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IZHAR OPPENHEIM

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GARLAND'S METHOD WITH BANACH COEFFICIENTS

IZHAR OPPENHEIM

We prove a Banach version of Garland's method of proving vanishing of cohomology for groups acting on simplicial complexes. The novelty of this new version is that our new condition applies to every reflexive Banach space.

This new version of Garland's method allows us to deduce several criteria for vanishing of group cohomology with coefficients in several classes of Banach spaces (uniformly curved spaces, Hilbertian spaces and L^p spaces).

Using these new criteria, we improve recent results regarding Banach fixed-point theorems for random groups in the triangular model and give a sharp lower bound for the conformal dimension of the boundary of such groups. Also, we derive new criteria for group stability with respect to p-Schatten norms.

1. Introduction

Let X be a locally finite, pure n-dimensional simplicial complex and Γ be a locally compact, unimodular group acting cocompactly and properly on X. Under the assumption that X is an affine building, Garland [1973] gave a local criterion for the vanishing of the equivariant k-th cohomology for any unitary representation of Γ and any $1 \leq k \leq n-1$. His approach was later generalized by Ballmann and Świątkowski [1997] to all simplicial complexes and this generalization is sometimes referred to as "Garland's method". There have been several generalizations of this method that considered the case where π is an isometric representation on a Banach space; see [Nowak 2015; Koivisto 2014; Oppenheim 2014]. However, all these generalizations gave somewhat weak results when applied to examples. For example, when considering vanishing of cohomology over L^p spaces, the results of [Nowak 2015] could not show vanishing of cohomology, for every $1 < p < \infty$, for $\tilde{A}_2$ groups nor for random groups (see Theorems 5.1 and 6.2 of that paper).

We note that Garland's original work referred to affine buildings, and in this set-up strong results regarding vanishing of cohomologies with Banach coefficients are known; see [Lafforgue 2009; Liao 2014; Lécureux et al. 2020] for results regarding vanishing of the first cohomology, and see [Oppenheim 2017; Lubotzky and Oppenheim 2020] for results regarding vanishing of higher cohomologies. However, much less is known when one considers the less-structured setting of a group acting on a simplicial complex without assuming the extra structure of an affine building.

Recently, the results for vanishing of the first cohomology of random groups with coefficients in Banach spaces were improved: First, Druțu and Mackay [2019] proved vanishing of the first cohomology

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for random groups for L^p spaces. Second, considering random groups in the triangular model, de Laat and de la Salle [2021] gave a criterion for vanishing of the first cohomology for a group acting on a 2-dimensional simplicial complex that was applicable to all uniformly curved Banach spaces (and in particular, to all L^p spaces).

The observation of de Laat and de la Salle was that one can get much stronger results than in previous works if the assumption of the spectral gap in the links is replaced with the assumption of a two-sided spectral gap (or equivalently with the assumption of contraction of the random walk operator). Using this insight and the ideas of [Nowak 2015], we rework Garland's method under the assumption of two-sided spectral gaps in the links and derive very general vanishing criteria that are applicable to all uniformly curved Banach spaces (and, in part, to all reflexive Banach spaces). We give two applications for our result:

Fixed-point properties of random groups. Applying our vanishing result to random groups in the triangular model improves on the results of de Laat and de la Salle when considering fixed-point properties with respect to L^p spaces. As a result, we derive a *sharp* lower bound for the conformal dimension of the boundary of these groups that was not achieved in previous works. Namely, in previous works [Druţu and Mackay 2019; de Laat and de la Salle 2021] it was shown that with high probability, this conformal dimension is contained in an interval between $C\sqrt{\log m}$ and $C' \log(m)$ (where m is a parameter of the model — see exact formulation below). Our work shows that in fact the conformal dimension is in an interval of the form $C'' \log m$ and $C' \log(m)$ and thus our result is sharp. We note that as far as we understand, the proof methods in [Druţu and Mackay 2019; de Laat and de la Salle 2021] cannot be improved to yield such a sharp bound.

Group stability with respect to p-Schatten norms. By a result of [De Chiffre et al. 2020], vanishing of the second cohomology for Hilbertian spaces implies stability with respect to p-Schatten norms (see definitions below). Thus, our new criteria for vanishing of the second cohomology gives new criteria for group stability.

1A. ***New criteria for vanishing of cohomology with Banach coefficients.*** In order to state our results, we will need the following notation. For every simplex $\tau \in X(k)$ define X_τ to be the link of τ and $M_\tau, A_\tau : \ell^2(X_\tau(0)) \to \ell^2(X_\tau(0))$ to be the following operators: M_τ is the orthogonal projection on the subspace of constant functions in $\ell^2(X_\tau(0))$ and A_τ in the random walk operator on the 1-skeleton of X_τ. With this notation, we prove the following:

Theorem 1.1. *Let $\mathbb{E}$ be a reflexive Banach space, X a locally finite, pure n-dimensional simplicial complex and Γ a locally compact, unimodular group acting cocompactly and properly on X.*

For $1 \leq k \leq n-1$, if

$$\max_{\tau \in X(k-1)} \|(A_\tau(I - M_\tau) \otimes \mathrm{id}_{\mathbb{E}})\|_{B(\ell^2(X_\tau(0);\mathbb{E}))} < \frac{1}{k+1},$$

then $H^k(X, \pi) = 0$ for every continuous isometric representation π of Γ on $\mathbb{E}$.

Remark 1.2. A result of the same flavor was given in [de Laat and de la Salle 2021, Theorem B] for the vanishing of the first cohomology for groups acting on 2-dimensional simplicial complexes. We note that our theorem improves on their Theorem B even when considering only vanishing of the first cohomology: First, our theorem holds for any reflexive Banach space, while Theorem B is only applicable for super-reflexive spaces. Second, in terms of parameters, the contraction condition of $A_\tau(I-M_\tau)\otimes \mathrm{id}_{\mathbb{E}}$ does not depend on the Banach space, but only on k (as opposed to Theorem B). Last, our proof is simpler in the regard that it does not use the p-Laplacian or any uniform convexity arguments.

Theorem 1.1 is easily applicable in the setting of uniformly curved Banach spaces (see Definition 2.1) such as (commutative and noncommutative) L^p spaces and more generally strictly θ-Hilbertian spaces (see the exact definition in Section 2C). Namely, for a uniformly curved Banach space we can deduce vanishing of cohomology based the fact that links are spectral expanders. Before stating these types of results, we recall the relevant terminology: Let (V, E) be a connected finite graph and let A be the random walk operator on this graph. Recall that A is a self-adjoint operator and has the eigenvalue 1 with multiplicity 1. For a constant λ, the graph (V, E) is called a *one-sided λ-spectral expander* if the second largest eigenvalue of A is $\leq \lambda$. The graph (V, E) is called a *two-sided λ-spectral expander* if the spectrum of A is contained in the interval $[-\lambda, \lambda]\cup\{1\}$.

Theorem 1.3 (informal; see Proposition 4.5 and Theorem 4.11 for explicit formulations). *Let $\mathbb{E}$ be a uniformly curved Banach space. There are positive constants $\{\lambda_k(\mathbb{E})>0 : k\in\mathbb{N}\}$ such that for every locally finite, pure n-dimensional simplicial complex X, every locally compact, unimodular group Γ acting cocompactly and properly on X the following hold*:

(1) *For every $1\leq k\leq n-1$, if there is $0<\lambda<\lambda_k(\mathbb{E})$ such that for every $\tau\in X(k-1)$ the one skeleton of X_τ is a two-sided λ-spectral expander, then $H^k(X,\pi)=0$ for every continuous isometric representation π of Γ on $\mathbb{E}$.*

(2) *For every $1\leq k\leq n-1$, if there is $0<\lambda<\lambda_k(\mathbb{E})$ such that for every $\tau\in X(n-2)$ the one skeleton of X_τ is a two-sided $\lambda/(1+(n-k-1)\lambda)$-spectral expander, then $H^k(X,\pi)=0$ for every continuous isometric representation π of Γ on $\mathbb{E}$.*

(3) *For every $1\leq k< n-1/\lambda_k(\mathbb{E})$, if there is $0<\lambda<\lambda_k(\mathbb{E})$ such that for every $\tau\in X(n-2)$, the one skeleton of X_τ is a one-sided $\lambda/(1+(n-k-1)\lambda)$-spectral expander, then $H^k(X,\pi)=0$ for every continuous isometric representation π of Γ on $\mathbb{E}$.*

Specifying the above to θ-Hilbertian spaces reads as follows:

Corollary 1.4. *Let X be a locally finite, pure n-dimensional simplicial complex such that all the links of X of dimension ≥ 1 are connected and Γ be a locally compact, unimodular group acting cocompactly and properly on X. Also let $0<\theta_0\leq 1$, $1\leq k\leq n-1$, $0<\lambda<(1/(2(k+1)))^{1/\theta_0}$ be constants. Define $\mathcal{E}_{\theta_0}$ to be the smallest class of Banach spaces that contains all strictly θ-Hilbertian Banach spaces for all $\theta_0\leq\theta\leq 1$ and is closed under subspaces, quotients, ℓ^2-sums and ultraproducts of Banach spaces.*

(1) *If for every $\tau\in X(k-1)$ the one skeleton of X_τ is a two-sided λ-spectral expander, then $H^k(X,\pi)=0$ for every $\mathbb{E}\in\mathcal{E}_{\theta_0}$ and every continuous isometric representation π of Γ on $\mathbb{E}$.*

(2) *If for every* $\tau \in \Sigma(n-2, \Gamma)$ *the* 1*-skeleton of* X_τ *is a two-sided* $\lambda/(1+(n-k-1)\lambda)$*-spectral expander, then* $H^k(X, \pi) = 0$ *for every* $\mathbb{E} \in \mathcal{E}_{\theta_0}$ *and every continuous isometric representation* π *of* Γ *on* $\mathbb{E}$.

(3) *If* $k \leq n - 1/\lambda$ *and for every* $\tau \in \Sigma(n-2, \Gamma)$ *the* 1*-skeleton of* X_τ *is a **one-sided*** $\lambda/(1+(n-k-1)\lambda)$*-spectral expander, then* $H^k(X, \pi) = 0$ *for every* $\mathbb{E} \in \mathcal{E}_{\theta_0}$ *and every continuous isometric representation* π *of* Γ *on* $\mathbb{E}$.

Remark 1.5. In all the results above, we gave criteria for vanishing of the equivariant cohomology. We recall that given that the simplicial complex X is aspherical, it holds that $H^k(X, \pi) = H^k(\Gamma, \pi)$ (the proof of this can be found for instance in [Brown 1982, Chapter 7, Section 7]), and thus under this additional assumption it follows that the criteria given above imply that $H^k(\Gamma, \pi) = 0$.

1B. ***Application to random groups.*** An immediate application of our criteria above is improving de Laat and de la Salle's results regarding random groups in the triangular model. The triangular model for random groups, denoted by $\mathcal{M}(m, d)$, is defined as follows: For a fixed density $d \in (0, 1)$, a group in $\mathcal{M}(m, d)$ is a finitely presented group of the form $\Gamma = \langle S \mid R \rangle$, where $|S| = m$ ($S \cap S^{-1} = \varnothing$) and R is a set of cyclically reduced relators of length 3 chosen uniformly among all subsets of cardinality $\lfloor (2m-1)^{3d} \rfloor$. A property P for groups is said to hold with overwhelming probability in this model if

$$\lim_{m \to \infty} \mathbb{P}(\Gamma \text{ in } \mathcal{M}(m, d) \text{ has P}) = 1.$$

Below, we will also use the binomial triangular model that is closely related to the triangular model. The binomial triangular model, denoted by $\Gamma(m, \rho)$, is defined as follows: a group in $\Gamma(m, \rho)$ is a finitely presented group of the form $\Gamma = \langle S \mid R \rangle$, where $|S| = m$, and R is a set of cyclically reduced relators of length 3, where each relator is chosen independently with probability ρ. We mention this model, since it is easier to analyze and the results of this analysis can be transferred to the model $\mathcal{M}(m, d)$.

The triangular model for random groups was introduced by Żuk [2003] who showed that when $d > \frac{1}{3}$, property (T) holds for groups in $\mathcal{M}(m, d)$ with overwhelming probability. De Laat and de la Salle [2021] (following [Druțu and Mackay 2019]) generalized the result of Żuk to the setting of uniformly curved Banach spaces. In order to explain this generalization, we recall that by a classical result of Delorme and Guichardet, a finitely generated discrete group Γ has property (T) if and only if it has property (FH), i.e., if and only if every affine isometric action of Γ on a Hilbert space admits a fixed point. Property (FH) is readily generalized to the Banach setting as follows: For a Banach space $\mathbb{E}$, a group Γ is said to have property ($F_{\mathbb{E}}$) if every continuous affine isometric action of Γ on $\mathbb{E}$ admits a fixed point. Also, a group Γ is said to have property (F_{L^p}) if it has property ($F_{\mathbb{E}}$) for every L^p space $\mathbb{E}$. De Laat and de la Salle [2021] showed that if $d > \frac{1}{3}$, then for every uniformly curved Banach space $\mathbb{E}$, property ($F_{\mathbb{E}}$) holds for groups in $\mathcal{M}(m, d)$ with overwhelming probability (their result is actually stronger — see Theorem 1.9 stated below).

Our results above are stated in the language of vanishing of the equivariant cohomology for groups acting on simplicial complexes. The connection between fixed-point properties and vanishing of cohomology readily follows from the following classical interpretation of group cohomology (see for instance the discussion in [Fernós et al. 2012, Section 2]):

Proposition 1.6. *Let Γ be a topological group and $\mathbb{E}$ be a Banach space. The group Γ has property* $(\mathrm{F}_{\mathbb{E}})$ *if and only if $H^1(\Gamma, \pi) = 0$ for every continuous isometric representation π of Γ on $\mathbb{E}$.*

The connection to our results regarding vanishing of cohomology described above is the following equivalence between fixed points and vanishing of the first cohomology. We recall that for a topological group Γ and a Banach space $\mathbb{E}$ the following are equivalent:

- $H^1(\Gamma, \pi) = 0$ for every continuous isometric representation π of Γ on $\mathbb{E}$.
- The group Γ has property $(\mathrm{F}_{\mathbb{E}})$.

Above, we discussed the vanishing of equivariant cohomology and not group cohomology, but as noted in Remark 1.5, this is equivalent in the case of groups acting on aspherical complexes. For a random group Γ in the model $\Gamma(m, \rho)$ (or in the model $\mathcal{M}(m, d)$), the Cayley complex of the group is a 2-dimensional simplicial complex that we will denote by X_Γ. We recall that the Cayley complex of a group is always simply connected and the action of a group on its Cayley complex is simply transitive on the vertices. In particular, since the group is finitely presented, the action is proper and cocompact. Thus, the vanishing of the first cohomology of Γ is equivalent to the vanishing of the first equivariant cohomology for the action of Γ on X_Γ. It follows that if we know that the links of X_Γ are two-sided spectral expanders, we can deduce property $(\mathrm{F}_{\mathbb{E}})$ for a uniformly curved Banach space $\mathbb{E}$ and a random group Γ in the model $\Gamma(m, \rho)$ by applying Theorem 1.3 stated above.

In [de Laat and de la Salle 2021], it was proven that the links of X_Γ for $\Gamma(m, \rho)$ are indeed two-sided spectral expanders:

Proposition 1.7 [de Laat and de la Salle 2021, Proposition 7.5]. *Let $\eta > 0$ be a constant. There is a constant $C > 0$ and a sequence $\{u_m\}_{m \in \mathbb{N}}$ tending to 0 such that the following holds: Let $m \in \mathbb{N}$ and $\rho \in (0, m^{-1.42})$. Also let Γ be a random group in the model $\Gamma(m, \rho)$ and X_Γ its Cayley complex. If*

$$\rho \geq \frac{(1+\eta)\log m}{8m^2},$$

then with probability $\geq 1 - u_m$, the link of every vertex of X_Γ is a $\sqrt{C/(\rho m^2)}$-two-sided spectral expander.

Combining this proposition with Theorem 1.3 above, we can reprove [de Laat and de la Salle 2021, Theorem 7.3]:

Theorem 1.8. *Let $\eta' > 0$ and $\rho \in (0, m^{-1.42})$ be constants and let C be the constant that appears in Proposition 1.7. Assume that*

$$\rho \geq \frac{(1+\eta')\log m}{8m^2},$$

and let Γ be a random group in the model $\Gamma(m, \rho)$. Then there is a sequence $\{u_m\}_{m \in \mathbb{N}}$ tending to 0 such that for uniformly curved Banach space $\mathbb{E}$ with $\lambda_1(\mathbb{E}) \geq \sqrt{C/(\rho m^2)}$ (where $\lambda_1(\mathbb{E})$ as in Theorem 1.3) it holds that Γ has property $(\mathrm{F}_{\mathbb{E}})$ with probability $\geq 1 - u_m$.

As in [de Laat and de la Salle 2021], using the fact that the fixed-point property passes to quotients, we can also recast this theorem in the triangular model (see further details in Section 7 of that paper) and reprove their Theorem C:

Theorem 1.9. *Let $0 < \eta < 2$, $d > \frac{1}{3} + (\log\log m - \log(2-\eta))/(3\log m)$ be constants. Also let Γ be a random group in the model $\mathcal{M}(m,d)$. Then there is a sequence $\{u_m\}_{m\in\mathbb{N}}$ tending to 0 and a constant $C > 0$ such that for uniformly curved Banach space $\mathbb{E}$ with*

$$\lambda_1(\mathbb{E}) \geq \sqrt{\frac{C}{(2m-1)^{3d-1}}}$$

(*where $\lambda_1(\mathbb{E})$ as in Theorem 1.3*) *it holds that Γ has property $(\mathrm{F}_{\mathbb{E}})$ with probability $\geq 1-u_m$.*

Combining this theorem with Corollary 4.7 leads to a stronger result than the one stated in [de Laat and de la Salle 2021] (and in [Druțu and Mackay 2019]) when considering L^p spaces. Namely, applying Corollary 4.7 yields the following:

Theorem 1.10. *Let $0 < \eta < 2$, $d > \frac{1}{3} + (\log\log m - \log(2-\eta))/(3\log m)$ be constants. Also let Γ be a random group in the model $\mathcal{M}(m,d)$. Then there is a sequence $\{u_m\}_{m\in\mathbb{N}}$ tending to 0 and a constant $C > 0$ such that for*

$$2 \leq p \leq \tfrac{1}{2}(3d-1)\log(2m-1) - \tfrac{1}{2}\log C$$

it holds that Γ has property (F_{L^p}) with probability $\geq 1-u_m$.

As a corollary, we improve the bound on the conformal dimension of random groups in the triangular model stated in [de Laat and de la Salle 2021, Corollary E]. Namely, by a theorem in [Bourdon 2016], if for a given $2 \leq p$, a hyperbolic group Γ has property (F_{L^p}), then the conformal dimension of $\partial_\infty \Gamma$ is at least p. Thus, we get:

Theorem 1.11. *Let $0 < \eta < 2$, $d > \frac{1}{3} + (\log\log m - \log(2-\eta))/(3\log m)$ be constants. Also let Γ be a random group in the model $\mathcal{M}(m,d)$. Then there is a sequence $\{u_m\}_{m\in\mathbb{N}}$ tending to 0 and a constant $C > 0$ such that*

$$\tfrac{1}{2}(3d-1)\log(2m-1) - \tfrac{1}{2}\log C \leq \mathrm{Confdim}(\partial_\infty \Gamma)$$

with probability $\geq 1-u_m$.

In particular, for $d \in \left(\frac{1}{3}, \frac{1}{2}\right)$ and a group Γ in $\mathcal{M}(m,d)$ it holds with overwhelming probability that

$$\tfrac{1}{2}(3d-1)\log(2m-1) - \tfrac{1}{2}\log C \leq \mathrm{Confdim}(\partial_\infty \Gamma).$$

Remark 1.12. The theorem above gives a sharp bound on the conformal dimension of the boundary. Indeed, in [Druțu and Mackay 2019, Proposition 10.6] it was shown that for $d \in \left(\frac{1}{3}, \frac{1}{2}\right)$ and a group Γ in $\mathcal{M}(m,d)$ it holds with overwhelming probability that

$$\mathrm{Confdim}(\partial_\infty \Gamma) \leq \frac{30}{2d-1}\log(2m-1).$$

1C. *Application to group stability.* Group stability has received much attention in recent years (see for instance [Glebsky and Rivera 2008; Arzhantseva and Păunescu 2015; Becker et al. 2019; Becker and Lubotzky 2020]) partly due to its connection to questions of group approximation; see for instance [De Chiffre et al. 2020]. In that work it was shown that, under some assumptions, group stability can be deduced for a group via the vanishing of its second cohomology. Another application of our work is providing a criterion for p-norm stability (stability with respect to the p-Schatten norm). In order to state this application, we first give the needed definitions and results from [De Chiffre et al. 2020].

Let Γ be a finitely presented group $\Gamma = \langle S \mid R \rangle$, with $R \subseteq \mathbb{F}_S$ the free group on S and $|R| < \infty$. Any map $\phi : S \to U(n)$ uniquely determines a homomorphism $\phi : \mathbb{F}_S \to U(n)$, which we will also denote by ϕ.

Given a distance dist_n on $U(n)$, the group Γ is called $\mathcal{G} = (U(n), \text{dist}_n)$-*stable* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $n \in \mathbb{N}$ if $\phi : S \to U(n)$ is a map with

$$\sum_{r \in R} \text{dist}_n(\phi(r), \text{id}_{U(n)}) < \delta,$$

then there exists a homomorphism $\tilde{\phi} : \Gamma \to U(n)$, with

$$\sum_{s \in S} \text{dist}_n(\phi(s), \tilde{\phi}(s)) < \varepsilon,$$

or equivalently, a map $\tilde{\phi} : S \to U(n)$, with $\sum_{r \in R} \text{dist}_n(\tilde{\phi}(r), \text{id}_{U(n)}) = 0$,

For $1 \leq p < \infty$, the Schatten p-norm on $M_n(\mathbb{C})$ is defined by $\|T\|_p = (\text{tr}\,|T|^p)^{1/p}$, where $|T| = \sqrt{T^*T}$. When $p = 2$, this is usually called the Frobenius norm. Define $\text{dist}_{n,p}$ to be the metric on $U(n)$ induced by this norm. Below, we will call a group Γ p-*norm stable* if it is stable with respect to $\mathcal{G} = (U(n), \text{dist}_{n,p})$.

We note that $(M_n(\mathbb{C}), \|\cdot\|_p)$ is a noncommutative L^p space and in particular, it is strictly θ-Hilbertian with $\theta = 2 - 2/p$ if $p \leq 2$ and $\theta = 2/p$ if $p \geq 2$. The discussion in [De Chiffre et al. 2020] implies the following criterion for p-norm stability (see also [García Morales and Glebsky 2022; Lubotzky and Oppenheim 2020]):

Theorem 1.13 [De Chiffre et al. 2020, Theorem 5.1, Remark 5.2]. *Let Γ be a finitely presented group and $0 < \theta_0 \leq 1$ be a constant. Define $\mathcal{E}_{\theta_0}$ to be the smallest class of Banach spaces that contains all strictly θ-Hilbertian Banach spaces for all $\theta_0 \leq \theta \leq 1$ and is closed under subspaces, ℓ^2-sums and ultraproducts of Banach spaces. If for every for every continuous isometric representation π of Γ on $\mathbb{E} \in \mathcal{E}_{\theta_0}$ it holds that $H^2(\Gamma, \pi) = 0$, then Γ is p-norm stable for every $1 + \theta_0/(2 - \theta_0) \leq p \leq 2/\theta_0$.*

Combining this theorem with Corollary 1.4 and Remark 1.5 immediately yields the following criterion for p-norm stability:

Theorem 1.14. *Let X be a locally finite, pure n-dimensional aspherical simplicial complex with $n \geq 3$ such that all the links of X of dimension ≥ 1 are connected and Γ be a finitely presented discrete group acting cocompactly and properly on X. Also let $0 < \theta_0 \leq 1$, $0 < \lambda < \left(\frac{1}{6}\right)^{1/\theta_0}$ be constants. Assume that* ***one*** *of the following holds*:

(1) *For every $\tau \in X(1)$, the one skeleton of X_τ is a two-sided λ-spectral expander.*

(2) *For every $\tau \in X(n-2)$, the 1-skeleton of X_τ is a two-sided $\lambda/(1 + (n-3)\lambda)$-spectral expander.*

(3) *It holds that $2 \leq n - 1/\lambda$ and for every $\tau \in X(n-2)$, the 1-skeleton of X_τ is a* ***one-sided*** *$\lambda/(1+(n-3)\lambda)$-spectral expander.*

Then Γ is p-norm stable for every $1 + \theta_0/(2 - \theta_0) \leq p \leq 2/\theta_0$.

Currently, we do not have new examples in which this theorem improves previous results. One can take X to be an affine building of a large dimension n, Γ a lattice of the full BN-pair group of X and apply Theorem 1.14(3) to deduce p-norm stability (where p depends on the thickness of the building and on n). However, as noted above, in the case where X is a classical affine building, stronger results are given in [Lubotzky and Oppenheim 2020].

Organization. In Section 2, we cover some needed preliminaries. In Section 3, we give the basic definitions regarding equivariant cohomology and prove a variation of Nowak's criterion for vanishing of cohomology. In Section 4, we prove our local criteria for vanishing of Banach cohomology.

2. Preliminaries

2A. ***Vector-valued ℓ^2 spaces.*** Given a finite set V, a function $m : V \to \mathbb{R}_+$ and a Banach space $\mathbb{E}$, we define the *vector-valued space* $\ell^2(V, m; \mathbb{E})$ to be the space of functions $\phi : V \to \mathbb{E}$, with the norm

$$\|\phi\|_{\ell^2(V,m;\mathbb{E})} = \left(\sum_{v\in V} m(v)|\phi(v)|^2\right)^{1/2},$$

where $|\cdot|$ is the norm of $\mathbb{E}$. We define $\ell^2(V, m) = \ell^2(V, m; \mathbb{C})$ and recall that $\ell^2(V, m)$ is also a Hilbert space with the inner-product

$$\langle \phi, \psi\rangle = \sum_{v\in V} m(v)\phi(v)\overline{\psi(v)}.$$

Let $T : \ell^2(V, m) \to \ell^2(V, m)$ be a linear operator and $T_{v,u} \in \mathbb{C}$ be the constants such that for every $\phi \in \ell^2(V, m)$ it holds that

$$(T\phi)(v) = \sum_{u\in V} T_{v,u}\phi(u).$$

Define $T \otimes \mathrm{id}_{\mathbb{E}} : \ell^2(V, m; \mathbb{E}) \to \ell^2(V, m; \mathbb{E})$ by the formula

$$((T \otimes \mathrm{id}_{\mathbb{E}})\phi)(v) = \sum_{u\in V} T_{v,u}\phi(u),$$

where $T_{v,u} \in \mathbb{C}$ are the same constants as above and $\phi \in \ell^2(V, m; \mathbb{E})$. We define $\|T \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))}$ to be the operator norm of $T \otimes \mathrm{id}_{\mathbb{E}}$.

Following [Pisier 2010], we call an operator $T : \ell^2(V, m) \to \ell^2(V, m)$ *fully contractive* if for every Banach space $\mathbb{E}$ it holds that $\|T \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))} \le 1$.

2B. ***Uniformly curved Banach spaces.*** Uniformly curved Banach spaces were introduced in [Pisier 2010]:

Definition 2.1. Let $\mathbb{E}$ be a Banach space. The space $\mathbb{E}$ is called uniformly curved if for every $0 < \varepsilon \le 1$ there is $\delta > 0$ such that, for every space $\ell^2(V, m)$ and every fully contractive linear operator $T : \ell^2(V, m) \to \ell^2(V, m)$, if $\|T\|_{B(\ell^2(V,m))} \le \delta$, then $\|T \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))} \le \varepsilon$.

The following theorem is due to [Pisier 2010]:

Theorem 2.2. *Every uniformly curved Banach space is super-reflexive and in particular reflexive.*

Given a monotone increasing function $\alpha : (0,1] \to (0,1]$ such that

$$\lim_{t \to 0^+} \alpha(t) = 0,$$

we define $\mathcal{E}_\alpha^{\text{u-curved}}$ to be the class of all (uniformly curved) Banach spaces $\mathbb{E}$ such that for every space $\ell^2(V,m)$ and every fully contractive linear operator $T : \ell^2(V,m) \to \ell^2(V,m)$, if $\|T\|_{B(\ell^2(V,m))} \leq \delta$, then $\|T \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))} \leq \alpha(\delta)$.

Proposition 2.3. *Let $T : \ell^2(V,m) \to \ell^2(V,m)$ be a linear operator and $L \geq 1,\ 0 < \delta \leq 1$ be constants such that*:

(1) $\|T\|_{B(\ell^2(V,m))} \leq \delta$.

(2) $\|T \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))} \leq L$ *for every Banach space* $\mathbb{E}$.

Then for every monotone increasing function $\alpha : (0,1] \to (0,1]$ such that $\lim_{t \to 0^+} \alpha(t) = 0$ and every $\mathbb{E} \in \mathcal{E}_\alpha^{\text{u-curved}}$ we have $\|T \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))} \leq L\alpha(\delta)$.

Proof. We note that $(1/L)T$ is a fully contractive operator such that

$$\left\| \frac{1}{L} T \right\|_{B(\ell^2(V,m))} \leq \frac{\delta}{L}.$$

Thus, by the definition of $\mathcal{E}_\alpha^{\text{u-curved}}$ it follows for every $\mathbb{E} \in \mathcal{E}_\alpha^{\text{u-curved}}$ that

$$\left\| \left(\frac{1}{L}\right) T \otimes \mathrm{id}_{\mathbb{E}} \right\|_{B(\ell^2(V,m;\mathbb{E}))} \leq \alpha\left(\frac{\delta}{L}\right),$$

and thus

$$\|T \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))} \leq L\alpha\left(\frac{\delta}{L}\right) \leq L\alpha(\delta),$$

where the last inequality is due to the fact that $L \geq 1$ and α is monotone increasing. □

We will also be interested in how $T \otimes \mathrm{id}_{\mathbb{E}}$ behaves under some operations; this is summed up in the following lemmas:

Lemma 2.4. *Let V be a finite set, T a bounded operator on $\ell^2(V,m)$ and $C > 0$ constant. Let $\mathcal{E} = \mathcal{E}(C)$ be the class of Banach spaces defined as*

$$\mathcal{E} = \{\mathbb{E} : \|T \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))} \leq C\}.$$

Then this class is closed under quotients, subspaces, ℓ^2-sums and ultraproducts of Banach spaces, i.e., preforming any of these operations on Banach spaces in $\mathcal{E}$ yields a Banach space in $\mathcal{E}$.

Proof. The fact that $\mathcal{E}$ is closed under quotients, subspaces and ultraproducts of Banach spaces was shown in [de la Salle 2016, Lemma 3.1]. The fact that $\mathcal{E}$ is closed under ℓ^2-sums is straightforward and left for the reader. □

Applying Lemma 2.4 on $\mathcal{E}_\alpha^{\text{u-curved}}$ defined above yields the following corollary:

Corollary 2.5. *For any monotone increasing function $\alpha : (0,1] \to (0,1]$ such that $\lim_{t \to 0^+} \alpha(t) = 0$, the class $\mathcal{E}_\alpha^{\text{u-curved}}$ defined above is closed under quotients, subspaces, ℓ^2-sums and ultraproducts of Banach spaces.*

2C. *Strictly θ-Hilbertian spaces.* Here we will describe a special class of uniformly curved Banach spaces that contains all (commutative and noncommutative) L^p spaces.

Two Banach spaces $\mathbb{E}_0, \mathbb{E}_1$ form a *compatible pair* $(\mathbb{E}_0, \mathbb{E}_1)$ if they are continuously linear embedded in the same topological vector space. The idea of complex interpolation is that given a compatible pair $(\mathbb{E}_0, \mathbb{E}_1)$ and a constant $0 \leq \theta \leq 1$, there is a method to produce a new Banach space $[\mathbb{E}_0, \mathbb{E}_1]_\theta$ as a "convex combination" of $\mathbb{E}_0$ and $\mathbb{E}_1$. We will not review this method here, and the interested reader can find more information on interpolation in [Bergh and Löfström 1976].

This brings us to consider the following definition due to [Pisier 1979]: a Banach space $\mathbb{E}$ is called *strictly θ-Hilbertian* for $0 < \theta \leq 1$ if there is a compatible pair $(\mathbb{E}_0, \mathbb{E}_1)$ with $\mathbb{E}_1$ a Hilbert space such that $\mathbb{E} = [\mathbb{E}_0, \mathbb{E}_1]_\theta$. Examples of strictly θ-Hilbertian spaces are L^p spaces and noncommutative L^p spaces (see [Pisier and Xu 2003] for definitions and properties of noncommutative L^p spaces), where in these cases $\theta = 2/p$ if $2 \leq p < \infty$ and $\theta = 2 - 2/p$ if $1 < p \leq 2$.

For our use, it will be important to bound the norm of an operator of the form $T \otimes \mathrm{id}_{\mathbb{E}}$ given that $\mathbb{E}$ is an interpolation space.

Lemma 2.6 [de la Salle 2016, Lemma 3.1]. *Let $(\mathbb{E}_0, \mathbb{E}_1)$ be a compatible pair, V be a finite set, $m : V \to \mathbb{R}_+$ be a function and $T \in B(\ell^2(V, m))$ be an operator. Then, for every $0 \leq \theta \leq 1$,*

$$\|T \otimes \mathrm{id}_{[\mathbb{E}_0,\mathbb{E}_1]_\theta}\|_{B(\ell^2(V,m;[\mathbb{E}_0,\mathbb{E}_1]_\theta))} \leq \|T \otimes \mathrm{id}_{\mathbb{E}_0}\|^{1-\theta}_{B(\ell^2(V,m;\mathbb{E}_0))} \|T \otimes \mathrm{id}_{\mathbb{E}_1}\|^{\theta}_{B(\ell^2(V,m;\mathbb{E}_1))},$$

where $[\mathbb{E}_0, \mathbb{E}_1]_\theta$ is the interpolation of $\mathbb{E}_0$ and $\mathbb{E}_1$.

This lemma has the following corollary that shows that strictly θ-Hilbertian spaces are uniformly curved (see also [de la Salle 2016, Lemma 3.1]):

Corollary 2.7. *Let $\mathbb{E}$ be a strictly θ-Hilbertian space with $0 < \theta \leq 1$, V be a finite set, $m : V \to \mathbb{R}_+$ be a function and $0 < \delta < 1$ be a constant. Assume that $T \in B(\ell^2(V, m))$ is a fully contractive operator such that $\|T\|_{B(\ell^2(V,m))} \leq \delta$. Then $\|T \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))} \leq \delta^\theta$.*

In other words, if $\mathbb{E}$ is a strictly θ-Hilbertian space with $0 < \theta \leq 1$, then for $\alpha(t) = t^\theta$ we have that $\mathbb{E} \in \mathcal{E}_\alpha^{\text{u-curved}}$.

Proof. For every Hilbert space $\mathbb{E}_1$ we have that $\|T \otimes \mathrm{id}_{\mathbb{E}_1}\|_{B(\ell^2(V,m;\mathbb{E}_1))} \leq \delta$ and thus the assertion stated above follows from Lemma 2.6. □

Corollary 2.8. *For a constant $0 < \theta_0 \leq 1$, define $\mathcal{E}_{\theta_0}$ to be the smallest class of Banach spaces that contains all strictly θ-Hilbertian Banach spaces for all $\theta_0 \leq \theta \leq 1$ and is closed under subspaces, quotients, ℓ^2-sums and ultraproducts of Banach spaces. Then, for every $0 < \theta_0 \leq 1$, we have that $\mathcal{E}_{\theta_0} \subseteq \mathcal{E}^{\text{u-curved}}_{\alpha(t)=t^{\theta_0}}$.*

Remark 2.9. A deep result of Pisier shows that the converse of the corollary above is "almost true" if one considers arcwise θ_0-Hilbertian spaces (see the definition in [Pisier 2010, Section 6]). Namely, by Corollary 6.7 of that work, for every $\theta_0 < \theta \leq 1$ it holds that every Banach space in $\mathcal{E}^{\text{u-curved}}_{\alpha(t)=t^\theta}$ is a subquotient of an arcwise θ_0-Hilbertian space. We will not define arcwise θ_0-Hilbertian spaces here and we will make no use of this fact.

2D. ***Random walks on finite graphs.*** Given a finite graph (V, E), a weight function on (V, E) is a function $m : E \to \mathbb{R}_+$ and (V, E) with a weight function is called a weighted graph. Given a weighted graph as above, we define, for every $v \in V$, $m(v) = \sum_{e \in E, v \in e} m(e)$ and $m(\varnothing) = \sum_{v \in V} m(v)$.

We also define $\ell^2(V, m)$ as in Section 2A above; i.e., $\ell^2(V, m)$ is the space of functions $\phi : V \to \mathbb{C}$ with an inner-product

$$\langle \phi, \psi \rangle = \sum_{\{v\} \in V} = m(v)\phi(v)\overline{\psi(v)}.$$

The *random walk* on (V, E) as above is the operator $A : \ell^2(V, m) \to \ell^2(V, m)$ defined as

$$(A\phi)(v) = \sum_{u \in V, \{u, v\} \in E} \frac{m(\{u, v\})}{m(v)} \phi(u).$$

We state without proof a few basic facts regarding the random walk operator:

(1) With the inner-product defined above, A is a self-adjoint operator and the eigenvalues of A lie in the interval $[-1, 1]$.

(2) The space of constant functions is an eigenspace of A with eigenvalue 1 and if (V, E) is connected, all other the other eigenfunctions of A have eigenvalues strictly less than 1.

(3) The graph (V, E) is bipartite if and only if -1 is an eigenvalue of A.

In the case where m is constant 1 on all the edges, for every vertex v, $m(v)$ is the valence of v and A is called the *simple random walk* on (V, E).

We define M to be the orthogonal projection on the space of constant functions: explicitly, for every $\phi \in \ell^2(V, m)$, $M\phi$ is the constant function

$$M\phi \equiv \frac{1}{m(\varnothing)} \sum_{v \in V} m(v)\phi(v).$$

We note that by the facts stated above, $AM = M$ and if (V, E) is connected and not bipartite, then $\|A(I - M)\| < 1$, where $\|\cdot\|$ denotes the operator norm. We recall the following definition of spectral expansion that appeared in the Introduction for nonweighted graphs:

Definition 2.10. Let (V, E) be a finite connected graph with a weight function m and $0 \leq \lambda < 1$ be a constant. The graph (V, E) is called a one-sided λ-spectral expander if the spectrum of $A(I - M)$ is contained in $[-1, \lambda]$. The graph (V, E) is called a two-sided λ-spectral expander if the spectrum of $A(I - M)$ is contained in $[-\lambda, \lambda]$ or equivalently if $\|A(I - M)\| \leq \lambda$.

Given a Banach space $\mathbb{E}$, we can consider the operator $(A(I - M)) \otimes \mathrm{id}_{\mathbb{E}} : \ell^2(V, m; \mathbb{E}) \to \ell^2(V, m; \mathbb{E})$.

Claim 2.11. *For every graph (V, E) and every Banach space $\mathbb{E}$, $\|(A(I - M)) \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))} \leq 2$.*

Proof. By the triangle inequality and linearity,

$$\|(A(I - M)) \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))} \leq \|A \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))} + \|A \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))} \|M \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))},$$

and therefore in order to prove the claim, it is enough to show that

$$\|A \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))} \leq 1, \quad \|M \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))} \leq 1.$$

Indeed, by the convexity of the function $|\cdot|^2$, for every $\phi \in \ell^2(V, m; \mathbb{E})$,

$$\begin{aligned}\|(A \otimes \mathrm{id}_{\mathbb{E}})\phi\|^2 &= \sum_{v \in V} m(v) \Bigg| \sum_{u \in V, \{u,v\} \in E} \frac{m(\{u, v\})}{m(v)} \phi(u) \Bigg|^2 \leq \sum_{v \in V} m(v) \sum_{u \in V, \{u,v\} \in E} \frac{m(\{u, v\})}{m(v)} |\phi(u)|^2 \\ &= \sum_{u \in V} |\phi(u)|^2 \sum_{v \in V, \{u,v\} \in E} m(\{u, v\}) = \sum_{u \in V} m(u) |\phi(u)|^2 = \|\phi\|^2\end{aligned}$$

and

$$\begin{aligned}\|(M \otimes \mathrm{id}_{\mathbb{E}})\phi\|^2 &= \sum_{v \in V} m(v) \Bigg| \frac{1}{m(\varnothing)} \sum_{u \in V} m(u)\phi(u) \Bigg|^2 \leq \sum_{v \in V} \sum_{u \in V} \frac{m(u)}{m(\varnothing)} |\phi(u)|^2 \\ &= \sum_{u \in V} m(u)|\phi(u)|^2 \sum_{v \in V} \frac{m(v)}{m(\varnothing)} = \sum_{u \in V} m(u)|\phi(u)|^2 = \|\phi\|^2. \qquad \square\end{aligned}$$

Combining this claim with Proposition 2.3 and Corollary 2.7 yields:

Corollary 2.12. *Let (V, E) be a connected finite graph with a weight function m and $0 < \lambda < 1$ be a constant such that (V, E) is a two-sided λ-spectral expander. For every monotone increasing function $\alpha : (0, 1] \to (0, 1]$ such that $\lim_{t \to 0^+} \alpha(t) = 0$ and every $\mathbb{E} \in \mathcal{E}_\alpha^{\text{u-curved}}$, we have*

$$\|(A(I - M)) \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))} \leq 2\alpha(\lambda).$$

In particular, for every $0 < \theta \leq 1$ and every strictly θ-Hilbertian space $\mathbb{E}$, we have

$$\|(A(I - M)) \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))} \leq 2\lambda^\theta.$$

2E. *Weighted simplicial complexes.* Let X be an n-dimensional simplicial complex. For $-1 \leq k \leq n$, we define $X(k)$ to be the k-dimensional faces of X and $X = \bigcup_k X(k)$. X is called *pure n-dimensional* if for every τ in X there is $\sigma \in X(n)$ such that $\tau \subseteq \sigma$. X is called *locally finite* if for every $\{v\} \in X(0)$, $|\{\sigma \in X(n) : v \in \sigma\}| < \infty$. Throughout this paper, we will always assume that X is pure n-dimensional and locally finite.

We define the weight function $m : \bigcup_{k=0}^n \mathrm{X}(k) \to \mathbb{R}$ inductively as follows:

$$\text{for all } \sigma \in \mathrm{X}(n), \quad m(\sigma) = 1,$$

and, for $0 \leq k \leq n - 1$ and $\tau \in X(k)$,

$$m(\tau) = \sum_{\sigma \in \mathrm{X}(k+1),\, \tau \subseteq \sigma} m(\sigma).$$

More explicitly,

$$\text{for all } \tau \in \mathrm{X}(k), \quad m(\tau) = (n - k)!\, |\{\sigma \in \mathrm{X}(n) : \tau \subseteq \sigma\}|.$$

In the case where X is finite, we also define $m(\varnothing) = \sum_{\{v\} \in X(0)} m(\{v\})$.

Given a simplex $\tau \in X(j)$, the *link of* τ is the subcomplex of X, denoted by X_τ, that is defined as

$$X_\tau = \{\eta \in X : \tau \cap \eta = \varnothing,\ \tau \cup \eta \in X\}.$$

We note that by the assumption that X is locally finite, it follows that X_τ is finite and by the assumption that X is pure n-dimensional, it follows that X_τ is pure $(n-j-1)$-dimensional (where j is the dimension of τ). The weight function on X_τ, denoted by m_τ, is defined as above:

$$\text{for all } \sigma \in \mathrm{X}_\tau(n-j-1), \quad m_\tau(\sigma) = 1,$$

and, for $0 \le k \le (n-j-1)-1$ and for $\eta \in X(k)$,

$$m_\tau(\eta) = \sum_{\sigma \in \mathrm{X}_\tau(k+1),\ \eta \subseteq \sigma} m_\tau(\sigma).$$

We observe that $m_\tau(\eta) = m(\tau \cup \eta)$: indeed, if $\eta \in \mathrm{X}_\tau(n-j-1)$, then $\tau \cup \eta \in X(n)$ and therefore

$$m_\tau(\eta) = 1 = m(\tau \cup \eta).$$

For $0 \le k \le (n-j-1)-1$ and $\eta \in X(k)$, the equality follows by induction:

$$m_\tau(\eta) = \sum_{\sigma \in \mathrm{X}_\tau(k+1),\ \eta \subseteq \sigma} m_\tau(\sigma) = \sum_{\sigma \in \mathrm{X}_\tau(k+1),\ \eta \subseteq \sigma} m(\tau \cup \sigma) = \sum_{\tau \cup \sigma \in \mathrm{X}((j+1)+k+1),\ \eta \subseteq \tau \cup \sigma} m(\tau \cup \sigma) = m(\eta).$$

2F. ***Group representations on Banach spaces.*** Let Γ be a locally compact group and $\mathbb{E}$ a Banach space. Let π be a representation $\pi : \Gamma \to B(\mathbb{E})$, where $B(\mathbb{E})$ are the bounded linear operators on $\mathbb{E}$. Throughout this paper we shall always assume π is continuous with respect to the strong operator topology without explicitly mentioning it. We recall that given π, the dual representation $\bar{\pi} : \Gamma \to B(\mathbb{E}^*)$ is defined as

$$\langle x, \bar{\pi}(g).y \rangle = \langle \pi(g^{-1}).x,\ y \rangle \quad \text{for all } g \in \Gamma,\ x \in \mathbb{E},\ y \in \mathbb{E}^*.$$

Observe that if π is an isometric representation, then $\bar{\pi}$ is an isometric representation: Indeed, for every $g \in \Gamma$,

$$\max_{x \in \mathbb{E},\ y \in \mathbb{E}^*,\ |x|=|y|=1} \langle x, \bar{\pi}(g).y \rangle = \max_{x \in \mathbb{E},\ y \in \mathbb{E}^*,\ |x|=|y|=1} \langle \pi(g^{-1}).x,\ y \rangle = |\pi(g^{-1}).x| = 1,$$

i.e., for every $g \in \Gamma$ and every $y \in \mathbb{E}^*$, if $|y| = 1$, then $|\bar{\pi}(g)y| = 1$ and it follows that $\bar{\pi}$ is isometric.

We remark that $\bar{\pi}$ might not be continuous for a general Banach space, but it is continuous for a large class of Banach spaces, called Asplund spaces:

Definition 2.13. A Banach space $\mathbb{E}$ is said to be an Asplund space if every separable subspace of $\mathbb{E}$ has a separable dual.

There are many examples of Asplund spaces and in particular every reflexive space is Asplund (see [Yost 1993] for an exposition on Asplund spaces). The reason we are interested in Asplund spaces is the following theorem of Megrelishvili:

Theorem 2.14 [Megrelishvili 1998, Corollary 6.9]. *Let Γ be a topological group and let π be a continuous representation of Γ on a Banach space $\mathbb{E}$. If $\mathbb{E}$ is an Asplund space, then the dual representation $\bar{\pi}$ is also continuous. In particular, if $\mathbb{E}$ is reflexive, then the dual representation $\bar{\pi}$ is continuous.*

3. Equivariant cohomology

Let X be a locally finite, pure n-dimensional simplicial complex with the weight function m defined above and Γ be a locally compact, unimodular group acting cocompactly and properly on X. Also let $\mathbb{E}$ be a reflexive Banach space and π be a continuous isometric representation.

Remark 3.1. By our assumption, $\mathbb{E}$ is reflexive and thus Asplund. Therefore, by Theorem 2.14, the assumption of continuity of π implies that $\bar{\pi}$ is also continuous.

Below, we will define the equivariant cohomology $H^k(X, \pi)$ and prove a general criterion for the vanishing of this cohomology. All the definitions below regarding cohomology already appeared in [Ballmann and Świątkowski 1997] for representations on Hilbert spaces and were generalized to the Banach setting in [Koivisto 2014]. The criterion for vanishing of cohomology appeared (in a somewhat different form) in [Nowak 2015] (and also in [Koivisto 2014]) and we claim no originality here.

In order to define the equivariant cohomology, we introduce the following notation (based on [Ballmann and Świątkowski 1997]):

(1) For $0 \leq k \leq n$, denote by $\Sigma(k)$ the set of ordered k-simplices (i.e., $\sigma \in \Sigma(k)$ is an ordered $(k+1)$-tuple of vertices that form a k-simplex in X).

(2) A map $\phi : \Sigma(k) \to \mathbb{E}$ is called *alternating* if, for every permutation $\gamma \in \mathrm{Sym}\{0, \dots, k\}$ and every $(v_0, \dots, v_k) \in \Sigma(k)$,
$$\phi((v_{\gamma(0)}, \dots, v_{\gamma(k)})) = \mathrm{sgn}(\gamma)\phi((v_0, \dots, v_k)).$$
Also, ϕ is called *equivariant* if, for every $g \in \Gamma$ and every $\sigma \in \Sigma(k)$,
$$\pi(g)\phi(\sigma) = \phi(g.\sigma).$$

(3) For $0 \leq k \leq n$, a k-cochain twisted by π is a map $\phi : \Sigma(k) \to \mathbb{E}$ that is both alternating and equivariant. We define $C^k(X, \pi)$ to be the space of all k-cochains twisted by π.

For $0 \leq k < n$, the *differential* $d_k : C^k(X, \pi) \to C^{k+1}(X, \pi)$ is given by
$$d_k\phi(\sigma) := \sum_{i=0}^{k+1} (-1)^i \phi(\sigma_i), \quad \sigma \in \Sigma(k+1),$$
where $\sigma_i = (v_0, \dots, \hat{v}_i, \dots, v_{k+1})$ for $(v_0, \dots, v_{k+1}) = \sigma \in \Sigma(k+1)$. By a standard computation $d_k \circ d_{k-1} = 0$, and we define the k-th cohomology as $H^k(X, \pi) = \mathrm{Ker}(d_k)/\,\mathrm{Im}(d_{k-1})$.

Remark 3.2. The reader should note that in the definition of the cohomology above, we made no use of the fact that $\mathbb{E}$ is a Banach space, and this definition applies in a much more general setting.

We define a norm on $C^k(X, \pi)$ in order to make it into a Banach space:

(1) We choose a set, denoted by $\Sigma(k, \Gamma) \subseteq \Sigma(k)$, of representatives for the action of Γ on $\Sigma(k)$. We note that by the equivariance assumption, $\phi \in C^k(X, \pi)$ is determined by its values on $\Sigma(k, \Gamma)$. We also note that by the assumption that the action of Γ is cocompact, $\Sigma(k, \Gamma)$ is a finite set.

(2) We extend the weight function m defined above to ordered simplices by forgetting the ordering; i.e., for every $(v_0, \dots, v_k) \in \Sigma(k)$, we define $m((v_0, \dots, v_k)) = m(\{v_0, \dots, v_k\})$.

(3) For a simplex $\sigma \in \Sigma(k)$, we define Γ_σ to be the pointwise stabilizer of σ; i.e., for $\sigma = (v_0, \dots, v_k)$, $g \in \Gamma_\sigma$ if and only if for every $0 \le i \le k$ it holds that $g.v_i = v_i$. We further define $|\Gamma_\sigma|$ to be the measure of Γ_σ with respect to the Haar measure of Γ. By the assumption that the action of Γ is proper, it follows that $|\Gamma_\sigma| < \infty$.

(4) We define a norm on $C^k(X, \pi)$ by

$$\|\phi\| = \left(\sum_{\sigma \in \Sigma(k,\Gamma)} \frac{m(\sigma)}{(k+1)!\,|\Gamma_\sigma|} |\phi(\sigma)|^2 \right)^{1/2},$$

where $|\cdot|$ denotes the norm of $\mathbb{E}$.

With the definitions above, $C^k(X, \pi)$ is a normed space and we leave it to the reader to verify that it is a Banach space (this is almost immediate due to (1) above).

Proposition 3.3. *The space $C^k(X, \pi)$ is reflexive.*

Proof. Define $\mathbb{E}^{\Sigma(k,\Gamma)} = \{\phi : \Sigma(k, \Gamma) \to \mathbb{E}\}$ with the norm

$$\|\phi\| = \left(\sum_{\sigma \in \Sigma(k,\Gamma)} \frac{m(\sigma)}{(k+1)!\,|\Gamma_\sigma|} |\phi(\sigma)|^2 \right)^{1/2}.$$

This is a reflexive Banach space, since it is a weighted ℓ^2 sum of $|\Sigma(k, \Gamma)|$ copies of $\mathbb{E}$. We note that $C^k(X, \pi)$ is a closed subspace of $\mathbb{E}^{\Sigma(k,\Gamma)}$ and thus it is also reflexive. $\square$

Choose $\Sigma'(k, \Gamma) \subseteq \Sigma(k, \Gamma)$ to be a set of representatives of the action of the permutation group $\mathrm{Sym}\{0, \dots, k\}$ on $\Sigma(k, \Gamma)$; i.e., for every $(v_0, \dots, v_k) \in \Sigma(k, \Gamma)$ there is a unique permutation $\gamma \in \mathrm{Sym}\{0, \dots, k\}$ such that $(v_{\gamma(0)}, \dots, v_{\gamma(k)}) \in \Sigma'(k, \Gamma)$. By definition all the cochains in $C^k(X, \pi)$ are equivariant and alternating and thus every map in $C^k(X, \pi)$ is uniquely determined by its values on $\Sigma'(k, \Gamma)$. However, it may be the case that not every map $\phi' : \Sigma'(k, \Gamma) \to \mathbb{E}$ can be extended to an equivariant and alternating map on $\Sigma(k)$. Below, we will give a necessary and sufficient condition for the existence of such an extension.

For $\sigma \in \Sigma(k)$, we define Γ_σ^+ and Γ_σ^- to be the subsets of Γ that (when restricted to σ) induce even and odd permutations on σ; i.e., for $\sigma = (v_0, \dots, v_k)$,

$$\Gamma_\sigma^+ = \{g \in \Gamma : g.(v_0, \dots, v_k) = (v_{\gamma(0)}, \dots, v_{\gamma(k)}),\ \gamma \in \mathrm{Sym}\{0, \dots, k\} \text{ and } \gamma \text{ is an even permutation}\},$$
$$\Gamma_\sigma^- = \{g \in \Gamma : g.(v_0, \dots, v_k) = (v_{\gamma(0)}, \dots, v_{\gamma(k)}),\ \gamma \in \mathrm{Sym}\{0, \dots, k\} \text{ and } \gamma \text{ is an odd permutation}\}.$$

We note that Γ_σ^+ is a subgroup of Γ and that,

$$\text{for all } g \in \Gamma_\sigma^+, \quad g.\Gamma_\sigma^+ = \Gamma_\sigma^+, \quad g.\Gamma_\sigma^- = \Gamma_\sigma^-,$$
$$\text{for all } g \in \Gamma_\sigma^-, \quad g.\Gamma_\sigma^+ = \Gamma_\sigma^-, \quad g.\Gamma_\sigma^- = \Gamma_\sigma^+.$$

Define the subspace $\mathbb{E}_{\sigma,\pi} \subseteq \mathbb{E}$ to be the subspace of vectors $x \in \mathbb{E}$ such that,

$$\text{for all } g \in \Gamma_\sigma^+, \quad \pi(g).x = x,$$
$$\text{for all } g \in \Gamma_\sigma^-, \quad \pi(g).x = -x.$$

Proposition 3.4. *A map $\phi' : \Sigma'(k,\Gamma) \to \mathbb{E}$ can be extended (uniquely) to a map $\phi \in C^k(X,\pi)$ if and only if for every $\sigma \in \Sigma'(k,\Gamma)$ it holds that $\phi'(\sigma) \in \mathbb{E}_{\sigma,\pi}$.*

Proof. Let $\phi' : \Sigma'(k,\Gamma) \to \mathbb{E}$ be some map.

Assume first that there is a map $\phi \in C^k(X,\pi)$ such that $\phi|_{\Sigma'(k,\Gamma)} = \phi'$. Let $(v_0,\dots,v_k) \in \Sigma'(k,\Gamma)$ and $g \in \Gamma_\sigma^+$. Also let $\gamma \in \mathrm{Sym}\{0,\dots,k\}$ such that γ is even and $g.(v_0,\dots,v_k) = (v_{\gamma(0)},\dots,v_{\gamma(k)})$. Then it holds that

$$\begin{aligned}
\pi(g).\phi'((v_0,\dots,v_k)) &= \pi(g).\phi((v_0,\dots,v_k)) \\
&= \phi(g.(v_0,\dots,v_k)) && \text{(since } \phi \text{ is equivariant)} \\
&= \phi((v_{\gamma(0)},\dots,v_{\gamma(k)})) \\
&= \phi((v_0,\dots,v_k)) = \phi'((v_0,\dots,v_k)) && \text{(since } \phi \text{ is alternating);}
\end{aligned}$$

i.e., $\pi(g).\phi'(\sigma) = \phi'(\sigma)$ for every $\sigma \in \Sigma'(k,\Gamma)$ and every $g \in \Gamma_\sigma^+$. By a similar computation, it follows that $\pi(g).\phi'(\sigma) = -\phi'(\sigma)$ for every $\sigma \in \Sigma'(k,\Gamma)$ and every $g \in \Gamma_\sigma^-$. Thus, for every $\sigma \in \Sigma'(k,\Gamma)$ it holds that $\phi'(\sigma) \in \mathbb{E}_{\sigma,\pi}$.

In the other direction, assume that for every $\sigma \in \Sigma'(k,\Gamma)$ it holds that $\phi'(\sigma) \in \mathbb{E}_{\sigma,\pi}$. For every $\gamma \in \mathrm{Sym}\{0,\dots,k\}$, every $g \in \Gamma$ and every $(v_0,\dots,v_k) \in \Sigma'(k,\Gamma)$, define

$$\phi(g.(v_{\gamma(0)},\dots,v_{\gamma(k)})) = \pi(g)\,\mathrm{sgn}(\gamma)\phi'((v_0,\dots,v_k)).$$

If we show that ϕ above is well-defined, it will follow from its definition that it is equivariant and alternating. Fix $\sigma = (v_0,\dots,v_k) \in \Sigma'(k,\Gamma)$ and let $\gamma,\gamma' \in \mathrm{Sym}\{0,\dots,k\}$, $g, g' \in \Gamma$ be such that

$$g.(v_{\gamma(0)},\dots,v_{\gamma(k)}) = g'.(v_{\gamma'(0)},\dots,v_{\gamma'(k)}).$$

Then

$$(v_{\gamma(\gamma')^{-1}(0)},\dots,v_{\gamma(\gamma')^{-1}(k)}) = (g^{-1}g').(v_0,\dots,v_k)$$

and therefore $g^{-1}g' \in \Gamma_\sigma^+ \cup \Gamma_\sigma^-$ and the sign of the permutation induced by $g^{-1}g'$ on σ is exactly $\mathrm{sgn}(\gamma(\gamma')^{-1}) = \mathrm{sgn}(\gamma)\,\mathrm{sgn}(\gamma')$. From the assumption that $\phi'((v_0,\dots,v_k)) \in \mathbb{E}_{\sigma,\pi}$ it follows that

$$\pi(g^{-1}g')\phi'((v_0,\dots,v_k)) = \mathrm{sgn}(\gamma)\,\mathrm{sgn}(\gamma')\phi'((v_0,\dots,v_k)),$$

or equivalently

$$\mathrm{sgn}(\gamma)\,\mathrm{sgn}(\gamma')\pi(g^{-1}g')\phi'((v_0,\dots,v_k)) = \phi'((v_0,\dots,v_k)).$$

Thus

$$\begin{aligned}\pi(g)\operatorname{sgn}(\gamma)\phi'((v_0,\dots,v_k)) &= \pi(g)\operatorname{sgn}(\gamma)\operatorname{sgn}(\gamma)\operatorname{sgn}(\gamma')\pi(g^{-1}g')\phi'((v_0,\dots,v_k))\\ &= \pi(g')\operatorname{sgn}(\gamma')\phi'((v_0,\dots,v_k)),\end{aligned}$$

and ϕ is well-defined. □

All the results above were stated for π, but since $\bar{\pi}$ is a representation of Γ on a reflexive Banach space, they pass automatically to $\bar{\pi}$; i.e., we can define $C^k(X,\bar{\pi})$ as above and by the same considerations it follows that $C^k(X,\bar{\pi})$ is also a reflexive Banach space. We also define $\bar{d}_k : C^k(X,\bar{\pi}) \to C^{k+1}(X,\bar{\pi})$ to be the differential defined as above.

The reason for considering $C^k(X,\bar{\pi})$ is that there is a natural coupling between $C^k(X,\pi)$ and $C^k(X,\bar{\pi})$: let $(\cdot,\cdot)$ denote the usual coupling between $\mathbb{E}$ and $\mathbb{E}^*$ and for $\phi \in C^k(X,\pi)$, $\psi \in C^k(X,\bar{\pi})$ define

$$\langle \phi,\psi\rangle := \sum_{\sigma\in\Sigma(k,\Gamma)} \frac{m(\sigma)}{(k+1)!\,|\Gamma_\sigma|}(\phi(\sigma),\psi(\sigma)).$$

With the above coupling, $C^k(X,\bar{\pi}) \subseteq (C^k(X,\pi))^*$. Actually, since $\mathbb{E}$ is reflexive, there is an isomorphism between $C^k(X,\bar{\pi})$ and $(C^k(X,\pi))^*$ (see [Koivisto 2014, Proposition 28]), but we will make no use of this fact. Given this coupling, we define $d_k^* : C^{k+1}(X,\bar{\pi}) \to C^k(X,\bar{\pi})$ to be the adjoint operator of d_k and $\bar{d}_k^* : C^{k+1}(X,\pi) \to C^k(X,\pi)$ to be the adjoint operator of $\bar{d}_k$.

We recall that for a Banach space $\mathbb{E}$, the duality mapping is a mapping $j : \mathbb{E} \to 2^{\mathbb{E}^*}$ defined as

$$j(x) = \{x^* \in \mathbb{E}^* : |x| = |x^*|,\ (x,x^*) = |x|^2\}$$

(the fact that the set defined by $j(x)$ is nonempty follows immediately from the Hahn–Banach theorem). By our assumption, $\mathbb{E}$ is reflexive and thus we also have the duality mapping $\bar{j} : \mathbb{E}^* \to 2^{\mathbb{E}} (= 2^{\mathbb{E}^{**}})$.

We define maps $J : C^k(X,\pi) \to 2^{C^k(X,\bar{\pi})}$ and $\bar{J} : C^k(X,\bar{\pi}) \to 2^{C^k(X,\pi)}$ by,

$$\begin{aligned}&\text{for all } \phi \in C^k(X,\pi), && J\phi = \{\psi \in C^k(X,\bar{\pi}) : \text{for all } \sigma \in \Sigma(k),\ \psi(\sigma) \in j(\phi(\sigma))\},\\ &\text{for all } \psi \in C^k(X,\bar{\pi}), && \bar{J}\psi = \{\phi \in C^k(X,\pi) : \text{for all } \sigma \in \Sigma(k),\ \phi(\sigma) \in \bar{j}(\psi(\sigma))\}.\end{aligned}$$

Proposition 3.5. *Let $X, \mathbb{E}, \pi, J, \bar{J}$ be as above and $\phi \in C^k(X,\pi)$, $\psi \in C^k(X,\bar{\pi})$. Then $J\phi$, $\bar{J}\psi$ are nonempty sets and,*

$$\begin{aligned}&\textit{for all } \phi^* \in J\phi, && \|\phi^*\|^2 = \|\phi\|^2 = \langle \phi,\phi^*\rangle,\\ &\textit{for all } \psi^* \in \bar{J}\psi, && \|\psi^*\|^2 = \|\psi\|^2 = \langle \psi^*,\psi\rangle.\end{aligned}$$

Proof. We will prove the assertions above only for $J\phi$, since the proof for $\bar{J}\psi$ is similar.

We will only show that $J\phi$ is nonempty: the fact that, for every $\phi^* \in J\phi$,

$$\|\phi^*\|^2 = \|\phi\|^2 = \langle \phi,\phi^*\rangle$$

follows from straightforward a computation that is left for to the reader.

Fix $\phi \in C^k(X,\pi)$. Choose $\Sigma'(k,\Gamma) \subseteq \Sigma(k,\Gamma)$ as above to be a set of representatives of the action of the permutation group $\mathrm{Sym}\{0,\dots,k\}$ on $\Sigma(k,\Gamma)$. By Proposition 3.4, it is enough to show that there is $\psi' : \Sigma'(k,\Gamma) \to \mathbb{E}^*$ such that for every $\sigma \in \Sigma'(k,\Gamma)$ it holds that $\psi'(\sigma) \in \mathbb{E}^*_{\sigma,\bar{\pi}}$ and $\psi'(\sigma) \in j(\phi(\sigma))$.

For every $\sigma \in \Sigma'(k,\Gamma)$, define $\varepsilon_\sigma : \Gamma_\sigma^+ \cup \Gamma_\sigma^- \to \{\pm 1\}$ as

$$\varepsilon_\sigma(g) = \begin{cases} 1, & g \in \Gamma_\sigma^+, \\ -1, & g \in \Gamma_\sigma^-. \end{cases}$$

Note that for every $g \in \Gamma_\sigma^+ \cup \Gamma_\sigma^-$, it follows that $\varepsilon_\sigma(g) = \varepsilon_\sigma(g^{-1})$ and that $\pi(g).\phi(\sigma) = \varepsilon_\sigma(g)\phi(\sigma)$.

Also, for every $\sigma \in \Sigma'(k,\Gamma)$, we choose some $x_\sigma^* \in j(\phi(\sigma))$ and define

$$\psi'(\sigma) = \frac{1}{|\Gamma_\sigma^+ \cup \Gamma_\sigma^-|} \int_{\Gamma_\sigma^+ \cup \Gamma_\sigma^-} \varepsilon_\sigma(g)\bar{\pi}(g).x_\sigma^* \, d\mu(g).$$

This integral is well-defined because by our assumptions the action of $\bar{\pi}$ is continuous and $\Gamma_\sigma^+, \Gamma_\sigma^-$ are compact sets.

Recall that for every $g' \in \Gamma_\sigma^+$ it holds that $g'.\Gamma_\sigma^+ = \Gamma_\sigma^+$, $g'.\Gamma_\sigma^- = \Gamma_\sigma^-$ and therefore for every $g'' \in \Gamma_\sigma^+ \cup \Gamma_\sigma^-$ it holds that $\varepsilon_\sigma((g')^{-1}g'') = \varepsilon_\sigma(g'')$. Also recall that the action of Γ preserves the Haar measure. Thus for every $g' \in \Gamma_\sigma^+$ and every $\sigma \in \Sigma'(k,\Gamma)$ it holds that

$$\begin{aligned}
\bar{\pi}(g').\psi'(\sigma) &= \frac{1}{|\Gamma_\sigma^+ \cup \Gamma_\sigma^-|} \int_{\Gamma_\sigma^+ \cup \Gamma_\sigma^-} \varepsilon_\sigma(g)\bar{\pi}(g'g).x_\sigma^* \, d\mu(g) \\
&= \frac{1}{|\Gamma_\sigma^+ \cup \Gamma_\sigma^-|} \int_{g'.\Gamma_\sigma^+ \cup g'.\Gamma_\sigma^-} \varepsilon_\sigma((g')^{-1}g'')\bar{\pi}(g'').x_\sigma^* \, d\mu(g'') \quad (\text{since } g'' = g'g) \\
&= \frac{1}{|\Gamma_\sigma^+ \cup \Gamma_\sigma^-|} \int_{\Gamma_\sigma^+ \cup \Gamma_\sigma^-} \varepsilon_\sigma(g'')\bar{\pi}(g'').x_\sigma^* \, d\mu(g'') = \psi'(\sigma).
\end{aligned}$$

Note that for every $g' \in \Gamma_\sigma^-$ it holds that $g'.\Gamma_\sigma^+ = \Gamma_\sigma^-$, $g'.\Gamma_\sigma^+ = \Gamma_\sigma^+$ and that for every $g'' \in \Gamma_\sigma^+ \cup \Gamma_\sigma^-$ it holds that $\varepsilon_\sigma((g')^{-1}g'') = -\varepsilon_\sigma(g'')$. Thus, by a computation similar to the one above, it follows that, for every $g' \in \Gamma_\sigma^-$ and every $\sigma \in \Sigma'(k,\Gamma)$,

$$\bar{\pi}(g').\psi'(\sigma) = -\psi'(\sigma)$$

and therefore $\psi'(\sigma) \in \mathbb{E}^*_{\sigma,\bar{\pi}}$.

We note that for every $\sigma \in \Sigma'(k,\Gamma)$ it holds that

$$\begin{aligned}
&(\phi(\sigma), \psi'(\sigma)) \\
&= \left(\phi(\sigma), \frac{1}{|\Gamma_\sigma^+ \cup \Gamma_\sigma^-|} \int_{\Gamma_\sigma^+ \cup \Gamma_\sigma^-} \varepsilon_\sigma(g)\bar{\pi}(g).x_\sigma^* \, d\mu(g)\right) \\
&= \frac{1}{|\Gamma_\sigma^+ \cup \Gamma_\sigma^-|} \int_{\Gamma_\sigma^+ \cup \Gamma_\sigma^-} \varepsilon_\sigma(g)(\phi(\sigma), \bar{\pi}(g).x_\sigma^*) \, d\mu(g) \\
&= \frac{1}{|\Gamma_\sigma^+ \cup \Gamma_\sigma^-|} \int_{\Gamma_\sigma^+ \cup \Gamma_\sigma^-} \varepsilon_\sigma(g)(\pi(g^{-1}).\phi(\sigma), x_\sigma^*) \, d\mu(g) \\
&= \frac{1}{|\Gamma_\sigma^+ \cup \Gamma_\sigma^-|} \int_{\Gamma_\sigma^+ \cup \Gamma_\sigma^-} (\varepsilon_\sigma(g))^2(\phi(\sigma), x_\sigma^*) \, d\mu(g) \quad (\text{since } \pi(g^{-1}).\phi(\sigma) = \varepsilon_\sigma(g^{-1})\phi(\sigma) = \varepsilon_\sigma(g)\phi(\sigma)) \\
&= \frac{1}{|\Gamma_\sigma^+ \cup \Gamma_\sigma^-|} \int_{\Gamma_\sigma^+ \cup \Gamma_\sigma^-} |\phi(\sigma)|^2 \, d\mu(g) = |\phi(\sigma)|^2,
\end{aligned}$$

and that

$$|\psi'(\sigma)| = \left| \frac{1}{|\Gamma_\sigma^+ \cup \Gamma_\sigma^-|} \int_{\Gamma_\sigma^+ \cup \Gamma_\sigma^-} \varepsilon_\sigma(g)\bar{\pi}(g).x_\sigma^* \, d\mu(g) \right|$$
$$\leq \frac{1}{|\Gamma_\sigma^+ \cup \Gamma_\sigma^-|} \int_{\Gamma_\sigma^+ \cup \Gamma_\sigma^-} |\bar{\pi}(g).x_\sigma^*| \, d\mu(g) = \frac{1}{|\Gamma_\sigma^+ \cup \Gamma_\sigma^-|} \int_{\Gamma_\sigma^+ \cup \Gamma_\sigma^-} |\phi(\sigma)| \, d\mu(g) = |\phi(\sigma)|,$$

and therefore $\psi'(\sigma) \in j(\phi(\sigma))$ as needed. □

Below, we will make use of changing the order of summation when calculating norms of maps $C^k(X, \pi)$ or coupling between maps of $C^k(X, \pi)$ and $C^k(X, \bar{\pi})$. For this, we will need the following: for $0 \leq l < k \leq n$ and $\tau \in \Sigma(l)$, $\sigma \in \Sigma(k)$, we write $\tau \subseteq \sigma$ if σ contains τ as a set (without respecting the ordering); i.e., for $\sigma = (v_0, \dots, v_k)$, $\tau = (w_0, \dots, w_l)$, we have $\tau \subseteq \sigma$ if $\{w_0, \dots, w_l\} \subseteq \{v_0, \dots, v_k\}$.

Proposition 3.6 [Ballmann and Świątkowski 1997, Lemma 1.3; Dymara and Januszkiewicz 2000, Lemma 3.3]. *For $0 \leq l < k \leq n$, let $f = f(\tau, \sigma)$ be a Γ-invariant function on the set of pairs (τ, σ), where $\tau \in \Sigma(l)$, $\sigma \in \Sigma(k)$ with $\tau \subseteq \sigma$. Then*

$$\sum_{\sigma \in \Sigma(k,\Gamma)} \sum_{\tau \in \Sigma(l),\, \tau \subseteq \sigma} \frac{f(\tau, \sigma)}{|\Gamma_\sigma|} = \sum_{\tau \in \Sigma(l,\Gamma)} \sum_{\sigma \in \Sigma(k),\, \tau \subseteq \sigma} \frac{f(\tau, \sigma)}{|\Gamma_\tau|}.$$

The reader should note that from now on we will use the above proposition to change the order of summation without mentioning it explicitly.

Proposition 3.7 (equivalent to [Ballmann and Świątkowski 1997, Propositions 1.5 and 1.6]).

(1) *The differential is a bounded operator and $\|d_k\| \leq \sqrt{k+2}$.*

(2) *We define $d_k^* : C^{k+1}(X, \bar{\pi}) \to C^k(X, \bar{\pi})$ to be the adjoint operator of d_k. Then*

$$d_k^* \phi(\tau) = \sum_{v \in \Sigma(0), v\tau \in \Sigma(k+1)} \frac{m(v\tau)}{m(\tau)} \phi(v\tau), \qquad \tau \in \Sigma(k),$$

where $v\tau = (v, v_0, \dots, v_k)$ for $\tau = (v_0, \dots, v_k)$.

Proof. (1) For every $\phi \in C^k(X, \pi)$ we have

$$\|d_k \phi\|^2 = \sum_{\sigma \in \Sigma(k+1,\Gamma)} \frac{m(\sigma)}{(k+2)!\, |\Gamma_\sigma|} \left| \sum_{i=0}^{k+1} (-1)^i \phi(\sigma_i) \right|^2 \leq \sum_{\sigma \in \Sigma(k+1,\Gamma)} \frac{m(\sigma)}{(k+2)!\, |\Gamma_\sigma|} (k+2) \sum_{i=0}^{k+1} |\phi(\sigma_i)|^2$$
$$= \sum_{\sigma \in \Sigma(k+1,\Gamma)} \frac{m(\sigma)}{(k+1)!\, (k+1)!\, |\Gamma_\sigma|} \sum_{\tau \in \Sigma(k), \tau \subset \sigma} |\phi(\tau)|^2$$
$$= \sum_{\tau \in \Sigma(k,\Gamma)} \frac{|\phi(\tau)|^2}{(k+1)!\, (k+1)!\, |\Gamma_\tau|} \sum_{\sigma \in \Sigma(k+1), \tau \subset \sigma} m(\sigma) = \sum_{\tau \in \Sigma(k,\Gamma)} \frac{(k+2)!\, m(\tau) |\phi(\tau)|^2}{(k+1)!\, (k+1)!\, |\Gamma_\tau|} = (k+2) \|\phi\|^2.$$

(2) For $\sigma \in \Sigma(k+1)$ and $\tau \subset \sigma$, $\tau \in \Sigma(k)$ denote by $[\sigma : \tau]$ the incidence coefficient of τ with respect to σ; i.e., if σ_i has the same vertices as τ then for every $\psi \in C^k(X, \pi)$ we have $[\sigma : \tau]\psi(\tau) = (-1)^i \psi(\sigma_i)$.

Take $\phi \in C^{k+1}(X, \bar{\pi})$ and $\psi \in C^k(X, \pi)$. We note that for every $\tau \in \Sigma(k)$, every $\sigma \in \Sigma(k+1)$ and every $g \in \Gamma$,

$$(\psi(g.\tau), \phi(g.\sigma)) = (\pi(g)\psi(\tau), \bar{\pi}(g)\phi(\sigma)) = (\psi(\tau), \phi(\sigma)),$$

and we will use this fact in equality $(*)$ below, in which we apply Proposition 3.6:

$$\begin{aligned}
\langle d\psi, \phi\rangle &= \sum_{\sigma\in\Sigma(k+1,\Gamma)} \frac{m(\sigma)}{(k+2)!\,|\Gamma_\sigma|}\left(\sum_{i=0}^{k+1}(-1)^i\psi(\sigma_i), \phi(\sigma)\right) \\
&= \sum_{\sigma\in\Sigma(k+1,\Gamma)} \frac{m(\sigma)}{(k+1)!\,(k+2)!\,|\Gamma_\sigma|}\left(\sum_{\tau\in\Sigma(k),\,\tau\subset\sigma}[\sigma:\tau]\psi(\tau), \phi(\sigma)\right) \\
&= \sum_{\sigma\in\Sigma(k+1,\Gamma)} \frac{1}{(k+1)!\,|\Gamma_\sigma|}\sum_{\tau\in\Sigma(k),\,\tau\subset\sigma} m(\tau)\left(\psi(\tau), \frac{[\sigma:\tau]m(\sigma)}{m(\tau)(k+2)!}\phi(\sigma)\right) \\
&= \sum_{\tau\in\Sigma(k,\Gamma)} \frac{m(\tau)}{(k+1)!\,|\Gamma_\tau|}\sum_{\sigma\in\Sigma(k+1),\,\tau\subset\sigma}\left(\psi(\tau), \frac{[\sigma:\tau]m(\sigma)}{m(\tau)(k+2)!}\phi(\sigma)\right) \qquad (*) \\
&= \sum_{\tau\in\Sigma(k,\Gamma)} \frac{m(\tau)}{(k+1)!\,|\Gamma_\tau|}\left(\psi(\tau), \sum_{\sigma\in\Sigma(k+1),\,\tau\subset\sigma}\frac{[\sigma:\tau]m(\sigma)}{m(\tau)(k+2)!}\phi(\sigma)\right) \\
&= \sum_{\tau\in\Sigma(k,\Gamma)} \frac{m(\tau)}{(k+1)!\,|\Gamma_\tau|}\left(\psi(\tau), \sum_{v\in\Sigma(0),\,v\tau\in\Sigma(k+1)}\frac{m(v\tau)}{m(\tau)}\phi(v\tau)\right). \qquad \square
\end{aligned}$$

We end this section by proving the following criterion for vanishing of cohomology that appeared in a different form in [Nowak 2015] (we claim no originality here):

Lemma 3.8. *Let $X, \Gamma, \mathbb{E}, \pi$ be as above and $1 \le k \le n-1$. If there is a constant $C < 1$ such that, for every $\phi \in C^k(X, \pi)$, $\psi \in C^k(X, \bar{\pi})$,*

$$|\langle d_k\phi, \bar{d}_k\psi\rangle| + |\langle \bar{d}_{k-1}^*\phi, d_{k-1}^*\psi\rangle| \ge |\langle \phi, \psi\rangle| - C\left(\frac{\|\phi\|^2 + \|\psi\|^2}{2}\right),$$

then $H^k(X, \pi) = H^k(X, \bar{\pi}) = 0$.

Before proving this lemma, we recall the following facts regarding adjoint operators (for proof of these facts, see for instance [Megginson 1998, Corollary 1.6.6, Theorem 3.1.22]):

Theorem 3.9. *Let $\mathbb{E}_1, \mathbb{E}_2$ be Banach spaces and $T : \mathbb{E}_1 \to \mathbb{E}_2$ be a bounded linear operator. Then:*

(1) *The following are equivalent:*
   (a) *T maps $\mathbb{E}_1$ onto $\mathbb{E}_2$.*
   (b) *T^* is an isomorphism from $\mathbb{E}_2^*$ onto a subspace of $\mathbb{E}_1^*$.*
   (c) *There is a constant $c > 0$ such that, for every $x \in \mathbb{E}_2^*$, $\|T^*x\| \ge c\|x\|$.*
   (d) *T^* is injective with a closed image.*

(2) *The following are equivalent:*
   (a) *T^* maps $\mathbb{E}_2^*$ onto $\mathbb{E}_1^*$.*

(b) *T is an isomorphism from $\mathbb{E}_1$ onto a subspace of $\mathbb{E}_2$.*

(c) *There is a constant $c > 0$ such that, for every $x \in \mathbb{E}_1$, $\|Tx\| \geq c\|x\|$.*

(d) *T is injective with a closed image.*

Using these facts, we can prove Lemma 3.8:

Proof. We will only prove that $H^k(X, \pi) = 0$; the proof for $H^k(X, \bar{\pi})$ is similar. We define d'_{k-1} to be the $k-1$ differential with range $\operatorname{Ker}(d_k)$, i.e., $d'_{k-1} : C^{k-1}(X, \pi) \to \operatorname{Ker}(d_k)$, and we also define $i : \operatorname{Ker}(d_k) \hookrightarrow C^k(X, \pi)$ to be the natural injection. Therefore $d_{k-1} = i \circ d'_{k-1}$. We similarly define $\bar{d}'_{k-1} : C^{k-1}(X, \bar{\pi}) \to \operatorname{Ker}(\bar{d}_k)$ and $\bar{i} : \operatorname{Ker}(\bar{d}_k) \hookrightarrow C^k(X, \bar{\pi})$ and with this notation $\bar{d}_{k-1} = \bar{i} \circ \bar{d}'_{k-1}$.

By the assumptions of the lemma, for every $\phi \in \operatorname{Ker}(d_k)$, taking $\psi = \phi^* \in J\phi$ (using Proposition 3.5) yields that

$$|\langle \bar{d}^*_{k-1}\phi, d^*_{k-1}\phi^*\rangle| \geq |\langle \phi, \phi^*\rangle| - C\left(\frac{\|\phi\|^2 + \|\phi^*\|^2}{2}\right) = (1 - C)\|\phi\|^2.$$

We note that by Proposition 3.7,

$$|\langle \bar{d}^*_{k-1}\phi, d^*_{k-1}\phi^*\rangle| \leq \|\bar{d}^*_{k-1}\phi\| \, \|d^*_{k-1}\phi^*\| \leq \|\bar{d}^*_{k-1}\phi\| \sqrt{k+2}\|\phi\|.$$

Thus, for every $\phi \in \operatorname{Ker}(d_k)$,

$$\|\bar{d}^*_{k-1}\phi\| \geq \frac{1-C}{\sqrt{k+2}}\|\phi\|.$$

This yields that $\bar{d}^*_{k-1} \circ i$ is injective with a closed image. By the notation above, $(\bar{d}'_{k-1})^* \circ \bar{i}^* \circ i$ is injective with a closed image, and therefore $\bar{i}^* \circ i : \operatorname{Ker}(d_k) \to (\operatorname{Ker}(\bar{d}_k))^*$ is injective with a closed image. Note that $\operatorname{Ker}(\bar{d}_k)$ is a closed subspace of a reflexive space (using Proposition 3.3) and thus $\operatorname{Ker}(\bar{d}_k)$ is reflexive and it follows that $(\operatorname{Ker}(\bar{d}_k))^*$ is reflexive as well. Therefore by Theorem 3.9, $i^* \circ \bar{i} = (\bar{i}^* \circ i)^* : (\operatorname{Ker}(d_k))^* \to \operatorname{Ker}(\bar{d}_k)$ is onto.

By a similar argument, for a given $\psi \in \operatorname{Ker}(\bar{d}_k)$, if we take $\phi = \psi^* \in \bar{J}\psi$, then

$$|\langle \bar{d}^*_{k-1}\psi^*, d^*_{k-1}\psi\rangle| \geq (1 - C)\|\psi\|^2,$$

which implies that

$$\|d^*_{k-1}\psi\| \geq \frac{1-C}{\sqrt{k+2}}\|\psi\|.$$

Arguing as above, we deduce from this inequality that $(d'_{k-1})^* \circ i^* \circ \bar{i}$ is injective with a closed image.

We showed above that $i^* \circ \bar{i}$ is onto and therefore it follows that $(d'_{k-1})^* : (\operatorname{Ker}(d_k))^* \to (C^{k-1}(X, \pi))^*$ is injective with a closed image. Thus applying Theorem 3.9 yields that d'_{k-1} is onto, i.e., $\operatorname{Im}(d_{k-1}) = \operatorname{Ker}(d_k)$, or in other words, $H^k(X, \pi) = 0$. □

Remark 3.10. As in [Ballmann and Świątkowski 1997], we can define the Laplacian operators as follows: $\Delta^+_k = \bar{d}^*_k d_k$, $\Delta^-_k = d_{k-1}\bar{d}^*_{k-1}$. With this notation, the condition in Lemma 3.8 can be reformulated as follows: there is a constant $C < 1$ such that, for every $\phi \in C^k(X, \pi)$, $\psi \in C^k(X, \bar{\pi})$,

$$|\langle \Delta^+_k\phi, \psi\rangle| + |\langle \Delta^-_k\phi, \psi\rangle| \geq |\langle \phi, \psi\rangle| - C\left(\frac{\|\phi\|^2 + \|\psi\|^2}{2}\right).$$

4. Local criteria for vanishing of Banach cohomology

Below, we will prove local criteria for vanishing of equivariant cohomology in the spirit of "Garland's method". The method is an adaption of [Ballmann and Świątkowski 1997], but unlike the case of Hilbert spaces, considered in that work, in which the condition for vanishing of cohomology requires a (one-sided) spectral gap in the links, here the condition for vanishing of cohomology will require a two-sided spectral gap in the same links.

Let $X, \Gamma, \mathbb{E}, \pi$ be as in Section 3 (recall that we assume that π is continuous and $\mathbb{E}$ is reflexive and thus $\bar{\pi}$ is also continuous). Given an ordered simplex $(v_0, \dots, v_j) = \tau \in \Sigma(j)$, the link of τ is simply the link of $\{v_0, \dots, v_j\}$ defined above. Below, we will only be interested in the 1-skeleton to the links: given $\tau \in \Sigma(j)$, the 1-*skeleton of* X_τ is the weighted graph, denoted by (V_τ, E_τ), defined as

$$V_\tau = \{v : \{v\} \in X_\tau(0)\}, \quad E_\tau = X_\tau(1),$$

with the weight function $m_\tau(\{u, v\}) = m(\tau \cup \{u, v\})$, where $\tau \cup \{u, v\}$ is defined by the abuse of notation of treating τ as a set (and forgetting the ordering); i.e.,

$$m((v_0, \dots, v_j) \cup \{u, v\}) = m(\{v_0, \dots, v_j, u, v\}).$$

Note that with this definition, $m_\tau(v) = m(\tau \cup \{v\})$.

On this weighted graph, we define $\ell^2(V_\tau, m_\tau)$, $\ell^2(V_\tau, m_\tau; \mathbb{E})$ and the operators A_τ, M_τ as in Section 2D. On $\ell^2(V_\tau, m_\tau; \mathbb{E})$ define a norm denoted by $\|\cdot\|_\tau$ as in Section 3; i.e., for $\phi \in \ell^2(V_\tau, m_\tau; \mathbb{E})$,

$$\|\phi\|_\tau = \left(\sum_{v \in V_\tau} m_\tau(v) |\phi(v)|^2 \right)^{1/2},$$

where $|\cdot|$ is the norm of $\mathbb{E}$. Also, define a coupling $\langle \cdot, \cdot \rangle_\tau$ between $\ell^2(V_\tau, m_\tau; \mathbb{E})$ and $\ell^2(V_\tau, m_\tau; \mathbb{E}^*)$ as follows: for $\phi \in \ell^2(V_\tau, m_\tau; \mathbb{E})$, $\psi \in \ell^2(V_\tau, m_\tau; \mathbb{E}^*)$,

$$\langle \phi, \psi \rangle_\tau = \sum_{v \in V_\tau} m_\tau(v) (\phi(v), \psi(v)),$$

where $(\cdot, \cdot)$ is the standard coupling between $\mathbb{E}$ and $\mathbb{E}^*$.

Given $\phi \in C^k(X, \pi)$ and $\tau \in \Sigma(k-1)$ we define the *localization of* ϕ *at* X_τ, denoted by $\phi_\tau \in \ell^2(V_\tau, m_\tau; \mathbb{E})$, as

$$\phi_\tau(v) = \phi(v\tau) \quad \text{for all } v \in V_\tau,$$

where $v\tau$ is the concatenation of v with τ, i.e., $v\tau = (v, v_0, \dots, v_{k-1})$ for $\tau = (v_0, \dots, v_{k-1})$. We note that by the definition of X_τ, $v\tau \in \Sigma(k)$ and therefore $\phi(v\tau)$ is well-defined.

The basic observation in [Garland 1973] was that the norm of cochains can be computed by considering their localizations. Below, we generalize this observation to the Banach setting. The calculations below are very similar to those of [Ballmann and Świątkowski 1997], but we included all the calculations, because we need localization results not only for the norms, but for the couplings.

Lemma 4.1. *Let* $1 \leq k \leq n-1$, $\phi \in C^k(X, \pi)$, $\psi \in C^k(X, \bar{\pi})$. *Then*

$$(k+1)!\,\langle \phi, \psi \rangle = \sum_{\tau \in \Sigma(k-1,\Gamma)} \frac{1}{|\Gamma_\tau|} \langle \phi_\tau, \psi_\tau \rangle_\tau,$$

$$(k+1)!\,\|\phi\|^2 = \sum_{\tau \in \Sigma(k-1,\Gamma)} \frac{1}{|\Gamma_\tau|} \|\phi_\tau\|_\tau^2,$$

$$(k+1)!\,\|\psi\|^2 = \sum_{\tau \in \Sigma(k-1,\Gamma)} \frac{1}{|\Gamma_\tau|} \|\psi_\tau\|_\tau^2.$$

Proof. All these equalities follow from the definition of the localization and Proposition 3.6 and thus we will only prove the first equality, leaving the other two for the reader. Fix $\phi \in C^k(X, \pi)$, $\psi \in C^k(X, \bar{\pi})$. Then

$$\begin{aligned}
\sum_{\tau \in \Sigma(k-1,\Gamma)} \frac{1}{|\Gamma_\tau|} \langle \phi_\tau, \psi_\tau \rangle_\tau &= \sum_{\tau \in \Sigma(k-1,\Gamma)} \frac{1}{|\Gamma_\tau|} \sum_{v \in V_\tau} m_\tau(v)(\phi_\tau(v), \psi_\tau(v)) \\
&= \sum_{\tau \in \Sigma(k-1,\Gamma)} \frac{1}{|\Gamma_\tau|} \sum_{v \in V_\tau} m(v\tau)(\phi(v\tau), \psi(v\tau)) \\
&= \sum_{\tau \in \Sigma(k-1,\Gamma)} \frac{1}{|\Gamma_\tau|} \frac{1}{(k+1)!} \sum_{\sigma \in \Sigma(k), \tau \subseteq \sigma} m(\sigma)(\phi(\sigma), \psi(\sigma)) \\
&= \sum_{\sigma \in \Sigma(k,\Gamma)} \frac{m(\sigma)}{(k+1)!\,|\Gamma_\sigma|} \sum_{\tau \in \Sigma(k-1,\Gamma)} (\phi(\sigma), \psi(\sigma)) \sum_{\tau \in \Sigma(k-1), \tau \subseteq \sigma} 1 \\
&= (k+1)!\,\langle \phi, \psi \rangle.
\end{aligned}$$

□

Lemma 4.2. *Let* $1 \leq k \leq n-1$, $\phi \in C^k(X, \pi)$, $\psi \in C^k(X, \bar{\pi})$. *Then*

$$\langle \bar{d}_{k-1}^* \phi, d_{k-1}^* \psi \rangle = \frac{1}{k!} \sum_{\tau \in \Sigma(k-1,\Gamma)} \frac{1}{|\Gamma_\tau|} \langle (M_\tau \otimes \mathrm{id}_{\mathbb{E}}) \phi_\tau, \psi_\tau \rangle_\tau.$$

Proof. By Proposition 3.7, for every $\tau \in \Sigma(k-1)$,

$$\bar{d}_k^* \phi(\tau) = \sum_{v \in V_\tau} \frac{m(v\tau)}{m(\tau)} \phi(v\tau), \quad d_k^* \psi(\tau) = \sum_{v \in V_\tau} \frac{m(v\tau)}{m(\tau)} \psi(v\tau).$$

We note that by definition $m_\tau(\emptyset) = m(\tau)$ and therefore, for every $\tau \in \Sigma(k-1)$,

$$\begin{aligned}
\langle (M_\tau \otimes \mathrm{id}_{\mathbb{E}}) \phi_\tau, \psi_\tau \rangle_\tau &= \sum_{v \in V_\tau} m_\tau(v) \left(\sum_{u \in V_\tau} \frac{m_\tau(u)}{m_\tau(\emptyset)} \phi_\tau(u), \psi_\tau(v) \right) \\
&= \sum_{v \in V_\tau} m(v\tau) \left(\sum_{u \in V_\tau} \frac{m(u\tau)}{m(\tau)} \phi(u\tau), \psi(v\tau) \right) \\
&= \left(\sum_{u \in V_\tau} \frac{m(u\tau)}{m(\tau)} \phi(u\tau), \sum_{v \in V_\tau} m(v\tau) \psi(v\tau) \right) = m(\tau)(\bar{d}_k^* \phi(\tau), d_k^* \psi(\tau)).
\end{aligned}$$

Therefore

$$\sum_{\tau\in\Sigma(k-1,\Gamma)} \frac{1}{|\Gamma_\tau|}\langle (M_\tau\otimes \mathrm{id}_{\mathbb{E}})\phi_\tau, \psi_\tau\rangle_\tau = \sum_{\tau\in\Sigma(k-1,\Gamma)} \frac{m(\tau)}{|\Gamma_\tau|}(\bar{d}_k^*\phi(\tau), d_k^*\psi(\tau))$$
$$= k!\,\langle \bar{d}_{k-1}^*\phi, d_{k-1}^*\psi\rangle. \qquad \square$$

Lemma 4.3. *Let* $1\le k\le n-1,\ \phi\in C^k(X,\pi),\ \psi\in C^k(X,\bar{\pi})$. *Then*

$$\langle d_k\phi, \bar{d}_k\psi\rangle = \langle\phi,\psi\rangle - \frac{1}{k!}\sum_{\tau\in\Sigma(k-1,\Gamma)}\frac{1}{|\Gamma_\tau|}\langle (A_\tau\otimes\mathrm{id}_{\mathbb{E}})\phi_\tau,\psi_\tau\rangle_\tau.$$

Proof. For $\eta=(v_0,\dots,v_{k+1})\in\Sigma(k+1)$ and $0\le i\ne j\le k+1$, define $\eta_i=(v_0,\dots,\hat{v}_i,\dots,v_{k+1})$ and $\eta_{i,j}=(v_0,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_{k+1})$. Then

$$(d_k\phi(\eta),\bar{d}_k\psi(\eta)) = \left(\sum_{i=0}^{k+1}(-1)^i\phi(\eta_i), \sum_{j=0}^{k+1}(-1)^j\psi(\eta_j)\right)$$
$$= \sum_{i=0}^{k+1}(\phi(\eta_i),\psi(\eta_i)) + \sum_{0\le i\ne j\le k+1}(-1)^{i+j}(\phi(\eta_i),\psi(\eta_j)).$$

We note that by the assumption that ϕ,ψ are alternating, changing the order of η_i in the first sum above does not change the coupling and therefore

$$\sum_{i=0}^{k+1}(\phi(\eta_i),\psi(\eta_i)) = \frac{1}{(k+1)!}\sum_{\sigma\in\Sigma(k),\,\sigma\subseteq\eta}(\phi(\sigma),\psi(\sigma)).$$

We also note that, for every $i\ne j$,

$$(\phi(\eta_i),\psi(\eta_j)) = (-1)^{i+j-1}(\phi(v_j\eta_{i,j}),\psi(v_i\eta_{i,j}))$$

(this can be shown by considering the cases $i<j$ and $j<i$; we leave the proof for the reader). Therefore

$$\sum_{0\le i\ne j\le k+1}(-1)^{i+j}(\phi(\eta_i),\psi(\eta_j)) = -\sum_{0\le i\ne j\le k+1}(\phi(v_j\eta_{i,j}),\psi(v_i\eta_{i,j}))$$
$$= \frac{1}{k!}\sum_{\tau\in\Sigma(k-1),\,\tau\subseteq\eta}\ \sum_{v,v\tau\subseteq\eta}\left(\sum_{u,u\ne v,\,uv\tau\subseteq\eta}\phi(u\tau),\psi(v\tau)\right),$$

where $uv\tau$ is the concatenation, i.e., if $\tau=(v_0,\dots,v_{k-1})$, then $uv\tau=(u,v,v_0,\dots,v_{k-1})$ (we recall that $uv\tau\subseteq\eta$ refers only to inclusion as sets without regarding the ordering). This yields

$$\langle d_k\phi,\bar{d}_k\psi\rangle = \sum_{\eta\in\Sigma(k+1,\Gamma)}\frac{m(\eta)}{(k+2)!\,|\Gamma_\eta|}\frac{1}{(k+1)!}\sum_{\sigma\in\Sigma(k),\,\sigma\subseteq\eta}(\phi(\sigma),\psi(\sigma))$$
$$-\sum_{\eta\in\Sigma(k+1,\Gamma)}\frac{m(\eta)}{(k+2)!\,|\Gamma_\eta|}\frac{1}{k!}\sum_{\tau\in\Sigma(k-1),\,\tau\subseteq\eta}\ \sum_{v,v\tau\subseteq\eta}\left(\sum_{u,u\ne v,\,uv\tau\subseteq\eta}\phi(u\tau),\psi(v\tau)\right). \quad (1)$$

We will calculate each one of the expressions above separately. First, by applying Proposition 3.6,

$$\begin{aligned}
&\sum_{\eta\in\Sigma(k+1,\Gamma)} \frac{m(\eta)}{(k+2)!\,|\Gamma_\eta|}\frac{1}{(k+1)!}\sum_{\sigma\in\Sigma(k),\sigma\subseteq\eta}(\phi(\sigma),\psi(\sigma))\\
&\qquad=\sum_{\sigma\in\Sigma(k,\Gamma)}\frac{1}{(k+1)!\,|\Gamma_\sigma|}(\phi(\sigma),\psi(\sigma))\sum_{\sigma\in\Sigma(k+1),\,\sigma\subseteq\eta}\frac{m(\eta)}{(k+2)!}\\
&\qquad=\sum_{\sigma\in\Sigma(k,\Gamma)}\frac{m(\sigma)}{(k+1)!\,|\Gamma_\sigma|}(\phi(\sigma),\psi(\sigma))=\langle\phi,\psi\rangle. \qquad (2)
\end{aligned}$$

Second, applying Proposition 3.6 to the second expression,

$$\begin{aligned}
&\sum_{\eta\in\Sigma(k+1,\Gamma)}\frac{m(\eta)}{(k+2)!\,|\Gamma_\eta|}\frac{1}{k!}\sum_{\tau\in\Sigma(k-1),\,\tau\subseteq\eta}\ \sum_{v,v\tau\subseteq\eta}\Bigg(\sum_{u,\,u\neq v,\,uv\tau\subseteq\eta}\phi(u\tau),\psi(v\tau)\Bigg)\\
&\quad=\frac{1}{k!}\sum_{\tau\in\Sigma(k-1,\Gamma)}\frac{1}{|\Gamma_\tau|}\sum_{\eta\in\Sigma(k+1),\tau\subseteq\eta}\frac{m(\eta)}{(k+2)!}\sum_{v,v\tau\subseteq\eta}\Bigg(\sum_{u,u\neq v,uv\tau\subseteq\eta}\phi(u\tau),\psi(v\tau)\Bigg)\\
&\quad=\frac{1}{k!}\sum_{\tau\in\Sigma(k-1,\Gamma)}\frac{1}{|\Gamma_\tau|}\sum_{\{v,u\}\in E_\tau}m_\tau(\{v,u\})\sum_{v\in\{v,u\}}\Bigg(\sum_{u\in\{v,u\},\,u\neq v}\phi_\tau(u),\psi_\tau(v)\Bigg)\\
&\quad=\frac{1}{k!}\sum_{\tau\in\Sigma(k-1,\Gamma)}\frac{1}{|\Gamma_\tau|}\sum_{v\in V_\tau}\Bigg(\sum_{u\in\{v,u\},u\neq v}m_\tau(\{v,u\})\phi_\tau(u),\psi_\tau(v)\Bigg)\\
&\quad=\frac{1}{k!}\sum_{\tau\in\Sigma(k-1,\Gamma)}\frac{1}{|\Gamma_\tau|}\sum_{v\in V_\tau}m_\tau(v)\Bigg(\sum_{u\in\{v,u\},\,u\neq v}\frac{m_\tau(\{v,u\})}{m_\tau(v)}\phi_\tau(u),\psi_\tau(v)\Bigg)\\
&\quad=\frac{1}{k!}\sum_{\tau\in\Sigma(k-1,\Gamma)}\frac{1}{|\Gamma_\tau|}\langle(A_\tau\otimes\mathrm{id}_{\mathbb{E}})\phi_\tau,\psi_\tau\rangle_\tau. \qquad (3)
\end{aligned}$$

Combining (1), (2), and (3) yields the needed equality. □

After these lemmas, we can prove a local criterion for cohomology vanishing that appeared as Theorem 1.1 in the Introduction:

Theorem 4.4. *Let X be a locally finite, pure n-dimensional simplicial complex with the weight function m defined above and Γ be a locally compact, unimodular group acting cocompactly and properly on X. For every reflexive Banach space $\mathbb{E}$ and every $1\le k\le n-1$, if*

$$\max_{\tau\in\Sigma(k-1,\Gamma)}\|(A_\tau(I-M_\tau)\otimes\mathrm{id}_{\mathbb{E}})\|_{B(\ell^2(V_\tau,m_\tau;\mathbb{E}))}<\frac{1}{k+1},$$

then for every continuous isometric representation π of Γ on $\mathbb{E}$ it holds that $H^k(X,\pi)=0$.

Proof. Let $\mathbb{E}$ be a reflexive Banach space and π be a continuous isometric representation of Γ on $\mathbb{E}$. Define

$$C'=\max_{\tau\in\Sigma(k-1,\Gamma)}\|(A_\tau(I-M_\tau)\otimes\mathrm{id}_{\mathbb{E}})\|_{B(\ell^2(V,m;\mathbb{E}))}.$$

Then by Lemma 4.3, for every $\phi \in C^k(X,\pi)$, $\psi \in C^k(X,\bar{\pi})$,

$$
\begin{aligned}
&\langle d_k\phi, \bar{d}_k\psi\rangle \\
&\quad = \langle \phi,\psi\rangle - \frac{1}{k!}\sum_{\tau\in\Sigma(k-1,\Gamma)} \frac{1}{|\Gamma_\tau|}\langle (A_\tau \otimes \mathrm{id}_{\mathbb{E}})\phi_\tau, \psi_\tau\rangle_\tau \\
&\quad = \langle \phi,\psi\rangle - \frac{1}{k!}\sum_{\tau\in\Sigma(k-1,\Gamma)} \frac{1}{|\Gamma_\tau|}\langle (A_\tau(I-M_\tau) \otimes \mathrm{id}_{\mathbb{E}})\phi_\tau, \psi_\tau\rangle_\tau - \frac{1}{k!}\sum_{\tau\in\Sigma(k-1,\Gamma)} \frac{1}{|\Gamma_\tau|}\langle (A_\tau M_\tau \otimes \mathrm{id}_{\mathbb{E}})\phi_\tau, \psi_\tau\rangle_\tau \\
&\quad = \langle \phi,\psi\rangle - \frac{1}{k!}\sum_{\tau\in\Sigma(k-1,\Gamma)} \frac{1}{|\Gamma_\tau|}\langle (A_\tau(I-M_\tau) \otimes \mathrm{id}_{\mathbb{E}})\phi_\tau, \psi_\tau\rangle_\tau - \frac{1}{k!}\sum_{\tau\in\Sigma(k-1,\Gamma)} \frac{1}{|\Gamma_\tau|}\langle (M_\tau \otimes \mathrm{id}_{\mathbb{E}})\phi_\tau, \psi_\tau\rangle_\tau \\
&\quad = \langle \phi,\psi\rangle - \frac{1}{k!}\sum_{\tau\in\Sigma(k-1,\Gamma)} \frac{1}{|\Gamma_\tau|}\langle (A_\tau(I-M_\tau) \otimes \mathrm{id}_{\mathbb{E}})\phi_\tau, \psi_\tau\rangle_\tau - \langle \bar{d}^*_{k-1}\phi, d^*_{k-1}\psi\rangle,
\end{aligned}
$$

where the second-to-last equality follows from the fact $A_\tau M_\tau = M_\tau$ and the last equality by Lemma 4.2. Thus

$$\langle d_k\phi, \bar{d}_k\psi\rangle + \langle \bar{d}^*_{k-1}\phi, d^*_{k-1}\psi\rangle = \langle \phi,\psi\rangle - \frac{1}{k!}\sum_{\tau\in\Sigma(k-1,\Gamma)} \frac{1}{|\Gamma_\tau|}\langle (A_\tau(I-M_\tau) \otimes \mathrm{id}_{\mathbb{E}})\phi_\tau, \psi_\tau\rangle_\tau .$$

Applying absolute value on this equation and using the triangle inequality,

$$
\begin{aligned}
&|\langle d_k\phi, \bar{d}_k\psi\rangle| + |\langle \bar{d}^*_{k-1}\phi, d^*_{k-1}\psi\rangle| \\
&\qquad \geq |\langle \phi,\psi\rangle| - \frac{1}{k!}\sum_{\tau\in\Sigma(k-1,\Gamma)} \frac{1}{|\Gamma_\tau|}|\langle (A_\tau(I-M_\tau) \otimes \mathrm{id}_{\mathbb{E}})\phi_\tau, \psi_\tau\rangle_\tau| \\
&\qquad \geq |\langle \phi,\psi\rangle| - \frac{1}{k!}\sum_{\tau\in\Sigma(k-1,\Gamma)} \frac{1}{|\Gamma_\tau|}\|(A_\tau(I-M_\tau) \otimes \mathrm{id}_{\mathbb{E}})\|_{B(\ell^2(V_\tau,m_\tau;\mathbb{E}))}\|\phi_\tau\|_\tau\|\psi_\tau\|_\tau \\
&\qquad \geq |\langle \phi,\psi\rangle| - \frac{1}{k!}\sum_{\tau\in\Sigma(k-1,\Gamma)} \frac{1}{|\Gamma_\tau|}C'\frac{\|\phi_\tau\|_\tau^2 + \|\psi_\tau\|_\tau^2}{2} \\
&\qquad = |\langle \phi,\psi\rangle| - (k+1)C'\frac{\|\phi\|^2+\|\psi\|^2}{2} \quad \text{(by Lemma 4.1)}.
\end{aligned}
$$

If we write $C = (k+1)C'$, then by our assumption $C < 1$ and we prove that

$$|\langle d_k\phi, \bar{d}_k\psi\rangle| + |\langle \bar{d}^*_{k-1}\phi, d^*_{k-1}\psi\rangle| \geq |\langle \phi,\psi\rangle| - C\left(\frac{\|\phi\|^2+\|\psi\|^2}{2}\right),$$

and by Lemma 3.8, $H^k(X,\pi) = H^k(X,\bar{\pi}) = 0$. □

Next, we will apply this theorem in the context of uniformly curved spaces:

Proposition 4.5. *Let X, Γ be as above and $\alpha : (0,1] \to (0,1]$ be a strictly monotone increasing function. Fix $1 \leq k \leq n-1$. If there is $\lambda < \alpha^{-1}(1/(2(k+1)))$ such that, for every $\tau \in \Sigma(k-1,\Gamma)$, the 1-skeleton of X_τ is a two-sided λ-spectral expander, then $H^k(X,\pi) = 0$ for every $\mathbb{E} \in \mathcal{E}_\alpha^{\text{u-curved}}$ and every continuous isometric representation π of Γ on $\mathbb{E}$.*

Proof. First, recall that, by Theorem 2.2, every $\mathbb{E} \in \mathcal{E}_{\alpha}^{\text{u-curved}}$ is reflexive. Second, by Corollary 2.12, for every $\tau \in \Sigma(k-1, \Gamma)$,

$$\|(A_\tau(I - M_\tau)) \otimes \mathrm{id}_{\mathbb{E}}\|_{B(\ell^2(V,m;\mathbb{E}))} \leq 2\alpha(\lambda) < 2\alpha\left(\alpha^{-1}\left(\frac{1}{2(k+1)}\right)\right) = \frac{1}{k+1}.$$

Therefore, the conditions of Theorem 4.4 are fulfilled and $H^k(X, \pi) = 0$ for every continuous isometric representation π of Γ on $\mathbb{E}$. □

As a result of this proposition we deduce the following vanishing result for strictly Hilbertian spaces that appeared in Corollary 1.4(1):

Corollary 4.6. *Let X, Γ be as above, $0 < \theta_0 \leq 1$ a constant. Define $\mathcal{E}_{\theta_0}$ to be the smallest class of Banach spaces that contains all strictly θ-Hilbertian Banach spaces for all $\theta_0 \leq \theta \leq 1$ and is closed under passing to quotients, subspaces, ℓ^2-sums and ultraproducts of Banach spaces. Fix $1 \leq k \leq n-1$. If there is $0 < \lambda < (1/(2(k+1)))^{1/\theta_0}$ such that for every $\tau \in \Sigma(k-1, \Gamma)$ the 1-skeleton of X_τ is a two-sided λ-spectral expander, then $H^k(X, \pi) = 0$ for every $\mathbb{E} \in \mathcal{E}_{\theta_0}$ and every continuous isometric representation π of Γ on $\mathbb{E}$.*

Proof. Corollary 2.8 states that $\mathcal{E}_{\theta_0} \subseteq \mathcal{E}_{\alpha(t)=t^{\theta_0}}^{\text{u-curved}}$. Thus the assertion follows directly from Proposition 4.5. □

Specializing this corollary to the case of vanishing of the L^p cohomology of a group acting on a 2-dimensional simplicial complex yields:

Corollary 4.7. *Let X be a locally finite, pure 2-dimensional simplicial complex such that all the links of X of dimension ≥ 1 are connected and Γ be a locally compact, unimodular group acting cocompactly and properly on X. Also let $p > 2$, $0 < \lambda < 1/2^p$ be constants. Assume that for every vertex $\{v\} \in X(0)$ the 1-skeleton of $X_{\{v\}}$ is a two-sided λ-spectral expander. Then for every $2 \leq p' \leq p$, every space $\mathbb{E}$ that is a commutative or noncommutative $L^{p'}$-space and every continuous isometric representation π of Γ on $\mathbb{E}$ it holds that $H^1(X, \pi) = 0$.*

Proof. As noted above, for $2 \leq p < \infty$, every (commutative or noncommutative) L^p-space is θ-Hilbertian with $\theta = 2/p$. Thus applying Corollary 4.6 with $k = 1$, $n = 2$ and $\theta = 2/p$ gives the stated result. □

The conditions for Proposition 4.5 and Corollary 4.6 can be deduced for all $1 \leq k \leq n-1$, based only on the 1-dimensional links of X. This is done via the following theorem [Oppenheim 2018, Theorem 1.4]:

Theorem 4.8. *Let Y be a finite, pure l-dimensional complex, where $l \geq 2$, such that (1-skeletons of) all the links of Y of dimension ≥ 1 are connected (including the 1-skeleton of Y). Define m_Y to be the weight function on Y, V_Y the vertices of the 1-skeleton of Y and A_Y, M_Y the operators associated with the random walk on this 1-skeleton. Let $-1 \leq \kappa_1 \leq 0 \leq \kappa_2 \leq 1/l$ be constants such that for every $\tau \in Y(l-2)$ the spectrum of A_τ is contained in $[\kappa_1, \kappa_2] \cup \{1\}$. Then the spectrum of the random walk on the 1-skeleton of Y is contained in $[\kappa_1/(1-(l-1)\kappa_1), \kappa_2/(1-(l-1)\kappa_2)] \cup \{1\}$. Equivalently, if there are $-1 \leq \lambda_1 \leq 0 \leq \lambda_2 \leq 1$ such that for every $\tau \in Y(l-2)$ the spectrum of A_τ is contained in $[\lambda_1/(1+(l-1)\lambda_1), \lambda_2/(1+(l-1)\lambda_2)] \cup \{1\}$, then the spectrum of the random walk on the 1-skeleton of Y is contained in $[\lambda_1, \lambda_2] \cup \{1\}$.*

Remark 4.9. In [Oppenheim 2018] this theorem is written in the language of spectral gaps of Laplacians, but as noted above the translation to the language of random walks is straightforward.

Observation 4.10. Theorem 4.8 is not symmetric, as it may appear at first glance: while the upper bound on the spectrum of A_Y deteriorates as l increases, the lower bound actually improves as l increases. In particular, it is always the case that the smallest eigenvalue of the 1-skeleton of every graph is ≥ -1. Thus, in the above theorem we can always take $\kappa_1 = -1$ and get that the spectrum of the random walk on the 1-skeleton of Y is contained in $[-1/l, 1]$.

Using Theorem 4.8, we deduce a criterion for the vanishing of all the cohomologies:

Theorem 4.11. *Let X be a locally finite, pure n-dimensional simplicial complex such that all the links of X of dimension ≥ 1 are connected and Γ be a locally compact, unimodular group acting cocompactly and properly on X. Also let $\alpha : (0, 1] \to (0, 1]$ be a strictly monotone increasing function, $1 \leq k \leq n-1$ and $0 < \lambda < \alpha^{-1}(1/(2(k+1)))$ be constants.*

(1) *If for every $\tau \in \Sigma(n-2, \Gamma)$ the 1-skeleton of X_τ is a two-sided $\lambda/(1+(n-k-1)\lambda)$-spectral expander, then $H^k(X, \pi) = 0$ for every $\mathbb{E} \in \mathcal{E}_\alpha^{\text{u-curved}}$ and every continuous isometric representation π of Γ on $\mathbb{E}$.*

(2) *If $k \leq n - 1/\lambda$ and for every $\tau \in \Sigma(n-2, \Gamma)$ the 1-skeleton of X_τ is a* ***one-sided*** *$\lambda/(1+(n-k-1)\lambda)$-spectral expander, then $H^k(X, \pi) = 0$ for every $\mathbb{E} \in \mathcal{E}_\alpha^{\text{u-curved}}$ and every continuous isometric representation π of Γ on $\mathbb{E}$.*

Proof. Let $1 \leq k \leq n-1$ and let $\eta \in \Sigma(k-1, \Gamma)$. If we let $Y = X_\eta$, then Y is a pure $(n-k)$-dimensional finite simplicial complex and, with the notation of Theorem 4.8,

$$\|A_\eta(I - M_\eta)\|_{B(\ell^2(V_\eta, m_\eta))} = \|A_Y(I - M_Y)\|_{B(\ell^2(V_Y, m_Y))}.$$

Note that the 1-dimensional links of Y are also 1-dimensional links of X. We also note that for every $\tau \in \Sigma(n-2, \Gamma)$, X_τ is a graph and A_τ is the simple random walk on this graph.

(1) Assume that there is $0 \leq \lambda < \alpha^{-1}(1/(2(k+1)))$ such that for every $\tau \in \Sigma(n-1, \Gamma)$ the 1-skeleton of X_τ is a two-sided $\lambda/(1+(n-k-1)\lambda)$-spectral expander. Applying Theorem 4.8 yields that for every $\eta \in \Sigma(k-1, \Gamma)$ the 1-skeleton of X_η is a two-sided λ-spectral expander and thus the conditions of Proposition 4.5 are fulfilled and therefore $H^k(X, \pi) = 0$ for every $\mathbb{E} \in \mathcal{E}_\alpha^{\text{u-curved}}$ and every continuous isometric representation π of Γ on $\mathbb{E}$.

(2) The proof is similar to case (1), but we use Observation 4.10 in order to bound the spectrum from below. We leave the details to the reader. □

Applying the theorem above for strictly θ_0-Hilbertian (with $\alpha(t) = t^{\theta_0}$) immediately yields the following corollary, which appeared in the Introduction as part of Corollary 1.4:

Corollary 4.12. *Let X be a locally finite, pure n-dimensional simplicial complex such that all the links of X of dimension ≥ 1 are connected and Γ be a locally compact, unimodular group acting cocompactly and properly on X. Also let $0 < \theta_0 \leq 1$, $1 \leq k \leq n-1$, $0 < \lambda < (1/(2(k+1)))^{1/\theta_0}$ be constants. Define*

$\mathcal{E}_{\theta_0}$ to be the smallest class of Banach spaces that contains all strictly θ-Hilbertian Banach spaces for all $\theta_0 \le \theta \le 1$ and is closed under subspaces, quotients, ℓ^2-sums and ultraproducts of Banach spaces.

(1) *If for every $\tau \in \Sigma(n-2, \Gamma)$ the 1-skeleton of X_τ is a two-sided $\lambda/(1+(n-k-1)\lambda)$-spectral expander, then $H^k(X, \pi) = 0$ for every $\mathbb{E} \in \mathcal{E}_{\theta_0}$ and every continuous isometric representation π of Γ on $\mathbb{E}$.*

(2) *If $k \le n - 1/\lambda$ and for every $\tau \in \Sigma(n-2, \Gamma)$ the 1-skeleton of X_τ is a **one-sided** $\lambda/(1+(n-k-1)\lambda)$-spectral expander, then $H^k(X, \pi) = 0$ for every $\mathbb{E} \in \mathcal{E}_{\theta_0}$ and every continuous isometric representation π of Γ on $\mathbb{E}$.*

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IZHAR OPPENHEIM: izharo@bgu.ac.il
Department of Mathematics, Ben-Gurion University of the Negev, Be'er Sheva, Israel

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