

ANALYSIS & PDE

Volume 16

No. 3

2023

STEFAN IVANOV, IVAN MINCHEV AND DIMITER VASSILEV

**SOLUTION OF THE QC YAMABE EQUATION ON A 3-SASAKIAN
MANIFOLD AND THE QUATERNIONIC HEISENBERG GROUP**

SOLUTION OF THE QC YAMABE EQUATION ON A 3-SASAKIAN MANIFOLD AND THE QUATERNIONIC HEISENBERG GROUP

STEFAN IVANOV, IVAN MINCHEV AND DIMITER VASSILEV

A complete solution to the quaternionic contact Yamabe equation on the qc sphere of dimension $4n+3$ as well as on the quaternionic Heisenberg group is given. A uniqueness theorem for the qc Yamabe problem in a compact locally 3-Sasakian manifold is shown.

1. Introduction

It is well known that the solution of the Yamabe problem on a compact Riemannian manifold is unique in the case of negative or vanishing scalar curvature. The proofs of these results, which rely on the maximum principle, extend readily to sub-Riemannian settings, such as the CR and quaternionic contact (abbreviated as qc) Yamabe problems, due to the subellipticity of the involved operators. The positive (scalar curvature) case presents considerable difficulties due to the possible nonuniqueness. Among these cases the corresponding round spheres play a special role due to their roles in the general existence theorem and because of the connections with the corresponding L^2 Sobolev-type embedding inequalities. Through the corresponding Cayley transforms, the sphere cases are equivalent to the problems of finding all solutions to the respective Yamabe equations on the flat models which are the Euclidean space or the relevant Heisenberg groups. All solutions of the latter equations were found in the Riemannian and CR sphere cases in [Obata 1971; Jerison and Lee 1988], respectively. The classification of all solutions of the Yamabe equation in the Euclidean setting can be handled alternatively by a reduction to a radially symmetric solution [Gidas et al. 1979; Talenti 1976]. As far as the rigidity question is concerned, Yamabe established a uniqueness result in every conformal class of an Einstein metric [Obata 1971].

In this paper we determine all solutions of the qc Yamabe equation on the $(4n+3)$ -dimensional round sphere and quaternionic Heisenberg group and establish a uniqueness result in every qc conformal class containing a 3-Sasakian metric.

We continue by giving a brief background and the statements of our results. It is well known that the sphere at infinity of any noncompact symmetric space M of rank 1 carries a natural Carnot–Carathéodory structure; see [Mostow 1973; Pansu 1989]. A quaternionic contact (qc) structure [Biquard 1999; 2000] appears naturally as the conformal boundary at infinity of the quaternionic hyperbolic space. Following Biquard, a qc structure on a real $(4n+3)$ -dimensional manifold M is a codimension-3 distribution H (*the horizontal distribution*) locally given as the kernel of an \mathbb{R}^3 -valued 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ such that the

MSC2020: 53C17 .

Keywords: Yamabe equation, quaternionic contact structures, Einstein structures, divergence formula, Folland–Stein inequality, Heisenberg group.

three 2-forms $d\eta_i|_H$ are the fundamental forms of a quaternionic Hermitian structure on H . The 1-form η is determined up to a conformal factor and the action of $SO(3)$ on \mathbb{R}^3 , and therefore H is equipped with a conformal class $[g]$ of quaternionic Hermitian metrics. To every metric in the fixed conformal class one can associate a linear connection with torsion preserving the qc structure, see [Biquard 2000; Duchemin 2006], which is called the Biquard canonical connection. For a fixed metric in the conformal class of metrics on the horizontal space one associates the horizontal Ricci-type tensor of the Biquard connection, which is called the qc Ricci tensor. This is a symmetric tensor [Biquard 2000] whose trace part defines the qc scalar curvature. Furthermore, it was shown in [Ivanov et al. 2014a] that the torsion endomorphism of the Biquard connection completely determines the trace-free part of the horizontal Ricci tensor. The vanishing of the latter tensor defines the class of qc Einstein manifolds. A basic example of a qc manifold is a 3-Sasakian space, which can be defined as a $(4n+3)$ -dimensional Riemannian manifold whose Riemannian cone is a hyper-Kähler manifold and the qc structure is induced from that hyper-Kähler structure. By [Ivanov et al. 2014a; Ivanov et al. 2016] the qc Einstein manifolds of positive qc scalar curvature are exactly the locally 3-Sasakian manifolds, up to a multiplication with a constant factor and a $SO(3)$ -matrix. In particular, every 3-Sasakian manifold has vanishing torsion endomorphism and is a qc Einstein manifold.

The quaternionic contact Yamabe problem on a compact qc manifold M is the problem of finding a metric \bar{g} in the qc conformal class $[g]$ of a fixed metric on the horizontal space H for which the qc scalar curvature is constant. We note that a qc conformal transformation of the contact form described in Definition 2.1 amounts to a conformal change of the horizontal metric. Another natural problem is to explore the possible uniqueness or nonuniqueness of such qc Yamabe metrics. Within a fixed qc conformal class, the questions reduce to the solvability and uniqueness of positive solutions of the quaternionic contact (qc) Yamabe equation

$$\mathcal{L}u \equiv 4 \frac{Q+2}{Q-2} \Delta u - u \text{ Scal} = -u^{2^*-1} \overline{\text{Scal}},$$

where Δ is the horizontal sub-Laplacian defined using the Biquard connection ∇ , $\Delta h = \text{tr}^g(\nabla^2 h)$, Scal and $\overline{\text{Scal}}$ are the qc scalar curvatures correspondingly of (M, η) and $(M, \bar{\eta})$, $\bar{\eta} = u^{4/(Q-2)}\eta$, and $2^* = 2Q/(Q-2)$, with $Q = 4n + 6$ — the homogeneous dimension.

Alternatively, one can view the problem as a variational problem whereby on a compact quaternionic contact manifold M with a fixed conformal class $[\eta]$ the qc Yamabe equation characterizes the nonnegative extremals of the qc Yamabe functional defined by

$$\Upsilon(u) = \int_M \left(4 \frac{Q+2}{Q-2} |\nabla u|^2 + \text{Scal} u^2 \right) dv_g, \quad \int_M u^{2^*} dv_g = 1, \quad 0 < u \in \mathcal{D}^{1,2}(M).$$

Here dv_g denotes the Riemannian volume form of the Riemannian metric on M obtained by extending in a natural way the horizontal metric associated to η , and $\mathcal{D}^{1,2}(M)$ stands for the L^2 homogeneous Sobolev space. Considering M equipped with a fixed qc structure, hence, a conformal class $[\eta]$, the Yamabe constant is defined as the infimum

$$\lambda(M) \equiv \lambda(M, [\eta]) = \inf \left\{ \Upsilon(u) : \int_M u^{2^*} dv_g = 1, 0 < u \in \mathcal{D}^{1,2}(M) \right\}. \tag{1-1}$$

The main result of [Wang 2007] is that the qc Yamabe equation has a solution on a compact qc manifold provided $\lambda(M) < \lambda(S^{4n+3})$, where S^{4n+3} is the standard unit sphere in the quaternionic space \mathbb{H}^{n+1} .

In this paper we consider the qc Yamabe problem on the unit $(4n+3)$ -dimensional sphere in \mathbb{H}^{n+1} . The standard 3-Sasaki structure on the sphere $\tilde{\eta}$ has a constant qc scalar curvature $\widetilde{\text{Scal}} = 16n(n+2)$ and vanishing trace-free part of its qc Ricci tensor; i.e., it is a qc Einstein space. The images under conformal quaternionic contact automorphisms are again qc Einstein structures and, in particular, have constant qc scalar curvature. In [Ivanov et al. 2014a] we conjectured that these are the only solutions to the Yamabe problem on the quaternionic sphere and proved it in dimension 7 in [Ivanov et al. 2010]. One of the main goals of this paper is to prove this conjecture in full generality.

Theorem 1.1. *Let $\eta = 2h\tilde{\eta}$ be a qc conformal transformation of the standard qc structure $\tilde{\eta}$ on a 3-Sasakian sphere of dimension $4n+3$. If η has constant qc scalar curvature, then up to a multiplicative constant η is obtained from $\tilde{\eta}$ by a conformal quaternionic contact automorphism.*

We note that Theorem 1.1, together with the results of [Ivanov et al. 2014a], allows the determination of all solutions of the qc Yamabe problem on the sphere and on the quaternionic Heisenberg group $\mathbf{G}(\mathbb{H})$. This complements the CR case where [Jerison and Lee 1988] characterized all nonnegative solutions of the CR Yamabe problem on the Heisenberg group and the corresponding odd-dimensional spheres.

Recall that the quaternionic Heisenberg group $\mathbf{G}(\mathbb{H})$ of homogeneous dimension $Q = 4n + 6$ is given by $\mathbf{G}(\mathbb{H}) = \mathbb{H}^n \times \text{Im } \mathbb{H}$ with the group law

$$(q_o, \omega_o) \circ (q, \omega) = (q_o + q, \omega + \omega_o + 2 \text{Im } q_o \bar{q}),$$

where $q = (t^a, x^a, y^a, z^a) \in \mathbb{H}^n$, $\omega = (x, y, z) \in \text{Im } \mathbb{H}$. The standard qc contact form in quaternion variables is

$$\tilde{\Theta} = (\tilde{\Theta}_1, \tilde{\Theta}_2, \tilde{\Theta}_3) = \frac{1}{2}(d\omega - q \cdot d\bar{q} + dq \cdot \bar{q}).$$

The corresponding sub-Laplacian is given by $\Delta_{\tilde{\Theta}} u = \sum_{a=1}^n (T_a^2 u + X_a^2 u + Y_a^2 u + Z_a^2 u)$, where T_a, X_a, Y_a, Z_a denote the left-invariant horizontal vector fields on $\mathbf{G}(\mathbb{H})$. Theorem 1.1 shows, in particular, the following.

Corollary 1.2. *If Φ satisfies the qc Yamabe equation on the quaternionic Heisenberg group $\mathbf{G}(\mathbb{H})$, that is,*

$$\frac{4(Q+2)}{Q-2} \Delta_{\tilde{\Theta}} \Phi = -S_{\Theta} \Phi^{2^*-1}$$

for some constant S_{Θ} , then up to a left translation $\Phi = (2h)^{-(Q-2)/4}$ and h is given by

$$h(q, \omega) = c_0 [(\sigma + |q + q_0|^2)^2 + |\omega + \omega_o + 2 \text{Im } q_o \bar{q}|^2] \tag{1-2}$$

for some fixed $(q_o, \omega_o) \in \mathbf{G}(\mathbb{H})$ and constants $c_0 > 0$ and $\sigma > 0$. Furthermore, the qc scalar curvature of Θ is $S_{\Theta} = 128n(n+2)c_0\sigma$.

The above corollary confirms the conjecture made after [Garofalo and Vassilev 2001, Theorem 1.1]. In Theorem 1.6 of the same paper the conjecture claim was verified on all groups of Iwasawa type, but with the assumption of partial symmetry of the solution. Here, with a completely different method

from [Garofalo and Vassilev 2001] we show that the symmetry assumption is superfluous in the case of the quaternionic Heisenberg group. The corresponding solutions on the 3-Sasakian sphere are obtained via the Cayley transform, see for example [Ivanov et al. 2010; 2012; 2014a; Ivanov and Vassilev 2011, Sections 2.3, 5.2.1] for an account and history. Finally, it should be observed that the functions (1-2) with $c_0 \in \mathbb{R}$ give all conformal factors for which Θ is also qc Einstein. It is worth mentioning that as a consequence of Theorem 1.1 and Corollary 1.2, we obtain that all solutions to the qc Yamabe equation are given by the functions which realize the equality case of the L^2 Folland–Stein inequality. The latter were characterized, by a different method, in [Ivanov et al. 2012], where the center of mass technique developed for the CR case in [Frank and Lieb 2012; Branson et al. 2013] was used.

A major step in the proof of Theorem 1.1 is the following result, where we solve the qc Yamabe problem on locally 3-Sasakian compact manifolds. By the results of [Ivanov et al. 2014a; 2016] a qc Einstein manifold is of constant qc scalar curvature; hence as far as the qc Yamabe equation is concerned only the uniqueness of solutions needs to be addressed. As mentioned earlier, the interesting case is when the qc scalar curvature is a positive constant; hence we focus exclusively on the locally 3-Sasakian case.

Theorem 1.3. *Let $(M, \bar{\eta})$ be a compact locally 3-Sasakian qc manifold of qc scalar curvature $16n(n+2)$. If $\eta = 2h\bar{\eta}$ is qc conformal to an $\bar{\eta}$ structure which is also of constant qc scalar curvature, then up to a homothety (M, η) is locally 3-Sasakian manifold. Furthermore, the function h is constant unless $(M, \bar{\eta})$ is the unit 3-Sasakian sphere.*

The proof of Theorem 1.3, presented in Section 5, consists of two steps. The first step is a divergence formula Theorem 4.1 which shows that if $\bar{\eta}$ is of constant qc curvature and is qc conformal to a locally 3-Sasakian manifold, then $\bar{\eta}$ is also a locally 3-Sasakian manifold. The general idea to search for such a divergence formula goes back to [Obata 1971] where the corresponding result on a Riemannian manifold was proved for a conformal transformation of an Einstein space. However, our result is motivated by the (sub-Riemannian) CR case where a formula of this type was introduced in the ground-breaking paper [Jerison and Lee 1988]. As far as the qc case is concerned in [Ivanov et al. 2014a, Theorem 1.2] a weaker result was shown, namely Theorem 1.3 holds provided the vertical space of η is integrable. In dimension 7, the $n = 1$ case, this assumption was removed in [Ivanov et al. 2010, Theorem 1.2] where the result was established with the help of a suitable divergence formula. It should be noted that in the 7-dimensional case the [3]-component of the traceless qc Ricci tensor vanishes, which decreases the number of torsion components. The general $n > 1$ case treated here presents new difficulties due to the extra nonzero torsion terms that appear in the higher dimensions, which complicate considerably the search for a suitable divergence formula.

The proof of the second part of Theorem 1.3 builds on, in the Riemannian case, ideas of Obata, who used that the gradient of the (suitably taken) conformal factor is a conformal vector field and the characterization of the unit sphere through its first eigenvalue of the Laplacian among all Einstein manifolds. We show a similar, although a more complicated relation between the conformal factor and the existence of an infinitesimal qc automorphism (qc vector field). Our divergence formula found in Theorem 4.1 involves a smooth function f , see (4-7), expressed in terms of the conformal factor and its horizontal gradient. Remarkably, we found that the horizontal gradient of f is precisely the horizontal part of the qc vector

field mentioned above and the sub-Laplacian of f is an eigenfunction of the sub-Laplacian with the smallest possible eigenvalue $-4n$ thus showing a geometric nature of f (see [Remark 5.3](#)). Then we use the characterization of the 3-Sasakian sphere by its first eigenvalue of the sub-Laplacian among all locally 3-Sasakian manifolds established in [[Ivanov et al. 2014b](#), Theorem 1.2] for $(n > 1)$ and in [[Ivanov et al. 2013](#), Corollary 1.2] for $n = 1$.

A few final comments are in order. As noted above, the connection between the two Obata theorems—the uniqueness up to homothety of the Yamabe metric in the conformal class of an Einstein metric on a compact Riemannian manifold distinct from the round sphere and the characterization of the sphere as the extremal in the Lichnerowicz–Obata inequality—are well known. Our inspiration for the proof of the second part of [Theorem 1.3](#) came from the slightly different approach taken in [[Bourguignon and Ezin 1987](#)]; see [[Ivanov and Vassilev 2015](#), Theorem 2.6]. The attempt to find an extension of this argument to the qc setting resulted in the argument presented here.

Remark 1.4. The above argument leading to the uniqueness result in [Theorem 1.3](#) can be applied not only in the qc case but also in the case of a pseudohermitian structure on a CR manifold. In particular, the argument presented here reveals the geometric nature of the function in Jerison and Lee’s divergence formula [[1988](#)]. Indeed, the real part of the function f defined in Proposition 3.1 of that paper determines a CR vector field and its CR-Laplacian is an eigenfunction of the CR-Laplacian with the smallest possible eigenvalue $-2n$. More details for the CR case can be found in [[Ivanov and Vassilev 2015](#), Section 5.2].

Convention 1.5. We use the following:

- $\{e_1, \dots, e_{4n}\}$ denotes an orthonormal basis of the horizontal space H .
- The capital letters X, Y, Z, \dots denote horizontal vectors in H .
- The summation convention over repeated vectors from the basis $\{e_1, \dots, e_{4n}\}$ will be used. For example, for a $(0, 4)$ -tensor P , $k = P(e_b, e_a, e_a, e_b)$ means $k = \sum_{a,b=1}^{4n} P(e_b, e_a, e_a, e_b)$.
- The triple (i, j, k) denotes any cyclic permutation of $(1, 2, 3)$.
- The horizontal divergence $\nabla^* P$ of a $(0, 2)$ -tensor field P on M with respect to the Biquard connection is defined to be the $(0, 1)$ -tensor field $\nabla^* P(\cdot) = (\nabla_{e_a} P)(e_a, \cdot)$.

2. Quaternionic contact manifolds and the qc Yamabe problem

In this section we will briefly review the basic notions of quaternionic contact geometry and recall some results from [[Biquard 2000](#); [Ivanov et al. 2014a](#)]; see [[Ivanov and Vassilev 2011](#)] for a more leisurely exposition. We also give some background on the qc Yamabe problem.

2A. qc manifolds. A quaternionic contact (qc) manifold (M, η, g, \mathbb{Q}) is a $(4n+3)$ -dimensional manifold M with a codimension-3 distribution H locally given as the kernel of a 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in \mathbb{R}^3 . In addition H has an $\text{Sp}(n) \text{Sp}(1)$ structure; that is, it is equipped with a Riemannian metric g and a rank-3 bundle \mathbb{Q} consisting of endomorphisms of H locally generated by three almost complex structures I_1, I_2, I_3 on H satisfying the identities of the imaginary unit quaternions, $I_1 I_2 = -I_2 I_1 = I_3$,

$I_1 I_2 I_3 = -\text{id}_{|H}$, which are hermitian compatible with the metric $g(I_s \cdot, I_s \cdot) = g(\cdot, \cdot)$ and the following contact condition holds:

$$2g(I_s X, Y) = d\eta_s(X, Y).$$

A special phenomena, noted in [Biquard 2000], is that the contact form η determines the quaternionic structure and the metric on the horizontal distribution in a unique way.

The transformations preserving a given quaternionic contact structure η , i.e., $\bar{\eta} = \mu\Psi\eta$ for a positive smooth function μ and an $\text{SO}(3)$ matrix Ψ with smooth functions as entries, are called *quaternionic contact conformal (qc conformal) transformations*. If the function μ is constant, $\bar{\eta}$ is called qc homothetic to η . The qc conformal curvature tensor W^{qc} , introduced in [Ivanov and Vassilev 2010], is the obstruction for a qc structure to be locally qc conformal to the standard 3-Sasakian structure on the $(4n+3)$ -dimensional sphere [Ivanov et al. 2014a; Ivanov and Vassilev 2010].

Definition 2.1. A diffeomorphism ϕ of a qc manifold $(M, [g], \mathbb{Q})$ is called a *conformal quaternionic contact automorphism (conformal qc automorphism)* if ϕ preserves the qc structure; i.e.,

$$\phi^*\eta = \mu\Phi \cdot \eta$$

for some positive smooth function μ and some matrix $\Phi \in \text{SO}(3)$ with smooth functions as entries and $\eta = (\eta_1, \eta_2, \eta_3)^t$ is a local 1-form considered as a column vector of three one forms as entries.

On a qc manifold with a fixed metric g on H there exists a canonical connection defined first by O. Biquard [2000] when the dimension $(4n + 3)$ greater than 7, and in [Duchemin 2006] for the 7-dimensional case. Biquard showed that there is a unique connection ∇ with torsion T and a unique supplementary subspace V to H in TM , such that:

- (i) ∇ preserves the decomposition $H \oplus V$ and the $\text{Sp}(n) \text{Sp}(1)$ structure on H ; i.e., $\nabla g = 0$, $\nabla\sigma \in \Gamma(\mathbb{Q})$ for a section $\sigma \in \Gamma(\mathbb{Q})$, and its torsion on H is given by $T(X, Y) = -[X, Y]_{|V}$.
- (ii) For $\xi \in V$, the endomorphism $T(\xi, \cdot)_{|H}$ of H lies in $(\mathfrak{sp}(n) \oplus \mathfrak{sp}(1))^\perp \subset \mathfrak{gl}(4n)$.
- (iii) The connection on V is induced by the natural identification φ of V with the subspace $\mathfrak{sp}(1)$ of the endomorphisms of H ; i.e., $\nabla\varphi = 0$.

This canonical connection is also known as *the Biquard connection*. When the dimension of M is at least 11 [Biquard 2000] also described the supplementary distribution V , which is (locally) generated by the so-called Reeb vector fields $\{\xi_1, \xi_2, \xi_3\}$ determined by

$$\eta_s(\xi_k) = \delta_{sk}, \quad (\xi_s \lrcorner d\eta_s)_{|H} = 0, \quad (\xi_s \lrcorner d\eta_k)_{|H} = -(\xi_k \lrcorner d\eta_s)_{|H}, \tag{2-1}$$

where \lrcorner denotes the interior multiplication. If the dimension of M is 7, Duchemin [2006] shows that if we assume, in addition, the existence of Reeb vector fields as in (2-1), then the Biquard result holds. Henceforth, by a qc structure in dimension 7 we shall mean a qc structure satisfying (2-1).

The fundamental 2-forms ω_s of the quaternionic contact structure Q are defined by

$$2\omega_s|_H = d\eta_s|_H, \quad \xi \lrcorner \omega_s = 0, \quad \xi \in V.$$

Notice that (2-1) are invariant under the natural $SO(3)$ action. Using the triple of Reeb vector fields, we extend the metric g on H to a metric h on TM by requiring $\text{span}\{\xi_1, \xi_2, \xi_3\} = V \perp H$ and $h(\xi_s, \xi_k) = \delta_{sk}$. The Riemannian metric h , as well as the Biquard connection, do not depend on the action of $SO(3)$ on V , but both change if η is multiplied by a conformal factor [Ivanov et al. 2014a]. Clearly, the Biquard connection preserves the Riemannian metric on TM , $\nabla h = 0$.

The properties of the Biquard connection are encoded in the torsion endomorphism $T_\xi \in (\mathfrak{sp}(n) + \mathfrak{sp}(1))^\perp$. We recall the $Sp(n) Sp(1)$ -invariant decomposition. An endomorphism Ψ of H can be decomposed with respect to the quaternionic structure (\mathbb{Q}, g) uniquely into four $Sp(n)$ -invariant parts $\Psi = \Psi^{+++} + \Psi^{+--} + \Psi^{-+-} + \Psi^{--+}$, where the superscript $+++$ means commuting with all three I_i , $+-$ indicates commuting with I_1 and anticommuting with the other two and etc. The two $Sp(n) Sp(1)$ -invariant components $\Psi_{[3]} = \Psi^{+++}$, $\Psi_{[-1]} = \Psi^{+--} + \Psi^{-+-} + \Psi^{--+}$ are determined by

$$\begin{aligned} \Psi = \Psi_{[3]} &\iff 3\Psi + I_1\Psi I_1 + I_2\Psi I_2 + I_3\Psi I_3 = 0, \\ \Psi = \Psi_{[-1]} &\iff \Psi - I_1\Psi I_1 - I_2\Psi I_2 - I_3\Psi I_3 = 0. \end{aligned}$$

With a short calculation one sees that the $Sp(n) Sp(1)$ -invariant components are the projections on the eigenspaces of the Casimir operator $\Upsilon = I_1 \otimes I_1 + I_2 \otimes I_2 + I_3 \otimes I_3$ corresponding, respectively, to the eigenvalues 3 and -1 ; see [Capria and Salamon 1988]. If $n = 1$ then the space of symmetric endomorphisms commuting with all I_s is 1-dimensional; i.e., the $[3]$ -component of any symmetric endomorphism Ψ on H is proportional to the identity, $\Psi_{[3]} = -(\text{tr } \Psi/4)\text{id}_H$. Note here that each of the three 2-forms ω_s belongs to its $[-1]$ -component, $\omega_s = \omega_{s[-1]}$, and constitutes a basis of the Lie algebra $\mathfrak{sp}(1)$.

2B. The torsion tensor. Decomposing the endomorphism $T_\xi \in (\mathfrak{sp}(n) + \mathfrak{sp}(1))^\perp$ into its symmetric part T_ξ^0 and skew-symmetric part b_ξ , $T_\xi = T_\xi^0 + b_\xi$, Biquard [2000] showed that the torsion T_ξ is completely trace-free, $\text{tr } T_\xi = \text{tr } T_\xi \circ I_s = 0$, its symmetric part has the properties

$$T_{\xi_i}^0 I_i = -I_i T_{\xi_i}^0, \quad I_2(T_{\xi_2}^0)^{+--} = I_1(T_{\xi_1}^0)^{-+-}, \quad I_3(T_{\xi_3}^0)^{-+-} = I_2(T_{\xi_2}^0)^{--+}, \quad I_1(T_{\xi_1}^0)^{--+} = I_3(T_{\xi_3}^0)^{+--}.$$

The skew-symmetric part can be represented as $b_{\xi_i} = I_i u$, where u is a traceless symmetric $(1, 1)$ -tensor on H which commutes with I_1, I_2, I_3 . Therefore we have $T_{\xi_i} = T_{\xi_i}^0 + I_i u$. If $n = 1$ then the tensor u vanishes identically, $u = 0$, and the torsion is a symmetric tensor, $T_\xi = T_\xi^0$.

The two $Sp(n) Sp(1)$ -invariant trace-free symmetric 2-tensors $T^0(X, Y) = g((T_{\xi_1}^0 I_1 + T_{\xi_2}^0 I_2 + T_{\xi_3}^0 I_3)X, Y)$, $U(X, Y) = g(uX, Y)$ on H , introduced in [Ivanov et al. 2014a], have the properties

$$\begin{aligned} T^0(X, Y) + T^0(I_1 X, I_1 Y) + T^0(I_2 X, I_2 Y) + T^0(I_3 X, I_3 Y) &= 0, \\ U(X, Y) = U(I_1 X, I_1 Y) = U(I_2 X, I_2 Y) = U(I_3 X, I_3 Y). \end{aligned} \tag{2-2}$$

In dimension 7 ($n = 1$), the tensor U vanishes identically, $U = 0$.

These tensors determine completely the torsion endomorphism of the Biquard connection due to the identity [Ivanov and Vassilev 2010, Proposition 2.3] $4T^0(\xi_s, I_s X, Y) = T^0(X, Y) - T^0(I_s X, I_s Y)$, which implies

$$4T(\xi_s, I_s X, Y) = 4T^0(\xi_s, I_s X, Y) + 4g(I_s u I_s X, Y) = T^0(X, Y) - T^0(I_s X, I_s Y) - 4U(X, Y).$$

2C. Curvature, torsion and qc Einstein structures. Quaternionic contact Einstein manifolds introduced in [Ivanov et al. 2014a], see [Ivanov et al. 2016; Ivanov and Vassilev 2011] for further details and a more leisurely exposition, play a crucial role in solving the Yamabe equation on the quaternionic sphere (see [Ivanov et al. 2010] for dimension 7).

Let $R = [\nabla, \nabla] - \nabla_{[\cdot, \cdot]}$ be the curvature of the Biquard connection ∇ . The Ricci tensor and the scalar curvature, called *qc Ricci tensor* and *qc scalar curvature*, respectively, are defined by

$$\begin{aligned} \text{Ric}(X, Y) &= g(R(e_a, X)Y, e_a), \\ \text{Scal} &= \text{Ric}(e_a, e_a) = g(R(e_b, e_a)e_a, e_b). \end{aligned}$$

According to [Biquard 2000] the Ricci tensor restricted to H is a symmetric tensor. If the trace-free part of the qc Ricci tensor is zero, we call the quaternionic structure a *qc Einstein manifold* [Ivanov et al. 2014a]. It is shown in that paper that the qc Ricci tensor is completely determined by the components of the torsion. Theorem 1.3, Theorem 3.12 and Corollary 3.14 in [Ivanov et al. 2014a] imply that on a qc manifold $(M^{4n+3}, g, \mathbb{Q})$ the qc Ricci tensor and the qc scalar curvature satisfy

$$\begin{aligned} \text{Ric}(X, Y) &= (2n + 2)T^0(X, Y) + (4n + 10)U(X, Y) + \frac{\text{Scal}}{4n}g(X, Y), \\ \text{Scal} &= -8n(n + 2)g(T(\xi_1, \xi_2), \xi_3). \end{aligned}$$

Hence, the qc Einstein condition is equivalent to the vanishing of the torsion endomorphism of the Biquard connection and in this case the qc scalar curvature is constant [Ivanov et al. 2014a; 2016]. If $\text{Scal} > 0$, the latter holds exactly when the qc structure is locally 3-Sasakian up to a multiplication by a constant and an $\text{SO}(3)$ -matrix with smooth entries. Recall that a $(4n+3)$ -dimensional Riemannian manifold (M, g) is called 3-Sasakian if the cone metric $g_N = t^2g + dt^2$ on $N = M \times \mathbb{R}^+$ is a hyper-Kähler metric; namely, it has holonomy contained in $\text{Sp}(n + 1)$. The 3-Sasakian manifolds are Einstein with positive Riemannian scalar curvature.

2D. qc conformal transformations. Let h be a positive smooth function on a qc manifold (M, η) . If $\eta = 2h\bar{\eta}$, we will say that the vector-valued 1-form η is qc conformal to $\bar{\eta}$. We will denote the objects related to $\bar{\eta}$ by overlining the same object corresponding to η . Thus,

$$d\bar{\eta} = -\frac{1}{2h^2}dh \wedge \eta + \frac{1}{2h}d\eta \quad \text{and} \quad \bar{g} = \frac{1}{2h}g.$$

The new triple $\{\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3\}$ is determined by the conditions defining the Reeb vector fields as $\bar{\xi}_s = 2h\xi_s + I_s \nabla h$, where ∇h is the horizontal gradient defined by $g(\nabla h, X) = dh(X)$. The components of the torsion tensor transform according to the following formulas from [Ivanov et al. 2014a, Section 5]:

$$\begin{aligned} \bar{T}^0(X, Y) &= T^0(X, Y) + h^{-1}[\nabla dh]_{[\text{sym}]_{[-1]}}(X, Y), \\ \bar{U}(X, Y) &= U(X, Y) + (2h)^{-1}[\nabla dh - 2h^{-1}dh \otimes dh]_{[3][0]}(X, Y), \end{aligned} \tag{2-3}$$

where the symmetric part is given by

$$[\nabla dh]_{[\text{sym}]}(X, Y) = \nabla dh(X, Y) + \sum_{s=1}^3 dh(\xi_s) \omega_s(X, Y)$$

and $[\]_{[0]}$ indicates the trace-free part of the $[3]$ -component of the corresponding tensor.

2E. The qc Yamabe problem. Under a qc conformal transformation, as described above, the qc scalar curvature changes according to the formula given in [Biquard 2000],

$$\overline{\text{Scal}} = 2h(\text{Scal}) - 8(n+2)^2 h^{-1} |\nabla h|^2 + 8(n+2)\Delta h. \quad (2-4)$$

Let $Q = 4n + 6$ be the so-called homogeneous dimension of M and $2^* = 2Q/(Q-2)$ the L^2 -Sobolev conjugate exponent. It will be suitable to take the conformal factor in the form $\bar{\eta} = u^{4/(Q-2)}\eta$, which turns (2-4) into the qc Yamabe equation

$$\mathcal{L}u \equiv 4\frac{Q+2}{Q-2}\Delta u - u \text{Scal} = -u^{2^*-1} \overline{\text{Scal}}, \quad (2-5)$$

where Δ is the horizontal sub-Laplacian, $\Delta h = \text{tr}^g(\nabla^2 h)$, Scal and $\overline{\text{Scal}}$ are the qc scalar curvatures correspondingly of (M, η) and $(M, \bar{\eta})$. Thus, within a fixed qc conformal class, the Yamabe problem is the question of the solvability of the quaternionic contact (qc) Yamabe equation (2-5).

From a variational point of view, the qc Yamabe equation (2-5) is essentially the Euler–Lagrange equation of the extremals of the L^2 case of the Sobolev-type embedding inequality determined by (1-1). By standard subelliptic regularity results, any nontrivial nonnegative weak solution $u \in \mathcal{D}^{1,2}(M)$ of (2-5) is smooth and positive. Hence the result of this article can also be interpreted as the characterization of all nonnegative weak solutions of (2-5) on any closed compact locally 3-Sasakian manifold.

It should be mentioned that the original motivation of the qc Yamabe equation comes from its connection with the determination of the norm and extremals in the L^2 Folland–Stein [1974] Sobolev-type embedding on the quaternionic Heisenberg group $\mathbf{G}(\mathbb{H})$. This problem was considered in the general setting of groups of Heisenberg type [Garofalo and Vassilev 2001; Vassilev 2006; 2000], where, in particular, the equality case was characterized completely in the space of functions with partial symmetry on groups of Iwasawa type. Later on, Frank and Lieb [2012], and independently, Branson, Fontana and Morpurgo [Branson et al. 2013] developed a method based on a center of mass technique which yielded the characterization of equality cases of several inequalities, including the L^2 Sobolev and Folland–Stein inequalities in the Euclidean and CR Heisenberg group cases. These results were extended to the quaternionic and octonionic settings in [Ivanov et al. 2012; Christ et al. 2016a; 2016b]. The current paper showed that similarly to the Riemannian and CR model flat cases, in the model qc cases the only critical level of the qc Yamabe functional restricted to nonnegative functions is its minimum.

3. qc conformal transformations of a qc Einstein manifold

Throughout this section h is a positive smooth function on a fixed qc Einstein manifold $(M, \bar{\eta}, \mathbb{Q})$ and $\eta = 2h\bar{\eta}$ is a qc structure which is qc conformal to $\bar{\eta}$. We assume, in addition, that the qc structure η is of

constant qc scalar curvature $\text{Scal} = 16n(n + 2)$ and hence equal to the qc scalar curvature of $\bar{\eta}$. We recall some formulas from [Ivanov et al. 2010] which will be used in the subsequent sections.

We begin by defining the vectors

$$A_i = I_i[\xi_j, \xi_k], \quad A = A_1 + A_2 + A_3. \tag{3-1}$$

We denote with the same letter the corresponding horizontal 1-forms, defined by $A_i(X) = g(A_i, X)$ etc. A short calculation, see [Ivanov et al. 2010, Lemma 3.1], gives the following expression of the 1-forms A_s and A in terms of h ,

$$\begin{aligned} A_i(X) = & -\frac{h^{-2}}{2}dh(X) - \frac{h^{-3}}{2}|\nabla h|^2dh(X) - \frac{h^{-1}}{2}(\nabla dh(I_j X, \xi_j) + \nabla dh(I_k X, \xi_k)) \\ & + \frac{h^{-2}}{2}(dh(\xi_j) dh(I_j X) + dh(\xi_k) dh(I_k X)) \\ & + \frac{h^{-2}}{4}(\nabla dh(I_j X, I_j \nabla h) + \nabla dh(I_k X, I_k \nabla h)). \end{aligned} \tag{3-2}$$

Thus, after summing, we have also

$$\begin{aligned} A(X) = & -\frac{3h^{-2}}{2}dh(X) - \frac{3h^{-3}}{2}|\nabla h|^2dh(X) \\ & - h^{-1} \sum_{s=1}^3 \nabla dh(I_s X, \xi_s) + h^{-2} \sum_{s=1}^3 dh(\xi_s) dh(I_s X) + \frac{h^{-2}}{2} \sum_{s=1}^3 \nabla dh(I_s X, I_s \nabla h). \end{aligned} \tag{3-3}$$

Second we consider the 1-forms

$$D_s(X) = -\frac{h^{-1}}{2}[T^0(X, \nabla h) + T^0(I_s X, I_s \nabla h)]. \tag{3-4}$$

For simplicity, using the musical isomorphism, we will denote by D_1, D_2, D_3 also the corresponding (horizontal) vector fields, defined by $g(D_i, X) = D_i(X)$. Let us consider in addition the form D defined as

$$D \stackrel{\text{def}}{=} D_1 + D_2 + D_3 = -h^{-1}T^0(X, \nabla h), \tag{3-5}$$

where the last equality follows from (2-2). Setting $\bar{T}^0 = 0$ in (2-3), we obtain from (3-4) the expressions (see [Ivanov et al. 2010; Ivanov and Vassilev 2010])

$$\begin{aligned} D_i(X) = & h^{-2} dh(\xi_i) dh(I_i X) \\ & + \frac{h^{-2}}{4}[\nabla dh(X, \nabla h) + \nabla dh(I_i X, I_i \nabla h) - \nabla dh(I_j X, I_j \nabla h) - \nabla dh(I_k X, I_k \nabla h)]. \end{aligned} \tag{3-6}$$

The equalities (3-5) and (3-6) yield [Ivanov et al. 2010, Lemma 4.2]

$$D(X) = \frac{h^{-2}}{4} \left(3\nabla dh(X, \nabla h) - \sum_{s=1}^3 \nabla dh(I_s X, I_s \nabla h) \right) + h^{-2} \sum_{s=1}^3 dh(\xi_s) dh(I_s X). \tag{3-7}$$

Finally, we define the 1-forms (and corresponding vectors)

$$F_s(X) = -h^{-1}T^0(X, I_s \nabla h).$$

From the definition of F_i and (3-4) we find

$$F_i(X) = -h^{-1}T^0(X, I_i\nabla h) = -D_i(I_iX) + D_j(I_iX) + D_k(I_iX). \tag{3-8}$$

From [Ivanov et al. 2014a, Theorem 4.8] we have that on a $(4n+3)$ -dimensional qc manifold with constant qc scalar curvature the following Bianchi identities hold:

$$\nabla^*T^0 = (n+2)A, \quad \nabla^*U = \frac{1-n}{2}A. \tag{3-9}$$

With the help of (3-9) the following divergence formulas were proved in [Ivanov et al. 2010, Lemmas 4.2 and 4.3]:

$$\nabla^*D = |T^0|^2 - h^{-1}g(dh, D) - h^{-1}(n+2)g(dh, A) \tag{3-10}$$

and

$$\begin{aligned} \nabla^*\left(\sum_{s=1}^3 dh(\xi_s)F_s\right) &= \sum_{s=1}^3 [\nabla dh(I_s e_a, \xi_s)F_s(I_s e_a)] \\ &+ h^{-1}\sum_{s=1}^3 [dh(\xi_s)dh(I_s e_a)D(e_a) + (n+2)dh(\xi_s)dh(I_s e_a)A(e_a)]. \end{aligned} \tag{3-11}$$

4. The divergence formula

This section contains our main technical result. As mentioned in the Introduction, we were motivated to seek a divergence formula of this type based on the Riemannian, CR and 7-dimensional qc cases of the considered problem. The main difficulty was to find a suitable vector field with nonnegative divergence containing the norm of the torsion. The fulfillment of this task was facilitated by the results of [Ivanov et al. 2014a]. In particular, similarly to the CR case, but unlike the Riemannian case, we were not able to achieve a proof based purely on the Bianchi identities; see [Ivanov et al. 2014a, Theorem 4.8]. Recall that the setting here is the same as in Section 3. Since

$$\text{Scal} = \overline{\text{Scal}} = 16n(n+2),$$

the Yamabe equation (2-4) gives

$$\Delta h = 2n - 4nh + h^{-1}(n+2)|\nabla h|^2. \tag{4-1}$$

Equation (2-3) in the case of a qc Einstein structure $\bar{\eta}$, $\bar{T}^0 = \bar{U} = 0$, and (4-1) motivate the definition of the symmetric $(0, 2)$ -tensors

$$D(X, Y) = -T^0(X, Y) = \frac{h^{-1}}{4}\left[3\nabla^2 h(X, Y) - \sum_{s=1}^3 \nabla^2 h(I_s X, I_s Y) + 4\sum_{s=1}^3 dh(\xi_s)\omega_s(X, Y)\right], \tag{4-2}$$

$$E(X, Y) = -2U(X, Y)$$

$$\begin{aligned} &= \frac{h^{-1}}{4}\left[\nabla^2 h(X, Y) + \sum_{s=1}^3 \nabla^2 h(I_s X, I_s Y)\right] - \frac{2h^{-2}}{4}\left[dh(X)dh(Y) + \sum_{s=1}^3 dh(I_s X)dh(I_s Y)\right] \\ &\quad - \frac{h^{-1}}{4}(2-4h+h^{-1}|\nabla h|^2)g(X, Y). \end{aligned} \tag{4-3}$$

The 1-form D defined in (3-5) and expressed in terms of h in (3-7) satisfies $D(X) = h^{-1} \mathbf{D}(X, \nabla h)$. Define, in addition, the 1-form E by the equation

$$E(X) = h^{-1} \mathbf{E}(X, \nabla h) = -2h^{-1} U(X, \nabla h) = \frac{h^{-2}}{4} \left[\nabla^2 h(X, \nabla h) + \sum_{s=1}^3 \nabla^2 h(I_s X, I_s \nabla h) + (-2 + 4h - 3h^{-1} |\nabla h|^2) dh(X) \right], \tag{4-4}$$

where the second and third equalities follow from (4-3).

Finally, in addition to the 1-forms D and E and the symmetric (0, 2)-tensors \mathbf{D} and \mathbf{E} , we define the (0, 3)-tensors \mathbb{D} and \mathbb{E} as

$$\mathbb{D}(X, Y, Z) = -\frac{h^{-1}}{8} \left[dh(X) T^0(Y, Z) + dh(Y) T^0(X, Z) + \sum_{s=1}^3 dh(I_s X) T^0(I_s Y, Z) + \sum_{s=1}^3 dh(I_s Y) T^0(I_s X, Z) \right], \tag{4-5}$$

$$\mathbb{E}(X, Y, Z) = \frac{h^{-1}}{8} \left[dh(X) \mathbf{E}(Y, Z) + dh(Y) \mathbf{E}(X, Z) + \sum_{s=1}^3 dh(I_s X) \mathbf{E}(I_s Y, Z) + \sum_{s=1}^3 dh(I_s Y) \mathbf{E}(I_s X, Z) \right]. \tag{4-6}$$

After this preparation we are ready to state the main result.

Theorem 4.1. *Suppose (M^{4n+3}, η) is a quaternionic contact structure conformal to a 3-Sasakian structure $(M^{4n+3}, \bar{\eta})$ with $\eta = 2h\bar{\eta}$. If $\text{Scal}_\eta = \text{Scal}_{\bar{\eta}} = 16n(n + 2)$, then with f given by*

$$f = \frac{1}{2} + h + \frac{h^{-1}}{4} |\nabla h|^2, \tag{4-7}$$

the following identity holds:

$$\nabla^* \left(f(D + E) + \sum_{s=1}^3 dh(\xi_s) I_s E + \sum_{s=1}^3 dh(\xi_s) F_s + 4 \sum_{s=1}^3 dh(\xi_s) I_s A_s - \frac{10}{3} \sum_{s=1}^3 dh(\xi_s) I_s A \right) = \left(\frac{1}{2} + h \right) (|T^0|^2 + |\mathbf{E}|^2) + 2h|\mathbb{D} + \mathbb{E}|^2 + h \langle QV, V \rangle. \tag{4-8}$$

Here, the matrix Q is given by

$$Q = \begin{bmatrix} \frac{5}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -2 & -2 & -2 \\ -\frac{1}{2} & \frac{5}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{10}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{5}{2} & -\frac{1}{2} & -\frac{2}{3} & \frac{10}{3} & -\frac{2}{3} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{5}{2} & -\frac{2}{3} & -\frac{2}{3} & \frac{10}{3} \\ -2 & \frac{10}{3} & -\frac{2}{3} & -\frac{2}{3} & \frac{22}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -2 & -\frac{2}{3} & \frac{10}{3} & -\frac{2}{3} & -\frac{2}{3} & \frac{22}{3} & -\frac{2}{3} \\ -2 & -\frac{2}{3} & -\frac{2}{3} & \frac{10}{3} & -\frac{2}{3} & -\frac{2}{3} & \frac{22}{3} \end{bmatrix}, \tag{4-9}$$

and $V = (E, D_1, D_2, D_3, A_1, A_2, A_3)$ with E, D_s, A_s defined, correspondingly, in (4-4), (3-4) and (3-1). In particular, Q is a positive definite matrix with eigenvalues $1, \frac{9}{2} \pm \frac{\sqrt{73}}{2}$ and $\frac{11}{2} \pm \frac{\sqrt{89}}{2}$.

Proof. For the sake of making some formulas more compact, in the proof we will use sometimes the notation $XY = g(X, Y)$ for the product of two horizontal vector fields X and Y and the similar abbreviation for horizontal 1-forms.

We begin by recalling (3-7), (4-4) and (3-3), which imply

$$A(X) = \frac{3E(X) - D(X)}{2} - h^{-1} \sum_{s=1}^3 \nabla^2 h(I_s X, \xi_s) + \frac{3h^{-2}}{2} \sum_{s=1}^3 dh(\xi_s)dh(I_s X) - \frac{3h^{-2}}{2} \left(\frac{1}{2} + h + \frac{1}{4}h^{-1}|\nabla h|^2 \right) dh(X). \quad (4-10)$$

Using the function f defined in (4-7), we write (4-10) in the form

$$2 \sum_{s=1}^3 \nabla^2 h(I_s X, \xi_s) = h(3E(X) - D(X) - 2A(X)) + 3h^{-1} \sum_{s=1}^3 dh(\xi_s)dh(I_s X) - 3h^{-1} f dh(X). \quad (4-11)$$

The sum of (3-7) and (4-4) yields

$$(E + D)(X) = h^{-2} \nabla^2 h(X, \nabla h) + h^{-2} \sum_{s=1}^3 dh(\xi_s)dh(I_s X) + \frac{h^{-2}}{4} (-2 + 4h - 3h^{-1}|\nabla h|^2) dh(X). \quad (4-12)$$

Using (4-7) and (4-12), we obtain

$$2\nabla_X f = h(E + D)(X) - h^{-1} \sum_{s=1}^3 dh(\xi_s)dh(I_s X) + h^{-1} f dh(X). \quad (4-13)$$

We calculate the divergences of E using first (4-4) to obtain

$$\nabla^* E = 2h^{-2} dh(e_a)U(e_a, \nabla h) - 2h^{-1} (\nabla_{e_a} U)(e_a, \nabla h) - 2h^{-1} U(e_a, e_b) \nabla^2 h(e_a, e_b).$$

Taking into account the Bianchi identity (3-9), (4-3) and (4-4) it follows

$$\begin{aligned} \nabla^* E &= (n - 1)h^{-1} A(\nabla h) + U(e_a, e_b)(-2h^{-1})[\nabla^2 h(e_a, e_b) - 2h^{-1} dh(e_a)dh(e_b)] + h^{-1} E(\nabla h) \\ &= |E|^2 + h^{-1} E(\nabla h) + (n - 1)h^{-1} A(\nabla h). \end{aligned} \quad (4-14)$$

Similarly, we have

$$\begin{aligned} -\nabla^* I_s E &= 2h^{-2} dh(e_a)U(I_s e_a, \nabla h) + 2h^{-1} (\nabla_{e_a} U)(e_a, I_s \nabla h) - 2h^{-1} U(I_s e_a, e_b) \nabla^2 h(e_a, e_b) \\ &= h^{-1} (1-n)A(I_s \nabla h) + U(I_s e_a, e_b)(-2h^{-1})[\nabla^2 h(e_a, e_b) - 2h^{-1} dh(e_a)dh(e_b)] + h^{-1} E(I_s \nabla h) \\ &= U(I_s e_a, e_b)U(e_a, e_b) - h^{-1} (1-n)dh(I_s e_a)A(e_a) = -h^{-1} (1-n)dh(I_s e_a)A(e_a), \end{aligned} \quad (4-15)$$

since $U(I_s e_a, e_b)U(e_a, e_b) = E(I_s \nabla h) = 0$ due to (2-2).

Now we are prepared to calculate the divergence of the first four terms. Using (3-10), (3-11), (4-14), (4-13), (4-15) and (4-11), we have

$$\begin{aligned}
 & \nabla_{e_a} \left[f(D+E)(e_a) - \sum_{s=1}^3 dh(\xi_s)E(I_s e_a) + \sum_{s=1}^3 dh(\xi_s)F_s(e_a) \right] \\
 &= \left(\frac{h}{2}(E+D)(e_a) - \frac{h^{-1}}{2} \sum_{s=1}^3 dh(\xi_s)dh(I_s e_a) + \frac{h^{-1}}{2} f dh(e_a) \right) (D+E)(e_a) \\
 &+ f \left[-h^{-1}D(\nabla h) - h^{-1}(n+2)A(\nabla h) + |T^0|^2 + |E|^2 + h^{-1}dh(e_a)E(e_a) - h^{-1}(1-n)dh(e_a)A(e_a) \right] \\
 &+ h^{-1}(1-n) \sum_{s=1}^3 dh(\xi_s)dh(I_s e_a)A(e_a) + \sum_{s=1}^3 \nabla^2 h(I_s e_a, \xi_s)E(e_a) + \sum_{s=1}^3 \nabla^2 h(I_s e_a, \xi_s)F_s(I_s e_a) \\
 &+ h^{-1} \sum_{s=1}^3 dh(\xi_s)dh(I_s e_a)D(e_a) + h^{-1}(n+2) \sum_{s=1}^3 dh(\xi_s)dh(I_s e_a)A(e_a) \\
 &= f(|T^0|^2 + |E|^2) + \frac{h}{2}|D+E|^2 + \frac{h}{2}(3E-D-2A)(e_a)E(e_a) \\
 &+ h^{-1} \left[\sum_{s=1}^3 dh(\xi_s)dh(I_s e_a) - f dh(e_a) \right] \left(\frac{1}{2}D(e_a) + 3A(e_a) \right) + \sum_{s=1}^3 \nabla^2 h(I_s e_a, \xi_s)F_s(I_s e_a). \quad (4-16)
 \end{aligned}$$

At this point we will use that for any smooth function h on a qc manifold with constant qc scalar curvature the following formulas hold true [Ivanov et al. 2010, Lemma 4.1]:

$$\begin{aligned}
 \nabla^* \left(\sum_{s=1}^3 dh(\xi_s)I_s A_s \right) &= \sum_{s=1}^3 \nabla dh(I_s e_a, \xi_s)A_s(e_a), \\
 \nabla^* \left(\sum_{s=1}^3 dh(\xi_s)I_s A \right) &= \sum_{s=1}^3 \nabla dh(I_s e_a, \xi_s)A(e_a).
 \end{aligned} \quad (4-17)$$

Applying (4-17) and (4-11) we obtain

$$\begin{aligned}
 & \nabla_{e_a} \left[f(D+E)(e_a) - \sum_{s=1}^3 dh(\xi_s)E(I_s e_a) + \sum_{s=1}^3 dh(\xi_s)F_s(e_a) - 2 \sum_{s=1}^3 dh(\xi_s)I_s A(e_a) \right] \\
 &= f(|T^0|^2 + |E|^2) + \frac{h}{2}|D+E|^2 + \frac{h}{2}(3E-D-2A)E - h(3E-D-2A)A \\
 &+ \frac{h^{-1}}{2} \left[\sum_{s=1}^3 dh(\xi_s)dh(I_s e_a) - f dh(e_a) \right] D(e_a) + \sum_{s=1}^3 \nabla^2 h(I_s e_a, \xi_s)F_s(I_s e_a). \quad (4-18)
 \end{aligned}$$

According to (3-8), the last term in (4-18) reads

$$\begin{aligned}
 \sum_{s=1}^3 \nabla^2 h(I_s e_a, \xi_s)F_s(I_s e_a) &= D_1(e_a)[\nabla^2 h(I_1 e_a, \xi_1) - \nabla^2 h(I_2 e_a, \xi_2) - \nabla^2 h(I_3 e_a, \xi_3)] \\
 &+ D_2(e_a)[-\nabla^2 h(I_1 e_a, \xi_1) + \nabla^2 h(I_2 e_a, \xi_2) - \nabla^2 h(I_3 e_a, \xi_3)] \\
 &+ D_3(e_a)[-\nabla^2 h(I_1 e_a, \xi_1) - \nabla^2 h(I_2 e_a, \xi_2) + \nabla^2 h(I_3 e_a, \xi_3)]. \quad (4-19)
 \end{aligned}$$

Using (4-19) we rewrite the last line in (4-18) as

$$\begin{aligned} & \left[\frac{h^{-1}}{2} \sum_{s=1}^3 dh(\xi_s)dh(I_s e_a) - \frac{h^{-1}}{2} f dh(e_a) \right] D(e_a) + \sum_{s=1}^3 \nabla^2 h(I_s e_a, \xi_s) F_s(I_s e_a) \\ &= D_1(e_a) \left[\nabla^2 h(I_1 e_a, \xi_1) - \nabla^2 h(I_2 e_a, \xi_2) - \nabla^2 h(I_3 e_a, \xi_3) + \frac{h^{-1}}{2} \sum_{s=1}^3 dh(\xi_s)dh(I_s e_a) - \frac{h^{-1}}{2} f dh(e_a) \right] \\ &+ D_2(e_a) \left[-\nabla^2 h(I_1 e_a, \xi_1) + \nabla^2 h(I_2 e_a, \xi_2) - \nabla^2 h(I_3 e_a, \xi_3) + \frac{h^{-1}}{2} \sum_{s=1}^3 dh(\xi_s)dh(I_s e_a) - \frac{h^{-1}}{2} f dh(e_a) \right] \\ &+ D_3(e_a) \left[-\nabla^2 h(I_1 e_a, \xi_1) - \nabla^2 h(I_2 e_a, \xi_2) + \nabla^2 h(I_3 e_a, \xi_3) \right. \\ &\quad \left. + \frac{h^{-1}}{2} \sum_{s=1}^3 dh(\xi_s)dh(I_s e_a) - \frac{h^{-1}}{2} f dh(e_a) \right]. \end{aligned} \tag{4-20}$$

The equalities (4-4), (3-6) and (3-2) imply

$$\nabla^2 h(I_2 X, \xi_2) + \nabla^2 h(I_3 X, \xi_3) = h(E - D_1 - 2A_1)(X) + h^{-1} \sum_{s=1}^3 dh(\xi_s)dh(I_s X) - h^{-1} f dh(X). \tag{4-21}$$

Subtracting two times (4-21) from (4-11) we obtain

$$\begin{aligned} & \nabla^2 h(I_1 e_a, \xi_1) - \nabla^2 h(I_2 e_a, \xi_2) - \nabla^2 h(I_3 e_a, \xi_3) + \frac{h^{-1}}{2} \sum_{s=1}^3 dh(\xi_s)dh(I_s e_a) - \frac{h^{-1}}{2} f dh(e_a) \\ &= \frac{h}{2} [-E - D + 4D_1 - 2A + 8A_1](e_a). \end{aligned} \tag{4-22}$$

The left-hand side of the above identity is the second line in (4-20). The other two lines are evaluated similarly and the formulas are obtained from the above by a cyclic rotation of {1, 2, 3}. A substitution of the resulting new form of (4-20) in (4-18) gives

$$\begin{aligned} & \nabla_{e_a} \left[f(D + E)(e_a) - \sum_{s=1}^3 dh(\xi_s)E(I_s e_a) + \sum_{s=1}^3 dh(\xi_s)F_s(e_a) - 2 \sum_{s=1}^3 dh(\xi_s)I_s A(e_a) \right] \\ &= f(|T^0|^2 + |E|^2) + \frac{4h}{2} [E^2 + A^2 + D_1^2 + D_2^2 + D_3^2 - 2AE + 2A_1 D_1 + 2A_2 D_2 + 2A_3 D_3]. \end{aligned} \tag{4-23}$$

In view of (4-17) for any nonzero constant c we calculate the divergences as

$$\begin{aligned} & \nabla_{e_a} \left(c \sum_{s=1}^3 dh(\xi_s)I_s A_s(e_a) - \frac{c}{3} \sum_{s=1}^3 dh(\xi_s)I_s A(e_a) \right) \\ &= \frac{c}{3} [2\nabla^2 h(I_1 e_a, \xi_1) - \nabla^2 h(I_2 e_a, \xi_2) - \nabla^2 h(I_3 e_a, \xi_3)] A_1(e_a) \\ &+ \frac{c}{3} [2\nabla^2 h(I_2 e_a, \xi_2) - \nabla^2 h(I_1 e_a, \xi_1) - \nabla^2 h(I_3 e_a, \xi_3)] A_2(e_a) \\ &+ \frac{c}{3} [2\nabla^2 h(I_3 e_a, \xi_3) - \nabla^2 h(I_2 e_a, \xi_2) - \nabla^2 h(I_1 e_a, \xi_1)] A_3(e_a). \end{aligned} \tag{4-24}$$

Subtracting (4-21) from twice (4-11) yields

$$2\nabla^2 h(I_1 e_a, \xi_1) - \nabla^2 h(I_2 e_a, \xi_2) - \nabla^2 h(I_3 e_a, \xi_3) = h[2D_1 - D_2 - D_3 + 4A_1 - 2A_2 - 2A_3](e_a). \tag{4-25}$$

Now, taking into account (4-25), (4-24) and (4-23) we obtain

$$\begin{aligned} \nabla^* \left[f(D+E)(X) - \sum_{s=1}^3 dh(\xi_s) E(I_s X) + \sum_{s=1}^3 dh(\xi_s) F_s(X) - 2 \sum_{s=1}^3 dh(\xi_s) I_s A(X) \right] \\ + \nabla^* \left[c \sum_{s=1}^3 dh(\xi_s) I_s A_s(X) - \frac{c}{3} \sum_{s=1}^3 dh(\xi_s) I_s A(X) \right] \\ = f(|T^0|^2 + |E|^2) + \frac{4h}{2} [E^2 + A^2 + D_1^2 + D_2^2 + D_3^2 - 2AE + 2A_1 D_1 + 2A_2 D_2 + 2A_3 D_3] \\ + h \frac{c}{3} [(2D_1 - D_2 - D_3 + 4A_1 - 2A_2 - 2A_3) A_1] \\ + h \frac{c}{3} [(2D_2 - D_1 - D_3 + 4A_2 - 2A_1 - 2A_3) A_2] \\ + h \frac{c}{3} [(2D_3 - D_2 - D_1 + 4A_3 - 2A_2 - 2A_1) A_3]. \end{aligned} \tag{4-26}$$

In the next lemma we use again the notation $XY = g(X, Y)$ for the product of two horizontal vector fields X and Y and the similar abbreviation for horizontal 1-forms.

Lemma 4.2. *For the (0, 3)-tensors \mathbb{D} and \mathbb{E} defined by (4-5) and (4-6) we have*

$$\begin{aligned} |\mathbb{D}|^2 &= \frac{h^{-2}}{8} |\nabla h|^2 |T^0|^2 - \frac{1}{4} \sum_{s=1}^3 |D_s|^2 + \frac{1}{2} (D_1 D_2 + D_1 D_3 + D_2 D_3), \\ |\mathbb{E}|^2 &= \frac{h^{-2}}{8} |\nabla h|^2 |E|^2 - \frac{1}{4} |E|^2, \quad \mathbb{D}\mathbb{E} = \frac{1}{4} \sum_{s=1}^3 E D_s. \end{aligned} \tag{4-27}$$

Consequently,

$$\begin{aligned} \frac{h^{-2}}{4} |\nabla h|^2 (|T^0|^2 + |E|^2) \\ = 2|\mathbb{D} + \mathbb{E}|^2 - \sum_{s=1}^3 E D_s + \frac{1}{2} |E|^2 + \frac{1}{2} \sum_{s=1}^3 |D_s|^2 - (D_1 D_2 + D_1 D_3 + D_2 D_3). \end{aligned} \tag{4-28}$$

Proof. We shall repeatedly apply (2-2) and the defining equations (4-5), (4-6), (3-1) and (3-5). We have

$$\begin{aligned} |\mathbb{D}|^2 &= \frac{h^{-2}}{8} |\nabla h|^2 |T^0|^2 + \frac{h^{-2}}{8^2} \left(2T^0(\nabla h, e_c) T^0(\nabla h, e_c) - 4 \sum_{s=1}^3 T^0(I_s \nabla h, e_c) T^0(I_s \nabla h, e_c) \right. \\ &\quad \left. + 2 \sum_{s,t=1}^3 T^0(I_s I_t \nabla h, e_c) T^0(I_t I_s \nabla h, e_c) \right) \\ &= \frac{h^{-2}}{8} |\nabla h|^2 |T^0|^2 + \frac{1}{4} \left(- \sum_{s=1}^3 D_s^2 + 2(D_1 D_2 + D_1 D_3 + D_2 D_3) \right), \end{aligned} \tag{4-29}$$

which is the first line of (4-27). For example, the third term in (4-29) is calculated as

$$\begin{aligned} & 2h^{-2} \sum_{s,t=1}^3 T^0(I_s I_t \nabla h, e_c) T^0(I_t I_s \nabla h, e_c) \\ &= 2h^{-2} \sum_{s=1}^3 [T^0(\nabla h, e_c) T^0(\nabla h, e_c) - 2T^0(I_s \nabla h, e_c) T^0(I_s \nabla h, e_c)] \\ &= 6|D|^2 - 12 \sum_{s=1}^3 D_s^2 + 8(D_1 D_2 + D_1 D_3 + D_2 D_3) = -6 \sum_{s=1}^3 D_s^2 + 20(D_1 D_2 + D_1 D_3 + D_2 D_3), \end{aligned}$$

recalling the definition (3-5).

Similarly, we obtain the second line of (4-27). The equality (4-28) follows from (4-27), which completes the proof of Lemma 4.2. □

Finally, the proof of Theorem 4.1 follows by letting $c = 4$ in (4-26) and using (4-28) and (3-1). □

5. Proof of Theorems 1.3 and 1.1

5A. Proof of Theorem 1.3. The first step of the proof relies on Theorem 4.1. By a homothety we can suppose that both qc scalar curvatures are equal to $16n(n + 2)$. Integrating the divergence formula of Theorem 4.1 and then using the divergence theorem established in [Ivanov et al. 2014a, Proposition 8.1] shows that the integral of the left-hand side is zero. Thus,

$$\int_M \left(\frac{1}{2} + h\right) (|T^0|^2 + |E|^2) + 2h|\mathbb{D} + \mathbb{E}|^2 + h\langle QV, V \rangle = 0,$$

which, due to the fact that the matrix Q (4-9) is nonnegative and taking into account (4-3), shows that the quaternionic contact structure η has vanishing torsion, i.e., it is also qc Einstein according to [Ivanov et al. 2014a, Proposition 4.2]. This proves the first part of Theorem 1.3.

To prove the second part, we develop a sub-Riemannian extension of the result of [Obata 1971], see also [Bourguignon and Ezin 1987] and the review [Ivanov and Vassilev 2015, Theorem 2.6], on the relation between the Yamabe equation and the Lichnerowicz–Obata first eigenvalue estimate. We begin by recalling some results from [Ivanov et al. 2014a, Section 7.2]. A vector field Q on a qc manifold (M, η) is a *qc vector field* if its flow preserves the qc structure,

$$\mathcal{L}_Q \eta = (vI + O) \cdot \eta,$$

where v is a smooth function and $O \in so(3)$ is a matrix-valued function with smooth entries; see [Ivanov et al. 2014a, Definition 7.7] and the discussion preceding it. In fact, taking into account [Ivanov et al. 2014a, Lemma 2.2; 2017, Lemma 5.1], a vector field Q on a qc manifold (M, η) is a qc vector field if its flow preserves the horizontal distribution $H = \ker \eta$. Since the exterior derivative d commutes with the Lie derivative \mathcal{L}_Q , any qc vector field Q satisfies

$$\mathcal{L}_Q g = vg, \quad \mathcal{L}_Q I = O \cdot I, \quad I = (I_1, I_2, I_3)^t,$$

which is equivalent to saying that the flow of Q preserves the conformal class $[g]$ of the horizontal metric and the quaternionic structure \mathbb{Q} on H . The function ν can be easily expressed in terms of the divergence (with respect to g) of the horizontal part Q_H of the vector field Q . Indeed, from [Ivanov et al. 2014a, Lemma 7.12] we have

$$g(\nabla_X Q_H, Y) + g(\nabla_Y Q_H, X) + 2\eta_s(Q)g(T_{\xi_s}^0 X, Y) = \nu g(X, Y);$$

hence

$$\nu = \frac{1}{2n} \nabla^* Q_H.$$

This gives a geometric interpretation for the quantity $(\nabla^* Q_H)$; namely, the flow of a qc vector field Q preserves a fixed metric $g \in [g]$ if and only if $\nabla^* Q_H = 0$.

As an infinitesimal version of the qc Yamabe equation, we obtain the following general fact concerning the divergence of a qc vector field.

Lemma 5.1. *Let (M, η) be a qc manifold. For any qc vector field Q on M we have*

$$\Delta(\nabla^* Q_H) = -\frac{n}{2(n+2)} Q(\text{Scal}) - \frac{\text{Scal}}{4(n+2)} \nabla^* Q_H,$$

where Scal , ∇^* , Δ and the projection Q_H correspond to the contact form η .

Proof. Suppose Q is a qc vector field and let ϕ_t be the corresponding (local) 1-parameter group of diffeomorphisms generated by its flow. Then

$$\phi_t^*(\eta) = \frac{1}{2h_t} \eta \quad \text{and} \quad \phi_t^*(g) = \frac{1}{2h_t} g$$

for some positive function h_t , depending smoothly on the parameter t . The qc scalar curvature Scal_t of the pull back contact form $\phi_t^*(\eta)$ is given by $\text{Scal}_t = \text{Scal} \circ \phi_t$. Then, formula (2-4) yields

$$\text{Scal} \circ \phi_t = 2h_t(\text{Scal}) - 8(n+2)^2 h_t^{-1} |\nabla h_t|^2 + 8(n+2)\Delta h_t. \tag{5-1}$$

Clearly, we have $h_0 = \frac{1}{2}$, and from

$$\frac{1}{2n} (\nabla^* Q_H) g = \mathcal{L}_Q g = \left. \frac{d}{dt} \right|_{t=0} \left(\frac{1}{2h_t} g \right) = -\frac{h'_0}{2h_0} g = -2h'_0 g$$

we obtain that

$$h'_0 = -\frac{1}{4n} \nabla^* Q_H,$$

where h'_0 denotes the derivative of h_t at $t = 0$. A differentiation at $t = 0$ in (5-1) gives the lemma. \square

Lemma 5.2. *Let (M, η) and $(M, \bar{\eta})$ be qc Einstein manifolds with equal qc scalar curvatures $16n(n+2)$. If η and $\bar{\eta}$ are qc conformal to each other, $\bar{\eta} = \eta/(2h)$ for some smooth positive function h , then*

$$Q = \frac{1}{2} \nabla f + \sum_{s=1}^3 dh(\xi_s) \xi_s \tag{5-2}$$

is a qc vector field on M , where the function f is defined in (4-7).

Proof. The assumption of the lemma implies that $E = D = D_s = A_s = 0$. Using (4-21), (4-22) and (4-13) we obtain $\nabla^2 h(I_s X, \xi_s) = -df(X)$; thus

$$\nabla^2 h(X, \xi_s) = df(I_s X). \tag{5-3}$$

It follows that

$$\sum_{s=1}^3 \nabla_X (dh(\xi_s)\xi_s) = \sum_{s=1}^3 df(I_s X)\xi_s.$$

As observed in the introduction to the section, it is enough to show that the flow of the vector field Q , defined by (5-2), preserves the horizontal distribution H . For any $X \in H$, we have

$$\begin{aligned} \mathcal{L}_Q(X) &= \frac{1}{2}[\nabla f, X] + \sum_{s=1}^3 [dh(\xi_s)\xi_s, X] \\ &= \frac{1}{2}\nabla_{\nabla f} X - \frac{1}{2}\nabla_X(\nabla f) - \sum_{s=1}^3 \omega_s(\nabla f, X)\xi_s + \sum_{s=1}^3 [dh(\xi_s)\nabla_{\xi_s} X - \nabla_X(dh(\xi_s)\xi_s) - dh(\xi_s)T_{\xi_s}(X)] \\ &= \frac{1}{2}\nabla_{\nabla f} X - \frac{1}{2}\nabla_X(\nabla f) + \sum_{s=1}^3 dh(\xi_s)\nabla_{\xi_s} X \in H. \end{aligned}$$

This completes the proof.

We note that, alternatively, using (5-3) a short calculation shows that Q satisfies the conditions of [Ivanov et al. 2014a, Corollary 7.9]. □

At this point we are ready to complete the proof of Theorem 1.3. Consider the qc vector field Q defined in Lemma 5.2. By Lemma 5.1, the function $\phi = \frac{1}{2}\Delta f$ is either an eigenfunction of the sub-Laplacian with eigenvalue $-4n$, $\Delta\phi = -4n\phi$, or it vanishes identically. In the first case, using the quaternionic contact version of the Lichnerowicz–Obata eigenfunction sphere theorem [Ivanov et al. 2013, Theorem 1.2; 2014b, Corollary 1.2] (see also [Baudoin and Kim 2014]), we conclude that (M, η) is the 3-Sasakian sphere. In the other case, we have that $\Delta f = 0$; hence

$$f = \frac{1}{2} + h + \frac{h^{-1}}{4}|\nabla h|^2 = \text{const.}$$

since M is compact. It follows that $h = \frac{1}{2}$ by considering the points where h achieves its minimum and maximum and taking into account the qc Yamabe equation (4-1). The proof of Theorem 1.3 is complete.

Remark 5.3. Lemma 5.2 provides also a certain geometric insight for the function f in (4-7). In fact, up to an additive constant, f is the unique function on M for which $Q_H = \frac{1}{2}\nabla f$ is the horizontal part of a qc vector field Q with vertical part $Q_V = dh(\xi_s)\xi_s$, $Q = Q_H + Q_V$. This assertion is an easy consequence of the computation given in the proof of Lemma 5.2. Moreover, it implies that on the 3-Sasakian sphere $\phi = \Delta f$ is an eigenfunction of the sub-Laplacian realizing the smallest possible eigenvalue $-4n$ on a compact locally 3-Sasakian manifold.

5B. Proof of Theorem 1.1. Theorem 1.1 is a direct corollary from Theorem 1.3. Alternatively, as in the proof of Theorem 1.3, we can use in the first step Theorem 4.1 which shows that the “new” structure is also qc Einstein. The second step of the proof of Theorem 1.1 follows then also by taking into account [Ivanov et al. 2014a, Theorem 1.2] where all locally 3-Sasakian structures of positive constant qc scalar curvature which are qc conformal to the standard 3-Sasakian structure on the sphere were classified (we note that this classification extends easily to the case when no sign condition of the new qc structure is assumed, see [Ivanov and Vassilev 2015]).

Acknowledgements

Ivanov was visiting University of Pennsylvania, Philadelphia during the writing of the paper. He thanks UPenn for providing the support and an excellent research environment. Ivanov is partially supported by the National Science Fund of Bulgaria, National Scientific Program “VIHREN”, Project No. KP-06-DV-7. Ivanov and Minchev are partially supported by Contract DH/12/3/12.12.2017 and Contract 80-10-161/05.04.2021 with the Sofia University “St. Kl. Ohridski”. Minchev is supported by a SoMoPro II Fellowship which is cofunded by the European Commission from “People” specific programme (Marie Curie Actions) within the EU Seventh Framework Programme on the basis of the grant agreement REA No. 291782. It is further cofinanced by the South-Moravian Region. Vassilev was partially supported by Simons Foundation grant no. 279381 and ARPA-E grant DE-AR0001202.

References

- [Baudoin and Kim 2014] F. Baudoin and B. Kim, “Sobolev, Poincaré, and isoperimetric inequalities for subelliptic diffusion operators satisfying a generalized curvature dimension inequality”, *Rev. Mat. Iberoam.* **30**:1 (2014), 109–131. [MR](#) [Zbl](#)
- [Biquard 1999] O. Biquard, “Quaternionic contact structures”, pp. 23–30 in *Quaternionic structures in mathematics and physics* (Rome, 1999), edited by S. Marchiafava et al., Sapienza Univ. Rome, 1999. [MR](#) [Zbl](#)
- [Biquard 2000] O. Biquard, *Métriques d’Einstein asymptotiquement symétriques*, *Astérisque* **265**, Soc. Math. France, Paris, 2000. [MR](#) [Zbl](#)
- [Bourguignon and Ezin 1987] J.-P. Bourguignon and J.-P. Ezin, “Scalar curvature functions in a conformal class of metrics and conformal transformations”, *Trans. Amer. Math. Soc.* **301**:2 (1987), 723–736. [MR](#) [Zbl](#)
- [Branson et al. 2013] T. P. Branson, L. Fontana, and C. Morpurgo, “Moser–Trudinger and Beckner–Onofri’s inequalities on the CR sphere”, *Ann. of Math. (2)* **177**:1 (2013), 1–52. [MR](#) [Zbl](#)
- [Capria and Salamon 1988] M. M. Capria and S. M. Salamon, “Yang–Mills fields on quaternionic spaces”, *Nonlinearity* **1**:4 (1988), 517–530. [MR](#) [Zbl](#)
- [Christ et al. 2016a] M. Christ, H. Liu, and A. Zhang, “Sharp Hardy–Littlewood–Sobolev inequalities on quaternionic Heisenberg groups”, *Nonlinear Anal.* **130** (2016), 361–395. [MR](#) [Zbl](#)
- [Christ et al. 2016b] M. Christ, H. Liu, and A. Zhang, “Sharp Hardy–Littlewood–Sobolev inequalities on the octonionic Heisenberg group”, *Calc. Var. Partial Differential Equations* **55**:1 (2016), art. id. 11. [MR](#) [Zbl](#)
- [Duchemin 2006] D. Duchemin, “Quaternionic contact structures in dimension 7”, *Ann. Inst. Fourier (Grenoble)* **56**:4 (2006), 851–885. [MR](#) [Zbl](#)
- [Folland and Stein 1974] G. B. Folland and E. M. Stein, “Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group”, *Comm. Pure Appl. Math.* **27** (1974), 429–522. [MR](#) [Zbl](#)

- [Frank and Lieb 2012] R. L. Frank and E. H. Lieb, “Sharp constants in several inequalities on the Heisenberg group”, *Ann. of Math.* (2) **176**:1 (2012), 349–381. [MR](#) [Zbl](#)
- [Garofalo and Vassilev 2001] N. Garofalo and D. Vassilev, “Symmetry properties of positive entire solutions of Yamabe-type equations on groups of Heisenberg type”, *Duke Math. J.* **106**:3 (2001), 411–448. [MR](#) [Zbl](#)
- [Gidas et al. 1979] B. Gidas, W. M. Ni, and L. Nirenberg, “Symmetry and related properties via the maximum principle”, *Comm. Math. Phys.* **68**:3 (1979), 209–243. [MR](#) [Zbl](#)
- [Ivanov and Vassilev 2010] S. Ivanov and D. Vassilev, “Conformal quaternionic contact curvature and the local sphere theorem”, *J. Math. Pures Appl.* (9) **93**:3 (2010), 277–307. [MR](#) [Zbl](#)
- [Ivanov and Vassilev 2011] S. P. Ivanov and D. N. Vassilev, *Extremals for the Sobolev inequality and the quaternionic contact Yamabe problem*, World Sci., Hackensack, NJ, 2011. [MR](#) [Zbl](#)
- [Ivanov and Vassilev 2015] S. Ivanov and D. Vassilev, “The Lichnerowicz and Obata first eigenvalue theorems and the Obata uniqueness result in the Yamabe problem on CR and quaternionic contact manifolds”, *Nonlinear Anal.* **126** (2015), 262–323. [MR](#) [Zbl](#)
- [Ivanov et al. 2010] S. Ivanov, I. Minchev, and D. Vassilev, “Extremals for the Sobolev inequality on the seven-dimensional quaternionic Heisenberg group and the quaternionic contact Yamabe problem”, *J. Eur. Math. Soc.* **12**:4 (2010), 1041–1067. [MR](#) [Zbl](#)
- [Ivanov et al. 2012] S. Ivanov, I. Minchev, and D. Vassilev, “The optimal constant in the L^2 Folland–Stein inequality on the quaternionic Heisenberg group”, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **11**:3 (2012), 635–652. [MR](#) [Zbl](#)
- [Ivanov et al. 2013] S. Ivanov, A. Petkov, and D. Vassilev, “The sharp lower bound of the first eigenvalue of the sub-Laplacian on a quaternionic contact manifold in dimension seven”, *Nonlinear Anal.* **93** (2013), 51–61. [MR](#) [Zbl](#)
- [Ivanov et al. 2014a] S. Ivanov, I. Minchev, and D. Vassilev, *Quaternionic contact Einstein structures and the quaternionic contact Yamabe problem*, Mem. Amer. Math. Soc. **1086**, Amer. Math. Soc., Providence, RI, 2014. [MR](#) [Zbl](#)
- [Ivanov et al. 2014b] S. Ivanov, A. Petkov, and D. Vassilev, “The sharp lower bound of the first eigenvalue of the sub-Laplacian on a quaternionic contact manifold”, *J. Geom. Anal.* **24**:2 (2014), 756–778. [MR](#) [Zbl](#)
- [Ivanov et al. 2016] S. Ivanov, I. Minchev, and D. Vassilev, “Quaternionic contact Einstein manifolds”, *Math. Res. Lett.* **23**:5 (2016), 1405–1432. [MR](#) [Zbl](#)
- [Ivanov et al. 2017] S. Ivanov, I. Minchev, and D. Vassilev, “Quaternionic contact hypersurfaces in hyper-Kähler manifolds”, *Ann. Mat. Pura Appl.* (4) **196**:1 (2017), 245–267. [MR](#) [Zbl](#)
- [Jerison and Lee 1988] D. Jerison and J. M. Lee, “Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem”, *J. Amer. Math. Soc.* **1**:1 (1988), 1–13. [MR](#) [Zbl](#)
- [Mostow 1973] G. D. Mostow, *Strong rigidity of locally symmetric spaces*, Ann. of Math. Stud. **78**, Princeton Univ. Press, 1973. [MR](#) [Zbl](#)
- [Obata 1971] M. Obata, “The conjectures on conformal transformations of Riemannian manifolds”, *J. Differential Geom.* **6**:2 (1971), 247–258. [MR](#) [Zbl](#)
- [Pansu 1989] P. Pansu, “Métriques de Carnot–Carathéodory et quasiisométries des espaces symétriques de rang un”, *Ann. of Math.* (2) **129**:1 (1989), 1–60. [MR](#) [Zbl](#)
- [Talenti 1976] G. Talenti, “Best constant in Sobolev inequality”, *Ann. Mat. Pura Appl.* (4) **110** (1976), 353–372. [MR](#) [Zbl](#)
- [Vassilev 2000] D. N. Vassilev, *Yamabe equations on Carnot groups*, Ph.D. thesis, Purdue University, 2000, available at <https://www.proquest.com/docview/275913953>.
- [Vassilev 2006] D. Vassilev, “Existence of solutions and regularity near the characteristic boundary for sub-Laplacian equations on Carnot groups”, *Pacific J. Math.* **227**:2 (2006), 361–397. [MR](#) [Zbl](#)
- [Wang 2007] W. Wang, “The Yamabe problem on quaternionic contact manifolds”, *Ann. Mat. Pura Appl.* (4) **186**:2 (2007), 359–380. [MR](#) [Zbl](#)

STEFAN IVANOV: ivanovsp@fmi.uni-sofia.bg

Faculty of Mathematics and Informatics, University of Sofia, Sofia, Bulgaria

and

Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia, Bulgaria

and

Department of Mathematics, University of Pennsylvania, Philadelphia, PA, United States

IVAN MINCHEV: minchev@fmi.uni-sofia.bg

Faculty of Mathematics and Informatics, University of Sofia, Sofia, Bulgaria

and

Department of Mathematics and Statistics, Masaryk University, Brno, Czech Republic

DIMITER VASSILEV: vassilev@unm.edu

Department of Mathematics and Statistics, University of New Mexico, Albuquerque, NM, United States

Analysis & PDE

msp.org/apde

EDITORS-IN-CHIEF

Patrick Gérard [Universit  Paris Sud XI, France](mailto:patrick.gerard@universite-paris-saclay.fr)
patrick.gerard@universite-paris-saclay.fr
Cl ment Mouhot [Cambridge University, UK](mailto:c.mouhot@dpmms.cam.ac.uk)
c.mouhot@dpmms.cam.ac.uk

BOARD OF EDITORS

Massimiliano Berti	Scuola Intern. Sup. di Studi Avanzati, Italy berti@sissa.it	Werner M�ller	Universit�t Bonn, Germany mueller@math.uni-bonn.de
Zbigniew Błocki	Uniwersytet Jagielloński, Poland zbigniew.blocki@uj.edu.pl	Gilles Pisier	Texas A&M University, and Paris 6 pisier@math.tamu.edu
Charles Fefferman	Princeton University, USA cf@math.princeton.edu	Igor Rodnianski	Princeton University, USA irod@math.princeton.edu
Isabelle Gallagher	Universit� Paris-Diderot, IMJ-PRG, France gallagher@math.ens.fr	Yum-Tong Siu	Harvard University, USA siu@math.harvard.edu
Colin Guillarmou	Universit� Paris-Saclay, France colin.guillarmou@universite-paris-saclay.fr	Terence Tao	University of California, Los Angeles, USA tao@math.ucla.edu
Ursula Hamenstaedt	Universit�t Bonn, Germany ursula@math.uni-bonn.de	Michael E. Taylor	Univ. of North Carolina, Chapel Hill, USA met@math.unc.edu
Vadim Kaloshin	University of Maryland, USA vadim.kaloshin@gmail.com	Gunther Uhlmann	University of Washington, USA gunther@math.washington.edu
Izabella Laba	University of British Columbia, Canada ilaba@math.ubc.ca	Andr�s Vasy	Stanford University, USA andras@math.stanford.edu
Anna L. Mazzucato	Penn State University, USA alm24@psu.edu	Dan Virgil Voiculescu	University of California, Berkeley, USA dvv@math.berkeley.edu
Richard B. Melrose	Massachusetts Inst. of Tech., USA rbm@math.mit.edu	Jim Wright	University of Edinburgh, UK j.r.wright@ed.ac.uk
Frank Merle	Universit� de Cergy-Pontoise, France merle@ihes.fr	Maciej Zworski	University of California, Berkeley, USA zworski@math.berkeley.edu
William Minicozzi II	Johns Hopkins University, USA minicozz@math.jhu.edu		

PRODUCTION

production@msp.org

Silvio Levy, Scientific Editor


See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2023 is US \$405/year for the electronic version, and \$630/year (+\$65, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online.

APDE peer review and production are managed by EditFlow[®] from MSP.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

  2023 Mathematical Sciences Publishers

ANALYSIS & PDE

Volume 16 No. 3 2023

Global regularity for the nonlinear wave equation with slightly supercritical power MARIA COLOMBO and SILJA HAFFTER	613
Subelliptic wave equations are never observable CYRIL LETROUIT	643
Quantitative Alexandrov theorem and asymptotic behavior of the volume preserving mean curvature flow VESA JULIN and JOONAS NIINIKOSKI	679
A simple nuclear C^* -algebra with an internal asymmetry ILAN HIRSHBERG and N. CHRISTOPHER PHILLIPS	711
Partial regularity of Leray–Hopf weak solutions to the incompressible Navier–Stokes equations with hyperdissipation WOJCIECH S. OŻAŃSKI	747
The Peskin problem with viscosity contrast EDUARDO GARCÍA-JUÁREZ, YOICHIRO MORI and ROBERT M. STRAIN	785
Solution of the qc Yamabe equation on a 3-Sasakian manifold and the quaternionic Heisenberg group STEFAN IVANOV, IVAN MINCHEV and DIMITER VASSILEV	839
Garland’s method with Banach coefficients IZHAR OPPENHEIM	861