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PARTIAL REGULARITY OF LERAY–HOPF WEAK SOLUTIONS TO THE INCOMPRESSIBLE NAVIER–STOKES EQUATIONS WITH HYPERDISSIPATION

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We show that if u is a Leray–Hopf weak solution to the incompressible Navier–Stokes equations with hyperdissipation $\alpha \in (1, \frac{5}{4})$ then there exists a set $S \subset \mathbb{R}^3$ such that u remains bounded outside of S at each blow-up time, the Hausdorff dimension of S is bounded above by $5 - 4\alpha$ and its box-counting dimension is bounded by $\frac{1}{3}(-16\alpha^2 + 16\alpha + 5)$. Our approach is inspired by the ideas of Katz and Pavlović (*Geom. Funct. Anal.* **12**:2 (2002), 355–379).

1. Introduction

We are concerned with the incompressible Navier–Stokes equations with hyperdissipation,

$$\begin{aligned} u_t + (-\Delta)^\alpha u + (u \cdot \nabla)u + \nabla p &= 0 \quad \text{in } \mathbb{R}^3, \\ \operatorname{div} u &= 0, \end{aligned} \tag{1-1}$$

where $\alpha \in (1, \frac{5}{4})$. The equations are equipped with an initial condition $u(0) = u_0$, where u_0 is given. We note that the symbol $(-\Delta)^\alpha$ is defined as the pseudodifferential operator with the symbol $(2\pi)^{2\alpha} |\xi|^{2\alpha}$ in the Fourier space, which makes (1-1) a system of pseudodifferential equations.

It is well known that the hyperdissipative Navier–Stokes equations (1-1) are globally well-posed for $\alpha \geq \frac{5}{4}$, which was proved by Lions [1969]; see also [Tao 2009]. The question of well-posedness for $\alpha < \frac{5}{4}$, including the case $\alpha = 1$ of the classical Navier–Stokes equations, remains open.

The first partial regularity result for the hyperdissipative (1-1) model was given by Katz and Pavlović [2002], who proved that the Hausdorff dimension of the singular set in space at the first blow-up time of a local-in-time strong solution is bounded by $5 - 4\alpha$ for $\alpha \in (1, \frac{5}{4})$. Recently Colombo et al. [2020] showed that if $\alpha \in (1, \frac{5}{4}]$, u is a suitable weak solution of (1-1) on $\mathbb{R}^3 \times (0, \infty)$ and

$$S' := \{(x, t) : u \text{ is unbounded in every neighbourhood of } (x, t)\}$$

denotes the singular set in space-time then $\mathcal{P}^{5-4\alpha}(S') = 0$, where \mathcal{P}^s denotes the s -dimensional parabolic Hausdorff measure. This is a stronger result than that of [Katz and Pavlović 2002] since it is concerned with the space-time singular set S' (rather than the singular set in space at the first blow-up), it is a statement about the Hausdorff measure of the singular set (rather than merely the Hausdorff dimension) and it includes the case $\alpha = \frac{5}{4}$ (in which case the statement, $\mathcal{P}^0(S') = 0$, means that the singular set is in fact empty, and so (1-1) is globally well-posed). The main ingredient of the notion of a “suitable weak

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solution” in the approach of [Colombo et al. 2020] is a local energy inequality, which is a generalisation of the classical local energy inequality in the Navier–Stokes equations (i.e., when $\alpha = 1$) to the case $\alpha \in (1, \frac{5}{4})$. The fractional Laplacian $(-\Delta)^\alpha$ is incorporated in the local energy inequality using a version of the extension operator introduced in [Caffarelli and Silvestre 2007]; see also [Yang 2013; Kwon and Ożański 2022; Colombo et al. 2020, Theorem 2.3]. Colombo et al. [2020] also showed a bound on the box-counting dimension of the singular set

$$d_B(S' \cap (\mathbb{R}^3 \times [t, \infty))) \leq \frac{1}{3}(-8\alpha^2 - 2\alpha + 15) \quad (1-2)$$

for every $t > 0$. We note that this bound reduces to 0 at $\alpha = \frac{5}{4}$ and converges to $\frac{5}{3}$ as $\alpha \rightarrow 1^+$, which is the bound that one can deduce from the classical result of [Caffarelli et al. 1982]; see [Robinson and Sadowski 2007] or Lemma 2.3 in [Ożański 2019] for a proof. We note that this bound (for the Navier–Stokes equations) has recently been improved by [Wang and Yang 2019] (to the bound $d_B(S) \leq \frac{7}{6}$).

Here, we build on the work of [Katz and Pavlović 2002], as their ideas offer an entirely different viewpoint on the theory of partial regularity of the Navier–Stokes equations (or the Navier–Stokes equations with hyper- and hypodissipation), as compared to the early work of Scheffer [1976a; 1976b; 1977; 1978; 1980] and the celebrated result of [Caffarelli et al. 1982], as well as alternative approaches of [Vasseur 2007; Lin 1998; Ladyzhenskaya and Seregin 1999] and numerous extensions of the theory, such as [Colombo et al. 2020; Tang and Yu 2015; Kwon and Ożański 2022]. Instead it is concerned with the dynamics (in time) of energy packets that are localised both in the frequency space and the real space \mathbb{R}^3 , and with studying how these packets move in space, as well as transfer the energy between the high and low frequencies. An important concept in this approach is the so-called *barrier* (see (3-23)), which, in a sense, quarantines a fixed region in space in a way that prevents too much energy flux entering the region. This property is essential in showing regularity at points outside of the singular set.

In order to state our results, we will say that u is a (global-in-time) *Leray–Hopf weak solution* of (1-1) if

- (i) it satisfies the equations in a weak sense, namely

$$\int_0^t \int (-u\varphi_t + (-\Delta)^{\alpha/2}u \cdot (-\Delta)^{\alpha/2}\varphi + (u \cdot \nabla)u \cdot \varphi) = \int u_0 \cdot \varphi - \int u(t) \cdot \varphi(t)$$

holds for all $t > 0$ and all $\varphi \in C_0^\infty([0, \infty) \times \mathbb{R}^3; \mathbb{R}^3)$, with $\operatorname{div} \varphi(s) = 0$ for all $s \geq 0$ (where we wrote $\int \equiv \int_{\mathbb{R}^3}$ for brevity),

- (ii) the strong energy inequality,

$$\frac{1}{2}\|u(t)\|^2 + \int_s^t \|(-\Delta)^{\alpha/2}u(\tau)\|^2 d\tau \leq \frac{1}{2}\|u(s)\|^2 \quad (1-3)$$

holds for almost every $s \geq 0$ (including $s = 0$) and every $t > s$. Here $\|\cdot\|$ denotes the $\|\cdot\|_{L^2(\mathbb{R})}$ norm.

We note that Leray–Hopf weak solutions admit *intervals of regularity*; namely for every Leray–Hopf weak solution there exists a family of pairwise disjoint intervals $(a_i, b_i) \subset (0, \infty)$ such that u coincides with some strong solution of (1-1) on each interval and

$$\mathcal{H}^{(5-4\alpha)/2\alpha} \left(\mathbb{R} \setminus \bigcup_i (a_i, b_i) \right) = 0; \quad (1-4)$$

see Theorem 2.6 and Lemma 4.1 in [Jiu and Wang 2014] for a proof. This is a generalisation of the corresponding statement in the case $\alpha = 1$ (i.e., in the case of the Navier–Stokes equations); see Section 6.4.3 in [Ożański and Pooley 2018] and Chapter 8 in [Robinson et al. 2016].

Given $u_0 \in L^2(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$ there exists at least one global-in-time Leray–Hopf weak solution (see Theorem 2.2 in [Colombo et al. 2020], for example). We denote by S the singular set in space of u at single blow-up times, namely

$$S := \bigcup_i S_i, \tag{1-5}$$

where

$$S_i := \left\{ x \in \mathbb{R}^3 : u \text{ is unbounded in } U \times \left(\frac{1}{2}(a_i + b_i), b_i \right) \text{ for any neighbourhood } U \text{ of } x \right\}$$

denotes the singular set. In particular, if $x \notin S$ then $\limsup_{t \rightarrow b_i^-} \|u(t)\|_{L^\infty(U)} \leq c_i$ for every i and $U \ni x$. The first of our main results is the following.

Theorem 1.1. *Let u be a Leray–Hopf weak solution of (1-1) with $\alpha \in (1, \frac{5}{4})$ and an initial condition $u_0 \in H^1(\mathbb{R}^3)$, and let $\varepsilon > 0$. There exists $C > 0$ and a family of collections \mathcal{B}_j of cubes $Q \subset \mathbb{R}^3$ of sidelength $2^{-j(1+\varepsilon)}$ such that*

$$\#\mathcal{B}_j \leq C 2^{j(5-4\alpha+\varepsilon)}$$

for each $j \in \mathbb{N}$, and

$$S \subset \limsup_{j \rightarrow \infty} \bigcup_{Q \in \mathcal{B}_j} Q. \tag{1-6}$$

In particular, $d_H(S) \leq 5 - 4\alpha$.

Here d_H stands for the Hausdorff dimension, and we recall that $\limsup_{j \rightarrow \infty} G_j := \bigcap_{k \geq 0} \bigcup_{j \geq k} G_j$ denotes the set of points belonging to infinitely many G_j 's. It is well known (see Lemma 3.1 in [Katz and Pavlović 2002], for example) that (1-6) implies that $d_H(S) \leq 5 - 4\alpha + \varepsilon$, from which the last claim of the theorem follows by sending $\varepsilon \rightarrow 0$.

We note that C might depend on ε , but it does not depend on the interval of regularity (a_i, b_i) , which gives us a control of the structure of the singular sets S_i that is uniform across blow-ups in time of a Leray–Hopf weak solution. This is an improvement of the result of Katz and Pavlović [2002], who obtained such control for a given strong solution, and so for each interval of regularity (a_i, b_i) of a Leray–Hopf weak solution their result implies existence of $C_i > 0$ such that $S_i \subset \limsup_{j \rightarrow \infty} \bigcup_{Q \in \mathcal{B}_j^{(i)}} Q$ for some collections $\mathcal{B}_j^{(i)}$ of cubes of sidelength $2^{-j(1+\varepsilon)}$ satisfying $\mathcal{B}_j^{(i)} \leq C_i 2^{j(5-4\alpha+\varepsilon)}$ for all j . One could therefore expect that the constants C_i become unbounded as i varies (for example in a scenario of a limit point of the set of blow-up times $\{b_i\}$), and Theorem 1.1 shows that it does not happen.

We note, however, that Theorem 1.1 does not estimate the dimension of the singular set at the blow-up time which is not an endpoint b_i of an interval of regularity (but instead a limit of a sequence of such b_i 's). In other words, if $x \notin S$, $U \ni x$ is a small open neighbourhood of x and $\{(a_i, b_i)\}_i$ is a collection of consecutive intervals of regularity of u , we show that $\sup_{U \times ((a_i+b_i)/2, b_i)} |u| = c_i < \infty$, but our result does not exclude the possibility that $c_i \rightarrow \infty$ as $i \rightarrow \infty$. It also does not imply boundedness of $|u(t)|$ at times t near the left endpoint a_i of any interval of regularity (a_i, b_i) . These issues are related to the

fact that inside the barrier we still have to deal with infinitely many energy packets (i.e., infinitely many frequencies and cubes in \mathbb{R}^3). Thus, supposing that the estimate on the energy packets inside the barrier breaks down at some t , we are unable to localize the packet (i.e., the frequency and the cube) on which the growth occurs near t , unless t is located inside an interval of regularity; see Step 1 of the proof of Theorem 3.7 for details.

The proof of Theorem 1.1 is inspired by the strategy of the proof of [Katz and Pavlović 2002], which we extend to the case of Leray–Hopf weak solutions and we use a more robust main estimate. The main estimate controls the time derivative of the L^2 norm of the Littlewood–Paley projection $P_j u$ combined with a cut-off in space (the *energy packet*); see (3-2). We show that such norm is continuous in time (regardless of putative singularities of a Leray–Hopf weak solution), which makes the main estimate valid for all $t > 0$. Inspired by [Katz and Pavlović 2002], we then define *bad cubes* and *good cubes* (see (3-15)) and show that we have a certain more-than-critical decay on a cube that is good and has some good ancestors. We then construct \mathcal{B}_j as a certain cover of bad cubes and prove (1-6).

Our second main result is concerned with the box-counting dimension. We let

$$S^{(k)} := \bigcup_{i \leq k} S_i. \quad (1-7)$$

Theorem 1.2. *Let u be as in Theorem 1.1. Then $d_B(S^{(k)}) \leq \frac{1}{3}(-16\alpha^2 + 16\alpha + 5)$ for every $k \in \mathbb{N}$.*

We prove the theorem by sharpening the argument outlined below Theorem 1.1. We recall that the box-counting dimension d_B is concerned with covering the given set by a collection of balls of radius r ,

$$d_B(K) := \limsup_{r \rightarrow 0} \frac{\log N(K, r)}{-\log r}, \quad (1-8)$$

where $N(K, r)$ denotes the minimal number of balls (or boxes) of radius r required to cover K . In this context, one can actually use the families \mathcal{B}_j from (1-6) to deduce that $d_B(S^{(k)}) \leq \frac{1}{9}(-64\alpha^3 + 96\alpha^2 - 48\alpha + 35)$ for every k , which we discuss in detail in Section 4. This is however a worse estimate than claimed in Theorem 1.2.

In fact, in Section 4 we improve this estimate by constructing refined families \mathcal{C}_j that, in a sense, give a more robust control of the low modes, which reduces the number of cubes required to cover the singular set and hence improve the bound on $d_B(S^{(k)})$. See the informal discussion following Proposition 4.1 for more insight about this improvement.

We note that we can only estimate $d_B(S^{(k)})$ (rather than $d_B(S)$) because of the localisation issue described above. To be more precise, for each sufficiently small $\delta > 0$ we can construct a family of cubes of sidelength $\delta > 0$ that covers the singular set when t approaches a singular time, and that has cardinality less than or equal to $\delta^{(-16\alpha^2 + 16\alpha + 5)/3 + \varepsilon}$ for any given $\varepsilon > 0$. This family can be constructed independently of the interval of regularity, but given x outside of this family we can show that the solution is bounded in a neighbourhood of x if the choice of (sufficiently small) δ is dependent on the interval of regularity. This gives the limitation to only finite number of intervals of regularity in the definition of $S^{(k)}$.

We note that the result of [Colombo et al. 2020] is stronger than our result in the sense that it is concerned with the space-time singular set S' (rather than the singular set S in space), it is concerned

with the parabolic Hausdorff measure of S' (rather than merely the bound on $d_H(S')$) and its estimate of $d_B(S')$ is sharper than our estimate on $d_B(S_k)$.

However, our result is stronger than [Colombo et al. 2020] in the sense that it applies to any Leray–Hopf weak solutions (rather than merely suitable weak solutions). In other words we do not use the local energy inequality, which is the main ingredient of [Colombo et al. 2020]. Also, our approach does not include any estimates of the pressure function. In fact we only consider the Leray projection of the first equation in (1-1), which eliminates the pressure. Furthermore, our approach can be thought of as an extension of the global regularity of (1-1) for $\alpha > \frac{5}{4}$. In fact, the following corollary can be proved almost immediately using our main estimate; see Section 3F.

Corollary 1.3. *If $\alpha > \frac{5}{4}$ then (1-1) is globally well-posed.*

We also point out that our estimate on the box-counting dimension, $d_B(S_k) \leq \frac{1}{3}(-16\alpha^2 + 16\alpha + 5)$, converges to $\frac{5}{3}$ as $\alpha \rightarrow 1^+$, just as (1-2).

Finally, we also correct a number of imprecisions appearing in [Katz and Pavlović 2002]; see for example Remark 3.4 and Step 1 of the proof of Theorem 3.7.

The structure of the article is as follows. In Section 2 we introduce some preliminary concepts including the Littlewood–Paley projections, paraproduct decomposition, and Bernstein inequalities, as well as a number of analytic tools that allow us to manipulate quantities involving cut-offs in both the real space and the Fourier space, which includes estimates of the errors when one moves a Littlewood–Paley projection across spatial cut-offs and vice versa. We prove the first result, Theorem 1.1, in Section 3. We prove Corollary 1.3 in Section 3F and we prove the second result, Theorem 1.2, in Section 4.

2. Preliminaries

Unless specified otherwise, all function spaces are considered on the whole space \mathbb{R}^3 . In particular $L^2 := L^2(\mathbb{R}^3)$. We do not use the summation convention. We will write $\partial_i := \partial_{x_i}$, $B(R) := \{x \in \mathbb{R}^3 : |x| \leq R\}$, $\int := \int_{\mathbb{R}^3}$, and $\|\cdot\|_p := \|\cdot\|_{L^p(\mathbb{R}^3)}$. We reserve the notation “ $\|\cdot\|$ ” for the L^2 norm, that is, $\|\cdot\| := \|\cdot\|_2$.

We denote any positive constant by c (whose value may change at each appearance). We point out that c might depend on u_0 and α , which we consider fixed throughout the article. As for the constants dependent on some parameters, we sometimes emphasise the parameters by using subscripts. For example, $c_{k,q}$ is any constant dependent on k and q .

We denote by $e(j)$ (a j -negligible error) any quantity that can be bounded (in absolute value) by $c_K 2^{-Kj}$ for any given $K > 0$.

We say that a differential inequality $f' \leq g$ on a time interval I is satisfied in the *integral sense* if

$$f(t) \leq f(s) + \int_s^t g(\tau) \, d\tau \quad \text{for every } t, s \in I \text{ with } t > s. \tag{2-1}$$

We recall that Leray–Hopf weak solutions are weakly continuous with values in L^2 . Indeed, it follows from part (i) of the definition that

$$\int u(t)\varphi \text{ is continuous for every } \varphi \in C_0^\infty(\mathbb{R}^3) \text{ with } \operatorname{div} \varphi = 0.$$

This is also true if $\operatorname{div} \varphi \neq 0$, as in this case one can apply the Helmholtz decomposition to write $\varphi = \phi + \nabla \psi$, where $\operatorname{div} \phi = 0$ (then $\int u(t)\phi$ is continuous and $\int u(t)\nabla \psi = 0$ since $u(t)$ is divergence-free). Thus, since part (ii) gives that $\{u(t)\}_{t \geq 0}$ is bounded in L^2 , weak continuity of $u(t)$ follows.

2A. Littlewood–Paley projections. Given $f \in L^1(\mathbb{R}^3)$, we denote by \hat{f} its Fourier transform, i.e.,

$$\hat{f}(\xi) := \int f(x)e^{-2\pi i x \cdot \xi} dx, \quad \xi \in \mathbb{R}^3,$$

and by \check{f} its inverse Fourier transform, i.e., $\check{f}(x) := \hat{f}(-x)$. Let $h \in C^\infty(\mathbb{R}; [0, 1])$ be any function such that $h(x) = 1$ for $x < 1$ and $h(x) = 0$ for $x > 2$. We set $p(x) := h(|x|) - h(2|x|)$, where $x \in \mathbb{R}^3$, we let

$$p_j(\xi) := p(2^{-j}\xi) \quad \text{for } j \in \mathbb{Z}, \tag{2-2}$$

and we let P_j (the j -th Littlewood–Paley projection) be the corresponding multiplier operator, that is,

$$\widehat{P_j f}(\xi) := p_j(\xi)\hat{f}(\xi).$$

By construction, $\operatorname{supp} p_j \subset B(2^{j+1}) \setminus B(2^{j-1})$. We note that $\sum_{j \in \mathbb{Z}} p_j = 1$, and so formally $\sum_{j \in \mathbb{Z}} P_j = \operatorname{id}$. We also define

$$\tilde{P}_j \equiv P_{j \pm 2} := \sum_{k=j-2}^{j+2} P_k, \quad P_{j-4, j+2} := \sum_{k=j-4}^{j+2} P_k, \quad P_{\leq j} := \sum_{k=-\infty}^j P_k, \quad P_{\geq j} := \sum_{k=j}^{\infty} P_k, \tag{2-3}$$

and analogously for $\check{p}_j, p_{j-4, j+2}, p_{\leq j}, p_{\geq j}$. By a direct calculation one obtains that

$$\check{p}_j(y) = 2^{3j}\check{p}(2^j y) \tag{2-4}$$

for all $j \in \mathbb{Z}, y \in \mathbb{R}^3$. In particular $\|\check{p}_j\|_1 = c$ and so, since $P_j f = \check{p}_j * f$ (where “ $*$ ” denotes the convolution), Young’s inequality for convolutions gives

$$\|P_j u\|_q \leq c\|u\|_q \tag{2-5}$$

for any $q \in [1, \infty]$. Moreover, given $K > 0$ there exists $c_K > 0$ such that

$$|\check{p}_j(y)| \leq c_K(2^j|y|)^{-2K}2^{3j}, \tag{2-6}$$

$$|\partial_i \check{p}_j(y)| \leq c_K(2^j|y|)^{-2K}2^{4j} \tag{2-7}$$

for all $j \in \mathbb{Z}, y \neq 0$ and $i = 1, 2, 3$. Indeed, the case $j = 0$ follows by noting that

$$e^{2\pi i y \cdot \xi} = (-4\pi^2|y|^2)^{-K} \Delta_\xi^K e^{2\pi i y \cdot \xi}$$

and calculating

$$|\check{p}(y)| = \left| \int p(\xi)e^{2\pi i y \cdot \xi} d\xi \right| = (4\pi^2|y|^2)^{-K} \left| \int \Delta^K p(\xi)e^{2\pi i y \cdot \xi} d\xi \right| \leq c_K|y|^{-2K} \int_{B(2)} |\Delta^K p| = c_K|y|^{-2K}$$

(and similarly $|\partial_i \check{p}(y)| \leq c_K|y|^{-2K}$), where we have integrated by parts $2K$ times, and the case $j \neq 0$ follows from (2-4). Using (2-6) and (2-7) we also get

$$\|\check{p}_j\|_{L^q(B(d)^c)} \leq C_{K,q}(d2^j)^{-2K+3/q}2^{3j(q-1)/q} \tag{2-8}$$

and

$$\|\partial_i \check{P}_j\|_{L^q(B(d)^c)} \leq C_{K,q} (d2^j)^{-2K+3/q} 2^{j(1+n(q-1)/q)}, \tag{2-9}$$

respectively, for any $K > 0, d > 0, i = 1, 2, 3, j \in \mathbb{Z}$ and $q \geq 1$. Indeed

$$\int_{\mathbb{R}^3 \setminus B(d)} |\check{P}_j(y)|^q dy \leq C_{K,q} 2^{-jq(2K-3)} \int_{|y| \geq d} |y|^{-2Kq} dy = C_{K,q} 2^{-jq(2K-3)} d^{-2Kq+3},$$

from which (2-8) follows (and (2-9) follows analogously). We note that the same is true when p is replaced by any compactly supported multiplier.

Corollary 2.1. *Let $\lambda \in C_0^\infty(\mathbb{R}^3)$ and, given $j \in \mathbb{Z}$, set $\lambda_j(\xi) := \lambda(2^{-j}\xi)$. Then, given $d > 0$,*

$$\|\check{\lambda}_j\|_{L^2(\mathbb{R}^3 \setminus B(d))} \leq c_K 2^{-j(2K-3)} d^{-2K+3/2}.$$

We will denote by T the Leray projection, that is,

$$\widehat{Tf}(\xi) := \left(I - \frac{\xi \otimes \xi}{|\xi|^2} \right) \hat{f}, \tag{2-10}$$

where $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, and I denotes the 3×3 identity matrix.

2B. Bernstein inequalities. Here we point out classical Bernstein inequalities on \mathbb{R}^3 :

$$\|P_j f\|_q \leq c 2^{3j(1/p-1/q)} \|P_j f\|_p, \tag{2-11}$$

$$\|P_{\leq j} f\|_q \leq c 2^{3j(1/p-1/q)} \|P_{\leq j} f\|_p \tag{2-12}$$

for any $1 \leq p \leq q \leq \infty$. We refer the reader to Lemma 2.1 of [Bahouri et al. 2011] for a proof.

2C. The paraproduct formula. Here we briefly describe the Bony decomposition formula, that is, we concern ourselves with a structure of a Littlewood–Paley projection of a product of two functions, $P_j(fg)$. One could obviously write $f = \sum_{k \in \mathbb{Z}} P_k f$ (and similarly for g) to obtain that

$$P_j(fg) = P_j \left(\sum_{k,m \in \mathbb{Z}} P_k f P_m g \right). \tag{2-13}$$

However, since functions p_j, p_k have pairwise disjoint supports for many pairs $j, k \in \mathbb{Z}$, one could speculate that some of the terms on the right-hand side of (2-13) vanish. This is indeed the case and

$$\begin{aligned} P_j(fg) &= P_j \left(P_{j \pm 2} f P_{\leq j-5} g + P_{\leq j-5} f P_{j \pm 2} g + P_{j-4, j+2} f P_{j \pm 4} g + \sum_{k \geq j+3} P_k f P_{k \pm 2} g \right) \\ &= P_j(K_{\text{loc,low}} + K_{\text{low,loc}} + K_{\text{loc}} + K_{\text{hh}}), \end{aligned} \tag{2-14}$$

which is also known as *Bony’s decomposition formula*. For the sake of completeness we prove the formula below. Heuristically speaking, $K_{\text{loc,low}}$ corresponds to interactions between local (i.e., around j) modes of f and low modes of g , $K_{\text{low,loc}}$ to interactions between low modes of f and local modes of g , K_{loc} to local interactions and K_{hh} to interactions between high modes; see Figure 1 for a geometric interpretation of (2-14). We now prove (2-14). For this it is sufficient to show that

$$P_j(P_k f P_m g) = 0 \quad \text{for } (k, m) \in R_1 \cup R_2 \cup R_3, \tag{2-15}$$

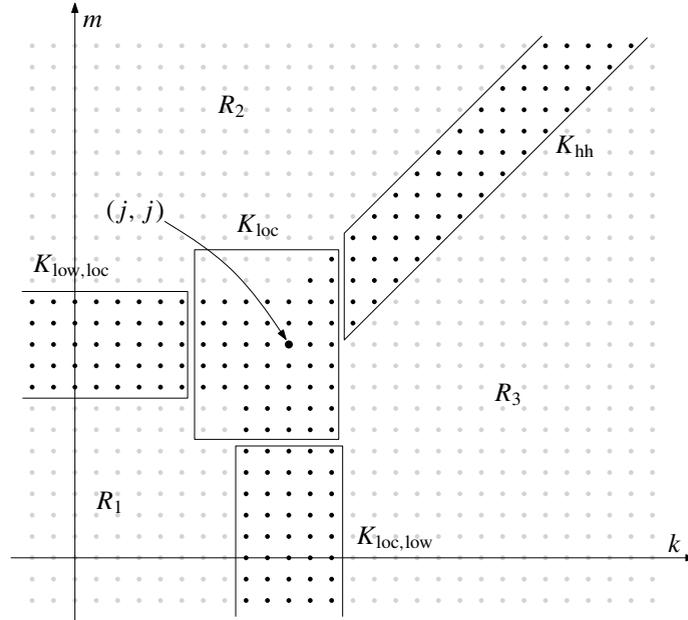


Figure 1. Sketch of the interpretation of the terms on the right-hand side of (2-14). The regions R_1, R_2, R_3 (consisting of grey dots) correspond to pairs (k, m) for which $P_j(P_k f P_m g)$ vanishes; see the discussion following (2-15).

where R_1, R_2, R_3 are as sketched in Figure 1. The Fourier transform of $w := P_j(P_k f P_m g)$ is

$$\hat{w}(\xi) = p_j(\xi) \int p_k(\eta) \hat{f}(\eta) p_m(\xi - \eta) \hat{g}(\xi - \eta) d\eta.$$

We can assume that $|\xi| \in (2^{j-1}, 2^{j+1})$ (as otherwise $p_j(\xi)$ vanishes) and that $|\eta| \in (2^{k-1}, 2^{k+1})$ (as otherwise $p_k(\eta)$ vanishes).

Case 1: $(k, m) \in R_1$. Suppose that $k \geq m$ (the opposite case is analogous). Then $j \geq k + 3$ (see Figure 1) and so

$$|\xi - \eta| \geq |\xi| - |\eta| \geq 2^{j-1} - 2^{k+1} \geq 2^{k+2} - 2^{k+1} = 2^{k+1} \geq 2^{m+1}.$$

Thus $p_m(\xi - \eta)$ vanishes.

Case 2: $(k, m) \in R_2 \cup R_3$. Suppose that $(k, m) \in R_2$ (the case $(k, m) \in R_3$ is analogous). Then $m \geq k + 3$ and $m \geq j + 3$ (see Figure 1) and so

$$|\xi - \eta| \leq |\xi| + |\eta| \leq 2^{j+1} + 2^{k+1} \leq 2 \cdot 2^{m-2} = 2^{m-1}.$$

Hence $p_m(\xi - \eta)$ vanishes as well, and so (2-15) follows.

2D. Moving bump functions across Littlewood–Paley projections. Here we show the following:

Lemma 2.2. *Let $\phi_1, \phi_2 : \mathbb{R}^3 \rightarrow [0, 1]$ be such that their supports are separated by at least $d > 2^{-j}$. Then*

$$\|\phi_1 P_j(\phi_2 f)\|_q \leq c_K (d2^j)^{-2K+3} \|f\|_q$$

for all $q \in [1, \infty]$, $j \in \mathbb{Z}$, $K > 0$ and $f \in L^q(\mathbb{R}^3)$. Furthermore, if $|\nabla\phi_2| \leq c d^{-1}$ then

$$\|\phi_1 P_j(\phi_2 \nabla f)\|_q \leq c_K (d2^j)^{-2K+3} 2^j \|f\|_q.$$

We will only use the lemma (and the corollary below) with $q = 2$ or $q = 1$.

Proof. We note that

$$\begin{aligned} \phi_1 P_j(\phi_2 f)(x) &= \phi_1(x) \int_{\text{supp } \phi_2} \check{p}_j(x-y) \phi_2(y) f(y) \, dy \\ &= \phi_1(x) \int_{\text{supp } \phi_2} \chi_{|x-y|>d} \check{p}_j(x-y) \phi_2(y) f(y) \, dy \end{aligned} \tag{2-16}$$

since the supports of ϕ_1, ϕ_2 are at least d apart. Thus using Young’s inequality for convolutions

$$\|\phi_1 P_j(\phi_2 f)\|_q \leq \|\check{p}_j\|_{L^1(B(d)^c)} \|\phi_2 f\|_q \leq c_K (d2^j)^{-2K+3} \|f\|_q$$

for any $K > 0$, where we used (2-8). This shows the first claim of the lemma. The second claim follows by replacing f by ∇f in (2-16), integrating by parts, and using Young’s inequality for convolutions to give

$$\begin{aligned} \|\phi_1 P_j(\phi_2 \nabla f)\|_q &\leq c \|\nabla \check{p}_j\|_{L^1(B(d)^c)} \|\phi_2 f\|_q + \|\check{p}_j\|_{L^1(B(d)^c)} \|\nabla \phi_2 f\|_q \\ &\leq c_K (d2^j)^{-2K+3} 2^j \|f\|_q, \end{aligned}$$

where we also used the assumption that $|\nabla\phi_2| \leq c d^{-1} < c 2^j$. □

In fact the same result is valid when P_j is replaced by the composition of P_j with any 0-homogeneous multiplier (e.g., the Leray projector).

Corollary 2.3. *Let M be a bounded, 0-homogeneous multiplier (i.e., $\widehat{Mf}(\xi) = m(\xi) \hat{f}(\xi)$, where $m(\lambda\xi) = m(\xi)$ for any $\lambda > 0$). Let $\phi_1, \phi_2 : \mathbb{R}^3 \rightarrow [0, 1]$ be such that their supports are separated by at least $d > 2^{-j}$. Then*

$$\|\phi_1 M P_j(\phi_2 \nabla f)\|_q \leq c_K (d2^j)^{-2K+3} 2^j \|f\|_q$$

for all $q \in [1, \infty]$, $j \in \mathbb{Z}$, $K > 0$ and $f \in L^q(\mathbb{R}^3)$.

2E. Moving Littlewood–Paley projections across spatial cut-offs. We say that $\phi \in C_0^\infty(\mathbb{R}^3)$ is a d -cutoff if $\text{diam}(\text{supp } \phi) \leq c d$ and $|D^l \phi| \leq c_l d^{-l}$ for any $l \geq 0$.

We denote by $e_d(j)$ any quantity that can be bounded (in absolute value) by $c_K 2^{cj} (d2^j)^{c-K}$ for any given $K > 0$. The point of such notation is that it will articulate the dependence of the size of the error in our main estimate (see Proposition 3.1) on both j and d .

In this section we show that, roughly speaking, we can move Littlewood–Paley projections P_j across d -cutoffs as long as $d > 2^{-j}$.

Lemma 2.4. *Given a d -cutoff ϕ , $q \in [1, \infty]$ and multiindices α, β , with $|\beta|, |\alpha| \leq 3$,*

$$\|(1 - \tilde{P}_j) D^\alpha (\phi P_j D^\beta f)\|_q \leq e_d(j) \|f\|_q$$

for every j .

Proof. We write $\phi = \phi_1 + \phi_2$, where

$$\begin{aligned}\hat{\phi}_1(\xi) &:= \chi_{|\xi| \leq 2^{j-2}} \hat{\phi}(\xi), \\ \hat{\phi}_2(\xi) &:= \chi_{|\xi| > 2^{j-2}} \hat{\phi}(\xi).\end{aligned}$$

Note that

$$\widehat{(\phi_1 P_j D^\beta f)}(\xi) = \int \hat{\phi}_1(\xi - \eta) p_j(\eta) (2\pi i)^{|\beta|} \eta^\beta \hat{f}(\eta) d\eta$$

is supported in $|\xi| \in (2^{j-2}, 2^{j+2})$ (as $\hat{\phi}_1(\xi - \eta)$ is supported in $\{|\xi - \eta| \leq 2^{j-2}\}$ and $p_j(\eta)$ is supported in $\{2^{j-1} < |\eta| < 2^{j+1}\}$). Since $\tilde{p}_j(\xi) = 1$ for such ξ , we obtain

$$\phi_1 P_j D^\beta f = \tilde{P}_j \phi_1 P_j D^\beta f, \quad (2-17)$$

and so it suffices to show that

$$\|(1 - \tilde{P}_j) D^\alpha (\phi_2 P_j D^\beta f)\|_q \leq e_d(j) \|f\|_q.$$

We will show that

$$\|\widehat{D^\alpha \phi_2}\|_1 \leq e_d(j) \quad (2-18)$$

for every $|\alpha| \leq 3$. Then the claim follows by writing

$$\begin{aligned}\|(1 - \tilde{P}_j) D^\alpha (\phi_2 P_j D^\beta f)\|_q &\leq \sum_{\alpha_1 + \alpha_2 = \alpha} \|D^{\alpha_1} \phi_2 P_j D^{\alpha_2 + \beta} f\|_q \\ &\leq \sum_{\alpha_1 + \alpha_2 = \alpha} \|D^{\alpha_1} \phi_2\|_\infty \|P_j D^{\alpha_2 + \beta} f\|_q \\ &\leq \sum_{|\alpha_1| \leq 3} \|\widehat{D^{\alpha_1} \phi_2}\|_1 \cdot 2^{6j} \|f\|_q \leq e_d(j) \|f\|_q.\end{aligned}$$

In order to see (2-18) we first note that

$$\begin{aligned}|\widehat{D^\alpha \phi_2}(\xi)| &\leq c |\xi|^{|\alpha|} \left| \int \phi_2(x) e^{-2\pi i x \cdot \xi} dx \right| \\ &= c |\xi|^{|\alpha|} (4\pi^2 |\xi|^2)^{-K} \left| \int \phi_2(x) \Delta^K e^{-2\pi i x \cdot \xi} dx \right| \\ &= c |\xi|^{|\alpha|} (4\pi^2 |\xi|^2)^{-K} \left| \int \Delta^K \phi_2(x) e^{-2\pi i x \cdot \xi} dx \right| \leq c_K |\xi|^{-2K + |\alpha|} d^{-2K+3}.\end{aligned}$$

Thus

$$\|\widehat{D^\alpha \phi_2}\|_1 = c \int_{|\xi| > 2^{j-2}} |\widehat{D^\alpha \phi_2}(\xi)| \leq c_K d^{-2K+3} \int_{|\xi| > 2^{j-2}} |\xi|^{-2K + |\alpha|} = c_K 2^{3j} (d 2^j)^{-2K+3},$$

which gives (2-18). \square

Similarly one can put the Littlewood–Paley projection “inside the cutoff”. In this case one can prove a statement similar to Lemma 2.4, but, since we will only need a version with no derivatives, we state a simplified statement.

Corollary 2.5. *Given a d -cutoff ϕ , $\|P_j(\phi(1 - P_{j\pm 2})f)\| \leq e_d(j)$ for every j .*

Proof. The claim follows using the same decomposition as above, $\phi = \phi_1 + \phi_2$. Since

$$\widehat{\phi(1 - P_{j\pm 2})f}(\xi) = \int \hat{\phi}_1(\xi - \eta)(1 - p_{j\pm 2}(\eta))\hat{f}(\eta) \, d\eta,$$

we see that (since $|\eta| \in (-\infty, 2^{j-2}) \cup (2^{j+2}, \infty)$) either $|\xi| \geq |\eta| - |\xi - \eta| \geq 2^{j+2} - 2^{j-2} \geq 2^{j+1}$ or $|\xi| \leq |\eta| + |\xi - \eta| \leq 2^{j-2} + 2^{j-2} = 2^{j-1}$. In any case $p_j(\xi) = 0$ and so $P_j(\phi_1(1 - \tilde{P}_j)f) = 0$. The part involving ϕ_2 can be estimated by $e_d(j)$ using the same argument as above. \square

2F. Cubes. We denote by Q any open cube in \mathbb{R}^3 . Given $a > 1$, we denote by aQ the cube with the same centre as Q and a times larger sidelength. We sometimes write $Q(x)$ to emphasise that cube Q is centred at a point $x \in \mathbb{R}^3$. Given an open cube Q of sidelength $d > 0$, we let $\phi_Q \in C_0^\infty(\mathbb{R}^3; [0, 1])$ be a d -cutoff such that

$$\phi_Q = 1 \text{ on } Q, \quad \text{supp } \phi_Q \subset \frac{7}{6}Q, \quad \text{and} \quad \|\nabla^k \phi_Q\|_\infty \leq C_k d^{-k}. \tag{2-19}$$

Note that

$$|\xi|^k |\hat{\phi}_Q(\xi)| \leq c_k d^{3-k} \quad \text{for } \xi \in \mathbb{R}^3, \tag{2-20}$$

which can be shown by a direct computation.

2G. Localised Bernstein inequalities. If Q is a cube of sidelength $d > 2^{-j}$ then

$$\|\phi_Q P_j f\|_q \leq c 2^{3j(1/2-1/q)} \|\phi_Q P_j f\| + e_d(j) \|f\|_q, \tag{2-21}$$

due to Lemma 2.4 and the classical Bernstein inequality (2-11).

2H. Absolute continuity. Here we state two lemmas that will help us (in Step 1 of the proof of Proposition 3.1) in proving the main estimate for Leray–Hopf weak solutions.

Lemma 2.6. *Suppose that $f : [a, b] \rightarrow \mathbb{R}$ is continuous and such that $f' \in L^1(a, b)$. Then*

$$f(t) = f(s) + \int_s^t f'(\tau) \, d\tau$$

for every $s, t \in (a, b)$.

Proof. This is elementary. \square

Lemma 2.7. *If $u(x, t)$ is weakly continuous in time on an interval (a, b) with values in $L^2(\mathbb{R}^3)$ then $P_j u$ is strongly continuous in time into $L^2(\Omega)$ on (a, b) for any bounded domain $\Omega \subset \mathbb{R}^3$.*

Proof. We note that

$$\|P_j u(t) - P_j u(s)\|_{L^2(\Omega)}^2 = \int_\Omega \left| \int \check{p}_j(x - y)(u(y, t) - u(y, s)) \, dy \right|^2 \, dx.$$

Weak continuity of $u(t)$ gives that the integral inside the absolute value converges to 0 as $t \rightarrow s$ (for any fixed x). Furthermore it is bounded by

$$\|\check{p}_j\| \|u(t) - u(s)\| \leq c_j,$$

where we used the Cauchy–Schwarz inequality and the fact that u is bounded in L^2 (a property of functions weakly continuous in L^2). Since the constant function c_j^2 is integrable on Ω , the claim of the lemma follows from the dominated convergence theorem. \square

3. The proof of the main result

In this section we prove Theorem 1.1; namely we will show that $d_H(S) \leq 5 - 4\alpha$, where S is the singular set in space of a Leray–Hopf weak solution (recall (1-5)). We will actually show that

$$d_H(S) \leq 5 - 4\alpha + \varepsilon$$

for any

$$\varepsilon \in (0, \min(\frac{1}{3}(4\alpha - 4), \frac{1}{20})). \tag{3-1}$$

We now fix such ε and we allow every constant (denoted by “ c ”) to depend on ε .

We say that a cube Q is a j -cube if it has sidelength $2^{-j(1-\varepsilon)}$. The reason for considering such “almost dyadic cubes” (rather than the dyadic cubes of sidelength 2^{-j}) is that $e_d(j) = e(j)$ for $d = 2^{-j(1-\varepsilon)}$ (which is not true for $d = 2^{-j}$). We say that a cover of a set is a j -cover if it consists only of j -cubes. We denote by $S_j(\Omega)$ any j -cover of Ω such that $\#S_j(\Omega) \leq c(\frac{1}{2} \text{diam}(\Omega))^{-j(1-\varepsilon)^3}$.

Moreover, given a j -cube and $k \in \mathbb{Z}$, we denote the k -cube concentric with Q by Q_k , that is,

$$Q_k := 2^{(j-k)(1-\varepsilon)} Q.$$

3A. The main estimate. Given a cube Q and $j \in \mathbb{Z}$ we let

$$u_{Q,j} := \|\phi_Q P_j u\|$$

and we write

$$u_{Q,j\pm 2} := \sum_{k=j-2}^{j+2} u_{Q,k}.$$

We point out that $u_{Q,j}$ is a function of time, which we will often skip in our notation.

We start with a derivation of an estimate for $u_{Q,j}$ for any $j \in \mathbb{Z}$ and any cube Q of sidelength $d > 16 \cdot 2^{-j}$.

Proposition 3.1 (main estimate). *Let u be a Leray–Hopf weak solution of the hyperdissipative Navier–Stokes equations (1-1) on the time interval $[0, \infty)$ and let $d > 16 \cdot 2^{-j}$. Then $u_{Q,j}$ is continuous on $[0, \infty)$ and*

$$\begin{aligned} \frac{d}{dt} u_{Q,j}^2 &\leq -c 2^{2\alpha j} u_{Q,j}^2 + c u_{Q,j} \left(2^j u_{3Q/2,j\pm 2} \sum_{\theta j \leq k \leq j-5} 2^{3k/2} u_{\max(Q_k, 3Q/2),k} \right. \\ &\quad \left. + 2^{5j/2} u_{3Q/2,j\pm 4}^2 + 2^{3j/2} \sum_{k \geq j+1} 2^k u_{3Q/2,k}^2 \right) + e_{\text{diss}} + \sum_{k \geq \theta j} e_d(k) \\ &= -G_{\text{diss}} + c u_{Q,j} (G_{\text{low,loc}} + G_{\text{loc}} + G_{\text{hh}}) + e_{\text{diss}} + e_{\text{v1}} + \sum_{k \geq \theta j} e_d(k) \end{aligned} \tag{3-2}$$

is satisfied in the integral sense (recall (2-1)) for any cube Q of side-length d and any $j \in \mathbb{Z}$, where

$$\theta := \frac{2}{3}(2\alpha - 1 - \varepsilon) \tag{3-3}$$

and

$$\begin{aligned} e_{\text{diss}} &:= c 2^{2\alpha j} (d2^j)^{-1} u_{3Q/2, j\pm 2}^2, \\ e_{\text{vl}} &:= c 2^{2\alpha j} 2^{-\varepsilon j} u_{3Q/2, j\pm 2}^2. \end{aligned}$$

Here $\max(Q_k, \frac{3}{2}Q)$ denotes the larger of the cubes $Q_k, \frac{3}{2}Q$, and G_{diss} should be thought of as the dissipation term, $G_{\text{low,loc}}$ the interaction between low (i.e., modes $k \leq j - 5$) and local modes (i.e., modes $j \pm 2$), G_{loc} the local interactions (i.e., including only the modes $j \pm 4$) and G_{hh} the interactions between high modes (i.e., modes $k \geq j$).

The role of the parameter θ is to separate the “very low” Littlewood projections from the “low” Littlewood–Paley projections. That is (roughly speaking), given $j \in \mathbb{N}$ we will not have to worry about the Littlewood–Paley projections P_k with $k < \theta j$ (i.e., they will be effortlessly absorbed by the dissipation at the price of the error term e_{vl} (“vl” here stands for “very low”); see (3-12)–(3-13) below for a detailed explanation), which is the reason why such modes are not included in $G_{\text{low,loc}}$. In fact $G_{\text{low,loc}}$ is (roughly speaking) the most dangerous term, as it represents, in a sense, the injection of energy from low scales to high scales, and we will need to use G_{diss} to counteract it; see Step 5 in the proof of Theorem 3.7.

The error term e_{diss} appearing in the estimate is the error appearing when estimating the dissipation term, and it cannot be estimated by $e_d(j)$. Its appearance is a drawback of the main estimate, but in our applications (in Theorems 3.3 and 3.7) it can be absorbed by G_{diss} .

Proof of Proposition 3.1. Recall (1-4) that a Leray–Hopf weak solution admits intervals of regularity.

Step 1: We show that it is sufficient to show (3-2) on each of the intervals of regularity.

On each interval of regularity (a, b) we apply the Leray projection (recall (2-10)) to the first equation of (1-1) to obtain

$$u_t + (-\Delta)^\alpha u + T[(u \cdot \nabla)u] = 0.$$

Multiplying by $P_j(\phi_Q^2 P_j u)$ and integrating in space we obtain (at any given time)

$$\frac{1}{2} \frac{d}{dt} u_{Q,j}^2 = - \int (-\Delta)^\alpha u P_j(\phi_Q^2 P_j u) - \int T[(u \cdot \nabla)u] P_j(\phi_Q^2 P_j u) =: I + J.$$

We note that $I, J \in L^1(0, T)$ for every $T > 0$. Indeed, by brutal estimates

$$\begin{aligned} |J| &= \left| \int \phi_Q P_j T[(u \cdot \nabla)u] \phi_Q^2 P_j u \right| \\ &\leq \|P_j T[(u \cdot \nabla)u]\|_1 \|P_j u\|_\infty \leq c \|u\| \|\nabla u\| \cdot 2^{3j/2} \|P_j u\| \leq c 2^{3j/2} \|\nabla u\| \end{aligned}$$

(where we used Bernstein inequality (2-11) in the third line), which is integrable on $(0, T)$ for every $T > 0$. That $I \in L^1(0, T)$ for every $T > 0$ is a consequence of Step 2 below. Thus, since $u(t)$ is weakly continuous with values in L^2 (recall Section 2), Lemma 2.6 gives that (3-2) is valid (in the integral sense) on $[0, \infty)$.

Thus it suffices to show that $I + J$ can be estimated by the right-hand side of (3-2).

Step 2: We show that $I \leq -G_{\text{diss}} + e_{\text{diss}} + e_d(j)$. (Note that this gives in particular that $I \in L^1(0, \infty)$, since (trivially) $u_{Q',j} \leq c$ for every cube Q' and every j .)

We write

$$\begin{aligned} I &= - \int \phi_Q (-\Delta)^\alpha P_j u \phi_Q P_j u \\ &= - \int (-\Delta)^\alpha \tilde{P}_j (\phi_Q P_j u) \phi_Q P_j u - \int (-\Delta)^\alpha (1 - \tilde{P}_j) (\phi_Q P_j u) \phi_Q P_j u - \int [\phi_Q, (-\Delta)^\alpha] P_j u \phi_Q P_j u \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

Note that, due to the Plancherel theorem

$$\begin{aligned} I_1 &= -c \int |\xi|^{2\alpha} \tilde{p}_j(\xi) |\hat{v}(\xi)|^2 d\xi \leq -c 2^{2\alpha j} \int \tilde{p}_j(\xi) |\hat{v}(\xi)|^2 d\xi \\ &= -c 2^{2\alpha j} \int \tilde{P}_j v \cdot v = -c 2^{2\alpha j} u_{Q,j}^2 + c 2^{2\alpha j} \int (1 - \tilde{P}_j) v \cdot v \\ &\leq -c 2^{2\alpha j} u_{Q,j}^2 + c 2^{2\alpha j} \|(1 - \tilde{P}_j)v\| = -G_{\text{diss}} + e_d(j), \end{aligned}$$

where we wrote $v := \phi_Q P_j u$ for brevity, and we used the fact that $\|v\| \leq c$ (recall (1-3)) in the last line, as well as Lemma 2.4 in the last equality.

Step 2.1: We show that $I_2 \leq e_d(j)$.

We write

$$I_2 \leq \|(-\Delta)^\alpha (1 - \tilde{P}_j) (\phi_Q P_j u)\| u_{Q,j},$$

and we will show that

$$\|(-\Delta)^\alpha (1 - \tilde{P}_j) (\phi_Q P_j u)\| \leq e_d(j). \quad (3-4)$$

(This completes this step as $u_{Q,j} \leq c$, as above.) Indeed, (3-4) follows in a way similar to Lemma 2.4 by taking the decomposition

$$\phi_Q = \phi_1 + \phi_2,$$

where

$$\begin{aligned} \hat{\phi}_1(\xi) &:= \chi_{|\xi| \leq 2^{j-2}} \hat{\phi}_Q(\xi), \\ \hat{\phi}_2(\xi) &:= \chi_{|\xi| > 2^{j-2}} \hat{\phi}_Q(\xi). \end{aligned}$$

We see that $\phi_1 P_j u = \tilde{P}_j (\phi_1 P_j u)$ (because of the supports in Fourier space, see (2-17)) and so it is sufficient to show that

$$\|(-\Delta)^\alpha (\phi_2 P_j u)\| \leq e_d(j)$$

(since $\|1 - \tilde{P}_j\| \leq 1$). Since the Fourier transform of $(-\Delta)^\alpha (\phi_2 P_j u)$ is

$$\begin{aligned} c|\xi|^{2\alpha} \int \hat{\phi}_2(\xi - \eta) p_j(\eta) \hat{u}(\eta) d\eta \\ \leq c \int |\xi - \eta|^{2\alpha} |\hat{\phi}_2(\xi - \eta) p_j(\eta) \hat{u}(\eta)| d\eta + c \int |\eta|^{2\alpha} |\hat{\phi}_2(\xi - \eta) p_j(\eta) \hat{u}(\eta)| d\eta, \end{aligned}$$

we obtain

$$\|(-\Delta)^\alpha (\phi_2 P_j u)\| \leq c \|u\| \int_{|\xi| > 2^{j-2}} |\xi|^{2\alpha} |\hat{\phi}_2(\xi)| d\xi + c \|\hat{\phi}_2\|_1 \|(-\Delta)^{2\alpha} P_j u\| \leq e_d(j),$$

where we used the Plancherel theorem, (2-18) and the fact that $\| |\cdot|^{2\alpha} \hat{\phi}_2(\cdot) \|_1 \leq e_d(j)$ (which follows in the same way as (2-18)).

Step 2.2: We show that $I_3 \leq e_{\text{diss}} + e_d(j)$.

We have

$$I_3 \leq \| [\phi_Q, (-\Delta)^\alpha] P_j u \|_{u_{Q,j}}.$$

For brevity we let $v := P_j(\phi_{3Q/2}u)$, $\phi := \phi_Q$ and

$$W := [\phi, (-\Delta)^\alpha]v.$$

We will show below that

$$\|W\| \leq c 2^{2\alpha j} (d 2^j)^{-1} u_{3Q/2,j} + e_d(j),$$

and we will show in Step 2.2c that

$$\|W\| = \|[\phi, (-\Delta)^\alpha] P_j u\| + e_d(j), \tag{3-5}$$

from which the claim of this step follows (and so, together with Step 2.1, finishes Step 2). Since

$$\widehat{W}(\xi) = c \int (|\eta|^{2\alpha} - |\xi|^{2\alpha}) \hat{\phi}(\xi - \eta) \hat{v}(\eta) \, d\eta,$$

we can decompose W by writing $f = \int_{|\eta-\xi| \leq 2^{j-3}} + \int_{|\eta-\xi| > 2^{j-3}}$, that is,

$$W = W_1 + W_2,$$

where

$$\widehat{W}_1(\xi) := c \int_{|\eta-\xi| \leq 2^{j-3}} (|\eta|^{2\alpha} - |\xi|^{2\alpha}) \hat{\phi}(\xi - \eta) \hat{v}(\eta) \, d\eta,$$

$$\widehat{W}_2(\xi) := c \int_{|\eta-\xi| > 2^{j-3}} (|\eta|^{2\alpha} - |\xi|^{2\alpha}) \hat{\phi}(\xi - \eta) \hat{v}(\eta) \, d\eta.$$

We will show (in Step 2.2b below) that $\|W_2\| \leq e_d(j)$. As for W_1 , since $\text{supp } p_j \subset \{|\eta| \in (2^{j-1}, 2^{j+1})\}$, note that

$$\text{supp } \widehat{W}_1 \subset \{|\xi| \in (2^{j-2}, 2^{j+2})\}. \tag{3-6}$$

Setting $f(z) := z^\alpha$ and expanding it in the Taylor series around $|\xi|^2$ we obtain

$$|\eta|^{2\alpha} - |\xi|^{2\alpha} = \sum_{k=1}^3 \frac{f^{(k)}(|\xi|^2)}{k!} (|\eta|^2 - |\xi|^2)^k + \frac{f^{(4)}(z_0)}{24} (|\eta|^2 - |\xi|^2)^4,$$

where z_0 belongs to the interval with endpoints $|\eta|^2$ and $|\xi|^2$ (and so in particular $z_0 \in [2^{2j-4}, 2^{2j+4}]$).

Writing

$$|\eta|^2 - |\xi|^2 = \sum_{i=1}^3 (\eta_i - \xi_i)(\eta_i + \xi_i)$$

and taking the k -th power we obtain

$$|\eta|^{2\alpha} - |\xi|^{2\alpha} = \sum_{k=1}^4 c_k f^{(k)}(z) \sum_{|\beta|=k, |\gamma_1|+|\gamma_2|=k} c_{\beta\gamma_1\gamma_2} (\eta - \xi)^\beta \eta^{\gamma_1} \xi^{\gamma_2},$$

where $z = |\xi|^2$ (for $k \leq 3$) or $z = z_0$ (for $k = 4$). Thus, noting that $|f^{(k)}(z)| \leq c 2^{j(2\alpha-2k)}$,

$$\begin{aligned}
|\widehat{W}_1(\xi)| &\leq c \sum_{k=1}^3 |f^{(k)}(|\xi|^2)| \sum_{|\beta|=k, |\gamma_1|+|\gamma_2|=k} |\xi|^{|\gamma_2|} \left| \int_{|\eta-\xi| \leq 2^{j-3}} (\xi - \eta)^\beta \widehat{\phi}(\xi - \eta) \eta^{\gamma_1} \widehat{v}(\eta) \, d\eta \right| \\
&\quad + c \sum_{|\beta|=4, |\gamma_1|+|\gamma_2|=4} |\xi|^{|\gamma_2|} \left| \int_{|\eta-\xi| \leq 2^{j-3}} f^{(k)}(z_0) (\xi - \eta)^\beta \widehat{\phi}(\xi - \eta) \eta^{\gamma_1} \widehat{v}(\eta) \, d\eta \right| \\
&\leq c \sum_{k=1}^3 \sum_{|\beta|=k, |\gamma_1|+|\gamma_2|=k} 2^{j(2\alpha-2k+|\gamma_2|)} \left| \int_{|\eta-\xi| \leq 2^{j-3}} (\xi - \eta)^\beta \widehat{\phi}(\xi - \eta) \eta^{\gamma_1} \widehat{v}(\eta) \, d\eta \right| \\
&\quad + c 2^{j(2\alpha-4)} \int_{|\eta-\xi| \leq 2^{j-3}} |\xi - \eta|^4 |\widehat{\phi}(\xi - \eta) \widehat{v}(\eta)| \, d\eta \\
&\leq c \sum_{k=1}^3 \sum_{|\beta|=k, |\gamma_1|+|\gamma_2|=k} 2^{j(2\alpha-2k+|\gamma_2|)} |\widehat{D^\beta \phi D^{\gamma_1} v}(\xi)| \\
&\quad + c \sum_{k=1}^3 \sum_{|\beta|=k, |\gamma_1|+|\gamma_2|=k} 2^{j(2\alpha-2k+|\gamma_2|)} \left| \int_{|\eta-\xi| > 2^{j-3}} (\xi - \eta)^\beta \widehat{\phi}(\xi - \eta) \eta^{\gamma_1} \widehat{v}(\eta) \, d\eta \right| \\
&\quad + c 2^{j(2\alpha-4)} \int_{|\eta-\xi| \leq 2^{j-3}} |\xi - \eta|^4 |\widehat{\phi}(\xi - \eta) \widehat{v}(\eta)| \, d\eta \\
&=: c \sum_{k=1}^3 \sum_{|\beta|=k, |\gamma_1|+|\gamma_2|=k} 2^{j(2\alpha-2k+|\gamma_2|)} |\widehat{D^\beta \phi D^{\gamma_1} v}(\xi)| + \text{Err}_1(\xi) + \text{Err}_2(\xi).
\end{aligned}$$

We will show below (in Step 2.2a below) that

$$\|\text{Err}_1\|, \|\text{Err}_2\| \leq c 2^{2\alpha j} (d 2^j)^{-1} u_{3Q/2, j\pm 2} + e_d(j).$$

This, together with the Plancherel identity gives

$$\begin{aligned}
\|W_1\| &\leq c \sum_{k=1}^3 \sum_{|\beta|=k, |\gamma_1|+|\gamma_2|=k} 2^{j(2\alpha-2k+|\gamma_2|)} \|D^\beta \phi D^{\gamma_1} v\| + c 2^{2\alpha j} (d 2^j)^{-1} u_{3Q/2, j\pm 2} + e_d(j) \\
&\leq c \sum_{k=1}^3 2^{2\alpha j} (d 2^j)^{-k} \|v\| + c 2^{2\alpha j} (d 2^j)^{-1} u_{3Q/2, j\pm 2} + e_d(j),
\end{aligned}$$

where we used the facts that $|\nabla^k \phi| \leq c d^{-k}$ for $k = 1, 2, 3$, and $\|D^{\gamma_1} v\| \leq c 2^{j|\gamma_2|} \|v\|$ (by applying Lemma 2.4). Since $d > 2^{-j}$ and

$$\|v\| \leq \|\phi_{3Q/2} \widetilde{P}_j u\| + e_d(j) = u_{3Q/2, j\pm 2} + e_d(j)$$

(where we applied Corollary 2.5), we thus arrive at

$$\|W_1\| \leq c 2^{2\alpha j} (d 2^j)^{-1} u_{3Q/2, j\pm 2} + e_d(j),$$

as required.

Step 2.2a: We show that $\|\text{Err}_1\| \leq e_d(j)$ and $\|\text{Err}_2\| \leq c 2^{2\alpha j} (d 2^j)^{-1} u_{3Q/2, j\pm 2} + e_d(j)$.

We focus on Err_1 first. We have

$$\begin{aligned}
 \text{Err}_1(\xi) &= c \sum_{k=1}^3 \sum_{|\beta|=k, |\gamma_1|+|\gamma_2|=k} 2^{j(2\alpha-2k+|\gamma_2|)} \left| \int_{|\eta-\xi|>2^{j-3}} (\xi-\eta)^\beta \hat{\phi}(\xi-\eta) \eta^{\gamma_1} \hat{v}(\eta) \, d\eta \right| \\
 &\leq c \sum_{k=1}^3 2^{j(2\alpha-k)} \int_{|\eta-\xi|>2^{j-3}} |\xi-\eta|^k |\hat{\phi}(\xi-\eta) \hat{v}(\eta)| \, d\eta \\
 &\leq c 2^{j(2\alpha-K)} \int_{|\eta-\xi|>2^{j-3}} |\xi-\eta|^K |\hat{\phi}(\xi-\eta) \hat{v}(\eta)| \, d\eta \\
 &\leq c_K 2^{j(2\alpha-K)} d^{1-K} \int_{|\eta-\xi|>2^j} |\xi-\eta|^{-2} |\hat{v}(\eta)| \, d\eta \\
 &\leq c_K 2^{j(2\alpha-1)} (d 2^j)^{(1-K)} \left(\int_{|\eta-\xi|>2^{j-5}} |\xi-\eta|^{-4} \, d\eta \right)^{1/2}
 \end{aligned}$$

for every $K > 3$, where we used (2-20) in the fourth line as well as the Cauchy–Schwarz inequality, (2-2) and the fact that $\|v\| \leq \|u\| \leq c$ (recall (1-3)) in the last line. Thus $\text{Err}_1(\xi) \leq e_d(j)$ for every $\xi \in \mathbb{R}^3$, and hence (since $|\xi| \leq 2^{j+2}$) also $\|\text{Err}_1\| \leq e_d(j)$.

As for Err_2 we write

$$\begin{aligned}
 \text{Err}_2(\xi) &= c 2^{j(2\alpha-4)} \int_{|\eta-\xi| \leq 2^{j-3}} |\xi-\eta|^4 |\hat{\phi}(\xi-\eta) \hat{v}(\eta)| \, d\eta \\
 &\leq c 2^{j(2\alpha-4)} d^{-1} \int_{|\eta-\xi| \leq 2^{j-3}} |\hat{v}(\eta)| \, d\eta \\
 &\leq c 2^{j(2\alpha-3/2)} (d 2^j)^{-1} \|v\| \\
 &= c 2^{j(2\alpha-3/2)} (d 2^j)^{-1} \|P_j \phi_{3Q/2} u\| \\
 &\leq c 2^{j(2\alpha-3/2)} (d 2^j)^{-1} u_{3Q/2, j\pm 2} + e_d(j),
 \end{aligned}$$

where we used (2-20) in the second line, the Cauchy–Schwarz inequality (as above) in the third line, and Corollary 2.5 in the last line. Thus

$$\|\text{Err}_2\| \leq c 2^{2\alpha j} (d 2^j)^{-1} u_{3Q/2, j\pm 2} + e_d(j),$$

as required.

Step 2.2b: We show that $\|W_2\| \leq e_d(j)$.

Indeed, since $|\xi|^{2\alpha} \leq c|\eta|^{2\alpha} + c|\xi-\eta|^{2\alpha}$, we obtain for any $K > 2\alpha$

$$\begin{aligned}
 |\widehat{W}_2(\xi)| &= \left| \int_{|\eta-\xi|>2^{j-5}} (|\eta|^{2\alpha} - |\xi|^{2\alpha}) \hat{\phi}(\xi-\eta) \hat{v}(\eta) \, d\eta \right| \\
 &\leq c \int_{|\eta-\xi|>2^{j-5}} |\eta|^{2\alpha} |\hat{\phi}(\xi-\eta) \hat{v}(\eta)| \, d\eta + c \int_{|\eta-\xi|>2^{j-5}} |\xi-\eta|^{2\alpha} |\hat{\phi}(\xi-\eta) \hat{v}(\eta)| \, d\eta \\
 &\leq c_K 2^{j(2\alpha-K)} \int_{|\eta-\xi|>2^{j-5}} |\xi-\eta|^K |\hat{\phi}(\xi-\eta) \hat{v}(\eta)| \, d\eta,
 \end{aligned}$$

where we used the inequality $1 < c_K |\xi-\eta|^K 2^{-jK}$, as well as $|\eta| \leq c 2^j$ inside the first integral in the second line and the inequality $1 \leq c_K |\xi-\eta|^{K-2\alpha} 2^{-j(K-2\alpha)}$ inside the second integral. Thus, using the

Plancherel identity and Young's inequality for convolutions

$$\begin{aligned}
\|W_2\| &= \|\widehat{W}_2\| \leq c_K 2^{j(2\alpha-K)} \|v\| \int_{|\eta|>2^{j-5}} |\eta|^K |\widehat{\phi}(\eta)| \, d\eta \\
&\leq c_K 2^{j(2\alpha-K)} \int_{|\eta|>2^{j-5}} |\eta|^{K+4} |\widehat{\phi}(\eta)| |\eta|^{-4} \, d\eta \\
&\leq c_K 2^{j(2\alpha-K)} d^{-(K+1)} \int_{|\eta|>2^{j-5}} |\eta|^{-4} \, d\eta \\
&= c_K 2^{2\alpha j} (d 2^j)^{-(K+1)},
\end{aligned}$$

as required, where we used (2-20) in the third inequality.

Step 2.2c: We show that $\|[\phi, (-\Delta)^\alpha] P_j (1 - \phi_{3Q/2}) u\| \leq e_d(j)$. (This implies (3-5).)

Indeed, letting (for brevity) $w := (1 - \phi_{3Q/2}) u$ and $q_j(\xi) := |\xi|^{2\alpha} p_j(\xi)$, we can write

$$\phi(-\Delta)^\alpha P_j w(x) = \phi(x) \int_{\{|x-y|\geq d/3\}} \check{q}_j(x-y) w(y) \, dy,$$

as in (2-16). Thus, since $\|\check{q}_j\|_{L^1(B(d/3)^c)} \leq e_d(j)$ (as in (2-8)), we can use Young's inequality for convolutions to obtain

$$\|\phi(-\Delta)^\alpha P_j w\| \leq \|\check{q}_j\|_{L^1(B(d/3)^c)} \|w\| \leq e_d(j). \quad (3-7)$$

On the other hand

$$\begin{aligned}
\|(-\Delta)^\alpha(\phi P_j w)\| &\leq \|(-\Delta)^\alpha \widetilde{P}_j(\phi P_j w)\| + \|(-\Delta)^\alpha(1 - \widetilde{P}_j)(\phi P_j w)\| \\
&\leq c 2^{2\alpha j} \|\phi P_j w\| + e_d(j) \leq e_d(j),
\end{aligned}$$

where we used (3-4) (applied with w instead of u) in the second line and Lemma 2.2 in the last line. This and (3-7) prove the claim.

Step 3: We show that $J \leq c u_{Q,j} (G_{\text{low,loc}} + G_{\text{loc}} + G_{\text{hh}}) + e_{\text{vl}} + \sum_{k \geq \theta j} e_d(k)$. (This together with Step 2 finishes the proof.)

We can rewrite J in the form

$$J = - \int \phi_Q P_j T[(u \cdot \nabla)u] \cdot (\phi_Q P_j u) = - \sum_{i,l,m} \int \phi_Q T_{mi} P_j (u_l \partial_l u_m) \phi_Q P_j u_i,$$

where we used the fact that “ T_{mi} ” and “ P_j ” are multipliers (so that they commute). (Recall that $\widehat{T}_{mi}(\xi) = (\delta_{mi} - \xi_m \xi_i |\xi|^{-2})$, see (2-10).) We now apply the paraproduct formula (2-14) to $P_j(u_l \partial_l u_m)$ to write

$$J = J_{\text{loc,low}} + J_{\text{low,loc}} + J_{\text{loc}} + J_{\text{hh}},$$

where each of $J_{\text{loc,low}}$, $J_{\text{low,loc}}$, J_{loc} , J_{hh} equals J except for the term $u_l \partial_l u_m$, which is replaced by the corresponding combination of the modes of u_l and $\partial_l u_m$, as in the paraproduct formula (see (3-8) and (3-10) below). We estimate J_{hh} in Step 3.1 below and $J_{\text{loc,low}}$, $J_{\text{low,loc}}$, J_{loc} in Step 3.2.

Step 3.1: We show that $J_{\text{hh}} \leq c u_{Q,j} G_{\text{hh}} + \sum_{k \geq j} e_d(k)$.

We write

$$\begin{aligned}
 J_{hh} &= - \sum_{i,l,m} \int \phi_Q T_{mi} P_j \left(\sum_{k \geq j+3} P_k u_l \tilde{P}_k \partial_l u_m \right) \phi_Q P_j u_i \\
 &\leq \| \phi_Q P_j u \|_\infty \sum_{i,l,m} \left\| \phi_Q T_{mi} P_j \sum_{k \geq j+3} (P_k u_l \tilde{P}_k \partial_l u_m) \right\|_1 \\
 &\leq c 2^{3j/2} u_{Q,j} \sum_{i,l,m} \left\| \phi_Q T_{mi} P_j \sum_{k \geq j+3} (P_k u_l \tilde{P}_k \partial_l u_m) \right\|_1 + e_d(j) \\
 &\leq c 2^{3j/2} u_{Q,j} \sum_{i,l,m} \left\| \phi_Q T_{mi} P_j \phi_{3Q/2}^3 \left(\sum_{k \geq j+3} P_k u_l \tilde{P}_k \partial_l u_m \right) \right\|_1 + e_d(j) \\
 &\leq c 2^{3j/2} u_{Q,j} \sum_{k \geq j+3} \| \phi_{3Q/2} P_k u \| \| \phi_{3Q/2}^2 \tilde{P}_k \nabla u \| + e_d(j), \tag{3-8}
 \end{aligned}$$

where, in the fourth line we applied Corollary 2.3 with $f := \sum_{k \geq j+3} P_k u_l \tilde{P}_k u_m$ and noted that $\text{supp } \phi \subset \frac{7}{3} Q$ is separated from $\text{supp}(1 - \phi_{3Q/2}^3)$ by at least $\frac{1}{3}d$. As for the third line, we used $P_j u = P_j \tilde{P}_j u$, (2-21) and (2-11) to write

$$\| \phi_Q P_j u \|_\infty \leq c 2^{3j/2} u_{Q,j} + e_d(j) \| \tilde{P}_j u \|_\infty \leq c 2^{3j/2} u_{Q,j} + e_d(j),$$

as well as noted that $e_d(j)$ multiplied by the (long) L^1 norm still gives $e_d(j)$, since we can brutally estimate this norm,

$$\begin{aligned}
 \left\| \phi_Q T_{mi} P_j \sum_{k \geq j+3} (P_k u_l \tilde{P}_k \partial_l u_m) \right\|_1 &\leq \| \phi_Q \| \left\| P_j \partial_l T_{mi} \sum_{k \geq j+3} (P_k u_l \tilde{P}_k u_m) \right\| \\
 &\leq c d^{3/2} 2^j \left\| P_j T_{mi} \sum_{k \geq j+3} (P_k u_l \tilde{P}_k u_m) \right\| \\
 &\leq c d^{3/2} 2^{5j/2} \left\| P_j \sum_{k \geq j+3} (P_k u_l \tilde{P}_k u_m) \right\|_1 \\
 &\leq c d^{3/2} 2^{5j/2} \sum_{k \geq j+3} \| P_k u_l \tilde{P}_k u_m \|_1 \leq c d^{3/2} 2^{5j/2} \sum_{k \geq j+1} \| P_k u \|^2 \\
 &\leq c d^{3/2} 2^{5j/2} \| u \|^2 \leq c d^{3/2} 2^{5j/2}
 \end{aligned}$$

for each i, l, m , where we used the Cauchy–Schwarz inequality in the first line, boundedness (in L^2) of the Leray projection (i.e., the fact that $|\widehat{T}_{mi}(\xi)| \leq 1$) and the Bernstein inequality (2-11) in the third line, (2-5) in the fourth line and the Cauchy–Schwarz inequality (twice) in the fifth line.

Noting that

$$\begin{aligned}
 \| \phi_{3Q/2}^2 \tilde{P}_k \nabla u \| &= \| P_{k \pm 2} (\phi_{3Q/2}^2 \nabla \tilde{P}_k u) \| + e_d(k) \\
 &\leq \| P_{k \pm 2} \nabla (\phi_{3Q/2}^2 \tilde{P}_k u) \| + 2 \| P_{k \pm 2} (\nabla \phi_{3Q/2} \phi_{3Q/2} \tilde{P}_k u) \| + e_d(k) \\
 &\leq c 2^k \| \phi_{3Q/2}^2 \tilde{P}_k u \| + c d^{-1} u_{3Q/2, k \pm 2} + e_d(k) \\
 &\leq c 2^k u_{3Q/2, k \pm 2} + e_d(k),
 \end{aligned}$$

where we used Lemma 2.4 in the first inequality, the fact that $\|\tilde{P}_k\| \leq 1$ and (2-19) in the third inequality, and the assumption $d > 2^{-j} > 2^{-k}$ in the last inequality, we obtain

$$J_{\text{hh}} \leq c 2^{3j/2} u_{Q,j} \sum_{k \geq j+1} 2^k u_{3Q/2,k}^2 + \sum_{k \geq j} e_d(k), \tag{3-9}$$

as required, where we also applied the Cauchy–Schwarz inequality in the first sum.

Step 3.2: We show that $J_{\text{loc,low}} + J_{\text{low,loc}} + J_{\text{loc}} \leq c u_{Q,j} (G_{\text{low,loc}} + G_{\text{loc}}) + e_{\text{vl}} + \sum_{k \geq \theta j} e_d(k)$. (This completes the proof of Step 3.)

We set

$$U_{lm} := \tilde{P}_j u_l \sum_{k \leq j-5} P_k u_m + \tilde{P}_j u_m \sum_{k \leq j-5} P_k u_l + \left(\sum_{k=j-4}^{j+2} P_k u_l \right) \left(\sum_{k=j-4}^{j+4} P_k u_m \right)$$

to write

$$\begin{aligned} J_{\text{loc,low}} + J_{\text{low,loc}} + J_{\text{loc}} &= - \sum_{i,l,m} \int \phi_Q T_{mi} P_j \partial_l U_{ml} \phi_Q P_j u_i \leq u_{Q,j} \sum_{i,l,m} \|\phi_Q T_{mi} P_j \partial_l U_{ml}\| \\ &= u_{Q,j} \sum_{i,l,m} \|\phi_Q T_{mi} P_j (\phi_{3Q/2}^3 \partial_l U_{ml})\| + e_d(j) \\ &\leq c u_{Q,j} \sum_{l,m} \|P_j (\phi_{3Q/2}^3 \partial_l U_{ml})\| + e_d(j) \\ &\leq c u_{Q,j} \sum_{l,m} (\|P_j \partial_l (\phi_{3Q/2}^3 U_{ml})\| + 3 \|P_j (\phi_{3Q/2}^2 \partial_l \phi_{3Q/2} U_{ml})\|) + e_d(j) \\ &\leq c 2^j u_{Q,j} \sum_{l,m} \|\phi_{3Q/2}^2 U_{ml}\| + e_d(j), \end{aligned} \tag{3-10}$$

where we applied Corollary 2.3 (with $q := 2$ and $f := U_{ml}$) in the third line, as well as (2-19) (as in the previous calculation) and the assumption $d > 2^{-j}$ in the last line.

We note that for each m, l

$$\|\phi_{3Q/2}^2 U_{ml}\| \leq 2 u_{3Q/2,j \pm 2} \left\| \phi_{3Q/2} \sum_{k \leq j-5} P_k u \right\|_{\infty} + \|\phi_{3Q/2} P_{j \pm 4} u\|_{\infty} u_{3Q/2,j \pm 4}. \tag{3-11}$$

Since we can estimate the above L^{∞} norm including the summation by writing

$$\sum_{k \leq j-5} = \sum_{k < \theta j} + \sum_{\theta j \leq k \leq j-5},$$

that is,

$$\begin{aligned} \left\| \phi_{3Q/2} \sum_{k \leq j-5} P_k u \right\|_{\infty} &\leq \left\| \phi_{3Q/2} \sum_{k < \theta j} P_k u \right\|_{\infty} + \sum_{\theta j \leq k \leq j-5} \|\phi_{\max(Q_k, 3Q/2)} P_k u\|_{\infty} \\ &\leq \|P_{\leq \theta j} u\|_{\infty} + c \sum_{\theta j \leq k \leq j-5} 2^{3k/2} u_{\max(Q_k, 3Q/2),k} + \sum_{k \geq \theta j} e_d(k) \\ &\leq c 2^{3\theta j/2} + c \sum_{\theta j \leq k \leq j-5} 2^{3k/2} u_{\max(Q_k, 3Q/2),k} + \sum_{k \geq \theta j} e_d(k), \end{aligned} \tag{3-12}$$

where we used the localised Bernstein inequality (2-21) in the second line (note that taking $\max(Q_k, \frac{3}{2}Q)$ is necessary since only then can we guarantee that the sidelength of such cube is greater than 2^{-k} , as required by (2-21)) and the Bernstein inequality (2-12) in the last line, we can plug it in (3-11) to get

$$\|\phi_{3Q/2}^2 U_{ml}\| \leq c u_{3Q/2, j\pm 2} 2^{3\theta j/2} + c u_{3Q/2, j\pm 2} \sum_{\theta j \leq k \leq j-5} 2^{3k/2} u_{3Q/2, k} + c 2^{3j/2} u_{3Q/2, j\pm 4}^2 + \sum_{k \geq \theta j} e_d(k),$$

where we used the assumption $d > 2^{-j+4}$ to apply the localised Bernstein inequality (2-21) again. Inserting this into (3-10) and using the fact that $\frac{3}{2}\theta = 2\alpha - 1 - \varepsilon$, we obtain

$$J_{loc, low} + J_{low, loc} + J_{loc} \leq c 2^{2\alpha j} 2^{-\varepsilon j} u_{3Q/2, j\pm 2}^2 + c 2^j u_{Q, j} u_{3Q/2, j\pm 2} \sum_{\theta j \leq k \leq j-5} 2^{3k/2} u_{3Q/2, k} + c 2^{5j/2} u_{Q, j} u_{3Q/2, j\pm 4}^2 + \sum_{k \geq \theta j} e_d(k), \quad (3-13)$$

as required (note the first term on the right-hand side is the “very low modes error”, e_{vl}). □

We now constrain ourselves to j -cubes. Given a j -cube Q we will write

$$u_Q := u_{Q, j}$$

for brevity. The above proposition then reduces to the following.

Corollary 3.2. *Let u be a Leray–Hopf weak solution of the Navier–Stokes equations (1-1) on the time interval $[0, \infty)$. Let Q be a j -cube with j large enough so that $2^{\varepsilon j} \geq 16$. Then*

$$\frac{d}{dt} u_Q^2 \leq -c 2^{2\alpha j} u_Q^2 + c u_Q \left(u_{3Q/2, j\pm 2} \sum_{\theta j \leq k \leq j-5} 2^{j+3k/2} u_{Q, k} + 2^{5j/2} u_{3Q/2, j\pm 4}^2 + \sum_{k \geq j+1} 2^{3j/2+k} u_{3Q/2, k}^2 \right) + c 2^{j(2\alpha-\varepsilon)} u_{3Q/2, j\pm 2}^2 + e(j). \quad (3-14)$$

Proof. We apply the estimate from Proposition 3.1 (which is valid due to the assumption $2^{\varepsilon j} > 16$). Since

$$e_{diss} \leq c 2^{j(2\alpha-\varepsilon)} u_{3Q/2, j\pm 2}^2$$

and

$$\sum_{k \geq \theta j} e_d(k) \leq c_K \sum_{k \geq \theta j} 2^{ck} 2^{\varepsilon k(c-K)} \leq c_K 2^{c\theta j + \varepsilon \theta j(c-K)} = e(j),$$

where K is taken large enough (to guarantee the summability of the geometric series), we arrive at (3-14), as required. □

3B. Good cubes and bad cubes. We now fix $u_0 \in H^1(\mathbb{R}^3)$ and a Leray–Hopf weak solution with initial data u_0 . We say that a cube Q is j -good if

$$\int_0^\infty \int_Q \sum_{k \geq j} 2^{2\alpha k} |P_k u|^2 \leq 2^{-j(5-4\alpha+\varepsilon)}. \quad (3-15)$$

We say that a j -cube is good if it is j -good. Otherwise we say that it is bad.

3C. Critical regularity on cubes with some good ancestors. We show that, for sufficiently large j , goodness of a j -cube and some of its ancestors guarantees critical regularity $(+\varepsilon)$ of u_Q on a smaller cube Q .

Theorem 3.3. *There exists $j_0 > 0$ (sufficiently large) such that whenever Q is a j -cube such that $j \geq j_0$ and each Q_{k-10} , $k \in [\theta j, j]$, is good then*

$$u_Q(t) < 2^{-(j/2)(5-4\alpha+\varepsilon)} \quad \text{for } t \in [0, T).$$

Remark 3.4. The above theorem appears in an imprecise form as Theorem 7.1 in [Katz and Pavlović 2002].¹ This is related to the somewhat unexpected way in which the dissipation error is handled in Lemma 6.3 in the same work. This lemma is in fact not needed, and it seems necessary to incorporate the dissipation error directly into the main estimate (in order to get around the imprecision), as in e_{diss} in (3-2).

Moreover the statement of Theorem 7.1 in [Katz and Pavlović 2002] suggests that goodness of only one cube is sufficient for the critical decay, which is not consistent with its proof (which uses goodness of the ancestors in the third line on p. 375).

Proof. Note that the claim is true for sufficiently small $t > 0$ since $u_0 \in H^1$, so that

$$\|P_j u_0\|^2 = \int p_j^2(\xi) |\hat{u}_0(\xi)|^2 d\xi \leq c 2^{-2j} \int |\xi|^2 |\hat{u}_0(\xi)|^2 d\xi \leq c 2^{-2j} \|u_0\|_{H^1}^2 < 2^{-j(5-4\alpha+\varepsilon)}$$

for sufficiently large j , and $u(t)$ remains bounded in H^1 for small $t > 0$. Suppose that the theorem is false, and let t_0 be the first time when it fails and Q a j -cube for which it fails. Then

$$u_Q(t) \leq 2^{-(j/2)(5-4\alpha+\varepsilon)} \quad \text{for } t \leq t_0, \tag{3-16}$$

with equality for $t = t_0$. Let $t_1 \in (0, t_0)$ be the last time when $u_Q(t_1) \leq \frac{1}{2} 2^{-(j/2)(5-4\alpha+\varepsilon)}$, so that

$$\frac{1}{2} 2^{-(j/2)(5-4\alpha+\varepsilon)} \leq u_Q(t) \leq 2^{-(j/2)(5-4\alpha+\varepsilon)} \quad \text{for } t \in (t_1, t_0). \tag{3-17}$$

Note that, since $\text{supp } \phi_{3Q/2} \subset \frac{7}{4}Q \subset Q_{j-1} \subset Q_{j-10}$ and Q_{j-10} is good,

$$\int_{t_1}^{t_0} \sum_{k \geq j-10} 2^{2\alpha k} u_{3Q/2,k}^2 \leq c \int_{t_1}^{t_0} \int_{Q_{j-10}} \sum_{k \geq j-10} 2^{2\alpha k} |P_k u|^2 \leq c 2^{-j(5-4\alpha+\varepsilon)},$$

and so in particular (recalling that $\alpha \in (1, \frac{5}{4})$)

$$\int_{t_1}^{t_0} u_{3Q/2,j \pm 4}^2 \leq c 2^{-j(5-2\alpha+\varepsilon)} \tag{3-18}$$

and

$$\int_{t_1}^{t_0} \sum_{k \geq j+1} 2^k u_{3Q/2,k}^2 \leq c 2^{j(1-2\alpha)} \int_{t_1}^{t_0} \sum_{k \geq j} 2^{2\alpha k} u_{3Q/2,k}^2 \leq c 2^{-j(4-2\alpha+\varepsilon)}. \tag{3-19}$$

¹The claim following “we must have” on p. 374 does not follow, as the assumption of the proof by contradiction is only on Q , rather than on every cube in its nuclear family.

Moreover, since Q_{k-10} is good for every $k \in [\theta j, j]$, we also have

$$\int_{t_1}^{t_0} u_{Q_k}^2 \leq c 2^{-k(5-2\alpha+\varepsilon)}$$

(as in (3-18)), and so

$$\int_{t_1}^{t_0} \sum_{\theta j \leq k \leq j-5} 2^{3k} u_{Q_k}^2 \leq c \sum_{\theta j \leq k \leq j-5} 2^{-k(2-2\alpha+\varepsilon)} \leq c 2^{-j(2-2\alpha+\varepsilon)}, \tag{3-20}$$

where we used the fact that $\alpha > 1$ and the fact that $\varepsilon > 0$ is small (recall (3-1)).²

Applying the main estimate (3-14) between t_1 and t_0 (and ignoring the first term on the right-hand side) and then utilizing (3-18)–(3-20) we obtain

$$\begin{aligned} 2^{-j(5-4\alpha+\varepsilon)} &= \frac{4}{3}(u_Q(t_0)^2 - u_Q(t_1)^2) \\ &\leq c \int_{t_1}^{t_0} u_Q \left(2^j u_{3Q/2, j\pm 2} \sum_{\theta j \leq k \leq j-5} 2^{3k/2} u_{3Q/2, k} + 2^{5j/2} u_{3Q/2, j\pm 4}^2 + 2^{3j/2} \sum_{k \geq j+1} 2^k u_{3Q/2, k}^2 \right) \\ &\quad + c 2^{j(2\alpha-\varepsilon)} \int_{t_1}^{t_0} u_{3Q/2, j\pm 2}^2 + e(j) \\ &\leq c 2^{-(j/2)(5-4\alpha+\varepsilon)} \left(2^j 2^{-(j/2)(5-2\alpha+\varepsilon)} 2^{-(j/2)(2-2\alpha+\varepsilon/2)} + 2^{5j/2} 2^{-j(5-2\alpha+\varepsilon)} + 2^{3j/2} 2^{-j(4-2\alpha+\varepsilon)} \right) \\ &\quad + c 2^{j(2\alpha-\varepsilon)} 2^{-j(5-2\alpha+\varepsilon)} \\ &\leq c 2^{-j(5-4\alpha+\varepsilon)} (2^{-3j\varepsilon/8} + 2^{-j\varepsilon/2} + 2^{-j\varepsilon/2} + 2^{-3j\varepsilon/2}) \\ &\leq c 2^{-j(5-4\alpha+\varepsilon)} 2^{-3j\varepsilon/8}, \end{aligned}$$

where, in the second inequality, we also used the Cauchy–Schwarz inequality and used the inequality $j \leq c 2^{j\varepsilon/4}$, as well as absorbed $e(j)$ (by writing, for example, $e(j) \leq c 2^{-j(5-4\alpha+2\varepsilon)}$ — recall the beginning of Section 2 for the definition of the j -negligible error $e(j)$). Thus

$$1 \leq c 2^{-j\varepsilon/4},$$

which gives a contradiction for sufficiently large j . □

3D. The singular set. Having defined good cubes and bad cubes, and observing that we have a “slightly more than critical” estimate on a cube that has some good ancestors (Theorem 3.3), we now characterize the singular set S in terms of its covers by bad cubes, and (in the next section) we show a much stronger (than critical) estimate outside S .

Let A_j denote the union of all bad j -cubes. Using Vitali covering lemma we can find a cover \mathcal{A}_j that covers A_j and such that

$$\#\mathcal{A}_j \leq c 2^{j(5-4\alpha+\varepsilon)}. \tag{3-21}$$

Indeed, the Vitali covering lemma gives a sequence of pairwise disjoint bad j -cubes $Q^{(l)}$ such that

$$A_j \subset \bigcup_l 5Q^{(l)}$$

²The restriction $\alpha > 1$ is used here, but $\alpha \geq 1$ would be sufficient by noting that $\sum_{k \geq \theta j} 2^{-k\varepsilon} \leq c 2^{-j\theta\varepsilon}$. Indeed, since $\theta > \frac{5}{8}$ (recall (3-3)), the last inequality of this proof would become $1 \leq c 2^{-j\varepsilon(\theta-1/2-1/8)}$, which still gives contradiction for large j .

However, since $\int_0^\infty \int |(-\Delta)^{\alpha/2} u|^2 \leq c$ (from the energy inequality, recall (1-3)),

$$\begin{aligned}
c &\geq \int_0^\infty \int |\xi|^{2\alpha} |\hat{u}(\xi)|^2 = \sum_{k \in \mathbb{Z}} \int_0^\infty \int p_k(\xi) |\xi|^{2\alpha} |\hat{u}(\xi)|^2 \\
&\geq c \sum_{k \geq j} 2^{2\alpha k} \int_0^\infty \int p_k(\xi)^2 |\hat{u}(\xi)|^2 = c \int_0^\infty \int \sum_{k \geq j} 2^{2\alpha k} |P_k u|^2 \\
&\geq c \sum_l \int_0^\infty \int_{Q^{(l)}} \sum_{k \geq j} 2^{2\alpha k} |P_k u|^2 \geq c \sum_l 2^{-j(5-4\alpha+\varepsilon)}, \tag{3-22}
\end{aligned}$$

where we used the Plancherel identity (twice, in the first and fourth lines), Tonelli's theorem (twice, in the second and fourth lines), and the fact that $Q^{(l)}$'s are pairwise disjoint in the fifth line. Thus

$$\#\{l\} \leq c 2^{j(5-4\alpha+\varepsilon)},$$

and so \mathcal{A}_j can be obtained by covering each of $5Q^{(l)}$ by at most 6^3 j -cubes.

In the remainder of this section we will show that there exists a (larger) j -cover \mathcal{B}_j of all bad j -cubes (i.e., of \mathcal{A}_j) with the same cardinality (i.e., satisfying (3-21), but with a larger constant) and the additional property that

$$\begin{aligned}
&\text{for any } x \text{ outside of } \mathcal{B}_j \text{ there exists } r \in (0, 2^{-10}) \\
&\text{such that } \partial(rQ_j(x)) \text{ does not touch any bad } k\text{-cube for any } k \geq j. \tag{3-23}
\end{aligned}$$

(Recall that $Q_j(x)$ denotes the j -cube centred at x .) We will refer to $\partial(rQ_j(x))$ as the *barrier*, and to (3-23) as the *barrier property*. We first discuss a simple geometric lemma.

Lemma 3.5 (geometric lemma). *Let $Q = Q(y)$, $Q' = Q'(x)$ be open cubes with sidelengths $2a$, $2b$, respectively. Then*

$$\partial(rQ) \text{ intersects } Q' \implies r \in [r_{Q'} - b/a, r_{Q'} + b/a],$$

where $r_{Q'} > 0$ is such that $x \in \partial(r_{Q'}Q)$.

Proof. We will write $\gamma := b/a$ for brevity. We split the reasoning into cases.

Case 1: $y \in \partial Q'$. Then $r_{Q'} = b/a$ (see Figure 2 (middle)), and so $r \geq r_{Q'} - b/a$ trivially. Moreover $\partial(rQ)$ intersects Q' if and only if $ra < 2b$ (see Figure 2 (middle)), that is, $r < 2b/a = r_{Q'} + b/a$, as required.

Case 2: $y \notin \bar{Q}'$. Then $r_{Q'} > b/a$ (which is clear by comparison with Case 1), and $\partial(rQ)$ intersects Q' if and only if

$$r_{Q'}a - b < ra < r_{Q'}a + b$$

(see Figure 2 (right)), as required.

Case 3: $y \in Q'$. Then $r_{Q'} < b/a$ and $\partial(rQ)$ intersects Q' if and only if

$$b - r_{Q'}a < ra < r_{Q'}a + b$$

(see Figure 2 (left)). The claim follows by ignoring the first of these two inequalities (and writing $r \geq 0 > r_{Q'} - b/a$ instead). \square

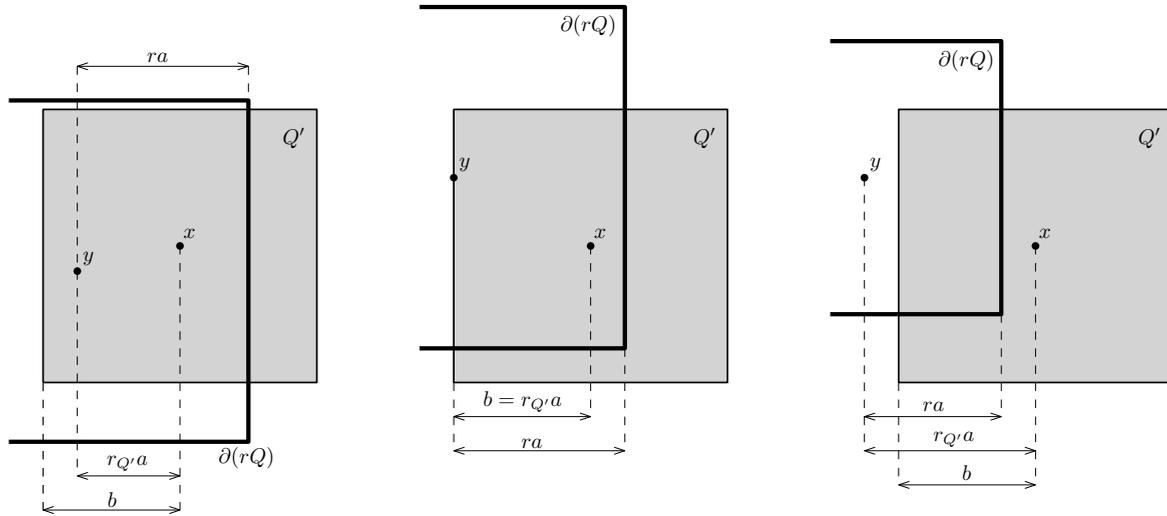


Figure 2. Sketch of the interpretation of Lemma 3.5.

We can now construct the j -cover satisfying the barrier property (3-23).

Lemma 3.6. *For every $j \geq 0$ there exists a j -cover \mathcal{B}_j of A_j such that $\#\mathcal{B}_j \leq c 2^{j(5-4\alpha+\varepsilon)}$ and the barrier property (3-23) holds.*

Proof. (Here we follow the argument from [Katz and Pavlović 2002, Section 8].) We will find a j -cover (also denoted by \mathcal{B}_j) of A_j such that

$$\begin{aligned} \text{for any } j\text{-cube } Q \text{ outside of } \mathcal{B}_j \text{ there exists } r \in (0, 2^{-10}) \\ \text{such that } \partial(rQ) \text{ does not touch any bad } k\text{-cube for any } k \geq j. \end{aligned} \quad (3-24)$$

(Here “outside” is a short-hand notation for “disjoint with every element of”.) The barrier property (3-23) is then recovered by replacing every j -cube $Q \in \mathcal{B}_j$ by $3Q$ and covering it by at most 4^3 j -cubes. Indeed, then for any x outside of such set we have that $Q_j(x)$ (the j -cube centred at x) is outside of \mathcal{B}_j and so the barrier property (3-23) follows from (3-24).

Step 1: We define *naughty* j -cubes.

We say that a j -cube Q is k -*naughty*, for $k \geq j$, if it intersects more than $\eta 2^{(k-j)(5-4\alpha+2\varepsilon)}$ elements of \mathcal{A}_k . Here $\eta \in (0, 1)$ is a universal constant, whose value we fix in Step 4 below. We say that a j -cube is *naughty* if it is k -naughty for any $k \geq j$. (Note that a bad cube is naughty. A good cube is not necessarily naughty, and vice versa.)

Step 2: For each $k \geq j$ we construct a j -cover $\mathcal{B}_{j,k}$ of all k -naughty j -cubes such that

$$\#\mathcal{B}_{j,k} \leq c\eta^{-1} 2^{j(5-4\alpha+\varepsilon)} 2^{\varepsilon(j-k)}. \quad (3-25)$$

(Note that $\mathcal{B}_{j,j}$ covers all j -naughty j -cubes, and so in particular all bad j -cubes.)

Let $Q^{(1)}$ be any k -naughty j -cube. Given $Q^{(1)}, \dots, Q^{(l)}$ let $Q^{(l+1)}$ be any k -naughty j -cube that is disjoint with each of $3Q^{(1)}, \dots, 3Q^{(l)}$. Note that then $3Q^{(1)}, \dots, 3Q^{(l)}$ contain all elements of \mathcal{A}_k that

$Q^{(1)}, \dots, Q^{(l)}$ intersect. This means that $Q^{(l+1)}$ intersects at least $\eta 2^{(k-j)(5-4\alpha+2\varepsilon)}$ “new” elements of \mathcal{A}_k (i.e., the elements that none of $Q^{(1)}, \dots, Q^{(l)}$ intersect). This means that such an iterative definition can go on for at most

$$L := \#\mathcal{A}_k \eta^{-1} 2^{(k-j)(5-4\alpha+2\varepsilon)} \leq c\eta^{-1} 2^{j(5-4\alpha+\varepsilon)} 2^{\varepsilon(j-k)}$$

steps, and then the family $\{3Q^{(1)}, \dots, 3Q^{(L)}\}$ covers all k -naughty j -cubes. We now cover each of $3Q^{(l)}$ ($l = 1, \dots, L$) by at most 4^3 j -cubes to obtain $\mathcal{B}_{j,k}$. (Note (3-25) then follows from the upper bound on L .)

Step 3: We define \mathcal{B}_j .

Let

$$\mathcal{B}_j := \bigcup_{k \geq j} \mathcal{B}_{j,k}.$$

By construction, \mathcal{B}_j covers all naughty j -cubes (and so, in particular, all bad j -cubes) and

$$\#\mathcal{B}_j \leq \sum_{k \geq j} \#\mathcal{B}_{j,k} \leq c\eta^{-1} 2^{j(5-4\alpha+\varepsilon)} \sum_{k \geq j} 2^{\varepsilon(j-k)} = c\eta^{-1} 2^{j(5-4\alpha+\varepsilon)},$$

as required (given η is fixed).

Step 4: We show that (3-24) holds for sufficiently small $\eta \in (0, 1)$. (This, together with the previous step, finishes the proof.)

Let Q be a j -cube disjoint with all elements of \mathcal{B}_j . Let us denote by $\mathcal{C}^k(Q)$ the collection of k -cubes Q' ($k \geq j$) from \mathcal{A}_k intersecting Q . Since Q is not naughty (as otherwise it would be covered by \mathcal{B}_j)

$$\#\mathcal{C}^k(Q) \leq \eta 2^{(k-j)(5-4\alpha+2\varepsilon)}.$$

Let $r_{Q'} \in (0, \infty)$ be such that $\partial(r_{Q'}Q)$ contains the centre of Q' . Applying Lemma 3.5 with $2a = 2^{-j(1-\varepsilon)}$ and $2b = 2^{-k(1-\varepsilon)}$ we obtain that

$$\partial(rQ) \text{ intersects } Q' \implies r \in [r_{Q'} - 2^{(1-\varepsilon)(j-k)}, r_{Q'} + 2^{(1-\varepsilon)(j-k)}].$$

Thus if $f_k(r)$ denotes the number of bad k -cubes that intersect $\partial(rQ)$ then

$$f_k(r) \leq \sum_{Q' \in \mathcal{C}^k(Q)} \chi_{[r_{Q'} - 2^{(1-\varepsilon)(j-k)}, r_{Q'} + 2^{(1-\varepsilon)(j-k)}]}(r).$$

Thus

$$\|f_k\|_{L^1(0, 2^{-10})} \leq 2\#\mathcal{C}^k(Q) 2^{(1-\varepsilon)(j-k)} \leq 2\eta 2^{(4\alpha-4-3\varepsilon)(j-k)},$$

and so letting $f := \sum_{k \geq j} f_k$ and recalling that $\alpha > 1$ and ε is small enough so that $4\alpha - 4 - 3\varepsilon > 0$ (see (3-1)) we obtain

$$\|f\|_{L^1(0, 2^{-10})} \leq \sum_{k \geq j} \|f_k\|_{L^1(0, 2^{-10})} \leq c\eta.$$

(This is the only place in the article where we need the assumption $\alpha > 1$; otherwise $\alpha \geq 1$ would be sufficient.) By choosing $\eta \in (0, 1)$ sufficiently small such that $c\eta < \frac{1}{2} 2^{-10}$ we see that $\|f\|_{L^1(0, 2^{-10})} < 2^{-10}$, and so there exists $r \in (0, 2^{-10})$ such that $f(r) = 0$ (recall that f takes only integer values). In other words there exists r such that $\partial(rQ)$ does not intersect any element of \mathcal{A}_k for any $k \geq j$, and so in particular any bad k -cube. □

We now let

$$E := \limsup_{j \rightarrow \infty} \bigcup_{Q \in \mathcal{B}_j} Q.$$

Observe that, since $\#\mathcal{B}_j \leq c 2^{j(5-4\alpha+\varepsilon)}$,

$$d_H(E) \leq 5 - 4\alpha + \varepsilon;$$

see, for example, Lemma 3.1 in [Katz and Pavlović 2002] for a proof.

3E. Regularity outside E . We now show that for every $x \notin E$ and every interval of regularity (a_i, b_i) there exists an open neighbourhood of x on which $u(t)$ remains bounded (as $t \in (\frac{1}{2}(a_i + b_i), b_i)$). This together with the above bound on $d_H(S)$ finishes the proof of Theorem 1.1.

Note that if $x \notin E$ then for sufficiently large j_0

$$x \notin Q \quad \text{for any } Q \in \mathcal{B}_j, \text{ for } j \geq j_0.$$

In particular

$$x \text{ does not belong to any bad } j\text{-cube for } j \geq j_0 \tag{3-26}$$

(since \mathcal{B}_j is a cover of all bad j -cubes), and for any $j_1 \geq j_0$ there exists $r = r(x, j_1) \in (0, 2^{-10})$ such that

$$\partial(rQ_{j_1}(x)) \text{ does not intersect any bad } k\text{-cube with } k \geq j_1 \tag{3-27}$$

(by the barrier property, (3-23)). The point is that the barrier can be constructed for any $j_1 \geq j_0$. This will be relevant for us, since in the proof of regularity below we will consider a j -cube with $j \geq j_1 \geq j_0/\theta^2$. Thus we will be able to deal with some of the low modes ($k \in [\theta j, j - 5]$) using (3-26) and others using (3-27). Indeed, for such modes we will have “cubes larger than j -cube” (i.e., Q_k with $k < j$) and we will obtain the critical decay on such cubes by either utilising the barrier property (3-27) (for cubes that are only “a little bit larger”, see Case 1 in Step 2 for details) or the fact that distant ancestors are large enough to contain x so that we can use (3-26). As for local and high modes (i.e., $k \geq j - 5$), we will use the barrier property (3-27) to obtain critical regularity for cubes located near the barrier, with more and more regularity on cubes located further away from the barrier towards the interior. In fact we can guarantee an arbitrary strong estimate for cubes located sufficiently far from the barrier, but we limit ourselves to the estimate $\lesssim 2^{-j(5-4\alpha+10)/2}$.

We now proceed to a rigorous version of the above explanation.

Theorem 3.7 (regularity outside E). *Let $x \notin E$. Given an interval of regularity (a_i, b_i) , there exists $c_i > 1$ and $j_1 = j_1(c_i) \in \mathbb{N}$ such that*

$$u_Q(t) < c_i 2^{-j\rho(Q)/2} \tag{3-28}$$

for all $t \in (\frac{1}{2}(a_i + b_i), b_i)$ and for every j -cube $Q \subset rQ_{j_1}(x)$, where $r \in (0, 2^{-10})$ is as in (3-27),

$$\rho(Q) := 5 - 4\alpha + \min(10, \frac{1}{10}\varepsilon\delta(Q))$$

and $\delta(Q)$ denotes the smallest $k \in \mathbb{N}$ such that Q_{j-k} intersects $\partial(rQ_{j_1}(x))$.

Note that the theorem gives no restriction on the range of j 's, but it is clear from the inclusion $Q \subset rQ_{j_1}(x)$ that $j \geq j_1 + 10$ (as $r < 2^{-10}$).

Proof. Since u is a strong solution in (a_i, b_i) , it is continuous in time in (a_i, b_i) with values in H^6 (recall (1-4)). Thus letting

$$c_i := 1 + c \|u(\frac{1}{2}(a_i + b_i))\|_{H^6}$$

we see that, for any j -cube Q , $u_Q(\frac{1}{2}(a_i + b_i)) \leq \|P_j u(\frac{1}{2}(a_i + b_i))\| < c_i 2^{-6j}$, and hence also $u_Q(t) < c_i$ for some $t > \frac{1}{2}(a_i + b_i)$ (due to the continuity of the H^6 norm). Thus the claim remains valid on some nonempty time interval following $\frac{1}{2}(a_i + b_i)$ (since $\rho(Q) \leq 5 - 4\alpha + 10 \leq 11$).

Since the interval of regularity (a_i, b_i) is fixed, from now on we will suppress the subindex “ i ”, for brevity.

We take j_0 sufficiently large so that (3-26) and the claims of Corollary 3.2 and Theorem 3.3 are valid (we will let j_0 even larger below). We let j_1 be the smallest integer such that

$$j_1 \geq (j_0 + 10)/\theta^2. \tag{3-29}$$

We also note that

$$\begin{aligned} \text{if } Q'(y) \text{ is a } k\text{-cube centred at } y \in rQ_{j_1}(x) \text{ and touching the barrier } \partial(rQ_{j_1}(x)) \\ \text{then } Q' \text{ is good if } k \geq j_0. \end{aligned} \tag{3-30}$$

Indeed, if $k \geq j_1$ then Q' is good by the barrier property (3-27). If $k < j_1$ then $Q' \supset rQ_{j_1}(x) \ni x$ (as the sidelength of $Q'(y)$ is more than 2^{10} times larger than the sidelength of $rQ_{j_1}(x) \ni y$), and so Q' is good by (3-26).

Suppose that the theorem is false and let $t_0 > \frac{1}{2}(a + b)$ be the first time when it fails. Then

$$u_{Q'}(t) \leq c 2^{-k\rho(Q')/2} \quad \text{for all } t \in [0, t_0] \text{ and all } k\text{-cubes } Q' \subset rQ_{j_1}(x) \tag{3-31}$$

and there exists a j -cube $Q \subset rQ_{j_1}(x)$ (for some $j \geq 0$) such that

$$u_Q(t_0) \geq 2^{-j\rho(Q)/2}. \tag{3-32}$$

We note that the existence of such Q is nontrivial, since there are infinitely many functions $u_{Q'}(t)$ for $Q' \subset rQ_{j_1}(x)$. In fact one can think of a scenario when all such $u_{Q'}$'s remain close to zero until t_0 with a sequence of $u_{Q'}$'s growing faster and faster past t_0 (in such scenario (3-31) holds but not (3-32)). We verify in Step 1 below that such a scenario does not happen (i.e., that such Q exists) as long as t_0 lies inside (a, b) .³

We now let $t_1 \in (0, t_0)$ be the last time such that $u_Q(t_1) = \frac{1}{2}2^{-j\rho(Q)/2}$. Then

$$u_Q(t) \in [\frac{1}{2}2^{-j\rho(Q)/2}, 2^{-j\rho(Q)/2}] \quad \text{for } t \in [t_1, t_0]. \tag{3-33}$$

The main estimate (3-14) gives

$$\begin{aligned} 2^{-j\rho(Q)} &= \frac{4}{3}(u_Q(t_0)^2 - u_Q(t_1)^2) \\ &\leq -c 2^{2\alpha j} \int_{t_1}^{t_0} u_Q^2 + c \int_{t_1}^{t_0} u_Q \left(2^j u_{3Q/2, j\pm 2} \sum_{\theta j \leq k \leq j-5} 2^{3k/2} u_{Q_k} + 2^{5j/2} u_{3Q/2, j\pm 4}^2 + 2^{3j/2} \sum_{k \geq j+1} 2^k u_{3Q/2, k}^2 \right) \\ &\quad + c \int_{t_1}^{t_0} 2^{2\alpha j} 2^{-j\epsilon} u_{3Q/2, j\pm 2}^2 + e(j), \end{aligned} \tag{3-34}$$

³This is the localisation issue that we referred to in the Introduction. This issue was ignored in [Katz and Pavlović 2002].

where we omitted the time argument in our notation. Note that we can write

$$e(j) \leq c 2^{-20j}$$

(recall the beginning of Section 2 for the definition of $e(j)$, the j -negligible error), so that it can be ignored (i.e., it can be absorbed into the left-hand side for sufficiently large j). We will estimate the terms appearing on the right-hand side of (3-34) in Steps 2–4 below, and we will conclude the proof in Step 5.

Step 1: We verify (3-32).

Let $m \in \mathbb{N}$. By definition of t_0 there exists $\tau \in (t_0, t_0 + 1/m)$ and a j -cube Q such that $u_Q(\tau) \geq c 2^{-j\rho(Q)/2}$. We claim that (3-32) holds for such Q if m is taken sufficiently large. Indeed, if it does not, then $2^{j\rho(Q)/2}u_Q(t_0) \leq 1$ for each m , and so

$$c - 1 \leq 2^{j\rho(Q)/2}(u_Q(\tau) - u_Q(t_0)) \leq 2^{11j/2} \|\phi_Q^2 P_j(u(\tau) - u(t_0))\| \leq c \|u(\tau) - u(t_0)\|_{H^6(\mathbb{R}^3)}^2$$

for all m , uniformly in j , and so continuity of u in time (on (a, b)) with values in H^6 gives a contradiction for sufficiently large m . (Note that, for simplicity, we have omitted the dependence of τ and Q on m in the notation above.)

Step 2: We observe that $\delta(Q) \geq 11$, so that in particular

$$\rho(Q) \geq 5 - 4\alpha + \varepsilon. \tag{3-35}$$

In order to see this, note that if $\delta(Q) \leq 10$ then Q_{j-10} touches $\partial(rQ_{j_1}(x))$. Thus (3-30) implies that Q_{k-10} is good for every $k \in [\theta j, j]$, since

$$k - 10 \geq \theta j - 10 \geq \theta j_1 - 10 \geq j_0$$

by our choice (3-29) of j_1 . Hence Theorem 3.3 gives that

$$2u_Q(t_0) < 2^{-j(5-4\alpha+\varepsilon)/2} \leq 2^{-j(5-4\alpha+\varepsilon\delta(Q)/10)/2} = 2^{-j\rho(Q)/2},$$

which contradicts (3-32).

Step 3: We show that

$$u_{Q_k}(t) \leq c 2^{-k(5-4\alpha+\varepsilon)/2}, \quad k \in [\theta j, j - 5],$$

$$u_{3Q/2,k}(t) \leq \begin{cases} c 2^{-j(\rho(Q)-2\varepsilon/5)/2}, & k \in [j - 4, \dots, j + 100/\varepsilon], \\ c 2^{-3j} 2^{-k(9-4\alpha)/2}, & k \geq j + 100/\varepsilon, \end{cases} \tag{3-36}$$

for $t \in (t_1, t_0)$.

Case 1: $k \in [\theta j, j - 5]$. If $\delta(Q_k) \geq 11$ then in particular $Q_k \subset rQ_{j_1}(x)$ and $\rho(Q_k) \geq 5 - 4\alpha + \varepsilon$, and so the claim follows from (3-31). If $\delta(Q_k) \leq 10$ then Q_{l-10} is good for every $l \in [\theta k, k]$ due to (3-30), since

$$l - 10 \geq \theta k - 10 \geq \theta^2 j - 10 \geq \theta^2 j_1 - 10 \geq j_0. \tag{3-37}$$

Therefore the claim follows from Theorem 3.3.

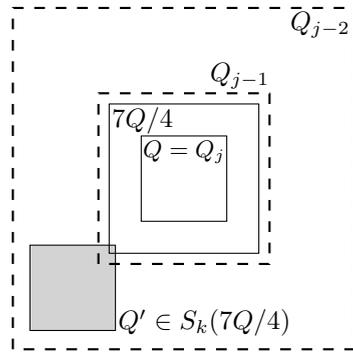


Figure 3. An illustration of (3-40) - note that each $Q' \in S_k(\frac{7}{4}Q)$ is of the same size as Q (as in the illustration) or smaller (as $k \geq j$).

Case 2: $k \in [j-4, \dots, j+100/\varepsilon]$. Then

$$\delta(Q_k) = \delta(Q) + k - j \geq \delta(Q) - 4 \geq 7, \quad (3-38)$$

where we used Step 2 in the last inequality. Hence $Q_k \subset rQ_{j_1}(x)$ and

$$\rho(Q_k) \geq \rho(Q) - \frac{2}{5}\varepsilon.$$

Thus since for $k \in [j-4, j-1]$ we have $\frac{3}{2}Q \subset Q_k$, (3-31) gives

$$u_{3Q/2,k} \leq 2^{-k\rho(Q_k)/2} \leq 2^{-k(\rho(Q)-2\varepsilon/5)/2} \leq c 2^{-j(\rho(Q)-2\varepsilon/5)/2},$$

as required. If $k \geq j$ we note that

$$u_{3Q/2,k} \leq \sum_{Q' \in S_k(7Q/4)} u_{Q'}, \quad (3-39)$$

where $S_k(\frac{7}{4}Q)$ denotes a cover of $\frac{7}{4}Q$ by k -cubes with $\#S_k(\frac{7}{4}Q) \leq c 2^{3(k-j)(1-\varepsilon)}$ (recall the beginning of Section 3). Since

$$Q'_j = 2^{-(j-k)(1-\varepsilon)} Q' \subset Q_{j-2} \quad \text{for every } Q' \in S_k(\frac{7}{4}Q), \quad (3-40)$$

see Figure 3, we obtain

$$\delta(Q') = \delta(Q'_j) + k - j \geq \delta(Q_{j-2}) = \delta(Q) - 2, \quad (3-41)$$

and so $\rho(Q') \geq \rho(Q) - \frac{1}{5}\varepsilon$. Therefore (3-31) gives

$$u_{Q'} \leq 2^{-k\rho(Q')/2} \leq 2^{-k(\rho(Q)-\varepsilon/5)/2} \leq c 2^{-j(\rho(Q)-2\varepsilon/5)/2},$$

and since $\#S_k(\frac{7}{4}Q) \leq c 2^{300(1-\varepsilon)/\varepsilon} = c$ (recall our constants may depend on ε) the claim follows by applying (3-39) above.

Case 3: $k \geq j+100/\varepsilon$. For such k we improve (3-41) by writing

$$\delta(Q') = \delta(Q'_j) + k - j \geq \delta(Q_{j-2}) + 100/\varepsilon = \delta(Q) + 100/\varepsilon - 2 > 100/\varepsilon \quad (3-42)$$

for any $Q' \in S_k(\frac{7}{4}Q)$ where we used Step 2 in the last inequality. This gives $\rho(Q') = 15 - 4\alpha$. Thus using (3-39) and the estimate $\#S_k(\frac{7}{4}Q) \leq c 2^{3(k-j)(1-\varepsilon)} \leq c 2^{3(k-j)}$ we arrive at

$$u_{3Q/2,k} \leq \sum_{Q' \in S_k(7Q/4)} u_{Q'} \leq \sum_{Q' \in S_k(7Q/4)} 2^{-k\rho(Q')/2} \leq c 2^{3(k-j)} 2^{-k\rho(Q')/2} = c 2^{-3j} 2^{-k(9-4\alpha)/2},$$

as required.

Step 4: We use the previous step to estimate the terms appearing on the right-hand side of the main estimate (3-34). Namely we show that

$$\begin{aligned} \sum_{\theta j \leq k \leq j-5} 2^{3k/2} u_{3Q/2,k} &\leq c 2^{3j/2} 2^{-j(5-4\alpha)/2} 2^{-j\varepsilon/2}, \\ u_{3Q/2,j\pm 4}^2 &\leq c 2^{-j\rho(Q)/2} 2^{-j(5-4\alpha)/2} 2^{-j\varepsilon/10}, \\ \sum_{k \geq j+1} 2^k u_{3Q/2,k}^2 &\leq c 2^j 2^{-j\rho(Q)/2} 2^{-j(5-4\alpha)/2} 2^{-j\varepsilon/10}. \end{aligned} \tag{3-43}$$

We note that, although the terms appearing on the right-hand side might look complicated, we write them in this form to articulate their roles. As for the factors $2^{3j/2}$ or 2^j , these are “bad factors” which, together with the corresponding factor in the main estimate (3-34), give $2^{5j/2}$. This should be compared against the factor $2^{2\alpha j}$ which is a “good factor” given by the dissipation (i.e., by the first term on the right-hand side of (3-34), which comes with a minus). This brings us to the factors of the form $2^{-j(5-4\alpha)}$ whose role is exactly to balance the “bad factor” against the “good factors”.

As for the factors $2^{-j\rho(Q)/2}$, we point out that together with the corresponding factor u_Q (which is bounded above and below by $2^{-j\rho(Q)/2}$ due to (3-33)) appearing in the basic estimate, one obtains $2^{-j\rho(Q)}$ as the common factor of all terms in (3-34).

Finally, the role of any factor involving ε is to make sure that the balance falls in our favour, namely that the resulting constant at all terms on the right-hand side of (3-34) (except for the first term), is smaller than the constant at the first term (the dissipation term). Writing the estimates in the form (3-43) also exposes the value of $5 - 4\alpha$, which is our desired bound on the Hausdorff dimension.

We now briefly verify (3-43). The first two of them follow from Step 3 by a simple calculation,

$$\sum_{\theta j \leq k \leq j-5} 2^{3k/2} u_{3Q/2,k} \leq c \sum_{\theta j \leq k \leq j-5} 2^{-k(2-4\alpha+\varepsilon)/2} \leq c 2^{-j(2-4\alpha+\varepsilon)/2} \tag{3-44}$$

and

$$u_{3Q/2,j\pm 4}^2 \leq c 2^{-j(\rho(Q)-2\varepsilon/5)} = c 2^{-j\rho(Q)/2} 2^{-j(\rho(Q)-4\varepsilon/5)/2} \leq c 2^{-j\rho(Q)/2} 2^{-j(5-4\alpha)/2} 2^{-j\varepsilon/10},$$

as required, where we used (3-35) in the last inequality. As for the third estimate in (3-43) we write

$$\sum_{k \geq j+1} = \sum_{j+1 \leq k \leq j+100/\varepsilon} + \sum_{k > j+100/\varepsilon},$$

and estimate each of the two sums separately,

$$\sum_{j+1 \leq k \leq j+100/\varepsilon} 2^k u_{3Q/2,k}^2 \leq c 2^j 2^{-j(\rho(Q)-2\varepsilon/5)} \leq c 2^j 2^{-j\rho(Q)/2} 2^{-j(5-4\alpha+\varepsilon/5)/2}$$

(recall that c might depend on ε), where we used (3-35) in the last inequality, and

$$\sum_{k>j+100/\varepsilon} 2^k u_{3Q/2,k}^2 \leq c 2^{-3j} \sum_{k>j+100/\varepsilon} 2^{-k(8-4\alpha)} \leq c 2^{-j(11-4\alpha)} \leq c 2^j 2^{-j\rho(Q)/2} 2^{-j(5-4\alpha+\varepsilon/5)/2},$$

where we used the inequality $11 - 4\alpha \geq -1 + \frac{1}{2}\rho(Q) + \frac{1}{2}(5 - 4\alpha) + \frac{1}{10}\varepsilon$ (a trivial consequence of the fact that $\rho(Q) \leq 5 - 4\alpha + 10$) in the last inequality.

Step 5: We conclude the proof.

Applying the estimates from the previous step into the main estimate (3-34) and recalling that $u_{3Q/2,j\pm 2}^2 \leq c 2^{-j(\rho(Q)-2\varepsilon/5)}$ (from Step 3) we obtain

$$\begin{aligned} & 2^{-j\rho(Q)} \\ & \leq -c 2^{2\alpha j} \int_{t_1}^{t_0} u_Q^2 + c \int_{t_1}^{t_0} u_Q (2^j 2^{-j(\rho(Q)-2\varepsilon/5)/2} 2^{3j/2} 2^{-j(5-4\alpha)/2} 2^{-j\varepsilon/2} + 2^{5j/2} 2^{-j\rho(Q)/2} 2^{-j(5-4\alpha)/2} 2^{-j\varepsilon/10} \\ & \quad + 2^{3j/2} 2^j 2^{-j\rho(Q)/2} 2^{-j(5-4\alpha)/2} 2^{-j\varepsilon/10}) \\ & \quad + 2^{2\alpha j} 2^{-j\varepsilon} \int_{t_1}^{t_0} 2^{-j(\rho(Q)-2\varepsilon/5)} \\ & = -c 2^{2\alpha j} \int_{t_1}^{t_0} u_Q^2 + c 2^{2\alpha j} \int_{t_1}^{t_0} u_Q (2^{-j\rho(Q)/2} (2^{-3j\varepsilon/10} + 2^{-j\varepsilon/10} + 2^{-j\varepsilon/10})) + c 2^{2\alpha j} 2^{-3j\varepsilon/5} \int_{t_1}^{t_0} 2^{-j\rho(Q)} \\ & \leq -c 2^{j(2\alpha-\rho(Q))} (t_0-t_1) (1-c 2^{-j\varepsilon/10}), \end{aligned}$$

where we used the lower bound $u_Q \geq \frac{1}{2} 2^{-j\rho(Q)/2}$ (see (3-33)) in the last line. Therefore if j_0 is sufficiently large so that

$$1 - c 2^{-j_1\varepsilon/10} > 0$$

(where c is the last constant appearing in the calculation above; recall also that j_1 is given by (3-29)), we obtain

$$1 \leq 0,$$

a contradiction. □

Corollary 3.8. *Given $x \notin E$ and an interval of regularity (a_i, b_i) there exists an open neighbourhood U of x such that*

$$\|u(t)\|_{L^\infty(U)} \text{ remains bounded for } t \in \left(\frac{1}{2}(a_i + b_i), b_i\right).$$

Proof. We fix an interval of regularity. By Theorem 3.7 there exists j_1 and $r \in (0, 2^{-10})$ such that

$$u_Q(t) \leq 2^{-j\rho(Q)/2}$$

for all $t \in [0, T)$ and all j -cubes $Q \subset r Q_{j_1}(x)$. Let $j_2 \in \mathbb{N}$ be the smallest number such that $\delta(Q) \geq 100/\varepsilon$ for every j -cube $Q \subset Q_{j_2}(x)$. (Note that the last condition implies also that $j \geq j_2$.) Then $\rho(Q) \geq 10$ for any such j -cube Q and so $u_Q \leq c 2^{-5j}$. We let

$$U := Q_{j_2+2}(x).$$

To show that $\|u(t)\|_{L^\infty(U)}$ remains bounded, we note that the localised Bernstein inequality (2-21) gives

$$\|\phi_Q P_j u\|_\infty \leq c 2^{3j/2} u_Q + e(j) \leq c 2^{-7j/2}$$

for every j -cube $Q \in S_j(U)$ with $j \geq j_2 + 2$. Hence

$$\|P_j u\|_{L^\infty(U)} \leq \sum_{Q \in S_j(U)} \|\phi_Q P_j u\|_\infty \leq c 2^{3(1-\varepsilon)(j-(j_2+2))} 2^{-7j/2} = c_{j_2} 2^{-j/2}$$

for such j and so

$$\begin{aligned} \|u\|_{L^\infty(U)} &\leq \|P_{\leq j_2+1} u\|_\infty + \sum_{j \geq j_2+2} \|P_j u\|_{L^\infty(U)} \\ &\leq c 2^{3j_2/2} \|P_{\leq j_2+1} u\| + c_{j_2} \sum_{j \geq j_2+2} 2^{-j/2} \leq c_{j_2}, \end{aligned}$$

as required, where we used the Bernstein inequality (2-12) in the second inequality. □

3F. Regularity for $\alpha > \frac{5}{4}$. Here we briefly verify Corollary 1.3. Letting $\varepsilon \in (0, 4\alpha - 5)$ we see that any j -cube ($j \geq 0$) satisfies

$$u_Q(t) \leq c \leq c 2^{-j(5-4\alpha+\varepsilon)}$$

for all $t \geq 0$. Thus any closed and sufficiently small surface $S \subset \mathbb{R}^3$ can be used as a barrier, and Theorem 3.7 (with $\partial(rQ_{j_1}(x))$ replaced by S) gives that $u_Q(t) < 2^{-j\rho(Q)/2}$ for all j -cubes Q located inside S and all $t \geq 0$ (provided u_0 is sufficiently smooth). Furthermore j_2 (from the proof of Corollary 3.8) can be chosen independently of x (i.e., depending only on how small S is), and consequently Corollary 3.8 gives boundedness of $\|u(t)\|_\infty$ in $t > 0$.

4. The box-counting dimension

Here we prove Theorem 1.2; namely that $d_B(S^{(k)}) \leq \frac{1}{3}(-16\alpha^2 + 16\alpha + 5)$, where $S^{(k)} := \bigcup_{i \leq k} S_i$ (recall (1-7)).

A bound on $d_B(S^{(k)})$ can in fact be obtained by examining the proof of Theorem 3.7 above. Namely, observing that the only consequence of $x \notin E$ that we used in its proof was that

$$x \notin Q \quad \text{for any } Q \in \mathcal{B}_k, \quad k \in [\theta^2 j_1 - 10, j_1], \tag{4-1}$$

where j_1 is taken sufficiently large. In fact, this allowed us to deduce that for a given j -cube $Q \subset rQ_{j_1}(x)$ the cube $Q_k = 2^{(j-k)(1-\varepsilon)}Q$ is good for such k 's (take $j_0 := \lfloor \theta^2 j_1 - 10 \rfloor$ and recall (3-26), (3-27) and (3-30)). This, in light of Theorem 3.3, gave us the ‘‘slightly more than critical’’ decay, which in turn enabled us to deduce better decay for cubes located further inside the barrier $rQ_{j_1}(x)$. Corollary 3.8 then deduced that $x \notin S$.

Using (4-1) we see that for sufficiently large j

$$\bigcup_{k \in \{\lfloor \theta^2 j - 10 \rfloor, \dots, j\}} \bigcup_{Q \in \mathcal{B}_k} Q$$

contains the singular set in space at a given blow-up time. Thus, covering each of the covers \mathcal{B}_k ($k \in \{\lfloor \theta^2 j - 10 \rfloor, \dots, j\}$) by at most

$$c 2^{3(j-k)(1-\varepsilon)} \#\mathcal{B}_k \leq c 2^{3(j-k)(1-\varepsilon)} 2^{k(5-4\alpha+\varepsilon)} = c 2^{3j(1-\varepsilon)} 2^{k(2-4\alpha+2\varepsilon)}$$

j -cubes we obtain a cover of the singular set by at most

$$c \sum_{k=\lfloor \theta^2 j - 10 \rfloor}^j 2^{3j(1-\varepsilon)} 2^{k(2-4\alpha+2\varepsilon)} \leq c 2^{j(3-3\varepsilon+\theta^2(2-4\alpha+2\varepsilon))} \\ = c 2^{j(-64\alpha^3+96\alpha^2(1+\varepsilon)-48\alpha(1+\varepsilon)^2+35+8\varepsilon^3+8\varepsilon^2-3\varepsilon)/9} \tag{4-2}$$

j -cubes, where we substituted $\theta = \frac{2}{3}(2\alpha - 1 - \varepsilon)$ (recall (3-3)) in the last line. In other words $N(S^{(m)}, r)$, the minimal number of r -balls required to cover $S^{(m)}$ (recall the definition (1-8) of the box-counting dimension), satisfies

$$N(S^{(m)}, r) \leq c r^{(-64\alpha^3+96\alpha^2(1+\varepsilon)-48\alpha(1+\varepsilon)^2+35+8\varepsilon^3+8\varepsilon^2-3\varepsilon)/9(1-\varepsilon)} \tag{4-3}$$

for sufficiently small r . This gives that

$$d_B(S^{(m)}) \leq \frac{1}{9}(-64\alpha^3 + 96\alpha^2 - 48\alpha + 35) \tag{4-4}$$

for every $m \in \mathbb{N}$. As noted in the Introduction, we point out that the required smallness of r for (4-3) to hold depends on the interval of regularity (a_i, b_i) . This is the reason why we only estimate $d_B(S^{(m)})$, rather than $d_B(S)$.

In what follows we present a sharper argument that allows one to get rid of one of θ 's in the first line of (4-2) to yield the following.

Proposition 4.1. *Given the interval of regularity (a_i, b_i) the set*

$$\bigcup_{k \in \{\lfloor \theta j - 10 \rfloor, \dots, j\}} \bigcup_{Q \in \mathcal{B}_k} Q$$

covers the singular set in space at time b_i if j is sufficiently large.

Assuming this proposition and letting \mathcal{C}_j be a j -cover of all elements of \mathcal{B}_k for $k = \lfloor \theta j - 10 \rfloor, \dots, j$, we obtain a j -cover of the singular set with

$$\#\mathcal{C}_j \leq c \sum_{k=\lfloor \theta j - 10 \rfloor}^j 2^{3(j-k)(1-\varepsilon)} \#\mathcal{B}_k \leq c 2^{j(3-3\varepsilon+\theta(2-4\alpha+2\varepsilon))} = c 2^{j(-16\alpha^2+16\alpha(1+\varepsilon)+5-17\varepsilon-4\varepsilon^2)/3},$$

which shows that $d_B(S^{(m)}) \leq \frac{1}{3}(-16\alpha^2 + 16\alpha + 5)$ for all $m \in \mathbb{N}$, by an argument analogous to that above. This is sharper than (4-4), and it proves Theorem 1.2. We note that if one was able to get rid of the other θ in (4-2), then one would obtain $d_B(S) \leq 5 - 4\alpha$, i.e., the same bound as for $d_H(S)$.

Before proceeding to the proof of Proposition 4.1, we comment on the main idea of Proposition 4.1 in an informal way.

Recall (3-37) that for each $k \in [\theta j, j - 5]$ we needed Q_{l-10} to be good for $l \in [\theta k, k]$, and deduced from the “ ε -better than critical” decay for u_{Q_k} (in Case 1 of Step 3 of the proof of Theorem 3.7, by using Theorem 3.3), which we have then plugged into the sum of the low modes of the main estimate (3-34) (in (3-44) above). However, looking closely at this term of the main estimate,

$$2^j \int_{t_1}^{t_0} u_Q u_{3Q/2, j \pm 2} \sum_{\theta j \leq k \leq j-5} 2^{3k/2} u_{Q_k},$$

we observe that it has a structure similar to the definition of a good cube (3-15). Indeed, ignoring u_Q and $u_{3Q/2, j \pm 2}$ for a moment we see that we could use (3-15) to estimate it. If that were possible, we would only need to require that Q_k (or rather Q_{k-10}) is good for $k \in [\theta j, j-5]$, and so we would end up with a saving of one θ . The only problem is that (3-15) is concerned with the time integral of a squared function, rather than the function itself, and so, applying the Cauchy–Schwarz inequality in the time integral we would obtain an additional factor of $(t_0 - t_1)^{-1/2}$; see the last term in (4-8) below. It turns out that this additional factor can be taken care of by absorbing a part of this term by the left-hand side (as in (4-9) below).

Proof of Proposition 4.1. We will show that if j_1 is sufficiently large then every x outside of \mathcal{C}_{j_1} is a regular point in the given interval of regularity (a, b) . We set

$$j_0 := \lfloor \theta j_1 - 10 \rfloor. \tag{4-5}$$

As in Theorem 3.7 we show that, for sufficiently large $j_1 = j_1(c_i)$,

$$u_Q(t) < c_i 2^{-j\rho(Q)} \tag{4-6}$$

for every $x \notin \bigcup_{Q \in \mathcal{C}_{j_1}} Q$, where c_i depends on the interval of regularity (a_i, b_i) . In fact, we can copy the entire proof of Theorem 3.7, except for Step 4, where we replace the estimate on the low modes (i.e., the first inequality in (3-43)) by

$$\sum_{k \in [\theta j, j-5]} 2^{3k/2} \int_{t_1}^{t_0} u_{Q_k} \leq c(t_0 - t_1) 2^{-j(2-4\alpha+\varepsilon)/2} + c(t_0 - t_1)^{1/2} 2^{-j(2-2\alpha+\varepsilon)/2}, \tag{4-7}$$

which we prove below. Given (4-7), we can plug it, together with the remaining two inequalities in (3-43), into the main estimate (3-34) (just as we did in Step 5 of the proof of Theorem 3.7 above) to yield

$$\begin{aligned} 2^{-j\rho(Q)} &= c(u_Q(t_0)^2 - u_Q(t_1)^2) \\ &\leq -c 2^{2\alpha j} \int_{t_1}^{t_0} u_Q^2 + c \int_{t_1}^{t_0} u_Q \left(2^j u_{3Q/2, j \pm 2} \sum_{\theta j \leq k \leq j-5} 2^{3k/2} u_{Q_k} + 2^{5j/2} u_{3Q/2, j \pm 4}^2 + 2^{3j/2} \sum_{k \geq j+1} 2^k u_{3Q/2, k}^2 \right) \\ &\quad + 2^{2\alpha j} 2^{-j\varepsilon} \int_{t_1}^{t_0} u_{3Q/2, j \pm 2}^2 + e(j) \\ &\leq -c 2^{2\alpha j} (t_0 - t_1) 2^{-j\rho(Q)} + c 2^{-j\rho(Q)} 2^{j(1+\varepsilon/5)} \int_{t_1}^{t_0} \sum_{\theta j \leq k \leq j-5} 2^{3k/2} u_{Q_k} \\ &\quad + c(t_0 - t_1) 2^{2\alpha j} 2^{-j\rho(Q)} (2^{j\varepsilon/10} + 2^{j\varepsilon/10} + 2^{3j\varepsilon/5}) \\ &\leq 2^{2\alpha j} (t_0 - t_1) 2^{-j\rho(Q)} (-c + c 2^{-j\varepsilon/10}) + c(t_0 - t_1)^{1/2} 2^{-j\rho(Q)} 2^{\alpha j} 2^{-3j\varepsilon/10}, \end{aligned} \tag{4-8}$$

where, in the last step, we applied (4-7) to estimate the low modes. At this point we obtain the same inequality as before (i.e., as in Step 5 of the proof of Theorem 3.7), except for the last term, which can be estimated using Young’s inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ to give

$$\frac{1}{2} 2^{-j\rho(Q)} + c 2^{2\alpha j} (t_0 - t_1) 2^{-j\rho(Q)} 2^{-3j\varepsilon/5}. \tag{4-9}$$

Absorbing the first term above on the left-hand side we obtain

$$1 \leq 2^{2\alpha j} (t_0 - t_1) (-c + c 2^{-j\epsilon/10}),$$

which gives a contradiction for sufficiently large j .

It remains to verify (4-7). To this end, if $\delta(Q_k) \geq 11$ then, as before, we can use the fact that the claim (4-6) remains valid until t_0 to obtain that

$$\sum_{\substack{k \in [\theta j, j-5] \\ \delta(Q_k) \geq 11}} 2^{3k/2} \int_{t_1}^{t_0} u_{Q_k} \leq c(t_0 - t_1) \sum_{\substack{k \in [\theta j, j-5] \\ \delta(Q_k) \geq 11}} 2^{3k/2} 2^{-k\rho(Q_k)/2} \leq c(t_0 - t_1) 2^{-j(2-4\alpha+\epsilon)/2},$$

where we used the fact that $\rho(Q_k) \geq 5 - 4\alpha + \epsilon$ in the last inequality.

If $\delta(Q_k) \leq 10$ then Q_{k-10} intersects the barrier $\partial(rQ_{j_1}(x))$, and so it is good as $k - 10 \geq \theta j - 10 \geq j_0$ (recall (3-30) and (4-5)). Thus since $\phi_{Q_k} \leq 1_{Q_{k-10}}$ (recall (2-19)) the definition (3-15) of a good cube gives

$$\int_{t_1}^{t_0} u_{Q_k}^2 \leq \int_{t_1}^{t_0} \int_{Q_{k-10}} |P_k u|^2 \leq c 2^{-k(5-2\alpha+\epsilon)}.$$

Hence

$$\begin{aligned} \sum_{k \in [\theta j, j-5], \delta(Q_k) \leq 10} 2^{3k/2} \int_{t_1}^{t_0} u_{Q_k} &\leq (t_0 - t_1)^{1/2} \sum_{\theta j, \dots, j-5, \delta(Q_k) \leq 10} 2^{3k/2} \left(\int_{t_1}^{t_0} u_{Q_k}^2 \right)^{1/2} \\ &\leq c(t_0 - t_1)^{1/2} \sum_{k \leq j-5} 2^{-k(2-2\alpha+\epsilon)/2} = c(t_0 - t_1)^{1/2} 2^{-j(2-2\alpha+\epsilon)/2}, \end{aligned}$$

as required. □

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