\theta) - b\boldsymbol{e}_t(\theta).$$

The trigonometric identities $\cos(a+b)=\cos(a)\cos(b)-\sin(a)\sin(b)$ and $\sin(a+b)=\sin(a)\cos(b)-\cos(a)\sin(b)$ further give

$$\boldsymbol{e}_r(\theta)\cdot\boldsymbol{e}_t\Big(\frac{\theta+\eta}{2}\Big) = \sin\Big(\frac{\theta-\eta}{2}\Big), \quad \boldsymbol{e}_t(\theta)\cdot\boldsymbol{e}_t\Big(\frac{\theta+\eta}{2}\Big) = \cos\Big(\frac{\theta-\eta}{2}\Big),$$

$$\boldsymbol{e}_r(\theta)\cdot\boldsymbol{e}_r\Big(\frac{\theta+\eta}{2}\Big) = \cos\Big(\frac{\theta-\eta}{2}\Big), \quad \boldsymbol{e}_t(\theta)\cdot\boldsymbol{e}_r\Big(\frac{\theta+\eta}{2}\Big) = -\sin\Big(\frac{\theta-\eta}{2}\Big).$$

These calculations imply the following computations for a circle that will be used frequently throughout the paper:

$$\partial_\theta \boldsymbol{X}_c(\theta)^\perp\cdot\Delta_\eta\boldsymbol{X}_c(\theta) = -R^2\sin\Big(\frac{\theta-\eta}{2}\Big), \tag{2-3}$$

$$\partial_\theta \boldsymbol{X}_c(\theta)\cdot\Delta_\eta\boldsymbol{X}_c(\theta) = R^2\cos\Big(\frac{\theta-\eta}{2}\Big), \tag{2-4}$$

$$\begin{aligned}\Delta_\eta\boldsymbol{X}_c(\theta)\otimes\Delta_\eta\boldsymbol{X}_c(\theta) &= \frac{a^2}{2}\begin{bmatrix}1-\cos(\theta+\eta) & -\sin(\theta+\eta)\\ -\sin(\theta+\eta) & 1+\cos(\theta+\eta)\end{bmatrix} \\ &\quad + \frac{b^2}{2}\begin{bmatrix}1+\cos(\theta+\eta) & \sin(\theta+\eta)\\ \sin(\theta+\eta) & 1-\cos(\theta+\eta)\end{bmatrix} + ab\begin{bmatrix}\sin(\theta+\eta) & -\cos(\theta+\eta)\\ -\cos(\theta+\eta) & -\sin(\theta+\eta)\end{bmatrix}.\end{aligned} \tag{2-5}$$

The matrices in the last line above have been simplified using the identities $\sin^2(a)=(1-\cos(2a))/2$, $\cos^2(a)=(1+\cos(2a))/2$, and $\sin(2a)=2\sin(a)\cos(a)$.

Next, we perform a Taylor expansion of the nonlinear terms around the a time-dependent uniformly parametrized circle (2-1) under the assumption that $|\Delta_\eta \boldsymbol{X}(\theta)| < 1$. First, we start with the magnitude of the curve

$$|\Delta \boldsymbol{X} + \Delta \boldsymbol{X}_c|^2 = 4R^2 \sin^2\left(\frac{\theta-\eta}{2}\right)\left(1 + \frac{2}{R^2}\Delta_\eta \boldsymbol{X}_c(\theta)\cdot\Delta_\eta \boldsymbol{X}(\theta) + \frac{1}{R^2}|\Delta_\eta \boldsymbol{X}(\theta)|^2\right).$$

Recalling the expression for $G(\Delta\boldsymbol{\mathcal{X}})$ in (1-9), we expand each term as

$$\begin{aligned}\log|\Delta \boldsymbol{X} + \Delta \boldsymbol{X}_c| &= \log\left(2R\left|\sin\left(\frac{\theta-\eta}{2}\right)\right|\right) + \frac{1}{2}\log\left(1 + \frac{2}{R^2}\Delta_\eta \boldsymbol{X}_c(\theta)\cdot\Delta_\eta \boldsymbol{X}(\theta) + \frac{1}{R^2}|\Delta_\eta \boldsymbol{X}(\theta)|^2\right)\\ &= \log\left(2R\left|\sin\left(\frac{\theta-\eta}{2}\right)\right|\right) + \frac{1}{R^2}\Delta_\eta \boldsymbol{X}_c(\theta)\cdot\Delta_\eta \boldsymbol{X}(\theta) + \mathcal{R}_1(\Delta_\eta \boldsymbol{X}(\theta)),\end{aligned} \tag{2-6}$$

where

$$\mathcal{R}_1(\Delta_\eta \boldsymbol{X}(\theta)) = \sum_{n=1}^{\infty}\sum_{\substack{m=0\\ n+m\geq 2}}^{n}\binom{n}{m}\frac{(-1)^{n-1}}{2nR^{2n}}(2\Delta_\eta \boldsymbol{X}_c(\theta)\cdot\Delta_\eta \boldsymbol{X}(\theta))^{n-m}|\Delta_\eta \boldsymbol{X}(\theta)|^{2m}. \tag{2-7}$$

We expand the denominator in the second term of (1-9) as

$$\frac{1}{|\Delta \boldsymbol{X} + \Delta \boldsymbol{X}_c|^2} = \frac{1}{4R^2\sin^2\left(\frac{\theta-\eta}{2}\right)}\left(1 - \frac{2}{R^2}\Delta_\eta \boldsymbol{X}_c(\theta)\cdot\Delta_\eta \boldsymbol{X}(\theta)\right) + \frac{1}{4R^2\sin^2\left(\frac{\theta-\eta}{2}\right)}\mathcal{R}_2(\Delta_\eta \boldsymbol{X}(\theta)), \tag{2-8}$$

with the notation

$$\mathcal{R}_2(\Delta_\eta \boldsymbol{X}(\theta)) = \sum_{n=1}^{\infty}\sum_{\substack{m=0\\ n+m\geq 2}}^{n}\binom{n}{m}\frac{(-1)^{n}}{R^{2n}}(2\Delta_\eta \boldsymbol{X}_c(\theta)\cdot\Delta_\eta \boldsymbol{X}(\theta))^{n-m}|\Delta_\eta \boldsymbol{X}(\theta)|^{2m}. \tag{2-9}$$

Therefore, we can write

$$\frac{(\Delta \boldsymbol{X} + \Delta \boldsymbol{X}_c)\otimes(\Delta \boldsymbol{X} + \Delta \boldsymbol{X}_c)}{|\Delta \boldsymbol{X} + \Delta \boldsymbol{X}_c|^2} = A_0 + A_L + A_N, \tag{2-10}$$

with

$$\begin{aligned}A_0 &= \frac{1}{R^2}\Delta_\eta \boldsymbol{X}_c(\theta)\otimes\Delta_\eta \boldsymbol{X}_c(\theta),\\ A_L &= -\frac{2}{R^4}\Delta_\eta \boldsymbol{X}_c(\theta)\cdot\Delta_\eta \boldsymbol{X}(\theta)\Delta_\eta \boldsymbol{X}_c(\theta)\otimes\Delta_\eta \boldsymbol{X}_c(\theta) + \frac{1}{R^2}\Delta_\eta \boldsymbol{X}_c(\theta)\otimes\Delta_\eta \boldsymbol{X}(\theta) + \frac{1}{R^2}\Delta_\eta \boldsymbol{X}(\theta)\otimes\Delta_\eta \boldsymbol{X}_c(\theta),\end{aligned}$$

and the nonlinear term is given by

$$\begin{aligned}A_N = \frac{1}{R^2}&\Delta_\eta \boldsymbol{X}(\theta)\otimes\Delta_\eta \boldsymbol{X}(\theta)\left(1 - \frac{2}{R^2}\Delta_\eta \boldsymbol{X}_c(\theta)\cdot\Delta_\eta \boldsymbol{X}(\theta) + \mathcal{R}_2(\Delta_\eta \boldsymbol{X}(\theta))\right)\\ &+ \frac{1}{R^2}\Delta_\eta \boldsymbol{X}_c(\theta)\otimes\Delta_\eta \boldsymbol{X}(\theta)\left(-\frac{2}{R^2}\Delta_\eta \boldsymbol{X}_c(\theta)\cdot\Delta_\eta \boldsymbol{X}(\theta) + \mathcal{R}_2(\Delta_\eta \boldsymbol{X}(\theta))\right)\\ &+ \frac{1}{R^2}\Delta_\eta \boldsymbol{X}(\theta)\otimes\Delta_\eta \boldsymbol{X}_c(\theta)\left(-\frac{2}{R^2}\Delta_\eta \boldsymbol{X}_c(\theta)\cdot\Delta_\eta \boldsymbol{X}(\theta) + \mathcal{R}_2(\Delta_\eta \boldsymbol{X}(\theta))\right)\\ &+ \frac{1}{R^2}\Delta_\eta \boldsymbol{X}_c(\theta)\otimes\Delta_\eta \boldsymbol{X}_c(\theta)\mathcal{R}_2(\Delta_\eta \boldsymbol{X}(\theta)).\end{aligned}$$

Joining the expansions (2-6) and (2-10), we split $G(\Delta\boldsymbol{\mathcal{X}})$ in (1-9) into zero-order, linear, and nonlinear parts in terms of $\boldsymbol{X}$ as follows:

$$G(\Delta\boldsymbol{\mathcal{X}}) = G_0(\Delta_\eta \boldsymbol{X}_c(\theta)) + G_L(\Delta_\eta \boldsymbol{X}_c(\theta), \Delta_\eta \boldsymbol{X}(\theta)) + G_N(\Delta_\eta \boldsymbol{X}_c(\theta), \Delta_\eta \boldsymbol{X}(\theta)), \tag{2-11}$$

where

$$G_0(\Delta_\eta \boldsymbol{X}_c(\theta)) = \frac{1}{4\pi}\Big(-\log\Big(2R\Big|\sin\Big(\frac{\theta-\eta}{2}\Big)\Big|\Big)I + \frac{1}{R^2}\Delta_\eta \boldsymbol{X}_c(\theta)\otimes\Delta_\eta \boldsymbol{X}_c(\theta)\Big), \tag{2-12}$$

$$\begin{aligned} G_L(\Delta_\eta \boldsymbol{X}_c(\theta), \Delta_\eta \boldsymbol{X}(\theta)) = \frac{1}{4\pi R^2}\Big(&-\Delta_\eta \boldsymbol{X}_c(\theta)\cdot\Delta_\eta \boldsymbol{X}(\theta) I \\ &-\frac{2}{R^2}\Delta_\eta \boldsymbol{X}_c(\theta)\cdot\Delta_\eta \boldsymbol{X}(\theta)\Delta_\eta \boldsymbol{X}_c(\theta)\otimes\Delta_\eta \boldsymbol{X}_c(\theta) \\ &+\Delta_\eta \boldsymbol{X}_c(\theta)\otimes\Delta_\eta \boldsymbol{X}(\theta)+\Delta_\eta \boldsymbol{X}(\theta)\otimes\Delta_\eta \boldsymbol{X}_c(\theta)\Big), \end{aligned} \tag{2-13}$$

$$\begin{aligned} G_N(\Delta_\eta \boldsymbol{X}_c(\theta), \Delta_\eta \boldsymbol{X}(\theta)) = &-\frac{1}{4\pi}\mathcal{R}_1(\Delta_\eta \boldsymbol{X}(\theta)) I \\ &+\frac{1}{4\pi R^2}\Delta_\eta \boldsymbol{X}(\theta)\otimes\Delta_\eta \boldsymbol{X}(\theta)\Big(1-\frac{2}{R^2}\Delta_\eta \boldsymbol{X}_c(\theta)\cdot\Delta_\eta \boldsymbol{X}(\theta)+\mathcal{R}_2(\Delta_\eta \boldsymbol{X}(\theta))\Big) \\ &+\frac{1}{4\pi R^2}\big(\Delta_\eta \boldsymbol{X}_c(\theta)\otimes\Delta_\eta \boldsymbol{X}(\theta)+\Delta_\eta \boldsymbol{X}(\theta)\otimes\Delta_\eta \boldsymbol{X}_c(\theta)\big) \\ &\qquad\times\Big(-\frac{2}{R^2}\Delta_\eta \boldsymbol{X}_c(\theta)\cdot\Delta_\eta \boldsymbol{X}(\theta)+\mathcal{R}_2(\Delta_\eta \boldsymbol{X}(\theta))\Big) \\ &+\frac{1}{4\pi R^2}\Delta_\eta \boldsymbol{X}_c(\theta)\otimes\Delta_\eta \boldsymbol{X}_c(\theta)\mathcal{R}_2(\Delta_\eta \boldsymbol{X}(\theta)). \end{aligned} \tag{2-14}$$

Consider the splitting of the solution $\boldsymbol{F}(\theta)$ to (1-13) into zero-order, linear, and nonlinear parts as

$$\boldsymbol{F}(\theta) = \boldsymbol{F}_0(\theta) + \boldsymbol{F}_L(\theta) + \boldsymbol{F}_N(\theta). \tag{2-15}$$

(We will prove bounds for these terms in Section 3B.) Introducing the splittings (2-11) and (2-15) in (1-10), we obtain

$$\boldsymbol{\mathcal{X}}_t(\theta) = \boldsymbol{\mathcal{O}}(\boldsymbol{X}_c)(\theta) + \boldsymbol{\mathcal{L}}(\boldsymbol{X}_c, \boldsymbol{X})(\theta) + \boldsymbol{\mathcal{N}}(\boldsymbol{X}_c, \boldsymbol{X})(\theta), \tag{2-16}$$

where we recall that $\boldsymbol{\mathcal{X}}(\theta) = \boldsymbol{X}(\theta) + \boldsymbol{X}_c(\theta)$ and we use the notation

$$\begin{aligned} \boldsymbol{\mathcal{O}}(\boldsymbol{X}_c)(\theta) &= \int_{\mathbb{S}} G_0(\Delta_\eta \boldsymbol{X}_c(\theta))\boldsymbol{F}_0(\eta)\,d\eta, \\ \boldsymbol{\mathcal{L}}(\boldsymbol{X}_c, \boldsymbol{X})(\theta) &= \int_{\mathbb{S}} G_0(\Delta_\eta \boldsymbol{X}_c(\theta))\boldsymbol{F}_L(\eta)\,d\eta + \int_{\mathbb{S}} G_L(\Delta_\eta \boldsymbol{X}_c(\theta), \Delta_\eta \boldsymbol{X}(\theta))\boldsymbol{F}_0(\eta)\,d\eta, \\ \boldsymbol{\mathcal{N}}(\boldsymbol{X}_c, \boldsymbol{X})(\theta) &= \int_{\mathbb{S}} \big(G_L(\Delta_\eta \boldsymbol{X}_c(\theta), \Delta_\eta \boldsymbol{X}(\theta))\boldsymbol{F}_L(\eta) + G_N(\Delta_\eta \boldsymbol{X}_c(\theta), \Delta_\eta \boldsymbol{X}(\theta))\boldsymbol{F}_0(\eta) \\ &\qquad + G_N(\Delta_\eta \boldsymbol{X}_c(\theta), \Delta_\eta \boldsymbol{X}(\theta))\boldsymbol{F}_L(\eta) + G(\boldsymbol{\mathcal{X}}(\theta) - \boldsymbol{\mathcal{X}}(\eta))\boldsymbol{F}_N(\eta)\big)\,d\eta. \end{aligned}$$

We have thus expanded the evolution equation (1-10) distinguishing the zero-order, linear, and nonlinear in $\boldsymbol{X}$ contributions.

2B. ***Linearized system.*** We proceed to show that the linearized system gives rise to a diffusion operator on $\boldsymbol{X}$. Since the linear structure is the same for any uniformly parametrized circle (see [Mori et al. 2019]), we will use now (2-1) with $a = 1$, $b = c = d = 0$ and $R = 1$ to simplify the computations, and for clarity of notation we will denote this circle by $\boldsymbol{X}_\star$.

We will now linearize (1-10) and (1-13). We first determine $\boldsymbol{F}_0$, the value of $\boldsymbol{F}$ at the steady state:

$$0 = \boldsymbol{\mathcal{O}}(\boldsymbol{X}_\star)(\theta) = \int_{\mathbb{S}} G_0(\Delta_\eta \boldsymbol{X}_\star(\theta))\boldsymbol{F}_0(\eta)\,d\eta, \tag{2-17}$$

$$\boldsymbol{F}_0(\theta) - 2A_\mu \boldsymbol{\mathcal{S}}_0(\boldsymbol{F}_0, \boldsymbol{X}_\star)(\theta) = 2A_e \widetilde{\boldsymbol{F}}_{\mathrm{el},0}(\theta), \tag{2-18}$$

where

$$\widetilde{\boldsymbol{F}}_{\mathrm{el},0}(\theta) = \partial_\theta^2 \boldsymbol{X}_\star(\theta)$$

and

$$\boldsymbol{\mathcal{S}}_0(\boldsymbol{F}_0, \boldsymbol{X}_\star)(\theta) = \frac{1}{\pi}\int_{\mathbb{S}} \partial_\theta \boldsymbol{X}_\star^\perp \cdot \Delta_\eta \boldsymbol{X}_\star(\theta)\Delta_\eta \boldsymbol{X}_\star(\theta)\Delta_\eta \boldsymbol{X}_\star(\theta)\cdot \frac{\boldsymbol{F}_0(\eta)}{2\sin\left(\frac{\theta-\eta}{2}\right)}\,d\eta.$$

Rewriting $\Delta_\eta \boldsymbol{X}_\star(\theta)\Delta_\eta \boldsymbol{X}_\star(\theta)\cdot \boldsymbol{F}_0(\eta) = \Delta_\eta \boldsymbol{X}_\star(\theta)\otimes\Delta_\eta \boldsymbol{X}_\star(\theta)\boldsymbol{F}_0(\eta)$, and recalling the computations (2-3) and (2-5), one finds that

$$\boldsymbol{\mathcal{S}}_0(\boldsymbol{F}_0, \boldsymbol{X}_\star)(\theta) = -\frac{1}{2\pi}\int_{\mathbb{S}} M(\theta,\eta)\boldsymbol{F}_0(\eta)\,d\eta,$$

where from (2-5) we have

$$M(\theta,\eta) = \Delta_\eta \boldsymbol{X}_\star(\theta)\otimes\Delta_\eta \boldsymbol{X}_\star(\theta) = \frac{1}{2}\begin{bmatrix} 1-\cos(\theta+\eta) & -\sin(\theta+\eta) \\ -\sin(\theta+\eta) & 1+\cos(\theta+\eta)\end{bmatrix}, \tag{2-19}$$

and therefore $\boldsymbol{F}_0$ is defined by

$$\boldsymbol{F}_0(\theta) + \frac{A_\mu}{\pi}\int_{\mathbb{S}} M(\theta,\eta)\boldsymbol{F}_0(\eta)\,d\eta = 2A_e\partial_\theta^2\boldsymbol{X}_\star(\theta).$$

Since $\partial_\theta^2\boldsymbol{X}_\star = -\boldsymbol{X}_\star$ and noting that

$$\frac{1}{\pi}\int_{\mathbb{S}} M(\theta,\eta)\boldsymbol{X}_\star(\eta)\,d\eta = -\boldsymbol{X}_\star(\theta),$$

it is easily seen that

$$\boldsymbol{F}_0(\theta) = \frac{2A_e}{1-A_\mu}\partial_\theta^2\boldsymbol{X}_\star(\theta). \tag{2-20}$$

Now, recalling (2-12), it can be checked that (2-20) satisfies in fact (2-17):

$$\begin{aligned} 4\pi\left(\frac{1-A_\mu}{2A_e}\right)&\int_{\mathbb{S}} G_0(\Delta_\eta\boldsymbol{X}_\star(\theta))\boldsymbol{F}_0(\eta)\,d\eta \\ &= -\int_{\mathbb{S}}\log\left(2\left|\sin\left(\frac{\theta-\eta}{2}\right)\right|\right)\partial_\eta^2\boldsymbol{X}_\star(\eta)\,d\eta + \int_{\mathbb{S}} M(\theta,\eta)\partial_\eta^2\boldsymbol{X}_\star(\eta)\,d\eta, \end{aligned}$$

so integration by parts in the first term yields (2-17).

We now proceed to compute the linear term $\boldsymbol{\mathcal{L}}(\boldsymbol{X}_\star, \boldsymbol{X})(\theta)$ in (2-16). For convenience, we write it as

$$\boldsymbol{\mathcal{L}}(\boldsymbol{X}_\star, \boldsymbol{X})(\theta) = \int_{\mathbb{S}} G_0(\Delta_\eta\boldsymbol{X}_\star)\boldsymbol{F}_L(\eta)\,d\eta + \int_{\mathbb{S}} (\nabla G(\Delta\boldsymbol{X}_\star)\boldsymbol{F}_0(\eta))\Delta\boldsymbol{X}\,d\eta, \tag{2-21}$$

where G_0 and G are defined in (2-12) and (1-9), respectively. To simplify the second integral above, note

$$\begin{aligned} \frac{\partial G_{ij}}{\partial x_1}(\Delta\boldsymbol{X}_\star)e_{\mathrm{r},j}(\eta) &= \frac{\partial G_{i1}}{\partial x_1}(\Delta\boldsymbol{X}_\star)e_{\mathrm{r},1}(\eta) + \frac{\partial G_{i2}}{\partial x_1}(\Delta\boldsymbol{X}_\star)e_{\mathrm{r},2}(\eta) \\ &= \frac{\partial G_{i1}}{\partial x_1}(\Delta\boldsymbol{X}_\star)\partial_\eta X_{\star,2} - \frac{\partial G_{i2}}{\partial x_1}(\Delta\boldsymbol{X}_\star)\partial_\eta X_{\star,1} \\ &= -\frac{\partial G_{i2}}{\partial x_2}(\Delta\boldsymbol{X}_\star)\partial_\eta X_{\star,2} - \frac{\partial G_{i2}}{\partial x_1}(\Delta\boldsymbol{X}_\star)\partial_\eta X_{\star,1} = \partial_\eta G_{i2}(\Delta\boldsymbol{X}_\star). \end{aligned}$$

Here $e_{\mathrm{r},j}$ is the j-th component of the vector e_{r} etc. Further, in the third equality above, we used the fact that the stokeslet is divergence-free:

$$\frac{\partial G_{i1}}{\partial x_1}+\frac{\partial G_{i2}}{\partial x_2}=0.$$

Likewise, we have

$$\frac{\partial G_{ij}}{\partial x_2}(\Delta \boldsymbol{X}_\star)e_{\mathrm{r},j}(\eta)=-\partial_\eta G_{i1}(\Delta \boldsymbol{X}_\star).$$

We thus have

$$\begin{aligned}\int_{\mathbb{S}}(\nabla G(\Delta \boldsymbol{X}_\star)\boldsymbol{F}_0(\eta))\Delta \boldsymbol{X}\,d\eta&=-\frac{2A_e}{1-A_\mu}\int_{\mathbb{S}}(\mathcal{R}^{-1}\partial_\eta G(\boldsymbol{X}_\star(\theta)-\boldsymbol{X}_\star(\eta)))\Delta \boldsymbol{X}\,d\eta\\&=-\frac{2A_e}{1-A_\mu}\int_{\mathbb{S}}(\mathcal{R}^{-1}G(\boldsymbol{X}_\star(\theta)-\boldsymbol{X}_\star(\eta)))\partial_\eta \boldsymbol{X}(\eta)\,d\eta\\&=\frac{2A_e}{1-A_\mu}\int_{\mathbb{S}}G(\boldsymbol{X}_\star(\theta)-\boldsymbol{X}_\star(\eta))(\mathcal{R}^{-1}\partial_\eta \boldsymbol{X}(\eta))\,d\eta.\end{aligned}$$

Since $G(\boldsymbol{X}_\star(\theta)-\boldsymbol{X}_\star(\eta))\overset{\text{def}}{=}G_0(\Delta_\eta \boldsymbol{X}_\star)$, equation (2-21) simplifies to

$$\boldsymbol{\mathcal{L}}(\boldsymbol{X}_\star,\boldsymbol{X})(\theta)=\int_{\mathbb{S}}G_0(\Delta_\eta \boldsymbol{X}_\star)\left(\boldsymbol{F}_L(\eta)+\frac{2A_e}{1-A_\mu}\mathcal{R}^{-1}\partial_\eta \boldsymbol{X}\right)d\eta. \tag{2-22}$$

This is our specification of the linearized operator.

We will now determine $\boldsymbol{F}_L$ as in (2-15), that is, the linear part of $\boldsymbol{F}$ in (1-13). We find

$$\boldsymbol{F}_L(\theta)+\frac{A_\mu}{\pi}\int_{\mathbb{S}}M(\theta,\eta)\boldsymbol{F}_L(\eta)\,d\eta=2A_e\partial_\theta^2\boldsymbol{X}-2A_\mu(\boldsymbol{Q}+\boldsymbol{S}), \tag{2-23}$$

where

$$\begin{aligned}Q_i&=-\int_{\mathbb{S}}\mathcal{T}_{ijk}(\Delta \boldsymbol{X}_\star)F_{0,k}(\eta)\mathcal{R}_{jl}^{-1}\partial_\theta X_l(\theta)\,d\eta,\\S_i&=-\int_{\mathbb{S}}\frac{\partial \mathcal{T}_{ijk}}{\partial x_m}(\Delta \boldsymbol{X}_\star)\Delta \boldsymbol{X}_m F_{0,k}(\eta)\mathcal{R}_{jl}^{-1}\partial_\theta X_{\star,l}(\theta)\,d\eta.\end{aligned}$$

Let us compute $\boldsymbol{Q}$. We start with

$$-\mathcal{T}_{ijk}(\Delta \boldsymbol{X}_\star)F_{0,k}=-\frac{2A_e}{1-A_\mu}\frac{\Delta X_{\star,i}\Delta X_{\star,j}\Delta X_{\star,k}e_{\mathrm{r},k}(\eta)}{\pi|\Delta \boldsymbol{X}_\star|^4}=\frac{2A_e}{1-A_\mu}\frac{\Delta X_{\star,i}\Delta X_{\star,j}}{2\pi|\Delta \boldsymbol{X}_\star|^2},$$

where we used

$$\frac{\Delta \boldsymbol{X}_\star\cdot \boldsymbol{e}_{\mathrm{r}}(\eta)}{|\Delta \boldsymbol{X}_\star|^2}=-\frac{1}{2}. \tag{2-24}$$

Therefore, we have

$$\begin{aligned}\boldsymbol{Q}&=\frac{A_e}{1-A_\mu}\frac{1}{\pi}\int_{\mathbb{S}}\frac{\Delta \boldsymbol{X}_\star\otimes\Delta \boldsymbol{X}_\star}{|\Delta \boldsymbol{X}_\star|^2}\,d\eta\mathcal{R}^{-1}\partial_\theta \boldsymbol{X}(\theta)\\&=\frac{A_e}{1-A_\mu}\frac{1}{\pi}\int_{\mathbb{S}}M(\theta,\eta)\,d\eta\mathcal{R}^{-1}\partial_\theta \boldsymbol{X}(\theta)=\frac{A_e}{1-A_\mu}\mathcal{R}^{-1}\partial_\theta \boldsymbol{X}(\theta),\end{aligned}$$

where we used (2-19) in the second equality. We next compute $\boldsymbol{S}$,

$$
\begin{aligned}
-\frac{\partial \mathcal{T}_{ijk}}{\partial x_m}&(\Delta X_\star)\Delta X_m F_{0,k}(\eta)\mathcal{R}^{-1}_{jl}\partial_\theta X_{\star,l}(\theta) \\
&= -\frac{1}{\pi}\frac{2A_e}{1-A_\mu}\left(\frac{\Delta X_i \Delta X_{\star,j}\Delta X_{\star,k} e_{\mathrm{r},j}(\theta)e_{\mathrm{r},k}(\eta)}{|\Delta \boldsymbol{X}_\star|^4}\right) \\
&\quad - \frac{2A_e/\pi}{1-A_\mu}\left(\frac{\Delta X_{\star,i}\Delta X_j \Delta X_{\star,k} e_{\mathrm{r},j}(\theta)e_{\mathrm{r},k}(\eta)}{|\Delta \boldsymbol{X}_\star|^4} + \frac{\Delta X_{\star,i}\Delta X_{\star,j}\Delta X_k e_{\mathrm{r},j}(\theta)e_{\mathrm{r},k}(\eta)}{|\Delta \boldsymbol{X}_\star|^4}\right) \\
&\quad + \frac{8A_e/\pi}{1-A_\mu}\left(\frac{\Delta X_{\star,i}\Delta X_{\star,j}\Delta X_{\star,k} e_{\mathrm{r},j}(\theta)e_{\mathrm{r},k}(\eta)\Delta X_{\star,m}\Delta X_m}{|\Delta \boldsymbol{X}_\star|^6}\right) \\
&= -\frac{1}{\pi}\frac{2A_e}{1-A_\mu} I - \frac{2A_e/\pi}{1-A_\mu} II + \frac{8A_e/\pi}{1-A_\mu} III.
\end{aligned}
$$

We simplify each term as follows:

$$
\begin{aligned}
I &= -\frac{1}{4}\Delta X_i, \\
II &= -\frac{\Delta X_{\star,i}\Delta X_j e_{\mathrm{r},j}(\theta)}{2|\Delta \boldsymbol{X}_\star|^2} + \frac{\Delta X_{\star,i}\Delta X_k e_{\mathrm{r},k}(\eta)}{2|\Delta \boldsymbol{X}_\star|^2} = -\frac{\Delta X_{\star,i}\Delta X_{\star,j}\Delta X_j}{2|\Delta \boldsymbol{X}_\star|^2}, \\
III &= -\frac{\Delta X_{\star,i}\Delta X_{\star,j}\Delta X_j}{4|\Delta \boldsymbol{X}_\star|^2},
\end{aligned}
$$

where above we made repeated use of (2-24) and

$$
\frac{\Delta \boldsymbol{X}_\star \cdot \mathcal{R}^{-1}\partial_\theta \boldsymbol{X}_\star(\theta)}{|\Delta \boldsymbol{X}_\star|^2} = \frac{\Delta \boldsymbol{X}_\star \cdot \boldsymbol{e}_{\mathrm{r}}(\theta)}{|\Delta \boldsymbol{X}_\star|^2} = \frac{1}{2}.
$$

Thus

$$
-\frac{\partial \mathcal{T}_{ijk}}{\partial x_m}(\Delta X_\star)\Delta X_m F_{0,k}(\eta)\mathcal{R}^{-1}_{jl}\partial_\theta X_{\star,l} = \frac{A_e/2\pi}{1-A_\mu}\left(\Delta X_i - \frac{2\Delta X_{\star,i}\Delta X_{\star,j}\Delta X_j}{|\Delta \boldsymbol{X}_\star|^2}\right).
$$

Substituting this back into the expression for $\boldsymbol{S}$ in (2-23), we have

$$
\begin{aligned}
\boldsymbol{S} &= \frac{A_e/2\pi}{1-A_\mu}\int_{\mathbb{S}} (I - 2M(\theta,\eta))\Delta \boldsymbol{X}\, d\eta \\
&= -\frac{A_e/2\pi}{1-A_\mu}\int_{\mathbb{S}} (I - 2M(\theta,\eta))\boldsymbol{X}(\eta)\, d\eta = \frac{A_e/2\pi}{1-A_\mu}(-\langle \boldsymbol{e}_{\mathrm{r}}, \boldsymbol{X}\rangle \boldsymbol{e}_{\mathrm{r}} + \langle \boldsymbol{e}_{\mathrm{t}}, \boldsymbol{X}\rangle \boldsymbol{e}_{\mathrm{t}}),
\end{aligned}
$$

where we used (2-19) in the first equality and we are using the notation (1-24) for the inner product.

Equation (2-23) thus reduces to

$$
\begin{aligned}
\boldsymbol{F}_L(\theta) &+ \frac{A_\mu}{\pi}\int_{\mathbb{S}} M(\theta,\eta)\boldsymbol{F}_L(\eta)\, d\eta \\
&= 2A_e\partial_\theta^2 \boldsymbol{X}(\theta) - \frac{2A_eA_\mu}{1-A_\mu}\left(\mathcal{R}^{-1}\partial_\theta \boldsymbol{X}(\theta) + \frac{1}{2\pi}(-\langle \boldsymbol{e}_{\mathrm{r}}, \boldsymbol{X}\rangle \boldsymbol{e}_{\mathrm{r}}(\theta) + \langle \boldsymbol{e}_{\mathrm{t}}, \boldsymbol{X}\rangle \boldsymbol{e}_{\mathrm{t}}(\theta))\right).
\end{aligned}
$$

We must solve the above equation for $\boldsymbol{F}_L$ in terms of $\boldsymbol{X}$. Suppose $\langle \boldsymbol{e}_{\mathrm{r}}, \boldsymbol{X}\rangle = \langle \boldsymbol{e}_{\mathrm{t}}, \boldsymbol{X}\rangle = 0$. Then, it is easily checked that

$$
\boldsymbol{F}_L(\theta) = 2A_e\partial_\theta^2 \boldsymbol{X}(\theta) - \frac{2A_eA_\mu}{1-A_\mu}\mathcal{R}^{-1}\partial_\theta \boldsymbol{X}(\theta). \tag{2-25}
$$

We may further compute $\boldsymbol{F}_L$ when $\boldsymbol{X}$ is either $\boldsymbol{e}_{\mathrm{r}}$ or $\boldsymbol{e}_{\mathrm{t}}$. Noting that

$$M(\theta,\eta)\boldsymbol{e}_{\mathrm{r}}(\eta)=\tfrac{1}{2}(\boldsymbol{e}_{\mathrm{r}}(\eta)-\boldsymbol{e}_{\mathrm{r}}(\theta)),\quad M(\theta,\eta)\boldsymbol{e}_{\mathrm{t}}(\eta)=\tfrac{1}{2}(\boldsymbol{e}_{\mathrm{t}}(\eta)+\boldsymbol{e}_{\mathrm{t}}(\theta)),$$

we see by an easy calculation that

$$\text{If } \boldsymbol{X}=\boldsymbol{e}_{\mathrm{r},t},\ \text{then } \boldsymbol{F}_L=-\frac{2A_e}{1-A_\mu}\boldsymbol{e}_{\mathrm{r},t}.$$

Note that

$$2A_e\partial_\theta^2\boldsymbol{e}_{\mathrm{r},t}-\frac{2A_eA_\mu}{1-A_\mu}\mathcal{R}^{-1}\partial_\theta\boldsymbol{e}_{\mathrm{r},t}=-\frac{2A_e}{1-A_\mu}\boldsymbol{e}_{\mathrm{r},t}.$$

This shows that the expression for $\boldsymbol{F}_L$ in (2-25) is in fact valid without the restriction $\langle\boldsymbol{e}_{\mathrm{r}},\boldsymbol{X}\rangle=\langle\boldsymbol{e}_{\mathrm{t}},\boldsymbol{X}\rangle=0$. Substituting (2-25) into (2-22) yields

$$\boldsymbol{\mathcal{L}}(\boldsymbol{X}_\star,\boldsymbol{X})(\theta)=2A_e\int_{\mathbb{S}}G_0(\Delta_\eta\boldsymbol{X}_\star)(\partial_\eta^2\boldsymbol{X}(\eta)+\mathcal{R}^{-1}\partial_\eta\boldsymbol{X}(\eta))\,d\eta. \tag{2-26}$$

Finally, since

$$G_0(\Delta_\eta\boldsymbol{X}_\star)=-\frac{1}{4\pi}\Big(\log\Big|2\sin\Big(\frac{\theta-\eta}{2}\Big)\Big|\Big)I+M(\theta,\eta)$$

and

$$\int_{\mathbb{S}}M(\theta,\eta)\Big(\partial_\eta^2\boldsymbol{X}(\eta)+\mathcal{R}^{-1}\partial_\eta\boldsymbol{X}(\eta)\Big)\,d\eta=\int_{\mathbb{S}}(\partial_\eta^2M(\theta,\eta)-\partial_\eta M(\theta,\eta)\mathcal{R}^{-1})\boldsymbol{X}(\eta)\,d\eta=0,$$

we have

$$\begin{aligned}\boldsymbol{\mathcal{L}}(\boldsymbol{X}_\star,\boldsymbol{X})(\theta)&=-\frac{A_e}{2\pi}\int_{\mathbb{S}}\log\Big|2\sin\Big(\frac{\theta-\eta}{2}\Big)\Big|(\partial_\eta^2\boldsymbol{X}(\eta)+\mathcal{R}^{-1}\partial_\eta\boldsymbol{X}(\eta))\,d\eta\\&=-\frac{A_e}{2\pi}\int_{\mathbb{S}}\frac{\partial_\eta\boldsymbol{X}(\eta)+\mathcal{R}^{-1}\boldsymbol{X}(\eta)}{2\tan\big(\frac{\theta-\eta}{2}\big)}\,d\eta,\end{aligned}$$

which is given by a Hilbert transform

$$\boldsymbol{\mathcal{L}}(\boldsymbol{X}_\star,\boldsymbol{X})(\theta)=-\frac{A_e}{2}\mathcal{H}(\partial_\eta\boldsymbol{X}(\eta)+\mathcal{R}^{-1}\boldsymbol{X}(\eta))(\theta). \tag{2-27}$$

Therefore, the system (2-16) can be written as

$$\boldsymbol{\mathcal{X}}_t(\theta)=-\frac{A_e}{2}(\Lambda\boldsymbol{X}(\theta)+\mathcal{H}\mathcal{R}^{-1}\boldsymbol{X}(\theta))+\boldsymbol{\mathcal{N}}(\boldsymbol{X}_c,\boldsymbol{X})(\theta). \tag{2-28}$$

Notice that $\boldsymbol{X}_c$ is a uniformly parametrized circle with time-dependent radius $R(t)$, as opposed to the $\boldsymbol{X}_*$ used in this subsection to obtain the linearization. We will use the system (2-28) to study the global-in-time dynamics of the Peskin problem in the rest of this paper.

2C. *Evolution of the $\dot{\mathcal{F}}^{1,1}_\nu$ norm of X.* We first notice that, because $\boldsymbol{X}(\theta)$ is real-valued, it must hold that $\widehat{\boldsymbol{X}}(-k)=\overline{\widehat{\boldsymbol{X}}}(k)$. Therefore, the norm (1-25) can be written in terms of positive frequencies alone

$$\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}=2\sum_{k\ge1}e^{\nu(t)k}k|\widehat{\boldsymbol{X}}(k)|=2\sum_{k\ge1}e^{\nu(t)k}k\sqrt{\widehat{X}_1\overline{\widehat{X}}_1(k)+\widehat{X}_2\overline{\widehat{X}}_2(k)}. \tag{2-29}$$

The system (2-28) in Fourier variables reads for $k \geq 0$ as

$$\widehat{\boldsymbol{\mathcal{X}}}_t(k) = -\frac{A_e}{2} L(k) \widehat{\boldsymbol{X}}(k) + \mathcal{F}(\boldsymbol{\mathcal{N}}(\boldsymbol{X}_c, \boldsymbol{X}))(k). \tag{2-30}$$

Here we recall that $\boldsymbol{\mathcal{X}} = \boldsymbol{X} + \boldsymbol{X}_c$. Further the diffusion matrix is given by

$$L(k) = \begin{bmatrix} k & -i\,\mathrm{sgn}(k) \\ i\,\mathrm{sgn}(k) & k \end{bmatrix}, \quad k \geq 1, \qquad L(0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The diagonalization of this matrix for $k \geq 1$ shows that

$$L(k) = P(k) \mathcal{D}(k) P(k)^{-1},$$

where for $k \geq 1$ we have

$$P(k) = \frac{1}{\sqrt{2}} \begin{bmatrix} -i\,\mathrm{sgn}(k) & 1 \\ 1 & -i\,\mathrm{sgn}(k) \end{bmatrix}, \quad P(k)^{-1} = \overline{P(k)}, \quad \mathcal{D}(k) = \begin{bmatrix} k+1 & 0 \\ 0 & k-1 \end{bmatrix}.$$

And when $k = 0$ we define

$$P(0) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad P(0)^{-1} = 2\overline{P(0)}, \quad \mathcal{D}(0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

This leads us to define the change of variables

$$\widehat{\boldsymbol{Y}}(k) \stackrel{\text{def}}{=} P(k)^{-1} \widehat{\boldsymbol{X}}(k), \quad \widehat{\boldsymbol{Y}}_c(k) \stackrel{\text{def}}{=} P(k)^{-1} \widehat{\boldsymbol{X}}_c(k), \tag{2-31}$$

with $\boldsymbol{\mathcal{Y}} \stackrel{\text{def}}{=} \boldsymbol{Y} + \boldsymbol{Y}_c$. The system (2-30) for $k \geq 0$ then becomes

$$\widehat{\boldsymbol{\mathcal{Y}}}_t(k) = -\frac{A_e}{2} \mathcal{D}(k) \widehat{\boldsymbol{Y}}(k) + P(k)^{-1} \mathcal{F}(\boldsymbol{\mathcal{N}}(\boldsymbol{X}_c, \boldsymbol{X}))(k). \tag{2-32}$$

The relationship between $\boldsymbol{X}$ and $\boldsymbol{Y}$ in space variables is given by the Hilbert transform (1-20), using also $\mathcal{H}^2(Y_j) = -Y_j$, as follows:

$$\boldsymbol{X}(\theta) = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathcal{H}Y_1(\theta) + Y_2(\theta) \\ Y_1(\theta) + \mathcal{H}Y_2(\theta) \end{bmatrix}, \quad \boldsymbol{Y}(\theta) = \frac{1}{\sqrt{2}} \begin{bmatrix} -\mathcal{H}X_1(\theta) + X_2(\theta) \\ X_1(\theta) - \mathcal{H}X_2(\theta) \end{bmatrix}. \tag{2-33}$$

Because, for $k \neq 0$, $P(k)$ is a unitary matrix, it holds that $\|P(k)\| = \|P(k)^{-1}\| = 1$, and therefore

$$|\widehat{\boldsymbol{Y}}(k)| = |\widehat{\boldsymbol{X}}(k)|, \quad k \neq 0,$$

and thus

$$\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu} = \|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{1,1}_\nu}. \tag{2-34}$$

We will use this norm equivalence several times in the following.

Notice that the first Fourier coefficient of a uniformly parametrized circle (2-1) is given by

$$\widehat{\boldsymbol{X}}_c(1) = \frac{1}{2} \begin{bmatrix} a+bi \\ -ia+b \end{bmatrix},$$

and in the $\boldsymbol{Y}$-variable

$$\widehat{\boldsymbol{Y}}_c(1)=\frac{1}{\sqrt{2}}\begin{bmatrix}0\\ a+bi\end{bmatrix}. \tag{2-35}$$

Note that $\mathcal{H}(\cos\theta)=\sin\theta$ and $\mathcal{H}(\sin\theta)=-\cos\theta$. Then from the transformation (2-33) uniformly parametrized circles (1-16) in the $\boldsymbol{\mathcal{Y}}$ variable are spanned by

$$\tilde{\boldsymbol{e}}_r(\theta)=\sqrt{2}\begin{bmatrix}0\\ \cos\theta\end{bmatrix},\quad \tilde{\boldsymbol{e}}_t(\theta)=\sqrt{2}\begin{bmatrix}0\\ -\sin\theta\end{bmatrix},\quad \tilde{\boldsymbol{e}}_1=\frac{1}{\sqrt{2}}\begin{bmatrix}0\\ 1\end{bmatrix},\quad \tilde{\boldsymbol{e}}_2=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 0\end{bmatrix}. \tag{2-36}$$

Further the second component of $\widehat{\boldsymbol{Y}}(1)$ becomes zero after the operation of $\mathcal{D}(1)$ is applied, which corresponds to the fact that uniformly parametrized circles are steady states. Therefore, we will split the curve $\boldsymbol{\mathcal{Y}}(\theta)=\boldsymbol{Y}_c(\theta)+\boldsymbol{Y}(\theta)$, with

$$\widehat{\boldsymbol{Y}}(0)=\begin{bmatrix}0\\ 0\end{bmatrix},\quad \widehat{Y}_2(1)=0,$$

since those frequencies are contained in the time-dependent circle (2-1). In other words, $\boldsymbol{Y}$ is the projection of $\boldsymbol{\mathcal{Y}}$ onto the orthogonal complement of the vector space spanned by (2-36). In fact, the system of equations (2-32) does not provide dissipation for the zero frequency of $\boldsymbol{\mathcal{Y}}$ nor for the second component of its first frequency (i.e., for uniformly parametrized circles). We thus can only expect decay for $\boldsymbol{Y}$. It is convenient then to write the equations of those frequencies in (2-32) separately:

$$\begin{aligned}
\widehat{\boldsymbol{\mathcal{Y}}}_t(0)&=\partial_t\widehat{\boldsymbol{Y}}_c(0)=P(0)^{-1}\mathcal{F}(\boldsymbol{\mathcal{N}}(\boldsymbol{X}_c,\boldsymbol{X}))(0),\\
\partial_t\widehat{\mathcal{Y}}_1(1)&=\partial_t\widehat{Y}_1(1)=-A_e\widehat{Y}_1(1)+\big(P(1)^{-1}\mathcal{F}(\boldsymbol{\mathcal{N}}(\boldsymbol{X}_c,\boldsymbol{X}))(1)\big)_1,\\
\partial_t\widehat{\mathcal{Y}}_2(1)&=\partial_t\widehat{Y}_{c,2}(1)=\big(P(1)^{-1}\mathcal{F}(\boldsymbol{\mathcal{N}}(\boldsymbol{X}_c,\boldsymbol{X}))(1)\big)_2,\\
\widehat{\boldsymbol{\mathcal{Y}}}_t(k)&=\widehat{\boldsymbol{Y}}_t(k)=-\frac{A_e}{2}\mathcal{D}(k)\widehat{\boldsymbol{Y}}(k)+P(k)^{-1}\mathcal{F}(\boldsymbol{\mathcal{N}}(\boldsymbol{X}_c,\boldsymbol{X}))(k),\quad k\geq 2.
\end{aligned} \tag{2-37}$$

Therefore, we study the evolution in time of $\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{1,1}_\nu}$, as in (2-29), which is given by

$$\begin{aligned}
\frac{d}{dt}\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{1,1}_\nu}&=\frac{d}{dt}\bigg(2\sum_{k\geq1}e^{\nu(t)k}k\sqrt{\widehat{Y}_1(k)\overline{\widehat{Y}}_1(k)+\widehat{Y}_2(k)\overline{\widehat{Y}}_2(k)}\bigg)\\
&=2\sum_{k\geq1}\nu'(t)k^2e^{\nu(t)k}|\widehat{\boldsymbol{Y}}(k)|+2\sum_{k\geq1}e^{\nu(t)k}k\frac{\partial_t\widehat{\boldsymbol{Y}}(k)^T\overline{\widehat{\boldsymbol{Y}}}(k)+\widehat{\boldsymbol{Y}}(k)^T\overline{\partial_t\widehat{\boldsymbol{Y}}}(k)}{2|\widehat{\boldsymbol{Y}}(k)|},
\end{aligned}$$

and introducing the time derivative (2-32), with $\boldsymbol{\mathcal{N}}=\boldsymbol{\mathcal{N}}(\boldsymbol{X}_c,\boldsymbol{X})=\boldsymbol{\mathcal{N}}(\boldsymbol{\mathcal{X}})$, we have

$$\begin{aligned}
\frac{d}{dt}\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{1,1}_\nu}=2\sum_{k\geq1}\nu'(t)k^2e^{\nu(t)k}|\widehat{\boldsymbol{Y}}(k)|-2A_e\sum_{k\geq1}e^{\nu(t)k}k\frac{(k+1)|\widehat{Y}_1(k)|^2+(k-1)|\widehat{Y}_2(k)|^2}{2|\widehat{\boldsymbol{Y}}(k)|}&\\
+2\sum_{k\geq1}e^{\nu(t)k}k\frac{(P(k)^{-1}\widehat{\boldsymbol{\mathcal{N}}(\boldsymbol{\mathcal{X}})}(k))^T\overline{\widehat{\boldsymbol{Y}}}(k)+\widehat{\boldsymbol{Y}}(k)^T\overline{(P(k)^{-1}\widehat{\boldsymbol{\mathcal{N}}(\boldsymbol{\mathcal{X}})}(k))}}{2|\widehat{\boldsymbol{Y}}(k)|}&.
\end{aligned}$$

Noticing that for $k\geq1$ we have

$$-A_ek\big((k+1)|\widehat{Y}_1(k)|^2+(k-1)|\widehat{Y}_2(k)|^2\big)\frac{1}{|\widehat{\boldsymbol{Y}}(k)|}=-A_ek(k-1)|\widehat{\boldsymbol{Y}}(k)|-2A_ek\frac{|\widehat{Y}_1(k)|^2}{|\widehat{\boldsymbol{Y}}(k)|},$$

we can then see a diffusion term coming from the linear part:

$$\frac{d}{dt}\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{1,1}_{\nu}} \le -A_e \sum_{k\ge 1} e^{\nu(t)k} k(k-1)|\widehat{\boldsymbol{Y}}(k)| - 2A_e \sum_{k\ge 1} e^{\nu(t)k} k \frac{|\widehat{Y}_1(k)|^2}{|\widehat{\boldsymbol{Y}}(k)|} + 2\sum_{k\ge 1} \nu'(t)k^2 e^{\nu(t)k}|\widehat{\boldsymbol{Y}}(k)| + 2\sum_{k\ge 1} e^{\nu(t)k} k|(P(k)^{-1}\widehat{\boldsymbol{\mathcal{N}}(\boldsymbol{\mathcal{X}})}(k))|. \tag{2-38}$$

The balance above does not include the control of $\widehat{\boldsymbol{Y}}_c$. We will show in Section 4A that the evolution of $\widehat{\boldsymbol{Y}}_c(0)$, that is, of the center, can be controlled by all the other frequencies. Moreover, the incompressibility condition (1-3) allows us to control $\widehat{Y}_{c,2}(1)$ as follows:

$$\begin{aligned} V_0 = \pi &= \frac{1}{2}\int_{-\pi}^{\pi} \boldsymbol{\mathcal{X}}(\theta)\wedge \partial_\theta \boldsymbol{\mathcal{X}}(\theta)\,d\theta = \frac{1}{2}\int_{-\pi}^{\pi} (\mathcal{X}_1\partial_\theta\mathcal{X}_2 - \mathcal{X}_2\partial_\theta\mathcal{X}_1)\,d\theta \\ &= \frac{1}{4}\int_{-\pi}^{\pi} \big((\mathcal{H}\mathcal{Y}_1+\mathcal{Y}_2)(\partial_\theta\mathcal{Y}_1+\Lambda\mathcal{Y}_2) - (\mathcal{Y}_1+\mathcal{H}\mathcal{Y}_2)(\Lambda\mathcal{Y}_1+\partial_\theta\mathcal{Y}_2)\,d\theta\big). \end{aligned} \tag{2-39}$$

Performing the products and taking into account the equalities

$$\int_{-\pi}^{\pi} \mathcal{H}\mathcal{Y}_i\Lambda\mathcal{Y}_j\,d\theta = \int_{-\pi}^{\pi} \mathcal{Y}_i\partial_\theta\mathcal{Y}_j\,d\theta, \quad \int_{-\pi}^{\pi} \mathcal{H}\mathcal{Y}_j\partial_\theta\mathcal{Y}_j\,d\theta = -\int_{-\pi}^{\pi} \mathcal{Y}_j\Lambda\mathcal{Y}_j\,d\theta,$$

we obtain

$$\begin{aligned} \pi &= \frac{1}{2}\int_{-\pi}^{\pi} (\mathcal{Y}_2\Lambda\mathcal{Y}_2 - \mathcal{Y}_1\Lambda\mathcal{Y}_1)\,d\theta = \pi(\widehat{\mathcal{Y}_2\Lambda\mathcal{Y}_2}(0) - \widehat{\mathcal{Y}_1\Lambda\mathcal{Y}_1}(0)) \\ &= \pi\sum_{k\in\mathbb{Z}} (|k|\widehat{\mathcal{Y}}_2(k)\widehat{\mathcal{Y}}_2(-k) - |k|\widehat{\mathcal{Y}}_1(k)\widehat{\mathcal{Y}}_1(-k)) \\ &= \pi\sum_{k\in\mathbb{Z}} |k|\big(|\widehat{Y}_{c,2}(k)|^2 + \widehat{Y}_{c,2}(k)\widehat{Y}_2(-k) + \widehat{Y}_2(k)\widehat{Y}_{c,2}(-k) + |\widehat{Y}_2(k)|^2 - |\widehat{Y}_1(k)|^2\big), \end{aligned}$$

where we have used that $\widehat{Y}_{c,1}(k) = 0$ for $k \neq 0$. We can also eliminate the terms $\widehat{Y}_{c,2}(k)\widehat{Y}_2(-k)$ and $\widehat{Y}_2(k)\widehat{Y}_{c,2}(-k)$, since $\widehat{Y}_2(1) = 0$ and $\widehat{Y}_{c,2}(k) = 0$ for $k \neq 0, \pm 1$. Therefore,

$$\frac{1}{2} = \frac{a^2+b^2}{2} + \sum_{k\ge 1} k(|\widehat{Y}_2(k)|^2 - |\widehat{Y}_1(k)|^2).$$

And so the incompressibility condition translates to the constraint

$$|\widehat{Y}_{c,2}(1)|^2 = \frac{a^2+b^2}{2} = \frac{R^2}{2} = \frac{1}{2} - \sum_{k\ge 1} k(|\widehat{Y}_2(k)|^2 - |\widehat{Y}_1(k)|^2). \tag{2-40}$$

Then, we can obtain an upper bound

$$\begin{aligned} |\widehat{Y}_{c,2}(1)|^2 &\le \frac{1}{2} + \sum_{k\ge 1} k(|\widehat{Y}_2(k)|^2 + |\widehat{Y}_1(k)|^2) = \frac{1}{2} + \sum_{k\ge 1} (k^{1/2}|\widehat{\boldsymbol{Y}}(k)|)^2 \\ &\le \frac{1}{2} + \Big(\sum_{k\ge 1} k^{1/2}|\widehat{\boldsymbol{Y}}(k)|\Big)^2 = \frac{1}{2} + \frac{1}{4}\|\boldsymbol{Y}\|^2_{\dot{\mathcal{F}}^{1/2,1}} \le \frac{1}{2} + \frac{1}{4}\|\boldsymbol{Y}\|^2_{\dot{\mathcal{F}}^{1/2,1}}, \end{aligned}$$

and analogously we find the lower bound

$$\frac{R^2}{2} = |\widehat{Y}_{c,2}(1)|^2 \ge \frac{1}{2} - \frac{1}{4}\|\boldsymbol{Y}\|^2_{\dot{\mathcal{F}}^{1/2,1}}. \tag{2-41}$$

Recalling the relationship between $\boldsymbol{X}$ and $\boldsymbol{Y}$ in (2-34), we finally obtain

$$\frac{1}{2}-\frac{1}{4}\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}^{1,1}_\nu} \le |\widehat{Y}_{c,2}(1)|^2 \le \frac{1}{2}+\frac{1}{4}\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}^{1,1}_\nu},$$

and, since $|\widehat{\boldsymbol{Y}}_c(1)|^2 = R^2/2 = |\widehat{\boldsymbol{X}}_c(1)|^2/2$,

$$1-\frac{1}{2}\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}^{1,1}_\nu} \le |\widehat{\boldsymbol{X}}_c(1)|^2 \le 1+\frac{1}{2}\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}^{1,1}_\nu},$$

so using the notation $R^2 = a^2+b^2$, we have

$$\frac{1}{\sqrt{1+\frac{1}{2}\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}^{1,1}_\nu}}} \le \frac{1}{R} \le \frac{1}{\sqrt{1-\frac{1}{2}\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}^{1,1}_\nu}}}. \tag{2-42}$$

The upper bound above motivates us to define

$$C_1 \overset{\text{def}}{=} C_1(\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}) = \frac{1}{\sqrt{1-\frac{1}{2}\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}^{1,1}_\nu}}}. \tag{2-43}$$

We will later use (2-42) to control the size of R when $\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}(t) \to 0$ as $t \to \infty$.

2D. ***Complete system.*** We finally summarize the final form of the system of equations that describes our problem. The system given by (1-10) and (1-13) for $\boldsymbol{\mathcal{X}}$ was replaced by (2-37) on the Fourier coefficients of the associated variable $\boldsymbol{\mathcal{Y}}$ from (2-31). We recall that we decompose $\boldsymbol{\mathcal{Y}}$ into a time-dependent circle $\boldsymbol{Y}_c$ plus the deviation from the circle given by $\boldsymbol{Y}$. In other words, we decompose $\boldsymbol{\mathcal{Y}}$ into its projection onto the vector space spanned by (2-36) represented by $\boldsymbol{Y}_c$ and its orthogonal complement represented by $\boldsymbol{Y}$. Therefore, recalling (2-35), we have

$$\widehat{\boldsymbol{Y}}(0)=0, \qquad \widehat{Y}_2(1)=0, \qquad \widehat{\boldsymbol{Y}}_c(k)=0, \quad k\neq 0,1, \qquad \widehat{Y}_{c,1}(1)=0. \tag{2-44}$$

Now, for $k=1$ and $k\ge 2$ separately, we have

$$\begin{aligned} \widehat{\partial_t Y_1}(1) &= -A_e\widehat{Y}_1(1) + (P(1)^{-1}\widehat{\boldsymbol{\mathcal{N}}(\boldsymbol{X}_c,\boldsymbol{X})}(1))_1, \\ \widehat{\partial_t \boldsymbol{Y}}(k) &= -\frac{A_e}{2}\mathcal{D}(k)\widehat{\boldsymbol{Y}}(k) + P(k)^{-1}\widehat{\boldsymbol{\mathcal{N}}(\boldsymbol{X}_c,\boldsymbol{X})}(k), \end{aligned} \tag{2-45}$$

where $\boldsymbol{X}_c$ and $\boldsymbol{X}$ are given in terms of $\boldsymbol{Y}_c$ and $\boldsymbol{Y}$ in (2-31). In the following paragraphs, we will write one or the other without distinction for simplicity of notation. The incompressibility condition (2-39) yielded (2-40). Thus in particular

$$\sqrt{\tfrac{1}{2}-\tfrac{1}{4}\|\boldsymbol{Y}\|^2_{\dot{\mathcal{F}}^{1,1}_\nu}} \le |\widehat{Y}_{c,2}(1)| \le \sqrt{\tfrac{1}{2}+\tfrac{1}{4}\|\boldsymbol{Y}\|^2_{\dot{\mathcal{F}}^{1,1}_\nu}}. \tag{2-46}$$

To close the system, notice that $\widehat{\boldsymbol{Y}}_c(0) = P(k)^{-1}\widehat{\boldsymbol{X}}_c(0)$ and, from (1-10), we have

$$\widehat{\partial_t \boldsymbol{X}_c}(0) = \frac{1}{2\pi}\int_{\mathbb{S}}\int_{\mathbb{S}} G(\Delta\boldsymbol{X}_c+\Delta\boldsymbol{X})\boldsymbol{F}(\eta)\,d\eta\,d\theta,$$

with $\boldsymbol{F}$ defined by (1-13). We can also write the equation for $\widehat{\boldsymbol{X}}_c(0)$ using (2-16) or (2-30) and recalling that the zero frequency of the linear part vanishes,

$$\widehat{\partial_t \boldsymbol{X}_c}(0) = \widehat{\boldsymbol{\mathcal{N}}(\boldsymbol{X}_c,\boldsymbol{X})}(0). \tag{2-47}$$

We notice that the evolution of the zero frequency $\widehat{\boldsymbol{Y}}_c(0)$, corresponding to the center, is decoupled from all the other equations (in terms of the $\widehat{\boldsymbol{Y}}_c(0)$-variable), because $\widehat{\boldsymbol{X}}_c(0)$ does not appear on the right-hand side of (1-10) and (1-13) and therefore also (2-47). This can be seen from the fact that in (1-10) G only depends on the difference $\Delta\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{X}}(\theta) - \boldsymbol{\mathcal{X}}(\eta)$ and in (1-13) the expression for $\boldsymbol{\mathcal{S}}$ only depends on $\partial_\theta \boldsymbol{\mathcal{X}}$ and $\Delta\boldsymbol{\mathcal{X}}$. In summary, the system to determine $\boldsymbol{\mathcal{Y}}$ (equivalently determining $\boldsymbol{\mathcal{X}}$ via (2-31)) consists of (2-44), (2-45), (2-40), and (2-47). That is, all together we have

$$\begin{aligned}
\widehat{\boldsymbol{Y}}(0) = 0, \qquad \widehat{Y}_2(1) = 0, \qquad \widehat{\boldsymbol{Y}}_c(k) &= 0, \quad k \neq 0, 1, \qquad \widehat{Y}_{c,1}(1) = 0,\\
\partial_t \widehat{\boldsymbol{Y}}_c(0) &= P(0)^{-1}\widehat{\boldsymbol{\mathcal{N}}(\boldsymbol{X}_c, \boldsymbol{X})}(0),\\
\partial_t \widehat{Y}_1(1) &= -A_e \widehat{Y}_1(1) + (P(1)^{-1}\widehat{\boldsymbol{\mathcal{N}}(\boldsymbol{X}_c, \boldsymbol{X})}(1))_1,\\
\partial_t \widehat{\boldsymbol{Y}}(k) &= -\frac{A_e}{2}\mathcal{D}(k)\widehat{\boldsymbol{Y}}(k) + P(k)^{-1}\widehat{\boldsymbol{\mathcal{N}}(\boldsymbol{X}_c, \boldsymbol{X})}(k), \quad k \geq 2,\\
|\widehat{Y}_{c,2}(1)|^2 &= \frac{1}{2} - \sum_{k\geq 1} k(|\widehat{Y}_2(k)|^2 - |\widehat{Y}_1(k)|^2),
\end{aligned} \tag{2-48}$$

with $\boldsymbol{F}$ given in (1-13), and $\boldsymbol{Y}$, $\boldsymbol{X}$ related by (2-31).

To prove Theorem 1.2 (see Section 4) we will use system (2-48) to obtain the energy balance (2-38) to show the decay of $\boldsymbol{Y}$. We will need to perform a priori estimates on the nonlinear terms, which in particular requires us to prove bounds for $\boldsymbol{F}$ due to the viscosity contrast. Those estimates are performed in the next section. The decay for $\boldsymbol{Y}$ will allow us to control the evolution of the zero frequency, that is, of the center.

3. A priori estimates

In this section we perform the a priori estimates on $\boldsymbol{X}$ and $\boldsymbol{F}$ that will be used in the proof of our main result, Theorem 1.2. First, in Proposition 3.1, we estimate the nonlinear terms in (2-16) in terms of $\boldsymbol{X}$ and $\boldsymbol{F}$. Next, in Section 3B, we obtain the a priori estimates for $\boldsymbol{F}$ in (1-13) in terms of $\boldsymbol{X}$. In order to get the result with critical regularity, we have to get uniform bounds for some Fourier multipliers given by principal values (see Lemma 3.2).

3A. *A priori estimates on X.*

Proposition 3.1. *Assume $\boldsymbol{F}_0, \boldsymbol{F}_L, \boldsymbol{F}_N \in \mathcal{F}^{0,1}_\nu$ and $\boldsymbol{X} \in \dot{\mathcal{F}}^{2,1}_\nu$. Then, the nonlinear term $\boldsymbol{\mathcal{N}} = \boldsymbol{\mathcal{N}}(\boldsymbol{X}_c, \boldsymbol{X})(\theta) = \boldsymbol{\mathcal{N}}(\boldsymbol{\mathcal{X}})$ in* (2-16) *satisfies the following estimate in $\dot{\mathcal{F}}^{1,1}_\nu$:*

$$\|\boldsymbol{\mathcal{N}}\|_{\dot{\mathcal{F}}^{1,1}_\nu} \leq 11\sqrt{2}D_1\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}\|\boldsymbol{F}_L\|_{\mathcal{F}^{0,1}_\nu} + \tfrac{147}{2}D_2\|\boldsymbol{F}_0\|_{\mathcal{F}^{0,1}_\nu}\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{2,1}_\nu} + \tfrac{9}{4}D_3\|\boldsymbol{F}_N\|_{\mathcal{F}^{0,1}_\nu}, \tag{3-1}$$

where $D_i = D_i(\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}, \nu_\infty) \approx 1$ are increasing functions of $\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}$ and ν_∞ such that

$$\lim_{\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}\to 0^+} D_i(\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}, 0) = 1$$

and are defined in (3-54).

In the proof, the following multiplier will come up frequently:

$$m(k,\eta) \overset{\text{def}}{=} \frac{1-\dfrac{\sin(k\eta/2)}{k\tan(\eta/2)}e^{-ik\eta/2}}{2\sin(\eta/2)}, \quad |k|\ge 1, \tag{3-2}$$

and we define $m(0,\eta)=0$.

Now let $n \ge 1$, $k=k_0$, $k_1,\dots,k_{2n}$ be integers that further satisfy $|k_j-k_{j+1}|\ge 1$ for all $j=0,1,\dots,2n-1$. We define the integral of type $I_n=I_n(k,k_1,\dots,k_{2n})$ by

$$I_n \overset{\text{def}}{=} \mathrm{pv}\int_{-\pi}^{\pi} m(k-k_1,\eta)\prod_{j=1}^{2n-1}\frac{\sin((k_j-k_{j+1})\eta/2)}{(k_j-k_{j+1})\sin(\eta/2)}e^{-i(k_1+k_{2n})\eta/2}\,d\eta. \tag{3-3}$$

We further define $I_n=0$ if $k_j=k_{j+1}$ for any $j=0,1,\dots,2n-1$. We will also consider the integral, $I'_n=I'_n(k_1,\dots,k_{2n})$, under the same conditions

$$I'_n \overset{\text{def}}{=} \mathrm{pv}\int_{-\pi}^{\pi}\frac{\sin((k_1+k_{2n})\eta/2)}{\sin(\eta/2)}\prod_{j=1}^{2n-1}\frac{\sin((k_j-k_{j+1})\eta/2)}{(k_j-k_{j+1})\sin(\eta/2)}\,d\eta. \tag{3-4}$$

We again define $I'_n=0$ if $k_j=k_{j+1}$ for any $j=1,\dots,2n-1$. In the proofs of the a priori estimates in this section we will frequently use the following lemma.

Lemma 3.2. *We recall* (3-2), (3-3) *and* (3-4). *Then, the following uniform bounds hold*:

$$|I_n(k,k_1,\dots,k_{2n})|\le 2\pi,$$
$$|I'_n(k_1,\dots,k_{2n})|\le 2\pi.$$

This lemma will be proven at the end of this section.

Proof of Proposition 3.1. We first take a derivative of $\boldsymbol{\mathcal{N}}(\boldsymbol{X}_c,\boldsymbol{X})(\theta)$ in (2-16) and let

$$\partial_\theta\boldsymbol{\mathcal{N}}(\boldsymbol{X}_c,\boldsymbol{X})(\theta)=\boldsymbol{\mathcal{N}}_1(\theta)+\boldsymbol{\mathcal{N}}_2(\theta)+\boldsymbol{\mathcal{N}}_3(\theta)+\boldsymbol{\mathcal{N}}_4(\theta), \tag{3-5}$$

where

$$\boldsymbol{\mathcal{N}}_1(\theta)=\int_{\mathbb{S}}\partial_\theta\big(G_L(\Delta_\eta\boldsymbol{X}_c(\theta),\Delta_\eta\boldsymbol{X}(\theta))\big)\boldsymbol{F}_L(\eta)\,d\eta,$$
$$\boldsymbol{\mathcal{N}}_2(\theta)=\int_{\mathbb{S}}\partial_\theta\big(G_N(\Delta_\eta\boldsymbol{X}_c(\theta),\Delta_\eta\boldsymbol{X}(\theta))\big)\boldsymbol{F}_0(\eta)\,d\eta,$$
$$\boldsymbol{\mathcal{N}}_3(\theta)=\int_{\mathbb{S}}\partial_\theta\big(G_N(\Delta_\eta\boldsymbol{X}_c(\theta),\Delta_\eta\boldsymbol{X}(\theta))\big)\boldsymbol{F}_L(\eta)\,d\eta,$$
$$\boldsymbol{\mathcal{N}}_4(\theta)=\int_{\mathbb{S}}\partial_\theta\big(G(\boldsymbol{\mathcal{X}}(\theta)-\boldsymbol{\mathcal{X}}(\eta))\big)\boldsymbol{F}_N(\eta)\,d\eta.$$

We will bound $\boldsymbol{\mathcal{N}}_i$ in $\mathcal{F}^{0,1}_\nu$ for $i=1,2,3,4$.

<u>$\boldsymbol{\mathcal{N}}_1$ estimates</u>: Taking a derivative in (2-13), we obtain

$$\boldsymbol{\mathcal{N}}_1(\theta)=\sum_{i=0}^{10}\boldsymbol{\mathcal{N}}_{1,i}(\theta), \tag{3-6}$$

and we proceed to bound each of these terms in $\mathcal{F}^{0,1}_\nu$. We note that each term $\mathcal{N}_{1,i}$ corresponds to when the derivative hits a different term inside (2-13). The terms $\mathcal{N}_{1,i}$ are written in (3-8), (3-15), (3-16), (3-19), and (3-20) in the following.

The first term $\mathcal{N}_{1,1}(\theta)$ is given by

$$\mathcal{N}_{1,1}(\theta) = -\frac{1}{4\pi R^2}\int_{\mathbb{S}} \partial_\theta \Delta_\eta \boldsymbol{X}_c(\theta)\cdot \Delta_\eta \boldsymbol{X}(\theta)\boldsymbol{F}_L(\eta)\,d\eta.$$

We first take the derivative of $\Delta_\eta \boldsymbol{X}_c(\theta)$ in (2-2) to obtain

$$\partial_\theta \Delta_\eta \boldsymbol{X}_c(\theta) = \frac{\partial_\theta \boldsymbol{X}_c(\theta) - \dfrac{(\boldsymbol{X}_c(\theta)-\boldsymbol{X}_c(\eta))}{2\tan((\theta-\eta)/2)}}{2\sin((\theta-\eta)/2)}.$$

Further define the operator $\mathcal{D}^2(\boldsymbol{X}_c)$ (and analogously $\mathcal{D}^2(\boldsymbol{X})$) to be $\partial_\theta \Delta_\eta \boldsymbol{X}_c(\theta)$ as above after taking the change of variables $\eta \leftarrow \theta-\eta$ as follows:

$$\mathcal{D}^2(\boldsymbol{X}_c)(\theta,\eta) \stackrel{\text{def}}{=} \frac{\partial_\theta \boldsymbol{X}_c(\theta) - \dfrac{\boldsymbol{X}_c(\theta)-\boldsymbol{X}_c(\theta-\eta)}{2\tan(\eta/2)}}{2\sin(\eta/2)}. \tag{3-7}$$

Then we make the change of variables $\eta \leftarrow \theta-\eta$ to obtain

$$\mathcal{N}_{1,1}(\theta) = -\frac{1}{4\pi R^2}\int_{\mathbb{S}} \mathcal{D}^2(\boldsymbol{X}_c)(\theta,\eta)^T \Delta_{\theta-\eta}\boldsymbol{X}(\theta)\boldsymbol{F}_L(\theta-\eta)\,d\eta, \tag{3-8}$$

where we used transpose notation instead of a dot for future convenience in the notation. We will also make extensive use of the identities

$$\begin{aligned}\widehat{\Delta_{\theta-\eta}\boldsymbol{X}}(k) &= \frac{1-e^{-ik\eta}}{2\sin(\eta/2)}\widehat{\boldsymbol{X}}(k) = \frac{\sin(k\eta/2)}{k\sin(\eta/2)}e^{-ik\eta/2}\widehat{\partial_\theta \boldsymbol{X}}(k),\\ \widehat{\Delta_{\theta-\eta}\boldsymbol{X}_c}(k) &= \frac{1-e^{-ik\eta}}{2\sin(\eta/2)}\widehat{\boldsymbol{X}}_c(k) = \frac{\sin(k\eta/2)}{k\sin(\eta/2)}e^{-ik\eta/2}\widehat{\partial_\theta \boldsymbol{X}_c}(k).\end{aligned} \tag{3-9}$$

We remark that both terms above are equal to 0 when $k=0$. We further have

$$\widehat{\mathcal{D}^2(\boldsymbol{X}_c)}(k) = m(k,\eta)\widehat{\partial_\theta \boldsymbol{X}_c}(k), \tag{3-10}$$

where $m(k,\eta)$ is given by (3-2).

Regarding the Fourier coefficients of the derivative of the circle (2-1) we have

$$\widehat{\partial_\theta \boldsymbol{X}_c}(k) = \frac{a+ib}{2}\delta_1(k)\begin{bmatrix} i\\ 1\end{bmatrix} - \frac{a-ib}{2}\delta_{-1}(k)\begin{bmatrix} i\\ -1\end{bmatrix}. \tag{3-11}$$

Taking Fourier transform in (3-8), we obtain

$$\begin{aligned}\widehat{\mathcal{N}}_{1,1}(k) &= -\frac{1}{4\pi R^2}\int_{\mathbb{S}} \widehat{\mathcal{D}^2(\boldsymbol{X}_c)}(k)^T * \widehat{\Delta_{\theta-\eta}\boldsymbol{X}}(k) * e^{-ik\eta}\widehat{\boldsymbol{F}}_L(k)\,d\eta\\ &= -\frac{1}{4\pi R^2}\int_{\mathbb{S}} \sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} \widehat{\mathcal{D}^2(\boldsymbol{X}_c)}(k-k_1)^T\,\widehat{\Delta_{\theta-\eta}\boldsymbol{X}}(k_1-k_2)e^{-ik_2\eta}\widehat{\boldsymbol{F}}_L(k_2)\,d\eta,\end{aligned}$$

and plugging in (3-9) and (3-10) we have

$$\widehat{\mathcal{N}}_{1,1}(k) = -\frac{1}{4\pi R^2}\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} \widehat{\partial_\theta \boldsymbol{X}_c}(k-k_1)^T \widehat{\partial_\theta \boldsymbol{X}}(k_1-k_2)\widehat{\boldsymbol{F}}_L(k_2)I_1(k,k_1,k_2),$$

with I_1 given by (3-3). By Lemma 3.2 we have $|I_1(k,k_1,k_2)|\le 2\pi$. Then we get

$$|\widehat{\mathcal{N}}_{1,1}(k)| \le \frac{1}{2R^2}\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} |\widehat{\partial_\theta \boldsymbol{X}_c}(k-k_1)^T \widehat{\partial_\theta \boldsymbol{X}}(k_1-k_2)||\widehat{\boldsymbol{F}}_L(k_2)|. \tag{3-12}$$

Then, it follows from (3-11) that

$$|\widehat{\partial_\theta \boldsymbol{X}_c}(k-k_1)| \le \frac{R}{\sqrt{2}}\delta_{1,-1}(k-k_1). \tag{3-13}$$

We will now also use the notation (1-22). In particular we have

$$|\widehat{\partial_\theta \boldsymbol{X}_c}(k-k_1)^T \widehat{\partial_\theta \boldsymbol{X}}(k_1-k_2)| \le \frac{\sqrt{2}}{2}R(\delta_1(k-k_1)+\delta_{-1}(k-k_1))|\widehat{\partial_\theta \boldsymbol{X}}(k_1-k_2)|.$$

Therefore, we can write

$$|\widehat{\mathcal{N}}_{1,1}(k)| \le \frac{\sqrt{2}}{4R}\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} \delta_{1,-1}(k-k_1)|\widehat{\partial_\theta \boldsymbol{X}}(k_1-k_2)||\widehat{\boldsymbol{F}}_L(k_2)|.$$

We multiply by $e^{\nu(t)k} = e^{\nu(t)(k-k_1)}e^{\nu(t)(k_1-k_2)}e^{\nu(t)k_2}$ to get

$$e^{\nu(t)k}|\widehat{\mathcal{N}}_{1,1}(k)| \le \frac{\sqrt{2}}{4R}\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} e^{\nu(t)(k-k_1)}\delta_{1,-1}(k-k_1)e^{\nu(t)(k_1-k_2)}|\widehat{\partial_\theta \boldsymbol{X}}(k_1-k_2)|e^{\nu(t)k_2}|\widehat{\boldsymbol{F}}_L(k_2)|,$$

so Young's inequality for convolutions and the estimate (2-42) yield the bound

$$\|\mathcal{N}_{1,1}\|_{\mathcal{F}^{0,1}_\nu} \le \frac{e^{\nu_\infty}\sqrt{2}\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}}{2\sqrt{1-\frac{1}{2}\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}^{1,1}_\nu}}}\|\boldsymbol{F}_L\|_{\mathcal{F}^{0,1}_\nu}. \tag{3-14}$$

This is our desired estimate for $\mathcal{N}_{1,1}$.

We now proceed to estimate $\mathcal{N}_{1,2}$ as

$$\mathcal{N}_{1,2}(\theta) = -\frac{1}{4\pi R^2}\int_{\mathbb{S}} \Delta_{\theta-\eta}\boldsymbol{X}_c(\theta)^T \boldsymbol{\mathcal{D}}^2(\boldsymbol{X})(\theta,\eta)\boldsymbol{F}_L(\theta-\eta)\,d\eta, \tag{3-15}$$

with Fourier transform given by

$$\widehat{\mathcal{N}}_{1,2}(k) = -\frac{1}{4\pi R^2}\int_{\mathbb{S}}\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} \widehat{\Delta_{\theta-\eta}\boldsymbol{X}_c}(k_1-k_2)^T \widehat{\boldsymbol{\mathcal{D}}^2(\boldsymbol{X})}(k-k_1)e^{-ik_2\eta}\widehat{\boldsymbol{F}}_L(k_2)\,d\eta.$$

Using again (3-9) and (3-10), we can write it as

$$\widehat{\mathcal{N}}_{1,2}(k) = -\frac{1}{4\pi R^2}\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} \widehat{\partial_\theta \boldsymbol{X}_c}(k_1-k_2)^T \widehat{\partial_\theta \boldsymbol{X}}(k-k_1)\widehat{\boldsymbol{F}}_L(k_2)I_1(k,k_1,k_2),$$

with I_1 given by (3-3). Using Lemma 3.2, we find that

$$|\widehat{\mathcal{N}}_{1,2}(k)| \le \frac{1}{2R^2} \sum_{k_1\in\mathbb{Z}} \sum_{k_2\in\mathbb{Z}} |\widehat{\partial_\theta \boldsymbol{X}_c}(k-k_1)^T \widehat{\partial_\theta \boldsymbol{X}}(k_1-k_2)|\,|\widehat{\boldsymbol{F}}_L(k_2)|,$$

so following the steps after (3-12) we conclude that

$$\|\mathcal{N}_{1,2}\|_{\mathcal{F}^{0,1}_\nu} \le \frac{e^{\nu_\infty}\sqrt{2}\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}}{2\sqrt{1-\frac{1}{2}\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}^{1,1}_\nu}}} \|\boldsymbol{F}_L\|_{\mathcal{F}^{0,1}_\nu}.$$

This completes our bound for $\mathcal{N}_{1,2}$.

The term $\mathcal{N}_{1,3}$ is given by

$$\mathcal{N}_{1,3}(\theta) = -\frac{1}{R^4}\int_{\mathbb{S}} \mathcal{D}^2(\boldsymbol{X}_c)(\theta,\eta)^T \Delta_{\theta-\eta}\boldsymbol{X}(\theta)\Delta_{\theta-\eta}\boldsymbol{X}_c(\theta)\otimes\Delta_{\theta-\eta}\boldsymbol{X}_c(\theta)\boldsymbol{F}_L(\theta-\eta)\frac{d\eta}{2\pi}, \tag{3-16}$$

and its Fourier transform by

$$\widehat{\mathcal{N}}_{1,3}(k) = -\frac{1}{2\pi\,\widehat{\partial_\theta \boldsymbol{X}_c}(k_2-k_3)R^4} \sum_{k_1\in\mathbb{Z}}\cdots\sum_{k_4\in\mathbb{Z}} \widehat{\partial_\theta \boldsymbol{X}_c}(k-k_1)^T\widehat{\partial_\theta \boldsymbol{X}}(k_1-k_2) \otimes\widehat{\partial_\theta \boldsymbol{X}_c}(k_3-k_4)\widehat{\boldsymbol{F}}_L(k_4) I_2(k,k_1,\dots,k_4),$$

with $I_2(k,k_1,\dots,k_4)$ given by (3-3). Since $|I_2(k,\dots,k_4)| \le 2\pi$ from Lemma 3.2, we have

$$|\widehat{\mathcal{N}}_{1,3}(k)| \le \frac{1}{R^4}\sum_{k_1\in\mathbb{Z}}\cdots\sum_{k_4\in\mathbb{Z}} |\widehat{\partial_\theta \boldsymbol{X}_c}(k-k_1)^T\widehat{\partial_\theta \boldsymbol{X}}(k_1-k_2)|\,\|\widehat{\partial_\theta \boldsymbol{X}_c}(k_2-k_3)\otimes\widehat{\partial_\theta \boldsymbol{X}_c}(k_3-k_4)\|\,|\widehat{\boldsymbol{F}}_L(k_4)|. \tag{3-17}$$

Expression (3-11) gives

$$\begin{aligned}\widehat{\partial_\theta \boldsymbol{X}_c}(k_2-k_3)\otimes\widehat{\partial_\theta \boldsymbol{X}_c}(k_3-k_4) &= \frac{(a+ib)^2}{4}\delta_1(k_2-k_3)\delta_1(k_3-k_4)\begin{bmatrix}-1 & i\\ i & 1\end{bmatrix} + \frac{(a-ib)^2}{4}\delta_{-1}(k_2-k_3)\delta_{-1}(k_3-k_4)\begin{bmatrix}-1 & -i\\ -i & 1\end{bmatrix}\\ &\quad -\frac{(a+ib)(a-ib)}{4}\delta_1(k_2-k_3)\delta_{-1}(k_3-k_4)\begin{bmatrix}-1 & -i\\ i & -1\end{bmatrix} - \frac{(a+ib)(a-ib)}{4}\delta_{-1}(k_2-k_3)\delta_1(k_3-k_4)\begin{bmatrix}-1 & i\\ -i & -1\end{bmatrix}.\end{aligned}$$

All the matrices above have norm equal to 2, so that

$$\|\widehat{\partial_\theta \boldsymbol{X}_c}(k_2-k_3)\otimes\widehat{\partial_\theta \boldsymbol{X}_c}(k_3-k_4)\| \le \frac{R^2}{2}\delta_{1,-1}(k_2-k_3)\delta_{1,-1}(k_3-k_4). \tag{3-18}$$

Introducing this bound, together with (3-13), back to (3-17), we find that

$$|\widehat{\mathcal{N}}_{1,3}(k)| \le \frac{\sqrt{2}}{4R}\sum_{k_1\in\mathbb{Z}}\cdots\sum_{k_4\in\mathbb{Z}} \delta_{1,-1}(k-k_1)|\widehat{\partial_\theta \boldsymbol{X}}(k_1-k_2)|\delta_{1,-1}(k_2-k_3)\delta_{1,-1}(k_3-k_4)|\widehat{\boldsymbol{F}}_L(k_4)|;$$

thus multiplication by the exponential $e^{\nu(t)k}$, Young's inequality and (2-42) yield that

$$\|\mathcal{N}_{1,3}\|_{\mathcal{F}^{0,1}_\nu} \le \frac{2\sqrt{2}e^{3\nu_\infty}\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}}{\sqrt{1-\frac{1}{2}\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}^{1,1}_\nu}}}\|\boldsymbol{F}_L\|_{\mathcal{F}^{0,1}_\nu}.$$

This completes our bound for $\mathcal{N}_{1,3}$.

The term $\boldsymbol{\mathcal{N}}_{1,4}$ is given by

$$\boldsymbol{\mathcal{N}}_{1,4}(\theta) = -\frac{1}{R^4}\int_{\mathbb{S}} \Delta_{\theta-\eta}\boldsymbol{X}_c(\theta)^T \boldsymbol{\mathcal{D}}^2(\boldsymbol{X})(\theta,\eta)\Delta_{\theta-\eta}\boldsymbol{X}_c(\theta)\otimes\Delta_{\theta-\eta}\boldsymbol{X}_c(\theta)\boldsymbol{F}_L(\theta-\eta)\frac{d\eta}{2\pi}. \tag{3-19}$$

We take the Fourier transform and write the result as

$$\widehat{\boldsymbol{\mathcal{N}}}_{1,4}(k) = -\frac{1}{2\pi R^4}\sum_{k_1\in\mathbb{Z}}\cdots\sum_{k_4\in\mathbb{Z}} \widehat{\partial_\theta \boldsymbol{X}_c}(k_1-k_2)^T \widehat{\partial_\theta \boldsymbol{X}}(k-k_1)\widehat{\partial_\theta \boldsymbol{X}_c}(k_2-k_3) \otimes\widehat{\partial_\theta \boldsymbol{X}_c}(k_3-k_4)\widehat{\boldsymbol{F}}_L(k_4)I_2(k,k_1,\dots,k_4),$$

with $I_2(k,k_1,\dots,k_4)$ given by (3-3). Since $|I_2|\le 2\pi$ by Lemma 3.2, comparing now with (3-17), we conclude that

$$\|\boldsymbol{\mathcal{N}}_{1,4}\|_{\mathcal{F}^{0,1}_\nu} \le \frac{2\sqrt{2}e^{3\nu_\infty}\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}}{\sqrt{1-\frac{1}{2}\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}^{1,1}_\nu}}}\|\boldsymbol{F}_L\|_{\mathcal{F}^{0,1}_\nu}.$$

This completes our estimate for $\boldsymbol{\mathcal{N}}_{1,4}$.

The remaining terms from $\mathcal{N}_1(\theta)$ in (3-6) are

$$\begin{aligned}
\boldsymbol{\mathcal{N}}_{1,5}(\theta) &= -\frac{1}{R^4}\int_{\mathbb{S}} \Delta_{\theta-\eta}\boldsymbol{X}_c(\theta)^T\Delta_{\theta-\eta}\boldsymbol{X}(\theta)\boldsymbol{\mathcal{D}}^2(\boldsymbol{X}_c)(\theta,\eta)\otimes\Delta_{\theta-\eta}\boldsymbol{X}_c(\theta)\boldsymbol{F}_L(\theta-\eta)\frac{d\eta}{2\pi},\\
\boldsymbol{\mathcal{N}}_{1,6}(\theta) &= -\frac{1}{R^4}\int_{\mathbb{S}} \Delta_{\theta-\eta}\boldsymbol{X}_c(\theta)^T\Delta_{\theta-\eta}\boldsymbol{X}(\theta)\Delta_{\theta-\eta}\boldsymbol{X}_c(\theta)\otimes\boldsymbol{\mathcal{D}}^2(\boldsymbol{X}_c)(\theta,\eta)\boldsymbol{F}_L(\theta-\eta)\frac{d\eta}{2\pi},\\
\boldsymbol{\mathcal{N}}_{1,7}(\theta) &= \frac{1}{4\pi R^2}\int_{\mathbb{S}} \boldsymbol{\mathcal{D}}^2(\boldsymbol{X}_c)(\theta,\eta)\otimes\Delta_{\theta-\eta}\boldsymbol{X}(\theta)\boldsymbol{F}_L(\theta-\eta)\,d\eta,\\
\boldsymbol{\mathcal{N}}_{1,8}(\theta) &= \frac{1}{4\pi R^2}\int_{\mathbb{S}} \Delta_{\theta-\eta}\boldsymbol{X}_c(\theta)\otimes\boldsymbol{\mathcal{D}}^2(\boldsymbol{X})(\theta,\eta)\boldsymbol{F}_L(\theta-\eta)\,d\eta,\\
\boldsymbol{\mathcal{N}}_{1,9}(\theta) &= \frac{1}{4\pi R^2}\int_{\mathbb{S}} \boldsymbol{\mathcal{D}}^2(\boldsymbol{X})(\theta,\eta)\otimes\Delta_{\theta-\eta}\boldsymbol{X}_c(\theta)\boldsymbol{F}_L(\theta-\eta)\,d\eta,\\
\boldsymbol{\mathcal{N}}_{1,10}(\theta) &= \frac{1}{4\pi R^2}\int_{\mathbb{S}} \Delta_{\theta-\eta}\boldsymbol{X}(\theta)\otimes\boldsymbol{\mathcal{D}}^2(\boldsymbol{X}_c)(\theta,\eta)\boldsymbol{F}_L(\theta-\eta)\,d\eta.
\end{aligned} \tag{3-20}$$

It is not hard to see that $\boldsymbol{\mathcal{N}}_{1,5}$ and $\boldsymbol{\mathcal{N}}_{1,6}$ are bounded exactly as $\boldsymbol{\mathcal{N}}_{1,3}$ in (3-16), since the bound (3-18) is also valid for $\boldsymbol{\mathcal{D}}^2(\boldsymbol{X}_c)(\theta,\eta)\otimes\Delta_{\theta-\eta}\boldsymbol{X}_c(\theta)$ or $\Delta_{\theta-\eta}\boldsymbol{X}_c(\theta)\otimes\boldsymbol{\mathcal{D}}^2(\boldsymbol{X}_c)(\theta,\eta)$.

We proceed then with $\boldsymbol{\mathcal{N}}_{1,7}$. Comparing with $\boldsymbol{\mathcal{N}}_{1,1}$ in (3-8), (3-12), we obtain

$$|\widehat{\boldsymbol{\mathcal{N}}}_{1,7}(k)| \le \frac{1}{2R^2}\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} \|\widehat{\partial_\theta \boldsymbol{X}_c}(k-k_1)\otimes\widehat{\partial_\theta \boldsymbol{X}}(k_1-k_2)\|\,|\widehat{\boldsymbol{F}}_L(k_2)|.$$

Using (3-11), we find that

$$\begin{aligned}
\|\widehat{\partial_\theta \boldsymbol{X}_c}(k-k_1)\otimes\widehat{\partial_\theta \boldsymbol{X}}(k_1-k_2)\| &\le \frac{R}{2}\delta_1(k-k_1)\left\|\begin{bmatrix}-1\\ i\end{bmatrix}\widehat{\partial_\theta \boldsymbol{X}}(k_1-k_2)\right\| + \frac{R}{2}\delta_{-1}(k-k_1)\left\|\begin{bmatrix}-1\\ -i\end{bmatrix}\widehat{\partial_\theta \boldsymbol{X}}(k_1-k_2)\right\|\\
&\le \frac{\sqrt{2}}{2}R\delta_{1,-1}(k-k_1)|\widehat{\partial_\theta \boldsymbol{X}}(k_1-k_2)|,
\end{aligned} \tag{3-21}$$

where in the last inequality we have used that the matrix norm (1-19) is bounded by the Frobenius norm. Therefore we conclude that

$$\|\boldsymbol{\mathcal{N}}_{1,7}\|_{\mathcal{F}_\nu^{0,1}} \le \frac{\sqrt{2}e^{\nu_\infty}\|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}}}{2\sqrt{1-\frac{1}{2}\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}_\nu^{1,1}}}}\|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}}.$$

The bound for $\boldsymbol{\mathcal{N}}_{1,8}$ follows in the same way as that of $\boldsymbol{\mathcal{N}}_{1,7}$:

$$\|\boldsymbol{\mathcal{N}}_{1,8}\|_{\mathcal{F}_\nu^{0,1}} \le \frac{\sqrt{2}e^{\nu_\infty}\|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}}}{2\sqrt{1-\frac{1}{2}\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}_\nu^{1,1}}}}\|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}}.$$

Finally, the bounds for $\boldsymbol{\mathcal{N}}_{1,9}$ and $\boldsymbol{\mathcal{N}}_{1,10}$ are the same as for $\boldsymbol{\mathcal{N}}_{1,7}$ and $\boldsymbol{\mathcal{N}}_{1,8}$ because

$$\begin{aligned}
&\|\widehat{\partial_\theta \boldsymbol{X}}(k-k_1)\otimes\widehat{\partial_\theta \boldsymbol{X}_c}(k_1-k_2)\| \\
&\qquad\le \frac{R}{2}\delta_1(k_1-k_2)\|\widehat{\partial_\theta \boldsymbol{X}}(k-k_1)[-1\ \ i]\| + \frac{R}{2}\delta_{-1}(k_1-k_2)\|\widehat{\partial_\theta \boldsymbol{X}}(k-k_1)[-1\ \ -i]\| \\
&\qquad\le \frac{\sqrt{2}}{2}R\delta_{1,-1}(k_1-k_2)|\widehat{\partial_\theta \boldsymbol{X}}(k-k_1)|.
\end{aligned}\tag{3-22}$$

Joining the bounds for $\boldsymbol{\mathcal{N}}_{1,1}$ to $\boldsymbol{\mathcal{N}}_{1,10}$, we obtain the bound for $\boldsymbol{\mathcal{N}}_1$ in (3-6) as

$$\|\boldsymbol{\mathcal{N}}_1\|_{\mathcal{F}_\nu^{0,1}} \le 11\sqrt{2}e^{3\nu_\infty}C_1\|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}}\|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}},\tag{3-23}$$

where C_1 is defined in (2-43). This completes our estimates for the $\boldsymbol{\mathcal{N}}_1$ term.

<u>$\boldsymbol{\mathcal{N}}_3$ estimates</u>: Taking a derivative in (2-14), we split $\boldsymbol{\mathcal{N}}_3$ as

$$\boldsymbol{\mathcal{N}}_3(\theta) = \sum_{i=1}^{11}\boldsymbol{\mathcal{N}}_{3,i},\tag{3-24}$$

where

$$\begin{aligned}
\boldsymbol{\mathcal{N}}_{3,1}(\theta) &= -\frac{1}{4\pi}\int_{\mathbb{S}}\partial_\theta\mathcal{R}_1(\Delta_\eta \boldsymbol{X}(\theta))\boldsymbol{F}_L(\eta)\,d\eta,\\
\boldsymbol{\mathcal{N}}_{3,2}(\theta) &= \frac{1}{4\pi R^2}\int_{\mathbb{S}}\partial_\theta\big(\Delta_\eta \boldsymbol{X}(\theta)\otimes\Delta_\eta \boldsymbol{X}(\theta)\big)\Big(1-\frac{2}{R^2}\Delta_\eta \boldsymbol{X}_c(\theta)^T\Delta_\eta \boldsymbol{X}(\theta)\Big)\boldsymbol{F}_L(\eta)\,d\eta,\\
\boldsymbol{\mathcal{N}}_{3,3}(\theta) &= -\frac{1}{2\pi R^4}\int_{\mathbb{S}}\Delta_\eta \boldsymbol{X}(\theta)\otimes\Delta_\eta \boldsymbol{X}(\theta)\partial_\theta\big(\Delta_\eta \boldsymbol{X}_c(\theta)^T\Delta_\eta \boldsymbol{X}(\theta)\big)\boldsymbol{F}_L(\eta)\,d\eta,\\
\boldsymbol{\mathcal{N}}_{3,4}(\theta) &= \frac{1}{4\pi R^2}\int_{\mathbb{S}}\partial_\theta\big(\Delta_\eta \boldsymbol{X}(\theta)\otimes\Delta_\eta \boldsymbol{X}(\theta)\big)\mathcal{R}_2(\Delta_\eta \boldsymbol{X}(\theta))\boldsymbol{F}_L(\eta)\,d\eta,\\
\boldsymbol{\mathcal{N}}_{3,5}(\theta) &= \frac{1}{4\pi R^2}\int_{\mathbb{S}}\Delta_\eta \boldsymbol{X}(\theta)\otimes\Delta_\eta \boldsymbol{X}(\theta)\partial_\theta\mathcal{R}_2(\Delta_\eta \boldsymbol{X}(\theta))\boldsymbol{F}_L(\eta)\,d\eta,\\
\boldsymbol{\mathcal{N}}_{3,6}(\theta) &= -\frac{1}{2\pi R^4}\int_{\mathbb{S}}\partial_\theta\big(\Delta_\eta \boldsymbol{X}_c(\theta)\otimes\Delta_\eta \boldsymbol{X}(\theta)+\Delta_\eta \boldsymbol{X}(\theta)\otimes\Delta_\eta \boldsymbol{X}_c(\theta)\big)\Delta_\eta \boldsymbol{X}_c(\theta)^T\Delta_\eta \boldsymbol{X}(\theta)\boldsymbol{F}_L(\eta)\,d\eta,\\
\boldsymbol{\mathcal{N}}_{3,7}(\theta) &= -\frac{1}{2\pi R^4}\int_{\mathbb{S}}\big(\Delta_\eta \boldsymbol{X}_c(\theta)\otimes\Delta_\eta \boldsymbol{X}(\theta)+\Delta_\eta \boldsymbol{X}(\theta)\otimes\Delta_\eta \boldsymbol{X}_c(\theta)\big)\partial_\theta\big(\Delta_\eta \boldsymbol{X}_c(\theta)^T\Delta_\eta \boldsymbol{X}(\theta)\big)\boldsymbol{F}_L(\eta)\,d\eta,\\
\boldsymbol{\mathcal{N}}_{3,8}(\theta) &= \frac{1}{4\pi R^2}\int_{\mathbb{S}}\partial_\theta\big(\Delta_\eta \boldsymbol{X}_c(\theta)\otimes\Delta_\eta \boldsymbol{X}(\theta)+\Delta_\eta \boldsymbol{X}(\theta)\otimes\Delta_\eta \boldsymbol{X}_c(\theta)\big)\mathcal{R}_2(\Delta_\eta \boldsymbol{X}(\theta))\boldsymbol{F}_L(\eta)\,d\eta,
\end{aligned}$$

$$\mathcal{N}_{3,9}(\theta) = \frac{1}{4\pi R^2}\int_{\mathbb{S}}\big(\Delta_\eta \boldsymbol{X}_c(\theta)\otimes\Delta_\eta \boldsymbol{X}(\theta)+\Delta_\eta \boldsymbol{X}(\theta)\otimes\Delta_\eta \boldsymbol{X}_c(\theta)\big)\partial_\theta\mathcal{R}_2(\Delta_\eta \boldsymbol{X}(\theta))\boldsymbol{F}_L(\eta)\,d\eta,$$
$$\mathcal{N}_{3,10}(\theta) = \frac{1}{4\pi R^2}\int_{\mathbb{S}}\partial_\theta\big(\Delta_\eta \boldsymbol{X}_c(\theta)\otimes\Delta_\eta \boldsymbol{X}_c(\theta)\big)\mathcal{R}_2(\Delta_\eta \boldsymbol{X}(\theta))\boldsymbol{F}_L(\eta)\,d\eta,$$
$$\mathcal{N}_{3,11}(\theta) = \frac{1}{4\pi R^2}\int_{\mathbb{S}}\Delta_\eta \boldsymbol{X}_c(\theta)\otimes\Delta_\eta \boldsymbol{X}_c(\theta)\partial_\theta\mathcal{R}_2(\Delta_\eta \boldsymbol{X}(\theta))\boldsymbol{F}_L(\eta)\,d\eta,$$

where $\mathcal{R}_1$ and $\mathcal{R}_2$ were defined in (2-7) and (2-9).

We proceed with $\mathcal{N}_{3,1}$ first. We take the derivative in (2-7) to obtain

$$\mathcal{N}_{3,1}(\theta) = \boldsymbol{O}_1(\theta)+\boldsymbol{O}_2(\theta)+\boldsymbol{O}_3(\theta), \tag{3-25}$$

where

$$\begin{aligned}\boldsymbol{O}_1(\theta) = -\frac{1}{8\pi}\int_{\mathbb{S}}\sum_{n\geq 1}\sum_{\substack{m=0\\ n+m\geq 2}}^{n-1}\binom{n}{m}\frac{(-1)^{n-1}(n-m)}{nR^{2n}}&(2\Delta_\eta \boldsymbol{X}_c(\theta)^T\Delta_\eta \boldsymbol{X}(\theta))^{n-m-1}\\ &\cdot 2\partial_\theta\Delta_\eta \boldsymbol{X}_c(\theta)^T\Delta_\eta \boldsymbol{X}(\theta)|\Delta_\eta \boldsymbol{X}(\theta)|^{2m}\boldsymbol{F}_L(\eta)\,d\eta,\\
\boldsymbol{O}_2(\theta) = -\frac{1}{8\pi}\int_{\mathbb{S}}\sum_{n\geq 1}\sum_{\substack{m=0\\ n+m\geq 2}}^{n-1}\binom{n}{m}\frac{(-1)^{n-1}(n-m)}{nR^{2n}}&(2\Delta_\eta \boldsymbol{X}_c(\theta)^T\Delta_\eta \boldsymbol{X}(\theta))^{n-m-1}\\ &\cdot 2\Delta_\eta \boldsymbol{X}_c(\theta)^T\partial_\theta\Delta_\eta \boldsymbol{X}(\theta)|\Delta_\eta \boldsymbol{X}(\theta)|^{2m}\boldsymbol{F}_L(\eta)\,d\eta,\\
\boldsymbol{O}_3(\theta) = -\frac{1}{4\pi}\int_{\mathbb{S}}\sum_{n\geq 1}\sum_{m=1}^{n}\binom{n}{m}\frac{(-1)^{n-1}m}{nR^{2n}}&(2\Delta_\eta \boldsymbol{X}_c(\theta)^T\Delta_\eta \boldsymbol{X}(\theta))^{n-m}\\ &\cdot|\Delta_\eta \boldsymbol{X}(\theta)|^{2(m-1)}\Delta_\eta \boldsymbol{X}(\theta)^T\partial_\theta\Delta_\eta \boldsymbol{X}(\theta)\boldsymbol{F}_L(\eta)\,d\eta.\end{aligned}$$

After performing the change of variables $\eta\leftarrow\theta-\eta$, we take Fourier transform of $\boldsymbol{O}_1(\theta)$ to obtain

$$\begin{aligned}\widehat{\boldsymbol{O}}_1(k) = -\frac{1}{8\pi}\int_{\mathbb{S}}\sum_{n\geq 1}\sum_{\substack{m=0\\ n+m\geq 2}}^{n-1}\binom{n}{m}\frac{(-1)^{n-1}(n-m)}{nR^{2n}}&*^{n-m-1}\,\widehat{2\Delta_{\theta-\eta}\boldsymbol{X}_c(\theta)^T\Delta_{\theta-\eta}\boldsymbol{X}(\theta)}\\ &*\widehat{2\mathcal{D}^2(\boldsymbol{X}_c)(\theta)^T\Delta_{\theta-\eta}\boldsymbol{X}}(k)*^m\,\widehat{\Delta_{\theta-\eta}\boldsymbol{X}(\theta)^T\Delta_{\theta-\eta}\boldsymbol{X}(\theta)}*\widehat{\boldsymbol{F}}_L(k)\,d\eta.\end{aligned}$$

Using (3-9) and (3-10), we rewrite it as

$$\begin{aligned}\widehat{\boldsymbol{O}}_1(k) = -\frac{1}{8\pi}\sum_{n\geq 1}\sum_{\substack{m=0\\ n+m\geq 2}}^{n-1}\binom{n}{m}\frac{(-1)^{n-1}(n-m)}{nR^{2n}}&\\ \sum_{k_1}\cdots\sum_{k_{k_{2n}}}\prod_{j=0}^{n-m-2}&2\widehat{\partial_\theta\boldsymbol{X}_c}(k_{2j+1}-k_{2j+2})^T\widehat{\partial_\theta\boldsymbol{X}}(k_{2j+2}-k_{2j+3})\\ &\cdot 2\widehat{\partial_\theta\boldsymbol{X}_c}(k-k_1)^T\widehat{\partial_\theta\boldsymbol{X}}(k_{2n-2m-1}-k_{2n-2m})\\ \prod_{j=n-m}^{n-1}&\widehat{\partial_\theta\boldsymbol{X}}(k_{2j}-k_{2j+1})^T\widehat{\partial_\theta\boldsymbol{X}}(k_{2j+1}-k_{2j+2})\widehat{\boldsymbol{F}}_L(k_{2n})I_n(k,k_1,\dots,k_{2n}),\end{aligned} \tag{3-26}$$

with $|I_n(k,k_1,\dots,k_{2n})|\leq 2\pi$ given by (3-3) and using Lemma 3.2. Above we are using the convention that $\prod_{j=j_1}^{j_2} f(j)\equiv 1$ if $j_2<j_1$. Recalling estimate (3-13), distributing the exponential factor $e^{\nu(t)k}$, and applying Young's inequality, we have

$$\|\boldsymbol{O}_1\|_{\mathcal{F}^{0,1}_\nu}\leq\frac{1}{4}\Bigg(\sum_{n\geq 1}\sum_{\substack{m=0\\ n+m\geq 2}}^{n-1}\binom{n}{m}\frac{(n-m)}{nR^{2n}}(2\sqrt{2})^{n-m}e^{(n-m)\nu(t)}R^{n-m}\|\boldsymbol{X}\|^{n-m}_{\dot{\mathcal{F}}^{1,1}_\nu}\|\boldsymbol{X}\|^{2m}_{\dot{\mathcal{F}}^{1,1}_\nu}\Bigg)\|\boldsymbol{F}_L\|_{\mathcal{F}^{0,1}_\nu},$$

which can be summed first in m to get

$$\|\boldsymbol{O}_1\|_{\mathcal{F}_\nu^{0,1}} \le \frac{1}{4}\sum_{n\ge 2}(2\sqrt{2})^n e^{n\nu(t)}\frac{\|\boldsymbol{X}\|^n_{\dot{\mathcal{F}}_\nu^{1,1}}}{R^n}\left(1+\frac{\|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}}}{2\sqrt{2}e^{\nu(t)}R}\right)^{n-1}\|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}}, \tag{3-27}$$

and then summed in n,

$$\|\boldsymbol{O}_1\|_{\mathcal{F}_\nu^{0,1}} \le \frac{2e^{2\nu_\infty}\left(1+\dfrac{\|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}}}{2\sqrt{2}e^{\nu_\infty}R}\right)}{1-2\sqrt{2}e^{\nu_\infty}\dfrac{\|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}}}{R}\left(1+\dfrac{\|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}}}{2\sqrt{2}e^{\nu_\infty}R}\right)}\frac{\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}_\nu^{1,1}}}{R^2}\|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}}.$$

Using estimate (2-42) and the notation (2-43), we conclude that

$$\|\boldsymbol{O}_1\|_{\mathcal{F}_\nu^{0,1}} \le 2e^{2\nu_\infty}C_2C_1^2\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}_\nu^{1,1}}\|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}}, \tag{3-28}$$

with

$$C_2 = \frac{1+\frac{1}{2\sqrt{2}}e^{-\nu_\infty}C_1\|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}}}{1-2\sqrt{2}e^{\nu_\infty}C_1\|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}}\left(1+\frac{1}{2\sqrt{2}}e^{-\nu_\infty}C_1\|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}}\right)}, \tag{3-29}$$

where C_1 was defined in (2-43).

We proceed with $\boldsymbol{O}_2$ in (3-25). We take Fourier transform and, recalling (3-9), we obtain

$$\begin{aligned}\widehat{\boldsymbol{O}}_2(k) = -\frac{1}{8\pi}\sum_{n\ge 1}\sum_{\substack{m=0\\ n+m\ge 2}}^{n-1}\binom{n}{m}\frac{(-1)^{n-1}(n-m)}{nR^{2n}}\\ \sum_{k_1}\cdots\sum_{k_{k_{2n}}}\prod_{j=0}^{n-m-2} 2\widehat{\partial_\theta \boldsymbol{X}_c}(k_{2j+1}-k_{2j+2})^T\widehat{\partial_\theta \boldsymbol{X}}(k_{2j+2}-k_{2j+3})\\ \cdot 2\widehat{\partial_\theta \boldsymbol{X}_c}(k_{2n-2m-1}-k_{2n-2m})^T\widehat{\partial_\theta \boldsymbol{X}}(k-k_1)\\ \prod_{j=n-m}^{n-1}\widehat{\partial_\theta \boldsymbol{X}}(k_{2j}-k_{2j+1})^T\widehat{\partial_\theta \boldsymbol{X}}(k_{2j+1}-k_{2j+2})\widehat{\boldsymbol{F}}_L(k_{2n})I_n(k,k_1,\dots,k_{2n}),\end{aligned} \tag{3-30}$$

again with $|I_n(k,k_1,\dots,k_{2n})|\le 2\pi$ from (3-3) and Lemma 3.2. Thus, comparing (3-30) with (3-26), we find the estimate for $\boldsymbol{O}_2$,

$$\|\boldsymbol{O}_2\|_{\mathcal{F}_\nu^{0,1}} \le 2e^{2\nu_\infty}C_2C_1^2\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}_\nu^{1,1}}\|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}}, \tag{3-31}$$

with C_2 defined in (3-29) and C_1 in (2-43).

Repeating these steps for $\boldsymbol{O}_3$, we obtain

$$\|\boldsymbol{O}_3\|_{\mathcal{F}_\nu^{0,1}} \le \frac{1}{2}\sum_{n\ge 1}\sum_{m=1}^{n}\binom{n}{m}\frac{m(2\sqrt{2})^{n-m}}{nR^{2n}}e^{\nu(t)(n-m)}R^{n-m}\|\boldsymbol{X}\|^{n-m}_{\dot{\mathcal{F}}_\nu^{1,1}}\|\boldsymbol{X}\|^{2(m-1)}_{\dot{\mathcal{F}}_\nu^{1,1}}\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}_\nu^{1,1}}\|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}},$$

which after summation in m the right side above becomes

$$\begin{aligned}\|\boldsymbol{O}_3\|_{\mathcal{F}_\nu^{0,1}} &\le \frac{1}{2}\sum_{n\ge 1}\frac{1}{nR^{2n}}n(2\sqrt{2}e^{\nu(t)}R\|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}}+\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}_\nu^{1,1}})^{n-1}\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}_\nu^{1,1}}\|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}}\\ &= \frac{1}{2}\sum_{n\ge 1}\frac{\|\boldsymbol{X}\|^{n-1}_{\dot{\mathcal{F}}_\nu^{1,1}}}{R^{n-1}}\left(2\sqrt{2}e^{\nu(t)}+\frac{\|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}}}{R}\right)^{n-1}\frac{\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}_\nu^{1,1}}}{R^2}\|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}},\end{aligned} \tag{3-32}$$

and after summation in n we have

$$\|\boldsymbol{O}_3\|_{\mathcal{F}_\nu^{0,1}} \le \frac{1}{2}\frac{1}{1-2\sqrt{2}e^{\nu_\infty}(\|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}}/R)(1+e^{-\nu_\infty}(\|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}}/2\sqrt{2}R))}\frac{\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}_\nu^{1,1}}}{R^2}\|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}}.$$

Introducing the bound for R in (2-42), we obtain

$$\|\boldsymbol{O}_3\|_{\mathcal{F}_\nu^{0,1}} \le \tfrac{1}{2}C_3C_1^2\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}_\nu^{1,1}}\|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}}, \tag{3-33}$$

and using C_2 in (3-29) and C_1 in (2-43) we have

$$C_3 = \frac{C_2}{1+\frac{1}{2\sqrt{2}}e^{-\nu_\infty}C_1\|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}}}. \tag{3-34}$$

Joining the bounds (3-28), (3-31), and (3-33), we find the estimate for $\boldsymbol{\mathcal{N}}_{3,1}$ from (3-25) as

$$\|\boldsymbol{\mathcal{N}}_{3,1}\|_{\mathcal{F}_\nu^{0,1}} \le \tfrac{9}{2}C_4C_1^2\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}_\nu^{1,1}}\|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}}, \tag{3-35}$$

with

$$C_4 = \tfrac{2}{9}\big(4e^{2\nu_\infty}C_2+\tfrac{1}{2}C_3\big). \tag{3-36}$$

This completes our desired estimate for $\boldsymbol{\mathcal{N}}_{3,1}$.

We continue with the next term $\boldsymbol{\mathcal{N}}_{3,2}$ from (3-24), which we split in two:

$$\boldsymbol{\mathcal{N}}_{3,2}(\theta) = \boldsymbol{O}_4(\theta)+\boldsymbol{O}_5(\theta),$$

where

$$\begin{aligned}\boldsymbol{O}_4(\theta) &= \frac{1}{4\pi R^2}\int_{\mathbb{S}}\partial_\theta\big(\Delta_\eta\boldsymbol{X}(\theta)\otimes\Delta_\eta\boldsymbol{X}(\theta)\big)\boldsymbol{F}_L(\eta)\,d\eta,\\ \boldsymbol{O}_5(\theta) &= -\frac{1}{2\pi R^4}\int_{\mathbb{S}}\partial_\theta\big(\Delta_\eta\boldsymbol{X}(\theta)\otimes\Delta_\eta\boldsymbol{X}(\theta)\big)\Delta_\eta\boldsymbol{X}_c(\theta)^T\Delta_\eta\boldsymbol{X}(\theta)\boldsymbol{F}_L(\eta)\,d\eta.\end{aligned}$$

The bounds for these terms follow in a similar way to that of $\boldsymbol{\mathcal{N}}_{1,2}$ from (3-15) and $\boldsymbol{\mathcal{N}}_{1,4}$ from (3-19), respectively. Taking into account that

$$\|\widehat{\partial_\theta\boldsymbol{X}}(k-k_1)\otimes\widehat{\partial_\theta\boldsymbol{X}}(k_1-k_2)\| \le |\widehat{\partial_\theta\boldsymbol{X}}(k-k_1)||\widehat{\partial_\theta\boldsymbol{X}}(k_1-k_2)|, \tag{3-37}$$

and Lemma 3.2, it is not hard to find that

$$|\widehat{\boldsymbol{O}}_4(k)| \le \frac{1}{R^2}\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}}|\widehat{\partial_\theta\boldsymbol{X}}(k-k_1)||\widehat{\partial_\theta\boldsymbol{X}}(k_1-k_2)||\widehat{\boldsymbol{F}}_L(k_2)|,$$

and recalling (3-13), we have

$$|\widehat{\boldsymbol{O}}_5(k)| \le \frac{\sqrt{2}}{R^3}\sum_{k_1\in\mathbb{Z}}\cdots\sum_{k_4\in\mathbb{Z}}|\widehat{\partial_\theta\boldsymbol{X}}(k-k_1)||\widehat{\partial_\theta\boldsymbol{X}}(k_1-k_2)|\delta_{1,-1}(k_2-k_3)|\widehat{\partial_\theta\boldsymbol{X}}(k_3-k_4)||\widehat{\boldsymbol{F}}_L(k_4)|.$$

Therefore,

$$\|\boldsymbol{O}_4\|_{\mathcal{F}_\nu^{0,1}} \le \frac{\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}_\nu^{1,1}}}{R^2}\|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}}, \qquad \|\boldsymbol{O}_5\|_{\dot{\mathcal{F}}_\nu^{1,1}} \le 2\sqrt{2}e^{\nu_\infty}\frac{\|\boldsymbol{X}\|^3_{\dot{\mathcal{F}}_\nu^{1,1}}}{R^3}\|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}};$$

thus

$$\|\boldsymbol{\mathcal{N}}_{3,2}\|_{\mathcal{F}_\nu^{0,1}} \le \left(1 + 2\sqrt{2}e^{\nu_\infty}\frac{\|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}}}{R}\right)\frac{\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}_\nu^{1,1}}}{R^2}\|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}},$$

so plugging in the estimate (2-42) yields that

$$\|\boldsymbol{\mathcal{N}}_{3,2}\|_{\mathcal{F}_\nu^{0,1}} \le C_5 C_1^2 \|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}_\nu^{1,1}}\|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}}, \tag{3-38}$$

with

$$C_5 = 1 + 2\sqrt{2}e^{\nu_\infty}C_1\|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}}. \tag{3-39}$$

This completes our estimate for $\boldsymbol{\mathcal{N}}_{3,2}$.

The Fourier transform of $\boldsymbol{\mathcal{N}}_{3,3}$ in (3-24) can be bounded as

$$|\widehat{\boldsymbol{\mathcal{N}}}_{3,3}(k)| \le \frac{\sqrt{2}}{2R^3}\sum_{k_1\in\mathbb{Z}}\cdots\sum_{k_4\in\mathbb{Z}}|\widehat{\partial_\theta \boldsymbol{X}}(k-k_1)|\,|\widehat{\partial_\theta \boldsymbol{X}}(k_1-k_2)|\delta_{1,-1}(k_2-k_3)|\widehat{\partial_\theta \boldsymbol{X}}(k_3-k_4)|\,|\widehat{\boldsymbol{F}}_L(k_4)|,$$

and thus

$$\|\boldsymbol{\mathcal{N}}_{3,3}\|_{\mathcal{F}_\nu^{0,1}} \le \sqrt{2}e^{\nu_\infty}\frac{\|\boldsymbol{X}\|^3_{\dot{\mathcal{F}}_\nu^{1,1}}}{R^3}\|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}},$$

which becomes

$$\|\boldsymbol{\mathcal{N}}_{3,3}\|_{\mathcal{F}_\nu^{0,1}} \le \sqrt{2}e^{\nu_\infty}C_1^3\|\boldsymbol{X}\|^3_{\dot{\mathcal{F}}_\nu^{1,1}}\|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}}. \tag{3-40}$$

Similarly, recalling (2-9), the estimate for $\boldsymbol{\mathcal{N}}_{3,4}$ in (3-24) is

$$\|\boldsymbol{\mathcal{N}}_{3,4}\|_{\mathcal{F}_\nu^{0,1}} \le \frac{\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}_\nu^{1,1}}}{R^2}\sum_{n\ge1}\sum_{\substack{m=0\\ n+m\ge2}}^{n}\binom{n}{m}\frac{(2\sqrt{2})^{n-m}e^{(n-m)\nu(t)}R^{n-m}}{R^{2n}}\|\boldsymbol{X}\|^{n-m}_{\dot{\mathcal{F}}_\nu^{1,1}}\|\boldsymbol{X}\|^{2m}_{\dot{\mathcal{F}}_\nu^{1,1}}\|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}},$$

which can be rewritten as

$$\begin{aligned}
&\|\boldsymbol{\mathcal{N}}_{3,4}\|_{\mathcal{F}_\nu^{0,1}}\\
&\quad\le \frac{\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}_\nu^{1,1}}}{R^2}\left(\sum_{n\ge1}\frac{(2\sqrt{2}e^{\nu(t)}R\|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}})^n}{R^{2n}}\left(1+\frac{\|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}}}{2\sqrt{2}e^{\nu(t)}R}\right)^n - \frac{2\sqrt{2}e^{\nu(t)}\|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}}}{R}\right)\|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}}\\
&\quad= \frac{2\sqrt{2}e^{\nu(t)}\|\boldsymbol{X}\|^3_{\dot{\mathcal{F}}_\nu^{1,1}}}{R^3}\left(\sum_{n\ge1}\left(\frac{2\sqrt{2}e^{\nu(t)}\|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}}}{R}\right)^{n-1}\left(1+\frac{\|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}}}{2\sqrt{2}e^{\nu(t)}R}\right)^n - 1\right)\|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}}\\
&\quad= \frac{2\sqrt{2}e^{\nu(t)}\|\boldsymbol{X}\|^3_{\dot{\mathcal{F}}_\nu^{1,1}}}{R^3}\left(\frac{\|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}}}{2\sqrt{2}e^{\nu(t)}R}\right.\\
&\qquad\qquad\left.+\left(1+\frac{\|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}}}{2\sqrt{2}e^{\nu(t)}R}\right)\sum_{n\ge2}\left(\frac{2\sqrt{2}e^{\nu(t)}\|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}}}{R}\left(1+\frac{\|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}}}{2\sqrt{2}e^{\nu(t)}R}\right)\right)^{n-1}\right)\|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}}.
\end{aligned}$$

Performing the sum in n and using estimate (2-42), we conclude that

$$\|\boldsymbol{\mathcal{N}}_{3,4}\|_{\mathcal{F}_\nu^{0,1}} \le 9C_6C_1^4\|\boldsymbol{X}\|^4_{\dot{\mathcal{F}}_\nu^{1,1}}\|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}}, \tag{3-41}$$

with

$$C_6=\frac{1}{9}\left(1+8e^{2\nu_\infty}\left(1+\frac{1}{2\sqrt{2}}e^{-\nu_\infty}C_1\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}\right)C_2\right), \tag{3-42}$$

where C_1, C_2 were defined in (2-43), (3-29). This completes our estimate for $\boldsymbol{\mathcal{N}}_{3,4}$.

To deal with the term $\boldsymbol{\mathcal{N}}_{3,5}$ in (3-24), we have to take a derivative in $\mathcal{R}_2$ from (2-9). This gives the splitting

$$\boldsymbol{\mathcal{N}}_{3,5}(\theta)=\boldsymbol{O}_6(\theta)+\boldsymbol{O}_7(\theta)+\boldsymbol{O}_8(\theta), \tag{3-43}$$

where

$$\begin{aligned}\boldsymbol{O}_6(\theta)=\frac{1}{4\pi R^2}\int_{\mathbb{S}}\sum_{n\ge1}\sum_{\substack{m=0\\ n+m\ge2}}^{n-1}\binom{n}{m}\frac{(-1)^n(n-m)}{R^{2n}}(\Delta_\eta\boldsymbol{X}(\theta)\otimes\Delta_\eta\boldsymbol{X}(\theta))&\\ \cdot(2\Delta_\eta\boldsymbol{X}_c(\theta)^T\Delta_\eta\boldsymbol{X}(\theta))^{n-m-1}2\partial_\theta\Delta_\eta\boldsymbol{X}_c(\theta)^T\Delta_\eta\boldsymbol{X}(\theta)|\Delta_\eta\boldsymbol{X}(\theta)|^{2m}\boldsymbol{F}_L(\eta)\,d\eta&,\end{aligned}$$

$$\begin{aligned}\boldsymbol{O}_7(\theta)=\frac{1}{4\pi R^2}\int_{\mathbb{S}}\sum_{n\ge1}\sum_{\substack{m=0\\ n+m\ge2}}^{n-1}\binom{n}{m}\frac{(-1)^n(n-m)}{R^{2n}}(\Delta_\eta\boldsymbol{X}(\theta)\otimes\Delta_\eta\boldsymbol{X}(\theta))&\\ \cdot(2\Delta_\eta\boldsymbol{X}_c(\theta)^T\Delta_\eta\boldsymbol{X}(\theta))^{n-m-1}2\Delta_\eta\boldsymbol{X}_c(\theta)^T\partial_\theta\Delta_\eta\boldsymbol{X}(\theta)|\Delta_\eta\boldsymbol{X}(\theta)|^{2m}\boldsymbol{F}_L(\eta)\,d\eta&,\end{aligned}$$

$$\begin{aligned}\boldsymbol{O}_8(\theta)=\frac{1}{2\pi R^2}\int_{\mathbb{S}}\sum_{n\ge1}\sum_{m=1}^{n}\binom{n}{m}\frac{(-1)^n m}{R^{2n}}(\Delta_\eta\boldsymbol{X}(\theta)\otimes\Delta_\eta\boldsymbol{X}(\theta))&\\ \cdot(2\Delta_\eta\boldsymbol{X}_c(\theta)^T\Delta_\eta\boldsymbol{X}(\theta))^{n-m}|\Delta_\eta\boldsymbol{X}(\theta)|^{2(m-1)}\Delta_\eta\boldsymbol{X}(\theta)^T\partial_\theta\Delta_\eta\boldsymbol{X}(\theta)\boldsymbol{F}_L(\eta)\,d\eta&.\end{aligned}$$

Comparing $\boldsymbol{O}_6$ and $\boldsymbol{O}_8$ to $\boldsymbol{O}_1$ and $\boldsymbol{O}_3$, respectively, in (3-25), and recalling the bounds (3-27), (3-32), together with (3-37), we find that

$$\|\boldsymbol{O}_6\|_{\mathcal{F}^{0,1}_\nu}\le\frac{\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}^{1,1}_\nu}}{2R^2}\sum_{n\ge2}\frac{(2\sqrt{2})^n e^{n\nu(t)}n}{R^n}\|\boldsymbol{X}\|^n_{\dot{\mathcal{F}}^{1,1}_\nu}\left(1+\frac{\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}}{2\sqrt{2}e^{\nu(t)}R}\right)^{n-1}\|\boldsymbol{F}_L\|_{\mathcal{F}^{0,1}_\nu},$$

$$\|\boldsymbol{O}_8\|_{\mathcal{F}^{0,1}_\nu}\le\frac{\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}^{1,1}_\nu}}{R^2}\sum_{n\ge1}\frac{n\|\boldsymbol{X}\|^{n-1}_{\dot{\mathcal{F}}^{1,1}_\nu}}{R^{n-1}}\left(2\sqrt{2}e^{\nu(t)}+\frac{\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}}{R}\right)^{n-1}\frac{\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}^{1,1}_\nu}}{R^2}\|\boldsymbol{F}_L\|_{\mathcal{F}^{0,1}_\nu},$$

which after summation in n the right side above becomes

$$\begin{aligned}\|\boldsymbol{O}_6\|_{\mathcal{F}^{0,1}_\nu}\le 4e^{2\nu_\infty}\left(1+\frac{\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}}{2\sqrt{2}e^{\nu_\infty}R}\right)\left(2-\frac{2\sqrt{2}e^{\nu_\infty}\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}}{R}\left(1+\frac{\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}}{2\sqrt{2}e^{\nu_\infty}R}\right)\right)&\\ \times\left(1-2\sqrt{2}e^{\nu_\infty}\frac{\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}}{R}\left(1+\frac{\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}}{2\sqrt{2}e^{\nu_\infty}R}\right)\right)^{-2}\frac{\|\boldsymbol{X}\|^4_{\dot{\mathcal{F}}^{1,1}_\nu}}{R^4}\|\boldsymbol{F}_L\|_{\mathcal{F}^{0,1}_\nu}&,\end{aligned}$$

$$\|\boldsymbol{O}_8\|_{\mathcal{F}^{0,1}_\nu}\le\left(1-2\sqrt{2}e^{\nu_\infty}\frac{\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}}{R}\left(1+\frac{\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}}{2\sqrt{2}e^{\nu_\infty}R}\right)\right)^{-2}\frac{\|\boldsymbol{X}\|^4_{\dot{\mathcal{F}}^{1,1}_\nu}}{R^4}\|\boldsymbol{F}_L\|_{\mathcal{F}^{0,1}_\nu}.$$

It is now clear that, for the same reason that the bound for $\boldsymbol{O}_2$ (3-30) was the same as that for $\boldsymbol{O}_1$ (3-26), the estimate for $\boldsymbol{O}_7$ is the same as the one for $\boldsymbol{O}_6$. Therefore, with (2-42), we conclude that

$$\|\boldsymbol{\mathcal{N}}_{3,5}\|_{\mathcal{F}^{0,1}_\nu}\le 17C_7C_1^4\|\boldsymbol{X}\|^4_{\dot{\mathcal{F}}^{1,1}_\nu}\|\boldsymbol{F}_L\|_{\mathcal{F}^{0,1}_\nu}, \tag{3-44}$$

with

$$C_7 = \frac{16}{17} e^{2\nu_\infty} C_2 C_3 \Big(1 - C_8^{-1}\sqrt{2} e^{\nu_\infty} \|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}} \Big(1 + \frac{1}{2\sqrt{2}} C_8^{-1} e^{-\nu_\infty} \|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}}\Big)\Big) + \frac{C_3^2}{17}, \tag{3-45}$$

where we note that C_7 is indeed increasing in $\|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}}$ as can be seen because the infinite sums in the upper bounds of $\|\boldsymbol{O}_6\|_{\mathcal{F}_\nu^{0,1}}$ and $\|\boldsymbol{O}_8\|_{\mathcal{F}_\nu^{0,1}}$ above are indeed increasing. Further above we are also using

$$C_8 \stackrel{\text{def}}{=} \sqrt{1 + \tfrac{1}{2}\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}_\nu^{1,1}}}, \tag{3-46}$$

and we are further using C_2 and C_3 from (3-29) and (3-34).

Recalling the bounds (3-21) and (3-22), the remaining terms $\boldsymbol{\mathcal{N}}_{3,6}$–$\boldsymbol{\mathcal{N}}_{3,11}$ can be estimated similarly, using also C_1, C_6 and C_7 from (2-43), (3-42) and (3-45), to obtain

$$\begin{aligned}
\|\boldsymbol{\mathcal{N}}_{3,6}\|_{\dot{\mathcal{F}}_\nu^{1,1}} &\le 8e^{2\nu_\infty} C_1^2 \|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}_\nu^{1,1}} \|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}},\\
\|\boldsymbol{\mathcal{N}}_{3,7}\|_{\dot{\mathcal{F}}_\nu^{1,1}} &\le 8e^{2\nu_\infty} C_1^2 \|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}_\nu^{1,1}} \|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}},\\
\|\boldsymbol{\mathcal{N}}_{3,8}\|_{\dot{\mathcal{F}}_\nu^{1,1}} &\le 18\sqrt{2} e^{\nu_\infty} C_6 C_1^3 \|\boldsymbol{X}\|^3_{\dot{\mathcal{F}}_\nu^{1,1}} \|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}},\\
\|\boldsymbol{\mathcal{N}}_{3,9}\|_{\dot{\mathcal{F}}_\nu^{1,1}} &\le 34\sqrt{2} e^{\nu_\infty} C_7 C_1^3 \|\boldsymbol{X}\|^3_{\dot{\mathcal{F}}_\nu^{1,1}} \|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}},\\
\|\boldsymbol{\mathcal{N}}_{3,10}\|_{\dot{\mathcal{F}}_\nu^{1,1}} &\le 18e^{2\nu_\infty} C_6 C_1^2 \|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}_\nu^{1,1}} \|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}},\\
\|\boldsymbol{\mathcal{N}}_{3,11}\|_{\mathcal{F}_\nu^{0,1}} &\le 34e^{2\nu_\infty} C_7 C_1^2 \|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}_\nu^{1,1}} \|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}}.
\end{aligned} \tag{3-47}$$

Therefore, from the splitting (3-24) and adding all the bounds (3-35), (3-38), (3-40), (3-41), (3-44), and (3-47), we conclude that

$$\|\boldsymbol{\mathcal{N}}_3\|_{\mathcal{F}_\nu^{0,1}} \le \tfrac{147}{2} C_9 C_1^2 \|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}_\nu^{1,1}} \|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}}, \tag{3-48}$$

where

$$\begin{aligned}
C_9 = \tfrac{2}{147}\big(\tfrac{9}{2}C_4 + C_5 + 16e^{2\nu_\infty} + 18e^{2\nu_\infty}C_6 + 34e^{2\nu_\infty}C_7 \\
+ (\sqrt{2} + 18\sqrt{2}C_6 + 34\sqrt{2}C_7)e^{\nu_\infty} C_1 \|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}} + (9C_6 + 17C_7)C_1^2 \|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}_\nu^{1,1}}\big),
\end{aligned} \tag{3-49}$$

with C_1, C_4, C_5, C_6, and C_7 defined in (2-43), (3-36), (3-39), (3-42), and (3-45).

$\boldsymbol{\mathcal{N}}_2$ estimates: It is clear from (3-5) that the previous estimate for $\boldsymbol{\mathcal{N}}_3$ in (3-48) is also valid for $\boldsymbol{\mathcal{N}}_2$, with $\|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}}$ replaced by $\|\boldsymbol{F}_0\|_{\mathcal{F}_\nu^{0,1}}$. Therefore we have

$$\|\boldsymbol{\mathcal{N}}_2\|_{\mathcal{F}_\nu^{0,1}} \le \tfrac{147}{2} C_9 C_1^2 \|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}_\nu^{1,1}} \|\boldsymbol{F}_0\|_{\mathcal{F}_\nu^{0,1}}, \tag{3-50}$$

with C_9 defined above in (3-49).

$\boldsymbol{\mathcal{N}}_4$ estimates: We split the term $\boldsymbol{\mathcal{N}}_4$ in (3-5) following the splitting (2-11):

$$\boldsymbol{\mathcal{N}}_4(\theta) = \boldsymbol{\mathcal{N}}_{4,1}(\theta) + \boldsymbol{\mathcal{N}}_{4,2}(\theta) + \boldsymbol{\mathcal{N}}_{4,3}(\theta),$$

where

$$\begin{aligned}\mathcal{N}_{4,1}(\theta) &= \int_{\mathbb{S}} \partial_\theta (G_0(\Delta_\eta \boldsymbol{X}_c(\theta)))\mathbf{F}_N(\eta)\, d\eta,\\ \mathcal{N}_{4,2}(\theta) &= \int_{\mathbb{S}} \partial_\theta \big(G_L(\Delta_\eta \boldsymbol{X}_c(\theta), \Delta_\eta \boldsymbol{X}(\theta))\big)\mathbf{F}_N(\eta)\, d\eta,\\ \mathcal{N}_{4,3}(\theta) &= \int_{\mathbb{S}} \partial_\theta \big(G_N(\Delta_\eta \boldsymbol{X}_c(\theta), \Delta_\eta \boldsymbol{X}(\theta))\big)\mathbf{F}_N(\eta)\, d\eta.\end{aligned}$$

We notice that the term $\mathcal{N}_{4,2}$ can be bounded exactly as $\mathcal{N}_1$ in (3-5), with $\|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}}$ replaced by $\|\boldsymbol{F}_N\|_{\mathcal{F}_\nu^{0,1}}$, that is, from (3-23) we have

$$\|\mathcal{N}_{4,2}\|_{\mathcal{F}_\nu^{0,1}} \le 11\sqrt{2}e^{3\nu_\infty} C_1 \|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}} \|\boldsymbol{F}_N\|_{\mathcal{F}_\nu^{0,1}},$$

with C_1 from (2-43). Analogously using the similarity between $\mathcal{N}_{4,3}$ and $\mathcal{N}_3$ in (3-48), we have

$$\|\mathcal{N}_{4,3}\|_{\mathcal{F}_\nu^{0,1}} \le \tfrac{147}{2} C_9 C_1^2 \|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}_\nu^{1,1}} \|\boldsymbol{F}_N\|_{\mathcal{F}_\nu^{0,1}},$$

where we recall C_9 from (3-49).

Now taking a derivative in (2-12), the term $\mathcal{N}_{4,1}$ can be written as

$$\begin{aligned}\mathcal{N}_{4,1}(\theta) = -\frac{1}{4\pi}\int_{\mathbb{S}} \frac{F_N(\eta)}{2\tan((\theta-\eta)/2)} + \frac{1}{4\pi R^2}\int_{\mathbb{S}} \partial_\theta \Delta_\eta \boldsymbol{X}_c(\theta) \otimes \Delta_\eta \boldsymbol{X}_c(\theta) \boldsymbol{F}_N(\eta)\, d\eta&\\ + \frac{1}{4\pi R^2}\int_{\mathbb{S}} \Delta_\eta \boldsymbol{X}_c(\theta) \otimes \partial_\theta \Delta_\eta \boldsymbol{X}_c(\theta) \boldsymbol{F}_N(\eta)\, d\eta&,\end{aligned}$$

and therefore, recalling (3-18), we have

$$\|\mathcal{N}_{4,1}\|_{\mathcal{F}_\nu^{0,1}} \le \big(\tfrac14 + 2e^{2\nu_\infty}\big)\|\boldsymbol{F}_N\|_{\mathcal{F}_\nu^{0,1}}.$$

We add the previous bounds to obtain

$$\|\mathcal{N}_4\|_{\mathcal{F}_\nu^{0,1}} \le \tfrac94 C_{10} \|\boldsymbol{F}_N\|_{\mathcal{F}_\nu^{0,1}}, \tag{3-51}$$

with

$$C_{10} = \tfrac49\big(\tfrac14 + 2e^{2\nu_\infty} + 11\sqrt{2}e^{3\nu_\infty} C_1\|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}} + \tfrac{147}{2} C_9 C_1^2 \|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}_\nu^{1,1}}\big), \tag{3-52}$$

with C_1, C_9 defined in (2-43) and (3-49). Combining the estimates (3-23), (3-50), (3-48), and (3-51), we conclude from (3-5) that

$$\|\mathcal{N}\|_{\dot{\mathcal{F}}_\nu^{1,1}} \le \tfrac{147}{2} C_9 C_1^2 \|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}} \|\boldsymbol{F}_0\|_{\mathcal{F}_\nu^{0,1}} \|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{2,1}} + 11\sqrt{2} C_{11} C_1 \|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}} \|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}} + \tfrac94 C_{10}\|\boldsymbol{F}_N\|_{\mathcal{F}_\nu^{0,1}},$$

where

$$C_{11} = \tfrac{1}{11\sqrt{2}}\big(11\sqrt{2}e^{3\nu_\infty} + \tfrac{147}{2} C_9 C_1 \|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}}\big), \tag{3-53}$$

and C_1, C_9 are defined in (2-43) and (3-49). Rename the constants

$$D_1 = C_{11}C_1, \quad D_2 = C_9 C_1^2, \quad D_3 = C_{10}, \tag{3-54}$$

to get the result (3-1), where C_1, C_9, C_{10}, and C_{11} are given in (2-43), (3-49), (3-52), and (3-53). □

3B. ***A priori estimates on $\boldsymbol{F}$.*** In this section we will obtain bounds for $\boldsymbol{F}_0$, $\boldsymbol{F}_L$, and $\boldsymbol{F}_N$ in $\mathcal{F}^{0,1}_\nu$.

Proposition 3.3. *Assume that $\boldsymbol{X} \in \dot{\mathcal{F}}^{2,1}_\nu$ and that $\boldsymbol{F}$ solves* (1-13). *Then, the functions $\boldsymbol{F}_0$ in* (3-58), *$\boldsymbol{F}_L$ in* (2-25) *and $\boldsymbol{F}_N = \boldsymbol{F} - \boldsymbol{F}_0 - \boldsymbol{F}_L$, satisfy the estimate*

$$\|\boldsymbol{F}_0\|_{\mathcal{F}^{0,1}_\nu} \le \sqrt{2} e^{\nu_\infty} C_8 \frac{2A_e}{1-A_\mu}, \tag{3-55}$$

where C_8 is defined in (3-46). *Further*

$$\|\boldsymbol{F}_L\|_{\mathcal{F}^{0,1}_\nu} \le 2A_e \left(1 + \frac{|A_\mu|}{1-A_\mu}\right) \|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{2,1}_\nu}, \tag{3-56}$$

$$\|\boldsymbol{F}_N\|_{\mathcal{F}^{0,1}_\nu} \le 1000\sqrt{2} A_e \frac{|A_\mu|(1+|A_\mu|)}{(1-A_\mu)^2(1+A_\mu)} D_4 \|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu} \|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{2,1}_\nu}, \tag{3-57}$$

where $D_4 = D_4(\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}; A_\mu, \nu_\infty)$ is an increasing function of $\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}$ as in (1-28) *such that*

$$\lim_{\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu} \to 0^+} D_4(\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}; 0, 0) = 1$$

and is defined in (3-78).

Proof. First, for a general circle the expression for $\boldsymbol{F}_0$ in (2-20) becomes

$$\boldsymbol{F}_0(\theta) = \frac{2A_e}{1-A_\mu} \partial_\theta^2 \boldsymbol{X}_c. \tag{3-58}$$

Similar to (3-13) using (3-11) we have for (3-55) that

$$\|\boldsymbol{F}_0\|_{\mathcal{F}^{0,1}_\nu} \le \sqrt{2} e^{\nu_\infty} R \frac{2A_e}{1-A_\mu} \le \sqrt{2} e^{\nu_\infty} C_8 \frac{2A_e}{1-A_\mu},$$

where C_8 is given by (3-46) and we used (2-42).

Further $\boldsymbol{F}_L$ is given by (2-25) and so we have

$$\|\boldsymbol{F}_L\|_{\mathcal{F}^{0,1}_\nu} \le 2A_e \|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{2,1}_\nu} + \frac{2|A_\mu|A_e}{1-A_\mu} \|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu},$$

which gives (3-56).

We proceed with the expansion of the nonlinear terms in (1-13). First, using (2-8), we write

$$\frac{1}{|\Delta \boldsymbol{X} + \Delta \boldsymbol{X}_c|^4} = \frac{1}{16R^4 \sin^4((\theta-\eta)/2)} \Big(1 - \frac{4}{R^2} \Delta_\eta \boldsymbol{X}_c(\theta)^T \Delta_\eta \boldsymbol{X}(\theta) + \mathcal{R}_3(\Delta_\eta \boldsymbol{X}(\theta))\Big),$$

where

$$\begin{aligned}\mathcal{R}_3(\Delta_\eta \boldsymbol{X}(\theta)) = -\frac{4}{R^2} \Delta_\eta \boldsymbol{X}_c(\theta)^T \Delta_\eta \boldsymbol{X}(\theta) \mathcal{R}_2(\Delta_\eta \boldsymbol{X}(\theta)) + 2\mathcal{R}_2(\Delta_\eta \boldsymbol{X}(\theta)) \\ + \frac{4}{R^4} (\Delta_\eta \boldsymbol{X}_c(\theta)^T \Delta_\eta \boldsymbol{X}(\theta))^2 + (\mathcal{R}_2(\Delta_\eta \boldsymbol{X}(\theta)))^2,\end{aligned} \tag{3-59}$$

and $\mathcal{R}_2(\Delta_\eta \boldsymbol{X}(\theta))$ is given in (2-9). Then, we use the above expansion to rewrite $\boldsymbol{\mathcal{S}}(\boldsymbol{F}, \boldsymbol{\mathcal{X}})(\theta)$ from (1-14) as

$$\boldsymbol{\mathcal{S}}(\boldsymbol{F}, \boldsymbol{\mathcal{X}})(\theta) = \int_{\mathbb{S}} K(\boldsymbol{X}_c, \boldsymbol{X})(\theta, \eta) \frac{\boldsymbol{F}(\theta-\eta)}{2\sin(\eta/2)}\, d\eta, \tag{3-60}$$

where

$$K(\boldsymbol{X}_c,\boldsymbol{X})(\theta,\eta)=\frac{1}{\pi R^4}(\partial_\theta\boldsymbol{\mathcal{X}}(\theta)^{\perp})^T\Delta_{\theta-\eta}\boldsymbol{\mathcal{X}}(\theta)\Delta_{\theta-\eta}\boldsymbol{\mathcal{X}}(\theta)\otimes\Delta_{\theta-\eta}\boldsymbol{\mathcal{X}}(\theta)\\ \times\Big(1-\frac{4}{R^2}\Delta_\eta\boldsymbol{X}_c(\theta)^T\Delta_\eta\boldsymbol{X}(\theta)+\mathcal{R}_3(\Delta_\eta\boldsymbol{X}(\theta))\Big),$$

and we recall the notation $\boldsymbol{\mathcal{X}}(\theta)=\boldsymbol{X}_c(\theta)+\boldsymbol{X}(\theta)$ and (2-2).

We will plug in the splitting for $\boldsymbol{F}$ in (2-15) into (1-13). We first introduce an analogous splitting for K as

$$K(\boldsymbol{X}_c,\boldsymbol{X})(\theta,\eta)=K_0(\boldsymbol{X}_c)(\theta,\eta)+K_L(\boldsymbol{X}_c,\boldsymbol{X})(\theta,\eta)+K_N(\boldsymbol{X}_c,\boldsymbol{X})(\theta,\eta). \tag{3-61}$$

After we remove the zero-order (2-18), and linear-order terms (2-23), then (1-13) for the nonlinear-order terms becomes the following equation for $\boldsymbol{F}_N$:

$$\boldsymbol{F}_N(\theta)-2A_\mu\int_{\mathbb{S}}K_0(\boldsymbol{X}_c)(\theta,\eta)\frac{\boldsymbol{F}_N(\theta-\eta)}{2\sin(\eta/2)}\,d\eta=\boldsymbol{J}(\boldsymbol{X},\boldsymbol{F}_N)(\theta), \tag{3-62}$$

with

$$\begin{aligned}\boldsymbol{J}(\boldsymbol{X},\boldsymbol{F}_N)(\theta)&=2A_\mu\int_{\mathbb{S}}\big(K_L(\boldsymbol{X}_c,\boldsymbol{X})(\theta,\eta)+K_N(\boldsymbol{X}_c,\boldsymbol{X})(\theta,\eta)\big)\frac{\boldsymbol{F}_N(\theta-\eta)}{2\sin(\eta/2)}\,d\eta\\ &\quad+2A_\mu\int_{\mathbb{S}}\big(K_L(\boldsymbol{X}_c,\boldsymbol{X})(\theta,\eta)+K_N(\boldsymbol{X}_c,\boldsymbol{X})(\theta,\eta)\big)\frac{\boldsymbol{F}_L(\theta-\eta)}{2\sin(\eta/2)}\,d\eta\\ &\quad+2A_\mu\int_{\mathbb{S}}K_N(\boldsymbol{X}_c,\boldsymbol{X})(\theta,\eta)\boldsymbol{F}_0(\theta-\eta)\frac{d\eta}{2\sin(\eta/2)},\end{aligned} \tag{3-63}$$

where the first term in $\boldsymbol{J}$ will be treated as a perturbation with $\boldsymbol{F}_0$ and $\boldsymbol{F}_L$ given in (3-58) and (2-25) respectively. Notice that K_0 is given by

$$K_0(\boldsymbol{X}_c)(\theta,\eta)=\frac{1}{\pi R^4}(\partial_\theta\boldsymbol{X}_c(\theta)^{\perp})^T\Delta_{\theta-\eta}\boldsymbol{X}_c(\theta)\Delta_{\theta-\eta}\boldsymbol{X}_c(\theta)\otimes\Delta_{\theta-\eta}\boldsymbol{X}_c(\theta),$$

where by (2-3) and (2-5) we have

$$\begin{aligned}(\partial_\theta\boldsymbol{X}_c(\theta)^{\perp})^T\Delta_{\theta-\eta}\boldsymbol{X}_c(\theta)&=-R^2\sin\Big(\frac{\eta}{2}\Big),\\ \Delta_{\theta-\eta}\boldsymbol{X}_c(\theta)\otimes\Delta_{\theta-\eta}\boldsymbol{X}_c(\theta)&=\frac{a^2}{2}\begin{bmatrix}1-\cos(2\theta-\eta)&-\sin(2\theta-\eta)\\-\sin(2\theta-\eta)&1+\cos(2\theta-\eta)\end{bmatrix}\\ &\quad+\frac{b^2}{2}\begin{bmatrix}1+\cos(2\theta-\eta)&\sin(2\theta-\eta)\\ \sin(2\theta-\eta)&1-\cos(2\theta-\eta)\end{bmatrix}+ab\begin{bmatrix}\sin(2\theta-\eta)&-\cos(2\theta-\eta)\\-\cos(2\theta-\eta)&-\sin(2\theta-\eta)\end{bmatrix}.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{S}}K_0(\boldsymbol{X}_c)(\theta,\eta)\frac{\boldsymbol{F}_N(\theta-\eta)}{2\sin(\eta/2)}\,d\eta&=-\frac{1}{4\pi}\int_{\mathbb{S}}\boldsymbol{F}_N(\theta-\eta)\,d\eta\\ &\quad-\frac{a^2-b^2}{4\pi R^2}\int_{\mathbb{S}}\begin{bmatrix}-\cos(2\theta-\eta)&-\sin(2\theta-\eta)\\-\sin(2\theta-\eta)&\cos(2\theta-\eta)\end{bmatrix}\boldsymbol{F}_N(\theta-\eta)\,d\eta\\ &\quad-\frac{ab}{2\pi R^2}\int_{\mathbb{S}}\begin{bmatrix}\sin(2\theta-\eta)&-\cos(2\theta-\eta)\\-\cos(2\theta-\eta)&-\sin(2\theta-\eta)\end{bmatrix}\boldsymbol{F}_N(\theta-\eta)\,d\eta.\end{aligned}$$

Then taking the Fourier transform we find that

$$\mathcal{F}\left(\int_{\mathbb{S}} K_0(\boldsymbol{X}_c)(\theta,\eta)\frac{\boldsymbol{F}_N(\theta-\eta)}{2\sin(\eta/2)}\,d\eta\right)(k)$$
$$= -\frac{1}{2}\widehat{\boldsymbol{F}}_N(0)\delta_0(k) + \frac{(a+ib)^2}{4R^2}\begin{bmatrix}1 & -i\\ -i & -1\end{bmatrix}\widehat{\boldsymbol{F}}_N(-1)\delta_1(k) + \frac{(a-ib)^2}{4R^2}\begin{bmatrix}1 & i\\ i & -1\end{bmatrix}\widehat{\boldsymbol{F}}_N(1)\delta_{-1}(k).$$

Equation (3-62) is then given on the Fourier side by the expressions

$$\begin{aligned}\widehat{\boldsymbol{F}}_N(0) &= \frac{1}{1+A_\mu}\widehat{\boldsymbol{J}(\boldsymbol{X},\boldsymbol{F}_N)}(0),\\ \widehat{\boldsymbol{F}}_N(k) &= \widehat{\boldsymbol{J}(\boldsymbol{X},\boldsymbol{F}_N)}(k), \quad k\ge 2,\end{aligned} \tag{3-64}$$

while for $k=1$ one has that

$$\widehat{\boldsymbol{F}}_N(1) - A_\mu\frac{(a+ib)^2}{2R^2}\begin{bmatrix}1 & -i\\ -i & -1\end{bmatrix}\widehat{\boldsymbol{F}}_N(-1) = \widehat{\boldsymbol{J}(\boldsymbol{X},\boldsymbol{F}_N)}(1),$$
$$\widehat{\boldsymbol{F}}_N(-1) - A_\mu\frac{(a-ib)^2}{2R^2}\begin{bmatrix}1 & i\\ i & -1\end{bmatrix}\widehat{\boldsymbol{F}}_N(1) = \widehat{\boldsymbol{J}(\boldsymbol{X},\boldsymbol{F}_N)}(-1),$$

which gives that

$$\begin{bmatrix}1-A_\mu^2/2 & -iA_\mu^2/2\\ iA_\mu^2/2 & 1-A_\mu^2/2\end{bmatrix}\widehat{\boldsymbol{F}}_N(1) = A_\mu\frac{(a+ib)^2}{2R^2}\begin{bmatrix}1 & -i\\ -i & -1\end{bmatrix}\widehat{\boldsymbol{J}(\boldsymbol{X},\boldsymbol{F}_N)}(-1) + \widehat{\boldsymbol{J}(\boldsymbol{X},\boldsymbol{F}_N)}(1),$$

and thus

$$\widehat{\boldsymbol{F}}_N(1) = \frac{A_\mu}{1-A_\mu^2}\frac{(a+ib)^2}{2R^2}\begin{bmatrix}1 & -i\\ -i & -1\end{bmatrix}\widehat{\boldsymbol{J}(\boldsymbol{X},\boldsymbol{F}_N)}(-1) + \frac{1}{1-A_\mu^2}\begin{bmatrix}1-A_\mu^2/2 & iA_\mu^2/2\\ -iA_\mu^2/2 & 1-A_\mu^2/2\end{bmatrix}\widehat{\boldsymbol{J}(\boldsymbol{X},\boldsymbol{F}_N)}(1).$$

Since we have

$$\left\|\begin{bmatrix}1 & -i\\ -i & -1\end{bmatrix}\right\| = 2, \quad \left\|\begin{bmatrix}1-A_\mu^2/2 & iA_\mu^2/2\\ -iA_\mu^2/2 & 1-A_\mu^2/2\end{bmatrix}\right\| = 1,$$

we obtain

$$\begin{aligned}|\widehat{\boldsymbol{F}}_N(1)| &\le \frac{|A_\mu|}{1-A_\mu^2}|\widehat{\boldsymbol{J}(\boldsymbol{X},\boldsymbol{F}_N)}(-1)| + \frac{1}{1-A_\mu^2}|\widehat{\boldsymbol{J}(\boldsymbol{X},\boldsymbol{F}_N)}(1)|\\ &= \frac{1+|A_\mu|}{(1-A_\mu)(1+A_\mu)}|\widehat{\boldsymbol{J}(\boldsymbol{X},\boldsymbol{F}_N)}(1)|,\end{aligned}$$

which together with (3-64) implies that

$$\|\boldsymbol{F}_N\|_{\mathcal{F}_\nu^{0,1}} \le \frac{1+|A_\mu|}{(1-A_\mu)(1+A_\mu)}\|\boldsymbol{J}(\boldsymbol{X},\boldsymbol{F}_N)\|_{\mathcal{F}_\nu^{0,1}}. \tag{3-65}$$

This is our estimate for $\boldsymbol{F}_N$.

<u>$\boldsymbol{J}(\boldsymbol{X},\boldsymbol{F}_N)$ estimate</u>: Notice that $\boldsymbol{J}(\boldsymbol{X},\boldsymbol{F}_N)$ corresponds to the nonlinear terms in $\boldsymbol{\mathcal{S}}(\boldsymbol{F},\boldsymbol{\mathcal{X}})$ except the one in the left-hand side of (3-62). For simplicity in notation, we are going to estimate $\boldsymbol{\mathcal{S}}(\boldsymbol{F},\boldsymbol{\mathcal{X}})$, and later extract from there the corresponding bounds for $\boldsymbol{J}(\boldsymbol{X},\boldsymbol{F}_N)$.

Consider the following splitting for $\boldsymbol{\mathcal{S}}(\boldsymbol{F},\boldsymbol{\mathcal{X}})$ from (1-14):

$$\boldsymbol{\mathcal{S}}(\boldsymbol{F},\boldsymbol{\mathcal{X}})(\theta)=\boldsymbol{\mathcal{S}}_1(\boldsymbol{F},\boldsymbol{\mathcal{X}})(\theta)+\boldsymbol{\mathcal{S}}_2(\boldsymbol{F},\boldsymbol{\mathcal{X}})(\theta)+\boldsymbol{\mathcal{S}}_3(\boldsymbol{F},\boldsymbol{\mathcal{X}})(\theta), \tag{3-66}$$

with

$$\boldsymbol{\mathcal{S}}_1(\boldsymbol{F},\boldsymbol{\mathcal{X}})(\theta)=\frac{1}{\pi R^4}\int_{\mathbb{S}}(\partial_\theta\boldsymbol{\mathcal{X}}(\theta)^\perp)^T\Delta_{\theta-\eta}\boldsymbol{\mathcal{X}}(\theta)\Delta_{\theta-\eta}\boldsymbol{\mathcal{X}}(\theta)\otimes\Delta_{\theta-\eta}\boldsymbol{\mathcal{X}}(\theta)\frac{\boldsymbol{F}(\theta-\eta)}{2\sin(\eta/2)}\,d\eta,$$

$$\begin{aligned}\boldsymbol{\mathcal{S}}_2(\boldsymbol{F},\boldsymbol{\mathcal{X}})(\theta)=-\frac{4}{\pi R^6}\int_{\mathbb{S}}&(\partial_\theta\boldsymbol{\mathcal{X}}(\theta)^\perp)^T\Delta_{\theta-\eta}\boldsymbol{\mathcal{X}}(\theta)\Delta_{\theta-\eta}\boldsymbol{\mathcal{X}}(\theta)\\&\otimes\Delta_{\theta-\eta}\boldsymbol{\mathcal{X}}(\theta)\Delta_\eta\boldsymbol{X}_c(\theta)^T\Delta_\eta\boldsymbol{X}(\theta)\frac{\boldsymbol{F}(\theta-\eta)}{2\sin(\eta/2)}\,d\eta,\end{aligned}$$

$$\boldsymbol{\mathcal{S}}_3(\boldsymbol{F},\boldsymbol{\mathcal{X}})(\theta)=\frac{1}{\pi R^4}\int_{\mathbb{S}}(\partial_\theta\boldsymbol{\mathcal{X}}(\theta)^\perp)^T\Delta_{\theta-\eta}\boldsymbol{\mathcal{X}}(\theta)\Delta_{\theta-\eta}\boldsymbol{\mathcal{X}}(\theta)\otimes\Delta_{\theta-\eta}\boldsymbol{\mathcal{X}}(\theta)\mathcal{R}_3(\Delta_\eta\boldsymbol{X}(\theta))\frac{\boldsymbol{F}(\theta-\eta)}{2\sin(\eta/2)}\,d\eta.$$

We take Fourier transform of $\boldsymbol{\mathcal{S}}_1(\boldsymbol{F},\boldsymbol{\mathcal{X}})$ to obtain

$$\begin{aligned}\widehat{\boldsymbol{\mathcal{S}}_1(\boldsymbol{F},\boldsymbol{\mathcal{X}})}(k)=\frac{1}{\pi R^4}\sum_{k_1\in\mathbb{Z}}\cdots\sum_{k_4\in\mathbb{Z}}&(\widehat{\partial_\theta\mathcal{X}}(k-k_1)^\perp)^T\widehat{\partial_\theta\mathcal{X}}(k_1-k_2)\widehat{\partial_\theta\mathcal{X}}(k_2-k_3)\\&\otimes\widehat{\partial_\theta\mathcal{X}}(k_3-k_4)\widehat{\boldsymbol{F}}(k_4)I_2'(k_1,\dots,k_4),\end{aligned}$$

where

$$\begin{aligned}|I_2'(k_1,\dots,k_4)|&=\left|\int_{\mathbb{S}}\prod_{j=1}^{3}\frac{\sin((k_j-k_{j+1})\eta/2)}{(k_j-k_{j+1})\sin(\eta/2)}\frac{e^{-i(k_1+k_4)\eta/2}}{2\sin(\eta/2)}\,d\eta\right|\\&=\left|\int_{\mathbb{S}}\frac{\sin((k_1+k_4)\eta/2)}{\sin(\eta/2)}\prod_{j=1}^{3}\frac{\sin((k_j-k_{j+1})\eta/2)}{(k_j-k_{j+1})\sin(\eta/2)}\,d\eta\right|.\end{aligned}$$

The integral I_2' turns out to be the previously defined integral in (3-4). In Lemma 3.2 we show that $|I_2'|\le 2\pi$. Using (3-11) and (3-13), we have

$$\begin{aligned}&|(\widehat{\partial_\theta\mathcal{X}}(k-k_1)^\perp)^T\widehat{\partial_\theta\mathcal{X}}(k_1-k_2)|\\&\quad\le\frac{R^2}{2}\big(\delta_1(k-k_1)\delta_{-1}(k_1-k_2)+\delta_{-1}(k-k_1)\delta_1(k_1-k_2)\big)\\&\qquad+\frac{R}{\sqrt{2}}\delta_{1,-1}(k-k_1)|\widehat{\partial_\theta\boldsymbol{X}}(k_1-k_2)|+\frac{R}{\sqrt{2}}\delta_{1,-1}(k_1-k_2)|\widehat{\partial_\theta\boldsymbol{X}}(k-k_1)|+|\widehat{\partial_\theta\boldsymbol{X}}(k-k_1)|\,|\widehat{\partial_\theta\boldsymbol{X}}(k_1-k_2)|,\end{aligned}$$

while recalling (3-18), (3-21) and (3-22), we obtain

$$\begin{aligned}&\|\widehat{\partial_\theta\mathcal{X}}(k_2-k_3)\otimes\widehat{\partial_\theta\mathcal{X}}(k_3-k_4)\|\\&\quad\le\frac{R^2}{2}\delta_{1,-1}(k_2-k_3)\delta_{1,-1}(k_3-k_4)+\frac{\sqrt{2}}{2}R\delta_{1,-1}(k_2-k_3)|\widehat{\partial_\theta\boldsymbol{X}}(k_3-k_4)|\\&\qquad\qquad+\frac{\sqrt{2}}{2}R|\widehat{\partial_\theta\boldsymbol{X}}(k_2-k_3)|\delta_{1,-1}(k_3-k_4)+|\widehat{\partial_\theta\boldsymbol{X}}(k_2-k_3)|\,|\widehat{\partial_\theta\boldsymbol{X}}(k_3-k_4)|.\end{aligned}$$

Therefore, Young's inequality for convolutions yields that

$$\|\boldsymbol{\mathcal{S}}_1(\boldsymbol{F},\boldsymbol{\mathcal{X}})\|_{\mathcal{F}^{0,1}_\nu}\le 2\bigg(e^{2\nu_\infty}+2\sqrt{2}e^{\nu_\infty}\frac{\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}}{R}+\frac{\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}^{1,1}_\nu}}{R^2}\bigg)\bigg(2e^{2\nu_\infty}+2\sqrt{2}e^{\nu_\infty}\frac{\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}}{R}+\frac{\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}^{1,1}_\nu}}{R^2}\bigg)\|\boldsymbol{F}\|_{\mathcal{F}^{0,1}_\nu},$$

which by (2-42) and notation (2-43) can be rewritten as

$$\|\boldsymbol{\mathcal{S}}_1(\boldsymbol{F},\boldsymbol{\mathcal{X}})\|_{\mathcal{F}^{0,1}_\nu} \le 2(2e^{4\nu_\infty}+6\sqrt{2}e^{3\nu_\infty}C_1\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}+11C_{12}C_1^2\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}^{1,1}_\nu})\|\boldsymbol{F}\|_{\mathcal{F}^{0,1}_\nu}, \tag{3-67}$$

with

$$C_{12}=\tfrac{1}{11}(11e^{2\nu_\infty}+4\sqrt{2}e^{\nu_\infty}C_1\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}+C_1^2\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}^{1,1}_\nu}), \tag{3-68}$$

and C_1 is defined in (2-43).

Following the same steps, for $\widehat{\boldsymbol{\mathcal{S}}_2(\boldsymbol{F},\boldsymbol{\mathcal{X}})}$, one finds that

$$\|\boldsymbol{\mathcal{S}}_2(\boldsymbol{F},\boldsymbol{\mathcal{X}})\|_{\mathcal{F}^{0,1}_\nu} \le 8\sqrt{2}e^{\nu_\infty}(2e^{4\nu_\infty}+6\sqrt{2}e^{3\nu_\infty}C_1\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}+11C_{12}C_1^2\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}^{1,1}_\nu})C_1\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}\|\boldsymbol{F}\|_{\mathcal{F}^{0,1}_\nu}.$$

We define

$$C_{13}=\tfrac{1}{2}(2e^{4\nu_\infty}+6\sqrt{2}e^{3\nu_\infty}C_1\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}+11C_{12}C_1^2\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}^{1,1}_\nu}), \tag{3-69}$$

so that

$$\|\boldsymbol{\mathcal{S}}_2(\boldsymbol{F},\boldsymbol{\mathcal{X}})\|_{\mathcal{F}^{0,1}_\nu} \le 16\sqrt{2}e^{\nu_\infty}C_{13}C_1\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}\|\boldsymbol{F}\|_{\mathcal{F}^{0,1}_\nu}. \tag{3-70}$$

This completes our $\widehat{\boldsymbol{\mathcal{S}}_2(\boldsymbol{F},\boldsymbol{\mathcal{X}})}$ estimate.

Next, we proceed with $\widehat{\boldsymbol{\mathcal{S}}_3(\boldsymbol{F},\boldsymbol{\mathcal{X}})}$ in (3-66). We split it accordingly to (3-59) as

$$\widehat{\boldsymbol{\mathcal{S}}_3(\boldsymbol{F},\boldsymbol{\mathcal{X}})}=\widehat{\boldsymbol{\mathcal{S}}_{3,1}(\boldsymbol{F},\boldsymbol{\mathcal{X}})}+\widehat{\boldsymbol{\mathcal{S}}_{3,2}(\boldsymbol{F},\boldsymbol{\mathcal{X}})}+\widehat{\boldsymbol{\mathcal{S}}_{3,3}(\boldsymbol{F},\boldsymbol{\mathcal{X}})}+\widehat{\boldsymbol{\mathcal{S}}_{3,4}(\boldsymbol{F},\boldsymbol{\mathcal{X}})},$$

with

$$\begin{aligned}
\boldsymbol{\mathcal{S}}_{3,1}(\boldsymbol{F},\boldsymbol{\mathcal{X}}) &= -\frac{4}{\pi R^6}\int_{\mathbb{S}}(\partial_\theta\boldsymbol{\mathcal{X}}(\theta)^\perp)^T\Delta_{\theta-\eta}\boldsymbol{\mathcal{X}}(\theta)\Delta_{\theta-\eta}\boldsymbol{\mathcal{X}}(\theta)\otimes\Delta_{\theta-\eta}\boldsymbol{\mathcal{X}}(\theta)\\
&\qquad\qquad\cdot\Delta_{\theta-\eta}\boldsymbol{X}_c(\theta)^T\Delta_{\theta-\eta}\boldsymbol{X}(\theta)\mathcal{R}_2(\Delta_\eta\boldsymbol{X}(\theta))\frac{\boldsymbol{F}(\theta-\eta)}{2\sin(\eta/2)}\,d\eta,\\
\boldsymbol{\mathcal{S}}_{3,2}(\boldsymbol{F},\boldsymbol{\mathcal{X}}) &= \frac{2}{\pi R^4}\int_{\mathbb{S}}(\partial_\theta\boldsymbol{\mathcal{X}}(\theta)^\perp)^T\Delta_{\theta-\eta}\boldsymbol{\mathcal{X}}(\theta)\Delta_{\theta-\eta}\boldsymbol{\mathcal{X}}(\theta)\otimes\Delta_{\theta-\eta}\boldsymbol{\mathcal{X}}(\theta)R_2(\Delta_\eta\boldsymbol{X}(\theta))\frac{\boldsymbol{F}(\theta-\eta)}{2\sin(\eta/2)}\,d\eta,\\
\boldsymbol{\mathcal{S}}_{3,3}(\boldsymbol{F},\boldsymbol{\mathcal{X}}) &= \frac{4}{\pi R^8}\int_{\mathbb{S}}(\partial_\theta\boldsymbol{\mathcal{X}}(\theta)^\perp)^T\Delta_{\theta-\eta}\boldsymbol{\mathcal{X}}(\theta)\Delta_{\theta-\eta}\boldsymbol{\mathcal{X}}(\theta)\otimes\Delta_{\theta-\eta}\boldsymbol{\mathcal{X}}(\theta)\\
&\qquad\qquad\cdot(\Delta_{\theta-\eta}\boldsymbol{X}_c(\theta)^T\Delta_{\theta-\eta}\boldsymbol{X}(\theta))^2\frac{\boldsymbol{F}(\theta-\eta)}{2\sin(\eta/2)}\,d\eta,\\
\boldsymbol{\mathcal{S}}_{3,4}(\boldsymbol{F},\boldsymbol{\mathcal{X}}) &= \frac{1}{\pi R^4}\int_{\mathbb{S}}(\partial_\theta\boldsymbol{\mathcal{X}}(\theta)^\perp)^T\Delta_{\theta-\eta}\boldsymbol{\mathcal{X}}(\theta)\Delta_{\theta-\eta}\boldsymbol{\mathcal{X}}(\theta)\otimes\Delta_{\theta-\eta}\boldsymbol{\mathcal{X}}(\theta)(R_2(\Delta_\eta\boldsymbol{X}(\theta)))^2\frac{\boldsymbol{F}(\theta-\eta)}{2\sin(\eta/2)}\,d\eta.
\end{aligned}$$

The procedure follows the steps used to bound $\boldsymbol{\mathcal{N}}_{3,4}(\theta)$ in (3-24) and (3-41), where the term $\mathcal{R}_2$ from (2-9) was also involved. After taking Fourier transform and using Lemma 3.2, Young's inequality for convolutions and summation in m and n gives

$$\begin{aligned}
\|\boldsymbol{\mathcal{S}}_{3,1}(\boldsymbol{F},\boldsymbol{\mathcal{X}})\|_{\mathcal{F}^{0,1}_\nu} &\le 144\sqrt{2}e^{\nu_\infty}C_6C_{13}C_1^3\|\boldsymbol{X}\|^3_{\dot{\mathcal{F}}^{1,1}_\nu}\|\boldsymbol{F}\|_{\mathcal{F}^{0,1}_\nu},\\
\|\boldsymbol{\mathcal{S}}_{3,2}(\boldsymbol{F},\boldsymbol{\mathcal{X}})\|_{\mathcal{F}^{0,1}_\nu} &\le 72C_6C_{13}C_1^2\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}^{1,1}_\nu}\|\boldsymbol{F}\|_{\mathcal{F}^{0,1}_\nu},\\
\|\boldsymbol{\mathcal{S}}_{3,3}(\boldsymbol{F},\boldsymbol{\mathcal{X}})\|_{\mathcal{F}^{0,1}_\nu} &\le 32e^{2\nu_\infty}C_{13}C_1^2\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}^{1,1}_\nu}\|\boldsymbol{F}\|_{\mathcal{F}^{0,1}_\nu},\\
\|\boldsymbol{\mathcal{S}}_{3,4}(\boldsymbol{F},\boldsymbol{\mathcal{X}})\|_{\mathcal{F}^{0,1}_\nu} &\le 324C_{13}C_6^2C_1^4\|\boldsymbol{X}\|^4_{\dot{\mathcal{F}}^{1,1}_\nu}\|\boldsymbol{F}\|_{\mathcal{F}^{0,1}_\nu}.
\end{aligned}$$

Joining the above bounds, we obtain

$$\|\boldsymbol{\mathcal{S}}_3(\boldsymbol{F},\boldsymbol{\mathcal{X}})\|_{\mathcal{F}_\nu^{0,1}} \le 104C_{13}C_{14}C_1^2\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}_\nu^{1,1}}\|\boldsymbol{F}\|_{\mathcal{F}_\nu^{0,1}}, \tag{3-71}$$

with

$$C_{14} = \tfrac{1}{104}(72C_6 + 32e^{2\nu_\infty} + 144\sqrt{2}e^{\nu_\infty}C_6C_1\|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}} + 324C_6^2C_1^2\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}_\nu^{1,1}}), \tag{3-72}$$

where C_1 and C_6 previously defined in (2-43) and (3-42), respectively. We combine the bounds (3-67), (3-70), and (3-71), and order them as

$$\|\boldsymbol{\mathcal{S}}(\boldsymbol{F},\boldsymbol{\mathcal{X}})\|_{\mathcal{F}_\nu^{0,1}} \le 4e^{4\nu_\infty}\|\boldsymbol{F}\|_{\mathcal{F}_\nu^{0,1}} + 28\sqrt{2}e^{5\nu_\infty}C_1\|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}}\|\boldsymbol{F}\|_{\mathcal{F}_\nu^{0,1}} + 222C_{15}C_1^2\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}_\nu^{1,1}}\|\boldsymbol{F}\|_{\mathcal{F}_\nu^{0,1}},$$

with

$$C_{15} = \tfrac{1}{222}(104C_{13}C_{14} + 8\sqrt{2}e^{\nu_\infty}(6\sqrt{2}e^{3\nu_\infty} + 11C_{12}C_1\|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}}) + 22C_{12}), \tag{3-73}$$

and C_1 in (2-43), C_{12} in (3-68), C_{13} in (3-69) and C_{14} in (3-72).

We remark that instead of (3-66), analogously to the splitting for K in (3-61) we can split $\boldsymbol{\mathcal{S}}(\boldsymbol{F},\boldsymbol{\mathcal{X}})(\theta)$ as

$$\boldsymbol{\mathcal{S}}(\boldsymbol{F},\boldsymbol{\mathcal{X}})(\theta) = \boldsymbol{\mathcal{S}}_0(\boldsymbol{F},\boldsymbol{\mathcal{X}})(\theta) + \boldsymbol{\mathcal{S}}_L(\boldsymbol{F},\boldsymbol{\mathcal{X}})(\theta) + \boldsymbol{\mathcal{S}}_N(\boldsymbol{F},\boldsymbol{\mathcal{X}})(\theta), \tag{3-74}$$

where from (3-60) and (3-61) we have

$$\boldsymbol{\mathcal{S}}_0(\boldsymbol{F},\boldsymbol{\mathcal{X}})(\theta) = \int_{\mathbb{S}} K_0(\boldsymbol{X}_c,\boldsymbol{X})(\theta,\eta)\frac{\boldsymbol{F}(\theta-\eta)}{2\sin(\eta/2)}\,d\eta.$$

Then $\boldsymbol{\mathcal{S}}_L(\boldsymbol{F},\boldsymbol{\mathcal{X}})(\theta)$ analogously contains $K_L(\boldsymbol{X}_c,\boldsymbol{X})(\theta,\eta)$ from (3-61) and $\boldsymbol{\mathcal{S}}_L$ is linear in $\boldsymbol{X}$. Then $\boldsymbol{\mathcal{S}}_N(\boldsymbol{F},\boldsymbol{\mathcal{X}})(\theta)$ similarly contains $K_N(\boldsymbol{X}_c,\boldsymbol{X})(\theta,\eta)$ from (3-61) and $\boldsymbol{\mathcal{S}}_N$ is nonlinear in $\boldsymbol{X}$. Then it is clear from the above that we have the estimates

$$\begin{aligned}
\|\boldsymbol{\mathcal{S}}_0(\boldsymbol{F},\boldsymbol{\mathcal{X}})\|_{\mathcal{F}_\nu^{0,1}} &\le 4e^{4\nu_\infty}\|\boldsymbol{F}\|_{\mathcal{F}_\nu^{0,1}},\\
\|\boldsymbol{\mathcal{S}}_L(\boldsymbol{F},\boldsymbol{\mathcal{X}})\|_{\mathcal{F}_\nu^{0,1}} &\le 28\sqrt{2}e^{5\nu_\infty}C_1\|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}}\|\boldsymbol{F}\|_{\mathcal{F}_\nu^{0,1}},\\
\|\boldsymbol{\mathcal{S}}_N(\boldsymbol{F},\boldsymbol{\mathcal{X}})\|_{\mathcal{F}_\nu^{0,1}} &\le 222C_{15}C_1^2\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}_\nu^{1,1}}\|\boldsymbol{F}\|_{\mathcal{F}_\nu^{0,1}}.
\end{aligned}$$

This splits the estimates into zero-order, linear-order, and nonlinear-order which is useful because of (3-63).

Next, recalling the definition of $\boldsymbol{J}(\boldsymbol{X},\boldsymbol{F}_N)$ from (3-63) and its relation with $\boldsymbol{\mathcal{S}}(\boldsymbol{F},\boldsymbol{\mathcal{X}})$ in (3-60) and (3-74), it follows that

$$\begin{aligned}
\|\boldsymbol{J}(\boldsymbol{X},\boldsymbol{F}_N)\|_{\mathcal{F}_\nu^{0,1}} &\le 56\sqrt{2}|A_\mu|C_{16}C_1\|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}}\|\boldsymbol{F}_N\|_{\mathcal{F}_\nu^{0,1}}\\
&\quad + 56\sqrt{2}|A_\mu|C_{16}C_1\|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}}\|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}} + 444|A_\mu|C_{15}C_1^2\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}_\nu^{1,1}}\|\boldsymbol{F}_0\|_{\mathcal{F}_\nu^{0,1}},
\end{aligned} \tag{3-75}$$

with

$$C_{16} = \frac{1}{28\sqrt{2}}(28\sqrt{2}e^{5\nu_\infty} + 222C_{15}C_1\|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}}). \tag{3-76}$$

Finally, bound (3-75) allows us to estimate $\boldsymbol{F}_N$ from (3-65),

$$\begin{aligned}
&\|\boldsymbol{F}_N\|_{\mathcal{F}_\nu^{0,1}}\\
&\quad\le \frac{56\sqrt{2}|A_\mu|(1+|A_\mu|)}{(1-A_\mu)(1+A_\mu)}C_{17}C_{16}C_1\|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}}\|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}} + \frac{444|A_\mu|(1+|A_\mu|)}{(1-A_\mu)(1+A_\mu)}C_{17}C_{15}C_1^2\|\boldsymbol{X}\|^2_{\dot{\mathcal{F}}_\nu^{1,1}}\|\boldsymbol{F}_0\|_{\mathcal{F}_\nu^{0,1}},
\end{aligned}$$

where

$$C_{17} = \left(1 - \frac{56\sqrt{2}|A_\mu|(1+|A_\mu|)}{(1-A_\mu)(1+A_\mu)} C_{16} C_1 \|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}}\right)^{-1}, \tag{3-77}$$

and the bounds for $\boldsymbol{F}_0$, $\boldsymbol{F}_L$ are given in (3-55) and (3-56). Substituting these bounds we find

$$\begin{aligned}\|\boldsymbol{F}_N\|_{\mathcal{F}_\nu^{0,1}} \le 112\sqrt{2}A_e \frac{|A_\mu|(1+|A_\mu|)(1-A_\mu+|A_\mu|)}{(1-A_\mu)^2(1+A_\mu)} C_{17} C_{16} C_1 \|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}} \|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{2,1}} \\ + 888\sqrt{2}A_e \frac{|A_\mu|(1+|A_\mu|)}{(1-A_\mu)^2(1+A_\mu)} e^{\nu_\infty} C_8 C_{17} C_{15} C_1^2 \|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{1,1}} \|\boldsymbol{X}\|_{\dot{\mathcal{F}}_\nu^{2,1}}.\end{aligned}$$

Defining

$$D_4 = \tfrac{1}{1000} C_1 C_{17}(112(1-A_\mu+|A_\mu|)C_{16} + 888 e^{\nu_\infty} C_8 C_1 C_{15}), \tag{3-78}$$

where C_1, C_8, C_{15}, C_{16}, and C_{17} are defined in (2-43), (3-46), (3-73), (3-76), and (3-77), we can write the estimate for $\boldsymbol{F}_N$ as (3-57). □

Proof of Lemma 3.2. Recalling (3-3) and (3-2) and using the odd part of the integral we can rewrite $I = I_n$ as

$$\begin{aligned} I &= -i \,\mathrm{pv}\!\int_{-\pi}^{\pi} \frac{\sin((k_1+k_{2n})\eta/2) - \frac{\sin((k-k_1)\eta/2)}{(k-k_1)\tan(\eta/2)}\sin((k+k_{2n})\eta/2)}{2\sin(\eta/2)} \prod_{j=1}^{2n-1} \frac{\sin((k_j-k_{j+1})\eta/2)}{(k_j-k_{j+1})\sin(\eta/2)}\, d\eta \\ &= -\frac{i}{2}(I' - I''),\end{aligned}$$

where

$$I' \stackrel{\text{def}}{=} \mathrm{pv}\!\int_{-\pi}^{\pi} \frac{\sin((k_1+k_{2n})\eta/2)}{\sin(\eta/2)} \prod_{j=1}^{2n-1} \frac{\sin((k_j-k_{j+1})\eta/2)}{(k_j-k_{j+1})\sin(\eta/2)}\, d\eta, \tag{3-79}$$

$$I'' \stackrel{\text{def}}{=} \mathrm{pv}\!\int_{-\pi}^{\pi} \cos(\eta/2)\frac{\sin((k+k_{2n})\eta/2)}{\sin(\eta/2)} \prod_{j=0}^{2n-1} \frac{\sin((k_j-k_{j+1})\eta/2)}{(k_j-k_{j+1})\sin(\eta/2)}\, d\eta. \tag{3-80}$$

Note that if $k_1+k_{2n}=0$ then $I'=0$ and if $k+k_{2n}=0$ then $I''=0$. We henceforth assume that $|k_1+k_{2n}| \ge 1$ and $|k+k_{2n}| \ge 1$. We will calculate (3-79) and then (3-80).

Notice that $\sin((k_j-k_{j+1})\eta/2) = \mathrm{sgn}(k_j-k_{j+1})\sin(|k_j-k_{j+1}|\eta/2)$ and, since $|k_j-k_{j+1}| \ge 1$, we rewrite the quotient in the product form as

$$\begin{aligned}\frac{\sin(|k_j-k_{j+1}|\eta/2)}{\sin(\eta/2)} &= \frac{e^{i|k_j-k_{j+1}|\eta/2} - e^{-i|k_j-k_{j+1}|\eta/2}}{e^{i\eta/2}-e^{-i\eta/2}} = \frac{e^{i|k_j-k_{j+1}|\eta/2}(1-e^{-i|k_j-k_{j+1}|\eta})}{e^{i\eta/2}(1-e^{-i\eta})} \\ &= e^{i(|k_j-k_{j+1}|-1)\eta/2} \sum_{m=0}^{|k_j-k_{j+1}|-1} e^{-i\eta m} = \sum_{m=0}^{|k_j-k_{j+1}|-1} e^{i(-2m+|k_j-k_{j+1}|-1)\eta/2}.\end{aligned}$$

We conclude that

$$\prod_{j=1}^{2n-1}\frac{\sin(|k_j-k_{j+1}|\eta/2)}{\sin(\eta/2)}=\prod_{j=1}^{2n-1}\sum_{m_j=0}^{|k_j-k_{j+1}|-1}e^{i(-2m_j+|k_j-k_{j+1}|-1)\eta/2}$$
$$=\sum_{m_1=0}^{|k_1-k_2|-1}\cdots\sum_{m_{2n-1}=0}^{|k_{2n-1}-k_{2n}|-1}e^{i(|k_1-k_2|+\cdots+|k_{2n-1}-k_{2n}|-2(m_1+\cdots+m_{2n-1})-2n)\eta/2}.$$

In particular following those calculations we can express the integrand of I' in (3-79) as

$$\frac{\sin((k_1+k_{2n})\eta/2)}{\sin(\eta/2)}\prod_{j=1}^{2n-1}\frac{\sin((k_j-k_{j+1})\eta/2)}{(k_j-k_{j+1})\sin(\eta/2)}$$
$$=\left(\operatorname{sgn}(k_1+k_{2n})\prod_{j=1}^{2n-1}\frac{1}{|k_j-k_{j+1}|}\right)\sum_{\substack{m_j=0\\1\le j\le 2n-1}}^{|k_j-k_{j+1}|-1}\ \sum_{m_{2n}=0}^{|k_1+k_{2n}|-1}e^{iB_1\eta/2},$$

where to be clear in the sum the m_j indicates a further summation over all $j\in\{1,\dots,2n-1\}$. Also we define B_1 above as

$$B_1=\sum_{j=1}^{2n-1}|k_j-k_{j+1}|+|k_1+k_{2n}|-2\sum_{j=1}^{2n}m_j-2n.$$

Notice that no matter what the sign of any of the terms inside the absolute values above is we always have $\sum_{j=1}^{2n-1}|k_j-k_{j+1}|+|k_1+k_{2N}|=2l$ for some integer l so that B_1 is an even integer. This holds because the sum contains two copies of every $k_1,\dots,k_{2n}$.

We further integrate as $\int_{-\pi}^{\pi}e^{iB_1\eta/2}\,d\eta=(4/B_1)\sin(B_1\pi/2)$, and we notice that since B_1 is an even integer, either $(4/B_1)\sin(B_1\pi/2)=0$ if $B_1\neq 0$ or $(4/B_1)\sin(B_1\pi/2)=2\pi$ when $B_1=0$. We can then find the following expression for I':

$$I'=2\pi\left(\operatorname{sgn}(k_1+k_{2n})\prod_{j=1}^{2n-1}\frac{1}{|k_j-k_{j+1}|}\right)\sum_{\substack{m_j=0\\1\le j\le 2n-1}}^{|k_j-k_{j+1}|-1}\ \sum_{m_{2n}=0}^{|k_1+k_{2n}|-1}1_{B_1=0}. \tag{3-81}$$

This is our calculation of the integral (3-79). We note, as a function of the single variable m_{2n}, that B_1 is decreasing and takes the value zero at most one time. Thus $\sum_{m_{2n}=0}^{|k_1+k_{2n}|-1}1_{B_1=0}\le 1$. We conclude that $|I'|\le 2\pi$. Note that (3-4) is exactly (3-79). So this proves the second estimate for (3-4) in Lemma 3.2.

We now calculate the integral (3-80), which is rather similar. We obtain

$$\prod_{j=0}^{2n-1}\frac{\sin(|k_j-k_{j+1}|\eta/2)}{\sin(\eta/2)}=\sum_{\substack{m_j=0\\0\le j\le 2n-1}}^{|k_j-k_{j+1}|-1}e^{i(|k_0-k_1|+\cdots+|k_{2n-1}-k_{2n}|-2(m_0+\cdots+m_{2n-1})-2n)\eta/2}.$$

Then we can express the integrand of I'' as

$$\cos(\eta/2)\frac{\sin((k+k_{2n})\eta/2)}{\sin(\eta/2)}\prod_{j=0}^{2n-1}\frac{\sin((k_j-k_{j+1})\eta/2)}{(k_j-k_{j+1})\sin(\eta/2)}$$
$$=\frac{1}{2}\left(\operatorname{sgn}(k+k_{2n})\prod_{j=0}^{2n-1}\frac{1}{|k_j-k_{j+1}|}\right)\sum_{\substack{m_j=0\\0\le j\le 2n-1}}^{|k_j-k_{j+1}|-1}\ \sum_{m_{2n}=0}^{|k+k_{2n}|-1}(e^{iB_2\eta/2}+e^{iB_3\eta/2}).$$

We define

$$B_2 = \sum_{j=0}^{2n-1} |k_j - k_{j+1}| + |k + k_{2n}| - 2\sum_{j=0}^{2n} m_j - 2n,$$

$$B_3 = \sum_{j=0}^{2n-1} |k_j - k_{j+1}| + |k + k_{2n}| - 2\sum_{j=0}^{2n} m_j - 2n - 2.$$

Similarly since $\sum_{j=0}^{2n} |k_j - k_{j+1}| + |k + k_{2n}|$ contains two copies of every $k_0, k_1, \ldots, k_{2n}$ then it is always an even integer. Therefore we conclude that B_2 and B_3 both are even integers.

We can then similarly find the following expression for I'':

$$I'' = \pi\left(\operatorname{sgn}(k + k_{2n}) \prod_{j=0}^{2n-1} \frac{1}{|k_j - k_{j+1}|}\right) \sum_{\substack{m_j=0\\ 0\le j\le 2n-1}}^{|k_j-k_{j+1}|-1} \sum_{m_{2n}=0}^{|k+k_{2n}|-1} (1_{B_2=0} + 1_{B_3=0}).$$

Then using the same argument as our upper bound estimate for I' we obtain that $|I''| \le 2\pi$. □

Remark 3.4. One can generally calculate the sum in (3-81) exactly. In particular the value of the sums in (3-81) can be seen as the number of nonnegative integer solutions to the equation

$$m_1 + \cdots + m_{2n} = \frac{1}{2}\sum_{j=1}^{2n-1} |k_j - k_{j+1}| + |k_1 + k_{2n}| - n,$$

with the restrictions that $0 \le m_j \le |k_j - k_{j+1}| - 1$ for $j = 1, \ldots, 2n-1$ and $0 \le m_{2n} \le |k_1 + k_{2n}| - 1$. This value can be calculated exactly using the inclusion-exclusion formula.

Alternatively, if $n = 1$ in (3-4) then one can calculate, on the region where $I_1' \ne 0$, that we have exactly

$$I_1' = 2\pi \frac{\min\{|k_1 - k_2|, |k_1 + k_2|\} \operatorname{sgn}(k_1 + k_2)}{|k_1 - k_2|}.$$

And this formula is consistent with our estimate in Lemma 3.2.

4. Proof of main theorem

This section is devoted to the proof of Theorem 1.2. In Section 4A we show the scheme of the proof for existence of solutions via a regularization argument. The main part consists in obtaining the a priori estimates, in particular the energy inequality from (1-30). Uniqueness is later proved in Section 4B.

4A. *Existence.* The proof follows a standard regularization argument. We will use a regularization of (1-10) and (1-13), written in the form of (2-48), and the a priori estimates of the previous section to find a weak solution in the sense of Definition 4.2 below. The regularity obtained for the solution will imply that the solution found is indeed a strong solution, which we prove later is unique.

Definition 4.1. For fixed $t \in [0, T]$ and $\boldsymbol{\phi}(t) \in W^{2,\infty}(\mathbb{S})$, we say that $\boldsymbol{\psi}(t) \in L^\infty(\mathbb{S})$ is a weak solution of

$$\boldsymbol{\psi}(\theta, t) + 2A_\mu \partial_\theta \boldsymbol{\phi}(\theta, t)^\perp \cdot \int_{\mathbb{S}} \mathcal{T}(\boldsymbol{\phi}(\theta, t) - \boldsymbol{\phi}(\eta, t)) \cdot \boldsymbol{\psi}(\eta, t)\, d\eta = 2A_e \partial_\theta^2 \boldsymbol{\phi}(\theta, t),$$

with $\mathcal{T}$ given by (1-11), if for any $\boldsymbol{\varphi} \in \mathcal{D}(\mathbb{S})$ it holds that

$$\int_{\mathbb{S}} \boldsymbol{\psi}(\theta,t)\cdot\boldsymbol{\varphi}(\theta)\,d\theta + 2A_\mu \int_{\mathbb{S}} \varphi_i(\theta) \int_{\mathbb{S}} \partial_\theta \phi_j(\theta,t)^{\perp} \mathcal{T}_{ijk}(\boldsymbol{\phi}(\theta,t)-\boldsymbol{\phi}(\eta,t))\psi_k(\eta,t)\,d\eta\,d\theta = 2A_e \int_{\mathbb{S}} \partial_\theta^2 \boldsymbol{\phi}(\theta,t)\cdot\boldsymbol{\varphi}(\theta).$$

Definition 4.2. We say that $\boldsymbol{\mathcal{X}} \in L^\infty([0,T];W^{1,\infty}(\mathbb{S}))\cap L^1([0,T];W^{2,\infty}(\mathbb{S}))$ is a weak solution of (1-10) if for almost every $t\in[0,T]$ the arc-chord condition (1-17) is satisfied, and if for any $\boldsymbol{\varphi}\in\mathcal{D}(\mathbb{S}\times[0,T])$ it holds that

$$\int_{\mathbb{S}} \boldsymbol{\mathcal{X}}(\theta,t)\cdot\boldsymbol{\varphi}(\theta,t)\,d\theta - \int_{\mathbb{S}} \boldsymbol{\mathcal{X}}_0(\theta)\cdot\boldsymbol{\varphi}(\theta,0)\,d\theta - \int_0^t\int_{\mathbb{S}} \boldsymbol{\mathcal{X}}(\theta,\tau)\cdot\partial_t\boldsymbol{\varphi}(\theta,\tau)\,d\theta d\tau = \int_0^t\int_{\mathbb{S}} \boldsymbol{\varphi}(\theta,\tau)\cdot\int_{\mathbb{S}} G(\boldsymbol{\mathcal{X}}(\theta,\tau)-\boldsymbol{\mathcal{X}}(\eta,\tau))\boldsymbol{F}(\eta,\tau)\,d\eta\,d\theta\,d\tau,$$

where G is defined in (1-9) and $\boldsymbol{F}\in L^1([0,T];L^\infty(\mathbb{S}))$ is the solution in the sense of Definition 4.1 of (1-13).

We will write $f_M = \mathcal{J}_M f$ for general f such as $f=\boldsymbol{X}$, $f=\boldsymbol{X}_c$, $f=\boldsymbol{Y}$ or $f=\boldsymbol{F}$, with $\mathcal{J}_M$ the high-frequency cut-off defined in (1-23). We start by considering a regularized version of system (1-10), (1-13) (where (1-10) is written in (2-28) with the linear and nonlinear terms apart). For each positive integer M, consider the regularized initial data $\boldsymbol{\mathcal{X}}_{0,M}$ and the corresponding solution $\boldsymbol{\mathcal{X}} = \boldsymbol{X}_M + \boldsymbol{X}_{M,c}$ to the regularized system

$$\begin{aligned} \partial_t \boldsymbol{\mathcal{X}}_M &= -\frac{A_e}{2}(\Lambda \boldsymbol{X}_M + \mathcal{H}\mathcal{R}^{-1}\boldsymbol{X}_M) + \mathcal{J}_M \boldsymbol{\mathcal{N}}(\boldsymbol{X}_{M,c},\boldsymbol{X}_M),\\ \boldsymbol{F}_M &= 2A_\mu \mathcal{J}_M \boldsymbol{\mathcal{S}}(\boldsymbol{F}_M,\boldsymbol{\mathcal{X}}_M) + 2A_e\partial_\theta^2\boldsymbol{\mathcal{X}}_M. \end{aligned} \tag{4-1}$$

We define correspondingly $\boldsymbol{\mathcal{Y}}_{0,M}$ and $\boldsymbol{\mathcal{Y}}_M = \boldsymbol{Y}_M + \boldsymbol{Y}_{M,c}$. We recall that (2-28) could be written in $\boldsymbol{\mathcal{Y}}$-variables as (2-48). The corresponding regularized system in these variables reads as follows:

$$\begin{aligned} &\widehat{\boldsymbol{Y}}_M(0)=0, \quad \widehat{Y}_{M,2}(1)=0, \qquad \widehat{\boldsymbol{Y}}_{M,c}(k)=0, \quad k\neq 0,1, \qquad \widehat{Y}_{M,c,1}(1)=0,\\ &\partial_t\widehat{\boldsymbol{Y}}_{M,c}(0) = P(0)^{-1}\widehat{\boldsymbol{\mathcal{N}}(\boldsymbol{X}_{M,c},\boldsymbol{X}_M)}(0),\\ &\partial_t\widehat{Y}_{M,1}(1) = -A_e\widehat{Y}_{M,1}(1) + (P(1)^{-1}\widehat{\boldsymbol{\mathcal{N}}(\boldsymbol{X}_{M,c},\boldsymbol{X}_M)}(1))_1,\\ &\partial_t\widehat{\boldsymbol{Y}}_M(k) = -\frac{A_e}{2}\mathcal{D}(k)\widehat{\boldsymbol{Y}}_M(k) + P(k)^{-1}\widehat{\boldsymbol{\mathcal{N}}(\boldsymbol{X}_{M,c},\boldsymbol{X}_M)}(k), \quad 2\le k\le M,\\ &|\widehat{Y}_{M,c,2}(1)|^2 = \frac{1}{2} - \sum_{1\le k\le M} k(|\widehat{Y}_{M,2}(k)|^2 - |\widehat{Y}_{M,1}(k)|^2), \end{aligned} \tag{4-2}$$

with $\boldsymbol{F}_M$ given by (4-1). Since $\boldsymbol{X}_{M,c}$ is a circle with radius satisfying (2-42), the chord arc condition (1-17) is clearly satisfied for $\|\boldsymbol{X}_M\|_{\dot{\mathcal{F}}_\nu^{1,1}}$ sufficiently small; we shall soon see that this in fact holds so long as $\|\boldsymbol{X}_M\|_{\dot{\mathcal{F}}_\nu^{1,1}} < k(A_\mu)$, which is defined in (4-9). Then, with the same size condition, $\boldsymbol{F}_M$ is estimated in terms of $\boldsymbol{X}_M$ as in Section 3B. Thus, with $\boldsymbol{F}_M$ solved in terms of $\boldsymbol{Y}_M$ using the transformation (2-31), we obtain an ODE of the form

$$\dot{\boldsymbol{Y}}_M = \mathcal{J}_M \mathcal{G}(\boldsymbol{Y}_M), \quad \boldsymbol{Y}_M(0) = \boldsymbol{Y}_{M,0},$$

for a certain nonlinear function $\mathcal{G}$. Notice that the ODE for $\widehat{\boldsymbol{Y}}_{M,c}(0)$ is decoupled from the rest because there are no zero modes in $\widehat{\boldsymbol{\mathcal{N}}(\boldsymbol{X}_{M,c}, \boldsymbol{X}_M)}(0)$. Therefore, Picard's theorem on Banach spaces yields the local existence of regularized solutions $\boldsymbol{Y}_M \in C^1([0,T_M); H^m_M)$, where $H^m_M = \{f \in H^m(\mathbb{S}) : \operatorname{supp}(\hat{f}) \subset [-M, M]\}$. Since the a priori energy estimate (4-11) holds for the regularized system, we have uniform bounds for $\boldsymbol{Y}_M$ in the space $L^\infty(\mathbb{R}_+; \dot{\mathcal{F}}^{1,1}_\nu) \cap L^1(\mathbb{R}_+; \dot{\mathcal{F}}^{2,1}_\nu)$. It is not hard to prove that $\boldsymbol{Y}_M$ forms a Cauchy sequence in $L^\infty([0,T]; \mathcal{F}^{0,1}_\nu)$, so that we have a candidate for solution. One can then apply a version of the Aubin–Lions lemma (see Corollary 6 of [Simon 1987]) to get the strong convergence, up to a subsequence, of the approximate problems in $L^2([0,T]; \dot{\mathcal{F}}^{1,1}_\nu)$. Next, since $\widehat{\boldsymbol{Y}}_M(m,t) \to \widehat{\boldsymbol{Y}}(m,t)$ as $M \to \infty$, for all $m \in \mathbb{Z}$ and almost every t, Fatou's lemma allows us to conclude that

$$\begin{aligned} M(t) &= \|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{1,1}_\nu}(t) + \frac{A_e}{4}\mathcal{C}\int_0^t \|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{2,1}_\nu}(\tau)\,d\tau \\ &\leq \liminf_{M\to+\infty}\left(\|\boldsymbol{Y}_M\|_{\dot{\mathcal{F}}^{1,1}_\nu}(t) + \frac{A_e}{4}\mathcal{C}\int_0^t \|\boldsymbol{Y}_M\|_{\dot{\mathcal{F}}^{2,1}_\nu}(\tau)\,d\tau\right) \leq \|\boldsymbol{Y}_0\|_{\dot{\mathcal{F}}^{1,1}}, \end{aligned}$$

so we obtain that the limit function $\boldsymbol{Y}$ belongs to $L^\infty([0,T]; \dot{\mathcal{F}}^{1,1}_\nu) \cap L^1([0,T]; \dot{\mathcal{F}}^{2,1}_\nu)$. Now, we claim that the strong convergence, up to a subsequence, of $\boldsymbol{X}_M \to \boldsymbol{X}$ in $L^\infty([0,T]; \dot{\mathcal{F}}^{1,1}_\nu) \cap L^1([0,T]; \dot{\mathcal{F}}^{2,1}_\nu)$ holds. The proof of this claim follows in fact along the lines of the proof of uniqueness (see Section 4B). This strong convergence immediately implies from (4-1), under the size constraint (1-29), the strong convergence $\boldsymbol{F}_M \to \boldsymbol{F}$ in $L^1([0,T]; \mathcal{F}^{0,1}_\nu)$. In fact, it suffices to consider $\boldsymbol{F}_{M_1}$ and $\boldsymbol{F}_{M_2}$, write their difference as

$$\begin{aligned} \boldsymbol{F}_{M_1} - \boldsymbol{F}_{M_2} = 2A_\mu(&\mathcal{J}_{M_1}\boldsymbol{\mathcal{S}}(\boldsymbol{F}_{M_1}, \boldsymbol{\mathcal{X}}_{M_1}) - \mathcal{J}_{M_2}\boldsymbol{\mathcal{S}}(\boldsymbol{F}_{M_1}, \boldsymbol{\mathcal{X}}_{M_1})) \\ &+ 2A_\mu(\mathcal{J}_{M_2}\boldsymbol{\mathcal{S}}(\boldsymbol{F}_{M_1}, \boldsymbol{\mathcal{X}}_{M_1}) - \mathcal{J}_{M_2}\boldsymbol{\mathcal{S}}(\boldsymbol{F}_{M_2}, \boldsymbol{\mathcal{X}}_{M_1})) \\ &\quad + 2A_\mu(\mathcal{J}_{M_2}\boldsymbol{\mathcal{S}}(\boldsymbol{F}_{M_2}, \boldsymbol{\mathcal{X}}_{M_1}) - \mathcal{J}_{M_2}\boldsymbol{\mathcal{S}}(\boldsymbol{F}_{M_2}, \boldsymbol{\mathcal{X}}_{M_2})) \\ &\quad\quad + 2A_e(\partial_\theta^2\boldsymbol{\mathcal{X}}_{M_1} - \partial_\theta^2\boldsymbol{\mathcal{X}}_{M_2}), \end{aligned}$$

and perform estimates similar to the ones in Section 3B to find that $\boldsymbol{F}_M$ forms a Cauchy sequence in $L^1([0,T]; \mathcal{F}^{0,1}_\nu)$. Since $\boldsymbol{X}_{M,c}$ is given in terms of $\boldsymbol{X}_M$, the above convergence holds for $\boldsymbol{\mathcal{X}}_M$. The strong convergence $\boldsymbol{\mathcal{X}}_M \to \boldsymbol{\mathcal{X}}$ in $L^\infty([0,T]; \dot{\mathcal{F}}^{1,1}_\nu)$ together with $\boldsymbol{F}_M \to \boldsymbol{F}$ in $L^1([0,T]; \mathcal{F}^{0,1}_\nu)$ yields $\boldsymbol{\mathcal{X}}$ as a solution to (1-10) in the sense of Definition 4.2. (Moreover, it is easy to check in (4-1) the strong convergence of the right-side terms in $L^1([0,T]; \dot{\mathcal{F}}^{1,1}_\nu)$.)

We refer to Section 5 of [Gancedo et al. 2020] for a similar approximation argument, including the instant generation of analyticity and the continuity in time. We include it here for completeness. From the strong convergence in $L^1([0,T]; \dot{\mathcal{F}}^{1,1}_\nu)$ of the right-hand side of (1-10), we must have that $\partial_t\boldsymbol{\mathcal{X}}_M \to \partial_t\boldsymbol{\mathcal{X}}$ in $L^1([0,T]; \dot{\mathcal{F}}^{1,1}_\nu)$. Consider $0 < t \leq t_1 < t_2$. Then,

$$\|\boldsymbol{\mathcal{X}}(t_2) - \boldsymbol{\mathcal{X}}(t_1)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(t)}} = \left\| \int_{t_1}^{t_2} \partial_t\boldsymbol{\mathcal{X}}(\tau)\,d\tau \right\|_{\dot{\mathcal{F}}^{1,1}_{\nu(t)}} \leq \int_{t_1}^{t_2} \|\partial_t\boldsymbol{\mathcal{X}}(\tau)\|_{\dot{\mathcal{F}}^{1,1}_{\nu(\tau)}}\,d\tau,$$

which from the fact that $\partial_t\boldsymbol{\mathcal{X}} \in L^1([0,T]; \dot{\mathcal{F}}^{1,1}_\nu)$ yields that the solution is analytic for all positive times, and $\boldsymbol{\mathcal{X}} \in C([\varepsilon, T]; \dot{\mathcal{F}}^{1,1}_\nu)$ for any $\varepsilon > 0$. Moreover, fix $\tilde{\nu}_m \in (0, \nu_\infty)$ and define $\tilde{\nu}(t)$ according to (1-27);

now given any $t_2 > 0$ choose $0 < t_1 < t_2$ close enough to t_2 that $\tilde{\nu}(t_2) < \nu(t_1)$. Thus, it holds that

$$\|\boldsymbol{\mathcal{X}}(t_2) - \boldsymbol{\mathcal{X}}(t_1)\|_{\dot{\mathcal{F}}^{1,1}_{\tilde{\nu}(t_2)}} \to 0 \quad \text{as } t_1 \to t_2,$$

and therefore we have $\boldsymbol{\mathcal{X}} \in C([0,T]; \dot{\mathcal{F}}^{1,1}_{\tilde{\nu}})$. Since $\tilde{\nu}_m \in (0, \nu_\infty)$ is an arbitrary number in an open interval, we conclude that $\boldsymbol{\mathcal{X}} \in C([0,T]; \dot{\mathcal{F}}^{1,1}_{\nu})$. Finally, the analyticity in space for all positive times implies that $\boldsymbol{\mathcal{X}} \in C([\varepsilon,T]; \dot{\mathcal{F}}^{s,1}_{\tilde{\nu}})$ for any $s \geq 0$, $\varepsilon > 0$, and $0 < \tilde{\nu} < \nu$. This regularity translates to $\boldsymbol{F}$ as well for $t \geq \varepsilon$. Therefore, one can consider $\partial_t \boldsymbol{\mathcal{X}}(t_2) - \partial_t \boldsymbol{\mathcal{X}}(t_1)$ for arbitrary $t_2, t_1 \geq \varepsilon$ to find in particular that $\partial_t \boldsymbol{\mathcal{X}} \in C((0,T]; \mathcal{F}^{0,1}_{\nu})$.

We have proven that $\boldsymbol{\mathcal{X}}$ is a strong solution in the sense of Definition 1.1 as claimed in Theorem 1.2. In Section 4B we prove that this solution is unique.

We now prove the global-in-time energy inequality in (1-30).

Proof of (1-30). Equations (2-45) show decay of the higher frequencies (2-38) if we are able to control the nonlinear terms, for which we will need the constraint (2-40). Indeed, using (2-38) and the inequality $k(k-1) \geq k^2/2$ for $k \geq 2$ implies

$$\frac{d}{dt}\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{1,1}_{\nu}} \leq -\Big(\frac{A_e}{4} - \nu'(t)\Big)\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{2,1}_{\nu}} + \|\boldsymbol{\mathcal{N}}(\boldsymbol{X}_c, \boldsymbol{X})\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}, \tag{4-3}$$

where we have used that $\|P(k)^{-1}\| = 1$, and we can choose $\nu'(t)$ as small as we need. The goal is thus to obtain a bound like

$$\|\boldsymbol{\mathcal{N}}(\boldsymbol{X}_c, \boldsymbol{X})\|_{\dot{\mathcal{F}}^{1,1}_{\nu}} \leq C(\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_{\nu}})\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{2,1}_{\nu}}, \tag{4-4}$$

with $C(\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}) \approx \|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}$.

We proceed to complete the nonlinear estimate (4-4) to obtain the adequate sign in the balance (4-3). We insert the a priori bounds on $\boldsymbol{F}$ given by (3-55), (3-56), and (3-57), into the estimate (3-1) to obtain

$$\begin{aligned}\|\boldsymbol{\mathcal{N}}\|_{\dot{\mathcal{F}}^{1,1}_{\nu}} \leq 22\sqrt{2}A_e \frac{1 - A_\mu + |A_\mu|}{1 - A_\mu} D_1 \|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_{\nu}} \|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{2,1}_{\nu}} + 147\sqrt{2}\frac{A_e}{1 - A_\mu} e^{\nu_\infty} D_2 C_8 \|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_{\nu}} \|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{2,1}_{\nu}} \\ + 2250\sqrt{2}A_e \frac{|A_\mu|(1+|A_\mu|)}{(1-A_\mu)^2(1+A_\mu)} D_3 D_4 \|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_{\nu}} \|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{2,1}_{\nu}},\end{aligned}$$

which finally gives the desired estimate

$$\|\boldsymbol{\mathcal{N}}\|_{\dot{\mathcal{F}}^{1,1}_{\nu}} \leq 169\sqrt{2}\frac{A_e}{1 - A_\mu} D_5 \|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_{\nu}} \|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{2,1}_{\nu}}, \tag{4-5}$$

where

$$D_5 = \frac{1}{169}\bigg(22(1 - A_\mu + |A_\mu|)D_1 + 147 e^{\nu_\infty} D_2 C_8 + 2250\frac{|A_\mu|(1+|A_\mu|)}{(1-A_\mu)(1+A_\mu)} D_3 D_4\bigg), \tag{4-6}$$

and C_8, D_1, D_2, D_3, D_4, are given by (3-46), (3-54), and (3-78). Recalling the equivalence (2-34) and inserting the above bound into (4-3), we obtain

$$\frac{d}{dt}\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{1,1}_{\nu}} \leq -A_e\bigg(\frac{1}{4} - \frac{\nu'(t)}{A_e} - 169\sqrt{2}\frac{D_5}{1 - A_\mu}\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}\bigg)\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{2,1}_{\nu}}. \tag{4-7}$$

Since $\nu'(t) = \nu_\infty/(1+t)^2$ and $\nu_\infty > 0$ in (1-27) can be chosen arbitrarily small, if the condition

$$1 - 676\sqrt{2}\,\frac{D_5(\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu})}{1-A_\mu}\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu} > 0 \tag{4-8}$$

holds initially, where D_5 is defined in (4-6), then the fact that D_5 decreases as $\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}$ decreases guarantees that this condition is propagated in time. For the same reasons, this condition can be stated as a smallness condition for $\|\boldsymbol{X}_0\|_{\dot{\mathcal{F}}^{1,1}}$ as

$$\|\boldsymbol{X}_0\|_{\dot{\mathcal{F}}^{1,1}} < k(A_\mu), \tag{4-9}$$

with k a function defined implicitly via (4-8) (see also Figure 1). Because D_5 is increasing on $\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}$, we have the explicit upper bound

$$k(A_\mu) < \frac{1-A_\mu}{676\sqrt{2}D_5(0)}, \tag{4-10}$$

where

$$\frac{1-A_\mu}{676\sqrt{2}D_5(0)} = \left(\frac{588\sqrt{2}}{1-A_\mu} + 88\sqrt{2}\left(1+\frac{|A_\mu|}{1-A_\mu}\right) + \frac{9\sqrt{2}|A_\mu|(1+|A_\mu|)}{(1-A_\mu)(1+A_\mu)}\left(112\left(1+\frac{|A_\mu|}{1-A_\mu}\right)+\frac{888}{1-A_\mu}\right)\right)^{-1}.$$

Then, for small enough $\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}$, the upper bound in (4-10) approximates the actual value of $k(A_\mu)$. Therefore,

$$\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{1,1}_\nu}(t) + \frac{A_e}{4}\mathcal{C}\int_0^t \|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{2,1}_\nu}(\tau)\,d\tau \le \|\boldsymbol{Y}_0\|_{\dot{\mathcal{F}}^{1,1}}, \tag{4-11}$$

with

$$\mathcal{C} = \mathcal{C}(\|\boldsymbol{X}_0\|_{\dot{\mathcal{F}}^{1,1}_\nu}, A_\mu, \nu_\infty) = 1 - 4\frac{\nu'(t)}{A_e} - 676\sqrt{2}\,\frac{D_5(\|\boldsymbol{X}_0\|_{\dot{\mathcal{F}}^{1,1}_\nu})}{1-A_\mu}\|\boldsymbol{X}_0\|_{\dot{\mathcal{F}}^{1,1}_\nu}. \tag{4-12}$$

Moreover, since $\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{1,1}_\nu} \le \|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{2,1}_\nu}$, the inequality (4-7) gives

$$\frac{d}{dt}\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{1,1}_\nu} \le -\frac{A_e}{4}\mathcal{C}\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{1,1}_\nu},$$

and thus

$$\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{1,1}_\nu} \le \|\boldsymbol{Y}_0\|_{\dot{\mathcal{F}}^{1,1}_\nu} e^{-(A_e/4)\mathcal{C}t}. \tag{4-13}$$

This completes the decay estimate.

The control of the zero frequency follows from (2-47) with

$$|\widehat{\boldsymbol{X}}_c(0)| \le |\widehat{\boldsymbol{X}}_{0,c}(0)| + \int_0^t |\widehat{\boldsymbol{\mathcal{N}}(\boldsymbol{X}_c,\boldsymbol{X})}(0)|\,d\tau. \tag{4-14}$$

Notice that the estimates of the nonlinear terms in $\mathcal{F}^{0,1}$ can be done as in Section 3A and yield the bound

$$|\widehat{\boldsymbol{\mathcal{N}}(\boldsymbol{X}_c,\boldsymbol{X})}(0)| \le \|\boldsymbol{\mathcal{N}}(\boldsymbol{X}_c,\boldsymbol{X})\|_{\mathcal{F}^{0,1}} \le A_e\frac{\widetilde{D}_5}{1-A_\mu}\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{2,1}_\nu},$$

where $\widetilde{D}_5$ is a constant that plays the role of D_5. Recalling (2-34) and the energy balance (4-11), we introduce this bound back to (4-14) to conclude

$$|\widehat{\boldsymbol{X}}_c(0)| \le |\widehat{\boldsymbol{X}}_{0,c}(0)| + \widetilde{C}\|\boldsymbol{X}_0\|^2_{\dot{\mathcal{F}}^{1,1}},$$

with

$$\widetilde{C} = \frac{\widetilde{D}_5}{(1 - A_\mu)\mathcal{C}}, \tag{4-15}$$

and D_5, $\mathcal{C}$ given in (4-6) and (4-12) respectively.

Finally, the decay (4-13) applied to (2-46) yields

$$\tfrac{1}{2} R(t)^2 = |\widehat{Y}_{c,2}(1)|^2 \to \tfrac{1}{2} \quad \text{as } t \to +\infty,$$

showing the exponentially fast convergence to a uniformly parametrized circle of area π. □

4B. *Uniqueness.* Consider two solutions $\boldsymbol{\mathcal{X}} = \boldsymbol{X}_c + \boldsymbol{X}$ and $\widetilde{\boldsymbol{\mathcal{X}}} = \widetilde{\boldsymbol{X}}_c + \widetilde{\boldsymbol{X}}$ with initial data $\boldsymbol{\mathcal{X}}_0$ and $\widetilde{\boldsymbol{\mathcal{X}}}_0$ in $\mathcal{F}^{1,1}$. Recalling the system in the $\boldsymbol{\mathcal{Y}}$-variables (2-32), we have

$$\frac{d}{dt}\|\boldsymbol{\mathcal{Y}} - \widetilde{\boldsymbol{\mathcal{Y}}}\|_{\dot{\mathcal{F}}^{1,1}_\nu} \le -\Big(\frac{A_e}{4} - \nu'(t)\Big)\|\boldsymbol{Y} - \widetilde{\boldsymbol{Y}}\|_{\dot{\mathcal{F}}^{2,1}_\nu} + \sqrt{2}\|\boldsymbol{\mathcal{N}}(\boldsymbol{X}_c, \boldsymbol{X}) - \boldsymbol{\mathcal{N}}(\widetilde{\boldsymbol{X}}_c, \widetilde{\boldsymbol{X}})\|_{\dot{\mathcal{F}}^{1,1}_\nu}, \tag{4-16}$$

and

$$|\widehat{\boldsymbol{Y}}_c(0) - \widehat{\widetilde{\boldsymbol{Y}}}_c(0)| \le |\widehat{\boldsymbol{Y}}_{0,c}(0) - \widehat{\widetilde{\boldsymbol{Y}}}_{0,c}(0)| + \int_0^t \big|P(0)^{-1}\widehat{\boldsymbol{\mathcal{N}}(\boldsymbol{X}_c, \boldsymbol{X})}(0) - P(0)^{-1}\widehat{\boldsymbol{\mathcal{N}}(\widetilde{\boldsymbol{X}}_c, \widetilde{\boldsymbol{X}})}(0)\big|\, d\tau. \tag{4-17}$$

Notice that, in comparison with (4-3), we are including in the left-hand side of (4-16) the terms corresponding to (the first frequency of) the circle part,

$$2|\widehat{Y}_{c,2}(1) - \widehat{\widetilde{Y}}_{c,2}(1)|.$$

Although these terms are neutral with respect to the dissipative linear operator, whenever they appear on the right-hand side, we will be able to absorb them by using Grönwall's lemma and (4-11) (which both $\boldsymbol{Y}$ and $\widetilde{\boldsymbol{Y}}$ satisfy). Notice further that since the nonlinear terms do not contain the zero frequency of $\boldsymbol{\mathcal{Y}}$, i.e., $\widehat{\boldsymbol{Y}}_c(0)$, equation (4-17) implies

$$|\widehat{\boldsymbol{Y}}_c(0) - \widehat{\widetilde{\boldsymbol{Y}}}_c(0)| = 0, \tag{4-18}$$

once we show from (4-16) that $\|\boldsymbol{\mathcal{Y}} - \widetilde{\boldsymbol{\mathcal{Y}}}\|_{\dot{\mathcal{F}}^{1,1}_\nu} = 0$. Thus we proceed to deal with (4-16).

The difference between the nonlinear terms in (4-16) is split in four, according to (3-5), so that we have

$$\begin{aligned}\|\boldsymbol{\mathcal{N}}(\boldsymbol{X}_c, \boldsymbol{X}) - \boldsymbol{\mathcal{N}}(\widetilde{\boldsymbol{X}}_c, \widetilde{\boldsymbol{X}})\|_{\dot{\mathcal{F}}^{1,1}_\nu} \le{}& \|\boldsymbol{\mathcal{N}}_1(\boldsymbol{X}_c, \boldsymbol{X}) - \boldsymbol{\mathcal{N}}_1(\widetilde{\boldsymbol{X}}_c, \widetilde{\boldsymbol{X}})\|_{\mathcal{F}^{0,1}_\nu} + \|\boldsymbol{\mathcal{N}}_2(\boldsymbol{X}_c, \boldsymbol{X}) - \boldsymbol{\mathcal{N}}_2(\widetilde{\boldsymbol{X}}_c, \widetilde{\boldsymbol{X}})\|_{\mathcal{F}^{0,1}_\nu} \\ &+ \|\boldsymbol{\mathcal{N}}_3(\boldsymbol{X}_c, \boldsymbol{X}) - \boldsymbol{\mathcal{N}}_3(\widetilde{\boldsymbol{X}}_c, \widetilde{\boldsymbol{X}})\|_{\mathcal{F}^{0,1}_\nu} + \|\boldsymbol{\mathcal{N}}_4(\boldsymbol{X}_c, \boldsymbol{X}) - \boldsymbol{\mathcal{N}}_4(\widetilde{\boldsymbol{X}}_c, \widetilde{\boldsymbol{X}})\|_{\mathcal{F}^{0,1}_\nu}.\end{aligned} \tag{4-19}$$

We start by explaining the estimate corresponding to the first subterm $\boldsymbol{\mathcal{N}}_{1,1}$ in detail (see (3-6)), and later we will explain the general procedure. We have

$$\boldsymbol{\mathcal{N}}_{1,1}(\boldsymbol{X}_c, \boldsymbol{X})(\theta) - \boldsymbol{\mathcal{N}}_{1,1}(\widetilde{\boldsymbol{X}}_c, \widetilde{\boldsymbol{X}})(\theta) = \boldsymbol{Q}_1 + \boldsymbol{Q}_2 + \boldsymbol{Q}_3 + \boldsymbol{Q}_4, \tag{4-20}$$

where

$$\begin{aligned}\boldsymbol{Q}_1 &= \Big(-\frac{1}{4\pi R^2} + \frac{1}{4\pi \widetilde{R}^2}\Big)\int_{\mathbb{S}} \partial_\theta \Delta_\eta \boldsymbol{X}_c(\theta)^T \Delta_\eta \boldsymbol{X}(\theta) \boldsymbol{F}_L(\eta)\, d\eta, \\ \boldsymbol{Q}_2 &= \frac{1}{4\pi \widetilde{R}^2}\int_{\mathbb{S}} \big(\partial_\theta \Delta_\eta \widetilde{\boldsymbol{X}}_c(\theta) - \partial_\theta \Delta_\eta \boldsymbol{X}_c(\theta)\big)^T \Delta_\eta \widetilde{\boldsymbol{X}}(\theta) \widetilde{\boldsymbol{F}}_L(\eta)\, d\eta,\end{aligned}$$

$$\boldsymbol{Q}_3 = \frac{1}{4\pi \widetilde{R}^2} \int_{\mathbb{S}} \partial_\theta \Delta_\eta \boldsymbol{X}_c(\theta)^T \big(\Delta_\eta \widetilde{\boldsymbol{X}}(\theta) - \Delta_\eta \boldsymbol{X}(\theta)\big) \widetilde{\boldsymbol{F}}_L(\eta)\, d\eta,$$
$$\boldsymbol{Q}_4 = \frac{1}{4\pi \widetilde{R}^2} \int_{\mathbb{S}} \partial_\theta \Delta_\eta \boldsymbol{X}_c(\theta)^T \Delta_\eta \boldsymbol{X}(\theta) \big(\widetilde{\boldsymbol{F}}_L(\eta) - \boldsymbol{F}_L(\eta)\big)\, d\eta.$$

For the first term, we need to estimate the difference between R and $\widetilde{R}$. Recalling (2-40), where $|\widehat{Y}_{c,2}(1)|^2 = R^2/2$ with $R^2 = a^2 + b^2$, we have

$$\begin{aligned} |R^2 - \widetilde{R}^2| &= \left| -2 \sum_{k \geq 1} k(|\widehat{Y}_2(k)|^2 - |\widehat{Y}_1(k)|^2) + 2 \sum_{k \geq 1} k(|\widehat{\widetilde{Y}}_2(k)|^2 - |\widehat{\widetilde{Y}}_1(k)|^2) \right| \\ &\leq 2 \sum_{k \geq 1} k\Big(\big| |\widehat{Y}_2(k)| - |\widehat{\widetilde{Y}}_2(k)| \big| (|\widehat{Y}_2(k)| + |\widehat{\widetilde{Y}}_2(k)|) + \big| |\widehat{Y}_1(k)| - |\widehat{\widetilde{Y}}_1(k)| \big| (|\widehat{Y}_1(k)| + |\widehat{\widetilde{Y}}_1(k)|) \Big). \end{aligned}$$

Further note for $j = 1, 2$ that

$$\big| |\widehat{Y}_j(k)| - |\widehat{\widetilde{Y}}_j(k)| \big| \leq |\widehat{Y}_j(k) - \widehat{\widetilde{Y}}_j(k)|,$$

and on the $\mathbb{S}$ domain we have $|\widehat{Y}_j(k)| \leq \|Y_j\|_{L^\infty(\mathbb{S})} \leq \|Y_j\|_{\dot{\mathcal{F}}^{0,1}}$. We conclude

$$|R^2 - \widetilde{R}^2| \leq (\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{0,1}} + \|\widetilde{\boldsymbol{Y}}\|_{\dot{\mathcal{F}}^{0,1}}) \|\boldsymbol{Y} - \widetilde{\boldsymbol{Y}}\|_{\dot{\mathcal{F}}^{1,1}}.$$

Therefore, using also (2-41), we obtain

$$\left| \frac{1}{R^2} - \frac{1}{\widetilde{R}^2} \right| = \left| \frac{\widetilde{R}^2 - R^2}{R^2 \widetilde{R}^2} \right| \leq \frac{\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{0,1}} + \|\widetilde{\boldsymbol{Y}}\|_{\dot{\mathcal{F}}^{0,1}}}{\sqrt{1 - \frac{1}{2}\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{\frac{1}{2},1}}} \sqrt{1 - \frac{1}{2}\|\widetilde{\boldsymbol{Y}}\|_{\dot{\mathcal{F}}^{\frac{1}{2},1}}}} \|\boldsymbol{Y} - \widetilde{\boldsymbol{Y}}\|_{\dot{\mathcal{F}}^{1,1}}.$$

In particular, for a constant $c(\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{1,1}}, \|\widetilde{\boldsymbol{Y}}\|_{\dot{\mathcal{F}}^{1,1}}) > 0$, we can write

$$\left| \frac{1}{R^2} - \frac{1}{\widetilde{R}^2} \right| \leq c(\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{1,1}}, \|\widetilde{\boldsymbol{Y}}\|_{\dot{\mathcal{F}}^{1,1}}) \|\boldsymbol{Y} - \widetilde{\boldsymbol{Y}}\|_{\dot{\mathcal{F}}^{1,1}}.$$

Then, the bound for $\boldsymbol{Q}_1$ follows as in the estimate for the term (3-8), we obtain

$$\|\boldsymbol{Q}_1\|_{\mathcal{F}_\nu^{0,1}} \leq c(\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}_\nu^{1,1}}, \|\widetilde{\boldsymbol{Y}}\|_{\dot{\mathcal{F}}_\nu^{1,1}}) \|\boldsymbol{F}_L\|_{\mathcal{F}_\nu^{0,1}} \|\boldsymbol{Y} - \widetilde{\boldsymbol{Y}}\|_{\dot{\mathcal{F}}^{1,1}},$$

and introducing the estimate for $\boldsymbol{F}_L$ from (3-56) we have

$$\|\boldsymbol{Q}_1\|_{\mathcal{F}_\nu^{0,1}} \leq A_e c(\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}_\nu^{1,1}}, \|\widetilde{\boldsymbol{Y}}\|_{\dot{\mathcal{F}}_\nu^{1,1}}, A_\mu, \nu_\infty) \|\boldsymbol{Y}\|_{\dot{\mathcal{F}}_\nu^{2,1}} \|\boldsymbol{Y} - \widetilde{\boldsymbol{Y}}\|_{\dot{\mathcal{F}}^{1,1}},$$

which is trivially bounded by

$$\|\boldsymbol{Q}_1\|_{\mathcal{F}_\nu^{0,1}} \leq A_e c(\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}_\nu^{1,1}}, \|\widetilde{\boldsymbol{Y}}\|_{\dot{\mathcal{F}}_\nu^{1,1}}, A_\mu, \nu_\infty) \|\boldsymbol{Y}\|_{\dot{\mathcal{F}}_\nu^{2,1}} \|\boldsymbol{\mathcal{Y}} - \widetilde{\boldsymbol{\mathcal{Y}}}\|_{\dot{\mathcal{F}}^{1,1}}.$$

It is now clear that (4-11) allows us to control this term by Grönwall's lemma in (4-16).

We proceed to estimate $\boldsymbol{Q}_2$ in (4-20). Following the steps in (3-8), but maintaining the difference between $\widetilde{\boldsymbol{X}}_c$ and $\boldsymbol{X}_c$ together, we find that

$$\begin{aligned} \|\boldsymbol{Q}_2\|_{\mathcal{F}_\nu^{0,1}} &\leq c(\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}_\nu^{1,1}}, \|\widetilde{\boldsymbol{Y}}\|_{\dot{\mathcal{F}}_\nu^{1,1}}, A_\mu, \nu_\infty) \|\widetilde{\boldsymbol{Y}}\|_{\dot{\mathcal{F}}_\nu^{2,1}} |\widehat{Y}_{c,2}(1) - \widehat{\widetilde{Y}}_{c,2}(1)| \\ &\leq A_e c(\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}_\nu^{1,1}}, \|\widetilde{\boldsymbol{Y}}\|_{\dot{\mathcal{F}}_\nu^{1,1}}, A_\mu, \nu_\infty) \|\widetilde{\boldsymbol{Y}}\|_{\dot{\mathcal{F}}_\nu^{2,1}} \|\boldsymbol{\mathcal{Y}} - \widetilde{\boldsymbol{\mathcal{Y}}}\|_{\dot{\mathcal{F}}^{1,1}}, \end{aligned}$$

and it is thus controlled in the same way. The bound for $\boldsymbol{Q}_3$ follows exactly as in (3-8) and has the same structure as the bound for $\boldsymbol{Q}_2$.

Finally, we are left with $\boldsymbol{Q}_4$, for which we have

$$\boldsymbol{Q}_4 \le c(\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{1,1}_\nu}, \|\widetilde{\boldsymbol{Y}}\|_{\dot{\mathcal{F}}^{1,1}_\nu})\|\widetilde{\boldsymbol{F}}_L - \boldsymbol{F}_L\|_{\mathcal{F}^{0,1}_\nu}.$$

We emphasize that the constant above is given *exactly* by the one for $\boldsymbol{\mathcal{N}}_{1,1}$ in (3-14). The estimate for $\widetilde{\boldsymbol{F}}_L - \boldsymbol{F}_L$ follows from (2-25) (compare to (3-56)); we have

$$\|\widetilde{\boldsymbol{F}}_L - \boldsymbol{F}_L\|_{\mathcal{F}^{0,1}_\nu} \le 2A_e\|\widetilde{\boldsymbol{X}} - \boldsymbol{X}\|_{\dot{\mathcal{F}}^{2,1}_\nu} + 2A_e\frac{|A_\mu|}{1-A_\mu}\|\widetilde{\boldsymbol{X}} - \boldsymbol{X}\|_{\dot{\mathcal{F}}^{1,1}_\nu}, \tag{4-21}$$

so, moving to the $\boldsymbol{Y}$-variable, we obtain

$$\boldsymbol{Q}_4 \le A_e c(\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{1,1}_\nu}, \|\widetilde{\boldsymbol{Y}}\|_{\dot{\mathcal{F}}^{1,1}_\nu})\|\widetilde{\boldsymbol{Y}} - \boldsymbol{Y}\|_{\dot{\mathcal{F}}^{2,1}_\nu} + A_e c(\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{1,1}_\nu}, \|\widetilde{\boldsymbol{Y}}\|_{\dot{\mathcal{F}}^{1,1}_\nu}, A_\mu, \nu_\infty)\|\widetilde{\boldsymbol{Y}} - \boldsymbol{Y}\|_{\dot{\mathcal{F}}^{1,1}_\nu},$$

and therefore trivially we have

$$\boldsymbol{Q}_4 \le A_e c(\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{1,1}_\nu}, \|\widetilde{\boldsymbol{Y}}\|_{\dot{\mathcal{F}}^{1,1}_\nu})\|\widetilde{\boldsymbol{Y}} - \boldsymbol{Y}\|_{\dot{\mathcal{F}}^{2,1}_\nu} + A_e c(\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{1,1}_\nu}, \|\widetilde{\boldsymbol{Y}}\|_{\dot{\mathcal{F}}^{1,1}_\nu}, A_\mu, \nu_\infty)\|\widetilde{\boldsymbol{\mathcal{Y}}} - \boldsymbol{\mathcal{Y}}\|_{\dot{\mathcal{F}}^{1,1}_\nu}.$$

Although in this section we are denoting by c all constants (possibly depending on $\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{1,1}_\nu}$, $\|\widetilde{\boldsymbol{Y}}\|_{\dot{\mathcal{F}}^{1,1}_\nu}$, A_μ, ν_∞), it is important to notice that the constant in front of the high-order term $\|\widetilde{\boldsymbol{Y}} - \boldsymbol{Y}\|_{\dot{\mathcal{F}}^{2,1}_\nu}$ is less than or equal to the one appearing in the nonlinear estimates from Section 3A. This will allow us to absorb these terms using the negative sign coming from the dissipative linear term without additional conditions on the initial data other than the one needed for the earlier existence proof.

In summary, so far we have obtained

$$\begin{aligned}\|\boldsymbol{\mathcal{N}}_{1,1}(\boldsymbol{X}_c, \boldsymbol{X}) - \boldsymbol{\mathcal{N}}_{1,1}(\widetilde{\boldsymbol{X}}_c, \widetilde{\boldsymbol{X}})\|_{\mathcal{F}^{0,1}_\nu} &\le A_e c(\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{1,1}_\nu}, \|\widetilde{\boldsymbol{Y}}\|_{\dot{\mathcal{F}}^{1,1}_\nu})\|\widetilde{\boldsymbol{Y}} - \boldsymbol{Y}\|_{\dot{\mathcal{F}}^{2,1}_\nu}\\ &\quad + A_e g(\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{1,1}_\nu}, \|\widetilde{\boldsymbol{Y}}\|_{\dot{\mathcal{F}}^{1,1}_\nu}, \|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{2,1}_\nu}, \|\widetilde{\boldsymbol{Y}}\|_{\dot{\mathcal{F}}^{2,1}_\nu}, A_\mu, \nu_\infty)\|\widetilde{\boldsymbol{\mathcal{Y}}} - \boldsymbol{\mathcal{Y}}\|_{\dot{\mathcal{F}}^{1,1}_\nu},\end{aligned}$$

where g is a function whose L^1-in-time norm is bounded independently of time in terms of the initial data $\|\boldsymbol{Y}_0\|_{\mathcal{F}^{0,1}}$, $\|\widetilde{\boldsymbol{Y}}_0\|_{\mathcal{F}^{0,1}}$. Therefore the second term above can be controlled in (4-16) after using the Grönwall inequality.

Following the same steps for all the terms corresponding to $\boldsymbol{\mathcal{N}}_1$ from (3-6), it is clear that one obtains

$$\begin{aligned}\|\boldsymbol{\mathcal{N}}_1(\boldsymbol{X}_c, \boldsymbol{X}) - \boldsymbol{\mathcal{N}}_1(\widetilde{\boldsymbol{X}}_c, \widetilde{\boldsymbol{X}})\|_{\mathcal{F}^{0,1}_\nu} &\le c(\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{1,1}_\nu}, \|\widetilde{\boldsymbol{Y}}\|_{\dot{\mathcal{F}}^{1,1}_\nu})\|\widetilde{\boldsymbol{F}}_L - \boldsymbol{F}_L\|_{\mathcal{F}^{0,1}_\nu}\\ &\quad + A_e g(\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{1,1}_\nu}, \|\widetilde{\boldsymbol{Y}}\|_{\dot{\mathcal{F}}^{1,1}_\nu}, \|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{2,1}_\nu}, \|\widetilde{\boldsymbol{Y}}\|_{\dot{\mathcal{F}}^{2,1}_\nu}, A_\mu, \nu_\infty)\|\widetilde{\boldsymbol{\mathcal{Y}}} - \boldsymbol{\mathcal{Y}}\|_{\dot{\mathcal{F}}^{1,1}_\nu},\end{aligned}$$

where we use the same letter g to denote another L^1-in-time function as explained above and the constant in front of $\|\widetilde{\boldsymbol{F}}_L - \boldsymbol{F}_L\|_{\mathcal{F}^{0,1}_\nu}$ is exactly given by the one in (3-23). Since the coefficient of the higher-order term in the bound (4-21) is smaller than the one in (3-56), we guarantee that

$$\begin{aligned}\|\boldsymbol{\mathcal{N}}_1(\boldsymbol{X}_c, \boldsymbol{X}) - \boldsymbol{\mathcal{N}}_1(\widetilde{\boldsymbol{X}}_c, \widetilde{\boldsymbol{X}})\|_{\mathcal{F}^{0,1}_\nu} &\le A_e c(\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{1,1}_\nu}, \|\widetilde{\boldsymbol{Y}}\|_{\dot{\mathcal{F}}^{1,1}_\nu})\|\widetilde{\boldsymbol{Y}} - \boldsymbol{Y}\|_{\dot{\mathcal{F}}^{2,1}_\nu}\\ &\quad + A_e g(\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{1,1}_\nu}, \|\widetilde{\boldsymbol{Y}}\|_{\dot{\mathcal{F}}^{1,1}_\nu}, \|\boldsymbol{Y}\|_{\dot{\mathcal{F}}^{2,1}_\nu}, \|\widetilde{\boldsymbol{Y}}\|_{\dot{\mathcal{F}}^{2,1}_\nu}, A_\mu, \nu_\infty)\|\widetilde{\boldsymbol{\mathcal{Y}}} - \boldsymbol{\mathcal{Y}}\|_{\dot{\mathcal{F}}^{1,1}_\nu}.\end{aligned} \tag{4-22}$$

Now, we realize that the same idea applies to the other nonlinear terms in (4-19). In $\mathcal{N}_2$ there are not high-order terms to absorb, since the expression of $\boldsymbol{F}_0$ (3-58) only depends on the circle part, which has to be controlled via Grönwall. The term $\mathcal{N}_3$ will provide an estimate like the one above for $\mathcal{N}_1$, where the constant in front of $\|\widetilde{\boldsymbol{Y}}-\boldsymbol{Y}\|_{\dot{\mathcal{F}}_\nu^{2,1}}$ is smaller than (3-48) for the same reasons given before. Finally, the same can be said for $\mathcal{N}_4$, but with an analogous estimate to (4-21) for the difference $\boldsymbol{F}_N-\widetilde{\boldsymbol{F}}_N$. It follows in the same way as the estimate (3-57), so we omit details to avoid repetition.

The final estimate for the difference of the nonlinear terms in (4-19) has then the form (4-22), with a coefficient of the highest-order norm smaller than the coefficient of the norm with the highest-order derivative in (4-5). Therefore, under condition (1-29), the highest-regularity terms in the nonlinear upper bound can be absorbed by the dissipation in (4-16) and thus

$$\frac{d}{dt}\|\mathcal{Y}-\widetilde{\mathcal{Y}}\|_{\dot{\mathcal{F}}_\nu^{1,1}} \le A_e g(\|\boldsymbol{Y}\|_{\dot{\mathcal{F}}_\nu^{1,1}}, \|\widetilde{\boldsymbol{Y}}\|_{\dot{\mathcal{F}}_\nu^{1,1}}, \|\boldsymbol{Y}\|_{\dot{\mathcal{F}}_\nu^{2,1}}, \|\widetilde{\boldsymbol{Y}}\|_{\dot{\mathcal{F}}_\nu^{2,1}}, A_\mu, \nu_\infty)\|\widetilde{\mathcal{Y}}-\mathcal{Y}\|_{\dot{\mathcal{F}}_\nu^{1,1}},$$

which provides for all time via Grönwall that

$$\|\mathcal{Y}-\widetilde{\mathcal{Y}}\|_{\dot{\mathcal{F}}_\nu^{1,1}} \le c(\|\boldsymbol{Y}_0\|_{\dot{\mathcal{F}}_\nu^{1,1}}, \|\widetilde{\boldsymbol{Y}}_0\|_{\dot{\mathcal{F}}_\nu^{1,1}}, A_e, A_\mu, \nu_\infty)\|\mathcal{Y}_0-\widetilde{\mathcal{Y}}_0\|_{\dot{\mathcal{F}}_\nu^{1,1}}.$$

We conclude that $\|\mathcal{Y}-\widetilde{\mathcal{Y}}\|_{\dot{\mathcal{F}}_\nu^{1,1}}=0$. Together with (4-18), this completes the proof. □

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SOLUTION OF THE QC YAMABE EQUATION ON A 3-SASAKIAN MANIFOLD AND THE QUATERNIONIC HEISENBERG GROUP

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A complete solution to the quaternionic contact Yamabe equation on the qc sphere of dimension $4n+3$ as well as on the quaternionic Heisenberg group is given. A uniqueness theorem for the qc Yamabe problem in a compact locally 3-Sasakian manifold is shown.

1. Introduction

It is well known that the solution of the Yamabe problem on a compact Riemannian manifold is unique in the case of negative or vanishing scalar curvature. The proofs of these results, which rely on the maximum principle, extend readily to sub-Riemannian settings, such as the CR and quaternionic contact (abbreviated as qc) Yamabe problems, due to the subellipticity of the involved operators. The positive (scalar curvature) case presents considerable difficulties due to the possible nonuniqueness. Among these cases the corresponding round spheres play a special role due to their roles in the general existence theorem and because of the connections with the corresponding L^2 Sobolev-type embedding inequalities. Through the corresponding Cayley transforms, the sphere cases are equivalent to the problems of finding all solutions to the respective Yamabe equations on the flat models which are the Euclidean space or the relevant Heisenberg groups. All solutions of the latter equations were found in the Riemannian and CR sphere cases in [Obata 1971; Jerison and Lee 1988], respectively. The classification of all solutions of the Yamabe equation in the Euclidean setting can be handled alternatively by a reduction to a radially symmetric solution [Gidas et al. 1979; Talenti 1976]. As far as the rigidity question is concerned, Yamabe established a uniqueness result in every conformal class of an Einstein metric [Obata 1971].

In this paper we determine all solutions of the qc Yamabe equation on the $(4n+3)$-dimensional round sphere and quaternionic Heisenberg group and establish a uniqueness result in every qc conformal class containing a 3-Sasakian metric.

We continue by giving a brief background and the statements of our results. It is well known that the sphere at infinity of any noncompact symmetric space M of rank 1 carries a natural Carnot–Carathéodory structure; see [Mostow 1973; Pansu 1989]. A quaternionic contact (qc) structure [Biquard 1999; 2000] appears naturally as the conformal boundary at infinity of the quaternionic hyperbolic space. Following Biquard, a qc structure on a real $(4n+3)$-dimensional manifold M is a codimension-3 distribution H (*the horizontal distribution*) locally given as the kernel of an $\mathbb{R}^3$-valued 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ such that the

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three 2-forms $d\eta_i|_H$ are the fundamental forms of a quaternionic Hermitian structure on H. The 1-form η is determined up to a conformal factor and the action of SO(3) on $\mathbb{R}^3$, and therefore H is equipped with a conformal class $[g]$ of quaternionic Hermitian metrics. To every metric in the fixed conformal class one can associate a linear connection with torsion preserving the qc structure, see [Biquard 2000; Duchemin 2006], which is called the Biquard canonical connection. For a fixed metric in the conformal class of metrics on the horizontal space one associates the horizontal Ricci-type tensor of the Biquard connection, which is called the qc Ricci tensor. This is a symmetric tensor [Biquard 2000] whose trace part defines the qc scalar curvature. Furthermore, it was shown in [Ivanov et al. 2014a] that the torsion endomorphism of the Biquard connection completely determines the trace-free part of the horizontal Ricci tensor. The vanishing of the latter tensor defines the class of qc Einstein manifolds. A basic example of a qc manifold is a 3-Sasakian space, which can be defined as a $(4n+3)$-dimensional Riemannian manifold whose Riemannian cone is a hyper-Kähler manifold and the qc structure is induced from that hyper-Kähler structure. By [Ivanov et al. 2014a; Ivanov et al. 2016] the qc Einstein manifolds of positive qc scalar curvature are exactly the locally 3-Sasakian manifolds, up to a multiplication with a constant factor and a SO(3)-matrix. In particular, every 3-Sasakian manifold has vanishing torsion endomorphism and is a qc Einstein manifold.

The quaternionic contact Yamabe problem on a compact qc manifold M is the problem of finding a metric $\bar{g}$ in the qc conformal class $[g]$ of a fixed metric on the horizontal space H for which the qc scalar curvature is constant. We note that a qc conformal transformation of the contact form described in Definition 2.1 amounts to a conformal change of the horizontal metric. Another natural problem is to explore the possible uniqueness or nonuniqueness of such qc Yamabe metrics. Within a fixed qc conformal class, the questions reduce to the solvability and uniqueness of positive solutions of the quaternionic contact (qc) Yamabe equation

$$\mathcal{L}u \equiv 4\,\frac{Q+2}{Q-2}\triangle u - u\,\mathrm{Scal} = -u^{2^*-1}\,\overline{\mathrm{Scal}},$$

where $\triangle$ is the horizontal sub-Laplacian defined using the Biquard connection ∇, $\triangle h = \mathrm{tr}^g(\nabla^2 h)$, Scal and $\overline{\mathrm{Scal}}$ are the qc scalar curvatures correspondingly of (M,η) and $(M,\bar{\eta})$, $\bar{\eta} = u^{4/(Q-2)}\eta$, and $2^* = 2Q/(Q-2)$, with $Q = 4n+6$ — the homogeneous dimension.

Alternatively, one can view the problem as a variational problem whereby on a compact quaternionic contact manifold M with a fixed conformal class $[\eta]$ the qc Yamabe equation characterizes the nonnegative extremals of the qc Yamabe functional defined by

$$\Upsilon(u) = \int_M \left(4\frac{Q+2}{Q-2}|\nabla u|^2 + \mathrm{Scal}\,u^2\right) dv_g, \quad \int_M u^{2^*}\,dv_g = 1, \quad 0 < u \in \mathcal{D}^{1,2}(M).$$

Here dv_g denotes the Riemannian volume form of the Riemannian metric on M obtained by extending in a natural way the horizontal metric associated to η, and $\mathcal{D}^{1,2}(M)$ stands for the L^2 homogeneous Sobolev space. Considering M equipped with a fixed qc structure, hence, a conformal class $[\eta]$, the Yamabe constant is defined as the infimum

$$\lambda(M) \equiv \lambda(M,[\eta]) = \inf\left\{\Upsilon(u) : \int_M u^{2^*}\,dv_g = 1,\ 0 < u \in D^{1,2}(M)\right\}. \tag{1-1}$$

The main result of [Wang 2007] is that the qc Yamabe equation has a solution on a compact qc manifold provided $\lambda(M) < \lambda(S^{4n+3})$, where S^{4n+3} is the standard unit sphere in the quaternionic space $\mathbb{H}^{n+1}$.

In this paper we consider the qc Yamabe problem on the unit $(4n+3)$-dimensional sphere in $\mathbb{H}^{n+1}$. The standard 3-Sasaki structure on the sphere $\tilde{\eta}$ has a constant qc scalar curvature $\widetilde{\mathrm{Scal}} = 16n(n+2)$ and vanishing trace-free part of its qc Ricci tensor; i.e., it is a qc Einstein space. The images under conformal quaternionic contact automorphisms are again qc Einstein structures and, in particular, have constant qc scalar curvature. In [Ivanov et al. 2014a] we conjectured that these are the only solutions to the Yamabe problem on the quaternionic sphere and proved it in dimension 7 in [Ivanov et al. 2010]. One of the main goals of this paper is to prove this conjecture in full generality.

Theorem 1.1. *Let $\eta = 2h\tilde{\eta}$ be a qc conformal transformation of the standard qc structure $\tilde{\eta}$ on a 3-Sasakian sphere of dimension $4n+3$. If η has constant qc scalar curvature, then up to a multiplicative constant η is obtained from $\tilde{\eta}$ by a conformal quaternionic contact automorphism.*

We note that Theorem 1.1, together with the results of [Ivanov et al. 2014a], allows the determination of *all* solutions of the qc Yamabe problem on the sphere and on the quaternionic Heisenberg group $\boldsymbol{G}(\mathbb{H})$. This complements the CR case where [Jerison and Lee 1988] characterized all nonnegative solutions of the CR Yamabe problem on the Heisenberg group and the corresponding odd-dimensional spheres.

Recall that the quaternionic Heisenberg group $\boldsymbol{G}(\mathbb{H})$ of homogeneous dimension $Q = 4n+6$ is given by $\boldsymbol{G}(\mathbb{H}) = \mathbb{H}^n \times \operatorname{Im}\mathbb{H}$ with the group law

$$(q_o, \omega_o) \circ (q, \omega) = (q_o + q, \omega + \omega_o + 2\operatorname{Im} q_o \bar{q}),$$

where $q = (t^a, x^a, y^a, z^a) \in \mathbb{H}^n$, $\omega = (x, y, z) \in \operatorname{Im}\mathbb{H}$. The *standard* qc contact form in quaternion variables is

$$\tilde{\Theta} = (\tilde{\Theta}_1, \tilde{\Theta}_2, \tilde{\Theta}_3) = \tfrac{1}{2}(d\omega - q \cdot d\bar{q} + dq \cdot \bar{q}).$$

The corresponding sub-Laplacian is given by $\triangle_{\tilde{\Theta}} u = \sum_{a=1}^{n} (T_\alpha^2 u + X_\alpha^2 u + Y_\alpha^2 u + Z_\alpha^2 u)$, where T_a, X_a, Y_a, Z_a denote the left-invariant horizontal vector fields on $\boldsymbol{G}(\mathbb{H})$. Theorem 1.1 shows, in particular, the following.

Corollary 1.2. *If Φ satisfies the qc Yamabe equation on the quaternionic Heisenberg group $\boldsymbol{G}(\mathbb{H})$, that is,*

$$\frac{4(Q+2)}{Q-2} \triangle_{\tilde{\Theta}} \Phi = -S_\Theta \Phi^{2^*-1}$$

for some constant S_Θ, then up to a left translation $\Phi = (2h)^{-(Q-2)/4}$ and h is given by

$$h(q, \omega) = c_0[(\sigma + |q + q_0|^2)^2 + |\omega + \omega_o + 2\operatorname{Im} q_o \bar{q}|^2] \tag{1-2}$$

for some fixed $(q_o, \omega_o) \in \boldsymbol{G}(\mathbb{H})$ and constants $c_0 > 0$ and $\sigma > 0$. Furthermore, the qc scalar curvature of Θ is $S_\Theta = 128n(n+2)c_0\sigma$.

The above corollary confirms the conjecture made after [Garofalo and Vassilev 2001, Theorem 1.1]. In Theorem 1.6 of the same paper the conjecture claim was verified on all groups of Iwasawa type, but with the assumption of partial symmetry of the solution. Here, with a completely different method from [Garofalo and Vassilev 2001] we show that the symmetry assumption is superfluous in the case of the quaternionic Heisenberg group. The corresponding solutions on the 3-Sasakian sphere are obtained via the Cayley transform, see for example [Ivanov et al. 2010; 2012; 2014a; Ivanov and Vassilev 2011, Sections 2.3, 5.2.1] for an account and history. Finally, it should be observed that the functions (1-2) with $c_0 \in \mathbb{R}$ give all conformal factors for which Θ is also qc Einstein. It is worth mentioning that as a consequence of Theorem 1.1 and Corollary 1.2, we obtain that all solutions to the qc Yamabe equation are given by the functions which realize the equality case of the L^2 Folland–Stein inequality. The latter were characterized, by a different method, in [Ivanov et al. 2012], where the center of mass technique developed for the CR case in [Frank and Lieb 2012; Branson et al. 2013] was used.

A major step in the proof of Theorem 1.1 is the following result, where we solve the qc Yamabe problem on locally 3-Sasakian compact manifolds. By the results of [Ivanov et al. 2014a; 2016] a qc Einstein manifold is of constant qc scalar curvature; hence as far as the qc Yamabe equation is concerned only the uniqueness of solutions needs to be addressed. As mentioned earlier, the interesting case is when the qc scalar curvature is a positive constant; hence we focus exclusively on the locally 3-Sasakian case.

Theorem 1.3. *Let $(M, \bar{\eta})$ be a compact locally 3-Sasakian qc manifold of qc scalar curvature $16n(n+2)$. If $\eta = 2h\bar{\eta}$ is qc conformal to an $\bar{\eta}$ structure which is also of constant qc scalar curvature, then up to a homothety (M, η) is locally 3-Sasakian manifold. Furthermore, the function h is constant unless $(M, \bar{\eta})$ is the unit 3-Sasakian sphere.*

The proof of Theorem 1.3, presented in Section 5, consists of two steps. The first step is a divergence formula Theorem 4.1 which shows that if $\bar{\eta}$ is of constant qc curvature and is qc conformal to a locally 3-Sasakian manifold, then $\bar{\eta}$ is also a locally 3-Sasakian manifold. The general idea to search for such a divergence formula goes back to [Obata 1971] where the corresponding result on a Riemannian manifold was proved for a conformal transformation of an Einstein space. However, our result is motivated by the (sub-Riemannian) CR case where a formula of this type was introduced in the ground-breaking paper [Jerison and Lee 1988]. As far as the qc case is concerned in [Ivanov et al. 2014a, Theorem 1.2] a weaker result was shown, namely Theorem 1.3 holds provided the vertical space of η is integrable. In dimension 7, the $n = 1$ case, this assumption was removed in [Ivanov et al. 2010, Theorem 1.2] where the result was established with the help of a suitable divergence formula. It should be noted that in the 7-dimensional case the [3]-component of the traceless qc Ricci tensor vanishes, which decreases the number of torsion components. The general $n > 1$ case treated here presents new difficulties due to the extra nonzero torsion terms that appear in the higher dimensions, which complicate considerably the search for a suitable divergence formula.

The proof of the second part of Theorem 1.3 builds on, in the Riemannian case, ideas of Obata, who used that the gradient of the (suitably taken) conformal factor is a conformal vector field and the characterization of the unit sphere through its first eigenvalue of the Laplacian among all Einstein manifolds. We show a similar, although a more complicated relation between the conformal factor and the existence of an infinitesimal qc automorphism (qc vector field). Our divergence formula found in Theorem 4.1 involves a smooth function f, see (4-7), expressed in terms of the conformal factor and its horizontal gradient. Remarkably, we found that the horizontal gradient of f is precisely the horizontal part of the qc vector field mentioned above and the sub-Laplacian of f is an eigenfunction of the sub-Laplacian with the smallest possible eigenvalue $-4n$ thus showing a geometric nature of f (see Remark 5.3). Then we use the characterization of the 3-Sasakian sphere by its first eigenvalue of the sub-Laplacian among all locally 3-Sasakian manifolds established in [Ivanov et al. 2014b, Theorem 1.2] for $(n > 1)$ and in [Ivanov et al. 2013, Corollary 1.2] for $n = 1$.

A few final comments are in order. As noted above, the connection between the two Obata theorems — the uniqueness up to homothety of the Yamabe metric in the conformal class of an Einstein metric on a compact Riemannian manifold distinct from the round sphere and the characterization of the sphere as the extremal in the Lichnerowicz–Obata inequality — are well known. Our inspiration for the proof of the second part of Theorem 1.3 came from the slightly different approach taken in [Bourguignon and Ezin 1987]; see [Ivanov and Vassilev 2015, Theorem 2.6]. The attempt to find an extension of this argument to the qc setting resulted in the argument presented here.

Remark 1.4. The above argument leading to the uniqueness result in Theorem 1.3 can be applied not only in the qc case but also in the case of a pseudohermitian structure on a CR manifold. In particular, the argument presented here reveals the geometric nature of the function in Jerison and Lee's divergence formula [1988]. Indeed, the real part of the function f defined in Proposition 3.1 of that paper determines a CR vector field and its CR-Laplacian is an eigenfunction of the CR-Laplacian with the smallest possible eigenvalue $-2n$. More details for the CR case can be found in [Ivanov and Vassilev 2015, Section 5.2].

Convention 1.5. We use the following:

- $\{e_1, \dots, e_{4n}\}$ denotes an orthonormal basis of the horizontal space H.
- The capital letters $X, Y, Z, \dots$ denote horizontal vectors in H.
- The summation convention over repeated vectors from the basis $\{e_1, \dots, e_{4n}\}$ will be used. For example, for a $(0, 4)$-tensor P, $k = P(e_b, e_a, e_a, e_b)$ means $k = \sum_{a,b=1}^{4n} P(e_b, e_a, e_a, e_b)$.
- The triple (i, j, k) denotes any cyclic permutation of $(1, 2, 3)$.
- The horizontal divergence $\nabla^* P$ of a $(0, 2)$-tensor field P on M with respect to the Biquard connection is defined to be the $(0, 1)$-tensor field $\nabla^* P(\,\cdot\,) = (\nabla_{e_a} P)(e_a, \cdot\,)$.

2. Quaternionic contact manifolds and the qc Yamabe problem

In this section we will briefly review the basic notions of quaternionic contact geometry and recall some results from [Biquard 2000; Ivanov et al. 2014a]; see [Ivanov and Vassilev 2011] for a more leisurely exposition. We also give some background on the qc Yamabe problem.

2A. ***qc manifolds.*** A quaternionic contact (qc) manifold $(M, \eta, g, \mathbb{Q})$ is a $(4n+3)$-dimensional manifold M with a codimension-3 distribution H locally given as the kernel of a 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in $\mathbb{R}^3$. In addition H has an $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$ structure; that is, it is equipped with a Riemannian metric g and a rank-3 bundle $\mathbb{Q}$ consisting of endomorphisms of H locally generated by three almost complex structures I_1, I_2, I_3 on H satisfying the identities of the imaginary unit quaternions, $I_1 I_2 = -I_2 I_1 = I_3$,

$I_1 I_2 I_3 = -\mathrm{id}_{|H}$, which are hermitian compatible with the metric $g(I_s\,\cdot\,, I_s\,\cdot\,) = g(\,\cdot\,,\,\cdot\,)$ and the following contact condition holds:

$$2g(I_s X, Y) = d\eta_s(X, Y).$$

A special phenomena, noted in [Biquard 2000], is that the contact form η determines the quaternionic structure and the metric on the horizontal distribution in a unique way.

The transformations preserving a given quaternionic contact structure η, i.e., $\bar{\eta} = \mu\Psi\eta$ for a positive smooth function μ and an SO(3) matrix Ψ with smooth functions as entries, are called *quaternionic contact conformal* (*qc conformal*) *transformations*. If the function μ is constant, $\bar{\eta}$ is called qc homothetic to η. The qc conformal curvature tensor W^{qc}, introduced in [Ivanov and Vassilev 2010], is the obstruction for a qc structure to be locally qc conformal to the standard 3-Sasakian structure on the $(4n+3)$-dimensional sphere [Ivanov et al. 2014a; Ivanov and Vassilev 2010].

Definition 2.1. A diffeomorphism ϕ of a qc manifold $(M, [g], \mathbb{Q})$ is called a *conformal quaternionic contact automorphism* (*conformal qc automorphism*) if ϕ preserves the qc structure; i.e.,

$$\phi^*\eta = \mu\Phi\cdot\eta$$

for some positive smooth function μ and some matrix $\Phi \in \mathrm{SO}(3)$ with smooth functions as entries and $\eta = (\eta_1, \eta_2, \eta_3)^t$ is a local 1-form considered as a column vector of three one forms as entries.

On a qc manifold with a fixed metric g on H there exists a canonical connection defined first by O. Biquard [2000] when the dimension $(4n+3)$ greater than 7, and in [Duchemin 2006] for the 7-dimensional case. Biquard showed that there is a unique connection ∇ with torsion T and a unique supplementary subspace V to H in TM, such that:

(i) ∇ preserves the decomposition $H \oplus V$ and the $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$ structure on H; i.e., $\nabla g = 0$, $\nabla\sigma \in \Gamma(\mathbb{Q})$ for a section $\sigma \in \Gamma(\mathbb{Q})$, and its torsion on H is given by $T(X, Y) = -[X, Y]_{|V}$.

(ii) For $\xi \in V$, the endomorphism $T(\xi,\,\cdot\,)_{|H}$ of H lies in $(\mathrm{sp}(n) \oplus \mathrm{sp}(1))^{\perp} \subset \mathrm{gl}(4n)$.

(iii) The connection on V is induced by the natural identification φ of V with the subspace sp(1) of the endomorphisms of H; i.e., $\nabla\varphi = 0$.

This canonical connection is also known as *the Biquard connection*. When the dimension of M is at least 11 [Biquard 2000] also described the supplementary distribution V, which is (locally) generated by the so-called Reeb vector fields $\{\xi_1, \xi_2, \xi_3\}$ determined by

$$\eta_s(\xi_k) = \delta_{sk}, \quad (\xi_s \lrcorner\, d\eta_s)_{|H} = 0, \quad (\xi_s \lrcorner\, d\eta_k)_{|H} = -(\xi_k \lrcorner\, d\eta_s)_{|H}, \tag{2-1}$$

where $\lrcorner$ denotes the interior multiplication. If the dimension of M is 7, Duchemin [2006] shows that if we assume, in addition, the existence of Reeb vector fields as in (2-1), then the Biquard result holds. Henceforth, by a qc structure in dimension 7 we shall mean a qc structure satisfying (2-1).

The fundamental 2-forms ω_s of the quaternionic contact structure Q are defined by

$$2\omega_{s|H} = d\eta_{s|H}, \qquad \xi \lrcorner\, \omega_s = 0, \quad \xi \in V.$$

Notice that (2-1) are invariant under the natural SO(3) action. Using the triple of Reeb vector fields, we extend the metric g on H to a metric h on TM by requiring $\operatorname{span}\{\xi_1, \xi_2, \xi_3\} = V \perp H$ and $h(\xi_s, \xi_k) = \delta_{sk}$. The Riemannian metric h, as well as the Biquard connection, do not depend on the action of SO(3) on V, but both change if η is multiplied by a conformal factor [Ivanov et al. 2014a]. Clearly, the Biquard connection preserves the Riemannian metric on TM, $\nabla h = 0$.

The properties of the Biquard connection are encoded in the torsion endomorphism $T_\xi \in (\mathrm{sp}(n)+\mathrm{sp}(1))^\perp$. We recall the $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$-invariant decomposition. An endomorphism Ψ of H can be decomposed with respect to the quaternionic structure $(\mathbb{Q}, g)$ uniquely into four $\mathrm{Sp}(n)$-invariant parts $\Psi = \Psi^{+++} + \Psi^{+--} + \Psi^{-+-} + \Psi^{--+}$, where the superscript $+++$ means commuting with all three I_i, $+--$ indicates commuting with I_1 and anticommuting with the other two and etc. The two $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$-invariant components $\Psi_{[3]} = \Psi^{+++}$, $\Psi_{[-1]} = \Psi^{+--} + \Psi^{-+-} + \Psi^{--+}$ are determined by

$$\begin{aligned} \Psi = \Psi_{[3]} \quad &\iff \quad 3\Psi + I_1\Psi I_1 + I_2\Psi I_2 + I_3\Psi I_3 = 0, \\ \Psi = \Psi_{[-1]} \quad &\iff \quad \Psi - I_1\Psi I_1 - I_2\Psi I_2 - I_3\Psi I_3 = 0. \end{aligned}$$

With a short calculation one sees that the $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$-invariant components are the projections on the eigenspaces of the Casimir operator $\Upsilon = I_1 \otimes I_1 + I_2 \otimes I_2 + I_3 \otimes I_3$ corresponding, respectively, to the eigenvalues 3 and -1; see [Capria and Salamon 1988]. If $n = 1$ then the space of symmetric endomorphisms commuting with all I_s is 1-dimensional; i.e., the [3]-component of any symmetric endomorphism Ψ on H is proportional to the identity, $\Psi_{[3]} = -(\operatorname{tr}\Psi/4)\mathrm{id}_{|H}$. Note here that each of the three 2-forms ω_s belongs to its [-1]-component, $\omega_s = \omega_{s[-1]}$, and constitutes a basis of the Lie algebra sp(1).

2B. ***The torsion tensor.*** Decomposing the endomorphism $T_\xi \in (\mathrm{sp}(n)+\mathrm{sp}(1))^\perp$ into its symmetric part T^0_ξ and skew-symmetric part b_ξ, $T_\xi = T^0_\xi + b_\xi$, Biquard [2000] showed that the torsion T_ξ is completely trace-free, $\operatorname{tr} T_\xi = \operatorname{tr} T_\xi \circ I_s = 0$, its symmetric part has the properties

$$T^0_{\xi_i} I_i = -I_i T^0_{\xi_i}, \quad I_2(T^0_{\xi_2})^{+--} = I_1(T^0_{\xi_1})^{-+-}, \quad I_3(T^0_{\xi_3})^{-+-} = I_2(T^0_{\xi_2})^{--+}, \quad I_1(T^0_{\xi_1})^{--+} = I_3(T^0_{\xi_3})^{+--}.$$

The skew-symmetric part can be represented as $b_{\xi_i} = I_i u$, where u is a traceless symmetric (1, 1)-tensor on H which commutes with I_1, I_2, I_3. Therefore we have $T_{\xi_i} = T^0_{\xi_i} + I_i u$. If $n = 1$ then the tensor u vanishes identically, $u = 0$, and the torsion is a symmetric tensor, $T_\xi = T^0_\xi$.

The two $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$-invariant trace-free symmetric 2-tensors $T^0(X, Y) = g((T^0_{\xi_1} I_1 + T^0_{\xi_2} I_2 + T^0_{\xi_3} I_3)X, Y)$, $U(X, Y) = g(uX, Y)$ on H, introduced in [Ivanov et al. 2014a], have the properties

$$\begin{aligned} T^0(X, Y) + T^0(I_1X, I_1Y) + T^0(I_2X, I_2Y) + T^0(I_3X, I_3Y) &= 0, \\ U(X, Y) = U(I_1X, I_1Y) = U(I_2X, I_2Y) &= U(I_3X, I_3Y). \end{aligned} \tag{2-2}$$

In dimension 7 ($n = 1$), the tensor U vanishes identically, $U = 0$.

These tensors determine completely the torsion endomorphism of the Biquard connection due to the identity [Ivanov and Vassilev 2010, Proposition 2.3] $4T^0(\xi_s, I_sX, Y) = T^0(X, Y) - T^0(I_sX, I_sY)$, which implies

$$4T(\xi_s, I_sX, Y) = 4T^0(\xi_s, I_sX, Y) + 4g(I_s u I_s X, Y) = T^0(X, Y) - T^0(I_sX, I_sY) - 4U(X, Y).$$

2C. ***Curvature, torsion and qc Einstein structures.*** Quaternionic contact Einstein manifolds introduced in [Ivanov et al. 2014a], see [Ivanov et al. 2016; Ivanov and Vassilev 2011] for further details and a more leisurely exposition, play a crucial role in solving the Yamabe equation on the quaternionic sphere (see [Ivanov et al. 2010] for dimension 7).

Let $R = [\nabla, \nabla] - \nabla_{[\,\cdot\,,\cdot\,]}$ be the curvature of the Biquard connection ∇. The Ricci tensor and the scalar curvature, called *qc Ricci tensor* and *qc scalar curvature*, respectively, are defined by

$$\begin{aligned} \operatorname{Ric}(X, Y) &= g(R(e_a, X)Y, e_a), \\ \operatorname{Scal} &= \operatorname{Ric}(e_a, e_a) = g(R(e_b, e_a)e_a, e_b). \end{aligned}$$

According to [Biquard 2000] the Ricci tensor restricted to H is a symmetric tensor. If the trace-free part of the qc Ricci tensor is zero, we call the quaternionic structure *a qc Einstein manifold* [Ivanov et al. 2014a]. It is shown in that paper that the qc Ricci tensor is completely determined by the components of the torsion. Theorem 1.3, Theorem 3.12 and Corollary 3.14 in [Ivanov et al. 2014a] imply that on a qc manifold $(M^{4n+3}, g, \mathbb{Q})$ the qc Ricci tensor and the qc scalar curvature satisfy

$$\begin{aligned} \operatorname{Ric}(X, Y) &= (2n+2)T^0(X, Y) + (4n+10)U(X, Y) + \frac{\operatorname{Scal}}{4n} g(X, Y), \\ \operatorname{Scal} &= -8n(n+2)g(T(\xi_1, \xi_2), \xi_3). \end{aligned}$$

Hence, the qc Einstein condition is equivalent to the vanishing of the torsion endomorphism of the Biquard connection and in this case the qc scalar curvature is constant [Ivanov et al. 2014a; 2016]. If $\operatorname{Scal} > 0$, the latter holds exactly when the qc structure is locally 3-Sasakian up to a multiplication by a constant and an SO(3)-matrix with smooth entries. Recall that a $(4n+3)$-dimensional Riemannian manifold (M, g) is called 3-Sasakian if the cone metric $g_N = t^2 g + dt^2$ on $N = M \times \mathbb{R}^+$ is a hyper-Kähler metric; namely, it has holonomy contained in $\operatorname{Sp}(n+1)$. The 3-Sasakian manifolds are Einstein with positive Riemannian scalar curvature.

2D. ***qc conformal transformations.*** Let h be a positive smooth function on a qc manifold (M, η). If $\eta = 2h\bar\eta$, we will say that the vector-valued 1-form η is qc conformal to $\bar\eta$. We will denote the objects related to $\bar\eta$ by overlining the same object corresponding to η. Thus,

$$d\bar\eta = -\frac{1}{2h^2}\, dh \wedge \eta + \frac{1}{2h}\, d\eta \quad \text{and} \quad \bar g = \frac{1}{2h} g.$$

The new triple $\{\bar\xi_1, \bar\xi_2, \bar\xi_3\}$ is determined by the conditions defining the Reeb vector fields as $\bar\xi_s = 2h\xi_s + I_s \nabla h$, where ∇h is the horizontal gradient defined by $g(\nabla h, X) = dh(X)$. The components of the torsion tensor transform according to the following formulas from [Ivanov et al. 2014a, Section 5]:

$$\begin{aligned} \overline{T}^0(X, Y) &= T^0(X, Y) + h^{-1}[\nabla dh]_{[\mathrm{sym}][-1]}(X, Y), \\ \overline{U}(X, Y) &= U(X, Y) + (2h)^{-1}[\nabla dh - 2h^{-1} dh \otimes dh]_{[3][0]}(X, Y), \end{aligned} \tag{2-3}$$

where the symmetric part is given by

$$[\nabla dh]_{[\mathrm{sym}]}(X,Y)=\nabla dh(X,Y)+\sum_{s=1}^{3} dh(\xi_s)\,\omega_s(X,Y)$$

and ${}_{[3][0]}$ indicates the trace-free part of the [3]-component of the corresponding tensor.

2E. ***The qc Yamabe problem.*** Under a qc conformal transformation, as described above, the qc scalar curvature changes according to the formula given in [Biquard 2000],

$$\overline{\mathrm{Scal}}=2h(\mathrm{Scal})-8(n+2)^2h^{-1}|\nabla h|^2+8(n+2)\triangle h. \tag{2-4}$$

Let $Q=4n+6$ be the so-called homogeneous dimension of M and $2^*=2Q/(Q-2)$ the L^2-Sobolev conjugate exponent. It will be suitable to take the conformal factor in the form $\bar{\eta}=u^{4/(Q-2)}\eta$, which turns (2-4) into the qc Yamabe equation

$$\mathcal{L}u\equiv 4\frac{Q+2}{Q-2}\triangle u-u\,\mathrm{Scal}=-u^{2^*-1}\,\overline{\mathrm{Scal}}, \tag{2-5}$$

where $\triangle$ is the horizontal sub-Laplacian, $\triangle h=\mathrm{tr}^g(\nabla^2 h)$, Scal and $\overline{\mathrm{Scal}}$ are the qc scalar curvatures correspondingly of (M,η) and $(M,\bar{\eta})$. Thus, within a fixed qc conformal class, the Yamabe problem is the question of the solvability of the quaternionic contact (qc) Yamabe equation (2-5).

From a variational point of view, the qc Yamabe equation (2-5) is essentially the Euler–Lagrange equation of the extremals of the L^2 case of the Sobolev-type embedding inequality determined by (1-1). By standard subelliptic regularity results, any nontrivial nonnegative weak solution $u\in\mathcal{D}^{1,2}(M)$ of (2-5) is smooth and positive. Hence the result of this article can also be interpreted as the characterization of all nonnegative weak solutions of (2-5) on any closed compact locally 3-Sasakian manifold.

It should be mentioned that the original motivation of the qc Yamabe equation comes from its connection with the determination of the norm and extremals in the L^2 Folland–Stein [1974] Sobolev-type embedding on the quaternionic Heisenberg group $\boldsymbol{G}(\mathbb{H})$. This problem was considered in the general setting of groups of Heisenberg type [Garofalo and Vassilev 2001; Vassilev 2006; 2000], where, in particular, the equality case was characterized completely in the space of functions with partial symmetry on groups of Iwasawa type. Later on, Frank and Lieb [2012], and independently, Branson, Fontana and Morpurgo [Branson et al. 2013] developed a method based on a center of mass technique which yielded the characterization of equality cases of several inequalities, including the L^2 Sobolev and Folland–Stein inequalities in the Euclidean and CR Heisenberg group cases. These results were extended to the quaternionic and octonionic settings in [Ivanov et al. 2012; Christ et al. 2016a; 2016b]. The current paper showed that similarly to the Riemannian and CR model flat cases, in the model qc cases the only critical level of the qc Yamabe functional restricted to nonnegative functions is its minimum.

3. qc conformal transformations of a qc Einstein manifold

Throughout this section h is a positive smooth function on a fixed qc Einstein manifold $(M,\bar{\eta},\mathbb{Q})$ and $\eta=2h\bar{\eta}$ is a qc structure which is qc conformal to $\bar{\eta}$. We assume, in addition, that the qc structure η is of constant qc scalar curvature $\mathrm{Scal} = 16n(n+2)$ and hence equal to the qc scalar curvature of $\bar\eta$. We recall some formulas from [Ivanov et al. 2010] which will be used in the subsequent sections.

We begin by defining the vectors

$$A_i = I_i[\xi_j, \xi_k], \quad A = A_1 + A_2 + A_3. \tag{3-1}$$

We denote with the same letter the corresponding horizontal 1-forms, defined by $A_i(X) = g(A_i, X)$ etc. A short calculation, see [Ivanov et al. 2010, Lemma 3.1], gives the following expression of the 1-forms A_s and A in terms of h,

$$\begin{aligned} A_i(X) = &-\frac{h^{-2}}{2} dh(X) - \frac{h^{-3}}{2}|\nabla h|^2 dh(X) - \frac{h^{-1}}{2}(\nabla dh(I_j X, \xi_j) + \nabla dh(I_k X, \xi_k)) \\ &+ \frac{h^{-2}}{2}(dh(\xi_j)\, dh(I_j X) + dh(\xi_k)\, dh(I_k X)) \\ &+ \frac{h^{-2}}{4}(\nabla dh(I_j X, I_j \nabla h) + \nabla dh(I_k X, I_k \nabla h)). \end{aligned} \tag{3-2}$$

Thus, after summing, we have also

$$\begin{aligned} A(X) = &-\frac{3h^{-2}}{2} dh(X) - \frac{3h^{-3}}{2}|\nabla h|^2 dh(X) \\ &- h^{-1}\sum_{s=1}^{3} \nabla dh(I_s X, \xi_s) + h^{-2}\sum_{s=1}^{3} dh(\xi_s)\, dh(I_s X) + \frac{h^{-2}}{2}\sum_{s=1}^{3} \nabla dh(I_s X, I_s \nabla h). \end{aligned} \tag{3-3}$$

Second we consider the 1-forms

$$D_s(X) = -\frac{h^{-1}}{2}[T^0(X, \nabla h) + T^0(I_s X, I_s \nabla h)]. \tag{3-4}$$

For simplicity, using the musical isomorphism, we will denote by D_1, D_2, D_3 also the corresponding (horizontal) vector fields, defined by $g(D_i, X) = D_i(X)$. Let us consider in addition the form D defined as

$$D \overset{\text{def}}{=} D_1 + D_2 + D_3 = -h^{-1} T^0(X, \nabla h), \tag{3-5}$$

where the last equality follows from (2-2). Setting $\overline{T}^0 = 0$ in (2-3), we obtain from (3-4) the expressions (see [Ivanov et al. 2010; Ivanov and Vassilev 2010])

$$\begin{aligned} D_i(X) = &\, h^{-2}\, dh(\xi_i)\, dh(I_i X) \\ &+ \frac{h^{-2}}{4}[\nabla dh\,(X, \nabla h) + \nabla dh\,(I_i X, I_i \nabla h) - \nabla dh\,(I_j X, I_j \nabla h) - \nabla dh\,(I_k X, I_k \nabla h)]. \end{aligned} \tag{3-6}$$

The equalities (3-5) and (3-6) yield [Ivanov et al. 2010, Lemma 4.2]

$$D(X) = \frac{h^{-2}}{4}\Big(3\nabla dh(X, \nabla h) - \sum_{s=1}^{3} \nabla dh(I_s X, I_s \nabla h)\Big) + h^{-2}\sum_{s=1}^{3} dh(\xi_s)\, dh(I_s X). \tag{3-7}$$

Finally, we define the 1-forms (and corresponding vectors)

$$F_s(X) = -h^{-1} T^0(X, I_s \nabla h).$$

From the definition of F_i and (3-4) we find

$$F_i(X) = -h^{-1}T^0(X, I_i\nabla h) = -D_i(I_iX) + D_j(I_iX) + D_k(I_iX). \tag{3-8}$$

From [Ivanov et al. 2014a, Theorem 4.8] we have that on a $(4n+3)$-dimensional qc manifold with constant qc scalar curvature the following Bianchi identities hold:

$$\nabla^* T^0 = (n+2)A, \quad \nabla^* U = \frac{1-n}{2}A. \tag{3-9}$$

With the help of (3-9) the following divergence formulas were proved in [Ivanov et al. 2010, Lemmas 4.2 and 4.3]:

$$\nabla^* D = |T^0|^2 - h^{-1}g(dh, D) - h^{-1}(n+2)g(dh, A) \tag{3-10}$$

and

$$\begin{aligned}\nabla^*\Big(\sum_{s=1}^3 dh(\xi_s)F_s\Big) &= \sum_{s=1}^3 [\nabla\, dh(I_se_a, \xi_s)F_s(I_se_a)] \\ &\quad + h^{-1}\sum_{s=1}^3 [dh(\xi_s)dh(I_se_a)D(e_a) + (n+2)\, dh(\xi_s)\, dh(I_se_a)\, A(e_a)]. \end{aligned}\tag{3-11}$$

4. The divergence formula

This section contains our main technical result. As mentioned in the Introduction, we were motivated to seek a divergence formula of this type based on the Riemannian, CR and 7-dimensional qc cases of the considered problem. The main difficulty was to find a suitable vector field with nonnegative divergence containing the norm of the torsion. The fulfillment of this task was facilitated by the results of [Ivanov et al. 2014a]. In particular, similarly to the CR case, but unlike the Riemannian case, we were not able to achieve a proof based purely on the Bianchi identities; see [Ivanov et al. 2014a, Theorem 4.8]. Recall that the setting here is the same as in Section 3. Since

$$\mathrm{Scal} = \overline{\mathrm{Scal}} = 16n(n+2),$$

the Yamabe equation (2-4) gives

$$\triangle h = 2n - 4nh + h^{-1}(n+2)|\nabla h|^2. \tag{4-1}$$

Equation (2-3) in the case of a qc Einstein structure $\bar\eta$, $\overline{T}^0 = \overline{U} = 0$, and (4-1) motivate the definition of the symmetric (0, 2)-tensors

$$\boldsymbol{D}(X,Y) = -T^0(X,Y) = \frac{h^{-1}}{4}\Big[3\nabla^2 h(X,Y) - \sum_{s=1}^3 \nabla^2 h(I_sX, I_sY) + 4\sum_{s=1}^3 dh(\xi_s)\omega_s(X,Y)\Big], \tag{4-2}$$

$$\begin{aligned}\boldsymbol{E}(X,Y) &= -2U(X,Y) \\ &= \frac{h^{-1}}{4}\Big[\nabla^2 h(X,Y) + \sum_{s=1}^3 \nabla^2 h(I_sX, I_sY)\Big] - \frac{2h^{-2}}{4}\Big[dh(X)dh(Y) + \sum_{s=1}^3 dh(I_sX)dh(I_sY)\Big] \\ &\quad - \frac{h^{-1}}{4}(2-4h+h^{-1}|\nabla h|^2)g(X,Y).\end{aligned}\tag{4-3}$$

The 1-form D defined in (3-5) and expressed in terms of h in (3-7) satisfies $D(X) = h^{-1}\boldsymbol{D}(X, \nabla h)$. Define, in addition, the 1-form E by the equation

$$\begin{aligned} E(X) &= h^{-1}\boldsymbol{E}(X, \nabla h) = -2h^{-1}U(X, \nabla h) \\ &= \frac{h^{-2}}{4}\Big[\nabla^2 h(X, \nabla h) + \sum_{s=1}^{3} \nabla^2 h(I_s X, I_s \nabla h) + (-2 + 4h - 3h^{-1}|\nabla h|^2) dh(X)\Big], \end{aligned} \tag{4-4}$$

where the second and third equalities follow from (4-3).

Finally, in addition to the 1-forms D and E and the symmetric $(0, 2)$-tensors $\boldsymbol{D}$ and $\boldsymbol{E}$, we define the $(0, 3)$-tensors $\mathbb{D}$ and $\mathbb{E}$ as

$$\begin{aligned} \mathbb{D}(X, Y, Z) = -\frac{h^{-1}}{8}\Big[& dh(X)T^0(Y, Z) + dh(Y)T^0(X, Z) \\ &+ \sum_{s=1}^{3} dh(I_s X)T^0(I_s Y, Z) + \sum_{s=1}^{3} dh(I_s Y)T^0(I_s X, Z)\Big], \end{aligned} \tag{4-5}$$

$$\begin{aligned} \mathbb{E}(X, Y, Z) = \frac{h^{-1}}{8}\Big[& dh(X)\boldsymbol{E}(Y, Z) + dh(Y)\boldsymbol{E}(X, Z) \\ &+ \sum_{s=1}^{3} dh(I_s X)\boldsymbol{E}(I_s Y, Z) + \sum_{s=1}^{3} dh(I_s Y)\boldsymbol{E}(I_s X, Z)\Big]. \end{aligned} \tag{4-6}$$

After this preparation we are ready to state the main result.

Theorem 4.1. *Suppose (M^{4n+3}, η) is a quaternionic contact structure conformal to a 3-Sasakian structure $(M^{4n+3}, \bar\eta)$ with $\eta = 2h\bar\eta$. If $\mathrm{Scal}_\eta = \mathrm{Scal}_{\bar\eta} = 16n(n+2)$, then with f given by*

$$f = \frac{1}{2} + h + \frac{h^{-1}}{4}|\nabla h|^2, \tag{4-7}$$

the following identity holds:

$$\begin{aligned} \nabla^*\Big(f(D+E) + \sum_{s=1}^{3} dh(\xi_s) I_s E + \sum_{s=1}^{3} dh(\xi_s)\, F_s + 4\sum_{s=1}^{3} dh(\xi_s) I_s A_s - \frac{10}{3}\sum_{s=1}^{3} dh(\xi_s)\, I_s A\Big) \\ = \Big(\frac{1}{2} + h\Big)(|T^0|^2 + |\boldsymbol{E}|^2) + 2h|\mathbb{D} + \mathbb{E}|^2 + h\langle QV, V\rangle. \end{aligned} \tag{4-8}$$

Here, *the matrix Q is given by*

$$Q = \begin{bmatrix} \frac{5}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -2 & -2 & -2 \\ -\frac{1}{2} & \frac{5}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{10}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{5}{2} & -\frac{1}{2} & -\frac{2}{3} & \frac{10}{3} & -\frac{2}{3} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{5}{2} & -\frac{2}{3} & -\frac{2}{3} & \frac{10}{3} \\ -2 & \frac{10}{3} & -\frac{2}{3} & -\frac{2}{3} & \frac{22}{3} & -\frac{2}{3} & -\frac{2}{3}, \\ -2 & -\frac{2}{3} & \frac{10}{3} & -\frac{2}{3} & -\frac{2}{3} & \frac{22}{3} & -\frac{2}{3} \\ -2 & -\frac{2}{3} & -\frac{2}{3} & \frac{10}{3} & -\frac{2}{3} & -\frac{2}{3} & \frac{22}{3} \end{bmatrix}, \tag{4-9}$$

and $V = (E, D_1, D_2, D_3, A_1, A_2, A_3)$ with E, D_s, A_s defined, *correspondingly*, *in* (4-4), (3-4) *and* (3-1). *In particular*, *Q is a positive definite matrix with eigenvalues* 1, $\frac{9}{2} \pm \frac{\sqrt{73}}{2}$ *and* $\frac{11}{2} \pm \frac{\sqrt{89}}{2}$.

Proof. For the sake of making some formulas more compact, in the proof we will use sometimes the notation $XY = g(X, Y)$ for the product of two horizontal vector fields X and Y and the similar abbreviation for horizontal 1-forms.

We begin by recalling (3-7), (4-4) and (3-3), which imply

$$A(X) = \frac{3E(X) - D(X)}{2} - h^{-1}\sum_{s=1}^{3}\nabla^2 h(I_s X, \xi_s) + \frac{3h^{-2}}{2}\sum_{s=1}^{3} dh(\xi_s)dh(I_s X) - \frac{3h^{-2}}{2}\Big(\frac{1}{2} + h + \frac{1}{4}h^{-1}|\nabla h|^2\Big)dh(X). \tag{4-10}$$

Using the function f defined in (4-7), we write (4-10) in the form

$$2\sum_{s=1}^{3}\nabla^2 h(I_s X, \xi_s) = h(3E(X) - D(X) - 2A(X)) + 3h^{-1}\sum_{s=1}^{3} dh(\xi_s)dh(I_s X) - 3h^{-1} f dh(X). \tag{4-11}$$

The sum of (3-7) and (4-4) yields

$$(E + D)(X) = h^{-2}\nabla^2 h(X, \nabla h) + h^{-2}\sum_{s=1}^{3} dh(\xi_s)dh(I_s X) + \frac{h^{-2}}{4}(-2 + 4h - 3h^{-1}|\nabla h|^2)dh(X). \tag{4-12}$$

Using (4-7) and (4-12), we obtain

$$2\nabla_X f = h(E + D)(X) - h^{-1}\sum_{s=1}^{3} dh(\xi_s)dh(I_s X) + h^{-1} f dh(X). \tag{4-13}$$

We calculate the divergences of E using first (4-4) to obtain

$$\nabla^* E = 2h^{-2}dh(e_a)U(e_a, \nabla h) - 2h^{-1}(\nabla_{e_a} U)(e_a, \nabla h) - 2h^{-1}U(e_a, e_b)\nabla^2 h(e_a, e_b).$$

Taking into account the Bianchi identity (3-9), (4-3) and (4-4) it follows

$$\begin{aligned}\nabla^* E &= (n-1)h^{-1}A(\nabla h) + U(e_a, e_b)(-2h^{-1})[\nabla^2 h(e_a, e_b) - 2h^{-1}dh(e_a)dh(e_b)] + h^{-1}E(\nabla h)\\ &= |\boldsymbol{E}|^2 + h^{-1}E(\nabla h) + (n-1)h^{-1}A(\nabla h).\end{aligned} \tag{4-14}$$

Similarly, we have

$$\begin{aligned}-\nabla^* I_s E &= 2h^{-2}dh(e_a)U(I_s e_a, \nabla h) + 2h^{-1}(\nabla_{e_a} U)(e_a, I_s\nabla h) - 2h^{-1}U(I_s e_a, e_b)\nabla^2 h(e_a, e_b)\\ &= h^{-1}(1-n)A(I_s\nabla h) + U(I_s e_a, e_b)(-2h^{-1})[\nabla^2 h(e_a, e_b) - 2h^{-1}dh(e_a)dh(e_b)] + h^{-1}E(I_s\nabla h)\\ &= U(I_s e_a, e_b)U(e_a, e_b) - h^{-1}(1-n)dh(I_s e_a)A(e_a) = -h^{-1}(1-n)dh(I_s e_a)A(e_a),\end{aligned} \tag{4-15}$$

since $U(I_s e_a, e_b)U(e_a, e_b) = E(I_s\nabla h) = 0$ due to (2-2).

Now we are prepared to calculate the divergence of the first four terms. Using (3-10), (3-11), (4-14), (4-13), (4-15) and (4-11), we have

$$\begin{aligned}
&\nabla_{e_a}\Big[f(D+E)(e_a)-\sum_{s=1}^{3}dh(\xi_s)E(I_se_a)+\sum_{s=1}^{3}dh(\xi_s)F_s(e_a)\Big]\\
&\quad=\Big(\frac{h}{2}(E+D)(e_a)-\frac{h^{-1}}{2}\sum_{s=1}^{3}dh(\xi_s)dh(I_se_a)+\frac{h^{-1}}{2}f\,dh(e_a)\Big)(D+E)(e_a)\\
&\qquad+f\big[-h^{-1}D(\nabla h)-h^{-1}(n+2)A(\nabla h)+|T^0|^2+|E|^2+h^{-1}dh(e_a)E(e_a)-h^{-1}(1-n)dh(e_a)A(e_a)\big]\\
&\qquad\quad+h^{-1}(1-n)\sum_{s=1}^{3}dh(\xi_s)dh(I_se_a)A(e_a)+\sum_{s=1}^{3}\nabla^2h(I_se_a,\xi_s)E(e_a)+\sum_{s=1}^{3}\nabla^2h(I_se_a,\xi_s)F_s(I_se_a)\\
&\qquad\qquad+h^{-1}\sum_{s=1}^{3}dh(\xi_s)dh(I_se_a)D(e_a)+h^{-1}(n+2)\sum_{s=1}^{3}dh(\xi_s)dh(I_se_a)A(e_a)\\
&\quad=f(|T^0|^2+|\boldsymbol{E}|^2)+\frac{h}{2}|D+E|^2+\frac{h}{2}(3E-D-2A)(e_a)E(e_a)\\
&\qquad+h^{-1}\Big[\sum_{s=1}^{3}dh(\xi_s)dh(I_se_a)-f\,dh(e_a)\Big]\Big(\frac{1}{2}D(e_a)+3A(e_a)\Big)+\sum_{s=1}^{3}\nabla^2h(I_se_a,\xi_s)F_s(I_se_a).
\end{aligned}\tag{4-16}$$

At this point we will use that for any smooth function h on a qc manifold with constant qc scalar curvature the following formulas hold true [Ivanov et al. 2010, Lemma 4.1]:

$$\begin{aligned}
\nabla^*\Big(\sum_{s=1}^{3}dh(\xi_s)I_sA_s\Big)&=\sum_{s=1}^{3}\nabla dh(I_se_a,\xi_s)A_s(e_a),\\
\nabla^*\Big(\sum_{s=1}^{3}dh(\xi_s)I_sA\Big)&=\sum_{s=1}^{3}\nabla dh(I_se_a,\xi_s)A(e_a).
\end{aligned}\tag{4-17}$$

Applying (4-17) and (4-11) we obtain

$$\begin{aligned}
&\nabla_{e_a}\Big[f(D+E)(e_a)-\sum_{s=1}^{3}dh(\xi_s)E(I_se_a)+\sum_{s=1}^{3}dh(\xi_s)F_s(e_a)-2\sum_{s=1}^{3}dh(\xi_s)I_sA(e_a)\Big]\\
&\quad=f(|T^0|^2+|\boldsymbol{E}|^2)+\frac{h}{2}|D+E|^2+\frac{h}{2}(3E-D-2A)E-h(3E-D-2A)A\\
&\qquad\qquad+\frac{h^{-1}}{2}\Big[\sum_{s=1}^{3}dh(\xi_s)dh(I_se_a)-f\,dh(e_a)\Big]D(e_a)+\sum_{s=1}^{3}\nabla^2h\,(I_se_a,\xi_s)F_s(I_se_a).
\end{aligned}\tag{4-18}$$

According to (3-8), the last term in (4-18) reads

$$\begin{aligned}
\sum_{s=1}^{3}\nabla^2h\,(I_se_a,\xi_s)F_s(I_se_a)&=D_1(e_a)[\nabla^2h(I_1e_a,\xi_1)-\nabla^2h(I_2e_a,\xi_2)-\nabla^2h(I_3e_a,\xi_3)]\\
&\quad+D_2(e_a)[-\nabla^2h(I_1e_a,\xi_1)+\nabla^2h(I_2e_a,\xi_2)-\nabla^2h(I_3e_a,\xi_3)]\\
&\qquad+D_3(e_a)[-\nabla^2h(I_1e_a,\xi_1)-\nabla^2h(I_2e_a,\xi_2)+\nabla^2h(I_3e_a,\xi_3)].
\end{aligned}\tag{4-19}$$

Using (4-19) we rewrite the last line in (4-18) as

$$\Big[\frac{h^{-1}}{2}\sum_{s=1}^{3}dh(\xi_s)dh(I_se_a)-\frac{h^{-1}}{2}f\,dh(e_a)\Big]D(e_a)+\sum_{s=1}^{3}\nabla^2h\,(I_se_a,\xi_s)F_s(I_se_a)$$

$$\begin{aligned}
&=D_1(e_a)\Big[\nabla^2h(I_1e_a,\xi_1)-\nabla^2h(I_2e_a,\xi_2)-\nabla^2h(I_3e_a,\xi_3)+\frac{h^{-1}}{2}\sum_{s=1}^{3}dh(\xi_s)dh(I_se_a)-\frac{h^{-1}}{2}f\,dh(e_a)\Big]\\
&\quad+D_2(e_a)\Big[-\nabla^2h(I_1e_a,\xi_1)+\nabla^2h(I_2e_a,\xi_2)-\nabla^2h(I_3e_a,\xi_3)+\frac{h^{-1}}{2}\sum_{s=1}^{3}dh(\xi_s)dh(I_se_a)-\frac{h^{-1}}{2}f\,dh(e_a)\Big]\\
&\quad+D_3(e_a)\Big[-\nabla^2h(I_1e_a,\xi_1)-\nabla^2h(I_2e_a,\xi_2)+\nabla^2h(I_3e_a,\xi_3)\\
&\qquad\qquad+\frac{h^{-1}}{2}\sum_{s=1}^{3}dh(\xi_s)dh(I_se_a)-\frac{h^{-1}}{2}f\,dh(e_a)\Big].
\end{aligned}\tag{4-20}$$

The equalities (4-4), (3-6) and (3-2) imply

$$\nabla^2h(I_2X,\xi_2)+\nabla^2h(I_3X,\xi_3)=h(E-D_1-2A_1)(X)+h^{-1}\sum_{s=1}^{3}dh(\xi_s)dh(I_sX)-h^{-1}f\,dh(X).\tag{4-21}$$

Subtracting two times (4-21) from (4-11) we obtain

$$\begin{aligned}
&\nabla^2h(I_1e_a,\xi_1)-\nabla^2h(I_2e_a,\xi_2)-\nabla^2h(I_3e_a,\xi_3)+\frac{h^{-1}}{2}\sum_{s=1}^{3}dh(\xi_s)dh(I_se_a)-\frac{h^{-1}}{2}f\,dh(e_a)\\
&\qquad=\frac{h}{2}[-E-D+4D_1-2A+8A_1](e_a).
\end{aligned}\tag{4-22}$$

The left-hand side of the above identity is the second line in (4-20). The other two lines are evaluated similarly and the formulas are obtained from the above by a cyclic rotation of $\{1, 2, 3\}$. A substitution of the resulting new form of (4-20) in (4-18) gives

$$\begin{aligned}
&\nabla_{e_a}\Big[f(D+E)(e_a)-\sum_{s=1}^{3}dh(\xi_s)E(I_se_a)+\sum_{s=1}^{3}dh(\xi_s)F_s(e_a)-2\sum_{s=1}^{3}dh(\xi_s)I_sA(e_a)\Big]\\
&\quad=f(|T^0|^2+|\boldsymbol{E}|^2)+\frac{4h}{2}[E^2+A^2+D_1^2+D_2^2+D_3^2-2AE+2A_1D_1+2A_2D_2+2A_3D_3].
\end{aligned}\tag{4-23}$$

In view of (4-17) for any nonzero constant c we calculate the divergences as

$$\begin{aligned}
&\nabla_{e_a}\Big(c\sum_{s=1}^{3}dh(\xi_s)I_sA_s(e_a)-\frac{c}{3}\sum_{s=1}^{3}dh(\xi_s)I_sA(e_a)\Big)\\
&\qquad=\frac{c}{3}[2\nabla^2h(I_1e_a,\xi_1)-\nabla^2h(I_2e_a,\xi_2)-\nabla^2h(I_3e_a,\xi_3)]A_1(e_a)\\
&\qquad\quad+\frac{c}{3}[2\nabla^2h(I_2e_a,\xi_2)-\nabla^2h(I_1e_a,\xi_1)-\nabla^2h(I_3e_a,\xi_3)]A_2(e_a)\\
&\qquad\quad+\frac{c}{3}[2\nabla^2h(I_3e_a,\xi_3)-\nabla^2h(I_2e_a,\xi_2)-\nabla^2h(I_1e_a,\xi_1)]A_3(e_a).
\end{aligned}\tag{4-24}$$

Subtracting (4-21) from twice (4-11) yields

$$2\nabla^2 h(I_1 e_a, \xi_1) - \nabla^2 h(I_2 e_a, \xi_2) - \nabla^2 h(I_3 e_a, \xi_3) = h[2D_1 - D_2 - D_3 + 4A_1 - 2A_2 - 2A_3](e_a). \tag{4-25}$$

Now, taking into account (4-25), (4-24) and (4-23) we obtain

$$\begin{aligned}
\nabla^*\Big[f(D+E)(X) - \sum_{s=1}^3 dh(\xi_s)E(I_sX) + \sum_{s=1}^3 dh(\xi_s)F_s(X) - 2\sum_{s=1}^3 dh(\xi_s)I_sA(X)\Big] \\
+\nabla^*\Big[c\sum_{s=1}^3 dh(\xi_s)I_sA_s(X) - \frac{c}{3}\sum_{s=1}^3 dh(\xi_s)I_sA(X)\Big] \\
= f(|T^0|^2+|\boldsymbol{E}|^2) + \frac{4h}{2}[E^2+A^2+D_1^2+D_2^2+D_3^2-2AE+2A_1D_1+2A_2D_2+2A_3D_3] \\
+h\frac{c}{3}[(2D_1-D_2-D_3+4A_1-2A_2-2A_3)A_1] \\
+h\frac{c}{3}[(2D_2-D_1-D_3+4A_2-2A_1-2A_3)A_2] \\
+h\frac{c}{3}[(2D_3-D_2-D_1+4A_3-2A_2-2A_1)A_3].
\end{aligned} \tag{4-26}$$

In the next lemma we use again the notation $XY = g(X, Y)$ for the product of two horizontal vector fields X and Y and the similar abbreviation for horizontal 1-forms.

Lemma 4.2. *For the* $(0, 3)$*-tensors* $\mathbb{D}$ *and* $\mathbb{E}$ *defined by* (4-5) *and* (4-6) *we have*

$$\begin{aligned}
|\mathbb{D}|^2 &= \frac{h^{-2}}{8}|\nabla h|^2|T^0|^2 - \frac{1}{4}\sum_{s=1}^3 |D_s|^2 + \frac{1}{2}(D_1D_2 + D_1D_3 + D_2D_3), \\
|\mathbb{E}|^2 &= \frac{h^{-2}}{8}|\nabla h|^2|\boldsymbol{E}|^2 - \frac{1}{4}|E|^2, \quad \mathbb{D}\mathbb{E} = \frac{1}{4}\sum_{s=1}^3 ED_s.
\end{aligned} \tag{4-27}$$

Consequently,

$$\begin{aligned}
\frac{h^{-2}}{4}|\nabla h|^2(|T^0|^2 + |\boldsymbol{E}|^2) \\
= 2|\mathbb{D}+\mathbb{E}|^2 - \sum_{s=1}^3 ED_s + \frac{1}{2}|E|^2 + \frac{1}{2}\sum_{s=1}^3 |D_s|^2 - (D_1D_2 + D_1D_3 + D_2D_3).
\end{aligned} \tag{4-28}$$

Proof. We shall repeatedly apply (2-2) and the defining equations (4-5), (4-6), (3-1) and (3-5). We have

$$\begin{aligned}
|\mathbb{D}|^2 &= \frac{h^{-2}}{8}|\nabla h|^2|T^0|^2 + \frac{h^{-2}}{8^2}\Big(2T^0(\nabla h, e_c)T^0(\nabla h, e_c) - 4\sum_{s=1}^3 T^0(I_s\nabla h, e_c)T^0(I_s\nabla h, e_c) \\
&\qquad + 2\sum_{s,t=1}^3 T^0(I_sI_t\nabla h, e_c)T^0(I_tI_s\nabla h, e_c)\Big) \\
&= \frac{h^{-2}}{8}|\nabla h|^2|T^0|^2 + \frac{1}{4}\Big(-\sum_{s=1}^3 D_s^2 + 2(D_1D_2 + D_1D_3 + D_2D_3)\Big),
\end{aligned} \tag{4-29}$$

which is the first line of (4-27). For example, the third term in (4-29) is calculated as

$$
\begin{aligned}
2h^{-2}\sum_{s,t=1}^{3} & T^0(I_sI_t\nabla h, e_c)T^0(I_tI_s\nabla h, e_c) \\
&= 2h^{-2}\sum_{s=1}^{3}[T^0(\nabla h, e_c)T^0(\nabla h, e_c) - 2T^0(I_s\nabla h, e_c)T^0(I_s\nabla h, e_c)] \\
&= 6|D|^2 - 12\sum_{s=1}^{3}D_s^2 + 8(D_1D_2 + D_1D_3 + D_2D_3) = -6\sum_{s=1}^{3}D_s^2 + 20(D_1D_2 + D_1D_3 + D_2D_3),
\end{aligned}
$$

recalling the definition (3-5).

Similarly, we obtain the second line of (4-27). The equality (4-28) follows from (4-27), which completes the proof of Lemma 4.2. □

Finally, the proof of Theorem 4.1 follows by letting $c = 4$ in (4-26) and using (4-28) and (3-1). □

5. Proof of Theorems 1.3 and 1.1

5A. ***Proof of Theorem 1.3.*** The first step of the proof relies on Theorem 4.1. By a homothety we can suppose that both qc scalar curvatures are equal to $16n(n+2)$. Integrating the divergence formula of Theorem 4.1 and then using the divergence theorem established in [Ivanov et al. 2014a, Proposition 8.1] shows that the integral of the left-hand side is zero. Thus,

$$
\int_M \left(\tfrac{1}{2}+h\right)(|T^0|^2 + |\boldsymbol{E}|^2) + 2h|\mathbb{D}+\mathbb{E}|^2 + h\langle QV, V\rangle = 0,
$$

which, due to the fact that the matrix Q (4-9) is nonnegative and taking into account (4-3), shows that the quaternionic contact structure η has vanishing torsion, i.e., it is also qc Einstein according to [Ivanov et al. 2014a, Proposition 4.2]. This proves the first part of Theorem 1.3.

To prove the second part, we develop a sub-Riemannian extension of the result of [Obata 1971], see also [Bourguignon and Ezin 1987] and the review [Ivanov and Vassilev 2015, Theorem 2.6], on the relation between the Yamabe equation and the Lichnerowicz–Obata first eigenvalue estimate. We begin by recalling some results from [Ivanov et al. 2014a, Section 7.2]. A vector field Q on a qc manifold (M,η) is a *qc vector field* if its flow preserves the qc structure,

$$
\mathcal{L}_Q\eta = (\nu I + O)\cdot\eta,
$$

where ν is a smooth function and $O \in so(3)$ is a matrix-valued function with smooth entries; see [Ivanov et al. 2014a, Definition 7.7] and the discussion preceding it. In fact, taking into account [Ivanov et al. 2014a, Lemma 2.2; 2017, Lemma 5.1], a vector field Q on a qc manifold (M,η) is a qc vector field if its flow preserves the horizontal distribution $H = \ker\eta$. Since the exterior derivative d commutes with the Lie derivative $\mathcal{L}_Q$, any qc vector field Q satisfies

$$
\mathcal{L}_Q g = \nu g, \quad \mathcal{L}_Q I = O\cdot I, \quad I = (I_1, I_2, I_3)^t,
$$

which is equivalent to saying that the flow of Q preserves the conformal class $[g]$ of the horizontal metric and the quaternionic structure $\mathbb{Q}$ on H. The function ν can be easily expressed in terms of the divergence (with respect to g) of the horizontal part Q_H of the vector field Q. Indeed, from [Ivanov et al. 2014a, Lemma 7.12] we have

$$g(\nabla_X Q_H, Y) + g(\nabla_Y Q_H, X) + 2\eta_s(Q) g(T^0_{\xi_s} X, Y) = \nu\, g(X, Y);$$

hence

$$\nu = \frac{1}{2n} \nabla^* Q_H.$$

This gives a geometric interpretation for the quantity $(\nabla^* Q_H)$; namely, the flow of a qc vector field Q preserves a fixed metric $g \in [g]$ if and only if $\nabla^* Q_H = 0$.

As an infinitesimal version of the qc Yamabe equation, we obtain the following general fact concerning the divergence of a qc vector field.

Lemma 5.1. *Let (M, η) be a qc manifold. For any qc vector field Q on M we have*

$$\Delta(\nabla^* Q_H) = -\frac{n}{2(n+2)} Q(\mathrm{Scal}) - \frac{\mathrm{Scal}}{4(n+2)} \nabla^* Q_H,$$

where Scal, ∇^*, Δ *and the projection Q_H correspond to the contact form η.*

Proof. Suppose Q is a qc vector field and let ϕ_t be the corresponding (local) 1-parameter group of diffeomorphisms generated by its flow. Then

$$\phi_t^*(\eta) = \frac{1}{2h_t}\eta \quad \text{and} \quad \phi_t^*(g) = \frac{1}{2h_t} g$$

for some positive function h_t, depending smoothly on the parameter t. The qc scalar curvature Scal_t of the pull back contact form $\phi_t^*(\eta)$ is given by $\mathrm{Scal}_t = \mathrm{Scal} \circ \phi_t$. Then, formula (2-4) yields

$$\mathrm{Scal} \circ \phi_t = 2h_t(\mathrm{Scal}) - 8(n+2)^2 h_t^{-1} |\nabla h_t|^2 + 8(n+2)\triangle h_t. \tag{5-1}$$

Clearly, we have $h_0 = \frac{1}{2}$, and from

$$\frac{1}{2n}(\nabla^* Q_H)\, g = \mathcal{L}_Q\, g = \frac{d}{dt}\Big|_{t=0} \Big(\frac{1}{2h_t} g\Big) = -\frac{h_0'}{2h_0} g = -2h_0' g$$

we obtain that

$$h_0' = -\frac{1}{4n} \nabla^* Q_H,$$

where h_0' denotes the derivative of h_t at $t = 0$. A differentiation at $t = 0$ in (5-1) gives the lemma. □

Lemma 5.2. *Let (M, η) and $(M, \bar\eta)$ be qc Einstein manifolds with equal qc scalar curvatures $16n(n+2)$. If η and $\bar\eta$ are qc conformal to each other, $\bar\eta = \eta/(2h)$ for some smooth positive function h, then*

$$Q = \tfrac{1}{2}\nabla f + \sum_{s=1}^{3} dh(\xi_s)\xi_s \tag{5-2}$$

is a qc vector field on M, where the function f is defined in (4-7).

Proof. The assumption of the lemma implies that $E = D = D_s = A_s = 0$. Using (4-21), (4-22) and (4-13) we obtain $\nabla^2 h(I_s X, \xi_s) = -df(X)$; thus

$$\nabla^2 h(X, \xi_s) = df(I_s X). \tag{5-3}$$

It follows that

$$\sum_{s=1}^{3} \nabla_X (dh(\xi_s)\xi_s) = \sum_{s=1}^{3} df(I_s X)\xi_s.$$

As observed in the introduction to the section, it is enough to show that the flow of the vector field Q, defined by (5-2), preserves the horizontal distribution H. For any $X \in H$, we have

$$\begin{aligned}
\mathcal{L}_Q(X) &= \tfrac{1}{2}[\nabla f, X] + \sum_{s=1}^{3} [dh(\xi_s)\xi_s, X] \\
&= \tfrac{1}{2}\nabla_{\nabla f} X - \tfrac{1}{2}\nabla_X(\nabla f) - \sum_{s=1}^{3} \omega_s(\nabla f, X)\xi_s + \sum_{s=1}^{3} [dh(\xi_s)\nabla_{\xi_s} X - \nabla_X(dh(\xi_s)\xi_s) - dh(\xi_s)T_{\xi_s}(X)] \\
&= \tfrac{1}{2}\nabla_{\nabla f} X - \tfrac{1}{2}\nabla_X(\nabla f) + \sum_{s=1}^{3} dh(\xi_s)\nabla_{\xi_s} X \in H.
\end{aligned}$$

This completes the proof.

We note that, alternatively, using (5-3) a short calculation shows that Q satisfies the conditions of [Ivanov et al. 2014a, Corollary 7.9]. □

At this point we are ready to complete the proof of Theorem 1.3. Consider the qc vector field Q defined in Lemma 5.2. By Lemma 5.1, the function $\phi = \frac{1}{2}\triangle f$ is either an eigenfunction of the sub-Laplacian with eigenvalue $-4n$, $\triangle\phi = -4n\phi$, or it vanishes identically. In the first case, using the quaternionic contact version of the Lichnerowicz–Obata eigenfunction sphere theorem [Ivanov et al. 2013, Theorem 1.2; 2014b, Corollary 1.2] (see also [Baudoin and Kim 2014]), we conclude that (M, η) is the 3-Sasakian sphere. In the other case, we have that $\Delta f = 0$; hence

$$f = \frac{1}{2} + h + \frac{h^{-1}}{4}|\nabla h|^2 = \text{const}.$$

since M is compact. It follows that $h = \frac{1}{2}$ by considering the points where h achieves its minimum and maximum and taking into account the qc Yamabe equation (4-1). The proof of Theorem 1.3 is complete.

Remark 5.3. Lemma 5.2 provides also a certain geometric insight for the function f in (4-7). In fact, up to an additive constant, f is the unique function on M for which $Q_H = \frac{1}{2}\nabla f$ is the horizontal part of a qc vector field Q with vertical part $Q_V = dh(\xi_s)\xi_s$, $Q = Q_H + Q_V$. This assertion is an easy consequence of the computation given in the proof of Lemma 5.2. Moreover, it implies that on the 3-Sasakian sphere $\phi = \triangle f$ is an eigenfunction of the sub-Laplacian realizing the smallest possible eigenvalue $-4n$ on a compact locally 3-Sasakian manifold.

5B. ***Proof of Theorem 1.1.*** Theorem 1.1 is a direct corollary from Theorem 1.3. Alternatively, as in the proof of Theorem 1.3, we can use in the first step Theorem 4.1 which shows that the "new" structure is also qc Einstein. The second step of the proof of Theorem 1.1 follows then also by taking into account [Ivanov et al. 2014a, Theorem 1.2] where all locally 3-Sasakian structures of positive constant qc scalar curvature which are qc conformal to the standard 3-Sasakian structure on the sphere were classified (we note that this classification extends easily to the case when no sign condition of the new qc structure is assumed, see [Ivanov and Vassilev 2015]).

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GARLAND'S METHOD WITH BANACH COEFFICIENTS

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We prove a Banach version of Garland's method of proving vanishing of cohomology for groups acting on simplicial complexes. The novelty of this new version is that our new condition applies to every reflexive Banach space.

This new version of Garland's method allows us to deduce several criteria for vanishing of group cohomology with coefficients in several classes of Banach spaces (uniformly curved spaces, Hilbertian spaces and L^p spaces).

Using these new criteria, we improve recent results regarding Banach fixed-point theorems for random groups in the triangular model and give a sharp lower bound for the conformal dimension of the boundary of such groups. Also, we derive new criteria for group stability with respect to p-Schatten norms.

1. Introduction

Let X be a locally finite, pure n-dimensional simplicial complex and Γ be a locally compact, unimodular group acting cocompactly and properly on X. Under the assumption that X is an affine building, Garland [1973] gave a local criterion for the vanishing of the equivariant k-th cohomology for any unitary representation of Γ and any $1 \leq k \leq n-1$. His approach was later generalized by Ballmann and Świątkowski [1997] to all simplicial complexes and this generalization is sometimes referred to as "Garland's method". There have been several generalizations of this method that considered the case where π is an isometric representation on a Banach space; see [Nowak 2015; Koivisto 2014; Oppenheim 2014]. However, all these generalizations gave somewhat weak results when applied to examples. For example, when considering vanishing of cohomology over L^p spaces, the results of [Nowak 2015] could not show vanishing of cohomology, for every $1 < p < \infty$, for $\tilde{A}_2$ groups nor for random groups (see Theorems 5.1 and 6.2 of that paper).

We note that Garland's original work referred to affine buildings, and in this set-up strong results regarding vanishing of cohomologies with Banach coefficients are known; see [Lafforgue 2009; Liao 2014; Lécureux et al. 2020] for results regarding vanishing of the first cohomology, and see [Oppenheim 2017; Lubotzky and Oppenheim 2020] for results regarding vanishing of higher cohomologies. However, much less is known when one considers the less-structured setting of a group acting on a simplicial complex without assuming the extra structure of an affine building.

Recently, the results for vanishing of the first cohomology of random groups with coefficients in Banach spaces were improved: First, Druțu and Mackay [2019] proved vanishing of the first cohomology

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for random groups for L^p spaces. Second, considering random groups in the triangular model, de Laat and de la Salle [2021] gave a criterion for vanishing of the first cohomology for a group acting on a 2-dimensional simplicial complex that was applicable to all uniformly curved Banach spaces (and in particular, to all L^p spaces).

The observation of de Laat and de la Salle was that one can get much stronger results than in previous works if the assumption of the spectral gap in the links is replaced with the assumption of a two-sided spectral gap (or equivalently with the assumption of contraction of the random walk operator). Using this insight and the ideas of [Nowak 2015], we rework Garland's method under the assumption of two-sided spectral gaps in the links and derive very general vanishing criteria that are applicable to all uniformly curved Banach spaces (and, in part, to all reflexive Banach spaces). We give two applications for our result:

Fixed-point properties of random groups. Applying our vanishing result to random groups in the triangular model improves on the results of de Laat and de la Salle when considering fixed-point properties with respect to L^p spaces. As a result, we derive a *sharp* lower bound for the conformal dimension of the boundary of these groups that was not achieved in previous works. Namely, in previous works [Druțu and Mackay 2019; de Laat and de la Salle 2021] it was shown that with high probability, this conformal dimension is contained in an interval between $C\sqrt{\log m}$ and $C' \log(m)$ (where m is a parameter of the model — see exact formulation below). Our work shows that in fact the conformal dimension is in an interval of the form $C'' \log m$ and $C' \log(m)$ and thus our result is sharp. We note that as far as we understand, the proof methods in [Druțu and Mackay 2019; de Laat and de la Salle 2021] cannot be improved to yield such a sharp bound.

Group stability with respect to p-Schatten norms. By a result of [De Chiffre et al. 2020], vanishing of the second cohomology for Hilbertian spaces implies stability with respect to p-Schatten norms (see definitions below). Thus, our new criteria for vanishing of the second cohomology gives new criteria for group stability.

1A. ***New criteria for vanishing of cohomology with Banach coefficients.*** In order to state our results, we will need the following notation. For every simplex $\tau \in X(k)$ define X_τ to be the link of τ and $M_\tau, A_\tau : \ell^2(X_\tau(0)) \to \ell^2(X_\tau(0))$ to be the following operators: M_τ is the orthogonal projection on the subspace of constant functions in $\ell^2(X_\tau(0))$ and A_τ in the random walk operator on the 1-skeleton of X_τ. With this notation, we prove the following:

Theorem 1.1. *Let $\mathbb{E}$ be a reflexive Banach space, X a locally finite, pure n-dimensional simplicial complex and Γ a locally compact, unimodular group acting cocompactly and properly on X.*

For $1 \leq k \leq n-1$, if

$$\max_{\tau \in X(k-1)} \|(A_\tau(I - M_\tau) \otimes \mathrm{id}_{\mathbb{E}})\|_{B(\ell^2(X_\tau(0);\mathbb{E}))} < \frac{1}{k+1},$$

then $H^k(X, \pi) = 0$ for every continuous isometric representation π of Γ on $\mathbb{E}$.

Remark 1.2. A result of the same flavor was given in [de Laat and de la Salle 2021, Theorem B] for the vanishing of the first cohomology for groups acting on 2-dimensional simplicial complexes. We note that our theorem improves on their Theorem B even when considering only vanishing of the first cohomology: First, our theorem holds for any reflexive Banach space, while Theorem B is only applicable for super-reflexive spaces. Second, in terms of parameters, the contraction condition of $A_\tau(I - M_\tau) \otimes \mathrm{id}_{\mathbb{E}}$ does not depend on the Banach space, but only on k (as opposed to Theorem B). Last, our proof is simpler in the regard that it does not use the p-Laplacian or any uniform convexity arguments.

Theorem 1.1 is easily applicable in the setting of uniformly curved Banach spaces (see Definition 2.1) such as (commutative and noncommutative) L^p spaces and more generally strictly θ-Hilbertian spaces (see the exact definition in Section 2C). Namely, for a uniformly curved Banach space we can deduce vanishing of cohomology based the fact that links are spectral expanders. Before stating these types of results, we recall the relevant terminology: Let (V, E) be a connected finite graph and let A be the random walk operator on this graph. Recall that A is a self-adjoint operator and has the eigenvalue 1 with multiplicity 1. For a constant λ, the graph (V, E) is called a *one-sided λ-spectral expander* if the second largest eigenvalue of A is $\leq \lambda$. The graph (V, E) is called a *two-sided λ-spectral expander* if the spectrum of A is contained in the interval $[-\lambda, \lambda] \cup \{1\}$.

Theorem 1.3 (informal; see Proposition 4.5 and Theorem 4.11 for explicit formulations). *Let $\mathbb{E}$ be a uniformly curved Banach space. There are positive constants $\{\lambda_k(\mathbb{E}) > 0 : k \in \mathbb{N}\}$ such that for every locally finite, pure n-dimensional simplicial complex X, every locally compact, unimodular group Γ acting cocompactly and properly on X the following hold*:

(1) *For every $1 \leq k \leq n-1$, if there is $0 < \lambda < \lambda_k(\mathbb{E})$ such that for every $\tau \in X(k-1)$ the one skeleton of X_τ is a two-sided λ-spectral expander, then $H^k(X, \pi) = 0$ for every continuous isometric representation π of Γ on $\mathbb{E}$.*

(2) *For every $1 \leq k \leq n-1$, if there is $0 < \lambda < \lambda_k(\mathbb{E})$ such that for every $\tau \in X(n-2)$ the one skeleton of X_τ is a two-sided $\lambda/(1+(n-k-1)\lambda)$-spectral expander, then $H^k(X, \pi) = 0$ for every continuous isometric representation π of Γ on $\mathbb{E}$.*

(3) *For every $1 \leq k < n - 1/\lambda_k(\mathbb{E})$, if there is $0 < \lambda < \lambda_k(\mathbb{E})$ such that for every $\tau \in X(n-2)$, the one skeleton of X_τ is a one-sided $\lambda/(1+(n-k-1)\lambda)$-spectral expander, then $H^k(X, \pi) = 0$ for every continuous isometric representation π of Γ on $\mathbb{E}$.*

Specifying the above to θ-Hilbertian spaces reads as follows:

Corollary 1.4. *Let X be a locally finite, pure n-dimensional simplicial complex such that all the links of X of dimension ≥ 1 are connected and Γ be a locally compact, unimodular group acting cocompactly and properly on X. Also let $0 < \theta_0 \leq 1$, $1 \leq k \leq n-1$, $0 < \lambda < (1/(2(k+1)))^{1/\theta_0}$ be constants. Define $\mathcal{E}_{\theta_0}$ to be the smallest class of Banach spaces that contains all strictly θ-Hilbertian Banach spaces for all $\theta_0 \leq \theta \leq 1$ and is closed under subspaces, quotients, ℓ^2-sums and ultraproducts of Banach spaces.*

(1) *If for every $\tau \in X(k-1)$ the one skeleton of X_τ is a two-sided λ-spectral expander, then $H^k(X, \pi) = 0$ for every $\mathbb{E} \in \mathcal{E}_{\theta_0}$ and every continuous isometric representation π of Γ on $\mathbb{E}$.*

(2) *If for every $\tau \in \Sigma(n-2, \Gamma)$ the* 1*-skeleton of X_τ is a two-sided $\lambda/(1+(n-k-1)\lambda)$-spectral expander*, *then $H^k(X, \pi) = 0$ for every $\mathbb{E} \in \mathcal{E}_{\theta_0}$ and every continuous isometric representation π of Γ on $\mathbb{E}$.*

(3) *If $k \leq n - 1/\lambda$ and for every $\tau \in \Sigma(n-2, \Gamma)$ the* 1*-skeleton of X_τ is a* ***one-sided*** *$\lambda/(1+(n-k-1)\lambda)$-spectral expander*, *then $H^k(X, \pi) = 0$ for every $\mathbb{E} \in \mathcal{E}_{\theta_0}$ and every continuous isometric representation π of Γ on $\mathbb{E}$.*

Remark 1.5. In all the results above, we gave criteria for vanishing of the equivariant cohomology. We recall that given that the simplicial complex X is aspherical, it holds that $H^k(X, \pi) = H^k(\Gamma, \pi)$ (the proof of this can be found for instance in [Brown 1982, Chapter 7, Section 7]), and thus under this additional assumption it follows that the criteria given above imply that $H^k(\Gamma, \pi) = 0$.

1B. ***Application to random groups.*** An immediate application of our criteria above is improving de Laat and de la Salle's results regarding random groups in the triangular model. The triangular model for random groups, denoted by $\mathcal{M}(m, d)$, is defined as follows: For a fixed density $d \in (0, 1)$, a group in $\mathcal{M}(m, d)$ is a finitely presented group of the form $\Gamma = \langle S \mid R \rangle$, where $|S| = m$ ($S \cap S^{-1} = \varnothing$) and R is a set of cyclically reduced relators of length 3 chosen uniformly among all subsets of cardinality $\lfloor (2m-1)^{3d} \rfloor$. A property P for groups is said to hold with overwhelming probability in this model if

$$\lim_{m \to \infty} \mathbb{P}(\Gamma \text{ in } \mathcal{M}(m, d) \text{ has P}) = 1.$$

Below, we will also use the binomial triangular model that is closely related to the triangular model. The binomial triangular model, denoted by $\Gamma(m, \rho)$, is defined as follows: a group in $\Gamma(m, \rho)$ is a finitely presented group of the form $\Gamma = \langle S \mid R \rangle$, where $|S| = m$, and R is a set of cyclically reduced relators of length 3, where each relator is chosen independently with probability ρ. We mention this model, since it is easier to analyze and the results of this analysis can be transferred to the model $\mathcal{M}(m, d)$.

The triangular model for random groups was introduced by Żuk [2003] who showed that when $d > \frac{1}{3}$, property (T) holds for groups in $\mathcal{M}(m, d)$ with overwhelming probability. De Laat and de la Salle [2021] (following [Druțu and Mackay 2019]) generalized the result of Żuk to the setting of uniformly curved Banach spaces. In order to explain this generalization, we recall that by a classical result of Delorme and Guichardet, a finitely generated discrete group Γ has property (T) if and only if it has property (FH), i.e., if and only if every affine isometric action of Γ on a Hilbert space admits a fixed point. Property (FH) is readily generalized to the Banach setting as follows: For a Banach space $\mathbb{E}$, a group Γ is said to have property ($\mathrm{F}_{\mathbb{E}}$) if every continuous affine isometric action of Γ on $\mathbb{E}$ admits a fixed point. Also, a group Γ is said to have property (F_{L^p}) if it has property ($\mathrm{F}_{\mathbb{E}}$) for every L^p space $\mathbb{E}$. De Laat and de la Salle [2021] showed that if $d > \frac{1}{3}$, then for every uniformly curved Banach space $\mathbb{E}$, property ($\mathrm{F}_{\mathbb{E}}$) holds for groups in $\mathcal{M}(m, d)$ with overwhelming probability (their result is actually stronger — see Theorem 1.9 stated below).

Our results above are stated in the language of vanishing of the equivariant cohomology for groups acting on simplicial complexes. The connection between fixed-point properties and vanishing of cohomology readily follows from the following classical interpretation of group cohomology (see for instance the discussion in [Fernós et al. 2012, Section 2]):

Proposition 1.6. *Let Γ be a topological group and $\mathbb{E}$ be a Banach space. The group Γ has property* $(\mathrm{F}_{\mathbb{E}})$ *if and only if $H^1(\Gamma, \pi) = 0$ for every continuous isometric representation π of Γ on $\mathbb{E}$.*

The connection to our results regarding vanishing of cohomology described above is the following equivalence between fixed points and vanishing of the first cohomology. We recall that for a topological group Γ and a Banach space $\mathbb{E}$ the following are equivalent:

- $H^1(\Gamma, \pi) = 0$ for every continuous isometric representation π of Γ on $\mathbb{E}$.
- The group Γ has property $(\mathrm{F}_{\mathbb{E}})$.

Above, we discussed the vanishing of equivariant cohomology and not group cohomology, but as noted in Remark 1.5, this is equivalent in the case of groups acting on aspherical complexes. For a random group Γ in the model $\Gamma(m, \rho)$ (or in the model $\mathcal{M}(m, d)$), the Cayley complex of the group is a 2-dimensional simplicial complex that we will denote by X_Γ. We recall that the Cayley complex of a group is always simply connected and the action of a group on its Cayley complex is simply transitive on the vertices. In particular, since the group is finitely presented, the action is proper and cocompact. Thus, the vanishing of the first cohomology of Γ is equivalent to the vanishing of the first equivariant cohomology for the action of Γ on X_Γ. It follows that if we know that the links of X_Γ are two-sided spectral expanders, we can deduce property $(\mathrm{F}_{\mathbb{E}})$ for a uniformly curved Banach space $\mathbb{E}$ and a random group Γ in the model $\Gamma(m, \rho)$ by applying Theorem 1.3 stated above.

In [de Laat and de la Salle 2021], it was proven that the links of X_Γ for $\Gamma(m, \rho)$ are indeed two-sided spectral expanders:

Proposition 1.7 [de Laat and de la Salle 2021, Proposition 7.5]. *Let $\eta > 0$ be a constant. There is a constant $C > 0$ and a sequence $\{u_m\}_{m \in \mathbb{N}}$ tending to 0 such that the following holds: Let $m \in \mathbb{N}$ and $\rho \in (0, m^{-1.42})$. Also let Γ be a random group in the model $\Gamma(m, \rho)$ and X_Γ its Cayley complex. If*

$$\rho \geq \frac{(1+\eta)\log m}{8m^2},$$

then with probability $\geq 1 - u_m$, the link of every vertex of X_Γ is a $\sqrt{C/(\rho m^2)}$-two-sided spectral expander.

Combining this proposition with Theorem 1.3 above, we can reprove [de Laat and de la Salle 2021, Theorem 7.3]:

Theorem 1.8. *Let $\eta' > 0$ and $\rho \in (0, m^{-1.42})$ be constants and let C be the constant that appears in Proposition 1.7. Assume that*

$$\rho \geq \frac{(1+\eta')\log m}{8m^2},$$

and let Γ be a random group in the model $\Gamma(m, \rho)$. Then there is a sequence $\{u_m\}_{m \in \mathbb{N}}$ tending to 0 such that for uniformly curved Banach space $\mathbb{E}$ with $\lambda_1(\mathbb{E}) \geq \sqrt{C/(\rho m^2)}$ (where $\lambda_1(\mathbb{E})$ as in Theorem 1.3) it holds that Γ has property $(\mathrm{F}_{\mathbb{E}})$ *with probability $\geq 1 - u_m$.*

As in [de Laat and de la Salle 2021], using the fact that the fixed-point property passes to quotients, we can also recast this theorem in the triangular model (see further details in Section 7 of that paper) and reprove their Theorem C:

Theorem 1.9. *Let $0 < \eta < 2$, $d > \frac{1}{3} + (\log\log m - \log(2-\eta))/(3\log m)$ be constants. Also let Γ be a random group in the model $\mathcal{M}(m, d)$. Then there is a sequence $\{u_m\}_{m\in\mathbb{N}}$ tending to 0 and a constant $C > 0$ such that for uniformly curved Banach space $\mathbb{E}$ with*

$$\lambda_1(\mathbb{E}) \geq \sqrt{\frac{C}{(2m-1)^{3d-1}}}$$

(where $\lambda_1(\mathbb{E})$ as in Theorem 1.3) it holds that Γ has property $(\mathrm{F}_{\mathbb{E}})$ with probability $\geq 1 - u_m$.

Combining this theorem with Corollary 4.7 leads to a stronger result than the one stated in [de Laat and de la Salle 2021] (and in [Druțu and Mackay 2019]) when considering L^p spaces. Namely, applying Corollary 4.7 yields the following:

Theorem 1.10. *Let $0 < \eta < 2$, $d > \frac{1}{3} + (\log\log m - \log(2-\eta))/(3\log m)$ be constants. Also let Γ be a random group in the model $\mathcal{M}(m, d)$. Then there is a sequence $\{u_m\}_{m\in\mathbb{N}}$ tending to 0 and a constant $C > 0$ such that for*

$$2 \leq p \leq \tfrac{1}{2}(3d-1)\log(2m-1) - \tfrac{1}{2}\log C$$

it holds that Γ has property (F_{L^p}) with probability $\geq 1 - u_m$.

As a corollary, we improve the bound on the conformal dimension of random groups in the triangular model stated in [de Laat and de la Salle 2021, Corollary E]. Namely, by a theorem in [Bourdon 2016], if for a given $2 \leq p$, a hyperbolic group Γ has property (F_{L^p}), then the conformal dimension of $\partial_\infty \Gamma$ is at least p. Thus, we get:

Theorem 1.11. *Let $0 < \eta < 2$, $d > \frac{1}{3} + (\log\log m - \log(2-\eta))/(3\log m)$ be constants. Also let Γ be a random group in the model $\mathcal{M}(m, d)$. Then there is a sequence $\{u_m\}_{m\in\mathbb{N}}$ tending to 0 and a constant $C > 0$ such that*

$$\tfrac{1}{2}(3d-1)\log(2m-1) - \tfrac{1}{2}\log C \leq \mathrm{Confdim}(\partial_\infty \Gamma)$$

with probability $\geq 1 - u_m$.

In particular, for $d \in \left(\frac{1}{3}, \frac{1}{2}\right)$ and a group Γ in $\mathcal{M}(m, d)$ it holds with overwhelming probability that

$$\tfrac{1}{2}(3d-1)\log(2m-1) - \tfrac{1}{2}\log C \leq \mathrm{Confdim}(\partial_\infty \Gamma).$$

Remark 1.12. The theorem above gives a sharp bound on the conformal dimension of the boundary. Indeed, in [Druțu and Mackay 2019, Proposition 10.6] it was shown that for $d \in \left(\frac{1}{3}, \frac{1}{2}\right)$ and a group Γ in $\mathcal{M}(m, d)$ it holds with overwhelming probability that

$$\mathrm{Confdim}(\partial_\infty \Gamma) \leq \frac{30}{2d-1}\log(2m-1).$$

1C. *Application to group stability.* Group stability has received much attention in recent years (see for instance [Glebsky and Rivera 2008; Arzhantseva and Păunescu 2015; Becker et al. 2019; Becker and Lubotzky 2020]) partly due to its connection to questions of group approximation; see for instance [De Chiffre et al. 2020]. In that work it was shown that, under some assumptions, group stability can be deduced for a group via the vanishing of its second cohomology. Another application of our work is providing a criterion for p-norm stability (stability with respect to the p-Schatten norm). In order to state this application, we first give the needed definitions and results from [De Chiffre et al. 2020].

Let Γ be a finitely presented group $\Gamma = \langle S \mid R\rangle$, with $R \subseteq \mathbb{F}_S$ the free group on S and $|R| < \infty$. Any map $\phi : S \to U(n)$ uniquely determines a homomorphism $\phi : \mathbb{F}_S \to U(n)$, which we will also denote by ϕ.

Given a distance dist_n on $U(n)$, the group Γ is called $\mathcal{G} = (U(n), \mathrm{dist}_n)$-*stable* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $n \in \mathbb{N}$ if $\phi : S \to U(n)$ is a map with

$$\sum_{r\in R} \mathrm{dist}_n(\phi(r), \mathrm{id}_{U(n)}) < \delta,$$

then there exists a homomorphism $\tilde{\phi} : \Gamma \to U(n)$, with

$$\sum_{s\in S} \mathrm{dist}_n(\phi(s), \tilde{\phi}(s)) < \varepsilon,$$

or equivalently, a map $\tilde{\phi} : S \to U(n)$, with $\sum_{r\in R} \mathrm{dist}_n(\tilde{\phi}(r), \mathrm{id}_{U(n)}) = 0$,

For $1 \leq p < \infty$, the Schatten p-norm on $M_n(\mathbb{C})$ is defined by $\|T\|_p = (\mathrm{tr}\,|T|^p)^{1/p}$, where $|T| = \sqrt{T^*T}$. When $p = 2$, this is usually called the Frobenius norm. Define $\mathrm{dist}_{n,p}$ to be the metric on $U(n)$ induced by this norm. Below, we will call a group Γ *p-norm stable* if it is stable with respect to $\mathcal{G} = (U(n), \mathrm{dist}_{n,p})$.

We note that $(M_n(\mathbb{C}), \|\cdot\|_p)$ is a noncommutative L^p space and in particular, it is strictly θ-Hilbertian with $\theta = 2 - 2/p$ if $p \leq 2$ and $\theta = 2/p$ if $p \geq 2$. The discussion in [De Chiffre et al. 2020] implies the following criterion for p-norm stability (see also [García Morales and Glebsky 2022; Lubotzky and Oppenheim 2020]):

Theorem 1.13 [De Chiffre et al. 2020, Theorem 5.1, Remark 5.2]. *Let Γ be a finitely presented group and $0 < \theta_0 \leq 1$ be a constant. Define $\mathcal{E}_{\theta_0}$ to be the smallest class of Banach spaces that contains all strictly θ-Hilbertian Banach spaces for all $\theta_0 \leq \theta \leq 1$ and is closed under subspaces, ℓ^2-sums and ultraproducts of Banach spaces. If for every for every continuous isometric representation π of Γ on $\mathbb{E} \in \mathcal{E}_{\theta_0}$ it holds that $H^2(\Gamma, \pi) = 0$, then Γ is p-norm stable for every $1 + \theta_0/(2 - \theta_0) \leq p \leq 2/\theta_0$.*

Combining this theorem with Corollary 1.4 and Remark 1.5 immediately yields the following criterion for p-norm stability:

Theorem 1.14. *Let X be a locally finite, pure n-dimensional aspherical simplicial complex with $n \geq 3$ such that all the links of X of dimension ≥ 1 are connected and Γ be a finitely presented discrete group acting cocompactly and properly on X. Also let $0 < \theta_0 \leq 1,\ 0 < \lambda < \left(\frac{1}{6}\right)^{1/\theta_0}$ be constants. Assume that **one** of the following holds*:

(1) *For every $\tau \in X(1)$, the one skeleton of X_τ is a two-sided λ-spectral expander.*

(2) *For every $\tau \in X(n-2)$, the 1-skeleton of X_τ is a two-sided $\lambda/(1 + (n-3)\lambda)$-spectral expander.*

(3) *It holds that $2 \leq n - 1/\lambda$ and for every $\tau \in X(n-2)$, the 1-skeleton of X_τ is a **one-sided** $\lambda/(1+(n-3)\lambda)$-spectral expander.*

Then Γ is p-norm stable for every $1 + \theta_0/(2 - \theta_0) \leq p \leq 2/\theta_0$.

Currently, we do not have new examples in which this theorem improves previous results. One can take X to be an affine building of a large dimension n, Γ a lattice of the full BN-pair group of X and apply Theorem 1.14(3) to deduce p-norm stability (where p depends on the thickness of the building and on n). However, as noted above, in the case where X is a classical affine building, stronger results are given in [Lubotzky and Oppenheim 2020].

Organization. In Section 2, we cover some needed preliminaries. In Section 3, we give the basic definitions regarding equivariant cohomology and prove a variation of Nowak's criterion for vanishing of cohomology. In Section 4, we prove our local criteria for vanishing of Banach cohomology.

2. Preliminaries

2A. ***Vector-valued ℓ^2 spaces.*** Given a finite set V, a function $m : V \to \mathbb{R}_+$ and a Banach space $\mathbb{E}$, we define the *vector-valued space* $\ell^2(V, m; \mathbb{E})$ to be the space of functions $\phi : V \to \mathbb{E}$, with the norm

$$\|\phi\|_{\ell^2(V,m;\mathbb{E})} = \left(\sum_{v\in V} m(v)|\phi(v)|^2\right)^{1/2},$$

where $|\cdot|$ is the norm of $\mathbb{E}$. We define $\ell^2(V, m) = \ell^2(V, m; \mathbb{C})$ and recall that $\ell^2(V, m)$ is also a Hilbert space with the inner-product

$$\langle \phi, \psi\rangle = \sum_{v\in V} m(v)\phi(v)\overline{\psi(v)}.$$

Let $T : \ell^2(V, m) \to \ell^2(V, m)$ be a linear operator and $T_{v,u} \in \mathbb{C}$ be the constants such that for every $\phi \in \ell^2(V, m)$ it holds that

$$(T\phi)(v) = \sum_{u\in V} T_{v,u}\phi(u).$$

Define $T \otimes \mathrm{id}_{\mathbb{E}} : \ell^2(V, m; \mathbb{E}) \to \ell^2(V, m; \mathbb{E})$ by the formula

$$((T \otimes \mathrm{id}_{\mathbb{E}})\phi)(v) = \sum_{u\in V} T_{v,u}\phi(u),$$

where $T_{v,u} \in \mathbb{C}$ are the same constants as above and $\phi \in \ell^2(V, m; \mathbb{E})$. We define $\|T \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))}$ to be the operator norm of $T \otimes \mathrm{id}_{\mathbb{E}}$.

Following [Pisier 2010], we call an operator $T : \ell^2(V, m) \to \ell^2(V, m)$ *fully contractive* if for every Banach space $\mathbb{E}$ it holds that $\|T \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))} \leq 1$.

2B. ***Uniformly curved Banach spaces.*** Uniformly curved Banach spaces were introduced in [Pisier 2010]:

Definition 2.1. Let $\mathbb{E}$ be a Banach space. The space $\mathbb{E}$ is called uniformly curved if for every $0 < \varepsilon \leq 1$ there is $\delta > 0$ such that, for every space $\ell^2(V, m)$ and every fully contractive linear operator $T : \ell^2(V, m) \to \ell^2(V, m)$, if $\|T\|_{B(\ell^2(V,m))} \leq \delta$, then $\|T \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))} \leq \varepsilon$.

The following theorem is due to [Pisier 2010]:

Theorem 2.2. *Every uniformly curved Banach space is super-reflexive and in particular reflexive.*

Given a monotone increasing function $\alpha : (0, 1] \to (0, 1]$ such that

$$\lim_{t\to 0^+} \alpha(t) = 0,$$

we define $\mathcal{E}_\alpha^{\text{u-curved}}$ to be the class of all (uniformly curved) Banach spaces $\mathbb{E}$ such that for every space $\ell^2(V, m)$ and every fully contractive linear operator $T : \ell^2(V, m) \to \ell^2(V, m)$, if $\|T\|_{B(\ell^2(V,m))} \le \delta$, then $\|T \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))} \le \alpha(\delta)$.

Proposition 2.3. *Let $T : \ell^2(V, m) \to \ell^2(V, m)$ be a linear operator and $L \ge 1,\ 0 < \delta \le 1$ be constants such that*:

(1) $\|T\|_{B(\ell^2(V,m))} \le \delta$.

(2) $\|T \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))} \le L$ *for every Banach space* $\mathbb{E}$.

Then for every monotone increasing function $\alpha : (0, 1] \to (0, 1]$ such that $\lim_{t\to 0^+} \alpha(t) = 0$ and every $\mathbb{E} \in \mathcal{E}_\alpha^{\text{u-curved}}$ we have $\|T \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))} \le L\alpha(\delta)$.

Proof. We note that $(1/L)T$ is a fully contractive operator such that

$$\left\|\frac{1}{L}T\right\|_{B(\ell^2(V,m))} \le \frac{\delta}{L}.$$

Thus, by the definition of $\mathcal{E}_\alpha^{\text{u-curved}}$ it follows for every $\mathbb{E} \in \mathcal{E}_\alpha^{\text{u-curved}}$ that

$$\left\|\left(\frac{1}{L}\right)T \otimes \mathrm{id}_{\mathbb{E}}\right\|_{B(\ell^2(V,m;\mathbb{E}))} \le \alpha\left(\frac{\delta}{L}\right),$$

and thus

$$\|T \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))} \le L\alpha\left(\frac{\delta}{L}\right) \le L\alpha(\delta),$$

where the last inequality is due to the fact that $L \ge 1$ and α is monotone increasing. □

We will also be interested in how $T \otimes \mathrm{id}_{\mathbb{E}}$ behaves under some operations; this is summed up in the following lemmas:

Lemma 2.4. *Let V be a finite set, T a bounded operator on $\ell^2(V, m)$ and $C > 0$ constant. Let $\mathcal{E} = \mathcal{E}(C)$ be the class of Banach spaces defined as*

$$\mathcal{E} = \{\mathbb{E} : \|T \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))} \le C\}.$$

Then this class is closed under quotients, subspaces, ℓ^2-sums and ultraproducts of Banach spaces, i.e., preforming any of these operations on Banach spaces in $\mathcal{E}$ yields a Banach space in $\mathcal{E}$.

Proof. The fact that $\mathcal{E}$ is closed under quotients, subspaces and ultraproducts of Banach spaces was shown in [de la Salle 2016, Lemma 3.1]. The fact that $\mathcal{E}$ is closed under ℓ^2-sums is straightforward and left for the reader. □

Applying Lemma 2.4 on $\mathcal{E}_\alpha^{\text{u-curved}}$ defined above yields the following corollary:

Corollary 2.5. *For any monotone increasing function $\alpha : (0, 1] \to (0, 1]$ such that $\lim_{t\to 0^+} \alpha(t) = 0$, the class $\mathcal{E}_\alpha^{\text{u-curved}}$ defined above is closed under quotients, subspaces, ℓ^2-sums and ultraproducts of Banach spaces.*

2C. ***Strictly θ-Hilbertian spaces.*** Here we will describe a special class of uniformly curved Banach spaces that contains all (commutative and noncommutative) L^p spaces.

Two Banach spaces $\mathbb{E}_0, \mathbb{E}_1$ form a *compatible pair* $(\mathbb{E}_0, \mathbb{E}_1)$ if they are continuously linear embedded in the same topological vector space. The idea of complex interpolation is that given a compatible pair $(\mathbb{E}_0, \mathbb{E}_1)$ and a constant $0 \leq \theta \leq 1$, there is a method to produce a new Banach space $[\mathbb{E}_0, \mathbb{E}_1]_\theta$ as a "convex combination" of $\mathbb{E}_0$ and $\mathbb{E}_1$. We will not review this method here, and the interested reader can find more information on interpolation in [Bergh and Löfström 1976].

This brings us to consider the following definition due to [Pisier 1979]: a Banach space $\mathbb{E}$ is called *strictly θ-Hilbertian* for $0 < \theta \leq 1$ if there is a compatible pair $(\mathbb{E}_0, \mathbb{E}_1)$ with $\mathbb{E}_1$ a Hilbert space such that $\mathbb{E} = [\mathbb{E}_0, \mathbb{E}_1]_\theta$. Examples of strictly θ-Hilbertian spaces are L^p spaces and noncommutative L^p spaces (see [Pisier and Xu 2003] for definitions and properties of noncommutative L^p spaces), where in these cases $\theta = 2/p$ if $2 \leq p < \infty$ and $\theta = 2 - 2/p$ if $1 < p \leq 2$.

For our use, it will be important to bound the norm of an operator of the form $T \otimes \mathrm{id}_{\mathbb{E}}$ given that $\mathbb{E}$ is an interpolation space.

Lemma 2.6 [de la Salle 2016, Lemma 3.1]. *Let $(\mathbb{E}_0, \mathbb{E}_1)$ be a compatible pair, V be a finite set, $m : V \to \mathbb{R}_+$ be a function and $T \in B(\ell^2(V, m))$ be an operator. Then, for every $0 \leq \theta \leq 1$,*

$$\|T \otimes \mathrm{id}_{[\mathbb{E}_0,\mathbb{E}_1]_\theta}\|_{B(\ell^2(V,m;[\mathbb{E}_0,\mathbb{E}_1]_\theta))} \leq \|T \otimes \mathrm{id}_{\mathbb{E}_0}\|^{1-\theta}_{B(\ell^2(V,m;\mathbb{E}_0))} \|T \otimes \mathrm{id}_{\mathbb{E}_1}\|^{\theta}_{B(\ell^2(V,m;\mathbb{E}_1))},$$

where $[\mathbb{E}_0, \mathbb{E}_1]_\theta$ is the interpolation of $\mathbb{E}_0$ and $\mathbb{E}_1$.

This lemma has the following corollary that shows that strictly θ-Hilbertian spaces are uniformly curved (see also [de la Salle 2016, Lemma 3.1]):

Corollary 2.7. *Let $\mathbb{E}$ be a strictly θ-Hilbertian space with $0 < \theta \leq 1$, V be a finite set, $m : V \to \mathbb{R}_+$ be a function and $0 < \delta < 1$ be a constant. Assume that $T \in B(\ell^2(V, m))$ is a fully contractive operator such that $\|T\|_{B(\ell^2(V,m))} \leq \delta$. Then $\|T \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))} \leq \delta^\theta$.*

In other words, if $\mathbb{E}$ is a strictly θ-Hilbertian space with $0 < \theta \leq 1$, then for $\alpha(t) = t^\theta$ we have that $\mathbb{E} \in \mathcal{E}_\alpha^{\text{u-curved}}$.

Proof. For every Hilbert space $\mathbb{E}_1$ we have that $\|T \otimes \mathrm{id}_{\mathbb{E}_1}\|_{B(\ell^2(V,m;\mathbb{E}_1))} \leq \delta$ and thus the assertion stated above follows from Lemma 2.6. □

Corollary 2.8. *For a constant $0 < \theta_0 \leq 1$, define $\mathcal{E}_{\theta_0}$ to be the smallest class of Banach spaces that contains all strictly θ-Hilbertian Banach spaces for all $\theta_0 \leq \theta \leq 1$ and is closed under subspaces, quotients, ℓ^2-sums and ultraproducts of Banach spaces. Then, for every $0 < \theta_0 \leq 1$, we have that $\mathcal{E}_{\theta_0} \subseteq \mathcal{E}^{\text{u-curved}}_{\alpha(t)=t^{\theta_0}}$.*

Remark 2.9. A deep result of Pisier shows that the converse of the corollary above is "almost true" if one considers arcwise θ_0-Hilbertian spaces (see the definition in [Pisier 2010, Section 6]). Namely, by Corollary 6.7 of that work, for every $\theta_0 < \theta \leq 1$ it holds that every Banach space in $\mathcal{E}^{\text{u-curved}}_{\alpha(t)=t^\theta}$ is a subquotient of an arcwise θ_0-Hilbertian space. We will not define arcwise θ_0-Hilbertian spaces here and we will make no use of this fact.

2D. ***Random walks on finite graphs.*** Given a finite graph (V, E), a weight function on (V, E) is a function $m : E \to \mathbb{R}_+$ and (V, E) with a weight function is called a weighted graph. Given a weighted graph as above, we define, for every $v \in V$, $m(v) = \sum_{e \in E, v \in e} m(e)$ and $m(\emptyset) = \sum_{v \in V} m(v)$.

We also define $\ell^2(V, m)$ as in Section 2A above; i.e., $\ell^2(V, m)$ is the space of functions $\phi : V \to \mathbb{C}$ with an inner-product

$$\langle \phi, \psi \rangle = \sum_{\{v\} \in V} = m(v)\phi(v)\overline{\psi(v)}.$$

The *random walk* on (V, E) as above is the operator $A : \ell^2(V, m) \to \ell^2(V, m)$ defined as

$$(A\phi)(v) = \sum_{u \in V, \{u, v\} \in E} \frac{m(\{u, v\})}{m(v)} \phi(u).$$

We state without proof a few basic facts regarding the random walk operator:

(1) With the inner-product defined above, A is a self-adjoint operator and the eigenvalues of A lie in the interval $[-1, 1]$.

(2) The space of constant functions is an eigenspace of A with eigenvalue 1 and if (V, E) is connected, all other the other eigenfunctions of A have eigenvalues strictly less than 1.

(3) The graph (V, E) is bipartite if and only if -1 is an eigenvalue of A.

In the case where m is constant 1 on all the edges, for every vertex v, $m(v)$ is the valence of v and A is called the *simple random walk* on (V, E).

We define M to be the orthogonal projection on the space of constant functions: explicitly, for every $\phi \in \ell^2(V, m)$, $M\phi$ is the constant function

$$M\phi \equiv \frac{1}{m(\emptyset)} \sum_{v \in V} m(v)\phi(v).$$

We note that by the facts stated above, $AM = M$ and if (V, E) is connected and not bipartite, then $\|A(I - M)\| < 1$, where $\|\cdot\|$ denotes the operator norm. We recall the following definition of spectral expansion that appeared in the Introduction for nonweighted graphs:

Definition 2.10. Let (V, E) be a finite connected graph with a weight function m and $0 \leq \lambda < 1$ be a constant. The graph (V, E) is called a one-sided λ-spectral expander if the spectrum of $A(I - M)$ is contained in $[-1, \lambda]$. The graph (V, E) is called a two-sided λ-spectral expander if the spectrum of $A(I - M)$ is contained in $[-\lambda, \lambda]$ or equivalently if $\|A(I - M)\| \leq \lambda$.

Given a Banach space $\mathbb{E}$, we can consider the operator $(A(I - M)) \otimes \mathrm{id}_{\mathbb{E}} : \ell^2(V, m; \mathbb{E}) \to \ell^2(V, m; \mathbb{E})$.

Claim 2.11. *For every graph (V, E) and every Banach space $\mathbb{E}$, $\|(A(I - M)) \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))} \leq 2$.*

Proof. By the triangle inequality and linearity,

$$\|(A(I - M)) \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))} \leq \|A \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))} + \|A \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))} \|M \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))},$$

and therefore in order to prove the claim, it is enough to show that

$$\|A \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))} \le 1, \quad \|M \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))} \le 1.$$

Indeed, by the convexity of the function $|\cdot|^2$, for every $\phi \in \ell^2(V, m; \mathbb{E})$,

$$\begin{aligned}\|(A \otimes \mathrm{id}_{\mathbb{E}})\phi\|^2 &= \sum_{v\in V} m(v)\Bigg|\sum_{u\in V,\{u,v\}\in E} \frac{m(\{u,v\})}{m(v)}\phi(u)\Bigg|^2 \le \sum_{v\in V} m(v) \sum_{u\in V,\{u,v\}\in E} \frac{m(\{u,v\})}{m(v)}|\phi(u)|^2 \\ &= \sum_{u\in V}|\phi(u)|^2 \sum_{v\in V,\{u,v\}\in E} m(\{u,v\}) = \sum_{u\in V} m(u)|\phi(u)|^2 = \|\phi\|^2\end{aligned}$$

and

$$\begin{aligned}\|(M \otimes \mathrm{id}_{\mathbb{E}})\phi\|^2 &= \sum_{v\in V} m(v)\Bigg|\frac{1}{m(\varnothing)}\sum_{u\in V} m(u)\phi(u)\Bigg|^2 \le \sum_{v\in V}\sum_{u\in V}\frac{m(u)}{m(\varnothing)}|\phi(u)|^2 \\ &= \sum_{u\in V} m(u)|\phi(u)|^2 \sum_{v\in V}\frac{m(v)}{m(\varnothing)} = \sum_{u\in V} m(u)|\phi(u)|^2 = \|\phi\|^2. \qquad \square\end{aligned}$$

Combining this claim with Proposition 2.3 and Corollary 2.7 yields:

Corollary 2.12. *Let (V, E) be a connected finite graph with a weight function m and $0 < \lambda < 1$ be a constant such that (V, E) is a two-sided λ-spectral expander. For every monotone increasing function $\alpha : (0, 1] \to (0, 1]$ such that $\lim_{t\to 0^+} \alpha(t) = 0$ and every $\mathbb{E} \in \mathcal{E}_\alpha^{\text{u-curved}}$, we have*

$$\|(A(I - M)) \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))} \le 2\alpha(\lambda).$$

In particular, for every $0 < \theta \le 1$ and every strictly θ-Hilbertian space $\mathbb{E}$, we have

$$\|(A(I - M)) \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))} \le 2\lambda^\theta.$$

2E. *Weighted simplicial complexes.* Let X be an n-dimensional simplicial complex. For $-1 \le k \le n$, we define $X(k)$ to be the k-dimensional faces of X and $X = \bigcup_k X(k)$. X is called *pure n-dimensional* if for every τ in X there is $\sigma \in X(n)$ such that $\tau \subseteq \sigma$. X is called *locally finite* if for every $\{v\} \in X(0)$, $|\{\sigma \in X(n) : v \in \sigma\}| < \infty$. Throughout this paper, we will always assume that X is pure n-dimensional and locally finite.

We define the weight function $m : \bigcup_{k=0}^n \mathrm{X}(k) \to \mathbb{R}$ inductively as follows:

$$\text{for all } \sigma \in \mathrm{X}(n), \quad m(\sigma) = 1,$$

and, for $0 \le k \le n-1$ and $\tau \in X(k)$,

$$m(\tau) = \sum_{\sigma\in \mathrm{X}(k+1),\, \tau\subseteq\sigma} m(\sigma).$$

More explicitly,

$$\text{for all } \tau \in \mathrm{X}(k), \quad m(\tau) = (n-k)!\,|\{\sigma \in \mathrm{X}(n) : \tau \subseteq \sigma\}|.$$

In the case where X is finite, we also define $m(\varnothing) = \sum_{\{v\}\in X(0)} m(\{v\})$.

Given a simplex $\tau \in X(j)$, the *link of* τ is the subcomplex of X, denoted by X_τ, that is defined as

$$X_\tau = \{\eta \in X : \tau \cap \eta = \varnothing,\ \tau \cup \eta \in X\}.$$

We note that by the assumption that X is locally finite, it follows that X_τ is finite and by the assumption that X is pure n-dimensional, it follows that X_τ is pure $(n-j-1)$-dimensional (where j is the dimension of τ). The weight function on X_τ, denoted by m_τ, is defined as above:

$$\text{for all } \sigma \in \mathrm{X}_\tau(n-j-1), \quad m_\tau(\sigma) = 1,$$

and, for $0 \le k \le (n-j-1)-1$ and for $\eta \in X(k)$,

$$m_\tau(\eta) = \sum_{\sigma \in \mathrm{X}_\tau(k+1),\ \eta \subseteq \sigma} m_\tau(\sigma).$$

We observe that $m_\tau(\eta) = m(\tau \cup \eta)$: indeed, if $\eta \in \mathrm{X}_\tau(n-j-1)$, then $\tau \cup \eta \in X(n)$ and therefore

$$m_\tau(\eta) = 1 = m(\tau \cup \eta).$$

For $0 \le k \le (n-j-1)-1$ and $\eta \in X(k)$, the equality follows by induction:

$$m_\tau(\eta) = \sum_{\sigma \in \mathrm{X}_\tau(k+1),\ \eta \subseteq \sigma} m_\tau(\sigma) = \sum_{\sigma \in \mathrm{X}_\tau(k+1),\ \eta \subseteq \sigma} m(\tau \cup \sigma) = \sum_{\tau \cup \sigma \in \mathrm{X}((j+1)+k+1),\ \eta \subseteq \tau \cup \sigma} m(\tau \cup \sigma) = m(\eta).$$

2F. *Group representations on Banach spaces.* Let Γ be a locally compact group and $\mathbb{E}$ a Banach space. Let π be a representation $\pi : \Gamma \to B(\mathbb{E})$, where $B(\mathbb{E})$ are the bounded linear operators on $\mathbb{E}$. Throughout this paper we shall always assume π is continuous with respect to the strong operator topology without explicitly mentioning it. We recall that given π, the dual representation $\bar{\pi} : \Gamma \to B(\mathbb{E}^*)$ is defined as

$$\langle x, \bar{\pi}(g).y \rangle = \langle \pi(g^{-1}).x,\ y \rangle \quad \text{for all } g \in \Gamma,\ x \in \mathbb{E},\ y \in \mathbb{E}^*.$$

Observe that if π is an isometric representation, then $\bar{\pi}$ is an isometric representation: Indeed, for every $g \in \Gamma$,

$$\max_{x \in \mathbb{E},\ y \in \mathbb{E}^*,\ |x|=|y|=1} \langle x, \bar{\pi}(g).y \rangle = \max_{x \in \mathbb{E},\ y \in \mathbb{E}^*,\ |x|=|y|=1} \langle \pi(g^{-1}).x,\ y \rangle = |\pi(g^{-1}).x| = 1,$$

i.e., for every $g \in \Gamma$ and every $y \in \mathbb{E}^*$, if $|y| = 1$, then $|\bar{\pi}(g)y| = 1$ and it follows that $\bar{\pi}$ is isometric.

We remark that $\bar{\pi}$ might not be continuous for a general Banach space, but it is continuous for a large class of Banach spaces, called Asplund spaces:

Definition 2.13. A Banach space $\mathbb{E}$ is said to be an Asplund space if every separable subspace of $\mathbb{E}$ has a separable dual.

There are many examples of Asplund spaces and in particular every reflexive space is Asplund (see [Yost 1993] for an exposition on Asplund spaces). The reason we are interested in Asplund spaces is the following theorem of Megrelishvili:

Theorem 2.14 [Megrelishvili 1998, Corollary 6.9]. *Let Γ be a topological group and let π be a continuous representation of Γ on a Banach space $\mathbb{E}$. If $\mathbb{E}$ is an Asplund space, then the dual representation $\bar{\pi}$ is also continuous. In particular, if $\mathbb{E}$ is reflexive, then the dual representation $\bar{\pi}$ is continuous.*

3. Equivariant cohomology

Let X be a locally finite, pure n-dimensional simplicial complex with the weight function m defined above and Γ be a locally compact, unimodular group acting cocompactly and properly on X. Also let $\mathbb{E}$ be a reflexive Banach space and π be a continuous isometric representation.

Remark 3.1. By our assumption, $\mathbb{E}$ is reflexive and thus Asplund. Therefore, by Theorem 2.14, the assumption of continuity of π implies that $\bar{\pi}$ is also continuous.

Below, we will define the equivariant cohomology $H^k(X,\pi)$ and prove a general criterion for the vanishing of this cohomology. All the definitions below regarding cohomology already appeared in [Ballmann and Świątkowski 1997] for representations on Hilbert spaces and were generalized to the Banach setting in [Koivisto 2014]. The criterion for vanishing of cohomology appeared (in a somewhat different form) in [Nowak 2015] (and also in [Koivisto 2014]) and we claim no originality here.

In order to define the equivariant cohomology, we introduce the following notation (based on [Ballmann and Świątkowski 1997]):

(1) For $0 \leq k \leq n$, denote by $\Sigma(k)$ the set of ordered k-simplices (i.e., $\sigma \in \Sigma(k)$ is an ordered $(k+1)$-tuple of vertices that form a k-simplex in X).

(2) A map $\phi : \Sigma(k) \to \mathbb{E}$ is called *alternating* if, for every permutation $\gamma \in \mathrm{Sym}\{0,\dots,k\}$ and every $(v_0,\dots,v_k) \in \Sigma(k)$,

$$\phi((v_{\gamma(0)},\dots,v_{\gamma(k)})) = \mathrm{sgn}(\gamma)\phi((v_0,\dots,v_k)).$$

Also, ϕ is called *equivariant* if, for every $g \in \Gamma$ and every $\sigma \in \Sigma(k)$,

$$\pi(g)\phi(\sigma) = \phi(g.\sigma).$$

(3) For $0 \leq k \leq n$, a k-cochain twisted by π is a map $\phi : \Sigma(k) \to \mathbb{E}$ that is both alternating and equivariant. We define $C^k(X,\pi)$ to be the space of all k-cochains twisted by π.

For $0 \leq k < n$, the *differential* $d_k : C^k(X,\pi) \to C^{k+1}(X,\pi)$ is given by

$$d_k\phi(\sigma) := \sum_{i=0}^{k+1} (-1)^i \phi(\sigma_i), \quad \sigma \in \Sigma(k+1),$$

where $\sigma_i = (v_0,\dots,\hat{v}_i,\dots,v_{k+1})$ for $(v_0,\dots,v_{k+1}) = \sigma \in \Sigma(k+1)$. By a standard computation $d_k \circ d_{k-1} = 0$, and we define the k-th cohomology as $H^k(X,\pi) = \mathrm{Ker}(d_k)/\mathrm{Im}(d_{k-1})$.

Remark 3.2. The reader should note that in the definition of the cohomology above, we made no use of the fact that $\mathbb{E}$ is a Banach space, and this definition applies in a much more general setting.

We define a norm on $C^k(X, \pi)$ in order to make it into a Banach space:

(1) We choose a set, denoted by $\Sigma(k, \Gamma) \subseteq \Sigma(k)$, of representatives for the action of Γ on $\Sigma(k)$. We note that by the equivariance assumption, $\phi \in C^k(X, \pi)$ is determined by its values on $\Sigma(k, \Gamma)$. We also note that by the assumption that the action of Γ is cocompact, $\Sigma(k, \Gamma)$ is a finite set.

(2) We extend the weight function m defined above to ordered simplices by forgetting the ordering; i.e., for every $(v_0, \dots, v_k) \in \Sigma(k)$, we define $m((v_0, \dots, v_k)) = m(\{v_0, \dots, v_k\})$.

(3) For a simplex $\sigma \in \Sigma(k)$, we define Γ_σ to be the pointwise stabilizer of σ; i.e., for $\sigma = (v_0, \dots, v_k)$, $g \in \Gamma_\sigma$ if and only if for every $0 \leq i \leq k$ it holds that $g.v_i = v_i$. We further define $|\Gamma_\sigma|$ to be the measure of Γ_σ with respect to the Haar measure of Γ. By the assumption that the action of Γ is proper, it follows that $|\Gamma_\sigma| < \infty$.

(4) We define a norm on $C^k(X, \pi)$ by
$$\|\phi\| = \left(\sum_{\sigma \in \Sigma(k,\Gamma)} \frac{m(\sigma)}{(k+1)!\,|\Gamma_\sigma|} |\phi(\sigma)|^2 \right)^{1/2},$$
where $|\cdot|$ denotes the norm of $\mathbb{E}$.

With the definitions above, $C^k(X, \pi)$ is a normed space and we leave it to the reader to verify that it is a Banach space (this is almost immediate due to (1) above).

Proposition 3.3. *The space $C^k(X, \pi)$ is reflexive.*

Proof. Define $\mathbb{E}^{\Sigma(k,\Gamma)} = \{\phi : \Sigma(k, \Gamma) \to \mathbb{E}\}$ with the norm
$$\|\phi\| = \left(\sum_{\sigma \in \Sigma(k,\Gamma)} \frac{m(\sigma)}{(k+1)!\,|\Gamma_\sigma|} |\phi(\sigma)|^2 \right)^{1/2}.$$
This is a reflexive Banach space, since it is a weighted ℓ^2 sum of $|\Sigma(k, \Gamma)|$ copies of $\mathbb{E}$. We note that $C^k(X, \pi)$ is a closed subspace of $\mathbb{E}^{\Sigma(k,\Gamma)}$ and thus it is also reflexive. $\square$

Choose $\Sigma'(k, \Gamma) \subseteq \Sigma(k, \Gamma)$ to be a set of representatives of the action of the permutation group $\mathrm{Sym}\{0, \dots, k\}$ on $\Sigma(k, \Gamma)$; i.e., for every $(v_0, \dots, v_k) \in \Sigma(k, \Gamma)$ there is a unique permutation $\gamma \in \mathrm{Sym}\{0, \dots, k\}$ such that $(v_{\gamma(0)}, \dots, v_{\gamma(k)}) \in \Sigma'(k, \Gamma)$. By definition all the cochains in $C^k(X, \pi)$ are equivariant and alternating and thus every map in $C^k(X, \pi)$ is uniquely determined by its values on $\Sigma'(k, \Gamma)$. However, it may be the case that not every map $\phi' : \Sigma'(k, \Gamma) \to \mathbb{E}$ can be extended to an equivariant and alternating map on $\Sigma(k)$. Below, we will give a necessary and sufficient condition for the existence of such an extension.

For $\sigma \in \Sigma(k)$, we define Γ_σ^+ and Γ_σ^- to be the subsets of Γ that (when restricted to σ) induce even and odd permutations on σ; i.e., for $\sigma = (v_0, \dots, v_k)$,
$$\Gamma_\sigma^+ = \{g \in \Gamma : g.(v_0, \dots, v_k) = (v_{\gamma(0)}, \dots, v_{\gamma(k)}),\ \gamma \in \mathrm{Sym}\{0, \dots, k\} \text{ and } \gamma \text{ is an even permutation}\},$$
$$\Gamma_\sigma^- = \{g \in \Gamma : g.(v_0, \dots, v_k) = (v_{\gamma(0)}, \dots, v_{\gamma(k)}),\ \gamma \in \mathrm{Sym}\{0, \dots, k\} \text{ and } \gamma \text{ is an odd permutation}\}.$$

We note that Γ_σ^+ is a subgroup of Γ and that,

$$\text{for all } g \in \Gamma_\sigma^+, \quad g.\Gamma_\sigma^+ = \Gamma_\sigma^+, \quad g.\Gamma_\sigma^- = \Gamma_\sigma^-,$$
$$\text{for all } g \in \Gamma_\sigma^-, \quad g.\Gamma_\sigma^+ = \Gamma_\sigma^-, \quad g.\Gamma_\sigma^- = \Gamma_\sigma^+.$$

Define the subspace $\mathbb{E}_{\sigma,\pi} \subseteq \mathbb{E}$ to be the subspace of vectors $x \in \mathbb{E}$ such that,

$$\text{for all } g \in \Gamma_\sigma^+, \quad \pi(g).x = x,$$
$$\text{for all } g \in \Gamma_\sigma^-, \quad \pi(g).x = -x.$$

Proposition 3.4. *A map $\phi' : \Sigma'(k, \Gamma) \to \mathbb{E}$ can be extended (uniquely) to a map $\phi \in C^k(X, \pi)$ if and only if for every $\sigma \in \Sigma'(k, \Gamma)$ it holds that $\phi'(\sigma) \in \mathbb{E}_{\sigma,\pi}$.*

Proof. Let $\phi' : \Sigma'(k, \Gamma) \to \mathbb{E}$ be some map.

Assume first that there is a map $\phi \in C^k(X, \pi)$ such that $\phi|_{\Sigma'(k,\Gamma)} = \phi'$. Let $(v_0, \dots, v_k) \in \Sigma'(k, \Gamma)$ and $g \in \Gamma_\sigma^+$. Also let $\gamma \in \operatorname{Sym}\{0, \dots, k\}$ such that γ is even and $g.(v_0, \dots, v_k) = (v_{\gamma(0)}, \dots, v_{\gamma(k)})$. Then it holds that

$$\begin{aligned}
\pi(g).\phi'((v_0, \dots, v_k)) &= \pi(g).\phi((v_0, \dots, v_k)) \\
&= \phi(g.(v_0, \dots, v_k)) && \text{(since } \phi \text{ is equivariant)} \\
&= \phi((v_{\gamma(0)}, \dots, v_{\gamma(k)})) \\
&= \phi((v_0, \dots, v_k)) = \phi'((v_0, \dots, v_k)) && \text{(since } \phi \text{ is alternating);}
\end{aligned}$$

i.e., $\pi(g).\phi'(\sigma) = \phi'(\sigma)$ for every $\sigma \in \Sigma'(k, \Gamma)$ and every $g \in \Gamma_\sigma^+$. By a similar computation, it follows that $\pi(g).\phi'(\sigma) = -\phi'(\sigma)$ for every $\sigma \in \Sigma'(k, \Gamma)$ and every $g \in \Gamma_\sigma^-$. Thus, for every $\sigma \in \Sigma'(k, \Gamma)$ it holds that $\phi'(\sigma) \in \mathbb{E}_{\sigma,\pi}$.

In the other direction, assume that for every $\sigma \in \Sigma'(k, \Gamma)$ it holds that $\phi'(\sigma) \in \mathbb{E}_{\sigma,\pi}$. For every $\gamma \in \operatorname{Sym}\{0, \dots, k\}$, every $g \in \Gamma$ and every $(v_0, \dots, v_k) \in \Sigma'(k, \Gamma)$, define

$$\phi(g.(v_{\gamma(0)}, \dots, v_{\gamma(k)})) = \pi(g) \operatorname{sgn}(\gamma) \phi'((v_0, \dots, v_k)).$$

If we show that ϕ above is well-defined, it will follow from its definition that it is equivariant and alternating. Fix $\sigma = (v_0, \dots, v_k) \in \Sigma'(k, \Gamma)$ and let $\gamma, \gamma' \in \operatorname{Sym}\{0, \dots, k\}$, $g, g' \in \Gamma$ be such that

$$g.(v_{\gamma(0)}, \dots, v_{\gamma(k)}) = g'.(v_{\gamma'(0)}, \dots, v_{\gamma'(k)}).$$

Then

$$(v_{\gamma(\gamma')^{-1}(0)}, \dots, v_{\gamma(\gamma')^{-1}(k)}) = (g^{-1}g').(v_0, \dots, v_k)$$

and therefore $g^{-1}g' \in \Gamma_\sigma^+ \cup \Gamma_\sigma^-$ and the sign of the permutation induced by $g^{-1}g'$ on σ is exactly $\operatorname{sgn}(\gamma(\gamma')^{-1}) = \operatorname{sgn}(\gamma) \operatorname{sgn}(\gamma')$. From the assumption that $\phi'((v_0, \dots, v_k)) \in \mathbb{E}_{\sigma,\pi}$ it follows that

$$\pi(g^{-1}g')\phi'((v_0, \dots, v_k)) = \operatorname{sgn}(\gamma) \operatorname{sgn}(\gamma')\phi'((v_0, \dots, v_k)),$$

or equivalently

$$\operatorname{sgn}(\gamma) \operatorname{sgn}(\gamma')\pi(g^{-1}g')\phi'((v_0, \dots, v_k)) = \phi'((v_0, \dots, v_k)).$$

Thus

$$\begin{aligned}\pi(g)\,\mathrm{sgn}(\gamma)\phi'((v_0,\dots,v_k)) &= \pi(g)\,\mathrm{sgn}(\gamma)\,\mathrm{sgn}(\gamma)\,\mathrm{sgn}(\gamma')\pi(g^{-1}g')\phi'((v_0,\dots,v_k))\\ &= \pi(g')\,\mathrm{sgn}(\gamma')\phi'((v_0,\dots,v_k)),\end{aligned}$$

and ϕ is well-defined. □

All the results above were stated for π, but since $\bar{\pi}$ is a representation of Γ on a reflexive Banach space, they pass automatically to $\bar{\pi}$; i.e., we can define $C^k(X,\bar{\pi})$ as above and by the same considerations it follows that $C^k(X,\bar{\pi})$ is also a reflexive Banach space. We also define $\bar{d}_k : C^k(X,\bar{\pi}) \to C^{k+1}(X,\bar{\pi})$ to be the differential defined as above.

The reason for considering $C^k(X,\bar{\pi})$ is that there is a natural coupling between $C^k(X,\pi)$ and $C^k(X,\bar{\pi})$: let $(\,\cdot\,,\cdot\,)$ denote the usual coupling between $\mathbb{E}$ and $\mathbb{E}^*$ and for $\phi \in C^k(X,\pi),\ \psi \in C^k(X,\bar{\pi})$ define

$$\langle \phi,\psi\rangle := \sum_{\sigma\in\Sigma(k,\Gamma)} \frac{m(\sigma)}{(k+1)!\,|\Gamma_\sigma|}(\phi(\sigma),\psi(\sigma)).$$

With the above coupling, $C^k(X,\bar{\pi}) \subseteq (C^k(X,\pi))^*$. Actually, since $\mathbb{E}$ is reflexive, there is an isomorphism between $C^k(X,\bar{\pi})$ and $(C^k(X,\pi))^*$ (see [Koivisto 2014, Proposition 28]), but we will make no use of this fact. Given this coupling, we define $d_k^* : C^{k+1}(X,\bar{\pi}) \to C^k(X,\bar{\pi})$ to be the adjoint operator of d_k and $\bar{d}_k^* : C^{k+1}(X,\pi) \to C^k(X,\pi)$ to be the adjoint operator of $\bar{d}_k$.

We recall that for a Banach space $\mathbb{E}$, the duality mapping is a mapping $j : \mathbb{E} \to 2^{\mathbb{E}^*}$ defined as

$$j(x) = \{x^* \in \mathbb{E}^* : |x| = |x^*|,\ (x,x^*) = |x|^2\}$$

(the fact that the set defined by $j(x)$ is nonempty follows immediately from the Hahn–Banach theorem). By our assumption, $\mathbb{E}$ is reflexive and thus we also have the duality mapping $\bar{j} : \mathbb{E}^* \to 2^{\mathbb{E}} (= 2^{\mathbb{E}^{**}})$.

We define maps $J : C^k(X,\pi) \to 2^{C^k(X,\bar{\pi})}$ and $\bar{J} : C^k(X,\bar{\pi}) \to 2^{C^k(X,\pi)}$ by,

$$\begin{aligned}&\text{for all } \phi \in C^k(X,\pi), \quad J\phi = \{\psi \in C^k(X,\bar{\pi}) : \text{for all } \sigma \in \Sigma(k),\ \psi(\sigma) \in j(\phi(\sigma))\},\\ &\text{for all } \psi \in C^k(X,\bar{\pi}), \quad \bar{J}\psi = \{\phi \in C^k(X,\pi) : \text{for all } \sigma \in \Sigma(k),\ \phi(\sigma) \in \bar{j}(\psi(\sigma))\}.\end{aligned}$$

Proposition 3.5. *Let $X, \mathbb{E}, \pi, J, \bar{J}$ be as above and $\phi \in C^k(X,\pi)$, $\psi \in C^k(X,\bar{\pi})$. Then $J\phi$, $\bar{J}\psi$ are nonempty sets and,*

$$\begin{aligned}&\textit{for all } \phi^* \in J\phi, \quad \|\phi^*\|^2 = \|\phi\|^2 = \langle\phi,\phi^*\rangle,\\ &\textit{for all } \psi^* \in \bar{J}\psi, \quad \|\psi^*\|^2 = \|\psi\|^2 = \langle\psi^*,\psi\rangle.\end{aligned}$$

Proof. We will prove the assertions above only for $J\phi$, since the proof for $\bar{J}\psi$ is similar.

We will only show that $J\phi$ is nonempty: the fact that, for every $\phi^* \in J\phi$,

$$\|\phi^*\|^2 = \|\phi\|^2 = \langle\phi,\phi^*\rangle$$

follows from straightforward a computation that is left for to the reader.

Fix $\phi \in C^k(X,\pi)$. Choose $\Sigma'(k,\Gamma) \subseteq \Sigma(k,\Gamma)$ as above to be a set of representatives of the action of the permutation group $\mathrm{Sym}\{0,\dots,k\}$ on $\Sigma(k,\Gamma)$. By Proposition 3.4, it is enough to show that there is $\psi' : \Sigma'(k,\Gamma) \to \mathbb{E}^*$ such that for every $\sigma \in \Sigma'(k,\Gamma)$ it holds that $\psi'(\sigma) \in \mathbb{E}^*_{\sigma,\bar{\pi}}$ and $\psi'(\sigma) \in j(\phi(\sigma))$.

For every $\sigma \in \Sigma'(k, \Gamma)$, define $\varepsilon_\sigma : \Gamma_\sigma^+ \cup \Gamma_\sigma^- \to \{\pm 1\}$ as

$$\varepsilon_\sigma(g) = \begin{cases} 1, & g \in \Gamma_\sigma^+, \\ -1, & g \in \Gamma_\sigma^-. \end{cases}$$

Note that for every $g \in \Gamma_\sigma^+ \cup \Gamma_\sigma^-$, it follows that $\varepsilon_\sigma(g) = \varepsilon_\sigma(g^{-1})$ and that $\pi(g).\phi(\sigma) = \varepsilon_\sigma(g)\phi(\sigma)$.

Also, for every $\sigma \in \Sigma'(k, \Gamma)$, we choose some $x_\sigma^* \in j(\phi(\sigma))$ and define

$$\psi'(\sigma) = \frac{1}{|\Gamma_\sigma^+ \cup \Gamma_\sigma^-|} \int_{\Gamma_\sigma^+ \cup \Gamma_\sigma^-} \varepsilon_\sigma(g)\bar{\pi}(g).x_\sigma^* \, d\mu(g).$$

This integral is well-defined because by our assumptions the action of $\bar{\pi}$ is continuous and $\Gamma_\sigma^+, \Gamma_\sigma^-$ are compact sets.

Recall that for every $g' \in \Gamma_\sigma^+$ it holds that $g'.\Gamma_\sigma^+ = \Gamma_\sigma^+,\ g'.\Gamma_\sigma^- = \Gamma_\sigma^-$ and therefore for every $g'' \in \Gamma_\sigma^+ \cup \Gamma_\sigma^-$ it holds that $\varepsilon_\sigma((g')^{-1}g'') = \varepsilon_\sigma(g'')$. Also recall that the action of Γ preserves the Haar measure. Thus for every $g' \in \Gamma_\sigma^+$ and every $\sigma \in \Sigma'(k, \Gamma)$ it holds that

$$\begin{aligned} \bar{\pi}(g').\psi'(\sigma) &= \frac{1}{|\Gamma_\sigma^+ \cup \Gamma_\sigma^-|} \int_{\Gamma_\sigma^+ \cup \Gamma_\sigma^-} \varepsilon_\sigma(g)\bar{\pi}(g'g).x_\sigma^* \, d\mu(g) \\ &= \frac{1}{|\Gamma_\sigma^+ \cup \Gamma_\sigma^-|} \int_{g'.\Gamma_\sigma^+ \cup g'.\Gamma_\sigma^-} \varepsilon_\sigma((g')^{-1}g'')\bar{\pi}(g'').x_\sigma^* \, d\mu(g'') \quad (\text{since } g'' = g'g) \\ &= \frac{1}{|\Gamma_\sigma^+ \cup \Gamma_\sigma^-|} \int_{\Gamma_\sigma^+ \cup \Gamma_\sigma^-} \varepsilon_\sigma(g'')\bar{\pi}(g'').x_\sigma^* \, d\mu(g'') = \psi'(\sigma). \end{aligned}$$

Note that for every $g' \in \Gamma_\sigma^-$ it holds that $g'.\Gamma_\sigma^+ = \Gamma_\sigma^-$, $g'.\Gamma_\sigma^+ = \Gamma_\sigma^+$ and that for every $g'' \in \Gamma_\sigma^+ \cup \Gamma_\sigma^-$ it holds that $\varepsilon_\sigma((g')^{-1}g'') = -\varepsilon_\sigma(g'')$. Thus, by a computation similar to the one above, it follows that, for every $g' \in \Gamma_\sigma^-$ and every $\sigma \in \Sigma'(k, \Gamma)$,

$$\bar{\pi}(g').\psi'(\sigma) = -\psi'(\sigma)$$

and therefore $\psi'(\sigma) \in \mathbb{E}_{\sigma,\bar{\pi}}^*$.

We note that for every $\sigma \in \Sigma'(k, \Gamma)$ it holds that

$$\begin{aligned} &(\phi(\sigma), \psi'(\sigma)) \\ &\quad = \left(\phi(\sigma), \frac{1}{|\Gamma_\sigma^+ \cup \Gamma_\sigma^-|} \int_{\Gamma_\sigma^+ \cup \Gamma_\sigma^-} \varepsilon_\sigma(g)\bar{\pi}(g).x_\sigma^* \, d\mu(g)\right) \\ &\quad = \frac{1}{|\Gamma_\sigma^+ \cup \Gamma_\sigma^-|} \int_{\Gamma_\sigma^+ \cup \Gamma_\sigma^-} \varepsilon_\sigma(g)(\phi(\sigma), \bar{\pi}(g).x_\sigma^*) \, d\mu(g) \\ &\quad = \frac{1}{|\Gamma_\sigma^+ \cup \Gamma_\sigma^-|} \int_{\Gamma_\sigma^+ \cup \Gamma_\sigma^-} \varepsilon_\sigma(g)(\pi(g^{-1}).\phi(\sigma), x_\sigma^*) \, d\mu(g) \\ &\quad = \frac{1}{|\Gamma_\sigma^+ \cup \Gamma_\sigma^-|} \int_{\Gamma_\sigma^+ \cup \Gamma_\sigma^-} (\varepsilon_\sigma(g))^2(\phi(\sigma), x_\sigma^*) \, d\mu(g) \quad (\text{since } \pi(g^{-1}).\phi(\sigma) = \varepsilon_\sigma(g^{-1})\phi(\sigma) = \varepsilon_\sigma(g)\phi(\sigma)) \\ &\quad = \frac{1}{|\Gamma_\sigma^+ \cup \Gamma_\sigma^-|} \int_{\Gamma_\sigma^+ \cup \Gamma_\sigma^-} |\phi(\sigma)|^2 \, d\mu(g) = |\phi(\sigma)|^2, \end{aligned}$$

and that

$$
\begin{aligned}
|\psi'(\sigma)| &= \left| \frac{1}{|\Gamma_\sigma^+ \cup \Gamma_\sigma^-|} \int_{\Gamma_\sigma^+ \cup \Gamma_\sigma^-} \varepsilon_\sigma(g)\bar{\pi}(g).x_\sigma^* \, d\mu(g) \right| \\
&\le \frac{1}{|\Gamma_\sigma^+ \cup \Gamma_\sigma^-|} \int_{\Gamma_\sigma^+ \cup \Gamma_\sigma^-} |\bar{\pi}(g).x_\sigma^*| \, d\mu(g) = \frac{1}{|\Gamma_\sigma^+ \cup \Gamma_\sigma^-|} \int_{\Gamma_\sigma^+ \cup \Gamma_\sigma^-} |\phi(\sigma)| \, d\mu(g) = |\phi(\sigma)|,
\end{aligned}
$$

and therefore $\psi'(\sigma) \in j(\phi(\sigma))$ as needed. □

Below, we will make use of changing the order of summation when calculating norms of maps $C^k(X, \pi)$ or coupling between maps of $C^k(X, \pi)$ and $C^k(X, \bar{\pi})$. For this, we will need the following: for $0 \le l < k \le n$ and $\tau \in \Sigma(l)$, $\sigma \in \Sigma(k)$, we write $\tau \subseteq \sigma$ if σ contains τ as a set (without respecting the ordering); i.e., for $\sigma = (v_0, \dots, v_k)$, $\tau = (w_0, \dots, w_l)$, we have $\tau \subseteq \sigma$ if $\{w_0, \dots, w_l\} \subseteq \{v_0, \dots, v_k\}$.

Proposition 3.6 [Ballmann and Świątkowski 1997, Lemma 1.3; Dymara and Januszkiewicz 2000, Lemma 3.3]. *For $0 \le l < k \le n$, let $f = f(\tau, \sigma)$ be a Γ-invariant function on the set of pairs (τ, σ), where $\tau \in \Sigma(l)$, $\sigma \in \Sigma(k)$ with $\tau \subseteq \sigma$. Then*

$$
\sum_{\sigma \in \Sigma(k,\Gamma)} \sum_{\tau \in \Sigma(l),\, \tau \subseteq \sigma} \frac{f(\tau, \sigma)}{|\Gamma_\sigma|} = \sum_{\tau \in \Sigma(l,\Gamma)} \sum_{\sigma \in \Sigma(k),\, \tau \subseteq \sigma} \frac{f(\tau, \sigma)}{|\Gamma_\tau|}.
$$

The reader should note that from now on we will use the above proposition to change the order of summation without mentioning it explicitly.

Proposition 3.7 (equivalent to [Ballmann and Świątkowski 1997, Propositions 1.5 and 1.6]).

(1) *The differential is a bounded operator and $\|d_k\| \le \sqrt{k+2}$.*

(2) *We define $d_k^* : C^{k+1}(X, \bar{\pi}) \to C^k(X, \bar{\pi})$ to be the adjoint operator of d_k. Then*

$$
d_k^* \phi(\tau) = \sum_{v \in \Sigma(0),\, v\tau \in \Sigma(k+1)} \frac{m(v\tau)}{m(\tau)} \phi(v\tau), \qquad \tau \in \Sigma(k),
$$

where $v\tau = (v, v_0, \dots, v_k)$ for $\tau = (v_0, \dots, v_k)$.

Proof. (1) For every $\phi \in C^k(X, \pi)$ we have

$$
\begin{aligned}
\|d_k \phi\|^2 &= \sum_{\sigma \in \Sigma(k+1,\Gamma)} \frac{m(\sigma)}{(k+2)!\,|\Gamma_\sigma|} \left| \sum_{i=0}^{k+1} (-1)^i \phi(\sigma_i) \right|^2 \le \sum_{\sigma \in \Sigma(k+1,\Gamma)} \frac{m(\sigma)}{(k+2)!\,|\Gamma_\sigma|} (k+2) \sum_{i=0}^{k+1} |\phi(\sigma_i)|^2 \\
&= \sum_{\sigma \in \Sigma(k+1,\Gamma)} \frac{m(\sigma)}{(k+1)!\,(k+1)!\,|\Gamma_\sigma|} \sum_{\tau \in \Sigma(k), \tau \subset \sigma} |\phi(\tau)|^2 \\
&= \sum_{\tau \in \Sigma(k,\Gamma)} \frac{|\phi(\tau)|^2}{(k+1)!\,(k+1)!\,|\Gamma_\tau|} \sum_{\sigma \in \Sigma(k+1), \tau \subset \sigma} m(\sigma) = \sum_{\tau \in \Sigma(k,\Gamma)} \frac{(k+2)!\,m(\tau)|\phi(\tau)|^2}{(k+1)!\,(k+1)!\,|\Gamma_\tau|} = (k+2)\|\phi\|^2.
\end{aligned}
$$

(2) For $\sigma \in \Sigma(k+1)$ and $\tau \subset \sigma$, $\tau \in \Sigma(k)$ denote by $[\sigma : \tau]$ the incidence coefficient of τ with respect to σ; i.e., if σ_i has the same vertices as τ then for every $\psi \in C^k(X, \pi)$ we have $[\sigma : \tau]\psi(\tau) = (-1)^i \psi(\sigma_i)$.

Take $\phi \in C^{k+1}(X, \bar{\pi})$ and $\psi \in C^k(X, \pi)$. We note that for every $\tau \in \Sigma(k)$, every $\sigma \in \Sigma(k+1)$ and every $g \in \Gamma$,

$$(\psi(g.\tau), \phi(g.\sigma)) = (\pi(g)\psi(\tau), \bar{\pi}(g)\phi(\sigma)) = (\psi(\tau), \phi(\sigma)),$$

and we will use this fact in equality $(*)$ below, in which we apply Proposition 3.6:

$$\begin{aligned}
\langle d\psi, \phi\rangle &= \sum_{\sigma\in\Sigma(k+1,\Gamma)} \frac{m(\sigma)}{(k+2)!\,|\Gamma_\sigma|}\biggl(\sum_{i=0}^{k+1}(-1)^i\psi(\sigma_i), \phi(\sigma)\biggr) \\
&= \sum_{\sigma\in\Sigma(k+1,\Gamma)} \frac{m(\sigma)}{(k+1)!\,(k+2)!\,|\Gamma_\sigma|}\biggl(\sum_{\tau\in\Sigma(k),\,\tau\subset\sigma}[\sigma:\tau]\psi(\tau), \phi(\sigma)\biggr) \\
&= \sum_{\sigma\in\Sigma(k+1,\Gamma)} \frac{1}{(k+1)!\,|\Gamma_\sigma|}\sum_{\tau\in\Sigma(k),\,\tau\subset\sigma} m(\tau)\biggl(\psi(\tau), \frac{[\sigma:\tau]m(\sigma)}{m(\tau)(k+2)!}\phi(\sigma)\biggr) \\
&= \sum_{\tau\in\Sigma(k,\Gamma)} \frac{m(\tau)}{(k+1)!\,|\Gamma_\tau|}\sum_{\sigma\in\Sigma(k+1),\,\tau\subset\sigma} \biggl(\psi(\tau), \frac{[\sigma:\tau]m(\sigma)}{m(\tau)(k+2)!}\phi(\sigma)\biggr) \qquad (*) \\
&= \sum_{\tau\in\Sigma(k,\Gamma)} \frac{m(\tau)}{(k+1)!\,|\Gamma_\tau|}\biggl(\psi(\tau), \sum_{\sigma\in\Sigma(k+1),\,\tau\subset\sigma} \frac{[\sigma:\tau]m(\sigma)}{m(\tau)(k+2)!}\phi(\sigma)\biggr) \\
&= \sum_{\tau\in\Sigma(k,\Gamma)} \frac{m(\tau)}{(k+1)!\,|\Gamma_\tau|}\biggl(\psi(\tau), \sum_{v\in\Sigma(0),\,v\tau\in\Sigma(k+1)} \frac{m(v\tau)}{m(\tau)}\phi(v\tau)\biggr). \qquad \square
\end{aligned}$$

We end this section by proving the following criterion for vanishing of cohomology that appeared in a different form in [Nowak 2015] (we claim no originality here):

Lemma 3.8. *Let $X, \Gamma, \mathbb{E}, \pi$ be as above and $1 \leq k \leq n-1$. If there is a constant $C < 1$ such that, for every $\phi \in C^k(X, \pi)$, $\psi \in C^k(X, \bar{\pi})$,*

$$|\langle d_k\phi, \bar{d}_k\psi\rangle| + |\langle \bar{d}_{k-1}^*\phi, d_{k-1}^*\psi\rangle| \geq |\langle\phi, \psi\rangle| - C\biggl(\frac{\|\phi\|^2+\|\psi\|^2}{2}\biggr),$$

then $H^k(X, \pi) = H^k(X, \bar{\pi}) = 0$.

Before proving this lemma, we recall the following facts regarding adjoint operators (for proof of these facts, see for instance [Megginson 1998, Corollary 1.6.6, Theorem 3.1.22]):

Theorem 3.9. *Let $\mathbb{E}_1, \mathbb{E}_2$ be Banach spaces and $T : \mathbb{E}_1 \to \mathbb{E}_2$ be a bounded linear operator. Then:*

(1) *The following are equivalent:*
   (a) *T maps $\mathbb{E}_1$ onto $\mathbb{E}_2$.*
   (b) *T^* is an isomorphism from $\mathbb{E}_2^*$ onto a subspace of $\mathbb{E}_1^*$.*
   (c) *There is a constant $c > 0$ such that, for every $x \in \mathbb{E}_2^*$, $\|T^*x\| \geq c\|x\|$.*
   (d) *T^* is injective with a closed image.*

(2) *The following are equivalent:*
   (a) *T^* maps $\mathbb{E}_2^*$ onto $\mathbb{E}_1^*$.*

(b) *T is an isomorphism from $\mathbb{E}_1$ onto a subspace of $\mathbb{E}_2$.*
(c) *There is a constant $c > 0$ such that, for every $x \in \mathbb{E}_1$, $\|Tx\| \geq c\|x\|$.*
(d) *T is injective with a closed image.*

Using these facts, we can prove Lemma 3.8:

Proof. We will only prove that $H^k(X, \pi) = 0$; the proof for $H^k(X, \bar{\pi})$ is similar. We define d'_{k-1} to be the $k-1$ differential with range $\mathrm{Ker}(d_k)$, i.e., $d'_{k-1} : C^{k-1}(X, \pi) \to \mathrm{Ker}(d_k)$, and we also define $i : \mathrm{Ker}(d_k) \hookrightarrow C^k(X, \pi)$ to be the natural injection. Therefore $d_{k-1} = i \circ d'_{k-1}$. We similarly define $\bar{d}'_{k-1} : C^{k-1}(X, \bar{\pi}) \to \mathrm{Ker}(\bar{d}_k)$ and $\bar{i} : \mathrm{Ker}(\bar{d}_k) \hookrightarrow C^k(X, \bar{\pi})$ and with this notation $\bar{d}_{k-1} = \bar{i} \circ \bar{d}'_{k-1}$.

By the assumptions of the lemma, for every $\phi \in \mathrm{Ker}(d_k)$, taking $\psi = \phi^* \in J\phi$ (using Proposition 3.5) yields that

$$|\langle \bar{d}^*_{k-1}\phi, d^*_{k-1}\phi^*\rangle| \geq |\langle \phi, \phi^*\rangle| - C\left(\frac{\|\phi\|^2 + \|\phi^*\|^2}{2}\right) = (1 - C)\|\phi\|^2.$$

We note that by Proposition 3.7,

$$|\langle \bar{d}^*_{k-1}\phi, d^*_{k-1}\phi^*\rangle| \leq \|\bar{d}^*_{k-1}\phi\| \, \|d^*_{k-1}\phi^*\| \leq \|\bar{d}^*_{k-1}\phi\| \sqrt{k+2}\|\phi\|.$$

Thus, for every $\phi \in \mathrm{Ker}(d_k)$,

$$\|\bar{d}^*_{k-1}\phi\| \geq \frac{1 - C}{\sqrt{k+2}}\|\phi\|.$$

This yields that $\bar{d}^*_{k-1} \circ i$ is injective with a closed image. By the notation above, $(\bar{d}'_{k-1})^* \circ \bar{i}^* \circ i$ is injective with a closed image, and therefore $\bar{i}^* \circ i : \mathrm{Ker}(d_k) \to (\mathrm{Ker}(\bar{d}_k))^*$ is injective with a closed image. Note that $\mathrm{Ker}(\bar{d}_k)$ is a closed subspace of a reflexive space (using Proposition 3.3) and thus $\mathrm{Ker}(\bar{d}_k)$ is reflexive and it follows that $(\mathrm{Ker}(\bar{d}_k))^*$ is reflexive as well. Therefore by Theorem 3.9, $i^* \circ \bar{i} = (\bar{i}^* \circ i)^* : (\mathrm{Ker}(d_k))^* \to \mathrm{Ker}(\bar{d}_k)$ is onto.

By a similar argument, for a given $\psi \in \mathrm{Ker}(\bar{d}_k)$, if we take $\phi = \psi^* \in \bar{J}\psi$, then

$$|\langle \bar{d}^*_{k-1}\psi^*, d^*_{k-1}\psi\rangle| \geq (1 - C)\|\psi\|^2,$$

which implies that

$$\|d^*_{k-1}\psi\| \geq \frac{1 - C}{\sqrt{k+2}}\|\psi\|.$$

Arguing as above, we deduce from this inequality that $(d'_{k-1})^* \circ i^* \circ \bar{i}$ is injective with a closed image.

We showed above that $i^* \circ \bar{i}$ is onto and therefore it follows that $(d'_{k-1})^* : (\mathrm{Ker}(d_k))^* \to (C^{k-1}(X, \pi))^*$ is injective with a closed image. Thus applying Theorem 3.9 yields that d'_{k-1} is onto, i.e., $\mathrm{Im}(d_{k-1}) = \mathrm{Ker}(d_k)$, or in other words, $H^k(X, \pi) = 0$. □

Remark 3.10. As in [Ballmann and Świątkowski 1997], we can define the Laplacian operators as follows: $\Delta^+_k = \bar{d}^*_k d_k$, $\Delta^-_k = d_{k-1}\bar{d}^*_{k-1}$. With this notation, the condition in Lemma 3.8 can be reformulated as follows: there is a constant $C < 1$ such that, for every $\phi \in C^k(X, \pi)$, $\psi \in C^k(X, \bar{\pi})$,

$$|\langle \Delta^+_k \phi, \psi\rangle| + |\langle \Delta^-_k \phi, \psi\rangle| \geq |\langle \phi, \psi\rangle| - C\left(\frac{\|\phi\|^2 + \|\psi\|^2}{2}\right).$$

4. Local criteria for vanishing of Banach cohomology

Below, we will prove local criteria for vanishing of equivariant cohomology in the spirit of "Garland's method". The method is an adaption of [Ballmann and Świątkowski 1997], but unlike the case of Hilbert spaces, considered in that work, in which the condition for vanishing of cohomology requires a (one-sided) spectral gap in the links, here the condition for vanishing of cohomology will require a two-sided spectral gap in the same links.

Let $X, \Gamma, \mathbb{E}, \pi$ be as in Section 3 (recall that we assume that π is continuous and $\mathbb{E}$ is reflexive and thus $\bar{\pi}$ is also continuous). Given an ordered simplex $(v_0, \dots, v_j) = \tau \in \Sigma(j)$, the link of τ is simply the link of $\{v_0, \dots, v_j\}$ defined above. Below, we will only be interested in the 1-skeleton to the links: given $\tau \in \Sigma(j)$, the *1-skeleton of* X_τ is the weighted graph, denoted by (V_τ, E_τ), defined as

$$V_\tau = \{v : \{v\} \in X_\tau(0)\}, \quad E_\tau = X_\tau(1),$$

with the weight function $m_\tau(\{u, v\}) = m(\tau \cup \{u, v\})$, where $\tau \cup \{u, v\}$ is defined by the abuse of notation of treating τ as a set (and forgetting the ordering); i.e.,

$$m((v_0, \dots, v_j) \cup \{u, v\}) = m(\{v_0, \dots, v_j, u, v\}).$$

Note that with this definition, $m_\tau(v) = m(\tau \cup \{v\})$.

On this weighted graph, we define $\ell^2(V_\tau, m_\tau)$, $\ell^2(V_\tau, m_\tau; \mathbb{E})$ and the operators A_τ, M_τ as in Section 2D. On $\ell^2(V_\tau, m_\tau; \mathbb{E})$ define a norm denoted by $\|\cdot\|_\tau$ as in Section 3; i.e., for $\phi \in \ell^2(V_\tau, m_\tau; \mathbb{E})$,

$$\|\phi\|_\tau = \left(\sum_{v \in V_\tau} m_\tau(v) |\phi(v)|^2 \right)^{1/2},$$

where $|\cdot|$ is the norm of $\mathbb{E}$. Also, define a coupling $\langle \cdot, \cdot \rangle_\tau$ between $\ell^2(V_\tau, m_\tau; \mathbb{E})$ and $\ell^2(V_\tau, m_\tau; \mathbb{E}^*)$ as follows: for $\phi \in \ell^2(V_\tau, m_\tau; \mathbb{E})$, $\psi \in \ell^2(V_\tau, m_\tau; \mathbb{E}^*)$,

$$\langle \phi, \psi \rangle_\tau = \sum_{v \in V_\tau} m_\tau(v) (\phi(v), \psi(v)),$$

where $(\cdot, \cdot)$ is the standard coupling between $\mathbb{E}$ and $\mathbb{E}^*$.

Given $\phi \in C^k(X, \pi)$ and $\tau \in \Sigma(k-1)$ we define the *localization of* ϕ *at* X_τ, denoted by $\phi_\tau \in \ell^2(V_\tau, m_\tau; \mathbb{E})$, as

$$\phi_\tau(v) = \phi(v\tau) \quad \text{for all } v \in V_\tau,$$

where $v\tau$ is the concatenation of v with τ, i.e., $v\tau = (v, v_0, \dots, v_{k-1})$ for $\tau = (v_0, \dots, v_{k-1})$. We note that by the definition of X_τ, $v\tau \in \Sigma(k)$ and therefore $\phi(v\tau)$ is well-defined.

The basic observation in [Garland 1973] was that the norm of cochains can be computed by considering their localizations. Below, we generalize this observation to the Banach setting. The calculations below are very similar to those of [Ballmann and Świątkowski 1997], but we included all the calculations, because we need localization results not only for the norms, but for the couplings.

Lemma 4.1. *Let* $1 \le k \le n-1$, $\phi \in C^k(X,\pi)$, $\psi \in C^k(X,\bar{\pi})$. *Then*

$$(k+1)!\,\langle \phi, \psi\rangle = \sum_{\tau\in\Sigma(k-1,\Gamma)} \frac{1}{|\Gamma_\tau|}\langle \phi_\tau, \psi_\tau\rangle_\tau,$$

$$(k+1)!\,\|\phi\|^2 = \sum_{\tau\in\Sigma(k-1,\Gamma)} \frac{1}{|\Gamma_\tau|}\|\phi_\tau\|_\tau^2,$$

$$(k+1)!\,\|\psi\|^2 = \sum_{\tau\in\Sigma(k-1,\Gamma)} \frac{1}{|\Gamma_\tau|}\|\psi_\tau\|_\tau^2.$$

Proof. All these equalities follow from the definition of the localization and Proposition 3.6 and thus we will only prove the first equality, leaving the other two for the reader. Fix $\phi \in C^k(X,\pi)$, $\psi \in C^k(X,\bar{\pi})$. Then

$$\begin{aligned}
\sum_{\tau\in\Sigma(k-1,\Gamma)} \frac{1}{|\Gamma_\tau|}\langle \phi_\tau, \psi_\tau\rangle_\tau &= \sum_{\tau\in\Sigma(k-1,\Gamma)} \frac{1}{|\Gamma_\tau|}\sum_{v\in V_\tau} m_\tau(v)(\phi_\tau(v), \psi_\tau(v)) \\
&= \sum_{\tau\in\Sigma(k-1,\Gamma)} \frac{1}{|\Gamma_\tau|}\sum_{v\in V_\tau} m(v\tau)(\phi(v\tau), \psi(v\tau)) \\
&= \sum_{\tau\in\Sigma(k-1,\Gamma)} \frac{1}{|\Gamma_\tau|}\frac{1}{(k+1)!}\sum_{\sigma\in\Sigma(k),\tau\subseteq\sigma} m(\sigma)(\phi(\sigma), \psi(\sigma)) \\
&= \sum_{\sigma\in\Sigma(k,\Gamma)} \frac{m(\sigma)}{(k+1)!\,|\Gamma_\sigma|}\sum_{\tau\in\Sigma(k-1,\Gamma)} (\phi(\sigma), \psi(\sigma)) \sum_{\tau\in\Sigma(k-1),\tau\subseteq\sigma} 1 \\
&= (k+1)!\,\langle \phi, \psi\rangle.
\end{aligned}$$

□

Lemma 4.2. *Let* $1 \le k \le n-1$, $\phi \in C^k(X,\pi)$, $\psi \in C^k(X,\bar{\pi})$. *Then*

$$\langle \bar{d}_{k-1}^*\phi, d_{k-1}^*\psi\rangle = \frac{1}{k!}\sum_{\tau\in\Sigma(k-1,\Gamma)} \frac{1}{|\Gamma_\tau|}\langle (M_\tau\otimes \mathrm{id}_{\mathbb{E}})\phi_\tau, \psi_\tau\rangle_\tau.$$

Proof. By Proposition 3.7, for every $\tau \in \Sigma(k-1)$,

$$\bar{d}_k^*\phi(\tau) = \sum_{v\in V_\tau} \frac{m(v\tau)}{m(\tau)}\phi(v\tau), \qquad d_k^*\psi(\tau) = \sum_{v\in V_\tau} \frac{m(v\tau)}{m(\tau)}\psi(v\tau).$$

We note that by definition $m_\tau(\varnothing) = m(\tau)$ and therefore, for every $\tau \in \Sigma(k-1)$,

$$\begin{aligned}
\langle (M_\tau\otimes \mathrm{id}_{\mathbb{E}})\phi_\tau, \psi_\tau\rangle_\tau &= \sum_{v\in V_\tau} m_\tau(v)\biggl(\sum_{u\in V_\tau}\frac{m_\tau(u)}{m_\tau(\varnothing)}\phi_\tau(u), \psi_\tau(v)\biggr) \\
&= \sum_{v\in V_\tau} m(v\tau)\biggl(\sum_{u\in V_\tau}\frac{m(u\tau)}{m(\tau)}\phi(u\tau), \psi(v\tau)\biggr) \\
&= \biggl(\sum_{u\in V_\tau}\frac{m(u\tau)}{m(\tau)}\phi(u\tau), \sum_{v\in V_\tau} m(v\tau)\psi(v\tau)\biggr) = m(\tau)(\bar{d}_k^*\phi(\tau), d_k^*\psi(\tau)).
\end{aligned}$$

Therefore

$$\sum_{\tau\in\Sigma(k-1,\Gamma)}\frac{1}{|\Gamma_\tau|}\langle(M_\tau\otimes\mathrm{id}_{\mathbb{E}})\phi_\tau,\psi_\tau\rangle_\tau=\sum_{\tau\in\Sigma(k-1,\Gamma)}\frac{m(\tau)}{|\Gamma_\tau|}(\bar{d}_k^*\phi(\tau),d_k^*\psi(\tau))$$
$$=k!\,\langle\bar{d}_{k-1}^*\phi,d_{k-1}^*\psi\rangle.\qquad\square$$

Lemma 4.3. *Let* $1\le k\le n-1$, $\phi\in C^k(X,\pi)$, $\psi\in C^k(X,\bar{\pi})$. *Then*

$$\langle d_k\phi,\bar{d}_k\psi\rangle=\langle\phi,\psi\rangle-\frac{1}{k!}\sum_{\tau\in\Sigma(k-1,\Gamma)}\frac{1}{|\Gamma_\tau|}\langle(A_\tau\otimes\mathrm{id}_{\mathbb{E}})\phi_\tau,\psi_\tau\rangle_\tau.$$

Proof. For $\eta=(v_0,\dots,v_{k+1})\in\Sigma(k+1)$ and $0\le i\ne j\le k+1$, define $\eta_i=(v_0,\dots,\hat{v}_i,\dots,v_{k+1})$ and $\eta_{i,j}=(v_0,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_{k+1})$. Then

$$(d_k\phi(\eta),\bar{d}_k\psi(\eta))=\left(\sum_{i=0}^{k+1}(-1)^i\phi(\eta_i),\sum_{j=0}^{k+1}(-1)^j\psi(\eta_j)\right)$$
$$=\sum_{i=0}^{k+1}(\phi(\eta_i),\psi(\eta_i))+\sum_{0\le i\ne j\le k+1}(-1)^{i+j}(\phi(\eta_i),\psi(\eta_j)).$$

We note that by the assumption that ϕ, ψ are alternating, changing the order of η_i in the first sum above does not change the coupling and therefore

$$\sum_{i=0}^{k+1}(\phi(\eta_i),\psi(\eta_i))=\frac{1}{(k+1)!}\sum_{\sigma\in\Sigma(k),\,\sigma\subseteq\eta}(\phi(\sigma),\psi(\sigma)).$$

We also note that, for every $i\ne j$,

$$(\phi(\eta_i),\psi(\eta_j))=(-1)^{i+j-1}(\phi(v_j\eta_{i,j}),\psi(v_i\eta_{i,j}))$$

(this can be shown by considering the cases $i<j$ and $j<i$; we leave the proof for the reader). Therefore

$$\sum_{0\le i\ne j\le k+1}(-1)^{i+j}(\phi(\eta_i),\psi(\eta_j))=-\sum_{0\le i\ne j\le k+1}(\phi(v_j\eta_{i,j}),\psi(v_i\eta_{i,j}))$$
$$=\frac{1}{k!}\sum_{\tau\in\Sigma(k-1),\,\tau\subseteq\eta}\ \sum_{v,v\tau\subseteq\eta}\left(\sum_{u,u\ne v,\,uv\tau\subseteq\eta}\phi(u\tau),\psi(v\tau)\right),$$

where $uv\tau$ is the concatenation, i.e., if $\tau=(v_0,\dots,v_{k-1})$, then $uv\tau=(u,v,v_0,\dots,v_{k-1})$ (we recall that $uv\tau\subseteq\eta$ refers only to inclusion as sets without regarding the ordering). This yields

$$\langle d_k\phi,\bar{d}_k\psi\rangle=\sum_{\eta\in\Sigma(k+1,\Gamma)}\frac{m(\eta)}{(k+2)!\,|\Gamma_\eta|}\frac{1}{(k+1)!}\sum_{\sigma\in\Sigma(k),\,\sigma\subseteq\eta}(\phi(\sigma),\psi(\sigma))$$
$$-\sum_{\eta\in\Sigma(k+1,\Gamma)}\frac{m(\eta)}{(k+2)!\,|\Gamma_\eta|}\frac{1}{k!}\sum_{\tau\in\Sigma(k-1),\,\tau\subseteq\eta}\ \sum_{v,v\tau\subseteq\eta}\left(\sum_{u,u\ne v,\,uv\tau\subseteq\eta}\phi(u\tau),\psi(v\tau)\right).\quad(1)$$

We will calculate each one of the expressions above separately. First, by applying Proposition 3.6,

$$\begin{aligned}
&\sum_{\eta\in\Sigma(k+1,\Gamma)} \frac{m(\eta)}{(k+2)!\,|\Gamma_\eta|}\frac{1}{(k+1)!}\sum_{\sigma\in\Sigma(k),\sigma\subseteq\eta}(\phi(\sigma),\psi(\sigma))\\
&\qquad= \sum_{\sigma\in\Sigma(k,\Gamma)}\frac{1}{(k+1)!\,|\Gamma_\sigma|}(\phi(\sigma),\psi(\sigma))\sum_{\sigma\in\Sigma(k+1),\,\sigma\subseteq\eta}\frac{m(\eta)}{(k+2)!}\\
&\qquad= \sum_{\sigma\in\Sigma(k,\Gamma)}\frac{m(\sigma)}{(k+1)!\,|\Gamma_\sigma|}(\phi(\sigma),\psi(\sigma)) = \langle\phi,\psi\rangle. \qquad (2)
\end{aligned}$$

Second, applying Proposition 3.6 to the second expression,

$$\begin{aligned}
&\sum_{\eta\in\Sigma(k+1,\Gamma)} \frac{m(\eta)}{(k+2)!\,|\Gamma_\eta|}\frac{1}{k!}\sum_{\tau\in\Sigma(k-1),\,\tau\subseteq\eta}\ \sum_{v,v\tau\subseteq\eta}\Bigg(\sum_{u,\,u\neq v,\,uv\tau\subseteq\eta}\phi(u\tau),\psi(v\tau)\Bigg)\\
&\quad=\frac{1}{k!}\sum_{\tau\in\Sigma(k-1,\Gamma)}\frac{1}{|\Gamma_\tau|}\sum_{\eta\in\Sigma(k+1),\tau\subseteq\eta}\frac{m(\eta)}{(k+2)!}\sum_{v,v\tau\subseteq\eta}\Bigg(\sum_{u,u\neq v,uv\tau\subseteq\eta}\phi(u\tau),\psi(v\tau)\Bigg)\\
&\quad=\frac{1}{k!}\sum_{\tau\in\Sigma(k-1,\Gamma)}\frac{1}{|\Gamma_\tau|}\sum_{\{v,u\}\in E_\tau}m_\tau(\{v,u\})\sum_{v\in\{v,u\}}\Bigg(\sum_{u\in\{v,u\},\,u\neq v}\phi_\tau(u),\psi_\tau(v)\Bigg)\\
&\quad=\frac{1}{k!}\sum_{\tau\in\Sigma(k-1,\Gamma)}\frac{1}{|\Gamma_\tau|}\sum_{v\in V_\tau}\Bigg(\sum_{u\in\{v,u\},u\neq v}m_\tau(\{v,u\})\phi_\tau(u),\psi_\tau(v)\Bigg)\\
&\quad=\frac{1}{k!}\sum_{\tau\in\Sigma(k-1,\Gamma)}\frac{1}{|\Gamma_\tau|}\sum_{v\in V_\tau}m_\tau(v)\Bigg(\sum_{u\in\{v,u\},\,u\neq v}\frac{m_\tau(\{v,u\})}{m_\tau(v)}\phi_\tau(u),\psi_\tau(v)\Bigg)\\
&\quad=\frac{1}{k!}\sum_{\tau\in\Sigma(k-1,\Gamma)}\frac{1}{|\Gamma_\tau|}\langle(A_\tau\otimes\mathrm{id}_{\mathbb{E}})\phi_\tau,\psi_\tau\rangle_\tau. \qquad (3)
\end{aligned}$$

Combining (1), (2), and (3) yields the needed equality. □

After these lemmas, we can prove a local criterion for cohomology vanishing that appeared as Theorem 1.1 in the Introduction:

Theorem 4.4. *Let X be a locally finite, pure n-dimensional simplicial complex with the weight function m defined above and Γ be a locally compact, unimodular group acting cocompactly and properly on X. For every reflexive Banach space $\mathbb{E}$ and every $1\le k\le n-1$, if*

$$\max_{\tau\in\Sigma(k-1,\Gamma)}\|(A_\tau(I-M_\tau)\otimes\mathrm{id}_{\mathbb{E}})\|_{B(\ell^2(V_\tau,m_\tau;\mathbb{E}))}<\frac{1}{k+1},$$

then for every continuous isometric representation π of Γ on $\mathbb{E}$ it holds that $H^k(X,\pi)=0$.

Proof. Let $\mathbb{E}$ be a reflexive Banach space and π be a continuous isometric representation of Γ on $\mathbb{E}$. Define

$$C'=\max_{\tau\in\Sigma(k-1,\Gamma)}\|(A_\tau(I-M_\tau)\otimes\mathrm{id}_{\mathbb{E}})\|_{B(\ell^2(V,m;\mathbb{E}))}.$$

Then by Lemma 4.3, for every $\phi \in C^k(X,\pi)$, $\psi \in C^k(X,\bar{\pi})$,

$$\begin{aligned}
&\langle d_k\phi, \bar{d}_k\psi\rangle \\
&\quad= \langle \phi,\psi\rangle - \frac{1}{k!}\sum_{\tau\in\Sigma(k-1,\Gamma)}\frac{1}{|\Gamma_\tau|}\langle (A_\tau \otimes \mathrm{id}_{\mathbb{E}})\phi_\tau, \psi_\tau\rangle_\tau \\
&\quad= \langle \phi,\psi\rangle - \frac{1}{k!}\sum_{\tau\in\Sigma(k-1,\Gamma)}\frac{1}{|\Gamma_\tau|}\langle (A_\tau(I-M_\tau) \otimes \mathrm{id}_{\mathbb{E}})\phi_\tau, \psi_\tau\rangle_\tau - \frac{1}{k!}\sum_{\tau\in\Sigma(k-1,\Gamma)}\frac{1}{|\Gamma_\tau|}\langle (A_\tau M_\tau \otimes \mathrm{id}_{\mathbb{E}})\phi_\tau, \psi_\tau\rangle_\tau \\
&\quad= \langle \phi,\psi\rangle - \frac{1}{k!}\sum_{\tau\in\Sigma(k-1,\Gamma)}\frac{1}{|\Gamma_\tau|}\langle (A_\tau(I-M_\tau) \otimes \mathrm{id}_{\mathbb{E}})\phi_\tau, \psi_\tau\rangle_\tau - \frac{1}{k!}\sum_{\tau\in\Sigma(k-1,\Gamma)}\frac{1}{|\Gamma_\tau|}\langle (M_\tau \otimes \mathrm{id}_{\mathbb{E}})\phi_\tau, \psi_\tau\rangle_\tau \\
&\quad= \langle \phi,\psi\rangle - \frac{1}{k!}\sum_{\tau\in\Sigma(k-1,\Gamma)}\frac{1}{|\Gamma_\tau|}\langle (A_\tau(I-M_\tau) \otimes \mathrm{id}_{\mathbb{E}})\phi_\tau, \psi_\tau\rangle_\tau - \langle \bar{d}^*_{k-1}\phi, d^*_{k-1}\psi\rangle,
\end{aligned}$$

where the second-to-last equality follows from the fact $A_\tau M_\tau = M_\tau$ and the last equality by Lemma 4.2. Thus

$$\langle d_k\phi, \bar{d}_k\psi\rangle + \langle \bar{d}^*_{k-1}\phi, d^*_{k-1}\psi\rangle = \langle \phi,\psi\rangle - \frac{1}{k!}\sum_{\tau\in\Sigma(k-1,\Gamma)}\frac{1}{|\Gamma_\tau|}\langle (A_\tau(I-M_\tau) \otimes \mathrm{id}_{\mathbb{E}})\phi_\tau, \psi_\tau\rangle_\tau.$$

Applying absolute value on this equation and using the triangle inequality,

$$\begin{aligned}
&|\langle d_k\phi, \bar{d}_k\psi\rangle| + |\langle \bar{d}^*_{k-1}\phi, d^*_{k-1}\psi\rangle| \\
&\qquad\geq |\langle \phi,\psi\rangle| - \frac{1}{k!}\sum_{\tau\in\Sigma(k-1,\Gamma)}\frac{1}{|\Gamma_\tau|}|\langle (A_\tau(I-M_\tau) \otimes \mathrm{id}_{\mathbb{E}})\phi_\tau, \psi_\tau\rangle_\tau| \\
&\qquad\geq |\langle \phi,\psi\rangle| - \frac{1}{k!}\sum_{\tau\in\Sigma(k-1,\Gamma)}\frac{1}{|\Gamma_\tau|}\|(A_\tau(I-M_\tau) \otimes \mathrm{id}_{\mathbb{E}})\|_{B(\ell^2(V_\tau,m_\tau;\mathbb{E}))}\|\phi_\tau\|_\tau\|\psi_\tau\|_\tau \\
&\qquad\geq |\langle \phi,\psi\rangle| - \frac{1}{k!}\sum_{\tau\in\Sigma(k-1,\Gamma)}\frac{1}{|\Gamma_\tau|}C'\frac{\|\phi_\tau\|_\tau^2 + \|\psi_\tau\|_\tau^2}{2} \\
&\qquad= |\langle \phi,\psi\rangle| - (k+1)C'\frac{\|\phi\|^2 + \|\psi\|^2}{2} \quad \text{(by Lemma 4.1).}
\end{aligned}$$

If we write $C = (k+1)C'$, then by our assumption $C < 1$ and we prove that

$$|\langle d_k\phi, \bar{d}_k\psi\rangle| + |\langle \bar{d}^*_{k-1}\phi, d^*_{k-1}\psi\rangle| \geq |\langle \phi,\psi\rangle| - C\left(\frac{\|\phi\|^2 + \|\psi\|^2}{2}\right),$$

and by Lemma 3.8, $H^k(X,\pi) = H^k(X,\bar{\pi}) = 0$. □

Next, we will apply this theorem in the context of uniformly curved spaces:

Proposition 4.5. *Let X, Γ be as above and $\alpha : (0,1] \to (0,1]$ be a strictly monotone increasing function. Fix $1 \leq k \leq n-1$. If there is $\lambda < \alpha^{-1}(1/(2(k+1)))$ such that, for every $\tau \in \Sigma(k-1,\Gamma)$, the 1-skeleton of X_τ is a two-sided λ-spectral expander, then $H^k(X,\pi) = 0$ for every $\mathbb{E} \in \mathcal{E}_\alpha^{\text{u-curved}}$ and every continuous isometric representation π of Γ on $\mathbb{E}$.*

Proof. First, recall that, by Theorem 2.2, every $\mathbb{E} \in \mathcal{E}_{\alpha}^{\text{u-curved}}$ is reflexive. Second, by Corollary 2.12, for every $\tau \in \Sigma(k-1, \Gamma)$,

$$\|(A_\tau(I - M_\tau)) \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))} \le 2\alpha(\lambda) < 2\alpha\left(\alpha^{-1}\left(\frac{1}{2(k+1)}\right)\right) = \frac{1}{k+1}.$$

Therefore, the conditions of Theorem 4.4 are fulfilled and $H^k(X, \pi) = 0$ for every continuous isometric representation π of Γ on $\mathbb{E}$. □

As a result of this proposition we deduce the following vanishing result for strictly Hilbertian spaces that appeared in Corollary 1.4(1):

Corollary 4.6. *Let X, Γ be as above, $0 < \theta_0 \le 1$ a constant. Define $\mathcal{E}_{\theta_0}$ to be the smallest class of Banach spaces that contains all strictly θ-Hilbertian Banach spaces for all $\theta_0 \le \theta \le 1$ and is closed under passing to quotients, subspaces, ℓ^2-sums and ultraproducts of Banach spaces. Fix $1 \le k \le n-1$. If there is $0 < \lambda < (1/(2(k+1)))^{1/\theta_0}$ such that for every $\tau \in \Sigma(k-1, \Gamma)$ the 1-skeleton of X_τ is a two-sided λ-spectral expander, then $H^k(X, \pi) = 0$ for every $\mathbb{E} \in \mathcal{E}_{\theta_0}$ and every continuous isometric representation π of Γ on $\mathbb{E}$.*

Proof. Corollary 2.8 states that $\mathcal{E}_{\theta_0} \subseteq \mathcal{E}_{\alpha(t)=t^{\theta_0}}^{\text{u-curved}}$. Thus the assertion follows directly from Proposition 4.5. □

Specializing this corollary to the case of vanishing of the L^p cohomology of a group acting on a 2-dimensional simplicial complex yields:

Corollary 4.7. *Let X be a locally finite, pure 2-dimensional simplicial complex such that all the links of X of dimension ≥ 1 are connected and Γ be a locally compact, unimodular group acting cocompactly and properly on X. Also let $p > 2$, $0 < \lambda < 1/2^p$ be constants. Assume that for every vertex $\{v\} \in X(0)$ the 1-skeleton of $X_{\{v\}}$ is a two-sided λ-spectral expander. Then for every $2 \le p' \le p$, every space $\mathbb{E}$ that is a commutative or noncommutative $L^{p'}$-space and every continuous isometric representation π of Γ on $\mathbb{E}$ it holds that $H^1(X, \pi) = 0$.*

Proof. As noted above, for $2 \le p < \infty$, every (commutative or noncommutative) L^p-space is θ-Hilbertian with $\theta = 2/p$. Thus applying Corollary 4.6 with $k = 1$, $n = 2$ and $\theta = 2/p$ gives the stated result. □

The conditions for Proposition 4.5 and Corollary 4.6 can be deduced for all $1 \le k \le n-1$, based only on the 1-dimensional links of X. This is done via the following theorem [Oppenheim 2018, Theorem 1.4]:

Theorem 4.8. *Let Y be a finite, pure l-dimensional complex, where $l \ge 2$, such that (1-skeletons of) all the links of Y of dimension ≥ 1 are connected (including the 1-skeleton of Y). Define m_Y to be the weight function on Y, V_Y the vertices of the 1-skeleton of Y and A_Y, M_Y the operators associated with the random walk on this 1-skeleton. Let $-1 \le \kappa_1 \le 0 \le \kappa_2 \le 1/l$ be constants such that for every $\tau \in Y(l-2)$ the spectrum of A_τ is contained in $[\kappa_1, \kappa_2] \cup \{1\}$. Then the spectrum of the random walk on the 1-skeleton of Y is contained in $[\kappa_1/(1-(l-1)\kappa_1), \kappa_2/(1-(l-1)\kappa_2)] \cup \{1\}$. Equivalently, if there are $-1 \le \lambda_1 \le 0 \le \lambda_2 \le 1$ such that for every $\tau \in Y(l-2)$ the spectrum of A_τ is contained in $[\lambda_1/(1+(l-1)\lambda_1), \lambda_2/(1+(l-1)\lambda_2)] \cup \{1\}$, then the spectrum of the random walk on the 1-skeleton of Y is contained in $[\lambda_1, \lambda_2] \cup \{1\}$.*

Remark 4.9. In [Oppenheim 2018] this theorem is written in the language of spectral gaps of Laplacians, but as noted above the translation to the language of random walks is straightforward.

Observation 4.10. Theorem 4.8 is not symmetric, as it may appear at first glance: while the upper bound on the spectrum of A_Y deteriorates as l increases, the lower bound actually improves as l increases. In particular, it is always the case that the smallest eigenvalue of the 1-skeleton of every graph is ≥ -1. Thus, in the above theorem we can always take $\kappa_1 = -1$ and get that the spectrum of the random walk on the 1-skeleton of Y is contained in $[-1/l, 1]$.

Using Theorem 4.8, we deduce a criterion for the vanishing of all the cohomologies:

Theorem 4.11. *Let X be a locally finite, pure n-dimensional simplicial complex such that all the links of X of dimension ≥ 1 are connected and Γ be a locally compact, unimodular group acting cocompactly and properly on X. Also let $\alpha : (0, 1] \to (0, 1]$ be a strictly monotone increasing function, $1 \leq k \leq n-1$ and $0 < \lambda < \alpha^{-1}(1/(2(k+1)))$ be constants.*

(1) *If for every $\tau \in \Sigma(n-2, \Gamma)$ the 1-skeleton of X_τ is a two-sided $\lambda/(1+(n-k-1)\lambda)$-spectral expander, then $H^k(X, \pi) = 0$ for every $\mathbb{E} \in \mathcal{E}_\alpha^{\text{u-curved}}$ and every continuous isometric representation π of Γ on $\mathbb{E}$.*

(2) *If $k \leq n - 1/\lambda$ and for every $\tau \in \Sigma(n-2, \Gamma)$ the 1-skeleton of X_τ is a* ***one-sided*** *$\lambda/(1+(n-k-1)\lambda)$-spectral expander, then $H^k(X, \pi) = 0$ for every $\mathbb{E} \in \mathcal{E}_\alpha^{\text{u-curved}}$ and every continuous isometric representation π of Γ on $\mathbb{E}$.*

Proof. Let $1 \leq k \leq n-1$ and let $\eta \in \Sigma(k-1, \Gamma)$. If we let $Y = X_\eta$, then Y is a pure $(n-k)$-dimensional finite simplicial complex and, with the notation of Theorem 4.8,

$$\|A_\eta(I - M_\eta)\|_{B(\ell^2(V_\eta, m_\eta))} = \|A_Y(I - M_Y)\|_{B(\ell^2(V_Y, m_Y))}.$$

Note that the 1-dimensional links of Y are also 1-dimensional links of X. We also note that for every $\tau \in \Sigma(n-2, \Gamma)$, X_τ is a graph and A_τ is the simple random walk on this graph.

(1) Assume that there is $0 \leq \lambda < \alpha^{-1}(1/(2(k+1)))$ such that for every $\tau \in \Sigma(n-1, \Gamma)$ the 1-skeleton of X_τ is a two-sided $\lambda/(1+(n-k-1)\lambda)$-spectral expander. Applying Theorem 4.8 yields that for every $\eta \in \Sigma(k-1, \Gamma)$ the 1-skeleton of X_η is a two-sided λ-spectral expander and thus the conditions of Proposition 4.5 are fulfilled and therefore $H^k(X, \pi) = 0$ for every $\mathbb{E} \in \mathcal{E}_\alpha^{\text{u-curved}}$ and every continuous isometric representation π of Γ on $\mathbb{E}$.

(2) The proof is similar to case (1), but we use Observation 4.10 in order to bound the spectrum from below. We leave the details to the reader. ☐

Applying the theorem above for strictly θ_0-Hilbertian (with $\alpha(t) = t^{\theta_0}$) immediately yields the following corollary, which appeared in the Introduction as part of Corollary 1.4:

Corollary 4.12. *Let X be a locally finite, pure n-dimensional simplicial complex such that all the links of X of dimension ≥ 1 are connected and Γ be a locally compact, unimodular group acting cocompactly and properly on X. Also let $0 < \theta_0 \leq 1$, $1 \leq k \leq n-1$, $0 < \lambda < (1/(2(k+1)))^{1/\theta_0}$ be constants. Define*

$\mathcal{E}_{\theta_0}$ to be the smallest class of Banach spaces that contains all strictly θ-Hilbertian Banach spaces for all $\theta_0 \le \theta \le 1$ and is closed under subspaces, quotients, ℓ^2-sums and ultraproducts of Banach spaces.

(1) *If for every $\tau \in \Sigma(n-2, \Gamma)$ the 1-skeleton of X_τ is a two-sided $\lambda/(1+(n-k-1)\lambda)$-spectral expander, then $H^k(X, \pi) = 0$ for every $\mathbb{E} \in \mathcal{E}_{\theta_0}$ and every continuous isometric representation π of Γ on $\mathbb{E}$.*

(2) *If $k \le n - 1/\lambda$ and for every $\tau \in \Sigma(n-2, \Gamma)$ the 1-skeleton of X_τ is a* ***one-sided*** *$\lambda/(1+(n-k-1)\lambda)$-spectral expander, then $H^k(X, \pi) = 0$ for every $\mathbb{E} \in \mathcal{E}_{\theta_0}$ and every continuous isometric representation π of Γ on $\mathbb{E}$.*

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