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EIGENVALUE BOUNDS FOR SCHRÖDINGER OPERATORS WITH RANDOM COMPLEX POTENTIALS

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We consider the Schrödinger operator perturbed by a random complex-valued potential. For this operator, we consider its eigenvalues situated in the unit disk. We obtain an estimate on the rate of accumulation of these eigenvalues to the positive half-line.

1. Introduction and main results

We study the behavior of eigenvalues of the operator $H = -\Delta + V$ acting on a Hilbert space $L^2(\mathbb{R}^d)$, where $d \geq 3$. The potential V is assumed to be a complex-valued function of the form

$$V(x) = \sum_{n\in\mathbb{Z}^d} \omega_n v_n \chi(x-n), \qquad v_n \in \mathbb{C}, \quad x \in \mathbb{R}^d,$$

where the ω_n are independent random variables taking values in the interval $[-1, 1]$ and χ is the characteristic function of the unit cube $[0, 1)^d$.

The probability space in our theorems is the set Σ of all infinite sequences $\omega = \{\omega_n\}_{n\in\mathbb{Z}^d}$. The probability measure is defined on Σ as the infinite product of corresponding measures on intervals $[-1, 1]$. Since ω_n can be viewed as a function on Σ whose value is equal to the n-th coordinate of ω, its expectation $\mathbb{E}[\omega_n]$ can be viewed as an integral over Σ. We impose the condition

$$\mathbb{E}[\omega_n] = 0$$

on ω_n guaranteeing oscillations of V. The coefficients v_n do not have to be real.

To formulate the main result, we set

$$\widetilde{V}(x) = \sum_{n\in\mathbb{Z}^d} |v_n| \chi(x-n).$$

Note that $\widetilde{V}$ is a nonnegative function such that $|V| \leq \widetilde{V}$.

Theorem 1.1. *Let $d \geq 3$, let $R_0 > 0$ and let $1 < \nu < q < 2$. Then the eigenvalues λ_j of the operator $-\Delta + V$ satisfy*

$$\mathbb{E}\Bigg[\sum_{|\lambda_j|<R_0^2} \operatorname{Im}\sqrt{\lambda_j}\,|\lambda_j|^{(q-1)/2}\Bigg] \leq C|R_0|^{q-\nu}\bigg(\int_{\mathbb{R}^d} |\widetilde{V}(x)|^p\,dx\bigg)^2, \tag{1.1}$$

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with

$$p = \frac{d}{2} + \frac{d-q}{2(d-2)}. \tag{1.2}$$

It is assumed that $\operatorname{Im}\sqrt{\lambda_j} \geq 0$. *The constant* C *in* (1.1) *depends only on* d, ν *and* q.

Theorem 1.1 is a particular case of the following statement, which has rather complicated looking conditions imposed on the parameters.

Theorem 1.2. *Let* $d \geq 3$, *and let* $R_0 > 0$. *Assume that the parameters* $\varkappa$ *and* p *obey the conditions*

$$\frac{\varkappa}{2p} + \frac{d-1}{2} < \varkappa < \frac{d+1}{2},$$

and

$$\max\{2, \varkappa\} \leq p < \min\left\{2\varkappa, \frac{d\varkappa}{2\varkappa - 1}\right\}.$$

Assume also that $\widetilde{V} \in L^p(\mathbb{R}^d)$. *Then the eigenvalues* λ_j *of the operator* $-\Delta + V$ *satisfy*

$$\mathbb{E}\left[\sum_{|\lambda_j| \leq R_0^2} \operatorname{Im}\sqrt{\lambda_j}\,|\lambda_j|^{(q-1)/2}\right] \leq C|R_0|^{q-2p-p(d-1)/\varkappa}\left(\int_{\mathbb{R}^d} |\widetilde{V}(x)|^p\,dx\right)^2, \quad q > 2p - \frac{p(d-1)}{\varkappa}. \tag{1.3}$$

It is assumed that $\operatorname{Im}\sqrt{\lambda_j} \geq 0$. *If* $\varkappa = \frac{1}{2}(d+1)$, *then* (1.3) *holds with* $p = \varkappa$. *The constant* C *in* (1.3) *depends only on* d, p, $\varkappa$ *and* q.

It is known that, if $v_n \in \mathbb{R}$, the eigenvalues λ_j obey the Lieb–Thirring estimate (see [Helffer and Robert 1990; Laptev and Weidl 2000; Lieb and Thirring 1976])

$$\sum_j |\lambda_j|^\gamma \leq C\int_{\mathbb{R}^d} |V(x)|^{d/2+\gamma}\,dx, \quad V = \overline{V}, \ d \geq 3, \ \gamma \geq 0. \tag{1.4}$$

Theorem 1.1 allows one to consider real potentials V for which the right-hand side of (1.4) is infinite, while the left-hand side is finite almost surely. Indeed, let $1 < 2\gamma = q < d/(d-1)$. Then the parameter p in (1.2) satisfies the inequality

$$p > \tfrac{1}{2}d + \gamma. \tag{1.5}$$

Similar results for real random potentials $V = \overline{V}$ were obtained by the author and Vainberg in [Safronov and Vainberg 2008]. However, there is a big difference between Theorem 1.1 and the results of that earlier work, since the only point of accumulation of eigenvalues of the operator H considered there is the point $\lambda = 0$. When one studies complex-valued potentials, the fact that the eigenvalues λ_j might accumulate to points other than $\lambda = 0$ should not be excluded. Examples of decaying complex potentials V such that eigenvalues of $H = -\Delta + V$ accumulate to points of the positive real line $\mathbb{R}_+$ are constructed in [Bögli 2017]. Because of the difference between the cases of real and complex potentials, it would be more appropriate to ask what new information Theorem 1.1 provides compared to [Frank 2018; Frank and Sabin 2017], rather than realize that this theorem does not follow from the Lieb–Thirring estimate even in the selfadjoint case.

The related result of [Frank and Sabin 2017] says that there is a constant C that depends on d, p and γ such that

$$\sum_j \operatorname{dist}(\lambda_j, \mathbb{R}_+)|\lambda_j|^{\gamma-1} \le C\left(\int_{\mathbb{R}^d} |V|^p\,dx\right)^{2\gamma/(2p-d)}, \tag{1.6}$$

under conditions on γ and p implying that $p < \gamma + \frac{1}{2}d$. One can now refer to (1.5) to conclude that our results do give new information about the distribution of eigenvalues in the complex plane.

The same conclusion could be made by an analysis of the results of [Frank 2018], where the eigenvalues in the disk

$$\mathbb{D}_V = \left\{z \in \mathbb{C} : |z|^{p-d/2} \le C_{p,d}\int |V|^p\,dx\right\}$$

are considered separately from the rest of the eigenvalues; here $p > \frac{1}{2}d$. R. Frank [2018] proves that under some restrictions on p,

$$\left(\sum_{\lambda_j \in \mathbb{D}_V} \operatorname{dist}(\lambda_j, \mathbb{R}_+)^\gamma\right)^\sigma \le C\int_{\mathbb{R}^d} |V|^p\,dx, \tag{1.7}$$

for γ equal to either p or $2p - d + \varepsilon$. The constants $C > 0$ and $\sigma > 0$ depend only on d and p in the first case but also on $\varepsilon > 0$ in the second. In its turn, $\varepsilon > 0$ belongs to the interval whose size depends on p. The observation we make is that $p < \gamma + \frac{1}{2}d$ in (1.7). On the other hand, in deterministic results, p simply can not be larger than $\gamma + \frac{1}{2}d$.

Theorem 1.1 gives information about the eigenvalues of H situated in a finite disk about the origin. The behavior of the eigenvalues outside of this disk is described below.

Theorem 1.3. *Let $d \ge 3$, let $R > 0$ and let $1 < \nu < q < 2$. Then the eigenvalues λ_j of the operator $-\Delta + V$ satisfy*

$$\mathbb{E}\left[\sum_{|\lambda_j| \ge R^2} \frac{\operatorname{Im}\sqrt{\lambda_j}(|\lambda_j| - R^2)}{|\lambda_j|R}\right] \le C|R|^{-\nu}\left(\int_{\mathbb{R}^d} |\widetilde{V}(x)|^p\,dx\right)^2,$$

with

$$p = \frac{d}{2} + \frac{d-q}{2(d-2)}.$$

It is assumed that $\operatorname{Im}\sqrt{\lambda_j} \ge 0$. *The constant C in (1.1) depends only on d, ν and q.*

According to Theorem 1.3, the condition $\widetilde{V} \in L^p$ implies that, for any $R > 0$,

$$\sum_{|\lambda_j| \ge R^2} |\operatorname{Im}\sqrt{\lambda_j}| < \infty \tag{1.8}$$

almost surely. Eigenvalues of H outside a finite disk about the point $z = 0$ were also studied in [Frank 2018]. However, in the theorems of that work the radius R of the disk depends on V. Moreover, when $d \ge 3$, these theorems guarantee convergence of $\sum_{|\lambda_j| \ge R^2} |\operatorname{Im}\lambda_j|^\alpha |\lambda_j|^{-\beta}$ for some $\alpha > 1$ and $\beta > 0$ rather than convergence of the series (1.8).

Theorem 1.3 immediately implies the following assertion.

Corollary 1.4. *Let $d \geq 3$, let $R > 0$ and let $1 < \nu < q < 2$. Then the eigenvalues λ_j of the operator $-\Delta + V$ satisfy*

$$\mathbb{E}\Big[\sum_{R^2 \leq |\lambda_j| \leq 2R^2} \operatorname{Im}\sqrt{\lambda_j}\,|\lambda_j|^{(\nu-1)/2}\Big] \leq C\Big(\int_{\mathbb{R}^d} |\widetilde{V}(x)|^p\,dx\Big)^2,$$

with

$$p = \frac{d}{2} + \frac{d-q}{2(d-2)}.$$

It is assumed that $\operatorname{Im}\sqrt{\lambda_j} \geq 0$. *The constant C in (1.1) depends only on d, ν and q.*

We also mention the article [Frank 2018] because Theorem 1.1 of that paper deals with the question about the shape of the domain containing all eigenvalues of H. In particular, it implies that the imaginary part of an eigenvalue tends to zero as the real part tends to infinity (in a quantitative way) once $V \in L^p$ with $p > \frac{1}{2}(d+1)$. Despite a vague visual resemblance of Corollary 1.4 to such a theorem, it does not give new information about the region containing all eigenvalues of H.

The next statement is an improvement of Theorem 1.1 for $3 \leq d \leq 5$ and $R_0 \leq 1$.

Theorem 1.5. *Let $3 \leq d \leq 5$, and let $0 < R_0 \leq 1$. Assume that τ_1 satisfies*

$$0 \leq \Big(\Big(\frac{d}{2} + \frac{(\eta-1)(d+1)}{7d} + \frac{d-\eta}{2(d-2)}\Big) - 2\Big)\tau_1 \leq \frac{(\nu-1)(d+1)}{7d},$$

with η and ν such that $1 < \nu < \eta < 2$. If $d = 3$, then we assume additionally that $8\nu + 9\eta < 26$. Let p, q and r be the numbers defined by

$$p = \frac{d}{7\tau_1}, \quad \frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{2} \quad \text{and} \quad \frac{1}{r} = \frac{1-\theta}{2p} + \frac{\theta}{2},$$

where θ is the solution of the equation

$$\tau_1(1-\theta) + \frac{\theta}{2}\Big(\frac{d}{2} + \frac{d-\eta}{2(d-2)}\Big) = 1.$$

Then the eigenvalues λ_j of the operator $-\Delta + V$ satisfy

$$\mathbb{E}\Big[\sum_{|\lambda_j| \leq R_0^2} \operatorname{Im}\sqrt{\lambda_j}\,|\lambda_j|^{(\sigma-1)/2}\Big] \leq C_{\tau_1,\sigma}|R_0|^{\sigma-\theta q\nu/2}\Big(\int_{\mathbb{R}^d} |\widetilde{V}(x)|^r\,dx\Big)^{2q/r}, \quad \sigma > \tfrac{1}{2}\theta q\nu.$$

Besides its dependence on d, the constant $C_{\tau_1,\sigma}$ in this inequality depends on a choice of the parameters τ_1 and σ.

Theorem 1.5 gives new information about eigenvalues of H. Even in the case $V = \overline{V}$, this theorem does not follow from the Lieb–Thirring estimates. It turns into Theorem 1.1 for dimensions $3 \leq d \leq 5$ once we set $\tau_1 = 0$. On the other hand, since it allows us to consider ratios σ/r smaller than ratios q/p allowed by Theorem 1.1, Theorem 1.5 is an improvement of Theorem 1.1 for dimensions $3 \leq d \leq 5$ and the values of the parameter $R_0 < 1$.

One of the difficulties we encountered in this paper is that our statements can not be derived by taking expectations in the inequalities obtained by Borichev, Golinskii and Kupin [Borichev et al. 2009]. The reason is that operators of the Birman–Schwinger type we are dealing with might have different properties for different ω. This difficulty was overcome through an application of the Joukowski transform to a half-plane with a removed semidisk and consecutive integration with respect to the radius.

Eigenvalue bounds for Schrödinger operators with complex potentials have been studied for a long time. First of all, one should mention the related work of B. Pavlov, who found sharp conditions on V guaranteeing that H has only finitely many eigenvalues in $\mathbb{C} \setminus \mathbb{R}_+$. In particular, this is true for the one dimensional operator on the half-line $\mathbb{R}_+$ (see [Pavlov 1966]) if

$$|V(x)| \leq Ce^{-c\sqrt{x}}, \quad \forall x \in \mathbb{R}_+,$$

for some constants C and $c > 0$.

In 2001, E. B. Davies posed a question whether the estimate

$$|\lambda| \leq \frac{1}{4}\left(\int_{\mathbb{R}} |V(x)|\, dx\right)^2, \quad d = 1,$$

that he and his collaborators established for any nonreal eigenvalue λ of H (see [Abramov et al. 2001; Davies and Nath 2002]) can be extended to higher dimensions. This question was nicely handled by R. Frank [2011]. It was shown that, if $0 < \gamma \leq \frac{1}{2}$ and $d \geq 2$, then there is a positive constant $C_{\gamma,d}$ such that

$$|\lambda|^\gamma \leq C_{\gamma,d} \int_{\mathbb{R}^d} |V(x)|^{d/2+\gamma}\, dx, \tag{1.9}$$

for any eigenvalue of H in $\mathbb{C} \setminus \mathbb{R}_+$. The technique of [Frank 2011] was further developed and combined with some complex analysis in [Frank and Sabin 2017], where the authors already give the estimate (1.6) on the rate of accumulation of eigenvalues to the positive half-line $\mathbb{R}_+$. Another bound of this type is the inequality (1.7) established in [Frank 2018].

Note also, that if one only considers eigenvalues outside of a cone

$$\Gamma_\varepsilon = \{z \in \mathbb{C} : \operatorname{Re} z \geq 0,\ |\operatorname{Im} z| \leq \varepsilon \operatorname{Re} z\}$$

(here $\varepsilon > 0$), then the Lieb–Thirring bound holds for these eigenvalues (see [Frank et al. 2006]):

$$\sum_{\lambda_j \notin \Gamma_\varepsilon} |\lambda_j|^\gamma \leq C_{\gamma,d,\varepsilon} \int_{\mathbb{R}^d} |V(x)|^{d/2+\gamma}\, dx, \quad \gamma \geq 1.$$

While we do not intend to describe all results related to the theory of operators with complex-valued potentials, we would like to mention the articles [Briet et al. 2021; Cuenin 2017; Cuenin et al. 2014; Demuth et al. 2009; Demuth and Katriel 2008; Hansmann 2011; 2017; Korotyaev 2020; Korotyaev and Laptev 2018; Korotyaev and Safronov 2020; Laptev and Safronov 2009; Pavlov 1967] in addition to those already mentioned, all of which could be viewed as valuable contributions in this area.

2. Preliminaries

Everywhere below, $\mathfrak{S}_p$ denotes the class of compact operators K obeying

$$\|K\|_{\mathfrak{S}_p}^p = \mathrm{Tr}(K^*K)^{p/2} < \infty, \qquad p > 1.$$

Note that if $K \in \mathfrak{S}_p$ for some $p > 1$, then $K \in \mathfrak{S}_q$ for $q > p$ and $\|K\|_q \le \|K\|_p$.

Let z_j be the eigenvalues of a compact operator $K \in \mathfrak{S}_n$ where $n \in \mathbb{N} \setminus \{0\}$. We define the n-th determinant of $I + K$ as

$$\det_n(I+K) = \prod_j (1+z_j)\exp\biggl(\sum_{m=1}^{n-1} \frac{(-1)^m z_j^m}{m}\biggr), \qquad n \ge 2,$$

$$\det(I+K) = \prod_j (1+z_j), \qquad n = 1.$$

There exists a constant $C_n > 0$ depending only on n such that

$$|\det_n(I+X)| \le e^{C_n\|X\|_{\mathfrak{S}_n}^n}, \qquad \forall X \in \mathfrak{S}_n.$$

Moreover, we have the following statement; see Proposition 2.1 of [Korotyaev and Safronov 2020].

Proposition 2.1. *Let $n \ge 2$. Then for any $n-1 \le p \le n$, there exists a constant $C_{p,n} > 0$ depending only on p and n such that*

$$|\det_n(I+X)| \le e^{C_{p,n}\|X\|_{\mathfrak{S}_p}^p}, \qquad \forall X \in \mathfrak{S}_p. \tag{2.1}$$

The way the eigenvalue bounds are obtained in [Korotyaev and Safronov 2020] uses applications of the following abstract result.

Theorem 2.2. *Let H_0 be a selfadjoint operator on a Hilbert space $\mathfrak{H}$. Let W_1 and W_2 be two bounded operators on $\mathfrak{H}$, and let $V = W_2W_1$. Assume that the function*

$$\mathbb{C}_+ \ni z \mapsto W_1(H_0 - z)^{-1}W_2 \in \mathfrak{S}_p, \quad 1 \le p < \infty,$$

is analytic in the upper half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$ and continuous up to the real line $\mathbb{R}$. Assume also that

$$\|W_1(H_0 - z)^{-1}W_2\|_{\mathfrak{S}_p}^p = o\biggl(\frac{1}{|z|}\biggr), \qquad \text{as } |z| \to \infty. \tag{2.2}$$

Then the eigenvalues λ_j of $H_0 + V$ in $\mathbb{C}_+$ satisfy

$$\sum_j \mathrm{Im}\,\lambda_j \le C_p \int_{-\infty}^{\infty} \|W_1(H_0 - \lambda - i0)^{-1}W_2\|_{\mathfrak{S}_p}^p \, d\lambda, \tag{2.3}$$

where C_p depends only on the parameter p.

Proof. The proof of this statement relies on Jensen's inequality for zeros of an analytic function, which is (also) justified in Proposition 3.11 of [Korotyaev and Safronov 2020]. ☐

Proposition 2.3. *Let $a(z)$ be an analytic function on $\mathbb{C}_+$ satisfying the condition*

$$a(z) = 1 + o\left(\frac{1}{|z|}\right) \quad \text{as } |z| \to \infty.$$

Assume that for some $\gamma > 0$,

$$\ln|a(\lambda + i\gamma)| \le f(\lambda), \quad \forall \lambda \in \mathbb{R}.$$

Then zeros of $a(z)$ situated above the line $\operatorname{Im} z = \gamma$ satisfy the inequality

$$\sum_j (\operatorname{Im} \lambda_j - \gamma)_+ \le \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\lambda)\, d\lambda. \tag{2.4}$$

The statement also holds for $\gamma = 0$, if $a(z)$ is continuous up to the real line $\mathbb{R}$.

The bound (2.3) follows from (2.1) and the estimate (2.4) with $\gamma = 0$ once we set

$$a(z) = \det_n(I + W_1(H_0 - z)^{-1} W_2)$$

and

$$f(\lambda) = C_{p,n} \| W_1(H_0 - \lambda - i0)^{-1} W_2 \|^p_{\mathfrak{S}_p}.$$

According to the Birman–Schwinger principle, z is an eigenvalue of $H_0 + V$ if and only if $a(z) = 0$ (multiplicities coincide). This completes the proof of Theorem 2.2. □

One of the tools used in the present paper is an interpolation. Interpolation has been also used to prove Theorem 1.2 of [Korotyaev and Safronov 2020], which can be generalized and formulated as follows.

Theorem 2.4. *Let (Ω, μ) be a space with an σ-finite measure μ such that $L^2(\Omega, \mu)$ is separable. Let H_0 be a selfadjoint operator on the Hilbert space $L^2(\Omega, \mu)$. Assume that the integral kernel of the operator e^{-itH_0} satisfies the estimate*

$$|e^{-itH_0}(x, y)| \le \frac{C}{t^{\varkappa}}, \quad \forall t > 0, \ \ \forall x, y \in \Omega,$$

for some $\varkappa > 0$. Let $V \in L^p(\Omega, \mu) \cap L^\infty(\Omega, \mu)$ for $p > \varkappa$ such that $p \ge 1$. Assume also that (2.2) holds for all W_1 and W_2 that belong to a class of functions dense in $L^{2p}(\Omega, \mu)$. Then eigenvalues of the operator $H = H_0 + V$ satisfy

$$\sum_j |\operatorname{Im} \lambda_j|^r \le C_{p,r} \left(\int_\Omega |V(x)|^p \, d\mu \right)^{r/p-\varkappa},$$

for any $r > \max\{2(p - \varkappa), 1\}$.

The proof of this result is a counterpart of the proof of Theorem 1.2 from [Korotyaev and Safronov 2020], with the only differences being that the value of the parameter $\varkappa$ in Theorem 1.2 of that work is $\frac{3}{2}$ and $\Omega = \mathbb{R}^3$. However, one can consider different $\varkappa$ as well as spaces Ω which are different from $\mathbb{R}^d$. Especially interesting are spaces of fractional dimensions for which $2\varkappa$ is not an integer.

Another object that we will work with is the operator

$$X(k) = |V|^{1/2}(-\Delta - z)^{-1} V (-\Delta - z)^{-1} V |V|^{-1/2}, \quad z = k^2, \ \ k \in \mathbb{C}_+.$$

If V is a bounded compactly supported function, then $X(k)$ is a trace class operator for $d \le 3$, and $X(k) \in \mathfrak{S}_p$ for $p > \frac{1}{4}d$ and $d \ge 4$. In this case, we set

$$D_n(k) = \det_n(I - X(k)), \quad n > \tfrac{1}{4}d, \ n \in \mathbb{N}.$$

Proposition 2.5. *Let V be a compactly supported function on $\mathbb{R}^d$. If a point $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ is an eigenvalue of $H = -\Delta + V$, then $D_n(k) = 0$ for $k = \sqrt{\lambda}$. The algebraic multiplicity of the eigenvalue λ does not exceed the multiplicity of the root of the function $D_n(\cdot)$.*

Proof. According to the Birman–Schwinger principle, a point λ is an eigenvalue of H if and only if -1 is an eigenvalue of $|V|^{1/2}(-\Delta - \lambda)^{-1} V |V|^{-1/2}$. Therefore, 1 is an eigenvalue of $X(k_0)$ with $k_0^2 = \lambda$. On the other hand, if 1 is an eigenvalue of $X(k_0)$, then $D_n(k_0) = 0$.

The statement about the multiplicity follows from the fact that an isolated eigenvalue of H whose multiplicity m is larger than 1 can be turned into m simple eigenvalues by an arbitrarily small perturbation of finite rank (which does not have to be a function). For any $\varepsilon > 0$ there is a finite rank operator K_ε such that $\|K_\varepsilon\| < \varepsilon$ and that all eigenvalues of $-\Delta + K_\varepsilon + V$ near λ are simple. Define now the function

$$d_\varepsilon(k) = \det_n(I - |V|^{1/2}(-\Delta + K_\varepsilon - z)^{-1} V (-\Delta + K_\varepsilon - z)^{-1} V |V|^{-1/2}),$$

analytic in the neighborhood of $k_0 = \sqrt{\lambda}$ for sufficiently small $\varepsilon > 0$. In this neighborhood of the point k_0, we have $d_\varepsilon(k) \to D_n(k)$ uniformly, as $\varepsilon \to 0$. Since the function $d_\varepsilon(k)$ has at least m zeros near k_0, the multiplicity of the zero of the function $D_n(k)$ at $k = k_0$ can not be smaller than m by the argument principle. □

3. Large values of $\operatorname{Re} \zeta$ without projections

The following proposition gives an important estimate for the integral kernel of $(-\Delta - z)^{-\zeta}$.

Proposition 3.1. *Let $d \ge 2$, and let $\frac{1}{2}(d-1) \le \operatorname{Re} \zeta \le \frac{1}{2}(d+1)$. The integral kernel of the operator $(-\Delta - z)^{-\zeta}$ satisfies the estimate*

$$|(-\Delta - z)^{-\zeta}(x, y)| \le \beta e^{\alpha (\operatorname{Im} \zeta)^2} |k|^{(d-1)/2 - \operatorname{Re} \zeta} |x - y|^{\operatorname{Re} \zeta - (d+1)/2}, \tag{3.1}$$

for $z \notin \mathbb{R}_+$. The positive constants β and α in this inequality depend only on d and $\operatorname{Re} \zeta$.

The proof of this proposition, as well as related references, can be found in [Frank and Sabin 2017].

Everywhere below, we use the notation $\chi_l(x) = \chi(x - l)$, where $l \in \mathbb{Z}^d$.

Corollary 3.2. *Let $\frac{1}{2}(d-1) \le \operatorname{Re} \zeta < \frac{1}{2}(d+1)$, where $d \ge 2$. Let $2 \le r < 2d/(2 \operatorname{Re} \zeta - 1)$. Suppose that W is a function of the form*

$$W(x) = \sum_{n \in \mathbb{Z}^d} w_n \chi(x - n), \quad w_n \in \mathbb{C}, \ x \in \mathbb{R}^d.$$

Then

$$\|W (-\Delta - z)^{-\zeta} \chi_l\|_{\mathfrak{S}_2} \le \beta e^{\alpha (\operatorname{Im} \zeta)^2} |k|^{(d-1)/2 - \operatorname{Re} \zeta} \|W\|_r, \tag{3.2}$$

for $z \notin \mathbb{R}_+$. The positive constants β and α in this inequality depend only on d and $\operatorname{Re}\zeta$. *If* $\operatorname{Re}\zeta = \frac{1}{2}(d+1)$ *and $d \geq 2$, then (3.2) holds with $r = 2$.*

Proof. It follows from (3.1) that

$$\|W(-\Delta - z)^{-\zeta}\chi_l\|^2_{\mathfrak{S}_2} \leq Ce^{2\alpha(\operatorname{Im}\zeta)^2}|k|^{(d-1)-2\operatorname{Re}\zeta}\sum_{n\in\mathbb{Z}^d}(|n-l|+1)^{2\operatorname{Re}\zeta-(d+1)}|w_n|^2.$$

A simple application of Hölder's inequality leads to (3.2). □

We need to turn (3.2) into a similar estimate for the $\mathfrak{S}_4$-norm of the operator corresponding to smaller values of $\operatorname{Re}\zeta$. For that purpose, we employ the inequality

$$\|W(-\Delta - z)^{-\zeta}\chi_l\| \leq \beta e^{\alpha(\operatorname{Im}\zeta)^2}\|W\|_\infty, \tag{3.3}$$

for $\operatorname{Re}\zeta = 0$.

By interpolation we obtain the following proposition from (3.2) and (3.3).

Proposition 3.3. *Let* $\frac{1}{2}(d-1) \leq \varkappa < \frac{1}{2}(d+1)$, *where $d \geq 2$. Let $2 \leq r < 2d/(2\varkappa - 1)$. Suppose that W is a function of the form*

$$W(x) = \sum_{n\in\mathbb{Z}^d} w_n\chi(x-n), \quad w_n \in \mathbb{C}, \quad x \in \mathbb{R}^d.$$

Then, for any $\operatorname{Re}\zeta = \tau \in (0, \varkappa]$ *and* $z \notin \mathbb{R}_+$,

$$\|W(-\Delta - z)^{-\zeta}\chi_l\|_{\mathfrak{S}_{2\varkappa/\tau}} \leq \beta e^{\alpha(\operatorname{Im}\zeta)^2}|k|^{((d-1)/(2\varkappa)-1)\tau}\|W\|_{r\varkappa/\tau}. \tag{3.4}$$

The positive constants β and α in this inequality depend only on d and τ. If $\varkappa = \frac{1}{2}(d+1)$ and $d \geq 2$, then (3.4) holds with $r = 2$.

Proof. Indeed, let $\operatorname{Re}\zeta_0 = \tau$, and let

$$A = \Omega|A|$$

be the polar decomposition of the operator

$$A = |W|^{\zeta_0/\tau}(-\Delta - z)^{-\zeta_0}\chi_l.$$

Consider the function

$$f(\zeta) = e^{\alpha\zeta^2}\operatorname{Tr}(|W|^{\zeta/\tau}(-\Delta - z)^{-\zeta}\chi_l|A|^{(2\varkappa-\zeta+i\operatorname{Im}\zeta_0)/\tau}\Omega^*).$$

If $\operatorname{Re}\zeta = 0$, then

$$|f(\zeta)| \leq C_1\|A\|^{2\varkappa/\tau}_{\mathfrak{S}_{2\varkappa/\tau}}.$$

If $\operatorname{Re}\zeta = \varkappa$, then

$$|f(\zeta)| \leq C_2|k|^{(d-1)/2-\varkappa}\|A\|^{\varkappa/\tau}_{\mathfrak{S}_{2\varkappa/\tau}}\|W\|^{\varkappa/\tau}_{r\varkappa/\tau}.$$

Consequently, by the three lines lemma,

$$|f(\zeta_0)| \leq C|k|^{\theta((d-1)/2-\varkappa)}\|W\|^{\theta\varkappa/\tau}_{r\varkappa/\tau}\|A\|^{(2-\theta)\varkappa/\tau}_{\mathfrak{S}_{2\varkappa/\tau}}, \quad \theta = \tau/\varkappa.$$

Put differently,

$$|e^{\alpha\zeta_0^2}|\|A\|_{\mathfrak{S}_{2\varkappa/\tau}}^{2\varkappa/\tau} \le C|k|^{\theta((d-1)/2-\varkappa)}\|W\|_{r\varkappa/\tau}^{\theta\varkappa/\tau}\|A\|_{\mathfrak{S}_{2\varkappa/\tau}}^{(2-\theta)\varkappa/\tau}, \quad \theta = \tau/\varkappa.$$

The latter inequality implies (3.4). □

In particular, once we set $r\varkappa/\tau = 4$, we obtain the following.

Corollary 3.4. *Let $\frac{1}{2}(d-1) \le \varkappa < \frac{1}{2}(d+1)$, where $d \ge 2$. Suppose that W is a function of the form*

$$W(x) = \sum_{n\in\mathbb{Z}^d} w_n \chi(x-n), \quad w_n \in \mathbb{C}, \ \ x \in \mathbb{R}^d.$$

Then

$$\|W(-\Delta - z)^{-\zeta}\chi_l\|_{\mathfrak{S}_4} \le \beta e^{\alpha(\operatorname{Im}\zeta)^2}|k|^{((d-1)/(2\varkappa)-1)\operatorname{Re}\zeta}\|W\|_4, \tag{3.5}$$

for any $\frac{1}{2}\varkappa \le \operatorname{Re}\zeta < \min\{\varkappa, d\varkappa/(4\varkappa-2)\}$ and $z \notin \mathbb{R}_+$. The positive constants β and α in this inequality depend only on d and $\operatorname{Re}\zeta$. If $\varkappa = \frac{1}{2}(d+1)$ and $d \ge 2$, then (3.5) holds with $\operatorname{Re}\zeta = \frac{1}{2}\varkappa$.

Let us now consider the operator

$$\mathfrak{X}(\zeta) = e^{\alpha_0\zeta^2} W(-\Delta - z)^{-\zeta} V(-\Delta - z)^{-\zeta} W,$$

where W is a *fixed* function independent of ω. The proof of the following proposition is based on the fact that $\mathbb{E}[\omega_n] = 0$.

Proposition 3.5. *Let $\frac{1}{2}(d-1) \le \varkappa < \frac{1}{2}(d+1)$, where $d \ge 2$. Let $\frac{1}{2}\varkappa \le \operatorname{Re}\zeta < \min\{\varkappa, d\varkappa/(4\varkappa-2)\}$. Assume that $\widetilde{V} \in L^2(\mathbb{R}^d)$, $W \in L^4(\mathbb{R}^d)$ and $\alpha_0 > 2\alpha$. Then*

$$(\mathbb{E}(\|\mathfrak{X}(\zeta)\|_{\mathfrak{S}_2}^2))^{1/2} \le C_{\operatorname{Re}\zeta} e^{(2\alpha-\alpha_0)(\operatorname{Im}\zeta)^2}|k|^{((d-1)/\varkappa-2)\operatorname{Re}\zeta}\|\widetilde{V}\|_2\|W\|_4^2. \tag{3.6}$$

If $\varkappa = \frac{1}{2}(d+1)$ and $d \ge 2$, then (3.6) holds with $\operatorname{Re}\zeta = \frac{1}{2}\varkappa$.

Proof. Obviously,

$$\mathbb{E}(\|\mathfrak{X}(\zeta)\|_{\mathfrak{S}_2}^2) = \mathbb{E}(\operatorname{Tr}\mathfrak{X}(\zeta)^*\mathfrak{X}(\zeta)) \le e^{2\alpha_0\operatorname{Re}\zeta^2}\sum_{l\in\mathbb{Z}^d}|v_l|^2\|W(-\Delta - z)^{-\zeta}\chi_l\|_{\mathfrak{S}_4}^2\|\chi_l(-\Delta - z)^{-\zeta}W\|_{\mathfrak{S}_4}^2.$$

Together with Corollary 3.4, this implies (3.6). □

Corollary 3.6. *Let $\frac{1}{2}(d-1) \le \varkappa < \frac{1}{2}(d+1)$, where $d \ge 2$. Let $\frac{1}{2}\varkappa \le \operatorname{Re}\zeta < \min\{\varkappa, d\varkappa/(4\varkappa-2)\}$. Assume that $\widetilde{V} \in L^2(\mathbb{R}^d)$, $W = \widetilde{V}^{1/2}$ and $\alpha_0 > 2\alpha$. Then*

$$(\mathbb{E}(\|\mathfrak{X}(\zeta)\|_{\mathfrak{S}_2}^2))^{1/2} \le C_{\operatorname{Re}\zeta} e^{(2\alpha-\alpha_0)(\operatorname{Im}\zeta)^2}|k|^{((d-1)/\varkappa-2)\operatorname{Re}\zeta}\|\widetilde{V}\|_2^2. \tag{3.7}$$

If $\varkappa = \frac{1}{2}(d+1)$ and $d \ge 2$, then (3.7) holds with $\operatorname{Re}\zeta = \frac{1}{2}\varkappa$.

4. An estimate for the square of the Birman–Schwinger operator

According to the observations that we made, if $W = \sqrt{\widetilde{V}}$, then $\mathfrak{X}(\zeta)$ is a function that obeys (3.7) for some rather large values of $\operatorname{Re}\zeta$, and it also obeys

$$\|\mathfrak{X}(\zeta)\| \le C\|\widetilde{V}\|_\infty^2,$$

for $\operatorname{Re}\zeta = 0$. To obtain our first result about eigenvalues, we can interpolate between these two cases. Let

$$\widetilde{X}(k) = W(-\Delta - z)^{-1}V(-\Delta - z)^{-1}W, \quad z = k^2, \ k \in \mathbb{C}_+,$$

where W is a fixed function independent of ω. What follows is the result of the interpolation (which does not work for $d = 2$).

Proposition 4.1. *Let $\frac{1}{2}(d-1) \le \varkappa < \frac{1}{2}(d+1)$, where $d \ge 3$. Let*

$$\max\{2, \varkappa\} \le p < \min\left\{2\varkappa, \frac{d\varkappa}{2\varkappa - 1}\right\}. \tag{4.1}$$

Let $W = \widetilde{V}^{1/2}$. Assume that $\widetilde{V} \in L^p(\mathbb{R}^d)$. Then

$$(\mathbb{E}(\|\widetilde{X}(k)\|_{\mathfrak{S}_p}^p))^{1/p} \le C|k|^{(d-1)/\varkappa - 2}\|\widetilde{V}\|_p^2. \tag{4.2}$$

If $\varkappa = \frac{1}{2}(d+1)$ and $d \ge 3$, then (4.2) holds with $p = \varkappa$.

Proof. Note that $X(k) = \mathfrak{X}(1)$. The logic of interpolation says that (4.2) holds for p defined as

$$p = 2/\theta, \quad \text{for } \theta \text{ such that } 1 = \theta\tau,$$

where $\frac{1}{2}\varkappa \le \tau < \min\{\varkappa, d\varkappa/(4\varkappa - 2)\}$. Of course, this interpolation works only if $\tau > 1$, which is impossible for $d = 2$. Observe that, with this notation, $p = 2\tau$.

Let

$$X(k) = \Omega|X(k)|$$

be the polar decomposition of the operator $X(k)$. Consider the function

$$f(\zeta) = e^{\alpha_0\zeta^2}\mathbb{E}(\operatorname{Tr}(|W|^\zeta(-\Delta - z)^{-\zeta}V_\zeta(-\Delta - z)^{-\zeta}|W|^\zeta|X(k)|^{2\tau-\zeta}\Omega^*)),$$

where

$$V_\zeta(x) := \sum_n \omega_n|v_n|^\zeta e^{i\arg v_n}\chi(x - n).$$

If $\operatorname{Re}\zeta = 0$, then

$$|f(\zeta)| \le C_1\mathbb{E}(\|X(k)\|_{\mathfrak{S}_{2\tau}}^{2\tau}).$$

If $\operatorname{Re}\zeta = \tau$, then

$$|f(\zeta)| \le C_2|k|^{((d-1)/\varkappa - 2)\tau}(\mathbb{E}(\|X(k)\|_{\mathfrak{S}_{2\tau}}^{2\tau}))^{1/2}\|\widetilde{V}\|_{2\tau}^{2\tau}.$$

Consequently, by the three lines lemma,

$$|f(1)| \le C|k|^{(d-1)/\varkappa - 2}\|\widetilde{V}\|_{2\tau}^2(\mathbb{E}(\|X(k)\|_{\mathfrak{S}_{2\tau}}^{2\tau}))^{1 - 1/(2\tau)}.$$

Put differently,

$$\mathbb{E}(\|X(k)\|_{\mathfrak{S}_{2\tau}}^{2\tau}) \le C|k|^{(d-1)/\varkappa-2}\|\widetilde{V}\|_{2\tau}^{2}(\mathbb{E}(\|X(k)\|_{\mathfrak{S}_{2\tau}}^{2\tau}))^{1-1/(2\tau)}.$$

The latter inequality implies (4.2) because $2\tau = p$. □

Now we can formulate and prove the following result.

Theorem 4.2. *Let $d \ge 3$, and let $1 < \nu < q < 2$. Assume that $W = |V|^{1/2}$. Then*

$$\mathbb{E}(\|X(k)\|_{\mathfrak{S}_p}^{p}) \le C|k|^{-\nu}\|\widetilde{V}\|_{p}^{2p}, \tag{4.3}$$

for p defined by

$$p = \frac{d(d-1)-q}{2(d-2)} = \frac{d}{2} + \frac{d-q}{2(d-2)}. \tag{4.4}$$

Proof. Observe that the assumption $\nu < q < 2$ leads to the inequalities

$$\frac{d+1}{2} < p < \frac{d(d-1)-\nu}{2(d-2)}. \tag{4.5}$$

We will show that the conditions of Proposition 4.1 are fulfilled for the parameter $\varkappa$ defined by

$$\varkappa = \frac{(d-1)p}{2p-\nu}.$$

The latter relation simply means that

$$\nu = \left(2 - \frac{(d-1)}{\varkappa}\right)p. \tag{4.6}$$

Consequently, (4.3) follows from (4.2). The second inequality in (4.5) implies

$$\varkappa > \frac{d(d-1)-\nu}{2(d-\nu)} > \frac{d-1}{2}, \tag{4.7}$$

while the first inequality in (4.5) combined with the condition $\nu < 2$ implies

$$\varkappa < \frac{d+1}{2}.$$

One can also see that the first inequality in (4.7) is equivalent to the estimate

$$p = \frac{\varkappa\nu}{2\varkappa-(d-1)} < \frac{d\varkappa}{2\varkappa-1}.$$

Finally, note that when $d \ge 3$, the condition $p < 2\varkappa$ follows from the fact that $\nu + q > 2$. □

5. Proof of Theorem 1.1

We will work with the function

$$d(z) = \det_n(I - X(k)), \quad n = [p]+1,$$

where z is related to k via the Joukowski mapping

$$z = \frac{R}{k} + \frac{k}{R}, \qquad R > 0,$$

which maps the set $\{k \in \mathbb{C} : \operatorname{Im} k > 0,\ |k| > R\}$ onto the upper half-plane $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Rather standard arguments lead to the estimate

$$\sum_j \operatorname{Im} z_j \le C \int_{-\infty}^{\infty} \ln|d(z)|\, dz, \tag{5.1}$$

where the z_j are the zeros of the function $d(z)$ situated in the upper half-plane $\mathbb{C}_+$. In fact, (5.1) could be established in the same way as Jensen's inequality for zeros of an analytic function on a unit disk. In (5.1) we assume that V is compactly supported. The relation (5.1) leads to the estimate

$$\sum_j \left(\frac{|k_j|^2 - R^2}{|k_j|^2 R}\right)_+ \operatorname{Im} k_j \le C\left(\int_{-\infty}^{\infty} \|X(k)\|^p_{\mathfrak{S}_p}\left(\frac{1}{R} - \frac{R}{k^2}\right)_+ dk + \int_0^{\pi} \|X(R\cdot e^{i\theta})\|^p_{\mathfrak{S}_p} \sin\theta\, d\theta\right).$$

Taking the expectation we obtain

$$\begin{aligned}\mathbb{E}&\left[\sum_j \frac{\operatorname{Im} k_j (|k_j|^2 - R^2)_+}{|k_j|^2 R}\right] \\ &\le C\left(\int_{-\infty}^{\infty} \mathbb{E}[\|X(k)\|^p_{\mathfrak{S}_p}]\left(\frac{1}{R} - \frac{R}{k^2}\right)_+ dk + \int_0^{\pi} \mathbb{E}[\|X(R\cdot e^{i\theta})\|^p_{\mathfrak{S}_p}] \sin\theta\, d\theta\right).\end{aligned} \tag{5.2}$$

Due to Theorem 4.2, the latter inequality leads to

$$\mathbb{E}\left[\sum_j \frac{\operatorname{Im} k_j (|k_j|^2 - R^2)_+}{|k_j|^2 R}\right] \le C|R|^{-\nu}\|\widetilde{V}\|_p^{2p}. \tag{5.3}$$

Now, suppose that we consider only the eigenvalues $\lambda_j = k_j^2$ that satisfy the inequality

$$|k_j| \le R_0.$$

Multiplying (5.3) by R^{q-1} and integrating with respect to R from 0 to R_0, we obtain

$$\mathbb{E}\left[\sum_{|k_j| \le R_0} \operatorname{Im} k_j |k_j|^{q-1}\right] \le C|R_0|^{q-\nu}\|\widetilde{V}\|_p^{2p}, \qquad q > \nu. \tag{5.4}$$

This implies Theorem 1.1. □

Theorem 1.2 can be proved in the same way. The only difference is that one needs to use Proposition 4.1 instead of Theorem 4.2.

Note also that (5.3) implies Theorem 1.3.

6. Operators of the Birman–Schwinger type

Let a, b and V be functions on $\mathbb{R}^d$. Define

$$A_\zeta = |a|^\zeta F V_\zeta F^* |b|^\zeta,$$

where F is the unitary Fourier transform operator. For any complex number z, we understand V_z as the sum

$$V_z(x) := \sum_n \omega_n |v_n|^z e^{i \arg v_n} \chi(x-n).$$

Note that the operator A_ζ can be viewed as a sum over the lattice $\mathbb{Z}^d$:

$$A_\zeta = \sum_{n \in \mathbb{Z}^d} A_{\zeta,n}, \tag{6.1}$$

where

$$A_{\zeta,n} = \omega_n |a|^\zeta F |v_n|^\zeta e^{i \arg v_n} \chi(\cdot - n) F^* |b|^\zeta.$$

We will show that while A_ζ might not be bounded at some points ω, it is still a compact operator almost surely if a, b and $\widetilde{V}$ are in L^2. We remind the reader that $\widetilde{V}$ was defined as the function

$$\widetilde{V}(x) = \sum_n |v_n| \chi(x-n).$$

Remark. Operators of the form $aFWF^*b$ do not have to be bounded for all a, b and W from L^2. Indeed, let

$$W(x) = (|x|+1)^{-s}, \quad \text{with } \tfrac{1}{2}d < s < \tfrac{2}{3}d,$$

and let

$$a(\xi) = b(\xi) = \begin{cases} |\xi|^{-3s/4} & \text{if } |\xi| \le 1, \\ 0 & \text{if } |\xi| > 1. \end{cases}$$

If $aFWF^*b$ was bounded, the operator $T = aF\sqrt{W}$ would be bounded as well. The latter is not true, simply because $T\psi \notin L^2$ for $\psi = W$ (the singularity of $T\psi$ at zero is $|\xi|^{3s/4-d}$).

Proposition 6.1. *Let $a \in L^2$, $b \in L^2$ and $\widetilde{V} \in L^2$. Let also $p \ge 2$. Then the sum (6.1) with* $\operatorname{Re}\zeta = 2/p$ *converges almost surely in $\mathfrak{S}_p$. Moreover,*

$$(\mathbb{E}[\|A_\zeta\|_{\mathfrak{S}_p}^p])^{1/p} \le (2\pi)^{-2d/p} \|a\|_2^{2/p} \|b\|_2^{2/p} \|\widetilde{V}\|_2^{2/p}, \quad \operatorname{Re}\zeta = 2/p. \tag{6.2}$$

Proof. We are going to prove (6.2) for one point ζ_0 such that $\operatorname{Re}\zeta_0 = 2/p$. For that purpose, we define the operator $K(\omega) = |A_{\zeta_0}|^{p/2}$. Then, obviously,

$$\beta := \mathbb{E}(\|K\|_{\mathfrak{S}_2}^2) = \mathbb{E}[\|A_{\zeta_0}\|_{\mathfrak{S}_p}^p].$$

Let $\Omega = \Omega(\omega)$ be the partially isometric operator appearing in the polar decomposition

$$A_{\zeta_0} = \Omega(\omega)|A_{\zeta_0}|.$$

We introduce the analytic function

$$f(\zeta) = \mathbb{E}[\operatorname{Tr} A_\zeta |K|^{2-\zeta} |K|^{i \operatorname{Im} \zeta_0} \Omega^*],$$

which will be treated by the three lines lemma. Since $\|A_\zeta\| \le 1$ for $\operatorname{Re} \zeta = 0$, and $\||K|^{i \operatorname{Im} \zeta_0} \Omega^*\| \le 1$, we obtain that

$$|f(\zeta)| \le \beta, \quad \text{for } \operatorname{Re} \zeta = 0. \tag{6.3}$$

On the other hand,

$$|f(\zeta)| \le (2\pi)^{-d} \beta^{1/2} \|\widetilde{V}\|_2 \|a\|_2 \|b\|_2, \quad \text{for } \operatorname{Re} \zeta = 1, \tag{6.4}$$

by an analogue of Hölder's inequality valid for Schatten classes. Indeed, for $\operatorname{Re} \zeta = 1$,

$$|f(\zeta)|^2 \le \mathbb{E}[\|A_\zeta\|_{\mathfrak{S}_2}^2] \cdot \mathbb{E}[\|K\|_{\mathfrak{S}_2}^2],$$

and

$$\mathbb{E}[\|A_\zeta\|_{\mathfrak{S}_2}^2] = \mathbb{E}[\operatorname{Tr} A_\zeta^* A_\zeta] = \sum_{n \in \mathbb{Z}^d} \mathbb{E}[\operatorname{Tr} A_{\zeta,n}^* A_{\zeta,n}] \le (2\pi)^{-2d} \|\widetilde{V}\|_2^2 \|a\|_2^2 \|b\|_2^2.$$

Using the three lines lemma, we obtain from (6.3) and (6.4) that

$$|f(\zeta)| \le (2\pi)^{-d \operatorname{Re} \zeta} \beta^{1-\operatorname{Re} \zeta/2} \|\widetilde{V}\|_2^{\operatorname{Re} \zeta} \|a\|_2^{\operatorname{Re} \zeta} \|b\|_2^{\operatorname{Re} \zeta}.$$

Note now that $f(\zeta_0) = \beta$. Consequently,

$$\beta^{1/p} \le (2\pi)^{-2d/p} \|\widetilde{V}\|_2^{2/p} \|a\|_2^{2/p} \|b\|_2^{2/p}. \qquad \square$$

Corollary 6.2. *Let T be a random operator of the form*

$$T = |a| F V F^* |b|,$$

with

$$V(x) := \sum_n \omega_n v_n \chi(x - n).$$

Let $a \in L^p$, $b \in L^p$, $v_n \in \ell^p$ and $p \ge 2$. Then

$$(\mathbb{E}[\|T\|_{\mathfrak{S}_p}^p])^{1/p} \le (2\pi)^{-2d/p} \|a\|_p \|b\|_p \|\widetilde{V}\|_p.$$

Proof. Observe that the functions $|a|^{p/2}$, $|b|^{p/2}$ and $\widetilde{V}^{p/2}$ belong to L^2. Therefore, according to the proposition, the $\mathfrak{S}_p$-norm of the operator

$$\widetilde{K} = |a|^{p\zeta/2} F V_{p\zeta/2} F^* |b|^{p\zeta/2}$$

obeys the inequality

$$(\mathbb{E}[\|\widetilde{K}\|_{\mathfrak{S}_p}^p])^{1/p} \le (2\pi)^{-2d/p} \||a|^{p/2}\|_2^{2/p} \||b|^{p/2}\|_2^{2/p} \|\widetilde{V}^{p/2}\|_2^{2/p}, \quad \operatorname{Re} \zeta = 2/p. \qquad \square$$

The following result is a very well-known bound obtained by E. Seiler and B. Simon [Seiler and Simon 1975]. Moreover, the reader can easily prove it using standard interpolation.

Proposition 6.3. *Let a and W be two functions from $L^p(\mathbb{R}^d)$ with $p \geq 2$. Let T be the operator*

$$T = aFW,$$

where F is the operator of the Fourier transform. Then

$$\|T\|_{\mathfrak{S}_p} \leq (2\pi)^{-d/p}\|a\|_p\|W\|_p, \qquad p \geq 2.$$

Corollary 6.4. *Let $q \geq p \geq 2$. Let T be a random operator of the form*

$$T = |a|FVF^*|b|,$$

with

$$V(x) := \sum_n \omega_n v_n \chi(x-n).$$

Let $a \in L^p$, $b \in L^q$ and $v_n \in \ell^p$. Then

$$(\mathbb{E}[\|T\|^q_{\mathfrak{S}_p}])^{1/q} \leq (2\pi)^{-d/p-d/q}\|a\|_p\|b\|_q\|\widetilde{V}\|_p.$$

Proof. According to Proposition 6.3,

$$\|T\|_{\mathfrak{S}_p} \leq (2\pi)^{-d/p}\|a\|_p\|b\|_\infty\|\widetilde{V}\|_p, \qquad p \geq 2.$$

On the other hand, according to Corollary 6.2,

$$(\mathbb{E}[\|T\|^p_{\mathfrak{S}_p}])^{1/p} \leq (2\pi)^{-2d/p}\|a\|_p\|b\|_p\|\widetilde{V}\|_p.$$

It remains to interpolate between the two cases. For that purpose, we introduce the function

$$f(\zeta) = \mathbb{E}[(\operatorname{Tr} K^p)^{(1+q-p)(1-\zeta)/p+\zeta(p-1)(q-p)/p^2} \operatorname{Tr} |a|FVF^*|b|^{q\zeta/p}K^{p-1}\Omega^*],$$

where $K = ||a|FVF^*|b||$ and Ω is the partially isometric operator appearing in the polar decomposition

$$|a|FVF^*|b| = \Omega K.$$

For convenience, we write

$$\beta := \mathbb{E}[(\operatorname{Tr} K^p)^{q/p}].$$

If $\operatorname{Re}\zeta = 0$, then by Hölder's inequality,

$$|f(\zeta)| \leq (2\pi)^{-d/p}\beta\|a\|_p\|\widetilde{V}\|_p.$$

If $\operatorname{Re}\zeta = 1$, then

$$|f(\zeta)| \leq \mathbb{E}[(\operatorname{Tr} K^p)^{(p-1)(q-p)/p^2}\||a|FVF^*|b|^{q/p}\|_{\mathfrak{S}_p}(\operatorname{Tr} K^p)^{(p-1)/p}],$$

which leads to

$$|f(\zeta)| \leq \beta^{1-1/p}(2\pi)^{-2d/p}\|a\|_p\|b\|_q^{q/p}\|\widetilde{V}\|_p.$$

Observe also that

$$f(p/q) = \beta.$$

Thus by the three lines lemma,

$$\beta \leq \beta^{1-1/q}(2\pi)^{-d/p-d/q}\|a\|_p\|b\|_q\|\widetilde{V}\|_p. \qquad \square$$

7. Large values of $\operatorname{Re}\zeta$

Let $0 < R \le 1$. Let $\chi_{0,k}$ be the characteristic function of the ball

$$\mathfrak{B} = \left\{\xi \in \mathbb{R}^d : |\xi| \le \frac{2|k|}{R}\right\},$$

and let $\chi_{1,k} = 1 - \chi_{0,k}$ be the characteristic function of its complement

$$\mathbb{R}^d \setminus \mathfrak{B} = \left\{\xi \in \mathbb{R}^d : |\xi| > \frac{2|k|}{R}\right\}.$$

We introduce the operators

$$P_{n,k} = F\chi_{n,k}F^*,$$

which are the spectral projections of $-\Delta$ corresponding to the intervals $[0, 4|k|^2/R^2]$ and $(4|k|^2/R^2, \infty)$.

Besides depending on the properties of $(-\Delta - z)^{-\zeta}$, the arguments of this paper also rely on the properties of the operators $P_{n,k}(-\Delta - z)^{-\zeta}$ for different values of ζ. In this section, we discuss relatively large values of $\operatorname{Re}\zeta$. The following proposition gives an important estimate for the integral kernel of $P_{n,k}(-\Delta - z)^{-\zeta}$.

Proposition 7.1. *Let $R \le 1$. Let $d \ge 2$, and let $\frac{1}{2}(d-1) < \operatorname{Re}\zeta \le \frac{1}{2}(d+1)$. The integral kernel of the operator $P_{j,k}(\Delta - z)^{-\zeta}$ satisfies the estimate*

$$|P_{j,k}(-\Delta - z)^{-\zeta}(x, y)| \le \beta e^{\alpha(\operatorname{Im}\zeta)^2}|k|^{(d-1)/2-\operatorname{Re}\zeta}|x-y|^{\operatorname{Re}\zeta-(d+1)/2}, \tag{7.1}$$

for $z \notin \mathbb{R}_+$ and $j = 0, 1$. The positive constants β and α in this inequality depend only on d and $\operatorname{Re}\zeta$.

Proof. Due to Proposition 3.1, it is sufficient to prove only one of the inequalities (7.1). Let us first estimate the integrals

$$\begin{aligned} I_n &= \int_{2^n|k|<R|\xi|<2^{n+1}|k|} \frac{e^{i\xi(x-y)}\,d\xi}{(|\xi|^2-k^2)^\zeta} = -|x-y|^{-2}\int_{2^n|k|<R|\xi|<2^{n+1}|k|} \frac{\Delta_\xi e^{i\xi(x-y)}\,d\xi}{(|\xi|^2-k^2)^\zeta} \\ &= |x-y|^{-2}\int_{\mathbb{S}_{2^{n+1}|k|/R}\cup\mathbb{S}_{2^n|k|/R}} \frac{\pm i(x-y)\xi e^{i\xi(x-y)}\,dS_\xi}{|\xi|(|\xi|^2-k^2)^\zeta} \\ &\qquad - \zeta|x-y|^{-2}\int_{2^n|k|<R|\xi|<2^{n+1}|k|} \frac{2i\xi(x-y)e^{i\xi(x-y)}\,d\xi}{(|\xi|^2-k^2)^{\zeta+1}}, \end{aligned} \tag{7.2}$$

for $n \ge 1$. We will show that

$$|I_n| \le \beta e^{\alpha(\operatorname{Im}\zeta)^2}(2^n|k|/R)^{(d-1)/2-\operatorname{Re}\zeta}|x-y|^{\operatorname{Re}\zeta-(d+1)/2}, \tag{7.3}$$

for some $\beta > 0$ and $\alpha > 0$. A priori,

$$|I_n| \le C_d e^{2\pi|\operatorname{Im}\zeta|}(2^n|k|/R)^{d-2\operatorname{Re}\zeta}, \tag{7.4}$$

but the representation (7.2) leads to

$$|I_n| \le C_d e^{2\pi|\operatorname{Im}\zeta|}(2^n|k|/R)^{d-2\operatorname{Re}\zeta-1}|x-y|^{-1}. \tag{7.5}$$

The first estimate (7.4) implies (7.3) for $2^n|k||x-y|<R$, because in this case,

$$|I_n|\le C_d e^{2\pi|\operatorname{Im}\zeta|}(2^n|k|/R)^{d-2\operatorname{Re}\zeta}(2^n|k||x-y|/R)^{\operatorname{Re}\zeta-(d+1)/2}.$$

The second inequality (7.5) implies (7.3) for $2^n|k||x-y|\ge R$, because $\frac{1}{2}(d+1)-\operatorname{Re}\zeta\le 1$ and, therefore,

$$(2^n|k|/R)^{d-2\operatorname{Re}\zeta-1}|x-y|^{-1}\le (2^n|k|/R)^{d-2\operatorname{Re}\zeta+\operatorname{Re}\zeta-(d+1)/2}|x-y|^{\operatorname{Re}\zeta-(d+1)/2}.$$

The estimates (7.3) imply (7.1) for $j=1$, because

$$P_{1,k}(-\Delta-z)^{-\zeta}(x,y)=(2\pi)^{-d}\sum_{n=1}^{\infty}I_n. \qquad \square$$

Corollary 7.2. *Let* $\frac{1}{2}(d-1)<\operatorname{Re}\zeta<\frac{1}{2}(d+1)$, *where* $d\ge 2$. *Let* $2\le r<2d/(2\operatorname{Re}\zeta-1)$. *Suppose that* W *is a function of the form*

$$W(x)=\sum_{n\in\mathbb{Z}^d}w_n\chi(x-n),\qquad w_n\in\mathbb{C},\quad x\in\mathbb{R}^d.$$

Then

$$\|WP_{j,k}(-\Delta-z)^{-\zeta}\chi_l\|_{\mathfrak{S}_2}\le\beta e^{\alpha(\operatorname{Im}\zeta)^2}|k|^{(d-1)/2-\operatorname{Re}\zeta}\|W\|_r,\tag{7.6}$$

for $z\notin\mathbb{R}_+$ *and* $j=0,1$. *The positive constants* β *and* α *in this inequality depend only on* d *and* $\operatorname{Re}\zeta$. *If* $\operatorname{Re}\zeta=\frac{1}{2}(d+1)$ *and* $d\ge 2$, *then* (7.6) *holds with* $r=2$.

Proof. It follows from (7.1) that

$$\|WP_{j,k}(-\Delta-z)^{-\zeta}\chi_l\|^2_{\mathfrak{S}_2}\le Ce^{2\alpha(\operatorname{Im}\zeta)^2}|k|^{(d-1)-2\operatorname{Re}\zeta}\sum_{n\in\mathbb{Z}^d}(|n-l|+1)^{2\operatorname{Re}\zeta-(d+1)}|w_n|^2.$$

A simple application of Hölder's inequality leads to (7.6). $\square$

On the other hand, we have the inequality

$$\|WP_{j,k}(-\Delta-z)^{-\zeta}\chi_l\|\le\beta e^{\alpha(\operatorname{Im}\zeta)^2}\|W\|_\infty,\tag{7.7}$$

for $\operatorname{Re}\zeta=0$.

By interpolation, we obtain the following from (7.6) and (7.7).

Proposition 7.3. *Let* $\frac{1}{2}(d-1)<\varkappa<\frac{1}{2}(d+1)$, *where* $d\ge 2$. *Let* $2\le r<2d/(2\varkappa-1)$. *Suppose that* W *is a function of the form*

$$W(x)=\sum_{n\in\mathbb{Z}^d}w_n\chi(x-n),\qquad w_n\in\mathbb{C},\quad x\in\mathbb{R}^d.$$

Then, for any $\operatorname{Re}\zeta=\tau\in(0,\varkappa)$, $z\notin\mathbb{R}_+$ *and* $j=0,1$,

$$\|WP_{j,k}(-\Delta-z)^{-\zeta}\chi_l\|_{\mathfrak{S}_{2\varkappa/\tau}}\le\beta e^{\alpha(\operatorname{Im}\zeta)^2}|k|^{((d-1)/(2\varkappa)-1)\tau}\|W\|_{r\varkappa/\tau}.\tag{7.8}$$

The positive constants β *and* α *in this inequality depend only on* d *and* τ. *If* $\varkappa=\frac{1}{2}(d+1)$ *and* $d\ge 2$, *then* (7.8) *holds with* $r=2$.

Proof. Indeed, let $\operatorname{Re}\zeta_0 = \tau$, and let

$$A = \Omega |A|$$

be the polar decomposition of the operator

$$A = |W|^{\zeta_0/\tau} P_{j,k}(-\Delta - z)^{-\zeta_0} \chi_l.$$

Consider the function

$$f(\zeta) = e^{\alpha\zeta^2} \operatorname{Tr}(|W|^{\zeta/\tau} P_{j,k}(-\Delta - z)^{-\zeta} \chi_l |A|^{(2\varkappa - \zeta + i \operatorname{Im}\zeta_0)/\tau} \Omega^*).$$

If $\operatorname{Re}\zeta = 0$, then

$$|f(\zeta)| \le C_1 \|A\|_{\mathfrak{S}_{2\varkappa/\tau}}^{2\varkappa/\tau}.$$

If $\operatorname{Re}\zeta = \varkappa$, then

$$|f(\zeta)| \le C_2 |k|^{(d-1)/2-\varkappa} \|A\|_{\mathfrak{S}_{2\varkappa/\tau}}^{\varkappa/\tau} \|W\|_{r\varkappa/\tau}^{\varkappa/\tau}.$$

Consequently, by the three lines lemma,

$$|f(\zeta_0)| \le C |k|^{\theta((d-1)/2-\varkappa)} \|W\|_{r\varkappa/\tau}^{\theta\varkappa/\tau} \|A\|_{\mathfrak{S}_{2\varkappa/\tau}}^{(2-\theta)\varkappa/\tau}, \quad \theta = \tau/\varkappa.$$

Put differently,

$$|e^{\alpha\zeta_0^2}| \|A\|_{\mathfrak{S}_{2\varkappa/\tau}}^{2\varkappa/\tau} \le C |k|^{\theta((d-1)/2-\varkappa)} \|W\|_{r\varkappa/\tau}^{\theta\varkappa/\tau} \|A\|_{\mathfrak{S}_{2\varkappa/\tau}}^{(2-\theta)\varkappa/\tau}, \quad \theta = \tau/\varkappa.$$

The latter inequality implies (7.8), and the proof is completed. □

In particular, once we set $r\varkappa/\tau = 4$, we obtain the following.

Corollary 7.4. *Let $\frac{1}{2}(d-1) < \varkappa < \frac{1}{2}(d+1)$, where $d \ge 2$. Suppose that W is a function of the form*

$$W(x) = \sum_{n \in \mathbb{Z}^d} w_n \chi(x - n), \quad w_n \in \mathbb{C}, \quad x \in \mathbb{R}^d.$$

Then

$$\|W P_{j,k}(-\Delta - z)^{-\zeta} \chi_l\|_{\mathfrak{S}_4} \le \beta e^{\alpha(\operatorname{Im}\zeta)^2} |k|^{((d-1)/(2\varkappa)-1)\operatorname{Re}\zeta} \|W\|_4, \tag{7.9}$$

for any $\frac{1}{2}\varkappa \le \operatorname{Re}\zeta < \min\{\varkappa, d\varkappa/(4\varkappa - 2)\}$, $z \notin \mathbb{R}_+$ and $j = 0, 1$. The positive constants β and α in this inequality depend only on d and $\operatorname{Re}\zeta$. If $\varkappa = \frac{1}{2}(d+1)$ and $d \ge 2$, then (7.9) holds with $\operatorname{Re}\zeta = \frac{1}{2}\varkappa$.

We will now discuss the properties of the random operators

$$X_{n,m}(\zeta) = e^{\alpha_0\zeta^2}(W P_{n,k}(-\Delta - z)^{-\zeta} V (-\Delta - z)^{-\zeta} P_{m,k} W).$$

Here W is a *fixed* function which does not depend on ω.

Proposition 7.5. *Let $\frac{1}{2}(d-1) < \varkappa < \frac{1}{2}(d+1)$, where $d \ge 2$. Let $\frac{1}{2}\varkappa \le \operatorname{Re}\zeta < \min\{\varkappa, d\varkappa/(4\varkappa - 2)\}$. Assume that $\widetilde{V} \in L^2(\mathbb{R}^d)$, $W \in L^4(\mathbb{R}^d)$ and $\alpha_0 > 2\alpha$. Then*

$$(\mathbb{E}(\|X_{n,m}(\zeta)\|_{\mathfrak{S}_2}^2))^{1/2} \le C_{\operatorname{Re}\zeta} e^{(2\alpha-\alpha_0)(\operatorname{Im}\zeta)^2} |k|^{((d-1)/\varkappa-2)\operatorname{Re}\zeta} \|\widetilde{V}\|_2 \|W\|_4^2. \tag{7.10}$$

If $\varkappa = \frac{1}{2}(d+1)$ and $d \ge 2$, then (7.10) holds with $\operatorname{Re}\zeta = \frac{1}{2}\varkappa$.

Proof. Obviously,

$$\begin{aligned}\mathbb{E}(\|X_{n,m}(\zeta)\|_{\mathfrak{S}_2}^2) &= \mathbb{E}(\operatorname{Tr} X_{n,m}(\zeta)^* X_{n,m}(\zeta)) \\ &\le e^{2\alpha_0 \operatorname{Re}\zeta^2} \sum_{l\in\mathbb{Z}^d} |v_l|^2 \|W P_{n,k}(-\Delta - z)^{-\zeta}\chi_l\|_{\mathfrak{S}_4}^2 \|\chi_l(-\Delta - z)^{-\zeta} P_{m,k} W\|_{\mathfrak{S}_4}^2.\end{aligned}$$

Together with Corollary 7.4, this implies (7.10). □

We will also study the spectral properties of the operator

$$Y(\zeta) = X_{0,0}(\zeta) + X_{0,1}(\zeta) + X_{1,0}(\zeta).$$

Corollary 7.6. *Let* $\frac{1}{2}(d-1) < \varkappa < \frac{1}{2}(d+1)$, *where* $d \ge 2$. *Let* $\frac{1}{2}\varkappa \le \operatorname{Re}\zeta < \min\{\varkappa, d\varkappa/(4\varkappa-2)\}$. *Assume that* $\widetilde{V} \in L^2(\mathbb{R}^d)$, $W = \widetilde{V}^{1/2}$ *and* $\alpha_0 > 2\alpha$. *Then*

$$(\mathbb{E}(\|Y(\zeta)\|_{\mathfrak{S}_2}^2))^{1/2} \le C_{\operatorname{Re}\zeta} e^{(2\alpha-\alpha_0)(\operatorname{Im}\zeta)^2} |k|^{((d-1)/\varkappa - 2)\operatorname{Re}\zeta} \|\widetilde{V}\|_2^2. \tag{7.11}$$

If $\varkappa = \frac{1}{2}(d+1)$ *and* $d \ge 2$, *then* (7.11) *holds with* $\operatorname{Re}\zeta = \frac{1}{2}\varkappa$.

8. Small values of Re ζ

The notations we use in this section are the same as in the previous one. In particular, the projections $P_{n,k}$ are the same as before. As was mentioned, the arguments of this paper rely on the properties of the operators $P_{n,k}(-\Delta - z)^{-\zeta}$ for different values of ζ. In this section, we discuss the case $0 \le \operatorname{Re}\zeta < 1$.

In the next two propositions, we discuss the properties of the random operators

$$X_{n,m}(\zeta) = e^{\alpha_0\zeta^2}(W P_{n,k}(-\Delta - z)^{-\zeta} V (-\Delta - z)^{-\zeta} P_{m,k} W),$$

for $\operatorname{Re}\zeta = \frac{1}{2}\gamma$ and $0 < \gamma < \frac{3}{2}$. Here W is a *fixed* function which does not depend on ω. The value of the parameter α_0 should be sufficiently large as in Corollary 7.6.

Later, we will also study the spectral properties of the operator

$$Y(\zeta) = X_{0,0}(\zeta) + X_{0,1}(\zeta) + X_{1,0}(\zeta).$$

However, the terms in this representation will be studied separately. A this point, we do not discuss $X_{1,1}(\zeta)$ at all.

Proposition 8.1. *Let* $d \ge 2$. *Let* $z \in \mathbb{C}\setminus\mathbb{R}_+$, *and let* $2 \le 2p < 3/\gamma$. *Assume that* $0 < R \le 1$. *If* $\operatorname{Re}\zeta = \frac{1}{2}\gamma$, $W \in L^{4p}$ *and* $\widetilde{V} \in L^{2p}$, *then* $X_{0,0}(\zeta) \in \mathfrak{S}_p$ *almost surely. Moreover,*

$$\mathbb{E}(\|X_{0,0}(\zeta)\|_{\mathfrak{S}_p}^p)^{1/p} \le C_{p,\gamma} e^{-\alpha_0|\operatorname{Im}\zeta|^2/2} \left(\frac{|k|}{R}\right)^{3d/(2p)-2\gamma} \|\widetilde{V}\|_{2p} \|W\|_{4p}^2. \tag{8.1}$$

Proof. This statement follows from Corollary 6.2 and Proposition 6.3. If $r = \frac{1}{2}q = 2p$, then $1/r + 2/q = 1/p$. Moreover, since

$$X_{0,0}(\zeta) = e^{\alpha_0\zeta^2}(W(-\Delta - z)^{-\zeta/3} P_{0,k}(-\Delta - z)^{-2\zeta/3} V (-\Delta - z)^{-2\zeta/3} P_{0,k}(-\Delta - z)^{-\zeta/3} W),$$

we obtain the estimate

$$\|\widetilde{X}_{0,0}(\zeta)\|_p$$
$$\le |e^{\alpha_0\zeta^2}|\cdot\|W(-\Delta-z)^{-\zeta/3}P_{0,k}\|_q\|\widetilde{P}_{0,k}(-\Delta-z)^{-2\zeta/3}V(-\Delta-z)^{-2\zeta/3}P_{0,k}\|_r\|P_{0,k}(-\Delta-z)^{-\zeta/3}W\|_q.$$

It remains to realize that

$$\begin{aligned}\Bigl(\int_{\mathbb{R}^d}\frac{\chi_{0,k}\,d\xi}{|(|\xi|^2-z)^{2\zeta/3}|^r}\Bigr)^{2/r} &\le \Bigl(\int_{|\xi|<2|k|}\frac{d\xi}{|(|\xi|^2-z)^{2\zeta/3}|^r}\Bigr)^{2/r}+c_{p,\gamma}e^{c|\operatorname{Im}\zeta|}\Bigl(\int_{|\xi|<2|k|/R}\frac{d\xi}{|\xi|^{2\gamma r/3}}\Bigr)^{2/r}\\ &\le C_{p,\gamma}e^{c|\operatorname{Im}\zeta|}\Bigl(\frac{|k|}{R}\Bigr)^{2(d-2r\gamma/3)/r}=C_{p,\gamma}e^{c|\operatorname{Im}\zeta|}\Bigl(\frac{|k|}{R}\Bigr)^{d/p-4\gamma/3}, \qquad \gamma r<3,\end{aligned}$$

while a similar argument shows that

$$\Bigl(\int_{\mathbb{R}^d}\frac{\chi_{0,k}\,d\xi}{|(|\xi|^2-z)^{\zeta/3}|^q}\Bigr)^{2/q}\le\widetilde{C}_{p,\gamma}e^{c|\operatorname{Im}\zeta|}\Bigl(\frac{|k|}{R}\Bigr)^{2(d-q\gamma/3)/q}=\widetilde{C}_{p,\gamma}e^{c|\operatorname{Im}\zeta|}\Bigl(\frac{|k|}{R}\Bigr)^{d/(2p)-2\gamma/3}. \qquad \square$$

Proposition 8.2. *Let $2\le d\le 5$. Let $z\in\mathbb{C}\setminus\mathbb{R}_+$, and let $2\le 2p<3/\gamma$. Assume that $4p\gamma>d$ and $0<R\le 1$. If $\operatorname{Re}\zeta=\frac{1}{2}\gamma$, $W\in L^{4p}$ and $\widetilde{V}\in L^{2p}$, then $X_{0,1}(\zeta)\in\mathfrak{S}_p$ for all ω. Moreover,*

$$\|X_{0,1}(\zeta)\|_{\mathfrak{S}_p}\le C_{p,\gamma}e^{-\alpha_0|\operatorname{Im}\zeta|^2/2}\Bigl(\frac{|k|}{R}\Bigr)^{d/p-2\gamma}\|\widetilde{V}\|_{2p}\|W\|_{4p}^2. \tag{8.2}$$

Proof. Since

$$X_{0,1}(\zeta)=e^{\alpha_0\zeta^2}(W(-\Delta-z)^{-\zeta/3}P_{0,k}(-\Delta-z)^{-2\zeta/3}VP_{1,k}(-\Delta-z)^{-\zeta}W),$$

we obtain the estimate

$$\|X_{0,1}(\zeta)\|_p\le|e^{\alpha_0\zeta^2}|\cdot\|W(-\Delta-z)^{-\zeta/3}P_{0,k}\|_{4p}\|P_{0,k}(-\Delta-z)^{-2\zeta/3}V\|_{2p}\|P_{1,k}(-\Delta-z)^{-\zeta}W\|_{4p}.$$

It remains to realize that

$$\Bigl(\int_{\mathbb{R}^d}\frac{\chi_{0,k}\,d\xi}{|(|\xi|^2-z)^{2\zeta/3}|^{2p}}\Bigr)^{1/(2p)}\le\widetilde{C}_{p,\gamma}e^{c|\operatorname{Im}\zeta|}\Bigl(\frac{|k|}{R}\Bigr)^{d/(2p)-2\gamma/3},$$

while

$$\Bigl(\int_{\mathbb{R}^d}\frac{\chi_{0,k}\,d\xi}{|(|\xi|^2-z)^{\zeta/3}|^{4p}}\Bigr)^{1/(4p)}\le\widetilde{C}_{p,\gamma}e^{c|\operatorname{Im}\zeta|}\Bigl(\frac{|k|}{R}\Bigr)^{d/(4p)-\gamma/3}.$$

Finally,

$$\Bigl(\int_{\mathbb{R}^d}\frac{\chi_{1,k}\,d\xi}{|(|\xi|^2-z)^{\zeta}|^{4p}}\Bigr)^{1/(4p)}\le 2e^{c|\operatorname{Im}\zeta|}\Bigl(\int_{|\xi|>2|k|/R}\frac{d\xi}{\bigl(\frac{3}{4}|\xi|^2\bigr)^{2\gamma p}}\Bigr)^{1/(4p)}\le\widetilde{C}_{p,\gamma}e^{c|\operatorname{Im}\zeta|}\Bigl(\frac{|k|}{R}\Bigr)^{d/(4p)-\gamma}. \qquad \square$$

Let us now talk about the operator $Y(\zeta)$. The study of this operator must be harder compared to the study of $X_{1,1}(\zeta)$ simply because $P_{1,k}(-\Delta-z)^{-\zeta}$ is bounded uniformly in z while this is not true about $P_{0,k}(-\Delta-z)^{-\zeta}$.

Corollary 8.3. *Let $2 \le d \le 5$. Let $|k| \ge R$ where $0 < R \le 1$. Let also $W = \sqrt{\widetilde{V}}$. Assume that $2 \le 2p < 3/\gamma$ and $4p\gamma > d$. If $\operatorname{Re}\zeta = \frac{1}{2}\gamma$ and $\widetilde{V} \in L^{2p}$, then*

$$\mathbb{E}(\|Y(\zeta)\|_{\mathfrak{S}_p}^p)^{1/p} \le C_{p,\gamma} e^{-\alpha_0 |\operatorname{Im}\zeta|^2/2} \left(\frac{|k|}{R}\right)^{3d/(2p)-2\gamma} \|\widetilde{V}\|_{2p}^2.$$

In particular, we can set $p = 1$ and prove the following statement.

Proposition 8.4. *Let $2 \le d \le 5$. Let $|k| \ge R$ where $0 < R \le 1$. Let also $W = \sqrt{\widetilde{V}}$. Assume that*

$$\tfrac{1}{8}d < \tfrac{1}{2}\gamma = \operatorname{Re}\zeta < \tfrac{3}{4}.$$

Then

$$\mathbb{E}(\|Y(\zeta)\|_{\mathfrak{S}_1}) \le C_{\operatorname{Re}\zeta} e^{-\alpha_0 |\operatorname{Im}\zeta|^2/2} \left(\frac{|k|}{R}\right)^{3d/2-4\operatorname{Re}\zeta} \|\widetilde{V}\|_2^2.$$

9. Another interpolation between small and large values of Re ζ

Let us recall two theorems that hold for the operator

$$Y(\zeta) = X_{0,0}(\zeta) + X_{0,1}(\zeta) + X_{1,0}(\zeta),$$

with $W = \widetilde{V}^{1/2}$. By small values of $\operatorname{Re}\zeta$ we mean the values that are considered in Corollary 8.3, which states that, for any $p \ge 1$ and $d/(8p) < \operatorname{Re}\zeta < 3/(4p)$,

$$\mathbb{E}(\|Y(\zeta)\|_{\mathfrak{S}_p}^p)^{1/p} \le C_{\operatorname{Re}\zeta,p} e^{-\alpha_0 |\operatorname{Im}\zeta|^2/2} \left(\frac{|k|}{R}\right)^{3d/(2p)-4\operatorname{Re}\zeta} \|\widetilde{V}\|_{2p}^2. \tag{9.1}$$

In this corollary, we had to assume that $2 \le d \le 5$ and $|k| \ge R$, where $0 < R \le 1$. One should also not forget that our assumptions about $\gamma = 2\operatorname{Re}\zeta$ imply that $\operatorname{Re}\zeta < \frac{3}{4}$.

In the next result, we only replace $4\operatorname{Re}\zeta$ by $d/(2p)$ in the right-hand side of (9.1).

Theorem 9.1. *Let $2 \le d \le 5$. Let $W = \widetilde{V}^{1/2}$. Let*

$$0 < \operatorname{Re}\zeta < \tfrac{3}{4}.$$

Assume that

$$\frac{d}{8\operatorname{Re}\zeta} < p < \frac{3}{4\operatorname{Re}\zeta}, \qquad p \ge 1,$$

and $0 < R \le 1$. Then

$$\mathbb{E}(\|Y(\zeta)\|_{\mathfrak{S}_p}^p)^{1/p} \le C_{\operatorname{Re}\zeta,p} e^{-\alpha_0 |\operatorname{Im}\zeta|^2/2} \left(\frac{|k|}{R}\right)^{d/p} \|\widetilde{V}\|_{2p}^2,$$

for $|k| \ge R$.

For the sake of simplicity, we choose

$$p = \frac{d}{7\operatorname{Re}\zeta}.$$

In this case, because of the assumption $p \ge 1$ that we made, we have to assume that

$$0 < \operatorname{Re}\zeta \le \tfrac{1}{7}d.$$

Note that $\frac{1}{7}d < \frac{3}{4}$. Thus, we can formulate the following assertion.

Corollary 9.2. *Let $2 \le d \le 5$. Let $0 < \operatorname{Re}\zeta \le \frac{1}{7}d$ and let $p = d/(7\operatorname{Re}\zeta)$. Assume that $0 < R \le 1$. Then*

$$\mathbb{E}(\|Y(\zeta)\|_{\mathfrak{S}_p}^p)^{1/p} \le C_{\operatorname{Re}\zeta,p} e^{-\alpha_0|\operatorname{Im}\zeta|^2/2}\left(\frac{|k|}{R}\right)^{d/p}\|\widetilde{V}\|_{2p}^2,$$

for $|k| \ge R$.

By the large values of $\operatorname{Re}\zeta$ we mean the values appearing in Corollary 7.6. We will use only a simpler version of this result.

Theorem 9.3. *Let $d \ge 3$. Let $1 < \nu < \eta < 2$. Let*

$$2\operatorname{Re}\zeta = \frac{d}{2} + \frac{d-\eta}{2(d-2)}. \tag{9.2}$$

Assume that $V \in L^2(\mathbb{R}^d)$ and $\alpha_0 > 2\alpha$. Then

$$(\mathbb{E}(\|Y(\zeta)\|_{\mathfrak{S}_2}^2))^{1/2} \le C_{\operatorname{Re}\zeta} e^{(2\alpha-\alpha_0)(\operatorname{Im}\zeta)^2}|k|^{-\nu/2}\|\widetilde{V}\|_2^2. \tag{9.3}$$

Proof. For $\operatorname{Re}\zeta$ defined in (9.2), the assumption $\nu < \eta < 2$ leads to the inequalities

$$\frac{d+1}{2} < 2\operatorname{Re}\zeta < \frac{d(d-1)-\nu}{2(d-2)}. \tag{9.4}$$

Let us now introduce the parameter $\varkappa$, setting

$$\varkappa = \frac{2(d-1)\operatorname{Re}\zeta}{4\operatorname{Re}\zeta - \nu}.$$

The latter relation simply means that

$$\nu = \left(2 - \frac{(d-1)}{\varkappa}\right)2\operatorname{Re}\zeta. \tag{9.5}$$

Thus (9.3) coincides with (7.11). Let us check that all conditions of Corollary 7.6 are fulfilled. The second inequality in (9.4) implies

$$\varkappa > \frac{d(d-1)-\nu}{2(d-\nu)} > \frac{d-1}{2}, \tag{9.6}$$

while the first inequality in (9.4) combined with the condition $\nu < 2$ implies that

$$\varkappa < \frac{d+1}{2}.$$

One can also see that the first inequality in (9.6) is equivalent to the estimate

$$2\operatorname{Re}\zeta = \frac{\varkappa\nu}{2\varkappa - (d-1)} < \frac{d\varkappa}{2\varkappa - 1}.$$

Finally, note that when $d \ge 3$, the condition $\operatorname{Re}\zeta < \varkappa$ follows from the fact that $\nu + \eta > 2$. Consequently, Corollary 7.6 implies Theorem 9.3. □

We interpolate between Corollary 9.2 and Theorem 9.3.

Theorem 9.4. *Let $3 \le d \le 5$. Assume that τ_1 satisfies*

$$0 \le \left(\left(\frac{d}{2} + \frac{(\eta-1)(d+1)}{7d} + \frac{d-\eta}{2(d-2)}\right) - 2\right)\tau_1 \le \frac{(\nu-1)(d+1)}{7d}, \tag{9.7}$$

with η and ν such that $1 < \nu < \eta < 2$. If $d = 3$, then we assume additionally that $8\nu + 9\eta < 26$. Let p, q and r be the numbers defined by

$$p = \frac{d}{7\tau_1}, \quad \frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{2} \quad \textit{and} \quad \frac{1}{r} = \frac{1-\theta}{2p} + \frac{\theta}{2}, \tag{9.8}$$

where θ is the solution of the equation

$$\tau_1(1-\theta) + \frac{\theta}{2}\left(\frac{d}{2} + \frac{d-\eta}{2(d-2)}\right) = 1. \tag{9.9}$$

Then

$$(\mathbb{E}(\|Y(1)\|^q_{\mathfrak{S}_q}))^{1/q} \le C_q \left(\frac{|k|}{R}\right)^{d(1-\theta)/p} |k|^{-\theta\nu/2} \|\widetilde{V}\|^2_r, \tag{9.10}$$

for $|k| \ge R$ and $0 < R \le 1$.

Proof. Observe that

$$\tau_1 < \begin{cases} \dfrac{2(\nu-1)(d+1)}{7(d-3)d} \le \dfrac{d}{7} & \text{if } d > 3, \\ \dfrac{8(\nu-1)}{21(2-\eta)} \le \dfrac{d}{7} & \text{if } 8\nu + 9\eta < 26 \text{ and } d = 3. \end{cases}$$

In both cases, τ_1 obeys

$$0 < \tau_1 \le \tfrac{1}{7}d.$$

Consider $Y(\zeta)$ for ζ running over the strip

$$\tau_1 \le \operatorname{Re}\zeta \le \frac{d}{4} + \frac{d-\eta}{4(d-2)}.$$

Since we have some information about the values of this function on the boundary of the strip, we obtain (9.10) by interpolation between Corollary 9.2 and Theorem 9.3. □

Remark. We need to explain why the parameters were selected as described in Theorem 9.4. The work with perturbation determinants requires convergence of integrals of the form

$$\int_\varepsilon^\infty \mathbb{E}(\|Y(1)\|^q_{\mathfrak{S}_q})\, dk, \quad \varepsilon > 0,$$

so we need the parameters to satisfy the condition

$$\frac{qd(1-\theta)}{p} - \frac{q\theta\nu}{2} < -1,$$

which is equivalent to the inequality

$$\tau_1(1-\theta) < \frac{\theta\nu}{14} - \frac{1}{7q} = \frac{\theta(\nu-1)}{14} - \frac{(1-\theta)\tau_1}{d},$$

implying that

$$\tau_1(1-\theta) < \frac{\theta(\nu-1)(d+1)}{14d}.$$

The latter can be written differently as

$$1-\frac{\theta}{2}\left(\frac{d}{2}+\frac{d-\eta}{2(d-2)}\right) < \frac{\theta(\nu-1)(d+1)}{14d}.$$

In other words,

$$2 < \theta\left(\frac{d}{2}+\frac{(\nu-1)(d+1)}{7d}+\frac{d-\eta}{2(d-2)}\right). \tag{9.11}$$

The condition that θ is large can be converted into an inequality showing that τ_1 is small. The relation (9.11) is satisfied if

$$\left(\left(\frac{d}{2}+\frac{(\nu-1)(d+1)}{7d}+\frac{d-\eta}{2(d-2)}\right)-2\right)\tau_1 < \frac{(\nu-1)(d+1)}{7d}.$$

Since $\eta > \nu$, this condition is obviously fulfilled if (9.7) holds.

In the next statement, we estimate the remainder $X_{1,1}(\zeta)$ for $\zeta = 1$.

Theorem 9.5. *Let $p > \frac{3}{4}d \geq 2$, and let $\zeta = 1$. Then*

$$\mathbb{E}[\|X_{1,1}(\zeta)\|^p_{\mathfrak{S}_{p/2}}]^{1/p} \leq C\left(\frac{|k|}{R}\right)^{-4}\|\widetilde{V}\|^2_p.$$

Proof. In this theorem, we deal with the operator

$$W(-\Delta-z)^{-1}P_{1,k}V(-\Delta-z)^{-1}P_{1,k}W.$$

On the one hand, we see that

$$\mathbb{E}[\|(-\Delta-z)^{-2/3}P_{1,k}V(-\Delta-z)^{-2/3}P_{1,k}\|^p_{\mathfrak{S}_p}]^{1/p} \leq C\left(\int_{|\xi|>2|k|/R}||\xi|^2-z|^{-2p/3}\,d\xi\right)^{2/p}\|\widetilde{V}\|_p,$$

which implies the inequality

$$\mathbb{E}[\|(-\Delta-z)^{-2/3}P_{1,k}V(-\Delta-z)^{-2/3}P_{1,k}\|^p_{\mathfrak{S}_p}]^{1/p} \leq C\left(\frac{|k|}{R}\right)^{-8/3}\|\widetilde{V}\|_p, \quad p > \tfrac{3}{4}d.$$

On the other hand,

$$\|W(-\Delta-z)^{-1/3}P_{1,k}\|^2_{\mathfrak{S}_{2p}} \leq C\left(\frac{|k|}{R}\right)^{-4/3}\|\widetilde{V}\|_p, \quad p > \tfrac{3}{4}d.$$

Consequently,

$$\mathbb{E}[\|W(-\Delta-z)^{-1}P_{1,k}V(-\Delta-z)^{-1}P_{1,k}W\|^p_{\mathfrak{S}_{p/2}}]^{1/p} \leq C\left(\frac{|k|}{R}\right)^{-4}\|\widetilde{V}\|^2_p, \quad p > \tfrac{3}{4}d. \qquad \square$$

The next statement follows by Hölder's inequality.

Corollary 9.6. *Let $q > \frac{3}{8}d \geq 1$, and let $\zeta = 1$. Then*

$$\mathbb{E}[\|X_{1,1}(\zeta)\|^q_{\mathfrak{S}_q}]^{1/q} \leq C\left(\frac{|k|}{R}\right)^{-4} \|\widetilde{V}\|^2_{2q}.$$

Surprisingly, q in (9.8) satisfies the inequality $q > \frac{3}{8}d \geq 1$. Thus, we obtain the following result.

Theorem 9.7. *Let $3 \leq d \leq 5$. Assume that τ_1 satisfies (9.7) with η and ν such that $1 < \nu < \eta < 2$. If $d = 3$, then we assume additionally that $8\nu + 9\eta < 26$. Let p, q and r be the numbers defined by*

$$p = \frac{d}{7\tau_1}, \quad \frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{2} \quad \text{and} \quad \frac{1}{r} = \frac{1-\theta}{2p} + \frac{\theta}{2}, \tag{9.12}$$

where θ is the solution of the equation

$$\tau_1(1-\theta) + \frac{\theta}{2}\left(\frac{d}{2} + \frac{d-\eta}{2(d-2)}\right) = 1. \tag{9.13}$$

Then

$$(\mathbb{E}(\|X(k)\|^q_{\mathfrak{S}_q}))^{1/q} \leq C_q\left[\left(\frac{|k|}{R}\right)^{d(1-\theta)/p} |k|^{-\theta\nu/2} + \left(\frac{|k|}{R}\right)^{-4}\right]\|\widetilde{V}\|^2_r,$$

for $|k| \geq R$ and $0 < R \leq 1$.

10. Proof of Theorem 1.5

Again, we work with the function

$$d(z) = \det_n(I - X(k)), \quad n = [q] + 1,$$

where z is related to k via the Joukowski mapping

$$z = \frac{R}{k} + \frac{k}{R}, \quad R > 0.$$

Standard arguments allow us to rewrite (5.2) with p replaced by q as

$$\mathbb{E}\left[\sum_j \frac{\operatorname{Im} k_j(|k_j|^2 - R^2)_+}{|k_j|^2 R}\right] \leq C\left(\int_{-\infty}^{\infty} \mathbb{E}[\|X(k)\|^q_{\mathfrak{S}_q}]\left(\frac{1}{R} - \frac{R}{k^2}\right)_+ dk + \int_0^{\pi} \mathbb{E}[\|X(R\cdot e^{i\theta})\|^q_{\mathfrak{S}_q}]\sin\theta\, d\theta\right),$$

where the k_j are defined as square roots of eigenvalues of H. Due to Theorem 9.7, the latter inequality yields

$$\mathbb{E}\left[\sum_j \frac{\operatorname{Im} k_j(|k_j|^2 - R^2)_+}{|k_j|^2 R}\right] \leq C|R|^{-\theta q\nu/2}\|\widetilde{V}\|^{2q}_r. \tag{10.1}$$

Now, suppose that we consider only the eigenvalues $\lambda_j = k_j^2$ that satisfy the inequality

$$|k_j| \leq R_0.$$

Multiplying (10.1) by $R^{\sigma-1}$ and integrating with respect to R from 0 to R_0, we obtain

$$\mathbb{E}\left[\sum_{|k_j|\leq R_0} \operatorname{Im} k_j |k_j|^{\sigma-1}\right] \leq C|R_0|^{\sigma-\theta q\nu/2}\|\widetilde{V}\|^{2q}_r, \quad \sigma > \tfrac{1}{2}\theta q\nu. \qquad \square$$

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